On the sigma value and sigma range of the join of a finite number of even cycles of the same order

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Abstract. Let \( c : V(G) \to \mathbb{N} \) be a vertex coloring of a simple, connected graph \( G \). For a vertex \( v \) of \( G \), the color sum of \( v \), denoted by \( \sigma(v) \), is the sum of the colors of the neighbors of \( v \). If \( \sigma(u) \neq \sigma(v) \) for any two adjacent vertices \( u \) and \( v \) of \( G \), then \( c \) is called a sigma coloring of \( G \). The sigma chromatic number of \( G \), denoted by \( \sigma(G) \), is the minimum number of colors required in a sigma coloring of \( G \). Let \( \max(c) \) be the largest color assigned to a vertex of \( G \) by a coloring \( c \). The sigma value of \( G \), denoted by \( \nu(G) \), is the minimum value of \( \max(c) \) over all sigma \( k \)-colorings \( c \) of \( G \) for which \( \sigma(G) = k \). On the other hand, the sigma range of \( G \), denoted by \( \rho(G) \), is the minimum value of \( \max(c) \) over all sigma colorings \( c \) of \( G \). In this paper, we determine the sigma value and the sigma range of the join of a finite number of even cycles of the same order. In particular, if \( n \geq 4 \) and \( n \) is even, then we will show that \( \rho(kC_n) = \nu(kC_n) = 2 \) if and only if (i) \( k \leq \left\lfloor \frac{n}{4} \right\rfloor + 1 \), whenever \( n \equiv 0 \) (mod 4), and (ii) \( k \leq \left\lfloor \frac{n-2}{4} \right\rfloor + 1 \), whenever \( n \equiv 2 \) (mod 4).

1. Introduction
Let \( G = (V(G), E(G)) \) be a simple, connected graph with vertex set \( V(G) \) and edge set \( E(G) \). A (vertex) coloring of \( G \) is a function \( c : V(G) \to \mathbb{N} \), where \( \mathbb{N} \) is the set of natural numbers and is often referred to as the set of colors. A coloring \( c \) is said to be a proper coloring if \( c(x) \neq c(y) \) whenever \( x \) and \( y \) are adjacent vertices in \( G \). The chromatic number of \( G \), denoted by \( \chi(G) \), is the minimum number of colors required in a proper coloring of \( G \).

A new type of coloring, called sigma coloring was introduced by Chartrand, Okamoto, and Zhang in [1]. Suppose \( c \) is a coloring of a graph \( G \) where adjacent vertices may possibly be assigned the same color. The color sum of a vertex \( v \) is given by \( \sigma_G(v) = \sum_{x \in N(x)} c(x) \), where \( N(x) \) is the neighborhood of \( x \). For simplicity, the color sum of \( v \) will also be denoted by \( \sigma(v) \) when the graph \( G \) is clear. We say that \( c \) is a sigma coloring of \( G \) if and only if \( \sigma(u) \neq \sigma(v) \) whenever \( u \) and \( v \) are adjacent vertices in \( G \). The sigma chromatic number of \( G \) is the minimum number of colors required in a sigma coloring of \( G \), and is denoted by \( \sigma(G) \). Thus, from the definition, a sigma coloring induces a proper coloring of a graph where each vertex \( v \) is assigned the color sum \( \sigma(v) \).

