Exact solutions to the family of Fisher’s reaction-diffusion equation using Elzaki homotopy transformation perturbation method

Adedapo C. Loyinmi1 | Timilehin K. Akinfe2

1Department of Mathematics, Tai Solarin University of Education, Ijagun, Ijebu Ode, Nigeria
2Department of Mathematics, Tai Solarin University of Education, Ijagun, Ijebu Ode, Nigeria

Correspondence
Adedapo C. Loyinmi, Department of Mathematics, Tai Solarin University of Education, Ijagun, Ijebu Ode, Ogun State, Nigeria.
Email: loyinmiac@tasued.edu.ng

In this research work, we propose an unprecedented hybrid algorithm which involves the coupling of a new integral transform, namely, the Elzaki transform and the well-known homotopy perturbation method called the Elzaki homotopy transformation perturbation method (EHTPM) to solve for the exact solution of three distinct types of Fisher’s equation, namely, the Fisher’s equation of two cases, the sixth-order Fisher’s equation and the nonlinear diffusion equation of the Fisher type. These equations are prominent in mathematical biology and highly applicable in genetic propagation, population dynamics, stochastic processes, combustion theory, as well as a prototype model for a spreading flame, and so on. The efficacy and authenticity of this method was established via convergence and error analysis, and shows that the solutions obtained from EHTPM are unique and convergent. The results of EHTPM when compared with the exact results, homotopy results, and results from the other existing literature via table of comparison, 3D plots, and the convergence plots validates that EHTPM is an efficient and reliable tool of providing exact solutions to a wider class of nonlinear partial differential equations in a simple and straightforward manner, with no discretization, linearization, computation of Adomian polynomials, and devoid of errors.

KEYWORDS
3D plots, convergence plots, Elzaki transform, genetic propagation, genetic theory, homotopy perturbation, multivariate series, nonlinear partial differential equations, reaction-diffusion equations, table of comparison exact results

1 INTRODUCTION

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.1
A partial differential equation (PDE) is an equation, which consists of a dependent variable (the unknown function) and several independent variables. PDEs come to play when there is more than one independent variable to be considered in a system. These PDEs could be linear or nonlinear as the case may be. An equation is said to linear its unknown function and derivatives are linear, and nonlinear if otherwise. The linear and nonlinear PDEs occur in many applications and play a prominent role in the field of engineering, and applied sciences, in which real-life processes and phenomenon are described by it.

As a result of the complexity of nature, virtually all processes and phenomenon in sciences and engineering are inherently nonlinear of which they are being described by nonlinear partial differential equation (NPDE) as there are several conditions and parameters to be considered in the system.

Studies on NPDEs have grown very wide in virtually all fields of science and engineering. However, consistent research being carried out by mathematicians and researchers in numerous fields of science and engineering still yields new useful results.

NPDEs have been widely studied by numerous researchers over the years and has become ubiquitous in nature, they can be classified into integrable and nonintegrable.

The integrable equations are those whose behavior are determined by their initial conditions and as a matter of fact can be solved by integrating them from those initial conditions. In this case, solutions are sometimes a sort of superposition of solutions. However, some nonlinear PDE are integrable after some symbolic transformations. These equations have numerous exact solutions being constructed to them, however, still needs the attention of mathematicians as there is no single best method for a model/problem.

The preference of a method to another lies in the rapid convergence to exact results, computational stress, and simplicity of the method. However, some methods can perform better on some models than others due to the nonlinearity of the model and the radius of convergence of such method. Bildik, Rajarama, Chakraverty, and so on buttressed this based on their convergence analysis on some iterative methods.

Amongst these integrable equations are the Benjamin-Ono equation relevant in internal waves in deep water, nonlinear Schrödinger equation applicable in optics and water waves, Kadomtsev-Petviashvili equation applied in shallow water waves, Korteweg-de-Vries equation applicable in shallow waves models, the Sine-Gordon equation applicable in solitons and quantum field theory, the nonlinear fifth-order Kaup-Kupershmidt equation, Davey Stewartson equation to mention a few.

On the contrary, nonintegrable equations, namely, the Ginzburg-Landau equation applied in the theory of conductivity, Fisher’s equation applied in genetic propagation, Burger-Huxley equation applicable in advection-diffusion models, and so on require special attention, methods and constructive algorithm to obtain their exact solution. They are known to have few or no exact solutions.

However, just like the integrable equations there are some numerous remarkable methods, which are used to obtain exact and explicit solutions of nonintegrable PDEs. The most remarkable ones are the Jacobi elliptic function method, Darboux transformation, the tanh-function method, Weierstrass elliptic function method, Hirota bilinear method to mention a few.

Exact solution rarely exists generally for nonlinear differential equations (ordinary and partial) and as a result of this, efforts have been made over the years by mathematicians in constructing semianalytical and numerical means, schemes or algorithms for solving them, namely, Taylor collocation method, Euler collocation methods, wavelet collocation methods, iterative differential quadrature method, variational iteration method, homotopy perturbation method (HPM), perturbation iteration method, Adomian decomposition method, new iterative method, homotopy analysis method, reduced differential transform method, and residual power series method.

The investigation and development of travelling/solitary wave solution plays a prominent role in nonlinear science and numerous soliton scientists have also endeavor to seek the solitary wave solution of numerous models including the Fisher’s equation.

In the quest of seeking exact solutions to model/equations, mathematicians have also come up with hybrid methods (coupling two distinct methods) to obtain exact solutions of some models, namely, Sumudu decomposition method, homotopy perturbation transformation (HPTM) method, homotopy-variational iteration method, Elzaki differential transform, Elzaki projected differential transform, Elzaki homotopy transformation perturbation method (EHTPM), Laplace-Adomian decomposition method, Elzaki iterative method, Laplace-variational iteration method, new iterative transform method, and homotopy analysis transform methods.
Very recently, Cherif and Djelloul\cite{56} coupled Elzaki integral transform and variational iteration method on PDEs of fractional order; Dhunde and Waghmare\cite{57} coupled the double Laplace transform with the new iterative method on NPDEs; Jena and Chakraverty\cite{58} coupled the Elzaki transform with the homotopy perturbation on time-fractional Navier-Stokes equation. Again in Reference 7, they implemented the residual power series approach on the fractional Black-Scholes option pricing equations, Alderremy et al\cite{2} proposed a novel decomposition method involving the Elzaki transform and Adomian decomposition method on nonlinear fractional differential equations, Singh and Sharma\cite{59} implemented EHTPM and HPTM method on nonlinear fractional PDEs as a comparative study of these two methods. Series solutions were obtained using all of these methods of which converges rapidly to the exact solutions of the problems/models being solved.

In this article, we have implemented an unprecedented hybrid method involving a new integral transform (Elzaki transform) which is a modification of Laplace and Sumudu transform and the well-known HPM on nondimensional form of the Fisher’s equation.

\[ u_t = u_{xx} + f(u), \]  

(1)

where \( f(u) = \alpha u(1 - u^\beta)(u - \alpha); 0 < \alpha < 1; x > 0 \).

Herein, we have considered three types of Fisher’s equation, namely, the Fisher’s equation of Case 1 and 2 (\( \alpha = 1, \alpha = 6 \)), the Fisher’s equation of order six (sixth order) (\( \alpha = 1, \beta = 6 \)), and the nonlinear diffusion of the Fisher type due to the variation in the reactive, diffusive, and mere constant (\( \alpha = \beta = 1, 0 < \alpha < 1 \)).

In this proposed hybrid method, the solution of the equations are obtained in form of a rapidly convergent series, only few iterations leads to highly accurate solutions and these iterations are computed easily and simultaneously with a straightforward process. The closed form of the series obtained can be determined easily via computational software such as Maple, Mathematica, and so on.

Before now numerous analytical, numerical, semianalytical, and solitary waves solution of the Fisher’s equation have been presented in literatures.

Equation (1) with \( \alpha = 1, \beta = 1, \) and \( \alpha = 0 \) was first studied analytically by Reference 60, and as then several analytical methods as well as numerical, semianalytical, and numerical methods have been constructed and implemented over the years on the equation and its typologies. See References 17, 18, 22, 29, 35, 37, 61-63.

Very recently, Agbavon et al\cite{64} studied the Fisher’s equation numerically by taking the diffusion term to be smaller than the reaction term, Mickens and Oyedoji\cite{65} presented travelling wave solutions to the modified Burgers and diffusionless Fisher’s equations, and so on. See References 66, 67.

Based on the nonlinearity of PDEs, there is no single best method for an equation. Some methods converge faster to exact solution on some models/problems while some do not. This is why nonlinear models still craves for variety of analytical, semianalytical, and numerical means to improve the convergence of the solutions obtained from these methods. This became the basis for the motivation of this research.

Elzaki transform is a new integral transform was invented by Tarig\cite{68} and was derived from the classical Fourier integral. It is a modified version of the Laplace and existing Sumudu transform. Based on the mathematical simplicity of the Elzaki transform, it facilitates the process of solving ordinary and PDEs in the time domain.\cite{68-71}

As a result, Elzaki integral transform is a powerful and efficient tool that has provided analytical/exact solution to some differential equations which Sumudu transform has failed to solve. See Reference 72.

Due to this efficiency and the fact that Elzaki transform is the modification of Laplace and Sumudu transform, we have preferred to use the Elzaki transform to other integral transforms coupled with the HPM as this is method unprecedented.

The transform is defined by

\[ T(v) = v \int_0^\infty f(t)e^{-t} \, dt. \]  

(2)

Herein, \( f(t) \) is a function of time.

The idea of HPM was introduced by He\cite{23} who merged the traditional perturbation technique with the homotopy in topology by constructing a convex homotopy, considering the unknown function to a be an infinite series of an imbedding parameter \( p \in [0, 1] \) (perturbing the unknown function); and then obtain the solution of the problem in series form converging to the exact solution. He used it to solve strongly nonlinear problems in applied sciences, such as the Duffing equations, the Ear-Drum equations, and so on.\cite{26-28}
Linear PDEs can be solved directly and exactly by integral transforms such as Laplace transforms, Sumudu transforms, Fourier transforms, Elzaki transforms, and so on.

Recently, Jena and Chakraverty using the Sumudu transformation method solved the Bagley-Torvik equations and so on.

However, the limitation of these integral transforms is the zero capability in solving nonlinear differential equations (ordinary and partial). In this case, we couple these integral transforms with a semianalytical/numerical method (hybrid method) so as to handle the nonlinear terms.

In this current work, Elzaki transform is totally incapable of handling nonlinear terms arising in the equations, this is why it has been coupled with the HPM (a semianalytical method) so as to handle the nonlinear terms, which arises in the equation.

Furthermore, Elzaki integral transform is an exact technique that provides exact/analytical solution to linear PDEs; the homotopy perturbation is a semianalytical technique, which provides rapidly convergent approximate solutions to linear and nonlinear PDEs. Merging these two methods (exact and semiexact method) would definitely yield a result, that is, a replica of the exact result.

The huge merit of this proposed hybrid method lies in coupling two powerful and efficient methods (Elzaki transform and HPM) to provide an exact solution to a nonlinear reaction-diffusion PDE with less computational work and the need of only initial condition(s).

