Nonlinearly Exponential Stability of Compressible Navier-Stokes System with Degenerate Heat-Conductivity

Bin Huang, Xiaoding Shi †
Department of Mathematics, Faculty of Science,
Beijing University of Chemical Technology,
Beijing 100029, P. R. China

Abstract

We study the large-time behavior of strong solutions to the one-dimensional, compressible Navier-Stokes system for a viscous and heat conducting ideal polytropic gas, when the viscosity is constant and the heat conductivity is proportional to a positive power of the temperature. Both the specific volume and the temperature are proved to be bounded from below and above independently of time. Moreover, it is shown that the global solution is nonlinearly exponentially stable as time tends to infinity. Note that the conditions imposed on the initial data are the same as those of the constant heat conductivity case ([Kazhikhov-Shelukhin. J. Appl. Math. Mech. 41 (1977); Kazhikhov. Boundary Value Problems for Hydrodynamical Equations, 50 (1981)] and can be arbitrarily large. Therefore, our result can be regarded as a natural generalization of the Kazhikhov’s ones for the constant heat conductivity case to the degenerate and nonlinear one.

Keywords: Compressible Navier-Stokes system; Degenerate heat-conductivity; Strong solutions; Nonlinearly exponential stability
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1 Introduction

We consider the compressible Navier-Stokes system, describing the one-dimensional motion of a viscous heat-conducting gas, written in the Lagrange variables (see [5,28])

\[ v_t = u_x, \]

\[ u_t + P_x = \left( \mu \frac{u_x}{v} \right)_x, \]

\[ \left( e + \frac{u^2}{2} \right)_t + (P u)_x = \left( \kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v} \right)_x, \]

where \( t > 0 \) is time, \( x \in (0, 1) \) denotes the Lagrange mass coordinate, and the unknown functions \( v > 0, u, \theta > 0, e > 0, \) and \( P \) are, respectively, the specific volume of the gas,

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†Email addresses: abinhuang@gmail.com (B. Huang), shixd@mail.buct.edu.cn (X. Shi)
fluid velocity, internal energy, absolute temperature, and pressure. In this paper, we concentrate on ideal polytropic gas, that is, \( P, e \) satisfy
\[ P = R\theta/v, \quad e = c_v\theta + \text{const.}, \] (1.4)
where both specific gas constant \( R \) and heat capacity at constant volume \( c_v \) are positive constants. For \( \mu \) and \( \kappa \), we consider the case where \( \mu \) and \( \kappa \) are proportional to (possibly different) powers of \( \theta \):
\[ \mu = \tilde{\mu}\theta^\alpha, \quad \kappa = \tilde{\kappa}\theta^\beta, \] (1.5)
with constants \( \tilde{\mu}, \tilde{\kappa} > 0 \) and \( \alpha, \beta \geq 0 \).

The system (1.1)-(1.5) is supplemented with the initial conditions:
\[(v, u, \theta)(x, t = 0) = (v_0, u_0, \theta_0)(x), \quad x \in (0, 1), \] (1.6)
and boundary ones:
\[ u(0, t) = u(1, t) = 0, \quad \theta_x(0, t) = \theta_x(1, t) = 0, \quad t \geq 0. \] (1.7)

One can deduce from the Chapman-Enskog expansion for the first level of approximation in kinetic theory that the viscosity \( \mu \) and heat conductivity \( \kappa \) are functions of temperature alone (\([6, 7]\)). In particular, if the intermolecular potential varies as \( r^{-a} \), with intermolecular distance \( r \), then \( \mu \) and \( \kappa \) are both proportional to the power \( (a + 4)/(2a) \) of the temperature, that is, (1.5) holds with \( \alpha = \beta = (a + 4)/(2a) \). Indeed, for Maxwellian molecules \( (a = 4) \), the dependence is linear, while for elastic spheres \( (a \to \infty) \), the dependence is like \( \theta^{1/2} \).

For constant coefficients \( (\alpha = \beta = 0) \) and large initial data, Kazhikhov and Shelukhin [18] first obtained the global existence of solutions in bounded domains. From then on, significant progress has been made on the mathematical aspect of the initial boundary value problems, see [1–4, 11, 12, 15] and the references therein. Moreover, much effort has been made to generalize this approach to other cases and in particular to models satisfying (1.5), which in fact has proved to be challenging especially for temperature dependence on \( \mu \). Motivated by the fact that in the case of isentropic flow a temperature dependence in the viscosity translates into a density dependence, there is a body of literature (see [4, 8–10, 16, 25] and the references therein) studying the case that \( \mu \) is independent of \( \theta \), and heat conductivity is allowed to depend on temperature in a special way with a positive lower bound and balanced with corresponding constitution relations.

When it comes to the physical case (1.5) with \( \alpha = \beta \), there is few results partially because of the possible degeneracy and strong nonlinearity in viscosity and heat diffusion introduced in such relations. As a first step in this direction, Jenssen-Karper [13] proved the global existence of a weak solution to (1.1)–(1.7) under the assumption that \( \alpha = 0 \) and \( \beta \in (0, 3/2) \). Later, for \( \alpha = 0 \) and \( \beta \in (0, \infty) \), Pan-Zhang [26] obtain the global strong solution under the condition that
\[ (v_0, u_0, \theta_0) \in H^1 \times H^2 \times H^2. \] (1.8)

Concerning the large-time behavior of the strong solutions to (1.1)–(1.7), Kazhikhov [17] (see also [11, 19, 21, 23, 27] among others) first obtains that for the case that \( \alpha = \beta = 0 \), the strong solution is nonlinearly exponentially stable as time tends to infinity. However, it should be mentioned here that the methods used there relies
heavily on the non-degeneracy of the heat conductivity $\kappa$ and cannot be applied directly to the degenerate and nonlinear case ($\alpha = 0, \beta > 0$). In fact, one of the main aims of this paper is to show that for $\alpha = 0$ and $\beta > 0$, the global strong solutions obtained by $[20]$ are indeed asymptotically stable as time tends to infinity. Moreover, we will improve the results of $[20]$ by relaxing their assumptions on the initial data. Then we state our main result as follows.

**Theorem 1.1** Suppose that

$$\alpha = 0, \quad \beta > 0,$$

and that the initial data $(v_0, u_0, \theta_0)$ satisfies

$$(v_0, \theta_0) \in H^1(0, 1), \quad u_0 \in H^1_0(0, 1),$$

and

$$\inf_{x \in (0, 1)} v_0(x) > 0, \quad \inf_{x \in (0, 1)} \theta_0(x) > 0.$$  

Then, the initial-boundary-value problem (1.1) - (1.7) has a unique strong solution $(v, u, \theta)$ satisfying

\[
\begin{align*}
&v, \theta \in L^\infty(0, \infty; H^1(0, 1)), \quad u \in L^\infty(0, \infty; H^1_0(0, 1)), \\
v_t &\in L^\infty(0, \infty; L^2(0, 1)) \cap L^2(0, \infty; H^1(0, 1)), \\
v_{xx}, \theta_{xx}, u_t, \theta_t, v_{xt}, u_{xx}, \theta_{xx} &\in L^2((0, 1) \times (0, \infty)).
\end{align*}
\]

Moreover, there exists some positive constants $C$ and $\eta_0 > 0$ such that for any $(x, t) \in (0, 1) \times (0, \infty),

$$C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C,$$

and that for any $t > 0$,

$$\left\|\left(v - \int_0^1 v_0dx, u, \theta - \int_0^1 \theta_0dx\right)(\cdot, t)\right\|_{H^1(0, 1)} \leq Ce^{-\eta_0t}. \quad (1.14)$$

A few remarks are in order.

