SOME EIGENVALUE PROBLEMS INVOLVING THE \((p(\cdot), q(\cdot))-\)LAPLACIAN

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ABSTRACT. In this work, we are concerned with a Robin and Neumann problem with \((p(\cdot), q(\cdot))-\)Laplacian. Under some appropriate conditions on the data involved in the elliptic problem, we prove the existence of solutions applying two versions of Mountain Pass theorem, Ekeland’s variational principle and Lagrange multiplier rule.

1. INTRODUCTION

The aim of this paper is to investigate the eigenvalue problem

\[
\begin{cases}
-\Delta_p u - \Delta_q u = \lambda |u|^{r-2} & \text{in } \Omega, \\
\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \frac{\partial u}{\partial \nu} + \beta_1 |u|^{p-2} + \beta_2 |u|^{q-2} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n, n \geq 2\), is a bounded smooth domain, and \(\frac{\partial u}{\partial \nu}\) is the outer unit normal derivative on \(\partial \Omega\). We study (1.1) in distinct situations, and the solutions \(u\) will be sought in the variable exponent Sobolev space \(W := W^{1,\mathcal{M}(\cdot)}(\Omega)\), where \(\mathcal{M} = \max\{p, q\}\).

The main results of this work are the following theorems.

**Theorem 1.1.** Assume that \(p, q, r \in C_+(\bar{\Omega})\), \(\alpha \in L^\infty(\Omega)\), \(\alpha^- > 0\), and \(\beta_1, \beta_2 \in L^\infty(\partial \Omega)\) with \(\beta_1^-, \beta_2^- > 0\).

(A) If \(r^+ < \min\{p^-, q^-\}\), then any \(\lambda > 0\) is an eigenvalue for problem (1.1). Moreover, for any \(\lambda > 0\) there exists a sequence \(\{u_k\}\) of nontrivial weak solutions for problem (1.1) such that \(u_k \to 0\) in \(W\).

(B) If \(r^- < \min\{p^-, q^-\}\) and \(r^+ < (\mathcal{M}^*)^-, \) then there exists \(\Lambda > 0\) such that any \(\lambda \in (0, \Lambda)\) is an eigenvalue for problem (1.1).

(C) If \(\mathcal{M}^+ < r^- \leq r^+ < (\mathcal{M}^*)^-\), then for any \(\lambda > 0\), the problem (1.1) possesses a nontrivial weak solution.

**Theorem 1.2.** Assume that \(p \in C_+(\bar{\Omega})\), \(r \equiv q, q > 2\) is a constant, \(\alpha \in L^\infty(\Omega)\), \(\alpha^- > 0\), and \(\beta_1 \equiv \beta_2 \equiv 0\).

(A) Suppose that \(p^+ < q\). Then the eigenvalue set of problem (1.1) is precisely \(\{0\} \cup \left\{\inf_{u \in C_q \setminus \{0\}} \left\{\int_{\Omega} |\nabla u|^q dx / \int_{\Omega} |u|^r dx, \infty\right\}\right\}\).

(B) Suppose that \(q < p^-\). Then the eigenvalue set of problem (1.1) is precisely \(\{0\} \cup \left\{\inf_{u \in C_p \setminus \{0\}} \left\{\int_{\Omega} |\nabla u|^p dx / \int_{\Omega} |u|^{p-q} dx, \infty\right\}\right\}\).

2. PRELIMINARIES

We first recall some facts on the variable exponent spaces \(L^{p(\cdot)}(\Omega)\) and \(W^{1,p(\cdot)}(\Omega)\). For more detail, see [6, 7]. Suppose that \(\Omega \subset \mathbb{R}^n\) is a bounded open domain with smooth boundary \(\partial \Omega\) and \(p \in C_+(\bar{\Omega})\), where

\[ C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) \mid \inf_{\Omega} p > 1 \right\}. \]

The variable exponent Lebesgue space \(L^{p(\cdot)}(\Omega)\) is defined by

\[ L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} |u|^p dx < \infty \right\}, \]

with the norm

\[ \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0 \mid \int_{\Omega} \left( \frac{|u|}{\tau} \right)^p dx \leq 1 \right\}. \]

The variable exponent Sobolev space \(W^{1,p(\cdot)}(\Omega)\) is defined by

\[ W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}, \]

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with the norm
\[ \|u\|_{W^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}. \]

Both \(L^p(\Omega)\) and \(W^{1,p}(\Omega)\) are separable, reflexive and uniformly convex Banach spaces, see \([7, 11]\).

**Proposition 2.1.** (see \([11]\) Theorem 2.1). For any \(u \in L^p(\Omega)\) and \(v \in L^{\frac{p}{p-1}}(\Omega)\), we have
\[ \int_\Omega |uv| \,dx \leq 2\|u\|_{L^p(\Omega)} \|v\|_{L^{\frac{p}{p-1}}(\Omega)}. \]

**Proposition 2.2.** (see \([6, 7]\)). If \(q \in C(\Omega)\) and
\[ 1 \leq q < p^* \quad \text{on } \bar{\Omega}, \]
then there is a compact embedding \(W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)\), where
\[ p^*(x) = \begin{cases} \frac{np(x)}{n - p(x)} & \text{if } p(x) < n, \\ \infty & \text{if } p(x) \geq n. \end{cases} \]

Now, we introduce a norm which will be used later. Let \(\beta \in L^\infty(\partial \Omega)\), with \(\beta^- := \text{ess inf}_{\partial \Omega} \beta > 0\), and \(u \in W^{1,p}(\Omega)\), define
\[ \|u\|_{p, \beta} = \inf \left\{ \tau > 0 \mid \int_\Omega \frac{\|\nabla u\|^p}{\tau} \,dx + \int_{\partial \Omega} \beta \frac{|u|^p}{\tau} \,d\sigma \leq 1 \right\}, \]
where \(d\sigma\) is the measure on the boundary \(\partial \Omega\). By \([5]\) Theorem 2.1, \(\| \cdot \|_{p, \beta}\) is also a norm on \(W^{1,p}(\Omega)\) which is equivalent to \(\| \cdot \|_{W^{1,p}(\Omega)}\).

