**Abstract**

On the basis of the S-wave phase shifts $\pi\pi$ scattering behaviour from the threshold up to dipion mass $m_{\pi\pi} = 1$ GeV, it is shown, that the linear correlation relates the S-wave phase shifts: $\delta^0_0(s) = \eta \delta^2_0(s)$, where $\eta = -4.65 \pm 0.05$. By using this correlation at the solution of the Roy equations, the accuracy of determination of S-wave lengths $a^0_0$ and $a^2_0$ is considerably improved: $a^0_0 = (0.220 \pm 0.006) m^{-1}_\pi$; $a^2_0 = (-0.0472 \pm 0.0013) m^{-1}_\pi$. The obtained result unambiguously witnesses in favor of the standard ChPT with large quark condensate.

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1 Introduction

The chiral symmetry of QCD Lagrangian is well known to be spontaneously broken down. Two theories - Chiral Perturbation Theory (ChPT) [1,2] and Generalised Chiral Perturbation Theory (GChPT) [3] based on QCD - can describe the strong interactions at low energies. The determinative factor in these theories is the existence of vacuum condensates violating chiral symmetry. These theories having the same form of the effective Lagrangian differ from each other by value of quark condensate and light quark masses. The fact determining the choice of the version is that the S-wave $\pi\pi$ scattering lengths $a_0^0$ and $a_0^2$ are very sensitive to the parameters of the model and consequently are the key parameters for unambiguous determination of the scenario of chiral symmetry violation. So, ChPT predicts the value $a_0^0=0.220$ and GChPT $a_0^2 = 0.263^1$.

During some time, despite large accumulated experimental material on scattering lengths, this choice was difficult to be made. The matter is that the experiment $K_{e4}$ [4] gave evidence in favor of GChPT, whereas most $\pi N \rightarrow \pi\pi N$ experiments inclined rather to ChPT. The situation has changed recently as the latest experiment $K_{e4}$ E865 [5] showed evidence for ChPT, i.e. for the version with strong quark condensate and small masses of light quarks. The present work shows that even without using the latest $K_{e4}$ results, based only on the information contained in experimental S-wave $\pi\pi$ scattering data array, it is possible to improve considerably the accuracy of determination of $a_0^0$ and $a_0^2$ and thus to choose the scenario of chiral symmetry violation for certain.

2 Roy equations

In our previous work [6] the Roy equations [7,8] were used to calculate the S-wave $\pi\pi$ scattering lengths. The experimental values of the S- and P-wave $\pi\pi$ phase shifts, obtained from an analysis of five charged channels, were presented by that time in all dipion mass region from the threshold up to $m_{\pi\pi} = 1$ GeV [9-22]. Based on this fact we could realize the consecutive solution technique of the Roy equations without using iterative procedures. Due to this approach the problem of convergence of the solutions was eliminated automatically and the process of calculation of scattering lengths $a_0^0$ and $a_0^2$ becomes absolutely clear. For the case of the charged pions, the Roy equations are given by:

$$\text{Ref}_l(s) = \lambda_l(s) + \frac{1}{\pi} \int_{4}^{51} \Psi(x, s) dx + \varphi_l(s)$$

(1)

Explicit expressions for $\Psi(x, s^2)$ are given in [6]. The corrections $\varphi_l(s)$, estimating the contributions from the higher waves ($l \geq 2$) and from the large mass region were adopted from [23]. According to the theory [8] subtractions $\lambda_l^0(s)$ are defined as follow:

$$\lambda_0^0(s) = a_0^0 + \frac{s - 4}{12} (2a_0^0 - 5a_0^2); \quad \lambda_0^2(s) = a_0^2 - \frac{s - 4}{24} (2a_0^0 - 5a_0^2)$$

(2)

The solution of the Roy equations (1) comprised some steps. First, we performed fitting for each phase shifts $\delta_l^j$ and obtained smooth curves adequately describing experimental

\footnote{The S-wave scattering lengths $a_0^0$ and $a_0^2$ are given in $m_{\pi}^{-1}$}

\footnote{Here and below, $s$ is the Mandelstam variable, $s = m_{\pi\pi}^2/m_{\pi}^2$}
Figure 1: S-wave $\pi\pi$ phase shifts. The solid curves represent the result of fitting in terms of expression (3).