The subsequent organization of this work is structured as follows: Section 2 provides a detailed description of the reaction-diffusion model (Fisher’s equation) and its applicability in numerous fields, Sections 3 and 4 gives explicit details and description of the two methods (Elzaki transform and HPM), Section 5 illustrates the application of the hybrid method EHTPM on the nondimensional form of the Fisher’s equation, Section 6 consists of the convergence and error analysis which includes the linkage between the two methods, the uniqueness of the solution obtained from the proposed method, and some convergence theorems, while the applicability of the hybrid method on the three types of Fisher’s equation is demonstrated in Section 7, and finally, results, discussion of results obtained, and the conclusion is given in Sections 8-10, respectively.

2 | THE FISHER’S EQUATION

The Fisher’s equation derived by Fisher has the form

\[
\begin{align*}
    u_t &= u_{xx} + f(u), \\
    f(u) &= u(1-u).
\end{align*}
\]

Equation (3) was investigated independently by Fisher, Kolmogorov, Petrovsky, and Piscunov in 1937, after that it was widely known as the Fisher’s equation. It is a prominent model in population dynamics, which describes the spatial spread of an advantageous variant form of a given gene (Allele).

The above Equation (3) became one of the pertinent equations of which researchers overtime considered the various forms of the equation with some other parameters. For example, a generalized form of Equation (3) applicable in biology, which is a nonlinear evolution equation that describes one-dimensional diffusion models for insect and animal dispersal and invasion takes the form

\[
    u_x = u^\mu(1-u^\nu) + (u^k u_x)_x,
\]

where \( t \) is the time, \( x \) is the spatial coordinate, \( u(x, t) \) is the population density, \( \mu, \nu, \) and \( k \) are positive parameters. \( u^\mu(1-u^\nu) \) represents the growth of the population while \( u^k \) characterizes the diffusion process depending on the population density \( u(x, t) \). The existence of the Fisher equation in different forms and different names (family of the Fisher’s equation) makes it an interesting equation to study.

When \( f(u) = a u(1-u^\beta)(u-a) \) in Equation (3), we have the nondimensional form of the Fisher’s equation given as

\[
    u_t = u_{xx} + a u(1-u^\beta)(u-a),
\]

\[
    0 < a < 1; \quad x > 0.
\]

where \( a \) and \( \beta \) are the reactive and diffusive constants.
This is a nonlinear model for a physical system involving linear diffusion and nonlinear growth. When \( \alpha = 1, \beta = 1, \) and \( a = 0 \) in Equation (5) we have the Fisher’s equation derived in 1937 by Ronald Fisher. This is the same as when \( k = 0, \mu = 1, \) and \( \varepsilon = 1 \) in Equation (4) and when \( f(u) = u(1 - u) \) in Equation (3).

The Fisher’s equation also describes the propagation of a virile mutant in an infinitely long habitat. It also represents a model equation the evolution of a neutron population in a nuclear reactor and a prototype model for a spreading flame.

The equation became one of the most important classes of nonlinear differential equations because of their applicability in chemical kinetics, logistic population growth models, neurophysiology, nuclear reactor theory, the spread of early farming in Europe, and so on.

### 3 | ELZAKI TRANSFORM

The Elzaki transform is a semiinfinite convergent integral of the form (Table A1).

\[
T(v) = v \int_0^\infty f(t)e^{-\frac{t}{v}} dt, \tag{6}
\]

or

\[
T(v) = v^2 \int_0^\infty f(\nu t)e^{-\nu t} dt. \tag{7}
\]

The function \( f \) is of exponential order in the set

\[
A = \{ f(t) : \exists m, k_1, k_2, > 0, | f(t) | < M e^{\frac{t}{v}} \}. \tag{8}
\]

If

\[
t \in (-1) \times [0, \infty),
\]

then

\[
E[f(t)] = T(v) = v \int_0^\infty f(t)e^{-\frac{t}{v}} dt.
\]

It is called modified Sumudu transform invented/introduced by Elzaki.

By proceeding, we have the transform of derivatives using integration by parts;

\[
E \left[ \frac{df}{dt} \right] = \frac{1}{v} T(x, v) - vf(0, 0), \tag{9}
\]

\[
E \left[ \frac{d^2f}{dt^2} \right] = \frac{1}{v^2} T(x, v) - f(x, 0) - \frac{df(x, 0)}{dt}, \tag{10}
\]

\[
E \left[ \frac{df}{dx} \right] = T'(x, v) = \frac{dT(X, 0)}{dx}, \tag{11}
\]

\[
E \left[ \frac{d^2f}{dx^2} \right] = T''(x, v) = \frac{d^2T(X, 0)}{dx^2}. \tag{12}
\]

Higher order derivatives with respect to \( t \) can be obtained by mathematical induction as

\[
E \left[ \frac{d^nf(x(t))}{dt^n} \right] \Rightarrow \frac{E[f(x(t))]}{v^n} - \sum_{k=0}^{n-1} v^{2-k} \frac{d^k f(x(0))}{dt^k}. \tag{13}
\]
4 | HOMOTOPY PERTURBATION METHOD

The HPM was proposed by He\textsuperscript{23} who is a Chinese professor of mathematics in 1998; he was able to couple the traditional perturbation method with the homotopy in topology of which he employed in solving some nonlinear breath taking problems.

Homotopy is a fundamental concept in topology and differential geometry.\textsuperscript{83} The concept of homotopy can be traced back to rules Poincare,\textsuperscript{84} a French mathematician.

Shortly speaking, a homotopy describes a kind of continuous variation or deformation in mathematics. For examples, a circle can be continuously deformed into the square or an ellipse, also the shape of a coffee cup can be continuously deformed into a doughnut.

However, the shape of a coffee cup cannot be distorted continuously into the shape of a football; essentially a homotopy defines a connection between different things in mathematics, which contains some characteristics in some aspects.\textsuperscript{24}

We consider two topological spaces \((X, \tau_1)\) and \((Y, \tau_2)\) of which \(f\) and \(g\) are continuous maps of the spaces \(X\) and \(Y\), \(F\) being homotopy to \(g\) means a deformation of \(f\) into \(g\) and if \((X, \tau_1)\) and \((Y, \tau_2)\) are continuous, the \(\forall \tau_1 \in X \exists f^{-1}(\tau_2) \in Y\) (this is surjection).

It is said that \(f\) is Homotopic to \(g\), if there is a continuous map.

\[
F : X \times [0, 1] \rightarrow Y.
\]

Such that,

\[
f(x, 0) = f(x) \quad \text{and} \quad f(x, 1) = g(x),
\]

with the unit interval \([0, 1] \forall x \in X\).

Then this map is called homotopy between \(f\) and \(g\)

4.1 | Illustration of the method

For the HPM, we consider a general equation of the type

\[
D(u) = 0, \quad (14)
\]

where \(D\) is any differential operator, A convex homotopy (deformation) \(H(u, p)\) is defined such that,

\[
H(u, p) = (1 - p)F(u) + pD(u), \quad (15)
\]

\(F(u)\) is a fundamental operator with known solution \(u_0\), which can be obtained easily.

It is clear that \(H(u, p) = 0\) as \(D(u) = 0\) and \(H(u, p)\) is a convex homotopy on \(D(u)\).

For the convex homotopy \(H(u, p)\) we have that \(H(u, p) = F(u), H(u, 1) = D(u)\).

This monotonic changing process of \(p\) from zero to unity (\(p = 1\)) indicates that the known solution deforms into the original problem \(D(u) = 0\) where \(p \in [0, 1]\) is taken as an expanding parameter.

In this method, we first use the imbedding parameter “\(p\)” as a “small parameter and assume that the solution of the Equation (14) can be written as power series in \(P\).

\[
U = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots + p^n u_n. \quad (16)
\]

By setting “\(p\)” to unity (\(p \rightarrow 1\)) in the Equation (16), we obtained that approximate solution of the Equation (14) to be

\[
U = \lim_{p \rightarrow 1} U = u_0 + u_1 + u_2 + \ldots + u_n, \quad (17)
\]

\[
\therefore U = \lim_{p \rightarrow 1} U = \sum_{k=0}^{n} u_k. \quad (18)
\]
5 | APPLICATION OF THE EHTPM TO THE NONDIMENSIONAL FORM OF THE FISHER’S REACTION-DIFFUSION EQUATION

Consider the nondimensional form of the Fisher’s equation in Equation (5) with an initial condition $u(x, 0) = f(x)$ and taking the Elzaki transform of the equation, we have

$$E\{u_t\} = E\{u_{xx} + \alpha(1 - u^\beta)(u - a)\}. \quad (19)$$

Subject to

$$u(x, 0) = f(x).$$

From the Elzaki transform of partial derivatives we have that

$$E\{u_t\} = E\{u_{xx} + \alpha(1 - u^\beta)(u - a)\}.$$

By taking the inverse Elzaki transform of the Equation (21) we have;

$$E^{-1}\{E\{u_{xx} + \alpha(1 - u^\beta)(u - a)\}\} = v^2f(x) + vE\{F\{u_{xx} + \alpha(1 - u^\beta)(u - a)\}\}.$$

We now apply the next method in the algorithm, which is the HPM on Equation (23);

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t). \quad (24)$$

Equation (23) becomes,

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) + p \left[ E^{-1}\left[vE\left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} + \alpha\left[1 - \left(\sum_{n=0}^{\infty} p^n u_n\right)^\beta\right]\left(\sum_{n=0}^{\infty} p^n u_n - a\right)\right]\right], \quad (25)$$

where,

$$\sum_{n=0}^{\infty} p^n H_n(n)$$

is a polynomial in terms of the embedding parameter $p \in [0, 1]$ called the He’s polynomial which represents the nonlinear terms. Then Equation (25) becomes;

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) + p \left[ E^{-1}\left[vE\left(\sum_{n=0}^{\infty} p^n H_n(t)\right)\right]\right]. \quad (26)$$
By opening up the series in Equation (26),

\[ p^0 u_0(x, t) + p^1 u_1(x, t) + p^2 u_2(x, t) + \cdots + p^n u_n(x, t) = G(x, t) + \left[ E^{-1} \left\{ \begin{array}{l} \nu E \left\{ pH_0(u) + p^2 H_1(u) + \cdots + p^n H_n(u) \right\} \end{array} \right] \right]. \] (27)

By comparing, the coefficients from the powers of \( p \), we have

\[ \begin{align*}
p^0 & : u_0(x, t) = G(x, t), \\
p^1 & : u_1(x, t) = E^{-1} [\nu^2 E] [H_0(u)]], \\
p^2 & : u_2(x, t) = E^{-1} [\nu^2 E] [H_1(u)]], \\
p^3 & : u_3(x, t) = E^{-1} [\nu^2 E] [H_2(u)]], \\
& \vdots \\
p^n & : u_n(x, t) = E^{-1} [\nu^2 E] [H_{n-1}(u)]].
\end{align*} \] (28)

This is the coupling of the Elzaki transform and the homotopy perturbation using the He’s polynomial48,49 on the Fisher’s reaction-diffusion equation.

