Asymptotically optimal detection of changes in stochastic models with switching regimes

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Abstract

This paper deals with the problem of asymptotically optimal detection of changes in regime-switching stochastic models. We need to divide the whole obtained sample of data into several sub-samples with observations belonging to different states of a stochastic models with switching regimes. For this purpose, the idea of reduction to a corresponding change-point detection problem is used. Both univariate and multivariate switching models are considered. For the univariate case, we begin with the study of binary mixtures of probabilistic distributions. In theorems 1 and 2 we prove that type 1 and type 2 errors of the proposed method converge to zero exponentially as the sample size tends to infinity. In theorem 3 we prove that the proposed method is asymptotically optimal by the rate of this convergence in the sense that the lower bound in the a priori informational inequality is attained for our method. Several generalizations to the case of multiple univariate mixtures of probabilistic distributions are considered. For the multivariate case, we first study the general problem of classification of the whole array of data into several sub-arrays of observations from different regimes of a multivariate stochastic model with switching states. Then we consider regression models with abnormal observations and switching sets of regression coefficients. Results of a detailed Monte Carlo study of the proposed method for different stochastic models with switching regimes are presented.

1. Introduction

In this paper the problem of the retrospective detection of changes in stochastic models with switching regimes is considered. Our main goal is to propose asymptotically optimal methods for detection and estimation of possible 'switches', i.e. random
and transitory departures from prevailing stationary regimes of observed stochastic models.

First, let us mention previous important steps into this field. Models with switching regimes have a long pre-history in statistics (see, e.g., Lindgren (1978)). A simple switching model with two regimes has the following form:

\[ Y_t = X_t \beta_1 + u_{1t} \quad \text{for the 1st regime} \]
\[ Y_t = X_t \beta_2 + u_{2t} \quad \text{for the 2nd regime}. \]

For models with endogenous switchings usual estimation techniques for regressions are not applicable. Goldfeld and Quandt (1973) proposed regression models with Markov switchings. In these models probabilities of sequential switchings are supposed to be constant. Usually they are described by the matrix of probabilities of switchings between different states.

Another modification of the regression models with Markov switchings was proposed by Lee, Porter (1984). The following transition matrix was studied:

\[ \Lambda = [p_{ij}]_{i,j=0,1}, \quad p_{ij} = P\{I_t = j | I_{t-1} = i\}. \]

Lee and Porter (1984) consider an example with railway transport in the US in 1880-1886s which were influenced by the cartel agreement. The following regression model was considered:

\[ \log P_t = \beta_0 + \beta_1 X_t + \beta_2 I_t + u_t, \]

where \( I_t = 0 \) or \( I_t = 1 \) in dependence of 'price wars' in the concrete period.

Cosslett and Lee (1985) generalized the model of Lee and Porter to the case of serially correlated errors \( u_t \).

Many economic time series occasionally exhibit dramatic breaks in their behavior, associated with events such as financial crises (Jeanne and Mason, 2000; Cerra, 2005; Hamilton, 2005) or abrupt changes in government policy (Hamilton, 1988; Sims and Zha, 2004; Davig, 2004). Abrupt changes are also a prevalent feature of financial data and empirics of asset prices (Ang and Bekaert, 2003; Garcia, Luger, and Renault, 2003; Dai, Singleton, and Wei, 2003).

The functional form of the 'hidden Markov model' with switching states can be written as follows:

\[ y_t = c_{st} + \phi y_{t-1} + \epsilon_t, \quad (i) \]
where \( s_t \) is a random variable which takes the values \( s_t = 1 \) and \( s_t = 2 \) obeying a two-state Markov chain law:

\[
Pr(s_t = j|s_{t-1} = i, s_{t-2} = k, \ldots, y_{t-1}, y_{t-2}, \ldots) = Pr(s_t = j|s_{t-1} = i) = p_{ij}. \quad (ii)
\]

A model of the form (1-2) with no autoregressive elements (\( \phi = 0 \)) appears to have been first analyzed by Lindgren (1978) and Baum, et al. (1980). Specifications that incorporate autoregressive elements date back in the speech recognition literature to Poritz (1982), Juang and Rabiner (1985), and Rabiner (1989). Markov-switching regressions were first introduced in econometrics by Goldfeld and Quandt (1973), the likelihood function for which was first calculated by Cossett and Lee (1985). General characterizations of moment and stationarity conditions for Markov-switching processes can be found in Tjostheim (1986), Yang (2000), Timmermann (2000), and Francq and Zakoian (2001).

A useful review of modern approaches to estimation in Markov-switching models can be found in Hamilton (2005).

However, the mechanism of Markov chain modeling is far not unique in statistical description of dependent observations. Besides Markov models, we can mention martingale and copula approaches to dealing with dependent data, as well as description of statistical dependence via different coefficients of 'mixing'. All of these approaches are interrelated and we must choose the most appropriate method for the concrete problem. In this paper we choose the mixing paradigm for description of statistical dependence.

Remark that \( \psi \)-mixing condition is imposed below in this paper in order to obtain the exponential rate of convergence to zero for type 1 and type 2 error probabilities (see theorems 1 and 2 below). Another alternative was to assume \( \alpha \)-mixing property which is always satisfied for aperiodic and irreducible countable-state Markov chains (see Bradley (2005)). Then we can obtain the hyperbolic rate of convergence to zero for type 1 and type 2 error probabilities. For the majority of practical applications, it is enough to assume \( r \)-dependence (for a certain finite number of lags \( r \geq 1 \)) of observations and state variables. Then all proofs become much shorter.

Now let us mention some important problems which lead to stochastic models with switching regimes.
Splitting mixtures of probabilistic distributions

In the simplest case we suppose that the d.f. of observations has the following form:

\[ F(x) = (1 - \epsilon)F_0(x) + \epsilon F_1(x), \]

where \( F_0(x) \) is the d.f. of ordinary observations; \( F_1(x) \) is the d.f. of abnormal observations; \( 0 \leq \epsilon < 1 \) is the probability of obtaining an abnormal observation.

We need to test the hypothesis of statistical homogeneity (no abnormal observations) of an obtained sample \( X^N = \{x_1, x_2, \ldots, x_N\} \). If this hypothesis is rejected then we need to estimate the share of abnormal observations (\( \epsilon \)) in the sample and to classify this sample into sub-samples of ordinary and abnormal observations.

**Estimation for regression models with abnormal observations**

The natural generalization of the previous model is the regression model with abnormal observations

\[ Y = X\beta + \epsilon, \]

where \( Y \) is the \( n \times 1 \) vector of dependent observations; \( X \) is the \( n \times k \) matrix of predictors; \( \beta \) is \( k \times 1 \) vector of regression coefficients; \( \epsilon \) id the \( n \times 1 \) vector of random noises with the d.f. of the following type:

\[ f_\epsilon(x) = (1 - \delta)f_0(x) + \delta f_1(x), \]

where \( 0 \leq \delta < 1 \) is the probability to obtain an abnormal observation; \( f_0(x) \) is the density function of ordinary observations; \( f_1(x) \) is the density function of abnormal observations. For example, in the model with Huber’s contamination [Huber, 1985]:

\[ f_0(\cdot) = \mathcal{N}(0, \sigma^2), \quad f_1(\cdot) = \mathcal{N}(0, \Lambda^2). \]

**Estimation for regression models with changing coefficients**

Regression models with changing coefficients is another generalization of the contamination model. Suppose a baseline model is described by the following regression:

\[ Y = X\alpha + \xi, \]

where the mechanism of a change is purely random:

\[ \alpha = \begin{cases} \beta & \text{with the probability } 1 - \epsilon \\ \gamma & \text{with the probability } \epsilon, \end{cases} \]
and $\beta \neq \gamma$.

We need again to test the hypothesis of statistical homogeneity of an obtained sample and to divide this sample into sub-samples of ordinary and abnormal observations if the homogeneity hypothesis is rejected.

The goal of this paper is to propose methods which can solve these problems effectively. Theoretically, we mean estimation of type 1 and type 2 errors in testing the statistical homogeneity hypothesis and with estimation of contaminations parameters in the case of rejectioning this hypothesis. Practically, we propose procedures for implementation of these methods for univariate and multivariate models.

The structure of this paper is as follows. First, we consider univariate models with switching effects. For binary mixtures of probabilistic distributions we prove theorem 1 about exponential convergence to zero of type 1 error in classification (to detect switches for a statistically homogenous sample) as the sample size $N$ tends to infinity; theorem 2 about exponential convergence to zero of type 2 error (vice versa, to accept stationarity hypothesis for a sample with switches); and theorem 3 which establishes the lower bound for the error of classification for binary mixtures. From theorems 2 and 3 we conclude that the proposed method is asymptotically optimal by the order of convergence to zero of the classification error.

Different generalizations of the proposed method for the case of univariate models with multiple switching regimes and for multivariate models with switching regimes are considered. Results of a detailed Monte Carlo study of the proposed method for different stochastic models with switching regimes are presented.

2. Univariate models

2.1. Binary mixtures

2.1.1. Problem statement and description of the detection/estimation method

Suppose the d.f. of the observations is the binary mixture

$$f(x) = (1 - \epsilon)f_0(x) + \epsilon f_0(x - h),$$

where $\epsilon, h$ are unknown.
The problem is to estimate parameters $\epsilon, h$ by the sample $X^N = \{x_i\}_{i=1}^N$, where all $x_i$ has the same d. f. $f(\cdot)$.

An ad hoc method of estimation of these parameters is as follows: ordinary and 'abnormal' observations are heuristically classified to two sub-samples and the estimate $\hat{\epsilon}$ is computed as the share of the size of sub-sample of abnormal observations in the whole sample size. Clear, this method is correct only for large values of $h$. However, this idea of two sub-samples can be used in construction of more subtle methods of estimation.

The estimation method is as follows:

1) From the initial sample $X^N$ compute the estimate of the mean value:

$$\theta_N = \frac{1}{N} \sum_{i=1}^n x_i$$

2) Fix the parameter $b > 0$ and classify observations as follows: if an observation falls into the interval $(\theta_N - b, \theta_N + b)$, then we place it into the sub-sample of ordinary observations, otherwise - to the sub-sample of abnormal observations.

3) Then for each $b > 0$ we obtain the following decomposition of the sample $X^N$ into two sub-samples

$$X_1(b) = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N_1}\}, \quad |\tilde{x}_i - \theta_N| < b,$$

$$X_2(b) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_2}\}, \quad |\hat{x}_i - \theta_N| \geq b$$

Denote by $N_1 = N_1(b)$, $N_2 = N_2(b)$, $N = N_1 + N_2$ the sizes of the sub-samples $X_1$ and $X_2$, respectively.

