Optimal Routing and Power Control for a Single Cell, Dense, Ad Hoc Wireless Network

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Abstract—We consider a dense, ad hoc wireless network, confined to a small region. The wireless network is operated as a single cell, i.e., only one successful transmission is supported at a time. Data packets are sent between source-destination pairs by multihop relaying. We assume that nodes self-organise into a multihop network such that all hops are of length \(d\) meters, where \(d\) is a design parameter. There is a contention based multiaccess scheme, and it is assumed that every node always has data to send, either originated from it or a transit packet (saturation assumption). In this scenario, we seek to maximize a measure of the transport capacity of the network (measured in bit-meters per second) over power controls (in a fading environment) and over the hop distance \(d\), subject to an average power constraint.

We first argue that for a dense collection of nodes confined to a small region, single cell operation is efficient for single user decoding transceivers. Then, operating the dense ad hoc wireless network (described above) as a single cell, we study the hop length and power control that maximizes the transport capacity for a given network power constraint. More specifically, for a fading channel and for a fixed transmission time strategy (akin to the IEEE 802.11 TXOP), we find that there exists an intrinsic aggregate bit rate (\(\Theta_{opt}\) bits per second, depending on the contention mechanism and the channel fading characteristics) carried by the network, when operating at the optimal hop length and power control. The optimal transport capacity is of the form \(d_{opt}^{\frac{1}{d}} \times \Theta_{opt}\) with \(d_{opt}\) scaling as \(P_t^{\frac{1}{d}}\), where \(P_t\) is the available time average transmit power and \(\eta\) is the path loss exponent. Under certain conditions on the fading distribution, we then provide a simple characterisation of the optimal operating point.

Index Terms—Multihop Relaying, Self-Organisation

I. INTRODUCTION

We consider a scenario in which there is a large number of stationary nodes (e.g., hundreds of nodes) confined to a small area (e.g., spatial diameter 30m), and organised into a multihop ad hoc wireless network. We assume that, traffic in the network is homogeneous and data packets are sent between source-destination pairs by multihop relaying with single user decoding and forwarding of packets, i.e., signals received from nodes other than the intended transmitter are treated as interference. A distributed multiaccess contention scheme is used in order to schedule transmissions; for example, the CSMA/CA based distributed coordination function (DCF) of the IEEE 802.11 standard for wireless local area networks (WLANs). We assume that all nodes can decode all the contention control transmissions (i.e., there are no hidden nodes), and only one successful transmission takes place at any time in the network. In this sense we say that we are dealing with a single cell scenario. We further assume that, during the exchange of contention control packets, pairs of communicating nodes are able to estimate the channel fade between them and are thus able to perform power control per transmission.

There is a natural tradeoff between using high power and long hop lengths (single hop direct transmission between the source-destination pair), versus using low power and shorter hop lengths (multihop communication using intermediate nodes), with the latter necessitating more packets to be transported in the network. The objective of the present paper is to study optimal routing, in terms of the hop length, and optimal power control for a fading channel, when the network (described above) is used in a multihop mode. Our objective is to maximise a certain measure of network transport capacity (measured in bit-meters per second; see Section IV), subject to a network power constraint. A network power constraint determines, to a first order, the lifetime of the network.

Situations and considerations such as those that we study could arise in a dense ad hoc wireless sensor network. Ad hoc wireless sensor networks are now being studied as possible replacements for wired measurement networks in large factories. For example, a distillation column in a chemical plant could be equipped with pressure and temperature sensors and valve actuators. The sensors monitor the system and communicate the pressure and temperature values to a central controller which in turn actuates the valves to operate the column at the desired operating point. Direct communication between the sensors and actuators is also a possibility. Such installations could involve hundreds of devices, organised into a single cell ad hoc wireless network because of the physical proximity of the nodes. There would be many flows within the network and there would be multihopping. We wish to address the question of optimal organisation of such an ad hoc network so as to maximise its transport capacity subject to a power constraint. The power constraint relates to the network life-
time and would depend on the application. In a factory situation, it is possible that power could be supplied to the devices, hence large power would be available. In certain emergencies, “transient” sensor networks could be deployed for situation management; we use the term “transient” as these networks are supposed to exist for only several minutes or hours, and the devices could be disposable. Such networks need to have large throughputs, but, being transient networks, the power constraint could again be loose. On the other hand sensor networks deployed for monitoring some phenomenon in a remote area would have to work with very small amounts of power, while sacrificing transport capacity. Our formulation aims at providing insights into optimal network operation in a range of such scenarios.

A. Preview of Contributions

We motivate the definition of the transport capacity of the network as the product of the aggregate throughput (in bits per second) and the hop distance (in meters). For random spatio-temporal fading, we seek the power control and the hop distance that jointly maximizes the transport capacity, subject to a network average power constraint. For a fixed data transmission time strategy (discussed in Section III-B), we show that the optimal power allocation function has a water pouring form (Section V-A). At the optimal operating point (hop distance and power control) the network throughput ($\Theta_{opt}$, in bits per second) is shown to be a fixed quantity, depending only on the contention mechanism and fading model, but independent of the network power constraint (Section V-B). Further, we show that the optimal transport capacity is of the form $d_{opt}(\bar{P}_t) \times \Theta_{opt}$, with $d_{opt}$ scaling as $\bar{P}_t^{1/\eta}$, where $\bar{P}_t$ is the available average transmit power, and $\eta$ is the power law path loss exponent (Theorem 5.2). Finally, we provide a condition on the fading density that leads to a simple characterisation of the optimal hop distance (Section V-C).