**Remark 1.1** For $\alpha = \beta = 0$, under the conditions (1.10) and (1.11), Kazhikhov and Shelukhin $[18]$ first obtained existence of global strong solutions to the initial-boundary-value problem (1.1) - (1.7). Later, Kazhikhov $[17]$ further proves that the strong solution is nonlinearly exponentially stable as time tends to infinity. Therefore, our Theorem 1.1 can be regarded as a natural generalization of the classical results $[17, 18]$ to the degenerate and nonlinear case that $\alpha = 0, \beta > 0$.

**Remark 1.2** As far as the existence of global strong solutions is concerned, our result also improves Pan and Zhang’s result $[20]$ where they need the initial data satisfy (1.8) which are stronger than (1.10).

We now make some comments on the analysis of this paper. The key step to study the large-time behavior of the global strong solutions is to get the time-independent lower and upper bounds of both $v$ and $\theta$ (see (2.12) and (2.50)). Compared with $[17][23]$, the main difficulty comes from the degeneracy and nonlinearity of the heat conductivity because of $\beta > 0$. Hence, to obtain (2.50), some new ideas are needed. The key observations are as follows: First, after using the standard energetic estimate (see (2.2))
and modifying the idea due to Kazhikhov [17], we obtain that the specific volume $v$ is bounded from above and below time-independently (see (2.12)). Then, although it seems difficult to obtain the uniform lower bound of $\theta$ at first, after observing that (see (2.28))

$$\int_0^T \max_{x \in [0,1]} \left| \theta^{1/2}(x,t) - \int_0^1 \theta^{1/2}(x,t) dx \right|^2 dt \leq C,$$

we prove that the $L^\infty(0, \infty; L^p)$-norm of $\theta^{-1}$ is bounded (see (2.20)), which in turn not only implies that (see (2.32) and (2.34))

$$\int_0^T \max_{x \in [0,1]} \left| \theta(x,t) - \int_0^1 \theta(x,t) dx \right|^2 dt + \int_0^T \int_0^1 u_x^2 dx dt \leq C,$$

but also yields that the $L^2((0,1) \times (0,T))$-norm of $\theta_x$ is bounded provided $\beta > 1$ (see (2.48)). Finally, for $\beta \in (0,1]$, we find that the $L^2((0,1) \times (0,T))$-norm of $\theta_x$ can be bounded by the $L^4(0,T; L^2((0,1)))$-norm of $u_x$ which plays an important role in obtaining the uniform bound on $L^2((0,1) \times (0,T))$-norm of both $\theta_x$ and $u_{xx}$ (see Lemma 2.8) for $\beta \in (0,1]$. The whole procedure will be carried out in the next section.

2 Proof of Theorem 1.1

We first state the following the local existence result which can be proved by using the principle of compressed mappings (c.f. [14,22,29]).

**Lemma 2.1** Let (1.9)–(1.11) hold. Then there exists some $T > 0$ such that the initial-boundary-value problem (1.1)–(1.7) has a unique strong solution $(v,u,\theta)$ satisfying

$$\begin{aligned}
&v, \theta \in L^\infty(0,T; H^1(0,1)), \quad u \in L^\infty(0,T; H^1_0(0,1)), \\
v_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)), \\
u_{tt}, \theta_t, v_{xt}, u_{xx}, \theta_{xx} \in L^2((0,1) \times (0,T)).
\end{aligned}$$

Then, the a priori estimates (see (2.12), (2.30), (2.32), (2.50), and (2.65) below) where the constants depend only on the data of the problem make it possible to continue the local solution to the whole interval $[0, \infty)$ and finish the proof of Theorem 1.1.

Next, without loss of generality, we assume that $\mu = \kappa = R = c_v = 1$ and that

$$\int_0^1 v_0 dx = 1, \quad \int_0^1 \left( \frac{u_0^2}{2} + \theta_0 \right) dx = 1. \quad (2.1)$$

Motivated by the second law of thermodynamics, one has the following standard energetic estimate embodying the dissipative effects of viscosity and thermal diffusion.

**Lemma 2.2** It holds that

$$\sup_{0 \leq t < \infty} \int_0^1 \left( \frac{u^2}{2} + (v - \ln v) + (\theta - \ln \theta) \right) dx + \int_0^T V(s) ds \leq E_0, \quad (2.2)$$

where

$$V(t) \triangleq \int_0^1 \left( \frac{\theta \theta_x^2}{v \theta_x^2} + \frac{u_x^2}{v \theta_x^2} \right) (x,t) dx,$$
and

\[ E_0 = \int_0^1 \left( \frac{u_0^2}{2} + (v_0 - \ln v_0) + (\theta_0 - \ln \theta_0) \right) \, dx. \]

**Proof.** It follows from (1.1), (1.3), (1.7), and (2.1) that for \( t > 0 \)

\[ \int_0^1 v(x,t) \, dx = 1, \quad \int_0^1 \left( \frac{u^2}{2} + \theta \right) (x,t) \, dx = 1. \]

(2.3)

Noticing that the energy equation (1.3) can be written as

\[ \theta_t + \frac{\theta}{v} u_x = \left( \frac{\theta^3 \theta_x}{v} \right)_x + \frac{u^2_x}{v} + \theta \frac{\theta^3 \theta_x^2}{v^2}, \]

(2.4)
multiplying (1.1) by \( 1 - v^{-1} \), (1.2) by \( u \), (2.4) by \( 1 - \theta^{-1} \), and adding them altogether, we get

\[ \left( \frac{u^2}{2} + (v - \ln v) + (\theta - \ln \theta) \right)_t + u_x v \theta + \theta \frac{\theta^3 \theta_x^2}{v^2} \]

which together with (1.7) yields (2.2) and finishes the proof of Lemma 2.2. \( \square \)

Next, we derive the following representation of \( v \) which is essential in obtaining the time-independent upper and lower bounds of \( v \).

**Lemma 2.3** We have the following expression of \( v \)

\[ v(x,t) = D(x,t)Y(t) + \int_0^t \frac{D(x,t)Y(t)\theta(x,\tau)}{D(x,\tau)Y(\tau)} \, d\tau, \]

(2.5)

where

\[ D(x,t) = v_0(x) \exp \left\{ \int_0^x (u(y,t) - u_0(y)) \, dy \right\} \]

\[ \times \exp \left\{ - \int_0^1 v \int_0^x u dy dx + \int_0^1 v_0 \int_0^x u_0 dy dx \right\}, \]

(2.6)

and

\[ Y(t) = \exp \left\{ - \int_0^1 \int_0^t (u^2 + \theta) \, dx \, ds \right\}. \]

(2.7)

**Proof.** First, denoting by

\[ \sigma = \frac{u_x}{v} - \frac{\theta}{v}, \]

(2.8)
it follows from (1.2) that

\[ \left( \int_0^x u dy \right)_t = \sigma - \sigma(0,t), \]

(2.9)

which implies

\[ v \sigma(0,t) = v \sigma - v \left( \int_0^x u dy \right)_t. \]
Integrating this in $x$ over $(0, 1)$ together with (2.3) and (2.8) yields

$$
\sigma(0, t) = \int_0^1 v \sigma dx - \int_0^1 v \left( \int_0^x u dy \right) dx
= \int_0^1 (u_x - \theta) dx - \left( \int_0^1 v \int_0^x u dy dx \right)_t + \int_0^1 u_x \int_0^x u dy dx
= - \left( \int_0^1 v \int_0^x u dy dx \right)_t - \int_0^1 (\theta + u^2) dx.
$$

Next, since $u_x = v_t$, we have

$$
\sigma = (\ln v)_t - \frac{\theta}{v},
$$

which together with (2.9) and (2.10) gives

$$
\left( \int_0^x u dy \right)_t = (\ln v)_t - \frac{\theta}{v} + \left( \int_0^1 v \int_0^x u dy dx \right)_t + \int_0^1 (\theta + u^2) dx.
$$

Integrating this over $(0, t)$ leads to

$$
v(x, t) = D(x, t) Y(t) \exp \left\{ \int_0^t \frac{\theta}{v} ds \right\},
$$

(2.11)

with $D(x, t)$ and $Y(t)$ as in (2.6) and (2.7) respectively.

