Greedy capped nonlinear Kaczmarz methods

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Abstract

To solve nonlinear problems, we construct two kinds of greedy capped nonlinear Kaczmarz methods by setting a capped threshold and introducing an effective probability criterion for selecting a row of the Jacobian matrix. The capped threshold and probability criterion are mainly determined by the maximum residual and maximum distance rules. The block versions of the new methods are also presented. We provide the convergence analysis of these methods and their numerical results behave quite well.

Keywords: Nonlinear Kaczmarz, Greed, Maximum residual rule, Maximum distance rule, Nonlinear problems

1 Introduction

Consider the nonlinear system

\[ f(x) = 0, \]  

where \( f : \mathbb{R}^n \to \mathbb{R}^m \) and \( x \in \mathbb{R}^n \) is an unknown variable. We assume throughout that \( f(x) = [f_1(x), \ldots, f_m(x)]^T \in \mathbb{R}^m \) is a continuously differentiable vector-valued function, and there exists a solution \( x_* \) satisfying (1), i.e., \( f(x_*) = 0 \). Such problem is fundamental in numerical computing arising from many areas, e.g., machine learning [1], differential equations [2] and optimization problems [3].
There have been many studies on solving nonlinear problems by iterative methods [4–6]. Among them the nonlinear Kaczmarz method [7–9] is a typical representative for the so-called single sample-based method and can be formulated as

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{f_{i_k}(\mathbf{x}_k)}{\|\nabla f_{i_k}(\mathbf{x}_k)\|_2^2} \nabla f_{i_k}(\mathbf{x}_k).$$

In the nonlinear Kaczmarz iteration scheme, each iteration is formed by projecting the current point $\mathbf{x}_k$ to the constraint set defined by $f_{i_k}(\mathbf{x}_k) + \nabla f_{i_k}(\mathbf{x}_k)^T (\mathbf{x} - \mathbf{x}_k) = 0$. After determining the iteration formula, it can be found that how to select the index $i_k$ is particularly important. At present, there are mainly three different rules for selecting $i_k$ [9] leading to three different methods: (1) Nonlinear randomized Kaczmarz (NRK) method, where $i_k$ is randomly selected from $[m]$ with probability of $p_{i_k} = \frac{|f_{i_k}(\mathbf{x}_k)|^2}{\|f(\mathbf{x}_k)\|_2^2}$; (2) Nonlinear Kaczmarz (NK) method, where $i_k$ cyclically picks value from $[m]$; (3) Nonlinear uniformly randomized Kaczmarz (NURK) method, where $i_k$ is randomly sampled from $[m]$ with equal probability. The theoretical analysis in [9] shows that the convergence factors of the NRK and NURK methods are the same, however, the NRK method performs better in terms of the number of iterations and computing time in most cases. It also usually outperforms the NK method. Upon close examination, we can find that the probability criterion used in the NRK method makes the method select the index corresponding to the larger entry of the residual vector as much as possible. However, it is possible for the NRK method to select an index with relatively small residual component at each iteration. This is inconsistent with the original intention of the NRK method and may lead to slower convergence.

In this paper, inspired by [10–12], where the capped threshold is introduced to ensure that relatively large component values are selected, we first propose the distance-residual capped nonlinear Kaczmarz (DR-CNK) method for solving the nonlinear problem (1). Here the capped threshold is mainly determined by the maximal distance rule which is discussed in [13] in detail. Similarly, the residual-distance capped nonlinear Kaczmarz (RD-CNK) method is also presented, whose capped threshold is mainly determined by the maximal residual rule [13]. We prove that the two new methods converge linearly in expectation with convergence factors being strictly smaller than that of the NRK and NURK methods presented in [9]. Furthermore, two block versions, i.e., the multiple samples-based methods, are also proposed to accelerate our new methods.

The rest of this paper is organized as follows. Section 2 provides some preliminaries. In Section 3, we propose the DR-CNK and RD-CNK methods and their convergence analysis are given in Section 4. The block versions of the new methods are presented in Section 5 and the relevant convergence theorems are proved in Section 6. Experimental results are shown in Section 7. Finally, we conclude the paper with some remarks.
2 Preliminaries

For a matrix \( A = (A_{i,j}) \in \mathbb{R}^{m \times n} \), \( \sigma_{\max}(A) \), \( \|A\|_2 \), \( \|A\|_F \), \( A^\dagger \), and \( A_\tau \) denote its largest singular value, spectral norm, Frobenius norm, Moore-Penrose pseudo-inverse, and the restriction onto the row indices in the set \( \tau \), respectively. We use \(|\tau|\), \( \mathbb{R}^k \), and \( \mathbb{E} \) to denote the number of elements of a set \( \tau \), the conditional expectation conditioned on the first \( k \) iterations, and the full expected value, respectively. In addition, we let \([m] := \{1, \cdots, m\}\) for an integer \( m \geq 1 \) and define
\[
h_2(A) = \inf_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},
\]
and
\[
f'(x) = [\nabla f_1(x), \cdots, \nabla f_m(x)]^T \in \mathbb{R}^{m \times n},
\]
which is the Jacobian matrix of \( f \) at \( x \).

In addition, the following facts are necessary throughout the paper.

