ON ALMOST SUBNORMAL SUBGROUPS IN DIVISION RINGS

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Abstract. Let \( D \) be a division ring with infinite center \( F \), and \( G \) an almost subnormal subgroup of \( D^* \). In this paper, we show that if \( G \) is locally solvable, then \( G \subseteq F \). Also, assume that \( M \) is a maximal subgroup of \( G \). It is shown that if \( M \) is non-abelian locally solvable, then \( [D : F] = p^2 \) for some prime number \( p \). Moreover, if \( M \) is locally nilpotent then \( M \) is abelian.

1. Introduction

There has been recently a great deal of interest in algebraic structure of the multiplicative group \( D^* \) of a division ring \( D \). For a self-contained account of the development on this area of study, we refer to \cite{14}. It is not much to say that the starting point is the Wedderburn Little Theorem stating that any finite division ring is a field. This famous theorem is the subject for many generalizations. Many authors paid attention on the question that how far \( D^* \), and more generally its subnormal subgroups, from being abelian. On this line, the well-known result of Hua states that if \( D^* \) is solvable then \( D \) is a field. Some special types of subnormal subgroups such as nilpotent, solvable, and locally nilpotent have been examined. For instance, Stuth \cite[Theorem 6 (iii)]{26} proved that every solvable subnormal subgroup of \( D^* \) is central, i.e, it is contained in the center \( F \) of \( D \). In \cite{17}, Huzurbazar showed that this result remains true if the word ‘solvable’ is replaced by ‘locally nilpotent’. In \cite[Theorem 2.3]{19}, the authors extended the result of Hua by proving that if \( D^* \) is locally solvable, then \( D \) is a field. It is clear that if \( G \) is locally nilpotent or solvable, then it is locally solvable. However, the converse does not hold: there exists a division ring which contains a locally solvable subgroup that is neither solvable nor locally nilpotent (see \cite[1.4.13]{25}). In view of \cite[Theorem 6 (iii)]{26} and \cite[Theorem 2.3]{19}, it is natural to ask whether every locally solvable subnormal subgroup of \( D^* \) is central (see \cite[Conjecture 1]{11}). It was shown in \cite{12} that the question has the positive answer in the case when \( D \) is algebraic over \( F \).

Recently, the positive answer to this question in general case have been obtained in \cite{18}. One of our purposes in this paper is to extend this result for almost subnormal subgroups. We shall prove that in a division ring \( D \) with center \( F \) every locally solvable almost subnormal subgroup is central, provided \( F \) is infinite.

Let us recall the notion of almost subnormal subgroups. Following Harley \cite{13}, a subgroup \( H \) of a group \( G \) is called almost subnormal in \( G \) if there is a finite sequence of subgroups

\[
H = H_0 \leq H_1 \leq \cdots \leq H_n = G,
\]

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Lemma 2.1. Let $D = E(G)$ be a division ring generated by its locally nilpotent subgroup $G$ and its division subring $E$ such that $E \leq C_D(G)$. Set $H = N_{D^*}(G)$. Denote the maximal 2-subgroup of $G$ by $Q$. Then one of the following holds:

2. Locally solvable almost subnormal subgroups are central

We need the following lemma from [33].

Lemma 2.1 (33 1.4). Let $D = E(G)$ be a division ring generated by its locally nilpotent subgroup $G$ and its division subring $E$ such that $E \leq C_D(G)$. Set $H = N_{D^*}(G)$. Denote the maximal 2-subgroup of $G$ by $Q$. Then one of the following holds:

in which either $[H_{i+1} : H_i]$ is finite or $H_i$ is normal in $H_{i+1}$ for $0 \leq i \leq n - 1$. Clearly if $H$ is a subnormal subgroup of $G$, then it is almost subnormal. But we do not have the converse. In [3] and [22], there are examples of almost subnormal subgroups in division rings that are not subnormal. Recently, almost subnormal subgroups in skew linear groups have been studied in [3], [7], [9], [22].

Note that in the case of arbitrary center $F$, T. H. Dung [1] has recently proved that any locally solvable almost subnormal subgroup $G$ of $D^*$ is central, provided $G$ is algebraic over $F$.

