An integral formula on the Heisenberg group

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Abstract

Let $\mathbb{H}^n$ denote the $(2n+1)$-dimensional (sub-Riemannian) Heisenberg group. In this note, we shall prove an integral identity (see Theorem[1.2]) which generalizes a formula obtained in the Seventies by Reilly. Some first applications will be given in Section 4.

Key words and phrases: Heisenberg groups; Sub-Riemannian geometry; hypersurfaces; Reilly’s Formula.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In the last years, the sub-Riemannian geometry of Carnot groups has become a rich research field in both Analysis and Geometric Measure Theory; see, for instance, [2], [5], [7], [8], [9], [11], [23], [18], [20], [29], but of course the list is far from being complete or exhaustive. General overviews of sub-Riemannian (or Carnot-Charathéodory) geometries are Gromov, [14], and Montgomery, [22].

In this paper, our ambient space is the so-called Heisenberg group $\mathbb{H}^n$, $n \geq 1$, which can be regarded as $\mathbb{C}^n \times \mathbb{R}$ endowed with a polynomial group law $\star : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{H}^n$. Its Lie algebra $\mathfrak{h}_n$ identifies with the tangent space $T_0\mathbb{H}^n$ at the identity $0 \in \mathbb{H}^n$. Later on, $(z, t) \in \mathbb{R}^{2n+1}$ will denote exponential coordinates of a generic point $p \in \mathbb{H}^n$. Take now a left-invariant frame $\mathcal{F} = \{X_1, Y_1, ..., X_n, Y_n, T\}$ for the tangent bundle $T\mathbb{H}^n$, where $X_i(p) := \frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i}$, $Y_i(p) := \frac{\partial}{\partial y_i} + \frac{\partial}{\partial t}$ and $T(p) := \frac{\partial}{\partial t}$. Denoting by $[\cdot, \cdot]$ the Lie bracket of vector fields, one has $[X_i, Y_j] = T$ for every $i = 1, ..., n$ and all other commutators vanish. Hence, $T$ is the center of $\mathfrak{h}_n$ and $\mathfrak{h}_n$ turns out to be nilpotent and stratified of step 2, i.e. $\mathfrak{h}_n = H \oplus H_2$ where $H := \text{span}_\mathbb{R}\{X_1, Y_1, ..., X_n, Y_n\} \subset T\mathbb{H}^n$ is the horizontal bundle and $H_2 = \text{span}_\mathbb{R}\{T\}$ is the 1-dimensional (vertical) subbundle of $T\mathbb{H}^n$ associated with the center of $\mathfrak{h}_n$. From now on, $\mathbb{H}^n$ will be endowed with the (left-invariant) Riemannian metric $h := \langle \cdot, \cdot \rangle$ which makes $\mathcal{F}$ an orthonormal frame.

Remark 1.1. Hereafter, the pair $(\mathbb{H}^n, h)$ will be thought of as a Riemannian manifold. By duality w.r.t. the metric $h$, we define a basis of left-invariant 1-forms for the cotangent bundle $T^*\mathbb{H}^n$. Therefore, we have $X_1^* = dx_1$, $dy_1 = Y_1$, ..., $X_i^* = dx_i$, $Y_i^* = dy_i$, ..., $X_n^* = dx_n$, $Y_n^* = dy_n$. Furthermore, one has $\theta := T^* = dt + \frac{1}{2} \sum_{i=1}^n (y_idx_i - x_idy_i)$, which is the contact form of $\mathbb{H}^n$. The Riemannian left-invariant volume form $\sigma_{2n+1}^\mathbb{H} = \wedge_{i=1}^{2n+1} (T^*\mathbb{H}^n)$ is defined by $\sigma_{2n+1}^\mathbb{H} := \left(\bigwedge_{i=1}^n dx_i \wedge dy_i\right) \wedge \theta$ and the measure, obtained by integration of $\sigma_{2n+1}^\mathbb{H}$, turns out to be the Haar measure of $\mathbb{H}^n$. 

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The metric $h = \langle \cdot , \cdot \rangle$ induces a corresponding metric $h_H$ on $H$, which is used to measure the length of horizontal curves. The natural distance in sub-Riemannian geometry is the Carnot-Carathéodory distance $d_{CC}$, defined by minimizing the (Riemannian) length of all (piecewise smooth) horizontal curves joining two different points. This definition makes sense because, in view of Chow’s Theorem, different points can always be joined by horizontal curves.

The stratification of $h_H$ is related with the existence of a 1-parameter group of automorphisms, called Heisenberg dilations, defined by $\delta_p(z, t) := (sz, s^2t)$, for every $p \equiv (z, t) \in \mathbb{R}^{2n+1}$. The intrinsic dilations play an important role in this geometry. In this regard, we stress that the integer $Q = 2n + 2$, which represents the “homogeneous dimension” of $\mathbb{H}^n$ (w.r.t. Heisenberg dilations), turns out to be the dimension of $\mathbb{H}^n$ as a metric space w.r.t. the $CC$-distance $d_{CC}$.

Another key notion is that of $H$-perimeter, but since we are dealing with smooth boundaries, we do not adopt the usual variational definition. So let $S \subset \mathbb{H}^n$ be a smooth hypersurface and let $v$ the (Riemannian) unit normal along $S$. The $H$-perimeter measure $\sigma^H_2$ is the $(Q - 1)$-homogeneous measure, w.r.t. Heisenberg dilations, given by $v \in S$, where $P_H : T\mathcal{O} \rightarrow H$ is the orthogonal projection operator onto $H$ and $\sigma^H_2$ is the Riemannian measure on $S$. We recall that the unit $H$-normal along $S$ is the normalized projection onto $H$ of the (Riemannian) unit normal $v$, i.e. $v := \frac{P_Hv}{\|P_Hv\|}$, and that the so-called characteristic set $C_S$ of $S$ is the zero set of the function $|P_Hv|$; see Section 2.2. The $H$-perimeter $\sigma^H_2$ is in fact the natural measure on hypersurfaces and it turns out to be equivalent, up to a density function called metric factor (see, for instance, [18]), to the spherical $(Q - 1)$-dimensional Hausdorff measure associated with $d_{CC}$ (or to any other homogeneous distance).

Below we shall prove a general integral identity, which generalizes to the sub-Riemannian setting of the Heisenberg group $\mathbb{H}^n$ a well-known formula, proved by Reilly (see [28]) in his work concerning Aleksandrov’s Theorem; for a very nice presentation of the original result we refer the reader to [17].

**Theorem 1.2** (Main result). Let $D \subset \mathbb{H}^n$ and let $S = \partial D$ be a $C^2$-smooth compact (closed) hypersurface. Let $\phi : D \rightarrow \mathbb{R}$ be a smooth solution to $\begin{cases} \Delta_H \phi = \psi & \text{on } D \\ \phi = \varphi & \text{on } S \end{cases}$.

Then

$$
\int_D \left\{ \psi^2 - \|Hess_H \phi\|_{c^2} + 2 \left\langle \left( grad_H (T\phi) , (grad_H \phi)^\perp \right) \right\rangle \right\} \sigma^H_2
$$

$$
= \int_S \left\{ 2 \frac{\partial \varphi}{\partial n_H} \left( L_{HS} \varphi - \frac{\varphi}{2} \frac{\partial \varphi}{\partial n_H} \right) - \mathcal{H}_H \left( \frac{\partial \varphi}{\partial n_H} \right)^2 - S_H (grad_{HS} \varphi, grad_{HS} \varphi) \right\} \sigma^H_2,
$$

We stress that:

- $Hess_H$ is the horizontal Hessian operator;
- the symbol $X^\perp$ (whenever $X \in H$) denotes a linear skew-symmetric map. More precisely, it is defined by setting $X^\perp := -C^H_{2n+1} X$, where $C^H_{2n+1} \in M_{2n \times 2n}(\mathbb{R})$ is given by formula (1); see below.
- $grad_H$ and $grad_{HS}$ denote the horizontal gradient and the horizontal tangent gradient, resp.;
- $L_{HS}$ denotes a 2nd order horizontal tangential operator, which plays the role of the classical Laplace-Beltrami operator in Riemannian geometry;
- $\sigma := \frac{\nu}{\|P_H\nu\|}$, where $\nu_T = (\nu, T)$;
- $\mathcal{H}_H$ is the horizontal mean curvature of $S$;
- $S_H$ is the symmetric part of the horizontal 2nd fundamental form of $S$.

In Section 4 we shall prove some direct applications of our main result.

