Spectral Subspace Dictionary Learning
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Abstract

Dictionary learning, the problem of recovering a sparsely used matrix \( D \in \mathbb{R}^{M \times K} \) and \( N \) independent \( K \times 1 \) \( s \)-sparse vectors \( X \in \mathbb{R}^{K \times N} \) from samples of the form \( Y = DX \), is of increasing importance to applications in signal processing and data science. Early papers on provable dictionary learning identified that one can detect whether two samples \( y_i, y_j \) share a common dictionary element by testing if their inner product (correlation) exceeds a certain threshold: \(|\langle y_i, y_j \rangle| > \tau\). These correlation-based methods work well when sparsity is small, but suffer from declining performance when sparsity grows faster than \( \sqrt{M} \); as a result, such methods were abandoned in the search for dictionary learning algorithms when sparsity is nearly linear in \( M \).

In this paper, we revisit correlation-based dictionary learning. Instead of seeking to recover individual dictionary atoms, we employ a spectral method to recover the subspace spanned by the dictionary atoms in the support of each sample. This approach circumvents the primary challenge encountered by previous correlation methods, namely that when sharing information between two samples it is difficult to tell which dictionary element the two samples share. We prove that under a suitable random model the resulting algorithm recovers dictionaries in polynomial time for sparsity linear in \( M \) up to log factors. Our results improve on the best known methods by achieving a decaying error bound in dimension \( M \); the best previously known results for the overcomplete (\( K > M \)) setting achieve polynomial time linear regime only for constant error bounds. Numerical simulations confirm our results.

1 Introduction

The problem of finding sparse representations for large datasets is of tremendous importance in data science and machine learning applications. Sparse representations have obvious advantages for data storage and processing, while offering insight into a dataset’s intrinsic structure. This sparse recovery problem can often be formulated as that of how to recover a sparse vector \( x \in \mathbb{R}^K \) from a dense sample of the form \( y = Dx \) for some known sparsely-used matrix \( D \) called the “dictionary.” Depending on applications, this dictionary can be known from physics or hand-designed, such as the wavelet bases used in image processing [e.g., Vetterli and Kovacevic, 1995]. Beyond numerous further applications in signal and image processing (see Elad [2010] for a summary of developments), sparse representations have been fruitfully applied in areas including computational neuroscience [Olshausen and Field, 1996b,a, 1997] and machine learning [Argyriou et al., 2006, Ranzato et al., 2007].

Yet as the tasks of understanding and compressing data become increasingly central to the needs of modern technology, there is a corresponding interest in methods which learn from data not only the underlying sparse codes but also the dictionary itself. This is the dictionary learning problem. Specifically, we aim to recover a sparsely used matrix \( D \in \mathbb{R}^{M \times K} \) from \( N \) measurements of the form \( y = Dx \), where \( x \in \mathbb{R}^K \) is sufficiently sparse (that is, \( x \) has significantly fewer than \( K \) nonzero entries). Written in matrix form, we seek to recover \( D \) from a matrix \( Y = DX \in \mathbb{R}^{M \times N} \) with the prior knowledge that rows of \( X \) are sparse. In most applications, practitioners are interested in recovering overcomplete dictionaries which

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satisfy $K > M$, as these allow for greater flexibility in basis selection and for sparser representation [e.g. Chen et al., 1998, Donoho et al., 2006].

1.1 Prior Work

Dictionary learning is typically formulated as a nonconvex optimization problem of finding the dictionary $D$ and matrix with sparse columns $X$ such that $X$ is as sparse as possible:

$$\text{Find } D, X \text{ minimizing } \|X\|_0 \text{ subject to } Y = DX.$$  

As a nonconvex optimization, finding solutions to this problem is computationally challenging. The most popular heuristic for solving the dictionary learning problem is alternating minimization. Alternating minimization algorithms rely on the fact that when $D$ or $X$ is known, the other can be solved using known methods, most frequently based on $\ell_1$ relaxations that make the problem convex [Candés et al., 2006]. Alternating minimization techniques thus alternate between a “sparse coding” step in which a guess for $D$ is fixed and the algorithm solves for $X$, and a “dictionary update” step in which $X$ is fixed and the dictionary is updated. This process is repeated until a convergence criterion is met. These algorithms often lack theoretical guarantees, though some recent work has found conditions under which particular alternating minimization algorithms converge to a global minimum in the dictionary learning setting [Chatterji and Bartlett, 2017, Agarwal et al., 2016].

In this paper, we are interested in dictionary learning algorithms with provable guarantees. Initial theoretical study of provable dictionary learning focused on the case when $s \sim \sqrt{M}$; this is a well-known recovery boundary even when the dictionary $D$ is known. Spielman et al. [2012] developed an algorithm that accurately recovers the dictionary in this sparsity regime, but their algorithm does not generalize to recovery of overcomplete dictionaries. Arora et al. [2013] and Agarwal et al. [2017] then independently introduced similar correlation and clustering methods, which enjoy similar theoretical guarantees for the $s \sim \sqrt{M}$ regime.

Candés et al. [2006] showed that when the dictionary $D$ is known and satisfies certain properties such as the restricted isometry property [Candes and Tao, 2005], it is possible to recover $x_i$ from $y_i = Dx_i$ when $x_i$ has linearly-many nonzeros in $M$. Accordingly, there was tremendous interest in determining whether this scaling was likewise possible when $D$ is unknown. In [Arora et al., 2014], the authors develop provable methods for recovering dictionaries with sparsity $s = O(M)$ up to logarithmic factors, but their method requires quasipolynomial running time. In a pair of papers, Sun, Qu, and Wright develop a polynomial-time method which can provably recover invertible dictionaries with $s = O(M)$ [Sun et al., 2017a,b]. However, this algorithm depends intimately on properties of orthogonal and invertible matrices and thus is limited to the case of complete dictionaries ($M = K$); moreover, their theoretical guarantees demand a high sample complexity of $K \gg M^8$.

More recently, Zhai et al. [2020b] introduced a method based on $\ell^4$ norm optimization. Despite many nice properties of this approach further elaborated by Zhai et al. [2020a], it remains limited to the complete case and the authors prove only the accuracy of a global optimum with no guarantee of convergence in arbitrary dimension.

For the overcomplete setting, Barak et al. [2015] developed a tensor decomposition method based on the sum-of-squares hierarchy that can recover overcomplete dictionaries with sparsity up to $s = O(M^{1-\delta})$ for any $\delta > 0$ in polynomial time, but this time tends to super-polynomial as $\delta \to 0$ and requires a constant error as $M \to \infty$. This and related methods have generally enjoyed the best theoretical guarantees for efficient dictionary learning in the overcomplete linear sparsity regime due to their impressive generality, especially after their runtime was improved to polynomial time by Ma et al. [2016] provided that the target error remains constant. Yet the requirement of constant error is strict—with these methods, even in sublinear sparsity regimes such as $s \sim M^{0.99}$, an inverse logarithmic decay in error requires superpolynomial time.

1.2 Intuition and Our Contribution

The correlation-based clustering method of Arora et al. [2014] offers an appealing intuition in the $s \sim \sqrt{M}$ regime: that pairs of elements which “look similar” in the sense of being highly correlated are likely to share support. Specifically, the method relies on the fact that for incoherent dictionaries $D$ (that is,
pursue an intermediate step of recovering spanning subspaces. We propose that instead of immediately attempting to recover dictionary elements, we first by only a logarithmic factor. This holds true even in our proposed “almost-linear” regime where sparsity differs from linear in dimension 

\[ |\langle d_{k_1}, d_{k_2} \rangle| \leq c/\sqrt{M} \text{ for } k_1 \neq k_2 \]  

and symmetric coefficients, then if \( s \ll \sqrt{M} \) then as \( M \to \infty \), the variance of \( \langle y_i, y_j \rangle \) —the correlation\(^1\)—will tend to zero. As a result, thresholding \( |\langle y_i, y_j \rangle| \) becomes a reliable indicator of whether \( y_i \) and \( y_j \) share a common dictionary element in their support. By constructing a graph whenever \( |\langle y_i, y_j \rangle| \geq \tau \) for some threshold \( \tau \) (say 1/2), one can determine which groups of \( y_i \)’s share the same common support element by applying overlapping clustering methods. One can then recover the individual dictionary vectors by a spectral method on each of the resulting clusters.

Yet correlation-based clustering cannot be performed with accuracy once sparsity exceeds \( \sqrt{M} \), as above this threshold correlation does not reliably indicate whether two samples share a common dictionary element. As a result, methods based on the correlation \( \langle y_i, y_j \rangle \) have not been widely employed in subsequent attempts to solve dictionary learning in the linear \( s \sim M \) sparsity regime, with practitioners instead turning to more technical machinery such as the sum-of-squares hierarchy [Barak et al., 2015] or Riemannian trust-regions [Sun et al., 2017a,b].

In our present work, we revisit correlation-based dictionary learning methods. By adopting a different approach which sidesteps the key challenges of previous correlation thresholding methods, we are able to apply these methods successfully even in the linear sparsity regime. In this paper, we introduce the \textbf{Spectral Subspace Dictionary Learning} (SSDL) algorithm for solving the overcomplete dictionary learning problem in the linear sparsity regime up to logarithmic factors. We show that for a suitable probabilistic model, the algorithm runs in polynomial time and results in an error which decays in \( M \).

In other words, our algorithm actively performs better in high dimensions compared to alternatives: the previous best known methods for the overcomplete linear regime, that is, those of [Ma et al., 2016], require super-polynomial (quasi-polynomial) runtime to achieve errors with even an inverse-logarithmic decay in \( M \). This holds true even in our proposed “almost-linear” regime where sparsity differs from linear in dimension by only a logarithmic factor.

Our method is a natural adaptation of the correlation-based approach of Arora et al. [2013] to the linear sparsity regime. We propose that instead of immediately attempting to recover dictionary elements, we first pursue an intermediate step of recovering \textit{spanning subspaces}, the subspaces \( S_i \) spanning the supporting dictionary elements of each sample \( y_i \). Once these subspaces are recovered, the individual dictionary elements can be recovered through pairwise comparison of subspaces to find their intersection. This can be interpreted as reversing the order of the algorithm of Arora et al. [2013]: whereas those authors proceeded by first detecting support information about \( X \) then using this to extract geometric information about \( D \), we propose first to recover information about the geometry of \( D \) in the form of spanning subspaces, then to use this subspace information to find shared support among columns of \( X \).

The primary advantage of this approach is that the subspace recovery step effectively recovers geometric information from all of \( y_i \)’s supporting dictionary elements at once. In particular, if \( y_i \) and \( y_j \) are highly correlated, we no longer need to concern ourselves with which particular element they share in their support. Accordingly, this approach does not require the cumbersome clustering method used in previous correlation-based methods. The resulting method is surprisingly simple to implement, and its iterative nature makes it highly parallelizable. We provide theoretical guarantees that the resulting method accurately recovers dictionaries with sparsity \( s \sim M \log^{-6+\eta}(M) \) for any \( \eta > 0 \), which depend for their proof only on variants of familiar concentration inequalities from high-dimensional probability [e.g., Vershynin, 2018, chapter 2].

To recover these subspaces, we employ a spectral method based on computing eigenvalues of a modified covariance matrix of the samples \( Y \). Specifically, for each \( j \) we examine the \textit{correlation-weighted covariance matrix} \( \Sigma_j \) defined as the covariance of the correlation-weighted samples \( \langle y_i, y_j \rangle y_i \). The resulting matrix will be weighted toward those elements which share support elements with \( y_j \), and as a result its \( s \) leading eigenvectors will span a subspace close to that spanned by the support elements of \( y_j \).

The resulting algorithm is conceptually simple, and proving its performance guarantees requires only relatively standard techniques from high-dimensional probability. We emphasize that, unlike many algorithms for dictionary learning with theoretical guarantees, SSDL requires no initialization; accordingly, it is an ideal candidate for use as an initializer for a subsequent refinement by an iterative method.

\(^1\)We note this is a slight abuse of notation as the vectors \( y_i, y_j \) may not be unit vectors.
1.3 Structure of Paper

In section 2, we technically specify the problem to be solved and introduce our notations, parameter scaling, and probabilistic model. In section 3, we motivate and detail the SSDL algorithm. Section 4 contains an overview of our main theoretical results, while section 5 sketches their proof, though we defer detailed proofs of most technical lemmas to the appendix. Lastly, section 6 contains the results of numerical experiments validating our results.

2 Parameter Scaling, Data Model, and Conventions

In this section, we explicitly define our parameter scaling and random model, as well as some notational conventions.

2.1 Problem Statement

**Definition 2.1 (Sparse Dictionary Learning).** Let $D = (d_1, d_2, \ldots, d_K)$ be an (unknown) $M \times K$ matrix with unit vector columns, called the dictionary. Let $x$ be an $s$-sparse random vector, and define the random vector $y = D x$. The sparse dictionary learning problem is:

Given $Y = DX$ where $X$ is a $K \times N$ matrix with columns $\{x_i\}_{i=1}^N$ i.i.d. copies of $x$, recover $D$.

It is clear from the definition that $D$ can only be recovered up to sign and permutation. Accordingly, we employ the following definition for comparing two dictionaries, due to Arora et al. [2013]: we say that two dictionaries are column-wise $\varepsilon$-close if their columns are close in Euclidean norm after an appropriate permutation and change of sign. In detail:

**Definition 2.2 (Column-wise $\varepsilon$-close [Arora et al., 2013]).** Two dictionaries $A = (a_1, a_2, \ldots, a_K)$ and $B = (b_1, b_2, \ldots, b_K)$ are column-wise $\varepsilon$-close if they have the same dimensions $M \times K$ and there exists a permutation $\pi$ of $\{1, \ldots, K\}$ and a $K$-element sequence $\theta_k \in \{-1, 1\}$ such that for all $k = 1, \ldots, K$,

$$\|a_k - \theta_k b_{\pi(k)}\|_2 \leq \varepsilon.$$  

2.2 Parameter Scaling

We denote the following parameters and their scaling:

**Definition 2.3 (Parameters and scaling).** We define $M$ to be the dimension of the samples $y_j$, $s$ the sparsity level, $K$ the dictionary size, $N$ the number of samples, $J$ the number of recovered subspaces, and $\ell$ the maximum intersection size. We assume the following parameter scaling (all parameters are assumed to grow at most polynomially in $M$):

- $0 < \gamma < \eta$ constant in $M$
- $s = C_s M \log^{-(6+\eta)}(M)$
- $K = C_K M \log^{4+\gamma}(M)$
- $N \geq \max\left\{\frac{s^{10}}{M^{3}}, \frac{K^2 s^4}{M^3}\right\} \sim \max\{M^4 \log^{-60-10\eta}(M), M^3 \log^{(-16+2\gamma-4\eta)}(M)\}$
- $J = K \log^3 K$
- $\ell = \left\lceil \frac{\log(2K)}{\log(K/s)} \right\rceil$, the smallest integer such that $(s/K)^\ell \leq \frac{1}{2K}$.
2.3 Data Model

We begin by defining the following distributions for our dictionary $D$ and sparsity pattern $X$:

**Definition 2.4 ($D$ distribution).** A random matrix $D \in \mathbb{R}^{M \times K}$ follows the $D$ distribution if its columns \( \{d_k\}_{k=1}^{K} \) are $K$ independent uniformly distributed unit vectors in $\mathbb{R}^{M}$.

