Large-$N$ Yang–Mills Theory as Classical Mechanics

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Abstract. To formulate two-dimensional Yang–Mills theory with adjoint matter fields in the large-$N$ limit as classical mechanics, we derive a Poisson algebra for the color-invariant observables involving adjoint matter fields. We showed rigorously in Ref. [15] that different quantum orderings of the observables produce essentially the same Poisson algebra. Here we explain, in a less precise but more pedagogic manner, the crucial topological graphical observations underlying the formal proof.

One major unsolved problem in physics is hadronic structure. We would like to explain, for instance, the momentum distributions of valence quarks, sea quarks and gluons inside a proton. We have accumulated a fairly large amount of experimental data on the distribution functions [1], but we have made relatively little advance in explaining them from the widely accepted fundamental theory of strong interaction, quantum chromodynamics (QCD).

The emergence of hadrons is a low-energy phenomenon of strong interaction. The strong coupling constant is large, and perturbative QCD fails very badly. Other approximations are needed. One widely studied approximation is the large-$N$ limit, in which the number of colors $N$ is taken to be infinitely large [2]. This approximation is believed to capture the essence of low-energy strong interaction phenomena. Indeed, ’t Hooft showed that if space–time is assumed to be two-dimensional only, then the meson spectrum displays itself as a Regge trajectory [3]. This attractive feature of the large-$N$ limit has drawn the attention of a large number of researchers for more than two decades. They want to build up a systematic theory of the large-$N$ limit to deal with baryons in addition to mesons in four-dimensional space–time.

One important feature of the large-$N$ limit is that the expectation value of a product of two observables $A$ and $B$ is the same as the product of the expectation values of these two observables. The difference is of order $1/N$ and so can be
omitted [4,5]:
\[ \langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + O(1/N). \]

In other words, there is no quantum fluctuation. The theory thus behaves like classical mechanics [6], and it should be possible for us to formulate the large-\(N\) limit of QCD as classical mechanics.

Formally speaking, we need three ingredients to build up a theory as classical mechanics [7]. The first is the notion of a manifold to describe the geometry of the phase space of positions and momenta. Dynamical variables are then functions on the manifold. The second is the notion of a Hamiltonian function \( H \), one of the dynamical variables. This function displays the physical features (e.g., symmetry) of the system, and governs the time evolution of it. How the Hamiltonian function governs the time evolution is determined by a Poisson algebra [8], the third ingredient of classical mechanics. To understand what a Poisson algebra is, we need a number of preliminary notions.

An algebra \( R \) is a linear space on a field \( K \) with a multiplication rule of any 2 vectors in \( R \) such that for any \( x, y \) and \( z \in R \) and \( a \in K \),

1. \( xy \in R \);
2. the vectors satisfy the distributive properties
   \[
   x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz; \quad \text{and}
   \]
3. \( a(xy) = (ax)y = x(ay) \).

An associative algebra is an algebra \( R \) such that for any \( x, y \) and \( z \in R \),
\[ x(yz) = (xy)z. \]

A Poisson algebra is an associative algebra \( R \) which is equipped with a bilinear map \( \{ \, , \} : R \times R \rightarrow R \), the Poisson bracket, with the following properties for any \( x, y \) and \( z \in R \):

1. skew-symmetry:
   \[ \{ y, x \} = -\{ x, y \}; \]
2. the Jacobi identity:
   \[ \{ x, \{ y, z \} \} + \{ y, \{ z, x \} \} + \{ z, \{ x, y \} \} = 0; \quad \text{and} \]
3. the Leibniz identity:
   \[ \{ xy, z \} = x\{ y, z \} + \{ x, z \} y. \]
How a dynamical variable $G(t)$ changes with the time $t$ is given by the equation