For \(p \in C(\bar{\Omega})\), we will write \(p^- := \text{inf}_{\Omega} p\) and \(p^+ := \text{sup}_{\Omega} p\). An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by the following

**Proposition 2.3.** (see \([5, Proposition 2.4]\)). Let \(\rho_{p, \beta}(u) := \int_\Omega |\nabla u|^p \,dx + \int_{\partial \Omega} \beta |u|^p \,d\sigma\). For \(u \in W^{1,p}(\Omega)\) we have
\[(i) \quad \|u\|_{p, \beta} < 1 (= 1, > 1) \Leftrightarrow \rho_{p, \beta}(u) < 1 (= 1, > 1), \]
\[(ii) \quad \|u\|_{p, \beta} \leq 1 \Rightarrow \|u\|_{p, \beta}^- \leq \rho_{p, \beta}(u) \leq \|u\|_{p, \beta}^+. \]
\[(iii) \quad \|u\|_{p, \beta} \geq 1 \Rightarrow \|u\|_{p, \beta}^+ \leq \rho_{p, \beta}(u) \leq \|u\|_{p, \beta}^-.

The Euler-Lagrange functional associated with \([11]\) is defined as \(\Phi_\lambda : W \to \mathbb{R}\),
\[ \Phi_\lambda(u) = \int_\Omega \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q \,dx + \int_{\partial \Omega} \frac{\beta_1}{p} |u|^p + \frac{\beta_2}{q} |u|^q \,d\sigma - \lambda \int_\Omega \frac{1}{r} |u|^r \,dx, \]
where \(p, q, r \in C_+(\bar{\Omega})\), \(r^- < (M^*)^-\), \(\alpha \in L^\infty(\Omega)\), \(\alpha^- > 0\), and \(\beta_1, \beta_2 \in L^\infty(\partial \Omega)\), with \(\beta_1^-, \beta_2^- > 0\) or \(\beta_1 \equiv \beta_2 \equiv 0\). Standard arguments imply that \(\Phi_\lambda \in C^1(W, \mathbb{R})\) and
\[ \langle \Phi'_\lambda(u), v \rangle = \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \nabla v \,dx + \int_{\partial \Omega} (\beta_1 |u|^{p-2} + \beta_2 |u|^{q-2}) uv \,d\sigma \]
for all \(u, v \in W\). Thus, the weak solutions of \([11]\) coincide with the critical points of \(\Phi_\lambda\). If such a weak solution exists and is nontrivial, then the corresponding \(\lambda\) is an eigenvalue of problem \([11]\).

Finally, we define \(L_{p, \lambda} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\perp\) by
\[ \langle L_{p, \lambda}(u), v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \,dx + \int_{\partial \Omega} \beta |u|^{p-2} uv \,d\sigma, \]
where \(\beta \in L^\infty(\partial \Omega)\) and \(\beta^- > 0\).

**Proposition 2.4.** (see \([9, Proposition 2.2]\)). \(L_{p, \lambda}\) is a mapping of type \((S_+), i.e., if \(u_k \to u \) in \(W^{1,p}(\Omega)\), and
\[ \lim_{k \to \infty} \langle L_{p, \lambda}(u_k) - L_{p, \lambda}(u), u_k - u \rangle \leq 0. \] Then \(u_k \to u \) in \(W^{1,p}(\Omega)\).
3. Main Results

We organize our main results into five sections. In Case 3.1 we prove Theorem 1.1 (A). In Case 3.2 we prove Theorem 1.1 (B). In Case 3.3 we prove Theorem 1.1 (C). In Case 3.4 we prove Theorem 1.2 (A). In Case 3.5 we prove Theorem 1.2 (B). In the first three sections we will follow [10], and in the last two sections we follow [12].

3.1. The case $r^+ < \min\{p^-, q^-\}$. We want to apply the symmetric mountain pass lemma in [10]. We start with the following

Definition 3.1. Let $X$ be a Banach space and $E$ a subset of $X$. $E$ is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set $E$ which does not contain the origin, we define a genus $\gamma(E)$ of $E$ by the smallest integer $k$ such that there exists an odd continuous mapping from $E$ to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a $k$, we define $\gamma(E) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let $\Gamma_k$ denote the family of closed symmetric subsets $E$ of $X$ such that $0 \notin E$ and $\gamma(E) \geq k$.

Assumption A. Let $X$ be an infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfy (A1) and (A2) below.

(A1) $I(u)$ is even, bounded from below, $I(0) = 0$, and $I(u)$ satisfies the Palais-Smale condition (PS).

(A2) For each $k \in \mathbb{N}$, there exists an $E_k \in \Gamma_k$ such that $\text{sup}_{u \in E_k} I(u) < 0$.

Theorem 3.1. (Symmetric mountain pass lemma) Under Assumption (A), either (i) or (ii) below holds.

(i) There exists a sequence $\{u_k\}$ such that $I'(u_k) = 0$, $I(u_k) < 0$, and $\{u_k\}$ converges to zero.

(ii) There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k \to \infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k \to \infty} I(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.

We will need the following result.

Lemma 3.1. (i) The functional $\Phi_{\lambda}$ satisfies the condition (A1).

(ii) The functional $\Phi_{\lambda}$ satisfies the condition (A2).

Proof. (i) It is clear that $\Phi_{\lambda}$ is even and $\Phi_{\lambda}(0) = 0$. Let $u \in \mathcal{W}$. Since $r^+ < \mathcal{M}^-$, by Young’s inequality, we have

$$|u|^r \leq \varepsilon|u|^\mathcal{M} + C_1(\varepsilon, r, \mathcal{M}) \quad \text{on } \Omega.$$ 

Then

$$\Phi_{\lambda}(u) \geq C_2 \left( \int_\Omega |\nabla u|^\mathcal{M} dx + \int_{\partial \Omega} |u|^\mathcal{M} d\sigma \right) - \frac{\lambda \alpha^+}{r^-} \left( \varepsilon \int_\Omega |u|^\mathcal{M} dx + C_1|\Omega| \right),$$

where $C_2 = \min \{1, \beta_1^{-1}, \beta_2^{-1} \}$.

