Hecke Transformations of Conformal Blocks in WZW Theory. I. KZB Equations for Non-Trivial Bundles

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Abstract. We describe new families of the Knizhnik–Zamolodchikov–Bernard (KZB) equations related to the WZW-theory corresponding to the adjoint $G$-bundles of different topological types over complex curves $\Sigma_{g,n}$ of genus $g$ with $n$ marked points. The bundles are defined by their characteristic classes – elements of $H^2(\Sigma_{g,n}, Z(G))$, where $Z(G)$ is a center of the simple complex Lie group $G$. The KZB equations are the horizontality condition for the projectively flat connection (the KZB connection) defined on the bundle of conformal blocks over the moduli space of curves. The space of conformal blocks has been known to be decomposed into a few sectors corresponding to the characteristic classes of the underlying bundles. The KZB connection preserves these sectors. In this paper we construct the connection explicitly for elliptic curves with marked points and prove its flatness.

Key words: integrable system; KZB equation; Hitchin system; characteristic class

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1 Introduction

The Knizhnik–Zamolodchikov–Bernard (KZB) equations [8, 9, 40] are a system of differential equations for conformal blocks in a conformal field theory. Here we consider the WZW theory of the level $k$, related to a simple complex Lie group $G$ and defined on a Riemann surface $\Sigma_{g,n}$ of genus $g$ with $n$ marked points $(z_1, z_2, \ldots, z_n)$. To describe this model, one should define a $G$-bundle over $\Sigma_{g,n}$. Topologically, the $G$-bundles are defined by their characteristic classes. Let $Z(G)$ be a center of $G$ and $G^{\text{ad}} = G/Z(G)$. The characteristic classes are obstructions to lift the $G^{\text{ad}}$-bundles to the $G$-bundles. They are elements of the cohomology group $H^2(\Sigma_g, Z(G)) \sim Z(G)$ [46].

If $\tilde{G}$ is the corresponding simply connected group (the universal covering with the natural group structure) and $\tilde{G} = G/Z'(G)$, then elements from $H^2(\Sigma_g, Z'(G))$ are obstruction to lift the $G$-bundles to the $\tilde{G}$-bundles. In particular, consider $G = \text{Spin}(N)$ and $SO(N) = \text{Spin}(N)/Z_2$. Then $H^2(\Sigma_g, Z_2) \sim Z_2$ defines the Stiefel–Whitney classes of the $SO(N)$-bundles over $\Sigma_g$.

For generic bundles the WZW theories were studied in [23, 35]. The aim of this paper is to define the KZB equations in these theories. The KZB equations have a large range of applications in mathematics. In particular, on the critical level they produce Hamiltonians of the quantum Hitchin system [30, 34, 43, 57], while in the classical limit they lead to the...
monodromy-preserving equations [32, 41, 44, 62, 66]. In this way, we obtain new classes of these systems.

The KZB equations are described in the following way. Consider the highest weight representations $V_{\mu_a}$ ($\mu_a$ are the highest weights) of $G$ attached to the marked points. For a positive integer $k$ define the integrable module $\hat{V}_{\mu_a}$ of level $k$ of the centrally extended loop group $D^\times \to G$, where $D^\times = D \setminus z_a$ is a punctured disk around the marked point $z_a$. The conformal blocks are linear functionals $\hat{V}_{[n]} \equiv \hat{V}_{\mu_1} \otimes \cdots \otimes \hat{V}_{\mu_n} \to \mathbb{C}$ satisfying some additional conditions (the Ward identities). Let $C_G(\hat{V}_{[n]})$ be a space of conformal blocks. This space depends on parameters – the complex structure of $\Sigma_{g,n}$, and in this way forms a bundle over the moduli space $\mathcal{M}_{g,n}$ of complex structures. There exists a projectively flat connection in this bundle (the KZB connection). Then the meaning of the KZB equations is that the conformal blocks are the horizontal sections of the KZB connection. The KZB equations were derived originally for the genus zero case by Knizhnik and Zamolodchikov [40] and were generalized later to arbitrary genus by Bernard [8, 9]. In subsequent years the KZB equations was studied in a number of works [4, 16, 22, 29, 33, 36].

If the cocenter $Z^\vee(G) = \text{Ker} \tilde{G} \to G$ is non-trivial then the integrable module is a sum of sectors, corresponding to the characteristic classes of the underlying bundles

$$\hat{V}_{\mu} = \bigoplus_{j=0}^{N-1} V^{(j)}_{\mu_j}, \quad N = \text{ord } Z(G).$$

In terms of the spectra the WZW theory this was studied essentially in [23]. Similarly, the conformal blocks are also a sum of different sectors. In each sector one can define the KZB connection.

The aim of this paper is to construct explicitly the KZB connections in all sectors of conformal blocks for the WZW theory defined on elliptic curves. The compatibility conditions (horizontality of the KZB connection) are verified explicitly.

The KZB connection in the trivial sector was studied in [24]. This construction is based on the classical dynamical $r$-matrix with the spectral parameter living on the elliptic curve. The $r$-matrices of this type related to the trivial sector were classified by Etingof and Varchenko [19]. Recently, we have classified the dynamical elliptic $r$-matrices as sections of some bundles of an arbitrary topological type over elliptic curves [46]. It turned out that the dynamical parameters of the $r$-matrices are elements of the moduli spaces of the bundles. It allows us to define the KZB connection in these cases.

Different approach to classification of elliptic $r$-matrices was proposed in [17, 18, 21] and the corresponding KZB connection was also constructed in [17, 18]. The staring point of last approach is an automorphism of the extended Dynkin diagram. In our construction we considered only those automorphisms that isomorphic to elements of the center $Z$. In this case we come to the same $r$-matrices and the KZB equations as in [17, 18]. For $A_n$, $D_n$ and $E_6$ algebras there exists another type automorphisms. So far the underlying vector bundle structure is unclear. It should be noted that in [17, 18] the derivation of the KZB equation is based on the representations of conformal blocks as twisted traces of intertwiners. We will come to this representation in the forthcoming paper where the Hecke transformation of conformal blocks will be considered (see below).

For the SL$(N, \mathbb{C})$ WZW model on elliptic curves the KZB equation in the similar to our form was described in [42]. The authors considered a particular type of bundles that lead to the Belavin–Drinfeld classical $r$-matrix. In this case the corresponding KZB equation has not dynamical parameter and similar to the KZ equation. However, if $N$ is not a prime number there exist $r$-matrices and the corresponding KZB equations intermediate between Felder and Belavin–Drinfeld cases.
In the subsequent paper we will describe the transformation operators that intertwine the different sectors (the Hecke transformations). It is worthwhile to notice that in the classical case these transformations provide a passage from the elliptic Calogero–Moser system to integrable Euler–Arnold top [48, 69, 70] (see also [47, 59, 70, 73]). For arbitrary characteristic classes these type of models were described in [45]. Different aspects and applications of the Hecke transformations to integrable systems and related topics (such as Painlevé–Schlesinger equations [2, 15, 50, 51, 60, 67, 68, 71], monopoles [14, 25, 31, 37, 39, 49, 58], quadratic Poisson structures [13, 74], applications to AGT conjecture [53, 54, 55] etc.) can be found in wide range of literature.

The paper has the following structure. In Section 2 we consider a general setting of the KZB equations related to arbitrary curves $\Sigma_{g,n}$ and arbitrary characteristic class of the bundles. In Section 3 the space of conformal blocks is described. In Section 4 we consider the genus one $N$, all simply-connected groups ($\text{SL}(N,\mathbb{C})$, $\text{Sp}_N$, $E_6$, $E_7$ and $\text{Spin}_N$ except $N = 4n$), the center is a cyclic group. For $\text{Spin}_{4N} \mathcal{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The adjoint group is the quotient group $G^{ad} = G/\mathcal{Z}$. Assume for simplicity that $\mathcal{Z}$ is a cyclic group $\mathbb{Z}_l$ of order $l$.

Let $K$ be a maximal compact subgroup of $G$ and $T$ is the Cartan torus of $K$. Consider the homomorphisms of $S^1 \to T$

$$e(\varphi) \to (e(\gamma_1 \varphi), e(\gamma_2 \varphi), \ldots, e(\gamma_l \varphi)) \in T, \quad e(\varphi) = \exp(2\pi i \varphi).$$

$P^\vee = \{ \gamma = (\gamma_1, \ldots, \gamma_l) \}$ is a coweight lattice in the Cartan subalgebra $\mathfrak{h}^K = \text{Lie}(T)$ and in $\mathfrak{h} \subset \mathfrak{g} = \text{Lie}(G)$. Let $Q^\vee$ be the coroot lattice ($Q^\vee \subseteq P^\vee$). The center $\mathcal{Z}(G)$ is isomorphic to the quotient group $\mathcal{Z} \sim P^\vee/Q^\vee$. In particular, if $\varpi^\vee \in P^\vee$ is a coweight such that $l\varpi^\vee \in Q^\vee$, then the $\mathcal{Z} \sim \mathbb{Z}_l$. It is generated by the element $e(l\varpi^\vee) = \exp(2\pi i l \varpi^\vee) \in T$. For $\text{Spin}_{4N}$ the center is generated by two coweights, corresponding to the left and right spinor representations.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\{ \alpha \} = R \subset \mathfrak{h}^*$ is the root system [12]. There is the root decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\alpha}, \quad \text{ad}_{X} \mathfrak{g}^{\alpha} = \langle X, \alpha \rangle \mathfrak{g}^{\alpha}, \quad X \in \mathfrak{h}.$$

$R$ is an union of positive and negative roots $R = R_+ \cup R_-$ with respect to some ordering in $\mathfrak{h}^*$. Let $\Pi = \{ \alpha_1, \ldots, \alpha_l \}$ be a basis of simple roots in $R$. The dual system $\Pi^\vee = \{ \alpha_1^\vee, \ldots, \alpha_l^\vee \}$ ($\langle \alpha_j, \alpha_k^\vee \rangle = \delta_{jk}$) forms a basis in $\mathfrak{h}$.

Let $t$ be a coordinate in $\mathbb{C}$. Define the loop group $L(G) = \{ \mathbb{C}^* \to G \} = \{ g(t) \}$ such that $g(t)$ has a finite order poles when $t \to 0$. In other words, $L(G)$ is the group of Laurent polynomials $L(G) = G \otimes \mathbb{C}[\lbrack t, t^{-1} \rbrack]$. There is a central extension $\hat{L}(\mathfrak{g})$ of $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lbrack t, t^{-1} \rbrack]$

$$\hat{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}K,$$

defined by a two-cocycle $c(X \otimes f, Y \otimes g) = (X, Y) \text{ Res}(gf^\prime dt)$.

The set of the affine roots if of the form: $R^{aff} = \{ \check{\alpha} = \alpha + n, \alpha \in R, n \in \mathbb{Z}, n \neq 0 \}$. Let $\{ \mathfrak{h}_n \}$ be the basis of simple coroots in $\mathfrak{h}$. Then the analog of the root decomposition for the loop algebra.
has the following form

\[ L(g) = g + \sum_{n \neq 0} \sum_{\alpha \in \Pi} h_\alpha t^n + \sum_{\alpha \in R^{aff}} g^\alpha, \quad g^\alpha = x_\alpha e_\alpha t^n. \]

Let \(-\alpha_0\) be the highest root \(-\alpha_0 \in R_+\). The system of simple affine roots is \(\hat{\Pi} = \Pi \cup (-\alpha_0 + 1)\). It is a basis in \(R^{aff}\). Consider the positive loop subalgebra

\[ L^+(g) = (b + g \otimes t \mathbb{C}[[t]]) , \quad (2.2) \]

where \(b = h \oplus \sum_{\alpha \in R^+} g^\alpha\) is the positive Borel subalgebra. Let also

\[ L^-(g) = (n_- + g \otimes t^{-1} \mathbb{C}[[t^{-1}]]) , \quad n_- = \sum_{\alpha \in R^-} g^\alpha. \quad (2.3) \]

Then \(\hat{L}(g) (2.1)\) is the direct sum

\[ \hat{L}(g) = L^-(g) \oplus L^+(g) \oplus \mathbb{C}K. \quad (2.4) \]

Each summand is a Lie subalgebra of \(\hat{L}(g)\). There are two types of the affine Weyl groups: \(W_P = W \rtimes P^\vee\) and \(W_Q = W \rtimes Q^\vee\), where \(W\) is the Weyl group of \(g\),

\[ W_P = \{ \hat{w} = wt^\gamma, \ w \in W, \ \gamma \in P^\vee \} , \quad W_Q = \{ \hat{w} = wt^\gamma, \ w \in W, \ \gamma \in Q^\vee \}. \quad (2.5) \]

They act on the root vectors as \(e^\delta = e_\alpha t^n \rightarrow e^\hat{\delta} = e_{\hat{w}(\alpha)} t^{n_{\hat{w}(x)}(\gamma, \alpha)}\). The loop groups \(L(G) = G \otimes \mathbb{C}[[t, t^{-1}]]\) have the Bruhat decomposition [61]. Define subgroups

\[ L^+(G) = \{ g_0 + g_1 t + \cdots \}, \quad g_j \in G, \quad g_0 = b \in B, \quad \text{is the positive Borel subgroup}, \quad (2.6) \]

\[ N^- (G) = \{ n_- + g_1 t^{-1} + \cdots \}, \quad n_- \in N_-, \quad \text{is the negative nilpotent subgroup}, \quad (2.7) \]

\[ N^+ (G) = \{ n_+ + g_1 t + \cdots \}, \quad n_+ \in N_+, \quad \text{is the positive nilpotent subgroup}. \quad (2.8) \]

The Bruhat decomposition takes the form

\[ L(G^{ad}) = \bigcup_{\hat{w} \in W_P} N^- (G^{ad}) \hat{w} L^+ (G^{ad}), \quad L(\tilde{G}) = \bigcup_{\hat{w} \in W_Q} N^- (\tilde{G}) \hat{w} L^+ (\tilde{G}). \quad (2.9) \]

For a loop \(g(t)\) in \(G^{ad}\) denote by \(\tilde{g}\) its lift to a map from \(S^1\) to \(\tilde{G}\). This map can be multivalued, after turning along the circle the value can be multiplied by some element of the center which we call the monodromy: \(g(e^{2\pi i t}) = e(\gamma) g(t), \ (e(x) = e^{2\pi i x})\). If \(\gamma \notin Q^\vee\) then \(\zeta = e(\gamma)\) is a non-trivial element of the center \(\mathcal{Z}\) and the map \(g(t)\) is well defined for \(G = G^{ad}\), but not for \(\tilde{G}\). In this way we have the representations

\[ L(G^{ad}) = \bigcup_{\gamma \in P^\vee} L_\gamma (G^{ad}), \quad L_\gamma (G^{ad}) = \{ g(e^{2\pi i t}) = e(\gamma) g(t) \}. \quad (2.10) \]

If \(\gamma_1 = \gamma_2 + \delta\) for any \(\delta \in Q^\vee\) then \(\gamma_1\) and \(\gamma_2\) lead to the same monodromies. We say in this case that \(L_{\gamma_1} (\tilde{G})\) and \(L_{\gamma_2} (\tilde{G})\) are equivalent. Then from (2.10) we have

\[ L(G^{ad}) = \bigcup_{\zeta \in \mathcal{Z}} L_\zeta (G^{ad}). \]
In particular, if the center $\mathcal{Z} \sim \mathbb{Z}_t$ is generated by a fundamental coweight $\varpi^\vee$, then

$$L(G^{\text{ad}}) = \bigoplus_{j=0}^{l-1} L_j(G^{\text{ad}}), \quad L_j(G^{\text{ad}}) = \{ g(e^{2\pi i j t}) = e(j\varpi^\vee)g(t) \},$$

(2.11)

and $L_j(G^{\text{ad}}) = e(j\varpi^\vee)(L(G)/\mathcal{Z})$.

