On String Field Theory and Effective Actions

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ABSTRACT

A truncation of string field theory is compared with the duality invariant effective action of $D = 4, N = 4$ heterotic strings to cubic order. The three string vertex must satisfy a set of compatibility conditions. Any cyclic three string vertex is compatible with the $D = 4, N = 4$ effective field theory. The effective actions may be useful in understanding the non-polynomial structure and the underlying symmetry of covariant closed string field theory, and in addressing issues of background independence. We also discuss the effective action and string field theory of the $N = 2$ string.

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1 Introduction

The low–energy effective action of string theory is derived from string field theory (SFT) by integrating out massive modes of the string field (see for example ref. [1] for a review). In SFT different classical solutions correspond to string propagation in different backgrounds. A property of string theory is the possibility to deform a given vacuum continuously. In four dimensions such deformations correspond to changing the vacuum expectation values of massless scalars with a flat effective potential.

Modes that are not massless in a given vacuum, may become light at some other points of the moduli space of string backgrounds. To obtain effective theories compatible with the global structure of the moduli space of backgrounds, one should integrate out only ultra–massive modes, that is, modes that remain massive in all backgrounds. Alternatively, one may use restrictions deduced from space–time supersymmetry in order to find the structure of the effective actions [2].

A (successful) covariant string field theory of closed strings has been discussed recently [3, 4]. Yet, the invariance principle that needs a non–polynomial nature of the covariant SFT and the issue of background independence are not clear. Under the weak assumptions of ref. [5], a covariant string field theory of closed strings is necessarily non–polynomial just as Einstein’s theory of gravity. The necessity of higher–point interactions may be an indication of the existence of a nontrivial invariance principle that needs a nonlinear expression to become manifest.

Background independence means that a SFT formulated around a given background is equivalent to a SFT formulated around any other backgrounds. In other words, different classical solutions of a given SFT should interpolate between string field theories formulated in different backgrounds. Practically, however, deforming a classical string field solution relating two backgrounds is hard (see for example ref. [6]). This is mainly due to the existence of an infinite number of string oscillation modes.

As any off–shell formulation, string field theory is not uniquely defined. The three string interaction is described by the three string vertex, and the vertex is defined by an off–shell formulation of the three–point function in the $2–d$ conformal field theory. A symmetric overlap three string vertex appears to be a canonical choice that is simple, but it would be interesting to examine the other possibilities. Even a symmetric vertex can be defined by
different maps of three semi–infinite cylinders (or unit discs) to the sphere.
(String field theories formulated with different vertices appear to be related by a field redefinition [7].)

In this paper we study the compatibility conditions of string field theory and effective actions of point particles to cubic order. This can be used as a ‘test’ for different formulations of SFT, and may be helpful in understanding the non–polynomial structure in covariant SFT of closed strings and its (possible) invariance principle, as well as addressing issues of background independence.

The effective actions discussed in this work are particular ones. We start in section 2 with the $D = 4$ heterotic string in toroidal backgrounds. In four dimensions the effective action is a gauged $N = 4$ supergravity with matter. The action is restricted as a consequence of space–time supersymmetry, and one is only left with the freedom to choose the set of matter multiplets and the gauge algebra. Keeping a one to one correspondence with the string states that may become light at some point of the moduli space of toroidal background, the number of matter multiplets is infinite, and an infinite dimensional gauge symmetry (called the ‘duality invariant string gauge algebra’) is introduced [2]. A short review of a completely duality invariant effective action of $N = 4$ heterotic strings is presented in section 2.

Once the structure constants of the gauge algebra are given, the scalar potential is fixed. Expanding the scalar potential in physical fields around zero cosmological constant minima gives rise to a non–polynomial potential. The quadratic term gives the correct mass spectrum of the corresponding string modes; this is described in ref. [2]. In this work we focus on the cubic terms and compare them to the cubic interaction in string field theory.

In section 3 we present the three string vertex and the string field in toroidal backgrounds. We then truncate the string field to modes that may become light somewhere in the moduli space of toroidal backgrounds. Compatibility with the effective action of $N = 4$ heterotic strings is then translated into a set of conditions, that should be satisfied by the lower modes of the three string vertex (namely, by the Neumann coefficients $N_{00}, N_{01}$ and $N_{11}$). In section 4 we prove that any cyclic three string vertex obey the conditions, and therefore, a cyclic vertex is compatible with the effective action.

In section 5 we discus the effective action of the $N = 2$ string in compari-
son with the $N = 2$ string field theory. In toroidal backgrounds, a truncation of the $N = 2$ string field to modes that may become light include, essentially, all the modes that may become physical in toroidal backgrounds. This is peculiar to $N = 2$ strings which, therefore, may be used as a simple laboratory to more complicated string theories.

In section 6 we present a discussion about this work and some open questions. Finally, in a set of appendices we present some useful formulas, and discuss two examples.

2 A Completely Duality Invariant Effective Action of $N = 4$ Heterotic Strings

In this section we present a short review of a completely duality invariant effective action. We then expand the scalar potential in physical fields to cubic order.

The construction of a completely duality invariant effective action of the $D = 4$, $N = 4$ heterotic string is presented in ref. [2] using the structure of gauged $N = 4$ supergravity with matter [8], obtained through superconformal methods by coupling a number of $N = 4$ vector multiplets to the $N = 4$ superconformal gauge multiplets. Here we present only some points relevant for the discussion in section 3; for more details we refer the reader to ref. [4].

The important point is that the form of a gauged $D = 4$, $N = 4$ supergravity (with at most two space–time derivatives) coupled to matter multiplets is almost uniquely fixed: it is completely determined by the knowledge of the gauge group [8]. Thus one has only the freedom to choose the right set of matter multiplets and the correct gauge group in order to figure out completely the low–energy effective action of the string.

The scalar fields which are relevant for $D = 4$ low–energy physics $Z_a^S$, $a = 1, ... 6$, $S = 1, ..., \infty$ will be labeled as follows:

\[
Z_a^b, \quad Z_a^I, \quad Z_a^p, \quad p \in \Gamma^{6,22}, \quad p^2 = 2, \quad a, b = 1, ..., 6, \quad I = 1, ..., 22, \quad (2.1)
\]

where $\Gamma^{6,22}$ is an even self–dual lorentzian lattice of signature $(6,22)$. The momentum $p = (p_L, p_R)$ has 6 left–handed components $p_a^L$, $a = 1, ..., 6$, and 22 right–handed components $p_I^R$, $I = 1, ..., 22$. This is the Narain lattice of
the string toroidal compactification [9]. The scalar product is lorentzian

\[ pq = - \sum_{a=1}^{6} p_L^a q_L^a + \sum_{I=1}^{22} p_R^I q_R^I, \]

(2.2)

where L (R) denote left–handed (right–handed) momenta. The indices \((b, I)\) refer to Cartan sub–algebra, while the \(p\) indices are lorentzian length two generalized roots (and therefore their number is infinite).

The structure constants of the duality invariant gauge algebra which are relevant for \(D = 4\) low–energy physics are

\[ f_{pqr} = \epsilon(p, q)\delta_{p+q+r, 0}, \quad f_{Ipq} = p_I\delta_{p+q, 0}, \quad f_{apq} = p_a\delta_{p+q, 0}, \]

(2.3)

The two cocycle \(\epsilon(p, q)\) satisfies the identities

\[ \epsilon(p, q)\epsilon(p + q, r) = \epsilon(p, q + r)\epsilon(q, r), \]
\[ \epsilon(p, q) = (-1)^{pq}\epsilon(q, p), \quad \epsilon(p, -p) = 1, \]
\[ \epsilon(p, q) = \epsilon(q, -p - q). \]

(2.4)

The structure constants \(f_{RST}\) in (2.3) are completely antisymmetric. Indices are lowered and raised with the metric \(\eta_{ST}\) defined by

\[ \eta_{ab} = -\delta_{ab}, \quad \eta_{IJ} = \delta_{IJ}, \quad \eta_{pq} = \delta_{p+q, 0}. \]

(2.5)

In the Poincaré gauge it turns out that six of the vector multiplets serve as “compensating” multiplets, absorbing redundant superconformal symmetries. The compensators \(Z^b_a\) in (2.1) can be solved in terms of the physical fields \(Z^I_a\) and \(Z^p_a\). This is done by using the \(SO(6)\) symmetry and the quadratic constraint (originated by selecting the Poincaré gauge and eliminating the Lagrange multiplier auxiliary fields)

\[ \eta_{ST}Z^SZ^T_b = -\delta_{ab}, \]

(2.6)

where \(\eta_{ST}\) is given in (2.5). Therefore, the fields \(Z^S_a\) can be regarded as the first six rows of an \(SO(6, \infty)\) matrix.