Finally, denoting by

$$
g = \int_0^t \frac{\theta}{v} ds,
$$

we have by using (2.11)

$$
g_t = \frac{\theta(x, t)}{v(x, t)} = \frac{\theta(x, t)}{D(x, t) Y(t) \exp\{g\}},
$$

which gives

$$
\exp\{g\} = 1 + \int_0^t \frac{\theta(x, \tau)}{D(x, \tau) Y(\tau)} d\tau.
$$

Putting this into (2.11) leads to (2.5). \hfill \Box

With Lemmas 2.3 and 2.2 at hand, we are in a position to prove the time-independent upper and lower bounds of $v$.

**Lemma 2.4** For any $(x, t) \in [0, 1] \times [0, +\infty)$, it holds

$$
C^{-1} \leq v(x, t) \leq C,
$$

(2.12)

where (and in what follows) $C$ denotes some generic positive constant depending only on $\beta, \|\nu - 1, u_0, \theta_0 - 1\|_{H^1(0, 1)}$, $\inf_{x \in [0, 1]} v_0(x)$, and $\inf_{x \in [0, 1]} \theta_0(x)$. 

Proof. First, since the function $x - \ln x$ is convex, Jensen’s inequality gives
\[
\int_0^1 \theta \, dx - \ln \int_0^1 \theta \, dx \leq \int_0^1 (\theta - \ln \theta) \, dx,
\]
which together with (2.2) and (2.3) leads to
\[
\bar{\theta}(t) \triangleq \int_0^1 \theta(x,t) \, dx \in [\alpha_1, 1],
\] (2.13)
where $0 < \alpha_1 < \alpha_2$ are two roots of
\[
x - \ln x = E_0.
\]

Next, both (2.3) and Cauchy’s inequality imply
\[
\left| \int_0^1 v \int_0^x u \, dy \, dx \right| \leq \int_0^1 v \left( \int_0^1 u^2 \, dy \right)^{1/2} \, dx \leq C,
\]
which combined with (2.13) gives
\[
C^{-1} \leq D(x,t) \leq C. \tag{2.14}
\]

Furthermore, one deduces from (2.3) that
\[
1 \leq \int_0^1 (u^2 + \theta) \, dx \leq 2,
\]
which yields that for any $0 \leq \tau < t < \infty$,
\[
e^{-2t} \leq Y(t) \leq 1, \quad e^{-2(t-\tau)} \leq \frac{Y(t)}{Y(\tau)} \leq e^{-(t-\tau)}. \tag{2.15}
\]

Next, it follows from (2.3) that
\[
\left| \theta^{\frac{\beta+1}{2}}(x,t) - \bar{\theta}^{\frac{\beta+1}{2}}(t) \right| \leq \frac{\beta + 1}{2} \left( \int_0^1 \frac{\theta^2 \theta^2}{\theta^2 v} \, dx \right)^{1/2} \left( \int_0^1 \theta v \, dx \right)^{1/2} \leq CV^{1/2}(t) \max_{x \in [0,1]} v^{1/2}(x,t),
\]
which together with (2.13) leads to
\[
\frac{\alpha_1}{4} - CV(t) \max_{x \in [0,1]} v(x,t) \leq \theta(x,t) \leq C + CV(t) \max_{x \in [0,1]} v(x,t), \tag{2.16}
\]
for all $(x,t) \in [0,1] \times [0,\infty)$.

Next, it follows from (2.3), (2.14), (2.15), and (2.16) that
\[
v(x,t) \leq C + C \int_0^t e^{-\tau} \max_{x \in [0,1]} \theta(x,\tau) \, d\tau \leq C + C \int_0^t e^{-\tau} \left( 1 + V(\tau) \max_{x \in [0,1]} v(x,\tau) \right) \, d\tau \leq C + C \int_0^t V(\tau) \max_{x \in [0,1]} v(x,\tau) \, d\tau,
\]
which together with the Gronwall inequality gives
\[
v(x, t) \leq C \tag{2.17}
\]
for all \((x, t) \in [0, 1] \times [0, +\infty)\). Combining this with (2.5), (2.14), (2.13), and (2.16) yields that
\[
v(x, t) \geq C \int_0^t e^{-2(t-\tau)} \min_{x \in [0,1]} \theta(x, \tau) d\tau
\geq C \int_0^t e^{-2(t-\tau)} \left( \frac{\alpha_1}{4} - CV(\tau) \right) d\tau
\geq \frac{C\alpha_1}{8} - \frac{C\alpha_1}{8}e^{-2t} - C \int_0^t e^{-2(t-\tau)} V(\tau) d\tau. \tag{2.18}
\]
Noticing that
\[
\int_0^t e^{-2(t-\tau)} V(\tau) d\tau = \int_0^{t/2} e^{-2(t-\tau)} V(\tau) d\tau + \int_{t/2}^t e^{-2(t-\tau)} V(\tau) d\tau
\leq e^{-t} \int_0^{\infty} V(\tau) d\tau + \int_{t/2}^t V(\tau) d\tau \to 0, \quad \text{as } t \to \infty,
\]
we deduce from (2.18) that there exists some \(\tilde{T} > 0\) such that
\[
v(x, t) \geq \frac{C\alpha_1}{16} \tag{2.19}
\]
for all \((x, t) \in [0, 1] \times [\tilde{T}, +\infty)\).

Finally, using (2.11), (2.13), and (2.15), we obtain that there exists some positive constant \(C\) such that
\[
v(x, t) \geq C^{-1}
\]
for all \((x, t) \in [0, 1] \times [0, \tilde{T}]\). Combining this, (2.19), and (2.17) gives (2.12) and finishes the proof of Lemma 2.4. \(\Box\)

To obtain the uniform (with respect to time) lower bound of the temperature, we need the following time-independent bound on the \(L^\infty(0, T; L^p)\)-norm of \(\theta^{-1}\).

**Lemma 2.5** For any \(p > 0\), there exists some positive constant \(C(p)\) such that
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta^{1-p} dx + \int_0^T \int_0^1 \theta^2 \frac{\theta^2}{\theta^{p+1}} dx dt + \int_0^T \int_0^1 u_x^2 \theta^{p} dx dt \leq C(p). \tag{2.20}
\]

**Proof.** First, it follows from (2.2) that (2.20) holds for \(p = 1\).
Next, for \( p \neq 1 \), multiplying (2.4) by \( 1/\theta^p \) and integration by parts gives
\[
\frac{1}{p-1} \left( \int_0^1 \theta_1 - p \, dx \right) + p \int_0^1 \frac{\theta^2_1}{\theta^p+1} \, dx + \int_0^1 \frac{u_2^x}{\theta^p} \, dx
\]
\[
= \int_0^1 \frac{(\theta_1 - p - 1) u_2}{\theta} \, dx + \int_0^1 \frac{u_2^x}{\theta} \, dx
\]
\[
\leq C(p) \int_0^1 |\theta_2^1 - 1| \left( \int_0^1 |u_2| \, dx + \left( \int_0^1 \theta_1 - p \, dx \right)^{1/2} \left( \int_0^1 \frac{u_2^x}{\theta^p} \, dx \right)^{1/2} \right)
\]
\[
+ \left( \int_0^1 \ln \, v \, dx \right)_t
\]
\[
\leq C(p) \max_{x \in [0,1]} |\theta_2^1 - 1|^2 + C(p) \left( \int_0^1 |u_2| \, dx \right)^2 + \frac{1}{2} \int_0^1 \frac{u_2^2}{\theta^p} \, dx
\]
\[
+ C(p) \max_{x \in [0,1]} |\theta_2^1 - 1|^2 \int_0^1 \theta_1 - p \, dx + \left( \int_0^1 \ln \, v \, dx \right)_t.
\]