$$\frac{dG(t)}{dt} = \{G,H\}.$$  

Making the assumption that space–time is two-dimensional, one of us introduced a formulation of large-$N$ Yang–Mills theory as classical mechanics several years ago [9]. As in the mesonic model 't Hooft studied, gluons are not dynamical objects in this classical mechanics; only quarks and anti-quarks are. The phase space turns out to an infinite-dimensional Grassmannian manifold [10]. (Briefly speaking, an infinite-dimensional Grassmannian manifold is a collection of subspaces of an infinite-dimensional vector space. A differential structure is conferred upon this collection to turn it into a manifold.) Dynamical variables are composed of quark and anti-quark fields. They are bilocal functions on the Grassmannian manifold. The Poisson bracket can be uniquely determined from the geometric properties of the Grassmannian. The Hamiltonian is chosen in such a way that upon quantization of this classical mechanics, i.e., if we retain terms of subleading orders in the large-$N$ limit, it will take exactly the form the action of Yang–Mills theory is conventionally written. The meson spectrum predicted by this classical mechanics is precisely the same as that obtained by 't Hooft. However, the model can be used to calculate structure functions of a baryon also [11]. As an initial attempt, the proton is assumed to be made up of valence quarks only. Sea quarks and gluons are omitted. In the infinite momentum frame, the transverse momenta of valence quarks can be neglected. Thus the Yang–Mills theory can be dimensionally reduced from four dimensions to two. Dimensional reduction is a good approximation for valence quarks carrying a large fraction of the total momentum of the proton, but not so good for the quarks carrying almost no momentum. Indeed, it turned out the momentum distribution functions predicted by this classical model of Yang–Mills theory agrees well with experimental data at high-momentum regime, but not so well near zero momentum.

How about sea quarks and gluons inside a hadron? The sea quark distribution function can actually be predicted within the same classical mechanical framework [9,12]; it is a matter of modifying the details of some approximations. However, gluon dynamics require a new set Poisson brackets and a new phase space. Since gluons carry about 20% of the total momentum of a proton [1], it is worthwhile to study them further. A Poisson algebra for gluons was constructed a few years ago [13]. This was achieved by a technique called deformation quantization [14,8]. Deformation quantization refers to the procedure of defining an algebra of smooth functions in such a way that when the functions are multiplied, it is as if we are multiplying suitably ordered operators these smooth functions represent. Any physical observable of gluons must involve creation and annihilation operators of gluons, each of which carries two color quantum numbers and a vector-valued linear momentum. The two color quantum numbers can be treated as the row and column indices of an $N \times N$ matrix, $N$ being the total number of colors. Any physical observable has
to be color-invariant. This implies that it has to be a polynomial of the traces of matrix products with a generic form

\[ f^I = \frac{1}{Nm^{2m+1}} \text{Tr} \eta^i_1 \eta^i_2 \cdots \eta^i_m, \]

where \( i_1, i_2, \ldots, i_m \) are quantum states other than colors of the gluons, \( I \) is the sequence \( i_1, i_2, \ldots, i_m \), and the factor of \( N \) is put to the left of the trace to ensure that the Poisson algebra of these \( f^I \)'s, to be introduced shortly, is well defined in the large-\( N \) limit. In each product, the operators have to satisfy a certain ordering. As an initial attempt, the operators are Weyl-ordered in Ref. [13]. Hence,

\[ \eta^i_1 = 1/2(a^i_\sigma + a^{i\dagger}_\sigma), \]
\[ \eta^{-i}_1 = i/2(a^i_\sigma - a^{i\dagger}_\sigma) \]

When we multiply two physical observables together, we need to rearrange the order of the creation and annihilation operators to make the resultant product consistent with the quantum ordering. As a result, multiplication of physical observables is still associative but no longer commutative. The commutator of two physical observables therefore provides us a Poisson bracket. In the case of Weyl-ordered quantum observables of gluons, the Poisson bracket is

\[ \{ f^I, f^J \}_W = 2i \sum_{r=1, \text{odd}}^{\infty} \sum_{\mu_1 < \mu_2 < \cdots < \mu_r} \left( -\frac{i\hbar}{2} \right)^r \tilde{\omega}^{j_1 j_2 \cdots j_r} f^I(\mu_1, \mu_2) f^J(\nu_2, \nu_1) f^I(\mu_2, \mu_3) f^J(\nu_3, \nu_2) \cdots f^I(\mu_r, \mu_1) f^J(\nu_1, \nu_r). \]  

