Embedding problem of linear fractional maps of $B_N$

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Abstract

This paper focuses on the embedding problem of linear fractional maps which explains when a linear fractional self-map can be a member of a semi-group of self-maps of $B_N$.

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1 Introduction

Let $B_N$ be the open unit ball of $\mathbb{C}^N$ and $H(B_N)$ the collection of holomorphic functions defined on $B_N$. Cowen and MacCluer discussed a class of self-map called linear fractional map based on the research of automorphism of $B_N$ in [8]. A linear fractional map $\varphi$ is given by

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D},$$

where $A \in \mathbb{C}^{N \times N}$, $B \in \mathbb{C}^N$, $C \in \mathbb{C}^N$ and $D \in \mathbb{C}$ satisfy several conditions. Properties of linear fractional maps have been deeply studied. We refer the reader to the excellent papers written by Cowen and MacCluer [7, 8], Bayart [1, 2], Bisi and Bracci [4], Jiang and Ouyang [9] etc.

The following theorem is the Denjoy-Wolff Theorem corresponding to the one dimensional case ([10]):

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Theorem 1.1 (Denjoy-Wolff Theorem on $B_N$) Suppose that $\varphi$ is a holomorphic self-map of $B_N$ without interior fixed points. There is a unique $w \in \partial B_N$ such that the iteration series $\{\varphi_n\}$ converges to $w$ uniformly on compact subsets of $B_N$.

$w$ in the above theorem is called the Denjoy-Wolff point of $\varphi$. According to Theorem 1.3 in [10], there is a positive number $\delta \in (0, 1]$ such that

$$0 < \liminf_{z \to w} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = \delta \leq 1,$$

$\delta$ will be defined as the boundary dilation coefficient of $\varphi$ at $w$, or simply the boundary dilation coefficient of $\varphi$. With the help of Denjoy-Wolff point and boundary dilation coefficient, the holomorphic self-maps of $B_N$ will be assert by the following definition:

Definition 1.2 Suppose $\varphi$ is a holomorphic self-map of $B_N$.

1. $\varphi$ is elliptic if $\varphi$ has at least one interior fixed point;
2. $\varphi$ is hyperbolic if $\varphi$ has no interior fixed point and boundary dilation coefficient $\delta \in (0, 1)$;
3. $\varphi$ is parabolic if $\varphi$ has no interior fixed point and boundary dilation coefficient $\delta = 1$.

A continuous semigroup $\{\varphi_t\}$ of $H(B_N)$ in $B_N$ is a continuous homomorphism from the additive semigroup of non-negative real numbers into the composition semigroup of all holomorphic self-maps of $B_N$ endowed with the compact-open topology (Definition 2.6). As for theories about semigroups in the unit disc $\mathbb{D}$ which has been deeply studied and applied in many different contexts, we refer the reader to Shoikhet [13] and references therein.

As to semigroup of linear fractional self-maps in the case of higher dimension, Bracci and co-workers [5] gave a complete classification up to conjugation of continuous semigroups of linear fractional self-maps of $B_N$, and in [6], they characterized the infinitesimal generator of a semigroup of linear fractional self-maps of $B_N$ and solved the embedding problem on $\mathbb{D}$ from a dynamical point of view.

In this paper, we considered the embedding problem for semigroups of linear fractional self-maps on $B_N$. This problem may be completely solved when the linear fractional self-map $\varphi$ is elliptic. In the non-elliptic cases, some sufficient conditions about the embedding problem were given when
ϕ is normal. According to these conditions, the embedding problem when the dimension $N = 2$ could be solved. These conditions also give an answer to the embedding problem when the linear fractional self-map was an automorphism.

2 Background material

Suppose that $A \in \mathbb{C}^{N \times N}$, the spectrum of $A$ will be denoted by $\sigma(A)$, and the spectra radius

$$\rho(A) = \max \{|\lambda| : \lambda \in \sigma(A)\}.$$ 

Let $A^H$ be the conjugate transpose of $A$, and $\|A\|$ be the spectra norm of $A$, i.e.

$$\|A\| = \rho(A^H A)^{\frac{1}{2}} = \sup_{x \in \mathbb{C}^N, |x| = 1} |Ax|.$$ 

Given a matrix $A \in \mathbb{C}^{N \times N}$,

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

may be defined as the exponential of $A$, where $A^0 = E$, $A^n = A \cdot \cdots \cdot A$, $n$ times repeated, for $n \geq 1$. The exponential of a matrix will always be defined.

There is a classic proposition about matrix analysis. ([3], P241, Theorem 71)

**Lemma 2.1** Given any invertible matrix $A \in \mathbb{C}^{N \times N}$, there exist a matrix $M$ such that $\exp(2\pi i M) = A$. If $M$ is triangularly blocked of some type, then so are the matrices $M$. The eigenvalues of any two such $M$ can only differ by integers, and there is a unique matrix $M$ whose eigenvalues have real parts in the half-open interval $[0, 1)$.

**Definition 2.2** Let $M \in \mathbb{C}^{N \times N}$, we say that $M$ is dissipative if for any $w \in \mathbb{C}^N$,

$$\Re w^H Mw \leq 0.$$ 

**Proposition 2.3** (Phillips-Lumer, [5]) For any $t \in \mathbb{R}$, $\|\exp(t M)\| \leq 1$ if and only if $M$ is dissipative.

It would be easier to verify whether $M$ is dissipative or not in the case that $M$ is normal.
Proposition 2.4 Let $M \in \mathbb{C}^{N \times N}$ be normal. Then $M$ is dissipative if and only if $\|\exp(M)\| \leq 1$.

Proof. Suppose firstly that $\|\exp(M)\| \leq 1$. Based on $MM^H = M^HM$, 

$$\exp(M)\exp(M)^H = \exp(M + M^H).$$

Since 

$$\|\exp(M)\|^2 = \rho\left(\exp(M)\exp(M)^H\right),$$

for any $\lambda \in \sigma(M + M^H)$, we see that 

$$|\exp(\lambda)| \leq 1,$$

therefore, $\Re \lambda \leq 0$. Further more, $\lambda \leq 0$ results from that $M + M^H$ is a hermitian matrix. As a consequence, for any $w \in \mathbb{C}^N$, we get 

$$w^H(M + M^H)w \leq 0,$$

or 

$$\Re w^HMw \leq 0.$$ 

Therefore $M$ is dissipative.

Otherwise, if $M$ is dissipative, for any $t \in \mathbb{R}$, 

$$\|\exp(tM)\| \leq 1$$

according to Phillips-Lumer’s theorem. Let $t = 1$, we obtain 

$$\|\exp(M)\| \leq 1.$$ 

We denoted by $LFT(B_N)$ the set of linear fractional self-maps of $B_N$. Let $V$ be an one dimensional affine subset of $\mathbb{C}^N$ and let 

$$S = B_N \cap V,$$

$S$ would be called a slice of $B_N$. The direction subspace of $S$ is defined by 

$$V_S := \text{span}\{s - s' : s, s' \in S\}.$$ 

For a collection of slices $\{S_j : j = 1, 2, \cdots, p\}$ of $B_N$, if the dimension of the subspace spanned by the corresponding direction subspaces $\{V_{S_1}, \cdots, V_{S_p}\}$ equals to $p$, then $\{S_j : j = 1, 2, \cdots, p\}$ is said to be linear independent.
For any $\varphi \in \text{LFT}(B_N)$ and any slice $S$ of $B_N$, if $\varphi(S) \subset S$, $S$ would be called an invariant slice of $\varphi$. Let

$$\#\text{inv}(\varphi) = \dim(\text{span}\{V_S : S \text{ is an invariant slice of } \varphi\}).$$

Then two conclusions are drawn: $\#\text{inv}(\varphi) = 0$ if and only if $\varphi$ has no invariant slice; $\#\text{inv}(\varphi) = 1$ if and only if $\varphi$ has only one invariant slice. (For more discussion about slice see [5].)

**Definition 2.5** Suppose $\varphi$ is a self-map of $B_N$, $z_0 \in B_N$ is a fixed point of $\varphi$. $L_U(\varphi,z_0)$ is defined as the unitary space of $\varphi$ at $z_0$ if

$$L_U(\varphi,z_0) = \bigoplus_{|\lambda|=1} \ker(d\varphi_{z_0} - \lambda E)^N.$$ 

The dimension of $L_U(\varphi,z_0)$ is termed the unitary index of $\varphi$ at $z_0$ which is denoted by $u(\varphi,z_0)$.

As Lemma 3.1 in [5] has shown, $u(\varphi,z_0) = u(\varphi,z_1)$ for any two fixed point of $\varphi$. Thus the unitary index of $\varphi$ is usually denoted by $u(\varphi)$.