The parameter $b$ is chosen so that the sub-samples $X_1$ and $X_2$ are separated in the best way. For this purpose, consider the following statistic:

$$\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_1} \tilde{x}_i - N_1 \sum_{i=1}^{N_2} \hat{x}_i).$$

4) Define the boundary $C > 0$ and compare it with the value $J = \max |\Psi_N(b)|$ on the set $b > 0$. If $J \leq C$ then we accept the hypothesis $H_0$ about the absence of abnormal observations; if, however, $J > C$ then the hypothesis $H_0$ is rejected and the estimates of the parameters $\epsilon$ and $h$ are constructed. Remark that our primary goal is to separate ordinary and abnormal observations in the sample. Evidently, classification errors must be small and therefore we have to require some kind of convergence of the estimate $\hat{\epsilon}_N$ to its true value $\epsilon$. 

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5) If $J > C$ then define the number $b_N^*$:

$$b_N^* \in \arg \max_{b>0} |\Psi_N(b)|.$$

Then

$$\epsilon_N^* = N_2(b_N^*)/N, \quad h_N^* = \theta_N/\epsilon_N^*.$$

are the nonparametric estimates for $\epsilon$ and $h$, respectively.

In the general case for construction of unbiased and consistent estimates of $\epsilon$ and $h$ we can use the following relationships:

$$\hat{\epsilon}_N \hat{h}_N = \theta_N \quad \frac{1 - \hat{\epsilon}_N}{\hat{\epsilon}_N} = \frac{f_0(\theta_N - b_N^* - \hat{h}_N) - f_0(\theta_N + b_N^* - \hat{h}_N)}{f_0(\theta_N + b_N^*) - f_0(\theta_N - b_N^*)}.$$

We will show that, under some conditions, the estimates $\hat{\epsilon}_N$ and $\hat{h}_N$ tend almost surely to the true values $\epsilon$ and $h$ as $N \to \infty$. The sub-sample of abnormal observations is $X_2(b_N^*)$.

### 2.1.2. Main results

Let us formulate the main assumptions.

**A1. Mixing conditions**

On the probability space $(\Omega, \mathcal{F}, P)$ let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $\sigma$-algebras from $\mathcal{F}$. Consider the following measure of dependence between $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\psi(\mathcal{H}_1, \mathcal{H}_2) = \sup_{A \in \mathcal{H}_1, B \in \mathcal{H}_2, P(A)P(B) \neq 0} \left| \frac{P(AB)}{P(A)P(B)} - 1 \right|.$$

Suppose $\{y_n\}$, $n \geq 1$ is a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. Denote by $\mathcal{F}_t = \sigma\{y_i : s \leq i \leq t\}$, $1 \leq s \leq t < \infty$ the minimal $\sigma$-algebra generated by random variables $y_i$, $s \leq i \leq t$. Define

$$\psi(n) = \sup_{t \geq 1} \psi(\mathcal{F}_t, \mathcal{F}_{t+n}).$$

We say that random sequence $\{y_n\}$ satisfies the $\psi$-mixing condition if the function $\psi(n)$ (which is also called the $\psi$-mixing coefficient) tends to zero as $n$ goes to infinity.

The $\psi$-mixing condition is satisfied in most practical cases. In particular, for a Markov chain (not necessarily stationary), if $\psi(n) < 1$ for a certain $n$, then $\psi(k)$ goes to zero at least exponentially as $k \to \infty$ (see Bradley, 2005, theorem 3.3).
A2. Cramer condition

We say that the sequence \( \{y_n\} \) satisfies the uniform Cramer condition if there exists \( T > 0 \) such that for each \( i \), \( \mathbb{E} \exp(t y_i) < \infty \) for \( |t| < T \).

For a centered sequence \( \{y_n\} \) this condition is equivalent to the following (see Petrov, 1987): there exist \( g > 0, H > 0 \) such that

\[
\mathbb{E} e^{t y_n} \leq e^{\frac{4t^2}{g}} , \quad |t| \leq H,
\]

for all \( n = 1, 2, \ldots \).

We assume that conditions A1 and A2 hold true everywhere in the paper.

For any \( x > 0 \) let us choose the number \( \gamma(x) \) from the following condition:

\[
\ln(1 + \gamma(x)) = \begin{cases} 
\frac{x^2}{4g} , & x \leq gH \\
\frac{xH}{4} , & x > gH,
\end{cases}
\]

where \( g, H \) are taken from the uniform Cramer condition.

For the chosen \( \gamma(x) \), let us find such integer \( \phi_0(x) \geq 1 \) from the \( \psi \)-mixing condition that \( \psi(l) \leq \gamma(x) \) for \( l \geq \phi_0(x) \).

Below we denote by \( \mathbb{P}_0(E_0), \mathbb{P}_\epsilon(E_\epsilon) \) measure (mathematical expectation) of the sequence \( X^N \) under the condition \( \epsilon = 0 \) or \( h = 0 \) (no 'abnormal' observations) and under the condition \( \epsilon h \neq 0 \).

In the following theorem the exponential upper estimate for type 1 error is obtained for the proposed method.

**Theorem 1.**

Let \( \epsilon = 0 \). Suppose the d.f. \( f_0(\cdot) \) is symmetric w.r.t. zero. Then for any \( C > 0 \) the following estimate holds:

\[
\mathbb{P}_0 \{ \max_{b > 0} |\Psi_N(b)| > C \} \leq 4\phi_0(CN/2) \exp(-L(C)N),
\]

where \( L(C) = \min \left( \frac{HC}{8\phi_0(CN/2)}, \frac{C^2}{16\phi^2_0(CN/2)g} \right) \), the constants \( g, H \) are taken from the uniform Cramer condition.

The proof of theorem 1 is given in the Appendix.

Now consider characteristics of this method in case \( \epsilon h \neq 0 \). Here we again assume that \( \mathbb{E}_0 x_i = 0, i = 1, \ldots, N \).
Put (for any fixed $\epsilon, h$)

$$r(b) = \int_{\epsilon h - b}^{\epsilon h + b} f(x)\,dx, \quad d(b) = \int_{\epsilon h - b}^{\epsilon h + b} f(x)\,dx$$

$$\Psi(b) = r(b) - \epsilon h d(b)$$

and consider the equation

$$f(\epsilon h + b) = f(\epsilon h - b). \quad (1)$$

In the following theorem type 2 error is studied.

**Theorem 2.**

Suppose all assumptions of theorem 1 are satisfied and there exists $r^* = \sup_b r(b)$. Suppose also that $f''(\cdot) \neq 0$ and continuous. Then for $0 < C < \max_b |\Psi(b)|$ we have

1) $P_{\epsilon}\{\max_b |\Psi_N(b)| \leq C\} \leq 4\phi_0(CN/2 + r^*) \exp(-L(\delta)N)$

where $\delta = \max_b |\Psi(b)| - C > 0$, $L(\delta) = \min(\frac{\delta^2}{16\phi_0^2g}, \frac{H\delta}{8\phi_0})$.

2) Suppose, moreover, that equation (1) has a unique root $b^*$ (for any fixed $\epsilon, h$). Then

$b^*_N \to b^*$ $P_{\epsilon}$-a.s. as $N \to \infty$;

3) The estimates $\hat{\epsilon}_N, \hat{h}_N$ converge $P_{\epsilon}$-a.s. to the true values of the parameters $\epsilon, h$, respectively, as $N \to \infty$.

The proof of theorem 2 is given in the Appendix.

**2.1.3. Recommendations for the choice of the threshold $C$**

For practical applications of the above obtained results we need to know the threshold $C$.

a) In order to compute this threshold, at least one training sample without switchings is needed.

b) For this sample we compute the threshold $C$ from the following empirical formula which follows from theorem 1:

$$C = C(N) \sim \sigma \sqrt{\frac{\phi_0(\cdot) |\ln \alpha|}{N}}$$

where $N$ is the sample size, $\sigma^2$ is the variation of $\phi_0$-dependent observations and $\alpha$ is the 1st type error level.
In other words we compute the dispersion $\sigma^2$ of observations and the integer $\phi_0$ (by the first zero lag of the autocorrelation function of the training sample). Then we compute the threshold $C$.

Let us give one example which explains how to do it in practice.

Consider the following model (without switchings)

\[ x(n) = \rho x(n - 1) + \sigma \xi_n, \quad n = 1, \ldots, N, \]

where $\xi_n$ are i.i.d.r.v.’s with the d.f. $N(0, 1)$, and replacing $\phi_0(\cdot)$ by $(1 - \rho)^{-1}$.

As a result, the following regression relationship for the threshold $C$ was obtained:

\[
\log(C) = -0.9490 - 0.4729 \log(N) + 1.0627 \log(\sigma) - 0.6502 \log(1 - \rho) - 0.2545 \log(1 - \alpha).
\]

(2)

Remark that $R^2 = 0.978$ for this relationship and its residuals are stationary at the error level 5%. The elasticity coefficient for the factor $N$ is close to its theoretical value $1/2$. The calibration coefficient $\exp(-0.949) = 0.3871$ here depends on the Gaussian d.f. of observations.

We have to note that in practice, we need to calibrate the above formula for the threshold $C$ using several homogenous samples.

**Examples**

The proposed method was tested in the following experiments.

In the first series of tests the following mixture model was studied:

\[ f_\epsilon(x) = (1 - \epsilon)f_0(x) + \epsilon f_0(x - h), \quad f_0(\cdot) = N(0, 1), \quad 0 \leq \epsilon < 1/2. \]

First, the critical thresholds of the decision statistic $\max_{b>0} |\Psi_N(b)|$ were computed. For this purpose we use the above formula for the threshold $C$ for the values $\alpha = 0.95$, $\rho = 0$, $\sigma = 1$. The threshold values $C$ in each experiment are presented in table 1.

| $N$ | 50 | 100 | 300 | 500 | 800 | 1000 | 1200 | 1500 | 2000 |
|-----|----|-----|-----|-----|-----|------|------|------|------|
| $\alpha = 0.95$ | 0.1681 | 0.1213 | 0.0710 | 0.0534 | 0.044 | 0.0380 | 0.038 | 0.037 | 0.034 | 0.029 |
| $\alpha = 0.99$ | 0.1833 | 0.1410 | 0.0869 | 0.0666 | 0.050 | 0.0471 | 0.0390 | 0.038 | 0.035 |

In the second series of tests the threshold value for $\alpha = 0.95$ was chosen as the critical threshold $C$ in experiments with non-homogenous samples (for $\epsilon \neq 0$). For
different sample sizes in 5000 independent trials of each test, the estimate of type 2 error \( w_2 \) (i.e. the frequency of the event \( \max_b |\Psi_N(b)| < C \) for \( \epsilon > 0 \)) and the estimate \( \hat{\epsilon} \) of the parameter \( \epsilon \) were computed. The results are presented in Table 2.