**Definition 1** ([14]) For \( i \in [m] \) and \( \forall x_1, x_2 \in \mathbb{R}^n \), there exists \( \eta_i \in [0, \eta) \) satisfying \( \eta = \max_i \eta_i < \frac{1}{2} \) such that
\[
|f_i(x_1) - f_i(x_2) - \nabla f_i(x_1)^T (x_1 - x_2)| \leq \eta_i |f_i(x_1) - f_i(x_2)|,
\]
then the function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is referred to satisfy the local tangential cone condition.

**Lemma 1** ([9]) If the function \( f \) satisfies the local tangential cone condition, then for \( i_k \in [m] \), \( \forall x_1, x_2 \in \mathbb{R}^n \) and the updating formula (2), we have
\[
\|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - (1 - 2\eta_{i_k}) \frac{|f_{i_k}(x_k)|^2}{\|\nabla f_{i_k}(x_k)\|_2^2}.
\]

**Lemma 2** ([13]) If the function \( f \) satisfies the local tangential cone condition, then for \( \forall x_1, x_2 \in \mathbb{R}^n \) and an index subset \( \tau \subseteq [m] \), we have
\[
\|f_\tau(x_1) - f_\tau(x_2)\|_2^2 \geq \frac{1}{1 + \eta^2} \|f'_\tau(x_1)(x_1 - x_2)\|_2^2.
\]

**Lemma 3** ([13]) If the function \( f \) satisfies the local tangential cone condition and a vector \( x_* \in \mathbb{R}^n \) satisfies \( f(x_*) = 0 \), then from the block iteration formula \( x_{k+1} = x_k - (f'_{\tau_k}(x_k))^\dagger f_{\tau_k}(x_k) \) with \( \tau_k \subseteq [m] \), we have
\[
\|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - \left( h_2((f'_{\tau_k}(x_k))^\dagger) - 2\eta^2\sigma_{\max}^2 \left( (f'_{\tau_k}(x_k))^\dagger \right) \right) \|f_{\tau_k}(x_k)\|_2^2.
\]

3 Single sample-based capped methods

Intuitively, at the \( k \)th iteration, to make \( f(x) \rightarrow 0 \) as fast as possible, we might want the larger entries in the vector \( f(x) \) to be preferentially annihilated as much as possible. In this way, the corresponding nonlinear Kaczmarz method
should converge quickly. With this in mind, we construct the DR-CNk method listed in Algorithm 1. Specifically, it first determines the index subset \( U_k \) in (4) by using the combination of the maximum and average distances, and then samples an index from the subset \( U_k \) with probability that is proportional to the corresponding residual. This means that the final selected iteration index \( i_k \) in the DR-CNk method is jointly determined by two greedy rules, namely, the distance rule for determining the index set \( U_k \) and the residual rule for extracting the index \( i_k \) from the set \( U_k \). Conversely, we can also use the maximum residual rule to construct an index subset \( I_k \), and then select an iteration index \( i_k \) from the set with the probability criterion determined by the distance, thus obtaining the RD-CNk method shown in Algorithm 2.

**Algorithm 1** The DR-CNk method

**Require:** The initial estimate \( x_0 \in \mathbb{R}^n \).

1. for \( k = 0, 1, 2, \cdots \) until convergence, do
2. Compute
   \[
   \varepsilon_k = \frac{1}{2} \left( \frac{1}{\| f(x_k) \|_2^2} \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\| \nabla f_i(x_k) \|_2^2} + \frac{1}{\| f'(x_k) \|_F^2} \right). \tag{3}
   \]
3. Determine the index subset
   \[
   U_k = \left\{ i \mid |f_i(x_k)|^2 \geq \varepsilon_k \| f(x_k) \|_2 \| \nabla f_i(x_k) \|_2^2 \right\}. \tag{4}
   \]
4. Compute the \( i \)-th entry \( \tilde{r}^{(i)}_k \) of the vector \( \tilde{r}_k \) according to
   \[
   \tilde{r}^{(i)}_k = \begin{cases} f_i(x_k), & \text{if } i \in U_k, \\ 0, & \text{otherwise}. \end{cases}
   \]
5. Select \( i_k \in U_k \) with probability \( \Pr(\text{row} = i_k) = \frac{|\tilde{r}^{(i_k)}_k|^2}{\| \tilde{r}_k \|_2^2} \).
6. Update \( x_{k+1} = x_k - \frac{f_{i_k}(x_k)}{\| \nabla f_{i_k}(x_k) \|_2^2} \nabla f_{i_k}(x_k) \).
7. end for

**Remark 1** The DR-CNk method is well defined as the index set \( U_k \) in (4) is always nonempty. This is because
\[
\max_{i \in [m]} \frac{|f_i(x_k)|^2}{\| \nabla f_i(x_k) \|_2^2} \geq \sum_{i=1}^m \frac{\| \nabla f_i(x_k) \|_2^2}{\| f'(x_k) \|_F^2} \frac{|f_i(x_k)|^2}{\| \nabla f_i(x_k) \|_2^2} = \frac{\| f(x_k) \|_2^2}{\| f'(x_k) \|_F^2}
\]
and then
\[
\frac{|f_{i_k}(x_k)|^2}{\| \nabla f_{i_k}(x_k) \|_2^2} = \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\| \nabla f_i(x_k) \|_2^2} \geq \frac{1}{2} \left( \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\| \nabla f_i(x_k) \|_2^2} + \frac{\| f(x_k) \|_2^2}{\| f'(x_k) \|_F^2} \right).
\]
Algorithm 2 The RD-CNK method

Require: The initial estimate $x_0 \in \mathbb{R}^n$.