**Definition 2.6** For any open subset $U$ of $\mathbb{C}^N$, the collectivity of holomorphic self-maps of $U$ is denote by $H(U,U)$. $\{\varphi_t : t \geq 0\} \subset H(U,U)$ is a semigroup of holomorphic self-map if

1. $\varphi_0 = id_U$, where $id_U : U \to U$ is the identity map;
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for ant $s, t \geq 0$;
3. $\varphi_t$ converges uniformly on compact subsets of $U$ when $t \to 0^+$.

Throughout this paper, $\{\varphi_t\}$ is called a semigroup for short.

An element of a semigroup $\{\varphi_t\}$ is said to be an iterate of $\{\varphi_t\}$. There are many special properties of semigroup $\{\varphi_t\}$, for instance,

- every iterate of $\{\varphi_t\}$ is an injection;
- if one of the iterates is an automorphism, then all of the iterates are automorphism;
- for any $z \in U$, the map $t \mapsto \varphi_t(z)$ is real analytic.

Semigroups whose iterates are linear fractional maps of $B_N$ have been discussed at [5]. The following is some of the conclusion:
Theorem 2.7 Suppose \( \{ \varphi_t \} \) is a semigroup of \( H(B_N, B_N) \). If there is a \( t_0 \in (0, +\infty) \) such that \( \varphi_{t_0} \) is an elliptic (hyperbolic or parabolic) self-map, then for any \( t \in (0, +\infty) \), \( \varphi_t \) is elliptic (hyperbolic or parabolic). Furthermore, if \( \varphi_{t_0} \) is non-elliptic, then all the iterates of \( \{ \varphi_t \} \) share the same Denjoy-Wolff point.

According to the above theorem, a semigroup of linear fractional self-maps can be divided into type of elliptic semigroup, type of hyperbolic semigroup and type of parabolic semigroup.

On the other hand, in every semigroup of linear fractional self-maps \( \{ \varphi_t \} \) of \( B_N \) there is a holomorphic map \( G : B_N \to \mathbb{C}^N \) such that for any \( t \geq 0 \),

\[
\frac{\partial \varphi_t}{\partial t} = G \circ \varphi_t.
\]

The above map \( G \) is said to be the infinitesimal generator of \( \{ \varphi_t \} \). Besides, one semigroup has only one infinitesimal generator and vice versa (c.f. [6]).

A linear fractional map \( \varphi \) can be embedded into a semigroup composed of linear fractional self-map \( \{ \varphi_t \} \) if \( \varphi \) is a iterate of \( \{ \varphi_t \} \). Simple computation shows that if \( \varphi : B_N \to B_N \) is conjugated to \( \psi : U \to U \), which means that there is a biholomorphic map \( \sigma \) between \( B_N \) and \( U \), such that

\[
\psi = \sigma \circ \psi \circ \sigma^{-1},
\]

\( \varphi \) can be embedded into some semigroup if and only if \( \psi \) can be embedded in to some semigroup. Therefore, the study of the embedding problem of semigroup will based on the meaning of conjugation.

3 Normal forms of linear fractional self-maps

3.1 The elliptic cases

Let \( \varphi \in LFT(B_N) \) be elliptic. \( \varphi \) has at least one interior fixed point. Therefore, \( \varphi \) is supposed to fix the origin throughout no more than one automorphism. In this case, for any \( z \in B_N \),

\[
\varphi(z) = \frac{Az}{\langle z, C \rangle + 1},
\]

where \( A \in \mathbb{C}^{N \times N}, \ C \in \mathbb{C}^N \). By \( F = Fix(\varphi) \) is meant the collection of fixed points of \( \varphi \). According to [12], \( F \) is the intersection of \( B_N \) and some subspace of \( \mathbb{C}^N \). The dimension of this subspace is denoted by \( p \). Let \( u = u(\varphi) \) which is the unitary index of \( \varphi \) at the origin.
Proposition 3.1 Let \( A \in \mathbb{C}^{N \times N} \), and \( \| A \| \leq 1 \). If \( \lambda \) is an eigenvalue of \( A \) with \( | \lambda | = 1 \), then the geometric multiplicity of \( \lambda \) is equal to 1.

**Proof.** According to Schur’s Triangularization Theorem (P508, [11]), there is a unitary matrix \( U \in \mathbb{C}^{N \times N} \) and an upper-triangular matrix

\[
T = \begin{bmatrix}
\lambda & a_{12} & \cdots & a_{1N} \\
0 & \lambda_2 & \ast & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \lambda_N
\end{bmatrix},
\]

such that

\[ A = U^H T U. \]

\( \| T \| \leq 1 \) for \( \| A \| \leq 1 \).

Suppose there is a subscript \( i \) such that \( a_{1i} \neq 0 \). Let

\[
z_\lambda = \left( \frac{\bar{\lambda}}{\sqrt{1 + |a_{1i}|^2}}, 0, \cdots, \frac{\bar{a}_{1i}}{\sqrt{1 + |a_{1i}|^2}}, \cdots, 0 \right)^T
\]

be a vector of \( \mathbb{C}^N \) with

\[ |z_\lambda| = 1 \]

and

\[
Tz_\lambda = \left( \frac{\bar{\lambda} \lambda}{\sqrt{1 + |a_{1i}|^2}} + \frac{\bar{a}_{1i}a_{1i}}{\sqrt{1 + |a_{1i}|^2}}, \cdots \right)^T = \left( \sqrt{1 + |a_{1i}|^2}, \cdots \right).
\]

Therefore

\[ \|Tz_\lambda\| \geq \sqrt{1 + |a_{1i}|^2}, \]

and the inequality above contradicts to the fact that \( \| T \| \leq 1 \). As a consequence, \( a_{1i} = 0 \) for all \( i = 2, \cdots, n \). This means that the proposition holds.

Proposition 3.2 Let \( \varphi \in \text{LFT} (B_N) \) be elliptic with unitary index \( u (\varphi) \geq 1 \). Then \( \varphi \) is conjugated to \( \psi \in \text{LFT} (B_N) \) with

\[
\psi \left( z', z'' \right) = \left( \Lambda z', A_1 z'' \right),
\]

where \( (z', z'') \in \mathbb{C}^u \times \mathbb{C}^{N-u} \cap B_N \), \( \Lambda \) is a diagonal and unitary matrix of order \( u \) and \( A_1 \) is a matrix of order \( N - u \) with spectral radius \( \rho (A_1) < 1 \), \( \| A_1 \| \leq 1 \).
Parts of the following idea come from the proof of Theorem 4.3 in [5].

**Proof.** Resulting from the previous discussion, we may assume that
\[ \varphi (z) = \frac{Az}{\langle z, C \rangle + 1}. \]

Simple computation indicates that the Jacobi matrix of \( \varphi \) at the origin is \( d\varphi_0 = A \). According to Schwartz’s lemma (see [12]), \( \|A\| \leq 1 \). As a consequence of Rudin’s version of the Julia–Wolff–Carathéodory theorem (see [12]), there is at least one eigenvalue of \( A \) which module equals 1 since \( u \geq 1 \).

Due to Proposition 3.1, there is a unitary matrix \( U \), such that
\[
U^H AU = \left[ \begin{array}{cc} \Lambda & 0 \\ 0 & A_1 \end{array} \right],
\]

where
\[
\Lambda = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_u
\end{bmatrix},
\]

and \( |\lambda_j| = 1 \) for \( j = 1, 2, \ldots, u \). \( A_1 \) is a matrix of order \( N - u \) with \( \|A_1\| \leq 1 \) and \( \rho(A_1) < 1 \).

Let
\[
\varphi_1 (z', z'') = U^H (\varphi(Uz)) = U^H \left( \frac{AUz}{\langle Uz, C \rangle + 1} \right)
\]
\[
= \frac{U^H AUz}{\langle z, U^H C \rangle + 1} = \frac{(\Lambda z', A_1 z'')}{\langle z, U^H C \rangle + 1},
\]

where \( (z', z'') \in \mathbb{C}^u \times \mathbb{C}^{N-u} \). Eventually, \( \varphi_1 \) is conjugated to \( \varphi \). We denote
\[
U^H C = (c', c'') \in \mathbb{C}^u \times \mathbb{C}^{N-u},
\]

thus
\[
\varphi_1 (z', z'') = \frac{(\Lambda z', A_1 z'')}{\langle z', c' \rangle + \langle z'', c'' \rangle + 1}.
\]

\( \varphi_1 \left( \{ (z', O) \in \mathbb{C}^u \times \mathbb{C}^{N-u} : |z'| < 1 \} \right) \) \( \subseteq \) \( \{ (z', O) \in \mathbb{C}^u \times \mathbb{C}^{N-u} : |z'| < 1 \} \), consequently
\[
\varphi_2 (z') \overset{\Delta}{=} \varphi_1 (z', O) = \frac{\Lambda z'}{\langle z', c' \rangle + 1}
\]

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is a self-map of the unit ball in \( \mathbb{C}^n \) and

\[
d(\varphi_2)_O = \Lambda.
\]

On the basis of Schwartz’s lemma on the ball, \( \varphi_2 \) is linear and as a consequence \( c' = 0 \). As a result,

\[
\varphi_1(z) = \frac{(\Lambda z', A_1 z'')}{(z'', c'')} + 1.
\]

