NOTES ON FIBONACCI PARTITIONS

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To my children Sergej, Jelena, and Maria with love

Abstract. Let \( f_1 = 1, f_2 = 2 \) and \( f_i = f_{i-1} + f_{i-2} \) for \( i > 2 \) be the sequence of Fibonacci numbers. Let \( \Phi_i(n) \) be the quantity of partitions of natural number \( n \) into \( h \) different Fibonacci numbers. In terms of Zeckendorf partitions I deduce a formula for the function \( \Phi(n; t) := \sum_{h \geq 1} \Phi_h(n)t^h \), and use it to analyze the functions \( F(n) := \Phi(n; 1) \) and \( \chi(n) := \Phi(n; -1) \). I obtain the least upper bound for \( F(n) \) when \( f_i \leq n < f_{i+1} \). In particular, it implies that \( F(n) \leq \sqrt{n+1} \) for any natural \( n \).

I prove also that \( |\chi(n)| \leq 1 \), and \( \lim_{n \to \infty} (\chi^2(1) + \chi^2(2) + \ldots + \chi^2(N)) = 0 \).

For any \( k \geq 2 \), I define a special finite set \( G(k) \) of solutions of the equation \( F(n) = k \); all solutions can be easily obtained from \( G(k) \). This construction uses a non-standard representation of rational numbers as certain continued fractions and provides with a canonical identification \( \prod_{k \geq 2} G(k) = G_+ \), where \( G_+ \) is the monoid freely generated by the positive rational numbers \( < 1 \).

Let \( \Psi(k) \) be the cardinality of \( G(k) \). I prove that, for \( i \geq 2k \) and \( k \geq 2 \), the interval \( [f_i, f_{i+1} - 2] \) contains exactly \( 2\Psi(k) \) solutions of the equation \( F(n) = k \) and offer a formula for the Dirichlet generating function of the sequence \( \Psi(k) \). In addition, I study the set of minimal solutions of the equation \( F(n) = k \) as \( k \) varies and I offer a conjecture on the distribution of such solutions.

1. Introduction

Most of the results of this article appeared initially as conjectures, based on computer experiments. Several conjectures I was unable to prove are formulated in what follows.

Let \( f_0 = f_1 = 1 \) and \( f_i = f_{i-1} + f_{i-2} \) for \( i \geq 2 \) be the sequence of Fibonacci numbers. A Fibonacci partition of a natural number \( n \) is a representation \( n = f_{i_1} + f_{i_2} + \ldots + f_{i_h} \), where \( 1 \leq i_1 < i_2 < \ldots < i_h \). The numbers \( f_{i_1}, f_{i_2}, \ldots, f_{i_h} \) are referred to as the parts of the Fibonacci partition.

Let \( \Phi_h(n) \) be the quantity of Fibonacci partitions of the natural number \( n \) with \( h \) parts. Define

\[
\Phi(n; t) := \sum_{h \geq 1} \Phi_h(n)t^h,
\]

and \( \Phi(0; t) := 1 \). The main Theorem 2.11 offers an explicit formula for \( \Phi(n; t) \) in terms of the Zeckendorf partition of \( n \), see Def.2.3 This formula is the foundation for all subsequent results of the article.

Furthermore, I study properties of the function \( F(n) := \Phi(n; 1) \) which counts the quantity of all Fibonacci partitions of \( n \), and the function \( \chi(n) := \Phi(n; -1) \) which counts the difference between the quantities of Fibonacci partitions of \( n \) with even and odd number of parts. Theorem 2.11 easily implies that

\[
\prod_{i=1}^{\infty} (1 - x^i) = 1 + \sum_{n=1}^{\infty} \chi(n)x^n,
\]

where \( \chi(n) = 0, \pm 1 \).

Let \( \mathbb{N} \) be the set of natural numbers (i.e., the positive integers), and let \( \mathbb{N}(k) \subset \mathbb{N} \) be the set of all solutions of the equation \( F(n) = k \). Theorem implies that \( F(n) = 1 \) whenever \( n = f_i - 1 \), where \( i \geq 1 \). The main goal of Section 3 is to offer an algorithm, which calculates the set \( \mathbb{N}(k) \), where \( k \geq 2 \).

Let me outline its construction. The main idea originated from Theorem 2.11 which may be interpreted by means of the representations of rational numbers as “nonstandard” continued fractions (observe minus sign in formula (10)). On \( \mathbb{N} \), I explicitly define, using this connection, a canonical action of the semigroup

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1 The “conventional” indexing of Fibonacci numbers, see [http://en.wikipedia.org/wiki/Fibonacci_number](http://en.wikipedia.org/wiki/Fibonacci_number) is different from indexing in this article. The formulations of main statements are more neat with our indexing than with “conventional” one, cf. [9], where the same problem is encountered.
such that \( n \in \mathbb{N} \setminus \mathbb{N}(1) \). I canonically define a fundamental domain \( G \subset \mathbb{N} \setminus \mathbb{N}(1) \) of this action. The elements of \( G \) are called the \textit{generating numbers}. Set
\[
G(k) := \{ n \in G \mid F(n) = k \}.
\]
Then \( G = \bigsqcup_{k \geq 2} G(k) \). The elements of \( G(k) \) are said to be the \textit{k-generating numbers}. The set of generating numbers possesses several remarkable properties; it has a natural structure of a noncommutative monoid canonically isomorphic to the \textit{monoid freely generated by the set of positive rational numbers} \(< 1\). Denote by \( \times \) the monoidal multiplication. Then, for \( n_1, n_2 \in G \), we have \( F(n_1 \times n_2) = F(n_1)F(n_2) \). Thus, \( G(k_1) \times G(k_2) \subset G(k_1k_2) \). Moreover, for any \( k \geq 2 \), the set \( G(k) \) is finite. Set
\[
\Psi(k) := |G(k)|.
\]
The algorithm presented in Section 3 makes it possible to build the sets \( G(k) \). Any \( n_0 \in G(k) \) obtained generates an infinite set of solutions of the equation \( F(n) = k \) as the \( H \)-orbit of \( n_0 \). For different numbers \( n_0 \in G(k) \), the corresponding orbits do not intersect, and their union coincides with the set \( \mathbb{N}(k) \). Therefore, since the definition of \( H \)-action is explicit, we can obtain explicit formulas for all solutions of the equation \( F(n) = k \) as soon as the set of \( k \)-generating numbers and their corresponding Zeckendorf partitions are known.

For example, for a prime number \( k \), all solutions are easy to find if we know the representation of any rational number \( \frac{m}{k} \), where \( m = 1, 2, \ldots, k - 1 \), as a continued fraction of the form \([10]\). It is interesting to note that in our approach a \textit{noncommutative object} the monoid \( G \) of generating numbers with multiplication operation \( \times \) quite naturally appears. Moreover, the monoidal structure of \( G \) allows us to completely describe the structure of the whole set of solutions of the equation \( F(n) = k \) as \( k \) varies.

Although in this article it is not used, in Theorem 3.9 I give a transparent description of \( G \):
\[
G = \left\{ 2|\phi| + l \mid l \in \mathbb{N} \text{ and } \phi := \frac{1}{2}(1 + \sqrt{5}) \right\}.
\]
In this form the sequence of generating numbers was known since at least 1992 (see [10], A003623). Auxiliary Section 4 contains some inequalities needed in the next two sections. In Section 5 first I show that the maximal \( k \)-generating number is equal to \( f_{2k} - 2 \). Using this and the \( H \)-action on \( \mathbb{N} \) I establish a “stabilization” of quantities of Fibonacci partitions in the following sense: if \( i \geq 2k \), then
\[
| \{ n \mid f_i - 1 \leq n < f_{i+1} - 1 \text{ and } F(n) = k \} | = \begin{cases} 1 & \text{if } k = 1, \\ 2\Psi(k) & \text{if } k > 1. \end{cases}
\]
In particular, this gives the asymptotic
\[
\left| \{ n \leq N \mid F(n) = k \} \right| \sim_{N \to \infty} 2\Psi(k) \log_\phi(N), \quad \text{where } k \geq 2.
\]
In Section 6 for all \( n \) such that \( f_i - 1 \leq n < f_{i+1} - 1 \), I establish the least upper bound
\[
F(n) \leq \begin{cases} f_{i+1} & \text{if } i \equiv 1 \mod 2, \\ 2f_{i-1} & \text{if } i \equiv 0 \mod 2, \end{cases}
\]
and list all the cases in which the equalities hold.

One corollary of this result is the inequality \( F(n) \leq \sqrt{n+1} \) valid for any natural \( n \), where the equality holds whenever \( n = f_i^2 - 1 \).

In addition, it turns out that if \( i \not\in \{1, 2, 3, 4, 6, 9\} \) and \( f_i - 1 \leq f_{i+1} - 1 \), the upper bound of \( F(n) \) is attained for exactly 2 values of \( n \) if \( i \equiv 1 \mod 2 \), and for exactly 4 values of \( n \) if \( i \equiv 0 \mod 2 \). The material of Section 7 concerns the function \( \chi(n) \). Since \( |\chi(n)| \leq 1 \), the quantity of natural numbers \( n \) such that \( n \leq N \) with \( \chi(n) = \pm 1 \) is equal to
\[
X(N) := \chi^2(1) + \chi^2(2) + \cdots + \chi^2(N).
\]
The main result of Section 8 (Theorem 7.1) establishes that \( \lim_{N \to \infty} \frac{X(N)}{N} = 0 \). This is a bit surprising since the natural numbers \( n \) with \( \chi(n) \neq 0 \) appear rather often. For example, \( X(f_{26}) = X(196418) = 46299 \). The proof is based only on Theorem 2.11. Additional information on the behaviour of sequence \( \chi(n) \) is contained in Remark 7.5.

Computing the minimal \( k \)-generating number, i.e., the minimal solution of the equation \( F(n) = k \), is a difficult task. Denote this solution by \( m_F(k) \). The main result of Section 8 states that there exists a uniquely defined proper subset \( P \subseteq \mathbb{N} \) such that, for any natural \( k \geq 2 \), there is a unique decomposition

\[
m_F(k) = m_F(k_1) \times m_F(k_2) \times \cdots \times m_F(k_{r(k)}),
\]

where \( k_1, k_2, \ldots, k_{r(k)} \in P \) and \( r(k) > 1 \) whenever \( k \notin P \).

The numbers — elements of \( P \) will be called \( F \)-primitive. The set \( P \) contains prime numbers, Fibonacci numbers, and certain other numbers. In general, the nature of \( P \) is completely obscure to me. To make it clearer, I analyzed the numerical data presented in [10], A013583. As a result, I was able to formulate in Section 8 a conjecture concerning distribution of the numbers \( m_F(k) \) when \( k \in P \). Based on this analysis, I can also conjecture that asymptotically we have

\[
|\{k \leq N \mid k \text{ is } F\text{-primitive}\}| \sim cN, \quad \text{where } c \approx 0.35.
\]

For several remarks on the graph of the function \( F \), see Section 8. The main goal of Section 8 is to offer a conjecture describing the boundary of the convex hull of the set of points \( \{(n, F(n)) \mid n \in [f_i - 1, f_{i+1} - 1]\} \).

In Section 8, I study the function \( \Psi(k) \), and some arithmetical functions naturally related to it. (As an aside, observe that \( \Psi(k) \) was first defined by R. Munafó in 1994 as the quantity of continental \( \mu \)-atoms of period \( k \) in the Mandelbrot set, see 10, A006874.)

I show how one can recursively calculate \( \Psi(k) \). Furthermore, in Theorem 10.3 I prove a formula which implies the following expression for the Dirichlet generating function of the sequence \( \Psi(k) \) in terms of the Riemann \( \zeta \)-function:

\[
1 + \sum_{k=2}^{\infty} \frac{\Psi(k)}{k^s} = \left(2 - \frac{\zeta(s-1)}{\zeta(s)}\right)^{-1}.
\]  

Additional arithmetical functions considered in Section 10 appear as generating functions for orbit sets of some naturally defined subgroups in the automorphism group of the monoid \( G \). In passing, three known sequences appear here: ordered Bell numbers, Bell numbers, and numbers of the unordered multiplicative partitions. This part of Section 10 does not contain results directly related to Fibonacci partitions, but is a collection of ad hoc remarks.

Most of the above-mentioned results were obtained at the end of 1994 and presented in 1995 as an internal report of Bern University (Report No. 329/FW 1, January 1995, pp. 129). In 2003, I put a revised version of the report in arXiv.

Here, essential additions to the 2003-version are the main result of Section 8, result of Section 8, and formula 1. Among other things, the result of Section 8 implies the inequality \( F(n) \leq \sqrt{n} + 1 \) in a more natural and clear way as compared with the earlier proof. I also give an explicit algorithm for deriving all solutions of the equation \( F(n) = k \); the 2003-version contains only an implicit form of the algorithm.

Two of the mentioned results were published earlier: the inequality \( |\chi(n)| \leq 1 \) was obtained by N. Robbins in [9]; Theorem 7.1 was obtained by F. Ardilla in [1]. My proofs of these claims are based on Theorem 2.11 and differ from the proofs in [9] and [1].

