Berry–Esseen smoothing inequality for the Wasserstein metric on compact Lie groups

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Abstract
We prove a general inequality estimating the distance of two probability measures on a compact Lie group in the Wasserstein metric in terms of their Fourier transforms. The result is close to being sharp. We use a generalized form of the Wasserstein metric, related by Kantorovich duality to the family of functions with an arbitrarily prescribed modulus of continuity. The proof is based on smoothing with a Fejér-like kernel, and a Fourier decay estimate for continuous functions. As a corollary, we show that the rate of convergence of random walks on semisimple groups in the Wasserstein metric is necessarily almost exponential, even without assuming a spectral gap. Applications to equidistribution and empirical measures are also given.

1 Introduction

If the Fourier transform of two Borel probability measures on \( \mathbb{R} \) are equal, then the measures themselves are also equal. The celebrated Berry–Esseen smoothing inequality is a quantitative form of this fundamental fact of classical Fourier analysis. Given two Borel probability measures \( \nu_1 \) and \( \nu_2 \) on \( \mathbb{R} \), let \( F_j(x) = \nu_j((\infty, x]) \), \( j = 1, 2 \), and define

\[
\delta_{\text{unit}}(\nu_1, \nu_2) = \sup_{x \in \mathbb{R}} |F_1(x) - F_2(x)|.
\]

**Theorem A** (Berry–Esseen smoothing inequality). Assume that \( |F_2(x) - F_2(y)| \leq K|x - y| \) for all \( x, y \in \mathbb{R} \) with some constant \( K > 0 \). Then for any real number \( T > 0 \),

\[
\delta_{\text{unit}}(\nu_1, \nu_2) \ll \frac{K}{T} + \int_{-T}^{T} \frac{|\hat{\nu}_1(t) - \hat{\nu}_2(t)|}{|t|} \, dt
\]

with a universal implied constant.
In the terminology of probability theory, \( F_j(x) \) is the distribution function of \( \nu_j \); the Fourier transform \( \hat{\nu}_j(t) = \int_{\mathbb{R}} e^{itx} \, d\nu_j(x) \) is the characteristic function of \( \nu_j \); finally, \( \delta_{\text{unif}} \) is the uniform metric (or Kolmogorov metric) on the set of probability distributions. For somewhat sharper forms of Theorem A see Petrov \[21]\ Chapter 5.1.

Similar smoothing inequalities are known for several other probability metrics on \( \mathbb{R} \), see Bobkov \[4]\ for a survey. Some, but not all require a smoothness assumption on one of the distributions; for instance, Theorem A is usually formulated on one of the distributions; for instance, Theorem A is usually formulated under the assumption that \( F_2 \) is differentiable and \( |F_2(x)| \leq K \). A common feature of such results is that the distance of \( \nu_1 \) and \( \nu_2 \) in some probability metric is bounded above by the sum of two terms depending on a free parameter \( T > 0 \): one term decays as \( T \) increases, and the other term depends on the Fourier transforms of \( \nu_1 \) and \( \nu_2 \) only on the interval \([-T, T]\).

Berry–Esseen type smoothing inequalities are known in other spaces as well. The first multidimensional version, an upper bound for the uniform metric on \( \mathbb{R}^d \) is due to von Bahr \[2\]. Niederreiter and Philipp proved an analogous result for two Borel probability measures \( \nu_1 \) and \( \nu_2 \) on the torus \( \mathbb{R}^d/\mathbb{Z}^d \). By identifying \( \mathbb{R}^d/\mathbb{Z}^d \) with the unit cube \([0, 1)^d\), we can define the uniform metric as \( \delta_{\text{unif}}(\nu_1, \nu_2) = \sup_{x \in [0, 1]^d} |\nu_1(B(x)) - \nu_2(B(x))| \), where \( B(x) = [0, x_1) \times \cdots \times [0, x_d) \). The Fourier transform is now \( \hat{\nu}_j(m) = \int_{\mathbb{R}^d/\mathbb{Z}^d} e^{-2\pi i (m, x)} \, d\nu_j(x), \; m \in \mathbb{Z}^d \). Let \( \mu_{\mathbb{R}^d/\mathbb{Z}^d} \) be the normalized Haar measure, and let \( \|m\|_\infty = \max_{1 \leq k \leq d} |m_k| \).

**Theorem B** (Niederreiter–Philipp \[19]\). Assume that \( \nu_2(B) \leq K \mu_{\mathbb{R}^d/\mathbb{Z}^d}(B) \) for all axis parallel boxes \( B \subseteq [0, 1)^d \) with some constant \( K > 0 \). Then for any integer \( M \geq 1 \),

\[
\delta_{\text{unif}}(\nu_1, \nu_2) \leq \frac{K}{M} + \sum_{0 < \|m\|_\infty \leq M} \frac{1}{\prod_{k=1}^d \max\{|m_k|, 1\}} \frac{|\hat{\nu}_1(m) - \hat{\nu}_2(m)|}{M}
\]

with an implied constant depending only on \( d \).

The goal of this paper is to prove a Berry–Esseen type smoothing inequality in more general compact groups. In this more general setting only those probability metrics remain meaningful whose definition does not rely on concepts such as axis parallel boxes and distribution functions. One of the most important such metrics is the \( p \)-Wasserstein metric \( W_p \). Given a compact metric space \((X, \rho)\) and two Borel probability measures \( \nu_1 \) and \( \nu_2 \) on \( X \), we define

\[
W_p(\nu_1, \nu_2) = \inf_{\vartheta \in \text{Coup}(\nu_1, \nu_2)} \int_{X \times X} \rho(x, y)^p \, d\vartheta(x, y) \quad (0 < p \leq 1),
\]

and

\[
W_p(\nu_1, \nu_2) = \inf_{\vartheta \in \text{Coup}(\nu_1, \nu_2)} \left( \int_{X \times X} \rho(x, y)^p \, d\vartheta(x, y) \right)^{1/p} \quad (1 < p < \infty).
\]
Here $\text{Coup}(\nu_1, \nu_2)$ is the set of couplings of $\nu_1$ and $\nu_2$; that is, the set of Borel probability measures $\vartheta$ on $X \times X$ with marginals $\vartheta(B \times X) = \nu_1(B)$ and $\vartheta(X \times B) = \nu_2(B)$, $B \subseteq X$ Borel. Recall that for any $p > 0$, $W_p$ is a metric on the set of Borel probability measures on $X$, and it generates the topology of weak convergence.

Respecting the philosophy of the Berry–Esseen inequality, we wish to find an upper bound to $W_p(\nu_1, \nu_2)$ depending on the Fourier transform of $\nu_1$ and $\nu_2$ only up to a certain “level”. For this reason we chose to work with compact Lie groups, where the theory of highest weights provides a suitable framework to formalize the meaning of “level”. More precisely, our main result applies to any compact, connected Lie group $G$; classical examples include $\mathbb{R}^d/\mathbb{Z}^d$, $U(d)$, $SU(d)$, $SO(d)$, $Sp(d)$ and $Spin(d)$. Let $n$ and $r$ be the dimension and the rank of $G$, and let $\hat{G}$ denote the unitary dual. Further, let $\lambda_\pi$ denote the highest weight of the representation $\pi \in \hat{G}$, and let $\|A\|_{\text{HS}} = \sqrt{\text{tr}(A^*A)}$ be the Hilbert–Schmidt norm of a matrix $A$.

For a more formal setup we refer to Section 2.1. Generalizing and sharpening our recent result on the torus $G = \mathbb{R}^d/\mathbb{Z}^d$ [6], in this paper we prove the following Berry–Esseen type smoothing inequality for $W_p$ on compact Lie groups.

**Theorem 1.** Let $\nu_1$ and $\nu_2$ be Borel probability measures on a compact, connected Lie group $G$. If $0 < p < 1$, then for any real number $M > 0$,

$$W_p(\nu_1, \nu_2) \ll \frac{1}{(1 - p)M^p} + M^{1-p} \left( \sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi|^{\frac{2}{1-p}} - 2 \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \right)^{1/2}. \quad (1)$$

In addition, for any real number $M > 0$,

$$W_1(\nu_1, \nu_2) \ll \frac{\log(M + 2)}{M} + \left( \sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi|^{\frac{2}{1-p}} - 2 \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \right)^{1/2}. \quad (2)$$

The implied constants depend only on $G$. The result holds without any smoothness assumption on $\nu_1$ and $\nu_2$. We do not know if the factors $1/(1 - p)$ in (1) and $\log(M + 2)$ in (2) are necessary; however, as the applications in Section 2.3 will show, the inequalities are optimal up to these factors. Our methods do not work when $p > 1$, and in this case finding a Berry–Esseen type smoothing inequality on compact groups remains open; the only result of this type we are aware of is due to Steinerberger [23], and applies on the torus $\mathbb{R}^d/\mathbb{Z}^d$ under strong smoothness assumptions (satisfied e.g. if $\nu_2$ is the
The reason is that our proof is based on Kantorovich duality for $W_p$; recall that the Kantorovich duality theorem states that for any $0 < p \leq 1$,

$$W_p(\nu_1, \nu_2) = \sup_{f \in \mathcal{F}_p} \left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right|,$$

where, with $\rho$ denoting the geodesic distance on $G$,

$$\mathcal{F}_p = \{ f : G \to \mathbb{R} : |f(x) - f(y)| \leq \rho(x, y)^p \text{ for all } x, y \in G \}.$$