Table 2.

| \( \epsilon = 0.1 \) | \( h = 2.0 \) | \( h = 1.5 \) |
|----------------|-------------|-------------|
| \( N \) | \( C \) | \( C \) | \( C \) | \( C \) | \( C \) | \( C \) | \( C \) |
| 300 | 0.0710 | 0.0710 | 0.0710 | 0.0710 | 0.0710 | 0.0710 | 0.0710 |
| 500 | 0.0534 | 0.0534 | 0.0534 | 0.0534 | 0.0534 | 0.0534 | 0.0534 |
| 800 | 0.044 | 0.044 | 0.044 | 0.044 | 0.044 | 0.044 | 0.044 |
| 1000 | 0.038 | 0.038 | 0.038 | 0.038 | 0.038 | 0.038 | 0.038 |
| 1200 | 0.029 | 0.029 | 0.029 | 0.029 | 0.029 | 0.029 | 0.029 |
| 2000 | 0.022 | 0.022 | 0.022 | 0.022 | 0.022 | 0.022 | 0.022 |
| 3000 | 0.020 | 0.020 | 0.020 | 0.020 | 0.020 | 0.020 | 0.020 |

\[ \epsilon \] \[ w_2 \] \[ \hat{\epsilon} \]
| \( N \) | \( \epsilon \) | \( \hat{\epsilon} \) | \( \hat{\epsilon} \) | \( \hat{\epsilon} \) | \( \hat{\epsilon} \) | \( \hat{\epsilon} \) | \( \hat{\epsilon} \) |
|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 300 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 |
| 500 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 800 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 1000 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 1200 | 0.62 | 0.62 | 0.62 | 0.62 | 0.62 | 0.62 | 0.62 |
| 2000 | 0.42 | 0.42 | 0.42 | 0.42 | 0.42 | 0.42 | 0.42 |
| 3000 | 0.16 | 0.16 | 0.16 | 0.16 | 0.16 | 0.16 | 0.16 |

2.1.4. Asymptotic optimality

Now consider the question about the asymptotic optimality of the proposed method in the class of all estimates of the parameter \( \epsilon \). The a priori theoretical lower bound for the estimation error of the parameter \( \epsilon \) in the model with i.i.d. observations with d.f. \( f_\epsilon(x) = (1 - \epsilon)f_0(x) + \epsilon f_1(x) \) is given in the following theorem.

**Theorem 3.** Let \( \mathcal{M}_N \) be the class of all estimates of the parameter \( \epsilon \). Then for any \( 0 < \delta < \epsilon \),

\[
\liminf_{N \to \infty} \inf_{\hat{\epsilon}_N \in \mathcal{M}_N} \sup_{0 < \epsilon < 1/2} \frac{1}{N} \ln P_\epsilon \{|\hat{\epsilon}_N - \epsilon| > \delta\} \geq -\delta^2 J(\epsilon),
\]

where \( J(\epsilon) = \int [(f_0(x) - f_1(x))^2/f_\epsilon(x)] dx \) is the generalized \( \kappa^2 \) distance between densities \( f_0(x) \) and \( f_1(x) \) and \( P_\epsilon \) is the measure corresponding to the density \( f_\epsilon(x) \).

**Proof.**

Remark that it suffices to consider consistent estimates of the parameter \( \epsilon \) (for non-consistent estimates the limit in the left hand of the above inequality is equal to zero). This class is not empty because of the method proposed in the paper.

Suppose \( \hat{\epsilon}_N \) is any consistent estimate of \( \epsilon \) and \( 0 < \delta < \delta' \). Consider the random variable \( \lambda_N = \lambda_N(x_1, \ldots, x_N) = I\{|\hat{\epsilon}_N - \epsilon| > \delta\} \), where \( I(A) \) is the indicator of the set \( A \).

Then for any \( d > 0 \):

\[
P_\epsilon \{|\hat{\epsilon}_N - \epsilon| > \delta\} = E_\epsilon \lambda_N \geq E_\epsilon (\lambda_N I\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}),
\]
where $f(X^N, \epsilon)$ is the likelihood function of the sample $X^N$ of observations with the density function $f_\epsilon(x)$, i.e.,

$$f(X^N, \epsilon) = \prod_{i=1}^{N} [(1 - \epsilon)f_0(x_i) + \epsilon f_1(x_i)].$$

Further,

$$\mathbb{E}_\epsilon(\lambda N \mathbb{I}\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}) \geq e^{-d} \mathbb{E}_{\epsilon+\delta'}(\lambda N \mathbb{I}\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}) \geq e^{-d} (\mathbb{P}_{\epsilon+\delta'}\{|\hat{\epsilon}_N - \epsilon| > \delta\} - \mathbb{P}_{\epsilon+\delta'}\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) > e^d\}).$$

Since $\hat{\epsilon}_N$ is a consistent estimate, $\mathbb{P}_{\epsilon+\delta'}\{|\hat{\epsilon}_N - \epsilon| > \delta\} \to 1$ as $N \to \infty$.

Let us consider the probability $\mathbb{P}_{\epsilon+\delta'}\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) > e^d\}$. We have

$$\ln \frac{f(X^N, \epsilon + \delta')}{f(X^N, \epsilon)} = \sum_{i=1}^{N} \ln (1 + \delta' \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)}) = \delta' \sum_{i=1}^{N} \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)} + o(\delta').$$

On the other hand,

$$\mathbb{E}_{\epsilon+\delta'} \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)} = \delta' \int \frac{(f_1(x_i) - f_0(x_i))^2}{f_\epsilon(x_i)} dx_i = \delta' J(\epsilon).$$

Therefore, choosing $d = N((\delta')^2 + \kappa) J(\epsilon)$, $\kappa = o((\delta')^2)$, we obtain

$$\mathbb{P}_{\epsilon+\delta'}\{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) > e^d\} \to 0 \text{ as } N \to \infty.$$

Thus,

$$\mathbb{P}_\epsilon\{|\hat{\epsilon}_N - \epsilon| > \delta\} \geq (1 - o(1)) e^{-N\delta^2 J(\epsilon)},$$

or

$$\liminf_{N \to \infty} \inf_{\hat{\epsilon}_N \in M_N} \sup_{0 < \epsilon < 1/2} \frac{1}{N} \ln \mathbb{P}_\epsilon\{|\hat{\epsilon}_N - \epsilon| > \delta\} \geq -\delta^2 J(\epsilon),$$

Theorem 3 is proved.

Comparing results of theorems 2 and 3 we conclude that the proposed method is asymptotically optimal by the order of convergence of the estimates of a mixture parameters to their true values.

2.1.5. Generalizations: non-symmetric distribution functions
Results obtained in theorems 1 and 2 can be generalized to the case of non-symmetric distribution functions. Suppose the d.f. \( f_0(\cdot) \) is asymmetric w.r.t. zero. Then we can modify the proposed method as follows.

1. From the initial sample \( X^N = \{x_1, \ldots, x_N\} \) compute the mean value \( \theta_N = \frac{1}{N} \sum_{i=1}^{N} x_i \) and the sample \( Y^N = \{y_1, \ldots, y_N\} \); \( y_i = x_i - \theta_N \). Then we divide the sample \( Y^N \) into two sub-samples \( I_1(b), I_2(b) \) as follows:

\[
y_i \in \begin{cases} 
I_1(b) = \{\hat{y}_1, \ldots, \hat{y}_{N_1(b)}\}, & -\phi(b) \leq y_i \leq b \\
I_2(b) = \{\hat{y}_1, \ldots, \hat{y}_{N_2(b)}\}, & y_i > b \text{ or } y_i < -\phi(b),
\end{cases}
\]

where the function \( \phi(b) \) is defined from the following condition:

\[
0 = \int_{-\phi(b)}^{b} y f_0(y) dy,
\]

\( f_0(y) = f_0(x - \theta_N), \) \( N = N_1(b) + N_2(b) \) and \( N_1(b), N_2(b) \) are sample sizes of \( I_1(b), I_2(b) \), respectively.

2. As before we compute the statistic

\[
\Psi_N(b) = \frac{1}{N_2(b)} \sum_{i=1}^{N_1(b)} \hat{y}_i - N_1(b) \sum_{i=1}^{N_2(b)} \hat{y}_i.
\]

3. Then the value \( J = \max_b |\Psi_N(b)| \) is compared with the threshold \( C \). If \( J \leq C \) then the hypothesis \( H_0 \) (no abnormal observations) is accepted; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected and the estimate of the parameter \( \epsilon \) is constructed.

4. For this purpose, define the value \( b_N^* \):

\[
b_N^* \in \arg \max_{b > 0} |\Psi_N(b)|.
\]

Then

\[
\epsilon_N^* = \frac{N_2(b_N^*)}{N}.
\]

Consider application of this method for the study of the classic \( \epsilon \)-contamination model:

\[
f_\epsilon(\cdot) = (1 - \epsilon)N(\mu, \sigma^2) + \epsilon N(\mu, \Lambda^2), \quad \Lambda^2 >> \sigma^2, \quad 0 \leq \epsilon < 1/2.
\]

For this model, the method described above has the form:

1. From the sample of observations \( X^N = \{x_1, \ldots, x_N\} \) the mean value estimate \( \hat{\mu} = \sum_{i=1}^{N} x_i/N \) was computed.
2. The sequence \( y_i = (x_i - \hat{\mu})^2, \ i = 1, \ldots, N \) and its empirical mean \( \theta_N = \sum_{i=1}^{N} y_i / N \) are computed.

3. Then for each \( b \in [0, B_{\max}] \), where \( B_{\max} \) is a certain a priori chosen maximal value of the parameter \( b \), the sample \( Y^N = \{y_1, \ldots, y_N\} \) is divided into two sub-samples in the following way: for \( \theta_N(1 - \phi(b)) \leq y_i \leq \theta_N(1 + b) \) put \( \tilde{y}_i = y_i \) (the size of the sub-sample \( N_1 = N_1(b) \)), otherwise put \( \hat{y}_i = y_i \) (the size of the sub-sample \( N_2 = N_2(b) \)). Here we choose the function \( \phi(b) \) from the following condition:

\[
\frac{\theta_N(1+b)}{\theta_N(1-\phi(b))} \int_{\theta_N(1-\phi(b))}^{\theta_N(1+b)} y f_0(y) dy = 0,
\]

where \( f_0(\cdot) = N(0, (1-\epsilon)^2 \sigma^2) \).