In another direction, maximal subgroups of $D^*$, and more generally of subnormal subgroups, were considered. The authors whose study in this direction often prefer to examine how big these maximal subgroups are, that is, how such subgroups reflect the algebraic structure of $D$. Some kinds of such subgroups such that abelian, nilpotent, and solvable subgroups were considered. It was shown in [5] that any nilpotent maximal subgroup of $D^*$ is abelian. The result was then generalized by the authors in [23] that any nilpotent maximal subgroup of a subnormal subgroup is also abelian. Unfortunately, the result is no longer true if the word ‘nilpotent’ is replaced by ‘solvable’. In fact, in the multiplicative group $\mathbb{H}^*$ of the division ring of real quaternions $\mathbb{H}$, the set $\mathbb{C}^* \cup \mathbb{C}^*j$ forms a non-abelian solvable maximal subgroup (see [2]). In general, it was showed that if the multiplicative group $D^*$ of a division ring $D$ contains a non-abelian solvable maximal subgroup, then $D$ is cyclic algebra of prime degree over the center $F$ (see [4]). In the beginning of 2019, the works in [6] and [20] show that this result can be extended to the subnormal subgroups of $D^*$. For precisely, assume that $G$ is a subnormal subgroup of $D^*$ and that $M$ is a maximal subgroup of $G$. It was shown that if $M$ is non-abelian solvable, then $[D : F] = p^2$ for some prime $p$. It is natural to ask that can we obtain the analogous results in the case ‘locally nilpotent’ or ‘locally solvable’? Relating to this question, the authors in [8] showed that if $M$ is a non-abelian locally solvable, then $D$ is cyclic of prime degree, provided $M'$ is algebraic over $F$. It was shown in [19] that this results still hold if the $i$-th derived subgroup $M^{(i)}$ of $M$ is algebraic over $F$ for some $i$, instead of $M'$. In this paper, we confirm that the answer is ‘yes’ in general setting. Let $G$ be an almost subnormal subgroup of $D^*$, and assume that $M$ is a maximal subgroup of $G$. We show that if $M$ is a non-abelian locally solvable maximal subgroup, then $D$ is cyclic of prime degree over $F$. Also, if $M$ is locally nilpotent maximal subgroup, then $M$ is abelian.

Through out this paper, for a ring $R$ with the identity $1 \neq 0$, the symbol $R^*$ stands for the group of units of $R$. If $D$ is a division ring with the center $F$ and $S \subseteq D$, then $F[S]$ and $F(S)$ denotes respectively the subring and the division subring of $D$ generated by $F \cup S$. For a group $G$ and a positive integer $i$, the symbol $G^{(i)}$ is the $i$-th derived subgroup of $G$. If $G$ is a group and $H, K$ are subgroups of $G$, then $H_K(G)$ is the normalizer of $H$ in $K$, that is $H_K(G) = K \cap N_G(H)$. If $A$ is a ring or a group, then $Z(A)$ denotes the center of $A$. 
(i) If \( G = Q \cdot C_G(Q) \) where \( Q \) is quaternion of order 8, then \( H/GE^* \cong \text{Sym}(3) \times Y \) for some abelian group \( Y \).

(ii) If \( Q \) is non-abelian with \( 8 < |Q| < \infty \), then \( H/GE^* \) has an abelian subgroup \( Y \) of index in \( H/GE^* \) at most 2.

(iii) In all other cases \( H/GE^* \) is abelian.

In the above lemma if we take \( E = F \), where \( F \) is the center of \( D \), then we have the following result.

**Lemma 2.2.** Let \( D \) be a division ring with center \( F \), and \( G \) a locally nilpotent subgroup of \( D^* \) such that \( F(G) = D \). If \( H = N_D(G) \), then \( H/GE^* \) is solvable. \( \Box \)

**Lemma 2.3** ([3 Theorem 3.10]). Let \( D \) be a division ring with infinite center \( F \), and \( K \) a division subring of \( D \). Assume that \( G \) is a non-central almost subnormal subgroup of \( D^* \). If \( g^{-1}Kg \subseteq K \) for any \( g \in G \), then either \( K \subseteq F \) or \( K = D \).

**Lemma 2.4.** Let \( D \) be a division ring with center \( F \). If \( G \) is a solvable almost subnormal subgroup of \( D^* \), then \( G \subseteq F \).

**Proof.** Trivially \( G \) satisfies a non-trivial group identity, and the result follows from [22, Theorem 2.2]. \( \Box \)

**Lemma 2.5.** Let \( G \) be a group. If \( G/Z(G) \) is locally finite, then so is \( G' \).

**Proof.** Let \( A = \langle a_1, a_2, \ldots, a_n \rangle \) be a finitely generated subgroup of \( G' \). Then we may write

\[
a_i = [x_{i1}, y_{i1}] \cdot [x_{i2}, y_{i2}] \cdots [x_{ik}, y_{ik}],
\]

where \( x_{ij}, y_{ij} \in G \) for \( 1 \leq i \leq n \). If \( N \) is the subgroup of \( G \) generated by all \( x_{ij}, y_{ij} \), then \( A \subseteq N \). By hypothesis, we conclude that \( NZ(G)/Z(G) \) is finite. It follows by Schur’s Theorem that \( N' = (NZ(G))' \) is finite ([21 10.1.3, p.278]); note that \( Z(G) \subseteq Z(NZ(G)) \). This implies that \( A \) is finite. \( \Box \)

**Lemma 2.6.** If \( D \) is a division ring such that \( D^{(k)} \) is locally nilpotent for some \( k \geq 1 \), then \( D \) is commutative.

**Proof.** Since \( D^{(k)} \) is normal in \( D^* \), by [28 Theorem 1], \( D^{(k)}/Z(D^{(k)}) \) is locally finite, so \( D^{(k+1)} \) is locally finite by Lemma 2.5. By [13 Theorem 8], we conclude that \( D^{(k+1)} \subseteq F \), which implies that \( D \) is solvable. By a result of Hua [10], it follows that \( D \) is commutative. \( \Box \)

**Proposition 2.7.** Let \( D \) be a division ring with infinite center \( F \). If \( G \) is a locally nilpotent almost subnormal subgroup of \( D^* \), then \( G \subseteq F \).

