Analysis of the Leakage Queue:

A Queueing Model for Energy Storage Systems with Self-discharge

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Abstract

Energy storage is a crucial component of the smart grid, since it provides the ability to buffer transient fluctuations of the energy supply from renewable sources. Even without a load, energy storage systems experience a reduction of the stored energy through self-discharge. In some storage technologies, the rate of self-discharge can exceed 50% of the stored energy per day. In this paper, we investigate the self-discharge phenomenon in energy storage using a queueing system model, which we refer to as leakage queue. When the average net charge is positive, we discover that the leakage queue operates in one of two regimes: a leakage-dominated regime and a capacity-dominated regime. We find that in the leakage-dominated regime, the stored energy stabilizes at a point that is below the storage capacity. Under suitable independence assumptions for energy supply and demand, the stored energy in this regime closely follows a normal distribution. We present two methods for computing probabilities of underflow and overflow at a leakage queue. The methods are validated in a numerical example where the energy supply resembles a wind energy source.

I. Introduction

With their ability to absorb the intermittency and uncertainty in renewable energy generation, energy storage systems facilitate the integration of renewable energy sources into the grid. There exists a wide variety of energy storage technologies, each offering a trade-off with
regards to storage capacity per unit of volume (energy density), delivered power per unit of volume (power density), scalability, and others [10]. For instance, compressed air energy storage systems have a high capacity but a low power density, which makes them suitable for long-term storage applications that do not need a fast response time. On the other end of the spectrum, supercapacitors have a high power density, but limited storage capacity, which serves applications that require fast responses. Hybrid energy storage systems seek to combine the advantages of different storage technologies in order to meet the need of a specific application [22]. Due to the generally high cost of energy storage technologies – the price of energy storage can be thousands of dollars per kilowatt hour – the economic viability of an energy storage system crucially depends on properly dimensioning the storage size. Over-provisioning of an energy storage system unnecessarily increases costs, while under-provisioning may render it ineffective. The need for tools to dimension energy storage systems has motivated the development of analytical methods. By modelling energy storage systems as finite-capacity queueing systems with stochastic arrivals (energy supply) and departures (energy demand), the vast queueing theory literature becomes available to the problem of storage sizing. However, a closer inspection reveals that energy storage systems are not automatically a good fit for a queueing theory analysis. For one, arrivals and departures in queueing analysis are often expressed as point processes that track arrival and departure events. Energy supply and demand, on the other hand, are better characterized by fluid-flow processes. Also, many queueing theory methods have been developed for job shop manufacturing and communication networks, where buffer provisioning is primarily concerned with preventing overflows. Here, it is generally required that the average service rate exceeds the average arrival rate. Differently, in energy storage the overriding concern is the prevention of buffer underflows (‘empty batteries’), and, consequently, the average arrival rate generally exceeds the average service rate.

Different from systems usually analyzed by queueing theoretic methods, the stored energy may shrink over time even if the system is inactive. The rate of leakage of the stored energy, referred to as self-discharge, results from chemical reactions, loss of thermal or kinetic energy, and other factors. It can be as low as 1% of the total charge per month for lithium-ion batteries.

1In addition to the price, the cost of energy storage also takes into account other factors such as the number of recharge cycles over the lifetime of a unit, the amount of energy that can be withdrawn in a single recharge cycle (depth of discharge), and ancillary costs.
and may exceed 50% per day for flywheels. Interestingly, there is no queueing analysis in the literature that accounts for the impact of self-discharge. The lack of analytical models for systems with self-discharge was made evident in recent performance studies of energy storage, which resorted to optimization methods when accounting for self-discharge. The main complication in a performance analysis of storage with self-discharge is that self-discharge adds a deterministic or stochastic process that runs concurrently with the conventional arrival and service processes. Since the quantity of self-discharge depends on the amount of stored energy, the self-discharge rate is not an independent process, but is coupled with the supply and demand processes.

In this paper we study the dynamics of queueing systems that model energy storage systems with self-discharge. For example, how does the self-discharge process interact with the processes for energy supply and energy demand? Is it possible to identify parameter regions where a storage system has a desirable behavior, that is, where overflows and underflows are rare events? We consider a queueing model, referred to as queue with leakage or leakage queue, where supply and demand are governed by stochastic processes and, additionally, in each time slot, the content of the queue is reduced by a factor $\gamma$, with $0 < \gamma < 1$. We consider leakage queues with finite and infinite capacity. Under the assumption of an on average positive net charge, i.e., the average supply exceeds the average demand, we make a number of discoveries, some of which are quite surprising:

- We find that a leakage queue with finite capacity has parameter regimes with distinct behaviors, which require different approaches for an analysis.
- In the parameter regime where the queue is rarely full or empty, a leakage queue is well approximated by an idealized infinite-capacity system that permits stored energy to become negative. We find that the distribution of the stored energy in this regime is close to Gaussian.
- A leakage queue with $\gamma > 0$ is stable for arbitrary supply distributions with a finite average. Stability is maintained for systems with arbitrarily large capacity. Moreover, convergence to the stable steady state occurs exponentially fast.
- The steady-state filling level can be precisely determined in a simple expression that only requires the averages of the supply and demand processes, as well as $\gamma$.

The goal of our study is to gain insight into the self-discharge phenomenon occurring in energy storage systems. To isolate the self-discharge effects to a maximum degree, we consider a
minimalist system model that only leaves the input, output, and leakage processes in place. Consequently, our model is not a high-fidelity model for energy storage or a particular storage technology. Also, we do not account for the potentially complex interactions between users and utilities in demand side management. Lastly, we do not address the scale at which energy storage is deployed, that is, whether it is utility scale, community scale, or household scale. (We note that our numerical examples use parameter ranges that apply to residential households.) By eliminating these factors and by considering a bare-bones model of energy storage, we are able to observe and quantify previously unreported dynamics in energy storage systems.

The remainder of the paper is structured as follows. In Section II we discuss the literature on the analysis of energy storage systems. In Section III we explore the dynamics of queueing systems with leakage. In Section IV we establish the stability of the leakage queue under a broad set of assumptions. In Section V we present two analytical approaches, each applicable to a specified regime of parameters, for deriving overflow and underflow probabilities. In Section VI we evaluate our analysis with random processes that mimic the behavior of a renewable energy source. We present conclusions in Section VII.

II. RELATED WORK

Energy storage plays a major role in many aspects of the smart grid, and, consequently, there is a extensive literature on their analysis. The electrical grid requires that power generation and demand load are continuously balanced. This becomes more involved with time-variable renewable energy sources and storage systems absorbing the variations from such sources. Smart grid approaches that take the perspective of a utility operator are concerned with placement, sizing, and control of energy storage systems with the goal to optimally balance power [41], [42], [43], reduce power generation costs [45], or operational costs [26]. Works in this area are frequently formulated as optimal control or optimization problems, with the objective to devise distributed algorithms that achieve a desired operating point.

Demand side management [15] takes the perspective of an energy user, and broadly refers to measures that encourage users to become more energy efficient. As one form of demand side management, demand response refers to methods for short-term reductions in energy consumption. By creating incentives to users, demand response seeks to match elastic demands with fluctuating renewable energy sources. In [7], [38], demand response is formulated as a utility maximization problem where dynamic pricing incentivizes individual users to benefit the overall
system. Studies on demand response apply a wide range of methods, from coordination between appliances [33], bounds on prediction errors [29], and game-theoretic approaches [30].

