On the long-time asymptotics of the modified Camassa-Holm equation with step-like initial data

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Abstract

We study the long time asymptotic behavior for the Cauchy problem of the modified Camassa-Holm (mCH) equation with step-like initial data

\[
\begin{align*}
  m_t + (m (u^2 - u_x^2))_x &= 0, \quad m = u - u_{xx}, \\
  u(x, 0) &= u_0(x) \to \begin{cases} A_1, & x \to +\infty, \\
  A_2, & x \to -\infty, \end{cases}
\end{align*}
\]

where \(A_1\) and \(A_2\) are two positive constants. Our main technical tool is the representation of the Cauchy problem with an associated matrix Riemann-Hilbert (RH) problem and the consequent asymptotic analysis of this RH problem. Based on the spectral analysis of the Lax pair associated with the mCH equation and scattering matrix, the solution of the step-like initial problem is characterized via the solution of a RH problem in the new scale \((y, t)\). We adopt double coordinates \((\xi, c)\) to divide the half-plane \(\{(\xi, c) : \xi \in \mathbb{R}, \ c > 0, \ \xi = y/t\}\) into four asymptotic regions. Further using the Deift-Zhou steepest descent method, we derive different long time asymptotic expansion of the solution \(u(y, t)\) in different space-time regions by the different choice of \(g\)-function. The corresponding leading asymptotic approximations are given with the slow/fast decay step-like background wave in genus-0 regions and elliptic waves in genus-2

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regions. The second term of the asymptotics is characterized by Airy function or parabolic cylinder model. Their residual error order is $O(t^{-1})$ or $O(t^{-2})$ respectively.

**Keywords:** Modified Camassa-Holm equation, step-like initial value, Riemann-Hilbert problem, steepest descent method, long time asymptotics, Airy functions, hyperelliptic functions.

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1 Introduction

The Camassa and Holm (CH) equation

\[ m_t + (um)_x + u_x m = 0, \quad m = u - u_{xx} \]

was first introduced by Camassa and Holm in [1] as a model for shallow water waves, but it already appeared earlier in a list by Fuchssteiner and Fokas [2]. The CH equation has attracted considerable interest and been studied extensively due to their rich mathematical structure and remarkable properties, such as peakon solutions, bi-Hamiltonian, algebro-geometric solutions, local and global well-posedness of the Cauchy problem [3–13].

It is observed that all nonlinear terms in the CH equation are quadratic. Over the last few years, various modifications and generalizations of the CH equation have been introduced. For example, Novikov applied the symmetry approach to classify integrable equations of the form

\[ (1 - \partial^2_x)u_t = F(u, u_x, u_{xx}, \cdots) \]

into two integrable CH-type equations with cubic nonlinearity [14]. One is the well-
known mCH equation

\[ m_t + \left( m \left( u^2 - u_x^2 \right) \right)_x = 0, \quad m = u - u_{xx}, \]  

(1.1)

and another one is called the Novikov equation

\[ m_t + (m_x u + 3mu_x) u = 0, \quad m = u - u_{xx}. \]  

(1.2)

In an equivalent form, the mCH equation was given by Fokas [15], Fuchssteiner [16], Olver and Rosenau [17] and Qiao [18], where the equation was derived from the two-dimensional Euler system and the M/W-shape solitons and peakon/cuspon solutions were presented. So the mCH equation (1.1) is also referred to as the Fokas-Olver-Rosenau-Qiao equation [19], but is mostly known as the mCH equation. The mCH equations have non-smooth solitons (called peakons) as solutions [18, 20, 21]. The stability and orbital stability of peakons for the mCH equation were further shown by Qu and Liu [22, 23]. The well-posedness for the Cauchy problem of the mCH equation (1.1) was studied [24–26]. The local well-posedness and the precise blow-up phenomena for the Cauchy problem of the mCH equation were discussed [27, 28]. The wave-breaking and peakons for the mCH equation were investigated by Gui, Liu, Olver and Qu [29]. The algebro-geometric quasiperiodic solutions were constructed by using algebro-geometric method [19]. With the aid of reciprocal transformation, Backlund transformation and nonlinear superposition formula for the mCH equation were given [30]. The local well-posedness for classical solutions and global weak solutions to the mCH equation (1.1) were considered in Lagrangian coordinates [31]. Applying the scaling transformation and taking parameter limit, the mCH equation (1.1) can reduce a short pulse equation [32–34].

Note that the soliton-type solutions of the mCH equation (1.1) vanishing at infinity are weak solutions in the form of peaked waves, which are orbitally stable [22, 23]. On the other hand, adding to the original mCH equation (1.1) a linear dispersion term \( \kappa u_x \) with \( \kappa > 0 \) leads to a form of the mCH equation [35–37]

\[ m_t + \left( m \left( u^2 - u_x^2 \right) \right)_x + \kappa u_x = 0, \quad m = u - u_{xx}, \]  

(1.3)
where $\kappa$ characterizes the effect of the linear dispersion. It can be shown that the mCH equation (1.1) on a nonzero background or the mCH equation (1.3) with decaying initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \; t > 0$$

(1.4)

may support smooth soliton solutions $[37–39]$. The mCH equation (1.3) admits a Lax pair and its smooth dark soliton solutions were obtained by the method of inverse scattering transformation method $[35, 38]$. By using a reciprocal transformation and the Hirota bilinear method, Matsuno obtained the smooth bright multisoliton solutions for the mCH equation (1.3) $[37, 39]$. Boutet de Monvel, Karpenko and Shepelsky first developed a RH approach to deal with the mCH equation (1.3) with nonzero boundary conditions $[40]$. They further present the results of the asymptotic analysis in the solitonless case for the two space-time regions $3/4 < \xi < 1$, $1 < \xi < 3$ with $\xi = y/t$ $[41]$. Xu and Fan applied Deift-Zhou steepest decedent method to obtain long-time asymptotic behavior of (1.3) with Schwartz initial value $[42]$. Very recently we applied the $\bar{\partial}$-steepest decedent method to obtain its long-time asymptotics to the mCH equation (1.3) with weighted Sobolev initial value $[43]$.

The implementation of the rigorous asymptotic analysis to step-like Cauchy problems for integrable equations started in the papers $[45, 46]$, which extended the methods from Deift, Venakides, and Zhou $[47]$. Since then, problems with step-like initial data have also been considered for a variety of integrable systems such as the KdV equation $[48, 49]$, the focusing and defocusing NLS equations $[50–56]$, the modified KdV equation $[57–61]$ and Camassa-Holm equation$[62]$ among many others. A wide range of important physical phenomena manifest themselves in the behavior of solutions of such problems for large times, e.g., collisionless and dispersive shock waves $[63]$, rarefaction waves $[56]$, modulated waves$[64]$, elliptic waves$[52]$ and so on. The main feature in the long-time behavior that distinguishes step-like initial conditions from decaying initial conditions is the formation of an oscillatory region that connects the different behavior at $x \to \pm\infty$ of the solution. These oscillatory regions are typically described by elliptic or hyperelliptic modulated waves.
Very recently, Karpenko, Shepelsky and Teschl develop the RH formalism to the mCH equation (1.1) with step-like initial data

\[ u(x, 0) = u_0(x) \rightarrow \begin{cases} A_1, & x \to +\infty, \\ A_2, & x \to -\infty, \end{cases} \]  

(1.5)

where \( u(x, t) \) sufficiently fast approaches its large-\( x \) limits. A representation for the solution of this problem was given in terms of the solution of an associated RH problem [44].

In this paper, we are interested in the long-time asymptotics of the mCH equation (1.1) with such step-like initial data (1.5). For convenience to study the long time asymptotic behavior of the mCH equation (1.1), we properly deal with this initial value. Without loss of generality, we assume that \( A_1 < A_2 \) in our initial value. Further noting that if \( u(x, t) \) is a solution of the mCH equation (1.1), then for an any constant \( c \neq 0 \), the function \( cu(x, -c^2t) \) also is a solution of the mCH equation (1.1). So we can use an equivalent scale transformation to \( u(x, t) \) such that \( A_2 = 1 \) and \( A_1 = 1/c \) with a constant \( c > 1 \). Finally we consider the following step-like initial value in our paper

\[ u(x, 0) = u_0(x) \rightarrow \begin{cases} 1/c, & x \to +\infty, \\ 1, & x \to -\infty. \end{cases} \]  

(1.6)

In this way, we find that the types of asymptotic expansions for the mCH equation (1.1) are closely related to the scope of two parameter \( \xi = y/t \) and \( c \). So in our paper we adopt double coordinates \( (\xi, c) \) to divide the upper half plane \( \{(\xi, c) : \xi \in \mathbb{R}, c > 1\} \) into four different space-time regions (see Figure 1), in which we will present different leading order asymptotic approximations for the mCH equation (1.1) with step-like initial value (1.6), see Theorem 1 in the section 7.

Our paper is arranged as follows. In Section 2, we study the eigenfunctions and the corresponding spectral functions associated with step-like initial value (1.6). Further we analyze their analyticity, symmetries and asymptotic to construct the RH problem for \( M(z) \) of step-like initial value problem, which will be used to analyze long-time asymptotics of the mCH equation in our paper. In Section 3 and section 4,
\[ \xi = \frac{3}{4} \]
\[ \xi = 1 - \frac{2}{c}(c^2 - 2) \]
\[ \xi = \xi_m \]
\[ \xi = 1 + \frac{2}{c} \]

**Figure 1:** Asymptotic approximations of the mCH equation in different space-time-\((\xi, c)\) regions, where the regions I and II corresponding to genus-0, they are slow-decay and fast-decay background regions, respectively; The Regions III and IV corresponding to genus-2 region, they are the first-type and second-type elliptic wave regions. Here \(\xi_m\) is the critical condition that under the case of Region III, the stationary phase point of \(g\)-function merge \(c\).
we construct the RH problem associated with the Regions I and II, further transform it into a model RH problem. In Sections 5 and 6, to analyze the RH problem in the regions III and IV, we introduce a $g$-function in genus two Riemann surface and transform the original RH problem to a hybrid RH problem $M^{(2)}(z)$, which is further decompose into a $M^{mod}(z)$ model problem and an inner local problems. The $M^{mod}(z)$ contributes to the leading term of the asymptotics and is given by Riemann theta functions attached to a hyperelliptic Riemann surface in subsection 5.2.1 and subsection 6.3 in different region. Finally, in Section 7, based on a series of transformations above, a decomposition formula for $M(z)$ from which we then obtain the long-time asymptotic behavior for the solutions of the Cauchy problem of the mCH equation (1.1) and (1.6) via a reconstruction formula. The main result is summarized in the Theorem 1.

2 Direct scattering and the RH problem

2.1 Spectral analysis on the Lax pair

The mCH equation (1.1) admits the Lax pair [40]

$$\Phi_x = X\Phi, \quad \Phi_t = T\Phi,$$

where

$$X = \frac{1}{2} \begin{pmatrix} -1 & zm \\ -zm & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} z^{-2} + \frac{u^2-u_x^2}{2} & -z^{-1}(u - u_x) - \frac{1}{2}(u^2 - u_x^2)m \\ z^{-1}(u - u_x) + \frac{1}{2}(u^2 - u_x^2)m & -z^{-2} - \frac{u^2-u_x^2}{2} \end{pmatrix}.$$

Since the Lax pair (2.1) admit spectral singularity at $z = \infty$ and $z = 0$, we should control the asymptotic behavior of the eigenfunction $\Phi$ as $z \to \infty$ and $z \to 0$ for any real constant $c$.

**Case I.** $z = \infty$.

We denote a matrix function relying on $c$

$$D_c(z) = \frac{1}{2} \begin{pmatrix} \phi_c(z) + \phi_c(z)^{-1} & \phi_c(z)^{-1} - \phi_c(z) \\ \phi_c(z)^{-1} - \phi_c(z) & \phi_c(z) + \phi_c(z)^{-1} \end{pmatrix},$$

(2.2)
where $\phi_c(z)$ is a branch function given by
\[
\phi_c(z) = \left( \frac{c+z}{c-z} \right)^{1/4} \sim e^{\frac{3i\pi}{4}} + \mathcal{O}(z^{-1}), \quad z \to \infty.
\]
Obviously,
\[
D_c(z+0i) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} D_c(z-0i), \quad z \in [-c,c],
\]
and $\phi_c(z+0i) = i\phi_c(z-0i)$. We define two gauge transformations
\[
\Psi^\pm(z) = D_{c_\pm}(z) \Phi(z),
\]
where $c_+ = c$, $c_- = 1$, the $\Psi^\pm(z)$ satisfy the following Lax pair
\begin{align}
\Psi_x^\pm &= -\frac{im}{2} \sqrt{z^2 - c_\pm^2} \sigma_3 \Psi^\pm + P^\pm \Psi^\pm, \quad (2.4) \\
\Psi_t^\pm &= i \sqrt{z^2 - c_\pm^2} \left( \frac{m(u^2 - u_x^2)}{2} + \frac{1}{c_\pm z} \right) \sigma_3 \Psi^\pm + L^\pm \Psi^\pm, \quad (2.5)
\end{align}
where
\[
P^\pm = i \frac{c_\pm m - 1}{2} \begin{pmatrix} c_\pm & z \\ -z & -c_\pm \end{pmatrix},
\]
\[
L^\pm = i \left( \frac{c_\pm(u^2 - u_x^2)(1-c_\pm m)}{2\sqrt{z^2 - c_\pm^2}} - \frac{u - 1/c_\pm}{\sqrt{z^2 - c_\pm^2}} \right) \sigma_3 + \frac{u_x}{c_\pm} \sigma_1
\]
\[
+ i \left( \frac{z(u^2 - u_x^2)(1-c_\pm m)}{2\sqrt{z^2 - c_\pm^2}} - \frac{c_\pm u - 1}{z\sqrt{z^2 - c_\pm^2}} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Further we introduce a transformation
\[
\mu^\pm(z) = \Psi^\pm(z)e^{itp_\pm(z)}\sigma_3,
\]
where
\[
tp_\pm(z) = \frac{\sqrt{z^2 - c_\pm^2}}{2} \left( \int_{+\infty}^x \left( m(s) - 1/c_\pm \right) ds + \frac{x}{c_\pm} - \frac{2t}{c_\pm z^2} + \frac{t}{c_\pm^2} \right). \quad (2.7)
\]
Then \( \mu^\pm(\mathbf{z}) \) admit asymptotics

\[
\mu^\pm(z) \sim I, \quad x \to \pm \infty,
\]

and satisfy a new Lax pair

\[
\begin{align*}
\mu^\pm_x &= -i \partial_{x}(tp^\pm)[\sigma_3, \mu^\pm] + P^\pm \mu^\pm, \\
\mu^\pm_t &= -i \partial_{t}(tp^\pm)[\sigma_3, \mu^\pm] + L^\pm \mu^\pm.
\end{align*}
\]

The above Lax pair can be written as a full derivative form, which is further integrated along \((\pm \infty, t) \to (x, t)\) and leads to two Volterra type integrals

\[
\mu^\pm(x, t; z) = I + \int_{\pm \infty}^{x} e^{i \sqrt{z^2-c_2^2 f_x m(v) dv}} [P^\pm \mu^\pm(s, t; z)] ds.
\]

Denote

\[
\Sigma_- = [-1, 1], \quad \Sigma_+ = [-c, c],
\]

then we can show that \( \mu^\pm(x, t; z) \) is analytical in \( \mathbb{C} \setminus \Sigma \pm \) respectively.

**Proposition 1.** The Jost functions \( \mu^\pm(z) \) admit two kinds of symmetries

\[
\mu^\pm(z) = \sigma_1 \mu^\pm(\bar{z})\sigma_1 = \sigma_2 \mu^\pm(-z)\sigma_2.
\]

Since \( D_{c^\pm}(z)^{-1}\Psi^\pm(z; x, t) \) are two fundamental matrix solutions of the Lax pair \((2.1)\), they satisfy a linear relation

\[
D_{c_+}(z)^{-1}\Psi^+(z; x, t) = D_{c_-}(z)^{-1}\Psi^-(z; x, t) S(z),
\]

where \( S(z) \) is a scattering matrix

\[
S(z) = \begin{pmatrix}
    s_{11}(z) & s_{12}(z) \\
    s_{21}(z) & s_{22}(z)
\end{pmatrix}, \quad \det S(z) = 1.
\]

Combing the transformation \((2.6)\) with the equation \((2.12)\) gives

\[
S(z) = e^{itp_+ \sigma_3(\mu^-)^{-1}D_{c_-}D_{c^+_1}^{-1}e^{-itp_+ \sigma_3}},
\]
which is analytical on $\mathbb{C} \setminus (\Sigma_+ \cup \Sigma_-)$. Defining two reflection coefficients by

\[ r_1(z) = \frac{s_{21}(z)}{s_{11}(z)}, \quad r_2(z) = \frac{s_{12}(z)}{s_{22}(z)}. \tag{2.14} \]

Let

\[ \tilde{\mu}^\pm(z) = D_{c_\pm}^{-1} \mu^\pm(z), \tag{2.15} \]

then the Volterra type integrals (2.10) are changed into

\[
\begin{aligned}
\tilde{\mu}^\pm(z) &= D_{c_\pm}^{-1} + \int_{\pm \infty}^z F(z) \left( X + \frac{mi \sqrt{z^2 - c_\pm^2}}{2} D_{c_\pm}^{-1} \sigma_3 D_{c_\pm} \right) \tilde{\mu}^\pm e^{-\frac{i}{2} \sigma_3 \sqrt{z^2 - c_\pm^2} \int_{\pm}^s m(\ell) d\ell} d\sigma_3 D_{c_\pm},
\end{aligned}
\tag{2.16}
\]

where

\[ F(z) = D_{c_\pm}^{-1} e^{\frac{i}{2} \sigma_3 \sqrt{z^2 - c_\pm^2} \int_{\pm}^s m(\ell) d\ell} D_{c_\pm} \]

is a analytical function on $\mathbb{C}$. Thus we give the following proposition

**Proposition 2.** The Jost functions $\tilde{\mu}^\pm(z)$ and the scattering matrix $S(z)$ have $-\frac{1}{4}$-weak singularity at $z = \pm 1$ and $z = \pm c$.

**Corollary 1.** We have asymptotics

\[ 1 - r_1(z)r_2(z) = O((z \pm c)^{1/2}), \quad z \to \pm c. \]

To construct the RH problem, we need to consider the jump of the Jost functions $\tilde{\mu}^\pm(z)$ on the cut $\Sigma_\pm$.

**Proposition 3.** The Jost functions $\mu^\pm(z)$ admit the jump relations

(i) For $z \in \Sigma_\pm$,

\[ \mu^\pm(z + 0i) = \sigma_1 \mu^\pm(z - 0i) \sigma_1. \tag{2.17} \]

(ii) Especially for $z \in [-1, 1]$,

\[ \tilde{\mu}^\pm_{11}(z + 0i) = i\tilde{\mu}^\pm_{12}(z - 0i), \quad \tilde{\mu}^\pm_{21}(z + 0i) = i\tilde{\mu}^\pm_{22}(z - 0i), \]

\[ s_{11}(z + 0i) = s_{22}(z - 0i), \quad s_{12}(z + 0i) = s_{21}(z - 0i). \]
(iii) For \( z \in [-c, c] \setminus [-1, 1] \), \( \tilde{\mu}^+(z) \) has same jump as above equation while \( \tilde{\mu}^-(z) \) has no jump. And
\[
s_{11}(z + 0i) = is_{12}(z - 0i), \quad s_{21}(z + 0i) = is_{22}(z - 0i). \tag{2.18}
\]

From (2.13), we have
\[
S(z) \sim e^{\frac{1}{2}H z}\sigma_3, \quad z \to \infty,
\tag{2.19}
\]
where \( H \) is conserved quantity with
\[
H = (1 - 1/c)x + (1/c^3 - 1)t + \int_{x}^{\infty} (m - 1)ds + \int_{x}^{\infty} (m - 1/c)ds.
\]

The zeros of \( s_{11}(z) \) and \( s_{22}(z) \) on \( \mathbb{C} \) are known to occur and they correspond to spectral singularities. They are excluded from our analysis in this paper. Thus, \( r_1(z) \) and \( r_2(z) \) are analytic in \( \mathbb{C} \setminus (\Sigma_- \cup \Sigma_+) \).

**Case II: \( z = 0 \).**

We rewrite the Lax pair (2.4)-(2.5) in the form
\[
\Psi_\pm^x = -\frac{i}{2c_\pm} \sigma_3 \Psi_\pm + P_0^\pm \Psi_\pm, \tag{2.20}
\]
\[
\Psi_\pm^t = i \sqrt{z^2 - c^2_\pm} \left( \frac{1}{2c^3_\pm} + \frac{1}{c_\pm z^2} \right) \sigma_3 \Psi_\pm + L_0^\pm \Psi_\pm, \tag{2.21}
\]
where
\[
P_0^\pm = iz \frac{c_\pm m - 1}{2c_\pm \sqrt{z^2 - c^2_\pm}} \begin{pmatrix} z & c_\pm \\ -c_\pm & -z \end{pmatrix},
\]
\[
L_0^\pm = L_\pm + i \sqrt{z^2 - c^2_\pm} \left( \frac{m(u_2^2 - u_1^2)}{2} + \frac{1}{c_\pm z} - \frac{1}{2c^3_\pm} + \frac{1}{c_\pm z^2} \right) \sigma_3.
\]

Making transformation
\[
\mu^\pm_0(z) = \Psi_\pm e^{iq^\pm_3}, \quad q^\pm = \frac{i}{2c_\pm} \left[ x - \left( \frac{1}{c_\pm^2} + \frac{2}{z^2} \right) t \right], \tag{2.22}
\]

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then $\mu_0^\pm(z)$ admit a new Lax pair

\[
\mu_{0,x}^\pm(z) = -\frac{i}{2c_\pm} \sqrt{z^2 - c_\pm^2} \left[ \sigma_3, \mu_0^\pm(z) \right] + P_0^\pm \mu_0^\pm(z),
\]

\[
\mu_{0,t}^\pm(z) = i \sqrt{z^2 - c_\pm^2} \left( \frac{1}{2c_\pm^2} + \frac{1}{c_\pm z^2} \right) [\sigma_3, \mu_0^\pm(z)] + L_0^\pm \mu_0^\pm(z).
\]

(2.23)

(2.24)

It also can be written in to two Volterra type integrals

\[
\mu_0^\pm(z) = I + \int_{\pm\infty}^x e^{\frac{i}{c_\pm \sigma_3} \sqrt{z^2 - c_\pm^2}(s-x)} \left[ P_0^\pm \mu_0^\pm(s;z) \right] ds.
\]

(2.25)

To reconstruct the potential $u(x,t)$, we take $z = 0$, then

\[
P_0^\pm(0) = 0, \quad \mu_0^\pm(0) = I.
\]

(2.26)

Expanding $\mu_0^\pm(z)$ at $z = 0$ gives

\[
\mu_0^\pm(z) = I + \frac{z}{2} \left( \int_{\pm\infty}^x (m - \frac{1}{c_\pm}) e^{s-x} ds - \int_{\pm\infty}^x (m - \frac{1}{c_\pm}) e^{s-x} ds \right) + O(z^2).
\]

Because $\Psi^\pm(z)$ admit same Lax pair (2.4)-(2.5), there exist two matrix function

\[
C_\pm(z) \text{ independent of } x \text{ and } t
\]

\[
\mu_0^\pm(z) e^{-i\varepsilon^\pm \sigma_3} C_\pm(z) = \mu^\pm(z) e^{-it\varepsilon^\pm \sigma_3}.
\]

(2.27)

Since

\[
q_\pm - tp_\pm = -\frac{1}{2} \sqrt{z^2 - c_\pm^2} \int_{\pm\infty}^x (m - 1/c_\pm) ds,
\]

taking the limits $x \to \pm\infty$, we obtain $C_\pm(z) \equiv I$. Then

\[
\mu^\pm(0+0i) = \exp \{ i(q_\pm(0+0i) - tp_\pm(0+0i))\sigma_3 \},
\]

(2.28)

which combines with $D_{\varepsilon}(0+0i) = -i\sigma_1$ implies that

\[
\tilde{\mu}^\pm(0+0i) = \left( \begin{array}{cc} 0 & i e^{-i(q_\pm(0+0i) - tp_\pm(0+0i))} \\ ie^{i(q_\pm(0+0i) - tp_\pm(0+0i))} & 0 \end{array} \right).
\]

(2.29)

Consequently,

\[
S(0+0i) = \exp \{ i(q_-(0+0i) - q_+(0+0i))\sigma_3 \}.
\]

(2.30)
2.2 Setting up a RH problem with step-like initial data

Define a sectionally analytical matrix

\[
M(z) \triangleq M(z; x, t) = \begin{cases} 
\left( \frac{\tilde{\mu}^-_1(z)}{s_{11}(z)} e^{-it(p_+ - p_-)} , \tilde{\mu}^+_2(z) \right), & \text{as } z \in \mathbb{C}^+, \\
\left( \tilde{\mu}^+_1(z) , \frac{\tilde{\mu}^-_2(z)}{s_{22}(z)} e^{it(p_+ - p_-)} \right), & \text{as } z \in \mathbb{C}^-, 
\end{cases}
\]  \tag{2.31}

where \( \tilde{\mu}^\pm_1(z) \) and \( \tilde{\mu}^\pm_2(z) \) denote the first and second column of \( \tilde{\mu}^\pm(z) \), respectively.

