On a compact finite-difference scheme of the third order of weak approximation

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Abstract. The stability and accuracy of compact difference schemes with artificial viscosities of the fourth divergence order are studied. These schemes have a third order both of classical approximation on smooth solutions and weak approximation on discontinuous solutions. As a result of the stability analysis of these schemes in the linear approximation, the optimal values of their viscosity coefficients were obtained. Test calculations are presented to demonstrate the advantages of the new compact scheme compared to the TVD and WENO schemes when calculating discontinuous solutions with shock waves.

1. Introduction

In the classical paper [1] it was shown that among linear schemes with two time layers there are no higher order monotonic schemes. Thus, the development of numerical methods for hyperbolic systems of conservation laws was largely aimed at overcoming this “Godunov prohibition”. As a result, various classes of both difference and projection schemes were developed, in which an increased order on smooth solutions and monotonicity were achieved due to nonlinear flux correction. The main classes of such schemes, which will be abbreviated as NFC (Nonlinear Flux Correction) ones are: MUSCL-schemes [2], TVD-schemes [3], WENO-schemes [4], DG-schemes [5], CABARET schemes [6]. The main advantage of these schemes is that they localize shocks with high accuracy in the absence of significant non-physical oscillations.

It was shown in [7-9] that NFC-schemes like TVD, WENO and DG ones (regardless of their accuracy on smooth solutions) have no more than the first order of integral convergence at intervals, which boundary lies in the area of the shock wave influence, which leads to a significant decrease in their accuracy in this area. One of the reasons for this accuracy decrease is due to the fact that the minimax flux correction procedure for these schemes leads to a decrease in the smoothness of difference flux, which in turn leads to a decrease in the approximation order of the $\varepsilon$-Hugoniot conditions on shock fronts [10]. However, the main reason for reducing the accuracy of NFC-schemes in the calculation of shock waves is that, having an increased order of classical approximation on smooth solutions, they provide no more than the first order of weak approximation on discontinuous solutions [11,12]. The concept of a weak finite-difference approximation of a hyperbolic system of conservation laws was introduced in [11], where a criterion for such an approximation was obtained for a class of piecewise continuous bounded functions. Sufficient conditions for weak approximation on continuous difference solutions were found in [12]. It was shown that among the explicit difference schemes with two time
layers there are no schemes of a higher order of weak approximation. Therefore, this is also holds for the most of NFC schemes used in practice.

In [12], it was proved that in symmetric compact difference schemes [13-15], the orders of classical approximation on smooth solutions and weak approximation on discontinuous solutions coincide. Since these schemes are unstable at strong discontinuities, a special method was proposed for stabilizing the simplest of them, which is three-layer in time and three-point in space. This method is associated with the addition of a special artificial viscosity, approximating the fourth spatial derivative in the lower temporal layer, which ensures the preservation of a higher order of weak approximation. The compact scheme modified in this way (in spite of the fact that it is not an NFC scheme) shows much higher accuracy in the areas of shock influence compared to the TVD scheme [3].

The main disadvantage of the compact scheme proposed in [12] is that it (despite the implicit implementation associated with the use of runs and iterations on nonlinearity) is obviously stable in the linear approximation only for the Courant numbers \( z \leq 0.5 \), and the maximum rapid damping of the numerical oscillations at the shock wave front occurs at \( z \leq 0.5 \). As a result, calculations using this scheme were carried out at \( z \leq 0.2 \), which is significantly lower than the characteristic Courant numbers, which are used in explicit NFC schemes. Therefore, the main goal of this paper is to develop a new compact scheme of a higher order of weak approximation, whose stability is not related to the above limitations on the Courant number.

2. Weak finite-difference approximation of conservation laws

Consider a quasilinear hyperbolic system of conservation laws

\[
\frac{\partial u}{\partial t} + f(u) = 0,
\]

where \( u(x,t) \) is the desired and \( f(u) \) is a given smooth vector functions. Piecewise continuous vector function \( u \) is a weak solution of the system (1) if for each test function \( \phi\in C^\infty_0 \) satisfies the following system of integral equations

\[
\int \left( u \phi_t + f(u) \phi_x \right) dx dt = 0.
\]

If in this formula and further the integration area is not indicated, it means that the integral is taken over the function \( \phi \) carrier. We approximate system (1) with a finite-difference scheme

\[
\Lambda_h[v_h(x,t), v_{h+1}, \ldots, v_{h+n}, h, \tau, \sigma] = 0,
\]

where \( \Lambda \) is the smooth generating function of the scheme, \( h \) and \( \tau = \sigma h \) are spatial and temporal grid steps, and \( \sigma \) is the constant parameter. Let us formulate the classical notion of a finite-difference approximation.

**Definition 1.** The difference scheme (3) approximates system (1) with \( k \)-th order if for each sufficiently smooth solution \( u(x,t) \) of the system the inequality

\[
\Lambda_h[u(x,t)] \leq O(h^k)
\]

holds and there are solutions for which this inequality becomes equality.