**Definition 2.5 ($\mathcal{X}(W)$ distribution).** Let $W$ be a symmetric random variable satisfying $|W| \in [c, C]$ almost surely for $0 < c \leq C$. A random vector $X \in \mathbb{R}^{K \times N}$ follows a $\mathcal{X}(W)$ distribution if:

- The supports $\Omega_i = \text{supp}(x_i)$ of each column $x_i$ of $X$ are independent, uniformly random $s$-element subsets of $[K]$.
- Nonzero entries of $X$ are i.i.d. copies of $W$.

We note that this definition implies columns of $X$ are independent when distributed according to a $\mathcal{X}(W)$ distribution. In our theoretical results, we assume that $D \sim D$ and $X \sim \mathcal{X}(W)$. As the extension to bounded symmetric random variables is trivial, in our theoretical results we assume $W = \pm 1$ with equal probability. This choice of particular distribution for $X$ is made for theoretical convenience and significantly simplifies the analysis, but we expect our results to hold with minimal modifications for the more commonly used Bernoulli-Gaussian model used by Spielman et al. [2012] and others.

Our result differs from many other provable results on dictionary learning in that we assume the dictionary $D$ to be a random matrix. The specific geometric properties required to recover $D$, which are reliably satisfied by a $D$-distributed random matrix, are outlined in definitions 5.2 and 5.7. Our treatment is similar to that of the well-known restricted isometry property [Candes and Tao, 2005], a deterministic property often assumed in recovery guarantees for compressed sensing (that is, when the dictionary is known). The restricted isometry property is known to hold for many types of random matrices, but no families of deterministic matrices for which it holds are yet known [e.g., Bandeira et al., 2013]. Accordingly, such recovery results implicitly make a random-matrix assumption for $D$.

2.4 Notation and conventions

Vectors are represented by boldface lowercase letters, while matrices will be written as boldface uppercase letters. Roman letters (both upper- and lowercase) will be used for both scalars and random variables depending on context. Calligraphic letters will be used to refer to sets or events. We will use the notation $|\mathcal{A}|$ for the number of elements in a finite set $\mathcal{A}$, and $\mathcal{A}^c$ for its complement. $\delta$ is reserved for the Dirac delta function: $\delta_{x \in \Omega_k}$ equals one for $k \in \Omega_k$ and zero otherwise. We will also employ the shorthand $[N] = \{1, \ldots, N\}$.

We use two matrix norms at different points in the text. The standard $l_2$ operator norm will be denoted by $\|\cdot\|_2$ while the Frobenius norm will be denoted $\|\cdot\|_F$. Vector norms always refer to the standard $l_2$ (Euclidean) norm, and will be denoted $\|\cdot\|_2$. We will use the notation $a \ll b$, where both $a$ and $b$ depend on $M$, to mean $\lim_{M \to \infty} \frac{a}{b} = 0$, where the norm in question may depend on context.

The index-free notation $y = Dx$ will refer to a generic independent copy drawn from the sampling distribution, used for index-independent properties of this distribution such as expectation, while we reserve the indexed notation $y_i$ to refer to a particular random vector in the sample $Y$. Given a sample $y_i$, its support, denoted $\Omega_i$, is defined as the set of indices of the dictionary vectors in its construction with nonzero coefficients:

$$\Omega_i := \text{supp}(x_i) = \{k \in [K] : x_{ik} \neq 0\}.$$  

The “support vectors” of $y_i$ refer to the dictionary elements indexed by $\Omega_i$; the set $\{d_k\}_{k \in \Omega_i}$. We use the notation $A - B$ for the relative complement of $B$ in $A$: $A - B = \{x : x \in A, x \notin B\}$. We denote the dimension of a vector subspace $S$ with the shorthand $\dim(S)$.

Throughout this text, “with high probability” means that an event occurs with probability converging to $1$ faster than any polynomial in $M$; often these will be bounds with the approximate form $M^{-\log M}$. We will frequently make use of the fact that under this definition, the union of polynomially-many events occurring with high probability also occurs with high probability. As our results are asymptotic in nature, we implicitly assume without statement that, where necessary, $M$ is sufficiently large for our results to hold.
3 Algorithm

In this section, we outline the key elements of the spectral subspace dictionary learning algorithm (SSDL). SSDL consists of two main steps: subspace recovery, wherein we aim to recover the subspaces spanned by the support vectors of each sample \(y_i\), and subspace intersection, which combines the information from subsets of the recovered subspaces to recover individual dictionary elements.

3.1 History and Motivation

The key concept underlying SSDL and its proof is the idea that for a dictionary with approximately orthogonal columns (typically \(|\langle d_k, d_m \rangle| \leq C/\sqrt{M}\) for \(k \neq m\)), given two different samples \(y_i\) and \(y_j\), the absolute inner product \(|\langle y_i, y_j \rangle|\) should be larger when they share an element in their support. This idea was used by Arora et al. [2013] for the case \(s \ll \sqrt{M}\). Indeed, since dictionary vectors are unit vectors, we have

\[
\langle y_i, y_j \rangle = \sum_{k \in \Omega_i} \sum_{m \in \Omega_j} x_{ik} x_{jm} \langle d_k, d_m \rangle = \sum_{k \in \Omega_i \cap \Omega_j} x_{ik} x_{jk} + \sum_{k \in \Omega_i - \Omega_j} \sum_{m \in \Omega_j - \Omega_i} x_{ik} x_{jm} \langle d_k, d_m \rangle.
\]

For a random dictionary, the inner product \(\langle d_k, d_m \rangle\) will be of order approximately \(1/\sqrt{M}\) with high probability. Thus if nonzero coefficients \(x_{ik}\) are bounded below, each term in the sum over the intersection \(\Omega_i \cap \Omega_j\) is much larger than the terms in the second sum. In many cases, particularly when \(s \ll \sqrt{M}\), the intersection \(\Omega_i \cap \Omega_j\) will contain at most one element, in which case the first sum is either 0 or \(\pm 1\). On the other hand, the second term will be a sum of approximately \(s^2\) random variables of magnitude \(1/\sqrt{M}\). In particular, if nonzero entries of \(x_i\) are sub-Gaussian and have mean zero, the Hanson-Wright inequality [Hanson and Wright, 1971] guarantees that with high probability (up to possible log factors),

\[
\left| \sum_{k \in \Omega_i - \Omega_j} \sum_{m \in \Omega_j - \Omega_i} x_{ik} x_{jm} \langle d_k, d_m \rangle \right| \leq \frac{\sqrt{|\Omega_i - \Omega_j|}}{\sqrt{M}} \frac{\sqrt{|\Omega_j - \Omega_i|}}{\sqrt{M}} \approx s \frac{M}{\sqrt{M}}.
\]

From this one can consider the following heuristic normal approximation:

\[
\langle y_i, y_j \rangle \sim \begin{cases} N(0, C^2 s^2/M), & |\Omega_i \cap \Omega_j| = 0 \\ N(\pm 1, C^2 s^2/M), & |\Omega_i \cap \Omega_j| = 1. \end{cases}
\]

(1)

Thus when \(s \ll \sqrt{M}\), the absolute inner product \(|\langle y_i, y_j \rangle|\) behaves close to an indicator function for whether \(y_i\) and \(y_j\) share support (the case \(|\Omega_i \cap \Omega_j| \geq 2\) occurs with negligible probability for \(s \ll \sqrt{M}\) and \(K \geq M\)).

Arora et al. [2013] used these inner products to construct a graph where \(i\) and \(j\) shared an edge if \(|\langle y_i, y_j \rangle|\) exceeded some threshold \(\tau\). Overlapping clustering methods were then used to recover overlapping communities \(C_k\) each corresponding to a dictionary element, after which the dictionary elements could be recovered by averaging or by taking the top eigenvalue of the community covariance \(\frac{1}{N} \sum_{i \in C_k} y_i y_i^T\).

However, when \(s \gg \sqrt{M}\), this approach breaks down, as the variance of the terms in 1 dominates the mean term, meaning \(|\langle y_i, y_j \rangle|\) can no longer be used as a reliable indicator of shared support. Indeed, in this case for any fixed threshold \(\tau\), \(P(\langle y_i, y_j \rangle \geq \tau)\) will tend to 1, making reliable community detection nearly impossible. This was perceived as a fundamental roadblock to generalizing the correlation-based techniques of Arora et al. [2013] beyond the \(s \sim \sqrt{M}\) barrier; accordingly, subsequent research on dictionary learning with theoretical guarantees used entirely different techniques.

In this work, our primary observation is that even in the \(s \gg \sqrt{M}\) regime, correlations still contain sufficient information on shared support for the dictionary to be recovered. Even though, in this regime, \(\langle y_i, y_j \rangle\) is dominated by terms originating from non-shared support elements, there is still a small bias in favor of shared support: \(|\langle y_i, y_j \rangle|\) will, on average, be larger when \(|\Omega_i \cap \Omega_j| > 0\).

As already noted, in this regime the correlations are not strong enough to directly infer the sparsity pattern as in [Arora et al., 2013]. Therefore in this work we propose an intermediate step: before attempting to recover the dictionary elements, for each sample \(y_i\) we recover its spanning subspace \(S_i\).
Definition 3.1 (Spanning Subspace). Given a sample \( y_i = Dx_i \), the spanning subspace of sample \( i \) is the subspace \( S_i \), defined as \( S_i = \text{span}\{d_k : k \in \text{supp}(x_i)\} \).

First recovering the spanning subspaces obviates any need to perform community detection on an unreliable connection graph, which was the immediate point of failure for the correlation-based method of Arora et al. [2013] in the \( s \gg \sqrt{M} \) setting.

### 3.2 Subspace Recovery

At a high level, given a sample \( y_j \), the subspace recovery step is a spectral method that constructs a matrix which, with high probability, will have lead \( s \) eigenvectors spanning a subspace close to the true spanning subspace \( S_j \). To estimate \( S_j \), we consider a statistic based on the classical estimator for the covariance of \( y \), the sample covariance matrix \( \hat{\Sigma} \):

\[
\hat{\Sigma} = \frac{1}{N} YY^T = \frac{1}{N} \sum_{i=1}^{N} y_i y_i^T.
\]

As long as \( E y = 0 \), it is easy to see that \( \hat{\Sigma} \) is an unbiased estimator (that is, \( E \hat{\Sigma} = Eyy^T \)); by the law of large numbers, then, \( \hat{\Sigma} \to Eyy^T \) in \( N \) almost surely (later we will use quantitative versions of this result; see, for instance, Vershynin [2018], theorems 4.7.1 and 5.6.1).

To find the subspace \( S_j \), though, we need our estimate to be biased in favor of directions spanned by the support elements of \( y_j \). Our goal is to weight the sample covariance matrix in such a way that a sample \( y_i \) is given more weight the larger the shared support between \( y_i \) and \( y_j \). At first, we employed a method based on the thresholding scheme of Arora et al. [2013], using the statistic

\[
\hat{\Sigma}_j^\tau := \frac{1}{N} \sum_{i \neq j} 1_{\{|\langle y_j, y_i \rangle| \geq \tau\}} \langle y_j, y_i \rangle^2 y_i y_i^T,
\]

where \( \tau \) was a fixed threshold parameter. Our idea was that, although the bias would be small, \( P\{|\langle y_j, y_i \rangle| \geq \tau\} \) would nonetheless be greater when \( y_i \) shares support with \( y_j \) even when \( s \gg \sqrt{M} \). This worked well in numerical simulations, but proved intractable for theoretical work.

Noting that the above is the sample covariance matrix for the random vector \( 1_{\{|\langle y_j, y \rangle| \geq \tau\}} Y \), we replaced this nonlinear thresholding function \( 1_{\{|\langle y_j, y \rangle| \geq \tau\}} \) with the quadratic weight \( \langle y_j, y \rangle^2 \). This change allowed for much cleaner computations by the linearity of expectations. Accordingly, we now introduce the key statistic of the subspace recovery step, the correlation-weighted covariance \( \Sigma_j \):

\[
\Sigma_j := E[|\langle y_j, y \rangle|^2 yy^T].
\]

and its sample version \( \hat{\Sigma}_j \):

\[
\hat{\Sigma}_j := \frac{1}{N} \sum_{i \neq j} \langle y_j, y_i \rangle^2 y_i y_i^T.
\]

We point out that \( \Sigma_j \) and \( \hat{\Sigma}_j \) are the covariance and the sample covariance estimator, respectively, for the random vector \( \langle y_j, y \rangle Y \). Not only is this a more theoretically tractable object, but it also resulted in an immediate improvement in the accuracy of our empirical simulations. As before, the idea is that samples \( y_i \) which share support elements with \( y_j \) will have a higher correlation and therefore the covariance will be “stretched” in favor of the directions spanned by the support elements of \( y_j \).

A major challenge is that when \( s \gg \sqrt{M} \) this bias remains small, with the result that \( \hat{\Sigma}_k \) will be close to the unweighted covariance matrix of \( y \), with only a small perturbation in the directions \( S_j \). However, this covariance can be accurately estimated by the sample covariance \( \hat{\Sigma} = \frac{1}{N} YY^T \). With this estimate in hand, we remove the \( DD^T \) component by “covariance projection” taking the orthogonal complement of \( \hat{\Sigma}_j \) (in the Frobenius sense) with respect to the unweighted sample covariance matrix \( \Sigma = \frac{1}{N} \sum_{i=1}^{N} y_i y_i^T = \frac{1}{N} YY^T \),
with the goal of leaving only the bias component. Specifically, we look at the lead \( s \) eigenvectors of the matrix

\[
\hat{\Sigma}_j^\text{proj} := \hat{\Sigma}_j - \frac{\langle \hat{\Sigma}_j, \hat{\Sigma} \rangle}{\|\hat{\Sigma}\|^2} \hat{\Sigma}.
\]

As sample covariance matrices, \( \hat{\Sigma}_0 \) and \( \hat{\Sigma} \) can be made arbitrarily close to their expectations with sufficiently large sample size, so this statistic will have spectral properties close to the bias matrix \( \sum_{k \in \Omega_j} d_k d_k^T \). We detail the precise process in algorithm 1.

**Algorithm 1: SSR**

**Input:** index \( j \), \( M \times N \) data matrix \( Y = (y_1, y_2, \ldots, y_N) \), est. covariance matrix \( \hat{\Sigma} = \frac{1}{N} YY^T \)

**Output:** \( s \)-dimensional subspace \( \hat{S}_j \)

1. **Correlation-Weighted Covariance:** Compute \( \hat{\Sigma}_j = \frac{1}{N} \sum_{i=1}^N \langle y_j, y_i \rangle^2 y_i y_i^T \)
2. **Covariance Projection:** Compute \( \hat{\Sigma}_j^\text{proj} = \hat{\Sigma}_j - \text{proj}_j(\hat{\Sigma}_j) \)
3. **Spectral Recovery:** Compute the leading \( s \) eigenvectors of \( \hat{\Sigma}_j \) and set \( \hat{S}_j \) equal to their span.