Then \( M(z) \) solves the following RH problem.

RHP 1. Find a matrix-valued function \( M(z) = M(z; x, t) \) which satisfies

- Analyticity: \( M(z) \) is analytical in \( \mathbb{C} \setminus \mathbb{R} \);
- Symmetry: \( M(z) = \sigma_2 M(-z) \sigma_2 = \sigma_1 M(\bar{z}) \sigma_1 \);
- Jump condition: \( M \) has continuous boundary values \( M_\pm(z) \) on \( \mathbb{R} \) and

\[
M_+(z) = M_-(z) \tilde{V}(z), \quad z \in \mathbb{R},
\]  \tag{2.32}

where

\[
\tilde{V}(z) = \begin{cases} 
\begin{pmatrix} 
1 - r_1 r_2 & -r_2 e^{-2itp_-} \\
r_1 e^{2itp_-} & 1 
\end{pmatrix}, & \text{as } z \in \mathbb{R} \setminus [-c, c], \\
\begin{pmatrix} 
0 & -r_2 (z - 0i) e^{-2itp_-} \\
r_1 (z + 0i) e^{2itp_-} & 1 
\end{pmatrix}, & \text{as } z \in [-c, c] \setminus [-1, 1], \\
\begin{pmatrix} 
0 & i \\
i & 0 
\end{pmatrix}, & \text{as } z \in [-1, 1], 
\end{cases}
\]

- Asymptotic behaviors

\[
M(z) = I + O(z^{-1}), \quad z \to \infty;
\]  \tag{2.33}

- Singularity: \( M(z) \) has singularity at \( z = \pm 1 \) with

\[
M(z) \sim \begin{cases} 
O(z \mp 1)^{1/4}, & \text{as } z \to \pm 1 \text{ in } \mathbb{C}^+, \\
O(z \mp 1)^{-1/4}, & \text{as } z \to \pm 1 \text{ in } \mathbb{C}^-,
\end{cases}
\]  \tag{2.34}

\[
M(z) \sim \begin{cases} 
O(z \mp 1)^{-1/4}, & \text{as } z \to \pm 1 \text{ in } \mathbb{C}^+,
\end{cases}
\]  \tag{2.35}
The solution of mCH equation (1.1) is difficult to reconstruct from the above RHP1, since \( p_-(x, t, z) \) is still unknown. Boutet de Monvel and Shepelsky proposed an idea to change the spatial variable \( x \) to variable \( y \) \[12, 13\]. Following this idea, we introduce a new scale

\[ y(x, t) = x - \int_x^{-\infty} (m(s) - 1) \, ds. \] (2.36)

The price to pay for this is that the solution of the initial problem can be given only implicitly or perimetrically. It will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale. By the definition of the new scale \( y(x, t) \), we define

\[ N(z) = N(z; y, t) \triangleq M(z; x(y, t), t). \] (2.37)

For convenience we denote \( \xi = y/t \) with

\[ p_- = \frac{\sqrt{z^2 - 1}}{2} (\xi - 1 - 2z^{-2}), \]

then we can get the RH problem for the new variable \((y, t)\).

**RHP 2.** Find a matrix-valued function \( N(z) \triangleq N(z; y, t) \) which satisfies

\[ \begin{align*}
\text{Analyticity: } & N(z) \text{ is meromorphic in } \mathbb{C} \setminus \mathbb{R}; \\
\text{Symmetry: } & N(z) = \sigma_2 N(-z) \sigma_2 = \sigma_1 N(z) \sigma_1; \\
\text{Jump condition: } & N \text{ has continuous boundary values } N_\pm(z) \text{ on } \mathbb{R} \text{ and}
\end{align*} \]

\[ N_+(z) = N_-(z) \tilde{V}(z), \quad z \in \mathbb{R}, \] (2.38)

where

\[ \tilde{V}(z) = \begin{cases} 
\begin{pmatrix} 1 - r_1 r_2 & -r_2 e^{-2itp_-} \\
r_1 e^{2itp_-} & 1 \end{pmatrix}, & \text{as } z \in \mathbb{R} \setminus [-c, c], \\
\begin{pmatrix} 0 & -r_2(z - 0i) e^{-2itp_-} \\
r_1(z + 0i) e^{2itp_-} & 1 \end{pmatrix}, & \text{as } z \in [-c, c] \setminus [-1, 1], \\
\begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix}, & \text{as } z \in [-1, 1].
\end{cases} \]
Asymptotic behaviors: \( N(z) = I + \mathcal{O}(z^{-1}), \; z \to \infty; \)

Singularity: \( N(z) \) has singularity at \( z = \pm 1 \) with

\[
N(z) \sim \left( \mathcal{O}(z \mp 1), \mathcal{O}(z \mp 1)^{-1/4} \right), \; z \to \pm 1 \text{ in } \mathbb{C}^+, \quad (2.39)
\]

\[
N(z) \sim \left( \mathcal{O}(z \mp 1)^{-1/4}, \mathcal{O}(z \mp 1)^{1/4} \right), \; z \to \pm 1 \text{ in } \mathbb{C}^-.
\quad (2.40)
\]

From the asymptotic behavior of the functions \( \tilde{\mu}^\pm (z) \) and (2.30), we have

\[
N(z) = \begin{pmatrix}
0 & i f_1 \\
if_1 & 0
\end{pmatrix} + iz \begin{pmatrix} f_2 & 0 \\
0 & f_3 \end{pmatrix} + \mathcal{O}(z^2), \; z \to 0 \in \mathbb{C}^+, \quad (2.41)
\]

where

\[
f_1 = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{x} (m-1) ds \right\},
\]

\[
f_2 = \frac{e^{\frac{i}{2} \int_{-\infty}^{x} (m-1) ds}}{2c} \left( \int_{-\infty}^{x} (cm-1) e^{-s} ds + e^{\frac{i}{2} \int_{-\infty}^{x} (cm-1) ds} \right),
\]

\[
f_3 = \frac{e^{-\frac{i}{2} \int_{-\infty}^{x} (m-1) ds}}{2} \left( -\int_{-\infty}^{x} (m-1) e^{-s} ds + e^{-\frac{i}{2} \int_{-\infty}^{x} (m-1) ds} \right).
\]

Thus,

\[
x(y, t) = y - 2 \ln (-i M_{21}(0 + 0 i)). \quad (2.42)
\]

Combining with (1.1), we arrive at following reconstruction formula

\[
u(x, t) = f_1(x, t) f_2(x, t) + f_1(x, t)^{-1} f_3(x, t), \quad (2.43)
\]

\[
\partial_x u(x, t) = -f_1(x, t) f_2(x, t) + f_1(x, t)^{-1} f_3(x, t), \quad (2.44)
\]

namely,

\[
u(x, t) = -N_{12}(0 + 0 i) \lim_{z \to 0} \frac{N_{11}(z) - N_{11}(0)}{z}
\]

\[
- N_{12}(0 + 0 i)^{-1} \lim_{z \to 0} \frac{N_{22}(z) - N_{22}(0)}{z}, \; z \in \mathbb{C}^+, \quad (2.45)
\]

\[
\partial_x u(x, t) = N_{12}(0 + 0 i) \lim_{z \to 0} \frac{N_{11}(z) - N_{11}(0)}{z}
\]

\[
- N_{12}(0 + 0 i)^{-1} \lim_{z \to 0} \frac{N_{22}(z) - N_{22}(0)}{z}, \; z \in \mathbb{C}^+. \quad (2.46)
\]
Moreover, the jump matrix $\tilde{V}(z)$ admits the following decomposition

On the interval $\mathbb{R} \setminus [-c, c]$,

$$
\tilde{V}(z) = \begin{pmatrix}
1 & -r_2 e^{-2itp_-} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1 & -r_2 e^{-2itp_-} \\
1 & 1
\end{pmatrix} \sigma_3 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
$$

(2.47)

On the interval $[-c, c] \setminus [-1, 1]$,

$$
\tilde{V}(z) = \begin{pmatrix}
1 & -r_2 (z - 0i) e^{-2itp_-} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r_1 (z + 0i) e^{-2itp_-} & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
r_1 (z - 0i) e^{2itp_-} & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r_1 (z + 0i) e^{-2itp_-} & 1
\end{pmatrix} \sigma_3 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
$$

The long-time asymptotics of RHP 2 is affected by the growth or decay of the exponential function $e^{\pm 2itp_-}$ with

$$
p_- = \frac{\sqrt{z^2 - 1}}{2} (\xi - 1 - 2z^{-2}), \quad dp_- = \frac{1}{2\sqrt{z^2 - 1}z^3} \left[ (\xi - 1)z^4 + 2z^2 - 4 \right].
$$

So to analyze the long-time asymptotics, we need control the real part of $2itp_-$. But we find that in some cases, this exponential function is not applicable. Its adaptation to problems with nonzero background has required the development of the g-function mechanism [47]. This mechanism is relevant when some entries of the jump matrix grow exponentially or oscillate as $t \to \infty$. The general idea consists in replacing the original phase function in the jump matrix. This new function $g(\xi, z)$ need to be analytic on $\mathbb{C}$ except cut, and satisfies that, after appropriately choosing triangular factorizations of the jump matrices and associated deformations of the original RH problem, the jumps containing exponentially growing entries, become constant matrices (independent of $z$, but dependent on $\xi$) of special structure whereas the other jumps decay exponentially to the identity matrix. And it must have same asymptotic properties as $z \to \infty$, $0 \in \mathbb{C}^+$ as $p_-:

$$
p_- = \frac{1 - \xi}{2} z + O(z^{-1}), \quad \frac{d}{dz}p_- = \frac{1 - \xi}{2} + O(z^{-2}), \quad z \to \infty;
$$

(2.48)

$$
p_- = \frac{i}{z^2} - \frac{i\xi}{2} + O(z), \quad \frac{d}{dz}p_- = \frac{-2i}{z^3} + O(z), \quad z \to 0 \in \mathbb{C}^+.
$$

(2.49)
The structure of the limiting RH problem is such that the problem can be solved explicitly in terms of Riemann theta functions and Abel integrals on Riemann surfaces associated with the limiting RH problem. For different ranges of the parameter $\xi = y/t$, different Riemann surfaces may appear [46, 50–54]. To find the g-function, we first consider the signature table and stationary phase points of $p_-$ in Figure 2.

(a): For the case $\xi < \frac{3}{4}$, there is no stationary point on $\mathbb{R}$;

(b): For the case $\frac{3}{4} < \xi < 1$ there are four stationary points on $\mathbb{R}$

$$\pm \lambda_1 = \pm \left(\frac{1 - \sqrt{4\xi - 3}}{1 - \xi}\right)^{1/2}, \quad \pm \lambda_2 = \pm \left(\frac{1 + \sqrt{4\xi - 3}}{1 - \xi}\right)^{1/2}. \quad (2.50)$$

(c): For the case $1 < \xi < 3$, there are two stationary points on $\mathbb{R}$

$$\pm \lambda_1 = \pm \left(\frac{\sqrt{4\xi - 3} - 1}{\xi - 1}\right)^{1/2}. \quad (2.51)$$

(d): For the case $3 < \xi$, there are two stationary points on $\mathbb{R}$

$$\pm \lambda_1 = \pm \sqrt{2(\xi - 1)^{-1/2}}. \quad (2.52)$$

These figures mean that in the case of subfigure (a), we can deal with the jump reserve $p_-$. But in the other region of $\xi$, the applicability of $p_-$ need to discussed. In the following section, we will study different g-functions respectively.

3 Slow-decay background region

In this section, we will discuss the applicability of $p_-$ in different region of $\xi$. The Region I contains the following three different cases:

(i) $\xi < \frac{3}{4}$; (ii) $1 < c \leq \lambda_1$, $\frac{3}{4} < \xi < 1$; (iii) $1 < c \leq \lambda_1$, $1 < \xi$,

where $\lambda_1 = \sqrt{\frac{1}{1 - \xi} - \sqrt{(\frac{1}{1 - \xi} - 4)\frac{1}{1 - \xi}}}$. As shown in Figure 2, in the case (i), the jump on $\mathbb{R} \setminus [-1, 1]$ is easy to deal with. Specially in the case (ii) and case (iii), we have that $c < \lambda_1$. 

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(i) The case $\xi < \frac{3}{4}$.

There has no stationary point, we define

$$\Sigma^\pm(\xi) = e^{\pm i \psi} \mathbb{R}^+ \cup e^{i(\pi \mp \psi)} \mathbb{R}^+, \quad \Omega = \mathbb{C} \setminus (\Omega^+ \cup \Omega^-),$$

$$\Omega^\pm(\xi) = \left\{ z; z = e^{\pm \psi} l, \ l \in \mathbb{R}, \ 0 < \phi < \psi \right\} \cup \left\{ z; z = e^{\pm \psi} l, \ l \in \mathbb{R}, \ \pi \psi < \phi < \pi \right\},$$

where $\psi$ is a small enough positive angle such that it is non-intersect with the curve $\text{Im}p_-(z) = 0$.

(ii) The case $1 < c \leq \lambda_1$, $\frac{3}{4} < \xi < 1$. 

Figure 2: In the white region, $\text{Im}p_- < 0$, while in another region, $\text{Im}p_- < 0$.
Let curve \( \Sigma_j^\pm, j = 1, 2 \) as Figure 4 shown. Near the point \((0, 0)\) \(\Sigma_1, \Sigma_2\) are the same as the case \(\xi < \frac{3}{4}\).

![Figure 4](image-url)

**Figure 4:** Figure of \(\Sigma^{(1)}\) in the case of \(1 < c \leq \lambda_1, \frac{3}{4} < \xi < 1\).

(iii) The case \(1 < c \leq \lambda_1, 1 < \xi\).

Let the curve \(\Sigma_j^\pm, j = 1, 2\) as Figure 5 showing. The only difference from the case (ii) is there is only two phase point \(\pm \lambda_1\).

![Figure 5](image-url)

**Figure 5:** Figure of \(\Sigma^{(1)}\) in the case of \(1 < c \leq \lambda_1, 1 < \xi\).

Moreover, in this region depending on \(\xi, c\), we introduce a piecewise matrix
interpolation function

\[
G(z; \xi, c) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -r_1 e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^+; \\
\begin{pmatrix} 1 & -r_2 e^{-2itp} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^-; \\
\begin{pmatrix} 1 & r_2 e^{-2itp} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^+; \\
\begin{pmatrix} 0 & 1 \\ r_1 e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^-; \\
I & \text{as } z \text{ in elsewhere,}
\end{cases}
\]

(3.1)

To deal with the jump on \( \mathbb{R} \), we denote a interval

\[
I(\xi) = \begin{cases} 
\emptyset, & \text{as } \xi < \frac{3}{4}; \\
[-\lambda_2, -\lambda_1] \cup [\lambda_1, \lambda_2], & \text{as } \frac{3}{4} < \xi < 1; \\
(-\infty, -\lambda_1] \cup [\lambda_1, +\infty) & \text{as } \xi > 1;
\end{cases}
\]

(3.2)

and introduce an auxiliary function \( \delta(z) = \delta(z; \xi, c) \), which satisfies the following scalar RH problem.

(a) \( \delta(z) \) is analytic on \( \mathbb{C} \setminus I(\xi) \);

(b) \( \delta_-(z) = \delta_+(z)(1 - r_1 r_2) \), \( z \in I(\xi) \), \( \delta_+(z) = \delta_+(z) \), \( z \in \mathbb{R} \setminus I(\xi) \);

(c) \( \delta(z) \to 1 \), as \( z \to \infty \in \mathbb{C} \setminus I(\xi) \).

The solution of the RH problem can given by

\[
\log \delta(z) = -\frac{1}{2\pi i} \int_{I(\xi)} \frac{\log(1 - r_1(s) r_2(s))}{s - z} ds.
\]

(3.3)

which has the following properties

\[
\delta(z) = \exp \{ I_0^1 \} \cdot (1 + z I_0^2) + \mathcal{O}(z^2), \ z \to 0 \in \mathbb{C}^+,
\]

where

\[
I_0^1 = -\frac{c}{2\pi} \int_{I(\xi)} \frac{\log(1 - r_1(s) r_2(s))}{s} ds,
\]

(3.4)

\[
I_0^2 = -\frac{c}{2\pi} \int_{I(\xi)} \frac{\log(1 - r_1(s) r_2(s))}{s^2} ds.
\]

(3.5)
Define a new matrix-valued function

\[ M^{(1)}(z; \xi, c) \triangleq N(z)G(z; \xi, c)\delta^{\sigma_3}, \]  

which then satisfies the following RH problem.

**RHP 3.** Find a matrix-valued function \( M^{(1)}(z) = M^{(1)}(z; \xi, c) \) which satisfies

- **Analyticity:** \( M^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \), where
  \[ \Sigma^{(1)}(\xi) = (\cup_{j=1}^{4}\Sigma_j(\xi)) \cup [-1, 1]; \]  

- **Symmetry:** \( M^{(1)}(z) = \sigma_2 M^{(1)}(-z) \sigma_2 = \sigma_1 \overline{M^{(1)}(\bar{z})} \sigma_1; \)

- **Jump condition:** \( M^{(1)} \) has continuous boundary values \( M^{(1)}_{\pm}(z) \) on \( \Sigma^{(1)}(\xi) \) and
  \[ M^{(1)}_{+}(z) = M^{(1)}_{-}(z)V^{(1)}(z), \quad z \in \Sigma^{(1)}; \]  

where

\[
V^{(1)}(z) = \begin{cases}
  \begin{pmatrix} 1 & 0 \\ r_1\delta^2e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \Sigma^+_1; \\
  \begin{pmatrix} 1 & -r_2\delta^{-2}e^{-2itp} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Sigma^-_1; \\
  \begin{pmatrix} 1 & -r_2\delta^{-2}e^{-2itp} \\ 0 & 1 - r_1r_2 \end{pmatrix}, & \text{as } z \in \Sigma^+_2; \\
  \begin{pmatrix} 1 & 0 \\ r_1\delta^2e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \Sigma^-_2; \\
  \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & z \in [-1, 1].
\end{cases}
\]  

- **Asymptotic behaviors:** \( M^{(1)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty; \)

- **Singularity:** \( M^{(1)}(z) \) has singularity at \( z = \pm 1 \) with
  \[ M^{(1)}(z) \sim \mathcal{O}(z \mp 1)^{-1/4}, \quad z \to \pm 1 \text{ in } \mathbb{C} \setminus \Sigma^{(1)}. \]  

We construct the solution \( M^{(1)}(z) \) as follow

\[
M^{(1)}(z) = \begin{cases}
  E(z; \xi, c)M^{\text{mod1}}(z; \xi, c), & z \notin U_{\pm\lambda_1} \cup U_{\pm\lambda_2}, \\
  E(z; \xi, c)M^{1, \pm}(z; \xi, c), & z \in U_{\pm\lambda_1}, \\
  E(z; \xi, c)M^{2, \pm}(z; \xi, c), & z \in U_{\pm\lambda_2},
\end{cases}
\]  

(3.11)
where we denote $U_{\pm \lambda_j}$ as the neighborhood of $\pm \lambda_j$
\[
U_{\pm \lambda_j} = \{ z : |z \pm \lambda_j| \leq \varrho \}. \quad (3.12)
\]
Here, $\varrho$ is a small positive constant such that $\varrho < \min \left\{ \frac{\zeta_j - 1}{3}, \frac{\zeta_j - c}{3}, \varepsilon \right\}$. In the case (i), the jump matrix exponentially decays to the identity matrix $I$ as $t \to \infty$ on $\Sigma_1$ and $\Sigma_2$ because of the absence of phase point, namely, $U(\pm \lambda_j) = \emptyset$. Although $z = 0$ is a pole of $p_-$, the exponential function decays to 0 at a speed of $e^{-C(\xi,c)t/|z|^2}$. So this case is trivial. Then the existence and uniqueness of $E(z;\xi,c)$ can be shown by a small-norm RH problem [65, 66] with
\[
E(z) = I + \mathcal{O}(e^{-C(\xi,c)t}) \quad (3.13)
\]
with $C(\xi,c)$ being a positive constant rely on $\xi$ and $c$. However, in the case(ii) and (iii), the phase points have contribution on $t \to \infty$. The jump matrix exponentially decaying to the identity matrix $I$ as $t \to \infty$ away from phase points.