Consider the quotient $\text{Fl}^{\text{aff}} = L(G^{\text{ad}})/L^+(G^{\text{ad}})$ [61]. It is called the affine flag variety. Let $\Sigma_\hat{w}$ be an $N^- (G^{\text{ad}})$-orbit of $\hat{w}$ in $\text{Fl}^{\text{aff}}$. This orbit is diffeomorphic to the intersection $N^- (G^{\text{ad}})\hat{w} = N^- (G^{\text{ad}}) \cap \hat{w}N^- (G^{\text{ad}})\hat{w}^{-1}$. Therefore, its codimension in $\text{Fl}^{\text{aff}}$ is the length $l(\hat{w})$ of $\hat{w}$. It is the number of negative affine roots which $\hat{w}$ transforms to positive ones (Theorem 8.7.2 in [61]). The Bruhat decomposition (2.9) defines the stratification of $\text{Fl}$:

$$\text{Fl}^{\text{aff}} = L(G^{\text{ad}})/L^+(G^{\text{ad}}) = \bigcup_{\hat{w} \in W_\rho} \Sigma_{\hat{w}}.$$  

(2.12)

### 2.2 Integrable modules

Consider a subset of dominant weights $P^+ = \{ \mu \in P | \langle \mu, \alpha^\vee \rangle \geq 0 \text{ for } \alpha^\vee \in \Pi^\vee \}$. Each dominant weights define a $\mathfrak{g}$-module $V_\mu$. It contains the highest weight vector (HWV) $v_\mu$ such that

$$Xv_\mu = \langle X, \mu \rangle v_\mu \quad \text{for} \quad X \in \mathfrak{h}, \quad g^\alpha v_\mu = 0 \quad \text{for} \quad \alpha \in R^+.$$  

Define the Verma module $V_\mu$ of $\hat{L}(\mathfrak{g})$ associated with $V_\mu$ [38]. Let $I_k = \{ \mu \in P^+ | \langle \mu, \alpha_0^\vee \rangle \leq k \}$ be a subset of dominant weights. Define the action of $L^+(\mathfrak{g})$ (2.2) on $V_\mu$: $(\mathfrak{g} \otimes t\mathbb{C}[[t]])V_\mu = 0$, $KV_\mu = k \text{Id}_t$, and $\mathfrak{b}$ acts on $V_\mu$ as described above. Then $V_\mu = U(\hat{L}(\mathfrak{g})) \otimes_{U(L(\mathfrak{g}))} V_\mu$ is induced, where $\mu \in I_k$. There is the isomorphism

$$V_\mu \sim U(L^-(\mathfrak{g})) \otimes_{\mathbb{C}} v_\mu.$$  

Let $E_{\alpha_0}$ be the root subspace in $\mathfrak{g}$ corresponding to $\alpha_0$. Consider the maximal submodule $S_\mu$ of $V_\mu$ generated by the singular vector

$$E_{\alpha_0} \otimes t^{-1}k^{-\langle \mu, \alpha_0 \rangle}v_\mu.$$  

(2.13)

The irreducible integrable module $\hat{V}_\mu$ is the quotient

$$\hat{V}_\mu = V_\mu/S_\mu.$$  

(2.14)

We identify the module $V_\mu$ with a submodule $V_\mu \otimes 1 \hookrightarrow V_\mu$. The integrable module $\hat{V}_\mu$ can be characterized in the following way: the subspace of $\hat{V}_\mu$ annihilated by the positive subalgebra $\mathfrak{g} \otimes \mathbb{C}[[t]]$ is isomorphic to the finite-dimensional $\mathfrak{g}$-module $V_\mu$

$$V_\mu \sim \{ v \in \hat{V}_\mu | (\mathfrak{g} \otimes t\mathbb{C}[[t]])v = 0 \}.$$  

The group $L(G)$ has a central extension $1 \to \mathbb{C}^* \to \hat{L}G \to LG \to 1$ corresponding to (2.1). The integrable module can be described in terms of $\hat{L}G$. The action of $L^+(G)$ on the HWV has the form

$$L^+(G)v_\mu = \chi_\mu(b)v_\mu, \quad \lambda v_\mu = e(k)v_\mu, \quad \lambda \in \text{the center } \mathbb{C}^*,$$

(2.15)

where $\chi_\mu(b)$ is the character of the Borel subgroup $B$. Then $\hat{V}_\mu$ is generated by the action of $N^-(G)$ (2.7) on $V_\mu$. 


In this way we describe only “the trivial sector” of the \( L(G) \)-module. Consider the Bruhat representation for \( L(G^{\text{ad}}) \) (2.9), and let \( \hat{w} = t^\gamma, \gamma \in P^\vee \). Define the Verma modules with the HWV \( t^\gamma v_\mu \),

\[
V_\mu(\gamma) = U(L^-(g)) \otimes_\mathbb{C} t^\gamma v_\mu. \tag{2.16}
\]

They have the singular vectors \( (E_{\alpha_0} \otimes t^{-1})^{k-(\mu,\alpha_0)+1} t^\gamma v_\mu \) (compare with (2.13)). Let \( S_{\mu,\gamma} \) be the maximal submodules generated by these singular vectors. Consider the quotient spaces

\[
\hat{V}_\mu(\gamma) = V_\mu(\gamma)/S_{\mu,\gamma}, \tag{2.17}
\]

and define their direct sum

\[
\hat{V}_\mu = \bigoplus_{\gamma \in P^\vee} \hat{V}_\mu(\gamma). \tag{2.18}
\]

We say that two subspaces \( \hat{V}_\mu(\gamma_1) \) and \( \hat{V}_\mu(\gamma_2) \) are equivalent if \( \gamma_1 = \gamma_2 + \delta \), where \( \delta \in Q^\vee \). This equivalence leads to the decomposition of \( \hat{V}_\mu \) (as a \( L(G^{\text{ad}}) \)-module) into a sum of \( l = \text{ord}(Z(G)) \) sectors,

\[
\hat{V}_\mu = \bigoplus_{\zeta \in Z} \hat{V}_\mu(\zeta). \tag{2.19}
\]

Notice that \( (t^\gamma v_\mu) \) is not the HWV with respect to \( L^+(g) \). However, it was proved in [23] that there exists a unique element \( \hat{w} = \hat{w}(\gamma) = t^\delta w \in W_Q \) such that \( t^\gamma \hat{w} v_\mu \) is the HWV. We demonstrate it below for \( L(SL(2, \mathbb{C})) \). The elements \( \hat{w} \) and \( \gamma \) represent the same element \( \zeta \in Z \). Then we define the Verma module

\[
V_\mu(\zeta) \sim U(L(g)^-) \otimes_\mathbb{C} (t^\gamma \hat{w} v_\mu). \tag{2.20}
\]

The vector \( (E_{\alpha_0} \otimes t^{-1})^{k-(\mu,\alpha_0)+1} (\gamma \hat{w} v_\mu) \) is singular and corresponds to the submodule \( S_{\mu,\gamma} \).

As in (2.14) we identify the integrable modules \( V_\mu(\zeta)/S_{\mu,\gamma} \) with \( \hat{V}_\mu(\zeta) \) (2.16).

Let \( V_\mu^* \) be the dual module. The Borel–Weil–Bott theorem for the loop group [61] states that \( V_\mu^* \) can be realized as the space of sections of a line bundle \( L_\mu \) over the affine flag variety (2.12). The line bundle is determined by the action \( L^+(G) \times \mathbb{C}^* \) on its sections as in (2.15),

\[
L_\mu = \{(g, \xi) \sim (gb, \chi_\mu(b^{-1}) \xi), \ g \in L(G), \ b \in L^+(G)\}. \tag{2.21}
\]

### 3 Conformal blocks and KZB equation in general case

#### 3.1 Moduli space of holomorphic \( G \)-bundles

Let \( \mathcal{P} \) be a principle \( G \)-bundle over a curve \( \Sigma_{g,n} \) of genus \( g \) with \( n \) marked points \( \vec{z} = (z_1, \ldots, z_n) \), \( n > 0 \), \( V \) is a \( G \)-module and \( E_G = \mathcal{P} \times_G V \) is the associated bundle. We consider the set of isomorphism classes of holomorphic \( G \)-bundles \( \mathcal{M}_{G,g,n} \) over \( \Sigma_{g,n} \) with the quasi-parabolic structures at the marked points [64]. They are defined in the following way. A \( G \)-bundle can be trivialized over small disjoint disks \( D = \bigcup_{a=1}^n D_a \) around the marked points and over \( \Sigma_{g,n} \setminus \vec{z} \). Therefore, \( \mathcal{P} \) is defined by the transition holomorphic functions on \( D^x = \bigcup_{a=1}^n (D_a^x) \) and \( D_a^x = D_a \setminus z_a \). If \( G(X) \) are the holomorphic maps from \( X \subset \Sigma_g \) to \( G \), then the isomorphism classes are defined as the double coset space

\[
\text{Bun}_G = G(\Sigma_{g,n} \setminus \vec{z})/G(D^x)/G(D) \sim \mathcal{M}_{G,g,n}. \tag{3.1}
\]
Let $t_a$ be a local coordinate in the disks $D_a$. Then $G(D) = \prod_{a=1}^n G(D_a) = \prod_{a=1}^n G \otimes \mathbb{C}[[t_a]]$ and

$$G(D^\times) = \prod_{a=1}^n L_a(G), \quad L_a(G) = G \otimes \mathbb{C}[[t_a, t_a^{-1}]]. \quad (3.2)$$

Let us fix $G$-flags at fibers over the marked points. The quasi-parabolic structure of the $G$-bundle means that $G(D)$ preserves these $G$-flags. In other words, $G(D_a) = L_a^+(G)$ (2.6). At the level of the Lie algebra Lie($G(D)$) = $\bigoplus_{a=1}^n L_a^+(g)$ (2.2). We discuss the Lie algebra $\mathfrak{g}_{\text{out}} = \text{Lie}(G(\Sigma_{g,n} \setminus \bar{z}))$ below.

Consider the one-point case $\bar{z} = z_0$ in (3.2). Let $g(t) \in G[[t, t^{-1}]] = G(D^\times_{z_0})$ be the transition function on the punctured disc $D^\times_{z_0}$ with the local coordinate $t$. This transition function defines a $G$-bundle. Its Lie algebra Lie($G(D^\times)$) = $g \otimes \mathbb{C}[[t, t^{-1}]]$ assumes the form (see (3.1))

$$\text{Lie}(G(D^\times)) = \mathfrak{g}_{\text{out}} \oplus T \text{Bun}_G \oplus L^+(g). \quad (3.3)$$

Introduce a new transition matrix $\tilde{g}(t) = t^\gamma g(t)$, where $\gamma \in P^\vee$ is an element of the coweight lattice. It defines a new bundle $\tilde{E}_G$. The passage from $E_G$ to $\tilde{E}_G$ is called the modification of the bundle $E_G$ at the point $z_0$. The modification amounts to the passage between different sectors of the integrable module attached at $z_0$. This phenomenon is consistent with the quasi-parabolic structure at $z_0$. In general, it can have an arbitrary direction. It means that $t^\gamma$ may be replaced by $\text{Ad}_f(t^\gamma)$, where $f \in G$. As it was mentioned in Section 2.2 there is a unique modification that preserves the HWV of the integrable module $\tilde{V}_\mu$ attached at $z_0$.

To be a $G$-bundle over $\Sigma_g$ the transition matrix $g$ should have a trivial monodromy $g(te^{2\pi i}) = g(t)$ around $w$. If $g(t)$ has a trivial monodromy and $\gamma$ belongs to the coroot sublattice $Q^\vee$, then $\tilde{g}(t)$ also has a trivial monodromy. Otherwise, the monodromy is an element of the center $Z(G)$. For example, let $\gamma = j\varpi^\vee$, where $\varpi^\vee$ generate the group $\mathbb{Z}_t$, i.e. $t\varpi^\vee \in Q^\vee$, while $j\varpi^\vee \notin Q^\vee$ for $j \neq 0, \text{mod}(l)$. In this case

$$g(te^{2\pi i}) = \zeta^j g(t), \quad \zeta = e(\varpi^\vee). \quad (3.4)$$

If $j \neq 0$ then $g(t)$ is not a transition matrix for the $G$-bundle. But it can be considered as a transition matrix for the $G^{\text{ad}}$-bundle, since $G^{\text{ad}} = G/Z$. In this case the $G$-bundle is topologically non-trivial and $\zeta$ represents the characteristic class of $E_G$. The characteristic class is an obstruction to lift $G^{\text{ad}}$-bundle to $G$-bundle. It is represented by an element $H^2(\Sigma_g, Z)$ [46].

Let $\tilde{g}(t) = g_j(t) = t^{j\varpi^\vee}$. Then the multiplication by $g_j(t)$ provides a passage in (2.11) from the trivial sector to the non-trivial sectors

$$g_j(t) \cdot L_0(G^{\text{ad}}) = L_j(G^{\text{ad}}).$$

In general, we have a decomposition of the moduli space (3.1) into sectors

$$M_{G^{\text{ad}}, g, 1} = \bigcup_{\gamma \in P^\vee} M^{(\gamma)}_{G^{\text{ad}}, g, 1},$$

$$M^{(\gamma)}_{G^{\text{ad}}, g, 1} = G(\Sigma_{g,n} \setminus w) \setminus G^{\text{ad}} \otimes \mathbb{C}[[t, t^{-1}]]/G \otimes \mathbb{C}[[t]],$$

$$G^{\text{ad}} \otimes \mathbb{C}[[t, t^{-1}]] = G^{\text{ad}} \otimes t^{\gamma} \mathbb{C}[[t, t^{-1}]]. \quad (3.5)$$

In particular, for $\Sigma_{0,1}$ ($CP^1 \sim \mathbb{C} \cup \infty$) and the marked point $z_1 = 0$ this representation is related to the Grothendieck description of the vector bundles over $CP^1$. Let $g_- \in G \otimes \mathbb{C}[z^{-1}]$, $g_+ \in G \otimes \mathbb{C}[z]$. Then $g(z) \in L(G) = (\mathbb{C}^* \to G) = \{g(z)\}$ has the Birkhoff decomposition [61]

$$g(z) = g_-z^\gamma g_+, \quad \gamma \in P^\vee. \quad (3.6)$$
It means that any vector bundle $E_G$ over $\mathbb{C}P^1$ is isomorphic to the direct sum of the line bundles $\bigoplus_{i=1}^t \mathcal{L}_i$, where $\mathcal{L}_i$ is defined by the transition function $z^{\gamma_i}$, $\gamma = (\gamma_1, \ldots, \gamma_t)$. If $\gamma \notin Q^\vee E_G$ then has a non-trivial characteristic class. In fact, the bundle with $\gamma \neq 0$ are unstable.

Two subsets $\mathcal{M}^{(\gamma_1)}_{G,g,1}$ and $\mathcal{M}^{(\gamma_2)}_{G,g,1}$ of the moduli space correspond to the vector bundles with the same characteristic class if $\gamma_1 = \gamma_2 + \beta$, $\beta \in Q^\vee$. Then the topological classification of the moduli spaces of the vector bundles by their characteristic classes follows from (3.5)

$$\mathcal{M}_{G,g,1} = \bigcup_{\zeta \in \mathcal{Z}} \mathcal{M}_{G,g,1}^{(\zeta)},$$

$$\mathcal{M}_{G,g,1}^{(\zeta)} = G((\Sigma_{g,n} \setminus w) \setminus G \otimes \mathbb{C}[[t, t^{-1}]]/G \otimes \mathbb{C}[[t]],$$

$$G(\zeta) \otimes \mathbb{C}[[t, t^{-1}]] = G \otimes t^{j\varpi^\vee} \mathbb{C}[[t, t^{-1}]], \quad \zeta = e(j\varpi^\vee).$$

(3.7)

Similar representation exists for the space $\mathcal{M}_{G,g,n}$.

### 3.2 Moduli of complex structures of curves

Let $\mathfrak{M}_g$ be the moduli space of complex structures of compact curves $\Sigma_g$ of genus $g$. The moduli space $\mathfrak{M}_{g,n}$ of the complex structures of curves with marked points is foliated over $\mathfrak{M}_g$ with fibers $\mathcal{U} \subset \mathbb{C}^n$ corresponding to the moving marked points.

An infinitesimal deformation of the complex structures is represented by the Beltrami $(-1, 1)$ differential $\mu(z, \bar{z}) = \mu \frac{\bar{\partial}}{\partial z} \otimes d\bar{z}$ on $\Sigma_{g,n}$. In this way $\mu$ is $(0, 1)$ form on $\Sigma$ taking values in $T^{(1,0)}(\mathfrak{M}_{g,n})$ and vanishing at the marked points. The basis in the tangent space $T(\mathfrak{M}_{g,n})$ is represented by the Dolbeault cohomology group $H^1(\Sigma_g, \Gamma(\Sigma_g \setminus \bar{z}) \otimes \bar{K})$, where $\bar{K}$ is the anti-canonical class.

