An upper bound for min-max angle of polygons

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Abstract

Let \( S \) be a set of \( n \) points in the plane, \( \wp(S) \) be the set of all simple polygons crossing \( S \), \( \gamma_P \) be the maximum angle of polygon \( P \in \wp(S) \) and \( \theta = \min_{P \in \wp(S)} \gamma_P \). In this paper, we prove that \( \theta \leq 2\pi - \frac{2\pi}{r+m} \) where \( m \) and \( r \) are the number of edges and inner points of the convex hull of \( S \), respectively. We also propose an algorithm to construct a polygon with the said upper bound on its angles. Constructing a simple polygon with angular constraint on a given set of points in the plane can be used for path planning in robotics. Moreover, we improve our upper bound on \( \theta \) and prove that this is tight for \( r = 1 \).

Keywords: Min-Max angle, Upper bound, Angular onion peeling, Sweep arc, Simple polygonization, Computational geometry

1. Introduction

An optimal polygonization of a set of points in the plane is a classical problem in computational geometry and has been applied to many fields such as image processing \[1,2\], pattern recognition \[3,4\], geographic information system \[5\], etc. Considering a set \( S \) of points in the plane, there are different numbers of simple polygons on \( S \). Enumerating and generating simple polygons on \( S \) has been the focus of many studies \[6,7,8,9,10\].

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Finding polygons with special properties over all polygonizations is of particular interest to researchers. The minimum and maximum area polygonization are NP-complete, as shown by Fekete [11, 12]. The problems of computing the simple polygons with minimum and maximum perimeters is the well-known NP-complete problems called TSP and max-TSP, respectively. There are many ongoing studies on approximation algorithm for minimum and maximum area polygonization [13, 14], TSP [15, 16] and max-TSP [17].

In some of these approaches the angles have been investigated in many problems over polygonization. The Angular-Metric TSP [18] is the problem of finding a tour on \( S \) minimizing the sum of the direction changes at each point. Fekete and Woeginger introduced Angle-Restricted Tour problem in [19]. For a set \( A \subseteq (-\pi, \pi] \) of angles, Angle-Restricted Tour is the problem of finding a simple or non-simple polygon on \( S \) where all angles of the polygon belong to \( A \). In [20] \( \alpha \)-concave hull refers to a simple polygon \( P \) with minimum area covering a set of points such that all angels of \( P \) are less than or equal to \( \pi + \alpha \).

Reflexivity, the smallest number of reflex vertices among all polygonizations of a set of points, is considered as a convexity measurement for those points. Arkin et al. [21] introduced the concept of reflexivity and presented lower and upper bounds for reflexivity of any set of \( n \) points. E. Ackerman et al. [22] improved the upper bound and proposed an algorithm to compute polygon with at most this number of reflex vertices in the time complexity of \( O(n \log n) \). In [23] a convexity measurement has been proposed for polyhedra.

Rorabaugh [24] investigated the min-max value of reflex angles in polygonizations as another convexity measurement for a set of points and derived an upper bound for their solution.

In [25], the upper bound \( 2\pi - \frac{2\pi}{r/m} \) is presented for min-max value of the angles in polygonization where \( m \) and \( r \) are the number of edges and inner points of the convex hull of \( S \), respectively. Here we improved this upper bound to \( 2\pi - \frac{2\pi}{r/m} \).

The rest of the paper is as follows: In the section 2, notations and definitions are presented. In section 3, the upper bound is derived and in section 4, we
Table 1: Notations of symbols

| Notation | Description |
|----------|-------------|
| $S$      | A set of points in the plane |
| $n$      | cardinality of $S$ |
| $s_i$    | $i$th point of $S$ $(1 \leq i \leq n)$ |
| $CH$     | convex hull of $S$ |
| $m$      | number of vertices of $CH$ |
| $IP$     | inner points of $CH$ |
| $r$      | cardinality of $IP$ |
| $P$      | a simple Polygon crossing $S$ |
| $V_P$    | vertices of $P$ |
| $E_P$    | edges of $P$ |
| $c_j$    | $j$th vertex of $CH$ $(1 \leq j \leq m)$ |
| $e_j$    | $j$th edge of $CH$ $(1 \leq j \leq m)$ |
| $s_is_j$ | an edge of $P$ with $s_i$ and $s_j$ as its end points $(1 \leq i, j \leq n, i \neq j)$ |
| $\wp(S)$ | set of all simple polygons crossing $S$ |
| $\alpha, \beta, \gamma, \theta$ | angles between 0 and $2\pi$ |

conclude the paper highlighting its achievements.

