TRANSFER EQUALS COMPREHENSION

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Abstract. Recently, conservative extensions of Peano and Heyting arithmetic in the spirit of Nelson’s syntactic approach to Nonstandard Analysis, have been proposed. We continue and extend this study with an eye on Reverse Mathematics-style results, formulating a suitable base theory along the way. In this way, we prove the equivalence between respectively the comprehension principles ($\exists_2^2$), ($S^2$), and ($E_2$), and the Transfer principle limited respectively to $\Pi^0_1$, $\Pi^1_1$, and $\Pi^\omega_1$-formulas.

1. Introduction

Recently, conservative extensions of Peano and Heyting arithmetic based on Nonstandard Analysis have been introduced ([7]). The authors of the latter use Nelson’s syntactic approach to Nonstandard Analysis pioneered in internal set theory ([14]). In Nelson’s framework, the language is extended with a new unary predicate ‘st($x$)’, read as ‘$x$ is standard’, governed by three new axioms, namely Idealization, Transfer, and Standardization. In the setting of [7], the Transfer principle is as follows:

$$(\forall st\exists t)[(\forall st\exists t)(\forall x)(\varphi(t,x)) \rightarrow (\forall x)(\varphi(t,x))]$$

(TP$_\forall$)

where $\varphi(t, \cdot)$ is internal, i.e. without the new predicate ‘st’, and involves no parameters but the (standard) ones shown, namely $t$.

The authors of [7] conjectured that adding TP$_\forall$ to their version of Peano arithmetic results in a conservative extension (See [7, p. 1992]). Although this conjecture turns out to be wrong in general as shown in Section 3.1, disallowing the parameters $t$ in TP$_\forall$ does result in the envisioned conservative extension. In particular, consider the following version of TP$_\forall$

$$(\forall x)[(\forall x)(\varphi(x)) \rightarrow (\forall x)(\varphi(x))]$$

(PF-TP$_\forall$)

where the internal formula $\varphi(x)$ does not involve parameters, i.e. all variables are shown. In Section 3.2 we show that PF-TP$_\forall$ gives rise to a conservative extension of various systems of arithmetic, including Peano arithmetic. In this way, we partially answer a question by Avigad from [3, p. 39], namely how much Transfer one can conservatively add to a given system.

Furthermore, the Transfer principle PF-TP$_\forall$ is extremely useful in proving Reverse Mathematics-style equivalences. The reader is referred to [15, 16] for an introduction and overview of Reverse Mathematics. In Section 4.1 we show that the sub-principle $\Pi^0_1$-TRANS of TP$_\forall$ is equivalent to arithmetical comprehension in the form of the functionals ($\exists^2$) and ($\mu^2$). What is more, in Section 4.2 we show that the sub-principle $\Pi^1_1$-TRANS of TP$_\forall$ is equivalent to $\Pi^1_1$-comprehension in the form of the Suslin functional ($S^2$) and related ($\mu_1$)-functional. The general case for ($E_2$) and $\Pi^\omega_1$-TRANS is proved in Section 4.3. While ($\exists^2$) and ($S^2$) correspond to two of the ‘Big Five systems’, the functional ($E_2$) constitutes full second-order comprehension from $Z_2$, and plays a central role in reverse topology ([9]). We discuss some foundational implications regarding ‘mathematical naturalness’ of these results in the conclusion in Section 5.
Finally, we point out that our results differ substantially from those proved in [10]: We prove *equivalences* between standard and nonstandard statements over a suitable base theory, while Keisler formulates systems of nonstandard arithmetic conservative over the Big Five systems of Reverse Mathematics.

2. Nonstandard Peano arithmetic

In this section, we briefly explain the nonstandard version of Peano Arithmetic as introduced and studied in [7].

Our starting point will be the system $\text{E-PA}^\omega$ of Peano arithmetic in all finite types, as formalised in [13, §3.3, p. 48]. This is the system called $\text{E-PA}^\omega_0$ in [17] and $\text{E-PA}^\omega$, in [18]. As to some basic properties of $\text{E-PA}^\omega$, only equality of natural numbers is a primitive notion; Equality at higher types is defined extensionally and we have axioms stating that extensional equality is a congruence. In addition, our version of $\text{E-PA}^\omega$ does not include product types. The price we have to pay for making this choice is that we often end up working with tuples of terms and variables of different types and we will have to adopt some conventions for how these ought to be handled. Fortunately, there are some well-known standard conventions here which we will follow (See [13, 17] or [7]).

The first nonstandard system that we will consider is $\text{E-PA}^\omega_{st}$. The language of $\text{E-PA}^\omega_{st}$ is obtained from that of $\text{E-PA}^\omega$ by adding unary predicates $\text{st}^\sigma$ as well as two new quantifiers $\forall^\text{st} x^\sigma$ and $\exists^\text{st} x^\sigma$ for every type $\sigma \in \mathcal{T}$. Formulas in the old language of $\text{E-PA}^\omega$, i.e. those not containing these new symbols, we will call *internal*; By contrast, general formulas from $\text{E-PA}^\omega_{st}$ will be called *external*. To distinguish clearly between internal and external formulas, we will adopt the following:

**IMPORTANT CONVENTION:** We follow Nelson [14] in using small Greek letters to denote internal formulas and capital Greek letters to denote formulas which can be external.

The system $\text{E-PA}^\omega_{st}$ is $\text{E-PA}^\omega$ plus the basic axioms $\text{EQ}$, $\mathcal{T}_{st}$, and $\text{IA}^\text{st}$.

**Definition 1** (Basic axioms).

- The axiom $\text{EQ}$ stands for the defining axioms of the external quantifiers:
  
  \[ \forall^\text{st} x \Phi(x) \iff \forall x (\text{st}(x) \rightarrow \Phi(x)), \]
  
  \[ \exists^\text{st} x \Phi(x) \iff \exists x (\text{st}(x) \land \Phi(x)). \]

- The axiom $\mathcal{T}_{st}$ consists of:
  
  1. The axioms $\text{st}(t) \land x = y \rightarrow \text{st}(y)$,
  2. The axiom $\text{st}(t)$ for each closed term $t$ in $\mathcal{T}$,
  3. The axioms $\text{st}(f) \land \text{st}(x) \rightarrow \text{st}(fx)$.
  4. The axiom $\text{st}(x) \land y \leq x \rightarrow \text{st}(y)$.

- The axiom $\text{IA}^\text{st}$ is the external induction axiom:

  \[ (\Phi(0) \land \forall^\text{st} x^0 (\Phi(x) \rightarrow \Phi(x+1))) \rightarrow \forall^\text{st} x^0 \Phi(x). \]  \hspace{1cm} \text{(IA}^\text{st})

In $\text{EQ}$ and $\text{IA}^\text{st}$, the expression $\Phi(x)$ is an arbitrary external formula in the language of $\text{E-PA}^\omega_{st}$, possibly with additional free variables. Besides external induction in the form of $\text{IA}^\text{st}$, the system $\text{E-PA}^\omega_{st}$ also contains the internal induction axiom

\[ \forall^\text{st} x^0 (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall^\text{st} x^0 \varphi(x), \]

simply because this is part of $\text{E-PA}^\omega$. Following our above convention, it is clear that this principle applies to internal formulas only.

It is easy to see that $\text{E-PA}^\omega_{st}$ is a conservative extension of $\text{E-PA}^\omega$: One gets an interpretation of $\text{E-PA}^\omega_{st}$ in $\text{E-PA}^\omega$ by declaring everything to be standard. For more information and results on $\text{E-PA}^\omega_{st}$, we refer to [7].
We finish this section with a remark on notation.