The complete action of a gauged \(N = 4\) supergravity coupled to matter is given in ref. [8]. Here we present only the scalar potential

\[ V(Z) = \frac{1}{4} Z^{QU} Z^{SV} (\eta^{TW} + \frac{2}{3} Z^{TW} f_{QST} f_{UVW}), \]

(2.7)
where
\[ Z^{QS} = \sum_a Z^Q_a Z^S_a. \] (2.8)

The simplest zero cosmological constant minima of the potential (2.7) lie in the Cartan sub-algebra. Let us expand the scalar fields around a vacuum expectation value (VEV) in the Cartan sub-algebra,
\[ Z^S_a = C^b_a \delta^S_b + C^I_I a + z^S_a, \] (2.9)
where \( C^b_a, C^I_I, a, b = 1, \ldots, 6, I = 1, \ldots, 22 \), are constants and the constraint (2.6) is satisfied. A rotation of \( Z^S_a \) to
\[ (Z')^S_a = Z^S_a M^S_a, \] (2.10)
where \( M \in SO(6, \infty) \), preserves the scalar quadratic constraint (2.6). Changing the VEV of the scalar fields to new ones in the Cartan sub-algebra can be done by a rotation in \( SO(6, 22) \subset SO(6, \infty) \).

Under the orthogonal transformation (2.10), the potential \( V \) transforms into
\[ V' = \frac{1}{4} (Z')^Q U (Z')^S V (\eta^T W + \frac{2}{3} (Z')^T W) f_{QST} f_{UVW} \]
\[ = \frac{1}{4} Z^Q U Z^S V (\eta^T W + \frac{2}{3} Z^T W) f'_{QST} f'_{UVW}, \] (2.11)
where
\[ f'_{QST} = (M^{-1})^T (M^{-1})_Q f_{Q'S'T'} M^S_a. \] (2.12)
Thus changing the VEV of the Higgs fields in the Cartan sub-algebra is equivalent to a transformation of the structure constants (2.3), given by an \( SO(6, 22) \) rotation of \( \Gamma^{6,22} \) (i.e. a rotation of the “momenta” labels \( p \)). This transformation is an isomorphism of the gauge algebra.

Using the field redefinition described above, one can always choose a zero cosmological constant minimum to lie at
\[ Z^b_a = \delta^b_a, \quad a, b = 1, \ldots, 6, \]
\[ Z^I_a = 0, \quad I = 1, \ldots, 22, \quad Z^p_a = 0 \quad \forall p \in \Gamma^{6,22}. \] (2.13)
Expanding the scalar potential (2.7) around the point (2.13) one finds the quadratic and cubic terms

\[
V(z) = \frac{1}{2} \sum_{p,a,b} [(p^ap^az^{-p}_b - p^a_p z^{-p}_b z^{-p}_a)] + \frac{1}{2} \sum_{p,q,r,a,b,c} \delta_{p+q+r,0} \epsilon(q,p)(p - r)^b \delta_{a,b,c} z^{-p}_a z^{-q}_b z^{-r}_c + \sum_{p,I,a,b,c} p^I p^a \delta_{a,b,c} z^{-p}_a - z^{-p}_I + o(z^4) \tag{2.14}
\]

The duality invariant string gauge algebra requires the introduction of extra fields, in order to complete the algebra defined by the structure constants (2.3) to a Lie algebra. These extra fields replace higher spin fields of the string spectrum, and they are ultra–massive for all backgrounds (namely, for any VEV of the scalar fields). We, therefore, have truncated out these fields by setting \(z = 0\) in \(V(z)\) for fields that are ultra–massive at all toroidal compactifications.

The quadratic term in (2.14) gives the correct mass spectrum of the corresponding string modes \([2]\). The field \(p^a z^{-p}_a\) in \(V(z)\) is the Goldstone boson. In the next section we will check the compatibility of the cubic terms in \(V(z)\) (2.14) with the three string vertex in string field theory.

3 The Three String Vertex and Effective Cubic Interaction

In this section we present the three string vertex and the string field in toroidal backgrounds. We then derive the effective cubic interaction for \(N = 4\) heterotic strings, and study compatibility conditions with the effective action described in section 2. Finally, we discuss two examples.

3.1 The Three String Vertex and the String Field in Toroidal Backgrounds

The three string vertex in string field theory (SFT) is derived from the off–shell definition of the three–point functions on the sphere. We regard the sphere as the complex plane with a point at \(\infty\), and introduce a coordinate
z. Any choice of three punctures are conformally equivalent to one another, so one can take \( z = 0, 1, \infty \) (for a cyclic vertex it is sometimes more convenient to choose different punctures; this is discussed in section 4). Correspondingly we introduce three local coordinates \( z_i(z)(i = 1, 2, 3) \) as three local functions of \( z \). We arrange that each puncture corresponds to the origin of a local coordinate, i.e.

\[
z_1(0) = 0, \quad z_2(1) = 0, \quad z_3(\infty) = 0.
\]

(3.1)

If the vertex is cyclic (namely, invariant under a cyclic transformation of 1,2,3), the coordinates \( z_2 \) and \( z_3 \) are defined in terms of \( z_1 \) as

\[
z_2(z) = z_1(1 - \frac{1}{z}), \quad z_3(z) = z_2(1 - \frac{1}{z}) = z_1(\frac{1}{1 - z}).
\]

(3.2)

We introduce a parameter \( a \) given by the expansion of \( z_1(z) \):

\[
z_1(z) = az + ...
\]

(3.3)

The off–shell three–point function is now defined to be

\[
V_{ijk} = <0 | h_3(V_i(0)) h_2(V_j(0)) h_1(V_k(0)) |0> \]

(3.4)

where \( h_i \) are the inverse maps

\[
h_i \equiv (z_i^{-1}, \overline{z_i}^{-1}).
\]

(3.5)

When the vertex operators \( V_i \) in (3.4) correspond to primary fields, the off–shell three–point function is

\[
V_{ijk} = \lim_{\epsilon \to 0} h'_3(\epsilon) d_i h'_2(\epsilon) d_j h'_1(\epsilon) d_k <0 | V_i(h_3(\epsilon)) V_j(h_2(\epsilon)) V_k(h_1(\epsilon)) |0> \]

(3.6)

where \( h' = |\partial h|^2 \), and \( d_i \) is the conformal weight of the vertex \( V_i \).

The cubic interaction in string field theory is given by

\[
S_3 = \frac{g}{3} \sum_{123} <V||\Psi >_1 |\Psi >_2 |\Psi >_3
\]

(3.7)

where \( |\Psi > \) is the string field, and \( \sum_{123} <V \) is the three string vertex. The vertex is constructed such that when contracted with three string field modes it gives the corresponding off–shell three–point function \( V_{ijk} \) in (3.4). In
toroidal background, the three string vertex can be given in the form (see for example [10]):

\[ 123 < V = 123 < 0 | e^{E_{123}} \epsilon(p_2, p_3)(...) \]  

The expression for \( E_{123} \) is:

\[ E_{123} = \frac{1}{2} \sum_{r,s=1,2,3} \sum_{n,m \geq 0} \left( \sum_a N_{nm}^{rs} \alpha_n^{a(r)} \alpha_m^{a(s)} + \sum_I N_{nm}^{rs} \bar{\alpha}_n^{I(r)} \bar{\alpha}_m^{I(s)} \right). \]  

(3.9)

The index \( a \) refers to left–moving coordinates and the index \( I \) refers to right–moving coordinates similar to the notation used for the effective action in section 2. The operators \( \alpha_n^{a(r)} \) in (3.9) are the modes of the left–handed coordinate \( X^n_L \) of the \( r \)'th string; the operators \( \bar{\alpha}_n^{I(r)} \) are the modes of the right–handed coordinate \( X^n_R \) of the \( r \)'th string. The modes obey the commutation relations

\[ [\alpha_n^{a(r)}, \alpha_m^{b(s)}] = n \eta^{ab} \delta^{rs} \delta_{n+m,0}, \quad [\bar{\alpha}_n^{I(r)}, \bar{\alpha}_m^{J(s)}] = n \eta^{IJ} \delta^{rs} \delta_{n+m,0} \]  

(3.10)

The zero mode operators get values in the even–self dual lorentzian lattice when acting on \( |0 >_{123} \) (see appendix A), namely,

\[ (\alpha_0^{a(r)}(p_r), \alpha_0^{I(r)}(p_r)) \equiv (p_L^{a(r)}, p_R^{I(r)}), \]  

(3.11)

where \( (p_L, p_R) \) is a lorentzian momentum in the Narain lattice of compactification. The two cocycle \( \epsilon(p_r, p_s) \) in (3.8) obeys the relations (2.4); the momenta \( p_r \) are the lorentzian momenta of the \( r \)'th string. Finally, the dots in (3.8) refer to delta functions of left–right level matching conditions \( (L_0 = \bar{L}_0) \), and momentum conservation \( (p_1 + p_2 + p_3 = 0) \), and to ghost factors which are not important for our discussion.