Another consequence will be discussed in Section 4.1. More precisely, we shall obtain the following formula:

$$
\int_{-\varepsilon}^{\varepsilon} ds \int_{S_t} \left( \mathcal{H}_H r^2 - \|S_H r\|_{c^2} + \frac{3n - 1}{2} (\sigma^H) r^2 \right) \sigma^H_2 = -\int_{S^* \cup S^-} \mathcal{H}_H \sigma^H_2 r^2,
$$
where $(\mathcal{H}_0)$, $S'_i$ and $\tau'$ denote, respectively, the horizontal mean curvature, the symmetric part of the horizontal 2nd fundamental form, and the (weighted) vertical part of the normal $\nu'$ of the hypersurface $S = \{x \in \mathbb{H}^n : f_t(x) = f(x, t) = 0 \; \forall \; t \in ]-\epsilon, \epsilon[\}$. More precisely, we are assuming that there is a foliation of a small spatial neighborhood of the (compact, closed hypersurface) $S := S_0$ by means of level sets of a smooth function $f : \mathbb{H}^n \times ]-\epsilon, \epsilon[ \to \mathbb{R}$ (say of class $C^3$) such that:

- $|\text{grad} f_t| \neq 0$ along $S$, for every $t \in ]-\epsilon, \epsilon[.$
- $|\text{grad}_t f_t| = 1$ at each non characteristic (abbreviated NC) point of $S_i$.

see, Corollary 4.9

As a final remark, we have to mention that, unfortunately, the original arguments of Reilly (or those in Li’ survey [17]) cannot be adapted to our context and, above all, it seems to be still a difficult problem to prove a generalized version of Aleksandrov’ Theorem in $\mathbb{H}^n$ for $n > 1$; see [29] for the case $n = 1$.

2. Preliminaries

2.1. Heisenberg group $\mathbb{H}^n$. The Heisenberg group $(\mathbb{H}^n, \star)$, $n \geq 1$, is a connected, simply connected, nilpotent and stratified Lie group of step 2 on $\mathbb{R}^{2n+1}$, w.r.t. a polynomial group law $\star$; see below. The Lie algebra $h_n$ of $\mathbb{H}^n$ is a $(2n + 1)$-dimensional real vector space henceforth identified with the tangent space $T_0\mathbb{H}^n$ at the identity $0 \in \mathbb{H}^n$. We adopt exponential coordinates of the 1st kind in such a way that every point $p \in \mathbb{H}^n$ can be written out as $p = \exp(x_1, y_1; \ldots, x_n, y_n, t)$. The Lie algebra $h_n$ can be described by means of a frame $\mathcal{F} := \{X_1, Y_1, \ldots, X_n, Y_n, T\}$ of left-invariant vector fields for $T\mathbb{H}^n$, where $X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}$, $Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}$, $i = 1, \ldots, n$, $T(p) := \frac{\partial}{\partial t}$, for every $p \in \mathbb{H}^n$. More precisely, if $[\cdot, \cdot]$ denote Lie brackets, then the only non trivial commuting relations are $[X_i, Y_i] = T$ for every $i = 1, \ldots, n$. In other words, $T$ is the center of $h_n$ and $h_n$ turns out to be a nilpotent and stratified Lie algebra of step 2, i.e. $h_n = H \oplus H_2$. The first layer $H$ is called horizontal whereas the complementary layer $H_2 := \text{span}_\mathbb{R}(T)$ is called vertical. A horizontal left-invariant frame for $H$ is given by $\mathcal{F}_H := \{X_1, Y_1, \ldots, X_n, Y_n, Y_0\}$. The group law $\star$ on $\mathbb{H}^n$ is determined by a corresponding operation $\odot$ on $h_n$, i.e. $\exp X \star \exp Y = \exp(X \odot Y)$ for every $X, Y \in h_n$, where $\odot : h_n \times h_n \to h_n$ is defined by $X \odot Y = X + Y + \frac{1}{2}[X, Y]$. Thus, for every $p = \exp(x_1, y_1; \ldots, x_n, y_n, t)$, $p' = \exp(x'_1, y'_1; \ldots, x'_n, y'_n, t') \in \mathbb{H}^n$ we have

$$p \star p' := \exp \left( x_1 + x'_1, y_1 + y'_1; \ldots, x_n + x'_n, y_n + y'_n, t + t' + \frac{1}{2} \sum_{i=1}^{n} (x_i y_i' - x_i' y_i) \right).$$

The inverse of any $p \in \mathbb{H}^n$ is given by $p^{-1} := \exp(-x_1, -y_1; \ldots, -x_n, -y_n, -t)$ and $0 = \exp(0, 0)$. Later on, we shall set $z := (x_1, y_1; \ldots, x_n, y_n) \in \mathbb{H}^{2n}$ and identify each point $p \in \mathbb{H}^n$ with its exponential coordinates $(z, t) \in \mathbb{R}^{2n+1}$.

**Definition 2.1.** We call sub-Riemannian metric $h_n$ any symmetric positive bilinear form on $H$. The CC-distance $d_{CC}(p, p')$ between $p, p' \in \mathbb{H}^n$ is defined by

$$d_{CC}(p, p') := \inf \int \sqrt{h_n(\gamma', \gamma')} dt,$$

where the inf is taken over all piecewise-smooth horizontal curves $\gamma$ joining $p$ to $p'$. We shall equip $T\mathbb{H}^n$ with the left-invariant Riemannian metric $h := \langle \cdot, \cdot \rangle$ making $\mathcal{F}$ an orthonormal -abbreviated o.n.- frame and assume $h_n := h|_H$.

By Chow’s Theorem it turns out that every couple of points can be connected by a horizontal curve, not necessarily unique, and for this reason $d_{CC}$ turns out to be a true metric on $\mathbb{H}^n$ whose topology is equivalent to the standard (Euclidean) topology of $\mathbb{R}^{2n+1}$; see [13], [22]. The so-called structural
constants (see [14] or [19, 20]) of $h_n$ are described by the skew-symmetric $(2n \times 2n)$-matrix

\[
C^{2n+1}_H := \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
. & . & . & . \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\]

This matrix is associated with the real valued skew-symmetric bilinear map $\Gamma_H : H \times H \to \mathbb{R}$ given by $\Gamma_H(X, Y) = \langle [X, Y], T \rangle$.

**Notation 2.2.** We shall set

- $z^\perp := -C^{2n+1}_H z = (-y_1, x_1, \ldots, -y_n, x_n) \in \mathbb{R}^{2n}$, $\forall z \in \mathbb{R}^{2n}$;
- $X^\perp := -C^{2n+1}_H X \quad \forall X \in H$.

Given $p \in \mathbb{H}^n$, we shall denote by $L_p : \mathbb{H}^n \to \mathbb{H}^n$ the left translation by $p$, i.e. $L_p p' = p \ast p'$, for every $p' \in \mathbb{H}^n$. The map $L_p$ is a group homomorphism and its differential $L_{p,*} : T_0 \mathbb{H}^n \to T_p \mathbb{H}^n$, $L_{p,*} = \frac{d}{dt} |_{t=0}$, is given by $L_{p,*} = \text{col}[X_1(p), Y_1(p), \ldots, X_n(p), Y_n(p), T(p)]$.

There exists a 1-parameter group of automorphisms $\delta_s : \mathbb{H}^n \to \mathbb{H}^n (s \geq 0)$, called Heisenberg dilations, defined by $\delta_s p := \exp \left( sz, s^2 t \right)$ for every $s \geq 0$, where $p = \exp (z, t) \in \mathbb{H}^n$. We recall that the homogeneous dimension of $\mathbb{H}^n$ is the integer $Q := 2n + 2$. By a well-known result of Mitchell (see, for instance, [22]), this number coincides with the Hausdorff dimension of $\mathbb{H}^n$ as metric space w.r.t. the CC-distance $d_{CC}$; see [13, 22].

We shall denote by $\nabla$ the unique left-invariant Levi-Civita connection on $\mathbb{H}^n$ associated with the metric $h = \langle \cdot, \cdot \rangle$. We observe that, for every $X, Y, Z \in \mathfrak{x} := \mathcal{C}^\infty(\mathbb{H}^n, \mathbb{H}^n)$ one has

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left( \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \right).
\]

For every $X, Y, Z \in \mathfrak{x}_H := \mathcal{C}^\infty(\mathbb{H}^n, H)$, we shall set $\nabla^\ast_X Y := \mathcal{P}_H (\nabla_X Y)$, where $\mathcal{P}_H$ denotes orthogonal projection onto $H$. The operation $\nabla^\ast$ is a vector-bundle connection later called $H$-connection; see [20] and references therein. It is not difficult to see that $\nabla^\ast$ is flat, compatible with the sub-Riemannian metric $h_n$ and torsion-free. These properties follow from the very definition of $\nabla^\ast$ and from the corresponding properties of the Levi-Civita connection $\nabla$.