The classic concept of approximation becomes vague for discontinuous solutions. At the same time, the accuracy of the difference approximation in the vicinity of the shock determines the accuracy with which the difference scheme transmits the Hugoniot conditions through shock. Therefore [12] offered a special notion of weak finite difference approximation on difference solutions \( v_h(x,t) \).

**Definition 2.** The difference scheme (3) weakly approximates system (1) with \( k \)-th order if for each its uniformly on \( h \) bounded solution \( v_h(x,t) \) and for each test function \( \phi(x,t)\in C^\infty_0 \) the inequality

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\[ \int \int (v_h \varphi_x + f(v_h) \varphi_h) \, dx \, dt \leq O(h^k) \]  

(5)

holds and there are difference solutions for which this inequality becomes equality.

This definition, unlike definition 1, retains its meaning in the neighborhood of the discontinuity line of the approximated exact solution. From the formula (5) it follows that if the difference scheme (3) satisfies the definition 2, then its difference solution \( v_h(x,t) \) is a weak solution of system (1) with an accuracy not less than \( O(h^k) \) and therefore it transmits the Hugoniot conditions with increased accuracy. Sufficient conditions for weak approximation by definition 2 were obtained in [12]. It follows from these conditions that among the explicit two-layer in time difference schemes there are no schemes of higher order of weak approximations. Moreover, the number of bilayer also includes such schemes in which the transition to the next time layer occurs as a result of applying the higher order Runge-Kutta method. It means that virtually all NFC-schemes, in particular, MUSCL, TVD, WENO and CABARET type schemes, have no more than the first order of weak approximation in the sense of definition 2.

3. Compact finite-difference schemes

Denote by \( T^{nrt}_{jh} \) a shift operator, whose action on each function \( u(x,t) \) is determined by the identity

\[ T^{nrt}_{jh} \circ u(x,t) = u(x + jh, t + n\tau). \]  

(6)

We will also use the following notations \( T^0_{jh} = T^0_{j\mu}, \ D^{nrt} = T^{nrt}_{0, E} = T^0_{0} \). A symmetric in time and space compact difference scheme, that approximates the system (1) with the \( k \)-th order, has the form [13-15]:

\[ A_\tau \circ v_h + A_h \circ f(v_h) = 0, \]  

(7)

where

\[ A_\tau = \sum \alpha_j T^\tau_{jh}, \ A_h = \sum \alpha_n T^{nrt}, \ A_i = \sum \beta_n T^{nrt} \]  

(8)

are linear difference operators with constant coefficients satisfying conditions

\[ \frac{\Delta}{h} \circ u = A_\tau \circ u + O(h^k), \ \frac{\Delta}{\tau} \circ u = A_i \circ u + O(\tau^k) \]  

(9)

for all sufficiently smooth functions \( u(x,t) \). In [12] it is shown that symmetric compact schemes have the following remarkable property: the orders of their classical and weak approximations coincide.

The simplest compact scheme (7), which is three-layer in time and three-point in space, is obtained with operators

\[ A_\tau = T_{h+} + 4E + T_{h-}, \ A_h = \frac{T_{h+} - T_{h-}}{2}, \ A_i = \frac{T_i + 4E + T^{-i}}{6}, \ A_\tau = \frac{T^{nrt} - T^{-nrt}}{2} \]  

(10)

and has a fourth approximation order. To ensure the stability of the scheme (7), (10) when calculating strong discontinuities, we will add artificial viscosities to its right-hand side

\[ (w_h)_i = -C_i \frac{h^3}{\tau} \left( \frac{\Delta}{h} \right)^4 \circ T^{nrt} \circ v_h, \ i = -1, 0, 1, \]  

(11)

where \( C_i > 0 \) are artificial viscosity coefficients,

\[ (\Delta)_i = (T_{h/2} - T_{-h/2}) \]  

(12)
is fourth order divergence operator. As a result, we obtain the following three-parameter family of compact difference schemes

\[ A_\tau \cdot \frac{\Delta}{\tau} \cdot v_h + A_\tau \cdot \frac{\Delta}{h} \cdot f(v_h) + \frac{h^4}{\tau} \sum_{i=1}^{C} \left( \frac{\Delta}{h} \right)^i \cdot T^i \cdot v_h = 0 \]

(13)

third order both classical and weak approximations [16].