II. MOTIVATION FOR SINGLE CELL OPERATION

In this context (a dense, ad hoc wireless network), the seminal paper by Gupta and Kumar [3] would suggest that each node should communicate with neighbours as close as possible while maintaining network connectivity. This maximises network transport capacity (in bit-meters per second), while minimising network average power. It has been observed by Dousse and Thiran in [4], that if, unlike [3], a practical model of bounded received power for finite transmitter power is used, then the increasing interference with an increasing density of simultaneous transmitters is not consistent with a minimum SINR requirement at each receiver. The following discussion motivates single cell operation for our framework.

Consider a planar wireless network with $n$ nodes in a square of fixed area $A$. Let $K(n)$ be the spatial reuse in the network (the number of simultaneous transmissions) and $r(K(n))$, the maximum transmitter-receiver separation. Denote by $P(K(n))$ the common power per transmitter (assumed to be the same for all nodes) satisfying a network average power constraint $\bar{P}$ (as in Section V). The maximum SINR achievable per link in such a network (with single user decoding receivers) is bounded above by $\frac{P(K(n))}{N+(K(n)-1)\frac{P(K(n))}{(2A)^{\frac{2}{\eta}}}}$, where $N$ is the receiver noise power and $I_{K(n)}$, the interference at a node due to spatial reuse. Using the finite (and fixed) area assumption, the minimum interference power from any simultaneous transmission is given by $\frac{P(K(n))}{(\sqrt{2A})^n}$. Hence, the SINR achievable over any link is bounded above by $\frac{P(K(n))}{N+(K(n)-1)\frac{P(K(n))}{(2A)^{\frac{2}{\eta}}}}$. The aggregate throughput in the network is now bounded above by $C(K(n))$,

$$C(K(n)) := K(n) \log \left(1 + \frac{P(K(n))}{N+(K(n)-1)\frac{P(K(n))}{(2A)^{\frac{2}{\eta}}}}\right)$$

Clearly, $C(K(n))$ is uniformly bounded above. Also, $r(K(n)) \leq \sqrt{2A}$. Hence, we see that the transport capacity achievable in the network (bounded above by $\sup_{K(n):K(n)\geq 2} C(K(n)) r(K(n))$) is finite, independent of the number of nodes and the network power $\bar{P}$. Further, we would expect the transmitter-receiver separation (bounded above by $r(K(n))$) to decrease to 0 as $K(n)$ increases to $\infty$ (finite area assumption). Hence, $\lim_{K(n) \rightarrow \infty} C(K(n)) r(K(n)) = 0$. This implies that there exists an optimal $K(n), 1 < K(n) < \infty$, which maximises the transport capacity in the network, i.e., the optimum spatial reuse is finite. Now, consider a simple TDMA scheme, without spatial reuse and with direct transmission between the source and the destination. For a transmit power $\bar{P}$, the TDMA schedule achieves $\Theta(\log(\bar{P}))$ transport capacity, i.e., an unbounded transport capacity as a function of the network power $\bar{P}$. As discussed above, with spatial reuse, the system becomes interference limited, and hence, becomes inefficient for large $\bar{P}$. More recently, El Gamal and Mammen [5] have shown that, if the transceiver energy and communication

$^1$ Note that $C(K(n)) \leq K(n) \log \left(1 + \frac{(2A)^{\frac{2}{\eta}}}{(K(n)-1)\frac{P(K(n))}{(2A)^{\frac{2}{\eta}}}}\right)$, independent of the transmit power $P(K(n))$. 
overheads at each hop is factored in, then the operating regime studied in [3] is neither energy efficient nor delay optimal. Fewer hops between the transmitter and receiver (and hence, less spatial reuse) reduce the overhead energy consumption and lead to a better throughput-delay tradeoff.

While optimal operation of the network might suggest using some spatial reuse (finite, as discussed above), coordinating simultaneous transmissions (in a distributed fashion), in a constrained area, is extremely difficult and the associated time, energy and synchronisation overheads have to be accounted for. In view of the above discussion, in this paper, we assume that the medium access control (MAC) is such that only one transmitter-receiver pair communicate at any time in the network.

A. Outline of the Paper

In Section III we describe the system model and in Section IV we motivate the objective. We study the transport capacity of a single cell multihop wireless network, operating in the fixed transmission time mode, in Section V. Section VI concludes the paper and discusses future work.

III. THE NETWORK MODEL

There is a dense collection of immobile nodes that use multiaccess multihop radio communication with single user decoding and packet forwarding to transport packets between various source-destination pairs.

