The \log \log growth of channel capacity for nondispersive nonlinear optical fiber channel in intermediate power range. Extension of the model.

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In our previous paper [1] we considered the optical channel modelled by the nonlinear Schrödinger equation with zero dispersion and additive Gaussian noise. We found per-sample channel capacity for this model. In the present paper we extend per-sample channel model by introducing the initial signal dependence on time and the output signal detection procedure. The proposed model is a closer approximation of the realistic communications link than the per-sample model where there is no dependence of the initial signal on time. For the proposed model we found the correlators of the output signal both analytically and numerically. Using these correlators we built the conditional probability density function. Then we calculated an entropy of the output signal, a conditional entropy, and the mutual information. Maximizing the mutual information we found the optimal input signal distribution, channel capacity, and their dependence on the shape of the initial signal in the time domain for the intermediate power range.

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I. INTRODUCTION.

Nonlinear communication channels have received a lot of attention in last twenty years due to the development of fiber optical communication systems. In these communication systems the Kerr nonlinearity in the optical fiber becomes important when one increases the power of transmitted signal. The problem of capacity finding was considered analytically and numerically in a series of papers, see e.g. [1–11] and references therein. In spite of a lot of publications this problem has not been solved for the case of arbitrary Kerr nonlinearity and the second dispersion parameter of an optical fiber. The nondispersive model is much simpler than the case with nonzero dispersion. The nondispersive model is much simpler than the case with nonzero dispersion and additive Gaussian noise. We found per-dispens channel capacity for this model. In the present paper we extend per-dispens channel model by introducing the initial signal dependence on time and the output signal detection procedure. The proposed model is a closer approximation of the realistic communications link than the per-dispens model where there is no dependence of the initial signal on time. For the proposed model we found the correlators of the output signal both analytically and numerically. Using these correlators we built the conditional probability density function. Then we calculated an entropy of the output signal, a conditional entropy, and the mutual information. Maximizing the mutual information we found the optimal input signal distribution, channel capacity, and their dependence on the shape of the initial signal in the time domain for the intermediate power range.

The per-sample model assumes that the input signal does not depend on time. In realistic communication channel the transmitted signal does depend on time. In the recent paper [12] the influence of the receiver, signal, and noise bandwidth on the autocorrelation function and the capacity was discussed within the filter-and-sample model for the channel with zero dispersion. In our opinion, one of the important results of the paper [12] is understanding that the conditional PDF depends significantly on the properties of the receiver.

In this paper we consider the nondispersive channel in the intermediate power range in the case where the initial signal depends on time and has the bandwidth much less than the noise bandwidth. We also introduce the detection procedure which takes into account the time resolution characteristics of the detector and we demonstrate the influence of the detector and the noise bandwidth on statistical properties of the channel. Therefore this paper is the generalization of the previous results of Refs. [1, 2] for the per-sample model to the time-dependent signal.

The paper is organized in the following way. In Sec. II we present the model of the signal propagation, the input signal and the receiver model. In Sec. III we obtain the conditional probability density function for the introduced model. In Sec. IV we present numerical results for the correlators and compare these results with analytical ones. And in Sec. V we calculate the optimal input signal distribution and the channel capacity in the intermediate power range. In the Conclusion we discuss our results.
II. MODELS OF THE SIGNAL PROPAGATION AND DETECTION

In our model the propagation of the signal \( \psi(z, t) \) is described by the stochastic nonlinear Schrödinger equation (NLSE) with zero dispersion:

\[
\partial_z \psi - i \gamma |\psi|^2 \psi = \eta(z, t), \tag{1}
\]

where \( \gamma \) is the Kerr nonlinearity coefficient, the function \( \psi(z, t) \) obeys the input and output conditions: \( \psi(z = 0, t) = X(t) \) and \( \psi(z = L, t) = Y(t) \), respectively. \( L \) is the length of the signal propagation, and \( \eta(z, t) \) is an additive complex noise with zero mean \( \langle \eta(z, t) \rangle = 0 \), and the correlation function in the frequency domain:

\[
\langle \eta(z, \omega) \eta^*(z', \omega') \rangle = 2\pi Q \delta(\omega - \omega') \theta \left( \frac{W'}{2} - |\omega| \right) \times \delta(z - z'), \tag{2}
\]

where the bar means complex conjugation, and \( Q \) is a power of the noise per unit length and per unit frequency, \( \theta(\omega) \) is the Heaviside theta-function, \( \delta(\omega) \) is the Dirac delta-function, \( W' \) is the bandwidth of the noise. The noise \( \eta(z, \omega) \) is not white due to limited bandwidth. In the time domain this correlator has the form

\[
\langle \eta(z, t) \eta^*(z', t') \rangle = Q \frac{W'}{2\pi} \text{sinc} \left( \frac{W'(t - t')}{2} \right) \delta(z - z'), \tag{3}
\]