**Definition 2.3.** For any $\psi \in \mathcal{C}^\infty(\mathbb{H}^n)$, the $H$-gradient of $\psi$ is the horizontal vector field $\text{grad}_H \psi \in \mathfrak{x}_H$ such that $\langle \text{grad}_H \psi, X \rangle = d\psi(X) = X\psi$ for every $X \in \mathfrak{x}_H$. The $H$-divergence $\text{div}_H X$ of $X \in \mathfrak{x}_H$ is defined, at each point $p \in \mathbb{H}^n$, by $\text{div}_H(X)(p) := \text{Trace} \left( Y \to \nabla^\ast_X Y \right) (p) (Y \in H_p)$. The $H$-Laplacian $\Delta_H$ is the 2nd order differential operator given by $\Delta_H \psi := \text{div}_H (\text{grad}_H \psi)$ for every $\psi \in \mathcal{C}^\infty(\mathbb{H}^n)$.

Having fixed a left-invariant Riemannian metric $h$ on $\mathbb{H}^n$, one defines by duality (w.r.t. the left-invariant metric $h$) a global co-frame $\mathcal{F}^* := \{X_1^*, Y_1^*, \ldots, X_n^*, Y_n^*, T^*\}$ of left-invariant 1-forms for the cotangent bundle $T^\ast \mathbb{H}^n$, where $X_i^* = dx_i$, $Y_i^* = dy_i (i = 1, \ldots, n)$, $\theta := T^* = dt + \frac{1}{2} \sum_{i=1}^n (y_i dx_i - x_i dy_i)$. The differential 1-form $\theta$ represents the contact form of $\mathbb{H}^n$. The Riemannian left-invariant volume form $\sigma_{R}^{2n+1}$$ \in \wedge^{2n+1}(T^\ast \mathbb{H}^n)$ is given by $\sigma_{R}^{2n+1} := (\wedge_{i=1}^n dx_i \wedge dy_i) \wedge \theta$ and the measure obtained by integrating $\sigma_{R}^{2n+1}$ is the Haar measure of $\mathbb{H}^n$.

### 2.2. Hypersurfaces

Let $S \subset \mathbb{H}^n$ be a hypersurface of class $C^r (r \geq 1)$ and let $\nu$ be the (Riemannian) unit normal along $S$. We recall that the Riemannian measure $\sigma_{R}^{2n} \in \wedge^{2n}(T^\ast S)$ on $S$ can be defined by contraction\footnote{Let $M$ be a Riemannian manifold. The linear map $\mathcal{J} : \mathcal{N}(T^\ast M) \to \mathcal{N}^{-1}(T^\ast M)$ is defined, for $X \in TM$ and $\omega' \in \mathcal{N}(T^\ast M)$, by $(\mathcal{J} X)\omega' = \omega'(X, Y, \ldots, Y)$; see, for instance, [10]. This operation is called contraction or interior product.} of the volume form $\sigma_{R}^{2n+1}$ with the unit normal $\nu$ along $S$, i.e. $\sigma_{R}^{2n, L} S := (\mathcal{J} \sigma_{R}^{2n+1})|_S$.

We say that $p \in S$ is a characteristic point if $\dim H_p = \dim(H_p \cap T_p S)$. The characteristic set of $S$ is the set of all characteristic points, i.e. $C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_p S)\}$. It is worth noticing that
$p \in C_S$ if, and only if, $|P_H v(p)| = 0$. Since $|P_H v(p)|$ is continuous along $S$, it follows that $C_S$ is a closed subset of $S$, in the relative topology. We stress that the $(Q - 1)$-dimensional Hausdorff measure of $C_S$ vanishes, i.e. $\mathcal{H}^{Q-1}(C_S) = 0$; see [2], [18].

**Remark 2.4.** Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth hypersurface. By using Frobenius’ Theorem about integrable distributions, it can be shown that the topological dimension of $C_S$ is strictly less than $(n + 1)$; see also [13]. For deeper results about the size of $C_S$ in $\mathbb{H}^n$, see [2], [3].

Throughout this paper we shall make use of a (smooth) homogeneous measure on hypersurfaces, called $H$-perimeter measure; see also [11], [12], [8], [9], [18], [19], [20], [26], [29].

**Definition 2.5 ($\sigma^{2n}_H$-measure).** Let $S \subset \mathbb{H}^n$ be a $C^1$-smooth non-characteristic (henceforth abbreviated as NC) hypersurface and let $v$ be the unit normal vector along $S$. The unit $H$-normal along $S$ is defined by $v_t := \frac{P_H v}{|P_H v|}$. Then, the $H$-perimeter form $\sigma^{2n}_H \in \wedge^{2n} (T^*S)$ is the contraction of the volume form $\sigma^{2n+1}_H$ of $\mathbb{H}^n$ by the horizontal unit normal $v_t$, i.e. $\sigma^{2n}_H L S := \left( v_t \cdot \sigma^{2n+1}_H \right) |S|$. 

If $C_S \neq \emptyset$ we extend $\sigma^{2n}_H$ up to $C_S$ by setting $\sigma^{2n}_H L C_S = 0$. It turns out that $\sigma^{2n}_H L S = |P_H v| \sigma^{2n}_H L S$.

Moreover, at each $p \in S \setminus C_S$ one has $H_p = \text{span}_\mathbb{R} (v_t(p)) \oplus H_p S$, where $H_p S := H_p \cap T_p S$. This allows us to define, in the obvious way, the associated subbundles $HS \subset TS$ and $v_t S$ called horizontal tangent bundle and horizontal normal bundle along $S \setminus C_S$, respectively. On the other hand, at each characteristic point $p \in C_S$, only $HS$ is well-defined and we have $H_p S = H_p$ for any $p \in C_S$.

**Definition 2.6.** If $U \subseteq S$ is an open set, we denote by $C^i_{HS}(U)$, $(i = 1, 2)$ the space of functions whose $HS$-derivatives up to the $i$-th order are continuous on $U$. We denote by $C^i_{HS}(U, HS)$, $(i = 1, 2)$ the space of functions with target in $HS$, whose $HS$-derivatives up to the $i$-th order are continuous on $U$.

Another important geometric object is given by $\sigma := \frac{\nu}{|P_H v|}$; see [19], [20], [9]. Although the function $\sigma$ is not defined at $C_S$, we have $\sigma \in L^1_{loc}(S, \sigma^{2n}_H)$.

The following definitions can be found in [20], for general Carnot groups. Below, unless otherwise specified, we shall assume that $S \subset \mathbb{H}^n$ is a $C^2$-smooth NC hypersurface (i.e. non-characteristic). Let $\nabla^T S$ be the connection on $S$ induced from the Levi-Civita connection $\nabla$ on $\mathbb{H}^n$. As for the horizontal connection $\nabla^H$, we define a “partial connection” $\nabla^H S$ associated with the subbundle $HS \subset TS$ by setting

$$\nabla^H X : = \mathcal{P}_{HS} \left( \nabla^T X \right)$$

for every $X, Y \in \mathfrak{x}^1_{HS}$, where $\mathcal{P}_{HS} : TS \rightarrow HS$ denotes the orthogonal projection operator of $TS$ onto $HS$. Starting from the orthogonal splitting $H = v_t S \oplus HS$, it can be shown that

$$\nabla^H X Y = \nabla^H Y - \left( \nabla^H X Y, v_t \right) v_t$$

for every $X, Y \in \mathfrak{x}^1_{HS}$.

**Definition 2.7.** If $\psi \in C^1_{HS}(S)$, we define the $HS$-gradient of $\psi$ to be the horizontal tangent vector field $\text{grad}_{HS} \psi \in \mathfrak{x}^1_{HS} := C(S, HS)$ such that $\langle \text{grad}_{HS} \psi, X \rangle = d\psi(X) = X \psi$ for every $X \in HS$. If $X \in \mathfrak{x}^1_{HS}$, the $HS$-divergence $\text{div}_{HS} X$ of $X$ is given, at each point $p \in S$, by $\text{div}_{HS} X(p) := \text{Trace} \left( Y \rightarrow \nabla^H Y X \right)(p)$ $(Y \in H_p S)$. Note that $\text{div}_{HS} X \in C(S)$. The $HS$-Laplacian $\Delta_{HS} : C^2_{HS}(S) \rightarrow C(S)$ is the second order differential operator given by $\Delta_{HS} \psi := \text{div}_{HS} (\text{grad}_{HS} \psi)$ for every $\psi \in C^2_{HS}(S)$. The horizontal 2nd fundamental form of $S$ is the bilinear map $B_H : \mathfrak{x}^1_{HS} \times \mathfrak{x}^1_{HS} \rightarrow C(S)$ defined by

$$B_H(X, Y) := \left( \nabla^H X Y, v_t \right)$$

for every $X, Y \in \mathfrak{x}^1_{HS}$.