Let us compare it with the Čech like construction of $T\mathfrak{M}_{g,n}$ as a double coset space. As above, consider small disks $D_a$ around marked points with local coordinates $t_a$. Let $\mathbb{C}[[t_a, t_a^{-1}]]/\partial_{t_a}$, $\mathbb{C}[[t_a]]/\partial_{t_a}$ be vector fields on $D_a^\times$ while $D_a$ and $\Gamma(\Sigma_g \setminus \bar{z})$ is a space of vector fields on $\Sigma_g \setminus \bar{z}$. The vector fields from the latter space can have poles of finite orders at the marked points. Then

$$T\mathfrak{M}_{g,n} = \Gamma(\Sigma_g \setminus \bar{z}) \big/ \bigoplus_{a=1}^n \mathbb{C}[[t_a, t_a^{-1}]]/\partial_{t_a} \big/ \bigoplus_{a=1}^n \mathbb{C}[[t_a]]/\partial_{t_a}. \quad (3.8)$$

This construction has the following relation to the Dolbeault description. We establish correspondence between $\zeta \in \bigoplus_{a=1}^n \mathbb{C}[[t_a, t_a^{-1}]]/\partial_{t_a}$ on $\bigcup_{a=1}^n D_a^\times$ and the Beltrami differential $\mu$. Let $\zeta_{\text{out}} \in \Gamma(\Sigma_g \setminus \bar{z})$, $\zeta_{\text{int}} \in \bigoplus_{a=1}^n \mathbb{C}[[t_a]]/\partial_{t_a}$. Consider two equations on $\bigcup_{a=1}^n D_a^\times$

$$\bar{\partial}\zeta_{\text{out}} = \mu, \quad \bar{\partial}\zeta_{\text{int}} = \mu,$$

where $\bar{\partial}|_{D_a^\times} = \partial_{t_a}$. On $D_a^\times$ $\bar{\partial}(\zeta_{\text{out}} - \zeta_{\text{int}}) = 0$ and, therefore, $\zeta_{\text{out}} - \zeta_{\text{int}}$ represents a Dolbeault cocycle. The first equation has solutions that can be continued on $\Sigma_g \setminus \bar{z}$ and the second – on $\bigcup_{a=1}^n D_a$. If $\zeta \in \bigoplus_{a=1}^n \mathbb{C}[[t_a, t_a^{-1}]]/\partial_{t_a}$ has continuations $\zeta_{\text{out}}$ and $\zeta_{\text{int}}$ then it corresponds to a trivial element of $T\mathfrak{M}_{g,n}$. On the other hand, $\bar{\partial}\zeta = \mu$ globally and, therefore, $\mu$ represents an exact Dolbeault cocycle. In this way the non-trivial vector fields $\zeta \in \bigoplus_{a=1}^n \mathbb{C}[[t_a, t_a^{-1}]]/\partial_{t_a}$ correspond to elements of $H^1(\Sigma_g, \Gamma(\Sigma_g \setminus \bar{z}) \otimes \bar{K})$. 
### 3.3 Definition of conformal blocks and coinvariants

Let us associate with $\Sigma_{g,n}$ the following set: integer $k$ and the weights $\vec{\mu} = (\mu_1, \ldots, \mu_n, \mu_a \in I_k)$ attached to the marked points $\vec{z} = (z_1, \ldots, z_n)$. The $\hat{L}(g)$-module (2.4)

$$\hat{V}^{[n]}_{\vec{z}, \vec{\mu}} = \bigotimes_{a=1}^{n} \hat{V}_{\mu_a}, \quad \vec{z} = (z_1, \ldots, z_n).$$

(3.9)

According to (2.19)

$$\hat{V}_{\mu_a} = \bigoplus_{\zeta_a \in \mathbb{Z}} \hat{V}_{\mu_a}(\zeta_a).$$

(3.10)

Coming back to (3.1) we define a Lie algebra $\mathfrak{g}_{\text{out}} = \text{Lie}(G(\Sigma_g \setminus D))$ as a Lie algebra of meromorphic functions on $\Sigma_{g,n}$ with poles at $\vec{z} = (z_1, \ldots, z_n)$ taking values in $g$. Let $(t_1, \ldots, t_n)$ are local coordinates in $D$. There is a homomorphism $O(\Sigma_g \setminus \vec{z}) \to \mathbb{C}[t_a, t_a^{-1}]$ for each $z_a$ providing the homomorphism of the Lie algebras $\mathfrak{g}_{\text{out}} \to \mathfrak{g} \otimes \mathbb{C}[t_a, t_a^{-1}]$. In this way $\mathfrak{g}_{\text{out}}$ acts on $\hat{V}^{[n]}_{\vec{z}, \vec{\mu}}$ as

$$(X \otimes f) \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_a v_1 \otimes \cdots \otimes (X \otimes f(t_a)) \cdot v_a \otimes \cdots \otimes v_n.$$

This is a Lie algebra action. Due to the residue theorem this homomorphism is lifted to the diagonal central extension

$$\mathfrak{g}_{\text{out}} \hookrightarrow \bigoplus_{a=1}^{n} \hat{L}_a(\mathfrak{g}), \quad \hat{L}_a(\mathfrak{g}) = (\mathfrak{g} \otimes \mathbb{C}[t_a, t_a^{-1}]) \oplus K, \quad K \to k.$$

In what follows we need a relation of $\hat{V}_\mu$ with the space of coinvariants. In general setting the coinvariants are defined in the following way. Let $\mathfrak{W}$ be a module of a Lie algebra $\mathfrak{t}$. The space of coinvariants $[V]_{\mathfrak{t}}$ is the quotient-space $[W]_{\mathfrak{t}} = W/\mathfrak{t} \cdot W$. In the case at hand we define the space of coinvariants with respect to the action of $\mathfrak{g}_{\text{out}}$:

$$\mathcal{H}(\vec{z}, \vec{\mu}) = [\hat{V}^{[n]}_{\vec{z}, \vec{\mu}}]_{\mathfrak{g}_{\text{out}}}, \quad ([V]_g = V/\mathfrak{g} \cdot V).$$

The space of conformal blocks $\mathcal{C}(\hat{V}^{[n]}_{\vec{z}, \vec{\mu}})$ is the dual space to the coinvariants. In other words, $\mathcal{C}(\hat{V}^{[n]}_{\vec{z}, \vec{\mu}})$ is the space of linear functionals on $\hat{V}^{[n]}_{\vec{z}, \vec{\mu}}$, invariant under $\mathfrak{g}_{\text{out}}$:

$$F : \hat{V}^{[n]}_{\vec{z}, \vec{\mu}} \to \mathbb{C}, \quad F(X \cdot v) = 0 \quad \text{for any} \quad X \in \mathfrak{g}_{\text{out}}.$$

Put it differently, the conformal blocks are $\mathfrak{g}_{\text{out}}$-invariant elements of the contragradient module $\hat{V}^{*[n]}_{\vec{z}, \vec{\mu}}$. For a single marked point case the conformal blocks are $\mathfrak{g}_{\text{out}}$ invariant sections of the line bundle $\mathcal{L}_\mu$ over the affine flag variety (2.21).

According to (3.9) and (3.10) the space $\hat{V}^{[n]}_{\vec{z}, \vec{\mu}}$ has the representation

$$\hat{V}^{[n]}_{\vec{z}, \vec{\mu}} = \bigotimes_{a=1}^{n} \bigoplus_{\zeta_a \in \mathbb{Z}} \hat{V}_{\mu_a}(\zeta_a).$$

In a similar way the conformal blocks are decomposed in subspaces corresponding to the characteristic classes of the bundles

$$\mathcal{C}(\hat{V}^{[n]}_{\vec{z}, \vec{\mu}}) = \bigotimes_{a=1}^{n} \bigoplus_{\zeta_a \in \mathbb{Z}} \mathcal{C}_a(\hat{V}_{\mu_a}(\zeta_a)),$$

where

$$\mathcal{C}_a = \{ F(\zeta_a) : \hat{V}_{\mu_a}(\zeta_a) \to \mathbb{C} \}. \quad (3.11)$$
3.4 Variation of the moduli space of complex structures

The space of conformal blocks \( \mathcal{C}(\hat{V}^{[n]}_{\vec{z},\vec{\mu}}) \) is a bundle over \( \mathcal{M}_{g,n} \). This bundle is equipped with the KZB connection that can be described as follows.

A stress-tensor \( T(z, \bar{z}) \) in general theories, defined on a surface \( \Sigma_{g,n} \), generates vector fields on \( \Sigma_{g,n} \). A dual object to \( T(z, \bar{z}) \) is the Beltrami differential \( \mu(z, \bar{z}) \). It means that there is a connection on the bundle of fields over \( \mathcal{M}_{g,n} \) (the Friedan–Shenker connection)

\[
\nabla_\mu F = \delta_\mu F + \int_{\Sigma} \mu TF. \tag{3.12}
\]

In conformal field theories the stress-tensor is a meromorphic projective structure on \( \Sigma_{g,n} \). The connection acting on the space of conformal blocks is projectively flat. The conformal blocks are horizontal sections of this bundle. The horizontality conditions are nothing else but the KZB equations for the conformal blocks. In general setting these equations are discussed in [33] (for the smooth curves) and in [22]. They have the form of non-stationary Schrödinger equations [36].

The connection (3.12) can be rewritten in a local form based on the representation (3.8). Let \( \bigcup_{a=1}^{n} D_a^x \subset \Sigma_g \) and \( \gamma_a \subset D_a^x \) is a small contour and \( \varsigma_a \) is a vector field in \( D_a^x \). Then (3.12) can be written as

\[
\nabla_{\varsigma_a} F = \partial_{\varsigma_a} F + \oint_{\gamma_a} \varsigma_a TF \tag{3.13}
\]

and the KZB equation assumes the form

\[
\nabla_\varsigma F = 0. \tag{3.14}
\]

At the marked points \( T \) has the second order poles, while \( \varsigma_a \in \mathbb{C}[\{t_a\}]\partial_{t_a} \) (3.8). Thereby, this integral produces \( \partial_{\varsigma_a} F \). On the other hand, the product \( TF \) is non-singular outside the disks \( D_a \). Then for \( \varsigma_a \in \Gamma(\Sigma_a \setminus \vec{z}) \) the integrals vanish. It means that the conformal blocks \( F \) are defined on \( \mathcal{M}_{g,n} \).

Consider a one point case and let \( t \) be a local coordinate on a punctured disk \( D^\times \). The stress-tensor in the local coordinate has the Fourier expansion \( T(t) = \sum_{n \in \mathbb{Z}} L_n t^{-n-2} \). The coefficients obey the Virasoro commutation relations \([L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\).

In the WZW model the stress-tensor is obtained from the currents by means of the Sugawara construction (see [7]). Let \( \{t_\alpha\} \) be a basis in \( \mathfrak{g} \), \( \{\bar{t}^{\beta}\} \) is the dual basis, and \( I_\alpha(t) = \sum_m t_{\alpha,m} t^{-m-1} \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \). Then

\[
T(t) = \frac{1}{2(k + h^\vee)} \sum_\alpha :I_\alpha(t)I^\alpha(t):;
\]

where \( h^\vee \) is the dual Coxeter number. The Fourier coefficients of \( T(t) \) take the form

\[
L_m = \frac{1}{2(k + h^\vee)} \sum_{p \in \mathbb{Z}} :t_{\alpha,-p} t_{\alpha+p+m}:. \tag{3.15}
\]

The normal ordering means placing to the right \( t_{\alpha,n}^a (t_{\alpha,n}) \) with \( n > 0 \). The Virasoro central charge is

\[
c = \frac{\dim \mathfrak{g}}{k + h^\vee}.
\]
The Virasoro algebra acts on $\mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}]]$ as

$$L_n \mapsto t^{n+1} \frac{d}{dt}. \quad (3.16)$$

This action is well defined because the action of the Sugawara tensor is well defined on the integrable modules. In particular, it follows from (3.16) that for the moving points equation (3.14) assumes the form

$$(\partial_{z_a} - L^a_{-1}) F = 0. \quad (3.17)$$

The restriction of $\nabla_i$ on $\mathcal{C}_a$ (3.11) yields a family of the KZB equations

$$\sum_a \nabla_a F(\xi_a) = 0. \quad (3.18)$$

In next section we construct these equations explicitly for the bundles over elliptic curves.

### 3.5 Variation of the moduli space of holomorphic bundles

#### 3.5.1 General construction

The moduli space of holomorphic bundles $\mathcal{M}_{G,g,n} = \text{Bun}_G$ (3.1) is foliated over the moduli space of complex structures $\mathfrak{M}_{g,n}$. Let us consider the dependence of the space of coinvariants $\mathcal{H}(\tilde{z}, \tilde{\mu})$ (conformal blocks $\mathcal{C}(V^{[n]})$) on the variations of the moduli of the bundles $\text{Bun}_G$. For simplicity consider the one-point case. Let $t_a$ be a local coordinate in $D^\times_a$, and $G_{\text{out}} = G(\Sigma_g \setminus z_a)$. Define the quotient

$$M_G = G_{\text{out}} \setminus G(D^\times_a), \quad G(D^\times_a) = G[[t_a, t_a^{-1}]]. \quad (3.19)$$

This space is the moduli space of $G$-bundles with a trivialization around $z_a$ (see (3.1)).

Let $\hat{V}_{\mu_a}$ be an integrable module (2.14) attached to $z_a$. Recall that $\hat{V}_{\mu_a}^*$ is the space of holomorphic sections $\Gamma(\mathcal{L}_{\mu_a})$ of the line bundle (2.21) over the affine flag variety (2.12). In these terms the space of conformal blocks has the following interpretation [6, 20]. Since $\hat{V}_{\mu_a}$ is the integrable representation, the group $G(D^\times)$ acts on $\hat{V}_{\mu_a}$. Thereby, the subgroup $G_{\text{out}}$ acts on $\hat{V}_{\mu_a}$ also. Due to (3.19) $G(D^\times_a)$ acts on $\mathcal{M}_G$ from the right. Therefore, $G(D^\times_a)$ acts on the sections $\hat{V}_{\mu_a} \otimes \mathcal{O}(M_G)$ of the trivial vector bundle $\hat{V}_{\mu_a} \times \mathcal{M}_G$

$$(g \cdot v)(x) = (gv)(xg), \quad g \in G[[t_a, t_a^{-1}]], \quad v \in \hat{V}_{\mu_a}. \quad (3.20)$$

Consider the space of the coinvariants

$$\hat{V}_{\mu_a} \otimes \mathcal{O}(M_G) / (\hat{V}_{\mu_a} \otimes \mathcal{O}(M_G)) \cdot \text{Stab}_x,$$

where $\text{Stab}_x$ is $\text{Lie}(G_x(D^\times)), G_x(D^\times) = \{ g | x \cdot g = x, x \in M_G \}$. In particular, $\text{Stab}_x = \mathfrak{g}_{\text{out}}$ for $x$ corresponding to $G_{\text{out}}$. The spaces of coinvariants are isomorphic for different choices of $x$. The dual space $\Gamma(\mathcal{L}_{\mu})/G_{\text{out}}$ is the space of conformal blocks. The quotient $\Gamma(\mathcal{L}_{\mu})/G_{\text{out}}$ is a space of sections of the line bundle over $\text{Bun}_G$ (3.1). It means that the space of conformal blocks is a non-Abelian generalization of the theta line bundles over the Jacobians.

#### 3.5.2 $\text{SL}(2, \mathbb{C})$-bundles over $\mathbb{C}P^1$

It is instructive to consider this construction for $\Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \infty$. This case was analyzed in details in [5] for the trivial $G$-bundles and $\gamma = 0$ in (3.6). Here we consider $G = \text{SL}(2, \mathbb{C})$-bundles with $\gamma \in \mathbb{P}^\vee$. Let $t_a = h, e, f$ be the Cartan–Chevalley basis in the Lie algebra $\text{sl}(2, \mathbb{C})$

$$[h, e] = 2e, \quad [h, f] = 2f, \quad [e, f] = h,$$
and \( t_\alpha(n) = t_\alpha t^n \). The Verma module \( \mathcal{V}_\mu \) is generated by \( L^- (\mathfrak{sl}(2, \mathbb{C})) = c \cdot f + \mathfrak{g} \otimes t^{-1} \mathbb{C}[t^{-1}] \) (2.20), (2.3). The HV \( v_\mu \) with the weight \( \mu \in P(\mathfrak{sl}(2, \mathbb{C})) \) is defined by the conditions \( hv_\mu = 2sv_\mu \), \( s = \frac{1}{2}(\mu, h) \in \frac{1}{2}\mathbb{Z} \), \( ev_\mu = 0 \), \( t_\alpha(n)v_\mu = 0 \) for \( n > 0 \). The singular vector \((et^{-1})^{k-1-2s}\) generates the submodule \( S_\mu \subset \mathcal{V}_\mu \), and the integrable module is the quotient \( \hat{\mathcal{V}}_\mu = \mathcal{V}_\mu / S_\mu \).

This form of \( \mathcal{V}_\mu \) defines a trivial sector in (2.19). Note that \( \mathcal{Z}(\mathfrak{SL}(2, \mathbb{C})) = \mathcal{Z}_2 = \{ \zeta = (0, 1) \} \). Therefore, there are two sectors in the integrable module (2.19). Consider the non-trivial sector corresponding to \( \zeta = 1 \). Let \( W_P = \{ \hat{w} \} \) be the Weyl group (2.5), \( \hat{w} = \mathcal{Z}_2 \rtimes \gamma \), where \( \gamma \) belongs to the weight lattice \( \gamma \in P^v(\mathfrak{sl}(2, \mathbb{C})) = P(\mathfrak{sl}(2, \mathbb{C})) = \frac{1}{2}\mathbb{Z} \). Since \( Q^v(\mathfrak{sl}(2, \mathbb{C})) = Q(\mathfrak{sl}(2, \mathbb{C})) \), \( \mathcal{Z} = P/Q \sim \mathcal{Z}_2 \), and \( \zeta = 1 \) corresponds to \( \gamma \not\in Q \). It means that \( \langle \gamma, h \rangle \) is odd. Then according to (2.19)

\[
\mathcal{V}_\mu(\zeta = 1) = U(L^-(\mathfrak{sl}(2, \mathbb{C}))) (t^\gamma v_\mu), \quad \gamma \not\in Q, \quad \hat{\mathcal{V}}_\mu(\zeta = 1) = \mathcal{V}_\mu(\zeta = 1)/S_\mu.
\]

As it was mentioned above, \( t^\gamma v_\mu \) is not the HWV. In other words, it is not annihilated by the positive nilpotent loop subalgebra (2.8) \( \text{Lie}(N^+(\mathfrak{SL}(2, \mathbb{C}))) = \{ n(t) = b \cdot e + \mathfrak{g} \otimes t\mathbb{C}[t], \ b \in \mathbb{C} \} \). In fact, we have

\[
n(t)t^\gamma v_\mu = t^\gamma \text{Ad}_{t^{-1}}(n(t))v_\mu.
\]

Let \( \langle \gamma, h \rangle = 2s > 0 \), and \( s \in \frac{1}{2} + \mathbb{Z} \). Then

\[
\text{Ad}_{t^{-1}}(n(t)) = \sum_{m \geq 0} (a_{m+1} \cdot ht^{m+1} + b_m \cdot et^{2s+m} + c_{m+1} \cdot ft^{-2s+m+1}).
\]

The terms \( c_{m+1} \cdot ft^{-2s+m+1} \) for \( m \leq 2s - 1 \) do not belong to \( \text{Lie}(N^+(\mathfrak{SL}(2, \mathbb{C}))) \). Multiply \( t^\gamma \) by \( wt^{\gamma_1} \in W_Q \), where \( \langle \gamma_1, h \rangle = -2s + 1 \) and \( w : e \leftrightarrow f \). This transformation preserves the sector. Now \( n(t) \) annihilates the vector \( wt^{\gamma_1} t^\gamma v_\mu \). Note \( (wt^{\gamma_1}) \) is uniquely defined by \( \gamma \). Thus, for any vector \( t^\gamma v_\mu \) we define a unique HWV from the same sector. It is a particular case of general theorem proved in [23].

