The Asymptotic Spectrum of LOCC Transformations

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Abstract—We study the exact, non-deterministic conversion of multipartite pure quantum states into one-another via local operations and classical communication (LOCC) and asymptotic entanglement transformation under such channels. In particular, we consider the maximal number of copies of any given target state that can be extracted exactly from many copies of any given initial state as a function of the exponential decay in the success probability, known as the converse error exponent. We give a formula for the optimal rate presented as an infimum over the asymptotic spectrum of LOCC conversion. A full understanding of exact asymptotic extraction rates between pure states in the converse regime thus depends on a full understanding of this spectrum. We present a characterization of spectral points and use it to describe the spectrum in the bipartite case. This leads to a full description of the spectrum and thus an explicit formula for the asymptotic extraction rate between pure bipartite states, given a converse error exponent. This extends the result on entanglement concentration in [1], where the target state is fixed as the Bell state. In the limit of vanishing converse error exponent, the rate formula provides an upper bound on the exact asymptotic extraction rate between two states, when the probability of success goes to 1. In the bipartite case, we prove that this bound holds with equality.

Index Terms—Quantum entanglement, local operations and classical communication, asymptotic entanglement transformation, error exponent, resource theory, asymptotic conversion rates.

I. INTRODUCTION

The primary objects of study in this paper are entangled $k$-partite pure states of finite dimensional quantum systems, represented by vectors in the tensor product of $k$ finite dimensional Hilbert spaces. Allowing for certain quantum operations of these systems yields a resource theory, viewing quantum states as resources and quantum operations as methods of extracting one resource from another. One important set of quantum operations are LOCC channels, which allow for local application of quantum channels, represented by completely positive maps, and the sending of classical information between parties. The overarching question is then; which states can be transformed to which via LOCC channels? And asymptotically: How many copies of the target state $|\phi\rangle$ can be extracted per copy of the resource state $|\psi\rangle$? Besides from distinguishing between the single- and multi-copy setting, one may also consider different notions of accuracy in the transformation. One may consider either deterministic or probabilistic transformations. One can also require exact transformation or allow for some error, in the sense that we proclaim success, if the output of the LOCC channel is sufficiently close to the target state, for some notion of distance. This distance which will often be the trace distance or be described by some loss in fidelity. In the multi-copy regime, we may consider various restriction on the asymptotic behavior of either the probability of successful transformation or on the fidelity between the output and the target state. Finally one may consider all mixed states, or restrict consideration only to pure states. In this paper we will consider exact LOCC transformations of pure states and describe the asymptotic relation between success probability and extractions rates. We will initially work with the multipartite case, and subsequently specialize to the bipartite case.

In the bipartite regime the single-copy, pure, exact, deterministic case is characterized completely by Nielsen’s theorem [2] stated in Theorem 5 below. Nielsen’s theorem was generalized to multipartite states in [3] and [4]. Also completely understood is the bipartite, single-copy, pure, exact, probabilistic case [5], [6]. In the bipartite, regime, the deterministic, fixed fidelity-error conversion rate between pure states is simply the ratio of the von Neumann entropies of the marginals [7] and the second-order asymptotics have been further studied in [8]. For mixed bipartite states, asymptotic conversion becomes irreversible and entanglement cost (when the resource state is maximally mixed) is the regularization of the entanglement of formation [9], [10].

The multipartite case is significantly more difficult for both pure and mixed states and has been studied extensively in recent decades (for a recent review on multipartite entanglement see [11]). Even the problem of deciding whether two states are equally entangled is highly nontrivial. Inequivalence can be certified with the help of invariant quantities [12]–[14], and in some cases normal forms are available [15]–[17] that can be used to decide equivalence. Another key factor in this difficulty is the lack of a unique maximally entangled
state (like the EPR pair for bipartite states) into which any other pure entangled state can be asymptotically reversibly converted. Instead one needs to consider maximally entangled sets of states [18], which however contain almost all states already for four qubits. The situation is further complicated by the subtle difference between LOCC transformations allowing finitely or infinitely many rounds as well as their closures [19]–[21]. For states within an SLOCC class [3] derived a necessary and sufficient condition for the possibility of a single-copy transformation via separable operations, which can be considered a generalisation of Nielsen’s theorem. For many-to-one SLOCC conversion, conversion is characterised by inclusion of entanglement clusters [22]. In the context of transforming asymptotically many copies, [23] investigated asymptotic SLOCC transformations and noted their connection to the problem of finding the exponent of matrix multiplication in algebraic complexity theory. This observation led to a series of works [24]–[30], and our result is also inspired by this connection.

In this paper we only consider exact LOCC transformations for pure states. The exact, vanishing failure-probability case is resolved in the bipartite case by Theorem 6 below, where it is shown to be the minimal ratio of $\alpha$-Rényi entropies for $\alpha \in [0, 1]$. Since exact, vanishing failure-probability is necessarily more restrictive than respectively SLOCC and fixed fidelity error, the rate must be smaller than respectively the log-ratio of local dimension and the ratio of local von Neumann entropies, corresponding to $\alpha = 0$ and $\alpha = 1$.

In [1] the optimal exact extraction rate was described as a function of the exponential behavior of the probability of successful transformation. This was done in the bipartite case, with target state fixed as the maximally entangled state (entanglement concentration). Note that this differs from requiring an approximate transformation with fidelity decreasing like $2^{-nr}$, also studied in [1], which is in general possible with a higher rate. If probability of failure behaves like $2^{-nr}$, where $n$ is the number of copies of the resource, $r$ is called the direct error exponent. If probability of success behaves like $2^{-nr}$, $r$ is called a converse error exponent. Given converse error exponent $r$, the following formula for the concentration rate was given in [1]:

$$E^*(r) = \inf_{\alpha \in [0, 1]} \frac{ra + \log \sum_i p_i^\alpha}{1 - \alpha}. \tag{1}$$

Here $(p_i)$ are the Schmidt Coefficients of the resource state.

For two multipartite pure states, $|\psi\rangle$ and $|\phi\rangle$, we let $E^*(r, \psi, \phi)$ be the number of copies of $|\phi\rangle$ that can be asymptotically extracted per copy of $|\psi\rangle$ with success probability behaving like $2^{-nr+\log(n)}$. We say that the optimal extraction rate from $|\psi\rangle$ to $|\phi\rangle$ with converse error exponent $r$ is $E^*(r, \psi, \phi)$. As a consequence of Theorem 2 below we show in (63) that for $k$ parties, the optimal extraction rate can be expressed as

$$E^*(r, \psi, \phi) = \inf_{f \in \Delta(S_k)} \frac{ra(f) + \log f(|\psi\rangle)}{\log f(|\phi\rangle)}. \tag{2}$$

Here $\Delta(S_k)$ is the set of real, LOCC-monotone functions on the set of unnormalized states, that are additive under direct sum, multiplicative under tensor product and normalized, and $a(f) = \log f(\sqrt{2}|0\ldots0\rangle)$. $\Delta(S_k)$ will be called the asymptotic LOCC spectrum. The definition of $\Delta(S_k)$ will be made explicit in section II. Due to (2), an explicit description of the $\Delta(S_k)$ would imply a complete understanding of the asymptotic extraction rates given a converse error exponent.

In Theorem 3 we present a characterization of the functions in $\Delta(S_k)$. Note that in the case where the target $|\phi\rangle$ is the normalized GHZ-state $|0\ldots0+1\ldots1\rangle$, we have $\log f(|\phi\rangle) = 1 - a(f)$, showing the resemblance between (1) and (2).

In the bipartite case the characterization from Theorem 3 below allows for an explicit description of $\Delta(S_2)$ (Theorem 4).

We present the following formula for the extraction rate between $|\psi_P\rangle$ and $|\psi_Q\rangle$ with converse error exponent $r$, generalizing the formula for $E^*(r)$:

$$E^*(r, |\psi_P\rangle, |\psi_Q\rangle) = \inf_{\alpha \in [0, 1]} \frac{ra + \log \sum_i p_i^\alpha}{\log \sum_i q_i^\alpha}. \tag{3}$$

When $|\psi_Q\rangle$ is an EPR-pair, we retrieve (1).

Taking the limit $r \to 0$ in (2), one obtains the optimal extraction rate when success probability is allowed to go to 0, but not exponentially fast. While it does not follow from the general theory, one might reasonably conjecture that this coincides with the extraction rate for success probability going to 1. In the bipartite case we show that this is indeed true. Expressed with Rényi entropies in Theorem 6, the formula for the optimal extraction rate between bipartite pure states with success probability going to 1 is

$$E(|\psi_P\rangle, |\psi_Q\rangle) = \min_{\alpha \in [0, 1]} \frac{H_\alpha(P)}{H_\alpha(Q)}. \tag{4}$$

This result much resembles the formula conjectured in [31, Example 8.26] and proven in [32], where the minimum is taken over all of $[0, \infty]$, describing the extraction rate under the condition, that the probability of success is identically 1 for sufficiently many copies.