**Notation 2** (Finite sequences in E-PA\(^\omega\)). An important notational matter is that inside E-PA\(^\omega\), we should be able to talk about finite sequences of objects of the same type, not to be confused with the metalinguistic notion of tuple we mentioned earlier. There are at least two ways of approaching this: First of all, as in [7], we could extend E-PA\(^\omega\) with types \(\sigma^*\) for finite sequences of objects of type \(\sigma\), add constants for the empty sequence and the operation of prepending an element to a sequence, as well as a list recursor satisfying the expected equations. Secondly, as in [8], we could exploit the fact that one can code finite sequences of objects of type \(\sigma\) as a single object of type \(\sigma\) in such a way that every object of type \(\sigma\) codes a sequence. Moreover, the standard operations on sequences, such as extracting their length or concatenating them, are given by terms in Gödel’s \(\mathcal{T}\). We choose the second option here, as this makes scaling down E-PA\(^\omega\) to weaker systems easier.

In fact, finite sequences are really stand-ins for finite sets in our setting, and we will often use set-theoretic notation for this reason, i.e. \(\emptyset\) for the code of the empty sequence, \(\cup\) for concatenation and \(\{x\}\) for the finite sequence of length 1 with sole component \(x\). For \(x\) and \(y\) of the same type we will write \(x \in y\) if \(x\) is equal to one of the components of the sequence coded by \(y\). Finally, if \(Y\) is of type \(\sigma\) \(\Rightarrow \tau\) and \(x\) is of type \(\tau\) we define \(Y[x]\) of type \(\tau\) as follows: \(Y[x] := \bigcup_{f \in \gamma} f(x)\).

3. Transfer

In this section, we prove our main result regarding the Transfer principle, namely that the latter limited to formulas without parameters leads to a conservative extension of the original system from [7] based on E-PA\(^\omega\).

**3.1. On a conjecture regarding Transfer.** In this section, we prove that the conjecture from [7] p. 197 is incorrect. First, we recall the principles I, HAC\(_{\text{int}}\) and TP\(_{\text{V}}\) from [7]. The former is a typed version of Nelson’s idealisation principle

\[
\forall^\mathit{st} y' \exists x \forall y \in y' \varphi(x, y) \rightarrow \exists x \forall^\mathit{st} y \varphi(x, y).
\]

Furthermore, HAC\(_{\text{int}}\) is a weak (‘Herbrandized’) version of the axiom of choice for internal formulas:

\[
\forall^\mathit{st} x \exists^\mathit{st} y \varphi(x, y) \rightarrow \exists^\mathit{st} Y \forall^\mathit{st} x \exists y \in Y[x] \varphi(x, y).
\]

(HAC\(_{\text{int}}\))

The term ‘Herbrandized’ is meant to indicate that the choice function \(Y\) does not provide a single witness, but only a finite list of candidates; This list contains an actual witness, but one may not be able to (effectively) find one. This setup is similar to the idea of a **Herbrand disjunction**, hence the name. Note that HAC\(_{\text{int}}\) could also have been formulated using the normal application operation:

\[
\forall^\mathit{st} x \exists^\mathit{st} y \varphi(x, y) \rightarrow \exists^\mathit{st} Y \forall^\mathit{st} x \exists y \in Y(x) \varphi(x, y).
\]

(HAC\(_{\text{int}}\))

This is readily seen to be equivalent to the above version. Finally, TP\(_{\text{V}}\) is the Transfer principle:

\[
\forall^\mathit{st} L \left( \forall^\mathit{st} x \varphi(x, L) \rightarrow \forall x \varphi(x, L) \right)
\]

(TP\(_{\text{V}}\))

where \(\forall^\mathit{st} L\) is supposed to quantify away all the remaining free variables in \(\varphi(x, L)\).

It is one of the main results of [7] that the system E-PA\(_{\text{int}}\) + I + HAC\(_{\text{int}}\) is conservative over E-PA\(^\omega\). As noted in the introduction, it was conjectured in [7] that this was not the strongest result possible, and that even E-PA\(_{\text{int}}\) + I + HAC\(_{\text{int}}\) + TP\(_{\text{V}}\) is conservative over E-PA\(^\omega\). This conjecture is unfortunately not true, as was independently observed by the two authors of this paper. To see this, consider the following statement:

\[
\forall x^0 \exists y^0 \varphi(x, y) \rightarrow \exists f^1 \forall x^0 \exists y \in f(x) \varphi(x, y),
\]

(1)
Lemma 3. There is a formula \( \varphi \) with two free variables \( x \) and \( y \) such that

\[
\text{E-PA}^* \vdash \forall x^0 \exists y^0 \varphi(x, y) \rightarrow \exists f^1 \forall x^0 \exists y \in f(x) \varphi(x, y).
\]

Proof. Suppose the statement (11) would be provable in E-PA\(^*\); then it would hold in the HEO-model of E-PA\(^*\) (See e.g. [17]). That is, we fix the hereditarily recursive functions as a model of E-PA\(^*\). Now apply (11) to

\[
\varphi_0(x, y) \equiv (\forall e^0 \leq x)(\exists z^0)(\{e\}(x) = z) \rightarrow y > \{e\}(x).
\]

Here, \( \{e\}(x) \) is the value of the \( e \)-th partial recursive function at \( x \), if it exists. The statement \( \forall x^0 \exists y^0 \varphi_0(x, y) \) is provable in E-PA\(^*\), so (11) would provide us with a function \( f \) such that \( \forall x^0 \exists y \in f(x) \varphi_0(x, y) \). Now define \( g(x) = \max\{y : y \in f(x)\} \) and note that \( g \) grows faster than any total recursive function. However, such functions do not exist in the HEO-model and we obtain a contradiction. \( \square \)

In conclusion, the conjecture from [7, p. 1997] is wrong in light of Lemma 3. As it happens, not even E-PA\(^*\) + HAC\(_{int}\) + TP\(_\varphi\) is conservative over E-PA\(^*\); in fact, we will see in Corollary 14 below that it is not even conservative over PA.

3.2. Parameter-free Transfer. In the previous section, we observed that the conjecture that E-PA\(^*\) + 1 + HAC\(_{int}\) + TP\(_\varphi\) is conservative over E-PA\(^*\) is false. Each false conjecture has a silver lining, however, and a slightly modified version of the conjecture, suggested by the second author, is true. Indeed, instead of the ‘full’ Transfer principle TP\(_\varphi\), consider the parameter-free version

\[
\forall x^0 \varphi(x) \rightarrow \forall x \varphi(x) \quad \text{(PF-TP\(_\varphi\))}
\]

where \( \varphi(x) \) is not supposed to have any free variables besides those in \( x \). In this section, we prove that E-PA\(_{st}\) + 1 + HAC\(_{int}\) + PF-TP\(_\varphi\) is conservative over E-PA\(_{st}\) using techniques similar to those in [7].

In particular, we make use of the \( S_{st}\)-interpretation for E-PA\(_{st}\) as introduced in [7, §7]. In the latter, the fact was used that in E-PA\(_{st}\), all the logical connectives can be defined using only \( \land, \lor, \forall, \exists, \forall^\ast \), etc. Thus, one only has to provide the clauses for the \( S_{st}\)-interpretation of these four connectives as in [7, p. 1989]. Of course, this makes for a slick proof of the soundness of the \( S_{st}\)-interpretation, but to compute the \( S_{st}\)-interpretation of a concrete formula, one really does not want to first rewrite the formula using this limited set of connectives, as practice shows that this rarely yields a manageable result.

For this reason, in this paper, we also regard \( \land, \forall^\ast, \exists^\ast, \exists \) as primitive and additionally determine suitable \( S_{st}\)-interpretations for these. We will only regard implication as a defined connective, with \( \Phi \rightarrow \Psi \) defined as \( \neg \Phi \lor \Psi \). As we will see, this approach does turn out to be feasible.

We now extend the aforementioned \( S_{st}\)-interpretation to the other connectives.