Theorem 1 thus estimates the difference of the integrals of $f$ with respect to $\nu_1$ and $\nu_2$ uniformly in $f \in \mathcal{F}_p$. From our methods it also follows (see (19)) that for any $f \in \mathcal{F}_p$,

$$\left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right| \leq \frac{1}{(1 - p)M^p} + \sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi| \frac{2}{\pi} \|f(\pi)\|_{\text{HS}} \cdot \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}},$$

with an implied constant depending only on $G$ if $0 < p < 1$, and the same estimate holds with $1/(1 - p)M^p$ replaced by $\log(M + 2)/M$ if $p = 1$. Hence for a given $f \in \mathcal{F}_p$ whose Fourier transform decays fast enough, the results of Theorem 1 can be improved. Fast Fourier decay follows e.g. from suitable smoothness assumptions on $f$, see Sugiura [25]. For instance, if $f \in \mathcal{F}_p$ is $2m$ times differentiable and $\Delta^m f \in \mathcal{F}_p$, then $|\lambda_\pi| \frac{2}{\pi} \|f(\pi)\|_{\text{HS}}$ can be replaced by $|\lambda_\pi| \frac{2}{\pi} \cdot 2^{-4m}$; see (20). Here $\Delta^m$ denotes the $m$-fold iteration of the Laplace–Beltrami operator. Note that Fourier decay rates play a role in classical Berry–Esseen type inequalities as well: the coefficient $|t|^{-1}$ (resp. $\prod_{k=1}^d \max\{|m_k|, 1\}^{-1}$) in Theorem A (resp. Theorem B) is explained by the fact that the Fourier transform of the indicator function of an interval (resp. axis parallel box) decays at this rate.

The most straightforward application of Theorem 1 is estimating the rate of convergence of random walks in the $W_p$ metric. Let $\nu^{*k}$ denote the $k$-fold convolution power of $\nu$, and let $\mu_G$ be the Haar measure on $G$. Recall that $\nu^{*k} \to \mu_G$ weakly as $k \to \infty$ if and only if the support of $\nu$ is contained neither in a proper closed subgroup, nor in a coset of a proper closed normal subgroup of $G$; see Stromberg [24]. Using a nonuniform spectral gap result of Varjú, we prove the following application of Theorem 1.

**Corollary 2.** Let $\nu$ be a Borel probability measure on a compact, connected, semisimple Lie group $G$. If $\nu^{*k} \to \mu_G$ weakly as $k \to \infty$, then for any $0 < p \leq 1$,

$$W_p(\nu^{*k}, \mu_G) \leq e^{-ck^{1/3}},$$

where the constant $c > 0$ and the implied constant depend only on $G$ and $\nu$. 
The condition of semisimplicity cannot be removed. The main motivation came from our recent paper [6] on quantitative ergodic theorems for random walks. Given independent, identically distributed $G$-valued random variables $\zeta_1, \zeta_2, \ldots$, with distribution $\nu$, we showed that for any $f \in F_p$ the sum $\sum_{k=1}^{N} f(\zeta_1 \zeta_2 \cdots \zeta_k)$ satisfies the central limit theorem and the law of the iterated logarithm, provided $\sum_{k=1}^{\infty} W_p(\nu^* k, \mu_G) < \infty$. Corollary [2] thus provides a large class of examples of random walks with fast enough convergence in $W_p$, and consequently to which our quantitative ergodic theorems apply. We do not know whether $W_p(\nu^* k, \mu_G) \ll e^{-ck^{1/3}}$ remains true for $p > 1$.

Another possible application is in uniform distribution theory, where the goal is finding finite sets $\{a_1, a_2, \ldots, a_N\} \subset G$ which make the integration error $|N^{-1} \sum_{k=1}^{N} f(a_k) - \int_G f \, d\mu_G|$ small for a suitable class of test functions. Applying Theorem [1] to $\nu_1 = N^{-1} \sum_{k=1}^{N} \delta_{a_k}$ (where $\delta_a$ is the Dirac measure concentrated at $a \in G$) and $\nu_2 = \mu_G$, we can quantitatively measure how well distributed a finite set is with respect to test functions $f \in F_p$. Note that in this case we have

$$\|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 = \frac{1}{N^2} \sum_{k, \ell=1}^{N} \chi_\pi(a_k^{-1} a_\ell),$$

where $\chi_\pi(x) = \text{tr} \pi(x)$ is the character of the representation $\pi \in \hat{G}$. Theorem [1] thus becomes an abstract version of the Erdős–Turán inequality, estimating the distance of a finite set from uniformity in terms of character sums. As an illustration, we will show that certain finite sets in SO(3) constructed by Lubotzky, Phillips and Sarnak using deep number theory are close to being optimal with respect to $W_p$.

A similar Erdős–Turán inequality on compact Riemannian manifolds with respect to sufficiently nice test sets was proved by Colzani, Gigante and Travaglini [12]. Steinerberger [22] estimated the $W_2$ distance of $N^{-1} \sum_{k=1}^{N} \delta_{a_k}$ from uniformity in terms of the Green function of the Laplace–Beltrami operator on a compact Riemannian manifold. Numerical results for certain finite point sets on the orthogonal group $O(d)$ and on Grassmannian manifolds were obtained by Pausinger [20].

The discussion above can be generalized from $F_p$ to the class of functions with an arbitrarily prescribed modulus of continuity, and we will actually work out the details in this generality. In particular, our results apply to any given $f \in C(G)$. The formal setup and notation are given in Section [2.1]; we state the general form of Theorem [1] with explicit constants in Section [2.2]; applications to random walks and to uniform distribution theory are discussed in more detail, and the proof of Corollary [2] is given in Section [2.3]. The proof of the main result, Theorem [3] is given in Section [3].
2 Results

2.1 Notation

Throughout the paper $G$ denotes a compact, connected Lie group with identity element $e \in G$ and Lie algebra $\mathfrak{g}$. Let $\exp : \mathfrak{g} \to G$ and $n = \dim G$ denote the exponential map and the dimension of $G$ as a real smooth manifold. Fix an Ad-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$, and let $|X| = \sqrt{(X,X)}$, $X \in \mathfrak{g}$. This inner product defines a Riemannian metric on $G$; let $\rho$ denote the corresponding geodesic metric on $G$. The Laplace–Beltrami operator on $G$ is $\Delta = \sum_{k=1}^n X_kX_k$ (as an element of the universal enveloping algebra of $\mathfrak{g}$), where $X_1, \ldots, X_n$ is an orthonormal base in $\mathfrak{g}$ with respect to $(\cdot, \cdot)$; this does not depend on the choice of the orthonormal base.

Let $r$ denote the rank of $G$, and fix a maximal torus $T$ in $G$ with Lie algebra $\mathfrak{t}$. Let $\mathfrak{t}^* = \text{Hom}(\mathfrak{t}, \mathbb{R})$ denote the dual vector space. The sets

$$\Gamma = \{X \in \mathfrak{t} : \exp(2\pi X) = e\},$$

$$\Gamma^* = \{\lambda \in \mathfrak{t}^* : \lambda(X) \in \mathbb{Z} \text{ for all } X \in \Gamma\}$$

are dual lattices of full rank in $\mathfrak{t}$ and $\mathfrak{t}^*$, respectively. The inner product on $\mathfrak{g}$ naturally defines an inner product on $\mathfrak{t}^*$, which we also denote by $(\cdot, \cdot)$; we also write $|\lambda|$ = $\sqrt{(\lambda, \lambda)}$, $\lambda \in \mathfrak{t}^*$. The weights will be considered elements of $\Gamma^*$; the character of $T$ corresponding to $\lambda \in \Gamma^*$ is $\exp(X) \mapsto e^{i\lambda(X)}$, $X \in \mathfrak{t}$. Let $R$ be the set of roots, and choose a set of positive roots $R^+$; we have $|R| = n - r$ and $|R^+| = (n - r)/2$. Let

$$C^+ = \{\lambda \in \mathfrak{t}^* : (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in R^+\}$$

be the dominant Weyl chamber; the set of dominant weights is thus $\Gamma^* \cap C^+$. The Weyl group of $G$ with respect to $T$ will be denoted by $W(G,T) = N_G(T)/T$.

