Ramanujan and Quantum Black Holes

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Abstract: Explorations of quantum black holes in string theory have led to fascinating connections with the work of Ramanujan on partitions and mock theta functions, which in turn relate to diverse topics in number theory and enumerative geometry. This article aims to explain the physical significance of these interconnections.

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1 Quantum Black Holes

A classical black hole is the region of spacetime which cannot send signals to faraway observers. It is black because even light cannot escape its strong gravity. To make these notions precise, consider the prototypical Schwarzschild spacetime with line element [1]

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(1.1)

where \( t \) is time and \( r, \theta, \phi \) are spherical coordinates. This pseudo-Riemannian metric is Ricci flat and hence satisfies Einstein equations without matter. The region \( r \leq 2M \) is the Schwarzschild black hole of mass \( M \). The boundary of this region at \( r = 2M \) is called the event horizon. It is a peculiar 3-surface that is both stationary (independent of time) and lightlike (its conormal \( dr \) has vanishing norm). Light emanating from inside the black hole cannot escape to faraway region at large \( r \) because it cannot overtake the event horizon moving at the speed of light.

One consequence of this unusual causal structure is the discovery of Bekenstein and Hawking [2, 3] that a black hole carries entropy \( S(M) \) given by

\[ S(M) = \frac{A(M)}{4}. \]  

(1.2)

This remarkable formula, valid in the limit of large area, implies a surprising connection between thermodynamics, geometry, and quantum mechanics. It poses two important questions.

1. In quantum theory, a system with entropy \( S(M) \) and mass \( M \) corresponds to an ensemble of vectors in an eigensubspace \( \mathcal{H}(M) \) of dimension \( d(M) \) with mass eigenvalue \( M \). Can one associate such a subspace with a black hole in the Hilbert space of quantum gravity and compute its dimension?
2. Can one define and compute [4, 5] a quantum generalization of (1.2) that is valid even for small area? Schematically, it is expected to have the form

\[ S = a_0 A + a_1 \log(A) + \frac{a_2}{A} + \ldots b_0 e^{-c_0 A} + \ldots \] (1.3)

with some coefficients \((a_0, b_0, c_0 \ldots)\).

Both questions have rich implications and have provided invaluable clues in the search for quantum gravity.

String theory offers the most promising approach to a consistent quantum theory of gravity. Equations of motion of string theory are generalizations of Einstein equations. Some of the ‘supersymmetric’ solutions of these equations are ten-dimensional product manifolds \(M_{10} = M_4(Q) \times \Sigma_6\) where \(\Sigma_6\) is a compact Ricci-flat six-dimensional manifold. The manifold \(M_4(Q)\) contains a black hole with a metric analogous to (1.1) but now specified by a vector of integral charges \(Q\) so that the mass is determined by \(Q\). The entropy \(S(Q)\) of this special class of black holes is determined entirely in terms of the charges and geometric properties of \(M_4(Q)\) by a formula analogous to (1.3).

The fundamental physical significance of entropy of these black holes stems from the Boltzmann relation

\[ d(Q) = \exp[S(Q)] \] (1.4)

which links a macroscopic geometric property of spacetime to the underlying microscopic Hilbert space of quantum gravity. It provides a window into the quantum structure of spacetime in much the same way entropy of gases revealed the quantum structure of matter.

Within the framework of string theory, a black hole in \(M_4(Q)\) corresponds to a multi-dimensional ‘membrane’ wrapping a homology cycle in \(\Sigma_6\). The Hilbert space \(\mathcal{H}(Q)\) is the space of states of this membrane. The integers \(d(Q)\), sometimes called the degeneracy, are then given by certain enumerative invariants of \(\Sigma_6\), which in turn are related to problems in combinatorics and number theory. This line of enquiry leads naturally to the work of Ramanujan as we illustrate below.

