Perfect orderings on Bratteli diagrams II:
general Bratteli diagrams

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Abstract

We continue our study of orderings on Bratteli diagrams started in [BKY12],
where Bratteli diagrams of finite rank were considered. We extend the notions of
languages, permutations (called correspondences in this paper), skeletons and as-
associated graphs to the case of general Bratteli diagrams, and show their relevance
to the study of perfect orderings - those that support Vershik maps; in particular,
perfect orderings with several extremal paths. A perfect ordering comes equipped
with a skeleton and a correspondence, and conversely, given a skeleton and corre-
spondence, we describe explicitly how to construct perfect orderings, by showing
that paths in the associated directed graphs determine the language of the order.
We describe an explicit algorithmic method to create perfect orderings on Bratteli
diagrams based on the study of certain relations between the entries of the dia-
gram’s incidence matrices and properties of the associated graphs, with the latter
relations characterizing diagrams which support perfect orderings. Also, we apply
the notions of skeletons and associated graphs, to give a new combinatorial proof
of the fact that diagrams supporting perfect orderings with $k$ maximal paths have
a copy of $\mathbb{Z}^{k-1}$ contained in their infinitesimal subgroup. Under certain condi-
tions, we show that a similar result holds if the diagram supports countably many
maximal paths. Our results are illustrated by numerous examples.

1 Introduction

This paper is a natural extension of our previous work [BKY12] that was devoted to the
study of orderings on Bratteli diagrams of finite rank. Here we drop the assumption on
the rank of the diagrams and consider general Bratteli diagrams, simple and non-simple ones. Our approach in these papers is based on the following point of view: for a fixed Bratteli diagram $B$, we consider the set $O_B$ of all possible orderings $\omega$ on $B$ and study conditions on $B$ that would lead to the existence of continuous dynamics on $B$ (Vershik maps, in other terms). Orderings that admit Vershik maps are called perfect and form a subset $P_B$ of $O_B$. If $B$ is simple, then such orderings always exist (for example, the so-called proper orders, which have a unique maximal path and a unique minimal path). Here we focus on orderings with non-unique maximal (minimal) paths. It is worth mentioning that there exist Bratteli diagrams (even with a relatively simple structure) that do not admit perfect orderings at all, that is $P_B = \emptyset$ [Med06], [BKY12]. We recall the following crucial fact that emphasizes the importance of perfect orderings: the class of Bratteli diagrams with perfect orderings is in a one-to-one correspondence with the set of aperiodic homeomorphisms of a Cantor set. Moreover, it is well known that perfectly ordered simple Bratteli diagrams correspond to minimal homeomorphisms [HPS92] and perfectly ordered non-simple Bratteli diagrams correspond to non-minimal aperiodic homeomorphisms of a Cantor set [Med06].

Our motivation for this work is the following (rather vaguely formulated) question. Given a Bratteli diagram $B$, how can one extract the most essential information from the structure of $B$ that would be responsible for the existence of perfect orderings on $B$? On the way to solving this problem, we introduce and study the fruitful concepts of skeletons, correspondences, and associated directed graphs. To explain how these concepts appear on the scene, we start with an ordered Bratteli diagram $(B, \omega)$. In this case, we have the sets $X_{\max}$ and $X_{\min}$ of infinite maximal and minimal paths in the path space $X_B$, and also maximal and minimal edges in each set $r^{-1}(v)$, where $v$ is a vertex in $B$. In general, the closed sets $X_{\max}$ and $X_{\min}$ can have any cardinality. When they are singletons the Vershik map $\varphi_{\omega}$ always exists, and we call such diagrams properly ordered. For finite rank diagrams, when the number of vertices at each level is bounded, these sets are finite. Furthermore, if $\varphi_{\omega}$ exists, then it establishes a one-to-one map from $X_{\max}$ onto $X_{\min}$. What is described here is a prototype of the notion of a skeleton $F$ together with the notion of correspondence $\sigma$ between the extremal paths which is defined by the Vershik map sending maximal paths into minimal ones. In this paper, we are also interested in the inverse problem: given an unordered Bratteli diagram $B$, and complete information about sets of designated extremal paths and extremal edges, together with a correspondence between the maximal and minimal paths, can this partial information can be extended to a perfect order $\omega$?

Another important tool for the study of perfect orderings on Bratteli diagrams is the notion of the sequence of directed graphs $(H_n)$ that are associated to skeletons and correspondences (for definitions in Section 3). It turns out that paths in $(H_n)$
can be used to generate perfect orderings on $B$. Conversely, if an order is perfect, then its language is generated by paths in the graphs $(\mathcal{H}_n)$. We show how the notions of skeletons and associated graphs can work by characterizing Bratteli diagrams that support perfect orders. Also, we apply these concepts to give a new proof of a known result from [GPS95] and [Put89] about subgroups of the infinitesimal group; our proof explicitly identifies the infinitesimals whose existence is guaranteed using the given skeleton and correspondence.

The paper is organized as follows. In Section 2 we recall the basic notions and definitions related to Bratteli diagrams. Although this paper is a continuation of [BKY12], we find it useful to include this material in the present paper for the reader’s convenience. Again we stress that the telescoping procedure is one of the most useful tools that we repeatedly apply in the paper without detailed explanations. When an order $\omega$ is given on a Bratteli diagram $B = (V, E)$, we can define the notion of a language $L(B, \omega, n)$ whose alphabet is the set of vertices $V_n$ for any $n \geq 1$. We note that, for finite rank diagrams, the language does not depend on level $n$ but, in the general case, we have to work with each level separately.

Section 3 contains the main notions we elaborate in the paper. They are skeletons, correspondences, and associated graphs. If $(B, \omega)$ is an ordered Bratteli diagram, then, in order to see the skeleton $F$ generated by $\omega$, one removes all edges from $B$ that are not maximal or minimal. This skeleton contains two sets $X_{\text{max}}(F)$ and $X_{\text{min}}(F)$ of infinite maximal and minimal paths. Moreover, by telescoping $B$, we can assume that the sources of all maximal and minimal edges are in the vertices through which maximal and minimal paths flow. Then a correspondence $\sigma : X_{\text{max}}(F) \rightarrow X_{\text{min}}(F)$ establishes a homeomorphism between the sets of maximal and minimal paths. This means that if we would like to define a perfect order on an unordered Bratteli diagram, then the first thing we need to do is to determine a skeleton, consisting of prospective maximal and minimal paths and maximal and minimal edges, and define a correspondence between these sets of extreme paths. Having this done, we are able to define a sequence of associated graphs $(\mathcal{H}_n)$, an important ingredient in construction of perfect orderings. In the case of orderings supporting infinitely many maximal paths, we require the sets $X_{\text{max}}(F)$ and $X_{\text{min}}(F)$ to be of the correct topological form, and this is reflected in the definition of a skeleton. We note that it is the concept of correspondence that is more complicated than the simpler notion of a permutation that we used in the case of finite rank diagrams. The maps in the correspondence can be set valued, and this makes the presentation more technical. Nevertheless, it seems rather unnatural for a correspondence to be badly behaved - examples exist, but they seem contrived.

In Section 4 we prove our main result, Theorem 4.6, finding the necessary and sufficient conditions on a Bratteli diagram that would guarantee the existence of a perfect
ordering on it. These conditions are of two different kinds. If a perfect order exists on the diagram then firstly, the associated graphs of its skeleton and correspondence must satisfy the property of positive strong connectedness, and secondly, a careful analysis reveals that the entries of incidence matrices must satisfy the so called balance conditions. Once we have precisely formulated these conditions on an unordered Bratteli diagram, we prove they are sufficient, by giving an algorithm allowing us to define a perfect order (such an order is not unique, in general). It turns out that a perfect order is obtained when one travels along an Eulerian path in the associated graphs $\mathcal{H}_n$.

The last section contains a new proof of known results stated and proved in [GPS95] which, in turn, are based on the earlier paper [Put89]: namely that if $G$ is the dimension group of a simple Bratteli diagram $B$, such that $B$ supports a perfect order with exactly $j$ maximal paths and $j$ minimal paths, then the infinitesimal subgroup of $G$ contains a subgroup isomorphic to $\mathbb{Z}^{j-1}$. Our combinatorial proof uses extensively the machinery of skeletons and associated graphs that we have developed, as well as our characterization of diagrams that support perfect orderings in Theorem 4.6. We also show that our proof can be extended to a work for a class of Bratteli diagrams that have countably many extremal paths, and believe that an appropriate version of these results holds for dimension groups of aperiodic diagrams. Here we mention recent results of [Han13], where given a dimension group whose infinitesimal subgroup contains $\mathbb{Z}^k$, concrete (equal row and column sum) Bratteli diagram representations of these dimension groups are found. Some of the examples in [Han13] can be shown to satisfy the conditions of Theorem 4.6. It would be interesting to characterize the dimension groups of diagrams that support perfect orderings.

We end with a few remarks. We find it useful to include a number of examples in the text that will help the reader to understand the concepts and statements. Our examples are mainly of simple diagrams, but the constructs extend to aperiodic diagrams in a similar fashion. The words “order” and “ordering” are mostly used as synonyms, although we often use the former for a specific order, and the latter for an arbitrary order chosen from $O_B$.

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2 Definitions and notation

2.1 Fundamentals of Bratteli diagrams

In this section, we collect the notation and basic definitions that are used throughout the paper; these definitions are mostly standard, but we include them to fix our notation. More information about Bratteli diagrams can be found in the papers [HPS92], [GPS95] and [Dur10].

Definition 2.1. A Bratteli diagram is an infinite graph \( B = (V, E) \) such that the vertex set \( V = \bigcup_{i \geq 0} V_i \) and the edge set \( E = \bigcup_{i \geq 1} E_i \) are partitioned into disjoint subsets \( V_i \) and \( E_i \) such that

(i) \( V_0 = \{ v_0 \} \) is a single point;

(ii) \( V_i \) and \( E_i \) are finite sets;

(iii) there exist a range map \( r \) and a source map \( s \) from \( E \) to \( V \) such that \( r(E_i) = V_i \), \( s(E_i) = V_{i-1} \), and \( s^{-1}(v) \neq \emptyset, r^{-1}(v') \neq \emptyset \) for all \( v \in V \) and \( v' \in V \setminus V_0 \).

The pair \((V_i, E_i)\) or just \( V_i \) is called the \( i \)-th level of the diagram \( B \). A finite or infinite sequence of edges \((e_i : e_i \in E_i)\) such that \( r(e_i) = s(e_{i+1}) \) is called a finite or infinite path respectively. We write \( e(v, v') \) to denote a finite path \( e = (e_1, ..., e_k) \) such that \( s(e) = v (= s(e_1)) \) and \( r(e) = v' (= r(e_k)) \), and let \( E(v, v') \) denote the set of all such paths. For a Bratteli diagram \( B \), let \( X_B \) denote the set of infinite paths starting at the top vertex \( v_0 \). We endow \( X_B \) with the topology generated by cylinder sets \( U(e_k, ..., e_n) := \{ x \in X_B : x_i = e_i, \ i = k, ..., n \} \), where \((e_k, ..., e_n)\) is a finite path in \( B \) from level \( k \) to level \( n \). Then \( X_B \) is a 0-dimensional compact metric space with respect to this topology. We assume throughout the paper that \( X_B \) has no isolated points for all considered Bratteli diagrams \( B \). A Bratteli diagram \( B = (V, E) \) is called simple if for any level \( n \) there is \( m > n \) such that \( E(v, w) \neq \emptyset \) for all \( v \in V_n \) and \( w \in V_m \). If for some \( d \) we have \( |V_n| \leq d \) for all \( n \geq 1 \), then \( B \) is called of finite rank. If \( B \) has finite rank, and \( d \) is the smallest integer such that there are infinitely many levels in \( B \) with exactly \( d \) vertices, then we then say \( B \) has rank \( d \). In this article we will consider general Bratteli diagrams, which are not necessarily of finite rank.

For a Bratteli diagram \( B \), the tail (cofinal) equivalence relation \( E \) on the path space \( X_B \) is defined as \( x \sim y \) if \( x_n = y_n \) for all \( n \) sufficiently large where \( x = (x_n), y = (y_n) \). Let \( X_{\text{per}} = \{ x \in X_B : |[x]| < \infty \} \). By definition, we see that \( X_{\text{per}} = \{ x \in X_B : \exists n > 0 (|r^{-1}(r(x_i))| = 1 \forall i \geq n) \} \). A Bratteli diagram \( B = (V, E) \) is called aperiodic if \( X_{\text{per}} = \emptyset \), i.e., every \( E \)-orbit is countably infinite. In this work, we study only aperiodic Bratteli diagrams.

We will constantly use the telescoping procedure for a Bratteli diagram. Roughly speaking, in order to telescope a Bratteli diagram, one takes a subsequence of levels...
\{n_k\} and considers the set of all finite paths between the new consecutive levels \{n_k\} and \{n_{k+1}\} as new edges. In particular, a Bratteli diagram \(B\) has rank \(d\) if and only if there is a telescoping \(B'\) of \(B\) such that \(B'\) has exactly \(d\) vertices at each level, and, for each \(d' < d\), there are at most finitely many levels of \(B\) with \(d'\) vertices. More information about the telescoping procedure can be found in many papers on Bratteli diagrams, for example, in \cite{GPS95}.

Given a Bratteli diagram \(B = (V, E)\), the incidence matrix \(F_n = (f_{v,w}^{(n)}), \ n \geq 1\), is a \(|V_{n+1}| \times |V_n|\) matrix whose entries \(f_{v,w}^{(n)}\) are equal to the number of edges between the vertices \(v \in V_{n+1}\) and \(w \in V_n\), i.e.,

\[
f_{v,w}^{(n)} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.
\]

For simplicity, we can assume that \(F_0 = (1, ..., 1)^T\) although this assumption is not crucial. Next we define a family of aperiodic diagrams that have an incidence matrix structure that is useful for our purposes.

**Definition 2.2.** We define the family \(\mathcal{A}\) of Bratteli diagrams, all of whose incidence matrices are of the form

\[
F_n := \begin{pmatrix}
A_n^{(1)} & 0 & \cdots & 0 & 0 \\
0 & A_n^{(2)} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_n^{(k)} & 0 \\
B_n^{(1)} & B_n^{(2)} & \cdots & B_n^{(k)} & C_n \end{pmatrix}
\] (1)

where

1. for \(1 \leq i \leq k\), there is a sequence \((d_i^{(n)})\) such that \(A_n^{(i)}\) is a \(d_i^{(n+1)} \times d_i^{(n)}\) matrix,
2. all matrices \(A_n^{(i)}, B_n^{(i)}\) and \(C_n\) are strictly positive, and
3. there is a sequence \((d^{(n)})\) such that \(C_n\) is a \(d^{(n+1)} \times d^{(n)}\) matrix.

### 2.2 Orders on a Bratteli diagram

Suppose that a linear order “\(>\)” is defined on every set \(r^{-1}(v), v \in \bigcup_{n \geq 1} V_n\). For each \(k, l\) with \(k < l\), this linear order defines the lexicographic order (denoted also by \(>)\) on the set \(E_{k+1} \circ \ldots \circ E_l(v) := \{(e_{k+1}, \ldots, e_l) : e_i \in E_i, r(e_i) = s(e_{i+1}), i = k + 1, \ldots, l - 1, r(e_l) = v\}\) of finite paths from vertices of levels \(V_k\) to \(v \in V_l\). For, \((e_{k+1}, \ldots, e_l) > (f_{k+1}, \ldots, f_l)\) if and only if there is an \(i\) with \(k + 1 \leq i \leq l\), such that \(e_j = f_j\) for \(i < j \leq l\) and \(e_i > f_i\). It follows that any two paths in \(X_B\) that are eventually equal are comparable and we use \(\omega\) to denote the corresponding partial order on \(X_B\).
Let $B$ be a Bratteli diagram; fix an order $\omega$ on $B$. We call a finite or infinite path $e = (e_i)$ maximal (minimal) if every $e_i$ is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$. Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ the sets of all maximal and minimal infinite paths from $X_B$, respectively. For a finite rank Bratteli diagram $B$, the sets $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ are always finite for any $\omega$, and if $B$ has rank $d$, then each of them have at most $d$ elements. An ordered Bratteli diagram $(B, \omega)$ is called properly ordered if the sets $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ are singletons. Finally, if $B'$ is a telescoping of $B$, and $\omega$ is an order on $B$, then $\omega$ naturally defines the (telescoped) order $\omega' = L(\omega)$ on $B'$.