Our results are inspired by the work of Strassen on the asymptotic restriction problem for tensors [33]. In that paper he establishes a characterization in terms of the asymptotic spectrum associated with the semiring of equivalence classes of tensors, equipped with the preorder induced by tensor restriction. We prove that the semiring $S_k$ of local unitary equivalence classes of pure unnormalized states, equipped with the preorder induced by LOCC convertibility satisfies similar properties, which leads to the characterization in (2).

II. THE ASYMPTOTIC LOCC SPECTRUM

In [33], Strassen considers the semiring of equivalence classes of tensors under invertible local linear transformations. This is a semiring with respect to direct sum and tensor products and the preorder, given by convertibility via local linear transformations, respects the algebraic structure of this semiring. By applying the spectral theorem [33, Theorem 2.3], one gets the asymptotic spectrum $\Delta(B)$ of tensors. First we recall the necessary definitions and state the theorem.

**Definition 1.** A commutative semiring $(S, +, \cdot)$ is a set $S$ with two binary, commutative, and associative operations
(+, ·) containing distinct additive and multiplicative identity elements 0, 1 ∈ S, satisfying the distributive law:

\[ a(b + c) = ab + ac. \]  

(5)

Note that what distinguishes a semiring from a ring, is that there is no guarantee of an additive inverse. In fact the semiring we will consider in this paper has no additive inverses, except for 0. In this paper all semirings are commutative and semiring shall therefore be understood to implicitly mean commutative semiring.

Definition 2. A preorder \( \leq \) on \( S \) is a binary relation which is transitive and reflexive (but not necessarily antisymmetric). We say that \((S, +, ·, \leq)\) is a preordered semiring, if \( \leq \) respects the algebraic structure on \( S \). That is, when \( a \leq b \) and \( c \leq d \):

\[ a + c \leq b + d \]  
\[ ac \leq bd. \]  
\[ (6) \quad (7) \]

Remark 1. Note that in order to show conditions (6) and (7) it suffices to show \( a + c \leq b + c \) and \( ac \leq bc \) whenever \( a \leq b \), since this implies \( a + c \leq b + c \leq b + d \) whenever \( a \leq b \) and \( c \leq d \), and similarly for the product.

One can always turn a semiring into a preordered semiring by defining \( \leq \) to be either the equality preorder \((x \leq y \iff x = y)\) or the other extreme preorder \((\forall x, y \in S : x \leq y)\). We shall only be interested in certain non-trivial preorders, namely semirings where \( \mathbb{N} \subset S \) and the preorder restricted to \( \mathbb{N} \) is the usual ordering of \( \mathbb{N} \).

Theorem 1 (Strassen, [33], see also [34, Theorem 2.2]). Let \((S, \leq)\) be a preordered semiring with \( \mathbb{N} \subset S \) satisfying the following:

1) \( \leq \) restricted to \( \mathbb{N} \) is the usual ordering of \( \mathbb{N} \).
2) For any \( a, b \in S\setminus\{0\} \) there is an \( r \in \mathbb{N} \) such that \( a \leq rb \).

Define the asymptotic preorder \( \leq_{\text{as}} \) on \( S \) by: \( a \leq_{\text{as}} b \) if and only if \( a^n \leq 2^{\frac{1}{n}}b^r \) for some integer-valued sequence \( x_n \in o(\mathbb{N}) \).

Then \((S, \leq_{\text{as}})\) is also a preordered semiring. Let \( \Delta(S) \) be the set

\[ \{ f \in \operatorname{Hom}(S, \mathbb{R}^+) | \forall a, b \in S : a \leq b \implies f(a) \leq f(b) \}. \]

That is \( \Delta(S) \) is set of order-preserving semiring homomorphisms from \( S \) to \( \mathbb{R}^+ \). Then

\[ a \leq_{\text{as}} b \iff \forall f \in \Delta(S) : f(a) \leq f(b). \]  
\[ (8) \]

Let \( \Delta(S) \) be equipped with the topology generated by the maps \( \hat{a} : \Delta(S) \to \mathbb{R} \), given by \( \hat{a} : f \mapsto f(a) \). That is, \( \Delta(S) \) is equipped with the coarsest topology making these maps continuous. Then \( \Delta(S) \) is a compact Hausdorff space and \( a \mapsto \hat{a} \) is a semiring homomorphism \( S \to C(\Delta(S)) \), which, by (8), respects both \( \leq_{\text{as}} \) and \( \leq \) on \( S \). \( \Delta(S) \) will be called the asymptotic spectrum of \( S \).

Remark 2. In this paper, we are interested in the asymptotic ordering of \( S \) and (8) is therefore the important property of the asymptotic spectrum. The topology on \( \Delta(S) \) will not play a role, but the fact that \( S \) maps into \( C(\Delta(S)) \) in an order preserving manner explains the use of the term spectrum.

The goal of this section is to study the semiring of local unitary orbits of unnormalized pure states with preorder defined by LOCC convertibility. We show that this is a preordered semiring and that the conditions of Theorem 1 are satisfied, yielding an LOCC spectrum.

Definition 3. We define unnormalized states to be positive elements

\[ \rho \in \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k) \otimes \text{Diag}(\mathbb{C}^X), \]

where \((\mathcal{H}_i)_{i=1}^k\) are finite dimensional Hilbert spaces, \( X \) is a finite set and \( \text{Diag}(\mathbb{C}^X) = \text{span}\{ |x⟩⟨x| \mid x \in X \} \) is the space of diagonal matrices acting on \( \mathbb{C}^X \). The cone of positive elements in \( \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k) \otimes \text{Diag}(\mathbb{C}^X) \) will be called a \( k \)-partite state space.

The states in Definition 3, are precisely the unnormalized \((k + 1)\)-partite quantum-classical states on the systems \( \mathcal{H}_1, \ldots, \mathcal{H}_k, \mathbb{C}^X \), where \( \mathbb{C}^X \) is the classical part and the other systems are quantum. The \( \mathcal{H}_i \)'s are to be viewed as physically separated quantum systems and \( X \) as a shared classical register. We will define our allowed LOCC-operations in such a way that the classical system can only be manipulated classically, and in such a way that each party has access only to their own system and to the classical register.

Notice that we are not demanding that the states are normalized to \( \text{Tr}(\rho) = 1 \). We associate to each unnormalized state, \( \rho \), the normalized state \( \frac{\rho}{\text{Tr}(\rho)} \). Since we will be dealing mostly with unnormalized states, we will use the term state to refer to an unnormalized state, unless otherwise stated. Physical processes will be modeled by completely positive trace non-increasing maps. Such a map can be turned into a trace preserving one by adding an extra Kraus operator which maps into a subspace orthogonal to the rest. This should be thought of as an undesirable measurement outcome which prevents the protocol from being successfully applied. However, it is often more convenient to work with the trace non-increasing version. When applied to an input state with trace 1, the trace of the output is the probability of success. For an unnormalized input, the success probability is the ratio of the traces.

Notation II.1. Given a multipartite system \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \) and a map \( K : \mathcal{H}_i \to \mathcal{H}'_i \) on the \( i \)'th system, we denote by \((K)_i\) the map acting by \( K \) on the \( i \)'th system and with the identity on all other systems. I.e.

\[ (K)_i = I_{\mathcal{H}_1} \otimes \cdots \otimes I_{\mathcal{H}_{i-1}} \otimes K \otimes I_{\mathcal{H}_{i+1}} \otimes \cdots \otimes I_{\mathcal{H}_k} \]  
\[ (9) \]

Definition 4. We define a one-step LOCC channel to be a map

\[ \Lambda : \mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_k) \otimes \text{Diag}(\mathbb{C}^X) \]

\[ \to \mathcal{B}(\mathcal{H}'_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}'_k) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_k) \otimes \text{Diag}(\mathbb{C}^X) \]

between two state spaces that is given by

\[ \Lambda : \rho \mapsto \sum_{j \in J} (K_j)_j \otimes |g(j)⟩⟨f(j)| \rho (|K_j^*⟩_j ⊗ |f(j)⟩ ⟨g(j)|). \]  
\[ (10) \]
Here $i \leq k$ is a positive integer, $J$ is a finite index set and $f : J \to \mathcal{X}$, $g : J \to \mathcal{Y}$ are maps. For each $j \in J$, $K_j : \mathcal{H}_i \to \mathcal{H}'_i$ is a linear map. These maps need to satisfy

$$\sum_{j \in J} K_j^* K_j \otimes |f(j)\rangle \langle f(j)| \leq I,$$

where $I$ denotes the identity operator on $\mathcal{H}_i \otimes \mathbb{C}^\mathcal{X}$. The operators $K_j$ are called the Kraus operators.

