Explicit Toric Metric on Resolved Calabi-Yau Cone

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Abstract

We present an explicit non-singular complete toric Calabi-Yau metric using the local solution recently found by Chen, Lü and Pope. This metric gives a new supergravity solution representing D3-branes.
D3-branes on the tip of toric Calabi-Yau cones have been extensively studied in connection with the AdS/CFT correspondence \cite{1}. It is natural to consider the deformations of cone metrics in order to explore non-conformal theories \cite{2,3,4,5,6,7}.

In this letter, we study a Calabi-Yau metric, i.e. Ricci-flat Kähler metric, constructed as the BPS limit of the six dimensional Euclideanised Kerr-NUT-AdS black hole metric \cite{8}. In the black hole with equal angular momenta, the corresponding Calabi-Yau metric is of the form

\[
g = \frac{(1-x)(1-z)}{3} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{(1-x)(x-z)}{f(x)} dx^2 + \frac{(1-z)(z-x)}{h(z)} dz^2 \tag{1}
\]

where

\[
f(x) = 2x^3 - 3x^2 + a, \quad h(z) = 2z^3 - 3z^2 + b \tag{2}
\]

with two parameters \(a\) and \(b\). We assume that the roots \(x_i\) (\(i = 1, 2, 3\)) of \(f(x) = 0\) are all distinct and real, and further they are ordered as \(z_1 < x_1 < x_2 < x_3\) for the smallest real root \(z_1\) of \(h(z) = 0\). In order to avoid a curvature singularity we take the coordinates \(x\) and \(z\) to lie in the region \(z_1 < x_1 \leq x \leq x_2 < 1\). Indeed, it is easy to see that such a singularity appears at \(z = x\).

For \(r \to \infty\) (\(z = -r^2/2\)), the metric tends to a cone metric \(dr^2 + r^2 \bar{g}\), where

\[
\bar{g} = \frac{1-x}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1-x}{2f(x)} dx^2 + \frac{f(x)}{18(1-x)} (d\beta + \cos \theta d\phi)^2 \tag{3}
\]

\[
+ \frac{1}{9} (d\psi - \cos \theta d\phi + x(d\beta + \cos \theta d\phi))^2.
\]

This metric yields the Sasaki-Einstein metric \(Y_{p,q}\) when we impose a suitable condition for the parameter \(a\) \cite{9}.

Next let us look at the geometry near \(z = z_1\). We introduce new coordinates given by

\[
\varphi_1 = -\frac{1}{2}(\psi - z_1 \beta), \quad \varphi_2 = \frac{1}{1 - z_1} (\psi + z_1 \beta), \quad \varphi_3 = \phi. \tag{4}
\]
Then the metric behaves as

\[ g \simeq du^2 + u^2 d\varphi_1^2 + g^{(4)}, \]

where \( u^2 = 2(z_1 - x)(z_1 - z)/(3z_1) \). Therefore, the periodicity of \( \varphi_1 \) should be \( 2\pi \) in order to avoid an orbifold singularity. The four dimensional Kähler metric \( g^{(4)} \) is given by

\[ g^{(4)} = \frac{(1 - x)(1 - z_1)}{3}(d\theta^2 + \sin^2 \theta d\varphi_3^2) + \frac{(1 - x)(x - z_1)}{f(x)} dx^2 \]

\[ + \frac{(1 - z_1)^2 f(x)}{9(1 - x)(x - z_1)}(d\varphi_2 - \cos \theta d\varphi_3)^2. \]

We now argue that by taking the special parameters \( a = (1/2) - (1/32)\sqrt{13} \) which corresponds to \( Y^{2,1} \), and \( b = (1/4)(137 + 37\sqrt{13}) \), the four dimensional space with metric \( g^{(4)} \) is a non-trivial \( S^2 \)-bundle over \( S^2 \), i.e. the first del Pezzo surface \( dP_1 \). To see this, introduce a radial coordinate \( y^2 = 2(x_i - z_1) \mid x - x_i \mid \mid 3x_i \mid \) on the \( (x, \varphi_2) \)-fibre space defined by fixing the \( S^2 \) coordinates \( \theta \) and \( \varphi_3 \) in (6). Then, the fibre metric near boundary \( x = x_i \) \((i = 1, 2)\) is written as

\[ dy^2 + \left( \frac{x_i(1 - z_1)}{x_i - z_1} \right)^2 y^2 d\varphi_2. \]

Using the values of \( a \) and \( b \), we have

\[ x_1 = \frac{1}{8}(1 - \sqrt{13}), \quad x_2 = \frac{1}{8}(7 - \sqrt{13}), \quad z_1 = -\frac{1}{2}(2 + \sqrt{13}), \]

and then \( x_i(1 - z_1)/(x_i - z_1) = \pm 1/2 \). The apparent singularities at \( x = x_i \) can be avoided by choosing the periodicity of \( \varphi_2 \) to be \( 4\pi \). Thus, the \((x, \varphi_2)\)-fibre space is topologically \( S^2 \). On the other hand, fixing the coordinate \( x \) in (6), we obtain a principal \( U(1) \)-bundle over \( S^2 \) with the Chern number

\[ \frac{1}{4\pi} \int_{S^2} d(- \cos \theta d\varphi_3) = 1. \]