Let $(B, \omega) = (V, E, \omega)$ be an ordered Bratteli diagram. We can define the Vershik map $\varphi = \varphi_\omega : X_B \setminus X_{\text{max}}(\omega) \to X_B \setminus X_{\text{min}}(\omega)$. Namely, if an infinite path $x = (x_1, x_2, \ldots)$ is not in $X_{\text{max}}(\omega)$, then $\varphi(x_1, x_2, \ldots) = (x_1^0, x_1^{0-1}, x_k, x_{k+1}, x_{k+2}, \ldots)$, where $k = \min\{n \geq 1 : x_n$ is not maximal}, $(\overline{x}_k)$ is the successor of $x_k$ in $r^{-1}(r(x_k))$, and $(x_1^0, \ldots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x}_k))$. If $\varphi_\omega$ can be extended to a homeomorphism $\varphi_\omega : X_B \to X_B$, then we say that $\omega$ is perfect. The set of all perfect orders on $B$ is denoted by $\mathcal{P}_B$. A proper order $\omega$ is obviously perfect. For the most natural orders, the sets $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ have empty interior. If $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ have empty interior and $\omega$ is perfect, then $\varphi_\omega$ can be extended to $X_B$, and this can be done in a unique way. A Bratteli diagram $B = (V, E)$ is called regular if for any ordering $\omega \in \mathcal{O}_B$ the sets $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ have empty interior. For example, all finite rank Bratteli diagrams are regular. In this article we assume that all diagrams are regular. Hence, given $(B, \omega)$ with $\omega \in \mathcal{P}_B$, the uniquely defined $\varphi_\omega$ is a homeomorphism of $X_B$, and we obtain the so called Bratteli-Vershik system $(X_B, \varphi_\omega)$ (called also an adic system). If $\omega$ is proper, we call the adic system $(X_B, \varphi_\omega)$ a proper adic system.

For a diagram $B$, let $\mathcal{O}_B$ denote the set of all orders on $B$ and $\mathcal{O}_B(j)$ denote the subset of all these orders which have $j$ minimal and $j$ maximal paths. Recall that $\mathcal{P}_B$ is the subset of all orders $\omega$ such that the corresponding adic map $\varphi_\omega$ is a homeomorphism.

### 2.3 The language of an ordered Bratteli diagram

If $V$ is a finite alphabet, let $V^+$ denote the set of nonempty words over $V$. We use the notation $W' \subseteq W$ to indicate that $W'$ is a subword of $W$. If $W_1, W_2, \ldots, W_n$, are words, then we let $\prod_{i=1}^n W_i$ refer to their concatenation.

Let $\omega$ be an order on a Bratteli diagram $B$. Fix a vertex $v \in V_n$ and some level $m < n$, consider the set $E(V_m, v) = \bigcup_{v' \in V_m} E(v', v)$ of all finite paths between vertices of level $m$ and $v$. This set can be ordered by $\omega$: $E(V_m, v) = \{e_1, \ldots, e_p\}$ where $e_i < e_{i+1}$ for $1 \leq i \leq p - 1$. Define the word $w(v, m, n) := s(e_1)s(e_2)\ldots s(e_p)$ over the alphabet.
If \( W = v_1 \ldots v_r \in V^+_n \), let \( w(W, n-1, n) := \prod_{i=1}^r w(v_i, n-1, n) \).

**Definition 2.3.** The *level-* \( n \) *language* \( \mathcal{L}(B, \omega, n) \) *of* \((B, \omega)\) *is*

\[
\mathcal{L}(B, \omega, n) := \{ W : W \subset w(v, n, N), \text{ for some } v \in V_N, N > n \}.
\]

Note that if \( B \) is simple and if \( W \subset w(v, n, N_0) \) for some \( v \in V_{N_0} \) where \( N_0 > n \), then \( W \subset w(v, n, N) \) for any \( N > N_0 \) and any \( v \in V_N \). If \( V_n = V \) for each \( n \), then the *language* \( \mathcal{L}(B, \omega) \) is

\[
\mathcal{L}(B, \omega) := \limsup_n \mathcal{L}(B, \omega, n).
\]

Continuing to assume that \( V_n = V \) for all \( n \), if a maximal (minimal) path \( M (m) \) goes through the same vertex \( v_M (v_m) \) at each level of \( B \), we will call this path *vertical.*

The following proposition characterizes when \( \omega \) is a perfect order on a finite rank Bratteli diagram, and was proved in \([BKY12, Proposition 3.2, Lemma 3.3]\) for finite rank diagrams.

**Proposition 2.4.** Let \((B, \omega)\) be an ordered Bratteli diagram.

1. Suppose \( |V_n| = d \) for each \( n \in \mathbb{N} \), and that the \( \omega \)-maximal and \( \omega \)-minimal paths \( M_1, \ldots, M_k \) and \( m_1, \ldots, m_{k'} \) are vertical passing through the vertices \( v_{M_1}, \ldots, v_{M_k} \) and \( v_{m_1}, \ldots, v_{m_{k'}} \) respectively. Then \( \omega \) is perfect if and only if

   - \( a \) \( k = k' \),
   - \( b \) there is a permutation \( \sigma \) of \( \{1, \ldots, k\} \) such that for each \( i \in \{1, \ldots, k\} \), \( v_{M_i} v_{m_j} \in \mathcal{L}(B, \omega) \) if and only if \( j = \sigma(i) \).

2. Let \( B' \) be a telescoping of \( B \). Then an order \( \omega \in \mathcal{P}_B \) if and only if the corresponding lexicographic order \( \omega' = L(\omega) \in \mathcal{P}_{B'} \).

We end this section with an example that answers negatively the following question that is related to statement 2 of Proposition 2.4. Let \( B \) be a Bratteli diagram and \( B' \) a telescoping of \( B \). Is it true that any perfect order on \( B' \) is obtained by telescoping of a perfect order on \( B \)?

**Example 2.5.** In this example, we define a stationary Bratteli diagram \( B \) such that for a telescopied diagram \( B' \) there is a perfect order \( \omega' \in \mathcal{P}_{B'} \) satisfying the condition \( \omega' \neq L(\omega) \) for any perfect order \( \omega \) on \( B \).

Let \( B \) be a stationary Bratteli diagram defined on the set of four vertices \( \{a, b, c, d\} \) by the incidence matrix

\[
F = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 \\
1 & 1 & 1 & 2 \\
\end{pmatrix}.
\]
Let $B'$ be the telescoped diagram with the incidence matrix $F' = F^2$. In order to define a perfect order $\omega'$, we let $w(a, n, n + 1) = (adbc)^6a$, $w(b, n, n + 1) = (bcad)^6b$, $w(c, n, n + 1) = (cadb)^6c$, and, finally, for $r^{-1}(d)$ we set $w(d, n, n + 1) = bca$. This appearance of $d^7$ prevents $\omega'$ from being a lexicographical order on $B'$ generated by any choice of $\omega$ on $B$. On the other hand, using Proposition 2.4, it is not hard to verify that $\omega' \in \mathcal{P}_{B'}$.

3 Skeletons on Bratteli diagrams

3.1 Skeletons and associated graphs on finite rank Bratteli diagrams

We first recall a few definitions and facts related to finite rank Bratteli diagrams from [BKY12] in order to clarify our constructions below. If $\omega$ is an order on a finite rank diagram $B$ then we can assume, by telescoping if necessary, that all extremal paths are vertical. Let $\tilde{V}$ and $V$ be the sets of maximal and minimal vertices respectively – these are the vertices through which a maximal or minimal path flows. By telescoping if necessary, we can also assume that

1. $|V_n| = \text{rank}(B)$ is constant in $n$, and
2. given a vertex $v \in V$, the maximal (minimal) edge in $E_n$ with range $v$ has the same source in $\tilde{V}$ ($V$), independent of $n$.

In other words, we can assume that $V = V_n$ and the structure of the sets of maximal and minimal edges is stationary.

For each $v \in \bigcup_{n=2}^{\infty} V_n$, let $\tilde{e}_v$ and $e_v$ represent the maximal and minimal edges respectively with range $v$. Based on the remark above, we have $s(\tilde{e}_v) \in \tilde{V}_n$ and $s(e_v) \in V_n$ for any $v \in V_{n+1}$, $n \geq 1$. Given $(B, \omega)$, we call the set $\mathcal{F}_\omega = (\tilde{V}, V, \{\tilde{e}_v, e_v : v \in V_n, n \geq 2\})$ the skeleton associated to $\omega$. If $\omega$ is a perfect order on $B$, it follows that $|\tilde{V}| = |\tilde{V}|$, and if $\sigma : \tilde{V} \to V$ is the permutation given by Proposition 2.4, we call $\sigma$ the accompanying permutation. If an ordered Bratteli diagram $(B, \omega)$ has been telescoped so that it has a skeleton, then we say that $(B, \omega)$ is well telescoped.

Conversely, given an unordered finite rank diagram $B$, we can define a skeleton $\mathcal{F} = (\tilde{V}, V, \{\tilde{e}_v, e_v : v \in V_n, n \geq 2\})$ where $\tilde{V}$ and $V$ are subsets of $V_n$ of the same cardinality (more detailed definition can be found in [BKY12]). Arbitrarily choosing a bijection $\sigma : \tilde{V} \to V$, we can consider the set of orders on $B$ which have $\mathcal{F}$ as skeleton and $\sigma$ as accompanying permutation.

Let

$$W_{\tilde{v}} := \{v \in V_{n+1} : s(\tilde{e}_v) = \tilde{v}\}$$

and

$$W'_{\overline{v}} := \{v \in V_{n+1} : s(e_v) = \overline{v}\}.$$
These sets, as $\tilde{v} \in \tilde{V}$ and $\pi \in \overline{V}$ vary, form two partitions $W$ and $W'$ of $V_n$ for each $n$. Recall that, due to the finiteness of the diagram’s rank, we can assume that, for any vertex $v \in V_n$, the sources of $\tilde{v}_v$ and $\pi_v$ do not depend on $n$; nor do the partitions $\{W_{\tilde{v}} : \tilde{v} \in \tilde{V}\}$ and $\{W'_{\pi} : \pi \in \overline{V}\}$. We use the notation $[\pi, \tilde{v}] := W'_{\pi} \cap W_{\tilde{v}}$.

**Definition 3.1.** Let $F$ be a skeleton on the finite rank $B$ with accompanying permutation $\sigma$. Let $H = (T, P)$ be the directed graph where the set $T$ of vertices of $H$ consists of partition elements $[\pi, \tilde{v}]$ of $W' \cap W$, and where there is an edge in $P$ from $[\pi, \tilde{v}]$ to $[\pi', \tilde{v}']$ if and only if $\pi' = \sigma(\tilde{v})$. We call $H$ the directed graph associated to $(F, \sigma)$.

**Example 3.2.** Suppose that $\tilde{V} = \overline{V} = \{a, b\}$, $\sigma(a) = a$, $\sigma(b) = b$, and the set of vertices of $H$ is $[a, a], [a, b], [b, a], [b, b]$. Then $H$ is illustrated in Figure 1. Note that we do not specify a skeleton here. In general, it is possible that for some skeletons one of the vertices $[b, a]$ or $[a, b]$ be degenerate, for example, if $W_{\tilde{a}} \cap W_{\tilde{b}} = \emptyset$, then the vertex $[a, b]$ is not present in $H$. The only other graph $H$ is the one associated with the permutation $\sigma(a) = b$, $\sigma(b) = a$; in this case $H$ is identical to that of Figure 1 except that the vertices $\{[a, a], [b, b], [a, b], [b, a]\}$ of the new graph are relabelled $\{[b, a], [a, b], [b, b], [a, a]\}$ respectively.

![Figure 1: The associated graph $H$ for $\omega \in P_B(2)$, $\sigma(a) = a$, $\sigma(b) = b$.](image)

Suppose that $B$ has skeleton $F$ and accompanying permutation $\sigma$. Then any path in $H$ corresponds to a family of words in $V^+$: for if $p_1p_2\ldots p_k$ is a path in $H$ where $p_i$ has source $[\overline{v}_i, \tilde{v}_i]$, and for each $i$, $v_i$ is any vertex in $[\overline{v}_i, \tilde{v}_i]$, then the path $p_1p_2\ldots p_k$ corresponds to the word $v_1v_2\ldots v_k$ (and many such words exist). Conversely, if the word $v_1v_2\ldots v_k$ is such that $v_i \in [\overline{v}_i, \tilde{v}_i]$ for each $i$, and $[\overline{v}_1, \tilde{v}_1][\overline{v}_2, \tilde{v}_2]\ldots [\overline{v}_k, \tilde{v}_k]$ is a path in $H$, then we say that $v_1v_2\ldots v_k$ corresponds to a valid path in $H$.

The following was shown in [BKY12, Remark 3.7, Lemma 3.8].

**Lemma 3.3.** Let $B$ be a finite rank Bratteli diagram, $F$ be a skeleton on $B$ and $\sigma : \tilde{V} \to \overline{V}$ be a bijection. Let $H$ be the associated directed graph. Suppose there exists an ordering $\omega$ on $B$ with skeleton and accompanying permutation $(F, \sigma)$, such that each
word in \( \mathcal{L}(B, \omega) \) corresponds to a valid path in \( \mathcal{H} \). Then \( \omega \) is perfect. Conversely, if \( \omega \) is a perfect ordering with skeleton and accompanying permutation \( (F, \sigma) \), then every word in \( \mathcal{L}(B, \omega) \) corresponds to a valid path in \( \mathcal{H} \).

### 3.2 Skeletons, associated graphs, and correspondences on Bratteli diagrams

If \( B \) is not of finite rank, the notion of a skeleton can be generalized, although the notation is more technical. Suppose that \( \omega \) is an order on \( B \). To each maximal path \( M \) and minimal path \( m \) we associate the sequences \( (v_n(M)) \) and \( (v_n(m)) \) of vertices that \( M \) and \( m \) pass through. For each \( n \), let \( \bar{V}_n := \{ v \in V_n : v = v_n(M) \text{ for some maximal path } M \} \); we call vertices in \( \bar{V}_n \) maximal vertices. In other words, for each \( \bar{v} \in \bar{V}_n \), there is at least one infinite maximal path passing through \( \bar{v} \). Similarly we can define \( \bar{V}_n \), the set of minimal vertices in \( V_n \).

**Proposition 3.4.** Let \( (B, \omega) \) be an ordered Bratteli diagram. Then there exists a telescoping \( (B', \omega') = ((V', E'), \omega') \) of \( (B, \omega) \) to a sequence of levels \( (n_k) \), such that, for every vertex \( v \in V' \), any maximal edge \( \bar{e}_v \in E'_k \) has source in \( \bar{V}_{k-1} \) and any minimal edge \( \bar{e}_v \) has source in \( \bar{V}_{k-1} \).

**Proof.** We use the idea of the proof of Proposition 2.8 from [HPS92]. Take \( n_1 = 1 \), write \( V_1 = \bar{V}_1 \cup (V_1 \setminus \bar{V}_1) \). Take a vertex \( v \in V_1 \setminus \bar{V}_1 \) and consider all edges \( e \) from \( E_{\max} \) such that \( s(e) = v \). We say that \( e \) is extendable if there is an edge \( e' \in E_{\max} \) such that \( r(e) = s(e') \). Let \( E_v \) be the set of all finite paths from \( E_{\max} \) consisting of extendable edges. Clearly, \( E_v \) is a finite set. Therefore we can find \( n_2 \) such that if \( f \) is a maximal finite path with source in \( V_1 \setminus \bar{V}_1 \), then \( r(f) \in V_n \) with \( n < n_2 \). Thus, all maximal paths with source in \( V_1 \) and range in \( V_{n_2} \) must have source in \( \bar{V}_1 \). We describe only the next step, the rest following by induction. Write \( V_{n_2} = \bar{V}_{n_2} \cup (V_{n_2} \setminus \bar{V}_{n_2}) \). Find an \( n_3 \) such that if \( f \) is a maximal finite path with source in \( V_{n_2} \setminus \bar{V}_{n_2} \), then \( r(f) \in V_n \) with \( n < n_3 \). Therefore all maximal paths with source in \( V_{n_2} \) and range in \( V_{n_3} \) must have source in \( \bar{V}_{n_2} \). Continue, and telescope \((B, \omega)\) via levels \( (n_k) \). Since maximal edges in \( E_{k_{\max}} \) in the telescoped diagram \((B', \omega')\) correspond to maximal paths between \( V_{n_{k-1}} \) and \( V_{n_k} \) in \( B \), the result follows. An identical argument yields the result for minimal edges. \(\square\)

Proposition 3.4 allows us to assert that, given an order \( \omega \) on \( B \), by telescoping if necessary, we can assume that each maximal edge \( \bar{e}_v \in E_n \) has source in \( \bar{V}_{n-1} \), similarly for each minimal edge. Given an unordered Bratteli diagram \( B \), we generalize the notion of a skeleton of a finite rank diagram in the following way.