**Proof.** Assume by contradiction that \( G \) is not contained in \( F \). Since \( G \) is an almost subnormal subgroup of \( D^* \), there is a finite chain of subgroups

\[
G = G_0 \leq G_1 \leq \cdots \leq G_n = D^*;
\]

for which either \( [G_{i+1} : G_i] \) is finite or \( G_i \leq G_{i+1} \), for \( 0 \leq i \leq n-1 \). If \( [G_1 : G_0] \) is finite, then \( C := \text{Core}_{G_1}(G_0) = \bigcap_{x \in G_1} x^{-1}G_0x \) is a normal subgroup of finite index in \( G_1 \) contained in \( G_0 \). It follows that \( s := [G_0 : C] \) is finite. If \( C \subseteq F \), then \( x^s y^s x^{-s} y^{-s} = 1 \) is a group identity of \( G_0 \). It follows by [22 Theorem 2.2] that \( G = G_0 \subseteq F \), a contradiction. Therefore \( C \) is not contained in \( F \). By replacing \( G_0 \) by \( C \) if necessary, we can assume that \( G_0 \) is normal in \( G_1 \).
Since \( G_0 \not\subseteq F \), it follows by Lemma 2.3 that \( F(G_0) = D \). If we set \( H = N_D(G_0) \), then Lemma 2.2 says that \( H/G_0F^* \) is solvable. Since \( G_0 \) is assumed to be normal in \( G_1 \), we conclude that \( G_1 \subseteq H \). It follows that \( G_1/G_1 \cap G_0F^* \cong G_1F^*/G_0F^* \) is also solvable. Thus, there exists an integer \( k_1 \) for which \( G^{(k_1)}_1 \subseteq G_0F^* \), which is a locally nilpotent group. If \( G^{(k_1)}_1 \subseteq F \), then \( G_1 \) is solvable. According to Lemma 2.4, we conclude that \( G_1 \subseteq F \), a contradiction. Therefore, we may assume that \( G^{(k_1)}_1 \not\subseteq F \).

Now \( G^{(k_1)}_1 \) is a non-central locally nilpotent normal subgroup of \( G_2 \). By replacing \( G_1 \) and \( G_0 \) by \( G_2 \) and \( G^{(k_1)}_1 \) in the preceding paragraph, we conclude that \( G^{(k_2)}_2 \) is locally nilpotent, for some \( k_2 \). By finite steps, we obtain that \( D^{(k_n)} \) is locally nilpotent for some \( k_n \). This fact together with Lemma 2.6 implies that \( D \) is commutative, a contradiction. \( \square \)

**Lemma 2.8.** Let \( D \) be a division ring with center \( F \), and \( G \) a subgroup of the multiplicative group \( D^* \) of \( D \). If \( G \) is locally finite, then \( F(G) = F[G] \).

**Proof.** Assume that \( 0 \neq x \in F[G] \). Then, \( x = f_1g_1 + f_2g_2 + \cdots + f_ng_n \) for some \( f_i \in F \) and \( g_i \in G \). The subgroup \( H = \langle g_1, g_2, \ldots, g_n \rangle \) is finite, so \( F[H] \) is a domain which is a finite dimensional vector space over \( F \). This implies that \( F[H] \) is a division ring, so \( F(H) = F[H] \). Hence, \( x^{-1} \in F[H] \subseteq F[G] \). Consequently, \( F(G) = F[G] \). \( \square \)

**Lemma 2.9.** Let \( D \) be a division ring with center \( F \), and \( G \) a locally finite subgroup of \( D^* \) for which \( F(G) = D \). Then, \( D \) is a locally finite division ring.

**Proof.** By Lemma 2.8 \( D = F[G] \), so for any finite subset \( \{x_1, x_2, \ldots, x_k\} \subseteq D \), we can write

\[
x_i = f_{i_1}g_{i_1} + f_{i_2}g_{i_2} + \cdots + f_{i_s}g_{i_s},
\]

where \( f_{i_j} \in F \) and \( g_{i_j} \in G \). Let \( H = \langle g_{i_j} : 1 \leq i \leq k, 1 \leq j \leq s \rangle \) be the subgroup of \( G \) generated by all \( g_{i_j} \). Since \( G/G \cap F^* \cong GF^*/F^* \) is locally finite, the group \( HF^*/F^* \) is finite. Let \( \{y_1, y_2, \ldots, y_t\} \) be a transversal of \( F^* \) in \( HF^* \) and set

\[
S = Fy_1 + Fy_2 + \cdots + Fy_t.
\]

Then, clearly \( S \) is a division ring that is finite dimensional over \( F \) containing \( \{x_1, x_2, \ldots, x_k\} \). It follows that \( F(x_1, x_2, \ldots, x_k) \) is a finite dimensional vector space over \( F \). \( \square \)

**Lemma 2.10.** Let \( D \) be a division ring with infinite center \( F \). If \( G \) is a locally finite almost subnormal subgroup of \( D^* \), then \( G \subseteq F \).