Figure 1(a) below shows a sigma coloring of \( G \) using three colors and the color sums are indicated above or below each vertex. However, Figure 1(b) shows that it is possible to have a sigma coloring of \( G \) using only 2 colors. Since there are adjacent vertices with the same degree, it is not possible to use only one color. Thus, \( \sigma(G) = 2 \).
For a graph $G$, if $\chi(G) = k$, then there is always a proper coloring of $G$ using elements of the set $\{1, 2, \ldots, k\}$. Although another set of colors may be used, we know that $k$ is the smallest among all the largest colors in proper colorings of $G$. On the other hand, such need not be the case in a sigma coloring of $G$. Suppose $\sigma(G) = k$ and let $c$ be a sigma coloring of $G$ using $k$ colors. Let $\max(c) = \max\{c(v)\mid v \in V(G)\}$ and $\min(c) = \min\{c(v)\mid v \in V(G)\}$. In [1], the following graph parameters associated to sigma colorings were defined. The sigma value of $G$ is the number $\nu(G) = \min\{\max(c)\mid c \text{ is a sigma } k - \text{coloring of } G\}$. On the other hand, the sigma range of $G$ is the number $\rho(G) = \min\{\max(c)\mid c \text{ is a sigma coloring of } G\}$. It was shown in [1] and [8] that $\sigma(G) \leq \rho(G) \leq \nu(G)$. Note that it is possible for a graph $G$ to have $\sigma(G) \neq \rho(G)$ and $\rho(G) \neq \nu(G)$ as shown in [1] and [8].

This paper discusses the sigma value and sigma range of joins of even cycles of the same order. If $G_1$ and $G_2$ are disjoint graphs, the join of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is a graph whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv\mid u \in V(G_1), v \in V(G_2)\}$. Studies on the sigma chromatic number of some families of graphs can be found in [2-6]. In [7], a coloring similar to a sigma coloring was also introduced. However, little is known about the two other graph parameters, $\nu(G)$ and $\rho(G)$. While it can be easily shown that for a cycle $C_n$, with $n \geq 3$, $\rho(C_n) = \nu(C_n) = 2$ if $n$ is even and $\rho(C_n) = \nu(C_n) = 3$ if $n$ is odd, the values of $\rho(C_n)$ and $\nu(C_n)$, where $kC_n$ is the join of $k$ cycles $C_n$ are yet to be determined. In this paper, we address this problem when $C_n$ is an even cycle. The methodology involves using the least and largest possible sums of colors in a sigma 2-coloring of $C_n$ using only the colors 1 and 2.

2. Known Results

The following observations and theorems will be used in this paper.

**Observation 2.1** ([1]). Let $G$ be a non trivial connected graph. Then $\sigma(G) = 1$ if and only if any two adjacent vertices of $G$ have different degrees.

**Theorem 2.1** ([1]). If $C_n$ is a cycle of order $n$, where $n \geq 3$, then

$$\sigma(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

**Theorem 2.2** ([2]). Suppose $c$ is a sigma $k-$coloring of $G + H$, where $G$ and $H$ are disjoint graphs. Then, the restricted colorings $c_1 = c|_{V(G)}$ and $c_2 = c|_{V(H)}$ are sigma colorings of $G$ and $H$, respectively, that use at most $k$ colors. Thus, $\sigma(G) \leq \sigma(G + H)$ and $\sigma(H) \leq \sigma(G + H)$.

**Theorem 2.3** ([1], [8]). For a non trivial connected graph $G$, $\sigma(G) \leq \rho(G) \leq \nu(G)$. 

![Image of graph colorings](image-url)
3. Minimal and Maximal Sigma Colorings of $C_n$

Let $n \in \mathbb{N}$ and $n$ be even. Also, let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_iv_{i+1} | 1 \leq i \leq n-1\} \cup \{v_1v_n\}$. From Theorem 2.1, we have $\sigma(C_n) = 2$. We give a sigma 2-coloring $c_1$ of $C_n$ using the colors 1 and 2. Let $c_1(v_i) = 2$ if $4 \mid i$ or $i = n$, and $c_1(v_i) = 1$, otherwise. The coloring $c_1$ of $C_n$ is illustrated in the diagram below.

![Diagram](image)

(a) $n \equiv 0 \pmod{4}$

(b) $n \equiv 2 \pmod{4}$

Figure 2. The coloring $c_1$ of $C_n$

Note that when $n \equiv 0 \pmod{4}$, the color sums of the vertices in $C_n$ are as follows:

$$\sigma_{c_1}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd} \end{cases}$$

When $n \equiv 2 \pmod{4}$, the color sums of the vertices in $C_n$ are as follows:

$$\sigma_{c_1}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd and } i \neq n-1 \\ 4, & \text{if } i = n-1. \end{cases}$$

Since adjacent vertices in $C_n$ have unequal color sums, $c_1$ is a sigma coloring of $C_n$. Since $c_1$ is a coloring that uses the colors 1 and 2, we then have the following result.