Consider the trivial \( G \)-bundles over \( \mathbb{C}P^1 \). It was proved in [5] that the conformal blocks are \( G \)-invariant functionals on the module \( \hat{\mathcal{V}}_\mu[1] \) satisfying some additional conditions. In particular, for \( n = 1 \)

\[
\dim(\mathfrak{C}(G)) = \begin{cases} 0, & \mu \neq 0, \\ 1, & \mu = 0, \end{cases}
\]

and for \( \hat{\mathcal{V}}[2]_{(0, \infty)(\mu_0, \mu_\infty)} \)

\[
\dim(\mathfrak{C}(G)) = \begin{cases} 0, & \mu_0 \neq \mu_\infty^*, \\ 1, & \mu_0 = \mu_\infty^*. \end{cases}
\]

Let us analyze the case of \( \mathfrak{SL}(2, \mathbb{C}) \)-bundles. It follows from the Bruhat decomposition (2.9) that there are two types of the \( \mathfrak{SL}(2, \mathbb{C}) \)-bundles over \( \mathbb{C}P^1 \) – the trivial, when \( \gamma \in Q \) in (3.5) is an element of the root lattice \( \gamma \in Q \), and non-trivial, when \( \gamma \not\in Q \). Note that the stable bundles correspond \( \gamma = 0 \). In the first case we deal with the adjoint bundles that can be lifted to the \( \mathfrak{SL}(2, \mathbb{C}) \)-bundles. In the second case there is an obstruction to lift these bundles to the \( \mathfrak{SL}(2, \mathbb{C}) \)-bundles.

Let \( z^{-1} \) be a local coordinate in a neighborhood of \( \infty \). The Lie algebra \( \mathfrak{g}_{\text{out}} \) assumes the form \( \mathfrak{g}_{\text{out}} = \mathfrak{sl}(2, \mathbb{C}) + z^{-1} \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[z^{-1}] \). Let \( n = 1 \) and \( z = 0 \) is the marked point with the attached integrable \( L(\mathfrak{sl}(2, \mathbb{C})) \)-module \( \hat{\mathcal{V}}_\mu \). We have \( \mathfrak{g}_{\text{out}}(v_\mu) = \hat{\mathcal{V}}_\mu \) for \( \mu \neq 0 \). But \( v_0 \not\in \mathfrak{g}_{\text{out}}(v_0) \) and \( v_0 \) is the coinvariant confirming (3.21).
Consider the integrable module generated by $z^\gamma v_\mu$, where, as above, $\langle \gamma, h \rangle = 2s > 0$, and $s \in \frac{1}{2} + \mathbb{Z}$. Then

$$\mathfrak{g}_{\text{out}} z^\gamma v_\mu = \text{Ad}_{z^\gamma}^{-1}(n(t)) = z^\gamma \sum_{m \geq 0} \left( a_{-m} z^{-m} \cdot h + b_{-m} z^{-2s-m} \cdot e + c_{-m} z^{2s-m} \right) \cdot f v_\mu.$$ 

Then the elements $b_{-m} z^{-2s-m} \cdot e(v_\mu)$ for $0 \leq m < 2s$ are not generated by $\mathfrak{g}_{\text{out}}$. Thus, if $\gamma \neq 0$, the space of coinvariants (and the space of conformal blocks) is non-empty for an arbitrary weights $\mu$. Its dimension depends on $\gamma$: $\text{dim}(\mathcal{C}(\text{SL}(2, \mathbb{C}))) = 2s$ (compare with (3.21)).

### 3.5.3 The form of connection

For conformal blocks we have (see (3.20))

$$F(x) = (gF)(xg), \quad g \in \text{Stab}_x.$$ 

Define the current $J(t_a) = (g^{-1}dg)(t_a) \in \mathfrak{g} \otimes \mathbb{C}[[t_a, t_a^{-1}]] \otimes \Omega^1(D_a^\times)$ for $g(t_a) \in G \otimes \mathbb{C}[[t_a, t_a^{-1}]]$. A local version of (3.20) is defined by the operator

$$\nabla_{u_a} = \partial_{u_a} + \oint_{\gamma_a} (g^{-1}dg)(t_a),$$

where $t_a$ is a generator of $\mathfrak{g}$ and $u_a$ is a coordinate of the tangent vector to $\text{Bun}_G$. The action of $\nabla_{u_a}$ on the conformal blocks is well defined because the conformal blocks are $\mathfrak{g}_{\text{out}}$-invariant. Therefore, they are horizontal with respect to this connection

$$\partial_{u_a} F + \oint_{\gamma_a} (g^{-1}dg)(t_a), t_a) F = 0. \quad (3.22)$$

If one takes $u$ from $\mathcal{M}_{G, g, n}^{(\mathbb{C})}$ (3.7) then (3.22) takes the form

$$\partial_{u_a(\zeta_a)} F(\zeta_a) + \oint_{\gamma_a} (g^{-1}dg)(t_a), t_a) F(\zeta_a) = 0,$$

where $F(\zeta_a) \in \mathcal{C}_a$ (3.11).

### 4 KZB equations related to elliptic curves and non-trivial bundles

#### 4.1 Moduli space of elliptic curves

We consider in details the genus one case $\Sigma_{1,n}$. Let $\Sigma_\tau = \mathbb{C}/\langle \tau, 1 \rangle$ be the elliptic curve with the modular parameter in the upper half-plane $\mathcal{H} = \{ \text{Im } m \tau > 0 \}$. For $n \in \mathbb{Z}$, $n \geq 1$ define the set of marked points $\vec{z} = (z_1, \ldots, z_n)$. Due to the $\mathbb{C}$ action on $\Sigma_\tau(z \to z + c)$, we assume that $\sum_a z_a = 0$. A big cell $\mathcal{M}_{1,n}^0$ in the Teichmüller space $\mathcal{M}_{1,n}$ is defined as

$$\mathcal{M}_{1,n}^0 = \left\{ (z_1, \ldots, z_n), \sum_a z_a = 0, z_k \neq z_j, \text{mod}(\langle \tau, 1 \rangle) \right\} \times \mathcal{H}.$$
4.2 Moduli space of holomorphic $G$-bundles over elliptic curves

For $G = GL_N$ the moduli space of holomorphic bundles was described by M. Atiyah [3]. For the trivial $G$-bundles, where $G$ is a complex simple group, it was done in [10, 11, 52]. Non-trivial $G$-bundles and their moduli spaces were considered in [26, 27, 28, 63]. We describe the moduli space of stable non-trivial holomorphic bundles over $\Sigma$ using an approach of [46].

Let $G$ be a complex simple Lie group. An universal cover $\tilde{G}$ of $G$ in all cases apart $G_2$, $F_4$ and $E_8$ has a non-trivial center $\mathcal{Z}(\tilde{G})$. The adjoint group is the quotient $G^{\text{ad}} = \tilde{G}/\mathcal{Z}(\tilde{G})$. For the cases $A_n$ (when $n \neq p l$ is non-prime) and $D_n$ the center $\mathcal{Z}(\tilde{G})$ has non-trivial subgroups $\mathbb{Z}_l \sim \mu_l = \mathbb{Z}/l\mathbb{Z}$. Assume that $(p, l)$ are co-prime. There exists the quotient-groups

$$G_I = \tilde{G}/\mathcal{Z}_I, \quad G_p = G_I/\mathcal{Z}_p, \quad G^{\text{ad}} = G_I/\mathcal{Z}(G_I), \quad (4.1)$$

where $\mathcal{Z}(G_I)$ is the center of $G_I$ and $\mathcal{Z}(G_I) \sim \mu_{s} = \mathcal{Z}(\tilde{G})/\mathcal{Z}_I$.

Following [56] we define a $G$-bundle $E_G = \mathcal{P} \times_G V$ by the transition operators $Q$ and $\Lambda_j$ acting on the sections of $s \in \Gamma(E_G)$ as

$$s(z + 1) = Q(z)s(z), \quad s(z + \tau) = \Lambda(z)s(z), \quad (4.2)$$

where $Q(z)$ and $\Lambda(z)$ take values in $\text{End}(V)$. Going around the basic cycles of $\Sigma$ we come to the equation

$$Q(z + \tau)\Lambda(z)Q(z)^{-1}\Lambda^{-1}(z + 1) = \text{Id}. \quad (4.3)$$

It follows from [56] that it is possible to choose the constant transition operators. Then we come to the equation

$$QAQ^{-1}\Lambda^{-1} = \text{Id}. \quad (4.4)$$

Replace (4.4) by the equation

$$QAQ^{-1}\Lambda^{-1} = \zeta \text{Id},$$

where $\zeta$ is a generator of the center $\mathcal{Z}(\tilde{G})$. In this case $(Q, \Lambda)$ are the clutching operators for $G^{\text{ad}}$-bundles, but not for $\tilde{G}$-bundles, and $\zeta$ plays the role of obstruction to lift the $G^{\text{ad}}$-bundle to the $G$-bundle. Here $\zeta = e(\varpi^\vee)$ is a generator of the center $\mathcal{Z}(\tilde{G})$, where $\varpi^\vee \in P^\vee$ is a fundamental coweight such that $N\varpi^\vee \in Q^\vee$ and $N = \text{ord}(\mathcal{Z}(\tilde{G})).$\(^2\)

Let $0 < j \leq N$. Consider a bundle with the space of sections with the quasi-periodicities

$$s(z + 1) = Qs(z), \quad s(z + \tau) = \Lambda_j s(z) \quad (4.5)$$

such that

$$QA_jQ^{-1}\Lambda_j^{-1} = \zeta^j \text{Id}. \quad (4.6)$$

If $j$ and $N$ are co-prime numbers then $\zeta^j$ generates $\mathcal{Z}(\tilde{G})$. In this case $Q$ and $\Lambda_j$ can serve as transition operators only for a $G^{\text{ad}} = \tilde{G}/\mathcal{Z}$-bundle, but not for $G$-bundle and $\zeta^j$ is an obstruction to lift $G^{\text{ad}}$-bundle to $G$-bundle.

The element $\zeta$ has a cohomological interpretation. It is called the characteristic class of $E_G$. It can be identified with elements of the group $H^2(\Sigma, \mathcal{Z}(\tilde{G}))$. This group classifies the of the characteristic classes of the bundles [46].

\(^2\)For the simplicity we assume here and in what follows that $\mathcal{Z} \sim \mathbb{Z}_l$. The case $\mathcal{Z}(\text{Spin}(4n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ can be considered in a similar way.
Consider, as above, a non-prime $N = pl$ and put $j = p$. Then $\zeta^j$ is a generator of the group $\mathbb{Z}_l$. In this case $Q$ and $\Lambda_j$ are transition operators for $G_l = \tilde{G}/\mathbb{Z}_l$-bundles (see (4.1)) and $\zeta^j$ is an obstruction to lift a $G_l$-bundle to a $\tilde{G}$-bundle.

The moduli space of stable holomorphic over $\Sigma_r$ with the sections (4.5) is defined as

$$\mathcal{M}_{G,1}^{(j)} = \text{(solutions of (4.6))/(conjugation)}. \quad (4.7)$$

For the stable bundles this description of the moduli space is equivalent to (3.1). In fact, the monodromy of $s(z)$ around $z = 0$ is the same as in (3.4). Similar to (3.7) we have

$$\mathcal{M}_{G,1} = \bigcup_{j=1}^{N} \mathcal{M}_{G,1}^{(j)}. \quad (4.8)$$

Assume that $Q$ is a semi-simple element and $Q \in H_{\tilde{G}}$ is a fixed Cartan subgroup of $\tilde{G}$. It means that we consider an open subset

$$\mathcal{M}_{G,1}^{(j)} \supset \left( \mathcal{M}_{G,1}^{(j)} \right)^0 \equiv \mathcal{M}_{0}^{(j)}(G) = \{ (Q \in H_{\tilde{G}}, \Lambda_j)/(\text{conjugation}) \}.$$  

In this case the solutions of (4.4) have the form [46]

$$Q = \exp \left( 2\pi i \rho^\vee / h \right), \quad \Lambda_j = \Lambda_0 V_j, \quad (4.9)$$

where $\rho^\vee$ is a half-sum of positive coroots, $h$ is the Coxeter number, $\Lambda_0$ is an element of the Weyl group defined by $\zeta^j$:

$$\zeta^j \to \Lambda_0, \quad (\Lambda_0)^l = \text{Id}.$$  

The element $\Lambda_0$ preserves the extended system of simple roots $\Pi^\text{ext} = \Pi \cup (\alpha_0)$, where $-\alpha_0$ is a maximal root [46, Proposition 3.1]. In this way $\Lambda_0$ is a symmetry of the extended Dynkin diagram of $g = \text{Lie} (\tilde{G})$, generated by $w^\vee$ [12].

Let $\tilde{H}_0 \subset H_{\tilde{G}}$ be the Cartan subgroup commuting with $\Lambda_0$. To describe $V_j$ consider the adjoint action $\lambda = \text{Ad}(\Lambda_0)$ on the Cartan subalgebra $\mathfrak{h} = \text{Lie}(H_{\tilde{G}})$. Let $\mathfrak{h}_0 = \text{Lie}(\tilde{H}_0)$ be the invariant subalgebra $(\lambda(\mathfrak{h}_0) = \mathfrak{h}_0)$. Then $V_j = \exp(2\pi i u)(u \in \mathfrak{h}_0)$ is an arbitrary element from $\tilde{H}_0$ defining the moduli space $\mathcal{M}_{0}^{(j)}(G)$.

There exists a basis $\Pi^\vee_j$ in $\mathfrak{h}_0$ such that $\Pi$ is a system of simple roots for a simple Lie subalgebra $\tilde{\mathfrak{g}}_0 \subset g$. For the list of these subalgebras see [46]. If $j = N$, we come to the trivial bundles (4.4). In this case $\Lambda_0 = \text{Id},\mathfrak{h}_0 = \mathfrak{h}$ and $\tilde{\mathfrak{g}}_0 = g$. Let $\tilde{Q}^\vee$ and $\tilde{P}^\vee$ be the coroot and the coweight lattices in $\tilde{\mathfrak{g}}_0$, and $\tilde{W}$ is the Weyl group corresponding to $\Pi$. Define the Bernstein–Schwarzman type groups [10, 11]. They are constructed by means of the lattices $\tilde{Q}^\vee$ or $\tilde{P}^\vee$. In the first case it is the semidirect products

$$\tilde{W}_{\text{BS}} = \tilde{W} \ltimes (\tau \tilde{Q}^\vee \oplus \tilde{Q}^\vee). \quad (4.10)$$

Then the moduli space of non-trivial $\tilde{G}$-bundles with the characteristic class $\zeta^j$ is the fundamental domain in $\tilde{\mathfrak{h}}^{(j)}_0$ under the action of $\tilde{W}_{\text{BS}}$

$$\mathcal{M}_{0}^{(j)}(\tilde{G}) = C^\text{sc}_j = \tilde{\mathfrak{h}}^{(j)}_0 / \tilde{W}_{\text{BS}} \quad (4.11)$$

is the moduli space of non-trivial $\tilde{G}$-bundles.

Consider $G^\text{ad}$-bundles. Define the semidirect product

$$W^\text{ad}_{\text{BS}} = \tilde{W} \ltimes (\tau \tilde{P}^\vee \oplus \tilde{P}^\vee). \quad (4.12)$$
A fundamental domain of this group in \( \tilde{\mathfrak{h}}_0^{(j)} \) is \( \mathfrak{c}^{ad} = \tilde{\mathfrak{h}}_0^{(j)}/\tilde{\mathbb{W}}_{BS}^{ad} \) and

\[
\mathcal{M}_0^{(j)}(G^{ad}) = C^{ad}_j = \tilde{\mathfrak{h}}_0^{(j)}/\tilde{\mathbb{W}}_{BS}^{ad}
\]

is the moduli space of the non-trivial \( G^{ad} \)-bundles. It is the moduli space of \( E_{G^{ad}} \)-bundles with characteristic class defined by \( \zeta \). In other words

\[
u \in \begin{cases} 
C^{nc}_j \text{ for } E_G \text{-bundles,} \\
C^{ad}_j \text{ for } E_{G^{ad}} \text{-bundles.}
\end{cases}
\]

### 4.3 The gauge Lie algebra for elliptic curves

Here we define the moduli space of holomorphic \( G \)-bundles coming back to the double coset construction (3.1). Recall, that the Lie algebra \( g_\text{out} = \text{Lie}(G(\Sigma_{\tau,n} \setminus \bar{z})) \) is a Lie algebra of meromorphic functions on \( \Sigma_{\tau,n} \) with poles at \( \bar{z} = (z_1, \ldots, z_n) \) and the quasi-periodicities (4.2), (4.3).