One can see that if the time difference \( t - t' = 2n\pi/W' \) then the correlator \( \langle \rangle \) is equal to zero, here \( n \) is integer. Thus we can solve equation \( \text{(1)} \) independently for parameters \( t_j = j\Delta \) for different integer \( j \), where \( \Delta = 2\pi/W' \) is the time grid spacing. Therefore instead of the continuous time model \( \text{(1)} \) we will consider the following discrete model:

\[
\partial_z \psi(z, t_j) - i\gamma |\psi(z, t_j)|^2 \psi(z, t_j) = \eta(z, t_j), \tag{4}
\]

for any time moment \( t_j \). It means that we obtain the set of independent time channels since the noise in these moments is not correlated. We present the input and output conditions in the discrete form as well: \( \psi(z = 0, t_j) = X(t_j) \) and \( \psi(z = L, t_j) = Y(t_j) \). Note that the solution \( \Phi(z, t_j) \) of the equation \( \text{(4)} \) with zero noise which obeys the input condition \( \Phi(z = 0, t_j) = X(t_j) \) has the form:

\[
\Phi(z, t_j) = X(t_j) e^{i\gamma z|X(t_j)|^2}. \tag{5}
\]

Below we assume that the frequency bandwidth \( W' \) of the noise is much broader than the frequency bandwidth \( W \) of the input signal \( X(t) \) and the frequency bandwidth \( \tilde{W} \) of the function \( \Phi(z = L, t) \).

In our model the input signal \( X(t) \) has the form:

\[
X(t) = \sum_{k = -N}^{N} C_k f(t - kT_0), \tag{6}
\]

where \( C_k \) are complex random coefficients with some probability density function \( P_X \{ \{ C \} \} \), \( \{ C \} = \{ C_{-N}, \ldots, C_N \} \); the pulse envelope \( f(t) \) is the real function which is normalized as \( \int_{-\infty}^{\infty} \frac{dt}{T_0} f^2(t) = 1 \). The pulse envelope \( f(t) \) has the following properties: the overlapping of the functions \( f(t - kT_0) \) and \( f(t - mT_0) \) for \( k \neq m \) is negligible: \( \int_{-\infty}^{\infty} df(t - kT_0)f(t - mT_0) \approx 0 \).

It means that the function \( f(t) \) has almost the finite support \([-T_0/2, T_0/2]\), and the input signal \( X(t) \) is defined on the interval \( T = (2N + 1)T_0 \). Thus the frequency support of the function \( X(t) \) is infinite. But we imply that \( \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \approx \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \), where \( X(\omega) \) is the Fourier transformation of \( X(t) \). The last relation means that \( T_0 W \gg 1 \).

In our consideration the average input signal power \( P \) is fixed:

\[
P = \int \left( \prod_{k = -N}^{N} d^2 C_k \right) P_X \{ \{ C \} \} \int_{-\infty}^{\infty} \frac{dt}{T} |X(t)|^2, \tag{7}
\]

where \( d^2 C_k = d\text{Re} C_k d\text{Im} C_k \), and the input signal probability density function \( P_X \{ \{ C \} \} \) is normalized as:

\[
\int \left( \prod_{k = -N}^{N} d^2 C_k \right) P_X \{ \{ C \} \} = 1. \tag{8}
\]

Using the properties of the function \( f(t - kT_0) \) we can rewrite equation \( \text{(7)} \):

\[
P = \int d^2 C_m P_X^{(m)} [C_m] |C_m|^2, \tag{9}
\]

where

\[
P_X^{(m)} [C_m] = \int \left( \prod_{k = -N, k \neq m}^{N} d^2 C_k \right) P_X \{ \{ C \} \}, \tag{10}
\]

and we imply that the distribution \( P_X^{(m)} [C_m] \) does not depend on \( m \).

Let us describe the output signal detection procedure. Our detector recovers the information which is carried by the coefficients \( \{ C_k \} \). First, the detector receives the signal \( \psi(z = L, t_j) \) at the discrete time moments \( t_j = j\Delta \), here \( j = -M, \ldots, M - 1 \), where \( M = T/(2\Delta) \gg N \). It means that the time resolution of the detector coincides with the time discretization \( \Delta \). Since \( \Delta \ll 1/W \) our detector can completely recover the input signal in noiseless case. Second, the detector removes the nonlinear phase to obtain the recovered input signal \( \tilde{X}(t) \) in the following form:

\[
\tilde{X}(t) = \psi(z = L, t_j) e^{-i\gamma L^2 \psi(z = L, t_j)^2}. \tag{11}
\]

And finally, using \( \tilde{X}(t) \) detector recovers the coefficients
\( \tilde{C}_k \) by projecting on the basis functions \( f(t - kT_0) \):
\[
\tilde{C}_k = \frac{1}{T_0} \int_{-\infty}^{\infty} dt f(t - kT_0) \tilde{X}(t)
\]
\[
\approx \frac{\Delta}{T_0} \sum_{j=-M}^{M-1} f(t_j - kT_0) \tilde{X}(t_j). \quad (12)
\]
One can check that in the case of zero noise \( \tilde{X}(t) = X(t) \) and \( \tilde{C}_k = C_k \).