Performance analysis of energy storage systems intends to support dimensioning of storage by providing metrics such as overflow and underflow probabilities, and the amount of stored energy in the steady state. Since detailed models of the circuit or electrochemical processes in an energy storage system, as given in [6], [20], [37], are not analytically tractable, energy storage systems are generally described by abstract models. Differential or difference equations have been used for detailed descriptions of the energy evolution in lithium-ion batteries [20] and flywheels [13], [14]. The suitability of queueing theory for analyzing the dynamics of energy storage has been pointed out in [1]. Interestingly, queueing theory was applied in the 1960s for analyzing storage properties of water reservoirs [8, Chp. III.5], and the fluid-flow analysis of queueing systems was known as ‘dam theory’ [34].

More recently, a fluid-flow interpretation of queueing theory, known as ‘network calculus’ [27], has been applied to energy storage systems. A deterministic analysis has been used in [28] to devise battery charging schedules that prevent batteries from running empty. Stochastic extensions of the network calculus have been applied to analyze energy storage in the presence of random, generally Markovian, energy sources [40], [47], [49]. In these works, the evolution of the stored energy is expressed using a time-dependent function for the backlog in a finite-capacity queueing system from [9]. Recent studies [14], [16], [17], [46], [50] have improved the fidelity of energy storage models by considering factors such as limited charging and discharging rates, charging and discharging inefficiencies, as well as self-discharge. In [16], the self-discharge is modeled by a constant rate function, whereas the other works [14], [17], [46], [50] use a proportional leakage ratio as described in Sec. I. Since queueing systems for energy storage systems with proportional self-discharge could not be solved analytically, the existing analyses resort to simulation and optimization methods. These provide numerical solutions, but do not easily give insight into parameter regimes and basic tradeoffs.

III. A QUEUEING MODEL FOR ENERGY STORAGE WITH SELF-DISCHARGE

We model an energy storage system as a finite queueing system, as shown in Fig. 1. The arrivals to the system consist of a time-varying energy supply from energy sources, the service process consists of the time-varying energy demand from customers, and the departures are the serviced demand. The stored energy and capacity, which correspond to the backlog and
Fig. 1: Queueing model of energy storage system.

A. Dynamics of the leakage queue

For the purpose of the analysis, we assume that the energy processes are discrete-time, fluid flow random processes, where we consider deterministic processes as a special case. The energy supplied when the storage is at capacity is considered wasted, and demand to an empty storage is considered lost. We use $a(n)$, $s(n)$, and $d(n)$ to denote the energy supply, energy demand, and serviced demand, respectively, in time slot $n$, measured in Wh. We define the net charge or drift of the system in slot $n$, denoted by $\delta(n)$, as the difference

$$\delta(n) = a(n) - s(n).$$

The amount of energy stored at time slot $n$, denoted by $B(n)$ and referred to as stored energy, corresponds to the backlog in the usual terminology of queueing theory. Alternative terms in
the energy storage literature are energy content, state of charge, or battery load. The maximum
amount of stored energy, referred to as storage capacity, is denoted by \( C \).

We assume that the queue has a fixed self-discharge ratio \( \gamma \) with \( 0 \leq \gamma < 1 \), with the
interpretation that, the stored energy at time \( n \), \( B(n) \), shrinks to \( (1 - \gamma)B(n) \) by time slot \( n+1 \).
The case \( \gamma = 0 \) refers to a system without self-discharge. Since the leakage ratio frequently
appears in the form \( 1 - \gamma \), we define the complementary leakage ratio \( \bar{\gamma} \) as
\[
\bar{\gamma} = 1 - \gamma.
\]

Then, the energy evolution of the storage system can be described in terms of a recursive equation
[17], [46] by
\[
B(n) = \min\{[\bar{\gamma}B(n - 1) + \delta(n)]^+, C\},
\]
where we use the notation \([x]^+ = \max\{x, 0\}\). We refer to a queueing system with this dynamics
as a queue with leakage or leakage queue. Descriptions of energy storage systems generally use
a fixed self-discharge ratio, even though the self-discharge may depend on the amount of stored
energy, temperature, or other factors. Here, the fixed self-discharge ratio represents a long-term
average [18].

There exist other types of queueing systems where admittance, service, or sojourn time are
functions of the backlog (stored energy), e.g., a \( G/G/\infty \) queue, a queue with discouraged arrivals
[24 Chp. 3.3], or a reneging queue [19]. To see how the leakage queue differs, let us consider
discrete-time fluid flow versions of these queues. Additionally ignoring overflows and underflows,
the change of the backlog of the different queues in a time slot is given by

Leakage queue:
\[
B(n) - B(n - 1) = a(n) - s(n) - \bar{\gamma}B(n - 1),
\]

\( G/G/\infty \) queue:
\[
B(n) - B(n - 1) = a(n) - s(n)B(n - 1),
\]

Discouraged arriv.:
\[
B(n) - B(n - 1) = \frac{a(n)}{B(n-1)+1} - s(n),
\]

Reneging queue:
\[
B(n) - B(n - 1) = a(n)\left(1 - \frac{B(n-1)}{C}\right) - s(n).
\]

Compared to a \( G/G/\infty \) system, the leakage queue has an additional process. Compared to queues
with discouraged arrivals or reneging, the backlog in the leakage queue does not throttle arrivals.

If time units are expressed in hours, a self-discharge of 5% per day for a full battery corre-
sponds to a leakage ratio of \( \gamma = 0.0021 \). This follows since the leakage in a day is \( 1 - \bar{\gamma}^{24} \).
Likewise, we have the correspondences

\[
\begin{align*}
10\% \text{ per day} & \sim \gamma = 0.0044, \\
20\% \text{ per day} & \sim \gamma = 0.0093, \\
50\% \text{ per day} & \sim \gamma = 0.0285.
\end{align*}
\]

Note that the leakage ratios depend on the length of the time slot.

We use \( l_o(n) \) and \( l_u(n) \) to denote the overflow and underflow processes, respectively, at the storage system. In the context of energy storage, \( l_o(n) \) is often referred to as the waste of power and \( l_u(n) \) is referred to as the loss of power. The processes are given by

\[
\begin{align*}
l_o(n) &= [\bar{\gamma}B(n-1) + \delta(n) - C]^+, \\
l_u(n) &= [-\bar{\gamma}B(n-1) - \delta(n)]^+.
\end{align*}
\]

The expression in Eq. (1) can be refined to consider other pertinent features of an energy storage system. For example, some types of storage, such as lithium-ion batteries, perform better if they are not fully charged. The depth of discharge \( \text{DoD} \) refers to the maximum level to which a battery should be charged, expressed as a percentage of the storage capacity. The charging rate \( \alpha_c \) and the discharging rate \( \alpha_d \) refer to bounds on the maximum power at which the storage can be charged and discharged. The charging efficiency \( \eta \) of the storage expresses the amount of energy that is lost in the charging process. In [16], these factors are taken into account and the evolution of the stored energy is represented by Eq. (1), where \( C \) is replaced with \( C \times \text{DoD} \), and \( \delta(n) \) is set to

\[
\delta(n) = \min\{[\delta(n)]^+, \alpha_c\} \eta - \min\{[\delta(n)]^+, \alpha_d\}.
\]

As shown in [16], by defining modified processes for supply and demand it is possible to express Eq. (3) in terms of a simpler energy evolution similar to Eq. (1).

