Biharmonic Hypersurfaces with Constant Scalar Curvature in $E_5^s$

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Abstract. In this paper, we obtain that every biharmonic non-degenerate hypersurfaces in semi-Euclidean space $E_5^s$ with constant scalar curvature of diagonal shape operator has zero mean curvature.

1. Introduction

In 1964, Eells and Sampson [16] introduced the notion of poly-harmonic maps as a natural generalization of the well-known harmonic maps. Thus, while $\phi : (M, g) \rightarrow (N, h)$ harmonic maps between Riemannian manifolds are critical points of the energy functional $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$, the biharmonic maps are critical points of the bienergy functional $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g$, where $\tau = \text{trace} \nabla d\phi$

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is the tension field of $\phi$.

The study of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in the middle of 1980s. In particular, he proved that biharmonic surfaces in Euclidean 3-spaces are minimal. There are many non-existence results in Euclidean spaces developed by I. Dimitric in his doctoral thesis [14] and paper [15]. Based on these results, B. Y. Chen [7] in 1991 posed the following well-known conjecture: The only biharmonic submanifolds of Euclidean spaces are the minimal ones. Also, the conjecture was later proved for hypersurfaces in Euclidean 4-spaces [22] and for hypersurfaces with three distinct principal curvatures in $E^5$ [23]. Recently, it was proved that Chen’s conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension [11] and for hypersurfaces in Euclidean spaces of arbitrary dimension with three distinct principal curvatures [20]. Also, it was shown that the conjecture is true for every biharmonic hypersurfaces in $E^5$ with zero scalar curvature [12]. The conjecture is a local problem and understanding local structure of biharmonic submanifolds to the point of minimality is a complex task. That may be the possible reason for the conjecture to be open till now in general so far. The global version of Chen’s conjecture for biharmonic submanifolds in Euclidean space was studied in [17]. There exist lots of examples of proper biharmonic submanifolds in spheres (see, for instance [2-6, 18-19]).

In contrast to the submanifolds of Euclidean spaces, Chen’s conjecture is not true always for submanifolds of pseudo-Euclidean spaces. For example, B. Y. Chen and S. Ishikawa [9, 10] obtained some examples of proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces $E^4_s$ for $s = 1, 2, 3$, (see also [8]). But for hypersurfaces in pseudo-Euclidean spaces, it is reasonable that Chen’s conjecture is also right. It was proved that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal [9, 10]. In [13], F. Defever et al. proved that the biharmonic conjecture is true for non-degenerate hypersurfaces of semi-Euclidean 4-spaces. A. Arvanitoyeorgos et al. [1] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal. Recently, it was proved that every biharmonic hypersurfaces with three distinct principal curvatures of diagonal shape operator in $E^5_s$ must be minimal [21]. It led us to investigate biharmonic hypersurface in semi-Euclidean 5-spaces with four distinct principal curvatures.

In this paper, we study biharmonic non-degenerate hypersurfaces of constant scalar curvature in semi-Euclidean spaces $E^5_s$ with diagonal shape operator.

2. Preliminaries

Let $(M^4_s, g)$, $r = 0, 1, 2, 3, 4$, be a 4-dimensional hypersurface isometrically immersed in a 5-dimensional semi-Euclidean space $(E^5_s, \bar{g})$, $s = 0, 1, 2, 3, 4, 5$ and $g = \bar{g} |_{M^4_s}$. We denote by $\xi$ unit normal vector to $M^4_s$ with $\bar{g}(\xi, \xi) = \varepsilon$, where $\varepsilon = \pm 1$, according as $M^4_s$ is pseudo-Riemannian or Riemannian hypersurface.

Let $\nabla$ and $\nabla$ denote linear connections on $E^5_s$ and $M^4_s$, respectively. Then, the
Gauss and Weingarten formulae are given by

\[(2.1) \quad \nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall \ X, Y \in \Gamma(TM^4),\]

\[(2.2) \quad \nabla_X \xi = -A_\xi X,\]

where \(h\) is the second fundamental form and \(A\) is the shape operator. It is well known that the second fundamental form \(h\) and shape operator \(A\) are related by

\[(2.3) \quad \nabla(h(X, Y), \xi) = g(A_\xi X, Y).\]

The mean curvature is given by

\[(2.4) \quad \varepsilon H = \frac{1}{4} \text{trace} A.\]

The Gauss and Codazzi equations are given by

\[(2.5) \quad R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY,\]

\[(2.6) \quad (\nabla_X A)Y = (\nabla_Y A)X,\]

respectively, where \(R\) is the curvature tensor and

\[(2.7) \quad (\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y),\]

for all \(X, Y, Z \in \Gamma(TM^4)\).

A biharmonic submanifold in a semi-Euclidean space is called proper biharmonic if it is not minimal. The necessary and sufficient condition for \(M^4\) to be biharmonic in \(E^5_s\) is

\[(2.8) \quad \triangle H + \varepsilon H \text{trace} A^2 = 0,\]

\[(2.9) \quad A(\text{grad} H) + 2\varepsilon H \text{grad} H = 0,\]

where \(H\) denotes the mean curvature. Also, the Laplace operator \(\triangle\) of a scalar valued function \(f\) is given by [8]

\[(2.10) \quad \triangle f = -\sum_{i=1}^{4} \epsilon_i(e_i e_i f - \nabla_{e_i} e_i f),\]

where \(\{e_1, e_2, e_3, e_4\}\) is an orthonormal local tangent frame on \(M^4\) and \(g(e_i, e_i) = \epsilon_i\).

A vector \(X\) in \(E^5_s\) is called spacelike, timelike or lightlike according as \(\nabla(X, X) > 0\), \(\nabla(X, X) < 0\) or \(\nabla(X, X) = 0\), respectively. A non-degenerate hypersurface \(M^4\) of \(E^5_s\) is called Riemannian or pseudo-Riemannian according as the induced metric on \(M^4\) from the indefinite metric on \(E^5_s\) is definite or indefinite. A shape operator of pseudo-Riemannian hypersurfaces is not diagonalizable always unlike the Riemannian hypersurfaces.
3. Biharmonic Non-Degenerate Hypersurfaces of Constant Scalar Curvature in $E_5$

We have the following cases.

(a) The case of four distinct principal curvatures

In this section, we study biharmonic non-degenerate hypersurfaces Riemannian or pseudo-Riemannian $M_4^r$ with diagonal shape operator. We also assume that mean curvature is not constant and $\text{grad} H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset $U$ of $M_4^r$, with $\text{grad}_p H \neq 0$ for all $p \in U$. From (2.9), it is easy to see that $\text{grad} H$ is an eigenvector of the shape operator $A$ with the corresponding principal curvature $-2 \epsilon H$. The $\text{grad} H$ can be spacelike or timelike. Without losing generality, we choose $e_1$ in the direction of $\text{grad} H$ and therefore shape operator $A$ of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

\[
(3.1) \quad A_H = \begin{pmatrix}
-2 \epsilon H & \lambda_2 \\
\lambda_3 & \lambda_4
\end{pmatrix}.
\]

The $\text{grad} H$ can be expressed as

\[
(3.2) \quad \text{grad} H = \sum_{i=1}^{4} e_i(H)e_i.
\]

As we have taken $e_1$ parallel to $\text{grad} H$, consequently

\[
(3.3) \quad e_1(H) \neq 0, e_2(H) = 0, e_3(H) = 0, e_4(H) = 0.
\]

We express

\[
(3.4) \quad \nabla e_i e_j = \sum_{k=1}^{4} \epsilon_k \omega^k_{ij} e_k, \quad i, j = 1, 2, 3, 4.
\]

Using compatibility conditions $\nabla e_i g(e_i, e_i) = 0$ and $\nabla e_i g(e_i, e_j) = 0$, we obtain

\[
(3.5) \quad \omega^i_{ki} = 0, \quad \omega^j_{ki} + \omega^i_{kj} = 0,
\]

for $i \neq j$, and $i, j, k = 1, 2, 3, 4$.