### III. STATISTICS OF \( \tilde{C}_k \)

In the previous paper [1] we obtained the conditional probability function \( P[Y|X] \) for the case where input \( X \) and output \( Y \) signals do not depend on time (per-sample conditional PDF). In the previous section we extend our model [1] by including detector procedure and time dependence of the input signal \( X(t) \). Our goal is to obtain conditional probability function \( P[\{\tilde{C}\}|\{C\}] \), i.e., the probability to detect the set of coefficients \( \{\tilde{C}\} \) if the transmitted set is \( \{C\} \). Using the function \( P[\{\tilde{C}\}|\{C\}] \) we can calculate the probability density function \( P_{\text{out}}[\{\tilde{C}\}] \) as
\[
P_{\text{out}}[\{\tilde{C}\}] = \int \prod_{k=-N}^{N} d^2 C_k P[\{\tilde{C}\}|\{C\}] P_X[\{C\}]. \quad (13)
\]
Since the propagation of the signal in the different time moments \( t_j \) is independent, and noise is not correlated, the conditional probability function \( P[Y(t)|X(t)] \), i.e., the probability density to obtain the output signal \( Y(t) \) for the given input signal \( X(t) \), can be presented in the factorized form:
\[
P[Y(t)|X(t)] = \prod_{j=-M}^{M-1} P_j[Y_j|X_j], \quad (14)
\]
where \( X_j = X(t_j) \), \( Y_j = Y(t_j) \), and \( P_j[Y_j|X_j] \) is per-sample conditional PDF obtained in Ref. [1]. The function \( P_j[Y_j|X_j] \) in the leading and next-to-leading order in parameter \( \sqrt{Q} \) can be deduced from the results of Ref. [1], where we have to replace parameter \( Q \) by \( Q/\Delta \):

\[
P_j[Y_j|X_j] = \Delta \exp \left\{ -\frac{\Delta}{QL(1 + \mu^2_{(j)/3})} \left( 1 - \frac{\mu_{(j)}/\rho_{(j)} \mu_{(j)}^2}{15(1 + \mu^2_{(j)/3})^2} \mu_{(j)}(15 + \mu^2_{(j)/3})x_{(j)} - \mu_{(j)}(4\mu^4_{(j)} + 25\mu^2_{(j)/3} + 225) x_{(j)}^3 + \left( 23\mu^4_{(j)} + 255\mu^2_{(j)/3} - 90 \right) x_{(j)}^2 y_{(j)} + \mu_{(j)} \left( 20\mu^4_{(j)} + 117\mu^2_{(j)/3} - 45 \right) x_{(j)} y_{(j)}^2 - 3 \left( 5\mu^4_{(j)} + 33\mu^2_{(j)/3} + 30 \right) y_{(j)}^3 \right) \right\}, \quad (15)
\]

Here \( \rho_{(j)} = |X_j| \), \( X_j = \rho_{(j)} e^{i\phi_{(j)}} \), \( \mu_{(j)} = \gamma L \rho^2_{(j)} \), and \( x_{(j)} = \text{Re} \left[ Y_j e^{-i\phi_{(j)}} - i\mu_{(j)} - \rho_{(j)} \right] \), \( y_{(j)} = \text{Im} \left[ Y_j e^{-i\phi_{(j)}} - i\mu_{(j)} - \rho_{(j)} \right] \). The expression (15) was obtained in Ref. [1] on the condition that the average input signal power \( P \) lies in the intermediate power range:
\[
\frac{QL}{\Delta} \ll P \ll \Delta/ (QL^3\gamma^2) , \quad (16)
\]
where \( P = 2\pi \int_0^\infty d\rho \rho^3 P[\rho] \), \( P[\rho] \) is the distribution function of the quantity \( \rho \), see Ref. [1]. Therefore, our consideration is restricted by the condition (16). The factorization of \( P[Y(t)|X(t)] \) in the form (14) means that there are \( 2M \) independent “sub-channels”. Note that, the signal \( X(t) \) is completely defined by \( 2N + 1 \) coefficients \( C_k \), i.e., there are only \( 2N + 1 \) independent \( X_j \), but all \( 2M \) quantities \( Y_j \) are independent. However, our detector reduces the function \( Y(t) \) to the set of \( 2N + 1 \) coefficients \( \{\tilde{C}_k\} \) by the procedure (11) and (12). Therefore we have to reduce the function \( P[Y(t)|X(t)] \) to the function \( P[\{\tilde{C}_k\}|\{C_k\}] \) by integrating over \( 2M - 2N - 1 \) redundant degrees of freedom. Using the conditional PDF \( P[Y(t)|X(t)] \) in the form (14) one can calculate all correlators of the coefficients \( \tilde{C}_k \): \( \langle \tilde{C}_{k_1} \rangle \), \( \langle \tilde{C}_{k_1} \tilde{C}_{k_2} \rangle \), \( \langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_n} \rangle \). Here
\[
\langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_n} \rangle = \int \prod_{j=-M}^{M-1} d^2 Y_j P[Y(t)|X(t)] \tilde{C}_{k_1} \ldots \tilde{C}_{k_n} , \quad (17)
\]
where \( d^2 Y_j = d\text{Re} Y_j d\text{Im} Y_j \), and \( \tilde{C}_k \) is defined in equation (12), and in the discrete form it reads:
\[
\tilde{C}_k = \frac{\Delta}{T_0} \sum_{j=-M}^{M-1} f(t_j - kT_0) Y_j e^{-i\gamma L|Y_j|^2} . \quad (18)
\]
To recover the function $P[\{\tilde{C}_k\}|\{C_k\}]$ in the leading approximation in parameter $Q$ it is necessary to know only three correlators: $\langle \tilde{C}_k \rangle$, $\langle \tilde{C}_k \tilde{C}_{m} \rangle$, $\langle \tilde{C}_k \tilde{C}_{m} \rangle$. After substitution of Eqs. 13, 15, and 18 to Eq. 17 and performing the integration we obtain in the leading order in the noise parameter $Q$:

$$\langle \tilde{C}_k \rangle = C_k - \frac{iC_k Q L^2 \gamma}{\delta} \left( 1 - \frac{i\gamma L |C_k|^2 n_k}{3} \right), \quad (19)$$

$$\langle (\tilde{C}_m - \langle \tilde{C}_m \rangle) (\tilde{C}_n - \langle \tilde{C}_n \rangle) \rangle = -i\delta_{m,n} \frac{C^2_m Q L^2 \gamma}{T_0} \left( n_4 - \frac{2n_6}{3} \gamma L |C_m|^2 \right), \quad (20)$$

$$\langle (\tilde{C}_m - \langle \tilde{C}_m \rangle) (\tilde{C}_n - \langle \tilde{C}_n \rangle) \rangle = \delta_{m,n} \frac{Q L}{T_0} \left( 1 + \frac{2n_6}{3} \gamma L^2 |C_m|^4 \right), \quad (21)$$

where $\delta_{m,n}$ is Kronecker symbol and

$$n_s = \int_{-T_0/2}^{T_0/2} dt f^s(t), \quad (22)$$

$$P_m[\tilde{C}_m|C_m] \approx \frac{T_0}{\pi Q L \sqrt{1 + \xi^2 \mu_m^2/3}} \exp \left[ -T_0 \left( 1 + \frac{4n_6 \mu_m^2}{3} \right) x_m^2 + 2x_m y_m \mu_m n_4 + y_m^2 \right]. \quad (24)$$

Here we have introduced the notations:

$$x_m = \text{Re} \left[ e^{-i\phi_m} \left\{ \tilde{C}_m - C_m + \frac{iC_m \gamma Q L}{T_0} \left( 1 - \frac{i\gamma L |C_m|^2 n_k}{3} \right) \right\} \right], \quad (25)$$

$$y_m = \text{Im} \left[ e^{-i\phi_m} \left\{ \tilde{C}_m - C_m + \frac{iC_m \gamma Q L}{T_0} \left( 1 - \frac{i\gamma L |C_m|^2 n_k}{3} \right) \right\} \right], \quad (26)$$

$$\phi_m = \arg C_m, \quad \mu_m = \gamma L |C_m|^2, \quad \xi^2 = (4n_6 - 3n_4^2). \quad (27)$$

Note that for the first correlator $\langle (\tilde{C}_k - C_k) \rangle$ is proportional to $QL/\Delta = QLW/(2\pi)$, i.e., it is proportional to the total noise power. Whereas the correlators (20) and (21) are proportional to $QL/T_0$ and do not depend on the discretization parameter $\Delta$ only in leading order in parameter $Q$ and depend on the parameter $\Delta$ in higher order corrections in parameter $Q$, see Appendix A.

Using the correlators (19)–(21) we obtain the conditional PDF $P[\tilde{C}|C]$ in the leading order in parameter $Q$:

$$P[\tilde{C}|C] = \prod_{m=-N}^{N} P_m[\tilde{C}_m|C_m], \quad (23)$$

where

$$\int d^2 \tilde{C}_m P_m[\tilde{C}_m|C_m] = 1. \quad (29)$$

Eq. (23) means that we have $2N + 1$ independent information channels, and the channel corresponding to the time slot $m$ is described by the function $P_m[\tilde{C}_m|C_m]$. The function $P_m[\tilde{C}_m|C_m]$ obeys the normalization condition

$$\int d^2 \tilde{C}_m P_m[\tilde{C}_m|C_m] = 1. \quad (29)$$

Since there are $2N + 1$ independent channels, we can choose the input signal distribution $P_X[\{C_m\}]$ in the factorized form:

$$P_X[\{C\}] = \prod_{k=-N}^{N} P_X^{(k)}[C_k], \quad (30)$$

and we can consider only one channel, say $m$-th channel. For this channel we can calculate the probability distribution function of the coefficients $\tilde{C}_m$:

$$P^{(m)}_{\text{out}}[\tilde{C}_m] = \int d^2 \tilde{C}_m P_m[\tilde{C}_m|C_m] P_X^{(m)}[C_m]. \quad (31)$$

We imply that the function $P_X^{(m)}[C_m]$ is a smooth function that changes on a scale $|C_m|^2 \sim P$ which is much