1: for $k = 0, 1, 2, \cdots$ until convergence, do
2: \hspace{1em} Compute \(\delta_k = \frac{1}{2} \left( \frac{\max_{i \in [m]} |f_i(x_k)|^2}{\|f(x_k)\|_2^2} + \frac{1}{m} \right)\).

3: Determine the index subset \(\mathcal{I}_k = \left\{ i \mid |f_i(x_k)|^2 \geq \delta_k \|f(x_k)\|_2^2 \right\}\).

4: Compute the \(i\)th entry \(\tilde{d}_k^{(i)}\) of the vector \(\tilde{d}_k\) according to \(\tilde{d}_k^{(i)} = \begin{cases} \frac{f_i(x_k)}{\|\nabla f_i(x_k)\|_2}, & \text{if } i \in \mathcal{I}_k, \\ 0, & \text{otherwise.} \end{cases}\)

5: Select \(i_k \in \mathcal{I}_k\) with probability \(Pr(\text{row} = i_k) = \frac{|\tilde{d}_k^{(i_k)}|^2}{\|\tilde{d}_k\|_2^2}\).

6: Update \(x_{k+1} = x_k - \frac{f_{i_k}(x_k)}{\|\nabla f_{i_k}(x_k)\|_2^2} \nabla f_{i_k}(x_k)\).

7: end for

imply \(i_k \in \mathcal{U}_k\).

Similarly, we can show that the index set \(\mathcal{I}_k\) in Algorithm 2 is also nonempty, that is, the RD-CNK method is also well defined.

In addition, if \(f(x) = Ax - b\), the DR-CNK and RD-CNK methods recover the greedy randomized Kaczmarz method [10] and the greedy randomized Motzkin-Kaczmarz method [12], respectively.

Remark 2 As in [11, 15], a relaxation parameter \(\theta \in [0, 1]\) can be introduced into the quantities \(\varepsilon_k\) in (3) and \(\delta_k\) in (5), thus obtaining

\(\varepsilon_k = \theta \frac{1}{\|f(x_k)\|_2^2} \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\|\nabla f_i(x_k)\|_2^2} + (1 - \theta) \frac{1}{\|f'(x_k)\|_F^2}\)

and

\(\delta_k = \theta \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\|f(x_k)\|_2^2} + (1 - \theta) \frac{1}{m}\).

Then, the corresponding relaxed greedy capped nonlinear Kaczmarz methods can be obtained.
4 Convergence analysis

Below, we present the convergence guarantees for the DR-CNK and RD-CNK methods.

**Theorem 1** If the nonlinear function $f$ satisfies the local tangential cone condition given in Definition 1, $\eta = \max_{i \in [m]} \eta_i < \frac{1}{2}$, and $f(x*) = 0$, then the iterations of the DR-CNK method in Algorithm 1 satisfy

$$
\mathbb{E} \left[ \|x_{k+1} - x_*\|^2 \right] \leq \left( 1 - \frac{1 - 2\eta}{1 + \eta^2} \epsilon_k \| f'(x_k) \| \right) \mathbb{E} \left[ \|x_k - x_*\|^2 \right].
$$

**Proof** From Lemma 1, we have

$$
\|x_{k+1} - x_*\|^2 \leq \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \frac{\|f_{i_k}(x_k)\|^2}{\|\nabla f_{i_k}(x_k)\|^2}.
$$

Taking expectation of both sides conditioned on $x_k$ gives

$$
\mathbb{E}^k \left[ \|x_{k+1} - x_*\|^2 \right] \leq \|x_k - x_*\|^2 - \mathbb{E}^k \left[ \left( 1 - 2\eta \right) \frac{\|f_{i_k}(x_k)\|^2}{\|\nabla f_{i_k}(x_k)\|^2} \right],
$$

$$
\leq \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \mathbb{E}^k \left[ \frac{\|f_{i_k}(x_k)\|^2}{\|\nabla f_{i_k}(x_k)\|^2} \right] = \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \sum_{i_k \in \mathcal{U}_k} \frac{\|\tilde{r}_{i_k}\|^2}{\|\tilde{r}_k\|^2} \frac{\|f_{i_k}(x_k)\|^2}{\|\nabla f_{i_k}(x_k)\|^2},
$$

which together with the definitions of $\epsilon_k$ in (3) and the index subset $\mathcal{U}_k$ in (4) leads to

$$
\mathbb{E}^k \left[ \|x_{k+1} - x_*\|^2 \right] \leq \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \sum_{i_k \in \mathcal{U}_k} \frac{\|\tilde{r}_{i_k}\|^2}{\|\tilde{r}_k\|^2} \epsilon_k \| f(x_k) \|^2,
$$

$$
= \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \epsilon_k \| f(x_k) - f(x_*) \|^2.
$$