The Neumann coefficients \( N_{nm}^{rs} \) (and their complex conjugates \( \bar{N}_{nm}^{rs} \)) in \( E_{123} \) (3.9) depend on the off–shell definition of the three point function (3.4), namely, they are defined in terms of the maps \( h \) (3.3). The Neumann coefficients \( N_{nm}^{rs} \) with \( n,m > 0 \) are invariant under \( SL(2,C) \) transformations of \( z \) (3.4). The coefficients \( N_{00}^{rs}, N_{0n}^{rs}, \) and \( N_{n0}^{rs} \) can be redefined by making use of momentum conservation. With the use of momentum conservation these coefficients are also \( SL(2,C) \) invariant. The formulas for the Neumann coefficients of the closed string vertex (as contour integrals over functions of \( h \) and \( h' \)) appear for example in [4]. For a cyclic vertex, the \( N \)'s are given in terms of one function \( h \) (see section 4).
In toroidal backgrounds, the string field $|\Psi>$ can be expanded in terms of the Fock space states

$$|\Psi> = |T> + |Z_L> + |Z_R> + |Z_{LR}> + ...$$  \hspace{1cm} (3.12)\

(Here we ignore the ghosts in $|\Psi>$.) The state $|T>$ corresponds to the tachyon mode

$$|T> = \sum_{p^2=0} T_p |p>,$$  \hspace{1cm} (3.13)\

where $p^2 = p_R^2 - p_L^2$. (The tachyon does not appear when we discuss the heterotic string). The states $|Z_L>$ correspond to the first left–moving excitations

$$|Z_L> = \sum_{a,p^2=2} Z^a_p \alpha_{-1}^a |p>,$$  \hspace{1cm} (3.14)\

and the states $|Z_R>$ correspond to the first right–moving excitations

$$|Z_R> = \sum_{I,p^2=-2} Z^I_p \bar{\alpha}_{-1}^I |p>.$$  \hspace{1cm} (3.15)\

(The states $|Z_R>$ do not appear when we discuss the heterotic string). Similarly, the states $|Z_{LR}>$ correspond to

$$|Z_{LR}> = \sum_{a,I,p^2=0} Z^a_I \alpha_{-1}^a \bar{\alpha}_{-1}^I |p>.$$  \hspace{1cm} (3.16)\

In the Fock space expansions (3.13)–(3.16) we have restricted to states satisfying the left–right level matching condition, and for that purpose we sum over momenta with particular lorentzian length appropriate to each mode. The modes which do not satisfy the left–right level matching condition are projected out anyway when contracted with $|V>_{123}$. The dots in (3.12) correspond to higher excitation modes of the string; we did not keep them as they do not appear in low–energy effective actions. In other words, we truncate massive modes that remain massive at all toroidal backgrounds.

To end this sub–section, we remark that a truncation of massive modes by setting them to zero is consistent if we deal with cubic interactions only. However, in case one is interested in higher order interactions of light modes, the propagation of ultra–massive modes usually contributes to the amplitudes. One should then integrate out the ultra–massive modes properly in order to derive the correct effective action.
3.2 The Effective Cubic Interaction and Compatibility Conditions for $N = 4$ Heterotic Strings

We are now ready to present the part of the cubic interaction (3.7) which is relevant for low-energy effective actions of $N = 4$ heterotic strings. After some straightforward calculations one finds

\[
S_3 = \frac{g}{3} \sum_{p_1, p_2, p_3, a, b, c} \delta_{p_1 + p_2 + p_3, 0} e^{E_{00}} \epsilon(p_2, p_3) \Gamma_{abc} \{ Z_{p_1}^a Z_{p_2}^b Z_{p_3}^c \}
+ \sum_{I, r = 1, 2, 3} (N_{10}^{1r} P_r^a Z_{p_1}^b Z_{p_3}^c + N_{10}^{2r} P_r^a Z_{p_1}^b Z_{p_2}^c + N_{10}^{3r} P_r^a Z_{p_1}^b Z_{p_2}^c Z_{p_3}^c),
\]

(3.17)

where the lorentzian length is $p_{r}^2 = 2$ in $Z_{p_r}^a$, and $p_{r}^2 = 0$ in $Z_{p_r}^a$, and

\[
\Gamma_{abc} = \sum_{r} (N_{11}^{31} N_{10}^{2r} P_r^a \eta^{ac} + N_{11}^{32} N_{10}^{1r} P_r^b \eta^{bc} + N_{11}^{33} N_{10}^{1r} P_r^c \eta^{ab})
+ \beta \sum_{r, s, t} N_{10}^{1r} N_{10}^{1s} N_{10}^{1t} P_r^a P_s^b P_t^c.
\]

(3.18)

In (3.18) $\beta = 1$ (0) for the bosonic (heterotic) string; this is explained later. Some formulas that are useful for the calculation of (3.17) are given in appendix A.

The factor $e^{E_{00}}$ in (3.17) is given by

\[
E_{00} = \frac{1}{2} \sum_{r, s} (N_{00}^{rs} P_{L(r)} P_{L(s)} + N_{00}^{rs} P_{R(r)} P_{R(s)}).
\]

(3.19)

For a cyclic vertex one can use momentum conservation to bring $N_{00}^{rs}$ into the form (see section 4):

\[
N_{00}^{rs} \equiv \delta^{rs} N_{00} = -\delta^{rs} \log a,
\]

(3.20)

where $a$ is defined in (3.3). One finds

\[
e^{E_{00}} = a^{-\frac{3}{2}} \sum_{r} P_{L(r)}^2 \ a^{-\frac{3}{2}} \sum_{r} P_{R(r)}^2.
\]

(3.21)
This ‘scaling–factor’ is a result of the $h'$ appearing in the off–shell three–point function (3.6) (we call it a scaling–factor because $a \to \lambda a$ under the scale transformation $h \to \lambda^{-1}h$).

Next we discuss briefly the property of a heterotic string field $|\Psi>$ (in the Neveu–Schwarz sector). The string field in (3.12) is presented for bosonic modes. For the heterotic string, however, the right–handed modes are bosonic while the left–handed modes correspond to a fermionic string. Therefore, the expansion of the string field in Fock space states includes fermionic modes as well. The Fock space of a fermionic string includes modes in different pictures (the picture presented in (3.12) is the 0–picture). In addition, the string vertex includes both bosonic and fermionic ghost factors. A proper cancelation of the ghost factors requires that a three–point function is computed with two modes in the $(-1)$–picture, and one mode in the 0–picture. One then finds that the only difference in the off–shell three–point function compared to a computation in the 0–picture is the cubic $ppp$ term in (3.18), namely: $\beta = 1$ for the bosonic string while $\beta = 0$ for the heterotic string.

To compare with the duality invariant effective action of $N = 4$ heterotic strings the following should be done. The left–moving index $a$ is split into an internal index $a = 1, \ldots, 6$ and a space–time index $\mu = 1, \ldots, 4$. Similarly, the right–moving index $I$ is split into the internal index $I = 1, \ldots, 22$ and a space–time index. The $D = 4$ space–time can be regarded as a decompactification limit of the toroidal background. The space–time dependence of the fields and the space–time derivatives then appear correctly after performing a Fourier transform from momentum space to coordinate space. Here we restrict the discussion to the scalar fields $Z$ with internal indices only. We are thus comparing the part of $S_3$ in (3.17) that is relevant for low–energy with the scalar potential $V(Z)$ (2.7). Moreover, we should restrict to fields $Z^{aI}_p$ with $p = 0$ only, since the fields $Z^{aI}_p$ with $p \neq 0$ remain ultra–massive in $D = 4$ (they may become light only in the decompactification limit of internal coordinates).

After doing all that, we find that manifest compatibility with the scalar potential (to cubic order) is translated into the following conditions for the Neumann coefficients (modulo redefinitions of $N^{rs}_{00}$ and $N^{rs}_{01}$ using momentum
conservation; this will be discussed in more detail in section 4):

\[ N_{11}^{12} N_{01}^{33} + \text{cyclic} = 0, \quad (3.22) \]

\[ N_{11}^{12} (N_{01}^{13} + N_{01}^{23}) + \text{cyclic} = 0, \quad (3.23) \]

\[(\bar{N}_{01}^{13} - \bar{N}_{01}^{23}) [N_{11}^{12} (N_{01}^{13} - N_{01}^{23}) + N_{11}^{23} (N_{01}^{12} - N_{01}^{21}) + N_{11}^{13} (N_{01}^{12} - N_{01}^{22})] + \text{cyclic} = 0, \quad (3.24) \]

\[ N_{11}^{12} (N_{01}^{23} - N_{01}^{13}) (\epsilon \bar{N}_{00} + \bar{N}_{01}^{23} - \bar{N}_{01}^{13}) + \text{cyclic} = 0, \quad (3.25) \]

\[ |a| = 1 \quad \text{(i.e. Re } N_{00} = 0), \quad (3.26) \]

where \(X(1,2,3) + \text{cyclic} \equiv X(1,2,3) + X(2,3,1) + X(3,1,2)\) (this is only done for the upper indices).

The first condition (3.22) means that a term \(p_b \delta^{ac} z_p^a z_p^b z_p^c \) is absent in \(V(z)\) (2.14) (the momenta \(p,q,r\) in (2.14) are replaced with \(p_1, p_2, p_3\), respectively). Such a term can always be removed by the use of momentum conservation \(p_2 = -p_1 - p_3\). In other words, the Neumann coefficients \(N_{01}^{rr}\) can be redefined to 0 by making use of momentum conservation \(p_1 = -p_2 - p_3\) for the term \(p_1 a^{(1)}\) in \(E_{123}\) etc., and therefore, condition (3.22) is always satisfied.

