Two non-nilpotent linear transformations that satisfy the cubic $q$-Serre relations

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Abstract

Let $\mathbb{K}$ denote an algebraically closed field with characteristic 0, and let $q$ denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let $A_q$ denote the unital associative $\mathbb{K}$-algebra defined by generators $x, y$ and relations

\[
\begin{align*}
 x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 &= 0, \\
y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 &= 0,
\end{align*}
\]

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$. We classify up to isomorphism the finite-dimensional irreducible $A_q$-modules on which neither of $x, y$ is nilpotent. We discuss how these modules are related to tridiagonal pairs.

Keywords. Tridiagonal pair, Leonard pair, $q$-Racah polynomial, quantum group, quantum affine algebra.

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1 Statement of the problem and results

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field with characteristic 0. We fix a nonzero scalar $q \in \mathbb{K}$ that is not a root of 1 and adopt the following notation:

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \ldots
\]

Definition 1.1 Let $A_q$ denote the unital associative $\mathbb{K}$-algebra defined by generators $x, y$ and relations

\[
\begin{align*}
 x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 &= 0, \\
y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 &= 0.
\end{align*}
\]

We call $x, y$ the standard generators for $A_q$.

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Remark 1.2  The equations (2), (3) are the cubic \( q \)-Serre relations \([32\text{ p. }11]\). 

We are interested in a certain class of \( \mathcal{A}_q \)-modules. To describe this class we recall a concept. Let \( V \) denote a finite-dimensional vector space over \( \mathbb{K} \). A linear transformation \( X : V \to V \) is said to be \textit{nilpotent} whenever there exists a positive integer \( n \) such that \( X^n = 0 \).

**Definition 1.3** Let \( V \) denote a finite-dimensional \( \mathcal{A}_q \)-module. We say this module is \textit{NonNil} whenever the standard generators \( x, y \) are not nilpotent on \( V \).

In this paper we will classify up to isomorphism the NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-modules.

Before we state our results, we explain why this problem is of interest. A recent topic of research concerns the Leonard pairs \([23, 37, 39, 12, 13, 14, 15, 16, 17, 48, 49, 50, 51, 53, 54] \) and the closely related tridiagonal pairs \([11, 2, 21, 25, 26, 31, 35, 36] \). A Leonard pair is a pair of semisimple linear transformations on a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other \([12\text{ Definition }1.1]\). The Leonard pairs are classified \([32, 49]\) and correspond to the orthogonal polynomials that make up the terminating branch of the Askey scheme \([28, 50]\). A tridiagonal pair is a mild generalization of a Leonard pair \([24\text{ Definition }1.1]\). For these objects the classification problem is open. Let \( V \) denote a NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-module. Then the standard generators \( x, y \) act on \( V \) as a tridiagonal pair; see Section 2 for the details.

We now summarize our classification. We begin with some comments. Let \( V \) denote a NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-module. As we will see, the standard generators \( x, y \) are semisimple on \( V \). Moreover there exist an integer \( d \geq 0 \) and nonzero scalars \( \alpha, \alpha^* \in \mathbb{K} \) such that the set of distinct eigenvalues of \( x \) (resp. \( y \)) on \( V \) is \( \{\alpha q^{d-i} | 0 \leq i \leq d\} \) (resp. \( \{\alpha^* q^{d-i} | 0 \leq i \leq d\} \)). We call the ordered pair \((\alpha, \alpha^*)\) the \textit{type} of \( V \). Replacing \( x, y \) by \( x/\alpha, y/\alpha^* \) the type becomes \((1, 1)\). Consequently it suffices to classify the NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-modules of type \((1, 1)\). As we will see, these objects are related to certain modules for the quantum affine algebra \( U_q(\widehat{\mathfrak{sl}}_2) \). This algebra is defined as follows.

**Definition 1.4** \([13\text{ p. }262]\) The quantum affine algebra \( U_q(\widehat{\mathfrak{sl}}_2) \) is the unital associative \( \mathbb{K} \)-algebra with generators \( e_i^\pm, K_i^\pm, i \in \{0, 1\} \) and the following relations:

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad (4)
\]

\[
K_0 K_1 = K_1 K_0, \quad (5)
\]

\[
K_i e_i^\pm K_i^{-1} = q^{\pm 2} e_i^\pm, \quad (6)
\]

\[
K_i e_j^\pm K_i^{-1} = q^{\pm 2} e_j^\pm, \quad i \neq j, \quad (7)
\]

\[
[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad (8)
\]

\[
[e_0^+, e_1^-] = 0, \quad (9)
\]

\[
(e_j^+)^3 e_i^+ - [3]_q (e_j^+)^2 e_j^+ e_i^+ + [3]_q e_j^+ e_j^+ (e_j^+)^2 - e_j^+ (e_j^+)^3 = 0, \quad i \neq j. \quad (10)
\]

We call \( e_i^\pm, K_i^\pm, i \in \{0, 1\} \) the \textit{Chevalley generators} for \( U_q(\widehat{\mathfrak{sl}}_2) \).
Remark 1.5 Corollary 3.2.6] There exists an injection of \( \mathbb{K} \)-algebras from \( \mathcal{A}_q \) to \( U_q(\widehat{\mathfrak{sl}}_2) \) that sends \( x \) and \( y \) to \( e_0^+ \) and \( e_1^+ \) respectively. Consequently \( \mathcal{A}_q \) is often called the positive part of \( U_q(\widehat{\mathfrak{sl}}_2) \).

The finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}}_2) \)-modules are classified up to isomorphism by V. Chari and A. Pressley [13]. In Section 3 we review this classification, and here mention just those aspects needed to state our results. Let \( \mathbb{K}[z] \) denote the \( \mathbb{K} \)-algebra consisting of the polynomials in an indeterminate \( z \) that have coefficients in \( \mathbb{K} \). Let \( V \) denote a finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}}_2) \)-module. By [13, Prop. 3.2] the actions of \( K_0 \) and \( K_1 \) on \( V \) are semisimple. Also by [13, Prop. 3.2] there exist an integer \( d \geq 0 \) and scalars \( \varepsilon_0, \varepsilon_1 \) chosen from \( \{1, -1\} \) such that

(i) the set of distinct eigenvalues of \( K_0 \) on \( V \) is \( \{\varepsilon_0 q^{d-2i} \mid 0 \leq i \leq d\} \);

(ii) \( K_0 K_1 - \varepsilon_0 \varepsilon_1 I \) vanishes on \( V \).

The ordered pair \( (\varepsilon_0, \varepsilon_1) \) is called the type of \( V \). Now assume \( V \) has type \( (1, 1) \). Following Drinfel’d [16], Chari and Pressley define a polynomial \( P = P_V \) in \( \mathbb{K}[z] \), called the Drinfel’d polynomial of \( V \) [13 Section 3.4]. By [13, p. 261] the map \( V \to P_V \) induces a bijection between the following two sets:

(i) The isomorphism classes of finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}}_2) \)-modules of type \( (1, 1) \);

(ii) The polynomials in \( \mathbb{K}[z] \) that have constant coefficient 1.

The main results of this paper are contained in the following two theorems and subsequent remark.

**Theorem 1.6** Let \( V \) denote a NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-module of type \( (1, 1) \). Then there exists a unique \( U_q(\widehat{\mathfrak{sl}}_2) \)-module structure on \( V \) such that the standard generators \( x \) and \( y \) act as \( e_0^+ + K_0 \) and \( e_1^+ + K_1 \) respectively. This \( U_q(\widehat{\mathfrak{sl}}_2) \)-module is irreducible type \( (1, 1) \) and \( P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0 \).

**Theorem 1.7** Let \( V \) denote a finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}}_2) \)-module of type \( (1, 1) \) such that \( P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0 \). Then there exists a unique \( \mathcal{A}_q \)-module structure on \( V \) such that the standard generators \( x \) and \( y \) act as \( e_0^+ + K_0 \) and \( e_1^+ + K_1 \) respectively. This \( \mathcal{A}_q \)-module is NonNil irreducible of type \( (1, 1) \).

**Remark 1.8** Combining Theorem 1.6 and Theorem 1.7 we obtain a bijection between the following two sets:

(i) The isomorphism classes of NonNil finite-dimensional irreducible \( \mathcal{A}_q \)-modules of type \( (1, 1) \);

(ii) The isomorphism classes of finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}}_2) \)-modules \( V \) of type \( (1, 1) \) such that \( P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0 \).
The plan for the paper is as follows. In Section 2 we describe how $A_q$-modules are related to tridiagonal pairs. In Section 3 we recall some facts about finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules, including the classification due to Chari and Pressley. In Section 4 we discuss the Drinfel’d polynomial. Sections 5–12 are devoted to proving Theorem 1.6 and Theorem 1.7.

2 $A_q$-modules and tridiagonal pairs

In this section we explain how a NonNil finite-dimensional irreducible $A_q$-module gives a tridiagonal pair. We will use the following concepts. Let $V$ denote a finite-dimensional vector space over $\mathbb{K}$ and let $X: V \to V$ denote a linear transformation. For $\theta \in \mathbb{K}$ we define $V_X(\theta) = \{v \in V \mid Xv = \theta v\}$.

