SOME REMARKS ON THE CHARACTERS OF THE GENERAL LIE SUPERAELGEBRA

R.C. ORELLANA AND MIKE ZABROCKI

Abstract. We compute an explicit formula the Hilbert (Poincaré) series for the ring of hook Schur functions.

Introduction

Berele and Regev [1] defined the ring of hook Schur functions and showed that these functions are the characters of the general Lie superalgebra. Since the introduction of the hook Schur functions they have been extensively studied with respect to their combinatorial properties [2], [3], [4].

We compute the Hilbert series of this ring by giving a generating function for the partitions which fit inside of a $(k, \ell)$-hook. Besides its interest as a purely combinatorial result, this formula should have applications to discovering algebraic properties of the ring of hook Schur functions.

1. Definitions and Notation

For $k, \ell \in \mathbb{N}$ let $x_1, \ldots, x_k; y_1, \ldots, y_\ell$ be two sets of commuting variables. In this note $\lambda, \mu, \nu$ will represent partitions and these partitions will be identified with their corresponding Young diagrams (geometric representation of partitions).

Let $n \in \mathbb{N}$. Define $H_n(k, \ell)$ to be the set of partitions $\lambda = (\lambda_1, \cdots, \lambda_m)$ such that $\lambda_{k+i} \leq \ell$ for $i = 1, \cdots, m-k$. This is equivalent to Young diagrams that fit inside the $(k, \ell)$-hook, see Figure 1.

Definition 1. (a)Let $\lambda$ be a partition of $n$. Then the Hook Schur function $HS_{\lambda}$ is defined as follows, for any $k, \ell$,

$$HS_{\lambda}(x_1, \ldots, x_k; y_1, \ldots, y_\ell) := \sum_{\mu < \lambda} s_\mu(x_1, \ldots, x_k)s_{\lambda'/\mu'}(y_1, \cdots, y_\ell),$$

where $s_\nu$ denotes the Schur function and $\lambda'/\mu'$ denotes the conjugate of the skew partition $\lambda/\mu$. 

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Lemma 2.

(b) \( \Lambda_{n}^{(k,\ell)} = \text{span}_{F} \{ HS_{\lambda}(x_{1}, \ldots , x_{n}; y_{1}, \ldots , y_{\ell}) \mid \lambda \in H_{n}(k, \ell) \} \) where \( H_{n}(k, \ell) \) is the subset of the partitions of \( n \) which fit in the \((k, \ell)\) hook.

Then form \( \Lambda^{(k,\ell)} = \bigoplus_{n \geq 0} \Lambda_{n}^{(k,\ell)} \).

These functions were defined in [1] as the characters of the general Lie superalgebra, \( p(V) \), where \( V \) is a vector space with an associated \( \mathbb{Z}/2\mathbb{Z} \) grading. Berele and Regev proved that \( \Lambda^{(k,\ell)} \) is a ring.

2. Hilbert Series of \( \Lambda^{(k,\ell)} \)

Computing this series is equivalent to the combinatorial problem of counting the number of Young diagrams which fit inside a \((k, \ell)\) hook. It is well-known that the generating function for partitions with less than or equal to \( k \) rows is given by \( G_{k}(t) = \prod_{i=1}^{k} \frac{1}{1-t^{i}} \) and that the generating function for the partitions which fit inside of a \( k \times \ell \) rectangle is given by

\[
\begin{bmatrix}
k + \ell \\
\ell
\end{bmatrix} = \frac{(t; t)_{k+\ell}}{(t^{2}; t)_{k}(t; t)_{\ell}}
\]

where \((a; x)_{n} = (1-a)(1-ax)(1-ax^{2}) \cdots (1-ax^{n-1})\).

Let \( G_{k,\ell}(t) \) be the generating function of the partitions that fit in the \((k, \ell)\) hook. Clearly we have that \( G_{k,\ell}(t) \) are symmetric with respect to the \( k \) and \( \ell \) parameters and it is quite easy to see combinatorially that this function must satisfy the recurrence

\[
G_{k,\ell}(t) = G_{k,\ell-1}(t) + t^{\ell(k+1)}G_{k}(t)G_{\ell}(t)
\]

since the right hand side represents the generating function of the partitions that fit inside of the \((k, \ell)\)-hook which do not contain the cell \((k+1, \ell)\), plus the generating function for the partitions in the hook which do contain the cell \((k+1, \ell)\).

