Research Article

An Existence Theorem for Fractional $q$-Difference Inclusions with Nonlocal Substrip Type Boundary Conditions

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Received 2 August 2014; Revised 22 September 2014; Accepted 22 September 2014

Academic Editor: Erdal Karapinar

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By employing a nonlinear alternative for contractive maps, we investigate the existence of solutions for a boundary value problem of fractional $q$-difference inclusions with nonlocal substrip type boundary conditions. The main result is illustrated with the aid of an example.

1. Introduction

In this paper, we consider the following boundary value problem of fractional $q$-difference inclusions with nonlocal and substrip type boundary conditions:

$$cD^\nu_q x(t) \in F(t, x(t)), \quad t \in [0,1], \quad 1 < \nu \leq 2, \quad 0 < q < 1,$$

$$x(0) = g(x), \quad x(\omega) = b \int_\delta^\omega x(s) \, dq^s, \quad 0 < \omega < \delta < 1,$$

where $cD^\nu_q$ denotes the Caputo fractional $q$-derivative of order $\nu$, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a given continuous function, and $b$ is a real constant. Here, we emphasize that the nonlocal conditions are regarded as more plausible than the standard initial conditions for the description of some physical phenomena. In (I), $g(x)$ may be understood as $g(x) = \sum_{j=1}^p \alpha_j x(t_j)$, where $\alpha_j$, $j = 1, \ldots, p$, are given constants and $0 < t_1 < \cdots < t_p \leq 1$. For more details, we refer to the work by Byszewski and Lakshmikantham [1, 2].

Boundary value problems with integral boundary conditions constitute an important class of problems and arise in the mathematical modeling of various phenomena such as heat conduction, wave propagation, gravitation, chemical engineering, underground water flow, thermoelasticity, and plasma physics. They include two-point, three-point, multipoint, and nonlocal boundary value problems.

The topic of fractional differential equations has been of great interest for many researchers in view of its theoretical development and widespread applications in various fields of science and engineering such as control, porous media, electromagnetic, and other fields [3, 4]. For some recent results and applications, we refer the reader to a series of papers ([5–13]) and the references cited therein.

Fractional $q$-difference ($q$-fractional) equations are regarded as fractional analogue of $q$-difference equations. Motivated by recent interest in the study of fractional-order differential equations, the topic of $q$-fractional equations has attracted the attention of many researchers. The details of some recent work on the topic can be found in ([14–20]). For notions and basic concepts of $q$-fractional calculus, we refer to a recent text [21].

The present work is motivated by a recent paper of the authors [22], where the problem (I) was considered for a single valued case. To the best of our knowledge, this is the first paper dealing with fractional $q$-difference inclusions in the given framework. Moreover, the main result of our paper can be regarded as an improvement and extension of some known results; see, for instance, [18, 19].

The paper is organized as follows. Section 2 contains some fundamental concepts of fractional $q$-calculus. In Section 3, we show the existence of solutions for the problem (I) by
means of the nonlinear alternative for contractive mappings. Finally, an example illustrating the applicability of our result is presented.

2. Preliminaries
First of all, we recall the notations and terminology for q-fractional calculus [21, 23, 24].

For a real parameter $q \in \mathbb{R} \setminus \{1\}$, a $q$-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \quad (2)$$

The $q$-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a; q)_0 = 1, \quad (a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i), \quad k \in \mathbb{N} \cup \{\infty\}. \quad (3)$$

The $q$-analogue of the exponent $(x - y)^k$ is

$$(x - y)^{(0)} = 1, \quad (x - y)^{(k)} = \prod_{j=0}^{k-1} (x - yq^j). \quad (4)$$

The $q$-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1 - q)^{(y-1)}}{1 - q} \Gamma(y), \quad (5)$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y + 1) = [y]_q \Gamma_q(y)$. 

Definition 1 (see [23]). Let $f$ be a function defined on $[0, 1]$. The fractional $q$-integral of the Riemann-Liouville type of order $\beta \geq 0$ is

$$I_q^\beta f(t) = \int_0^t \frac{(t - s)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) \, dq \, s$$

is equivalent to an integral equation:

$$I_q^\beta I_q^\gamma f(t) = I_q^{\beta + \gamma} f(t); \quad \gamma, \beta \in \mathbb{R}^+. \quad (8)$$

Further, it has been shown in Lemma 6 of [24] that

$$I_q^\beta ((x)^{(\sigma)}) = \frac{\Gamma_q(\sigma + 1)}{\Gamma_q(\beta + \sigma + 1)} (x)^{(\beta + \sigma)}, \quad (9)$$

$$0 < x < a, \quad \beta \in \mathbb{R}^+, \quad \sigma \in (-1, \infty).$$

Before giving the definition of fractional $q$-derivative, we recall the concept of $q$-derivative.

