Monochromatic and Heterochromatic
Subgraph Problems
in a Randomly Colored Graph

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Abstract. Let $K_n$ be the complete graph with $n$ vertices and $c_1, c_2, \ldots, c_r$ be $r$ different colors. Suppose we randomly and uniformly color the edges of $K_n$ in $c_1, c_2, \ldots, c_r$. Then we get a random graph, denoted by $K_n^c$. In the paper, we investigate the asymptotic properties of several kinds of monochromatic and heterochromatic subgraphs in $K_n^c$. Accurate threshold functions in some cases are also obtained.

Keywords: monochromatic, heterochromatic, threshold function

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1 Introduction

The study of random graphs was begun by P. Erdős and A. Rényi in the 1960s [7–9] and now has a comprehensive literature [3,6].

The most frequently encountered probabilistic model of random graph is $G_{n,p(n)}$, where $0 \leq p(n) \leq 1$. It consists of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ in which the edges are chosen independently and with probability $p(n)$. As $p(n)$ goes from zero to one the random graph $G_{n,p(n)}$ evolves from empty to full.

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P. Erdős and A. Rényi discovered that for many natural properties $A$ of graphs there was a narrow range in which $\Pr[\mathcal{G}_{n,p(n)} \text{ has property } A]$ moves from near zero to near one. So we introduce the following important definition ([5], page 14).

**Definition 1.1** A function $p(n)$ is a threshold function for property $A$ if the following two conditions are satisfied:

1. If $p'(n) \ll p(n)$, then $\lim_{n \to \infty} \Pr[\mathcal{G}_{n,p'(n)} \text{ has property } A] = 0$.

2. If $p'(n) \gg p(n)$, then $\lim_{n \to \infty} \Pr[\mathcal{G}_{n,p'(n)} \text{ has property } A] = 1$.

In general, if $\Pr[\mathcal{G}_{n,p(n)} \text{ has property } A] \to 0$, we say almost no $\mathcal{G}_{n,p(n)}$ has property $A$. Conversely, if $\Pr[\mathcal{G}_{n,p(n)} \text{ has property } A] \to 1$, we say almost every $\mathcal{G}_{n,p(n)}$ has property $A$.

In this article, we introduce the following probabilistic model of random graphs. Let $K_n$ be the complete graph with vertex set $V = \{1, 2, \ldots, n\}$ and $c_1, c_2, \ldots, c_r$ be $r = r(n)$ different colors. We now send $c_1, c_2, \ldots, c_r$ to the edges of $K_n$ randomly and equiprobably, which means each edge is colored in $c_i (1 \leq i \leq r)$ with probability $\frac{1}{r}$. Thus we get a random graph $\mathcal{K}_n$. The probability space $(\Omega, \mathcal{F}, \Pr)$ of $\mathcal{K}_n$ has a simple form: $\Omega$ has $r^{(n)}$ elements and each one has probability $\frac{1}{r^{(n)}}$ to appear.

The subgraph of $\mathcal{K}_n$ with vertices $1, 2, \ldots, n$ and the edges that have color $c_i$ is denote by $\mathcal{G}_i$. Obviously, it is just the random graph $\mathcal{G}_{n,p(n)}$ ([3], page 34), where $p(n) = \frac{1}{r}$.

Matching, clique and tree are three kinds of important subgraphs. As to their definitions, please refer to [2]. A $k$-matching is a matching of $k$ independent edges. A $k$-clique is a clique of $k$ vertices. Similar, a $k$-tree is a tree of $k$ vertices. In a $k$-matching ($k$-clique, $k$-tree), if all of the edges are in a same color, we call it a monochromatic $k$-matching ($k$-clique, $k$-tree); On the other hand, if any two of edges are of different colors, we call it a heterochromatic $k$-matching ($k$-clique, $k$-tree).

Having a monochromatic $k$-matching, $k$-clique or $k$-tree or a heterochromatic $k$-matching, $k$-clique or $k$-tree are all properties of $\mathcal{K}_n$. We want to investigate these properties and obtain the threshold functions for them.
Two properties will be especially demonstrated: monochromatic $k$-matching and heterochromatic $k$-matching. For the others, the methods are similar and we list the results in Section 4.