**Proposition 3.1.** Let $n \geq 4$ be even. Then, $\rho(C_n) = \nu(C_n) = 2$.

From the definition, the coloring $c_1$ maximizes the number of times that the smallest color, 1, is assigned to the vertices of $C_n$ and minimizes the number of times that the largest color, 2, is assigned. If $S(n) = \sum_{x \in V(C_n)} c(x)$, where $c$ is a sigma coloring of $C_n$, then $c_1$ is a sigma coloring giving the smallest possible sum $S(n)$. We say that a sigma coloring of $C_n$ is a *minimal sigma coloring* of $C_n$ if the corresponding value of $S(n)$ is minimal. On the other hand, a *maximal sigma coloring* of $C_n$ is one that yields the largest value of $S(n)$ over all sigma colorings using the colors 1 and 2. A maximal sigma coloring of $C_n$ can be constructed by interchanging the colors 1 and 2 in $c_1$. A maximal coloring $c_2$ may also be given as follows: $c_2(v_i) = 1$ if $i \equiv 3 \pmod{4}$ or $i = n-1$, and $c_2(v_i) = 2$, otherwise.
When \( n \equiv 0 \pmod{4} \), the color sums of the vertices in \( C_n \) are as follows:

\[
\sigma_{c_2}(v_i) = \begin{cases} 
3, & \text{if } i \text{ is even} \\
4, & \text{if } i \text{ is odd}.
\end{cases}
\]

When \( n \equiv 2 \pmod{4} \), the color sums of the vertices in \( C_n \) are as follows:

\[
\sigma_{c_2}(v_i) = \begin{cases} 
2, & \text{if } i = n - 2 \\
3, & \text{if } i \text{ is even and } i \neq n - 2 \\
4, & \text{if } i \text{ is odd}.
\end{cases}
\]

In either case, adjacent vertices in \( C_n \) have unequal color sums. Thus, \( c_2 \) is a sigma coloring of \( C_n \). Note that \( c_2(C_n) = \{1, 2\} \) and \( c_2 \) maximizes the number of times that the largest color, 2, is assigned to the vertices of \( C_n \). Thus, \( c_2 \) will yield the highest possible value of \( S(n) \) using the colors 1 and 2.

Now, let \( T(n) \) and \( M(n) \) denote the sum of colors of all the vertices of \( C_n \) in a minimal and a maximal sigma 2-coloring of \( C_n \), respectively, using the colors 1 and 2. Then the values of \( T(n) \) and \( M(n) \) are given as follows:

\[
T(n) = \begin{cases} 
5 \left( \frac{n}{4} \right), & \text{if } n \equiv 0 \pmod{4} \\
5 \left\lfloor \frac{n}{4} \right\rfloor + 3, & \text{if } n \equiv 2 \pmod{4},
\end{cases}
\]

and

\[
M(n) = \begin{cases} 
7 \left( \frac{n}{4} \right), & \text{if } n \equiv 0 \pmod{4} \\
7 \left\lfloor \frac{n}{4} \right\rfloor + 3, & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

4. Main Results

We will use the definitions of \( S(n) \), \( M(n) \), and \( T(n) \) for a cycle \( C_n \), as given in Section 3. The first result shows that for every integer \( k \) between the sum \( T(n) \) obtained from a minimal coloring and the sum \( M(n) \) obtained from a maximal coloring, there exists a sigma 2-coloring of \( C_n \) using the colors 1 and 2 such that the corresponding sum of colors is equal to \( k \).