- All nodes use the same contention mechanism with the same parameters (e.g., all nodes use IEEE 802.11 DCF with the same back-off parameters).
- We assume that nodes send control packets (such as RTS/CTS in IEEE 802.11) with a constant power (i.e., power control is not used for the control packets) during contention, and these control packets are decodable by every node in the network. As in IEEE 802.11, this can be done by using a low rate, robust modulation scheme and by restricting the diameter of the network. This is the “single cell” assumption, also used in [1], and implies that there can be only one successful ongoing transmission at any time.
- During the control packet exchange, each transmitter learns about the channel “gain” to its intended receiver, and decides upon the power level that is used to transmit its data packet. For example, in IEEE 802.11, the channel gain to the intended receiver could be estimated during the RTS/CTS control packet exchange. Such channel information can then be used by the transmitter to do power control. In our paper, we assume that such channel estimation and power control is possible on a transmission-by-transmission basis.
- In this work, we model only an average power constraint and not a peak power constraint.
- Saturation assumption: We assume that the traffic is homogeneous in the network and all the nodes have data to send at all times; these could be locally generated packets or transit packets. In [7], the authors study the problem of load balancing in dense multihop wireless networks with arbitrary traffic requirements. In our work, we do not restrict to straight line paths, and permit such a load balancing routing strategy as in [7], which ensures that the load and the channel access pattern are identical for all the nodes.

Data packets are sent between source-destination pairs by multihop relaying. Based on the dense network and traffic homogeneity assumption, we further make the following assumption.

- The nodes self-organise so that all hops are of length \(d\), i.e., a one hop transmission always traverses a distance of \(d\) meters. This hop distance, \(d\), will be one of our optimisation variables.

For a random node deployment, the hop distance that maximises the system throughput need not be the same for every node and every flow. However, the approximation holds good for a homogeneous network with large number of nodes. Further, it will be practically infeasible to optimize every hop in a dense setup with hundreds of nodes.

A. Channel Model: Path Loss, Fading and Transmission Rate

The channel gain between a transmitter-receiver pair for a hop is assumed to be a function of the hop length \(d\) and the multipath fading “gain” \(h\). The path loss for a hop distance \(d\) is given by \(\frac{d^\eta}{d^\eta\text{, where } \eta}\), where \(\eta\) is the path loss exponent, chosen depending on the propagation characteristics of the environment (see, for e.g., [8]). This variation of path loss with \(d\) holds for \(d > d_0\), the far field reference distance; we will assume that this inequality holds (i.e., \(d > d_0\)), and will justify this assumption in the course of the analysis (see Theorem 5.2).

We assume a flat and slow fading channel with additive white Gaussian noise of power \(\sigma^2\). We assume that for each transmitter-receiver pair, the channel gain due to multipath fading may change from transmission to transmission, but remains constant over any packet transmission duration. Since successive transmissions can take place between randomly selected pairs of nodes (as per the outcome of the distributed contention mechanism) we are actually
modeling a spatio-temporal fading process. We assume that this fading process is stationary in space and time with some given marginal distribution $H$. Let the cumulative distribution of $H$ be $A(h)$ (with a p.d.f. $a(h)$), which by our assumption of spatio-temporal stationarity of fading is the same for all transmitter-receiver pairs and for all transmissions. We assume that the channel coherence time, $\tau_c$, applicable to all the links in the network, upper bounds every data transmission duration in the network. Further, we assume that $H$ and $\tau_c$ are independent of the hop distance $d$.

When a node transmits to another node at a distance $d$ (in the transmitting antenna’s far field), using transmitter power $P$, with channel power gain due to fading, $h$, then we assume that the transmission rate given by Shannon’s formula is achieved over the transmission burst; i.e., the transmission rate is given by

$$C = W \log \left(1 + \frac{hP\alpha}{\sigma^2d^\alpha}\right)$$

where $W$ is the signal bandwidth and $\alpha$ is a constant accounting for any fixed power gains between the transmitter and the receiver. Note that this requires that the transmitter has available several coding schemes of different rates, one of which is chosen for each channel state and power level.

**B. Fixed Transmission Time Strategy**

We consider a fixed transmission time scheme, where all data transmissions are of equal duration, $T (< \tau_c)$ secs, independent of the bit rate achieved over the wireless link. This implies that the amount of data that a transmitter sends during a transmission opportunity is proportional to the achieved physical link rate. Upon a successful control packet exchange, the channel (between the transmitter, that “won” the contention, and its intended receiver) is reserved for a duration of $T$ seconds independent of the channel state $h$. This is akin to the “TxOP” (transmission opportunity) mechanism in the IEEE 802.11 standard. Thus, when the power allocated during the channel state $h$ is $P(h)$, $C(h)T$ bits are sent across the channel, where $C(h) = W \log \left(1 + \frac{P(h)\alpha}{\sigma^2d^\alpha}\right)$. When $P(h) = 0$, we assume that the channel is left idle for the next $T$ seconds. The transmitter does not relinquish the channel immediately, and the channel reserved for the transmitter-receiver pair (for example, by the RTS/CTS signalling) is left empty for the duration of $T$ seconds.

The optimality of a fixed transmission time scheme, for throughput, as compared to a fixed packet length scheme, can be formally established (see Appendix D), we only provide an intuition here. When using fixed packet lengths, a transmitter may be forced to send the entire packet even if the channel is poor, thus taking longer time and more power. On the other hand, in a fixed transmission time scheme, we send more data when the channel is good and limit our inefficiency when the channel is poor.