We observe $\theta$ is an eigenvalue of $X$ if and only if $V_X(\theta) \neq 0$, and in this case $V_X(\theta)$ is the corresponding eigenspace. Observe the sum $\sum_{\theta \in \mathbb{K}} V_X(\theta)$ is direct. Moreover this sum is equal to $V$ if and only if $X$ is semisimple.

Lemma 2.1 Let $V$ denote a finite-dimensional vector space over $\mathbb{K}$. Let $X: V \to V$ and $Y: V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:

(i) The expression $X^3Y - [3]_q X^2 Y X + [3]_q XY X^2 - Y X^3$ vanishes on $V_X(\theta)$.

(ii) $Y V_X(\theta) \subseteq V_X(q^2 \theta) + V_X(\theta) + V_X(q^{-2} \theta)$.

Proof: For $v \in V_X(\theta)$ we have

$$(X^3 Y - [3]_q X^2 Y X + [3]_q XY X^2 - Y X^3)v = (X^3 - \theta[3]_q X^2 + \theta^2[3]_q X - \theta^3 I) Y v$$

Since $Xv = \theta v$

$$= (X - q^2 \theta I)(X - \theta I)(X - q^{-2} \theta I) Y v,$$

where $I: V \to V$ is the identity map. The scalars $q^2 \theta, \theta, q^{-2} \theta$ are mutually distinct since $\theta \neq 0$ and since $q$ is not a root of 1. The result follows. \qed

We now recall the concept of a tridiagonal pair.

Definition 2.2 [24] Definition 1.1] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a tridiagonal pair on $V$, we mean an ordered pair of linear transformations $A: V \to V$ and $A^*: V \to V$ that satisfy the following four conditions:

(i) Each of $A, A^*$ is semisimple.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of $A$ such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (11)$$

where $V_{-1} = 0, V_{d+1} = 0$. 

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(iii) There exists an ordering \( V^*_0, V^*_1, \ldots, V^*_\delta \) of the eigenspaces of \( A^* \) such that
\[
AV^*_i \subseteq V^*_i + V^*_i + V^*_{i+1} \quad (0 \leq i \leq \delta),
\]
where \( V^*_{-1} = 0, V^*_{\delta+1} = 0 \).
(iv) There does not exist a subspace \( W \) of \( V \) such that \( AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V \).

Note 2.3 According to a common notational convention, \( A^* \) denotes the conjugate transpose of \( A \). We are not using this convention. In a tridiagonal pair \( A, A^* \) the linear transformations \( A \) and \( A^* \) are arbitrary subject to (i)–(iv) above.

Note 2.4 In Section 1 we mentioned the concept of a Leonard pair. A Leonard pair is the same thing as a tridiagonal pair for which the \( V_i, V^*_i \) all have dimension 1 [43, Lemma 2.2].

We refer the reader to [21, 43] for the basic theory of tridiagonal pairs. For connections to distance-regular graphs see [6, p. 260], [12], [17], [24] Example 1.4], [31], [38], [40], [41]. For connections to statistical mechanics see [7], [8], [9], [10], [15], [53, Section 34] and the references therein. For related topics see [3], [4], [5], [18], [19], [20], [21], [22], [29], [30], [33], [55], [56], [57], [58].

We recall a few basic facts about tridiagonal pairs. Let \( A, A^* \) denote a tridiagonal pair on \( V \) and let \( d, \delta \) be as in Definition 2.2(ii), (iii). By [24] Lemma 4.5] we have \( d = \delta \); we call this common value the diameter of \( A, A^* \). An ordering of the eigenspaces of \( A \) (resp. \( A^* \)) will be called standard whenever it satisfies (11) (resp. (12)). We comment on the uniqueness of the standard ordering. Let \( V_0, V_1, \ldots, V_\delta \) denote a standard ordering of the eigenspaces of \( A \). Then the ordering \( V_\delta, V_{\delta-1}, \ldots, V_0 \) is standard and no other ordering is standard. A similar result holds for the eigenspaces of \( A^* \). An ordering of the eigenvalues of \( A \) (resp. \( A^* \)) will be called standard whenever the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^* \)) is standard.

Definition 2.5 Let \( d \) denote a nonnegative integer and let \( \theta_0, \theta_1, \ldots, \theta_d \) denote a sequence of scalars taken from \( \mathbb{K} \). We call this sequence a \( q \)-string whenever there exists a nonzero scalar \( \alpha \in \mathbb{K} \) such that \( \theta_i = \alpha q^{d-2i} \) for \( 0 \leq i \leq d \).

Definition 2.6 A tridiagonal pair \( A, A^* \) is said to be \( q \)-geometric whenever (i) there exists a standard ordering of the eigenvalues of \( A \) which forms a \( q \)-string; and (ii) there exists a standard ordering of the eigenvalues of \( A^* \) which forms a \( q \)-string.

We refer the reader to [11, 21, 25, 26, 43] for information about \( q \)-geometric tridiagonal pairs.

Theorem 2.7 [43, Lemma 4.8] Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. Let \( A : V \to V \) and \( A^* : V \to V \) denote linear transformations. Then the following are equivalent:

(i) \( A, A^* \) is a \( q \)-geometric tridiagonal pair on \( V \).
(ii) There exists a NonNil irreducible $A_q$-module structure on $V$ such that the standard generators $x, y$ act as $A, A^*$ respectively.

Proof: (i) $\Rightarrow$ (ii): We first show

$$A^3A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0. \quad (13)$$

Since the tridiagonal pair $A, A^*$ is $q$-geometric, there exists a standard ordering $\theta_0, \theta_1, \ldots, \theta_d$ of the eigenvalues for $A$ which forms a $q$-string. For $0 \leq i \leq d$ let $V_i$ denote the eigenspace of $A$ associated with $\theta_i$. Then the ordering $V_0, V_1, \ldots, V_d$ is standard and therefore satisfies (11). By this and since $\theta_0, \theta_1, \ldots, \theta_d$ is a $q$-string we find

$$A^* V_A(\theta) \subseteq V_A(q^2 \theta) + V_A(\theta) + V_A(q^{-2} \theta)$$

for each eigenvalue $\theta$ of $A$. Invoking Lemma 2.4 we find that in equation (13) the expression on the left vanishes on each eigenspace of $A$. These eigenspaces span $V$ since $A$ is semisimple, and equation (13) follows. Interchanging $A, A^*$ in the above argument we find

$$A^3 A - [3]_q A^2 A^* A + [3]_q A A^* A^2 - AA^* A^3 = 0. \quad (14)$$

By (13), (14) there exists an $A_q$-module structure on $V$ such that $x, y$ act as $A, A^*$ respectively. This $A_q$-module is irreducible in view of Definition 2.2(iv). This $A_q$-module is NonNil since each of $A, A^*$ has all eigenvalues nonzero.

(ii) $\Rightarrow$ (i): The field $\mathbb{K}$ is algebraically closed so it contains all the eigenvalues for $A$. Since $V$ has finite positive dimension, $A$ has at least one eigenvalue. Since $x$ acts on $V$ as $A$ and since this action is not nilpotent, $A$ has at least one nonzero eigenvalue $\theta$. The scalars $\theta, \theta q^2, \theta q^4, \ldots$ are mutually distinct since $q$ is not a root of unity, so these scalars are not all eigenvalues for $A$. Consequently there exists a nonzero $\eta \in \mathbb{K}$ such that $V_A(\eta) \neq 0$ and $V_A(\eta q^2) = 0$. There exists an integer $d \geq 0$ such that $V_A(\eta q^{-2i})$ is nonzero for $0 \leq i \leq d$ and zero for $i = d + 1$. We abbreviate $V_i = V_A(\eta q^{-2i})$ for $0 \leq i \leq d$. By construction $\sum_{i=0}^d V_i$ is $A$-invariant. By Lemma 2.3 we find

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0, V_{d+1} = 0$. Consequently $\sum_{i=0}^d V_i$ is $A^*$-invariant. Since the $A_q$-module structure on $V$ is irreducible and since $\sum_{i=0}^d V_i \neq 0$ we find $\sum_{i=0}^d V_i = V$. Now $A$ is semisimple and Definition 2.2(ii) holds. Interchanging $A, A^*$ in the above argument we find $A^*$ is semisimple and Definition 2.2(iii) holds. Definition 2.2(iv) is satisfied since the $A_q$-module $V$ is irreducible. We have now shown that $A, A^*$ is a tridiagonal pair on $V$. From the construction this tridiagonal pair is $q$-geometric.

Corollary 2.8 Let $V$ denote a NonNil finite-dimensional irreducible $A_q$-module. Then the standard generators $x, y$ are semisimple on $V$. Moreover there exist an integer $d \geq 0$ and nonzero scalars $\alpha, \alpha^* \in \mathbb{K}$ such that the set of distinct eigenvalues of $x$ (resp. $y$) on $V$ is $\{\alpha q^{-2i} \mid 0 \leq i \leq d\}$ (resp. $\{\alpha^* q^{-2i} \mid 0 \leq i \leq d\}$).

Proof: Immediate from Definition 2.6 and Theorem 2.7. \qed
Definition 2.9 Let $V$ denote a NonNil finite-dimensional irreducible $A_q$-module. By the type of $V$ we mean the ordered pair $(\alpha, \alpha^*)$ from Corollary 2.8. By the diameter of $V$ we mean the scalar $d$ from Corollary 2.8.