**Proof.** Assume by contradiction that \( G \not\subseteq F \). It follows by Lemma 2.3 that \( F(G) = D \). In view of Lemma 2.9 we conclude that \( D \) is a locally finite division ring. According to [22] Theorem 4.3, we conclude that \( G \) contains a non-cyclic free subgroup. But this is impossible since \( G \) is locally finite. \( \square \)

**Lemma 2.11.** Every locally solvable periodic group is locally finite.

**Proof.** Let \( G \) be a locally solvable periodic group, and \( H \) a finitely generated subgroup of \( G \). Then, \( H \) is solvable with derived series of length \( n \geq 1 \), say,

\[
1 = H^{(n)} \subseteq H^{(n-1)} \subseteq \cdots \subseteq H' \subseteq H.
\]
We shall prove that $H$ is finite by induction on $n$. For if $n = 1$, then $H$ is a finitely generated periodic abelian group, so it is finite. Suppose $n > 1$. It is clear that $H/H'$ is a finitely generated periodic abelian group, so it is finite. Hence, $H'$ is finitely generated. By induction hypothesis, $H'$ is finite, and as a consequence, $H$ is finite. □

**Remark 1.** For a group $G$, let $\tau(G)$ be the unique maximal periodic normal subgroup of $G$, and $B(G)$ a subgroup of $G$ such that $B(G)/\tau(G)$ is the Hirsch-Plotkin radical of $G/\tau(G)$. It is clear that both $\tau(G)$ and $B(G)$ are characteristic subgroups of $G$. The group $B(G)$ contains the Hirsch-Plotkin radical of $G$. So, if $B(G)$ is locally nilpotent, then it is coincided with the Hirsch-Plotkin radical of $G$. If $G$ is locally solvable, then by Lemma 2.11 $\tau(G)$ is locally finite. Moreover, if $G$ is locally nilpotent, then $\tau(G)$ is the set of all elements of finite order of $G$ (see [24, 12.1.1]).

**Lemma 2.12** ([24, Point 20]). Let $R = F[G]$ be an algebra over the field $F$ that is a domain, where $G$ is a locally solvable subgroup of the group of units of $R$ such that $B(G) = F^* \cap G$. Then, $R$ is an Ore domain. Moreover, if $D$ is the skew field of fractions of $R$, then $N_{D^*}(G) = GF^*$.

**Lemma 2.13.** Let $D$ be a division ring with infinite center $F$. If $G$ is a locally solvable almost subnormal subgroup of $D^*$, then $B(G) \subseteq F$.

**Proof.** It follows by Lemma 2.10 that $\tau(G)$ is contained in $F$. Take any finitely generated subgroup $H$ of $B(G)$. Since $B(G)/\tau(G)$ is locally nilpotent, it follows that $H/\tau(G)$ is nilpotent. Thus, we have $[[H, H], \ldots, H] \subseteq \tau(G) \subseteq F$, from which it follows that $H$ is nilpotent. Therefore, $B(G)$ is a locally nilpotent group. Since $B(G)$ is normal in $G$, it is an almost subnormal subgroup of $D^*$. If follows by Proposition 2.7 that $B(G) \subseteq F$. □

**Lemma 2.14.** Let $D$ be a division ring with infinite center $F$. If $G$ is a locally solvable non-central almost subnormal subgroup of $D^*$, then $F(G) = D$ and $N_{D^*}(G) = GF^*$.

**Proof.** By Lemma 2.3 we conclude that $F(G) = D$. If we set $R = F[G]$, then $R$ is an Ore domain by [30, Corollary 24]. Thus, the skew field of fractions of $R$ is coincided with $D$. It follows by Lemma 2.13 that $B(G) \subseteq F \cap G$, hence $B(G)/\tau(G) \subseteq (F \cap G)/\tau(G)$. Since $(F \cap G)/\tau(G)$ is an abelian normal subgroup of $G/\tau(G)$, the maximality of $B(G)/\tau(G)$ in $G/\tau(G)$ implies that $B(G)/\tau(G) = (F \cap G)/\tau(G)$, from which we have $B(G) = F \cap G$. Finally, by Lemma 2.12 we have $N_{D^*}(G) = GF^*$. □

**Theorem 2.15.** Let $D$ be a division ring with infinite center $F$. If $G$ is a locally solvable almost subnormal subgroup of $D^*$, then $G \subseteq F$.

**Proof.** Assume by contradiction that $G \not\subseteq F$. Since $G$ is an almost subnormal subgroup of $D^*$, there is a finite chain of subgroups

$$G = G_0 \leq G_1 \leq \cdots \leq G_n = D^*,$$

for which either $[G_{i+1} : G_i]$ is finite or $G_i \trianglelefteq G_{i+1}$, for $0 \leq i \leq n - 1$. By the same argument used in the first paragraph of the proof of Proposition 2.7 we may assume that $G_0$ is normal in $G_1$. It follows by Lemma 2.14 that $N_{D^*}(G_0) = G_0 F^*$, which
is a locally solvable group. Since $G_1 \subseteq N_{D^*}(G_0)$, we conclude that $G_1$ is a locally solvable group. After finite steps, we obtain that $D^*$ is locally solvable. According to [10] Theorem 2.1, we conclude that $D$ is commutative, a contradiction. □

3. Locally solvable maximal subgroups

Let $R$ be a ring, $S$ a subring of $R$, and $G$ a subgroup of the group of units of $R$ normalizing $S$ such that $R = S[G]$. Suppose that $N = G \cap S$ is a normal subgroup of $G$ and $R = \bigoplus_{t \in T} tS$, where $T$ is some (and hence any) transversal of $T$ to $G$. Then, we say that $R$ is a crossed product of $S$ by $G/\tau$ (see [31] or [25, p.23]). For the convenience of readers, we gather a number of theorems of B. A. F Wehrfritz, which will be used in the proofs of our results.

