Stability and oscillation of linear delay differential equations

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Abstract
There is a close connection between stability and oscillation of delay differential equations. For the first-order equation
\[ x'(t) + c(t)x(\tau(t)) = 0, \quad t \geq 0, \]
where \( c \) is locally integrable of any sign, \( \tau(t) \leq t \) is Lebesgue measurable, \( \lim_{t \to \infty} \tau(t) = \infty \), we obtain sharp results, relating the speed of oscillation and stability. We thus unify the classical results of Myshkis and Lillo. We also generalise the \( 3/2 \)-stability criterion to the case of measurable parameters, improving \( 1 + 1/e \) to the sharp \( 3/2 \) constant.

Keywords: stability, first-order delay differential equation, oscillation, asymptotic behavior, oscillating coefficient
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1. Introduction

The first sharp stability condition for the non-autonomous equation with one delay term
\[ x'(t) + c(t)x(\tau(t)) = 0, \quad t \geq 0, \quad (1) \]
where \( c : [0, +\infty) \to \mathbb{R} \) and \( \tau : [0, +\infty) \to \mathbb{R} \) are continuous, \( \tau(t) \leq t \), goes back to the work of Myshkis \[28\], \[29\]. It states that if \( \inf_{t \geq 0} c(t) > 0, \forall t \in [0, +\infty) \) and
\[ \sup_{t \in [0, +\infty)} (t - \tau(t)) \cdot \sup_{t \in [0, +\infty)} c(t) < \frac{3}{2}, \quad (2) \]
all solutions of (1) tend to zero as \( t \to \infty \). Lillo \[26\], Yorke \[30\], Yoneyama \[33\], Gusarenko and Domoshnitskii \[16\], successively extended criterion (2), proving, among other conditions, that (2) can be replaced (when \( c(t) > 0, \forall t \in [0, +\infty) \) and the function \( t \mapsto \int_{\tau(t)}^{t} c(s)ds \) is continuous) by \( \limsup_{t \to \infty} \int_{\tau(t)}^{t} c(s)ds < \frac{3}{2} \).

Note that the constant \( \frac{3}{2} \) is sharp. According to the results of Ladas, Sficas and Stavroulakis \[25\], Malygina \[27\], Győri and Hartung \[20\], it can however be improved to \( \frac{\pi}{2} \) when the function \( t \mapsto \int_{\tau(t)}^{t} c(s)ds \) is sufficiently close to a constant.

For a non-negative coefficient \( c \), any positive solution of (1) is non-increasing and, for \( \int_{0}^{\infty} c(s)ds = \infty \), tends to zero as \( t \to \infty \), and any negative solution is non-decreasing. A function \( x : [0, +\infty) \to \mathbb{R} \) is called oscillatory if it has arbitrarily large zeros. For studying asymptotics of a solution of (1) with a non-negative coefficient, it is essentially sufficient to consider oscillatory solutions.

Myshkis also studied (\[28\] Theorem 12, \[29\] Theorem 12), supposing either \( c(t) \geq 0, \forall t \in [0, +\infty) \) or \( c(t) \leq 0, \forall t \in [0, +\infty) \) and
\[ \tau_{\max} := \sup_{t \in [0, +\infty)} (t - \tau(t)) < \infty, \quad (3) \]
solutions that have at least one zero in every interval \([t, t + \tau_{\max}]\) of length \( \tau_{\max} \). Such solutions are usually referred to as rapidly oscillating, in contrast to those for which there are always long enough (exceeding
the delay) intervals, where \( x(t) \) keeps its sign. These solutions are known as slowly oscillating. For (1) with non-negative coefficients, existence of a slowly oscillating solution implies oscillation of all solutions, see Agarwal et al. [1, Theorem 2.23, p.51]. As for (1) with non-positive coefficients, all oscillatory solutions are rapidly oscillating, see the summary in Myshkis [28, p.85].

If
\[
\tau_{\text{max}} \cdot \sup_{t \in [0, +\infty)} |c(t)| < 2,
\]
all rapidly oscillating solutions tend to zero as \( t \to \infty \). An example is given in Myshkis [28], showing that 2 is the best possible constant in (1).

Lillo [26], based on previous results by Buchanan [10], extended Myshkis’ results and proved that in the case \(-1 \leq c(t) \leq 0, \forall t \in [0, +\infty)\), the inequality
\[
\tau_{\text{max}} < 2.75 + \ln 2
\]
implies that all oscillatory solutions of (1) tend to zero.

We note that condition (2) of Myshkis [28, 29], Lillo [26], Yorke [36], condition (1) of Myshkis [28, 29], condition (5) of Lillo [26] are all three related to certain limit-case periodic solutions of (1).

As for the case of (1) with a sign-changing coefficient and multiple delays
\[
x'(t) + \sum_{i=1}^{n} c_i(t)x(\tau_i(t)) = 0
\]
the study of stability of solutions is considerably more complex.

The first asymptotic results were obtained in the case of “a small delay”, by comparison with the solutions of some simpler equations, see paragraph 10 in [28], as well as Uvarov [32], Driver, Sasser and Slater [13] and Driver [14]. Similar asymptotic results were later obtained concerning (3) with \( n = 1 \) by Győri and Pituk [18, 21], Pituk and Röst [31] and with \( n = 2 \) by Haddock and Sacker [22], Atkinson and Haddock [1], Győri [17], Arino and Pituk [2], Diblík [11]. For \( n \geq 2 \), Haddock and Kuang [23] generalise the results of [1, 2, 11], and Faria and Huang [15] generalise those of [28, 32, 13, 14, 17]. Furthermore, Krisztin [24], So, Yu and Chen [31], Yoneyama and Sugie [33, 35], using Yorke’s method, extend the 3/2-condition to multiple delays.

The technique of Azbelev, Berezansky, and Rakhmatullina [4] was applied by Gusarenko and Domoshnitskii [16], and later by Berezansky and Braverman [6, 7, 8]. In [6, 7, 8], equation (3) with measurable parameters was treated as a perturbation of some canonical stable equation of type (1), with further application of the Bohl-Perron theorem (W-method). A similar method was developed in Győri, Hartung and Turi [19] and applied to the case of a system of linear equations (the vector case with a matrix coefficient).

Among the various results that were obtained, is the following [8]. Assume that \( \sup_{t \in [0, +\infty)} (t - \tau_i(t)) < \infty, \liminf_{t \to \infty} \sum_{k=1}^{n} c_k(t) > 0 \) and
\[
\limsup_{t \to \infty} \int_{\min_{i=1,\ldots,n} \tau_i(t)}^{t} \left( \sum_{i=1}^{n} c_i(s) \right) ds < 1 + \frac{1}{e}
\]
Then all solutions of (6) tend to zero. In Berezansky and Braverman [6, 7, 8] the extension of such results to a sign-changing coefficient, as well as proving the 3/2-criterion for measurable parameters has been set as an open problem.

In the present paper, we first prove the 3/2-stability result in the case of measurable parameters. Combining the techniques of Lillo [26] and Yorke [36], we generalise the notion of rapid oscillation and also introduce and describe a certain real function \( A \), such that, the limit-case periodic functions corresponding to the results of Myshkis [28, Theorem 12], [24, Theorem 12] and Lillo [26], can be represented as points on its graph. We thus obtain sharp stability conditions, relating the speed of oscillation and stability, with no sign conditions implied on the coefficient.
2. The 3/2-stability criterion for a non-negative coefficient

Let us introduce some relevant definitions and notations. In the following, we assume that \( c : [0, +\infty) \to \mathbb{R} \) is Lebesgue measurable and locally integrable, i.e. satisfies \( \int_0^t |c(u)| \, du < \infty, \forall t \in [0, +\infty) \) and \( \tau : [0, +\infty) \to \mathbb{R} \) is Lebesgue measurable with
\[
\tau(t) \leq t \quad \text{and} \quad \lim_{t \to \infty} \tau(t) = \infty.
\]
Denote for any \( t \geq 0 \)
\[
\tau_{\min}(t) := \inf_{v \geq t} \tau(v)
\]
The relation (7) guarantees that there exists \( t_1 \geq 0 \) such that \( \tau_{\min}(t_1) > 0. \)

Throughout the following, we will assume
\[
\rho > \inf \{ t > 0 : \tau_{\min}(t) > t_1 \}
\]
In the literature, a solution of (1) is usually a function \( x : (t_1, t_2] \cup [t_2, t_3) \to \mathbb{R} \) where \(-\infty \leq t_1 \leq t_2 \leq t_3 \leq \infty, \) sufficiently regular on \((t_1, t_2]\) so that it satisfies (1) (everywhere, or almost everywhere, depending also on the assumptions on the functions \( c(t) \) and \( \tau(t) \)) on \([t_2, t_3)\). In this paper, we are concerned exclusively with the behavior near \(+\infty\) and not with existence/uniqueness of solutions, so by (8), it suffices to study the restriction of \( x \) on \([\tau_{\min}(t_0), +\infty)\) for any sufficiently large \( t_0 \). We call a function \( x : [\tau_{\min}(t_0), +\infty) \to \mathbb{R} \), where \( t_0 \in [\rho, +\infty) \), a solution of (1) if it is absolutely continuous on \([\tau_{\min}(t_0), +\infty)\) and satisfies (1) almost everywhere (a.e.) on \([t_0, +\infty)\).