An LOCC protocol is a finite sequence of composable one-step LOCC channels and the composition is an LOCC channel.

In the above formulation of LOCC we allow for a finite, yet unbounded number of rounds of communication and local operations. For simplicity, our results concern this scenario, but we expect the techniques to be applicable more generally, e.g. to compare LOCC with its closure in the asymptotic limit. For a review on different notions of LOCC see [19].

The reader may have noticed that the definition we use differs from that found elsewhere in the literature (although the idea of sharing every measurement result with every party is already present in [35, Footnote 20]). In our formulation the local channels do not depend explicitly on earlier rounds, but instead act jointly on the local system and the shared classical register, which is supposed to store the required measurement results. Conveniently, this also removes the need to model the passing of classical messages between the parties. One can think of a one-step LOCC channel in our sense as the act of reading the classical register ($f$) and $g$ is the identity map on $\mathcal{Y}$:

$$g : \mathcal{Y} \to \mathcal{Y}.$$

An LOCC protocol is a finite sequence of composable one-step LOCC channels and the composition is an LOCC channel.

Definition 5. We say that a one-step LOCC channel, $\Lambda$, is remembering if the Kraus operators are indexed over $J \subseteq \mathcal{Y}$ and $g$ is the identity map on $\mathcal{Y}$. That is

$$\Lambda : \rho \mapsto \sum_{y \in \mathcal{Y}} \left((K_y)_i \otimes |y\rangle \langle f(y)|\right) \rho \left((K_y^*)_i \otimes |f(y)\rangle \langle y|\right).$$

(12)

Given two states $\rho_1$ and $\rho_2$, we say that $\rho_2$ can be extracted from $\rho_1$ under LOCC and write $\rho_1 \overset{\text{LOCC}}{\longrightarrow} \rho_2$ if there exists an LOCC channel $\Lambda$ such that $\Lambda(\rho_1) = \rho_2$. Under the identification $B(\mathcal{H}_1) \otimes \cdots \otimes B(\mathcal{H}_k) \simeq B(\mathcal{H}_1) \otimes \cdots \otimes B(\mathcal{H}_k) \otimes \text{Diag}(\mathbb{C})$ we shall also consider positive elements of the former as states.

To any vector $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ we associate the pure state $|\psi\rangle \langle \psi| \in B(\mathcal{H}_1) \otimes \cdots \otimes B(\mathcal{H}_k)$ and we write $|\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle$ if the corresponding statement is true for their respective states.

Remark 3. Note that we allow for trace non-increasing completely positive maps. So $|\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle$ means that we can convert $|\psi\rangle$ to $|\phi\rangle$ with success probability $\langle \phi| \phi \rangle$.

Definition 6. Given $k \in \mathbb{N}$ and finite dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}'_1, \ldots, \mathcal{H}_k, \mathcal{H}'_k$ we say that $|\phi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ and $|\psi\rangle \in \mathcal{H}'_1 \otimes \cdots \otimes \mathcal{H}'_k$ are locally unitarily equivalent (or LU-equivalent), if there exist partial isometries $U_i : \mathcal{H}_i \to \mathcal{H}'_i$ such that $|\psi\rangle = (U_1 \otimes \cdots \otimes U_k) |\phi\rangle$

and

$$|\phi\rangle = (U_1^* \otimes \cdots \otimes U_k^*) |\psi\rangle.$$

Let $S_k$ denote the set of local unitary equivalence classes, commonly abbreviated as the LU classes.

The above definition is a slight generalization of the standard definition of LU-equivalence for pure states, where the $U_i$’s are normally required to be unitaries. But up to embedding states into larger spaces, the definitions are equivalent in the following sense: $|\psi\rangle$ and $|\phi\rangle$ are LU-equivalent in the sense of Definition 6 if and only if there exist subspaces $\mathcal{K}_i \subset \mathcal{H}_i$, $\mathcal{K}'_i \subset \mathcal{H}'_i$ and $|\psi_0\rangle \in \bigotimes_i \mathcal{K}_i$, $|\phi_0\rangle \in \bigotimes_i \mathcal{K}'_i$ such that $|\psi\rangle$ and $|\phi\rangle$ are images of $|\psi_0\rangle$ and $|\phi_0\rangle$ under the inclusion maps and $|\psi_0\rangle$ is LU-equivalent to $|\phi_0\rangle$ via unitaries. The subspaces $\mathcal{K}_i$ and $\mathcal{K}'_i$, will be the support of the marginal density operators of $|\psi\rangle$ and $|\phi\rangle$.

In the bipartite case $k = 2$, each LU equivalence class is uniquely represented by its ordered Schmidt coefficients. In the case $k \geq 3$ characterizing LU classes is a highly non-trivial task. For a characterization of LU classes in the $k$-qubit case, i.e. when each local systems are 2-dimensional, see [17].

Note that for any two representatives, $[|\psi\rangle] = [|\phi\rangle]$, of an element of $S_k$, the partial isometries witnessing this equivalence define $k$-step LOCC channels mapping one to the other and back; $|\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle \overset{\text{LOCC}}{\longrightarrow} |\psi\rangle$. In other words, states that are locally unitarily equivalent are also LOCC-equivalent. The following preorder is therefore well-defined:

$$[|\psi\rangle] \geq [|\phi\rangle] \text{ iff } |\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle.$$

By [36, Corollary 1], LOCC equivalence also implies local unitary equivalence. So the above preorder is in fact a partial order. This is not of importance for the theory to work, but still worth noting.

When $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ and $|\phi\rangle \in \mathcal{H}'_1 \otimes \cdots \otimes \mathcal{H}'_k$ we may take direct sum and tensor product to get new $k$-partite states

$$|\psi\rangle \oplus |\phi\rangle \in (\mathcal{H}_1 \oplus \mathcal{H}'_1) \otimes \cdots \otimes (\mathcal{H}_k \oplus \mathcal{H}'_k)$$

and

$$|\psi\rangle \otimes |\phi\rangle \in (\mathcal{H}_1 \otimes \mathcal{H}'_1) \otimes \cdots \otimes (\mathcal{H}_k \otimes \mathcal{H}'_k).$$

The direct sum, $|\psi\rangle \oplus |\phi\rangle$, is the sum of the images of $|\psi\rangle$ and $|\phi\rangle$ under the natural inclusions into $(\mathcal{H}_1 \oplus \mathcal{H}'_1) \otimes \cdots \otimes (\mathcal{H}_k \oplus \mathcal{H}'_k)$.

Both sum and product respect local unitary equivalence, turning $(S_k, \oplus, \otimes)$ into a semiring. We wish to apply
Theorem 1 to \((S_k, \otimes, \leq)\). For this purpose, what remains to be shown is that \((S_k, \otimes, \leq)\) is a preordered semiring and that conditions 1 and 2 of Theorem 1 are satisfied. We start out by showing that it is a preordered semiring. (7) is immediate, so we proceed to proving (6), which is done in Proposition 2.

We say that a state \(\rho \in B(H_1) \otimes \cdots \otimes B(H_4) \otimes \Diag(C^X)\) is conditionally pure if it can be written in the form
\[
\rho = \sum_{x \in \mathcal{X}} |\phi_x\rangle \langle \phi_x| \otimes |x\rangle \langle x| .
\] (13)
It is intuitively clear that any erasure of classical information can be deferred until the very end of a protocol. This means that in each step the classical register is updated in such a way that the previous values as well as any new measurement result can be read off from the new value (this is what we call remembering, see Definition 5), except for the last step, where the unnecessary information is erased. For completeness, we provide a proof of this fact in Proposition 1 below, within our framework. The proof is by induction, using the following lemma in the induction step.

**Lemma 1.** Let \(\Lambda = \Lambda_2 \circ \Lambda_1\) be a two-step LOCC channel. Then \(\Lambda = \Lambda'_2 \circ \Lambda'_1\), for some two-step LOCC protocol \((\Lambda'_2, \Lambda'_1)\) where \(\Lambda'_1\) is remembering.