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Note: The text above is a translation from a document that seems to be discussing a complex mathematical topic, possibly related to signal processing or information theory, given the use of notations like $\tilde{C}_k$, $C_k$, $Q$, $\gamma$, etc. The exact context and application of these concepts would depend on the full document, which is not provided here. The translation aims to convey the mathematical essence and structure of the content, focusing on the core equations and notations.
greater than $QL/\Delta$:

$$P \gg QL/\Delta \gg QL/T_0.$$  

(32)

In other words, the signal power is much greater than the noise power in the channel. The variation scale of the function $P_m(C_m|C_m)$ in the variable $C_m$ is of order of $\sqrt{QL/T_0}$ therefore we can use Laplace’s method \cite{10} for the calculation of the integral (31). Performing the integration in the leading order in parameter $Q$ we obtain

$$P_{\text{out}}^{(m)}[C_m] = P_{\text{in}}^{(m)}[C_m],$$

(33)

for details see Appendix C in Ref. \cite{1}. The result \cite{1} implicates that the statistics of the coefficients $C_m$ coincides with the statistics of the coefficients $\tilde{C}_m$.

IV. NUMERICAL CALCULATIONS OF THE CORRELATORS

In order to verify analytical results we performed numerical simulations of pulse propagation through nonlinear nondispersive optical fiber and calculated correlators \cite{19, 20, 21}. For these purposes we solve numerically Eq. (1) for fixed input signal $X(t)$ and for different realizations of the noise $\eta(z, t)$. Then we numerically perform the detection procedure described by Eqs. (11), (12). Finally, we average the coefficients $C_k$ and their quadratic combinations over noise realizations. In our simulations we use two numerical methods of the solution of Eq. (1): the split-step Fourier method and Runge-Kutta method of the fourth order. The results are presented in the following subsections. We have checked that the numerical results do not depend on the numerical method and these results are consistent with analytical ones for different realizations of the form $f(t)$ of the input pulse.

For numerical simulation we choose the following realistic channel parameters. The duration of one pulse is $T_0 = 10^{-18}$ sec; fiber length is equal to $L = 800$ km; Kerr nonlinearity parameter is $\gamma = 1.25 \text{ (km}\times\text{W})^{-1}$.

A. Split-step Fourier method

Equation (11) was integrated numerically over $z$ from 0 up to communication line length $L$ using split-step Fourier method \cite{19, 18}:

$$\psi(z + h, t) = \psi(z, t) \exp \left( i \gamma |\psi(z, t)|^2 h \right) + \tilde{F}_- \delta Q_h,$$

(34)

where $\psi(z, t)$ stands for numerical solution of (11), $h$ is a step size of $z$-mesh, $\tilde{F}_-$ denotes discrete inverse Fourier transform. The quantity $\delta Q_h$ stands for the noise addition per step $h$ which is made in frequency domain according to

$$\psi(z, \omega_j) \to \psi(z, \omega_j) + \sqrt{hQ/\tau} \frac{\eta_X + i\eta_Y}{\sqrt{2}},$$

(35)

where $j = 0, \ldots, 2M - 1$ stands for index of $\omega$-mesh, $2M$ is the number of $t$- and $\omega$-mesh points, $T$ is the total width of $t$-mesh, we choose $T = 64T_0$, see Eq. (25) below; $\eta_X$ and $\eta_Y$ are independent standard Gauss random numbers with zero mean and $\sigma^2 = 1$, additive noise level is $Q = 10^{-21}$ W/(km$\times$Hz).

The input signal for $z = 0$ has the form

$$\psi(z = 0, t) = X(t) = \sum_{k=1}^{64} C_k f(t - kT_0),$$

(36)

here we use the pulse envelope of the Gaussian form:

$$f(t) = \sqrt{\frac{T_0}{T_1\sqrt{\pi}}} \exp \left( -\frac{t^2}{2T_1^2} \right),$$

(37)

where $T_1 = T_0/10 = 10^{-11}$ sec stands for the characteristic time scale of the function $f(t)$. Pulse intersection is negligible. For such pulses coefficients $n_k$ defined in Eq. (22) are $n_4 = T_3/T_1 \approx 3.989$, $n_6 = (T_3/T_1)^2 \approx 18.38$, $n_8 = (T_3/T_1)^3 \approx 89.79$, $\xi \approx 5.08$.

In the numerical simulation we vary the average power $\frac{1}{64} \sum_{k=1}^{64} |C_k|^2$ of the input signal from 0.0177 mW up to 4.43 mW. It corresponds to the variation of the peak power $|C_k|^2 f^2(0)$ from 0.1 mW up to 25 mW.

Simulations are performed for different $t$-meshes (different grid spacing $\Delta$), i.e., for different noise bandwidths and fixed noise parameter $Q$. These meshes differ from each other by time grid spacing $\Delta = T/(2M)$: $\Delta_1 = 9.77 \times 10^{-14}$ sec, $\Delta_2 = 1.95 \times 10^{-13}$ sec and $\Delta_3 = 3.91 \times 10^{-13}$ sec. These grid spacings determine the width of the noise: $1/\Delta_1 = 10.26$ THz, $1/\Delta_2 = 5.12$ THz and $1/\Delta_3 = 2.56$ THz.