2. Preliminaries

Let $S = \{s_1, s_2, ..., s_n\}$ be a set of points in the plane and $CH$ be the convex hull of $S$. The vertices and edges of $CH$ are denoted by $V_{CH} = \{c_1, c_2, ..., c_m\}$ and $E_{CH} = \{e_1, e_2, ..., e_m\}$, respectively. Furthermore, let $IP = \{a_1, a_2, ..., a_r\}$ be the inner points of $CH$ where $r = n - m$. Table 1 shows more notations that are used in the rest of the paper. A polygon $P$ crossing $S$ is specified by a closed chain of vertices $P = (p_1, p_2, ..., p_n, p_1)$ such that $S = V_P = \{p_1, p_2, ..., p_n\}$.

Let $e = AB$ be a line segment. The minor arc $\hat{AB}$ with measure equal to $\alpha$ is denoted by $s_\alpha$, and the major arc $\hat{AB}$ with measure equal to $\beta = 2\pi - \alpha$ is denoted by $S_\beta$. We denote the minor and major segments on $e$ by $m_\alpha$ and $m_\beta$. 

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\(M^\beta_e\), respectively (see Fig. 1). Also, we use the concept of "Sweep Arc" in our algorithm which is defined in [25, Section 4]. A sweep arc on \(e\) is a minor arc \(s^0_e\) where it expands to the major arc \(S^2\pi_e\).

![Figure 1: The notations of minor arc, major arc, minor segment and major segment on \(e\)](image)

3. Min-Max Angle

In this section we present two upper bounds for \(\theta\). Let us first present a lemma followed by a theorem.

**Lemma 1.** Let \(l = c_1c_2\) be a line segment and \(S\) be a set of \(n\) points inside the \(M^\beta_{\max}\), where \(\beta_{\max} = 2\pi - \frac{4\pi}{m}\) for an integer number \(m\). Assume that \(t\) points \(\{s_1, s_2, ..., s_t\}\) are met by the sweep arc on \(l\) and \(P = (c_1, s_1, s_2, ..., s_t, c_2, c_1)\) is a simple polygon such that all internal angles of \(s_i\) are greater than or equal to \(2\pi m\). Let \(x\) be \((t + 1)\)th point met by the sweep arc. There exists an edge \(ab\) of \(P\) such that \(\hat{a}xb\) is greater than or equal to \(\frac{2\pi}{t+1,m}\).

**Proof.** We prove the lemma by induction on \(t\). When \(t = 0\), the chain \(P = (c_1, s_1, s_2, ..., s_t, c_2, c_1)\) is a line segment \(c_1c_2\). Therefore, we consider both cases \(t = 0\) and \(t = 1\) as the base cases.

**Base case \((t = 0)\).** Let \(x\) be the first point that the sweeping arc meets. We construct the polygon by connecting \(x\) to \(c_1\) and \(c_2\). Since the maximum measure of the arc is \(\beta_{\max}\), the internal angle of \(\gamma = c_1xc_2\) in the triangle \(\triangle c_1xc_2\) is greater than or equal to \(\frac{2\pi}{m}\) (see Fig. 2).
Base case \((t = 1)\). Let \(s_1\) be the first point that the sweeping arc meets and \(x\) be the second one. Also, let \(e_1 = c_1s_1\) and \(e_2 = s_1c_2\) be two edges of \(P = (c_1, s_1, c_2, c_1)\). The edges \(e_1\) and \(e_2\) divide the sweeping arc into 3 parts; the arc \(B_1\) where \(e_1\) is visible but \(e_2\) is not visible from all the points on it, the arc \(B_2\) where \(e_2\) is visible but \(e_1\) is not visible from all the points on it, and finally the arc \(B_3\) where \(e_1\) and \(e_2\) are both visible from all the points on it (see Fig. 3).

Figure 2: \(\hat{x}\) is greater than \(\frac{2\pi}{m}\).