The horizontal mean curvature is the trace of $B_H$, i.e. $H_H := \text{Tr} B_H$.

Unless $n = 1$, $B_H$ is not symmetric; see [20]. Therefore, it is convenient to represent $B_H$ as a sum of two operators, one symmetric, one skew-symmetric, i.e. $B_H = S_H + A_H$. It turns out that $A_H = \frac{1}{3} \sigma C^2_{HS} |_{HS}$; see [20]. The linear operator $C^2_{HS}$ only acts on horizontal tangent vectors and hence we shall set $C^2_{HS} := C^2_{HS} |_{HS}$. We have the identity $\|A_H\|^2_{C^0} = \frac{1}{2} \sigma^2$; see [20], Example 4.11.
p. 470. This can be proved by means of an adapted o.n. frame \( \mathcal{F} \) along \( S \). Furthermore, we observe that \( \nu^T_{\mathcal{H}} \in \text{Ker}A_H \), where \( \nu^T_{\mathcal{H}} = -C^T_{\mathcal{H},1-1,1}v^T_{\mathcal{H}} \).

**Definition 2.8.** Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth NC hypersurface. We call adapted frame along \( S \) any o.n. frame \( \mathcal{F} := \{\tau_1, ..., \tau_{2n+1}\} \) for \( \mathbb{H}^n \) such that:

\[
\tau_1|_S = \nu^T_{\mathcal{H}}, \quad H_pS = \text{span}_\mathbb{R}\{\tau_2(p), ..., \tau_{2n}(p)\} \quad \text{for every } p \in S, \quad \tau_{2n+1} := T.
\]

Furthermore, we shall set \( I_H := \{1, 2, 3, ..., 2n\} \) and \( I_{HS} := \{2, 3, ..., 2n\} \).

**Lemma 2.9** (see [20], Lemma 3.8). Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth NC hypersurface and fix \( p \in S \). We can always choose an adapted o.n. frame \( \mathcal{F} := \{\tau_1, ..., \tau_{2n+1}\} \) along \( S \) such that \( \langle \nabla_X \tau_i, \tau_j \rangle = 0 \) at \( p \) for every \( i, j \in I_{HS} \) and every \( X \in H_pS \).

**Lemma 2.10.** Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth NC hypersurface. Then

\[
(2) \quad \Delta_{HS} \phi = \Delta_H \phi + \mathcal{H}_H \frac{\partial \phi}{\partial v_H} - \langle \text{Hess}_H \phi \nu^T_{\mathcal{H}}, \nu^T_{\mathcal{H}} \rangle \quad \forall \phi \in C^\infty(\mathbb{H}^n).
\]

**Proof.** Using an adapted frame \( \mathcal{F} \), we compute

\[
\begin{align*}
\Delta_H \phi &= \sum_{i \in I_H} (\tau_1^{(2)} - \nabla^H_{\tau_1,\tau_1})(\phi) \\
&= \tau_1^{(2)}(\phi) - \langle \nabla^H_{\nu^T_{\mathcal{H}}} \tau_1 \rangle(\phi) + \sum_{i \in I_{HS}} \left( \langle \tau_1^{(2)} - \nabla^H_{\nu^T_{\mathcal{H}}} \tau_1 \rangle(\phi) - \langle \nabla^H_{\tau_1,\tau_1} \rangle \tau_1(\phi) \right) \\
&= \tau_1^{(2)}(\phi) - \langle \nabla^H_{\nu^T_{\mathcal{H}}} \tau_1 \rangle(\phi) + \Delta_{HS} \phi - \mathcal{H}_H \tau_1(\phi).
\end{align*}
\]

Note that the first identity comes from the usual invariant definition of the Laplace operator on Riemannian manifolds (or vector bundles); see [15]. Now we claim that

\[
\tau_1^{(2)}(\phi) - \langle \nabla^H_{\nu^T_{\mathcal{H}}} \tau_1 \rangle(\phi) = \langle \text{Hess}_H \phi \tau_1, \tau_1 \rangle.
\]

Assuming \( \tau_1 = \sum_{i \in I_H} A_i^1X_i \) yields

\[
\tau_1^{(2)}(\phi) = \sum_{i \in I_H} \tau_1(A_i^1X_i(\phi)) = \sum_{i, j \in I_{HS}} \left( \tau_1(A_i^1X_i(\phi)) + A_i^1A_j^1X_j(Y_i(\phi)) \right).
\]

Since \( \nabla^H_{\tau_1,\tau_1} \tau_1 = \sum_{i, j \in I_{HS}} \left( \tau_1(A_i^1X_i) + A_i^1A_j^1\nabla^H_{X_i,\tau_1}X_j \right) \) and \( \nabla^H_{X_i,\tau_1}X_j = 0 \), the claim follows because

\[
(3) \quad \tau_1^{(2)}(\phi) - \langle \nabla^H_{\nu^T_{\mathcal{H}}} \tau_1 \rangle(\phi) = \sum_{i, j \in I_{HS}} A_i^1A_j^1X_j(Y_i(\phi)) = \langle \text{Hess}_H \phi \tau_1, \tau_1 \rangle.
\]

\[\square\]

**Definition 2.11** (Horizontal tangential operators). Let \( S \subset \mathbb{H}^n \) be a NC hypersurface. We shall denote by \( \mathcal{D}_{HS} : \mathfrak{X}_{HS} \longrightarrow \mathcal{C}(S) \) the 1st order differential operator given by

\[
\mathcal{D}_{HS}(X) := \text{div}_S X + \sum \left( C^T_{\mathcal{H},1-1,1}v^T_{\mathcal{H}} X, X \right) = \text{div}_S X - \sum \left( \nu^T_{\mathcal{H}} X, X \right) \quad \forall X \in \mathfrak{X}_{HS}(HS) := C^1_{HS}(S, HS).
\]

Moreover, we shall denote by \( \mathcal{L}_{HS} : C^2_{HS}(S) \longrightarrow \mathcal{C}(S) \) the 2nd order differential operator given by

\[
\mathcal{L}_{HS} \varphi := \mathcal{D}_{HS}(\text{grad}_H \varphi) = \Delta_{HS} \varphi - \sum \frac{\partial \varphi}{\partial v^T_{\mathcal{H}}} \quad \forall \varphi \in C^2_{HS}(S).
\]

Note that \( \mathcal{D}_{HS}(\varphi X) = \varphi \mathcal{D}_{HS} X + \langle \text{grad}_H \varphi, X \rangle \) for every \( \varphi \in C^1_{HS}(S) \). These definitions are motivated by Theorem 3.17 in [20], which was proved first for NC hypersurfaces with boundary; see [19] [20]. Actually, it holds true even in case of non-empty characteristic sets. A simple way to formulate this claim is based on the next:
**Definition 2.12.** Let \( X \in C^1_{hs} (S \setminus C_{S, HS}) \) and set \( \alpha_X := (X \downarrow \alpha^{2n}_H)|_S \). We say that \( X \) is admissible (for the horizontal divergence formula) if the differential forms \( \alpha_X \) and \( d\alpha_X \) are continuous on all of \( S \). We say that \( \phi \in C^2_{hs} (S \setminus C_S) \) is admissible if \( grad_{hs} \phi \) is admissible for the horizontal divergence formula.

If the differential forms \( \alpha_X \) and \( d\alpha_X \) are continuous on all of \( S \), then Stokes formula holds true; see, for instance, [31]. In particular, we stress that: (i) if \( X \in C^1_{hs} (S, HS) \), then \( X \) is admissible; (ii) if \( \phi \in C^2_{hs} (S) \), then \( \phi \) is admissible. The following holds:

**Theorem 2.13.** Let \( S \) be a compact hypersurface of class \( C^2 \) without boundary. Then

\[
\int_S ^{D_{hs}} X \sigma^{2n}_H = - \int_S H_{\alpha} (X, \nu) \sigma^{2n}_H \quad \forall \ X \in \mathfrak{X}^1_h.
\]

Note that, if \( X \in \mathfrak{X}^1_{hs} \), the first integral on the right hand side vanishes and, in this case, the formula is referred as “horizontal divergence formula”.