Conditions (3.23)–(3.25) specify the correct relative factors between different cubic terms in \(V(z)\). The second condition (3.23) means that the relative factor between the terms \(p_1^b \delta^{ac} z_p^a z_p^b z_p^c\) and \(p_3^b \delta^{ac} z_p^a z_p^b z_p^c\) in \(V(z)\) is correct. However, this relative factor can always be fixed by using momentum conservation and the properties (2.4) of the two cocycle which lead to the identity

\[ \sum_{p_1, p_2, p_3, a, b, c} \delta_{p_1 + p_2 + p_3, 0} \epsilon(p_2, p_1)(p_1 + p_3)^b \delta^{ac} z_p^a z_p^b z_p^c = 0. \quad (3.27) \]

A three string vertex should therefore satisfy only the conditions (3.24)–(3.26). The third condition (3.24) means that the relative factor between the terms \(z_p^a z_p^b\) and \(z_p^b z_p^a\) in (2.14) is correct, and the fourth condition (3.25) means that the relative factor between the terms \(z_p^a z_p^b z_p^c\) and \(z_p^c z_p^a z_p^b\) in \(V(z)\) is correct. (In deriving condition (3.25) we have used the identity (3.27)).
Conditions (3.24), (3.25) are satisfied by any cyclic three string vertex (this is shown in section 4). The last condition, however, is somewhat special: cyclic vertices may violate only the condition \( |a| = 1 \) (3.26). In other words, after solving \( p_{R(r)}^2 \) in terms of \( p_{L(r)}^2 \), the term (3.21) gives rise to a factor

\[
|a| = \sum_{r=1}^{3} p_{L(r)}^2,
\]

which is, apparently, the only difference between effective cubic interactions derived from different cyclic three string vertices. However, this factor can be eliminated by performing an appropriate field redefinition; this is demonstrated within the examples below.

### 3.3 Examples

In this sub-section we discuss two examples. The first is the Witten vertex [12]; it corresponds to the choice (see for example ref. [5])

\[
z_1(z) = i\alpha(1 + t)^{3/2} - (1 - t)^{3/2} = \frac{3\sqrt{3}}{4}z + ..., \quad t = i\sqrt{3}z, \quad z = \frac{2}{z - 2}. \tag{3.29}
\]

This is the natural extension of the covariant open string field theory vertex. With the standard normalization \( \alpha = 1 \), the entire sphere is covered by three unit discs \( \{|z| \leq 1, i = 1, 2, 3\} \), where \( z_i \) are given in (3.1), (3.2). With this normalization \( a = \frac{3\sqrt{3}}{4} \) in (3.3).

The Neumann coefficients for Witten vertex were computed for the open string in [13], and can be computed for the closed string following the discussion in section 4 below (see appendix B). In both cases the Neumann coefficients are

\[
N_{00}^{rs} = \delta^{rs} \log\left(\frac{4}{3\sqrt{3}}\right), \tag{3.30}
\]

\[
N_{01}^{21} = N_{01}^{32} = N_{01}^{13} = -N_{01}^{12} = -N_{01}^{23} = -N_{01}^{31} = \frac{2}{3\sqrt{3}}, \quad N_{01}^{rr} = 0. \tag{3.31}
\]

---

2 When \( p_L \) includes space–time momenta in addition to internal momenta, then the factor (3.28) is 1 on–shell.

3 This was pointed out to me by B. Zwiebach.
The Neumann coefficients (3.30), (3.31), (3.32) obey the conditions (3.22)–(3.25). However, condition (3.26) is not satisfied because $|a| \neq 1$ in (3.29).

Inserting the Neumann coefficients in $S_3$ (3.17), and truncating the string field to the modes $Z_a^p, Z^I_{p=0} \equiv Z^I_a$, one finds that the cubic interaction in a SFT based on Witten vertex is

$$S_3(Z) = \frac{g'}{3} \frac{1}{2} \sum_{p_1+p_2+p_3,a,b,c} |a|^{-2} \sum_{r=1}^3 \delta_{L(r)}^{-1} \left( Z_a^{p_1} Z_b^{p_2} Z_c^{p_3} \right. + \left. \sum_{p,I,a,b,c} |a|^{-2/3} p^I \delta_{abc} Z_c^{p} (Z_a^{p} Z^I_b - Z_a^{p} Z^I_b) \right).$$

(3.33)

Here we have used the lorentzian scalar product (2.2) and the values $p^2 = 2$ in $Z_a^p$ in order to solve $p^2_R$ in terms of $p^2_L$, and consequently $g' = \frac{9}{3} g$. Except for the factor $|a|^{-2} \sum_{r=1}^3 \delta_{L(r)}^{-1}$ (in the second sum in (3.33) $|a|^{-2} \sum_{r=1}^3 \delta_{L(r)}^{-1} = |a|^{-2} p^2_L$), $S_3(Z)$ (3.33) equals to the cubic term in the scalar potential $V(z)$ (2.14).

The extra factor $|a|^{-2} \sum_{r=1}^3 \delta_{L(r)}^{-1}$ can be eliminated by an appropriate field redefinition. This is done as follows: the truncated action derived from SFT is of the form

$$S(Z) = \frac{1}{2} \sum_p p^2 Z_p Z_{-p} + \frac{g}{3} \sum_{p,q} f(p, q) e^{-A(p^2 + q^2 + (p+q)^2)} Z_p Z_q Z_{-p-q}.$$  

(3.34)

By defining

$$Z_p = z_p - \frac{g}{3} \sum_q \left[ f(p, q) e^{-A(p^2 + q^2 + (p+q)^2)} - 1 \right] p^2 z_q z_{-p-q} \quad \text{for } p \neq 0,$$

$$Z_0 = z_0,$$

(3.35)

the action (3.34) becomes

$$S(z) = \frac{1}{2} \sum_p p^2 z_p z_{-p} + \frac{g}{3} \sum_{p,q} f(p, q) z_p z_q z_{-p-q} + o(z^4).$$

(3.36)
After performing this field redefinition, the action derived from SFT (3.33), equals to the scalar potential $V(z)$ in (2.14) to cubic order.

Nevertheless, the simplest way to get rid of the factor $|a|^{-\sum_{r=1}^{3} p_{r}(v)}$ is to define the three string vertex with a map $z_{1}(z) = e^{i\phi} z + ...$, namely, $|a| = 1$. A simple vertex of this type is the SCSV vertex [14]; it is defined by the map

$$z_{1}(z) = z.$$  \hspace{1cm} (3.37)

The map $z_{3}(z) = h_{2}(z) = \frac{1}{z^{2}}$ in (3.2),(3.3) is the $SL(2,C)$ transformation sending $\infty \rightarrow 0, \ 0 \rightarrow 1, \ 1 \rightarrow \infty$. This vertex is cyclic but not symmetric under all permutations. The Neumann coefficients for the open string were computed in [15], and can be computed for the closed string following the discussion in section 4 below (see appendix C). For the closed string the Neumann coefficients are

$$N_{01}^{rr} = N_{11}^{rr} = 0, \quad N_{00}^{rs} = \delta^{rs} \log i,$$

$$N_{01}^{21} = N_{01}^{32} = N_{01}^{13} = i, \quad N_{01}^{12} = N_{01}^{23} = N_{01}^{31} = 0,$$

$$N_{11}^{12} = N_{11}^{13} = N_{11}^{23} = -1. \hspace{1cm} (3.38)$$

It turns out that only conditions (3.22) and (3.23) are violated; but these are not important, as explained in sub–section 3.2.

The SCSV vertex is an example of a cyclic three string vertex that satisfies conditions (3.24),(3.25). Therefore, it is compatible with the $N = 4$ supergravity coupled to matter effective field theory. In the next section we show that all the cyclic vertices are compatible with the $N = 4$ effective action.

4 A Cyclic Vertex is Compatible with Effective Actions

In this section we prove that any cyclic three string vertex obey the conditions (3.24),(3.25). Therefore, a cyclic vertex is compatible with the effective action.

We start with a symmetric vertex, namely, a vertex that is invariant under any permutation of the three strings. In particular, the vertex is cyclic. To derive the conditions on the functions $h_{i}$ in (3.3), it would be simpler to
work with the three punctures at $z = 1, \rho, \bar{\rho}$, where $\rho = e^{2\pi i/3}$. In this case, a cyclic permutation $P_{123}$ of the punctures is given by a multiplication by $\rho$, and therefore, $h_1, h_2, h_3$ are given in terms of a single function $h$:

$$z = h_1(z_1) \equiv h(z_1), \quad z = h_2(z_2) = \rho h(z_2), \quad z = h_3(z_3) = \bar{\rho} h(z_3),$$

such that

$$h_1(0) = 1, \quad h_2(0) = \rho, \quad h_3(0) = \bar{\rho}. \quad (4.2)$$

It is convenient to define a function $O(z)$ and expansion parameters $a_1, a_2$ by

$$h(z) = e^{O(z)}, \quad (4.3)$$

where

$$O(z) = a_1 z + a_2 z^2 + o(z^3),$$

$$h(z) = 1 + a_1 z + \frac{1}{2} a_1^2 z^2 + a_2 z^2 + o(z^3). \quad (4.4)$$

The group of permutations $S_3$ is generated by the cyclic transformation $P_{123}$ and one permutation, say $P_{23}$. The map $P_{23}$ exchanges $\rho$ with $\bar{\rho}$ leaving the point $z = 1$ fixed. This map is

$$P_{23} : \quad z \to z' = \frac{1}{z}. \quad (4.5)$$

The choice of $h_1$ cannot be arbitrary; a constraint comes from $P_{23}$. In terms of $z_1$ the map $P_{23}$ must be a phase rotation. Since $P_{23}^2 = 1$, the only choice of phase is by a minus sign, and therefore $h$ satisfies

$$h(z) = \frac{1}{h(-z)}. \quad (4.6)$$

The condition (4.6) implies that $O(z)$ is an odd function, and in particular $a_2 = 0$.