(1)

In this equation, \( \tilde{\omega}^{ij} \) is an anti-symmetric constant tensor. \( I(\mu_1, \mu_2) f(\nu_2, \nu_1) \) is the sequence \( \mu_1+1, \mu_1+2, \ldots, \mu_2-1, \nu_2+1, \nu_2+2, \ldots, \nu_1-1 \). Notice that we sum over all possible sets of values of \( \mu_1, \mu_2, \ldots, \mu_r \) such that they are strictly increasing, and all possible sets of values of \( \nu_1, \nu_2, \ldots, \nu_r \) such that they are strictly decreasing up to a cyclic permutation. Eq.(1) looks complicated, but we can actually visualize it in Fig. 1. We will call this Poisson algebra \( \mathcal{W} \).

Nevertheless, in many practical applications, we need quantum observables made up of normal-ordered rather than Weyl-ordered operators. We thus need to apply the above idea of deformation quantization to derive a Poisson bracket of normal-ordered observables [15]. In this case, A color-invariant observable has the form

\[ \phi^I = \frac{1}{N^{n/2+1}} \text{Tr} z^{i_1} z^{i_2} \cdots z^{i_n}, \]

where

\[ z_{\sigma}^{i_\rho} = a_{\sigma}^{i_\rho} \text{ and } z_{\sigma}^{-i_\rho} = a_{\sigma}^{i\dagger_\rho} \]

for \( i > 0 \). The Poisson bracket for these operators turns out to be
FIGURE 1. (a) A typical color-invariant observable $f^I$. Each solid circle represents an $\eta^I$. Notice the cyclic symmetry of the figure. (b) A simplified diagrammatic representation of $f^I$. We use the capital letter $I$ to denote the whole sequence $i_1, i_2, \ldots, i_m$. (c) A typical term in $\{f^I, f^J\}_W$. This is a product of the color-invariant observables $f^I(\mu_1, \mu_2) J(\nu_3, \nu_5)$, $f^I(\mu_2, \mu_3) J(\nu_4, \nu_6)$, $f^I(\mu_3, \mu_4) J(\nu_1, \nu_5)$, and $f^I(\mu_5, \mu_1) J(\nu_3, \nu_2)$. We can identify these color-invariant observables by their vertices. For example, $f^I(\mu_1, \mu_2) J(\nu_2, \nu_1)$ can be described as a ‘loop with vertices $i_{\mu_1}$, $i_{\mu_2}$, $j_{\nu_2}$ and $j_{\nu_1}$’, though none of these vertices belong to $f^I(\mu_1, \mu_2) J(\nu_2, \nu_1)$.

$\{\phi^I, \phi^J\}_N = \sum_{r=1}^{\infty} \sum_{\mu_1 < \mu_2 < \ldots < \mu_r} \sum_{(\nu_1 > \nu_2 > \ldots > \nu_r)} h^r \gamma_{i_{\mu_1} j_{\nu_1}} \ldots \gamma_{i_{\mu_r} j_{\nu_r}} \cdot \phi^I(\mu_1, \mu_2) J(\nu_2, \nu_1) \phi^I(\mu_2, \mu_3) J(\nu_3, \nu_2) \ldots \phi^I(\mu_r, \mu_1) J(\nu_1, \nu_r) - (I \leftrightarrow J), \tag{2}$

where $\gamma_{\mu^\nu} = 0$ unless $\mu < 0$ and $\nu > 0$. We will call this Poisson algebra $N$.