Finally, we state some useful Green’s formulas:

(i) \( \int_S \mathcal{L}_{hs} \varphi \sigma^{2n}_H = 0 \) for every \( \varphi \in C^2_{hs} (S) \);

(ii) \( \int_S \psi \mathcal{L}_{hs} \varphi \sigma^{2n}_H = \int_S \langle grad_{hs} \varphi, grad_{hs} \psi \rangle \sigma^{2n}_H \) for every \( \varphi, \psi \in C^2_{hs} (S) \);

(iii) \( \int_S \mathcal{L}_{hs} \left( \frac{\partial^2}{\partial \nu^2} \right) \sigma^{2n}_H = \int_S \mathcal{L}_{hs} \varphi \sigma^{2n}_H + \int_S |grad_{hs} \varphi|^2 \sigma^{2n}_H = 0 \) for every \( \varphi \in C^2_{hs} (S) \).

### 3. Proof of Theorem 2.2

**Proof.** Below we shall make use of the fixed left-invariant frame \( \mathcal{F} = \{X_1, X_2, \ldots, X_{2n}, X_{2n+1}\} \), where we have set \( X_{2i} := Y_i \) for every \( i = 1, \ldots, n \) and \( X_{2n+1} := T \). First, we compute

\[
\frac{1}{2} \Delta_i |grad_{\mu} \phi|^2 = \frac{1}{2} \sum_{i,j \in I_H} X_i X_j (X_i \phi)^2
\]

\[
= \sum_{i,j \in I_H} X_i (X_j \phi X_i \phi) + \sum_{i,j \in I_H} (X_i \phi X_j \phi)
\]

\[
= \sum_{i,j \in I_H} \left( (X_i X_j \phi)^2 + X_i X_j (X_i \phi) X_j \phi \right)
\]

\[
= \sum_{i,j \in I_H} \left( \phi^2_{ij} + \Delta_i (X_j \phi) X_j \phi \right).
\]

Moreover, we have

\[
\Delta_i (X_j \phi) = \sum_{i \in I_H} X_i (X_j \phi
\]

\[
= \sum_{i \in I_H} X_i (X_j \phi + [X_i, X_j] (\phi))
\]

\[
= \sum_{i \in I_H} X_j X_i (X_i \phi) + [X_i, X_j] (X_i \phi) + X_i ([X_i, X_j] (\phi))
\]

\[
= \sum_{i \in I_H} X_j \Delta_i \phi + C^{2n+1}_{i,j} TX_i \phi + C^{2n+1}_{i,j} X_i (T \phi)
\]

\[
= \sum_{i \in I_H} X_j \Delta_i \phi + C^{2n+1}_{i,j} X_i (T \phi) + C^{2n+1}_{i,j} X_i (T \phi)
\]

\[
= X_j \Delta_i \phi - 2 \left( C^{2n+1}_{i,j} grad_{\mu} (T \phi), X_j \right).
\]
From these computations, we infer the formula

\[ \frac{1}{2} \Delta_H |\text{grad}_H \phi|^2 = \|\text{Hess}_H \phi\|^2_{\ell^2} + \left\langle \text{grad}_H (\Delta_H \phi), \text{grad}_H \phi \right\rangle - 2 \left\{ C_H^{2n+1} \text{grad}_H (T \phi), \text{grad}_H \phi \right\} = \|\text{Hess}_H \phi\|^2_{\ell^2} + \left\langle \text{grad}_H \psi, \text{grad}_H \phi \right\rangle + 2 \left\langle T (\text{grad}_H \phi), C_H^{2n+1} \text{grad}_H \phi \right\rangle. \]  

(5)

Hereafter we will use the hypothesis \( \phi|_S = \varphi \). By applying the usual Divergence Theorem, we have

\[ \int_D \left\langle \text{grad}_H \phi + \langle \text{grad}_H \psi, \text{grad}_H \phi \rangle \right\rangle \sigma^n_{\mathcal{R}} = \int_S \psi \frac{\partial \varphi}{\partial v_H} \sigma^n_{\mathcal{R}} \]

and we get that

\[ \int_D \langle \text{grad}_H \psi, \text{grad}_H \phi \rangle \sigma^n_{\mathcal{R}} = - \int_D \psi^2 \sigma^n_{\mathcal{R}} + \int_S \psi \frac{\partial \varphi}{\partial v_H} \sigma^n_{\mathcal{R}}. \]

Set now \( \chi := \frac{\text{grad}_H \phi^2}{2} \). By integrating (5) along \( D \) and using the last identity, we obtain

\[ \int_D \Delta_H \chi \sigma^n_{\mathcal{R}} = \int_D \left( \|\text{Hess}_H \phi\|^2_{\ell^2} - \psi^2 + 2 \left\langle T \text{grad}_H \phi, C_H^{2n+1} \text{grad}_H \phi \right\rangle \right) \sigma^n_{\mathcal{R}} + \int_S \psi \frac{\partial \varphi}{\partial v_H} \sigma^n_{\mathcal{R}}. \]

(6)

So let \( \mathcal{F}_H = \{\tau_1(\nu_H), \tau_2, \ldots, \tau_{2n}\} \) be a horizontal frame for \( H \) adapted to \( S \) and let us compute

\[ \int_D \Delta_H \chi \sigma^n_{\mathcal{R}} = \int_S \left\langle \text{grad}_H \chi, \nu_H \right\rangle \sigma^n_{\mathcal{R}} = \int_S \left\langle \text{grad}_H \left( \frac{|\text{grad}_H \varphi|^2}{2} \right), \nu_H \right\rangle \sigma^n_{\mathcal{R}} = \sum_{j \in \mathcal{H}} \int_S \tau_j(\varphi) \frac{\partial \tau_j(\varphi)}{\partial v_H} \sigma^n_{\mathcal{R}} = \int_S \left( \frac{\partial \varphi}{\partial v_H} \frac{\partial^2 \varphi}{\partial v_H^2} + \sum_{j \in \mathcal{H}} \tau_j(\varphi) \frac{\partial \tau_j(\varphi)}{\partial v_H} \right) \sigma^n_{\mathcal{R}}. \]

(7)

Using the identity \( \Delta_H \varphi = \Delta_H \varphi + \mathcal{H}_H \frac{\partial \varphi}{\partial v_H} - \left( \text{Hess}_H (\varphi) \nu_H, \nu_H \right) \) (see Lemma\(^{2,10}\)) and the fact that

\[ \left( \frac{\partial^2 \varphi}{\partial v_H^2} - \nabla^H_{\nu_H} \nu_H \right)(\varphi) = \left( \text{Hess}_H (\varphi) \nu_H, \nu_H \right), \]

yields

\[ \int_S \left( \frac{\partial \varphi}{\partial v_H} \frac{\partial^2 \varphi}{\partial v_H^2} + \sum_{j \in \mathcal{H}} \tau_j(\varphi) \frac{\partial \tau_j(\varphi)}{\partial v_H} \right) \sigma^n_{\mathcal{R}} = \int_S \nu_H(\varphi) \left( \Delta_H \varphi - \Delta_H \varphi + \mathcal{H}_H \nu_H(\varphi) + \nabla^H_{\nu_H} \nu_H(\varphi) \right) \sigma^n_{\mathcal{R}}. \]

(8)
The second integrand of the right-hand side of (7) can be computed by means of a frame $\mathcal{F}$ satisfying Lemma 2.9 at a fixed point $p \in S$. More precisely, we have

$$A := \sum_{j \in \text{iHS}} \int_S \tau_j(\varphi) \left( \nu_h(\tau_j(\varphi)) \right) \sigma_n^{2n}$$

$$= \sum_{j \in \text{iHS}} \int_S \tau_j(\varphi) \left( \nu_h(\tau_j(\varphi)) + \left( \nabla^H_{\nu_h} \tau_j - \nabla^H_{\tau_j} \nu_h \right)(\varphi) \right) \sigma_n^{2n}$$

$$= \sum_{j \in \text{iHS}} \int_S \tau_j(\varphi) \left( \nu_h(\tau_j(\varphi)) + \sum_{l \in \text{lHS}} \left( \nabla^H_{\nu_h} \tau_j, \tau_l \right) \tau_l(\varphi) - \left( \nabla^H_{\tau_j} \nu_h, \tau_l \right) \tau_l(\varphi) \right) \sigma_n^{2n}$$

$$= \sum_{j \in \text{iHS}} \int_S \left\{ \left( \nabla^H_{\nu_h} \varphi, \nabla^H_{\tau_j} \nu_h(\varphi) \right) \right\} - \left\{ \left( \nabla^H_{\nu_h} \nu_h, \nabla^H_{\tau_j} \varphi \right) \nu_h(\varphi) + B_h \left( \nabla^H_{\nu_h} \varphi, \nabla^H_{\tau_j} \varphi \right) \right\} \sigma_n^{2n}.$$  