Figure 3: The edges \(e_1\) and \(e_2\) divide the sweeping arc into 3 parts: \(B_1, B_2\) and \(B_3\).

case 1. If \(x\) is placed on \(B_1\), the angle \(c_1\hat{x}s_1\) is greater than \(\gamma = c_1\hat{c}c_2\) and the angle \(\gamma\) is greater than or equal to \(\frac{2\pi}{m}\). Hence, the angle \(c_1\hat{x}s_1\) is greater than \(\frac{2\pi}{m}\). So, we consider the edge \(c_1s_1\) as the desired edge \(ab\) such that \(\hat{a}\hat{a}\hat{b}\) is greater than or equal to \(\frac{\pi}{m}\).

case 2. If \(x\) is placed on \(B_2\), the angle \(s_1\hat{x}c_2\) is greater than \(\gamma\) and the angle \(\gamma\) is greater than or equal to \(\frac{2\pi}{m}\). Hence, the angle \(s_1\hat{x}c_2\) is greater than \(\frac{2\pi}{m}\). So, we consider the edge \(s_1c_2\) as the desired edge \(ab\) such that \(\hat{a}\hat{a}\hat{b}\) is greater than or equal to \(\frac{\pi}{m}\).

case 3. If \(x\) is placed on \(B_3\), the maximum of \(c_1\hat{x}s_1\) and \(s_1\hat{x}c_2\) is greater than \(\frac{\gamma}{2}\). Since \(\gamma\) is greater than \(\frac{2\pi}{m}\), the maximum of \(c_1\hat{x}s_1\) and \(s_1\hat{x}c_2\) is
greater than \( \frac{2\pi}{m} \). Hence, if \( c_1 \bar{s}_1 \) is greater than \( s_1 \bar{c}_2 \), the edge \( c_1 \bar{s}_1 \) is considered as \( \bar{a} \bar{b} \), otherwise, the edge \( \bar{s}_1 \bar{c}_2 \) is considered as \( \bar{a} \bar{b} \).

**Induction assumption.** Let \( y \) be \( k \)th point that the sweeping arc meets. There exists an edge \( \bar{a} \bar{b} \) of \( P = (c_1, s_1, s_2, \ldots, s_{k-1}, c_2, c_1) \) such that \( \widehat{ayb} \) is greater than or equal to \( \frac{2\pi}{m} \). We show that there exists an edge \( \bar{a} \bar{b} \) of \( P \) such that \( \widehat{ayb} \) is greater than or equal to \( \frac{2\pi}{(k+1)m} \). Here, three cases need to be examined:

**case 1.** All edges of \( P \) except \( c_1 \bar{c}_2 \) are visible from \( x \). Let \( e_1 = c_1 \bar{s}_1 \), \( e_2 = s_1 \bar{s}_2 \), \( \ldots \), and \( e_{k+1} = s_{k} \bar{c}_2 \) be edges of \( P \), \( \beta_i \) be the angle subtended by \( e_i \) at the point \( x \) and \( \beta_M \) be the maximum one. Since the angle \( \gamma \) is greater than or equal to \( \frac{2\pi}{m} \) and \( \sum_{i=1}^{k+1} \beta_i = \gamma \), we have \( \beta_M > \frac{2\pi}{(k+1)m} \). Let \( e \) be the edge that corresponds to \( \beta_M \). So, the edge \( e \) is considered as \( \bar{a} \bar{b} \) such that \( \widehat{AXB} \) is greater than or equal to \( \frac{2\pi}{(k+1)m} \) (see Fig. 4).

**Figure 4:** All edges \( e_i \) are visible from \( x \).

**case 2.** There exists an edge \( e = \bar{c} \bar{d} \) of \( P \) such that both endpoints of \( e \) are not visible from \( x \). We obtain a polygon \( P' \) from \( P \) by contracting \( \bar{c} \bar{d} \) the edge \( e \), i.e. \( P' = P/e \). Since \( P' \) has \( k + 1 \) vertex points, by induction assumption, there exists an edge \( e' = \bar{a} \bar{b} \) of \( P' \) such that \( \widehat{AXB} \) is greater than or equal to \( \frac{2\pi}{k \cdot m} \). The polygon \( P'' \) is obtained from \( P \) by removing
the edge $\overline{ab}$ and adding two edges $\overline{ax}$ and $\overline{xb}$. Since the two end points of $e = \overline{cd}$ are invisible from $x$, contracting and splitting $e$ has no effect on the measure of the angle $\hat{axb}$. Hence, the angle $\hat{axb}$ in $P''$ is greater than or equal to $\frac{2\pi}{(k+1)m}$ (see Fig. 5).

