A Nonlinear Elliptic PDE with Two Sobolev-Hardy
Critical Exponents

YanYan Li∗ and Chang-Shou Lin

Abstract

In this paper, we consider the following PDE involving two Sobolev-Hardy critical exponents,
\[ \Delta u + \lambda \frac{u^{2^*(s_1) - 1}}{|x|^{s_1}} + \frac{u^{2^*(s_2) - 1}}{|x|^{s_2}} = 0 \text{ in } \Omega, \]
\[ u = 0 \text{ on } \Omega, \]
where \( 0 \leq s_2 < s_1 \leq 2 \), \( \lambda \neq 0 \in \mathbb{R} \) and \( 0 \in \partial \Omega \). The existence (or nonexistence) for least-energy solutions has been extensively studied when \( s_1 = 0 \) or \( s_2 = 0 \). In this paper, we prove that if \( 0 < s_2 < s_1 < 2 \) and the mean curvature of \( \partial \Omega \) at \( 0 H(0) < 0 \), then (1.1) has a least-energy solution. Therefore, this paper has completed the study of (0.1) for the least-energy solutions. We also prove existence or nonexistence of positive entire solutions of (0.1) with \( \Omega = \mathbb{R}^N \) under different situations of \( s_1, s_2 \) and \( \lambda \).

1 Introduction

Let \( 0 \leq s \leq 2, 2^*(s) = \frac{2(N-s)}{N-2} \) and \( L^{2^*(s)}(\mathbb{R}^N) \) denote the space of \( f \) with \( \int |f|^{2^*(s)} \, dx < +\infty \). It is well known that the inclusion \( H^1_0(\Omega) \hookrightarrow L^{2^*(s)}(\mathbb{R}^N) \) is a family of non-compact embeddings. In this paper, we want to study the combined effect of two such Sobolev-Hardy critical exponents on a nonlinear partial differential equation. More precisely, we consider
\[ \Delta u + \lambda \frac{u^{2^*(s_1) - 1}}{|x|^{s_1}} + \frac{u^{2^*(s_2) - 1}}{|x|^{s_2}} = 0 \text{ in } \Omega, \]
\[ u(x) > 0 \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial \Omega, \]
where \( 0 \leq s_2 < s_1 \leq 2 \) and \( \lambda \in \mathbb{R} \). Throughout the paper, \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) with \( 0 \in \partial \Omega \).

Our motivation for studying equation (1.1) comes from the celebrated Caffarelli-Kohn-Nirenberg inequality \[ [4, 11] \]: there exists a constant \( C \) such that for any \( u \in C_0^\infty(\mathbb{R}^N) \), the inequality
\[ \int_{\mathbb{R}^N} |x|^{-b_9} u^q \, dx \leq C \int_{\mathbb{R}^N} |x|^{-2q} |\nabla u|^2 \, dx \]

*Partially supported by NSF grant DMS-0701545.
holds, where $-\infty < a < \frac{-2N}{2-N-2(b-a)}$, $0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. Let $D^{1,2}_a(\Omega)$ be the completion of $C^\infty_0(\Omega)$ with the norm $\|u\|_a^2 = \int_{\Omega} |x|^{-2a} |\nabla u|^2 dx$, and set

$$S(a, b; \Omega) = \inf_{u \in D^{1,2}_a(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \lambda |x|^{-2a} |\nabla u|^2 dx}{\left(\int_{\Omega} |x|^{-bq} |u|^q dx\right)^{\frac{2}{s}}}.$$ 

Naturally, we ask whether the best constant $S(a, b; \Omega)$ holds, where $-\infty < a < \frac{-2N}{2-N-2(b-a)}$, $0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. Let $D^{1,2}_a(\Omega)$ be the completion of $C^\infty_0(\Omega)$ with the norm $\|u\|_a^2 = \int_{\Omega} |x|^{-2a} |\nabla u|^2 dx$, and set

$$S(a, b; \Omega) = \inf_{u \in D^{1,2}_a(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \lambda |x|^{-2a} |\nabla u|^2 dx}{\left(\int_{\Omega} |x|^{-bq} |u|^q dx\right)^{\frac{2}{s}}}.$$ 

For the past twenty years, this problem has been extensively studied. For recent development, we refer the readers to [1, 5, 6, 11, 12, 13, 14, 15, 22, 23, 24, 19] and the references therein.

When 0 $\in \partial \Omega$, this problem was first studied by Ghoussoub-Kang [11] and Ghoussoub-Robert [15], also see [11]. In [11], among other things, Chern and the second author of this paper proved the following theorem.

**Theorem A.** Suppose 0 $\in \partial \Omega$ and the mean curvature $H(0) < 0$. Then the best constant $S(a, b; \Omega)$ can be achieved in $D^{1,2}_a(\Omega)$ if $a, b, q$ satisfy one of the following conditions:

(i) $a < b < a + 1$ and $N \geq 3$,

(ii) $b = a > 0$ and $N \geq 4$.

When $a = 0$ and $0 < b < 1$, Theorem A was first proved by Ghoussoub and Robert [15]. The proof of Theorem A in [11] is to make use of a transformation: $u(x) = |x|^{-a} v(x)$. Straightforward computations give

$$\int_{\Omega} |x|^{-2a} |\nabla u|^2 dx = \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} \frac{u^2}{|x|^2} dx,$$

where $\lambda = a(N - 2 - a)$. Then $S(a, b; \Omega)$ is equal to the following best constant:

$$S(a)(\Omega) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} \frac{|u|^2}{|x|^2} dx}{\left(\int_{\Omega} \frac{|u|^s}{|x|^s} dx\right)^{\frac{2}{s}}} \quad (1.2)$$

where $\lambda = a(N - 2 - a)$ and $s = (b-a)q \in [0, 2)$ if $b < a + 1$. Note that if $b = a + 1$, thus $s = 2$ and the question for the best constant is a linear problem. Hence, we always exclude the case $b = a + 1$. By (1.2), Theorem A is equivalent to saying that equation (1.1) has a solution provided that either (i) $N \geq 3$, $\lambda < \frac{(\frac{N-2}{2})^2}{2}$, $0 < s_2 < s_1 = 2$, or (ii) $N \geq 4$, $0 < \lambda < \frac{(\frac{N-2}{2})^2}{2}$, $s_1 = 2$ and $s_2 = 0$.

To study equation (1.1), we consider the nonlinear functional $\Phi$:

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{p_1 + 1} \int_{\Omega} \frac{(u^+)^{p_1+1}}{|x|^{s_1}} dx - \frac{1}{p_2 + 1} \int_{\Omega} \frac{(u^+)^{p_2+1}}{|x|^{s_2}} dx$$

for $u \in H^1_0(\Omega)$, where for the simplicity of notations, we let $p_1 = 2^*(s_1) - 1$ and $p_2 = 2^*(s_2) - 1$. It is easy to see that there is positive constants $\rho_0, c_0 > 0$ such that

$$\Phi(u) \geq c_0 \text{ if } \|u\|_{H^1_0} = \rho_0.$$

Note that $p_2 > p_1$ because $s_1 > s_2$. Thus, no matter what the sign of $\lambda$ is, there is $u_0 \in H^1_0(\Omega)$ such that $\Phi(u_0) \leq 0$. Set

$$c_* = \inf_{P \subset P} \max_{w \in P} \Phi(w),$$

(1.3)
where $\mathcal{P}$ is the class of continuous paths in $H^1_0(\Omega)$ connecting 0 and $u_0$. We note that since $p_2 > p_1$, the function $t \rightarrow \Phi(tu)$ has the unique maximum for $t \geq 0$. Furthermore, we have

$$c_* = \inf_{u \in H^1_0(\Omega)} \max_{u \neq 0} \Phi(tu).$$

It is well-known that due to the non-compact embedding of $H^1_0 \hookrightarrow L^{2^*(s)}(\frac{dx}{|x|^s})$, $\Phi$ does not satisfy the Palais-Smale condition. Therefore, in general $c_*$ might not be a critical value for $\Phi$. As usual, if $c_*$ is a critical value, and $u$ is a critical point of $\Phi$ with $\Phi(u) = c_*$, then $u$ is called a least-energy solution.

When $s_2 = 0$, equation (1.4) becomes

$$\Delta u + \lambda \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + u^{\frac{N+s_2}{N-2}} = 0 \quad \text{in} \quad \Omega.$$ (1.4)

When $\lambda < 0$, the best constant $S_\lambda(\Omega)$ of (1.2) always satisfies

$$S_\lambda(\Omega) = S_0(\Omega) = S_N,$$

where $S_N$ is the Sobolev best constant. Thus, $S_\lambda(\Omega)$ can not be attained in $H^1_0(\Omega)$, and as a consequence, $c_*$ could not be a critical value of $\Phi$. In fact, for $0 \leq s_1 < 2$, it is not difficult to see that the constant $c_*$ of (1.3) is always equal to $\frac{1}{N}S_N^2$ and $c_*$ is never a critical value for $\Phi$. Thus, there exist no least-energy solutions for equation (1.4) when $\lambda < 0$. However, when $\lambda > 0$, $0 < s_1 < 2$ and $s_2 = 0$, the following theorem was proved in [17].

**Theorem B.** Suppose $N \geq 4$ and $0 \in \partial\Omega$ with $H(0) < 0$. Then equation (1.4) has a solution, provided that $\lambda > 0$, and $0 < s_1 < 2$.

In summary, equation (1.4) has been studied for either $s_1 = 2$ or $s_2 = 0$. The purpose of this paper is to study the remaining cases for equation (1.1). The following is one of our main theorems.