For each average power of the signal and each mesh step size $\Delta$ we simulate propagation of the signal for different realizations of the noise and then average obtained results for correlators over realizations. The total number of noise realizations for fixed $X(t)$, see Eq. (25), is determined by the necessary statistic relative error and is chosen as $5.0 \times 10^4$. This number of the realizations corresponds to the statistic relative error for correlators \cite{20} and \cite{21} on the level of 0.2% (since the total number of pulses is $64 \times 5.0 \times 10^4 = 3.2 \times 10^6$). We performed simulations on $z$-meshes with different number of points (100, 200, 400, 800) and checked out that the results do not depend on step size $h$.

In Figs. 15 we present the results for different time grid spacing $\Delta$ as a function of input signal power. In Fig. 15 the results are presented for the grid spacings $\Delta_1$ and $\Delta_3$ because the results for $\Delta_1$ and $\Delta_2$ almost coincide. One can see that numerical and analytical results are in a good agreement up to 3 mW at least. However the difference between numerical and analytical results for the smallest time grid spacing $\Delta_1$ is maximal. Decreasing of the parameter $\Delta$ means the increasing of the spectral bandwidth of the noise. This increasing results in
The difference between numerical and analytical approximation of large signal-to-noise ratio in the form (15). This form was derived in the approximation consisting of input signal power $|C_k|^2$ for $f(t)$ from Eq. (37). The noise power parameter is $Q = 10^{-21} \text{ W}/(\text{km} \times \text{Hz})$. Dashed dotted, dashed, and solid lines correspond to analytic representation (19) for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively. Circles, squares, and diamonds correspond to numerical results for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively.

Figure 3: The real part of the correlator (20) multiplied by $(-1)$ as a function of input signal power $|C_m|^2$ for $f(t)$ from Eq. (37). The noise power parameter is $Q = 10^{-21} \text{ W}/(\text{km} \times \text{Hz})$. Dashed dotted, dashed, and solid lines correspond to analytic representation (20) with NLO-corrections (A1) for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively. Circles, squares, and diamonds correspond to numerical results for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively.

Figure 4: The imaginary part of the correlator (20) multiplied by $(-10)$ as a function of input signal power $|C_m|^2$ for $f(t)$ from Eq. (37). The noise power parameter is $Q = 10^{-21} \text{ W}/(\text{km} \times \text{Hz})$. Dashed dotted, dashed, and solid lines correspond to analytic representation (20) with NLO-corrections (A1) for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively. Circles, squares, and diamonds correspond to numerical results for time grid spacings $\Delta_1$, $\Delta_2$, $\Delta_3$, respectively.

B. Runge-Kutta method

For the equation (11) the time $t$ is the incoming parameter. Thus the simulation consists in the solution of ordinary differential equation with various initial conditions determined by the real pulse shape $f(t)$, the amplitude $C_m$, and independent random noise functions $\eta(z,t)$. In the second method we used pulse envelopes of the form

$$f_n(t) = A_n \cos^n(\pi t/T_0)$$

for $n = 2,4$, $t \in [-T_0/2, T_0/2]$; $A_2 = \sqrt{83}$ and $A_4 = \sqrt{1283}$. We choose the time discretization parameter $\Delta = T_0/64$. The random noise was realized as the tele-
The noise power parameter reads as $Q = 10^{-21} \text{ W/(km Hz)}$. Dashed dotted, dashed, and solid lines correspond to analytic representation (21) with NLO-corrections (A3) for time grid spacings $\Delta_1, \Delta_3$, respectively. Circles and squares correspond to numerical results for time grid spacings $\Delta_1, \Delta_3$, respectively.

The analytical results in comparison with the numerical results are presented in Figs. 5-7 for different pulse shapes and average power. The numerical results are presented with statistic errors on the level of three standard deviations. One can see that numerical and analytical results are in a good agreement as well.

V. ENTROPIES AND MUTUAL INFORMATION

Now we proceed to the calculation of the output signal entropy

$$H[\tilde{C}_m] = - \int d^2 \tilde{C}_m P_{out}^{(m)}[\tilde{C}_m] \log P_{out}^{(m)}[\tilde{C}_m]$$

conditional entropy

$$H[\tilde{C}_m|C_m] = - \int d^2 \tilde{C}_m d^2 C_m P_{m}[\tilde{C}_m|C_m] \times P_{X}^{(m)}[C_m] \log P_{m}[\tilde{C}_m|C_m]$$
... and the mutual information
\[ I_{P_X}^{(m)} = H[\tilde{C}_m] - H[\tilde{C}_m|C_m]. \] (41)

Our calculations of the entropies [22], [23], and the mutual information [24] are similar to calculations of the entropies and the mutual information for per-sample channel, see Sec.III and Sec.IV of Ref. [1]. Therefore we will not repeat the similar calculations here and present only the final results:

\[ H[\tilde{C}_m] = H[C_m] = \int d^2 C_m P_X^{(m)}[C_m] \log P_X^{(m)}[C_m], \] (42)

\[ H[\tilde{C}_m|C_m] = 1 + \log \left[ \frac{\pi Q L}{T_0} \right] + \frac{1}{2} \int d^2 C_m P_X^{(m)}[C_m] \log \left[ 1 + \xi^2 \gamma^2 L^2 |C_m|^4 \right]. \] (43)