Choosing $\varepsilon = \frac{r^- C_1}{2 \Delta \alpha^+}$, we have

$$\Phi_{\lambda}(u) \geq \frac{C_2}{2} \left( \int_\Omega |\nabla u|^\mathcal{M} dx + \int_{\partial \Omega} |u|^\mathcal{M} d\sigma \right) - \frac{\lambda \alpha^+ C_1}{r^-} |\Omega|.$$ 

Therefore, $\Phi_{\lambda}$ is bounded from below and coercive. It remains to show that the functional $\Phi_{\lambda}$ satisfies the (PS) condition to complete the proof. Let $\{u_k\} \subset \mathcal{W}$ be a sequence such that

$$\{\Phi_{\lambda}(u_k)\} \text{ is bounded and } \Phi_{\lambda}'(u_k) \to 0 \text{ in } \mathcal{W}^*.$$ 

Then, by the coercivity of $\Phi_{\lambda}$, the sequence $\{u_k\}$ is bounded in $\mathcal{W}$. By the reflexivity of $\mathcal{W}$ and Proposition 2.2, for a subsequence still denoted $\{u_k\}$, we have

$$u_k \to u \text{ in } \mathcal{W} \quad \text{and} \quad u_k \to u \text{ in } L^{r^+}(\Omega).$$

Therefore,

$$\langle \Phi_{\lambda}'(u_k) - \Phi_{\lambda}'(u), u_k - u \rangle \to 0 \quad \text{and} \quad \int_\Omega \alpha (|u_k|^{-2} u_k - |u|^{-2} u) (u_k - u) dx \to 0.$$

Thus

$$\langle (L_{p, \beta_1} + L_{q, \beta_2})(u_k) - (L_{p, \beta_1} + L_{q, \beta_2})(u), u_k - u \rangle \to 0.$$
On the other hand, we have \(|a|^{-2}a - |b|^{-2}b \cdot (a - b) \geq 0\) for all \(a, b \in \mathbb{R}^m \setminus \{0\}\) with \(m \geq 1\) and \(\sigma > 1\). Consequently
\[
\begin{align*}
 u_k &\to u \text{ in } W^{1,p}(\Omega), \quad \limsup_{k \to \infty} (L_{p,\beta_1}(u_k) - L_{p,\beta_1}(u), u_k - u) \leq 0, \\
 u_k &\to u \text{ in } W^{1,q}(\Omega), \quad \limsup_{k \to \infty} (L_{q,\beta_2}(u_k) - L_{q,\beta_2}(u), u_k - u) \leq 0.
\end{align*}
\]
Therefore, according to Lemma 2.4, \(u_k \to u\) in \(\mathcal{W}\). The proof is complete.

(ii) Let \(\{\varphi_k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}^n)\) be such that \(\text{supp}(\varphi_k) \subset \Omega\), \(\{\varphi_k \neq 0\} \neq \emptyset\), and \(\text{supp}(\varphi_j) \cap \text{supp}(\varphi_k) = \emptyset\) if \(j \neq k\). Take \(F_k = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\}\), it is clear that \(\dim F_k = k\).

Denote

\[
\ell_k = \inf_{u \in S_{\beta_1} \cap F_k} \int_{\Omega} |u|^p \, dx > 0 \quad \text{and} \quad \mu_k = \sup_{u \in S_{\beta_1} \cap F_k} \int_{\Omega} |\nabla u|^q \, dx > 0,
\]

where \(S_{\beta_1} = \{u \in \mathcal{W} \mid |u|_{p,\beta_1} = 1\}\). Write \(E_k(t) = t (S_{\beta_1} \cap F_k)\) for \(0 < t < 1\). Obviously, \(\gamma(E_k(t)) = k\), for all \(t \in (0, 1)\). We deduce, \(E_k(t) \in \Gamma_k\).

Let \(k \in \{1, 2, \ldots\}\). Now, we show that there exists \(t_0 \in (0, 1)\) such that
\[
\sup_{u \in E_k(t_0)} \Phi(\lambda) < 0.
\]

Indeed, we have \(\lambda > \frac{r^+}{\alpha - \ell_k} \left( \frac{1}{p} t_0^{p^-} + \frac{\mu_k}{q} t_0^{q^-} \right)\) for some \(t_0 \in (0, 1)\). Then,
\[
\begin{align*}
\sup_{u \in E_k(t_0)} \Phi(\lambda) &= \sup_{v \in S_{\beta_1} \cap F_k} \Phi(\lambda)(t_0 v) \\
&= \sup_{v \in S_{\beta_1} \cap F_k} \int_{\Omega} \frac{t_0^p}{p} |\nabla v|^p + \frac{t_0^q}{q} |\nabla v|^q \, dx - \lambda \int_{\Omega} \frac{\alpha t_0 r^+}{r} |v|^r \, dx \\
&\leq \frac{1}{p} t_0^{p^-} + \frac{\mu_k}{q} t_0^{q^-} - \frac{\lambda \alpha \ell_k r^+}{r} t_0^{r^+} < 0.
\end{align*}
\]

This completes the proof.

\[\square\]

**Proof of Theorem 1.1** \((\mathbb{A})\). By Lemma 3.1 and Theorem 3.1, \(\Phi_\lambda\) admits a sequence of nontrivial weak solutions \(\{u_k\}\) such that for any \(k\), we have
\[
u_k \neq 0, \quad \Phi_\lambda(u_k) = 0, \quad \Phi_\lambda(u_k) \leq 0, \quad \text{and} \ u_k \to 0 \text{ in } \mathcal{W}.
\]

\[\square\]

### 3.2. The case \(r^- < \min\{p^-, q^-\}\) and \(r^+ < (M^*)^-\).

Since \(r^+ < (M^*)^-\), from Proposition 2.2, it follows that \(\mathcal{W}\) is continuously embedded in \(L^{r^+}(\Omega)\).

Write
\[
C_* := \sup_{u \in \mathcal{W} \setminus \{0\}} \frac{||u||_{L^{r^+}(\Omega)}}{||u||_{M, 1}} < \infty.
\]

To prove the Theorem 1.1 \((\mathbb{B})\), we use the Ekeland’s variational principle (see [3]):

**Theorem 3.2.** (Ekeland Principle-weak form). Let \((X, d)\) be a complete metric space. Let \(\Phi : X \to \mathbb{R} \cup \{\infty\}\) be lower semicontinuous and bounded below. Then given \(\varepsilon > 0\) there exist \(u_\varepsilon \in X\) such that
\[
\Phi(u_\varepsilon) \leq \inf_X \Phi + \varepsilon,
\]
and
\[
\Phi(u_\varepsilon) < \Phi(u) + \varepsilon d(u, u_\varepsilon), \quad \text{for all } u \in X \text{ with } u \neq u_\varepsilon.
\]

We have the following auxiliary

**Lemma 3.2.**

(i) There is \(\Lambda > 0\) so that for any \(\lambda \in (0, \Lambda)\) there exist \(\rho, a > 0\) such that \(\Phi_\lambda(u) \geq a\) for any \(u \in \mathcal{W}\) with \(||u||_{M, 1} = \rho\).