**Theorem 1.1.** Suppose $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$. Then equation (1.1) has a least-energy solution if

$$N \geq 3, \lambda \in \mathbb{R} \quad \text{and} \quad 0 < s_2 < s_1 < 2.$$
We note that if solutions are assumed to be in \( H^1_0(\mathbb{R}_+^N) \), Theorem 1.3 with \( s_1 = 2 \) has been proved in [17]. The authors of [17] employed the method of moving planes to prove Theorem 1.3 where the behavior of \( u \) at \( \infty \) is needed. One way to find asymptotic behavior is to apply the Kelvin transform to \( u \):

\[
\hat{u}(y) = \left( \frac{1}{|y|} \right)^{N-2} u \left( \frac{y}{|y|^2} \right) \quad \text{for} \quad |y| < 1.
\]

It is a straightforward computation to show that \( \hat{u}(y) \) satisfies

\[
\Delta \hat{u} + \lambda \frac{\hat{u}^{2^*(s_1)-1}(y)}{|y|^{s_1}} + \hat{u}^{\frac{N+2}{N-2}}(y) = 0 \quad \text{in} \quad B_1 \cap \mathbb{R}_+^N.
\]

But \( \hat{u} \) is no longer contained in \( H^1_{\text{loc}}(\mathbb{R}_+^N) \), i.e., the integration of \( \nabla \hat{u} \) might be \( +\infty \) in any neighborhood of 0. In this case, the origin 0 is called a nonremovable singularity of \( \hat{u} \). It is a really interesting question: What is the asymptotic behavior of \( \hat{u} \) near the singularity? Previously, this kind of problems have been studied:

\[
\Delta u + g(y,u) + u^{\frac{N+2}{2}} = 0 \quad \text{in} \quad 0 < |y| < 1.
\]

Under the monotonicity assumption of \( u \):

\[
g(y,t)t^{-\frac{N+2}{N-2}} \quad \text{is decreasing for large} \ t > 0,
\]

it was proved that \( u(y) = O(|y|^{-\frac{N+2}{2}}) \) near 0. See [7, 8, 9, 10]. For our case,

\[
g(y,u) = \lambda \frac{u^{2^*(s_1)-1}}{|y|^{s_1}} \quad \text{and} \quad \lambda < 0.
\]

Then \( g(y,t)t^{-\frac{N+2}{N-2}} \) is increasing in \( t > 0 \). Hence, the methods in [7, 8, 9, 10] can not work for our nonlinearity. We should address this asymptotic problem later.

Our proof of Theorem 1.3 employs the idea of the method of moving spheres, a variant of the method of moving planes. The method of moving planes has been developed through the works by A.D. Alexandrov, Serrin [26], and Gidas, Ni and Nirenberg [16]. Here, we will not require any assumption on the behavior of solutions at \( \infty \), by taking some advantage of the upper half space \( \mathbb{R}_+^N \), while compared to \( \mathbb{R}^N \). We think this proof might be useful in other problems also. See [21, 20, 18] for some related results.

This paper is organized as follows. In Section 2, we will prove Theorem 1.3 and a generalization of it. In Section 3, we will employ a blowing-up argument to prove Theorem 1.2. This kind of arguments have been developed for studying the nonlinear equation involving the Sobolev critical exponent, see [7, 8, 9, 10, 19]. The existence of least-energy solutions of equation (1.1) with \( 0 < s_2 < s_1 < 2 \) are obtained in Section 4. In final section, we discuss a perturbed equation of equation (1.1) for the case \( \lambda < 0, 0 = s_2 < s_1 < 2 \).

## 2 Nonexistence of Entire Solutions

In this section, we begin with a proof of Theorem 1.3. We first make a remark about regularity of \( u(x) \). It is shown that \( u \in C^\alpha(\mathbb{R}_+^N) \), for any \( \alpha \in (0,1) \). For a proof, see [11] and [17].

If \( u = 0 \) at some point of \( \mathbb{R}_+^N \), then \( u \equiv 0 \) by the strong maximum principle. Hence, we will always assume that

\[
u(x) > 0.
\]
Lemma 2.1. Let \( u(x) \) be a positive solution of equation (1.4). Suppose \( u \in H^1_{\text{loc}}(\mathbb{R}^N_+) \). Then \( \frac{\partial u}{\partial x^N} > 0 \) in \( \mathbb{R}^N_+ \).

Before giving a proof of Lemma 2.1, we apply Lemma 2.1 to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose \( u(x) \) is a positive solution of equation (1.4).

We claim \( u \) is uniformly bounded in any compact set of \( \mathbb{R}^N \). Suppose the contrary, then there exist \( \bar{x}_i \in \mathbb{R}^N \), such that

\[
u(\bar{x}_i) \to \infty \quad \text{as} \quad i \to \infty.
\]

By the monotonicity of \( u \) in \( x_N \)-direction, we may assume that

\[|\bar{x}_i| \to \infty.
\]

Consider

\[v_1(x) = (1 - |x - \bar{x}_i|)^{N-2} u(x), \quad |x - \bar{x}_i| \leq 1.
\]

For some \(|x_i - \bar{x}_i| < 1\),

\[v_1(x_i) = \max_{|x-x_i|\leq 1} v_1(x),
\]

here we have used the fact that \(v_1(x) = 0\) for \(|x - \bar{x}_i| = 1\).

Let \( \sigma_i = \frac{1}{2}(1 - |x_i - \bar{x}_i|) > 0 \).

Then

\[
(2\sigma_i)^{\frac{N-2}{2}} u(x_i) = v_1(x_i) \geq v_1(\bar{x}_i) = u(\bar{x}_i) \to \infty.
\]

It follows that

\[R_i := \sigma_i u(x_i)^{\frac{N-2}{2}} \to \infty.
\]

Since

\[v_1(x_i) \geq v_1(x) \geq \sigma_i^{\frac{N-2}{2}} u(x), \quad \forall \ x \in B_{\sigma_i}(x_i),
\]

we see that

\[u(x) \leq 2^{\frac{N-2}{2}} u(x_i), \quad \forall \ x \in B_{\sigma_i}(x_i).
\]

Consider

\[w_1(y) := \frac{1}{u(x_i)} u \left( x_i + \frac{y}{u(x_i)^{\frac{N-2}{2}}} \right), \quad |y| < R_i = \sigma_i u(x_i)^{\frac{N-2}{2}} \to \infty.
\]

Then

\[w_1(0) = 1, \quad \text{and} \quad w_1(y) \leq 2^{\frac{N-2}{2}}, \quad \forall \ |y| < R_i.
\]

Using the equation satisfied by \( u_i \), we have

\[
\Delta w_1(y) - \frac{1}{u(x_i)^{\frac{N+2}{N-2}}} \left| \frac{y}{u(x_i)^{\frac{N-2}{2}}} \right|^{N-2} u(x_i)^{\frac{N+2}{N-2}} + w_1(y)^{\frac{N+2}{N-2}} = 0, \quad |y| < R_i.
\]

Since \(|\bar{x}_i| \to \infty\), it is clear that

\[|x_i + \frac{y}{u(x_i)^{\frac{N-2}{2}}}| \geq |\bar{x}_i| - |x_i - \bar{x}_i| - \sigma_i \geq |\bar{x}_i| - 2 \to \infty, \quad \text{uniformly for} \ |y| < R_i.
\]
Given the bound of \( w_i \), we know from standard elliptic estimates that on every compact subset of \( \mathbb{R}^N \), \( \{ w_i \} \) is bounded in \( C^3 \) norm. After passing to a subsequence, we have,

\[
w_i \to w \quad \text{in } C^3_{\text{loc}}(\mathbb{R}^N).
\]

Given the above estimates, and the equation of \( w_i \), we have

\[
\Delta w + w^{\frac{N+2}{N-2}} = 0, \quad \text{on } \mathbb{R}^N,
\]

and

\[
w(0) = 1, \quad w \geq 0 \quad \text{on } \mathbb{R}^N.
\]

By the strong maximum principle, \( w > 0 \) on \( \mathbb{R}^N \).

By the classification theorem of Caffarelli-Gidas-Spruck, \( w(y) = C_N \left( \frac{\mu}{1 + \mu^2 |y - y_0|^2} \right)^{\frac{N}{N-2}} \), (2.2)

where \( \mu > 0 \) and \( y_0 \in \mathbb{R}^N \).

But we know from the monotonicity of \( w_i \), \( w \) must be monotone in \( y_N \)-direction. This is a contradiction and the claim is proved.

Let \( u_j(x', x_N) = u(x', x_N + r_j) \) where \( r_j < r_{j+1} \to +\infty \) as \( j \to +\infty \). By Lemma 2.1, \( u_j(x) < u_{j+1}(x) \). Since \( u(x) \) is uniformly bounded, \( u_j(x) \to u_\infty(x) \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \), where \( u_\infty(x) \) is a positive solution to

\[
\Delta u_\infty + u_\infty^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N.
\]

Again, Lemma 2.1 yields a contradiction to (2.2). Hence, the proof of Theorem 1.3 is complete. □

**Proof of Lemma 2.1** The proof uses the method of moving spheres, a variant of the method of moving planes which are developed through the works of Alexandrov, Serrin [26], and Gidas, Ni and Nirenberg [16]. We also make use of the “narrow domain idea” from Berestycki and Nirenberg [2].