Further, considering Lemma 2, we obtain

$$
\mathbb{E}^k \left[ \|x_{k+1} - x_*\|^2 \right] \leq \|x_k - x_*\|^2 - \left( 1 - 2\eta \right) \epsilon_k \frac{1}{1 + \eta^2} \| f'(x_k) (x_k - x_*) \|^2,
$$

$$
\leq \|x_k - x_*\|^2 - \frac{1 - 2\eta}{1 + \eta^2} \epsilon_k \| f'(x_k) \| \| x_k - x_* \|_2^2,
$$

$$
= \left( 1 - \frac{1 - 2\eta}{1 + \eta^2} \epsilon_k h^2(f'(x_k)) \right) \| x_k - x_* \|_2^2.
$$

So, the desired result (7) can be deduced by taking expectation on both sides and using the tower rule of expectation. \(\square\)

**Remark 3** According to $\epsilon_k$ in (3), we have

$$
\epsilon_k \| f'(x_k) \| F^2 = \frac{1}{2} \left( \frac{\| f'(x_k) \| F^2}{\| f(x_k) \|_2^2} \max_{i \in [m]} \frac{| f_i(x_k) |^2}{\| \nabla f_i(x_k) \|_2^2} + 1 \right).
$$
\[ \frac{1}{2} \left( \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\|\nabla f_i(x_k)\|^2_2} / \|f(x_k)\|^2_{F'} + 1 \right) \]

\[ = \frac{1}{2} \left( \max_{i \in [m]} \frac{|f_i(x_k)|^2}{\|\nabla f_i(x_k)\|^2_2} / \sum_{i \in [m]} \frac{\|\nabla f_i(x_k)\|^2_2}{\|f'(x_k)\|^2_{F'}} + 1 \right) \]

\[ \geq 1. \]

That is,

\[ \varepsilon_k \geq \frac{1}{\|f'(x_k)\|^2_{F'}}. \]

Then, it holds that

\[ \frac{1 - 2\eta}{1 + \eta^2} \varepsilon_k h^2 (f'(x_k)) > \frac{1 - 2\eta}{(1 + \eta)^2} \frac{h^2 (f'(x_k))}{\|f'(x_k)\|^2_{F',m}}, \]

which implies

\[ \rho_{\text{DR-CNK}} = 1 - \frac{1 - 2\eta}{1 + \eta^2} \varepsilon_k h^2 (f'(x_k)) < 1 - \frac{1 - 2\eta}{(1 + \eta)^2} \frac{h^2 (f'(x_k))}{\|f'(x_k)\|^2_{F',m}} = \rho_{\text{NRK}} = \rho_{\text{NURK}}. \]

Here \( \rho_{\text{NRK}} \) and \( \rho_{\text{NURK}} \) are respectively the convergence factors of the NRK and NURK methods provided in [9]. So, we can conclude that the convergence factor of the DR-CNK method is strictly smaller than that of the NRK and NURK methods.

**Theorem 2** If the nonlinear function \( f \) satisfies the local tangential cone condition given in Definition 1, \( \eta = \max_{i \in [m]} \eta_i < \frac{1}{2} \), and \( f(x_*) = 0 \), then the iterations of the RD-CNK method in Algorithm 2 satisfy

\[ \mathbb{E} \left[ \|x_{k+1} - x_*\|^2_2 \right] \leq \left( 1 - \frac{1 - 2\eta}{1 + \eta^2} \max_{i \in [m]} \|\nabla f_i(x_k)\|^2_2 \right) \mathbb{E} \left[ \|x_k - x_*\|^2_2 \right]. \]

**Proof** Following an analogous proof process to the DR-CNK method, we can get the inequality

\[ \mathbb{E}^k \left[ \|x_{k+1} - x_*\|^2_2 \right] \leq \|x_k - x_*\|^2_2 - (1 - 2\eta) \mathbb{E}^k \left[ \|f_i(x_k)\|^2 \right]/\|\nabla f_i(x_k)\|^2_2 \]

\[ \leq \|x_k - x_*\|^2_2 - (1 - 2\eta) \max_{i \in [m]} \|\nabla f_i(x_k)\|^2_2 \mathbb{E}^k \left[ |f_i(x_k)|^2 \right] \]

\[ \leq \|x_k - x_*\|^2_2 - \frac{1 - 2\eta}{\max_{i \in [m]} \|\nabla f_i(x_k)\|^2_2} \sum_{i_k \in I_k} \left\| d_{i_k} \right\|^2_2 |f_i(x_k)|^2, \]

and from the definitions of \( \delta_k \) in (5) and the index subset \( I_k \) in (6), we can further obtain

\[ \mathbb{E}^k \left[ \|x_{k+1} - x_*\|^2_2 \right] \leq \|x_k - x_*\|^2_2 - \frac{1 - 2\eta}{\max_{i \in [m]} \|\nabla f_i(x_k)\|^2_2} \sum_{i_k \in I_k} \left\| d_{i_k} \right\|^2_2 \delta_k \|f(x_k)\|^2_2 \]
which together with Lemma 2 leads to

\[
\mathbb{E}^k \left[ \| x_{k+1} - x^* \|_2^2 \right] \leq \| x_k - x^* \|_2^2 - \frac{(1 - 2\eta) \delta_k}{\max_{i \in [m]} \| \nabla f_i(x_k) \|_2} \left( 1 + \frac{\eta^2}{2} \right) \| f'(x_k)(x_k - x^*) \|_2^2 \leq \| x_k - x^* \|_2^2 - \frac{1 - 2\eta}{1 + \eta^2} \frac{\delta_k}{\max_{i \in [m]} \| \nabla f_i(x_k) \|_2} \frac{h_2^2(f'(x_k))}{2} \| x_k - x^* \|_2^2.
\]