The Poisson algebras $W$ and $N$ look very different. Are they intrinsically different Poisson algebras, or are they the same Poisson algebra with different expressions? This question can be answered by looking for a Poisson morphism between $W$ and $N$. Let $R_1$ and $R_2$ be two Poisson algebras. A Poisson morphism is a mapping $F : R_1 \rightarrow R_2$ such that it preserves

1. vector addition:
   \[ F(x + y) = F(x) + F(y); \]

2. scalar multiplication:
   \[ F(kx) = kF(x); \]

3. vector multiplication:
   \[ F(xy) = F(x)F(y); \] and
4. the Poisson bracket:

\[ F(\{x, y\}_1) = \{F(x), F(y)\}_2 \]

for any \( k \in K, x \) and \( y \in R_1 \). Here \( \{, \}_1 \) and \( \{, \}_2 \) are the Poisson brackets of \( R_1 \) and \( R_2 \), respectively. If there exists a Poisson morphism between two Poisson algebras, then they are effectively the same Poisson algebra.

In our case, a Poisson morphism \( F : W \rightarrow N \text{ does exist} \). The reader can find a thoroughly rigorous formulation and proof of this Poisson morphism in Ref. [15]. Roughly speaking, the mapping \( F \) is accomplished in two steps. Consider the color-invariant observable \( f_{i_1 \cdots i_{22}} \in W \) in Fig. 2. The first step involves splitting the big all-encompassing loop into a number of smaller loops. The straight lines joining the solid circles and cutting off the big loop have to be within the loop. Moreover, no two straight lines can cross each other. Each solid circle \( \eta^i \) which is not touched by any straight line is now identified as a linear combination of solid circles \( \eta'^i \) in \( N \) by the formula

\[ \eta'^i = \begin{cases} 
\frac{1}{2}(z^i + z^{-i}) & \text{if } i > 0; \\
\frac{3}{2}(z^i - z^{-i}) & \text{if } i < 0.
\end{cases} \]

The resultant diagram represents the product of the color-invariant observables in \( N \), each of which is represented by a smaller loop in the resultant diagram. For instance, the product in Fig. 2(a) is

\[ T^{i_4}_{i_{12}} T^{i_{7}}_{i_{12}} T^{i_{8}}_{i_{17}} T^{i_{10}}_{i_{16}} T^{i_{12}}_{i_{14}} \phi^{i_{12}}_{i_{13}} \phi^{i_{5}}_{i_{6}} \phi^{i_{9}}_{i_{19}} \phi^{i_{11}}_{i_{15}} \phi^{i_{13}}_{i_{13}} \phi^{i_{18}}_{i_{19}} \phi^{i_{20}}, \]

where \( T^{ij} \) is a polynomial of \( C^{ij} \) and \( C^{ji} \), where in turn \( C^{ij} \) is a polynomial of \( \gamma^{ij} \),
Now notice a crucial observation. Identify \( \omega \) factor \( G \) loop \( j \) obtained by choosing a smaller loop in \( I \) with vertices \( i \) due to the Poisson bracket. The contracted pairs are \( T_{ij} \) and \( C \) partitions of \( f \) in \( N \) and \( F \) in \( W \). Add up all such possible replacements. The sum will precisely be \( G \) up to a

\[ \gamma^{-i,j}, \gamma^{i,-j} \text{ and } \gamma^{-i,-j} \]  

Fig. 3. A typical term in \( \{F(f^I), F(f^J)\}_N \) or \( F(\{f^I, f^J\}_W) \).

Why is this \( F \) a Poisson morphism? The most non-trivial statement we need to show is that \( F \) preserves the Poisson bracket. In other words, We need to show that each term in \( \{F(f^I), F(f^J)\}_N \) is a term in \( \{f^I, f^J\}_W \), and vice versa. Consider Fig. 3. The big oval-shaped object on the left is \( f^I \), and the one on the right is \( f^J \). The dotted lines inside \( f^I \) divide it into a product of color-invariant observables in \( N \). So do the dotted lines inside \( f^J \). A typical term in \( \{F(f^I), F(f^J)\}_N \) is obtained by choosing a smaller loop in \( I \), and a smaller loop in \( J \), and perform the contractions illustrated in Fig. 1. In Fig. 3, we have chosen the smaller loop with vertices \( i_{p11(-1)}, i_{p11(1)}, i_{p20(-1)}, i_{p20(1)}, i_{p30(-1)} \) and \( i_{p30(1)} \) in \( I \), and the smaller loop \( j_{p21(-1)}, j_{p21(1)}, j_{p30(-1)}, j_{p30(1)}, j_{p30(-3)}, j_{p30(3)}, j_{p30(-4)} \) and \( j_{p30(4)} \) in \( J \). (See the explanation of the jargons here in the caption of Fig. 1.) There are 7 contractions due to the Poisson bracket. The contracted pairs are \( i_{p10} \) and \( j_{p10} \), \( i_{p11} \) and \( j_{p11} \), \( i_{p20} \) and \( j_{p20} \), \( i_{p21} \) and \( j_{p21} \), \( i_{p22} \) and \( j_{p22} \), \( i_{p30} \) and \( j_{p30} \). According to Eq. (1), these contractions produce in this term of \( \{F(f^I), F(f^J)\}_N \) a constant factor \( G \) which the reader can divine is a polynomial of \( C^{\pm i_{p10}\pm j_{p10}}, \ldots, C^{\pm i_{p30}\pm j_{p30}} \).