First, we extend the 3/2 stability criterion (2) of Myshkis [28], [29], Lillo [26], Yorke [36], to the case of measurable parameters. A generalisation of Yorke’s technique [36, Proposition 4.2], [36, Lemma 4.3], is applied.

**Theorem 2.1.** Let \( c(t) \geq 0, \forall t \geq 0 \) and
\[
\sup_{t \geq \rho} \int_{\tau(t)}^{t} |c(\zeta)| \, d\zeta \leq \frac{3}{2}.
\]
Then all oscillatory solutions of (1) are bounded. If the inequality in (9) is strict, all oscillatory solutions tend to zero.

Define
\[
q := \max \left\{ \sup_{u \in [\rho, +\infty)} \int_{\tau(u)}^{u} c(s) \, ds, 1 \right\}
\]
where \( \rho \) is defined in (8). Without loss of generality we can assume that at time \( t \) we have
\[
\sup_{v \in \mathbb{R}^2_{\min}(t), t} |x(v)| > 0, \quad x(t) = 0.
\]
If \( \sup_{v \in \mathbb{R}^2_{\min}(t), t} |x(v)| = 0 \) for any sufficiently large zero \( t \), we obtain for any \( \varepsilon > 0 \), similarly to the proof below, \( \forall v \geq t, \ |x(v)| \leq (q - \frac{1}{2}) \varepsilon. \) That is, \( x(v) = 0, v \geq \tau_{\min}(t) \) and the conclusion of the theorem holds. We note that this also follows for locally bounded \( c \), from [7, p.11] and [1, Theorem B.1].

It suffices to prove that
\[
\forall u \geq t, \ |x(u)| \leq (q - \frac{1}{2}) \sup_{v \in \mathbb{R}^2_{\min}(t), t} |x(v)|.
\]
If the strict inequality in (9) holds, \( q - \frac{1}{2} \in (0, 1) \), and repeating this process (\( t := \xi \) is arbitrary) to the next zeros \( \xi_k \) chosen such that \( \xi_{k-1} < \tau_{\min}^2(\xi_k) \), which is possible by (7), one obtains
\[
|x(u)| \leq \left( q - \frac{1}{2} \right)^k \sup_{s \in \mathbb{R}^2_{\min}(t), t} |x(s)|, \ u \geq \xi_k.
\]
which would imply \( \lim_{u \to \infty} x(u) = 0 \).

Assume the contrary and denote, where the set in the right-hand side is non-empty,

\[
z := \inf \left\{ u \geq t : |x(u)| > \left( q - \frac{1}{2} \right) \sup_{r \in [\tau_{\min}^2, t]} |x(r)| \right\}
\]

\[
\tilde{t} := \sup \{ u \in [t, z] : x(u) = 0 \}
\]

Obviously, \( \tilde{t} \geq t \) and as \( q - \frac{1}{2} \leq 1 \)

\[
\sup_{u \in [\tau_{\min}^2, z]} |x(u)| = \sup_{u \in [\tau_{\min}^2, t]} |x(u)|
\]

(11)

We can assume without loss of generality that \( x(z) > 0 \), i.e.

\[
x(z) = \left( q - \frac{1}{2} \right) \sup_{u \in [\tau_{\min}^2, t]} |x(u)|.
\]

By the definition of \( z \), there is a decreasing sequence of \( u_n > z \) such that

\[
u_n > u_{n+1}, \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} u_n = z, \quad x(u_n) > x(z) > 0.
\]

Then \( \forall n \in \mathbb{N} \), integrating (11), we get

\[
0 < x(u_n) - x(z) = -\int_z^{u_n} c(w)x(\tau(w))dw.
\]

(12)

Since \( c(w) \geq 0 \) we can deduce from (12) that

\[
(\forall n \in \mathbb{N}) \quad \exists r_n \in (z, u_n) : x(\tau(r_n)) < 0.
\]

As the sequence \( u_n > z \) tends to \( z^+ \), we can extract a subsequence \( r_{k(n)} \) such that

\[
r_{k(n)} > r_{k(n+1)} > z, \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} r_{k(n)} = z, \quad x(\tau(r_{k(n)})) < 0.
\]

Also, we have

\[
\int_z^{r_{k(n)}} c(w)dw > 0,
\]

as \( c(t) \geq 0 \) and otherwise for sufficiently large \( n \), we would have

\[
\left| \int_z^{u_n} c(w)x(\tau(w))dw \right| \leq \left( \text{ess sup}_{w \in [z, u_n]} |x(\tau(w))| \right) \int_z^{u_n} c(w)dw = 0,
\]

which contradicts (12). Now, \( x \) is positive on \( (\tilde{t}, z] \), and by continuity, \( x \) is non-negative on \( [\tilde{t}, r_{k(n)}] \) for sufficiently large \( n \). As \( x(\tau(r_{k(n)})) < 0 \), we have \( \tau(r_{k(n)}) < \tilde{t} \). By the definition of \( q \),

\[
\int_{\tilde{t}}^{z} c(w)dw < \int_{\tau(r_{k(n)})}^{r_{k(n)}} c(w)dw \leq q.
\]

(13)
For $v \in [\tilde{t}, z]$, the following inequalities hold, (we use (1), (11) and the definition of $q$),
\[
x'(v) \leq c(v) \min \{ |x(\tau(v))|, -x(\tau(v)) \}
\]
\[
\leq c(v) \min \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \begin{cases} 0 & \text{if } \tau(v) \geq \tilde{t} \\ \int_{\tau(v)}^{\tilde{t}} c(w)|x(\tau(w))|dw & \text{if } \tau(v) \leq \tilde{t} \end{cases}
\]
\[
\leq c(v) \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \begin{cases} 0 & \text{if } \tau(v) \geq \tilde{t} \\ \int_{\tau(v)}^{\tilde{t}} c(w)dw & \text{if } \tau(v) \leq \tilde{t} \end{cases}
\]
\[
\leq c(v) \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \begin{cases} 0 & \text{if } \tau(v) \geq \tilde{t} \\ q - \int_{\tilde{t}}^{v} c(w)dw & \text{if } \tau(v) \leq \tilde{t} \end{cases}
\]
\[
\leq c(v) \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \begin{cases} 0 & \text{if } \tau(v) \geq \tilde{t} \\ 1, & \text{otherwise} \end{cases}
\]

Now we evaluate $x(z) = \left(q - \frac{1}{2}\right) \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)|:
\[
x(z) = \int_{\tilde{t}}^{z} x'(u)du \leq \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \int_{\tilde{t}}^{z} c(v) \min \left[1, q - \int_{\tilde{t}}^{v} c(w)dw \right] dv.
\]

Taking into account (13), with the change of variable $r = \int_{\tilde{t}}^{v} c(w)dw$, we get
\[
\left(q - \frac{1}{2}\right) \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| = x(z) \leq \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \int_{[0, f_{\tilde{t}}^z c(u)du]} \min(1, q - r)dr
\]
\[
< \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \int_{0}^{q} \min(1, q - r)dr
\]
\[
= \sup_{u \in [\tau^2_{\min}(t), t]} |x(u)| \left(q - \frac{1}{2}\right),
\]

since
\[
\int_{0}^{q} \min(1, q - r) dr = \int_{0}^{q-1} 1dr + \int_{q-1}^{q} (q - r) dr = \int_{0}^{q-1} 1dr + \left(qr - \frac{r^2}{2}\right)_{r=q-1}^{q} = q - \frac{1}{2}.
\]

The inequality $x(z) < x(z)$ is a contradiction, which completes the proof of (10). This implies boundedness if (11) holds and convergence to zero if the inequality in (9) is strict.

**Theorem 2.2.** Let $c(t) \geq 0, \forall t \geq 0$. Then all nonoscillatory solutions of (1) are monotone and have a limit for $t \to \infty$. Furthermore, if the integral $\int_{0}^{\infty} c(v)dv$ diverges, they tend to zero as $t \to \infty$.

