NOTE ON WERMUTH’S THEOREM ON COMMUTING OPERATOR EXPONENTIALS

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Abstract. We apply Wermuth’s theorem on commuting operator exponentials to show that if \( A, B \in B(X) \), \( X \) being Banach space and \( A \) of \( 2\pi i \)-congruence free spectrum, then \( e^A B = B e^A \) if and only if \( AB = BA \). We employ this observation to provide alternative proof of similar result by Chaban and Mortad, applicable for \( X \) being a Hilbert space.

1. Introduction. Wermuth’s theorem

It is well known that if two elements \( a, b \) of noncommutative unital Banach algebra commute, then their exponentials \( e^a \) and \( e^b \) also commute. Although the converse statement is wrong in general, it was shown by E. Wermuth that if \( A \) and \( B \) are bounded operators on Banach space satisfying additional condition of being of \( 2\pi i \)-congruence free spectrum, then the opposite implication also remains true.

Definition 1. Let \( S \subset \mathbb{C} \) and let \( z \in \mathbb{C} \) be arbitrarily chosen. We say that \( S \) is \( z \)-congruence free if and only if no two different elements \( s_1, s_2 \in S \) exists such that \( s_1 = s_2 \mod z \). Equivalently,

\[
\forall s_1, s_2 \in S : s_1 - s_2 \neq kz, \quad k \in \mathbb{Z} \setminus \{0\}.
\] (1.1)

Let \( B(X) \) be a Banach algebra of all bounded linear endomorphisms over Banach space \( X \). The Wermuth’s theorem is formulated for pairs of operators \( A, B \in B(X) \) and involves \( 2\pi i \)-congruence freedom of both spectra \( \sigma(A), \sigma(B) \) and can be stated as follows:

Theorem 1 (Wermuth). Let \( A, B \in B(X) \) and let both \( \sigma(A), \sigma(B) \subset \mathbb{C} \) to be \( 2\pi i \)-congruence free. Then, \( e^A e^B = e^B e^A \) if and only if \( AB = BA \).

Original formulation \([1]\) of Wermuth’s theorem concerned finite dimensional matrix algebra \( M_n(\mathbb{C}) \) and then was generalized \([2]\) to the case of \( B(X) \) for any Banach space \( X \). Later on, several results concerning commutativity of exponentials (or their functions) in noncommutative unital algebras emerged (see e.g. \([3–7]\)), with or without making explicit use of \( 2\pi i \)-congruence freedom hypothesis; in particular, interesting result was obtained by Chaban and Mortad in \([8]\) for a case of \( C^* \)-algebra \( B(H) \) for Hilbert space \( H \), which roughly says that if \( A \in B(H) \) is normal operator of “well-behaved” spectrum, then \( e^A B = B e^A \) if and only if \( AB = BA \). In this paper, we formulate a similar result in the Banach space setting in Section 2. This allows us to we present alternative proof of theorem by Chaban and Mortad in Section 3.

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2. The main result

We start with a simple observation of “shrinking”, or “rescaling” property of bounded subsets of complex plane (lemma 1 below). Our main result is then formulated as Theorem 2.

Lemma 1. For every bounded nonempty set \( U \subset \mathbb{C} \) and every \( z \in \mathbb{C} \setminus \{0\} \), there exists such \( \tau > 0 \) small enough, that set \( tU = \{tw : w \in U\} \) is \( z \)-congruence free for every \( t \in (0, \tau) \).

Proof. Let \( U \subset \mathbb{C} \) be nonempty and bounded and let

\[
\Delta = \sup_{z_1, z_2 \in U} |z_1 - z_2|
\]

be its diameter. If \( \Delta > 0 \), define

\[
\tau = \frac{|z|}{\Delta}.
\]

Then, for every \( t \in (0, \tau) \) and every pair of complex numbers \( z_1, z_2 \in U \) we have

\[
t|z_1 - z_2| < \tau \Delta = |z|.
\]

Let \( tU = \{tz : z \in U\} \) and take any \( w_1, w_2 \in tU \). As \( w_1 = tz_1 \) and \( w_2 = tz_2 \) for some \( z_1, z_2 \in U \), equation (2.3) implies

\[
|w_1 - w_2| = t|z_1 - z_2| < |z|,
\]

which automatically results in

\[
w_1 - w_2 \neq kz, \quad k \in \mathbb{Z} \setminus \{0\}
\]

for every \( w_1, w_2 \in tU \), i.e. \( w_1 \neq w_2 \) (mod \( z \)) and \( tU \) is \( z \)-congruence free for any \( t \in (0, \tau) \). On the other hand, if \( \Delta = 0 \), i.e. \( U = \{z_0\} \) is a singleton, then set \( tU \) is automatically \( z \)-congruence free for any \( t \in (0, \infty) \) as \( w_1 = w_2 = tz_0 \) and

\[
w_1 - w_2 = 0 \neq kz, \quad k \in \mathbb{Z} \setminus \{0\}.
\]

Then, one can take any \( \tau \in (0, \infty) \).

\( \square \)

Theorem 2. Let \( A, B \in B(X) \) and let \( \sigma(A) \) be \( 2\pi i \)-congruence free. Then, \( e^A B = Be^A \) if and only if \( AB = BA \).