**IV. Multihop Transport Capacity**

Let $d$ denote the common hop length and $\{P(h)\}$ a power allocation policy, with $P(h)$ denoting the transmit power used when the channel state is $h$. We take a simple model for the random access channel contention process. The channel goes through successive contention periods. Each period can be either an idle slot, or a collision period, or a successful transmission with probabilities $p_i, p_c$ and $p_s$ respectively. Under the node saturation assumption, the aggregate bit rate carried by the system, $\Theta_T(\{P(h)\}, d)$, for the hop distance $d$ and power allocation $\{P(h)\}$, is given by (see (1))

$$\Theta_T(\{P(h)\}, d) := \frac{p_s(\int_0^\infty L(h) dA(h))}{p_tT_i + p_cT_c + p_s(T_o + T)}$$

where $L(h) := C(h)T$ (and $C(h)$ is a function of $\{P(h)\}$ and $d$). $T_i, T_c, T_o$ are the average time overheads associated with an idle slot, collision and data transmission. For example, in IEEE 802.11 with the RTS/CTS mechanism being used, a collision takes a fixed time independent of the data transmission rate. We note that $p_i, p_c, p_s, T_i, T_o, T_c$ depend only on the parameters of the distributed contention mechanism (MAC protocol) and the channel, and not on any of the decision variables that we consider.

With $\Theta_T(\{P(h)\}, d)$ defined as in (1), we consider $\Theta_T(\{P(h)\}, d) \times d$ as our measure of transport capacity of the network. This measure can be motivated in several ways. $\Theta_T(\{P(h)\}, d)$ is the rate at which bits are transmitted by the network nodes. When transmitted successfully, each bit traverses a distance $d$. Hence, $\Theta_T(\{P(h)\}, d) \times d$ is the rate of spatial progress of the flow of bits in the network (in bit-meters per second). Viewed alternatively, it is the weighted average of the end-to-end flow throughput with respect to the distance traversed. Suppose that a flow $i$ covers a distance $D_i$ with $\frac{D_i}{d}$ hops (assumed to be an integer for this argument). Let $\beta_i \Theta_T(\{P(h)\}, d)$ be the fraction of throughput of the network that belongs to flow $i$. Then, $\frac{\beta_i \Theta_T(\{P(h)\}, d) \times D_i}{d}$ is the end-to-end throughput for flow $i$ and $\sum \frac{\beta_i \Theta_T(\{P(h)\}, d) \times D_i}{d} = \beta \Theta_T(\{P(h)\}, d) \times d$ is the end-to-end throughput for all flows in bit-meters per second. Summing over all the flows, we have $\Theta_T(\{P(h)\}, d) \times d$, the aggregate end-to-end flow throughput in bit-meters per second.

With the above motivation, our aim in this paper is to maximise the quantity $\Theta_T(\{P(h)\}, d) \times d$ over the hop
distance \( d \) and over the power control \( \{P(h)\} \), subject to a network average power constraint, \( \bar{P} \). We use a network power constraint that accounts for the energy used in data transmission as well as the energy overheads associated with communication. The network average power, \( \mathcal{P}(\{P(h)\}) \), is given by,

\[
\mathcal{P}(\{P(h)\}) := \frac{p_iE_i + p_cE_c + p_s(E_o + T \int_0^\infty P(h) \, dA(h))}{p_iT_i + p_cT_c + p_s(T_o + T)}
\]  

(2)

\( E_i, E_c \) and \( E_o \) correspond to the energy overheads associated with an idle period, collision and successful transmission. Thus, \( E_i \) denotes the total energy expended in the network over an idle slot, \( E_c \) denotes the total average energy expended by the colliding nodes, as well as the idle energy of the idle nodes, and \( E_o \) denotes the average energy expended in the successful contention negotiation between the successful transmitter-receiver pair, the receive energy at the receiver (in the radio and in the packet processor), and the idle energy expended by all the other nodes over the time \( T_o + T \). We assume that \( E_i, E_c \) and \( E_o \) depend only on the contention mechanism and not on the decision variables \( d \) and \( \{P(h)\} \).

V. OPTIMISING THE TRANSPORT CAPACITY

For a given \( \{P(h)\} \) and \( d \), and the corresponding throughput \( \Theta_T(\{P(h)\}, d) \), the transport capacity in bit-meters per second, which we will denote by \( \psi(\{P(h)\}, d) \), is given by

\[
\psi(\{P(h)\}, d) := \Theta_T(\{P(h)\}, d) \times d
\]

Maximizing \( \psi(\cdot, \cdot) \) involves optimizing over \( d \), as well as \( \{P(h)\} \). However, we observe that, it would not be possible to vary \( d \) with fading, as routes cannot vary at the fading time scale. Hence, we propose to optimize first over \( \{P(h)\} \) for a given \( d \), and then optimize over \( d \), i.e., we seek to solve the following problem,

\[
\max_{d} \max_{\{P(h)\}: \mathcal{P}(\{P(h)\}) \leq \bar{P}} \psi(\{P(h)\}, d) =: \bar{\psi}(\{P(h)\}, d)
\]  

(3)