Having obtained \( u_0(x, t), u_1(x, t), u_2(x, t) \) up to desired iteration;

Then the solution according to homotopy \( (p \to 1) \) is given by

\[ U(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots + u_n(x, t). \] (29)

Equation (29) results to a Taylor series of two variables of which its closed form can be determined.

6 | CONVERGENCE AND ERROR ANALYSIS OF EHTPM

6.1 | Linkage of EHTPM

Herein, the convergence of the proposed hybrid method is studied, of which we the connection, relationship, and the point of linkage of Elzaki transform and HPM was analyzed, the uniqueness and convergence of solution obtained by the proposed hybrid method was also analyzed.

Consider the general form of a NPDE,

\[ LU(x, t) + NU(x, t) = g(x, t). \] (30)

\( L \) and \( N \) are linear and nonlinear differential operators, respectively, \( g(x, t) \) is the source term that determines the homogeneity of the equation.

By taking the Elzaki transform of Equation (30),

\[ LT(x, v) + NT(x, v) = g(x, v). \] (31)

Let \( g(x, v) \) be denoted by \( g(r) \), \( r \in \Omega \)

\[ LT(x, v) + NT(x, v) = g(r). \] (32)

By constructing a convex homotopy \( \overline{w}(r, p) : \Omega \times [0, 1] \to \mathbb{R}^n \) on Equation (32) satisfying

\[ H(\overline{w}, p) = (1 - p) [L(\overline{w}) - L(T_0)] + p [A(\overline{w}) - g(r)] = 0, \] (33)

\[ p \in [0, 1], r \in \Omega. \]
\[ A(T) = LT(x, v) + NT(x, v) \]
\[ \therefore A(\overline{w}) = L(\overline{w}) + N(\overline{w}). \quad (34) \]

By putting Equation (34) into Equation (33), we have,
\[ H(\overline{w}, p) = L(\overline{w}) - L(T_0) - pL(\overline{w}) + pL(T_0) + pL(\overline{w}) + pN(\overline{w}) - pg(r) = 0. \quad (35) \]
\[ \therefore H(\overline{w}, p) = L(\overline{w}) - L(T_0) + pL(T_0) + p[N(\overline{w}) - g(r)] = 0. \quad (36) \]

From the concept of homotopy, when \( p = 0, \overline{w} = T \), and \( g(r) = g(x, v) \), then;
\[ H(\overline{w}, 0) = H(T, 0) = L(T) - L(T_0) = 0 \]
\[ \Rightarrow L(T) = L(T_0), \]
when \( p = 1, \overline{w} = T \), and \( g(r) = g(x, v) \), then;
\[ H(\overline{w}, 1) = H(T, 1) = L(T) + N(T) - g(x, v) = 0 \]
\[ \Rightarrow LT + NT = g(x, v). \quad (37) \]
Equation (37) gives us the previous Equation (31) that was decomposed.

Taking the inverse Elzaki transform of Equation (37), we have;
\[ LU(x, t) + NU(x, t) = g(x, t). \]

This implies that the monotonic increase of the imbedding parameter \( p \in [0, 1] \) from 0 to 1 deforms the problem to the exact solution.

From Equation (36), we can write that;
\[ L(\overline{w}) = L(T_0) - p[g(r) - L(T_0) - N(\overline{w})]. \quad (38) \]

Applying the inverse linear operator on Equation (38),
\[ \overline{w} = T_0 + p \left[ L^{-1}g(r) - L^{-1}N(\overline{w}) - T_0 \right]. \quad (39) \]
We assume the solution of \( \overline{w}(x, v) \) in terms of a convergent series of the form
\[ \overline{w} = \overline{w}_0 + p\overline{w}_1 + p^2\overline{w}_2 + p^3\overline{w}_3 + \cdots \]
\[ \therefore \overline{w} = \sum_{k=0}^{\infty} p^k\overline{w}_k. \quad (40) \]

Put Equation (40) in Equation (39), it becomes;
\[ \sum_{k=0}^{\infty} p^k\overline{w}_k = T_0 + p \left[ L^{-1}g(r) - L^{-1}N \left( \sum_{k=0}^{\infty} p^k\overline{w}_k \right) - T_0 \right]. \quad (41) \]

If \( p \to 1 \), we can obtain the solution of Equation (32) as;
\[ T = \lim_{p \to 1} \overline{w}_k = \lim_{p \to 1} \left( \sum_{k=0}^{\infty} p^k\overline{w}_k \right). \]
We can then write Equation (41) as;

\[ T = L^{-1}g(t) - L^{-1}N(\bar{w}_0 + \bar{w}_1 + \bar{w}_2 + \cdots) \]

\[ \therefore T(x, v) = L^{-1}g(t) - L^{-1}N(\bar{w}_0(x, v) + \bar{w}_1(x, v) + \bar{w}_2(x, v) + \cdots), \quad (42) \]

\[ T(x, v) = E(U(x, t)), \]

\[ \bar{w}_k(x, v) = E(\bar{w}_k(x, t)). \]

We can easily obtain the solution of Equation (30) by taking the inverse Elzaki transform of Equation (42) above as:

\[ U(x, t) = L^{-1}g(x, t) - L^{-1}N\left(\sum_{k=0}^{\infty} \bar{w}_k(x, t)\right). \quad (43) \]

### 6.2 Uniqueness of solution

In this section, we would show that the solution to the family of Fisher’s equation which is a nonlinear PDE obtained by EHTPM is unique.

We consider a Banach space of continuous functions \( A[0, T] \) with the supremum norm also \( U(x, t), U_n(x, t) \in A[0, T] \) all through this section.

The uniqueness theorem proved here is similar to that of Reference 85, by considering the general fractional nonlinear PDE of the form;

\[ \frac{\partial^n U}{\partial t^n} + LU + NU = g(x, t). \quad (44) \]

**Theorem 6.2.1.** (The uniqueness theorem)

The solution of the Equation (44) obtained by EHTPM is unique, whenever \( 0 < \gamma < 1 \).

**Proof.** Consider the PDE in Equation (44) above,

\[ \frac{\partial^n U}{\partial t^n} + LU + NU = g(x, t). \]

Let \( L \) and \( N \) satisfy the Lipschitz condition and by taking the Elzaki transform of the PDE;

\[ E\left[ \frac{\partial^n U}{\partial t^n} + LU + NU \right] = E[g(x, t)]. \]

From the Elzaki transform of partial derivatives\(^{24,71,86}\)

\[ \therefore \frac{T(x, v)}{v^n} = \sum_{k=0}^{n-1} v^{2-n+k} \frac{\partial^k U(x, 0)}{\partial t^k} + E[g(x, t) - LU - NU], \quad (45) \]

\[ T(x, v) = E[U(x, t)] \]

\[ \Rightarrow E[U(x, t)] = v^k \sum_{k=0}^{n-1} v^k \frac{\partial^k U(x, 0)}{\partial t^k} + v^n E[g(x, t) - LU - NU]. \quad (46) \]

By taking the inverse Elzaki transform of Equation (46);

\[ \therefore U(x, t) = \sum_{k=0}^{n-1} \frac{k!}{k!} \frac{\partial^k U(x, 0)}{\partial t^k} + E^{-1}[v^n E[g(x, t) - LU - NU]]. \quad (47) \]
Suppose we have two distinct solutions \( U(x, t) \) and \( W(x, t) \), we can write that

\[
|U - W| = \left| \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k U(x, 0)}{\partial t^k} + E^{-1} [\nu^E[g(x, t) - LU - NU]] - \left( \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k W(x, 0)}{\partial t^k} + E^{-1} [\nu^E[g(x, t) - LW - NW]] \right) \right|. \tag{48}
\]

We can write from triangle inequality that;

\[
|U - W| \leq \left| \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k U(x, 0)}{\partial t^k} - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k W(x, 0)}{\partial t^k} \right| + |E^{-1} [\nu^E[-LU - LW - NU - NW]]|, \tag{49}
\]

\[
\Rightarrow |U - W| \leq |E^{-1} [\nu^E[L(U - W) + N(U - W)]]|. \tag{50}
\]

By applying the convolution theorem,

\[
|U - W| \leq \int_0^t \left( |L(U) - L(W)| + |N(U) - N(W)| \right) \left| \frac{(t - \tau)^n}{n!} \right| d\tau. \tag{51}
\]

As \( L \) and \( N \) satisfies the Lipschitz condition, then \( L \) is a bounded operator with \( |L(U) - L(W)| \leq \mu |U - W| \), and \( N \) is given such that \( |N(U) - N(W)| \leq \epsilon |U - W| \) for \( \epsilon > 0 \).

Equation (51) can now be written as

\[
|U - W| \leq \int_0^t \left( |\mu(U - W)| + |\epsilon(U - W)| \right) \left| \frac{(t - \tau)^n}{n!} \right| d\tau, \tag{52}
\]

\[
\mu, \epsilon > 0.
\]

Applying the mean value theorem for integrals on the inequality Equation (52), Let \( M = \max(t - \tau)^n \) with \( t \in [0, T] \), then Equation (52) becomes

\[
|U - W| \leq [(\mu + \epsilon)]|U - W|MT. \tag{53}
\]

Let \( (\mu + \epsilon)MT = \gamma \), Equation (53) becomes

\[
|U - W| \leq \gamma |U - W|, \tag{54}
\]

\[
\therefore (1 - \gamma)|U - W| \leq 0. \tag{55}
\]

This means that \( U = W \) whenever \( \gamma < 1 \) and \( \gamma \in (0, 1) \).

Hence, the solution is unique.

\[\blacksquare\]

6.3 \quad **Convergence theorems**

In this section, we would study some convergence theorems on the series solution obtained to the problem via proposed method (EHTPM) and prove them. We justify that this series solution is convergent and hence to the exact solution to the problem from our convergence analysis.

**Theorem 6.3.1.** (Banach fixed-point theorem)

Let \( X \) be a Banach space with the nonlinear map \( T : X \to X \) and assume that.

\[
\|T[u] - T[\bar{u}]\| \leq \epsilon \|u - \bar{u}\|, u, \bar{u} \in X, \quad 0 < \epsilon < 1.
\]
We say that $T$ is a unique fixed point and the sequence generated by EHTPM is regarded as $u_{n+1} = Tu_n$ with an arbitrary choice $u_0 \in X$, converges to the fixed point of $T$ and we have:

$$
\|u_n - u_{n+1}\| \leq \|u_1 - u_0\| \sum_{k=0}^{v-2} \varepsilon^k. \tag{56}
$$

The above theorem is prerequisite for the following analysis that would be carried out and can be deduced from the Banach fixed-point theorem.$^9$

**Theorem 6.3.2.** Let $X$ be a Banach space denoted with a suitable norm $\|\cdot\|$ over which the sequence of partial sums (series) $\sum_{k=0}^{\infty} u_k$ is defined.

Assume that the initial guess $U_0 = u_0$ remains inside the ball $B_r(u)$ of the solution $u(x, t)$.

