Distributed proximal gradient algorithm for non-smooth non-convex optimization over time-varying networks

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Abstract

This note studies the distributed non-convex optimization problem with non-smooth regularization, which has wide applications in decentralized learning, estimation and control. The objective function is the sum of different local objective functions, which consist of differentiable (possibly non-convex) cost functions and non-smooth convex functions. This paper presents a distributed proximal gradient algorithm for the non-smooth non-convex optimization problem over time-varying multi-agent networks. Each agent updates local variable estimate by the multi-step consensus operator and the proximal operator. We prove that the generated local variables achieve consensus and converge to the set of critical points with convergence rate $O(1/T)$. Finally, we verify the efficacy of proposed algorithm by numerical simulations.

Index Terms

distributed proximal gradient algorithm, non-smooth non-convex optimization, time-varying communication

I. INTRODUCTION

Motivated by many problems in signal processing and machine learning over networks, distributed non-smooth non-convex optimization has attracted significant attention. In this problem setup, each node in the network only knows local function information and communicates with its neighbors to solve the global optimization problem. One fundamental model for distributed non-smooth non-convex optimization, arising from optimization problems such as Lasso [1], SVM [2], and optimizing neural networks [3], is that each local objective function of a node is the summation of a (non-convex) differentiable function and a non-smooth convex function ($l_1$ norm or indicator function). Although the research on distributed optimization has made significant progress on non-smooth convex problems [4]–[8], distributed non-smooth non-convex optimization is still challenging.

Researchers have made great achievements in centralized and parallel algorithms for non-smooth non-convex optimization problems [9]–[11]. For instance, [10] developed a proximal alternating linearized minimization algorithm with global convergence under Kurdyka-Lojasiewicz property. [11] extended the two blocks of objective function in [10] to multiple blocks and introduced extrapolation to accelerate the block prox-linear method. When the proximal operator does not have an analytic solution or exactly solving the proximal operator is time-consuming, [12], [13] studied some inexact proximal gradient algorithms for non-convex optimization. With the explosion of data and the development of distributed network systems,

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[14], [15] developed some asynchronous parallel methods with considerations of unreliable communication links. However, with privacy or security considerations, it is necessary to design fully distributed algorithms for large-scale non-smooth non-convex optimization.

In recent years, some distributed discrete-time algorithms [16]-[20] have been proposed for non-smooth non-convex optimization over multi-agent networks. Over time-invariant graphs, [16], [17] proposed distributed proximal gradient algorithms for (non-smooth) non-convex optimization with convergence to consensus stationary solutions. However, time-invariant graphs are difficult and expensive to hold for practical multi-agent networks. Over time-varying networks, [19] developed a distributed discrete-time algorithm with successive convex approximation and dynamic consensus mechanism. If agents only have noisy observations of local functions, [20] proposed a distributed stochastic approximation algorithm over time-varying graphs without requiring objective functions be convex and Lipschitz continuous. However, the diminishing step-sizes in existing algorithms hinder the convergence performance. This paper studies a distributed algorithm with a constant step-size for non-smooth non-convex optimization over time-varying communication graphs.

The contributions of this paper are summarized as follows.

- The paper proposes one distributed proximal gradient algorithm for non-smooth non-convex optimization over time-varying multi-agent networks. The proposed algorithm adopts the multi-step consensus stage to make local variable estimates closer to each other and extends the recent distributed proximal algorithm [16] over time-invariant graphs to time-varying network graphs. What’s more, the proposed algorithm owns a constant step-size, overcoming the shortage of diminishing step-sizes that hinder the convergence performance [19], [20].
- We provide complete and rigorous convergence proofs for the proposed distributed proximal gradient algorithm. The proposed algorithm over time-varying graphs has a same convergence rate $O(\frac{1}{T})$ as the algorithm over time-invariant graphs [16]. To the best of our knowledge, for non-smooth non-convex optimization problems over time-varying graphs, this is the first convergence result showing the rate of convergence of distributed algorithms without using successive convex approximation.

The remainder of the paper is organized as follows. The preliminary mathematical notations, graph theory and proximal operator are introduced in section II. The optimization problem description and the design of a distributed solver are provided in section III. The convergence performance of the proposed algorithm is proved theoretically in section IV. The numerical simulations are provided in section V and the conclusion is made in section VI.

II. Preliminaries

A. Mathematical notations & graph theory

We write $\mathbb{R}$ as the set of real numbers, $\mathbb{N}$ as the set of natural numbers, $\mathbb{R}^n$ as the set of $n$-dimensional real column vectors and $\mathbb{R}^{n \times m}$ as the set of $n$-by-$m$ real matrices, respectively. We denote $v'$ as the transpose of a vector $v$. In addition, $\|\cdot\|$ denotes the Euclidean norm, $\langle \cdot, \cdot \rangle$ denotes the inner product, which is defined by $\langle a, b \rangle = a'b$ and $\lceil a \rceil$ deontes the smallest integer greater than real number $a$. The vectors in this paper are column vectors unless otherwise stated. For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, $\nabla g(x)$ denotes the gradient of function $g$ with respect to $x$. The $\varepsilon$-subdifferential of a convex function $h$ at $x$ is the set of vectors $y$ such that $h(z) - h(x) \geq y^T(z - x) - \varepsilon$ for all $z$.

The dynamic communication among $m$ agents over time-varying undirected topology is often modeled as $G(\mathcal{V}, \mathcal{E}(t), A(t))$, where $\mathcal{V} = \{1, \ldots, m\}$ is a finite nonempty node set with $i$ representing $i$th node, $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ is the time-varying edge set. The adjacent matrix is denoted by $A(t) = [a_{ij}(t)] \in \mathbb{R}^{m \times m}$ such that $a_{ij}(t) = a_{ji}(t) > 0$ if $\{i, j\} \in \mathcal{E}(t)$ and the elements $a_{ij}(t) = 0$ otherwise. Note that adjacent matrices of undirected graphs are symmetric matrices. If an edge $\{i, j\} \in \mathcal{E}(t)$, then node $j$ is called a neighbor of node $i$. 

B. Proximal Operator

For a proper non-differentiable convex function \( h : \mathbb{R}^n \to (-\infty, \infty] \) and a scalar \( \alpha > 0 \), the proximal operator is defined as

\[
\text{prox}_{\alpha, h}(x) = \arg\min_{z \in \mathbb{R}^n} h(z) + \frac{1}{2\alpha} \| z - x \|^2.
\]  

(1)

The minimum is attained at a unique point \( y = \text{prox}_{\alpha, h}(x) \), which means the proximal operator is a single-valued map. In addition, it follows from the optimality condition for convex optimization problems that

\[
0 \in \partial h(y) + \frac{1}{\alpha} (y - x),
\]

(2)

where the set \( \partial h(y) \) is the subdifferential of non-differentiable function \( h \) at \( y \). The following proposition presents some properties of the proximal operator.