Lemma 2.10 For all nonzero $\alpha, \alpha^* \in \mathbb{K}$ there exists a $\mathbb{K}$-algebra automorphism of $A_q$ such that

$$x \to \alpha x, \quad y \to \alpha^* y.$$ 

Proof: This is immediate from Definition 1.1. $\square$

Remark 2.11 Given a NonNil finite-dimensional irreducible $A_q$-module, we can change its type to any other type by applying an automorphism from Lemma 2.10.

3 Finite dimensional $U_q(\hat{\mathfrak{sl}}_2)$-modules

In this section we recall some facts about finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-modules, including the classification due to Chari and Pressley [13].

We begin with some notation. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $d$ denote a nonnegative integer. By a decomposition of $V$ with diameter $d$ we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of nonzero subspaces of $V$ such that

$$V = \sum_{i=0}^{d} U_i \quad \text{(direct sum)}.$$ 

We do not assume the spaces $U_0, U_1, \ldots, U_d$ have dimension 1. For notational convenience we define $U_{-1} = 0$ and $U_{d+1} = 0$.

Lemma 3.1 [13, Prop. 3.2] Let $V$ denote a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module. Then there exist unique scalars $\varepsilon_0, \varepsilon_1$ in $\{1, -1\}$ and a unique decomposition $U_0, \ldots, U_d$ of $V$ such that

$$(K_0 - \varepsilon_0 q^{2i-d} I) U_i = 0, \quad (K_1 - \varepsilon_1 q^{d-2i} I) U_i = 0 \quad (15)$$

for all $i = 0, 1, \ldots, d$. Moreover, for $0 \leq i \leq d$ we have

$$e_0^+ U_i \subseteq U_{i+1}, \quad e_1^+ U_i \subseteq U_{i+1}, \quad (16)$$

$$e_0^- U_i \subseteq U_{i-1}, \quad e_1^- U_i \subseteq U_{i-1}. \quad (17)$$

Definition 3.2 The ordered pair $(\varepsilon_0, \varepsilon_1)$ in Lemma 3.1 is the type of $V$ and $d$ is the diameter of $V$. The sequence $U_0, \ldots, U_d$ is the weight space decomposition of $V$ (relative to $K_0$ and $K_1$).

Lemma 3.3 [13, Prop. 3.3] For any choice of scalars $\varepsilon_0, \varepsilon_1$ from $\{1, -1\}$, there exists a $\mathbb{K}$-algebra automorphism of $U_q(\hat{\mathfrak{sl}}_2)$ such that

$$K_i \to \varepsilon_i K_i, \quad e_i^+ \to \varepsilon_i e_i^+, \quad e_i^- \to e_i^-$$

for $i \in \{0, 1\}.$
Remark 3.4 Given a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module, we can alter its type to any other type by applying an automorphism from Lemma 3.3.

We now recall the evaluation modules.

Lemma 3.5 [13, Section 4] There exists a family of finite-dimensional irreducible type $(1, 1)$ $U_q(\hat{\mathfrak{sl}}_2)$-modules

$$V(d, a) \quad d = 1, 2, \ldots \quad 0 \neq a \in \mathbb{K}$$

with the following property. The module $V(d, a)$ has a basis $v_0, v_1, \ldots, v_d$ such that

$$
\begin{align*}
K_0 v_i &= q^{2i-d} v_i \quad (0 \leq i \leq d), \\
e_0^- v_i &= q^{i}[d - i + 1]q v_i - 1 \quad (1 \leq i \leq d), \\
e_0^+ v_i &= q^{-1} a [i + 1] q v_{i+1} \quad (0 \leq i \leq d - 1), \\
e_1^- v_i &= e_1^+ v_{i+1} \quad (0 \leq i \leq d - 1), \\
e_1^+ v_i &= [d - i + 1]q v_{i-1} \quad (1 \leq i \leq d), \\
e_1^+ v_0 &= 0.
\end{align*}
$$

Definition 3.6 [13, Definition 4.2] The $U_q(\hat{\mathfrak{sl}}_2)$-module $V(d, a)$ from Lemma 3.5 is called an evaluation module. The scalar $a$ is called the evaluation parameter.

Remark 3.7 [13, Remark 4.2] Referring to Lemma 3.5 and Definition 3.6, two evaluation modules $V(d, a)$ and $V(d', a')$ are isomorphic if and only if $(d, a) = (d', a').$

We now recall how the tensor product of two $U_q(\hat{\mathfrak{sl}}_2)$-modules becomes a $U_q(\hat{\mathfrak{sl}}_2)$-module. We will use the following fact.

Lemma 3.8 [13, p. 263] $U_q(\hat{\mathfrak{sl}}_2)$ has the following Hopf algebra structure. The comultiplication $\Delta$ satisfies

$$
\begin{align*}
\Delta(e_i^+) &= e_i^+ \otimes K_i + 1 \otimes e_i^+, \\
\Delta(e_i^-) &= e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-, \\
\Delta(K_i) &= K_i \otimes K_i.
\end{align*}
$$

The counit $\varepsilon$ satisfies

$$
\varepsilon(e_i^+) = 0, \quad \varepsilon(K_i) = 1.
$$

The antipode $S$ satisfies

$$
S(K_i) = K_i^{-1}, \quad S(e_i^+) = -e_i^+ K_i^{-1}, \quad S(e_i^-) = -K_i e_i^-.
$$

Combining Lemma 3.8 with [14, p. 110] we routinely obtain the following.
Lemma 3.9  Let \( V, W \) denote \( U_q(\widehat{\mathfrak{sl}_2}) \)-modules. Then the tensor product \( V \otimes W = V \otimes_W W \) has the following \( U_q(\widehat{\mathfrak{sl}_2}) \)-module structure. For \( v \in V \), for \( w \in W \) and for \( i \in \{0, 1\} \),

\[
\begin{align*}
e^+_{i}(v \otimes w) &= e^+_i v \otimes K_i w + v \otimes e^+_i w, \\
e^-_{i}(v \otimes w) &= e^-_i v \otimes w + K^{-1}_i v \otimes e^-_i w, \\
K_i(v \otimes w) &= K_i v \otimes K_i w.
\end{align*}
\]

Definition 3.10  \([14, \text{p. 110}]\) There exists a one dimensional \( U_q(\widehat{\mathfrak{sl}_2}) \)-module on which each element \( z \in U_q(\widehat{\mathfrak{sl}_2}) \) acts as \( \varepsilon(z)I \), where \( \varepsilon \) is from Lemma 3.8 and \( I \) is the identity map. In particular on this module each of \( e^+_0, e^+_1 \) vanishes and each of \( K_0^{\pm 1}, K_1^{\pm 1} \) acts as \( I \). This module is irreducible, with diameter 0 and type \((1, 1)\). This module is unique up to isomorphism. We call this module the trivial \( U_q(\widehat{\mathfrak{sl}_2}) \)-module.

We now state Chari and Pressley’s classification of the finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}_2}) \)-modules. In view of Remark 3.2 it suffices to consider the modules of type \((1, 1)\).

Theorem 3.11  \([13, \text{Theorem 4.8, Theorem 4.11}]\) Each nontrivial finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}_2}) \)-module of type \((1, 1)\) is isomorphic to a tensor product of evaluation modules. A tensor product of evaluation modules

\[
V(d_1, a_1) \otimes V(d_2, a_2) \otimes \cdots \otimes V(d_N, a_N)
\]

is irreducible if and only if

\[
a_i a_j^{-1} \not\in \{q^{d_i + d_j}, q^{d_i + d_j - 2}, \ldots, q^{d_i - d_j} + 2\}, \quad (1 \leq i, j \leq N; \ i \neq j)
\]

and in this case it is type \((1, 1)\).

We end this section with a comment.

Lemma 3.12  Let \( V \) denote a nontrivial finite-dimensional irreducible \( U_q(\widehat{\mathfrak{sl}_2}) \)-module of type \((1, 1)\). Write \( V \) as a tensor product of evaluation modules:

\[
V = V(d_1, a_1) \otimes V(d_2, a_2) \otimes \cdots \otimes V(d_N, a_N).
\]

Let \( U_0, \ldots, U_d \) denote the weight space decomposition of \( V \) from Lemma 3.1. Then

\[
\sum_{i=0}^{d} \dim(U_i) z^i = \prod_{j=1}^{N}(1 + z + z^2 + \cdots + z^{d_j}).
\]

Proof: Combine Lemma 3.5 and Lemma 3.9 \( \square \)
4 The Drinfel’d polynomial

In this section we recall the Drinfel’d polynomial associated with a finite-dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1, 1)$.

**Definition 4.1** Let $V$ denote a finite-dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1, 1)$, and let $U_0, \ldots, U_d$ denote the corresponding weight space decomposition from Lemma 3.1. From Lemma 3.12 we find $\dim(U_0) = 1$. Pick an integer $i \geq 0$. By Lemma 3.1 we find $U_0$ is an eigenspace for $(e_1^+)^i(e_0^+)^i$; let $\sigma_i$ denote the corresponding eigenvalue. We observe $\sigma_i = 0$ if $i > d$.

We will use the following notation. With reference to (1) we define

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q \quad n = 0, 1, 2, \ldots$$

(18)

We interpret $[0]_q! = 1$.