Then, the series solution $\sum_{k=0}^{\infty} u_k$ converges if $\exists \varepsilon > 0$ such that

$$
\|u_{n+1}\| \leq \varepsilon \|u_n\|
$$

**Proof.** We define a sequence of partial sums as:

$$
T_0 = u_0,
T_1 = u_0 + u_1,
T_2 = u_0 + u_1 + u_2,
T_3 = u_0 + u_1 + u_2 + u_3,
\vdots
T_n = u_0 + u_1 + u_2 + u_3 + \cdots + u_n.
$$

Next is that we would have to show that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $X$.

To show this, we consider the relation that

$$
\|T_{n+1} - T_n\| = \|u_{n+1}\| \leq \varepsilon \|u_n\| \leq \varepsilon^2 \|u_{n-1}\| \leq \varepsilon^3 \|u_{n-2}\| \leq \cdots \leq \varepsilon^{n+1} \|u_0\|, \tag{57}
$$

for $n = 0, 1, 2, \ldots$

For every $n, k \in \mathbb{N}$, $n \geq k$, we have

$$
\|T_n - T_k\| = \|(T_{n-1} - T_{n-2}) + (T_{n-2} - T_{n-3}) + \cdots + (T_{k+1} - T_k)\|. \tag{58}
$$

From triangle inequality, we have,

$$
\|(T_{n-1} - T_{n-2}) + (T_{n-2} - T_{n-3}) + \cdots + (T_{k+1} - T_k)\| \leq \|T_{n-1} - T_{n-2}\| + \|T_{n-2} - T_{n-3}\| + \cdots + \|T_{k+1} - T_k\|.
$$

and,

$$
\|T_n - T_{n-1}\| + \|T_{n-1} - T_{n-2}\| + \cdots + \|T_{k+1} - T_k\| \leq \varepsilon^n \|u_0\| + \varepsilon^{n-1} \|u_0\| + \cdots + \varepsilon^k \|u_0\| = \frac{1 - \varepsilon^{n-k}}{1 - \varepsilon} \varepsilon^k \|u_0\|
$$

$$
\vdots \|T_n - T_k\| \leq \frac{1 - \varepsilon^{n-k}}{1 - \varepsilon} \varepsilon^k \|u_0\|. \tag{59}
$$

Showing that the sequence is bounded and we can obtain for $0 < \varepsilon < 1$ that;

$$
\lim_{n,k \to \infty} \|T_n - T_k\| = 0.
$$

This proves that the sequence of partial sum (series) generated by EHTPM is Cauchy and hence convergent.
6.3.1 Maximum truncation error

From $\|T_n - T_k\|$ and for $n \geq k$,

$$\|T_n - T_k\| \leq \frac{1 - \epsilon^{n-k}}{1 - \epsilon} \epsilon^{k+1} \|u_0\| .$$

As $0 < \epsilon < 1$; then consequently for $n \geq k$, $1 - \epsilon^{n-k} < 1$,

$$\therefore \|T_n - T_k\| \leq \frac{\epsilon^{k+1}}{1 - \epsilon} \|u_0\| .$$

Showing that $\{T_n\}_{n=0}^\infty$ is bounded.

Let $E_K = \frac{\epsilon^{k+1}}{1 - \epsilon}$,

$$\therefore \|T_n - T_k\| = E_K \|u_0\| .$$

Remark. The parameter $E_K$ is the maximum truncation error of $u(x, t)$.

7 APPLICATION OF THE METHOD (EHTPM)

7.1 Fisher’s equation

Case 1:

Consider the nondimensional Fisher’s equation in Equation (5) with $\alpha = \beta = 1$, equal diffusive and reactive constant such that

$$u_t = u_{xx} + u(1 - u). \quad (60)$$

Subject to

$$u(x, 0) = \lambda .$$

By taking the Elzaki transform of the Equation (60)

$$E[u_t] = E[u_{xx} + u(1 - u)], \quad (61)$$

$$E[u_t] = \frac{E[u(x, t)]}{v} - vu(x, 0),$$

$$\therefore \frac{E[u(x, t)]}{v^2} = vu(x, 0) + E[u_{xx} - u^2] + E[u]. \quad (62)$$

From this Equation (62) above,

$$\frac{E[u(x, t)]}{v} - E[u(x, t)] = vu(x, 0) + E[u_{xx} - u^2], \quad (63)$$

$$\therefore E[u(x, t)] \left( \frac{1}{v} - 1 \right) = vu(x, 0) + E[u_{xx} - u^2]. \quad (64)$$
By rearranging all terms appropriately,

\[ E[u(x, t)] = \frac{v^2}{1-v}u(x, 0) + \frac{v}{1-v}E[u_{xx} - u^2]. \quad (65) \]

By taking the inverse Elzaki of the Equation (65);

\[ E^{-1}[E[u(x, t)]] = E^{-1} \left[ \frac{v^2}{1-v}u(x, 0) \right] + E^{-1} \left[ \frac{v^2}{1-v}E[u_{xx} - u^2] \right]. \quad (66) \]

\[ E^{-1} \left[ \frac{v^2}{1-v}u(x, 0) \right] = u(x, 0)E^{-1} \left[ \frac{v^2}{1-v} \right] = u(x, 0)e^t. \]

\[ \therefore u(x, t) = \lambda e^t + E^{-1} \left[ \frac{v^2}{1-v}E[u_{xx} - u^2] \right]. \quad (67) \]

Now by applying HPM to Equation (67),

Let

\[ u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = \sum_{n=0}^{\infty} p^n u_n, \quad (68) \]

\[ \therefore u(x, t) = \lambda e^t + p \left[ E^{-1} \left[ \frac{v^2}{1-v} \right] \left[ \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)^2 \right] \right]. \quad (69) \]

The arising nonlinear term would be denoted by \( \sum_{n=0}^{\infty} p^n H_n(n) \)

\[ \Rightarrow \left[ \sum_{n=0}^{\infty} p^n H_n(n) \right] = \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)^2. \quad (70) \]

By replacing Equation (70) into Equation (69) we have;

\[ \sum_{n=0}^{\infty} p^n u_n(x, t) = \lambda e^t + p \left[ E^{-1} \left[ \frac{v^2}{1-v} \right] \left\{ \sum_{n=0}^{\infty} p^n H_n(u) \right\} \right]. \quad (71) \]

The first few He’s polynomials from the above computed are

\[ H_0(u) = u_{0xx} - u_0^2; \]
\[ H_1(u) = u_{1xx} - 2u_0 u_1; \]
\[ H_2(u) = u_{2xx} - u_1^2 - 2u_0 u_2; \]
\[ H_3(u) = u_{3xx} - 2u_1 u_2 - 2u_0 u_3; \]
\[ \vdots \]

\[ H_0(u) = -\lambda^2 e^{t}; \]
\[ H_1(u) = 2\lambda^3 (e^{t} - e^{2t}); \]
\[ H_2(u) = -3\lambda^4 e^{3t} + 6\lambda^4 e^{3t} - 3\lambda^4 e^{2t}; \]
\[ \vdots \]
By comparing the powers of $p$ in Equation (71) we have

\[ \begin{align*}
    p^0 : & \quad u_0(x, t) = \lambda e^t, \\
    p^1 : & \quad u_1(x, t) = E^{-1} [v E[H_0(u)]] , \\
    p^2 : & \quad u_2(x, t) = E^{-1} [v E[H_1(u)]] , \\
    p^3 : & \quad u_3(x, t) = E^{-1} [v E[H_2(u)]] , \\
    & \vdots \\
    p^n : & \quad u_n(x, t) = E^{-1} [v E[H_{n-1}(u)]] .
\end{align*} \tag{73} \]

Solving Equation (73) accordingly, we obtain the respective solutions as:

\[ u_0(x, t) = u(x, 0) = \lambda e^t. \]

From Equation (73),

\[ u_1(x, t) = E^{-1} \left[ \frac{v^2}{1 - v} E[H_0(u)] \right] = E^{-1} \left[ \frac{v^2}{1 - v} E[-\lambda^2 e^{2t}] \right]. \]

We factor out and leave all function of $t$ in the bracket;

\[ u_1(x, t) = -\lambda^2 E^{-1} \left[ \frac{v^2}{1 - v} E[e^{2t}] \right] = -\lambda^2 E^{-1} \left[ \frac{v^3}{(1 - v)(1 - 2v)} \right], \]

\[ \therefore u_1(x, t) = -\lambda^2 (e^t - 1). \]

Similarly,

\[ u_2(x, t) = -\lambda^3 (e^t - 1)^2; \quad u_3(x, t) = -\lambda^4 (e^t - 1)^3. \]

For $t = \ln \left( \frac{1 + \frac{x}{\lambda}}{x} \right)$, with $\gamma \in (0, 1)$ we have the relations;

\[ \| T_1 - T_0 \| = \| u_1 \| = \| -\lambda^2 (e^t - 1) \|, \]

\[ \Rightarrow \| u_1 \| = \| \lambda e^t \| (\lambda (e^t - 1)) \leq \| \lambda e^t \| \| \gamma \| u_0 \| ; \quad \gamma = -\lambda (e^t - 1); \]

\[ \| T_2 - T_1 \| = \| u_2 \| = \| \lambda^2 (e^t - 1)^2 \| = \| \lambda (\lambda (e^t - 1))^2 \|, \]

\[ \therefore \| T_2 - T_1 \| = \| \lambda (\lambda (e^t - 1))^2 \| \leq \| \lambda e^t \| \| \gamma^2 \| u_0 \| ; \]

\[ \| T_3 - T_2 \| = \| u_3 \| = \| \lambda^4 (e^t - 1)^3 \| = \| \lambda (\lambda (e^t - 1))^3 \| \leq \| \lambda e^t \| \| \gamma^3 \| u_0 \| , \]

\[ \therefore \]

Hence consider,

\[ \| T_n - T_m \| = \| T_n - T_{n-1} + T_{n-1} - T_{n-2} + T_{n-2} - T_{n-3} + \cdots + T_{m+1} - T_m \|
\leq \| T_n - T_{n-1} \| + \| T_{n-1} - T_{n-2} \| + \| T_{n-2} - T_{n-3} \| + \cdots + \| T_{m+1} - T_m \|, \tag{74} \]

\[ \| T_n - T_{n-1} \| + \| T_{n-1} - T_{n-2} \| + \| T_{n-2} - T_{n-3} \| + \cdots + \| T_{m+1} - T_m \| = \| u_n \| + \| u_{n-1} \| + \| u_{n-2} \| + \cdots + \| u_{m+1} \|. \]
Consequently,

\[ \| u_n \| + \| u_{n-1} \| + \| u_{n-2} \| + \cdots + \| u_{m+1} \| \leq \gamma^{m+1}(1 + \gamma^1 + \gamma^2 + \gamma^3 + \cdots + \gamma^{n-m-1})\| u_0 \| \leq \frac{\gamma^{m+1}}{1 - \gamma} \| u_0 \|. \]

\[ \therefore \| T_n - T_m \| \leq \frac{\gamma^{m+1}}{1 - \gamma} \| u_0 \|. \]

Thus,

\[ \lim_{n,m \to \infty} T_n - T_m = 0. \]

Showing that the sequence \( \{ T_n \} \) is Cauchy and hence convergent.

