Exact accelerated solitons by orbiting charged space debris

A. Mukherjee *, 1 S. P. Acharya †, 2 and M. S. Janaki ‡

1Department of Theoretical Physics and Quantum Technologies , NUST “MISIS”, Moscow, Russia
2Saha Institute of Nuclear Physics, Kolkata, India

We consider the nonlinear wave field induced by the orbiting charged space debris in the plasma environment generated at Low Earth Orbital (LEO) region. The generated nonlinear ion acoustic wave is shown to be governed by the forced Korteweg-de Vries (KdV) equation with the forcing function dependent on the charged space debris. When the forcing debris function depends self consistently on the nonlinear wave field, the forced KdV equation turns to be an integrable system for a specific condition. Exact accelerated solitons (velocity of the soliton changes where as amplitude remains constant) has been derived for the first time in such system and its integrability properties are explored. The amplitude of the solitonic source debris function varies with time and its shape changes over its propagation. Hence its detection by some remote means may pave a new way of measurement in this subject. For a more general condition, the approximate analytic soliton solution of the forced equation is also derived to show its features.

I. INTRODUCTION

Research in space plasma physics increased drastically after the launch of Sputnik-1 [1], the first artificial Earth satellite, on 4 October 1957 and still remains one of the important research fields in plasma science and technology. In recent times, the near-Earth space has become a laboratory for studying plasma physics. Scientists have sent rockets and satellites to the ionosphere [2] to gather scientific data because it becomes impossible to create on earth a laboratory as vast and variable as the ionosphere. In this regard, the research on the dynamics of space debris [3] has gained its importance.

The inactive space objects like dead satellites, destroyed spacecrafts, meteoroids and other inactive materials resulting from many natural phenomena are collectively called “space debris” [3]. Their sizes vary from microns to centimeters depending on the conditions [4]. These are found substantially in the Low Earth Orbital (LEO) and Geosynchronous Earth Orbital (GEO) region with high velocities like 10km/s [5]. As these objects are immersed in a plasma medium and exposed to many interplanetary radiations including solar radiations, they get charged due to photo emission, electron and ion collection, secondary electron emission [6] etc. This results in a significant effect on other active space assets [7, 8]. Due to their high velocities particularly in the GEO region, even micron-sized objects can harm the running spacecrafts a lot while centimeter-sized objects can also puncture it [4]. In the LEO region, the critical situation has already reached where the debris population continues to increase due to debris-debris collisions [9]. Thus, in order to get rid of these deteriorating effects, the dynamical study of charged space debris is becoming important. The active debris removal (ADR) becomes a challenging problem in the twenty-first century due to their enormous number in a complicated plasma environment particularly in the GEO region [10]. Recently many attempts have been made for the ADR in the LEO region like electrodynamics tethers for the LEO applications [11], electric sails [12], space debris pushers [13], robotic docking [14] etc. Various explorations on tracking the complex network of the space debris orbits have also been done by many space agencies in order to look for admissible orbits and prevent collisions between the spacecrafts and debris objects.

There also exist many indirect tracking methods for the debris problem. Recently, a new detection method of charged debris objects in LEO region has been discussed by Sen et al[15] where the generated nonlinear ion acoustic solitons are modelled by forced KdV equation. They have discussed few exact line solitons of the equation where the debris field also have the solitonic form. They carried out a numerical computation taking the forcing term to have Gaussian form and evaluated “precursor solitons”. Such solitons which are emitted periodically can be detected and give an indirect evidence of the existence of charged debris [15]. Many studies on the forced KdV equation have been done in plasmas considering different forms of source function[16–18]. In superthermal plasmas [17], plasmas with $k$ distributed particles [16] the solitary wave structure of forced KdV equation has been studied. In Thomas-Fermi plasmas, periodic, quasiperiodic and chaotic structures in the presence of source debris term are also studied [19].