In particular for the S-wave phase shifts expansion (3) was used:

$$\delta_0^I(s) = \frac{2}{\sqrt{s}} (C_1^I q + C_2^I q^3 + C_3^I q^5 + C_4^I q^7)$$

(3)

Where $q = \frac{1}{2}\sqrt{s - 4}$ - is c.m. pion momentum and $C_k^I$ - are free parameters ($I=0,2$; $k = 1 \div 4$). Experimental values of phase shifts and fitting curves (3) are shown in Fig.1. Then, the resulting smooth curve $\delta_0^I(s)$ were input into the Roy equations (1) to find subtractions $\lambda_0^I(s)$. At the conclusive stage we carried out fitting of the subtractions $\lambda_0^I(s)$ using terms (2) and determined the S-wave $\pi\pi$ lengths. The resulting subtractions $\lambda_0^I(s)$ with fitting straight lines (2) are shown in Fig.2. Note that the phase shifts $\delta_0^2$ from "electronic experiment" [18] were not used. This question was considered in [6] in detail.

We stress that the linear relations were obtained indeed when fitting subtractions $\lambda_0^I(s)$. For us, it is a proof of that all the calculation steps of the solving the Roy equations and also all the preliminary work comprising the fitting phase shifts $\delta_0^I(s)$ in terms (3) were carried out correctly. In our previous study [6] we obtained:

$$a_0^0 = 0.240 \pm 0.023; \quad a_0^2 = -0.034 \pm 0.013$$

(4)

with correlation coefficient $r=0.945$. We stress that in [6] we made use of all available experimental data, carried out the consequent analysis, but resulting uncertainties of $a_0^0$
and $a_0^2$ were so large that they did not allow to choose between ChPT and GChPT.

3 Correlation between $\delta_0^0(s)$ and $\delta_0^2(s)$

The analysis which has been carried out showed that the general statistics contained in the available data for the phase shifts is enough to improve significantly the accuracy of determination of the S-wave scattering lengths and thus to choose correct ChPT version. But the strong correlation between $a_0^0$ and $a_0^2$, which is present in any method of the solution of the Roy equations, in a latent form may be, prevents one to make the above and results in large uncertainties of the values $a_0^0$ and $a_0^2$.

It should be noted, that the correlation between $a_0^0$ and $a_0^2$ is present in both scenarios ChPT and GChPT [24] due to causes of fundamental nature. The principal conclusion can be made that using model-independent analysis, based solely on the Roy equations, it is impossible to increase the accuracy of definition of the S-wave $\pi\pi$ lengths $a_0^0$ and $a_0^2$ without removing beforehand the correlation between them. Therefore we need some additional constraint on $a_0^0$ and $a_0^2$. And we shall demonstrate that such constraint can be retrieved from the available data.

Let us analyse our previous result (4). As the correlation coefficient $r$ is very close to one,
Figure 3: The ratio of the S-wave phase shifts $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$. The straight line represents the constant $\eta = -4.65$.

it implies the values $a_0^0$ and $a_0^2$ are related by linear dependence. Then, as $\delta_0^0(s) \propto a_0^0 q$ near the threshold, the phase shifts $\delta_0^0(s)$ and $\delta_0^2(s)$ must be related by linear dependence too. In this way the simplest hypothesis, i.e. the hypothesis about proportionality phase shifts into some area above the threshold, must be verify. For this the ratio $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$ was analysed for the available experimental data. As the phase shifts $\delta_0^0(s_i)$ and $\delta_0^2(s_j)$ were measured mainly at different energy values, the smoothed curve (Fig.1) representing the fitting function (3) was used for calculation of the phase shifts $\delta_0^2$ at the points $s = s_i$, where the phase shifts $\delta_0^0$ were measured. Thus, the ratio of the S-wave phase shifts was calculated as $\xi(s_i) = \delta_0^0(s_i)/\delta_0^2(s = s_i)$. The values $\xi(s_i)$ are given in Fig.3.

The uncertainties $\sigma_\xi$ were calculated by the standard rule of propagation of errors and finally they were defined both by errors of phase shifts $\delta_0^0(s_i)$ and $\delta_0^2(s_j)$.