Next, we claim that for any real number \( q \), there exists a positive constant \( C(q) \) such that
\[
1 - \bar{\theta}^q \leq C(q)V^{1/2}(t). \tag{2.22}
\]

Indeed, standard calculation gives
\[
1 - \bar{\theta}^q = \int_0^1 \frac{d}{d\eta} \left( \left( \int_0^1 (\theta + \eta \frac{u_2^2}{2}) \, dx \right)^q \right) \, d\eta
\]
\[
= q \int_0^1 \left( \int_0^1 (\theta + \eta \frac{u_2^2}{2}) \, dx \right)^{q-1} \, d\eta \cdot \int_0^1 \frac{u_2^2}{2} \, dx
\]
\[
\leq C(q) \int_0^1 u_2^2 \, dx,
\]
where in the last inequality we have used
\[
\alpha_1 \leq \int_0^1 \theta \, dx \leq \int_0^1 (\theta + \eta \frac{u_2^2}{2}) \, dx \leq 1,
\]
due to (2.3) and (2.13). Furthermore,
\[
\int_0^1 u_2^2 \, dx \leq \max_{x \in [0,1]} |u| \left( \int_0^1 u_2^2 \, dx \right)^{1/2} \leq C \int_0^1 |u_2| \, dx, \tag{2.24}
\]
and
\[
\int_0^1 |u_2| \, dx \leq \left( \int_0^1 \frac{u_2^2}{v\theta} \, dx \right)^{1/2} \left( \int_0^1 v\theta \, dx \right)^{1/2}
\]
\[
\leq C \left( \int_0^1 \frac{u_2^2}{v\theta} \, dx \right)^{1/2} \leq CV^{1/2}(t). \tag{2.25}
\]

Combining all these estimates (2.23)–(2.25) gives (2.22).
Then, it follows from (2.22) and (2.13) that for $\beta \in (0, 1)$,
\[
\max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - 1 \right| \leq \max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - \bar{\theta}^{\frac{1}{\beta}} \right| + \max_{x \in [0, 1]} \left| \bar{\theta}^{\frac{1}{\beta}} - 1 \right|
\leq C \int_0^1 \theta^{-\frac{1}{\beta}} |\theta_x| \, dx + CV^{1/2}(t)
\leq C \left( \int_0^1 \theta^{\beta-2} \theta^2_x \, dx \right)^{1/2} \left( \int_0^1 \theta^{1-\beta} \, dx \right)^{1/2} + CV^{1/2}(t)
\leq CV^{1/2}(t),
\]
and that for $\beta \geq 1$,
\[
\max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - 1 \right| \leq \max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - \bar{\theta}^{\frac{1}{\beta}} \right| + \max_{x \in [0, 1]} \left| \bar{\theta}^{\frac{1}{\beta}} - 1 \right|
\leq C \max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - \bar{\theta}^{\frac{1}{\beta}} \right| + CV^{1/2}(t)
\leq C \int_0^1 \theta^{\frac{1}{\beta}-1} |\theta_x| \, dx + CV^{1/2}(t)
\leq CV^{1/2}(t).
\]

It thus follows from (2.26), (2.27), and (2.2) that
\[
\int_0^T \max_{x \in [0, 1]} \left| \theta^{\frac{1}{\beta}} - 1 \right|^2 \, dt \leq C.
\]

Finally, noticing that for $p \in (0, 1)$,
\[
\int_0^1 \theta^{1-p} \, dx \leq \int_0^1 \theta \, dx + 1 \leq C,
\]
and that both (2.2) and (2.3) imply
\[
\sup_{0 \leq t < \infty} \int_0^1 |\ln v| \, dx \leq C,
\]
after using (2.25), (2.2), (2.28), and Gronwall’s inequality, we obtain (2.20) from (2.21) and finish the proof of Lemma 2.5.

Next, using Lemma 2.5, we have the following estimate on the $L^\infty(0, T; L^2)$-norm of $v_x$.

**Lemma 2.6** There exists a positive constant $C$ such that
\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 \, dx + \int_0^T \int_0^1 v_x^2 (\theta + 1) \, dx \, dt \leq C,
\]
for any $T \geq 0$.

**Proof.** First, choosing $p = \beta$ in (2.20) gives
\[
\int_0^T \int_0^1 \theta^{-1} \theta_x^2 \, dx \, dt \leq C,
\]
which together with (2.13) implies
\[
\int_0^T \max_{x \in [0, 1]} (\theta(x, t) - \bar{\theta}(t))^2 dt \leq C \int_0^T \int_0^1 \theta^{-1} \theta_x^2 dx \int_0^1 \theta dx dt \leq C. \tag{2.32}
\]

Next, integrating the momentum equation (1.2) multiplied by \( u \) over \([0, 1]\) with respect to \( x \), we obtain after integrating by parts
\[
\frac{1}{2} \left( \int_0^1 u^2 dx \right)_t + \int_0^1 \frac{u^2}{v} dx \\
= \int_0^1 \frac{\theta}{v} u_x dx \\
= \int_0^1 \frac{(\theta - \bar{\theta})}{v} u_x dx + (\bar{\theta} - 1) \int_0^1 \frac{u_x}{v} dx + \int_0^1 \frac{u_x}{v} dx \\
\leq C \max_{x \in [0, 1]} (\theta - \bar{\theta})^2 + CV(t) + \left( \int_0^1 \ln v dx \right)_t,
\]
where in the last inequality we have used (2.22) and (2.25). Combining this with (2.2), (2.29), and (2.32) yields
\[
\int_0^T \int_0^1 u_x^2 dx dt \leq C. \tag{2.34}
\]

Next, using (1.1), we rewrite momentum equation (1.2) as
\[
\left( \frac{v_x}{v} \right)_t = u_t + \left( \frac{\theta}{v} \right)_x,
\]
due to
\[
\left( \frac{v_t}{v} \right)_x = \left( \frac{v_x}{v} \right)_t.
\]

Multiplying (2.35) by \( \frac{v_x}{v} \) leads to
\[
\frac{1}{2} \left[ \left( \frac{v_x}{v} \right)^2 \right]_t = \frac{v_x}{v} u_t + \frac{v_x}{v} \left( \frac{\theta}{v} \right)_x \\
= \left( \frac{v_x}{v} u \right)_t - u(\ln v)_{xt} + \frac{v_x \theta_x}{v^2} - \frac{v_x^2 \theta}{v^3} \\
= \left( \frac{v_x}{v} u \right)_t - [u(\ln v)]_x + \frac{u_x^2}{v} + \frac{v_x \theta_x}{v^2} - \frac{v_x^2 \theta}{v^3}.
\]