History. Let me shortly outline some earlier publications on Fibonacci partitions. Mainly, they appeared in Fibonacci Quarterly during 1960s. Almost all of them (e.g., [6] [8] [2]) deal with the calculation of \( F(n) \) for some particular values of \( n \) like \( n = f_i, f_i - 1, f_i^2 - 1, f_2 f_{2i+1}, \) etc. In [8], the solutions of the equation \( F(n) = k \) for \( k = 1, 2, 3 \) were also obtained. During that time the only general result on \( F(n) \) was presented in the article by Carlitz [2], where a recurrent formula for \( F(n) \) was established by means of the Zeckendorf partition. But, as Carlitz notes, it is too complicated for practical usage. Later on, two other general results were obtained in the already quoted articles [9] and [1]. To the best of my knowledge, the cited articles cover all currently published essential information on Fibonacci partitions.

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explained his result, N. Robbins for sending me a draft of his article [9], R. P. Stanley who pointed me to the paper [1], and A. M. Vershik for useful discussions.

**Notation:**
- \( \mathbb{Z} \) – the ring of integer numbers.
- \( \mathbb{N} \) – the commutative semigroup (by multiplication) of natural (positive integer) numbers.
- \( \mathbb{Q}_{>0} \) – the set of positive rational numbers.
- \( \mathbb{Q}_{(0,1)} \) – the set of positive rational numbers \(< 1\).
- \( |M| \) – the cardinality of the set \( M \).
- \( M_1 \sqcup M_2 \) – the disjoint union of sets \( M_1 \) and \( M_2 \).
- \( \varphi(n) \) – the quantity of naturals \( r < n \) relatively prime to \( n \) (the Euler totient function).
- \([a]\) and \([a]\) denote the maximal integer \( \leq a \) and the minimal integer \( \geq a \), respectively.

For a finite set of numbers \( I \), define \( I^- := \min(I) \) and \( I^+ := \max(I) \).

For any \( a \in \mathbb{Z} \), define \( \beta(a) := \begin{cases} 0 & \text{if } a \equiv 0 \pmod{2}, \\ 1 & \text{if } a \equiv 1 \pmod{2}. \end{cases} \)

\( m \cdot a \) for any \( m \in \mathbb{N} \) is a shorthand for \( a, \ldots, a \), where \( a \) is repeated \( m \) times.

\([a, b] := \{n \in \mathbb{Z} \mid a, b \in \mathbb{Z} \text{ and } a \leq n \leq b\} \)

### 2. Quantity of Fibonacci partitions

In this section we formulate and prove the main result of the paper – Theorem 2.11. First, let us introduce necessary definitions.

**Definition 2.1.** A (strict) partition is a finite ordered set \( I = \langle i_1, i_2, \ldots, i_h \rangle \) of natural numbers such that \( i_1 < i_2 < \ldots < i_h \). These numbers are called parts of the partition \( I \). The numbers \( \|I\| = i_1 + i_2 + \ldots + i_h \) and \( |I| = h \) are called the degree and length of \( I \), respectively.

**Definition 2.2.** The union \( I_1 \sqcup I_2 \) of partitions \( I_1 \) and \( I_2 \) with \( I^+_1 < I^-_2 \) is the partition whose set of parts is the union of the sets of parts of \( I_1 \) and \( I_2 \).

**Definition 2.3.** A 2-partition is a partition \( \langle i_1, i_2, \ldots, i_h \rangle \) such that \( i_{r+1} - i_r \geq 2 \) for all \( r \) such that \( 1 \leq r < h \). The set of all 2-partitions is denoted by \( \mathcal{P} \).

**Definition 2.4.** A 2-partition \( \langle i_1, i_2, \ldots, i_h \rangle \) is called simple if \( \beta(i_1) = \beta(i_2) = \ldots = \beta(i_h) \).

For any 2-partition \( I \), there exists a unique decomposition \( I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_r \), called the canonical decomposition of \( I \), where \( I_1, I_2, \ldots, I_r \) are simple 2-partitions and \( \beta(I_m) \neq \beta(I_{m+1}) \) for all \( m \) such that \( 1 \leq m < r \).

For example, \( (3, 5, 10, 12) = (3, 5) \sqcup (10, 12) \).

**Definition 2.5.** For any 2-partition \( I = \langle i_1, i_2, \ldots, i_h \rangle \), define a vector \( a(I) = (a_1, a_2, \ldots, a_h) \), where

\[
a_l = \begin{cases} \left\lfloor \frac{i_l-1}{2} \right\rfloor + 1 & \text{if } l = 1, \\ \left\lfloor \frac{i_l-1}{2} \right\rfloor + 1 & \text{if } 1 < l \leq h. \end{cases}
\]

Note that \( a_1 \geq 1 \) and \( a_l \geq 2 \) if \( l \geq 2 \). For example, \( a((3, 5, 10, 12)) = (2, 2, 3, 2) \).

**Definition 2.6.** Let \( I \) be a 2-partition, let \( I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_r \) be the canonical decomposition, and let \( a(I) = (a_1(I), a_2(I), \ldots, a_r(I)) \), where \( |\alpha_m(I)| = |I_m| \) for any \( m = 1, 2, \ldots, r \).

A polyvector of \( I \) is an expression of the form

\[
a(I) = \alpha_1(I) \times \alpha_2(I) \times \cdots \times \alpha_r(I).
\]
Proof. If $\alpha((3, 5, 10, 12)) = \alpha((3, 5) \cup (10, 12)) = (2, 2) \times (3, 2)$.
Let $I$ be a 2-partition and let $\alpha(I) = \alpha_1(I) \times \alpha_2(I) \times \cdots \times \alpha_r(I)$ be the corresponding polyvector. Set
\[
\Delta(I; t) = \Delta(\alpha_1(I); t) \cdot \Delta(\alpha_2(I); t) \cdots \Delta(\alpha_r(I); t),
\]
where for any vector $A = (a_1, a_2, \ldots, a_m)$ with natural coordinates,
\[
\Delta(A; t) = \Delta(a_1, a_2, \ldots, a_m; t) := \left| \begin{array}{cccc}
\xi(a_1; t) & t^{a_2+1} & 0 & 0 \\
1 & \xi(a_2; t) & t^{a_3+1} & 0 \\
 & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & \xi(a_{m-1}; t) & t^{a_m+1} \\
0 & 0 & \ldots & 0 & 1 & \xi(a_m; t)
\end{array} \right|.
\]
and for natural $a$ we have
\[
\xi(a; t) := t + t^2 + \ldots + t^a.
\]
The polynomial $\Delta(A; t)$ can be defined recursively by the formulas
\[
\Delta(a_1; t) := \xi(a_1; t), \quad \Delta(a_1, a_2; t) := \xi(a_2; t)\Delta(a_1; t) - t^{a_2+1}, \\
\Delta(a_1, \ldots, a_m; t) := \xi(a_m; t)\Delta(a_1, \ldots, a_{m-1}; t) - t^{a_m+1}\Delta(a_1, \ldots, a_{m-2}; t) \quad \text{for } m > 2. \tag{3}
\]
Now let us turn to Fibonacci partitions and their connection with 2-partitions.

**Definition 2.7.** A Fibonacci partition is a partition all whose parts are Fibonacci numbers. For any partition $I = \langle i_1, i_2, \ldots, i_k \rangle$, define $f_I := \langle f_{i_1}, f_{i_2}, \ldots, f_{i_k} \rangle$.

The set of all Fibonacci partitions of degree $n$ is denoted by $\mathcal{F}(n)$.

**Definition 2.8.** A Zeckendorf partition of degree $n$ is any partition $f_I \in \mathcal{F}(n)$, where $I$ is a 2-partition.

It is almost self-evident that for any natural $n$, there exists a unique Zeckendorf partition of degree $n$ (see [3], Sec.6.6). We will denote this partition by $f_{Z(n)}$, where $Z(n) := \langle z(n, 1), z(n, 2), \ldots, z(n, \lambda(n)) \rangle$ is a 2-partition.

Thus, $n = \|f_{Z(n)}\|$. For example,
\[
Z(333) = \langle 3, 5, 10, 12 \rangle \quad \text{since} \quad 333 = f_3 + f_5 + f_{10} + f_{12}.
\]

**Definition 2.9.** A natural $n$ is called $f$-simple if the 2-partition $Z(n)$ is simple.

**Definition 2.10.** Let $M$ be a set of partitions of equal degree, and let $m_l$ be the quantity of these partitions of length $l$. The polynomial $G_M(t) := \sum_l m_l t^l$ is called the generating function of $M$.

Define
\[
\Phi(0; t) = 1 \quad \text{and} \quad \Phi(n; t) = G_{\mathcal{F}(n)}(t) \quad \text{for integer } n > 0.
\]
The following statement is the main result of the section:

**Theorem 2.11.** For $n$ natural, let $\alpha(Z(n)) = \alpha_1(n) \times \alpha_2(n) \times \cdots \times \alpha_r(n)$ be the polyvector of the 2-partition $Z(n)$. Then
\[
\Phi(n; t) = \Delta(\alpha_1(n); t) \cdot \Delta(\alpha_2(n); t) \cdots \Delta(\alpha_r(n); t). \tag{4}
\]

For example,
\[
\Phi(333; t) = \Delta(2, 2; t) \cdot \Delta(3, 2; t) = t^4 + 2t^5 + 4t^6 + 4t^7 + 3t^8 + t^9.
\]

Proof of Theorem 2.11 is based on several lemmas. For brevity, until the end of this section we will write, as a rule, $\lambda$ instead of $\lambda(n)$.

**Lemma 2.12.** The maximal part of any partition from $\mathcal{F}(n)$ is equal to either $f_{z(n, \lambda)}$, or $f_{z(n, \lambda) - 1}$.

*Proof.* If $\langle f_{i_1}, f_{i_2}, \ldots, f_{i_k} \rangle \in \mathcal{F}(n)$ and $f_{i_k} \leq f_{z(n, \lambda) - 2}$, then
\[
n = f_{i_1} + \ldots + f_{i_k} \leq f_1 + f_2 + \ldots + f_{z(n, \lambda) - 2} = f_{z(n, \lambda)} - 2 < n,
\]
which is impossible. \(\square\)
Lemma 2.13. Any partition from $\mathbb{F}(f_i)$ is of the form

$$
\langle f_{i-2s}, f_{i-(2s-1)}, f_{i-(2s-3)} \ldots, f_{i-3} + f_{i-1} \rangle,
$$
where $s = 0, 1, \ldots, \left\lfloor \frac{i-1}{2} \right\rfloor$.

In particular, $\Phi(f_i; t) = \xi \left( \left\lfloor \frac{i-1}{2} \right\rfloor + 1\right) t$.

Proof. This follows by induction on $s$ from Lemma 2.12. $\square$

Lemma 2.14. For any partition $\langle f_1, f_2, \ldots, f_n \rangle \in \mathbb{F}(n)$ and any $m$ such that $1 \leq m \leq \lambda$, there exists an $s$ such that $1 \leq s \leq h$ and $\langle f_{i_s}, f_{i_{s+1}}, \ldots, f_{i_h} \rangle \in \mathbb{F}(f_z(n,m) + \ldots + f_z(n,\lambda))$.

Proof. The induction on $\lambda$ shows that it suffices to consider only the case where $m = \lambda$. By Lemma 2.12 either $i_h = z(n, \lambda)$, or $i_h = z(n, \lambda) - 1$. If $i_h = z(n, \lambda)$, then $s = h$. Let $i_h = z(n, \lambda) - 1$ and let $s \geq 1$ be the maximal number such that $i_{s+1} = i_s + 1$ and $i_{m+1} - i_m = 2$ for all $m$ such that $s < m < h$. Then $f_{i_s} + \ldots + f_{i_h} = f_z(n,\lambda)$. $\square$

Lemma 2.15. Let $z(n, 1) > m \geq 0$. Set

$$
F_m(n) := \{ f_1 \in \mathbb{F}(n) \mid I^- > m \}.
$$

Then

$$
G_{F_m(n)}(t) = \Phi \left( f_z(n,1) + f_z(n,2) - m + \ldots + f_z(n,\lambda) - m \right) \cdot t.
$$

Proof. For $\langle f_1, \ldots, f_n \rangle \in F_m(n)$, set $\nu_m (\langle f_1, \ldots, f_n \rangle) = (f_{i_1-m}, \ldots, f_{i_h-m})$. Let us check that

$$
\nu_m (\langle f_1, \ldots, f_n \rangle) \in \mathbb{F}(f_z(n,1) + f_z(n,2) - m + \ldots + f_z(n,\lambda) - m).
$$

By Lemma 2.14 there exists an $s \geq 1$ such that $f_{i_s} + \ldots + f_{i_h} = f_z(n,\lambda)$. For $s = 1$, the claim follows from Lemma 2.13. For $s > 1$, we use the induction on $s$. The inductive hypothesis shows that

$$
f_{i_1-m} + \ldots + f_{i_{s-1}-m} = f_z(n,1) - m + \ldots + f_z(n,\lambda-1) - m \quad \text{and} \quad f_{i_s-m} + \ldots + f_{i_h-m} = f_z(n,\lambda) - m.
$$

This gives the claim required. Thus, the map

$$
\nu_m : \mathbb{F}_m(n) \rightarrow \mathbb{F}(f_z(n,1) + f_z(n,2) - m + \ldots + f_z(n,\lambda) - m)
$$

is well-defined and injective. Hence, $\nu_m$ is bijective. $\square$

Lemma 2.16. Let $\langle f_z(n,1), \ldots, f_z(n,\lambda1), \ldots, f_z(n,\lambda) \rangle$ and $n_1 = f_z(n,1) + \ldots + f_z(n,\lambda)$, where $1 \leq \lambda < \lambda$. If $\beta(z(n,1) + 1) \neq \beta(z(n,\lambda1))$, then

$$
\Phi(n; t) = \Phi(n_1; t) \cdot \Phi(f_z(n,\lambda1) - z(n,\lambda1) + 1 + \ldots + f_z(n,\lambda) - z(n,\lambda1) + 1; t).
$$

Proof. For $f_1 = \langle f_1, f_2, \ldots, f_n \rangle \in \mathbb{F}(n)$, let $f_{\lambda1} = \langle f_1, f_{\lambda+1}, \ldots, f_n \rangle \in \mathbb{F}(n - n_1)$. Such an $s$ exists by Lemma 2.12. Then $f_1 = \langle f_i, f_{i_2}, \ldots, f_{i_h} \rangle \in \mathbb{F}(n)$. By Lemma 2.12, either $i_s = z(n, \lambda1)$, or $i_s = z(n, \lambda1) - 1$. Since $i_s > i_{s-1}$, then $i_s > z(n, \lambda1) - 1$.