Figure 5: The edge $e$ of $P$ is invisible from $x$. Contracting $e$ leads to construct $P'$ from $P$. The polygon $P''$ obtained from $P$ and $P'$.

case 3. There exists an edge $e = \overline{cd}$ of $P$ such that one endpoint of $e$ is not visible from $x$ (see Fig. 6). We obtain a polygon $P' = P/e$ from $P$ by contracting the edge $e$. Since $P'$ has $k + 1$ vertex points, by induction assumption, there exists an edge $e' = \overline{ab}$ of $P'$ such that $\hat{axb}$ is greater than or equal to $\frac{2\pi}{km}$. The polygon $P''$ is obtained from $P$ by removing the edge $\overline{ab}$ and adding two edges $\overline{ax}$ and $\overline{xb}$.

Figure 6: The vertex $s_1$ is visible and $c_1$ is invisible from $x$.

If either $a$ or $b$ in $P''$ be an endpoint of $e$, contracting and splitting $e$ has an effect on the measure of the angle $\hat{axb}$ (see Fig. 7). In other words, the angle $\hat{axb}$ in $P''$ is not equal to the angle $\hat{axb}$ in $P'$. It is clear that the angle $\hat{axb}$ in $P''$ is greater than the angle $\hat{axb}$ in $P'$. Since the angle $\hat{axb}$ in $P'$ is greater than or equal to $\frac{2\pi}{km}$, the angle $\hat{axb}$ in $P''$ is greater than or equal to $\frac{2\pi}{(k+1)m}$.  

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Figure 7: $\hat{\alpha}_{xb}$ in polygon $P''$ is greater than $\hat{\alpha}_{xb}$ in polygon $P'$. Also, if both points $a$ or $b$ in $P''$ are not the endpoints of $e$, contracting and splitting $e$ has no effect on the measure of the angle $\hat{\alpha}_{xb}$ (see fig. 8). In other words, the angle $\hat{\alpha}_{xb}$ in $P''$ is equal to the angle $\hat{\alpha}_{xb}$ in $P'$. Hence, the angle $\hat{\alpha}_{xb}$ in $P''$ is greater than or equal to $2\pi (k+1)$.

Figure 8: $\hat{\alpha}_{xb}$ in polygon $P''$ is equal to $\hat{\alpha}_{xb}$ in polygon $P'$.

**Theorem 1.** Let $l = c_1c_2$ be a line segment and $S$ be a set of $n$ points inside the $M_{l_{\beta_{\max}}}$, such that $\beta_{\max} = 2\pi - \frac{4\pi}{m}$ for an integer number $m$ (see Fig. 9 a). There exists a chain $(s_1, s_2, ..., s_n)$ on $S$ such that all internal angles of $s_i$ in the polygon $(c_1, s_1, s_2, ..., s_n, c_2, c_1)$ are greater than or equal to $\frac{2\pi}{n.m}$ (see Fig. 9 b).

**Proof.** We prove theorem 1 by constructing the polygon $(c_1, s_1, s_2, ..., s_n, c_2, c_1)$, using the following algorithm which is a modified version of that originally presented in [25]:

**Algorithm 1 (Modified Sweep Arc Algorithm).**

1. Sweep the arc $c_1c_2$ from $s_i^0$ to $S_{l_{\beta_{\max}}}^0$. 

□
2. Let $x_1$ be the first point which is met by the sweep arc. Construct $P = (c_1, x_1, c_2, c_1)$ as the desired polygon.

3. Set $i = 2$.

4. Let $P = (c_1, s_1, s_2, ..., s_{i-1}, c_2, c_1)$ be the constructed polygon inside the sweep arc and $x_i$ be the $i$th point which is met by the sweep arc.

5. Assume that $e_1 = \overline{c_1 s_1}$, $e_2 = \overline{s_1 s_2}$, ..., and $e_i = \overline{s_{i-1} c_2}$ are the edges of $P$. If $e_j$ is visible from $x$, set $\beta_j = \text{The angle subtended by } e_j \text{ at the point } x$, otherwise set $\beta_j = 0$.

6. Let $\beta_M = \max_{1 \leq j \leq i} \beta_j$ and $e = \overline{ab}$ be the edge that corresponds to $\beta_M$.

7. Remove the edge $e$ from $P$ and add two edges $\overline{ax_i}$ and $\overline{x_i b}$ to construct the desired polygon.