All these studies till now discuss either numerical or perturbative soliton solutions. Exact solutions are discussed in [15] which are line solitons with constant amplitude and velocity. But in real physical situations, the velocity of both

* Electronic mail: abhikmukherjeesinp15@gmail.com
† Electronic mail: siba.acharya@saha.ac.in and siba.acharya39@gmail.com
‡ Electronic mail: ms.janaki@saha.ac.in
ion acoustic soliton and the source term may vary with time showing acceleration. Again, since source debris function induces such variations in nonlinear ion acoustic waves, hence they may be self consistently connected to each other. In this work we will discuss about the exact accelerated solitons induced self consistently by the shape changing debris solitons which is not studied till now as far as our knowledge goes. In that case the debris function depends on the ion acoustic wave in such a way that the forced evolution equation becomes an integrable system. Such ion acoustic solitons accelerate depending on the debris field whereas their amplitudes remain constant. On the other hand, both amplitude and velocity of the solitonic debris function change over time. Such variations in both the solitons may lead to a new way of detection of charged debris objects by optical means. For a more general condition, an approximate analytic solution is also derived to show its properties.

The paper is organized as follows. In section-II, the derivation of the evolution equation in presence of charged debris is shown. Exact accelerated solitons are derived from the forced yet integrable nonlinear evolution equation in section-III. In section-IV, we discuss the integrable properties of the evolution equation. A more general yet approximate soliton solution for small debris term is derived in section-V. Conclusive remarks and future direction of research are stated in section-VI followed by acknowledgements and bibliography.

II. DERIVATION OF THE NONLINEAR EVOLUTION EQUATION

In this context, we consider the finite amplitude nonlinear ion acoustic waves in the Low Earth Orbital (LEO) region due to the passage of charged debris. This region which we consider is populated by a low temperature low density plasma. We consider the propagation of ion acoustic waves, where the ion species is treated as the cold species i.e. the ion pressure is neglected and the electrons are assumed to obey a Boltzmann distribution. The basic normalized system of equations in this system in (1+1) dimensions are given by:

\[
\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0 \tag{1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \tag{2}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} - e\phi + n = S(x,t), \tag{3}
\]

where the following normalizations have been used:

\[x \rightarrow x/\lambda_d; \quad t \rightarrow \frac{C_s}{\lambda_d^2} t; \quad n \rightarrow \frac{n}{n_0}; \quad u \rightarrow \frac{u}{C_s}; \quad \phi \rightarrow \frac{e\phi}{k_BT_e}\]

where \(\lambda_d\) is the electron Debye length, \(C_s\) is the ion acoustic speed, \(k_B\) is the Boltzmann constant and \(n_0\) is the equilibrium ion density.

Equations (1), (2) and (3) represent ion continuity equation, ion momentum equation and Poisson equation respectively, where \(n, u, \phi\) denote the density and velocity of the ion species and the electrostatic potential respectively. In the RHS of equation (3), the term \(S(x,t)\) represents a charge density source arising from the charged debris.

We introduce two stretched variables as:

\[\xi = \epsilon^{1/2}(x - v_p t); \quad \tau = \epsilon^{3/2} t, \tag{4}\]

where, \(v_p\) is a constant and \(\epsilon\) is a small perturbation parameter which we will use later.

In this work, we consider the same system taken by Sen et al. [15] but our treatment is totally different. In their analytic treatment, they have considered the solution and the source term \(S\) to be of the form of a line soliton, where the amplitude and velocity are constants. But in this work we do not follow such restrictions, because in reality the amplitude or the velocity of both the debris source term and the soliton can vary with time. Also such variations of the source wave and nonlinear ion acoustic wave may be self consistently connected to each other since the charged debris can induce nonlinear excitations in the medium.