Let us take for simplicity the case (4.4) and apply the decomposition (A.1) corresponding the characteristic class defined by \( \zeta \) to the Lie algebra \( g_\text{out} \):

\[
g_\text{out} = \bigoplus_{k=0}^{l-1} g_k, \quad g_0 = g'_0 \oplus \tilde{g}_0, \quad \text{Ad}_{\Lambda}(g_k(z)) = e \left( \frac{k}{l} \right) g_k(z).
\]

Consider the quasi-periodicity conditions (4.2). The GS-basis is diagonal under \( \text{Ad}_\Lambda \) and \( \text{Ad}_\varphi \) actions (A.6)–(A.8). We should find functions on \( \Sigma_\tau \setminus D \) that have the same phase-factors and pole singularities at \( \bar{z} \). To define \( g_k \) and \( g'_0 \) we use the functions \( \phi \) (B.1), and \( \varphi^{k,m}_\alpha \) (B.4). They have the needed quasi-periodicities (B.16), (B.17) and poles at \( z = 0 \) (B.7), (B.9). Then we find

\[
g_k = \left\{ \sum_{a=1}^{n} \sum_{m=0}^{K(a,k)} \left( \sum_{\alpha \in \Pi} x_{a,m,a}^k \partial_z^m \phi \left( \frac{k}{l}, z - z_a \right) \mathfrak{h}_\alpha + \sum_{\alpha \in R} y^k_{\alpha,m,a} \varphi^{k,m}_\alpha(u, z - z_a) t^k_{\alpha} \right) \right\},
\]

\[
g'_0 = \left\{ \sum_{a=1}^{n} \sum_{\alpha \in R} \sum_{m=0}^{K(a,\alpha)} \mathfrak{g}^{0,m}_\alpha(u, z - z_a) t^0_{\alpha} \right\}.
\]

Similarly, from (B.10), (B.11), (B.12), (B.14), (B.15) we have

\[
\tilde{g}_0 = \left\{ \sum_{a=1}^{n} \left( \sum_{\alpha \in \Pi} x^0_{\alpha,0} + \sum_{m=1}^{K(a,\alpha)} x^0_{a,m,a} E_m(z - z_a) \right) h_\alpha \right.
\]

\[
+ \left. \sum_{\alpha \in R} \sum_{m=0}^{K(a,\alpha)} \mathfrak{g}^{0,m}_\alpha(u, z - z_a) E_\alpha \right\}.
\]

Then \( g_\text{out} \) has the correct quasi-periodicities and has poles of orders \( K(a,m), K(a,\alpha) \) at \( z_a, a = 1, \ldots, n \). In this last expression (due to the residue theorem) from (B.14) we assume that

\[
\sum_{a=1}^{n} x^0_{\alpha,1,a} = 0.
\]

Let us unify the last two expression (4.15) and (4.16) in a single formula,

\[
g_0 = \left\{ \sum_{a=1}^{n} \left( \sum_{\alpha \in \Pi} x^0_{\alpha,0} + \sum_{m=1}^{K(a,\alpha)} x^0_{a,m,a} E_m(z - z_a) \right) h_\alpha \right. \]
Notice that the constant terms conclude from (4.21) that locally the action on $G$ except the two types of modes describing the moduli space: general case (3.3).

Define the Lie algebra with the loose condition (4.17):

\[
\mathfrak{g}_k \sim \sum_{a=1}^{n} \sum_{\alpha, \beta \in \Pi} x_{a,0,a}^k \left( \cdots + \frac{1}{z - z_a} + E_1 \left( \frac{k}{l} \right) + \sum_{b \neq a} \phi \left( \frac{k}{l}, z_a - z_b \right) + \right) \mathfrak{g}_\alpha^k \\
+ y_{a,0,a}^k \left( \cdots + \frac{1}{z - z_a} + E_1 \left( \langle u + \kappa \tau, \alpha \rangle + \frac{k}{l} \right) + 2\pi i \langle \kappa, \alpha \rangle \right) \\
+ \sum_{b \neq a} \phi_{\alpha,0}^k (u, z_b - z_a) + \cdots \right) \mathfrak{g}_\alpha.
\]

\[
\mathfrak{g}_0 \sim \sum_{a \in \Pi} \left( \cdots + x_{a,1,a}^0 \left( \frac{1}{z - z_a} + \sum_{b \neq a} E_1 (z_b - z_a) + x_{a,0,a}^0 + \cdots \right) \mathfrak{h}_\alpha \\
+ \sum_{a \in R} y_{a,0,a}^0 \left( \cdots + \frac{1}{z - z_a} + E_1 \left( \langle u + \kappa \tau, \alpha \rangle \right) + 2\pi i \langle \kappa, \alpha \rangle \right) \\
+ \sum_{b \neq a} \phi_{\alpha}^0 (u, z_b - z_a) + \cdots \right) \mathfrak{h}_\alpha.
\]

Here “\cdots” means the terms of order $o(z - z_a)^{-1}$ and $o(1)$. For $\mathfrak{g}_{\text{int}} = \text{Lie}(G(U_D))$ we have local expansions in neighborhoods of the marked points

\[
\mathfrak{g}_{\text{int}} = \left\{ X = \sum_{a=1}^{n} \left( b_a + \sum_{j>0} y_j^a (z - z_a)^j \right), \ b_a \in \mathfrak{b}_0, \ y_j^a \in \mathfrak{g} \right\}.
\]

Define the Lie algebra with the loose condition (4.17)

\[
\mathfrak{g}'_{\text{out}} = \mathfrak{g}_{\text{out}} \quad \text{with} \quad \sum_{a=1}^{n} x_{a,1,a}^0 \in \mathbb{C}
\]

and let $\mathfrak{n}^- = \sum_{\alpha \in R^+} \mathfrak{g}^{-\alpha}$. Then the Lie algebra $\text{Lie}(G(D^\times))$ has the form (compare with the general case (3.3))

\[
\text{Lie} \left( G(D^\times) \right) = \mathfrak{g}'_{\text{out}} \oplus \left( \oplus_{a=1}^{n} \mathfrak{n}_a^+ \right) \oplus \mathfrak{g}_{\text{int}} \\
= \mathfrak{g}_{\text{out}} \oplus \left( \sum_{a=1}^{n} \sum_{\alpha \in \Pi} x_{a,1,a}^0 \mathfrak{h}_\alpha \right) \oplus \left( \oplus_{a=1}^{n} \mathfrak{n}_a^- \right) \oplus \mathfrak{g}_{\text{int}}.
\]

Notice that the constant terms $\mathfrak{n}_a^-$ come from the constant terms $c(m,k)$ in (B.9). We can conclude from (4.21) that locally the action on $G(D^\times)$ by $G_{\text{out}} = G(\Sigma_{1,n} \setminus \mathbb{Z})$ from the left and by $G_{\text{int}} = \prod_{a=1}^{n} G(D_a)$ from the right absorbs almost all negative and positive modes of $G(D^\times)$ except the two types of modes describing the moduli space:
The vector \( u = \sum_{a=1}^{n} \sum_{\alpha \in \tilde{\Pi}} x_{a,1,\alpha}^0 h_{\alpha} \in \mathfrak{h}_0 \). It defines an element of the moduli space \( M_{G,1} \).

- The Lie algebras \( \mathfrak{n}_a^-, a = 1, \ldots, n \). They are the tangent spaces to the flag varieties attached the marked points coming from the quasi-parabolic structure of the bundle.

### 4.4 Conformal blocks

In this section we define connections on the space of conformal blocks and derive the KZB equations in a similar way as it was done for the trivial characteristic classes in [24]. The derivation is based on the representation of the moduli space of bundles as the double coset space (3.1) in a given sector of the decomposition (3.5). In other words, the characteristic class (defined by \( j = 0, \ldots, l-1 \) in (4.7)) is fixed and we deal with \( G_{\text{out}} \)\( /G_{\gamma}^\text{ad} \otimes \mathbb{C}[[t_a, t_a^{-1}]] / G_{\text{int}} \), where \( G_{\text{out}} = G(\Sigma_{\tau,n} \setminus \mathcal{Z}) \) and \( G_{\text{int}} = G \otimes \mathbb{C}[[t_a]] \) were described above.

Let us write down the Virasoro generators (3.15) using the GS-basis (A.3), (A.4), (A.5)

\[
L_m^a = \frac{1}{2(k + h^+)} \sum_{p \in \mathbb{Z}} \sum_{q = 0}^{l-1} \left( \sum_{\alpha \in R} : \alpha \partial_{\alpha}^2 \Phi^q(p + m) : + \sum_{\alpha \in \tilde{\Pi}} : \Phi^q(-p) \Phi^{-q}(p + m) : \right). \tag{4.22}
\]

Consider the integrable modules attached to the marked points \( \hat{V}_{\gamma}^{[n]} \) (3.9) and the corresponding conformal blocks. They satisfy the equations (3.14), (3.18), (3.22). For elliptic curve they assume the form:

- The moving points (3.17):

\[
(\partial_\alpha - L_{-1}^a) F = 0, \quad \partial_\alpha = \frac{1}{k + h^+} \partial_{z_\alpha}, \quad t_\alpha = z - z_\alpha. \tag{4.23}
\]

- The vector field corresponding to the deformation of the moduli \( \tau \) of the elliptic curve \( \Sigma_{\tau,n} \):

\[
\left( \partial_\tau - \frac{1}{2\pi i} E_1(z) \partial_z \right) F = 0. \tag{4.24}
\]

This action follows from (3.13) and the operator algebra

\[ T(z') F(z) = E_1(z' - z) \partial_z F(z) + \text{analitic part}. \]

- The invariance with respect to the action of \( \mathfrak{g}_{\text{out}} \) (3.22):

\[
(l \partial_{u_{-\alpha}} + E_1(z) \delta_{-\alpha}^0) F = 0, \quad \alpha \in \tilde{\Pi}, \quad u = \{u_\alpha\}, \quad \alpha \in \tilde{\Pi}, \tag{4.25}
\]

where \( \delta_{-\alpha}^0 \) are the Cartan generators (A.5). Notice that this operator is well defined on \( M_G \).

The vector field (4.24) is defined on the universal curve \( \mathcal{H} \times \mathbb{C} / \langle \tau, 1 \rangle \setminus \mathcal{H} \times 0 \), since it is invariant under the lattice shifts \( \langle \tau, 1 \rangle \). The \( \tau \) deformation can be defined in the non-holomorphic form as \( \partial_\tau + \frac{z - z_\alpha}{z - z_\alpha} \partial_z \).

The invariance with respect to \( \text{Lie}(G_{\text{out}}) \) (4.14), (4.18) means that

\[
\varphi_{\alpha}^k (u_\alpha, z - z_\alpha) t_\alpha^k F = 0, \quad \alpha \in R, \forall k, \tag{4.26}
\]

\[
\varphi_{\alpha}^k (0, z - z_\alpha) \delta_{-\alpha}^k F = 0, \quad \alpha \in \tilde{\Pi}, \quad k \neq 0.
\]
Now using (4.19), (4.20) and (4.26) we write down the annihilation condition \( g_{\text{out}} F = 0 \) in in the basis \( t^{k,c}_{\alpha}(m) = 1 \otimes \cdots \otimes 1 \otimes t^{k,c}_{\alpha}(m) \otimes 1 \otimes \cdots \otimes 1 \) (on the \( c \)-th place):

\[
\begin{align*}
& \left( t^{k,a}_{\alpha}(-1) + \left( E_1 \left( u_\alpha + (\kappa, \alpha) \tau + \frac{k}{l} \right) + 2\pi i (\kappa, \alpha) \right) t^{k,a}_{\alpha}(0) \\
& + \sum_{c \neq a} \varphi^k_{\alpha} (u_\alpha, z_c - z_a) t^{k,c}_{\alpha}(0) \right) F = 0, \\
& \left( \delta f^{k,a}_{\alpha}(-1) + \left( E_1 (\kappa, \alpha) \tau + \frac{k}{l} \right) + 2\pi i (\kappa, \alpha) \right) \delta f^{k,a}_{\alpha}(0) \\
& + \sum_{c \neq a} \varphi^k_{\alpha} (0, z_c - z_a) \delta f^{k,c}_{\alpha}(0) \right) F = 0 \\
\end{align*}
\]

for \( \alpha \in R, \forall k \) and \( \alpha \in \tilde{\Pi}, k \neq 0 \) correspondingly. In the same way (4.24) and (4.25) assume the form

\[
\begin{align*}
& \left( \partial_\tau + \frac{1}{2\pi i} L^a_{-2} + \frac{1}{2\pi i} \sum_{c \neq a} E_1 (z_c - z_a) L^a_{-1} \right) F = 0, \\
& \left( l\partial_{u_\alpha} + \delta f^0_{\alpha}(-1) + \sum_{c \neq a} E_1 (z_c - z_a) \delta f^0_{\alpha}(0) \right) F = 0, \quad \alpha \in \tilde{\Pi}. \\
\end{align*}
\]

Now we are ready to evaluate the Virasoro generators, i.e. to express them in terms of zero modes of the loop algebra \( t^{k,c}_{\alpha}(0) \equiv t^{k,c}_{\alpha} \) only. As we have found above the positive modes of the loop algebra act on \( F \) by zero \( t^{k,a}_{\alpha}(m) F = 0, m \in \mathbb{Z}_+ \). Therefore, from (4.22) we have

\[
(k + h^\vee) L^a_{-1} = \sum_{q=0}^{l-1} \left( \sum_{\alpha \in R} t^{q,a}_{\alpha}(-1) t^{-q,a}_{\alpha}(0) + \sum_{\alpha \in \tilde{\Pi}} \delta f^{q,a}_{\alpha}(-1) h^{-q,a}_{\alpha}(0) \right) \quad \text{on } F
\]

and

\[
(k + h^\vee) L^a_{-2} = \sum_{q=0}^{l-1} \left( \sum_{\alpha \in R} t^{q,a}_{\alpha}(-2) t^{-q,a}_{\alpha}(0) + \sum_{\alpha \in \tilde{\Pi}} \delta f^{q,a}_{\alpha}(-2) h^{-q,a}_{\alpha}(0) \right) \\
+ \frac{1}{2} \sum_{q=0}^{l-1} \left( \sum_{\alpha \in R} t^{q,a}_{\alpha}(-1) t^{-q,a}_{\alpha}(-1) + \sum_{\alpha \in \tilde{\Pi}} \delta f^{q,a}_{\alpha}(-1) h^{-q,a}_{\alpha}(-1) \right) \quad \text{on } F.
\]

In order to find \( L^a_{-1} \) one need to substitute \( t^{k,a}_{\alpha}(-1), \delta f^{k,a}_{\alpha}(-1) \) from (4.27) and \( \delta f^0_{\alpha}(-1) \) from (4.28) into (4.29)

\[
-(k + h^\vee) L^a_{-1} = i \sum_{\alpha \in \tilde{\Pi}} h^0_{\alpha}(0) \partial_{u_\alpha} \\
+ \sum_{q=0}^{l-1} \sum_{\alpha \in R} |q|^2 \left( E_1 \left( u_\alpha + (\kappa, \alpha) \tau + \frac{q}{l} \right) + 2\pi i (\kappa, \alpha) \right) t^{q,a}_{\alpha}(0) t^{-q,a}_{\alpha}(0) \\
+ \sum_{c \neq a} \varphi^q_{\alpha} (u_\alpha, z_c - z_a) t^{q,c}_{\alpha}(0) t^{-q,a}_{\alpha}(0)
\]
\[ + \sum_{q=0}^{l-1} \sum_{\alpha \in \Pi} \left( E_1 \left( \langle \kappa, \alpha \rangle \tau + \frac{q}{l} \right) + 2\pi i \langle \kappa, \alpha \rangle \right) \delta^{q,a}_0(0) h^{-q,a}_{-\alpha}(0) \]

\[ + \sum_{\alpha \neq \alpha} \varphi^q_\alpha(0, z_c - z_a) \delta^{q,c}_\alpha(0) h^{-q,a}_{-\alpha}(0) \],

where \( \varphi^0_\alpha(0, z_c - z_a) = E_1(z_c - z_a) \). The first term in the last line vanishes due to skew-symmetry with respect to \( \alpha, q \to -\alpha, -q \). The similar term in the second line does not vanish because \([t_{q,a}^0(0), t_{-q,a}^0(0)] = \frac{p_\alpha}{\sqrt{l}} \exp \left( -2\pi i \frac{q}{l} \right) h^0_{\alpha,a} \) [46]. Therefore,

\[ \sum_{q=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \left( E_1 \left( u_{\alpha} + \langle \kappa, \alpha \rangle \tau + \frac{q}{l} \right) + 2\pi i \langle \kappa, \alpha \rangle \right) t^{q,a}_\alpha(0) t^{-q,a}_{-\alpha}(0) \]

\[ = \frac{1}{2} \sum_{q=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \left( E_1 \left( u_{\alpha} + \langle \kappa, \alpha \rangle \tau + \frac{q}{l} \right) + 2\pi i \langle \kappa, \alpha \rangle \right) \frac{p_\alpha}{\sqrt{l}} \exp \left( -2\pi i \frac{q}{l} \right) h^0_{\alpha,a} \]

\[ = l \sum_{\alpha \in \Pi} h^0_{\alpha,a}(0) \partial u_{\alpha} \left\{ \log \prod_{q=0}^{l-1} \prod_{\alpha \in R} q \right\} \frac{p_\alpha |\alpha|^2}{2\sqrt{l}} \exp \left( -2\pi i \frac{q}{l} \right) . \]

The term

\[ \frac{1}{2} \sum_{q=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \exp \left( -2\pi i \frac{q}{l} \right) h^0_{\alpha,a} \]

vanishes because of summation over \( q \). Notice also that the obtained scalar expression does not depend on \( \{z_c\} \). Then, the equation (4.23) gives

\[ \left( \partial_{u} + l \sum_{\alpha \in \Pi} h^0_{\alpha,a}(0) \partial u_{\alpha} + \sum_{q=0}^{l-1} \sum_{\alpha \in R} \left( \sum_{\alpha \neq \alpha} \varphi^q_\alpha(0, z_c - z_a) t^{q,c}_\alpha(0) t^{-q,a}_{-\alpha}(0) \right) \right) \tilde{F} = 0 , \]

where

\[ \tilde{F} = F \prod_{q=0}^{l-1} \prod_{\alpha \in R} q \frac{p_\alpha |\alpha|^2}{2\sqrt{l}} \exp \left( -2\pi i \frac{q}{l} \right) . \]

This is the first set of equations in (4.41). In order to obtain the second one (the KZB connection \( \nabla_{\tau} \) along \( \tau \)) one should use (4.30). It is needed to compute \( L^2_{-2} \). The later arises from the local expansion of (B.4) for \( k = 1 \). Then the following identities should be used

\[ \partial_{z} \phi(u, z) = \phi(u, z) \left( E_1(z + u) - E_1(z) \right) = f(u, z) + (E_1(u) - E_1(z)) \phi(u, z) , \]

where \( f(u, z) = \partial_{u} \phi(u, z) \) for \( t(-2)t(0) \)-terms and

\[ \phi(u, z - z_a) \phi(-u, z - z_b) = -\phi(u, z - z_a) \phi(u, z_b - z) \]

\[ = -\phi(u, z_b - z_a) \left( E_1(u) + E_1(z - z_a) + E_1(z_b - z) - E_1(u + z_b - z_a) \right) \]

\[ = f(u, z_b - z_a) + \phi(u, z_b - z_a) \left( E_1(z - z_b) - E_1(z - z_a) \right) \]

for \( t(-1)t(-1) \)-terms. On the other hand \( \nabla_{\tau} \) is a unique flat connection for given \( \nabla_{u} \) (4.37). The final answer is given below in Section 4.6. This answer is verified in Appendix C.
4.5 Classical $r$-matrix

The construction of the KZB connection is based on the classical dynamical elliptic $r$-matrix defined as sections of bundles over elliptic curves [13, 47, 74]. For trivial $G$-bundles our list coincides with the elliptic $r$-matrices were defined in [19]. A more general class of elliptic $r$-matrices was constructed in [17, 18]. The latter classification includes our list though it was derived from different postulates.