From Codazzi equation (2.6), we have $g((\nabla e_i A)e_j, e_j) = g((\nabla e_j A)e_i, e_j)$ and then using (2.7), (3.1) and (3.4), we obtain

\[
(3.6) \quad e_j e_i(\lambda_j) = (\lambda_i - \lambda_j) \omega^i_{ji},
\]
and similarly, \( g((\nabla_{e_i} A)e_j, e_k) = g((\nabla_{e_j} A)e_i, e_k) \) gives
\[
(\lambda_i - \lambda_j)\omega^j_{ki} = (\lambda_k - \lambda_j)\omega^j_{ik},
\]
respectively, for distinct \( i, j, k = 1, 2, 3, 4 \).

Since \( \lambda_1 = -2\varepsilon H \), from (3.3), we get
\[
e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, e_4(\lambda_1) = 0.
\]
Also, it is easy to show that
\[
[e_i, e_j](\lambda_1) = 0, \quad i, j = 2, 3, 4,
\]
which gives
\[
\omega^1_{ij} = \omega^1_{ji},
\]
for \( i \neq j \) and \( i, j = 2, 3, 4 \).

Now, we show that \( \lambda_j \neq \lambda_1, j = 2, 3, 4 \). In fact, if \( \lambda_j = \lambda_1 \) for \( j \neq 1 \), then from (3.6), we find
\[
\varepsilon_j e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega^1_{jj} = 0,
\]
which contradicts the first expression of (3.8).

Since \( M^4 \) has four distinct principal curvatures, from (2.4), we obtain that
\[
\lambda_2 + \lambda_3 + \lambda_4 = 6\varepsilon H.
\]
Putting \( i \neq 1, j = 1 \) in (3.6) and using (3.8) and (3.5), we find
\[
\omega^1_{ii} = 0, \quad i = 1, 2, 3, 4.
\]
Putting \( k = 1, j \neq i \), and \( i, j = 2, 3, 4 \) in (3.7), and using (3.9), we get
\[
\omega^1_{ij} = \omega^1_{ji} = \omega^j_{i1} = \omega^j_{1i} = 0, \quad j \neq i \text{ and } i, j = 2, 3, 4.
\]

Thus, we have the following:

**Lemma 3.1.** Let \( M^4, r = 0, 1, 2, 3, 4 \), be a biharmonic non-degenerate hypersurface of non-constant mean curvature with four distinct principal curvatures in semi-Euclidean space \( E^5_s \), \( s = 0, 1, 2, 3, 4, 5 \), having the shape operator given by (3.1) with respect to suitable orthonormal frame \( \{e_1, e_2, e_3, e_4\} \). Then, we obtain
\[
\nabla_{e_i} e_i = 0, \quad i = 1, 2, 3, 4,
\]
\( \nabla_{e_i} e_1 = \omega^1_{i1} e_i, \quad i = 2, 3, 4, \)

(3.15)

\( \nabla_{e_i} e_i = \omega^j_i e_j, \quad i = 2, 3, 4, \)

(3.16)

\( \nabla_{e_i} e_j = \sum_{i \neq j, k} \omega^k_{ij} e_k, \quad i, j = 2, 3, 4, \) and \( i \neq j, \)

(3.17)

where \( \omega^i_{ij} \) satisfy (3.5) and (3.6) for \( i, j = 1, 2, 3, 4. \)

Evaluating Riemannian curvatures, using Lemma 3.1, Gauss equation and comparing the coefficients with respect to an orthonormal frame \( \{e_1, e_2, e_3, e_4\} \), we find the following:

- \( X = e_1, Y = e_2, Z = e_1, \)

\[ e_1 (\omega^1_{22}) e_2 - (\omega^1_{22})^2 = -2 \varepsilon e_1 H \lambda_2. \]

(3.18)

- \( X = e_1, Y = e_3, Z = e_1, \)

\[ e_1 (\omega^1_{33}) e_3 - (\omega^1_{33})^2 = -2 \varepsilon e_1 H \lambda_3. \]

(3.19)

- \( X = e_1, Y = e_4, Z = e_1, \)

\[ e_1 (\omega^1_{44}) e_4 - (\omega^1_{44})^2 = -2 \varepsilon e_1 H \lambda_4. \]

(3.20)

- \( X = e_1, Y = e_2, Z = e_2, \)

\[ e_1 (\omega^2_{22}) e_2 - \omega^2_{22} \omega^1_{22} = 0. \]

(3.21)

- \( X = e_1, Y = e_3, Z = e_3, \)

\[ e_1 (\omega^3_{33}) e_3 - \omega^3_{33} \omega^1_{33} = 0. \]

(3.23)

- \( X = e_1, Y = e_4, Z = e_4, \)

\[ e_1 (\omega^4_{44}) e_4 - \omega^4_{44} \omega^1_{44} = 0. \]

(3.25)
• \(X = e_2, Y = e_3, Z = e_2\),

\[
e_2(\omega^2_{33}) + e_3(\omega^2_{22}) - e_1\omega^1_{12}\omega^1_{33} - e_4\omega^4_{12}\omega^4_{33} - e_2(\omega^2_{22})^2 - e_3(\omega^3_{33})^2 + (\omega^3_{13}\omega^3_{23} - \omega^3_{34}\omega^3_{14} - \omega^3_{32}\omega^3_{24})\epsilon_4 = e_2\epsilon_2\lambda_2\lambda_3.
\]

(3.27)

• \(X = e_2, Y = e_4, Z = e_2\),

\[
e_3(\omega^3_{22}) + e_3\omega^3_{22}\omega^3_{33} - e_2\omega^2_{22}\omega^2_{12} = 0.
\]

(3.28)

\[
e_3(\omega^3_{22}) + e_3\omega^3_{22}\omega^3_{33} - e_2\omega^2_{22}\omega^2_{12} = 0.
\]

(3.29)

• \(X = e_3, Y = e_4, Z = e_3\),

\[
e_4(\omega^4_{22}) + e_4\omega^4_{22}\omega^4_{33} - e_2\omega^2_{22}\omega^2_{12} = 0.
\]

(3.30)

\[
e_4(\omega^4_{22}) + e_4\omega^4_{22}\omega^4_{33} - e_2\omega^2_{22}\omega^2_{12} = 0.
\]

(3.32)

• \(X = e_2, Y = e_3, Z = e_3\),

\[
e_3(\omega^3_{22}) + e_1\omega^1_{13}\omega^1_{33} - e_2\omega^2_{23}\omega^2_{13} - e_3(\omega^3_{33})^2 - e_4(\omega^4_{44})^2 + (\omega^3_{34}\omega^3_{24} - \omega^3_{32}\omega^3_{24} - \omega^3_{34}\omega^3_{24})\epsilon_3 = e_2\epsilon_3\lambda_3\lambda_4.
\]

(3.33)

\[
e_4(\omega^4_{22}) + e_4\omega^4_{22}\omega^4_{33} - e_2\omega^2_{22}\omega^2_{12} = 0.
\]