### 2 Colored Partitions

A simple example is \(\Sigma_6 = K3 \times T^2\) where \(K3\) is a Ricci-flat 4-manifold (‘Kummer surface’) and \(T^2\) is a 2-torus. The integer \(d(Q)\) in this case equals the Euler character of the symmetric product of \(n\) copies of \(K3\), where \(n\) is a particular integral norm of the vector \(Q\). Recall that the Euler character \(\chi_1\) of a single copy \(K3\) is 24 and \(K3\) has only even cohomology, that is, it admits only even harmonic forms. Given this topological data, the problem of computing the Euler character \(\chi_n\) of \(Sym^n(K3)\) is equivalent [6, 7] to the problem of finding the number of partitions \(p_{24}(n)\) of a positive integer \(n\) using integers of 24 different colors, a problem close to Ramanujan’s work. The solution is given [8] in terms of \(q\)-coefficients of a partition function:

\[ Z(\tau) = \frac{1}{\Delta(\tau)}, \quad (q := e^{2\pi i \tau}) \] (2.1)
where $\Delta(\tau)$ is the Ramanujan cusp form of weight 12. The partition function is thus a modular form of weight $-12$:

$$Z\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-12} Z(\tau)$$

for all

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}) ,$$

and admits a Fourier expansion

$$Z(\tau) = \sum_{n=0}^{\infty} C(n) q^n ,$$

It is easy combinatorics to see that

$$p_{24}(n) = C(n) \quad (n > 0) .$$

### 3 Hardy-Ramanujan Formula

The modular properties of $Z(\tau)$ imply that $C(n)$ admits the Hardy-Ramanujan-Rademacher expansion [9] and equals

$$\sum_{c=1}^{\infty} \left(\frac{2\pi}{c}\right)^{14} K(n, -1, c) I_{13} \left(\frac{z}{c}\right)$$

with $z = 4\pi \sqrt{n}$, where

$$I_{13}(z) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{t^{14}} \exp[t + \frac{z^2}{4t}]$$

is a modified Bessel function and

$$K(n, m, c) := \sum_{d \in \mathbb{Z}/c\mathbb{Z} \atop da \equiv 1 \mod c} e^{2\pi i \left(\frac{n^2}{c} + \frac{ma}{c}\right)}$$

is the Kloosterman sum for $n, m, c \in \mathbb{Z}$.

For a large class of other black holes, the integers $d(Q)$ are similarly related [10] to more complicated invariants such as the Donaldson-Thomas invariants. They are given in terms of Fourier coefficients of a variety of modular objects such as Jacobi or Siegel forms, and admit an expansion [11, 12] that generalizes (3.2).

Remarkably, the Hardy-Ramanujan formula is an exact convergent expansion of an integer in terms of analytic functions. This is just what is needed to verify the Boltzmann relation (1.4) where $d(Q)$ is an integer given by a counting problem but $S(Q)$ is an analytic function determined by the geometry of spacetime. Its leading asymptotics valid for large charges has exponential growth. For example, the large $z$ asymptotics gives

$$d(n) \sim I(z) \sim e^z \sim e^{4\pi \sqrt{n}} , \quad n \gg 1 \quad (3.2)$$
This is in accordance with leading, large area entropy of black holes and the Boltzmann relation. Using methods of supersymmetric localization it has become possible to compute both sides of equation (1.4) in a number of examples \cite{13–18} to find nontrivial agreement. This verification of Boltzmann relation even beyond the large area approximation constitutes one of the important successes of string theory.

4 Mock Jacobi Forms

In an interesting class of examples, the partition function $Z(\tau, z)$ depends on an additional variable $z$. It is modular in $\tau$ and transforms like a meromorphic Jacobi form of index $m$ under the ‘elliptic’ transformations

$$z \rightarrow z + \lambda \tau + \mu, \quad \lambda, \mu \in \mathbb{Z},$$

with double poles at $z = 0$ and its images. The Fourier coefficients $d(Q)$ are no longer uniquely defined but depend on the choice of the $z$-contour. This seems to contradict (1.4) because entropy of $S(Q)$ of the corresponding black hole is uniquely defined and does not suffer from any ambiguities.