4.5.1 Axiomatic description of $r$-matrices

The classical dynamical $r$-matrix is a meromorphic one form $r = r(u, z) dz$, $(u \in \tilde{h}_0)$ on $\mathbb{C}$ taking values in $g \otimes g$ that satisfies the following conditions:

1. $r(z)$ has a pole at $z = 0$ and

   \[ \text{Res}_{z=0} r(z) = C_2 = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 t^k_{\alpha} \otimes t^{-k}_{\alpha} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \Phi^k_{\alpha} \otimes \eta^{-k}_{\alpha}, \]

   where $t^k_{\alpha}$, $\Phi^k_{\alpha}$, $\eta^{-k}_{\alpha}$ are generators of the GS basis in $g$ (see Appendix A). If $V$ is a $g$-module, then $C_2$ acts by the permutation on $V \otimes V$.

2. Behavior under the shifts by the generators of the lattice $\mathbb{Z} \oplus \tau \mathbb{Z}$:

   \[ r(z + 1) = \text{Ad}_{Q} r(z), \quad r(z + \tau) = 2\pi i \sum_{\alpha \in \Pi} \Phi^0_{\alpha} \otimes \eta^0_{\alpha} + \text{Ad}_{\Lambda^0} r(z), \] (4.31)

   where the Ad-action is taken with respect to the first factor in $g \otimes g$. Here $Q = e(\kappa)$, $\Lambda^{(j)} = \Lambda_{\alpha} e(u)$ (see (4.9)), $\{ \eta^0_{\alpha} \}$ ($\{ \Phi^0_{\alpha} \}$) is the simple coroot basis (the dual basis) in the invariant subalgebra $\tilde{g}_0$ (see Appendix A). It means that $r$ is a connection in the $g \otimes g$-bundle over $\Sigma_{\tau}$.

3. The classical dynamical Yang–Baxter equation (CDYBE). It follows from 1 that $r(z)$ can be represented as

   \[ r(z) = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \Phi^k_{\alpha}(z) |\alpha|^2 t^k_{\alpha} \otimes t^{-k}_{\alpha} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \Phi^k_{\alpha}(z) \Phi^k_{\alpha} \otimes \eta^{-k}_{\alpha}. \]

   Then $r(z)$ is a solution of CDYBE:

   \[ [r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] \]

   \[ -\sqrt{t} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 t^k_{\alpha} \otimes t^{-k}_{\alpha} \otimes h_0^0 \partial_1 \Phi^k_{\alpha}(u, z - w) \]

   \[ -\frac{|\alpha|^2}{2} t^k_{\alpha} \otimes t^{-k}_{\alpha} \otimes h_0^0 \partial_1 \Phi^k_{\alpha}(u, z - x) \]

   \[ + \frac{|\alpha|^2}{2} h_0^0 \otimes t^k_{\alpha} \otimes t^{-k}_{\alpha} \partial_1 \Phi^k_{\alpha}(u, w - x) = 0, \quad z_{ij} = z_i - z_j, \] (4.32)

   where $\partial_1$ is the differentiation with respect to the first argument.

4. The unitarity

   \[ r^{12}(u, z) + r^{21}(u, -z) = 0. \]
5. The zero weight condition

\[ [X \otimes 1 + 1 \otimes X, r(u, z)] = 0, \quad X \in \mathfrak{h}. \]

**Lemma 1.** • Any \( r' \)-matrix satisfying 1–5 has the form

\[ r'(u, z) = r(u, z) + \delta r(u), \]

where

\[ r(u, z) = r_\delta(u, z) + r_R(z), \]

\[ r_R(u, z) = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} r^k_\alpha(u, z), \quad r^k_\alpha(u, z) = |\alpha|^2 \varphi^k_\alpha(u, z) t^k_\alpha \otimes t^{-k}_\alpha, \]

\[ r_\delta(z) = \sum_{k=0}^{l-1} \sum_{\alpha \in H} r^0_\alpha(z), \quad r^0_\alpha(z) = \varphi^0_\alpha(z) \tilde{f}^k_\alpha \otimes h^{-k}_\alpha, \]

satisfies 1–5 and \( \varphi^k_\beta(z) \) is defined in (B.3), \( \varphi^k_0(z) = \phi(k/l, z), \varphi_0^0(z) = E_1(z). \)

- \( \delta r(u) \in \mathfrak{h}_0 \otimes \mathfrak{h}_0 \)

\[ \delta r(u) = \sum_{\alpha, \beta \in \mathfrak{h}} A_{\alpha \beta} \tilde{f}^0_\alpha \otimes h^0_\beta, \quad A_{\alpha \beta} = -A_{\beta \alpha}. \]

- \( \delta r(u) \) is generated by the gauge transformation

\[ \delta r(u) = -i \sum_{\alpha \in \mathfrak{h}} (\partial_{u_\alpha} f)^{-1} \otimes h^0_\alpha, \quad (\partial_{u_\alpha} f)^{-1} \in \mathfrak{h}_0, \quad f = f(u) \in \tilde{H}_0, \]

where \( \tilde{H}_0 \) is a Cartan subgroup of the invariant subgroup \( \tilde{G}_0 \subset G \) (see Table 1 in [46]).

**Proof.** It follows from the properties of the functions \( \varphi^k_\alpha(u, z), \varphi^0_\alpha(u, z) \) described in the Appendix B that \( r(u, z) \) satisfy 1 and 2. It was proved in [46] that it is a solution of the CDYBE. This sum is a classical dynamical \( r \)-matrix corresponding to a non-trivial characteristic class defined by (4.31). The conditions 4 and 5 can be checked as well. The conditions 1–3, 5 define the \( r \)-matrix up to a constant \((z\text{-independent})\) Cartan term \( \delta r \). Then it follows from 4 that \( A_{\alpha \beta} \) is antisymmetric.

Next we wish to prove that locally \( A_{\alpha \beta} = -l(\partial_{u_\alpha} (f)^{-1})_\beta \) for some \( f \in \tilde{H}_0 \). The twisted \( r \)-matrix must satisfy the CDYB equation. Plugging \( r + \delta r \) into (4.32) we see that the “commutator” part vanishes identically since \([r^{ab}, \delta r^{ac}] + [r^{ab}, \delta r^{bc}] \equiv 0\) due to

\[ \begin{bmatrix} t^{k,a}_\alpha, h^0_\beta \end{bmatrix} \otimes t^{-k,a}_\alpha + t^{k,a}_\alpha \otimes \begin{bmatrix} t^{-k,a}_\alpha, h^0_\beta \end{bmatrix} = 0. \]

The “derivative” part of (4.32) yields \( \partial_{u_\alpha} A_{\beta \gamma} + \partial_{u_\alpha} A_{\alpha \beta} + \partial_{u_\beta} A_{\gamma \alpha} = 0 \) or

\[ dA = 0, \quad A = \sum_{\alpha, \beta \in \mathfrak{h}} A_{\alpha \beta} du_\alpha \wedge du_\beta \in \mathcal{M}_G. \]

The term \( \delta r \) is called the dynamical twist of the \( r \)-matrix. The statement follows from the Poincaré lemma.
4.5.2 $r$-matrices as sections of bundles over moduli spaces

Consider the behavior of the $r$-matrix (4.33) under the action of lattices $\tau\tilde{Q}^\vee \oplus \tilde{Q}^\vee$ (4.10) and $\tau\tilde{P}^\vee \oplus \tilde{P}^\vee$ (4.12) on the dynamical parameter $u$. It follows from (B.2), (B.3) and (B.16) that the $r$-matrices have distinct type of quasi-periodicities with respect $u \in \tilde{h}_0$. Let $\beta^\vee \in \tilde{\Pi}^\vee$ be a simple coroot, corresponding to the invariant algebra $\tilde{\tau}$.

As it was established the part of connection related to the moving points coincides with the $4.6$ KZB connection related to elliptic curves (4.36).

Then we find

$$r^k_\alpha(u + \beta^\vee, z) = r^k_\alpha(u, z), \quad r^k_\alpha(u + \tau \beta^\vee, z) = e(-n_{\alpha, \beta}z)r^k_\alpha(u, z).$$

Let $\Xi^\vee$ be a basis of fundamental co-weights dual to the basis $\Pi$, and $\tilde{\omega}^\vee$ is a fundamental coweight in $\tilde{P}^\vee$. Since $\tilde{P}^\vee$ is a sublattice of $P^\vee$, the weight $\tilde{\omega}^\vee$ can be decomposed in the basis of the fundamental co-weights $\tilde{\omega}^\vee = \sum_{\nu \in \Xi^\vee} n_{\nu}^\omega \nu^\vee$, where $n_{\nu}^\omega \in \mathbb{Z}$. As above we find

$$r^k_\alpha(u + \tilde{\omega}^\vee, z) = r^k_\alpha(u, z), \quad r^k_\alpha(u + \tau \tilde{\omega}^\vee, z) = e(-n_{\nu}^\omega \delta_{(\nu^\vee, \alpha)}z)r^k_\alpha(u, z).$$

On the other hand, due to the $\Lambda$-invariance of $\tilde{Q}^\vee$, we have $\langle \beta^\vee, \lambda^m(\alpha) \rangle = \langle \beta^\vee, \alpha \rangle$. Therefore, $\text{Ad}_{\exp(2\pi i \beta^\vee)} E_{\lambda^m(\alpha)} = e(\langle \alpha, \beta^\vee \rangle) E_{\lambda^m(\alpha)}$. Then from (A.3) we find that

$$\text{Ad}_{\exp(-2\pi i \beta^\vee z)} t^a_\alpha = e(-n_{\alpha, \beta}z)t^a_\alpha.$$

Similarly, due to $\Lambda$-invariance of $\tilde{P}^\vee$, we have also

$$\text{Ad}_{\exp(-2\pi i \tilde{\omega}^\vee z)} t^a_\alpha = e(-n_{\nu}^\omega \delta_{(\nu^\vee, \alpha)}z)t^a_\alpha.$$

Since the Cartan part $r_\beta$ of the $r$-matrix does not depend on $u$ we come to the relations

$$r(u + \beta^\vee, z) = r(u, z), \quad r(u + \tau \beta^\vee, z) = \text{Ad}_{\exp(-2\pi i \beta^\vee z)} r(u, z),$$

$$r(u + \tilde{\omega}^\vee, z) = r(u, z), \quad r(u + \tau \tilde{\omega}^\vee, z) = \text{Ad}_{\exp(-2\pi i \tilde{\omega}^\vee z)} r(u, z).$$

In all cases the adjoint actions $\text{Ad}_k$ act on the first component of the tensor product and play the role of the clutching operators.

Let $x(k, \alpha) = \langle u, \alpha + \kappa \tau \rangle + k/l$. Then $r(u, z)$ is singular when $x(k, \alpha) \to 0$ (see (B.7) and (B.2))

$$r(u, z) = |\alpha|^2 e(\langle \kappa, \alpha \rangle z) \left( \frac{1}{x(k, \alpha)} + O(1) \right) t^k_\alpha \otimes t^{-k}_\alpha.$$

It means that $r(u, z)$ are sections of the bundles over the moduli spaces $C^c_j$ (4.11), or $C^{sc}_j$ (4.13) with sections taking values in $g \otimes g$ with the quasi-periodicities (4.34), (4.35) and with the singularities (4.36).

4.6 KZB connection related to elliptic curves

As it was established the part of connection related to the moving points coincides with the introduced above $r$-matrix. Here we prove that this connection is flat. Consider the following differential operators

$$\nabla_a = \partial_a + \hat{\partial}^a + \sum_{c \neq a} r^{ac},$$

$$\nabla_\tau = 2\pi i \partial_\tau + \Delta + \frac{1}{2} \sum_{b, d} f^{bd},$$

where $r^{ac}$ and $f^{bd}$ are $C^\infty$-functions.
From the definition it follows that \( r \) and, therefore
\[ r = -\sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_{k}^{\alpha}(u, z_a - z_c) t_{\alpha}^{k} a t_{-\alpha}^{-k}, c + \sum_{k=0}^{l-1} \sum_{\alpha \in R} \varphi_{0}^{\alpha}(u, z_a - z_c) \delta_{\alpha}^{k} a \delta_{\alpha}^{-k}, c, \]

\[ f = -\sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 f_{\alpha}^{k}(u, z_a - z_c) t_{\alpha}^{k} a t_{-\alpha}^{-k}, c + \sum_{k=0}^{l-1} \sum_{\alpha \in R} f_{0}^{k}(u, z_a - z_c) \delta_{\alpha}^{k} a \delta_{\alpha}^{-k}, c, \]

where \( t_{\alpha}^{k} \) = 1 \( \otimes \) \( \cdots \) 1 \( \otimes \) 1 \( \otimes \) \( \cdots \) 1 (with \( t_{\alpha}^{k} \) on the \( \alpha \)-th place) and similarly for the generators \( \delta_{\alpha}^{k} \) and \( \delta_{\alpha}^{-k} \). The following short notations are used here
\[ \hat{\varphi} = l \sum_{\alpha \in \Pi} b_{\alpha}^{0} a \partial_{\alpha}, \quad \Delta = \frac{l}{2} \sum_{\alpha \in \Pi} \sum_{s=0}^{l-1} \partial_{\alpha a} \partial_{u_{\alpha s} a} \]

and
\[ \varphi_{\alpha}^{k}(u, z) = e^{2\pi i(\kappa, \alpha)z} \phi \left( (u + \kappa \tau, \alpha) + \frac{k}{l}, z \right), \]
\[ f_{\alpha}^{k}(u, z) = e^{2\pi i(\kappa, \alpha)z} f \left( (u + \kappa \tau, \alpha) + \frac{k}{l}, z \right). \] (4.38)

From the definition it follows that \( r^{ac} = -r^{ca} \) and \( f^{ac} = f^{ca} \). Following (B.6) and (B.7) we put
\[ \varphi_{\alpha}^{0}(z) = E_{1}(z), \] (4.39)
\[ f_{\alpha}^{0}(z) = \rho(z) = \frac{1}{2}(E_{1}^{2}(z) - \varphi(z)). \] (4.40)

Notice that
\[ f_{\alpha}^{k}(u, 0) = -E_{2} \left( (u + \kappa \tau, \alpha) + \frac{k}{l} \right) = -\varphi \left( (u + \kappa \tau, \alpha) + \frac{k}{l} \right) - 2\eta_{1} \]

and, therefore
\[ f^{cc} = -\sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_{\alpha}^{k} c t_{\alpha}^{k}, c - \sum_{k=0}^{l-1} \sum_{\alpha \in R} \delta_{\alpha}^{k} c \delta_{\alpha}^{-k}, c - 2l\eta_{1} C_{2}^{c}, \]

where \( C_{2}^{c} \) is the Casimir operator acting on the \( c \)-th component. Recall that we study the following system of differential equations
\[ \nabla_{a} F = 0, \quad a = 1, \ldots, n, \quad \nabla_{\tau} F = 0. \] (4.41)

There are two types of the compatibility conditions of KZB equations (4.41)
\[ [\nabla_{a}, \nabla_{b}] F = 0, \quad a, b = 1, \ldots, n, \quad [\nabla_{a}, \nabla_{\tau}] F = 0, \quad a = 1, \ldots, n. \] (4.42)

It is important to mention that the solutions of (4.41) \( F \) are assumed to satisfy the following condition
\[ \left( \sum_{c=1}^{n} b_{\alpha}^{0, c} \right) F = 0, \quad \text{for any } \alpha \in \tilde{\Pi}. \] (4.43)
Proposition 1. The upper equations in (4.42) \([\nabla_a, \nabla_b] = 0\) are valid for the \(r\)-matrix (4.33) on the space of solutions of (4.41) satisfying (4.43). They follow from the classical dynamical Yang–Baxter equations

\[
[r^{ab}, r^{ac}] + [r^{ab}, r^{bc}] + [r^{ac}, r^{bc}] + [\hat{\varphi}^{a}, r^{bc}] + [\hat{\varphi}^{c}, r^{ab}] + [\hat{\varphi}^{b}, r^{ca}] = 0. \tag{4.44}
\]

Proposition 2. The lower equations in (4.42) \([\nabla_a, \nabla_r] = 0\) are valid for the \(r\)-matrix (4.38) on the space of solutions of (4.41) satisfying (4.43).