Thus, by taking the full expectation on both sides, we get the desired result (8).

\[ \square \]

**Remark 4** Since \( \max_{i \in [m]} \| \nabla f_i(x_k) \|_2^2 \leq \| f'(x_k) \|_2^2 \) and \( \delta_k = \frac{1}{2} \left( \max_{i \in [m]} \frac{\| f_i(x_k) \|_2^2}{\| f(x_k) \|_2^2} + \frac{1}{m} \right) \geq \frac{1}{m} \), we have

\[
\rho_{\text{RD-CNK}} = 1 - \frac{1 - 2\eta}{1 + \eta^2} \frac{\delta_k h_2^2(f'(x_k))}{\max_{i \in [m]} \| \nabla f_i(x_k) \|_2^2} < 1 - \frac{1 - 2\eta}{(1 + \eta)^2} \frac{h_2^2(f'(x_k))}{\| f'(x_k) \|_2^2 m} = \rho_{\text{NRK}} = \rho_{\text{NURK}},
\]

which means that the convergence factor of the RD-CNK method is strictly smaller than that of the NRK and NURK methods presented in [9].

## 5 Multiple samples-based capped methods

After determining the index subsets \( \mathcal{U}_k \) in Algorithm 1 and \( \mathcal{I}_k \) in Algorithm 2, we can directly project the current iteration \( x_k \) onto the solution space of these subsets leading to the corresponding block methods, i.e., the distance-based block capped nonlinear Kaczmarz (DB-CNK) method listed in Algorithm 3 and the residual-based block capped nonlinear Kaczmarz (RB-CNK) method shown in Algorithm 4.

**Remark 5** Since the updating index \( i_k \) in the DR-CNK method belongs to the subset \( \mathcal{U}_k \) which is directly used in the DB-CNK method, we can deduce that the DB-CNK method must converge at least as fast as the DR-CNK method. We can also obtain a similar relationship between the RB-CNK and RD-CNK methods as the index \( i_k \) used in the latter also belongs to the subset \( \mathcal{I}_k \) which is directly used in the former.

In addition, a parameter \( \xi \in (0, 1] \) can be introduced into \( \epsilon_k \) and \( \delta_k \) and then obtaining

\[
\epsilon_k = \xi \frac{1}{\| f(x_k) \|_2^2} \max_{i \in [m]} \frac{| f_i(x_k) |^2}{\| \nabla f_i(x_k) \|_2^2} \quad \text{and} \quad \delta_k = \xi \frac{\max_{i \in [m]} | f_i(x_k) |^2}{\| f(x_k) \|_2^2}.
\]
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Algorithm 3 The DB-CNK method

Require: The initial estimate \(x_0 \in \mathbb{R}^n\).

1: for \(k = 0, 1, 2, \cdots\) until convergence, do
2: \(\varepsilon_k = \frac{1}{2} \left( \frac{1}{\|f(x_k)\|_2} \max_{i \in \{m\}} |f_i(x_k)|^2 \right) \left( \frac{1}{\|\nabla f_i(x_k)\|_2^2} + \frac{1}{\|f'(x_k)\|_F^2} \right)\).
3: Determine the index subset \(\mathcal{U}_k = \{i| f_i(x_k) \geq \varepsilon_k \|f(x_k)\|_2 \sqrt{\|\nabla f_i(x_k)\|_2^2} \} \).
4: Update \(x_{k+1} = x_k - (f'_{\mathcal{U}_k}(x_k)) \dagger f_{\mathcal{U}_k}(x_k)\).
5: end for

Algorithm 4 The RB-CNK method

Require: The initial estimate \(x_0 \in \mathbb{R}^n\).

1: for \(k = 0, 1, 2, \cdots\) until convergence, do
2: \(\delta_k = \frac{1}{2} \left( \frac{\max_{i \in \{m\}} |f_i(x_k)|^2}{\|f(x_k)\|_2^2} + \frac{1}{m} \right)\).
3: Determine the index subset \(\mathcal{I}_k = \{i| f_i(x_k) \geq \delta_k \|f(x_k)\|_2 \} \).
4: Update \(x_{k+1} = x_k - (f'_{\mathcal{I}_k}(x_k)) \dagger f_{\mathcal{I}_k}(x_k)\).
5: end for

as discussed in [12, 16]. Further, if \(|\mathcal{U}_k| = |\mathcal{I}_k| = 1\), the DB-CNK and RB-CNK methods will respectively reduce to the MD-NK and MR-NK methods presented in [13].

6 Convergence analysis

In this section, we establish the convergence theorems of the DB-CNK and RB-CNK methods.

