The Rudin-Keisler ordering of P-points under $b = c$

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Abstract

M. E. Rudin proved under CH that for each P-point there exists another P-point strictly RK-greater \[5\]. Assuming $p = c \[2\]$, A. Blass showed the same, and proved that each RK-increasing $\omega$-sequence of P-points is upper bounded by a P-point, and that there is an order embedding of the real line into the class of P-points with respect to the RK-(pre)ordering \[1\]. In the present paper the results cited above are proved under a (weaker) assumption $b = c$.

A. Blass also asked in \[1\] what ordinals can be embedded in the set of P-points and pointed out, that such an ordinal may not be greater then $c^+$. In the present paper the question is answered showing (under $b = c$) that there is an order embedding of $c^+$ into P-points.

A free ultrafilter $u$ is a P-point iff for each partition $(V_n)_{n < \omega}$ of $\omega$ there exists a set $U \in u$ such that either $U \subset V_n$ for some $n < \omega$ or else $U \cap V_n$ is finite for all $n < \omega$. A filter $F$ is said to be RK-greater than a filter $G$ (in symbols, $F \geq_{RK} G$) if there exists a map $h$ such that $h^{-1}(G) \subset F$.

Let $W = \{W_n : n < \omega\}$ (1) be a partition of a subset of $\omega$ into infinite sets. A filter $K$ is a called contour if there exists a partition $W$ such that $W \in K$ iff there is a cofinite set $I \subset \omega$ such that $W \cap W_n$ is cofinite on $W_n$ for each $n \in I$. Then we call $K$ a contour of $W$ and denote $K = \int W$.

A fundamental property used in the present paper is the following reformulation of \[6\] Proposition 2.1

\[1\]Actually this results were stated under MA, but the proofs works also under $p = c$, which was pointed out by A. Blass in \[2\].

\[\ Asterisk\ Key\ words:\ P-points,\ Rudin-Keisler\ ordering,\ b = c;\ 2010\ MSC:\ 03E05;\ 03E17\]

\[\ Footnote\ See\ \[3\] for a systematic presentation of contours.]
Proposition 1. A free ultrafilter is a P-point iff it includes no contour.

We found a link between contours and subfamilies of \( \omega \), which allows us to interpret the behavior of (un)bounded families of functions in terms of the behavior of filters with respect to contours. This approach lead us to a surprisingly short and easy proofs of the theorems mentioned in the abstract. In a recent paper [4] D. Raghavan and S. Shelah proved (under \( p = c \)) that there is an order-embedding of \( \mathcal{P}(\omega)/fin \) into the set of P-points ordered by \( \geq_{RK} \), and gave a short review of earlier results concerning embeddings of different orders into the class of P-points.

Recall that if \( f, g \in \omega^\omega \) then we say that \( g \) dominates \( f \) (and write \( f \geq^* g \)) if \( f(n) \leq g(n) \) for almost all \( n < \omega \). We say that a family \( F \) of \( \omega^\omega \) functions is unbounded if there is no \( g \in \omega^\omega \) that dominates all functions \( f \in F \). The minimal cardinality of unbounded family is the bounding number \( b \). We also say that a family \( F \subset \omega^\omega \) is dominating if for each \( g \in \omega^\omega \) there is \( f \in F \) that dominates \( g \). The pseudointersection number \( p \) is a minimal cardinality of a free filter without pseudointersection i.e. without a set that is almost contained in each element of the filter. Finally an ultrafilter number \( u \) is the minimal cardinality of a base of free ultrafilter. It is well known that \( p \leq b \leq u \leq c \), and that there are models for which \( p < b \), for both see for example [8].

The family of sets has strong finite intersection property (sfip) if each finite subfamily has infinite intersection.

We say that two families of sets \( \mathcal{A}, \mathcal{B} \) mesh (and write \( \mathcal{A} \# \mathcal{B} \)) if \( \mathcal{A} \cup \mathcal{B} \) has the sfip. If \( \mathcal{A} = \{ A \} \), then we abridge \( \{ A \} \# \mathcal{B} \) to \( A \# \mathcal{B} \). If \( \mathcal{A} \) has the sfip, then by \( \langle \mathcal{A} \rangle \) we denote the filter generated by \( \mathcal{A} \).

Let \( A \) be an infinite subset of \( \omega \). A filter \( \mathcal{F} \) (on \( \omega \)) is called cofinite on \( A \) whenever \( U \in \mathcal{F} \) iff \( A \setminus U \) is finite. It is said to be cofinite if it is cofinite on some \( A \). The class of cofinite filters is denoted by \( \mathcal{Cof} \).