(3.35)

• \(X = e_2, Y = e_3, Z = e_3\),

\[
e_2(\omega^2_{33}) + e_2\omega^2_{33}\omega^2_{12} - e_3\omega^3_{33}\omega^3_{13} = 0.
\]

(3.36)

• \(X = e_2, Y = e_4, Z = e_4\),

\[
e_2(\omega^2_{33}) + e_2\omega^2_{33}\omega^2_{12} - e_3\omega^3_{33}\omega^3_{13} = 0.
\]

(3.37)

• \(X = e_3, Y = e_4, Z = e_4\),

\[
e_2(\omega^2_{33}) + e_2\omega^2_{33}\omega^2_{12} - e_3\omega^3_{33}\omega^3_{13} = 0.
\]

(3.38)

\[
e_2(\omega^2_{33}) + e_2\omega^2_{33}\omega^2_{12} - e_3\omega^3_{33}\omega^3_{13} = 0.
\]

(3.39)

• \(X = e_3, Y = e_4, Z = e_4\),

\[
e_3(\omega^3_{22}) + e_3\omega^3_{22}\omega^3_{33} - e_4\omega^4_{44}\omega^4_{14} = 0.
\]

(3.40)
\begin{equation}
e_3(\omega^2_{44}) + \epsilon_3\omega^3_{44}\omega^2_{33} - \epsilon_4\omega^3_{44}\omega^2_{44} = 0.
\end{equation}

From (3.1) and Gauss equation, the scalar curvature \(\rho\) is given by
\begin{equation}
\rho = 12H^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2.
\end{equation}

Using (2.5), (2.8), (2.10), (3.1) and Lemma 3.1, we find
\begin{equation}
-\epsilon_1e_1(H) + (\epsilon_2\omega^1_{22} + \epsilon_3\omega^1_{33} + \epsilon_4\omega^1_{44})e_1e_1(H) + \epsilon H(16H^2 - \rho) = 0.
\end{equation}

From (3.3) and Lemma 3.1, we obtain
\begin{equation}
e_i e_1(H) = 0, \quad i = 2, 3, 4.
\end{equation}

For constant scalar curvature \(\rho\), using (3.43) and (3.44), we get
\begin{equation}
e_i(\epsilon_2\omega^1_{22} + \epsilon_3\omega^1_{33} + \epsilon_4\omega^1_{44}) = 0, \quad i = 2, 3, 4.
\end{equation}

Also, using Lemma 3.1, it is easy to see that
\begin{equation}
[e_1, e_i] = \epsilon_i\omega^i_{ij}e_i, \quad i = 2, 3, 4.
\end{equation}

Now, we have

**Lemma 3.2.** Let \(M^4, r = 0, 1, 2, 3, 4\), be a biharmonic non-degenerate hypersurface of constant scalar curvature with four distinct principal curvatures in semi-Euclidean space \(E^s_n, s = 0, 1, 2, 3, 4, 5\), having the shape operator given by (3.1) with respect to suitable orthonormal frame \(\{e_1, e_2, e_3, e_4\}\). Then, \(e_i(\lambda_j) = 0\), for \(i, j = 2, 3, 4\), and \(i \neq j\).

**Proof.** Operating with \(e_2\) on both sides of (3.42), (3.11) and using (3.6), we find
\begin{equation}
(\lambda_2 - \lambda_4)^2\omega^2_{44}e_4 + (\lambda_2 - \lambda_3)^2\omega^2_{33}e_3 = 0.
\end{equation}

Differentiating (3.47) along \(e_1\) and using (3.6), (3.23), (3.25), we get
\begin{equation}
[-2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4)\omega^2_{22} + \epsilon_4(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_4)\omega^1_{44}]\omega^2_{44}e_4 + [-2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\omega^2_{22} + \epsilon_3(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_3)\omega^1_{33}]\omega^2_{33}e_3 = 0.
\end{equation}

and using (3.47) in the above equation, we get
\begin{equation}
[2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega^2_{22} + \epsilon_4(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_4)\omega^1_{44} - \epsilon_3(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_3)\omega^1_{33}]\omega^2_{44} = 0.
\end{equation}

Similarly, acting with \(e_1\) and \(e_2\) on (3.11), successively and using (3.6), (3.45), (3.36), (3.38) and (3.47), subsequently, we obtain
\begin{equation}
[e_2(\lambda_1 - \lambda_3)\omega^1_{22} + \epsilon_4(\lambda_3 - \lambda_2)\omega^1_{44} + \epsilon_3(\lambda_2 - \lambda_4)\omega^1_{33}]\omega^2_{44} = 0.
\end{equation}
Equations (3.48) and (3.49) show that either \( \omega^2_{24} \), or the expression between square brackets, has to vanish. We now prove that \( \omega^2_{24} \) has to be zero. In fact, if \( \omega^2_{24} \neq 0 \), then the expressions between square brackets has to be zero:

\[
2e_2(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\omega^2_{22} + e_4(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3)\omega^2_{44} - e_3(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)\omega^2_{33} = 0.
\]

(3.50)

\[
e_2(\lambda_4 - \lambda_3)\omega^1_{22} + e_4(\lambda_3 - \lambda_4)\omega^1_{44} + e_3(\lambda_2 - \lambda_4)\omega^1_{33} = 0.
\]

(3.51)

Eliminating \( \omega^1_{22} \) from (3.50) and (3.51), we get

\[
(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(e_3\omega^1_{33} + e_4\omega^1_{44}) = 0,
\]

(3.52)

which shows that

\[
e_3\omega^1_{33} + e_4\omega^1_{44} = 0.
\]

(3.53)

If (3.53) is true, then using it to eliminate \( \omega^1_{33} \), from (3.50) and (3.51), we find

\[
2e_2(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\omega^2_{22} + e_4[(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3) + (2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)]\omega^2_{44} = 0.
\]

(3.54)

\[
e_2(\lambda_4 - \lambda_3)\omega^1_{22} + e_4(\lambda_3 + \lambda_4 - 2\lambda_2)\omega^1_{44} = 0.
\]

(3.55)

From (3.54) and (3.55), we obtain

\[
(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) = 0,
\]

(3.56)

which is contradiction of the fact that principal curvatures are distinct. Therefore, \( \omega^2_{24} = 0 \), which gives \( \omega^2_{33} = 0 \) in view of (3.47). Consequently, \( e_2(\lambda_3) = e_2(\lambda_4) = 0 \).