Thus, any partition $f_1 \in \mathbb{F}(n)$ can be uniquely written as an element $(f_1, f_{\lambda1})$ of the set

$$
M(n, \lambda1) := \mathbb{F}(n) \times \mathbb{F}(n_{\lambda1} - 1(n - m)).
$$

This gives an embedding $u : \mathbb{F}(n) \rightarrow M(n, \lambda1)$. The formula

$$
|f_1, f_{\lambda1}| = |f_1| + |f_{\lambda1}|
$$

defines a length function on $M(n, \lambda1)$. By Lemma 2.15, we see that the corresponding generating function coincides with the right side of formula 5.

Let $(f_1, f_{\lambda1}) \in M(n, \lambda1)$. Lemma 2.13 implies that $\beta(i_s) = \beta(z(n, \lambda1))$. If $\beta(i_s) \neq \beta(z(n, \lambda1))$, then $J^+_{\lambda1} > z(n, \lambda1)$. Since $J^+_{\lambda1} \leq z(n, \lambda1)$, then $J_\lambda \cup J_\lambda^+ \in \mathbb{F}(n)$. Hence the map $u$ is bijective. Therefore, the generating functions of the sets $M(n, \lambda1)$ and $\mathbb{F}(n)$ coincide. $\square$

Lemma 2.17. Let $a = \frac{1}{2} (z(n, \lambda) - z(n, \lambda - 1)) + 1$, where $n$ is an $f$-simple number. Then

$$
\Phi(n; t) = \Phi \left( n - f_z(n,\lambda1) - \mid 1 + t^a \Phi( n - f_z(n,\lambda1) - f_z(n,\lambda1); t) \right).
$$
Proof. Let
\[ M(n) = \mathbb{F}(n - f_{z(n, \lambda)}) \times \mathbb{F}(z(n, \lambda) - 1) \left( f_{z(n, \lambda)} \right). \]
Define an imbedding \( u : \mathbb{F}(n) \to M(n) \) as in the proof of Lemma 2.16. Applying Lemma 2.15, we see that the generating function of \( M(n) \) is equal to
\[ \Phi(n - f_{z(n, \lambda)}; t) \cdot \Phi(f_{z(n, \lambda)} - z(n, \lambda - 1); t). \]
There is a unique partition \( f_A \in \mathbb{F}(z(n, \lambda) - 1) \left( f_{z(n, \lambda)} \right) \) such that \( A^- = z(n, \lambda): \)
\[ A = \langle z(n, \lambda), z(n, \lambda) + 1, z(n, \lambda) + 3, \ldots, z(n, \lambda) - 3, z(n, \lambda) - 1 \rangle. \]
Define \( N(n) := \{(f_j \cup f_{z(n, \lambda)}), f_A) \in M(n) \mid f_j \in \mathbb{F}(n - f_{z(n, \lambda)} - f_{z(n, \lambda) - 1}) \}. \)
The element \((f_j, f_{z(n, \lambda)}) \in M(n)\) belongs to the set \( u(\mathbb{F}(n)) \) if and only if \((f_j, f_{z(n, \lambda)}) \notin N(n)\).
That is, \( u(\mathbb{F}(n)) = M(n) \setminus N(n) \). Hence, \( \Phi(n; t) \), the generating function of \( u(\mathbb{F}(n)) \), is the difference of generating functions of the sets \( M(n) \) and \( N(n) \).
Since \( |A| = a \), it follows that \( \mu + 1 \Phi(n - f_{z(n, \lambda)} - f_{z(n, \lambda) - 1}; t) \) is the generating function of \( N(n) \).

Proof of Theorem 2.11. First, let \( n \) be an \( f \)-simple number and \( a(n) = (a_1, a_2, \ldots, a_\lambda) \). Let us show that then \( \Phi(n; t) = \Delta(a_1, a_2, \ldots, a_\lambda; t) \). For \( \lambda = 1 \), this follows from Lemma 2.13. This lemma also implies that \( \Phi(f_{z(n, \lambda)} - z(n, \lambda - 1); t) = \xi(a; t) \). By induction on \( \lambda \) we can rewrite formula (6) as
\[ \Phi(n; t) = \xi(a, \lambda; t) \Delta(a_1, a_2, \ldots, a_\lambda - 1; t) - \mu^{\lambda + 1} \Delta(a_1, a_2, \ldots, a_\lambda - 2; t). \]
Since the right-hand side of this formula coincides with the right-hand side of (3), we obtain the claim. In general case, formula (4) follows from Lemma 2.10 by induction on \( r \).

Define \( F(0) = \chi(0) = 1 \) and for \( n > 0 \) we set
\[ F(n) := \Phi(n; 1) \quad \text{and} \quad \chi(n) := \Phi(n; -1). \]

Corollary 2.18. (See also [9].) The values of \( \chi(n) \) are \( 0, \pm 1 \) for any \( n \in \mathbb{N} \).

Proof. It suffices to show that, for a vector \( A = (a_1, \ldots, a_m) \) with natural coordinates, \( \Delta(A; -1) = 0, \pm 1 \).
For \( m = 1, 2 \), this is clear.

Let \( m > 2 \). Set for brevity \( x(m) := \Delta(a_1, \ldots, a_m; -1) \). Then formula (3) shows that
\[ x(m) = \begin{cases} x(m - 2) & \text{if } \beta(a_m) = 0, \\ -x(m - 1) - x(m - 2) & \text{if } \beta(a_m) = 1. \end{cases} \]
To check that values of \( x(m) \) are \( 0, \pm 1 \) only, let us perform induction on \( m \). For \( \beta(a_m) = 0 \), the claim is clear.
For \( \beta(a_m) = \beta(a_m - 1) = 1 \), the claim follows as well, since
\[ x(m) = -x(m - 1) - x(m - 2) = x(m - 2) + x(m - 3) - x(m - 2) = x(m - 3). \]
If \( \beta(a_m) = 1 \) and \( \beta(a_m - 1) = 0 \), then the claim also follows by induction, because
\[ x(m) = -x(m - 1) - x(m - 2) = -x(m - 3) - x(m - 2) = \Delta(a_1, a_2, \ldots, a_{m - 2}, a_{m - 1} + 1; -1). \]

Lemma 2.19. We have \( n \in [f_i, f_{i+1} - 1] \) if and only if \( \lambda(n) = i \).

Proof. The claim obviously follows from the easily verifiable formula
\[ f_{2 - \beta(i)} + f_{4 - \beta(i)} + \ldots + f_{i-2} + f_i = f_{i+1} - 1. \]
This Lemma and Theorem 2.11 imply the next statement, to be used in Sections 7 and 9.

Corollary 2.20. Let \( n \in [f_a, f_a + 1 - 1] \), where \( 0 \leq a \leq i - 2 \). Then
\[ \Phi(n + f_i; t) = \begin{cases} \xi(1 + \frac{a+1}{2}; t) \cdot \Phi(n; t) & \text{if } \beta(a) \neq \beta(i), \\ \xi(1 + \frac{a+2}{2}; t) \cdot \Phi(n; t) - t^{\frac{a+1}{2}} \cdot \Phi(n - f_a; t) & \text{if } \beta(a) = \beta(i). \end{cases} \]

The following statement shows that on the interval \([f_i - 1, f_{i+1} - 1]\) the function \( \Phi \) is “symmetric” in the following sense:
Lemma 2.21. If \( n \in [f_i - 1, f_{i+1} - 1] \), where \( i \geq 1 \), then
\[
\Phi(n; t) = t^{f_i} \cdot \Phi(f_{i+2} - 2 - n; t^{-1}).
\]
In particular, for \( t = 1 \) and for \( t = -1 \), we, respectively, obtain the formulas:
\[
F(n) = F(f_{i+2} - 2 - n) \quad \text{and} \quad \chi(n) = (-1)^i \chi(f_{i+2} - 2 - n).
\]

Proof. Let \( \rho_i \) be the reflection of the interval \([f_i - 1, f_{i+1} - 1]\) with respect to its center:
\[
\rho_i(n) = f_{i+2} - 2 - n.
\]
Since \( f_1 + f_2 + \ldots + f_i = f_{i+2} - 2 \), we see that each Fibonacci partition \( f_1 \) of \( n \in [f_i - 1, f_{i+1} - 1] \) with \( m \) parts, where \( 1 \leq m < i \), corresponds to the Fibonacci partition \( f_{i'} \) of \( f_{i+2} - 2 - n \) with \( i - m \) parts, where \( I' = \{1, 2, \ldots, i\} \setminus I \). This implies the formula required. \( \square \)

3. Solutions of the equation \( F(n) = k \) and generating numbers

The main goal of this section is to present an algorithm to find all solution of the equation \( F(n) = k \). Our main tool here is the notion of generating numbers (Def.3.6).

Definition 3.1. Let \( \mathcal{A} \) be the set of vectors \((a_1, a_2, \ldots, a_m)\) with natural coordinates such that \( a_1 \geq 1 \) while \( a_i \geq 2 \) for all \( i \geq 2 \), and let \( \mathcal{A}_+ \subset \mathcal{A} \) be the set of vectors with \( a_1 \geq 2 \).

Any expression of the form \( A = A_1 \times A_2 \times \ldots \times A_r \), where \( A_1 \in \mathcal{A} \) and \( A_k \in \mathcal{A}_+ \), if \( k > 1 \) is said to be a polyvector. (By definition \( A = \emptyset \) if \( r = 0 \).) The set of polyvectors is denoted by \( \mathcal{A} \).

Obviously, \( \mathcal{A} \) coincides with the set of polyvectors of 2-partitions (see Def.2.4). Define
\[
D(\emptyset) = 1 \quad \text{and} \quad D(A) = \Delta(A; 1) \quad \text{for} \quad A \in \mathcal{A}.
\]

Lemma 3.2. Let \( A = (a_1, \ldots, a_m) \in \mathcal{A} \). Then
\[\pi(A) := \frac{D(a_2, \ldots, a_m)}{D(a_1, a_2, \ldots, a_m)}\]
is an irreducible fraction, \( \pi: \mathcal{A} \rightarrow \mathbb{Q}_{>0} \) is a one-to-one map, and \( \pi(A) < 1 \) if and only if \( A \in \mathcal{A}_+ \).

It is easy to see that, for \( A = (a_1, a_2, \ldots, a_m) \), we have
\[
\pi(A) = \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \ldots - \frac{1}{a_{m-1}} - \frac{1}{a_m}.
\]

To prove Lemma 3.2 it suffices to repeat (with obvious modifications) the proofs of the corresponding claims on representations of rational numbers by continued fractions from [4], Ch.X.

Let us only note that, in the decomposition \( \frac{1}{k} = \pi(a_1, \ldots, a_m) \), the numbers \( a_i \) can be calculated recursively, starting from the values \( l_1 = l \) and \( k_1 = k \), by using the formulas
\[
a_i = \left\lfloor \frac{k_i}{l_i} \right\rfloor, \quad l_{i+1} = l_i \cdot a_i - k_i, \quad k_{i+1} = l_i \quad \text{if} \quad l_i \neq 0.
\]
The calculation terminates when \( l_{i+1} = 0 \).

Define
\[
\Gamma := \{ g_1 \times g_2 \times \ldots \times g_r \mid g_i \in \mathbb{Q}_{>0} \quad \text{and} \quad g_i \in \mathbb{Q}_{(0,1)} \quad \text{for} \quad i \geq 2; \quad r \in \mathbb{N} \}
\]
and a map \( \pi: \mathcal{A} \rightarrow \Gamma \) by the formula
\[
\pi(A_1 \times \ldots \times A_r) = \pi(A_1) \times \ldots \times \pi(A_r).
\]

Lemma 3.2 shows that \( \pi \) is bijective. Adding the maps \( Z \) and \( \alpha \) defined in Section 2, we obtain a sequence
\[
\mathbb{N} \xrightarrow{Z} \mathbb{P} \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\pi} \Gamma,
\]
where the maps \( Z \) and \( \pi \) are bijective, and \( \alpha \) is a two-sheeted map.

Theorems 2.11 and Lemma 3.2 imply
Proposition 3.3. Let \( \Pi : \mathbb{N} \to \Gamma \) be a map defined by the formula
\[
\Pi(n) := \pi(a(Z(n))).
\]
Define
\[
\Gamma(k) := \left\{ \frac{l_1}{k_1} \times \frac{l_2}{k_2} \times \ldots \times \frac{l_r}{k_r} \in \Gamma \mid k_1k_2\ldots k_r = k; \ r \in \mathbb{N} \right\} \subset \Gamma.
\]
Then \( F(n) = k \) for \( n \in \mathbb{N} \) if and only if \( \Pi(n) \in \Gamma(k) \).