We define the bivariate process \( \Delta_\gamma(m, n) \) as

\[
\Delta_\gamma(m, n) = \sum_{k=m+1}^{n} \delta(k)\bar{\gamma}^{n-k}.
\]

For \( \gamma = 0 \), \( \Delta_\gamma(m, n) \) is the cumulative net charge process. With this definition, we present our first result, which is an explicit non-recursive expression for the stored energy in a queue with leakage. The result, presented in the next theorem, extends the backlog equation by Cruz and Liu for finite-capacity queues [9] to leakage queues.
Theorem 1. Let $B(n)$ be the stored energy in a leakage queue with finite capacity $C$ and leakage ratio $\gamma$, as in Eq. (1). Then

$$B(n) = \min_{0 \leq m \leq n} \left\{ \max_{m \leq j \leq n} \left\{ C_m \bar{\gamma}^{n-m} \mathbb{1}_{j=m} + \Delta (j, n) \right\} \right\},$$

where $\mathbb{1}_{j=m}$ is the indicator function that evaluates to 1 if $j = m$, and to 0 otherwise, and $C_m$ is defined as

$$C_m = \begin{cases} B(0) & \text{if } m = 0, \\ C & \text{if } m > 0. \end{cases}$$

Eq. (4) implies that the effect of the initial charge $C_0$ vanishes as time increases. By taking $C \to \infty$, we immediately get for a leakage queue with infinite capacity that

$$B(n) = \max_{0 \leq j \leq n} \left\{ C_0 \bar{\gamma}^{n-m} \mathbb{1}_{j=0} + \Delta (j, n) \right\}.$$

Proof. We first argue that

$$B(n) \leq \min_{0 \leq m \leq n} \left\{ \max_{m \leq j \leq n} \left\{ C_m \bar{\gamma}^{n-m} \mathbb{1}_{j=m} + \Delta (j, n) \right\} \right\}. \quad (5)$$

Let $m$ be an arbitrary time slot with $0 \leq m \leq n$. If $B(j) > 0$ for all $j$ with $m < j \leq n$, then

$$B(j) \leq \bar{\gamma} B(j-1) + \delta(j), \quad \text{for } m < j \leq n,$$

which implies

$$B(n) \leq \bar{\gamma}^{n-m} B(m) + \Delta (m, n).$$

Otherwise, there exists $j$ with $m < j \leq n$ such that $B(j) = 0$, which implies

$$B(n) \leq \Delta (j, n).$$

In either case, since $B(m) \leq C_m$, it follows that

$$B(n) \leq \max_{m \leq j \leq n} \left\{ C_m \bar{\gamma}^{n-m} \mathbb{1}_{j=m} + \Delta (j, n) \right\}. \quad (6)$$

Since $m$ was arbitrary, and $B(m) \leq C_m$, this establishes Eq. (5).

To complete the proof, it suffices to find one value of $m$ that produces equality in Eq. (6). Choose $m = 0$ if $B(j) < C$ for all $j = 1, \ldots, n$. Otherwise, choose $m$ to be the index of the last time slot up to $n$ with $B(m) = C$. Since no overflow occurs in time slots $j = m + 1, \ldots, n$, the recursion in Eq. (1) yields

$$B(j) = [\bar{\gamma} B(j-1) + \delta(j)]^+, \quad \text{for } m < j \leq n.$$

Since $B(m) = C_m$ by the choice of $m$, it follows that Eq. (6) holds with equality. \qed
Numerous analytical methods are available for estimating the overflow probability at a buffered link. These methods were developed for applications of queueing theory in telecommunications and manufacturing, e.g., [21]. However, in application areas such as multimedia streaming and energy storage, underflow is a more serious concern than overflow. By developing dual models where the roles of underflow and overflow events are switched, the existing know-how for computing overflow probabilities can be leveraged for the computation of underflow probabilities [1], [4]. We follow this approach by presenting a dual system for a leakage queue. Since the dual system is not a physical system, we resort to conventional queueing terminology and talk about arrivals, service, and backlog.

We refer to the leakage queue in Fig. 1 as the original system. The dual system is a leakage queue with the same capacity $C$ and leakage ratio $\gamma$. Arrivals and service at the dual system, denoted by $a'(n)$ and $s'(n)$, are defined as $a'(n) = \gamma C + s(n)$ and $s'(n) = a(n)$, with $\delta'(n) = a'(n) - s'(n)$. We denote by $B'(n)$ the backlog process of the dual system. The overflow and underflow processes of the dual system, denoted by $l'_o(n)$ and $l'_u(n)$, are as in Eq. (2), where we replace $\delta(n)$ by $\delta'(n)$ and $B(n)$ by $B'(n)$. Fig. 2 illustrates the queueing model of the dual system. With this definition, the backlog $B'(n)$ of the dual system satisfies the recursion

$$B'(n) = \min\{[\gamma B'(n-1) + \delta'(n)]^+, C\}.$$  \hspace{1cm} (7)

Duality of the original and the dual system is established by the following lemma.
Lemma 1. Given a queue with leakage as shown in Fig. 1 and the dual system shown in Fig. 2. If \( B(0) + B'(0) = C \), then the backlog in the original system and the dual system satisfy

\[
B(n) + B'(n) = C
\]

for all \( n > 0 \).

From the lemma it follows immediately that \( l'_o(n) = l_u(n) \) and \( l'_u(n) = l_o(n) \), as long as the dual system is properly initialized. Hence, we can obtain the underflow probability in the original system by computing the overflow probability in the dual system.

Proof. We proceed by induction. The base case is covered by the assumption that \( B(0) + B'(0) = C \). For the inductive step, suppose that \( B(n-1) + B'(n-1) = C \) for some \( n > 0 \). In particular, \( 0 \leq B'(n-1) \leq C \). We rewrite Eq. (7) in terms of \( C - B'(n) \) and apply the identity

\[
C - B'(n) = \min \{ [x]^+, C \} = \min \{ [C - x]^+, C \}
\]

to obtain

\[
C - B'(n) = C - \min \{ [\gamma B'(n-1) + \delta(n)]^+, C \}
\]

\[
= \min \{ [C - \gamma B'(n-1) - \delta'(n)]^+, C \}
\]

\[
= \min \{ [\gamma (C - B'(n-1)) + \gamma C - \delta'(n)]^+, C \}.
\]

Since \( C - B'(n-1) = B(n-1) \) by the inductive hypothesis, and \( \gamma C - \delta'(n) = \delta(n) \), it follows that

\[
C - B'(n) = \min \{ [\gamma B(n-1) + \delta(n)]^+, C \}.
\]

We conclude with Eq. (7) that \( C - B'(n) = B(n) \).

We can exploit the dual system to obtain an alternate expression for the backlog.

Corollary 1. The backlog in a leakage queue with capacity \( C \) and leakage ratio \( \gamma \) is given by

\[
B(n) = \max \{ \min \{ C_0 \gamma^n \mathbb{1}_{j=m=0} + C \gamma^{n-j} \mathbb{1}_{j>m} + \Delta_j(j,n) \} \}_{0 \leq m \leq n, m \leq j \leq n} \}
\]

Proof. Write \( B(n) = C - B'(n) \) and apply Theorem 1 to the dual system.
C. Two regimes for the analysis

In this subsection, we make observations that will prove crucial for the analysis of leakage queues. Throughout, we will work with a net charge that is positive on average, since the storage system will be mostly empty otherwise. We find that the leakage queue operates in two regimes with fundamentally different behaviors. In one regime, the stored energy is stable at a point below the storage capacity. Here, the leakage queue behaves similarly to a reference system, which has infinite capacity and allows the stored energy to become negative. In the other regime, the stored energy is generally close to the capacity. Here, the leakage queue is similar to a conventional finite-capacity queueing system in overload.