Similarly, using some mathematical manipulations, the above convergence analysis applies to the subsequent solutions obtained via EHTPM.

Then the solution to the Fisher’s equation according to homotopy from Equation (29) is given as:

\[ U(x, t) = \lambda e^t - \lambda^2 (e^t(e' - 1)) + \lambda^3 (e^t(e' - 1)^2) - \lambda^4 (e^t(e' - 1)^3) + \cdots. \quad (75) \]

Using the computer computation tool, the above multivariate series solution converges to the closed form.

\[ \therefore u(x, t) = \frac{\lambda e^t}{1 - \lambda + \lambda e^t}. \quad (76) \]

### 7.2 Fisher’s equation

**Case 2:**

Consider the nondimensional Fisher’s equation in Equation (5) with \( \alpha = 6, \beta = 1, \gamma = 0 \) such that

\[ u_t = u_{xx} + 6u(1 - u). \quad (77) \]

Subject to

\[ u(x, 0) = \frac{1}{(1 + e^x)^2}. \]

By taking the Elzaki transform of the Equation (77)

\[ E[u_t] = E[u_{xx} + 6u(1 - u)], \quad (78) \]

\[ E[u_t] = \frac{E[u(x, t)]}{v} - v u(x, 0), \]

\[ \therefore E[u(x, t)] = v^2 u(x, 0) + vE[u_{xx} + 6u(1 - u)]. \quad (79) \]

By taking the Elzaki transform of (79);

\[ E^{-1}[E[u(x, t)]] = E^{-1}[v^2 u(x, 0)] + E^{-1}[vE[u_{xx} + 6u(1 - u)]], \quad (80) \]

\[ E^{-1}[v^2 u(x, 0)] = u(x, 0)E^{-1}[v^2]. \]
\[ u(x, t) = u(x, 0) + E^{-1} [vE[u_{xx} + 6u(1 - u)]] \]  
\[ \therefore u(x, t) = u(x, 0) + E^{-1} [vE[u_{xx} + 6u(1 - u)]] \]  

Now by applying HPM to Equation (82), 
Let 
\[ u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = \sum_{n=0}^{\infty} p^n u_n, \]  
\[ \therefore u(x, t) = \frac{1}{(1 + e^{e^x})^2} + p \left[ E^{-1} \left[ vE \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} + 6 \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \left[ 1 - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right] \right]. \]  

The arising nonlinear term would be denoted by \( \sum_{n=0}^{\infty} P^n H_n(n) \) 
\[ \Rightarrow \left[ \sum_{n=0}^{\infty} p^n H_n(n) \right] = \left[ \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} + 6 \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \left[ 1 - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right]. \]  

By replacing Equation (85) into Equation (84) with the initial condition we have; 
\[ \sum_{n=0}^{\infty} p^n u_n(x, t) = \frac{1}{(1 + e^{e^x})^2} + p \left[ E^{-1} \left[ vE \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) \right] \right]. \]  

The first few He’s polynomials from the above computed are 
\[ H_0(u) = u_{0xx} + 6u_0 - 6u_0^2, \]
\[ H_1(u) = u_{1xx} + 6u_1 - 12u_0u_1, \]
\[ H_2(u) = u_{2xx} + 6u_2 - u_1^2 - 12u_0u_2, \]
\[ H_3(u) = u_{3xx} - 12u_1u_2 + 6u_3, \]
\[ \vdots \]
\[ H_0(u) = \frac{10e^x}{(1 + e^{e^x})^3}; H_1(u) = \frac{-50e^x + 50e^{2x} + 100e^{3x}}{(1 + e^{e^x})^5}t; \]
\[ H_2(u) = \frac{250e^{2x}(e^{2x} - 3e^x + 1)}{(1 + e^{e^x})^6}t^2; H_3(u) = \frac{250(30e^{4x} - 159e^{3x} + 75e^{2x} - 21e^x + 3)}{3(1 + e^{e^x})^7}. \]

By comparing the powers of “p” in Equation (86) we have 
\[ p^0 : u_0(x, t) = \frac{1}{(1 + e^{e^x})^2}, \]
\[ p^1 : u_1(x, t) = E^{-1} [vE[H_0(u)]]; \]
\[ p^2 : u_2(x, t) = E^{-1} [vE[H_1(u)]]; \]
\[ p^3 : u_3(x, t) = E^{-1} [vE[H_2(u)]]; \]
\[ \vdots \]
\[ p^n : u_n(x, t) = E^{-1} [vE[H_{n-1}(u)]]. \]  

(87)
Solving Equation (87) accordingly, we obtain the respective solutions of the equation as:

\[ u_0(x, t) = \frac{1}{(1 + e^x)^2}; u_1(x, t) = \frac{10e^x}{(1 + e^x)^3} t; u_2(x, t) = \frac{25e^x(-1 + e^{2x})}{(1 + e^x)^4} t^2; \]
\[ u_3(x, t) = \frac{125(-1 + 7e^x - 4e^{2x})}{3(1 + e^x)^5} t^3, \ldots \]

Then the solution to the Fisher’s equation according to homotopy is given as:

\[ u(x, t) = \frac{1}{(1 + e^x)^2} + \frac{10e^x}{(1 + e^x)^3} t + \frac{25e^x(-1 + e^{2x})}{(1 + e^x)^4} t^2 + \frac{125(-1 + 7e^x - 4e^{2x})}{3(1 + e^x)^5} t^3 + \ldots. \] (88)

Using the computer computation tool, the above multivariate series solution converges to the closed form.

\[ \therefore u(x, t) = \frac{1}{(1 + e^{x-3t^2})^2}. \] (89)

This is the exact solution of the equation. The convergence of the solution here is further illustrated with a convergence plot in Figure 6.

### 7.3 The sixth-order Fisher’s equation

Consider the nondimensional Fisher’s equation in Equation (3) with \( \alpha = 1, \beta = 6, a = 0 \) such that

\[ u_t = u_{xx} + u(1 - u^6). \] (90)

Subject to

\[ u(x, 0) = \frac{1}{\sqrt[3]{1 + e^{(3/2)x}}}. \]

By taking the Elzaki transform of the equation

\[ E[u_t] = E[u_{xx} + u(1 - u^6)], \] (91)
\[ E[u_t] = \frac{E[u(x, t)]}{v} - vu(x, 0), \]
\[ \therefore E[u(x, t)] = v^2u(x, 0) + vE\{u_{xx} + u(1 - u^6)\}. \] (92)

By taking the inverse Elzaki of Equation (92);

\[ E^{-1}[E[u(x, t)]] = E^{-1}[v^2u(x, 0)] + E^{-1}[vE\{u_{xx} + u(1 - u^6)\}]. \] (93)
\[ E^{-1}[v^2u(x, 0)] = u(x, 0)E^{-1}[v^2], \]
\[ \therefore u(x, t) = u(x, 0) + E^{-1}[vE\{u_{xx} + u(1 - u^6)\}]. \] (94)
\[ \therefore u(x, t) = u(x, 0) = \frac{1}{\sqrt[3]{1 + e^{(3/2)x}}} + E^{-1}[vE\{u_{xx} + u(1 - u^6)\}]. \] (95)
Now by applying HPM to Equation (95), let

\[ u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = \sum_{n=0}^{\infty} p^n u_n. \]  

\[ \therefore u(x, t) = \frac{1}{\sqrt{(1 + e^{(3/2)x})}} + p \left[ E^{-1} \left[ vE \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right]. \]  

The arising nonlinear term would be denoted by \( \sum_{n=0}^{\infty} p^n H_n(n) \)

\[ \Rightarrow \left[ \sum_{n=0}^{\infty} p^n H_n(n) \right] = \left[ \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] + \left[ \sum_{n=0}^{\infty} p^n u_n(x, t) \right] \left[ 1 - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)^6 \right]. \]

By replacing Equation (98) into Equation (97) we have;

\[ \sum_{n=0}^{\infty} p^n u_n(x, t) = \frac{1}{\sqrt{(1 + e^{(3/2)x})}} + p \left[ E^{-1} \left[ vE \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) \right] \right]. \]

The first few He's polynomials from the above computed are

\[ H_0(u) = u_{0xx} + u_0 - u_0^2, \]
\[ H_1(u) = u_{1xx} + u_1 - 7u_0^5 u_1, \]
\[ H_2(u) = u_{2xx} + u_2 - 7u_0^6 u_2 - 21u_0^5 u_1^2, \]
\[ H_3(u) = u_{3xx} + u_3 - 42u_0^6 u_1 u_2 - 35u_0^5 u_1^3, \]
\[ \vdots \]

\[ H_0(u) = \frac{\frac{5}{4} e^{\frac{1}{2}x}}{(1 + e^{\frac{1}{2}x})} t; H_1(u) = \frac{25e^{\frac{3}{2}x}}{16(1 + e^{\frac{3}{2}x})} t; H_2(u) = \frac{125e^{-\frac{1}{2}x} (9e^{-3x} - 18e^{-\frac{5}{2}x} + 1)}{128(1 + e^{\frac{1}{2}x})^\frac{10}{3}} t^2. \]

By comparing the powers of “\( p \)” in Equation (99) we have

\[ p^0 : u_0(x, t) = \frac{1}{\sqrt{(1 + e^{(3/2)x})}} , \]
\[ p^1 : u_1(x, t) = E^{-1}[vE[H_0(u)]], \]
\[ p^2 : u_2(x, t) = E^{-1}[vE[H_1(u)]], \]
\[ p^3 : u_3(x, t) = E^{-1}[vE[H_2(u)]], \]
\[ \vdots \]
\[ p^n : u_n(x, t) = E^{-1}[vE[H_{n-1}(u)]]. \]

Solving Equation (100) accordingly,
We obtain the respective solutions of the equation as:

\[ u_0(x, t) = \frac{1}{\sqrt{1 + e^{\left(\frac{3}{2}\right)}x}}; \quad u_1(x, t) = \frac{5e^{\frac{3}{2}x}}{4\left(1 + e^{\frac{3}{2}x}\right)^{\frac{3}{4}}} t; \quad u_2(x, t) = \frac{25e^{\frac{3}{2}x} \left(e^{\frac{3}{2}x} - 3\right)}{16\left(1 + e^{\frac{3}{2}x}\right)^{\frac{7}{4}}} t^2 \ldots \]

Then the solution to the Fisher’s equation according to homotopy is given as:

\[ u(x, t) = \frac{1}{\sqrt{1 + e^{\left(\frac{3}{2}\right)}x}} + \frac{5e^{\frac{3}{2}x}}{4\left(1 + e^{\frac{3}{2}x}\right)^{\frac{3}{4}}} t + \frac{25e^{\frac{3}{2}x} \left(e^{\frac{3}{2}x} - 3\right)}{16\left(1 + e^{\frac{3}{2}x}\right)^{\frac{7}{4}}} t^2 + \ldots \]  
\[(101)\]

Using the computer computation tool, the above multivariate series solution converges to the closed form.

\[ u(x, t) = \frac{1}{2} \tanh \left[ \frac{3}{4} \left( x - \frac{5}{2} t \right) \right] + \frac{1}{2} \]
\[(102)\]

This is in excellent agreement with the result in Wazwaz and Gorguis\textsuperscript{29} and the exact solution of the sixth-order Fisher’s equation.

