CUNTZ-KRIEGER ALGEBRAS ASSOCIATED WITH HILBERT 
C*-QUAD MODULES OF COMMUTING MATRICES

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Abstract. Let $\mathcal{O}_{H^A,B}^\kappa$ be the C*-algebra associated with the Hilbert C*-quad module arising from commuting matrices $A, B$ with entries in $\{0, 1\}$. We will show that if the associated tiling space $X^\kappa_{A,B}$ is transitive, the C*-algebra $\mathcal{O}_{H^A,B}^\kappa$ is simple and purely infinite. In particular, for two positive integers $N, M$, the K-groups of the simple purely infinite C*-algebra $\mathcal{O}_{H^{[N],[M]}}^\kappa$ are computed by using the Euclidean algorithm.

1. Introduction

In [11], the author has introduced a notion of C*-symbolic dynamical system, which is a generalization of a finite labeled graph, a λ-graph system and an automorphism of a unital C*-algebra. It is denoted by $(A, \rho, \Sigma)$ and consists of a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital C*-algebra $A$ such that $\rho_\alpha(Z_A) \subset Z_A, \alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ where $Z_A$ denotes the center of $A$. It provides a subshift $\Lambda_\rho$ over $\Sigma$ and a Hilbert C*-bimodule $H_\rho^A$ over $A$ which gives rise to a C*-algebra $\mathcal{O}_\rho$ as a Cuntz-Pimsner algebra ([11], cf. [6], [19]). In [13] and [14], the author has extended the notion of C*-symbolic dynamical system to C*-textile dynamical system which is a higher dimensional analogue of C*-symbolic dynamical system. The C*-textile dynamical system $(A, \rho, \eta, \Sigma_\rho, \Sigma_\eta, \kappa)$ consists of two C*-symbolic dynamical systems $(A, \rho, \Sigma_\rho)$ and $(A, \eta, \Sigma_\eta)$ with a common unital C*-algebra $A$ and a commutation relation between the endomorphisms $\rho$ and $\eta$ through a map $\kappa$ stated below. Set

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\}$$
$$\Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$  

We assume that there exists a bijection $\kappa : \Sigma^{\rho\eta} \rightarrow \Sigma^{\eta\rho}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \quad (1.1)$$

A C*-textile dynamical system provides a two-dimensional subshift and a multi structure of Hilbert C*-bimodule that has multi right actions and multi left actions and multi inner products. Such a multi structure of Hilbert C*-bimodule is called a Hilbert C*-quad module, denoted by $H^\rho_\kappa$. In [14], the author has introduced a C*-algebra associated with the Hilbert C*-quad module. The C*-algebra $\mathcal{O}_{H^\rho_\kappa}^\kappa$ has been constructed in a concrete way from the structure of the Hilbert C*-quad module $H^\rho_\kappa$ by a two-dimensional analogue of Pimsner’s construction of C*-algebras from Hilbert C*-bimodules. It is generated by the quotient images of creation operators on two-dimensional analogue of Fock Hilbert module by module maps of compact operators. As a result, the C*-algebra has been proved to have
Let us denote by $H$ introduced by Nasu ([16]). We then have a $C$-structure of the Hilbert $C^*$-quad module ([14]). Assume that both $A$ and $B$ are essential, which means that they have no rows or columns identically to zero vector. They yield directed graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$ with a common vertex set $V = \{v_1, \ldots, v_N\}$ and edge sets $E_A$ and $E_B$ respectively, where the edge set $E_A$ consist of $A(i,j)$-edges from the vertex $v_i$ to the vertex $v_j$ and $E_B$ consist of $B(i,j)$-edges from the vertex $v_i$ to the vertex $v_j$. We then have two $C^*$-symbolic dynamical systems $(A_N, \rho^A, E_A)$ and $(A_N, \rho^B, E_B)$ with $A_N = \mathbb{C}^N$. Denote by $s(e), r(e)$ the source vertex and the range vertex of an edge $e$. Put

$$\Sigma^{AB} = \{ (a, b) \in E_A \times E_B \mid r(a) = s(b) \},$$
$$\Sigma^{BA} = \{ (a, \beta) \in E_B \times E_A \mid r(a) = s(\beta) \}.$$ 

Assume that the commutation relation

$$AB = BA$$  \hspace{1cm} (1.2)

holds. We may take a bijection $\kappa : \Sigma^{AB} \rightarrow \Sigma^{BA}$ such that $s(\alpha) = s(a), r(b) = r(\beta)$ holds. We may take a bijection $\kappa : \Sigma^{AB} \rightarrow \Sigma^{BA}$ such that $s(\alpha) = s(a), r(b) = r(\beta)$ for $\kappa(\alpha, b) = (a, \beta)$, which we fix. This situation is called an LR-textile system introduced by Nasu ([16]). We then have a $C^*$-textile dynamical system (see [14])

$$(A_N, \rho^A, \rho^B, E_A, E_B, \kappa).$$

Let us denote by $\mathcal{H}_\kappa^{A,B}$ the associated Hilbert $C^*$-quad module defined in [14]. We set

$$E_\kappa = \{ (a, b, a, \beta) \in E_A \times E_B \times E_B \times E_A | \kappa(a, b) = (a, \beta) \}. \hspace{1cm} (1.3)$$

Each element of $E_\kappa$ is called a tile. Let $X_{A,B}^\kappa \subset (E_\kappa)^{\mathbb{Z}^2}$ be the two-dimensional subshift of the Wang tilings of $E_\kappa$ (cf. [24]). It consists of the two-dimensional configurations $x : \mathbb{Z}^2 \rightarrow E_\kappa$ compatible to their boundary edges on each tile, and is called the subshift of the tiling space for the specification $\kappa : \Sigma^{AB} \rightarrow \Sigma^{BA}$. We say that $X_{A,B}^\kappa$ is transitive if for two tiles $\omega, \omega' \in E_\kappa$, there exists $(\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X_{A,B}^\kappa$ such that $\omega_{0,0} = \omega, \omega_{i,j} = \omega'$ for some $(i, j) \in \mathbb{Z}^2$ with $j < 0 < i$. We set

$$\Omega_\kappa = \{ (\alpha, a) \in E_A \times E_B | s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B \}. \hspace{1cm} (1.4)$$

and define two $|\Omega_\kappa| \times |\Omega_\kappa|$-matrices $A_\kappa$ and $B_\kappa$ with entries in $\{0, 1\}$ by

$$A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \kappa(a, b) = (a, \beta) \text{ for some } \beta \in E_A, \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (1.5)$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$, and

$$B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \kappa(a, b) = (a, \beta) \text{ for some } b \in E_B, \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (1.6)$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. Put the matrix

$$H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}. \hspace{1cm} (1.7)$$
It has been proved in [14] that the $C^*$-algebra $O_{H_{\kappa},A,B}$ associated with the Hilbert $C^*$-quad module $H_{\kappa}^{A,B}$ is isomorphic to the Cuntz-Krieger algebra $O_{H_{\kappa}}$ for the matrix $H_{\kappa}$. In this paper, we first show the following theorem.

**Theorem 1.1** (Theorem 2.9). The subshift $X_{A,B}^\kappa$ of the tiling space is transitive if and if the matrix $H_{\kappa}$ is irreducible. In this case, $H_{\kappa}$ satisfies condition (I), so that the $C^*$-algebra $O_{H_{\kappa},A,B}$ is simple and purely infinite.

We will second see the following theorem.

**Theorem 1.2** (Theorem 2.10). If the matrix $A$ or $B$ is irreducible, the matrix $H_{\kappa}$ is irreducible and satisfies condition (I), so that the $C^*$-algebra $O_{H_{\kappa},A,B}$ is simple and purely infinite.

