Global Regularity of 2D Generalized MHD Equations with Magnetic Diffusion

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Abstract

Abstract: This paper is concerned with the global regularity of the 2D (two-dimensional) generalized magnetohydrodynamic equations with only magnetic diffusion $\Lambda^{2\beta} b$. It is proved that when $\beta > 1$ there exists a unique global regular solution for this equations. The obtained result improves the previous known one which requires that $\beta > \frac{3}{2}$.

1 Introduction

Consider the Cauchy problem of the following two-dimensional generalized magnetohydrodynamic equations:

$$\begin{cases}
u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \nu \Lambda^{2\alpha} u, \\ b_t + u \cdot \nabla b = b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ b(x,0) = b_0(x)
\end{cases}$$

(1.1)

for $x \in \mathbb{R}^2$ and $t > 0$, where $u = u(x,t)$ is the velocity, $b = b(x,t)$ is the magnetic, $p = p(x,t)$ is the pressure, and $u_0(x), b_0(x)$ with $\text{div} u_0(x) = \text{div} b_0(x) = 0$ are the initial velocity and magnetic, respectively. Here $\nu, \kappa, \alpha, \beta \geq 0$ are nonnegative constants and $\Lambda$ is defined by

$$\hat{\Lambda} f(\xi) = |\xi| \hat{f}(\xi),$$

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where ∧ denotes the Fourier transform. In the following sections, we will use the inverse Fourier transform ∨.

The global regularity of the 2D GMHD equations (1.1) has attracted a lot of attention and there have been extensive studies (see [4]-[13]). It follows from [9] that the problem (1.1) has a unique global regular solution if

\[ \alpha \geq 1, \quad \beta > 0, \quad \alpha + \beta \geq 2. \]

Tran, Yu and Zhai [7] got a global regular solution under assumptions that

\[ \alpha \geq 1/2, \quad \beta \geq 1 \quad \text{or} \quad 0 \leq \alpha < 1/2, \quad 2\alpha + \beta > 2 \quad \text{or} \quad \alpha \geq 2, \quad \beta = 0. \]

Recently, it was shown in [6] that if \( 0 \leq \alpha < 1/2, \beta \geq 1, \quad 3\alpha + 2\beta > 3 \), then the solution is global regular. In particular, when \( \alpha = 0, \beta > \frac{3}{2} \), the solution is global regular. This was proved independently in [11], [12]. Meanwhile, Fan, Nakamura and Zhou [4] used properties of the heat equation and presented a global regular solution when \( 0 < \alpha < \frac{1}{2}, \beta = 1 \).

In this paper, we aim at getting the global regular solution of (1.1) when \( \nu = 0, \kappa > 0 \) and \( \beta > 1 \). For simplicity, we let \( \kappa = 1 \). That is, we consider

\[ \begin{aligned}
 u_t + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b, \\
 b_t + u \cdot \nabla b &= b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \\
 \nabla \cdot u &= \nabla \cdot b = 0, \\
 u(x, 0) &= u_0(x), \quad b(x, 0) = b_0(x).
\end{aligned} \]

(1.2)

Let \( \omega = -\partial_2 u_1 + \partial_1 u_2 \) and \( j = -\partial_2 b_1 + \partial_1 b_2 \) represent the vorticity and the current respectively. We will prove that \( \omega, j \in L^2(0, T; L^\infty) \) and obtain the global regularity of the solution by the BKM type criterion in [2]. To this end, we will take advantage of the approaches used in [4] and [6] to deal with the higher regularity estimates of \( j \). More precisely, using the equation satisfied by the current \( j \), we will obtain the estimates of \( \|\Lambda^\gamma j\|_{L^2}^2 (t) + \int_0^t \|\Lambda^{\beta+r} j\|_{L^2}^2 \leq C \) with \( r = \beta - 1 \). Using the singular integral representation of \( \Lambda^\delta j \) with some \( \delta > 0 \), we will obtain the estimate \( \|\nabla j\|_{L^2(0, T; L^\infty(\mathbb{R}^2))} \). Then we get the estimates of \( \|\omega\|_{L^2(0, T; L^\infty(\mathbb{R}^2))} \) using the particle trajectory method. It should be noted that after the paper is finished, at the almost same time, Cao, Wu and Yuan obtain the similar result independently using a different method (see [8]). In comparison with result obtained in [5], it is not required that \( \|\nabla j_0\|_{L^\infty} < \infty \) in our result.

The main result of this paper is stated as follows.