The proofs of these statements are given in the Appendix C.

Let us also remark that the non-trivial trigonometric and rational limits of the above formulae can be obtained via procedures described in [1, 65, 72].

### A Generalized Sine (GS) basis in simple Lie algebras

Let \(Z\) be a subgroup of the center \(Z(\tilde{G})\) of \(\tilde{G}\), and consider a quotient group \(G = \tilde{G}/Z\). Assume for simplicity that \(Z(\tilde{G})\) is cyclic. The case \(\text{Spin}(4n)\) where \(Z(G) = \mu_2 \times \mu_2\) can be treated similarly.

Let us take an element \(\zeta \in Z(\tilde{G})\) of order \(l\), generating \(Z\). It defines uniquely an element \(\Lambda_0\) from the Weyl group \(W\) (see [12, 46]). It is a symmetry of the corresponding extended Dynkin diagram and \((\Lambda_0)^l = \text{Id}\). \(\Lambda_0\) generates a cyclic group \(\mu_l = (\Lambda_0, (\Lambda_0)^2, \ldots, (\Lambda_0)^l = 1)\) isomorphic to a subgroup of \(Z(\tilde{G})\). Note that \(l\) is a divisor of \(\text{ord}(Z(\tilde{G}))\). Consider the action of \(\Lambda_0\) on \(g\). Since \((\Lambda_0)^l = \text{Id}\) we have a \(l\)-periodic gradation

\[
g = \bigoplus_{n=0}^{l-1} g_n, \quad \lambda(g_n) = \omega^a g_n, \quad \omega = \exp \frac{2\pi i}{l}, \quad \lambda = \text{Ad}_{\Lambda_0}, \tag{A.1}
\]

\[
[g_n, g_{n+b}] = g_{n+b} \mod l,
\]

where \(g_0\) is a subalgebra \(g_0 \subset g\) and the subspaces \(g_n\) are its representations. Since \(Q\) and \(\Lambda\) commute in the adjoint representations the root subspaces \(g_a\) are their common eigenspaces.

**GS-basis.** Here we shortly reproduce the construction of the GS-basis following [46]. Since \(\Lambda_0 \in W\) it preserves the root system \(R\). Define the quotient set \(\mathcal{T}_l = R/\mu_l\). Then \(R\) is represented as a union of \(\mu_l\)-orbits \(R = \bigcup_{\tau=1}^l \mathcal{O}\). We denote by \(\mathcal{O}(\bar{\beta})\) an orbit starting from the root \(\bar{\beta}\)

\[
\mathcal{O}(\bar{\beta}) = \{\beta, \lambda(\beta), \ldots, \lambda^{l-1}(\beta)\}, \quad \bar{\beta} \in \mathcal{T}_l.
\]

The number of elements in an orbit \(\mathcal{O}\) (the length of \(\mathcal{O}\)) is \(l/p_\alpha = l_{\alpha}\), where \(p_\alpha\) is a divisor of \(l\). Let \(\nu_\alpha\) be a number of orbits \(\mathcal{O}_\alpha\) of the length \(l_{\alpha}\). Then \(\mathcal{O} = \sum \nu_\alpha l_{\alpha}\). Notice that if \(\mathcal{O}(\bar{\beta})\) has length \(l_{\beta} (l_{\beta} \neq 1)\), then the elements \(\lambda^{k\beta}\) and \(\lambda^{k+l_{\beta}\bar{\beta}}\) coincide. First, transform the root basis \(\mathcal{E} = \{E_\beta, \beta \in R\}\) in \(\mathcal{L}\). Define an orbit in \(\mathcal{E}\)

\[
\mathcal{E}_\beta = \{E_\beta, E_{\lambda(\beta)}, \ldots, E_{\lambda^{l-1}(\beta)}\}
\]

corresponding to \(\mathcal{O}(\bar{\beta})\). Again \(\mathcal{E} = \bigcup_{\bar{\beta} \in \mathcal{T}_l} \mathcal{E}_\beta\). For \(\mathcal{O}(\bar{\beta})\) define the set of integers

\[
J_{p_\alpha} = \{a = mp_\alpha \mid m \in \mathbb{Z}, \ a \text{ is defined mod } l\}, \quad p_\alpha = \frac{l}{l_{\alpha}}. \tag{A.2}
\]

Let \(E_\alpha \ (\alpha \in R)\) be the root basis of \(g\). “The Fourier transform” of the root basis on the orbit \(\mathcal{O}(\bar{\beta})\) is defined as

\[
t^\beta_\alpha = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{ma} E_{\lambda^m(\beta)}, \quad \omega = \exp \frac{2\pi i}{l}, \quad a \in J_{\beta}. \tag{A.3}
\]
Almost the same construction exists in \( \hat{\mathfrak{h}} \). Again let \( \Lambda_0 \) generates the group \( \mu_1 \). Since \( \Lambda_0 \) preserves the extended Dynkin diagram, its action preserves the extended coroot system \( \Pi^\text{ext} = \Pi \cup \check{\alpha}_0 \) in \( \hat{\mathfrak{h}} \). Consider the quotient \( K_l \equiv \Pi^\text{ext} / \mu_1 \). Define an orbit \( \mathcal{H}(\check{\alpha}) \) of length \( l_\alpha = l / p_\alpha \) in \( \Pi^\text{ext} \) passing through \( H_\alpha \in \Pi^\text{ext} \)

\[
\mathcal{H}(\check{\alpha}) = \{ H_\alpha, H_{\lambda(\alpha)}, \ldots, H_{\lambda^{l-1}(\alpha)} \}, \quad \check{\alpha} \in K_l = \Pi^\text{ext} / \mu_1.
\]

The set \( \Pi^\text{ext} \) is a union of \( \mathcal{H}(\check{\alpha}) \):

\[
(\Pi^\vee)^\text{ext} = \cup_{\check{\alpha} \in K_l} \mathcal{H}(\check{\alpha}).
\]

Define “the Fourier transform”

\[
h_\alpha^c = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mc} H_{\lambda^m(\alpha)}, \quad \omega = \exp \left( \frac{2\pi i}{l} \right), \quad c \in J_\alpha \quad (\text{see (A.2)}).
\]

The basis \( h_\alpha^c \) \( (c \in J_\alpha, \check{\alpha} \in K_l) \) is over-complete in \( \hat{\mathfrak{h}} \). Namely, let \( \mathcal{H}(\check{\alpha}_0) \) be an orbit passing through the minimal coroot \( \{ H_{\alpha_0}, H_{\lambda(\alpha_0)}, \ldots, H_{\lambda^{l-1}(\alpha_0)} \} \). Then the element \( h_0^- \) is a linear combination of elements \( h_\alpha^0 \) \( (\alpha \in \Pi) \) and we should exclude it from the basis. We replace the basis \( \Pi^\vee \) in \( \hat{\mathfrak{h}} \) by

\[
h_\alpha^c, \quad c \in J_\alpha, \quad \begin{cases} \alpha \in \check{K}_l = K_l \setminus \mathcal{H}(\check{\alpha}_0), & c = 0, \\ \check{\alpha} \in K_l, & c \neq 0. \end{cases}
\]

As before there is a one-to-one map \( \Pi^\vee \leftrightarrow \{ h_\alpha^c \} \). The elements \( (h_\alpha^a, t_\beta^a) \) form GS basis in \( \mathfrak{g}(l-a) \) \( (A.1) \). The dual basis is generated by elements \( \check{h}_\alpha^a \)

\[
(\check{\delta}_\alpha^a, h_\beta^a) = \delta(a+b,0(\text{mod}\ l)) \delta_{\alpha,\beta}, \quad \check{\delta}_\alpha^a = \sum_{\beta \in \Pi} (A_{a,\beta}^a)^{-1} h_\beta^{-a}, \quad h_\beta^a = \sum_{\alpha \in \Pi} (A_{a,\beta}) \check{\delta}_\alpha^a, \quad (A.5)
\]

where

\[
A_{a,\beta}^a = \frac{2}{|\beta|} \sum_{s=0}^{l-1} \omega^{-sa} a_{\beta,\lambda^s(\alpha)}
\]

and \( a_{\alpha,\beta} \) is the Cartan matrix of \( \mathfrak{g} \).

The \( \lambda \)-invariant subalgebra \( \mathfrak{g}_0 \) contains the subspace

\[
V = \left\{ \sum_{\beta \in T^l} a_{\beta} t_\beta^0, \quad a_{\beta} \in \mathbb{C} \right\}.
\]

Then \( \mathfrak{g}_0 \) is a sum of \( \check{\mathfrak{g}}_0 \) and \( V \)

\[
\mathfrak{g}_0 = \check{\mathfrak{g}}_0 \oplus V.
\]

In the invariant simple algebra \( \check{\mathfrak{g}}_0 \) instead of the basis \( (h_\alpha^0, t_\beta^0) \) we can use the Chevalley basis and incorporate it in the GS-basis

\[
\{ h_\alpha^0, t_\beta^0 \} \rightarrow \{ \check{\mathfrak{g}}_0 = \{ H_\alpha, \check{\alpha} \in \check{\Pi}, E_\beta, \check{\beta} \in \check{R} \}, \quad V = \{ t_\beta^0, \check{\beta} \in T' \}, \}
\]

where \( \check{\Pi} \) is a system of simple roots constructed by the averaging of the \( \lambda \) action on \( \Pi^\text{ext} \), and \( \check{R} \) is a system of roots of \( \check{\mathfrak{g}}_0 \) generated by \( \check{\Pi} \). We have the following action of the adjoint operators on the GS-basis:

\[
\text{Ad}_\Lambda (t_\beta^0) = e \left( \langle \check{u}, \beta \rangle - \frac{c}{l} \right) t_\beta^0, \quad \text{Ad}_\Lambda (h_\alpha^c) = e \left( -\frac{c}{l} \right) h_\alpha^c, \quad e(x) = \exp(2\pi ix). \quad (A.6)
\]
In addition,
\[
\text{Ad}_Q(b^c_\beta) = b^c_\beta, \quad \text{Ad}_Q(H_\alpha) = H_\alpha, \quad (A.7)
\]
\[
\text{Ad}_Q(e_\beta) = e(\langle \kappa, \beta \rangle) e_\beta, \quad \text{Ad}_Q(E_\alpha) = e(\kappa, \tilde{\alpha}) E_\alpha. \quad (A.8)
\]

There are also the evident relations
\[
\text{Ad}_\Lambda(E_\alpha) = e(\langle \tilde{u}, \tilde{\alpha} \rangle) E_\alpha, \quad \text{Ad}_\Lambda(H_\alpha) = H_\alpha, \quad \tilde{u} \in \tilde{h}.
\]

In particular,
\[
\text{Ad}_\Lambda(t^c_\beta) = e \left( \langle \tilde{u}, \beta \rangle - \frac{c}{l} \right) t^c_\beta, \quad \text{Ad}_\Lambda(h^c_\beta) = e \left( -\frac{c}{l} \right) h^c_\beta, \quad e(x) = \exp(2\pi i x).
\]

**Commutation relations in the GS basis:**
\[
\left[ t^a_{\alpha}, t^b_{\beta} \right] = \begin{cases} \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{bs} C_{\alpha, \lambda} \lambda^s t^{a+b}_{\alpha + \lambda \beta}, & \alpha \neq -\lambda \beta, \\ \frac{p_{\alpha}}{\sqrt{l}} \omega^{sb} t^{a+b}_{\alpha}, & \alpha = -\lambda \beta, \end{cases}
\]
\[
\left[ h^k_{\alpha}, t^m_{\beta} \right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{2(\alpha, \lambda \beta)}{(\alpha, \alpha)} t^{k+m}_{\beta},
\]
\[
\left[ \varphi^k_{\alpha}, t^m_{\beta} \right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{(\alpha, \alpha)}{2} (\tilde{\alpha}, \lambda \beta) t^{k+m}_{\beta}.
\]

**B Elliptic functions**

The basic function is the theta-function
\[
\vartheta(z|\tau) = q^{\frac{z^2}{2}} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)}.
\]

It is a holomorphic function on \( \mathbb{C} \) with simple zeroes at the lattice \( \tau \mathbb{Z} + \mathbb{Z} \) and the quasi-periodicities
\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}} e^{-2\pi i z} \vartheta(z).
\]

Define the ratio of the theta-functions
\[
\phi(u, z) = \frac{\vartheta(u + z) \vartheta'(0)}{\vartheta(u) \vartheta(z)}, \quad (B.1)
\]

Then
\[
\phi(u, z) = \phi(z, u), \quad \phi(-u, -z) = -\phi(u, z). \quad (B.2)
\]

**Related functions:**
\[
\varphi^m_{\beta}(u, z) = e(\langle \kappa, \beta \rangle z) \varphi \left( (u + \kappa \tau, \beta) + \frac{m}{l}, z \right), \quad (B.3)
\]
\[
\varphi^{m,k}_{\beta}(u, z) = \partial_z e(\langle \kappa, \beta \rangle z) \varphi \left( (u + \kappa \tau, \beta) + \frac{m}{l}, z \right), \quad \varphi^{m,0}_{\beta} = \varphi^m_{\beta}, \quad (B.4)
\]
\[
f(u, z) = \partial_u \phi(u, z), \quad (B.5)
\]
\[
f(u, z) = \phi(u, z)(E_1(u + z) - E_1(u)), \quad (B.6)
\]
\( \phi(u, z) \) has a pole at \( z = 0 \) and
\[
\phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2} \left( E_1^2(u) - \varphi(u) \right) + \cdots.
\] (B.7)

Similarly,
\[
\varphi_\beta^m(u, z) = \frac{1}{z} + E_1 \left( (u + \kappa \tau, \beta) + \frac{m}{\tau} \right) + 2\pi i (\kappa, \beta) + \frac{z}{2} \left( E_1^2(u) - \varphi(u) \right) + \cdots,
\] where \( E_1 \) is (B.10). It follows from this expansion that
\[
\varphi_\beta^m(u, z) = -\frac{1}{z^2} + \frac{1}{2} \left( E_1^2(u) - \varphi(u) \right) + \cdots,
\] (B.8)

\[
\varphi_\beta^m(u, z) = (-1)^k \frac{1}{z^{k+1}} + c(m, k) + \cdots.
\]

In other words \( \varphi_\beta^m(u, z) \) has not poles of order less than \( k + 1 \).

The Eisenstein functions:
\[
E_1(z|\tau) = \partial_z \log \vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z + \cdots,
\] (B.10)

where
\[
\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m \tau + n)^2} = \frac{24}{\tau^2} \eta'(\tau), \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n > 0} (1 - q^n),
\]
\[
E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1,
\] (B.11)

and more general for \( k > 2 \)
\[
E_k(z|\tau) = (-\partial_z)^{k+1} \log \vartheta(z|\tau), \quad E_k(z|\tau) \sim \frac{1}{z^k} + \cdots.
\] (B.12)

Relation to the Weierstrass functions:
\[
\zeta(z, \tau) = E_1(z, \tau) + 2\eta_1(\tau) z, \quad \varphi(z, \tau) = E_2(z, \tau) - 2\eta_1(\tau).
\]

Quasi-periodicity:
\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q \frac{1}{2} e^{-2\pi i z} \vartheta(z),
\] (B.13)
\[
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i,
\] (B.14)
\[
E_k(z + 1) = E_k(z), \quad E_k(z + \tau) = E_k(z), \quad k > 1,
\] (B.15)
\[
\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i u} \phi(u, z),
\] (B.16)
\[
\varphi_\beta^m(u, z + 1) = e(\langle \kappa, \beta \rangle) \varphi_\beta^m(u, z),
\]
\[
\varphi_\beta^m(u, z + \tau) = e \left( -\langle u, \beta \rangle - \frac{m}{\tau} \right) \varphi_\beta^m(u, z),
\] (B.17)
\[
f(u, z + 1) = f(u, z), \quad f(u, z + \tau) = e^{-2\pi i u} f(u, z) - 2\pi i \phi(u, z).
\]

The following identities are also used here
\[
2\pi i \partial_z \phi(u, z) = \partial_z \partial_u \phi(u, z) = \partial_z f(u, z)
\]
and for the functions (4.38) this identity takes the form

\[ 2\pi i \partial_r \phi^m_\alpha(z) = \partial_z f^k_\alpha(z). \] (B.18)

Fay identity:

\[ \phi(u_1, z_1) \phi(u_2, z_2) - \phi(u_1 + u_2, z_1) \phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2) \phi(u_1, z_1 - z_2) = 0. \]

Differentiating over \( u_2 \) we find

\[
\phi(u_1, z_1) f(u_2, z_2) - \phi(u_1 + u_2, z_1) f(u_2, z_2 - z_1)
= \phi(u_2, z_2 - z_1) f(u_1 + u_2, z_1) + \phi(u_1, z_1 - z_2) f(u_1 + u_2, z_2).
\]

Substituting here

\[
u_1 = \langle u + \kappa \tau, \alpha + \beta \rangle + \frac{k + m}{l}, \quad u_2 = -\langle u + \kappa \tau, \beta \rangle - \frac{m}{l},
\]

\[
z_1 = z_a - z_c = z_{ac}, \quad z_2 = z_b - z_c = z_{bc},
\]

and multiplying by appropriate exponential factor we can rewrite it in the form

\[
\varphi^k_\alpha(z_{ac}) f^m_\beta(z_{ab}) - \varphi^m_\beta(z_{ab}) f^k_\alpha(z_{ac}) + \varphi^{k+m}_{\alpha+\beta}(z_{ac}) f^k_\alpha(z_{cb}) - \varphi^{k+m}_{\alpha+\beta}(z_{ac}) f^{-m}_{\beta}(z_{bc}) = 0. \] (B.19)