Let $N,M$ be positive integers with $N,M > 1$. They give $1 \times 1$ commuting matrices $A = [N], B = [M]$. We will present K-theory formulae for the $C^*$-algebras $O_{H_{\kappa}}^{[N],[M]}$ with exchanging specification $\kappa$. The directed graph $G_A$ associated to the matrix $A = [N]$ is a graph consists of $N$-self directed loops denoted by $E_A$ with a vertex denoted by $v$. Similarly the directed graph $G_B$ consists of $M$-self directed loops denoted by $E_B$ with the vertex $v$. We fix a specification $\kappa : E_A \times E_B \rightarrow E_B \times E_A$ defined by exchanging $\kappa(\alpha, \alpha) = (a, \alpha)$ for $(\alpha, a) \in E_A \times E_B$. We will have the following K-theory formulae for the $C^*$-algebra $O_{H_{\kappa}}^{[N],[M]}$. In its computation, the Euclidean algorithm will be used.

**Theorem 1.3.** For integers $1 < N \leq M \in \mathbb{N}$ and a specification $\kappa$ of exchanging directed $N$-loops and $M$-loops, the $C^*$-algebra $O_{H_{\kappa}}^{[N],[M]}$ is a simple purely infinite Cuntz-Krieger algebra whose $K$-groups are

$$K_0(O_{H_{\kappa}}^{[N],[M]}) \cong 0,$$

$$K_1(O_{H_{\kappa}}^{[N],[M]}) \cong \begin{array}{c}
\mathbb{Z}/(N-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N-1)\mathbb{Z} \\
\oplus \mathbb{Z}/(M-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(M-1)\mathbb{Z} \\
\oplus \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/[k_1,k_2,\ldots,k_{j+1}]/(M-1)(M+N-1)\mathbb{Z}
\end{array}$$

where $d = (N-1,M-1)$ the greatest common divisor of $N-1$ and $M-1$, and the sequence $k_0,k_2,\ldots,k_{j+1}$ is the successive integral quotients of $M-1$ by $N-1$ by the Euclidean algorithm, and the integer $[k_1,k_2,\ldots,k_{j+1}]$ is defined by inductively

$[k_0] = 1, \quad [k_1] = k_1, \quad [k_1,k_2] = 1 + k_1k_2, \quad \ldots,$

$[k_1,k_2,\ldots,k_{j+1}] = [k_1,k_2,\ldots,k_j]k_{j+1} + [k_1,\ldots,k_{j-1}]$.

We remark that the $C^*$-algebras studied in this paper are different from the higher rank graph algebras studied by A. Kumjian–D. Pask [7], G. Robertson–T. Steger [21], V. Deaconu [3], and etc.

Throughout the paper, we denote by $\mathbb{N}$ and by $\mathbb{Z}_+$ the set of positive integers and the set of nonnegative integers respectively.
2. Transitivity of the tilings $X^*_A,B$ and simplicity of $\mathcal{O}_{A,B}$

Let $\Sigma$ be a finite set. The two-dimensional full shift over $\Sigma$ is defined to be

$$\Sigma^{\mathbb{Z}^2} = \{(x_{i,j})_{(i,j)\in \mathbb{Z}^2} \mid x_{i,j} \in \Sigma\}.$$ 

An element $x \in \Sigma^{\mathbb{Z}^2}$ is regarded as a function $x : \mathbb{Z}^2 \to \Sigma$ which is called a configuration on $\mathbb{Z}^2$. For a vector $m = (m_1, m_2) \in \mathbb{Z}^2$, let $\sigma^m : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$ be the translation along vector $m$ defined by

$$\sigma^m((x_{i,j})_{(i,j)\in \mathbb{Z}^2}) = (x_{i+m_1,j+m_2})_{(i,j)\in \mathbb{Z}^2}.$$ 

A subset $X \subset \Sigma^{\mathbb{Z}^2}$ is said to be translation invariant if $\sigma^m(X) = X$ for all $m \in \mathbb{Z}^2$. It is obvious to see that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is translation invariant if and only if $X$ is invariant only both horizontally and vertically, that is, $\sigma^{(1,0)}(X) = X$ and $\sigma^{(0,1)}(X) = X$. For $k \in \mathbb{Z}_+$, put

$$[-k,k]^2 = \{(i,j) \in \mathbb{Z}^2 \mid -k \leq i,j \leq k\} = [-k,k] \times [-k,k].$$

A metric $d$ on $\Sigma^{\mathbb{Z}^2}$ is defined by for $x, y \in \Sigma^{\mathbb{Z}^2}$ with $x \neq y$

$$d(x, y) = \frac{1}{2k} \text{ if } x(0,0) = y(0,0),$$

where $k = \max\{k \in \mathbb{Z}_+ \mid x_{[-k,k]^2} = y_{[-k,k]^2}\}$. If $x(0,0) \neq y(0,0)$, put $k = -1$ on the above definition. If $x = y$, we set $d(x, y) = 0$. A two-dimensional subshift $X$ is defined to be a closed, translation invariant subset of $\Sigma^{\mathbb{Z}^2}$ (cf. [9, p.467]). A two-dimensional subshift $X$ is said to have the diagonal property if for $(x_{i,j})_{(i,j)\in \mathbb{Z}^2}, (y_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X$, the conditions $x_{i,j} = y_{i,j}, x_{i+1,j-1} = y_{i+1,j-1}$ imply $x_{i,j-1} = y_{i,j-1}, x_{i+1,j} = y_{i+1,j}$ (see [13]). The diagonal property has the following property: for $x \in X$ and $(i,j) \in \mathbb{Z}^2$, the configuration $x$ is determined by the diagonal line $(x_{i+n,j-n})_{n \in \mathbb{Z}}$ through $(i,j)$.

We henceforth go back to our previous situation of $C^*$-textile dynamical system $(\Lambda_\kappa, \rho^A, \rho^B, E_A, E_B, \kappa)$ coming from $N \times N$ commuting matrices $A$ and $B$ with specification $\kappa$ as in Section 1. We always assume that both matrices $A$ and $B$ are essential. It yields a two-dimensional subshift $X^*_A,B$ as follows: Let $\Sigma$ be the set $E_{\kappa}$ of tiles defined in (1.3). For $\omega = (\alpha, b, a, \beta) \in E_{\kappa}$, define maps $t (= \text{top}), b (= \text{bottom}) : E_{\kappa} \to E_A$ and $l (= \text{left}), r (= \text{right}) : E_{\kappa} \to E_B$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b$$

as in the following figure:

```
  o --t(\omega)--> o
  |     \       |
  |   a(\omega) \     \ b(\omega)
  o --r(\omega)--> o
```

A configuration $(\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in E^*_{\kappa}$ is said to be paired if the conditions

$$t(\omega_{i,j}) = b(\omega_{i,j+1}), \quad r(\omega_{i,j}) = l(\omega_{i+1,j}), \quad l(\omega_{i,j}) = r(\omega_{i-1,j}), \quad b(\omega_{i,j}) = t(\omega_{i,j-1})$$

hold for all $(i,j) \in \mathbb{Z}^2$. Let $X^*_{A,B}$ be the set of all paved configurations $(\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in E^*_{\kappa}$ . It consists of the Wang tilings of the tiles of $E_{\kappa}$ (see [24]). The following proposition is easy.
Proposition 2.1. \( X_\kappa^{A,B} \) is a two-dimensional subshift having diagonal property.