**return** \( \hat{S}_j \)

Naturally, the subspace \( \hat{S}_j \) recovered by algorithm 1 will only approximately match the true subspace \( S_j \). For this reason we introduce the following metric on \( s \)-dimensional subspaces:

**Definition 3.2 (Subspace Distance).** Let \( S, \hat{S} \) be two \( s \)-dimensional subspaces of \( \mathbb{R}^M \), and let \( F \) and \( \hat{F} \) be orthonormal bases for \( S, \hat{S} \) respectively. We define the subspace distance \( D \) between \( S \) and \( \hat{S} \) as

\[
D(S, \hat{S}) = \sup_{z \in S, \|z\|_2 = 1} \|z - \text{proj}_S(z)\|_2 = \|F - \hat{F}\hat{F}^T F\|_2.
\]

In section 4.1, we demonstrate that the recovered \( \hat{S}_j \) is close to the true \( S_j \) for each \( j \) simultaneously with high probability. We also show in theorem 4.5 that with high probability, up to logarithmic factors only the first \( O(K) \) subspaces are required to find every dictionary vector via subspace intersection.

### 3.3 Subspace Intersection

Since the eigenvectors returned by this spectral method are orthonormal, they do not correspond to dictionary elements directly, but instead form a basis for a subspace \( \hat{S}_i \) close to the true subspace \( S_i \). Thus an additional **subspace intersection step** is needed to recover actual dictionary elements from the estimated subspaces.

To motivate our subspace intersection algorithm, we note that if the subspaces \( S_i \) were known exactly, there is a particularly simple algorithm to find a dictionary element when subspaces \( S_i \) are known exactly. Since \( S_i = \text{span}\{d_k\}_{k \in \Omega_i} \), \( S_i \cap S_j = \text{span}\{d_k\}_{k \in \Omega_i \cap \Omega_j} \) (almost surely). It follows that if \( \dim(S_i \cap S_j) = 1 \) exactly, then \( S_i \cap S_j = \text{span}\{d_k\} \) where \( k \) is the unique element in \( \Omega_i \cap \Omega_j \).

Letting \( F_i \) and \( \hat{F}_j \) be orthonormal basis matrices for \( S_i \) and \( S_j \), respectively, we can write an element in \( S_i \) as \( F_i \mathbf{v} \) for \( \mathbf{v} \in \mathbb{R}^s \). Since the matrix for projection onto subspace \( S_j \) is \( F_j F_j^T \), it follows that \( \dim(S_i \cap S_j) = \dim(\ker(F_i - F_j F_j^T F_i)) \); denote this matrix \( P_{ij} \). Then any \( \mathbf{v} \) in the kernel of \( P_{ij} \) corresponds to a vector \( F_i \mathbf{v} \) in \( S_i \cap S_j \). Then if \( \dim(\ker(P_{ij})) = 1 \), we can easily recover a basis for the intersection \( S_i \cap S_j \).

As long as \( s^2/K \) is small, \( |S_i \cap S_j| \) will typically have either 0 or 1 element, so an intersection between two subspaces will rarely have dimension above one. When \( s^2/K \gg 1 \)—typically the case in our setting—we will instead need to perform subspace intersection on more than two subspaces. It is easy to see that

\[
E[|\Omega_1 \cap \Omega_2 \cap \ldots \Omega_{\ell}|] = \frac{s^{\ell+1}}{K^{\ell}}.
\]

Therefore, for \( \ell = \frac{\log s}{\log K/s} \), we have \( E[|\Omega_1 \cap \Omega_2 \cap \ldots \Omega_{\ell}|] \leq 1 \) (this bound is made precise in theorem 4.5).

In the case when we only know approximate subspaces \( \hat{S}_i \), \( P_{ij} \) will almost surely have trivial kernel, so we relax the condition \( \dim(\ker(P_{ij})) = 1 \) to the condition that, given some small threshold \( \tau \), \( P_{ij} \) has
conclude that the entire SSDL process takes $O$ element. Accordingly, with high probability the subspace intersection step to take probability, one only needs to check fewer than $s$ is uniformly distributed among $O$ eigenvectors, which both have order $O$ can be computed in time adds To compute the correlation-weighted covariance $\hat{\Sigma}$, we have $\hat{\Sigma}_{ij} = \frac{1}{N} \sum_{l=1}^{N} y_{il} y_{jl}^T$, where $y_{il}$ is the $l$th element of $y_i$. Define the approximate subspace intersection

**Definition 3.3** (Approximate Subspace Intersection). Let $S_i$ and $S_j$ be subspaces of $\mathbb{R}^M$ with respective orthonormal basis matrices $F_i, F_j$. Denote by $P_{ij} = (I - F_i F_j^T) F_i$ the projection matrix of $S_i$ onto $S_j^\perp$. The approximate subspace intersection of $S_i$ onto $S_j$ with threshold $\tau$ is the subspace $A_\tau(S_i, S_j)$ of $\mathbb{R}^M$ defined as the span of all singular vectors of $P_{ij}$ corresponding to sufficiently small singular values:

$$A_\tau(S_i, S_j) = \text{span}\{v : v \text{ is a singular vector of } P_{ij} \text{ corresponding to a singular value } \sigma_v \leq \tau\},$$

with the convention that $\text{span}(\emptyset) = \{0\}$.

In algorithm 2, we detail the approximate subspace intersection algorithm for a fixed list of $\ell$ subspaces $S_1, \ldots, S_\ell$. In theorem 4.5, we show that with high probability that to recover all $K$ dictionary elements it suffices to consider only the non-overlapping intersections $\bigcap_{p=1}^{\ell} S_{(j-1)+p}$ for $j = 1, \ldots, K \log^3 K / \ell$. To recover an entire dictionary, then, we first employ subspace recovery to learn the subspaces $\{\hat{S}_1, \ldots, \hat{S}_J\}$ for the first $J = K \log^3 K$ samples, then take the intersection of each consecutive set of $\ell$ subspaces. (Duplicates, those estimated dictionary elements which are close to one another based on absolute inner product, can be handled by rejecting duplicates or averaging them together.)

**Algorithm 2: SSI, Approximate Subspace $\ell$-fold Intersection**

**Input:** List of subspaces $S_1, \ldots, S_\ell$, threshold $\tau < 1$

**Output:** Estimated dictionary element, or **False** if no element is found.

$S \leftarrow S_1$

for $i \in \{2, \ldots, \ell\}$

$A = A_\tau(S, \hat{S}_i)$

if $\dim(A) = 0$ then

return False

else if $\dim(A) = 1$ then

Set $d$ a basis of $A$

return $d$

else if $\dim(A) \geq 2$ then

$S = A$

return False

3.4 Time Complexity

To compute the correlation-weighted covariance $\hat{\Sigma}_j$ for a single $j$, the single subspace recovery algorithm adds $N$ matrices of the form $\langle y_j, y_i \rangle^2 y_i y_j^T$, each of which can be computed in $O(M^2)$ time, meaning $\hat{\Sigma}_j$ can be computed in time $O(NM^2)$. Finding the top $s$ eigenvalues and eigenvectors then takes an additional $O(sM^2)$ operations, meaning the entire subspace recovery step for a single sample $y_i$ can be completed in $O(NM^2)$ time. As only the first $K \log^3 K$ subspaces are required to find every dictionary vector via subspace intersection, computing all subspaces will take time $O(NKM^2)$ (again, up to log factors).

The runtime of the each subspace intersection step is dominated by matrix multiplication and finding eigenvectors, which both have order $O(M^3)$. Under the assumption that the support of each sample is uniformly distributed among $s$-element subsets of $\{1, \ldots, K\}$, we show in theorem 4.5 that with high probability, one only needs to check fewer than $O(K \log^3 K)$ intersections in order to recover each dictionary element. Accordingly, with high probability the subspace intersection step to take $O(KM^3)$ time, so we conclude that the entire SSDL process takes $O(NKM^2 + KM^3)$ with high probability, up to log factors.
4 Main Results

We are now ready to present our main result, which states that most dictionaries $D \sim \mathcal{D}$ can be recovered with sparsity linear in $M$ up to a logarithmic factor:

**Theorem 4.1.** As in 2.3, fix parameters $0 < \gamma < \eta$ and set $s = C_s M \log^{-(6+\eta)}(M)$, $K = C_K \log^{4+\gamma}(M)$, and $N = \max \left\{ \frac{s^6}{M^2}, \frac{K^2 s^4}{M^2} \right\}$. Suppose that $Y = DX$ with $D \sim \mathcal{D}$ and $X \sim \mathcal{X}(W)$. Then SSDL recovers a dictionary $\hat{D}$ that is column-wise $o(1)$-close to $D$ with high probability in $M$: $\hat{D}$ and $D$ are the same size and there exists $\varepsilon = \varepsilon(M) = o(1)$, a sequence of signs $\theta$, and a permutation $\pi$ of $\{1, \ldots, K\}$ such that for all $k = 1, \ldots, K$:

$$||d_k - \theta_k \hat{d}_{\pi(k)}||_2 < \varepsilon.$$ 

Here the implied rate of convergence of $\varepsilon$ depends on $C_s, C_k, \eta$, and $\gamma$.

We note that this theorem can be adapted to the case where $s = C_s M^{1-\gamma}$ and $K = C_K M^{1+\gamma}$; we restrict our proof to the case that $s$ and $K$ are linear up to logarithmic factors, as this is the most challenging regime for the dictionary learning problem. We also point out that theorem 4.1 indicates that the sample complexity required by SSDL is lower in these easier sparsity regimes.

We now outline the strategy for our proof. First, we fix a single element in the sample, which we will denote $y_0$; due to the symmetry of the $\mathcal{D}$ and $\mathcal{X}(W)$ distributions, without loss of generality, we may assume $\Omega_0 = \{1, 2, \ldots, s\}$ and $x_{01} = \ldots = x_{0s} = 1$. Under these assumptions, $y_0 = \sum_{k=1}^s d_k$. We then prove that the subspace recovery algorithm recovers the subspace $S_0$ with error tending to 0 with high probability, using standard concentration of measure results (Hoeffding’s, Bernstein’s inequalities) along with $\varepsilon$-net arguments for controlling the norms of random matrices. Since there are only polynomially-many samples, by a union bound we can recover all subspaces $\{S_i\}_{i=1}^N$ accurately with high probability.

We then demonstrate that given close enough estimates for these subspaces, the subspace intersection step recovers individual dictionary elements with error tending to 0 with high probability (and correctly returns nothing when no common dictionary element is present). These results follow from the near-orthogonality of vectors in high dimensions [e.g. Vershynin, 2018] along with Weyl’s theorem and the Davis-Kahan Theorem on the continuity of eigenvalues and invariant subspaces, respectively, of matrices under perturbation [Weyl, 1912, Davis and Kahan, 1970].

4.1 Guarantees for Subspace Recovery

We proceed with an overview of our guarantees for the subspace recovery step. To prove the result for fixed $x_0$, we first consider the expectation of the correlation-weighted covariance $\hat{\Sigma}_0$ conditional on a fixed dictionary $D$ and fixed support set $\Omega_0$. We show that, with high probability, $D$ is such that $E[\hat{\Sigma}_0 | D]$ consists of a matrix with leading $s$-eigenvectors which approximately span $S_0$, plus a multiple of the dictionary covariance matrix which will be removed in the covariance projection step. Specifically, we show that:

$$E[\hat{\Sigma}_0 | D] = \frac{s}{K} \left( \frac{s}{K} v_0 v_0^T + \sum_{k \in \Omega_0} Z_k d_k d_k^T + Z_D DD^T + o(1) \right)$$

(2)

Here $v_0 = \sum_{k=1}^K \langle y_0, d_k \rangle d_k$ is a random vector close to $y_0$ in direction. Specifically, $v_0/||v_0||_2 = y_0/||y_0|| + o(1)$, $Z_k = 1 + 2 \langle \hat{y}_0^k, d_k \rangle$ are random variables bounded below by 1/2 with high probability, and $Z_D$ is a random variable, the particular value of which is not important as it will be completely removed in covariance projection. The $o(1)$ term is a random matrix with operator norm tending to 0 in $M$.

It follows that once the $DD^T$ term is removed, $E[\hat{\Sigma}_0 | D]$ consists of three further terms: a large rank-one matrix generated by a vector tending to $y_0/||y_0||_2$ when normalized, a sum of $s$ constant-order rank-one matrices generated by vectors spanning $S_0$, and a matrix with operator norm tending to 0. Accordingly, with high probability the leading $s$ eigenvectors of $E[\hat{\Sigma}_0 | D]$ will span a subspace $\hat{S}_0$ close to $S_0$:

**Theorem 4.2.** Fix parameters $0 < \gamma < \eta$ and set $s = C_s M \log^{-(6+\eta)}(M)$, $K = C_K M \log^{4+\gamma}(M)$. Suppose that $Y = DX$ with $D \sim \mathcal{D}$ and $X \sim \mathcal{X}(W)$. For $D, Y$ distributed as in 2.3, define $S_0 = \text{span} \{d_k\}_{k \in \Omega_0}$ and $\hat{S}_0 = \frac{1}{K} \sum_{i=1}^N \langle y_0, y_i \rangle^2 y_i y_i^T$. Then the span of the leading $s$ eigenvectors of $E[\hat{\Sigma}_0 | D] - \frac{1}{||DD^T||_F} F_D DD^T$ tends to $S_0$ in probability as $M \to \infty$ (in the sense of 3.2).
4.1.1 Sample complexity

Since the sample covariance matrix is an unbiased estimator of the true covariance matrix, the law of large numbers will guarantee that for fixed dictionary $\mathbf{D}$, $\hat{\Sigma}_j \rightarrow E[\Sigma_j] \mathbf{D}$ almost surely with enough samples $N$. In this section we state explicit high-probability bounds on the convergence of $\hat{\Sigma}_0$ to its expectation.

In particular, we claim that as long as number of samples satisfies $N \geq \max \left\{ \frac{s^{10}}{M^6}, \frac{K^2 s^4}{M^3} \right\}$, the empirically observed matrix $\hat{\Sigma}_0$ will have the same spectral properties as its expectation with high probability. This translates to a worst-case complexity of order $M^4$. As these results are largely applications of known technical results, we defer details of the proof to the appendix. Specifically, we have:

Theorem 4.3. Let $N$ be at most polynomial in $M$ with $N \geq \max \left\{ \frac{s^{10}}{M^6}, \frac{K^2 s^4}{M^3} \right\}$. Then with high probability

$$\hat{\Sigma}_0^{proj} = \hat{\Sigma}_0 - \text{proj}_{\hat{S}_0}(\hat{\Sigma}_0) = \frac{s}{K} \left( \frac{s}{K} \mathbf{v}_0 \mathbf{v}_0^T + \sum_{k \in \Omega_0} Z_k \mathbf{d}_k \mathbf{d}_k^T + o(1) \right),$$

where $Z_k$ are random variables satisfying $Z_k > 1/2$, and the $o(1)$ term is a matrix with $o(1)$ measured in operator norm. It follows that, as $M \to \infty$, the subspace spanned by the lead $s$ eigenvalues of $\text{proj}_{\hat{S}_0}(\hat{\Sigma}_0)$ will tend to $\mathcal{S}_0$ with high probability.

The result follows from known results on covariance estimation along with straightforward—if lengthy—bounds using the triangle inequality.