We are now ready to calculate the Neumann coefficients. For the closed string vertex, $N_{00}^{rs}$, $N_{01}^{rs}$, $N_{11}^{rs}$ are given by (see for example [4])

$$N_{00}^{rr} = \log h'_r(0),$$

$^{4}$ This was suggested to me by K. Ranganathan and B. Zwiebach.
\[ N_{rs}^{rs} = \log(h_r(0) - h_s(0)), \quad \text{for } r \neq s, \]
\[ N_{rs}^{rs} = N_{rs}^{sr} = \oint \frac{dz}{2\pi i z} \frac{h'_r(z) - i}{h_r(0) - h_s(z)}, \]
\[ N_{rs}^{rs} = \oint \frac{dz}{2\pi i z} \frac{h'_s(z)}{2\pi i w} \frac{h'_s(w) \left(h_r(z) - h_s(w)\right)^2 - 1}{(h_r(z) - h_s(w))^2}, \quad (4.7) \]

where each contour surrounds the origin of the integration variable. Using (4.1), (4.2) and (4.4) one finds for a symmetric vertex

\[ N_{00}^{11} = \log a_1, \quad N_{00}^{22} = \log a_1 + \frac{2\pi i}{3}, \quad N_{00}^{33} = \log a_1 - \frac{2\pi i}{3}, \]
\[ N_{00}^{21} = N_{00}^{12} + \pi i = \log \sqrt{3} + \frac{5\pi i}{6}, \quad N_{00}^{23} = N_{00}^{32} + \pi i = \log \sqrt{3} + \frac{\pi i}{2}, \]
\[ N_{00}^{13} = N_{00}^{31} + \pi i = \log \sqrt{3} + \frac{\pi i}{6}, \quad (4.8) \]

\[ N_{01}^{rr} = -i \frac{1}{2} a_1, \quad (4.9) \]
\[ N_{01}^{21} = N_{01}^{13} = N_{01}^{32} = a_1 \frac{e^{-\pi i / 3}}{\sqrt{3}}, \]
\[ N_{01}^{12} = N_{01}^{31} = N_{01}^{23} = a_1 \frac{e^{-2\pi i / 3}}{\sqrt{3}}, \quad (4.10) \]
\[ N_{11}^{rs} = \frac{a_1^2}{3} \quad \text{for } r \neq s \quad (4.11) \]

In (4.8)–(4.11) we have used the numerical values \(1 - \rho = |1 - \rho e^{-\pi i / 6}| \) and \(|1 - \rho| = \sqrt{3}\). The parameter \(a_1\) in (4.8)–(4.11) is defined in eq. (4.4). In deriving the result (4.9) we have used the fact that \(a_2 = 0\) in (4.4); the other results do not depend on the invariance of the vertex under \(P_{23}\) (i.e., cyclic symmetry is sufficient to derive all but eq. (4.9); we will discuss the case \(a_2 \neq 0\) later).

To compare with the Neumann coefficients derived for Witten vertex given in (3.30), (3.31), and (3.32), we have to redefine the \(N's\) in (4.8)–(4.11) by using \(p_1 + p_2 + p_3 = 0\). (We know that the Neumann coefficients

\footnote{There is a factor of \(-i\) in \(N_{01}\) and a factor of \(-1\) in \(N_{11}\) relative to ref. [4]; it comes from a relative \(i\) factor in the definition of \(\alpha_1\) and \(\alpha_0\).}
must be invariant under $SL(2, C)$ transformations, up to a redefinition of $N_{00}^{rs}$ and $N_{01}^{rs}$ by making use of momentum conservation \cite{11}). For $N_{00}^{rs}$ one finds

$$E_{00} = \frac{1}{2} \sum_{r,s=1,2,3} \left( N_{00}^{rs} p_{L(r)} p_{L(s)} + \tilde{N}_{00}^{rs} p_{R(r)} p_{R(s)} \right)$$

and therefore, the redefined $N_{00}$ are

$$N_{00}^{rs} \equiv \delta^{rs} N_{00} = \delta^{rs} \log a_1 \sqrt{3}.$$  \hfill (4.12)

We have thus recovered eq. (3.20) if we identify

$$a = \frac{\sqrt{3}}{a_1}. \quad (4.14)$$

(The parameter $a$ is defined by expending $z_1 = h_1^{-1}(z)$ \cite{33}, and therefore is proportional to the inverse of $a_1$. An illustration of the relation between the parameters $a$ and $a_1$ is given for Witten vertex in appendix B.)

In the same way, by using $p_1 + p_2 = -p_3$, $p_1 + p_3 = -p_2$, $p_2 + p_3 = -p_1$ we can redefine $N_{01}^{rs}$ s.t. $N_{01}^{rs} = 0$. One finds

$$\sum_{r,s=1,2,3} N_{01}^{rs} p_r \alpha_1^{(s)} = \frac{1}{2a} \{ \alpha_1^{(1)} (p_2 - p_3) + \alpha_1^{(2)} (p_3 - p_1) + \alpha_1^{(3)} (p_1 - p_2) \}, \quad (4.15)$$

and therefore, the redefined $N_{01}$ are

$$N_{01}^{21} = N_{01}^{13} = N_{01}^{32} = -N_{01}^{12} = -N_{01}^{23} = N_{01}^{31} = -N_{01}^{22} = \frac{a_1}{2\sqrt{3}} = \frac{1}{2} a^{-1} = \frac{1}{2} e^{N_{00}}, \quad (4.16)$$

The Neumann coefficients $N_{01}$ are therefore related to $N_{00}$. The $N_{11}$’s are also related to $N_{00}$:

$$N_{11}^{12} = N_{11}^{13} = N_{11}^{23} = \frac{a_1^2}{3} = \frac{1}{a^2} = e^{2N_{00}}. \quad (4.17)$$
Using the redefined Neumann coefficient \((4.13), (4.16), \) and \((4.17)\) one finds that the conditions \((3.22)–(3.25)\) are satisfied. We thus conclude that any symmetric three string vertex is compatible with the cubic interaction of the effective action.

We end this section showing that in fact any cyclic three string vertex is compatible with the effective action. In the following we show that the absence of a \(z^2\) term in \(O(z)\) \((1.4)\) (namely, \(a_2 = 0\)) is necessary only for the redundant conditions \((3.22)\) and \((3.23)\) to hold. If \(a_2 \neq 0\), one finds that the only difference with respect to the symmetric vertex is in \(N_{rr}^{01}\) (before using momentum conservation):

\[
N_{rr}^{01} = -i \oint \frac{dz}{2\pi i} \frac{h_r'(z)}{h_r(z) - h_r(0)} = -i \oint \frac{dz}{2\pi i} \frac{1}{a_1 z^2} (a_1 + (a_1^2 + 2a_2)z + ...) (1 - (1/2a_1 + a_2/a_1)z + ...)
\]

\[= -i \left( \frac{1}{2}a_1 + \frac{a_2}{a_1} \right). \]

\((4.18)\)

As before, \(N_{rs}^{rr}\) can be redefined using momentum conservation s.t. \(N_{rr}^{01} = 0\). One finds

\[
\text{redefined } N_{01}^{rr} : \\
N_{01}^{21} = N_{01}^{32} = N_{01}^{13} = \frac{1}{2} e^{N_{00}} + \frac{ia_2}{a_1},
\]

\[N_{01}^{12} = N_{01}^{23} = N_{01}^{31} = -\frac{1}{2} e^{N_{00}} + \frac{ia_2}{a_1},
\]

\(N_{01}^{rr} = 0. \) \((4.19)\)

(In appendix C it is shown that for the SCSV vertex \(a_1 = i\sqrt{3}, a_2 = i\frac{\sqrt{3}}{2}, \) and using \((4.19)\) one recovers the Neumann coefficients given in section 3 \((3.38)\).) Finally, the Neumann coefficients of a cyclic vertex given by \((4.13), (4.17) \) and \((4.13)\) satisfy conditions \((3.24)\) and \((3.25)\), and compatibility with the effective action is guaranteed.
5 Effective Action and String Field Theory of the $N = 2$ String

In this section we sketch the construction of the three $N = 2$ closed string vertex and derive from it the effective action of $N = 2$ strings in toroidal backgrounds. The following discussing is far from complete as we ignore the ghost dependence and picture changing. Yet, it is sufficient for comparison with the effective action.

The critical $N = 2$ string has perhaps the simplest vertex operator algebra of any string theory. This is because, unlike other string theories, it contains a finite number of physical degrees of freedom [16, 17]. In the toroidal background $T^{2,2}$, physical operators are specified by the lorentzian momenta $p$ [18]:

$$p = (p_{L1}, p_{R1}; p_{L2}, p_{R2}) \in \Gamma_{4;4}$$

(5.1)

where $\Gamma_{4;4}$ is an even self–dual lorentzian lattice with signature $(4; 4)$; $p_{II}$ is a complex vector, $I = L(R)$ for left–handed (right–handed) momenta, and $i = s(t)$ for space–like (time–like) components.