Theorem 3 If the nonlinear function \(f\) satisfies the local tangential cone condition given in Definition 1, \(\eta = \max_{i \in \{m\}} \eta_i < \frac{1}{2}\), \(f(x^*) = 0\) and \(\alpha = h_2^2\left((f'_{\mathcal{U}_k}(x_k))^\dagger\right) - 2\sigma_{\max}^2\left((f'_{\mathcal{U}_k}(x_k))^\dagger\right) > 0\), where
\(h_2^2((f'_{\mathcal{I}_k}(x_k))^\dagger) = \min_{\mathcal{U}_k} h_2^2((f'_{\mathcal{U}_k}(x_k))^\dagger)\) and \(\sigma_{\max}^2\left((f'_{\mathcal{I}_k}(x_k))^\dagger\right) = \max_{\mathcal{U}_k} \sigma_{\max}^2\left((f'_{\mathcal{U}_k}(x_k))^\dagger\right)\),
then the iterations of the DB-CNK method in Algorithm 3 satisfy
\[\|x_{k+1} - x^*\|_2^2 \leq \left(1 - \alpha \min_{i \in \{m\}} \|\nabla f_i(x_k)\|_2 \|\mathcal{U}_k\| \varepsilon_k \frac{1}{1 + \eta^2 h_2^2(f'(x_k))} \right) \|x_k - x^*\|_2^2.\]
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Proof According to Lemma 3, the definition of \( \alpha \), and Algorithm 3, we have
\[
\|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - \left( h_2^2((f'_{I_k}(x_k))^\dagger) - 2\eta\sigma_{\text{max}}^2 \left( (f'_{I_k}(x_k))^\dagger \right) \right) \|f_{I_k}(x_k)\|_2^2
\]
\[
\leq \|x_k - x_*\|_2^2 - \alpha \|f_{I_k}(x_k)\|_2^2
\]
\[
= \|x_k - x_*\|_2^2 - \alpha \sum_{j \in \mathcal{U}_k} \frac{|f_j(x_k)|^2}{\|\nabla f_j(x_k)\|_2^2} \|\nabla f_j(x_k)\|_2^2
\]
\[
\leq \|x_k - x_*\|_2^2 - \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 \sum_{j \in \mathcal{U}_k} \frac{|f_j(x_k)|^2}{\|\nabla f_j(x_k)\|_2^2}
\]
\[
\leq \|x_k - x_*\|_2^2 - \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 |\mathcal{U}_k|\varepsilon_k \|f(x_k)\|_2^2,
\]
which together with Lemma 2 yields
\[
\|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 |\mathcal{U}_k|\varepsilon_k \frac{h_2^2(f'(x_k))}{1 + \eta^2} > 0,
\]
\[
\leq \left( 1 - \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 |\mathcal{U}_k|\varepsilon_k \frac{h_2^2(f'(x_k))}{1 + \eta^2} \right) \|x_k - x_*\|_2^2.
\]
So, the desired result (9) is obtained.

Remark 6 Since 0 \leq \|x_{k+1} - x_*\|_2^2 and \( \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 |\mathcal{U}_k|\varepsilon_k \frac{h_2^2(f'(x_k))}{1 + \eta^2} > 0 \), we have
\[
0 \leq \|x_k - x_*\|_2^2 - \alpha \min_{i \in [m]} \|\nabla f_i(x_k)\|_2^2 |\mathcal{U}_k|\varepsilon_k \frac{h_2^2(f'(x_k))}{1 + \eta^2} \|x_k - x_*\|_2^2 < \|x_k - x_*\|_2^2,
\]
which implies that the convergence factor of the DB-CNk method is smaller than 1. Similarly, we can get that the convergence factor of the RB-CNk method presented in Theorem 4 is also smaller than 1.

Theorem 4 If the nonlinear function \( f \) satisfies the local tangential cone condition given in Definition 1, \( \eta = \max_{i \in [m]} \eta_i < \frac{1}{2} \), \( f(x_*) = 0 \) and \( \beta = h_2^2((f'_{I}(x_k))^\dagger) - 2\eta\sigma_{\text{max}}^2 \left( (f'_{I}(x_k))^\dagger \right) > 0 \), where
\[
h_2^2((f'_{I}(x_k))^\dagger) = \min_{I_k} h_2^2((f'_{I_k}(x_k))^\dagger)
\]
and \( \sigma_{\text{max}}^2 \left( (f'_{I_k}(x_k))^\dagger \right) = \max_{I_k} \sigma_{\text{max}}^2 \left( (f'_{I_k}(x_k))^\dagger \right) \),
then the iterations of the RB-CNk method in Algorithm 4 satisfy
\[
\|x_{k+1} - x_*\|_2^2 \leq \left( 1 - \beta |\mathcal{I}_k|\delta_k \frac{1}{1 + \eta^2} h_2^2(f'(x_k)) \right) \|x_k - x_*\|_2^2.
\]