4.2 Subspace Intersection

In this section, we present guarantees stating that that with high probability, the subspace intersection step rejects groups of subspaces which do not contain a unique dictionary element in their intersection. If they do intersect, then subspace intersection returns a vector close to the true vector. Specifically, we have the following theorem:

Theorem 4.4. Let $\ell = \frac{\log(2K)}{\log(K/s)}$, and let $\mathcal{J}$ be a collection of at most polynomially-many $\ell$-element subsets of $[N]$. With high probability as $M \to \infty$, the following holds uniformly for every $\mathcal{I} \in \mathcal{J}$:

If $\bigcap_{i \in \mathcal{I}} \mathcal{S}_i = \text{span}(\mathbf{d}_k)$, then $\mathbf{d}$ will be returned by algorithm 2 with $\tau = 1/2$ and will satisfy $\min_{i \in \{1, \ldots, K\}} \{ \| \mathbf{d} - t \mathbf{d}_k \|_2 \} = o(1)$. Moreover, if $\dim \left( \bigcap_{i \in \mathcal{I}} \mathcal{S}_i \right) \neq 1$, the algorithm returns False.

4.2.1 All $k$ appear as $\ell$-fold intersection with $K \log^3 K$ subspaces

We conclude by showing that it is possible to isolate each dictionary element by looking at only a polynomial number of $\ell$-element intersections, guaranteeing we can recover every column in the dictionary in polynomial time. Specifically, for $\ell = \frac{\log(2K)}{\log(K/s)}$ and $J = K \log^3 K$, we claim that with high probability, scanning over the first $J/\ell$ disjoint $\ell$-element subsets is sufficient to recover every dictionary element at least once.

Theorem 4.5. Let $\ell$ be the smallest integer such that $(s/K)^\ell < 1/2K$, and assume $\Omega_i$, $i = 1, \ldots, K$ are uniformly distributed among $s$-element subsets of $\{1, \ldots, K\}$. For positive integer $J$ and $j \in [J/\ell]$, define the non-intersecting $\ell$-fold intersections $\Omega_j^\ell$ as

$$\Omega_j^\ell = \bigcap_{p=1}^\ell \Omega_{(j-1)\ell+p}.$$

Then as long as $J \geq K \log^3 K$, with high probability for every $k \in [K]$ there exists a $j \in [J/\ell]$ such that $\Omega_j^\ell = \{ k \}$.

The proof of theorem 4.5 follows by fixing $k$ and noting that since the sets $\Omega_j^\ell$ do not overlap, they will be independent, then calculating the probability that none of them contain $k$ as a unique element of intersection; a union bound completes the proof. Details are in the appendix.
Theorem 4.5 guarantees that subspace intersection will recover every dictionary element at least once, while requiring only $K \log^2 K$ intersections. We know from the previous section that with high probability, this dictionary element will be recovered accurately by theorem 2. Collectively, these results complete the proof of theorem 4.1.

5 Proofs of Theoretical Guarantees

In this section, we provide sketches of the proof for the results outlined in the previous section.

5.1 Proofs for Subspace Intersection

We begin with a proof of theorem 4.2. The continuity of invariant subspaces under perturbation [Weyl, 1912, Davis and Kahan, 1970] means the theorem follows from the following technical result:

Lemma 5.1. Fix parameters as in 4.2. Then the following holds with high probability:

$$E[\tilde{\Sigma}_0|D] = \frac{s}{K} \left( \frac{s}{K} \mathbf{v}_0 \mathbf{v}_0^T + \sum_{k \in \Omega_0} Z_k \mathbf{d}_k \mathbf{d}_k^T + Z_{\mathbf{D}} \mathbf{D} \mathbf{D}^T + o(1) \right)$$

where \( \mathbf{v}_0 \) is a random vector satisfying \( \frac{\mathbf{v}_0}{\|\mathbf{v}_0\|_2} = \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2} + o(1) \), \( Z_k \) are random variables with \( Z_k > 1/2 \), and \( Z_{\mathbf{D}} \) is a random variable. Moreover, covariance projection yields:

$$E[\tilde{\Sigma}_0|D] - \frac{\langle E[\tilde{\Sigma}_0|D], \mathbf{D} \mathbf{D}^T \rangle}{\|\mathbf{D} \mathbf{D}^T\|_F^2} \mathbf{D} \mathbf{D}^T = \frac{s}{K} \left( \frac{s}{K} \mathbf{v}_0 \mathbf{v}_0^T + \sum_{k \in \Omega_0} Z_k \mathbf{d}_k \mathbf{d}_k^T + o(1) \right)$$

To prove our results, we define a “good event” \( \mathcal{G}_0 \) for fixed \( \mathbf{x}_0 \), which occurs with high probability. \( \mathcal{G}_0 \) describes sufficient geometric properties of \( \mathbf{D} \) for the dictionary to be recovered successfully. This allows us to prove separately that \( \mathcal{G}_0 \) occurs with high probability, after which we can treat these as deterministic properties while proving our main results.

Many of the following facts can be inferred heuristically using the standard approximation that in high dimensions, uniformly distributed unit vectors are nearly distributed as \( N(0, \frac{1}{M} I) \) random vectors. Specifically, since the \( N(0, \frac{1}{M} I) \) distribution is rotationally invariant, we can assume the existence of a collection \( \mathbf{w}_1, \ldots, \mathbf{w}_K \) of i.i.d. \( N(0, \frac{1}{M} I) \) random vectors such that \( \mathbf{d}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|_2} \) for all \( k = 1, \ldots, K \). It was shown by Stam [1982] that the \( N(0, \frac{1}{M} I) \) and uniform distributions converge in total variation; we will only use weaker properties that can be derived from standard concentration of measure results.

Before defining \( \mathcal{G}_0 \), we introduce the useful notation \( \tilde{\mathbf{y}}_0 = \mathbf{y}_0 - \delta_{k \in \Omega_0} \mathbf{d}_k \). We now define \( \mathcal{G}_0 \) rigorously:

Definition 5.2 (Good Event). The good event for \( \mathbf{x}_0 \), denoted \( \mathcal{G}_0 \), is the event that the following conditions hold simultaneously. We use \( c \) and \( C \) for small and large positive constants respectively; constants vary between list items.

\[ \mathcal{G}_0.1 \ |\mathbf{y}_0|_2 = \sqrt{s} + o(\sqrt{s}) \]

\[ \mathcal{G}_0.2 \text{ Defining } \tilde{\mathbf{y}}_0^k = \mathbf{y}_0 - \delta_{k \in \Omega_0} \mathbf{d}_k, \text{ we have } \sup_k |\langle \tilde{\mathbf{y}}_0^k, \mathbf{d}_k \rangle| \leq \frac{C \sqrt{s} \log M}{\sqrt{M}} \text{ for all } k \]

\[ \mathcal{G}_0.3 \sup_{k \neq m} |\langle \mathbf{d}_k, \mathbf{d}_m \rangle| \leq \frac{C \log M}{\sqrt{M}} \]

\[ \mathcal{G}_0.4 \text{ For all } k \in \Omega_0, 1/2 < |\langle \mathbf{y}_0, \mathbf{d}_k \rangle| < C \]

\[ \mathcal{G}_0.5 \text{ For all } k, \left| \sum_{m \neq k} \langle \mathbf{y}_0, \mathbf{d}_m \rangle \langle \mathbf{d}_m, \mathbf{d}_k \rangle \right| \leq \frac{C \sqrt{k} \log M}{M} \]

\[ \mathcal{G}_0.6 \frac{c K}{\sqrt{M}} \leq \|\mathbf{D} \mathbf{D}^T\|_F \leq CK/\sqrt{M} \]

\[ \mathcal{G}_0.7 \frac{c K^2}{M} \leq \sum_{k=1}^K |\langle \mathbf{y}_0, \mathbf{d}_k \rangle|^2 \leq \frac{CK^2}{M}, \text{ and the same inequality holds for } \tilde{\mathbf{y}}_0^k \text{ in place of } \mathbf{y}_0. \]
\[ G_0.8 \quad \frac{c_K}{M} \leq \|DD^T\|_2 \leq \frac{c_K}{M} \]

\[ G_0.9 \quad \left\| \sum_{k=1}^{K} \langle y_0^k, d_k \rangle^2 d_k d_k^T \right\|_2 \leq \frac{c K^2 \log^2 M}{M^2} \]

\[ G_0.10 \quad \text{Let } \text{proj}_{\gamma_0}(d_k) \text{ denote the orthogonal projection of } d_k \text{ onto the complement of } y_0. \text{ Then} \]

\[ \left\| \sum_{k=1}^{K} \langle y_0, d_k \rangle \text{proj}_{\gamma_0}(d_k) \right\|_2 \leq \frac{C \sqrt{K} \log^2 M}{\sqrt{M}}. \]

We have the following lemma, with proof in the appendix:

**Lemma 5.3.** The good event \( G_0 \) occurs with high probability.

With lemma 5.3 in hand, we begin our proof of theorem 4.2 via lemma 5.1. The proof turns on the following calculation, which decomposes the expectation \( E[\hat{\Sigma}_j|D] \) into four terms: a large leading eigenvector close to \( y_0 \) in direction, a rank-\( s \) “signal” term with eigenvectors spanning \( S_0 \), a random-variable multiple of the dictionary covariance \( DD^T \), and a “noise” term with small operator norm.

**Lemma 5.4.** For fixed \( \Omega_0 \),

\[
\frac{K}{s} E[\langle y_0, y \rangle^2 yy^T|D] = \frac{s - 1}{K - 1} v_0 v_0^T + \left( 1 - \frac{2(s - 1)}{K - 1} \right) \sum_{k \in \Omega_0} [1 + 2 \langle \hat{y}_0^k, d_k \rangle] d_k d_k^T + \left( 1 - \frac{2(s - 1)}{K - 1} \right) \sum_{k=1}^{K} \langle \hat{y}_0^k, d_k \rangle^2 d_k d_k^T + \left( \frac{s - 1}{K - 1} \sum_{k=1}^{K} \langle y_0, d_k \rangle^2 \right) DD^T \tag{5}
\]

where \( v_0 = \sum_{k=1}^{K} \langle y_0, d_k \rangle d_k \) and \( \hat{y}_0^k = y_0 - \delta_k \in \Omega_0 d_k \).

Here the first row consists of signal terms, which form a matrix with \( s \)-leading eigenvalues approximately spanning the subspace \( S_0 \). By contrast, the second row consists of terms which will ultimately not affect this subspace information: they either have small magnitude (the third term) or they will be removed by covariance projection (the fourth term). We include the proof of this lemma as the computations involved are instructive regarding the composition of \( \hat{\Sigma}_j \):

**Proof.** As all expectations in this lemma are conditional on \( D \) and \( y_0 \), we will not write this explicitly in the proof.

We can expand:

\[
E \langle y_0, y \rangle^2 yy^T = E \sum_{k_1 \in \Omega} \sum_{k_2 \in \Omega} \sum_{k_3 \in \Omega} \sum_{k_4 \in \Omega} x_{k_1} x_{k_2} x_{k_3} x_{k_4} \langle y_0, d_{k_1} \rangle \langle y_0, d_{k_2} \rangle \langle y_0, d_{k_3} \rangle d_{k_4} d_{k_4}^T.
\]

Since \( E x_{ik} = 0 \) for all \( i, k \), terms in the above expectation will be nonzero only when the indices are paired.

Accordingly,

\[
E \langle y_0, y \rangle^2 yy^T = E \sum_{k \in \Omega} \sum_{m \in \Omega} x_k^2 x_m^2 \langle y_0, d_k \rangle^2 d_m d_m^T + E \sum_{k \in \Omega} \sum_{m \in \Omega, m \neq k} x_k^2 x_m^2 \langle y_0, d_k \rangle \langle y_0, d_m \rangle d_k d_m^T
\]

\[
= E \sum_{k=1}^{K} \sum_{m=1}^{K} \mathbb{1}_{\{k, m\} \subseteq \Omega} \langle y_0, d_k \rangle^2 d_m d_m^T + E \sum_{k=1}^{K} \sum_{m \neq k} \mathbb{1}_{\{k, m\} \subseteq \Omega} \langle y_0, d_k \rangle \langle y_0, d_m \rangle d_k d_m^T
\]

\[
= \sum_{k=1}^{K} \sum_{m=1}^{K} \mathbb{P}(\{k, m\} \subseteq \Omega) \langle y_0, d_k \rangle^2 d_m d_m^T + \sum_{k=1}^{K} \sum_{m \neq k} \mathbb{P}(\{k, m\} \subseteq \Omega) \langle y_0, d_k \rangle \langle y_0, d_m \rangle d_k d_m^T
\]

\[
= \frac{s}{K} \sum_{k=1}^{K} \langle y_0, d_k \rangle^2 d_k d_k^T + \frac{s(s - 1)}{K(K - 1)} \sum_{k=1}^{K} \sum_{m \neq k} \langle y_0, d_k \rangle^2 d_m d_m^T + \frac{s(s - 1)}{K(K - 1)} \sum_{k=1}^{K} \sum_{m \neq k} \langle y_0, d_k \rangle \langle y_0, d_m \rangle d_k d_m^T.
\]
Next, we complete the square by transferring part of the first term into the other two:

\[ E(y_0, y)^2 y y^T = \left( \frac{s}{K} - \frac{2s(s-1)}{K(K-1)} \right) \left( \sum_{k=1}^K (y_0, d_k)^2 d_k d_k^T \right) + \]
\[ \frac{s(s-1)}{K(K-1)} \left( \sum_{k=1}^K (y_0, d_k)^2 \right) \left( \sum_{k=1}^K d_k d_k^T \right) + \frac{s(s-1)}{K(K-1)} \left( \sum_{k=1}^K (y_0, d_k) d_k \right) \left( \sum_{k=1}^K (y_0, d_k) d_k^T \right)^T \]  

(6)

Lastly, we note that for \( k \in \Omega_0 \), \( (y_0, d_k) = 1 + (y_0 - d_k, d_k) = 1 + \langle \hat{y}_0, d_k \rangle \), while for \( k > s \), \( \hat{y}_0^k = 0 \). Accordingly we can substitute

\[ \sum_{k=1}^K (y_0, d_k)^2 d_k d_k^T = \sum_{k=1}^s \left[ 1 + 2 \langle \hat{y}_0, d_k \rangle \right] d_k d_k^T + \sum_{k=1}^K \langle \hat{y}_0, d_k \rangle^2 d_k d_k^T. \]

The result follows by making these substitutions in 6, factoring out \( s/K \), and noting that \( \sum_{k=1}^K d_k d_k^T = DD^T \)
and \( v_0 = \sum_{k=1}^K (y_0, d_k) d_k \) by definition.

We proceed by controlling the signal terms:

\[ \frac{s-1}{K-1} v_0 v_0^T + \sum_{k \in \Omega_0} \left[ 1 + 2 \langle \hat{y}_0, d_k \rangle \right] d_k d_k^T. \]

It follows immediately from \( \mathcal{G}_0.2 \) that, on \( \mathcal{G}_0 \), the coefficients in the second term are bounded below by \( 1/2 \) as stated in theorem 4.2. It remains to show that \( v_0/\|v_0\|_2 \) is approximately \( y_0/\|y_0\|_2 \):

**Lemma 5.5.** Let \( v_0 = \sum_{k=1}^K (y_0, d_k) d_k \), and assume that \( \mathcal{G}_0 \) holds. Then \( \langle v_0, \frac{y_0}{\|y_0\|_2} \rangle \geq \frac{C \sqrt{\kappa} \sqrt{\log^2 M}}{\sqrt{K}} \). Moreover, for any unit vector \( z \) in the orthogonal complement of \( y_0 \), \( |\langle v_0, z \rangle| \leq \frac{C \sqrt{\kappa} \sqrt{\log^2 M}}{\sqrt{M}} \). It follows that

\[ \frac{v_0}{\|v_0\|_2} \approx \frac{y_0}{\|y_0\|_2} + O \left( \frac{C \sqrt{\kappa} \sqrt{\log^2 M}}{\sqrt{K}} \right). \]

**Proof.** We expand the expression to yield

\[ \langle v_0, \frac{y_0}{\|y_0\|_2} \rangle = \sum_{k=1}^K \frac{(y_0, d_k)^2}{\|y_0\|_2}. \]

The first claim follows from the above expression and \( \mathcal{G}_0.1, \mathcal{G}_0.7 \).