Theorem 2.13 allows us to integrate by parts the first integrand and, since the boundary term vanishes, we get that

$$\int_S D^H_{\nu_h}(\nu_h(\varphi)\nabla^H_{\nu_h} \varphi) \sigma_n^{2n} = \int_S \left( \text{div}^H (\nu_h(\varphi)\nabla^H_{\nu_h} \varphi) - \sigma \left( \nu_h^+ (\nu_h(\varphi)\nabla^H_{\nu_h} \varphi) \right) \right) \sigma_n^{2n} = 0$$

and hence

$$\int_S \left( \nabla^H_{\nu_h} \varphi, \nabla^H_{\tau_j} \nu_h(\varphi) \right) \sigma_n^{2n} = - \int_S \left( \nu_h(\varphi)\Delta^H_H \varphi - \sigma \nu_h(\varphi)\nu_h^+(\varphi) \right) \sigma_n^{2n}.$$  

Therefore

$$A = \int_S \left( \nu_h(\varphi) \left( \Delta^H_H \varphi - \sigma \nu_h^+(\varphi) \right) + \left( \nabla^H_{\nu_h} \nu_h, \nabla^H_{\tau_j} \varphi \right) \right) + B_h \left( \nabla^H_{\nu_h} \varphi, \nabla^H_{\tau_j} \varphi \right) \sigma_n^{2n}.$$  

Finally, by making use of (6), (8) and (9) we obtain

$$\int_S \left( \left( \nabla^H_{\nu_h} \varphi \right)^2 - 2 \nu_h(\varphi) \left( \Delta^H_H \varphi - \frac{\sigma \nu_h^+(\varphi)}{2} \right) + \mathcal{H}_H(\nu_h(\varphi))^2 + S_H(\nabla^H_{\nu_h} \varphi, \nabla^H_{\tau_j} \varphi) \right) \sigma_n^{2n}$$

$$= \int_D \left( \|\text{Hess}_H \phi\|_{L^2}^2 - \psi^2 + 2 \left( \text{T}(\nabla^H_{\nu_h} \phi), C_H^{2n+1} \nabla^H_{\nu_h} \phi \right) \right) \sigma_n^{2n+1},$$

which is equivalent to the thesis, once we note that

$$\mathcal{L}_H \varphi + \frac{\text{sgn} \nu_h^+(\varphi)}{2} = \Delta^H_H \varphi - \frac{\sigma \nu_h^+(\varphi)}{2}.$$  

This achieves the proof.

4. SOME APPLICATIONS

Let us begin with the following:

**Remark 4.1.** Let $\phi \in C^2(\mathbb{H}^n)$. In general, the horizontal Hessian $\text{Hess}_H \phi$ of $\phi$ is not symmetric. However, we may consider its standard decomposition

$$\text{Hess}_H \phi = \frac{\text{Hess}_H \phi + \text{Hess}_H^\text{sym} \phi}{2} + \frac{\text{Hess}_H \phi - \text{Hess}_H^\text{sym} \phi}{2} := \text{Hess}_H^\text{sym} \phi + \text{Hess}_H^\text{skew} \phi$$

where $\text{Hess}_H^\text{sym} \phi$ denotes the symmetric part of $\text{Hess}_H \phi$ and $\text{Hess}_H^\text{skew} \phi$ denotes its skew-symmetric part. It is not difficult to see that

$$\text{Hess}_H^\text{skew} \phi = -\frac{T \phi}{2} C_{H}^{2n+1}.$$
Corollary 4.3. Indeed note that, by its own definition, the $i$-th row of $\text{Hess}^{\text{skew}}_H \phi$ is given by $\frac{\text{grad}_H (X_i \phi) - X_i (\text{grad}_H \phi)}{2}$, and one has $[X_j, X_i](\phi) = (X_j X_i - X_i X_j)(\phi) = C_i^0 \delta_{j}^{n+1} T \phi$. Therefore

$$\|\text{Hess}_H \phi\|_{G_0}^2 = \|\text{Hess}^{\text{sym}}_H \phi\|_{G_0}^2 + \frac{n}{2} (T \phi)^2.$$  

The last identity just says that the Gram norm of a matrix is the sum of the Gram norm of its symmetric part with the Gram norm of its skew-symmetric part. Finally, it is elementary to see that $\|C_i^0 \delta_{j}^{n+1}\|_{G_0} = 2n$.

By applying Theorem 1.2 together with Newton’s inequality we deduce an interesting inequality.

**Corollary 4.2.** Under the same hypotheses of Theorem 1.2, the following holds:

$$\frac{2n - 1}{2n} \int_D \psi^2 \sigma_{R}^{2n+1} \geq \int_D \left( \frac{n}{2} (T \phi)^2 - 2 \left\langle \text{grad}_H (T \phi), (\text{grad}_H \phi)^\perp \right\rangle \right) \sigma_{R}^{2n+1}$$

$$+ \int_S \left( 2 \frac{\partial \varphi}{\partial v_H} \left( \text{L}_H \phi - \frac{\varphi \cdot \varphi}{2} \frac{\partial \varphi}{\partial v_H^\perp} \right) - \text{H}_H \left( \frac{\partial \varphi}{\partial v_H} \right)^2 - s_H (\text{grad} \hspace{1mm} v_H \varphi, \text{grad}_H \varphi) \right) \sigma_H^{2n}.$$  

with equality if, and only if, $\text{Hess}^{\text{sym}}_H \phi = \frac{\psi}{2n} \text{Id}_H$.

Note that $\text{Id}_H \equiv 1_{2n \times 2n} \in \mathcal{M}_{2n \times 2n}(\mathbb{R})$.

**Proof.** Using Newton’s inequality yields

$$\|\text{Hess}_H \phi\|_{G_0}^2 = \|\text{Hess}^{\text{sym}}_H \phi\|_{G_0}^2 + \frac{n}{2} (T \phi)^2$$

$$\geq \frac{\text{Tr}^2 (\text{Hess}^{\text{sym}}_H \phi)}{2n} + \frac{n}{2} (T \phi)^2$$

$$= \frac{\Delta_H \phi}{2n} + \frac{n}{2} (T \phi)^2$$

$$= \frac{\psi^2}{2n} + \frac{n}{2} (T \phi)^2.$$  

As it is well-known, one has equality in this inequality if, and only if, $\text{Hess}^{\text{sym}}_H \phi = \frac{\psi}{2n} \text{Id}_H$. From this argument and Theorem 1.2 we easily get (10) and the thesis follows. \qed

The next three corollaries will follow from Theorem 1.2 by making appropriate choices of the “test function” $\varphi : D \rightarrow \mathbb{R}$.

**Corollary 4.3.** Let $D \subset \mathbb{R}^n$ and let $S = \partial \overline{D}$ be a $C^2$-smooth compact (closed) hypersurface. Then

$$\int_S \left\{ \text{H}_H \left\langle V, v_H \right\rangle^2 - s_H (V, v_H) \right\} \sigma_H^{2n} = 3 \int_S \varphi \left\langle V, v_H \right\rangle \left\langle V, v_H^\perp \right\rangle \sigma_H^{2n}.$$  

**Proof.** Let $V \in \mathfrak{X}_H$ be a constant left-invariant vector field and take $\varphi = \left\langle V, x_H \right\rangle$. Then, we have

- $\text{grad}_H \phi = V$;
- $T \phi = 0$;
- $\Delta_H \phi = 0$;
- $\text{Hess}_H \phi = 0_{2n \times 2n}$;
- $\frac{\partial \varphi}{\partial v_H} = \left\langle V, v_H \right\rangle$;
- $\frac{\partial \varphi}{\partial v_H^\perp} = \left\langle V, v_H^\perp \right\rangle$;
- $\text{grad}_H \varphi = V_H = V - \left\langle V, v_H \right\rangle v_H$;
- $\Delta_H \varphi = \text{div}_H (\text{grad}_H \varphi) = \text{H}_H \left\langle V, v_H \right\rangle$;
- $\mathcal{L}_H \varphi = \left\langle (\text{H}_H v_H - \varphi v_H^\perp), V \right\rangle = \text{H}_H \left\langle V, v_H \right\rangle - \varphi \left\langle V, v_H^\perp \right\rangle$. 