We define

\[
z_t = \left( \sqrt{1 - t^2 |c''|^2} e_1, -tc'' \right),
\]

then for all \( t \in [0, 1] \),

\[
|z_t|^2 = \left( \sqrt{1 - t^2 |c''|^2} \right)^2 + t^2 |c''|^2 = 1,
\]

and

\[
\varphi_1(z_t) = \frac{\left( \sqrt{1 - t^2 |c''|^2} \Lambda e_1' - t A_1 c'' \right)}{(-tc'', c'')} + 1 = \frac{\left( \sqrt{1 - t^2 |c''|^2} \Lambda e_1' - t A_1 c'' \right)}{1 - t |c''|^2}.
\]

We obtain

\[
|\varphi_1(z_t)|^2 \geq \frac{1 - t^2 |c''|^2}{(1 - t |c''|^2)^2},
\]

\[
\frac{d}{dt} \left( \frac{1 - t^2 |c''|^2}{(1 - t |c''|^2)^2} \right) = 2 \frac{|c''|^2}{(t |c''|^2 - 1)^3} (t - 1).
\]

Consequently for all \( t \in [0, 1] \), \( \frac{1 - t^2 |c''|^2}{(1 - t |c''|^2)^2} \) is decreasing for \( t \). Therefore when \( t > 0 \),

\[
|\varphi_1(z_t)|^2 > \lim_{t \to 0} \frac{1 - t^2 |c''|^2}{(1 - t |c''|^2)^2} = 1.
\]
It is impossible since \( \varphi_1 (B_N) \subset B_N \) and \( \varphi_1 \) is holomorphic. As a consequence, \( e'' = O \) and 
\[
\varphi_1 (z) = (Az', A_1z'').
\]

\[\blacksquare\]

**Proposition 3.3** Let \( \varphi \) be a elliptic linear fractional self-map of \( B_N \). Suppose that \( \varphi \) has a unique interior fixed point and unitary index \( u(\varphi) = 0 \). Then \( \varphi \) is conjugated to \( \psi \in \text{LFT}(B_N) \) with 
\[
\hat{A}z \over \delta \left( \langle z, (\hat{A}^H - E) e_1 \rangle + 1 \right),
\]
where \( \hat{A} \) is a matrix of order \( N \) with \( \rho (\hat{A}) < 1 \), \( \| \hat{A} \| \leq 1 \) and \( \delta \in [0, 1] \).

**Proof.** Without loss of generality, we suppose that the unique interior fixed point of \( \varphi \) is the origin. For any \( z \in B_N \), 
\[
\varphi (z) = \frac{Az}{\langle z, C \rangle + 1},
\]
where \( A \in \mathbb{C}^{N \times N} \), \( C \in \mathbb{C}^N \). As a result, \( \| A \| \leq 1 \). Due to the previous lemma, \( \varphi \) has at least one invariant slice if there is a eigenvalue of \( A \) such that \( |\lambda| = 1 \). This conclusion is not established since \( u = 0 \). Therefore, \( \rho (A) < 1 \), both \( A - E \) and \( A^H - E \) are invertible and there exists a vector \( V \in \mathbb{C}^N \) such that 
\[
C = (A^H - E) V.
\]
On the other hand, there is a unitary matrix \( U \) such that 
\[
U^H V = \delta e_1,
\]
where \( \delta = |V|, e_1 = (1, 0, \cdots, 0)^T \). Let \( \hat{\varphi} (z) = U^H \circ \varphi \circ U (z), \hat{A} = U^H A U \), then 
\[
\hat{\varphi} (z) = \frac{U^H A U z}{\langle U z, C \rangle + 1} = \frac{U^H A U z}{\langle U z, (A^H - E) V \rangle + 1} = \frac{U^H A U z}{\langle z, U^H (A^H - E) V \rangle + 1} = \frac{U^H A U z}{\langle z, (U^H A U^H - E) U^H V \rangle + 1} = \frac{\hat{A} z}{\delta \left( \langle z, (\hat{A}^H - E) e_1 \rangle + 1 \right)}.
\]
Simple computation shows that
\[ \hat{\varphi}_n(z) = \frac{\hat{A}^n z}{\delta \left( z, \left( \left( \hat{A}^n \right)^H - E \right) e_1 \right) + 1}. \]

Since \( \hat{\varphi}_n(B_N) \subset B_N \) for every \( n \in \mathbb{N} \),
\[ \left| \delta \left( \left( \hat{A}^n \right)^H - E \right) e_1 \right| \leq 1. \]
\( \hat{A}_n \to O \) when \( n \) tends to infinity since \( A \) and \( \hat{A} \) have the same spectrum. This conclusion leads to \( \delta \in [0, 1] \).

### 3.2 Non-elliptic cases

The Siegel half-plane domain of \( \mathbb{C}^N \) is defined by
\[ \mathbb{H}^N = \left\{ (u_1, u') \in \mathbb{C} \times \mathbb{C}^{N-1} : \text{Im} u_1 > |u'|^2 \right\}. \]
\( \mathbb{H}^N \) is biholomorphic with \( B_N \) via the Cayley transformation:
\[ \sigma(z_1, z') = \left( \frac{1 + z_1}{1 - z_1}, \frac{iz'}{1 - z_1} \right). \]

Every linear fractional map on \( B_N \) is conjugated to a linear fractional map on \( \mathbb{H}^N \) since \( \sigma \) is a linear fractional map. Besides, the following lemma can be drawn.

**Lemma 3.4** Suppose \( \varphi \in \text{LFT}(B_N) \) be non-elliptic with boundary dilation coefficient \( \alpha = \alpha(\varphi) \). Then \( \varphi \) is conjugated to a self-map \( \tilde{\varphi} \) of \( \mathbb{H}^N \) with
\[ \tilde{\varphi}(z, w) = (\lambda z + 2i \langle w, a \rangle + b, Mw + c), \quad (z, w) \in \mathbb{H}^N, \]
where \( c \in \mathbb{C}, b, d \in \mathbb{C}^{N-1}, M \in \mathbb{C}^{(N-1) \times (N-1)} \). Conversely, such a map is a self-map of \( \mathbb{H}^N \) if and only if
\[ (P1) \quad Q := \lambda I - M^H M \text{ is a hermitian positive semi-definite matrix}; \]
\[ (P2) \quad \text{Im} (b) - |c|^2 \geq \langle Q^+(M^*c - a), M^*c - a \rangle \text{ where } Q^+ \text{ is the pseudo-inverse of } Q \text{ (for more details about pseudo-inverse, we refer to [11], p422)}; \]
\[ (P3) \quad QQ^+(M^*c - a) = M^*c - a. \]
Proof. This lemma is a modified version of theorem 4.1 of [5]. The only difference is (P3). In fact, the corresponding condition is $M^*c - a$ belongs to the space spanned by the columns of $Q$ in Theorem 4.1 of [5]. That is to say, there is a vector $x \in \mathbb{C}^{N-1}$ such that

$$Qx = M^*c - a.$$ 

According to [11], this equation has at least one solution if and only if

$$QQ^+ (M^*c - a) = M^*c - a$$

holds. ■

The following lemma about normal forms of parabolic linear fractional self-maps results from section 2 of [1].

**Lemma 3.5** Let $\varphi \in \text{LFM} (B_N)$ be parabolic. Then $\varphi$ is conjugated to $

$$\psi \in \text{LFM} (\mathbb{H}^N)$$

with

$$\psi (z, u, v, w) = (z + 2i \langle u, a \rangle + 2i \langle w, c \rangle + b, u + a, Dv, Aw),$$

(1)

where

1. $D$ is diagonal, $\sigma (D) \subset \partial \mathbb{D} \setminus \{1\}$;
2. $Q = I - A^H A$ is a hermitian positive semi-definite matrix;
3. $\text{Im} (b) - |a|^2 \geq \langle Q^+ c, c \rangle$;
4. $QQ^+ c = c$.

Sometimes there are no $u$ or no $v$ or no $w$ appeared in (1). Furthermore, due to theorem 4.4 of [5], we may assume that $a = 0$ if $\varphi$ has at least one invariant slice.

**Theorem 3.6** Let $\varphi \in \text{LFT} (B_N)$ be hyperbolic. Then $\varphi$ is conjugated to $\psi \in \text{LFT} (\mathbb{H}^N)$ with

$$\psi (z, u, v, w) = \left(\lambda z + b, \sqrt{\lambda} u, \sqrt{\lambda} Dv, \sqrt{\lambda} Aw + c\right),$$

where

1. $D$ is diagonal, $\sigma (D) \subset T \setminus \{1\}$;
2. Both $Q = I - A^H A$ and $P = I - AA^H$ are hermitian positive semi-definite matrix;
3. $\text{Im} \ (b) \geq \langle P^+ c, c \rangle$ where $P^+$ is the pseudo-inverse of $Q$;

4. $QQ^+ A^H c = A^H c$.