Now, we claim \( e_3(\lambda_2) = e_3(\lambda_4) = 0 \). To prove this operating with \( e_3 \) on both sides of (3.42), (3.11) and using (3.6), we find

\[
(\lambda_3 - \lambda_4)^2\omega^2_{24}e_4 + (\lambda_3 - \lambda_2)^2\omega^2_{22}e_2 = 0.
\]

(3.57)

Differentiating (3.57) along \( e_1 \) and using (3.6), (3.21) and (3.26), we get

\[
[-2e_3(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_4)\omega^1_{33} + e_4(2\lambda_1 + \lambda_3 - 3\lambda_4)(\lambda_3 - \lambda_4)\omega^2_{44}]\omega^3_{24}e_4 + [-2e_3(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)\omega^1_{33} + e_2(2\lambda_1 + \lambda_3 - 3\lambda_2)(\lambda_3 - \lambda_2)\omega^2_{22}]\omega^3_{22}e_2 = 0.
\]

Using (3.57) in the above equation, we get

\[
[2e_3(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\omega^1_{33} + e_4(2\lambda_1 + \lambda_3 - 3\lambda_4)(\lambda_3 - \lambda_2)\omega^2_{44} - e_2(2\lambda_1 + \lambda_3 - 3\lambda_2)(\lambda_3 - \lambda_4)\omega^2_{22}]\omega^3_{24}e_2 = 0.
\]

(3.58)
Similarly, acting with $\epsilon_4$ and $\epsilon_3$ on (3.11), successively and using (3.6), (3.45), (3.28), (3.40) and (3.57), subsequently we obtain

\begin{equation}
(3.59) \quad [\epsilon_3(\lambda_1 - \lambda_2)\omega_{33}^1 + \epsilon_4(\lambda_2 - \lambda_3)\omega_{14}^1 + \epsilon_2(\lambda_3 - \lambda_4)\omega_{22}^1]\omega_{44}^2 = 0.
\end{equation}

Equations (3.58) and (3.59) show that either $\omega_{44}^2$, or the expression between square brackets, has to vanish. We now prove that $\omega_{44}^2$, has to be zero. In fact, if $\omega_{44}^2 \neq 0$, then the expressions between square brackets has to be zero:

\begin{equation}
(3.60) \quad 2\epsilon_3(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^2 + \epsilon_4(2\lambda_1 + \lambda_3 - 3\lambda_4)(\lambda_3 - \lambda_2)\omega_{14}^2 - \epsilon_2(2\lambda_1 + \lambda_3 - 3\lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^2 = 0.
\end{equation}

\begin{equation}
(3.61) \quad \epsilon_3(\lambda_4 - \lambda_2)\omega_{33}^1 + \epsilon_4(\lambda_2 - \lambda_3)\omega_{44}^1 + \epsilon_2(\lambda_3 - \lambda_4)\omega_{22}^1 = 0.
\end{equation}

Eliminating $\omega_{33}^1$ from (3.60) and (3.61), we get

\begin{equation}
(3.62) \quad (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)(\epsilon_2\omega_{22}^1 + \epsilon_4\omega_{44}^1) = 0,
\end{equation}

which shows that

\begin{equation}
(3.63) \quad \epsilon_2\omega_{22}^1 + \epsilon_4\omega_{44}^1 = 0.
\end{equation}

If (3.63) is true, then using it to eliminate $\omega_{22}^1$, from (3.60) and (3.61), we find

\begin{equation}
(3.64) \quad 2\epsilon_3(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^2 + \epsilon_4([2\lambda_1 + \lambda_3 - 3\lambda_4](\lambda_3 - \lambda_2)
+ (2\lambda_1 + \lambda_3 - 3\lambda_2)(\lambda_3 - \lambda_4)]\omega_{44}^2 = 0.
\end{equation}

\begin{equation}
(3.65) \quad \epsilon_3(\lambda_4 - \lambda_2)\omega_{33}^1 + \epsilon_4(\lambda_2 + \lambda_4 - 2\lambda_3)\omega_{44}^1 = 0.
\end{equation}

From (3.64) and (3.65), we obtain

\begin{equation}
(3.66) \quad (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4) = 0,
\end{equation}

which is a contradiction of the fact that principal curvatures are distinct. Therefore, $\omega_{44}^2 = 0$, which gives $\omega_{22}^1 = 0$ in view of (3.57). Consequently, $\epsilon_3(\lambda_2) = \epsilon_3(\lambda_4) = 0$.

Now, we claim $\epsilon_4(\lambda_2) = \epsilon_4(\lambda_3) = 0$. To prove this acting $\epsilon_4$ on both sides of (3.42), (3.11) and using (3.6), we find

\begin{equation}
(3.67) \quad (\lambda_4 - \lambda_3)^2\omega_{33}^2\epsilon_3 + (\lambda_4 - \lambda_2)^2\omega_{22}^1\epsilon_2 = 0.
\end{equation}

Differentiating (3.67) along $\epsilon_1$ and using (3.6), (3.22) and (3.24), we get

\begin{equation}
[-2\epsilon_4(\lambda_1 - \lambda_4)(\lambda_4 - \lambda_3)\omega_{44}^2 + \epsilon_3(2\lambda_1 + \lambda_4 - 3\lambda_3)(\lambda_4 - \lambda_3)\omega_{33}^2\epsilon_3]
+ [-2\epsilon_4(\lambda_1 - 
- \lambda_4)(\lambda_4 - \lambda_3)\omega_{44}^2 + \epsilon_2(2\lambda_1 + \lambda_4 - 3\lambda_3)(\lambda_4 - \lambda_3)\omega_{22}^1\epsilon_2] = 0
\end{equation}
and using (3.67) in the above equation, we get

\[
2\epsilon_4(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)\omega^1_{14} + \epsilon_3(2\lambda_1 + \lambda_4 - 3\lambda_3)(\lambda_4 - \lambda_2)\omega^1_{53} - \epsilon_2(2\lambda_1 + \lambda_4 - 3\lambda_2)(\lambda_4 - \lambda_3)\omega^1_{22}\omega^1_{33} = 0.
\]

Similarly, acting with \(\epsilon_1\) and \(\epsilon_4\) on (3.11), successively and using (3.6), (3.45), (3.31), (3.34) and (3.67), subsequently we obtain

\[
\epsilon_4(\lambda_3 - \lambda_2)\omega^1_{34} + \epsilon_3(\lambda_2 - \lambda_4)\omega^1_{13} + \epsilon_2(\lambda_4 - \lambda_3)\omega^1_{12} = 0.
\]

Equations (3.68) and (3.69) show that either \(\omega^1_{33}\), or the expression between square brackets, has to vanish. We now prove that \(\omega^1_{33}\) has to be zero. In fact, if \(\omega^1_{33} \neq 0\) then the expressions between square brackets has to be zero:

\[
2\epsilon_4(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)\omega^1_{14} + \epsilon_3(2\lambda_1 + \lambda_4 - 3\lambda_3)(\lambda_4 - \lambda_2)\omega^1_{53} - \epsilon_2(2\lambda_1 + \lambda_4 - 3\lambda_2)(\lambda_4 - \lambda_3)\omega^1_{22} = 0.
\]

Eliminating \(\omega^1_{14}\) from (3.70) and (3.71), we get

\[
(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)(\epsilon_2\omega^1_{22} + \epsilon_3\omega^1_{33}) = 0,
\]

which shows that

\[
\epsilon_2\omega^1_{22} + \epsilon_3\omega^1_{33} = 0.
\]

If (3.73) is true, then using it to eliminate \(\omega^1_{22}\), from (3.70) and (3.71), we find

\[
2\epsilon_4(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)\omega^1_{14} + \epsilon_3[(2\lambda_1 + \lambda_4 - 3\lambda_3)(\lambda_4 - \lambda_2) + (2\lambda_1 + \lambda_4 - 3\lambda_2)(\lambda_4 - \lambda_3)]\omega^1_{33} = 0.
\]

(3.75)

\[
\epsilon_4(\lambda_3 - \lambda_2)\omega^1_{34} + \epsilon_3(\lambda_2 + \lambda_3 - 2\lambda_4)\omega^1_{33} = 0.
\]

From (3.74) and (3.75), we obtain

\[
(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3) = 0,
\]

which is a contradiction of the fact that principal curvatures are distinct. Therefore, \(\omega^1_{33} = 0\), which gives \(\omega^1_{22} = 0\) in view of (3.67). Consequently, \(\epsilon_4(\lambda_2) = \epsilon_4(\lambda_3) = 0\), which completes the proof. \(\square\)

Now, we have:

**Lemma 3.3.** Let \(M^4_r, r = 0, 1, 2, 3, 4,\) be a biharmonic non-degenerate hypersurface of constant scalar curvature with four distinct principal curvatures in semi-Euclidean space \(E^5_s, s = 0, 1, 2, 3, 4, 5,\) having the shape operator given by (3.1) with respect to suitable orthonormal frame \(\{e_1, e_2, e_3, e_4\}\). Then, we have

\[
(\lambda_3 - \lambda_4)\epsilon_2\omega^1_{22} + (\lambda_4 - \lambda_2)\epsilon_3\omega^1_{33} + (\lambda_2 - \lambda_3)\epsilon_4\omega^1_{44} = 0
\]
or

$$\omega^i_{kj} = 0,$$

for $i \neq j \neq k$, and $i, j, k = 2, 3, 4$.