(ii) There exists \(\xi \in \mathcal{W}\) such that \(\xi \geq 0, \xi \neq 0\) and \(\Phi_\lambda(t\xi) < 0\), for \(t > 0\) small enough.
Proof. (i) Fix $\rho \in (0, \min \{C_*^{-1}, 1\})$, where $C_*$ is given by (3.3). Hence, if $u \in W$ with $\|u\|_{M,1} = \rho$, we have $\|u\|_{L^r(\Omega)} \leq C_* \rho < 1$ and

$$
\Phi_\lambda(u) \geq \frac{\min\{1, \beta_1^-, \beta_2^--\}}{\max\{p^+, q^+\}} \left( \int |\nabla u|^M dx + \int_{\partial \Omega} |u|^M d\sigma \right) - \frac{\lambda \alpha}{r^-} \int_{\Omega} |u|^r dx
$$

$$
\geq \frac{\min\{1, \beta_1^-, \beta_2^--\}}{\max\{p^+, q^+\}} \|u\|_{M,1}^{r+} - \lambda \frac{\alpha}{r^-} \|u\|_{M,1}^{r-} = \rho^{r-} \left( \frac{\min\{1, \beta_1^-, \beta_2^--\}}{\max\{p^+, q^+\}} \rho^{M^+-r-} - \frac{\lambda \alpha}{r^-} C_*^{-r-} \right).
$$

Choose

$$
\Lambda = \frac{\min\{1, \beta_1^-, \beta_2^--\}}{\max\{p^+, q^+\}} \rho^{2(M^+-r-)}.
$$

Then, if $\lambda \in (0, \Lambda)$:

$$
\Phi_\lambda(u) \geq \rho^{M^+} \frac{\min\{1, \beta_1^-, \beta_2^--\} (1 - \rho^{M^+-r-})}{\max\{p^+, q^+\}} > 0.
$$

This completes the proof.

(ii) Hence $r^- < \min\{p^-, q^-\}$, there exists $\varepsilon > 0$ such that

$$
r^- + \varepsilon < \min\{p^-, q^-\}.
$$

Since $r \in C(\bar{\Omega})$, there is an open set $U \subset \Omega$ for which

$$
|r - r^-| < \varepsilon, \quad \text{in } U.
$$

Thus,

$$
r < r^- + \varepsilon < \min\{p^-, q^-\}, \quad \text{in } U.
$$

Take $\xi = 1$. For all $t \in (0, 1)$, we obtain

$$
\Phi_\lambda(t\xi) \leq \frac{\beta_1^+[\partial \Omega]}{p^-} t^{p-} + \frac{\beta_2^+[\partial \Omega]}{q^-} t^{q-} - \frac{\lambda \alpha}{r^+} t^{r+} + \varepsilon
$$

$$
= \left( \frac{\beta_1^+[\partial \Omega]}{p^-} t^{p-} - \varepsilon + \frac{\beta_2^+[\partial \Omega]}{q^-} t^{q-} - \varepsilon + \frac{\lambda \alpha}{r^+} t^{r+} \right) t^{r-}.
$$

Then, for any $t > 0$ small enough, we have

$$
\Phi_\lambda(t\xi) < 0.
$$

The proof is complete.

Proof of Theorem 1.1. By Lemma 5.2 (i), we have

$$
\inf_{\partial B_\rho} \Phi_\lambda > 0,
$$

where $\partial B_\rho = \{u \in W : \|u\|_{M,1} = \rho\}$ and $B_\rho = \{u \in W : \|u\|_{M,1} < \rho\}$.

Using the estimate (3.4), it follows that

$$
\Phi_\lambda(u) \geq -\frac{\lambda \alpha}{r^-} (C_* \rho)^{r^-} \quad \text{for } u \in B_\rho.
$$

Hence, by Lemma 5.2 (ii),

$$
-\infty < \inf_{B_\rho} \Phi_\lambda < 0.
$$

Let

$$
0 < \varepsilon < \inf_{\partial B_\rho} \Phi_\lambda - \inf_{B_\rho} \Phi_\lambda.
$$

Then, by applying Ekeland’s variational principle to the functional

$$
\Phi_\lambda : \overline{B_\rho} \to \mathbb{R},
$$

there exists $u_\varepsilon \in \overline{B_\rho}$ such that

$$
\Phi_\lambda (u_\varepsilon) \leq \inf_{B_\rho} \Phi_\lambda + \varepsilon,
$$

$$
\Phi_\lambda (u_\varepsilon) < \Phi_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{M,1}, \quad \text{for all } u \in \overline{B_\rho} \text{ with } u \neq u_\varepsilon.
$$
Since \( \Phi_\lambda (u) \leq \inf_{B_r} \Phi_\lambda + \varepsilon < \inf_{\partial B_r} \Phi_\lambda \), we deduce \( u \in B_r \). Then, for \( t > 0 \) small enough and \( v \in B_1 \), we have
\[
\frac{\Phi_\lambda (u_\varepsilon + tv) - \Phi_\lambda (u_\varepsilon)}{t} + \varepsilon \|v\|_{M,1} > 0.
\]
As \( t \to 0^+ \), we obtain
\[
\langle \Phi'_\lambda (u_\varepsilon), v \rangle + \varepsilon \|v\|_{M,1} \geq 0, \quad \text{for all } v \in B_1.
\]
Hence, \( \|\Phi'_\lambda (u_\varepsilon)\|_{W^*} \leq \varepsilon \). We deduce that there exists a sequence \( \{u_k\} \subset B_r \) such that
\[
\Phi_\lambda (u_k) \to \inf_{B_r} \Phi_\lambda \quad \text{and} \quad \Phi'_\lambda (u_k) \to 0 \quad \text{in } W^*.
\]
As in the proof of Lemma 3.1 (i), for a subsequence, we obtain \( u_k \to u \) in \( W \). Thus, by (3.5) we have
\[
\Phi_\lambda (u) = \inf_{B_r} \Phi_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda (u) = 0.
\]
So finishes up the proof.

3.3. The case \( M^+ < r^- \leq r^+ < (M^*)^- \). We will prove the Theorem (1) using the Mountain Pass Theorem (see (13)).