Integrating (2.36) over \([0, 1] \times [0, T]\), one has
\[
\sup_{0 \leq t \leq T} \int_0^1 \left[ \frac{1}{2} \left( \frac{v_x}{v} \right)^2 - \frac{v_x}{v} u \right] dx + \int_0^T \int_0^1 \frac{v_x^2 \theta}{v^3} dx dt \\
\leq C + \int_0^T \int_0^1 \frac{v_x \theta_x}{v^2} dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{v_x^2 \theta}{v^3} dx dt + C \int_0^T \int_0^1 \theta^{-1} \theta_x^2 dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{v_x^2 \theta}{v^3} dx dt,
\]
due to (2.31). This in particular implies
\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx + \int_0^T \int_0^1 v_x^2 \theta dx dt \leq C, \tag{2.37}
\]
due to the following simply fact:
\[
\int_0^1 \frac{v_x}{v} u dx \leq \frac{1}{4} \int_0^1 \left( \frac{v_x}{v} \right)^2 dx + C.
\]
Finally, it follows from (2.37) that
\[
\bar{\theta} \int_0^1 v_x^2 dx = \int_0^1 v_x^2 (\bar{\theta} - \theta) dx + \int_0^1 v_x^2 \theta dx 
\leq \frac{\bar{\theta}}{2} \int_0^1 v_x^2 dx + \frac{1}{2\theta} \int_0^1 v_x^2 (\theta - \bar{\theta})^2 dx + \int_0^1 v_x^2 \theta dx
\leq \frac{\bar{\theta}}{2} \int_0^1 v_x^2 dx + C \max_{\theta \in [0,1]} (\theta - \bar{\theta})^2 + \int_0^1 v_x^2 \theta dx,
\]
which together with (2.32) and (2.37) leads to
\[
\int_0^T \int_0^1 v_x^2 dx dt \leq C.
\]
Combining this with (2.37) gives (2.30) and finishes the proof of Lemma 2.6.

For further uses, we need the following estimate on the \(L^2((0,1) \times (0,T))\)-norm of \(\theta_x\) for \(\beta \in (0,1]\).

**Lemma 2.7** If \(0 < \beta \leq 1\), there exists a positive constant \(C\) such that
\[
\int_0^T \int_0^1 \theta_x^2 dx dt \leq C + C \int_0^T \left( \int_0^1 u_x^2 dx \right)^2 dt, \tag{2.38}
\]
for any \(T > 0\).

**Proof.** Multiplying (2.4) by \(\theta^{1-\frac{\beta}{2}}\) and integration by parts gives
\[
\frac{2}{4-\beta} \left( \int_0^1 \theta^{2-\frac{\beta}{2}} dx \right)_t + \frac{(2-\beta)}{2} \int_0^1 \frac{\theta^{\frac{\beta}{2}} \theta^2}{v} dx 
= - \int_0^1 \frac{\theta^{2-\frac{\beta}{2}}}{v} u_x dx + \int_0^1 \frac{\theta^{1-\frac{\beta}{2}} u_x^2}{v} dx 
= \int_0^1 \frac{(\theta^{2-\frac{\beta}{2}} - \bar{\theta}^{2-\frac{\beta}{2}})}{v} u_x dx + \left( 1 - \bar{\theta}^{2-\frac{\beta}{2}} \right) \int_0^1 \frac{u_x^2}{v} dx
- \int_0^1 \frac{u_x dx}{v} + \int_0^1 \frac{\theta^{1-\frac{\beta}{2}} u_x^2}{v} dx
\leq C \int_0^1 \left| \theta^{2-\frac{\beta}{2}} - \bar{\theta}^{2-\frac{\beta}{2}} \right| u_x |dx| + CV(t) - \left( \int_0^1 \ln v dx \right)_t + \int_0^1 \frac{\theta^{1-\frac{\beta}{2}} u_x^2}{v} dx,
\tag{2.39}
\]
where in the last inequality we have used (2.22). Direct calculation yields that

\[
\int_0^1 |\theta^2 - \bar{\theta}^2| u_x^2 \, dx \\
\leq C \max_{x \in [0,1]} |\theta^{1 - \frac{\beta}{2}} - \bar{\theta}^{1 - \frac{\beta}{2}}| \left( \int_0^1 \left( \theta^{2 - \frac{\beta}{2}} + 1 \right) \, dx \right)^{1/2} \left( \int_0^1 u_x^2 \, dx \right)^{1/2} \\
\leq C \left( \int_0^1 \theta^{1 - \frac{\beta}{2}} \theta_x \, dx \right)^2 + C \int_0^1 \left( \theta^{2 - \frac{\beta}{2}} + 1 \right) \, dx \int_0^1 u_x^2 \, dx \\
\leq C \int_0^1 \theta^{2 - \frac{\beta}{2}} dx \int_0^1 \theta^{1 - \frac{\beta}{2}} \, dx + C \int_0^1 \left( \theta^{2 - \frac{\beta}{2}} + 1 \right) \, dx \int_0^1 u_x^2 \, dx \\
\leq C \int_0^1 \theta^{1 - \frac{\beta}{2}} dx + C \int_0^1 \left( \theta^{2 - \frac{\beta}{2}} + 1 \right) \, dx \int_0^1 u_x^2 \, dx, 
\]

and that for any \( \delta > 0 \)

\[
\int_0^1 \frac{\theta^{1 - \frac{\beta}{2}} u_x^2}{v} \, dx \\
\leq C \left( \max_{x \in [0,1]} |\theta^{1 - \frac{\beta}{2}} - \bar{\theta}^{1 - \frac{\beta}{2}}| + 1 \right) \int_0^1 u_x^2 \, dx \\
\leq C \int_0^1 \theta^{1 - \frac{\beta}{2}} \theta_x \, dx \int_0^1 u_x^2 \, dx + C \int_0^1 u_x^2 \, dx \\
\leq \delta \int_0^1 \left( \theta^{-1} + \theta^{\frac{\beta}{2}} \right) \theta_x^2 \, dx + C(\delta) \left( \int_0^1 u_x^2 \, dx \right)^2 + C \int_0^1 u_x^2 \, dx. 
\]

Putting (2.40) and (2.41) into (2.39), choosing \( \delta \) suitably small, and using (2.34), (2.31), and the Gronwall inequality, one obtains

\[
\int_0^1 \theta^{2 - \beta/2} \, dx + \int_0^T \int_0^1 \theta^{\beta/2} \theta_x^2 \, dx dt \leq C + C \int_0^T \left( \int_0^1 u_x^2 \, dx \right)^2 dt, 
\]

which together with (2.2) implies

\[
\int_0^T \int_0^1 \theta_x^2 \, dx dt \leq \int_0^T \int_0^1 \left( \theta^{\beta/2} + \theta^{\beta/2} \right) \theta_x^2 \, dx dt \\
\leq C + \int_0^T \int_0^1 \theta^{\beta/2} \theta_x^2 \, dx dt \\
\leq C + C \int_0^T \left( \int_0^1 u_x^2 \, dx \right)^2 dt. 
\]

This gives (2.38) and finishes the proof of Lemma 2.7. \( \square \)

Then, we have the following uniform estimate on the \( L^2((0,1) \times (0,T)) \)-norm of \( u_t \) and \( u_{xx} \).