**Proposition II.1.** [27] Let \( h : \mathbb{R}^n \to (-\infty, \infty] \) be a closed proper convex function. For a scalar \( \alpha > 0 \) and \( x \in \mathbb{R}^n \), let \( y = \text{prox}_{\alpha, h}(x) \).

(a) The relationship \( h(u) \geq h(y) + \frac{1}{\alpha} \langle x - y, u - y \rangle \) holds for all \( u \in \mathbb{R}^n \).

(b) For \( x, \hat{x} \in \mathbb{R}^n \),

\[
\| \text{prox}_{\alpha, h}(x) - \text{prox}_{\alpha, h}(\hat{x}) \| \leq \| x - \hat{x} \|.
\]

(c) The vector \( y \) can be written as \( y = x - \alpha z \), where \( z \in \partial h(y) \).

(d) We have \( \frac{1}{\alpha} (x - y) \in \partial h(y) \).

When there exist errors in the computation of proximal operators, we denote the inexact proximal operator by \( \text{prox}^{\varepsilon}_{\alpha, h}(\cdot) \). Let \( x_k \) denote the variable at iteration \( k \) and \( \varepsilon_k \) denote the error in the proximal objective function. Then, the inexact proximal operator at iteration \( k \) is

\[
x_k \in \text{prox}^{\varepsilon_k}_{\alpha, h}(y) \triangleq \left\{ \hat{x} \Big| \left\| \hat{x} - y \right\|^2 + h(\hat{x}) \leq \varepsilon_k + \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2\alpha} \| x - y \|^2 + h(x) \right\} \right\}.
\]

(3)

III. Problem Description and Distributed Solver

Consider a multi-agent system composed of \( m \) agents, which are interconnected by a time-varying communication network. We aim to design a distributed algorithm for the multi-agent system to solve the following optimization problem

\[
\min_x f(x) = \frac{1}{m} \sum_{i=1}^{m} (g_i(x) + h(x)),
\]

(4)

where \( x \in \mathbb{R}^n \) is the decision variable, function \( g_i \) is a differentiable (possibly non-convex) local cost function, and function \( h \) is a non-smooth and convex regular function. For each agent \( i \) in the network, \( x_i \in \mathbb{R}^n \) is the local estimate of variable \( x \).

**Remark III.1.** The non-convexity of function \( g_i \) makes it difficult to design an efficient convergent algorithm with rigorous proofs of optimality. In this paper, we prove that the proposed algorithm converges to the set of critical points. Although there exist some centralized works, it is not straightforward to extend them to distributed cases since the convergence may not hold with the influence of distributed nature.

Through this paper, we assume that the following standard assumptions hold for the optimization problem \( f \).

**Assumption III.1.** (a) For each agent \( i \), \( g_i \) is continuously differentiable and has a Lipschitz-continuous gradient with constant \( L > 0 \),

\[
\| \nabla g_i(x) - \nabla g_i(y) \| \leq L \| x - y \|,
\]

(5)
which implies that
\[ g_i(x) \leq g_i(y) + \langle \nabla g_i(y), x - y \rangle + \frac{L}{2} \| x - y \|^2. \] (6)

(b) The regular function \( h \) is convex.
(c) There exists a scalar \( G_g \) such that for each agent \( i, \| \nabla g_i(x) \| < G_g. \)
(d) There exists a scalar \( G_h \) such that for each sub-gradient \( z \in \partial h(x), \| z \| < G_h. \)
(e) The optimization problem owns at least one optimal solution \( x^* \).

Then, we propose the following distributed proximal gradient algorithm for solving (4). For \( i \in \{1, \ldots, m\}, k = 1, \ldots, \)
\[ q_{i,k+1} = x_{i,k} - \alpha \nabla g_i(x_{i,k}), \quad \] (7a)
\[ v_{i,k+1} = \sum_{j=1}^{m} \lambda_{ij,k+1} q_{j,k+1}, \quad \] (7b)
\[ x_{i,k+1} = \text{prox}_{\alpha,h}(v_{i,k+1}), \quad \] (7c)
where \( \alpha < \frac{1}{L} \) is a constant step-size which is also used in the proximal operator, \( \lambda_{ij,k} \) is the \((i,j)\)th element of matrix \( \Phi(t(k) + k, t(k)) \),
\[ \lambda_{ij,k} = [\Phi(t(k) + k, t(k))]_{ij}, \]
where \( t(k) \) is the total number of communication steps before iteration \( k \) and \( \Phi \) is a transition matrix, which is defined as
\[ \Phi(t, s) = A(t)A(t-1)\cdots A(s+1)A(s), \quad t > s \geq 0, \]
where \( A(t) \) is the adjacent matrix at time \( t \).

Before analyzing the behavior of proposed algorithm (7) over a time-varying network, we introduce the following assumption.

Assumption III.2. Consider the undirected time-varying network with adjacent matrices \( A(t) = [a_{ij}(t)], t = 1, 2, \cdots \)
(a) For each \( t \), the adjacent matrix \( A(t) \) is doubly stochastic.
(b) There exists a scalar \( \eta \in (0, 1) \) such that \( a_{ii}(t) \geq \eta \) for all \( i \in \{1, \cdots, m\} \). In addition, \( a_{ij}(t) \geq \eta \) if \( \{i, j\} \in \mathcal{E}(t) \) and \( a_{ij}(t) = 0 \) otherwise.
(c) The time-varying graph sequence \( \mathcal{G}_k \) is uniformly connected, which means that agent \( j \) receives information from \( i \) for infinitely many \( t \). Moreover, there exists an integer \( B \geq 1 \) such that agent \( i \) sends its information to all other agents at least once every \( B \) consecutive time slots.

Remark III.2. In this assumption, part (a) guarantees that the variable estimates of neighbors impose an equal influence on the local variable estimate. Part (b) means that each agent gives significant weight to its current estimate and the estimates received from its neighbors. Part (c) states that the time-varying network is capable of exchanging information between any pair of agents in bounded time.

Remark III.3. Over time-varying graphs, the proposed updating (7b) represents that agents perform \( k \) rounds of communication steps at iteration \( k \), which may be expensive as iteration number \( k \) increases. However, when the time-varying graph is periodic, the updating (7b) is easy to compute due to the fact that \( \lambda_{ij,k} \) is also periodic, which has been investigated in [22]. What’s more, if the bounded intercommunication interval \( B \) is known, the number of communication steps taken at iteration \( k \) is significant reduced and the convergence performance is further improved, which has been discussed in [23].
IV. Main Result

In this section, we present theoretical proofs for the convergence properties of proposed distributed algorithm. Let \( \bar{x}_k \equiv \frac{1}{m} \sum_{i=1}^{m} x_{i,k} \), \( \bar{v}_k \equiv \frac{1}{m} \sum_{i=1}^{m} v_{i,k} \), and \( z_k \equiv \text{prox}_{\alpha, h}(\bar{v}_k) \). The following lemma states that the update of the average variable is viewed as an inexact centralized proximal gradient algorithm with the errors controlled by multiple communications at each iteration.