From here we obtain:

\[
\phi(b) = 1 - \frac{b}{e^b - 1}.
\]

4. For any \( b \in [0, B_{\max}] \), the following statistic is computed:

\[
\Psi_N(b) = \frac{1}{N^2} \left( N_2 \sum_{i=1}^{N_1} \tilde{y}_i - N_1 \sum_{i=1}^{N_2} \hat{y}_i \right).
\]

where \( N = N_1 + N_2, N_1 = N_1(b), N_2 = N_2(b) \) are sizes of sub-samples of ordinary and abnormal observations, respectively.

5. Then, as above, the threshold \( C > 0 \) is chosen and compared with the value \( J = \max_b |\Psi_N(b)| \). If \( J \leq C \) then the hypothesis \( H_0 \) (no abnormal observations) is accepted; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected and the estimate of the parameter \( \epsilon \) is constructed as follows.

Define the value \( b_N^* \):

\[
b_N^* \in \arg \max_{b > 0} |\Psi_N(b)|.
\]

Then

\[
\epsilon_N^* = \frac{N_2(b_N^*)}{N}.
\]

Remark. For estimation of the threshold, we use the approach described in 2.1.3.

In experiments the critical values of the statistic \( \max_b |\Psi_N(b)| \) were computed. For this purpose, as above, for homogenous samples (for \( \epsilon = 0 \)), \( \alpha \)-quantiles of the decision statistic \( \max_b |\Psi_N(b)| \) were computed (\( \alpha = 0.95, 0.99 \)). The results obtained in 5000 trials of each test are presented in table 3.
Table 3.

| $N$  | 50  | 100 | 300 | 500  | 800  | 1000 | 1200 | 1500 | 2000 |
|------|-----|-----|-----|------|------|------|------|------|------|
| 0.95 | 0.3031 | 0.2330 | 0.1419 | 0.1252 | 0.1244 | 0.1146 | 0.1107 | 0.1075 |
| 0.99 | 0.3699 | 0.2862 | 0.1947 | 0.1543 | 0.1436 | 0.1331 | 0.1269 | 0.1190 | 0.1157 |

The quantile value for $\alpha = 0.95$ was chosen as the critical threshold $C$ in experiments with non-homogenous samples (for $\epsilon \neq 0$). For different sample sizes in 5000 independent trials of each test, the estimate of type 2 error $w_2$ (i.e. the frequency of the event $\max_b |\Psi_N(b)| < C$ for $\epsilon > 0$) and the estimate $\hat{\epsilon}$ of the parameter $\epsilon$ were computed. The results are presented in tables 4 and 5.

Table 4.

| $\Lambda = 3.0$ | $\epsilon = 0.05$ |
|-----------------|------------------|
| $N$             | $C$              | $w_2$ | $\hat{\epsilon}$ |
| 300             | 0.1570           | 0.27  | 0.064             |
| 500             | 0.1419           | 0.15  | 0.056             |
| 800             | 0.1252           | 0.06  | 0.052             |
| 1000            | 0.1244           | 0.04  | 0.05              |

Table 5.

| $\Lambda = 5.0$ | $\epsilon = 0.01$ |
|-----------------|------------------|
| $N$             | $C$              | $w_2$ | $\hat{\epsilon}$ |
| 1000            | 0.1244           | 0.25  | 0.0135            |
| 1200            | 0.1146           | 0.20  | 0.013             |
| 1500            | 0.1107           | 0.15  | 0.012             |
| 2000            | 0.1075           | 0.10  | 0.011             |
| 3000            | 0.1019           | 0.04  | 0.010             |

2.2. Multiple switchings

Suppose we obtain the data $X^N = \{x_1, \ldots, x_N\}$, where the d.f. of an observation $x_i$ can be written as follows:

$$f(x_i) = (1 - \epsilon_1 - \cdots - \epsilon_k) f_0(x_i - h_1) + \epsilon_1 f_0(x_i - h_2) + \cdots + \epsilon_k f_0(x_i - h_{k+1}),$$

where $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_k \geq 0$, $0 \leq \epsilon_1 + \cdots + \epsilon_k < 1$, $|h_1| < |h_2| < \cdots < |h_{k+1}|$.

Suppose that the d.f. $f_0(x)$ is symmetric w.r.t. $x = 0$ and $\min_{1 \leq i \leq k} (|h_{i+1}| - |h_i|) \geq B > 0$. 

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Our goal is to test the hypothesis \( \epsilon_s = 0, s = 1, \ldots, k \) (no switches) and in case this hypothesis is rejected to estimate the number of switches \( k \geq 1 \) and the parameters of the model \( \epsilon_i, i = 1, \ldots, k \), and \( h_j, j = 1, \ldots, k + 1 \). In this section we denote by \( E_i, i = 0, 1, \ldots, k \), the mathematical expectation of random variables corresponding to the d.f. with shift \( h_i(h_0 \overset{\text{def}}{=} 0) \).

This model has the following sense. In the case of a binary switching we have ordinary and abnormal observations. In the case of multiple switchings abnormal observations are from different classes. The idea to use the sample mean as a reference point of the above described method is no more valid, because in case of many classes it can be greatly biased towards the maximal \( |h_i| \). Instead, we use the reference points from the histogram of the obtained sample. Concretely, we do as follows.

1. Construct the histogram \( \text{hist}_N(t) \) of data by the whole sample \( X^N \) obtained. Find \( \arg \max_t \text{hist}_N(t) \). An arbitrary point from this set is assumed to be the reference point \( \theta_N \) used in the following algorithm for a binary switching model.

1.1. Fix the parameter \( b > 0 \) and classify observations as follows: if an observation falls into the interval \((\theta_N - b, \theta_N + b)\), then we place it into the sub-sample of ordinary observations, otherwise - to the sub-sample of abnormal observations.

1.2. Then for each \( b > 0 \) we obtain the following decomposition of the sample \( X^N \) into two sub-samples

\[
X_1(b) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_1}\}, \quad |\hat{x}_i - \theta_N| < b, \\
X_2(b) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_2}\}, \quad |\hat{x}_i - \theta_N| \geq b
\]

Denote by \( N_1 = N_1(b), N_2 = N_2(b), N = N_1 + N_2 \) the sizes of the sub-samples \( X_1 \) and \( X_2 \), respectively.

The parameter \( b \) is chosen so that the sub-samples \( X_1 \) and \( X_2 \) are separated in the best way. For this purpose, consider the following statistic:

\[
\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_1} \hat{x}_i - N_1 \sum_{i=1}^{N_2} \hat{x}_i).
\]

1.3. Define the boundary \( C > 0 \) and compare it with the value \( J = \max_b |\Psi_N(b)| \) on the set \( 0 < b \leq B \). If \( J \leq C \) then we accept the hypothesis \( H_0 \) about the absence of abnormal observations; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected and the estimates of the parameters \( \epsilon \overset{\text{def}}{=} (\epsilon_1 + \cdots + \epsilon_k) \) and \( h_1 \) are constructed. Remark that our primary goal is to separate ordinary and abnormal observations in the sample.
Evidently, classification errors must be small and therefore we have to require some kind of convergence of the estimate \( \hat{\epsilon}_N \) to its true value \( \epsilon \).

1.4. Define the number \( b^*_N \):

\[
b^*_N \in \arg \max_{0 < b \leq B} |\Psi_N(b)|.
\]

Then

\[
\hat{\epsilon}_N = N_2(b^*_N)/N.
\]

2. As a result, we obtain two classes of observations at the first step (ordinary and abnormal data) and the estimate \( \hat{\epsilon}_N \) of the sum \( \epsilon = (\epsilon_1 + \cdots + \epsilon_k) \), as well as the estimate of the average \( E_0 x_i \).

3. Then we remove all found 'ordinary' observations from the sample and repeat steps 1 and 2. As a result, we obtain the estimate \( \hat{\epsilon}_1 \) of the parameter \( \epsilon_1 \), as well as the estimate of the average \( E_1 x_i \).

4. So we proceed further until a sub-sample without switches is obtained (i.e. the decision threshold \( C \) is not exceeded). As a result, we obtain the estimate \( \hat{k}_N \) of the number of classes \( k \), as well as the estimates of the parameters \( \epsilon_1 > \cdots > \epsilon_k > 0 \) and averages \( E_0 x_i, E_1 x_i, \ldots, E_k x_i \).

We see that this method is based upon reduction to the case of a binary switching model. In this case we characterize the quality of a method by the performance criteria of the right estimation of the number of classes (i.e. \( \hat{k}_N = k \)) and the accuracy of estimation (e.g., \( \max_i |\hat{\epsilon}_i - \epsilon_i| \) in the case \( \hat{k}_N = k \)). So we must use the following performance criterion:

\[
P_\epsilon \{(\hat{k}_N \neq k) \cup \left( (\max_i |\hat{\epsilon}_i - \epsilon_i| > \delta) \cap (\hat{k}_N = k) \right) \}.
\]

However, we see that the crucial thing is to correctly estimate the number of classes \( k \). The estimates of the parameters \( h_1, \ldots, h_{k+1} \) are assumed to be the reference points at each step of the above described recurrent procedure. Then consistent estimates of \( \epsilon_i \) can be obtained by some of standard methods (e.g., the method of moments). Therefore we use the following performance criterion:

\[
P_\epsilon \{\hat{k}_N \neq k\}.
\]

Remark that the 1st type error for multiple switchings can be estimated like in the binary case (we do not formulate this result). As to the 2nd type error (i.e. the
probability that we stop at the 1st step of the method because the decision threshold
is not exceeded) just observe that a binary switch is a particular case of the general
multiple switching situation (when all $\epsilon_i$ beginning from $i = 2$ are equal to zero).

Therefore

$$P_\epsilon\{\text{2nd type error, multiple switches}\} \leq P_\epsilon\{\text{2nd type error, binary case}\} \leq L_1 \exp(-L(\delta)N),$$

for $0 \leq \delta \leq \max_{0 \leq b \leq B_{\max}} |\Psi(b)| - C$, where $L(\delta) = \min\left(\frac{\delta^2}{16\phi_0 g}, \frac{H\delta}{8\phi_0}\right)$, $L_1 = 4\phi_0$.