**Lemma 2.8** There exists a positive constant \( C \) such that

\[
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 \, dx + \int_0^T \int_0^1 (u_t^2 + u_{xx}^2) \, dx dt \leq C. \quad (2.42)
\]

for any \( T \geq 0 \).
Proof. First, we rewrite the momentum equation (1.2) as

\[ u_t - u_{xx} = - \frac{u_xx}{v} - \frac{\theta_x}{v} + \frac{\theta v_x}{v^2} \tag{2.43} \]

Multiplying both sides of (2.43) by \( u_{xx} \) and integrating the resultant equality in \( x \) over \([0, 1]\) lead to

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u_{x}^2 dx + \int_0^1 \frac{u_{x}^2}{v} \, dx \leq \left| \int_0^1 \frac{u_x v_x}{v^2} u_{xx} \, dx \right| + \left| \int_0^1 \frac{\theta_x}{v} u_{xx} \, dx \right| + \left| \int_0^1 \frac{\theta v_x}{v^2} u_{xx} \, dx \right| \tag{2.44}
\]

Direct computation yields that for any \( \delta > 0 \),

\[
\int_0^1 \left( u_{x}^2 v_x^2 + v_x^2 \theta^2 + \theta_x^2 \right) \, dx \\
\leq C \max_{x \in [0,1]} u_{x}^2 + C \max_{x \in [0,1]} (\theta - \bar{\theta})^2 + C \int_0^1 u_{x}^2 \, dx + \int_0^1 \theta_x^2 \, dx \tag{2.45}
\]

\[
\leq \delta \int_0^1 u_{x}^2 \, dx + C(\delta) \int_0^1 u_{x}^2 \, dx + C \max_{x \in [0,1]} (\theta - \bar{\theta})^2 + C \int_0^1 u_{x}^2 \, dx + \int_0^1 \theta_x^2 \, dx,
\]

where in the last inequality we have used

\[
\max_{x \in [0,1]} u_{x}^2 \leq \int_0^1 \left| u_{x}^2 \right| \, dx \\
\leq 2 \left( \int_0^1 u_{xx}^2 \, dx \right)^{1/2} \left( \int_0^1 u_{x}^2 \, dx \right)^{1/2} \tag{2.46}
\]

\[
\leq \delta \int_0^1 u_{x}^2 \, dx + C(\delta) \int_0^1 u_{x}^2 \, dx,
\]

due to \( \int_0^1 u_{x} \, dx = 0 \). Putting (2.45) into (2.44) and choosing \( \delta \) suitably small yields

\[
\int_0^1 u_{x}^2 \, dx + \int_0^T \int_0^1 u_{xx}^2 \, dx \, dt \leq C + C \int_0^T \int_0^1 \theta_x^2 \, dx \, dt \tag{2.47}
\]

due to (2.32), (2.34), and (2.30).

Next, on the one hand, if \( \beta > 1 \), choosing \( p = \beta - 1 \) in (2.20) gives

\[
\int_0^T \int_0^1 \theta_x^2 \, dx \, dt \leq C, \tag{2.48}
\]

which along with (2.47) gives

\[
\sup_{0 \leq t \leq T} \left[ \int_0^1 u_{x}^2 \, dx + \int_0^T \int_0^1 u_{xx}^2 \, dx \, dt + \int_0^T \int_0^1 \theta_x^2 \, dx \, dt \right] \leq C. \tag{2.49}
\]
On the other hand, if \( \beta \in (0, 1] \), it follows from (2.47), (2.38), (2.34), and Gronwall’s inequality that (2.49) still holds.

Finally, it follows from (2.43), (2.49), (2.45), (2.32), (2.34), and (2.30) that
\[
\int_0^T \int_0^1 u_x^2 dt dx \leq C,
\]
which together with (2.49) gives (2.52) and finishes the proof of Lemma 2.8.

Now, we can prove the uniform lower and upper bounds of the temperature \( \theta \).

**Lemma 2.9** There exists a positive constant \( C \) such that for any \((x, t) \in [0, 1] \times [0, T]\)
\[
C^{-1} \leq \theta(x, t) \leq C.
\]

**Proof.** First, for \( p > \beta + 1 \), multiplying (2.4) by \( \theta^{p-1} \theta_t \) and integrating the resultant equality in \( x \) over \((0, 1)\) leads to
\[
\frac{1}{p} \left( \int_0^1 \theta^p dx \right)_t + (p - 1) \int_0^1 \frac{\theta^{p+\beta-2} \theta_t^2}{v} dx
\leq \int_0^1 \frac{\theta^{p-1} u_x^2}{v} dx - \int_0^1 \frac{\theta^{p-1} u_x v}{v} dx
\leq C \max_{x \in [0, 1]} u_x^2 \int_0^1 \theta^{p-1} dx + C \int_0^1 |(\theta^{p-1}) u_x| dx - \int_0^1 \frac{u_x}{v} dx
\leq C \max_{x \in [0, 1]} (u_x^2 + |\theta - 1|^2) \left( 1 + \int_0^1 \theta^{p} dx \right) - \left( \int_0^1 \ln v dx \right)_t.
\]

It follows from (2.49), (2.46), and (2.34) that
\[
\int_0^T \max_{x \in [0, 1]} u_x^2 dt + \int_0^T \int_0^1 \theta_x^2 dx dt \leq C (p),
\]
which together with (2.22) shows
\[
\int_0^T \max_{x \in [0, 1]} |\theta - 1|^2 dt \leq C \int_0^T \left( \max_{x \in [0, 1]} |\theta - \bar{\theta}|^2 + \max_{x \in [0, 1]} |\bar{\theta} - 1|^2 \right) dt
\leq C \int_0^T \int_0^1 \theta_x^2 dx dt + C \int_0^T V(t) dt \leq C.
\]

Combining (2.51) - (2.53) with the Gronwall inequality gives
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta^p dx + \int_0^T \int_0^1 \theta^{p+\beta-2} \theta_x^2 dx dt \leq C(p).
\]

Next, multiplying (2.4) by \( \theta^d \theta_t \) and integrating the resultant equality over \((0, 1)\)
yields
\[
\begin{align*}
\int_0^1 \theta^2 \beta_x^2 dx + \int_0^1 \frac{\theta^2 + 1 \theta_t u_x}{v} dx \\
= \int_0^1 \frac{\theta^2 \theta_t}{v} \bigg( \frac{\theta^2 \theta_x}{v} \bigg)_x dx + \int_0^1 \frac{\theta^2 \theta_t u_x^2}{v} dx
\end{align*}
\]
\[
= - \int_0^1 \frac{\theta^2 \theta_x}{v} \bigg( \frac{\theta^2 \theta_t}{v} \bigg)_x dx + \int_0^1 \frac{\theta^2 \theta_t u_x^2}{v} dx
\]
\[
= - \int_0^1 \frac{\theta^2 \theta_x}{v} \bigg( \frac{\theta^2 \theta_t}{v} \bigg)_x dx + \int_0^1 \frac{\theta^2 \theta_t u_x^2}{v} dx
\]
\[
= - \frac{1}{2} \bigg( \int_0^1 \frac{\theta^2 \theta_x}{v} dx \bigg)_t - \frac{1}{2} \int_0^1 \frac{(\theta^2 \theta_x)^2 u_x}{v^2} dx + \int_0^1 \frac{\theta^2 \theta_t u_x^2}{v} dx,
\]
which gives
\[
\begin{align*}
\int_0^1 \theta^2 \beta_x^2 dx + \frac{1}{2} \left( \int_0^1 \frac{\theta^2 \theta_x}{v} dx \right)_t \\
= - \frac{1}{2} \int_0^1 \frac{\theta^2 \theta_x}{v} u_x dx - \int_0^1 \frac{\theta^2 \theta_t u_x}{v} dx + \int_0^1 \frac{\theta^2 \theta_t u_x^2}{v} dx
\end{align*}
\]
\[
\leq C \max_{x \in [0,1]} |u_x| \int_0^1 \left( \frac{\theta^2 \theta_x}{v} \right)^2 dx + \frac{1}{2} \int_0^1 \theta^2 \theta_t^2 dx + C \int_0^1 \theta^2 + u_x^2 dx
\]
\[
+ C \int_0^1 \theta^2 \beta_x^2 dx
\]
\[
\leq C \left( \int_0^1 \left( \frac{\theta^2 \theta_x}{v} \right)^2 dx \right) + C \max_{x \in [0,1]} u_x^2 + C \max_{x \in [0,1]} u_x^4 + \frac{1}{2} \int_0^1 \theta^2 \beta_x^2 dx
\]
due to (2.54).