**Proof.** Using Lemma 3.2 in (3.16) and (3.17), we get

$$\nabla e_2 e_3 = \omega^4_{23} e_4 e_4,$$

and

$$\nabla e_3 e_2 = \omega^4_{32} e_4 e_4,$$

respectively. Using it along with (2.5) and Lemma 3.1 to evaluate $g(R(e_1, e_2)e_3, e_4)$, and $g(R(e_1, e_3)e_2, e_4)$, we obtain

$$e_1(\omega^4_{23}) - \epsilon_2 \omega^1_{22} \omega^4_{33} = 0,$$

and

$$e_1(\omega^4_{32}) - \epsilon_3 \omega^1_{33} \omega^4_{22} = 0,$$

respectively.

Putting $j = 4, k = 2, i = 3$ in (3.7), we get

$$\lambda_3 - \lambda_4 \omega^4_{23} = (\lambda_2 - \lambda_4) \omega^4_{32}.$$

Differentiating (3.79) with respect to $e_1$ and simplifying, we get

$$\omega^4_{32}(\epsilon_2 \omega^1_{22} - \epsilon_4 \omega^1_{44}) = \omega^4_{23}(\epsilon_3 \omega^1_{33} - \epsilon_4 \omega^1_{44}).$$

Now, (3.79) and (3.80) are homogeneous system of equations in two variables $\omega^4_{32}$ and $\omega^4_{23}$ having either non trivial solution or trivial solution. If it has trivial solution only, then, we have $\omega^4_{32} = 0$ and $\omega^4_{23} = 0$.

If it has non-trivial solution also, then the determinant formed by coefficients of $\omega^4_{32}$ and $\omega^4_{23}$ in (3.79) and (3.80) will be zero, i.e.,

$$(\lambda_3 - \lambda_4)\epsilon_2 \omega^1_{22} + (\lambda_4 - \lambda_2)\epsilon_3 \omega^1_{33} + (\lambda_2 - \lambda_3)\epsilon_4 \omega^1_{44} = 0.$$

Similarly, we can prove that either $\omega^4_{32} = \omega^4_{23} = \omega^2_{33} = \omega^2_{34} = 0$ or the determinant formed by their coefficients is zero. This completes the proof of the Lemma. □

Then we have the following two cases:

**Case A:** Let (3.78) holds, then using Lemma 3.2 in (3.27), (3.30) and (3.33), we find

$$-\epsilon_1 \omega^1_{22} \omega^1_{33} = \epsilon_2 \epsilon_3 \lambda_2 \lambda_3,$$

$$-\epsilon_1 \omega^1_{22} \omega^1_{44} = \epsilon_2 \epsilon_4 \lambda_2 \lambda_4,$$

and

$$-\epsilon_1 \omega^1_{33} \omega^1_{44} = \epsilon_3 \epsilon_4 \lambda_3 \lambda_4,$$

respectively.
respectively.

Using (3.11), (3.18)–(3.20) and (3.42), we get

\[
3e_1e_1(H) = -e_1(H)\left\{ \frac{e_1(x_2)}{\lambda_2 - \lambda_1} + \frac{e_1(x_3)}{\lambda_3 - \lambda_1} + \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right\} = -e_1H(2AH^2 - \rho).
\]

From (3.43) and (3.84), we find

\[
\frac{\epsilon^2(\lambda_2)}{\lambda_2 - \lambda_1} + \frac{\epsilon^2(\lambda_3)}{\lambda_3 - \lambda_1} + \frac{\epsilon^2(\lambda_4)}{\lambda_4 - \lambda_1} - 2e_1(H)\left\{ \frac{e_1(x_2)}{\lambda_2 - \lambda_1} + \frac{e_1(x_3)}{\lambda_3 - \lambda_1} + \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right\} = e_1H[2AH^2(1 + 2\varepsilon) + \rho(1 + 3\varepsilon)].
\]

Using (3.81)–(3.83) in (3.85), we obtain

\[
e_1e_1(H)\left\{ \frac{e_1(x_2)}{\lambda_2 - \lambda_1} + \frac{e_1(x_3)}{\lambda_3 - \lambda_1} + \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right\} = H^3(6 - 18\varepsilon) - \frac{\rho H(1 + 8\varepsilon)}{4} + 3\lambda_2\lambda_3\lambda_4.
\]

Using (3.43) and (3.86), we have

\[
e_1e_1(H) = H^3(6 - 2\varepsilon) - \frac{\rho H(1 + 12\varepsilon)}{4} + 3\lambda_2\lambda_3\lambda_4.
\]

On the other hand, using (3.6) and (3.81)–(3.83), we have

\[
e_1(\lambda_2\lambda_3\lambda_4) = \lambda_2\lambda_3\lambda_4\left\{ \frac{e_1(x_2)}{\lambda_2 - \lambda_1} + \frac{e_1(x_3)}{\lambda_3 - \lambda_1} + \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right\} - 6\varepsilon e_1H\left( \frac{e_1(x_2)}{\lambda_2 - \lambda_1} \right)\left( \frac{e_1(x_3)}{\lambda_3 - \lambda_1} \right)\left( \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right)
\]

Using (3.18)–(3.20) and (3.81)–(3.83), we find

\[
12\varepsilon H\lambda_2\lambda_3\lambda_4 = e_1\left( \frac{e_1(x_2)}{\lambda_2 - \lambda_1} \right)\left( \frac{e_1(x_3)}{\lambda_3 - \lambda_1} \right)\left( \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right).
\]

Also, from (3.81)–(3.83), we obtain

\[
(\lambda_2\lambda_3\lambda_4)^2 = -e_1\left( \frac{e_1(x_2)}{\lambda_2 - \lambda_1} \right)\left( \frac{e_1(x_3)}{\lambda_3 - \lambda_1} \right)\left( \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right)^2.
\]

Differentiating (3.90) along \(e_1\), and using (3.88)–(3.89), we get

\[
e_1(\lambda_2\lambda_3\lambda_4) = 2\lambda_2\lambda_3\lambda_4\left( \frac{e_1(x_2)}{\lambda_2 - \lambda_1} \right)\left( \frac{e_1(x_3)}{\lambda_3 - \lambda_1} \right)\left( \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right).
\]

Acting with \(e_1\) on both sides of (3.86) and using (3.18)–(3.20), (3.86), (3.87) and (3.91), we find

\[
e_1(H)\left\{ (6 - 54\varepsilon)H^2 - \frac{(5 + 8\varepsilon)}{4}\rho \right\} = \left\{ \frac{e_1(x_2)}{\lambda_2 - \lambda_1} + \frac{e_1(x_3)}{\lambda_3 - \lambda_1} + \frac{e_1(x_4)}{\lambda_4 - \lambda_1} \right\} \frac{H^3(12 - 20\varepsilon) - \frac{\rho H(2 + 20\varepsilon)}{4}}{4}.
\]