We know that the $q$-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0, \quad (10)$$

$$(D_q f)(0) = \lim_{t \to 0^+} (D_q f)(t).$$

Furthermore,

$$D_q^n f = f, \quad D_q^n f = D_q (D_q^{n-1} f), \quad n = 1, 2, 3, \ldots. \quad (11)$$

Definition 3 (see [21]). The Caputo fractional $q$-derivative of order $\beta > 0$ is defined by

$${}^c D_q^\beta f(t) = I_q^{[\beta] - \beta} D_q^\beta f(t), \quad (12)$$

where $[\beta]$ is the smallest integer greater than or equal to $\beta$.

Next, we recall some properties involving Riemann-Liouville $q$-fractional integral and Caputo fractional $q$-derivative ([21, Theorem 5.2]):

$$I_q^\beta \, {}^c D_q^\beta f(t) = f(t) - \sum_{k=0}^{[\beta]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0^+), \quad \forall t \in (0, a], \quad \beta > 0; \quad (13)$$

$${}^c D_q^\beta I_q^\beta f(t) = f(t), \quad \forall t \in (0, a], \quad \beta > 0.$$  

Lemma 4 (see [22]). Let $y \in C([0, 1], \mathbb{R})$. Then, the following problem

$${}^c D_q^\nu x(t) = y(t), \quad 1 < \nu \leq 2,$$

$$x(0) = y_0, \quad x(\omega) = b \int_0^1 x(s) \, dq \, s, \quad x_0 \in \mathbb{R}, \quad t \in [0, 1], \quad (14)$$

is equivalent to an integral equation:

$$x(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t \frac{(t - s)^{(\nu-1)}}{\Gamma_q(\nu)} y(s) \, dq \, s$$

$$+ \frac{t}{\Gamma_q(\nu)} \int_0^1 \left( \int_0^t (s - q\omega)^{(\nu-1)} y(u) \, dq \, u \right) \, dq \, s$$

$$- \int_0^\omega (\omega - q\omega)^{(\nu-1)} y(s) \, dq \, s$$

$$+ y_0 \left[ 1 + \frac{t}{\Gamma_q(\nu)} (b (1 - \delta) - 1) \right], \quad (15)$$
where
\[ \theta = \omega - \frac{b(1 - \delta^2)}{1 + q} \neq 0. \]  

3. Existence Results

First of all, we outline some basic definitions and results for multivalued maps [25, 26].

For a normed space \( (X, \| \cdot \|) \), let \( \mathcal{P}_c(X) = \{ Y \in \mathcal{P}(X) : Y \) is closed \}, \( \mathcal{P}_b(X) = \{ Y \in \mathcal{P}(X) : Y \) is bounded \}, \( \mathcal{P}_p(X) = \{ Y \in \mathcal{P}(X) : Y \) is compact and convex \}, \( \mathcal{P}_{cp,c}(X) = \{ Y \in \mathcal{P}(X) : Y \) is compact and convex \}. A multivalued map \( G : X \to \mathcal{P}(X) \)

(i) is convex (closed) valued if \( G(x) \) is convex (closed) for all \( x \in X \);

(ii) is bounded on bounded sets if \( G(\mathbb{B}) = \cup \{ G(x) : x \in \mathbb{B} \} \) is bounded in \( X \) for all \( \mathbb{B} \in \mathcal{P}_b(X) \) (i.e., \( \sup_{x \in \mathbb{B}} |G(x)| < \infty \));

(iii) is called upper semicontinuous (u.s.c.) on \( X \) if, for each \( x_0 \in X \), the set \( G(x_0) \) is a nonempty closed subset of \( X \) and, for each open set \( N \) of \( X \) containing \( G(x_0) \), there exists an open neighborhood \( N_0 \) of \( x_0 \) such that \( G(N_0) \subseteq N \);

(iv) is said to be completely continuous if \( G(\mathbb{B}) \) is relatively compact for every \( \mathbb{B} \in \mathcal{P}_b(X) \);

(v) is said to be measurable if, for every \( y \in \mathbb{R} \), the function

\[ t \mapsto d \left( y, G(t) \right) = \inf \{ |y - z| : z \in G(t) \} \]  

is measurable;

(vi) has a fixed point if there is \( x \in X \) such that \( x \in G(x) \). The fixed point set of the multivalued operator \( G \) will be denoted by \( \text{Fix} G \).

