Black Hole Greybody Factors and Absorption of Scalars by Effective Strings

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Abstract

We compute the greybody factors for classical black holes in a domain where two kinds of charges and their anticharges are excited by the extra energy over extremality. We compare the result to the greybody factors expected from an effective string model which was earlier shown to give the correct entropy. In the regime where the left and right moving temperatures are much smaller than the square root of the effective string tension, we find a non-trivial greybody factor which agrees with the effective string model. However, if the temperatures are comparable with the square root of the effective string tension, the greybody factors agree only at the leading order in energy. Nevertheless, there are several interesting relations between the two results, suggesting that a modification of the effective string model might lead to better agreement.
1. Introduction

In recent times there has been considerable progress in understanding the properties of black holes from a microscopic fundamental theory of gravity – string theory. The state content of nonperturbative string theory appears to be just correct to give the entropy of extremal black holes, when we estimate the number of states of string theory carrying the same charges as the black hole [1]. This result on the entropy extends to a variety of situations where a black hole is close to extremality [2]. The entropy of certain near-extremal branes sometimes agrees with the Bekenstein-Hawking entropy up to factors of order unity [3] whose origin was recently explained in [4]. These developments follow earlier suggestions of Russo and Susskind [5] that the large number of states of a string with given mass might be related to the large entropy of black holes with the same mass, and calculations of Sen [6] which showed such agreement up to a numerical coefficient for elementary string states corresponding to extreme black holes.

Somewhat surprisingly, the rates of exciting and de-exciting the states of string theory that correspond to a black hole agree, at low energies, with the rates of absorption and Hawking radiation for the black hole. The string calculation is carried out at weak coupling, while the black hole with a classical horizon is a good description for strong coupling. There is no a priori reason for this agreement at different values of the coupling; nevertheless it appears to work in a remarkably large class of low energy interactions. The leading term for minimally coupled scalars in 5 dimensions was computed in [7], for the string state and for the classical black hole, and the above mentioned agreement was found. (It had been shown in [8], [9] that the two rates should agree up to a constant of proportionality.) The agreement at leading order for charged scalars in 4 and 5 dimensions was demonstrated in [10]. In [11] such calculations were extended (in 5 dimensions) to the case where the emitted energy is comparable to the appropriately defined left and right moving temperatures, which yields agreement for the greybody factor of the black hole in this domain. Such greybody factors agree also in 4 dimensions [12], and for the absorption of the non-minimally coupled ‘fixed scalars’ [13,14,15] in 5 dimensions [16] and in 4 dimensions [17].

In the above calculations the model used for the effective description of the stringy black holes was that of a single long string that absorbs and emits gravitons by coupling them to its vibration modes [18]. These modes are assumed to be the lowest energy excitations in the system, which implies that the moduli of the spacetime are chosen to suppress the excitations of other kinds of charges and anti-charges. What happens when the moduli are such that the excitation of one other kind of charge pair becomes relevant?

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2 In [18] it was shown that the elementary string quantized by the Polyakov prescription leads to the same cross sections at leading order in energy.
The simplest case of a second excitation which becomes relevant is that the string which carried the vibrations (the ‘momentum’ and ‘antimomentum’ modes) can also be excited to give additional pairs of ‘strings-antistrings’. But the creation of a string-antistring pair can be thought of as a segment of the original string that bends over to run for a while in the opposite direction in target space, after which it runs in the forward direction again. Thus, such ‘bends’ are equivalent to the string-antistring pairs.

When we quantize the vibrations of the D-string as a 1-dimensional gas of massless particles living on the D-string, then we are in the limit where these vibrations are small amplitude, long wavelength transverse displacements of the string. Thus the ‘bends’ that might correspond to the string-antistring excitations are not included. But if we quantize the string by the Polyakov prescription, then all configurations of the string are included, and so we expect that both the momentum-antimomentum and the string-antistring excitations are taken into account.

Indeed it was shown in \[4\] that using a Polyakov quantized effective string the entropy of the 5 dimensional black hole was correctly reproduced in the domain where one of the charges (given by the 5-D-branes) is large, but the momentum, antimomentum, string winding and string antiwinding excitations are all comparable.\(^3\) The tension of this effective string was found to be equal to that of a D-string divided by the number of the 5-branes.\(^4\) In \[24\] it was shown that if we assume that a Polyakov quantized string has an effective central charge of 6, then its low energy absorption cross section for minimal scalars equals the area of the horizon of the corresponding black hole, which is classically the correct cross section for the black hole \[25\]. This leading order in energy calculation was carried out in more detail and extended to the 4-dimensional case in \[26\].

In this paper we investigate the absorption and emission of minimally coupled scalars from a black hole which has two charges and the nonextremality parameter all comparable to each other. This means that we can excite two kinds of branes and antibranes, and thus this case cannot be covered by a model where we consider small transverse vibrations of an effective string. For the 5-dimensional case we use the model where we have a collection of 5-D-branes wrapped on a 5-torus, and an effective string quantized by the Polyakov prescription \[4\] lives inside the 5-branes. For the 4-dimensional case we also assume, following \[27\], that the physics is given by a central charge \(c = 6\) effective string. In this case the effective string arises at a triple intersection of the 5-branes in M-theory \[27\], so that the geometrical picture is more complicated.

\(^3\) A description of extremal black holes using a similar non-critical string theory was worked out in \[21\].