Next, it follows from (2.46) and (2.42) that
\[
\int_0^T \max_{x \in [0,1]} u_x^4 dt \leq C,
\]
which together with (2.55), the Gronwall inequality, (2.52), and (2.54) leads to
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \theta^2 \theta_x \right)^2 dx + \int_0^T \int_0^1 \theta^2 \theta_t^2 dx dt \leq C.
\]
This in particular gives
\[
\max_{x \in [0,1]} \left| \theta^{\beta+1} - \bar{\theta}^{\beta+1} \right| \leq (\beta + 1) \int_0^1 \theta^3 |\theta_x| dx
\]
\[
\leq C \left( \int_0^1 \frac{\theta^2}{v} \left( \frac{\theta^2 \theta_x}{v} \right)^2 \right)^{1/2} \leq C,
\]
which implies that for all \((x, t) \in [0, 1] \times [0, \infty),\)
\[
\theta(x, t) \leq C.
\]
Next, it follows from (2.22) that
\[
\int_0^T \int_0^1 \left( \theta^{\beta+2} - 1 \right)^2 \, dx \, dt
\leq 2 \int_0^T \int_0^1 \left( \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right)^2 \, dx \, dt + C \int_0^T V(t) \, dt
\leq C \int_0^T \int_0^1 \left( \theta^{\beta+1} |\theta_x| \right)^2 \, dx \, dt + C
\leq C \int_0^T \int_0^1 \theta^{\beta-2} \theta_x^2 \, dx \, dt + C \leq C,
\] (2.59)
where in the last inequality we have used (2.58). Combining this, (2.57), and (2.58) in particular gives
\[
\int_0^1 \left( \int_0^1 \left( \theta^{\beta+2} - 1 \right)^2 \, dx \right) \, dt
= 2 \int_0^T \int_0^1 \left( \theta^{\beta+2} - 1 \right) \left( \theta^{\beta+2} \right)_t \, dx \, dt
\leq C \int_0^T \int_0^1 \left( \theta^{\beta+2} - 1 \right)^2 \, dx \, dt + C \int_0^T \int_0^1 \theta^{2\beta+2} \theta_t^2 \, dx \, dt
\leq C + C \int_0^T \int_0^1 \theta^{\beta-2} \theta_x^2 \, dx \, dt \leq C,
\]
which together with (2.59) leads to
\[
\lim_{t \to \infty} \int_0^1 \left( \theta^{\beta+2} - 1 \right)^2 \, dx = 0.
\] (2.60)
Then, we claim that
\[
\lim_{t \to \infty} (1 - \bar{\theta}) = 0,
\] (2.61)
which combined with (2.60) gives
\[
\lim_{t \to \infty} \int_0^1 \left( \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right)^2 \, dx = 0.
\] (2.62)
It thus follows from (2.57) and (2.58) that
\[
\max_{x \in [0,1]} \left( \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right)^2
\leq C \int_0^1 \left| \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right| \left| (\theta^{\beta+2})_x \right| \, dx
\leq C \left( \int_0^1 \left( \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right)^2 \, dx \right)^{1/2} \left( \int_0^1 \left( \theta^{\beta} \theta_x \right)^2 \, dx \right)^{1/2}
\leq C \left( \int_0^1 \left( \theta^{\beta+2} - \bar{\theta}^{\beta+2} \right)^2 \, dx \right)^{1/2},
\]
which together with (2.62) and (2.61) implies that there exists some $T_0 > 0$ such that
\[
\theta(x,t) \geq 1/2,
\] (2.63)
for all \((x, t) \in [0, 1] \times [T_0, \infty)\). Moreover, it follows from [26, Lemma 2.2] that there exists some constant \(C \geq 2\) such that
\[
\theta(x, t) \geq C^{-1},
\]
for all \((x, t) \in [0, 1] \times [0, T_0]\). Combining this, (2.63), and (2.58) gives (2.50).

Finally, it remains to prove (2.61). Indeed, it follows from (2.33) and (2.58) that
\[
\frac{1}{2} \left( \int_0^1 u^2(x) \, dx \right)_t + \int_0^1 \frac{u^2}{v} \, dx = \int_0^1 \frac{\theta}{v} u_x \, dx \leq C \int_0^1 |u_x| \, dx,
\]
which yields that there exists some constant \(C\) such that for any \(N > 0\) and \(s, t \in [N, N + 1]\)
\[
\int_0^1 u^2(x, t) \, dx - \int_0^1 u^2(x, s) \, dx \leq C \int_N^{N+1} \int_0^1 |u_x| \, dx \, d\tau.
\]
Integrating this with respect to \(s\) over \((N, N + 1)\) and using (2.24) yields that
\[
\sup_{N \leq t \leq N+1} \int_0^1 u^2(x, t) \, dx \leq C \left( \int_N^{N+1} \int_0^1 u_x^2 \, dx \, d\tau \right)^{1/2}. \tag{2.64}
\]
Letting \(N \to \infty\) in (2.64) gives
\[
\lim_{t \to \infty} \int_0^1 u^2(x, t) \, dx = 0,
\]
due to (2.34). Combining this with (2.3) gives (2.61) and finishes the proof of Lemma 2.9. □

Next, we have the following uniform estimate on the \(L^2((0, 1) \times (0, T))\)-norm of \(\theta_t\) and \(\theta_{xx}\).

**Lemma 2.10** There exists a positive constant \(C\) such that
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta_t^2 \, dx + \int_0^T \int_0^1 \left( \theta_t^2 + \theta_{xx}^2 \right) \, dx \, dt \leq C. \tag{2.65}
\]

**Proof.** First, both (2.50) and (2.57) lead to
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta_t^2 \, dx + \int_0^T \int_0^1 \theta_t^2 \, dx \, dt \leq C. \tag{2.66}
\]

Next, it follows from (2.3) that
\[
\frac{\theta^3 \theta_x}{v} = \frac{\theta^3 \theta_x v_x}{v^2} = \frac{u_x^2}{v} + \frac{\theta u_x}{v} + \theta_t,
\]
which together with (2.30), (2.50), (2.34), (2.56), and (2.66) gives
\[
\int_0^T \int_0^1 \left( \theta^3 \theta_x \right)_x^2 \, dx \, dt \leq C \int_0^T \max_{x \in [0, 1]} \left( \theta^3 \theta_x \right)^2 \int_0^1 u_x^2 \, dx \, dt + C \leq C \int_0^T \max_{x \in [0, 1]} \left( \theta^3 \theta_x \right)^2 \, dt + C. \tag{2.67}
\]
Since \( \theta_x(0, t) = 0 \), we get by \( (2.52) \) and \( (2.50) \),
\[
\int_0^T \max_{x \in [0, 1]} \theta^2 \theta_x^2 \, dt \leq \int_0^T \int_0^1 \left\| \left( \theta^2 \theta_x \right)_x \right\|^2 \, dx \, dt
\leq C(\delta) + \delta \int_0^T \int_0^1 \left\| \theta^2 \theta_x \right\|^2 \, dx \, dt,
\]
which together with \( (2.67) \) and \( (2.50) \) implies
\[
\int_0^T \max_{x \in [0, 1]} \theta^2 \theta_x^2 \, dt \leq C.
\] \( (2.68) \)

Finally, since
\[
\theta_{xx} = \frac{(\theta^2 \theta_x)_x}{\theta} - \frac{3 \theta_x}{\theta},
\]
it follows from \( (2.65), (2.50), \) and \( (2.57) \) that
\[
\int_0^T \int_0^1 \theta^2_{xx} \, dx \, dt \leq C \int_0^T \int_0^1 \left\| \left( \theta^2 \theta_x \right)_x \right\|^2 \, dx \, dt + C \int_0^T \max_{x \in [0, 1]} \theta^2 \theta_x \int_0^1 \theta^2 \, dx \, dt
\leq C + C \sup_{0 \leq t \leq T} \int_0^1 \theta^2 \, dx \int_0^T \max_{x \in [0, 1]} \theta^2 \, dt
\leq C,
\]
which together with \( (2.66) \) gives \( (2.65) \) and finishes the proof of Lemma 2.10.