**Proof.** Let \(A_r = B(H'_1) \otimes \cdots \otimes B(H'_1) \otimes \Diag(C^X)\) for \(r = 0, 1, 2\) be the three state spaces in question; \(A_0 \xrightarrow{\Lambda'_1} A_1 \xrightarrow{\Lambda'_2} A_2\). Let \(A_r\) be defined as in Definition 4 by \(f_r, g_r, r, \tau_r\) and \((K'_r)_{j \in J_r}\), for \(r = 1, 2\). We first enlarge the register, \(X'_1\) of \(A_1\), using \(\mathcal{X}\)
\[
J = \mathcal{X} = \{(j_1, j_2) \in J_1 \times J_2 | f_2(j_2) = g_1(j_1)\} \quad (14)
\]
and let \(A'_1 = B(H'_1) \otimes \cdots \otimes B(H'_1) \otimes \Diag(C^X)\). Define the remembering channel \(\Lambda'_1 : A_0 \rightarrow A'_1\) by the index map \(f'_1 : X \rightarrow \mathcal{X}_0\) given as \(f'_1 : (j_1, j_2) \mapsto f_1(j_1)\). The Kraus operators for \(\Lambda'_1\) are \((K'_1)_{x \in X}\), where \(K_{1(j_1,j_2)} = K_1^{j_1}\). Define \(\Lambda'_2\) in a similar manner: The Kraus operators \((K'_2)_{x \in X}\) are indexed over \(X\) with \(K_{1(j_1,j_2)} = K_2^{j_2}\) and applied via the index maps \(f'_2 = \text{id} : X \rightarrow X\) and \(g'_2 : (j_1, j_2) \mapsto g_2(j_2)\).

\[
A'_2 \circ \Lambda'_1(\rho) = \sum_{(j_1,j_2) \in \mathcal{X}} \left[ (K'_2)_{j_2} (K'_1)_{j_1} h_i \otimes |g_2(j_2)\rangle \langle f_1(j_1)| \right] \otimes |x\rangle \langle x| .
\]
\[
= \Lambda_2 \circ \Lambda_1(\rho) .
\] (15)

**Remark 4.** Note that for the construction in the proof of Lemma 1, if \(X' = X_2\) is a one-point set and \(A_1 = A_2\) and \(\Lambda_2 = I_{A_1}\) is the identity channel, then \(\Lambda'_2 = \Tr_{\Diag(C^X)}\) is just the partial trace of the register.

**Proposition 1.** Given a channel \(\Lambda\) for which the final space state has a one-point register, there exists an LOCC protocol \((\Lambda_n, \ldots, \Lambda_1)\) consisting of remembering one-step LOCC channels, such that
\[
\Lambda = \Tr_{\Diag(C^X)} \circ \Lambda_n \circ \cdots \circ \Lambda_1 .
\] (16)

Here \(X_i\) is the \(i\)th register and \(\Tr_{\Diag(C^X)}\) is the partial trace of the final register.

**Proof.** Let \(\Lambda\) be the composition of an \(n\)-step LOCC protocol and let \(\Lambda_n\) be the final state space. Then \(\Lambda = I_{A_n} \circ \Lambda\).

By applying Lemma 1 \(n\) times and by Remark 4
\[
\Lambda = \Tr_{\Diag(C^X)} \circ \Lambda_n \circ \cdots \circ \Lambda_1 ,
\] (17)
where \(\Lambda_i\) is a remembering one-step channel for \(i = 1, \ldots, n\).

**Proposition 2.** Let \(|\phi_1\rangle, |\phi_2\rangle\) and \(|\psi\rangle\) be some pure states, then
\[
|\phi_1\rangle \xrightarrow{\text{LOCC}} |\phi_2\rangle \quad \Rightarrow \quad |\phi_1\rangle \otimes |\psi\rangle \xrightarrow{\text{LOCC}} |\phi_2\rangle \otimes |\psi\rangle .
\]

**Proof.** Assume that \(|\phi_1\rangle \xrightarrow{\text{LOCC}} |\phi_2\rangle\) and let \(|\psi\rangle \in H_1 \otimes \cdots \otimes H_k\) be some pure state. By Proposition 1,
\[
|\phi_2\rangle |\psi\rangle = \Tr_{\Diag(C^X)} \circ \Lambda_n \circ \cdots \circ \Lambda_1 |\phi_1\rangle |\psi\rangle \quad (18)
\]
for some remembering protocol \((\Lambda_n, \ldots, \Lambda_1)\). This implies that
\[
\Lambda_n \circ \cdots \circ \Lambda_1 |\phi_1\rangle |\psi\rangle = \sum_{y \in \mathcal{X}_1} a_y |\phi_2\rangle |\phi_2\rangle \otimes |y\rangle \langle y| \quad (19)
\]
for some \(a_y \geq 0\) with \(\sum_y a_y = 1\). It suffices to show that there exists some LOCC channel \(\Lambda'\) such that
\[
\Lambda' |\phi_1\rangle |\psi\rangle = \sum_{y \in \mathcal{X}_1} a_y |\phi_2\rangle |\phi_2\rangle \otimes |y\rangle \langle y| . \quad (20)
\]
This is shown by induction on \(n\). For \(n = 0\) it is trivial. Assume that it is possible for \((n - 1)\)-step protocols. Let \((K_x)_{x \in \mathcal{X}_1}\) be the Kraus operators for \(\Lambda_1\) acting on system \(i\).

\[
\Lambda_1 |\phi_1\rangle |\psi\rangle = \sum_{x \in \mathcal{X}_1} |\phi_1\rangle |\phi_1\rangle \otimes |x\rangle \langle x| \quad (21)
\]
where \(\{|x\rangle\}_{x \in \mathcal{X}_1}\) is a partition of \(\mathcal{X}_1\). Let \(c_x = \sum_{y \in \mathcal{X}_1} a_y\). By the induction hypothesis there exist channels \((\Lambda'_x)_{x \in \mathcal{X}_1}\) such that
\[
\Lambda'_x \left[ |\phi_x\rangle \otimes \sqrt{c_x} |\psi\rangle |\phi_x\rangle \otimes \sqrt{c_x} |\psi\rangle \otimes |x\rangle \langle x| \right] = \sum_{y \in \mathcal{X}_1} a_y |\phi_2\rangle |\phi_2\rangle \otimes |y\rangle \langle y| . \quad (23)
\]
Define \(\Lambda'_1\) by the Kraus operators \(K'_x = K_x \otimes \sqrt{c_x} I\) for each \(x \in \mathcal{X}_1\), where \(I\) is the identity operator on \(H_i\). Then
\[
\Lambda'_1 |\phi_1\rangle |\psi\rangle = \sum_{x \in \mathcal{X}_1} |\phi_x\rangle \otimes \sqrt{c_x} |\psi\rangle |\phi_x\rangle \otimes \sqrt{c_x} |\psi\rangle \otimes |x\rangle \langle x| . \quad (24)
\]
Let \(\tilde{\Lambda}\) be the LOCC channel
\[
\tilde{\Lambda} : \sum_{x \in \mathcal{X}_1} \rho_x \otimes |x\rangle \langle x| \mapsto \sum_{x \in \mathcal{X}_1} \Lambda'_x \left[ \rho_x \otimes |x\rangle \langle x| \right] \quad (25)
\]
Now
\[
\tilde{\Lambda} \circ \Lambda' \left| \phi_1 \oplus \psi \right> \left< \phi_1 \oplus \psi \right| \\
= \tilde{\Lambda} \sum_{x \in X} \left| \phi_x \oplus \sqrt{c_x} \psi \right> \left< \phi_x \oplus \sqrt{c_x} \psi \right| \otimes |x\rangle \langle x|
\]
\[
= \sum_{x \in X} \sum_{y \in X} a_{xy} \left| \phi_2 \oplus \psi \right> \left< \phi_2 \oplus \psi \right| \otimes |y\rangle \langle y|
\]
\[
= \sum_{y \in Y} a_{y} \left| \phi_2 \oplus \psi \right> \left< \phi_2 \oplus \psi \right| \otimes |y\rangle \langle y|. \quad (26)
\]

By Remark 1 it follows that equation (6) holds for \((S_k, \oplus, \otimes, \leq)\), which is therefore a preordered semiring.

It remains to be shown that conditions 1 and 2 in Theorem 1 are satisfied. The multiplicative unit in \(S_k\) is represented by the pure state \([0 \ldots 0] \in \mathbb{C}^d\otimes k\) and the additive unit is represented by the zero-vector \(0 \in \mathbb{C}^d\otimes k\). \(S_k\) embeds into \(S_k\) in the following sense: An integer \(d \in \mathbb{N}\) is represented in \(S_k\) by the \(d\)-level, \(k\)-partite, unnormalized GHZ state
\[
|\text{GHZ}_d\rangle = \sum_{i=0}^{d-1} |i \ldots i\rangle \in (\mathbb{C}^d\otimes k). \quad (27)
\]
If \(d_1 \geq d_2\), then the one-step trace reducing LOCC channel, defined by the single Kraus operator \(K = \sum_{i=0}^{d_2-1} \frac{|i\rangle \langle i|}{\text{LOCC}}\) acting on the first system, witnesses \(|\text{GHZ}_{d_1}\rangle \rightarrow |\text{GHZ}_{d_2}\rangle\).

And since LOCC channels never increase the trace, we have \(|\text{GHZ}_{d_1}\rangle \otimes |\text{GHZ}_{d_2}\rangle \iff d_1 \geq d_2\), so condition 1 holds.