## 2 Monochromatic $k$-Matchings in $K^r_n$

Let $k$ be an integer. Obviously, in $K^r_n$, there are altogether

$$q = \frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2k+2}{2}}{k!}$$

sets of $k$ independent edges. Arrange them in an order and the $i$-th one is denoted by $M_i$.

Let $A_i$ be the event that the edges in $M_i$ are monochromatic and $X_i$ be the indicator variable for $A_i$. That is,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ happens} \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Then the random variable

$$X = X_1 + X_2 + \cdots + X_q$$

denotes the number of monochromatic $k$-matchings in $K^r_n$.

For each $1 \leq i \leq q$,

$$E(X_i) = Pr[X_i = 1] = \frac{r}{r^k}.$$  

From the linear of the expectation [1],

$$E(X) = E(X_1 + X_2 + \cdots + X_q)$$

$$= \frac{r}{r^k} q$$

$$= \frac{n!}{(n - 2k)!(2k)!r^{k-1}}. \quad (2.2)$$

By careful calculation, the following assertions (*) for (2.2) are true, which will be used later:
1. If \( r \) is fixed, then for every \( 1 \leq k \leq \frac{n}{2} \), \( E(X) \to \infty \).

2. If \( k \) is fixed and \( r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}} \), then \( E(X) \to \infty \);

3. If \( k \) is fixed and \( r \gg (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}} \), then \( E(X) \to 0 \);

4. If \( k \) is fixed and \( r = c^{(0)} (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}} \), where \( c^{(0)} > 0 \) is a constant, then \( E(X) \to \frac{1}{(c^{(0)})^{k-1}} \).

Though \( k \) and \( r \) can be both functions of \( n \), if they are both variables, the situation becomes very complicated. So we illustrate monochromatic \( k \)-matching problem from three aspects: \( r \) is fixed, \( k \) is fixed and \( k = \left\lfloor \frac{n}{2} \right\rfloor \). The last case is the perfect matching case or the nearly perfect matching case. Since we focus on the asymptotic properties, we will not distinguish \( \left\lfloor \frac{n}{2} \right\rfloor \) from \( \frac{n}{2} \). That is, we always suppose \( n \) is an even.

2.1 \( r \) is fixed

Assertion (*) 1 says that \( E(X) \to \infty \) for every \( 1 \leq k \leq \frac{n}{2} \) if \( r \) is fixed. We certainly expect that \( Pr[X > 0] \to 1 \) holds. In fact, it does.

**Theorem 2.1** If \( r \geq 1 \) is fixed, then almost every \( K^r_n \) has a monochromatic \( k \)-matching for any \( 1 \leq k \leq \frac{n}{2} \).

**Proof.** We have mentioned in Section 1 that the subgraph \( G_t \) of \( K^r_n \) is actually the random graph \( G_{n,p(n)} \), where \( p(n) = \frac{1}{r} \). There is a result saying that the threshold function for \( G_{n,p} \) has a perfect matching is \( \frac{\log n}{n} \) ( [6], page 85). If \( r \) is fixed, then \( \frac{1}{r} \gg \frac{\log n}{n} \), which implies that almost every \( G_t \) has a perfect matching. Then almost every \( K^r_n \) has a monochromatic \( k \)-matching for every \( 1 \leq k \leq \frac{n}{2} \).

2.2 \( k \) is fixed

In this case, we prove the following theorem.
Theorem 2.2 If $k$ is fixed ($k = 1$ is a trivial case so suppose $k \geq 2$), then

$$\lim_{n \to \infty} Pr[X > 0] = \begin{cases} 0 & \text{if } r \gg \left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}}, \\ 1 & \text{if } r \ll \left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}}. \end{cases}$$ (2.3)

That is to say, \( \left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}} \) is the threshold function for the property that \( K^r_n \) has a monochromatic \( k \)-matching.