Theorem 3.3. (Ambrosetti-Rabinowitz). Let \( I \in C^1(X, \mathbb{R}) \) be satisfying the (PS) condition on the real Banach space \( X \). Let \( u_0, u_1 \in X, c_0 \in \mathbb{R} \) and \( R > 0 \) be such that

(i) \( \|u_1 - u_0\| > R \),
(ii) \( \max \{I(u_0), I(u_1)\} < c_0 \leq I(v) \) for all \( v \) such that \( \|v - u_0\| = R \).

Then \( I \) has a critical point \( u \) with \( I(u) = c, c \geq c_0 \); the critical value \( c \) is defined by
\[
c = \inf_{p \in K} \sup_{t \in [0,1]} I(p(t)),
\]
where \( K \) denotes the set of all continuous maps \( p : [0,1] \to X \) with \( p(0) = u_0 \) and \( p(1) = u_1 \).

We need the following

Lemma 3.3. (i) There exist \( \eta, b > 0 \) such that \( \Phi_\lambda (u) \geq b \) for any \( u \in W \) with \( \|u\|_W = \eta \).
(ii) There is \( \zeta \in W \) such that \( \|\zeta\|_W > \eta \) and \( \Phi_\lambda (\zeta) < 0 \), where \( \eta \) is given in (i).
(iii) The functional \( \Phi_\lambda \) satisfies the condition (PS).

Proof. (i) Assume that \( \|u\|_{M,1} \leq \min \{C^{-1}, 1\} \), where \( C \) is given by (3.3). Then
\[
\Phi_\lambda (u) \geq \frac{\min \{1, \beta_1^+, \beta_2^+\}}{\max \{p^+, q^+\}} \left( \int |\nabla u|^M dx + \int_{\partial \Omega} |u|^M d\sigma \right) - \frac{\lambda \alpha^+ C^{-\gamma}}{r^-} \|u\|_{M,1}^{r^-}.
\]
Since \( M^+ < r^- \), assertion (i) follows.

(ii) Take \( \xi = 1 \). For \( t > 1 \), we have
\[
\Phi_\lambda (t\xi) \leq \frac{\beta_1^+ |\partial \Omega|}{p^-} t^{p^-} + \frac{\beta_2^+ |\partial \Omega|}{q^-} t^{q^-} - \frac{\lambda \alpha^- |\Omega|}{r^+} t^{r^-}.
\]
Since \( \max \{p^+, q^+\} < r^- \), for \( t > 1 \) large enough, there is \( \zeta = t\xi \) such that \( \|\zeta\|_W > \eta \) and \( \Phi_\lambda (\zeta) < 0 \). This completes the proof.

(iii) Let \( \{u_k\} \subset W \) be a sequence such that \( \sup_k \Phi_\lambda (u_k) < \infty \) and \( \Phi'_\lambda (u_k) \to 0 \) in \( W^* \). First we prove that \( \{u_k\} \) is bounded. We argue by contradiction. Let \( \varepsilon > 0 \) be so that \( r^- > M^+ + \varepsilon \), and suppose
\[
\|u_k\|_{M,1} \to \infty, \quad (M^+ + \varepsilon) \|u_k\|_{M,1} \geq -\langle \Phi'_\lambda (u_k), u_k \rangle, \quad \text{and} \quad \|u_k\|_{M,1} > 1 \text{ for any } k.
\]
We know from [8, Theorem 6.2.29] that

\[ \text{Theorem 3.4.} \]

\[ \text{The Case} \]

\[ \text{3.4.} \]

\[ y \]

\[ \text{surjective operator}. \]

Then there exist \( C \) such that

\[ \text{Lemma 3.4.} \]

\[ \text{for all} \]

\[ \lambda > \sigma \in \mathbb{N} \]

\[ \text{where} \]

\[ \text{following expression} \]

\[ \text{Consequently}, \]

\[ \text{max} \{p^+, q^+\} < \mathcal{M}^+ + \varepsilon \text{ and } \mathcal{M}^- > 1, \text{ we have a contradiction. So, the sequence} \{u_k\} \text{ is bounded in } \mathcal{W} \]

\[ \text{and similar arguments as those used in the proof of Lemma 3.1 (i)} \text{ completes the proof.} \]

\[ \text{Proof of Theorem 1.1 \textbf{(ii)}}. \]

From Lemma 3.3 (i) and (ii), we deduce that \( \Phi_\lambda \) has a critical point \( u \) with

\[ \Phi_\lambda(u) = \inf_{\gamma \in K, t \in [0,1]} \sup_{\gamma(t)} \Phi_\lambda(\gamma(t)) \geq b, \]

where \( K = \{ \gamma \in C([0,1], \mathcal{W}) \mid \gamma(0) = 0 \text{ and } \gamma(1) = \zeta \}. \) This completes the proof.

\[ \text{\\}\]

\[ \text{3.4. The Case} \quad p^+ < q \quad \text{and} \quad q > 2. \]

Let us recall the following theorem (see [14]).

\[ \text{Theorem 3.4. (Lagrange multiplier rule)} \]

\[ \text{Let } X \text{ and } Y \text{ be real Banach spaces and let } f : D \to \mathbb{R}, G : D \to Y \text{ be } C^1 \text{ functions on the open set } D \subset X. \text{ If } u \text{ is a minimum point of } f \text{ on } \{ x \in D \mid G(x) = 0 \}, \text{ and } G'(u) \text{ is a surjective operator. Then there exist } y^* \in Y^* \text{ such that} \]

\[ f'(u) + y^* \circ G'(u) = 0. \]

Observe that \( p^+ < q \) implies \( \mathcal{W} = W^{1,q}(\Omega) \). Define

\[ \mathcal{C}_q := \left\{ u \in W^{1,q}(\Omega) \mid \int_\Omega \alpha |u|^{q-2}u dx = 0 \right\}. \]

We know from [8] Theorem 6.2.29 that

\[ \sigma_1 := \inf_{u \in \mathcal{C}_q \setminus \{0\}} \frac{\int_\Omega |\nabla u|^q dx}{\int_\Omega \alpha |u|^q dx} > 0. \]

Inspired by [2] Section 2.3.2] will be to consider the restriction of \( \Phi_\lambda \) to the Nehari-type manifold defined by

\[ \mathcal{N}_\lambda := \left\{ u \in \mathcal{C}_q \setminus \{0\} \mid \langle \Phi_\lambda'(u), u \rangle = 0 \right\}, \]

where \( \lambda > \sigma_1 \). In Lemma 3.4 below we prove that \( \mathcal{N}_\lambda \) is nonempty. We recall that the functional \( \Phi_\lambda \) has the following expression

\[ \Phi_\lambda(u) = \int_\Omega \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q dx - \lambda \int_\Omega \alpha |u|^q dx. \]  \[ (3.6) \]

