Uniqueness of static spherically symmetric vacuum solutions in the IR limit of nonrelativistic quantum gravity

Tomohiro Harada, Umpei Miyamoto and Naoki Tsukamoto
Department of Physics, Rikkyo University, Toshima, Tokyo 175-8501, Japan
E-mail: harada@rikkyo.ac.jp

Abstract. We investigate static spherically symmetric vacuum solutions in the IR limit of projectable nonrelativistic quantum gravity, including the renormalisable quantum gravity recently proposed by Horava, with an emphasis on the uniqueness of the solutions. It is found that the projectability condition plays an important role. Without the cosmological constant, the spacetime is uniquely given by the Schwarzschild solution. With the cosmological constant, the spacetime is uniquely given by the Kottler (Schwarzschild-(anti) de Sitter) solution for the entirely vacuum spacetime. However, in addition to the Kottler solution, the static spherical and hyperbolic universes are uniquely admissible for the locally empty region, for the positive and negative cosmological constants, respectively. This implies that static spherically symmetric entirely vacuum solutions would not admit the freedom to reproduce the observed flat rotation curves of galaxies.

1. Introduction
Horava [1] proposed a quantum gravity theory, which is supposed to be power-counting renormalisable and ghost free. This theory is called the Horava-Lifshitz gravity. Subsequently, a variety of vacuum solutions have been discovered. However, it does not seem so clear how unique these solutions are. The uniqueness of the solutions depends on the conditions we impose for the solutions. For simplicity, we now consider static, spherically symmetric vacuum spacetimes with and without the cosmological constant in infrared (IR) limit of the Horava-Lifshitz gravity. This article is based on Harada, Miyamoto and Tsukamoto [2], and therein the full list of the relevant references is provided.

2. Horava-Lifshitz gravity
The Horava-Lifshitz gravity is featured with the following characters. The theory is essentially based on the “3+1” view of the spacetime. There is a preferred spacetime foliation of the spacetime with which the theory is given in a physically meaningful manner. The gravitational fields consist of a three-metric $g_{ij}$, a shift $N_i$ and a lapse $N$. The line element in the four dimensional spacetime is given by

$$ds^2 = -Nc^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

The Lagrangian is covariant against the foliation preserving diffeomorphisms $\tilde{x}^i = \tilde{x}^i(t, x^j)$ and $\tilde{t} = \tilde{t}(t)$. The kinetic term of the Lagrangian consists of the quadratic form of the
first time derivative of $g_{ij}$. The theory is at the fixed point in the ultraviolet (UV) limit against the anisotropic scaling $x \rightarrow bx$ and $t \rightarrow b^\gamma t$, where the gravitational coupling constant is nondimensional with respect to this scaling. If we additionally impose the detailed balance condition and the relevant deformation to recover the Einstein-Hilbert action with the cosmological constant in the IR limit, the gravitational Lagrangian is fixed to the following form:

$$L_g = \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2(1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2w^4} \left(C_{ij} - \frac{\mu w^2}{2} R_{ij}\right) \left(C^{ij} - \frac{\mu w^2}{2} R^{ij}\right),$$

where $\kappa$, $\lambda$, $\mu$, $\Lambda_W$ and $w$ are constants, $K_{ij}$ is the extrinsic curvature and $C_{ij}$ is the Cotton tensor, a symmetric tensor made of the third spatial derivative of $g_{ij}$. Since the theory is covariant against the foliation-preserving diffeomorphisms, it is natural to restrict the lapse function to $N$ to a function only of $t$. This is called the projectability condition. By imposing this condition, the constraints constitute a Lie algebra and there appears dark matter as an integration constant [3].

Hereafter we will concentrate on the IR limit of the Hořava-Lifshitz gravity with $\lambda = 1$, where only the terms with the lowest order spatial derivatives in the potential are left. From the comparison with the action of general relativity, we obtain

$$\alpha = \frac{1}{16\pi G c}, \quad \xi = c^2 \alpha, \quad \sigma = -2\Lambda c^2 \alpha.$$

3. Static spherically symmetric vacuum solution in the IR limit

We assume that the space is static and spherically symmetric, where the line element is given by

$$g_{ij}dx^i dx^j = e^{2\omega(r)} dt^2 + r^2( d\theta^2 + \sin^2 \theta d\phi^2).$$

The lapse and shift should be in the form $N = \text{const}$ and $N^i/N = (v(r), 0, 0)$.