3.1 A model RH problem on cuts

The jump matrix exponentially decays to the identity matrix $I$ as $t \to \infty$ on $\Sigma_1^\pm$, which finally leads to the model RH problem:

**RHP 4.** Find a matrix-valued function $M^{\text{mod1}}(z)$ which satisfies

- **Analyticity:** $M^{\text{mod1}}(z)$ is holomorphic in $\mathbb{C} \setminus [-1,1]$;
- **Jump condition:** $M^{\text{mod1}}$ has continuous boundary values $M^{\text{mod1}}_\pm(z)$ on $[-1,1]$ and
\[
M^{\text{mod1}}_+(z) = M^{\text{mod1}}_-(z)V^{\text{mod1}}(z), \quad z \in [-1,1], \quad (3.14)
\]
where
\[
V^{\text{mod1}}(z) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad z \in [-1,1]; \quad (3.15)
\]
- **Asymptotic behaviors:** $M^{\text{mod1}}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty$;
- **Singularity:** $M^{\text{mod1}}(z)$ has singularity at $z = \pm 1$ with:
\[
M^{\text{mod1}}(z) \sim \mathcal{O}(z \mp c)^{-1/4}, \quad z \to \pm 1 \text{ in } \mathbb{C} \setminus \mathbb{R}. \quad (3.16)
\]
The solution of this model RH problem can be given by

\[ M^{\text{mod1}}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1(z) + \phi_1(z)^{-1} & \phi_1(z) - \phi_1(z)^{-1} \\ \phi_1(z) - \phi_1(z)^{-1} & \phi_1(z) + \phi_1(z)^{-1} \end{pmatrix}, \]  

(3.17)

where \( \phi_1(z) = \left( \frac{1+z}{1-z} \right)^{1/4} \). As \( z \to 0 \in \mathbb{C}^+ \),

\[ M^{\text{mod1}}(z) = \sqrt{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \frac{zi}{\sqrt{2}} I + O(z^2) \]  

(3.18)

### 3.2 Localized RH problem near phase points

As \( t \to +\infty \), we consider to reduce the RHP 3 to a model RH problem whose solution can be given explicitly in terms of parabolic cylinder functions on every contour \( \Sigma_{j,\pm}^{(0)} = \Sigma^{(1)} \cap U(\pm \lambda_j) \) respectively. And we only give the details of \( \Sigma_{1,+}^{(0)} \), the model of other critical point can be constructed similar. We denote \( \hat{\Sigma}_{1,+}^{(0)} \) as the contour \( \{ z = \lambda_1 + le^{\pm i}, \; l \in \mathbb{R} \} \) oriented from \( \Sigma_{1,+}^{(0)} \), and \( \Sigma_{j,\pm}^{(0)} \) is the extension of \( \Sigma_{j,\pm} \) respectively. And for \( z \) near \( \lambda_1 \), note that \( p''(\lambda_1) > 0 \), so we rewrite phase function \( p_- \) as

\[ p_-(z) = p_-(\lambda_1) + (z - \lambda_1)2p''(\lambda_1) + O((z - \lambda_1)^3). \]  

(3.19)

Consider following local RH problem

**RHP 5.** Find a matrix-valued function \( M^{1,+}(z) \) with following properties

- Analyticity: \( M^{1,+}(z) \) is analytical in \( \mathbb{C} \setminus \hat{\Sigma}_{1,+}^{(0)} \).
\textbf{Jump condition:} $M^{1,+}(z)$ has continuous boundary values $M^{1,\pm}_-\big|_{\Sigma_+^{(0)}}$ and

$$M^{1,+}_-(z) = M^{1,+}_-(z)V^{1,+}(z), \quad z \in \hat{\Sigma}_+^{(0)}, \tag{3.20}$$

where jump matrix $V^{1,+}(z)$ is given by (see Figure 6)

$$V^{1,+}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_1^2 e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \hat{\Sigma}_1; \\ \begin{pmatrix} 1 & -r_2^2 e^{-2itp} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \hat{\Sigma}_2; \\ \begin{pmatrix} 1 & -r_3^2 e^{-2itp} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \hat{\Sigma}_3; \\ \begin{pmatrix} 1 & 0 \\ r_4^2 e^{2itp} & 1 \end{pmatrix}, & \text{as } z \in \hat{\Sigma}_4; \end{cases}. \tag{3.21}$$

\textbf{Asymptotic behaviors:} $M^{1,+}(z) = I + O(z^{-1}), \quad z \to \infty.$

RHP 5 does not possess the symmetry condition, because it is a local model and will only be used for bounded values of $z$. In order to motivate the model, let $\zeta = \zeta(z)$ denote the rescaled local variable

$$\zeta(z) = 2t^{1/2} \sqrt{p''(\lambda_1)}(z - \lambda_1). \tag{3.22}$$

This map is a conformal bijection maps $U(\lambda_1)$ to an expanding neighborhood of $\zeta = 0$. We choose the branch which maps the upper half plane to the lower half plane. Moreover, we denote:

$$r_{\lambda_1}^{\pm} = r_1(\pm \lambda_1) \delta_{\lambda_1}^2 (\pm z_2) e^{2itp} (\pm z_2) (4t p''(\lambda_1))^{\nu(\pm \lambda_1)}. \tag{3.23}$$

where

$$\nu(\lambda_1) = \frac{1}{2\pi} \log(1 - r_1 r_2(\lambda_1)).$$

Through this change of variable, the jump $V^{1,+}(z)$ approximates to the jump of a parabolic cylinder model problem as follow:

\textbf{RHP 6.} Find a matrix-valued function $M^{pc}(\zeta; \xi)$ with following properties:
Analyticity: $M^{pc}(\zeta; \xi)$ is analytical in $\mathbb{C} \setminus \Sigma^{pc}$ with $\Sigma^{pc} = \{Re^{\varphi i}\} \cup \{Re^{(\pi-\varphi)i}\}$ shown in Figure 7;

Jump condition: $M^{pc}$ has continuous boundary values $M^{pc}_\pm$ on $\Sigma^{pc}$ and

$$M^{pc}_+(\zeta; \xi) = M^{pc}_-(\zeta; \xi)V^{pc}(\zeta), \quad \zeta \in \Sigma^\zeta,$$

where

$$V^{pc}(\zeta; \xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r^+_\lambda e^{2i\nu(\lambda)e^{\frac{\varphi}{2}}} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+e^{\varphi i}, \\ \begin{pmatrix} 1 & 0 \\ -\bar{r}^+_\lambda \zeta^{-2i\nu(\lambda)e^{-\frac{\varphi}{2}}} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+e^{-\varphi i}, \end{cases}$$

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ r^+_\lambda e^{-2i\nu(\lambda)e^{\frac{\varphi}{2}}} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+e^{(-\pi+\varphi)i}, \\ \begin{pmatrix} 1 & 0 \\ -\bar{r}^+_\lambda e^{-2i\nu(\lambda)e^{-\frac{\varphi}{2}}} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+e^{(\pi-\varphi)i}. \end{cases}$$

Asymptotic behaviors: $M^{pc}(\zeta; \xi) = I + M^{pc}_1\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty.$

Then the solution of the RHP 5 can be given by (for example, see [68] Theorem A.1-A.6)

$$M^{i,\pm}(z) = I + \frac{t^{-1/2}i}{z \mp \lambda_j \sqrt{p''(\pm \lambda_j)}} \begin{pmatrix} 0 & [M^{i,pc}_1]_{21} \\ -[M^{i,pc}_1]_{12} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}).$$

26
While the RHP 6 has an explicit solution, which is expressed in terms of solutions of the parabolic cylinder equation
\[
\frac{\partial^2 D_a(z)}{\partial z^2} + \left(\frac{1}{2} - \frac{z^2}{2} + a\right) D_a(z) = 0.
\]
A derivation of this result is given in [69]. Substitute above consequence into (3.26) and obtain
\[
M_j^{\pm}(z) = I + \frac{t^{-1/2}}{z \mp \lambda_j} \frac{i}{2 \sqrt{p''_{\pm}(\pm \lambda_j)}} \begin{pmatrix} 0 & \tilde{\beta}_{12}^{j,\pm} \\ \tilde{\beta}_{21}^{j,\pm} & 0 \end{pmatrix} + O(t^{-1}),
\]
where
\[
\tilde{\beta}_{12}^{j,\pm} = \frac{\sqrt{2\pi} e^{\frac{i}{2} \pi \nu(\pm \lambda_j)} e^{\frac{i}{2} \pi i}}{r_{\pm,\lambda_j} \Gamma(-i\nu(\pm \lambda_j))}, \quad \tilde{\beta}_{21}^{j,\pm} = -\nu(\pm \lambda_j), \quad j = 1, 2.
\]
We finally obtain
\[
\text{Proposition 4. As } t \to +\infty,
M_j^{\pm}(z) = I + \frac{t^{-1/2}}{z \mp \lambda_j} \frac{i}{2 \sqrt{p''_{\pm}(\pm \lambda_j)}} \begin{pmatrix} 0 & \tilde{\beta}_{12}^{j,\pm} \\ \tilde{\beta}_{21}^{j,\pm} & 0 \end{pmatrix} + O(t^{-1}), \quad j = 1, 2,
\]
where
\[
A_{j,\pm}(\xi) = \frac{i}{2 \sqrt{p''_{\pm}(\pm \lambda_j)}} \begin{pmatrix} 0 & \tilde{\beta}_{12}^{j,\pm} \\ \tilde{\beta}_{21}^{j,\pm} & 0 \end{pmatrix}.
\]

3.3 The small norm RH problem for error function

In this subsection, we consider the error matrix-function \(E(z; \xi, c)\) in this region.

\[
\text{RHP 7. Find a matrix-valued function } E(z; \xi, c) \text{ with following properties:}
\]
\[
\text{Analyticity: } E(z; \xi, c) \text{ is analytical in } \mathbb{C} \setminus \Sigma^E, \text{ where}
\Sigma^E = \partial \Omega_{\xi} \cup \left[\Sigma^{(1)} \setminus (\Omega_{\xi} \cup [-1, 1])\right], \quad \Omega_{\xi} = (\cup_{j=1,2} U_{\pm,\lambda_j}) \cup U_{\pm,1};
\]
\[
\text{Asymptotic behaviors:}
E(z; \xi, c) \sim I + O(z^{-1}), \quad |z| \to \infty;
\]
\[
|E(z; \xi, c)| = O(1) \quad \text{on } \mathbb{C} \setminus \Sigma^E,
\]
\[
\text{where}
\Omega_{\xi} = \partial \Omega_{\xi} \cup \left[\Sigma^{(1)} \setminus (\Omega_{\xi} \cup [-1, 1])\right], \quad \Omega_{\xi} = (\cup_{j=1,2} U_{\pm,\lambda_j}) \cup U_{\pm,1};
\]
\[
|E(z; \xi, c)| = O(1) \quad \text{on } \mathbb{C} \setminus \Sigma^E,
\]
\[
E(z; \xi, c) \sim I + O(z^{-1}), \quad |z| \to \infty;
\]
\[
|E(z; \xi, c)| = O(1) \quad \text{on } \mathbb{C} \setminus \Sigma^E,
\]
\[
E(z; \xi, c) \sim I + O(z^{-1}), \quad |z| \to \infty;
\]
\[
|E(z; \xi, c)| = O(1) \quad \text{on } \mathbb{C} \setminus \Sigma^E,
\]
Jump condition: \( E(z; \xi, c) \) has continuous boundary values \( E_{\pm}(z; \xi, c) \) on \( \Sigma^E \) satisfying
\[
E_{+}(z; \xi, c) = E_{-}(z; \xi, c)V^E(z),
\]
where the jump matrix \( V^E(z) \) is given by
\[
V^E(z) = \begin{cases} 
M_{\text{mod}1}(z_-)V^{(1)}(z)M_{\text{mod}1}(z_+)^{-1}, & z \in \Sigma^E \setminus \partial U_\xi, \\
M_{\text{mod}}(z)M_{\text{mod}1}(z)^{-1}, & z \in \partial U_\xi,
\end{cases}
\]
(3.32)

We will show that for large times, the error function \( E(z; \xi, c) \) solves following small norm RH problem.

Out of \( U_\xi \), the jump \( V^E \) has the following estimates
\[
\| V^E(z) - I \|_p \lesssim \exp\{-tK_p\}, z \in \Sigma^E \setminus U_\xi, \quad p \in [1, \infty].
\]
(3.33)

For \( z \in \partial U_\xi \), \( M_{\text{mod}}(z) \) is bounded, so by using (3.30), we find that
\[
|V^E(z) - I| = \mathcal{O}(t^{-1/2}).
\]
(3.34)

Therefore, the existence and uniqueness of the RHP 7 can shown by using a small-norm RH problem [65, 66]. Moreover, according to Beal-Coifman theory, the solution of the RHP 7 can be given by
\[
E(z; \xi, c) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s))(V^E(s) - I)}{s - z} ds,
\]
(3.35)

where the \( \varpi \in L^\infty(\Sigma^E) \) is the unique solution of following equation
\[
(1 - C_E)\varpi = C_E(I),
\]
(3.36)

and \( C_E \) is a integral operator: \( L^\infty(\Sigma^E) \to L^2(\Sigma^E) \) defined by
\[
C_E(f)(z) = C_{-}(f(V^E(z) - I)),
\]
(3.37)

where the \( C_{-} \) is the usual Cauchy projection operator on \( \Sigma^E \)
\[
C_{-}(f)(s) = \lim_{z \to \Sigma^E_{-}} \frac{1}{2\pi i} \int_{\Sigma^E} \frac{f(s)}{s - z} ds.
\]
(3.38)
By (3.34), we have
\[
\| C_E \| \leq \| C_- \| \| V^E(z) - I \|_2 \lesssim O(t^{-1/2}),
\]
(3.39)
which implies that \( 1 - C_E \) is invertible for sufficiently large \( t \). So \( \varpi \) exists and is unique. Besides,
\[
\| \varpi \|_{L^\infty(\Sigma^E)} \lesssim \frac{\| C_E \|}{1 - \| C_E \|} \lesssim t^{-1/2}.
\]
(3.40)
In order to reconstruct the solution \( u(y,t) \) of (1.3), we need the asymptotic behavior of \( E(z;\xi,c) \) as \( z \to 0 \in \mathbb{C}^+ \) and the long time asymptotic behavior of \( E(0) \). Note that when we estimate its asymptotic behavior, from (3.35) and (??) we only need to consider the calculation on \( \partial U_\xi \) because it approach zero exponentially on other boundary.

**Proposition 5.** As \( z \to 0 \in \mathbb{C}^+ \), we have
\[
E(z;\xi,c) = E(0) + E_1 z + O(z^2),
\]
(3.41)
where
\[
E(0) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s} ds,
\]
(3.42)
with long time asymptotic behavior
\[
E(0) = I + t^{-1/2}H^{(0)} + O(t^{-1}),
\]
(3.43)
where
\[
H^{(0)} = \sum_{p = \pm \lambda_j, j = 1, 2} \frac{M^{mod}(p) A_{j,\pm}(\xi) M^{mod}(p)^{-1}}{p}.
\]
(3.44)
Here \( A_{j,\pm}(\xi) \) is given by (3.30). And
\[
E_1 = \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s^2} ds = t^{-1/2}H^{(1)} + O(t^{-1}),
\]
where
\[
H^{(1)} = \sum_{p = \pm 2} \frac{M^{mod}(p) A_{\pm}(\xi) M^{mod}(p)^{-1}}{p^2}.
\]
(3.45)
Proof. Substitute the long time asymptotic behavior of $V^E, \varpi(s)$ and Proposition 4 into $2\pi i(E(0) - I)$:

$$
\int_{\Sigma^E} \frac{(I + \varpi(s))(V^E - I)}{s} ds \\
= \int_{\partial U(\xi)} \frac{M^{mod1}(s)(M^j,\pm(s) - I)M^{mod1}(s)^{-1}}{s} ds + O(t^{-1}) \\
= t^{-1/2} \int_{\partial U(\xi)} \frac{M^{mod1}(s)A_{j,\pm}(\xi)M^{mod1}(s)^{-1}}{s(z \mp \lambda_j)} ds + O(t^{-1}).
$$

(3.46)

Then by residue theorem we finally arrive at the result.

4 Fast-decay background region

The Region II is corresponding to the case $\xi > 1 + 2/c$. In this case, we introduce a new scalar function

$$
X(z) = \sqrt{z^2 - c^2}, \quad \theta_+(z) = X(z) \left(\frac{1 - \xi}{2} + \frac{1}{cz^2}\right),
$$

(4.1)

where $X(z)$ is analytic on $\mathbb{C} \setminus [-c, c]$ and take the single-valued analytic branch such that $X(z) \in i\mathbb{R}$ on $[-c, c]_+$. Then

$$
\frac{d\theta_+}{dz} = \frac{1}{z^3X(z)} \left(\frac{\xi - 1}{2} - \frac{z^2}{c} + 2c\right).
$$

(4.2)

Further define $\lambda_1 = \sqrt{\frac{2}{\xi(\xi - 1)}} \in (0, 1)$ satisfying $\frac{\xi - 1}{2} + \frac{1}{e\lambda_1^2} = 0$. The sign of the imaginary part Im$\theta_+$ is shown in Figure 8.

Define

$$
\Sigma^\pm = \left\{-1 + e^{\pm \frac{2\pi i}{c\lambda_1^2}}\mathbb{R}^+\right\} \cup \left\{1 + e^{\pm \frac{2\pi i}{c\lambda_1^2}}\mathbb{R}^+\right\},
$$

(4.3)

$$
\Omega^\pm = \left\{z; z = -1 + e^{\pm \psi i}l, l \in \mathbb{R}^+, \ 3\pi/4 < \psi < \pi\right\} \cup \left\{z; z = 1 + e^{\pm \psi i}l, l \in \mathbb{R}^+, \ 0 < \psi < \pi/4\right\},
$$

(4.4)

$$
\overline{\Omega} = \mathbb{C} \setminus (\Omega^+ \cup \Omega^-).
$$

(4.5)
Figure 8: In the case $\xi > 1 + \frac{2}{c^2}$, $\operatorname{Im} \theta_+(z) > 0$ in yellow region while $\operatorname{Im} \theta_+(z) < 0$ in white region. And critical line $\operatorname{Im} \theta(z) = 0$ is black solid line.

Obviously,

\begin{align*}
p_- - \theta_+ &= O(z^{-1}), \text{ as } z \to \infty, \quad (4.6) \\
p_- - \theta_+ &= -\frac{ci}{2}(\xi - 1) + \frac{i}{2e^2} - \frac{i\xi}{2} + O(z^2), \text{ as } z \to 0 \in \mathbb{C}^+. \quad (4.7)
\end{align*}

So we can use $\theta_+$ to replace $p_-$ in the exponential function. And we will utilize these factorizations to deform the jump contours, so that the oscillating factor $e^{\pm 2i\theta_+}$ are decaying in corresponding region respectively.

Figure 9: The region of $\Omega^\pm$ and $\overline{\Omega}$. And $\Sigma^{(1)} = \Sigma^+ \cup \Sigma^- \cup [-c, c]$. The same as Figure 8, yellow region means $\operatorname{Im} \theta_+(z) > 0$ while white region means $\operatorname{Im} \theta_+(z) < 0$.

Similarly as in the above section, in this region of $\xi$, we introduce a piecewise
matrix interpolation function

\[
G(z) = G(z; \xi, c) = \begin{cases} 
\left( \begin{array}{cc} 1 & r_2 e^{-2i\theta_+} \\ 0 & 1-r_1 r_2 \end{array} \right), & \text{as } z \in \Omega^+; \\
\left( \begin{array}{cc} 1 & 0 \\ r_1 e^{2i\theta_+} & 1 \end{array} \right), & \text{as } z \in \Omega^-; \\
I & \text{as } z \text{ in elsewhere,}
\end{cases}
\]  

(4.8)

Note that \(1 - r_1 r_2 = 0\), so the matrix function \(G(z)\) bring a new \(-\frac{1}{4}\)-singularity on \(z = \pm c\). We define the new matrix-valued function \(M^{(1)}(z)\)

\[
M^{(1)}(z) \triangleq M^{(1)}(z; y, t, c) = N(z)e^{it(p_--\theta_+)\sigma_3}G(z),
\]  

(4.9)

which then satisfies the following RH problem.

**RHP 8.** Find a matrix-valued function \(M^{(1)}(z)\) which satisfies

\begin{itemize}
  \item **Analyticity:** \(M^{(1)}(z)\) is meromorphic in \(\mathbb{C} \setminus (\Sigma^{(1)} \cup \mathbb{R})\), where
  \[
  \Sigma^{(1)} = \Sigma_1 \cup \Sigma_2 \cup [-c, c],
  \]  
  shown in Figure 9;
  \item **Symmetry:** \(M^{(1)}(z) = \sigma_2 M^{(1)}(-z) \sigma_2 = \sigma_1 M^{(1)}(\overline{z}) \sigma_1\);
  \item **Jump condition:** \(M^{(1)}\) satisfies the jump condition
  \[
  M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z), \quad z \in \Sigma^{(1)} \cup \mathbb{R}, \quad z \in \Sigma^{(1)} \cup \mathbb{R},
  \]  
  (4.11)
\end{itemize}

where

\[
V^{(1)}(z) = \begin{cases} 
\left( \begin{array}{cc} 1 & r_2 e^{-2i\theta_+} \\ 0 & 1-r_1 r_2 \end{array} \right), & \text{as } z \in \Sigma_1, \\
\left( \begin{array}{cc} 1 & 0 \\ r_1 e^{2i\theta_+} & 1 \end{array} \right), & \text{as } z \in \Sigma_2, \\
(1 - r_1 r_2)^{\sigma_3}, & \text{as } z \in \mathbb{R} \setminus [-c, c], \\
\left( \begin{array}{cc} 0 & -r_2(z - 0i) \\ r_1(z + 0i) & 0 \end{array} \right), & \text{as } z \in [-c, c] \setminus [-1, 1], \\
\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), & \text{as } z \in [-1, 1],
\end{cases}
\]  

(4.12)
Asymptotic behaviors: $M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty$;

Singularity: $M^{(1)}(z)$ has singularity at $z = \pm 1, \pm c$ with:

$$M^{(1)}(z) \sim \left( O(z \mp 1)^{1/4}, O(z \mp 1)^{-1/4} \right), \quad z \to \pm 1 \text{ in } \mathbb{C}^\pm,$$

$$M^{(1)}(z) \sim \left( O(1), O(z \mp c)^{-1/2} \right), \quad z \to \pm c \text{ in } \mathbb{C}^+,$$

$$M^{(1)}(z) \sim \left( O(z \mp c)^{-1/2}, O(1) \right), \quad z \to \pm c \text{ in } \mathbb{C}^-.$$

To deal with the jump on $\mathbb{R}$, we give an introduction of an auxiliary function $\delta(z) = \delta(z; \xi, c)$, which relies on $\xi$ and admits the following jump condition:

$$\delta_-(z) = \delta_+(z)(1 - r_1 r_2), \quad z \in \mathbb{R} \setminus [-c, c];$$

$$\delta_-(z)\delta_+(z) = i[r_2]_-, \quad z \in [-c, c] \setminus [-1, 1];$$

$$\delta_-(z)\delta_+(z) = 1, \quad z \in [-1, 1].$$

Define the function

$$\log \delta(z) = \frac{X(z)}{2\pi i} \left( \int_{-c}^{-1} + \int_{c}^{1} \frac{\log(i[r_2]_- (s))}{(s - z)[X]_+(s)} \, ds - \frac{X(z)}{2\pi i} \int_{\mathbb{R} \setminus [-c,c]} \frac{\log(1 - r_1(s)r_2(s))}{(s - z)X(s)} \, ds. \right)$$

Proposition 6. The scalar function $\delta(z)$ satisfies the following properties

(a) $\delta(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$;

(b) $\delta(z)$ has singularity at $z = \pm c$ with

$$\delta(z) = O(z - p)^{\mp 1/4}, \quad z \to p \in \mathbb{C}^\pm \setminus \mathbb{R}, \quad p = c, -c. \quad (4.16)$$

(c) As $z \to \infty \in \mathbb{C} \setminus \mathbb{R}$, $\delta(z)$ has limit $\delta(\infty)$ with

$$\log \delta(\infty) = -\frac{1}{2\pi i} \left( \int_{-c}^{-1} + \int_{c}^{1} \frac{\log(i[r_2]_- (s))}{[X]_+(s)} \, ds \right)$$

$$+ \frac{1}{2\pi i} \int_{\mathbb{R} \setminus [-c,c]} \frac{\log(1 - r_1(s)r_2(s))}{X(s)} \, ds. \quad (4.17)$$

(d) As $z \to 0 \in \mathbb{C}^+$,

$$\delta(z) = \exp \left\{ I_3^1 \right\} \cdot (1 + z I_3^2) + O(z^2). \quad (4.18)$$
Here,
\[ I_1^\delta = \frac{c}{2\pi} \left( \int_{-c}^{1-c} + \int_{c}^{1} \right) \frac{\log(i[r_2](s))}{s[X]_+(s)} ds - \frac{c}{2\pi} \int_{\mathbb{R}\setminus[-c,c]} \frac{\log(1 - r_1(s) r_2(s))}{s X(s)} ds, \quad (4.19) \]
\[ I_2^\delta = \frac{c}{2\pi} \left( \int_{-c}^{1-c} + \int_{c}^{1} \right) \frac{\log(i[r_2](s))}{s^2[X]_+(s)} ds - \frac{c}{2\pi} \int_{\mathbb{R}\setminus[-c,c]} \frac{\log(1 - r_1(s) r_2(s))}{s^2 X(s)} ds. \quad (4.20) \]

**Proof.** The proof of (a), (c) and (d) is trivial. And the proof of (b) is similar as [55] and Appendix C in [54]. \( \square \)

We give a new transformation
\[ M^{(2)} = \delta(\infty)^{-\sigma_3} M^{(1)} \delta^{\sigma_3}, \quad (4.21) \]
which then satisfies the following RH problem.

**RHP 9.** Find a matrix-valued function \( M^{(2)}(z) \) which satisfies

- **Analyticity:** \( M^{(2)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \), where
  \[ \Sigma^{(1)} = \Sigma^+ \cup \Sigma^- \cup [-c,c], \quad (4.22) \]
  shown in Figure 9;

- **Symmetry:** \( M^{(2)}(z) = \sigma_2 M^{(2)}(-z) \sigma_2 = \sigma_1 M^{(2)}(\bar{z}) \sigma_1 \);

- **Jump condition:** \( M^{(2)} \) has continuous boundary values \( M^{(2)}_\pm(z) \) on \( \Sigma^{(1)} \) and
  \[ M^{(2)}_+(z) = M^{(2)}_-(z)V^{(2)}(z), \quad z \in \Sigma^{(1)}, \quad (4.23) \]
  where
  \[ V^{(2)}(z) = \begin{cases} 
    \left( \begin{array}{cc}
    1 & -r_2 e^{-2it\theta} \\
    0 & \frac{1 - r_1 r_2}{1 - r_1 r_2}
    \end{array} \right), & \text{as } z \in \Sigma^+, \\
    \left( \begin{array}{cc}
    1 & 0 \\
    r_2 e^{2it\theta} & 1 - r_1 r_2
    \end{array} \right), & \text{as } z \in \Sigma^-, \\
    \left( \begin{array}{cc}
    0 & i \\
    i & 0
    \end{array} \right), & \text{as } z \in [-c,c],
  \end{cases} \quad (4.24) \]

- **Asymptotic behaviors:** \( M^{(2)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty \);

- **Singularity:** \( M^{(2)}(z) \) has singularity at \( z = \pm c \) with
  \[ M^{(2)}(z) \sim \mathcal{O}(z \mp c)^{-1/4}, \quad z \to \pm c \text{ in } \mathbb{C} \setminus \mathbb{R}. \quad (4.25) \]
The jump matrix exponentially decays to the identity matrix $I$ as $t \to \infty$ on $\Sigma^\pm_1$, which finally leads to the model RH problem.