We proceed by proving that condition 2 holds:

**Proposition 3.** For any non-zero pure states \(|\psi\rangle\) and \(|\phi\rangle\), there is a \(d \in \mathbb{N}\) such that
\[
|\text{GHZ}_d\rangle \otimes |\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle. \quad (28)
\]

**Proof.** It is well known that a sufficiently large supply of GHZ states can be used to prepare any entangled state as follows: One party locally constructs the normalized \(|\phi\rangle\), the parties then convert GHZ states to EPR pairs [37], [38] and use quantum teleportation [39], [40, s. 6.5.3] to distribute the normalized version of \(|\phi\rangle\). Furthermore \(|\psi\rangle \overset{\text{LOCC}}{\longrightarrow} \|\psi\| |\text{GHZ}_1\rangle\).

So for sufficiently large \(d\)
\[
|\text{GHZ}_d\rangle \otimes |\psi\rangle \overset{\text{LOCC}}{\longrightarrow} \frac{1}{\|\psi\|} |\phi\rangle \otimes |\psi\rangle \overset{\text{LOCC}}{\longrightarrow} \|\psi\| |\text{GHZ}_1\rangle. \quad (29)
\]

To complete the proof, we adjust the norm with the help of additional unnormalized GHZ states. So one increases \(d\) to \(dn\) for large enough \(n\) and traces out the GHZ states not used for teleportation:
\[
|\text{GHZ}_{dn}\rangle \otimes |\psi\rangle = |\text{GHZ}_n\rangle \otimes |\text{GHZ}_d\rangle \otimes |\psi\rangle \overset{\text{LOCC}}{\longrightarrow} \|\psi\| |\text{GHZ}_n\rangle \otimes |\phi\rangle \overset{\|\psi\|}{\longrightarrow} \sqrt{n} \|\psi\| |\phi\rangle. \quad (30)
\]

And for \(n > \frac{\|\psi\|^2}{\|\phi\|^2}\)
\[
\sqrt{n} \|\psi\| |\phi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle. \quad (31)
\]

Theorem 1 now applies to \(S_k\).

**Theorem 2.** Let \(\Delta(S_k)\) be the set of order preserving semiring homomorphisms \(S_k \rightarrow \mathbb{R}^+\). Then
\[
[|\psi\rangle] \geq [|\phi\rangle] \iff \forall f \in \Delta(S_k) : f(|\psi\rangle) \geq f(|\phi\rangle). \quad (32)
\]
We call \(\Delta(S_k)\) the asymptotic LOCC spectrum.

Concretely \([|\psi\rangle] \geq [|\phi\rangle]\) means that
\[
|\text{GHZ}_2\rangle \otimes \alpha \rightarrow |\psi\rangle \otimes \alpha \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle \otimes \alpha, \quad (33)
\]
where \(|\text{GHZ}_2\rangle = |0 \ldots 0\rangle + |1 \ldots 1\rangle\) is the unnormalized two-level GHZ state. In other words; to extract \(n\) copies of \(|\phi\rangle\), we need \(n\) copies of \(|\psi\rangle\), a proportionally vanishing number of GHZ states and the success probability decays as \(2\alpha \log (\|\psi\|^2 - \|\phi\|^2) + o(n)\). Since we only need a proportionally vanishing amount of GHZ states we may, assuming that \(|\psi\rangle\) is globally entangled, i.e. that tracing out any number of subsystems always leaves a mixed state, extract these GHZ states from \(|\phi\rangle\) without further cost in the asymptotic limit. Indeed, one can show that when \(|\psi\rangle\) is globally entangled, \(|\psi\rangle \otimes k \overset{\text{LOCC}}{\longrightarrow} x |\text{GHZ}_2\rangle\) for some \(x > 0\), e.g. by extracting EPR-pairs (see [26, Lemma 4]) and using teleportation. That is, for any globally entangled \(|\psi\rangle\)
\[
E^*(r, \psi, \phi) = \sup \left\{ r \in \mathbb{R}^+ \mid 2^{(2r + o(n))} |\psi\rangle \otimes n \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle \otimes \tau n \right\}. \quad (34)
\]

This implies
\[
E^*(r, \psi, \phi) = \sup \left\{ r \in \mathbb{R}^+ \mid \forall f \in \Delta(S_k) : f(2^r |\psi\rangle) \geq f(|\phi\rangle)^r \right\}. \quad (35)
\]

**III. Spectral Points**

The goal of this section is to prove Theorem 3 below, which establishes a condition for a semiring homomorphism \(f : S_k \rightarrow \mathbb{R}^+\) to be monotone (i.e. order preserving) and hence defines a point in \(\Delta(S_k)\).

**Theorem 3.** Let \(f : S_k \rightarrow \mathbb{R}^+\) be a monotone semiring homomorphism. Then \(f\) is monotone if and only if there is an \(a \in [0, 1]\) such that \(f(\sqrt{a} |\psi\rangle) = p a^f\) for all \(p > 0\) and
\[
f(|\phi\rangle) \geq \left( f((\Pi_i)|\phi\rangle) \right)^{1/a} + f((1 - \Pi_i)|\phi\rangle)^{1\alpha}, \quad (36)
\]
for any \(|\phi\rangle\) \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k, i \in \{1, \ldots, k\}\) and orthogonal projection \(\Pi \in B(\mathcal{H}_i)\).

In the above theorem, and for the duration of this section, the case \(a = 0\) should be interpreted in the sense of \(a \rightarrow 0\). That is, (36) becomes
\[
f(|\phi\rangle) \geq \max \{ f((\Pi_i)|\phi\rangle), f((1 - \Pi_i)|\psi\rangle) \}. \quad (37)
\]

**Proposition 4.** Let \(f : S_k \rightarrow \mathbb{R}^+\) be a monotone semiring homomorphism. There is an \(\alpha \geq 0\) such that
\[
f(\sqrt{\alpha}|\phi\rangle) = p^\alpha f(|\phi\rangle) \quad (38)
\]
for each \(|\phi\rangle\) and each \(p > 0\).
Proof. Since \( p \mapsto f((\sqrt{p}|0\ldots0\rangle)) \) is multiplicative, nondecreasing, sends 0 to 0 and 1 to 1, it follows from the solution to the Cauchy functional equation that
\[
f((\sqrt{p}|0\ldots0\rangle)) = p^\alpha
\]
for all \( p > 0 \) and some \( \alpha \geq 0 \). Therefore
\[
f((\sqrt{p}|\phi\rangle) = f((\sqrt{p}|\phi\rangle\otimes|0\ldots0\rangle)) = f(|\phi\rangle) f((\sqrt{p}|0\ldots0\rangle)) = p^\alpha f(|\phi\rangle) f(|0\ldots0\rangle) = p^\alpha f(|\phi\rangle).
\]
For the proof of Theorem 3 we introduce the following extension of a monotone homomorphism \( f : S_k \to \mathbb{R}^+ \) to conditionally pure states. Given \( f \) such that \( f((\sqrt{p}|0\ldots0\rangle)) = p^\alpha \) for some \( \alpha > 0 \), we define
\[
f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) = \left(\sum_{x \in X'} f(|\phi_x\rangle)^{1/\alpha}\right)^\alpha
\]
and if \( \alpha = 0 \), we define
\[
f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) = \max_{x \in X'} f(|\phi_x\rangle).
\]

**Proposition 5.** The extension of \( f \) is multiplicative under tensor product.

Proof. For \( \alpha > 0 \)
\[
f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) \otimes \left(\sum_{y \in Y'} |\psi_y\rangle_y \otimes |y\rangle_y\right)
= f\left(\sum_{x \in X'} (|\phi_x\rangle_x \otimes |\psi_y\rangle_y) \otimes |x\rangle_x |y\rangle_y\right)
= \left(\sum_{x \in X'} f(|\phi_x\rangle)^{1/\alpha} f(|\psi_y\rangle)^{1/\alpha}\right)^\alpha
= \left(\sum_{x \in X'} f(|\phi_x\rangle)^{1/\alpha}\right)^\alpha \left(\sum_{y \in Y'} f(|\psi_y\rangle)^{1/\alpha}\right)^\alpha
= f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) f\left(\sum_{y \in Y'} |\psi_y\rangle_y \otimes |y\rangle_y\right).
\]

If \( \alpha = 0 \), then
\[
f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) \otimes \left(\sum_{y \in Y'} |\psi_y\rangle_y \otimes |y\rangle_y\right)
= \max_{x \in X'} f(|\phi_x\rangle) f(|\psi_y\rangle) = \max_{x \in X'} f(|\phi_x\rangle) \max_{y \in Y'} f(|\psi_y\rangle)
= f\left(\sum_{x \in X'} |\phi_x\rangle_x \otimes |x\rangle_x\right) f\left(\sum_{y \in Y'} |\psi_y\rangle_y \otimes |y\rangle_y\right).
\]

The following proposition is true for general LOCC channels, but since it is not needed in full generality, we shall prove it only for remembering LOCC channels.