Consider any nonoscillatory solution $x$ such that $x(t) > 0, t \geq a$. Assume that for the constant $b$ we have
\[
b > \inf \{ v > 0 : \tau_{\min}(v) \geq a \}. \tag{14}
\]

Now, using (11) and (12)
\[
x'(t) = -c(t)x(\tau(t)) \leq 0, t \geq b. \tag{15}
\]

So $x$ is nonincreasing and positive on $[b, +\infty)$. This proves that it has a finite limit at infinity.

Assuming
\[
\int_{0}^{\infty} c(v)dv = \infty \tag{16}
\]
we will show that $\lim_{t \to \infty} x(t) = 0$. Assume the contrary, i.e. that $\lim_{t \to \infty} x(t) > 0$. Then for some $c \geq b$, taking into account (7), we have

$$x(\tau(v)) \geq (1/2) \lim_{t \to \infty} x(t), \forall v \geq c \tag{17}$$

Now, using (15), (16), (17),

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left( x(c) + \int_c^t x'(v)dv \right) = \lim_{t \to \infty} \left( x(c) - \int_c^t \xi(v)x(\tau(v))dv \right) \leq \liminf_{t \to \infty} \left( x(c) - (1/2) \lim_{u \to \infty} x(u) \int_c^t \xi(v)dv \right) = -\infty$$

This contradicts $\lim_{t \to \infty} x(t) > 0$. Therefore, $\lim_{t \to \infty} x(t) = 0$ is proved.

3. Stability in the case of sign-changing $c(t)$

We start with extending the notion of rapid oscillation, in a way that enables us to calculate the "speed" of oscillation precisely.

**Definition 3.1.** We call a function $x : [t_0, +\infty) \to \mathbb{R}$, $\ell$-rapidly oscillating where $\ell \in [0, +\infty)$ if it has arbitrarily large zeros and for some $t_1 \geq t_0, \forall A, B \in [t_1, +\infty)$ with $A < B$ the following implication holds

$$[\forall v \in (A, B), x(v) \neq 0] \implies \int_A^B |c(v)|dv \leq \ell.$$  

In other words, $\ell$-rapid oscillation means that in any non-oscillation segment, the integral of the absolute value of the coefficient does not exceed a prescribed value $\ell$.

**Theorem 3.2.** Let $c(t) \leq 0, t \geq 0$ and $\sup_{t \geq t_0} \int_{\tau(t)}^t |c(\zeta)|d\zeta < \infty$, where $\rho$ satisfies (8). Then all oscillatory solutions of (11) are $\sup_{t \geq t_0} \int_{\tau(t)}^t |c(\zeta)|d\zeta - \ell$-rapidly oscillating.

Let $x(t)$ be a non-trivial solution of (11). Consider two points $\phi > \varphi$ such that $x(\varphi) = x(\phi) = 0$ and $x(v) \neq 0, v \in (\varphi, \phi)$. Without loss of generality, we can assume $x(v) > 0, v \in (\varphi, \phi)$. Then for $v \in (\varphi, \phi)$ arbitrarily close to $\phi$

$$0 > x(\phi) - x(v) = -\int_v^\phi c(u)x(\tau(u))du. \tag{18}$$

As $c(u) \leq 0, u \in [\varphi, \phi]$, by (18) we can assume

$$(\forall v \in (\varphi, \phi)) \exists \omega_v \in (v, \phi) : x(\tau(\omega_v)) < 0. \tag{19}$$

Consider the sequence $v_n$ with $v_0 := \omega_{\frac{\phi-\varphi}{2}}$ and $v_{n+1} := \omega_{\frac{v_n+\phi}{2}}$. Obviously, by (19),

$$\phi > v_{n+1} > \frac{v_n + \phi}{2}, n \in \mathbb{N} \text{ and } \lim_{n \to \infty} v_n = \phi, \tag{20}$$

$$x(\tau(v_n)) < 0. \tag{21}$$

Inequality (21) implies (we recall that $x(v) > 0, v \in (\varphi, \phi)$)

$$\tau(v_n) < \varphi. \tag{22}$$
Now, using (22) we have

\[
\sup_{t \geq 0} \int_{\tau(t)} \xi d\zeta \geq \int_{\tau(v_n)} |c(u)| du \\
= -\int_{v_n}^{\phi} |c(u)| du + \int_{\tau(v_n)}^{\phi} |c(u)| du \\
\geq -\int_{v_n}^{\phi} |c(u)| du + \int_{\phi}^{\phi} |c(u)| du
\]

In virtue of (20) we can let the \( v_n \) tend to \( \phi \) and obtain

\[
\sup_{t \geq 0} \int_{\tau(t)} \xi d\zeta \geq \int_{\phi}^{\phi} |c(u)| du.
\]

This concludes the proof.

**Theorem 3.3.** Let \( \ell \in [0, 2] \). Then all \( \ell \)-rapidly oscillating solutions of (1) are bounded. Also, if \( \ell \in [0, 2) \), all \( \ell \)-rapidly oscillating solutions of (1) tend to zero.

Let \( x(t) \) be an \( \ell \)-rapidly oscillating solution of (1). If \( \ell = 2 \) then set \( q := 2 \). Otherwise (if \( \ell \in [0, 2) \)), fix \( q \in (\ell, 2) \). As any \( \ell \)-rapidly oscillating function is \((\ell+\varepsilon)\)-rapidly oscillating, \( \forall \varepsilon > 0 \), we can assume that \( \ell > 1 \) and at time \( t \) we have

\[
\sup_{v \in [\tau_{\min}^2(t), t]} |x(v)| > 0, \quad x(t) = 0.
\]

If \( \sup_{v \in [\tau_{\min}^2(t), t]} |x(v)| > 0 \), for any sufficiently large zero \( t \), we obtain for any \( \varepsilon > 0 \), similarly to the proof below, \( \forall u \geq t, \ |x(u)| \leq \max \{ q - 1, 1 - \frac{1}{2}(q - \ell) \} \varepsilon \). That is, \( x(u) = 0, u \geq \tau_{\min}^2(\xi_1) \) and the conclusion of the theorem holds. We note that this also follows for locally bounded \( c \), from [5, p.11] and [1, Theorem B.1].

We will prove that

\[
|x(u)| \leq \max \{ q - 1, 1 - \frac{1}{2}(q - \ell) \} \sup_{v \in [\tau_{\min}^2(t), t]} |x(v)|, \quad \forall u \geq t.
\]

(23)

Obviously if \( \ell \in [0, 2) \) we have \( q - 1 \in (0, 1), 1 - \frac{1}{2}(q - \ell) \in (0, 1) \). Then, repeating this process (\( t := \xi_1 \) is arbitrary) to the next zeros \( \xi_k \) chosen such that \( \xi_{k-1} < \tau_{\min}^2(\xi_k) \), one obtains

\[
|x(u)| \leq \left( \max \{ q - 1, 1 - \frac{1}{2}(q - \ell) \} \right)^k \sup_{v \in [\tau_{\min}^2(\xi_k), t]} |x(v)|, \quad u \geq \xi_k,
\]

which would imply \( \lim_{u \to \infty} x(u) = 0 \). Assume the contrary and define, where the set in the right-hand side is non-empty,

\[
z = \inf \left\{ v \in [t, +\infty) : |x(v)| > \max \left( q - 1, 1 - \frac{1}{2}(q - \ell) \right) \sup_{u \in [\tau_{\min}^2(t), t]} |x(u)| \right\}
\]

\[
\tilde{\ell} = \sup \left\{ v \in [t, z] : x(v) = 0 \right\}
\]

\[
y = \inf \left\{ v \in [z, +\infty) : x(v) = 0 \right\}
\]

Obviously, \( \tilde{\ell} \geq t \) and as \( \max \{ q - 1, 1 - \frac{1}{2}(q - \ell) \} \leq 1 \)

\[
\sup_{u \in [\tau_{\min}^2(t), z]} |x(u)| = \sup_{u \in [\tau_{\min}^2(t), \tilde{\ell}]} |x(u)| = \sup_{u \in [\tau_{\min}^2(t), t]} |x(u)|
\]

(24)
As $x$ is $\ell$-rapidly oscillating,
\[
\int_{t}^{y} |c(u)| \, du \leq \ell \leq \frac{1}{2} (q + \ell) = q - \frac{1}{2} (q - \ell).
\] (25)

Also, for $v \in [\tilde{t}, y)$, integrating (11), using (24), we get
\[
|v(t)| \leq \max \left\{ \sup_{u \in [\tilde{t}, y]} |x(u)|, \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)| \right\} \int_{w}^{y} |c(p)| \, dp.
\]

Consider a point $w \in (z, y)$: $|x(w)| = \sup_{v \in [\tilde{t}, y]} |x(v)|$. Then,
\[
\max \left\{ |x(w)|, \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)| \right\} \int_{w}^{y} |c(p)| \, dp \geq |x(w)|,
\] (26)