Now consider the event $\{k_N \neq k\} = \{k_N < k\} \cup \{k_N > k\}$.

The event $\{k_N < k\}$ means that at a certain recurrent step of the above described
procedure a sub-sample of remaining observations (after eliminations of all previous
sub-samples) is considered to be "pure" (i.e. without switches) but in reality it contains
some more switches. The probability of this event is less than the 2nd type error at
this step of the procedure. Therefore,

$$P_\epsilon\{k_N < k\} \leq L_1 \exp(-L(\delta)N),$$

for $0 \leq \delta = \max_{0 \leq b \leq B_{\max}} |\Psi(b)| - C$, where $L(\delta) = \min\left(\frac{\delta^2}{16\phi_0 g}, \frac{H\delta}{8\phi_0}\right)$, $L_1 = 4\phi_0$.

The event $\{k_N > k\}$ means that finally some more switches are detected in the
obtained sample than in reality. The probability of this event is less than the 1st type
error at the final step of the above recurrent procedure:

$$P_\epsilon\{k_N > k\} \leq L_1 \exp(-L(C)N),$$

where $L(C) = \min\left(\frac{C^2}{16\phi_0 g}, \frac{HC}{8\phi_0}\right)$, $L_1 = 4\phi_0$.

Therefore the following theorem holds.

**Theorem 4.**

Suppose $0 < C < \max_{0 \leq b \leq B_{\max}} |\Psi(b)|$. Then the 2nd type error probability is estimated
from above as follows:

$$P_\epsilon\{\text{2nd type error}\} \leq L_1 \exp(-L(\delta)N),$$

where $0 \leq \delta = \max_{0 \leq b \leq B_{\max}} |\Psi(b)| - C$, $L(\delta) = \min\left(\frac{\delta^2}{16\phi_0 g}, \frac{H\delta}{8\phi_0}\right)$, $L_1 = 4\phi_0$.

Moreover, the estimate of the number of switchings $\hat{k}_N$ converges a.s. to the true
value of $k$ as $N \to \infty$ and

$$P_\epsilon\{k_N \neq k\} \leq L_1(\exp(-L(\delta)N) + \exp(-L(C)N)),$$

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where $0 \leq \delta = \max_{0 \leq b \leq B_{max}} |\Psi(b)| - C$ and $L(C) = \min(\frac{C^2}{16\phi^2_0 g}, \frac{HC}{8\phi_0})$, $L_1 = \min(\frac{\delta^2}{16\phi^2_0 g}, \frac{H\delta}{8\phi_0})$, $L_1 = 4\phi_0$.

**Example**

Let us consider the following example. Suppose we have the model with three classes of observations:

$$f(x_i) = (1 - \epsilon_1 - \epsilon_2) f_0(x_i - h_1) + \epsilon_1 f_0(x_i - h_2) + \epsilon_2 f_0(x_i - h_3), \quad i = 1, \ldots, N,$$

where $f_0(\cdot) = N(0,1)$; $x_i$ are i.r.v.’s.

The problem is to estimate the unknown number of classes $k = 3$, parameters $h_1, h_2, h_3$, and $\epsilon_1, \epsilon_2$ by the sample $X^N = \{x_1, \ldots, x_N\}$.

Concretely, in this model the following parameters were chosen:

$$\epsilon_1 = 0.3; \quad \epsilon_2 = 0.15$$

$$h_1 = 1, \quad h_2 = 3, \quad h_3 = 7.$$

For estimation of the decision threshold, the above empirical formula (2) can be used:

$$\log(C) = -0.9490 - 0.4729 \log(N) + 1.0627 \log(\sigma) - 0.6502 \log(1 - \rho) - 0.2545 \log(1 - \alpha).$$

Again remark that the elasticity coefficient for the factor $N$ is close to its theoretical value $-0.5$.

In experiments we estimated the number of classes $\hat{k}_N$ and the corresponding error $\hat{e}r_N = P_{\epsilon}\{\hat{k}_N \neq k\}$.

The following results were obtained (each cell of this table is the average in 1000 replications):

**Table 6.**

| $N$   | 100  | 200  | 300  | 500  | 700  | 1000 | 1500 |
|-------|------|------|------|------|------|------|------|
| $\hat{e}r_N$ | 0.116 | 0.090 | 0.070 | 0.048 | 0.036 | 0.016 | 0.010 |
3. Multivariate models

3.1. Multivariate classification

Binary mixtures

Now let us consider the multivariate classification problem with binary mixtures. Suppose multivariate observations are of the following type:

\[ X^N = \{X^n \}_{n=1}^N, \quad X^n = (x^n_1, \ldots, x^n_k). \]

The multivariate density function of the vector \( X^n \) is

\[ f(X^n) = (1 - \epsilon)f_0(X^n) + \epsilon f_1(X^n), \]

where \( f_0(\cdot) \), \( f_1(\cdot) \) are the d.f.'s of ordinary and abnormal observations, respectively; the d.f. \( f_0(\cdot) \) is supposed to be symmetric w.r.t. its mean vector.

First, let us consider the case \( E_1(X^n) = a \neq 0 \), i.e. changes in mean of abnormal observations. Remark that the baseline "change-in-mean" problem is usually considered in many methods of 'cluster analysis' in which different distances between multivariate 'points' of characteristics (even without references to density functions and mathematical expectations of observations) are considered.

The method can be formulated in analogy with the univariate case:

1) From the initial sample \( X^N \) compute the estimate of the mean value:

\[ \theta_N = \frac{1}{N} \sum_{i=1}^N X^i. \]

2) Fix the parameter \( b > 0 \) and classify observations as follows:

if \( \|X^i - \theta_N\| \leq b \), then we place \( X^i \) into the sub-sample of ordinary observations \( \{\tilde{Y}^i\} \);

if \( \|X^i - \theta_N\| > b \), then we place \( X^i \) into the sub-sample of abnormal observations \( \{\hat{Y}^i\} \).

As a result, for each \( b > 0 \) we obtain the decomposition of the sample \( X^N \) into sub-samples of ordinary and abnormal observations. Suppose the size of ordinary sub-sample is \( N_1(b) \) and the size of abnormal sub-sample is \( N_2(b) \).

3) The parameter \( b \) can be chosen in order to separate the sub-samples of ordinary and abnormal observations (\( \{\tilde{Y}^i\} \) and \( \{\hat{Y}^i\} \), respectively) in the best way. For this
purpose, consider the following statistic:

$$\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_1} \tilde{Y}_i - N_1 \sum_{i=1}^{N_2} \hat{Y}_i).$$

4) Define the boundary $C > 0$ and compare it with the value $J = \max_b \|\Psi_N(b)\|$ on the set $b > 0$. If $J \leq C$ then we accept the hypothesis $H_0$ about the absence of abnormal observations; if, however, $J > C$ then the hypothesis $H_0$ is rejected and the estimates of the parameters $\epsilon$ and $a$ are constructed.

Remark that our primary goal is to separate ordinary and abnormal observations in the sample. Evidently, classification errors must be small and therefore we have to require some kind of convergence of the estimate $\hat{\epsilon}_N$ to its true value $\epsilon$.

5) Define the number $b^*_N$:

$$b^*_N \in \arg \max_{b > 0} \|\Psi_N(b)\|.$$  

Then

$$\epsilon^*_N = N_2(b^*_N)/N, \quad a^*_N = \theta_N/\epsilon^*_N.$$  

are the nonparametric estimates for $\epsilon$ and $a$, respectively.

Our main results in this case are analogous to the univariate situation.

**Theorem 5.**

Suppose $\epsilon = 0$ and the d.f. $f_0(\cdot)$ is symmetric w.r.t. its mean vector. Then for any $C > 0$ the following upper estimate for the probability of the 1st type error holds:

$$P_0\{\max_{b > 0} \|\Psi_N(b)\| > C\} \leq 4\phi_0(CN/2) \exp(-L(C)N),$$

where $L(C) = \min \left( \frac{HC}{8\phi_0(CN/2)}, \frac{C^2}{16\phi_0^2(CN/2)g} \right)$, the constants $g, H$ are taken from the uniform Cramer condition.

For the 2nd type error we can formulate the following result.

**Theorem 6.**

Suppose all assumptions of theorem 5 are satisfied and there exists $r^* = \sup_b r(b)$. Suppose also that $f''(\cdot) \neq 0$ and continuous. Then for $0 < C < \max_b |\Psi(b)|$ we have

$$P_\epsilon\{\max_b \|\Psi_N(b)\| \leq C\} \leq 4\phi_0(CN/2 + r^*) \exp(-L(\delta)N)$$
where $\delta = \max_b \| \Psi(b) \| - C > 0$, $L(\delta) = \min(\frac{\delta^2}{16\phi_0^2g}, \frac{H\delta}{8\phi_0})$.

2) Suppose, moreover, that equation (*) has a unique root $b^*$. Then $b^*_N \to b^*$ $\mathbb{P}_e$-a.s. as $N \to \infty$;

This method deals with binary mixtures of multivariate d.f.’s. Its generalization to multiple classes of multivariate d.f.’s can be obtained in analogy with the previous section.

**Multiple switches**

In this case the multivariate density function of the vector $X^n$ is

$$f(X^n) = (1 - \epsilon_1 - \cdots - \epsilon_k)f_0(X^n - h_1) + \epsilon_1 f_0(X^n - h_2) + \cdots + \epsilon_k f_0(X^n - h_{k+1})$$

where $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_k \geq 0$, $0 \leq \epsilon_1 + \cdots + \epsilon_k < 1$, $\|h_1\| < \|h_2\| < \cdots < \|h_{k+1}\|$. Suppose that the d.f. $f_0(x)$ is symmetric w.r.t. $x = 0$ and $\min_{1 \leq i \leq k} (\|h_{i+1}\| - \|h_i\|) \geq B > 0$.

In order to estimate the number of classes $k$, as well as parameters $\epsilon_i$ we do as follows:

From the sample of initial multivariate observations

$$X^n = (x^1_n, \ldots, x^k_n), \quad n = 1, \ldots, N.$$ we build the sample of their Euclidean norms:

$$Y_n = \|X^n\| = \sqrt{(x^1_n)^2 + \cdots + (x^k_n)^2}, \quad n = 1, \ldots, N.$$

1. Construct the histogram $\text{hist}_N(t)$ of data by the whole sample $Y^N = (Y_1, \ldots, Y_N)$. Find $\arg \max_t \text{hist}_N(t)$. An arbitrary point from this set is assumed to be the reference point $\theta_N$ used in the following algorithm for a binary switching model.