**RHP 10.** Find a matrix-valued function $M^{\text{modc}}(z)$ which satisfies

- **Analyticity:** $M^{\text{modc}}(z)$ is holomorphic in $\mathbb{C} \setminus [-c,c]$;
- **Jump condition:** $M^{\text{modc}}$ has continuous boundary values $M^{\text{modc}}_\pm(z)$ and
  $$M^{\text{modc}}_+(z) = M^{\text{modc}}_-(z)V^{\text{modc}}(z), \quad z \in [-c,c],$$
  where
  $$V^{\text{modc}}(z) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad z \in [-c,c];$$
- **Asymptotic behaviors**
  $$M^{\text{modc}}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty;$$
- **Singularity:** $M^{\text{modc}}(z)$ has singularity at $z = \pm c$ with
  $$M^{\text{modc}}(z) \sim \mathcal{O}(z \mp c)^{-1/4}, \quad z \to \pm c \text{ in } \mathbb{C} \setminus \mathbb{R}.$$ 

We can construct the solution of model RH problem

$$M^{\text{modc}}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_c(z) + \phi_c(z)^{-1} & \phi_c(z) - \phi_c(z)^{-1} \\ \phi_c(z) - \phi_c(z)^{-1} & \phi_c(z) + \phi_c(z)^{-1} \end{pmatrix},$$

as $z \to 0 \in \mathbb{C}^+$,

$$M^{\text{modc}}(z) = \sqrt{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \frac{zi}{\sqrt{2}c}I + \mathcal{O}(z^2).$$

Considering transformation

$$E(z) = M^{(2)}(M^{\text{modc}}(z))^{-1},$$

which has jump matrix exponentially decaying to the identity matrix $I$ as $t \to \infty$ on $\Sigma^\pm_1$. Then its existence and uniqueness can be shown by a small-norm RH problem with

$$E(z) = I + \mathcal{O}(t^{-2}).$$
5 The first-type genus-2 elliptic wave region

In the Region III, we need to introduce a new $g$-function defined on genus 2 Riemann surface which has real branch points $\pm 1, \pm c$ and $\pm z_0$ with $1 < z_0 < c$. And the canonical homology basis $\{a_j, b_j\}_{j=1}^2$ is shown in Figure 10. Note that this region contains two cases

$3/4 < \xi < 1: \quad (1) \ 2 < c^2 < 4, \ 1 - \frac{2(c^2 - 2)}{c^4} < \xi < 1; \quad (2) \ c^2 > 4, \ \xi_m < \xi < 1$;

$1 < \xi < +\infty: \quad (3) \ c^2 < 2, \ 1 + \frac{2(2-c^2)}{c^4} < \xi < 1 + 2/c; \quad (4) \ c^2 > 2, \ 1 < \xi < 1 + 2/c$.

In this two different cases, $g$ has different property. So after we proving the basic property of $g$, we will discuss this two different cases separately. Here, the definition of $\xi_m$ is means the critical condition of $\xi$ that stationary phase point $z_2$ of the $g$-function in case (1) merge $c$. The existence of $\xi_m$ is given in the Appendix B.

![Figure 10: The canonical homology basis $\{a_j, b_j\}_{j=1}^2$ of the genius 2 Riemann surface.](image)

5.1 Constructing the $g$-function

To construct the $g$-function, we first introduce:

$$Y(z) = \left[ \frac{z^2 - \xi}{(z^2 - 1)(z^2 - c^2)} \right]^{1/2},$$

where $z_0$ is a real number in $(1, c)$ and the branch of the square root is such that $Y(z) \in i\mathbb{R}^+$ for $z \in [z_0, c]$. And the $dg$ is the derivative of $g$-function

$$dg = \frac{Y(z)}{z^3} \left[ \frac{1 - \xi}{2} z^4 - \frac{c}{z_0} \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) z^2 + \frac{2c}{z_0} \right] dz.$$  (5.2)

Here, $dg$ is a meromorphic differential defined on the 2-genus Riemann surface, with $dg$ on the upper sheet and $-dg$ on the lower sheet. Similarly, simply calculation
shows that
\[
\frac{dg}{dz} - \frac{dp_-}{dz} = O(z^{-2}), \text{ as } z \to \infty;
\]
\[
\frac{dg}{dz} - \frac{dp_-}{dz} = O(z), \text{ as } z \to 0 \in \mathbb{C}^+.
\]

Denote \( \Sigma^{\text{mod}} \) as the union of three branch cuts:
\[
\Sigma^{\text{mod}} = [-c, -z_0] \cup [-1, 1] \cup [z_0, c].
\] (5.3)

Thus the g-function is given by
\[
g(z) = g(z; \xi, c) = \int_{\xi}^{z} dg, \quad z \in \mathbb{C} \setminus \Sigma^{\text{mod}}.
\] (5.4)

**Proposition 7.** There exist a real number \( z_0 = z_0(\xi, c) \) in \((1, c)\) such that the function \( g(z) \) defined above has the following properties

(a) The a-period of \( g(z) \) is zero and the b-period of \( g(z) \) is real;

(b) \( g(z) \) satisfies the following jump conditions across \([-c, c]\):
\[
g_-(z) + g_+(z) = 0, \quad z \in (z_0, c),
\] (5.5)
\[
g_-(z) - g_+(z) = 0, \quad z \in (1, z_0) \cup (-z_0, -1),
\] (5.6)
\[
g_-(z) + g_+(z) = B_1, \quad z \in (-1, 1),
\] (5.7)
\[
g_-(z) + g_+(z) = B_2, \quad z \in (-c, -z_0),
\] (5.8)

where \( B_j = B_j(\xi, c) = \frac{1}{2} \oint_{b_j} dg \) is real;

(c) \( g(z) \) has another phase point \( z_1 = z_1(\xi) \in (z_0, c) \) which is the solution of equation \( \frac{\xi - 1}{2} z^4 + \frac{c}{z_0} \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) z^2 - \frac{2c}{z_0} = 0 \);

(a) In Case (1), (2) with \( c^2 > 2, 1 > \xi \), \( g(z) \) has another phase point \( z_2 = z_2(\xi) \in (c, +\infty) \), \( z_2 > z_1 \), which also is a solution of equation \( \frac{\xi - 1}{2} z^4 + \frac{c}{z_0} \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) z^2 - \frac{2c}{z_0} = 0 \). When \( c > 2 \), as \( \xi \to \xi_m < 1 - \frac{2(c^2-2)}{c^2} \), \( z_2(\xi) \) decreases to \( c \).
Proof. From the symmetry of $\text{d}g$, $a_2$-period of $g(z)$ is zero. Rewrite the function $Y(z)$ as $Y(z; z_0)$. Let $F(s)$ be a function defined on $\mathbb{R}$ with

$$F(s) = \int_s^c \frac{[\text{Y}]_+(z; s)}{z^3} \left[ \frac{\xi - 1}{2} z^4 + \frac{c}{s} \left( 1 + \frac{1}{c^2} - \frac{1}{s^2} \right) z^2 - \frac{2c}{s} \right] \text{d}z. \quad (5.9)$$

Then we have $F(c) = 0$ and

$$F(1) = \int_1^c \frac{1}{z^3} \left[ (z^2 - c^2)^{-1/2} \right] \left( \frac{\xi - 1}{2} z^4 + \frac{z^2}{c} - 2c \right) \text{d}z = -\theta_+(1^+) - \theta_-(1^+).$$

And we consider the $s$-derivative of $F$ at $s = c$,

$$\frac{dF}{ds}(c) = -\frac{i}{c^3} (c^2 - 1)^{-1/2} \left( \frac{\xi - 1}{2} c^4 + c^2 - 2 \right). \quad (5.10)$$

In the case $\xi < 1$, obviously, $F(1) \in i\mathbb{R}^-$. Thus, when $c^2 > 2$, $\xi > \frac{2(c^2 - 2)}{c^4} + 1$, $\frac{dF}{ds}(c) \in i\mathbb{R}^-$. And in the case $1 < \xi < \frac{2}{c} + 1$, from the property of $\theta_+$ in above section, we have that $F(1) \in i\mathbb{R}^+$. While when $c^2 < 2$, $1 + \frac{2(c^2 - 2)}{c^4} < \xi < \frac{2}{c} + 1$ and $c^2 > 2, 1 < \xi < \frac{2}{c} + 1$, we have that $\frac{dF}{ds}(c) \in i\mathbb{R}^+$. So there must exist $z_0 \in (1, c)$, such that $F(z_0) = 0$. This also implies that there exist $z_1 \in (z_0, c)$ such that $f(z_1^2) = 0$ with

$$f(x) = \frac{\xi - 1}{2} x^2 + \frac{c}{z_0} \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) x - \frac{2c}{z_0}.$$

By simply calculating the $a_1$-period of $g(z)$ can be zero and the both $b$-period are real. Obviously, $f(0) < 0$. So in the $\xi > 1$ case, $f(x)$ only has one zero $z_1$ on $\mathbb{R}^+$. And in the $\xi < 1$ case, we note another real solution of $f(z) = 0$ is $z_2^2$.

In addition, in the $\xi < 1$ case, simple calculation gives that

$$\frac{\partial F}{\partial s}(s; \xi) = -\int_s^c \frac{z(1 - \xi)}{2 \sqrt{(z^2 - 1)(z^2 - c^2)(z^2 - s^2)}} s^3 (s^2 - z_1^2)(s^2 - z_2^2) \text{d}z, \quad (5.11)$$

$$\frac{\partial F}{\partial (1 - \xi) / 2}(s; \xi) = \int_s^c \frac{z \sqrt{z^2 - s^2}}{\sqrt{(z^2 - 1)(z^2 - c^2)}} \text{d}z. \quad (5.12)$$

So when $\xi$ decrease from $1$, $z_0(\xi)$ increase in $(1, c)$ while $z_2$ as a solution of $f(x) = 0$ decrease. When $z_2$ merge $c$, we denote this critical condition as $\xi_m$. \hfill \Box

Next, because $g$ will have different sign table in $\xi > 1$ and $\xi < 1$, we will discuss $g$-function according to it. Denote constant

$$g(\infty) = \lim_{z \to \infty} g(z) - p_-(z). \quad (5.13)$$
5.2 Opening the jump in the region $1 < \xi < +\infty$

In this region, we give the signature table of $\text{Im}g$ is given in Figure 11. To open the jump contour $\mathbb{R}$, we define

\[
\Sigma^+_1 = \left\{ z = -z_1 + e^{\pm \frac{3\pi i}{4}} \mathbb{R}^+ \right\} \cup \left\{ z = z_1 + e^{\pm \frac{\pi i}{4}} \mathbb{R}^+ \right\},
\]

\[
\Sigma^+_2 = \left\{ z = -z_0 + e^{\pm \psi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\} \cup \left\{ z = z_0 + e^{(\pi \mp \psi)i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\}
\]

\[
\mathcal{U} \left\{ z = 1 + e^{\psi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\} \cup \left\{ z = -1 + e^{(\pi - \psi)i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\},
\]

where $\psi < \frac{\pi}{4}$ is chosen as a small enough positive constant such that $\Sigma^+_3$ is contained in the region of $\text{Im}g > 0$. And further define opened domains

\[
\Omega^+_1 = \left\{ z = -z_1 + e^{\pm \phi i} l, \ l \in \mathbb{R}^+, \ \pi > \phi > \frac{3\pi}{4} \right\}
\]

\[
\cup \left\{ z = z_1 + e^{\pm \phi i} l, \ l \in \mathbb{R}^+, \ 0 < \phi < \frac{\pi}{4} \right\},
\]

\[
\Omega^+_2 = \left\{ z = -z_0 + e^{\pm \phi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}), \ 0 < \phi < \psi \right\}
\]

\[
\cup \left\{ z = z_0 + e^{\pm \phi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}), \ \pi > \phi > \pi \mp \psi \right\},
\]

\[
\cup \left\{ z = 1 + e^{\pm \phi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}), \ 0 < \phi < \psi \right\}
\]

\[
\cup \left\{ z = -1 + e^{\pm \phi i} l, \ l \in (0, \frac{z_0 - 1}{2 \cos \psi}), \ \pi > \phi > \pi \mp \psi \right\},
\]

Now we use $g$ to replace $p_-$ in the exponential function. And we will utilize these factorizations to deform the jump contours, so that the oscillating factor $e^{\pm 2i\theta}$ are decaying in corresponding region respectively.
In this region of $\xi, c$, we introduce a piecewise matrix interpolation function

$$G(z) = G(z; \xi, c) = \begin{cases} \begin{pmatrix} 1 & \frac{r_1e^{-2itg}}{1-r_1e^{2itg}} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^+; \\ \begin{pmatrix} 1 & 0 \\ \frac{r_1e^{-2itg}}{1-r_1e^{2itg}} & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^-; \\ \begin{pmatrix} 1 & 0 \\ -r_1e^{2itg} & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^+; \\ \begin{pmatrix} 1 & -r_2e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^-; \\ I & \text{as } z \text{ in elsewhere}, \end{cases}$$

(5.14)

Same as above section, $G(z)$ bring a new singularity. We define the new matrix-valued function $M^{(1)}(z)$

$$M^{(1)}(z) \triangleq e^{itg(\infty)\sigma_3}N(z)e^{it(p-g)\sigma_3}G(z),$$

(5.15)

which then satisfies the following RH problem.

**RHP 11.** Find a matrix-valued function $M^{(1)}(z)$ which satisfies

- **Analyticity:** $M^{(1)}(z)$ is meromorphic in $\mathbb{C} \setminus (\Sigma^{(1)} \cup (-\infty, -z_1) \cup (z_1, \infty))$, where

$$\Sigma^{(1)} = \left( \bigcup_{j=1}^2 \Sigma_j^\pm \right) \cup [-1, 1] \cup [-z_1, -z_0] \cup [z_0, z_1],$$

(5.16)

see Figure 12;
\textbf{Symmetry:} \( M^{(1)}(z) = \sigma_2 M^{(1)}(-z) \sigma_2 = \sigma_1 M^{(1)}(\bar{z}) \sigma_1 \);

\textbf{Jump condition:} \( M^{(1)} \) has continuous boundary values \( M^{(1)}_\pm(z) \) on the contour \( \Sigma^{(1)} \cup (-\infty,-z_1) \cup (z_1,\infty) \) and

\[
M^{(1)}_+(z) = M^{(1)}_-(z) \hat{V}^{(1)}(z), \quad z \in \Sigma^{(1)} \cup (-\infty,-z_1) \cup (z_1,\infty),
\]

where

\[
\hat{V}^{(1)}(z) = \begin{cases}
(1 - r_1 r_2)^\sigma_3, & \text{as } z \in \mathbb{R} \setminus [-c,c], \\
\begin{pmatrix}
1 & -r_2 e^{-2itg} \\
0 & 1 - r_1 r_2
\end{pmatrix}, & \text{as } z \in \Sigma^+_1, \\
\begin{pmatrix}
1 & 0 \\
r_1 e^{2itg} & 1
\end{pmatrix}, & \text{as } z \in \Sigma^-_1, \\
\begin{pmatrix}
1 & 0 \\
r_1 e^{2itg} & 1
\end{pmatrix}, & \text{as } z \in \Sigma^+_2, \\
\begin{pmatrix}
1 & -r_2 e^{-2itg} \\
0 & 1
\end{pmatrix}, & \text{as } z \in \Sigma^-_2, \\
\begin{pmatrix}
0 & -r_2(z - 0i) e^{-itB_2} \\
r_1(z + 0i) e^{itB_2} & 0
\end{pmatrix}, & \text{as } z \in [-c,-z_1], \\
\begin{pmatrix}
0 & -r_2(z - 0i) e^{-itB_2} \\
r_1(z + 0i) e^{itB_2} & e^{it(g_+ - g_-)}
\end{pmatrix}, & \text{as } z \in [-z_1,-z_0], \\
\begin{pmatrix}
0 & ie^{-itB_1} \\
0 & ie^{itB_1}
\end{pmatrix}, & \text{as } z \in [-1,1], \\
\begin{pmatrix}
0 & -r_2(z - 0i) \\
r_1(z + 0i) e^{-2itg_+} & e^{-2itg_+}
\end{pmatrix}, & \text{as } z \in [z_0,z_1], \\
\begin{pmatrix}
0 & -r_2(z - 0i) \\
r_1(z + 0i) & 0
\end{pmatrix}, & \text{as } z \in [z_1,c],
\end{cases}
\]

\textbf{Asymptotic behaviors:}

\[
M^{(1)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty;
\]
\[ M^{(1)}(z) \sim \mathcal{O}(z^\mp 1)^{-1/4}, \ z \to \pm 1 \text{ in } \mathbb{C} \setminus \Sigma^{(1)}, \quad (5.20) \]
\[ M^{(1)}(z) \sim \left( \mathcal{O}(1), \mathcal{O}(z \mp c)^{-1/2} \right), \ z \to \pm c \text{ in } \mathbb{C}^+, \quad (5.21) \]
\[ M^{(1)}(z) \sim \left( \mathcal{O}(z \mp c)^{-1/2}, \mathcal{O}(1) \right), \ z \to \pm c \text{ in } \mathbb{C}^-. \quad (5.22) \]

To deal with the jump on \( \mathbb{R} \), we give an introduction of an auxiliary function \( D(z) \), which admits the following jump condition:
\[ D_-(z) = D + (z)(1 - r_1 r_2), \quad z \in \mathbb{R} \setminus [-c, c]; \]
\[ D_-(z)D_+(z) = i[r_2]_-, \quad z \in [-c, c] \setminus [-z_0, z_0]; \]
\[ D_-(z)D_+(z) = 1, \quad z \in [-1, 1]. \]

Define \( Y_3(z) = (z^2 - 1)(z^2 - c^2)Y(z) \), and
\[
\log D(z) = \frac{Y_3(z)}{2\pi i} \left( \int_{-c}^{-z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]_-(s))}{(s - z)[Y_3]_+(s)} ds
- \frac{Y_3(z)}{2\pi i} \int_{\mathbb{R} \setminus [-c, c]} \frac{\log(1 - r_1(s)r_2(s))}{(s - z)Y_3(s)} ds. \quad (5.23)
\]

**Proposition 8.** The scalar function \( D(z) \) satisfies the following properties

(a) \( D(z) \) is analytic on \( \mathbb{C} \setminus ((-\infty, -z_0) \cup [-1, 1] \cup (z_0, \infty)) \);

(b) \( D(z) \) has singularity at \( z = \pm c \) with:
\[ D(z) = \mathcal{O}(z - p)^{\mp 1/4}, \quad z \to p \in \mathbb{C}^\pm \setminus \mathbb{R}, \quad p = c, -c. \quad (5.24) \]

(c) As \( z \to \infty \in \mathbb{C} \setminus \mathbb{R} \), \( D(z) \) has limit \( D_\infty(z) \) with
\[
\log D_\infty(z) = -\frac{1}{2\pi i} \left( \int_{-c}^{-z_0} + \int_{c}^{z_0} \right) \frac{\log(i[r_2]_-(s))}{[X]_+(s)} \left( z^2 + zs + s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds
+ \frac{1}{2\pi i} \int_{\mathbb{R} \setminus [-c, c]} \frac{\log(1 - r_1(s)r_2(s))}{X(s)} \left( z^2 + zs + s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds.
\]

(d) As \( z \to 0 \in \mathbb{C}^+ \),
\[ D_\infty(z) = D_\infty(0) \left( 1 + D_\infty^{(1)} z \right) + \mathcal{O}(z^2), \quad (5.25) \]
\[ D(z) = D(0) \left( 1 + D^{(1)} z \right) + \mathcal{O}(z^2), \quad (5.26) \]
where

\[
\log(D_\infty(0)) = \frac{1}{2\pi i} \left( \int_{-c}^{-z_0} + \int_z^{z_0} \right) \frac{\log(|r_2(s)|)}{|X(s)|} \left( \frac{1 + c^2 + z_0^2}{2} - s^2 \right) ds
\]

\[
+ \frac{1}{2\pi i} \int_{R \setminus [-c,c]} \frac{\log(1 - r_1(s) r_2(s))}{X(s)} \left( s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds,
\]

\[
D^{(1)} = -\frac{1}{2\pi i} \left( \int_{-c}^{-z_0} + \int_z^{z_0} \right) s \frac{\log(|r_2(s)|)}{|X(s)|} ds + \frac{1}{2\pi i} \int_{R \setminus [-c,c]} \frac{s \log(1 - r_1(s) r_2(s))}{X(s)} ds,
\]

\[
\log(D(0)) = \frac{c_2 \alpha}{2\pi} \left( \int_{-c}^{-z_0} + \int_z^{z_0} \right) \frac{\log(|r_2(s)|)}{|X(s)|} ds - \frac{c_2 \alpha}{2\pi} \int_{R \setminus [-c,c]} \frac{\log(1 - r_1(s) r_2(s))}{s Y_3(s)} ds,
\]

\[
D^{(1)} = \frac{c_2 \alpha}{2\pi} \left( \int_{-c}^{-z_0} + \int_z^{z_0} \right) \frac{\log(|r_2(s)|)}{s^2 Y_3(s)} ds - \frac{c_2 \alpha}{2\pi} \int_{R \setminus [-c,c]} \frac{\log(1 - r_1(s) r_2(s))}{s^2 Y_3(s)} ds.
\]

By using \( D(z) \), we define a new matrix function

\[
M^{(2)} = D^{-\sigma_3} M^{(1)} D^{\sigma_3},
\]

which then satisfies the following RH problem.

**RHP 12.** Find a matrix-valued function \( M^{(2)}(z) \) which satisfies

- **Analyticity:** \( M^{(2)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \);
- **Symmetry:** \( M^{(2)}(z) = \sigma_2 M^{(2)}(-z) \sigma_2 = \sigma_1 M^{(2)}(\bar{z}) \sigma_1 \);
- **Jump condition:** \( M^{(2)} \) has continuous boundary values \( M^{(2)}_{\pm}(z) \) on \( \Sigma^{(1)} \) and

\[
M^{(2)}_{+}(z) = M^{(2)}_{-}(z) V^{(2)}(z), \quad z \in \Sigma^{(1)},
\]

(5.27)
where

\[
V^{(2)}(z) = \begin{cases}
\begin{pmatrix}
1 & -r_2 D^{-2} e^{-2 i t g} \\
0 & 1
\end{pmatrix}, & \text{as } z \in \Sigma_1^+, \\
\begin{pmatrix}
1 & 0 \\
r_1 D^2 e^{2 i t g} & 1
\end{pmatrix}, & \text{as } z \in \Sigma_1^-, \\
\begin{pmatrix}
1 & 0 \\
r_1 D^2 e^{2 i t g} & 1
\end{pmatrix}, & \text{as } z \in \Sigma_2^+; \\
\begin{pmatrix}
1 & -r_2 D^{-2} e^{-2 i t g} \\
0 & 1
\end{pmatrix}, & \text{as } z \in \Sigma_2^-, \\
\begin{pmatrix}
0 & ie^{-it B_2} \\
ine^{it B_2} & 0
\end{pmatrix}, & \text{as } z \in [-c, -z_1], \\
\begin{pmatrix}
0 & ie^{-it B_2} \\
ine^{it B_2} & 0
\end{pmatrix}, & \text{as } z \in [-z_1, -z_0], \\
\begin{pmatrix}
0 & ie^{-it B_1} \\
ine^{it B_1} & 0
\end{pmatrix}, & \text{as } z \in [-1, 1], \\
\begin{pmatrix}
0 & i D e^{-2 i t g} \\
i & 0
\end{pmatrix}, & \text{as } z \in [z_0, z_1], \\
\begin{pmatrix}
0 & i D e^{-2 i t g} \\
i & 0
\end{pmatrix}, & \text{as } z \in [z_1, c],
\end{cases}
\]

- Asymptotic behaviors: \( M^{(2)}(z) = I + O(z^{-1}), \ z \to \infty; \)
- Singularity: \( M^{(2)}(z) \) has singularity at \( z = \pm c \) with

\[
M^{(2)}(z) \sim O(z \mp c)^{-1/4}, \ z \to \pm c, \ \pm 1 \ in \ \mathbb{C} \setminus \mathbb{R}. \tag{5.30}
\]

Away from \( \mathbb{R} \), the jump \( \dot{V}^{(2)}(z) \) exponentially approaches the identity matrix as \( t \to \infty \). So we expect to only consider the jump on \( \mathbb{R} \). To arrive at this goal, we denote \( U(\xi) \) as the union set of neighborhood of \( \pm z_0 \):

\[
U(\xi) = U(\pm z_0), \ U(\pm z_0) = \{z : |z \mp z_0| \leq \varrho\}. \tag{5.31}
\]

Here, \( \varrho \) is a small positive constant such that \( \varrho < \min\{\frac{z_0 - 1}{3}, \frac{z_1 - z_0}{3}\} \). In the case of [43], \( \theta(z_1) \) is in \( \mathbb{R} \) such that as \( t \to \infty \), the term \( \theta''(z_1)(z - z_1)^2 \) in the Taylor expansion of \( \theta \) at \( z = z_1 \) is dominating. But in this paper, the phase point \( \pm z_1 \) is on the cut
with $\text{Im}(z_1)_+ < 0$ which means the exponential function in $\hat{V}^{(2)}(z)$ also decays exponentially on $\pm z_1$. In fact, it is also decays exponentially on $(z_0, z_1] \cup [-z_1, z_0)$.