8. Set $i = i + 1$. If $i \leq n$, then go to 4, otherwise exit.

Based on Lemma 1: \( \forall j \in \{1, 2, ..., i\} \) the angles $\hat{s}_j$ in $P$ are greater than or equal to $\frac{2\pi}{i}$. In step 4 of the algorithm. Therefore, when $i = n$, the angles $\hat{s}_j$ in $P$ are greater than or equal to $\frac{2\pi}{n}$. $\square$

It is proved in [25, Lemma 3] that all angles of the mentioned polygon $P$ are greater than or equal to $\frac{2\pi}{r}$. Here, based on theorem 1 we increase this bound to $\frac{2\pi}{r}$. This yields us the following corollaries:

**Corollary 1.** Let $S$ be a set of points in the plane, $CH$ be the convex hull of $S$ and $m$ and $r$ be the number of edges and inner points of $CH$, respectively. If we replace algorithm 1 of [25, Theorem 2, Step 2.a of Algorithm 2] by the modified sweep arc algorithm, the upper bound $2\pi - \frac{2\pi}{r,m}$ is achieved for $\theta$. 

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Remark 1. Based on corollary 1, in the case of $r = 1$, $2\pi - \frac{2\pi}{n-1}$ is an upper bound for $\theta$ over all simple polygons crossing $S$. It is noteworthy that this bound is tight in this case. The tightness is achieved when the inner point is at the center of a regular $n$-gons, as illustrated in Fig. 10.

Figure 10: Maximum angle of each polygon crossing these points is equal to $2\pi - \frac{2\pi}{6}$

The following corollary improved the upper bound to $2\pi - \frac{2\pi}{d.m}$ where $d$ is depth of angular onion peeling on $S$ which is defined in [25].

**Corollary 2.** Let $S$ be a set of points in the plane, $CH$ be the convex hull of $S$, $m$ be the cardinality of edges of $CH$ and $d$ be the depth of angular onion peeling on $S$. If we replace algorithm 1 of [25], Theorem 3, Step 2.b of Algorithm 3] by the modified sweep arc algorithm, the upper bound $2\pi - \frac{2\pi}{d.m}$ is achieved for $\theta$.

Since the time complexity of modified sweep arc algorithm is $O(r)$, those of both modified algorithm 2 and 3 are $O(n \log n + rm)$. Note that the modified algorithm 2 and 3 are those proposed in [23] in which the algorithm 1 is replaced by the modified sweep arc algorithm. Based on corollary 1, the modified algorithm 2 constructs a polygon such that its internal angles are less than or equal to $2\pi - \frac{2\pi}{r.m}$. Based on corollary 2, this bound is improved to $2\pi - \frac{2\pi}{d.m}$ using modified algorithm 3. When $S$ is a set of $n$ points in the plane and the convex hull of $S$ has $n-1$ edges, the depth of angular onion peeling on $S$ is equal to 1. Hence, the upper bound for $\theta$ is equal to $2\pi - \frac{2\pi}{1(n-1)}$ which confirms the remark [1].

Computing $\alpha$-concave hull on a set $S$ of points is an NP-complete problem [20]. For all $\alpha > \theta$, $\alpha$-concave hull crosses all points of $S$. So, the polygon
computed by modified algorithm 3 is an $\alpha$-polygon \cite{20} which approximates $\alpha$-concave hull of $S$. The following corollary shows the relation between $\alpha$-concave hull and the computed upper bound.

**Corollary 3.** Let $S$ be a set of points in the plane, $CH$ be the convex hull of $S$, $m$ be the cardinality of edges of $CH$ and $d$ be the depth of angular onion peeling on $S$. For all $\alpha > 2\pi - \frac{2\pi}{d.m}$, there always exists an $\alpha$-concave hull $P$ on $S$ such that $P$ crosses all points of $S$.

Coverage path planning is a fundamental problem in the field of robotics. There are many limitation factors in order to plan a path for a robot to cover (or visit) all points of a set of points, such as robot rotation angle. The following corollary presents the essential relation between path planning in robotics and our upper bounds on $\theta$.

**Corollary 4.** Let $S$ be a set of $n$ points in the plane, $CH$ be the convex hull of $S$, $m$ be the cardinality of edges of $CH$ and $d$ be the depth of angular onion peeling on $S$. If the robot rotation angle is greater than $2\pi - \frac{2\pi}{d.m}$, there always exists a path for the robot to cover $S$. As stated before, this path can be found in $O(n \log n + rm)$.

4. Conclusion

The major problem investigated in this paper is that of finding a simple polygon with angular constraint on a given set of points in the plane. We derived the upper bounds for min-max value of angles over all simple polygons crossing the given set of points. We also presented algorithms to compute the polygons thereby satisfying the derived upper bounds. In addition to the theoretical results, this bound is an important achievement in the field of robotic.

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