A FORMULATION OF QUANTUM FIELD THEORY
REALIZING A SEA OF INTERACTING DIRAC PARTICLES

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Abstract. In this survey article, we explain a few ideas behind the fermionic projector approach and summarize recent results which clarify the connection to quantum field theory. The fermionic projector is introduced, which describes the physical system by a collection of Dirac states, including the states of the Dirac sea. Formulating the interaction by an action principle for the fermionic projector, we obtain a consistent description of interacting quantum fields which reproduces the results of perturbative quantum field theory. We find a new mechanism for the generation of boson masses and obtain small corrections to the field equations which violate causality.

1. Introduction and Motivation

In order to give the negative-energy solutions of the Dirac equation a meaningful physical interpretation, Dirac proposed that in the vacuum all states of negative energy should be occupied by particles forming the so-called Dirac sea \[ [6, 7] \]. His idea was that the homogeneous and isotropic Dirac sea configuration of the vacuum should not be accessible to measurements, but deviations from this uniform configuration should be observable. Thus particles are described by occupying additional states having positive energy, whereas “holes” in the Dirac sea can be observed as anti-particles. Moreover, Dirac noticed in [7] that deviations from the uniform sea configuration may also be caused by the interaction with an electromagnetic field. In order to analyze this effect, he first considered a formal sum over all vacuum sea states

\[ R(t, \vec{x}; t', \vec{x}') = \sum_{l \text{ occupied}} \Psi_l(t, \vec{x}) \overline{\Psi_l(t', \vec{x}')} . \]  

He found that this sum diverges if the space-time point \((t, \vec{x})\) lies on the light cone centered at \((t', \vec{x}')\) (i.e. if \((t-t')^2 = |\vec{x} - \vec{x}'|^2\)). Next, he inserted an electromagnetic potential into the Dirac equation,

\[ (i\partial + eA(t, \vec{x}) - m)\Psi_l(t, \vec{x}) = 0 . \]

He proceeded by decomposing the resulting sum (1.1) as

\[ R = R_a + R_b , \]  

where \(R_a\) is again singular on the light cone, whereas \(R_b\) is a regular function. The dependence of \(R_a\) and \(R_b\) on the electromagnetic potential can be interpreted as describing a “polarization of the Dirac sea” caused by the non-uniform motion of the sea particles in the electromagnetic field.

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When setting up an interacting theory, one faces the problem that the total charge density of the sea states is given by the divergent expression

\[ \sum_{l \text{ occupied}} e \bar{\Psi}_l(t, \vec{x}) \gamma^0 \Psi_l(t, \vec{x}). \]

Thus the Dirac sea has an infinite charge density, making it impossible to couple it to a Maxwell field. Similarly, the Dirac sea has an infinite negative energy density, leading to divergences in Einstein’s equations. Thus before formulating the field equations, one must get rid of the infinite contribution of the Dirac sea to the current and the energy-momentum tensor.

In the standard perturbative description of quantum field theory (QFT), this is accomplished by subtracting infinite counter terms (for a more detailed discussion also in connection to renormalization see Section 2 below). Then in the resulting theory, the Dirac sea is no longer apparent. Therefore, it is a common view that the Dirac sea is merely a historical relic which is no longer needed in modern QFT. However, this view is too simple because removing the Dirac sea by infinite counter terms entails conceptual problems. The basic shortcoming can already be understood from the representation (1.2) of the Dirac sea in an electromagnetic field. Since the singular term \( R_a \) involves \( \mathcal{A} \), the counter term needed to compensate the infinite charge density of the Dirac sea must depend on the electromagnetic potential. But then it is no longer clear how precisely this counter term is to be chosen. In particular, should the counter term include \( R_b \), or should \( R_b \) not be compensated and instead enter the Maxwell equations? More generally, in a given external field, the counter terms involve the background field, giving a lot of freedom in choosing the counter terms. In curved space-time, the situation is even more problematic because the counter terms depend on the choice of coordinates. Taking the resulting arbitrariness seriously, one concludes that the procedure of subtracting infinite charge or energy densities is not a fully convincing concept. Similarly, infinite counter terms are also needed in order to treat the divergences of the Feynman loop diagrams. Dirac himself was uneasy about these infinities, as he expressed later in his life in a lecture on quantum electrodynamics [8, Chapter 2]:

“I must say that I am very dissatisfied with the situation, because this so-called good theory does involve neglecting infinities which appear in its equations . . . in an arbitrary way. This is not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small – not neglecting it just because it is infinitely great and you do not want it!”

The dissatisfaction about the treatment of the Dirac sea in perturbative QFT was my original motivation for trying to set up a QFT where the Dirac sea is not handled by infinite counter terms, but where the states of the Dirac sea are treated on the same footing as the particle states all the way, thus making Dirac’s idea of a “sea of interacting particles” manifest. The key step for realizing this program is to describe the interaction by a new type of action principle, which has the desirable property that the divergent terms in (1.1) drop out of the equations, making it unnecessary to subtract any counter terms. This action principle was first introduced in [14]. More recently, in [15] it was analyzed in detail for a system of Dirac seas in the simplest possible configuration referred to as a single sector. Furthermore, the connection to
entanglement and second quantization was clarified in [17]. Putting these results together, we obtain a consistent formulation of QFT which is satisfying conceptually and reproduces the results of perturbative QFT. Moreover, our approach gives surprising results which go beyond standard QFT, like a mechanism for the generation of boson masses and small corrections to the field equations which violate causality. The aim of the present paper is to explain a few ideas behind the fermionic projector approach and to review the present status of the theory.