First of all, if we define \( \Phi \rightarrow \Psi \) as \( \neg \Phi \lor \Psi \), this means that, if \( \forall^\ast x \exists^\ast y \varphi(x, y, a) \) is the \( S_{st}\)-interpretation of \( \Phi(a) \) and \( \forall^\ast x \exists^\ast y \psi(y, v, b) \) is the \( S_{st}\)-interpretation of \( \Psi(b) \), then we have:

\[
(\Phi(a) \rightarrow \Psi(b))^{S_{st}} \equiv \forall^\ast x \exists^\ast y \exists^\ast v \quad (\exists y \in \exists^\ast x \varphi(x, y, a) \rightarrow \psi(y, v, b)),
\]

modulo some applications of classical logic in the internal matrix, of course.

where \( \varphi \) is an internal formula containing only the variables \( x \) and \( y \) free. This statement, with all shown quantifiers relative to ‘st’, easily follows from HAC\(_{int}\), and applying TP\(_\varphi\) immediately yields (11). However, we have the following lemma.
Furthermore, the clauses for the Krivine negative translation $\Phi^{Kr} : \equiv \neg \Phi_{Kr}$ for the extended language are:

\[
\begin{align*}
(\Phi \land \Psi)_{Kr} & \equiv \Phi_{Kr} \lor \Psi_{Kr}, \\
(\forall^e z \Phi(x))_{Kr} & \equiv \exists^e z \Phi_{Kr}(z), \\
(\exists^e z \Phi(x))_{Kr} & \equiv \forall^e z \neg \Phi_{Kr}(z), \\
(\exists z \Phi(z))_{Kr} & \equiv \forall z \neg \Phi_{Kr}(z),
\end{align*}
\]

This means that, if we would define:

\[
\begin{align*}
(\Phi(\bar{y}) \land \Psi(\bar{b}))_{St} & :\equiv \forall^e y, b \exists^e y, b \varphi(\bar{y}, \bar{b}) \land \psi(\bar{y}, \bar{b}), \\
(\forall^e z \Phi(y))_{St} & :\equiv \forall^e y, z' \exists^e y, z \forall z \in z' \exists y \in y' \varphi(\bar{y}, y', z, \bar{z}), \\
(\exists^e z \Phi(y))_{St} & :\equiv \forall^e y, z' \exists^e y, z \forall z \in z' \exists y \in y' \varphi(\bar{y}, y', z, \bar{z}), \\
(\exists z \Phi(z))_{St} & :\equiv \forall z \neg \varphi(\bar{y}, y', z, \bar{z}).
\end{align*}
\]

then the soundness proof in [1] p. 190] still works. Indeed, we only have to check that [1] Lemma 7.3 extends to $\forall$, $\exists^e$ and $\exists$ as defined, and that the negative translation of the EQ-axiom is provable using that very same EQ-axiom in $E-HA^{non}_n$.

However, we can do better than the above ‘primitive’ interpretation. Indeed, we add yet another universal quantifier $\forall^e z$ to $E-PA^\omega$ with clause:

\[
(\forall^e z \Phi(\bar{y}, z))_{St} :\equiv \forall^e y, z' \exists^e y, z \forall z \in z' \exists y \in y' \varphi(\bar{y}, y', z, \bar{z}).
\]

As it turns out, our new simpler interpretation is equivalent to [2].

**Lemma 4.** The equivalence $\forall^e z \Phi(z) \leftrightarrow \forall^e z \Phi(z)$ is $S_{St}$-interpretable.

**Proof.** To improve readability, we will ignore the fact that we have to work with tuples and drop the underlining. The $S_{St}$-interpretation of $\forall^e z \Phi(z) \rightarrow \forall^e z \Phi(z)$ is

\[
\forall^e z, x, y, z' \exists^e z', u, Y \left( \exists y' \in V[z', u] \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \rightarrow \varphi(x, y, z) \right),
\]

i.e. we need to find terms $R, S, T$ such that

\[
\forall z, x, y, z' \exists y \in T[z, x, V'] \left( \exists y' \in V[z', x] \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \rightarrow \varphi(x, y, z) \right)
\]

is provable in $E-PA^\omega$. We start by putting $R[z, x, V'] = \{ \{ z \} \}$ and $S[z, x, V'] = \{ x \}$. Then we need to find a term $T$ such that

\[
\forall z, x, y, z' \exists y \in T[z, x, V'] \left( \exists y' \in V[z', x] \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \rightarrow \varphi(x, y, z) \right)
\]

becomes provable in $E-PA^\omega$. Here, we can put $T[z, x, V'] = \bigcup V'[\{ z \}, x, \{ \curvearrowright \}].$

The $S_{St}$-interpretation of the other direction $\forall^e z \Phi(z) \rightarrow \forall^e z \Phi(z)$ is

\[
\forall^e z', u, Y \exists^e u' \in R[z', u, Y], z \in S[z', u, Y], x \in T[z', u, Y] \left( \exists y \in Y[z, x] \varphi(\bar{y}, y, z) \rightarrow \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \right),
\]

i.e. we need to find terms $R, S, T$ such that $E-PA^\omega$ proves

\[
\forall z', u, Y \exists y' \in R[z', u, Y], z \in S[z', u, Y], x \in T[z', u, Y] \left( \exists y \in Y[z, x] \varphi(\bar{y}, y, z) \rightarrow \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \right),
\]

Here we start by putting $S[z', u, Y] = z'$ and $T[z', u, V'] = \{ u \}$, and what remains is to find a term $R$ such that $E-PA^\omega$ proves

\[
\forall z', u, Y \exists y' \in R[z', u, Y], z \in z' \left( \exists y \in Y[z, u] \varphi(\bar{y}, y, z) \rightarrow \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \right),
\]

or

\[
\forall z', u, Y \exists y' \in R[z', u, Y] \left( \forall z \in z' \exists y \in Y[z, u] \varphi(\bar{y}, y, z) \rightarrow \forall x \in z' \exists y \in y' \varphi(\bar{u}, v, \bar{x}) \right).
\]

Finally, we put $R[z', u, Y] = \{ \bigcup_{z \in z'} Y[z, u] \}$, and we are done. \(\square\)
In light of the previous lemma, we can use our simpler interpretation \[\box{3}\] instead of \[\box{2}\], leading to the following theorem.

**Theorem 5.** Let \(\Phi(a)\) be a formula in the language of \(E-\text{PA}_{\text{int}}^\omega\) and suppose \(\Phi(a)^{\text{int}} \equiv \forall x \exists y \varphi(x, y, a)\). If \(\Delta_{\text{int}}\) is a collection of internal formulas and
\[
E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}} + \text{PF-TP}_\psi + \Delta_{\text{int}} \vdash \Phi(a),
\]
then
\[
E-\text{PA}^\omega + \Delta_{\text{int}} \vdash \exists y \forall a \exists y \in t[x] \varphi(x, y, a).
\]

**Proof.** We proceed by induction on the derivation of \(\Phi(a)\) in \(E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}} + \text{PF-TP}_\psi + \Delta_{\text{int}}\). We can re-use a number of results from \[7\]. To this end, consider \[7,77\]. The latter states that whenever \(E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}} + \Delta_{\text{int}} \vdash \Phi(a)\), then there are closed terms \(t\) in G"odel's \(T\) such that \(E-\text{PA}^\omega \vdash \forall x \exists y < t[x] \varphi(x, y, a)\). Note that \(t\) in no way depends on the parameters \(a\). The previous derivation implies in particular that \(E-\text{PA}^\omega \vdash \exists y \forall a \exists y \in t[x] \varphi(x, y, a)\). So whenever \(\Phi(a)\) is an axiom of \(E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}}\), we have \(E-\text{PA}^\omega \vdash \exists y \forall a \exists y \in t[x] \varphi(x, y, a)\). Therefore, it only remains to consider the inference rules and the new axiom \(\text{PF-TP}_\psi\).