**Theorem 4.1.** Let \( n \geq 4 \) be even. For every integer \( k \) such that \( T(n) \leq k \leq M(n) \), there exists a sigma 2-coloring \( c \) of the vertices of \( C_n \) using the colors 1 and 2 such that \( S(n) = k \).
Proof. Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \), \( E(C_n) = \{v_iv_{i+1}|1 \leq i \leq n-1\} \cup \{v_1v_n\} \), and let \( s = S(n) - T(n) \). If \( s = 0 \), then we use the coloring \( c_1 \) presented in Section 3 to color the vertices of \( C_n \). Suppose \( s \geq 1 \). We have the following cases:

Case 1: Suppose \( s \) is an even number and \( s \neq M(n) - T(n) \). Define a coloring \( c_3 \) of \( C_n \) by

\[
c_3(v_i) = \begin{cases} 
2, & \text{if } i \equiv 1, 2 \pmod{4}, \text{ where } 2 \leq i \leq 2s + 2 \text{ and } i \neq 2s + 1 \\
c_1(v_i), & \text{otherwise},
\end{cases}
\]

where \( c_1 \) is the minimal sigma coloring of \( C_n \) given in Section 3.

Subcase 1.1: If \( n \equiv 0 \pmod{4} \), then the color sums of the vertices of \( C_n \) are given by

\[
\sigma_{c_3}(v_i) = \begin{cases} 
2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s \leq i \leq n \\
3, & \text{if } i \text{ is even for } 4 \leq i \leq 2s - 2 \text{ or } i \text{ is odd and } 2s + 5 \leq i \leq n \\
4, & \text{if } i \text{ is odd for } 1 \leq i \leq 2s + 3.
\end{cases}
\]

Subcase 1.2: If \( n \equiv 2 \pmod{4} \), then the color sums of the vertices of \( C_n \) are given as follows:

\[
\sigma_{c_3}(v_i) = \begin{cases} 
2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s \leq i \leq n \\
3, & \text{if } i \text{ is even and } 4 \leq i \leq 2s - 2 \text{ or } i \text{ is odd and } 2s + 5 \leq i \leq n - 3 \\
4, & \text{if } i \text{ is odd and } 1 \leq i \leq 2s + 3, \text{ or } i = n - 1.
\end{cases}
\]

Case 2: Suppose \( s \) is an odd number and \( s \neq M(n) - T(n) \). Define a coloring \( c_4 \) of the vertices of \( C_n \) by

\[
c_4(v_i) = \begin{cases} 
2, & \text{if } i \equiv 1, 2 \pmod{4}, \text{ where } 2 \leq i \leq 2s \\
c_1(v_i), & \text{otherwise},
\end{cases}
\]

where \( c_1 \) is the minimal sigma coloring of \( C_n \) given in Section 3.

Subcase 2.1: If \( n \equiv 0 \pmod{4} \), then the color sums of the vertices of \( C_n \) is given by

\[
\sigma_{c_4}(v_i) = \begin{cases} 
2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s + 4 \leq i \leq n \\
3, & \text{if } i \text{ is even and } 4 \leq i \leq 2s + 2, \text{ or } i \text{ is odd and } 2s + 5 \leq i \leq n \\
4, & \text{if } i \text{ is odd and } 1 \leq i \leq 2s + 3.
\end{cases}
\]

Subcase 2.2: If \( n \equiv 2 \pmod{4} \), then the color sums of the vertices of \( C_n \) are given as follows:

\[
\sigma_{c_4}(v_i) = \begin{cases} 
2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s + 2 \leq i \leq n \\
3, & \text{if } i \text{ is even and } 4 \leq i \leq 2s, \text{ or } i \text{ is odd and } 2s + 3 \leq i \leq n - 3 \\
4, & \text{if } i = n - 1, \text{ or } i \text{ is odd and } 1 \leq i \leq 2s + 1.
\end{cases}
\]

Case 3: If \( s = M(n) - T(n) \), then \( k = M(n) \). We color the vertices of \( C_n \) using the maximal sigma coloring \( c_2 \) given in Section 2.