For each \( x \in C([0,1], \mathbb{R}) \), define the set of selections of \( F \) by

\[ S_{F,x} := \{ v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1] \} \].  

Definition 5. A multivalued map \( F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is said to be a Carathéodory function if

(i) \( t \mapsto F(t, x) \) is measurable for each \( x \in \mathbb{R} \);

(ii) \( x \mapsto F(t, x) \) is upper semicontinuous for almost all \( t \in [0,1] \).

Further, a Carathéodory function \( F \) is called \( L^1 \)-Carathéodory if

(iii) for each \( \alpha > 0 \), there exists \( \varphi_a \in L^1([0,1], \mathbb{R}^+) \) such that

\[ \| F(t,x) \| \leq \varphi_a(t) \]  

for all \( |x| \leq \alpha \) and for a.e. \( t \in [0,1] \).

We define the graph of \( G \) to be the set \( \text{Gr}(G) = \{(x, y) \in X \times Y, y \in G(x)\} \) and recall two results for closed graphs and upper-semicontinuity.

Lemma 6 (see [25, Proposition 1.2]). If \( G : X \to \mathcal{P}_d(Y) \) is u.s.c., then \( \text{Gr}(G) \) is a closed subset of \( X \times Y \), that is, for every sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq X \) and \( \{y_n\}_{n \in \mathbb{N}} \subseteq Y \), if \( n \to \infty \), \( x_n \to x_* \), \( y_n \to y_* \), and \( y_n \in G(x_n) \), then \( y_* \in G(x_*) \). Conversely, if \( G \) is completely continuous and has a closed graph, then it is upper semicontinuous.

Lemma 7 (see [27]). Let \( X \) be a separable Banach space. Let \( F : [0,1] \times X \to \mathcal{P}_{cp,c}(X) \) be an \( L^1 \)-Carathéodory function. Then, the operator

\[ \Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \]

\[ x \mapsto (\Theta \circ S_F)(x) = \Theta \left( S_{F,x} \right) \]

is a closed graph operator in \( C([0,1], X) \times C([0,1], X) \).

Definition 8. A function \( f \in AC^1([0,1], \mathbb{R}) \) is called a solution of problem (1) if there exists a function \( \Psi \in L^1([0,1], \mathbb{R}) \) with \( f(t) = f(t, x(t)) \), a.e. on \( [0,1] \) such that \( D^1 \Psi(x(t)) = f(t) \), a.e. on \( [0,1] \) and \( x(0) = g(x) \) and \( x(\omega) = b \int_0^t \psi(s) \, ds \).

To prove our main result in this section we will use the following form of the nonlinear alternative for contractive maps [28, Corollary 3.8].

Theorem 9. Let \( X \) be a Banach space and let \( D \) be a bounded neighborhood of \( 0 \in X \). Let \( \chi_1 : X \to \mathcal{P}_{cp,c}(X) \) and \( \chi_2 : \overline{D} \to \mathcal{P}_{cp,c}(X) \) be two multivalued operators such that

(a) \( \chi_1 \) is contraction,

(b) \( \chi_2 \) is u.s.c and compact.

Then, if \( \chi = \chi_1 + \chi_2 \), either

(i) \( \chi \) has a fixed point in \( \overline{D} \)

(ii) there is a point \( u \in \partial D \) and \( \lambda \in (0,1) \) with \( u \in \lambda \chi(u) \).

Theorem 10. Assume that

(H1) \( F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R}) \) is \( L^1 \)-Carathéodory multivalued map;

(H2) there exists a continuous nondecreasing function \( \psi : [0, \infty) \to (0, \infty) \) and a function \( p \in C([0,1], \mathbb{R}^+) \) such that

\[ \| F(t,x) \| := \sup \{ |y| : y \in F(t,x) \} \leq p(t) \psi(\|x\|) \]

for each \( (t,x) \in [0,1] \times \mathbb{R} \);

(H3) \( g : C([0,1], \mathbb{R}) \to \mathbb{R} \) is a continuous function satisfying the condition

\[ |g(u) - g(v)| \leq \ell \| u - v \| \]

for all \( u, v \in C([0,1], \mathbb{R}), \ell > 0 \).