With regard to the inference rules of \(E-\text{PA}_{\text{int}}^\omega\) in \[7\], we can take a formalisation in which Modus Ponens is the only inference rule. Again, we momentarily ignore the fact that we are working with tuples. So if \(\Phi(a)^{\text{int}} \equiv \forall x \exists y \varphi(x, y, a)\) and \(\Psi(b)^{\text{int}} \equiv \forall x \forall v \psi(x, v, b)\), then we have to prove that from \(E-\text{PA}^\omega \vdash \exists X, V, a \forall Y, u \exists x \in X[Y, u] \exists v \in V[Y, u] (\exists y \in Y[x] \varphi(x, y, a) \rightarrow \psi(u, v, b))\) and from
\[
E-\text{PA}^\omega \vdash \exists t \forall a, x \exists y \in t[x] \varphi(x, y, a),
\]
follows that
\[
E-\text{PA}^\omega \vdash \exists s \forall b, u \exists v \in s[u] \psi(u, v, b).
\]
Proving this is easy: Reasoning in \(E-\text{PA}^\omega\), let \(X, V, t\) be as in the premises and define \(s\) such that \(s[u] = V[t, u]\).

Finally, it remains to consider \(\text{PF-TP}_\psi\). For this, we need to show that
\[
E-\text{PA}^\omega \vdash \exists t \exists y \in t (\varphi(y) \rightarrow \forall z \varphi(z))
\]
or
\[
E-\text{PA}^\omega \vdash \exists s (\varphi(s) \rightarrow \forall z \varphi(z))
\]
But the latter is a classical tautology, known as the so-called Drinker’s Principle. \(\square\)

**Remark 6.** Note that in the last step of the proof, if \(\varphi\) had an additional standard parameter \(x\), we would need to prove
\[
E-\text{PA}^\omega \vdash \exists t \forall x \exists y \in t[x] (\varphi(x, y) \rightarrow \forall z \varphi(z, x)). \tag{4}
\]
Here, we would need the provable existence of a kind of ‘Herbrandized Skolem function’. This seems impossible without some form of choice or comprehension. Thus, the big difference between general and parameter-free transfer seems to derive from the fact that the provable existence of genuine Skolem functions is not so harmless (indeed, it a kind of axiom of choice), while the provable existence of Skolem constants is just logic.

**Corollary 7.** The system \(E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}} + \text{PF-TP}_\psi\) is conservative over \(E-\text{PA}^\omega\). In fact, if
\[
E-\text{PA}_{\text{int}}^\omega + I + \text{HAC}_{\text{int}} + \text{PF-TP}_\psi \vdash \forall x \exists y \varphi(x, y, a),
\]
then \(E-\text{PA}^\omega \vdash \forall x \exists y \varphi(x, y, a)\).

**Proof.** Follows from the previous theorem and the fact that \((\forall x \exists y \varphi(x, y, a))^{\text{int}}\) is \(\forall x \exists y \varphi(x, y, a)\) for internal \(\varphi(x, y, a)\). \(\square\)
Clearly, the above conservation proof does not need the full strength of Peano arithmetic. Indeed, careful inspection of the above proofs reveals that instead of Peano Arithmetic, we can use *Elementary Function Arithmetic* (EFA), also referred to as $I\Delta_0 + \text{EXP}$.

Thus, let $E\text{-}EFA^\omega_{st}$ be the analogue of $E\text{-}PA^\omega_{st}$ with first-order strength EFA, similar to Avigad’s ERA$^\omega$ or Kohlenbach’s system $E\text{-}G_3A^\omega$ (See [3] p. 31] and [13, p. 55]). In particular, the system $E\text{-}EFA^\omega_{st}$ is $E\text{-}G_3A^\omega$ plus EQ and $\mathcal{T}_{st}$.

**Corollary 8.** The system $E\text{-}EFA^\omega_{st} + \text{I + HAC}_{int} + \text{PF-TP}_\psi$ is conservative over $E\text{-}EFA^\omega$. If we add QF-AC$^{1,0}$, we obtain a $\Pi^0_2$-conservative extension of EFA.

For brevity, we denote the system $E\text{-}EFA^\omega_{st}$ simply by EFA$^\omega$. Due to its central role in Reverse Mathematics, we also list this conservation result for the system $\text{PRA}$. The system $E\text{-}PRA^\omega_{st}$ is $E\text{-}PRA^\omega$ from [12] plus EQ and $\mathcal{T}_{st}$.

**Corollary 9.** The system $E\text{-}PRA^\omega_{st} + \text{I + HAC}_{int} + \text{PF-TP}_\psi$ is conservative over $E\text{-}PRA^\omega$. If we add QF-AC$^{1,0}$, we obtain a $\Pi^0_2$-conservative extension of $\text{PRA}$.

These corollaries are a considerable strengthening of Avigad’s earlier results [3]. In the next section, we show that PF-TP$\psi$ is also very useful in practice.

### 3.3. Functionals, functions, and constants.

In this section, we show how PF-TP$\psi$ allows us to provide a unique and standard description of the functionals from the comprehension principles from the introduction. This discussion is essential to obtaining the equivalences in Section 4.4.

By way of example, consider the following sentence which introduces the so-called $\mu$-operator. The latter plays an important, if not central, role in e.g. [1, 5, 6, 12].

$$(\exists \mu^2)[(\forall f^1)((\exists x^0) f(x) = 0 \rightarrow f(\mu(f)) = 0)].$$

($\mu^2$)

Often, it is convenient to add a symbol $\mu_0$ to the language and adopt the axiom:

$$[(\forall f^1)((\exists x^0) f(x) = 0 \rightarrow f(\mu_0(f)) = 0)];$$

($\mu_2^0$)

This approach is taken in e.g. [4] p. 935, [4.5], [11 §2.5] and [5] for the comprehension principles from Sections 4.4 and 4.5. Let us now consider both principles in the light of PF-TP$\psi$ where the language of $E\text{-}PA^\omega_{st}$ is enriched by the symbol $\mu_2^0$.

First of all, since the formula in square brackets in ($\mu^2$) is internal and does not involve parameters other than $\mu$ itself, we obtain

$$(\exists \xi^2)[(\forall f^1)((\exists x^0) f(x) = 0 \rightarrow f(\zeta(f)) = 0)],$$

via the contraposition of PF-TP$\psi$. In other words, if $\mu$ is in ($\mu^2$), we may assume it is standard. Furthermore, expressing that $\mu(f)$ with $\mu$ from ($\mu^2$) is the least $k^0$ such that $f(k) = 0$, requires trivial changes to $\mu$ and ($\mu^2$), as follows:

$$(\exists \tilde{\mu}^2)[(\forall f^1)((\exists x^0) f(x) = 0 \rightarrow f(\tilde{\mu}(f)) = 0 \land (\forall k < \tilde{\mu}(f))(f(k) \neq 0)].$$

(6)

In particular, $\tilde{\mu}$ can be defined from $\mu$ in $E\text{-}PA^\omega_{st}$. Applying again PF-TP$\psi$, we obtain

$$(\exists \tilde{\xi}^2)[(\forall f^1)((\exists x^0) f(x) = 0 \rightarrow f(\tilde{\xi}(f)) = 0 \land (\forall k < \tilde{\xi}(f))(f(k) \neq 0)],$$

i.e. the associated $\tilde{\xi}(f)$ is also minimal. Hence, we must have $\tilde{\mu}(f) = \tilde{\xi}(f)$ for any $f^1$, i.e. $\tilde{\mu} = \tilde{\xi}$. Hence, the functional $\tilde{\mu}$ was standard and uniquely determined to begin with by Definition 11. For this reason, we can add the axiom $\text{st} (\mu_0)$ to $E\text{-}PA^\omega_{st}$ without any problems.

Secondly, by the previous, PF-TP$\psi$ essentially guarantees that the functional $\mu$ from ($\mu^2$) is standard and unique, and we also have $\mu^2 \rightarrow (\mu^2)^\omega$. In light of Corollary 7, the first-order strength of our systems is already determined by the
standard world, suggesting that \(\mu^2\text{st} \rightarrow \mu^2\text{st}\) should also hold. Unfortunately, the formula in square brackets in \(\mu^2\text{st}\) contains a parameter, namely \(\mu\), and we cannot apply PF-TP\(\gamma\). By contrast, \(\mu^2\text{st}\) does not contain parameters, i.e. \(\mu^2\) \(\leftrightarrow\) \(\mu^2\text{st}\) via PF-TP\(\gamma\), illuminating the advantage of adding \(\mu_0\) to the language.