The definition of \( \alpha \) shows that the sets
\[
\mathbb{P}_a := \{ I \in \mathbb{P} \mid \beta(I^\perp) = a \}, \quad \text{where } a = 0, 1,
\]
are the sheets of the map \( \alpha \) interchanged by the involution \( \sigma : \mathbb{P} \to \mathbb{P} \), defined by the formula
\[
\sigma\{i_1,i_2,\ldots,i_q\} := \begin{cases} \{i_1 + 1,i_2 + 1,\ldots,i_q + 1\} & \text{if } \beta(i_1) = 1, \\ \{i_1 - 1,i_2 - 1,\ldots,i_q - 1\} & \text{if } \beta(i_1) = 0. \end{cases}
\]

In what follows we use the following notation:
\[
N(k) := \{ n \in \mathbb{N} \mid F(n) = k \},
\]
\[
N_a := Z^{-1}(\mathbb{P}_a), \quad N_a(k) = N(k) \cap N_a, \quad \mathbb{P}_a(k) = Z(N_a(k)), \quad \text{where } a = 0, 1.
\]
Then
\[
N = N_0 \cup N_1, \quad N(k) = N_0(k) \cup N_1(k) \quad \text{and} \quad \mathbb{P}(k) = \mathbb{P}_0(k) \cup \mathbb{P}_1(k).
\]

Since \( Z \) is a bijective map, the involution \( \sigma : \mathbb{P} \to \mathbb{P} \) induces an involution \( \sigma : \mathbb{N} \to \mathbb{N} \) by the formula
\[\sigma(n) := f_{\sigma(Z(n))} \] From Theorem 2.11 it follows that \( F(n) = F(\sigma(n)) \).

Because \( N(k) = N_0(k) \cup N_1(k) \), to find all solutions of the equation \( F(n) = k \) it suffices to compute the set \( N_1(k) \). For any natural \( k \), the sequence (12) obviously induces the sequence of bijective maps
\[
N_1(k) \xrightarrow{\mathcal{L}_1} P_1(k) \xrightarrow{\pi} A(k) \xrightarrow{\sigma} \Gamma(k), \tag{13}
\]
where \( A(k) := \pi^{-1}(\Gamma(k)) \). To obtain 2-partition from \( N_1(k) \) corresponding to \( g = g_1 \times g_2 \times \ldots \times g_r \in \Gamma(k) \), we can use the next two-step algorithm:

Step 1: Compute \( A(g) = \pi^{-1}(g_1) \times \pi^{-1}(g_2) \times \ldots \times \pi^{-1}(g_r) \in A \) using formulas (11).

Step 2: Compute \( \alpha^{-1}(A(g)) \in P_1(k) \). Then the required Fibonacci partition is \( f_{\alpha^{-1}(A(g))} \in N_1(k) \).

For any vector \( A = (a_1,a_2,\ldots,a_m) \in A \), define
\[
d(A) := a_1 + a_2 + \ldots + a_m - m, \\
v(A) := (a_1 + 1,a_2 + 1,\ldots,a_m + 1), \\
e(A) := (2d(a_1) + 1,2d(a_2) + 1,\ldots,2d(a_1,a_2,\ldots,a_m) + 1).
\]

The following directly verifiable statement computes \( \alpha^{-1}(A) \in \mathbb{P}_1 \) for any \( A \in A \).

Lemma 3.4. Let \( A = A_1 \times A_2 \times \ldots \times A_r \in A \). Then
\[
\alpha^{-1}(A) = \varepsilon(A_1) \cup v^{-2d(A_1)+1}(\varepsilon(A_2)) \cup \ldots \cup v^{-2d(A_1)+2d(A_2)+\ldots+2d(A_{r-1})+r-1}(\varepsilon(A_r)) \tag{14}
\]
is the canonical decomposition of \( \alpha^{-1}(A) \) in \( P_1 \).

In particular, if \( \alpha(Z(n)) = A_1 \times A_2 \times \ldots \times A_r \), then
\[
z(n,\lambda(n)) = 2d(A_1) + 2d(A_2) + \ldots + 2d(A_r) + r. \tag{15}
\]

For example, let us find all solutions of the equation \( F(n) = 1 \). It is obvious that \( \Gamma(1) = \mathbb{N} \). For \( a \in \mathbb{N} \), we have \( \pi^{-1}(a) = (1,(a - 1) \times 2) \). From formula (13) we obtain \( \alpha^{-1}(\pi^{-1}(a)) = (1,3,\ldots,2a - 1) \in \mathbb{P}_1(1) \).

Now, formulas
\[
f_1 + f_3 + \ldots + f_{2a-1} = f_{2a} - 1, \quad \sigma(f_{2a} - 1) = f_2 + f_4 + \ldots + f_{2a} = f_{2a+1} - 1 \tag{16}
\]
and Theorem 2.11 show that
\[
N(1) = \{ f_i - 1 \mid i = 1, 2, 3, \ldots \}.
\]

The above algorithm certainly is not sufficient to get all solutions of the equation \( F(n) = k \) because the set \( \Gamma(k) \) is infinite. Nevertheless, for any \( k \geq 2 \), one can canonically define a finite set of solutions.
called the $k$-generating numbers, from which all solutions can be obtained by a regular procedure. To
describe this set of numbers we need some definitions.

Define an action of the (additive) semigroup $\mathbb{Z}_{\geq 0}$ on the set $\Gamma$ by setting

$$[a](g_1 \times g_2 \times \cdots \times g_r) = (g_1 + a) \times g_2 \times \cdots \times g_r,$$

where $a \in \mathbb{Z}_{\geq 0}$. Define an involution $\tau : \Gamma \to \Gamma$ by the formulas

$$\tau(l) = l,
\tau(g_1 \times g_2 \times \cdots \times g_r) = 1 \times g_1 \times g_2 \times \cdots \times g_r,
\tau(l \times g_1 \times g_2 \times \cdots \times g_r) = (l - 1 + g_1) \times g_2 \times \cdots \times g_r,
\tau((l + g_1) \times g_2 \times \cdots \times g_r) = (l + 1) \times g_1 \times g_2 \times \cdots \times g_r,$$

where $l \in \mathbb{N}$, $g_1, g_2, \ldots, g_r \in \mathbb{Q}_{(0,1)}$, and $r \geq 1$. Let $T(\tau)$ be the group with two elements generated by $\tau$.

It is trivial to check that $[a] \cdot \tau = \tau \cdot [a]$. Therefore, an action of the semigroup $H_0 := \mathbb{Z}_{\geq 0} \times T(\tau)$ on $\Gamma$ is canonically defined; obviously, $H_0(\Gamma(k)) \subset \Gamma(k)$. The next claim easily follows from the definitions.

**Lemma 3.5.** The canonical action of the semigroup $H_0$ on the set $\Gamma^* := \Gamma \setminus \Gamma(1)$ is free and

$$\Gamma_+ := \{g_1 \times g_2 \times \cdots \times g_r \in \Gamma \mid g_1, g_2, \ldots, g_r \in \mathbb{Q}_{(0,1)}\} \subset \Gamma^*$$

is a fundamental domain of this action. That is, $H_0(\Gamma_+) = \Gamma^*$ and if $g \in \Gamma_+$, then $h(g) \not\in \Gamma_+$ for any nonunit element $h \in H_0$.

Using the sequence of the bijective maps

$$N_1 \xrightarrow{\pi} P_1 \xrightarrow{\alpha} A \xrightarrow{\sigma} \Gamma,$$

we transfer the action of $[a]$ and $\tau$ on $N_1$. Further, using the decomposition $N = N_0 \sqcup N_1$ and the involution $\sigma : N \to N$ we extend the actions of $[a]$ and $\tau$ on $N$ by the formulas

$$[a](n) = \sigma \cdot [a] \cdot \sigma(n),
\tau(n) = \sigma \cdot \tau \cdot \sigma(n),$$

where $n \in N_0$. Thus, we see that an action of the semigroup $H := \mathbb{Z}_{\geq 0} \times T(\tau) \times T(\sigma)$
on the set $N$ is canonically defined. Here $T(\tau)$ and $T(\sigma)$ are the groups, each with two elements, generated by the involutions $\tau$ and $\sigma$, respectively.

**Definition 3.6.** Any number $n \in N_1$, such that $z(n, 1) \geq 3$, is said to be a generating number. If, in addition, $F(n) = k$, then $n$ is said to be a $k$-generating number.

By $G$ and $G(k)$ we denote the set of all generating and the set of $k$-generating numbers, respectively.

For example, $f_{2k-1}$ is a $k$-generating number for $k > 1$ since $F(f_{2k-1}) = k$; so $G(k) \neq \emptyset$ for any $k > 1$. The first 20 generating numbers $n$ and the corresponding values of $F(n)$ are as follows (see [10], A003623):

| $n$ | 3  | 8  | 11 | 16 | 21 | 24 | 29 | 32 | 37 | 42 | 45 | 50 | 55 | 58 | 63 | 66 | 71 | 76 | 79 | 84 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $F(n)$ | 2  | 3  | 3  | 4  | 4  | 5  | 5  | 4  | 6  | 6  | 6  | 5  | 7  | 7  | 8  | 8  | 7  | 8  | 7  | 8  | 7  |

The definitions imply

**Lemma 3.7.** A number $n \in N_1$ is a generating number whenever $\Pi(n) \in \Gamma_+$, and $n \in N_1$ is a $k$-generating number whenever $\Pi(n) \in \Gamma_+(k) := \Gamma(k) \cap \Gamma_+$. In particular, the map $\Pi : G(k) \to \Gamma_+(k)$ is bijective.

Since $\Gamma_+(k)$ is a finite set, $G(k)$ is a finite set as well. Define

$$\Psi(k) := |\Gamma_+(k)| = |G(k)|.$$

For some details on the function $\Psi(k)$, see Section [10].

The set $\Gamma_+$ has a structure of a monoid freely generated by the set $\mathbb{Q}_{(0,1)}$ with the multiplication $\times$. It is obvious that $\Gamma_+(k_1) \times \Gamma_+(k_2) \subset \Gamma_+(k_1k_2)$. Therefore, Lemma [3.7] implies that the set of generating numbers $G = \coprod_{k \geq 2} G(k) \subset \mathbb{N}$ also has a canonical structure of a monoid isomorphic to $\Gamma_+$. Denote
the multiplication in $G$ by $\times$ as well. Then $G(k_1) \times G(k_2) \subset G(k_1k_2)$, and the map $F : G \to \mathbb{N}$ is a homomorphism of monoids.

In other words, we can $x$-multiply the generating solutions $n_1$ and $n_2$ of the equations $F(n) = k_1$ and $F(n) = k_2$ and obtain, for $n_1 \neq n_2$, two different generating solutions $n_1 \times n_2$ and $n_2 \times n_1$ of the equation $F(n) = k_1k_2$. For example, $F(3) = 2$ and $F(8) = 3$. Then $3 \times 8 = 36$ and $8 \times 3 = 42$ are generating solutions of the equation $F(n) = 6$.

Lemma 3.5 and Lemma 3.7 directly imply

**Theorem 3.8.** The semigroup $H \simeq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ canonically acts on the set $\mathbb{N}$ so that, for any $h \in H$, we have $F(h(n)) = F(n)$.

The restriction of the $H$-action on the set $\mathbb{N} \setminus \{1\}$ is free, and the set $G \subset \mathbb{N} \setminus \{1\}$ of generating numbers is a fundamental domain of this restricted action. In particular, for any $k \geq 2$, we have $N(k) = H(G(k))$.

It is convenient to describe the $H$-action on $n \in \mathbb{N} \setminus \{1\}$ explicitly. Define

$$\theta(n) := \begin{cases} 1 & \text{if } z(n, 1) \geq 3, \\ \min\{ m : z(n, m) - z(n, m - 1) \geq 3 \} & \text{if } z(n, 1) < 3 \text{ and } 1 < m \leq \lambda(n), \end{cases}$$

$$Z'(n) := (z(n, \theta(n)), z(n, \theta(n) + 1), \ldots, z(n, \lambda(n))).$$

For any partition $I = (i_1, i_2, \ldots, i_q)$ and any integer $r$, we set $I + r := (i_1 + r, i_2 + r, \ldots, i_q + r)$. Thanks to formulas (16) we have (see formula (2) for notation)

$$n = f_{2\theta(n) - \beta(z(n, 1)) - 1} + 1 + f_{Z'(n)} \quad \text{and} \quad \sigma(n) = \begin{cases} f_{2\theta(n) - 1} + 1 & \text{if } \beta(z(n, 1)) = 1, \\ f_{2\theta(n) - 2} + 1 & \text{if } \beta(z(n, 1)) = 0. \end{cases}$$

If $n \in \mathbb{N} \setminus \{1\}$, then from the definitions we obtain formulas

$$[a](n) = f_{2\theta(n) - 2 + a} - 1 + \|f_{Z'(n)}\|,$$

$$\tau(n) = \begin{cases} f_{2\theta(n) - 1} + 1 & \text{if } \beta(z(n, \theta(n))) = 1, \\ f_{2\theta(n) - 4} + 1 & \text{if } \beta(z(n, \theta(n))) = 0. \end{cases}$$

Since $[a] \cdot \sigma = \sigma \cdot [a]$ and $\tau \cdot \sigma = \sigma \cdot \tau$, formulas (17)-(19) completely describe the $H$-action on $\mathbb{N} \setminus \{1\}$.