2. Perturbative Quantum Field Theory and its Shortcomings

Let us revisit the divergences in (1.1) in the context of modern QFT. Historically, Dirac’s considerations were continued by Heisenberg [26], who analyzed the singularities of $R_a$ in more detail and used physical arguments involving conservation laws and the requirement of gauge invariance to deduce a canonical form of the counter terms in Minkowski space. This result was then taken up by Uehling and Serber [36, 35] to deduce corrections to the Maxwell equations which are now known as the one-loop vacuum polarization. A more systematic analysis became possible by covariant perturbation theory as developed following the pioneering work of Schwinger, Feynman and Dyson (see for example [34, 12, 10]). In the resulting formulation of the interaction in terms of Feynman diagrams, one can compute the loop corrections and the $S$-matrix of a scattering process, in excellent agreement with experiments. Moreover, the procedure of subtracting infinite counter terms was replaced by the renormalization program, which can be outlined as follows (for details cf. [31] or [4]): In order to get rid of the divergences of the Feynman diagrams, one first regularizes the theory. Then one shows that the regularization can be removed if at the same time the coupling constants and masses in the theory are suitably rescaled. Typically, the coupling constants and the masses diverge as the ultraviolet regularization is removed, but in such a way that the effective theory obtained in the limit has finite effective coupling constants and finite effective masses. The renormalization program is carried out order by order in perturbation theory. Clearly, the procedure is not unique as there is a lot of freedom in choosing the regularization. A theory is called renormalizable if this freedom can be described to all orders in perturbation theory by a finite number of empirical constants.

Despite its overwhelming success, the present formulation of QFT suffers from serious shortcomings. A major technical problem is that, despite considerable effort (see for example [21]), one has not succeeded in rigorously constructing an interacting QFT in Minkowski space. In particular, the renormalized perturbation series of quantum electrodynamics makes sense only as a formal power expansion in the coupling constant. A more conceptual difficulty is that the covariant perturbation expansion makes statements only on the scattering matrix. This makes it possible to compute the asymptotic in- and out-states in a scattering process. But it remains unclear what the quantum field is at intermediate times, while the interaction takes place. Moreover, one needs free asymptotic states to begin with. But under realistic conditions, the system interacts at all times, so that there are no asymptotic states. What does the quantum field mean in this situation? For example, if one tries to formulate the theory in a fixed time-dependent background field, then there are no plane-wave solutions to perturb from, so that standard perturbation theory fails. If one tries to include gravity, the equivalence principle demands that the theory should be covariant under general coordinate transformations. But the notion of free states distinguishes
specific coordinate systems, in which the free states are represented by plane waves. A related difficulty is entailed in the notion of the “Feynman propagator”, defined by the conditions that positive and negative frequencies should propagate to the future and past, respectively. Again the notion “frequency” refers to an observer, explaining why Feynman’s frequency conditions are not invariant under general coordinate transformations. To summarize, present QFT involves serious conceptual difficulties if one wants to go beyond the computation of the scattering matrix and tries to understand the dynamics of the quantum field at intermediate times or considers systems which for large times do not go over to a free field theory in Minkowski space.

In order to understand these conceptual difficulties in more detail, it is a good starting point to disregard the divergences caused by the interaction and to consider free quantum fields in an external field. In this considerably simpler setting, there are several approaches to construct quantum fields, as we now outline. Historically, quantum fields in an external field were first analyzed in the Fock space formalism. Klaus and Scharf [28, 29] considered the Fock representation of the electron-positron field in the presence of a static external field. They noticed that the Hamiltonian needs to be regularized by suitable counter terms which depend on the external field. Thus the simple method of the renormalization program of removing the regularization while adjusting the bare masses and coupling constants no longer works. Similar to the explanation in Section [1], one needs to subtract infinite counter terms which necessarily involve the external field. Klaus and Scharf also realized that the Fock representation in the external field is in general inequivalent to the standard Fock representation of free fields in Minkowski space (see also [30, 27]). This result shows that a perturbation expansion about the standard Fock vacuum necessarily fails.

In the time-dependent setting, Fierz and Scharf [13] proposed that the Fock representation should be adapted to the external field as measured by a local observer. Then the Fock representation becomes time and observer dependent. This implies that the distinction between particles and anti-particles no longer has an invariant meaning, but it depends on the choice of an observer. In this formulation, the usual particle interpretation of quantum states only makes sense for the in- and outgoing scattering states, but it has no invariant meaning for intermediate times. For a related approach which allows for the construction of quantum fields in the presence of an external magnetic field see [5]. In all the above approaches, the Dirac sea leads to divergences, which must be treated by an ultraviolet regularization and suitable counter terms.