(22)
there exists a number $M > 0$ such that
\[ \frac{(1 - \ell k_0) M}{\|p\| \psi (M) \mu_0} > 1, \]  
(23)
with $\ell k_0 < 1$, where
\[ \mu_0 := \frac{1}{\Gamma_q (v + 1)} + \frac{1}{|\delta|} \left( \frac{|b| (1 - \delta + i)}{\Gamma_q (v + 2)} + \frac{\omega^\nu}{\Gamma_q (v + 1)} \right), \]
(24)
\[ k_0 := 1 + \frac{1}{|\delta|} |b| (1 - \delta) - 1|.
Then, the problem (1) has at least one solution on $[0, 1]$.

Proof. To transform the problem (1) into a fixed point problem, we define an operator $F : C([0, 1], \mathbb{R}) \to P(C([0, 1], \mathbb{R}))$ as
\[ F (x) = \left\{ h \in C ([0, 1], \mathbb{R}) : h (t) = \int_0^t \left( t - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \right. \]
\[ + \frac{t}{\delta} \left\{ b \int_0^1 \left( \int_0^t \left( s - qu \right)^{(v-1)} \Gamma_q (v) f (u) du \right) du \right. \]
\[ - \int_0^w \left( \omega - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \left\} \right. \]
\[ + \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \]
\[ t \in [0, 1], f \in S_{F, x}. \]
(25)

Next, we introduce two operators $F_1 : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ and $F_2 : C([0, 1], \mathbb{R}) \to P(C([0, 1], \mathbb{R}))$ as follows:
\[ F_1 (x) (t) = \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \quad t \in [0, 1], \]
\[ F_2 (x) = \left\{ h \in C ([0, 1], \mathbb{R}) : h (t) = \int_0^t \left( t - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \right. \]
\[ + \frac{t}{\delta} \left\{ b \int_0^1 \left( \int_0^t \left( s - qu \right)^{(v-1)} \Gamma_q (v) f (u) du \right) du \right. \]
\[ - \int_0^w \left( \omega - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \left\} \right. \]
\[ + \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \]
\[ t \in [0, 1], f \in S_{F, x}. \]
(26)

Observe that $F = F_1 + F_2$. We will show that the operators $F_1$ and $F_2$ satisfy all the conditions of Theorem 9 on $[0, 1]$. For the sake of clarity, we split the proof into a number of steps and claims.

Step 1. $F_1$ is a contraction on $C([0, 1], \mathbb{R})$. This is a consequence of (H3). Indeed, we have
\[ \|F_1 (x) - F_1 (y)\| \leq \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] \|g (x) - g (y)\| \]
\[ + \frac{1}{|\delta|} \left| b \right| \|f (u)\| du \right) du \right. \]
\[ - \int_0^w \left( \omega - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \left\} \right. \]
\[ + \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \]
\[ t \in [0, 1], f \in S_{F, x}. \]
(25)

Next, we introduce two operators $F_1 : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ and $F_2 : C([0, 1], \mathbb{R}) \to P(C([0, 1], \mathbb{R}))$ as follows:
\[ F_1 (x) (t) = \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \quad t \in [0, 1], \]
\[ F_2 (x) = \left\{ h \in C ([0, 1], \mathbb{R}) : h (t) = \int_0^t \left( t - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \right. \]
\[ + \frac{t}{\delta} \left\{ b \int_0^1 \left( \int_0^t \left( s - qu \right)^{(v-1)} \Gamma_q (v) f (u) du \right) du \right. \]
\[ - \int_0^w \left( \omega - qs \right)^{(v-1)} \Gamma_q (v) f (s) ds \left\} \right. \]
\[ + \left[ 1 + \frac{t}{\delta} (b (1 - \delta) - 1) \right] g (x), \]
\[ t \in [0, 1], f \in S_{F, x}. \]
(26)
Let \( t_1, t_2 \in [0, 1] \) with \( t_1 < t_2 \) and \( x \in B_\rho \). Then, for each \( h \in \mathcal{F}_2(x) \), we obtain

