Odd Vertex equitable even labeling of cyclic snake related graphs

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Abstract

Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{1, 3, ..., q\}$ if $q$ is odd or $A = \{1, 3, ..., q + 1\}$ if $q$ is even. A graph $G$ is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, ..., 2q$ where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. Here, we prove that the graph $nC_4$-snake, $CS(n_1, n_2, ..., n_k)$, $n_i \equiv 0(\text{mod} 4), n_i \geq 4$, be a generalized $kC_n$-snake, $TOQS_n$ and $TOQS_n$ are odd vertex equitable even graphs.

Keywords : vertex equitable labeling, vertex equitable graph, odd vertex equitable even labeling, odd vertex equitable even graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. Let $G(V,E)$ be a graph with $p$ vertices and $q$ edges. We follow the basic notations and terminology of graph theory as in [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively.

The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan in [16] and further studied in [5]-[14]. Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0, 1, 2, \ldots, \lfloor \frac{q}{2} \rfloor \}$. A graph $G$ is said to be vertex equitable if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$, where $v_f(a)$ is the number of vertices $v$ with $f(v) = a$ for $a \in A$. The vertex labeling $f$ is known as vertex equitable labeling. A graph $G$ is said to be a vertex equitable if it admits a vertex equitable labeling.

Motivated by the concept of vertex equitable labeling of graphs, Jeyanthi, Maheswari and Vijaya Lakshmi defined a new labeling namely *odd vertex equitable even labeling* [15]. A graph $G$ with $p$ vertices and $q$ edges and $A = \{1, 3, \ldots, q\}$ if $q$ is odd or $A = \{1, 3, \ldots, q + 1\}$ if $q$ is even. A graph $G$ is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$ where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. In [15] they proved that the graphs like path, $P_n$, $P_n \cup K_{1,n-2}$ ($n \geq 3$), $K_{2,n}$, $T_p$-tree, a ladder $L_n$, arbitrary super subdivision of any path $P_n$ are odd vertex equitable even graphs.

Also they proved that the graphs $K_{1,n}$ is an odd vertex equitable even graph iff $n \leq 2$, the graph $G = K_{1,n} \cup K_{1,n-2}$ ($n \geq 3$) is an odd vertex equitable even graph and cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0 \text{ or } 1 \pmod{4}$. In addition, they proved that if every edge of a graph $G$ is an edge of a triangle, then $G$ is not an odd vertex equitable even graph.
We use the following definitions in the subsequent section.

**Theorem 1.1.** The cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0$ or 1 ($\mod 4$).

**Theorem 1.2.** Let $G_1(p_1, q_1), G_2(p_2, q_2), \ldots, G_m(p_m, q_m)$ be an odd vertex equitable even graphs with $\sum_{i=1}^{m-1} q_i$ is even, $q_m$ is even or odd and $u_i, v_i$ be the vertices of $G_i(1 \leq i \leq m)$ labeled by 1, if $q_i$ is odd or $q_i + 1$ if $q_i$ is even. Then the graph $G$ obtained by identifying $v_1$ with $u_2$ and $v_2$ with $u_3$ and $v_3$ with $u_4$ and so on until we identify $v_{m-1}$ with $u_m$ is also an odd vertex equitable even graph.

**Definition 1.3.** The corona $G_1 \odot G_2$ of the graphs $G_1$ and $G_2$ is defined as a graph obtained by taking one copy of $G_1$ (with $p$ vertices) and $p$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex of the $i^{th}$ copy of $G_2$.

**Definition 1.4.** Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \diamond G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by identifying one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$.