When both $p_L \neq 0$ and $p_R \neq 0$ the only vertex operators that may become physical at some point in the moduli space of toroidal backgrounds are

$$V_p(X, \bar{X}) = e^{i(p \cdot \bar{X} + \bar{p} \cdot X)}, \quad p^2 = 0,$$

(5.2)

where $p^2$ is defined by the lorentzian scalar product (2.2), namely, $p$ is a non–zero null vector of the $(4; 4)$ lattice. In (5.2) the dot product is

$$p \cdot \bar{X} \equiv p_L \bar{X}_L + p_R \bar{X}_R,$$

(5.3)

where $\bar{p}$ is the complex conjugate of $p$, $p \bar{X}$ is the lorentzian scalar product (2.2), and $X_L^i$ and $X_R^i$ are the $N = 2$ (holomorphic and anti-holomorphic) chiral superfields

$$X_L^i(Z, \theta^-) = x_L^i(Z) + \psi_L^i(Z)\theta^-,$$

$$X_R^i(\bar{Z}, \bar{\theta}^-) = x_R^i(\bar{Z}) + \psi_R^i(\bar{Z})\bar{\theta}^-,$$

$$Z = z - \theta^+\theta^-, \quad \bar{Z} = \bar{z} - \bar{\theta}^+\bar{\theta}^-.$$

(5.4)

The superfield $\bar{X}$ is obtained from $X$ by complex conjugation (in field space) together with interchanging $\theta^+ \leftrightarrow \theta^-$. 

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The upper component of the $N = 2$ superfield $V_p(X, \bar{X})$ (5.2) is

$$V_p(z, \bar{z}) = V_p^L(z, \bar{z}) = (ip_L \partial \bar{x}_L - i\bar{p}_L \partial x_L - (p_L \bar{\psi}_L)(\bar{p}_L \psi_L)) (ip_R \partial \bar{x}_R - i\bar{p}_R \partial x_R - (p_R \bar{\psi}_R)(\bar{p}_R \psi_R)) e^{i(p \cdot z + \bar{p} \cdot \bar{z})}. \quad (5.5)$$

The operator (5.5) is on–shell when both $p_L^2 = 0$ and $p_R^2 = 0$. Given $p \in \Gamma^{4;4}$ there is a point in the space of toroidal backgrounds where $V_p$ is on–shell if $f p^2 = 0$ (see ref. [18]), and therefore, the operators (5.2) are the set of all operators with $p_L \neq 0$ and $p_R \neq 0$ that may become physical.

When $p = 0$ there are discrete states [19, 18]

$$V_{ij}(z, \bar{z}) = \frac{1}{2} (\partial x^i \bar{\partial} \bar{x}^j + \bar{\partial} x^i \partial \bar{x}^j). \quad (5.6)$$

The discrete states (5.6) generate deformations of the metric moduli, and are on–shell at any background. They can be formed from off–shell $V_p$'s by choosing an appropriate normalization as $p$ goes to zero [18].

In addition to $V_{ij}$, there are discrete states that appear in particular toroidal backgrounds [18]. They can be formed from off–shell $V_p$'s by choosing an appropriate normalization as $p_R \to 0$ while $p_L^2 = 0, p_L \neq 0$ (and similarly for $p_L \to 0$ while $p_R^2 = 0, p_R \neq 0$). They are given by the vertex operators

$$J_i^{(p_L, 0)} = (ip_L \partial \bar{x}_L - i\bar{p}_L \partial x_L - (p_L \bar{\psi}_L)(\bar{p}_L \psi_L)) \bar{\partial} x^i \ e^{i(p_L \bar{x}_L + \bar{p}_L x_L)};$$

$$\bar{J}_i^{(0, p_R)} = \partial x^i_L (ip_R \partial \bar{x}_R - i\bar{p}_R \partial x_R - (p_R \bar{\psi}_R)(\bar{p}_R \psi_R)) e^{i(p_R \bar{x}_R + \bar{p}_R x_R)}. \quad (5.7)$$

The discrete states in (5.7) give rise to exact gauge symmetries.

We are now ready to expand the $N = 2$ string field in terms of all the Fock space states that may become physical modes in some toroidal background. In the 0–picture:

$$|\Phi > = \sum_{p^2=0} \phi_p V_p |0 > + \sum_{ij} G^{ij} V_{ij} |0 > + \sum_{i, p^2=0} A_{ip} J_i^p |0 > + \sum_{i, p^2=0} \bar{A}_{ip} \bar{J}_i^p |0 >. \quad (5.8)$$

Here $J_i^p$ and $\bar{J}_i^p$ are the extensions of (5.7) to off–shell momenta $p$. The terms involving the gauge fields $A_{ip} (\bar{A}_{ip})$ are summed over all momenta $p \in \Gamma^{4;4}$ that may become of the form $(p_L, 0), p_L^2 = 0, p_L \neq 0 ((0, p_R), p_R^2 = 0, p_R \neq 0)$.
at some point of the moduli space of toroidal backgrounds; those are also the set of non–zero null vectors of $\Gamma^{4,4}$, and therefore we sum over $p^2 = 0$ in (5.8).\footnote{We can extend the sums in (5.8) to all $p \in \Gamma^{4,4}$; the modes with $p^2 \neq 0$ will then be eliminated by the $\delta(L_0 - \bar{L}_0, 0)$ in the three string vertex. With this extension we have a one to one correspondence between modes of the string field and elements of the volume–preserving diffeomorphism algebra of $T^{4,4}$ (see ref. [18]).}

Following the discussion in section 3, the three $N = 2$ string vertex is determined by the parameter $a$ in (3.3) (because higher oscillations of the $N = 2$ string are never physical), and the cubic interaction for a cyclic vertex is

$$S_3(N = 2) = S_{\phi\phi} + S_{G\phi\phi} + S_{A\phi\phi},$$

with

$$S_{\phi\phi} = \frac{1}{3} \sum_{p_1, p_2, p_3} a^{-\frac{1}{2}} \sum_{r=1}^{3} p_r^2 \bar{a} - \frac{1}{2} \sum_{r=1}^{3} p_r^2 \delta_{p_1 + p_2 + p_3, 0}$$

and

$$S_{G\phi\phi} = \frac{1}{4} \sum_p a^{-p_L^2 \bar{a} - p_R^2} (p_L^L \bar{p}^L_R + \bar{p}^L_L p^L_R) G_{ij} \phi_{ij} \phi_{-p},$$

and

$$S_{A\phi\phi} = \frac{1}{3} \sum_{p_1, p_2, p_3} a^{-\frac{1}{2}} \sum_{r=1}^{3} p_r^2 \bar{a} - \frac{1}{2} \sum_{r=1}^{3} p_r^2 \delta_{p_1 + p_2 + p_3, 0}$$

and

$$\times \frac{1}{2} [(p_L^L \bar{p}^L_R - p_L^L \bar{p}^L_R) G_{ij} \phi_{ij} \phi_{-p} + (p_R^R \bar{p}^R_R - p_R^R \bar{p}^R_R) G_{ij} \phi_{ij} \phi_{-p}], \quad (5.9)$$

The sums in (5.9) are only over the momenta appropriate to each of the fields $\phi_p, A_{ip}, \bar{A}_{ip}$ as discussed above. The first term in (5.9) arises from the off–shell three point function \footnote{More precisely, one calculates tree–level correlation functions by inserting $V_p$ at $n$ points on the sphere, and integrating over their positions modulo global superconformal transformations. One then finds that two of the $V_p$’s in (5.10) are in the $(-1, -1)$–picture, namely, $V_p^{-1,-1} \equiv c e^{-\phi_+} e^{-\bar{\phi}_-} e^{-\bar{\phi}_+} e^{-\phi_-} e^{i(p \cdot \bar{p} + \bar{p} \cdot p)}$, where $c$ is the spin $-1$ ghost of the $(b, c)$ system, and $\phi_+, \phi_-$ are the two scalars used in bosonizing the spin $3/2, -1/2$ ghosts ($\beta^\pm, \gamma^\pm$) of the $N = 2$ string [24].}
\[ \times \delta_{p_1+p_2+p_3,0}(p_L^2\bar{p}_L^3 - p_L^3\bar{p}_L^3)(p_R^2\bar{p}_R^3 - p_R^3\bar{p}_R^3). \] (5.10)

The second term in (5.9) arises from the off-shell three point function

\[ V_{(ij),p,-p} = <0|h_3(V_{ij}(0))h_2(V_p(0))h_1(V_{-p}(0))|0> = \frac{1}{2}a^{-3/2}\bar{p}_L^2\bar{p}_L^2(p_L^2\bar{p}_L^2 + \bar{p}_L^2p_L^2). \] (5.11)

The third term in (5.9) arises from the off-shell three point functions

\[ V_{(ip),p_2,p_3} = <0|h_3(J_{ip_1}(0))h_2(V_{ip_2}(0))h_1(V_{ip_3}(0))|0> = \frac{1}{2}a^{-3/2}\bar{p}_L^2\bar{p}_L^2\delta_{p_1+p_2+p_3,0}(p_L^2\bar{p}_L^3 - p_L^3\bar{p}_L^3)p_{R1}^i. \]

\[ V_{(ip),p_2,p_3} = <0|h_3(J_{ip_1}(0))h_2(V_{ip_2}(0))h_1(V_{ip_3}(0))|0> = \frac{1}{2}a^{-3/2}\bar{p}_L^2\bar{p}_L^2\delta_{p_1+p_2+p_3,0}p_{R1}^i(p_R^2\bar{p}_R^3 - p_R^3\bar{p}_R^3). \] (5.12)