We derive briefly the evolution equation for nonlinear ion acoustic waves in the presence of source term \(S\) using reductive perturbation technique (RPT) [20]. Each dependent variable is expanded in series of a small parameter \(\epsilon\) as:

\[n = 1 + \epsilon n_1 + \epsilon^2 n_2 + O(\epsilon^3) \tag{5}\]
\[ u = \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3) \] (6)

\[ \phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + O(\epsilon^3) \] (7)

As a general approximation, we assume that the amplitude of the source function varies slowly with time. After scaling, we get

\[ S = \epsilon^2 a(\tau) S_1(\xi, \tau). \] (8)

Accordingly the differential operators are expressed in terms of the stretched variables as

\[ \frac{\partial}{\partial x} = \epsilon \frac{1}{2} \frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial t} = -\epsilon^1/2 v_p \frac{\partial}{\partial \xi} + \epsilon^3/2 \frac{\partial}{\partial \tau} \] (9)

Putting these expanded and stretched variables in equation (1) and collecting powers of \( \epsilon \) we get:

\[ O(\epsilon^{3/2}) : -v_p \frac{\partial n_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} = 0 \] (10)

\[ O(\epsilon^{5/2}) : -v_p \frac{\partial n_2}{\partial \xi} + \frac{\partial n_1}{\partial \tau} + \frac{\partial u_2}{\partial \xi} + \frac{\partial}{\partial \xi} (n_1 u_1) = 0 \] (11)

Similarly using (2), we get

\[ O(\epsilon^{3/2}) : -v_p \frac{\partial u_1}{\partial \xi} + \frac{\partial \phi_1}{\partial \xi} = 0, \] (12)

\[ O(\epsilon^{5/2}) : -v_p \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} = 0 \] (13)

From equation (3), we get:

\[ O(\epsilon) : -\phi_1 + n_1 = 0 \] (14)

\[ O(\epsilon^2) : \frac{\partial^2 \phi_1}{\partial \xi^2} - \phi_2 - \frac{\phi_1^2}{2} + n_2 = a(\tau) S_1 \] (15)

Now equations (10), (12) and (14) give

\[ n_1 = u_1 = \phi_1, \quad v_p^2 = 1 \] (16)

From equations (11) and (13), we get

\[ \frac{\partial n_2}{\partial \xi} = \left\{ \frac{\partial n_1}{\partial \tau} + \frac{\partial n_2}{\partial \xi} + \frac{\partial}{\partial \xi} (u_1^2) \right\} \] (17)

\[ \frac{\partial u_2}{\partial \xi} = \left\{ \frac{\partial n_1}{\partial \tau} + n_1 \frac{\partial n_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} \right\} \] (18)

Putting these relations into equation (15) and differentiating wrt \( \xi \) we get,

\[ n_1 \tau + n_1 n_1 \xi + \frac{1}{2} n_1 n_{1 \xi} \xi = F_\xi, \quad F = \frac{1}{2} a(\tau) S_1(\xi, \tau) \] (19)
which is nothing but the forced Korteweg-de Vries (KdV) equation, where subscripts denote partial derivatives. Changing to new set of variables $n_1 = 3U, \tau = -2T, \xi = X, F = -F_1$ we get a more convenient form:

$$U_T - 6UU_X - U_{XXX} = F_{1X}, \quad F_1 = \frac{1}{2}a(T)S_1(X,T) \tag{20}$$

Though (20) is not exactly solvable in general yet for a specific choice of dependent variables it miraculously turns into an integrable system. If we assume that the source function and the ion acoustic waves are self consistently related to each other such that $U \approx S_1$ then (20) can be reduced to

$$U_T - 6UU_X - U_{XXX} = -\frac{1}{2}a(T)U_X = G_X, \quad G = -\frac{1}{2}a(T)U \tag{21}$$

which is an integrable equation having rich analytic properties. Instead of constant velocity for line solitons, (21) admits accelerated solitons, controlled by the self consistent source debris function. We will investigate the novel features of equation (21) in the following sections.