As an alternative to working with Fock spaces, one can use the so-called point splitting renormalization method, which is particularly useful for renormalizing the expectation value of the energy-momentum tensor [3]. Similar to the above procedure of Dirac and Heisenberg for treating the charge density of the Dirac sea, the idea is to replace a function of one variable $T(x)$ by a two-point distribution $T(x, y)$, and to take the limit $y \to x$ after subtracting suitable singular distributions which take the role of counter terms. Analyzing the singular structure of the counter terms leads to the so-called Hadamard condition (see for example [20]). Reformulating the Hadamard condition for the two-point function as a local spectral condition for the wave front set [32] turns out to be very useful for the axiomatic formulation of free quantum fields in curved space-time. As in the Fock space formalism, in the point splitting approach the particle interpretation depends on the observer. This is reflected mathematically by the fact that the Hadamard condition specifies the two-point distribution only up
to smooth contributions. For a good introduction to free quantum fields in curved space-time we refer to the recent book [2].

We again point out that in all the above papers on quantum fields in an external field or in curved space-time, only free fields are considered. The theories are not set up in a way where it would be clear how to describe an additional interaction in terms of Feynman diagrams. Thus it is fair to say that the formulation of a background independent interacting perturbative QFT is an open and apparently very difficult problem. All the methods so far suffer from the conceptual difficulty that to avoid divergences, one must introduce infinite counter terms ad-hoc.

3. AN ACTION PRINCIPLE FOR THE FERMIONIC PROJECTOR IN SPACE-TIME

In order to introduce the fermionic projector approach, we now define our action principle on a formal level (for the analytic justification and more details see [15, Chapter 2]). Similar to \((1.1)\), we describe our fermion system for any points \(x\) and \(y\) in Minkowski space by the so-called kernel of the fermionic projector

\[
P(x, y) = -\sum_{\text{occupied}} \Psi_l(x) \overline{\Psi_l(y)},
\]

where by the occupied states we mean the sea states except for the anti-particle states plus the particle states. For any \(x\) and \(y\), we introduce the closed chain \(A_{xy}\) by

\[
A_{xy} = P(x, y) P(y, x).
\]

It is a \(4 \times 4\)-matrix which can be considered as a linear operator on the Dirac wave functions at \(x\). For such a linear operator \(A\) we define the spectral weight \(|A|\) by

\[
|A| = \sum_{i=1}^{4} |\lambda_i|,
\]

where \(\lambda_1, \ldots, \lambda_4\) are the eigenvalues of \(A\) counted with algebraic multiplicities. We define the Lagrangian \(L\) by

\[
L_{xy}[P] = |A_{xy}^2| - \frac{1}{4} |A_{xy}|^2.
\]

Integrating over space-time, we introduce the functionals

\[
S[P] = \iint L_{xy}[P] \, d^4x \, d^4y \quad \text{and} \quad T[P] = \iint |A_{xy}|^2 \, d^4x \, d^4y.
\]

Our action principle is to minimize \(S\) for fixed \(T\),

under variations of the wave functions \(\Psi_l\) which preserve the normalization with respect to the space-time inner product

\[
<\Psi|\Phi> = \int \overline{\Psi(x)} \Phi(x) \, d^4x.
\]

The action principle \((3.5)\) is the result of many thoughts and extensive calculations carried out over several years. The considerations which eventually led to this action principle are summarized in [14, Chapter 5]. Here we only make a few comments. We first note that the factor \(1/4\) in \((3.3)\) is merely a convention, as the value of this factor can be arbitrarily changed by adding to \(S\) a multiple of the constraint \(T\). Our convention has the advantage that for the systems under consideration here, the Lagrange multiplier of the constraint vanishes, making it possible to disregard the
constraint in the following discussion. Next, we point out that taking the absolute value of an eigenvalue of the closed chain is a non-linear (and not even analytic) operation, so that our Lagrangian is not quadratic. As a consequence, the corresponding Euler-Lagrange equations are nonlinear. Our Lagrangian has the property that it vanishes if $A$ is a multiple of the identity matrix. Furthermore, it vanishes if the eigenvalues of $A$ form a complex conjugate pair. These properties are responsible for the fact that the singularities on the light cone discussed in the introduction drop out of the Euler-Lagrange equations. Moreover, it is worth noting that the action involves only the fermionic wave functions, but no bosonic fields appear at this stage. The interaction may be interpreted as a direct particle-particle interaction of all the fermions, taking into account the sea states. We finally emphasize that our action involves neither coupling constants nor any other free parameters.

Clearly, our setting is very different from the conventional formulation of physics. We have neither a fermionic Fock space nor any bosonic fields. Although the expression (3.1) resembles the two-point function, the $n$-point functions are not defined in our setting. More generally, it seems inappropriate and might even be confusing to use notions from QFT, which have no direct correspondence here. Thus one should be willing to accept that we are in a new mathematical framework where we describe the physical system on the fundamental level by the fermionic projector with kernel (3.1). The connection to QFT is not obvious at this stage, but will be established in what follows.

We finally remark that our approach of working with a nonlinear functional on the fermionic states has some similarity to the “non-linear spinor theory” by Heisenberg et al [9], which was controversially discussed in the 1950s, but did not get much attention after the invention of renormalization. We point out that our action (3.5) is completely different from the equation $\partial / \Psi \pm l^2 \gamma^5 \gamma^j \Psi \left( \overline{\Psi} \gamma^j \gamma^5 \Psi \right) = 0$ considered in [9]. Thus there does not seem to be a connection between these approaches.