Differentiating again (3.92) along \(e_1\) and using (3.18)–(3.20), (3.81)–(3.83), (3.42) and (3.87), we obtain
\[ 
\epsilon_1 \left\{ H^3(6 - 2\varepsilon) - \frac{\rho H(1+12\varepsilon)}{4} + \frac{3\lambda_2 \lambda_3 \lambda_4}{4} \right\} \{(6 - 54\varepsilon)H^2 - \frac{(5+8\varepsilon)}{4}\rho\} + 2e_1^2(H)(6 - 54\varepsilon)H = \]
\[ 
\epsilon_1(\rho - 24H^2)\left\{ H^3(12 - 20\varepsilon) - \frac{\rho H(2+20\varepsilon)}{4} \right\} + \left\{ \frac{\epsilon_1(\lambda_1)}{\lambda_1} + \frac{\epsilon_1(\lambda_1)}{\lambda_1} + \frac{\epsilon_1(\lambda_1)}{\lambda_1} \right\} \{(3H^2(12 - 20\varepsilon) - \frac{\rho H(2+20\varepsilon)}{4})e_1(H),
\]

which on using (3.86), gives

\[ 
\epsilon_1 \left\{ H^3(6 - 2\varepsilon) - \frac{\rho H(1+12\varepsilon)}{4} + \frac{3\lambda_2 \lambda_3 \lambda_4}{4} \right\} \{(6 - 54\varepsilon)H^2 - \frac{(5+8\varepsilon)}{4}\rho\} + 2e_1^2(H)(6 - 54\varepsilon)H = \epsilon_1(\rho - 24H^2)\left\{ H^3(12 - 20\varepsilon) - \frac{\rho H(2+20\varepsilon)}{4} \right\} + \epsilon_1 \left\{ 3H^2(12 - 20\varepsilon) - \frac{\rho H(2+20\varepsilon)}{4} \right\} \{(3H^2(6 - 18\varepsilon) - \frac{\rho H(1+8\varepsilon)}{4}) + \frac{3\lambda_2 \lambda_3 \lambda_4}{4}. \]

From (3.86) and (3.87), we have

\[ 
\epsilon_1 e_1^2(H)\{(6 - 54\varepsilon)H^2 - \frac{(5+8\varepsilon)}{4}\rho\} = \left\{ H^3(12 - 20\varepsilon) - \frac{\rho H(2+20\varepsilon)}{4} \right\} \{(3H^2(6 - 18\varepsilon) - \frac{\rho H(1+8\varepsilon)}{4}) + \frac{3\lambda_2 \lambda_3 \lambda_4}{4}. \]

Then we have following cases:

1. For spacelike normal vector \( \xi \): In this case \( \varepsilon = 1 \). Then eliminating \( e_1^2(H) \) using (3.93) and (3.94), we get

\[ 
\epsilon_1 \left\{ 8(\rho - 24H^2)(16H^3 + 11\rho H) + 2(48H^2 + 11\rho)(48H^3 - 9\rho H + 3\lambda_2 \lambda_3 \lambda_4) \right\} + (192H^2 + 13\rho) = (16H^3 - 13\rho H + 3\lambda_2 \lambda_3 \lambda_4)(192H^2 + 13\rho)^2 + 788H(48H^3 + 9\rho H - 3\lambda_2 \lambda_3 \lambda_4)(16H^3 + 11\rho H),
\]

or

\[ 
\lambda_2 \lambda_3 \lambda_4(-18432H^4 + 26784\rho H^2 + 351\rho^2) = 884736H^7 + 37136\rho^2 H^3 + 1521\rho^3 H - 370176\rho H^5.
\]

On the other hand, differentiating (3.94) along \( e_1 \) and using (3.91) and (3.87), we obtain

\[ 
\epsilon_1 \left\{ 8(\rho + 12H^2)(16H^3 + 11\rho H) + (192H^2 + 13\rho)(-32H^3 - 22\rho H + 6\lambda_2 \lambda_3 \lambda_4) + 2(48H^2 + 11\rho)(-48H^3 - 9\rho H + 3\lambda_2 \lambda_3 \lambda_4) \right\} + (192H^2 + 13\rho) = 2(16H^3 - 13\rho H + 3\lambda_2 \lambda_3 \lambda_4)(192H^2 + 13\rho)^2 - 768H(48H^3 + 9\rho H - 3\lambda_2 \lambda_3 \lambda_4)(16H^3 + 11\rho H),
\]

which on solving, gives

\[ 
\lambda_2 \lambda_3 \lambda_4(9216H^4 - 4464\rho H^2 + 429\rho^2) = 1179648H^7 - 81408\rho H^5 - 27248\rho^2 H^3 + 377\rho^3 H.
\]

Now, eliminating \( \lambda_2 \lambda_3 \lambda_4 \) from equations (3.96) and (3.98), we get

\[ 
f(H, \rho) = 2H\left(-260091\rho^5 - 430404H^2\rho^4 - 227385216H^4\rho^3 - 1819201536H^6\rho^2 + 20228603900H^8\rho - 14948499460H^{10}\right) = 0.
\]
Now, \( f(H, \rho) \) is a polynomial in \( H \) with constant coefficients and as a real function satisfying a polynomial equation with constant coefficients must be constant, we get that \( H \) is constant.

**(ii)** As above, for timelike normal vector \( \xi \), we get the same result that \( H \) must be constant.

**Case B**: Let (3.77) hold. Then, from (3.5) and (3.7), we find

\[
(3.99) \quad (\lambda_2 - \lambda_3)\omega_3^4 = (\lambda_4 - \lambda_3)\omega_4^3 = (\lambda_2 - \lambda_4)\omega_3^4.
\]

Using (3.99) and (3.5), we get

\[
(3.100) \quad \omega_3^4\omega_2^3 + \omega_4^3\omega_2^4 + \omega_4^3\omega_3^3 = 0.
\]

Adding (3.27), (3.30), (3.33) and using (3.11), (3.42), (3.100) and Lemma 3.2, we get

\[
(3.101) \quad -\epsilon_2\epsilon_3\omega_1^2\omega_3^1 - \epsilon_2\epsilon_4\omega_2^1\omega_4^1 - \epsilon_4\epsilon_3\omega_4^1\omega_3^1 = \epsilon_1(12H^2 + \frac{\rho}{2}).
\]

Using (3.5), (3.7) and (3.100), we obtain

\[
(3.102) \quad \lambda_2\omega_3^3\omega_3^3 + \lambda_3\omega_4^3\omega_2^3 + \lambda_4\omega_4^3\omega_3^3 = 0.
\]

Multiplying (3.27), (3.30), (3.33) by \( \lambda_2\epsilon_3 \), \( \lambda_3\epsilon_2 \), and \( \lambda_2\epsilon_4 \epsilon_3 \) respectively, and adding these equations and using (3.102), we find

\[
(3.103) \quad \epsilon_2\epsilon_3\lambda_4\omega_2^1\omega_3^1 + \epsilon_2\epsilon_4\lambda_3\omega_1^2\omega_4^1 + \epsilon_4\epsilon_3\lambda_2\omega_2^1\omega_3^3 = -3\epsilon_1\lambda_2\lambda_3\lambda_4.
\]