Let $\hat{G}$ be the unitary dual of $G$. For any $\pi \in \hat{G}$, let $d_\pi$ and $\lambda_\pi$ denote the dimension and the highest weight of $\pi$. The map $\pi \mapsto \lambda_\pi$ is a bijection from $\hat{G}$ to the set of dominant weights $\Gamma^* \cap C^+$. Let $\kappa_\pi \geq 0$ denote the negative Laplace eigenvalue of $\pi$; that is, $\Delta\pi = -\kappa_\pi \pi$ where $\Delta$ acts entrywise. Recall that

$$\kappa_\pi = |\lambda_\pi|^2 + 2(\lambda_\pi, \rho^+) \quad \text{and} \quad d_\pi = \frac{\prod_{\alpha \in R^+}(\lambda_\pi + \rho^+, \alpha)}{\prod_{\alpha \in R^+}(\rho^+, \alpha)},$$

where $\rho^+ = \sum_{\alpha \in R^+} \alpha/2$ is the half-sum of positive roots; in particular,

$$|\lambda_\pi|^2 \leq \kappa_\pi \leq |\lambda_\pi|^2 + O(|\lambda_\pi|) \quad \text{and} \quad d_\pi \ll |\lambda_\pi|^{(n-r)/2}.$$

Let $\mu_G$ (resp. $\mu_T$) denote the normalized Haar measure on $G$ (resp. $T$). The Fourier transform of a function $f : G \to \mathbb{C}$ is $\hat{f}(\pi) = \int_G f(x)\pi(x)^* \, d\mu_G(x)$, $\pi \in \hat{G}$;
that of a Borel probability measure \( \nu \) on \( G \) is
\[
\hat{\nu}(\pi) = \int_G \pi(x)^* \, d\nu(x), \quad \pi \in \hat{G}.
\]
Here \( \pi(x)^* \) denotes the adjoint of \( \pi(x) \), and the integrals are taken entrywise.

Let \( g : [0, \infty) \to [0, \infty) \) be a nondecreasing and subadditive\(^1\) function such that
\[
\lim_{t \to 0^+} g(t) = 0,
\]
and define
\[
W_g(\nu_1, \nu_2) = \inf_{\varphi \in \text{Coup}(\nu_1,\nu_2)} \int_{G \times G} g(\rho(x,y)) \, d\varphi(x,y),
\]
where \( \text{Coup}(\nu_1,\nu_2) \) is the set of couplings, as before. Letting
\[
\mathcal{F}_g = \{ f : G \to \mathbb{R} : |f(x) - f(y)| \leq g(\rho(x,y)) \text{ for all } x, y \in G \},
\]
the Kantorovich duality theorem states
\[
W_g(\nu_1, \nu_2) = \sup_{f \in \mathcal{F}_g} \left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right|.
\]
Note that \( W_g \) is a metric on the set of Borel probability measures on \( G \) and it generates the topology of weak convergence, unless \( g \) is constant zero. In the special case \( g(t) = t^p, 0 < p \leq 1 \) we write \( W_p \) (resp. \( \mathcal{F}_p \)) instead of \( W_g \) (resp. \( \mathcal{F}_g \)). We mention that given \( f \in C(G) \), the function
\[
g_f(t) = \sup\{|f(x) - f(y)| : x, y \in G, \rho(x,y) \leq t\}
\]
is nondecreasing and subadditive, and \( \lim_{t \to 0^+} g_f(t) = 0 \); in fact, \( g_f \) is the smallest \( g \) for which \( f \in \mathcal{F}_g \).

**Remark.** Kantorovich duality is usually stated for \( g(t) = t \), i.e. for Lipschitz functions. To see the general case, note that \( g(\rho(x,y)) \) is another metric on \( G \) generating the topology of \( G \), unless \( g \) is constant zero; the subadditivity of \( g \) is needed for the triangle inequality. Kantorovich duality for Lipschitz functions in the \( g(\rho(x,y)) \) metric thus implies Kantorovich duality for \( W_g \) as claimed. Further, since the usual 1-Wasserstein metric with respect to \( g(\rho(x,y)) \) generates the topology of weak convergence, so does \( W_g \).

### 2.2 Berry–Esseen inequality on compact Lie groups

Fix dual bases\(^2\) \( \beta_1, \ldots, \beta_r \) and \( \alpha_1, \ldots, \alpha_r \) in the lattices \( \Gamma \) and \( \Gamma^* \), respectively; that is, \( \alpha_k(\beta_\ell) = \delta_{k,\ell} \). By a lattice box in \( \Gamma^* \) of size \( L \in \mathbb{N} \) we mean a set of the form
\[
\left\{ v_0 + \sum_{k=1}^r n_k \alpha_k : n_1, \ldots, n_r \in [0, L-1] \cap \mathbb{Z} \right\}
\]

\(^1\)That is, \( g(t+u) \leq g(t) + g(u) \) for all \( t, u \geq 0 \).

\(^2\)The set of roots might not span \( \Gamma^* \), since we did not assume that \( G \) is semisimple.
with some \( v_0 \in \Gamma^* \). Since the dominant Weyl chamber \( C^+ \) is a closed, convex cone in \( t^* \) with a nonempty interior, the set \( \{ \lambda \in t^* : |\lambda| \leq M/2 \} \cap C^+ \) contains a lattice box in \( \Gamma^* \) of size \( L \gg M \) with an implied constant depending only on \( G \).

The main result of the paper is the following Berry–Esseen type inequality.

**Theorem 3.** Let \( \nu_1 \) and \( \nu_2 \) be Borel probability measures on a compact, connected Lie group \( G \). Let \( g : [0, \infty) \to [0, \infty) \) be nondecreasing and subadditive such that \( \lim_{t \to 0^+} g(t) = 0 \), and let

\[
\psi(t) = |W(G, T)| \left( \sum_{k=1}^{r} \int_0^{t/2} g \left( \frac{2\pi|\beta_k|}{t} x \right) \frac{\sin^2(\pi x)}{x^2} \, dx \right),
\]

\[
\phi(t) = \inf_{0 < c < 2(\sqrt{m+n} - n)} \frac{n}{1 - c - c^2/(4n)} \cdot g \left( \frac{c}{t} \right).
\]

Then for any real number \( M > 0 \),

\[
W_\rho(\nu_1, \nu_2) \leq \psi(L) + \phi(M) \left( \sum_{0 < |\lambda| \leq M} \frac{d_\pi}{\kappa_\pi} \| \hat{\nu}_1(\pi) - \hat{\nu}_2(\pi) \|_{HS}^2 \right)^{1/2}
\]

provided \( \{ \lambda \in t^* : |\lambda| \leq M/2 \} \cap C^+ \) contains a lattice box in \( \Gamma^* \) of size \( L \in \mathbb{N} \).

If \( g(t) = t^p \) with some \( 0 < p \leq 1 \), then choosing e.g. \( c = (\sqrt{17} - 3)/2 \) (this is optimal in the worst case \( p \to 0, \ n = 1 \)) yields \( \phi(M) \leq 3n^{3/2-p}M^{1-p} \). In addition, for all \( L \in \mathbb{N} \),

\[
\psi(L) \leq \begin{cases} 
|W(G, T)| \frac{\sum_{k=1}^{r} |\beta_k|^p}{1-p} L^{-p} & \text{if } 0 < p < 1, \\
|W(G, T)| \left( \sum_{k=1}^{r} \pi |\beta_k| \right)^{2+\log(L)} / L & \text{if } p = 1.
\end{cases}
\]

This follows directly from the definition of \( \psi \); for a detailed proof we refer to [6, Proposition 3]. As observed, we can always choose \( L \gg M \), therefore Theorem 3 implies Theorem 1.

We mention that \( \psi(t) \ll g(1/t) \), and consequently \( \psi(L) \ll g(1/M) \) holds for a large class of \( g \). This is the case for instance if \( t^{-p} g(t) \) is nondecreasing for some \( 0 < p < 1 \); simply use \( g(2\pi|\beta_k|x/t) \leq (2\pi|\beta_k|x)^p g(1/t) \), and extend the integration to \([0, \infty)\). Note that this covers all cases when \( g(t) \) is larger than any positive power of \( t \) as \( t \to 0^+ \), i.e. when the prescribed modulus of continuity is weaker than Hölder continuity. As we shall see in Section 2.3.3 in the case \( \psi(t) \ll g(1/t) \) Theorem 3 is sharp up to a constant factor depending on \( G \) and \( g \).
2.3 Applications

2.3.1 Spectral gaps and random walks

Given Borel probability measures $\nu_1$ and $\nu_2$ on $G$, let $\nu_1 \ast \nu_2$ denote their convolution, and let $\nu_1^*(B) = \nu_1(B^{-1})$, $B \subseteq G$ Borel. If $\zeta_1$ and $\zeta_2$ are independent $G$-valued random variables with distribution $\nu_1$ and $\nu_2$, then $\nu_1 \ast \nu_2$ (resp. $\nu_1^*$) is the distribution of $\zeta_1 \zeta_2$ (resp. $\zeta_1^{-1}$).

Let $L^2_0(G, \mu_G)$ be the orthogonal complement of the space of constant functions in $L^2(G, \mu_G)$; that is, the set of all $f \in L^2(G, \mu_G)$ with $\int_G f \, d\mu_G = 0$. Given a Borel probability measure $\nu$ on $G$, let $T_\nu : L^2_0(G, \mu_G) \to L^2_0(G, \mu_G)$, $(T_\nu f)(x) = \int_G f(xy) \, d\nu(y)$ be its associated Markov operator. Observe that $T_{\nu_1 \ast \nu_2} = T_{\nu_1} T_{\nu_2}$ and $T_{\nu^*} = T_{\nu^*}$; in particular, $T_\nu$ is self-adjoint (resp. normal) if and only if $\nu = \nu^*$ (resp. $\nu \ast \nu^* = \nu^* \ast \nu$).