\(^4\) A rescaling of the string tension was also advocated for the NS-NS charged black holes in \[22,23\].
By studying the minimally coupled massless scalar in the background of a black hole corresponding to the D-brane or M-brane bound state, we derive the absorption cross-section using the methods of semi-classical General Relativity. This gives us explicit formulae for its frequency dependence. We define the left and right moving temperatures which, in the effective string interpretation, characterize the thermal distributions of the left and right movers. If \( T_L, T_R \ll \sqrt{T_{\text{eff}}^\text{eff}} \), then the GR greybody factor has a non-trivial dependence on \( \omega/T_L \) and \( \omega/T_R \) which exactly agrees with our calculation on the effective string side.

If, however, \( T_{L,R} \) are of the order of the effective string scale, then the situation is more complicated. At leading order in the energy of the incident quantum we recover the result that the cross section for absorption equals the area of the horizon. But the next correction in energy yields a difference between the classical greybody factor and what the simplest effective string calculation, that carried out in \([24]\), gives. This suggests that in this domain of parameters some new effects, such as the splitting and joining of effective strings, should be taken into account. It is curious that if we set a certain parameter \( \beta \) in the classical greybody factor to zero, then the agreement with the effective string greybody factor found from the results of \([24]\) is restored. It is hoped that the nature of the deviation between the classical case and the simplest string model will lead to an improved understanding of the excitation spectrum of the 5-brane - string bound state system.

We also extend our calculations to the case of charged scalar emission, and to 4-dimensional black holes, where all the conclusions are qualitatively similar to the \( D = 5 \) case.

The plan of the paper is as follows. In section 2 we write down the wave equation for massless scalars in the background of the 5-dimensional black hole and compute the classical absorption cross section in the domain of parameters mentioned above. In section 3 we compute the cross section expected from the Polyakov quantization of an effective string. Section 4 extends the results to the charged case, and section 5 to the 4-dimensional case. Section 6 compares the classical and string greybody factors. Section 7 is a discussion, with some conjectures.

2. Classical absorption by \( D = 5 \) black holes with one large charge

We would like to study minimally coupled massless scalars moving in the background of a \( D = 5 \) black hole with 3 \( U(1) \) charges. The metric of such a black hole is \([28,29,30]\)

\[
ds_5^2 = -f^{-2/3}h dt^2 + f^{1/3} \left( h^{-1} dr^2 + r^2 d\Omega_3^2 \right),
\]

where

\[
h(r) = \left( 1 - \frac{r_0^2}{r^2} \right), \quad f(r) = \left( 1 + \frac{r_0^2}{r^2} \right) \left( 1 + \frac{r_1^2}{r^2} \right) \left( 1 + \frac{r_2^2}{r^2} \right).
\]
The \( l \)-th partial wave of a minimally coupled massless scalar satisfies the radial equation,

\[
\frac{h}{r^3} \frac{d}{dr} \left( hr^3 \frac{dR_l}{dr} \right) + \left[ f \omega^2 - h \frac{l(l+2)}{r^2} \right] R_l(r) = 0 .
\]  \hspace{1cm} (2)

In this paper we are concerned with the following range of parameters,\(^5\)

\[
r_0, r_1, r_n \ll r_5 .
\]  \hspace{1cm} (3)

It is convenient to introduce hyperbolic angles \( \sigma_1 \) and \( \sigma_n \) defined by

\[
r_n^2 = r_0^2 \sinh^2 \sigma_n , \quad r_1^2 = r_0^2 \sinh^2 \sigma_1 .
\]

In order to find approximate solutions of (2) we will match the solutions in region I, where \( r \ll r_5 \), to solutions in region II, where \( r \gg r_0, r_1, r_n \). Because of the condition (3), the inner and outer regions have a large overlap, so that a reliable matching is possible.

In region I, (2) simplifies to

\[
\left[ (hr^3 \partial_r)^2 + (\omega r_5)^2 (r^2 + r_1^2)(r^2 + r_n^2) - hl(l+2)r^4 \right] R_l(r) = 0 .
\]  \hspace{1cm} (4)

In terms of the variable \( z = h(r) \), this becomes

\[
z \frac{d}{dz} \frac{dR}{dz} + \left[ D + \frac{C}{(1-z)} + \frac{E}{(1-z)^2} \right] R_l = 0 ,
\]  \hspace{1cm} (5)

where

\[
D = \frac{1}{4} (\omega r_5)^2 \frac{r_1^2 r_n^2}{r_0^2} = \frac{1}{4} (\omega r_5)^2 \sinh^2 \sigma_1 \sinh^2 \sigma_n ,
\]

\[
C = \frac{1}{4} \left[ (\omega r_5)^2 \frac{r_1^2 + r_n^2}{r_0^2} + l(l+2) \right] = \frac{1}{4} \left[ (\omega r_5)^2 (\sinh^2 \sigma_1 + \sinh^2 \sigma_n) + l(l+2) \right] ,
\]  \hspace{1cm} (6)

\[
E = \frac{1}{4} \left( (\omega r_5)^2 - l(l+2) \right) .
\]

Remarkably, (5) may be reduced to a hypergeometric equation by a substitution of the form

\[
R_l = z^\alpha (1-z)^\beta F(z) .
\]  \hspace{1cm} (7)

After some algebra we find that, if \( \alpha \) and \( \beta \) satisfy

\[
E + \beta(\beta - 1) = 0 , \quad \alpha^2 + D + C + E = 0 ,
\]  \hspace{1cm} (8)