Define

\[
x_R := (0, \ldots, 0, -R).
\]

Let

\[
u_{x_R, \lambda}(y) := \left( \frac{\lambda}{|y - x_R|} \right)^{N-2} u \left( x_R + \frac{\lambda^2(y - x_R)}{|y - x_R|^2} \right)
\]

be the Kelvin transformation of \( u \) with respect to the ball \( B_\lambda(x_R) \) with center \( x_R \) and radius \( \lambda > 0 \). By direct computations, we have for \( y \in B_\lambda(x_R) \cap \mathbb{R}^N_+ \),

\[
\Delta v_{x_R, \lambda}(y) - \left( \frac{\lambda}{|y - x_R|} \right)^{2s} \frac{u^2_{x_R, \lambda}(y)}{|x_R + \frac{\lambda^2(y - x_R)}{|y - x_R}|^s} + u_{x_R, \lambda}^{\frac{N+2}{N-2}} = 0.
\]

We want to show that

\[
u_{x_R, \lambda}(y) \geq u(y) \quad \forall y \in B_\lambda(x_R) \cap \mathbb{R}^N_+, \forall \lambda > R. \quad (2.3)
\]

To prove (2.3), we first claim

\[
\left( \frac{\lambda}{|y - x_R|} \right)^{2s} \frac{1}{|x_R + \frac{\lambda^2(y - x_R)}{|y - x_R}|^s} \leq \frac{1}{|y|^s}. \quad (2.4)
\]


for \( y \in B_\lambda(x_R) \cap \mathbb{R}^N_+ \), \( \forall \lambda > R \).

For \( y \in B_\lambda(x_R) \cap \mathbb{R}^N_+ \), we write

\[
\frac{y - x_R}{|y - x_R|} = \theta = (\theta_1, \ldots, \theta_N), \quad |y - x_R| = r.
\]

Then

\[
\mu_1(\theta) < r < \lambda, \tag{2.5}
\]

where \( \mu_1(\theta) \) is determined by

\[x_R + \mu_1(\theta)\theta \in \partial \partial \mathbb{R}^N_+ .\]

Namely,

\[
\mu_1(\theta) = \frac{R}{\theta_N}.
\]

(2.4) is equivalent to

\[
\left(\frac{\lambda}{r}\right)^{2s} \frac{1}{|x_R + \frac{\lambda^2}{r}\theta|^s} \leq \frac{1}{|x_R + r\theta|^s}.
\]

This is equivalent to

\[
\left(\frac{\lambda^2}{r}ight)^2 \frac{1}{|x_R + \frac{\lambda^2}{r}\theta|^2} \leq r^2 \frac{1}{|x_R + r\theta|^2} . \tag{2.6}
\]

For \( r \) satisfying (2.5), we have

\[
\frac{\lambda^2}{R} > \frac{\lambda^2}{r} > r > \mu_1(\theta).
\]

Let

\[
\eta(\mu) := \mu^2 \frac{1}{|x_R + \mu\theta|^2}, \quad \mu > \mu_1(\theta).
\]

In order to prove (2.6), we only need to prove

\[
\eta'(\mu) \leq 0, \quad \mu_1(\theta) < \mu < \frac{\lambda^2}{R} \tag{2.7}
\]

This follows from the following calculations, for \( \mu > \mu_1(\theta) \),

\[
|x_R + \mu\theta|^4 \eta'(\mu) = 2\mu|x_R + \mu\theta|^2 - \mu^2 \frac{d}{d\mu}(|x_R + \mu\theta|^2) = 2\mu R\theta_N (\mu_1 - \mu) < 0.
\]

We have proved (2.7), and therefore proved (2.4). It follows that

\[
-\Delta u_{x_R, \lambda} + \frac{1}{|y|^s} u_{x_R, \lambda}^{2(s)-1} \geq u_{x_R, \lambda}(y)\frac{2s}{s+2}, \quad \text{in } B_\lambda(x_R) \cap \mathbb{R}^N_+.
\]

Thus

\[
-\Delta (u_{x_R, \lambda} - u) + \frac{1}{|y|^s} \left( u_{x_R, \lambda}^{2(s)-1} - u_{x_R, \lambda}^{2s}-1 \right) \geq u_{x_R, \lambda} \frac{N+2}{N} - u \frac{N+2}{N}, \quad \text{in } B_\lambda(x_R) \cap \mathbb{R}^N_+. \tag{2.8}
\]

Write

\[
w_\lambda = u_{x_R, \lambda} - u, \quad w_\lambda^- = \max\{0, -w_\lambda\}. \tag{2.9}
\]
We first require that $R < \lambda_0(R) < 2R$, then for $R < \lambda < \lambda_0(R)$, we have

$$|x_R + \frac{x^2(y - x_R)}{|y - x_R|^2}| \leq |x_R| + \frac{x^2}{R} \leq 5R, \quad \forall y \in B_\lambda(x_R) \cap \mathbb{R}^N_+.$$  

Multiply $w^-_\lambda$ to the inequality \(\ref{eq:1} \) and integrate by parts on $B_\lambda(x_R) \cap \mathbb{R}^N_+$, we have, using $w_\lambda \geq 0$ on $\partial(B_\lambda(x_R) \cap \mathbb{R}^N_+)$,

$$\int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} |\nabla w^-_\lambda|^2 \, dy \leq \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} (|\nabla w^-_\lambda|^2 - \frac{1}{|y|} (u_{x_R, \lambda}^{2^*(s)-1} - u^{2^*(s)-1}) w^-_\lambda) \, dy$$

$$\leq \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} \left( \frac{N+2}{u_{x_R, \lambda}^{2^*(s)-1} - u^{2^*(s)-1}} \right) w^-_\lambda \, dy$$

$$\leq \frac{N + 2}{N - 2} \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} (\max\{u_{x_R, \lambda}, u\})^{\frac{N}{N-2}} (w^-_\lambda)^2 \, dy$$

$$\leq \frac{N + 2}{N - 2} \sup_{B_{x_R}(0) \cap \mathbb{R}^N_+} u^{\frac{N}{N-2}} \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} (w^-_\lambda)^2 \, dy$$

$$\leq C(N)|B_\lambda(x_R) \cap \mathbb{R}^N_+|^\frac{1}{N} \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} |\nabla w^-_\lambda|^2 \, dy.$$  

Now we can choose $\lambda_0(R) > R$ but very close to $R$, then $|B_\lambda(x_R) \cap \mathbb{R}^N_+|$ is small, and we have

$$\int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} |\nabla w^-_\lambda|^2 \, dy \leq \frac{1}{2} \int_{B_\lambda(x_R) \cap \mathbb{R}^N_+} |\nabla w^-_\lambda|^2 \, dy.$$  

This implies $\nabla w^-_\lambda = 0$ in $B_\lambda(x_R) \cap \mathbb{R}^N_+$ and therefore, since $w^-_\lambda = 0$ on $\partial B_\lambda(x_R) \cap \mathbb{R}^N_+$, $w^-_\lambda = 0$ in $B_\lambda(x_R) \cap \mathbb{R}^N_+$. Step 1 is established.

Define

$$\tilde{\lambda}(R) := \sup\{\mu \mid \mu > R, \text{ and } u_{x_R, \lambda}(y) \geq u(y), \forall y \in B_\lambda(x_R) \cap \mathbb{R}^N_+, \forall R < \lambda < \mu\}.$$  

By Step 1, $\tilde{\lambda}(R)$ is well defined and $R < \tilde{\lambda}(R) \leq \infty$.

**Step 2.** $\tilde{\lambda}(R) = \infty$ for all $R > 0$.

We establish Step 2 by contradiction. Suppose that $\tilde{\lambda} \equiv \tilde{\lambda}(R) < \infty$ for some $R > 0$. Then

$$u_{x_R, \tilde{\lambda}}(y) \geq u(y), \forall y \in B_\tilde{\lambda}(x_R) \cap \mathbb{R}^N_+.$$  

Since $u_{x_R, \tilde{\lambda}} > u$ on $B_\tilde{\lambda}(x_R) \cap \partial\mathbb{R}^N_+$, we have, by the strong maximum principle,

$$u_{x_R, \tilde{\lambda}}(y) > u(y), \forall y \in B_\lambda(x_R) \cap \mathbb{R}^N_+.$$  

For $\delta > 0$ small, and the value to be fixed below, let

$$K := \{y \in B_\lambda(x_R) \cap \mathbb{R}^N_+ \mid \text{dist}(y, \partial(B_\lambda(x_R) \cap \mathbb{R}^N_+)) \geq \delta\}.$$
Then
\[ b := \min_K w_\lambda > 0, \]
where we have used the notation \(2.9\).

Consider \(\lambda < \lambda < \lambda + \epsilon\), where the value of \(\epsilon = \epsilon(\delta) < \delta\) is chosen so that
\[ w_\lambda > \frac{b}{2}, \quad \text{on } K, \quad \forall \lambda < \lambda < \lambda + \epsilon. \]  
(2.10)

Multiplying \(2.9\) by \(w_\lambda^-\) and integrating by parts on \((B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K\) leads to, as before,
\[
\int_{(B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K} |\nabla w_\lambda^-|^2 dy \\
\leq \int_{(B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K} (|\nabla w_\lambda^-|^2 - \frac{1}{|y|^s}(2^{*}(s) - 1)u_{x_R, \lambda}^2 \nu - u_{x_R, \lambda}^{2^*(s)-1})w_\lambda^- dy \\
\leq C ||(B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K||^{\frac{s}{2}} \int_{(B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K} |\nabla w_\lambda^-|^2 dy.
\]

Now we can fix the value of \(\delta\) so that \(C ||(B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K||^{\frac{s}{2}} < \frac{1}{2}\), and we obtain as before \(w_\lambda^- = 0\) on \((B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K\), i.e.
\[ u_{x_R, \lambda}(y) \geq u(y), \quad \forall y \in (B_\lambda(x_R) \cap \mathbb{R}^N_+) \setminus K, \quad \forall \lambda < \lambda < \lambda + \epsilon. \]
This and \(2.10\) contradicts to the definition of \(\lambda(R)\). Step 2 is established.