**Proof.** Let $\varphi \in LFM (B_N, B_N)$ be hyperbolic. According to theorem 4 in [5], $\varphi$ is conjugated to a self-map $\tilde{\varphi}$ of $\mathbb{H}^N$ with

$$\tilde{\varphi} (z, W) = (\lambda z + 2i \langle W, a \rangle + b, MW + c), \ (z, W) \in \mathbb{H}^N,$$

where $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{N-1}$, $A \in \mathbb{C}^{(N-1) \times (N-1)}$ satisfy the following conditions:

1. $Q_M := \lambda I - M^H M$ is a Hermit semi-definite matrix;
2. $\text{Im} \ (b) - |c|^2 \geq \langle Q_M^+ (M^* c - a), M^* c - a \rangle$ where $Q^+$ is the pseudo-inverse of $Q$,
3. $Q_M Q_M^+ (M^* c - a) = M^* c - a$.

We may assume that

$$\frac{1}{\sqrt{\lambda}} M = \begin{pmatrix} I & D \\ \lambda I - M^H M & A \end{pmatrix}$$

since

$$\left\| \left( \frac{1}{\sqrt{\lambda}} M \right) \left( \frac{1}{\sqrt{\lambda}} M \right)^H \right\| \leq 1,$$

where $D$ is diagonal with $\sigma(D) \subset \partial \mathbb{D} \setminus \{1\}$ and $\rho(A) < 1$, $\|A\| \leq 1$. That is why $\tilde{\varphi}$ could be written as the following form:

$$\tilde{\varphi} (z, W) = (\lambda z + 2i \langle u, a_1 \rangle + 2i \langle v, a_2 \rangle + 2i \langle w, a_3 \rangle + b, \sqrt{\lambda} u + c_1, \sqrt{\lambda} D v + c_2, \sqrt{\lambda} A w + c_3).$$

Let $\tau$ be an automorphism of $\mathbb{H}^N$ which preserves $\infty$ with

$$\tau (z, W) = (z + 2i \langle W, \gamma \rangle + \beta, W + \gamma)$$

and

$$\tau^{-1} (z, W) = \left( z - 2i \langle W, \gamma \rangle - \beta + 2i |\gamma|^2, W - \gamma \right).$$

We set $\psi_1 = \tau \circ \tilde{\varphi} \circ \tau^{-1}$, then

$$\psi_1 (z, W) = (\lambda z + 2i \langle W, \tilde{a} \rangle + \tilde{b}, \sqrt{\lambda} u + (1 - \sqrt{\lambda}) \gamma_1 + c_1, \sqrt{\lambda} D v + (I - \sqrt{\lambda} D) \gamma_2 + c_2, \sqrt{\lambda} A w + (I - \sqrt{\lambda} A) \gamma_3 + c_3)$$

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for some $\tilde{a}, \tilde{b}$. Choose $\gamma_1$ and $\gamma_2$ such that 

$$(1 - \sqrt{\lambda}) \gamma_1 + c_1 = 0 \quad \text{and} \quad (I - \sqrt{\lambda}D) \gamma_2 + c_2 = 0,$$

we can obtain

$$\psi_1(z, W) = \left(\lambda z + 2i \langle u, \tilde{a}_1 \rangle + 2i \langle v, \tilde{a}_2 \rangle + 2i \langle w, \tilde{a}_3 \rangle + \tilde{b}, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}Aw + \tilde{c}\right).$$

Let

$$\tilde{Q} = \sqrt{\lambda} \left( \begin{array}{cc} I & D \\ A & \lambda \end{array} \right) \left( \begin{array}{cc} I & D \\ A & \lambda \end{array} \right)^H = \lambda \left( \begin{array}{cc} 0 & 0 \\ 0 & I - A^H A \end{array} \right).$$

$\tilde{Q}$ is hermitian semi-definite positive matrix since $\psi_1$ is a self-map of $\mathbb{H}^N$ fixing $\infty$. As a result, $I - A^H A$ is hermitian semi-definite positive. We may get $\tilde{a}_1 = 0, \tilde{a}_2 = 0$ based on (3) and

$$\tilde{Q}^+ = \frac{1}{\lambda} \left( \begin{array}{cc} 0 & 0 \\ 0 & (I - A^H A)^+ \end{array} \right),$$

Let $\tau_1$ be an automorphism of $\mathbb{H}^N$ with

$$\tau_1(z, u, v, w) = (z + 2i \langle w, \gamma \rangle + \beta_1, u, v, w + \gamma),$$

$$\tau_1^{-1}(z, u, v, w) = \left(z - 2i \langle w, \gamma \rangle - \beta_1 + 2i |\gamma|^2, u, v, w - \gamma\right).$$

Let $\psi = \tau \circ \psi_1 \circ \tau^{-1}$, thus

$$\psi(z, u, v, w) = \tau \circ \psi_1 \left(z - 2i \langle w, \gamma \rangle - \beta_1 + 2i |\gamma|^2, u, v, w - \gamma\right)$$

$$= \tau \left(\lambda z + 2i \langle w, \tilde{a}_3 + \sqrt{\lambda}A^H \gamma - \lambda \gamma \rangle + \tilde{b}_1, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}A(z - \gamma) + \tilde{c}_1\right)$$

We may choose $\gamma$ such that $\tilde{a}_3 + \sqrt{\lambda}A^H \gamma - \lambda \gamma = 0$ since $\rho(A) < 1$. If we write $\tilde{b}_1$ as $b$ and $\tilde{c}_1$ as $c$ again, the result will be

$$\psi(z, u, v, w) = \left(\lambda z + b, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}A + c\right).$$

According to (1), (2) and (3), we obtain

- Both $Q = I - A^H A$ and $P = I - AA^H$ are hermitian positive semi-definite matrix;
• \( \text{Im} b - |c|^2 \geq \langle Q^+ A^H c, A^H c \rangle; \)

• \( QQ^+ A^H c = A^H c. \)

Let \( A = U \Sigma V \) be the singular value decomposition of \( A \), namely \( U \) and \( V \) are unitary, \( \Sigma \) is diagonal and the coefficients on the diagonal of \( \Sigma \) are non-negative. Let

\[
\Sigma = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},
\]

where \( 1 \notin \sigma(B) \). According to this decomposition,

\[
Q = V^H \begin{bmatrix} 0 & 0 \\ 0 & I - B^2 \end{bmatrix} V, \quad Q^+ = V^H \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} V,
\]

\[
P = U \begin{bmatrix} 0 & 0 \\ 0 & I - B^2 \end{bmatrix} U^H, \quad P^+ = U \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} U^H,
\]

and

\[
QQ^+ A^H c = V^H \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} U^H c.
\]

Therefore, \( QQ^+ A^H c = A^H c \) is equivalent to

\[
\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^H c = 0.
\]

Consequently,

\[
\langle Q^+ A^H c, A^H c \rangle + |c|^2 = c^H U \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} V V^H \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} V V^H \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} U^H c + |c|^2
\]

\[
= c^H U \begin{bmatrix} 0 & 0 \\ 0 & B^2 (I - B^2)^{-1} \end{bmatrix} U^H c + c^H U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^H c
\]

\[
= c^H U \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} U^H c
\]

\[
= \langle P^+ c, c \rangle.
\]

The proof of the theorem above actually has proved the following corollary:
Corollary 3.7 Let $\varphi \in \text{LFT}(B_N)$ be hyperbolic. Then $\varphi$ is conjugated to $\psi \in \text{LFT}(\mathbb{H}^N)$ with

$$\psi(z, u, v, w) = \left(\lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda} u, \sqrt{\lambda} D v, \sqrt{\lambda} A w + c \right),$$

where

1. $D$ is diagonal, $\sigma(D) \subset \partial D \setminus \{1\}$;
2. $Q = I - A^H A$ is hermitian positive semi-definite matrix;
3. $\text{Im} (b) - |c|^2 \geq \left\langle Q^+ (A^H c - \frac{a}{\sqrt{\lambda}}), A^H c - \frac{a}{\sqrt{\lambda}} \right\rangle$ where $Q^+$ is the pseudo-inverse of $Q$;
4. $QQ^+ \left( A^H c - \frac{a}{\sqrt{\lambda}} \right) = A^H c - \frac{a}{\sqrt{\lambda}}$.

In the case of parabolic automorphism, the following two corollaries are formed according to Proposition 4.3 in [5] and Lemma 3.5.

Corollary 3.8 Let $\varphi$ be a parabolic automorphism of $B_N$. Then $\varphi$ is conjugated to $\psi \in \text{Aut}(\mathbb{H}^N, \mathbb{H}^N)$ with

$$\psi(z, u, v) = \left( z + 2i \langle u, a \rangle + i |a|^2 + b, u + a, D v \right)$$

where $b$ is a real number and $D$ is diagonal, $\sigma(D) \subset \partial D \setminus \{1\}$.