For the second part, it suffices to show that the projection of \( v_0 \) on the orthogonal complement of \( y_0 \) has norm bounded by \( \frac{C \sqrt{\kappa} \sqrt{\log^2 M}}{\sqrt{M}} \), which follows immediately from \( \mathcal{G}_0.10 \). As \( \sqrt{K/M} \gg \log^2 M \), this term is small compared to the \( y_0 \) component, so the result follows. \( \square \)

### 5.2 Covariance Projection

From \( \mathcal{G}_0.9 \), since \( Ks/M^2 \ll 1/\log^2 M \) we know that \( \left( 1 - \frac{2(s-1)}{K-1} \right) \sum_{k=1}^K (\hat{y}_0^k, d_k)^2 d_k d_k^T \) (the noise term from 5) has small operator norm. This completes the proof of equation 3 in lemma 4.2, which says:

\[ \frac{K}{s} E[\hat{S}_0|D|] = \frac{s}{K} v_0 v_0^T + \sum_{k \in \Omega_0} Z_k d_k d_k^T + Z_D D D^T + o(1) \]  

(7)

To complete the proof of lemma 5.1, we wish to show that Frobenius projection of this matrix onto \( E y y^T = \frac{K}{s} D D^T \) removes the \( D D^T \) term in equation 7 while not contributing more than an \( o(1) \) factor elsewhere. Since projection is scale-invariant, it suffices to prove this for projection onto \( DD^T \) in place of \( E y y^T \). We prove the following lemma:
Lemma 5.6. Recall that $E[\hat{\Sigma}_0|D] = E[(y_0, y)^2 yy^T|D]$. On $G_0$,

$$\frac{\langle K \hat{\Sigma}_0 | D \rangle, DD^T \rangle}{\|DD^T\|^2_F} = \left( \frac{s}{K} \sum_{k=1}^{K} \langle y_0, d_k \rangle^2 \right) DD^T + o(1)$$

where the $o(1)$ bound is in terms of the operator norm.

This proof follows from $G_0$ and the triangle inequality; a detailed proof is deferred to the appendix. This completes the proof of lemma 5.1, from which theorem 4.2 immediately follows.

5.3 Sample Complexity

Next, we prove theorem 4.3. For this proof, we require that the following additional properties of $D$ hold; we call this the “better event.” This should be considered as an addendum to $G_0$ containing further geometric properties of the dictionary $D$ needed for empirical recovery.

Definition 5.7 (Better Event). The better event for $x_0$, denoted $B_0$, is the event that $G_0$ holds, and the following conditions hold simultaneously. We use $c$ and $C$ for large and small (positive) constants respectively; constants vary between list items.

1. $\frac{cK\sqrt{\log M}}{\sqrt{M}} \leq \left\| \sum_{k=1}^{K} \langle y_0, d_k \rangle d_k \right\|_2 \leq \frac{CK\sqrt{\log^2 M}}{\sqrt{M}}$
2. $\sqrt{E[\langle y_0, \hat{\Sigma}_0 D \rangle]^2} \geq \frac{Cs^{3/2}}{\sqrt{M}}$
3. $\|E[\hat{\Sigma}_0 D]\|_2 \leq \frac{Cs^3 \log^M M}{M^2}$
4. $\|E[\hat{\Sigma}_0 D]\|_F \leq \frac{Cs^3 \log^M M}{M^{3/2}}$

As with $G_0$, we have the following lemma, proved in the appendix:

Lemma 5.8. The better event $B_0$ occurs with high probability.

These properties allow us to apply the following theorem (Vershynin [2018], theorem 5.6.1) on covariance estimation, versions of which are well-known in the literature:

Theorem 5.9 (Vershynin [2018], Theorem 5.6.1, General Covariance Estimation (Tail Bound)). Let $z$ be a random vector in $\mathbb{R}^M$. Assume that, for some $\kappa \geq 1$,

$$\|z\|_2 \leq \kappa \sqrt{E[\|z\|_2^2]}$$

almost surely. Then, for every positive integer $N$, $\left\{ z_i \right\}_{i=1}^{N}$ i.i.d. copies of $z$, and $t \geq 0$, we have:

$$\frac{\|Ezz^T - \frac{1}{N} \sum_{i=1}^{N} z_i z_i^T\|_2}{\|Ezz^T\|_2} \leq C \left( \frac{\kappa^2 M (\log M + t)}{N} + \frac{\kappa^2 M (\log M + t)}{N} \right)$$

with probability at least $1 - 2 \exp(-t)$.

This theorem allows us to conclude that for sample size $N \sim M^4$ in the worst case, with high probability, the empirical correlation-weighted covariance matrix $\hat{\Sigma}^{proj}_0$ will be $o(1)$-close to its expectation. Accordingly, $\hat{\Sigma}^{proj}_0$ has lead $s$ eigenvalues which span a subspace converging to $S_0$. Accordingly, the subspace recovery step 1 recovers $S_0$ accurately with high probability. This completes the proof of theorem 4.3; having proven it for a fixed sample $y_0$, the result immediately holds for all samples $y_j$ simultaneously via a union bound.
5.4 Guarantees for subspace intersection

We now prove theorem 4.5, which states that the intersection step with close enough estimated subspaces \( \hat{S}_1, \ldots, \hat{S}_r \) accurately approximates the intersection of true subspaces \( S_1, \ldots, S_t \). The result follows from the following lemma:

**Lemma 5.10.** Suppose \( \Omega_1, \Omega_2 \) are at-most-\( s \)-element subsets of \( \{1, 2, \ldots, K \} \) such that \( \Omega_1 \cap \Omega_2 = \emptyset \). Let \( S_1 = \text{span}\{d_k\}_{k \in \Omega_1} \) and \( S_j = \text{span}\{d_m\}_{m \in \Omega_2} \). Then with high probability,

\[
\inf_{v_1 \in S_1, v_2 \in S_2} \frac{\|v_1 - v_2\|_2}{\|v_1\|_2 \cdot \|v_2\|_2} \geq \sqrt{2} - \frac{C \sqrt{s} \log M}{\sqrt{M}}.
\]

The proof of this lemma can be found in the appendix, and relies on the fact that the vectors in a random \( D \)-distributed dictionary are nearly orthogonal. It follows from lemma 5.10 that for large enough \( M \), two subspaces \( S_i \) and \( S_j \) contain vectors closer than a constant threshold if and only if they share support (\( |\Omega_i \cap \Omega_j| \geq 1 \)). Since this result holds with high probability, this holds for all pairs \( i, j \) simultaneously. Since this holds for pairwise intersections, the analogous result automatically holds for \( \ell \)-wise intersections as well.

The proof of theorem 4.4 now follows almost immediately. From lemma 5.10, the theorem would hold if the estimated subspaces \( \hat{S}_j \) were to be replaced by the true subspaces \( S_j \). Yet from theorem 4.3, we know that with high probability, \( \hat{S}_j \rightarrow S_j \) uniformly for all \( j \) by the continuity of invariant subspaces of matrices under perturbation ([Weyl, 1912, Davis and Kahan, 1970]). The proof of theorem 4.1 then follows from theorem 4.5, which indicates we recover each dictionary element at least once with a polynomial number of intersections.

6 Numerical Simulations

In this section, we supplement our theoretical results with numerical simulations. All code used in these simulations, including an open-source Python implementation of SSDL, is publicly available at [https://github.com/sew347/spectral_dict_learn](https://github.com/sew347/spectral_dict_learn). We consider two performance metrics. The first is convergence in angle: how close is \( \langle \hat{d}, d \rangle \) to 1. This measure can always be converted to the error in \( L_2 \) norm by the identity \( \|\hat{d} - d\|^2 = 2 - 2\langle \hat{d}, d \rangle \) (up to possible differences in sign). The second main metric is the proportion of false recoveries: a false recovery occurs when subspace intersection either returns a vector when a dictionary does not exist in the intersection of true subspaces, or when subspace intersection fails to return a vector when one is in the intersection of true subspaces.

To test our theoretical hypotheses, in figure 1, we show the maximum sparsity that can be tolerated by SDL while remaining above a certain accuracy threshold. Specifically, this figure shows the highest sparsity for which our test returns average angular accuracy above 0.95 with false recovery proportion below 0.08. Tests were run on the first 50 subspaces over 5 dictionaries in each dimension. To ensure a nontrivial number of overlaps, the support of each of the first 50 samples was seeded so that \( 1 \in \Omega_1, \Omega_2, 2 \in \Omega_3, \Omega_4, \ldots, 25 \in \Omega_5 \). In this test \( K = 2M \), and \( N \) was pegged to \( N = 30000 \) for \( M = 500 \) then allowed to grow as different powers of \( M \). The results confirm the findings of theorem 4.3: sparsity growth is linear when \( N \sim M^4 \), but slower than linear when \( N \sim M^3 \) or \( N \sim M^2 \).

7 Conclusion

We introduced SSDL as an efficient method for recovering dictionaries from high-dimensional samples even in the linear sparsity regime. In this regime, SSDL achieves decaying errors in \( M \) in polynomial time, improving on the best-known provable alternatives. Reproducible numerical simulations validate these results.

Our initial research on SSDL suggests several avenues for future research. First, we suspect it may be possible to reduce the sample complexity from approximately \( M^4 \) to closer to \( M^3 \) by replacing the covariance projection step with a more sophisticated method for controlling the dominant term in \( \Omega \). This is because the
As predicted by theorem 4.3, \( N \sim M^4 \) allows for linear sparsity growth in \( M \), while fewer samples result in sublinear growth.

The worst term in the error in theorem 4.3 comes from estimating a Frobenius rather than operator norm, which is known to have worse sample complexity \( (N \sim M^2) \) than estimation in the operator norm \( (N \sim M) \). We are also interested in investigating to what degree the uniformity assumption in the generation of support sets \( \Omega_j \) can be relaxed, to allow for more general sampling distributions. As our method involves computation of a fourth-order statistic, it bears some similarity to the recently introduced \( \ell_4 \)-based dictionary learning methods; future work will seek to study these connections in greater detail.

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## A Technical Proofs

In this section, we include proofs of technical lemmas which we omitted from the main text.

### A.1 Proof of lemma 5.3

We begin with the proof of our probability result, that $G_0$ occurs with high probability:

**Lemma 5.3.** The good event $G_0$ occurs with high probability.

We will make use of the following lemma:

**Lemma A.1.** Let $\sigma > 0$ and let $w$ be an $N(0, I)$ random vectors. Then with high probability, $\sqrt{M} - \log M < ||w||_2 < \sqrt{M} + \log M$.  

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Proof. By definition, entries of \( w \), denoted \( w_m \) for \( m = 1, \ldots, M \), are i.i.d. normal random variables with variance 1. It follows that \( \|w\|_2^2 = \sum_{m=1}^{M} w_m^2 \) will be distributed as a chi-squared random variable with \( M \) degrees of freedom. Such a variable obeys the following concentration inequalities [Laurent and Massart, 2000]:

\[
P \left( \|w\|_2^2 \geq M + \sqrt{Mt} + 2t \right) \leq e^{-t}\]
\[
P \left( \|w\|_2^2 \leq M - \sqrt{Mt} \right) \leq e^{-t}\]

which for \( t \leq M \) yields the symmetric bound:

\[
P \left( \|w\|_2^2 - M \geq 3\sqrt{Mt} \right) \leq e^{-t}\]

To convert this to a bound on \( \|w\|_2 \), we use the fact that for any \( z, \delta \geq 0, |z - 1| \geq \delta \) implies \( |z^2 - 1| \geq \max\{\delta, \delta^2\} \). Accordingly,

\[
P \left( \frac{1}{\sqrt{M}} \|w\|_2 - 1 \geq \delta \right) \leq P \left( \frac{1}{M} \|w\|_2^2 - 1 \geq \max\{\delta, \delta^2\} \right).
\]

Setting \( \delta = \log M / \sqrt{M} \) and applying 9 with \( t = (\log^2 M) / 9 \leq M \), we have

\[
P \left( \|w\|_2 - \sqrt{M} \leq \log M \right) \leq P \left( \|w\|_2^2 - M \leq \sqrt{M} \log M \right) \leq e^{- (\log^2 M) / 9}.
\]

which proves the result. \( \square \)

We now prove lemma 5.3:

Proof. Recall that the rotational invariance of the normal and uniform distributions guarantees the existence of a collection of i.i.d. random vectors \( w_k \sim N(0, \frac{1}{M} I) \) such that \( d_k = w_k / \|w_k\|_2 \). By lemma A.1, \( 1/2 < \|w_k\|_2^2 < 2 \) for all \( k \) with high probability, so throughout the proof we assume we reside on this event.

\( G_{0.1} \). We write

\[
y_0 = \sum_{k=1}^{s} d_k = \sum_{k=1}^{s} \frac{w_k}{\|w_k\|_2}.
\]

Accordingly, we may write

\[
y_0 = \sum_{k=1}^{s} w_k + \sum_{k=1}^{s} \left( \frac{1}{\|w_k\|} - 1 \right) w_k.
\]

As a sum of \( s \) independent \( N(0, \frac{1}{M} I) \), we know \( \sum_{k=1}^{s} w_k \sim N(0, \frac{s}{M} I) \). Then by A.1, we know that with high probability

\[
\sqrt{s} - \sqrt{s} \log M \sqrt{M} \leq \left\| \sum_{k=1}^{s} w_k \right\|_2 \leq \sqrt{s} + \sqrt{s} \log M \sqrt{M}.
\]

By similar logic, with high probability \( \left| \frac{1}{\|w_k\|} - 1 \right| \leq \frac{\log M}{\sqrt{M}} \). It follows from the triangle inequality that

\[
\sum_{k=1}^{s} \left( \frac{1}{\|w_k\|} - 1 \right) w_k \leq \sum_{k=1}^{s} \left| \frac{1}{\|w_k\|} - 1 \right| \|w_k\|_2 \leq s \log M \sqrt{M} \ll \sqrt{s}.
\]

It follows that \( \|y_0\| = \sqrt{s} \) up to a lower-order correction.