A relation between sets and functions. Let \( W \) be a partition [1]. For each \( n < \omega \), let \( (w^n_k)_{k<\omega} \) be an increasing sequence such that

\[
W_n = \{ w^n_k : k < \omega \}.
\]

For each \( f \in \omega^\omega \) and \( m < \omega \), let

\[
E_W(f, m) := \{ w^n_k : f(n) \leq k, m \leq n \}
\]

If \( F \in \bigcup W \), then, by definition, there exists a least \( n_F < \omega \) such that \( W_n \setminus F \) is finite for each \( n \geq n_F \). Now, for each \( n \geq n_F \), there exists a minimal \( k_n < \omega \) such that \( w^n_k \in F \) for each \( k \geq k_n \). Let \( \mathcal{F}_F \) denote the set of those functions \( f \), for which

\[
n \geq n_F \implies f(n) = k_n.
\]
Then \( E_W(f, n_F) \) is the same for each \( f \in F \). Sure enough, \( E_W(f_F, n_F) \in \bigcap W \).

Conversely, for every function \( f \in \omega \), define a family \( W_f \) of subsets of \( \omega \) as follows: \( F \in W_f \) if there is \( n_F < \omega \) such that \( F = E_W(f_F, n_F) \). Therefore,

**Proposition 2.** The family \( \bigcup_{f \in \omega} W_f \) is a base of \( \bigcap W \).

Let \( \mathcal{A} \) be a family of sets, and let \( \mathcal{F} \) be a filter. We say that \( \mathcal{A} \) is an external quasi-subbase (EQ-subbase) of \( \mathcal{F} \) if there exists a countable family \( \mathcal{B} \) such that \( A \cup \mathcal{B} \) has the sfip and \( \mathcal{F} \subset \langle A \cup \mathcal{B} \rangle \).

Let \( \mathcal{W} \) be a partition (1), then for each \( i < \omega \), let \( \tilde{W}_i := \bigcup_{n \geq i} W_n \) and \( \tilde{W} := \{ \tilde{W}_i : i < \omega \} \).

**Proposition 3.** Let \( \mathcal{W} \) be a partition and let \( \mathcal{A} \) be a family with the sfip. Then the following expressions are equivalent

1. \( \mathcal{A} \) is an EQ-subbase of \( \bigcap W \),
2. there exists a set \( D \) such that \( A \cup \tilde{W} \cup \{ D \} \) has the sfip, and \( \bigcap W \subset \langle A \cup \tilde{W} \cup \{ D \} \rangle \).

**Proof:** 2 \( \Rightarrow \) 1 is evident. We will show 1 \( \Rightarrow \) 2. Suppose that, on the contrary, \( 1 \land \neg 2 \), and let \( \mathcal{B} \) be a witness. Taking finite intersections \( \bigcap_{i \leq n} B_i \) instead of \( B_n \), we obtain a decreasing sequence, so that, without loss of generality, we assume that \( \mathcal{B} = \{ B_n \}_{n < \omega} \) is decreasing. By \( \neg 2 \), for each \( n \) there exists \( A_n \in \bigcap W \) such that \( A_n \notin \langle A \cup \tilde{W} \cup \{ B_n \} \rangle \).

Without loss of generality, for each \( n \) there is \( k(n) \geq n \) such that \( A_n \cap W_i \) is empty for all \( i < k(n) \) and \( W_i \setminus A_n \) is finite for all \( i \geq k(n) \). Define \( A_\infty := \bigcup_{i < \omega} \left( \bigcap_{n \leq k(n) \leq i} A_n \cap W_i \right) \) and note that \( A_\infty \notin \bigcap W \).

We will show that \( A_\infty \notin \langle A \cup \mathcal{B} \cup \tilde{W} \rangle \supset \langle A \cup \mathcal{B} \rangle \). To this aim, it suffices to show that \( A_\infty \notin \langle A \cup \tilde{W} \cup \{ B_n \} \rangle \) for each \( n < \omega \). Indeed, note that \( A_\infty \subset A_n \cup \tilde{W}^{c}_{k(n)} \) for each \( n < \omega \). From \( (A_n \cup \tilde{W}^{c}_{k(n)}) \cap \tilde{W}_{k(n)} \subset A_n \notin \langle A \cup \tilde{W} \cup \{ B_n \} \rangle \), we infer that \( (A_n \cup \tilde{W}^{c}_{k(n)}) \notin \langle A \cup \tilde{W} \cup \{ B_n \} \rangle \), and so \( A_\infty \notin \langle A \cup \tilde{W} \cup \{ B_n \} \rangle \).