In each of the cases above, we have shown that no two adjacent vertices of \( C_n \) have equal color sums. Thus, the colorings are sigma 2-colorings of \( C_n \) using the colors 1 and 2. In addition, the coloring strategy ensures that the number of vertices of \( C_n \) whose colors change from color 1 in \( c_1 \) to color 2 in \( c_3 \) (or \( c_4 \)) is exactly \( s \) and \( S(n) = T(n) + s = k \).
Lemma 4.2. Let $n$ be an even integer, $n \geq 4$, and let $c$ be any sigma 2-coloring of $C_n$ using the colors 1 and 2. If $S(n) \neq T(n)$, then there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 4$. If $S(n) \neq M(n)$, then there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 2$.

Proof. Suppose $c$ is a sigma 2-coloring of $C_n$ using the colors 1 and 2 such that $S(n) \neq T(n)$. Then, at least one block, say $B$, must be a sigma 2-coloring of $C_n$. Let $S(n)$ and $M(n)$ be the restriction of $c$ to each copy of $C_n$. Since $S(n)$ is a sigma 2-coloring of $C_n$ using the colors 1 and 2, there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 2$.

If the colors used in a 2-coloring of $C_n$ alternate among the vertices, then we say that the coloring is alternating.

Lemma 4.3. Let $c$ be a sigma 2-coloring of $C_n$ using the colors 1 and 2, and let $CS(C_n) = \{\sigma(u) : u \in V(C_n)\}$.

(i) If $c$ is alternating, then $CS(C_n) = \{2, 4\}$.

(ii) Suppose $c$ is not alternating and $n \equiv 0 \pmod{4}$.

- If $S(n) = T(n)$, then $CS(C_n) = \{2, 3\}$.
- If $S(n) = M(n)$, then $CS(C_n) = \{3, 4\}$.
- If $T(n) < S(n) < M(n)$, then $CS(C_n) = \{2, 3, 4\}$.

(iii) Suppose $c$ is not alternating and $n \equiv 2 \pmod{4}$. Then, $CS(C_n) = \{2, 3, 4\}$.

Proof. Suppose $c$ is a sigma 2-coloring of $C_n$ using the colors 1 and 2. Clearly, if $c$ is alternating, then $CS(C_n) = \{2, 4\}$ regardless of the congruence class of $n$.

Suppose $c$ is not alternating. If $n \equiv 0 \pmod{4}$ and $S(n) = T(n)$, then by definition of minimal coloring and by recalling the values of $\sigma(c_1(v))$ given in Section 3, we have $CS(C_n) = \{2, 3\}$. Likewise, if $S(n) = M(n)$, then by definition of maximal coloring and by recalling the values of $\sigma(c_2(v))$ given in Section 3, we have $CS(C_n) = \{3, 4\}$. On the other hand, if $T(n) < S(n) < M(n)$, then by Lemma 4.2, the color sums 2 and 4 are in $CS(C_n)$. Furthermore, since $c$ is not alternating, $C_n$ contains a block of four consecutive vertices having three vertices with color 1 or with color 2. Hence, one of these vertices must have a color sum equal to 3. Thus, $CS(C_n) = \{2, 3, 4\}$ and this is true regardless of the congruence class of $n$. In the case that $n \equiv 2 \pmod{4}$ and $S(n) = T(n)$ or $S(n) = M(n)$, we have $CS(C_n) = \{2, 3, 4\}$ as shown in Section 3.

The next result considers sigma colorings of the join of two even cycles $C_n$.

Lemma 4.4. Let $G = 2C_n = C_n + C_n$, where $n$ is even and $n \geq 8$. Suppose $c$ is a coloring of $G$ using the colors 1 and 2 such that the restriction of $c$ to each copy of $C_n$ in $G$ is a sigma 2-coloring of $C_n$. Suppose $S_i(n)$ denotes the sum of colors of the vertices restricted to the $i$th copy of $C_n$, where $i \in \{1, 2\}$, and assume without loss of generality that $S_2(n) \geq S_1(n)$. Then, $c$ is a sigma 2-coloring of $G$ if and only if $S_2(n) \geq S_1(n) + 3$.