**Lemma IV.1.** Suppose Assumptions [III.1] and [III.2] hold. The average variable satisfies

\[
\bar{x}_{k+1} \in \text{prox}_{\alpha, h}^{\epsilon_{k+1}}(\bar{x}_k - \alpha [\nabla g(\bar{x}_k) + e_{k+1}]),
\]

\[
e_{k+1} = \frac{1}{m} \sum_{i=1}^{m} \left( \nabla g_i(x_{i,k}) - \nabla g_i(\bar{x}_k) \right),
\]

\[
\epsilon_{k+1} = \| \bar{x}_k - z_{k+1} \| \left( G_h + \frac{1}{\alpha} \| z_{k+1} - \bar{v}_k \| \right) + \frac{1}{2\alpha} \| \bar{x}_{k+1} - z_{k+1} \|^2,
\]

where the inexact proximal operator \( \text{prox}_{\alpha, h}^{\epsilon}(\cdot) \) is defined in (3), \( \nabla g(\bar{x}_k) \equiv \frac{1}{m} \sum_{i=1}^{m} \nabla g_i(\bar{x}_k) \), \( G_h \) is defined in Assumption [III.1](d), and error sequences \( \{e_k\} \) and \( \{\epsilon_k\} \) satisfy

\[
\|e_{k+1}\| \leq \frac{L}{m} \sum_{i=1}^{m} \|x_{i,k} - \bar{x}_k\|,
\]

\[
\epsilon_{k+1} \leq \frac{2G_h}{m} \sum_{i=1}^{m} \|v_{i,k} - \bar{v}_k\| + \frac{1}{2\alpha} \left( \frac{1}{m} \sum_{i=1}^{m} \|v_{i,k} - \bar{v}_k\| \right)^2.
\]

**Proof.** By taking the average of (7a) and (7b),

\[
\bar{v}_{k+1} = \bar{x}_k - \alpha (\nabla g(\bar{x}_k) + e_{k+1}),
\]

where

\[
e_{k+1} = \frac{1}{m} \sum_{i=1}^{m} \left( \nabla g_i(x_{i,k}) - \nabla g_i(\bar{x}_k) \right).
\]

Because of the Lipschitz-continuity of the gradient of \( g_i(x) \),

\[
\|e_k\| \leq \frac{L}{m} \sum_{i=1}^{m} \|x_{i,k} - \bar{x}_k\|.
\]

Let

\[
\bar{z}_{k+1} = \text{prox}_{\alpha, h}(\bar{v}_{k+1}) = \text{argmin}_x \left\{ h(x) + \frac{1}{2\alpha} \|x - \bar{v}_{k+1}\|^2 \right\}
\]

denote the result of the exact proximal operator. In addition, \( \bar{x}_{k+1} = \frac{1}{m} \sum_{i=1}^{m} x_{i,k+1} = \frac{1}{m} \sum_{i=1}^{m} \text{prox}_{\alpha, h}(v_{i,k+1}) \). Then, the result of the proximal operator in the distributed algorithm can be seen as an approximation of \( z_{k+1} \). We next relate \( \bar{z}_{k+1} \) and \( \bar{x}_{k+1} \) by formulating the latter as an inexact proximal operator with error \( \epsilon_{k+1} \). A simple algebraic expansion gives

\[
h(\bar{x}_{k+1}) + \frac{1}{2\alpha} \|\bar{x}_{k+1} - \bar{v}_{k+1}\|^2 \leq h(\bar{z}_{k+1}) + G_h \|\bar{x}_{k+1} - z_{k+1}\| + \frac{1}{2\alpha} \left\{ \|z_{k+1} - \bar{v}_{k+1}\|^2 + 2(\bar{z}_{k+1} - \bar{v}_{k+1}, \bar{x}_{k+1} - z_{k+1}) + \|\bar{x}_{k+1} - z_{k+1}\|^2 \right\}
\]

\[
= \min_{z \in \mathbb{R}^d} \left\{ h(z) + \frac{1}{2\alpha} \|z - \bar{v}_{k+1}\|^2 \right\} + \|\bar{x}_{k+1} - z_{k+1}\| \left( G_h + \frac{1}{\alpha} \|z_{k+1} - \bar{v}_{k+1}\| \right) + \frac{1}{2\alpha} \|\bar{x}_{k+1} - z_{k+1}\|^2,
\]

where in the inequality, we used the convexity of \( h(x) \) and the bound on the subgradient \( \partial h(\bar{x}_{k+1}) \) to obtain \( h(\bar{x}_{k+1}) \leq h(\bar{z}_{k+1}) + G_h \|\bar{x}_{k+1} - z_{k+1}\| \), and in the equality, we used the fact that by definition, \( z_{k+1} \) is the optimizer of \( h(x) + \frac{1}{2\alpha} \|x - \bar{v}_{k+1}\|^2 \).
With this expression, we can write
\[ \bar{x}_{k+1} \in \operatorname{prox}_{\alpha, h}^{\varepsilon_{k+1}}(\bar{v}_{k+1}), \]
where
\[ \varepsilon_{k+1} = \|\bar{x}_{k+1} - z_{k+1}\| \left( G_h + \frac{1}{\alpha} \|z_{k+1} - \bar{v}_{k+1}\| \right) + \frac{1}{2\alpha} \|\bar{x}_{k+1} - z_{k+1}\|^2. \]

By definition, \( z_{k+1} = \operatorname{prox}_{\alpha, h}(\bar{v}_{k+1}) \) also implies \( \frac{1}{\alpha}(\bar{v}_{k+1} - z_{k+1}) \in \partial_h(z_{k+1}) \), and therefore its norm is bounded by \( G_h \). As a result,
\[ \varepsilon_{k+1} \leq 2G_h \|\bar{x}_{k+1} - z_{k+1}\| + \frac{1}{2\alpha} \|\bar{x}_{k+1} - z_{k+1}\|^2. \]

Combined with the nonexpensiveness of the proximal operator,
\[ \|\bar{x}_{k+1} - z_{k+1}\| \leq \frac{1}{m} \sum_{i=1}^{m} \|\operatorname{prox}_{\alpha, h}(v_{i,k}) - \operatorname{prox}_{\alpha, h}(\bar{v}_{k+1})\| \]
\[ \leq \frac{1}{m} \sum_{i=1}^{m} \|v_{i,k+1} - \bar{v}_{k+1}\|, \]
we obtain the desired results. \( \square \)

The next lemma shows that polynomial-geometric sequences are summable, which is vital for the convergence analysis of error sequences.