Let \( e_\omega, \omega \in E_\kappa \) be the standard basis of \( C^{|E_\kappa|} \). Put the projection \( E_\omega = \rho_b \circ \rho_a(1) = \rho_\alpha(1) \) for \( \omega = (\alpha, b, a, \beta) \in E_\kappa \). We set

\[
\mathcal{H}_\kappa^{A,B} = \sum_{\omega \in E_\kappa} e_\omega \otimes E_\omega A_N.
\]

Then \( \mathcal{H}_\kappa^{A,B} \) has a natural structure of not only Hilbert \( C^* \)-right module over \( A_N \) but also two other Hilbert \( C^* \)-bimodule structure, called Hilbert \( C^* \)-quad module. By two-dimensional analogue of Pimsner’s construction of Hilbert \( C^* \)-bimodule algebra ([19]), we have introduced a \( C^* \)-algebra \( \mathcal{O}_{H_\kappa^{A,B}} \) (see [14] for detail construction).

Let \( \Omega_\kappa \) be the subset of \( E_A \times E_B \) defined in (1.4). We define two \( |\Omega_\kappa| \times |\Omega_\kappa| \)-matrices \( A_\kappa \) and \( B_\kappa \) with entries in \( \{0, 1\} \) as in (1.5) and (1.6). The matrices \( A_\kappa \) and \( B_\kappa \) represent the concatenations of edges as in the following figures respectively:

\[
\begin{array}{cccc}
  o & \rightarrow & \alpha & \rightarrow & o & \rightarrow & \delta \\
  a & \downarrow & b & \downarrow & & & \\
  o & \rightarrow & o & \rightarrow & o & \rightarrow & o
\end{array}
\]

if \( A_\kappa((\alpha, a), (\delta, b)) = 1 \),

and

\[
\begin{array}{cccc}
  o & \rightarrow & \alpha & \rightarrow & o \\
  a & \downarrow & & \downarrow & b & \downarrow & d & \downarrow & o & \rightarrow & o
\end{array}
\]

if \( B_\kappa((\alpha, a), (\beta, d)) = 1 \).

Let \( H_\kappa \) be the \( 2|\Omega_\kappa| \times 2|\Omega_\kappa| \) matrix defined in (1.7). We have proved the following result in [14].

Theorem 2.2. The \( C^* \)-algebra \( \mathcal{O}_{H_\kappa^{A,B}} \) associated with Hilbert \( C^* \)-quad module \( \mathcal{H}_\kappa^{A,B} \) defined by commuting matrices \( A, B \) and a specification \( \kappa \) is isomorphic to the Cuntz-Krieger algebra \( \mathcal{O}_{H_\kappa} \) for the matrix \( H_\kappa \). Its K-groups \( K_*(\mathcal{O}_{H_\kappa}) \) are computed as

\[
\begin{align*}
K_0(\mathcal{O}_{H_\kappa}) &= \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n, \\
K_1(\mathcal{O}_{H_\kappa}) &= \text{Ker}(A_\kappa + B_\kappa - I_n) \text{ in } \mathbb{Z}^n,
\end{align*}
\]

where \( n = |\Omega_\kappa| \).

We will study a relationship between transitivity of the tiling space \( X_\kappa^{A,B} \) and simplicity of the \( C^* \)-algebra \( \mathcal{O}_{H_\kappa^{A,B}} \). An essential matrix with entries in \( \{0, 1\} \) is said to satisfy condition (I) (in the sense of [2]) if the shift space defined by the topological Markov chain for the matrix is homeomorphic to a Cantor discontinuum. The condition is equivalent to the condition that every loop in the associated directed graph has an exit ([8]). It is a fundamental result that a Cuntz-Krieger algebra is simple and purely infinite if the underlying matrix is irreducible and satisfies condition (I) ([2]). We will find a condition of the two-dimensional subshift \( X_\kappa^{A,B} \) of the tiling space under which the matrix \( H_\kappa \) is irreducible and satisfies...
condition (I). Hence the condition yields the simplicity and purely infiniteness of the algebra $O_{\mathcal{H}_{A,B}}$.

We are assuming that both of the matrices $A$ and $B$ are essential. Then we have

**Lemma 2.3.** Both of the matrices $A_\kappa$ and $B_\kappa$ are essential.

**Proof.** For $(\alpha, a) \in \Omega_\kappa$, by definition of $\Omega_\kappa$, there exist $\beta \in E_A$ and $b \in E_B$ such that $\kappa(\alpha, b) = (\alpha, \beta)$. Since $A$ is essential, one may take $\beta_1 \in E_A$ such that $s(\beta_1) = r(b)(= s(\beta))$. Hence $(\beta, \beta_1) \in \Sigma^{BA}$. Put $(\alpha_1, b_1) = \kappa^{-1}(\beta, \beta_1) \in \Sigma^{AB}$ so that $(\alpha_1, b_1) \in \Omega_\kappa$ and $A_\kappa((\alpha, a), (\alpha_1, b)) = 1$ as in the following figure:

For $(\delta, b) \in \Omega_\kappa$ there exists $\alpha \in E_A$ such that $r(\alpha) = s(\delta)(= s(b))$ because $A$ is essential. Hence $(\alpha, b) \in \Sigma^{AB}$. Put $(\alpha, \beta) = \kappa(\alpha, b)$ so that $(\alpha, a) \in \Omega_\kappa$ and $A_\kappa((\alpha, a), (\delta, b)) = 1$ as in the following figure:

Therefore one sees that $A_\kappa$ is essential, and similarly that $B_\kappa$ is essential. \hfill \Box

Hence we have

**Proposition 2.4.** The matrix $H_\kappa$ is essential and satisfies condition (I).

**Proof.** By the previous lemma, both of the matrices $A_\kappa$ and $B_\kappa$ are essential. Hence every row of $A_\kappa$ and of $B_\kappa$ has at least one 1. Since

$$H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix},$$

every row of $H_\kappa$ has at least two 1’s. This implies that a loop in the directed graph associated to the matrix $H_\kappa$ must has an exit so that the matrix $H_\kappa$ satisfies condition (I). \hfill \Box

For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, and $C, D = A$ or $B$, we have

$$[C_\kappa D_\kappa]((\alpha, a), (\alpha', a')) = \sum_{(\alpha_1, a_1) \in \Omega_\kappa} C_\kappa((\alpha, a), (\alpha_1, a_1))D_\kappa((\alpha_1, a_1), (\alpha', a')).$$

Hence $[A_\kappa A_\kappa]((\alpha, a), (\alpha', a')) \neq 0$ if and only if there exists $(\alpha_1, a_1) \in \Omega_\kappa$ such that $\kappa(\alpha, a_1) = (\alpha, \beta)$ for some $\beta \in E_A$ and $\kappa(\alpha_1, a') = (a_1, \beta_1)$ for some $\beta_1 \in E_A$ as in the following figure:

\[\begin{array}{cccc}
\circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\alpha_1} \circ \\
\downarrow a & & \downarrow b & \downarrow b_1 \\
\circ & \xrightarrow{\beta} & \circ & \xrightarrow{\beta_1} \circ
\end{array}\]
And also \([A_\kappa B_\kappa)((a, a'), (a', a')) \neq 0\) if and only if there exists \((\alpha_1, a_1) \in \Omega_\kappa\) such that \(\kappa(\alpha, a_1) = (a, \beta)\) for some \(\beta \in E_A\) and \(\kappa(\alpha_1, b_1) = (a_1, \alpha')\) for some \(b_1 \in E_B\) as in the following figure:

\[
\begin{array}{c}
\circ \stackrel{\alpha}{\rightarrow} \circ \\
a \downarrow \alpha_1 \downarrow b_1 \\
\circ \stackrel{\beta}{\rightarrow} \circ \\
a' \downarrow
\end{array}
\]