By Step 2, we have
\[ u_{x_R, R+a}(y) \geq u(y), \quad \forall y \in B_{R+a}(x_R) \cap \mathbb{R}^N_+, \quad \forall R, a > 0. \]  
(2.11)

It follows, for every \(y \in \mathbb{R}^N_+\), and every \(a > y_n\),
\[ u(y) \leq \lim_{R \to \infty} u_{x_R, R+a}(y) = u(y_1, \cdots, y_{N-1}, 2a - y_N). \]

The above implies
\[ u(y_1, \cdots, y_{N-1}, s) \leq u(y_1, \cdots, y_{N-1}, t), \quad \forall 0 < s < t. \]

We have proved
\[ \frac{\partial u}{\partial x_N} \geq 0, \quad \text{in } \mathbb{R}^N_+. \]

Applying \(\frac{\partial}{\partial x_N}\) to the equation of \(u\) leads to
\[
-\Delta (\frac{\partial u}{\partial x_N}) + \left( \frac{2^*(s) - 1}{|x|^s} u^{2^*(s)-2} - \frac{N+2}{N-2} u \frac{1}{|x|^s} \right) \frac{\partial u}{\partial x_N} + \frac{\partial}{\partial x_N} \left( \frac{1}{|x|^s} \right) u = 0, \quad \text{in } \mathbb{R}^N_+.
\]

By the strong maximum principle, we have \(\frac{\partial u}{\partial x_n}\) is always zero or strictly positive. But \(u = 0\) on the boundary on \(\mathbb{R}^N_+\) and positive in \(\mathbb{R}^N_+\), so we must have \(\frac{\partial u}{\partial x_n} > 0\) in \(\mathbb{R}^N_+\). Lemma 2.11 is established.

The main theorem in this section is the following generalization of Theorem 1.3.
Theorem 2.2. Let $s_i \in (0, 2]$, $P_i \in \mathbb{R}^{N-1}$ and let $u(x) \geq 0$ be a solution of
\begin{equation}
\begin{aligned}
\Delta u - \sum_{i=1}^{l} \frac{u^{2^{(s_i)-1}}}{|x - P_i|^{s_i}} + u^{\frac{N+z}{z}} = 0 \quad \text{in } \mathbb{R}^N, \\
u(x) = 0 \quad \text{on } \partial \mathbb{R}_+^N.
\end{aligned}
\tag{2.12}
\end{equation}
Suppose $u \in L^\infty(\mathbb{R}_+^N) \cap H^1_{loc}(\mathbb{R}_+^N)$. Then $u(x) \equiv 0$.

**Proof.** The main step is to show that $\frac{\partial u}{\partial x} \geq 0$ as did in Lemma 2.1. This second proof could work for the general situation of (2.12), but the boundedness of $u$ is required! The proof is divided into several steps.

**Step 1.** $u(x) \to 0$ as $|x| \to +\infty$. Suppose not. We may assume there are $x_j \to +\infty$, $u(x_j) \geq C > 0$ for some positive constant $C$. Let $u_j(x) = u(x + x_j)$. By elliptic estimates, $u_j(x)$ is bounded in $C^2$ in any compact set of $\mathbb{R}_+^N$. By passing to a subsequence, we may assume $u_j(x) \to u(x)$ in $C^2_{loc}(\mathbb{R}_+^N)$ and $u(x)$ satisfies
\begin{equation}
\begin{aligned}
\Delta u(x) + u^{\frac{N+z}{z}} = 0 \quad \text{in } \mathbb{R}_+^N, \\
u(x) = 0 \quad \text{on } \partial \mathbb{R}_+^N.
\end{aligned}
\tag{2.13}
\end{equation}

But it is well-known that (2.13) has no positive solutions. Thus, $u \equiv 0$ in $\mathbb{R}_+^N$ which contradicts to $u(0) \geq C > 0$. So, Step 1 is proved.

**Step 2.** We claim for any $\lambda > 0$,
\[ u(y^{\lambda}) > u(y) \quad \text{for } x \in \Sigma_\lambda = \{ (y_1, y_2, \cdots, y_N) | 0 \leq y_N < \lambda \}, \]
where $y^{\lambda} = (y_1, \cdots, y_{N-1}, 2\lambda - y_N)$. This step is a standard application of the method of moving planes. We give a sketch of proofs for the sake of completeness. Let
\[ w_\lambda(y) = u(y^{\lambda}) - u(y). \]
Then we have
\begin{align*}
\Delta w_\lambda(y) &= \sum_{j=1}^{l} \frac{1}{|y - P_j|^{s_j}} \left( u^{2^{(s_j)-1}}(y^{\lambda}) - u^{2^{(s_j)-1}}(y) \right) + u^{\frac{N+z}{z}}(y^{\lambda}) - u^{\frac{N+z}{z}}(y) \\
&= \sum_{j=1}^{l} \left( \frac{1}{|y^{\lambda} - P_j|^{s_j}} - \frac{1}{|y^{\lambda}|^{s_j}} \right) u^{2^{(s_j)-1}}(y^{\lambda}) \leq 0 \quad \text{in } \Sigma_\lambda.
\end{align*}
Thus $w_\lambda(y)$ satisfies
\[ \Delta w_\lambda(y) + (C_1(y) + C_2(y))w_\lambda(y) \leq 0 \quad \text{in } \Sigma_\lambda, \]
where
\[ C_1(y) \leq 0, \quad \text{and} \quad C_2(y) = \frac{u^{\frac{N+z}{z}}(y^{\lambda}) - u^{\frac{N+z}{z}}(y)}{u(y^{\lambda}) - u(y)}. \]
By Step 1, $C_2(y) = o(1)$ as $|y| \to +\infty$ and $y \in \Sigma_\lambda$.  

10
To prove $w_\lambda(y) > 0$ in $\Sigma_\lambda$ for $\lambda$ small, we consider the comparison function, 

$$v(y) = 1 - y_N^2, \quad 0 \leq y_N \leq \lambda,$$

and let 

$$\overline{w}_\lambda(y) = \frac{w_\lambda(y)}{v(y)}, \quad \text{i.e., } w_\lambda(y) = \overline{w}_\lambda(y)v(y).$$

Thus, $\overline{w}_\lambda$ satisfies 

$$\Delta \overline{w}_\lambda(y) + 2 \frac{\nabla v(y)}{v(y)} \cdot \nabla \overline{w}_\lambda(y) + \left(C_1(y) + C_2(y) - \frac{4}{v(y)}\right)\overline{w}_\lambda(y) \leq 0,$$  \hspace{1cm} \text{(2.14)}$$

Choose $\lambda$ small such that 

$$\frac{N + 2}{N - 2} \overline{w}_\lambda(y) \leq 2, \quad 0 \leq y_N \leq \lambda.$$

Now suppose the set $\left\{y \mid w_\lambda(y) < 0\right\} \neq \emptyset$. 

Because $\overline{w}_\lambda \geq 0$ on $\partial \Sigma_\lambda$ and $\lim_{y_N \to +\infty} \overline{w}_\lambda(y) = 0$, it is easy to see the minimum of $\overline{w}_\lambda$ can be achieved. Let $\overline{y} \in \Sigma_\lambda$ such that 

$$\overline{w}_\lambda(\overline{y}) = \inf_{y \in \Sigma_\lambda} \overline{w}_\lambda(y) < 0.$$ 

Since $\overline{w}_\lambda(\overline{y}) < 0$, 

$$C(\overline{y}) \leq \frac{N + 2}{N - 2} \overline{w}_\lambda(\overline{y}) \leq 2.$$ 

By applying the maximum principle, \text{(2.14)} yields 

$$0 < \left(C_1(\overline{y}) + C_2(\overline{y}) - \frac{4}{v(\overline{y})}\right)\overline{w}_\lambda(\overline{y}) \leq 0,$$

which is a contradiction. Hence, $w_\lambda(y) > 0 \forall y \in \Sigma_\lambda$.

Let 

$$\overline{x} = \sup \left\{\lambda \mid w_\lambda(y) > 0 \forall y \in \Sigma_\lambda, 0 < \mu \leq \lambda\right\}.$$ 

We claim $\overline{x} = +\infty$. Otherwise, we have 

$$\begin{cases} 
\overline{w}_\lambda(y) > 0 & \forall y \in \Sigma_{\overline{x}} \\
\frac{\partial \overline{w}_\lambda}{\partial y_N}(y', \lambda) < 0. 
\end{cases}$$ \hspace{1cm} \text{(2.15)}$$

by the strong maximum principle and Hopf boundary point lemma. By the definition of $\overline{x}$, there are $\lambda_j \downarrow \overline{x}$ such that $\left\{y \mid w_{\lambda_j}(y) < 0, y \in \Sigma_{\lambda_j}\right\} \neq \emptyset$. Set 

$$w_{\lambda_j}(y) = \overline{w}_{\lambda_j}(y)v(y),$$

where 

$$v(y) = (\overline{x} + 1)^2 - y_N^2, \quad y \in \Sigma_{\lambda_j}.$$ 

Then $\overline{w}_{\lambda_j}$ satisfies 

$$\Delta \overline{w}_{\lambda_j}(y) + 2 \nabla \log v(y) \cdot \nabla \overline{w}_{\lambda_j}(y) + \left(C_1(y) + C_2(y) - \frac{4}{v(y)}\right)\overline{w}_{\lambda_j} \leq 0$$ \hspace{1cm} \text{(2.16)}$$

11
Suppose $\overline{\mathcal{w}}_{\lambda_j}(\overline{y}_j) = \inf_{y \in \Sigma_j} \overline{\mathcal{w}}_{\lambda_j}(y) < 0$. By (2.15), we have $|\overline{y}_j| \to +\infty$. Note that

$$C_2(\overline{y}_j) \leq \frac{N + 2}{N - 2} u_{\overline{y}_j} \to 0 \quad \text{as } j \to +\infty.$$ 

Again, by the maximum principle, (2.16) yields a contradiction. Therefore, Step 2 is proved. Obviously, the conclusion of Lemma 2.1 follows immediately from Step 2. Therefore, Step 2 is proved.