**Theorem 1.1.** Let \( \beta > 1 \) and assume that \((u_0, b_0) \in H^\rho \) with \( \rho > \max\{2, \beta\} \). Then for any \( T > 0 \), the Cauchy problem (1.2) has a unique regular solution

\[ (u, b) \in L^\infty([0, T]; H^\rho(\mathbb{R}^2)) \quad \text{and} \quad b \in L^2([0, T]; H^{\rho+\beta}(\mathbb{R}^2)). \]

**Remark 1.1.** When \( \alpha = 0, \beta > \frac{3}{2}, \rho > 2 \), the result has been obtained in [6], [11] and [12].

## 2 Preliminaries

Let us first consider the following equation

\[ \begin{aligned}
 v_t + \Lambda^{2\beta} v &= f, \\
 v(x, 0) &= v_0(x).
\end{aligned} \]

Similiar to the heat equation, we can get

\[ v(x, t) = \int_{\mathbb{R}^2} t^{-\frac{1}{2}} h \left( \frac{x - y}{t^{\frac{1}{2}}} \right) v_0(y) dy + \int_0^t \int_{\mathbb{R}^2} (t - s)^{-\frac{1}{2}} h \left( \frac{x - y}{(t - s)^{\frac{1}{2}}} \right) f(y, s) dy ds, \]

(2.3)
where \( h(x) = \left( e^{-|x|^2} \right)^\vee (x) \) and it has the similar properties as the heat kernel.

**Lemma 2.1.** Let \( l \) be a nonnegative integer and \( \eta \geq 0 \), then
\[
\| \nabla^l h \|_{L^1} + \| \Lambda^\eta h \|_{L^1} \leq C. \tag{2.4}
\]

**Proof.** First, we give the proof of the estimates of \( \nabla^l h \).
\[
\| \nabla^l h \|_{L^1} = C \sup_{|\gamma| = l} \int_{\mathbb{R}^2} |\nabla^\gamma e^{-|\xi|^2} e^{i x \cdot \xi} d\xi| d x
\]
\[
= C \sup_{|\gamma| = l} \int_{|x| \leq 1} |\nabla^\gamma e^{-|\xi|^2} e^{i x \cdot \xi} d\xi| d x + C \int_{|x| \geq 1} |\nabla^\gamma e^{-|\xi|^2} e^{i x \cdot \xi} d\xi| d x
\]
\[
\leq C + C \sup_{|\gamma| = l} \int_{|x| \geq 1} (1 + |x|^2)^{-2} |\nabla^\gamma (e^{-|\xi|^2} (1 - \Delta) e^{i x \cdot \xi} d\xi) d x
\]
\[
\leq C + C \sup_{|\gamma| = l} \int_{|x| \geq 1} (1 + |x|^2)^{-2} |\nabla (1 - \Delta) e^{i x \cdot \xi} d\xi) d x
\]
\[
\leq C.
\]

Next, we start to estimate \( \Lambda^\eta h \) and let \( l > \eta \).
\[
\| \Lambda^\eta h \|_{L^1} = \left\| \sum_{k \geq 0} \Delta_k \Lambda^\eta h \right\|_{L^1}
\]
\[
\leq \| \Delta_{-1} \Lambda^\eta h \|_{L^1} + \sum_{k \geq 0} \| \Delta_k \Lambda^\eta h \|_{L^1}
\]
\[
\leq C \| h \|_{L^1} + C \sum_{k \geq 0} 2^{k(-l+\eta)} \| \Delta_k \nabla^l h \|_{L^1}
\]
\[
\leq C + C \sum_{k \geq 0} 2^{k(-l+\eta)} \| \nabla^l h \|_{L^1}
\]
\[
\leq C,
\]
where we use the nonhomogeneous Littlewood-Paley decompositions \( Id = \sum_k \Delta_k \) and Bernstein-Type inequalities (see [1]).

\[\Box\]

Now, let \( \omega = \nabla^l \cdot u = -\partial_2 u_1 + \partial_1 u_2 \) and \( j = \nabla^l \cdot b = -\partial_2 b_1 + \partial_1 b_2 \), and applying \( \nabla^l \cdot \) to the equations (1.1), we obtain the following equations for \( \omega \) and \( j \):
\[
\omega_t + u \cdot \nabla \omega = b \cdot \nabla j, \tag{2.5}
\]
\[
j_t + u \cdot \nabla j = b \cdot \nabla \omega + T (\nabla u, \nabla b) - \Lambda^\beta j, \tag{2.6}
\]
where
\[
T (\nabla u, \nabla b) = 2\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 b_2 (\partial_1 b_2 + \partial_2 b_2).
\]
The estimates for \( \omega, j \) are obtained in [7] and [6], which is presented in the following lemma.