We illustrate the different regimes with the aid of a numerical example drawn from an energy storage system with a photo-voltaic (PV) energy source and constant demand. We use the PV
energy generation for a residential rooftop system, which is based on an hourly data set of the typical solar irradiance in Los Angeles for the month of July [31]. The resulting solar energy is calculated with the System Advisor Model (SAM) software [32], where solar panels are scaled so that the average energy supply per hour is 1 kWh. The average demand per hour is assumed to be constant and set to 800 Wh, which is 80% of the supply. This is approximately the average power consumption per household in New Zealand [48].

For calibration, we first consider a queue without leakage, that is, $\gamma = 0$. In Fig. 3 we depict the energy content for systems with storage capacity $C = 10$ kWh and $C = 40$ kWh, where we assume that the storage is initially empty. Observe that the data captures the diurnal pattern of solar energy. As expected, once the storage fills up, the stored energy is always close to the storage capacity.

Before we discuss how the outcome changes in the presence of self-discharge, we introduce
a reference system that differs from the leakage queue in two ways. First, the reference system has infinite storage capacity \((C = \infty)\). Second, the stored energy is allowed to take negative values, where a negative occupancy can be thought of as an energy deficit. The dynamics of the stored energy in the reference system, denoted by \(B^r\), simplify to

\[
B^r(n) = \gamma B^r(n - 1) + \delta(n). \tag{8}
\]

Solving the recursion yields the formula

\[
B^r(n) = B(0)\gamma^n + \sum_{m=0}^{n} \Delta\gamma(m, n). \tag{9}
\]

If \(\delta(n)\) describes a stationary process with \(\delta(n) = \mathcal{D} \delta\) for all \(n\), where \(\mathcal{D}\) indicates equality in distribution, the expected value is

\[
E[B^r(n)] = \gamma E[B^r(n - 1)] + E[\delta].
\]

If there exists a steady state for \(B^r\), then, for \(n \to \infty\), the expected stored energy, denoted by \(E[B^r]\), is given by

\[
E[B^r] = \lim_{n \to \infty} E[B^r(n)] = \frac{E[\delta]}{\gamma}. \tag{9}
\]

In the next section, we will prove that \(B^r(n)\) always converges to a steady state, with expected value \(E[B^r]\).

We now re-compute the numerical example from Fig. 3 with a leakage ratio of \(\gamma = 0.0093\). With the given supply and demand we obtain that \(E[B^r] = 21.5\) kWh, where, for the data set, \(E[B^r]\) is the average net charge divided by \(\gamma\). Fig. 4 shows the stored energy for the reference system and the finite-capacity leakage queue. Note that the reference system initially takes negative values for \(B^r(n)\). In Fig. 4(a), where \(C > E[B^r]\), the stored energy in the reference system tracks the energy in the finite-capacity leakage queue with a high degree of accuracy. In Fig. 4(b), we show the results for \(C < E[B^r]\). Here, the stored energy in the finite-capacity leakage queue is very different from that of the reference system. In fact, the dynamics of the leakage queue resemble that of the finite-capacity queue without leakage (shown in Fig. 3(b)).

In the next sections, we will establish that our observations in the numerical example extend to other supply and demand distributions. It turns out that all leakage queues with \(\gamma > 0\) operate in one of two modes, which we will refer to as capacity-dominated regime and leakage-dominated regime, with very different characteristics.
Fig. 5: Average stored energy as a function of the storage capacity $C$.

- **Leakage-dominated regime** ($C > E[B^r]$): This regime, illustrated in Fig. 4(a), is characterized by an average stored energy below the storage capacity. This is unlike a conventional finite capacity queueing systems (with $\gamma = 0$), where the storage is full or close to full for $E[\delta] > 0$.

- **Capacity-dominated regime** ($C < E[B^r]$): In this regime, illustrated in Fig. 4(b), the stored energy is always close to the storage capacity $C$, which necessarily results in a high probability of overflow. The system behaves similarly to a conventional finite-capacity queueing system without leakage.

We will study the regimes in detail in Section V where we find that we must use different analysis methods for each regime.

An interesting property of leakage queues is that increasing the storage capacity much beyond $E[B^r]$ does not result in significant benefits. To emphasize this, we consider the same data set as used for Figs. 3 and 4 and compute the average stored energy as a function of the storage capacity. In Fig. 5 we show the results for self-discharge ratios of $\gamma = 0, 0.0093, 0.0285$. Without self-discharge ($\gamma = 0$), the average stored energy is always close to the capacity. For $\gamma > 0$, on the other hand, the average storage approaches a constant even as the capacity goes to infinity. The value of this constant is close to the stored energy in the reference system. To our knowledge,
this feature of energy storage systems with self-discharge has not received any attention.

Obviously, our observations of the stability of the leakage queue and the characterization of the leakage-dominated regime by the reference system are limited to the depicted data set. In the next sections, we will try to corroborate our findings for general random processes.

IV. STABILITY AND CONVERGENCE

In this section, we prove that the leakage queue defined by Eq. (1) converges for $0 < \gamma < 1$ to a unique steady state as $n \to \infty$. The net charge $\delta(n)$ for $n = 0, 1, \ldots$ is assumed to be an i.i.d. sequence of random variables of finite expectation $E[\delta]$. Assuming that $E[\delta] > 0$, we characterize the steady-state distribution of the stored energy, $B$, in terms of the steady state of the reference system $B^r$ and the capacity $C$. In the leakage-dominated regime, where $C > E[B^r]$, we prove that the steady state is close to that of the reference system. In the capacity-dominated regime, where $C < E[B^r]$, the expected drift in the dual system is negative. The steady state resembles that of a stable queue without leakage.

The stability and convergence results extend to leakage queues with infinite capacity. Obviously, a leakage queue of infinite capacity is always leakage-dominated.

A. Stability

A leakage queue of finite capacity $C$ is stable by definition, since $0 \leq B(n) \leq C$ for all $n \geq 0$. We will derive bounds on the distribution of $B(n)$ that do not depend on the value of $C$.

By the recursive definition from Eq. (1), we have $B(n) \leq \tilde{\gamma} B(n-1) + [\delta(n)]^+$. Solving the recursion, and then using the i.i.d. assumption on the drift, we obtain

$$B(n) \leq B(0) \tilde{\gamma}^n + \sum_{m=1}^{n} [\delta(m)]^+ \tilde{\gamma}^{n-m}$$

$$= \mathcal{D} B(0) \tilde{\gamma}^n + \sum_{m=0}^{n-1} [\delta(m)]^+ \tilde{\gamma}^m =: Y(n).$$

In particular,

$$E[B(n)] \leq E[Y(n)] = \frac{1 - \tilde{\gamma}^n}{\gamma} E[[\delta]^+] .$$

As $n \to \infty$, the random variables converge to

$$Y := \sum_{m=0}^{\infty} \tilde{\gamma}^m [\delta(m)]^+ .$$
By monotone convergence, $Y$ has finite mean, and is almost surely finite. By construction, $Pr(B(n) > x) \leq Pr(Y > x)$ holds for all $n \geq 0$. Since $B(n)$ is nonnegative, it follows that the leakage queue is stable.

For the reference system, a similar argument, using dominated convergence, shows that $B^r(n)$ converges in distribution to

$$B^r := \sum_{m=0}^{\infty} \delta(m)\bar{\gamma}^m.$$  \hspace{1cm} (10)

Since this series converges absolutely almost surely, the reference system is stable as well. We will refer to $B^r$ as the steady-state distribution of the reference system. Note that the expected value of Eq. (10) is equal to Eq. (9), justifying our earlier use of the notation $E[B^r].$

**B. Convergence to steady state**

We next show that the stored energy in a leakage queue converges in distribution to a steady state.