To calculate the optimal input signal distribution \( P_{opt}^{(m)}[C_m] \) we calculate the mutual information substituting Eqs. (42) and (43) to Eq. (41) then we variate the mutual information over \( P_X^{(m)}[C_m] \) with taking into account the normalization condition (5) and the fixed average power (9). Assuming the variation of the mutual information to be zero, we obtain the equation for the optimal input signal distribution \( P_{opt}^{(m)}[C_m] \). We solve the equation and obtain (for details of the similar calculations for per-sample channel see the Sec.III of Ref. [1]):

\[ P_{opt}^{(m)}[C_m] = N_0 \frac{e^{-\lambda_0 |C_m|^2}}{\sqrt{1 + \xi^2 \gamma^2 L^2 |C_m|^4}}. \] (44)

where parameters \( N_0 = N_0(P, \xi \gamma) \) and \( \lambda_0 = \lambda_0(P, \xi \gamma) \) are functions of the power \( P \) and modified nonlinearity parameter \( \xi \gamma \) by virtue of the relations (compare with Eqs. (46) and (47) of Ref. [1]):

\[ \int d^2 C_m P_{opt}^{(m)}[C_m] = \int_0^\infty d\rho \frac{2\pi N_0 \rho e^{-\lambda_0 \rho^2}}{\sqrt{1 + \xi^2 \gamma^2 L^2 \rho^4}} = 1. \] (45)
There is no simple analytical form for $800$ km; Gaussian shape (37) of $m$ sample model in Ref. [1] but with modification of the Kerr The result (47) is similar to that obtained for the per- linear channel at large signal-to-noise ratio, the second Eq. (47) corresponds to the Shannon’s result [19] for the information for small and large dimensionless nonlinearity parameter $\xi_\gamma L P$. For $\log \xi_\gamma L P \ll 1$ and $P \ll \Delta/(QL^3\xi^2\gamma^2)$. Here $B = 2e^{-\gamma_{E}}$, $\gamma_{E} \approx 0.5772$ is the Euler constant. Note that the asymptotics (49) is obtained with accuracy $1/\log^2(\xi_\gamma L P)$, see the Sec. IV of Ref. [1].

VI. CONCLUSION

In the present paper we use results obtained in Ref. [1] for per-sample model to calculate the informational characteristics of the channel where the input signal $X(t)$ depends on time, see Eq. (5). For this channel the information is carried by coefficients $C_k$. In the process of the signal propagation the input signal is transformed by the Kerr nonlinearity and the noise in the channel. To recover the transmitted information we introduce the detection procedure which removes the nonlinearity effects, see Eq. (11), and then projects $X(t)$ on the basis functions, see Eq. (12), to obtain the coefficients $\tilde{C}_k$. Using the conditional probability density function for per-sample model obtained in Ref. [1] we calculate the correlators of the coefficients $\tilde{C}_k$, see Eqs. (13)–(21). We demonstrate that these correlators depend on the noise bandwidth parameter $\Delta$. We also perform the numerical calculations of these correlators using two different methods and show that the numerical and analytical results are in agreement. Using obtained results for correlators we find the conditional probability density function $P\{\tilde{C}_k\}\{C_k\}$ in the leading and next-to-leading orders in parameter $QL/(\Delta P)$. Then we calculate the informational entropies and the mutual information for the channel in leading order in the parameter $QL/(T_0 P)$. We perform variation of the mutual information over the input signal distribution function and obtain the optimal input signal distribution function which maximizes the mutual information. We calculate the channel capacity in the leading order in parameter $QL/(T_0 P)$ and demonstrate that the capacity depends on the pulse envelope through one parameter $\xi$, see Eq. (28). The capacity grows as $\log P$ for sufficiently large average power $P$: $(\xi_\gamma Q L)^{-1} \ll P \ll \Delta/(QL^3\xi^2\gamma^2)$. Note that the same asymptotics was obtained for per-sample model, therefore taking into account the time dependance of the pulse envelope does not change the asymptotics behavior and modifies only the nonlinearity parameter $\gamma \to \xi_\gamma$. 

Figure 11: Shannon capacity and the mutual information $I_{p_{\text{opt}}(m)}$ for the parameters $Q = 10^{-21}$ W/(km×Hz); $L = 800$ km; $\gamma = 1.25$ (km×W)$^{-1}$; $T_0 = 10^{-10}$ sec, and for the Gaussian shape (37) of $f(t)$. The black dotted line corresponds to the Shannon limit $\log \left( \frac{\rho_n}{QL} \right)$, the black solid line corresponds to $I_{p_{\text{opt}}(m)}$, see Eq. (47), the black dashed dotted line corresponds to the asymptotics (49) for large $\gamma L P$. 

$$P = \int d^2 C_m P_{\text{opt}}(m) |C_m|^2 = \int_0^\infty \frac{2\pi N_0 \rho^3 e^{-\lambda_0 \rho^2}}{\sqrt{1 + \xi^2 \gamma^2 L^2 \rho^4 / 3}} \, d\rho.$$ (46)