### III. EXACT ACCELERATED SOLITONS

As we know, there is no general exact solution of equation (20), hence it can be solved numerically. Sen et al. [15] have discussed a few exact line solitons of the forced KdV(fKdV) equation taking the solitonic forcing function. Many studies in plasmas have been done considering different forms of source perturbations [16–18, 25]. In superthermal plasmas [17], plasmas with $\kappa$ distributed particles [16] the solitary wave structures of the fKdV equation have been explored. Periodic, quasiperiodic and chaotic structures in the presence of source term in Thomas Fermi plasmas are also investigated [19]. Using Gaussian type source functions, the fKdV equation is solved numerically in [15] and generation of pre-cursor solitons is discussed as well. Such solitons which are emitted periodically can be detected and give an indirect evidence of the existence of charged debris [15].

All these studies till now discuss either approximate or numerical soliton solutions. Exact solutions are discussed in [15] which are line solitons with constant amplitudes and velocities. But, in actual perturbative conditions, the velocities of both ion acoustic solitons and source solitons may vary with time showing acceleration. Again, since the source debris term induces such variations in the nonlinear ion acoustic waves hence they may be related to each other. In order to build a more relevant mathematical model of the system, we derive the exact accelerated solitons generated self consistently by charged space debris which is not studied till now as far as our knowledge goes. For more general treatment, the approximate analytic soliton solutions are derived in next section.

We start from equation (21) and show that it admits exact accelerated soliton solutions of constant amplitude. Introducing a new reference frame $\bar{X} = X + b(T), \bar{T} = T$ and choosing the function $b(T) = -\frac{1}{2}\int a(T)dT$ we get from equation (21)

$$U_{\bar{T}} - 6UU_{\bar{X}} - U_{\bar{X}\bar{X}} = 0, \tag{22}$$

which is the standard unforced KdV equation in this new moving frame which has the one soliton solution

$$U = A \sech^2\left\{\sqrt{A} \left(\frac{\bar{X}}{2} + \bar{T}\right)\right\}, \tag{23}$$

$A$ being constant. Now back boosting to the original co-ordinates $X, T$ we get the exact solution of (21) as

$$U = A \sech^2\left\{\sqrt{A} \left(X + AT - \frac{1}{2}\int a(T)dT\right)\right\}, \tag{24}$$

showing accelerating features due to the presence of the integral term in the argument. Choosing different functional forms of the amplitude of the source debris function $a(T)$, we get different types of accelerating solutions which are shown below. For $a(T) = \text{constant}$, the one soliton (24) converges to the constant amplitude and constant velocity line soliton which is discussed in [15].
(a)(i) 1 soliton for $a(T) = T$

(a)(ii) 1 soliton for $a(T) = 2 \cos (T/2)$

(b)(i) Source function $G = -\frac{1}{2} a(T) U = -\frac{1}{2} T U$

(b)(ii) Source function $G = -\frac{1}{2} a(T) U = -\cos (T/2) U$

FIG.1: The different functional forms of $a(T)$ causes the phase of the solitary wave to change which causes acceleration in $X, T$ plane, whereas the amplitude remains constant. One the other hand both amplitude and velocity of the source debris function $G$ varies with time. This change in amplitude of the charged space debris may be may be detected and can provide an indirect evidence it’s existence. The constant $A$ is taken $= 1$.

Thus, the exact accelerated solitons for ion acoustic wave in presence of charged debris term is derived for the first time as far as our knowledge goes. The velocity of the ion acoustic soliton changes over time showing accelerating/decelerating features whereas its amplitude remains constant. Whereas both the amplitude and velocity of the source debris term, which is also solitonic nature, changes over time showing shape changing effects.

The kind of exact soliton solutions which we have derived is also derived in a different context by Kundu[22, 23] where he has found an integrable nonholonomic deformation of the KdV equation. Forming AKNS type Lax matrices for this deformed KdV equation he found exact N-soliton solutions for the basic as well as the deforming field. Such kind of solitons exhibit accelerated motion controlled by the forcing function.