**Theorem 2.** Let $\delta(n)$ be an i.i.d. sequence of finite mean, $C > 0$, and $0 < \gamma < 1$. Then the stored energy $B(n)$ in a queue with leakage ratio $\gamma$, storage capacity $C$, and drift $\delta(n)$ converges in distribution to a steady state that does not depend on the initial condition.

The key observation is that Eq. (1) defines a contraction in a suitable metric on the space of probability distributions. Given two random variables $X_1, X_2$, with cumulative distribution functions (CDF) $F_1(x) = Pr(X_1 \leq x)$ and $F_2(x) = Pr(X_2 \leq x)$. Their Kantorovich-Rubinstein distance is defined by

$$d(F_1, F_2) = \int_{-\infty}^{\infty} |F_1(x) - F_2(x)| \, dx.$$  

With a slight abuse of notation, we write $d(X_1, X_2)$ in place of $d(F_1, F_2)$ for the Kantorovich-Rubinstein distance between the distributions of $X_1$ and $X_2$. The following technical lemma provides the necessary estimates.

**Lemma 2.** Let $X_1$ and $X_2$ be random variables of finite mean. Then

1) $d(\alpha X_1, \alpha X_2) = \alpha d(X_1, X_2)$ for every $\alpha > 0$.

2) $d([X_1]^+, [X_2]^+) \leq d(X_1, X_2)$.

3) $d(\min\{X_1, C\}, \min\{X_2, C\}) \leq d(X_1, X_2)$ for every $C \in \mathbb{R}$.

4) $d(X_1 + Y, X_2) \leq d(X_1, X_2)$ for every random variable $Y$ of finite mean that is independent of $X_1$ and $X_2$. 


Proof. For the first claim, we use that the CDF of \( \alpha X_i \) is
\[
Pr(\alpha X_i \leq x) = F_i(\alpha^{-1} x), \quad i = 1, 2
\]
and compute
\[
d(\gamma X_1, \gamma X_2) = \int_{-\infty}^{\infty} |F_1(\alpha^{-1} x) - F_2(\alpha^{-1} x)| \, dx
\]
\[
= \alpha \int_{-\infty}^{\infty} |F_1(y) - F_2(y)| \, dy
\]
\[
= \alpha d(X_1, X_2).
\]
The next two claims are immediate from the facts that the CDF of \([X_i]^+\) is \(F_i 1_{x>0}\), and the CDF of \(\min\{X_i, C\}\) is \(F_i 1_{x\leq C}\). For the final claim, let \(\mu\) be the probability distribution of \(Y\). Then the CDF of \(X_i + Y\) is \(F_i * \mu(x) = \int F_i(x-y) \, d\mu(y)\) for \(i = 1, 2\). Therefore
\[
d(X_1 + Y, X_2 + Y) = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F_i(x-y) - F_2(x-y) \, d\mu(y) \right| \, dx
\]
\[
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_1(x-y) - F_2(x-y)| \, d\mu(y) \, dx
\]
\[
= d(X_1, X_2).
\]
We have used the triangle inequality, and then applied Fubini’s theorem.

Proof of Theorem 2. Let \(\Psi\) be the transformation that maps the distribution of \(B(n-1)\) to the distribution of \(B(n)\) according to Eq. (1). Explicitly,
\[
\Psi(X) = \min\{[\bar{\gamma} X + \delta]^+, C\}, \quad (11)
\]
where \(\delta\) is independent of \(X\). By Lemma 2,
\[
d(\Psi(X_1), \Psi(X_2)) \leq d(\bar{\gamma} X_1, \bar{\gamma} X_2) = \bar{\gamma} d(X_1, X_2).
\]
By Banach’s contraction mapping theorem, \(\Psi\) has a unique fixed point, which we denote by \(B\). Moreover, by induction,
\[
d(B(n), B) \leq \bar{\gamma}^n d(B(0), B),
\]
proving convergence to the steady state.

The last step of the proof allows us to strengthen Theorem 2.

Corollary 2. The convergence of \(B(n)\) to the steady state in Theorem 2 occurs exponentially fast.
The same argument shows the convergence of a leakage queue of infinite capacity to its steady state. The only change is that Eq. (11) should be replaced by $\Psi(X) = [\bar{\gamma}X + \delta]^+$. The proof of the theorem also yields an estimate for the distance of the steady state of the stored energy from the steady state of the reference system, given in Eq. (10).

**Corollary 3.** The steady state distributions of the stored energy in the leakage queue and the reference system satisfy

$$d(B, B^r) \leq \frac{1}{\gamma} \left( \int_{-\infty}^{0} Pr(B \leq x) \, dx + \int_{C}^{\infty} Pr(B > x) \, dx \right).$$

**Proof.** By Theorem 2, the stored energy $B(n)$ in the leakage queue converges to the steady state regardless of the choice of the initial condition $B(0)$. Let us use the steady state of the reference system as an initial condition for $B(n)$, that is, $B(0) = \mathcal{D} B^r$. It is a consequence of the contraction mapping theorem that the distance to the steady state is bounded by

$$d(B, B(n)) \leq \frac{\bar{\gamma}^n}{\gamma} d(B(1), B(0)).$$

We set $n = 0$ and proceed to estimate $d(B(1), B(0))$. Due to our choice of the initial state, Eq. (1) yields

$$B(1) = \mathcal{D} \Psi(B) = \min\{[B]^+, C\},$$

because $\bar{\gamma}B^r + \delta = \mathcal{D} B^r$ by the definition of the reference system. Therefore

$$d(B(1), B(0)) = \int_{-\infty}^{0} Pr(B \leq x) \, dx + \int_{C}^{\infty} Pr(B > x) \, dx,$$

completing the proof.

V. PROBABILISTIC BOUNDS

In this section, we quantify the underflow and overflow probabilities $Pr(l_u > 0)$ and $Pr(l_o > 0)$ at the leakage queue. Based on Sec. III, we expect the reference system to provide a good approximation for the leakage queue in the capacity-dominated regime where the storage capacity is large enough to absorb random variations of power supply and demand. For the capacity-dominated regime, we offer a separate martingale analysis. As in Sec. IV, we assume that the net charge $\delta(n)$ is an i.i.d. process with finite mean. We now additionally assume that $\delta(n)$ has finite variance.
A. Gaussian analysis

In Sec. IV we showed that the stored energy in the reference system $B_r(n)$ has a unique steady state, given by Eq. (10). In the special case where $\delta(k)$ follows a normal distribution, the reference system $B_r$ is also Gaussian, with mean and variance in the steady state given by

$$E[B_r] = \frac{E[\delta]}{\gamma}, \quad \text{Var}[B_r] = \frac{\text{Var}[\delta]}{1 - \gamma^2}.$$  \hspace{1cm} (12)

Let $B$ be the steady state of the corresponding leakage queue. If $C > E[B_r]$, we expect the stored energy $B$ to be well-approximated by $B_r$, see Corollary 3. In particular, underflow and overflow probabilities should be small, and satisfy

$$Pr(l_u > 0) \approx Pr(B_r < 0) = \Phi \left( \frac{E[B_r]}{\sqrt{\text{Var}[B_r]}} \right),$$  \hspace{1cm} (13)

$$Pr(l_o > 0) \approx Pr(B_r > C) = \Phi \left( \frac{C - E[B_r]}{\sqrt{\text{Var}[B_r]}} \right),$$  \hspace{1cm} (14)

where $\Phi$ is the standard normal CDF.