The metric \( g^{(4)} \) can be regarded as a metric on the associated \( S^2 \)-bundle of the principal \( U(1) \)-bundle. The associated bundle is non-trivial since the Chern number is odd, and hence the total space is the \( dP_1 \).
Let us describe the Calabi-Yau metric \([11]\) from the point of view of toric geometry. The metric has an isometry \(T^3\), locally generated by the Killing vector fields, \(\partial/\partial \psi\), \(\partial/\partial \phi\) and \(\partial/\partial \beta\). The symplectic (Kähler) form \(\omega\) is given by

\[
\omega = \frac{1}{3} \left( d\psi \wedge d(x + z) + d\phi \wedge d((1 - x)(1 - z) \cos \theta) + d\beta \wedge d(xz) \right). \tag{10}
\]

Using the following generators of the \(T^3\) action \([10]\),

\[
\begin{align*}
\frac{\partial}{\partial \phi_1} &= \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \beta}, \\
\frac{\partial}{\partial \phi_2} &= \frac{\partial}{\partial \phi} + 3\ell \frac{\partial}{\partial \beta}, \\
\frac{\partial}{\partial \phi_3} &= -6\ell \frac{\partial}{\partial \beta},
\end{align*} \tag{11-13}
\]

one has Darboux coordinates \((\xi^1, \xi^2, \xi^3)\) on which the symplectic form takes the standard form \(\omega = d\xi^i \wedge d\phi_i\):

\[
\begin{align*}
\xi^1 &= \frac{1}{3} (1-x)(1-z)(1-\cos \theta), \\
\xi^2 &= -\frac{\ell}{2} (2xz + 1) - \frac{1}{3}(1-x)(1-z) \cos \theta, \\
\xi^3 &= \ell(2xz + 1)
\end{align*} \tag{14-16}
\]

with \(\ell^{-1} = -5 + 2\sqrt{13}\). For the range of variables: \(0 \leq \theta \leq \pi, x_1 \leq x \leq x_2, z \leq z_1\), where \(x_1, x_2\) and \(z_1\) are given by \([10]\), we find the Delzant polytope (see Fig.1)

\[
P = \{ \xi = (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3 \mid (\xi, v_a) \geq \lambda_a, \ a = 1, 2, \ldots, 5 \}. \tag{17}
\]

Here, each \(v_a\) is a primitive element of the lattice \(\mathbb{Z}^3 \subset \mathbb{R}^3\) and an inward-pointing normal vector to the two dimensional face of \(P\). Explicitly, the set of five vectors \(v_a = (1, w_a)\) can be chosen as (see Fig. 2)

\[
w_1 = (-1, -2), \ w_2 = (0, 0), \ w_3 = (-1, 0), \ w_4 = (-2, -1), \ w_5 = (-1, -1). \tag{18}
\]

The constants \(\lambda_a\) are given by

\[
\begin{align*}
\lambda_1 &= \frac{1}{72} (1 - 5\sqrt{13}), \\
\lambda_3 &= \frac{1}{216} (29 + 17\sqrt{13}), \\
\lambda_5 &= \frac{1}{54} (31 + 7\sqrt{13}),
\end{align*} \tag{19}
\]
and $\lambda_2 = \lambda_4 = 0$. The inner products $(\xi, v_a)$ are evaluated as

$$
(\xi, v_1) = \frac{2}{9}(1 + \sqrt{13})(x - x_1)(x_1 - z) + \lambda_1,
$$

$$
(\xi, v_2) = \frac{1}{3}(1 - x)(1 - z)(1 - \cos \theta),
$$

$$
(\xi, v_3) = \frac{2}{27}(7 + \sqrt{13})(x_2 - x)(x_2 - z) + \lambda_3,
$$

$$
(\xi, v_4) = \frac{1}{3}(1 - x)(1 - z)(1 + \cos \theta),
$$

$$
(\xi, v_5) = \frac{2}{27}(-2 + \sqrt{13})(x - z_1)(z_1 - z) + \lambda_5.
$$

Thus, we see that the five faces $F_a = \{\xi \in \mathbb{R}^3 \mid (\xi, v_a) = \lambda_a\}$ correspond to degeneration surfaces at $x = x_1$, $\theta = 0$, $x = x_2$, $\theta = \pi$ and $z = z_1$, respectively.

Finally, we note that a D3-brane solution can be constructed from the Calabi-Yau
metric given by $g$ (11) with the special parameters $a, b$:

$$g^{(10)} = H^{-1/2} g^{(3+1)} + H^{1/2} g,$$  \hspace{1cm} (21)

$$F_5 = (1 + *_{10}) \text{vol}_{(3+1)} \wedge dH^{-1}.$$  \hspace{1cm} (21)

We find the warp factor $H$ as a harmonic function $\triangle_g H = 0$:

$$H(z) = -\frac{7 - 2\sqrt{13}}{27} L^4 \log \left( \frac{8z^3 - 12z^2 + 137 + 37\sqrt{13}}{(2z + 2 + \sqrt{13})^3} \right) - \frac{(10 - 2\sqrt{13})L^4}{27\sqrt{6} + 2\sqrt{13}} \left( \frac{\pi}{2} - \arctan \left( \frac{-4z + 5 + \sqrt{13}}{3\sqrt{6} + 2\sqrt{13}} \right) \right),$$  \hspace{1cm} (22)

where the constant $L$ is given by

$$L^4 = \frac{4\pi^4 g_s(\alpha')^2 N}{\text{Vol}(Y_{2,1})} = 16(-46 + 13\sqrt{13})\pi g_s(\alpha')^2 N.$$  \hspace{1cm} (24)

For large $-z = r^2/2$, the warp factor behaves as

$$H = \frac{L^4}{r^4} + O(1/r^6),$$  \hspace{1cm} (25)

while near $z = z_1$

$$H \simeq \frac{2(7 - 2\sqrt{13})L^4}{27} \log(z_1 - z).$$  \hspace{1cm} (26)

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