**Proposition 6.** If \( f \) is monotone, then the extension is monotone under remembering LOCC channels.

Proof. First assume \( \alpha > 0 \) and start with the case where the initial state is pure:
\[
|\psi\rangle\langle\psi| \xrightarrow{\text{LOCC}} \sum_{i \in I} P(i) |\phi_i\rangle_i \langle i|_i.
\]

Here the \( |\phi_i\rangle \)'s are normalized and \( P : I \to \mathbb{R}^+ \) is a map.

Given \( n \in \mathbb{N} \) we say that a probability measure \( Q : I \to \mathbb{R}^+ \) is an \( n \)-type, if \( nQ(i) \in \mathbb{N} \) for each \( i \in I \). Given an \( n \)-type \( Q \), we say that a sequence in \( I^n \) is of type \( Q \), if \( i \) appears \( nQ(i) \) times. The type class \( T^n_Q \subset I^n \) is the set of sequences of type \( Q \). Given any \( n \)-type \( Q \):
\[
|\psi\rangle\langle\psi| \xrightarrow{\text{LOCC}} \sum_{i \in I} \bigwedge_{j=1}^n P(a_j) |\phi_{a_j}\rangle_{a_j} \otimes |a\rangle |a\rangle
\]
\[
= \sum_{a \in P} \prod_{j=1}^n P(a_j) |\phi_{a_j}\rangle_{a_j} \otimes |a\rangle |a\rangle.
\]

The last LOCC transformation is the projection onto the multi-indices of type \( Q \) followed by a unitary reshuffling of indices and a partial trace on the classical register. \( H(Q) = -\sum_i Q(i) \log Q(i) \) is the Shannon entropy of \( Q \) and \( D(Q||P) = \sum_i Q(i) \log \frac{Q(i)}{P(i)} \) is the relative entropy. Since the last expression in (42) is a pure state we can apply monotonicity of \( f \) on pure states to get
\[
f(|\psi\rangle)^n \geq f\left(\sum_{i \in I} P(i|Q^i) \right)^{\frac{n}{nQ(i)}} \prod_{i \in I} f(|\phi_i\rangle)^n Q(i).
\]

Since \( |T^n_Q| \geq 2^n H(Q) - |I| \log(n+1) \), [41, Lemma 2.3], this implies, by taking the \( n \)-th root;
\[
f(|\psi\rangle) \geq \left(2^{-D(Q||P) + \sum_i Q(i) \log f(|\phi_i\rangle)^{1/\alpha}}\right) 2^{-|I| \frac{\log(n+1)}{n}}.
\]
Let $Z = \sum_{i \in I} P(i) f(|\phi_i|)^{1/\alpha}$ and let $P_\phi$ be the probability distribution $P_\phi(i) = \frac{P(i) f(|\phi_i|)^{1/\alpha}}{Z}$. Then

$$-D(Q\|P) + \sum_i Q(i) \log f(|\phi_i|)^{1/\alpha} = -D(Q\|ZP_\phi)$$

$$= \log Z - D(Q\|P_\phi).$$

Using (45), (44) becomes

$$f(|\psi|) \geq 2^{(\log Z - D(Q\|P_\phi))^{\alpha} - \alpha l_{\log(n+1)}}.$$  (46)

For each $n \in \mathbb{N}$, let $Q_n$ be an $n$-type with supp $Q_n = \text{supp } P_\phi$ such that $\lim_n Q_n = P_\phi$. Then $D(Q_n\|P) \to D(P_\phi\|P_\phi) = 0$. Inserting $Q_n$ in (46) and letting $n \to \infty$ yields

$$f(|\psi|) \geq Z^\alpha = \left[\sum_{i \in I} P(i) f(|\phi_i|)^{1/\alpha}\right]^\alpha = \left(\sum_{i \in I} P(i)|\phi_i||i\rangle\langle i|\right)^\alpha.$$  (47)

showing that the extension is monotone under remembering one-step LOCC channels applied to pure states. We use this result to generalize to remembering one-step LOCC channels on conditionally pure states:

$$\sum_{j \in J} |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j| \xrightarrow{\text{LOCC}} \sum_{j \in J} \sum_{i \in I_j} |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |i\rangle\langle i|.$$  (48)

By restricting the protocol to only the Kraus operators acting on $|\psi_j\rangle$ one gets

$$|\psi_j\rangle\langle\psi_j| \xrightarrow{\text{LOCC}} \sum_{i \in I_j} |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |i\rangle\langle i|.$$  (49)

Therefore

$$f\left(\sum_{j \in J} |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j|\right)$$

$$= \left(\sum_{j \in J} f(|\psi_j|)\right)^{\alpha}$$

$$\geq \left(\sum_{j \in J} f(|\phi_{i,j}|)\right)^{\alpha}$$

$$= \left(\sum_{j \in J} \sum_{i \in I_j} |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |i\rangle\langle i|\right)^\alpha.$$  (50)

For the case $\alpha = 0$, note that

$$|\psi_j\rangle\langle\psi_j| \xrightarrow{\text{LOCC}} \sum_{i \in I_j} |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |i\rangle\langle i|$$

implies $|\psi_j\rangle \xrightarrow{\text{LOCC}} |\phi_{i,j}\rangle$ for each $i$, which by monotonicity of $f$ on pure states implies $f(|\psi_j|) \geq \max_i f(|\phi_{i,j}|)$. Therefore

$$f\left(\sum_{j \in J} |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j|\right)$$

$$= \max_j f(|\psi_j|)$$

$$\geq \max_{i,j} f(|\phi_{i,j}|)$$

$$= f\left(\sum_{j \in J} \sum_{i \in I_j} |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |i\rangle\langle i|\right).$$

\[\square\]

**Lemma 2.** Let $f : S_k \to \mathbb{R}^+$ be a semiring homomorphism with $f(\sqrt{p_0 \ldots p_n}) = p_i^\alpha$ for some $\alpha \in [0, 1]$ which satisfies (36) for any choice of pure state and orthogonal projection. Then

$$f(|\psi|) \geq f((A_i|\psi|)^{1/\alpha} + f((B_i|\psi|)^{1/\alpha})^{\alpha}$$  (51)

for any $|\psi| \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k$, $i \in \{1, \ldots, k\}$ and $A, B \in B(\mathcal{H}_i)$ with $A^*A + B^*B = I$.

**Proof.** Consider the operator $U = \begin{bmatrix} A & B \\ \sqrt{1 - A^*A - B^*B} \end{bmatrix}$.

$\mathcal{H}_i \to \mathcal{H}_i^3$. This is an isometry, so $f(|\psi|) = f(|\psi|)$, where $|\psi| = (U_i|\psi|)$. Let $\Pi : \mathcal{H}_i^3 \to \mathcal{H}_i$ be the projection onto the first summand. Then $\langle \Pi_i|\psi| \rangle = \langle A_i|\psi| \rangle$ and $\langle (I - \Pi_i)|\psi| \rangle = \langle (B_i|\psi| \rangle$, so

$$f(|\psi|) = f(|\psi|) \geq f((\Pi_i|\psi|)^{1/\alpha} + f((I - \Pi_i)|\psi|)^{1/\alpha})^{\alpha}$$

$$\geq f((A_i|\psi|)^{1/\alpha} + f((B_i|\psi|)^{1/\alpha})^{\alpha}.$$  (52)

\[\square\]

**Proof of Theorem 3.** Suppose $f$ is monotone, then by Proposition 4 there is an $\alpha \geq 0$ such that $f(\sqrt{p_0 \ldots p_n}) = p_i^\alpha f(|\psi|)$ for all $|\psi|$ and $p > 0$. Consider the extension of $f$ to conditionally pure states. Let $|\psi|, i$, and $\Pi$ be given in the statement of the theorem, then

$$|\psi\rangle\langle\psi| \xrightarrow{\text{LOCC}} (\Pi_i|\psi|\langle\phi|\langle\Pi_i \otimes |0\rangle\langle0|)$$

$$+ (I - \Pi_i|\psi|\langle\phi|\langle(I - \Pi_i) \otimes |1\rangle\langle1|, \quad (53)$$

so by monotonicity of the extension of $f$ we get

$$f(|\psi|) \geq f((\Pi_i|\psi|)\langle\phi|\langle\Pi_i \otimes |0\rangle\langle0|)$$

$$+ (I - \Pi_i|\psi|\langle\phi|\langle(I - \Pi_i) \otimes |1\rangle\langle1|^\alpha$$

When $|\psi| = |0 \ldots 0\rangle + |1 \ldots 1\rangle$, $\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $I - \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we get by (36)

$$2 = f(|\psi|) \geq f((I - \Pi_1|\psi|)\langle\phi|\langle(I - \Pi_1) \otimes |1\rangle\langle1|^\alpha$$

$$= f(|0 \ldots 0\rangle)\langle\phi|\langle1 \ldots 1\rangle)\langle\phi|\langle1 \ldots 1\rangle\langle\phi|\langle1 \ldots 1\rangle\langle\phi|^\alpha = 2^\alpha,$$  (54)

showing that $\alpha \leq 1$. This concludes the proof of the “only if” statement.