**Definition 4.2** [13, Section 3.4] Let $V$ denote a finite-dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1, 1)$. We define a polynomial $P_V \in \mathbb{K}[z]$ by

$$P_V = \sum_{i=0}^{\infty} \frac{(-1)^i \sigma_i q^{i} z^i}{[i]_q!},$$

(19)

where the scalars $\sigma_i$ are from Definition 4.1. We observe that $P_V$ has degree at most the diameter of $V$. Moreover $P_V$ has constant coefficient $\sigma_0 = 1$. We call $P_V$ the Drinfel’d polynomial of $V$.

**Note 4.3** The polynomial $P_V$ from Definition 4.2 is the same as the polynomial $P$ from [13, Theorem 3.4], provided $P$ is normalized to have constant coefficient 1. This is explained in [13, p. 268].

The Drinfel’d polynomial has the following properties.

**Theorem 4.4** [13, Corollary 4.2, Proposition 4.3] Let $V$ denote a finite-dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1, 1)$. If $V = V(d, a)$ is an evaluation module then

$$P_V = (1 - q^{d-1} a z)(1 - q^{d-3} a z) \cdots (1 - q^{1-d} a z).$$

If $V = V_1 \otimes \cdots \otimes V_N$ is a tensor product of evaluation modules then $P_V = \prod_{i=1}^{N} P_{V_i}$.

**Theorem 4.5** [13, p. 261] The map $V \rightarrow P_V$ induces a bijection between the following two sets:

(i) The isomorphism classes of finite-dimensional irreducible $U_q(\widehat{sl}_2)$-modules of type $(1, 1)$;

(ii) The polynomials in $\mathbb{K}[z]$ that have constant coefficient 1.
5 From NonNil $A_q$-modules to $U_q(\widehat{sl}_2)$-modules

In this section we cite some results from [26] that will be of use in our proof of Theorem 1.6. The following theorem is a minor adaption [26, Theorem 3.3] and [26, Theorem 13.1]; we will sketch the proof in order to motivate what comes later in the paper.

**Theorem 5.1** [26, Theorem 3.3, Theorem 13.1] Let $V$ denote a NonNil finite-dimensional irreducible $A_q$-module of type $(1,1)$. Then there exists a unique $U_q(\widehat{sl}_2)$-module structure on $V$ such that the standard generators $x$ and $y$ act as $e_0^+ + K_0$ and $e_1^+ + K_1$ respectively. This $U_q(\widehat{sl}_2)$-module is irreducible and type $(1,1)$.

**Sketch of Proof:** For notational convenience let $A : V \rightarrow V$ and $A^* : V \rightarrow V$ denote the linear transformations such that $x$ and $y$ act on $V$ as $A$ and $A^*$ respectively. By Theorem 2.7 the pair $A, A^*$ is a $q$-geometric tridiagonal pair on $V$. Let $d$ denote the diameter of this tridiagonal pair. By Definition 2.6 and since the $A_q$-module $V$ has type $(1,1)$, the sequence $q^d, q^{d-2}, \ldots, q^{-d}$ is a standard ordering of the eigenvalues for each of $A, A^*$. For $0 \leq i \leq d$ let $V_i$ (resp. $V_i^*$) denote the eigenspace of $A$ (resp. $A^*$) associated with the eigenvalue $q^{2i-d}$ (resp. $q^{d-2i}$). Moreover for $0 \leq i \leq d$ we define

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

By [26, Lemma 4.2] the sequence $U_0, \ldots, U_d$ is a decomposition of $V$. Therefore there exists a linear transformation $K : V \rightarrow V$ such that for $0 \leq i \leq d$, $U_i$ is an eigenspace for $K$ with eigenvalue $q^{2i-d}$. For $0 \leq i \leq d$ we define

$$W_i = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_d),$$
$$W_i^* = (V_d^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

Then $W_0, \ldots, W_d$ and $W_0^*, \ldots, W_d^*$ are decompositions of $V$ [26, Lemma 4.2]. Therefore there exist linear transformations $B : V \rightarrow V$ and $B^* : V \rightarrow V$ such that for $0 \leq i \leq d$, $W_i$ (resp. $W_i^*$) is an eigenspace for $B$ (resp. $B^*$) with eigenvalue $q^{2i-d}$ (resp. $q^{d-2i}$). By [26, Theorem 7.1] we have

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \quad (20)$$
$$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = I, \quad (21)$$
$$\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = I, \quad (22)$$
$$\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = I \quad (23)$$

and by [26, Theorem 10.1] we have

$$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I, \quad (24)$$
\[\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I, \quad (25)\]
\[\frac{qKA^* - q^{-1}A*K}{q - q^{-1}} = I, \quad (26)\]
\[\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I. \quad (27)\]

By construction we have
\[A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \quad (28)\]
\[A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^*2 - AA^{*3} = 0 \quad (29)\]

and by [26, Theorem 12.1] we have
\[B^3B^* - [3]_q B^2B^*B + [3]_qBB^*B^2 - B^*B^3 = 0, \quad (30)\]
\[B^{*3}B - [3]_q B^{*2}BB^* + [3]_q B^*BB^{*2} - BB^{*3} = 0. \quad (31)\]

In order to connect the above equations with the defining relations for \(U_q(\widehat{\mathfrak{sl}}_2)\) we make a change of variables. Define
\[R = A - K, \quad L = A^* - K^{-1}, \quad (32)\]
\[r = \frac{I - KB^*}{q(q - q^{-1})^2}, \quad l = \frac{I - K^{-1}B}{q(q - q^{-1})^2}, \quad (33)\]

so that
\[A = K + R, \quad A^* = K^{-1} + L, \quad (34)\]
\[B = K - q(q - q^{-1})^2 Kl, \quad B^* = K^{-1} - q(q - q^{-1})^2 K^{-1}r. \quad (35)\]

Evaluating (20)–(31) using (32), (33) we routinely obtain
\[KRK^{-1} = q^2 R, \quad KLik^{-1} = q^{-2} L, \quad (36)\]
\[KrK^{-1} = q^2 r, \quad KlK^{-1} = q^{-2} l, \quad (37)\]
\[rL - Lr = \frac{K - K^{-1}}{q - q^{-1}}, \quad lR - Rl = \frac{K^{-1} - K}{q - q^{-1}}, \quad (38)\]
\[lL = Ll, \quad rR = Rr, \quad (39)\]
\[0 = R^3 L - [3]_q R^2 LR + [3]_q RLR^2 - LR^3, \quad (40)\]
\[0 = L^3 R - [3]_q L^2 RL + [3]_q LRL^2 - RL^3. \quad (41)\]
\[0 = r^3 l - [3]_q r^2 lr + [3]_q rlr^2 - rl^3, \quad (42)\]
\[0 = l^3 r - [3]_q l^2 rl + [3]_q lrl^2 - rl^3. \quad (43)\]

Consequently \(V\) becomes a \(U_q(\widehat{\mathfrak{sl}}_2)\)-module on which the Chevalley generators act as follows:

| generator | \(e^+_0\) | \(e^+_1\) | \(e^-_0\) | \(e^-_1\) | \(K_0\) | \(K_1\) | \(K^{-1}_0\) | \(K^{-1}_1\) |
|---|---|---|---|---|---|---|---|---|
| action on \(V\) | \(R\) | \(L\) | \(l\) | \(r\) | \(K\) | \(K^{-1}\) | \(K^{-1}\) | \(K\) |
By the construction $x$ and $y$ act on $V$ as $e_0^+ + K_0$ and $e_1^+ + K_1$ respectively. By this and since the $A_q$-module $V$ is irreducible we find the $U_q(\hat{sl}_2)$-module $V$ is irreducible. By the definition of $K$ and since $K_0, K_1$ act on $V$ as $K, K^{-1}$ respectively we find the $U_q(\hat{sl}_2)$-module $V$ is type $(1,1)$. We have now proved all the assertions of the theorem except uniqueness. The proof of uniqueness is given in [26, Section 14]. □

Remark 5.2 The equations (20)–(31) give essentially the equitable presentation of $U_q(\hat{sl}_2)$ [26], [27], [52].

6 From $U_q(\hat{sl}_2)$-modules to NonNil $A_q$-modules, I

Let $V$ denote a finite-dimensional irreducible $U_q(\hat{sl}_2)$-module of type $(1,1)$. In this section we show that there exists a NonNil $A_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $e_0^+ + K_0$ and $e_1^+ + K_1$ respectively. We discuss some of the basic properties of this $A_q$-module.

Lemma 6.1 There exists a homomorphism of $\mathbb{K}$-algebras from $A_q$ to $U_q(\hat{sl}_2)$ that sends $x$ and $y$ to $e_0^+ + K_0$ and $e_1^+ + K_1$ respectively.

Proof: Using the defining relations for $U_q(\hat{sl}_2)$ we routinely find that $e_0^+ + K_0$ and $e_1^+ + K_1$ satisfy the cubic $q$-Serre relations. □

Assumption 6.2 Throughout this section $V$ will denote a finite-dimensional irreducible $U_q(\hat{sl}_2)$-module of type $(1,1)$. We let $U_0, \ldots, U_d$ denote the weight space decomposition of $V$, from Lemma 3.1. We let $A : V \to V$ and $A^* : V \to V$ denote the linear transformations that act as $e_0^+ + K_0$ and $e_1^+ + K_1$ respectively. By Lemma 6.1 $V$ is an $A_q$-module on which $x, y$ act as $A, A^*$ respectively.