A difference scheme (13) in which the artificial viscosity with the coefficient \( C_{c1} = 1/96 \) is added only on the lower time layer, i.e. \( C_o = C_i = 0 \), was considered in [12] and for Courant numbers \( z \leq 0.2 \) showed a significant higher accuracy in the calculation of shock waves, compared with the TVD scheme [3]. However, with Courant numbers \( z > 0.7 \) the scheme (13) with such viscosity coefficients becomes unstable. Spectral analysis of the scheme (13) in a linear approximation shows that in order to increase its stability, it is necessary to introduce artificial viscosity with a coefficient \( C_i > 0 \) on the upper temporal layer and that similar viscosity should not be added to the middle temporal layer. Test calculations have shown that the optimum localization of the shock in the calculation according to the compact scheme (13) with the Courant number \( z = 1 \) is achieved with the artificial viscosity coefficients \( C_{c1} = 1/96, C_o = 0, C_i = 1/10 \), which are used in the calculations given below.

4. Results of numerical calculations.

As a specific hyperbolic system (1), we choose the first approximation of the shallow water equations, for which

\[ \mathbf{u} = \begin{pmatrix} H \\ q \end{pmatrix}, \quad f(\mathbf{u}) = \begin{pmatrix} q \\ q^2/H + g H^2/2 \end{pmatrix}, \]

(14)

where \( H \) and \( q \) are the liquid depth and flow rate, and \( g \) is the gravity acceleration. Consider for system (1), (14) the Cauchy problem with periodic initial data

\[ H(x,0) = 2\sin \frac{\pi(2x+5)}{X} + 3, \quad q(x,0) = 0, \]

(15)

where \( X = 10 \) is the period length. Figures 1-4 show the calculation results obtained for the problem (1), (14), (15) using three finite-difference schemes: TVD-scheme [3], WENO5-scheme [4] and new compact scheme proposed in this paper. Since the exact solution of this problem is symmetric with respect to a point \( X/2 = 5 \), the numerical results are given on the segment \([0,5]\) of the half-period length. The calculations were carried out on a rectangular difference grid \( x_j = jh, t_{n+1} = t_n + \tau, t_0 = 0 \), with a constant spatial step \( h \) and with a variable time step \( \tau_n \) chosen from the stability condition

\[ \tau_n = \frac{zh}{\max_j \left( |v_j^n| + c_j^n \right)}, \quad f_j^n = f(x_j, t_n), \]

(16)

where \( v = q/H \) is the fluid velocity, \( c = \sqrt{gH} \) is the propagation velocity of small disturbances, and \( z \) is the Courant number (\( z = 0.9 \) for TVD and WENO5 schemes, and \( z = 1 \) for compact scheme).

The exact solution of problem (1), (14), (15) is modeled by numerical calculation using the TVD-scheme [3] on a fine grid with a spatial step \( h = 0.00025 \). In this solution (which is shown by solid lines in figures 1a-3a), as a result of a gradient catastrophe, shock waves arise, which subsequently interact many times. The numerical results for the given problem on a grid with the spatial step \( h = 0.1 \) are shown in figures 1a-3a. These results are obtained using the WENO and nonmonotonic compact schemes. Numerical solutions obtained by TVD-scheme are visually close to those obtained by WENO-scheme, therefore we do not depict them here. Figures 1b-3b show the errors
\[ \delta \hat{R}(x_i, x_j, t) = \log | R_e(x_i, x_j, t) - R_n(x_i, x_j, t) | \]

(17)

for the calculation of integrals

\[ R(x_i, x_j, t) = \int_{x_i}^{x_j} r(x, t) dx \]

(18)

of the r-invariant \( r = v - 2c \), where \( R_e \) is the “exact” value of the integral (18) obtained on a fine grid and \( R_n \) is its numerical value obtained by the Simpson formula on a grid with the spatial step \( h = 0.0025 \). Integration segment boundary \( x = 0 \) for figure 2b and \( x = 5 \) for figures 1b, 3b. Errors (17) not given in the intervals \([x_i, x_j]\) intersecting with the neighborhood of the shock (its boundary is shown in the figures by a dotted line), inside which there is no convergence of the difference solution. The calculation of errors (17) was performed on a grid with a spatial step \( h = 0.00025 \). The results of these computations are presented for every 40th spatial node \( j = 40i \). In figures, circles show the calculation results obtained by the TVD scheme, squares – by the WENO scheme and dots – by the compact scheme.

Calculations show that before the gradient catastrophe, when the exact solution is smooth, the WENO-scheme demonstrates higher accuracy compared to the TVD and compact schemes of a lower order of the classical approximation. However, after the formation of shock waves (figures 1b-3b), the accuracy of the TVD and WENO schemes in the areas of the shock influence becomes comparable and significantly lower than the accuracy of the compact scheme of a higher order of weak approximation. This is explained by the fact that the TVD and WENO schemes are explicit and two-layer in time and therefore have no more than the first order of weak approximation when calculating discontinuous solutions with shock.

\[ \text{Figure 1. The fluid depth (a) and the error in calculating the integral of the r-invariant at time } T = 1.25 \text{(b).} \]
Figure 2. The fluid depth (a) and the error in calculating the integral of the r-invariant at time $T = 2.2$ (b).

Figure 3. The fluid depth (a) and the error in calculating the integral of the r-invariant at time $T = 3$ (b).

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