Thirdly, while the aforementioned equivalence seems satisfying, we will prove

\[
\mu^2\text{st} \rightarrow \Pi_1^0\text{-TRANS} \rightarrow \mu^2\text{st},
\]

in Section 4.1, where the exact definition of \(\Pi_1^0\text{-TRANS}\) is not relevant here. However, there does not seem to be a way of proving \(\mu^2\) \(\rightarrow\) \(\mu^2\text{st}\) or \(\mu^2\text{st} \rightarrow \mu^2\).

In conclusion, the addition of standard function symbols like \(\mu_0\) to the language allows us to prove (using PF-TP\(\gamma\)) the formula (7) in which the nonstandard principle \(\Pi_1^0\text{-TRANS}\) follows from the standard principle \(\mu^2\text{st}\). Given the obvious difference in scope between these principles, (7) is already impressive.

We now discuss how we can obtain \((\mu^2) \leftrightarrow (\mu^2)\text{st}\), to reflect the already proved equivalence \((\mu^2) \leftrightarrow (\mu^2)\text{st}\). Recall that we established above that the functional \(\mu\) is standard and unique, assuming it exists as in (8). This observation gives rise to the following formula, where \(M(\mu)\) is the formula in square brackets in (6).

\[
(\forall^\text{st}\Xi)[M^\text{st}(\Xi) \rightarrow (\forall^\text{st}f^1)(\Xi(f) =_0 \mu_0(f))].
\]

We assume that this formula has been added to \(\text{E-PA}_\omega^\omega\) as a defining axiom for \(\mu_0\). The advantage of this approach is that if \(\mu^2\text{st}\) holds, then by (8):

\[
(\forall^\text{st}f^1)((\exists^\text{st}x^0)f(x) = 0 \rightarrow f(\mu_0(f)) = 0),
\]

yielding

\[
(\forall^\text{st}f^1,x^0)(f(x) = 0 \rightarrow f(\mu_0(f)) = 0),
\]

Now, this formula does not involve parameters (again in contrast to \(M(\mu)\)), i.e. we may apply PF-TP\(\gamma\) to (9), and we get \(\mu^2\text{st}\), which immediately implies \(\mu^2\).

Thus, we have proved that \(\mu^2\text{st} \leftrightarrow \mu^2\text{st}\) in our extended system involving (8), thanks to PF-TP\(\gamma\). As a ‘bonus’, for the functional \(\varphi\) from \((\exists^2)^\text{st}\) in Section 4.1, it is easy to prove \((\forall^\text{st}f^1)[\varphi(f) = 0 \leftrightarrow f(\mu_0(f)) = 0]\), i.e. we can make \((\exists^2)^\text{st}\) parameter-free by defining \(\varphi_0(f) := f(\mu_0(f))\).

In general, we extend the language of \(\text{E-PA}_\omega^\omega\) by new symbols \(\nu_0\) and \(\xi_0\), which are meant to correspond to the functionals \((\nu)\) and \((\xi)\) from the next section in the same way as \(\mu_0\) relates to \(\mu\). We extend \(\text{E-PA}_\omega^\omega\) by \(\text{st}(\xi_0^2)\land\text{st}(\nu_0^1)\), and the obvious analogues of (8). The symbols \(S_0\) and \(T_0\) for \((S_2)\) and \((\xi_2)\) can be defined in terms of \(\nu_0\) and \(\xi_0\) in the same way as \(\varphi_0\) is defined from \(\mu_0\). We assume similar modifications for \(\text{EFA}_\omega^\omega\) and \(\text{E-PRA}_\omega^\omega\) are made. To ‘future-proof’ these systems, one can similarly extend them to accommodate the study of e.g. the functional from uniform weak König’s lemma, the fan functional (See [12] for both), et cetera.

Note that our modifications still result in conservative extensions: The axiom (8) is trivially true in models where the \(\mu\)-operator is missing, like HEO; If the standard \(\mu\)-operator is available in a model \(\mathcal{M}\), say as \(\mu\), one simply interprets \(\mu_0\) as \(\mu\text{st}\) to make (8) hold in \(\mathcal{M}\).

4. Equivalences between Transfer and comprehension principles

In this section, we prove the equivalences between the Transfer and comprehension principles mentioned in the introduction.
4.1. The $\Pi^0_1$-Transfer principle and $\langle 3^2 \rangle$. In this section, we study the relation between $\Pi^0_1$-TRANS and arithmetical comprehension ($\langle 3^2 \rangle$), both defined below. The latter axiom is the functional version of ACA₀, the third ‘Big Five’ system of Reverse Mathematics as studied in [16, III].

\[(\exists \varphi^z)(\forall f^z)((\varphi(f) = 0 \leftrightarrow (\exists x^0) f(x) = 0)). \quad (3^2)\]

\[(\forall z \tau^z)((\forall x^z)(f^{(0 \times \tau^x)}(z)(\forall x^z)f(x, z) \neq 0 \rightarrow (\forall x^0)f(x, z) \neq 0)). \quad (\Pi^0_1, TRANS)\]

For clarity, we have made the standard parameters $z^\tau$ in $\Pi^0_1$-TRANS explicit. Recall that $\varphi^z$ for internal $\varphi$ is the latter formula with ‘st’ appended to all quantifiers (except bounded numerical ones). We have the following theorem.

**Theorem 10.** In EFA$^{st}$ + HAC$_{int}$ [PF-TP$_\omega$] we have $\Pi^0_1$-TRANS $\leftrightarrow \langle 3^2 \rangle$.

**Proof.** Assuming $\Pi^0_1$-TRANS, we have

\[(\forall z \tau, f^{(0 \times \tau)})(\forall M, N \in \Omega)[(\forall x^0 \leq M)f(x, z) \neq 0 \leftrightarrow (\forall x^0 \leq N)f(x, z) \neq 0],\]

where ‘$M \in \Omega$’ is symbolic notation for $\neg\exists\mathbf{st}(M^0)$. Bounded minimisation yields

\[(\forall z^\tau, f^{(0 \times \tau)})(\exists k)(\forall M, N \geq k)[(\forall x^0 \leq M)f(x, z) \neq 0 \leftrightarrow (\forall x^0 \leq N)f(x, z) \neq 0]. \quad (10)\]

By HAC$_{int}$, we obtain standard $F^{(\tau \times (0 \times \tau) \rightarrow 0^0)}$ such that

\[(\forall z \tau, f^{(0 \times \tau)})(\forall M, N \geq F(z, f))[[(\forall x^0 \leq M)f(x, z) \neq 0 \leftrightarrow (\forall x^0 \leq N)f(x, z) \neq 0]. \quad (11)\]

Note that HAC$_{int}$ actually provides a functional outputting (on input standard $f$ and $z$) a finite sequence of natural numbers of which at least one witnesses $k$ in (10). Taking the maximum of this finite sequence yields the functional $F$ as in (11).