Proof. For notation purposes, denote the $i$th copy of $C_n$ in $G = C_n + C_n$ by $C_{n_i}$ and let $c_{|C_{n_i}}$ be the restriction of $c$ to $C_{n_i}$. From the assumption, $S_i(n) = \sum_{v \in V(C_{n_i})} c(v)$ and $S_1(n) \leq S_2(n)$. In the following, we list all possible conditions between $S_2(n)$ and $S_1(n)$ and show that only the condition $S_2(n) \geq S_1(n) + 3$ will give a sigma 2-coloring of $G$ using the colors 1 and 2.

Case 1: Suppose $S_2(n) = S_1(n)$. 


Subcase 1.1: If $S_1(n) \neq M(n_1)$, then since $n = n_1 = n_2$, we have $S_2(n) \neq M(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 2 + S_1(n) = \sigma(v)$.

Subcase 1.2: If $S_1(n) = M(n_1)$, then $S_2(n) = M(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 4 + S_2(n) = 4 + S_1(n) = \sigma(v)$.

Case 2: Suppose $S_2(n) = S_1(n) + 1$.

Subcase 2.1: If $c|_{C_{n_2}}$ is alternating, then $c|_{C_{n_1}}$ is not. By Lemma 4.3, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 3 + S_2(n) = 3 + S_1(n) = \sigma(v)$.

Subcase 2.2: If $c|_{C_{n_2}}$ is not alternating, then $c|_{C_{n_1}}$ is alternating. By Lemma 4.3, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 2 + S_1(n) = \sigma(v)$.

Case 3: Suppose $S_2(n) = S_1(n) + 2$. Then, $S_1(n) \neq M(n_1)$ and $S_2(n) \neq T(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 4 + S_1(n) = \sigma(v)$.

Case 4: Suppose $S_2(n) \geq S_1(n) + 3$. Then, for any adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, we have $\sigma(u) \geq 2 + S_2(n) \geq 4 + S_1(n) > 4 + S_1(n) \geq \sigma(v)$.

We note that in each of Cases 1 to 3, there are two vertices $u$ and $v$ which are adjacent in the join $G = 2C_n$ such that $\sigma(u) = \sigma(v)$. This means that $c$ is not a sigma coloring in each of these cases. Finally, we should note that only Case 4 yields a sigma 2-coloring of $C_n$. This proves the lemma.

The result below deals with the main problem of this study which is to determine the sigma value and sigma range of the join of a finite number of even cycles of the same order.

**Theorem 4.5.** Let $G = kC_n$, where $n \geq 8$ is even and $k \geq 1$. Then,

$$\nu(G) = 2$$

if and only if

$$k \leq \begin{cases} \left\lceil \frac{n}{6} \right\rceil + 1, & \text{if } n \equiv 0 \pmod{4} \\ \left\lceil \frac{n-2}{6} \right\rceil + 1, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

**Proof.** Since $n$ is even, we have either $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$. Considering the values of $T(n)$ and $M(n)$ as given in equations (1) and (2) in Section 3, it follows that

$$\left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1 = \begin{cases} \left\lceil \frac{n}{6} \right\rceil + 1, & \text{if } n \equiv 0 \pmod{4} \\ \left\lceil \frac{n-2}{6} \right\rceil + 1, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Hence, we will prove that $\nu(G) = 2$ if and only if $k \leq \left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1$. As in the proof of Lemma 4.4, we denote the $i$th copy of $C_n$ in $G$ by $C_{n_i}$, $c|_{C_{n_i}}$ the restriction of $c$ to $C_{n_i}$, and $S_i(n) = \sum_{v \in V(C_{n_i})} c(v)$.