**Definition 3.5.** Let \( B \) be a Bratteli diagram. Suppose that we have two closed, nowhere dense sets of infinite paths \( \{M_\alpha : \alpha \in I\} \), and \( \{m_\beta : \beta \in J\}, |I| = |J| \), that
satisfy the following property: if \( v_N(M_\alpha) = v_N(M'_\alpha) \), then \( v_n(M_\alpha) = v_n(M'_\alpha) \) for each \( n < N \). In other words, if two paths \( M_\alpha \) and \( M'_\alpha \) have the same range \( \tilde{v} \) at level \( N \), then their initial segments up to level \( N \) coincide. A similar condition holds for paths \( m_\beta \) and \( m_\beta' \).

Let \( n \in \mathbb{N} \). Define

\[
\tilde{V}_n := \{ \tilde{v} : \tilde{v} = v_n(M_\alpha) \text{ for some } \alpha \in I \} \quad \text{and} \quad \nabla_n := \{ \nabla : \nabla = v_n(m_\beta) \text{ for some } \beta \in J \}.
\]

Let

\[
\{ \tilde{e}_v : v \in V_{n+1} \} \quad \text{and} \quad \{ \nabla_v : v \in V_{n+1} \}
\]

be sets of edges such that \( r(\tilde{e}_v) = r(\nabla_v) = v \), \( s(\tilde{e}_v) \in \tilde{V}_n \) and \( s(\nabla_v) \in \nabla_n \). We assume that, for any \( n \), if \( v \in \tilde{V}_n \cap \nabla_n \), then \( \tilde{e}_v \neq \nabla_v \). We also assume that if \( n \in \mathbb{N} \), \( \alpha \in I \) and \( \beta \in J \), then \( M_\alpha \in U(\tilde{e}_v) \) whenever \( v = v_n(M_\alpha) \) and \( m_\beta \in U(\nabla_v) \) whenever \( v = v_n(m_\beta) \).

Then we call \( \mathcal{F} = (\tilde{V}_{n-1}, \nabla_{n-1}, \{ \tilde{e}_v, \nabla_v : v \in V_n \} : n \geq 2) \) the skeleton associated to \( \{ M_\alpha : \alpha \in I \} \) and \( \{ m_\beta : \beta \in J \} \).

Vertices in the sets \( \tilde{V}_n \) and \( \nabla_n \) are called maximal and minimal vertices respectively. Paths \( M_\alpha \), \( \alpha \in I \), are called maximal and form the set \( X_{\text{max}}(\mathcal{F}) \), and paths \( m_\beta \), \( \beta \in J \), are called minimal and form the set \( X_{\text{min}}(\mathcal{F}) \).

**Remark 3.6.** Note that if we are given an order \( \omega \) on \( B \), and \( |X_{\text{max}}(\omega)| = |X_{\text{min}}(\omega)| \), then we can define the skeleton generated by \( \omega \), by letting \( \{ M_\alpha : \alpha \in I \} = X_{\text{max}}(\omega) \), \( \{ m_\beta : \beta \in J \} = X_{\text{min}}(\omega) \), and using Proposition 3.4 to generate the appropriate telescoping and set of extremal edges \( \{ \tilde{e}_v, \nabla_v : v \in V_n, n \geq 2 \} \). It is clear that if \( \mathcal{F} \) is defined by such an order \( \omega \), then \( X_{\text{max}}(\mathcal{F}) = X_{\text{max}}(\omega) \) and \( X_{\text{min}}(\mathcal{F}) = X_{\text{min}}(\omega) \). We will not be concerned with orders \( \omega \) such that \( |X_{\text{max}}(\omega)| \neq |X_{\text{min}}(\omega)| \), as our aim is to characterize perfect orders and such a condition would prevent an order from being perfect. We also note that the requirement that \( X_{\text{max}}(\mathcal{F}) \) and \( X_{\text{min}}(\mathcal{F}) \) be closed is natural: after all, we are introducing skeletons to build orders, and in that case the latter sets must be closed. The regularity of \( B \) leads to the requirement that the sets \( X_{\text{max}}(\omega) \) and \( X_{\text{min}}(\omega) \) are nowhere dense.

**Example 3.7.** Suppose that \( V_0 = \{ v_0 \} \), and for \( n \geq 1 \), \( V_n = \tilde{V}_n = \nabla_n = \{ v_1, \ldots, v_n \} \), and all incidence matrix entries are at least two. Suppose also that for \( v_i \in V_{n+1} \), we choose \( \tilde{e}_i \neq \nabla_i \) and define

\[
s(\tilde{e}_i) = \begin{cases} v_i & \text{if } i \neq n + 1 \\ v_n & \text{if } i = n + 1 \end{cases}, \quad \text{and} \quad s(\nabla_i) = \begin{cases} v_i & \text{if } i \neq n + 1 \\ v_1 & \text{if } i = n + 1 \end{cases}.
\]

In Figure 2 we have drawn (only) the extremal edges in \( B \), with dashed lines representing maximal edges and solid lines representing minimal edges. Consider the sets of infinite paths \( \{ M_\alpha : \alpha \in \mathbb{N} \cup \{ \infty \} \} \) whose edges consist of the identified maximal edges, so
that $M_1$ passes vertically through the vertex $v_1$ at all levels, and for $i > 1$, $M_i$ passes through vertices $v_1, v_2, \ldots, v_{i-1}, v_i$, and then goes down vertically through $v_i$. Finally, $M_\infty$ passes through vertices $v_1, v_2, v_3, \ldots$. Similarly consider the set of infinite paths \( \{m_\beta : \beta \in \mathbb{N}\} \) whose edges consist of the identified minimal edges, so that $m_1$ passes vertically through $v_1$, and for $i > 1$, $m_i$ passes through $v_1$ exactly $i-1$ times, then jumps to $v_i$ and goes down vertically through $v_i$. It is straightforward to verify that the sets \( \{M_\alpha : \alpha \in \mathbb{N} \cup \{\infty\}\} \) and \( \{m_\beta : \beta \in \mathbb{N}\} \) are both countable, closed, and nowhere dense, and that $\mathcal{F} = (\tilde{V}_{n-1}, \overline{V}_{n-1}, \{\tilde{e}_v, \overline{e}_v : v \in V_n\} : n \geq 2)$ is the skeleton associated to \( \{M_\alpha : \alpha \in \mathbb{N} \cup \{\infty\}\} \) and \( \{m_\beta : \beta \in \mathbb{N}\} \).

**Example 3.8.** Suppose that for $n \geq 1$, $V_n = \tilde{V}_n = \overline{V}_n = \{v_1, \ldots, v_{2^n}\}$, and all incidence matrix entries are at least two. We label each vertex $v_i \in V_n$ using a binary string $(x_1, \ldots, x_n)$ of length $n$ which denotes $i$'s binary expansion, starting with the least significant digit. For example, we label vertex $v_5 \in V_4$ with the string $(1010)$. Suppose that for $v_i \in V_{n+1}$,

\[
(s(\tilde{e}_{(x_1, \ldots, x_{n+1})})) = s(\overline{e}_{(x_1, \ldots, x_{n+1})}) = (x_1, \ldots, x_n).
\]

In this case the sets \( \{M_\alpha : \alpha \in \{0,1\}^\mathbb{N}\} \) and \( \{m_\alpha : \alpha \in \{0,1\}^\mathbb{N}\} \) are uncountable.

Note that the previous examples illustrate the fact that when defining a skeleton, we do not need complete information about $B$. In particular, at this point, we need to know very little about the transition matrices $(F_n)$ of $B$. The skeleton $\mathcal{F}$ is simply a constrained set of choices for all extremal edges when building an order. Next we discuss the conditions that allow us to complete the construction of an order so that it is perfect. In particular, we will now discuss a way of building a homeomorphism $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$ which is amenable to being extended to an adic (Vershik) map.

Suppose that $\omega$ is an order on $B$. Let $\sigma_n : \tilde{V}_n \to 2^{\overline{V}_n}$ be defined by $\tau \in \sigma_n(\tilde{v})$ if and only if $\tilde{e}_\tau \in \mathcal{L}(B, \omega, n)$. If $\omega$ is perfect, then for any sequence $(\tilde{v}_n) = (v_n(M_\alpha))$, there is a unique sequence $(\overline{\tau}_n) = (v_n(m_\beta))$ with $\overline{\tau}_n \in \sigma_n(\tilde{v}_n)$ for each $n$. If a perfect order $\omega$ has a finite number of extremal paths, we can say more. In that case, for all large enough $n$, and all maximal $\tilde{v} \in \tilde{V}_n$, $\sigma_n(\tilde{v})$ is an element, not a subset, of $\overline{V}_n$ - i.e. each maximal vertex can only be followed by a unique minimal vertex in $\mathcal{L}(B, \omega, n)$. One can then say that for all large $n$, $|\tilde{V}_n| = |\overline{V}_n|$ and $\sigma_n : \tilde{V}_n \to \overline{V}_n$ is a bijection. In the case of finite rank, the sets $\tilde{V}_n$ and $\overline{V}_n$, and the maps $\sigma_n : \tilde{V}_n \to \overline{V}_n$ could be taken to be equal for all $n$, and in that case we called $\sigma := (\sigma_n)$ a permutation.

In the general case of a perfect order with infinitely many extremal paths, though, this fact - that for any sequence $(\tilde{v}_n) = (v_n(M_\alpha))$, there is a unique sequence $(\overline{\tau}_n) = (v_n(m_\beta))$ with $\overline{\tau}_n \in \sigma_n(\tilde{v}_n)$ for each $n$ - does not generally imply that $(\tilde{v}_n) = (v_n(M_\alpha))$
Figure 2: The minimal and maximal edge structure for Example 3.7: solid lines are minimal edges, dashed lines are maximal.
is eventually a sequence of singletons. The main obstacle is that one can have pairs of distinct maximal paths that agree on an arbitrarily large initial segment. For, suppose that $\varphi_\omega(M) = m$, and assume that $M'$ is another maximal path that coincides with $M$ for the first $n$ segments till the vertex $v_n$. By continuity of $\varphi_\omega$, the minimal path $m' = \varphi_\omega(M')$ must be close to $m$, but it can be that $\overline{v}_n = \overline{v}_n(m) \neq \overline{v}_n(m') = \overline{w}_n$. So, we see that not only $\overline{v}_n v_n \in L(B, \omega, n)$ but also $\overline{v}_n w_n \in L(B, \omega, n)$, that is $\{\overline{v}_n, \overline{w}_n\} \in \sigma_n(\overline{v}_n)$. One can build such orders on any Bratteli diagram: see Example 3.

We use these observations to make the following definition, which generalizes the concept of a permutation for finite rank diagrams. Some notation: if $\sigma : \overline{V} \to 2V$ and $v \in V$, we define $\sigma^{-1}(v) := \{\overline{v} : \overline{v} \in \sigma(v)\}$.

**Definition 3.9.** Let $F$ be a skeleton for an unordered Bratteli diagram $B$. Suppose that $\sigma = (\sigma_n)_n$ is a sequence of maps where for each $n$, $\sigma_n : \overline{V}_n \to 2V$, such that for each $n$, $\bigcup_{\overline{v} \in \overline{V}_n} \sigma_n(\overline{v}) = \overline{V}_n$, and

1. $\sigma$ is composition consistent: let $M(n, N, \overline{v})$ and $m(n, N, \overline{v})$ denote the maximal and minimal paths from level $n$ to level $N > n$ with range $\overline{v}$ and $\overline{m}$ respectively. If $\overline{v} \in \sigma_N(\overline{v})$, then $s(m(n, N, \overline{v})) \in \sigma_n(s(M(n, N, \overline{v})))$,

2. for any $M \in X_{\max}(F)$, there is a unique $m \in X_{\min}(F)$ with $(v_n(m))_n \in \prod_n \sigma_n(v_n(M))$,

3. for any $m \in X_{\min}(F)$, there is a unique $M \in X_{\max}(F)$ with $(v_n(M))_n \in \prod_n \sigma_n^{-1}(v_n(m))$, and

4. the bijection $\sigma : X_{\max}(F) \to X_{\min}(F)$ defined using properties 2 and 3 is a homeomorphism.

Then we say that $\sigma = (\sigma_n)_n$ is a correspondence associated to $F$.

**Example 3.10.** We continue with the skeleton defined in Example 3.7 and Figure 2. Let $\sigma$ be defined by

$$\sigma_n(v_i) = \begin{cases} \{v_{i+1}\} & \text{if } 1 \leq i \leq n - 1 \\ \{v_1\} & \text{if } i = n \end{cases}$$

for each $n \geq 1$; then one can verify that $\sigma$ is composition consistent. Since each $\sigma_n : \overline{V}_n \to \overline{V}_n$ is in fact a point map, this means that items (2) and (3) of Definition 3.9 are satisfied. The homeomorphism $\sigma : X_{\max}(F) \to X_{\min}(F)$ satisfies $\sigma(M_\infty) = m_1$, and for $i \geq 1$, $\sigma(M_i) = m_{i+1}$.

**Example 3.11.** We continue with the skeleton defined in Example 3.8. Let ‘+1’ denote addition with carry, so that for example, $(1010) + 1 = (0110)$. Let $\sigma = (\sigma_n)_n$ be
defined by
\[
\sigma_n((x_1, \ldots, x_n)) = \begin{cases} 
(x_1, \ldots, x_n) + 1 & \text{if } (x_1, \ldots, x_n) \neq (1, \ldots, 1) \\
(0, \ldots, 0) & \text{if } (x_1, \ldots, x_n) = (1, \ldots, 1)
\end{cases}
\]
for each \(n \geq 1\); then one can verify that \(\sigma\) is composition consistent. Since each \(\sigma_n : \tilde{V}_n \to V_n\) is in fact a point map, this means that items (2) and (3) of Definition 3.9 are satisfied. Note that the bijection \(\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})\) satisfies \(\sigma(M_{111\ldots}) = m_{000\ldots}\), and \(\sigma(M_{x_1x_2\ldots}) = m_{y_1y_2\ldots}\), where \((y_1y_2\ldots) = (x_1x_2\ldots) + (100\ldots)\); in other words, \(\sigma\) is thus the odometer map.

**Example 3.12.** It seems to be difficult to find examples of skeletons and accompanying correspondences where the maps \(\sigma_n\) are not eventually point maps. If both \(v_N(m)\) and \(v_N(m')\) both belong to \(\sigma_N(v_N(M))\), the composition consistency condition forces \(v_n(m)\) and \(v_n(m')\) to belong to \(\sigma_n(v_n(M))\) for \(n < N\), making it hard for points (2) and (3) of the definition of a correspondence to be satisfied. Here is one example, illustrated in Figure 3. The vertex structure of this diagram is as in Example 3.7. We define, for each \(n\), \(\sigma_n(v_1) = \{v_2, v_3\}\), \(\sigma_n(v_i) = v_{i+1}\) for \(i = 2, \ldots, n - 1\), and \(\sigma_n(v_n) = v_1\). Then \((\sigma_n)_n\) defines a correspondence, with \(\sigma(M_n) = m_{n-1}\) for \(n \geq 1\), and \(\sigma(M_0) = m_\infty\).

In the case where \(\mathcal{F}\) is the skeleton associated to an ordered diagram \((B, \omega)\), we will always take \(\sigma_n\) to be that defined by \(\mathcal{L}(B, \omega, n)\), as discussed in the paragraph preceding Definition 3.9. Namely, given an order \(\omega\) with skeleton \(\mathcal{F}\), we define \(\tilde{\omega} \in \sigma_n(\tilde{\mathcal{F}})\) if and only if \(\tilde{\omega} \in \mathcal{L}(B, \omega, n)\). Whether or not \(\sigma = (\sigma_n)\) is a correspondence depends on whether \(\omega\) is perfect, as seen in the following proposition:

**Proposition 3.13.** Let \(\omega\) be an order on a regular Bratteli diagram \(B\) with skeleton \(\mathcal{F}\) and accompanying maps \(\sigma = (\sigma_n)\). Then \(\omega\) is perfect if and only if \(\sigma\) is a correspondence.