Then, by definition,

\[ \int_\Omega |\nabla u|^p + |\nabla u|^q dx = \lambda \int_\Omega \alpha |u|^q dx, \]  \[ (3.7) \]

for all \( u \in \mathcal{N}_\lambda \). Furthermore, for all \( u \in \mathcal{N}_\lambda \), we have

\[ \Phi_\lambda(u) = \int_\Omega \frac{q-p}{qp} |\nabla u|^p dx. \]  \[ (3.8) \]

Consequently,

\[ m_\lambda := \inf_{u \in \mathcal{N}_\lambda} \Phi_\lambda(u) \geq 0. \]

We have the following result needed later.

\[ \text{Lemma 3.4.} \]

\[ (i) \quad \mathcal{N}_\lambda \neq \emptyset. \]
(ii) Every minimizing sequence for $\Phi_\lambda$ on $\mathcal{N}_\lambda$ is bounded in $W^{1,q}(\Omega)$, i.e., if $\{u_k\} \subset \mathcal{N}_\lambda$ and $\Phi_\lambda(u_k) \to m_\lambda$, then $\sup_k \|u_k\|_{W^{1,q}(\Omega)} < \infty$.

(iii) $m_\lambda > 0$.

(iv) There exists $u_\Phi \in \mathcal{N}_\lambda$ such that $\Phi_\lambda(u_\Phi) = m_\lambda$.

Proof. (i) Since $\lambda > \inf_{u \in C_q \setminus \{0\}} \int_\Omega |\nabla u|^q dx$, there exists $u \in C_q \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^q dx < \lambda \int_\Omega |u|^q dx.$$ 

Hence, there exist $t_1, t_2 \in \mathbb{R}$ so that $0 < t_1 < 1 < t_2$ and

$$\int_\Omega t_2^{p-q} |\nabla u|^p dx < \lambda \int_\Omega |u|^q dx - \int_\Omega |\nabla u|^q dx < t_1^{p-q} \int_\Omega |\nabla u|^p dx.$$

Thus, we conclude that there exists $t \in (t_1, t_2)$ for which

$$\int_\Omega t^{p-q} |\nabla u|^p dx = \lambda \int_\Omega |u|^q dx - \int_\Omega |\nabla u|^q dx.$$ 

Therefore, $tu \in \mathcal{N}_\lambda$.

(ii) Let $\{u_k\}$ be a minimizing sequence. We argue by contradiction. Assume that $\|u_k\|_{W^{1,q}(\Omega)} \to \infty$. Then, by (3.6), it follows that $\int_\Omega |u_k|^q dx \to \infty$. Set $v_k := \frac{u_k}{\|u_k\|_{L^q(\Omega)}}$. Since $\int_\Omega |\nabla u_k|^q dx < \lambda \alpha^+ \int_\Omega |u_k|^q dx$, we deduce $\int_\Omega |\nabla v_k|^q dx < \lambda \alpha^+$. Thus, $\{v_k\}$ is bounded in $W^{1,q}(\Omega)$. From the reflexivity of $W^{1,q}(\Omega)$ and Propositions 2.2, it follows that there exist $v_0 \in W^{1,q}(\Omega)$ and a subsequence (still denoted $\{u_k\}$) such that $v_k \rightharpoonup v_0$ in $W^{1,q}(\Omega)$ (hence in $W^{1,q}(\Omega)$ as well), and $v_k \to v_0$ in $L^q(\Omega)$. Hence, by Lebesgue’s dominated convergence, $v_0 \in C_q$.

By (3.3), we obtain

$$\int_\Omega |\nabla v_k|^p dx \to 0.$$ 

Next, since $v_k \to v_0$ in $W^{1,p(\cdot)}(\Omega)$, we infer that

$$\int_\Omega |\nabla v|^p dx \leq \liminf_{k \to \infty} \int_\Omega |\nabla v_k|^p dx = 0$$

and consequently $v_0$ is a constant function. From $v_0 \in C_q$, $v_0 = 0$. It follows that $v_k \to 0$ in $L^q(\Omega)$, which contradicts the fact that $\|v_k\|_{L^q(\Omega)} = 1$ for all $k$. Consequently, $\{u_k\}$ must be bounded in $W^{1,q}(\Omega)$.

(iii) Assume by contradiction that $m_\lambda = 0$. Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence. By (ii), we know that $\{u_k\} \subset C_q \setminus \{0\}$ is bounded in $W^{1,q}(\Omega)$. It follows that there exists $u_0 \in W^{1,q}(\Omega)$ such that (on a subsequence, again denoted $\{u_k\}$) one has $u_k \rightharpoonup u_0$ in $W^{1,q}(\Omega)$, and $u_k \to u_0$ in $L^q(\Omega)$. Therefore, $u_0 \in C_q$ and

$$\int_\Omega |\nabla u_0|^p dx \leq \liminf_{k \to \infty} \int_\Omega |\nabla u_k|^p dx = 0,$$

because (3.3). Consequently $u_0 = 0$, since $u_0 \in C_q$.

Write $v_k := u_k/\|u_k\|_{L^q(\Omega)}$. Then, by (3.7) and $q > p^-$,

$$\int_\Omega |\nabla v|^p dx \to 0.$$ 

From $\int_\Omega |\nabla u_k|^q dx < \lambda \alpha^+ \int_\Omega |u_k|^q dx$, we have $\{v_k\} \subset C_q$ is bounded in $W^{1,q}(\Omega)$. It follows that, for a subsequence still denoted $\{u_k\}$, there exists $v_0 \in C_q$ such that $v_k \rightharpoonup v_0$ in $W^{1,q}(\Omega)$ and $v_k \to v_0$ in $L^q(\Omega)$. Next, we see

$$\int_\Omega |\nabla v_0|^p dx \leq \liminf_{k \to \infty} \int_\Omega |\nabla v_k|^p dx = 0,$$

and consequently $v_0$ is a constant function. In fact, $v_0 = 0$. Thus, $v_k \to 0$ in $L^q(\Omega)$, which contradicts the fact that $\|v_k\|_{L^q(\Omega)} = 1$ for all $k$. Consequently, $m_\lambda$ is positive, as asserted.