Remark 6.3 Referring to Assumption 6.2, the $A_q$-module $V$ is not necessarily irreducible. We will address this issue in Proposition 12.1.

Lemma 6.4 With reference to Assumption 6.2 the following hold for $0 \leq i \leq d$:

(i) The element $e_0^+$ acts on $U_i$ as $A - q^{2i-d}I$.

(ii) The element $e_1^+$ acts on $U_i$ as $A^* - q^{d-2i}I$.

Proof: (i) The element $e_0^+$ acts on $U_i$ as $A - K_0$. By (15) and since $V$ has type $(1,1)$ we find $K_0$ acts on $U_i$ as $q^{2i-d}I$. The result follows. (ii) Similar to the proof of (i) above. □

Lemma 6.5 With reference to Assumption 6.2 the following hold for $0 \leq i \leq d$:
(i) \((A - q^{2i-d}I)U_i \subseteq U_{i+1}\).
(ii) \((A^* - q^{d-2i}I)U_i \subseteq U_{i-1}\).

**Proof:** (i) Combine Lemma 6.4(i) with the inclusion on the left in (16).
(ii) Combine Lemma 6.4(ii) with the inclusion on the right in (17).

**Lemma 6.6** With reference to Assumption 6.2, each of \(A, A^*\) is semisimple with eigenvalues \(q^{-d}, q^{2-d}, \ldots, q^{d}\). Moreover for \(0 \leq i \leq d\) the dimension of the eigenspace for \(A\) (resp. \(A^*\)) associated with \(q^{2i-d}\) (resp. \(q^{d-2i}\)) is equal to the dimension of \(U_i\).

**Proof:** We first display the eigenvalues of \(A\). Observe that the scalars \(q^{2i-d}\) \((0 \leq i \leq d)\) are mutually distinct since \(q\) is not a root of unity. Recall that \(U_0, \ldots, U_d\) is a decomposition of \(V\). By Lemma 6.5(i) we see that, with respect to an appropriate basis for \(V\), \(A\) is represented by a lower triangular matrix that has diagonal entries \(q^{-d}, q^{2-d}, \ldots, q^{d}\) with \(q^{2i-d}\) appearing \(\dim(U_i)\) times for \(0 \leq i \leq d\). Hence for \(0 \leq i \leq d\) the scalar \(q^{2i-d}\) is a root of the characteristic polynomial of \(A\) with multiplicity \(\dim(U_i)\). We now show \(A\) is semisimple. To do this we show that the minimal polynomial of \(A\) has distinct roots. By Lemma 6.6(i) we find \(\prod_{i=0}^{d}(A - q^{2i-d}I)\) vanishes on \(V\). By this and since \(q^{2i-d}\) \((0 \leq i \leq d)\) are distinct we see that the minimal polynomial of \(A\) has distinct roots. Therefore \(A\) is semisimple. We have now proved our assertions concerning \(A\); our assertions concerning \(A^*\) are similarly proved.

**Corollary 6.7** With reference to Assumption 6.2, the \(A_q\)-module \(V\) is NonNil.

**Proof:** Immediate from Lemma 6.6

**Definition 6.8** With reference to Assumption 6.2 for \(0 \leq i \leq d\) we let \(V_i\) (resp. \(V_i^*\)) denote the eigenspace of \(A\) (resp. \(A^*\)) associated with the eigenvalue \(q^{2i-d}\) (resp. \(q^{d-2i}\)). We observe that \(V_0, \ldots, V_d\) (resp. \(V_0^*, \ldots, V_d^*\)) is a decomposition of \(V\).

**Lemma 6.9** With reference to Assumption 6.2 and Definition 6.8 the following hold for \(0 \leq i \leq d\):

(i) \(V_i + \cdots + V_d = U_i + \cdots + U_d\),

(ii) \(V_0^* + \cdots + V_i^* = U_0 + \cdots + U_i\).

**Proof:** (i) Let \(X_i = \sum_{j=i}^{d} U_j\) and \(X_i' = \sum_{j=i}^{d} V_j\). We show \(X_i = X_i'\). Define \(T_i = \prod_{j=i}^{d}(A - q^{2j-d}I)\). Then \(X_i' = \{v \in V \mid T_iv = 0\}\), and \(T_iX_i = 0\) by Lemma 6.5(i), so \(X_i \subseteq X_i'\). Now define \(S_i = \prod_{j=0}^{i-1}(A - q^{2j-d}I)\). Observe that \(S_iV = X_i'\), and \(S_iV \subseteq X_i\) by Lemma 6.5(i), so \(X_i' \subseteq X_i\). By these comments \(X_i = X_i'\).

(ii) Similar to the proof of (i) above.
Lemma 6.10 With reference to Assumption 6.2 and Definition 6.8 the following hold:

(i) For $0 \leq i \leq d$,
\[ A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \]
where $V_{-1} = 0$, $V_{d+1} = 0$.

(ii) For $0 \leq i \leq d$,
\[ AV^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1}, \]
where $V^*_{-1} = 0$, $V^*_{d+1} = 0$.

Proof: The elements $A, A^*$ satisfy the cubic $q$-Serre relations by Lemma 6.1 and Assumption 6.2. The result follows from this, Lemma 2.1, and Definition 6.8. \hfill \Box

Remark 6.11 With reference to Assumption 6.2, the pair $A, A^*$ is not necessarily a tridiagonal pair on $V$, since $V$ might be reducible as an $A_q$-module.

7 The projections $F_i, E_i, E^*_i$

Definition 7.1 With reference to Assumption 6.2 and Definition 6.8 we define the following for $0 \leq i \leq d$:

(i) We let $F_i : V \to V$ denote the linear transformation that satisfies both
\[ (F_i - I)U_i = 0, \]
\[ F_iU_j = 0 \quad \text{if} \quad j \neq i, \quad (0 \leq j \leq d). \]
We observe $F_i$ is the projection from $V$ onto $U_i$.

(ii) We let $E_i : V \to V$ denote the linear transformation that satisfies both
\[ (E_i - I)V_i = 0, \]
\[ E_iV_j = 0 \quad \text{if} \quad j \neq i, \quad (0 \leq j \leq d). \]
We observe $E_i$ is the projection from $V$ onto $V_i$.

(iii) We let $E^*_i : V \to V$ denote the linear transformation that satisfies both
\[ (E^*_i - I)V^*_i = 0, \]
\[ E^*_iV^*_j = 0 \quad \text{if} \quad j \neq i, \quad (0 \leq j \leq d). \]
We observe $E^*_i$ is the projection from $V$ onto $V^*_i$.

Proposition 7.2 With reference to Assumption 6.2 and Definition 7.1 the following hold for $0 \leq i \leq d$:
Proof: (i) It suffices to show \(F_iE_i - I\) vanishes on \(U_i\) and \(E_iF_i - I\) vanishes on \(V_i\). We will use the following notation. For \(0 \leq F\) our preliminary comments we routinely find (Lemma 7.3). The following formulae will be useful.

\[(i) \quad \text{The linear transformations} \quad U_i \rightarrow V_i \quad V_i \rightarrow U_i \]
\[u \rightarrow E_iu \quad v \rightarrow F_iv\]

are bijections, and moreover, they are inverses.

(ii) The linear transformations

\[(ii) \quad \text{The linear transformations} \quad U_i \rightarrow V_i^* \quad V_i^* \rightarrow U_i \]
\[u \rightarrow E_i^*u \quad v \rightarrow F_iv\]

are bijections, and moreover, they are inverses.

Proof: (i) It suffices to show \(F_iE_i - I\) vanishes on \(U_i\) and \(E_iF_i - I\) vanishes on \(V_i\). We will use the following notation. For \(0 \leq j \leq d\) recall \(U_j + \cdots + U_d = V_j + \cdots + V_d\); let \(W_j\) denote this common sum. We set \(W_{d+1} = 0\). By the construction \(W_i = U_i + W_{i+1}\) (direct sum) and \(W_i = V_i + W_{i+1}\) (direct sum). Also \((I - F_i)W_i = W_{i+1}\) and \((I - E_i)W_i = W_{i+1}\). We now show \(F_iE_i - I\) vanishes on \(U_i\). Pick \(u \in U_i\). Using \(F_iE_i - I = (F_i - I)E_i + E_i - I\) and our preliminary comments we routinely find \((F_iE_i - I)u \in W_{i+1}\). But \((F_iE_i - I)u \in U_i\) by construction and \(U_i \cap W_{i+1} = 0\) so \((F_iE_i - I)u = 0\). We now show \(E_iF_i - I\) vanishes on \(V_i\). Pick \(v \in V_i\). Using \(E_iF_i - I = (E_i - I)F_i + F_i - I\) and our preliminary comments we routinely find \((E_iF_i - I)v \in W_{i+1}\). But \((E_iF_i - I)v \in V_i\) by construction and \(V_i \cap W_{i+1} = 0\) so \((E_iF_i - I)v = 0\). We have now shown \(F_iE_i - I\) vanishes on \(U_i\) and \(E_iF_i - I\) vanishes on \(V_i\). Consequently the given maps are inverses. Each of these maps has an inverse and is therefore a bijection.