Previously, Sen et al. [15] have discussed few exact line solitons which move with constant amplitude and velocity
taking the solitonic forcing function. Few approximate analysis in plasmas have been done considering different nature of debris term [16–18] exploring solitary wave structures[17], [16]. In case of our exact result, the amplitude of the solitonic source perturbation varies with time which may pave a new way of detection of charged debris. Using the mathematical theory developed in [23, 23] we will show in next section that (21) is an integrable system having rich analytic beauties.

IV. INTEGRABLE PROPERTIES

In this section we will explore the integrable properties of (21) that is existence of Lax pair, infinite number of conserved quantities, higher soliton solutions etc. The mathematical formulation of such kind of systems are discussed in [22, 23] where the integrable nonholonomic deformation makes the forced KdV equation to be a completely integrable system. This type of integrable deformation is a relatively recent discovery which leads to the recently discovered integrable sixth order KdV[27, 28]. It is also shown in [23] that such deformations in KdV equation shows even a richer picture showing two-fold integrable hierarchy. Our equation (21) follows such integrability conditions where a self consistent solitonic forcing compensates the usual loss of energy inevitable in such system. The integrable mathematical properties of (21) are given below.

A. Lax pair

Existence of Lax pair operators is one of the necessary conditions of an integrable system. Our equation (21) contains the AKNS-type Lax pair $M(\lambda), N(\lambda)$ to formulate the linear problem [24] $\psi_X = M\psi, \psi_T = N\psi$, where the Lax operators satisfy the flatness condition

$$M_T - N_X + [M, N] = 0$$

and $\lambda$ is the spectral parameter[22]. Lax operators $M$ and $N = N_{\text{KdV}} + N_D$ are given by,

$$M = \begin{bmatrix} i\lambda & iU \\ i & -i\lambda \end{bmatrix}, \quad N_{\text{KdV}} = \begin{bmatrix} (-4i\lambda^3 + 2iU\lambda + U_X) & (-4iU\lambda^2 - 2U_X\lambda + iU_{XX} + 2iU^2) \\ (-4i\lambda^2 + 2U) & (4i\lambda^3 - 2iU\lambda - U_X) \end{bmatrix}$$

$$N_D = \left(\frac{1}{2\lambda}\right) \begin{bmatrix} i\frac{G_X}{G} + iG \\ i\frac{G_X}{G} - iG \end{bmatrix} \begin{bmatrix} G_X - \frac{G}{2} \\ -G_X - \frac{G}{2} \end{bmatrix}, \quad e_X = iUG_X.$$  

It can be checked that if $M$ and $N$ matrices are inserted in (25) then it will give (21) along with a nonholonomic constraint $G_{XXX} + 4UG_X + 2U_X G = 0$, which has to be satisfied by the wave field $U$ and forcing function $G$. Details of such integrable nonholonomic deformations are discussed in [22, 23].

B. Conserved quantities

The local conservation law of a (1+1) dimensional partial differential equation takes the form

$$\frac{\partial \rho}{\partial T} + \frac{\partial J}{\partial X} = 0,$$

which has to be satisfied by all solutions. The quantities $\rho(X, T)$ and $J(X, T)$ are called local conserved density and current density respectively. We will show that (21) has infinitely many conserved quantities $C_n$ which is a signature of integrable system.

From (21) we get the first conservation law which is nothing but conservation of mass where

$$\rho_1 = U, \quad J_1 = -U_{XX} - 3U^2 + a(T)U.$$  

Similarly, the second conservation law (conservation of momentum) is can be derived with

$$\rho_2 = U^2, \quad J_2 = -4U^3 - 2UU_{XX} + U_X^2 + a(T)U^2.$$  

Similarly we can derive infinite number of conserved densities from (21) showing integrable features. We can see from the above analysis that the conserved densities remain the same as in case of unforced KdV equation whereas the
current densities explicitly contain the forcing debris function. One important thing must be stated from Liouville integrability[24] that the conserved quantities $C_n, n = 1, 2, \ldots, \text{must commute i.e.,} \{C_m, C_n\} = 0$, as a necessary condition. From (28), we can see that if the wave profile $U$ and its derivatives vanish at space infinity, we can evaluate conserved quantities as

$$C_1 = \int U dX, \quad C_2 = \int U^2 dX, \quad C_3 = \int (4U^3 - 2U^2_X) dX,$$

and so on. Here $C_3$ is the Hamiltonian of the evolution equation (21).