**Examples.** Let us apply Theorem 3.8 to find all solutions of the equation $\sum_{n=1}^{k} F(n) = k$, where $k = 2, 3, 4$. In these examples, to compute the set $G(k)$ by means of the set $\Gamma_+ (k)$ we use formula (14).

Since $\Gamma_+ (2) = \{ \frac{1}{2} \}$, we have $G(2) = \{ f_3 \}$. Then formulas (18) and (19) show that any solution $n \in \mathbb{N} \setminus \{1\}$ of the equation $F(n) = 2$ can be uniquely represented in one of the forms:

either $[a](f_3) + f_2a - 1 + f_{2a+3},$ or $[a](\tau(f_3)) = [a](f_2 - 1 + f_4) = f_{2(a+1)} - 1 + f_{2a+4},$

where $a \geq 0$. Applying involution $\sigma$ to these numbers we see that the set of solutions of the equation $F(n) = 2$ coincides with the set of numbers

$$f_i - 1 + f_{i+3}, \quad f_i + 1 - f_{i+4}, \quad \text{where } i \in \mathbb{Z}_{\geq 0}.$$

Let $k = 3$ or $k = 4$. Because $\Gamma_+ (3) = \{ \frac{1}{3}, \frac{4}{3} \}$ and $\Gamma_+ (4) = \{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4} \times \frac{1}{4} \}$, we, respectively, have $G(3) = \{ f_5, f_3 + f_5 \}$ and $G(4) = \{ f_7, f_5 + f_7, f_3 + f_5 + f_7 \}$. Repeating the similar procedure as for $k = 2$, we see that the set of solutions of the equation $F(n) = 3$ coincides with the set of numbers

$$f_i - 1 + f_{i+5}, \quad f_i + 1 - f_{i+6}, \quad f_i - 1 + f_{i+3} + f_{i+5}, \quad f_i + 1 - f_{i+4} + f_{i+6},$$

and the set of solutions of the equation $F(n) = 4$ coincides with the set of numbers

$$f_i - 1 + f_{i+7}, \quad f_i - 1 + f_{i+8}, \quad f_i - 1 + f_{i+3} + f_{i+5} + f_{i+7}, \quad f_i + 1 - f_{i+4} + f_{i+6} + f_{i+8},$$

$$f_i - 1 + f_{i+3} + f_{i+6}, \quad f_i + 1 - f_{i+4} + f_{i+7},$$

where $i \in \mathbb{Z}_{\geq 0}$.

The set of generating numbers has the following compact description:

**Theorem 3.9.** The natural number $n$ is a generating number if and only if $n = 2[l \phi] + l$, where $l \in \mathbb{N}$.

\[\text{Footnote: } \begin{align*}
\text{The solutions of the equation } F(n) = k \text{ for } k = 1, 2, 3 \text{ were earlier obtained in } \text{[6]}.
\end{align*}\]
Proof. Using the known formula \(|l| f = f_{z(l,1)+1} + \ldots + f_{z(l,q)+1} - \beta(z(l,1))\) (see \([3]\), p. 339), we obtain

\[
2|l| + l = f_{z(l,1)+3} + \ldots + f_{z(l,q)+3} - 2\beta(z(l,1)).
\]

Since \(f_{z(l,1)+3} - 2 = f_3 + f_5 + \ldots + f_{z(l,1)+2} \geq 2\), we have

\[
2|l| + l = \begin{cases} 
\frac{f_{z(l,1)+3} + \ldots + f_{z(l,q)+3}}{2} & \text{if } \beta(z(l,1)) = 0, \\
\frac{f_3 + f_5 + \ldots + f_{z(l,1)+2} + f_{z(l,2)+3} + \ldots + f_{z(l,q)+3}}{2} & \text{if } \beta(z(l,1)) = 1.
\end{cases}
\]  

Hence \(2|l| + l \in \mathbb{G}\).

Conversely, let \(n \in \mathbb{G}\). Set

\[
u(n) = \begin{cases} 
\langle z(n,1) - 3, z(n,2) - 3, \ldots, z(n, \lambda(n)) - 3 \rangle & \text{if } z(n,1) > 3, \\
\langle 2a - 1, z(n,a + 1) - 3, \ldots, z(n, \lambda(n)) - 3 \rangle & \text{if } Z(n) = \langle 3, 5, \ldots, 2a + 1, z(n,a + 1), \ldots, z(n, \lambda(n)) \rangle,
\end{cases}
\]

where \(z(n,a + 1) > 2a + 3\). Then, for \(l = f_{\nu(n)}\), we have \(n = 2|l| + l\) by formula (20).

4. Estimates of \(D(A)\)

Here we establish an auxiliary statement to be used in the next two sections. The proof of it is based on the following formal identities:

\[
D(a_1, \ldots, a_s, k \ast 2) = D(a_1, \ldots, a_s) + kD(a_1, \ldots, a_{s-1}, a_s - 1),
\]  

(21)

\[
D(a_1, \ldots, a_m) = D(a_1, \ldots, a_{m-1}, 2) + (a_m - 2)D(a_1, \ldots, a_{m-1}).
\]  

(22)

Lemma 4.1. Let \(A = (a_1, \ldots, a_m) \in \mathbb{A}_+\). Then

\[
d(A) + 1 \leq D(A) \leq f_{d(A)+1}.
\]  

(23)

The left inequality in (23) becomes an equality if and only if \(m = 1\) or \(A = (2, 2, \ldots, 2)\).

The right inequality in (23) becomes an equality if and only if \(a_1 \leq 3\), \(i_m \leq 3\) and \(a_2 = \ldots = a_{m-1} = 3\).

Proof. First, consider the left inequality. For \(m = 1\) or \(A = (2, 2, \ldots, 2)\), the claim is clear. For \(m > 1\), let us perform induction on \(m\). To begin with, assume that \(A = (a_1, \ldots, a_s, k \ast 2)\), where \(a_s > 2\).

Since \(a_1 \geq 2\) and \(a_s - 1 \geq 2\), then formula (21) and the inductive hypothesis imply that

\[
D(A) \geq (k + 1)(a_1 + \ldots + a_s) + 1 + a_1 + \ldots + a_s + k - s + 1 = d(A) + 1.
\]

Let \(a_m > 2\). Formula (22), the inductive hypothesis, and the already considered case \(a_m = 2\) show that

\[
D(A) \geq (a_m - 1)(a_1 + \ldots + a_{m-1} - m - 2) + 1 + a_1 + \ldots + a_m - m + 1 = d(A) + 1.
\]

The left inequality is proved.

To prove the right inequality we use an obvious claim:

\[
f_{a+1} + l f_a \leq f_{a+l+1}, \quad \text{where equality holds if and only if } l = 0, 1.
\]  

(24)

If \(A = (2, 2, \ldots, 2)\), then \(D(A) = d(A) + 1 \leq f_{d(A)+1}\), where the equality holds whenever \(A = (2)\) or \(A = (2, 2)\). Otherwise we use the induction on \(m\).

Let \(A = (a_1, \ldots, a_s, k \ast 2)\), where \(a_s > 2\). Then formula (21) and the inductive hypothesis show that the inequality of lemma is a corollary of the inequality (24) when \(a = a_1 + a_2 + \ldots + a_s - s\) and \(l = k\); this inequality turns into an equality only for \(k = 1\). In this case, the inductive hypothesis implies also that \(a_1 \leq 3\) and \(a_2 = \ldots = a_{m-1} = 3\).

Let \(a_m > 2\). Then formula (22), the inductive hypothesis, and the already considered case \(a_m = 2\) show that the right inequality in (24) is a corollary of inequality (24) for \(a = a_1 + \ldots + a_{m-1} - m - 2\) and \(s = a_m - 2\). Moreover, the inductive hypothesis also implies that the right inequality in (24) turns into an equality only when \(a_1 \leq 3\) and \(a_2 = \ldots = a_m = 3\). \(\square\)
5. Stabilization of Fibonacci partitions

In this section we are using the results of Sections 3 and 4 to show that the function $F(n)$ stabilizes in the following sense:

**Theorem 5.1.** The quantity of numbers $n \in \mathbb{N}$ such that $f_i - 1 \leq n < f_{i+1} - 1$ and $F(n) = k$ does not depend on $i$ for $i \geq 2k$. It is equal to $2\Psi(k)$ if $k > 1$, and to 1 if $k = 1$.

**Lemma 5.2.** Let $n \in \mathbb{G}$ and $\alpha(Z(n)) = A_1 \times A_2 \times \ldots \times A_r$. Then $n \leq f_{2d(A_1)} + f_{2d(A_2)} + \ldots + f_{2d(A_r)} + r + 1 - 2$.

**Proof.** Formula (15) shows that $z(n, \lambda(n)) = 2d(A_1) + 2d(A_2) + \ldots + 2d(A_r) + r$. Since $n \in \mathbb{G}$ we have $n = f_z(n,1) + f_z(n,2) + \ldots + f_z(n,\lambda(n)) \leq f_3 + f_5 + \ldots + f_z(n,\lambda(n)) = f_{2d(A_1)} + f_{2d(A_2)} + \ldots + f_{2d(A_r)} + r + 1 - 2$. □

**Lemma 5.3.** The maximal $k$-generating number is equal to $f_{2k} - 2$.

**Proof.** First, note that $f_{2k} - 2$ is a $k$-generating number since

$$\Pi(f_{2k} - 2) = \Pi(f_3 + f_5 + \ldots + f_{2k-1}) = \frac{k-1}{k} \in \Gamma_+.$$ 

Let $n$ be the maximal $k$-generating number and $\alpha(Z(n)) = A_1 \times \ldots \times A_r$. From Theorem 2.11 and Lemma 4.1 we obtain

$$k = D(A_1) \cdot \ldots \cdot D(A_r) \geq (d(A_1) + 1) \cdot \ldots \cdot (d(A_r) + 1) \geq d(A_1) + \ldots + d(A_r) + r.$$ 

This inequality and Lemma 5.2 show that $n \leq f_{2d(A_1)} + \ldots + f_{2d(A_r)} + r + 1 - 2 \leq f_{2k-r+1} - 2 \leq f_{2k} - 2$. Since $f_{2k} - 2$ is a $k$-generating number, it follows that $n = f_{2k} - 2$ in view of the choice of $n$. □

**Proof of Theorem 5.1.** For $k = 1$, the statement follows from the above description of $\mathbb{G}(1)$. Let $k \geq 2$.

In the proof we use the action of the semigroup $H$ on $\mathbb{N} \setminus \mathbb{N}(1)$, defined in Section 3. Remind that $H$ is generated by the elements $[a]$, where $a \in \mathbb{Z}_{\geq 0}$, and by the involutions $\tau$ and $\sigma$; see formulas (14)–(15) for their action on $\mathbb{N} \setminus \mathbb{N}(1)$. For brevity, set

$$V(m) := [f_m - 1, f_{m+1} - 1].$$

**Lemma 2.19** Lemma 5.3 and formulas (18), (19) imply that for any $n \in \mathbb{G}(k)$ and any $m \geq 2k - 1$, there exists a unique $a = a(n,m) \in \mathbb{Z}_{\geq 0}$ such that either number $[a][n]$ or number $[a][\tau(n)]$ belongs to the set $V(m)$. Let $B_k(m)$ be the set of such numbers obtained for all $n \in \mathbb{G}(k)$. From Theorem 3.8 it follows that

$$B_k(m) = N_1(k) \cap V(m).$$

Then $N_0(k) \cap V(m+1) = \sigma(B_k(m))$. Thus, for any $m \geq 2k$, we have

$$N(k) \cap V(m) = \sigma(B_k(m-1)) \cup B_k(m),$$

where $\sigma(B_k(m-1)) \cap B_k(m) = \emptyset$. Since $|B_k(m)| = |\mathbb{G}(k)|$ for $m \geq 2k - 1$, then $|N(k) \cap V(m)| = 2\Psi(k)$ for any $m \geq 2k$, as claimed. □

6. The upper bounds for $F(n)$

In this section, for any $i \in \mathbb{N}$, we obtain the least upper bound for $F(n)$ when $n \in [f_i, f_{i+1} - 1]$.

**Lemma 6.1.** Let $i_1, i_2, \ldots, i_r$ be the natural numbers $\geq 2$. Then

$$f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_r} \leq \begin{cases} f_{i_1 + i_2 + \ldots + i_r - \frac{r+1}{2}} & \text{if } \beta(r) = 1, \\ 2f_{i_1 + i_2 + \ldots + i_r - \frac{r-1}{2}} & \text{if } \beta(r) = 0. \end{cases}$$ (25)

These inequalities turn into equalities only if either $r = 1$, or one of the following cases takes place:

$r = 2$ and either $i_1 = 2$, or $i_2 = 2$;
$r = 3$ and $i_1 = i_2 = i_3 = 2$;
$r = 4$ and $i_1 = i_2 = i_3 = i_4 = 2$. 


Proof. The well known formula (see [3], Sec.6.6)
\[ f_{a+b} = f_a f_b + f_{a-1} f_{b-1}, \]  
where \( a, b \in \mathbb{N} \),
\[ f_{a+b-1} \leq f_a f_b \leq \frac{1}{2} f_{a+b+1}. \]  
The left inequality in (27) turns into an equality only if either \( a = 1 \), or \( b = 1 \), and the right inequality in (27) turns into an equality only when \( a = b = 2 \).