At the decompactification limit the target–space is \( R^{2,2} \), and \( p_L^i = p_R^i \equiv p^i \) is a continuous space–time momentum. After performing a Fourier transform from momentum space to coordinate space \( S_3(N = 2) \) becomes

\[ S_{3}^{\text{dec}} = \int d^2x_1d^2x_2 \frac{1}{2}G^{ij}(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi)(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi) + \frac{1}{3}\eta^{ij}(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi)(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi)(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi) + \frac{1}{3}\eta^{ij}(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi)(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi)(e^{-\log|\eta|\eta^{kl}\partial_k\partial_l}\phi). \] (5.13)

The factors \( e^{-\log|\eta|\eta^{kl}\partial_k\partial_l} \) in (5.13) can be eliminated by an appropriate field redefinition (see explanation in section 3) \( ^8 \). Such a field redefinition

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\( ^8 \) The field redefinition (3.35) is singular in the decompactification case due to the factor \( \frac{1}{p^4} \), but an appropriate regular field redefinition can be done by replacing (3.35) with \( Z_p = z_p - \sum_q \left( e^{-\Delta p^2}p^2 \right) z_q z_{-p-q}[f(p,q)e^{A(p^2+(p+q)^2)} + f(q,p)e^{-A(p^2+(p+q)^2)} + f(-p-q,q)] + o(z^3). \) The factor \( e^{-\Delta p^2}p^2 \) is not singular in such a field redefinition.
introduces higher order terms in the new field $\phi$. After eliminating the exponents, and for $G^{i\bar{j}} = \eta^{i\bar{j}}$, one can get rid of the third term in (5.13) by an additional field redefinition

$$\phi \rightarrow \phi + \frac{1}{3} \eta^{i\bar{j}} \phi \partial_i \partial_j \phi.$$  \hspace{1cm} (5.14)

The field redefinition (5.14) also introduces higher order terms in the new field $\phi$. Now, dropping four and higher order terms in (5.13) one recovers the effective action presented in [17]:

$$S_{\text{eff}} = \int d^2 x_1 d^2 x_2 \left( \frac{1}{2} \eta^{i\bar{j}} \partial_i \phi \partial_j \phi + \frac{1}{3} \epsilon^{i\bar{j}} \epsilon^{\bar{k}l} \phi \partial_i \phi \partial_l \phi \partial_j \phi \right).$$ \hspace{1cm} (5.15)

The gauge fields do not appear explicitly in (5.13),(5.15); they can be introduced via a Kähler transformation [18]

$$\partial \phi \rightarrow \partial \phi + A, \hspace{0.5cm} \bar{\partial} \phi \rightarrow \bar{\partial} \phi + \bar{A}.$$ \hspace{1cm} (5.16)

In Minkowski background the transformation (5.16) introduces total derivatives.

So far, we have discussed only the three–point functions. It turns out that four–point and probably also higher–point amplitudes of the operators $V_p$ (5.2) vanish [17, 21, 20]. This fact is already predicted by the effective action (5.15) and it is thus believed to be correct to all orders. However, in the action $S_{\text{dec}}^3$ (5.13) we get extra terms of the form $\eta...\eta \phi...\phi$. These terms should be canceled against higher order interactions of (off–shell) $V_p$’s and discrete states $V_{i\bar{j}}$, when including the higher string vertices of the $N = 2$ closed string field theory.

Before ending this section, we remark that the kinetic term in the effective action (5.15) can be derived, alternatively, from the quadratic term of the $N = 2$ string field theory. Schematically,

$$12 < R |Q| \Phi >_1 |\Phi >_2 \sim \sum_p (p_L p_L + p_R p_R) \phi_p \phi_{-p},$$ \hspace{1cm} (5.17)

where $|R >_{12}$ is the two string reflector, and $Q$ is the $N = 2$ BRST operator.
6 Summary and Discussion

In this section we summarize the main points and present a discussion about this work. The structure of the scalar potential in gauged $D = 4$, $N = 4$ supergravity coupled to matter is fixed once the gauge algebra is given. This was used in order to derive a non-trivial check for the compatibility of SFT and effective actions to cubic order. It would be interesting to check the compatibility of higher order interactions as well. For that purpose one needs to work with the four (and higher) closed string vertex, and integrate out massive modes to derive an effective action. Assuming that the effective action is compatible with SFT to all orders we hope to use $D = 4$ supergravity to study some important problems in SFT. This is left for a future work; here we discuss some of the open problems.

By inserting the quadratic constraint (2.6) into the scalar potential (2.7), $V(Z)$ becomes non-polynomial in the physical fields. The structure of $V(Z)$ might be used to understand the non-polynomiality of closed string field theory. Moreover, the effective action of $N = 4$ heterotic strings is completely duality invariant when the gauge algebra is the lorentzian lattice algebra of the Narain lattice $\Gamma^{6,22}$. The infinite dimensional gauge symmetry is spontaneously broken at any point in the moduli space of toroidal backgrounds. The generalized duality transformations are residual discrete symmetries of the broken gauge algebra $\Gamma^{22}$. As expected, a similar phenomenon occurs in string field theory: target-space duality as a symmetry of string field theory was discussed recently for the $\alpha = p^+ \text{HIKKO}$ string field theory in toroidal backgrounds in ref. [10].

The nature of the infinite gauge algebra suggests that (for toroidal backgrounds) the underlying invariance principle of SFT has to do with an algebra defined on the Narain torus of compactification. Such is the case also for the $N = 2$ strings, for which an underlying off-shell algebra isomorphic to $v\text{diff}(T^4,4)$ was suggested [18].

Another important point is the question of background independence. The effective action of $N = 4$ heterotic strings is manifestly background independent, namely, it is independent of the toroidal background around which it is formulated. Different backgrounds correspond to different classical solutions, and are interpolated by changing expectation values of scalar fields at the minimum of the potential. The infinite gauge algebras in different backgrounds are isomorphic, although there are different unbroken gauge
groups in different backgrounds. We expect a similar phenomenon to occur in string field theory. However, in SFT the spin one ultra–massive gauge fields are replaced with the higher spin stringy modes, and revealing the underlying symmetry is harder.

We have shown that quadratic and cubic interactions in SFT of the fields $Z_p^a$ and $Z_p^{ai}$ are compatible with the $N = 4$ supergravity effective action for any cyclic vertex. Including the fields $Z_p^{ai}$, $p \neq 0$ in the cubic interaction derived from SFT (3.33) (in addition to $p = 0$) gives some interesting results. For example, it gives the leading order of a $D = 10$ supergravity in the decompactification limit, as was speculated in [2]. It would be interesting to study further the structure of cubic interactions that involve these fields.

A cyclic vertex is defined by a single map $h(z) = e^{O(z)}$, where $O(z) = a_1 z + a_2 z^2 + o(z^3)$ (4.4). The parameter $a_1$ in the three string vertex gives rise to a scale factor $e^{-\log(\sqrt{3}|a_1|) \sum_{r=1}^3 p_{L(r)}^2}$ in the cubic interaction. Although apparently this factor is not compatible with effective actions, there is a (non–linear) field redefinition (3.35) which gets rid of it to cubic order, and therefore, a cyclic three string vertex is compatible with effective actions for any scale factor. It would be interesting to study compatibility of the quadratic and cubic interactions using a general three string vertex.

The parameter $a_1$ specify the scaling factor of the map from the three unit discs (or semi–infinite cylinders) to the sphere. An overlap Witten vertex (namely, the three discs cover precisely the complete sphere) has a scale factor $a_1 = 4/3$ (see appendix B (B.3)). There are good reasons to believe that an overlap vertex is preferable, but we did not find a restriction like that arising from cubic order effective actions.

For the $N = 2$ string we have included in the string field $|\Phi>$ (5.8) all the modes that may become physical somewhere in the moduli space of toroidal backgrounds. Therefore, assuming this is all one needs for a consistent string field theory, the $N = 2$ string field theory is equivalent to its effective action. We have found that the effective action in Minkowski background agrees with the covariant closed string field theory to cubic order if we make an appropriate field redefinition. The non–linear field redefinition introduces higher order interactions. The higher order interactions found by studying the three string vertex must be canceled when including the higher string
vertices of the (non–polynomial) covariant string field theory.

Finally, it is important to understand better the role of the discrete states in $N = 2$ string field theory, and the way they fit into a background independent formulation. A similar difficulty arises when studying effective actions of the $d = 2$ string [23]. It may also be interesting to generalize this work to curved backgrounds, both for the $N = 2$ strings and the $N = 1$ heterotic string.

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Appendix A - Useful Formulas

In this appendix we present some useful formulas that help in particular in deriving eq. (3.17).