A more mathematically detailed way to find conserved quantities of integrable equations can be found from Inverse scattering transform analysis. One can systematically evaluate them using the space Lax operator in Ricatti equation which is discussed in detail in [24].

C. Higher soliton solutions

Being an integrable system, equation (21) can be solved to produce exact higher soliton solutions via many analytic methods like Inverse scattering transform, Hirota bilinearization technique etc. In Section-III, we discussed about one soliton solution (24) which shows acceleration instead of constant velocity, due to presence of the debris function $a(T)$ in its argument. In the same way we can find the exact two soliton solution given by

$$U = \frac{(k_2^2 - k_1^2)}{2} \left[ k_2 \text{coth}^2(\eta_2 / 2) + k_1^2 \text{sech}^2(\eta_1 / 2) \right],$$

$$\eta_1 = k_1 (X - \frac{1}{2} \int a(T) dT) + \omega_1 T + \eta_1^0,$$

$$\eta_2 = k_2 (X - \frac{1}{2} \int a(T) dT) + \omega_2 T + \eta_2^0,$$

where $k_{1,2}, \omega_{1,2}, \eta_{1,2}^0$ are constant parameters. In Figure-2, we show how the structure of two soliton of (21) changes (showing acceleration) due to the presence of $a(T)$ whereas the amplitude remains constant. In the same way, we can find recursively the $N$ soliton solution of (21) which we do not show here due to notational complexity.

![Figure 2](image1.png)

(b)(i) Two soliton when $a(T) = 0$ \hspace{1cm} (b)(ii) Two soliton when $a(T) = T$

FIG.2: The phase of two soliton $U$ changes due to $a(T)$ showing acceleration compared to two soliton without forcing, whereas the amplitude remains constant. Constant parameters are taken as $k_{1,2} = 2.2, 3.2, \omega_{1,2} = 1.9, 1.1, \eta_{1,2} = 0.5, 0.7$.

It can also be proved that our integrable equation (21) has infinitely many generalized symmetries and a recursion operator which are the signatures of integrability.
V. APPROXIMATE SOLITARY WAVE SOLUTION

In earlier sections we have discussed about the integrable equation (21) showing an exact accelerated soliton. Such kind of soliton is obtained when the source debris function is related self consistently with the ion acoustic wave. It is quite surprising to see that though there exists a forcing function by the source debris object in the forced KdV equation (21) yet the equation retains its integrable nature. This is possible because the source debris function is self consistently related to the ion acoustic wave. The charged debris function is determined by the dynamics of the wave \( U \), which in turn is forced self-consistently by the forcing field.

On the other hand, if the debris term depends differently on the ion acoustic soliton then the system looses its integrability, exactness. Nevertheless, we can get approximate soliton solution for some cases. We will discuss such kind solutions in this section.

We know in absence of any forcing term, KdV equation is a completely integrable system having infinite number of conserved quantities. If we consider the momentum conservation law of KdV equation it will look like,

\[
\frac{dI}{dT} = 0, \quad I = \int_{-\infty}^{\infty} U^2 \, dX
\]  

(35)

For our general nonlinear evolution equation (20) the dynamical equation of \( I \) results

\[
\frac{dI}{dT} = -\int_{-\infty}^{\infty} a(T)S_1 X \, U \, dX, \quad I = \int_{-\infty}^{\infty} U^2 \, dX.
\]  

(36)