The convergence of the solution here is further illustrated with a convergence plot in Figure 9.

### 7.4 Nonlinear diffusion equation of the Fisher type

Consider the nondimensional Fisher’s equation in Equation (5) with \( \alpha = 1, \beta = 6, 0 < a < 1 \) such that

\[ u_t = u_{xx} + u(1 - u)(u - a). \]  
\[(103)\]

Subject to

\[ u(x, 0) = \frac{1}{1 + e^{-\frac{1}{\sqrt{3}}x}}. \]

By taking the Elzaki transform of the Equation (103)

\[ E[u_t] = E[u_{xx} + u(1 - u)(u - a)]. \]  
\[(104)\]

\[ E[u_t] = \frac{E[u(x, t)]}{v} - vu(x, 0), \]

\[ \therefore E[u(x, t)] = v^2 u(x, 0) + vE\{u_{xx} + u(1 - u)(u - a)\}. \]  
\[(105)\]

By taking the inverse Elzaki of Equation (105);

\[ E^{-1}[E[u(x, t)]] = E^{-1}[v^2 u(x, 0)] + E^{-1}[vE\{u_{xx} + u(1 - u)(u - a)\}], \]
\[(106)\]

\[ E^{-1}[v^2 u(x, 0)] = u(x, 0)E^{-1}[v^2]. \]

\[ \therefore u(x, t) = u(x, 0) + E^{-1}[vE\{u_{xx} + u(1 - u)(u - a)\}]. \]  
\[(107)\]
\[ u(x, t) = \frac{1}{1 + e^{\frac{-\sqrt{3}}{\sqrt{x}}}} + E^{-1}[vE[u_{xx} + u(1-u)(u-a)]]. \quad (108) \]

Now by applying HPM to Equation (108), let

\[ u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = \sum_{n=0}^{\infty} p^n u_n, \quad (109) \]

\[ \therefore u(x, t) = \frac{1}{1 + e^{\frac{-\sqrt{3}}{\sqrt{x}}}} + p \left[ E^{-1} \left[ vE \left[ \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} + \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right] \right]. \quad (110) \]

The arising nonlinear term would be denoted by \( \sum_{n=0}^{\infty} P^n H_n(n) \)

\[ \Rightarrow \left[ \sum_{n=0}^{\infty} p^n H_n(n) \right] = \left[ \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} + \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \left[ 1 - \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right]^{6}. \quad (111) \]

By replacing Equation (111) into Equation (110) with the initial condition we have;

\[ \sum_{n=0}^{\infty} p^n u_n(x, t) = \frac{1}{1 + e^{\frac{-\sqrt{3}}{\sqrt{x}}}} + p \left[ E^{-1} \left[ vE \left\{ \sum_{n=0}^{\infty} p^n H_n(n) \right\} \right] \right]. \quad (112) \]

The first few He's polynomials from the above computed are

\[ H_0(u) = u_{0xx} + u_0^2 + au_0^2 - u_0^3 - au_0, \]
\[ H_1(u) = u_{1xx} + 2u_0u_1 + 2u_0u_1a - au_1 - 3u_0^2u_1, \]
\[ H_2(u) = u_{2xx} + u_1^2 + 2u_0u_2 + 4au_0u_2 - au_2 - 3u_0^2u_2 - 3u_0u_2^2, \]
\[ H_3(u) = u_{3xx} + 2u_1u_2 + 2u_1u_2a - au_3 - u_1^3 - 6u_0u_1u_2, \]
\[ \vdots \]

\[ H_0(u) = \frac{e^{-\frac{\sqrt{3}}{\sqrt{x}}}(1-2a)}{\left(1 + e^{-\frac{\sqrt{3}}{\sqrt{x}}}\right)}; H_1(u) = \frac{e^{-\frac{\sqrt{3}}{\sqrt{x}}}(1-2a)^2}{\left(1 + e^{-\frac{\sqrt{3}}{\sqrt{x}}}\right)^2} \cdot \frac{1}{4\left(1 + e^{-\frac{\sqrt{3}}{\sqrt{x}}}\right)} \cdot \ldots \]

By comparing the powers of "p" in Equation (112) we have

\[ p^0 : u_0(x, t) = \frac{1}{1 + e^{\frac{-\sqrt{3}}{\sqrt{x}}}}, \]
\[ p^1 : u_1(x, t) = E^{-1}[vE[H_0(u)]], \]
\[ p^2 : u_2(x, t) = E^{-1}[vE[H_1(u)]], \]
\[ p^3 : u_3(x, t) = E^{-1}[vE[H_2(u)]], \]
\[ \vdots \]
\[ p^n : u_n(x, t) = E^{-1}[vE[H_{n-1}(u)]]. \quad (113) \]

Solving Equation (113) accordingly,
We obtain the respective solutions of the equation as:

\[
\begin{align*}
    u_0(x, t) &= \frac{1}{1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}}; \\
    u_1(x, t) &= \frac{e^{-\frac{\sqrt{2}}{\sqrt{x}}}(1 - 2a)}{\left(1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)^2} t; \\
    u_2(x, t) &= \frac{e^{-\frac{\sqrt{2}}{\sqrt{x}}}(1 - 2a)^2 \left(-1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)}{8\left(1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)^3} t^2 + \cdots
\end{align*}
\]

Then the solution to the Fisher's equation according to homotopy is given as:

\[
\begin{align*}
    u(x, t) = \frac{1}{1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}} + \frac{e^{-\frac{\sqrt{2}}{\sqrt{x}}}(1 - 2a)}{\left(1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)^2} t + \frac{e^{-\frac{\sqrt{2}}{\sqrt{x}}}(1 - 2a)^2 \left(-1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)}{8\left(1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}\right)^3} t^2 + \cdots
\end{align*}
\]

(114)

Using the computer computation tool, the above multivariate series solution converges to the closed form.

\[
\begin{align*}
    u(x, t) = \frac{1}{1 + e^{-\frac{\sqrt{2}}{\sqrt{x}}}} \left(1 - \frac{1}{2}a\right)^t.
\end{align*}
\]

(115)

This is in excellent agreement with the result in Wazwaz and Gorguis\textsuperscript{29} and the exact solution of the generalized Fisher's equation.

The convergence of the solution here is further illustrated with a convergence plot in Figure 12.

8 | RESULTS

In the section, we present the results (EHTPM series solution) obtained from our proposed hybrid method (EHTPM) and compare these results via tables, convergence plots and 3D plots with the exact solutions (closed form solutions).

Results were presented for the Fisher's equation of Case 1 and 2, the sixth-order Fisher's equation, and the nonlinear diffusion equation of the Fisher type using the proposed EHTPM and HPM\textsuperscript{22} with table of comparison (Tables 1-4).

8.1 | Convergence and 3D plots

Convergence and 3D Plots are shown in Figures 1-11.

**TABLE 1** The comparison of exact, EHTPM, and HPM results at \( \lambda = 0.5 \)

| Case 1 (Fisher's equation) | \( t \) | Exact | EHTPM | HPM | EHTPM error = |\( |\text{Exact} - \text{EHTPM}| \) | HPM error = |\( |\text{Exact} - \text{HPM}| \) |
|----------------------------|-------|-------|-------|-----|----------------|----------------|----------------|
| 0.1 | 0.5249791875 | 0.5249791874 | 0.5499791875 | 0.0000000001 | 0.025 |
| 0.2 | 0.5498339973 | 0.5498339974 | 0.5998340000 | 0.0000000001 | 0.0500000027 |
| 0.3 | 0.5744425169 | 0.5744425170 | 0.6494425625 | 0.0000000001 | 0.0750000456 |
| 0.4 | 0.5986876602 | 0.5986876673 | 0.6986880000 | 0.0000000071 | 0.100003398 |
| 0.5 | 0.6224593310 | 0.6224596051 | 0.7956620000 | 0.000000274 | 0.173202669 |
| 0.6 | 0.6456563062 | 0.6456624829 | 0.8432043125 | 0.000006176 | 0.1975480063 |

*Note:* For each value of \( t = 0.1, 0.2, 0.3, 0.4, 0.5, \) and 0.6 at fourth iteration with their absolute errors.

*Abbreviations:* EHTPM, Elzaki homotopy transformation perturbation method; HPM, homotopy perturbation method.
### Table 2 Case 2 (Fisher’s equation)

| $x$  | $t$  | Exact       | EHTPM       | HPM          | EHTPM error = | HPM error = |
|------|------|-------------|-------------|--------------|---------------|--------------|
|      |      |             |             |              | $|\text{Exact} - \text{EHTPM}|$ | $|\text{Exact} - \text{EHTPM}|$ |
| $x = 1$ | 0.1  | 0.1425369566 | 0.1425369577 | 0.1425369577 | $1.1 \times 10^{-9}$ | $1.1 \times 10^{-9}$ |
|       | 0.2  | 0.2500000000 | 0.2500000088 | 0.2500000088 | $1.1 \times 10^{-6}$ | $1.1 \times 10^{-6}$ |
|       | 0.3  | 0.3874556189 | 0.3874556109 | 0.3874556109 | $8.0 \times 10^{-9}$ | $8.0 \times 10^{-9}$ |
|       | 0.4  | 0.5344466455 | 0.5337326285 | 0.5337326285 | $7.1 \times 10^{-5}$ | $7.1 \times 10^{-5}$ |
|       | 0.5  | 0.6684280243 | 0.6684281570 | 0.6684281570 | $1.3 \times 10^{-8}$ | $1.3 \times 10^{-8}$ |
| $x = 2$ | 0.1  | 0.03327907174 | 0.03327907149 | 0.03327907149 | $0.00000000025$ | $0.00000000025$ |
|       | 0.2  | 0.07232948815 | 0.07232948843 | 0.07232948843 | $0.00000000028$ | $0.00000000028$ |
|       | 0.3  | 0.1425369566  | 0.1425369580  | 0.1425369580  | $0.00000000014$ | $0.00000000014$ |
|       | 0.4  | 0.2500000000  | 0.2500000026  | 0.2500000026  | $0.00000000026$ | $0.00000000026$ |
|       | 0.5  | 0.3874556189  | 0.3874556109  | 0.3874556109  | $0.00000000026$ | $0.00000000026$ |
| $x = 3$ | 0.1  | 0.00575446348 | 0.00575446349 | 0.00575446349 | $0.00000000001$ | $0.00000000001$ |
|       | 0.2  | 0.0142093362  | 0.01420933638 | 0.01420933638 | $0.00000000024$ | $0.00000000024$ |
|       | 0.3  | 0.03327907174 | 0.03327907194 | 0.03327907194 | $0.00000000020$ | $0.00000000020$ |
|       | 0.4  | 0.07232948815 | 0.07232948819 | 0.07232948819 | $0.00000000004$ | $0.00000000004$ |
|       | 0.5  | 0.1425369566  | 0.1425369572  | 0.1425369572  | $0.00000000006$ | $0.00000000006$ |

Note: We present the exact, EHTPM, and HPM results at $x = 1, x = 2, x = 3$ for each value of $t = 0.1, 0.2, 0.3, 0.4, 0.5$ with their absolute errors for the Case 2 of the Fisher’s equation.

