ON THE CONJUGACY SEPARABILITY
OF GENERALIZED FREE PRODUCTS OF GROUPS

E. A. Ivanova

It is proved that generalized free product of two finite $p$-groups is a conjugacy $p$-separable group if and only if it is residually finite $p$-groups. This result is then applied to establish some sufficient conditions for conjugacy $p$-separability of generalized free product of infinite groups.

1. A group $G$ is called conjugacy separable (conjugacy $p$-separable) group if whenever elements $a$ and $b$ of $G$ are not conjugate in $G$, there is a homomorphism $\varphi$ of $G$ onto finite (respectively, finite $p$-group) $X$ such that elements $a\varphi$ and $b\varphi$ are not conjugate in $X$.

It is easy to see that a conjugacy separable group is residually finite group and a conjugacy $p$-separable group is residually finite $p$-groups. In general inverse statements are not true. But in some cases residually finite group (residually finite $p$-groups) is a conjugacy separable (respectively, conjugacy $p$-separable) group too. For example, J. Dyer [7] have proved that a free product with amalgamated subgroups of two finite groups is conjugacy separable group (in 1963 G. Baumslag [5] showed that any such group is residually finite).

Not every free product with amalgamated subgroups of two finite $p$-groups must be residually finite $p$-groups. G.Higman [9] have obtained necessary and sufficient conditions for such groups to be residually finite $p$-groups. The question arises whether these conditions are enough for such group to be a conjugacy $p$-separable group.

An element $g$ of a group $G$ is called $C_{fp}$-separable if for any $a \in G$ such that elements $a$ and $g$ are not conjugate in $G$, there exists a homomorphism $\varphi$ of $G$ onto finite $p$-group $X$ such that $a\varphi$ and $g\varphi$ are not conjugate in $X$. So a group $G$ is conjugacy $p$-separable if and only if every element $g \in G$ is $C_{fp}$-separable.

It was proved in [1] that if a free product with amalgamated subgroups of two finite $p$-groups is a residually finite $p$-group then every infinite order element $g \in G$ is $C_{fp}$-separable.

In fact the following generalization of this statement holds:
Theorem 1. Suppose \( G = (H * K; A = B, \varphi) \) is a free product of two finite \( p \)-groups \( H \) and \( K \) with amalgamated via isomorphism \( \varphi \) subgroups \( A \) and \( B \). \( G \) is a conjugacy \( p \)-separable group if and only if \( G \) is a residually finite \( p \)-groups.

Applying this result and using a standard technique we proved the following theorem:

Theorem 2. Suppose that \( H \) and \( K \) are conjugacy \( p \)-separable groups, \( A \leq H \) and \( B \leq K \) are central subgroups and every finite \( p \)-index subgroup of \( A \) and of \( B \) is \( p \)-separable in \( H \) and in \( K \) respectively. Then \( G = (H * K; A = B, \varphi) \) is a conjugacy \( p \)-separable group.

(Recall that a subset \( M \) of a group \( G \) is called \( p \)-separable if for every \( a \in G \), \( a \notin M \), there is a homomorphism \( \varphi \) of \( G \) onto finite \( p \)-group \( X \) such that \( a\varphi \notin M\varphi \).)

The description of conjugacy \( p \)-separable finitely generated nilpotent groups has been given in [2]. It follows from the theorem 2 that the following statement holds:

Theorem 3. Suppose \( G = (H * K; A = B, \varphi) \) is a free product with amalgamated subgroups of two finitely generated nilpotent groups \( H \) and \( K \), \( H \) and \( K \) are conjugacy \( p \)-separable groups, \( A \) and \( B \) are \( p' \)-isolated central subgroups of \( H \) and \( K \) respectively. Then \( G \) is a conjugacy \( p \)-separable group.

(Recall that if \( p \) is a prime then a subgroup \( X \) of a group \( Y \) is called \( p \)-isolated if \( y^p \in X \) implies \( y \in X \) for every \( y \in Y \). A subgroup \( X \) of a group \( Y \) is called \( p' \)-isolated if \( X \) is a \( q \)-isolated for every prime \( q \neq p \).)