\[
\left| (\mathcal{F}_1x)(t_2) - (\mathcal{F}_1x)(t_1) \right|
\leq \frac{1}{\Gamma_q(\nu)} \int_0^t \left[ (t_2 - qs)^{(\nu-1)} - (t_1 - qs)^{(\nu-1)} \right] |f(s)| \, ds
\]

\[
+ \frac{1}{\Gamma_q(\nu)} \int_{t_1}^{t_2} (t_2 - qs)^{(\nu-1)} |f(s)| \, ds
\]

\[
+ \frac{\|\rho\|}{\Gamma_q(\nu)} \int_0^t \left[ I_{\nu}^{1} \left( \frac{t_2 - qs}{\Gamma_q(\nu)} \right) |f(s)| \, ds \right]
\]

\[
+ \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} |f(s)| \, ds
\]

which is independent of \( x \) and tends to zero as \( t_2 - t_1 \to 0 \). Therefore, it follows by the Arzelà-Ascoli theorem that \( \mathcal{F}_2 : C([0, 1], \mathbb{R}) \to \mathcal{P}(C[0, 1], \mathbb{R}) \) is completely continuous.

**Claim 3.** \( \mathcal{F}_2 \) has a closed graph. Let \( x_n \to x_*, h_n \in \mathcal{F}_2(x_n) \), and \( h_n \to h_* \). Then, we need to show that \( h_* \in \mathcal{F}_2(x_*) \).

Associated with \( h_n \in \mathcal{F}_2(x_n) \), there exists \( f_n \in S_{F,x_n} \) such that, for each \( t \in [0, 1] \),

\[
h_n(t) = \int_0^t \frac{(t - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_n(s) \, ds
\]

\[
+ \frac{t}{\vartheta} \left\{ b \int_0^1 \frac{I_{\nu}^{1} (s - qu)^{(\nu-1)}}{\Gamma_q(\nu)} f_n(u) \, du \right\}
\]

\[
- \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_n(s) \, ds
\]

Then, we have to show that there exists \( f_* \in S_{F,x_*} \) such that, for each \( t \in [0, 1] \),

\[
h_* (t) = \int_0^t \frac{(t - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (s) \, ds
\]

\[
+ \frac{t}{\vartheta} \left\{ b \int_0^1 \frac{I_{\nu}^{1} (s - qu)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (u) \, du \right\}
\]

\[
- \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (s) \, ds
\]

(33)

Let us consider the continuous linear operator \( \Theta : L^1([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}) \) defined by

\[
f \mapsto \Theta(f)(t) = \int_0^t \frac{(t - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f(s) \, ds
\]

\[
+ \frac{t}{\vartheta} \left\{ b \int_0^1 \frac{I_{\nu}^{1} (s - qu)^{(\nu-1)}}{\Gamma_q(\nu)} f(u) \, du \right\}
\]

\[
- \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f(s) \, ds
\]

(34)

Observe that

\[
\left| h_n(t) - h_* (t) \right|
\]

\[
= \left| \int_0^t \frac{(t - qs)^{(\nu-1)}}{\Gamma_q(\nu)} (f_n(s) - f_* (s)) \, ds \right|
\]

\[
+ \frac{t}{\vartheta} \left\{ b \int_0^1 \frac{I_{\nu}^{1} (s - qu)^{(\nu-1)}}{\Gamma_q(\nu)} (f_n(u) - f_* (u)) \, du \right\}
\]

\[
- \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} (f_n(s) - f_* (s)) \, ds \right]\to 0,
\]

(35)

as \( n \to \infty \). Thus, it follows by Lemma 7 that \( \Theta S_{F,x_*} \) is a closed graph operator. Further, we have \( h_n(t) \in \Theta(S_{F,x_n}) \). Since \( x_n \to x_* \), therefore, we have

\[
h_* (t) = \int_0^t \frac{(t - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (s) \, ds
\]

\[
+ \frac{t}{\vartheta} \left\{ b \int_0^1 \frac{I_{\nu}^{1} (s - qu)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (u) \, du \right\}
\]

\[
- \int_0^\infty \frac{(\omega - qs)^{(\nu-1)}}{\Gamma_q(\nu)} f_* (s) \, ds
\]