**Lemma IV.2.** [23, Proposition 3] Let \( \gamma \in (0, 1) \), and let
\[ P_{k,N} = \{c_N k^N + \cdots + c_1 k + c_0 | c_j \in \mathbb{R}, j = 0, \cdots, N \} \]
denote the set of all \( N \)-th order polynomials of \( k \), where \( N \in \mathbb{N} \). Then for every polynomial \( p_k \in P_{k,N} \),
\[ \sum_{k=1}^{\infty} p_k \gamma^k < \infty. \]

The result of this Lemma for \( P_{k,N} = k^{N} \) will be particularly useful for the analysis in the following sections. Hence, we make the definition
\[ S_N^\gamma \triangleq \sum_{k=1}^{\infty} k^{N} \gamma^k < \infty. \] (11)

Before proving the summability of error sequences \( \{\|e_k\|\} \) and \( \{\varepsilon_k\} \), recursive expressions of the generated iterative variables are given in the next proposition.

**Proposition IV.1.** Under Assumptions [III.1] and [III.2] for each iteration \( k \geq 2, \)
(a) \[ \sum_{i=1}^{m} \|q_{i,k+1}\| \leq \sum_{i=1}^{m} \|q_{i,k}\| + \alpha m (G_g + G_h), \]
(b) \[ \sum_{i=1}^{m} \|x_{i,k} - x_{i,k-1}\| \leq 2m \Gamma \sum_{i=1}^{k-1} \gamma^i \sum_{i=1}^{m} \|q_{i,l}\| + (k-1) \alpha m (G_g + G_h), \]
(c) \[ \|x_{i,k} - \bar{x}_k\| \leq 2 \Gamma \gamma^k \sum_{i=1}^{m} \|q_{i,k}\|. \]

**Proof.** By (7c) and Proposition [II.1](c), there exists \( z_{i,k} \in \partial h(x_{i,k}) \) such that
\[ x_{i,k} = v_{i,k} - \alpha z_{i,k}. \] (12)

Since function \( h \) has bounded subgradients by Assumption [III.1](d),
\[ \|x_{i,k} - v_{i,k}\| \leq \alpha G_h. \] (13)
(a) Taking norm of (7a) and summing over $i$,
\[ \sum_{i=1}^{m} \|q_{i,k}\| = \sum_{i=1}^{m} \|x_{i,k-1} - \alpha \nabla g_i(x_{i,k-1})\| \]
\[ \leq \sum_{i=1}^{m} \|x_{i,k-1}\| + \alpha m G_g. \]  
(14)

It follows from (13) and $\|x_{i,k-1}\| - \|v_{i,k-1}\| \leq \|x_{i,k-1} - v_{i,k-1}\|$ that $\|x_{i,k-1}\| \leq \|v_{i,k-1}\| + \alpha G_h$. Since $v_{i,k-1}$ is a convex combination of $\{q_{j,k-1}\}_{j=1}^{m}$ by (7b),
\[ \sum_{i=1}^{m} \|v_{i,k-1}\| \leq \sum_{i=1}^{m} \|q_{i,k-1}\|. \]  
(15)

Substituting the above two inequalities in (14),
\[ \sum_{i=1}^{m} \|q_{i,k}\| \leq \sum_{i=1}^{m} \|q_{i,k-1}\| + \alpha m (G_g + G_h). \]

(b) By (12) and the proposed algorithm (7), $x_{i,k} = v_{i,k} - \alpha z_{i,k} = \sum_{j=1}^{m} \lambda_{ij,k} (x_{j,k-1} - \alpha \nabla g_j(x_{j,k-1})) - \alpha z_{i,k}$. Then,
\[ \sum_{i=1}^{m} \|x_{i,k} - x_{i,k-1}\| \]
\[ \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij,k} \|x_{j,k-1} - x_{i,k-1}\| + \alpha m (G_g + G_h). \]  
(16)

Next, consider the term $\sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij,k} \|x_{j,k-1} - x_{i,k-1}\|$. By the nonexpansiveness of the proximal operator,
\[ \|x_{j,k-1} - x_{i,k-1}\| \leq \|v_{j,k-1} - v_{i,k-1}\|. \]  
(17)

In addition, the bound of the distance between iterates $v_{i,k}$ and $\bar{v}_k$ satisfies
\[ \|v_{i,k} - \bar{v}_k\| = \left\| \sum_{j=1}^{m} \lambda_{ij,k} q_{j,k} - \frac{1}{m} q_{j,k} \right\| \]
\[ \leq \sum_{j=1}^{m} \lambda_{ij,k} - \frac{1}{m} \left\| q_{j,k} \right\| \]
\[ \leq \Gamma \gamma^k \sum_{j=1}^{m} \|q_{j,k}\|, \]  
(18)

where the last inequality follows from Proposition 1 [24], and $\Gamma = 2^{1+\frac{\eta B_0}{1-\eta B_0}}$, $\gamma = (1 - \eta B_0) \frac{1}{B_0}$, $B_0 = (m - 1) B$, $\eta$ is the lower bound in Assumption III.2(b), $B$ is the intercommunication interval bound in Assumption III.2(c). Then,
\[ \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij,k} \|v_{j,k-1} - v_{i,k-1}\| \]
\[ \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij,k} \left( \|v_{i,k-1} - \bar{v}_k\| + \|v_{j,k-1} - \bar{v}_k\| \right) \]
\[ \leq 2m \Gamma \gamma^{k-1} \sum_{j=1}^{m} \|q_{j,k}\|. \]  
(19)
where the last inequality follows from (18). Substituting (17) and (19) to (16),
\[
\sum_{i=1}^{m} \| x_{i,k} - x_{i,k-1} \| \\
\leq 2m\Gamma^{k-1} \sum_{j=1}^{m} \| q_{j,k-1} \| + \alpha m(G_g + G_h)
\]
(20)
\[
\leq 2m\Gamma \sum_{l=1}^{k-1} \gamma^l \sum_{i=1}^{m} \| q_{i,l} \| + (k-1)\alpha m(G_g + G_h).
\]
(c) By the definition of \( x_k \triangleq \frac{1}{m} \sum_{j=1}^{m} x_{j,k} \),
\[
\sum_{i=1}^{m} \| x_{i,k} - \frac{1}{m} \sum_{j=1}^{m} x_{j,k} \| \\
= \sum_{i=1}^{m} \| \frac{1}{m} \sum_{j=1}^{m} (x_{i,k} - x_{j,k}) \| \\
\leq \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} \| v_{i,k} - v_{j,k} \| \\
\leq 2\Gamma\gamma^k \sum_{i=1}^{m} \| q_{i,k} \|,
\]
where the first inequality is from nonexpansiveness of the proximal operator and the last inequality follows from (18).

By Proposition IV.1 and Lemma 1 in [23], there is a polynomial bound on \( \sum_{i=1}^{m} \| q_{i,k} \| \), which is stated in the following lemma. The proof is omitted since it is similar to the proof of Lemma 1 in [23].