Taking into account that
\[
|x(w)| > |x(z)| = \max \left( q - 1, 1 - \frac{1}{2} (q - \ell) \right) \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)| \geq (q - 1) \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)|,
\]
using (26) we have
\[
\int_{w}^{y} |c(u)| \, du > q - 1.
\] (27)

Using (26), (27) one has
\[
\int_{t}^{z} |c(u)| \, du \leq \int_{t}^{y} |c(u)| \, du - \int_{w}^{y} |c(u)| \, du < 1 - (1/2) (q - \ell)
\] (28)

Also, $\forall v \in [\tilde{t}, z]$, we obtain
\[
|x'(v)| \leq |c(v)||x(\tau(v))| \leq |c(v)| \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)|.
\]

Now we calculate
\[
|x(z)| = \max(q - 1, 1 - (1/2)(q - \ell)) \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)|.
\]

Integrating (11) and taking into account (24), we get
\[
|x(z)| \leq \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)| \int_{t}^{z} |c(p)| \, dp.
\]
Applying (28),
\[
|x(z)| \leq \sup_{u \in [\tau_{2 \min}(t), \tilde{t}]} |x(u)| \left( 1 - \frac{1}{2} (q - \ell) \right).
\]

Thus
\[
|x(z)| < \max \left( q - 1, 1 - \frac{1}{2} (q - \ell) \right) \sup_{v \in [\tau_{2 \min}(t), \tilde{t}]} |x(v)|,
\]
which contradicts to the definition of $z$. The contradiction proves (23).

Let us introduce a function which allows to get sharp results relating oscillation and stability. We will show that this function describes the limit-case periodic solutions of (11), which mark the border from solutions of (11) that tend to zero, to solutions of (11) that have unknown, possibly unbounded behavior. We
Lemma 3.4. The function $\Lambda$ defined in (29) is strictly decreasing, and $\Lambda([1, 2]) = [2, 3 - \sqrt{2} - \ln (\sqrt{2} - 1)]$. Further, the function $\sigma(s) := \Lambda(s) + s$, $s \in [1, 2]$ attains a global minimum at $s = 9/8$.

We compute $\Lambda'$ and $\sigma'$

\[
\Lambda'(s) = \left(2 + s - \sqrt{2s} - \ln \left(\sqrt{2s} - 1\right)\right)',
\]

\[
= 1 - \frac{1}{\sqrt{2s}} - \frac{1}{\sqrt{2s} \sqrt{2s} - 1}
\]

\[
= \frac{1}{\sqrt{2s}} \left(\sqrt{2s} - 1 - \frac{1}{\sqrt{2s} - 1}\right)
\]

\[
= \frac{1}{\sqrt{2s}} \frac{(\sqrt{2s} - 1)^2 - 1}{\sqrt{2s} - 1} < 0, \ s \in [1, 2).
\]

Also

\[
\sigma'(s) = 2 - \frac{1}{\sqrt{2s}} - \frac{1}{\sqrt{2s} \sqrt{2s} - 1}
\]

\[
= \frac{1}{\sqrt{2s}} \left(2\sqrt{2s} - 1 - \frac{1}{\sqrt{2s} - 1}\right)
\]

\[
= \frac{1}{\sqrt{2s}} \left(2\sqrt{2s} - 1\right) \left(\sqrt{2s} - 1\right) - \frac{1}{\sqrt{2s} - 1}
\]

\[
= \frac{1}{\sqrt{2s}} \frac{4s - 3\sqrt{2s}}{\sqrt{2s} - 1}
\]

\[
= \frac{2\sqrt{2s} - 3}{\sqrt{2s} - 1}
\]

\[
> 0, \ s \in (9/8, 2],
\]

\[
= 0, \ s = 9/8,
\]

\[
< 0, \ s \in [1, 9/8).
\]

Lemma 3.5. For each fixed $s \in [1, 2]$, the solution $\psi_s$ of the differential equation

\[
\psi'_s(t) = \min \{1, \max (s - t, \psi_s(t))\} , t \in [0, \Lambda(s) - 1], \ \psi_s(0) = 0 \tag{30}
\]

is strictly increasing and satisfies $\psi_s(\Lambda(s) - 1) = 1$.

Consider the sequence of functions

\[
\iota_{n+1}(t) := \int_0^t \min \{1, \max (s - t, \iota_n(u))\} \, du, t \in [0, \Lambda(s) - 1], \iota_0(t) \equiv 0
\]

where $n = 0, 1, 2, \ldots$. We have by induction, for any fixed $t \in [0, \Lambda(s) - 1]$, $\iota_{n+1}(t) \geq \iota_n(t)$ . By the dominated convergence theorem $(\Lambda(s) - 1 \geq \iota_n(t), \forall t \in [0, \Lambda(s) - 1], n = 0, 1, 2, \ldots)$, the pointwise limit function $\iota_\infty(t) := \lim_{n \to \infty} \iota_n(t)$, $t \in [0, \Lambda(s) - 1]$ solves (30). In view of the Lipschitzian nature of the function $h \mapsto \min \{1, \max (s - t, h)\}$ , this solution is unique. Alternatively, one could assume existence and calculate the solution as below, subsequently verifying that it is in fact a solution.
Evidently this solution of (30) satisfies $\psi_s(t) > 0$, and $\psi_s'(t) > 0$ for $t \in (0, \Lambda(s) - 1]$. There are two possible cases, $s < 2$ (case i) and $s = 2$ (case ii).

i) Case $s < 2$. If $t \in [0, s - 1]$, by the definition in (30) (taking into account $\psi_s(t) \leq \int_0^t 1 = t < s - (s - 1) = 1$)

$$\psi_s'(t) = 1,$$

so

$$\psi_s(s - 1) = s - 1.$$

Now, as $s - 1 < s - (s - 1) = 1$ there exists an interval $(s - 1, s - 1 + \varepsilon), \varepsilon > 0$ where $\psi_s'(t) = s - t$. Also the function $t \mapsto s - t$ is strictly decreasing and $\psi_s$ is strictly increasing. So either $s - t > \psi_s(t), t \in (s - 1, \Lambda(s) - 1)$ or there is a point $c_s$ such that

$$c_s := \sup\{t \in [s - 1, \Lambda(s) - 1] : u \in [s - 1, t] \implies \psi_s'(u) = s - u\} \in (s - 1, \Lambda(s) - 1)$$

$$\psi_s(c_s) = s - 1 + \int_{s-1}^{c_s} (s-t)dt = s - c_s$$

The following equalities are equivalent

$$s - 1 + \int_{s-1}^{c_s} (s-t)dt = s - c_s$$

$$c_s - 1 + sc_s - s^2 - s - (1/2)c_s^2 + (1/2)s^2 - s + 1/2 = 0$$

$$c_s + sc_s - (1/2)c_s^2 + -(1/2)s^2 - 1/2 = 0$$

$$(1/2)c_s^2 - (1 + s)c_s + (1/2)(s^2 + 1) = 0$$

Solving the equation $(1/2)c_s^2 - (1 + s)c_s + (1/2)(s^2 + 1) = 0$, we have

$$c_s = (1 + s) \pm \sqrt{2s}$$

Notice that if $c_s$ is well-defined, $c_s \geq (1 + s) - \sqrt{2s}$. Also, by the above calculation, we have $\psi_s((1 + s) - \sqrt{2s}) = s - ((1 + s) - \sqrt{2s}) = \sqrt{2s} - 1 < 1$. So $c_s$ is in fact well-defined and equals $(1 + s) - \sqrt{2s}$. Therefore (we recall the function $t \mapsto s - t$ is strictly decreasing and $\psi_s$ is strictly increasing), for $t \in [(1 + s) - \sqrt{2s}, \Lambda(s) - 1]$, we have

$$\max(s - t, \psi_s(t)) = \psi_s(t).$$

Defining

$$d_s := \sup\{t \in [(1 + s) - \sqrt{2s}, \Lambda(s) - 1] : u \in [(1 + s) - \sqrt{2s}, t] \implies \psi_s(u) < 1\},$$

we have by (30),

$$\psi_s'(t) = \psi_s(t), t \in [(1 + s) - \sqrt{2s}, d_s].$$

Equation (31) immediately implies

$$\psi_s(t) = \left(\sqrt{2s} - 1\right) \exp\left(t - (1 + s) + \sqrt{2s}\right).$$

Now, by (32) and definition (29),

$$d_s = \Lambda(s) - 1$$

and

$$\psi_s(\Lambda(s) - 1) = 1.$$

ii) Case $s = 2$. Here $\Lambda(2) = 2$ and $2 - t > 1, \forall t \in [0, \Lambda(2) - 1]$. Hence by (30),

$$\psi_s'(t) = 1, \ t \in [0, 1].$$

Integrating (33) we have $\psi_s(t) = t, \forall t \in [0, 1].$
Theorem 3.6. Assume that for some fixed $s \in [1, 2]$ the following inequalities hold
\[
\sup_{t \geq 0} \int_{\tau(t)}^{t} |c(\zeta)| d\zeta \leq s, \quad \ell \leq \Lambda(s),
\] (34)
where $\Lambda(s)$ is denoted in (29) and $\rho$ satisfies (3). Then all $\ell$-rapidly oscillating solutions of (1) are bounded, and if the strict inequality $\ell < \Lambda(s)$ holds, they tend to zero.