Similarly \([B_\kappa A_\kappa)((a, a'), (a', a')) \neq 0\) if and only if there exists \((\alpha_1, a_1) \in \Omega_\kappa\) such that \(\kappa(\alpha, b) = (a, a_1)\) for some \(b \in E_B\) and \(\kappa(\alpha_1, a') = (a_1, \beta_1)\) for some \(\beta_1 \in E_A\) as in the following figure:

\[
\begin{array}{c}
\circ \stackrel{\alpha}{\rightarrow} \circ \\
a \downarrow \alpha_1 \downarrow a' \\
\circ \stackrel{\beta}{\rightarrow} \circ \\
a' \downarrow
\end{array}
\]

And also \([B_\kappa B_\kappa)((\alpha, a), (\alpha', a')) \neq 0\) if and only if there exists \((\alpha_1, a_1) \in \Omega_\kappa\) such that \(\kappa(\alpha, b) = (a, \alpha_1)\) for some \(b \in E_B\) and \(\kappa(\alpha_1, b_1) = (a_1, \alpha')\) for some \(b_1 \in E_B\) as in the following figure:

\[
\begin{array}{c}
\circ \stackrel{\alpha}{\rightarrow} \circ \\
a \downarrow b \\
\circ \stackrel{\alpha_1}{\rightarrow} \circ \\
a_1 \downarrow b_1 \\
\circ \stackrel{\alpha'}{\rightarrow} \circ \\
a' \downarrow
\end{array}
\]

**Lemma 2.5.** \(A_\kappa B_\kappa = B_\kappa A_\kappa\).

**Proof.** For \((\alpha, a), (\alpha', a') \in \Omega_\kappa\), we have \([A_\kappa B_\kappa)((\alpha, a), (\alpha', a')) = m\) if and only if there exist \((\alpha_i, a'_i) \in \Omega_\kappa, i = 1, \ldots, m\) such that \(\kappa(\alpha, a'_i) = (a, \beta_i)\) for some \(\beta_i \in E_A\) and \(\kappa(\alpha_i, b_i) = (a'_i, \alpha')\) for some \(b_i \in E_B\) as in the following figure:

\[
\begin{array}{c}
\circ \stackrel{\alpha}{\rightarrow} \circ \stackrel{\alpha_1}{\rightarrow} \circ \\
a \downarrow \alpha_1 \downarrow b_1 \\
\circ \stackrel{\beta_i}{\rightarrow} \circ \stackrel{\alpha'}{\rightarrow} \circ \\
a' \downarrow
\end{array}
\]
Put $(a_i, \beta'_i) = \kappa(\beta_i, a')$. We then have $(\beta_i, a_i) \in \Omega_{\kappa}$ as in the following figure:

```
  o ---------------------- o
  |                       |
  a |     a'                 |
  |     |                   |
  o --------> o
  |                       |
  β_i |     α'                 |
  |     |                   |
  o --------> o
  |                       |
  a_i |     a'                 |
  |     |                   |
  o --------> o
  |                       |
  β'_i |     α'                 |
```

If $(\beta_i, a_i) = (\beta_j, a_j)$ in $\Omega_{\kappa}$, then we have $\beta_i = \beta_j$ so that $a'_i = a'_j$ and hence $\alpha_i = \alpha_j$. Therefore we have $[B_{\kappa}A_{\kappa}]((\alpha, a), (\alpha', a')) = m$. \hfill \Box

**Lemma 2.6.** The following four conditions are equivalent.

(i) The matrix $H_{\kappa}$ is irreducible.

(ii) For $(\alpha, a), (\alpha', a') \in \Omega_{\kappa}$, there exist $n, m \in \mathbb{Z}_+$ such that

\[
A_{\kappa}(A_{\kappa} + B_{\kappa})^{n}((\alpha, a), (\alpha', a')) > 0, \quad B_{\kappa}(A_{\kappa} + B_{\kappa})^{m}((\alpha, a), (\alpha', a')) > 0.
\]

(iii) The matrix $A_{\kappa} + B_{\kappa}$ is irreducible.

(iv) For $(\alpha, a), (\alpha', a') \in \Omega_{\kappa}$, there exists a paved configuration $(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X^{A,B}_{\kappa}$ such that

\[
t(\omega_{0,0}) = \alpha, \quad l(\omega_{0,0}) = a, \quad t(\omega_{i,j}) = \alpha', \quad l(\omega_{i,j}) = a'
\]

for some $(i, j) \in \mathbb{Z}^2$ with $j < 0 < i$.

**Proof.** (i) $\iff$ (ii): The identity

\[
H_{\kappa}^{n} = \begin{bmatrix}
A_{\kappa}(A_{\kappa} + B_{\kappa})^{n} & A_{\kappa}(A_{\kappa} + B_{\kappa})^{n} \\
B_{\kappa}(A_{\kappa} + B_{\kappa})^{n} & B_{\kappa}(A_{\kappa} + B_{\kappa})^{n}
\end{bmatrix}
\]

implies the equivalence between (i) and (ii).

(ii) $\implies$ (iii): Suppose that for $(\alpha, a), (\alpha', a') \in \Omega_{\kappa}$, there exists $n \in \mathbb{Z}_+$ such that

\[
A_{\kappa}(A_{\kappa} + B_{\kappa})^{n}((\alpha, a), (\alpha', a')) > 0
\]

so that

\[
(A_{\kappa} + B_{\kappa})^{n+1}((\alpha, a), (\alpha', a')) > 0.
\]

Hence the matrix $A_{\kappa} + B_{\kappa}$ is irreducible.

(iii) $\implies$ (ii): As $A_{\kappa}$ and $B_{\kappa}$ are both essential, for $(\alpha, a), (\alpha', a') \in \Omega_{\kappa}$ there exists $(\alpha_1, a_1), (\alpha_2, a_2) \in \Omega_{\kappa}$ such that

\[
A_{\kappa}((\alpha, a), (\alpha_1, a_1)) = 1, \quad B_{\kappa}((\alpha, a), (\alpha_2, a_2)) = 1.
\]

Since $A_{\kappa} + B_{\kappa}$ is irreducible, there exist $a, m \in \mathbb{Z}_+$ such that

\[
(A_{\kappa} + B_{\kappa})^{n}((\alpha_1, a_1), (\alpha', a')) > 0, \quad (A_{\kappa} + B_{\kappa})^{m}((\alpha_2, a_2), (\alpha', a')) > 0.
\]

Hence we have

\[
A_{\kappa}(A_{\kappa} + B_{\kappa})^{n}((\alpha, a), (\alpha', a')) > 0, \quad B_{\kappa}(A_{\kappa} + B_{\kappa})^{m}((\alpha, a), (\alpha', a')) > 0.
\]

(ii) $\implies$ (iv): For $(\alpha, a), (\alpha', a') \in \Omega_{\kappa}$, take $(\alpha_1, a_1) \in \Omega_{\kappa}$ and $\beta \in E_A$ such that $\kappa(\alpha, a_1) = (\alpha, \beta)$. By (ii), there exists $m \in \mathbb{Z}_+$ with $B_{\kappa}(A_{\kappa} + B_{\kappa})^{m}((\alpha, a), (\alpha', a')) > 0$. One may take $b' \in E_B$ and $\beta' \in E_A$ satisfying $\kappa(\alpha', b') = (\alpha', \beta')$, so that there
exists a paved configuration \((\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X_{A,B}^\kappa\) such that \(\omega_{0,0} = (\alpha, a_1, a, \beta)\) and \(\omega_{i,j} = (\alpha', b, a', \beta')\) for some \((i, j) \in \mathbb{Z}^2\) with \(j < 0 < i\) as in the following figure:

\[
\begin{array}{c}
\circ \xrightarrow{\alpha} \circ \xrightarrow{\alpha_1} \circ \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\circ \xrightarrow{\beta} \circ \xrightarrow{\beta_1} \circ \\
\end{array}
\]

(iv) \(\implies\) (ii): The assertion is clear. \(\square\)

**Definition.** A two-dimensional subshift \(X_{A,B}^\kappa\) is said to be **transitive** if for two tiles \(\omega, \omega' \in E_\kappa\) there exists a paved configuration \((\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X_{A,B}^\kappa\) such that \(\omega_{0,0} = \omega\) and \(\omega_{i,j} = \omega'\) for some \((i, j) \in \mathbb{Z}^2\) with \(j < 0 < i\).