1.1. Fix the parameter $b > 0$ and classify observations as follows:

if $\|Y_i - \theta_N\| \leq b$, then we place $Y_i$ into the sub-sample of ordinary observations ($Y^*_N$);

if $\|Y_i - \theta_N\| > b$, then we place $Y_i$ into the sub-sample of abnormal observations ($\hat{Y}^*_N$).
1.2. Then for each $b > 0$ we obtain the following decomposition of the sample $Y^N$ into two sub-samples

$$Y_1(b) = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_{N_1}\}, \quad \|\tilde{Y}_i - \theta_N\| < b,$$

$$Y_2(b) = \{\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_{N_2}\}, \quad \|\hat{Y}_i - \theta_N\| \geq b$$

Denote by $N_1 = N_1(b)$, $N_2 = N_2(b)$, $N = N_1 + N_2$ the sizes of the sub-samples $Y_1$ and $Y_2$, respectively.

The parameter $b$ is chosen so that the sub-samples $Y_1(b)$ and $Y_2(b)$ are separated in the best way. For this purpose, consider the following statistic:

$$\Psi_N(b) = \frac{1}{N^2} \left( N_2 \sum_{i=1}^{N_1} \tilde{Y}_i - N_1 \sum_{i=1}^{N_2} \hat{Y}_i \right).$$

1.3. Define the boundary $C > 0$ and compare it with the value $J = \max \|\Psi_N(b)\|$ on the set $0 < b \leq B$. If $J \leq C$ then we accept the hypothesis $H_0$ about the absence of abnormal observations; if, however, $J > C$ then the hypothesis $H_0$ is rejected and the estimates of the parameters $\epsilon = (\epsilon_1 + \cdots + \epsilon_k)$. Remark that our primary goal is to separate ordinary and abnormal observations in the sample. Evidently, classification errors must be small and therefore we have to require some kind of convergence of the estimate $\hat{\epsilon}_N$ to its true value $\epsilon_1 + \cdots + \epsilon_k$.

1.4. Define the number $b^*_N$:

$$b^*_N \in \arg \max_{0 < b \leq B} \|\Psi_N(b)\|.$$ 

Then

$$\epsilon^*_N = \frac{N_2(b^*_N)}{N}.$$ 

2. As a result, we obtain two classes of observations at the first step (ordinary and abnormal data) and the estimate $\hat{\epsilon}_N$ of the sum $\epsilon_1 + \cdots + \epsilon_k$.

3. Then we remove all found 'ordinary' observations from the sample and repeat steps 1 and 2. As a result, we obtain the estimate $\hat{\epsilon}_1$ of the parameter $\epsilon_1$.

4. So we proceed further until a sub-sample without switches is obtained (i.e. the decision threshold $C$ is not exceeded). As a result, we obtain the estimate $\hat{k}_N$ of the number of classes $k$, as well as the estimates of the parameters $\epsilon_1 > \cdots > \epsilon_k > 0$.

Again we remark that the 1st type error for multiple switchings can be estimated like in the binary case (we do not formulate this result). As to the 2nd type error (i.e. the probability that we stop at the 1st step of the method because the decision
threshold is not exceeded) just observe that a binary switch is a particular case of the
general multiple switching situation (when all $\epsilon_i$ beginning from $i = 2$ are equal to
zero.

Therefore

$$P_{\epsilon}\{2nd\ \text{type error, multiple switches}\} \leq P_{\epsilon}\{2nd\ \text{type error, binary case}\} \leq L_1 \exp(-L(\delta)N),$$

for $0 \leq \delta \leq \max_{0 \leq b \leq B} \|\Psi(b)\| - C$, where $L(\delta) = \min(\frac{\delta^2}{16\phi_0^2g}, \frac{H\delta}{8\phi_0})$, $L_1 = 4\phi_0$.

Now consider the event $\{\hat{k}_N \neq k\} = \{k_N < k\} \cup \{k_N > k\}$.

The event $\{k_N < k\}$ means that at a certain recurrent step of the above described
procedure a sub-sample of remaining observations (after eliminations of all previous
sub-samples) is considered to be "pure" (i.e. without switches) but in reality it contains
some more switches. The probability of this event is less than the 2nd type error at
this step of the procedure. Therefore,

$$P_{\epsilon}\{k_N < k\} \leq L_1 \exp(-L(\delta)N),$$

for $0 \leq \delta \leq \max_{0 \leq b \leq B} \|\Psi(b)\| - C$, where $L(\delta) = \min(\frac{\delta^2}{16\phi_0^2g}, \frac{H\delta}{8\phi_0})$, $L_1 = 4\phi_0$.

The event $\{k_N > k\}$ means that finally some more switches are detected in the
obtained sample than in reality. The probability of this event is less than the 1st type
error at the final step of the above recurrent procedure:

$$P_{*}\{k_N > k\} \leq L_1 \exp(-L(C)N),$$

where $L(C) = \min(\frac{C^2}{16\phi_0^2g}, \frac{HC}{8\phi_0})$, $L_1 = 4\phi_0$.

Therefore the following theorem holds.

**Theorem 7.**

Suppose $0 < C < \max_{0 \leq b \leq B} \|\Psi(b)\|$. Then the 2nd type error probability is estimated
from above as follows:

$$P_{\epsilon}\{2nd\ \text{type error}\} \leq L_1 \exp(-L(\delta)N),$$

where $0 \leq \delta \leq \max_{0 \leq b \leq B} \|\Psi(b)\| - C$, $L(\delta) = \min(\frac{\delta^2}{16\phi_0^2g}, \frac{H\delta}{8\phi_0})$, $L_1 = 4\phi_0$.

Moreover, the estimate of the number of switchings $\hat{k}_N$ converges a.s. to the true
value of $k$ as $N \to \infty$ and

$$P_{\epsilon}\{k_N \neq k\} \leq L_1(\exp(-L(\delta)N) + \exp(-L(C)N)).$$
where $0 \leq \delta = \max_{0 \leq b \leq B} \|\Psi(b)\| - C$ and $L(C) = \min \left( \frac{C^2}{16\phi_0^2}, \frac{HC}{8\phi_0} \right)$, $L(\delta) = \min \left( \frac{\delta^2}{16\phi_0^2}, \frac{H\delta}{8\phi_0} \right)$, $L_1 = 4\phi_0$.

**Example**

Suppose we have the model with three classes of multivariate Gaussian observations:

$$f(x_i) = (1 - \epsilon_1 - \epsilon_2) f_0(x_i - h_1) + \epsilon_1 f_0(x_i - h_2) + \epsilon_2 f_0(x_i - h_3), \quad i = 1, \ldots, N,$$

where $f_0(\cdot)$ has the multivariate Gaussian d.f. with the vector of means $\mu = (\mu_1, \mu_2)'$ and the covariance matrix $\text{Cov}(x_i) = \begin{pmatrix} 0.745 & -0.07 \\ -0.07 & 0.51 \end{pmatrix}$.

The problem is to estimate the unknown number of classes $k = 3$, parameters $h_1, h_2, h_3$, and $\epsilon_1, \epsilon_2$ by the sample $X^N = \{x_1, \ldots, x_N\}$.

Concretely, in this model the following parameters were chosen:

$$\epsilon_1 = 0.3; \quad \epsilon_2 = 0.15$$

$$h_1 = (0 0)', \quad h_2 = (1 2)', \quad h_3 = (2 3)'.$$

We take the norm of the vectors $x_i$ and so reduce this problem to the univariate case considered earlier in this paper.

For estimation of the decision threshold the above formula (2) can be used:

$$\log(C) = -0.9490 - 0.4729*\log(N) + 1.0627*\log(\sigma) - 0.6502*\log(1 - \rho) - 0.2545*\log(1 - \alpha).$$

Again we remark that the main problem is to estimate the number of classes $\hat{k}_N$ (estimation of $h_i$ and $\epsilon_j$ can be done with the help of some standard methods for the given model structure).

In experiments we estimated the number of classes $\hat{k}_N$ and the corresponding error $\hat{\epsilon}_N = P\{\hat{k}_N \neq k\}$.

The following results were obtained (each cell of this table is the average in 1000 independent trials of the test):

**Table 7.**

| $N$ | 100  | 200  | 300  | 500  | 700  | 1000 | 1500 |
|-----|------|------|------|------|------|------|------|
| $\hat{\epsilon}_N$ | 0.991 | 0.910 | 0.707 | 0.189 | 0.049 | 0.020 | 0.004 |
3.2. Switching regressions

The following model of observations was considered:

\[ y_i = X \beta_i + u_i = X(\zeta_i \beta_0 + (1 - \zeta_i) \beta_1) + u_i, \]

where

- \( y \) is a \( N \times 1 \) vector of dependent observations;
- \( X \) is a \( N \times k \) matrix of predictors;
- \( u \) is a \( N \times 1 \) vector of centered random noises;
- \( \beta_i \) is a \( k \times 1 \) vector of model coefficients, \( \zeta_i \) is a Bernoulli distributed r.v. (independent from \( u_i \)) with two states: 1 with the probability \( (1 - \epsilon) \) and 0 with the probability \( \epsilon \) for a certain unknown parameter \( 0 < \epsilon < 1 \). Here \( \beta_0 \neq \beta_1 \).

In terms, we suppose that regression coefficients of this model can change (switch) from the level \( \beta_0 \) to \( \beta_1 \) and the mechanism of this change is purely random. We need to test the hypothesis about the absence of switchings for each coefficient (\( \epsilon = 0 \)) and in the case of rejection of this hypothesis to construct the estimate of the parameter \( \epsilon > 0 \).

For solving this problem, consider the OLS estimate of the vector \( \beta_i \) (here and below ‘ is the symbol of transposition):

\[ \hat{\beta}_i = (X'X)^{-1}X'y_i = \zeta_i \beta_0 + (1 - \zeta_i) \beta_1 + (X'X)^{-1}X'u_i. \]

Since the sequence of noises \( u \) is centered, the problem is reduced to the above considered problem of detection of switches in the mean of an observed random vector. The matrix of predictors \( X \) influences only the random component.