The resolution of this puzzle has to do with the existence of multi-centered black holes. Equations of string theory admit a two-centered black hole solution that depends on the moduli of $\Sigma_6$. Locally, each center looks like a single black hole but the two centers are bound together. The distance between them is fixed by the charges and the moduli, and diverges as one approaches a co-dimension one ‘wall’ in the moduli space. The solution no longer exists on the other side of the wall. For a given charge $Q$, the enumerative ‘invariants’ $d(Q, \mu)$ now have a mild dependence on the moduli. They are invariant in a given chamber in the moduli space but jump upon crossing a wall separating two chambers because on one side the counting includes the two-centered black holes but on the other side it does not. This is known as the wall-crossing phenomenon \cite{19, 20}.

The ambiguity in defining the Fourier coefficients of $Z(\tau, z)$ now has a nice physical interpretation \cite{21–23}. The choice of the Fourier contour depends on the moduli $\mu$ of $\Sigma_6$ in a specific way. Crossing a wall in the moduli space corresponds to crossing a pole of the Fourier integrand. The residue at the pole gives the difference in the enumerative invariants $d(Q, \mu)$ on the two sides of the wall.

There is a special attractive chamber in the moduli space which admits only single-centered black holes as solutions. Let us denote the moduli in this chamber by $\mu^*$. One can now pose a more refined question whether one can compute $d(Q, \mu^*)$ and if it satisfies (1.4). The answer to this question naturally leads us into the realm of mock modular forms \cite{24} and is provided by the following theorem \cite{25}. The partition function admits a unique decomposition

$$Z(\tau, z) = Z^F(\tau, z) + Z^P(\tau, z),$$

such that the polar part has the form

$$Z^P(\tau, z) = \frac{p_{24}(m + 1)}{\Delta(\tau)} A_m(\tau, z)$$

(4.3)
where
\[
A_m(\tau, z) = \sum_{s \in \mathbb{Z}} q^{ms^2 + sy} y^{2ms + 1} (1 - q^s y)^2 \tag{4.4}
\]
is called an Appell-Lerch sum. It admits a completion obtained a correction term that is nonholomorphic in \( \tau \):
\[
\hat{A}(\tau, z) = A(\tau, z) + A^*_m(\tau, z) \tag{4.5}
\]
which transforms as a Jacobi form. The completion satisfies
\[
\sqrt{8\pi i m} \tau^{3/2} \frac{\partial}{\partial \bar{\tau}} \hat{A}_m(\tau, z) \tag{4.6}
\]
\[= - \sum_{\ell \mod 2m} \vartheta_{m, \ell}(\tau) \vartheta_{m, \ell}(\tau, z).\]

It also implies that \( Z^F(\tau, z) \) by itself is not modular but admits a modular completion \( \hat{Z}^F(\tau, z) \) which transforms like a Jacobi form and satisfies a ‘holomorphic anomaly equation’ similar to (4.6). Such an object is called a (mixed) mock Jacobi form. It turns out that Ramanujan’s mock theta functions are closely related to another type of mock Jacobi forms very similar to the ones that appear in the context of quantum black holes.

Using these ingredients one obtains a beautifully consistent picture [25]. The degeneracy \( d(Q, \mu^*) \) of single-centered black holes is given by the Fourier coefficients of the mock Jacobi form which are independent of the moduli. They are uniquely defined by the charges and satisfy the Boltzmann relation (1.4). The degeneracy of multi-centered black holes is given by the Fourier coefficients of \( Z^F(\tau, z) \). These do depend on the choice of the contour and hence indirectly on the moduli, and jump upon crossing walls in the moduli space consistent with the wall-crossing phenomenon.

It is remarkable that the mathematical ideas and tools created by Ramanujan a century ago have now come to have deep applications in quantum gravity.

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