2. The proof of theorem 1 is a certain modification of the J. Dyer’s proof of her result in [8].

Graph \( \Gamma \) is a system of two sets \( V = V(\Gamma) \) (the set of vertexes) and \( E = E(\Gamma) \) (the set of edges) and of three mappings : \( E \to E, \ o : E \to V \) and \( t : E \to V \) such that \( o(\overline{e}) = t(e), t(\overline{e}) = o(e), \overline{e} \neq e \) and \( \overline{\overline{e}} = e \) for every \( e \in E \). The edge \( \overline{e} \) is called inverse to edge \( e \), the vertex \( o(e) \in V \) is called the origin of edge \( e \), the vertex \( t(e) \in V \) is called the end of edge \( e \in E \). If \( o(e) = u \) and \( t(e) = v \) then we write \( e = (u,v) \).

The group graph is a pair \((G, \Gamma)\) of connected graph \( \Gamma \) and mapping \( G \). The mapping \( G \) associates every vertex \( v \in V(\Gamma) \) with group \( G_v \).
and every edge $e \in E(\Gamma)$, $e = (u,v)$ with group $G_e$ and two mappings

$$\rho_e : G_e \rightarrow G_u \text{ and } \tau_e : G_e \rightarrow G_v$$

such that $G_e = G_e$, $\rho_e = \tau_e$ and $\tau_e = \rho_e$.

The groups $G_v$ and $G_e$ are called vertex group and edge group of a group graph $(G, \Gamma)$ respectively.

Suppose $(G, \Gamma)$ is a group graph and $T$ is a maximal tree of $\Gamma$. Let $X_v$ be a set of generators and $R_v$ be a set of relations of the vertex group $G_v$ for every vertex $v \in V = V(\Gamma)$ (and if $v_1 \neq v_2$ then $X_{v_1} \cap X_{v_2} = \emptyset$).

The fundamental group $\pi(G, \Gamma)$ of a group graph $(G, \Gamma)$ is a group with generators $\bigcup_{v \in V} X_v$ and $t_e$, $e \in E(\Gamma) \setminus E(T)$, and relations $\bigcup_{v \in V} R_v$ and

$$g = g(\rho_e^{-1}\tau_e), \quad e \in E(T), \quad g \in G_e \rho_e,$$
$$t_e^{-1}gt_e = g(\rho_e^{-1}\tau_e), \quad e \in E(\Gamma) \setminus E(T), \quad g \in G_e \rho_e,$$
$$t_e = t_e^{-1}, \quad e \in E(\Gamma) \setminus E(T).$$

It is possible to prove that the group $\pi(G, \Gamma)$ does not depend on a choice of vertex groups presentations and maximal tree $T$.

It is well known ([6, 10, 11]) that every finite extension of a free group is isomorphic to a fundamental group of a group graph with finite vertex groups.

To prove the theorem 1 the following result is also required ([1]):

**Proposition 2.1.** Suppose $H \leq G$ is a subnormal finite $p$-index subgroup. If $h \in H$ is a $C_{fp}$-separable in group $H$ then $h$ is $C_{fp}$-separable in group $G$.

Now we are ready to prove the theorem 1.

Suppose $G = (H \ast K; A = B, \varphi)$ is a free product of two finite $p$-groups $H$ and $K$ with amalgamated via isomorphism $\varphi$ subgroups $A$ and $B$. The necessary conditions in theorem are evident.

Let $G$ be a residually finite $p$-groups. Then ([4, lemma 2.1]) $G$ is an extension of a free group $F$ by finite $p$-group. In view of mentioned above result from [1] to prove that $G$ is a conjugacy $p$-separable it is enough to show that whenever $a$ and $b$ are finite order elements and not conjugate in $G$, there is a homomorphism $\psi$ of $G$ onto a finite $p$-group $X$ such that $a\varphi$ and $b\varphi$ are not conjugate in $X$.

Since subgroup of $G$ which is generated by $F$ and $a$ is a subnormal in $G$ it follows from proposition 2.1 that we can assume that group $G$ is generated by $F$ and $a$. Then the quotient group $G/F$ is cyclic and
therefore if \( aF \neq bF \) then natural homomorphism of \( G \) onto \( G/F \) is required.