(iv) Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence. By (ii), $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$. Thus, there exists $u_\Phi \in C_q$ such that (on a subsequence, again denoted $\{u_k\}$) one has $u_k \rightharpoonup u_\Phi$ in $W^{1,q}(\Omega)$, and $u_k \to u_\Phi$ in $L^q(\Omega)$. We deduce

$$\Phi_\lambda(u_\Phi) \leq \liminf_{k \to \infty} \Phi_\lambda(u_k) = m_\lambda.$$ (3.9)
If \( u_\Phi = 0 \), arguing as in the proof of (iii), we are led to a contradiction. Consequently \( u_\Phi \in C_q \setminus \{ 0 \} \). Now, from (3.7), we deduce
\[
\int_\Omega |\nabla u_\Phi|^p + |\nabla u_\Phi|^q dx \leq \lambda \int_\Omega |u_\Phi|^q dx.
\]
If we have equality here, then \( u_\Phi \in N_\lambda \), and everything is done. Assume the contrary, i.e.,
\[
\int_\Omega |\nabla u_\Phi|^p + |\nabla u_\Phi|^q dx < \lambda \int_\Omega |u_\Phi|^q dx.
\] (3.10)
Let \( t > 0 \) be such that \( tu_\Phi \in N_\lambda \) (see the proof of (i)). From this, (3.10) and our condition \( p^+ < q \), one can infer that \( t \in (0, 1) \). Finally, by (3.8) and (3.9), we have
\[
0 < m_\lambda \leq \Phi_\lambda(tu_\Phi) \leq t^{p^-} \liminf_{k \to \infty} \Phi_\lambda(u_k) = t^{p^-} m_\lambda < m_\lambda,
\]
which is impossible. Hence, relation (3.10) cannot be valid, and consequently we must have \( u_\Phi \in N_\lambda \). Therefore, by (3.9), \( \Phi_\lambda(u_\Phi) = m_\lambda \).

Proof of Theorem 1.2 (A). Steep 1. Let \( u_\Phi \in N_\lambda \) be the minimizer found in Lemma 3.4 (iv). In fact \( u_\Phi \) is a solution of the minimization problem \( \min_{u \in W^{1,q}(\Omega) \setminus \{ 0 \}} \Phi_\lambda(u) \), under restrictions
\[
G_1(u) := \int_\Omega |\nabla u|^p + |\nabla u|^q dx - \lambda \int_\Omega |u|^q dx = 0, \tag{3.11}
\]
\[
G_2(u) := \int_\Omega |u|^{q-2} u dx = 0. \tag{3.12}
\]
We choose \( X = W^{1,q}(\Omega) \), \( Y = \mathbb{R}^2 \), \( D = W^{1,q}(\Omega) \setminus \{ 0 \} \), \( f = \Phi_\lambda \), \( G = (G_1, G_2) \). Obviously, the dual \( Y^* \) can be identified with \( \mathbb{R}^2 \). All the conditions from the statement of Theorem 3.4 are met, including the surjectivity condition on \( G'(u_\Phi) \), which means that for any pair \( (\zeta_1, \zeta_2) \in \mathbb{R}^2 \), there is a \( w \in W^{1,q}(\Omega) \) such that \( \langle G_1'(u_\Phi), w \rangle = \zeta_1 \), \( \langle G_2'(u_\Phi), w \rangle = \zeta_2 \). Indeed, choosing \( w = au_\Phi + b \) with \( a, b \in \mathbb{R} \), we obtain
\[
\left\{ \begin{array}{l}
a \int_\Omega p|\nabla u_\Phi|^p + q|\nabla u_\Phi|^q dx - \lambda aq \int_\Omega |u_\Phi|^q dx = \zeta_1, \\
b(q-1) \int_\Omega |u_\Phi|^{q-2} dx = \zeta_2,
\end{array} \right.
\]
which yields
\[
a \int_\Omega (p-q)|\nabla u_\Phi|^p dx = \zeta_1, \quad b(q-1) \int_\Omega |u_\Phi|^{q-2} dx = \zeta_2.
\]
Thus, \( a \) and \( b \) can be uniquely determined, hence \( G'(u_\Phi) \) is surjective, as asserted. Consequently, Theorem 3.4 is applicable to our minimization problem. Specifically, there exist some constants \( c, d \in \mathbb{R} \) such that
\[
\int_\Omega \left( (|\nabla u_\Phi|^{p-2} + |\nabla u_\Phi|^{q-2}) \nabla u_\Phi \nabla v dx - \lambda \int_\Omega |u_\Phi|^{q-2} u v dx \right)
\]
\[
+ c \left[ \int_\Omega (p|\nabla u_\Phi|^{p-2} + q|\nabla u_\Phi|^{q-2}) \nabla u_\Phi \nabla v dx - \lambda q \int_\Omega |u_\Phi|^{q-2} u v dx \right]
\]
\[
+ d(q-1) \int_\Omega |u_\Phi|^{q-2} v dx = 0,
\]
for all \( v \in W^{1,q}(\Omega) \). Testing with \( v = 1 \) above, in view of (3.12), yields
\[
d(q-1) \int_\Omega |u_\Phi|^{q-2} dx = 0,
\]
then \( d = 0 \). Next, testing with \( v = u_\Phi \) and using (3.11), we deduce
\[
c \int_\Omega (p-q)|\nabla u_\Phi|^p dx = 0,
\]
which implies \( c = 0 \). Therefore, for all \( v \in W^{1,q}(\Omega) \),
\[
\int_\Omega \left( (|\nabla u_\Phi|^{p-2} + |\nabla u_\Phi|^{q-2}) \nabla u_\Phi \nabla v dx - \lambda \int_\Omega |u_\Phi|^{q-2} u v dx \right) = 0,
\]
i.e., \( \lambda \) is an eigenvalue of problem (1.1).
Assume that Lemma 3.5.