Corollary 3.9 Let $\varphi$ be a hyperbolic automorphism of $B_N$, then $\varphi$ is conjugated to $\psi \in \text{Aut}(\mathbb{H}^N, \mathbb{H}^N)$ with

$$\psi(z, W) = \left( \lambda z + b, \sqrt{\lambda} U W \right)$$

where $(z, W) \in \mathbb{C} \times \mathbb{C}^{N-1}$, $b$ is a real number and $U$ is a unitary matrix.

4 Embedding problems

4.1 The elliptic case

Owing to the previous discussions, we would consider the embedding problem of the normal forms only.
**Theorem 4.1** Let $\varphi \in LFT(B_N)$ be elliptic with
\[
\varphi (z', z'') = (\Lambda z', Az'')
\]
where $\Lambda$ is a diagonal unitary matrix and $\rho(A) < 1$, $\|A\| \leq 1$. $\varphi$ can be embedded into a semigroup of linear fractional maps of $B_N$ if and only if there is a dissipative matrix $M$ and $\sigma(M)$ is a subset of the left half plane such that
\[
\exp(M) = A.
\]

**Proof.** We can find a real diagonal matrix $\Theta$ such that $\Lambda$ can be written as
\[
\Lambda = \exp(i\Theta),
\]
for $\Lambda$ is a diagonal unitary matrix.

Suppose firstly that there is a dissipative matrix $M$ such that $\exp(M) = A$. Hence,
\[
\varphi (z', z'') = (\exp(i\Theta) z', \exp(M) z'').
\]
Let
\[
\varphi_t (z', z'') = (\exp(it\Theta) z', \exp(tM) z'').
\]
$\varphi_t (B_N) \subset B_N$ since $\|\exp(it\Theta)\| = 1$ and $\|\exp(tM)\| \leq 1$. Moreover, $\varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$. As a result, $\{\varphi_t\}$ is a semigroup of $B_N$ with $\varphi_1 = \varphi$.

On the other hand, if $\varphi$ can be embedded into a semigroup of linear fractional self-maps $\{\varphi_t\}$, $\varphi_t$ is conjugated to the following linear fractional map owning to Theorem 3.2 of [5]:
\[
\psi_t (z', z'') = \left( \exp \left( it\hat{\Theta} \right) z', \exp \left( t\hat{M} \right) z'' \right),
\]
where $\hat{\Theta}$ is a real diagonal matrix, $\hat{M}$ is dissipative and all eigenvalues locates on the left half plane. Suppose that the corresponding index of $\varphi$ is $t_0$. Thus $\varphi$ is conjugated to
\[
\psi_{t_0} (z', z'') = \left( \exp \left( it_0\hat{\Theta} \right) z', \exp \left( t_0\hat{M} \right) z'' \right).
\]
There are unitary matrixes $U$ and $V$ such that
\[
\Lambda = U^H \exp \left( it_0\hat{\Theta} \right) U \quad \text{and} \quad A = V^H \exp \left( t_0\hat{M} \right) V = \exp \left( t_0V^H \hat{M}V \right),
\]
for every automorphism of $B_N$ which fixes the origin is a unitary transformation. Let $M = t_0V^H \hat{M}V$. Therefore,
\[
A = \exp(M)
\]
where $M$ is dissipative and all eigenvalues locate on the left half plane. ■

We will make use of the following generalization of Berkson-Porta’s criterion due to Aharonov, Elin, Reich and Shoikhet (see Theorem 1.3, [1]):

**Lemma 4.2** Let $F : B_N \to \mathbb{C}^N$ be holomorphic. $F$ is the infinitesimal generator of a semigroup of holomorphic self-maps of $B_N$ fixing the origin if and only if

$$F(z) = -Q(z)z$$

where $Q(z)$ is a matrix of order $N$ with holomorphic entries such that

$$\Re \langle Q(z), z \rangle \geq 0.$$  

Due to the above discussion, we can prove the following theorem.

**Theorem 4.3** Let $\varphi \in \text{LFT}(B_N)$ with unique interior fixed point and

$$\varphi(z) = \frac{Az}{\delta \langle z, (A^H - E)e_1 \rangle + 1},$$

where $\delta \in [0, 1]$, $e_1 = (1, 0, \cdots, 0)^T$. $\varphi$ can be embedded into a semigroup of linear fractional self-map of $B_N$ if and only if there is a matrix of order $N$ such that $A = \exp(M)$, and

$$\Re \left[ \langle Mz, z \rangle - \delta \langle Mz, e_1 \rangle |z|^2 \right] \geq 0.$$  

**Proof.** Due to [12], $A$ and $\varphi$ share the same fixed point. As a consequence, 1 is not an eigenvalue of $A$ since $\varphi$ has only one interior fixed point.

If $\varphi$ can be embedded into $\{\varphi_t\}$ which is a semigroup of linear fractional self-maps of $B_N$. Without loss of generality, we may assume that $\varphi = \varphi_1$. Due to Theorem 3.2 of [5], there is a matrix $M$ such that

$$\varphi_t(z) = \frac{\exp(tM)z}{\delta \langle z, (\exp(tM)^H - E)e_1 \rangle + 1}.$$  

$A = \exp(A)$ for $\varphi = \varphi_1$. Easy computation gives

$$\frac{d}{dt} \varphi_t(z) = (M - \delta \langle M \varphi_t(z), e_1 \rangle E) \varphi_t(z).$$

Thus

$$F(z) = (M - \delta \langle Mz, e_1 \rangle E)z.$$
is the infinitesimal generator of \{\varphi_t\}. By Lemma 4.2 we obtain
\[
\text{Re} \langle (M - \delta \langle M z, e_1 \rangle E) z, z \rangle = \text{Re} \left[ \langle M z, z \rangle - \delta \langle M z, e_1 \rangle \|z\|^2 \right] \geq 0.
\]
On the other hand, if there is a matrix \( M \) such that \( A = \exp(M) \), and
\[
\text{Re} \left[ \langle M z, z \rangle - \delta \langle M z, e_1 \rangle |z|^2 \right] \geq 0.
\]
then \( F(z) = (M - \delta \langle M z, e_1 \rangle E) z \) is the infinitesimal generator of some semigroup. Denote this semigroup by \{\varphi_t\}, then
\[
\frac{d}{dt} \varphi_t(z) = F(\varphi_t(z)).
\]
Hence we have the following differential equalities
\[
\begin{cases}
\frac{d}{dt} \varphi_t(z) = F(\varphi_t(z)) \\
\varphi_0(z) = z
\end{cases},
\]
such an initial problem has unique solution according to the theory of differential equation. Due to the previous discussion,
\[
\varphi_t(z) = \frac{\exp(tM) z}{\delta \left( z, \left( \exp(tM)H - E \right) e_1 \right) + 1}
\]
is just the solution of this equation. Therefore \{\varphi_t\} is a semigroup of linear fractional self-maps of \( B_N \) with
\[
\varphi_1(z) = \varphi(z).
\]

4.2 The parabolic cases
The embedded problem of non-elliptic case is much more complicated than the cases of elliptic ones. Some known conclusion can be found in [5].

For any \( \alpha, \beta \in \mathbb{C}, a \in \mathbb{C}^p, D \in \mathbb{C}^{q \times q}, A \in \mathbb{C}^{r \times r} \), let
\[
\tau(z, W) = (z + 2i \langle u, a \rangle + \beta, u + a, Dv, w),
\]
\[
\rho(z, W) = (z + 2i \langle w, c \rangle + \alpha, u, v, Aw).
\]
Easy computation shows that
\[
\tau \circ \rho = \rho \circ \tau.
\]
(2)
**Theorem 4.4** Let $\psi \in LFT(\mathbb{H}^N)$ be parabolic with
\[
\psi (z, u, v, w) = (z + 2i \langle w, c \rangle + b, u, v, Aw),
\]
where $\rho(A) < 1$ and $\|A\| \leq 1$. Let
\[
\exp(M) = A,
\]
and
\[
c_t = \left( I - \exp(M)^H \right)^{-1} \left( I - \exp(tM)^H \right) c,
\]
\[
Q_t = I - \exp(tM)^H \exp(tM),
\]
\[
b_t = tb.
\]

If
1. $M$ is dissipative;
2. for any $t \geq 0$, $t \Im b \geq \lambda^{-t} \langle Q_t^+ c_t, c_t \rangle$,
3. $Q_t Q_t^+ c_t = c_t$,

then $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$.