\(^5\) The solution of the higher partial wave equations in the regime \( r_0, r_n \ll r_1, r_5 \) was found by Maldacena and Strominger (private communication).
then the equation for $F(z)$ becomes
\[
z(1 - z)\frac{d^2F}{dz^2} + [(2\alpha + 1)(1 - z) - 2\beta z]\frac{dF}{dz} - [(\alpha + \beta)^2 + D]F = 0 \tag{9}
\]
which is the hypergeometric equation! In general, the solution to
\[
z(1 - z)\frac{d^2F}{dz^2} + [C - (1 + A + B)z]\frac{dF}{dz} - ABF = 0 \tag{10}
\]
which satisfies $F(0) = 1$, is the hypergeometric function $F(A, B; C; z)$. Thus, the solution in the inner region is
\[
R_I = z^\alpha(1 - z)^\beta F(\alpha + \beta + i\sqrt{D}, \alpha + \beta - i\sqrt{D}; 1 + 2\alpha; z). \tag{11}
\]
Since we are interested in an incoming wave at the horizon, we choose
\[
\alpha = -i\sqrt{D} + C + E = -i\frac{\omega r_5}{2} \cosh \sigma_1 \cosh \sigma_n.
\]
We will chose $\beta$ to be the smaller of the two roots of the quadratic equation for $\omega r_5 < l + 1$,
\[
2\beta = 1 - \sqrt{1 - 4E} = 1 - \sqrt{(l + 1)^2 - (\omega r_5)^2}.
\]
First we present the calculation in the regime $\omega r_5 < l + 1$, which is of primary interest to us. The modifications to $\omega r_5 > l + 1$, where $\beta$ becomes complex, will be discussed in the Appendix.

Using the asymptotics of the hypergeometric functions for $z \to 1$, we find that, for large $r$,
\[
R_I \to \left(\frac{r_0}{r}\right)^{2\beta} \frac{\Gamma(1 + 2\alpha)\Gamma(1 - 2\beta)}{\Gamma(1 + \alpha - \beta - i\sqrt{D})\Gamma(1 + \alpha - \beta + i\sqrt{D})} \tag{12}
\]
In the outer region (region II), (2) simplifies to
\[
r^{-3} \frac{d}{dr} r^3 \frac{dR}{dr} + \omega^2 + \frac{(\omega r_5)^2 - l(l + 2)}{r^2} R_I(r) = 0 , \tag{13}
\]
which is easily solved in terms of the Bessel functions. The dominant solution, which matches to the asymptotic form in region I, is
\[
R_{II} = 2A \rho^{-1} J_{1 - 2\beta}(\rho) , \quad \rho = \omega r .
\]
For small $r$ this approaches
\[
A \frac{\rho}{\Gamma(2 - 2\beta)} \left(\frac{2}{\omega r}\right)^{2\beta} .
\]
Matching $R_{II}$ to $R_I$, we find that

$$A = \left(\frac{\omega r_0}{2}\right)^{2\beta}(1-2\beta)\frac{\Gamma(1+2\alpha)\Gamma^2(1-2\beta)}{\Gamma(1+\alpha-\beta-i\sqrt{D})\Gamma(1+\alpha-\beta+i\sqrt{D})}$$  \hspace{1cm} (14)$$

The absorption cross-section may now be obtained using the method of fluxes. The flux per unit solid angle is

$$F = \frac{1}{2i}\left(R^* hr^3 \partial_r R - \text{c.c.}\right).$$  \hspace{1cm} (15)$$

The absorption probability is the ratio of the incoming flux at the horizon to the incoming flux at infinity,

$$P = \frac{\mathcal{F}_h}{\mathcal{F}_{\text{incoming}}} = \frac{\pi \omega^3}{2} r_0^2 \cosh \sigma_1 \cosh \sigma_n |A|^{-2}. \hspace{1cm} (16)$$

While this expression is quite complicated in general, it simplifies for the energy low enough that $\omega r_5 \ll l + 1$. Now $\beta \approx -l/2$ and $A$ may be expressed in the following form,

$$A = (1 + l)(l!)^2(\omega r_0/2)^{-l} \frac{\Gamma \left(1 - i\frac{\omega}{2\pi T_H}\right)}{\Gamma \left(1 + \frac{l}{2} - i\frac{\omega}{4\pi T_L}\right)} \frac{\Gamma \left(1 + \frac{l}{2} - i\frac{\omega}{4\pi T_R}\right)}{\Gamma \left(1 + \frac{l}{2} + i\frac{\omega}{4\pi T_R}\right)}$$  \hspace{1cm} (17)$$

where the left and right temperatures are

$$\frac{1}{T_L} = 2\pi r_5 \cosh(\sigma_1 - \sigma_n), \quad \frac{1}{T_R} = 2\pi r_5 \cosh(\sigma_1 + \sigma_n).$$  \hspace{1cm} (18)$$

As we show later, these are precisely the temperatures on the fractionated D-string moving within the 5-branes. Since the inverse Hawking temperature is

$$\frac{1}{T_H} = 2\pi r_5 \cosh \sigma_1 \cosh \sigma_n$$

we have the relation $[^{13}]$

$$\frac{2}{T_H} = 1 \frac{1}{T_L} + \frac{1}{T_R}. \hspace{2cm}$$

Let us note that the absorption probability for a partial wave with $l > 0$ is suppressed compared to the $s$-wave by a power of the small quantity $\omega r_0$, as long as $\omega r_5 < l + 1$. Thus, for $\omega r_5 < 2$ the $s$-wave dominates the absorption $[^6]$. Now the absorption cross-section is related to the $s$-wave absorption probability by

$$\sigma_{abs} = \frac{4\pi}{\omega^3} P_{l=0} = 2\pi^2 r_5^2 \cosh \sigma \cosh \delta |A_{l=0}|^{-2}. \hspace{1cm} (19)$$

[^6]: For $\omega r_5 \geq 2$ the $l = 1$ partial wave is no longer suppressed compared to $l = 0$. It is interesting that $\omega = 2/r_5$ is precisely the energy required to create the first massive state of the effective string, whose rest energy is $\sqrt{8\pi T^*_{\text{eff}}}$. In general, the $l$-th partial wave becomes important for $\omega r_5 \geq l + 1$.  