**Remark 4.** Let \( \mathcal{A} \) be an EQ-subbase of \( \bigcap W \). Then there exists a partition \( \mathcal{V} \) such that \( \bigcap V \subset \langle A \cup \tilde{V} \rangle \).

**Proof:** Take \( V_i \) to be the \( i \)-th non-finite \( W_j \cap D \) (For \( D \) from Proposition \ref{Proposition 2}).

**Theorem 5.** Let \( (\mathcal{A}_\alpha)_{\alpha < \beta < \beta} \) be an increasing sequence of families of sets such that \( \mathcal{A}_\alpha \) is not an EQ-subbase of any contour. Then \( \bigcup_{\alpha < \beta} \mathcal{A}_\alpha \) is not an EQ-subbase of any contour.
Proof: Suppose not. Let $\beta$, $\{A_\alpha\}_{\alpha<\beta}$ be like in assumptions, and let $\int W$ be a contour and $B$ be a countable family of sets such that $\int W \subset \langle A \cup B \rangle$. By Proposition 3 and Remark 1 without loss of generality, we may assume that $B = \{B_n\}_{n<\omega} = \int W$ is decreasing. Denote $C_\alpha := \langle A_\alpha \cup B \rangle$. Clearly, $C_\alpha$ does not include $\int W$ for each $\alpha < \beta$; thus, for each $\alpha < \beta$, there exists a set $D_\alpha \in \int W$ such that $D_\alpha \notin C_\alpha$. Let $g_\alpha \in f D_\alpha$ for each $\alpha < \beta$. Since $\beta < \omega$, the family $\{g_\alpha\}_{\alpha<\beta}$ is bounded by some function $g$. Let $G \in W_g$. We will show that $G \notin \bigcup \alpha<\beta C_\alpha$ so that $G \notin C_\alpha$ for each $\alpha < \beta$. Suppose not, and let $\alpha_0$ be a witness. By construction, there exists $n_0 < \omega$ such that $G \subset C_{n_0} \cup D_{n_0}$. Since $(D_{n_0} \cup B_{n_0}) \cap B_{n_0} \subset D_{n_0} \notin C_{n_0}$, thus $D_{n_0} \cup B_{n_0} \notin C_{n_0}$ and so $G \notin C_{n_0}$. 

Let $A$ be a family of sets with the sfip. We will say that $A$ has a $u$-property if $\langle A \cup B \rangle$ is a free ultrafilter only for families $B$ of cardinality $\geq u$.

Remark 6 (Folclore?). Let $F$ be a filter such that there is a map $f \in \omega^\omega$ that $f(F) \in \text{Cof}$. Then $F$ has $u$-property.

Proof: Let $f \in \omega^\omega$ be a function, such that $f(F) \in \text{Cof}$, if $\langle F \cup A \rangle$ is an ultrafilter thus $f(\langle F \cup A \rangle)$ is a free ultrafilter (base) and so $\{f(A) : A \in A\}$ has a cardinality of at least $u$. 

Remark 7. Let $F$ be a filter on $\omega$ and let $f \in \omega^\omega$ be a function, such that

$$\limsup_{n \in F} \text{card}(f^{-1}(n)) = \infty$$

(4)

for each $F \in F$. Then if $\langle f^{-1}(F) \cup A \rangle$ is an ultrafilter, then $\text{card}(A) \geq u$.

Proof.: For each $n < \omega$, let $(k_n)_{i<\text{card}(f^{-1}(n))}$ be an enumeration of $f^{-1}(n)$. Define function $h \in \omega^\omega$ such that $h(k_n) = i$ for each $n, i < \omega$. Clearly, $h(A \cup f^{-1}(p)) \in \text{Cof}$ and so we are in assumptions of Remark 6.

Remark 8 (Folclore). Let $Y, Z$ be families with the sfip of subsets of $\omega$, which are not ultrafilter-bases. Let $h \in \omega^\omega$. Then there are sets $Y, Z$ such that $Y \# Y$, $Z \# Z$ and $h(Z) \# Y$.

Proof.: Take any $O$ such that $O \# Y$ and $O^c \# Y$.

If $h^{-1}(O) \# Z$ then $Y = O$, $Z = (h^{-1}(O))^c$;

if $h^{-1}(O^c) \# Z$ then $Y = O^c$, $Z = h^{-1}(O)$;

if $h^{-1}(O) \# Z$ and $h^{-1}(O^c) \# Z$ then $Y = O$, $Z = (h^{-1}(O))^c$. 