Proof. We only need to address the “\( \Rightarrow \)” direction. Assume \( e^A B = Be^A \). Then, \( e^A \) commutes also with every analytic function of \( B \), so in particular, for all \( t \in \mathbb{R} \),

\[
e^A e^{tB} - e^{tB} e^A = 0.
\]

As \( \sigma(B) \) is a nonempty bounded subset of \( \mathbb{C} \), lemma 1 invoked for \( U = \sigma(B) \) guarantees that there exists such \( \tau > 0 \) that \( t\sigma(B) \) is \( 2\pi i \)-congruence free for any \( t \in (0, \tau) \). Therefore, operator \( tB \) is of \( 2\pi i \)-congruence free spectrum. By virtue of Wermuth’s result (theorem 1),

\[
e^A e^{tB} - e^{tB} e^A = 0 \quad \Rightarrow \quad A \cdot tB - tB \cdot A = t(AB - BA) = 0
\]

for every \( t \in (0, \tau) \), so \( A \) and \( B \) commute. Necessity is then clear.

\( \square \)
Remark 1. The assumption of $2\pi i$-congruence freedom of $\sigma(A)$ cannot be neglected, as the following (canonical) counterexample in $M_2(\mathbb{C})$ shows: let us choose $A$ and $B$ as, say,

$$A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

for some $a \in \mathbb{R}$. It is easy to verify by direct algebra that $e^A = -I$, so $e^A$ commutes with (any) $B$; however,

$$AB - BA = \begin{pmatrix} 2\pi a & 0 \\ 0 & -2\pi a \end{pmatrix}$$

which does not vanish unless $a = 0$ and matrices $A$ and $B$ do not commute in general. This is not surprising, as one easily shows $\sigma(A) = \{\pi i, -\pi i\}$ and hence, $\sigma(A)$ is not $2\pi i$-congruence free.

3. Relation to result of Chaban and Mortad

As we mentioned in the Introduction, a similar result concerning $C^*$-algebra $B(H)$, $H$ being a Hilbert space, was obtained some time ago by Chaban and Mortad [8]. For any $T \in B(H)$ we define its unique Cartesian decomposition of a form

$$T = \text{Re} T + i \text{Im} T,$$

where $\text{Re} T$ and $\text{Im} T$, called the real part and the imaginary part of $T$, respectively, are self-adjoint and bounded and given by

$$\text{Re} T = \frac{1}{2}(T + T^*), \quad \text{Im} T = \frac{1}{2i}(T - T^*).$$

**Theorem 3 (Chaban and Mortad).** Let $A, B \in B(H)$ be such that $A$ is normal and $\sigma(\text{Im} A) \subset (0, \pi)$, we have

$$e^A B = B e^A \iff AB = BA.$$ **(3.3)**

This theorem is then proved by means of methods different to ours by employing e.g. Fuglede theorem and without making direct references to $2\pi i$-congruence freedom. However, one can easily show that the $2\pi i$-congruence freedom is in fact the case here which allows to formulate alternative proof of the above result by directly applying theorem 2 (and hence Wermuth’s theorem in consequence).

**Lemma 2.** Let $A \in B(H)$ satisfy assumptions of theorem 3, i.e. $A$ is normal and $\text{Im} A \subset (0, \pi)$. Then, $\sigma(A)$ is $2\pi i$-congruence free.

**Proof.** If $A$ is normal, i.e. $AA^* = A^*A$, then the Cartesian decomposition of $A$ constitutes of a pair $(\text{Re} A, \text{Im} A)$ of commuting normal self-adjoint bounded operators. In such case one can show, applying the Gelfand transform, that spectrum $\sigma(A)$ satisfies

$$\sigma(A) \subset \sigma(\text{Re} A) + i \sigma(\text{Im} A).$$

For convenience, let us enclose $\sigma(A)$ by a rectangle in $\mathbb{C}$. As $\text{Re} A = (\text{Re} A)^*$ and $\|\text{Re} A\| < \infty$, its spectrum is a bounded subset of $\mathbb{R}$ and one can define

$$J = \inf \{[x_1, x_2] \subset \mathbb{R} : \sigma(\text{Re} A) \subset [x_1, x_2]\}$$

i.e. $J$ is the smallest interval containing the whole spectrum of $\text{Re} A$. Then,

$$\sigma(A) \subset J + i(0, \pi).$$

**(3.6)**
Take any two \( \lambda_1, \lambda_2 \in J + i(0, \pi) \), \( \lambda_k = a_k + ib_k \) for \( a_k \in J, b_k \in (0, \pi), k \in \{1, 2\}; \) there are two possible cases:

1. If it happens that \( a_1 = a_2 \), then
   \[
   |\lambda_1 - \lambda_2| = |b_1 - b_2| \leq \sup_{b_1, b_2 \in (0, \pi)} |b_1 - b_2| = \pi < 2\pi,
   \]
   hence \( \lambda_1 - \lambda_2 = i(b_1 - b_2) \neq 2k\pi i \) for every \( b_1, b_2 \in (0, \pi) \) and every \( k \in \mathbb{Z} \setminus \{0\} \).

2. If, on the other hand \( a_1 \neq a_2 \), then automatically
   \[
   \lambda_1 - \lambda_2 = a_1 - a_2 + i(b_1 - b_2) \neq 2k\pi i, \quad k \in \mathbb{Z} \setminus \{0\}.
   \]
In consequence, set \( J + i(0, \pi) \) is \( 2\pi i \)-congruence free; the same can be then stated about \( \sigma(\text{Im} A) \) as its subset. \( \square \)

Finally, we present an alternative proof of theorem by Chaban and Mortad as a straightforward corollary of the above observation:

**Proof of theorem 3.** If \( \sigma(\text{Im} A) \subset (0, \pi) \), then \( \sigma(A) \) is \( 2\pi i \)-congruence free by lemma 2. Hence, theorem 2 applies. \( \square \)

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