We start with a trivial estimate for $W_p(\nu, \mu_G)$ in terms of $T_\nu$. It is not difficult to see that $q_\nu := \|T_\nu\|_{op} = \sup_{\pi \in \hat{G} \neq \pi_0} \|\hat{\nu}(\pi)\|_{op}$, where $\pi_0 \in \hat{G}$ denotes the trivial representation, and $\| \cdot \|_{op}$ is the operator norm. Let $f \in F_p$ with $\int_G f \, d\mu_G = 0$ be arbitrary, and note that $|T_\nu f| \geq |(T_\nu f)(e)|/2$ on the ball centered at $e$ with radius $r = (|(T_\nu f)(e)|/2)^{1/p}$, we have

$$
|T_\nu f| \geq \left( \frac{(T_\nu f)(e)}{2} \right)^2 \mu_G(B(e, r)) \gg |(T_\nu f)(e)|^{2+n/p},
$$

$$
|T_\nu f|^2 \leq \|T_\nu\|^2_{op} \cdot \|f\|^2 \ll \|T_\nu\|^2_{op}.
$$

Therefore $|\int_G f \, d\nu| = |(T_\nu f)(e)| \ll q_\nu^{2p/(n+2p)}$, and consequently

$$
W_p(\nu, \mu_G) \ll q_\nu^{2p/(n+2p)}. \tag{3}
$$

We now deduce an almost sharp improvement on the trivial estimate (3). Recall
that \( \|A\|_{\text{HS}} \leq \sqrt{t} \|A\|_{\text{op}} \) for any \( d_\pi \times d_\pi \) matrix \( A \). With \( \nu_1 = \nu \) and \( \nu_2 = \mu_G \),

\[
\sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi|^{-2} \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \lesssim \sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi|^{-2} d_\pi \|\hat{\nu}(\pi)\|_{\text{op}}^2
\]

\[
\ll \sum_{\pi \in \hat{G}, 0 < |\lambda_\pi| \leq M} |\lambda_\pi|^{n-r-2} q_\nu^2
\]

\[
\ll \begin{cases} 
q_\nu^2 & \text{if } n = 1, \\
(\log(M + 2))q_\nu^2 & \text{if } n = 2, \\
M^{n-2} q_\nu^2 & \text{if } n \geq 3.
\end{cases}
\]  

(4)

Optimizing the value of the free parameter \( M > 0 \), in dimension \( n \geq 3 \) Theorem 1 thus gives

\[
W_p(\nu, \mu_G) \ll \begin{cases} 
(1 - p)^{2p/n - 1} q_\nu^{2p/n} & \text{if } 0 < p < 1, \\
(\log(2 + 1/q_\nu))^{1-2/n} q_\nu^{2/n} & \text{if } p = 1
\end{cases}
\]

with implied constant depending only on \( G \). Using Theorem 3 instead, we get

\[
W_p(\nu, \mu_G) \ll g(q_\nu^{2/n}) \text{ provided } \psi(t) \ll g(1/t).
\]

Similar estimates can be deduced in dimensions \( n = 1 \) and 2. Clearly \( q_\nu \leq 1 \), and \( q_\nu q_{\nu_1 \cdots \nu_2} \leq q_\nu^2 q_{\nu_2}^2 \); in particular, (5) gives an upper bound for \( W_p(\nu_1 \cdots \nu_N, \mu_G) \) in terms of \( \prod_{k=1}^{N} q_\nu^2 \).

We say that \( \nu \) has a spectral gap, if the spectral radius of \( T_\nu \) is strictly less than 1; note that this is a direct generalization of Cramér’s condition in classical probability theory. Assuming \( T_\nu \) is normal, having a spectral gap is equivalent to \( q_\nu < 1 \); for general \( T_\nu \), it is equivalent to \( q_{\nu, m} < 1 \) for some integer \( m \geq 1 \). Deciding whether a given \( \nu \) has a spectral gap is a highly nontrivial problem. Generalizing results of Bourgain and Gamburd [8], [9] on \( SU(2) \) and \( SU(d) \), Benoist and Saxcé considered a Borel probability measure \( \nu \) on a compact, connected, simple Lie group \( G \). They proved [3, Theorem 3.1] that if the support of \( \nu \) is not contained in any proper closed subgroup, and each element of the support (as a matrix) has algebraic entries, then \( \nu \) has a spectral gap. The same authors also conjectured that the condition that the matrix entries are algebraic can be dropped.

Using (3) (or even just (4)), \( W_p(\nu^{\ast k}, \mu_G) \to 0 \) exponentially fast as \( k \to \infty \) whenever \( \nu \) has a spectral gap. Corollary 2 is thus basically an unconditional (i.e. not assuming the conjecture of Benoist and Saxcé), weaker form of this fact. In contrast to the (semi)simple case, \( W_p(\nu^{\ast k}, \mu_G) \to 0 \) polynomially fast for certain finitely supported measures \( \nu \) on the torus \( \mathbb{R}^d / \mathbb{Z}^d \) [6].

So far we have only discussed the relationship between \( W_p(\nu, \mu_G) \) and the spectral gap of \( \nu \). Theorem 1, however, provides a quantitative relationship between \( W_p(\nu, \mu_G) \) and the spectrum of the self-adjoint operator \( T_\nu^* T_\nu \) itself. Indeed, by the Peter–Weyl theorem \( L^2_0(G, \mu_G) = \bigoplus_{\pi \in \hat{G}, \pi \neq \pi_0} V_\pi, \) where \( V_\pi \) is the vector space
spanned by the entries of \( \pi(x) \). Since \((T_\nu \pi)(x) = \pi(x) \hat{\nu}(\pi)^*\), the action of \( T_\nu \) on \( V_\pi \) is determined by \( \hat{\nu}(\pi) \); in particular, \( d_\nu \| \hat{\nu}(\pi) \|_{\text{HS}}^2 \) is simply the sum of all spectrum points of \( T_\nu^* T_\nu \) on \( V_\pi \). The proof of Corollary 2 is based on this quantitative relationship.

**Proof of Corollary 2.** It will be enough to prove the claim for \( p = 1 \). Indeed, the general case follows from \( W_p(\nu_1, \nu_2) \leq W_1(\nu_1, \nu_2)^p, \) \( 0 < p \leq 1 \); this can be seen directly from the definition of \( W_p \) and the Hölder inequality.

Varjú [27, Theorem 6] proved that for any Borel probability measure \( \vartheta \) on \( G \) and any \( M > 0 \),

\[
1 - \max_{0 < |\lambda_\pi| \leq M} \| \hat{\nu}(\pi) \|_{\text{op}} \geq c_0 \left( 1 - \max_{0 < |\lambda_\pi| \leq M_0} \| \hat{\nu}(\pi) \|_{\text{op}} \right) \frac{1}{\log^A(M + 2)}, \tag{6}
\]

where the constants \( c_0, M_0 > 0 \) and \( 1 \leq A \leq 2 \) depend only on the group \( G \); in fact, the exact value of \( A \) was also given. Since \( \nu^* k \to \mu_G \) weakly, we have \( \hat{\nu}(\pi)^k = \hat{\nu}^k(\pi) \to 0 \) for all \( \pi \neq \pi_0 \), and hence the spectral radius of \( \hat{\nu}(\pi) \) is less than \( 1 \). It follows that for any \( \pi \in \hat{G} \) with \( 0 < |\lambda_\pi| \leq M_0 \), we have \( \| \hat{\nu}(\pi)^m \|_{\text{op}} < 1 \) with some positive integer \( m = m(G, \nu) \); in particular,

\[
b = b(G, \nu) := c_0 \left( 1 - \max_{0 < |\lambda_\pi| \leq M_0} \| \hat{\nu}(\pi)^m \|_{\text{op}} \right) > 0.
\]

Applying (6) to \( \vartheta = \nu^* m \), we get that for any positive integer \( k \) and any \( M > 0 \),

\[
\max_{0 < |\lambda_\pi| \leq M} \| \nu^{*k}(\pi) \|_{\text{op}} \leq \left( \max_{0 < |\lambda_\pi| \leq M} \| \hat{\nu}(\pi)^m \|_{\text{op}} \right)^{\lfloor k/m \rfloor} \leq \left( 1 - \frac{b}{\log^A(M + 2)} \right)^{(k-m)/m} \leq e^{-b(k-m)/(m \log^A(M+2))}.
\]

Hence

\[
\sum_{\pi \in \hat{G}} |\lambda_\pi|^{\frac{n-r}{r} - 2} \| \nu^{*k}(\pi) \|^2_{\text{HS}} \leq \sum_{\pi \in \hat{G}} |\lambda_\pi|^{\frac{n-r}{r} - 2} d_\pi \| \nu^{*k}(\pi) \|^2_{\text{op}} \leq \sum_{\pi \in \hat{G}} |\lambda_\pi|^{n-r-2} e^{-b(k-m)/(m \log^A(M+2))} \leq M^n e^{-b(k-m)/(m \log^A(M+2))}.
\]
The first factor is actually $1$, $\log(M + 2)$, $M^{n-2}$ in the cases $n = 1$, $n = 2$, $n \geq 3$, but this will not play an important role. Theorem 1 thus gives that for any $M > 0$,

$$W_1(\nu^k, \mu_G) \ll \frac{\log(M + 2)}{M} + M^{n/2} e^{-b(k-m)/(2m \log^4(M+2))}.$$ 

Choosing $\log^4 M = b(k - m)/(2mn)$, we deduce

$$W_1(\nu^k, \mu_G) \ll k^{\frac{1}{n+1}} \exp \left( -\frac{n}{2} \left( \frac{b(k - m)}{2mn} \right)^\frac{1}{n+1} \right),$$

and the claim follows with any $0 < c < \frac{n}{2} \cdot \left( \frac{b}{2mn} \right)^\frac{1}{n+1}$. \hfill \Box

**Remark.** Assume $g : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing and concave, and $\lim_{t \rightarrow 0^+} g(t) = 0$; concavity implies, and is only slightly stronger than subadditivity. Then $W_g(\nu_1, \nu_2) \leq g(W_1(\nu_1, \nu_2))$ follows from the Jensen inequality. In particular, the general form of the conclusion of Corollary 2 is $W_g(\nu^k, \mu_G) \ll g(e^{-ck^{1/3}})$.