Suppose \( aF = bF \). Since \( F \) is a torsion-free so the orders of \( a \) and \( b \) are equal to \( p^n \) \((n \geq 1)\). By remark above the group \( G \) is isomorphic to a fundamental group of a group graph and its vertex subgroups can be embedded into the cyclic group of order \( p^n \). D. Dayer [8] showed that in this case there is a homomorphism \( \psi \) of \( G \) onto fundamental group \( H = \pi(\mathcal{H}, \Gamma) \) of a group graph \( (\mathcal{H}, \Gamma) \) such that

1) the graph \( \Gamma \) has only two vertexes \( u \) and \( v \); 
2) the vertex groups \( H_u \) and \( H_v \) are cyclic of order \( p^n \) and they are generated by elements \( x = a\varphi \) and \( y = b\varphi \) respectively; 
3) the order of every edge group \( H_e \) is less than \( p^n \).

Thus the generators of \( H \) are \( x, y \) and \( t_e \), where edge \( e \in E(\Gamma) \) is not equal to some fixed edge, the relations of \( H \) are:

a) \( x^{p^n} = 1, y^{p^n} = 1 \); 
b) \( x^r = y^s \), where elements \( x^r \in H_u \) and \( y^s \in H_v \) have the same order which is less than \( p^n \); 
c) \( t_e^{-1}h_1t_e = h_2 \), where the elements \( h_1 \in H_u \) and \( h_2 \in H_v \) have the same order which is less than \( p^n \).

By condition 3) every element \( x^r, h_1 \in H_u, y^s, h_2 \in H_v \) from relations b) and c) belongs to subgroup \( K_u \) or \( K_v \) respectively, where orders of \( K_u \) and \( K_v \) are equal to \( p^k, k < n \). Therefore relations b) and c) are trivial in \( H \) modulo \( N \) where \( N \) is a normal closure of \( K_u \) and \( K_v \). Thus the quotient-group \( H/N \) is a free product of finite cyclic groups \( H_u/K_u = (x) \) and \( H_v/K_v = (y) \), which orders are equal to \( p^{n-k} \), and the set of infinite cyclic groups \( (t_e) \). The composition of \( \psi \), natural homomorphism of \( H \) onto \( H/N \) and an evident homomorphism of \( H/N \) onto direct product of the groups \( H_u/K_u \) and \( H_v/K_v \) is required homomorphism of \( G \). The theorem 1 is proved.

3. Suppose \( H \) and \( K \) are groups, \( A \) is a subgroup of \( H \), \( B \) is a subgroup of \( K \), \( \varphi : A \to B \) is a isomorphism.

Every element \( x \) from \( G = (H \ast K; \ A = B, \ \varphi) \) can be presented as \( x = x_1x_2 \ldots x_n \), where each \( x_1, x_2, \ldots, x_n \) is from factor \( H \) or \( K \) and if \( n > 1 \) then \( x_i \) and \( x_{i+1} \) are from different factors for every \( i = 1, \ldots, n-1 \) (therefore they do not belong to \( A \) and \( B \)). This presentation is called reduced form of \( x \) and the number \( n \) (that doesn’t depend of choice of such presentation) is called the length of \( x \). An element \( x \in G \) is called
cyclically reduced if either its length $n$ equals to 1 or $n > 1$ and elements $x_1$ and $x_n$ in its reduced form are from different factors $H$ and $K$. In this case the expression $u_i = x_ix_{i+1}\ldots x_nx_1\ldots x_{i-1}$ is reduced for each $i = 1, 2, \ldots, n$. The element $u_i$ is called a cyclic permutation of $x$ (if $n = 1$ then $x$ is a unique cyclic permutation of $x$).

When amalgamating subgroups $A$ and $B$ are central then the general conditions for two elements from $G$ to be conjugate ([3]) can be simplified:

**Proposition 3.1.** Suppose $G = (H \ast K; A = B, \varphi)$ is a free product of two groups with amalgamated central subgroups $A$ and $B$. For every $g \in G$ there is a cyclically reduced $x \in G$ such that $g$ and $x$ are conjugate. Suppose $x \in G$ and $y \in G$ are cyclically reduced. Then $x$ and $y$ are conjugate in $G$ if and only if their length are equal and either they are from one factor $H$ or $K$ and conjugate in it, or their lengths more then 1 and one of these elements equals to the cyclic permutation of another.