Abbreviations: EHTPM, Elzaki homotopy transformation perturbation method; HPM, homotopy perturbation method.

### Table 3 The sixth-order Fisher’s equation

| $x$  | $t$  | Exact       | EHTPM       | EHTPM error = | HPM error = |
|------|------|-------------|-------------|---------------|--------------|
|      |      |             |             | $|\text{Exact} - \text{EHTPM}|$ | $|\text{Exact} - \text{HPM}|$ |
| $x = 1$ | 0.1  | 0.6258048411 | 0.6258048411 | 0.0000000000  | 0.0000000000 |
|       | 0.2  | 0.6845750433 | 0.6845750439 | 0.0000000006  | 0.0000000006 |
|       | 0.3  | 0.7412818057 | 0.7412818056 | 0.0000000001  | 0.0000000001 |
|       | 0.4  | 0.7937005260 | 0.7937005264 | 0.0000000004  | 0.0000000004 |
|       | 0.5  | 0.8399823314 | 0.8399823312 | 0.0000000002  | 0.0000000002 |
| $x = 2$ | 0.1  | 0.4072565019 | 0.4072565025 | 0.0000000006  | 0.0000000006 |
|       | 0.2  | 0.4568490786 | 0.4568490795 | 0.0000000009  | 0.0000000009 |
|       | 0.3  | 0.510401200 | 0.5104011202 | 0.0000000002  | 0.0000000002 |
|       | 0.4  | 0.5671464273 | 0.5671464271 | 0.0000000002  | 0.0000000002 |
|       | 0.5  | 0.6258048411 | 0.6258116411 | 0.0000000068  | 0.0000000068 |
| $x = 3$ | 0.1  | 0.2514918357 | 0.2514918356 | 0.0000000001  | 0.0000000001 |
|       | 0.2  | 0.2842933964 | 0.2842933962 | 0.0000000002  | 0.0000000002 |
|       | 0.3  | 0.3210317575 | 0.3210317529 | 0.0000000028  | 0.0000000028 |
|       | 0.4  | 0.3619693339 | 0.3619692659 | 0.0000000068  | 0.0000000068 |
|       | 0.5  | 0.4072565019 | 0.4072565013 | 0.0000000006  | 0.0000000006 |

Note: We present the exact, EHTPM and HPM results at $x = 1, x = 2, x = 3$ or each value of $t = 0.1, 0.2, 0.3, 0.4, 0.5$ with their absolute errors for the sixth-order Fisher’s equation. The exact results were obtained from the general solution obtained by Wang 38 using the nonlinear transformation as:

$$u(x, t) = \left( \frac{1}{2} \tanh \left( \frac{-\beta/2}{\sqrt{2\beta + 4}(x - \frac{\beta + 4}{\sqrt{2\beta + 4}}t + \frac{b}{2}) + \frac{1}{2}} \right) \right)^{\frac{1}{2}}$$  \hspace{1cm} (116)

For $\beta = 6$.

Abbreviations: EHTPM, Elzaki homotopy transformation perturbation method; HPM, homotopy perturbation method.
| $x$ | $t$ | Exact     | EHTPM        | EHTPM error = $|\text{Exact} - \text{EHTPM}|$ | HPM error = $|\text{Exact} - \text{HPM}|$
|-----|-----|-----------|--------------|---------------------------------|---------------------------------|
| 1   | 0.1 | 0.6741700549 | 0.6741700556 | 0.0000000007                  | 0.0000000007                  |
|     | 0.2 | 0.6785479545 | 0.6785479553 | 0.0000000008                  | 0.0000000008                  |
|     | 0.3 | 0.6828947004 | 0.6828947010 | 0.0000000006                  | 0.0000000006                  |
|     | 0.4 | 0.6872097630 | 0.6872097639 | 0.0000000009                  | 0.0000000009                  |
|     | 0.5 | 0.6914926343 | 0.6914926349 | 0.0000000006                  | 0.0000000006                  |
| 2   | 0.1 | 0.8075569884 | 0.8075569907 | 0.0000000023                  | 0.0000000023                  |
|     | 0.2 | 0.8106460584 | 0.8106460605 | 0.0000000021                  | 0.0000000021                  |
|     | 0.3 | 0.8136969821 | 0.8136969842 | 0.0000000021                  | 0.0000000021                  |
|     | 0.4 | 0.8167098629 | 0.8167098651 | 0.0000000022                  | 0.0000000022                  |
|     | 0.5 | 0.8196848174 | 0.8196848192 | 0.0000000018                  | 0.0000000018                  |
| 3   | 0.1 | 0.8948549056 | 0.8948549074 | 0.0000000018                  | 0.0000000018                  |
|     | 0.2 | 0.8967218917 | 0.8967218936 | 0.0000000019                  | 0.0000000019                  |
|     | 0.3 | 0.8985594848 | 0.8985594869 | 0.0000000021                  | 0.0000000021                  |
|     | 0.4 | 0.9003680153 | 0.9003680169 | 0.0000000016                  | 0.0000000016                  |
|     | 0.5 | 0.9021478138 | 0.9021478150 | 0.0000000012                  | 0.0000000012                  |

Note: We present the exact, EHTPM, and HPM results at $x = 1$, $x = 2$, $x = 3$ for each value of $t = 0.1, 0.2, 0.3, 0.4, 0.5$ and $a = 0.3$ with their absolute errors for the nonlinear diffusion of the Fisher type.

Abbreviations: EHTPM, Elzaki homotopy transformation perturbation method; HPM, homotopy perturbation method.

**Table 4** The nonlinear diffusion equation of the Fisher type.

**Figure 1** Solution plots of the exact solution/closed form solution in Equation (76) (Case 1).

**Figure 2** Solution plots of the multivariate series obtained in Equation (75) using the proposed Elzaki homotopy transformation perturbation method (Case 1).
**FIGURE 3** Convergence plots of the multivariate series obtained in Equation (75) using the proposed Elzaki homotopy transformation perturbation method (Case 1) with $x = 1$

**FIGURE 4** Solution plots of the exact/closed form solution in Equation (89) (Case 2)

**FIGURE 5** Solution plots of the multivariate series obtained in Equation (88) using the proposed Elzaki homotopy transformation perturbation method (Case 2)
**FIGURE 6** Convergence plots of the multivariate series obtained in Equation (88) using the proposed Elzaki homotopy transformation perturbation method (Case 2).

**FIGURE 7** Solution plots of the exact/closed form solution in Equation (102) (Sixth-order Fisher’s equation).

**FIGURE 8** Solution plots of the multivariate series obtained in Equation (101) using the proposed Elzaki homotopy transformation perturbation method (sixth-order Fisher’s equation).
**Figure 9** Convergence plots of the multivariate series obtained in Equation (101) using the proposed Elzaki homotopy transformation perturbation method (sixth-order Fisher’s equation) with $x = 1$

**Figure 10** Solution plots of the exact/closed form solution in Equation (115) (nonlinear diffusion equation of the Fisher type)

**Figure 11** Solution plots of the multivariate series obtained in Equation (114) using the proposed Elzaki homotopy transformation perturbation method (nonlinear diffusion equation of the Fisher type)
9 | DISCUSSION OF FINDINGS

In this research, we have proposed a hybrid method that involves the coupling of a new integral transform “Elzaki transform” and a well-known semianalytic method “HPM” in obtaining the exact solution of a reaction-diffusion equation prominent in the field of genetic propagation theory.

We have considered three types of Fisher’s equation, namely, the Fisher’s equation of Case 1 and 2 with distinct initial conditions, the sixth-order Fisher’s equation, and the nonlinear diffusion equation of the Fisher type of which the exact solution of these problems were presented in multivariate series form and closed form which corresponds and as a matter of fact a replica of the exact solution as regards convergence.

Convergence analyses of the proposed were carried out showing that the solution obtained from EHTPM is unique and the series solution generated using EHTPM is Cauchy, hence convergent generally.

The comparison tables which consists of HPM and EHTPM results were presented and it was discovered that results obtained from HPM alone\textsuperscript{22} (a semianalytic technique/method) was not convergent for the three types of Fisher’s equation considered while the coupling of Elzaki (an exact method/technique for linear PDEs) with homotopy (EHTPM) converges rapidly for family of Fisher’s equation being considered. This shows the advantage of EHTPM over HPM.\textsuperscript{22}

The solution plots in 3D and convergence plots of the respective solutions obtained using EHTPM shows that the proposed method agreed excellently with the exact solution of the problems.

10 | CONCLUSION

We have implemented an unprecedented hybrid method called the EHTPM on three typologies of a reaction-diffusion model as a result of variation in the model parameters. This model is a prominent model in the field of genetic propagation, population dynamics, and stochastic processes.

The convergence analysis of the proposed method has been carried out rigorously and this has shown that the series solution obtained using EHTPM is unique and as a matter of fact converges rapidly to the exact solution of the nonlinear PDE.

Using EHTPM, exact solutions has been successfully obtained in form of a rapidly convergent series to the Fisher’s equation of two cases, the sixth-order Fisher’s equation, and the nonlinear diffusion equation of the Fisher type.

These solutions have been compared via comparison tables with existing literature,\textsuperscript{22,29} and this comparison shows an excellent match with a more rapid convergence than using only HPM. Furthermore, it was discovered that HPM results were unable to converge to the exact solution for the Fisher’s equation of Case 1, while the results obtained using EHTPM converges completely to the exact solution at the fourth iteration.

As a result, we speculate that Elzaki homotopy transformation is an efficient tool of providing solutions to NPDEs and totally capable of providing solutions to a wide class of NPDEs including the Fisher’s equation.
In addition, EHTPM is an efficient asymptotic alternative in providing exact solutions to models arising in the field of genetic theory, nonlinear dynamics, population dynamics, stochastic processes, and so on.

In conclusion, this proposed method can be brought into the classroom, to provide exact solutions to NPDEs and models similar to that of Fisher’s equation.

ACKNOWLEDGEMENTS
With all sense of gratitude, the authors would like to thank the anonymous reviewers, for their constructive recommendations toward making this research article scientifically sound. In addition, Mr & Mrs Akinfe, Mrs Adetoun Loyinmi for their loving support.

CONFLICT OF INTERESTS
Authors of this research have declared no conflict of interest relevant to this article.

AUTHOR CONTRIBUTIONS
Adedapo Loyinmi was responsible for the conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing original draft, writing review, and editing. Timilehin Akinfe was responsible for conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing original draft, writing review, and editing.