Let us recall, the well-known theorem, see for example [II, Corollary 1].

Theorem 9. Let $u$ be an ultrafilter. If $f(u) =_{RK} u$, then there exists $U \in u$, such that $f$ is one-to-one on $U$. 

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M. E. Rudin proved in [5] that, under CH, for each P-point there exists
an RK-strictly greater P-point. Few years later in [1, Theorem 6] A. Blass
proved that theorem under \( p = c \).

**Theorem 10.** \((b = c)\) If \( p \) is a P-point, then there is a set \( \Omega \) of P-points, of
cardinality \( b \), such that \( u >_{RK} p \) for each \( u \in \Omega \), and that elements of \( \Omega \) are
Rudin-Keisler incomparable.

**Proof.** Let \( f \in \omega^\omega \) be a finite-to-one function such that \( \{ f \} \) for all \( F \in p \). We
define a family \( \mathcal{A} \) as follows: \( A \in \mathcal{A} \) iff there exist \( i < \omega \) and \( P \in p \) such that
card \((f^{-1}(n) \setminus A) < i \) for each \( n \in P \). Then Theorem 9 insures us that
the ultrafilters we are building are strictly RK-greater than \( p \).

We claim that \( \{ f^{-1}(p) \} \cup \mathcal{A} \) does not include any EQ-subbase of a con-
tour. Suppose not, and take a witness \( \mathcal{W} \). By Remark 4 without loss of
generality, we may assume that \( \mathcal{W} \subset \{ f^{-1}(p) \} \cup \mathcal{A} \cup \mathcal{W} \). Consider two cases:

- **Case 1:** There exists a sequence \((B_n)_{n<\omega}\) and a strictly increasing \( k \in \omega^\omega \) such that \( B_n \subset W_{k(n)} \), \( f(B_n) \notin p \) and \( P \# f^{-1}(p) \cup \mathcal{A} \), where \( B := \bigcup_{n<\omega} B_n \). Take a sequence \((f(\bigcup_{i<n} B_n))_{n<\omega}\). This is an increasing sequence,
and clearly, \( \bigcup_{n<\omega} f(\bigcup_{i<n} B_n) = f(B) \in p \). Make a partition of \( f(B) \) by
\( f(\bigcup_{i<n} B_n) \setminus f(\bigcup_{i<n} B_n) \), for \( i < \omega \). Since \( p \) is a P-point, there exists
\( P \in p \), such that \( P \cap (f(\bigcup_{i<n} B_n) \setminus f(\bigcup_{i<n} B_n)) \) is finite for all \( i < \omega \), and
thus \( f^{-1}(P) \cap B_n \) is finite for all \( i < \omega \). Therefore \( f^{-1}(P) \cap W_n \cap B \) is finite,
and thus \( (f^{-1}(P) \cap B)^c \in \mathcal{W} \), which means that \( \mathcal{W} \not\in \#(f^{-1}(p) \cup \mathcal{B}) \).

- **Case 2:** Otherwise, define sets \( V_i = W_i, V_j = W_j \cap f^{-1}(\bigcap_{k<j} f(W_k)) \), and
note that \( \bigcup_{i<\omega} V_i \in (\mathcal{A} \cup f^{-1}(p)) \). Then \((f(V_i))_{i<\omega}\) is a decreasing sequence,
and since \( f \) is finite-to-one, \((f(V_i)) \setminus f(V_{i+1}))_{i<\omega}\) is a partition of \( f(V_i) \in p \).
Since \( p \) is a P-point, there exists \( P \in p \) such that \( P_i = (f(V_i)) \setminus f(V_{i+1})) \cap P \) is
finite for each \( i < \omega \). Let \( g : \omega \to \omega \) be defined by \( g(i) := E(\frac{i+1}{2}) \), where
\( E(x) \) stands for the integer part of \( x \). Let \( R := \bigcup_{i<\omega} (f^{-1}(P_i) \cap \bigcup_{j \in (g(i),...,\omega)} V_i) \).

Note that \( R \cap V_i \) is finite for each \( i < \omega \), and that
\[ \lim \sup_{n \in \hat{P}} \card(f^{-1}(n) \cap R) = \infty \]
for all \( \hat{P} \in p \) and \( \hat{P} \subset P \). Thus \( R^c \not\in (\mathcal{A} \cup f^{-1}(p) \cup \mathcal{W}) \), but on the other
hand \( R^c \in \mathcal{W} \).