### 2.3.2 Uniform distribution theory

Next, we consider applications in uniform distribution theory. It is not difficult to see e.g. directly from the definition of $W_g$, that for any given nonempty finite set $A \subset G$ and any $g$ as in Section 2.1,

$$\inf_{\text{supp } \nu \subseteq A} W_g(\nu, \mu_G) = \int_G g(\text{dist}(A, x)) \, d\mu_G(x), \quad (7)$$

where the supremum is over all probability measures $\nu$ whose support is contained in $A$, and $\text{dist}(A, \cdot)$ denotes distance from the set $A$. Indeed, the infimum is attained when for any $a \in A$, $\nu(\{a\})$ is the Haar measure of the Voronoi cell

$$\{x \in G : \text{dist}(A, x) = \rho(a, x)\}.$$ 

In this case the optimal transport plan from $\nu$ to $\mu_G$ is to simply spread $\nu(\{a\})$ evenly over the given Voronoi cell. Recall that open balls $B(x, r)$ in $G$ of radius $0 < r < \text{diam } G$ satisfy $r^n \ll \mu_G(B(x, r)) \ll r^n$. A standard ball packing argument (using e.g. the “3r covering lemma” of Vitali) shows that the optimal distance from a probability measure supported on at most $N$ points to the Haar measure is

$$g(N^{-1/n}) \ll \inf_{|\text{supp } \nu| \leq N} W_g(\nu, \mu_G) \ll g(N^{-1/n}) \quad (8)$$

with implied constants depending only on $G$. In particular, (7) and (8) hold for $W_p$, $0 < p \leq 1$. We mention that both estimates hold also for any $1 < p < \infty$ with
\( W_p \) replaced by \( W_p^\ast \). For a detailed proof in the case \( 1 < p < \infty \) see Kloeckner [15]; the proof for \( 0 < p \leq 1 \) and for more general \( g \) is identical. We refer to the same paper for far reaching generalizations (e.g. to more general measures on Riemannian manifolds).

Lubotzky, Phillips and Sarnak [17], [18] considered the problem of finding well distributed finite sets in SO(3), and consequently, on the sphere \( S^2 \). For any \( N \) such that \( 2N - 1 \) is a prime congruent to 1 modulo 4, they constructed a symmetric set \( \{a_1, a_2, \ldots, a_{2N}\} \subset SO(3) \) for which the probability measure \( \nu_N = (2N)^{-1} \sum_{k=1}^{2N} \delta_{a_k} \) satisfies \( q_{\nu_N} = \sqrt{2N - 1}/N \); this spectral gap is in fact optimal among all symmetric sets of size \( 2N \). Since \( SO(3) \) has dimension \( n = 3 \), (5) yields

\[
W_p(\nu_N, \mu_{SO(3)}) \ll \begin{cases} (1 - p)^{2p/3 - 1} N^{-p/3} & \text{if } 0 < p < 1, \\ (\log N)^{1/3} N^{-1/3} & \text{if } p = 1 \end{cases}
\]

with a universal implied constant. By (8), this is optimal up to the factors \((1 - p)^{2p/3 - 1}\) and \((\log N)^{1/3}\). Note that the trivial estimate (3) only yields \( W_p(\nu_N, \mu_{SO(3)}) \ll N^{-p/(3+2p)} \). More generally, we have \( W_g(\nu_N, \mu_{SO(3)}) \ll g(N^{-1/3}) \) provided \( \psi(t) \ll g(1/t) \).

Clozel [11] proved a similar optimal (up to a constant factor) spectral gap estimate in terms of the size of a finite set in \( U(d) \). Less precise estimates on more general compact homogeneous spaces were obtained by Oh [16].

### 2.3.3 Empirical measures

Finally, we address the optimality of Theorems [1] and [3] we do so by deducing a simple estimate on the mean rate of convergence of empirical measures. Let \( \nu \) be an arbitrary Borel probability measure on \( G \), and let \( \zeta_1, \zeta_2, \ldots, \zeta_N \) be independent, identically distributed \( G \)-valued random variables with distribution \( \nu \). The probability measure \( \tau_N := N^{-1} \sum_{k=1}^{N} \delta_{\zeta_k} \) is called the corresponding empirical measure. Theorem [1] gives an estimate for \( W_p(\tau_N, \nu) \) — a random variable! — as follows. Let \( E_\pi = E_{\pi}(\zeta_1) = \hat{\nu}(\pi)^\ast \). With \( \nu_1 = \tau_N \) and \( \nu_2 = \nu \) we then have

\[
\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi) = \frac{1}{N} \sum_{k=1}^{N} (\pi(\zeta_k) - E_\pi)^\ast,
\]

and by independence, the “variance” satisfies

\[
E \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|^2_{HS} = \frac{1}{N^2} \sum_{k=1}^{N} E \text{tr} (\pi(\zeta_k)^\ast \pi(\zeta_k) - E_\pi^\ast E_\pi) \leq \frac{d_\pi}{N}.
\]

In the last step we used that \( \pi(x) \) is unitary. Following the steps in (4), in dimension \( n \geq 3 \) Theorem [1] gives that for any \( 0 < p < 1 \) and any \( M > 0 \),

\[
E_W(\tau_N, \nu) \leq \sqrt{E_W(\tau_N, \nu)^2} \ll \frac{1}{(1 - p)M^p} + M^{1-p} \sqrt{\frac{M^{n-2}}{N}}.
\]
and a similar estimate holds for $p = 1$. Optimizing the value of $M > 0$, we finally obtain that in dimension $n \geq 3$,

$$\mathbb{E} W_p(\nu_N, \nu) \ll \left\{ \begin{array}{ll} (1 - p)^{2p/n - 1} N^{-p/n} & \text{if } 0 < p < 1, \\
(\log N)^{1-2/n} N^{-1/n} & \text{if } p = 1 \\
\end{array} \right.$$  

with an implied constant depending only on $G$. More generally, we have $\mathbb{E} W_g(\nu_N, \nu) \ll g(N^{-1/n})$ provided $\psi(t) \ll g(1/t)$. Thus (8) shows that Theorem 1 is indeed sharp up to the factors $1/(1 - p)$ in (1) and $\log(M + 2)$ in (2); similarly, in the case $\psi(t) \ll g(1/t)$, Theorem 3 is sharp up to a constant factor depending only on $G$ and $g$. Note that the only compact, connected Lie groups in dimension $n = 1$ and $n = 2$ are $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$, and the optimality of Theorem 1 on these groups up to the same factors follows from results in [6].

The estimate in (9) is neither new, nor fully optimal. The rate of convergence of empirical measures in $W_p$ on more general metric spaces was studied by Bach and Weed [1], and by Boissard and Le Gouic [5]. Instead of Fourier methods, they used a sequence of partitions of the metric space, each refining its predecessor to construct transport plans. It follows e.g. from [5, Corollary 1.2] that $\mathbb{E} W_p(\nu_N, \nu)$ satisfies the fully optimal upper bound $\ll N^{-p/n}$ for all $0 < p \leq 1$. We refer to [1] for improvements for measures $\nu$ supported on sets of lower dimension than the ambient space.

### 3 Proof of Theorem 3

The proof of Berry–Esseen type inequalities are usually based on smoothing with an approximate identity whose Fourier transform has bounded support. For instance, in the proof of Theorem A this Fourier transform is the “rooftop function” $\max\{1 - |t|/T, 0\}$, supported on $[-T, T]$. The proof of Theorem B uses the discrete version $\prod_{k=1}^d \max\{1 - |m_k|/(M + 1), 0\}$, supported on $[-M, M]^d$; in the setting of the torus this is known as the Fejér kernel.

Our proof of Theorem 3 follows the same idea. Let $B_L \subset \Gamma^* \cap C^+$ be a lattice box of size $L \in \mathbb{N}$. Consider the function

$$F_L(x) = \frac{1}{|B_L|} \sum_{\lambda \in \hat{G}} \sum_{\pi \in B_L} \chi_{\pi}(x)^2,$$

where $\chi_{\pi}(x) = \text{tr} \pi(x)$ is the character of $\pi$, and $|B_L| = L^r$ is the cardinality of $B_L$. Clearly $F_L \geq 0$, and by the orthonormality of the characters, $\int_G F_L \, d\mu_G = 1$.