First, suppose $\nu(G) = 2$. Then, by definition, there exists a sigma 2-coloring $c$ of $kC_n$ using colors 1 and 2. Now, $c|_{C_{n_i}}$ as well as $c|_{C_{n_i} + C_{n_j}}$ are also sigma colorings for $1 \leq i, j \leq k$, by Theorem 2.2. By permuting the position of the cycles, if necessary, we can assume without loss of generality that the values of $S_i(n)$, for $1 \leq i \leq k$, are nondecreasing. By Lemma 4.4, we must have $S_{i+1}(n) \geq S_i(n) + 3$ for each $1 \leq i \leq k - 1$. As a consequence, $S_k(n) - S_1(n) \geq 3(k-1)$. Since $S_k(n) \leq M(n)$ and $S_1(n) \geq T(n)$, it follows that $S_k(n) - S_1(n) \leq M(n) - T(n)$. Combining
the inequalities, we obtain $3(k - 1) \leq M(n) - T(n)$, and so, $k \leq \frac{M(n) - T(n)}{3} + 1$. Since $k$ is an integer, we have $k \leq \left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1$.

Conversely, suppose $k \leq \left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1$. Consider the sequence $a_1 = T(n), a_2 = T(n) + 3, \ldots, a_k = T(n) + 3(k - 1)$. Note that the last equation yields $k = \left\lfloor \frac{a_k - T(n)}{3} \right\rfloor + 1$, and since $k \leq \left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1$, then we must have $a_k \leq M(n)$. Clearly, $a_i$ is an increasing sequence and $T(n) \leq a_i \leq M(n)$ for $1 \leq i \leq k$. By Theorem 4.1, there exists a sigma 2-coloring of $C_n$ using colors 1 and 2 such that $S_i(n) = a_i$. Since $a_{i+1} = a_i + 3$ for each $1 \leq i \leq k - 1$, then by applying Lemma 4.4 repeatedly, it follows that $c$ is a sigma 2-coloring of $kC_n$ using the colors 1 and 2. Consequently, $\nu(G) = 2$.

By Observation 2.1, the sigma chromatic number of a connected graph is 1 if and only if every two adjacent vertices of $G$ have different degrees. Since this is not the case for the join $kC_n$ of $k$ cycles with $n \geq 4$, then we must have $\sigma(kC_n) \geq 2$. Since by Theorem 2.3, $\sigma(kC_n) \leq \rho(kC_n) \leq \nu(kC_n)$, we have the following corollary.

**Corollary 4.6.** Let $G = kC_n$, where $n \geq 8$ is even and $k \geq 1$. Then,

$$\rho(G) = 2$$

if and only if

$$k \leq \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{4} \\ \left\lfloor \frac{n-2}{6} \right\rfloor + 1, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

**Example 1.** Suppose $n = 18$. By Theorem 4.5, we have $\nu(kC_{18}) = 2$ if and only if $k \leq 3$. Using Theorem 4.1 and the strategy of coloring $kC_n$ in the proof of Theorem 4.5, we give a sigma 2-coloring of $3C_{18}$ using the colors 1 and 2. For simplicity, we omit the edges joining vertices between different copies of $C_{18}$ in Figure 4.

![Figure 4. Sigma 2-coloring of 3C_{18}](image-url)
Observe that when \( k = 4 \), then in order to still have a sigma \( 2 \)-coloring of \( 4C_{18} \), \( S_4(n) \geq S_3(n) + 3 = 32 \) by Lemma 4.4. Since \( M(n) = 31 \) and \( M(n) \) is the sum of colors in a maximal coloring of \( C_{18} \), it follows that no sigma coloring of \( 4C_{18} \) using only the colors 1 and 2 will exist. As a consequence, \( \nu(4C_{18}) \neq 2 \).

5. Conclusion
In this paper, we considered the sigma value and sigma range in relation to the join of even cycles of the same order. While \( \rho(C_{n}) = \nu(C_{n}) = 2 \) when \( n \) is even, we determined necessary and sufficient conditions so that \( \rho(kC_{n}) \) and \( \nu(kC_{n}) \) are still equal to 2.

A problem that can be investigated further is that of determining \( \rho(kC_{n}) \) and \( \nu(kC_{n}) \) when \( n \) is odd. One might be interested also to consider other families of graphs whose sigma value and sigma range are yet unknown.

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