Taking the limit \( m = 0, \beta = 0 \) and using the expansion

\[ \phi(z, u) \sim \frac{1}{u} + E_1(z) + u \rho(z) + \cdots, \]

we find

\[ \varphi^k_\alpha(z_{ac}) \rho(z_{ab}) - E_1(z_{ab}) f^k_\alpha(z_{ac}) + \varphi^k_\alpha(z_{ab}) f^k_\alpha(z_{bc}) - \varphi^k_\alpha(z_{ac}) \rho(z_{cb}) = \frac{1}{2} \partial_u f^k_\alpha(z_{ac}). \] (B.20)

More Fay identities:

\[
\varphi^k_\alpha(z_{ac}) f^m_\beta(z_{ac}) - \varphi^m_\beta(z_{ac}) f^k_\alpha(z_{ac}) = \varphi^{k+m}_{\alpha+\beta}(z_{ac}) (\varphi^k_\alpha - \varphi^m_\beta), \] (B.21)

\[
\varphi^m_\beta(z_{ac}) f^{-m}_{\beta}(z_{ac}) - f^m_{\beta}(z_{ac}) = \varphi^{m}_{\beta}(z_{ac}) E_2^m_{\beta}, \] (B.22)

\[
\varphi^k_\beta(z_{ac}) \varphi^k_\beta(z_{ac}) \rho(z_{ac}) + E_1(z_{ac}) f^k_\beta(z_{ac}) = \frac{1}{2} \partial_u f^k_\beta(z_{ac}). \] (B.23)

The last one follows from

\[
\partial_u \phi(u, z) = \phi(u, z) (E_1(z + u) - E_1(u))
\]

and

\[
(E_1(z + u) - E_1(u) - E_1(z))^2 = \varphi(z) + \varphi(u) + \varphi(z + u) \]

C Proofs of Propositions 1 and 2

Proof of Proposition 1.

\[
[\nabla_a, \nabla_b] = [\partial_{za}, r^{ba}] + [\hat{\partial}^a, r^{ba}]
+ \sum_{c \neq a, b} [\hat{\partial}^c, r^{bc}] - [\partial_{zb}, r^{ab}] - [\hat{\partial}^b, r^{ab}] - \sum_{c \neq a, b} [\hat{\partial}^b, r^{ac}] + \sum_{d \neq b} \sum_{a \neq c} [r^{ac}, r^{bd}]. \] (C.1)
First, notice that
\[ \partial_{za} + \partial_{zb}, r^{ab} = 0. \]

Secondly,
\[ [\hat{\partial}^a, r^{ba}] - [\hat{\partial}^b, r^{ab}] = -[\hat{\partial}^a + \hat{\partial}^b, r^{ab}] = \sum_{c \neq a, b} [\hat{\partial}^c, r^{ab}] - \left[ \sum_c \hat{\partial}^c, r^{ab} \right]. \]

Thirdly,
\[ \sum_{d \neq b} \sum_{a \neq c} [r^{ac}, r^{bd}] = \sum_{c \neq a, b} \left( [r^{ac}, r^{ba}] + [r^{ab}, r^{bc}] + [r^{ac}, r^{hc}] \right). \]

Therefore,
\[ [\nabla_a, \nabla_b] = \sum_{c \neq a, b} \text{CDYB}^{abc} - \left[ \sum_c \hat{\partial}^c, r^{ab} \right], \quad \text{(C.2)} \]

where
\[ \text{CDYB}^{abc} = [r^{ab}, r^{ac}] + [r^{ab}, r^{hc}] + [\hat{\partial}^a, r^{bc}] + [\hat{\partial}^b, r^{ac}] + [\hat{\partial}^c, r^{ab}]. \]

and \[ [\sum_c \hat{\partial}^c, r^{ab}] = 0. \]

**Proof of Proposition 2.**

1. \[ [\delta_{za} f^{ac}] = 2 \pi i \partial_r r^{ac} \]

2. \[ \left[ \Delta, \sum_{c \neq a} r^{ac} \right] = \frac{1}{2} \sum_{c \neq a} \sum_{b \in R} \sum_{m=0}^{l-1} |\beta|^2 \partial_u f^m_\beta (z_a - z_c) h^{0,a}_{\beta} t_{\beta}^{m,a} t^{r-m,c} \]

3. Terms \[ \left[ \hat{\partial}^a, \frac{1}{2} f^{bc} \right] \]

3.1. Terms \[ \left[ \hat{\partial}^a, \frac{1}{2} f^{ac} \right] \] for \( c \neq a \):
Therefore we get
\[
\left[ \hat{\partial}^a, \sum_{c \neq a} f^{ac} \right] - \left[ \Delta, \sum_{c \neq a} r^{ac} \right] = \frac{1}{2} \sum_{c \neq a} \sum_{\alpha, \beta} \sum_{m=0}^{l-1} |\alpha|^2 \partial_{\alpha} f^m_{\alpha} (z_a - z_c) \left[ b^0_{\alpha} t^m_{a} + t^m_{a} \right]. \tag{C.5}
\]

3.2. Terms \( \left[ \hat{\partial}^a, \frac{1}{2} f^{aa} \right] \): \n\[
\left[ \hat{\partial}^a, \frac{1}{2} f^{aa} \right] = - \frac{1}{2} \sum_{m=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \partial_{\alpha} \phi^m_{\alpha} b^{0,a}_m t^{m,a}_\alpha - \beta
\]
\[= - \frac{1}{4} \sum_{m=0}^{l-1} \sum_{\beta \in R} |\beta|^2 E^m_{2\beta} b^{0,a}_\beta t^{m,a}_\beta + t^{m,a}_\beta. \tag{C.6}
\]

3.3. Terms \( \left[ \hat{\partial}^a, \frac{1}{2} f^{cc} \right] \) for \( c \neq a \):
\[
\left[ \hat{\partial}^a, \sum_{c \neq a} \frac{1}{2} f^{cc} \right] = - \frac{1}{4} \sum_{c \neq a} \sum_{m=0}^{l-1} \sum_{\beta \in R} |\beta|^2 E^m_{2\beta} b^{0,a}_\beta t^{m,a}_\beta + t^{m,a}_\beta. \tag{C.7}
\]

4. Terms \([r, f]\):
\[
\left[ \sum_{c \neq a} r^{ac}, \frac{1}{2} \sum_{b,d} f^{bd} \right] = \frac{1}{2} \sum_{b,c \neq a} \left( [r^{ac}, f^{ab}] + [r^{ac}, f^{bc}] + [r^{ab}, f^{ac}] + [r^{ab}, f^{bc}] \right) + \sum_{c \neq a} \left( [r^{ac}, f^{ac}] + \frac{1}{2} [r^{ac}, f^{aa}] + \frac{1}{2} [r^{ac}, f^{cc}] \right).
\]

4.1. Terms \([r, f]\) for \( b, c \neq a \) and \( b \neq c \):
\[
\frac{1}{2} \sum_{b,c \neq a} \left( [r^{ac}, f^{ab}] + [r^{ac}, f^{bc}] + [r^{ab}, f^{ac}] + [r^{ab}, f^{bc}] \right)
\]
\[= \frac{1}{2} \sum_{b,c \neq a} \sum_{k,m,s=0}^{l-1} \sum_{\alpha, \beta \in R} |\alpha|^2 |\beta|^2 \omega^{k,s} C^{k,s,\alpha,\beta} \left( \varphi^k_{\alpha} (z_a) f^m_{\beta} (z_b) - \varphi^m_{\beta} (z_a) f^k_{\alpha} (z_c) \right) + \varphi^{k+m}_{a+\beta} (z_a) f^k_{\alpha} (z_b) - \varphi^{k+m}_{a+\beta} (z_a) f^k_{\alpha} (z_c) \right) + \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_b) - \varphi^{m}_{a} (z_a) f^k_{\alpha} (z_c) + \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_b) - \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_c) \right) + 2 \left( \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_b) - \varphi^{m}_{a} (z_a) f^k_{\alpha} (z_b) \right) + \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_b) - \varphi^{k+m}_{a} (z_a) f^m_{\alpha} (z_c) \right) \cdot \tag{C.8}
\]
Almost all terms in this big sum vanish due to the Fay identity (B.19), except the terms with \( m = 0 \). Using (B.20) for these terms in the expression (C.8) we get

\[
- \frac{l}{4} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_{\alpha} f^k_{\alpha}(z_{ac}) t^k_{\alpha} b^0_{\alpha} t^k_{-\alpha} - \frac{l}{4} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_{\alpha} f^k_{\alpha}(z_{ac}) t^k_{\alpha} b^0_{\alpha} t^k_{-\alpha}
\]

\[
+ \frac{l}{2} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_{\alpha} f^k_{\alpha}(z_{bc}) t^k_{\alpha} b^0_{\alpha} t^k_{-\alpha}.
\]

Finally, using the symmetry in summation over \( c \) and \( b \) we obtain

\[
- \frac{l}{2} \sum_{a,b,c} \left( [r^{ac}, f^{ab}] + [r^{ac}, f^{bc}] + [r^{ab}, f^{ac}] + [r^{ab}, f^{bc}] \right)
\]

\[
= \frac{l}{2} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 |\beta|^2 \left( \varphi^k_{\alpha}(z_{ac}) f^m_{\beta}(z_{ac}) - \varphi^m_{\beta}(z_{ac}) f^k_{\alpha}(z_{ac}) \right) + \frac{l}{2} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 |\beta|^2 \left( \varphi^k_{\alpha}(z_{bc}) f^m_{\beta}(z_{bc}) - \varphi^m_{\beta}(z_{bc}) f^k_{\alpha}(z_{bc}) \right)
\]

\[
- \frac{l}{2} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 |\beta|^2 \left( -\varphi^m_{-\beta}(z_{ac}) f^{m+k}_{\alpha}(z_{ac}) - \varphi^{m+k}_{\beta}(z_{ac}) f^{m}_{\alpha}(z_{ac}) \right) + \frac{l}{2} \sum_{a,b,c} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 |\beta|^2 \left( -\varphi^m_{-\beta}(z_{bc}) f^{m+k}_{\alpha}(z_{bc}) - \varphi^{m+k}_{\beta}(z_{bc}) f^{m}_{\alpha}(z_{bc}) \right)
\]

Notice that the second term here cancels the expression (C.4).

4.2. Terms \( [r^{ac}, f^{ac}] \):
\[ - f_0^{-m}(z_{ac}) \Phi_{\beta}^{m+k}(z_{ac}) \left[ h_{\beta}^{m.c}, t_{\beta}^{-m-k,a} \right] + t_{\beta}^{k.c}. \] (C.10)

4.3. Terms \( [r^{ac}, f^{aa}] \) and \( [r^{ac}, f^{cc}] \) for \( c \neq a \):

\[
\left[ \sum_{c \neq a} r^{ac}, f^{aa} \right] = -\frac{1}{4} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\alpha|^2 |\beta|^2 \Phi_{\alpha+\beta}^k (z_{ac}) (\Phi_{\alpha}^k - \Phi_{\beta}^k) \\
\times \omega^{-ms}_{C \alpha+\beta} \left[ k, a, m,a, \beta \right] + t_{\alpha+\beta}^{-k-m,c} \\
- \frac{1}{4} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\beta|^2 \Phi_{\alpha}^k (z_{ac}) (\Phi_{\beta}^m - \Phi_{\beta}^{m+k}) \left[ h_{\beta}^{k,m,a}, t_{\beta}^{-m,a} \right] + \\
+ \frac{1}{2} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\beta|^2 \Phi_{\beta}^k (z_{ac}) (\Phi_{\beta}^m - \Phi_{\beta}^{m+k}) \left[ h_{\beta}^{-m,a}, t_{\beta}^{m,k,a} \right] + t_{\beta}^{-k,c}. \] (C.11)

\[
\left[ \sum_{c \neq a} r^{ac}, f^{cc} \right] = -\frac{1}{4} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\alpha|^2 |\beta|^2 \Phi_{\alpha+\beta}^k (z_{ac}) (\Phi_{\alpha}^k - \Phi_{\beta}^m) \\
\times \omega^{ms}_{C \alpha+\beta} \left[ t_{\alpha}^{-k,a}, \beta \right] + t_{\alpha+\beta}^{-k+m,c} \\
+ \frac{1}{4} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\beta|^2 \Phi_{\alpha}^k (z_{ac}) (\Phi_{\beta}^m - \Phi_{\beta}^{k+m}) \left[ h_{\beta}^{k,a}, t_{\beta}^{-k-m,c} \right] + \\
- \frac{1}{2} \sum_{c \neq a} \sum_{k, m=0 \beta \in R} |\beta|^2 \Phi_{\beta}^k (z_{ac}) (\Phi_{\beta}^m - \Phi_{\beta}^{m+k}) \left[ h_{\beta}^{m,c}, t_{\beta}^{-m-k,c} \right] + t_{\beta}^{k,a}. \] (C.12)

The first two lines in (C.10) are canceled by first lines in (C.11) and (C.12) due to identity (B.21). Next, the sum of the third line in (C.10), the second line in (C.12), the sum of fourth line in (C.10) and the second line in (C.11) are vanished due to (B.21) for all values of summation parameters except \( m = 0 \). For \( k = 0 \) these sums give

\[
- \frac{1}{4} \sum_{c \neq a} \sum_{m=0 \beta \in R} |\beta|^2 \left( f_0^{-m}(z_{ac}) \Phi_{\beta}^m (z_{ac}) - \Phi_{\beta}^m (z_{ac}) f_0^{-m}(z_{ac}) \right) \\
\times \left( h_{\beta}^{0,a} \left[ t_{\beta}^{-m,c}, t_{\beta}^{m,c} \right] + t_{\beta}^{0,c} \left[ h_{\beta}^{-m,a}, t_{\beta}^{-m,a} \right] \right) \]

(B.22)

\[
= \frac{1}{4} \sum_{c \neq a} \sum_{m=0 \beta \in R} |\beta|^2 \Phi_{\alpha}^0 (z_{ac}) (\Phi_{\beta}^m - \Phi_{\beta}^m (z_{ac})) \\
\times \left( h_{\beta}^{0,a} \left[ t_{\beta}^{-m,c}, t_{\beta}^{m,c} \right] + t_{\beta}^{0,c} \left[ h_{\beta}^{-m,a}, t_{\beta}^{-m,a} \right] \right) \]

and this is exactly what we need to compensate (C.6) and (C.7). (Note that (C.7) cancels by the first term here and (C.6) cancels by the second one due to (4.43).)

Finally, the last two lines in (C.10) are canceled by the list lines in (C.11) and (C.12) for all values of summation parameters except \( m = 0 \). For \( m = 0 \) the sum of these terms equals

\[
- \frac{1}{2} \sum_{c \neq a} \sum_{k=0 \beta \in R} |\beta|^2 \left( \Phi_{\alpha}^0 (z_{ac}) f_\beta^k (z_{ac}) - f_\beta^0 (z_{ac}) \Phi_{\beta}^k (z_{ac}) + \Phi_{\beta}^k (z_{ac}) \Phi_{\beta}^k \right) \\
\times \left( h_{\beta}^{0,a} \left[ t_{\beta}^{-k,c}, t_{\beta}^{k,c} \right] + t_{\beta}^{0,c} \left[ h_{\beta}^{-k,a}, t_{\beta}^{-k,a} \right] \right) \]

(C.13)

Notice that due to (4.39) and (4.40) \( f_\beta^0 (z_{ac}) = \rho (z_{ac}) \) and \( \Phi_{\alpha}^0 (z_{ac}) = E_1 (z_{ac}) \). Then using (B.23) we can simplify the expression (C.13)

\[
- \frac{1}{4} \sum_{c \neq a} \sum_{k=0 \beta \in R} |\beta|^2 \partial_{\beta} f_\beta^k (z_{ac}) \left( h_{\beta}^{0,a} \left[ t_{\beta}^{-k,c}, t_{\beta}^{k,c} \right] + t_{\beta}^{0,c} \left[ h_{\beta}^{-k,a}, t_{\beta}^{-k,a} \right] \right). \] (C.14)
Finally, we have nonzero terms from (C.5), first term in (C.9) and (C.14). All these terms are proportional to $\partial_u f^k_{\beta}$. Summing them up we find

$$ l \sum_{c \neq a}^{l-1} \sum_{k=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \partial_u f^k_{\beta}(z_{ac}) \left( -\frac{1}{4} [h^{0,a}_\beta, t^{k,a}_\beta] + t^{-k,c}_\beta + \frac{1}{4} [h^{0,c}_\beta, t^{-k,c}_\beta] + t^{k,a}_\beta + \frac{1}{2} \sum_{b \neq a,c} t^{k,a}_\beta t^{-k,c}_\beta \right) $$

$$ + \frac{1}{2} \sum_{b \neq a,c} t^{k,a}_\beta t^{-k,c}_\beta h^{0,b}_\beta + \frac{1}{2} \sum_{b \neq a,c} t^{k,a}_\beta t^{-k,c}_\beta h^{0,b}_\beta $$

$$ = l \sum_{c \neq a}^{l-1} \sum_{k=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \partial_u f^k_{\beta}(z_{ac}) \left( -\frac{1}{4} [h^{0,a}_\beta, t^{k,a}_\beta] + t^{-k,c}_\beta + \frac{1}{4} [h^{0,c}_\beta, t^{-k,c}_\beta] + t^{k,a}_\beta + \frac{1}{2} \sum_{b \neq a,c} t^{k,a}_\beta t^{-k,c}_\beta h^{0,b}_\beta \right) $$

$$ = 0. $$



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