**Proof.** Let
\[
\psi_t (z, u, v, w) = (u + 2i \langle w, c \rangle + b_t, u, v, \exp(tM) w).
\]
$\psi_t$ is a self-map of $\mathbb{H}^N$ for every $t \geq 0$ according to Theorem 3.4 for any $t \geq 0$. When $t = 0$, we have
\[
c_0 = 0, b_0 = 0, \exp(0M) = E.
\]
Thereby,
\[
\psi_0 (z, u, v, w) = (z, u, v, w).
\]
Direct computation shows that for any $s, t \geq 0$,
\[
\psi_s \circ \psi_t = \psi_t \circ \psi_s = \psi_{s+t}.
\]
That $\psi_t$ converges uniformly on compact subset of $\mathbb{H}^N$ is clear. As a consequence, $\{\psi_t\}$ is a semigroup of $\mathbb{H}^N$. Due to $\psi_1 = \psi$, $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$. ■

We will consider the case that the matrix $B$ showed up at the previous theorem is normal. Up to a conjugation with an automorphism, we may suppose that $B$ is diagonal.
Lemma 4.5 Let
\[ h(u, v, t) = \frac{1 + \exp(-2tu) - 2\exp(-tu)\cos(vt)}{(1 - \exp(-2tu)) t}. \]
Then
\[ h(u, v, t) \leq \lim_{t \to 0^+} h(u, v, t) = \frac{1}{2u} (u^2 + v^2) \]
for any \( u > 0, v \geq 0, t \geq 0 \).

Proof. Rewrite \( h(u, v, t) \) by
\[ h(u, v, t) = \frac{(1 - \exp(-tu))}{(1 + \exp(-tu)) t} + \frac{2\exp(-tu)(1 - \cos(vt))}{(1 - \exp(-2tu)) t} \]
\[ \Delta = h_1(u, t) + h_2(u, v, t). \]
We obtain
\[ \frac{\partial h_1}{\partial t} = \frac{1}{t^2(e^{-tu} + 1)^2} \left( e^{2(-tu)} + 2tue^{-tu} - 1 \right). \]
Let
\[ r(x) = \exp(-2x) + 2x \exp(-x) - 1. \]
Then
\[ r'(x) = -2e^{-x} (x + e^{-x} - 1). \]
Denote by \( x_0 \) any one of the solutions of \( r'(x) = 0 \), then
\[ e^{-x_0} = 1 - x_0, \]
and
\[ r(x_0) = (1 - x_0)^2 + 2x_0 (1 - x_0) - 1 = -x_0^2 \leq 0, \]
For any \( x \in (0, +\infty) \), \( r(x) \leq 0 \) since \( r(0) = 0, \lim_{x \to \infty} r(x) = -1 \) and \( r(x_0) \leq 0 \). As a consequence, \( h_1(u, t) \) is decreasing when \( u \) fixed. Therefore,
\[ h_1(u, t) \leq \lim_{t \to 0^+} h_1(u, t). \]
Rewrite \( h_2 \) by
\[ h_2(u, v, t) = \frac{2t\exp(-tu)}{(1 - \exp(-2tu))} \cdot \frac{(1 - \cos(vt))}{t^2} \]
\[ = \frac{u^2 2tu \exp(-tu)}{v^2} \cdot \frac{(1 - \cos(vt))}{(vt)^2}. \]
One has
\[
\frac{d}{dx} \left( \frac{2x \exp(-x)}{1 - \exp(-2x)} \right) = -2 \frac{e^{-3x}}{(e^{2(-x)} - 1)^2} (xe^{2x} + 1 + x - e^{2x}),
\]
and
\[
\frac{d}{dx} (xe^{2x} + 1 + x - e^{2x}) = 2xe^{2x} - e^{2x} + 1.
\]
For all \( x \geq 0, 2xe^{2x} - e^{2x} + 1 \geq 0 \) because \( 2xe^{2x} - e^{2x} + 1 = 0 \) has only one solution \( x = 0 \) and
\[
\lim_{x \to +\infty} (2xe^{2x} - e^{2x} + 1) = +\infty.
\]
As a result, for any \( x \in (0, +\infty), \)
\[
\frac{d}{dx} \left( \frac{2x \exp(-x)}{1 - \exp(-2x)} \right) \leq 0.
\]
Consequently,
\[
\frac{2x \exp(-x)}{1 - \exp(-2x)} \leq \lim_{x \to 0} \frac{2x \exp(-x)}{1 - \exp(-2x)} = 1.
\]
Moreover,
\[
\frac{1 - \cos x}{x^2} = \frac{2 \sin^2 \frac{x}{2}}{x^2} \leq \frac{2(\frac{x}{2})^2}{x^2} = \frac{1}{2}.
\]
Thus for all \( t > 0, \)
\[
\frac{2tu \exp(-tu)}{(1 - \exp(-2tu))} \leq \lim_{t \to 0} \frac{2tu \exp(-tu)}{(1 - \exp(-2tu))} = 1,
\]
and
\[
\frac{(1 - \cos (vt))}{(vt)^2} \leq \lim_{t \to 0} \frac{(1 - \cos (vt))}{(vt)^2} = \frac{1}{2}.
\]
As a consequence, for any \( t \geq 0, \)
\[
h_2(u, v, t) \leq h_2(u, v, 0).
\]
Therefore, when \( t \geq 0, \) we obtain
\[
h(u, v, t) = \frac{|1 - \exp ((-u + iv) t)|^2}{(1 - \exp(-2ut)) t} \leq \lim_{t \to 0} h(u, v, t) = \frac{1}{2u} (u^2 + v^2).
\]
\[
\square
\]
Corollary 4.6  Let $\psi \in \text{LFT} \left( \mathbb{H}^N \right)$ be parabolic with
\[
\psi(z, u, v, w) = (z + 2i \langle w, c \rangle + b, u, v, Aw)
\]
where $A = \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $|\lambda_j| < 1$ for $j = 1, 2, \cdots, r$. Let $\lambda_j = \exp (-u_j + iv_j)$ where $u_j > 0, v_j \geq 0$ for $j = 1, 2, \cdots, r$. If
\[
\text{Im } b \geq c^H \Theta c
\]
where
\[
\Theta = \text{diag} \left( \frac{1}{2u_1} \frac{u_1^2 + v_1^2}{|1 - \lambda_1|^2}, \cdots, \frac{1}{2u_r} \frac{u_r^2 + v_r^2}{|1 - \lambda_r|^2} \right),
\]
then $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$.

Proof. Let
\[
M = \text{diag} (-u_1 + iv_1, \cdots, -u_r + iv_r).
\]
Then
\[
A = \exp (M).
\]
Denote by
\[
c_t = (I - \exp \left( M^H \right))^{-1} \left( I - \exp \left( tM^H \right) \right) c, \\
Q_t = I - \left( \exp \left( tM \right) \right)^H \exp \left( tM \right), \\
b_t = tb.
\]
Since both $A$ and $M$ are normal matrix and $\|B\| = \|\exp (M)\|$, 
\[
\|\exp (M)\| \leq 1.
\]
According to proposition 2.4 for any $t \geq 0$, 
\[
\|\exp (tM)\| \leq 1.
\]
Therefore $Q_t$ is hermitian positive semi-definite and
\[
Q_t^+ = Q_t^{-1} = 
\begin{bmatrix}
\frac{1}{1-e^{-2u_1}} & & & \\
& \frac{1}{1-e^{-2u_2}} & & \\
& & \ddots & \\
& & & \frac{1}{1-e^{-2u_r}} \\
\end{bmatrix}.
\]
Besides,
\[ c_t = (E - \exp (M^H))^{-1} (E - \exp (tM^H)) c \]
\[ = \begin{bmatrix}
\frac{1 - e^{t(-u_1 - iv_1)}}{1 - e^{-u_1 - iv_1}} & \cdots & \frac{1 - e^{t(-u_r - iv_r)}}{1 - e^{-u_r - iv_r}}
\end{bmatrix} c. \]

As a result,
\[ c_t^H Q_t^+ c_t = c^H \Theta_t c, \]
where
\[ \Theta_t = \text{diag} \left( \frac{1 - e^{t(-u_1 + iv_1)}}{|1 - \lambda_1|^2(1 - e^{-2u_1})}, \cdots, \frac{1 - e^{t(-u_r + iv_r)}}{|1 - \lambda_r|^2(1 - e^{-2u_r})} \right). \]

Denote by
\[ b = (\beta_1, \beta_2, \cdots, \beta_p)^T, \]
then
\[ v_t^H Q_t^+ b_t = \sum_{j=1}^{p} \frac{|1 - e^{t(-u_j + iv_j)}|^2}{|1 - \lambda_j|^2(1 - e^{-2u_j})} |\beta_j|^2. \]

Let
\[ g_{\lambda_j} (t) = \frac{1}{t} \frac{|1 - e^{t(-u_j + iv_j)}|^2}{|1 - \lambda_j|^2(1 - e^{-2u_j})}. \]

According to Lemma 4.5, for \( j = 1, 2, \cdots, r, \)
\[ \sup_{t \geq 0} g_{\lambda_j} (t) = \frac{1}{2u_j} (u_j^2 + v_j^2) \frac{1}{|1 - \lambda_j|^2}. \]

Further more,
\[ \sup_{t \geq 0} \left\{ \frac{1}{t} c_t^H Q_t^+ c_t \right\} = \sum_{j=1}^{p} \frac{1}{2u_j} (u_j^2 + v_j^2) \frac{1}{|1 - \lambda_j|^2} |\beta_j|^2 \]
\[ = c^H \Theta c. \]

Consequently
\[ c_t^H Q_t^+ c_t \leq t \text{ Re } c. \]

Due to Theorem 4.4, \( \psi \) can be embedded into a semigroup of \( \mathbb{H}^N \).
Lemma 4.7 Let $\psi$ be a parabolic automorphism of $\mathbb{H}^N$ with

$$\psi(z, u, v) = \left( z + 2i \langle u, a \rangle + i |a|^2 + b, u + a, Dv \right)$$

where $b$ is a real number and $D$ is diagonal, $\sigma(D) \subset T \setminus \{1\}$. Then $\varphi$ can be embedded into a semigroup of $B_N$.