Due to the projectability condition, the Hamiltonian constraint from the variation $\delta N(t)$ is a nonlocal equation given by

$$\int dr r^2 e^{\omega} \left[ -\alpha(K_{ij}K^{ij} - K^2) + R + \sigma \right] = 0.$$ 

On the other hand, the momentum constraint from the variation $\delta N^i(t, x^j)$ is local as follows:

$$-2\frac{\omega' v}{r} = 0,$$

where the prime denotes the derivative with respect to $r$. The evolution equations from the variation $\delta g_{ij}(t, x^k)$ are complicated partial differential equations including the sixth order spatial derivatives in general. However, under the present settings, they become much simpler. The $(r, r)$ and $(\theta, \theta)$ components of the evolution equations become

$$\alpha \frac{v'}{r} \left( 2v' + \frac{v}{r} + 4\omega' v \right) + \xi \frac{1 - e^{-2\omega}}{r^2} + \frac{1}{2} \sigma = 0,$$

and

$$\alpha \left[ \left( v' + 2\omega v + \frac{v}{r}\right)' v + (\omega' v + v') \left( v' + \omega' v + \frac{v}{r}\right) + \left( \frac{v'}{r} \right)' \right] + \xi \frac{\omega'}{r} e^{-2\omega} + \frac{1}{2} \sigma = 0,$$

respectively, and there is no remaining independent equation in the evolution equations.

It is now easy to solve these field equations. The momentum constraint leads to $\omega' = 0$ or $v = 0$. 

2
3.1. Case (a): $\omega' = 0$

Putting $\omega = \omega_0 = \text{const}$, the evolution equation is integrated to give

$$v^2 = -\frac{\xi}{\alpha} (1 - e^{-2\omega_0}) - \frac{1}{6} \frac{\sigma}{\alpha} r^2 + \frac{C}{r},$$

where $C$ is a constant of integration. The right-hand side must be positive. This solution satisfy the Hamiltonian constraint because the integrand of the integral Hamiltonian constraint vanishes identically.

If we make the following foliation-nonpreserving coordinate transformation:

$$N dt = e^{-\omega_0}dT + \frac{e^{2\omega_0} v}{e^{2\omega_0} v^2} dr,$$

we obtain the standard form of the Schwarzschild-(A)dS metric, which is not projectable:

$$ds^2 = \left[ 1 - \frac{1}{3} \Lambda r^2 - \frac{2GM}{c^2 r} \right] dT^2 + \left[ 1 - \frac{1}{3} \Lambda r^2 - \frac{2GM}{c^2 r} \right]^{-1} dr^2 + r^2 d\Omega^2,$$

where we put $C = 2GM$. Note that the full theory is not covariant against the above coordinate transformation. The IR limit can be justified for sufficiently large $r$. This is not a solution for $\lambda \neq 1$.

3.2. Case (b): $v = 0$

The evolution equation leads to

$$e^{2\omega} = \left( 1 + \frac{\sigma}{2\xi} r^2 \right)^{-1}.$$

Although the integrand of the integral Hamiltonian constraint does not vanish for $\sigma \neq 0$, this solution is acceptable as a local solution, if its nonvanishing contribution to the global Hamiltonian constraint can be compensated by that from the nonempty or nonstatic region.

The metric is given by

$$ds^2 = -dt^2 + \frac{1}{1 - \Lambda r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

This is Minkowski for $\Lambda = 0$, static spherical universe (Einstein’s static universe) for $\Lambda > 0$ and static hyperbolic universe for $\Lambda < 0$. The IR limit is justified for sufficiently small $|\Lambda|$. It turns out that this is a solution for any value of $\lambda$.

4. Summary

The uniqueness of static, spherically symmetric solutions in the IR limit of Horava-Lifshitz gravity is investigated. If the Hamiltonian constraint is to be satisfied, the solution must be uniquely Schwarzschild ($\Lambda = 0$) or Schwarzschild-(A)dS ($\Lambda \neq 0$). A local solution may not satisfy the Hamiltonian constraint if we take into account the region outside our observable universe. In spite of this relaxation, there is no additional solution for $\Lambda = 0$, while only a static, homogeneous and isotropic universe is added for $\Lambda \neq 0$. The Schwarzschild ($\Lambda = 0$) and the Schwarzschild-(A)dS ($\Lambda \neq 0$) are subject to $\lambda = 1$, while the static, homogeneous and isotropic universe is a solution for any value of $\lambda$. Within this framework, there is no dark matter which explains the observed flat rotation curve of galaxies. See Harada, Miyamoto and Tsukamoto [2] for further details.

References

[1] Horava P., 2009 Phys. Rev. D79, 084008
[2] Harada T., Miyamoto U. and Tsukamoto N., to appear in Int. J. Mod. Phys. D, Preprint arXiv:0911.1187
[3] Mukohyama S., 2009 Phys. Rev. D80, 064005