(36)

for some \( f_* \in S_{F,x_*} \). Hence, \( \mathcal{F}_2 \) has a closed graph (and therefore has closed values). In consequence, the operator \( \mathcal{F}_2 \) is compact valued.
Thus, the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy hypotheses of Theorem 9 and hence, by its application, it follows that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda \mathcal{F}_1(x) + \lambda \mathcal{F}_2(x)$ for $\lambda \in (0, 1)$, then there exists $f \in S_{F_{\lambda}x}$ such that $x = \lambda \mathcal{F}(x)$, that is,

$$x(t) = \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_\alpha(q)} f(s) \, dq \, s \quad + \frac{t}{\beta} \left[ b \left( \int_0^t \frac{(s-qs)^{(\alpha-1)}}{\Gamma_\alpha(v)} f(u) \, du \right) \right] \, dq \, s \quad + \lambda \left[ 1 + \frac{t}{\beta} (b_1 - \delta) - 1 \right] g(x) ; \quad t \in [0, 1].$$

(37)

In consequence, we have

$$|x(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_\alpha(v)} |f(s)| \, dq \, s \quad + \frac{1}{|\beta|} \left[ b \left( \int_0^t \frac{(s-qs)^{(\alpha-1)}}{\Gamma_\alpha(v)} |f(u)| \, du \right) \right] \, dq \, s \quad + \lambda \left[ 1 + \frac{1}{|\beta|} b_1 (1 - \delta) - 1 \right] \ell \|x\|$$

$$\leq \|p\| \psi(\|x\|) \left[ \frac{1}{\Gamma_\alpha(v+1)} \right] \quad + \frac{1}{|\beta|} \left[ b \left( \frac{1 - \delta^{\omega-1}}{\Gamma_\alpha(v+2)} + \frac{\omega^\eta}{\Gamma_\alpha(v+1)} \right) \right] \quad + \lambda \left[ 1 + \frac{1}{|\beta|} b_1 (1 - \delta) - 1 \right] \ell \|x\|$$

$$\leq \|p\| \psi(\|x\|) \mu_0 + k_0 \ell \|x\|. \quad (38)$$

If condition (ii) of Theorem 9 holds, then there exist $\lambda \in (0, 1)$ and $x \in \partial B_M$ with $x = \lambda \mathcal{F}_1(x)$, where $B_M = \{ x \in C([0, 1], \mathbb{R}) : \|x\| \leq M \}$. Then, $x$ is a solution of $x = \lambda \mathcal{F}(x)$ with $\|x\| = M$. Now, by the last inequality, we get

$$\frac{(1 - \ell k_0) M}{\|p\| \psi(M) \mu_0} \leq 1, \quad (39)$$

which contradicts (23). Hence, $\mathcal{F}$ has a fixed point on $[0, 1]$ by Theorem 9, and consequently the problem (1) has a solution. This completes the proof.

Example 11. Consider the following $q$-fractional boundary value problem:

$$\frac{c D_q^{1/2}}{q} x(t) \in F(t, x(t)), \quad t \in [0, 1],$$

$$x(0) = 1 + \frac{1}{5} \tan^{-1}\left( x(\frac{1}{4}) \right), \quad x\left(\frac{1}{4}\right) = \frac{1}{5} \int_0^{1/4} x(s) \, dq \, s. \quad (40)$$

Here, $v = 3/2, q = 1/2, b_1 = 1/5, \omega = 1/4, \delta = 3/4, \ell = 1/15,$ and

$$F(t, x) = \frac{3}{(3 + t)^2} \frac{|x|}{1 + |x|} \quad + \frac{\cos x}{(4 + t)^2} \quad + \frac{1}{2} \quad + \frac{1}{10}. \quad (41)$$

Obviously,

$$\sup \{ |u| : u \in F(t, x) \} \leq \frac{3}{(3 + t)^2} \quad + \frac{1}{(4 + t)^2} \quad + \frac{1}{2} \quad = p(t) \psi(\|x\|). \quad (42)$$

With the given values, it is found that $\vartheta = 0.191667, \mu_0 = 1.3990, k_0 = 3.9564, \psi(\|x\|) = 1, \text{ and } M > 1.7026 \text{ by } (H_3).$ Thus, all the conditions of Theorem 10 hold. In consequence, the conclusion of Theorem 10 applies to the boundary value problem (40).

Disclosure

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of KAU.

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