**Proof.** Suppose that \(\omega\) is perfect. The fact that \((\sigma_n)\) is composition consistent follows from the definition of the level \(n\) languages \(\mathcal{L}(B, \omega, n)\). If it is the case that for distinct minimal paths \(m\) and \(m'\), and a maximal path \(M\), the two sequences \((v_n(m))\) and \((v_n(m'))\) belong to \((\sigma_n(v_n(M)))\) then we can build two sequences of paths \((x_n)\) and \((y_n)\), both converging to \(M\), where \(\varphi_\omega(x_n) \to m\) and \(\varphi_\omega(y_n) \to m'\), contradicting continuity of \(\varphi_\omega\). Thus for each maximal path \(M\), there is a unique \(m \in X_{\min}(\omega)\) with \((v_n(m))_n \in \prod_n \sigma_n(v_n(M))\), and the continuity of \(\varphi_\omega\) implies that in fact \(m = \varphi_\omega(M)\). Similarly for any \(m \in X_{\min}(\omega)\), there is a unique \(M = \varphi^{-1}_\omega(m) \in X_{\max}(\omega)\) with \((v_n(M))_n \in \prod_n \sigma^{-1}_n(v_n(m))\). Thus \(\sigma : X_{\max}(\omega) \to X_{\min}(\omega)\) coincides with \(\varphi_\omega : \omega \to \omega\).
Figure 3: The minimal and maximal edge structure for Example 3.12
$X_{\text{max}}(\omega) \to X_{\text{min}}(\omega)$, and the fact that $\varphi_{\omega}$ is a homeomorphism and $X_{\text{min}}(\omega)$, $X_{\text{max}}(\omega)$ are both closed implies that $\sigma : X_{\text{max}}(\omega) \to X_{\text{min}}(\omega)$ is a homeomorphism.

Conversely, suppose that $\sigma$ is a correspondence. The Vershik map $\varphi_{\omega}$ is well defined everywhere, and continuous, outside the sets of extreme paths. We use $\sigma$ to define $\varphi_{\omega}$ on $X_{\text{max}}(\omega)$, so that $\sigma$ equals the restriction of $\varphi_{\omega}$ to $X_{\text{max}}(\omega)$, a similar statement holding for $\sigma^{-1}$. As $\sigma$ is a correspondence, $\varphi_{\omega}$ is continuous on $X_{\text{max}}(\omega)$; thus to check continuity of $\varphi_{\omega}$, it is sufficient to consider a convergent sequence $(x_n)$ of non-maximal paths, where $x_n \to M$ with $M$ maximal. We claim that $(\varphi_{\omega}(x_n))$ converges to some minimal sequence $m$ (and in fact this $m$ does not depend on the choice of $(x_n)$). Suppose not. Then for two subsequences $(y_n)$ and $(y'_n)$ of $(x_n)$, we have $\varphi_{\omega}(y_n) \to m$ and $\varphi_{\omega}(y'_n) \to m'$ for two paths $m \neq m'$, which are necessarily minimal paths.

Since each $y_n$ is not maximal, this implies that for some subsequence $(n_k)$, $v_{n_k}(m) \in \sigma_{n_k}(v_{n_k}(M))$, which implies, by composition consistency, that $v_n(m) \in \sigma_n(v_n(M))$ for each $n \geq 1$. Similarly, as each $y'_n$ is not maximal for some subsequence $(n'_k)$, $v_{n'_k}(m') \in \sigma_{n'_k}(v_{n'_k}(M))$, which implies that $v_n(m') \in \sigma_n(v_n(M))$ for each $n$. Since we have assumed that $\sigma$ is a correspondence, this contradicts the fact that there is a unique minimal element $m = \sigma(M)$ such that $(v_n(m))_n \in \prod_n(\sigma_n(v_n(M)))$. Compactness ensures the continuity of $\varphi_{\omega}^{-1}$.

\[ \square \]

Proposition 3.13 tells us that behind every perfect order $\omega$ on a diagram $B$, there is an underlying skeleton $F$ and correspondence $\sigma$. More generally, a correspondence accompanying a skeleton $F$ will contain the information that allows us to extend the partial definition of orders using $F$ to construct perfect orders; now we discuss how to do this. We introduce the notion of graphs $(H_n)$ associated to $(F, \sigma)$. An analogous notion was introduced, for finite rank diagrams, in Section 3.2 of [BKY12].

Suppose that $B$ is a Bratteli diagram with skeleton $F = (\tilde{V}_{n-1}, \tilde{V}_{n-1}, \{\tilde{e}_v, \tilde{e}_v : v \in V_n\} : n \geq 2)$ associated to the set $\{M_\alpha : \alpha \in I\}$ of maximal paths and the set $\{m_\beta : \beta \in J\}$ of minimal paths. For any vertices $\tilde{v} \in \tilde{V}_{n-1}$ and $\overline{v} \in \overline{V}_{n-1}$, we set

\[ W_{\tilde{v}}(n) = \{ w \in V_n : s(\tilde{e}_w) = \tilde{v} \}, \quad W'_{\overline{v}}(n) = \{ w \in V_n : s(\overline{e}_w) = \overline{v} \}. \quad (2) \]

where $n \geq 2$. It is obvious that $W(n) = \{ W_{\tilde{v}}(n) : \tilde{v} \in \tilde{V}_{n-1}\}$ and $W'(n) = \{ W'_{\overline{v}}(n) : \overline{v} \in \overline{V}_{n-1}\}$ form two partitions of $V_n$. The intersection of $W(n)$ and $W'(n)$ is the partition $W''(n) \cap W(n)$ whose elements are non-empty sets $W'_{\overline{v}}(n) \cap W_{\tilde{v}}(n)$ where $(\overline{v}, \tilde{v}) \in \overline{V}_{n-1} \times \tilde{V}_{n-1}$. We shall use the notation $[\overline{v}, \tilde{v}, n] := W'_{\overline{v}}(n) \cap W_{\tilde{v}}(n)$ for shorthand.

**Definition 3.14.** Let $F$ be a skeleton on $B$ with an associated correspondence $\sigma$. Let $H_n = (T_n, P_n)$ be the directed graph where the set $T_n$ of vertices of $H_n$ will consist of partition elements $[\overline{v}, \tilde{v}, n]$ of $W(n) \cap W'(n)$, and where there is an edge in $P_n$ from
Proposition 3.15. Suppose that $F$ is a skeleton on $B$, with a correspondence $\sigma$. Let $(H_n)_{n}$ be the sequence of directed graphs associated to $(F, \sigma)$.

1. If the perfect order $\omega$ has associated skeleton and correspondence $(F, \sigma)$, then words in $L(B, \omega, n)$ correspond to paths in $H_n$.

2. Let $\omega$ be defined using the skeleton $F$ and correspondence $\sigma$, and where for each $n$, all words in $L(B, \omega, n)$ correspond to paths in $H_n$. Then $\omega$ is perfect.

Proof. For a given perfect order $\omega$, the map $\sigma_{n}$ is defined using the language $L(B, \omega, n)$. If $vw \in L(B, \omega, n)$, where $v \in [\pi, \tilde{v}, n]$ and $w \in [\pi', \tilde{v}', n]$, then $\tilde{v} \tilde{v}' \in L(B, \omega, n - 1)$, so that $\pi' \in \sigma_{n-1}(\tilde{v})$. Thus $vw$ corresponds to a path in $H_n$. The argument for longer words in $L(B, \omega, n)$ is similar.

To prove the second statement, take non-maximal paths $(x_n)$ converging to a maximal $M$. We shall show that in fact $\varphi_{\omega}(x_n) \rightarrow \sigma(M)$. This implies that $\varphi_{\omega}$ is continuous, and also that $\varphi_{\omega} : X_{\text{max}}(\omega) \rightarrow X_{\text{min}}(\omega)$ can be defined coinciding with $\sigma : X_{\text{max}}(F) \rightarrow X_{\text{min}}(F)$. Suppose that $x_n$ agrees with $M$ to level $k_n$. Then, since $\sigma$ is composition consistent, $v_j(\varphi_{\omega}(x_n)) \in \sigma_j(v_j(M))$ for each $j \leq k_n$. For some subsequence $n_i$, $\varphi_{\omega}(x_{n_i}) \rightarrow m$ where $m$ is a minimal path. This implies that $(v_j(m)) \in \prod_{j=1}^{\infty} \sigma(v_j(M))$, and by conditions (2) and (4) of Definition 3.9, $m = \sigma(M)$. Since any subsequence of $(\varphi_{\omega}(x_n))$ has a subsequence that converges to $m$, it follows that $\varphi_{\omega}(x_n) \rightarrow m$.

Example 3.16. We continue Examples 3.7 and 3.10; illustrated in Figure 4 is the graph $H_n$ associated to this skeleton and correspondence. Let $w(n) = v_1 \ldots v_n$; then $w(n)$ is generated from a path in $H_n$. Suppose that all words $w(v, n, n + 1)$ are defined using $w(n)$, subject to the constraints of the skeleton $F$ defined in Example 3.7, for example, $w(v_1, n, n + 1)$ must both start and end with $v_1$. Then, provided that the incidence matrices $(F_n)$ of $B$ allow us, we can define a perfect order $\omega$ with skeleton $F$ as in Example 3.7 and accompanying correspondence $\sigma$ as in Example 3.10. For
example, the $v_1$-indexed row of $F_n$ must be of the form $(\alpha_n + 1, \alpha_n, \ldots, \alpha_n, \beta_n)$ where
\(\alpha_n\) and \(\beta_n\) are positive integers. This will be further elucidated in Theorem 4.6.

Finally we state and prove the analogue of Proposition 3.10 in [BKY12]. Recall that a directed graph is strongly connected if for any two vertices \(v, v'\), there are paths from \(v\) to \(v'\), and also from \(v'\) to \(v\). If at least one of these paths exist, then \(G\) is weakly connected. Recall also the definition of the family \(A\) (Definition 2.2).

**Proposition 3.17.** Let \(B\) be a Bratteli diagram, and suppose \(\omega\) is a perfect ordering on \(B\) that defines the skeleton \(F_\omega\), the correspondence \(\sigma\) and the sequence of associated graphs \((H_n)_n\).

1. If \(B\) is simple, then \(H_n\) is strongly connected for any \(n\).
2. If \(B \in A\), \(H_n\) is weakly connected for any \(n\).

**Proof.** We prove (1) - the proof of (2) is similar (if we focus on \(w(v, n - 1, n)\) where \(v\) is the vertex which indexes the strictly positive row in \(F_n\)). In the case of simple diagrams, we can assume that all entries of \(F_n\) are positive for each \(n\).

Take two vertices \([v_1, v_1, n]\) and \([v_3, v_3, n]\) in \(H_n\). If \(v_2 \in \sigma_{n-1}(v_1)\), then there is some vertex \(\overline{v} \in V_n\) such that \(s(\overline{v}) = v_2\). Let \(\overline{v} \in [v_2, v_2, n]\). Clearly there is an edge from \([v_1, v_1, n]\) to \([v_2, v_2, n]\) in \(H_n\).

Let \(v \in V_{n+1}\) be such that \(s(e_v) = \overline{v}\). Let \(v_3 \in [v_3, v_3, n]\). Since \(B\) is simple, \(f_{v_3}^{(n)} > 0\); this means that \(\overline{v} \ldots v_3\) is a prefix of \(w(v, n, n + 1)\). This implies that there is path in \(H_n\) from \([v_2, v_2, n]\) to \([v_3, v_3, n]\). \(\square\)

**Remark 3.18.** It is not hard to see that the converse statement to Proposition 3.17 is not true. There are examples of non-simple diagrams of finite rank whose associated graphs are strongly connected.

Note also that the assumption that \(\omega\) is perfect is crucial. Moreover, there are examples of \emph{simple} finite rank Bratteli diagrams and skeletons none of whose associated
graphs are strongly connected. Indeed, let $B$ be a stationary diagram with $V = \{a, b, c\}$ with the skeleton $F = \{M_a, M_b, m_a, m_b; \overline{e_c}, \overline{\tau_c}\}$ where $s(\overline{e_c}) = b, s(\overline{\tau_c}) = a$. Let $\sigma(a) = a, \sigma(b) = b$. Constructing the associated graph $H$, we see that there is no path from $[b, b]$ to $[a, a]$. It can be also shown that there is no perfect ordering $\omega$ such that $F = F_{\omega}$. This observation complements Proposition 3.17 by stressing the importance of the strong connectedness of $H_n$ for the existence of perfect orderings.

4 Characterizing Bratteli diagrams that support perfect orders

In this section we characterize Bratteli diagrams that support perfect, non-proper orders via their incidence matrices. Our main result is Theorem 4.6 which extends a similar result proved in [BKY12, Theorem 3.19] for finite rank diagrams. We define the class $P^*_B$, a set of perfect orders whose language properties are similar to those of orders in $P_B(j)$, $j$ finite, and for whom a refined version of Theorem 4.6 holds, namely Corollary 4.9.

The intuition behind the proof of Theorem 4.6 is the following idea. If a diagram $B$ is to support a perfect, well-telescoped order $\omega$, then $\omega$ would define a skeleton $F$ and correspondence $\sigma$. The correspondence intrinsically contains the information about the languages defined by $\omega$, and this is further expressed with the sequence $(H_n)$ of directed graphs, in that words in $L(B, \omega, n)$ must correspond to paths in $H_n$. Words in $L(B, \omega, n)$ are generated by the orders placed on the edges in $r^{-1}(u)$, where $u \in V_m$ and $m > n$. To define $w(u, n, n + 1)$ for $u \in V_{n+1}$, then, we use $H_n$. The edge structure of $H_n$ implies that if a word $vw$ lies in $w(u, n, n + 1)$ with $v$ belonging to $W_{\overline{v}}(n)$, then $w$ must belong to $W_{\overline{v}}(n)$ for some $\overline{v} \in \sigma_{n-1}(\overline{v})$ - this is the statement of Lemma 4.2. This means that every time we leave a vertex in $H_n$ of the form $[*, v, n]$, we must go to a vertex of the form $[\overline{v}, *, n]$ for some $\overline{v} \in \sigma_{n-1}(\overline{v})$. Thus the $u$-th row of the $n$-th incidence matrix $F_n$ must have a ‘balance’ between entries $f_{u,v}^{(n)}$, where $v \in W_{\overline{v}}(n)$, and entries $f_{u,v'}^{(n)}$, where $v' \in \bigcup_{v \in \sigma_{n-1}(\overline{v})} W_{\overline{v}}(n)$. This is more precisely stated in Corollaries 4.3 and 4.4. It turns out that these ‘balance’ requirements, the system of relations (10), along with the system (9), are also sufficient.

First we define the class $P^*_B$, a class of perfect orders that naturally generalizes the class of perfect orders with finitely many extremal paths, and also introduce/re-introduce notation we shall need.

Definition 4.1. Let $\omega$ be a perfect order with skeleton $F$ and correspondence $\sigma = (\sigma_n)$. Suppose that $\omega$ satisfies the following conditions: for each maximal path $M$ with $\overline{v_n} = v_n(M), \sigma_n(\overline{v_n}) \in \nabla_n$ for all $n$ sufficiently large, and for each minimal path $m$
with $\overline{v}_n = v_n(m), \sigma_n(\overline{v}_n)^{-1} \in \tilde{V}_n$ for all $n$ sufficiently large. The set of all such perfect orders is denoted by $\mathcal{P}_B^\ast$.

In fact, it seems (see the comment in Example 3.12) that it is unnatural for a perfect order to not belong to $\mathcal{P}_B^\ast$. All perfect orders with finitely many maximal paths belong to $\mathcal{P}_B^\ast$, i.e. $\mathcal{P}_B \cap \mathcal{O}_B(j) \subseteq \mathcal{P}_B^\ast$ for each finite $j$. Also, suppose the perfect order $\omega$ is such that for each maximal path $M_\alpha$, and each minimal path $m_\beta$, there exist neighborhoods $U(M_\alpha)$ and $U(m_\beta)$ such that no other maximal paths belong to $U(M_\alpha)$, and no other minimal paths are in $U(m_\beta)$; then $\omega \in \mathcal{P}_B^\ast$. Note that such an order can have at most countably many extremal paths, since the correspondence $M_\alpha \rightarrow U(M_\alpha)$ is injective, and the set of clopen sets is countable.