4. **Intrinsic Formulation in a Discrete Space-Time**

Our action principle has the nice feature that it does not involve the differentiable, topological or causal structure of the underlying Minkowski space. This makes it possible to drop these structures, and to formulate our action principle intrinsically in a discrete space-time. To this end, we simply replace Minkowski space by a finite point set $M$. To every space-time point we associate the *spinor space* as a four-dimensional complex vector space endowed with an inner product of signature $(2, 2)$, again denoted by $\overline{\Psi} \Phi$. A *wave function* $\Psi$ is defined as a function which maps every space-time point $x \in M$ to a vector $\Psi(x)$ in the corresponding spinor space. For a (suitably orthonormalized) finite family of wave functions $\Psi_1, \ldots, \Psi_f$ we then define the kernel of the fermionic projector in analogy to (3.1) by

$$P(x, y) = - \sum_{i=1}^{f} \Psi_i(x) \overline{\Psi_i(y)}.$$  

Now the action principle can be introduced again by (3.4)–(3.6) if we only replace the space-time integrals by sums over $M$.

The formulation in discrete space-time is a possible approach for physics on the Planck scale. The basic idea is that the causal and metric structure should be induced on the space-time points by the fermionic projector as a consequence of a spontaneous
symmetry breaking effect. In non-technical terms, this *structure formation* can be understood by a self-organization of the wave functions as described by our action principle. More specifically, a discrete notion of causality is introduced as follows:

**Definition 4.1. (causal structure)** Two space-time points \( x, y \in M \) are called *time-like* separated if the spectrum of the product \( P(x, y)P(y, x) \) is real. Likewise, the points are *spacelike* separated if the spectrum of \( P(x, y)P(y, x) \) forms two complex conjugate pairs having the same absolute value.

We refer the reader interested in the spontaneous structure formation and the connection between discrete and continuum space-times to the survey paper [16] and the references therein. The only point of relevance for what follows is that in the discrete formulation, our action principle is finite and minimizers exist. Thus there is a fundamental setting where the physical equations are intrinsically defined and have regular solutions without any divergences.

5. **Bosonic Currents Arising from a Sea of Interacting Dirac Particles**

In preparation for analyzing our action principle, we need a systematic method for describing the kernel of the fermionic projector in position space. In the vacuum, the formal sum in (3.1) is made precise as the Fourier integral of a distribution supported on the lower mass shell,

\[
P_{\text{sea}}(x, y) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k + m) \delta(k^2 - m^2)} \Theta(-k^0) e^{-ik(x-y)}
\]  

(5.1)

(where \( \Theta \) is the Heaviside function). In order to introduce particles and anti-particles, one occupies (suitably normalized) positive-energy states or removes states of the sea,

\[
P(x, y) = P_{\text{sea}}(x, y) - \frac{1}{2\pi} \sum_{k=1}^{n_f} \bar{\Psi}_k(y) \Psi_k(x) + \frac{1}{2\pi} \sum_{l=1}^{n_a} \bar{\Phi}_l(y) \Phi_l(x).
\]  

(5.2)

Next we want to modify the physical system so as to describe a general interaction. To this end, it is useful to regard \( P(x, y) \) as the integral kernel of an operator \( P \) on the wave functions, i.e.

\[
(P\Psi)(x) := \int P(x, y) \Psi(y) \, d^4 y.
\]

Since we want to preserve the normalization of the fermionic states with respect to the inner product (3.6), the interacting fermionic projector \( \tilde{P} \) can be obtained from the vacuum fermionic projector \( P \) by the transformation

\[
\tilde{P} = UPU^{-1}
\]

with an operator \( U \) which is unitary with respect to the inner product (3.6). The calculation

\[
0 = U(i\partial - m)PU^{-1} = U(i\partial - m)U^{-1}\tilde{P}
\]

shows that \( \tilde{P} \) is a solution of the Dirac equation

\[
(i\partial + \mathcal{B} - m)\tilde{P} = 0 \quad \text{where} \quad \mathcal{B} := iU\partial U^{-1} - i\partial.
\]

This consideration shows that we can describe a general interaction by a potential \( \mathcal{B} \) in the Dirac equation, provided that \( \mathcal{B} \) is an operator of a sufficiently general form. It can be a multiplication or differential operator, but it could even be a nonlocal operator. The usual bosonic potentials correspond to special choices of \( \mathcal{B} \). This point
of view is helpful because then the bosonic potentials no longer need to be considered as fundamental physical objects. They merely become a technical device for describing specific variations of the Dirac sea.

In order to clarify the structure of $\tilde{P}$ near the light cone, one performs the so-called causal perturbation expansion and the light-cone expansion. For convenience omitting the tilde, one gets in analogy to (1.2) a decomposition of the form

$$P_{\text{sea}}(x, y) = P_{\text{sing}}(x, y) + P_{\text{reg}}(x, y),$$

where $P_{\text{sing}}(x, y)$ is a distribution which is singular on the light cone and can be expressed explicitly by a series of terms involving line integrals of $\mathcal{B}$ and its partial derivatives along the line segment $\overline{xy}$. The contribution $P_{\text{reg}}$, on the other hand, is a smooth function which is noncausal in the sense that it depends on the global behavior of $\mathcal{B}$ in space-time. It can be decomposed further into so-called low-energy and high-energy contributions which have a different internal structure.