**Lemma IV.3.** Under Assumptions III.1 and III.2 for the proposed algorithm (7), there exist non-negative scalars \( C_q = C_q(q_1,2,\cdots,q_m,2), C_q^1 = C_q^1(m,1,C_q, C_q^2), C_q^2 = C_q^2(m,\alpha,G_g,G_h) \) such that for iteration \( k \geq 2 \),
\[
\sum_{i=1}^{m} \| q_{i,k} \| \leq C_q + C_q^1 k + C_q^2 k^2.
\]

**Proof.** We proceed by induction on \( k \). First, we show that the result holds for \( k = 2 \) by choosing \( C_q = \sum_{i=1}^{m} \| q_{i,2} \| \). It suffices to show that, given the initial points \( x_{i,1}, \sum_{i=1}^{m} \| q_{i,2} \| \) is bounded. Indeed, by (14),
\[
\sum_{i=1}^{m} \| q_{i,2} \| \leq \sum_{i=1}^{m} \| x_{i,1} \| + \alpha m G_g \leq \sum_{i=1}^{m} \| q_{i,1} \| + \alpha m (G_g + G_h) < \infty,
\]
where the second inequality holds because of (12) and (15). Therefore, \( C_q = \sum_{i=1}^{m} \| q_{i,2} \| < \infty \) is a valid choice.

We scale Proposition IV.1(a) to
\[
\sum_{i=1}^{m} \| q_{i,k+1} \| \leq \sum_{i=1}^{m} \| q_{i,k} \| + \alpha m (G_g + G_h) + \sum_{i=1}^{m} \| x_{i,k} - x_{i,k-1} \|.
\]
(21)
Now suppose the result holds for some positive integer \( k \geq 2 \). We show that it also holds for \( k + 1 \).
Substituting the induction hypothesis for \( k \) into Proposition IV.1(b), we have

\[
\sum_{i=1}^{m} ||x_{i,k} - x_{i,k-1}|| \leq 2m\Gamma \sum_{l=1}^{k-1} \gamma^{l}(C_{q} + C_{q}^{1}l + C_{q}^{2}l^{2}) + (k-1)\alpha m(G_{g} + G_{h}).
\]

By Lemma IV.2 and (11), there exist constants \( S_{0}, S_{1}, S_{2} \) such that

\[
\sum_{l=1}^{\infty} \gamma^{l}(C_{q} + C_{q}^{1}l + C_{q}^{2}l^{2}) \leq C_{q}S_{0} + C_{q}^{1}S_{1} + C_{q}^{2}S_{2}.
\]

Then, by induction hypothesis,

\[
\sum_{i=1}^{m} ||q_{i}^{k+1}|| \leq C_{q} + C_{q}^{1}k + C_{q}^{2}k^{2} + \alpha m(G_{g} + G_{h}) + 2m\Gamma(C_{q}S_{0} + C_{q}^{1}S_{1} + C_{q}^{2}S_{2}) + (k-1)\alpha m(G_{g} + G_{h}).
\]

Comparing coefficients, we see that the right-hand side can be bounded by \( C_{q} + C_{q}^{1}(k+1) + C_{q}^{2}(k+1)^{2} \) if \( \alpha m(G_{g} + G_{h}) < 2C_{q}^{2} \) for the coefficient of \( k \), and 

\[
2m\Gamma(C_{q}S_{0} + C_{q}^{1}S_{1} + C_{q}^{2}S_{2}) \leq C_{q} + C_{q}^{2}
\]

for the constant coefficient. Therefore, the induction hypothesis holds for \( k+1 \) if we take

\[
C_{q} = \sum_{i=1}^{m} ||q_{i,2}||, \\
C_{q}^{1} = \frac{2m\Gamma C_{q}S_{0} + (2m\Gamma S_{2} - 1)C_{q}^{2}}{2m\Gamma S_{1} - 1}, \\
C_{q}^{2} = \frac{\alpha m}{2}(G_{g} + G_{h}).
\]

Now, by the Lemma IV.3 and Proposition IV.1, the boundedness of summabilities (defined in Lemma IV.2) of error sequences \( \{||e_{k}||\} \) and \( \{\varepsilon_{k}\} \) is proved in the following proposition.

**Proposition IV.2.** Under Assumptions III.1 and III.2 for sequences \( \{e_{k}\} \) and \( \{\varepsilon_{k}\} \) defined in (9), \( \sum_{k=1}^{\infty} ||e_{k}|| < \infty \), \( \sum_{k=1}^{\infty} \varepsilon_{k} < \infty \) and \( \sum_{k=1}^{\infty} \sqrt{\varepsilon_{k}} < \infty \).

**Proof.** By Lemma IV.2, it suffices to show that these error sequences are polynomial-geometric sequence.

(a) By Proposition IV.1(c),

\[
\frac{1}{m} \sum_{i=1}^{m} ||x_{i,k} - \bar{x}_{k}|| \leq 2\Gamma \gamma^{k} \sum_{i=1}^{m} ||q_{i,k}||.
\]

It follows from (9a) and (22) that

\[
||e_{k+1}|| \leq 2L\Gamma \gamma^{k} \sum_{i=1}^{m} ||q_{i,k}||.
\]

In addition, by Lemma IV.3 there exists \( \sum_{i=1}^{m} ||q_{i,k}|| \leq C_{q} + C_{q}^{1}k + C_{q}^{2}k^{2} \) such that

\[
||e_{k}|| \leq 2mL\Gamma \gamma^{k-1}(C_{q} + C_{q}^{1}(k-1)^{2} + C_{q}^{2}(k-1)^{2}),
\]

which implies that \( \{||e_{k}||\} \) is a polynomial-geometric sequence.

(b) It follows from (18), (9b) and Lemma IV.3 that

\[
\varepsilon_{k} \leq 2G_{h}\Gamma \gamma^{k}(C_{q} + C_{q}^{1}k + C_{q}^{2}k^{2}) + \frac{1}{2\alpha}[\Gamma \gamma^{k}(C_{q} + C_{q}^{1}k + C_{q}^{2}k^{2})]^{2}.
\]
Using the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all nonnegative real numbers $a, b$,

$$\sqrt{\varepsilon_k} \leq \sqrt{2G_1} \sqrt{\gamma^k} \left( \sqrt{C_q} + \sqrt{C_q^1} k + \sqrt{C_q^2} k^2 \right) + \frac{1}{\sqrt{2\alpha}} \Gamma \gamma^k (C_q + C_q^1 k + C_q^2 k^2).$$

Therefore, both sequences $\{\varepsilon_k\}$ and $\{\sqrt{\varepsilon_k}\}$ are polynomial-geometric sequences. \hfill \Box

Next, we prove that all local variables achieve consensus and converge to the average.

**Theorem IV.1.** Under Assumptions [III.1] and [III.2] $\lim_{k \to \infty} ||x_{i,k} - \bar{x}_k|| = 0$ for all $i = 1, \ldots, m$.