Let $\omega$ be a perfect order on $B$. Recall that $\omega$ generates the skeleton $\mathcal{F}_\omega = (\tilde{V}_n-1, \overline{V}_n-1, \{\overline{e}_v, \overline{\tau}_v : v \in V_n\})$, $n > 1$ and two partitions $W(n) = \{W_v(n) : \overline{v} \in \tilde{V}_n-1\}$ and $W'(n) = \{W'_\overline{v}(n) : \overline{\tau} \in \overline{V}_n-1\}$ of $V_n$. Moreover, we have also a sequence of correspondences $\sigma_n : \tilde{V}_n \rightarrow 2^{\tilde{V}_n}, n \geq 1$, defined by $\omega$. We recall also the notation used for maximal (minimal) paths: if $M$ is a maximal path then it determines uniquely a sequence of maximal vertices ($\tilde{v}_n = v_n(M)$).

Let $E(V_n, u)$ be the set of all finite paths between vertices of level $n$ and a vertex $u \in V_m$ where $m > n$. The symbols $\overline{e}(V_n, u)$ and $\overline{e}(V_n, u)$ are used to denote the maximal and minimal finite paths in $E(V_n, u)$, respectively; if $m = n + 1$ so that $u \in V_{n+1}$, then we revert to the shorter notation $\overline{e}_u$ and $\overline{\tau}_u$. Fix a maximal and minimal vertex $\overline{v} \in \tilde{V}_n-1$ and $\overline{\tau} \in \overline{V}_n-1$ respectively. Denote $E(W_\overline{v}(n), u) = \{e \in E(V_n, u) : s(e) \in W_\overline{v}(n), r(e) = u\}$ and $E(W'_\overline{v}(n), u) = \{e \in E(V_n, u) : s(e) \in W'_\overline{v}(n), r(e) = u\}$. Clearly, the sets $E(W_\overline{v}(n), u) : \overline{v} \in V$ and $E(W'_\overline{v}(n), u) : \overline{\tau} \in V$ form two partitions of $E(V_n, u)$. It may happen that the maximal finite path $\overline{e}(V_n, u)$ has its source in $W_\overline{v}(n)$. In this case, we define $\overline{E}(W_\overline{v}(n), u) = E(W_\overline{v}(n), u) \setminus \{\overline{e}(V_n, u)\}$. Otherwise, $\overline{E}(W_\overline{v}(n), u) = E(W_\overline{v}(n), u)$. Similarly we define the set $\overline{E}(W'_\overline{v}(n), u)$ using the minimal finite path $\overline{\tau}(V_n, u)$.

In the next few results we assume that a perfect order $\omega$ has attached its skeleton $\mathcal{F}$, correspondence $\sigma$ and partitions ($W(n)$) and ($W'(n)$). Also, for brevity we shall abuse notation: if $e$ is an edge, (or a set of edges), we will write $\varphi_\omega(e)$ instead of the more correct $\varphi_\omega(U(e))$.

**Lemma 4.2.** Suppose $(B, \omega)$ is a well telescoped ordered Bratteli diagram with $\omega \in \mathcal{P}_B$. Let $\overline{v} \in \tilde{V}_{n-1}$. If $u \in V_m$, $m > n$, and $e \in \overline{E}(W_\overline{v}(n), u)$, then $\varphi_\omega(e) \in \bigcup_{\overline{\tau} \in \sigma_{n-1}(\overline{v})} \overline{E}(W_\overline{\tau}(n), u)$.

**Proof.** Extend $e$ to a path $e^\ast$ in $\overline{E}(\overline{v}, u)$ by concatenating the maximal edge in $E(\overline{v}, s(e))$ to $e$. Similarly, extend $\varphi_\omega(e)$ to a path $e^{**}$ in $\overline{E}(\overline{\tau}, u)$ by concatenating the minimal
edge in \( E(\overline{\sigma}, \varphi_\omega(e)) \) to \( \varphi_\omega(e) \). Since \( \varphi_\omega(e^*) = (e^*)^* \), this means that \( \overline{e} \overline{v} \subseteq w(u, n-1, m) \), so that \( \overline{e} \overline{v} \in \mathcal{L}(B, \omega, n-1) \). By definition of \( \sigma_{n-1} \), \( \overline{v} \in \sigma_{n-1}(\overline{v}) \).

\[ \square \]

The following corollary can be easily deduced from Lemma 4.2.

**Corollary 4.3.** Let \((B, \omega)\) be a well telescoped ordered Bratteli diagram with \( \omega \in \mathcal{P}_B \). Then for any \( n \geq 2 \), \( \overline{v} \in \overline{V}_{n-1} \) and \( u \in V_m \), \( m > n \), we have

\[
\sum_{\overline{v} \in \sigma_{n-1}(\overline{v})} |\overline{E}(W_{\overline{v}}(n), u) \cap \varphi_\omega^{-1}(\overline{E}(W_{\overline{v}}'(n), u))| = |\overline{E}(W_{\overline{v}}(n), u)|. \tag{3}
\]

Also if \( \overline{v} \in \overline{V}_{n-1} \), then

\[
\sum_{\overline{v} \in \sigma_{n-1}(\overline{v})} |\overline{E}(W_{\overline{v}}(n), u) \cap \varphi_\omega^{-1}(\overline{E}(W_{\overline{v}}'(n), u))| = |\overline{E}(W_{\overline{v}}'(n), u)|. \tag{4}
\]

We can refine the statement of Corollary 4.3 in some special cases:

**Corollary 4.4.** Let \((B, \omega)\) be a well telescoped ordered Bratteli diagram with \( \omega \in \mathcal{P}_B \).

1. Suppose that \( \omega \in \mathcal{P}_B \cap \mathcal{O}_B(j) \). Then there exists an \( n_0 \) such that for any \( n \geq n_0 \), any vertex \( \overline{v} \in \overline{V}_{n-1} \), any \( m > n \), and any \( u \in V_m \), \( \sigma_{n-1}(\overline{v}) \) is a singleton and one has

\[
|\overline{E}(W_{\overline{v}}(n), u)| = |\overline{E}(W_{\sigma_{n-1}(\overline{v})}(n), u)|. \tag{5}
\]

2. Suppose that \( \omega \in \mathcal{P}_B^* \). Then for any maximal path \( M \) (and hence any sequence \((\overline{v}_n = v_n(M))\)), there exists an \( n_0 \) such that for any \( n \geq n_0 \), \( \sigma_n(\overline{v}_n) \) is a singleton, and for any \( m > n > n_0 \) and \( u \in V_m \), one has

\[
|\overline{E}(W_{\overline{v}_{n-1}}(n), u)| = |\overline{E}(W_{\sigma_{n-1}(\overline{v}_{n-1})}(n), u)|. \tag{6}
\]

Given the incidence matrices \((F_n)\) for \( B \), where \( F_n = \{(f_{u,w}^{(n)}): u \in V_{n+1}, w \in V_n\} \), we define the sequences of modified incidence matrices \((\overline{F}_n)\) and \((\overline{F}_n')\) as in [BKYY12]. Namely, define \( \overline{F}_n = (\overline{f}_{u,w}^{(n)}) \) and \( \overline{F}_n' = (\overline{f}_{u,w}^{(n)\prime}) \) by the following rule (here \( w \in V_n \), \( u \in V_{n+1} \) and \( n \geq 1 \)):

\[
\overline{f}_{u,w}^{(n)} = \begin{cases} f_{u,w}^{(n)} - 1 & \text{if } \overline{u} \in E(w, u) \\
 f_{u,w}^{(n)} & \text{otherwise}, \end{cases} \tag{7}
\]

and

\[
\overline{f}_{u,w}^{(n)\prime} = \begin{cases} f_{u,w}^{(n)} - 1 & \text{if } \sigma_u \in E(w, u) \\
 f_{u,w}^{(n)} & \text{otherwise}. \end{cases} \tag{8}
\]

Relation (3) implies that for each \( u \in V_{n+1} \) and each \( \overline{v} \in \overline{V}_{n-1} \), if \( w \in W_{\overline{v}}(n) \), then each \( \overline{f}_{u,w}^{(n)} \) can be written as

\[
\overline{f}_{u,w}^{(n)} = \sum_{\overline{v} \in \sigma_{n-1}(\overline{v})} \overline{f}_{u,w}^{(n)}_{\overline{v}, \overline{v}} \tag{9}
\]
and relation (10) says that for each $u \in V_{n+1}$ and each $\overline{v} \in \overline{V}_{n-1}$,

$$\sum_{\overline{w} \in W_{\overline{v}}(n)} \sum_{w \in W_{\overline{v}}(n)} \overline{f}_{u,w,\overline{w}}^{(n)} = \sum_{w' \in W_{\overline{v}}^{(n)}} \overline{f}_{u,w'}^{(n)}. \quad (10)$$

We will refer to the relations in (10) above as the balance relations. If $\omega$ is perfect and has finitely many extremal paths, then the balance relations have the following form: for $n > n_0$, $\overline{v} \in \overline{V}_{n-1}$ and $u \in V_{n+1}$:

$$\sum_{w \in W_{\overline{v}}(n)} \overline{f}_{u,w}^{(n)} = \sum_{w' \in W_{\overline{v}}^{(n)}} \overline{f}_{u,w'}^{(n)}, \quad u \in V_{n+1}. \quad (11)$$

If $\omega \in \mathcal{P}^*_B$ and the maximal path $M$ is given, then there exists $n_0$ such that for each $n > n_0$, if $\overline{v}_{n-1}(M) = \overline{v}$ and $u \in V_{n+1}$, then relation (11) is satisfied.

The content of Theorem 4.6 is that given a skeleton and correspondence on $B$, relations (9) and (10) are sufficient conditions on the incidence matrices of a Bratteli diagram, in order that it supports a perfect order $\omega$. Our proof is constructive in that given a diagram, skeleton and correspondence, we use an algorithm to define, for each $u \in V_{n+1}$ and $n \in \mathbb{N}$, the word $w(u, n, n + 1)$ - i.e. we order the set $r^{-1}(u)$. We do this by constructing a path in $\mathcal{H}_n$ that starts in $[\overline{v}_0, \overline{v}_0, n]$, where $s(\overline{v}_u) \in [\overline{v}_0, \overline{v}_0, n]$, terminates at $[\overline{v}_f, \overline{v}_f, n]$, where $s(\overline{v}_u) \in [\overline{v}_f, \overline{v}_f, n]$, and passes through each vertex in $\mathcal{H}_n$ a prescribed number of times that we now make precise.

Proposition 3.17 tells us that we have to assume that the directed graphs $\mathcal{H}_n$ are strongly connected. We make clear what we mean by this as follows. Fix $n \in \mathbb{N}$ and $u \in V_{n+1}$. If $[\overline{v}, \overline{v}, n] \in \mathcal{H}_n$, we associate a number $P_u([\overline{v}, \overline{v}, n]) := \sum_{w \in [\overline{v}, \overline{v}, n]} \overline{f}_{u,w}^{(n)}$ to the vertex $[\overline{v}, \overline{v}, n]$. This crossing number represents the number of times that we will have to pass through the vertex $[\overline{v}, \overline{v}, n]$ when we define an order on $r^{-1}(u)$, and here we emphasize that if we terminate at $[\overline{v}, \overline{v}, n]$, we do not consider this final visit as contributing to the crossing number - this is why we use the terms $\overline{f}_{u,w}^{(n)}$ and not $f_{u,w}^{(n)}$. We say that $\mathcal{H}_n$ is positively strongly connected if for each $u \in V_{n+1}$, the set of vertices $\{[\overline{v}, \overline{v}, n] : P_u([\overline{v}, \overline{v}, n]) > 0\}$, along with all the relevant edges of $\mathcal{H}_n$, form a strongly connected subgraph of $\mathcal{H}_n$. If $s(\overline{v}_u) \in [\overline{v}, \overline{v}, n]$ we shall call this vertex in $\mathcal{H}_n$ the terminal vertex, as when defining the order on $r^{-1}(u)$, we need a path that ends at this vertex (although it can obviously go through this vertex several times - in fact precisely $P_u([\overline{v}, \overline{v}, n])$ times).

Example 4.5. In this example we drop the dependence on $n$ and consider the stationary diagram $B = (V, E)$ that was used above in Example 2.5. Suppose that $V = \{a, b, c, d\}$, $\overline{V} = \overline{V} = \{a, b, c\}$, with $a \in [a, a]$, $b \in [b, b]$, $c \in [c, c]$ and $d \in [b, a]$. Let
\[ \sigma(a) = b,\ \sigma(b) = c\] and \[\sigma(c) = a.\] Suppose that the incidence matrix \( F \) of \( B \) is

\[
F := \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 \\
1 & 1 & 1 & 2
\end{pmatrix}
\]

Then if \( u = d \), \( P_d([a, a]) = 0 \), and the remaining three vertices \([b, b], [c, c]\) and \([b, a]\) do not form a strongly connected subgraph of \( \mathcal{H} \), - for example there is no path from \([c, c]\) to \([b, a]\). Hence for us \( \mathcal{H} \) is not positively strongly connected.

Note also that although the rows of this incidence matrix satisfy the balance relations, there is no way to define an order on \( r^{-1}(d) \) so that the resulting global order is perfect. The lack of positive strong connectivity of the graph \( \mathcal{H} \) is precisely the impediment.

The following theorem is our main result, extending a similar statement proved in [BKY12].

**Theorem 4.6.** Let \( B \) be a Bratteli diagram with incidence matrices \( (F_n) \). Let \( \mathcal{F} \) be a skeleton on \( B \) and \( \sigma \) an associated correspondence, such that the graphs \( \mathcal{H}_n \) are all positively strongly connected. Suppose that the collection of natural numbers

\[
\{ \overline{f}_{u,v}^{(n)} : u \in V_{n+1}, w \in W_{\overline{v}}(n), v \in \sigma_{n-1}(\overline{v}) \}, \quad \overline{v} \in \overline{V}_{n-1},
\]

is such that relations (9) and (10) are satisfied.

Then there exists a perfect order \( \omega \) on \( B \) having \((\mathcal{F}, \sigma)\) as associated skeleton and correspondence. Conversely, suppose that a perfect \( \omega \) has accompanying skeleton and correspondence \((\mathcal{F}, \sigma)\). Then there exists a set of natural numbers as in (12) such that relations (9) and (10) hold.

**Proof.** As the preceding discussion deals with the necessity a perfect order having to satisfy relations (9) and (10), we prove here only the sufficiency of these relations.

Our goal is to define a linear order on \( r^{-1}(u) \) for each \( u \in V_{n+1} \) and \( n > 1 \) - in other words to define \( w(u, n, n+1) \) - so that the corresponding partial ordering \( \omega \) on \( B \) is perfect. Recall that each set \( r^{-1}(u) \) contains two pre-selected edges \( \overline{e}_u, \overline{e}_u \) and they should be the maximal and minimal edges in \( r^{-1}(u) \) after defining \( w(u, n, n+1) \).

Our proof is based on an inductive procedure that is applied to each row of the incidence matrices. We first describe in details the first step of the algorithm that will be applied repeatedly. It will be seen from our proof that for given \( B, \mathcal{F} \) and \( \sigma \), neither is the word \( w(u, n, n+1) \) that we define unique, nor will our algorithm give all possible valid words.
We will first consider the particular case when the associated graphs \( \mathcal{H} = (\mathcal{H}_n) \) defined by \( \mathcal{F} \) do not have loops. After that, we will modify the construction to include possible loops in the algorithm.