Now define $\varphi(f, z)$ as the characteristic function of $[(\forall x^0 \leq F(z, f))f(x, z) \neq 0]$. Then (11) combined with $\Pi^0_1$-TRANS, yields

\[(\forall z \tau, f^{(0 \times \tau)})(\varphi(f(\cdot, z), z) = 1 \leftrightarrow (\forall x^0)f(x, z) \neq 0),\]

immediately establishing $\langle 3^2 \rangle$. Now let $\varphi$ be as in the latter and consider:

\[(\forall z \tau^x)((\forall \tau^x)(\forall \tau^x)\varphi(f (\cdot, z), z) \neq 0 \rightarrow (\forall x^0)f(x, z) \neq 0]. \quad (12)\]

The quantifier inside the square brackets can be brought to the front, and $\varphi$ may be replaced by the corresponding symbol $\varphi_0$ (defined in Section 3.3), yielding:

\[(\forall z \tau^x)((\forall \tau^x)(\forall \tau^x)\varphi_0(f (\cdot, z), z) \neq 0 \rightarrow f(x, z) \neq 0]. \quad (13)\]

Applying [PF-TP$_\omega$] and pushing the aforementioned universal quantifier ‘back in’,

\[(\forall z \tau^x)((\forall \tau^x)(\forall \tau^x)\varphi_0(f (\cdot, z), z) \neq 0 \rightarrow (\forall x^0)f(x, z) \neq 0]. \quad (14)\]

By the definition of $\varphi$, for standard $z, f$, we have

\[(\forall x^0)f(x, z) \neq 0 \rightarrow \varphi_0(f (\cdot, z), z) \neq 0, \quad (15)\]

and combining with the previous, we obtain

\[(\forall z \tau^x)((\forall \tau^x)(\forall \tau^x)\varphi_0(f (\cdot, z), z) \neq 0 \rightarrow (\forall x^0)f(x, z) \neq 0],\]

which is exactly $\Pi^0_1$-TRANS. □

Define $\langle 3^2 \rangle$ as $\langle 3^2 \rangle$ without existential quantifier but with $\varphi_0$ from Section 3.3.

---

1. Fix infinite $K$ and obtain the least $k \leq K$ such that for all $N, M$ with $k \leq N, M \leq K$, the formula in square brackets in [16] holds. This number must be finite and we obtain [16].

2. Note that HAC$_{int}$ $\rightarrow$ (QF-AC1.0)$^{st}$, yielding $(\langle 3^2 \rangle)^{st}$ $\rightarrow$ $(\mu^2)^{st}$ by [11] Prop. 3.4 and Cor. 3.5.
Corollary 11. In the system from the theorem, we can prove \((\exists^0_2) \leftrightarrow (\exists^0_2)^{st} \rightarrow \Pi^0_1\text{-TRANS} \rightarrow (\exists^2)^{st}\) without using \([5]\).

Furthermore, note the complimentary role played by \((\exists^2)^{st}\) and \(\text{PF-TP}_2\). The former allows us to remove existential quantifiers from the tautology
\[
(\forall^s z \forall^s f (0 \times \tau = 0) \left[ (\forall^s x) f(x, z) = 0 \rightarrow (\forall^s x) f(x, z) = 0 \right],
\]
to obtain \([12]\), to which we may apply parameter-free Transfer, to obtain \(\Pi^0_1\text{-TRANS}\).

Recall from Section 3.3 the following principle
\[(\exists^0_2)(\forall f^1)((\exists^0 x)f(x) = 0 \rightarrow f(\mu(f)) = 0).
\]
Note that by \([11]\) Prop. 3.4 and Cor. 3.5, the existence of the functional \((\mu^2)\) is equivalent to that of \((\exists^2)\), over Kohlenbach’s base theory \(\text{RCA}^0\) from \([12]\).

Corollary 12. In \(\text{EFA}^{st} + \text{HAC}_{\text{int}} + \text{PF-TP}_2\) we have \(\Pi^0_1\text{-TRANS} \leftrightarrow (\mu^2) \leftrightarrow (\mu^2)^{st}\).

If we add \(\text{QF-AC}^{1,0}\), we have \(\Pi^0_1\text{-TRANS} \leftrightarrow (\exists^2) \leftrightarrow (\mu^2)^{st}\).

Proof. Assume \(\Pi^0_1\text{-TRANS}\), obtain \((\exists^2)^{st}\) and derive \((\mu^2)^{st}\) as in \([11]\) Cor. 3.5. Indeed, \((\text{QF-AC}^{1,0})^{st}\) is immediate from \(\text{HAC}_{\text{int}}\) and hence applying the former to
\[(\forall^s f^1)(\exists^s x^0) [\varphi(f) = 0 \rightarrow f(x) = 0]\]
yields \((\mu^2)^{st}\) from \((\exists^2)^{st}\). Now \((\mu^2)^{st}\) immediately yields
\[(\forall^s f^1)(\forall^s x^0) [f(x) = 0 \rightarrow f(\mu(f)) = 0], \tag{16}\]
For standard \(f\), ‘\(f(\mu(f)) = 0\)’ is equivalent to ‘\((\forall^s y^0)(y = \mu(f) \rightarrow f(y) = 0)\)’, i.e.
\[
(\forall^s f^1)(\forall^s x^0) \left[ f(x) = 0 \rightarrow (\forall^s y^0)(y = \mu(f) \rightarrow f(y) = 0) \right],
\]
and the extra quantifier involving \(y\) can also be brought outside of the square brackets, yielding:
\[
(\forall^s f^1)(\forall^s x^0)(\forall^s y^0) \left[ f(x) = 0 \rightarrow (y = \mu(f) \rightarrow f(y) = 0) \right], \tag{17}\]
As \((\mu^2)^{st}\) is given, replace \(\mu\) in \((17)\) by the corresponding symbol \(\mu_0\) and note that there are no parameters in the resulting formula. Hence, we may apply \(\text{PF-TP}_2\) to \((17)\), and this is easily seen to yield \((\mu^2)\). Now assume the latter and note that
\[
(\exists^s \mu^2)(\forall^s f^1)((\exists^0 x)f(x) = 0 \rightarrow f(\mu(f)) = 0), \tag{18}\]
by the contraposition of \(\text{PF-TP}_2\) as the formula inside square brackets in \((18)\) is internal. Now since \(\mu(f)\) is standard for standard \(f\), we obtain \(\Pi^0_1\text{-TRANS}\). The remaining result is immediate by \([11]\) Cor. 3.5. \(\square\)

Note that applying \(\text{PF-TP}_2\) to \((\exists^2)\) does not immediately yield \(\Pi^0_1\text{-TRANS}\), in contrast to \((\mu^2)\), as is clear from the last part of the proof. Hence, \((\mu^2)\) is more suitable for our purposes, whereas \((\exists^2)\) requires an instance of the axiom of choice. Since \(\text{QF-AC}^{1,0}\) is included in e.g. \(\text{RCA}^0\), this is a weak requirement.

Finally, recall that \(\text{ACA}_0\) can be bootstrapped from \(\Sigma^0_1\)-comprehension as shown in \([16]\) III]. In the same way, we note that \(\Pi^0_2\text{-TRANS}\) can be bootstrapped to \(\Pi^0_{\omega\times}\text{-TRANS} = \bigcup_{k \in \omega} \Pi^0_k\text{-TRANS}\), where
\[
(\forall^s f^0)(\exists^s n^2) \ldots (Q^s n_k)(f(n_1, \ldots, n_k, \vec{x}) = 0 \quad (\Pi^0_k\text{-TRANS})
\]
\[
\leftrightarrow (\forall^0 n^0)(\exists^s n^2) \ldots (Q n_k)(f(n_1, \ldots, n_k, \vec{x}) = 0,
\]
for any standard \(f\) and \(\vec{x}\) of suitable type.
4.2. The $\Pi^1_2$-Transfer Principle and $(S^2)$. In this section, we study the relation between $\Pi^1_2\text{-TRANS}$ and the Suslin functional $(S^2)$, both defined below. The Suslin functional is the functional version of $\Pi^1_2\text{-ACA}_0$, the strongest ‘Big Five’ system of Reverse Mathematics, and is studied in e.g. [6,12,19].

\[
(\forall^* x^1)(\exists^* y^1)(\exists^* z^0)f(\overline{y}, z) = 0 \quad (\Pi^1_2\text{-TRANS})
\]

Applying $\Pi^1_2\text{-TRANS}$ hence \((\exists^* x^0)f(\overline{y}, z) = 0\) \((\Pi^1_2\text{-TRANS})\)

Again, we made the parameters in the Transfer principle explicit. By the Kleene normal form theorem (See e.g. [16, V.5.4]), any $\Pi^1_2$-formula in $\Sigma_2$, the language of second-order arithmetic, can be brought in the normal form as shown in $\Pi^1_2\text{-TRANS}$, given $\text{ACA}_0$. By Theorem 10, the latter is available if $\Pi^1_2\text{-TRANS}$ is.