Proof. From Markov’s inequality \([4]\)

$$Pr[X > 0] \leq E(X)$$

and assertion (*) 3, we have

$$Pr[X > 0] \to 0 \text{ if } r \gg \left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}}.$$ 

For the other half, we estimate \( \frac{\Delta}{(E(X))^2} \), where \( \Delta = \sum_{i \sim j} Pr[A_i \cap A_j] \). \( A_i(A_j) \) denotes the event that the edges in \( M_i(M_j) \) are monochromatic and \( i \sim j \) means the ordered pair of \( A_i \) and \( A_j \) that are not independent from each other.

Our goal is to prove that if \( r \ll \left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}} \), then \( \frac{\Delta}{(E(X))^2} \to 0 \).

Because

\[
\Delta = \sum_{i \sim j} Pr[A_i \cap A_j]
\]

\[
= \sum_{s=1}^{k-1} \sum_{(i,j)_s} \frac{r^{2k-s}}{s!(k-s)!} \left(\frac{n!}{(n-2s)!}\right)^{\frac{1}{k-1}}
\]

(\( (i,j)_s \) means the ordered pair of \( M_i \) and \( M_j \) that have \( s \) common edges)

\[
\leq \sum_{s=1}^{k-1} \frac{\left(\frac{n!}{(n-2s)!}\right)^{\frac{1}{k-1}}}{s!(k-s)!} \sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!}
\]

\[
= \frac{n!}{2^{2k}(n-2k)!(n-2k)!} \sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!}
\]

then we have

$$\frac{\Delta}{(E(X))^2} \leq \frac{k!}{n!} \sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!}(k-s)!.$$ (2.4)
If \( r \ll \left( \frac{n!}{(n-2k)!2^k k!} \right)^{\frac{1}{k-1}} \sim \left( \frac{1}{2^k k!} \right)^{\frac{1}{k-1}} n^{\frac{2k}{k-1}} \), then there are 3 possible cases: (i) \( r \ll n^2 \), (ii) \( r = c^{(1)} n^2 \), where \( c^{(1)} > 0 \) is a constant and (iii) \( n^2 \ll r \ll \left( \frac{n!}{(n-2k)!2^k k!} \right)^{\frac{1}{k-1}} \).

In case (i),
\[
\sum_{s=1}^{k-1} \frac{(n - 2s)!2^s r^s}{s!(k - s)!(k - s)!} = (1 + o(1)) \frac{2(n - 2)!}{(k - 1)!(k - 1)!} \tag{2.5}
\]

Then submit (2.5) to (2.4), we get
\[
\frac{\Delta}{(E(X))^2} \leq 2(1 + o(1)) \frac{k^2}{n(n - 1)} \to 0. \tag{2.6}
\]

In case (ii)
\[
\sum_{s=1}^{k-1} \frac{(n - 2s)!2^s r^s}{s!(k - s)!(k - s)!} = c^{(2)} \frac{2(n - 2)!}{(k - 1)!(k - 1)!}, \tag{2.7}
\]
where \( c^{(2)} \) is a sufficiently large constant.

Then submit (2.7) to (2.4), we get
\[
\frac{\Delta}{(E(X))^2} \leq 2c^{(2)} \frac{k^2}{n(n - 1)} \to 0. \tag{2.8}
\]

In case (iii)
\[
\sum_{s=1}^{k-1} \frac{(n - 2s)!2^s r^s}{s!(k - s)!(k - s)!} = (1 + o(1)) \frac{(n - 2k + 2)!2^{k-1} r^{k-2}}{(k - 1)!}. \tag{2.9}
\]

Then submit (2.9) to (2.4), we get
\[
\frac{\Delta}{(E(X))^2} \leq c^{(3)} \frac{n^{2k(k-2)}}{n^{2k-2}} \to 0, \tag{2.10}
\]
where \( c^{(3)} \) is a sufficiently large constant.