We evaluate the accuracy of this approximation for a leakage queue with Gaussian net charge, by comparing Eqs. (13) and (14) with simulations of the leakage queue. For the simulations, we compute averages over multiple repetitions of long simulation runs. We consider a storage system with a size up to 50 kWh, which covers a reasonable range for residential energy storage systems [44]. We assume a self-discharge of 20% per day ($\gamma = 0.0093$) or 50% per day ($\gamma = 0.0285$), and also consider a system without self-discharge ($\gamma = 0$). The energy supply and demand in a time slot of one hour are set so that $E[a] = 1$ kWh and $E[s] = 0.8$ kWh, respectively. The standard deviation is set to $\sigma_\delta = 1$ kWh.

Fig. 6(a) depicts the underflow probability computed from Eq. (13) as a function of the storage capacity, and compares it with simulations. Since the expression for $Pr(B_r < 0)$ in Eq. (13) does not depend on $C$, the analysis yields a straight line. The simulations of a system without leakage show that the underflow probability decreases exponentially in $C$. For systems with leakage, on the other hand, the underflow probability becomes eventually constant in the leakage-dominated region ($C > E[B_r]$). Hence, increasing the storage capacity further will not reduce the underflow probability.
Fig. 6: Underflow and overflow with Gaussian net charge \( E[\delta] = 200 \text{ Wh}, \sigma_\delta = 1 \text{ kWh}, \)
self-discharge per day: 0\%, 20\%, or 50\%).
In Fig. 6(b) we consider the wasted energy due to overflows. For the system without leakage, the overflow probability quickly settles at a value that does not depend on the storage capacity. Here, the storage is mostly full and the overflow compensates for the excess supply compared to the demand. Leakage queues with non-zero leakage show a dramatically different behavior. The analytical estimate for $Pr(l_o > 0)$ from Eq. (14) decreases faster than exponentially in $C$, which is also reflected in the simulations. We conclude that in leakage queues the overflow probability can be reduced arbitrarily by increasing the storage capacity.

Both plots in Fig. 6 show that the Gaussian analysis can provide good estimates of the underflow and overflow probabilities when the system is in the leakage-dominated regime ($C > E[B^r]$). Even if $\delta(n)$ does not follow a normal distribution, the Gaussian analysis provides good estimates (see Subsec. VI-A). To understand why this is the case, let us consider general supply and demand processes that are i.i.d. with arbitrary distributions. By Eq. (10), the stored energy $B^r$ in the steady state of the reference system is the sum of independent random variables $\delta(m)\bar{\gamma}^m$. In the limit $\gamma \to 0$, these random variables become i.i.d. In analogy with the Central Limit Theorem, one should expect that the distribution of $B^r$ approaches a Gaussian. The following result states that a suitably normalized version of $B^r$ converges to the normal distribution.

**Theorem 3.** Let $\delta(n)$ be a sequence of i.i.d. random variables with finite mean and variance. For $0 < \gamma < 1$, let $B^r$ be given by Eq. (10). Then, as $\gamma \to 0$,

$$Z := \frac{B^r - E[B^r]}{\sqrt{\text{Var}[B^r]}}$$

converges in distribution to a standard normal random variable.

**Proof.** It suffices to show that the characteristic function $E[e^{i\theta Z}]$ converges to $e^{-\theta^2/2}$, the characteristic function of the standard normal distribution, for every $\theta \in \mathbb{R}$ [12, Theorem 3.3.6]. Let $\mathcal{X}(\theta)$ be the characteristic function of the normalized random variable $\frac{\delta - E[\delta]}{\sqrt{\text{Var}[\delta]}}$. By the i.i.d. assumption,

$$E[e^{i\theta Z}] = \prod_{k=0}^{\infty} \mathcal{X} \left( \theta \sqrt{1 - \bar{\gamma}^2 \bar{\gamma}^k} \right).$$

Since $\delta$ has finite variance, $\mathcal{X}$ is twice differentiable at zero, with $\mathcal{X}(0) = 1$, $\mathcal{X}'(0) = 0$, and $\mathcal{X}''(0) = 1$. Using the Taylor expansion of $\mathcal{X}$ about zero, a routine estimate for the product (see, for example, [12, Exercise 3.1.1]) shows that $E[e^{i\theta Z}] \to e^{-\theta^2/2}$ as $\gamma \to 0$. \[\square\]
The theorem, in combination with Corollary 3 justifies Eq. (13) and Eq. (14) for general i.i.d. arrival and service processes of finite mean and variance, so long as $\gamma$ is sufficiently small and $C > E[B^r]$. If $\gamma$ is not close to zero, additional moments of $B^r$ may be used to approximate its distribution. Since $B^r$ is a sum of independent random variables, each of its cumulants can be computed directly from the corresponding cumulant of $\delta$, using Eq. (10). In particular, the skewness of $B^r$ is given by

$$\text{Skew}[B^r] = \frac{(1 - \bar{\gamma}^2)^{3/2}}{1 - \bar{\gamma}^3} \text{Skew}[\delta].$$  \hspace{1cm} (15)$$

For non-zero skewness, a skew-normal distribution, fitted to the mean, variance, and skewness of $B^r$, provides a better approximation to $\Pr(B^r > C)$ and $\Pr(B^r < 0)$ than a Gaussian [2]. This approximation can be inserted in place of the Gaussian on the right hand sides of Eq. (13) and Eq. (14). We will illustrate the benefits of using the skew normal approximation in Subsec. VI-A.

B. Martingale analysis

We have seen that the Gaussian analysis from the previous subsection provides a good level of accuracy in the leakage-dominated regime, but much less so in the capacity-dominated regime. Next, we present a martingale analysis for estimating the underflow probability in the capacity-dominated regime. Note that, in the capacity-dominated regime, the overflow probability is of less interest since the stored energy is mostly close to or at capacity.

Recall that at $E[\delta] > 0$, a leakage queue in the capacity-dominated behaves similarly to a finite-capacity queueing system. In the queueing literature, the overflow probability of finite-capacity queues has been studied extensively. We take advantage of the available methods for buffer overflow by resorting to the dual system presented in Subsec. III-B and applying the overflow analysis in [25], which is based on the Kingman-Ross delay bounds [23], [36].

**Theorem 4.** Consider a leakage queue with a net charge given by an i.i.d. process $\delta(n)$ and leakage ratio $0 < \gamma < 1$. Assume that the moment-generating function $M(\theta) = E[e^{\theta \delta}]$ exists for at least some $\theta < 0$, and let

$$\theta^* = \sup\{\theta > 0 : E[e^{\theta^* (\gamma C - \delta)}] \leq 1\}.$$  

If $C < E[B^r]$, then $\theta^* > 0$ and the underflow probability in the steady state is bounded by

$$\Pr(l_u > 0) \leq e^{-\theta^* C}. \hspace{1cm} (16)$$
If, moreover, \( a(n) \) and \( s(n) \) are independent processes, then

\[
Pr(l_u > 0) \leq \frac{e^{-\theta^* C}}{\inf_x E[e^{\theta^*(s-x)}|s > x]}.
\]

(17)

We will use the theorem in the special case where \( s(n) = s_0 + s_1(n) \), where \( s_0 \) is a constant and \( s_1(n) \) is exponentially distributed. In that case, the denominator of (17) can be evaluated exactly, resulting in

\[
Pr(l_u > 0) \leq e^{-\theta^* C}(1 - \theta^* E[s_1]).
\]

The following proof extends a recent martingale analysis of a fork-join system [35] to a leakage queue.