**Theorem 2.7.** The subshift \(X_{A,B}^\kappa\) of the tiling space is transitive if and only if the matrix \(H_\kappa\) is irreducible.

**Proof.** Assume that the matrix \(H_\kappa\) is irreducible. Hence the condition (iv) in Lemma 2.6 holds. Let \(\omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa\) be two tiles. Since \(A\) is essential, there exists \(\beta_1 \in E_A\) such that \(r(\beta)(= r(b)) = s(\beta_1)\), so that \((b, \beta_1) \in \Sigma^{B,A}\). One may take \((a_1, b_1) \in \Sigma^{A,B}\) such that \(\kappa(a_1, b_1) = (b, \beta_1)\) and hence \((a_1, b) \in \Omega_\kappa\) as in the following figure:

\[
\begin{array}{c}
\circ \xrightarrow{\alpha} \circ \xrightarrow{\alpha_1} \circ \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\circ \xrightarrow{\beta} \circ \xrightarrow{\beta_1} \circ \\
\end{array}
\]

For \((a_1, b), (\alpha', a') \in \Omega_\kappa\), by (iv) in Lemma 2.6, there exists \((\omega_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X_{A,B}^\kappa\) such that \(t(\omega_{0,0}) = a_1, l(\omega_{0,0}) = b, t(\omega_{i,j}) = \alpha', l(\omega_{i,j}) = a'\) for some \((i, j) \in \mathbb{Z}^2\) with \(j < 0 < i\). Since \(X_{A,B}^\kappa\) has diagonal property, there exists a paved configuration \((\omega'_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X_{A,B}^\kappa\) such that \(\omega'_{0,0} = \omega, \omega'_{i,j} = \omega'\). Hence \(X_{A,B}^\kappa\) is transitive.

Conversely assume that \(X_{A,B}^\kappa\) is transitive. For \((\alpha, a), (\alpha', a') \in \Omega_\kappa\), there exist \(b, b' \in E_B\) and \(\beta, \beta' \in E_A\) such that \(\omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa\). It is clear that the transitivity of \(X_{A,B}^\kappa\) implies the condition (iv) in Lemma 2.6, so that \(H_\kappa\) is irreducible. \(\square\)

**Lemma 2.8.** If \(A\) or \(B\) is irreducible, \(X_{A,B}^\kappa\) is transitive.
Proof. Suppose that the matrix \( A \) is irreducible. For two tiles \( \omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa \), there exist concatenated edges \((\beta, \beta_1, \ldots, \beta_n, \alpha')\) in the graph \( G_A \) for some edges \( \beta_1, \ldots, \beta_n \in E_A \). Since \( X_A^{\kappa} \) has diagonal property, there exists a configuration \((\omega_i,j)_{i \in \mathbb{Z}} \in X_A^{\kappa} \) such that \( \omega' = \omega_{i,j} \) for some \( i > 0, j = -1 \). Hence \( X_A^{\kappa} \) is transitive. \( \square \)

Since the \( C^* \)-algebra \( \mathcal{O}_{H_\kappa} \) is isomorphic to the Cuntz-Krieger algebra \( \mathcal{O}_{H_\kappa} \) by [14], we see the following theorems.

**Theorem 2.9.** The subshift \( X_A^{\kappa} \) of the tiling space is transitive if and if the matrix \( H_\kappa \) is irreducible. In this case, \( H_\kappa \) satisfies condition (I). Hence if the subshift \( X_A^{\kappa} \) of the tiling space is transitive, the \( C^* \)-algebra \( \mathcal{O}_{H_\kappa} \) is simple and purely infinite.

By Lemma 2.8, we have

**Theorem 2.10.** If the matrix \( A \) or \( B \) is irreducible, the matrix \( H_\kappa \) is irreducible and satisfies condition (I), so that the \( C^* \)-algebra \( \mathcal{O}_{H_\kappa} \) is simple and purely infinite.

3. The algebra \( \mathcal{O}_{H^{[N],[M]}} \) for two positive integers \( N, M \)

Let \( N, M \) be positive integers with \( N, M > 1 \). They give \( 1 \times 1 \) commuting matrices \( A = [N], B = [M] \). We will present K-theory formulae for the \( C^* \)-algebras \( \mathcal{O}_{H^{[N],[M]}} \) with exchanging specification \( \kappa \). In the computations below, we will use Euclidean algorithm to find order of the torsion part of the \( K_0 \)-group. The directed graph \( G_A \) for the matrix \( A = [N] \) is a graph consists of \( N \)-self directed loops with a vertex denote by \( v \). The \( N \)-self directed loops are denoted by \( E_A \). Similarly the directed graph \( G_B \) for \( B = [M] \) consists of \( M \)-self directed loops denoted by \( E_B \) with the vertex \( v \). We fix a specification \( \kappa : E_A \times E_B \rightarrow E_B \times E_A \) defined by exchanging \( \kappa(\alpha, a) = (\alpha, a) \) for \( (\alpha, a) \in E_A \times E_B \). Hence \( \Omega_\kappa = E_A \times E_B \) so that \( |\Omega_\kappa| = |E_A| \times |E_B| = N \times M \). We then know \( A_\kappa((\alpha, a), (\beta, b)) = 1 \) if and only if \( b = a \) and \( B_\kappa((\alpha, a), (\beta, d)) = 1 \) if and only if \( \beta = \alpha \) as in the following figures respectively.

\[
\begin{array}{ccc}
\circ & \rightarrow & \circ \\
\downarrow & \alpha \downarrow & \delta \\
\circ & \rightarrow & \circ \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\circ & \rightarrow & \circ \\
\downarrow & \alpha \downarrow & \beta \\
\circ & \rightarrow & \circ \\
\end{array}
\]

In [14], the K-groups for the case \( N = 2 \) and \( M = 3 \) have been computed such that

\[
K_0(\mathcal{O}_{H^{[2],[3]}}) \cong \mathbb{Z}/8\mathbb{Z}, \quad K_1(\mathcal{O}_{H^{[2],[3]}}) \cong 0.
\]

We will generalize the above computations.