Lemma 3.1 ([27, 2.5]). Let $R = F[G]$ be an $F$-algebra, where $F$ is a field and $G$ is a locally nilpotent group of units of $R$, such that for every finite subset $X$ of $R$ there is a finitely generated subgroup $Y$ of $G$ with $F[Y]$ prime and containing $X$. Let $\tau(G)$ be the unique maximal periodic normal subgroup of $G$, and $Z/\tau(G)$ the center of $G/\tau(G)$. Then, $R$ is a crossed product of $F[Z]$ by $G/Z$.

Lemma 3.2 ([25, 7]). Let $R = F[G]$ be an $F$-algebra, where $F$ is a field and $G$ is a locally solvable subgroup of the group of units of $R$ such that for every infinite subgroup $X$ of $G$ the left annihilator of $X - 1$ in $R$ is $\{0\}$. Let $B(G)$ be a subgroup of $G$ such that $B(G)/\tau(G)$ is the Hirsch-Plotkin radical of $G/\tau(G)$. Then $R$ is a crossed product of $F[B(G)]$ by $G/B(G)$.

Lemma 3.3 ([31, 3.2]). Let $R$ be a ring, $J$ a subring of $R$, and $H \leq K$ subgroups of the group of units of $R$ normalizing $J$ such that $R$ is the ring of right quotients of $J[H] \leq R$ and $J[K]$ is a crossed product of $J[B]$ by $K/B$ for some normal subgroup $B$ of $K$. Then $K = HB$.

For a group $G$, we denote by $Z_2(G)$, which is defined by $Z_2(G)/Z(G) = Z(G/Z(G))$, the second center of $G$.

Lemma 3.4 ([31, Theorem 1.1(c)]). Let $G$ be a $(P, L)\mathfrak{A}$-subgroup of $\text{GL}_n(D)$ such that the subalgebra $F[N]$ of $M_n(D)$ is a prime ring for every characteristic subgroup $N$ of $G$. Denote by $\tau(G)$ the unique maximal locally finite normal subgroup of $G$. If $\tau(G) \subsetneq Z_2(G)$, then $F[G]$ is a crossed product of $F[A]$ by $G/A$, for some abelian characteristic subgroup of $G$.

Remark 2. In the above lemma, the notation $(P, L)\mathfrak{A}$ stands for a class of groups, in which $\mathfrak{A}$ denotes the class of abelian groups and $P$ and $L$ the poly and local operators ([31]). It is clear that the class of groups $(P, L)\mathfrak{A}$ contains that of locally solvable groups.
(iii) In all other cases, \( H = AH_1 \).

**Lemma 3.6.** Let \( D \) be a division ring with infinite center \( F \), and \( G \) an almost subnormal subgroup of \( D^* \). Assume that \( M \) is a non-abelian locally solvable maximal subgroup of \( G \). If \( A \trianglelefteq M \), then either \( A \) is abelian or \( F(A) = D \).

**Proof.** Since \( A \trianglelefteq M \), we have \( M \subseteq N_G(F(A)^*) \subseteq G \). The maximality of \( M \) in \( G \) implies that either \( N_G(F(A)^*) = M \) or \( N_G(F(A)^*) = G \). If the first case occurs, then \( A \trianglelefteq F(A)^* \cap G \) is almost subnormal in \( F(A)^* \) contained in \( M \). Since \( M \) is locally solvable, so is \( A \). It follows by Theorem 2.14 that \( A \) is contained in the center of \( F(A) \), so \( A \) is abelian. If \( N_G(F(A)^*) = G \), then \( F(A) \) is a division subring of \( D \) normalized by \( G \). It follows by Lemma 3.8 that either \( A \trianglelefteq F \) (and hence \( A \) is abelian) or \( F(A) = D \).

The proof of the following proposition is a simple modification of the proof of [X, Theorem 3.3], so it should be omitted.

**Proposition 3.7.** Let \( D \) be a division ring with infinite center \( F \), and \( G \) an almost subnormal subgroup of \( D^* \). If \( M \) is a non-abelian metabelian maximal subgroup of \( G \), then \( [D : F] < \infty \).

**Lemma 3.8.** Let \( H \trianglelefteq G \) be groups. If \( H \) is a characteristic subgroup of \( G \), then so is \( C_G(H) \).