For a given \( d \) and power allocation \( \{P(h)\} \), define the time average transmission power, \( \bar{P}_t(\{P(h)\}, d) \), and the time average overhead power, \( \bar{P}_o \), as

\[
\bar{P}_t(\{P(h)\}, d) := \frac{p_s(\int_0^\infty P(h) \, dA(h))T}{p_iT_i + p_cT_c + p_s(T_o + T)}
\]

\[
\bar{P}_o := \frac{p_iE_i + p_cE_c + p_sE_o}{p_iT_i + p_cT_c + p_s(T_o + T)}
\]

Observe that \( \bar{P}_o \) does not depend on \( \{P(h)\} \) and \( d \). Now, the network power constraint can be viewed as

\[
\bar{P}_t(\{P(h)\}, d) \leq \bar{P} - \bar{P}_o
\]

where the right hand side is independent of \( \{P(h)\} \) or \( d \). \( \bar{P}_t := \bar{P} - \bar{P}_o \), is the time average transmitter power constraint for the network.

A. Optimization over \( \{P(h)\} \) for a fixed \( d \)

Consider the optimization problem (from (3))

\[
\max_{\{P(h)\}: \mathcal{P}(\{P(h)\}) \leq \bar{P}} \psi(\{P(h)\}, d)
\]  

(4)

The denominators of \( \Theta_T(\cdot, \cdot) \) in (1) and of \( \mathcal{P} \) in (2) are independent of \( d \) and the power control \( \{P(h)\} \). Thus, with \( d \) fixed, the optimization problem simplifies to maximizing \( \int_0^\infty L(h) \, dA(h) \) or,

\[
\int_0^\infty \log \left( 1 + \frac{P(h)h\alpha}{\sigma^2 d^n} \right) \, dA(h)
\]

subject to the power constraint,

\[
\int_0^\infty P(h) \, dA(h) \leq \bar{P}_t`
\]

where \( \bar{P}_t' \) is given by,

\[
\bar{P}_t' := \frac{(p_iT_i + p_cT_c + p_s(T_o + T))}{p_iT_i + p_cT_c + p_s(T_o + T)} \bar{P}_t
\]

\( \bar{P}_t' \) is the average transmit power constraint averaged only over the transmission periods (successful contention slots).

Without a peak power constraint, this is a well-known problem whose optimal solution has the water-pouring form (see [2]). The optimal power allocation function \( \{P(h)\} \) is given by

\[
P(h) = \left( \frac{1}{\lambda} - \frac{d^n \sigma^2}{h \alpha} \right)^+
\]

where \( \lambda \) is obtained from the power constraint equation

\[
\int_{\lambda \sigma^2 \alpha}^\infty a(h)P(h)dh = \bar{P}_t'
\]

The optimal power allocation is a nonrandomized policy, where a node transmits with power \( P(h) \) every time the channel is in state \( h \) (whenever \( P(h) > 0 \)), or leaves the channel idle for \( h \) such that \( P(h) = 0 \).

B. Optimization over \( d \)

By defining \( \xi(h) := \frac{P(h)}{\bar{P}_o} \), the problem of maximizing the throughput over power controls, for a fixed \( d \), becomes

\[
\max \int_0^\infty \log \left( 1 + \frac{\alpha h \xi(h)}{\sigma^2} \right) a(h)dh
\]

subject to

\[
\int_0^\infty \xi(h)a(h)dh \leq \frac{\bar{P}_t'}{\bar{P}_o}
\]
Observe that $\bar{P}'$ and $d$ influence the optimization problem only as $\frac{\bar{P}'}{\bar{d}^{\nu}}$. Denoting by $\Gamma\left(\frac{\bar{P}'}{\bar{d}^{\nu}}\right)$ the optimal value of this problem, the problem of optimisation over the hop-length, $d$, now becomes

$$\max_d \ d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right) \tag{5}$$

**Theorem 5.1:** In the problem defined by (5), the objective function $d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right)$, when viewed as a function of $d$, is continuously differentiable. Further, when the channel fading is random variable, $H$, has a finite mean ($\mathbb{E}(H) < \infty$), then

1) $\lim_{d \to 0} d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right) = 0$ and,

2) if in addition, $\eta \geq 2$, $\frac{d}{\eta} a\left(\frac{1}{\eta}\right)$ is continuously differentiable and $P(H > h) = O\left(\frac{1}{\eta^2}\right)$ for large $h$, then, $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right) = 0$.

**Proof:** The proofs of continuous differentiability of $d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right)$, 1) and 2) are provided in Appendix A. ■

**Remarks 5.1:**

1) Under the conditions proposed in Theorem 5.1 it follows that $d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right)$ is bounded over $d \in [0, \infty)$ and achieves its maximum in $d \in (0, \infty)$.

2) When the objective function (5) is unbounded, the optimal solution occurs at $d = \infty$ (follows from the continuity results).

3) We note that, in practice, $\eta \geq 2$.

Let $d_0$ be the far field reference distance (discussed in Section III-A).

**Theorem 5.2:** The following hold for the problem in (5).

1) Without the constraint $d > d_0$, the optimum hop distance $d_{\text{opt}}$ scales as $(\bar{P}'_{\text{opt}})^{\frac{1}{\eta}}$.

2) There is a value $\bar{P}'_{\text{tmin}}$ such that, for $\bar{P}' > \bar{P}'_{\text{tmin}}$, $d_{\text{opt}} > d_0$, and the optimal solution obeys the scaling shown in 1).