**Remark.** We emphasize that $F_L$, while inspired by it, is not a natural analogue of the Fejér kernel on $G$; indeed, we do not use all possible dominant weights. The
main technical advantage of $F_L$ is that it factors at a crucial stage of the proof; see equation (14) below. Using natural analogues of the Fejér kernel would entail a complicated combinatorial argument to handle exponential sums over dilated polytopes, and would only yield an improvement in Theorem 3 by a constant factor depending on $G$. For convergence properties of natural analogues of the Fejér kernel we refer to [10], [14] and [26].

Clearly,
\[
\left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right| \leq 2\|f - f \ast F_L\|_{\infty} + \left| \int_G f \ast F_L \, d\nu_1 - \int_G f \ast F_L \, d\nu_2 \right|,
\]

where $f \ast F_L$ denotes convolution. Our goal is to find an upper estimate of the right hand side which is uniform in $f \in F_g$; by Kantorovich duality, the same upper estimate will hold for $W_g(\nu_1, \nu_2)$. We bound the first term in (10) in Section 3.1; prove a decay estimate for the Fourier transform of $f$ in Section 3.2; bound the second term and finish the proof of Theorem 3 in Section 3.3.

### 3.1 Approximation in supremum norm

In this section we bound the supremum norm $\|f - f \ast F_L\|_{\infty}$ appearing in (10). Similar estimates for $p$-Hölder functions on classical groups with various versions of the Fejér kernel were given by Gong [14]. We shall need the celebrated *Weyl integral formula* and *Weyl character formula*, which we now recall. Let $\delta_G : T \to \mathbb{R}$ be
\[
\delta_G(\exp(X)) = \prod_{\alpha \in \mathbb{R}} (e^{i\alpha(X)} - 1) = \prod_{\alpha \in \mathbb{R}^+} |e^{i\alpha(X)} - 1|^2 \quad (X \in t).
\]

The Weyl integral formula [7, p. 338] states that for any central function $\varphi \in L^1(G, \mu_G)$ we have
\[
\int_G \varphi \, d\mu_G = \frac{1}{|W(G, T)|} \int_T \varphi \cdot \delta_G \, d\mu_T.
\]

We also record for future reference that for any $\varphi \in L^1(T, \mu_T)$,
\[
\int_T \varphi(t) \, d\mu_T(t) = \int_{t/\Gamma} \varphi(\exp(2\pi X)) \, d\mu_{t/\Gamma}(X),
\]

where $\mu_{t/\Gamma}$ is the normalized Haar measure on $t/\Gamma$. This follows from the fact that $X \mapsto \exp(2\pi X)$ from $t/\Gamma$ to $T$ is an isomorphism of compact commutative groups, and hence, a measure preserving map.
The Weyl character formula [7, p. 356] expresses an arbitrary character \( \chi_\pi \) as

\[
\chi_\pi(\exp(X)) = \frac{\sum_{w \in W(G,T)} \text{sign}(w) e^{i\lambda_\pi(X^w)} e^{i\rho^+(X^w-X)}}{\prod_{\alpha \in R^+} (1 - e^{-i\alpha(X)})} \quad (X \in t).
\]

Here \( \text{sign}(w) \) is \( \pm 1 \) depending on whether \( w \) is the product of an even or odd number of reflections, and \((w, X) \mapsto X^w \) is the canonical action of \( W(G,T) \) on \( t \). We emphasize that this form of the Weyl character formula holds even without assuming \( \rho^+ \in \Gamma^* \), as \( \exp(X) \mapsto e^{i\rho^+(X^w-X)} \) is well defined.

**Proposition 4.** For any \( f \in F_g \) and any \( L \in \mathbb{N} \),

\[
2\|f - f \ast F_L\|_\infty \leq \psi(L).
\]

**Proof.** Recall that the geodesic metric \( \rho \) is translation invariant both from the left and from the right. Since \( f \in F_g \), we have

\[
\|f - f \ast F_L\|_\infty = \sup_{x \in G} \left| \int_G (f(x) - f(xy^{-1})) F_L(y) \, d\mu_G(y) \right|
\]

\[
\leq \int_G g(\rho(e, y)) F_L(y) \, d\mu_G(y)
\]

\[
= \frac{1}{|W(G, T)|} \int_T g(\rho(e, t)) F_L(t) \delta_G(t) \, d\mu_T(t). \quad (12)
\]

In the last step we used the Weyl integral formula. By the definition of \( F_L \) and the Weyl character formula, with \( t = \exp(2\pi X) \), \( X \in t \),

\[
F_L(t) \delta_G(t) = \frac{\delta_G(t)}{|B_L|} \left| \sum_{w \in W(G,T)} \text{sign}(w) \frac{e^{2\pi i\rho^+(X^w)} \prod_{\alpha \in R^+} (1 - e^{-2\pi i\alpha(X)}) \sum_{\lambda_\pi \in B_L} e^{2\pi i\lambda_\pi(X^w)}}{\prod_{\lambda_\pi \in B_L} (1 - e^{-2\pi i\lambda_\pi(X)})^2} \right|^2
\]

\[
\leq \frac{\delta_G(t)}{|B_L|} |W(G, T)| \left| \sum_{w \in W(G,T)} \frac{1}{\prod_{\lambda_\pi \in B_L} (1 - e^{-2\pi i\lambda_\pi(X)})^2} \sum_{\lambda_\pi \in B_L} e^{2\pi i\lambda_\pi(X^w)} \right|^2
\]

\[
= \frac{|W(G, T)|}{|B_L|} \left| \sum_{w \in W(G,T)} \sum_{\lambda_\pi \in B_L} e^{2\pi i\lambda_\pi(X^w)} \right|^2.
\]

Applying the integral transformation (11) in (12), we thus get

\[
\|f - f \ast F_L\|_\infty \leq \frac{1}{|B_L|} \left| \sum_{w \in W(G,T)} \int_{t/T} g(\rho(e, \exp(2\pi X))) \sum_{\lambda_\pi \in B_L} e^{2\pi i\lambda_\pi(X^w)} \, d\mu_G(X) \right|^2.
\]
Here \( \rho(e, \exp(2\pi X)) \), the lattice \( \Gamma \) and \( \mu_t/\Gamma \) are invariant under the action \( X \mapsto X^w \) of the Weyl group. Therefore the terms in the previous sum are equal, and we have

\[
\| f - f * F_L \|_{\infty} \leq \frac{|W(G,T)|}{|B_L|} \int_{\Gamma} g(\rho(e, \exp(2\pi X))) \left| \sum_{\pi \in \hat{G}} e^{2\pi i \lambda_\pi(X)} \right|^2 d\mu_t/\Gamma(X). \tag{13}
\]

Recall that \( \beta_1, \ldots, \beta_r \) and \( \alpha_1, \ldots, \alpha_r \) are dual bases in \( \Gamma \) and \( \Gamma^* \). Let us parametrize \( X \in t/\Gamma \) as \( X = x_1 \beta_1 + \cdots + x_r \beta_r \), \( x \in [-1/2, 1/2]^r \). Since the exponential map is a geodesic, \( \rho(e, \exp(2\pi X)) \leq 2\pi |X| \) for all \( X \in t \). By the subadditivity of \( g \), in (13) we have

\[
g(\rho(e, \exp(2\pi X))) \leq g(2\pi |X|) \leq \sum_{k=1}^r g(2\pi |x_k\beta_k|).
\]

We chose \( B_L \) to be a lattice box in \( \Gamma^* \), therefore in (13) we also have

\[
\left| \sum_{\pi \in \hat{G}} e^{2\pi i \lambda_\pi(X)} \right|^2 = \left| \sum_{n_1, \ldots, n_r = 0}^{L-1} e^{2\pi i (n_1 x_1 + \cdots + n_r x_r)} \right|^2. \tag{14}
\]

Hence (13) yields

\[
\| f - f * F_L \|_{\infty} \leq \frac{|W(G,T)|}{|B_L|} \sum_{k=1}^r \int_{[-1/2, 1/2]^r} g(2\pi |x_k\beta_k|) \prod_{\ell=1}^r \left| \sum_{n_k = 0}^{L-1} e^{2\pi i n_k x_{\ell}} \right|^2 \, dx_k
\]

\[
= \frac{|W(G,T)|}{|B_L|} \sum_{k=1}^r \int_{[-1/2, 1/2]}^{1/2} g(2\pi |x_k\beta_k|) \sum_{n_k = 0}^{L-1} e^{2\pi i n_k x_k} \left| \sum_{n_k = 0}^{L-1} e^{2\pi i n_k x_k} \right|^2 \, dx_k
\]

\[
= 2|W(G,T)| \sum_{k=1}^r \int_{0}^{1/2} g(2\pi |\beta_k| x_k) \frac{\sin^2(L\pi x_k)}{L \sin^2(\pi x_k)} \, dx_k
\]

\[
\leq \frac{|W(G,T)|}{2} \sum_{k=1}^r \int_{0}^{1/2} g(2\pi |\beta_k| x_k) \frac{\sin^2(L\pi x_k)}{L x_k^2} \, dx_k.
\]

By a simple integral transformation, the right hand side is \( \psi(L)/2 \). \( \square \)

### 3.2 Decay of the Fourier transform

We prove a decay estimate for the Fourier transform in somewhat greater generality than what we need.
Proposition 5. Assume that $f \in L^1(G, \mu_G)$ satisfies

$$\left( \int_G |f(\rho h) - f(\rho)|^2 \, d\mu_G(\rho) \right)^{1/2} \leq g(\rho(h,e))$$

for all $h \in G$ with some nondecreasing function $g : [0, \infty) \to [0, \infty)$. Then for any real number $M > 0$,

$$\sum_{\pi \in \hat{G}} d_\pi \kappa_\pi \|\hat{f}(\pi)\|_{HS}^2 \leq \inf_{0 < c < 2(\sqrt{n^2 + n} - n)} \frac{n}{1 - c - c^2/(4n)} \cdot g\left(\frac{c}{nM}\right)^2.$$

If $g(t) = t^p$ with some $0 < p \leq 1$, we can choose e.g. $c = (\sqrt{17} - 3)/2$ yielding

$$\sum_{\pi \in \hat{G}} d_\pi \kappa_\pi \|\hat{f}(\pi)\|_{HS}^2 \leq 9n^{3-2p}M^{2-2p}.$$ (15)

In the special case $p = 1$ the factor 9 can be removed, since the optimal choice is then to let $c \to 0$ (and $M \to \infty$). An estimate similar to (15) has recently been proved by Daher, Delgado and Ruzhansky [13], with an unspecified implied constant in the place of $9n^{3-2p}$. Our main improvement is that this implied constant does not depend on $f$; a crucial feature in the study of the $p$-Wasserstein metric.