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For $\omega r_5 \ll 1$ we find

$$\sigma_{\text{abs}} = 2\pi^2 r_5 r_0^2 \cosh \sigma_1 \cosh \sigma_n \frac{\omega}{2(T_L + T_R)} \frac{e^{\frac{\omega}{T_R}} - 1}{(e^{\frac{\omega}{T_L}} - 1)(e^{\frac{\omega}{T_R}} - 1)}$$

(20)

which can be written as

$$\sigma_{\text{abs}} = \pi^3 r_5^2 r_0^2 (1 + \sinh^2 \sigma_1 + \sinh^2 \sigma_n) \frac{\omega}{(e^{\frac{\omega}{T_L}} - 1)(e^{\frac{\omega}{T_R}} - 1)}.$$  

(21)

This greybody factor has form similar to that found in [13]; we discuss the relations between what we have and the results in [13] in section 5.

From detailed balance it follows that the differential rate of Hawking emission is

$$d\Gamma = \frac{\pi^3 r_5^2 r_0^2 (1 + \sinh^2 \sigma_1 + \sinh^2 \sigma_n) \omega}{(e^{\frac{\omega}{T_L}} - 1)(e^{\frac{\omega}{T_R}} - 1)} \frac{d^4k}{(2\pi)^4}$$

(22)

In the next section, we compare (19) with the analysis of absorption by highly excited strings.

### 3. The effective string calculation

The effective string is taken to have a mass spectrum that resembles that of the Polyakov quantized elementary string, though we take $c = 6$ rather than $c = 12$ in the light cone gauge quantization [21]. While this is not likely to be an exact quantization of the string, we take it as an approximate description for large excitation levels. Thus, we assume that the mass levels are given by

$$m^2 = \left(\frac{2\pi R n_w T_{\text{eff}} + \frac{n_p}{R}}{R}\right)^2 + 8\pi T_{\text{eff}} N_R = \left(\frac{2\pi R n_w T_{\text{eff}} - \frac{n_p}{R}}{R}\right)^2 + 8\pi T_{\text{eff}} N_L$$

(23)

In the effective string calculation the absorption of the scalar is described by a 3-string vertex: the initial string absorbs a massless string state and yields a more energetic string. Let the incoming scalar have energy $\omega$. Since it does not carry any charge, the numbers $n_w, n_p$ are not altered by the absorption of the scalar. Thus we have

$$2m\delta m = 8\pi T_{\text{eff}} \delta N_R = 8\pi T_{\text{eff}} \delta N_L$$

(24)

The scalar contributes one left oscillator $\alpha_i$ and one right oscillator $\tilde{\alpha}_j$, with

$$i = j = \delta N_R = \delta N_L = \frac{m}{4\pi T_{\text{eff}}} \omega$$

(25)
We are interested in averaging the absorption rate over all initial string states of a given mass. Methods for performing such a calculation were developed in [24]. The initial state of the excited string is assumed to be populated by oscillators in a thermal form [24], and these contribute Bose enhancement factors to the calculation of the absorption cross-section. Following the methods of [24], we find that the absorption cross-section, which is the absorption rate minus the emission rate, is

\[
\sigma_{\text{abs}} = \frac{T_{\text{eff}}}{m\omega} (2\pi \kappa_5 \delta N_L)^2 \frac{e^{\beta^*_L \delta N_L + \beta^*_R \delta N_R} - 1}{(e^{\beta^*_L \delta N_L} - 1) (e^{\beta^*_R \delta N_R} - 1)}
\]

where

\[
\beta^*_{R,L} = \frac{\pi}{\sqrt{N_{R,L}}},
\]

and \(\kappa_5\) is the 5-dimensional gravitational constant. For the right movers, the factor in the exponent is

\[
\beta^*_R \delta N_R = \frac{m\omega}{4T_{\text{eff}} \sqrt{N_R}}.
\]

To compare with (22), we have to identify this with \(\omega^2 T_R\). Thus,

\[
T_R = 2T_{\text{eff}} \frac{\sqrt{N_R}}{m}.
\]

Following Maldacena, we will identify the effective string tension as the D-string tension divided by the number of 5-branes, \(n_5\),

\[
T_{\text{eff}} = \frac{1}{2\pi g n_5} = \frac{1}{2\pi r_5^2}
\]

where we have set \(\alpha' = 1\). Using

\[
m = \frac{RV r_0^2}{2g^2} (\cosh(2\sigma_1) + \cosh(2\sigma_n)) = \frac{2m^2 r_0^2}{\kappa_5^2} (1 + \sinh^2 \sigma_1 + \sinh^2 \sigma_n)
\]

\[
n_w = \frac{n_5 V r_0^2}{2g} \sinh(2\sigma_1)
\]

\[
n_p = \frac{R^2 V r_0^2}{2g^2} \sinh(2\sigma_n)
\]

we find

\[
[m^2 - (2\pi R n_w T_{\text{eff}} + \frac{n_p}{R})^2]^{1/2} = \frac{VR r_0^2}{2g^2} 2 \cosh(\sigma_1 + \sigma_n).
\]

Since

\[
\frac{\sqrt{N_R}}{m} = (\frac{1}{8\pi T_{\text{eff}}})^{1/2} [m^2 - (2\pi R n_w T_{\text{eff}} + \frac{n_p}{R})^2]^{1/2} \frac{1}{m}
\]

\[
8
\]
and
\[(\cosh(2\sigma_1) + \cosh(2\sigma_n)) = 2 \cosh(\sigma_1 + \sigma_n) \cosh(\sigma_1 - \sigma_n)\] (34)

we finally have
\[T_R = \frac{1}{2\pi r_5 \cosh(\sigma_1 + \sigma_n)}.\] (35)

Performing an analogous comparison for the left-movers, we find
\[T_L = \frac{1}{2\pi r_5 \cosh(\sigma_1 - \sigma_n)}.\] (36)

Thus, \(T_L\) and \(T_R\) are in agreement with the temperatures (18) found in the GR greybody factor in the limit \(\omega r_5 \ll 1\). In this limit, the cross section (26) is in complete agreement with the GR result, (21).