By substituting the previous calculations into the identity of Theorem 1.2 we get that the left hand side of the identity vanishes. Therefore, one has

\[
\int_S \left\{ 2 \langle V, \nu_H \rangle \left( \mathcal{H}_H \langle V, \nu_H \rangle - \frac{3}{2} \sigma \langle V, \nu_H^1 \rangle \right) - \mathcal{H}_H \langle V, \nu_H^1 \rangle^2 - S_H \langle V_{HS}, V_{HS} \rangle \right\} \sigma_H^{2n} = 0.
\]

Hence

\[
\int_S \left\{ \mathcal{H}_H \langle V, \nu_H \rangle^2 - 3 \sigma \langle V, \nu_H \rangle \langle V, \nu_H^1 \rangle - S_H \langle V_{HS}, V_{HS} \rangle \right\} \sigma_H^{2n} = 0,
\]

which is equivalent to the thesis. □

**Corollary 4.4.** Let \( D \subset \mathbb{H}^n \) and let \( S = \partial D \) be a \( C^2 \)-smooth compact (closed) hypersurface. Then

\[
\mathcal{V} \mathcal{O} \mathcal{L}^{2n+1}(D) = \frac{1}{2n} \left\{ \int_S 3 \int_S \sigma \langle x_H, \nu_H \rangle \langle x_H, \nu_H^1 \rangle \sigma_H^{2n} - \int_S \left\{ \mathcal{H}_H \langle x_H^1, \nu_H \rangle^2 - S_H \langle x_{HS}, x_{HS} \rangle \right\} \sigma_H^{2n} \right\}.
\]

**Proof.** Let \( \varphi = 2t \). Then, we have

- \( \text{grad}_H \varphi = x_H^1 \),
- \( T \varphi = 2 \),
- \( \Delta_H \varphi = 0 \),
- \( \text{Hess}_H \varphi = -C_H^{2n+1} \),
- \( \text{grad}_{HS} \varphi = x_{HS}^1 = x_H^1 - \langle x_H^1, \nu_H \rangle \nu_H \),
- \( \Delta_{HS} \varphi = \text{div}_{HS} (\text{grad}_{HS} \varphi) = \mathcal{H}_H \langle x_H^1, \nu_H \rangle \),
- \( L_{HS} \varphi = \mathcal{H}_H \langle x_H^1, \nu_H \rangle - \sigma \langle x_H, \nu_H \rangle \).

We also stress that

\[
L_{HS} \varphi = -\left( \left( \mathcal{H}_H \nu_H^1 + \sigma \nu_H \right), x_H \right) = -\left( \left( \mathcal{H}_H \nu_H - \sigma \nu_H^1 \right) \right), x_H \right) = \left( \left( \mathcal{H}_H \nu_H - \sigma \nu_H^1 \right), x_H \right).
\]

Hence, using Theorem 1.2 yields

\[
- \int_D \| c_H^{2n+1} \|^2 \| \sigma_H^{2n} = -2n \mathcal{V} \mathcal{O} \mathcal{L}^{2n+1}(D)
\]

\[
= \int_S \left\{ 2 \langle x_H, \nu_H \rangle \left( \mathcal{H}_H \langle x_H^1, \nu_H \rangle - \sigma \langle x_H, \nu_H \rangle - \frac{\sigma}{2} \langle x_H, \nu_H \rangle \right) - \mathcal{H}_H \langle x_H^1, \nu_H \rangle^2 - S_H \langle x_{HS}, x_{HS} \rangle \right\} \sigma_H^{2n}
\]

\[
= \int_S \left\{ \mathcal{H}_H \langle x_H^1, \nu_H \rangle^2 - 3 \sigma \langle x_H, \nu_H \rangle \langle x_H^1, \nu_H \rangle - S_H \langle x_{HS}, x_{HS} \rangle \right\} \sigma_H^{2n},
\]

which implies the thesis. □

**Corollary 4.5.** Let \( D \subset \mathbb{H}^n \) and let \( S = \partial D \) be a \( C^2 \)-smooth compact (closed) hypersurface. Then

\[
\mathcal{V} \mathcal{O} \mathcal{L}^{2n+1}(D) = \frac{1}{2n(2n-1)} \left\{ \int_S 3 \int_S \sigma \langle x_H, \nu_H \rangle \langle x_H, \nu_H^1 \rangle \sigma_H^{2n} - \int_S \left\{ \mathcal{H}_H \langle x_H, \nu_H \rangle^2 - S_H \langle x_{HS}, x_{HS} \rangle \right\} \sigma_H^{2n} \right\}.
\]

**Proof.** Let \( \varphi = \frac{\rho^2}{2} \), where \( \rho := \| x_H \|_{\text{iso}} = \sqrt{\sum_{i=1}^{2n} x_i^2} \). Then, we have

- \( \text{grad}_H \varphi = x_H \),
- \( T \varphi = 0 \),
- \( \Delta_H \varphi = 2n \),
- \( \text{Hess}_H \varphi = 1_{2n \times 2n} \),
- \( \text{grad}_{HS} \varphi = x_{HS} = x_H - \langle x_H, \nu_H \rangle \nu_H \),
- \( \Delta_{HS} \varphi = \text{div}_{HS} (\text{grad}_{HS} \varphi) = (h - 1) + \mathcal{H}_H \langle x_H, \nu_H \rangle \),
- \( L_{HS} \varphi = (2n - 1) + \left( \left( \mathcal{H}_H \nu_H - \sigma \nu_H^1 \right), x_H \right) = (2n - 1) + \mathcal{H}_H \langle x_H, \nu_H \rangle - \sigma \langle x_H, \nu_H \rangle \).
Thus, substituting these computations into the identity of Theorem 1.2 yields
\[
\int_D \left( (2n)^2 - 2n \right) \sigma_{n+1}^{2n+1} = 2n(2n-1)^2 \text{Vol}^{2n+1}(D)
\]
\[
= \int_S \left\{ 2 \langle x_h, \nu_h \rangle (2n-1) + H_h (\langle x_h, \nu_h \rangle - \sigma \langle x_h, \nu_h \rangle - \frac{1}{2} \sigma (\langle x_h, \nu_h \rangle) - H_h (\langle x_h, \nu_h \rangle) - S_h (x_{HS}, x_{HS}) \right\} \sigma_H^{2n}
\]
\[
= \int_S (2(2n-1) \langle x_h, \nu_h \rangle + H_h (\langle x_h, \nu_h \rangle) - 3 \sigma (\langle x_h, \nu_h \rangle) (\langle x_h, \nu_h \rangle) - S_h (x_{HS}, x_{HS}) \} \sigma_H^{2n}.
\]

Since \( \int_D \text{div}_x x \sigma_{n+1}^{2n+1} = 2n \text{Vol}^{2n+1}(D) = \int_S \langle x_h, \nu_h \rangle \sigma_H^{2n} \), we get that
\[
2n(2n-1)^2 \text{Vol}^{2n+1}(D) + \int_S (H_h (\langle x_h, \nu_h \rangle) - 3 \sigma (\langle x_h, \nu_h \rangle) (\langle x_h, \nu_h \rangle) - S_h (x_{HS}, x_{HS}) \} \sigma_H^{2n} = 0.
\]
This can be rewritten as
\[
\text{Vol}^{2n+1}(D) = \frac{1}{2n(2n-1)} \int_S (\langle x_h, \nu_h \rangle - 3 \sigma (\langle x_h, \nu_h \rangle) (\langle x_h, \nu_h \rangle) + S_h (x_{HS}, x_{HS}) \} \sigma_H^{2n},
\]
which is equivalent to the thesis.

Another interesting formula can be obtained by using jointly both Corollary 4.4 and Corollary 4.5.

**Corollary 4.6.** Let \( n > 1 \), let \( D \subset \mathbb{H}^n \) and let \( S = \partial D \) be a \( C^2 \)-smooth compact (closed) hypersurface. Then
\[
\text{Vol}^{2n+1}(D) = \frac{1}{4n(n-1)} \int_S (\langle x_h, \nu_h \rangle - 3 \sigma (\langle x_h, \nu_h \rangle) (\langle x_h, \nu_h \rangle) + S_h (x_{HS}, x_{HS}) \} \sigma_H^{2n}.
\]

**Proof.** Immediate.

### 4.1. An integral formula for the horizontal mean curvature

Let \( D \subset \mathbb{H}^n \) be a smooth, say \( C^3 \), bounded domain (i.e. open and connected) and assume that there exists a (global) defining function \( f : \mathbb{H}^n \rightarrow \mathbb{R} \) for \( D \). This means that
- \( D = \{ x \in \mathbb{H}^n : f(x) < 0 \} \),
- \( D^c = \mathbb{H}^n \setminus D = \{ x \in \mathbb{H}^n : f(x) > 0 \} \),
- \( \text{grad} \ f \neq 0 \) at every point \( x \in \partial D \).