Conversely, suppose $f$ is a homomorphism satisfying equation (36). By Lemma 2, $f$ satisfies (51). Consider the extension of $f$ to conditionally pure states. By Lemma 1 we need only check that $f$ is monotone under remembering one-step channels and monotone when tracing out the register of a state of the form $\sum_i a_i|\phi\rangle\langle\phi| \otimes |i\rangle\langle i|$. $f$ is monotone under the latter, since

$$\left(\sum_i f(\sqrt{a_i}|\phi|)\right)^{\alpha} = \left(\sum a_i\right)^{\alpha} f(|\phi|)$$

$$= f\left(\sum_i \sqrt{a_i}|\phi|\right).$$  (55)
For monotonicity under remembering one-step channels, we first consider pure states, that is, we need to show:

\[ f(\langle \phi \rangle) \geq \left( \sum_{i=1}^{d} f\left(K_i \mid \phi \right) \right)^{1/\alpha} \tag{56} \]

whenever \( \sum_{i} K_i^* K_i \leq I \). Assume for the sake of induction that (56) is true for \( d - 1 \) and let \((K_i)_{i=1}^{d}\) be Kraus operators with \( \sum_{i} K_i^* K_i \leq I \). Set

\[ A = \sum_{i=1}^{d} K_i^* K_i, \quad B = K_d, \tag{57} \]

and

\[ \tilde{K}_i = K_i A^{-1} \quad i = 1, \ldots, d - 1. \tag{59} \]

Here \( A^{-1} \) denotes the Moore–Penrose pseudoinverse. Since

\[ \sum_{i=1}^{d-1} \tilde{K}_i^* \tilde{K}_i = A^{-1} \sum_{i=1}^{d-1} K_i^* K_i A^{-1} = A^{-1} A^2 A^{-1} \leq I, \tag{60} \]

we may apply the induction hypothesis to the operators \((\tilde{K}_i)_{i=1}^{d-1}\) and the vector \(A_j |\phi\) to obtain

\[
\begin{align*}
&f(\langle \phi \rangle) \\
&\geq \left( f(A_j |\phi\rangle)^{1/\alpha} + f(B_j |\phi\rangle)^{1/\alpha} \right)^\alpha \\
&\geq \left( \left( \sum_{i=1}^{d-1} f((\tilde{K} A)_j |\phi\rangle)^{1/\alpha} \right)^{1/\alpha} + f(B_j |\phi\rangle)^{1/\alpha} \right)^\alpha \\
&= \left( \sum_{i=1}^{d-1} f((K_i)_j |\phi\rangle)^{1/\alpha} + f((K_d)_j |\phi\rangle)^{1/\alpha} \right)^\alpha \\
&= \left( \sum_{i=1}^{d} f((K_i)_j |\phi\rangle)^{1/\alpha} \right)^\alpha, \tag{61} \end{align*}
\]

finishing the induction step.

Just like in the proof of Proposition 6, this extends to conditionally pure states. \( \square \)

Note that for \( \alpha = 0 \) an LOCC spectral point is in fact a point in the asymptotic spectrum of tensors in the sense of [28] and [33]. For \( \alpha = 1 \) there is just one spectral point, the norm squared:

**Proposition 7.** Let \( f : S_k \to \mathbb{R} \) be a monotone semiring homomorphism with \( f(\sqrt{p}|0\ldots0\rangle) = pf(|0\ldots0\rangle) \) for \( p > 0 \), then

\[ f(\langle \phi \rangle) = \langle \phi |\phi\rangle \].

**Proof.** Given \( |\phi\rangle \) of norm 1 we have \( \frac{1}{\sqrt{d}} |\text{GHZ}_d\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \xrightarrow{\text{LOCC}} |0\ldots0\rangle \) for sufficiently large \( d \). Furthermore

\[ f\left( \frac{1}{\sqrt{d}} |\text{GHZ}_d\rangle \right) = \frac{1}{d} \sum_{i=0}^{d} f(|i\ldots i\rangle) \]

\[ = \frac{1}{d} \sum_{i=0}^{d} f(|0\ldots0\rangle) = f(|0\ldots0\rangle), \tag{62} \]

showing that \( f(|\phi\rangle) = f(|0\ldots0\rangle) = 1 \). So \( f(\sqrt{p}|\phi\rangle) = p \) for \( p > 0 \).

In light of Proposition 4, provided that \( |\psi\rangle \) is globally entangled, we get a concrete formula for the extraction rate with converse error exponent \( r \):

\[
E^*(r, \psi, \phi) = \sup \left\{ r \in \mathbb{R}^+ : \forall f \in \Delta(S_k) : f(2^{r/2} |\psi\rangle) \geq f(|\phi\rangle)^r \right\} \\
= \sup \left\{ r \in \mathbb{R}^+ : \forall f \in \Delta(S_k) : \right\}
= \inf \left\{ f \in \Delta(S_k) : \frac{r a(f) + \log f(|\phi\rangle)}{-\log f(|\phi\rangle)} \right\}. \tag{63} \]

Here \( a(f) = \log f(\sqrt{2}|0\ldots0\rangle) \) is the \( \alpha \) from Theorem 3. For resources which are not globally entangled, the formula expresses the extraction rate, provided a proportionately vanishing amount of entanglement shared between each pair of parties.

**IV. Example: Bipartite States and \( \Delta(S_2) \)**

When \( k = 2 \), we may, by the Schmidt decomposition, write any element in \( S_2 \) as a finite direct sum of terms of the form \( \sqrt{p}|00\rangle \). Therefore any monotone semiring homomorphism, \( f \), is entirely determined by the value of \( a(f) \in [0, 1] \): For \( |\phi\rangle = |\psi\rangle = \sum_i \sqrt{p_i} |i\rangle \) a monotone semiring homomorphism, \( f \), must be given by

\[ f(\langle \phi \rangle) = \sum_i p_i^\alpha = \text{Tr}\left[ \left( \text{Tr}_2 |\phi\rangle \langle \phi| \right)^\alpha \right], \tag{64} \]

where \( \text{Tr}_2 \) is the partial trace of the second system.

The question to answer is then: For which \( \alpha \in [0, 1] \) does \( f_\alpha : |\phi\rangle \mapsto \text{Tr}_2 \left[ \left( \text{Tr}_2 |\phi\rangle \langle \phi| \right)^\alpha \right] \) satisfy equation (36). The answer is all of them.

**Theorem 4.** \( \Delta(S_2) = \{ f_\alpha | \alpha \in [0, 1] \} \) where

\[ f_\alpha : |\phi\rangle \mapsto \text{Tr}_2 \left[ \left( \text{Tr}_2 |\phi\rangle \langle \phi| \right)^\alpha \right]. \]

**Proof.** When \( \alpha = 0 \), \( f_\alpha(|\psi\rangle) \) is the Schmidt rank, which is monotone. Assume instead that \( \alpha \in (0, 1] \). Let \( |\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) and \( \Pi \in \mathbb{B}(\mathbb{C}^d) \) be an orthogonal projection. It suffices to verify (36) for projections acting on the first system. Let \( X \in \mathbb{B}(\mathbb{C}^d) \) be such that

\[ |\phi\rangle = \sum_{i=1}^{d} X |i\rangle \otimes |i\rangle. \]

Since the coefficients of \(|\phi\rangle\) are the square roots of the eigenvalues of \( \text{Tr}_2 |\phi\rangle \langle \phi| \) and

\[ \text{Tr}_2 |\phi\rangle \langle \phi| = \sum_{i=1}^{d} X |i\rangle \langle i| \sim X^* X, \]

(36) is equivalent to

\[ \left[ \text{Tr}(X^* X)^\alpha \right]^{1/\alpha} \geq \left[ \text{Tr}(\Pi X^* X \Pi)^\alpha \right]^{1/\alpha} \]

\[ + \left[ \text{Tr}((I - \Pi) X^* (I - \Pi))^\alpha \right]^{1/\alpha}. \]
Since \(YY^* \) and \(Y^*Y \) always have the same eigenvalues we may formulate it instead as
\[
[\text{Tr}(X^a X^a)]^{1/\alpha} \geq [\text{Tr}(X^a \Pi X^a)]^{1/\alpha} + [\text{Tr}(X^a (I - \Pi) X^a)]^{1/\alpha}.
\]

For \(\alpha = 1\) this inequality holds since \(X^a X = X^a \Pi X = X^a \Pi X + X^a (I - \Pi) X\). For \(\alpha \in (0, 1]\) it follows from [42, Proposition 3.7].  