Remark 2.3. If $P_i = P_j \forall i \neq j$, then the proof of Lemma 2.1 still holds. Hence, in this case, the boundedness assumption is not necessary for the conclusion of Theorem 2.2.

3 Existence of Entire Solutions

In this section, we will give a proof of Theorem 2.2. To prove Theorem 2.2, we choose a convex domain $\Omega$ with $0 \in \partial \Omega$ and consider the following equation. For any small $\varepsilon > 0$

$$\begin{cases}
\Delta u + \frac{\lambda u_{p_1(\varepsilon)}(c)}{|x|^{s_1}} + \frac{u_{p_2(\varepsilon)}(c)}{|x|^{s_2}} = 0 & \text{in } \Omega, \\
u(x) > 0 & \text{in } \Omega \text{ and } u(x) = 0 \text{ on } \partial \Omega,
\end{cases} \quad (3.1)$$

where $p_1(\varepsilon) = 2^*(s_1) - 1 - \varepsilon$ and $p_2(\varepsilon) = 2^*(s_2) - 1 - \left(\frac{2-s_1}{s_1-1}\right)\varepsilon$.

For $\varepsilon > 0$, we let

$$\Phi_\varepsilon(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{p_1(\varepsilon) + 1} \int_\Omega \frac{|u_{p_1(\varepsilon)}(c) + 1}{|x|^{s_1}} dx - \frac{1}{|p_2(\varepsilon) + 1|} \int_\Omega \frac{u_{p_2(\varepsilon) + 1}}{|x|^{s_2}} dx$$

for $u \in H^1_0(\Omega)$ and $c^*_\varepsilon = \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi_\varepsilon(w)$,

where $\mathcal{P}$ is the class of all continuous paths in $H^1_0(\Omega)$ connecting 0 and some $u_0$ such that $\Phi_\varepsilon(u_0) \leq 0$. It is easy to see that $c^*_\varepsilon \leq C$ for some constant $C$ independent of $\varepsilon$. Since for $\varepsilon > 0$, $\Phi_\varepsilon$ satisfies the P-S condition, it is known that $c^*_\varepsilon$ is a critical point of $\Phi_\varepsilon$, i.e., there exists a solution $u_\varepsilon \in H^1_0(\Omega)$ with $\Phi_\varepsilon(u_\varepsilon) = c^*_\varepsilon$. Thus,

$$\begin{cases}
\frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{\lambda}{p_1(\varepsilon) + 1} \int_\Omega \frac{u_{p_1(\varepsilon)+1}(c)}{|x|^{s_1}} dx - \frac{1}{|p_2(\varepsilon) + 1|} \int_\Omega \frac{u_{p_2(\varepsilon)+1}}{|x|^{s_2}} dx = c^*_\varepsilon, \\
\int_\Omega |\nabla u_\varepsilon|^2 dx - \lambda \int_\Omega \frac{u_{p_1(\varepsilon)+1}}{|x|^{s_1}} dx - \int_\Omega \frac{u_{p_2(\varepsilon)+1}}{|x|^{s_2}} dx = 0.
\end{cases} \quad (3.2)$$

From (3.2), we have

$$\left(1 - \frac{1}{p_1(\varepsilon) + 1}\right) \int_\Omega |\nabla u_\varepsilon|^2 dx + \left(1 - \frac{1}{p_2(\varepsilon) + 1}\right) \int_\Omega \frac{u_{p_2(\varepsilon)+1}}{|x|^{s_2}} dx = c^*_\varepsilon.$$

By noting both of $\frac{1}{2} - \frac{1}{p_1(\varepsilon)+1}$ and $\frac{1}{p_1(\varepsilon)+1} - \frac{1}{p_2(\varepsilon)+1}$ are positive, we have

$$\int_\Omega |\nabla u_\varepsilon|^2 dx + \int_\Omega \frac{u_{p_1(\varepsilon)+1}}{|x|^{s_1}} dx + \int_\Omega \frac{u_{p_2(\varepsilon)+1}}{|x|^{s_2}} dx \leq C < +\infty. \quad (3.3)$$
Therefore, by passing to a subsequence if necessary, we might assume \( u_\varepsilon \to u \) in \( H^1_0(\Omega) \) as \( \varepsilon \to 0 \). If \( u \not\equiv 0 \), then \( u \) is a solution of
\[
\begin{cases}
\Delta u + \lambda \frac{u^{p_1}}{|x|^s_1} + \frac{u^{p_2}}{|x|^{s_2}} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{3.4}
\]
However, the standard Pohozaev identity yields that equation \((3.4)\) has no positive solutions because both of \( p_1 \) and \( p_2 \) are critical exponents. Thus \( u \equiv 0 \) and \( u_\varepsilon(x) \) must blow up as \( \varepsilon \to 0 \).

Before proceeding further, we will briefly discuss the regularity of \( u \) at 0. Because \( s_1 < 2 \), we can prove that
\[
\begin{cases}
u \in C^2(\Omega) & \text{if } s_1 < 1 + \frac{2}{n}, \\
u \in C^{1,\beta}(\Omega) & \text{if } 0 < \beta < \frac{n(2-s)}{n-2} \text{ if } s_1 > 1 + \frac{2}{n}, \\
u \in C^{1,\beta}(\Omega) & \text{for all } 0 < \beta < \frac{n}{2} \text{ if } s_1 > 1 + \frac{2}{n},
\end{cases}
\tag{3.5}
\]
see [17].

Let \( \tilde{u}_\varepsilon(x_\varepsilon) = \max_{\Omega} u_\varepsilon(x) = m_\varepsilon \) and \( k_\varepsilon = m_\varepsilon \frac{p_2^{(s_1-1)}}{s_2} \).

By direct computations, we have
\[
k_\varepsilon = m_\varepsilon \frac{\frac{1}{k_\varepsilon} + \frac{s_2}{s_1-2}}{s_2-2}.
\tag{3.6}
\]
First, we claim
\[
|x_\varepsilon| = O(k_\varepsilon).
\tag{3.6}
\]
Suppose not. By passing to a subsequence if necessary, we may assume
\[
\lim_{\varepsilon \to 0} \frac{|x_\varepsilon|}{k_\varepsilon} = +\infty.
\]
By scaling, we set
\[
\tilde{v}_\varepsilon(y) = \frac{u_\varepsilon(x_\varepsilon + r_\varepsilon y)}{m_\varepsilon} \quad \text{in } \Omega_\varepsilon,
\]
where
\[
\Omega_\varepsilon = \{ y \in \mathbb{R}^N \mid x_\varepsilon + r_\varepsilon y \in \Omega \}, \quad \text{and} \quad r_\varepsilon = |x_\varepsilon| \frac{2-s_1}{s_2} = \left( \frac{|x_\varepsilon|}{k_\varepsilon} \right)^{2-s_2}.
\]
By equation \((3.1)\), \( \tilde{v}_\varepsilon(y) \) satisfies
\[
\begin{cases}
\Delta \tilde{v}_\varepsilon + \lambda \left( \frac{k_\varepsilon}{|x_\varepsilon|} \right)^{s_1-s_2} \frac{\tilde{v}_\varepsilon^{p_1(s)}}{|y_\varepsilon + \frac{|y_\varepsilon + |x_\varepsilon| y_\varepsilon|}{|x_\varepsilon|}|^{p_1(s)}} + \frac{\tilde{v}_\varepsilon^{p_2(s)}}{|y_\varepsilon + \frac{|y_\varepsilon + |x_\varepsilon| y_\varepsilon|}{|x_\varepsilon|}|^{p_2(s)}} = 0 & \text{in } \Omega_\varepsilon, \\
y_\varepsilon = \frac{x_\varepsilon}{|x_\varepsilon|} \quad \text{and} \quad \tilde{v}_\varepsilon(y) \leq \tilde{v}_\varepsilon(0) = 1.
\end{cases}
\]
Let \( \Omega_\varepsilon \to H \) as \( \varepsilon \to 0 \), where either \( H = \mathbb{R}^N \) or \( H \) is a closed half space of \( \mathbb{R}^N \). Note \( \left( \frac{k_\varepsilon}{|x_\varepsilon|} \right)^{s_1-s_2} \) and \( \frac{x_\varepsilon}{|x_\varepsilon|} \) tend to 0 as \( \varepsilon \to 0 \). Then by applying elliptic estimates, \( \tilde{v}_\varepsilon \) converges to \( \tilde{v} \) in \( C^2_{loc}(H) \), where
\[
\Delta \tilde{v} + \tilde{v}^{p_2} = 0 \quad \text{in } H.
\]

If $H$ is a half space of $\mathbb{R}^N$, then $v$ also satisfies $v = 0$ on $\partial H$. Since $p_2^* = \frac{2(N-s_2)}{N-2} - 1 < \frac{N+2}{N-s_2}$, $v(y) \equiv 0$ in $H$ no matter $H$ is $\mathbb{R}^N$ or a half space. But it yields a contradiction to $v(0) = 1$. Thus, the claim is proved.

After (3.6) is established, we set

$$v_\varepsilon(y) = m_\varepsilon^{-1}u_\varepsilon(x_\varepsilon + k_\varepsilon y).$$

Then $v_\varepsilon(y)$ satisfies

$$\Delta v_\varepsilon + \frac{v_\varepsilon^{p_2(c)}}{\varepsilon^2 + y^2} + \frac{v_\varepsilon^{p_2(c)}}{\varepsilon^2 + y^2} = 0 \quad \text{in } \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{ y \in \mathbb{R}^N \mid x_\varepsilon + k_\varepsilon y \in \Omega \}$. Since $\frac{x_\varepsilon}{k_\varepsilon}$ is bounded, without loss of generality, we may assume $\frac{x_\varepsilon}{k_\varepsilon} \to y_0$. Therefore, $\Omega_\varepsilon \to H$ as $\varepsilon \to 0$, where $H$ is a half space of $\mathbb{R}^N$ with $-y_0 \in \partial H$ and by the elliptic estimates, $v_\varepsilon(y) \to v(y)$ in $C^2_{\text{loc}}(H)$. Clearly, $v$ satisfies

$$\begin{cases} \Delta v + \frac{u^{p_2(c)}}{|x|^{s_2}} + \frac{u^{p_2(c)}}{|x|^{s_2}} = 0 & \text{in } H, \\ v = 0 & \text{on } \partial H. \end{cases}$$

(3.7)

Since $v(0) = 1$, we have $y_0 \neq 0$. By a linear transformation of $y$, $H$ can be map onto $\mathbb{R}^N_+$ and $v$ is an entire solution of equation (1.1) with $\Omega = \mathbb{R}^N_+$. This completes the proof of the existence part of Theorem 1.2.