**Proof.** For any \( \varphi \in \text{Aut}(G) \) and \( x \in C_G(H) \), our task is to show that \( \varphi(x) \in C_G(H) \). Take an arbitrary element \( h \in H \). Since \( \varphi(H) = H \), there exists \( h' \in H \) such that

\[
\varphi(h') = h.
\]

Now, we have

\[
h\varphi(x) = \varphi(h')\varphi(x) = \varphi(h'x) = \varphi(xh') = \varphi(x)\varphi(h') = \varphi(x)h,
\]

which implies that \( \varphi(x) \in C_G(H) \).

**Theorem 3.9.** Let \( D \) be a division ring with infinite center \( F \), and \( G \) an almost subnormal subgroup of \( D^* \). If \( M \) is a non-abelian solvable maximal subgroup of \( G \), then the following hold:

(i) There exists a maximal subfield \( K \) of \( D \) such that \( K/F \) is a finite Galois extension with \( \text{Gal}(K/F) \cong M/K^* \cap G \cong \mathbb{Z}_p \) and \( [D : F] = p^2 \), for some prime number \( p \).

(ii) The subgroup \( K^* \cap G \) is the FC-center. Also, \( K^* \cap G \) is the Hirsch-Plotkin radical of \( M \). For any \( x \in M \setminus K \), we have \( x^p \in F \) and \( D = F[M] = \bigoplus_{i=1}^{p} Kx^i \).

**Proof.** First, we prove that \([D : F] < \infty \). Since \( M \) is non-abelian, Lemma 3.6 says that \( F(M) = D \). Also, we may suppose that \( M \) is solvable with derived length \( s \geq 2 \). Therefore, there exists such a series

\[
1 = M^{(s)} \subseteq M^{(s-1)} \subseteq M^{(s-2)} \subseteq \cdots \subseteq M' \subseteq M.
\]

If we set \( A = M^{(s-2)} \), then \( A \) is a non-abelian metabelian normal subgroup of \( M \). In view of Lemma 3.6, we conclude that \( F(A) = D \). It follows that \( Z(A) = F^* \cap A \) and \( F = C_D(A) \). Set \( H = N_D(A), B = C_A(A'), K = F(Z(B)), H_1 = H \cap K^*, \) and \( T = \tau(B) \) to be the maximal periodic (and hence locally finite) normal subgroup of \( B \). Since \( A' \) is a characteristic subgroup of \( A \), by Lemma 3.8 \( B \) is a characteristic subgroup of \( A \). By Remark 1 \( T \) is a characteristic subgroup of \( B \), hence \( T \) is also
a characteristic subgroup of $A$. It is clear that $H_1$ is an abelian group, so in view of Lemma 3.5, we have three possible cases:

**Case 1:** $T$ is not abelian.

Again by Lemma 3.5, we have $F(T) = D$. The local finiteness of $T$ together with Lemma 2.9 implies that $D$ is a locally finite division ring. Since $M$ is solvable, it contains no non-cyclic free subgroups. By [9, Theorem 3.1], it follows that $[D : F] < \infty$.

**Case 2:** $T$ is abelian and contains an element $x$ of order 4 not in the center of $B = C_A(A')$.

It is clear that $x$ is not contained in $F$. Because $x$ is of finite order, the field $F(x)$ is algebraic over $F$. It was proved (see the proof of [32, Theorem 1.1, p. 132]) that $\langle x \rangle$ is a 2-primary component of $T$, hence it is a characteristic subgroup of $T$. Consequently, $\langle x \rangle$ is a normal subgroup of $M$. Thus, all elements of the set $x^M := \{m^{-1}xm | m \in M\} \subseteq F(x)$ have the same minimal polynomial over $F$. This implies $|x^M| < \infty$, so $x$ is an $FC$-element, and consequently, $[M : C_M(x)] < \infty$. Setting $C = Core_M(C_M(x))$, then $C \subseteq M$ and $[M : C]$ is finite. By Lemma 3.6, either $C$ is abelian or $F(C) = D$. The latter case implies that $x \in F$, a contradiction. Thus, $C$ is abelian. If we set $K = F(C)$, then the finiteness of $M/C$ implies that $K$ is a subfield of $D$, over which $D$ is finite dimensional. This fact yields that $[D : F] < \infty$.

**Case 3:** $H = AH_1$.

Since $A' \subseteq H_1 \cap A$, we have $H/H_1 \cong A/A \cap H_1$ is abelian, and hence $H' \subseteq H_1$. Since $H_1$ is abelian, $H'$ is also abelian too. Moreover, $M \subseteq H$, it follows that $M'$ is also abelian. In other words, $M$ is a metabelian group, and the conclusions follow from Proposition 3.7.

By what we have proved, we conclude that $n := [D : F] < \infty$. Since $M$ is solvable, it contains no non-cyclic free subgroups. In view of [9, Theorem 3.1], we have $F[M] = D$, there exists a maximal subfield $K$ of $D$ containing $F$ such that $K/F$ is a Galois extension, $NG(K^*) = M$, $K^* \cap G$ is the Fitting normal subgroup of $M$ and it is the $FC$-center, and $M/K^* \cap G \cong Gal(K/F)$ is a finite simple group of order $[K : F]$. Since $M/K^* \cap G$ is solvable and simple, one has $M/K^* \cap G \cong Gal(K/F) \cong \mathbb{Z}_p$ for some prime number $p$. Therefore, $[K : F] = p$ and $[D : F] = p^2$.