Summarizing (2.6) (2.8) and (2.10), we end the proof of \( \frac{\Delta}{(E(X))^2} \to 0 \) with the condition \( r \ll \left( \frac{n!}{(n-2k)!2^k k!} \right)^{\frac{1}{k-1}} \).
A corollary of the Chebyshev’s inequality [4] asserts that if \( E(X) \to \infty \) and \( \Delta = \sigma((E(X))^2) \), then almost surely \( X > 0( [1], \text{page 46}) \). So from assertion (*) 2 and the above discuss, we obtain

\[
Pr[X > 0] \to 1 \text{ if } r \ll \left( \frac{n!}{(n - 2k)!2^k r!} \right)^{\frac{1}{k - 1}}.
\]

From the definition of the threshold function (Definition [1,1]), we can say that \( \left( \frac{n!}{(n - 2k)!2^k r!} \right)^{\frac{1}{k - 1}} \) is the threshold function for the property that \( K_n^r \) has a monochromatic \( k \)-matching.

2.3 \( k = \frac{n}{2} \)

When \( k = \frac{n}{2} \), a monochromatic \( k \)-matching is a monochromatic perfect matching.

Replace \( k \) with \( \frac{n}{2} \) in (2.2), we have

\[
E(X) = \frac{n!}{(\frac{n}{2})!\frac{n}{2} r^{\frac{n}{2} - 1}}.
\] (2.11)

By calculation of (2.11), we get \( E(X) \to 0 \) if \( r \geq \frac{n}{c(4)r} \), where \( c(4) < e \) is a constant; \( E(X) \to \infty \) if \( r \leq \frac{n}{e} \).

The following assertion is true as a direct corollary of Markov’s inequality and the threshold function for the property that \( G_n,p \) having a perfect matching ( [6], page 85). Here we omit its proof.

**Theorem 2.3** If \( r \geq \frac{n}{c(4)r} \), where \( c(4) < e \) is a constant, then almost no \( K_n^r \) has a monochromatic perfect matching. On the other hand, if \( r \leq \frac{n}{\log n + c(5)(n)} \), where \( c(5)(n) \to \infty \), then almost every \( K_n^r \) has a monochromatic perfect matching.

3 Heterochromatic \( k \)-Matchings in \( K_n^r \)

Following the symbols in the previous section, let \( B_i \) be the event that the edges in \( M_i \) are heterochromatic and \( Y_i \) be the indicator variable for the
event $B_i$. That is,
\[ Y_i = \begin{cases} 
1 & \text{if } B_i \text{ happens}, \\
0 & \text{otherwise}.
\end{cases} \] (3.1)

Then for each $1 \leq i \leq q$,
\[ \Pr[Y_i = 1] = \frac{(r)^k}{r^k}. \]

Then the random variable
\[ Y = Y_1 + Y_2 + \cdots + Y_q \]
denotes the number of heterochromatic $k$-matchings in $K_n^r$.

From the linear of the expectation [1],
\[
E(Y) = E(Y_1 + Y_2 + \cdots + Y_q) \\
= \frac{r}{r^k q} \\
= \frac{n!}{(n-2k)!2^k k! (r-k)!r^k}. \] (3.2)

Since $r \geq k$ is a necessary condition in the heterochromatic $k$-matching problem, we have the following assertion for $E(Y)$ by calculation of (3.2).

**Lemma 3.1** For every $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k$, $E(Y) \to \infty$, where $0 < \epsilon < 1$ is a constant that can be arbitrarily small.

The main result of this section is the following theorem:

**Theorem 3.2** If $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k$, where $0 < \epsilon < 1$ is a constant that can be arbitrarily small, then almost every $K_n^r$ contains a heterochromatic $k$-matching.

**Proof.** Similar to Theorem 2.2 for heterochromatic $k$-matchings, the following estimate is for $\Delta' = \sum_{i \neq j} \Pr[B_i \cap B_j]$. 