Without loss of generality, we can assume that $\ell > \Lambda(s) - 1$.

If $\ell < \Lambda(s)$, let the constant $\alpha \in (1 + \ell - \Lambda(s), 1)$ be a solution of
\[
\alpha - \psi_s(\ell - \alpha) = 0.
\]

Such a solution exists, as the function $j(\alpha) := \alpha - \psi_s(\ell - \alpha), \alpha \in [1 + \ell - \Lambda(s), 1]$ is continuous and $j(1 + \ell - \Lambda(s))j(1) = (\ell - \Lambda(s)) (1 - \psi_s(\ell - 1)) < 0$.

Otherwise (if $\ell = \Lambda(s)$), we set $\alpha = 1$.

Without loss of generality, we can assume at time $t$ we have
\[
\sup_{u \in [\tau_{\min}(t), t]} |x(u)| > 0, \quad x(t) = 0.
\]
If $\sup_{u \in [\tau_{\min}(t), t]} |x(u)| = 0$ for any sufficiently large zero $t$, we obtain for any $\varepsilon > 0$, similarly to the proof below, $\forall u \geq t, \quad |x(u)| \leq \psi_s(\ell - \alpha)\varepsilon$. That is, $x(u) = 0, u \geq \tau_{\min}(t)$ and the conclusion of the theorem holds.

We note that this also follows for locally bounded $c$, from [5, p.11] and [1, Theorem B.1].

It suffices to prove that
\[
\forall u \geq t, \quad |x(u)| \leq \psi_s(\ell - \alpha) \sup_{\zeta \in [\tau_{\min}^2(t), t]} |x(\zeta)|.
\] (35)

If the strict inequality $\ell < \Lambda(s)$ holds, by Lemma 3.5, we have $\psi_s(\ell - \alpha) \in (0, 1)$, and repeating this process ($t := \xi_1$ is arbitrary) to the next zeros $\xi_k$ chosen such that $\xi_{k-1} < \tau_{\min}^2(\xi_k)$, which is possible by (7), one obtains
\[
|x(u)| \leq (\psi_s(\ell - \alpha))^k \sup_{\zeta \in [\tau_{\min}^2(t), t]} |x(\zeta)|, \quad u \geq \xi_k,
\]
which would imply $\lim_{u \to \infty} x(u) = 0$.

Assume the contrary that (35) is not satisfied and define
\[
z = \inf \left\{ \zeta \in [t, +\infty) : |x(\zeta)| > \psi_s(\ell - \alpha) \sup_{v \in [\tau_{\min}^2(t), t]} |x(v)| \right\},
\]
\[
\hat{t} = \sup \{ \zeta \in [t, z] : x(\zeta) = 0 \},
\]
\[
y = \inf \{ \zeta \in [z, +\infty) : x(\zeta) = 0 \}.
\]

We have $\hat{t} \geq t$ and as $\psi_s(\ell - 1) \leq 1$
\[
\sup_{\zeta \in [\tau_{\min}^2(t), t]} |x(\zeta)| = \sup_{\zeta \in [\tau_{\min}^2(t), z]} |x(\zeta)|.
\] (36)

As $x$ is $\ell$-rapidly oscillating,
\[
\int_{\hat{t}}^{y} |c(\zeta)| d\zeta \leq \ell.
\] (37)

Also, for $\eta \in (\hat{t}, y)$, integrating (1), using (35),
\[
|x(\eta)| \leq \int_{\eta}^{y} |c(\zeta)||x(\tau(\zeta))| d\zeta \leq \max \left( \sup_{\zeta \in [t, y]} |x(v)|, \sup_{\zeta \in [\tau_{\min}^2(t), t]} |x(\zeta)| \right) \int_{\eta}^{y} |c(\zeta)| d\zeta.
\] (38)
Consider a point \( w \in (z, y) \) such that \( |x(w)| = \sup_{\zeta \in [t, y]} |x(\zeta)| \). Then
\[
x(w) = \sup_{\zeta \in [t, y]} |x(\zeta)| > |x(z)| = \psi_s(\ell - \alpha) \sup_{\zeta \in \tau_{\text{min}}(t), t} |x(\zeta)|.
\] (39)

By (35),
\[
\max \left( |x(w)|, \sup_{\zeta \in \tau_{\text{min}}(t), t} |x(\zeta)| \right) \int_{w}^{y} |c(\zeta)|d\zeta \geq |x(w)|.
\] (40)

Now, using (39), (40) we can assume
\[
\int_{w}^{y} |c(\zeta)|d\zeta > \psi_s(\ell - \alpha)
\] (41)

From (41) and (37), using the definition of \( \alpha \)
\[
\int_{t}^{\bar{z}} |c(\zeta)| d\zeta \leq \int_{t}^{w} |c(\zeta)| d\zeta - \int_{t}^{y} |c(\zeta)| d\zeta < \ell - \alpha.
\] (42)

Also, \( \forall \zeta \in [\bar{t}, z] \), integrating (11) we have (taking into account (36))
\[
|x'(\zeta)| = |c(\zeta)||x(\tau(\zeta))| \leq |c(\zeta)| \min \left( \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)|, \left\{ \int_{\tau(\zeta)}^{t} |c(u)||x(\tau(u))|du \right\} \right) \text{ if } \tau(\zeta) < \bar{t}
\] and
\[
\leq |c(\zeta)| \min \left( \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)|, \max \left\{ \sup_{u \in [\bar{t}, \zeta]} |x(u)|, \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)| \right\} \right)
\] (43)

Also, by (34),
\[
\int_{\tau(\zeta)}^{\zeta} |c(u)| du - \int_{t}^{\zeta} |c(u)| du \leq \sup_{\zeta \geq i} \int_{\tau(\zeta)}^{\zeta} |c(u)| du - \int_{t}^{\zeta} |c(u)| du \leq s - \int_{t}^{\zeta} |c(u)| du
\]

Thus
\[
|x'(\zeta)| \leq |c(\zeta)| \min \left( \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)|, \max \left\{ \sup_{u \in [\bar{t}, \zeta]} |x(u)|, \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)| \right\} \right) - \int_{t}^{\zeta} |c(v)| dv \right) \right)
\] (43)

Now, either \( s = 2 \) (case i) or \( s \in [1, 2] \) (case ii).

**Case i** We have \( \psi_s(\ell - \alpha) = \ell - \alpha \) and integrating (14), using (12)
\[
|x(z)| \leq \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)| \int_{t}^{s} |c(\zeta)| d\zeta < \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)| (\ell - \alpha) = \sup_{u \in [\tau_{\text{min}}^2(t), t]} |x(u)| \psi_s(\ell - \alpha)
\]
we have a contradiction.

**Case ii** We have \( \ell - \alpha \in (0, \Lambda(s) - 1) \).

Below, we assume that
\[
(1 + s) - \sqrt{2s} \leq \int_{t}^{s} |c(v)| dv < \ell - \alpha < \Lambda(s) - 1
\] (44)

The proof is similar otherwise and is based on the proof of Lemma (5.5). More precisely, as intermediate steps, we get estimates of appropriate integrals that imply the proof for \( \int_{t}^{s} |c(s)| ds \in [0, (1 + s) - \sqrt{2s}] \) or
$\ell - \alpha \in (0, (1 + s) - \sqrt{2s}]$. Alternatively, one could use a change of variables for (43), and prove that if two functions $\alpha, \beta$ with $\alpha(0) = \beta(0) = 0$ solve the equality (30) and the corresponding inequality, respectively, then $\alpha(t) \geq \beta(t), t \in [0, \Lambda(s) - 1]$. Here we present a direct proof. That is, we will show that

$$\sup_{u \in [t, v]} |x(u)| \leq \sup_{u \in [\tau_{\min}(t), t]} |x(u)| \int_{t}^{v} |c(u)| du, \quad v \in [\tilde{t}, z].$$

We have

$$\int_{t}^{z} |c(\zeta)| d\zeta > (1 + s) - \sqrt{2s} > s - 1.$$

We can introduce $z_1 < z_2, z_j \in [\tilde{t}, z], j = 1, 2,$ such that that

$$\int_{t}^{z_1} |c(\zeta)| d\zeta = s - 1, \quad z_2 := \inf \{r \in [\tilde{t}, z] : \int_{t}^{r} |c(\zeta)| d\zeta = (1 + s) - \sqrt{2s}\}.$$