To prove Lemma 6.1 let us perform induction on \( r \). For \( r = 1 \), the claim is trivial.

Applying formula (26) to \( a = i_1 \), \( b = i_2 \) and to \( a = 2 \), \( b = i_1 + i_2 - 2 \), we see that
\[ f_{i_1} f_{i_2} = 2 f_{i_1+i_2-2} - (f_{i_1-1} f_{i_2-1} - f_{i_1+i_2-3}) \leq 2 f_{i_1+i_2-2}. \]  
The inequality (28) follows from the left inequality in (27), where \( a = i_1 - 1 \) and \( b = i_2 - 1 \); and (28) turns into an equality only if either \( i_1 = 2 \), or \( i_2 = 2 \). This proves Lemma for \( r = 2 \).

Let \( r > 2 \). For \( \beta(r) = 0 \), the inductive hypothesis gives the inequality required, since
\[ f_{i_1} \cdot \ldots \cdot f_{i_{r-1}} \cdot f_i \leq f_{i_1+\ldots+i_{r-1}+1} \cdot f_i \leq 2 f_{i_1+\ldots+i_r-2}. \]  
For \( \beta(r) = 1 \), the inequality required implied by the inductive hypothesis and the right inequality in (27):
\[ f_{i_1} \cdot \ldots \cdot f_{i_{r-1}} \cdot f_i \leq 2 f_{i_1+\ldots+i_{r-1}+1} \cdot f_i \leq f_{i_1+\ldots+i_r-1}. \]  
The equalities in (26) are also covered by the last two inequalities. \( \square \)

**Theorem 6.2.** Let \( n \in [f_i, f_{i+1} - 1] \) and \( i \geq 3 \). Then
\[ F(n) \leq \begin{cases} f_{i+1} & \text{if } \beta(i) = 1, \\ 2 f_{i+1}^{r-1} & \text{if } \beta(i) = 0 \text{ and } i \neq 2, \\ 1 & \text{if } i = 2. \end{cases} \]  
Let \( M(i) \subseteq [f_i, f_{i+1} - 1] \) be the set of numbers \( n \) for which inequalities (26) become equalities. For \( i \geq 5 \), define the set of numbers
\[ M_i := \begin{cases} \left\{ f_2^2 - 1, f_2^2 + f_2 - 1 \right\} & \text{if } \beta(i) = 1, \\ \left\{ f_2^2 f_2 + 2 \cdot (-1)^{i+1} - 1, f_2^2 + f_2 - 1, \\ f_2^2 + 2 f_2 f_2 - 1, f_2 f_2 + 2 \cdot (-1)^{i+1} - 1 \right\} & \text{if } \beta(i) = 0. \end{cases} \]  
The set \( M(i) = M_i \) if \( i \not\in \{1, 2, 3, 4, 6, 9, 12\} \), and
\[ M(1) = \{1\}, \quad M(2) = \{2\}, \quad M(3) = \{3\}, \quad M(4) = \{5, 6, 16\}, \quad M(9) = M(9) \cap \{71\}, \quad M(12) = M(12) \cup \{304\}. \]  
Proof. We say that vector \( A = (a_1, a_2, \ldots, a_q) \in A_+ \) is special if \( a_1 \leq 3 \) and \( a_2 = \ldots = a_q = 3 \). Since, for any special vector, we have
\[ d(A) = \begin{cases} 2q - 1 & \text{if } A = (2, (q - 1) \ast 3), \\ 2q & \text{if } A = (q \ast 3). \end{cases} \]  
it follows that for any vector \( A \in A_+ \), there exists a unique special vector \( A_0 \) such that \( d(A) = d(A_0) \). Let
\[ \alpha(Z(n)) = A_1 \times A_2 \times \ldots \times A_r, \quad d(n) = d(A_1) + d(A_2) + \ldots + d(A_r), \quad r(n) = r. \]  
By formula (15)
\[ n \in [f_i, f_{i+1} - 1] \quad \text{if and only if} \quad 2d(n) + r(n) = i. \]  
Let \( \hat{A} : = Z^{-1} \cdot A_1 \times A_2 \times \ldots \times A_r \). Since \( d(\hat{A}) = d(n) \) and \( r(\hat{A}) = r(n) \), it follows that \( \hat{A} \in [f_i, f_{i+1} - 1] \). Moreover, by the right inequality in (26) and Theorem 2.11 we have
\[ F(n) \leq F(\hat{A}) = f_{d(A_1)+1} \cdot f_{d(A_2)+1} \cdot \ldots \cdot f_{d(A_r)+1}. \]  
Thus, we can assume that \( n = \hat{A} \), i.e., all vectors \( A_k \) are special.
Let $A(i)$ be a special vector such that (use formula (30))
\[
d(A(i)) = \begin{cases} 
    d(A_1) + d(A_2) + \ldots + d(A_r) + \frac{i-1}{2} = \frac{i+1}{2} & \text{if } \beta(i) = 1, \\
    d(A_1) + d(A_2) + \ldots + d(A_r) + \frac{i}{2} = \frac{i+2}{2} & \text{if } \beta(i) = 0. 
\end{cases}
\]
(32)

Define the number $n(i) \in [f_i, f_{i+1} - 1]$ by the formula
\[
n(i) := \begin{cases} 
    Z^{-1} \alpha^{-1}(A(i)) & \text{if } \beta(i) = 1, \\
    Z^{-1} \alpha^{-1}(2 \times A(i)) & \text{if } \beta(i) = 0. 
\end{cases}
\]

Let us show that $F(n) \leq F(n(i))$. Indeed, the assumption on $n$ and formulas (31) and (32) imply that this inequality is equivalent to the inequality
\[
f_{d(A_1)+1} \cdot f_{d(A_2)+1} \cdot \ldots \cdot f_{d(A_r)+1} \leq f_{d(A(i))+1} = \begin{cases} 
    f_{d(A_1)+d(A_2)+\ldots+d(A_r)+\frac{i-1}{2}+1} = \frac{f_{i+1}}{2} & \text{if } \beta(i) = 1, \\
    2 f_{d(A_1)+d(A_2)+\ldots+d(A_r)+\frac{i}{2}-1} = 2 f_i^{-1} & \text{if } \beta(i) = 0.
\end{cases}
\]

The substitution $d(A_k) = i_k - 1$ turns it into the inequality (29). Thus, (29) is proved.

The same reasoning and Lemma 4.1 show that for any $n \in M(i)$, we have
\[
\alpha(Z(n)) = \begin{cases} 
    A & \text{if } \beta(i) = 1, \\
    (2) \times A \text{ or } A \times (2) & \text{if } \beta(i) = 0,
\end{cases}
\]
where $A$ is of one of the following forms
\[
(2, 3, \ldots, 3), \quad (3, 3, \ldots, 3), \quad (2, 3, \ldots, 3, 2), \quad (2) \times (2) \times (2).
\]

Namely, formula (30) implies that if $\beta(i) = 1$, then
\[
A = \begin{cases} 
    (2) \times (2) \times (2) & \text{if } i = 9, \\
    (2) \times (2) \times (2) & \text{if } i = 12
\end{cases}
\]
If $i \neq 9, 12$ and if $\beta(i) = 1$, then
\[
A = \begin{cases} 
    (2, \frac{i-3}{2} * 3) \text{ or } (\frac{i-3}{2} * 3, 2) & \text{if } i \equiv 3 \mod 4, \\
    (\frac{i-1}{2} * 3) \text{ or } (2, \frac{i-2}{2} * 3, 2) & \text{if } i \equiv 1 \mod 4,
\end{cases}
\]
and if $\beta(i) = 0$, then
\[
A = \begin{cases} 
    (2, \frac{i-6}{3} * 3) \text{ or } (\frac{i-6}{3} * 3, 2) & \text{if } i \equiv 2 \mod 4, \\
    (\frac{i-1}{3} * 3) \text{ or } (2, \frac{i-2}{3} * 3, 2) & \text{if } i \equiv 0 \mod 4.
\end{cases}
\]

Since the polyvectors $\alpha(Z(n))$ for all $n \in M(i)$ are now known, we can find the corresponding $n$ by a routine calculation, thanks to formula (30) and easily verifiable identities
\[
f_3 + f_7 + \ldots + f_{4q-1} = f_{2q}^2 - 1, \quad f_5 + f_9 + \ldots + f_{4q+1} = f_{2q+1}^2 - 1, \\
f_2 + f_6 + \ldots + f_{4q-2} = f_{2q-1} \cdot f_{2q}, \quad f_4 + f_8 + \ldots + f_{4q} = f_{2q} \cdot f_{2q+1} - 1.
\]

As a result, we obtain the claimed formulas for $M_i$.

When $i = 9$ or $i = 12$, we should add $f_3 + f_6 + f_9 = 71$ and $f_3 + f_6 + f_9 + f_{12} = 304$, respectively, to the numbers constituting the sets $M_i$. This completes the proof.

**Corollary 6.3.** If $i \notin \{1, 2, 3, 4, 6, 9, 12\}$, then
\[
|M(i)| = \begin{cases} 
    2 & \text{if } \beta(i) = 1, \\
    4 & \text{if } \beta(i) = 0.
\end{cases}
\]

**Corollary 6.4.** For $n \in \mathbb{N}$, we have $F(n) \leq \sqrt{n+1}$, where the equality holds whenever $n = f_i^2 - 1$. 
Theorem 7.1. Let \( \beta(i) = 1 \) and \( i = 2m \). For \( m = 1, 2, 3, 4 \), the claim is trivial. Thanks to the formula \((33)\) and Theorem 6.2 to finish the proof it suffices to show that
\[
4f_{m-1}^2 < f_m f_{m+1} + 2 \cdot (-1)^{m+1} \quad \text{for} \quad m \geq 5.
\]
Using the Cassini formula \( f_m f_{m-1} = f_m f_{m-2} + (-1)^{m+1} \) (see [3], Sec. 6.6), we obtain
\[
f_m f_{m+1} - 4f_{m-1}^2 + 2 \cdot (-1)^{m+1} = f_m (f_m - f_{m-2}) + 2(-1)^m = f_m f_{m-5} + 2(-1)^m > 0. \quad \square
\]

7. How often does \( \chi(n) \) vanish

In this section we use Theorem 2.11 to prove the next statement – the main result of paper [1]:

**Theorem 7.1.** Let \( E(a) := \{ n \mid 0 < n \leq a \text{ and } \chi(n) = 0 \} \). Then \( \lim_{a \to \infty} \frac{E(a)}{a} = 1. \)

First we establish several auxiliary formulas.

**Lemma 7.2.** Let \( n \in [f_{i-3} + f_i - 1, f_{i-2} + f_i - 1] \), where \( i \geq 3 \). Then \( \chi(n) = 0. \)

**Proof.** Lemma follows from formula \((7)\), where \( t = -1 \) and \( a = i - 3 \). \( \square \)

**Lemma 7.3.** Let \( n \in [0, f_i - 1] \) and \( s \geq i \geq 1 \). Then \( \chi(n) = \chi(n + f_s + f_{s+2}). \)

**Proof.** It suffices to assume that \( n \) is an \( f \)-simple number. Let \( Z(n) = \{ i_1, i_2, \ldots, i_m \} \) and \( \alpha(n) = (a_1, a_2, \ldots, a_m) \).

By Lemma 2.12 we have \( i_m \leq i - 1 \). If \( i_m = i - 1 \) and \( s = i \), then
\[
n + f_i + f_{i+2} = f_i + \cdots + f_{i-1} + f_{i+3}.
\]
Therefore, \( \alpha(n + f_i + f_{i+2}) = (a_1, \ldots, a_{m-1}, a_m + 2) \). Then the claim follows from an obvious identity
\[
\chi(a_1, \ldots, a_{m-1}, a_m + 2) = \chi(a_1, \ldots, a_{m-1}, a_m).
\]
Let \( s - i_m \geq 2 \) and \( a = \lceil \frac{s - i_m}{2} \rceil + 1 \). Then
\[
\alpha(n + f_s + f_{s+2}) = \begin{cases} (a_1, \ldots, a_m, a, 2) & \text{if } \beta(s) = \beta(i_m), \\ (a_1, \ldots, a_m) \times (a, 2) & \text{if } \beta(s) \neq \beta(i_m). \end{cases}
\]
Now, for \( \beta(s) = \beta(i_m) \), the claim follows from the equality \( \xi(2; -1) = 0 \) and formula \( (3) \). For \( \beta(s) \neq \beta(i_m) \), it follows from the equality \( \chi(a, 2) = 1 \). \( \square \)

**Lemma 7.4.** Let
\[
h(i) := | \{ n \mid 0 \leq n \leq f_i - 1 \text{ and } \chi(n) = 0 \} | .
\]
Then \( h(0) = h(1) = h(2) = h(3) = 0 \), \( h(4) = 1 \), and
\[
h(i) = f_{i-5} + 1 + h(i - 1) + 2h(i - 4) \quad \text{if } \quad i \geq 5.
\]

**Proof.** For \( i < 5 \), the claim is trivial. Let \( i \geq 5 \).

The quantity of \( n \in [f_{i-1} - 1, f_i - 1] \) with \( \chi(n) = 0 \) is equal to \( h(i) - h(i - 1) \). By Lemma 7.2 we have \( \chi(n) = 0 \) if \( n \in [f_{i-4} + f_{i-1} - 1, f_{i-3} + f_{i-1}] \). The quantity of such numbers \( n \) is equal to \( f_{i-5} + 1 \).