\( G_{0.2} \). By definition of \( w_k \), we know that

\[
| \langle y_0^k, d_k \rangle | = | \langle \tilde{y}_0^k, w_k / \|w_k\|_2 \rangle | = \| \tilde{y}_0^k \|_2 \| w_k \|_2 \langle \tilde{y}_0^k, w_k \rangle.
\]
We know that $||w_k||_2 > 1/\sqrt{2}$ for all $k$. For $k \notin \Omega_0$, we know that $\tilde{y}_0^k = y_0$ and therefore by $G_{0.1}$, $||\tilde{y}_0^k||_2 \leq C\sqrt{s}$. For $k \in \Omega_0$, we have the same result after applying concentration of the norm to $\tilde{y}_0^k$ for each $k$ along with a union bound. Thus,

$$|\langle \tilde{y}_0^k, d_k \rangle| = \frac{||\tilde{y}_0^k||_2}{||w_k||_2} \left| \frac{\langle \tilde{y}_0^k, w_k \rangle}{||\tilde{y}_0^k||_2} \right| \leq C\sqrt{5} \left| \frac{\langle \tilde{y}_0^k, w_k \rangle}{||\tilde{y}_0^k||_2} \right| .$$

By rotational invariance of the normal distribution, for each $k$, $\langle \tilde{y}_0^k, w_k \rangle$ is distributed the same as $\langle e_1, w_k \rangle$ where $e_1$ is the unit vector with 1 in the first entry and zeros elsewhere. Since $w_k \sim N(0, \frac{1}{M}I)$, it follows that $\langle \tilde{y}_0^k, w_k \rangle \sim N(0, 1/M)$. It is known (see [Vershynin, 2018], proposition 2.1.2) that a normally distributed random variable $Z_\sigma \sim N(0, \sigma^2)$ obeys the concentration inequality:

$$P(Z_\sigma \geq t) \leq \frac{\sigma}{\sqrt{2\pi t}} \exp \left( \frac{-t^2}{2\sigma^2} \right)$$

Applying this to $\langle \tilde{y}_0^k, w_k \rangle$, we have $\left| \frac{\langle \tilde{y}_0^k, w_k \rangle}{||\tilde{y}_0^k||_2} \right| \leq \frac{C\sqrt{\log M}}{\sqrt{M}}$ with high probability; it follows that $\left| \langle \tilde{y}_0^k, d_k \rangle \right| \leq \frac{C\sqrt{\log M}}{\sqrt{M}}$ with high probability. $G_{0.2}$ then follows from a union bound over $k$.

$G_{0.3}$ This also follows from the Gaussian concentration inequality 10.

$G_{0.4}$ This result follows from $G_{0.2}$ and the fact that for $k \in \Omega_0$, $\langle y_0, d_k \rangle = 1 + \langle \tilde{y}_0^k, d_k \rangle$.

$G_{0.5}$ This can be proved by applying $G_{0.2}$ and Hoeffding’s inequality for fixed $k$ followed by a union bound.

$G_{0.6}$ For this, we write:

$$||DD^T||^2_k = \sum_{k=1}^{K} \sum_{m=1}^{K} tr(d_k^T d_m^T d_m^T d_k^T) = \sum_{k=1}^{K} \sum_{m=1}^{K} (d_k, d_m)^2 = \sum_{k=1}^{K} \sum_{m \neq n} (d_k, d_m)^2$$

For fixed $k$, we have

$$\sum_{m \neq n} (d_k, d_m)^2 = \sum_{m \neq n} \frac{(w_k, w_m)^2}{||w_k||_2 ||w_m||_2}$$

which implies

$$\frac{1}{2} \sum_{m \neq n} (w_k, w_m)^2 \leq \sum_{m \neq n} \langle d_k, d_m \rangle^2 \leq 2 \sum_{m \neq n} (w_k, w_m)^2 .$$

By independence $\sum_{m \neq n} (w_k, w_m)^2$ is $1/M$ times a chi-squared random variable with $K - 1$ degrees of freedom. We can then adapt the chi-squared concentration inequalities 8 to yield

$$P \left( \sum_{m \neq n} \langle w_k, w_m \rangle^2 \geq \frac{1}{M} \left( K - 1 + \sqrt{(K-1)t + 2t} \right) \right) \leq e^{-t}$$

$$P \left( \sum_{m \neq n} \langle w_k, w_m \rangle^2 \leq \frac{1}{M} \left( (K-1) - \sqrt{(K-1)t} \right) \right) \leq e^{-t} .$$

It follows that for fixed $k$, $\frac{K}{M} \leq \sum_{m \neq k} (d_k, d_m)^2 \leq \frac{CK}{M}$ with high probability. A union bound over all $k$ gives the same bound for all $k$ simultaneously with high probability, after which the result follows immediately from equation 11.

$G_{0.7}$ This follows from $G_{0.1}$, $G_{0.4}$, and the same chi-squared concentration inequalities.

$G_{0.8}$ By the variational characterization of the operator norm for symmetric matrices, we know that

$$||DD^T||_2 = \sup_{||z||=1} z^TDD^T z.$$
We employ an \( \varepsilon \)-net argument [e.g. Vershynin, 2018]. We let \( \mathcal{M} \) be a 1/4-net on the unit sphere; that is, \( \mathcal{M} \) is a finite set such that every point on the unit sphere is at most Euclidean distance 1/4 from a point in \( \mathcal{M} \). By known results on nets on the sphere, we can choose \( \mathcal{M} \) such that \( |\mathcal{M}| \leq 9^M \).

Moreover,

\[
\sup_{||z||=1} z^T \mathcal{D} \mathcal{D}^T z \leq 2 \max_{z \in \mathcal{M}} z^T \mathcal{D} \mathcal{D}^T z.
\]

We now fix \( z \in \mathcal{M} \). We have

\[
z^T \mathcal{D} \mathcal{D}^T z = \sum_{k=1}^K \langle z, d_k \rangle^2 = \sum_{k=1}^K \frac{1}{||w_k||^2} \langle z, w_k \rangle^2 \leq 2 \sum_{k=1}^K \langle z, w_k \rangle^2.
\]

By chi-squared concentration, then, \( P \left( z^T \mathcal{D} \mathcal{D}^T z \geq \frac{K}{M} \geq 8K/M \right) \leq e^{-K} \).

Unfixing \( z \) by a union bound, we have

\[
P \left( \max_{z \in \mathcal{M}} z^T \mathcal{D} \mathcal{D}^T z \geq 8K/M \right) \leq 9^M e^{-K}.
\]

Since \( K \) grows algebraically faster than \( M \) by 2.3, we conclude that \( \|\mathcal{D} \mathcal{D}^T\|_2 \leq \frac{C K}{M} \) with high probability. Identical reasoning provides the matching lower bound.

\( G_{0.9} \).

We denote \( B = \sum_{k=1}^K \langle \tilde{y}_k, d_k \rangle^2 d_k d_k^T \). To bound this term, we use another \( \varepsilon \)-net bound. As before, we choose \( \mathcal{M} \) to be a 1/4-net of the sphere such that \( |\mathcal{M}| \leq 9^M \).

We now consider a fixed \( z \in \mathcal{M} \). We first note that by \( G_{0.2} \), \( \langle \tilde{y}_k, d_k \rangle^2 \leq \frac{C s \log^2 M}{M} \) for all \( k \). Thus by \( G_{0.2} \) and lemma A.1:

\[
z^T B z = \sum_{k=1}^K \langle \tilde{y}_k, d_k \rangle^2 \left( \langle z, d_k \rangle^2 - 1/M \right) = \sum_{k=1}^K \frac{\langle \tilde{y}_k, w_k \rangle^2}{||w_k||^2} \frac{\langle z, w_k \rangle^2}{||w_k||^2} \leq \frac{C s \log^2 M}{M} \sum_{k=1}^K \langle z, w_k \rangle^2
\]

This sum will be distributed as \( 1/M \) times a chi-squared random variable with \( K \) degrees of freedom. By our now-familiar chi-squared concentration inequalities 8, we have

\[
P \left( \sum_{k=1}^K \langle z, w_k \rangle^2 \geq \frac{K}{M} + \frac{\sqrt{Kt}}{M} + \frac{2t}{M} \right) \leq e^{-t}.
\]

Unfixing \( z \) by a union bound over all \( z \in \mathcal{M} \), we have

\[
P \left( \max_{z \in \mathcal{M}} \left( \sum_{k=1}^K \langle z, w_k \rangle^2 \right) \geq \frac{K}{M} + \frac{\sqrt{Kt}}{M} + \frac{2t}{M} \right) \leq 9^M e^{-t}.
\]

Thus choosing \( t = M \log^2 M \), we have with high probability that for all \( z \in \mathcal{M} \),

\[
\sum_{k=1}^K \langle z, w_k \rangle^2 \leq \frac{K}{M} + \sqrt{\frac{K \log^2 M}{M}} + 2 \log^2 M \leq \frac{4K}{M}.
\]

Plugging this into 12, we conclude the result.

\( G_{0.10} \).

We will make use of the following theorem from Gross [2011]:

**Theorem A.2** (Vector Bernstein Inequality, Theorem 12 [Gross, 2011]). Let \( z_1, \ldots, z_N \) be mean zero random vectors with norm bounded by \( C \) almost surely. Then for \( V = \sum_{i=1}^N E||z_i||^2 \) and any \( t \leq V/C \),

\[
P \left( \left\| \sum_{i=1}^N z_i \right\|_2 \geq \sqrt{V} + t \right) \leq \exp \left( \frac{t^2}{4V} \right).
\]
We aim to show that, with high probability,

\[
\left\| \sum_{k=1}^{K} \langle \mathbf{y}_0, \mathbf{d}_k \rangle \text{proj}_{\mathbf{y}_0} \mathbf{d}_k \right\|_2 \leq \frac{\sqrt{Ks \log M}}{\sqrt{M}}.
\]

We have

\[
\sum_{k=1}^{K} \langle \mathbf{y}_0, \mathbf{d}_k \rangle \text{proj}_{\mathbf{y}_0} \mathbf{d}_k = \sum_{k=1}^{s} \langle \mathbf{y}_0, \mathbf{d}_k \rangle \text{proj}_{\mathbf{y}_0} \mathbf{d}_k + \sum_{k=s+1}^{K} \langle \mathbf{y}_0, \mathbf{d}_k \rangle \text{proj}_{\mathbf{y}_0} \mathbf{d}_k.
\]

By the triangle inequality, it suffices to bound the norms of these sums separately. We show details for the second sum; the first is similar, with some accounting for the fact that \( k \in \Omega_0 \).

For the second sum, we know \( k \notin \Omega_0 \). By rotational invariance, \( \langle \mathbf{y}_0, \mathbf{d}_k \rangle \) and \( \text{proj}_{\mathbf{y}_0} \mathbf{d}_k \) are nearly independent: specifically, \( \langle \mathbf{y}_0, \mathbf{d}_k \rangle \) will be independent of the normalized projection \( \text{proj}_{\mathbf{y}_0} \mathbf{d}_k \| \text{proj}_{\mathbf{y}_0} \mathbf{d}_k \|_2 \); denote this vector \( \mathbf{z}_k \). We note that the Pythagorean theorem gives \( \| \text{proj}_{\mathbf{y}_0} \mathbf{d}_k \|_2 = \sqrt{1 - \langle \mathbf{y}_0, \| \mathbf{y}_0 \|, \mathbf{d}_k \rangle^2} \), so setting \( a_k = \langle \mathbf{y}_0, \mathbf{d}_k \rangle \sqrt{1 - \langle \mathbf{y}_0, \| \mathbf{y}_0 \|, \mathbf{d}_k \rangle^2} \), we can write

\[
\sum_{k=s+1}^{K} \langle \mathbf{y}_0, \mathbf{d}_k \rangle \text{proj}_{\mathbf{y}_0} \mathbf{d}_k = \sum_{k=s+1}^{K} a_k \mathbf{z}_k
\]

where \( a_k \) and \( \mathbf{z}_k \) are independent. By \( G_0 \), we know \( |a_k| \leq C \sqrt{s} \log M / \sqrt{M} \) with high probability. On this event, we condition on \( a_k \); then \( a_k \mathbf{z}_k \) are mean-zero random vectors with \( \| a_k \mathbf{z}_k \|_2 \leq C \sqrt{s} \log M / \sqrt{M} \) and

\[
\sum_{k=s+1}^{K} E \| a_k \mathbf{z}_k \|_2^2 = \sum_{k=s+1}^{K} a_k^2 \leq \frac{Ks \log^2 M}{M}.
\]

Accordingly we can apply theorem A.2 with \( t = \frac{\sqrt{Ks \log^2 M}}{\sqrt{M}} \) to conclude that, with high probability,

\[
\left\| \sum_{k=s+1}^{K} a_k \mathbf{z}_k \right\|_2 \leq \frac{C \sqrt{Ks \log^{3/2} M}}{\sqrt{M}}.
\]

\[\square\]

### A.2 Proof of lemma 5.6

We now provide a detailed proof of 5.6:

**Lemma 5.6.** Recall that \( E[\hat{S}_0 | D] = E[\langle \mathbf{y}_0, \mathbf{y} \rangle \mathbf{y}^T | D] \). On \( G_0 \),

\[
\left\langle \frac{K}{s} E[\hat{S}_0 | D], DD^T \right\rangle_F DD^T = \left( \frac{s}{K} \sum_{k=1}^{K} \langle \mathbf{y}_0, \mathbf{d}_k \rangle^2 \right) DD^T + o(1)
\]

where the \( o(1) \) bound is in terms of the operator norm.

**Proof.** We note that by \( G_0 \) and \( G_0 \), we have

\[
\left\| DD^T \right\|_F \leq C
\]

Therefore by lemma 5.4, equation 7 and the triangle inequality, it suffices to show that:

\[
\left\| \left( \frac{s}{K} \mathbf{v}_0 \mathbf{v}_0^T, DD^T \right)_F \right\| + \left\| \sum_{k \in \Omega_0} \left( 1 + 2 \langle \hat{y}_0^k, \mathbf{d}_k \rangle \mathbf{d}_k \mathbf{d}_k^T + DD^T \right) \right\| + \left\| \sum_{k=1}^{K} \langle \mathbf{y}_0^k, \mathbf{d}_k \rangle^2 \mathbf{d}_k \mathbf{d}_k^T, DD^T \right\| = o(K)
\]

\[\text{(13)}\]
We consider each term individually, beginning with the second. We write:

\[
\sum_{k \in \Omega_0} (1 + 2 \langle y_0^k, d_k \rangle) d_k d_k^T, DD^T \right) = \sum_{k \in \Omega_0} d_k d_k^T, DD^T \right) + 2 \sum_{k \in \Omega_0} \langle y_0^k, d_k \rangle d_k d_k^T, DD^T \right) \tag{14}
\]

We bound the \( \sum_{k \in \Omega_0} d_k d_k^T \) term by writing

\[
\sum_{k \in \Omega_0} \sum_{m=1}^{K} \langle d_k, d_m \rangle^2 = \sum_{k \in \Omega_0} 1 + \sum_{k \in \Omega_0} \sum_{m \neq k} \langle d_k, d_m \rangle^2.
\]

which by the triangle inequality and \( G_{0.3} \) will be bounded by \( s + K s \frac{\log^2 M}{M^2} = o(K) \). To bound the latter term in 14, we have

\[
\sum_{k \in \Omega_0} \langle y_0^k, d_k \rangle d_k d_k^T, DD^T \right) = \sum_{k \in \Omega_0} \langle y_0^k, d_k \rangle + \sum_{k \in \Omega_0} \sum_{m \neq k} \langle y_0^k, d_k \rangle \langle d_k, d_m \rangle^2.
\]

By \( G_{0.2}, G_{0.3} \), and the triangle inequality, this is bounded by \( s \times \frac{C \sqrt{s} \log M}{\sqrt{M}} + K s \times \frac{\sqrt{s} \log^3 M}{M^2} = o(K) \).