**Lemma 2.2.** Assume that \( \alpha = 0, \beta > 1 \), \( r = \beta - 1 \), and \( k \geq \beta \). Let \( u_0, b_0 \in H^k \). For any \( T > 0 \), we have
\[
\| \omega \|_{L^2}^2 (t) + \| j \|_{L^2}^2 (t) + \int_0^t \| \Lambda^\beta j \|_{L^2}^2 d\tau \leq C (T),
\]
\[
\| \Lambda^\beta j \|_{L^2}^2 (t) + \int_0^t \| \Lambda^{\beta + r} j \|_{L^2}^2 d\tau \leq C (T).
\]
3 The Proof of Theorem 1.1

In this section, we will prove our main result Theorem 1.1. The proof is divided into three steps.

Step 1: \( \omega \in L^\infty(0, T; L^p(\mathbb{R}^2)), j \in L^p(0, T; \mathbb{R}^2) \) for any \( 2 < p < \infty \).

First, the second equation in (1.1) can be rewritten as

\[
 b_t + \Lambda^{2\beta} b = \sum_{i=1}^{2} \partial_i(b_i u - u b)
\]

Due to (2.3), we have

\[
b(x, t) = \int_{\mathbb{R}^2} t^{-\frac{1}{3}} h \left( \frac{x - y}{t^{\frac{1}{3}}} \right) b_0(y) dy + \int_{0}^{t} \int_{\mathbb{R}^2} (t - s)^{-\frac{1}{3}} h \left( \frac{x - y}{(t - s)^{\frac{1}{3}}} \right) \sum_{i=1}^{2} \partial_i(b_i u - u b)(y, s) dy ds.
\]

It follows from Lemma 2.2 that \( b \in L^\infty(0, T; L^p(\mathbb{R}^2)) \) and \( u \in L^\infty(0, T; L^p(\mathbb{R}^2)) \) for any \( 2 \leq p < \infty \). Thanks to Lemma 2.1, we can get

\[
\|\nabla b\|_{L^p(0, T; L^p(\mathbb{R}^2))} \leq C(T) \|\nabla b_0\|_{L^p(\mathbb{R}^2)} + C(T) \int_{0}^{T} \left( \int_{\mathbb{R}^2} \left| t^{-\frac{1}{3}} (\nabla h) \left( \frac{\cdot}{t^{\frac{1}{3}}} \right) \right|_{L^1(\mathbb{R}^2)}^2 dt \right)^{\frac{1}{2}}
+ C(T) \left( \int_{0}^{T} \int_{\mathbb{R}^2} \left| t^{-\frac{1}{3}} (\nabla^2 h) \left( \frac{\cdot}{t^{\frac{1}{3}}} \right) \right|_{L^1(\mathbb{R}^2)} dt \right)^{\frac{1}{2}}
\leq C(T) \|\nabla b_0\|_{L^p(\mathbb{R}^2)} + C(T) \|b \cdot \nabla u - u \cdot \nabla b\|_{L^2(0, T; L^p(\mathbb{R}^2))}.
\]

(3.10)

for any \( 2 \leq p < \infty \).

Multiplying (2.5) by \( |\omega|^{p-2} \omega \) \((p > 2)\), and integrating with respect to \( x \), we get

\[
\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p \leq \int_{\mathbb{R}^2} |b| |\nabla j| |\omega|^{p-1} dx,
\]

Thus, we have

\[
\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^p}^2 \leq \|b\|_{L^\infty} \|\nabla j\|_{L^p} \|\omega\|_{L^p}^{p-1}
\]

and

\[
\|\omega\|_{L^p}^2 \leq C \|\omega(x, 0)\|_{L^p}^2 + C \int_{0}^{t} (\|\nabla j\|_{L^p}^2 + \|\omega\|_{L^p}^2) ds
\leq C + C \int_{0}^{t} (\|\nabla^2 b\|_{L^p}^2 + \|\omega\|_{L^p}^2) ds
\]

(3.10)

\[
\leq C + C \int_{0}^{t} (\|b \cdot \nabla u - u \cdot \nabla b\|_{L^p}^2 + \|\omega\|_{L^p}^2) ds
\leq C + C \int_{0}^{t} (\|\nabla b\|_{L^p}^2 \|\omega\|_{L^\infty}^2 + \|\omega\|_{L^p}^2) ds
\leq C + C \int_{0}^{t} (1 + \|\nabla b\|_{L^p}) \|\omega\|_{L^p}^2 ds.
\]

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This, combining with the Gronwall’s inequality, leads to \( \omega \in L^\infty(0, T; L^p(\mathbb{R}^2)) \) for any \( 2 < p < \infty \).