**Theorem 13.** In $\text{EFA}^* + \text{HAC}_{\text{int}} + \text{PF-TP}$ we have $\Pi^1_2\text{-TRANS} \leftrightarrow (S^2)^{st}$.

**Proof.** First assume $\Pi^1_2\text{-TRANS}$ and note that the latter implies $\Pi^0_2\text{-TRANS}$, and hence $(\exists^* x^{\text{st}})^{\text{st}}$ by Theorem 10. Thus, let $\varphi$ be such that

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(h(\cdot, z)) = 0 \iff (\exists^* x^0)h(x, z) = 0], \quad (19)
\]

Applying $\Pi^1_2\text{-TRANS}$ to (19) for fixed standard $z^\tau$, we easily obtain

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(h(\cdot, z)) = 0 \iff (\exists^* x^0)h(x, z) = 0],
\]

Combining with $\Pi^1_2\text{-TRANS}$, we have that

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0],
\]

and we rename the standard variable ‘$y$’ for purposes of clarity, yielding:

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0]. \quad (20)
\]

Bringing out the (only) standard quantifier inside the square brackets of (20),

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0]. \quad (21)
\]

Applying $\text{HAC}_{\text{int}}$ to (21), we obtain standard $K(\tau \times ((0 \times \tau) \rightarrow 0)^{st})$, i.e. $K(z, f)$ is a finite sequence of type 1-objects, and (21) becomes

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0].
\]

Bringing the bounded quantifier inside the square brackets, we obtain

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0].
\]

Trivially, the reverse implication of the implication is also valid, i.e.

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[\varphi(f(\overline{y}, \cdot, z)) = 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0].
\]

Now define $1 - S(z, f)$ as the characteristic function of the left-hand side of the previous equivalence. Note that $(\forall^* x^1)\in K(z, f))\varphi(f(\overline{y}, \cdot, z)) = 0$ is equivalent to the bounded formula $\forall^0_k < |K(z, f))\varphi(f(\overline{y}, \cdot, z)) = 0$, where $(\cdot)_k$ is the $k$-th projection function, i.e. $(x)_k$ the $k$-th entry of the sequence $x$ for $k < |x|$, as defined in [7, Section 2.3].

Now assume $(S^2)^{st}$, and note that we obtain $(\exists^2)^{st}$, yielding $\Pi^0_2\text{-TRANS}$ by Theorem 10. Let $\varphi$ and $S$ be as in $(\exists^2)^{st}$ and $(S^2)^{st}$ and consider:

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[S(f(\cdot, z)) \neq 0 \iff (\forall^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0],
\]

which holds by the definition of the Suslin functional. As in the proof of Theorem 10, the universal quantifier inside the square brackets is brought outside, and we replace $S$ by the associated functional symbol $S_0$, yielding

\[
(\forall^* x^1)(\exists^* y^1)^{\text{st}}[S(f(\cdot, z)) \neq 0 \iff (\exists^* x^0)\varphi(f(\overline{y}, \cdot, z)) = 0],
\]
which contains no parameters and for which the square-bracketed formula is internal, i.e. we can apply $\text{PF-TP}_\nu$ yielding
\[(\forall z)(\forall f^0)(\exists x)(f^0(\overline{f}(x), z) = 0).\]
Using $\Pi^1_0$-TRANS and the definition of the Suslin functional, we easily obtain the reverse implication, i.e. for standard $z, f$, we have
\[S_0(f(\cdot, z)) = 0 \Leftrightarrow (\forall g^1)(\exists x^0)(f(\overline{g}(x), z) = 0),\]
and the definition of the Suslin function now implies $\Pi^1_2$-TRANS. □

Note that a corollary similar to Corollary 11 can be proved for $\Pi^1_2$-TRANS. Furthermore, if we would have ‘$(\forall^\ast g^1)$’ instead of ‘$(\forall g^1)$’ in (21), we could not apply HACint, as the formula in square brackets must be internal, i.e. without ‘st’, to apply HACint. In other words, $\Pi^1_2$-TRANS is essential to obtain $(S^2)^{st}$.

**Corollary 14.** The system $E-PA^\omega_{\text{int}} + \text{HAC}_{\text{int}} + \text{TP}_\nu$ is not a conservative extension of Peano Arithmetic.

**Proof.** Using the standard Suslin functional provided by the theorem, one can prove the standard totality, i.e. $(\forall^\ast x^0)(\exists x^0)(y = f(x))$, of functions $f$ of which PA cannot prove the totality. Applying Transfer for $\Pi^1_2$-formulas finishes the proof. □

Analogous to $(\mu^2)$ from the previous section, one defines:
\[\exists (\forall^\ast f^1)(\forall^\ast g^1)(S(f) = 0) \rightarrow (\forall^\ast x^0)(f(\overline{g}(x)) = 0)\]
This functional is called $(\mu_1)$ in [1] [8]. As in Corollary 12, we prove the following.

**Corollary 15.** In the system from the theorem, $\Pi^1_2$-TRANS $\leftrightarrow (\nu)^{st} \leftrightarrow (\nu)$. If we also assume QF-AC$^{1,1}$, then we have $\Pi^1_2$-TRANS $\leftrightarrow (\nu) \leftrightarrow (S^2)$.

**Proof.** First of all, using some coding, it is clear that $(\nu)^{st} \rightarrow (\mu)^{st}$, from which $(\nu)^{st} \rightarrow (S^2)^{st}$ immediately follows. Secondly, assume $(S^2)^{st}$, which yields $(\exists^2)^{st}$, and hence $(\forall^\ast f^1)(S(f) = 0) \rightarrow (\exists^2 x^0)(f(\overline{g}(x)) = 0),\]
and bringing the existential quantifier outside, we obtain
\[\exists (\forall^\ast f^1)(\exists^2 x^0)(S(f) = 0) \rightarrow (\forall^\ast x^0)(f(\overline{g}(x)) = 0)\]
Applying HACint to (23) now immediately yields $(\nu)^{st}$. Thirdly, the latter implies
\[\exists (\forall^\ast f^1, g^1)(\forall^\ast x^0)(f(\overline{g}(x)) = 0) \rightarrow (\forall^\ast x^0)(f(\overline{g}(x)) = 0)\]
Furthermore, for standard $f^1$ and $x^0$, ‘$f(\overline{g}(f)x) = 0$’ is equivalent to the formula
\[(\forall^\ast x^0)(\forall^\ast x^0)(f(\overline{g}(x)) = 0) \rightarrow (\forall^\ast x^0)(f(\overline{g}(x)) = 0)\]
which immediately yields $(\nu)^{st}$. Hence, $\Pi^1_2$-TRANS. The latter principle follows from $(\nu)^{st} \rightarrow (\mu)^{st} \rightarrow (S^2)^{st}$. Hence, (24) yields the formula
\[(\forall^\ast f^1, g^1)(\forall^\ast x^0)(f(\overline{g}(x)) = 0) \rightarrow (\forall^\ast x^0)(f(\overline{g}(x)) = 0)\]
where we used $\Pi^1_2$-TRANS inside the square brackets twice. As $(\nu)^{st}$ is given, we may replace $\nu$ in (24) by $\nu_0$. Thus, we may apply PF-TP$\nu$ to the resulting formula, which immediately yields $(\nu)$. Fourth, consider the latter and note that we may take $\nu$ to be standard by PF-TP$\nu$. Hence, $\nu(f)$ is standard for standard $f$ and $\Pi^1_2$-TRANS follows. Finally, to obtain the last part of the corollary, $(S^2)$ implies (23) without ‘st’ and applying QF-AC$^{1,1}$ now yields $(\nu)$. The latter also implies $(\exists^2)$, and hence we obtain $(\nu) \rightarrow (S^2)$. □

3Note that HACint actually provides a standard functional $F^{1→1}$ such that $F(f)$ is a list of potential witnesses to the formula in square brackets in (23). However, this formula is decidable as $(\exists^2)^{st}$ (and hence $\Pi^1_2$-TRANS) easily follows from $(S^2)^{st}$.
Note that applying PF-TP\(\varphi\) to \((S^2)\) does not immediately yield \(\Pi^1_1\)-TRANS, in contrast to \((\nu)\), as is clear from the second part of the proof. Hence, \((\nu)\) is more suitable for our purposes, whereas \((S^2)\) requires an instance of the axiom of choice.