Now, $\forall v \in [\tilde{t}, z_1]$, integrating inequality (43)

$$\sup_{u \in [t, v]} |x(u)| \leq \sup_{u \in [\tau_{\min}(t), t]} |x(u)| \int_{t}^{v} |c(\zeta)| d\zeta$$

For $v = z_1$, we have

$$\sup_{u \in [t, z_1]} |x(u)| \leq (s - 1) \sup_{u \in [\tau_{\min}(t), t]} |x(u)|.$$ 

Now, as $s - 1 < 1$ there exists an interval $(z_1, z_1 + \varepsilon)$, where $\varepsilon > 0, \int_{z_1}^{z_1+\varepsilon} |c(s)| ds > 0$, where

$$\sup_{u \in [\tau_{\min}(t), t]} |x(u)| \left(s - \int_{z_1}^{v} |c(u)| du\right) > \sup_{u \in [t, v]} |x(u)|, v \in (z_1, z_1 + \varepsilon)$$

Integrating (43) on the interval $(z_1, z_1 + \varepsilon)$, using (48), (49),

$$|x(v)| \leq |x(z_1)| + \sup_{u \in [\tau_{\min}(t), t]} |x(u)| \int_{z_1}^{v} |c(q)| \min \left(1, \max \left[1, \frac{\sup_{u \in [t, q]} |x(u)|}{\sup_{u \in [\tau_{\min}(t), t]} |x(u)|} \right] \right) dq$$

$$\leq \sup_{u \in [\tau_{\min}(t), t]} |x(u)| (s - 1) + \sup_{u \in [\tau_{\min}(t), t]} |x(u)| \int_{z_1}^{v} |c(q)| \left(s - \int_{t}^{q} |c(u)| du\right) dq,$$

Making a change of variable

$$r = \int_{t}^{q} |c(u)| du,$$

we get

$$|x(v)| \leq \sup_{u \in [\tau_{\min}(t), t]} |x(u)| \left[s - 1 + \int_{s-1}^{\int_{t}^{q} |c(u)| ds} (s - r) dr\right].$$

We will show that

$$\sup_{u \in [t, v]} |x(u)| < \left(s - \int_{t}^{v} |c(u)| du\right) \sup_{u \in [\tau_{\min}(t), t]} |x(u)|, v \in [z_1, z_2]$$

Assume the contrary, i.e. that there points $v \in [z_1, z_2]$ such that

$$\sup_{u \in [t, v]} |x(u)| \geq \left(s - \int_{t}^{v} |c(u)| du\right) \sup_{u \in [\tau_{\min}(t), t]} |x(u)|$$

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and that the following constant is well-defined

\[ \varpi := \inf \{ v \in [z_1, z_2] : \sup_{u \in [t, v]} |x(u)| \geq \left( s - \int_t^v |c(u)|du \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)| \}. \]  

(52)

Notice that by the definition of \( \varpi \) in (52) and of \( z_2 \) in (40), and also (49), we have \( \kappa := \int_t^v |c(u)|du \in (s - 1, (1 + s) - \sqrt{2s}) \). However, it is obvious that \( \forall \kappa \in (s - 1, (1 + s) - \sqrt{2s}) \),

\[ (s - \kappa) - [s - 1] - \int_{[s-1,\kappa]} (s - r)dr = (1/2)\kappa^2 - (1 + s)\kappa + (1/2)(s^2 + 1) = \frac{1}{2} \left( \kappa - (1 + s) + \sqrt{2s} \right) \left( \kappa - (1 + s) - \sqrt{2s} \right) > 0 \]

By the above computation and (50),

\[ \sup_{u \in [t, \infty]} |x(u)| < \left( s - \int_t^\infty |c(u)|du \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)|. \]  

(53)

Equation (53) contradicts the definition of \( \varpi \) in (52). This proves (51). By (51) and (50), using the change of variable \( r = \int_t^u |c(u)|du \),

\[ \sup_{u \in [t, z_2]} |x(u)| \leq \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)| \left( s - 1 + \int_{[s-1,(1+s)-\sqrt{2s}]} (s - r)dr \right) \]  

(54)

We now compute the integral in (54),

\[ [s - 1] + \int_{[s-1,(1+s)-\sqrt{2s}]} (s - r)dr = s - 1 + s \left( (1 + s) - \sqrt{2s} - (s - 1) \right) - (1/2) \left[ \left( (1 + s) - \sqrt{2s} \right)^2 - (s - 1)^2 \right] = \sqrt{2s} - 1 \]

Hence, (54) can be written as

\[ \sup_{u \in [t, z_2]} |x(u)| \leq \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)| \left( \sqrt{2s} - 1 \right). \]  

(55)

Now, \( \forall p \in [z_2, z] \), (we use (53), (49) and the definition of \( z_2 \) in (40))

\[ |x'(p)| \leq |c(p)| \max \left( s - \int_t^p |c(u)|du \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)|, \sup_{u \in [t, p]} |x(u)| \]

\[ \leq |c(p)| \max \left( \sqrt{2s} - 1 - \int_{z_2}^p |c(u)|du \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)|, \sup_{u \in [t, p]} |x(u)| \]

\[ \leq |c(p)| \max \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)|, \sup_{u \in [t, z_2]} |x(u)|, \sup_{u \in [z_2, p]} |x'(u)|du \]

\[ \leq |c(p)| \max \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)|, \left( \sqrt{2s} - 1 \right) \sup_{u \in [t, z_2]} |x(u)|, \sup_{u \in [z_2, p]} |x'(u)|du \]

\[ \leq |c(p)| \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\text{min}}^s(t), t]} |x(u)| + \int_{z_2}^p |x'(u)|du \].

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Therefore, the following five inequalities are valid:

\[
| x'(p) | - | c(p) | \int_{z_2}^{p} | x'(u) | du \leq | c(p) | (\sqrt{2s} - 1) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |,
\]

\[
| x'(p) | \exp \left( - \int_{0}^{p} | c(u) | du \right) - | c(p) | \exp \left( - \int_{0}^{p} | c(u) | du \right) \int_{z_2}^{p} | x'(u) | du
\leq | c(p) | \exp \left( - \int_{0}^{p} | c(u) | du \right) \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |,
\]

\[
\left[ \exp \left( - \int_{0}^{p} | c(u) | du \right) \int_{z_2}^{p} | x'(u) | du \right] \leq \left[ - \exp \left( - \int_{0}^{p} | c(u) | du \right) \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) | \right]^{'}, \tag{56}
\]

Integrating (56), we get

\[
\exp \left( - \int_{0}^{p} | c(u) | du \right) \int_{z_2}^{p} | x'(u) | du
\leq \left( \exp \left\{ - \int_{0}^{p} | c(u) | du \right\} - 1 \right) \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |.
\]

Thus

\[
\int_{z_2}^{p} | x'(u) | du \leq \left( \exp \left\{ \int_{z_2}^{p} | c(u) | du \right\} - 1 \right) \left( \sqrt{2s} - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |. \tag{57}
\]

Note that by (44) and the definition of \( z_2 \) in (46) we have

\[
\int_{z_2}^{z} | c(u) | du < \ell - \alpha - (1 + s) + \sqrt{2s}. \tag{58}
\]

Obviously, in view of (57), (59), (47), (55), inequality (15) holds. Using (58) and (57) for \( p = z \),

\[
\int_{z_2}^{z} | x'(u) | du < \left( \sqrt{2s} - 1 \right) \left( \exp \left[ \ell - \alpha - (1 + s) + \sqrt{2s} \right] - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |. \tag{59}
\]

Finally, by (55), (59), Lemma 5.5, and the definition of \( z \),

\[
| x(z) | = \psi_{z} (\ell - \alpha) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |,
\]

\[
\leq \int_{z_2}^{z} | x'(u) | du + \sup_{u \in [I, z_2]} | x(u) |
\leq \left( \sqrt{2s} - 1 \right) \left( \exp \left[ \ell - \alpha - (1 + s) + \sqrt{2s} \right] - 1 \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) | + \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) | \left( \sqrt{2s} - 1 \right)
= \left( \sqrt{2s} - 1 \right) \left( \exp \left[ \ell - \alpha - (1 + s) + \sqrt{2s} \right] \right) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |
= \psi_{z} (\ell - \alpha) \sup_{u \in [\tau_{\min}^2(t), t]} | x(u) |
\]

So we obtain a contradiction, which concludes the proof of (55) and of the theorem.
Theorem 3.7. Let the assumptions of Theorem 3.3 (or 3.6) be satisfied, and also there exist finite $C, D > 0$ such that $\forall t \geq t_1$,
\[
|t - \tau^2_{\min}(t)| \leq C, \quad \int_{\tau^2_{\min}(t)}^t |c(z)|dz \geq D.
\] (60)
Then all $\ell$-rapidly-oscillating solutions of (1) tend to zero exponentially, i.e. there exist positive constants $M > 0$ and $\gamma > 0$, $\delta > 0$ such that
\[
|x(t)| \leq M \sup_{u \in [t_1, t_1 + \delta]} |x(u)|e^{-\gamma(t-t_1)}.
\] (61)

Fix the constant $\beta := \left\lfloor \frac{r}{p} \right\rfloor + 1$ (obviously $\beta > \frac{r}{p}$), where $\left\lfloor \cdot \right\rfloor$ denotes the floor function. Suppose that $r - p \geq C\beta + C + 1$, where $p, r \in [t_1, +\infty)$ are arbitrary. Then $\tau^2_{\min}(r) \geq r - C\beta$ and (60) implies that
\[
\int_{r-C\beta}^r |c(z)|dz \geq D\beta > \ell
\] (62)
By Definition 4 and (62), $x(t)$ has at least one zero in $(r - C\beta, r)$, which we will denote as $w$.