The form of the dependence $\xi(s)$ up to $s=42$, i.e. up to $m_{3\pi}=900\text{MeV}$, makes one think that $\xi(s)$ is really a constant in this energy region. It witnesses in favor of the hypothesis being verified. It was calculated by fitting $\xi(s) \equiv \eta \text{ const}$ for the interval $S = 11 \div 42.5$: $\eta = -4.65 \pm 0.05$; $\chi^2 = 75$; $\langle \chi^2 \rangle = 80$. So, the proposed hypothesis is proved by statistics and consequently we can conclude that in the wide enough region the ratio of the S-wave
Figure 4: The $\pi\pi$ scattering lengths. The ellipse with the centre as a triangle indicates our previous results [6]; the ellipse with the centre as a rhomb shows the result of the present paper. The cross shows the latest ChPT result [27]; the square shows GChPT result [3].

phase shifts does not depend on energy. Thus the new correlation take place for this energy region:

$$\delta_0^0(s) = \eta \delta_0^2(s)$$

where $\eta = -4.65 \pm 0.05$. As stated above, from the fact of strong correlation of the scattering lengths follows the linear dependence of the phase shifts near the threshold and such dependence is found in some region above the threshold. Based on these facts we suppose that the phase shifts proportionality keeps constant for all the energy region up to $s=42.5$. So we extrapolate the relation (5) down to the threshold and assume that the factor of proportionality $\eta$ keeps its value in close vicinity to the threshold. The fitting $\xi(s) \equiv \eta \text{--const}$ for the interval $s = 4 \div 42.5$ gets naturally the same value of $\eta$, because statistical weights of the points near the threshold are insignificant. In general, vast uncertainties of the values $\xi(s)$ near the threshold (Fig.3) are caused by the fact that
the phase shifts $\delta_0^0(s)$ and $\delta_2^0(s)$ have large relative errors in that region. And apparently this situation will not change in the nearest future. If the relation (5) is true near the threshold, as we assume, then we obtain the new constraint on the scattering lengths:

$$a_0^0 = \eta a_0^2$$

(6)

This relation can be used to improve the accuracy of determination of the S-wave lengths.

4 Calculation of scattering lengths $a_0^0$ and $a_0^2$

First, the estimation of the values $a_0^0$ and $a_0^2$ was made without using the Roy equations. Two equations, the correlation (6) and the equation of the "universal curve" [25] were solved together. As a result we had:

$$a_0^0 = 0.20 \pm 0.02; \quad a_0^2 = -0.042 \pm 0.005$$

(7)

It is necessary to note, that in the work [26] in which the "universal curve" was introduced, attempt also was made to estimate independently the ratio $a_0^0/a_0^2$ for further calculating the S-wave lengths $a_0^0$ and $a_0^2$.

Then, the correlation (6) was used as a supplementary condition when solving the Roy equations. And we obtained at the end:

$$a_0^0 = 0.220 \pm 0.006; \quad a_0^2 = -0.0472 \pm 0.0013$$

(8)

The final result (8) together with our previous result (4) as well as the theoretical predictions of ChPT and GChPT are presented in Fig.4.

5 Conclusion

1) On the basis of the consequent statistical analysis of the S-wave phase shifts $\pi\pi$ scattering behaviour the new correlation (5) has been obtained which relates the S-wave phase shifts $\delta_0^0(s)$ and $\delta_0^2(s)$ from the threshold up to $m_{\pi\pi} = 900\,\text{MeV}$. We believe that this correlation has value in itself for understanding the mechanism of strong interactions.

2) The new constraint (6) on the S-wave scattering lengths $a_0^0$ and $a_0^2$ has been computed by extrapolation of the relation (5) to the threshold.

3) The constraint (6) has been used as a supplementary condition when solving the Roy equations (1). As the result the exact values of S-wave scattering lengths have been calculated: $a_0^0 = 0.220 \pm 0.006; \quad a_0^2 = -0.0472 \pm 0.0013$.

4) Comparison with the results of the work [27]: $a_0^0 = 0.220 \pm 0.005; \quad a_0^2 = -0.0444 \pm 0.0010$, where ChPT calculations were supplemented with the phenomenological representations [25], shows that one more argument is found for the standard ChPT [1,2] with small light quark masses and large quark condensate.

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