Let \( I_n \) be the \( n \times n \) identity matrix and \( E_n \) the \( n \times n \) matrix whose entries are all 1’s. For an \( N \times N \)-matrix \( C = [c_{i,j}]_{i,j=1}^N \) and an \( M \times M \)-matrix \( D = [d_{k,l}]_{k,l=1}^M \),
denote by $C \otimes D$ the $NM \times NM$ matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \ldots & c_{1N}D \\ c_{21}D & c_{22}D & \ldots & c_{2N}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}D & c_{N2}D & \ldots & c_{NN}D \end{bmatrix}.$$ 

Hence we have

$$E_N \otimes I_M = \begin{bmatrix} I_M & I_M & \ldots & I_M \\ I_M & I_M & \ldots & I_M \\ \vdots & \vdots & \ddots & \vdots \\ I_M & I_M & \ldots & I_M \end{bmatrix}, \quad I_N \otimes E_M = \begin{bmatrix} E_M & 0 & \ldots & 0 \\ 0 & E_M & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & E_M \end{bmatrix}.$$

We denote by $E_{[N]} = \{\alpha_1, \ldots, \alpha_N\}, E_{[M]} = \{a_1, \ldots, a_M\}$. As $\Omega_\kappa = E_{[N]} \times E_{[M]}$, the basis of $\mathbb{C}^N \otimes \mathbb{C}^M$ are ordered lexicographically from left as in the following way:

$$(\alpha_1, a_1), \ldots, (\alpha_1, a_M), (\alpha_2, a_1), \ldots, (\alpha_2, a_M), \ldots, (\alpha_N, a_1), \ldots, (\alpha_N, a_M) \quad (3.1)$$

Let $A_\kappa$ and $B_\kappa$ be the matrices defined in the previous section for the matrices $A = [N], B = [M]$ with exchanging specification $\kappa$. The following lemma is direct.

**Lemma 3.1.** The matrices $A_\kappa, B_\kappa$ are written as

$$A_\kappa = E_N \otimes I_M, \quad B_\kappa = I_N \otimes E_M$$

along the ordered basis (3.1). Hence we have

$$A_\kappa + B_\kappa - I_{NM} = \begin{bmatrix} E_M & I_M & \ldots & I_M \\ I_M & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_M \\ I_M & \ldots & I_M & E_M \end{bmatrix}. \quad (3.2)$$

We denote by $H_0$ the matrix $A_\kappa + B_\kappa - I_{NM}$. By Theorem 2.2, the $K$-groups of the algebra $O_{H_0}^{[N],[M]}$ are given by the kernel $\text{Ker}(H_0)$ and the cokernel $\text{Coker}(H_0)$ of the matrix $H_0$ in $\mathbb{Z}^{NM}$. We will transform $H_0$ preserving isomorphism classes of the groups $\text{Ker}(H_0)$ and $\text{Coker}(H_0)$ in $\mathbb{Z}^{NM}$ by the following operations called elementary operations on the matrix.

(A) Exchange two rows or two columns.
(B) Multiply a row or column by $-1$.
(C) Add an integer multiple of one row to another row, or of one column to another column.
(D) Add a row vector obtained by multiplication of an invertible matrix over $\mathbb{Z}$ of one row to another row, or of one column to another column.

The isomorphism classes of the groups of its kernel and its cokernel do not change under the elementary operations on the matrix. We will successively apply the above elementary operations to the matrix $H_0$ to obtain a diagonal matrix as in the following way.

1. Add the minus of the $(i+1)$-th row to the $i$-th row in order for $i = 1, \ldots, N-1$ in $H_0$ to obtain the matrix below denoted by $H_1$:
$$H_1 = \begin{bmatrix}
E_M - I_M & I_M - E_M & 0 & \cdots & 0 \\
0 & E_M - I_M & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & I_M - E_M & 0 \\
0 & \cdots & 0 & E_M - I_M & I_M - E_M \\
I_M & \cdots & \cdots & I_M & E_M
\end{bmatrix}$$

(2) Add the $i$-th row to the $(i+1)$-th row in order for $i = 1, \ldots, N-1$ in $H_1$ to obtain the matrix below denoted by $H_2$:

$$H_2 = \begin{bmatrix}
E_M - I_M & I_M - E_M & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & I_M - E_M & 0 \\
E_M - I_M & 0 & \cdots & 0 & I_M - E_M \\
E_M & I_M & \cdots & I_M & I_M
\end{bmatrix}$$

(3) Add the $E_M - I_M$ multiplication of the $N$-th row to the $(N-1)$-th row in $H_2$ to obtain the matrix below denoted by $H_3$:

$$H_3 = \begin{bmatrix}
E_M - I_M & I_M - E_M & 0 & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
E_M - I_M & 0 & \cdots & 0 & I_M - E_M & 0 \\
E_M - I_M & E_M - I_M & \cdots & E_M - I_M & 0 \\
E_M & I_M & \cdots & I_M & I_M
\end{bmatrix}$$

(4) Add the $i$-th row to the $(i-1)$-th row in order for $i = N-1, \ldots, 2$ in $H_3$ to obtain the matrix below denoted by $H_4$:

$$H_4 = \begin{bmatrix}
p_M(N-1) & 0 & \cdots & \cdots & 0 \\
p_M(N-2) & E_M - I_M & \ddots & \ddots & \vdots \\
p_M(2) & \vdots & \ddots & \ddots & \vdots \\
p_M(1) & E_M - I_M & \cdots & E_M - I_M & 0 \\
E_M & I_M & \cdots & I_M & I_M
\end{bmatrix}$$

where $p_M(i) = E_M^2 + (i-1)E_M - iI_M = (E_M + iI_M)(E_M - I_M)$ for $i = 1, \ldots, N-1$.

(5) Add the minus of the $j$-th column to the $(j-1)$-th column in order for $j = N, \ldots, 3$ and the $-E_M$ multiplication of the second column to the first column
in $H_4$ to obtain the matrix below denoted by $H_5$:

$$H_5 = \begin{bmatrix}
  p_M(N-1) & 0 & \cdots & 0 \\
p_M(N-2) & E_M - I_M \\
  \vdots & \ddots & \ddots & \vdots \\
p_M(2) & & E_M - I_M & 0 \\
p_M(1) & E_M - I_M & \cdots & E_M - I_M & 0 \\
  0 & 0 & \cdots & 0 & I_M
\end{bmatrix}.$$  

(6) Add the minus of the $(N-1)$-th column to the $j$-th column in order for $j = N-2, \ldots, 2$ and the $-(E_M + I_M)$ multiplication of the $(N-1)$-th column to the first column in $H_5$ to obtain the matrix below denoted by $H_6$:

$$H_6 = \begin{bmatrix}
p_M(N-1) & 0 & \cdots & 0 \\
p_M(N-2) & E_M - I_M \\
  \vdots & 0 & \ddots & \vdots \\
p_M(2) & & E_M - I_M & 0 \\
  0 & 0 & \cdots & 0 & I_M
\end{bmatrix}.$$  

(7) Add the $-(E_M + (N-j)I_M)$ multiplication of the $j$-th column to the first column in order for $j = N-1, \ldots, 2$ in $H_6$ to obtain the diagonal matrix below denoted by $H_7$:

$$H_7 = \begin{bmatrix}
p_M(N-1) & 0 & \cdots & 0 \\
  0 & E_M - I_M \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & E_M - I_M & 0 \\
  0 & \cdots & 0 & I_M
\end{bmatrix}.$$  

As $E_M^2 = M E_M$, we have $p_M(N-1) = (M + N - 2)E_M - (N-1)I_M$. We thus have

**Lemma 3.2.**

$$\text{Ker}(A_{\kappa} + B_{\kappa} - I_{NM}) \text{ in } \mathbb{Z}^{NM} \cong 0$$

and

$$\text{Coker}(A_{\kappa} + B_{\kappa} - I_{NM}) \text{ in } \mathbb{Z}^{NM}$$

$$\cong \left(\mathbb{Z}^{M}/(E_M - I_M)\mathbb{Z}^{M} \oplus \cdots \oplus \mathbb{Z}^{M}/(E_M - I_M)\mathbb{Z}^{M}\right)^{(N-2)} \oplus \mathbb{Z}^{M}/((M + N - 2)E_M - (N-1)I_M)\mathbb{Z}^{M}.$$  

**Proof.** It is straightforward to see that the matrix $A_{\kappa} + B_{\kappa} - I_{NM}$ is invertible by the formula (3.2). Since

$$\text{Coker}(A_{\kappa} + B_{\kappa} - I_{NM}) \text{ in } \mathbb{Z}^{NM} \cong \mathbb{Z}^{NM}/H_7\mathbb{Z}^{NM},$$

the formula for the cokernel is obvious.  