Hermitian conjugation of the oscillators $\alpha$ and $\bar{\alpha}$ in (3.9) is given by

$$\alpha^+_n = \alpha_{-n}, \quad \bar{\alpha}^+_n = \bar{\alpha}_{-n}.$$  \hspace{1cm} (A.1)

The commutation relations of the oscillators with $E^+_{123}$ (the hermitian conjugate of (3.9)) are:

$$[\alpha^a_a(r), E^+_{123}] = \sum_s \sum_{m>0} n N^r_{nm} \alpha^a_m + N^r_{n0} \alpha^a_0,$$

$$[\bar{\alpha}^I_I(r), E^+_{123}] = \sum_s \sum_{m>0} \bar{N}^r_{nm} \bar{\alpha}^I_m + \bar{N}^r_{n0} \bar{\alpha}^I_0,$$  \hspace{1cm} (A.2)

where $n > 0$ in (A.2). Also

$$[\alpha^a_a(r), e^{E^+_{123}}] = [\alpha^a_a(r), E^+_{123}] e^{E^+_{123}},$$

$$[\bar{\alpha}^I_I(r), e^{E^+_{123}}] = [\bar{\alpha}^I_I(r), E^+_{123}] e^{E^+_{123}},$$  \hspace{1cm} (A.3)

where $n > 0$ in (A.3).
The zero modes $\alpha_0$ and $\bar{\alpha}_0$ are operators which pick up values in the Narain lattice of compactification in the following way:

$$1 < p | 2 < p | 3 < p | \alpha_0^{a(r)} | 0 >_{123} = \delta_{p_1 + p_2 + p_3, 0} P_{L(r)}^a,$$
$$1 < p | 2 < p | 3 < p | \bar{\alpha}_0^{I(r)} | 0 >_{123} = \delta_{p_1 + p_2 + p_3, 0} P_{R(r)}^I.$$

(A.4)

(In the literature $\alpha_0$ is sometimes identified with $i\rho$, and this gives rise to relative $i$ factors in the formulas used in different works.)

We end this appendix with a simple symmetry of the Neumann coefficients, which is used repeatedly in the calculations:

$$N_{nm}^{rs} = N_{mn}^{sr}. \quad (A.5)$$

Appendix B - Witten Vertex

In this appendix we derive the map $h(z)$ in eq. (4.1) for Witten vertex and compute the Neumann coefficients.

The three string Witten vertex is a symmetric overlap vertex, namely, the images of the three unit discs mapped to the sphere by $h_1$, $h_2$ and $h_3$ in (4.1) cover precisely the complete sphere. With the three punctures at $z = 1, \rho, \bar{\rho}$, where $\rho = e^{2\pi i/3}$, the map $h(z)$ is manifestly cyclic symmetric, and thus map the unit disc $|z| \leq 1$ onto one third of the complex plane containing the puncture at $z = 1$ (see figure 1), namely

$$h(z) = \left(\frac{1 + z}{1 - z}\right)^{\frac{2}{3}}. \quad (B.1)$$

Expanding in $z$

$$h(z) = 1 + \frac{4}{3}z + \frac{8}{9}z^2 + o(z^3), \quad (B.2)$$

and comparing with the parametrization $h(z) = 1 + a_1 z + (\frac{1}{2}a_1^2 + a_2)z^2 + o(z^3)$ in (4.4) one finds

$$a_1 = \frac{4}{3}, \quad a_2 = 0. \quad (B.3)$$

From eq. (1.23) one finds $a = \frac{3\sqrt{3}}{4}$, which is compatible with the relation (1.14).

Inserting (B.3) in (1.13), (1.17) and (4.19) one finds the Neumann coefficients presented in section 3 (3.30), (3.31) and (3.32).
Appendix C - SCSV Vertex

In this appendix we derive the map \( h(z) \) in eq. (4.1) for SCSV vertex and compute the Neumann coefficients.

The SCSV vertex is a (non–symmetric) cyclic vertex defined by the map \( h(z) = z \) when the three punctures are at \( z = 0, 1, \infty \). To derive the SCSV vertex with the three punctures at \( z = 1, \rho, \bar{\rho} \), where \( \rho = e^{2\pi i/3} \), we act on \( h(z) \) with the \( SL(2, \mathbb{C}) \) transformation which maps \( 0 \to 1, 1 \to \rho \) and \( \infty \to \bar{\rho} \). One finds

\[
 h(z) = \frac{1 + \rho z}{1 + \bar{\rho} z}, \tag{C.1}
\]

and expanding in \( z \)

\[
 h(z) = 1 + i\sqrt{3} z + (i\sqrt{3}/2 - \frac{3}{2}) z^2 + o(z^3). \tag{C.2}
\]

Comparing with the parametrization \( h(z) = 1 + a_1 z + (\frac{1}{2} a_1^2 + a_2) z^2 + o(z^3) \) in (4.4) one finds

\[
 a_1 = i\sqrt{3}, \quad a_2 = i\frac{\sqrt{3}}{2}. \tag{C.3}
\]

Inserting (C.3) in (4.13), (4.17) and (4.19) one finds the Neumann coefficients

\[
 N_{11}^{rs} = \frac{a_1^2}{3} = -1, \quad \text{for } r \neq s
\]

\[
 N_{01}^{21} = N_{01}^{32} = N_{01}^{13} = \frac{a_1}{2\sqrt{3}} + \frac{ia_2}{a_1} = i,
\]

\[
 N_{01}^{21} = N_{01}^{23} = N_{01}^{31} = -\frac{a_1}{2\sqrt{3}} + \frac{ia_2}{a_1} = 0,
\]

\[
 N_{00}^{rs} = \delta^{rs}\log\frac{a_1}{\sqrt{3}} = \delta^{rs}\log i, \tag{C.4}
\]

and the rest of the Neumann coefficients are 0. The \( N^s \)s in (C.4) are the Neumann coefficients presented in section 3 (3.38).
References

[1] M.B. Green, J.H. Schwarz and E. Witten, ‘Superstring theory’ (Cambridge Univ. Press, Cambridge, 1987).

[2] A. Giveon and M. Porrati, Phys. Lett. B246 (1990) 54;
A. Giveon and M. Porrati, Nucl. Phys. B355 (1991) 422.

[3] M. Saadi and B. Zwiebach, Ann. Phys. 192 (1989) 213;
T. Kugo, H. Kunitomo and K. Suehiro, Phys. Lett. B226 (1989) 48;
B. Zwiebach, Mod. Phys. Lett. A5 (1990) 2753;
H. Sonoda and B. Zwiebach, Nucl. Phys. B331 (1990) 592;
B. Zwiebach, Comm. Math. Phys. 136 (1991) 83.

[4] T. Kugo and K. Suehiro, Nucl. Phys. B337 (1990) 434.

[5] H. Sonoda and B. Zwiebach, Nucl. Phys. B336 (1990) 185.

[6] A. Sen, Nucl. Phys. B345 (1990) 551; B347 (1990) 270;
S. Mukherji and A. Sen, Nucl. Phys. B363 (1991) 639;
A. Sen, preprint TIFR/TH/91–39, 1991.

[7] A. Eskin, Phys. Lett. B206 (1988) 612.

[8] M. de Roo, Nucl. Phys. B255 (1985) 515;
M. de Roo, Phys. Lett. B156 (1985) 331;
M. de Roo and P. Wagemans, Nucl. Phys. B262 (1985) 644;
E. Bergshoeff, I.G. Koh and E. Sezgin, Phys. Lett. B155 (1985) 71;
P. Wagemans, Phys. Lett. B206 (1988) 241.

[9] K.S. Narain, Phys. Lett. B169 (1986) 369;
K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B289 (1987) 414.

[10] T. Kugo and B. Zwiebach, preprint IASSNS–HEP–92/3, 1992.

[11] A. LeClair, M.E. Peskin and C.H. Preitschopf, Nucl. Phys. B317 (1989) 411.

[12] E. Witten, Nucl. Phys. B268 (1986) 253.
[13] D.J. Gross and A. Jevicki, Nucl. Phys. B283 (1987) 1.

[14] S. Sciuto, Nuovo Cim. Lett. 2 (1969) 411;
    L. Caneschi, A. Schwimmer and G. Veneziano, Phys. Lett. B30 (1969) 351.

[15] A. LeClair, Nucl. Phys. B297 (1988) 603.

[16] M. Ademollo et. al., Phys. Lett. B62 (1976) 105;
    A. D’Adda and F. Lizzi, Phys. Lett. B191 (1987) 85;
    J.D. Cohn, Nucl. Phys. B284 (1987) 349;
    M. Covri, V.A. Kostelecky and P. Moxhay, Phys. Rev. D39 (1988) 1611;
    S. Mathur and S. Mukhi, Nucl. Phys. B302 (1988) 130;
    N. Ohta and S. Osabe, Phys. Rev. D39 (1989) 1641.

[17] H. Ooguri and C. Vafa, Nucl. Phys. B361 (1991) 469.

[18] A. Giveon and A. Shapere, preprint CLNS–92/1139, IASSNS–HEP–92/14, 1992.

[19] J. Bienkowska, preprint EFI 91–65, 1991.

[20] M. Li, preprint UCSBTH–92–14, 1992.

[21] M. Bonini, E. Gava and R. Iengo, Mod. Phys. Lett. A6 (1991) 795.

[22] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301;
    A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B238 (1990) 57.

[23] E. Witten, preprint IASSNS–HEP–91/51, 1991;
    I. Klebanov and A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 3273;
    E. Witten and B. Zwiebach, preprint IASSNS–HEP–92/4, 1992.
FIGURE CAPTIONS

Figure 1:
The maps $h_1, h_2, h_3$ from the coordinate systems $z_1, z_2, z_3$ to the $z$–plane for Witten vertex with punctures at $z = 1, \rho, \bar{\rho}, \rho = e^{\frac{2\pi i}{3}}$. 