(ii) Similar to the proof of (i) above. \(\square\)

The following formulae will be useful.

**Lemma 7.3** With reference to Assumption 0.2 and Definition 7.1 for \(0 \leq i \leq d\) we have

\[E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - q^{2j-d}I}{q^{2i-d} - q^{2j-d}}, \quad (34)\]

\[E_i^* = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A^* - q^{d-2j}I}{q^{d-2i} - q^{d-2j}}. \quad (35)\]

Proof: Concerning (34), let \(E_i'\) denote the expression on the right in that line. Using Definition 0.8 we find \((E_i' - I)V_i = 0\) and \(E_i'V_j = 0\) \((0 \leq j \leq d, j \neq i)\). By this and Definition 7.1(ii) we find \(E_i = E_i'\). We have now proved (34). The proof of (35) is similar. \(\square\)

**8 How \(E_0, E_0^*, P_V\) are related**

Our goal in this section is to prove the following result.
Theorem 8.1 With reference to Assumption 6.2 and Definition 7.1, for \( u \in U \) we have
\[
E_0^* E_0 u = P_V(q^{-1}(q - q^{-1})^{-2})u, 
\]
where \( P_V \) is from Definition 4.2.

We will use the following lemma.

Lemma 8.2 With reference to Assumption 6.2 and Definition 7.1 the following hold for \( 0 \leq i, j \leq d \):

(i) The action of \( E_i \) on \( U_j \) is zero if \( i < j \) and coincides with
\[
\frac{1}{(q^{2i-d} - q^{2j-d})(q^{2i-d} - q^{2j+2-d}) \cdots (q^{2i-d} - q^{2i-2-d})}
\]
times
\[
\sum_{h=0}^{d-i} \frac{(e_0^+)_{i-j+h}}{(q^{2i-d} - q^{2i+2-d})(q^{2i-d} - q^{2i+4-d}) \cdots (q^{2i-d} - q^{2i+2h-d})}
\]
if \( i \geq j \).

(ii) The action of \( E_i^* \) on \( U_j \) is zero if \( i > j \) and coincides with
\[
\frac{1}{(q^{d-2i} - q^{d-2j})(q^{d-2i} - q^{d-2j+2}) \cdots (q^{d-2i} - q^{d-2i-2})}
\]
times
\[
\sum_{k=0}^{i} \frac{(e_1^+)_{j-i+k}}{(q^{d-2i} - q^{d-2i+2})(q^{d-2i} - q^{d-2i+4}) \cdots (q^{d-2i} - q^{d-2i+2k})}
\]
if \( i \leq j \).

Proof: (i) Pick \( u \in U_j \). We find \( E_i u \). First assume \( i < j \). Observe \( U_j \subseteq V_j + \cdots + V_d \) by Lemma 6.9(i) and \( E_i \) is zero on \( V_j + \cdots + V_d \) so \( E_i u = 0 \). Next assume \( i = j \). By Lemma 6.9(i) and since \( E_i u \in V_i \) we find \( E_i u \in U_i + \cdots + U_d \). Consequently there exist \( u_s \in U_s \) \( (i \leq s \leq d) \) such that \( E_i u = \sum_{s=i}^{d} u_s \). Using \( (A - q^{2i-d}I)E_i = 0 \) we find
\[
0 = (A - q^{2i-d}I)E_i u
\]
\[
= (A - q^{2i-d}I) \sum_{s=i}^{d} u_s
\]
\[
= \sum_{s=i}^{d} (e_0^+ + q^{2s-d} - q^{2i-d})u_s
\]
in view of Lemma 6.4(i). Rearranging terms in (37) we find \( 0 = \sum_{s=i}^{d} u'_s \) where
\[
u'_s = e_0^+ u_{s-1} + (q^{2s-d} - q^{2i-d})u_s \quad (i + 1 \leq s \leq d).
\]
Since \( u'_s \in U_s \) for \( i + 1 \leq s \leq d \) and since \( U_0, \ldots, U_d \) is a decomposition we find \( u'_s = 0 \) for \( i + 1 \leq s \leq d \). Consequently

\[
 u_s = (q^{2i-d} - q^{2s-d})^{-1} e_0^+ u_{s-1} \quad (i + 1 \leq s \leq d).
\]

By Proposition 7.2(i) and since \( u_i = F_i E_i u \) we find \( u_i = u \). From these comments we get the result for \( i = j \). Now assume \( i > j \). Define

\[
 v = \frac{(A - q^{2i-2d} I)(A - q^{2i-4d} I) \cdots (A - q^{2j-d} I) u}{(q^{2i-d} - q^{2i-2d})(q^{2i-d} - q^{2i-4d}) \cdots (q^{2i-d} - q^{2j-d})}.
\] (38)

Using \( E_i A = q^{2i-d} E_i \) we find \( E_i v = E_i u \). In order to get \( E_i u \) we compute \( E_i v \). By Lemma 6.4(i) and Lemma 6.5(i), the numerator on the right in (38) is equal to \((e_0^+)^{i-j} u \) and contained in \( U_i \). Now \( v \in U_i \) in view of (38). We can now get \( E_i v \) using our above discussion concerning the case \( i = j \). By these comments we obtain the result for \( i > j \).

(ii) Similar to the proof of (i) above.

\[
 \Box
\]

\textit{Proof of Theorem 8.1:} By Lemma 8.2(i) (with \( i = 0, j = 0 \)) we find

\[
 E_0 u = \sum_{h=0}^{d} \frac{(e_0^+)^h u}{(q^d - q^{2d}) (q^d - q^{d-2}) \cdots (q^d - q^{2h-d})}. \quad (39)
\]

Pick an integer \( h \) \((0 \leq h \leq d)\). By (16) we have \((e_0^+)^h u \in U_h\). By this and Lemma 8.2(ii) (with \( i = 0, j = h \)) we find

\[
 E_0^* (e_0^+)^h u = \frac{(e_0^+)^h (e_0^+)^h u}{(q^d - q^{d-2}) (q^d - q^{d-4}) \cdots (q^d - q^{d-2h})}. \quad (40)
\]

By Definition 4.1

\[
 (e_0^+)^h (e_0^+)^h u = \sigma_h u. \quad (41)
\]

To obtain (36) we multiply each term in (39) on the left by \( E_0^* \) and evaluate the results using (40), (41), and (16), (18), (19).

\[
 \Box
\]

9 The shape of a tridiagonal pair

In this section we return to the concept of a tridiagonal pair, discussed in Section 2. We will obtain a result concerning these objects that will be of use in our proof of Theorem 1.7. This result might be of independent interest.

We will use the following notation. Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension and let \( A, A^* \) denote a tridiagonal pair on \( V \). Let \( V_0, \ldots, V_d \) (resp. \( V_0^*, \ldots, V_d^* \)) denote a standard ordering of the eigenspaces of \( A \) (resp. \( A^* \)). By [24 Corollary 5.7], for \( 0 \leq i \leq d \) the spaces \( V_i, V_i^* \) have the same dimension; we denote this common dimension by \( \rho_i \). By the construction \( \rho_i \neq 0 \). By [24 Corollary 5.7] and [24 Corollary 6.6] the sequence \( \rho_0, \rho_1, \ldots, \rho_d \) is symmetric and unimodal; that is \( \rho_i = \rho_{d-i} \) for \( 0 \leq i \leq d \) and \( \rho_{i-1} \leq \rho_i \) for \( 1 \leq i \leq d/2 \). We call the sequence \( (\rho_0, \rho_1, \ldots, \rho_d) \) the \textit{shape vector} of \( A, A^* \).
Theorem 9.1 Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension and let $A, A^*$ denote a $q$-geometric tridiagonal pair on $V$. Let $(\rho_0, \rho_1, \ldots, \rho_d)$ denote the corresponding shape vector. Then there exists a nonnegative integer $N$ and positive integers $d_1, d_2, \ldots, d_N$ such that
\[ \sum_{i=0}^{d} \rho_i z^i = \prod_{j=1}^{N} (1 + z + z^2 + \cdots + z^{d_j}). \]

Proof: By Theorem 2.7 there exists a NonNil irreducible $A_q$-module structure on $V$ such that the standard generators $x, y$ act as $A, A^*$ respectively. Let $(\alpha, \alpha^*)$ denote the type of this $A_q$-module. Replacing $A, A^*$ by $A/\alpha, A^*/\alpha^*$ the type becomes $(1, 1)$ and the shape vector is unchanged. By Theorem 5.1 there exists an irreducible type $(1, 1)$ $U_q(\widehat{sl}_2)$-module structure on $V$ such that $x$ and $y$ act as $e^+_0 + K_0$ and $e^+_1 + K_1$ respectively. Let $U_0, \ldots, U_d$ denote the corresponding weight space decomposition of $V$. By Lemma 6.6 we find $\rho_i = \dim(U_i)$ for $0 \leq i \leq d$. By this and Lemma 3.12 we get the result. \qed

10 A basis for $A_q$

In this section we cite some results from [25] that we will use in our proof of Theorem 1.7. We view the $\mathbb{K}$-algebra $A_q$ as a vector space over $\mathbb{K}$. We display a basis for this vector space.

Recall $x, y$ are the standard generators for $A_q$.