Formally, we need to introduce the following vector \( I = (1, 1, \ldots, 1) \) (\( N \) units) and consider

\[ \tilde{\beta}_i = [\zeta_i \beta_0 + (1 - \zeta_i) \beta_1] I + (X'X)^{-1}X'u_i I. \]

Then the \( (k \times n) \) matrix \( \tilde{\beta}_i \) consists of \( N \) columns of \( k \times 1 \) vectors with means \( \beta_0 \) and \( \beta_1 \) changing in a purely random manner. Each component \( j = 1, \ldots, k \) of these vectors \( \tilde{\beta}_i^j \), \( i = 1, \ldots, N \) is therefore a univariate random sequence

\[ \tilde{\beta}_i^j = [\zeta_i \beta_0^j + (1 - \zeta_i) \beta_1^j]_i + \xi_i^j, \quad i = 1, \ldots, N, \]

where

\[ \xi_i^j = ((X'X)^{-1}X'u I)_i^j. \]
So the problem of detection of changes in regression coefficients is reduced to the above considered problem of detection switches in the mean value of a univariate random sequence. Remark that the uniform Cramer and the \( \psi \)-mixing conditions are still satisfied for the process \( \xi_i^j, i = 1, \ldots, N \). As \( E u_i \equiv 0 \) we get that there exist constants \( g_1 > 0, H_1 > 0 \) such that

\[
E e^{t \xi_i^j} \leq e^{\frac{1}{2} g_1 t^2}, \quad |t| \leq H_1,
\]

for all \( i = 1, \ldots, N, j = 1, \ldots, k \). Moreover, we choose the number \( m_0(\cdot) \) from the \( \psi \)-mixing condition for \( \xi_i^j, i = 1, \ldots, N \): for any chosen number \( \gamma(x) > 0 \): \( \psi(l) \leq \gamma(x) \) for \( l \geq m_0(x) \).

For testing the hypothesis of no switches we again consider the decision statistic \( \Psi_N(b) \) and compare the maximum of its module with the decision threshold \( C > 0 \). Then the following theorem holds:

**Theorem 8.**

Suppose \( \epsilon = 0 \), the d.f. of \( u_i \) is symmetric w.r.t. zero anf the \( \psi \)-mixing and the uniform Cramer conditions for \( \xi_i^j, i = 1, \ldots, N \) are satisfied. Then for any threshold \( C > 0 \) the following upper estimate for the 1st type error probability holds:

\[
P_0 \{ \max_{b > 0} |\Psi_N(b)| > C \} \leq 4 m_0(C N/2) \exp(-L(C)N),
\]

where \( L(C) = \min \left( \frac{H_1 C}{8 m_0(C N/2)}, \frac{C^2}{16 m_0(C N/2) g_1} \right) \), the constants \( g_1, H_1 \) are taken from the uniform Cramer condition.

In order to consider the 2nd type error we just remark that the considered switching regression model is equivalent to the following specification of a model with the binary switches in mean:

\[
f_{\tilde{\beta}^j}(x) = (1 - \epsilon) f_{\xi_j^j}(x - \beta_0^j) + \epsilon f_{\xi_j^j}(x - \beta_1^j).
\]

Denote \( h^j = \beta_1^j - \beta_0^j \neq 0 \) and consider the value

\[
r_{\tilde{\beta}^j}(b) = \int_{\beta_0^j + ch^j - b}^{\beta_0^j + ch^j + b} f_{\tilde{\beta}^j}(x) x \, dx.
\]

Then the following theorem holds.
Theorem 9.
Suppose all assumptions of theorem 8 are satisfied and there exists \( r^* = \sup_b r_{\beta_i'}(b) \).
Suppose also that \( f'_{\xi} (\cdot) \neq 0 \) and continuous. Then for \( 0 < C < \max_b |\Psi_{\beta_i'}(b)| \) we have
\[
P_{\epsilon}\{\max_b |\Psi_N(b)| \leq C\} \leq 4\phi_0(CN/2 + r^*) \exp(-L(\delta)N)
\]
where \( \delta = \max_b |\Psi_{\beta_i'}(b)| - C > 0 \), \( L(\delta) = \min(\frac{\delta^2}{16m_0^2g_1}, \frac{H_1\delta}{8m_0}) \).

Example
In the following example the regression model with one deterministic predictor was considered:
\[
y_i = c_1 + c_2 * i + u_i, \quad u_i \sim N(0; 1), \quad i = 1, \ldots, n.
\]
\[
\xi \sim U[0; 1]
\]
\[
\beta = [c_1; c_2] = \begin{cases} 
\beta_1, & \epsilon_1 < \xi \leq 1 \\
\beta_2, & 0 \leq \xi \leq \epsilon_1
\end{cases}
\]

Table 8.

| \( \epsilon = 0.05 \) | \( \beta_1 = [1; 1], \beta_2 = [1; 2] \) |
|---|---|---|---|
| \( N \) | 300 | 500 | 800 | 1000 |
| \( C \) | 0.07 | 0.05 | 0.04 | 0.03 |
| \( w_2 \) | 0.87 | 0.59 | 0.14 | 0.04 |
| \( \hat{\epsilon} \) | 0.08 | 0.059 | 0.052 | 0.05 |

Table 9.

| \( \epsilon = 0.1 \) | \( \beta_1 = [1; 1], \beta_2 = [1; 1.5] \) |
|---|---|---|---|
| \( N \) | 300 | 500 | 800 | 1000 |
| \( C \) | 0.07 | 0.05 | 0.04 | 0.03 |
| \( w_2 \) | 0.83 | 0.65 | 0.13 | 0.0 |
| \( \hat{\epsilon} \) | 0.15 | 0.12 | 0.102 | 0.10 |

Conclusion
In this paper problems of the retrospective detection/estimation of 'abnormal' observations were considered. The detection/estimation method was proposed. It was
proved that type 1 and type 2 errors of the proposed method converge to zero exponenti- 
ally as the sample size $N$ tends to infinity. The asymptotic optimality of the proposed 
method follows from theorem 3. In this theorem the theoretical lower bound for the 
error of estimation of the model’s parameters was established. This bound is attained 
for the proposed method (by the order of convergence to zero of the estimation error).

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Proofs of theorems.

Proof of theorem 1.
First, let us prove the following inequality:

\[ \max_{b > 0} P_0\{|\Psi_N(b)| > C\} \leq L_1 \exp(-L_2(C)N), \]

where \( L_1, L_2(C) \) are some positive constant and function not depending on \( N \).

For the statistic \( \Psi_N(b) \) we can write:

\[ \Psi_N(b) = (N \sum_{i=1}^{N_1} \tilde{x}_i - N \sum_{i=1}^{N} x_i) / N^2. \]

Then

\[ P_0\{|\Psi_N(b)| > C\} \leq P_0\{\sum_{i=1}^{N_1} \tilde{x}_i > C/2 \} + P_0\{N_1 \sum_{i=1}^{N} x_i > C/2 \}. \]

Further,

\[ P_0\{\sum_{i=1}^{N_1} \tilde{x}_i > C/2 \} = P_0\{\sum_{i=1}^{N_1} \tilde{x}_i > C/2 \} + P_0\{\sum_{i=1}^{N_1} \tilde{x}_i < -C/2 \}. \]

For any \( x > 0 \), let us choose the number \( \gamma(x) \) from the following condition:

\[ \ln(1 + \gamma(x)) = \begin{cases} 
\frac{x^2}{4g}, & x \leq gH \\
\frac{xH}{4}, & x > gH,
\end{cases} \]

where \( g, H \) are taken from the uniform Cramer condition.

For the chosen \( \gamma(x) \), let us find such integer \( \phi_0(x) \geq 1 \) from the \( \psi \)-mixing condition that \( \psi(l) \leq \gamma(x) \) for \( l \geq \phi_0(x) \). Take \( x = CN/2 \) and denote \( \phi_0(CN/2) = \phi_0(\cdot), \ \gamma(CN/2) = \gamma(\cdot) \).

For some fixed \( n \) denote \( S_n = \sum_{i=1}^{n} \tilde{x}_i \) and estimate the probability \( P_0\{S_n > CN/2\} \).

Consider the following decomposition of \( S_n \) into groups of weakly dependent terms:

\[ S_n = S_n^{(1)} + S_n^{(2)} + \ldots + S_n^{(\phi_0(\cdot))}, \]

\[ S_n^{(i)} = \tilde{x}_{i} + \tilde{x}_{i+\phi_0} + \ldots + \tilde{x}_{i+\phi_0([n/\phi_0(\cdot)])}. \]

The number of terms within each group is no less than \([n/\phi_0(\cdot)]\) and no more than \([n/\phi_0(\cdot)] + 1\) and the \( \psi \)-mixing coefficient between terms within each group is no more than \( \gamma(\cdot) \).
Then
\[ P_0 \{ S_n > \frac{C}{2} N \} \leq \sum_{i=1}^{\phi_0(\cdot)} P_0 \{ S_n^{(i)} > \frac{CN}{2\phi_0(\cdot)} \} \]
\[ \leq \phi_0(\cdot) \max_{1 \leq i \leq \phi_0(\cdot)} P_0 \{ |S_n^{(i)}| \geq \frac{CN}{2\phi_0(\cdot)} \}. \]

Consider \( Z_k^{(i)} \) defined by
\[ Z_k^{(i)} = k \sum_{j=0}^{k} \tilde{x}(i + \phi_0(\cdot) j) \]
and obtain the exponential upper estimate for
\[ P_0 \{ Z_k^{(i)} > x \}, \forall x > 0. \]

In virtue of Chebyshev's inequality, we have
\[ P_0 \{ Z_k^{(i)} > x \} \leq e^{-tx} \cdot E_0 e^{tZ_k^{(i)}}, \forall t > 0. \]

From \( \psi \)-mixing condition (see Ibragimov, Linnik, 1971) and choosing \( \gamma(\cdot) \) we have
\[ E_0^{tZ_k^{(i)}} \leq (1 + \gamma(\cdot))^k E_0 \exp(t\tilde{x}(i))E_0 \exp(t\tilde{x}(i + \phi_0)) \ldots E_0 \exp(t\tilde{x}(i + \phi_0 k)). \]
Therefore, for \( 0 \leq t \leq H \)
\[ E_0 \exp(tS_n) \leq (1 + \gamma(\cdot))^N \exp(\frac{1}{2} t^2 g N). \]

Hence,
\[ P_0 \{ S_n(x) > \frac{C}{2} N \} \leq \phi_0(\cdot) (1 + \gamma(\cdot))^N \exp \left( \frac{N}{2} (t^2 g - Ct/\phi_0(\cdot)) \right). \]