Thus, the jump matrix $V^{(2)}(z)$ uniformly goes to $I$ on $\Sigma^{(1)} \setminus U(\xi)$. So outside the $U(\xi)$ there is only exponentially small error (in $t$) by completely ignoring the jump condition of $M^{(2)}(z)$. And this proposition enlightens us to construct the solution $M^{(2)}(z)$ as follow

$$M^{(2)}(z) = M^{(2)}(z; \xi, c) = \begin{cases} E(z; \xi, c)M^{\text{mod}}(z; \xi, c) & z \notin U(\xi) \\ E(z; \xi, c)M^{\text{lo},+}(z; \xi, c) & z \in U(z_0) \\ E(z; \xi, c)M^{\text{lo},-}(z; \xi, c) & z \in U(-z_0) \end{cases}.$$  \hfill (5.32)

Here $M^{\text{mod}}(z)$ is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 5.2.1. $M^{\text{lo},\pm}(z)$ are local model of $\pm z_0$ which solution can be expressed in terms of Airy functions shown in Subsection 5.2.2. And $E(z; \xi, c)$ is the error function, which will be discussed in subsection 5.2.3. by the small-norm RH problem theory.

### 5.2.1 Model RH problem on Riemann surface

We consider the following model RH problem with its jump matrix on $\mathbb{R}$.

**RHP 13.** Find a matrix-valued function $M^{\text{mod}}(z)$ with following identities:

- **Analyticity:** $M^{\text{mod}}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{\text{cut}}$, with

  $$\Sigma^{\text{cut}} = [-c, -z_0] \cup [-1, 1] \cup [z_0, c];$$

- **Asymptotic behaviors:** $M^{\text{mod}}(z) \sim I + \mathcal{O}(z^{-1})$, $|z| \to \infty$;

- **Jump condition:** $M^{\text{mod}}(z)$ satisfies the jump relation

  $$M_+^{\text{mod}}(z) = M_-^{\text{mod}}(z)V^{\text{mod}}(z), \quad z \in \Sigma^{\text{cut}},$$

where the jump matrix $V^{\text{mod}}(z)$ is given by

$$V^{\text{mod}}(z) = \begin{cases} \begin{pmatrix} 0 & i e^{-itB_2} \\ i e^{itB_2} & 0 \end{pmatrix}, & \text{as } z \in [-c, -z_0], \\ \begin{pmatrix} 0 & i e^{-itB_1} \\ i e^{itB_1} & 0 \end{pmatrix}, & \text{as } z \in [-1, 1], \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \text{as } z \in [z_0, c]. \end{cases}$$  \hfill (5.33)
Singularity: $M^{\text{mod}}(z)$ has singularity at $z = \pm c$ with

$$M^{\text{mod}}(z) \sim \mathcal{O}(z \mp p)^{-1/4}, \quad z \to p = \pm c, \pm 1, \pm z_0 \in \mathbb{C} \setminus \mathbb{R}. \quad (5.34)$$

The solution $M^{\text{mod}}$ of the model RH problem can be characterized by $\vartheta$ function on the Riemann surface with genus-2. We define

$$\mathcal{N}(z) = \frac{1}{2} \begin{pmatrix} \kappa(z) + \kappa(z)^{-1} & \kappa(z) - \kappa(z)^{-1} \\ \kappa(z) - \kappa(z)^{-1} & \kappa(z) + \kappa(z)^{-1} \end{pmatrix}, \quad (5.35)$$

where

$$\kappa(z) = \left[ \frac{(z-c)(z-1)(z+z_0)}{(z-z_0)(z+1)(z+c)} \right]^{1/4}, \quad z \in \mathbb{C} \setminus \Sigma^{\text{cut}},$$

$$\kappa(z) = 1 + \mathcal{O}(z^{-2}), \quad z \to \infty.$$ 

Let $\omega_i, a_i, b_i, i = 1, 2$ denote the standard holomorphic differentials, canonical $a, b$ periods on the genus 2 Riemann surface $\mathcal{M}$ which is covered by two sheets of $\mathbb{C} \setminus \Sigma^{\text{cut}}$. And matrix $\tilde{B} \in GL_2(\mathbb{C})$, $\tilde{B}_{ij} = \oint_{b_j} \omega_i, \quad i, j = 1, 2$. Considering the Abel map

$$\mathcal{A} : \mathcal{M} \to \mathbb{C}^2/\tilde{B}M + N, \quad M, N \in \mathbb{Z}^2 \quad (5.36)$$

$$P \mapsto \left( \int_c^P \omega_1, \int_c^P \omega_2 \right)^T. \quad (5.37)$$

The $\vartheta$ function is defined by

$$\vartheta(z) = \sum_{n \in \mathbb{Z}^2} \exp(\pi i \langle Bn, n \rangle + 2\pi i \langle n, z \rangle) \quad (5.38)$$

which satisfies

$$\vartheta(z \pm e_j) = \vartheta(z), \quad (5.39)$$

$$\vartheta(z \pm Be_j) = \exp(\mp 2\pi i u_j - \pi i B_{jj}) \vartheta(z). \quad (5.40)$$

According to [67], there is a constant $\mathcal{K} \in \mathbb{C}^2$ such that for arbitrary divisor $\mathcal{P}_0$,

$$\vartheta(\mathcal{A}(P) - \mathcal{A}(\mathcal{P}_0) - \mathcal{K}) := \vartheta(\mathcal{A}(P) - K)$$

has two zeros $P_1, P_2$ on $\mathcal{M}$ with $P_1 + P_2 = \mathcal{P}_0$. 

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Observing if $P_1, P_2$ is zeros of $N_{11}, N_{22}$ only if $P_1', P_2'$ is zeros of $N_{12}, N_{21}$ where $P_1', P_2'$ are the same point of $P_1, P_2$ on the other sheet. Therefore, $\vartheta(\mathcal{A}(P) - K), \vartheta(\mathcal{A}(P) + K)$ have the same zeros as $N_{11}, N_{22}$ and $N_{12}, N_{21}$ have respectively at the time of $P_0 = P_1 + P_2$. We then can show that the RHP $\text{(13)}$ admits the solution

$$M^{\text{mod}}(z) = F(\infty) \begin{pmatrix} N_{11} \frac{\vartheta(A(z)-K+C)}{\vartheta(A(z)-K)} & N_{12} \frac{\vartheta(-A(z)-K+C)}{\vartheta(A(z)+K)} \\ N_{21} \frac{\vartheta(A(z)+K+C)}{\vartheta(A(z)+K)} & N_{22} \frac{\vartheta(-A(z)+K+C)}{\vartheta(-A(z)+K)} \end{pmatrix},$$

(5.41)

where

$$F(\infty) = \frac{1}{2} \text{diag} \left( \frac{\vartheta(A(z) - K)}{\vartheta(A(z) - K + C)}, \frac{\vartheta(A(z) - K)}{\vartheta(A(z) - K - C)} \right).$$

Noting that as $z \to 0 \in \mathbb{C}^+$,

$$N(z) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + z \frac{\sqrt{2}}{4} \left( \frac{1}{z_0} - \frac{1}{c} - 1 \right) \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} + \mathcal{O}(z^2),$$

(5.42)

then by using (5.41), we have

$$M^{\text{mod}}(z) = M^{\text{mod}}(0) + M^{\text{mod}}_1 z + \mathcal{O}(z^2),$$

(5.43)

where

$$M^{\text{mod}}_1 = \frac{\sqrt{2}}{4} P(\infty) \left( \frac{1}{z_0} - \frac{1}{c} - 1 \right) \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} G(0) + \frac{\sqrt{2}}{2} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} G_z(0),$$

and

$$G(z) = \begin{pmatrix} \vartheta(A(z) - K + C) / \vartheta(A(z) - K) & \vartheta(-A(z) - K + C) / \vartheta(A(z) + K) \\ \vartheta(A(z) + K + C) / \vartheta(A(z) + K) & \vartheta(-A(z) + K + C) / \vartheta(-A(z) + K) \end{pmatrix}.$$

5.2.2 Localized RH problem near phase points

Although the exponential function in $\hat{V}^{(2)}(z)$ decays exponentially on $\Sigma^{(1)} \setminus \Sigma^{\text{cut}}$ as $t \to \infty$. This decay is not uniform with respect to $z$ as $z$ approaches $\Sigma^{\text{cut}}$. Thus, on the parts of the contour $\Sigma^{(1)}$, that lie near $\Sigma^{\text{cut}}$, we need to introduce local solutions that are better approximations of $M^{(2)}(z)$ than $M^{\text{mod}}(z)$ is. Using these local approximations, we can derive appropriate error estimates as well as higher order asymptotics beyond the $O(1)$ term. In this subsection, we only give the details
of the model around $z_0$. We consider $M^{lo,+}(z)$ here as an example. Firstly, we denote $P(z_0)$ is the neighborhood of $z_0$ in the Riemann surface corresponding to $U(z_0)$. And let $\omega$ be the local coordinate of $P(z_0)$. It is an analytic homeomorphism. Observing $g$ is an holomorphic function in $P(z_0)$. And $z_0$ is its zero of order 3. Therefore, there exist a holomorphic homeomorphism $f_+$ on $P(z_0)$ such that

$$g(\omega) = -\frac{2i}{3}f_+^3(\omega). \quad (5.44)$$

Here, on the complex plane, because of $g_+((z_0,c)) \subset i\mathbb{R}^-$, we can choose $f_+ \in \mathbb{R}^+$ in $z \in (z_0,c) \cap U(z_0)$. Let

$$\lambda_+(\omega) = t^2 f_+^2(\omega). \quad (5.45)$$

Because of $g(\omega) = -g(-\omega)$, $z - z_0 \mapsto \lambda_+$ is a holomorphic homeomorphism from $U(z_0)$ to a neighborhood of zero. And

$$\frac{4}{3} \lambda_+^\frac{2}{3} = 2i t g_+, \quad z \in (z_0,c) \cap U(z_0), \quad (5.46)$$

where $(\cdot)^\frac{2}{3}$ is the same as in the Airy model in the Appendix A.

From the definition of $\lambda_+$, we have

$$\frac{4}{3} \lambda_+^\frac{2}{3} = 2i t g, \quad \text{Im} z > 0, \quad \frac{4}{3} \lambda_+^\frac{2}{3} = -2i t g, \quad \text{Im} z < 0. \quad (5.47)$$

We choose the jump contour $\Sigma_3, \Sigma_4$ satisfying

$$\lambda_+: \Sigma_3 \cap U(z_0) \mapsto e^{\frac{2i\pi}{3}} \mathbb{R}^+, \quad (5.48)$$
$$\lambda_+: \Sigma_4 \cap U(z_0) \mapsto e^{\frac{4i\pi}{3}} \mathbb{R}^+. \quad (5.49)$$

Define $M^{lo,+}(z)$ as follow

$$M^{lo,+}(z) = M^{mod} H(z; z_0)^{-1} N^{-1} \frac{2a}{3} \lambda_+^\frac{2}{3} m^{A_1}(\lambda_+) H(z; z_0). \quad (5.50)$$

The definition of $N$ comes from (A.9) and

$$H(z; z_0) = \begin{cases} \sigma_3^2 r_1^2, & z - z_0 \in \mathbb{C}^+ \cap U(z_0), \\ \sigma_3^2 r_2^2, & z - z_0 \in \mathbb{C}^- \cap U(z_0). \end{cases} \quad (5.51)$$
Moreover, \( M^{\text{mod}} H(z; z_0)^{-1} N^{-1} \lambda_+^{\frac{2\sigma_3}{3}} \) is a analytic and invertible function in \( U(z_0) \).

Similarly,
\[
M^{lo,-}(z) = M^{\text{mod}} H(z; z_0)^{-1} N^{-1} \lambda_+^{\frac{2\sigma_3}{3}} m^A(\lambda_-) H(z; z_0).
\] (5.52) 

with
\[
H(z; z_0) = \begin{cases} 
\exp{itB_2 \sigma_3 D^\sigma_3 r_1^{\sigma_3}}, & z - z_0 \in C^+ \cap U(-z_0), \\
\exp{-itB_2 \sigma_3 D^\sigma_3 r_2^{\sigma_3}}, & z - z_0 \in C^- \cap U(-z_0), 
\end{cases}
\] (5.53) 

\[
g = -\frac{2i}{3} f^3 + \frac{B_2}{2}, \quad \lambda_- = t^\frac{2}{3} f^2,
\] (5.54) 

where \( f_- \in \mathbb{R}^+ \) on \( z \in (-c, -z_0) \cap U(-z_0) \) and \( M^{\text{mod}} H(z; z_0)^{-1} N^{-1} \lambda_+^{\frac{2\sigma_3}{3}} \) is a analytic and invertible function in \( U(-z_0) \). Moreover, as \( z \to z_0 \) in \( C \setminus [z_0, c] \), \( \lambda_+ \) has expending
\[
\lambda_+ = (\tilde{A}_+)^{\frac{2}{3}} (z - z_0) + \frac{2\tilde{B}_+}{3\tilde{A}_+^\frac{1}{3}} (z - z_0)^2 + O((z - z_0)^3),
\] (5.55) 

\[
\lambda_- = (\tilde{A}_-)^{\frac{2}{3}} (z + z_0) + \frac{2\tilde{B}_-}{3\tilde{A}_-^\frac{1}{3}} (z + z_0)^2 + O((z + z_0)^3),
\] (5.56) 

where
\[
\tilde{A}_+ = \frac{t}{2} \left( \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \right)^{\frac{1}{2}} \left( \frac{\xi - 1}{2} z_0 + \frac{c}{z_0^2} \left( 1 + \frac{1}{c^2} \right) - \frac{3c}{z_0} \right),
\]
\[
\tilde{B}_+ = \frac{3t}{10} \left[ -\frac{1}{2} \left( \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \right)^{\frac{1}{2}} \left( \frac{\xi - 1}{2} z_0 + \frac{c}{z_0^2} \left( 1 + \frac{1}{c^2} \right) - \frac{3c}{z_0} \right) \right] + \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \left( \frac{\xi - 1}{2} \frac{z_0^2}{z_0} \left( 1 + \frac{1}{c^2} \right) + c \right),
\]
\[
\tilde{A}_- = -\frac{i}{2} \left( \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \right)^{\frac{1}{2}} \left( -\frac{\xi - 1}{2} z_0 + \frac{c}{z_0^2} \left( 1 + \frac{1}{c^2} \right) - \frac{3c}{z_0} \right),
\]
\[
\tilde{B}_- = \frac{3t}{10} \left[ -\frac{i}{2} \left( \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \right)^{\frac{1}{2}} \left( \frac{\xi - 1}{2} z_0 + \frac{c}{z_0^2} \left( 1 + \frac{1}{c^2} \right) - \frac{3c}{z_0} \right) \right] + \frac{2z_0}{(z_0^2 - 1)(c^2 - z_0^2)} \left( \frac{\xi - 1}{2} + \frac{c}{z_0^2} \left( 1 - \frac{1}{c^2} \right) - \frac{6c}{z_0^2} \right). \] (5.57)
5.2.3 The small norm RH problem for error function

In this subsection, we consider the error matrix-function $E(z; \xi, c)$.

**RHP 14.** Find a matrix-valued function $E(z; \xi, c)$ with following properties:

- **Analyticity:** $E(z; \xi, c)$ is analytical in $\mathbb{C} \setminus \Sigma^E$, where
  \[
  \Sigma^E = \partial U_\xi \cup \left[ \Sigma^{(2)} \setminus (U_\xi \cup [-c, -z_1] \cup [-1, 1] \cup [z_1, c]) \right];
  \]

- **Asymptotic behaviors:**
  \[
  E(z; \xi, c) \sim I + O(z^{-1}), \quad |z| \to \infty;
  \]

- **Jump condition:** $E(z; \xi, c)$ has continuous boundary values $E_\pm(z; \xi, c)$ on $\Sigma^E$ satisfying
  \[
  E_+(z; \xi, c) = E_-(z; \xi, c)V^E(z),
  \]
  where the jump matrix $V^E(z)$ is given by
  \[
  V^E(z) = \begin{cases} 
  M^{mod}(z)V^{(2)}(z)M^{mod}(z)^{-1}, & z \in \Sigma^E \setminus \partial U_\xi, \\
  M^{lo, \pm}(z)M^{mod}(z)^{-1}, & z \in \partial U_{\pm z_0},
  \end{cases}
  \]

which is shown in Figure 13.

The above RH problem, by (5.50), (5.52) and (A.8), satisfies

\[
\| V^E(z) - I \|_2 \lesssim O(t^{-1}).
\]

![Figure 13: The jump contour $\Sigma^E$ for the $E(z; \xi, c)$. The red circles are $U(\xi)$.](image-url)
Similar with the discussion in Section 3.3, the RHP (14) admits a unique solution, which can be given by
\[
E(z; \xi, c) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E(s) - I)}{s - z} ds,
\] (5.60)
where the \( \varpi \in L^\infty(\Sigma^E) \) is the unique solution of following equation
\[
(1 - C_E) \varpi = C_E (I). \tag{5.61}
\]

In order to reconstruct the solution \( u(y, t) \) of (1.3), we need the asymptotic behavior of \( E(z; \xi, c) \) as \( z \to 0 \in \mathbb{C}^+ \) and the long time asymptotic behavior of \( E(0) \). Note that when we estimate its asymptotic behavior, we only need to consider the calculation on \( \partial U(\xi) \) because it approach zero exponentially on other boundary. For convenience, we denote
\[
F^\pm = M^{mod} H(z; \pm z_0)^{-1} N^{-1} f^\pm_{\mp} = [F^\pm_{ij}]_{2 \times 2}, \tag{5.62}
\]
with \( (F^\pm)^{-1} \triangleq [F^{\pm,*}_{ij}]_{2 \times 2} \).

**Proposition 9.** As \( z \to 0 \in \mathbb{C}^+ \), we have
\[
E(z; \xi, c) = E(0) + E_1 z + \mathcal{O}(z^2), \tag{5.63}
\]
where
\[
E(0) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s} ds, \tag{5.64}
\]
with long time asymptotic behavior
\[
E(0) = I + t^{-1} H^{(0)} + \mathcal{O}(t^{-2}). \tag{5.65}
\]

And
\[
H^{(0)} = H^{(0)}(\xi, c) = \sum_{p = \pm z_0} \frac{d}{dz} \left( \frac{1}{z} F^\pm \left( \begin{array}{cc}
0 & -\frac{5}{48} \tilde{A}_-^\mp \\
0 & 0
\end{array} \right) (F^\pm)^{-1} \right)(p) + \sum_{p = \pm z_0} \frac{1}{p} \left( F^\pm \left( \begin{array}{cc}
0 & \frac{5}{30} \tilde{B}_\pm \tilde{A}_\pm^{-\frac{5}{3}} \\
-\frac{5}{48} \tilde{A}_\pm^{-\frac{1}{3}} & 0
\end{array} \right) (F^\pm)^{-1} \right)(p).
\]

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Here, $\tilde{B}_\pm$, $\tilde{A}_\pm$ is shown in (5.57). And

$$E_1 = \frac{1}{2\pi i} \int_{\Sigma E} \frac{(I + \varpi(s))(V^E - I)}{s^2} ds,$$  \hspace{1cm} (5.66)

satisfying long time asymptotic behavior condition

$$E_1 = t^{-1}H^{(1)} + O(t^{-2}),$$ \hspace{1cm} (5.67)

where

$$H^{(1)} = H^{(1)}(\xi, c) = \sum_{p = \pm z_0} d \left( \frac{1}{z^2} F^\pm \left( \begin{array}{cc} 0 & -\frac{5i}{48} \tilde{A}_\pm^{-\frac{7}{3}} \\ 0 & 0 \end{array} \right) (F^\pm)^{-1} \right) (p)$$

and

$$+ \sum_{p = \pm z_0} \frac{1}{p^2} \left( F^\pm \left( \begin{array}{cc} 0 & \frac{5}{36} \tilde{B}_\pm \tilde{A}_\pm^{-\frac{7}{3}} \\ -\frac{7i}{48} \tilde{A}_\pm^{-\frac{4}{3}} & 0 \end{array} \right) (F^\pm)^{-1} \right) (p).$$

Proof. By using expansion of $V^E$ and $\varpi(s) = O(t^{-1})$, we have

$$\int_{\Sigma E} \frac{(I + \varpi(s))(V^E - I)}{s} ds$$

$$= \frac{1}{t} \int_{\partial U(\pm z_0)} \frac{M^{mod}H(z; \pm z_0)^{-1}m_1^{Ai}H(z; \pm z_0)(M^{mod})^{-1}}{s f^4 \pm} ds + O(t^{-2}).$$ \hspace{1cm} (5.68)

Rewrite the as

$$M^{mod}H(z; \pm z_0)^{-1}m_1^{Ai}H(z; \pm z_0)(M^{mod})^{-1} = F^\pm \tilde{f}_{\pm} \tilde{f}^{\frac{\alpha_4}{4}} N m_1^{Ai} N^{-1} \tilde{f}_{\pm} \tilde{f}^{\frac{\alpha_4}{4}} (F^\pm)^{-1}.$$ \hspace{1cm}

Here, from (5.57) and the definition of $N$, $m_1^{Ai}$ in Appendix A, as $z \to z_0 \in \mathbb{C}^+$, we have

$$f^\pm \tilde{f}_{\pm} \tilde{f}^{\frac{\alpha_4}{4}} N m_1^{Ai} N^{-1} \tilde{f}_{\pm} \tilde{f}^{\frac{\alpha_4}{4}} =$$

$$\frac{1}{(z + z_0)^2} \left( \begin{array}{cc} 0 & -\frac{5i}{48} \tilde{A}_\pm^{-\frac{4}{3}} \\ 0 & 0 \end{array} \right) + \frac{1}{(z + z_0)} \left( \begin{array}{cc} 0 & \frac{5}{36} \tilde{B}_\pm \tilde{A}_\pm^{-\frac{7}{3}} \\ -\frac{7i}{48} \tilde{A}_\pm^{-\frac{4}{3}} & 0 \end{array} \right) + O(1).$$ \hspace{1cm} (5.69)

Then by residue theorem we finally arrive at the result. □
5.3 Opening the jump in the region $3/4 < \xi < 1$

This region include two cases: (1) $2 < c^2 < 4$, $1 - \frac{2(c^2 - 2)}{c^2} < \xi < 1$; (2) $c^2 > 4$, $\xi_m < \xi < 1$. We introduce a $g$ function by

$$dg = \frac{Y(z)}{z^3} \left[ \frac{1 - \xi}{2} z^4 - c \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) z^2 + \frac{2c}{z_0} \right] dz$$

will have another zero on $\mathbb{R}$ except on cut. It means $g$ have three pairs of stationary phase points on $\mathbb{R}$, which will gives more contribution as $t \to \infty$. Consider the equation

$$\frac{1 - \xi}{2} z^4 - c \left( 1 + \frac{1}{c^2} - \frac{1}{z_0^2} \right) z^2 + \frac{2c}{z_0} = 0.$$

It has two pairs of zeros on $\mathbb{R}$: $\pm z_1 \in (z_0, c)$, $\pm z_2$. Note that, $z_1^2 z_2^2 = \frac{4c}{z_0(1-\xi)}$. For a given $c$, when $\xi$ decreases from $1$, $z_2$ as a function of $\xi$ decreases from $+\infty$. We denote $\xi_m$ as the critical condition of $\xi$ that stationary phase point $z_2$ merge $c$. Under this case, the sign table of $\text{Im}g$ has the following figure:

```
| Img < 0 | Img < 0 | Img > 0 | Img < 0 | Img > 0 |
|--------|--------|--------|--------|--------|
| -z_2   | -c    | -z_0  | 0      | 1      |
| Img > 0 | Img > 0 | Img < 0 | Img > 0 | Img < 0 |
```

Figure 14: In the purpler region, $\text{Img}>0$ while in the white region, $\text{Img}<0$.