For any $x \in M \setminus K$, if $x^p \notin F$, then by the fact that $F[M] = D$, we conclude that $C_M(x^p) \neq M$. Moreover, since $x^p \in K^* \cap G$, it follows that $\langle x, K^* \cap G \rangle \leq C_M(x^p)$. In other words, $C_M(x^p)$ is a subgroup of $M$ strictly containing $K^* \cap G$. Because $M/K^* \cap G$ is simple, we have $C_M(x^p) = M$, a contradiction. Therefore $x^p \in F$. Furthermore, since $x^p \in K$ and $[D : K]_r = p$, we conclude $D = \bigoplus_{i=1}^{p-1} Kx_i$. It remains to prove that $K^* \cap G$ is the Hirsch-Plotkin radical of $M$. Note that we are in the case that $K^* \cap G \subseteq M$, we conclude that $K^* \cap G = K^* \cap M$. Thus, we have $M/K^* \cap M \cong MK^*/K^* \cong \mathbb{Z}_p$. Let $H$ be the Hirsch-Plotkin radical of $M$. Then $HHK^*/K^* \leq MK^*/K^*$, thus either $H \subseteq K^*$ or $HHK^* = MK^*$. The first case implies that $H \subseteq K^* \cap G$, so $H = K^* \cap G$; we are done. If the second case occurs, then $F[HHK^*] = F[M] = D$. It follows that $HHK^*$ is a locally nilpotent absolutely irreducible subgroup of $D^*$, hence it is center-by-locally finite by [28, Theorem 1]. It is clear that the center of $HHK^*$ is contained in $F^*$. This yields that $HHK^*/F^*$ is locally finite, from which it follows that $K/F$ is a non-trivial radical Galois.
extension. According to [21:15.13], we conclude that $D$ is algebraic over a finite subfield. But then Jacobson’s Theorem ([21:13.11]) says that $D$ is commutative, a contradiction.

\begin{theorem}
Let $D$ be a division ring with infinite center $F$, and $G$ an almost subnormal subgroup of $D^*$. If $M$ is a locally nilpotent maximal subgroup of $G$, then $M$ is abelian.
\end{theorem}

\begin{proof}
Assume by contradiction that $M$ is non-abelian. We claim that $[D : F] < \infty$. Let $T = \tau(M)$ be the unique maximal periodic (hence locally finite) normal subgroup of $M$, and $Z/T$ be the center of $M/T$. Since $T$ is normal in $M$, it follows by Lemma 3.6 that either $F(T) = D$ or $T$ is abelian. If the first case occurs, then Lemma 3.6 says that $D$ is a locally finite division ring. According to [21: Theorem 3.1], we have $[D : F] < \infty$; we are done. Now assume that $T$ is abelian. This fact implies that $Z$ is a solvable group with the derived series length of $s \geq 1$, say. Since $T$ and $Z/T$ is a characteristic subgroup of $M$ and $M/T$ respectively, we conclude that $Z$ is a characteristic subgroup of $M$. Again in view of Lemma 3.6, either $F(Z) = D$ or $Z$ is abelian. Let us consider the following two possible cases:

\textit{Case 1: $F(Z) = D$.}

Since $D$ is non-commutative, it follows that $Z$ is non-abelian. Thus, we may suppose that it is solvable with derived length $s \geq 2$. Therefore, there exists such a series

$$1 = Z^{(s)} \trianglelefteq Z^{(s-1)} \trianglelefteq \cdots \trianglelefteq Z \trianglelefteq M.$$ 

If we set $A = Z^{(s-2)}$, then $A$ is a non-abelian metabelian normal subgroup of $M$. In view of Lemma 3.6, we have $F(A) = D$. It follows that $Z(A) = F^* \cap A$ and $F = C_D(A)$. Set $H = N_{D^*}(A)$, $B = C_A(A')$, $K = F(Z(B))$, $H_1 = H \cap K^*$, and $T_1 = \tau(B)$ to be the unique maximal periodic normal subgroup of $B$. Then $H_1$ is an abelian group, and $T_1$ is an abelian (contained in $T$) characteristic subgroup of $B$, and hence of $A$ (Remark 1). It follows that $T_1$ is normal in $M$. By replacing $T$ by $T_1$ in the proof of Theorem 3.9 we also have $[D : F] < \infty$.

\textit{Case 2: $Z$ is abelian.}

Let $N$ be the maximal subgroup of $M$ with respect to the property: $N$ is an abelian normal subgroup of $M$ containing $Z$. We shall show that $M/N$ is a simple group. For, let $P$ be a normal subgroup of $M$ properly containing $N$. Clearly $P$ is non-abelian. By Lemma 3.6, we have $F(P) = D$. Since $P$ is locally nilpotent, it follows by [20: Corollary 24] that $F[P]$ is an Ore domain whose skew field of fractions is coincided with $D$. In view of Lemma 3.6, we conclude that $F[M]$ is a crossed product of $F[Z]$ by $M/Z$. It follows by Lemma 3.8 that $M = PZ = P$; recall that $Z \subseteq P$. This fact shows that $M/Z$ is simple. Note that $M \neq Z$ since $M$ is non-abelian. Because $M/Z$ is locally nilpotent, we conclude that $M/Z$ is finite of prime order ([21: 12.5.2, p.367]). If we set $L = F(Z)$, then $L$ is a subfield of $D$, over which $D$ is finite dimensional. This fact yields $[D : F] < \infty$.