**Proof.** By Proposition [V.1] (c),

$$\|x_{i,k} - \bar{x}_k\| \leq 2\Gamma \gamma^k \sum_{i=1}^{m} \|q_{i,k}\|.$$ 

By Lemma [IV.2] and Lemma [IV.3] $\sum_{k=1}^{\infty} ||x_{i,k} - \bar{x}_k||$ is bounded. Then, by monotone convergence theorem and Cauchy condition, we obtain

$$\lim_{k \to \infty} ||x_{i,k} - \bar{x}_k|| = 0.$$

Therefore, local variables achieve consensus and converge to the average $\bar{x}_k$ as $k \to \infty$. \hfill \Box

The next vital lemma characterizes $\partial_{\varepsilon_k} h(x_k)$, which is the $\varepsilon_k$-subdifferential of $h$ at $x_k$. The proof has been studied in Lemma 2 of [25].

**Lemma IV.4.** If $\bar{x}_k$ is an $\varepsilon_k$-optimal solution to (1) in the sense of (5) with $y = \bar{x}_{k-1} - \alpha(\nabla g(\bar{x}_{k-1}) + e_k)$, then there exists $p_k \in \mathbb{R}^n$ such that $\|p_k\| \leq \sqrt{2\alpha \varepsilon_k}$ and

$$\frac{1}{\alpha}(\bar{x}_{k-1} - \bar{x}_k - \alpha \nabla g(\bar{x}_{k-1}) - \alpha e_k - p_k) \in \partial_{\varepsilon_k} h(\bar{x}_k).$$

Now, motivated by the work [12], we are ready to discuss the convergence performance of the proposed algorithm. For a convex and closed set $\mathcal{K} \subset \mathbb{R}^n$, define $m(\mathcal{K}) = \min_{x \in \mathcal{K}} \|x\|$.

**Theorem IV.2.** Suppose Assumptions [III.1] and [III.2] hold. $\lim_{k \to \infty} m(\nabla g(\bar{x}_k) + \partial_{\varepsilon_k} h(\bar{x}_k)) = 0$ for distributed proximal gradient algorithm (7).

**Proof.** Let $g_k = \nabla g(\bar{x}_k) + e_{k+1}$. It follows from (8) and the definition of inexact proximal operator (3) that

$$\frac{1}{2\alpha} \|\bar{x}_{k+1} - \bar{x}_k + \alpha g_k\|^2 + h(\bar{x}_{k+1}) \leq \varepsilon_{k+1} + \min_x \left\{ \frac{1}{2\alpha} \|x - \bar{x}_k + \alpha g_k\|^2 + h(x) \right\}.$$ 

Equivalently,

$$\frac{1}{2\alpha} \|\bar{x}_{k+1} - \bar{x}_k\|^2 + \langle \bar{x}_{k+1} - \bar{x}_k, g_k \rangle + h(\bar{x}_{k+1}) \leq \varepsilon_{k+1} + \min_x \left\{ \frac{1}{2\alpha} \|x - \bar{x}_k\|^2 + \langle x - \bar{x}_k, g_k \rangle + h(x) \right\}.$$ 

Take $x = \bar{x}_k$ in the right-hand side of the above equation. We obtain

$$\langle g_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{1}{2\alpha} \|\bar{x}_{k+1} - \bar{x}_k\|^2 + h(\bar{x}_{k+1}) \leq h(\bar{x}_k) + \varepsilon_{k+1}. \tag{25}$$
By (6) and (25),

\[
\begin{aligned}
f(x_{k+1}) &= g(x_{k+1}) + h(x_{k+1}) \\
&\leq g(x_k) + \langle \nabla g(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\
&+ h(x_k) - \langle \nabla g(x_k) + e_{k+1}, x_{k+1} - x_k \rangle \\
&- \frac{1}{2\alpha} \|x_{k+1} - x_k\|^2 + \varepsilon_{k+1} \\
&= f(x_k) - \left( \frac{1}{2\alpha} - \frac{L}{2} \right) \|x_{k+1} - x_k\|^2 \\
&+ \varepsilon_{k+1} - \langle e_{k+1}, x_{k+1} - x_k \rangle,
\end{aligned}
\]

where the first inequality is due to the convexity of function \(g\) and the proximal operator relationship (25). By summing the inequality over \(k = 1, \ldots, T\),

\[
f(x_{T+1}) \leq f(x_1) - \left( \frac{1}{2\alpha} - \frac{L}{2} \right) \sum_{k=1}^{T} \|x_{k+1} - x_k\|^2 \\
+ \sum_{k=1}^{T} \varepsilon_{k+1} + \sum_{k=1}^{T} \|e_{k+1}\| \|x_{k+1} - x_k\|.
\]

By rearranging the terms,

\[
\sum_{k=1}^{T} \|x_{k+1} - x_k\|^2 \\
\leq \frac{1}{2\alpha} - \frac{L}{2} \left( f(x_1) - f(x_{T+1}) \right) + \frac{1}{2\alpha} - \frac{L}{2} \sum_{k=1}^{T} \varepsilon_{k+1} \\
+ \frac{1}{2\alpha} - \frac{L}{2} \sum_{k=1}^{T} \|e_{k+1}\| \|x_{k+1} - x_k\|,
\]

(26)

where \(\frac{1}{2\alpha} - \frac{L}{2} > 0\) because the step-size satisfies \(\alpha < \frac{1}{L}\).

Then, consider the last term \(\sum_{k=1}^{T} \|e_{k+1}\| \|x_{k+1} - x_k\|\). It follows from the inequality (20),

\[
\|x_{k+1} - x_k\| \\
= \frac{1}{m} \sum_{i=1}^{m} \|x_{i,k+1} - x_{i,k}\| \\
\leq 2\Gamma (\gamma^k \sum_{j=1}^{m} \|q_{j,k}\|) + \alpha (G_g + G_h).
\]

(27)

Then, by (23) and (27),

\[
\|e_{k+1}\| \|x_{k+1} - x_k\| \leq 4mL^2 \gamma^{2k} \left( \sum_{i=1}^{m} \|q_{i,k}\| \right)^2 \\
+ 2\alpha mL \Gamma (G_g + G_h) \gamma^k \sum_{i=1}^{m} \|q_{i,k}\|,
\]

(28)
where \( \sum_{i=1}^{m} \|q_{i,k}\| \leq C_q + C_q^1 k + C_q^2 k^2 \) and the right hand side is a polynomial-geometric sequence. By Lemma IV.2
\[
\sum_{k=1}^{T} \|e_{k+1}\| \|\bar{x}_{k+1} - \bar{x}_k\| < \infty.
\]
(29)