**Definition 1.5.** [1] A $kC_n$ -snake is defined as a connected graph in which all the $k$-blocks are isomorphic to the cycle $C_n$ and the block-cut point graph is a path. Let $P$ be the path of minimum length that contains all the cut vertices of a $kC_n$ -snake. Barrientos proved that any $kC_n$ -snake is represented by a string $s_1, s_2, \ldots, s_{k-2}$ of integers of length $k - 2$ where the $i^{th}$ integer, $s_i$ on the string is the distance between $i^{th}$ and $(i+1)^{th}$ cut vertices on the path $P$ from one extreme and is taken from $S_n = \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}$. The strings obtained for both extremes are assumed to be the same. Then there are at most $\left\lfloor \frac{n}{2} \right\rfloor^{k-2}$ non isomorphic $kC_n$ -snakes. For example, the string of a $10C_4$ -snake is shown in Figure 1.1 is $2, 2, 1, 2, 1, 1, 2, 1$. A $kC_n$ -snake is said to be linear if each integer of its string is $\left\lfloor \frac{n}{2} \right\rfloor$. 
A $nC_k$-snake is said to be linear if each integer of its string is $\left\lfloor \frac{k}{2} \right\rfloor$. The linear $nC_4$-snake graph with diagonal vertices $u_{1j}$ $(1 \leq j \leq n + 1)$, left to the diagonal vertices $v_{1j}$ $(1 \leq j \leq n)$ and right to the diagonal vertices $w_{1j}$ $(1 \leq j \leq n)$ is denoted by $QS_n$. For example, a linear $3C_4$-snake graph $QS_3$ is shown in Figure 1.2.

**Figure 1.1:** An embedding of $10C_4$-snake

**Figure 1.2:** A linear $3C_4$-snake $QS_3$

**Definition 1.6.** A generalized $kC_n$-snake is defined as a connected graph in which each block is isomorphic to a cycle $C_n$ for some $n$ and the block-cut point graph is a path. It is denoted by $CS(n_1, n_2, ..., n_k)$ where $B_1, B_2, ..., B_k$ are the consecutive blocks and $B_i$ is isomorphic to $C_{n_i}$. By applying the same methods used to obtain the strings of a $kC_n$-snake, we can show that any generalized $kC_n$-snake can also be represented by a string of integers $s_1, s_2, ..., s_{k-2}$ of length $k - 2$ where $s_{i-1} \in S_{n_i}$. 
Definition 1.7. [4] Let $T$ be a tree and $u_0$ and $v_0$ be the two adjacent vertices in $T$. Let $u$ and $v$ be the two pendant vertices of $T$ such that the length of the path $u_0 - u$ is equal to the length of the path $v_0 - v$. If the edge $u_0v_0$ is deleted from $T$ and $u$ and $v$ are joined by an edge $uv$, then such a transformation of $T$ is called an elementary parallel transformation (or an ept) and the edge $u_0v_0$ is called transformable edge. If by the sequence of ept’s, $T$ can be reduced to a path, then $T$ is called a $T_p$-tree (transformed tree) and such sequence regarded as a composition of mappings (ept’s) denoted by $P$, is called a parallel transformation of $T$. The path, the image of $T$ under $P$, is denoted as $P(T)$. A $T_p$-tree and the sequence of two ept’s reducing it to a path are illustrated in Figure 1.3.

![Figure 1.3](image)

2. Main Results

In this section, we prove that $nC_4$-snake, $CS(n_1, n_2, ..., n_k)$, $n_i \equiv 0 (\text{mod} 4)$, $n_i \geq 4$, be a generalized $kC_n$-snake, $TOQS_n$ and $TOQS_n$ are odd vertex equitable even graphs.

Theorem 2.1. The $nC_4$-snake is an odd vertex equitable even graph.

Proof. Let $G$ be a $nC_4$-snake with $n$ blocks and $G_i = C_4$, $1 \leq i \leq n - 1$ and $u_i, v_i$ be the vertices with labels 1 and $q + 1$ respectively. By Theorem 1.2, $nC_4$ admits an odd vertex equitable even labeling. An example for odd vertex equitable even labeling of $3C_4$-snake is shown in Figure 2.1.
Theorem 2.2. Let $G = CS(n_1, n_2, ..., n_k), n_i \equiv 0(\text{mod}4), n_i \geq 4$ be a generalized $kC_n$-snake with its strings $s_1, s_2, ..., s_{k-2}$ where $s_i \in \{1\}, 1 \leq i \leq k$. Then $G$ is an odd vertex equitable even graph.

Proof. By Theorem 1.1, the cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0(\text{mod}4)$. By Theorem 1.2, $CS(n_1, n_2, ..., n_k), n_i \equiv 0(\text{mod}4)$, is an odd vertex equitable even graph. An example for odd vertex equitable even labeling of $CS(8, 4, 12)$ is shown in Figure 2.2.
Theorem 2.3. If $T$ be a $T_p$ -tree on $m$ vertices, then the graph $\hat{T}\tilde{QS}_n$ is an odd vertex equitable even graph.