We range all contours in a sequence \((f \mathcal{W})_{\alpha < b}\) and \( \omega^\omega \) in a sequence
\((f_\beta)_{\beta < b}\). We will build a family \( \{(f_\alpha^\beta)_{\alpha < b}\}_{\beta < b} \) of increasing b-sequences
\( (f_\alpha^\beta)_{\alpha < b} \) of filters such that:
1) Each $\mathcal{F}_\alpha^\beta$ is generated by $\mathcal{A}$ together with some family of sets of cardinality $< b$;

2) $\mathcal{F}_0^\beta = \mathcal{A}$ for each $\beta < b$;

3) For each $\alpha, \beta < b$, there exists $F \in \mathcal{F}_{\alpha+1}^\beta$ such that $F^c \in \bigcap \mathcal{W}_\alpha$;

4) For every limit $\alpha$ and for each $\beta$, let $\mathcal{F}_\alpha^\beta = \bigcup_{\gamma < \alpha} \mathcal{F}_\gamma^\beta$;

5) For each $\alpha, \gamma < \alpha, \beta_1, \beta_2 < \alpha$, there exists a set $F \in \mathcal{F}_{\alpha+1}^\beta_1$ such that $(f_\gamma(F))^c \in \mathcal{F}_{\alpha+1}^\beta_2$.

The existence of such families follows by a standard induction with sub-inductions using Theorem 5 and Remark 8 for Condition 5. It follows from the proof of Remark 7 that $\mathcal{F}_\alpha^\beta$ is not an ultrafilter-base for each every $\alpha$ and $\beta$. It suffices now to take for each $\beta < b$, any ultrafilter extending $\bigcup_{\beta < b} \mathcal{F}_\beta^\alpha$ and note that, by Proposition 1, it is a P-point.

A. Blass proved [1, Theorem 7] also that, under $p = c$, each RK-increasing sequence of P-points is upper bounded by a P-point.

**Theorem 11.** $(b = c)$ If $(p_n)_{n < \omega}$ is an RK-increasing sequence of P-points, then there exists a P-point $u$ such that $u >_{RK} p_n$ for each $n < \omega$.

**Proof.** Let $f_n$ be a finite-to-one function that witnesses $p_{n+1} >_{RK} p_n$. For each natural number $m$, for each $P \in p_m$ define $B_m^P \subset \omega \times \omega$ by:

$$B_m^P \cap (\omega \times \{n\}) := \begin{cases} f_{m-1}^{-1} \circ f_{m-2}^{-1} \circ \ldots \circ f_n^{-1}(P), & \text{if } n > m; \\ P, & \text{if } n = m; \\ \emptyset, & \text{if } n < m. \end{cases}$$

Now let $B_m := \{B_m^P : P \in p_m\}$ and let $\mathcal{B} := \bigcup_{m < \omega} B_m$. Clearly, $\mathcal{B}$ is a filter, and each ultrafilter which extends $\mathcal{B}$ is RK-greater then each $p_n$.

For each $m < \omega$ consider a function $f_m : \bigcup_{n \geq m}(\omega \times \{n\}) \to \omega \times \{m\}$, such that

$f_m \upharpoonright (\omega \times \{n\}) = f_{m-1} \circ \ldots \circ f_{n-1}$ for $n > m$;

$f_m \upharpoonright (\omega \times \{n\}) = \text{id}_{\omega \times \{m\}}$ for $n = m$.

A proof analogous to the proof of Theorem 10 (for $p_m$ and $f_m$) shows that no $\mathcal{B}_n$ includes any EQ-subbase of contour. Hence, by Theorem 3, $\mathcal{B}$ does not include a EQ-subbase of any contour. A proof analogous to Remark 7 shows that $\mathcal{B}$ has a $u$-property.

Let $(\bigcap \mathcal{W}_\alpha)_{\alpha < b}$ be a sequence of all contours. We will build an increasing $b$-sequence of filters $\mathcal{F}_\alpha$ such that:

1) $\mathcal{F}_0 = \mathcal{B}$.

2) For each $\alpha$, there is such $F \in \mathcal{F}_{\alpha+1}$ that $F^c \in \bigcap \mathcal{W}_\alpha$;

3) For a limit $\alpha$, $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$.

The rest of the proof is an easier version of the final part of the proof of Theorem 10. ■
Fact 13. Let \( (p_n)_{n<\omega} \) be an RK-increasing sequence of P-points. Then there exists a family \( \mathcal{U} \) of P-points of cardinality \( \mathfrak{b} \) such that \( u >_{RK} p_n \) for each \( u \in \mathcal{U} \), and that elements of \( \mathcal{U} \) are RK-incomparable.