3) For $\bar{P}' > \bar{P}'_{\text{tmin}}$, the optimum power control $\{P(h)\}$ is of the water pouring form and scales as $\bar{P}'$.

4) For $\bar{P}' > \bar{P}'_{\text{tmin}}$, the optimal transport capacity scales as $(\bar{P}'_{\text{opt}})^{\frac{1}{\eta}}$.

**Proof:**

1) Let $d_{\text{opt}}$ be optimal for $\bar{P}' > 0$. We claim that, for $x > 0$, $x^{\frac{1}{\eta}} d_{\text{opt}}$ is optimal for the power constraint $x \bar{P}'$. For suppose this was not so, it would mean that there exists $d > 0$ such that

$$\left(\frac{x^{\frac{1}{\eta}} d_{\text{opt}}}{(x^{\frac{1}{\eta}} d_{\text{opt}})^{\eta}}\right) < d \Gamma\left(\frac{x \bar{P}'}{(x^{\frac{1}{\eta}} d_{\text{opt}})^{\eta}}\right)$$

or, equivalently,

$$\left(\frac{x^{\frac{1}{\eta}} d_{\text{opt}}}{(x^{\frac{1}{\eta}} d_{\text{opt}})^{\eta}}\right) < d \Gamma\left(\frac{x \bar{P}'}{(x^{\frac{1}{\eta}} d_{\text{opt}})^{\eta}}\right)$$

which contradicts the hypothesis that $d_{\text{opt}}$ is optimal for $\bar{P}'$.

2) With the path loss model $\frac{\bar{P}'}{\bar{d}^{\nu}}$, we see that for $d < d_0$, the received power is scaled more than the transmitted power $P$, due to the factor $\frac{1}{d^{\nu}}$, and an $\bar{d}^{\nu}$ factor in $\alpha$, i.e., the model over-estimates the received power and the transport capacity. Hence, the achievable transport capacity for $d < d_0$ is definitely less than $d \times \Gamma\left(\frac{\bar{P}'}{d^{\nu}}\right)$. The result now follows from the scaling result in 1).

3) It follows from 1) that, if $\bar{P}'$ scales by a factor $x$, then the optimum $d$ scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}'}{\bar{d}^{\nu}}$ is unchanged. Hence the optimal $\{\xi(h)\}$ is unchanged, which means that $\{P(h)\}$ must scale by $x$. The water pouring form is evident.

4) Again, by 1) and 2), if $\bar{P}'$ scales by a factor $x$, then the optimum $d$ scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}'}{\bar{d}^{\nu}}$ is unchanged. Thus $\Gamma\left(\frac{\bar{P}'}{\bar{d}^{\nu}}\right)$ is unchanged, and the optimal transport capacity scales as the optimum $d$, i.e., by the factor $x^{\frac{1}{\eta}}$.

**Remarks 5.2:**

The above theorem yields the following observations for the fixed transmission time model.

1) As an illustration, with $\eta = 3$, in order to double the transport capacity, we need to use $2^3$ times the $\bar{P}'$. This would result in a considerable reduction in network lifetime, assuming the same battery energy.

2) We observe that as the power constraint $\bar{P}'$ scales, the optimal bit rate carried in the network, $\Gamma\left(\frac{\bar{P}'}{\bar{d}^{\nu}}\right)$, stays constant, but the optimal transport capacity increases since the optimal hop length increases. Further, because of the way the optimal power control and the optimal hop length scale together, the nodes transmit at the same physical bit rate in each fading state; see the proof of Theorem 5.2 part 3).

**C. Characterisation of the Optimal $d$**

By the results in Theorem 5.1 we can conclude that the optimal solution of the maximisation in (5) lies in the set of points for which the derivative of $d \times \Gamma\left(\frac{\bar{P}'}{\bar{d}^{\nu}}\right)$ is zero. For a fixed $\bar{P}'$, define $\pi(d) := \frac{\bar{P}'}{d^{\nu}}$. Differentiating $d \times \Gamma(\pi(d))$, we obtain, (see Appendix A)

$$\frac{\partial}{\partial d} (d \Gamma(\pi(d))) = \Gamma(\pi(d)) - \eta \pi(d) \lambda(\pi(d))$$
where $\lambda(\pi)$ is the Lagrange multiplier for the optimisation problem that yields $\Gamma(\pi(d))$. Since $d$ appears only via $\pi(d)$, we can view the right hand side as a function only of $\pi$. We are interested in the zeros of the above expression. Clearly, $\pi = 0$ is a solution. The solution $\pi = 0$ corresponds to the case $d = \infty$; However, we are interested only in solutions of $d$ in $(0, \infty)$, and hence, we seek positive solutions of $\pi$ of

$$\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$$  \hfill (6)

Remarks 5.3: In Appendix A we consider a continuously distributed fading random variable $H$ with p.d.f. $a(h)$. The analysis can be done for a discrete valued fading distribution as well, and we provide this analysis in Appendix C. The following example then illustrates that, in general, the function $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$ can have multiple solutions. Consider a fading distribution that takes two values: $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. The function has 3 non-trivial stationary points.