Proof. Let $T$ be a $T_p$ -tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$ where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, ..., P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges.

Now denote the vertices of $P(T)$ successively by $u'_1, u'_2, ..., u'_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u_{i1}, u_{i2}, ..., u_{i(n+1)}, v_1, v_2, ..., v_k$ and $w_1, w_2, ..., w_k (1 \leq i \leq m)$ be the vertices of $i^{th}$ copy of $P_n$ with $u_{i(n+1)} = u'_i$.

Then $V(\hat{T}\tilde{QS}_n) = \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n + 1 \text{ with } u_{i(n+1)} = u'_i\} \cup \{u'_i, v_{ij}, w_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(\hat{T}\tilde{QS}_n) = \{e'_i = u'_i u_{i+1}' : 1 \leq i \leq m - 1\} \cup E(\tilde{QS}_n)$.

Here $|V(\hat{T}\tilde{QS}_n)| = m(3n + 1)$ and $|E(\hat{T}\tilde{QS}_n)| = 4mn + m - 1$.

Let $A = \{1, 3, ..., 4mn + m - 1\}$.

Define a vertex labeling $f : V(\hat{T}\tilde{QS}_n) \rightarrow A$ as follows:

For $1 \leq i \leq m, 1 \leq j \leq n + 1$ $f(u_{ij}) = \begin{cases} (4n + 1)(i - 1) + 4j - 3 & \text{if } i \text{ is odd} \\ (4n + 1)i - (4j - 3) & \text{if } i \text{ is even} \end{cases}$

For $1 \leq i \leq m, 1 \leq j \leq n$.

$f(v_{ij}) = f(u_{ij})$, $f(w_{ij}) = \begin{cases} (4n + 1)(i - 1) + 4j - 1 & \text{if } i \text{ is odd} \\ (4n + 1)i - (4j - 1) & \text{if } i \text{ is even} \end{cases}$

For the vertex labeling $f$, the induced edge labeling $f^*$ is as follows:

For $1 \leq i \leq m - 1$ $f^*(e'_i) = 2(4n + 1)i$.

The induced edge labels of $\tilde{QS}_n$ are $2(4n + 1)(i - 1) + 2j(1 \leq i \leq m, 1 \leq j \leq 2n)$ if $i$ is odd and $2(4n + 1)(i - 1) + 2j(1 \leq i \leq m, 1 \leq j \leq 2n)$ if $i$ is even.

Let $v_iv_j$ be a transformed edge in $T$ for some indices $i, j, 1 \leq i \leq j \leq m$. 

Let $P_1$ be the ept that deletes the edge $v_iv_j$ and adds an edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$.

Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$.

Therefore, $i$ and $j$ are of opposite parity, that is, $i$ is odd and $j$ is even or vice-versa.

The induced label of the edge $v_iv_j$ is given by $f^*(v_iv_j) = f^*(v_{i}v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) = 2(4n + 1)(i + t)$ and

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_{i}v_{i+t+1}) = 2(4n + 1)(i + t).$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$. It can be verified that the induced edge labels of $TOQS_n$ are $2, 4, 6, ..., 8mn + 2m - 2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence, $TOQS_n$ is an odd vertex equitable even graph.

An example for odd vertex equitable even labeling of $TOQS_2$ where $T$ is a $T_p$-tree on 8 vertices is shown in Figure 2.3.
Theorem 2.4. Let $T$ be a $T_p$-trees on $m$ vertices. Then the graph $T\tilde{O}QS_n$ is an odd vertex equitable even graph.

Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$ where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, ..., P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Now denote the vertices of $P(T)$ successively by $u_1', u_2', ..., u_m'$ starting from one pendant vertex of $P(T)$ right up to the other one.