For our computations we use mainly the following \( \ell \)-binomial identities.

\[
(1) \quad (1-t^{i})\begin{bmatrix}k \\ i \end{bmatrix} = (1-t^{k-i+1})\begin{bmatrix}k \\ i-1 \end{bmatrix}
\]

\[
(2) \quad \begin{bmatrix}a \\ b \end{bmatrix} = \begin{bmatrix}a-1 \\ b-1 \end{bmatrix} + t^{b}\begin{bmatrix}a-1 \\ b \end{bmatrix}
\]

Lemma 2. Let \( A^{k,\ell}(t) \) be defined by the recurrence

\[
A^{k,\ell}(t) = (1-t^{k+\ell})A^{k,\ell-1}(t) + t^{\ell(k+1)}\begin{bmatrix}k + \ell \\ \ell \end{bmatrix}
\]

with \( A^{k,0}(t) = 1 \). Then

\[
A^{k,\ell}(t) = 1 + \sum_{i=1}^{\ell} t^{i(k+\ell+1)}\begin{bmatrix}k \\ i \end{bmatrix} \sum_{j=0}^{\ell-i} t^{j(k-i+1)}\begin{bmatrix}i + j - 1 \\ j \end{bmatrix}
\]

Proof. We proceed by showing that \( A^{k,\ell}(t) \) given in equation \((3)\) satisfies the relation \( A^{k,\ell} - A^{k,\ell-1} + t^{k+\ell}A^{k,\ell-1} = t^{\ell(k+1)}\begin{bmatrix}k + \ell \\ \ell \end{bmatrix} \). The full details of this calculation are not especially enlightening, hence we leave them to the
reader. We give an outline of this proof by stating that as an intermediate step for the left hand side of this equation, one has

\[
A^{k,\ell}(t) - A^{k,\ell-1}(t) + t^{k+\ell} A^{k,\ell-1}(t) = \\
t^{\ell(k+1)} + \sum_{i=1}^{\ell-2} t^{(i+1)(k+\ell)} \left[ \begin{array}{c} k \\ i \end{array} \right] \sum_{j=0}^{\ell-i-2} t^{(j+1)(k-i)} \left[ \begin{array}{c} i + j \\ j \end{array} \right] \\
+ \sum_{i=1}^{\ell-2} t^{(i+1)(k+\ell)} \left[ \begin{array}{c} k \\ i \end{array} \right] \sum_{j=0}^{\ell-2-i} \mu_{j(k-i)} \left( t^j \left[ \begin{array}{c} i + j - 1 \\ j \end{array} \right] - \left[ \begin{array}{c} i + j \\ j \end{array} \right] \right) \\
+ \sum_{i=1}^{\ell-2} t^{(i+1)(k+\ell)} \left[ \begin{array}{c} k \\ i \end{array} \right] t^{(\ell-i-1)(k-i+1)} \left[ \begin{array}{c} \ell - 2 \\ \ell - 1 - i \end{array} \right] \\
+ \sum_{i=1}^{\ell} t^{\ell(k+1)+i^2} \left[ \begin{array}{c} k \\ i \end{array} \right] \left[ \begin{array}{c} \ell - 1 \\ \ell - i \end{array} \right] + t^{\ell(k+\ell)} \left[ \begin{array}{c} k \\ \ell - 1 \end{array} \right] \left[ \begin{array}{c} \ell - 2 \\ 0 \end{array} \right] \\
\]

This may be reduced by repeated applications of (3) until one arrives at

(4) \[
= t^{\ell(k+1)} \sum_{i=0}^{\ell} t^{i^2} \left[ \begin{array}{c} k \\ i \end{array} \right] \left[ \begin{array}{c} \ell \\ i \end{array} \right] = t^{\ell(k+1)} \left[ \begin{array}{c} k + \ell \\ \ell \end{array} \right] \\
\]

\[\square\]

**Theorem 3.** The Hilbert series of \( \Lambda^{(k,\ell)} \) is given by

\[
G_{k,\ell}(t) = A^{k,\ell}(t) \prod_{i=1}^{k+\ell} \frac{1}{1-t^i} \\
\]

**Proof.** Take the recurrence for \( G_{k,\ell}(t) \) and divide by \( G_{k+\ell}(t) \). By Lemma 3, \( A^{k,\ell}(t) \) satisfies the desired recurrence which gives us our result. \[\square\]

It would be interesting to have a purely combinatorial proof of this identity by showing that any partition that fits inside of a \((k, \ell)\)-hook is isomorphic to a partition with less than or equal to \(k + \ell\) rows and some other object counted by \( A^{k,\ell}(t) \).

**References**

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