The desired bound on the last term in 13 follows from a similar expansion and bound on \( \langle y_0^k, d_k \rangle^2 \) and \( \langle d_k, d_m \rangle^2 \) by \( G_{0.2}, G_{0.3} \), and the triangle inequality.

Lastly, we turn to the first term in 13. We have

\[
\langle \frac{s}{K} v_0 v_0^T, DD^T \right) = \frac{s}{K} \sum_{k=1}^{K} \langle v_0 v_0^T, d_k d_k^T \rangle = \frac{s}{K} \sum_{k=1}^{K} \langle v_0, d_k \rangle^2.
\]

We bound \( \langle v_0, d_k \rangle \) by expanding \( v_0 \):

\[
| \langle v_0, d_k \rangle | = \sum_{m=1}^{K} \langle y_0, d_m \rangle \langle d_m, d_k \rangle \leq | \langle y_0, d_k \rangle | + \sum_{m \neq k} \langle y_0, d_m \rangle \langle d_m, d_k \rangle.
\]

By \( G_{0.4} \), and \( G_{0.5} \), \( \langle v_0, d_k \rangle \) is bounded by \( c \) for \( k \in \Omega_0 \) and by \( \frac{C \sqrt{s} \log M}{M} \) otherwise. It follows that

\[
\frac{s}{K} \sum_{k=1}^{K} \langle v_0, d_k \rangle^2 \leq \frac{s}{K} \left( C s + \frac{C K^2 s \log^2 M}{M^2} \right) \leq \frac{C s^2}{K} + \frac{K s^2 \log^2 M}{M^2} = o(K)
\]

as desired.

\[ \square \]

### A.3 Proof of 4.3

In this section, we prove theorem 4.3:

**Theorem 4.3.** Let \( N \) be at most polynomial in \( M \) with \( N \geq \max \left\{ \frac{s^2}{M^2}, \frac{K^2 s^4}{M^2} \right\} \). Then with high probability

\[
\hat{\Sigma}_0^{\text{proj}} = \hat{\Sigma}_0 - \text{proj}_{\hat{S}_0}(\hat{\Sigma}_0) = \frac{s}{K} \left( \frac{s}{K} v_0 v_0^T + \sum_{k \in \Omega_0} Z_k d_k d_k^T + o(1) \right),
\]

where \( Z_k \) are random variables satisfying \( Z_k > 1/2 \), and the \( o(1) \) term is a matrix with \( o(1) \) measured in operator norm. It follows that, as \( M \to \infty \), the subspace spanned by the lead \( s \) eigenvalues of \( \text{proj}_{\hat{S}_0}(\hat{\Sigma}_0) \) will tend to \( S_0 \) with high probability.

We start by proving lemma 5.8:
Lemma 5.8. The better event $B_0$ occurs with high probability.

Proof. Since $G_0$ holds with high probability, we may assume we are on this event.

Recalling that $\left| \sum_{k=1}^{K} (y_0, d_k) d_k \right|_2 = \|v_0\|_2$ from lemma 5.5, the lower bound in $B_1$ follows immediately from lemma 5.5. For the upper bound, we apply $G_0.1$, $G_0.7$, and computations from lemma 5.5 to conclude $\|v_0\|_2 \leq \frac{CK\sqrt{T\log^2 M}}{M}$. For $B_2$, we assume $B_1$ holds and repeat the computations of lemma 5.4, replacing outer products with inner products. This yields

$$E[\| (y_0, y) y^\|_2^2 | D] = \left( \frac{s}{K} - \frac{2(s-1)}{K(K-1)} \right) \sum_{k=1}^{K} (y_0, d_k)^2 + \frac{s(s-1)}{K(K-1)} \left( K \sum_{k=1}^{K} (y_0, d_k)^2 + \left\| \sum_{k=1}^{K} (y_0, d_k) d_k \right\|_2^2 \right)$$

$$= \left( \frac{s(s-1)}{K(K-1)} + \frac{s}{K} - \frac{2(s-1)}{K(K-1)} \right) \sum_{k=1}^{K} (y_0, d_k)^2 + \frac{s(s-1)}{K(K-1)} \left\| \sum_{k=1}^{K} (y_0, d_k) d_k \right\|_2^2. \tag{15}$$

$G_0.7$ gives $\sum_{k=1}^{K} (y_0, d_k)^2 \geq \frac{c_s^2}{M}$, while by $B_1$, $\| \sum_{k=1}^{K} (y_0, d_k) d_k \|_2^2 \geq \frac{c_s^2}{M^2}$. We conclude that $E[\| (y_0, y) y^\|_2^2 | D] \geq \frac{c_s^3}{M} \implies \sqrt{E[\| (y_0, y) y^\|_2^2 | D]} \geq \frac{c_s^{3/2}}{\sqrt{M}}$.

Next, we prove $B_3$. By theorem 4.2, lemma 5.4, and the triangle inequality, we have

$$\| E[\widetilde{S}_0 | D] \|_2 \leq \frac{s}{K} \| v_0 v_0^T \|_2 + \left\| \sum_{k \in \Omega_0} (1 + 2 \langle \hat{y}_0^k, d_k \rangle) d_k d_k^T \right\|_2 + \left( \frac{s}{K} \sum_{k \in \Omega_0} (y_0, d_k)^2 \right) \| DD^T \|_2 + o(1). \tag{15}$$

Since $\| v_0 v_0^T \|_2 = \| v_0 \|_2^2$, $B_1$ gives $\| v_0 v_0^T \|_2 \leq \frac{CK^2 s \log^4 M}{M^2}$. We know from $G_0.8$ and $G_0.7$ that

$$\left( \sum_{k=1}^{K} (y_0, d_k)^2 \right) \| DD^T \|_2 \leq \frac{CK^2 s}{M^2}.$$

To bound $\left\| \sum_{k \in \Omega_0} (1 + 2 \langle \hat{y}_0^k, d_k \rangle) d_k d_k^T \right\|_2$, we first note that by $G_0.2$, the $\hat{y}_0^k$ terms are of lower order and can be safely ignored. We bound $\sum_{k \in \Omega_0} d_k d_k^T$ using an $\varepsilon$-net argument like that in 4.2. For any fixed $z$, by chi-squared concentration 8 we have

$$P \left( \sum_{k \in \Omega_0} (z, d_k)^2 \geq \frac{1}{M} \left( s + \sqrt{sM \log M + 2M \log^2 M} \right) \right) \leq \exp(-M \log^2 M).$$

by concentration for chi-squared random variables. Thus by a union bound over $9^M$ elements in a 1/4-net over the unit ball, with high probability,

$$\sup_{\| z \| = 1} \sum_{k \in \Omega_0} z^T d_k d_k^T z \leq \frac{s + \sqrt{sM \log M + 2M \log^2 M}}{M} \leq C \log^2 M.$$

It follows that $\left\| \sum_{k \in \Omega_0} (1 + 2 \langle \hat{y}_0^k, d_k \rangle) d_k d_k^T \right\|_2 \leq C \log^2 M$ with high probability.

Plugging these bounds back into 15, we get

$$\| E[\widetilde{S}_0 | D] \|_2 \leq \frac{s}{K} \left( \frac{s}{K} \frac{CK^2 s \log^4 M}{M^2} + C \log^2 M + \frac{s}{K} \frac{CK^2 s}{M^2} + o(1) \right) \leq \frac{Cs^3 \log^4 M}{M^2} \times \frac{M}{M^2}.$$

$B_4$ immediately follows from $B_3$ and the fact that the Frobenius norm is bounded above by $\sqrt{M}$ times the operator norm. \qed
To prove theorem 4.3, we will employ 5.9. Before applying this theorem, we describe one further event that depends on the realizations of the coefficients $x_1, \ldots, x_N$ in addition to the dictionary $D$ and base sample $y_0$:

**Lemma A.3.** Let $\mathcal{X}_i$ denote the event that $\|\langle y_0, y_i \rangle y_i \| \leq \frac{Cs^{3/2} \log M}{\sqrt{M}}$. Then $\mathcal{X}_i$ occurs with high probability.

**Proof.** Since $\mathcal{B}_0$ occurs with high probability, we assume we are on this event. By the same logic used to prove $\mathcal{G}_0.1$, we have $\|y_i\| \leq C \sqrt{s}$ with high probability. Therefore,

$$\|\langle y_0, y_i \rangle y_i \|_2 \leq \|\langle y_0, y_i \rangle \|_2 \leq C |\langle y_0, y_i \rangle| \sqrt{s}.$$  

We now control the term $|\langle y_0, y_i \rangle|$

$$\langle y_0, y_i \rangle = \sum_{k \in \Omega_i} x_{ik} \langle y_0, d_k \rangle = \sum_{k \in \Omega_0 - \Omega_i} x_{ik} + \sum_{k \in \Omega_i} x_{ik} \langle y_0, d_k \rangle.$$  

The magnitude of the first term will depend on the size of the intersection $\Omega_0 \cap \Omega_i$. It is easy to see that $E[|\Omega_0 \cap \Omega_i|] = s^2/K$. Therefore, using a Chernoff bound\(^2\) [e.g., Vershynin, 2018, theorem 2.3.1] we see that $|\Omega_0 \cap \Omega_i| \leq 2s^2 \log M/K$ with high probability. Then by $\mathcal{G}_0.2$ and Hoeffding’s inequality, with high probability we have that

$$|\langle y_0, y_i \rangle| \leq C \log M \left( \frac{s \log M}{\sqrt{K}} + \frac{s}{\sqrt{M}} \right) \leq \frac{Cs \log M}{\sqrt{M}},$$  

completing the proof. \(\square\)

We can now prove the following quantitative estimate on convergence of the sample covariance matrix $\hat{\Sigma}_0$ to its expectation:

**Lemma A.4.** Suppose that $\mathcal{B}_0$ holds. Then with high probability,

$$\|\hat{\Sigma}_0 - E[\hat{\Sigma}_0 | D]\|_2 \leq \frac{Cs^3 \log^6 M}{M^{3/2} \sqrt{N}}.$$  

**Proof.** In this lemma, the expectations conditioned on $D$ are implied and we do not write them explicitly. We are interested in estimating the true covariance matrix of the random vector $\langle y_0, y \rangle y$ by its sample covariance $\frac{1}{N} \sum_{i=1}^N \langle y_0, y_i \rangle^2 y_i y_i^T$. Given our high-probability bounds from lemma A.3 and B.2, we have the following high-probability bound:

$$\|\langle y_0, y_i \rangle y_i \| \leq C \log M \sqrt{E[\|\langle y_0, y \rangle y \|_2^2]}.$$  

We would like to apply theorem 5.9 with $\kappa = C \log M$, but since this is only with high probability, not almost surely, we cannot apply theorem 5.9 directly. Instead, recalling that $\mathcal{X}_i$ is the event that $\|\langle y_0, y_i \rangle y_i \| \leq \frac{Cs^{3/2} \log M}{\sqrt{M}}$, we define the truncated random vectors $z_i = \mathbb{1}_{\mathcal{X}_i} \langle y_0, y_i \rangle y_i$. Since $\bigcup_{i=1}^N \mathcal{X}_i$ occurs with high probability, with high probability $z_i = \langle y_0, y_i \rangle y_i$ for all $i$. Noting that $E[\|z\|_2^2] \leq E[\|\langle y_0, y \rangle y \|_2^2]$ by definition, we have that almost surely, $\|z\|_2 \leq C \log M \sqrt{E[\|z\|_2^2]}$. Therefore, we may apply theorem 5.9 to the sample covariance of $z_i$ with $\kappa = C \log M$, yielding

$$\left\| \frac{1}{N} \sum_{i=1}^N z_i z_i^T - E z_i z_i^T \right\|_2 \leq \sqrt{CM(\log^3 M + t \log^2 M)} \times \|E \hat{\Sigma}_0\|_2$$

with probability at least $1 - \exp(-t)$, where we again used that $E[\|z\|_2^2] \leq E[\|\hat{\Sigma}_0\|_2^2]$. Choosing $t = \log^2 M$ and applying B.3, we have with high probability that

$$\left\| \frac{1}{N} \sum_{i=1}^N z_i z_i^T - E z_i z_i^T \right\|_2 \leq \sqrt{CM \log^4 M \times \frac{s^3 \log^4 M}{M^2}} \leq \frac{Cs^3 \log^6 M}{M^{3/2} \sqrt{N}}.$$  

\(^2\)Strictly speaking, as elements in $\Omega_i$ are not chosen independently, a Chernoff bound cannot be applied directly. However, the negative correlation of sampling with replacement guarantees better concentration properties than if elements of $\Omega_i$ were chosen independently with probability $s/K$; see Dubhashi and Ranjan [1996] for details.

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Yet we have already noted that with high probability, \( z_i = \langle y_0, y_i \rangle y_i \) for all \( i \). Therefore, with high probability,
\[
\left\| \frac{1}{N} \sum_{i=1}^{N} \langle y_0, y_i \rangle^2 y_i y_i^T - E[z_i z_i^T] \right\|_2 \leq \frac{C s^3 \log^6 M}{M^{3/2} \sqrt{N}}.
\]

(16)

Lastly, we show that \( \|E[\hat{\Sigma}_0] - E[z_i z_i^T]\|_2 \) is small. We write
\[
E[\hat{\Sigma}_0] - E[z_i z_i^T] = E[(1 - \mathbb{1}_{\mathcal{X}_i}) \langle y_0, y \rangle^2 yy^T].
\]

Since \( y_0 \) and \( y \) are sums of \( s \) unit vectors, we have \( \| \langle y_0, y \rangle^2 yy^T \| \leq s^6 \), so the above becomes
\[
\|E[\hat{\Sigma}_0] - E[z_i z_i^T]\|_2 \leq s^6 (1 - \mathbb{P}(\mathcal{X}_i)).
\]

Since \( \mathcal{X}_i \) occurs with high probability, this term decays faster than any polynomial in \( M \). Thus \( E[z_i z_i^T] \) equals \( E[\hat{\Sigma}_0] \) up to a correction decaying faster than any polynomial in \( M \). Applying this in equation 16 by the triangle inequality, we conclude the result.