Lemma 2.21 implies that
\[
| \{ n \in [f_{i-1} - 1, f_{i-4} + f_{i-1} - 2] \mid \chi(n) = 0 \} | = | \{ n \in [f_{i-3} + f_{i-1}, f_i - 1] \mid \chi(n) = 0 \} | .
\]

\footnote{This claim appeared also in [9] in the proof of Theorem 9.}
Let \( n \in [f_{i-3} + f_{i-1}, f_i - 1] \). Clearly, \( n = n' + f_{i-3} + f_{i-1} \), where \( n' \in [0, f_{i-4} - 1] \). Lemma 4.3 shows that \( \chi(n) = \chi(n') \). Thus,
\[
| \{ n \in [f_{i-1} - 1, f_{i-4} + f_{i-1} - 2] \cup [f_{i-3} + f_{i-1}, f_i - 1] | \chi(n) = 0 \} | = 2h(i-4).
\]
Collecting all together we obtain the asserted recurrence for \( h(i) \).

**Proof of Theorem 7.1.** The sequence \( E(a) \) does not decrease as \( a \) grows, and its subsequence \( E(f_i - 1) = h(i) \) is increasing. Therefore, it suffices to check that \( \lim_{i \to \infty} \frac{h(i)}{i} = 1. \)

Let \( H(t) := \sum_{i=0}^{\infty} h(i)t^i \). From Lemma 7.4 it follows that
\[
H(t) = \frac{1}{1 - t - t^2} + \frac{1}{2(t - 1)} - \frac{1}{14(t + 1)} + \frac{8t^2 - 2t + 3}{7(2t^3 - 2t^2 + 2t - 1)}.
\]
In a standard way (see [3], Sec.7.3) this implies that
\[
h(i) = f_i - 1 + \frac{1}{t} \left( 3 + \beta(i) + a_1 t_1^i + a_2 t_2^i + a_3 t_3^i \right),
\]
where \( t_1, t_2, t_3 \) are the roots of the equation \( t^3 - 2t^2 + 2t - 2 = 0 \) and \( a_1, a_2, a_3 \) are some constants. Let \( t_1 \) be the real root while \( t_2 \) and \( t_3 \) be the complex conjugate roots. Then
\[
t_1 = \frac{u^2 + 2u - 2}{3u} \quad \text{and} \quad t_{2,3} = \frac{2 + 4u - u^2}{6u} \pm \frac{2 + u^2}{6u} \sqrt{-3}, \quad \text{where} \quad u = \sqrt[3]{3 + 3\sqrt{33}}.
\]
It is easy to verify that \( |t_{2,3}| < \phi \approx 1.62, \quad (t_1 \approx 1.54, |t_{2,3}| \approx 1.13) \).

Now, Theorem 7.1 follows from the asymptotic \( f_i \sim \frac{\sqrt{5}}{\sqrt{8} \phi^{i+1}} \), see [3], Sec.6.6.

**Remark 7.5.** Lemma 4.3 shows that there exist arbitrary long intervals of consequent numbers \( n \) with \( \chi(n) = 0 \). Contrariwise, one can show that the quantity of the consequent numbers \( n \) with \( \chi(n) \neq 0 \) does not exceed 4. Moreover, only the following sequences of \( n \) are realized:

\[
\{1\}, \{-1\}, \{1, -1\}, \{-1, 1\}, \{1, 1, -1\}, \{-1, -1, 1\}, \{1, -1, -1\}, \{-1, 1, 1\}, \{-1, 1, 1\}, \{-1, 1, 1\}.
\]

8. **On minimal solutions of the equation** \( F(n) = k \) **as** \( k \) **varies**

In this section I analyze the problem of finding the minimal solution of the equation \( F(n) = k \) for \( k > 1 \) from the point of view developed in Section 3. The main result of the section is Theorem 8.4.

**Definition 8.1.** Denote by \( m_F(k) \) the minimal solution of the equation \( F(n) = k \). We say that the number \( k > 1 \) is \( F\)-**primitive** if \( m_F(k) \) is \( f \)-**simple** number. In particular, for any \( F\)-**primitive** number \( k \) where is a uniquely defined number \( m(k) \) such that \( \Pi(m_F(k)) = \frac{m(k)}{k} \).

Let us give some examples of the \( F\)-**primitive** numbers. For any prime \( p \), any generating solution of the equation \( F(n) = p \) is \( f \)-**simple** by Proposition 3.3. Therefore, all primes are \( F\)-**primitive**. But many composite numbers are \( F\)-**primitive** as well. For example, Corollary 6.3 implies that \( m_F(f_i) = f_i^2 - 1 \). The number \( f_i^2 - 1 \) is \( f \)-**simple** since
\[
\Pi(f_i^2 - 1) = \begin{cases} \pi \left( \frac{2}{\pi}, \left\lfloor \frac{i+1}{2} \right\rfloor \ast 3 \right) = \frac{f_{i+1}}{f_i} & \text{if } i \equiv 0 \mod 2, \\ \pi \left( \left\lfloor \frac{i+1}{2} \right\rfloor \ast 3 \right) = \frac{f_{i+1}}{f_i} & \text{if } i \equiv 1 \mod 2. \end{cases}
\]
Therefore, any Fibonacci number \( f_i \) for \( i \geq 2 \) is \( F\)-**primitive**. The first \( F\)-**primitive** numbers \( k > 1 \) and the corresponding minimal values \( m(k) \) and \( m_F(k) \) are as follows:

| \( k \) | 2 | 3 | 5 | 7 | 8 | 11 | 13 | 17 | 18 | 19 | 21 | 23 | 27 | 29 | 31 | 34 | 37 |
|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| \( m(k) \) | 1 | 1 | 3 | 4 | 3 | 4 | 8 | 11 | 7 | 11 | 8 | 14 | 11 | 17 | 21 | 13 | 14 |
| \( m_F(k) \) | 3 | 8 | 24 | 58 | 63 | 152 | 168 | 406 | 401 | 435 | 440 | 1011 | 1066 | 1050 | 1160 | 1155 | 2647 |
Remark 8.2. Conjecturally, the numbers $f_i + f_{i+2}$, called the Lucas numbers, are $F$-primitive if $i > 1$. Moreover, $\Pi(m_F(f_i + f_{i+2})) = \Pi(m_F(f_i)) \oplus \Pi(m_F(f_{i+2}))$, where $\oplus$ denotes the Farey sum (see [4]). Then

$$m_F(f_i + f_{i+2}) = f_i^2 + f_{2i+5} - 1.$$ 

Most likely, the numbers $f_i + f_{i+3} + f_{i+5}$ for $i \geq 1$ are also $F$-primitive.

Formula (14) and the definition of $\Pi$ imply the “monotonicity” of the multiplication $\times$:

Lemma 8.3. For any $n, n_1, n_2 \in \mathbb{G}$ such that $n_1 < n_2$ we have $n_1 \times n < n_2 \times n$.

Theorem 8.4. For any natural $k \geq 2$ let

$$\Pi(m_F(k)) = \frac{l_1}{k_1} \times \frac{l_2}{k_2} \times \cdots \times \frac{l_r}{k_r} \in \Gamma_+.$$ 

Then the numbers $k_1, k_2, \ldots, k_r$ are $F$-primitive and $l_i = m(k_i)$ for any $i \in [1, r]$. Thus

$$m_F(k) = m_F(k_1) \times m_F(k_2) \times \cdots \times m_F(k_r).$$ 

Proof. The claim follows from Lemma 8.3 by induction on $r$. \hfill \square

For example,

$$m_F(6) = m_F(2) \times m_F(3) = 37, \quad m_F(54) = m_F(3) \times m_F(18) = 4456,$$

$$m_F(57) = m_F(19) \times m_F(3) = 4616, \quad m_F(96) = m_F(2) \times m_F(2) \times m_F(8) \times m_F(3) = 12993.$$

Theorem 6.2 easily implies that $m_F(2f_i) = m_F(2) \times m_F(f_i) = 3 \times (f_i^2 - 1) = f_i f_{i+3} - 2 \beta(i)$.

The following questions naturally arise:

1. How to describe the set of $F$-primitive numbers?
2. How to find $m(k)$ for an $F$-primitive number $k$ (for example, when $k$ is prime)?
3. When for a sequence of $F$-primitive numbers $k_1, k_2, \ldots, k_r$ we have

$$m_F(k_1 k_2 \cdots k_r) = m_F(k_1) \times m_F(k_2) \times \cdots \times m_F(k_r)?$$

I have plotted the graph of linear approximation of the frequencies distribution (histogram) of rational numbers $\frac{m(k)}{k}$, where $k$ runs over the set of $F$-primitive numbers from the segment $[1, 5000]$. This segment contains 1764 such numbers, 669 of them are primes. To build the plot, the sequence of minimal solutions of the equation $F(n) = k$ from [10], A013583, was quite useful.

![Figure 1. The frequency distribution of rationals $\Pi(m_F(k)) = \frac{m(k)}{k}$, where $k$ runs over the set of $F$-primitive numbers in interval $[1, 5000]$.](image-url)
The graph shows that set $\mathbb{P}$ of rational numbers $\frac{m^{(k)}}{k}$, where $k$ runs over the set of all $F$-primitive numbers, has a complicated structure. Probably, this set has infinite number of limit points.

**Conjecture 8.5.** The set $\mathbb{P}$ is invariant with respect to the reflection of the interval $[0, 1]$ with respect to point $\frac{1}{2}$. The numbers $\phi^{-2}$ and $\phi^{-1}$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$, are the limit points of $\mathbb{P}$.

9. **On the graph of the function $F$**

Many of the claims in this article are the result of attempting to explain the behavior of the graph of $F$. Let me make several remarks on this. The graph on the interval $[1, f_{19} - 1]$ is presented below:

**Figure 2.** The graph of the function $F(n)$, where $n \in [1, f_{19} - 1]$.

Theorem 2.11 implies that $F(n) = 1$ whenever $n = f_i - 1$, where $i \geq 1$. The graph of $F$ consists of a sequence of increasing “waves”: the $i$th wave $W_i$ for $i \geq 2$ is the set of points in the plane with coordinates $(n, F(n))$, where $f_i \leq n < f_{i+1}$; by definition, $W_0 = (0, 1), W_1 = (1, 1)$. For $f_a \leq n < f_{a+1}$, where $a \leq i - 2$, formula (7) for $t = 1$ shows that

$$F(n + f_i) = \begin{cases} \frac{i-a+1}{2} \cdot F(n) & \text{if } a \not\equiv i \mod 2, \\ \frac{i-a+2}{2} \cdot F(n) - F(n - f_a) & \text{if } a \equiv i \mod 2. \end{cases} \quad (34)$$

Let $S_i(n, F(n)) := (n + f_i, F(n + f_i))$. It is easy to see that

$$W_i = S_i(W_0 \cup W_1 \cup \cdots \cup W_{i-2}), \quad \text{where } i \geq 2. \quad (35)$$

This allows us to build the sets $W_i$ inductively, starting from the set $W_0 \cup W_1$. Formulas (35) and (34) show that the $i$th wave, where $i \geq 2$, is similar to the union of the $i - 2$ previous waves. These formulas can also be used to explain some visible symmetries inside each wave.

Lemma 2.21 shows that the function $F$ is invariant with respect to the reflection $\rho_i$ of the interval $[f_i - 1, f_{i+1} - 1]$ with respect to its center.

The next conjecture can provide a much more precise upper bound for $F(n)$ when $n \in [f_i - 1, f_{i+1} - 1]$ as compared with the estimate in Theorem 6.2.
Conjecture 9.1. Let $C_i$ be the convex hull of the set of points $(n, F(n))$, where $n \in [f_i - 1, f_{i+1} - 1]$ and $i > 7$. Set

$$b_1(i, q) := f_i - 1 + f_q^2, \quad b_2(i, q) := f_i - 1 + f_qf_{q+1} + 2 \cdot (-1)^q,$$

$$B_r(i) := \left\{ b_r(i, q), \rho_i(b_r(i, q)) \mid 1 \leq q \leq \left\lceil \frac{i}{2} \right\rceil - 2 \right\},$$

where $r = 1, 2$ and $\rho_i(n) = f_{i+2} - 2 - n$.

Then $C_i$ coincides with the convex hull of the set of points $(c, F(c))$, where $c \in B_1(i) \cup \{ f_i - 1, f_{i+1} - 1 \}$ if $\beta(i) = 1$, and $c \in B_1(i) \cup B_2(i) \cup \{ f_i - 1, f_{i+1} - 1 \}$ if $\beta(i) = 0$.

10. Variations on the theme of function $\Psi(k)$

The material of this section concerns the function $\Psi(k) = |G(k)|$ as well as several additional functions similar to it. In particular, from the formulas we will obtain a recurrent expression for $\Psi(k)$ and a formula for the Dirichlet generating function of the sequence $\Psi(k)$ will follow.