**Remark.** Suppose $v$ is a positive entire solution of (1.1). Then the Kelvin transformation $\hat{v}(y) = |y|^{2-n}v(\frac{y}{|y|^2})$ is also a positive entire solution. By the regularity (3.5), $|\hat{v}(y)| \leq C|y|$ for $|y| < 1$. Thus,

$$|v(y)| \leq C|y|^{1-n} \quad \text{for } |y| \geq 1. \quad (3.8)$$

By the standard gradient estimate, we have

$$|\nabla v(y)| \leq |y|^{-n} \quad \text{for } |y| \geq 1. \quad (3.9)$$

By using the well-known method of moving sphere, it can be proved that after a suitable scaling, $v(y) = \hat{v}(y)$. Since the argument is standard now, the proof is omitted here.

**Corollary 3.1.** There exists an entire solution $v$ of equation (1.1) with $\Omega = \mathbb{R}^N_+$ such that the critical value $\Phi(v) = \inf\{\Phi(u) \mid u \text{ is an entire solution of (1.1)}\}$.

**Proof.** We first note that

$$\{ \Phi(u) \mid u \text{ is a positive entire solution of (1.1)} \} \neq \emptyset,$$

because

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dy = \lambda \int_{\mathbb{R}^N_+} \frac{u^{2^*(s_1)}}{|x|^{s_1}} dy + \int_{\mathbb{R}^N_+} \frac{u^{2^*(s_2)}}{|x|^{s_2}} dy$$

$$\leq C \left( \left( \int_{\mathbb{R}^N_+} |\nabla u|^2 dy \right)^{2^*(s_1)} + \left( \int_{\mathbb{R}^N_+} |\nabla u|^2 dy \right)^{2^*(s_2)} \right)$$

implies $\|\nabla u\| \geq c_0$ for some constant $c_0 > 0$. 

14
Suppose \( v_j \) is a sequence of positive entire solutions of
\[
\begin{align*}
\Delta v_j + \lambda \frac{v_j^{p_1}}{|y|^{p_1}} + \frac{v_j^{p_2}}{|y|^{p_2}} &= 0 \quad \text{in } \mathbb{R}^N_+,
\quad \left| v_j \right| = 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{align*}
\]
(3.10)
such that \( \Phi(v_j) \downarrow \inf \{ \Phi(u) \mid u \text{ is an entire solution of (4.1)} \} \). By the remark above, we can assume \( v(y) = v_j(y) \). By (3.2) again, \( \|\nabla v_j\|_{L^2(\mathbb{R}^N)} \leq C \) for some constant \( C > 0 \). Let \( v_j \to v \) in \( H^1_0(\mathbb{R}^N_+) \). If \( v \neq 0 \), then
\[
\lim_{j \to +\infty} \Phi(v_j) = \left( \frac{1}{2} - \frac{1}{p_1 + 1} \right) \lim_{j \to +\infty} \int_{\mathbb{R}^N_+} |\nabla v_j|^2 \, dy + \frac{1}{p_1 + 1} \int_{\mathbb{R}^N_+} v_j^{p_1+1} \, dy \geq \left( \frac{1}{2} - \frac{1}{p_1 + 1} \right) \int_{\mathbb{R}^N_+} |\nabla v|^2 \, dy + \frac{1}{p_1 + 1} \int_{\mathbb{R}^N_+} v^{p_1+1} \, dy = \Phi(v).
\]
Then it yields the conclusion of Corollary 3.1.

If \( v_j \to 0 \), then \( \max_{|y| \leq 2} v_j(y) \to +\infty \), because \( v_j(y) = v_j(y) \) implies
\[
\frac{1}{2} \Phi(v_j) = \frac{1}{2} \int_{B_1} |\nabla v_j|^2 \, dy - \lambda \frac{v_j^{p_1+1}}{p_1+1} \int_{B_1} \frac{v_j^{p_1+1}}{|y|^{p_1+1}} \, dy - \frac{1}{p_2+1} \int_{B_2} \frac{v_j^{p_2+1}}{|y|^{p_2+1}} \, dy
\]
\[
= \left( \frac{1}{2} - \frac{1}{p_1 + 1} \right) \int_{B_1} |\nabla v_j|^2 \, dy + \frac{1}{p_1 + 1} \int_{B_2} v_j^{p_1+1} \, dy,
\]
By the proof of Theorem 4.2, we see that \( v_j \) blows up at \( y = 0 \) and the scaling \( w_j(y) \):
\[
w_j(y) = \frac{v_j(x_j + k_j y)}{v_j(x_j)} \to w,
\]
where \( v_j(x_j) = \max_{|y| \leq 2} v_j(y) \to \infty, k_j = m_j \to \infty \), and \( w \) is also a positive entire solution of equation (4.1). Thus,
\[
\lim_{j \to +\infty} \frac{1}{2} \Phi(v_j) \geq \left( \frac{1}{2} - \frac{1}{p_1 + 1} \right) \int_{\mathbb{R}^N_+} |\nabla w|^2 \, dy + \frac{1}{p_1 + 1} \int_{\mathbb{R}^N_+} w^{p_1+1} \, dy = \Phi(w),
\]
which yields
\[
\inf \{ \Phi(u) \mid u \text{ is an entire solution of equation (4.1)} \} = 0,
\]
a contradiction. Hence, \( v_j \to 0 \), and Corollary 3.1 is proved.

\section{Proof of Theorem 1.1}

Let \( v(y) \) be a least-energy solution of
\[
\begin{align*}
\Delta v + \lambda \frac{v^{p_1}}{|y|^{p_1}} + \frac{v^{p_2}}{|y|^{p_2}} &= 0 \quad \text{in } \mathbb{R}^N_+, \quad v(y) > 0 \quad \text{in } \mathbb{R}^N_+ \quad \text{and} \quad v(y) = 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{align*}
\]
(4.1)
and
\[
c_1 = \Phi(v) = \frac{1}{2} \int_{\mathbb{R}^N_+} |\nabla v|^2 \, dy - \frac{\lambda}{2^*(s_1)} \int_{\mathbb{R}^N_+} \frac{v^{2^*(s_1)}}{|y|^{s_1}} \, dy - \frac{1}{2^*(s_2)} \int_{\mathbb{R}^N_+} \frac{v^{2^*(s_2)}}{|y|^{s_2}} \, dy.
\]
(4.2)
Lemma 4.1. Suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \partial \Omega$. If $H(0) < 0$, then there exists a nonnegative function $v_0 \in H^1_0(\Omega) \setminus \{0\}$ such that

$$\max_{t \geq 0} \Phi(t v_0) < c_1.$$ 

Proof. Without loss of generality, we may assume that in a neighborhood of 0, $\partial \Omega$ can be represented by $x_n = \varphi(x')$, where $x' = (x_1, \cdots, x_{N-1})$, $\varphi(0) = 0$, $\nabla' \varphi(0) = 0$, $\nabla' = (\partial_1, \cdots, \partial_{N-1})$, and the outer normal of $\partial \Omega$ at 0 is $-e_N = (0, \cdots, 0, -1)$. Define

$$\phi(x) := (x', x_N - \varphi(x')).$$

We choose a small positive number $r_0$ so that there exist neighborhoods of 0, $U$ and $\tilde{U}$, such that $\phi(U) = B_{r_0}(0)$, $\phi(U \cap \Omega) = B^+_r(0)$, $\phi(\tilde{U}) = B^+_{r_0}(0)$ and $\phi(\tilde{U} \cap \Omega) = B^+_r(0)$. Here, we adopt the notation:

$$B^+_r(0) = B_{r_0} \cap \mathbb{R}^n_+ \text{ for } r_0 > 0.$$ 

Let $\eta \in C_0^\infty(U)$ be a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$. Set

$$u_\varepsilon(x) := \eta(x) v_\varepsilon(x) := \eta(x) \varepsilon^{- \frac{N-2}{2}} v\left(\frac{\phi(x)}{\varepsilon}\right) \text{ for } x \in \Omega.$$ 

For $t \geq 0$, we have

$$\Phi(t u_\varepsilon) = \frac{t^2}{2} \int_{\Omega} \vert \nabla u_\varepsilon \vert^2 \, dx - \frac{\lambda t^{2s_1}}{2^{s_1}} \int_{\Omega} u_\varepsilon^{2s_1} \, dx - \frac{t^{2s_2}}{2} \int_{\Omega} u_\varepsilon^{2s_2} \, dx \text{ for } u \in H^1_0(\Omega).$$

(4.3)

In what follows, we estimate each integral on the right-hand side of (4.3). Basically, the computation will be similar to Lemma 2.2 in [13]. For the sake of completeness, we will sketch the proof here. We refer the readers to [13] for details of computation.