![Plot of $d \times \Gamma\left(\frac{1}{\lambda}\right)$ (linear scale) vs. $d$ (log scale) for a channel with two fading states $h_1, h_2$. The fading gains are $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. The function has 3 non-trivial stationary points.](image)

The uniqueness result guarantees that a distributed implementation of the optimization problem, if it converges, provides us with the proof of this theorem.

We conclude from the above discussion that it is difficult to characterise the optimal solution when there are multiple stationary points. Hence we seek conditions for a unique positive stationary point, which must then be the maximising solution. In Appendix A we have shown that the equation characterising the stationary points, $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$, can be rewritten as

$$\int_0^1 (\log(y) - \eta(y - 1)) \frac{\lambda^2}{y^2} f\left(\frac{\lambda}{y}\right) dy = 0$$  \hfill (7)

for $f(x) := a\left(\frac{a x}{\pi}\right) e^{-\frac{a x}{\pi}}$. The density of the random variable $X := a\frac{h}{\pi}$. Notice that $\pi$ does not appear in this expression. The solution directly yields the Lagrange multiplier of the throughput maximisation problem for the optimal value of hop length. The following theorem guarantees the existence of at most one solution of $d$.

**Theorem 5.4:** If for any $\lambda_1 > \lambda_2 > 0$, $\frac{f(d)}{f(0)}$ is a strictly monotonic decreasing function of $y$, then the objective function $d \times \Gamma\left(\frac{\lambda}{\pi}\right)$ has at most one stationary point $d_{\text{opt}}, 0 < d_{\text{opt}} < \infty$.

**Proof:** The proof follows from Lemmas A.1 and A.2 in Appendix A.

**Corollary 5.1:** If $H$ has an exponential distribution and $\eta \geq 2$, then the objective in the optimisation problem of (5) has a unique stationary point $d_{\text{opt}} \in (0, \infty)$, which achieves the maximum.

**Proof:** $a(h)$ is of the form $\mu e^{-\mu h}$. From Theorem 5.1 we see that $\lim_{d \to 0} d \times \Gamma\left(\frac{\lambda}{\pi}\right) = 0$ and $\lim_{d \to \infty} d \times \Gamma\left(\frac{\lambda}{\pi}\right) = 0$. And, the monotonicity hypothesis in Theorem 5.4 holds for $a(h)$.

Remarks 5.4: 1) Hence, for $\eta \geq 2$, for the Rayleigh fading model there exists a unique stationary point which corresponds to the optimal operating point.

2) For $P' > P'_{\text{min}}$, and for the conditions in Theorem 5.1 and 5.4 let $\pi_{\text{opt}}$ denote the unique stationary point of (6). Then define $\Gamma(\pi_{\text{opt}}) = \Theta_{\text{opt}}$. It follows from Theorem 5.2 that the optimal transport capacity takes the form $\left(\frac{P'_{\text{opt}}}{\pi_{\text{opt}}}ight)^{\frac{1}{\gamma}}\Theta_{\text{opt}}$, where $\Theta_{\text{opt}}$ depends on $a(h)$ and the MAC parameters but not on $P$ (or $P'$).

3) Figure 2 numerically illustrates our results for the Rayleigh fading distribution and $\eta = 2$. Scaling $P'_{\text{opt}}$ by 4 scales the transport capacity from 2.3 to 4.6, i.e., by $\sqrt{2}$ by 4 and similarly for scaling $P'_{\text{opt}}$ by 9. The uniqueness result guarantees that a distributed implementation of the optimization problem, if it converges,
shall converge to the unique stationary point, which is the optimal solution.

VI. CONCLUSION

In this paper we have studied a problem of optimal power control and self-organisation in a single cell, dense, ad hoc multihop wireless network. The self-organisation is in terms of the hop distance used when relaying packets between source-destination pairs.

We formulated the problem as one of maximising the transport capacity of the network subject to an average power constraint. We showed that, for a fixed transmission time scheme, there corresponds an intrinsic aggregate packet carrying capacity at which the network operates at the optimal operating point, independent of the average power constraint. We also obtained the scaling law relating the optimal hop length to the power constraint, and hence relating the optimal transport capacity to the power constraint (see Theorem 5.2). Because of the way the power control and the optimal hop length scale, the optimal physical bit rate in each fading state is invariant with the power constraint. In Theorem 5.4 we provide a characterisation of the optimal hop distance for cases in which the fading density satisfies a certain monotonicity condition.

One motivation for our work is the optimal operation of sensor networks. If a sensor network is supplied with external power, or if the network is not required to have a long life-time, then the value of the power constraint, \( \bar{P} \), can be large, and a long hop distance will be used, yielding a large transport capacity. On the other hand, if the sensor network runs on batteries and needs to have a long life-time then \( \bar{P} \) would be small, yielding a small hop length. In either case, the optimal aggregate bit rate carried by the network would be the same.

In [6], the author studies the problem of developing a distributed algorithm for nodes to adapt themselves towards the optimal operating point. They first propose a distance discretization technique in which the hop distance on the critical geometric graph is used as a distance measure. Using the distance approximation, the author then develops a distributed algorithm aimed to maximize the transport capacity of the network in the sense of the framework presented in this paper.