Proof.: Just combine Theorem 10 with Theorem 11. ■

In [1] A. Blass asked (Question 4) what ordinals can be embedded in the set of P-points and pointed out that such an ordinal can not be greater then \( \mathfrak{c}^+ \). The question was mentioned also by Raghavan and Shelah in [2].

We prove that, under \( \mathfrak{b} = \mathfrak{c} \), there is an embedding of \( \mathfrak{c}^+ \) into P-points.

To this end we need some, probably known, facts. We say that a sub set for each

Fact 14. If an ordinal number \( \alpha \) can be sparsely embedded in \( \omega^* \) of nondecreasing functions less than any function \( f \in \omega^* \), then \( \alpha \) can be sparsely embedded as nondecreasing functions between any sparse pair of functions \( g <^* h \in \omega^* \).

Proof: Without loss of generality, \( f \) is nondecreasing. Let \( (f_\beta)_{\beta<\alpha} \) be an embedding of \( \omega^* \) under \( f \). Clearly it suffice to proof that there is an embedding under \( s \) defined by \( s(n) := h(n) - g(n) \) if \( h(n) \geq g(n) \) and \( s(n) = 0 \) otherwise.

Define a sequence \( (k(n))_{n<\omega} \) by \( k(0) := \min \{ m : s(i) \geq f(0) \text{ for all } i \geq m \} \), \( k(n+1) := \min \{ m : m > k(n) \text{ & } s(i) \geq f(n+1) \text{ for all } i \geq m \} \).

Define \( g_\alpha \) as follows: \( g_\alpha(n) = f_\alpha(m) \) iff \( k(n) \leq m < k(n+1) \).

Fact 15. For each \( \gamma < \mathfrak{b}^+ \), there exists a strictly \( <^* \)-increasing sparse sequence \( F = (f_\alpha)_{\alpha<\mathfrak{b}^+} \subset \omega^* \) of nondecreasing functions.

Proof. Clearly by Facts 13, 14 first ordinal number which may not be embedded as sparse sequence in \( \omega^* \) under \( \text{id}_\omega \) is a limit number or a successor of limit number. Again, by Facts 13 and 14 the set of ordinals less then mentioned in both cases limit number is closed under \( \mathfrak{b} \) sums, thus this number is not less then \( \mathfrak{b}^+ \).

We denote by \( \text{succ} \) the class of successor ordinals.
Lemma 16. For each $\gamma < b^+$, for each P-point $p$ there exists an RK-increasing sequence $\{p_\alpha : \alpha < \gamma, \alpha \in \text{succ} \cup \{0\}\}$ of P-points, such that $p_0 = p$.

Proof: Note that $\text{cof}(\gamma) \leq b$. Consider a set of pairwise disjoint trees $T_n$, such that each $T_n$ has a minimal element, each element of $T_n$ has exactly $n$ immediate successors and each branch has the highest $\omega$.

Let $\{f_\alpha\}_{\alpha < \gamma, \alpha \in \text{succ} \cup \{0\}} \subset \omega^\omega$ be sparse, strictly $\ast$-increasing sequence (there is some by Fact 15). For each $\text{succ} \cup \{0\} \ni \alpha < \gamma$, define

$$X_\alpha := \bigcup_{n < \omega} \text{Level}_{f_\alpha(n)} T_n.$$ 

For each $\text{succ} \cup \{0\} \ni \alpha < \beta$, define

$$f_\beta^\alpha : \bigcup_{f_\alpha(n) < f_\beta(n)} \text{Level}_{f_\beta(n)} T_n \to \bigcup_{f_\alpha(n) < f_\beta(n)} \text{Level}_{f_\beta(n)} T_n,$$

that agrees with the order of trees $T_n$ for $n < \omega$ such that $f_\alpha(n) < f_\beta(n)$. Note that $\text{dom} f_\beta^\alpha$ is co-finite on $X_\beta$ for each $\text{succ} \cup \{0\} \ni \alpha < \beta$.

Let $p = p_0$ be a P-point on $X_0 = \bigcup_{n < \omega} \text{Level}_0 T_n$. We will work by recursion building a filter $p_\beta$ on $X_\beta$ for each $\text{succ} \cup \{0\} \ni \alpha < \beta$.