By (60), $\tau^2_{\min}(w) > w - C\beta - 1 > p$. Therefore, using Theorem 3.3 (or 3.6), noting that $p < \tau^2_{\min}(w) \leq w < r$
\[
\sup_{u \in [r, +\infty)} |x(u)| \leq \sup_{u \in [w, +\infty)} |x(u)| \leq d \sup_{u \in [\tau^2_{\min}(w), w]} |x(u)| \leq d \sup_{u \in [p, r]} |x(u)|,
\]
where $d := \max \left\{ q - 1, 1 - \frac{1}{2}(q - \ell) \right\}$, (or $\psi_0(\ell - \alpha)$). Next, by induction on $\left\lfloor \frac{r-(t_1 + C\beta + C + 1)}{C\beta + C + 1} \right\rfloor$, one obtains
\[
\sup_{u \in [t_1, +\infty)} |x(u)| \leq \left( \sup_{u \in [t_1, t_1 + C\beta + C + 1]} |x(u)| \right) d^1 + \left( \frac{r_{\tau^2_{\min}(w)}}{w} - 1 \right) \leq \left( \sup_{u \in [t_1, t_1 + C\beta + C + 1]} |x(u)| \right) d^1 \left( \frac{r_{\tau^2_{\min}(w)}}{w} - 1 \right).
\]
Thus (61) is satisfied with
\[
|x(t)| \leq M \sup_{u \in [t_1, t_1 + \delta]} |x(u)|e^{-\gamma(t-t_1)}, \quad M = d^{-1}, \quad \gamma = -\frac{\ln d}{C\beta + C + 1}, \quad \delta = C\beta + C + 1,
\]
which concludes the proof.

4. Discussion, examples and open problems

We now introduce some limit-case periodic functions, solutions of (1), which are described by the $\Lambda(s)$ function, and also illustrate the sharpness of Theorem 3.6

Example 4.1. Consider an arbitrary $s \in [1, 2]$.

For $s \in [1, 2]$ consider the $\Lambda(s)$-periodic function
\[
x_s(t) := \psi_s(t) \quad t \in [0, \Lambda(s) - 1]
\]
x_s(t) = 1 - (t - \Lambda(s) + 1) \quad t \in [\Lambda(s) - 1, \Lambda(s)]
\]
x_s(t) = x_s(t + \Lambda(s)) \quad t \in \mathbb{R}

The function $x_s$ is $\Lambda(s)$-rapidly oscillating and a solution of (1) with
\[
|c_s(t)| = 1, \quad t \geq 0, \quad \sup_{t \geq p \geq \tau^s_{\min}(t)} \int_{\tau^s_{\min}(t)}^t |c_s(\zeta)|d\zeta = s.
\]

Here $c_s$ is a $\Lambda(s)$-periodic function defined on $[0, \Lambda(s)]$ as
\[
c_s(t) = \begin{cases} 
1, & t \in [0, \Lambda(s) - 1), \\
-1, & t \in [\Lambda(s) - 1, \Lambda(s)]
\end{cases}
\]
In other words \( x \) and \( \tau \) satisfies
\[
\tau_s(t) = \begin{cases} 
-1, & t \in [0, s - 1), \\
\tau - s, & t \in [s - 1, (s + 1) - \sqrt{2s}], \\
t, & t \in [(s + 1) - \sqrt{2s}, \Lambda(s) - 1], \\
\Lambda(s) - 1, & t \in [\Lambda(s) - 1, \Lambda(s)].
\end{cases}
\]
and \( \tau_s(t + \Lambda(s)) = \tau_s(t) + \Lambda(s), t \in \mathbb{R} \)

In other words \( x_s(t) \) satisfies almost everywhere
\[
\begin{align*}
x_s'(t) &= x_s(-1), & t \in [0, s - 1] \\
x_s'(t) &= x_s(t - s), & t \in [s - 1, (s + 1) - \sqrt{2s}] \\
x_s'(t) &= x_s(t), & t \in [(s + 1) - \sqrt{2s}, \Lambda(s) - 1] \\
x_s'(t) &= -x_s(\Lambda(s) - 1), & t \in [\Lambda(s) - 1, \Lambda(s)] \\
x_s'(t) &= x_s'(t + \Lambda(s)), & t \in \mathbb{R}
\end{align*}
\]

For \( s = 2 \), consider the 2-periodic function
\[
\begin{align*}
x_2(t) &= : \psi_2(t) & t \in [0, 1] \\
x_2(t) &= 2 - t & t \in [1, 2] \\
x_2(t) &= x_2(t + 2) & t \in \mathbb{R}
\end{align*}
\]
The function \( x_2 \) is 2-rapidly oscillating and a solution of (11) with \( |c_2(t)| = 1, t \geq 0, \sup_{t \geq \rho} \int_{\tau_2(t)}^{t} |c_2(\zeta)|d\zeta = 2. \)
Here \( c_2 \) is a 2-periodic function defined on \([0, 2]\) as
\[
c_2(t) = \begin{cases} 
1, & t \in [0, 1), \\
-1, & t \in [1, 2), 
\end{cases}
\]
and \( \tau_2 \) satisfies
\[
\tau_2(t) = \begin{cases} 
-1, & t \in [0, 1), \\
1, & t \in [1, 2].
\end{cases}
\]
and \( \tau_2 + 2 = \tau_2 + 2, t \in \mathbb{R} \)

In other words \( x_2(t) \) satisfies almost everywhere
\[
\begin{align*}
x_2'(t) &= x_2(-1), & t \in [0, 1] \\
x_2'(t) &= -x_2(1), & t \in [1, 2] \\
x_2'(t) &= x_2'(t + 2), & t \in \mathbb{R}
\end{align*}
\]
Hence the bound \( \Lambda(s) \) in Theorem 3.6 is sharp.

Next, let us explore certain \( \Lambda(s) \)-rapidly oscillating solutions of (11) with a non-positive coefficient \( c(t) \leq 0, \forall t \geq 0. \)

**Example 4.2.** Consider an arbitrary \( s \in [1, 2]. \)

For \( s \in [1, 2] \) consider the periodic function
\[
\begin{align*}
y_s(t) &= : \psi_s(t) & t \in [0, \Lambda(s) - 1] \\
y_s(t) &= 1 - (t - \Lambda(s) + 1) & t \in [\Lambda(s) - 1, \Lambda(s)] \\
y_s(t) &= -y_s(t + \Lambda(s)) & t \in \mathbb{R}
\end{align*}
\]
the function \( y_s \) is a \( \Lambda(s) \)-rapidly oscillating solution of (11) with
\[
c_s(t) = -1, \forall t \geq 0, \quad \sup_{t \geq \rho} \int_{\tau_s(t)}^{t} |c_s(\zeta)|d\zeta = s + \Lambda(s).
\]
Here $\tau_s$ satisfies

$$\tau_s(t) = \begin{cases} 
-1 - \Lambda(s), & t \in [0, s - 1), \\
-1 + \frac{1}{s - \Lambda(s)}, & t \in [s - 1, s + 1 - \sqrt{2s}], \\
-1, & t \in [s + 1 - \sqrt{2s}, \Lambda(s) - 1], \\
\Lambda(s), & t \in [\Lambda(s) - 1, \Lambda(s)], \\
-1, & t \in [\Lambda(s) - 1, \Lambda(s)]. 
\end{cases}$$