Definition 10.1 Let $n$ denote a nonnegative integer. By a word of length $n$ in $A_q$ we mean an expression of the form
\[ a_1 a_2 \cdots a_n \]
where $a_i = x$ or $a_i = y$ for $1 \leq i \leq n$. We interpret the word of length 0 as the identity element of $A_q$. We say this word is trivial.

Definition 10.2 Let $a_1 a_2 \cdots a_n$ denote a word in $A_q$. Observe there exists a unique sequence $(i_1, i_2, \ldots, i_s)$ of positive integers such that $a_1 a_2 \cdots a_n$ is one of $x^{i_1} y^{i_2} x^{i_3} \cdots y^{i_s}$ or $x^{i_1} y^{i_2} x^{i_3} \cdots x^{i_s}$ or $y^{i_1} x^{i_2} y^{i_3} \cdots y^{i_s}$ or $y^{i_1} x^{i_2} y^{i_3} \cdots x^{i_s}$. We call the sequence $(i_1, i_2, \ldots, i_s)$ the signature of $a_1 a_2 \cdots a_n$.

Example 10.3 Each of the words $yx^2 y^2 x$, $xy^2 x^2 y$ has signature $(1, 2, 2, 1)$.

Definition 10.4 Let $a_1 a_2 \cdots a_n$ denote a word in $A_q$ and let $(i_1, i_2, \ldots, i_s)$ denote the corresponding signature. We say $a_1 a_2 \cdots a_n$ is reducible whenever there exists an integer $\eta$ $(2 \leq \eta \leq s - 1)$ such that $i_{\eta-1} \geq i_\eta < i_{\eta+1}$. We say a word is irreducible whenever it is not reducible.

Example 10.5 A word of length less than 4 is irreducible. The only reducible words of length 4 are $xyx^2$ and $yxy^2$.  

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In the following lemma we give a necessary and sufficient condition for a given nontrivial word in $A_q$ to be irreducible.

**Lemma 10.6** [25 Lemma 2.28] Let $a_1a_2\cdots a_n$ denote a nontrivial word in $A_q$ and let $(i_1, i_2, \ldots, i_s)$ denote the corresponding signature. Then the following are equivalent:

(i) The word $a_1a_2\cdots a_n$ is irreducible.

(ii) There exists an integer $t$ ($1 \leq t \leq s$) such that

$$i_1 < i_2 < \cdots < i_t \geq i_{t+1} \geq \cdots \geq i_{s-1} \geq i_s.$$ 

**Theorem 10.7** [25 Theorem 2.29] The set of irreducible words in $A_q$ forms a basis for $A_q$.

11 From $U_q(\hat{sl}_2)$-modules to NonNil $A_q$-modules, II

In this section we return to the situation of Assumption 6.2. Referring to that assumption let $W$ denote an irreducible $A_q$-submodule of $V$. Our next goal is to show that $W$ contains the space $V_0$ from Definition 6.8.

**Definition 11.1** With reference to Assumption 6.2, let $W$ denote an irreducible $A_q$-submodule of $V$. Observe that $W$ is the direct sum of the nonzero spaces among $E_0W, \ldots, E_dW$, where the $E_i$ are from Definition 7.1(ii). We define

$$r = \min\{i \mid 0 \leq i \leq d, E_iW \neq 0\}.$$ 

We call $r$ the endpoint of $W$.

**Lemma 11.2** With reference to Assumption 6.2, let $W$ denote an irreducible $A_q$-submodule of $V$ and let $r$ denote the endpoint of $W$. Then $\dim(E_rW) = 1$.

**Proof:** By construction $W$ is an irreducible $A_q$-module. This module is NonNil by Lemma 6.6. By this and Theorem 2.7 the pair $A|_W, A^*|_W$ is a $q$-geometric tridiagonal pair on $W$. Let $s$ denote the diameter of $A|_W, A^*|_W$. Then $E_rW, E_{r+1}W, \ldots, E_sW$ is a standard ordering of the eigenspaces of $A|_W$. Applying Theorem 9.1 to $A|_W, A^*|_W$ we find $\dim(E_rW) = 1$. \qed

**Lemma 11.3** With reference to Assumption 6.2, let $W$ denote an irreducible $A_q$-submodule of $V$ and let $r$ denote the endpoint of $W$. Pick $v \in E_rW$ and write $u = F_rv$. Then $e_1^+u = 0$.

**Proof:** Observe $u \in U_r$ by Definition 7.1(i). We assume $r \geq 1$; otherwise $e_1^+u = 0$ since $e_1^+U_0 = 0$. Observe $e_1^+u \in U_{r-1}$ by (17). In order to show $e_1^+u = 0$ we show $e_1^+u \in U_r + \cdots + U_d$. Since $A^*$ acts on $V$ as $e_1^+ + K_1$ we have

$$e_1^+u = A^*v - K_1v + e_1^+(u - v).$$ 

We are going to show that each of the three terms on the right in (42) is contained in $U_r + \cdots + U_d$. By the definition of $r$ we have $W = E_rW + \cdots + E_dW$ so $W \subseteq V_r + \cdots + V_d$ in
Proof: We may assume $v \neq 0$; otherwise the result is trivial. Define

$$
\Delta_i = (A^* - q^{d-2r}I)(A^* - q^{d-2r-2}I) \cdots (A^* - q^{d-2r-2i+2}I).
$$

(44)

Since $\Delta_i$ is a polynomial in $A^*$ we find $\Delta_i W \subseteq W$. In particular $\Delta_i v \in W$ so $E_r \Delta_i v \in E_r W$. The vector $v$ spans $E_r W$ by Lemma 11.2 so there exists $t_i \in \mathbb{K}$ such that $E_r \Delta_i v = t_i v$. By this and since $E_r v = v$ we find $E_r (\Delta_i - t_i I) v = 0$. Now $(\Delta_i - t_i I) v \in E_{r+1} W + \cdots + E_d W$ in view of Definition 11.1. Observe $E_{r+1} W + \cdots + E_d W \subseteq V_{r+1} + \cdots + V_d$ where the $V_j$ are from Definition 6.8. By these comments and Lemma 6.9(i) we find $(\Delta_i - t_i I) v \in E_{r+1} W + \cdots + E_d W$. Consequently $F_r (\Delta_i - t_i I) v = 0$. Recall $F_r v = u$ so

$$
F_r \Delta_i v = t_i u.
$$

(45)

We now evaluate $F_r \Delta_i v$. Observe $v = E_r u$ by Proposition 7.2(ii) and since $u = F_r v$. By Lemma 6.3(ii) (with $i = r$, $j = r$) there exist nonzero scalars $\gamma_h \in \mathbb{K}$ ($0 \leq h \leq d - r$) such that $v = \sum_{h=0}^{d-r} \gamma_h (e_0^+)^h u$. For $0 \leq h \leq d - r$ we compute $F_r \Delta_i (e_0^+)^h u$. Keep in mind $(e_0^+)^h u \in U_{r+h}$ by the inclusion on the left in (16). First assume $h < i$. Using Lemma 6.3(ii) and (14) we find $\Delta_i (e_0^+)^h u$ is contained in $U_{r+h-i} + \cdots + U_{r-1}$ so $F_r \Delta_i (e_0^+)^h u = 0$. Next assume $h = i$. By Lemma 6.3(ii) and (14) we find $(\Delta_i - (e_0^+)^i)(e_0^+)^i u$ is contained in $U_{r+1} + \cdots + U_{r+i}$. By this and since $(e_0^+)^i (e_0^+)^i u \in U_r$ we find $F_r \Delta_i (e_0^+)^i u = (e_0^+)^i (e_0^+)^i u$. Next assume $h > i$. Using Lemma 6.3(ii) and (14) we find $\Delta_i (e_0^+)^h u$ is contained in $U_{r+h-i} + \cdots + U_{r+h}$. By this and since $h > i$ we find $F_r \Delta_i (e_0^+)^h u = 0$. By these comments we find $F_r \Delta_i v = \gamma_i (e_0^+)^i (e_0^+)^i u$. Combining this and (15) we obtain (13).

We will use the following result.

Lemma 11.5 [11, Theorem 13.1] Let $B$ denote the subalgebra of $U_q(\widehat{sl}_2)$ generated by $e_0^+$, $e_+^+$, $K_0^{+1}$, $K_1^{+1}$. Then with reference to Assumption 6.2, $V$ is irreducible as a $B$-module.

Proposition 11.6 With reference to Assumption 6.2, let $W$ denote an irreducible $A_q$-submodule of $V$. Then $W$ contains the space $V_0$ from Definition 6.8.
Proof: We first show \( r = 0 \), where \( r \) is the endpoint of \( W \). Pick a nonzero \( v \in E_rW \) and write \( u = F_r v \). Observe \( 0 \neq u \in U_r \) by Proposition 5.2(i). By Lemma 11.3 and the inclusion on the left in (36),

\[
e_1^+ u = 0, \quad (e_0^+)^{d-r+1} u = 0.
\]

By Lemma 11.4,

\[
(e_0^+)i(e_1^+)j u \in \text{Span}(u) \quad (0 \leq i \leq d - r).
\]

Let \( \tilde{W} \) denote the subspace of \( V \) spanned by all vectors of the form

\[
(e_0^+)i_1(e_1^+)i_2(e_0^+)i_3(e_1^+)i_4 \cdots (e_0^+)i_n u,
\]

where \( i_1, i_2, \ldots, i_n \) ranges over all sequences such that \( n \) is a nonnegative even integer, and \( i_1, i_2, \ldots, i_n \) are integers satisfying \( 0 \leq i_1 < i_2 < \cdots < i_n \leq d - r \). Observe \( u \in \tilde{W} \) so \( \tilde{W} \neq 0 \).