To the best of our knowledge, the exact strength of QF-AC\(^{2,1}\) is unknown.

4.3. The \(\Pi^1_{\infty}\)-Transfer principle and \((E_2)\). In this section, we study the relation between \(\Pi^1_{\infty}\)-TRANS and the functional \((E_2)\), both defined below. The latter functional constitutes the (full) comprehension axiom of \(Z_2\), and the latter system is also denoted \(\Pi^1_{\infty}\)-CA\(_0\). The functional \((E_2)\) plays a central role in Hunter’s study of reverse topology \([9]\).

\[
(\forall^\ast z^\ast)(\forall^\ast \varphi^\ast(1\times\tau\rightarrow 0))\left[\left(\forall^\ast g^\ast\right)(\varphi(g, z) = 0 \rightarrow (\forall^\ast g^\ast)(\varphi(g, z) = 0)\right]. \quad (\Pi^1_{\infty}\text{-TRANS})
\]

\[
(\exists \varphi^3)(\forall \varphi^2)[T(\varphi) = 0 \leftrightarrow (\exists g^1)(\varphi(g) \neq 0)]. \quad (E_2)
\]

Note that \(\Pi^1_{\infty}\text{-TRANS}\) and \((E_2)\) are different from \(\Pi^1_1\)-TRANS and \((S^2)\) in that in the former, the type 1-objects \(g\) are not restricted to type 0-inputs \(\overline{T}\). Thus, repeatedly applying \((E_2)\) (resp. \(\Pi^1_{\infty}\)-TRANS) allows one to decide (resp. transfer) any formula in \(\Pi^1_{\infty} \equiv \bigcup_{k \in \mathbb{N}} \Pi^1_k\).

**Theorem 16.** In \(\text{EFA}^\ast + \text{HAC}_{\text{int}} + \text{PF-TP}_\varphi\) we have \(\Pi^1_{\infty}\text{-TRANS} \leftrightarrow (E_2)^{st}\).

**Proof.** The proof is rather similar to that of Theorem 13, hence we do not spell out all details. Firstly, assume \(\Pi^1_{\infty}\text{-TRANS}\) and note that

\[
(\forall^\ast z^\ast)(\forall^\ast \varphi^\ast(1\times\tau\rightarrow 0))\left[\left(\exists^\ast h^1\right)(\varphi(h, z) = 0 \rightarrow (\forall^\ast g^1)(\varphi(g, z) = 0)\right].
\]

By HAC\(_{\text{int}}\), there is a standard \(F^{(\tau \times (1\times\tau)\rightarrow 0)}\rightarrow 1^\ast\) such that

\[
(\forall^\ast z^\ast)(\forall^\ast \varphi^\ast(1\times\tau\rightarrow 0))(\exists h^1 \in F(z, \varphi))[\varphi(h, z) = 0 \rightarrow (\forall^\ast g^1)(\varphi(g, z) = 0),
\]

and bringing in the existential quantifier and noting the reverse implication,

\[
(\forall^\ast z^\ast)(\forall^\ast \varphi^\ast(1\times\tau\rightarrow 0))\left[\left(\forall h^1 \in F(z, \varphi)\right)(\varphi(h, z) = 0 \leftrightarrow (\forall^\ast g^1)(\varphi(g, z) = 0),
\]

Finally, to obtain \((E_2)^{st}\), define \(1 - T(\varphi)\) as the characteristic function of the formula \((\forall^\ast k^0 < |F(z, \varphi)|)[\varphi(F(x, \varphi), \overline{k}, z) = 0], which is clearly equivalent to \((\forall h^1 \in F(z, \varphi))(\varphi(h, z) = 0)\).

Now assume \((E_2)^{st}\) and note that the latter implies \((\forall^\ast \varphi^2)(\forall^\ast g^1)[T(\varphi) \neq 0 \rightarrow \varphi(g) \neq 0]\), where we may replace \(T\) by the associated function symbol \(T_0\). Finally, apply \(\text{PF-TP}_\varphi\) to the resulting formula to obtain \((\forall^\ast \varphi^2)(\forall^\ast g^1)[T_0(\varphi) \neq 0 \rightarrow (\forall^\ast g^1)(\varphi(g) \neq 0]\). By the definition of \((E_2)^{st}\), we have \((\forall^\ast g^1)(\varphi(g) \neq 0) \rightarrow T_0(\varphi) \neq 0\) for standard \(\varphi^2\), and \(\Pi^1_{\infty}\text{-TRANS}\) follows. \(\square\)

Note that a corollary similar to Corollary 11 can be proved for \(\Pi^1_{\infty}\)-TRANS.

We do not consider the obvious generalizations of the previous theorem to \((E_k)\) and \(\Pi^1_{\infty}\)-TRANS. Analogous to \((\mu^2)\) and \((\nu)\) from the previous sections, we define:

\[
(\exists \xi^2)(\forall \varphi^2)\left[\left(\exists g^1\right)(\varphi(g) = 0) \rightarrow \varphi(\xi(\varphi)) = 0\right]. \quad (\xi)
\]

Similar to Corollary 12 and 13, we can then prove the following.

**Corollary 17.** In the system from the theorem, \(\Pi^1_{\infty}\text{-TRANS} \leftrightarrow (E_2)^{st} \leftrightarrow (\xi)\). If we also assume QF-AC\(^{2,1}\), then we have \(\Pi^1_{\infty}\text{-TRANS} \leftrightarrow (\xi) \leftrightarrow (E_2)\).

**Proof.** Similar to the proof of Corollary 13. \(\square\)

To the best of our knowledge, the exact strength of QF-AC\(^{2,1}\) is unknown.
5. Conclusion

In this paper, we proved the equivalence between respectively the comprehension principles \((\exists^2), (S^2),\) and \((\varepsilon_2),\) and the Transfer principle limited respectively to \(\Pi^0_1, \Pi^1_1,\) and \(\Pi^\infty_1,\) formulas, and this over a weak base theory based on \(\text{EFA}.\)

Now, while \((\exists^2)\) and \((S^2)\) correspond to two of the ‘Big Five systems’, the functional \((\varepsilon_2)\) constitutes full second-order comprehension from \(Z_2,\) and plays a central role in reverse topology, as explored in [9]. This leads us to the following foundational claim regarding ‘mathematical naturalness’ of logical systems, as qualified by Simpson in [16, I.12] as follows:

From the above it is clear that the five basic systems \(\text{RCA}_0, \text{WKL}_0,\) \(\text{ACA}_0, \text{ATR}_0, \Pi^1_1\text{-CA}_0\) arise naturally from investigations of the Main Question. The proof that these systems are mathematically natural is provided by Reverse Mathematics.

Given the equivalences proved in Sections 4.1 and 4.2, it would seem that \(\Pi^0_1\text{-TRANS}\) and \(\Pi^1_1\text{-TRANS}\) also count as (ordinary-) mathematically natural, in direct contrast to the view certain people hold regarding Nonstandard Analysis. The same can be said for \(\Pi^\infty_1\text{-TRANS}\) from Section 4.3 in the context of ‘non-ordinary’ mathematics, like the topology discussed in [9, §3.4]. In other words, Nonstandard Analysis never looked so standard.

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