Let’s remind also the following notion ([5]). Subgroups $R \leq H$ and $S \leq K$ are called $(A, B, \varphi)$-compatible if $(A \cap R)\varphi = B \cap S$. If normal subgroups $R \leq H$ and $S \leq K$ are $(A, B, \varphi)$-compatible then the mapping $\varphi_{R,S} : AR/R \to BS/S$, where $(aR)\varphi_{R,S} = (a\varphi)S$ ($a \in A$), is an isomorphism of subgroup $AR/R \leq H/R$ onto subgroup $BS/S \leq K/S$. Thus there is a free product

$$G_{R,S} = (H/R \ast K/S; AR/R = BS/S, \varphi_{R,S})$$

of the groups $H/R$ and $K/S$ with amalgamated via isomorphism $\varphi_{R,S}$ subgroups $AR/R$ and $BS/S$. Natural homomorphisms of the group $H$ onto quotient group $H/R$ and of the group $K$ onto quotient group $K/S$ can be extended to a homomorphism $\rho_{R,S}$ of the group $G = (H \ast K; A = B, \varphi)$ onto the group $G_{R,S}$.

**Proposition 3.2.** Suppose that $H$ and $K$ are conjugacy $p$-separable groups, $A \leq H$ and $B \leq K$ are central subgroups and subgroups $A$ and $B$ and every finite $p$-index subgroup of $A$ and $B$ are $p$-separable in group $H$ and $K$ respectively. Then for every finite $p$-index normal subgroups $M \leq H$ and $N \leq K$ there are $(A, B, \varphi)$-compatible finite $p$-index normal subgroups $R \leq H$ and $S \leq K$ such that $R \leq M$ and $S \leq N$.

**Proof.** Suppose $M$ contains a finite $p$-index (in $A$) subgroup $U \leq A$. Then subgroup $U$ is $p$-separable in the group $H$ and therefore the quotient
group $H/U$ is residually finite $p$-groups. Also since the quotient group $A/U$ is a finite subgroup of the quotient group $H/U$, therefore there is a finite $p$-index normal subgroup $R/U$ of the group $H/U$ such that $R/U \cap A/U = 1$. Then $R$ is a finite $p$-index normal subgroup of the group $H$ and $R \cap A = U$. We can consider also that $R \leq M$. The similar reasoning is fair and for the group $K$.

Suppose $M$ and $N$ are finite $p$-index normal subgroups of the groups $H$ and $K$ respectively, $U = (M \cap A) \cap (N \cap B)\varphi^{-1}$ and $V = (M \cap A)\varphi \cap (N \cap B)$. Then there are finite $p$-index normal subgroups $H$ and $K$ of the groups $R$ and $S$ respectively such that $R \leq M$, $R \cap A = U$, $KS \leq N$ and $S \cap B = V$. Since $U\varphi = V$ the subgroups $R$ and $S$ are required.

Now we are ready to prove the theorem 2. Suppose $H$ and $K$ are conjugacy $p$-separable groups, $G = (H \ast K; A = B, \varphi)$ is a free product of the groups $H$ and $K$ with amalgamated central subgroups $A$ and $B$.

Using standard methods of the proof for the free product of two groups with amalgamated subgroups to be residually finite $p$-groups it is easy to receive the following statement:

**Proposition 3.3.** Suppose $H$ and $K$ are residually finite $p$-groups, $A \leq H$ and $B \leq K$ are central subgroups and subgroups $A$ and $B$ and every finite $p$-index subgroup of $A$ and $B$ are $p$-separable in group $H$ and $K$ respectively. Then $G = (H \ast K; A = B, \varphi)$ is a residually finite $p$-groups.

It follows from the theorem 1 that if $R \leq H$ and $S \leq K$ are $(A, B, \varphi)$-compatible finite $p$-index normal subgroups then $G_{R,S}$ is a conjugacy $p$-separable group (it is proved in [9] that $G_{R,S}$ is a residually finite $p$-groups). Therefore it is enough to prove that whenever $x \in G$ and $y \in G$ are not conjugate in $G$ there are $(A, B, \varphi)$-compatible finite $p$-index normal subgroups $R \leq H$ and $S \leq K$ such that $x\rho_{R,S}$ and $y\rho_{R,S}$ are not conjugate in $G_{R,S}$.

Suppose $f \in G$ and $g \in G$ are not conjugate in $G$. Since proposition 3.1 we can assume without generality loss that $f$ and $g$ are cyclically reduced. Let’s consider some cases.