Similarly as the above section, we define the following contour relying on $\xi, c$:

$$\Sigma_1^\pm = \left\{ -z_1 + e^{(\pi+\psi)i} l, l \in (0, \frac{z_1 - z_2}{2 \cos \psi}) \right\} \cup \left\{ z_1 + e^{\pm \psi i} l, l \in (0, \frac{z_1 - z_2}{2 \cos \psi}) \right\}$$

$$\cup \left\{ z_2 + e^{(\pi+\psi)i} l, l \in (0, \frac{z_1 - z_2}{2 \cos \psi}) \right\} \cup \left\{ -z_2 + e^{\pm \psi i} l, l \in (0, \frac{z_1 - z_2}{2 \cos \psi}) \right\},$$

$$\Sigma_2^\pm = \left\{ -z_0 + e^{\pm \psi i} l, l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\} \cup \left\{ z_0 + e^{(\pi+\psi)i} l, l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\}$$

$$\cup \left\{ 1 + e^{\pm \psi i} l, l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\} \cup \left\{ -1 + e^{(\pi+\psi)i} l, l \in (0, \frac{z_0 - 1}{2 \cos \psi}) \right\}$$

$$\cup \left\{ -z_2 + e^{(\pi+\psi)i} \mathbb{R}^+ \right\} \cup \left\{ z_2 + e^{\pm \psi i} \mathbb{R}^+ \right\}.$$
Here $\psi \leq \frac{\pi}{4}$ is a small enough positive constant such that $\Sigma_j^\pm$, $j = 1, 2$ are contained in the region of $\text{Im} g > 0$. And similarly as above equation, $\Omega_j^\pm$ is a closed region, the edge of which is made up by $\Sigma_j^\pm$, $j = 1, 2$ and $\mathbb{R}$.

Figure 15: The region of $\Omega_j^\pm$, $j = 1, 2$. The same as Figure 14, purple region means $\text{Im} g(z) > 0$ while white region means $\text{Im} g(z) < 0$.

In this region of $\xi, c$, we introduce a piecewise matrix interpolation function

$$G(z) = \begin{cases} 
\begin{pmatrix} 1 & \frac{r_2e^{-2itg}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^+; \\
\begin{pmatrix} 1 & 0 \\ -r_1e^{2itg} & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^+; \\
\begin{pmatrix} 1 & -r_2e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^-; \\
I, & \text{as } z \text{ in elsewhere},
\end{cases} \quad (5.70)$$

Same as above section, $G(z)$ bring a new singularity. To deal with the jump on $\mathbb{R}$, we introduce an auxiliary function $D(z)$, which admits the following jump condition

$$D_-(z) = D_+(z)(1 - r_1r_2), \quad z \in [-z_2, z_2] \setminus [-c, c];$$

$$D_-(z)D_+(z) = i[r_2]_-, \quad z \in [-c, c] \setminus [-z_0, z_0];$$

$$D_-(z)D_+(z) = 1, \quad z \in [-1, 1].$$
Define \( Y_3(z) = (z^2 - 1)(z^2 - c^2)Y(z) \), and
\[
\log D(z) = \frac{Y_3(z)}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]-(s))}{(s-z)Y_3(s)} ds - \frac{Y_3(z)}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{(1 - r_1(s)r_2(s))}{(s-z)Y_3(s)} ds. \quad (5.71)
\]

**Proposition 10.** The scalar function \( D(z) \) satisfies the following properties
(a) \( D(z) \) is analytic on \( \mathbb{C} \setminus ((-z_2, -z_0) \cup [-1, 1] \cup (z_0, z_2)) \);
(b) \( D(z) \) has singularity at \( z = \pm c \) with
\[
D(z) = \mathcal{O}(z - p)^{\pm 1/4}, \quad z \to p \in \mathbb{C} \setminus \mathbb{R}, \quad p = c, -c. \quad (5.72)
\]
(c) As \( z \to \infty \in \mathbb{C} \setminus \mathbb{R} \), \( D(z) \) has limit \( D_{\infty}(z) \) with
\[
\log D_{\infty}(z) = -\frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]-(s))}{[X]_+(s)} \left( z^2 + zs + s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds
+ \frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(1 - r_1(s)r_2(s))}{X(s)} \left( z^2 + zs + s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds.
\]
(d) As \( z \to 0 \in \mathbb{C}^+ \),
\[
D_{\infty}(z) = D_{\infty}(0) \left( 1 + D_{\infty}^{(1)} z \right) + \mathcal{O}(z^2), \quad (5.73)
D(z) = D(0) \left( 1 + D^{(1)} z \right) + \mathcal{O}(z^2), \quad (5.74)
\]
where
\[
\log D_{\infty}(0) = \frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]-(s))}{[X]_+(s)} \left( \frac{1 + c^2 + z_0^2}{2} - s^2 \right) ds
+ \frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(1 - r_1(s)r_2(s))}{X(s)} \left( s^2 - \frac{1 + c^2 + z_0^2}{2} \right) ds,
D_{\infty}^{(1)} = -\frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{s \log(i[r_2]-(s))}{[X]_+(s)} ds + \frac{1}{2\pi i} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{s \log(1 - r_1(s)r_2(s))}{X(s)} ds,
\]
\[
\log D(0) = \frac{c_{z_0}}{2\pi} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]-(s))}{s[Y_3]_+(s)} ds - \frac{c_{z_0}}{2\pi} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(1 - r_1(s)r_2(s))}{sY_3(s)} ds,
D^{(1)} = \frac{c_{z_0}}{2\pi} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(i[r_2]-(s))}{s^2[Y_3]_+(s)} ds - \frac{c_{z_0}}{2\pi} \left( \int_{-c}^{z_0} + \int_{z_0}^{c} \right) \frac{\log(1 - r_1(s)r_2(s))}{s^2Y_3(s)} ds.
\]
(e) As \( z \to \pm z_2 \), there is

\[
\log D(z) = \mp \frac{\log(1 - r_1 r_2)}{2\pi i} \log(z \mp z_2) + \mathcal{O}(1), \quad z \to \pm z_2,
\]

where the logarithm function is analytic in \( U(\pm z_2) \setminus (\pm z_2, \pm z_2 \mp \epsilon) \) respectively with some positive number \( \epsilon \). Further, let

\[
\nu(z_2) = \pm \frac{1}{2\pi} \log(1 - r_1 r_2),
\]

Then we have

\[
D(z) = D_{\pm z_2}(z) \left( z \mp z_2 \right)^{\nu(\pm z_2)}, \tag{5.75}
\]

where \( D_{\pm z_2}(z) \) are bounded analytic functions in \( U(\pm z_2) \setminus (\pm z_2, \pm z_2 \mp \epsilon) \) respectively. And have continued boundary extension. Further, \( |D_{\pm z_2}(z) - D_{\pm z_2}(z_2)|/|z - z_2| \) is bounded on \( U(\pm z_2) \).

Denote

\[
\Sigma^{(2)} = \left( \bigcup_{j=1}^{2} \Sigma_{j}^{\pm} \right) \cup [-1, 1] \cup [-c, -z_0] \cup [z_0, c]. \tag{5.76}
\]

Through \( D(z) \) and \( G(z) \), in this region of \( \xi, c \), same as above subsection we give series of transformations:

\[
N \to M^{(1)} \to M^{(2)} = D^{-\sigma_3} e^{itg(\infty)} \sigma_3 N e^{it(p - g)} \sigma_3 D\sigma_3, \tag{5.77}
\]

which then satisfies the following RH problem.

**RHP 15.** Find a matrix-valued function \( M^{(2)}(z) \) which satisfies

- **Analyticity:** \( M^{(2)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(2)} \);
- **Symmetry:** \( M^{(2)}(z) = \sigma_2 M^{(2)}(-z) \sigma_2 = \sigma_1 M^{(2)}(\bar{z}) \sigma_1 \);
- **Jump condition:** \( M^{(2)} \) has continuous boundary values \( M^{(2)}_{\pm}(z) \) on \( \Sigma^{(2)} \) and

\[
M^{(2)}_{+}(z) = M^{(2)}_{-}(z) V^{(2)}(z), \quad z \in \Sigma^{(2)}, \tag{5.78}
\]
where

\[
V^{(2)}(z) = \begin{cases}
(1 - r_1 D^{-z} e^{-2 \imath t \varphi_0}) & \text{as } z \in \Sigma^+_1, \\
(1 - r_1 D^2 e^{2 \imath t \varphi_0}) & \text{as } z \in \Sigma^+_1, \\
(1 - r_1 D^2 e^{2 \imath t \varphi_0}) & \text{as } z \in \Sigma^+_2, \\
(1 - r_2 D^2 e^{-2 \imath t \varphi_0}) & \text{as } z \in \Sigma^-_2, \\
(0, i e^{-i t B_2}) & \text{as } z \in [-c, -z_1], \quad (5.79) \\
(0, i e^{i t B_2}) & \text{as } z \in [-z_1, -z_0], \\
(0, i e^{-i t B_1}) & \text{as } z \in [-1, 1], \\
(0, i D^z e^{-2 i t g_+}) & \text{as } z \in [z_0, z_1], \\
(0, i) & \text{as } z \in [z_1, c],
\end{cases}
\]

- Asymptotic behavior: \( M^{(2)}(z) = I + O(z^{-1}), \quad z \to \infty; \)
- Singularity: \( M^{(2)}(z) \) has singularity at \( z = \pm c \) with:

\[
M^{(2)}(z) \sim O(z \mp c)^{-1/4}, \quad z \to \pm c, \quad \pm 1 \text{ in } \mathbb{C} \setminus \mathbb{R}. \quad (5.80)
\]

Except the cut away from \( \mathbb{R} \), the jump \( V^{(2)}(z) \) exponentially approaches the identity matrix as \( t \to \infty \). So we expect to only consider the jump on \( \mathbb{R} \). However, different from above section, in this region, \( g \) has another pair of stationary phase points on \( \mathbb{R} \). So in this case, we denote \( U(\xi) \) as the union set of neighborhood of \( \pm z_0 \) and \( \pm z_2 \):

\[
U(\xi) = U(\pm z_0) \cup U(\pm z_2), \quad U(\pm z_j) = \{z : |z \mp z_j| \leq \varrho\}, \quad j = 0, 2. \quad (5.81)
\]

Here, \( \varrho \) is a small positive constant such that \( \varrho < \min \left\{ \frac{z_0 - 1}{3}, \frac{z_1 - 2z_0}{3}, \frac{z_2 - c}{3}, \epsilon \right\} \). Thus, the jump matrix \( V^{(2)}(z) \) uniformly goes to \( I \) on \( \Sigma^{(2)} \setminus U(\xi) \). So outside the \( U(\xi) \) there
is only exponentially small error (in $t$) by completely ignoring the jump condition of $M^{(2)}(z)$. And this proposition enlightens us to construct the solution $M^{(2)}(z)$ as follow:

$$M^{(2)}(z) = \begin{cases} 
E(z; \xi, c)M^{\text{mod}}(z; \xi, c), & z \notin U(\xi), \\
E(z; \xi, c)M^0, \pm (z; \xi, c), & z \in U(\pm z_0), \\
E(z; \xi, c)M^{\text{mod}}(z; \xi, c)M^{2, \pm}(z; \xi, c), & z \in U(\pm z_2).
\end{cases}$$

Here, same as $M^{\text{mod}}(z)$ is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 5.2.1. The difference is $M^{j, \pm}(z)$ are local model of $\pm z_j, j = 0, 2$. When $j = 0$, same as above subsection, its solution can be expressed in terms of Airy functions shown in Subsection 5.2.2. But when $j = 2$, its solution can be expressed in terms of parabolic cylinder shown in Subsection 5.3.1. And $E(z; \xi, c)$ is the error function, which will be discussed in subsection 5.3.2.

### 5.3.1 Localized RH problem near phase points

As $t \to +\infty$, we consider to reduce the RHP 15 to a model RH problem whose solution can be given explicitly in terms of parabolic cylinder functions on every contour $\Sigma_+^{(0)} = \Sigma^{(2)} \cap U(\pm z_2)$ respectively. And we only give the details of $\Sigma_+^{(0)}$, the model of other critical point can be constructed similar. We denote $\hat{\Sigma}_+^{(0)}$ as the contour $\{z = z_2 + le^{\pm \varphi}, l \in \mathbb{R}\}$ oriented from $\Sigma_+^{(0)}$, and $\hat{\Sigma}_j$ is the extension of $\Sigma_j$ respectively. And for $z$ near $z_2$, note that $g''(z_2) > 0$, so we rewrite phase function $g$ as

$$g(z) = g(z_2) + (z - z_2)^2 \frac{g''(z_2)}{2} + O((z - z_2)^3). \quad (5.82)$$

Consider following local RH problem:

**RHP 16.** Find a matrix-valued function $M^{2, +}(z)$ with following properties:

- **Analyticity:** $M^{2, +}(z)$ is analytical in $\mathbb{C} \setminus \hat{\Sigma}_+^{(0)}$;
- **Jump condition:** $M^{2, +}(z)$ has continuous boundary values $M^{2, +}_\pm$ on $\hat{\Sigma}_+^{(0)}$ and

$$M^{2, +}_+(z) = M^{2, +}_-(z)V^{2, +}(z), \quad z \in \hat{\Sigma}_+^{(0)}, \quad (5.83)$$
where the jump matrix $V^{2,+}(z)$ is given in Figure 16.

- Asymptotic behaviors: $M^{2,+}(z) = I + O(z^{-1})$, $z \to \infty$.

RHP 16 does not possess the symmetry condition shared by preceding RH problem, because it is a local model and will only be used for bounded values of $z$. In order to motivate the model, let $\zeta = \zeta(z)$ denote the rescaled local variable

$$\zeta(z) = 2^{1/2} \sqrt{g''(z_2)} (z - z_2).$$

This map is a conformal bijection maps $U(z_2)$ to an expanding neighborhood of $\zeta = 0$. We choose the branch which maps the upper half plane to the lower half plane. Moreover, we denote:

$$r_{z_2}^\pm = r_1(\pm z_2)D_{z_2}^2(\pm z_2)e^{2itg(\pm z_2)}(4tg''(\pm z_2))^{i\nu(\pm z_2)}.$$  \hspace{1cm} (5.85)

where, $D_{z_2}$ and $\nu(z_2)$ are defined in (5.75).

Through this change of variable, the jump $V^{2,+}(z)$ approximates to the jump of a parabolic cylinder model problem as follow:

RHP 17. Find a matrix-valued function $M^{pc}(\zeta; \xi)$ with following properties:

- Analyticity: $M^{pc}(\zeta; \xi)$ is analytical in $\mathbb{C} \setminus \Sigma^{pc}$ with $\Sigma^{pc} = \{\Re e^{\varphi i}\} \cup \{\Re e^{(\pi-\varphi)i}\}$;

- Jump condition: $M^{pc}$ has continuous boundary values $M^\pm_{pc}$ on $\Sigma^{pc}$ and

$$M^+_p(\zeta; \xi) = M^-_{pc}(\zeta; \xi)V^{pc}(\zeta), \quad \zeta \in \Sigma^{\zeta},$$

\hspace{1cm} (5.86)
where

\[ V^{pc}(\zeta; \xi) = \begin{cases} 
    \begin{pmatrix} 
    1 & 0 \\
    r^+_z e^{2i\nu(z_2)} e^{\frac{i}{2} \zeta^2} & 1
    \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{\varphi i}, \\
    \begin{pmatrix} 
    1 & 0 \\
    -\overline{r}_z e^{-2i\nu(z_2)} e^{-\frac{i}{2} \zeta^2} & 1
    \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{-\varphi i}, \\
    \begin{pmatrix} 
    1 & 0 \\
    -|r_z + z_2|^2 e^{\frac{i}{2} \zeta^2} & 1
    \end{pmatrix}, & \zeta \in \mathbb{R}^+ e^{(\pi-\varphi)i}.
\end{cases} \tag{5.87} \]

\[ M^{pc}(\zeta; \xi) = I + M^{pc}_1 \zeta^{-1} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty. \]

In a similar derivation to the RHP6, we can get

\[ M^{2, \pm}(z) = I + t^{-1/2} \frac{i}{z + z_2} \frac{\sqrt{g''(\pm z_2)}}{2 \sqrt{g''(\pm z_2)}} \begin{pmatrix} 0 & \tilde{\beta}^{\pm}_{12} \\
\tilde{\beta}^{\pm}_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}). \tag{5.88} \]

where

\[ \tilde{\beta}^{\pm}_{12} = \frac{\sqrt{2\pi} e^{-\frac{1}{2} \pi \nu(\pm z_2)} e^{\frac{\pi}{4} i}}{r^+_z \Gamma(-i\nu(\pm z_2))}, \quad \tilde{\beta}^{\pm}_{21} = -\nu(\pm z_2). \tag{5.89} \]

We finally obtain

**Proposition 11.** \( M^{2, \pm}(z) \) admits the following asymptotic expansion

\[ M^{2, \pm}(z) = I + t^{-1/2} \frac{A^{\pm}(\xi)}{1 + z_2} + \mathcal{O}(t^{-1}), \quad t \to +\infty, \tag{5.90} \]

where

\[ A^{\pm}(\xi) = \frac{i}{2 \sqrt{g''(z_2)}} \begin{pmatrix} 0 & \tilde{\beta}^{\pm}_{12} \\
\tilde{\beta}^{\pm}_{21} & 0 \end{pmatrix}. \tag{5.91} \]

### 5.3.2 The small norm RH problem for error function

In this subsection, we consider the error matrix-function \( E(z; \xi, c) \) in this region.
RHP 18. Find a matrix-valued function $E(z;\xi,c)$ with following properties:

- **Analyticity:** $E(z;\xi,c)$ is analytical in $\mathbb{C} \setminus \Sigma^E$, where

  $$\Sigma^E = \partial U(\xi) \cup \left[ \Sigma^{(2)} \setminus (U(\xi) \cup [-c,-z_1] \cup [-1,1] \cup [z_1,c]) \right];$$

- **Asymptotic behaviors:** $E(z;\xi,c) \sim I + \mathcal{O}(z^{-1}), \ |z| \to \infty$;

- **Jump condition:** $E(z;\xi,c)$ is continuous $E_\pm(z;\xi,c)$ on $\Sigma^E$ satisfying

  $$E_+(z;\xi,c) = E_-(z;\xi,c) V^E(z),$$

where the jump matrix $V^E(z)$ is given by

$$V^E(z) = \begin{cases} 
M^{\text{mod}}(z) V^{(2)}(z) M^{\text{mod}}(z)^{-1}, & z \in \Sigma^E \setminus \partial U(\xi), \\
M^{lo,\pm}(z) M^{\text{mod}}(z)^{-1}, & z \in \partial U_{\pm z_0}, \\
M^{\text{mod}}(z) M^{2,\pm}(z) M^{\text{mod}}(z)^{-1}, & z \in \partial U_{\pm z_2},
\end{cases} \tag{5.92}$$

which is shown in Figure 17.

![Figure 17: The jump contour $\Sigma^E$ for the $E(z;\xi,c)$](image)

Figure 17: The jump contour $\Sigma^E$ for the $E(z;\xi,c)$. The red circles are $U(\xi)$.

Similar with the discussion in Section 3.3, the RHP 18 satisfies

$$|V^E(z) - I| = \mathcal{O}(t^{-1/2}). \tag{5.93}$$

and admits a unique solution, which can be given by

$$E(z;\xi,c) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E(s) - I)}{s - z} ds, \tag{5.94}$$

where the $\varpi \in L^\infty(\Sigma^E)$ is the unique solution of following equation

$$(1 - C_E) \varpi = C_E(I). \tag{5.95}$$
In order to reconstruct the solution $u(y,t)$ of (1.3), we need the asymptotic behavior of $E(z;\xi,c)$ as $z \to 0 \in \mathbb{C}^+$ and the long time asymptotic behavior of $E(0)$.

**Proposition 12.** As $z \to 0 \in \mathbb{C}^+$, we have

$$E(z;\xi,c) = E(0) + E_1 z + \mathcal{O}(z^2),$$

where

$$E(0) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s} ds,$$

with long time asymptotic behavior

$$E(0) = I + t^{-1/2} H(0) + \mathcal{O}(t^{-2}).$$

And

$$H(0) = H(0)(\xi,c) = \sum_{p = \pm z} \frac{M_{\text{mod}}(p) A_{\pm}(\xi) M_{\text{mod}}(p)^{-1}}{p}.$$

Here, $\tilde{B}_\pm, \tilde{A}_\pm$ is shown in (5.57). And

$$E_1 = \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s^2} ds,$$

satisfying long time asymptotic behavior condition

$$E_1 = t^{-1/2} H(1) + \mathcal{O}(t^{-1}),$$

where

$$H(1) = H(0)(\xi,c) = \sum_{p = \pm z} \frac{M_{\text{mod}}(p) A_{\pm}(\xi) M_{\text{mod}}(p)^{-1}}{p^2}.$$

**Proof.** Substitute the long time asymptotic behavior of $V^E$, $\varpi(s)$ and Proposition 11 into $2\pi i (E(0) - I)$:

$$\int_{\Sigma^E} \frac{(I + \varpi(s)) (V^E - I)}{s} ds = \int_{\partial U(\pm z_2)} \frac{M_{\text{mod}}(s) (M^2_{\pm}(s) - I) M_{\text{mod}}(s)^{-1}}{s} ds + \mathcal{O}(t^{-1})$$

$$= \frac{1}{s(z \mp z_2)} M_{\text{mod}}(s) A_{\pm}(\xi) M_{\text{mod}}(s)^{-1} ds + \mathcal{O}(t^{-1}).$$

$$= t^{-1/2} \int_{\partial U(\pm z_2)} \frac{M_{\text{mod}}(s) A_{\pm}(\xi) M_{\text{mod}}(s)^{-1}}{s(z \mp z_2)} ds + \mathcal{O}(t^{-1}).$$
Then by residue theorem we finally arrive at the result.

6 The second-type genus-2 elliptic wave region

The Region IV is corresponding to the case $\frac{3}{4} < \xi < \xi_m$, $c > 2$. Here, as denote in above section, $\xi_m$ is the critical condition that stationary phase point $z_2$ merge $c$. Similarly, we need to construct new $g$-functions defined on genus 2 Riemann surface which has real branch points $\pm1$, $\pm z_1$ and $\pm z_2$ with $z_1 < z_2$. Different from above section, in Region III, $z_1$, $z_2$ both the stationary phase points of $g$. And the canonical homology basis $\{a_j, b_j\}_{j=1}^2$ is shown in Figure 10. In this two different cases, $g$ has different property. So after we proving the basic property of $g$, we will discuss separately.