ORCID
Adedapo C. Loyinmi © https://orcid.org/0000-0002-6171-4256
Timilehin K. Akinfe © https://orcid.org/0000-0002-5308-7053

REFERENCES
1. Definitions, meaning and types of differential equations, Wikipedia Web site. https://en.wikipedia.org/wiki/Differential_equation
2. Alderremy AA, Elzaki TM, Chamekh M. Modified Adomian decomposition method to solve generalized Emden–Fowler systems for singular IVP. *Math Probl Eng*. 2019;2019:6097095. https://doi.org/10.1155/2019/6097095.
3. Elzaki TM, Chamekh M. Solving nonlinear fractional differential equations using a new decomposition method. *Univ J Appl Math Comput*. 2018;6:27-35.
4. Al-Shaeer MJARA. Solutions to nonlinear partial differential equations by Tan-Cot method. *IJSR J Math*. 2013;5(3):6-11.
5. Kumbinarasaiah S. Numerical solution of partial differential equations using Laguerre wavelets collocation method. *Int J Manage Technol Eng*. 2019;9(1):3635-3639.
6. Rajarama J, Chakraverty S. Residual power series method for solving time-fractional model of vibration equation of large membranes. *J Appl Comput Mech*. 2018;5:603-615. https://doi.org/10.22055/jacm.2018.26668.1347.
7. Jena RM, Chakraverty S. A new iterative method based solution for fractional Black-Scholes option pricing equations (BSOPE). *SN Appl Sci*. 2019;3(1):95. https://doi.org/10.1007/s42452-018-0106-8.
8. Aziz I, Siraj-ul-Islam I, Asif M. Haar wavelet collocation method for three-dimensional elliptic partial differential equations. *Comput Math Appl*. 2017;73:2023-2034. https://doi.org/10.1016/j.camwa.2017.02.034.
9. Bildik N. General convergence analysis for the perturbation iteration technique. *Turk J Math Comput Sci*. 2017;6:1-9.
10. Ren Shi J, Vong DS. High accuracy error estimates of a Galerkin finite element method for nonlinear time fractional diffusion equation. *Numer Methods Partial Differ Eq*. 2019;1-18. https://doi.org/10.1002/num.22428.
11. List of Nonlinear partial differential equations, Wikipedia Web site. https://en.wikipedia.org/wiki/List_of_nonlinear_partial_differential_equations
12. Integrable systems, Wikipedia Web site. https://en.wikipedia.org/wiki/Integrable_system
13. Wazwaz A-M. *Partial Differential Equations and Solitary Waves Theory*. Berlin, Germany: Springer; 2009.
14. Deniz S, Bildik N. A note on stability analysis of Taylor collocation method. *Celal Bayar Üniversitesi Fen Bilimleri Dergisi*. 2017;13:149-153. https://doi.org/10.18466/cbeyarfb.302660.
15. Deniz S. Applications of Taylor collocation method and Lambert W function to the systems of delay differential equations. *Turkish J Math Comput Sci*. 2013;3:1-20130333.
16. Bildik N, Tosun M, Deniz S. Euler matrix method for solving complex differential equations with variable coefficients in rectangular domains. *Int J Appl Phys Math*. 2017;7(1):69-78. https://doi.org/10.17706/ijapm.2017.7.1.69-78.
17. Mittal RC, Rajni RR. A study of one dimensional nonlinear diffusion equations by Bernstein polynomial based differential quadrature method. *J Math Chem*. 2017;55:673-695.
18. Matinfar M, Ghanbari M. Solving the Fisher’s equation by means of variational iteration method. *Int J Contemp Math Sci*. 2009;4(7):343-348.
19. He JH. Variational iteration method for autonomous ordinary differential systems. Appl Math Comput. 2000;114(2–3):115-123. https://doi.org/10.1016/S0096-3003(99)00104-6.
20. He JH. Variational iteration method – a kind of non-linear analytical technique: some examples. Int J Non-linear Mech. 1999;34(4):699-708. https://doi.org/10.1016/S0020-7462(98)00048-1.
21. Soori M. The variational iteration method for the Newell-Whitehead-Segel equation. Zenodo. 2016:1-9. https://doi.org/10.5281/zenodo.167857.
22. Agurseven D, Ozis T. An analytical study for Fisher type equations by using homotopy perturbation method. Comput Math Appl. 2010;60:602-609.
23. He JH. Homotopy perturbation technique. Comput Methods Appl Mech Eng. 1999;178:257-262. https://doi.org/10.1016/S0045-7825(99)00018-3.
24. MES Ahmed. Application of Homotopy Perturbation Method to Linear and Nonlinear Partial Differential Equations [PhD thesis]. Sudan, North Africa: Sudan University of Science and Technology, College of Graduate Studies, 2016.
25. Cherif MH, Belghaba K, Ziane D. Homotopy perturbation method for solving the fractional Fisher’s equation. J Anal Appl. 2016;10:9-16.
26. He J-H. Homotopy perturbation method: a new nonlinear analytical technique. Appl Math Comput. 2003;135:73-79. https://doi.org/10.1016/S0096-3003(01)00312-5.
27. He J-H. Application of homotopy perturbation method to non-linear wave equations. Chaos Soliton Fract. 2005;26:695-700. https://doi.org/10.1016/j.chaos.2005.03.006.
28. He J-H. Homotopy perturbation method for solving boundary value problems. Phys Lett A. 2006;350(2):87-88. https://doi.org/10.1016/j.physleta.2005.10.005.
29. Wazwaz AM, Gorguis A. An analytical study of Fisher's equation by using Adomian decomposition method. Appl Math Comput. 2004;154:609-620.
30. Shukur AM. Adomian decomposition for certain space-time fractional partial differential equations. IOSR J Math. 2015;11(1):55-65.
31. Loyinmi AC, Erinle-Ibrahim LM, Adeyemi AE. The new iterative method (NIM) for solving telegraphic equation. J Niger Assoc Math Phys. 2017;43:31-36.
32. Wang K, Liu S. Application of new iterative transform method and modified fractional homotopy analysis transform method for fractional Fornberg-Whitehead equation. J Nonlinear Sci Appl. 2016;9(5):2419-2433. https://doi.org/10.22436/jnsa.009.05.42.
33. Jena RM, Chakraverty S, Jena SK. Dynamic response analysis of fractionally damped beams subjected to external loads using homotopy analysis method (HAM). J Appl Comput Mech. 2019;5:355-366. https://doi.org/10.22055/JACM.2019.27592.1419.
34. Loyinmi AC, Lawal OW, Sottin DO. Reduced differential transform method for solving partial integro-differential equation. J Niger Assoc Math Phys. 2017;43:37-42.
35. Yıldırım K, Ibis B, Bayram M. New solutions of the nonlinear Fisher type equations by the reduced differential transform. Nonlinear Sci Lett A. 2012;3:29-36.
36. Malfliet W. Solitary wave solutions of non-linear wave equations. Am J Phys. 1992;60(1992):650-654.
37. Tan Y, Xu H, Liao S. Explicit series solution of travelling waves with a front of Fisher equation. Chaos Soliton Fract. 2007;31:462-472.
38. Wang XY. Exact and explicit solitary wave solutions for the generalized Fishers equation. Phys Lett A. 1988;131:227-279. https://doi.org/10.1016/0375-9601(88)90027-8.
39. Khan AT, Naher H. Solitons and periodic solutions of the Fisher equation with nonlinear ordinary differential equation as auxiliary equation. J Appl Math Stat. 2018;6(6):244-252.
40. Bildik N, Deniz S. The use of Sumudu decomposition method for solving predator-prey systems. Math Sci Lett. 2016;5:285-289. https://doi.org/10.18576/msl/050310.
41. Priyanka C, Karthikeyan N. Solving nonlinear partial differential equations by using Sumudu decomposition method. Int J Eng Dev Res. 2017;6(3):589-591.
42. Ramadan M A-I, Al-luhaibi MS. Application of Sumudu decomposition method for solving nonlinear wave-like equations with variable coefficients. Electron J Math Anal Appl. 2016;4(1):116-124.
43. Johnston SJ, Jafari H, Moshokoa SP, Ariyan VM, Baleanu D. Laplace homotopy perturbation method for Burgers equation with space- and time-fractional order. Open Phys. 2016;14(1):247-252. https://doi.org/10.1515/openphys-2016-0023.
44. Jin L. Application of variational iteration method and homotopy perturbation to the modified Kawahara equation. Math Comput Modell. 2009;49(3–4):573-578. https://doi.org/10.1016/j.mcm.2008.06.017.
45. Elzaki TM. Solution to nonlinear differential equations using mixture of Elzaki transform and differential transform method. Int Math Forum. 2012;7(13):631-638.
46. Khaloua A. and Kadem A. Mixed of Elzaki transform and projected differential transform method for nonlinear wave-like equations with variable coefficients, 2018 https://doi.org/10.20944/preprints201808.0088.v1
47. Tarig E. Projected differential transform method and Elzaki transform for solving system of nonlinear partial differential equations. World Appl Sci J. 2014;32(9):1974-1979. https://doi.org/10.5829/idosi.wasj.2014.32.09.1253.
48. Elzaki T, Biazar J. Homotopy perturbation and Elzaki transform for solving system of nonlinear partial differential equations. World Appl Sci J. 2013;24(7):944-948.
49. Elzaki T, Eman Hilal MA. Homotopy perturbation and Elzaki transform for solving system of nonlinear partial differential equations. Math Theory Model. 2012;2(3):33-42.
50. Khan M, Hussain M, Jafari H, Khan Y. Application of Laplace decomposition method to solve nonlinear coupled partial differential equations. World Appl Sci J. 2013;9:13-19.
87. Kreyszig E. *Further Applications: Banach Fixed Point Theorems*, Erwin Kreyszig, *Introductory Functional Analysis with Applications*. New York, NY: Wiley Classic Libraries; 1989:299-321.

**How to cite this article:** Loyinmi AC, Akinfe TK. Exact solutions to the family of Fisher’s reaction-diffusion equation using Elzaki homotopy transformation perturbation method. *Engineering Reports*. 2020;2:e12084. https://doi.org/10.1002/eng2.12084

## APPENDIX

| $f(t)$   | $E[f(t)] = T(t)$ |
|----------|-----------------|
| $1$      | $v^2$           |
| $t$      | $v^3$           |
| $t^n$    | $n! v^{n+2}$    |
| $e^{at}$ | $v^2 (1-a)^{1/2}$ |
| $te^{at}$| $v^2 (1-a)^{1/2}$ |
| $\sin at$| $v^2 (1-a)^{1/2}$ |
| $\cos at$| $v^2 (1-a)^{1/2}$ |
| $\sinh at$| $v^2 (1-a)^{1/2}$ |
| $\cosh at$| $v^2 (1-a)^{1/2}$ |
| $e^{at} \sin bt$ | $v^2 (1-a)^{1/2} b^{1/2}$ |
| $e^{at} \cos bt$ | $v^2 (1-a)^{1/2} b^{1/2}$ |
| $t \sin at$ | $v^2 a^{1/2}$ |
| $\frac{a^{n-1} t}{\Gamma(a)}$, $a > 0$ | $v^{n+2}$ |
| $\frac{a^{n-1} t^{n}}{(1-a)^{1/2}}$, $n = 1, 2, ...$ | $v^{n+2}$ |
| $J_0(at)$ | $v^2 e^{-a^2}$ |
| $H(t-a)$ | $v^2 e^{-\frac{a^2}{2}}$ |
| $\delta(t-a)$ | $v^2 e^{-\frac{a^2}{2}}$ |

**Table A1** Elzaki table of transform for some functions