Taking the maximum of the right hand w.r.t. \( 0 \leq t \leq H \) and taking into account the choice of \( \gamma(\cdot) \) we have
\[ P_0 \left\{ \sum_{i=1}^{n} \tilde{x}_i > \frac{C}{2} N \right\} \leq \phi_0 \left\{ \begin{array}{ll}
\exp(-\frac{C^2 N}{16\phi_0^2(\cdot)}), & 0 < t < gH, \\
\exp(-\frac{C^2 N}{8\phi_0(\cdot) g}), & t > gH
\end{array} \right. \]
As this estimate does not depend of \( n \), we get
\[ \max_{b > 0} P_0 \{ |\Psi_N(b)| > C \} \leq 4\phi_0(\cdot) \exp(-L(C)N), \]
where
\[ L(C) = \min \left( \frac{HC}{8\phi_0(\cdot)}, \frac{C^2}{16\phi_0^2(\cdot) g} \right). \]

Note that we obtained the uniform (w.r.t. the parameter \( b \)) exponential upper estimate for the first type error. Therefore, the same upper estimate is valid for the probability:
\[ P_0 \{ \max_{b > 0} |\Psi_N(b)| > C \} \leq P_0 \{ \max_{b > 0} \left| \sum_{i=1}^{N_i} \tilde{x}_i \right| > \frac{C}{2} N \} + P_0 \{ \left| \sum_{i=1}^{N} x_i \right| > \frac{C}{2} N \}. \]
In fact, consider the r.v. $U_N(\omega) = \max_{b > 0} |\sum_{i=1}^{N_1} \tilde{x}_i|$ and define

$$\tau_N(\omega) = \min\{1 \leq n \leq N : |\sum_{i=1}^{n} \tilde{x}_i| = U_N\}.$$ 

Then

$$P_0\{U_N > CN/2\} = P_0\{\sum_{k=1}^{N} |\sum_{i=1}^{k} \tilde{x}_i| I\{\tau_N = k\} > CN/2\} \leq P_0\{|\sum_{i=1}^{k_{\max}(\omega)} \tilde{x}_i| > CN/2\};$$

where for any $\omega \in \Omega$: $|\sum_{i=1}^{k_{\max}(\omega)} \tilde{x}_i| = \max_{1 \leq i \leq N} |\sum_{i=1}^{k} \tilde{x}_i|.$

As above, we obtain the uniform upper estimate for the probability

$$P_0\{|\sum_{i=1}^{k_{\max}(\omega)} \tilde{x}_i| > CN/2\} \leq 4\phi_0(\cdot) \exp(-L(C)N),$$

where

$$L(C) = \min \left( \frac{HC}{8\phi_0(\cdot)}, \frac{C^2}{16\phi_0^2(\cdot)g} \right).$$

Theorem 1 is proved.

**Proof of theorem 2.**

Consider the main statistic:

$$\Psi_N(b) = \left( N \sum_{i=1}^{N_1(b)} \bar{x}_i - N_1(b) \sum_{i=1}^{N} x_i \right) / N^2.$$

We have

$$\frac{1}{N} \mathbb{E}_x \sum_{i=1}^{N_1} \tilde{x}_i = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_x \left( \sum_{i=1}^{n} \tilde{x}_i | N_1 = n \right) P_x \{N_1 = n\} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_x(\tilde{x}_i | \theta_N - b < \tilde{x}_i < \theta_N + b) P_x \{N_1 = n\} = \frac{1}{N} \left( \mathbb{E}_x N_1 \right) \int_{eh-b}^{eh+b} f(x) dx / \int_{eh-b}^{eh+b} f(x) dx = \int_{eh-b}^{eh+b} f(x) dx,$$

as $N \to \infty$

Here we used the relation

$$\frac{1}{N} \mathbb{E}_x N_1 = \frac{1}{N} \mathbb{E}_x \sum_{k=1}^{N} k \mathbb{I}(|x_k - \theta_N| \leq b) \to \int_{eh-b}^{eh+b} f(x) dx \text{ as } N \to \infty.$$
Therefore, using the latter relations, taking into account the law of large numbers and the relation

\[ \mathbb{E}_\epsilon x_i = \epsilon h \]

we have

\[ \mathbb{E}_\epsilon \Psi_N(b) \to \Psi(b) \quad \text{as} \ N \to \infty, \]

where \( \Psi(b) = r(b) - \epsilon h d(b) \).

For any \( C > 0 \) we can write:

\[
P_\epsilon \{ |\Psi_N(b) - \Psi(b)| > C \} \leq P_\epsilon \{ \frac{N_1(b)}{N} \sum_{i=1}^{N_1(b)} \tilde{x}_i - N r(b) | > \frac{C}{2} N \} + P_\epsilon \{ \frac{N_1(b)}{N} \sum_{i=1}^{N} x_i - N \epsilon h d(b) | > \frac{C}{2} N \}. \tag{3} \]

Consider the first term in the right hand:

\[
P_\epsilon \{ \sum_{i=1}^{N_1(b)} \tilde{x}_i - N r(b) | > \frac{C}{2} N \} = P_\epsilon \{ \sum_{i=1}^{N_1(b)} \tilde{x}_i > \frac{C}{2} N + N r(b) \} + P_\epsilon \{ \sum_{i=1}^{N_1(b)} \tilde{x}_i < - \frac{C}{2} N + N r(b) \}. \tag{4} \]

Analogously theorem 1, we put \( x = C N/2 + r(b) N \), find \( \phi_\alpha(x) \overset{\text{def}}{=} \phi_\alpha(\cdot) \) corresponding to this \( x \), decompose the sum \( \sum_{i=1}^{N_1(b)} \tilde{x}_i \) into \( \phi_\alpha(\cdot) \) groups of weakly dependent components and for each of these groups use Chebyshev’s inequality.

Using considerations analogous to those in theorem 1, finally, for large enough \( N \) we obtain:

\[
P_\epsilon \{ \sum_{i=1}^{N_1(b)} \tilde{x}_i > \frac{C}{2} N + N r(b) \} \leq \phi_\alpha(\cdot) \begin{cases} \exp(-\frac{C^2 N}{16 \phi_\alpha^2(\cdot) g}), & 0 < t < gH, \\ \exp(-\frac{C H N}{8 \phi_\alpha(\cdot)}), & t > gH \end{cases} \]

The second term in the right hand of (4) is estimated from above in the same way.

As to the second term in the right hand of (3), since \( N_1(b) \leq N \) for any \( \omega \), we obtain an analogous exponential upper estimate for it.

Again remark that we obtained the uniform (w.r.t. \( b > 0 \)) exponential upper estimate for the error probability. Therefore as in theorem 1 we can prove the following exponential estimate:

\[
P_\epsilon \{ \max_b |\Psi_N(b) - \Psi(b)| > C \} \leq 4 \phi_\alpha(\cdot) \begin{cases} \exp(-\frac{C^2 N}{16 \phi_\alpha^2(\cdot) g}), & 0 < C < gH, \\ \exp(-\frac{C H N}{8 \phi_\alpha(\cdot)}), & C > gH \end{cases} \]

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For type 2 error we can write:

\[ P\{ \max_b |\Psi_N(b)| < C \} \leq P\{ \max_b |\Psi_N(b) - \Psi(b)| > \max_b |\Psi(b)| - C \} \]

\[ \leq 4\phi_0(\cdot) \begin{cases} \exp\left( -\frac{N}{16} \frac{\delta^2}{\phi_0(\cdot)g} \right), & 0 < \delta \leq gH, \\ \exp\left( -\frac{NH\delta}{8\phi_0(\cdot)} \right), & \delta > gH, \end{cases} \]

where \( \delta = \max_b |\Psi(b)| - C \).

This completes the proof of 1).

As to the proof of 2), remark that the function \( \Psi(b) = E_N\Psi_N(b) \) satisfies the reversed Lipschitz condition in a neighborhood of \( b^* \).

In fact, we have \( \Psi(b^*) = 0, \Psi'(b^*) = 0 \) and \( \Psi''(b^*) = (f(\epsilon h + b^*) - f(\epsilon h - b^*)) + b^*(f'(\epsilon h + b^*) - f'(\epsilon h - b^*)) = 2(b^*)^2 f''(u) \neq 0, \) where \( 0 \leq u = u(b^*) \leq b^* \). Therefore in a small neighborhood of \( b^* \) we obtain:

\[ |\Psi(b) - \Psi(b^*)| = (b^*)^2 |f''(u(b^*))| (b - b^*)^2 \geq C(b - b^*)^2, \]

for a certain \( C = C(b^*) > 0 \).

Now for any \( 0 < \kappa < 1 \) consider the event \( |b_N - b^*| > \kappa \). Then

\[ P\{ |b_N - b^*| > \kappa \} \leq P\{ \max_b |\Psi_N(b_N) - \Psi(b^*)| > \frac{1}{2} C\kappa^2 \} \leq 4 \phi_0(\cdot) \exp(-L(C)N), \]

where \( L(C) = \min\left( \frac{C^2\kappa^4}{8\phi_0^2(\cdot)g}, \frac{HC\kappa^2}{16\phi_0(\cdot)} \right) \).

From this inequality it follows that \( b_N \to b^* \) \( P\)-a.s. as \( N \to \infty \).

Then

\[ \epsilon_N = N_2(b_N)/\kappa, \quad h_N = \theta_N/\epsilon_N \]

are the nonparametric estimates for \( \epsilon \) and \( h \), respectively.

In general these estimates are asymptotically biased and non-consistent. For construction of consistent estimates of \( \epsilon \) and \( h \), we need information about the d.f. \( f_0(\cdot) \).

These consistent estimates can be obtained from the following system of equations:

\[ \frac{\epsilon_N\hat{h}_N}{\epsilon_N} = \theta_N \]

\[ 1 - \epsilon_N = \frac{f_0(\theta_N - b_N - \hat{h}_N) - f_0(\theta_N + b_N - \hat{h}_N)}{f_0(\theta_N + b_N) - f_0(\theta_N - b_N)}. \]

The estimates \( \epsilon_N \) and \( \hat{h}_N \) are connected with the estimate \( b_N \) of the parameter \( b^* \) via this system of deterministic algebraic equations. Therefore the rate of convergence
\( \hat{\epsilon}_N \to \epsilon \) and \( \hat{h}_N \to h \) is determined by the rate of convergence of \( b_N \) to \( b^* \) (which is exponential w.r.t. \( N \)). So we conclude that \( \hat{\epsilon}_N \to \epsilon \) and \( \hat{h}_N \to h \) \( \mathbb{P}_\epsilon \)-a.s. as \( N \to \infty \).

Theorem 2 is proved.