4. Scalars carrying Kaluza-Klein charge

The case of charged scalars propagating in the non-extremal \(D = 5\) black hole background was studied in [13]. It was found that the wave equation for a scalar of energy \(k_0\) and charge \(k_5\) is obtained from that for a neutral scalar of energy \(\omega\) by the following replacement of parameters:
\[\omega \to \omega', \quad \sigma_n \to \sigma'_n,\]
where
\[\omega' = \sqrt{k_0^2 - k_5^2}, \quad e^{\sigma'_n} = e^{\sigma_n} \frac{k_0 - k_5}{\omega'}\]

We wish to see if this relation also exists between the cases of charged and neutral absorption in the effective string calculation. We let the incoming scalar carry the \(U(1)\) charge corresponding to the momentum component \(k_5\), which is along the direction in which the effective string carries momentum and winding. The absorption of the charged scalar is still given by the 3-string vertex, but now \(\delta N_L\) is no longer equal to \(\delta N_R\). In varying the relations (23), we have to include the variation \(\delta n_p\), whose relation to the absorbed charge is
\[k_5 = \frac{\delta n_p}{R}\]
while \(k_0 = \delta m\). Thus, we find
\[2mk_0 = 8\pi T_{eff} \delta N_R + 2k_5 \left(2\pi R n_w T_{eff} + \frac{n_p}{R}\right)\] (37)
\[2mk_0 = 8\pi T_{eff} \delta N_L - 2k_5 \left(2\pi R n_w T_{eff} - \frac{n_p}{R}\right)\] (38)

Solving for \(\delta N_{L,R}\), and using (27), we arrive at
\[\beta'_{L} \delta N_L = \pi r_0 \left(k_0 \cosh(\sigma_1 - \sigma_n) + k_5 \sinh(\sigma_1 - \sigma_n)\right)\]
\[ \beta^*_R \delta N_R = \pi r_0 (k_0 \cosh(\sigma_1 + \sigma_n) - k_5 \sinh(\sigma_1 + \sigma_n)) \]

Now it is not hard to see that
\[ \beta^*_R \delta N_R = \omega' \frac{R}{2T_R} , \quad \beta^*_L \delta N_L = \omega' \frac{R}{2T_L} \]

We then find that the effective string absorption cross section for charged scalars can be written as
\[ \sigma_{\text{abs}} = \pi^3 r_5^2 r_0^2 (1 + \sinh^2 \sigma_1 + \sinh^2 \sigma'_n) \frac{\omega' \left( e^{\frac{\omega'}{T_H}} - 1 \right)}{\left( e^{\frac{\omega'}{T_L}} - 1 \right) \left( e^{\frac{\omega'}{T_R}} - 1 \right)} \]

where
\[ \frac{1}{T_L'} = 2\pi r_5 \cosh(\sigma_1 - \sigma'_n) , \quad \frac{1}{T_R'} = 2\pi r_5 \cosh(\sigma_1 + \sigma'_n) . \]

and
\[ \frac{1}{T_H'} = \frac{1}{2T_L'} + \frac{1}{2T_R'} . \]

This is in agreement with the GR absorption cross section in the limit \( \omega r_5 \ll 1 \).

5. \( D = 4 \) black holes with two large charges

In this section we turn to minimally coupled massless scalars moving in the background of a \( D = 4 \) black hole with 4 \( U(1) \) charges. 3 of the charges may be taken to be the same as those in the \( D = 5 \) case: namely the D-5-brane charge, the D-1-brane charge along one of the compact directions, and the momentum charge along this same direction. The fourth charge can be taken to arise from Kaluza-Klein monopoles [31,32]. A different picture, which arises upon embedding the \( D = 4 \) black holes into M-theory, is to view them as triply intersecting 5-branes, wrapped over \( T^7 \) [27]. The radii \( r_1, r_2, r_3 \) are determined by the numbers of 5-branes positioned in (12345), (12367) and (14567) planes respectively. We are concerned here with the situation where \( r_2 \) and \( r_3 \) are large compared to the other radii. As shown in [33], in this regime the Bekenstein-Hawking entropy coincides with the entropy of a gas of effective strings with central charge \( c = 6 \) and tension
\[ T_{\text{eff}} = \frac{1}{8\pi r_2 r_3} . \]

These strings originate from the (12345) 5-branes wrapped over the torus.