Again using the fact that the Frobenius norm is bounded above by \( \sqrt{M} \) times the operator norm, we immediately get the following bound for the Frobenius norm:

**Corollary A.5.** With high probability,
\[
\| \Sigma - E[\hat{\Sigma}_0] \|_F \leq \frac{Cs^3 \log^6 M}{M \sqrt{N}}.
\]

Using essentially the same techniques, we can derive an analogous bound for the unweighted sample covariance:

**Corollary A.6.** Recall that \( \hat{\Sigma} = \frac{1}{N} YY^T = \frac{1}{N} \sum_{i=1}^{N} y_i y_i^T \) with \( E[\hat{\Sigma}] = E[yy^T] = \frac{1}{K} DD^T \). Then with high probability,
\[
\left\| \frac{1}{N} YY^T - E[yy^T] \right\|_2 = \left\| \hat{\Sigma} - \frac{s}{K} DD^T \right\|_2 \leq \frac{Cs \log^4 M}{\sqrt{M} \sqrt{N}}
\]

and
\[
\left\| \frac{1}{N} YY^T - E[yy^T] \right\|_F = \left\| \hat{\Sigma} - \frac{s}{K} DD^T \right\|_F \leq \frac{Cs \log^4 M}{\sqrt{N}}.
\]

**Proof.** To apply theorem 5.9, we first bound the norms of the random vectors \( y_i \) with high probability. This follows from the same arguments used in \( G_0.1 \) for \( \|y_0\| \), which gives \( \|y_i\|_2 \leq C \sqrt{s} \). The expectation can be computed as follows, noting that independence and symmetry guarantees \( E[x_k x_m (d_k, d_m) = 0 for k \neq m] \):
\[
E[\|y_i\|^2] = E[\langle y, y \rangle] = \sum_{k \in \Omega} \sum_{m \in \Omega} E[x_k x_m (d_k, d_m)] = \sum_{k \in \Omega} E x_k^2 + \sum_{k \neq m} E[x_k x_m (d_k, d_m)] = s.
\]

Then with high probability, for all \( i \), \( \|y_i\| \leq C \sqrt{E[\|y_i\|^2]} \). By means of a similar truncation trick to that used in lemma A.4, we may apply theorem 5.9 with \( k = C \), which gives
\[
\left\| \hat{\Sigma} - \frac{s}{K} DD^T \right\|_2 \leq \sqrt{CM \log M + t} \left\| \frac{s}{K} DD^T \right\|_2
\]

with probability at least \( 1 - 2 \exp(-t) \). Substituting \( t = \log^2 M \) and \( \frac{s}{K} \|DD^T\|_2 \leq \frac{s \log^2 M}{M} \) from \( G_0.8 \), it follows that with high probability,
\[
\left\| \hat{\Sigma} - \frac{s}{K} DD^T \right\|_2 \leq \frac{Cs \log M}{\sqrt{M} \sqrt{N}},
\]

and therefore that
\[
\left\| \hat{\Sigma} - \frac{s}{K} DD^T \right\|_F \leq \frac{Cs \log M}{\sqrt{N}}
\]

with high probability as well. \( \square \)
With these estimates in hand, we can turn to the final proof of theorem 4.3:

**Proof of Theorem 4.3.** All expectations in the following are implicitly conditioned on D. We assume that \( B_0 \) holds, along with lemma A.4 and its corollaries. From theorem 4.2, it suffices to show that for \( N \gg \max \left\{ \frac{s}{M}, K^2 s^4 \right\} \), we have

\[
\| \hat{\Sigma}_0 - \text{proj}_\Sigma(\hat{\Sigma}_0) - (E \hat{\Sigma}_0 - \text{proj}_{DD^T}(E \hat{\Sigma}_0)) \|_2 = o(s/K).
\]

By the triangle inequality this is bounded by

\[
\leq \| \hat{\Sigma}_0 - E \hat{\Sigma}_0 \|_2 + \| \text{proj}_{DD^T}(E \hat{\Sigma}_0) - \text{proj}_{DD^T}(\hat{\Sigma}_0) \|_2 + \| \text{proj}_{DD^T}(\hat{\Sigma}_0) - \text{proj}_\Sigma(\hat{\Sigma}_0) \|_2.
\]

From lemma A.4, we know that \( \| \hat{\Sigma}_0 - \Sigma \|_2 \leq \frac{Cs^3 \log^6 M}{M^{3/2} \sqrt{N}} \). We now expand the second term in 17:

\[
\| \text{proj}_{DD^T}(E \hat{\Sigma}_0) - \text{proj}_{DD^T}(\hat{\Sigma}_0) \|_2 = \left\| \langle DD^T, E \hat{\Sigma}_0 - \hat{\Sigma}_0 \rangle_F \right\|_{\| DD^T \|_F^2} \leq \| E \hat{\Sigma}_0 - \hat{\Sigma}_0 \|_F \| DD^T \|_F
\]

by the Cauchy-Schwarz inequality. By lemma A.4, \( G_{0.8}, \) and \( G_{0.6}, \) we know that

\[
\| E \hat{\Sigma}_0 - \hat{\Sigma}_0 \|_F \| DD^T \|_F \leq \frac{Cs^3 \log^6 M}{M^{3/2} \sqrt{N}} \times \frac{C \sqrt{M}}{K} \times \frac{CK}{M} \leq \frac{Cs^3 \log^6 M}{M^{3/2} \sqrt{N}}.
\]

The final term in 17 can be controlled as follows:

\[
\| \text{proj}_{DD^T}(\hat{\Sigma}_0) - \text{proj}_\Sigma(\hat{\Sigma}_0) \|_2 = \left\| \left( \hat{\Sigma}_0, DD^T \right)_F - \left( \hat{\Sigma}_0, \Sigma \right)_F \right\|_{\| \Sigma \|_F^2} \leq \left\| \hat{\Sigma}_0, \Sigma - \left( DD^T, \hat{\Sigma}_0 \right)_F \right\|_{\| DD^T \|_F^2} \leq \left\| \hat{\Sigma}_0, DD^T \right\|_{\| DD^T \|_F^2}
\]

By lemma A.4, then, we can substitute \( \| E \hat{\Sigma}_0 \|_F \) and \( \| E \hat{\Sigma}_0 \|_F \) for \( \| \hat{\Sigma}_0 \|_F \) and \( \| \hat{\Sigma}_0 \|_F \) respectively, up to a constant factor. We know from B.3 and B.4, \( \| E \hat{\Sigma}_0 \|_F \leq \frac{Cs^3 \log^4 M}{M^{3/2}} \) while \( \| E \hat{\Sigma}_0 \|_F \leq \frac{Cs^3 \log^4 M}{M^{3/2}} \). Then by \( G_{0.6}, \)

\[
\| \hat{\Sigma}_0, DD^T \|_{\| DD^T \|_F^2} \leq \frac{Cs^3 \log^4 M}{M^{3/2}} \times \frac{Cs^3 \log^4 M}{M^2} \times \frac{CM}{K} \leq \frac{Cs^6 \log^8 M}{K^2 M^{5/2}}.
\]

To bound the first term in equation 18, we use the triangle inequality, which yields

\[
\left\| DD^T - \left( \frac{DD^T}{\| DD^T \|_F^2} \right) \hat{\Sigma}_0 \right\|_2 \leq \frac{K}{s} \left( \left\| \frac{s}{K} DD^T - \hat{\Sigma}_0 \right\|_2 + \left\| \frac{\hat{\Sigma}_0}{\| DD^T \|_F^2} DD^T \right\|_2 - 1 \right) \| \hat{\Sigma}_0 \|_2.
\]

By corollary A.6 and \( G_{0.8}, \) we have \( \| \hat{\Sigma}_0 \|_2 \leq \frac{s^2}{\hat{\Sigma}_0} \times \frac{CK}{M} = \frac{Cs}{\hat{\Sigma}_0} \). By the same corollary and \( G_{0.6}, \) we have \( \| \hat{\Sigma}_0 \|_F \geq \frac{s^2}{\hat{\Sigma}_0} \times \frac{CK^2}{M} = \frac{Cs^2}{\hat{\Sigma}_0} \). Thus, noting that \( \| a \| - \| b \| \leq \| a - b \| \) for any norm, we have:

\[
\left\| \frac{\hat{\Sigma}_0}{\| DD^T \|_F^2} DD^T \right\|_2 - 1 \leq \left\| \frac{\hat{\Sigma}_0}{\| DD^T \|_F^2} DD^T \right\|_2 - \left\| \frac{\hat{\Sigma}_0}{\| DD^T \|_F^2} \right\|_2 \leq \frac{Cs \log M}{\sqrt{N}} \times \frac{CM}{K} \leq \frac{CM \log M}{K \sqrt{N}}
\]

where in the second to last inequality we again used corollary A.6. Combining the pieces in 19, we have

\[
\left\| DD^T - \left( \frac{DD^T}{\| DD^T \|_F^2} \right) \hat{\Sigma}_0 \right\|_2 \leq \frac{K}{s} \left( \frac{Cs \log M}{\sqrt{N}} + \frac{CM \log M}{K \sqrt{N}} \times \frac{Cs}{M} \right) \leq \frac{CK \log M}{\sqrt{M \sqrt{N}}}.\]
Plugging back into 18, we have
\[ \| \text{proj}_{\mathcal{D}^r} (E\hat{\Sigma}_0) - \text{proj}_{\mathcal{D}^r} (\hat{\Sigma}_0) \|_2 \leq \frac{C K \log M}{\sqrt{M} \sqrt{N}} \times \frac{C s^6 \log^8 M}{K^2 M^{3/2}} \leq \frac{C s^6 \log^9 M}{K M^3 \sqrt{N}}. \]

Combining our estimates for the three terms, we know that
\[ \| \hat{\Sigma}_0 - \text{proj}_{\hat{\Sigma}} (\hat{\Sigma}_0) - (E\hat{\Sigma}_0 - \text{proj}_{\mathcal{D}^r} (E\hat{\Sigma}_0)) \|_2 \leq \frac{C s^3 \log^6 M}{M^{3/2} \sqrt{N}} + \frac{C s^3 \log^8 M}{M^{3/2} \sqrt{N}} + \frac{C s^6 \log^9 M}{K M^3 \sqrt{N}}. \]

This will be \( o(s/K) \) provided \( N \gg \max\left\{ \frac{10}{M^2}, \frac{K^2 s^4}{M^3} \right\} \), as desired. Since \( B_0 \) and lemma A.4 hold with high probability, this completes the proof.

\[ \square \]

### A.4 Proof of lemma 5.10

We now prove lemma 5.10:

**Lemma 5.10.** Suppose \( \Omega_1, \Omega_2 \) are at-most-\( s \)-element subsets of \( \{1, 2, \ldots, K\} \) such that \( \Omega_1 \cap \Omega_2 = \emptyset \). Let \( \mathcal{S}_1 = \text{span}\{d_k\}_{k \in \Omega_1} \) and \( \mathcal{S}_j = \text{span}\{d_m\}_{m \in \Omega_2} \). Then with high probability,

\[ \inf_{\| v_1 \|_2 = 2, \| v_2 \|_2 = 1} \| v_1 - v_2 \|_2 \geq \sqrt{2} - \frac{C \sqrt{s} \log M}{\sqrt{M}}. \]

**Proof:** Without loss of generality, we may assume \( |\Omega_1| = |\Omega_2| = s \). We can write \( v_1 = \sum_{k \in \Omega_1} a_k d_k / \| \sum_{k \in \Omega_1} a_k d_k \|_2 \) and likewise \( v_2 = \sum_{m \in \Omega_2} b_m d_m / \| \sum_{m \in \Omega_2} b_m d_m \|_2 \) where \( \sum_k a_k^2 = \sum_m b_m^2 = 1 \). We first bound the norms below:

\[ \left\| \sum_{k \in \Omega_1} a_k d_k \right\|_2^2 = \left\langle \sum_{k \in \Omega_1} a_k d_k, \sum_{k \in \Omega_1} a_k d_k \right\rangle = 1 - \sum_{k \in \Omega_1} a_k \sum_{m \in \Omega_2, m \neq k} a_m \langle d_k, d_m \rangle. \]

Applying Hoeffding’s inequality to the inner sums, we have that \( \left| \sum_{m \neq k} a_m \langle d_k, d_m \rangle \right| \leq \frac{C \log M}{\sqrt{M}} \) for each \( k \).

By the triangle inequality, then,

\[ \sum_{k=1}^{K} \sum_{m \neq k} a_m \langle d_k, d_m \rangle \leq \sum_{k=1}^{K} |a_k| \frac{C \log M}{\sqrt{M}} \leq \frac{C \sqrt{s} \log M}{\sqrt{M}}. \]

where we use the fact that \( \sum_{k=1}^{s} |a_k| \) subject to \( \sum_{k=1}^{s} a_k^2 = 1 \) is maximized at \( \sqrt{s} \) with all \( |a_k| = 1/\sqrt{s} \).

We proceed to bound \( \| v_1 - v_2 \|_2^2 \):

\[ \| v_1 - v_2 \|_2^2 = 2 - \frac{2 \left\langle \sum_{k \in \Omega_1} a_k d_k, \sum_{m \in \Omega_2} b_m d_m \right\rangle}{\| \sum_{k \in \Omega_1} a_k d_k \|^2 \| \sum_{m \in \Omega_2} b_m d_m \|^2} \geq 2 - \frac{2 \sum_{k \in \Omega_1, m \in \Omega_2} \langle a_k d_k, b_m d_m \rangle}{2 - C \sqrt{s} \log M / \sqrt{M}} \]

\[ \geq 2 - \frac{2}{2 - C \sqrt{s} \log M / \sqrt{M}} \sum_{k \in \Omega_1} a_k \sum_{m \in \Omega_2, m \neq k} b_m \langle d_k, d_m \rangle. \]

Using exactly the same techniques as for the norm, we have that

\[ \left| \sum_{k \in \Omega_1} \sum_{m \in \Omega_2} b_m \langle d_k, d_m \rangle \right| \leq \frac{C \sqrt{s} \log M}{\sqrt{M}} \]

with high probability. The result follows by taking the square root and expanding in a power series. \[ \square \]
A.5 Proof of theorem 4.5

Lastly, we prove theorem 4.5:

**Theorem 4.5.** Let \( \ell \) be the smallest integer such that \((s/K)\ell < 1/2K\), and assume \( \Omega_i, i = 1, \ldots, K \) are uniformly distributed among \( s \)-element subsets of \( \{1, \ldots, K\} \). For positive integer \( J \) and \( j \in [J/\ell] \), define the non-intersecting \( \ell \)-fold intersections \( \Omega^\ell_j \) as

\[
\Omega^\ell_j = \bigcap_{p=1}^{\ell} \Omega_{(j-1)\ell + p}.
\]

Then as long as \( J \geq K \log^3 K \), with high probability for every \( k \in [K] \) there exists a \( j \in [J/\ell] \) such that \( \Omega^\ell_j = \{k\} \).

**Proof.** We begin by fixing \( k \) and then computing the probability that \( k \) is the unique element of intersection for a fixed \( \Omega^\ell_j \). We know that \( P(k \in \Omega^\ell_j) = (s/K)\ell \), while the probability that another element is in the intersection is bounded by \( K(s/K)\ell \), so we have

\[
P\left( \bigcap_{j \in I_i} \Omega_i = \{k\} \right) \geq \left( \frac{s}{K} \right)^\ell \left( 1 - K\left( \frac{s}{K} \right)^\ell \right) \geq \frac{1}{2} \left( \frac{s}{K} \right)^\ell.
\]

We now unfix the set \( I \) as follows. We divide \( \{1, 2, \ldots, N\} \) into disjoint \( \ell \)-element subsets \( I_i \). We define the random variables \( H_i \) to be indicators for the events \( \left\{ \bigcap_{j \in I_i} \Omega_i = \{k\} \right\} \). Since the sets \( I_i \) do not overlap, these are \( J/\ell \) independent Bernoulli random variables with success probability at least \( (s/K)^\ell/2 \). Since \((s/K)^\ell < 1/2K\), it follows that

\[
P\left( \sum_{i=1}^{J/\ell} H_i = 0 \right) \leq \left( 1 - \frac{1}{2} \left( \frac{s}{K} \right)^\ell \right)^{J/\ell} \leq \left( 1 - \frac{1}{4K} \right)^{N/\ell} \leq \exp\left( -\frac{J}{4K\ell} \right).
\]

Since \( J \geq K \log^3 K \) and \( \ell = \left\lceil \log(2K) / \log(K/s) \right\rceil \), we conclude the above bound tends to zero faster than any polynomial, which still holds when unfixing \( k \) by a union bound. This completes the proof. \( \square \)