In addition, by Proposition IV.2 \( \sum_{k=1}^{T} \varepsilon_{k+1} < \infty \). It follows from (26) and (29) that \( \sum_{k=1}^{T} \|\bar{x}_{k+1} - \bar{x}_k\|^2 < \infty \). It follows from monotone convergence theorem and Cauchy condensation test that
\[
\lim_{k \to \infty} \|\bar{x}_{k+1} - \bar{x}_k\| = 0.
\]
(30)

By Lemma IV.4 there exists \( p_{k+1} \) such that \( \|p_{k+1}\| \leq \sqrt{2\varepsilon_{k+1}} \) and
\[
0 \leq \frac{1}{\alpha} (\bar{x}_k - \bar{x}_{k+1} - p_{k+1}) - \nabla g(\bar{x}_k) - e_{k+1} + \nabla g(\bar{x}_{k+1})
- \nabla g(\bar{x}_{k+1}) - \partial e_{k+1} h(\bar{x}_{k+1}).
\]

Then,
\[
\frac{1}{\alpha} (\bar{x}_k - \bar{x}_{k+1} - p_{k+1}) - \nabla g(\bar{x}_k) - e_{k+1} + \nabla g(\bar{x}_{k+1})
\in \nabla g(\bar{x}_{k+1}) + \partial e_{k+1} h(\bar{x}_{k+1}).
\]
(31)

The left hand side of (31) satisfies
\[
\left\| \frac{1}{\alpha} (\bar{x}_k - \bar{x}_{k+1} - p_{k+1}) - \nabla g(\bar{x}_k) - e_{k+1} + \nabla g(\bar{x}_{k+1}) \right\|
\leq \left( \frac{1}{\alpha} + L \right) \|\bar{x}_k - \bar{x}_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha} + \|e_{k+1}\|},
\]
(32)

where we utilize \( \|p_k\| \leq \sqrt{2\varepsilon_k} \) in Lemma IV.4 and (5) in Assumption III.1. Then, by (30) and Proposition IV.2
\[
\lim_{k \to \infty} \left( \frac{1}{\alpha} + L \right) \|\bar{x}_k - \bar{x}_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha} + \|e_{k+1}\|}
= 0.
\]
(33)

Therefore, it follows from (31) and (33) that
\[
\lim_{k \to \infty} m(\nabla g(\bar{x}_k) + \partial e_{k+1} h(\bar{x}_k)) = 0.
\]

\[\square\]

Remark IV.1. Combining Theorems IV.1 and IV.2 we obtain that for all agent \( i \), the generated variable sequence \( x_{i,k} \) converges to the set of critical points and there exists a subsequence of \( x_{i,k} \) converging to one critical point of non-convex optimization problem (4).

Next, we consider the convergence rate of the proposed algorithm. By (31) and (32),
\[
\frac{1}{T} \sum_{k=1}^{T} \min_{d_k \in \partial e_k h(\bar{x}_k)} \|\nabla g(\bar{x}_k) + d_k\|
\leq \frac{1}{T} \sum_{k=1}^{T} \left( \frac{1}{\alpha} + L \right) \|\bar{x}_k - \bar{x}_{k-1}\| + \sqrt{\frac{2\varepsilon_k}{\alpha} + \|e_k\|}.
\]
(34)
Hence, we analyze $\frac{1}{T} \sum_{k=1}^{T} \|\bar{x}_k - \bar{x}_{k-1}\|^2$ to provide the convergence rate of (7) in the non-convex setting.

At first, we provide one related lemma, whose proof is provided in Lemma 1 of [25].

**Lemma IV.5.** Assume that the non-negative sequence $u_k$ satisfies the following recursion for all $k \geq 1$:

$$u_k^2 \leq S_k + \sum_{i=1}^{k} \lambda_i u_i,$$

with an increasing sequence $S_k$, $S_1 \geq u_1^2$ and $\lambda_i \geq 0$. Then for all $k \geq 1$,

$$u_k \leq \frac{1}{2} \sum_{i=1}^{k} \lambda_i + (S_k + \left(\frac{1}{2} \sum_{i=1}^{k} \lambda_i\right)^2)^{\frac{1}{2}}.$$

Then, we are ready to analyze the convergence rate of $\frac{1}{T} \sum_{k=1}^{T} \|\bar{x}_k - \bar{x}_{k-1}\|^2$.

**Theorem IV.3.** With Assumptions [III.1 and III.2] the convergence rate of the sequence $\frac{1}{T} \sum_{k=1}^{T} \|\bar{x}_k - \bar{x}_{k-1}\|^2$ is $O\left(\frac{1}{T}\right)$.

**Proof.** Recall that $\frac{1}{\alpha} (\bar{x}_k - \bar{x}_{k+1} - \alpha \nabla g(\bar{x}_k) - \alpha e_{k+1} - p_{k+1}) \in \partial_{\varepsilon_{k+1}} h(\bar{x}_{k+1})$ in (31). By (6) and the definition of $\varepsilon_k$-subdifferential,

$$f(\bar{x}_{k+1}) = g(\bar{x}_{k+1}) + h(\bar{x}_{k+1})$$

$$\leq g(\bar{x}_k) + \langle \nabla g(\bar{x}_k), \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L}{2} \|\bar{x}_{k+1} - \bar{x}_k\|^2 + h(\bar{x}_k)$$

$$- \left\langle \nabla g(\bar{x}_k) + e_{k+1} + \frac{1}{\alpha} (\bar{x}_{k+1} - \bar{x}_k + p_{k+1}), \bar{x}_{k+1} - \bar{x}_k \right\rangle + \varepsilon_{k+1}$$

$$= f(\bar{x}_{k}) - \frac{1}{\alpha} \|\bar{x}_{k+1} - \bar{x}_k\|^2 + \frac{L}{2} \|\bar{x}_{k+1} - \bar{x}_k\|^2$$

$$- \left\langle e_{k+1} + \frac{1}{\alpha} p_{k+1}, \bar{x}_{k+1} - \bar{x}_k \right\rangle + \varepsilon_{k+1}$$

$$\leq f(\bar{x}_{k}) - \left(\frac{1}{\alpha} - \frac{L}{2}\right) \|\bar{x}_{k+1} - \bar{x}_k\|^2$$

$$+ \left(\|e_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha}}\right) \|\bar{x}_{k+1} - \bar{x}_k\| + \varepsilon_{k+1}.$$  

By summing the above inequality over $k = 1, \ldots, T - 1$,

$$f(\bar{x}_T) \leq f(\bar{x}_0) - \left(\frac{1}{\alpha} - \frac{L}{2}\right) \sum_{k=1}^{T-1} \|\bar{x}_{k+1} - \bar{x}_k\|^2 + \sum_{k=1}^{T-1} \varepsilon_{k+1}$$

$$+ \sum_{k=1}^{T-1} \left(\|e_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha}}\right) \|\bar{x}_{k+1} - \bar{x}_k\|. \quad (35)$$

Then, by rearranging and scaling,

$$\|\bar{x}_T - \bar{x}_{T-1}\|^2 \leq \frac{1}{\alpha - \frac{L}{2}} \left( f(\bar{x}_0) - f(\bar{x}^*) + \sum_{k=1}^{T} \varepsilon_k \right)$$

$$+ \sum_{k=1}^{T} \left( \|e_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha}}\right) \|\bar{x}_{k+1} - \bar{x}_k\|,$$

$$+ \sum_{k=1}^{T} \left( \|e_{k+1}\| + \sqrt{\frac{2\varepsilon_{k+1}}{\alpha}}\right) \|\bar{x}_{k+1} - \bar{x}_k\|,$$
where $x^*$ is the optimal solution of optimization problem.