For simplicity, we here omit all details and only mention two points which are important for the physical understanding. First, one should keep in mind that the distribution $P_{\text{sea}}$ as defined by the causal perturbation expansion distinguishes a unique reference state, even if $\mathcal{B}$ is time dependent. Thus the decomposition (5.2) yields a globally defined picture of particles and anti-particles, independent of a local observer. Second, it is crucial for the following constructions that the line integrals appearing in $P_{\text{sing}}$ also involve partial derivatives of $\mathcal{B}$. In the case when $\mathcal{B} = \mathcal{A}$ is an electromagnetic potential (or similarly a general gauge field), one finds that $P_{\text{sing}}$ involves the electromagnetic field tensor and the electromagnetic current. More specifically, the contribution to $P_{\text{sing}}$ involving the electromagnetic current takes the form

$$-e \frac{1}{16\pi^3} \int_0^1 (\alpha - \alpha^2) \gamma_k (\partial^k A^l - \Box A^l) \lvert_{\alpha y + (1-\alpha)x} \lim_{\varepsilon \searrow 0} \log \left( \frac{(y-x)^2 + i\varepsilon (y^0 - x^0)}{(y-x)^2 + i\varepsilon (y^0 - x^0)} \right).$$

The appearance of this contribution to the fermionic projector can be understood similar to the “polarization of the Dirac sea” mentioned in the introduction as being a result of the non-uniform motion of the sea particles in the electromagnetic field. This contribution influences the closed chain (3.2) and thus has an effect on our action principle (3.5). In this way, the electromagnetic current also enters the corresponding Euler-Lagrange equations. In general terms, one can say that in our formulation, the bosonic currents arise in the physical equations only as a consequence of the collective dynamics of the particles of the Dirac sea.

6. THE CONTINUUM LIMIT, THE FIELD EQUATIONS

We now outline the method for analyzing our action principle for the fermionic projector (5.3). Since $P_{\text{sing}}$ is a distribution which is singular on the light cone, the pointwise product $P(x, y)P(y, x)$ is ill-defined. Thus in order to make mathematical sense of the Euler-Lagrange equations corresponding to our action principle, we need to introduce an ultraviolet regularization. Such a regularization is not a conceptual problem because the setting in discrete space-time in Section 4 can be regarded as a special regularization. Thus in our approach, a specific, albeit unknown regularization should have a fundamental significance. Fortunately, the details of this regularization are not needed for our analysis. Namely, for a general class of regularizations of the vacuum Dirac sea (for details see [15, Chapter 3] or [14, Chapter 4]), the Euler-Lagrange equations have a well-defined asymptotic behavior when the regularization is
removed. In this limit, the Euler-Lagrange equations give rise to differential equations involving the particle and anti-particle wave functions as well as the bosonic potentials and currents, whereas the Dirac sea disappears. This construction is subsumed under the notion \textit{continuum limit}.

In the recent paper [15], the continuum limit was analyzed in detail for systems which in the vacuum are described in generalization of (5.1) by a sum of Dirac seas,

$$P_{\text{sea}}(x, y) = \sum_{\beta=1}^{g} \int \frac{d^4k}{(2\pi)^4} \left( \frac{\not{k} + m_\beta}{\sqrt{2}} \right) \delta(k^2 - m_\beta^2) \Theta(-k^0) e^{-ik(x-y)}. \quad (6.1)$$

Such a configuration is referred to as a \textit{single sector}. The parameter $g$ can be interpreted as the number of generations of elementary particles. It turns out that in the case $g = 1$ of one Dirac sea, the continuum limit gives equations which are only satisfied in the vacuum, in simple terms because the logarithm in current terms like (5.4) causes problems. In order to get non-trivial differential equations, one must assume that there are exactly \textit{three generations} of elementary particles. In this case, the logarithms in the current terms of the three Dirac seas can compensate each other, as is made precise by a uniquely determined so-called local axial transformation. Analyzing the possible operators $B$ in the corresponding Dirac equation in an exhaustive way (including differential and nonlocal operators), one finds that the dynamics is described completely by an \textit{axial potential} $A_a$ coupled to the Dirac spinors. We thus obtain the coupled system

$$
(i\partial + \gamma^5 A_a - m)\Psi = 0, \quad C_0 j^k_a - C_2 A^k_a = 12\pi^2 J^k_a, \quad (6.2)
$$

where $j_a$ and $J_a$ are the axial currents of the gauge field and the Dirac particles,

$$
j^k_a = \partial^k A_a^l - \Box A^k_a \quad (6.3)
$$

$$
J^l_a = \sum_{k=1}^{n_f} \Psi_k \gamma^5 \gamma^l \Psi_k - \sum_{l=1}^{n_f} \Phi_l \gamma^5 \gamma^l \Phi_l. \quad (6.4)
$$

As in (5.2), the wave functions $\Psi_k$ and $\Phi_l$ denote the occupied particle and anti-particle states, respectively. The constants $C_0$ and $C_2$ in (6.2) are empirical parameters which take into account the unknown microscopic structure of space-time. For a given regularization method, these constants can be computed as functions of the fermion masses.