Moreover, if \( u \) is a related eigenfunction of (1.1), then testing with \( v = 1 \) in (2.2), we deduce \( \int_{\Omega} |u|^{q-2}u \, dx = 0 \). Hence, \( u \in C_\lambda \setminus \{0\} \). Next we prove that any \( \lambda \in (0, \sigma_1) \) is not an eigenvalue of (1.1). We argue by contradiction. Then there exist \( \lambda \in (0, \sigma_1) \) and \( u \in C_\lambda \setminus \{0\} \) such that
\[
\int_{\Omega} |\nabla u|^p + |\nabla u|^q \, dx = \lambda \int_{\Omega} u^q \, dx.
\]

Hence,
\[
0 \leq (\sigma_1 - \lambda) \int_{\Omega} u^q \, dx \leq \int_{\Omega} |\nabla u|^q \, dx - \lambda \int_{\Omega} u^q \, dx < \int_{\Omega} |\nabla u|^p + |\nabla u|^q \, dx - \lambda \int_{\Omega} u^q \, dx = 0.
\]

This is a contradiction.

\[\square\]

3.5. **The Case** \( q < p^- \) and \( q > 2 \). We have that \( q < p^- \) implies \( \mathcal{W} = W^{1,p(\cdot)}(\Omega) \). Define
\[
C := \left\{ u \in W^{1,p(\cdot)}(\Omega) \mid \int_{\Omega} |u|^{q-2}u \, dx = 0 \right\}.
\]

Note that \( C \subset C_q \), then
\[
\sigma_2 = \inf_{u \in C \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^q \, dx}{\int_{\Omega} |u|^q \, dx} > 0.
\]

(3.13)

We need the following result.

**Theorem 3.5.** (see [13] Theorem 1.38). Let \( X \) be a reflexive Banach space and \( I : C \subset X \to \mathbb{R} \) be weakly lower semicontinuous and assume

(i) \( C \) is a nonempty bounded weakly closed set in \( X \) or

(ii) \( C \) is a nonempty weakly closed set in \( X \) and \( I \) is weakly coercive on \( C \).

Then

(a) \( \inf_{u \in C} I(u) > -\infty \); 

(b) there is at least one \( u_0 \in C \) such that \( I(u_0) = \inf_{u \in C} I(u) \).

Moreover, if \( u_0 \) is an interior point of \( C \) and \( I \) is Gateaux differentiable at \( u_0 \), then \( I'(u_0) = 0 \).

Recall that a functional \( I : C \subset X \to \mathbb{R} \) is weakly lower semicontinuous at \( u_0 \in C \) if for every sequence \( \{u_k\} \subset C \) for which \( u_k \to u_0 \) it follows that \( I(u_0) \leq \lim \inf_{k \to \infty} I(u_k) \).

**Lemma 3.5.** Assume that \( \lambda > \sigma_2 \).

(i) \( \Phi_\lambda : C \to \mathbb{R} \) is weakly coercive, i.e., \( \Phi_\lambda(u) \to \infty \) as \( \|u\|_{W^{1,p(\cdot)}(\Omega)} \to \infty \) on \( C \).

(ii) The functional \( \Phi_\lambda : C \to \mathbb{R} \) has a global minimum point, say \( w_\Phi \in C \), such that \( \Phi_\lambda(w_\Phi) < 0 \).

**Proof.** Let \( u \in C \). By (3.13) and Hölder’s inequality, we have
\[
\sigma_2 \int_{\Omega} |u|^q \, dx \leq \int_{\Omega} |\nabla u|^q \, dx \leq C_1(\Omega, p, q) \|\nabla u\|_{L^q(\Omega)}^{q-2} \|u\|_{L^q(\Omega)}^q,
\]

\[
\leq C_1 \max \left\{ \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{q}{p}}, \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \right\}. \tag{3.14}
\]

Since \([u] := \|\nabla u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\) is equivalent to \(\|u\|_{W^{1,p(\cdot)}(\Omega)}\), from (3.14),
\[
C_3 \|u\|_{W^{1,p(\cdot)}(\Omega)} \leq [u] \leq C_2(\alpha, \sigma_2, \Omega, p, q) \max \left\{ \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \right\}.
\]

where \(C_3\) is a suitable positive constant. Then, if \( \|u\|_{W^{1,p(\cdot)}(\Omega)} > C_2/C_3 \),
\[
\Phi_\lambda(u) \geq \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \left[ \frac{1}{p} \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{1-\frac{2}{p}} - \frac{C_1}{\sigma_2} \right] + \int_{\Omega} \frac{1}{q} |\nabla u|^q \, dx.
\]

Hence, the conclusion follows.
The assumptions of Theorem 3.5 are satisfied. Thus, there is a global minimum point of $\Phi_\lambda : C \to \mathbb{R}$, say $w_\Phi \in C$. On the other hand, since $\lambda > \sigma_2$, there exists $w \in C \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla w|^q dx - \lambda \int_{\Omega} |w|^q dx < 0.$$  

Then, for small enough $t > 0$,

$$\Phi_\lambda(tw) \leq \Phi_\lambda(tw) \leq \frac{t^q}{q} \left( \int_{\Omega} q^p \frac{t^p - q}{p} |\nabla w|^p dx + \int_{\Omega} |\nabla w|^q dx - \lambda \int_{\Omega} |w|^q dx \right) < 0.$$  

In particular, this shows that $w_\Phi \neq 0$.

\[ \square \]

**Proof of Theorem 1.2** (B). Following the Step 2 of the proof of Theorem A we have that any $\lambda \in (0, \sigma_2]$ is not an eigenvalue of (1.1).

Now, assume that $\lambda > \sigma_2$. Let $w_\Phi$ be given by Lemma 3.5 (ii). Thus, $w_\Phi$ is a solution of the minimization problem $\min_{u \in W^{1,p}((\Omega))} \Phi_\lambda(u)$, under the restriction

$$G(u) := \int_{\Omega} \alpha |u|^{q-2} u dx = 0.$$  

From Theorem 3.4 with $X = W^{1,p}((\Omega))$, $Y = \mathbb{R}$, $D = W^{1,p}((\Omega))$, for some $a \in \mathbb{R}$ we have

$$\left[ \int_{\Omega} (|\nabla w_\Phi|^{p-2} + |\nabla w_\Phi|^{q-2}) \nabla w_\Phi \nabla v dx - \lambda \int_{\Omega} \alpha |w_\Phi|^{q-2} w_\Phi v dx + a(q-1) \int_{\Omega} \alpha |w_\Phi|^{q-2} v dx = 0, \right.$$  

for all $v \in W^{1,p}((\Omega))$. Testing with $v = 1$ above, we deduce

$$\alpha(q-1) \int_{\Omega} \alpha |w_\Phi|^{q-2} v dx,$$  

which yields $\alpha = 0$. Therefore, $\lambda$ is an eigenvalue of problem (1.1).

\[ \square \]

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