Step 2: \( \nabla j \in L^2(0, T; L^\infty(\mathbb{R}^2)) \).

Similar to [6], we apply \( \Lambda^\delta(0 < \delta < \min\{2\beta - 2, \rho - 2\}) \) on both sides of (2.6) to obtain

\[
\begin{align*}
(A^\delta j)_t + A^{2\beta} A^\delta j &= -A^\delta (u \cdot \nabla j) + A^\delta (b \cdot \nabla \omega) + A^\delta (T(\nabla u, \nabla b)).
\end{align*}
\]

(3.11)

Thanks to Lemma 2.2 and Step 1, we have that \( u_j, b_\omega, \) and \( T(\nabla u, \nabla b) \in L^p(0, T; \mathbb{R}^2) \) for any \( 2 < p < \infty \). In the same way as in Step 1, we have

\[
\begin{align*}
A^\delta(j, x, t) &= \int_{\mathbb{R}^2} t^{-\frac{\beta}{2}} h \left( \frac{x - y}{t^\frac{\beta}{2}} \right) A^\delta j_0(y) dy \\
&+ \int_0^t \int_{\mathbb{R}^2} (t - s)^{-\frac{\beta}{2}} h \left( \frac{x - y}{(t - s)^\frac{\beta}{2}} \right) (-A^\delta (u \cdot \nabla j) + A^\delta (b \cdot \nabla \omega)) (y, s) dy ds \\
&+ \int_0^t \int_{\mathbb{R}^2} (t - s)^{-\frac{\beta}{2}} h \left( \frac{x - y}{(t - s)^\frac{\beta}{2}} \right) A^\delta (T(\nabla u, \nabla b)) (y, s) dy ds
\end{align*}
\]

and

\[
\begin{align*}
\| \nabla A^\delta j \|_{L^2(0, T; L^p(\mathbb{R}^2))} &\leq C \| A^\delta j_0 \|_{L^p(\mathbb{R}^2)} \left( \int_0^T \left\| t^{-\frac{\beta}{2}} (\nabla h) \left( \frac{\cdot}{t^\frac{\beta}{2}} \right) \right\|^2_{L^1(\mathbb{R}^2)} dt \right)^{\frac{1}{2}} \\
&+ C \| u_j \|_{L^2(0, T; L^p(\mathbb{R}^2))} \left( \int_0^T \left\| t^{-\frac{\beta}{2}} (\Lambda^\delta \nabla^2 h) \left( \frac{\cdot}{t^\frac{\beta}{2}} \right) \right\|_{L^1(\mathbb{R}^2)} dt \right) \\
&+ C \| b_\omega \|_{L^2(0, T; L^p(\mathbb{R}^2))} \left( \int_0^T \left\| t^{-\frac{\beta}{2}} (\Lambda^\delta \nabla^2 h) \left( \frac{\cdot}{t^\frac{\beta}{2}} \right) \right\|_{L^1(\mathbb{R}^2)} dt \right) \\
&+ C \| T(\nabla u, \nabla b) \|_{L^2(0, T; L^p(\mathbb{R}^2))} \left( \int_0^T \left\| t^{-\frac{\beta}{2}} (\Lambda^\delta \nabla h) \left( \frac{\cdot}{t^\frac{\beta}{2}} \right) \right\|_{L^1(\mathbb{R}^2)} dt \right)
\end{align*}
\]

for any \( 2 < p < \infty \). So we can choose \( \delta \) small and \( p \) large enough such that \( \nabla j \in L^2(0, T; L^\infty(\mathbb{R}^2)) \) and \( \| A^\delta j_0 \|_{L^p} \leq C \| j_0 \|_{H^p} \).

Step 3: \( \omega \in L^\infty(0, T; L^\infty) \).

Because of the estimates of the step 2, and the following equation

\[
\omega_t + u \cdot \nabla \omega = b \cdot \nabla j,
\]

we can prove that \( \omega \in L^\infty(0, T; L^\infty) \) by using the particle trajectory method. By taking advantage of the BKM type criterion for global regularity (see [2]), we finish the proof of Theorem 1.1.

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