We will next compute the following groups to compute $\text{Coker}(A_\kappa + B_\kappa - I_{NM})$ in $\mathbb{Z}^{NM}$.

(i) $\mathbb{Z}^{M}/(E_M - I_M)\mathbb{Z}^M$.
(ii) $\mathbb{Z}^{M}/((M + N - 2)E_M - (N - 1)I_M)\mathbb{Z}^M$

(i) As the matrix $E_M - I_M$ is of the form
\[
\begin{bmatrix}
0 & 1 & \ldots & 1 \\
1 & 0 & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
1 & \ldots & 1 & 0
\end{bmatrix}
\]
by the same operations (1), (2) to get the matrix $H_2$ from $H_0$, the matrix $E_M - I_M$ goes to the matrix
\[
\begin{bmatrix}
-1 & 1 & 0 & \ldots & 0 \\
-1 & 0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 1 & 0 \\
-1 & 0 & \ldots & 0 & 1 \\
0 & 1 & \ldots & 1 & 1
\end{bmatrix}
\]
Add the minus of the $i$-th row to the $M$-th row in order for $i = 1, \ldots, M - 1$, we have the matrix
\[
\begin{bmatrix}
-1 & 1 & 0 & \ldots & 0 \\
-1 & 0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 1 & 0 \\
-1 & 0 & \ldots & 0 & 1 \\
M - 1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]
Add the $j$-th column to the first column for $j = 2, \ldots, M$, we have the matrix
\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
M - 1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]
which goes to the diagonal matrix with diagonal entries $[1, 1, \ldots, 1, M - 1]$ by exchanging rows. Hence we see that
\[
\mathbb{Z}^{M}/(E_M - I_M)\mathbb{Z}^M \cong \mathbb{Z}/(M - 1)\mathbb{Z}. \quad (3.3)
\]

(ii) Put $e = (M + N - 2) - (N - 1) = M - 1$ and $f = M + N - 2$. Then we have
\[
(M + N - 2)E_M - (N - 1)I_M = \begin{bmatrix}
e & f & \ldots & f \\
f & e & \ddots & \\
\vdots & \ddots & \ddots & f \\
f & \ldots & f & e
\end{bmatrix}. \quad (3.4)
\]
By a similar manner to the preceding operations from $H_1$ to $H_5$, one obtains the following matrix denoted by $L_2$ from the matrix (3.4)

$$L_2 = \begin{bmatrix}
e - f & f - e & 0 & \ldots & 0 \\
e - f & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & f - e & 0 \\
e - f & 0 & \ldots & 0 & f - e \\
e & f & \ldots & f & f
\end{bmatrix}.$$  

Add the $j$-th column to the first column for $j = 2, \ldots, M$ to obtain the matrix below denoted by $L_3$:

$$L_3 = \begin{bmatrix}
0 & f - e & 0 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & f - e & 0 \\
0 & 0 & \ldots & 0 & f - e \\
e + (M - 1)f & f & \ldots & f & f
\end{bmatrix}.$$  

Exchange columns to obtain the matrix below denoted by $L_4$:

$$L_4 = \begin{bmatrix}
f - e & 0 & \ldots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & f - e & 0 & 0 \\
0 & \ldots & 0 & f - e & 0 \\
f & \ldots & f & f & e + (M - 1)f
\end{bmatrix}.$$  

Add the minus of the $j$-th column to the $(j - 1)$-th column in order for $j = 2, \ldots, M - 1$ to obtain the matrix below denoted by $L_5$:

$$L_5 = \begin{bmatrix}
f - e & 0 & \ldots & \ldots & 0 \\
e - f & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & e - f & f - e & 0 \\
0 & \ldots & 0 & f & e + (M - 1)f
\end{bmatrix}.$$  

Add the $i$-th row to the $i + 1$-th row in order for $i = 1, \ldots, M - 2$ to obtain the matrix below denoted by $L_6$:

$$L_6 = \begin{bmatrix}
f - e & 0 & \ldots & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & f - e & 0 \\
0 & \ldots & 0 & f & e + (M - 1)f
\end{bmatrix}.$$  

Put the $2 \times 2$ matrix $L_{(N,M)}$ by setting

$$L_{(N,M)} = \begin{bmatrix} f - e & 0 \\ f & e + (M - 1)f \end{bmatrix}.$$  

As $f - e = N - 1$, we have the following lemma with (3.3).
Lemma 3.3.

(i) \( \mathbb{Z}^M / (E_M - I_M) \mathbb{Z}^M \cong \mathbb{Z} / (M - 1) \mathbb{Z} \).

(ii) \( \mathbb{Z}^M / ((M + N - 2)E_M - (N - 1)I_M) \mathbb{Z}^M \cong \mathbb{Z} / (N - 1) \oplus \cdots \oplus \mathbb{Z} / (N - 1) \oplus \mathbb{Z}^2 / L(N,M) \mathbb{Z}^2 \).

It remains to compute the group \( \mathbb{Z}^2 / L(N,M) \mathbb{Z}^2 \). Put \( n = N - 1, m = M - 1 \). As \( f - e = n \) and \( f = m + n \), we have \( e + (M-1)f = (M-1)(M+N-1) = m(m+n+1) \) so that

\[
L(N,M) = \begin{bmatrix}
    n & 0 \\
    m & m(m+n+1)
\end{bmatrix}.
\]

Add the minus of the first row to the second row in \( L(N,M) \) to obtain the matrix below denoted by \( L_{n,m} \):

\[
L_{n,m} = \begin{bmatrix}
    n & 0 \\
    m & m(m+n+1)
\end{bmatrix}.
\]

We may assume that \( M \geq N \) and hence \( m \geq n \).

If \( m \) is divided by \( n \) and hence there exists \( k \in \mathbb{N} \) such that \( m = nk \), by adding the \(-k\) multiplication of the first row to the second row in \( L_{n,m} \), the matrix goes to the diagonal matrix:

\[
\begin{bmatrix}
    n & 0 \\
    0 & m(m+n+1)
\end{bmatrix} = \begin{bmatrix}
    N-1 & 0 \\
    0 & (M-1)(M+N-1)
\end{bmatrix}.
\]

Hence we have

\[
\mathbb{Z}^2 / L_{(N,M)} \mathbb{Z}^2 \cong \mathbb{Z} / (N - 1) \mathbb{Z} \oplus \mathbb{Z} / (M - 1)(M + N - 1) \mathbb{Z}.
\]

Otherwise, by the Euclidean algorithm, we have lists of integers \( r_0, r_1, \ldots, r_j \) and \( k_0, k_1, \ldots, k_{j+1} \) for some \( j \in \mathbb{N} \) such that

\[
\begin{align*}
    m &= nk_0 + r_0, \quad 0 < r_0 < n, \\
    n &= r_0 k_1 + r_1, \quad 0 < r_1 < r_0, \\
    r_0 &= r_1 k_2 + r_2, \quad 0 < r_2 < r_1, \\
    \ldots \\
    r_{j-2} &= r_{j-1} k_j + r_j, \quad 0 < r_j < r_{j-1}, \\
    r_{j-1} &= r_j k_{j+1}, \quad 0 = r_{j+1}
\end{align*}
\]

where \( r_j = (m,n) \) the greatest common divisor of \( m \) and \( n \). Put \( g = m(m+n+1) \).