Using (3.100), (3.77), (3.7) and (3.5), we get

\[
(3.104) \quad \epsilon_2\omega_2^1\omega_3^1\omega_4^3 + \epsilon_3\omega_3^3\omega_3^2\omega_4^2 + \epsilon_4\omega_4^1\omega_3^2\omega_3^2 = 0.
\]

Multiplying (3.27), (3.30), (3.33) by \( \epsilon_2\lambda_3 \), \( \epsilon_2\lambda_4 \), and \( \epsilon_2\epsilon_4\omega_2^1 \) respectively, and adding these equations and using (3.104), we get

\[
(3.105) \quad \epsilon_4\lambda_2\lambda_3\omega_4^1 + \epsilon_3\lambda_2\lambda_4\omega_3^1 + \epsilon_2\lambda_3\lambda_4\omega_2^1 = -3\epsilon_1\omega_2^1\omega_3^3\omega_3^1.
\]

Differentiating (3.11) along \( e_1 \) and using (3.6), we have

\[
(3.106) \quad \epsilon_2\lambda_2\omega_2^2 + \epsilon_3\lambda_3\omega_3^1 + \epsilon_4\lambda_4\omega_4^1 = 6\epsilon_1(H) - 2\epsilon H(\epsilon_2\omega_2^1 + \epsilon_3\omega_3^1 + \epsilon_4\omega_4^1).
\]

Again, differentiating (3.42) along \( e_1 \) and eliminating \( e_1(H) \) using (3.106) and (3.6), we obtain

\[
(3.107) \quad \lambda_2^2\omega_2^1\epsilon_2 + \lambda_3^2\omega_3^1\epsilon_3 + \lambda_4^2\omega_4^1\epsilon_4 = 4H^2(\epsilon_2\omega_2^1 + \epsilon_3\omega_3^1 + \epsilon_4\omega_4^1).
\]
Also, from (3.11) and (3.42), we find

\[(3.108) \quad \lambda_2 \lambda_3 = 12H^2 + \frac{p}{2} - \lambda_4 (6cH - \lambda_4),\]

\[(3.109) \quad \lambda_2 \lambda_4 = 12H^2 + \frac{p}{2} - \lambda_3 (6cH - \lambda_3),\]

\[(3.110) \quad \lambda_3 \lambda_4 = 12H^2 + \frac{p}{2} - \lambda_2 (6cH - \lambda_2).\]

Now, multiplying (3.108), (3.109) and (3.110) by \(\epsilon_4 \omega_{14}^1, \epsilon_3 \omega_{33}^1\) and \(\epsilon_2 \omega_{22}^1\), respectively and adding these equations and using (3.106) and (3.107), we get

\[(3.111) \quad \epsilon_4 \lambda_2 \lambda_3 \omega_{14}^1 + \epsilon_3 \lambda_2 \lambda_4 \omega_{33}^1 + \epsilon_2 \lambda_3 \lambda_4 \omega_{22}^1 = (\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1)(28H^2 + \frac{p}{2}) - 36He_1(H).\]

Equating (3.105) and (3.111), we get

\[(3.112) \quad -3\epsilon_1 \omega_{22}^1 \omega_{33}^1 \omega_{14}^1 = (\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1)(28H^2 + \frac{p}{2}) - 36He_1(H).\]

Using (3.6), (3.106) and (3.111), we find

\[(3.113) \quad e_1(\lambda_2 \lambda_3 \lambda_4) = (\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1)(\lambda_2 \lambda_3 \lambda_4 + 56H^3\epsilon + \rho H) - 36He_1(H).\]

Eliminating \(e_1e_1(H)\) form (3.43) and (3.84), and using (3.6), (3.101) and (3.103), we get

\[(3.114) \quad -4e_1e_1(H)(\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1) = 3\lambda_2 \lambda_3 \lambda_4 - 24H^3(1 + 7\epsilon) + \rho H(1 - 2\epsilon).\]

Using (3.43) and (3.114), we obtain

\[(3.115) \quad -4e_1e_1(H) = 3\lambda_2 \lambda_3 \lambda_4 - 8H^3(1 + 9\epsilon) - \rho H(1 - \epsilon).\]

Now, using (3.11), (3.18)~(3.20) and (3.101), we find

\[(3.116) \quad e_1(\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1) = -12H^2 e_1 + (\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1)^2 + 2e_1(12H^2 + \frac{p}{2}).\]

Differentiating (3.114) and using (3.113), (3.115) and (3.116), we obtain

\[(3.117) \quad (\epsilon_2 \omega_{22}^1 + \epsilon_3 \omega_{33}^1 + \epsilon_4 \omega_{14}^1)(-8H^3(14 + 63\epsilon) + \rho H(2 - 3\epsilon) + 3\lambda_2 \lambda_3 \lambda_4)\]

\[= e_1(H)(-12H^2(2 + 60\epsilon) + \rho(5 - 2\epsilon)).\]

Using (3.114) and (3.117), we get

\[(3.118) \quad -9(\lambda_2 \lambda_3 \lambda_4)^2 + \lambda_2 \lambda_3 \lambda_4 \left\{ (408 + 2016\epsilon)H^3 + (9 + 15\epsilon)\rho H \right\}

\[+ (544\epsilon - 1352)\rho H^4 + (7\epsilon - 8)\rho^2 H^2 - (30912 + 87360)H^6 \]

\[= 4e_1 e_1(H)(\rho(5 - 2\epsilon) - 12H^2(60\epsilon + 2)).\]
Differentiating (3.118) and using (3.113), (3.115) and (3.117), we obtain

\[
(\lambda_2 \lambda_3 \lambda_4)^2 \{ (45\epsilon - 27)\rho + (3672 + 22032\epsilon)H^2 \} + \lambda_2 \lambda_3 \lambda_4 \{ (-1800576 - 4076352\epsilon)H^5 + (-23040 + 4896\epsilon)\rho H^3 + (-12 - 54\epsilon)\rho^2 H \}
\]

\[
+ (225437184\epsilon + 144883200)H^8 + (1510656\epsilon + 160960)\rho H^6 + (384\epsilon + 8808)\rho^2 H^4 + (-69 + 89)\rho^3 H^2 = -96\epsilon c_1^2(H)H(60\epsilon + 2)
\]

\[
(14 + 63) + \rho H(2 - 3\epsilon) + 3\lambda_2 \lambda_3 \lambda_4.
\]

Differentiating (3.112) and using (3.11), (3.114) (3.18) we obtain

\[
36\epsilon c_1^2(H) = -6H\lambda_2 \lambda_3 \lambda_4 (7 - 3\epsilon)\epsilon \epsilon (12 - 30\epsilon)\rho \]

\[
+ \rho H(20 + 37\epsilon) + \epsilon^2.
\]