First we compute the classical absorption. The metric of such a black hole is [34,33,35]
\[ ds_5^2 = -f^{-1/2} h dt^2 + f^{1/2} \left( h^{-1} dr^2 + r^2 d\Omega_2^2 \right) , \]
where
\[ h(r) = \left(1 - \frac{r_0}{r}\right), \quad f(r) = \left(1 + \frac{r_1}{r}\right)\left(1 + \frac{r_2}{r}\right)\left(1 + \frac{r_3}{r}\right)\left(1 + \frac{r_n}{r}\right) \]

The \( l \)-th partial wave of a minimally coupled massless scalar satisfies the radial equation,

\[
\frac{h}{r^2} \frac{d}{dr} \left( h r^2 \frac{dR}{dr} \right) + \left[ f \omega^2 - h \frac{l(l+1)}{r^2} \right] R_l(r) = 0 .
\] (44)

We are concerned with the range of parameters
\[ r_0, r_1, r_n \ll r_2, r_3 . \] (45)

It is convenient to introduce hyperbolic angles \( \sigma_1 \) and \( \sigma_n \) defined by
\[ r_n = r_0 \sinh^2 \sigma_n , \quad r_1 = r_0 \sinh^2 \sigma_1 . \]

In order to find approximate solutions of (2) we will again match the solutions in the inner and outer regions.

In region I, where \( r \ll r_2, r_3 \), it is convenient to use the variable \( z = h(r) \). The equation assumes the form (3), with the parameters now defined by

\[
D = \omega^2 r_2 r_3 \sinh^2 \sigma_1 \sinh^2 \sigma_n , \quad C = \omega^2 r_2 r_3 \left( \sinh^2 \sigma_1 + \sinh^2 \sigma_n \right) + l(l+1) , \quad E = \omega^2 r_2 r_3 - l(l+1) .
\] (46)

Thus, the solution in the inner region is given by (11), with
\[
\alpha = -i \omega \sqrt{r_2 r_3} \cosh \sigma_1 \cosh \sigma_n \\
2 \beta = 1 - \sqrt{(2l+1)^2 - 4 \omega^2 r_2 r_3} .
\]

We will present the calculation in the regime \( 2 \omega \sqrt{r_2 r_3} < 2l + 1 \).

In the outer region \( r \gg r_0, r_1, r_n \), (44) simplifies to

\[
\rho^{-2} \frac{d}{d\rho} \rho^2 \frac{dR_l}{d\rho} + \left[ 1 + \frac{\omega (r_2 + r_3)}{\rho} + \frac{\omega^2 r_2 r_3 - l(l+1)}{\rho^2} \right] R_l = 0 ,
\] (47)

where we have defined \( \rho = \omega r \). This equation can be solved in terms of the Coulomb functions. The dominant solution, which matches to the asymptotic form in region I, is
\[ R_{II} = A \rho^{-1} F_{-\beta}(\rho) . \]
For small $\rho$

$$F_{-\beta} \approx C_{-\beta}(\eta)\rho^{1-\beta}$$

where

$$\eta = -\frac{1}{2}(r_2 + r_3)\omega$$

$$C_L(\eta) = \frac{2^L e^{-\pi L/2} |\Gamma(L + 1 + i\eta)|}{\Gamma(2L + 2)}$$

Matching $R_{II}$ to the asymptotic form of $R_I$, given in (12), we find that

$$A = \frac{(\omega r_0)^{\beta}}{C_{-\beta}(\eta)} \frac{\Gamma(1 + 2\alpha)\Gamma(1 - 2\beta)}{\Gamma(1 + \alpha - \beta - i\sqrt{D})\Gamma(1 + \alpha - \beta + i\sqrt{D})}$$

For $2\omega\sqrt{r_2r_3} < 3$, the s-wave dominates the absorption, and we find

$$\sigma_{abs} = 4\pi r_0 \sqrt{r_2r_3} \omega r_0 \cosh \sigma_1 \cosh \sigma_n |A_{l=0}|^{-2}.$$  

Let us now consider the effective string calculation, assuming that the string again has $c = 6$. The relations (24) - (25) hold again. Using (42), and the results in [27,33,36], we find that the relations (29) - (31) are replaced by

$$m = \frac{\pi r_0}{\kappa_4^2} (\cosh(2\sigma_1) + \cosh(2\sigma_n))$$

$$2\pi R n_w T_{eff} = \frac{\pi r_0}{\kappa_4^2} \sinh(2\sigma_1)$$

$$\frac{n_p}{R} = \frac{\pi r_0}{\kappa_4^2} \sinh(2\sigma_n)$$

where $\kappa_4$ is the gravitational constant in $D = 4$.

We then find

$$T_R = \frac{1}{4\pi \sqrt{r_2r_3} \cosh(\sigma_1 + \sigma_n)}, \quad T_L = \frac{1}{4\pi \sqrt{r_2r_3} \cosh(\sigma_1 - \sigma_n)}$$

From (26) it follows that the absorption cross section predicted by the simplest effective string analysis is

$$\sigma_{abs} = 2\pi \sqrt{r_2r_3} \omega r_0 [\cosh(2\sigma_1) + \cosh(2\sigma_n)] \frac{\omega \left(e^{\frac{\omega}{T_R}} - 1\right)}{\left(e^{\frac{\omega}{T_L}} - 1\right) \left(e^{\frac{\omega}{T_R}} - 1\right)}$$

This is in agreement with the $\omega \sqrt{r_2r_3} \ll 1$ limit of the classical cross section (51).

In the limit $\omega \to 0$ we recover

$$\sigma_{abs} = 4\pi \sqrt{r_2r_3} r_0 \cosh \sigma_1 \cosh \sigma_n$$

which is the area of the horizon.
6. Comparing classical and ‘effective string’ greybody factors

We now wish to carefully compare the greybody factors obtained from the classical calculation and from the effective string calculation. We carry out the discussion below for the 5-dimensional case. The 4-dimensional case is essentially similar.

First we consider the limit where $r_1 \gg r_n$, which implies $\sigma_1 \gg \sigma_n$. We can now further consider the limit $\omega r_5 \ll 1$. Then

$$\beta \approx \frac{(\omega r_5)^2}{4}$$

and we can thus ignore $\beta$ in the expression (14) for the amplitude $A$. Then the classical greybody factor reduces to (21), which equals the result (26) from the effective string calculation.