Let $R \subset \beta \cap (\text{succ} \cup \{0\})$ cofinal with $\beta - 1$, and of order type less than or equal to $b$. For each function $f_\alpha^\beta$ ($\alpha \in R$), we proceed like in the proof of the Theorem 10, obtaining a family $\{(f_\alpha^\beta)^{-1}(p_\alpha) \cup A_\alpha\}$. By construction, the union of these families

$$C := \bigcup_{\alpha \in R} \{(f_\alpha^\beta)^{-1}(p_\alpha) \cup A_\alpha\}$$

has the sfip.

To show that the $C$ is not a EQ-subbase of any contour for each $\alpha < \beta$, $\alpha \in R$, proceed like in the of the claim in the proof of Theorem 10 for a function $f_\alpha^\beta$, where $f_0 := \text{id}_{X_0}$. If for some $\alpha \in R$ we are in Case 1, then the following proof is as in Case 1 for $f_\alpha^\beta$ and a P-point $p_\alpha$, and we are done. If not, then for each $\alpha \in R$, we are in Case 2, and it suffices to conclude the proof for $f_\alpha^\beta$ and for a P-point $p_0$ like in Case 2 of the claim of Theorem 10.

Finally, as at the end of the proof of Theorem 11, we list all contours in $b$-sequence and, by recursion, we add, preserving sfip, to the family $C$ the sets, the complements of which belong to the listed contours. By Theorem 5 the process will pass through all steps $< b$. By Proposition 11 family $C$ with added sets may be extend only to the P-point ultrafilters. 

As an immediate corollary of the Lemma 16 we have the following
Theorem 17. \((b = c)\) For each P-point \(p\) there exists an RK-increasing sequence \(\{p_\alpha\}_{\alpha < c^+}\) of P-points, such that \(p_0 = p\). 

By Theorem 11 each embedding of \(\omega\) into P-points is upper bounded by a P-point, by Theorem 17 there is an embedding of \(c^+\), which, clearly, is not upper bounded, so:

Question 18. What is a minimal ordinal \(\alpha\) such that there exists an unbounded embedding of \(\alpha\) into P-points.

A. Blass also proved [1, Theorem 8] that, under \(p = c\), there is an order embedding of the real line in the set of P-points. We will prove the same fact, but under \(b = c\). Our proof is based on the original idea of the set \(X\) of A. Blass. Therefore we quote the beginning of his proof, and then use our machinery.

Theorem 19. \((b = c)\) There exists an order embedding of the real line in the set of P-points.

Proof: ————- begining of quotation ————-

Let \(X\) be a set of all functions \(x : \mathbb{Q} \to \omega\) such that \(x(r) = 0\) for all but finitely many \(r \in \mathbb{Q}\); here \(\mathbb{Q}\) is the set of rational numbers. As \(X\) is denumerable, we may identify it with \(\omega\) via some bijection. For each \(\xi \in \mathbb{R}\), we define \(h_\xi : X \to X\) by

\[
h_\xi(x)(r) := \begin{cases} x(r) & \text{if } r < \xi, \\ 0 & \text{if } r \geq \xi. \end{cases}
\]

Clearly, if \(\xi < \eta\), then \(h_\xi \circ h_\eta = h_\eta \circ h_\xi = h_\xi\). Embedding of \(\mathbb{R}\) into P-points will be defined by \(\xi \mapsto D_\xi = h_\xi(D)\) for a certain ultrafilter \(D\) on \(X\). If \(\xi < \eta\), then

\[
D_\xi = h_\xi(D) = h_\xi \circ h_\eta(D) = h_\xi(D_\eta) \leq D_\eta.
\]

We wish to choose \(D\) in such a way that

(a) \(D_\xi \not\supseteq D_\eta\) (therefore, \(D_\xi < D_\eta\) when \(\xi < \eta\)), and

(b) \(D_\xi\) is a P-point.

Observe that it will be sufficient to chose \(D\) so that

(a') \(D_\xi \not\supseteq D_\eta\) when \(\xi < \eta\) and both \(\xi\) and \(\eta\) are rational, and

(b') \(D\) is a P-point.

Indeed, (a') implies (a) because \(\mathbb{Q}\) is dense in \(\mathbb{R}\). If (a) holds, then \(D_{\xi-1} < D_\xi\), so \(D_\xi\) is a nonprincipal ultrafilter \(\leq D\); hence (b') implies (b).
Claim (a') means that for all $\xi < \eta \in \mathbb{Q}$ and all $g : X \to X$, $D_{\eta} \neq g(D_{\xi}) = gh_{\xi}(D_{\eta})$. By Theorem 3, this is equivalent to $\{ x : gh_{\xi}(x) = x \} \not\subseteq D_{\eta}$, or by definition of $D_{\eta}$,

$$\{ x : gh_{\xi}(x) = h_{\eta}(x) \} = h_{\eta}^{-1}(\{ x : gh_{\xi}(x) = x \}) \not\subseteq D.$$  

(a’)

Now we proceed to construct a P-point $D$ satisfying (a”) for all $\xi < \eta \in \mathbb{Q}$ and for all $g : X \to X$; this will suffice to establish theorem.