Let $k = p_1^{a_1}p_2^{a_2} \cdots p_r^{a_r}$ be the prime decomposition of $k \in \mathbb{N}$ and $\deg(k) := n_1 + n_2 + \ldots + n_r$. Denote by $\Gamma_+(k, m)$ the set of words $\frac{k}{k_1} \times \frac{k_1}{k_2} \times \ldots \times \frac{k_r}{k_m} \in \Gamma_+(k)$. Then

$$\Gamma_+ = \prod_{k=2}^{\infty} \prod_{m=1}^{\deg(k)} \Gamma_+(k, m)$$

is a disjoint union of nonempty subsets. Set

$$\psi(k, m) := \begin{cases} |\Gamma_+(k, m)| & \text{if } m \leq \deg(k), \\ 0 & \text{if } m > \deg(k) \end{cases} \quad \text{and} \quad \Psi(k; t) := \sum_{m=1}^{\infty} \psi(k, m)t^m.$$

Remark 10.1. In Section [10] by abuse of notation I denote the function of two arguments by the same symbol $\Psi$ used to denote the function of one argument $\Psi(k) := \Psi(k; 1)$.

Theorem 10.2. The function $\Psi(k; t)$ is expressed by the recursive formula

$$\Psi(k; t) = t \sum_{a > 1, a | k} \Psi \left( \frac{k}{a}; t \right) \varphi(a),$$

where $\Psi(1; t) = 1$.

Proof. Let $a > 1$ and $\Gamma_+^{(a)}(k, m) := \left\{ \frac{k_1}{k_1} \times \frac{k_1}{k_2} \times \ldots \times \frac{k_m}{k_m} \in \Gamma_+(k) \mid k_m = a \right\}$. Then $\Gamma_+^{(a)}(k, m) \neq \emptyset$ if and only if $a | k$, and

$$|\Gamma_+^{(a)}(k, m)| = \sum_{1 \leq b < a, \gcd(a, b) = 1} \left| \Gamma_+ \left( \frac{k}{a}, m - 1 \right) \times \frac{b}{a} \right| = \left| \Gamma_+ \left( \frac{k}{a}, m - 1 \right) \right| \varphi(a). \quad (36)$$

Since $\Gamma_+^{(a_1)}(k, m) \cap \Gamma_+^{(a_2)}(k, m) = \emptyset$ for $a_1 \neq a_2$, it follows that

$$|\Gamma_+(k, m)| = \sum_{a > 1, a | k} |\Gamma_+^{(a)}(k, m)|. \quad (37)$$

To finish the proof we substitute expression (36) in (37) and apply the definition of $\Psi(k; t)$. \hfill \Box

Theorem 10.3. We have

$$1 + \sum_{k=2}^{\infty} \frac{\Psi(k; t)}{k^s} = \left( 1 + t - t \frac{\zeta(s - 1)}{\zeta(s)} \right)^{-1}. \quad (38)$$

Proof. (Cf. [5], p.138.) Let

$$L(s, \varphi) = 1 + \sum_{k=2}^{\infty} \frac{\varphi(k)}{k^s}.$$
For any \( m \geq 1 \), we obviously have
\[
\sum_{k=2}^{\infty} \frac{\psi(k, m)}{k^s} = (L(s, \varphi) - 1)^m.
\]
Therefore,
\[
1 + \sum_{k=2}^{\infty} \frac{\Psi(k; t)}{k^s} = 1 + \sum_{k=2}^{\infty} \sum_{m=1}^{\infty} \frac{\psi(k, m)t^m}{k^s} = 1 + \sum_{m=1}^{\infty} t^m (L(s, \varphi - 1))^m = (1 + t - tL(s, \varphi))^{-1}.
\]
Now, the claim required follows from the classical formula \( L(s, \varphi) = \frac{\zeta(s-1)}{\zeta(s)} \) (see [1], p.250).

Formula (38) for \( t = 1 \) turns into formula (1). For \( t = -1 \), formula (38) gives, after straightforward calculations, the expression
\[
\Psi(k; -1) = \mu(k) \varphi(k),
\]
where \( \mu \) is the Möbius function (see [4], p.234).

The sequence \( \Psi(k) \) rapidly grows with the growth of the quantity of prime factors of \( k \). Let \( p \) be a prime, let \( p, p_2, \ldots, p_r \) be different primes. One can show that
\[
\Psi(p^n) = (p - 1)(2p - 1)^{n-1}
\]
and
\[
\Psi(p_1p_2 \cdots p_r) = B(r) (p_1 - 1)(p_2 - 1) \cdots (p_r - 1),
\]
where \( B(0) = 1 \) and
\[
B(r + 1) = \sum_{i=0}^{r} \binom{r+1}{i} B(i).
\]
The numbers \( B(r) \in \{1, 3, 13, 75, 541, 4683, 47293, 545835, \ldots\} \), where \( r \geq 1 \), are known as ordered Bell numbers (see [11], p.175). The number \( B(r) \) is the quantity of ways to represent the set \( \{1, 2, \ldots, r\} \) as an ordered union of non-intersecting subsets.

**Definition 10.4.** The words \( \frac{k_1}{k_1} \times \frac{k_2}{k_2} \times \cdots \times \frac{k_m}{k_m} \) and \( \frac{\nu_1}{\nu_1} \times \frac{\nu_2}{\nu_2} \times \cdots \times \frac{\nu_m}{\nu_m} \) from the set \( \Gamma_+(k, m) \) will be referred to as

- \( \omega \)-equivalent if \( k_1 = k'_1, k_2 = k'_2, \ldots, k_m = k'_m \);

- \( t \)-equivalent if \( \left\{ \frac{k_1}{k_1}, \frac{k_2}{k_2}, \ldots, \frac{k_m}{k_m} \right\} = \left\{ \frac{\nu_1}{\nu_1}, \frac{\nu_2}{\nu_2}, \ldots, \frac{\nu_m}{\nu_m} \right\} \) as sets;

- \( s \)-equivalent if \( \{k_1, k_2, \ldots, k_m\} = \{k'_1, k'_2, \ldots, k'_m\} \) as sets;

- \( \ast \)-equivalent if they are \( t \)-equivalent, and if \( k_i = k'_i \), then either \( l_i = l'_i \), or \( l_i \cdot l'_i \equiv 1 \) mod \( k_i \).

Let us extend each of these relations to \( \Gamma_+ \) component-wise. It is easy to see that, for any of the relations introduced, the elements of \( \Gamma_+ \) are equivalent if and only if they belong to an orbit of the action of their corresponding subgroups in the automorphism group of \( \Gamma_+ \).

Each \( \omega \)-equivalence class from the set \( \Gamma_+(k, m) \) contains a unique word \( \frac{k_1}{k_1} \times \frac{k_2}{k_2} \times \cdots \times \frac{k_m}{k_m} \), where \( k_1k_2 \cdots k_m = k \), and the order of factors is essential. Any such decomposition is called an ordered multiplicative partition of \( k \) of length \( m \). Let \( \hat{\psi}(k, m) \) be the quantity of such partitions. Define
\[
\hat{\Psi}(k; t) := \sum_{m=1}^{\infty} \hat{\psi}(k, m)t^m.
\]
Arguing exactly as in the proof of Theorem 10.3 where \( \varphi \) is replaced with 1, we obtain the formula
\[
1 + \sum_{k=2}^{\infty} \frac{\hat{\psi}(k; t)}{k^s} = (1 + t - t\zeta(s))^{-1},
\]
which is well known at least for \( t = 1 \), see [5].

Each \( t \)-equivalence class from the set \( \Gamma_+(k, m) \) contains a unique word \( \frac{k_1}{k_1} \times \frac{k_2}{k_2} \times \cdots \times \frac{k_m}{k_m} \), such that \( 1 < k_1 \leq k_2 \leq \cdots \leq k_m \), and if \( k_i = k_{i+1} \), then \( l_i \leq l_{i+1} \) for \( 1 \leq i < m \). The set of these classes has a natural structure of a free commutative monoid \( \Gamma'_+ \), generated by the set \( Q_{(0,1)} \).

Denote the set of its elements with \( k_1k_2 \cdots k_r = k \) by \( \Gamma'_+(k) \) and define \( \Psi_+(k) := |\Gamma'_+(k)| \). One can think about \( \Psi_+ \) as about a “commutative” analog of \( \Psi \). To obtain a generating function for the sequence \( \Psi_+(k) \), let us first make a general observation.
For any function $f : \mathbb{N} \to \mathbb{N}$, consider the decomposition
\[
\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^s}\right)^{-f(k)} = 1 + \sum_{k=2}^{\infty} \tilde{f}(k) k^s.
\]

One can interpret the number $\tilde{f}(k)$ as the number of ways to write $k$ as the product $k = k_1 k_2 \cdots k_r$, where $1 < k_1 \leq k_2 \leq \cdots \leq k_r$, each factor $k_i$ has $f(k_i)$ different colors, and differently colored products are considered different. One can show that, for a prime $p$,
\[
\tilde{f}(p^a) = \sum_{a_1 + 2a_2 + \cdots + a_s = a} \prod_{i=1}^{\infty} \left( f(p^i) + a_i - 1 \right),
\]
where $a_1, a_2, \ldots, a_s, \ldots$ are the non-negative integers.

Let $f(k) = \varphi(k)$ be the Euler totient function. Then, obviously,
\[
\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^s}\right)^{-\varphi(k)} = 1 + \sum_{k=2}^{\infty} \Psi_c(k) k^s.
\]

One can show that, for different primes $p_1, p_2, \ldots, p_r$, we have
\[
\Psi_c(p_1 p_2 \cdots p_r) = B_c(r) (p_1 - 1)(p_2 - 1)\cdots(p_r - 1),
\]
where $B_c(0) = 1$ and
\[
B_c(r + 1) = \sum_{i=0}^{r} \binom{r}{i} B_c(i).
\]

The numbers $B_c(r) \in \{1, 2, 5, 15, 52, 203, 877, 4140, \ldots\}$, where $r \geq 1$, are known as Bell numbers (see [11], p.20). The number $B_c(r)$ is the quantity of ways to represent the set $\{1, 2, \ldots, r\}$ as a union of non-intersecting subsets.

I was unable to find a general expression for $\Psi_c(p^a)$. Here are the first several ones:
\[
\begin{align*}
\Psi_c(p) &= p - 1, \\
\Psi_c(p^2) &= \frac{p(p - 1)}{2!} \cdot 3, \\
\Psi_c(p^3) &= \frac{p(p - 1)}{3!} \cdot (13p - 5), \\
\Psi_c(p^4) &= \frac{p(p - 1)}{4!} \cdot (73p^2 - 45p + 14), \\
\Psi_c(p^5) &= \frac{p(p - 1)}{5!} \cdot (501p^3 - 414p^2 + 111p - 54), \\
\Psi_c(p^6) &= \frac{p(p - 1)}{6!} \cdot (4051p^4 - 4130p^3 + 1445p^2 - 190p + 264).
\end{align*}
\]

For the case of $s$-equivalence, we set $f(k) = 1$ for any $k \geq 2$. Then we obtain the known formula
\[
\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^s}\right)^{-1} = 1 + \sum_{k=2}^{\infty} \hat{\Psi}_c(k) k^s,
\]
where $\hat{\Psi}_c(k)$ is the quantity of the decompositions $k = k_1 k_2 \cdots k_m$, where $1 < k_1 \leq k_2 \leq \cdots \leq k_m$. Such a decomposition is called an unordered multiplicative partition of $k$. For example, $\tilde{\Psi}_c(p^n)$ for a prime $p$ is equal to the quantity of all integer partitions of $n$. This evident claim follows from formula (39) as well.

Finally, the generating function for the number of $s$-equivalent classes, has the form
\[
\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^s}\right)^{-\varphi_s(k)} = 1 + \sum_{k=2}^{\infty} \varphi_s(k) k^s,
\]
where $\varphi_s(k)$ is defined as follows. Let $(\mathbb{Z}/k\mathbb{Z})^*$ denote the multiplicative group of the ring $\mathbb{Z}/k\mathbb{Z}$. Then $\varphi_s(k)$ is the quantity of the invariant subsets under the inversion operation in the group $(\mathbb{Z}/k\mathbb{Z})^*$. 

This evident claim follows from formula (39) as well.
NOTES ON FIBONACCI PARTITIONS

The function $\varphi_*$ is multiplicative. That is, $\varphi_*(k_1 k_2) = \varphi_*(k_1) \varphi_*(k_2)$ for relatively prime $k_1$ and $k_2$. For any prime $p$ and any $a \in \mathbb{N}$, the structure of the group $(\mathbb{Z}/p^a\mathbb{Z})^*$ is well known (see [7]). It is not difficult to check the formula

$$\varphi_*(p^a) = \begin{cases} 1 & \text{if } p = 2 \text{ and } a = 1, \\ 2 & \text{if } p = 2 \text{ and } a = 2, \\ 2^{a-2} + 2 & \text{if } p = 2 \text{ and } a > 2, \\ \frac{1}{2} p^{a-1} (p-1) + 1 & \text{if } p > 2. \end{cases}$$

This allows us to easily compute $\varphi_*(n)$ for any natural $n$.

Calculations show that the first 20 values of the functions $\Psi(k), \Psi_c(k), \Psi_*(k)$ are as follows (see [10], A006874, A007896, A007898):

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| $\Psi(k)$ | 1 | 1 | 2 | 3 | 4 | 6 | 9 | 10 | 12 | 18 | 24 | 27 | 38 | 44 | 72 | 51 | 76 | 84 | 108 | 111 |
| $\Psi_c(k)$ | 1 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 8 | 10 | 12 | 12 | 16 | 16 | 19 | 18 | 24 | 16 | 20 | 18 |
| $\Psi_*(k)$ | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 7 | 7 | 6 | 12 | 7 | 8 | 12 | 16 | 9 | 15 | 10 | 18 | |

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