Using (3.120) to eliminate \( c_1^2(H) \) from (3.118) and (3.119), we obtain

\[
(\lambda_2 \lambda_3 \lambda_4)^2 a_1 + \lambda_2 \lambda_3 \lambda_4 b_1 = c_1,
\]

and

\[
(\lambda_2 \lambda_3 \lambda_4)^2 a_2 + \lambda_2 \lambda_3 \lambda_4 b_2 = c_2,
\]

respectively, where

\[a_1 = -162,\]

\[b_1 = 6H[(888 - 4032\epsilon + \epsilon_2 \epsilon_3 \epsilon_4 (4320 + 144\epsilon))H^2 + (43 + 17\epsilon + \epsilon_2 \epsilon_3 \epsilon_4 (12 - 30\epsilon))\rho],\]

\[c_1 = - (772992\epsilon + 3097728)H^6 - (11472\epsilon + 34608)\rho H^4
\]

\[- (556\epsilon - 172)\rho^2 H^2 - \rho^3(2\epsilon - 5),\]

\[a_2 = 9(15\epsilon - 9)\rho + (1000 + 624\epsilon + \epsilon_2 \epsilon_3 \epsilon_4 (2880 + 96\epsilon))H^2],\]

\[b_2 = 6H[(158406 - 1383840\epsilon + \epsilon_2 \epsilon_3 \epsilon_4(-185472 - 731136\epsilon))H^4 + (7376
\]

\[+ 1160\epsilon + \epsilon_2 \epsilon_3 \epsilon_4 (2736 - 4224\epsilon))\rho H^2 + (-2 + 93\epsilon)\rho^2],\]

\[c_2 = - H^2[(-157914624\epsilon + 38270672)H^6 + (-9505792\epsilon - 4894784)\rho H^4
\]

\[+ (-63616\epsilon - 89848)\rho^2 H^2 + (249\epsilon - 437)\rho^3] .\]

Now, eliminating \((\lambda_2 \lambda_3 \lambda_4)^2\) and \(\lambda_2 \lambda_3 \lambda_4\) from (3.121) and (3.122), we obtain

\[
(a_1c_2 - a_2c_1)^2 - (c_1b_2 - c_2b_1)(a_1b_2 - a_2b_1) = 0,
\]

which is a polynomial equation in \(H\) of degree 16 with constant coefficients. Now a real function satisfying a polynomial equation with constant coefficients must be constant and therefore \(H\) is constant, which is a contradiction.

(b) The case of three distinct principal curvatures
Suppose that $M$ is a biharmonic hypersurfaces with three distinct principal curvatures and constant scalar curvature with diagonal shape operator. We also assume that mean curvature is not constant and $\nabla H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset $U$ of $M_r^4$, with $\nabla p H \neq 0$ for all $p \in U$. From (2.9), it is easy to see that $\nabla H$ is an eigenvector of the shape operator $A$ with the corresponding principal curvature $-2\varepsilon H$.

Without losing generality, we choose $e_1$ in the direction of $\nabla H$ and therefore shape operator $A$ of the hypersurface will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

\[
A_H = \begin{pmatrix}
-2\varepsilon H & \lambda & \lambda \\
\lambda & \lambda & \lambda_4 \\
\lambda & \lambda & \lambda_4
\end{pmatrix}.
\]

From (3.11), (3.42) and (3.123), we get

\[
2\lambda + \lambda_4 = 6\varepsilon H.
\]

\[
2\lambda^2 + \lambda_4^2 = 12H^2 - \rho.
\]

We have the following cases:

(i) For spacelike normal vector $\xi$: In this case $\varepsilon = 1$. From (3.124) and (3.125), we find

\[
e_1(\lambda) = e_1(\lambda_4) = 2e_1(H).
\]

and

\[
e_4(\lambda) = e_4(\lambda_4) = 0.
\]

Now, equations (3.18), (3.20) and (3.33) reduce to

\[
e_1\left(\frac{e_1(\lambda)}{\lambda + 2H}\right) - \left(\frac{e_1(\lambda)}{\lambda + 2H}\right)^2 = -2\varepsilon_1 H\lambda,
\]

\[
e_1\left(\frac{e_1(6H - 2\lambda)}{8H - 2\lambda}\right) - \left(\frac{e_1(6H - 2\lambda)}{8H - 2\lambda}\right)^2 = -2\varepsilon_1 H(6H - 2\lambda),
\]

and,

\[
\left(\frac{e_1(\lambda)}{\lambda + 2H}\right)\left(\frac{e_1(6H - 2\lambda)}{8H - 2\lambda}\right) = -\varepsilon_1 \lambda(6H - 2\lambda).
\]
Using (3.126) in (3.128), (3.129) and (3.130), we find

\[ e_1 e_1(H) - \left( \frac{6e_1^2(H)}{\lambda + 2H} \right) = -\epsilon_1 H \lambda (\lambda + 2H), \]

(3.131)

\[ e_1 e_1(H) - \left( \frac{6e_1^2(H)}{8H - 2\lambda} \right) = -\epsilon_1 H(6H - 2\lambda)(8H - 2\lambda), \]

(3.132)

and,

\[ 4e_1^2(H) = -\epsilon_1 \lambda (\lambda + 2H)(6H - 2\lambda)(8H - 2\lambda). \]

(3.133)

From (3.131) and (3.132), we get

\[ 3e_1^2(H) = -\epsilon_1 H(\lambda + 2H)(\lambda - 8H)(\lambda - 4H). \]

(3.134)

Eliminating \( e_1^2 \) from (3.133) and (3.134), we obtain

\[ 3\lambda^2 - 8\lambda H - 8H^2 = 0. \]

(3.135)

On solving (3.135), we get \( \lambda = \left( \frac{4 \pm 2\sqrt{10}}{3} \right) H \), which gives \( e_1(\lambda) = \left( \frac{4 \pm 2\sqrt{10}}{3} \right) e_1(H) \), thus contradicting (3.126).

(ii) Proceeding as above, for timelike normal vector \( \xi \), we get a contradiction.

(c) The case of two distinct principal curvatures

Suppose that \( M \) is a nonminimal biharmonic hypersurface with two distinct principal curvatures and constant scalar curvature with shape operator diagonal. From (2.9), it is easy to see that \( \text{grad} H \) is an eigenvector of the shape operator \( A \) with the corresponding principal curvature \( -2\epsilon H \). Without losing generality, we choose \( e_1 \) in the direction of \( \text{grad} H \) and therefore shape operator \( A \) of hypersurfaces will take the following form with respect to a suitable frame \( \{e_1, e_2, e_3, e_4\} \)

\[
A = \begin{pmatrix} -2\epsilon H & \lambda \\
\lambda & \lambda \end{pmatrix}.
\]

(3.136)

From (3.11) and (3.136), we get

\[ \lambda = 2\epsilon H. \]

(3.137)

Also, from (3.6) and (3.137), we obtain

\[ \epsilon_2 \omega_{22}^1 = \epsilon_3 \omega_{33}^1 = \epsilon_4 \omega_{44}^1 = \frac{e_1(H)}{2H}. \]

(3.138)
Also, from (2.5), $R(e_1, e_2, e_1, e_2)$ shows that

\[(3.139)\]
\[e_1(\omega_{22}^1)e_2 = (\omega_{22}^1)^2 - \epsilon_14H^2.\]

Using (3.138) and (3.139), we find

\[(3.140)\]
\[\epsilon_1e_1e_1(\epsilon) = \frac{3\epsilon_1^2(H)}{2H} - 8H^3.\]

On the other hand, from (2.8), (2.10), (3.136), and (3.138), we have

\[(3.141)\]
\[\epsilon_1e_1e_1(\epsilon) = \frac{3\epsilon_1^2(H)}{2H} + 16H^3.\]

From (3.140) and (3.141), we get that $H$ must be zero, which is a contradiction.

Combining (a), (b) and (c), we have:

**Theorem 3.4.** Every biharmonic non-degenerate hypersurfaces $M^4_r$, $r = 0, 1, 2, 3, 4, \ldots$ of constant scalar curvature with diagonal shape operator in semi-Euclidean space $E^5_s$, $s = 0, 1, 2, 3, 4, 5$ has zero mean curvature.

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