Proof. Let

$$\psi_t(z, u, v) = \left( z + 2i \langle u, ta \rangle + it^2 |a|^2 + tb, u + ta, \exp(it\Theta)v'' \right),$$

then clearly, for every $t > 0$, $\psi_t$ is an automorphism of $\mathbb{H}^N$ and \{\psi_t\} is a semigroup of $\mathbb{H}^N$. Therefore $\psi$ can be embedded into some semigroup of $B_N$.

Theorem 4.8 Let $\psi \in LFT(\mathbb{H}^N)$ be parabolic with

$$\psi(z, u, v, w) = (z + 2i \langle u, a \rangle + 2i \langle w, c \rangle + b, u + a, Dv, Aw)$$

where $A = \text{diag}(\lambda_1, \ldots, \lambda_r)$ and $|\lambda_j| < 1$ for $j = 1, 2, \ldots, r$. Let $\lambda_j = \exp(-u_j + iv_j)$ where $u_j > 0, v_j \in [0, 2\pi)$ for $j = 1, 2, \ldots, r$. If

$$\text{Im} b - |a|^2 \geq c^H \Theta c$$

where

$$\Theta = \text{diag} \left( \frac{1}{2u_1} \frac{(u_1^2 + v_1^2)}{|1 - \lambda_1|^2}, \ldots, \frac{1}{2u_r} \frac{(u_r^2 + v_r^2)}{|1 - \lambda_r|^2} \right),$$

then $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$.

Proof. Let

$$\tau_{\psi}(z, W) = (z + 2i \langle u, a \rangle + \beta_{\psi}, u + a, Dv, w),$$

$$\rho_{\psi}(z, W) = (z + 2i \langle w, c \rangle + \alpha_{\psi}, u, v, Aw),$$

with $b = \alpha_{\psi} + \beta_{\psi}$ and $\text{Im} (\beta_{\psi}) = |a|^2$, $\alpha_{\psi} \in \mathbb{R}$. Then $\tau_{\psi}$ is a parabolic automorphism and according to Lemma 4.7 $\tau_{\psi}$ can be embedded into $\tau_{\psi,t}$ with

$$\tau_{\psi,t}(z, u, v, w) = \left( z + 2i \langle u, ta \rangle + it^2 |a|^2 + t \text{Re} \beta_{\psi}, u + ta, \exp(t\Theta_D), w \right).$$
According to Corollary 4.6, $\rho_\psi$ can be embedded into a semigroup $\rho_{\psi,t}$ of $\mathbb{H}^N$ with

$$
\rho_{\psi,t}(z,u,v,w) = (z + 2i \langle w, c_t \rangle + \alpha_{\psi,t}, u, v, \exp(t\Theta_A)w)
$$

for some suitable $c_t, \alpha_{\psi,t}$ and $\exp(t\Theta_A)$. Let

$$
\psi_t = \tau_{\psi,t} \circ \rho_{\psi,t},
$$

then

$$
\psi_{t+s} = \tau_{\psi,t+s} \circ \rho_{\psi,t+s}
= \tau_{\psi,t} \circ \tau_{\psi,s} \circ \rho_{\psi,s} \circ \rho_{\psi,t}
= \tau_{\psi,s} \circ \tau_{\psi,t} \circ \rho_{\psi,t} \circ \rho_{\psi,t} = \psi_s \circ \psi_t
= \psi_t \circ \psi_s.
$$

As a consequence, $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$. ■

4.3 The hyperbolic case

Just as Theorem 4.4 in the case of hyperbolic, we have:

**Proposition 4.9** Let $\psi \in \text{LFT}(\mathbb{H}^N)$ with

$$
\psi(z,u,v,w) = (\lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}Aw),
$$

where $\lambda > 1$, $D$ is diagonal and $\sigma(D) \subset T \setminus \{1\}$. Suppose $A$ is non-singular and there exists a matrix $M$ such that

$$
\exp(M) = B.
$$

Denoted by

$$
\lambda_t = \lambda^t,
A_t = \exp(tM),
\alpha_t = \left(\lambda - \sqrt{\lambda}A^H\right)^{-1} \left(\lambda_t - \sqrt{\lambda_t}A^H_t\right) a,
b_t = \frac{1 - \lambda_t}{1 - \lambda},
Q_t = E - A^H_t A_t.
$$

If
1. $Q_t$ is Hermitian positive semi-definite;

2. for any $t \geq 0$,
   \[ \text{Im} b_t \geq \frac{1}{\lambda_t} \langle Q_t^+ a_t, a_t \rangle; \]

3. for any $t \geq 0$, $Q_t Q_t^+ a_t = a_t$,

then $\psi$ can be embedded into a semigroup of $\mathbb{H}^N$.

The proof of the above theorem is just the same with Theorem 4.4, we omit it here.

**Lemma 4.10** Let
\[
h(\lambda, u, v, t) = \frac{1}{1 - e^{-\lambda t}} \cdot \left( \frac{e^{t(u + iv)} - 1}{1 - e^{2ut}} \right)^2.
\]
Then
\[
h(\lambda, u, v, t) \leq \lim_{t \to 0^+} h(\lambda, u, v, t) = -\frac{1}{\lambda(2u + \lambda)} (u^2 + v^2)
\]
for any $\lambda + 2u < 0$, $\lambda > 0$, $v \geq 0$ and $u < 0$.

**Proof.** Since
\[
h(\lambda, u, v, t) = \frac{1}{1 - e^{-\lambda t}} \cdot \frac{(1 - \exp(ut))^2 + 2 \exp(ut)(1 - \cos vt)}{(1 - e^{2ut})},
\]
let
\[
h_1(u, \lambda, t) = \ln \left( \frac{1}{1 - e^{-\lambda t}} \cdot \frac{(1 - \exp(ut))^2}{1 - e^{2ut}} \right),
\]
then
\[
\frac{d}{dt} h_1(u, \lambda, t) = \frac{d}{dt} \ln \left( \frac{1}{1 - e^{-\lambda t}} \cdot \frac{(1 - \exp(ut))^2}{1 - e^{2ut}} \right)
\]
\[
= \frac{1}{t} \left( -\frac{2ut \exp(ut)}{1 - \exp(ut)} + \frac{-\lambda t \exp(-\lambda t)}{1 - \exp(-\lambda t)} + \frac{(\lambda + 2u) t \exp(\lambda + 2u) t}{1 - \exp(\lambda + 2u) t} \right).
\]
Denote
\[
g(x) = \frac{x \exp(x)}{1 - \exp(x)},
\]
then
\[ g''(x) = \frac{d^2}{dx^2} \left( \frac{x \exp(x)}{1 - \exp(x)} \right) = - \frac{e^x}{(e^x - 1)^3} (x - 2e^x + xe^x + 2), \]

thus when \( x \in (-\infty, 0) \), we get
\[ g''(x) \leq 0, \]
as a consequence \( g \) is convex, and
\[ g(ut) \geq \frac{1}{2} (g(-\lambda t) + g((2u + \lambda)t)). \]

Therefore when \( t \in (0, +\infty) \), \( h_1 \) is decrease. Consequently,
\[ \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t}e^{2ut})} \leq \lim_{t \to 0} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t}e^{2ut})} \leq - \frac{2u^2}{\lambda (2u + \lambda)}. \]

Notice that when \( t \in (0, +\infty) \), the inequality
\[ \exp(ut) t^2 \leq (1 - \exp(ut))^2 \]
holds, so
\[ \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) t^2}{(1 - e^{\lambda t}e^{2ut})} \leq - \frac{2u^2}{\lambda (2u + \lambda)}. \]

On the other hand,
\[ \frac{(1 - \cos vt)}{t^2} \leq \lim_{t \to 0} \frac{(1 - \cos vt)}{t^2} = \frac{1}{2} v^2, \]
thus
\[ \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t}e^{2ut})} \leq - \frac{2u^2}{\lambda (2u + \lambda)} \cdot \frac{1}{2} v^2 = - \frac{u^2 v^2}{\lambda (2u + \lambda)}. \]