Proof From Lemma 3, the definition of \( \beta \), and Algorithm 4, we get
\[
\|x_{k+1} - x_*\|_2^2 \leq \|x_k - x_*\|_2^2 - \left( h_2^2((f'_{I_k}(x_k))^\dagger) - 2\eta\sigma_{\text{max}}^2 \left( (f'_{I_k}(x_k))^\dagger \right) \right) \|f_{I_k}(x_k)\|_2^2
\]
\[
\leq \|x_k - x_*\|_2^2 - \beta \|f_{I_k}(x_k)\|_2^2.
\]
\[ = \|x_k - x_\star\|_2^2 - \beta \sum_{j \in I_k} |f_j(x_k)|^2 \]
\[ \leq \|x_k - x_\star\|_2^2 - \beta |I_k| \delta_k \|f(x_k)\|_2^2, \]
which together with Lemma 2 yields
\[ \|x_{k+1} - x_\star\|_2^2 \leq \|x_k - x_\star\|_2^2 - \beta |I_k| \delta_k \frac{1}{1 + \eta^2} \|f'(x_k) (x_k - x_\star)\|_2^2 \]
\[ \leq \left( 1 - \beta |I_k| \delta_k \frac{1}{1 + \eta^2} h^2_f(f'(x_k)) \right) \|x_k - x_\star\|_2^2. \]
So, the desired result (10) is obtained. \[ \square \]

7 Experimental results

In this section, we mainly compare our new methods, i.e., the DR-CNK, RD-CNk, DB-CNk and RB-CNk methods, with the existing methods for solving Brown almost linear function and generalized linear model (GLM) in terms of the iteration numbers (denoted as “IT”) and computing time in seconds (denoted as “CPU”). The IT and CPU here are respectively the average of the IT and CPU over 10 runs of the algorithm and all experiments terminate once \( \|f(x_k)\|_2^2 < 10^{-6} \) or the number of iterations exceeds 200000.

7.1 Brown almost linear function

The function \([9, 17]\) is expressed as follows
\[ f_k(x) = x^{(k)} + \sum_{i=1}^n x^{(i)} - (n + 1), \quad 1 \leq k < n; \]
\[ f_k(x) = (\prod_{i=1}^n x^{(i)}) - 1, \quad k = n. \]

Here we only compare our methods with the NRK method because the authors in [9] have compared the NRK method with other methods in detail and concluded that the NRK method performed better in most cases. All experiments are start at \( x_0 = 0.5 \ast \text{ones}(n, 1) \).

We list the iteration numbers and computing time for the DR-CNk, RD-CNk, DB-CNk, RB-CNk and NRK methods in Tables 1 and 2. They show that our new methods vastly outperform the NRK method. In particular, the time speedup of the single sample-based methods, i.e., the DR-CNk and RD-CNk methods, against to the NRK method is maintained at about 10 times for most cases, and the time speedup of the multiple samples-based methods, i.e., the DB-CNk and RB-CNk methods, against to the NRK method can even reach 200 times, as can be seen in the last two rows of Table 2. Overall, the numerical results indicate that our new greedy capped schemes are better than the randomized strategy used in the NRK method.
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### Table 1

| $m \times n$ | NRK   | DR-CNK | RD-CNK | DB-CNK | RB-CNK |
|-------------|-------|--------|--------|--------|--------|
| 50 $\times$ 50 | 4780.2 | 755.2  | 755    | 1      | 1      |
| 100 $\times$ 100 | 16218 | 1308   | 1308   | 1      | 1      |
| 150 $\times$ 150 | 33764 | 1902   | 1902   | 1      | 1      |
| 200 $\times$ 200 | 3750.2 | 2506.4 | 1      | 1      |
| 250 $\times$ 250 | 84874 | 4371.8 | 4371.8 | 1      | 1      |
| 300 $\times$ 300 | 117100 | 3750   | 1      | 1      |
| 350 $\times$ 350 | 157190 | 4992.4 | 4992.4 | 1      | 1      |

### Table 2

| $m \times n$ | NRK   | DR-CNK | RD-CNK | DB-CNK | RB-CNK |
|-------------|-------|--------|--------|--------|--------|
| 50 $\times$ 50 | 0.2766 | 0.0656 | 0.0578 | 0.0078 | 0.0094 |
| 100 $\times$ 100 | 0.6078 | 0.1453 | 0.0969 | 0.0172 | 0.0125 |
| 150 $\times$ 150 | 1.1656 | 0.2141 | 0.1359 | 0.0063 | 0.0078 |
| 200 $\times$ 200 | 2.0438 | 0.2969 | 0.1953 | 0.0063 | 0.0078 |
| 250 $\times$ 250 | 3.0078 | 0.3391 | 0.2359 | 0.0078 | 0.0422 |
| 300 $\times$ 300 | 4.1422 | 0.3984 | 0.2797 | 0.0266 | 0.0156 |
| 350 $\times$ 350 | 5.7625 | 0.4609 | 0.3359 | 0.0266 | 0.0187 |
| 400 $\times$ 400 | 7.5219 | 0.4922 | 0.3922 | 0.0500 | 0.0375 |