For clarity, we point out that the Dirac current (6.4) involves only the particle and anti-particle states of the system, but not the states forming the Dirac sea. The reason is that the contributions by the sea states cancel each other in our action principle. As a consequence, only the deviations from the completely filled sea configuration contribute to the Dirac current. In the continuum limit, pair creation is described following Dirac’s original idea by removing a sea state and occupying instead a particle state. To avoid confusion, we mention that the wave functions $\Psi_k$ and $\Phi_l$ need to be suitably orthonormalized. Taking this into account, the sum of the one-particle currents in (6.4) is indeed the same as the expectation value of the current operator computed for the Hartree-Fock state obtained by taking the wedge product of the wave functions $\Psi_k$ and $\Phi_l$.

We finally remark that more realistic models are obtained if one describes the vacuum instead of (6.1) by a direct sum of several sectors. The larger freedom in perturbing the resulting Dirac operator gives rise to several effective gauge fields, which couple...
to the fermions in a specific way. As shown in [14, Chapters 6-8], this makes it possible to realize the gauge groups and couplings of the standard model. The derivation of the corresponding field equations is work in progress.

7. A NEW MECHANISM FOR THE GENERATION OF BOSON MASSES

The term $C_2 A_k^a$ in (6.2) gives the axial field a rest mass $M = \sqrt{C_2/C_0}$. This bosonic mass term is surprising, because in standard gauge theories a boson can be given a mass only by the Higgs mechanism of spontaneous symmetry breaking. We now explain how the appearance of the mass term in (6.2) can be understood on a non-technical level (for more details see [15, §6.2 and §8.5]).

In order to see the connection to gauge theories, it is helpful to consider the behavior of the Dirac operator and the fermionic projector under gauge transformations. We begin with the familiar gauge transformations of electrodynamics, for simplicity in the case $m = 0$ of massless fermions. Thus assume that we have a pure gauge potential $A = \partial \Lambda$ with a real function $\Lambda(x)$. This potential can be inserted into the Dirac operator by the transformation

$$i\partial / \rightarrow e^{i\Lambda(x)}i\partial e^{-i\Lambda(x)} = i\partial + (\partial \Lambda),$$

showing that the electromagnetic potential simply describes the phase transformation $\Psi(x) \rightarrow e^{i\Lambda(x)}\Psi(x)$ of the wave functions. Since the multiplication operator $U = e^{i\Lambda}$ is unitary with respect to the inner product (3.6), it preserves the normalization of the fermionic states. Thus in view of (3.1), the kernel of the fermionic projector simply transforms according to

$$P(x, y) \rightarrow e^{i\Lambda(x)}P(x, y) e^{-i\Lambda(y)}.$$ 

When forming the closed chain (3.2), the phase factors drop out. This shows that our action principle is gauge invariant under the local $U(1)$-transformations of electrodynamics.

We next consider an axial potential $A_a$ as appearing in (6.2). A pure gauge potential $A_a = \partial \Lambda$ can be generated by the transformation

$$i\partial / \rightarrow e^{i\gamma^5\Lambda(x)}i\partial e^{-i\gamma^5\Lambda(x)} = i\partial + \gamma^5(\partial \Lambda),$$

suggesting that the kernel of the fermionic projector should be transformed according to

$$P(x, y) \rightarrow e^{-i\gamma^5\Lambda(x)}P(x, y) e^{-i\gamma^5\Lambda(x)}.$$ 

The main difference compared to the electromagnetic case is that now the transformation operator $U = e^{-i\gamma^5\Lambda(x)}$ is not unitary with respect to the inner product (3.6). This leads to the technical complication that we need to be concerned about the normalization of the fermionic states. More importantly, the phases no longer drop out of the closed chain, because

$$A_{xy} \rightarrow \left(e^{-i\gamma^5\Lambda(x)} P(x, y) e^{-i\gamma^5\Lambda(x)} \right) \left(e^{-i\gamma^5\Lambda(y)} P(y, x) e^{-i\gamma^5\Lambda(x)} \right) \left(e^{-i\gamma^5\Lambda(y)} P(y, x) e^{-i\gamma^5\Lambda(x)} \right) e^{-i\gamma^5\Lambda(x)} P(x, y) e^{-2i\gamma^5\Lambda(y)} P(y, x) e^{-i\gamma^5\Lambda(x)}.$$ 

This shows that in general, our action is not invariant under axial gauge transformations. As a consequence, the appearance of the axial potential in the field equations does not contradict gauge invariance.
Figure 1. A fermionic loop diagram (left) and a bosonic loop diagram (right).

A more detailed analysis shows that the above axial transformation indeed changes only the phases of the eigenvalues $\lambda_i$ of the closed chain, and these phases drop out when taking their absolute values as appearing in the closed chain. But repeating the above argument in the case $m > 0$ of massive fermions, one finds additional contributions proportional to $m^2 A_a$ which even affect the absolute values $|\lambda_i|$. These contributions are responsible for the bosonic mass term in the field equations.