Add the \(-k_0\) multiplication of the first row to the second row in \( L_{n,m} \) to obtain the matrix below denoted by \( L_{n,m}(0) \):

\[
L_{n,m}(0) = \begin{bmatrix}
    n & 0 \\
    r_0 & g
\end{bmatrix}.
\]

Add the \(-k_1\) multiplication of the second row to the first row in \( L_{n,m}(0) \) to obtain the matrix below denoted by \( L_{n,m}(1) \):

\[
L_{n,m}(1) = \begin{bmatrix}
    r_1 & -k_1 g \\
    r_0 & g
\end{bmatrix}.
\]
Add the \(-k_2\) multiplication of the first row to the second row in \(L_{n,m}(1)\) to obtain the matrix below denoted by \(L_{n,m}(2)\):

\[
L_{n,m}(2) = \begin{bmatrix} r_1 & -k_1 g \\ r_2 & (1 + k_1 k_2) g \end{bmatrix}.
\]

We continue these procedures as follows. Add the \(-k_{2i-1}\) multiplication of the second row to the first row in \(L_{n,m}(2i-2)\) to obtain the matrix denoted by \(L_{n,m}(2i-1)\) to obtain the matrix denoted by \(L_{n,m}(2i)\) for \(i = 1, 2, \ldots\). The algorithm stops at \(j + 1 = 2i - 1\) or \(j + 1 = 2i\) for some \(i \in \mathbb{N}\). We set

\[
[k_0] = 1, \quad [k_1] = k_1, \quad [k_1, k_2] = 1 + k_1 k_2, \quad [k_1, k_2, k_3] = [k_1, k_2] k_3 + [k_1], \quad \ldots
\]

\[
[k_1, k_2, \ldots, k_{j+1}] = [k_1, k_2, \ldots, k_j] k_{j+1} + [k_1, \ldots, k_{j-1}].
\]

Then we have

\[
L_{n,m}(1) = \begin{bmatrix} r_1 & -[k_1] g \\ r_0 & g \end{bmatrix}, \quad L_{n,m}(2) = \begin{bmatrix} r_1 & -[k_1] g \\ r_2 & [k_1, k_2] g \end{bmatrix},
\]

and inductively

\[
L_{n,m}(2i-1) = \begin{bmatrix} r_{2i-1} & -[k_1, k_2, \ldots, k_{2i-1}] g \\ r_{2i-2} & [k_1, k_2, \ldots, k_{2i-2}] g \end{bmatrix},
\]

\[
L_{n,m}(2i) = \begin{bmatrix} r_{2i-1} & -[k_1, k_2, \ldots, k_{2i-1}] g \\ r_{2i} & [k_1, k_2, \ldots, k_{2i}] g \end{bmatrix}
\]

for \(i = 1, 2, \ldots\). We denote by \(d\) the greatest common divisor \((m, n)\) of \(m\) and \(n\), so that \(d = r_j\). Take \(m_0 \in \mathbb{Z}\) such that \(m = m_0 d\). Put \(g_0 = m_0 (m + n + 1)\) so that \(g = gcd\).

We have two cases.

Case 1: \(j + 1 = 2i - 1\) for some \(i \in \mathbb{N}\). We have

\[
L_{n,m}(j + 1) = \begin{bmatrix} r_{j+1} & -[k_1, k_2, \ldots, k_{j+1}] g \\ r_j & [k_1, k_2, \ldots, k_j] g \end{bmatrix} = \begin{bmatrix} 0 & -[k_1, k_2, \ldots, k_{j+1}] g \\ d & [k_1, k_2, \ldots, k_j] g_0 d \end{bmatrix}.
\]

Add the \(-[k_1, k_2, \ldots, k_j] g_0\) multiplication of the first column to the second column in the above matrix \(L_{n,m}(j + 1)\), and then exchange the rows to obtain the matrix below

\[
\begin{bmatrix} d & 0 \\ 0 & -[k_1, k_2, \ldots, k_{j+1}] g \end{bmatrix}.
\]

Case 2: \(j + 1 = 2i\) for some \(i \in \mathbb{N}\). We have

\[
L_{n,m}(j + 1) = \begin{bmatrix} r_j & -[k_1, k_2, \ldots, k_j] g \\ r_{j+1} & [k_1, k_2, \ldots, k_{j+1}] g \end{bmatrix} = \begin{bmatrix} d & -[k_1, k_2, \ldots, k_j] g_0 d \\ 0 & [k_1, k_2, \ldots, k_{j+1}] g \end{bmatrix}.
\]

Add the \([k_1, k_2, \ldots, k_j] g_0\) multiplication of the first column to the second column in the above matrix \(L_{n,m}(j + 1)\), and then exchange the rows to obtain the matrix below

\[
\begin{bmatrix} d & 0 \\ 0 & [k_1, k_2, \ldots, k_{j+1}] g \end{bmatrix}.
\]

We reach the following lemma.

**Lemma 3.4.**

\[
\mathbb{Z}^2 / L(N, M) \mathbb{Z}^2 \cong \mathbb{Z}/d \mathbb{Z} \oplus \mathbb{Z}/[k_1, k_2, \ldots, k_{j+1}] g \mathbb{Z}.
\]
Therefore we have

**Theorem 3.5.** For positive integers $1 < N \leq M \in \mathbb{N}$ and a specification $\kappa$ of exchanging $N$-loops and $M$-loops in a graph with one vertex, the $C^*$-algebra $O_{\mathcal{H}_N^{[N],[M]}}$ is a simple purely infinite Cuntz-Krieger algebra whose $K$-groups are

$$K_1(O_{\mathcal{H}_N^{[N],[M]}}) \cong 0,$$

$$K_0(O_{\mathcal{H}_N^{[N],[M]}}) \cong \mathbb{Z}/(N-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N-1)\mathbb{Z}$$

$$\oplus \mathbb{Z}/(M-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(M-1)\mathbb{Z}$$

$$\oplus \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/[k_1, k_2, \ldots, k_{j+1}](M-1)(M+N-1)\mathbb{Z}$$

where $d = (N-1, M-1)$ is the greatest common divisor of $N-1$ and $M-1$, the sequence $k_0, k_2, \ldots, k_{j+1}$ of integers is the list of the successive integral quotients of $M-1$ by $N-1$ in the Euclidean algorithm, and the integer $[k_1, k_2, \ldots, k_{j+1}]$ is defined by inductively

$$[k_0] = 1, \quad [k_1] = k_1, \quad [k_1, k_2] = 1 + k_1k_2, \quad \ldots$$

$$[k_1, k_2, \ldots, k_{j+1}] = [k_1, k_2, \ldots, k_{j}]k_{j+1} + [k_1, \ldots, k_{j-1}].$$

For the case $N = 2$ and $M \geq 2$, we have $d = 1, r_0 = 0$. We understand $[k_1, \ldots, k_{j+1}] = [k_0] = 1$ so that we have

$$[k_1, \ldots, k_{j+1}](M-1)(M+N-1) = 1 \times (M-1)(M+1) = M^2 - 1.$$

Hence

$$K_0(O_{\mathcal{H}_2^{[2],[M]}}) \cong \mathbb{Z}/(M^2 - 1)\mathbb{Z}. \quad (3.5)$$

If in particular, $M = 3$, the formula (3.5) is already seen in [14].

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