![Figure 18: The canonical homology basis $\{a_j, b_j\}_{j=1}^2$ of the genius 2 Riemann surface.](image)

6.1 Constructing the $g$-function

To construct the $g$-function, we first introduce

$$Z(z) = \left[ \frac{(z^2 - z_1^2)(z^2 - z_2^2)}{(z^2 - 1)} \right]^{1/2}, \quad (6.1)$$

Here the branch of the square root is such that $Z(z) \in i\mathbb{R}^+$ for $z \in [z_1, z_2]$. And $z_1$, $z_2$ admit:

$$\frac{1 - \xi}{2} = \frac{1}{z_1 z_2} \left( 1 - \frac{1}{z_1^2} - \frac{1}{z_2^2} \right). \quad (6.2)$$

d$g$ is the derivative of $g$-function:

$$dg = \frac{Z(z)}{z^3} \left[ \frac{1 - \xi}{2} z^2 - \frac{2}{z_1 z_2} \right] dz. \quad (6.3)$$


The document contains a mathematical discussion on a meromorphic differential defined on a 2-genus Riemann surface, with differential forms $d^\ast g$ on the upper sheet and $-d^\ast g$ on the lower sheet. The g-function is given by

$$g(z) = \int_{z_2}^z d^\ast g, \quad z \in \mathbb{C} \setminus \Sigma^{mod}.$$ (6.4)

**Proposition 13.** There exist a pair of real numbers $z_1 = z_1(\xi, c)$, $z_2 = z_2(\xi, c)$ in $(1, c)$ such that the function $g(z)$ defined above has the following properties:

(a) The $a$-period of $g(z)$ is zero and the $b$-period of $g(z)$ is in $\mathbb{R}$;
(b) the sign of $\text{Im} g$ has the same property in Figure 19;
(c) $g(z)$ satisfies the following jump conditions across $[z_2, z_1]$:

$$g_-(z) + g_+(z) = 0, \quad z \in (z_1, z_2),$$ (6.5)

$$g_-(z) - g_+(z) = 0, \quad z \in (1, z_1) \cup (-z_1, -1),$$ (6.6)

$$g_-(z) + g_+(z) = B_1, \quad z \in (-1, 1),$$ (6.7)

$$g_-(z) + g_+(z) = B_2, \quad z \in (-z_2, -z_1),$$ (6.8)

here, $B_j = B_j(\xi) = \frac{1}{2} \oint_{b_j} dg$ is real;

(d) $g(z)$ has another phase point $z_0 = z_0(\xi) \in (z_1, z_2)$ which is the solution of equation

$$\frac{\xi - 1}{2} z^2 - \frac{2}{z_1 z_2} = 0.$$

(e) As $\xi \to \xi_m$, we have $z_2 \to c$, while as $\xi \to \frac{3}{4}$, $z_1, z_2 \to 2$.

![Figure 19: In the purpler region, $\text{Im} g > 0$ while in the white region, $\text{Im} g < 0$.](image)

**Proof.** Denote $\eta = -\frac{2}{z_1 z_2}$. Thus, (6.2) gives

$$z_1^2 + z_2^2 = \frac{4}{\eta^2} \left(1 + \frac{1 - \xi}{\eta}\right).$$
Then the $a_2$-period of $g$ equals to zero if and only if $F(\eta, \xi) = 0$ with
\[
F(\eta, \xi) = \int_{z_1}^{z_2} \frac{Z(z)}{z^3} \left[ \frac{1 - \xi}{2} z^2 - \frac{2}{z_1 z_2} \right] dz.
\]
When $\xi = \frac{3}{4}$, $F(\eta, \xi) = 0$ has solution $(-\frac{1}{2}, \frac{3}{4})$, and on the other end $\xi = \xi_m$, $F(\eta, \xi) = 0$ has solution as shown in Proposition 7: $(-\frac{2}{\cos(\xi_m)}, \xi_m)$. And
\[
\frac{\partial F(\eta, \xi)}{\partial \eta} = \int_{z_1}^{z_2} \frac{z}{(z^2 - 1)(z^2 - z_1^2)(z^2 - z_2^2)} \left[ 1 + (1 - \xi) \left( \frac{2}{\eta^2} + \frac{2}{\eta^4}(1 - \xi) \right) \right] dz.
\]
Consider the function $f(x) = x^4 + 2(1 - \xi)x + 3(1 - \xi)^2$. Note that, (6.2) implies $-\eta > 1 - \xi$, so simple calculation gives that $f(\eta) > f(\xi - 1) > 0$. So $\frac{\partial F(\eta, \xi)}{\partial \eta} \neq 0$, which give the existence of solution $\eta$.

\section*{6.2 Opening the jump contour}

Similarly as the above section, we define the following contour relying on $\xi, c$
\[
\Sigma^\pm = \left\{ -z_1 + e^{\pm \psi i} l, \ l \in (0, \frac{z_1 - 1}{2 \cos \psi}) \right\} \cup \left\{ z_1 + e^{(\pi + \psi) i} l, \ l \in (0, \frac{z_1 - 1}{2 \cos \psi}) \right\} \\
\cup \left\{ 1 + e^{\pm \psi i} l, \ l \in (0, \frac{z_1 - 1}{2 \cos \psi}) \right\} \cup \left\{ -1 + e^{(\pi + \psi) i} l, \ l \in (0, \frac{z_1 - 1}{2 \cos \psi}) \right\} \\
\cup \{-z_2 + e^{(\pi + \psi) i} \mathbb{R}^+\} \cup \{z_2 + e^{\pm \psi i} \mathbb{R}^+\}
\]
The $\frac{\pi}{4} \geq \psi$ is a small enough positive constant such that $\Sigma^\pm$ are contained in the region of $\text{Im} g > 0$. And similarly as above equation, $\Omega^\pm$ is a closed region, the edge of which is made up by $\Sigma^\pm$ and $\mathbb{R}$.

In this region of $\xi, c$, we introduce a piecewise matrix interpolation function
\[
G(z) = G(z; \xi, c) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-r_1 e^{2igt} & 1 \end{pmatrix}, & \text{as } z \in \Omega^+; \\
\begin{pmatrix} 1 & -r_2 e^{-2igt} \\
0 & 1 \end{pmatrix}, & \text{as } z \in \Omega^-; \\
I & \text{as } z \text{ in elsewhere},
\end{cases}
\end{cases}
\]
Same as above section, $G(z)$ bring a new singularity. To deal with the jump on $\mathbb{R}$, we give a introduction of an auxiliary function $D(z)$, which admits the following jump
Figure 20: The region of $\Omega^\pm$. The same as Figure 14, purple region means $\text{Im}(z) > 0$ while white region means $\text{Im}(z) < 0$.

condition:

$$D_-(z)D_+(z) = i[r_2]_-, \quad z \in [-z_2, z_2] \setminus [-z_1, z_1];$$

$$D_-(z)D_+(z) = 1, \quad z \in [-1, 1].$$

Define

$$Z_3(z) = (z^2 - 1)Z(z),$$

$$\log D(z) = \frac{Z_3(z)}{2\pi i} \left( \int_{-z_2}^{-z_1} + \int_{z_2}^{z_1} \right) \frac{\log(i[r_2]_-(s))}{(s - z)[Z_3]_+(s)} ds.$$  \hspace{1cm} (6.10) \hspace{1cm} (6.11)

**Proposition 14.** The scalar function $D(z)$ satisfies the following properties:

(a) $D(z)$ is analytic on $\mathbb{C} \setminus ((-z_2, -z_1) \cup [-1, 1] \cup (z_1, z_2))$;

(b) $D(z)$ has singularity at $z = \pm c$ with:

$$D(z) = \mathcal{O}(z - p)^{-1/4}, \quad z \to p \in \mathbb{C}^\pm \setminus \mathbb{R}, \quad p = c, -c.$$ \hspace{1cm} (6.12)

(c) As $z \to \infty \in \mathbb{C} \setminus \mathbb{R}$, $D(z)$ has limit $D_\infty(z)$ with

$$\log D_\infty(z) = -\frac{1}{2\pi i} \left( \int_{-z_2}^{-z_1} + \int_{z_2}^{z_1} \right) \frac{\log(i[r_2]_-(s))}{[X]_+(s)} \left( z^2 + zs + s^2 - \frac{1 + z_2^2 + z_1^2}{2} \right) ds.$$ \hspace{1cm} (6.13)

(d) As $z \to 0 \in \mathbb{C}^+$,

$$D_\infty(z) = D_\infty(0) \left( 1 + D^{(1)}_\infty z \right) + \mathcal{O}(z^2),$$

$$D(z) = D(0) \left( 1 + D^{(1)} z \right) + \mathcal{O}(z^2),$$
where

\[
\log(D_\infty(0)) = \frac{1}{2\pi i} \left( \int_{z_2}^{z_1} + \int_{-z_2}^{-z_1} \right) \log(i[r_2]_-(s)) \left( \frac{1 + z_2^2 + z_1^2 - s^2}{2} \right) ds,
\]

(6.14)

\[
D^{(1)}_\infty = -\frac{1}{2\pi i} \left( \int_{z_2}^{z_1} + \int_{-z_2}^{-z_1} \right) s \log(i[r_2]_-(s)) \left( \frac{|X|_+(s)}{X} \right) ds,
\]

(6.15)

\[
\log(D(0)) = \frac{c_{z_1}}{2\pi} \left( \int_{z_2}^{z_1} + \int_{-z_2}^{-z_1} \right) \frac{\log(i[r_2]_-(s))}{s[Z_3]_+(s)} ds,
\]

(6.16)

\[
D^{(1)} = \frac{c_{z_1}}{2\pi} \left( \int_{z_2}^{z_1} + \int_{-z_2}^{-z_1} \right) \frac{\log(i[r_2]_-(s))}{s^2[Z_3]_+(s)} ds.
\]

(6.17)

Denote

\[
\Sigma^{(2)} = \left( \bigcup_{j=1}^2 \Sigma^+_j \right) \cup [-1, 1] \cup [-z_2, -z_1] \cup [z_1, z_2].
\]

(6.18)

Through \( D(z) \) and \( G(z) \), in this region of \( \xi, c \), same as above subsection we give series of transformations: Thus, we can give the same transformation in this case with

\[
N(z) \to M^{(1)}(z) \to M^{(2)}(z) = \begin{cases} E(z; \xi, c) M^{mod}(z; \xi, c) & z \notin U(\xi) \\ E(z; \xi, c) M^{j, pm}(z; \xi, c) & z \in U(\pm z_j), \ j = 1, 2, \end{cases}
\]

where \( M^{mod}(z) \) is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 6.3. \( M^{j, pm}(z) \) are local model of \( \pm z_j \) which solution can be expressed in terms of Airy functions similarly in Subsection 5.2.2.

And \( E(z; \xi, c) \) is the error function, which has jump contour in Figure 21, and it is similarly in subsection 5.2.3. We obtain its asymptotic property directly as follow:

\[
E(z; \xi, c) = E(0) + E_1 z + \mathcal{O}(z^2),
\]

(6.19)
where
\[ E(0) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s))(V^E - I)}{s} ds, \]  
with long time asymptotic behavior
\[ E(0) = I + t^{-1}H^{(0)} + O(t^{-2}). \]

And
\[ H^{(0)} = H^{(0)}(\xi, c) = \sum_{p = \pm z_j, j = 1, 2} \frac{d}{dz} \left( \frac{1}{z} F^\pm \left( \begin{array}{cc} 1 & -\frac{5i}{48} \tilde{A}^j_\pm \frac{1}{2} \\ 0 & 0 \end{array} \right) (F^\pm)^{-1} \right) \] 
\[ + \sum_{p = \pm z_j, j = 1, 2} \frac{1}{p} \left( F^\pm \left( \begin{array}{cc} 1 & -\frac{7i}{48} \tilde{A}^j_\pm \frac{1}{2} \\ 0 & 0 \end{array} \right) (F^\pm)^{-1} \right) \] 

where \( \tilde{B}^j_\pm \) and \( \tilde{A}^j_\pm \) are shown in (6.25). Further, \( E_1 \) admits the following asymptotic expansion
\[ E_1 = t^{-1}H^{(1)} + O(t^{-2}), \]  
where
\[ H^{(1)} = H^{(1)}(\xi, c) = \sum_{p = \pm z_j, j = 1, 2} \frac{d}{dz} \left( \frac{1}{z} F^\pm \left( \begin{array}{cc} 1 & -\frac{5i}{48} \tilde{A}^j_\pm \frac{1}{2} \\ 0 & 0 \end{array} \right) (F^\pm)^{-1} \right) \] 
\[ + \sum_{p = \pm z_j, j = 1, 2} \frac{1}{p} \left( F^\pm \left( \begin{array}{cc} 1 & -\frac{7i}{48} \tilde{A}^j_\pm \frac{1}{2} \\ 0 & 0 \end{array} \right) (F^\pm)^{-1} \right) \] 

Here we denote
\[ F^\pm = M^{mod}H(z; \pm z_0)^{-1}N^{-1}J^\pm = [F^\pm]_{2 \times 2}, \]
with \( (F^\pm)^{-1} \triangleq [F^\pm,]^*_{2 \times 2} \).
\[ \frac{3}{2} \mathrm{itg}(z) = \tilde{A}^1_+ \sqrt{z - z_1^3} + \tilde{B}^1_+ \sqrt{z - z_1^4} + O(\sqrt{z - z_1^5}), \]
where the jump matrix \( V \).

- \( z_1, z_2, -z_2 \) respectively.

\[
\tilde{A}_+^1 = \frac{t}{2} \left( \frac{2z_1(z_2^2 - z_1^2)}{(z_1^2 - 1)} \right)^{1/2} \left( \frac{1 - \xi}{z_1} - \frac{4}{z_1^2 z_2} \right), \\
\tilde{B}_+^1 = \frac{3it}{10} \left( \frac{2z_1(z_2^2 - z_1^2)}{(z_1^2 - 1)} \right)^{1/2} \left[ \frac{1 - \xi}{z_1^2} + \frac{12}{z_1 z_2} + \left( \frac{1 - \xi}{z_2} - \frac{4}{z_1 z_2^2} \right) \left( \frac{1}{4z_2} + \frac{z_1}{4 z_1^2 - z_2^2} - \frac{2z_1}{z_1^2 - 1} \right) \right], \\
\tilde{A}_+^2 = \frac{it}{2} \left( \frac{2z_2(z_1^2 - z_2^2)}{(z_2^2 - 1)} \right)^{1/2} \left( \frac{1 - \xi}{z_2} - \frac{4}{z_1 z_2^2} \right), \\
\tilde{B}_+^2 = \frac{3it}{10} \left( \frac{2z_2(z_1^2 - z_2^2)}{(z_2^2 - 1)} \right)^{1/2} \left[ \frac{1 - \xi}{z_2^2} + \frac{12}{z_1 z_2^2} + \left( \frac{1 - \xi}{z_1} - \frac{4}{z_1 z_2^2} \right) \left( \frac{1}{4z_1} + \frac{z_2}{4 z_1^2 - z_2^2} - \frac{2z_2}{z_1^2 - 1} \right) \right], \\
\tilde{A}_-^1 = -i\tilde{A}_+^1, \quad \tilde{B}_-^1 = i\tilde{B}_+^1, \quad \tilde{A}_-^2 = i\tilde{A}_+^2, \quad \tilde{B}_-^2 = -i\tilde{B}_+^2.
\]

\section{6.3 Model RH problem on Riemann surface}

Similarly to Subsection 5.2.1, we arrive at the following model RH problem

\textbf{RHP 19.} Find a matrix-valued function \( M^{\text{mod}}(z) \) with following identities:

- \textbf{Analyticity:} \( M^{\text{mod}}(z) \) is analytical in \( \mathbb{C} \setminus \Sigma^{\text{cut}} \), with

\[
\Sigma^{\text{cut}} = [-z_2, -z_1] \cup [-1, 1] \cup [z_1, z_2];
\]

- \textbf{Asymptotic behaviors:} \( M^{\text{mod}}(z) \sim I + \mathcal{O}(z^{-1}), \quad |z| \to \infty; \)

- \textbf{Jump condition:} \( M^{\text{mod}}(z) \) satisfies the jump relation

\[
M^{\text{mod}}_+(z) = M^{\text{mod}}_-(z)V^{\text{mod}}(z), \quad z \in \Sigma^{\text{cut}},
\]

where the jump matrix \( V^{\text{mod}}(z) \) is given by

\[
V^{\text{mod}}(z) = \begin{cases} 
\begin{pmatrix} 0 & i e^{-itB_2} \\ i e^{itB_2} & 0 \end{pmatrix}, & \text{as } z \in [-z_2, -z_1], \\
\begin{pmatrix} 0 & i e^{-itB_1} \\ i e^{itB_1} & 0 \end{pmatrix}, & \text{as } z \in [-1, 1], \\
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \text{as } z \in [z_1, z_2];
\end{cases}
\]

- \textbf{Singularity:} \( M^{\text{mod}}(z) \) has singularity at \( z = \pm z_2 \) with:

\[
M^{\text{mod}}(z) \sim \mathcal{O}(z \mp p)^{-1/4}, \quad z \to p = \pm z_2, \quad \pm 1, \quad \pm z_1 \text{ in } \mathbb{C} \setminus \mathbb{R}. \quad (6.27)
\]
\( M^{\text{mod}} \) can be derived by the so-called \( \vartheta \) function on the Riemann surface of genus 2. To construct the model RH problem \( M^{\text{mod}} \), we further need to let
\[
\kappa(z) = \left[ \frac{(z - z_2)(z - 1)(z + z_1)}{(z - z_1)(z + 1)(z + z_2)} \right]^{1/2}, \quad z \in \mathbb{C} \setminus \Sigma^{\text{cut}}, \tag{6.28}
\]
\[
\kappa(z) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{6.29}
\]
\[
\mathcal{N}(z) = \frac{1}{2} \begin{pmatrix} \kappa + \kappa^{-1} & \kappa - \kappa^{-1} \\ \kappa - \kappa^{-1} & \kappa + \kappa^{-1} \end{pmatrix}, \tag{6.30}
\]
Then there is a constant \( \mathcal{K} \in \mathbb{C}^2 \) satisfies for arbitrary divisor \( \mathcal{P}_0 \)
\[
\vartheta(A(P) - A(P_0) - \mathcal{K}) := \vartheta(A(P) - \mathcal{K})
\]
has \( g = 2 \) zeros \( P_1, P_2 \) on \( M \) with \( P_1 + P_2 = P_0 \). Observing if \( P_1, P_2 \) is zeros of \( \mathcal{N}_{11}, \mathcal{N}_{22} \) only if \( P_1', P_2' \) are zeros of \( \mathcal{N}_{12}, \mathcal{N}_{21} \) where \( P_1', P_2' \) are the same point of \( P_1, P_2 \) on the other sheet. Therefore, \( \vartheta(A(P) - K), \vartheta(A(P) + K) \) have the same zeros as \( \mathcal{N}_{11}, \mathcal{N}_{22} \) and \( \mathcal{N}_{12}, \mathcal{N}_{21} \) have respectively at the time of \( P_0 = P_1 + P_2 \). We then can show that the RHP 19 admits the solution
\[
M^{\text{mod}}(z) = F(\infty) \begin{pmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{21} & \mathcal{N}_{22} \end{pmatrix} \begin{pmatrix} \vartheta(A(\infty) - K + C) & \vartheta(-A(\infty) - K + C) \\ \vartheta(A(\infty) - K) & \vartheta(-A(\infty) - K) \end{pmatrix}, \tag{6.31}
\]
where
\[
F(\infty) = \frac{1}{2} \text{diag} \left( \frac{\vartheta(A(\infty) - K)}{\vartheta(A(\infty) - K + C)}, \frac{\vartheta(A(\infty) - K)}{\vartheta(A(\infty) - K - C)} \right).
\]
Noting that as \( z \to 0 \in \mathbb{C}^+, \)
\[
\mathcal{N}(z) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + z \frac{\sqrt{2}}{4} \left( \frac{1}{z_1} - \frac{1}{z_2} - 1 \right) \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} + \mathcal{O}(z^2), \tag{6.32}
\]
then we have
\[
M^{\text{mod}}(z) = M^{\text{mod}}(0) + M^{\text{mod}}_1 z + \mathcal{O}(z^2), \tag{6.33}
\]
where
\[
M^{\text{mod}}_1 = \frac{\sqrt{2}}{4} P(\infty) \begin{pmatrix} 1 - \frac{1}{z_1} - 1 \\ \frac{1}{z_1} - \frac{1}{z_2} - 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} G(0) + \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} G_z(0),
\]
and
\[
G(z) = \begin{pmatrix} \vartheta(A(z) - K + C) & \vartheta(-A(z) - K + C) \\ \vartheta(A(z) + K + C) & \vartheta(-A(z) + K + C) \end{pmatrix}. \]
7 Long-time asymptotics for the mCH equation

In this section, we give our main result on the long-time asymptotics of the mCH equation (1.1), which is discuss as follows.