In simple terms, the bosonic mass arises because the corresponding potential does not describe a local symmetry of our system. More specifically, an axial gauge transformation changes the relative phase of the left- and right-handed components of the fermionic projector. This relative phase does change the physical system and is thus allowed to enter the physical equations. In order to get a closer connection to the Higgs mechanism, one can say that the axial gauge symmetry is spontaneously broken by the states of the Dirac sea, because they distinguish the relative phase of the left- and right-handed components of the fermionic projector.

8. The Vacuum Polarization

We now describe how the one-loop vacuum polarization arises in the fermionic projector approach and compare the situation with perturbative QFT. For the derivation of the field equations in Section 6, we considered the singular contribution $P^{\text{sing}}(x,y)$ in (5.3), but we disregarded the noncausal contribution $P^{\text{reg}}$. Analyzing the latter contribution in the continuum limit gives rise to correction terms to the field equations (6.2) of the form

$$- f[0] * j^k_a + 6 f[2] * A^k_a, \quad (8.1)$$

where $f[\nu]$ are explicit Lorentz invariant distributions and the star denotes convolution (see [15, Theorem 8.2]). These corrections can already be understood in Dirac’s decomposition (1.2) as the “polarization effect” as described by the regular function $R_b$. In the static situation, the term $- f[0] * j^k_a$ reduces to the axial analogue of the well-known Uehling potential [36] (see [15, §8.2]), whereas the term $6 f[2] * A^k_a$ can be regarded as a correction to the bosonic mass term. We have thus reproduced the standard vacuum polarization, which is described in more modern language by the Feynman diagram involving one fermion loop in Figure 1 (left).

The connection to the Uehling correction in standard QFT can be understood most easily by going back to the original papers [7, 26, 36]. Heisenberg starts from Dirac’s decomposition (1.2). Motivated by symmetry considerations and physical arguments, he gives a procedure for disregarding the singularities, so that only a regular contribution remains. This regular contribution gives rise to the Uehling potential. Similarly, the starting point in [15] is the decomposition of the fermionic projector (5.3).
The main difference is that now the singular terms are not disregarded or removed, but they are carried along all the way. However, the singular terms drop out of the Euler-Lagrange equations corresponding to our action principle \( (3.5) \). In this way, all divergences disappear. The remaining finite contributions to \( P^{\text{sing}} \) give rise to the bosonic current and mass terms in the resulting field equations \( (6.2) \), whereas \( P^{\text{reg}} \) describes the vacuum polarization. The main advantage of the fermionic projector approach is that no counter terms are needed. The back-reaction of the Dirac sea on the electromagnetic field is finite, no divergences occur. Moreover, as we do not need counter terms, the setting immediately becomes background independent. It is to be expected (although it has not yet been worked out in detail) that the singularities of the fermionic projector will also drop out of the Euler-Lagrange equations if one sets up the theory in curved space-time.

In modern QFT, the vacuum polarization is still described as in the original papers, with the only difference that the singularities are now removed more systematically by a normal ordering of the field operators. In the interacting situation, the subtle point is to choose the correct “dressing” of the electrons. This means that one must distinguish a subspace of the Fock space as describing the Dirac sea; then the normal ordering is performed with respect to this subspace. In [1] a quantized Dirac field is considered which interacts with a Coulomb field and a magnetic field. It is shown that the resulting Hamiltonian is positive, provided that the atomic numbers and the fine structure constant are not too big. However, the chosen dressing has the shortcoming that polarization effects are suppressed. A more careful analysis is given in the series of papers [23, 21, 25, 22], where the vacuum state is constructed for a system of Dirac particles with electrostatic interaction in the Bogoliubov-Dirac-Fock approximation, and the question of renormalization is addressed. The conclusion of this analysis is that for mathematical consistency, one must take into account all the states forming the Dirac sea. Furthermore, the interaction “mixes” the states in such a way that it becomes impossible to distinguish between the particle states and the states of the Dirac sea. Thus, despite the use of a very different mathematical framework, the physical picture in these papers is quite similar to that of the fermionic projector approach.

9. General Loop Diagrams

So far, we only considered a Feynman diagram involving a fermion loop. Let us now consider how to obtain Feynman diagrams which involve bosonic loops: In the continuum limit, the system is described by the partial differential equations \( (6.2) \). Here the bosonic potential \( A_a \) is not quantized; it is simply a classical field. But the system \( (6.2) \) is nonlinear, and as shown in [15, §8.4], treating this nonlinearity perturbatively gives rise to the bosonic loop diagram in Figure 1 (right), as well as higher order bosonic loop diagrams. Taking the corrections \( (8.1) \) into account, one also gets the diagrams with fermion loops. In this way, one gets all the usual Feynman diagrams. But there are also differences. Since the analysis of the diagrams has not yet been carried out systematically, we merely state the potential effects as open problems:

- It is not clear whether the usual divergences of the bosonic loop diagram in Figure 1 (right) can be associated with a singularity of the fermionic projector which drops out of our action principle (similar to the explanation for the fermionic loop diagram in Section 8). More generally, it is an open problem whether the bosonic loop diagrams necessary diverge. In particular, it seems
promising to try to avoid the divergences completely by a suitable choice of
the bosonic Green’s function. This analysis might reveal a connection to the
“causal approach” by Epstein and Glaser \[11\] and Scharf \[33\].