By the change of the variable $\frac{\phi(x)}{\varepsilon} = y$, we get

$$\int_{\Omega} \vert \nabla u_\varepsilon \vert^2 \, dx = \int_{U \cap \Omega} \eta^2 \vert \nabla v \vert^2 \, dx - \int_{U \cap \Omega} \eta(\Delta \eta) v^2 \, dx$$

$$= \int_{\mathbb{R}^n_+} \vert \nabla v(y) \vert^2 \, dy - 2 \int_{B^+_r} \eta (\phi^{-1}(\varepsilon y))^2 \partial_N v(y) \nabla' v(y) \cdot (\nabla' \varphi)(\varepsilon y') \, dy + O(\varepsilon^2).$$

By using integration by parts and equation (4.1), the second term can be estimated as the following.

$$- 2 \int_{B^+_r} \eta (\phi^{-1}(\varepsilon y))^2 \partial_N v(y) \nabla' v(y) \cdot (\nabla' \varphi)(\varepsilon y') \, dy$$

$$= \frac{2}{\varepsilon} \int_{B^+_r} \eta (\phi^{-1}(\varepsilon y))^2 \partial_N v(y) \sum_{i=1}^{N-1} \partial_i v(y) \varphi(\varepsilon y') + O(\varepsilon^2)$$

$$= \frac{1}{\varepsilon} \int_{B^+_r} \eta (\phi^{-1}(\varepsilon y))^2 \partial_N \left[(\partial_N v(y))^2 \right] \varphi(\varepsilon y') \, dy$$

$$- \frac{2\lambda}{2^{s_1}} \int_{B^+_r} \eta (\phi^{-1}(\varepsilon y))^2 \frac{\partial_N [v(y)^{2s_1}]}{|y|^{s_1}} \varphi(\varepsilon y') \, dy.$$
and similarly, we have

\[ -\frac{2}{2^*(s_2)} \int_{B^+_{\frac{\epsilon}{2}}} \eta \left( \phi^{-1}(\epsilon y) \right)^2 \frac{\partial_{N} \left[ v(y)2^*(s_2) \right]}{|y|^{s_2}} \varphi(\epsilon y') dy + O(\epsilon^2) \]

\[ =: I_1 + I_2 + I_3 + O(\epsilon^2). \]

Since \( \partial \Omega \) is \( C^2 \) at 0, \( \varphi \) can be expanded as

\[ \varphi(y') = \sum_{i=1}^{N-1} \alpha_i y_i^2 + o(1)|y'|^2. \]

Hence,

\[ I_1 = \frac{1}{\epsilon} \int_{B^+_{\frac{\epsilon}{2}} \cap \partial R^N_+} \eta \left( \phi^{-1}(\epsilon y') \right)^2 \partial_{N} v(y',0) \varphi(\epsilon y') dy' \]

\[ = \varepsilon \sum_{i=1}^{N-1} \alpha_i \int_{R^{N-1}} (\partial_N v(y',0))^2 y_i^2 dy' (1 + o(1)) + O(\varepsilon^2) = K_1 H(0) \varepsilon (1 + o(1)) + O(\varepsilon^2), \]

where

\[ K_1 = \int_{R^{N-1}} |\partial_N v(y',0)|^2 |y'|^2 dy', \]

\[ I_2 = -\frac{2 \lambda s_1}{2^*(s_1) \varepsilon} \int_{B^+_{\frac{\epsilon}{2}}} \eta \left( \phi^{-1}(\epsilon y) \right)^2 v(y)2^*(s_1) y_N \varphi(\epsilon y') dy \]

\[ = -K_2 H(0) \varepsilon (1 + o(1)) \varepsilon + O(\varepsilon^2), \]

where

\[ K_2 = \frac{2 \lambda s_1}{2^*(s_1)} \int_{R^N} v(y)2^*(s_1) |y'|^2 y_N dy, \]

and

\[ I_3 = -\frac{2 \varepsilon_2}{2^*(s_2) \varepsilon} \int_{B^+_{\frac{\epsilon}{2}}} \eta \left( \phi^{-1}(\epsilon y) \right)^2 v(y)2^*(s_2) |y'|^2 y_N \varphi(\epsilon y') dy \]

\[ = -K_3 H(0) \varepsilon (1 + o(1)) + O(\varepsilon^2), \]

where

\[ K_3 = \frac{2 \varepsilon_2}{2^*(s_2)} \int R^N v(y)2^*(s_2) |y'|^2 y_N dy. \]

By (3.8) and (3.9), \( K_i, i = 1, 2, 3 \), are finite. Therefore, we have

\[ \int_{\Omega} |\nabla u_\epsilon|^2 dx = \int_{R^N} |\nabla v|^2 dy + \varepsilon H(0)(K_1 - K_2 - K_3)(1 + o(1)) + O(\varepsilon^2), \]

and similarly, we have

\[ \lambda \int_{\Omega} \frac{u_\epsilon^{2^*(s_1)}}{|x|^{s_1}} dx = \lambda \int_{R^N} \frac{v^{2^*(s_1)}}{|y|^{s_1}} dy - \frac{s \lambda}{\varepsilon} \int_{B^+_{\frac{\epsilon}{2}}} \frac{v(y)2^*(s_1)y_N \varphi(\epsilon y')}{|y|^{2^*+s_1}} dy + O(\varepsilon^2) \]

17
Therefore, and where positive constant defined by (4.2). Then Lemma 4.1 yields

\[ P \in \mathcal{P} \]

Thus Lemma 4.1 is proved.

We are now in a position to prove Theorem 1.1

\[ \Phi(\epsilon H) = (1) = \epsilon H \]

\[ \Phi(1) = \epsilon H \]

\[ f(1) = f_2(t) + O(\epsilon^2), \]

where

\[ f_1(t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dy - \frac{\lambda t^2(s_1)}{2} \int_{\mathbb{R}^N} |v|^{2n} dy - \frac{t^2(s_2)}{2} \int_{\mathbb{R}^N} v^2 \nu dy. \]

Since \( 2^*(s_2) > 2^*(s_1) \), \( \Phi(tu_c) \) has the unique maximum. Note that

\[ \max_{t \geq 0} f_1(t) = f_1(1) = c_1. \]

Hence, the maximum of \( \Phi(tu_c) \) occurs at \( t_c = 1 + o(1) \). By noting that

\[ f_2(t) = \epsilon H(0) \left[ \frac{t^2}{2} (K_1 - K_2 - K_3 + o(1)) + \frac{t^2(s_1)}{2} K_2 (1 + o(1)) + \frac{t^2(s_2)}{2} K_3 (1 + o(1)) \right], \]

and \( f_2(1) = \epsilon H(0) K_1 (1 + o(1)) < 0 \). Hence, we have

\[ \max_{t \geq 0} \Phi(tu_c) = \Phi(t_c u_c) \leq f_1(t_c) + f_2(t_c) < f_1(t_c) \leq f_1(1) = c_1. \]

Thus Lemma 4.1 is proved.

We are now in a position to prove Theorem 1.1

**Proof of Theorem 1.1.** As before, we let for small positive \( \epsilon \),

\[ \Phi(\epsilon u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{p_1(\epsilon) + 1} \int_{\Omega} \frac{(u^+)^{p_1(\epsilon) + 1}}{|x|^{s_1}} dx - \frac{1}{p_2(\epsilon) + 1} \int_{\Omega} \frac{(u^+)^{p_2(\epsilon) + 1}}{|x|^{s_2}} dx, \]

where \( p_1(\epsilon) = 2^*(s_1) - 1 - \epsilon \) and \( p_2(\epsilon) = 2^*(s_2) - 1 - \left( \frac{2 - s_2}{s_2 - s_1} \right) \epsilon \). By Lemma 4.1, there exists \( u_0 \in H_0^1(\Omega) \) such that \( \Phi(\epsilon u_0) \leq 0 \) and

\[ c_\epsilon^* = \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(\epsilon w), \]

where \( \mathcal{P} \) is the class of all continuous paths in \( H_0^1(\Omega) \) connecting 0 with \( u_0 \). Let \( c_1 \) be the positive constant defined by (4.2). Then Lemma 4.1 yields

\[ c_\epsilon^* \leq c_1, \quad (4.4) \]
provided that $\varepsilon \in [0, \varepsilon_0]$ for some small $\varepsilon_0 > 0$. For $\varepsilon > 0$, there exists a solution $u_\varepsilon \in H^1_0(\Omega)$ of
\[
\Delta u_\varepsilon + \frac{u^{p_1(\varepsilon)}}{|x|^{s_1}} + \frac{u^{p_2(\varepsilon)}}{|x|^{s_2}} = 0 \quad \text{in } \Omega,
\]
\[
u_\varepsilon = 0, \quad \text{on } \partial \Omega,
\]
and $\Phi_\varepsilon(u_\varepsilon) = c_\varepsilon$. Similar to (3.2), we have
\[
\int_\Omega |\nabla u_\varepsilon|^2 dx \leq C_1,
\]
for some constant $C_1$ independent of $\varepsilon$. By passing to a subsequence if necessary, we may assume
\[
u_\varepsilon \rightharpoonup u \quad \text{in } H^1_0(\Omega).
\]
If $u \neq 0$, then clearly $u$ is a solution of (1.1) and Theorem 1.1 is proved. So it remains to prove $u \equiv 0$ in $\Omega$.

Suppose $u \equiv 0$. As in Section 3, there exists $x_\varepsilon \in \Omega$ such that
\[
u_\varepsilon(x_\varepsilon) = \max_{x \in \Omega} u_\varepsilon(x) = m_\varepsilon \to +\infty,
\]
and after a linear transformation on $y$, we have
\[
v_\varepsilon(y) = \frac{u_\varepsilon(x_\varepsilon + k_\varepsilon y)}{m_\varepsilon} \to v(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N_+),
\]
where $k_\varepsilon = m_\varepsilon^{-\frac{s_1}{2} + \frac{s_2}{2s_2}}$ and $v$ satisfies
\[
\Delta v + \frac{v^{p_1}}{|y|^{s_1}} + \frac{v^{p_2}}{|y|^{s_2}} = 0 \quad \text{in } \mathbb{R}^N_+,
\]
v $\equiv 0$ on $\mathbb{R}^N_+$.