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\[ \Delta' = \sum_{i \sim j} \Pr[B_i \cap B_j] \]
\[ = \sum_{s=1}^{k-1} \sum_{(i,j)} \frac{(r)_k! (r-s)_{k-s} (k-s)!}{r^{2k-s}} \]
(where \((i,j)\) means the ordered pair of \(M_i\) and \(M_j\) that have \(s\) common edges)
\[ \leq \sum_{s=1}^{k-1} \frac{\binom{n}{s} \binom{n-2}{s} \ldots \binom{n-2(s-1)}{2}}{s!} \frac{\binom{n-2s}{2} \binom{n-2(s+1)}{2} \ldots \binom{n-2(k-1)}{2}}{(k-s)!} \frac{(r)_k! (r-s)_{k-s} (k-s)!}{r^{2k-s}} \]
\[ = \frac{n!r!}{2^{2k} (n-2k)! (n-2k)! (r-k)! (r-k)! r^{2k}} \sum_{s=1}^{k-1} \frac{(n-2s)! (r-s)! 2^s r^s}{s! (k-s)! (k-s)!} . \]

Then
\[ \frac{\Delta'}{(E(Y))^2} \leq \frac{k! k!}{n! r!} \sum_{s=1}^{k-1} \frac{(n-2s)! (r-s)! (2r)^s}{s! (k-s)! (k-s)!} . \] (3.3)

By careful calculation of (3.3), we get if \(k \ll n\), then
\[ \frac{\Delta'}{(E(Y))^2} \leq \frac{k! k!}{n! r!} (1 + o(1)) \frac{2r! (n-2)!}{(k-1)! (k-1)!} \]
\[ = (1 + o(1)) \frac{k^2}{n (n-1)} \to 0. \] (3.4)

From (3.4), Lemma 3.1 and the assertion that if \(E(Y) \to \infty\) and \(\Delta' = o((E(Y))^2)\), then almost surely \(Y > 0\) ([1], page 46), we have
\[ \Pr[Y > 0] \to 1, \]
which finishes the proof. \(\blacksquare\)

**Remark 3.3** As a corollary of Theorem 3.2, if one of \(k\) and \(r(\geq k)\) is fixed, then almost every \(K_n^r\) has a heterochromatic \(k\)-matching. The only left case that we can not deal with is that \(k = c^{(6)} n\), where \(0 < c^{(6)} \leq 1/2\) is a constant.
4 Results on Other Subgraphs

Completely similar to Section 2 and Section 3, we can study monochromatic $k$-clique, $k$-tree and heterochromatic $k$-clique, $k$-tree in $K^r_n$. We list our results here.

**Theorem 4.1** If $r$ is fixed, then

$$\lim_{n \to \infty} Pr[K^r_n \text{ contains a monochromatic } k\text{-clique}] = \begin{cases} 0 & \text{if } k \geq 2\log_r n, \\ 1 & \text{if } k \leq \frac{\log_r n}{1.704 \times 10^9}. \end{cases}$$

**Theorem 4.2** If $k$ is fixed, then

$$\lim_{n \to \infty} Pr[K^r_n \text{ contains a monochromatic } k\text{-clique}] = \begin{cases} 0 & \text{if } r \gg n^{\binom{k}{2}^{-1}}, \\ 1 & \text{if } r \leq n^{\frac{1}{2k!}} n^{\binom{k}{2}^{-1}}. \end{cases}$$

That is to say, $n^{\binom{k}{2}^{-1}}$ is the threshold function for the property that $K^r_n$ has a monochromatic $k$-clique.

**Theorem 4.3** If $r \geq n^{4+\epsilon}$, where $\epsilon > 0$ is a constant that can be arbitrarily small, then for every $k \leq n$, there almost surely exists a heterochromatic $k$-clique in $K^r_n$.

**Theorem 4.4** If $k$ is fixed, then

$$\lim_{n \to \infty} Pr[K^r_n \text{ contains a monochromatic } k\text{-tree}] = \begin{cases} 0 & \text{if } r \gg k^{\binom{n}{k}^{-1}}, \\ 1 & \text{if } r \leq \frac{k}{n} \binom{n}{k}^{1/2}. \end{cases}$$

**Theorem 4.5** If $r \geq c^{(7)} n$, where $c^{(7)} > 1$ is a constant, then almost no $K^r_n$ contains a monochromatic spanning tree.

**Theorem 4.6** If $r$ is fixed, then almost every $K^r_n$ contains a monochromatic $k$-tree for any $2 \leq k \leq n$.

**Theorem 4.7** If $2 \leq k \leq \log n$ and $r \geq k-1$, then almost every $K^r_n$ contains a heterochromatic $k$-tree.
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