By what we have proved, we conclude that $n := [D : F] < \infty$. We know that $D \otimes_F D^{op} \cong M_n(D)$. Therefore, we may view $M$ as a subgroup of $GL_n(F)$ to conclude that it is a solvable group. According to Theorem 3.9, we conclude that $M$ is distinguish from its Hirsch-Plotkin radical, which contrasts to the fact that $M$ is locally nilpotent.
\end{proof}
Theorem 3.11. Let $D$ be a division ring with infinite center $F$, and $G$ an almost
subnormal subgroup of $D^*$. If $M$ is a non-abelian locally solvable maximal subgroup
of $G$, then the following hold:

(i) There exists a maximal subfield $K$ of $D$ such that $K/F$ is a finite Galois
extension with $\text{Gal}(K/F) \cong M/K^* \cap G \cong \mathbb{Z}_p$ and $[D:F] = p^2$, for some
prime number $p$.
(ii) The subgroup $K^* \cap G$ is the FC-center. Also, $K^* \cap G$ is the Hirsch-Plotkin
radical of $M$. For any $x \in M \setminus K$, we have $x^p \in F$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.

Proof. First, we show that $[D:F] < \infty$. Since $M$ is non-abelian, Lemma 3.6 says
that $F(M) = D$. Let $T = \tau(M)$ be the unique maximal periodic normal subgroup
of $M$. If $T$ is non-abelian, then by the same arguments used in the beginning of
the proof of Theorem 3.10 we conclude that $[D:F] < \infty$; we are done. We may
therefore suppose that $T$ is abelian. There are two possible cases.

Case 1: $T \not\subseteq F$.

From the field theory, $F(T)$ is an algebraic field extension of $F$. Take $x \in T \setminus F$, and set $x^M := \{m^{-1}xm | m \in M\}$. Since $F(T)$ is normalized by $M$, we have
$x^M \subseteq F(T)$. Thus, the all elements of the set $x^M$ have the same minimal polynomial
over $F \subseteq F(T)$, hence $x^M$ is finite. In other words, $x$ is an $F$-element of $M$. By
the same argument used in Case 2 of the proof of Theorem 3.10 we conclude that $[D:F] < \infty$.

Case 2: $T \subseteq F$.

Let $B = B(M)$ be the subgroup of $M$ such that $B/T$ is the Hirsch-Plotkin
radical of the group $G/T$. As we see in the proof of Lemma 2.13, the group $B$ is
locally nilpotent. It follows that $B$ is the Hirsch-Plotkin radical of $M$ (Remark 11). If $M = B$, then $M$ is locally nilpotent and thus it is abelian by Theorem 3.10
a contradiction. We may therefore assume that $M \neq B$. We claim that $M/B$ is
a simple group. In fact, let $C$ be a normal subgroup of $M$ properly containing $C$. It is clear that $C$ is non-abelian. It follows by Lemma 3.6 that $F(C) = D$. According to Lemma 3.2, we conclude that $F[M]$ is a crossed product of $F[B]$ by $M/B$. Moreover, by [30] Corollary 24, it follows that $F(C)$ is an Ore domain whose
skew field of fractions is coincided with $D$. Therefore, we may apply Lemma 3.3
to conclude that $M = BC$, which implies that $M = C$. Thus, the group $M/B$ is
simple, as claimed. Again by [24] 12.5.2, p.367], we have $M/B$ is a finite group
of prime order. In view of Lemma 3.6, either $B$ is abelian or $F(B) = D$. Let us
consider the following two subcases.

Subcase 2.1: $B$ is abelian.

If we set $L = F(B)$, then $L$ is a subfield of $D$. Since $F(M) = D$ and $|M/B| < \infty$, we have $[D:L]_r < \infty$. This implies that $[D:F] < \infty$, and we are done.

Subcase 2.2: $F(B) = D$.

Since we are in the case $T = \tau(M) \subseteq F^* \cap M \subseteq \mathbb{Z}_2(M)$, we may apply Lemma 3.4
and Remark 2 to conclude that $F[M]$ is a crossed product of $F[A]$ by $M/A$, for
some abelian characteristic subgroup $A$ of $M$. Since $AT/T$ is an abelian normal subgroup of $M/T$, we have $AT/T \subseteq B/T$; recall that $B/T$ is the Hirsch-Plotkin radical of $M/T$. This implies that $A \subseteq B$. By Lemma 3.3, we conclude that $M = BA = B$. This contrasts to the fact that $B$ is a proper subgroup of $M$.

By what we have proved, we conclude that $n := [D : F] < \infty$. We know that $D \otimes_F D^{op} \cong M_n(F)$. Thus, we may view $M$ as a subgroup of $GL_n(F)$, to conclude that $M$ is solvable. The results follow by Theorem 3.9.

□

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