If we assume a small debris function \( S_1 \) which depends on the acoustic wave in such a way that \( S_1 = f(U) \) then we can get an approximate solution the amplitude of the solitary wave following[26] which varies very slowly with time. As an example, we assume \( S_1X = \delta U \), where \( \delta \) is a small parameter. We assume the solution as

\[
U = \frac{3A(T)}{2} \text{Sech}^2\left\{\frac{\sqrt{A(T)}}{2} (X - \frac{A(T)}{2} T)\right\},
\]  

(37)

where the amplitude \( A(T) \) varies slowly with time. Using (37) in (36) and solving we get,

\[
A(T) = A(0) \exp \left[ -\frac{2\delta}{3} \int a(T) dT \right]
\]  

(38)

Thus amplitude of the solitary wave varies slowly with time depending on the small parameter \( \delta \). In the following figures we see such variations for a given function \( a(T) \).

![Figure 3](image.png)

**FIG.3:** The variation of amplitude (figure (a)) and the wave profile (figure (b)) for the choice of \( a(T) = -\cos(T) \), \( A(0) = 1, \delta = 0.1 \). Here the amplitude and velocity of the solitary wave changes very slowly depending on the debris parameter \( \delta \).
Thus we get an approximate solitary wave solution for the ion acoustic wave with small debris term. For different functional dependence of the debris function $S_1$ on the wave profile $U$, we get different variation law of the amplitude $A(T)$ of the solitary wave. Here both amplitude and velocity of both the soliton and the debris function varies over time. Hence we can call it an approximate accelerated solitary wave induced by wave dependent debris field. Though the treatment is more general but the solution is approximate. On the other hand the exact accelerated soliton solution given in Section-III has more novel features which is the outcome of integrable nonholonomic deformation of forced KdV equation\[22, 23\].

Thus throughout our whole analysis, we have discussed about accelerated solitons (exact as well as approximate) that have different features compared to line solitons. In actual space plasma environment where every physical parameter changes, the localized ion density pattern also changes. To model this physical system, it is inaccurate to use line solitons having constant amplitude and velocity. This is because the localized density structures may bend, twist or turn in $X-T$ plane depending on physical circumstances. In this regard our exact as well as approximate solutions may be useful for building more accurate mathematical model.

**VI. CONCLUSIONS**

We can conclude as, we have obtained the evolution equation for the nonlinear wave field induced by the orbiting charged space debris in the plasma environment generated at Low Earth Orbital (LEO) region. The generated nonlinear ion acoustic wave is shown to be governed by the forced Korteweg-de Vries (KdV) equation with the forcing function dependent on the charged space debris. In earlier scientific literatures in this field, either numerical or perturbative soliton solutions have been discussed. Exact solutions are discussed in [15] which are line solitons with constant amplitude and velocity. But in real physical conditions where every physical parameter changes, the localized ion density pattern also changes. To model this physical system, it is inaccurate to use line solitons having constant amplitude and velocity. This is because the localized density structures may bend, twist or turn in $X-T$ plane depending on physical circumstances. Hence a solitonic form with variable parameters is necessary to model such system. For a specific self consistent condition between the ion acoustic wave and charged debris, the evolution equation turns to be an integrable system. As is well known that any forcing function usually spoils the integrability forbidding general analytic solution. However, our equation (21) retains its integrability in spite of the presence forcing function. This is because of the fact that a self consistent solitonic forcing compensates the usual loss of energy inevitable in such system. The evolution equation (21) is solved to yield exact accelerated soliton. The velocity of the ion acoustic soliton changes over time showing accelerating/decelerating features whereas its amplitude remains constant. Whereas both the amplitude and velocity of the solitonic debris term changes over time showing shape changing effects which may pave a new way of detection of charged debris. For a more general condition an approximate solitary wave solution is found where the amplitude varies slowly with time. Hence, our exact as well as approximate results may be useful for building more accurate mathematical model in this subject.

**VII. ACKNOWLEDGMENTS**

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