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Claim 1: $\mathcal{A}$ is not a EQ-subbase of any contour.

Proof: By Theorem 5 it suffices to prove that $\bigcup_{i < \omega} \mathcal{A}_i$ is not a EQ-subbase for any contour, for each $n < \omega$. Suppose not, and take (by Remark 4) witnesses: $i_0$ and $\mathcal{W}$ such that $\mathcal{W} \subseteq \bigcup_{i < i_0} \mathcal{A}_i \cup \overline{\mathcal{W}}$. For each $n < \omega$ consider a condition:

$$\exists x_n : \forall i < i_0 : (x_n \in h_{\xi_i}(\overline{\mathcal{W}}_i) \& \text{card}(h_{\eta_i}(h_{\xi_i}^{-1}(x_n)) \cap h_{\eta_i}(\overline{\mathcal{W}}_i)) > n) \quad (S_n)$$

Case 1: $S_n$ is fulfilled for all $n < \omega$. Then for each $n < \omega$, $j < n$, choose $x_{nj} \in \overline{\mathcal{W}}_n$ such that $h_{\xi_j}(x_{nj}) = x_n$ and $h_{\eta_i}(x_{nj}) \neq h_{\eta_j}(x_{nj})$ for $j_0 \neq j_1$. Define $E := \bigcup_{n < \omega} \bigcup_{j < n} \{ x_{nj} \}$. Clearly $E \subseteq \mathcal{W}$, but $E \not\subseteq \bigcup_{i < i_0} \bigcup_{i \in G} \mathcal{A}_i \cup \bigcup_{i < m} \mathcal{W}_i$ for any choice of finite family $G \subseteq X$ for any $m < \omega$.

Case 2: $S_n$ is not fulfilled for some $n_0 < \omega$. Then there exist functions $\{ g_{n,i} \}_{n \leq n_0, i < i_0} \subseteq X$ such that $\overline{\mathcal{W}}_1 \subseteq \bigcup_{n \leq n_0} \bigcup_{i < i_0} \{ A_{g_{n,i},\xi,i} \} \cup \bigcup_{n \leq n_0} \mathcal{W}_n$, i.e.

$$\mathcal{W} \cap \bigcup_{i < i_0} \mathcal{A}_i \cup \overline{\mathcal{W}}.$$

Claim 2: $\mathcal{A}$ has a u-property.

Proof: For $x \in X$ let $m(x) := \text{card}(\{ g \in \mathbb{Q} : x(g) \neq 0 \})$. Define $H_i = \{ x \in X : m(x) = i \}$. Clearly $\bigcup_{i \geq \omega} H_i \# \mathcal{A}$ for each $n < \omega$. So we may add $\mathcal{H} = \{ \bigcup_{i \geq \omega} H_i, n < \omega \}$ to $\mathcal{A}$ and since $\mathcal{A}$ is not a EQ-subbase for any contour, $\mathcal{A} \cup \mathcal{H}$ is not a EQ-subbase for any contour. Consider a function $s \in X$, such that $s(x) = i$ if and only if $x \in H_i$. The existence of function $s$ shows that we are in the assumptions of Remark 6.

Now it suffices to conclude like in the previous proofs.

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We define $\mathcal{B}$ to be a minimal cardinality of families $\mathcal{B}$ (of subsets of $\omega$) for which there exists family $\mathcal{A}$ (of subsets of $\omega$) such that

- $\mathcal{A} \cup \mathcal{B}$ has the sfip,
- $\mathcal{A}$ is not a EQ-subbase of any contour,
- $\mathcal{A} \cup \mathcal{B}$ includes some contour.

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Theorem 20. [4, Theorem 5.2] Each contour has a base of cardinality $\mathfrak{d}$.

By Theorems 5 and 20, $b \leq c \leq \mathfrak{d}$. While we look again at our proofs, we see that they still work under $c = u = \kappa$ (Theorems 10, 19), or even under $c = \kappa$, Theorems 11 (for one element family $\mathcal{U}$), 17 (for $b^+$ in the place of $c^+$).

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