From now on we will set \( S := \partial D \). The outward unit normal along \( S \) is given by \( \nu = \frac{\text{grad} \ f}{|\text{grad} \ f|} \). It will be useful to replace the defining function \( f \) for \( D \) with the function \( \tilde{f} = \frac{f}{|\text{grad} \ f|} \). In fact, this new function has the remarkable feature that \( |\text{grad}_H \tilde{f}| = 1 \) along \( S \). However, in general, the function \( \tilde{f} \) is just of class \( C^2 \) (i.e. one order of differentiability less smooth than \( f \)) on \( S \setminus C_S \) and fails to be smooth only at \( C_S \).

Thus, applying Theorem 1.2 to the function \( \phi \equiv \tilde{f} \) yields the formula:
\[
\int_D \left( (\Delta_H \phi)^2 - ||\text{Hess}_H \phi||_g^2 + 2 \langle \text{grad}_H (T \phi), (\text{grad}_H \phi)^\perp \rangle \right) \sigma_n^{2n+1} = - \int_S H_H \sigma_n^{2n}.
\]

Note that we have used \( \text{grad} \ \phi = \nu \) (which implies \( \text{grad}_H \phi = 0 \) and \( L_{HS} \phi = 0 \)) where \( \phi = \phi|_S \).

Now let \( S \) be a compact \( C^2 \)-smooth embedded hypersurface. A similar formula can be obtained when we consider a foliation of a small spatial neighborhood of \( S \). More precisely, let \( f : \mathbb{H}^n \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R} \) be a \( C^3 \)-smooth function such that:
- \( S_t = \{ x \in \mathbb{H}^n : f_t(x) = 0 \ \forall \ t \in [-\varepsilon, \varepsilon] \} \),
- \( |\text{grad} f_t| \neq 0 \) along \( S_t \) for every \( t \in [-\varepsilon, \varepsilon] \),
- \( |\text{grad}_H f_t| = 1 \) at each NC point of \( S_t \).

Moreover, let \( D := \{ x \in \mathbb{H}^n : f_t(x) \in [-\varepsilon, \varepsilon] \} \) and set \( S^\pm := \{ x \in \mathbb{H}^n : f_t(x) = \pm \varepsilon \} \).

We again apply Theorem 1.2 to the function \( \phi \equiv f \) and we similarly get
\[
\int_D \left( (\Delta_H \phi)^2 - ||\text{Hess}_H \phi||_g^2 + 2 \langle \text{grad}_H (T \phi), (\text{grad}_H \phi)^\perp \rangle \right) \sigma_n^{2n+1} = - \int_{S^+ \cup S^-} H_H \sigma_n^{2n}.
\]
where we have used \( \text{grad} \varphi = \nu \) (so that \( \text{grad}_{H^2} \varphi = 0 \) and \( L_{H^2} \varphi = 0 \)) and \( \varphi = \phi|_S \). The previous assumptions allow us to say something more. But before this, we need the following corollary of the classical Coarea formula:

**Proposition 4.7.** Let \( D \subset \mathbb{H}^n \) be a smooth domain and let \( \phi \in C^1(D) \). Then

\[
\int_D \psi |\text{grad}_H \phi(x)| \sigma_{H^2}^{2n+1}(x) = \int_R d_s \left( \int_{\phi^{-1}[s] \cap D} \psi \sigma_{H}^{2n} \right).
\]

Applying this formula yields

\[
\int_D \left( (\Delta_H \phi)^2 - \|\text{Hess}_H \phi\|_{G^1}^2 + 2 \left( \text{grad}_H (T \phi), (\text{grad}_H \phi)^\perp \right) \right) \sigma_{H^2}^{2n+1}
\]

\[
= \int \int d_s \left( \int_{S,=\phi^{-1}\{s\} \cap D} \left( (\mathcal{H}_S)^2 - \|\mathcal{I}_{H} \nu'_{H}\|_{G^1}^2 + 2 \frac{\partial \sigma'}{\partial \nu'_{H} \perp} \right) \sigma_{H^2}^{2n} \right)
\]

where \( \nu'_{H} \) is the unit \( H \)-normal along \( S \), and \( (\mathcal{H}_S) \), denotes the \( H \)-mean curvature of \( S \). Furthermore

\[
\|\mathcal{I}_{H} \nu'_{H}\|_{G^1}^2 = \|B'_{H} + \nabla_{H} \nu'_{H}\|_{G^1}^2
\]

\[
= \|S'_{H}\|_{G^1}^2 + \frac{n - 1}{2} (\sigma')^2 + (\sigma')^2 \|\nu'_{H}\|_{G^1}^2
\]

\[
= \|S'_{H}\|_{G^1}^2 + \frac{n + 1}{2} (\sigma')^2,
\]

where we have used \( B'_{H} = S'_{H} + A'_{H} \) together with the identity \( \nabla_{H} \nu'_{H} = -\sigma' C_{H^2}^{2n+1} \nu'_{H} \); see, for instance, [21]. Hence

\[
\int_{-\epsilon}^{\epsilon} d_s \int_{S,=\phi^{-1}\{s\} \cap D} \left( (\mathcal{H}_S)^2 - \|S'_{H}\|_{G^1}^2 + 2 \frac{\partial \sigma'}{\partial \nu'_{H} \perp} - \frac{n + 1}{2} (\sigma')^2 \right) \sigma_{H^2}^{2n} = -\int_{S^+ \cup S^-} \mathcal{H}_S \sigma_{H^2}^{2n},
\]

where \( II_S(\nu'_{H}, \sigma_{H^2}^{2n}) \) is nothing but the second variation formula of the \( H \)-perimeter \( (\sigma_{H^2}^{2n})_I \) of \( S \) for a variation \( \theta \) having variation vector \( W_t = \frac{d}{dt} \theta = \nu'_{H} \); see [20, 21].

At this point, we may apply another integral formula to each integral over \( S \). We stress that we are assuming that each \( S_t \) is a compact closed hypersurface, at least of class \( C^2 \).

**Lemma 4.8.** Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth compact hypersurface without boundary. Then

\[
\int_S \left( \frac{\partial \sigma}{\partial \nu'_{H}} - n \sigma^2 \right) \sigma_{H^2}^{2n} = 0,
\]

whenever \( \sigma \nu'_{H} \) is admissible (for the horizontal divergence formula).

**Proof.** We have

\[
\int_S D_{H^2} (\sigma \nu'_{H}) \sigma_{H^2}^{2n} = \int_S \left( \text{div}_{H^2} (\sigma \nu'_{H}) - \sigma (\nu'_{H}, \nu'_{H}) \right) \sigma_{H^2}^{2n} = \int_S \left( \text{div}_{H^2} (\sigma \nu'_{H}) - \sigma^2 \right) \sigma_{H^2}^{2n} = 0.
\]

Since

\[
\text{div}_{H^2} (\sigma \nu'_{H}) = \frac{\partial \sigma}{\partial \nu'_{H}} + \sigma \text{div}_{H^2} (\nu'_{H})
\]

\[
= \frac{\partial \sigma}{\partial \nu'_{H}} - \sigma \text{Tr} (B_{H} (\cdot, C_{H^2}^{2n+1})),
\]

where \( C_{H^2}^{2n+1} = C_{H^2}^{2n+1} \big|_{H^2} \), we get that

\[
\int_S \left( \frac{\partial \sigma}{\partial \nu'_{H}} - \sigma^2 \right) \sigma_{H^2}^{2n} = \int_S \left( \text{Tr} (B_{H} (\cdot, C_{H^2}^{2n+1})) \right) \sigma \sigma_{H^2}^{2n}.
\]
Since $\text{Tr}(B_H(\cdot, C^{2n+1}_H)) = (n-1)\sigma$, the thesis follows; see [20][21].

Finally, by using [14] and the last lemma, we have proved the following:

**Corollary 4.9.** Under the previous assumptions, the following holds:

$$\int_{-\epsilon}^{\epsilon} ds \int_{S^1} (\mathcal{H}_n)^2 - \|\mathcal{T}'\|^2_{L^2} + \frac{3n-1}{2}(\sigma r)^2 \sigma^{2n} = -\int_{S^1 \cup S^1} \mathcal{H}_n \sigma^{2n}.$$

**Proof.** Immediate. □

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