But
\[ \lim_{t \to 0} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t}e^{2ut})} = - \frac{v^2}{\lambda (2u + \lambda)}. \]
therefore for any \( t \in (0, \infty) \), one has
\[
 h(\lambda, u, v, t) \leq h(\lambda, u, v, 0) = \lim_{t \to 0} \frac{1}{1 - e^{-\lambda t}} \cdot \frac{(1 - \exp(ut))^2 + 2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})} = \frac{-1}{\lambda(2u + \lambda)} (u^2 + v^2).
\]

**Theorem 4.11** Let \( \psi \in LFT(\mathbb{H}^N) \) be hyperbolic with
\[
\psi(z, u, v, w) = \left( \lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda u}, \sqrt{\lambda Dv}, \sqrt{\lambda Aw} \right),
\]
where \( A = \text{diag} (\lambda_1, \cdots, \lambda_r) \) and \( |\lambda_j| < 1 \) for \( j = 1, 2, \cdots, r \). Let
\[
\lambda_j = \exp(-u_j + iv_j)
\]
with \( u_j > 0 \) and \( v_j \in [0, 2\pi) \). If
\[
\text{Im} \, b \geq \langle \Theta a, a \rangle
\]
with
\[
\Theta = \text{diag} \left( \frac{\ln \lambda + u_1}{2u_1 \ln \lambda} \frac{1}{|\lambda - \sqrt{\lambda} \lambda_1|^2}, \cdots, \frac{\ln \lambda + u_r}{2u_r \ln \lambda} \frac{1}{|\lambda - \sqrt{\lambda} \lambda_r|^2} \right),
\]
then \( \psi \) can be embedded into a semigroup of \( \mathbb{H}^N \).

**Proof.** Let
\[
M = \text{diag} (-u_1 + iv_1, \cdots, -u_r + iv_r),
\]
then
\[
A = \exp(M).
\]
Denote by
\[
\lambda_t = \lambda^t, \quad A_t = \exp(tM),
\]
\[
a_t = \left( \lambda - \sqrt{\lambda} A^H_t \right)^{-1} \left( \lambda_t - \sqrt{\lambda} A^H_t \right) a,
\]
\[
b_t = \frac{1 - \lambda t}{1 - \lambda} b,
\]
\[
Q_t = E - A^H_t A_t.
\]
Then
\[ \langle Q^+ t, a_t \rangle = a^H \text{diag} (\alpha_1 (t), \ldots, \alpha_s (t)) a, \]
where
\[ \alpha_j (t) = \frac{\left| \lambda^t - \sqrt{\lambda} e^{t(-u_j - iv_j)} \right|^2}{(1 - e^{-2t u_j}) \left| \lambda - \sqrt{\lambda} e^{-u_j - iv_j} \right|^2} = \frac{\lambda^2 t \left| 1 - e^{-t\left( \frac{\ln \lambda}{2} + u_j \right) + iv_j} \right|^2}{(1 - e^{-2t u_j}) \left| \lambda - \sqrt{\lambda} \lambda_j \right|^2}. \]

Notice that according to Lemma [4.10] for \( \frac{\ln \lambda}{2} + u_j > 0, v_j \geq 0 \) and \( t \geq 0, \)
\[ \lambda^t \left| 1 - e^{-t\left( \frac{\ln \lambda}{2} + u_j \right) + iv_j} \right|^2 = \frac{1}{\ln \lambda \cdot (-2u_j)} \left[ \left( \frac{\ln \lambda}{2} + u_j \right)^2 + v_j^2 \right] \]
\[ \leq \frac{1}{2u_j \ln \lambda} \left[ \left( \frac{\ln \lambda}{2} + u_j \right)^2 + v_j^2 \right], \]
Thus we get
\[ \frac{1}{\lambda^t (\lambda^t - 1)} \langle Q^+ t, a_t \rangle \leq b. \]

Our conclusion follows from Proposition [4.9] □

4.4 The case of dimensional 2 and automorphisms

When considering our question on \( \mathbb{C}^2 \), our conclusion would looked very simple.

Let \( \varphi \in \text{LFT} (B_2) \) be parabolic, then \( \varphi \) is conjugated to
\[ \psi_1 (u_1, u_2) = (u_1 + 2ibu_2 + c, \lambda u_2) \]
or
\[ \psi_2 (u_1, u_2) = (u_1 + c, e^{i\theta} u_2) \]
or
\[ \psi_3 (u_1, u_2) = (u_1 + 2iau_2 + c, u_2 + a), \]
where \( a, b, c \in \mathbb{C} \) and \( \lambda \in (0, 1) \). In the first case, let
\[ \lambda = \exp (-\mu + iv) \]
where $\mu > 0$ and $v \in [0, 2\pi)$. If

$$\text{Im}\ c \geq \frac{|b|^2 (\mu^2 + v^2)}{\mu |1 - \lambda|^2}.$$  

Then $\varphi$ can be embedding into a semigroup of $\mathbb{B}_2$. Since $\psi_2$ and $\psi_3$ can always be embedded into a semigroup, in the second and the third cases, $\varphi$ can always be embedded into a semigroup.

When $\varphi \in LFT(B_2)$ is hyperbolic, according to the proof of Theorem 3.6, $\varphi$ is conjugated to

$$\psi_1 (u_1, u_2) = \left(\lambda u_1 + 2i \langle u_2, b \rangle + c, \sqrt{\lambda} \alpha u_2 \right)$$

or

$$\psi_2 (u_1, u_2) = (\lambda u_1 + a, u_2 + b).$$

Let $\alpha = e^{\beta + i\gamma}$. $\psi_1$ can be embedded into a semigroup of $\mathbb{H}^2$ if

$$\text{Im}\ c \geq \frac{\lambda - 1}{2\beta \ln \lambda} \left(\frac{\ln \lambda}{\lambda} + \beta \right)^2 + \frac{\gamma^2}{|b|^2}.$$  

When concern about $\psi_2$, we have the following theorem:

**Theorem 4.12** Let $\psi \in LFT(\mathbb{H}^2)$ with

$$\psi (u_1, u_2) = (\lambda u_1 + a, u_2 + b).$$

If

$$\text{Im}\ a \geq \frac{(\lambda - 1)}{\ln^2 \lambda} |b|^2,$$

then $\psi$ can be embedded into a semigroup of $\mathbb{H}^2$.

**Proof.** It is easy to verify that

$$\frac{d}{dt} \left(\frac{\lambda^t - 1}{t}\right) = \frac{1}{t^2} \left(\lambda^t \ln \lambda - \lambda^t + 1\right)$$

and

$$\frac{d}{dt} \left(\lambda^t \ln \lambda - \lambda^t + 1\right) = t \lambda^t \ln^2 \lambda,$$

thus for any $t > 0$, we obtain

$$\frac{\lambda^t - 1}{t} \geq \lim_{t \to 0} \frac{\lambda^t - 1}{t} = \ln \lambda.$$
As a consequence,
\[ \frac{t^2}{(\lambda t - 1)^2} \leq \frac{1}{\ln^2 \lambda} \]

Let
\[ \psi_t(u_1, u_2) = \left( \lambda^t u_1 + \frac{\lambda^t - 1}{\lambda - 1} a, u_2 + tb \right), \]

For any \( t \geq 0 \),
\[ \text{Im} \left( \frac{\lambda^t - 1}{\lambda - 1} a \right) \geq \frac{\lambda^t - 1}{\lambda - 1} \cdot \frac{\lambda - 1}{\ln^2 \lambda} |b|^2 \geq |tb|^2. \]

According to Theorem 3.6, \( \psi_t \) is a self-map of \( \mathbb{H}^2 \) and thus \( \{ \psi_t \} \) is a semi-group of \( \mathbb{H}^2 \). Hence \( \psi \) can be embedded into a semigroup of \( \mathbb{H}^2 \). ■

When \( \varphi \) is an elliptic automorphism, \( \varphi \) is conjugated to a unitary transformation of \( B_N \) and therefore \( \varphi \) can always be embedded into a semigroup of \( B_N \). In the non-elliptic case, if \( \varphi \) is a parabolic automorphism with at least one invariant slice, then by Theorem 4.3 of [5], there exists a unitary matrix \( U \), such that \( \varphi \) is conjugated to
\[ \psi(u', u'') = (z' + ic, Uz''), \]
where \( \alpha \) is the boundary dilation coefficient of \( \varphi \), \( c \in \mathbb{R} \), thus by Theorem 4.6 \( \varphi \) can be embedded into a semigroup. If \( \varphi \) is a hyperbolic automorphism, then by Proposition 3.9 \( \varphi \) is conjugated to
\[ \psi_1(u', u'') = \frac{1}{\alpha} (z' + ic, \sqrt{\alpha} Uz''), \]
where \( U \) is a unitary matrix, thus by Theorem 4.9 \( \varphi \) can be embedded into a semigroup of \( B_N \). Together with Lemma 4.7, we obtain

**Corollary 4.13** Let \( \varphi \) be an automorphism of \( B_N \), then \( \varphi \) can be embedded into a semigroup of \( B_N \).

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