and $\tau_s(t + \Lambda(s)) = \tau_s(t) + \Lambda(s), t \in \mathbb{R}$

leading to the equation

$$y'_s(t) = y_s(-1 - \Lambda(s)), \quad t \in [0, s - 1],$$

$$y'_s(t) = y_s(t - s - \Lambda(s)), \quad t \in [s - 1, s + 1 - \sqrt{2s}],$$

$$y'_s(t) = y_s(t - \Lambda(s)), \quad t \in [s + 1 - \sqrt{2s}, \Lambda(s) - 1],$$

$$y'_s(t) = y_s(-1), \quad t \in [\Lambda(s) - 1, \Lambda(s)],$$

$$y'_s(t) = -y'_s(t + \Lambda(s)), \quad t \in \mathbb{R}.$$

For $s = 2$, consider the periodic function

$$y_2(t) := \psi_2(t) \quad t \in [0, 1]$$

$$y_2(t) = 2 - t \quad t \in [1, 2]$$

$$y_2(t) = -y_2(t + 2) \quad t \in \mathbb{R}$$

The function $y_2$ is 2-rapidly oscillating and a solution of (1) with $c_2(t) = -1, t \geq 0, \sup_{t \geq 0} f^{t}_{\tau_2(t)} |c_2(\zeta)|d\zeta = 4$. Here $\tau_2$ satisfies

$$\tau_2(t) = \begin{cases} 
-3, & t \in [0, 1), \\
-1, & t \in [1, 2). 
\end{cases} \quad \text{and} \quad \tau_2(t + 2) = \tau_2(t) + 2, t \in \mathbb{R}$$

In other words $y_2(t)$ satisfies almost everywhere

$$y'_2(t) = y_2(-3), \quad t \in [0, 1]$$

$$y'_2(t) = y_2(-1), \quad t \in [1, 2]$$

$$y'_2(t) = -y'_2(t + 2), \quad t \in \mathbb{R}$$

Recalling the function $\sigma(s) = s + \Lambda(s)$ of Lemma 3.4 we notice that the function $y_s$, which is a variation of $x_s$ in Example 4.1 (we have $|y_s(t)| = x_s(t), t \in \mathbb{R}$) so that it satisfies (1) with $0 \leq -c_s(s) \leq 1, t \geq 0$, has $\tau_{\max} = \sigma(s) = s + \Lambda(s)$.

Consider the following periodic functions, which were instrumental in the results of Myshkis [28], [29] and Lillo [26],

$$f(t) = 1 - t, t \in [0, 3/2]$$

$$f(t) = -1/2 - \int_{0}^{t - 3/2} (1 - u)du, t \in [3/2, 5/2]$$

$$f(t) = -f(t + 5/2), t \in \mathbb{R}$$

$$g(t) = 1 - t, t \in [0, 9/8]$$

$$g(t) = -1/8 - \int_{0}^{t - 9/8} (1 - u)du, t \in [9/8, 13/8]$$

$$g(t) = -1/2 \exp(t - 13/8), t \in [13/8, 13/8 + \ln 2]$$

$$g(t) = -g(t + \ln 2 + 13/8), t \in \mathbb{R}$$

The function $f$ (which first appeared in Myshkis [28], [29]) solves (1) with and $0 \leq c(t) \leq 1, t \geq 0$ and $\tau_{\max} = 3/2$, and the function $g$ solves (1) with $0 \leq -c(s) \leq 1, t \geq 0$ and $\tau_{\max} = 2.75 + \ln 2$. Lillo in [26] proved the following:
Let \( \Gamma \) denote the class of solutions \( \gamma(t) \) of (1) with \( 0 \leq c(t) \leq 1 \), \( t \geq 0 \) and \( \tau_{\text{max}} = 3/2 \) which satisfy

\[
\gamma(t) = \begin{cases} 
(-1)^i M_i, & t \in [t_{2i-1}, t_{2i}], \\
(-1)^i M_i f(t - t_{2i}), & t \in [t_{2i}, t_{2i+1}],
\end{cases}
\]

where

\[
M_{i+1} \geq M_i \geq \ldots \geq \lim_{i \to \infty} M_i > 0,
\]
\[
t_i \leq t_{i+1}, \quad \text{and} \quad \lim_{i \to \infty} t_i = +\infty.
\]

Suppose \( z(t) \) is an oscillatory solution of (1) with \( \tau_{\text{max}} = 3/2 \), and \( 0 \leq c(t) \leq 1 \), \( t \geq 0 \) which does not tend to zero. Then for some \( \gamma_z \in \Gamma \) which depends on \( z \), we have:

\[
(\forall \varepsilon > 0) \quad \exists \nu > 0 : t > \nu \implies |z(t) - \gamma_z(t)| < \varepsilon
\]

Also, for the case of \( 0 \leq -c(t) \leq 1 \), \( t \geq 0 \) and \( \tau_{\text{max}} = 2.75 + \ln 2 \), Buchanan [9] proved that a certain class of oscillatory solutions of (1) are, in the above sense, asymptotic to \( g \). Concerning the connection between the function \( g \) and the asymptotic behavior of (1) with \( c(t) \leq 0, t \geq 0 \) we refer to the various remarks and comparison theorems in Lillo [26].

By Lemma 3.4 we have that the function \( \sigma(s) = s + \Lambda(s) \) (which is equal to the \( \tau_{\text{max}} \) of \( y_s \) in Example 4.1) attains its minimum at \( 9/8 \). In other words, Lillo's constant in [26] is equal to the global minimum of the delay of our Examples \( y_s \)

\[
2.75 + \ln 2 = \min_{s \in [1,2]} \sigma(s)
\]

and we also have equality of the corresponding limit-case periodic solutions of (1)

\[
g(t+1) = y_{9/8}(t + \ln 2 + 13/8), \quad t \in \mathbb{R}
\]

Furthermore, we note that condition (4) and Theorem 3.3 are related to the limit case \( x_2(t) \) (defined in Example 4.11), as well as to the unbounded functions in Myshkis [28, paragraph 8].

We therefore propose the following conjectures.

**Conjecture 4.3.** Assume that \( c(t) \leq 0, t \geq 0 \) and \( \sup_{t \geq \rho} \int_{\tau(t)}^t |c(\zeta)|d\zeta < 2.75 + \ln 2 \), where \( \rho \) satisfies (8). Then all oscillatory solutions of (1) tend to zero.

**Conjecture 4.4.** Fix an arbitrary \( s \in [1,2] \). Under the assumption

\[
\sup_{t \geq \rho} \int_{\tau(t)}^t |c(\zeta)|d\zeta = s, \quad |c(\zeta)| \leq 1, \quad \forall \zeta \geq 0,
\]

where \( \rho \) satisfies (8), all solutions of (1) that are \( \Lambda(s) \)-rapidly oscillating and nonnegative (and do not tend to zero), are asymptotic (in the sense of Lillo [26] and Buchanan [3], [10]) to \( x_s \).

**Conjecture 4.5.** Under the assumption

\[
|c(\zeta)| \leq 1, \quad \forall \zeta \geq 0,
\]

all solutions of (1) that are 2-rapidly oscillating and nonnegative (and do not tend to zero), are asymptotic (in the sense of Lillo [26] and Buchanan [3], [10]) to \( x_2 \).

Moreover, the following problem remains open.

**Problem 4.6.** Using the relationship between the definition of \( \Lambda(s) \) and the functions \( \psi_s, x_s \) of Lemma 3.3, Example 4.1, extend the definition of \( \Lambda(s) \), as well as Theorem 3.6 to \( s \in (1/e, 1) \).
Presumably for $s$ close to $1/e$ (a constant related to oscillation of ..., see Myshkis [28, Theorem 29], we can only approximate $\Lambda(s)$. The results of Myshkis [28, Theorem 29], [29] and Domshlak and Aliev [12, Theorem 8] concerning the distance between zeros could be useful in approximating $\Lambda(s)$ for $s \in (1/e, 1)$, as well as calculating the oscillation speed of solutions of oscillatory equations.

Plotting $\Lambda(s), s \in (1, 2)$ (in red, connecting the points when the delay is equal to one and two) and extending the plot as the constant 2 of Theorem 3.3 we obtain Fig. 1. The points (2, 2) and (9/8, ln 2 + 13/8) described by the results of Myshkis [28, paragraph 8] and Lillo [26, p.9], are connected by the graph of $\Lambda(s)$. We have "connected the dots" between these two well-known celebrated results.

Figure 1: The graph summarizes the findings of this paper, relating the speed of oscillation in the sense of Definition 3.1 and the maximum delay, in the sense of $\sup_{t \geq \rho} \int_t^T |c(\zeta)| d\zeta$. The periodic solutions of $[1]$, $x_s, y_s$ of Examples 4.1, 4.2 correspond to each point of the graph of $\Lambda(s)$.

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