Proof. Let \( B'(n) \) be the dual system introduced in Subsec. III-B. Let \( \delta'(n) \) be the net charge in the dual system, as defined in Subsec. III-B and let \( M' (\theta) = E[e^{\theta \delta'}] \) be its moment-generating function. Since \( M' \) is a convex function, the set \( \{ \theta \in \mathbb{R} | M'(\theta) \leq 1 \} \) is an interval containing zero. By definition, \( \theta^* \) is the right endpoint of that interval. The assumption that \( E[Br] > C \) implies, by Eq. (12), that

\[
E[\delta'] = \gamma C - E[\delta] < 0.
\]

It follows that \( \theta^* > 0 \).

Suppose the dual system is started with \( B'(0) = 0 \). By Theorem 1, we have for every \( n > 0 \),

\[
B'(n) = \min_{0 \leq m \leq n} \left\{ \max_{m \leq j \leq n} \{ C\gamma^{n-m} \mathbb{1}_{j=m>0} + \Delta'_\gamma(j,n) \} \right\}
\leq \max_{0 \leq j \leq n} \{ \Delta'_\gamma(j,n) \}
= \max_{0 \leq m \leq n} \left\{ \sum_{k=0}^{m-1} \delta'(k)\gamma^k \right\},
\]

where

\[
\Delta'_\gamma(m,n) = \sum_{k=m+1}^{n} \delta'(k)\gamma^{n-k}.
\]

In the second line, we have set \( m = 0 \), and in the third line, we have used the i.i.d. assumption. Taking \( n \to \infty \), we obtain for the steady state

\[
Pr(l_u > 0) = Pr(l'_o > 0) \leq Pr \left( \max_{m \geq 0} \sum_{k=0}^{m} \delta'(k)\gamma^k > C \right).
\]

Note that the steady state does not depend on the choice of \( B(0) \) by Theorem 2.
Define for $m \geq 0$
\[ z(m) = e^{\theta^* \sum_{k=0}^{m} \delta'(k) \gamma^k}. \]

Then
\[ z(m) = e^{\theta^* \delta'(m) \gamma^m} z(m - 1), \]
\[ z(0) = e^{\theta^* \delta'(0)}. \]

Since $\delta'(m)$ is independent of $z(m - 1)$, we have
\[ E[z(m) \mid z(m - 1)] = E[e^{\theta^* \delta'(m) \gamma^m} z(m - 1)] \leq z(m - 1). \]

Therefore, $z(m)$ is a supermartingale. By Doob’s inequality for positive supermartingales,
\[ Pr(l_u > 0) \leq Pr(\max_{m \geq 0} z(m) > e^{\theta^* C}) \leq Pr(z(0) > e^{\theta^* C}) \leq e^{-\theta^* C}. \]

In the last step, we have used Markov’s inequality and the fact that $E[z(0)] = 1$. This proves the claim in Eq. (16).

The bound can be sharpened by a stopping-time argument, as follows. For $k \geq 0$, let $t_k$ be the minimum of $k$ and the number of the first time slot where $z$ exceeds $e^{\theta^* C}$. Clearly, each $t_k$ is a random variable of finite expectation, and $t_k \leq t_{k+1}$ for all $k$. Since $\max_{m \geq 0} z(m) > e^{\theta^* C}$ if and only if $\lim_{k \to \infty} t_k < \infty$, it follows that
\[ Pr(l_u > 0) \leq \lim_{k \to \infty} Pr(t_k < k). \]

By the optional stopping theorem and the positivity of $z(m)$,
\[ 1 = E[z(0)] \geq E[z(t_k)] \geq E[z(t_k) \mid t_k < k] Pr(t_k < k) \]
for each $k \geq 0$, that is,
\[ Pr(t_k < k) \leq \frac{1}{E[z(t_k) \mid t_k < k]}. \]
Fig. 7: Underflow in the capacity-dominated regime. ($E[\delta] = 200$ Wh, $\sigma_\delta = 50\sqrt{2}$ Wh and self-discharge per day of 0%, 20%, or 50%).

Now, using the method of [36] we find that

$$E[z(t_k) \mid t_k < k] = E[z(t_k) \mid z(t_k) > e^{\theta^*C}]$$

$$= e^{\theta^*C} E\left[e^{\theta^* \left(\sum_{j=0}^{t_k} \delta'(j)\gamma_j - C\right)} \left| \sum_{j=0}^{t_k} \delta'(j)\gamma_j > C\right]\right]$$

$$\geq e^{\theta^*C} \inf_{x \in \mathbb{R}} E\left[e^{\theta^* (\delta'(0) - x)} \mid \delta'(0) > x\right],$$

where the last line follows by conditioning on $t_k$ and on the value of $C - \sum_{j=1}^{t_k} \delta'(j)$. If $a(n)$ and $s(n)$ are independent one can condition also on $a(0)$ to obtain

$$E[z(t_k) \mid t_k < k] \geq e^{\theta^*C} \inf_{x \in \mathbb{R}} E\left[e^{\theta^* (s - x)} \mid s > x\right]. \tag{20}$$

The proof of Eq. (17) is concluded by inserting Eq. (20) into Eq. (19) and then using Eq. (18).

The theorem does not always provide accurate estimates in the capacity-dominated regime. The bounds on the loss probability are acceptable when $\gamma$ is close to zero and the randomness of the system is limited.

In Fig. 7 we evaluate Theorem 4 for a leakage queue with Gaussian supply and demand processes as in Subsec. V-A. We consider a storage capacity of $\leq 2$ kWh, which is clearly in
the capacity-dominated regime. For the parameters used in Subsec. V-A, the martingale bound does not result in good estimates. The results improve when we reduce the randomness of the stored energy process. In Fig. 7 we have done this by reducing the standard deviations to $\sigma_a = \sigma_s = 50$ Wh, resulting in $\sigma_\delta = 50\sqrt{2}$ Wh. In this case, the analytical results provide useful upper bounds on the underflow probabilities. Note that a storage system in the capacity-dominated regime with a high degree of randomness will result in high overflow probabilities as well as high underflow probabilities. While it is not apparent that such a parameter region is of interest for deployed energy storage systems, analytical methods that provide good bounds in this region remain an open problem.

VI. Evaluation of a Wind Energy Model

We next consider a leakage queue with a supply process that resembles a wind energy source. Our objectives are twofold. First, we want to see if the reference system remains useful in the context of more realistic, and more complex, random processes. Second, we want to evaluate the accuracy of our analysis. We use the wind speed process from [39], which models wind speed as an i.i.d. process with a Weibull distribution with density function

$$f_V(v) = \frac{k}{c} \left( \frac{v}{c} \right)^{k-1} e^{-\left( \frac{v}{c} \right)^k},$$

where $c$ and $k$, respectively, are the scale and shape parameters of the Weibull distribution. As in [39], we set the shape parameter to $k = 3$ in order to bound the degree of randomness of the wind speed. The scale parameter factor is set to $c = 7$ m/s, which results in an average wind speed of 6.25 m/s.

We consider a wind turbine with a rated power of $P_r = 1$ kW, which is comparable to a micro wind turbine for a residential home [3]. The output power of wind turbines, denoted by $P_w$ and expressed in kW/m$^2$ is a function of the wind speed $v$. Wind turbines are activated only when the wind speed is above a lower threshold (cut-in speed) and below an upper threshold (cut-out speed). The rated speed is the wind speed at which the wind turbine generates its rated
Using the power model from [5], we obtain

\[
P_w = \begin{cases} 
0 & v < v_{ci}, \\
\alpha v^3 - \beta P_r & v_{ci} \leq v \leq v_r, \\
P_r & v_r \leq v \leq v_{co}, \\
0 & v_{co} \leq v.
\end{cases}
\]  
(21)

where \(v_{ci}, v_r,\) and \(v_{co},\) respectively, are the cut-in, rated, and cut-out wind speeds, and \(\alpha\) and \(\beta\) are calculated such that Eq. (21) is continuous at \(v_{ci}\) and \(v_r,\) i.e., \(\alpha = \frac{P_r}{v_r^3 - v_{ci}^3}\) and \(\beta = \frac{v_{ci}^3}{v_r^3 - v_{ci}^3}.\)

The actual power from the wind turbine is given by [5]

\[
a(n) = P_w \cdot A_w \cdot \eta_w,
\]

where \(A_w\) and \(\eta_w\) are the total swept area and the efficiency of the wind turbine, respectively. The parameters of the wind turbine are summarized in Table I. With these parameters we obtain a supply process with \(E[a] = 1\) kWh and \(\sigma_a = 1050\) Wh.