**Case 1.** The lengths of $f$ and $g$ are not equal.

Since the subgroups $A$ and $B$ are $p$-separable in the groups $H$ and $K$ respectively there are finite $p$-index normal subgroups $M \leq H$ and $N \leq K$ such that all factors in the reduced forms of $f$ and $g$ (it is fair
only for the elements of length 1) are not from AM and BN respectively. Since proposition 3.2 there are \((A, B, \varphi)\)-compatible finite \(p\)-index normal subgroups \(R \leq H\) and \(S \leq K\) such that \(R \leq M\) and \(S \leq N\). Then \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are cyclically reduced in the group \(G_{R,S}\), the lengths of \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are equal to the lengths of \(f\) and \(g\) respectively and different. Therefore since proposition 3.1 \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are not conjugate in \(G_{R,S}\).

**Case 2.** The lengths of \(f\) and \(g\) are equal to 1, \(f\) and \(g\) are from different factors \(H\) and \(K\).

Suppose \(f \in H \setminus A\) and \(g \in K \setminus B\). Since \(A\) and \(B\) are \(p\)-separable subgroups in \(H\) and \(K\) respectively there are finite \(p\)-index normal subgroups \(M \leq H\) and \(N \leq K\) such that \(f \notin AM\) and \(g \notin BN\). At that time since proposition 3.2 there are \((A, B, \varphi)\)-compatible finite \(p\)-index normal subgroups \(R \leq H\) and \(S \leq K\) such that \(R \leq M\) and \(S \leq N\). Then \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are from different factors \(H/R\) and \(K/S\) of \(G_{R,S}\) and since proposition 3.1 the elements \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are not conjugate in \(G_{R,S}\).

**Case 3.** The lengths of \(f\) and \(g\) are equal to 1, \(f\) and \(g\) are both from one factor \(H\) or \(K\).

Suppose \(f \in H\) and \(g \in H\). Since \(f\) and \(g\) are not conjugate in \(H\) and \(H\) is a conjugacy \(p\)-separable group there is a finite \(p\)-index normal subgroup \(M \leq H\) such that \(fM\) and \(gM\) are not conjugate in \(H/M\). At that time since proposition 3.2 there are \((A, B, \varphi)\)-compatible finite \(p\)-index normal subgroups \(R \leq H\) and \(S \leq K\) such that \(R \leq M\) and \(S \leq N\). Then \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are both from the factor \(H/R\) of the group \(G_{R,S}\) and the elements \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are not conjugate in \(H/R\). Therefore since proposition 3.1 \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are not conjugate in \(G_{R,S}\).

**Case 4.** The lengths of \(f\) and \(g\) are equal and more than 1.

Suppose \(g_1, g_2, \ldots, g_r\) \((r\) is a length of \(g\)) are all cyclic permutations of \(g\). Since the subgroups \(A\) and \(B\) are \(p\)-separable in the groups \(H\) and \(K\) respectively there are finite \(p\)-index normal subgroups \(R_0 \leq H\) and \(S_0 \leq K\) such that all factors in the reduced forms of \(f\) and \(g\) are not from \(AR_0\) and \(BS_0\) respectively. Since \(f\) is not equal to \(g_1, g_2, \ldots, g_r\) and \(G\) is a residually finite \(p\)-groups (proposition 3.3) there is a finite \(p\)-index normal subgroup \(N \leq G\) such that \(fN\) is not equal to \(g_1N, g_2N, \ldots, g_rN\). Suppose \(R = R_0 \cap N\) and \(S = S_0 \cap N\). Then \(f_\rho_{R,S}\) and \(g_\rho_{R,S}\) are cyclically reduced in \(G_{R,S}\), their lengths are equal to \(r\) and \(f_\rho_{R,S}\) is not equal to \(g_1\rho_{R,S}, g_2\rho_{R,S}, \ldots, g_r\rho_{R,S}\). Since every cyclic permutation
of $g_{R,S}$ is equal to one of the elements $g_1\rho_{R,S}$, $g_2\rho_{R,S}$, ..., $g_r\rho_{R,S}$ the elements $f\rho_{R,S}$ and $g\rho_{R,S}$ are not conjugate in $G_{R,S}$ (proposition 3.1).

The theorem 2 is proved.

Since every $p'$-isolated subgroup of a finitely generated nilpotent group is $p$-separable the theorem 3 immediately follows from the theorem 2.

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