Proof of Proposition 5. We follow ideas in [13]. For the sake of simplicity, we shall think about $\pi \in \hat{G}$ as a $d_\pi \times d_\pi$ unitary matrix-valued function on $G$. For any matrix $A \in \mathbb{C}^{d_\pi \times d_\pi}$ let $\|A\|_{op} = \sup\{|Av| : v \in \mathbb{C}^{d_\pi}, |v| = 1\}$ and $\|A\|_{HS} = \sqrt{\text{tr}(A^*A)}$ denote the operator norm and the Hilbert–Schmidt norm, respectively. The operator norm is submultiplicative; further, for all $A, B \in \mathbb{C}^{d_\pi \times d_\pi}$ we have $\|AB\|_{HS} \leq \|A\|_{op} \cdot \|B\|_{HS}$, and the Cauchy–Schwarz inequality $|\text{tr}(A^*B)| \leq \|A\|_{HS} \cdot \|B\|_{HS}$.

One readily verifies the identity

$$(\pi(h) - I_{d_\pi}) \hat{f}(\pi) = \int_G (f(\rho h) - f(\rho)) \pi(\rho) \, d\mu_G(\rho),$$

where $I_{d_\pi}$ denotes the $d_\pi \times d_\pi$ identity matrix. By the Parseval formula and the assumption on $f$, for any $h \in G$ we have

$$\sum_{\pi \in \hat{G}} d_\pi \text{tr} \left( (\pi(h) - I_{d_\pi})^* (\pi(h) - I_{d_\pi}) \hat{f}(\pi) \hat{f}(\pi)^* \right) = \int_G |f(\rho h) - f(\rho)|^2 \, d\mu_G(\rho) \leq g(\rho(h,e))^2.$$
Since the exponential map is a geodesic, we have \( \rho(\exp(uX), e) \leq |uX| \) for all \( X \in \mathfrak{g} \) and \( u \in \mathbb{R} \). For any \( h = \exp(uX) \) the previous estimate thus yields
\[
\sum_{\pi \in \hat{G}} d_{\pi} \text{tr} \left( (\pi(h) - I_{d_x})^* (\pi(h) - I_{d_x}) \hat{f}(\pi) \hat{f}(\pi)^* \right) \leq g(|uX|)^2. \tag{16}
\]

Next, we wish to find a lower estimate. For any \( X \in \mathfrak{g} \) let
\[
d_{\pi}(X) = d_{\pi}(\exp(uX) | u=0) \in \mathbb{C}^{d_x \times d_x}
\]
denote the derived representation of \( \pi \).

**Lemma 1** (Taylor expansion of degree 1). For any \( X \in \mathfrak{g} \) and any \( u \in \mathbb{R} \),
\[
\|\pi(\exp(uX)) - I_{d_x} - u \cdot d_{\pi}(X)\|_{\text{op}} \leq \frac{u^2}{2} \|d_{\pi}(X)\|_{\text{op}}^2.
\]

**Proof of Lemma 1**. We simply apply the usual Taylor formula to the matrix-valued function \( F(u) = \pi(\exp(uX)) \). Since \( \pi \) is a homomorphism, we have \( F'(u) = \pi(\exp(uX)) d_{\pi}(X) \). First, note that for any \( u \in \mathbb{R} \),
\[
\|\pi(\exp(uX)) - I_{d_x} - u \cdot d_{\pi}(X)\|_{\text{op}} = \left\| \int_0^u \pi(\exp(yX)) d_{\pi}(X) dy \right\|_{\text{op}}
\]
\[
\leq \int_0^{|u|} \|\pi(\exp(yX))\|_{\text{op}} \cdot \|d_{\pi}(X)\|_{\text{op}} dy
\]
\[
= |u| \cdot \|d_{\pi}(X)\|_{\text{op}}.
\]

We used the fact that \( \pi(\exp(yX)) \) is a unitary matrix and thus has operator norm 1. Therefore
\[
\|\pi(\exp(uX)) - I_{d_x} - u \cdot d_{\pi}(X)\|_{\text{op}} = \left\| \int_0^u (\pi(\exp(yX)) - I_{d_x}) d_{\pi}(X) dy \right\|_{\text{op}}
\]
\[
\leq \int_0^{|u|} \|\pi(\exp(yX)) - I_{d_x}\|_{\text{op}} \cdot \|d_{\pi}(X)\|_{\text{op}} dy
\]
\[
\leq \int_0^{|u|} |y| \cdot \|d_{\pi}(X)\|_{\text{op}}^2 dy
\]
\[
= \frac{u^2}{2} \|d_{\pi}(X)\|_{\text{op}}^2.
\]

**Lemma 2** (Sugiura). For any \( X \in \mathfrak{g} \), we have \( \|d_{\pi}(X)\|_{\text{op}} \leq |\lambda_\pi| \cdot |X| \).

\[19\]
Proof of Lemma 2. In [25, Theorem 2] Sugiura stated and proved the estimate 
\[ \|d\pi(X)\|_{HS} \leq \sqrt{d_x} |\lambda_x| \cdot |X| \]. His proof is based on the fact that with some \( d_x \times d_x \) unitary matrix \( U \), we have \( Ud\pi(X)U^* = \text{diag}(i\lambda(X) : \lambda \in W(\pi)) \), where \( W(\pi) \) is the set of weights of \( \pi \). Further, we have \( |\lambda| \leq |\lambda_x| \) for all \( \lambda \in W(\pi) \). Hence Sugiura’s proof in fact yields the slightly stronger claim of Lemma 2.

Lemma 3. Let \( X_1, \ldots, X_n \) be an orthonormal base in \( g \). For any \( u \in \mathbb{R} \), the points \( h_k = \exp(uX_k) \) satisfy

\[
\sum_{k=1}^{n} (\pi(h_k) - I_{d_x})^* (\pi(h_k) - I_{d_x}) = u^2 \kappa_\pi I_{d_x} + E
\]

with some \( E \in \mathbb{C}^{d_x \times d_x} \), \( \|E\|_{op} \leq (u^2/2)\|d\pi(X_k)\|_{op}^2 \). Therefore

\[
\sum_{k=1}^{n} (\pi(h_k) - I_{d_x})^* (\pi(h_k) - I_{d_x}) = u^2 \sum_{k=1}^{n} d\pi(X_k)^* d\pi(X_k) + E
\]

where

\[
\|E\|_{op} = \left\| \sum_{k=1}^{n} (u \cdot d\pi(X_k)E_k + E_k^* u \cdot d\pi(X_k) + E_k^* E_k) \right\|_{op}
\]

\[
\leq \sum_{k=1}^{n} \left( 2|u| \cdot \|d\pi(X_k)\|_{op} \cdot \frac{u^2}{2} \|d\pi(X_k)\|_{op}^2 + \frac{u^4}{4} \|d\pi(X_k)\|_{op}^4 \right)
\]

By Lemma 2, the previous estimate yields \( \|E\|_{op} \leq n|u|^3|\lambda_x|^3 + n(u^4/4)|\lambda_x|^4 \). On the other hand, we have \( d\pi(X)^* = -d\pi(X) \), and by the definition of the Laplace–Beltrami operator,

\[
\sum_{k=1}^{n} d\pi(X_k)^* d\pi(X_k) = -\sum_{k=1}^{n} d\pi(X_k)d\pi(X_k) = -(\Delta\pi)(e) = \kappa_\pi I_{d_x}.
\]
We now finish the proof of Proposition 5. Recall that $|\lambda_\pi|^2 \leq \kappa_\pi$. From Lemma 3 we deduce that for any $u \in \mathbb{R},$

$$\text{tr} \left( \sum_{k=1}^{n} (\pi(h_k) - I_{d_\pi})^* (\pi(h_k) - I_{d_\pi}) \hat{f}(\pi) \hat{f}(\pi)^* \right)$$

$$= \text{tr} \left( u^2 \kappa_\pi \hat{f}(\pi) \hat{f}(\pi)^* \right) + \text{tr} \left( E \hat{f}(\pi) \hat{f}(\pi)^* \right)$$

$$\geq u^2 \kappa_\pi \|\hat{f}(\pi)\|_{\text{HS}}^2 - \|E \hat{f}(\pi)\|_{\text{HS}} \cdot \|\hat{f}(\pi)\|_{\text{HS}}$$

$$\geq u^2 \kappa_\pi \|\hat{f}(\pi)\|_{\text{HS}}^2 - \|E\|_{\text{op}} \cdot \|\hat{f}(\pi)\|_{\text{HS}}^2$$

$$\geq \|\hat{f}(\pi)\|_{\text{HS}}^2 \left( u^2 \kappa_\pi - n |u| |\lambda_\pi|^3 - n \frac{u^2}{4} |\lambda_\pi|^4 \right)$$

$$\geq \|\hat{f}(\pi)\|_{\text{HS}}^2 \cdot u^2 \kappa_\pi \left( 1 - n |u| \cdot |\lambda_\pi| - n \frac{u^2}{4} |\lambda_\pi|^2 \right).$$