**TABLE I: Parameters of the wind turbine.**

| Notation | Definition                  | Value   |
|----------|-----------------------------|---------|
| \(P_r\)  | Rated power                 | 1 kW    |
| \(v_{ci}\) | Cut-in wind speed           | 3 m/s   |
| \(v_{co}\) | Cut-out wind speed          | 25 m/s  |
| \(v_r\)  | Rated wind speed            | 12 m/s  |
| \(A_w\)  | Total swept area            | 10.8 m² |
| \(\eta_w\) | Wind turbine efficiency     | 50%     |

The demand process is set as the sum of a constant demand of 750 Wh and an i.i.d. exponential random value with average 50 Wh, resulting in \(E[s] = 800\) Wh and \(\sigma_s = 50\) Wh. We consider energy storage systems with a significant self-discharge, with leakage ratios \(\gamma = 0.0093\) (20% per day) and \(\gamma = 0.0285\) (50% per day), which is within the range of supercapacitors or flywheels. As before, we use 1 hour for the length of a time slot.
A. Impact of self-discharge

In Fig. 8, we show the underflow and overflow probability as a function of the size of the energy storage system for different leakage ratios. We depict the results of the Gaussian analysis and the simulations. Note that the average net charge $E[\delta]$ as well as the leakage ratios match the examples in Sec. V. Thus, by comparing Fig. 8 with Fig. 6, we can gauge the impact of replacing the Gaussian energy supply and demand with distributions that are quite different from a normal distribution. The results of the analysis and the simulations corroborate our earlier observations from Subsec. V-A. The underflow and overflow probabilities of a leakage queue are significantly different from those of a conventional finite-capacity queueing system ($\gamma = 0$). Different from a conventional queue, the underflow probability of a leakage queue does not vanish when the storage capacity is increased. The reverse is true for the overflow probability. As seen in Subsec. V-A, the conventional queueing system is generally at capacity with (an eventually) constant overflow probability. As for the accuracy of the Gaussian analysis, the estimates of the underflow and overflow probabilities are good in the leakage-dominated regime ($C > E[B^r]$). We conclude that the accuracy of the Gaussian analysis does not deteriorate when moving to non-Gaussian distributions for supply and demand.
Fig. 8: Underflow and overflow probabilities with wind energy source. ($E[a] = 1\ kWh, \ E[s] = 800\ Wh, \ \sigma_a = 1050\ Wh, \ \sigma_s = 50\ Wh$. The self-discharge per day is 0%, 20%, or 50%).
In Fig. 9, we present the underflow probabilities computed with the martingale analysis for a capacity-dominated regime where $C \leq 5 \text{kWh}$. Recall that, in this range, the overflow probability is always close to one. Since the martingale method does not provide usable bounds unless the underflow and the overflow probabilities are small, we reduce the randomness of the stored energy process $B(n)$ by reducing the demand process to $E[s] = 400 \text{ Wh}$ (and keeping $\sigma_s = 50 \text{ Wh}$.) Note that the expected values are selected to match those used in Fig. 6. Comparing Fig. 9 and Fig. 6, we see that the accuracy of the martingale analysis is comparable or even improved.

**B. Validation of the reference system**

We return to the main finding of this paper, which is the distinct behavior of the leakage queue in the leakage-dominated regime. In Subsec. III-C, we used empirical data to show that the reference system provides an accurate characterization of the leakage queue in the leakage-dominated regime. Further, in Subsec. V-A, we proved that the reference system approaches a normal distribution when $\gamma$ is small. Our final evaluation validates that the leakage queue in the leakage-dominated regime is well described by a normal distribution even if the supply and demand processes are not Gaussian.
We work with the same supply and demand processes described at the beginning of this section. We consider energy storage systems with $\gamma = 0.0093$ (20% self-discharge per day), and, therefore, $E[B^r] = 21.5$ kWh. Fig. 10(a) presents the CDF for the stored energy obtained from simulations of the leakage queue with capacities set to $C = 10, 20, 40$ kWh. The queue is leakage-dominated if $C = 40$ kWh, and capacity-dominated otherwise. We compare the distribution of the stored energy of the Gaussian approximation of the reference system (using Eq. (13)) with those obtained for a simulated leakage queue. We observe that for $C = 40$ kWh, the simulation closely follows the normal distribution of the reference system. In the capacity-dominated regime ($C = 10, 20$ kWh), the distributions of the stored energy do not resemble at all a normal distribution.

The characteristics of the distributions of the stored energy become more evident in a quantile-quantile (Q-Q) plot, where we compare the quantiles of the normal distribution with those of the simulations in 5% increments. In Fig. 10(b) we provide the quantiles of the normal distribution on the horizontal axis. The diagonal line, shown as a thin solid line, therefore depicts the normal distribution. The thick solid line (in gray) has the results for the skew normal distribution, using the skewness for $B^r$ from Eq. (15). The values obtained from the simulations are shown as colored data points. The closer the data points are to the diagonal, the better the match is with the normal distribution. We observe that the stored energy in a leakage-dominated regime ($C = 40$ kWh) is very close to the diagonal. The match is further improved with the skew normal distribution. The capacity-dominated regime ($C = 10, 20$ kWh) is obviously poorly matched with a normal distribution. Even, for $C = 20$ kWh, when the storage capacity is close to $E[B^r]$, the Q-Q plot is far from the diagonal.
Fig. 10: Comparison of distributions of the stored energy in a leakage queue. ($\gamma = 0.0093$, $C = 10, 20, 40$ kWh, $E[a] = 1$ kWh, $E[s] = 800$ Wh, $\sigma_a = 1050$ Wh, $\sigma_s = 50$ Wh).
VII. Conclusions

We presented an analysis of a queueing model for an energy storage system with self-discharge. The model, referred to as leakage queue, has, in addition to a supply and a demand process, a self-discharge process that removes storage content proportionally to the filling level. We identified two distinct parameter regimes for the leakage queue, which we called leakage-dominated regime and capacity-dominated regime. In the leakage-dominated regime, the queue settles in a steady state below the storage capacity. In the capacity-dominated regime, the leakage queue resembles a conventional finite-capacity queueing system. We presented analytical methods for computing probabilities of underflow and overflow, and evaluated their accuracy. Extensions of our work should address a relaxation of the discrete-time assumption to a continuous-time system as well as a relaxation of the i.i.d. assumptions to general stationary processes. The empirical solar irradiance data used in Subsec. III-C suggests that the regimes of the leakage queue manifest themselves even for non-stationary supply processes. It is also desirable to improve the martingale analysis for systems where the randomness of the system is high, and both underflow and overflow events occur frequently. Other directions for future research are an experimental confirmation of our findings on leakage queues in an actual energy storage system with self-discharge. An interesting question is how to take advantage of the observed effects of self-discharge to improve the design and control of energy storage systems.

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