Note that whenever $\sigma_1 \gg \sigma_n$ we have

$$\cosh(\sigma_1 \pm \sigma_n) \approx \frac{1}{2} e^{\sigma_1 \pm \sigma_n}$$

so that we can approximate the cosh functions in the classical greybody factor by exponential. Indeed, the error in the replacement of the cosh function by the exponential arises in $\sigma_{abs}$ in the form

$$(\omega r_5)^2 \cosh^2(\sigma_1 \pm \sigma_n) - (\omega r_5)^2 \frac{1}{4} e^{\sigma_1 \pm \sigma_n} \approx \frac{1}{2} (\omega r_5)^2$$

so that if we wish to ignore $\beta$ then we must also ignore the difference between the cosh function and the exponential.

In this limit $r_1 \gg r_n$ we then find that our result reduces to the result for the greybody factor in [13], where the parameters were taken to satisfy $r_0, r_n \ll r_1, r_5$. We can also compute from our results the case $r_1 \ll r_n$, which has a similar treatment. The greybody factors resulting here again agree with the effective string calculation, and are related to the case $r_n \ll r_1$ by duality.

Now we address the case $r_1 \sim r_n$, which implies $\cosh(\sigma_1 \pm \sigma_n) \sim 1$. First consider the limit $\omega \to 0$. Then $\beta \to 0$, and both the classical cross section (13) and the effective string cross section (29) reduce to the area $A$ of the horizon. Now we are interested in the lowest corrections, which are terms of order $\omega^2$. These arise from three different sources:

(a) The corrections that come from the $\omega/(4\pi T_{L,R})$ terms contained both in the classical result and in the effective string result - these are of the form

$$\sigma_{abs} \to A[1 + C_1 (\omega r_5)^2 \cosh^2(\sigma_1 \pm \sigma_n)]$$

where $C_1$ is a constant of order unity.
(b) The corrections that arise in the classical result (but are not present in the effective string result) due to the term $\beta$ in the $\Gamma$ functions in (14). These are of the form

$$\sigma_{\text{abs}} \to A[1 + C_2(\omega r_5)^2]$$

(62)

where $C_2$ is of order unity.

(c) The correction that comes from the first factor in (14) (which is not present in the string result):

$$\sigma_{\text{abs}} \to A(\frac{\omega r_0}{2})^{-4\beta} \approx A e^{-((\omega r_5)^2 \log(\frac{\omega r_0}{2}))} \approx A[1 - (\omega r_5)^2 \log(\frac{\omega r_0}{2})]$$

(63)

Note that

$$\omega r_0 = (\omega r_5)(r_0/r_5) \ll 1$$

(64)

so that

$$|\log(\frac{\omega r_0}{2})| \gg 1$$

(65)

Thus we see that the leading correction comes from the last source (c). Note that we are probing a domain where $\omega r_5 = \sqrt{\alpha_{\text{eff}}}\omega$ is no longer infinitesimal, so that the dynamics on the effective string scale comes into play. (When $r_1$ and $r_n$ are both comparable to $r_0$, the temperatures on the effective string are sufficiently high that nontrivial greybody factors arise only for $\omega r_5$ non-infinitesimal.)

7. Discussion

We have computed the greybody factors in the classical geometry and in the effective string model. Quantizing the string by the Polyakov method incorporates the duality symmetry expected of the greybody factors, which interchanges the string winding and the momentum charges. This symmetry interchanges the 1-brane charge with the momentum charge and is part of the S-duality, but from the point of view of the effective string it is T-duality. Thus the string calculation covers the domains where the winding dominates over momentum, where the momentum dominates over winding, and where they are comparable. The left and right temperatures of the string have the form (19), which reduce to the results of [13] in the limit where the winding charge is much larger than the momentum charge, and its T-dual result when the momentum charge is much larger than the winding charge. In these domains, the left and right temperatures are much smaller than $1/r_5$, and for scalar energy comparable to $T_L, T_R$ the GR greybody factor is reproduced by an effective string calculation.

Our GR calculation is also valid when $T_L$ and $T_R$ are comparable to $1/r_5$. Now the greybody factor contains no energy scale $\ll r_5$. While there is agreement, as expected, at the leading order in energy (the cross section is the area of the horizon in both cases), there
are deviations between the two calculations at the next order in the energy of the incident quantum. The form of the deviations is suggestive of the fact that some modification of the effective string model would yield agreement with the classical result at higher orders in the energy. One such modification that may be necessary is to include the splitting of the effective string because its string coupling is of order 1. If the parameter $\beta$ in the classical greybody factor (14) is set to zero, we obtain the same greybody factors that the effective string predicts. The leading correction when $\beta$ is nonzero is given by the term (63). This term is reminiscent of the ‘world sheet cut-off’ dependent factor that must be multiplied with the naive expression of a vertex operator in string theory. Since we have an effective string with a noncritical central charge, it may be that such a factor survives in final expressions for the amplitude.

We observed that, even in the limit $r_5 \gg r_0, r_1, r_n$, the higher angular momentum partial waves can contribute to the classical absorption cross section for $\omega r_5$ of order 1. It is suggestive that the $l = 1$ partial wave begins to contribute at exactly the energy that is needed to create the first massive state of the effective string.