Let $\bar{x}_0 = \bar{x}_1$, then $u_1^2 = 0$ and $S_1 \geq u_1^2$. By Lemma IV.5

$$\|\bar{x}_T - \bar{x}_{T-1}\|$$

$$\leq \frac{1}{2} \sum_{k=1}^{T} \lambda_k + \left( S_T + \left( \frac{1}{2} \sum_{k=1}^{T} \lambda_k \right)^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \sum_{k=1}^{T} \frac{1}{\alpha - \frac{T}{2}} \left( \sqrt{\frac{2\varepsilon_k}{\alpha}} + \|e_k\| \right)$$

$$+ \left( \frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + \frac{1}{\alpha - \frac{T}{2}} \sum_{k=1}^{T} \varepsilon_k \right)$$

$$+ \left( \frac{1}{\alpha - \frac{T}{2}} \left( \sqrt{\frac{2\varepsilon_k}{\alpha}} + \|e_k\| \right)^2 \right)^{\frac{1}{2}}.$$ 

Because $A_k$ and $B_k$ are increasing sequences, $\forall k \leq T$,

$$\|\bar{x}_k - \bar{x}_{k-1}\|$$

$$\leq A_T + \left( \frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + B_T + A_T^2 \right)^{\frac{1}{2}}$$

$$\leq 2A_T + \sqrt{\frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + \sqrt{B_T}}.$$ 

(36)

By (35) and (36),

$$\sum_{k=1}^{T} \|\bar{x}_k - \bar{x}_{k-1}\|^2$$

$$\leq \frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + B_T$$

$$+ 2A_T \left( 2A_T + \sqrt{\frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + \sqrt{B_T}} \right)$$

$$\leq \left( 2A_T + \sqrt{\frac{1}{\alpha - \frac{T}{2}} (f(\bar{x}_0) - f(x^*)) + \sqrt{B_T}} \right)^2.$$ 

Since $A_T$ and $B_T$ are upper bounded by Proposition IV.2, we obtain \{\frac{1}{T} \sum_{k=1}^{T} \|\bar{x}_k - \bar{x}_{k-1}\|^2 \} = O(1/T).

By (34) and Theorem IV.3, the proposed distributed algorithm has a convergence rate $O(\frac{1}{T})$ for the distributed non-convex optimization problem (4).

Remark IV.2. We establish the convergence rate of the proposed algorithm with respect to the number of communication steps. Let $t$ be the total number of communication steps taken. Since the proposed algorithm takes $k$ communication steps in iteration $k$, the total number of communication steps to
execute iterations \(1, \ldots, T\) is \(\sum_{k=1}^{T} k = \frac{T(T+1)}{2}\). Then, when \(t\) communication steps are taken, the number of iterations completed is \(T\) such that \(\frac{T(T+1)}{2} < t\), or equivalently, \(T = \lceil -1 + \sqrt{\frac{t+1}{2}} \rceil\). Hence, \(\frac{1}{T} \sum_{k=1}^{T} \min_{d_k \in \partial h(x_k)} \|\nabla g(x_k) + d_k\| = O(1/\sqrt{t})\), where \(t\) is the total number of communication steps taken.

V. SIMULATION

We apply algorithm (7) to learn the black-box binary classification problem [27], which is to find the optimal predictor \(x \in \mathbb{R}^n\) by solving the following problem:

\[
\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \frac{1}{1 + \exp(l_ia_i'x)} + \lambda_1\|x\|_1 + \lambda_2\|x\|_2^2,
\]

(37)

where \(a_i \in \mathbb{R}^n\), \(l_i \in \{-1, 1\}\) and \(\{a_i, l_i\}_{i=1}^{m}\) denotes the set of training samples. In this experiment, we use the publicly available real datasets\(^1\) which are summarized in Table I. The proposed distributed algorithm (7) is applied over ten-agent networks to solve the problem (37). Meanwhile, \(\lambda_1 = \lambda_2 = 5 \times 10^{-4}\). For the underlying network, we take time-invariant undirected connected graphs and periodic time-varying graphs, respectively.

1. Time-invariant undirected connected graphs: Over the undirected connected multi-agent networks, we apply (7) (denoted by ‘meth1’) and the algorithm in [16] (denoted by ‘meth2’) to solve (37). Both algorithms take the same constant step-size.

2. Time-varying undirected graphs: Over periodic time-varying graphs, which are generated randomly and satisfy Assumption III.2, we apply (7) (denoted by ‘meth1-s’) and the algorithm in [19] (denoted by ‘meth3-s’) to solve (37). The algorithm ‘meth3-s’ takes a diminishing step-size in [19], and the proposed algorithm ‘meth1-s’ takes a constant step-size.

Define \(D(\bar{x}) = \sum_{i=1}^{10} x_i' \sum_{j=1}^{10} a_{ij}(x_i - x_j)\). For the dataset \(a9a\) over time-invariant graphs, the trajectories of \(D(\bar{x})\) are shown in Fig. 1, which implies that the variable estimates of different agents achieve consensus. In addition, it is seen that the trajectory generated by (7) owns a better consistent performance than those generated by [16]. For the datasets \(a9a\) and \(covtype.binary\) over time-varying graphs, the convergence trajectories are shown in Figs. 2(a) and 2(b), respectively. The trajectories of the norm

\(^1\)a9a and covtype.binary are from the website www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/.
of gradient converging to zero imply that variable estimates converge to critical points of optimization problem (37). It is seen that over time-varying undirected graphs, ‘meth1-s’ owns a better convergence performance than ‘meth3-s’ proposed in [19]. Hence, the proposed distributed algorithm (7) is able to solve the binary classification problem (37) efficiently.

VI. CONCLUSION

This paper has proposed a distributed proximal gradient algorithm for the large-scale non-smooth non-convex optimization problem over time-varying multi-agent networks. The proposed algorithm has made use of multi-step consensus stage to improve the consistent performance of the proposed distributed algorithm. In addition, this paper has provided complete and rigorous theoretical convergence analysis. In binary classification tests, the proposed algorithm has a better convergence performance compared with existing distributed algorithms. One future research direction is to develop distributed stochastic algorithms for distributed non-convex optimization over multi-agent networks.

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