- The main difference of the perturbation expansion in the fermionic projector
approach is that instead of working with the Feynman propagator, the nor-
malization conditions for the sea states enforce a non-trivial combinatorics of
operator products involving different types of Green’s functions and funda-
mental solutions (for details see \[19\]). This difference has no influence on
the singularities of the resulting Feynman diagrams, and thus we expect that the
renormalizability of the theory is not affected. But the higher-loop radiative
corrections should depend on the detailed combinatorics, giving the hope to
obtain small deviations from standard QFT which might be tested experimen-
tally.

10. Violation of Causality

As explained in Section \[8\], the correction terms in \((8.1)\) can also be understood in
the framework of standard QFT via fermionic loop diagrams (like in Figure \[1\](left)).
However, the detailed analysis of the correction terms in position space as carried out
in \[15, Chapter 8 and Appendix D\] reveals an underlying structure which is not ap-
parent in the usual description in momentum space. Namely, the correction term \((8.1)\)
violates causality in the sense that the future can influence the past! To higher order in
the bosonic potential, even space-time points with spacelike separation can influence
each other. At first sight, a violation of causality seems worrisome because it contra-
dicts experience and seems to imply logical inconsistencies. However, these non-causal
correction terms are only apparent on the Compton scale, and furthermore they are
too small for giving obvious contradictions to physical observations. But they might
open the possibility for future experimental tests. For a detailed discussion of the
causality violation we refer to \[15, §8.2 and §8.3\].

In order to understand how the violation of causality comes about, it is helpful to
briefly discuss the general role of causality in the fermionic projector approach. We first
point out that in discrete space-time, causality does not arise on the fundamental level.
But for a given minimizer of our action principle, Definition \[4.1\] gives us the notion of a
“discrete causal structure.” This notion is compatible with our action principle in the
sense that space-time points \(x\) and \(y\) with spacelike separation do not influence each
other via the Euler-Lagrange equations. This can be seen as follows: According to our
definition, for such space-time points the eigenvalues of the closed chain all have the
same absolute value. Using the specific form of the Lagrangian \((3.3)\), this implies that
the Lagangian and its first variation vanish. This in turn implies that \(A_{xy}\) drops out
of the Euler-Lagrangian equations. We conclude that our action principle is “causal” in
the sense that no spacelike influences are possible. But at this stage, no time direction
is distinguished, and therefore there is no reason why the future should not influence
the past.

The system of hyperbolic equations \((6.2)\) obtained in the continuum limit is causal in
the sense that given initial data has a unique time evolution. Moreover, we have finite
propagation speed, meaning that no information can travel faster than the speed of
light. Thus in the continuum limit we recover the usual notion of causality. However,
the fermionic projector \(P_{\text{sea}}\) is not defined via an initial value problem, but it is a global
object in space-time (see \[14, Chapter 2\]). As a consequence, the contribution \(P_{\text{reg}}\)
in (5.3) is noncausal in the sense that the future influences the past. Moreover, to higher order in the bosonic potential the normalization conditions for the fermions give rise to nonlocal constraints. As a consequence, the bosonic potential may influence $P(x, y)$ even for spacelike distances.

11. Entanglement and Second Quantization

Taking the wedge product of the one-particle wave functions,
$$\Psi_1 \wedge \cdots \wedge \Psi_f,$$
and considering the continuum limit, we obtain a system of classical bosonic fields coupled to a fermionic Hartree-Fock state. Although this setting gives rise to the Feynman diagrams, it is too restrictive for describing all quantum effects observed in nature. However, as shown in [17], the framework of the fermionic projector also allows for the description of general second quantized fermionic and bosonic fields. In particular, it is possible to describe entanglement.

The derivation of these results is based on the assumption that space-time should have a non-trivial microstructure. In view of our concept of discrete space-time, this assumption seems natural. Homogenizing the microstructure, one obtains an effective description of the system by a vector in the fermionic or bosonic Fock space. This concept, referred to as the microscopic mixing of decoherent subsystems, is worked out in detail in [17]. In [18], the methods and results are discussed with regard to decoherence phenomena and the measurement problem.

12. Conclusions and Outlook

Combining our results, we obtain a formulation of QFT which is consistent with perturbative QFT but has surprising additional features. First, we find a new mechanism for the generation of masses of gauge bosons and obtain new types of corrections to the field equations which violate causality. Moreover, our model involves fewer free parameters, and the structure of the interaction is completely determined by our action principle. Before one can think of experimental tests, one clearly needs to work out a more realistic model which involves all elementary particles and includes all interactions observed in nature. As shown in [14] Chapters 6–8, a model involving 24 Dirac seas is promising because the resulting gauge fields have striking similarity to the standard model. Furthermore, the underlying diffeomorphism invariance gives agreement with the equivalence principle of general relativity. Thus working out the continuum limit of this model in detail will lead to a formulation of QFT which is satisfying conceptually and makes quantitative predictions to be tested in future experiments.

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