7.2 GLM

The regularized GLM has the form

$$\min_{w \in \mathbb{R}^d} P(w) \overset{\text{def}}{=} \frac{1}{p} \sum_{i=1}^{p} \phi_i (a_i^T w) + \frac{\lambda}{2} \|w\|^2,$$

where $\phi_i(t) = \ln (1 + e^{-y_it})$ is the logistic loss, $y_i \in \{-1, 1\}$ is the $i$th target value, $a_i \in \mathbb{R}^d$ is $i$th data sample, and $w \in \mathbb{R}^d$ is the parameter to optimize. By adopting the equivalent transformation discussed in [7], we can get the following nonlinear problem:

$$f(x) \overset{\text{def}}{=} \left[ \frac{1}{\lambda p} A \alpha - w \right] = 0,$$

where $f : \mathbb{R}^{p+d} \to \mathbb{R}^{p+d}$, $x = \left[ \begin{array}{c} \alpha \\ w \end{array} \right] \in \mathbb{R}^{p+d}$, $A \overset{\text{def}}{=} \left[ a_1, \ldots, a_p \right] \in \mathbb{R}^{d \times p}$, $\Phi(w) \overset{\text{def}}{=} \left[ \phi_1 (a_1^T w), \ldots, \phi_p (a_p^T w) \right]^T \in \mathbb{R}^p$ and $\alpha = -\Phi(w)$. For the problem, we only compare our methods with the sketched Newton-Raphson (SNR) [7] to further illustrate the advantages of our greedy capped sampling over uniform sampling.

We use datasets in the experiments for GLM taken from [18] on https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/ and the scaled versions are applied if provided. They have disparate properties, either ill or well conditioned, dense or sparse; see details in Table 3. Note that in Table 3, C.N is the condition number of the data matrix $A$, the smoothness constant...
Table 3 Details of the data sets for GLM.

| dataset    | dimension (d) | samples (p) | $L$     | C.N    | density |
|------------|---------------|-------------|---------|--------|---------|
| fourclass  | 2             | 862         | 0.0824  | 1.0737 | 0.9959  |
| german.numer| 24            | 1000        | 2.1113  | 15.4082| 0.9584  |
| heart      | 13            | 270         | 0.6973  | 7.0996 | 0.9624  |
| ionosphere | 34            | 351         | 1.5290  | 2.4485e+17 | 0.8841 |
| diabetes   | 8             | 768         | 0.5740  | 8.2105 | 0.9985  |
| sonar      | 60            | 208         | 3.2282  | 89.9388| 0.9999  |
| w1a        | 300           | 2477        | 0.6224  | 3.4303e+34 | 0.0382 |
| w2a        | 300           | 3470        | 0.6347  | 1.0416e+34 | 0.0388 |
| w3a        | 300           | 4912        | 0.6436  | 3.9777e+33 | 0.0388 |

$L \overset{\text{def}}{=} \lambda_{\text{max}}(AA^T) 4p + \lambda$ and

\[
\text{density} = \frac{\text{number of nonzero of a } d \times p \text{ data matrix } A}{dp}
\]

For all methods, we use $\lambda = \frac{1}{p}$ as the regularization parameter and let the initial value $x_0$ be zero, i.e., $w_0 = 0 \in \mathbb{R}^d$ and $\alpha_0 = 0 \in \mathbb{R}^p$.

For the SNR method, different parameter settings will lead to different methods, in which the Kaczmarz-TCS and Block TCS methods are discussed in detail as the main representatives [7, 19]. They are single-sample and multi-sample methods, respectively. So, we will compare the single-sample methods, i.e., the DR-CN, RD-CN, and Kaczmarz-TCS methods, and the multi-sample methods, i.e., the DB-CN, RB-CN, and Block TCS methods, separately.

The parameters used in the example are $\gamma = 1$, $\tau_d = d$, and $\tau_p = 150$, and the Bernoulli parameter $b = \frac{p}{(p+\tau_p)} - 0.11$. In the DB-CN and RB-CN methods, we adopt the same iterative framework as the Block TCS method. Specifically, the least norm solution is computed directly for the first $d$ rows of $f$, while the greedy capped strategies in Algorithms 3 and 4 are used for the last $p$ rows of $f$. In doing so, we can not only use the structure of the nonlinear function $f$, but also directly compare the relationship between greedy capped sampling and uniform sampling.

We show the results of our methods compared to the SNR method, i.e., the Kaczmarz-TCS and Block TCS methods, on GLM in Figures 1 to 3. In the first two figures, we test six datasets for single-sample methods and find that the performance of our DR-CN and RD-CN methods is not much different, but in most cases, the former is slightly better than the latter in terms of iteration numbers and computing time, and both methods are more efficient than the Kaczmarz-TCS method. In Figure 3, we test three datasets for multi-sample methods and get that our RB-CN and DB-CN methods perform about the same, and both methods outperform the Block TCS method. All these figures imply that the new greedy capped strategies are feasible and outperform the uniform sampling.
8 Concluding remarks

This paper proposes two greedy capped nonlinear Kaczmarz methods, i.e., the DR-CNK and RD-CNk methods, which make up for the deficiency of the NRK method, that is, the new methods will definitely not extract the index corresponding to the small component. The theoretical analysis shows that the convergence factors of our two methods are strictly smaller than that of the
NRK method, which is also verified by a large number of numerical examples. Further, we present their block versions for acceleration.

Considering the efficiency of the capped threshold, other threshold strategies can be further explored. In addition, although the new block methods perform well, we cannot determine the specific size of the index set in each iteration, which may lead to extreme cases. That is, either there is only one index in the set, or all indices, which is inconsistent with the original intention of the block sampling iteration. Therefore, how to design a more reasonable index set is also worth further discussion.

**Declarations**

**Ethical Approval.** Not Applicable

**Availability of supporting data.** The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Competing Interests.** The authors declare that they have no conflict of interest.

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