(17)

Let $M > 0$ and $0 < c < 2(\sqrt{n^2 + n} - n)$ be arbitrary, and choose $u = c / (nM)$. For any $0 < |\lambda_\pi| \leq M$ we then have

$$1 - n |u| \cdot |\lambda_\pi| - n \frac{u^2}{4} |\lambda_\pi|^2 \geq 1 - c - \frac{c^2}{4n} > 0,$$

and thus (16) and (17) imply

$$ng \left( \frac{c}{nM} \right)^2 \geq \sum_{\pi \in \hat{G}} \sum_{0 < |\lambda_\pi| \leq M} d_{\pi, \tau} \text{tr} \left( \sum_{k=1}^{n} (\pi(h_k) - I_{d_\pi})^* (\pi(h_k) - I_{d_\pi}) \hat{f}(\pi) \hat{f}(\pi)^* \right)$$

$$\geq \sum_{\pi \in \hat{G}} \sum_{0 < |\lambda_\pi| \leq M} d_{\pi} \|\hat{f}(\pi)\|_{\text{HS}}^2 \left( \frac{c}{nM} \right)^2 \kappa_\pi \left( 1 - c - \frac{c^2}{4n} \right).$$

Since $0 < c < 2(\sqrt{n^2 + n} - n)$ was arbitrary, the claim follows. \qed

### 3.3 The smoothed functions

In this section we estimate the second term in (11), and then finish the proof of Theorem 3.

**Proposition 6.** If $B_L \subseteq \{ \lambda \in t^* : |\lambda| \leq M/2 \} \cap C^+$, then for any $f \in L^1(G, \mu_G)$,

$$\left| \int_G f * F_L \, d\nu_1 - \int_G f * F_L \, d\nu_2 \right| \leq \sum_{\pi \in \hat{G}} \sum_{0 < |\lambda_\pi| \leq M} d_{\pi, \tau} \|\hat{f}(\pi)\|_{\text{HS}} \cdot \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}.$$

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However, we only need the trivial upper bound
\[ \chi_\pi(\exp(X)) = \sum_{\lambda \in W(\pi)} e^{\lambda(X)} \ (X \in \mathfrak{t}). \tag{18} \]
Further, \(|\lambda| \leq |\lambda_\pi|\) for all \(\lambda \in W(\pi)\).

By construction, \(F_L\) is a finite linear combination of functions \(\chi_{\pi_1} \overline{\chi_{\pi_2}}\) with \(|\lambda_{\pi_1}|, |\lambda_{\pi_2}| \leq M/2\). Note that \(\chi_{\pi_1} \overline{\chi_{\pi_2}} = \text{tr} (\pi_1 \otimes \overline{\pi_2})\), where \(\overline{\pi_2}\) is the contragredient of \(\pi_2\). The unitary representation \(\pi_1 \otimes \overline{\pi_2}\) decomposes into finitely many irreducible ones, hence \(\chi_{\pi_1} \overline{\chi_{\pi_2}} = \sum_{\pi \in \hat{G}} m_\pi \chi_\pi\) with some nonnegative integers \(m_\pi\), only finitely many of which are nonzero. Using (18), this means that
\[
\sum_{\lambda \in W(\pi_1)} \sum_{\eta \in W(\pi_2)} e^{i(\lambda-\eta)(X)} = \sum_{\pi \in \hat{G}} m_\pi \sum_{\zeta \in W(\pi)} e^{i\zeta(X)} \ (X \in \mathfrak{t}).
\]
These are classical trigonometric polynomials in the coordinates of \(X \in \mathfrak{t}\), it is thus not difficult to see that for any \(\zeta \in \Gamma^*\) which appears with a nonzero coefficient on the right hand side of the previous formula, there exist (not necessarily unique) weights \(\lambda \in W(\pi_1), \eta \in W(\pi_2)\) such that \(\zeta = \lambda - \eta\). In particular, \(|\zeta| \leq |\lambda_{\pi_1}| + |\lambda_{\pi_2}| \leq M\), and hence \(|\lambda_\pi| \leq M\) whenever \(m_\pi > 0\). Therefore \(\chi_{\pi_1} \overline{\chi_{\pi_2}}\), and consequently also \(F_L\), is a finite linear combination of characters \(\chi_\pi\) with \(|\lambda_\pi| \leq M\); say, \(F_L = \sum_{\pi \in \hat{G}, |\lambda_\pi| \leq M} c_\pi \chi_\pi\) with some coefficients \(c_\pi\). Computing \(c_\pi\) in general is a difficult problem, related to the Clebsch–Gordan coefficients. However, we only need the trivial upper bound
\[
|c_\pi| = \left| \int_G F_L \cdot \overline{\chi_\pi} \ d\mu_G \right| \leq \|\chi_\pi\|_\infty \int_G F_L \ d\mu_G = d_\pi,
\]
which implies
\[
\left| \int_G f * F_L \ d\nu_1 - \int_G f * F_L \ d\nu_2 \right| \leq \sum_{\pi \in \hat{G}, |\lambda_\pi| \leq M} d_\pi \left| \int_G f * \chi_\pi \ d\nu_1 - \int_G f * \chi_\pi \ d\nu_2 \right|
\]
\[
= \sum_{\pi \in \hat{G}, |\lambda_\pi| \leq M} d_\pi \left| \text{tr} \left( \widehat{f(\pi)}^* (\widehat{\nu_1(\pi)} - \widehat{\nu_2(\pi)}) \right) \right|
\]
\[
\leq \sum_{\pi \in \hat{G}, |\lambda_\pi| \leq M} d_\pi \|\widehat{f(\pi)}\|_{\text{HS}} \cdot \|\widehat{\nu_1(\pi)} - \widehat{\nu_2(\pi)}\|_{\text{HS}}.
\]
\[\square\]
Proof of Theorem 3. Using the estimate (10) and Propositions 4 and 6, we get that for any \( f \in \mathcal{F}_g \) and any real number \( M > 0 \),

\[
\left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right| \leq \psi(L) + \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} d_{\pi} \|\hat{f}(\pi)\|_{\text{HS}} \cdot \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}
\]

provided \( \{ \lambda \in t^* : |\lambda| \leq M/2 \} \cap C^+ \) contains a lattice box in \( \Gamma^* \) of size \( L \in \mathbb{N} \).

From the Cauchy–Schwarz inequality and Proposition 5 we further deduce

\[
\left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right| \\
\leq \psi(L) + \left( \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} d_{\pi} \kappa_{\pi} \|\hat{f}(\pi)\|_{\text{HS}}^2 \right)^{1/2} \left( \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} \frac{d_{\pi}}{\kappa_{\pi}} \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \right)^{1/2}
\]

\[
\leq \psi(L) + \phi(M) \left( \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} \frac{d_{\pi}}{\kappa_{\pi}} \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \right)^{1/2}.
\]

By Kantorovich duality, the same upper bound holds for \( W_g(\nu_1, \nu_2) \).

Finally, we prove a remark made after Theorem 1. Assume \( f \in \mathcal{F}_g \) is \( 2m \) times differentiable, and \( \Delta^m f \in \mathcal{F}_g \). Recall that \( \widehat{\Delta f}(\pi) = \kappa_{\pi} \hat{f}(\pi) \). By Proposition 5

\[
\sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} d_{\pi} \kappa_{2m+1} \|\hat{f}(\pi)\|_{\text{HS}}^2 = \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} d_{\pi} \kappa_{\pi} \|\Delta^m f(\pi)\|_{\text{HS}}^2 \leq \phi(M)^2,
\]

therefore from (19) and the Cauchy–Schwarz inequality we similarly get

\[
\left| \int_G f \, d\nu_1 - \int_G f \, d\nu_2 \right| \leq \psi(L) + \phi(M) \left( \sum_{\pi \in \hat{G} \atop 0 < |\lambda| \leq M} \frac{d_{\pi}}{\kappa_{2m+1}} \|\hat{\nu}_1(\pi) - \hat{\nu}_2(\pi)\|_{\text{HS}}^2 \right)^{1/2}
\]

provided \( \{ \lambda \in t^* : |\lambda| \leq M/2 \} \cap C^+ \) contains a lattice box in \( \Gamma^* \) of size \( L \in \mathbb{N} \).

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