Case I: There are no loops in the graphs \( \mathcal{H}_n \). To begin with, we take some \( u \in V_{n+1} \) and consider the \( u \)-th rows of matrices \( \overline{\mathbf{F}}_n \) and \( \overline{\mathbf{F}}_n \). They coincide with the row \((f_{u,v_1}^{(n)}, ..., f_{u,v_d}^{(n)})\) of the matrix \( \mathbf{F}_n \) except only one entry either corresponding to \(|E(s(\overline{v}_u), u)|\) and \(|E(s(\overline{e}_u), u)|\) in \( \overline{\mathbf{F}}_n \) and \( \overline{\mathbf{F}}_n \), respectively. Take \( \overline{v}_u \) and assign the number 0 to it, i.e. \( \overline{v}_u \) is the minimal edge in \( r^{-1}(u) \). Let \([\overline{v}_0, \overline{v}_0, n]\) be the vertex\(^1\) of \( \mathcal{H}_n \) such that \( s(\overline{v}_u) \in [\overline{v}_0, \overline{v}_0, n] \). Consider the set

\[
\{ \overline{f}_{u,w}^{(n)} : w \in [\overline{v}, \overline{v}, n] : \overline{v} \in \sigma_{n-1}(\overline{v}_0) \},
\]

and let \( \overline{f}_{u,w}^{(n)} \) be the maximum of this set, where \( w' \in [\overline{v}_1, \overline{v}_1, n] \). If there are several entries that are the maximal value, we chose one arbitrarily amongst them. Take any edge \( e_1 \in E(w', u) \). In the case where \( \overline{e}_u \in E(w', u) \), we choose \( e_1 \neq \overline{e}_u \). Assign the number 1 to \( e_1 \) so that \( e_1 \) becomes the successor of \( e_0 = \overline{v}_u \).

Two edges were labeled in the above procedure, \( e_0 \) and \( e_1 \). We may think of this step as if these edges were ‘removed’ from the set of all edges in \( r^{-1}(u) \). In the collection of relations \( (10) \) we have worked with the relation defined by \( u \) and \( \overline{v}_1 \). On the left hand side, the entry \( \overline{f}_{u,s(\overline{v}_u), \overline{v}_1}^{(n)} \) was reduced by 1, and on the right hand side, \( \overline{f}_{u,n}^{(n)} \) was reduced by 1. We need to verify that neither side was reduced by more than 1, i.e. we claim the remaining non-enumerated edges satisfy the relation

\[
\sum_{\overline{v} \in \sigma_{n-1}(\overline{v}_1)} \sum_{w \in W_{\overline{v}}^{(n)}} \overline{f}_{u,w}^{(n)} - 1 = \sum_{w' \in W_{\overline{v}}^{(n)}} \overline{f}_{u,w'}^{(n)} - 1.
\]

(13)

The choice of \( w' \in [\overline{v}_1, \overline{v}_1, n] \) actually means that we take the edge from \([\overline{v}_0, \overline{v}_0, n] \) to \([\overline{v}_1, \overline{v}_1, n] \) in the associated graph \( \mathcal{H}_n \). Note that \( \overline{v}_1 \notin \overline{\sigma}_{n-1}(\overline{v}_1) \), otherwise there would be a loop at \([\overline{v}_1, \overline{v}_1, n] \) in \( \mathcal{H}_n \), a contradiction to our assumption. This is why there is exactly one edge removed in each side of \( (13) \) so that our resulting row still satisfies \( (10) \). This completes the first step of the construction.

We are now at the vertex \([\overline{v}_1, \overline{v}_1, n] \) in \( \mathcal{H}_n \). To repeat the above procedure, note that we now have a ‘new’, reduced \( u \)-th row of \( \mathbf{F}_n \) - namely, the entry \( f_{u,n}^{(n)} \) has been reduced by one. Thus the crossing numbers of the vertices of \( \mathcal{H}_n \) change (one crossing number is reduced by one). Also note that in this new reduced row, \( f_{u,w'}^{(n)} = f_{u,w'}^{(n)} - 1 \); in other words, with each step of this algorithm the row we are working with changes, and the vertex \( w \) such that \( f_{u,w}^{(n)} = f_{u,w}^{(n)} - 1 \) changes. For, the vertex such that \( f_{u,w}^{(n)} = f_{u,w}^{(n)} - 1 \)

\(^1\)The same word ‘vertex’ is used in two meanings: for elements of the set \( T_0 \) of the graph \( \mathcal{H}_0 \) and for elements of the set \( V_n \) of the Bratteli diagram \( B \). To avoid any possible confusion, we point out explicitly what vertex is meant in the context.
belongs to the vertex in \( H_n \) where we currently are, and this changes at every step of the algorithm. Let us assume that all crossing numbers are still positive for the time being to describe the second step of the algorithm.

We apply the above described procedure again, this time to \( w' = s(e_1) \), to show how we should proceed to complete the next step. Consider the set

\[ \{ \tilde{f}^{(n)}_{u, w'} : w \in [\bar{v}, \tilde{v}, n] : \bar{v} \in \sigma_{n-1}(\tilde{v}_1) \}, \]

and let \( \tilde{f}^{(n)}_{u, w''} \) be the maximum of this set, where \( w'' \in [\bar{v}_3, \tilde{v}_3, n] \). Once again, if there are several entries that are the maximal value, we chose one arbitrarily amongst them.

Take any edge \( e_2 \in E(w'', u) \). In the case where \( \tilde{e}_u \in E(w'', u) \), we choose \( e_2 \neq \tilde{e}_u \).

Assign the number 2 to \( e_2 \) so that \( e_2 \) becomes the successor of \( e_1 \).

We note that in the collection of relations (9), indexed by the vertices \( u, \tilde{v}_1 \) and \( w' = s(e_1) \), one entry was 'removed' from each side of the relation: on the right hand side, the entry \( \tilde{f}^{(n)}_{u, w', \bar{v}_2} \) was reduced by 1.

In the collection of relations (10) we have worked with the relation defined by \( u \) and \( \bar{v}_2 \). On the left hand side, the entry \( \tilde{f}^{(n)}_{u, w', \bar{v}_2} \) was reduced by 1, and on the right hand side, \( \tilde{f}^{(n)}_{u, w''} \) was reduced by 1. As we saw in (13), the relevant relation in (10) becomes

\[
\sum_{\bar{v}, \bar{v}_2 \in \sigma_{n-1}(\bar{v})} \sum_{w \in W_{\bar{v}}(n)} \tilde{f}^{(n)}_{u, w, \bar{v}} - 1 = \sum_{w' \in W_{\bar{v}_2}(n)} \tilde{f}^{(n)}_{u, w'} - 1. \quad (14)
\]

We remark also that the choice that we made of \( w'' \) (or \( e_2 \)) allows us to continue the existing path (in fact, the edge) in \( H_n \) from \([\bar{v}_0, \tilde{v}_0, n]\) to \([\bar{v}_1, \tilde{v}_1, n]\) with the edge from \([\bar{v}_1, \tilde{v}_1, n]\) to \([\bar{v}_2, \tilde{v}_2, n]\), where \( \tilde{v}_2 \) is defined by the property that \( s(e_2) \in [\bar{v}_2, \tilde{v}_2, n] \).

This process can be continued. At each step we apply the following rules:

(i) the edge \( e_i \), that must be chosen next after \( e_{i-1} \), is taken from the set \( E(w^*, u) \) where \( w^* \) is such that \( f^{(n)}_{u, w^*} \) is maximal amongst \( f^{(n)}_{u, w} \), as \( w \) runs over \([\bar{v}, \tilde{v}, n]\) where \( \bar{v} \in \sigma_n(\tilde{v}_i) \), and

(ii) the edge \( e_i \) is always taken not equal to \( \tilde{e}_u \) unless no more edges except \( \tilde{e}_u \) are left.

After every step of the construction, we see that the following statements hold.

(i) Relations (9), (with \( \bar{v} = \tilde{v}_i \)) and (10) (with \( \bar{v} = \bar{v}_i \)) remain true when we treat them as the number of non-enumerated edges left in \( r^{-1}(u) \). In other words, when a pair of vertices \( \tilde{v}_i \) and \( \bar{v}_i \) is considered, we reduce by 1 each side of the relevant relations.

(ii) The used procedure allows us to build a path \( p \) from the starting vertex \([\bar{v}_0, \tilde{v}_0, n]\) going through other vertices of the graph \( H_n \) according to the choice we make at each step. We need to guarantee that at each step, we are able to move to a vertex in \( H_n \) whose crossing number is still positive (unless we are at the terminal stage). As long
as the crossing numbers of vertices in $H_n$ are positive, there is no concern. Suppose thought that we land at a (non-terminal) vertex $[\pi, \tilde{v}, n]$ in $H_n$ whose crossing number is one (and this is the first time this happens). When we leave this vertex, to go to $[\pi', \tilde{v}', n]$, the crossing number for $[\pi, \tilde{v}, n]$ will become 0 and therefore it will no longer be a vertex of $H_n$. Thus at this point, with each step, the graph $H_n$ is also changing (being reduced). We need to ensure that there is a way to continue the path out of $[\pi', \tilde{v}', n]$. Since

$$\sum_{w \in W_{\pi'}(n)} \tilde{f}^{(n)}_{u,w} \geq P_u[\pi', \tilde{v}', n] \geq 1,$$

then for some $\pi \in \sigma_{n-1}(\tilde{v}')$,

$$\sum_{\tilde{v} : \pi \in \sigma_{n-1}(\tilde{v})} \sum_{w \in W_{\pi}(n)} \tilde{f}^{(n)}_{u,w,\pi} \geq 1,$$

so that by the balance relations, $\sum_{w' \in W_{\pi'}(n)} \tilde{f}^{(n)}_{u,w'} \geq 1$. If the crossing number of all the vertices $[\pi, *, n]$ have been reduced to 0, then this means that $\sum_{w' \in W_{\pi'}(n)} \tilde{f}^{(n)}_{u,w'} = 1$, this tells us that we have to move into the terminal vertex for the last time. Then the balance relations, which continue to be respected, ensure we are done. Otherwise, the balance relations guarantee that $\sum_{w' \in W_{\pi'}(n)} \tilde{f}^{(n)}_{u,w'} > 1$, which means there is a valid continuation of our path out of $[\pi', \tilde{v}', n]$ and to a new vertex in $H_n$, and we are not at the end of the path. It is these balance relations which always ensure that the path can be continued until it reaches its terminal vertex.

(iii) In accordance with (i), the $u$-th row of $F_n$ is transformed by a sequence of steps in such a way that entries of the obtained rows form decreasing sequences. These entries show the number of non-enumerated edges remaining after the completed steps. It is clear that, by the rule used above, we decrease the largest entries first. It follows from the simplicity of the diagram that, for sufficiently many steps, the set $\{s(e_i)\}$ will contain all vertices $v_1, ..., v_d$ from $V_n$. This means that the transformed $u$-th row consists of entries which are strictly less than those of $F_n$. After a number of steps the $u$-th row will have a form where the difference between any two entries is $\pm 1$. After that, this property will remain true.

(iv) It follows from (iii) that we finally obtain that all entries of the resulting $u$-th row are zeros or ones. We apply the same procedure to enumerate the remaining edges from $r^{-1}(u)$ such that the number $|r^{-1}(u)| - 1$ is assigned to the edge $e_u$. This means that we have constructed the word $W_u = s(\pi_u)s(e_1) \cdots s(e_u)$, ie we have ordered $r^{-1}(u)$.

Looking at the path $p$ that is simultaneously built in $H_n$, we see that the number of times this path comes into and leaves a vertex $[\pi, \tilde{v}, n]$ is precisely that vertex’s crossing number $P_u([\pi, \tilde{v}, n])$. The path $p$ is an Eulerian path of $H_n$ that finally arrives to the vertex of $H_n$ defined by $s(\tilde{v}_u)$.
Case II: there is a loop in $H_n$. To deal with this case, we have to refine the described procedure to avoid a possible situation when the algorithm cannot be finished properly. Suppose that the graph $H_n$ has some loops.

We start as in Case I, and continue until we have arrived to a vertex $[\tilde{v}_1, \tilde{v}_1, n]$, where, for the first time, $[\tilde{v}_1, \tilde{v}_1, n]$ has a successor $[\tilde{v}_2, \tilde{v}_2, n]$ with a loop, ie $\tilde{v}_2 \in \sigma_n(\tilde{v}_2)$. If $[\tilde{v}_2, \tilde{v}_2, n]$ has crossing number zero, - ie it is the terminal vertex - and we are not at the terminal stage of defining the order, we ignore this vertex and continue as in Case I. If $[\tilde{v}_2, \tilde{v}_2, n]$ has a positive crossing number, i.e. $P_u([\tilde{v}_2, \tilde{v}_2, n]) > 0$, then at this point, we continue the path to $[\tilde{v}_2, \tilde{v}_2, n]$, and then traverse this loop ($\sum_{w \in [\tilde{v}_2, \tilde{v}_2, n]} \tilde{f}^{(n)}_{u, u, \tilde{v}_2} - 1$) times. This means we are traversing this loop enough times that it is effectively no longer part of the resulting $H_n$ that we have at the end of this step - we will no longer need, or even be able, to traverse the loop.

Looking at the relation
\[
\sum_{\tilde{v}: \tilde{v}_2 \in \sigma_{n-1}(\tilde{v})} \sum_{w \in W_{\tilde{v}}(n)} \tilde{f}^{(n)}_{u, u, \tilde{v}_2} = \sum_{w' \in W_{\tilde{v}_2}(n)} \tilde{f}^{(n)}_{u, u, w'}, \tag{15}
\]
we see that by the time we have arrived at the vertex $[\tilde{v}_2, \tilde{v}_2, n]$, traversed it exactly ($\sum_{w \in [\tilde{v}_2, \tilde{v}_2, n]} \tilde{f}^{(n)}_{u, u, \tilde{v}_2} - 1$) times, and left it, we see that we have removed $\sum_{w \in [\tilde{v}_2, \tilde{v}_2, n]} \tilde{f}^{(n)}_{u, u, \tilde{v}_2}$ from each side of [15]. We consequently enumerate all edges whose source lies in $[\tilde{v}_2, \tilde{v}_2, n]$ in any arbitrary order.

We also need to ensure that once we have ‘removed’ the loop at $[\tilde{v}_2, \tilde{v}_2, n]$ from the graph $H_n$, we do not disrupt future movement of our path, i.e. we do not disconnect $H_n$ in a damaging way. To see this, suppose we have a loop at $[\tilde{v}_2, \tilde{v}_2, n]$ whose crossing number is positive. If $[\tilde{v}_1, \tilde{v}_1, n]$ is a (non-looped) vertex with a positive crossing number which has $[\tilde{v}_2, \tilde{v}_2, n]$ as a successor, then for some $[\tilde{v}_3, \tilde{v}_3, n] \neq [\tilde{v}_2, \tilde{v}_2, n]$ with $\tilde{v}_3 \in \sigma_{n-1}(\tilde{v}_1)$, the vertex $[\tilde{v}_3, \tilde{v}_3, n]$ will (if we are not at the terminal stage) have a positive crossing number. This is because of our discussion above concerning [15]: the crossing number at the looped vertex appears on both sides, and cancels. So if $[\tilde{v}_1, \tilde{v}, n]$ has a positive crossing number, this contributes positive values to the left hand side of [15]; and so there is some vertex $[\tilde{v}_3, \tilde{v}_3, n]$ with a positive value on the right hand side. All this means that we are able to continue our path out of the looped vertex $[\sigma_n(\tilde{v}), \tilde{v}, n]$ when we arrive there at some future stage.

We now revert to the old procedure. We are at the vertex $[\tilde{v}_2, \tilde{v}_2, n]$ in $H_n$. If none of its successors have a loop, we revert to the algorithm in Case I, and continue with that algorithm until we reach a vertex in $H_n$ one of whose successors has a loop, and then repeat the procedure described in Case II. We continue until we have defined the order on $r^{-1}(u)$. To summarize the general procedure, we notice that, constructing the Eulerian path $p$, the following rule is used: as soon as $p$ arrives before a loop around a vertex in $H_n$, then $p$ makes as many loops around that vertex as needed so that this
loop never needs or can be used again. Then $p$ leaves the looped vertex and proceeds to a vertex according to the procedure in Case I, or, if there is as a follower a vertex with a loop, according to the procedure in Case II.

As noticed above, the fact that all edges $e$ from $r^{-1}(u)$ are enumerated is equivalent to defining a word formed by the sources of $e$. In our construction, we obtain the word $w(u, n, n+1) = s(e_1) \cdots s(e_j) \cdots \cdot s(\bar{e}_u)$.

Applying these arguments to every vertex $u$ of the diagram, we define an ordering $\omega$ on $B$. That $\omega$ is perfect follows from Lemma 3.15: we chose $\omega$ to have skeleton $F$, and for each $n$, constructed all words $w(v, n, n+1)$ to correspond to paths in $H_n$. The result follows.