Also by a direct computation, we have
\[
\liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^2 dx \geq \int_{\mathbb{R}^N_+} |\nabla v|^2 dy, \quad (4.5)
\]
and
\[
\lim_{\varepsilon \to 0} \int_\Omega \frac{u^{p_2(\varepsilon) + 1}}{|x|^{s_2}} dx \geq \int_{\mathbb{R}^N_+} \frac{v^{p_2 + 1}}{|y|^{s_2}} dy. \quad (4.6)
\]
By (3.2), we have
\[
c^*_\varepsilon = \left(1 - \frac{1}{p_1(\varepsilon) + 1}\right) \int_{\Omega} |\nabla u_\varepsilon|^2 dx + \left(1 - \frac{1}{p_2(\varepsilon) + 1}\right) \int_{\Omega} \frac{u^{p_2(\varepsilon) + 1}}{|x|^{s_2}} dx.
\]
Thus, (4.5) and (4.6) yields
\[
c^* = \lim_{\varepsilon \to 0} c^*_\varepsilon = \left(1 - \frac{1}{2^*(s_1)}\right) \int_{\mathbb{R}^N_+} |\nabla v|^2 dy + \left(1 - \frac{1}{2^*(s_2)}\right) \int_{\mathbb{R}^N_+} \frac{v^{2^*(s_2)}}{|y|^{s_2}} dy = c_1,
\]
which contradicts to (4.4). Hence, $u \neq 0$, and then Theorem 1.1 is proved.
5 The Case $s_2 = 0$

As discussed in Introduction, equation (1.1) with $\lambda < 0$ has no least-energy solutions. In this section, we will consider a perturbed equation from equation (1.1):

$$\begin{cases}
\Delta u - \frac{u^{2^*(s)-1}}{|x|^s} + u^p + u^{\frac{N+2}{N-2}} = 0 & \text{in } \Omega, \\
u(x) > 0 & \text{in } \Omega \text{ and } u(x) = 0 \text{ on } \partial \Omega.
\end{cases} \quad (5.1)$$

**Theorem 5.1.** Suppose $2^*(s) - 1 < p < \frac{N+2}{N-2}$ and $N \geq 4$. Then equation (5.1) has a positive solution.

**Proof.** Let

$$\Phi(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2^*(s)} \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s} \, dx - \frac{1}{p+1} \int_\Omega (u^+)^{p+1} \, dx - \frac{N-2}{2N} \int_\Omega (u^+)^{\frac{2N}{N-2}} \, dx.$$

Choose $0 \neq x_0 \in \Omega$ and

$$v_\mu(x) = \phi(x) \left( \frac{1}{1 + \mu^2 |x - x_0|^2} \right)^{\frac{N-2}{2}},$$

for large $\mu > 0$, where $\phi(x)$ is a cut-off function near $x_0$. Then it is not difficult to show that

$$\sup_{t \geq 0} \Phi(tv_\mu) = \Phi(t_0 v_\mu) < \frac{1}{N} \frac{\hat{S}}{\hat{N}}$$

provided that $2^*(s) - 1 < p < \frac{N+2}{N-2}$ and $\mu$ is sufficiently large. Let $u_0 = t_0 v_\mu$, and

$$c_* = \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w),$$

where $\mathcal{P}$ is the class of continuous paths in $H_0^1(\Omega)$ connecting 0 and $u_0$. Then it is easy to see

$$0 < c_* < \frac{1}{N} \frac{\hat{S}}{\hat{N}}. \quad (5.2)$$

We claim: $c_*$ is a critical value for $\Phi$. By the deformation lemma (see lemma in []), there exists $u_j \in H_0^1(\Omega)$ such that

$$\Phi(u_j) = c_* (1 + o(1)), \quad \int_\Omega |\nabla u_j|^2 \, dx + \int_\Omega \frac{(u_j^+)^{2^*(s)}}{|x|^s} \, dx - \int_\Omega (u_j^+)^{p+1} \, dx - \int_\Omega (u_j^+)^{\frac{2N}{N-2}} \, dx = o(1) \|u_j\|_{H_0^1}, \quad (5.3)$$

where $o(1) \to 0$ as $j \to +\infty$. By (5.2) and (5.3), we have

$$\int_\Omega |\nabla u_j|^2 \, dx \leq C$$

for some constant $C$ independent of $j$. By passing to a subsequence if necessary, we assume

$$\begin{cases} 
 u_j \rightharpoonup u & \text{in } H_0^1(\Omega), \\
u_j \to u & \text{in } L^{p+1}(\Omega).
\end{cases}$$

If $u \neq 0$, then it is easy to see $u$ is a solution of (5.1) and $\Phi(u) = c_*$. Hence, it remains to show that $u \neq 0$. We prove it by contradiction.
Now suppose $u \equiv 0$, and set
\[ A = \lim_{j \to +\infty} \int_{\Omega} |\nabla u_j|^2 \, dx, \quad B = \lim_{j \to +\infty} \int_{\Omega} \frac{(u_j^+)^{2^*(s)}}{|x|^s} \, dx \quad \text{and} \quad C = \lim_{j \to +\infty} \int_{\Omega} (u_j^+)^{\frac{2N}{N-2}} \, dx. \]

Then (5.3) implies
\[ c_* = \frac{A}{2} + \frac{B}{2^*(s)} - \frac{N-2}{2N} C, \tag{5.4} \]
and
\[ C = A + B. \tag{5.5} \]

By the Sobolev inequality, (5.5) implies
\[ C = A + B \geq A \geq S_N C^{\frac{1}{4-N}}, \]
Thus, we have
\[ C \geq S_N^\frac{N}{4-N} \quad \text{and} \quad A \geq S_N C^{\frac{1}{4-N}} \geq S_N^\frac{N}{4-N}. \]

Then (5.4) yields
\[ c_* = \left( \frac{1}{2} - \frac{N-2}{2N} \right) A + \left( \frac{1}{2^*(s)} - \frac{N-2}{2N} \right) B \geq \frac{1}{N} S_N^\frac{N}{4-N}, \]
a contradiction to (5.2). Hence, Theorem 5.1 is proved.

References

[1] T. Bartsch, S. Peng and Z. Zhang, Existence and non-existence of solutions to elliptic equations related to the Caffarelli-Kohn-Nirenberg inequalities, Calc. Var. Partial Differential Equations 30 (2007), no. 1, 113–136.
[2] H. Berestycki and L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geom. Phys. 5 (1988), no. 2, 237–275.
[3] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983), no.4, 437–477.
[4] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, Compositio Math. 53 (1984), no. 3, 259–275.
[5] D. Cao and P. Han, Solutions to critical equation with multi-singular inverse square potentials, J. Differential Equations 224 (2006), no. 2, 332–372.
[6] F. Catrina and Z. Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001), no.2, 229–258.
[7] C. C. Chen and C. S. Lin, A spherical Harnack inequality for singular solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), (2002), no.3-4, 713–738.
[8] C. C. Chen and C. S. Lin, Estimates of the conformal scalar curvature equation via the method of moving planes, Comm. Pure Appl. Math. 50 (1997), no.10, 971–1017.
[9] C. C. Chen and C. S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes. II, J. Differential Geom. 49 (1998), no. 1, 115–178.
[10] C. C. Chen and C. S. Lin, Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent, Duke Math. J. 78 (1995), no. 2, 315–334.
[11] J. L. Chern and C.S. Lin, Minimizers of Caffarelli-Kohn-Nirenberg inequalities on domains with the singularity on the boundary, Arch. Rational Mech. Anal. 197 (2010), 401–432.
[12] K. S. Chou and C. W. Chu, On the best constant for a weighted Sobolev-Hardy inequality, J. London Math. Soc. (2) 48 (1993), no. 1, 137–151.
[13] J. Dolbeault, M. J. Esteban and G. Tarantello, The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions, Ann. Sc. Norm. Super. Pisa Cl. Sci (5) 7 (2008), no. 2, 313–341.
[14] N. Ghoussoub and X. S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no.6, 767–793.
[15] N. Ghoussoub and F. Robert, The effect of curvature on the best constant in the Hardy-Sobolev inequalities, Geom. Funct. Anal. 16 (2006), no.6, 1201–1245.
[16] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.
[17] C. H. Hsia, C. S. Lin and H. Wadade, Revisiting an idea of Brézis and Nirenberg, to appear in J. Funct. Anal.
[18] Q. Jin, Y. Y. Li and H. Xu, Symmetry and Asymmetry: The method of moving spheres, Advances in Differential Equations 13 (2008), 601-640.
[19] Y. Y. Li, Prescribing scalar curvature on $S^n$ and related problems. I, J. Differential Equations 120 (1995), no. 2, 319–410.
[20] Y. Y. Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, Journal d’Analyse Mathematique 90 (2003), 27-87.
[21] Y. Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995), 383-417.
[22] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118 (1983), no.2, 349–374.
[23] C. S. Lin, Interpolation inequalities with weights, Comm. Partial Differential Equations 11 (1986), no. 14, 1515–1538.
[24] C. S. Lin and Z. Q. Wang, Symmetry of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities, Proc. Amer. Math. Soc. 132 (2004), no.6, 1685–1691.
[25] R. Musina, Ground state solutions of a critical problem involving cylindrical weights, Nonlinear Anal. 68 (2008), no. 12, 3972–3986.
[26] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. 43 (1971), 304–318.

YanYan Li
Department of Mathematics
Rutgers University
Piscataway, NJ 08854, USA
yyli@math.rutgers.edu

Chang-Shou Lin
Taïda Institute for Mathematical Sciences
Department of Mathematics
National Taiwan University
Taipei 106, Taiwan
cslin@math.ntu.edu.tw

22