Let $u_1, u_2, ..., u_{i(n+1)}, v_1, v_2, ..., v_m$ and $w_1, w_2, ..., w_m(1 \leq i \leq m)$ be the vertices of $i^{th}$ copy of $P_n$. □
Then \( V(T\overline{QS}_n) = \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n + 1\} \cup \{u_i, v_i, w_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \) and \( E(T\overline{QS}_n) = E(QS_n) \cup \{e'_i = u_{i}u_{i+1} : 1 \leq i \leq m-1\} \cup \{e''_i = u_{i}u_{i(n+1)} : 1 \leq i \leq m\}. \)

Here \( |V(T\overline{QS}_n)| = m(3n + 2) \) and \( |E(T\overline{QS}_n)| = 4mn + 2m - 1. \)

Let \( A = \{1, 3, ..., 4mn + 2m - 1\}. \)

Define a vertex labeling \( f : V(T\overline{QS}_n) \rightarrow A \) as follows:

For \( 1 \leq i \leq m, 1 \leq j \leq n+1 \)

\[
 f(u_{ij}) = \begin{cases} 
 (4n + 2)(i - 1) + 4j - 3 & \text{if } i \text{ is odd} \\ 
 (4n + 2)i - (4j - 3) & \text{if } i \text{ is even}
\end{cases}
\]

For \( 1 \leq i \leq m, 1 \leq j \leq n \)

\[
 f(v_{ij}) = f(u_{ij}), \quad f(w_{ij}) = \begin{cases} 
 (4n + 2)(i - 1) + 4j - 1 & \text{if } i \text{ is odd} \\ 
 (4n + 2)i - (4j - 1) & \text{if } i \text{ is even}
\end{cases}
\]

\[
 f(u'_i) = \begin{cases} 
 (4n + 2)i - 1 & \text{if } i \text{ is odd} \\ 
 (4n + 2)i - (4n + 1) & \text{if } i \text{ is even}
\end{cases}
\]

For the vertex labeling \( f \), the induced edge labeling \( f^* \) is as follows:

For \( 1 \leq i \leq m - 1 \)

\[
 f^*(e'_i) = 2(4n + 2)i,
\]

For \( 1 \leq i \leq m \)

\[
 f^*(e''_i) = \begin{cases} 
 2(4n + 2)i - 2 & \text{if } i \text{ is odd} \\ 
 2(4n + 2)(i - 1) + 2 & \text{if } i \text{ is even}
\end{cases}
\]

The induced edge labels of \( QS_n \) are \( 2(4n+2)(i-1)+2j \) (\( 1 \leq i \leq m, 1 \leq j \leq 2n \)) if \( i \) is odd and \( 2(4n+2)(i-1)+2j \) (\( 1 \leq i \leq m, 1 \leq j \leq 2n \)) if \( i \) is even.

Let \( v_iv_j \) be a transformed edge in \( T \) for some indices \( i, j, 1 \leq i \leq j \leq m \).

Let \( P_t \) be the ept that deletes the edge \( v_iv_j \) and adds an edge \( v_{i+t}v_{j-t} \) where \( t \) is the distance of \( v_i \) from \( v_{i+t} \) and the distance of \( v_j \) from \( v_{j-t} \).

Let \( P \) be a parallel transformation of \( T \) that contains \( P_t \) as one of the constituent epts. Since \( v_{i+t}v_{j-t} \) is an edge in the path \( P(T) \), it follows that \( i + t + 1 = j - t \) which implies \( j = i + 2t + 1 \).

Therefore, \( i \) and \( j \) are of opposite parity, that is, \( i \) is odd and \( j \) is even or vice-versa.

The induced label of the edge \( v_iv_j \) is given by \( f^*(v_iv_j) = f^*(v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) = 2(4n + 2)(i + t) \) and
Odd vertex equitable even labeling of cyclic make related graphs

\[ f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_{i+t+1}) = 2(4n+2)(i+t). \]

Therefore, \( f^*(v_iv_j) = f^*(v_{i+t}v_{j-t}) \).

It can be verified that the induced edge labels of \( \tilde{T}_{\text{OQS}} \) are \( 2, 4, 6, \ldots, 8mn+4m-2 \) and \( |v_f(a) - v_f(b)| \leq 1 \) for all \( a, b \in A \).

Hence, \( \tilde{T}_{\text{OQS}} \) is an odd vertex equitable even graph.

An example for odd vertex equitable even labeling of \( \tilde{T}_{\text{OQS}}_2 \) where \( T \) is a \( T_p \)-tree on 8 vertices is shown in Figure 2.4.

![Figure 2.4](image-url)
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