For the region I, we have sequence of transformations

\[ N(z) \rightarrow M^{(1)}(z) \rightarrow M^{modI}(z), \quad (7.1) \]

from which we obtain

\[ N(z) = E(z; \xi, c)M^{modI}(z) \delta^{-\sigma_3} G(z)^{-1} = \sqrt{2} (I + H^{(0)} t^{-\frac{1}{2}}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \exp(-I^1_3 \sigma_3) \]
\[ + z \sqrt{2} (I + H^{(0)} t^{-\frac{1}{2}}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \exp(-I^1_3 \sigma_3) I^2_3 \sigma_3 + z \frac{i}{\sqrt{2}} (I + H^{(0)} t^{-\frac{1}{2}}) \exp(-I^1_3 \sigma_3) \]
\[ + z \sqrt{2} H^{(1)} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \exp(-I^1_3 \sigma_3) + O(z^2) + O(t^{-1}), \]

where the \( I^1_3 \) and \( I^2_3 \) are given by (3.4)-(3.4), \( H^{(0)}, H^{(1)} \) is in Prop. 5. We further calculate

\[ -i [N(0)]_{12} = \sqrt{2} e^{-I^1_2} (1 + H^{(0)}_{11} t^{-\frac{1}{2}}) + O(t^{-1}), \]
\[ -i \lim_{z \to 0 \in C^+} \frac{[N(z)]_{11} - [N(0)]_{11}}{z} = \frac{e^{-I^1_1}}{\sqrt{2}} + \sqrt{2} H^{(1)}_{12} e^{-I^1_3} (I + H^{(0)}_{11}) t^{-\frac{1}{2}} + O(t^{-1}), \]
\[ -i \lim_{z \to 0 \in C^+} \frac{[N(z)]_{22} - [N(0)]_{22}}{z} = \sqrt{2} H^{(1)}_{21} e^{-I^1_3} + \frac{e^{I^1_1}}{\sqrt{2}} + (\sqrt{2} H^{(0)}_{21} e^{-I^1_3} I^2_3) t^{-\frac{1}{2}} + O(t^{-1}). \]

Substituting above estimates into (2.42) and (2.45) leads to

\[ u(x, t) = u(y(x, t), t) = \frac{1}{2} + 2 H^{(1)}_{12} e^{2I^1_3} - H^{(1)}_{21} e^{-2I^1_3} \]
\[ + \left(-\frac{1}{2} H^{(0)}_{22} + H^{(0)}_{21} I^2_3 + 2 H^{(0)}_{12} I^2_3 + H^{(0)}_{11} \frac{5}{2} + H^{(1)}_{21} e^{-2I^1_3} + 2 H^{(1)}_{12} e^{2I^1_3} \right) t^{-\frac{1}{2}} + O(t^{-1}), \]
\[ x = y - 2 \ln(\sqrt{2} (1 + H^{(0)}_{22} t^{-\frac{1}{2}}) e^{-I^1_3}) + O(t^{-1}). \]

For the region II, we have done the sequence of transformations

\[ N(z) \rightarrow M^{(1)}(z) \rightarrow M^{(2)}(z) \rightarrow M^{modc}(z), \quad (7.2) \]

from which we find that

\[ N(z) = \delta(\infty)^{\sigma_3} E(z; \xi, c)M^{modc}(z) \delta(z)^{-\sigma_3} G(z)^{-1} e^{-it(p-\theta_+)} \sigma_3, \quad (7.3) \]
which definition is in Prop. 6, RHP 10, (4.33), (4.8), (4.7) and (4.1) respectively. To reconstruct \( u(x, t) \) by using (2.45), in above equation (7.3) we take \( z \to 0 \) in \( \mathbb{C}^+ \), we obtain that

\[
N(z) = \delta(\infty) \sigma_3 N^\text{mod}(z) e^{-iI_3^1 \sigma_3} (I - I_3^2 \sigma_3 z) e^{-it(p_- - \theta_\pm)(0^+) \sigma_3} + O(z^2) + O(t^{-2})
\]

\[
= \sqrt{2} \delta(\infty) \sigma_1 e^{-iI_3^1 \sigma_3} e^{-it(p_- - \theta_\pm)(0^+) \sigma_3}
\]

\[
+ z \delta(\infty) \sigma_3 \left( \frac{1}{c\sqrt{2}} e^{-iI_3^1 \sigma_3} - \sqrt{2} I_3^2 e^{-iI_3^1 \sigma_3 \sigma_2} \right) e^{-it(p_- - \theta_\pm)(0^+) \sigma_3} + O(z^2) + O(t^{-2}),
\]

which implies that

\[
-i[N(0)]_{12} = \sqrt{2} \delta(\infty) e^{1/2 + it(p_- - \theta_\pm)(0^+)} ,
\]

\[
-i \lim_{z \to 0 \in \mathbb{C}^+} \frac{[N(z)]_{11} - [N(0)]_{11}}{z} = -\frac{i \delta(\infty)}{c\sqrt{2}} e^{-1/2 - it(p_- - \theta_\pm)(0^+)},
\]

\[
-i \lim_{z \to 0 \in \mathbb{C}^+} \frac{[N(z)]_{22} - [N(0)]_{22}}{z} = -\frac{i \delta(\infty)}{c\sqrt{2}} e^{1/2 + it(p_- - \theta_\pm)(0^+)}.\]

Substituting above estimates into (2.42) and (2.45) leads to

\[
u(x, t) = u(y(x, t), t) = -\frac{i}{e} \left( \delta(\infty)^2 + \frac{1}{2} \right) + O(t^{-2}),
\]

\[x(y, t) = y - 2 \ln \left( \sqrt{2} \delta(\infty) e^{1/2 + it(p_- - \theta_\pm)(0^+)} \right) + O(t^{-2}).\]

For the regions III and IV, their solving processes are similar, we take \( \xi > 1 \) part of region III as an example. The sequence of transformations is

\[
N(z) \rightarrow M^{(1)}(z) \rightarrow M^{(2)}(z) = \begin{cases}
E(z; \xi, c) M^\text{mod}(z; \xi, c) , & z \notin U(\xi), \\
E(z; \xi, c) M^{\text{lo},+}(z; \xi, c) , & z \in U(z_0) , \\
E(z; \xi, c) M^{\text{lo},-}(z; \xi, c) , & z \in U(-z_0) ,
\end{cases}
\]

which gives

\[
N(z) = e^{-itg(\infty) \sigma_3} D_\infty(z) \sigma_3 E(z; \xi, c) M^\text{mod}(z) D(z)^{-\sigma_3} G(z)^{-1} e^{-it(p_- - g) \sigma_3}.
\]

To take \( z \to 0 \) in \( \mathbb{C}^+ \), further using the asymptotic expansion in (5.13), Prop.8,
Prop. 7, (5.41) and (5.14), we obtain
\[ N(z) = e^{-itg(\infty)\sigma_3}D_\infty(0)^{\sigma_3}M^{mod}(0+)D(0)^{-\sigma_3}e^{-it(p_-g)(0)\sigma_3}
+ t^{-1}e^{-itg(\infty)\sigma_3}D_\infty(0)\sigma_3H(0)M^{mod}(0+)D(0)^{-\sigma_3}e^{-it(p_-g)(0)\sigma_3}
+ ze^{-itg(\infty)\sigma_3}D_\infty(0)^{\sigma_3}(D^{(1)}\sigma_3M^{mod}(0+)D(0)^{-\sigma_3} + M_1^{mod}D(0)^{-\sigma_3}
-M^{mod}(0+)D(0)^{-\sigma_3}D^{(1)}\sigma_3)e^{-it(p_-g)(0)\sigma_3}
+ zt^{-1}e^{-itg(\infty)\sigma_3}D_\infty(0)^{\sigma_3}H^{(1)}M^{mod}(0+)D(0)^{-\sigma_3}e^{-it(p_-g)(0)\sigma_3} + O(z^2) + O(t^{-2}). \]

Substituting above equation into (2.42) and (2.45) leads to
\[ u(x, t) = u(y(x, t), t) = u_{g, D, \xi}(y, t) + t^{-1}E(\xi) + O(t^{-2}), \quad (7.4) \]

where we denote
\[ u_{g, D, \xi}(y, t) = -e^{-2itg(\infty)D_\infty(0)^{2}[M^{mod}(0+)][H^{(1)}]_{11} + [H^{(1)}]_{12}[M_1^{mod}]_{22}}
- e^{2itg(\infty)D_\infty(0)^{-2}[M^{mod}(0+)][H^{(1)}]_{21}\left[\left(D^{(1)} - D^{(1)}\right)[M^{mod}(0+)]_{12} + [M_1^{mod}]_{11}\right]
- e^{-itg(\infty)D_\infty(0)^{-2}H^{(1)}[M^{mod}(0+)][H^{(1)}]_{21}\left[\left(D^{(1)} - D^{(1)}\right)[M^{mod}(0+)]_{12} + [M_1^{mod}]_{11}\right]
+ e^{itg(\infty)D_\infty(0)^{-2}H^{(1)}[M^{mod}(0+)][H^{(1)}]_{22}\left[\left(D^{(1)} - D^{(1)}\right)[M^{mod}(0+)]_{22} + [M_1^{mod}]_{22}\right]. \quad (7.5) \]

and
\[ E(\xi) = -e^{-2itg(\infty)D_\infty(0)^{2}[M^{mod}(0+)][H^{(1)}]_{11} + [H^{(1)}]_{12}[M_1^{mod}]_{22}}
- e^{-itg(\infty)D_\infty(0)^{-2}H^{(1)}[M^{mod}(0+)][H^{(1)}]_{21}\left[\left(D^{(1)} - D^{(1)}\right)[M^{mod}(0+)]_{12} + [M_1^{mod}]_{11}\right]
- e^{-itg(\infty)D_\infty(0)^{-2}H^{(1)}[M^{mod}(0+)][H^{(1)}]_{22}\left[\left(D^{(1)} - D^{(1)}\right)[M^{mod}(0+)]_{22} + [M_1^{mod}]_{22}\right]. \quad (7.6) \]

Moreover, we have
\[ x(y, t) = y - 2\ln\left\{-ie^{-itg(\infty)+it(p_-g)(0)}D_\infty(0)D(0)[M^{mod}(0+)]_{12}\right\}
+ 2it\left[\frac{H^{(1)}}{11}[M^{mod}(0+)]_{12} + \frac{H^{(1)}}{12}[M^{mod}(0+)]_{22}\right]^{-1} + O(t^{-2}), \quad (7.7) \]

where \( H^{(0)} \) and \( H^{(1)} \) is in Prop. 9 and Prop. 12 corresponding to different case of \( \xi > 1 \) and \( \xi < 1 \) part of region III.

Finally, summing up above results gives our main theorem in this paper.
Theorem 1. Let \( u(x, t) \) be the solution for the initial-value problem (1.1) and (1.6), then there exist a large constant \( T_1 = T_1(\xi), \xi = \frac{y}{t} \), such that for all \( T_1 < t \to \infty \), the long time asymptotics of the mCH equation (1.1) are given as follows.

\[ \text{The region I: } (i) \xi < \frac{2}{3}; (ii) 1 < c \leq \lambda_1, \frac{2}{3} < \xi < 1; (iii) 1 < c \leq \lambda_1, 1 < \xi, \]

which is a slow decay step-like background constant region with genus-0. We have asymptotic expansion

\[ u(x, t) = u(y(x, t), t) = \frac{1}{2} + 2H_{12}^{(1)}e^{2I_3} - H_{21}^{(1)}e^{-2I_3} \]

\[ + \left( \frac{1}{2}H_{22}^{(0)} + H_{21}^{(0)}I_3^2 + 2H_{12}^{(0)}i\delta + H_{11}^{(0)}\left( \frac{5}{2} + H_{21}^{(1)}e^{-2I_3} + 2H_{12}^{(1)}e^{2I_3} \right) \right) t^{-\frac{1}{2}} + O(t^{-1}), \]

\[ x = y - 2\ln \left( \sqrt{2}(1 + H_{22}^{(0)}t^{-\frac{1}{2}})e^{-I_3} \right) + O(t^{-1}), \]

where the \( I_3^2 \) and \( I_3^2 \) are given by (3.4)-(3.4), \( H^{(0)} \) and \( H^{(1)} \) are given in Proposition 5.

\[ \text{The region II: } \xi > 1 + 2/c, \]

which a fast decay step-like background constant region with genus-0. We have asymptotic expansion

\[ u(x, t) = u(y(x, t), t) = -ic^{-1} (\delta(\infty)^2 + 1/2) + O(t^{-2}), \]

\[ x(y, t) = y - 2\ln \left( \delta(\infty)e^{I_3^2 + ia(y,t)} \right) + O(t^{-2}), \]

where \( a(y, t) = -\frac{1}{2}(c + 1)y + \frac{it}{2} (c^{-2} + c) \), \( I_3^2 \) and \( \delta(\infty) \) are given in Prop. 6.

\[ \text{The region III: Genus-2 elliptic wave region.} \]

1. \( 1 < \xi < 1 + \frac{2}{c}, c > \sqrt{2}; (ii) 1 + \frac{2}{c}(2 - c^2) < \xi < 1 + \frac{2}{c}, \sqrt{2} < c < 2, \]

we have asymptotic expansion

\[ u(y(x, t), t) = u_{g,D}(y, t) + t^{-1}\mathcal{E}(\xi) + O(t^{-2}), \]

\[ x(y, t) = y - 2\ln \left( -ie^{-i\xi(\infty) + it(\xi)\delta(\infty)D(0)D(0)M^{mod(0^+)}[0]_{12}} \right) \]

\[ + 2i\left[ H^{(0)}[M^{mod(0^+)}[0]]_{12} + [H^{(0)}]_{12}[M^{mod(0^+)}]_{22}t^{-1} + O(t^{-2}), \right] \]

where \( u_{g,D}(y, t), \mathcal{E}(\xi), \xi, g(\xi), g(z), D(0), D(0), M^{mod}, H^{(0)} \) and \( H^{(1)} \) are show in (7.5), (7.6), (5.13), Prop.7, Prop.8, (5.41) and Prop.9, respectively.
(iii) \( \xi_m < \xi < 1, c > 2 \), we have asymptotic expansion

\[
\begin{align*}
 u(y(x,t), t) &= u_g, D, \xi(y, t) + t^{-1/2} E(\xi) + O(t^{-1}), \\
x(y, t) &= y - 2 \ln \left( -ie^{i\theta(0)} + i(\theta(0) - g(0)) D(0) [M^{mod}(0^+)]_{12} \right) \\
&\quad + 2i \frac{[H^{(0)}]_{11}[M^{mod}(0^+)]_{12} + [H^{(0)}]_{12}[M^{mod}(0^+)]_{22}t^{-1/2} + O(t^{-1}),}
\end{align*}
\]

where \( u_g, D, \xi(y, t), E(\xi), g(\infty), g(z), D(0), D^{mod}, H^{(0)} \) and \( H^{(1)} \) are show in (7.5), (7.6), (5.13), (5.41), Prop. 7, Prop. 10 and Prop. 12, respectively. Although has same sign, \( E(\xi), H^{(0)} \) and \( H^{(1)} \) represent the contribution of the pairs of stationary phase point out of cut via parabolic cylinder model.

\[\text{◮} \text{ The region IV: } \frac{3}{4} < \xi < \xi_m, 2 < c, \text{ which is a genus-2 elliptic wave region.} \]

We have asymptotic expansion

\[
\begin{align*}
 u(y(x,t), t) &= u_g, D, \xi(y, t) + t^{-1} E(\xi) + O(t^{-2}), \\
x(y, t) &= y - 2 \ln \left( -ie^{i\theta(\infty)} + it(\theta(0) - g(0)) D(0) [M^{mod}(0^+)]_{12} \right) \\
&\quad + 2i \frac{[H^{(0)}]_{11}[M^{mod}(0^+)]_{12} + [H^{(0)}]_{12}[M^{mod}(0^+)]_{22}t^{-1} + O(t^{-2}).}
\end{align*}
\]

where \( u_g, D, \xi(y, t), E(\xi), g(\infty), D(0), D(z), D^{mod}, H^{(0)} \) and \( H^{(1)} \) are show in (7.5), (7.6), Prop. 14, Prop. 13 and Prop. 15, respectively. Although has same sign, \( E(\xi), H^{(0)} \) and \( H^{(1)} \) represent the common contribution of two local Airy Model of two pairs of stationary phase points.

A Appendix. The RH model for Airy function

In this appendix, we recall the standard model RH problem of Airy function that is used in our paper. Let \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \subset \mathbb{C} \) denote the rays

\[
\begin{align*}
\Gamma_1 := \{ z e^{2\pi i} | z \in \mathbb{R}^+ \}, & \quad \Gamma_2 := \{ -z | z \in \mathbb{R}^+ \}, \\
\Gamma_3 := \{ z e^{4\pi i} | z \in \mathbb{R}^+ \}, & \quad \Gamma_4 := \{ z | z \in \mathbb{R}^+ \}.
\end{align*}
\]

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The corresponding open sectors are given as follows

\[ S_1 = \{ z | \arg z \in (0, 2\pi/3) \}, S_2 = \{ z | \arg z \in (2\pi/3, \pi) \}, \]
\[ S_3 = \{ z | \arg z \in (\pi, 4\pi/3) \}, S_4 = \{ z | \arg z \in (4\pi/3, 2\pi) \} \]  

(A.3)

(A.4)

Let \( \chi = e^{\frac{2i\pi}{3}} \) and the function \( m^{Ai}(z) \) for \( z \in \mathbb{C} \setminus \Gamma \) by

\[
m^{Ai}(z) = A(z) \begin{cases} 
  e^{\frac{2i\pi}{3}\sigma_3}, & z \in S_1, \\
  \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) e^{\frac{2i\pi}{3}\sigma_3}, & z \in S_2, \\
  \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) e^{\frac{2i\pi}{3}\sigma_3}, & z \in S_3, \\
  e^{\frac{2i\pi}{3}\sigma_3}, & z \in S_4.
\end{cases}
\]

(A.5)

\[
A(z) = \begin{cases} 
  (Ai(z) & Ai(\chi^2z) \\
  Ai'(z) & \chi^2Ai'(\chi^2z) \\
  Ai(z) & -\chi^2Ai(\chi z) \\
  Ai'(z) & -Ai'(\chi z)
\end{cases} \begin{pmatrix} e^{-\frac{\pi}{6}\sigma_3}, & \text{Im} z > 0, \\
  e^{-\frac{\pi}{6}\sigma_3}, & \text{Im} z < 0,
\end{pmatrix}
\]

(A.6)

The asymptotic behavior of \( m^{Ai}(z) \) as \( z \to \infty \) can be shown as

\[
N^{-1}z^{\frac{\sigma_3}{4}}m^{Ai}(z) = I + \sum_{j=1}^{\infty} \frac{m_j^{Ai}}{z^{\frac{\sigma_3}{4}}}, \quad z \to \infty,
\]

(A.8)

Lemma 1. \( m^{Ai} : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2 \times 2} \) is a matrix valued analytic function and satisfies the jump condition

\[
m^{Ai}(z) = m^{Ai}(z)v^{Ai}(z),
\]

where

\[
v^{Ai}(z) = \begin{cases} 
  \left( \begin{array}{cc} 1 & 0 \\ -e^{\frac{2i\pi}{3}} & 1 \end{array} \right), & z \in \Gamma_1 \cup \Gamma_3, \\
  \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), & z \in \Gamma_2, \\
  \left( \begin{array}{cc} 1 & -e^{\frac{4i\pi}{3}} \\ 0 & 1 \end{array} \right), & z \in \Gamma_4.
\end{cases}
\]

(A.7)
where
\[ m_j^{\text{Ai}} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} N^{-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & -i \end{array} \right) \left( \frac{3}{2} \right)^j \left( \begin{array}{cc} (\frac{1}{\sqrt{2}})^j u_j & u_j \\ -(\frac{1}{\sqrt{2}})^j v_j & v_j \end{array} \right) e^{-\frac{i\pi}{4} \sigma_3}, \] (A.9)

and
\[ N = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right), \quad u_j = \frac{(2j+1)(2j+3)...(6j-1)}{(216)^j j!}, \quad v_j = \frac{6j+1}{1-6j} u_j. \]

**B Appendix. The existence of \( \xi_m \) in elliptic region**

All discussion in the follows is under the case
\[ \frac{3}{4} < \xi < 1, \quad \eta - (\eta(\eta - 4))^{1/2} < e^{2}, \quad \eta := (1 - \xi)^{-1}. \]

In a compact region
\[ \Omega = \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{x^2} + \frac{1}{y^2} + \frac{xy}{\eta^2} \leq 1, 0 < x \leq y \right\}, \]
we consider the function
\[ F(x, y) = \int_x^y \frac{1}{z^3} \left[ \frac{z^2 - x^2}{(z^2 - 1)(y^2 - z^2)} \right]^{1/2} \left[ \frac{\xi - 1}{2} z^4 + \frac{y}{x} \left( 1 + \frac{1}{y^2} - \frac{1}{x^2} \right) z^2 - \frac{2y}{x} \right] \, dz \]
which is continuous and differential, moreover satisfies that
\[ F(x, y) \equiv 0, \quad (x, y) \in L := \left\{ (x, y) : x = y, \frac{2}{x^2} + \frac{x^2}{2\eta} < 1 \right\}. \]

Therefore two positive phase points \( z_1 \) and \( z_2 \) can be defined as
\[ z_1^2 z_2^2 = \frac{4\eta y}{x}, \quad z_1^2 + z_2^2 = \frac{2\eta y}{x} \left( 1 + \frac{1}{y^2} - \frac{1}{x^2} \right). \]

Then \( \max\{z_1, z_2\} \geq y \) leads to the following two cases
\[ \frac{xy^3}{4\eta} \leq 1 \quad \text{or} \quad \frac{xy^3}{4\eta} > 1, \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{xy}{2\eta} \leq 1. \] (B.1)

While \( \min\{z_1, z_2\} > x \) implies \( \frac{x^5}{4\eta y} < 1 \). Especially for \( (x, y) \in \Omega, \max\{z_1, z_2\} \geq y \) and \( \max\{z_1, z_2\} = y \) if and only if \( (x, y) \in \partial\Omega \setminus L \).
Without loss of generality, we let \(0 < z_1 < z_2\). Direct calculation shows that

\[
F_x = \frac{1}{2x^3} \xi (x^2 - z_1^2)(x^2 - z_2^2) \int_x^y z \frac{z}{\sqrt{(z^2 - 1)(z^2 - x^2)(y^2 - z^2)}} dz,
\]
\[
F_y = -\frac{1}{x}(1 - \frac{1}{x^2} - \frac{1}{y^2} - \frac{(1 - \xi)xy}{2}) \int_x^y \frac{z^2 - x^2}{y^2 - z^2} \sqrt{\frac{z^2 - 1}{(z^2 - x^2)(y^2 - z^2)}} dz.
\]

Obviously, \(F(x, y) = 0\) in the interior of \(\Omega\) implies \(x < z_1 < y\) then \(F_x > 0\). Because \(F_x \neq 0\) and \(F_y \neq 0\) in the interior of \(\Omega\), the curve \(F(x, y) = 0\) in the interior of must approach to the boundary of \(\Omega\).

Meanwhile, we have

\[
F_x(x, x) = F_y(x, x) = -\frac{1}{\sqrt{x^2 - 1}x^3} \left( \frac{\xi - 1}{2} x^4 + x^2 - 2 \right) < 0, \quad (x, y) \in \Omega_1.
\]

which implies \(F(x, y) = 0\) in the interior of \(\Omega\) won’t approach to \(L\). Moreover, the point \((\eta + (\eta\eta - 4)^{1/2})^{1/2}, (\eta + (\eta(\eta - 4))^{1/2})^{1/2}\) \(\in \partial \Omega\) is away from \(\{(x, y)|\frac{x^5}{\eta y} < 1\}\), which implies the curve won’t approach to this point either.

Considering the boundary \(\partial \Omega \setminus \{(x, y)|x = y\}\), in another word,

\[
\{ \frac{1}{x^2} + \frac{1}{y^2} + \frac{xy}{2} = 1\} | 0 < x \leq y \}.
\]

It can be a curve with a positive parameter \(w \in \{w_0^{-1} < w < w_0|w_0 + w_0^{-1} = \sqrt{\eta}, w_0 > 1\}\) as

\[
x = w \sqrt{\eta} \left( \sqrt{1 - w\eta^{-1/2} + w^{-1}\eta^{-1/2}} - \sqrt{1 - w\eta^{-1/2} - w^{-1}\eta^{-1/2}} \right), \quad (B.2)
\]
\[
y = w \sqrt{\eta} \left( \sqrt{1 - w\eta^{-1/2} + w^{-1}\eta^{-1/2}} + \sqrt{1 - w\eta^{-1/2} - w^{-1}\eta^{-1/2}} \right). \quad (B.3)
\]

Directly calculating can derive

\[
\frac{\partial F}{\partial w} = \frac{2 \left( 6\sqrt{\eta}w^2 - 4\eta w + \frac{2\sqrt{\eta}}{w} \right)}{\sqrt{1 - \frac{w}{\sqrt{\eta}} + \frac{1}{\sqrt{\eta}} \sqrt{1 - \frac{w}{\sqrt{\eta}} - \frac{1}{\sqrt{\eta}}}} \int_x^y \frac{(w - \sqrt{\eta})z^2 + 2\sqrt{\eta}}{z \sqrt{(z^2 - 1)(z^2 - x^2)(y^2 - z^2)}} dz.
\]

The above integral is negative in \(w \in \{w_0^{-1} < w < w_0|w_0 + w_0^{-1} = \sqrt{\eta}, w_0 > 1\}\), \(F(w_0^{-1}) = F(w_0) = 0\) and the equation

\[
6\sqrt{\eta}w^2 - 4\eta w + \frac{2\sqrt{\eta}}{w^2} = 0
\]

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admits two solutions between $w_0^{-1}$ and $w_0$, which implies there is one unique zero of $F(w)$ in $w \in \{w_0^{-1} < w < w_0 | w_0 + w_0^{-1} = \sqrt{\eta}, w_0 > 1\}$ written as $w = w_m$. And $\frac{dF}{dw}(w_m) \neq 0$ implies there must exist one unique branch of curve $F(x, y) = 0$ approaches to the point $(x(w_m), y(w_m))$. And another possible point is

$$\left((\eta - (\eta(\eta - 4))^{1/2})^{1/2}, (\eta - (\eta(\eta - 4))^{1/2})^{1/2}\right) = (x(w_0^{-1}), y(w_0^{-1})).$$

For $y$ decrease to $(\eta - (\eta(\eta - 4))^{1/2})^{1/2} = y(w_0^{-1})$,

$$F(x(w), y(w)) < 0, \quad F_y(y(w)) < 0$$

implies there exist $x = x(y)$ satisfies $F(x, y) = 0$. And every $x$ satisfies $F(x, y) = 0$ has $\frac{dF}{dx} > 0$ implies the zeros is unique. Therefore, there must exist a unique curve satisfies $F(x, y) = 0$ from $(x(w_0^{-1}), y(w_0^{-1}))$ to $(x(w_m), y(w_m))$ in $\Omega$. Note $y(w_m)$ as $c_m$. In another word, for every $\xi < 1, c \in (\eta + (\eta(\eta - 4))^{1/2})^{1/2}, c_m(\xi))$, there exist a $z_0 = z_0(c)$ satisfies $F(z_0, c) = 0$ and $z_2(z_0, c_m) = c_m$ when $c$ comes to $c_m$. Let $\xi_m(c) = c_m^{-1}(c)$. It is well-defined because $c_m = y(w_m)$ is continuous with respect to $\xi$, which implies $c_m(\xi)$ is a surjection to $c^2 > 4$. Then the existence of $\xi_m$ has been proved.

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