In our calculation the effective string was taken to behave as a single multi-wound string that can absorb and emit quanta. But at least for a string in free space (i.e. not bound to 5-branes) such behavior is true only in certain domains of coupling-length space \[38\]. If the unexcited string is too short, for instance, it responds to an incoming graviton by splitting rather than moving to a higher excited state \[20\]. In the present case the effective string is likely to be strongly coupled, as there is no small parameter that governs its coupling. Thus we may need to take into account multi-string processes to get better agreement with the classical greybody factors.

In this quantization of the string the excitation levels are described by $N_L, N_R$ which cannot be associated with either winding or momentum excitations alone, but represent some combination of the two. This is natural in view of the T-duality of the excitation spectrum. One would like to formulate all results on the excitation of branes in such duality invariant ways. For example, we know that when $n_w$ D-strings come close to each other, they are described by an $SU(n_w)$ gauge theory. What happens when these D-strings also carry a total of $n_p$ units of momentum along their winding direction? If the strings are very long, and the momentum density correspondingly dilute, we can imagine that the effective physics is still governed by $SU(n_w)$ gauge theory. But if the momentum density is high, it is more convenient to perform dualities that interchange the winding and momentum charges. In the dual picture there are $n_p$ strings, and in some domain of parameters they

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It was noted in \[37\] that for black holes other than the supersymmetric 5 and 4 dimensional ones considered above, an effective string model does not hold for the absorption cross sections beyond leading order in the energy.
are described by an $SU(n_p)$ gauge theory. What then can we say of the general description of the situation with both winding and momentum charges present?

In [39] it was argued that, in the case of $n_w$ 0-branes on $n_p$ 4-branes which is related to our case by T-duality, the moduli space is $T_4 \times T_4^{n_wn_p} / S^{n_wn_p}$, where $T_4$ is the 4-torus. At the tip of the cone generated by the symmetrization, we have a similar structure to that which would arise in the moduli space of $n_wn_p$ 0-branes, which are described by $SU(n_pn_w)$. Thus it is possible that at high density of excitations (the limit where one expects to make a black hole) the effective description involves the gauge group $SU(n_pn_w)$, and the groups $SU(n_w)$, $SU(n_p)$ arise only in certain limits of moduli.

Where can such a large group come from? Note that the product $n_pn_w$ represents the number of fractional momentum modes that arise from $n_p$ units of momentum on $n_w$ strings. Equivalently, we can T-dualize to obtain $n_w$ 0-branes joined by $n_wn_p$ open string segments. If these momentum modes or open strings interact with each other, then they might yield the large gauge group suggested above. Thus, it is possible that when the brane charges are increased in a compact volume, at some point the more complicated gauge symmetry arises. There are open fundamental strings that run between the D-strings, but there can also be open D-strings that connect these open fundamental strings because, by duality, a D-string can terminate on a fundamental string. This may lead to a complex nested structure of gauge groups, especially at the coupling that is strong enough to make the D-string tension comparable to the fundamental string tension. Note that the black hole horizon area depends on the product $n_5n_pn_w$, so that it is natural to consider this number of elementary constituents in trying to estimate the bound state size for branes or to realize the idea of ‘holography’ [40] in this model.

8. Appendix: absorption by $D = 5$ black holes for $\omega r_5 \geq l + 1$

For $\omega r_5 \geq l + 1$, the matching has to be done more carefully. In the inner region, the large $r$ asymptotic is

\[
R_I \rightarrow \left(\frac{r_0}{r}\right)^{2\beta} \frac{\Gamma(1 + 2\alpha)\Gamma(1 - 2\beta)}{\Gamma(1 + \alpha - \beta - i\sqrt{D})\Gamma(1 + \alpha - \beta + i\sqrt{D})}
+ \left(\frac{r_0}{r}\right)^{2(1-\beta)} \frac{\Gamma(1 + 2\alpha)\Gamma(2\beta - 1)}{\Gamma(\alpha + \beta - i\sqrt{D})\Gamma(\alpha + \beta + i\sqrt{D})}
\]

Now $\beta$ and $1 - \beta$ are complex conjugates,

\[
\beta = \frac{1 - ix}{2}, \quad x = \sqrt{(\omega r_5)^2 - (l + 1)^2}
\]

In the outer region the solution is

\[
R_{II} = 2\rho^{-1}(AJ_{ix}(\rho) + BJ_{-ix}(\rho)), \quad \rho = \omega r
\]
Matching the two terms separately, we find

\[ A = ix(\omega r_0/2)^{1-ix} \frac{\Gamma \left( 1 - i \frac{\omega}{2\pi T_H} \right) \Gamma^2(ix)}{\Gamma \left( \frac{1+ix}{2} - i \frac{\omega}{4\pi T_L} \right) \Gamma \left( \frac{1+ix}{2} - i \frac{\omega}{4\pi T_R} \right)}, \]

\[ B = -ix(\omega r_0/2)^{1+ix} \frac{\Gamma \left( 1 - i \frac{\omega}{2\pi T_H} \right) \Gamma^2(-ix)}{\Gamma \left( \frac{1-ix}{2} - i \frac{\omega}{4\pi T_L} \right) \Gamma \left( \frac{1-ix}{2} - i \frac{\omega}{4\pi T_R} \right)}. \]

The reflection coefficient for the \( l \)-th partial wave is

\[ R_l = \frac{A e^{\pi x/2} + B e^{-\pi x/2}}{A e^{-\pi x/2} + B e^{\pi x/2}}. \]

In the limit of large \( \omega r_5 \), it is possible to show that the reflection probability satisfies the following bound,

\[ |R_l|^2 \leq 4e^{-2\pi \omega r_5}. \]

Thus, for each partial wave, we find that the reflection probability vanishes exponentially at high energies. This agrees with the intuition that, at high energies, black holes are good absorbers.

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