The capacity of one channel $m$, i.e., the mutual information calculated using the optimal input signal distribution (44) reads

$$C = I_{p_{\text{opt}}(m)} = \log \left( \frac{PT_0}{\pi e Q L} \right) + \lambda_0 - \log [P N_0].$$ (47)

One can see that the first term in the right-hand side of Eq. (47) corresponds to the Shannon’s result (44) for the linear channel at large signal-to-noise ratio, the second and third terms are related with the nonlinearity impact. The result (47) is similar to that obtained for the per-sample model in Ref. [1] but with modification of the Kerr nonlinearity parameter $\gamma$ for per-sample model to parameter $\xi_\gamma$ for the present model, where $\xi = \sqrt{4n_0 - 3\lambda_0^2}$. There is no simple analytical form for $N_0$ and $\lambda_0$, see the Secs. III and IV of Ref. [1], therefore we present below the analytical results for the asymptotics of the mutual information for small and large dimensionless nonlinearity parameter $\xi_\gamma L P$ and the numerical calculations in Fig. (11).

Performing the substitution $\gamma \to \xi_\gamma$ in the results of the Sec.III and Sec IV of Ref. [1] we arrive at following asymptotics of the mutual information for small and large dimensionless nonlinearity parameter $\gamma L P$:

$$I_{p_{\text{opt}}(X)} \approx \log \left( \frac{PT_0}{QL} \right) - \frac{\xi^2 \gamma^2 L^2 P^2}{3},$$ (48)
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Appendix A: Correlators (20) and (21) with NLO corrections

Let us present the correlator (20) with next-to-leading (NLO) corrections in the noise power.

\[
\langle (\tilde{C}_m - \langle \tilde{C}_m \rangle) (\tilde{C}_n - \langle \tilde{C}_n \rangle) \rangle = \delta_{m,n}
\]

\[
\langle (\tilde{C}_m - C_m) (\tilde{C}_m - C_m) \rangle - \left( \frac{Q L^2 \gamma}{\Delta} \right)^2 C_m^2 \{ -1 + \frac{n_2}{9} \gamma L |C_m|^4 + i \frac{2n_4}{3} \gamma L |C_m|^2 \} = \delta_{m,n} \left( \frac{Q L^2 \gamma}{T_0} \left[ - \frac{2n_6}{3} \gamma |C_m|^2 + \frac{n_4}{n_2} 2 \gamma L C_m^4 + \frac{58n_6}{15} \gamma L |C_m|^2 \right] \right).
\]

(A1)

Here we have used the relation

\[
\langle (\tilde{C}_m - \langle \tilde{C}_m \rangle) (\tilde{C}_m - \langle \tilde{C}_m \rangle) \rangle = \langle (\tilde{C}_m - C_m) (\tilde{C}_m - C_m) \rangle - \langle \tilde{C}_m - C_m \rangle^2
\]

(A2)

the result (19) for \( \langle \tilde{C}_m - C_m \rangle \) and the calculation of \( \langle (\tilde{C}_m - C_m) (\tilde{C}_m - C_m) \rangle \) on the base of next-to-leading order result for \( P[Y|X] \) in Ref. [2].

In a similar manner it is easy to calculate the following corrections to correlator (21) from the results obtained in the leading order.

\[
\langle (\tilde{C}_m - \langle \tilde{C}_m \rangle) (\tilde{C}_n - \langle \tilde{C}_n \rangle) \rangle = \delta_{m,n} \left( \frac{Q L^2 \gamma}{T_0} \right)^2 \left[ \frac{n_4}{n_2} 2 \gamma L C_m^4 + \frac{58n_6}{15} \gamma L |C_m|^2 \right].
\]

(A3)

Note that these NLO results (A1) and (A3) contain the time discretization parameter \( \Delta \) related with the noise bandwidth \( W' = \frac{2 \pi}{\Delta} \). The relative importance of the NLO corrections in correlators (A1) and (A3) is governed by the dimensionless parameter \( \left( \frac{Q L^2 \gamma}{T_0} \right) \gamma L P \), i.e., it increases linearly for large and increasing \( P \). To demonstrate the importance of these corrections for our numerical results we present the Fig. (12) where for the noise power parameter \( Q = 5.94 \times 10^{-21} \) W/(km×Hz) the imaginary part of the leading order contribution (20) and the next-to-leading order corrections (A1) are presented together with the numerical results (Runge-Kutta method) for the envelope form \( f_2(t) = \sqrt{\frac{E}{2}} \cos^2(\pi t/T_0) \).

One can see that our calculations, i.e., Eq. (24) and formulae of the Sec. V based on the leading order results (10)–(21) are in a good agreement with the numerical calculations up to the average power of order of 4 mW for given noise and channel parameters.

Figure 12: The imaginary part of the correlator (20) multiplied by \((-10)\) as a function of input signal power \( |C_m|^2 \) for \( f_2(t) = \sqrt{\frac{E}{2}} \cos^2(\pi t/T_0) \) in the leading order (20), see black dashed-dotted line, and with the next-to-leading order corrections (A1), see solid line. The noise power parameter
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