We are going to show \( \tilde{W} = V \) and \( \tilde{W} \subseteq U_r + \cdots + U_d \). We now show \( \tilde{W} = V \). To do this we show that \( \tilde{W} \) is \( B \)-invariant, where \( B \) is from Lemma 11.5. To begin, we show that \( \tilde{W} \) is invariant under each of \( e_0^+, e_1^+ \). By Remark 1.5 there exists an \( A_q \)-module structure on \( V \) such that \( x, y \) act as \( e_0^+, e_1^+ \) respectively. With respect to this \( A_q \)-module structure we have \( \tilde{W} = A_0 u \) in view of Lemma 10.6 Theorem 7.7 and (46), (47). It follows that \( \tilde{W} \) is invariant under each of \( e_0^+, e_1^+ \). We now show that \( \tilde{W} \) is invariant under each of \( K_0^{\pm 1}, K_1^{\pm 1} \). By (16), (17) the vector (38) is contained in \( U_{r+i} \) where \( i = \sum_{h=1}^n h_i (-1)^h \). Therefore the vector (38) is a common eigenvector for \( K_0, K_1 \). Recall \( \tilde{W} \) is spanned by the vectors (38) so \( \tilde{W} \) is invariant under each of \( K_0^{\pm 1}, K_1^{\pm 1} \). By our above comments \( \tilde{W} \) is \( B \)-invariant. By this and Lemma 11.5 we find \( \tilde{W} = V \). We now show \( \tilde{W} \subseteq U_r + \cdots + U_d \). We mentioned above that the vector (38) is contained in \( U_{r+i} \) where \( i = \sum_{h=1}^n h_i (-1)^h \). From the construction \( 0 \leq i \leq d - r \) so \( U_{r+i} \subseteq U_r + \cdots + U_d \). Therefore the vector (38) is contained in \( U_r + \cdots + U_d \) so \( \tilde{W} \subseteq U_r + \cdots + U_d \). We have shown \( \tilde{W} = V \) and \( \tilde{W} \subseteq U_r + \cdots + U_d \). Now \( r = 0 \) since \( U_0, \ldots, U_d \) is a decomposition of \( V \). Now \( E_0 W \neq 0 \) by Definition 11.1. We have \( \dim(V_0) = \dim(U_0) \) by Lemma 1.6 and \( \dim(U_0) = 1 \) by Lemma 3.12 so \( \dim(V_0) = 1 \). We have \( 0 \neq E_0 W \subseteq V_0 \) so \( E_0 W = V_0 \). But \( E_0 W \subseteq W \) by (34) so \( V_0 \subseteq W \). \( \square \)

12 The classification

In this section we give the proof of Theorem 4.6 and Theorem 4.7. These proofs depend on the following Proposition.

Proposition 12.1 With reference to Assumption 6.2 the following are equivalent:

(i) \( V \) is irreducible as an \( A_q \)-module.

(ii) \( P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0 \).
Proof: (i) ⇒ (ii) We assume $P_V(q^{-1}(q - q^{-1})^{-2}) = 0$ and get a contradiction. Define

$$W_i = (V_0 + \cdots + V_i) \cap (V_{i+1}^* + \cdots + V_d^*) \quad (0 \leq i \leq d - 1),$$

where the $V_j, V_j^*$ are from Definition 6.8. Further define $W = W_0 + \cdots + W_{d-1}$. We are going to show that $W$ is an $A_q$-submodule of $V$. We begin, we first show $AW \subseteq W$. For $0 \leq i \leq d - 1$ we have $(A - q^{2i-d}I) \sum_{j=i}^{i-1} V_j = \sum_{j=i}^{i-1} V_j$ by Definition 6.8 and $(A - q^{2i-d}I) \sum_{j=i+1}^{d} V_j^* = \sum_{j=i+1}^{d} V_j^*$ by Lemma 6.10(ii). By these comments

$$(A - q^{2i-d}I)W_i \subseteq W_{i-1} \quad (1 \leq i \leq d - 1), \quad (A - q^{-d}I)W_0 = 0$$

and it follows $AW \subseteq W$. We now show $A^*W \subseteq W$. For $0 \leq i \leq d - 1$ we have $(A^* - q^{d-2i-2}I) \sum_{j=0}^{i} V_j \subseteq \sum_{j=0}^{i} V_j$ by Lemma 6.10(i) and $(A^* - q^{d-2i-2}I) \sum_{j=i+1}^{d} V_j^* = \sum_{j=i+1}^{d} V_j^*$ by Definition 6.8. By these comments

$$(A^* - q^{d-2i-2}I)W_i \subseteq W_{i+1} \quad (0 \leq i \leq d - 2), \quad (A^* - q^{-d}I)W_{d-1} = 0$$

and it follows $A^*W \subseteq W$. So far we have shown that $W$ is an $A_q$-submodule of $V$. We now show $W \neq V$. For $0 \leq i \leq d - 1$ we have $W_i \subseteq V_i^* + \cdots + V_i^*$ so $W_i \subseteq V_i^* + \cdots + V_i^*$. It follows $W \subseteq V_1^* + \cdots + V_d^*$ so $W \neq V$. We now show $W \neq 0$. To do this we display a nonzero vector in $W_0$. Pick a nonzero $u \in U_0$. Applying Theorem 8.1 we find $E_0^*E_0u = 0$. Write $v = E_0u$. Then $0 \neq v \in V_0$ by Proposition 7.2(i). Observe $E_0^*v = 0$ so $v \in V_1^* + \cdots + V_d^*$. From these comments $v \in W_0$. We have displayed a nonzero vector $v$ contained in $W_0$. Of course $W_0 \subseteq W$ so $W \neq 0$. We have now shown that $W$ is an $A_q$-submodule of $V$ and $W \neq V$, $W \neq 0$. This contradicts our assumption that $V$ is irreducible as an $A_q$-module. We conclude $P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0$.

(ii) ⇒ (i) Let $W$ denote an irreducible $A_q$-submodule of $V$. We show $W = V$. Define $W'_i = W \cap U_i$ for $0 \leq i \leq d$ and put $W' = \sum_{i=0}^{d} W'_i$. By the construction $W' \subseteq W$. We are going to show $W' = V$. To do this we show $W' \neq 0$ and $W'$ is $B$-invariant, where $B$ is from Lemma 11.5. We now show $W' \neq 0$. To do this we display a nonzero vector in $W'_0$. By Proposition 11.6 we find $V_0 \subseteq W$. Pick a nonzero $v \in V_0$ and write $u = F_0v$. By Proposition 7.2(i) we find $0 \neq u \in U_0$ and $v = E_0u$. Combining this last equation with Theorem 8.1 we find $u = \alpha^{-1}E_0^*v$ where $\alpha = P_V(q^{-1}(q - q^{-1})^{-2})$. We have $v \in W$ by construction and $E_0^*W \subseteq W$ by (35) so now $u \in W$. By the above comments $0 \neq u \in W'_0$. Of course $W'_0 \subseteq W'$ so $W' \neq 0$. We now show that $W'$ is $B$-invariant. To begin, we show $K_0^{-1}W' \subseteq W'$. By (15) $K_0 - q^{2i-d}I$ vanishes on $U_i$ for $0 \leq i \leq d$. Therefore $K_0^{-1}W'_i \subseteq W'_i$ for $0 \leq i \leq d$ so $K_0^{-1}W' \subseteq W'$. By this and since $K_0K_1 - I$ vanishes on $V$ we find $K_1^{-1}W' \subseteq W'$. We show $e_0^*W' \subseteq W'$. From Lemma 6.4(ii) and the inclusion on the left in (16) we find

$$e_0^*W'_i \subseteq W'_{i+1} \quad (0 \leq i \leq d - 1), \quad e_0^*W'_d = 0$$

and it follows $e_0^*W' \subseteq W'$. We show $e_1^*W' \subseteq W'$. From Lemma 6.4(ii) and the inclusion on the right in (17) we find

$$e_1^*W'_i \subseteq W'_{i-1} \quad (1 \leq i \leq d), \quad e_1^*W'_0 = 0$$

and it follows $e_1^*W' \subseteq W'$. By our above comments $W'$ is $B$-invariant. We have now shown that $W'$ is nonzero and $B$-invariant so $W' = V$ by Lemma 11.5. Recall $W' \subseteq W$ so $W = V$.\[\]
The result follows.

It is now a simple matter to prove Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6: Combine Theorem 5.1 and the implication (i) ⇒ (ii) in Proposition 12.1.

Proof of Theorem 1.7: By Corollary 6.7 there exists a NonNil \( A_q \)-module structure on \( V \) such that \( x \) and \( y \) act as \( e_0^+ + K_0 \) and \( e_1^+ + K_1 \) respectively. By construction this \( A_q \)-module is unique. This \( A_q \)-module is irreducible by the implication (ii) ⇒ (i) in Proposition 12.1. This \( A_q \)-module is type (1, 1) by Lemma 6.6.

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