**Example 4.7.** We continue with Examples 3.7 and 3.10, defining an order on $r^{-1}(v_2)$ where $v_2 \in V_3$, if $(f_{v_2,v_1}^{(3)}, f_{v_2,v_2}^{(3)}, f_{v_2,v_3}^{(3)}) = (1, 2, 1)$. In what follows we drop the superscript (3). This simple example illustrates why loop in the graphs $H_n$ can cause a problem. The graph $H = H_3$ is shown in Figure 5; recall that $v_1 \in [v_1, v_1], v_2 \in [v_2, v_2]$ and $v_3 \in [v_1, v_2]$. Since all the maps $\sigma_n$ are point maps, the system of relations (9) becomes trivial. The balance relations (10) become

\[
\tilde{f}_{v_2,v_1} = \tilde{f}_{v_2,v_2}, \quad \text{and} \quad \tilde{f}_{v_2,v_2} + \tilde{f}_{v_2,v_3} = \tilde{f}_{v_2,v_1} + \tilde{f}_{v_2,v_3}
\]

respectively, and our vector $(1, 2, 1)$ satisfies these constants. The only valid choice of an ordering of $r^{-1}(v_2)$ obtained using our algorithm is $w(v_2, 2, 3) = v_2v_3v_1v_2$, and in fact this is the only valid ordering possible.

**Example 4.8.** We continue with Example 3.12 and illustrate how to define an order on $r^{-1}(v_1)$ where $v_1 \in V_4$, if $(f_{v_1,v_1}^{(4)}, f_{v_1,v_2}^{(4)}, f_{v_1,v_3}^{(4)}, f_{v_1,v_4}^{(4)}) = (4, 2, 2, 3)$. In what follows we drop the superscript (4). The graph $H = H_4$ is shown in Figure 6; recall that $v_1 \in [v_1, v_1], v_2 \in [v_2, v_1], v_3 \in [v_2, v_2]$ and $v_4 \in [v_3, v_3]$. The (nontrivial part of the) system of relations (9) become

\[
\tilde{f}_{v_1,v_1} = \tilde{f}_{v_1,v_1,v_2} + \tilde{f}_{v_1,v_1,v_3} \quad \text{and} \quad \tilde{f}_{v_1,v_2} = \tilde{f}_{v_1,v_2,v_2} + \tilde{f}_{v_1,v_2,v_3},
\]

![Figure 5: The graph $H_3$ for Example 4.7](image-url)
and the balance relations (10) become, with \( \bar{v} = v_1, v_2 \) and \( v_3 \)

\[
\bar{f}_{v_1,v_4} = \bar{f}_{v_1,v_1}, \quad \bar{f}_{v_1,v_1,v_2} + \bar{f}_{v_1,v_2,v_2} = \bar{f}_{v_1,v_1} + \bar{f}_{v_1,v_2}, \quad \text{and} \quad \bar{f}_{v_1,v_1,v_3} + \bar{f}_{v_1,v_2,v_2} + \bar{f}_{v_1,v_2,v_3} = \bar{f}_{v_1,v_4}
\]

respectively. If we let

\[
\bar{f}_{v_1,v_1} = \bar{f}_{v_1,v_1,v_2} + \bar{f}_{v_1,v_1,v_3} = 2 + 1 \quad \text{and} \quad \bar{f}_{v_1,v_2} = \bar{f}_{v_1,v_2,v_2} + \bar{f}_{v_1,v_2,v_3} = 2 + 0,
\]

then the balance relations are satisfied. A valid choice of an ordering on \( r^{-1}(v_1) \) obtained using our algorithm is \( w(v_1, 3, 4) = v_1 v_2^3 v_3 v_4 v_1 v_3 v_4 v_1 v_4 v_1 \); the other is \( w(v_1, 3, 4) = v_1 v_2^3 v_3 v_1 v_4 v_1 v_3 v_4 v_1 \). Note that there are other valid choices of orderings on \( r^{-1}(v_1) \), but they are not achieved with this algorithm.

As a corollary we identify the conditions needed so that a perfect order in \( P_B^* \) is supported by \( B \). Note that one can talk of a skeleton and correspondence \( \mathcal{F} \) and \( \sigma \) of being capable of generating orders that belong to \( P_B^* \): either all perfect orders \( \omega \) having skeleton \( (\mathcal{F}, \sigma) \) belong to \( P_B^* \), or none of them do.

**Corollary 4.9.** Let \( B \) be a Bratteli diagram with incidence matrices \((F_n)\). Let \( \mathcal{F} \) be a skeleton on \( B \) and \( \sigma \) an associated correspondence that can generate orders in \( P_B^* \), and suppose that all associated graphs \( \mathcal{H}_n \) are positively strongly connected. Suppose also that for each \( M \) (and hence each sequence \((\bar{v}_n = v_n(M))\), there exists an \( n_0 \) such that for any \( n \geq n_0 \), any \( m > n \), and any \( u \in V_m \), the entries of incidence matrices \((F_n)\) satisfy condition (11). Then there is a perfect ordering \( \omega \) on \( B \) such that \( \mathcal{F} = \mathcal{F}_\omega \) and the Vershik map \( \varphi_\omega \) satisfies the relation \( \varphi_\omega = \sigma \) on \( X_{\max}(\mathcal{F}) \).

In the remaining part of this section, we will consider the class of Bratteli diagrams \( \mathcal{A} \) that is close, by its structure, to diagrams of finite rank. We refer to the notation in Definition 2.2.
Let the columns of $A_n^{(i)}$ be indexed by vertices $V_n^{(i)}$. Define $X_B^{(i)} = \{ x = (x_n) \in X_B : s(x_n) \in V_n^{(i)} \text{ for each } n \}$, and let $X_B^{(a)} = X_B \setminus \bigcup_i X_B^{(i)}$. We will call $X_B^{(i)}$ the $i$-th minimal component of $X_B$ (here minimality is considered with respect to the tail equivalence relation $\mathcal{E}$). If $B \in A$, and has incidence matrices of the form as in \[\] we will say that $B$ has $k$ minimal components.

Given a skeleton $F$ on $B$, let $X_{\max}^{(i)}(F) = X_{\max}(F) \cap X_B^{(i)}$; define $X_{\min}^{(i)}(F)$ similarly. Also, if $\omega$ is an order on $B$, define $X_{\max}^{(i)}(\omega)$ and $X_{\min}^{(i)}(\omega)$ analogously. Note that for any statement that we make about a skeleton, we can make an analogous statement about an order - simply consider the skeleton associated with the well-telescoped order.

The following lemma is straightforward.

**Lemma 4.10.** Let $B \in A$. Then

1. If $F$ is a skeleton on $B$, then, for each $i$, the sets $X_{\max}^{(i)}(F)$ and $X_{\min}^{(i)}(F)$ are closed.

2. If $\omega$ is a perfect order on $B$, then, for each $i$, $\varphi_\omega : X_{\max}^{(i)}(\omega) \to X_{\min}^{(i)}(\omega)$ is a homeomorphism.

We use Proposition [3.17] to prove the following generalization of Proposition 3.25 in [BKYT2].

**Proposition 4.11.** Let $B \in A$ have $k$ minimal components. Suppose that $C_n$ is a $d \times d$ matrix where $1 \leq d \leq k - 1$. If $k = 2$, then there are perfect orderings on $B$ only if $C_n = (1)$ for all but finitely many $n$. If $k > 2$, then there is no perfect ordering on $B$.

**Proof.** We first claim that in $\mathcal{H}_n$, there are $k$ connected components of vertices $T_n^{(1)}, \ldots, T_n^{(k)}$, such that there are no edges from vertices in $T_n^{(i)}$ to vertices in $T_n^{(j)}$ if $i \neq j$. To see this, if $1 \leq i \leq k$, let $T_n^{(i)} = \{ [\overline{v_i}, \overline{n}], \overline{v} \in V_n^{(i)}, \overline{v} \in V_n^{(i)} \}$. By Lemma 4.10 for large $n$, if $\overline{v} \in V_n^{(i)}$ it is not possible that $\overline{v} \in \sigma_n(\overline{v})$ if $\overline{v} \notin V_n^{(i)}$.

If $\omega$ is a perfect order on $B$, then by Proposition 3.17 the graphs $\mathcal{H}_n$ are weakly connected. The only way that this can happen is if there are $k - 1$ remaining vertices in $\mathcal{H}_n$ (so that $d$ must equal $k - 1$), each have one incoming edge from one of the components $T_n^{(i)}$, and one outgoing edge into one of the components $T_n^{(j)}$. Each of these remaining vertices in $\mathcal{H}_n$ corresponds to exactly one vertex in $V_n \setminus \bigcup_{i=1}^k V_n^{(i)}$. Thus at least one of the components, say $T_n^{(1)}$, has no incoming edges. Take now a vertex $t$, not belonging to any of the $T_n^{(j)}$’s, such that if $v \in t$, then $s(\overline{v}) \in T_n^{(p)}$, where $p \neq 1$. Since the row in $F_n$ corresponding to $v$ is strictly positive, we have a contradiction as there is no path from $T_n^{(p)}$ to $T_n^{(1)}$. \(\square\)
5 The infinitesimal subgroup of diagrams that support perfect orders

We will use our results from Section 4 to give an alternative proof of the following result that was proved in [GPS95] (Corollary 2). We recall that one can associate the so called dimension group with each Bratteli diagram $B = (V, E)$. Let $(F_n)$ be the sequence of incidence matrices of $B$, then the dimension group $G$ is defined as the inductive limit:

$$G := \lim_{n \to \infty} \mathbb{Z} |V_n| \xrightarrow{F_n} \mathbb{Z} |V_{n+1}|.$$ 

If $G$ is a simple dimension group, the elements $g$ of $G$ of the infinitesimal subgroup $Inf(G)$ are defined by the relation: $\tau(g) = 0$ for every trace $\tau$ on $G$.

**Theorem 5.1.** Let $B$ be a simple Bratteli diagram and $G$ its dimension group. Suppose $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(j)$, that is there is a perfect order $\omega$ on $B$ with exactly $j$ maximal paths and $j$ minimal paths where $j \geq 2$. Then $Inf(G)$, the infinitesimal subgroup of $G$, contains a subgroup isomorphic to $\mathbb{Z}^{j-1}$.

**Proof.** (1) In the proof, we will use the notation defined in Sections 3 and 4. This means that we will freely operate with such objects as the skeleton $F_\sigma$, correspondence $\sigma = (\sigma_n)$, sets of maximal and minimal vertices $V_n, \overline{V}_n$, partitions $W(n), W'(n)$, sets $E(W_v(n), u), \overline{E}(W'_v(n), u)$, maximal and minimal finite paths $\overline{e}_v, \overline{e}(V_n, u), \overline{e}(V_n, u)$ (defined just before Lemma 4.2), etc.

Fix a level $n$ such that $n > n_0$ where $n_0$ is as in the first statement of Corollary 4.3, so that $|\overline{V}_n| = |V_n| = j$ for all $i \geq n - 1$, and take a maximal vertex $\overline{v}^* \in \overline{V}_{n-1}$. We construct a sequence of vectors $(\varepsilon_{\overline{v}^*}^{(n+k)})_{k \geq 1}$ with $\varepsilon_{\overline{v}^*}^{(n+k)} \in \mathbb{Z} |V_{n+k}|$ as follows. Take first a vertex $v \in V_{n+1}$ and set

$$\varepsilon_{\overline{v}^*}^{(n+1)}(v) := \begin{cases} 
-1 & \text{if } s(\overline{v}_n) \in W'_{\sigma_{n-1}(\overline{v}^*)}(n), s(\overline{e}_v) \notin W_{\overline{v}^*}(n), \\
1 & \text{if } s(\overline{v}_n) \notin W'_{\sigma_{n-1}(\overline{v}^*)}(n), s(\overline{e}_v) \in W_{\overline{v}^*}(n), \\
0 & \text{otherwise}
\end{cases}$$

(16)

to obtain the $v$-th entry of $\varepsilon_{\overline{v}^*}^{(n+1)}$. In general, let $v$ be any vertex from $V_{n+k}$. Then we define $\varepsilon_{\overline{v}^*}^{(n+k)}$ as follows:

$$\varepsilon_{\overline{v}^*}^{(n+k)}(v) := \begin{cases} 
-1 & \text{if } s(\overline{v}(V_n, v)) \in W'_{\sigma_{n-1}(\overline{v}^*)}(n), s(\overline{e}(V_n, v)) \notin W_{\overline{v}^*}(n), \\
1 & \text{if } s(\overline{v}(V_n, v)) \notin W'_{\sigma_{n-1}(\overline{v}^*)}(n), s(\overline{e}(V_n, v)) \in W_{\overline{v}^*}(n), \\
0 & \text{otherwise}.
\end{cases}$$

(17)

(2) We will show that for any $k \geq 1$

$$F_{n+k} \varepsilon_{\overline{v}^*}^{(n+k)} = \varepsilon_{\overline{v}^*}^{(n+k+1)}.$$
To prove (18), we use another representation of entries of the vector $\varepsilon^{(n+k)}_{\bar{v}^*}(v)$. Indeed, since $\omega$ is perfect we have that relation (11) in Section 4 holds. Also, if $F(n, n + k) = F_{n+k-1} \circ \cdots \circ F_n$, $k \geq 1$, and $u \in V_{n+k}$, then relation (11) becomes

$$
\sum_{w \in W_0(n)} \tilde{f}_{u,w}^{(n, n+k)} = \sum_{w' \in W'_{n-1}^*(\bar{v})} \tilde{f}_{u',w'}^{(n, n+k)}, \quad u \in V_{n+k}.
$$

It is straightforward to check that

$$
\varepsilon^{(n+k)}_{\bar{v}^*}(v) = \sum_{w \in W_0^*(n)} f_{v,w}^{(n,n+k)} - \sum_{w' \in W'_{n-1}^*(\bar{v}^*)} f_{v,w'}^{(n,n+k)}, \quad v \in V_{n+k}.
$$

Then (18) can be proved by induction. Indeed, compute for $k = 1$

$$
F_{n+1} \varepsilon^{(n+1)}_{\bar{v}^*}(v) = \sum_{u \in V_{n+1}} f_{v,u}^{(n+1)} \left( \sum_{w \in W_0^*(n)} f_{v,w}^{(n)} - \sum_{w' \in W'_{n-1}^*(\bar{v}^*)} f_{v,w'}^{(n)} \right)
= \sum_{w \in W_0^*(n)} f_{v,w}^{(n,n+2)} - \sum_{w' \in W'_{n-1}^*(\bar{v}^*)} f_{v,w'}^{(n,n+2)}
= \varepsilon^{(n+2)}_{\bar{v}^*}(v), \quad v \in V_{n+2}.
$$

The induction step can be computed in a similar way. We omit the details.

(3) It follows from relation (18) that every vector $\varepsilon^{(n+1)}_{\bar{v}^*}, \bar{v}^* \in \overline{V}_{n-1}$, generates the element $g_{\bar{v}^*} = (\varepsilon^{(n+1)}_{\bar{v}^*}, \varepsilon^{(n+2)}_{\bar{v}^*}, \varepsilon^{(n+3)}_{\bar{v}^*}, \ldots)$ of the dimension group $G$ of $B$.

Next, since $W(n)$ and $W'(n)$ constitute partitions of $V_n$, we have

$$
\sum_{\bar{v} \in \overline{V}_{n-1}} \varepsilon^{(n+k)}_{\bar{v}}(v) = \sum_{\bar{v} \in \overline{V}_{n-1}} \left( \sum_{w \in W_0^*(n)} f_{v,w}^{(n,n+k)} - \sum_{w' \in W'_{n-1}^*(\bar{v}^*)} f_{v,w'}^{(n,n+k)} \right)
= \sum_{w \in V_n} f_{v,w}^{(n,n+k)} - \sum_{w' \in V_n} f_{v,w'}^{(n,n+k)}
= 0
$$

for any $k$ and $v$. That is the vectors $\{\varepsilon^{(n+k)}_{\bar{v}} : \bar{v} \in \overline{V}_{n-1}\}$ are linearly dependent.