Now notice a crucial observation. Identify \( \bar{\omega}^{ij} \) with a certain polynomial of \( C^{ij} \) and \( C^{ji} \). Replace an odd number of \( C^{ij} \)’s in \( G \) with \( \bar{\omega}^{ij} \)’s, and the remaining \( C^{ij} \)’s with \( T^{ij} \)’s. Add up all such possible replacements. The sum will precisely be \( G \) up to a
multiplicative constant. Consequently, we can draw some of these contractions as dotted lines because they are $T$’s, and others as thick lines to show that they are $\tilde{\omega}$’s. In Fig. 3, $C_{i^0_{10}j^0_{10}}, C_{i^0_{20}j^0_{20}}$ and $C_{i^0_{30}j^0_{30}}$ are replaced with $\tilde{\omega}_{i^0_{10}j^0_{10}}, \tilde{\omega}_{i^0_{20}j^0_{20}}$ and $\tilde{\omega}_{i^0_{30}j^0_{30}}$, respectively, whereas $C_{i^0_{11}j^0_{11}}, C_{i^0_{21}j^0_{21}}, C_{i^0_{22}j^0_{22}}$ and $C_{i^0_{23}j^0_{23}}$ are replaced with $T_{i^0_{11}j^0_{11}}, T_{i^0_{21}j^0_{21}}, T_{i^0_{22}j^0_{22}}$ and $T_{i^0_{23}j^0_{23}}$, respectively.

Now notice another crucial observation. This diagram can be reproduced by computing the Poisson bracket in $W$ first and mapping the resultant expression with $F$ later. The Poisson bracket in $W$ produces a product of three color-invariant observables. The first one $L_1$ is characterized with the vertices $i_{\rho_{10}}, i_{\rho_{20}}, j_{\rho_{20}}$ and $j_{\rho_{10}}$; the second one $L_2$ with the vertices $i_{\rho_{30}}, i_{\rho_{10}}, j_{\rho_{30}}$ and $j_{\rho_{20}}$; and the third one $L_3$ with the vertices $i_{\rho_{30}}, i_{\rho_{10}}, j_{\rho_{10}}$ and $j_{\rho_{30}}$. The mapping $F$ projects this product to $N$. This is done, as usual, by splitting these 3 loops into smaller loops with dotted lines so that no two lines cross each other inside any of these 3 loops. Notice that in Fig. 3, if we flip the dotted line $i_{\rho_{11}(\sim)}i_{\rho_{11}(1)}$ to within $L_1$, and the dotted lines $i_{\rho_{20}(\sim)}i_{\rho_{20}(1)}$ and $j_{\rho_{21}(\sim)}j_{\rho_{21}(1)}$ to within $L_2$, no two dotted lines will cross each other. Therefore, we obtain the same term.

Conversely, any term $F(\{f^I, f^J\}_W)$ can be seen as a term in $\{F(f^I), F(f^J)\}_N$ by a similar diagrammatic argument. Thus $F$ is indeed a Poisson morphism.

This Poisson algebra may have other interesting mathematical properties. We hope that we can use this Poisson algebra to solve gluon dynamics in the future.

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