Note that the topology on \(\Delta(S_2)\) as described in Theorem 1 is the Euclidean topology on \([0, 1]\), such that \(\Delta(S_2)\) can topologically be identified with the unit interval.

Since \(\Delta(S_2)\) is known we get the following formula for the asymptotic extraction rate between normalized states given converse error exponent \(r\). \(H_a(P) = \frac{1}{1 - \alpha} \log \sum_i p_i^\alpha\) is the \(\alpha\)-Rényi entropy.

\[
E^*(r, \psi_P, \psi_Q) = \inf_{\alpha \in [0, 1]} \frac{r\alpha + \log \sum_i p_i^\alpha}{\log \sum_i q_i^\alpha} = \inf_{\alpha \in [0, 1]} \frac{r\alpha}{1 - \alpha} + \frac{H_a(P)}{H_a(Q)}.
\]

(65)

When \(|\psi_Q \rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|\) is the maximally mixed state we retrieve the result [1, eq. (114)]:

\[
E^*(r, \psi_P, \psi_Q) = \inf_{\alpha \in [0, 1]} \frac{r\alpha + \log \sum_i p_i^\alpha}{\log \sum_i q_i^\alpha} = \inf_{\alpha \in [0, 1]} \frac{r\alpha}{1 - \alpha} + \frac{H_a(P)}{H_a(Q)}.
\]

(66)

V. Bipartite States and Success Probability Going to 1

So far we have considered optimal extraction rates where the success probability is allowed to go to 0. Setting \(r = 0\) in equation (65) gives the optimal extraction rate between the two states, where the success rate is allowed to go to 0, but not exponentially fast. This is a good candidate for the optimal extraction rate, when we demand that the success probability goes to 1. Indeed, as we show in Theorem 6, setting \(r = 0\) in (65) yields the extraction rate between the two states when demanding that success probability goes to 1. For this purpose we make use of the Nielsen’s Theorem on LOCC convertibility between bipartite states. The methods of this section are not related to the asymptotic spectrum.

**Theorem 5** (Nielsen, [2]). Let \(P = (p_i)_{i=1}^{d_1}\) and \(Q = (q_i)_{i=1}^{d_2}\) be two probability distributions, where \(p_i\) and \(q_i\) are ordered non-increasingly. Then
\[
|\psi_P \rangle \xrightarrow{\text{LOCC}} |\psi_Q \rangle \iff P \preceq Q.
\]

(67)

Here \(P \preceq Q\) means that \(Q\) majorizes \(P\), i.e.
\[
\sum_{i=1}^N p_i \leq \sum_{i=1}^N q_i
\]

(68)

for all \(N\).

**Proposition 8.** Let \(P = (p_i)_{i=1}^{d_1}\) and \(Q = (q_i)_{i=1}^{d_2}\) be two probability distributions which are ordered non-increasingly, and assume that
\[
\inf_{\alpha \in [0, 1]} \log f_a(|\psi_P \rangle) = \inf_{\alpha \in [0, 1]} \log \sum_i p_i^\alpha = \min_{\alpha \in [0, 1]} H_a(P) > 1.
\]

(69)

Then for sufficiently large \(n\)
\[
\sqrt{n} |\psi_P \rangle^{\otimes n} \xrightarrow{\text{LOCC}} |\psi_Q \rangle^{\otimes n}
\]

(70)

for some sequence of \(x_n \geq 1\) with \(x_n \to 1\). That is; one can asymptotically transform \(n\) copies of \(|\psi_P \rangle\) to \(n\) copies of \(|\phi \rangle\) with probability of success going to 1 as \(n \to \infty\).

**Proof.** Assume that we have proven the statement of Proposition 8 for all non-uniform \(P\) and \(Q\). Then for general \(P\) and \(Q\) satisfying (69), we let \(P'\) and \(Q'\) be non-uniform probability distributions with \(P \preceq P'\) and \(Q' \preceq Q\), such that \(\min_{\alpha \in [0, 1]} H_a(P) > \min_{\alpha \in [0, 1]} H_a(P') > 1\). Then by Theorem 5 and Proposition 8 for \(P'\) and \(Q'\), the statement follows for \(P\) and \(Q\). So without loss of generality we can assume that \(P\) and \(Q\) are non-uniform.

\[
|\psi_P \rangle^{\otimes n} = |\psi_P^{\otimes n} \rangle = \sum_{I \in [d]n} \sqrt{p_I} |II\rangle,
\]

(71)

where \(p_I = \prod_{i=1}^n p_{I_i}\). From (69) we conclude that \(H(P) > H(Q)\). Let \(V^* > -H(P)\) be chosen such that [32, Proposition 3.5] applies. Set \(t_n = 2^{V^*}\) and note that
\[
|\psi_P \rangle^{\otimes n} \xrightarrow{\text{LOCC}} \sum_{I \in [d]n} \min(\sqrt{p_I}, t_n) |II\rangle.
\]

(72)

Let \(x_n = \left(\sum_{I \in [d]n} \min(p_I, t_n)\right)^{-1}\) such that
\[
|\eta_n \rangle = \sqrt{x_n} \sum_{I \in [d]n} \min(\sqrt{p_I}, t_n) |II\rangle
\]

(73)

is normalized and
\[
\sqrt{x_n} |\psi_P \rangle^{\otimes n} \xrightarrow{\text{LOCC}} |\eta_n \rangle.
\]

(74)

The proof is complete when it is shown that \(x_n \to 1\) and \(|\eta_n \rangle \xrightarrow{\text{LOCC}} |\psi_Q \rangle^{\otimes n}\) for large \(n\). First note that
\[
\sum_{I \in [d]n} \min(p_I, t_n) \geq 1 - \sum_{I \in [d]n} p_{I_n} > 0
\]

(75)

By [32, Proposition 2.6, eq. (26)]
\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{I \in [d]n} p_{I_n} < 0,
\]

(76)

which implies
\[
\lim_{n \to \infty} \sum_{I \in [d]n} p_{I_n} = 0.
\]

(77)

So \(x_n \to 1\).

Let \(\tilde{P}_n = (\tilde{p}_{n,I})_{I \in [d]n} = (x_n \min(p_I, t_n))_{I \in [d]n}\) be the probability distribution, such that \(|\eta_n \rangle = |\psi_{\tilde{P}_n} \rangle\). By Theorem 5, it must be shown that \(\tilde{P}_n \preceq Q^{\otimes n}\) for large \(n\). Note that \(\tilde{P}_n \preceq Q^{\otimes n}\). For the remainder of this proof, given a probability distribution \(X\), let \(X^k(i)\), denote the \(i\)th largest
value of $X$. By [32, Proposition 3.5], we have for large $n$ and for all $N$ such that $P^{⊙n}\downarrow(N) \leq 2^{nV^+}$
\[
\sum_{i=1}^{N-1} \tilde{P}_n^\downarrow(i) \leq \sum_{i=1}^{N-1} P^{⊙n}\downarrow(i) \leq \sum_{i=1}^{N-1} Q^{⊙n}\downarrow(i). \quad (78)
\]
Let $N^*$ be the largest number such that $P^{⊙n}\downarrow(N^*) > 2^{nV^+}$. By (78)
\[
\sum_{i=1}^{N^*} \tilde{P}_n^\downarrow(i) \leq \sum_{i=1}^{N^*} Q^{⊙n}\downarrow(i), \quad (79)
\]
and since $\tilde{P}_n^\downarrow(i)$ is constant for $i \in [N^*]$, we have
\[
\sum_{i=1}^{N} \tilde{P}_n^\downarrow(i) \leq \sum_{i=1}^{N} Q^{⊙n}\downarrow(i) \quad (80)
\]
for all $N \leq N^*$.

**Corollary 1.** Given $n, m \in \mathbb{N}$ with
\[
\frac{m}{n} > \inf_{\alpha \in [0, 1]} \log f_\alpha(|\psi\rangle).
\]
Then
\[
\sqrt{x_k} |\psi\rangle^{⊙nk} \xrightarrow{\text{LOC}} |\phi\rangle^{⊙nk} \quad (82)
\]
for some sequence $x_k \rightarrow 1$.

Let $E(\psi, \phi)$ denote the optimal rate at which one can extract $|\phi\rangle$ exactly from $|\psi\rangle$ with chance of success going to 1 asymptotically. Combining Corollary 1 and equation (65) with $r = 0$, yields respectively a lower and upper bound on $E(\psi, \phi)$, which can be summed up as:

**Theorem 6.** Given probability distributions $P$ and $Q$
\[
E(\psi_P, \psi_Q) = \min_{\alpha \in [0, 1]} \frac{H_\alpha(P)}{H_\alpha(Q)}. \quad (83)
\]

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