On the other hand, we claim that any subset of this set containing $j - 1$ vectors is linearly independent over $\mathbb{Z}$. To simplify our notation we consider the set of vectors $\{\varepsilon^{(n+1)}_{\bar{v}} : \bar{v} \in \overline{V}_{n-1}\}$ only, since the case for the vectors $\{\varepsilon^{(n+k)}_{\bar{v}} : \bar{v} \in \overline{V}_{n-1}\}$ where $k > 1$ is considered similarly. Suppose that $\overline{V}_{n-1} = \{\bar{v}_1, \ldots, \bar{v}_j\}$. Consider the $|V_{n+1}| \times j$-matrix whose $i$-th column is the vector $\varepsilon^{(n+1)}_{\bar{v}_i}$. We fix a vertex $u \in V_{n+1}$ and look at the $u$-th row $R_u$ of this matrix; this is the row formed by the $u$-th entries of the vectors $\{\varepsilon^{(n+1)}_{\bar{v}_i} : 1 \leq i \leq j\}$. The definition of the entries of the row $R_u$ (see (16)) shows that either they are all 0 or each value 1 and $-1$ is taken exactly once, and
all remaining entries in this row are zero. Note that since this property holds for any vertex $u$, we have another proof of the fact that the sum of all vectors $\{\varepsilon^{(n+1)}_v\}$ is zero. Also, for every $\tilde{v}$, the strong connectivity of the graph $H_n$ implies that we have $|\{u \in V_{n+1} : \varepsilon^{(n+1)}_{u,\tilde{v}} = 1\}| \geq 1$ and $|\{u \in V_{n+1} : \varepsilon^{(n+1)}_{u,\tilde{v}} = -1\}| \geq 1$ (but these sets may be of different cardinalities).

We pick any vector $\varepsilon^{(n+1)}_{\tilde{v}}$ in the set $\{\varepsilon^{(n+1)}_v\}$ and show that the set of remaining vectors $\{\varepsilon^{(n+1)}_{v_i}\} \setminus \varepsilon^{(n+1)}_{\tilde{v}}$ is linearly independent. Indeed, assume that

$$\sum_{\tilde{v}_i \neq \tilde{v}} m_{\tilde{v}_i} \varepsilon^{(n+1)}_{\tilde{v}_i} = 0;$$

we shall show that all $m_{\tilde{v}_i}$'s in (20) must be equal to $m$. If one assumes that $m \neq 0$ we obtain two contradictory equalities $\sum_{\tilde{v} \neq \tilde{v}_0} \varepsilon^{(n+1)}_{\tilde{v}} = 0$ and $\sum_{\tilde{v} \neq \tilde{v}_0} \varepsilon^{(n+1)}_{\tilde{v}} = -\varepsilon^{(n+1)}_{\tilde{v}_0}$. Pick any two maximal vertices $\tilde{v}_p$ and $\tilde{v}_t$ that are not equal to $\tilde{v}$. Firstly, by the strong connectivity of $H_n$, there exist vertices $[\tilde{v}_p, \tilde{v}_p]$ and $[\tilde{v}_t, \tilde{v}_t]$ in $H_n$ that do not have loops. Secondly, also by the strong connectivity of $H_n$, we can find a path from $[\tilde{v}_p, \tilde{v}_p]$ to $[\tilde{v}_t, \tilde{v}_t]$, and we can also assume that for any vertex along this path, there are no loops (otherwise these vertices can be removed from the path, and we still have a valid path). Finally, note that if there is an edge from $[\tilde{v}', \tilde{v}']$ to $[\tilde{v}, \tilde{v}']$ and there is no loop at $[\tilde{v}', \tilde{v}']$, then $m_{\tilde{v}'} = m_{\tilde{v}}$. The result follows.

(4) It remains to show that the elements $g_{\tilde{v}} \tilde{v} \in \tilde{V}_{n-1}$, belong to the infinitesimal subgroup $Inf(G)$. Since $G$ is a simple dimension group, it suffices to check that $\tau(g_{\tilde{v}}) = 0$ for any trace $\tau$ on $G$ or, equivalently, for any invariant measure $\mu$.

Fix a probability $\varphi_\omega$-invariant measure $\mu$ on the path space of the diagram $B$. Consider the vector $p^{(n)} = (p_v^{(n)}) \in \mathbb{R}^{|V_n|}$ whose entries are $\mu$-measures of a finite path (cylinder set) with source $v_0$ and range $v$. This means that, in particular, $\mu(\tilde{c}(v_0, w)) = p^{(n)}_w$, $w \in W^{(n)}$ and $\mu(\tilde{c}(v_0, w')) = p^{(n)}_{w'}$, $w' \in W^{(n-1)}$. As noticed in [BKMS10], we have $F_n^{T} p^{(n+1)} = p^{(n)}$, $\forall n \in \mathbb{N}$, or

$$\sum_{v \in V_{n+1}} f^{(n)}_{v, u} p^{(n+1)}_v = p^{(n)}_u, w \in V_n.$$

Compute $\langle p^{(n+1)}, \varepsilon^{(n+1)}_\tilde{v} \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product:

$$\langle p^{(n+1)}, \varepsilon^{(n+1)}_\tilde{v} \rangle = \langle F_n^{T} p^{(n+2)}, \varepsilon^{(n+1)}_\tilde{v} \rangle = \langle p^{(n+2)}, F_n^{T} \varepsilon^{(n+1)}_\tilde{v} \rangle = \langle p^{(n+2)}, \varepsilon^{(n+2)}_\tilde{v} \rangle = \cdots = \langle p^{(n+j)}, \varepsilon^{(n+j)}_\tilde{v} \rangle$$

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for any $j$. On the other hand,

\[
\langle p^{(n+1)}, \varepsilon^{(n+1)} \rangle = \sum_{v \in V_{n+1}} p_v^{(n+1)} \left( \sum_{w \in W_v(n)} f_{v,w}^{(n)} - \sum_{w' \in W_{\sigma_{n-1}(v)}'(n)} f_{v,w'}^{(n)} \right) = \sum_{w \in W_v(n)} \sum_{v \in V_{n+1}} f_{v,w}^{(n)} - \sum_{w' \in W_{\sigma_{n-1}(v)}'(n)} \sum_{v \in V_{n+1}} f_{v,w'}^{(n)} = \sum_{w \in W_v(n)} p_w^{(n)} - \sum_{w' \in W_{\sigma_{n-1}(v)}'(n)} p_{w'}^{(n)} = 0
\]

because

\[
\sum_{w \in W_v(n)} p_w^{(n)} = \mu( \bigcup_{w \in W_v(n)} \bar{e}(v_0, w)) = \mu(\varphi_w( \bigcup_{w \in W_v(n)} \bar{e}(v_0, w))) = \mu( \bigcup_{w' \in W_{\sigma_{n-1}(v)}'(n)} \bar{e}(v_0, w') = \sum_{w' \in W_{\sigma_{n-1}(v)}'(n)} p_{w'}^{(n)}.
\]

This proves that $g_{\bar{v}} \in \text{Inf}(G)$. \hfill \Box

**Remark 5.2.** In the simpler case when $B$ is of finite rank, then for each $n$, $\varepsilon^{(n)}_{\bar{v}_*} = \varepsilon_{\bar{v}_*}$, each of the latter $j$ vectors correspond to an infinitesimal, and $\varepsilon_{\bar{v}_*} = -\sum_{\bar{v} \neq \bar{v}_*} \varepsilon_{\bar{v}}$, while $\{\varepsilon_{\bar{v}} : \bar{v} \neq \bar{v}_*\}$ is a linearly independent set, so that there are $j-1$ identified copies of $\mathbb{Z}$ in the infinitesimal subgroup of $\text{dim}(B)$.

**Example 5.3.** Let

\[
F_n = \begin{pmatrix}
  f_{aa}^{(n)} & f_{ab}^{(n)} & \alpha^{(n)} & \alpha^{(n)} \\
  f_{ba}^{(n)} & f_{bb}^{(n)} & \beta^{(n)} & \beta^{(n)} \\
  f_{ca}^{(n)} & f_{cb}^{(n)} & \gamma^{(n)} + 1 & \gamma^{(n)} \\
  f_{da}^{(n)} & f_{db}^{(n)} & \delta^{(n)} + 1 & \delta^{(n)} + 1 \\
\end{pmatrix}; \quad (21)
\]

then there exist orders $\omega$ on $B$ that belong to $\mathcal{P}_B \cap \mathcal{O}_B(2)$, and such that the associated graph and correspondence is as in Figure 4, where $a \in [a, a]$, $b \in [b, b]$, $c \in [a, b]$ and $d \in [b, a]$. In this case,

\[
\varepsilon_a = \begin{pmatrix}
  0 \\
  0 \\
  -1 \\
  1 \\
\end{pmatrix} \text{ and } \varepsilon_b = \begin{pmatrix}
  0 \\
  0 \\
  1 \\
  -1 \\
\end{pmatrix}; \quad (22)
\]

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so that \( F_n \varepsilon_a = \varepsilon_a \) as claimed and \( \varepsilon_a \) corresponds to an element of \( \text{Inf}(G) \).

**Example 5.4.** One can vary the given skeleton and correspondence in Theorem 5.1 to maximize the number of copies of \( \mathbb{Z} \) that one can find in \( \text{Inf}(G) \). For example, if a finite rank simple diagram \( B \) supports orders in \( \omega \in \mathcal{P}_B(2) \) that can have either of the two possible graphs described in Example 3.2, this means that there is a copy of \( \mathbb{Z} \times \mathbb{Z} \) in \( \text{Inf}(G) \). The incidence matrices of this diagram must have a very restrictive structure. For example, if \( B \) has rank 4, then the incidence matrices must be of the form

\[
F_n = \begin{pmatrix}
  a_n^{(1)} + 1 & a_n^{(2)} & a_n^{(1)} & a_n^{(2)} \\
  b_n^{(1)} & b_n^{(2)} + 1 & b_n^{(1)} & b_n^{(2)} \\
  c_n^{(1)} & c_n^{(2)} + 1 & c_n^{(1)} & c_n^{(2)} \\
  d_n^{(1)} & d_n^{(2)} + 1 & d_n^{(1)} & d_n^{(2)}
\end{pmatrix}
\]  

(23)

Bratteli diagrams with these incidence matrices have 2 orders in \( \mathcal{P}_B \cap \mathcal{O}_B(2) \), each with a different associated graph \( \mathcal{H} \). This implies that they have (at least) two independent infinitesimals, corresponding to the elements.

\[
\varepsilon = \begin{pmatrix}
  0 \\
  0 \\
  -1 \\
  1
\end{pmatrix}
\quad \text{and} \quad
\varepsilon' = \begin{pmatrix}
  1 \\
  -1 \\
  0 \\
  0
\end{pmatrix}
\]  

(24)

Next we extend Theorem 5.1 to diagrams supporting perfect orders that belong to \( \mathcal{P}_B^* \). This result overlaps Corollary 3 in [GPS95].

**Theorem 5.5.** Let \( B \) be a simple Bratteli diagram and \( G \) its dimension group. Suppose \( \omega \) belongs to \( \mathcal{P}_B^* \cup \bigcup_{j=1}^\infty \mathcal{O}_B(j) \). Then \( \text{Inf}(G) \), the infinitesimal subgroup of \( G \), contains as a subgroup the free abelian group on countably many generators.

**Proof.** The proof is similar to that of Theorem 5.1. The second statement of Corollary 4.3 tells us that for each maximal path \( M \), with \( \overline{v}_n = v_n(M) \), there exists some level \( n_0 \) such that if \( n \geq n_0 - 1, \sigma_n(\overline{v}_n) \) is a singleton. We construct a sequence of vectors \( \varepsilon_M^{(n_0+k)}(k \geq 1) \) with \( \varepsilon_M^{(n+k)}(k \geq 1) \in \mathbb{Z}^{V_{n+k}} \) as follows. Take first a vertex \( v \in V_{n_0+1} \) and set

\[
\varepsilon_M^{(n_0+1)}(v) := \begin{cases}
-1 & \text{if } s(\overline{v}_v) \in W_{\sigma_{n_0-1}(\overline{v}_{n_0-1})}(n_0), \ s(\overline{v}_v) \notin W_{\overline{v}_{n_0-1}}(n_0), \\
1 & \text{if } s(\overline{v}_v) \notin W_{\sigma_{n_0-1}(\overline{v}_{n_0-1})}(n_0), \ s(\overline{v}_v) \in W_{\overline{v}_{n_0-1}}(n_0), \\
0 & \text{otherwise}
\end{cases}
\]

to obtain the \( v \)-th entry of \( \varepsilon_M^{(n_0+1)} \). In general, let \( v \) be any vertex from \( V_{n_0+k} \). Then
we define \( \varepsilon^{(n_0+k)}_M \) as follows:

\[
\varepsilon^{(n_0+k)}_M(v) := \begin{cases} 
-1 & \text{if } s(\tau(V_{n_0}, v)) \in W'_{\sigma_{n_0-1}(\tilde{v}_{n_0-1})}(n_0), \ s(\tilde{c}(V_{n_0}, v)) \notin W_{\tilde{v}_{n_0-1}}(n_0), \\
1 & \text{if } s(\tau(V_{n_0}, v)) \notin W'_{\sigma_{n_0-1}(\tilde{v}_{n_0-1})}(n_0), \ s(\tilde{c}(V_{n_0}, v)) \in W_{\tilde{v}_{n_0-1}}(n_0), \\
0 & \text{otherwise.}
\end{cases}
\]

As in (2) of Theorem \( 5.1 \), we can show that for any \( k \geq 1 \)

\[ F_{n_0+k}^{(n_0+k)} = \varepsilon^{(n_0+k+1)}_M. \] (25)

This means that we can define, from relation \( 25 \), the element \( g_M = (\varepsilon^{(n_0+1)}_M, \varepsilon^{(n_0+2)}_M, \varepsilon^{(n_0+3)}_M, \ldots) \) of the dimension group \( G \) of \( B \). In this way, we get a countably infinite collection of elements \( \{g_M : M \text{ maximal}\} \).

The argument that the collection \( \{g_M : M \text{ maximal}\} \) generates a free abelian group, as the case of (3) of Theorem \( 5.1 \) depends on the strong connectivity of the graphs \( \mathcal{H}_n \). Take a finite set of maximal paths \( (M_1, \ldots, M_k) \) and suppose that there is a linear relation

\[
\sum_{i=1}^{k} m_i g_{M_i} = 0,
\]

where the \( m_i \)'s are nonzero. Let \( \tilde{v}_n^i = \tilde{v}_n(M_i) \) and \( \tilde{\pi}_n^i = \tilde{\pi}_n(M_i) \), and choose an \( N \) large enough so that \( \sigma_n(\tilde{v}_n^i) \) is a singleton for each \( n \geq N - 1 \) and \( 1 \leq i \leq k \). Consider the \( |V_{N+1}| \times k \)-matrix whose \( i \)-th column is the vector \( \varepsilon^{(N+1)}_{M_i} \). We fix a vertex \( u \in V_{N+1} \) and look at the \( u \)-th row \( R_u \) of this matrix; this is the row formed by the \( u \)-th entries of the vectors \( \{\varepsilon_{M_i}^{(N+1)}\}, 1 \leq i \leq k \). The definition of the entries of the row \( R_u \) shows that apart from at most one occurrence of 1 and of \(-1\), they are all 0. Note that unlike the case in the proof of (3) of Theorem \( 5.1 \) it is possible that only one of the values 1, \(-1\) appear in any row \( R_u \). Also, for every \( \tilde{v} \), the strong connectivity of the graph \( \mathcal{H}_n \) implies that we have \( |\{u \in V_{n+1} : \varepsilon_{u, \tilde{v}}^{(n+1)} = 1\}| \geq 1 \) and \( |\{u \in V_{n+1} : \varepsilon_{u, \tilde{v}}^{(n+1)} = -1\}| \geq 1 \).

The linear relation implies that \( \sum_{i=1}^{k} m_i \varepsilon^{(N+1)}_{M_i} = 0 \). It follows that if a 1 occurs in the row \( R_u \) a \(-1\) must also occur; i.e. if a 1 occurs in \( R_u \), then \( u \in [\tilde{\pi}_N, \tilde{\pi}_N^k] \) for some \( 1 \leq i, j \leq k \). Otherwise - if \( R_u \) consists only of zeros - \( u \in [\tilde{\pi}, \tilde{\pi}] \) with \( \tilde{\pi} \notin \{\tilde{\pi}_N^1, \ldots, \tilde{\pi}_N^k\} \) and \( \tilde{v} \notin \{\tilde{v}_N^1, \ldots, \tilde{v}_N^k\} \). Thus, we have partitioned \( \mathcal{H}_{N+1} \) into two disconnected sets of vertices, contradicting its strong connectivity.

The proof that each \( g_M \) is an infinitesimal is now very similar to part (4) of the proof of Theorem \( 5.1 \).

\[ \square \]

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