Finally, we have the following nonlinearly exponential stability of the strong solutions.

**Lemma 2.11** There exist some positive constants \( C \) and \( \eta_0 \) both depending only on \( \beta, \| (v_0 - 1, u_0, \theta_0 - 1) \|_{H^1(0, 1)}, \inf \_{x \in [0, 1]} v_0(x), \) and \( \inf \_{x \in [0, 1]} \theta_0(x) \) such that
\[
\| (v - 1, u, \theta - 1)(\cdot, t) \|_{H^1(0, 1)} \leq C e^{-\eta_0 t}.
\] \( (2.69) \)

**Proof.** Noticing that all the constants \( C \) in Lemmas 2.6, 2.8, and 2.10 are independent of \( T \), we have
\[
\int_0^\infty \left( \left\| \frac{d}{dt} v_x(\cdot, t) \right\|^2_{L^2} + \left\| \frac{d}{dt} u_x(\cdot, t) \right\|^2_{L^2} + \left\| \frac{d}{dt} \theta_x(\cdot, t) \right\|^2_{L^2} \right) \, dt \leq C,
\] \( (2.70) \)
where we have used
\[
\int_0^1 u_x u_{xx} \, dx = - \int_0^1 u_t u_{xx} \, dx.
\]
It thus follows from \( (2.70) \) that
\[
\lim_{t \to \infty} \| (v_x, u_x, \theta_x)(\cdot, t) \|_{L^2(0, 1)} = 0,
\]
which in particular implies
\[
\lim_{t \to \infty} \| (v - 1, u, \theta - 1)(\cdot, t) \|_{H^1(0, 1)} = 0.
\]

Therefore, since we know that the temperature remains bounded from above and below independently of time and the solution becomes small in \( H^1 \)-norm for large time \( t \), we can conclude the solution decays to the constant state exponentially as \( t \to \infty \), that is, \( (2.69) \) holds (c.f. [24]).
References

[1] Amosov, A. A., Zlotnik, A. A. Global generalized solutions of the equations of the one-dimensional motion of a viscous heat-conducting gas. Soviet Math. Dokl. 38 (1989), 1-5.

[2] Amosov, A. A., Zlotnik, A. A. Solvability "in the large" of a system of equations of the one-dimensional motion of an inhomogeneous viscous heat-conducting gas. Math. Notes 52 (1992), 753-763.

[3] Amosov, A. A., Zlotnik, A. A. On the stability of generalized solutions of equations of one-dimensional motion of a viscous heat-conducting gas, Sib. Math. J. 38 (1997) 663-684.

[4] Antontsev, S. N., Kazhikhov, A. V., Monakhov, V. N. Boundary Value Problems in Mechanics of Nonhomogeneous Fluids. Amsterdam, New York: North-Holland, 1990.

[5] Batchelor, G. K. An Introduction to Fluid Dynamics. London: Cambridge Univ. Press, 1967

[6] Cercignani, C., Illner, R., Pulvirenti, M. The Mathematical Theory of Dilute Gases, Appl. Math. Sci., vol.106, Springer-Verlag, New York, 1994.

[7] Chapman, S., Colwing, T. G. The Mathematical Theory of Nonuniform Gases, 3rd ed., Cambridge Math. Lib., Cambridge University Press, Cambridge, 1990.

[8] Hsiao, L. Quasilinear Hyperbolic Systems and Dissipative Mechanisms, World Scientific Publisher Co., 1997.

[9] Hsiao, L., Jiang, S. Nonlinear hyperbolic-parabolic coupled systems, Handbook of Differential Equations: Evolutionary Equations, Elsevier, 1, 287-384, 2004.

[10] Jiang, S. On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas, Math. Z., 216, 317-336, 1994.

[11] Jiang, S. Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain. Comm. Math. Phys. 178 (1996), 339-374.

[12] Jiang, S. Large-time behavior of solutions to the equations of a viscous polytropic ideal gas. Amni Mat. Pura Appl. 175 (1998), 253-275.

[13] Jenssen, H. K., Karper, T. K. One-dimensional compressible flow with temperature dependent transport coefficients, SIAM J. Math. Anal., 42 (2010), 904-930.

[14] Kanel, Y. I. On a model system of equations of one-dimensional gas motion. Differential Equations, 4 (1968), 374-380.

[15] Kawashima, S. Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications. Proc. R. Soc. Edinb. A 106 (1987), 169-194.

[16] Kawohl, B. Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas, J. Diff. Eqns., 58, 76-103, 1985.
[17] Kazhikhov, A. V. To a theory of boundary value problems for equations of one-dimensional nonstationary motion of viscous heat-conduction gases, in: Boundary Value Problems for Hydrodynamical Equations, No. 50, Institute of Hydrodynamics, Siberian Branch Acad. USSR, 1981, pp. 37-62, in Russian.

[18] Kazhikhov, A. V., Shelukhin, V. V. Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas. J. Appl. Math. Mech. 41 (1977), 273-282.

[19] Nagasawa, T. On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary. J. Diff. Eqs. 65 (1986), 49-67.

[20] Nagasawa, T. On the asymptotic behavior of the one-dimensional motion of the polytropic ideal gas with stress-free condition. Quart. Appl. Math. 46 (1988), 665-679.

[21] Nagasawa, T. On the one-dimensional free boundary problem for the heat-conductive compressible viscous gas. In: Mimura, M., Nishida, T. (eds.) Recent Topics in Nonlinear PDE IV, Lecture Notes in Num. Appl. Anal. 10, Amsterdam, Tokyo: Kinokuniya/North-Holland, 1989, pp. 83-99.

[22] Nash, J. Le probleme de Cauchy pour les équations différentielles d’un fluide général, Bull. Soc. Math. France, 90, 487-497, 1962.

[23] Nishida, T. Equations of motion of compressible viscous fluids. In: Nishida, T., Mimura, M., Fujii, H. (eds.) Pattern and Waves, Amsterdam, Tokyo: Kinokuniya/North-Holland, 1986, pp. 97-128.

[24] Okada, M.; Kawashima, S. On the equations of one-dimensional motion of compressible viscous fluids, J. Math. Kyoto Univ., 23 (1983), 55-71.

[25] Pan, R. Global smooth solutions and the asymptotic behavior of the motion of a viscous, heatconductive, one-dimensional real gas, J. Part. Diff. Eqs., 11, 273-288, 1998.

[26] Pan, R., Zhang, W. Compressible Navier-Stokes equations with temperature-dependent heat conductivities, Commun. Math. Sci. 13 (2015), 401-425.

[27] Qin, Y. Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors, Operator Theory, Advances and Applications, Vol 184. Basel, Boston, Berlin: Birkhäuser, 2008

[28] Serrin, J. Mathematical principles of classical fluid mechanics. In: Flügge, S., Truesdell, C. (eds.), Handbuch der Physik. VIII/1, Berlin-Heidelberg-NewYork: Springer-Verlag, 1972, pp. 125-262.

[29] Tani, A. On the first initial-boundary value problem of compressible viscous fluid motion, Publications of the Research Institute for Mathematical Sciences, 13, 193-253, 1977.