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APPENDIX

A. Stationary Points of \( d \times \Gamma(\pi(d)) \)

Recall that we defined \( \pi(d) := \frac{P_d}{\sigma^2} \). Further, \( \Gamma(\pi(d)) \) was defined as

\[
\Gamma(\pi(d)) := \max \int_0^\infty \log \left(1 + \frac{\alpha h P(h)}{\sigma^2} \right) a(h)dh \tag{8}
\]

where the maximum is over all power controls \( \{P(h)\} \) satisfying the constraint

\[
\int_0^\infty P(h) a(h)dh \leq \pi(d) \tag{9}
\]

For ease of notation, let us use the substitution \( x := \frac{\alpha h}{\sigma^2} \). Write \( \xi(x) := \xi\left(\frac{\alpha h}{\sigma^2}\right) = \frac{P(h)}{\sigma^2} \) and \( f(x) := a \left(\frac{x^2 + \alpha^2}{\alpha^2}\right)^{-\frac{x^2}{\alpha^2}} \). Note that \( f(\cdot) \) is the probability density of the random variable \( X := \frac{\alpha^2}{\sigma^2} \). Then, equations (8) and (9) can be rewritten as

\[
\Gamma(\pi) = \max \int_0^\infty \log(1 + x\xi(x)) f(x)dx
\]

Fig. 2. Plot of \( P(d) \Gamma(\pi(d)) \) (linear scale) vs. \( \pi(=\frac{P_d}{\sigma^2}) \) (log scale) for a fading channel (with exponential distribution). We consider 3 power levels \( P_1, 4P_1 \) and \( 9P_1 \) and \( \eta = 2 \). The function has a unique optimum \( \pi_{opt}(\pi_{opt} \approx 0.2) \) for all the 3 cases.
and
\[ \int_0^\infty \xi(x)f(x)dx \leq \pi \]

This optimisation problem is one of maximising a convex functional of \( \{\xi(x)\} \), subject to a linear constraint. The optimal solution of the problem has water-pouring form, and the optimal solution is given by,
\[ \xi(x) = \left( \frac{1}{\lambda(\pi)} - \frac{1}{x} \right)^+ \]
where \( \lambda(\pi) \) is obtained from
\[ \int_0^\infty \left( \frac{1}{\lambda(\pi)} - \frac{1}{x} \right) f(x)dx = \pi \]

Further, the derivative of the optimum value \( \Gamma(\pi) \), w.r.t. \( \pi \), i.e., \( \frac{\partial \Gamma(\pi)}{\partial \pi} = \lambda(\pi) \) (see Aubin [2]).

Let us now reintroduce the dependence on \( d \), and consider the problem of maximizing \( d \times \Gamma(\pi(d)) \) over \( d \). Differentiating \( d \times \Gamma(\pi(d)) \) w.r.t. \( d \), we get,
\[ \frac{\partial}{\partial d} (d \times \Gamma(\pi(d))) = \Gamma(\pi(d)) + d \frac{\partial \Gamma(\pi(d))}{\partial d} \]
\[ = \Gamma(\pi(d)) + d \frac{\partial \Gamma(\pi(d))}{\partial \pi} \frac{\partial \pi}{\partial d} \]
\[ = \Gamma(\pi(d)) + d \Gamma(\pi(d)) \frac{\partial \pi}{\partial d} \]
where \( \Gamma'(\pi) := \frac{\partial \Gamma(\pi)}{\partial \pi} \). Substituting \( \Gamma'(\pi) = \lambda(\pi) \), we have,
\[ \frac{\partial}{\partial d} (d \Gamma(\pi(d))) = \Gamma(\pi(d)) - \eta \pi \lambda(\pi(d)) \]
(10)
The stationary points of \( d \times \Gamma(\pi(d)) \) are now obtained by equating the right hand side of (10) to zero. Note that since \( d \) appears in this equation only as \( \pi(d) \), we need only study the roots of the equation
\[ \Gamma(\pi) - \eta \pi \lambda(\pi) = 0 \]
(11)

We now proceed to obtain a characterisation of the stationary points. Substituting the optimal solution in the expression of \( \Gamma(\pi) \) and \( \lambda(\pi) \), and suppressing the argument \( \pi \) in \( \lambda(\pi) \), we get,
\[ \Gamma(\pi) = \int_\lambda^\infty \log \left( \frac{x}{\lambda} \right) f(x)dx \]
(12)
with \( \lambda \) being given by
\[ \pi = \int_\lambda^\infty \left( \frac{1}{\lambda} - \frac{1}{x} \right) f(x)dx \]
(13)

Using the substitution \( z = \frac{1}{x}, l = \frac{1}{\pi}, \) and defining \( g(z) = \frac{1}{\pi} f \left( \frac{1}{z} \right) \), (12) and (13) becomes,
\[ \Gamma(\pi) = \int_0^l \log \left( \frac{1}{z} \right) g(z)dz \]
(14)
with \( l \) (actually, \( l(\pi) \)) being given by
\[ \pi = \int_0^l (l - z) g(z)dz \]
(15)

We note that \( g(\cdot) \) is the density of the random variable \( Z := \frac{1}{X} = \frac{x^2}{\alpha H} \).

We will use the following definitions for convenience. For a function \( t(\cdot) \) of the random variable \( Z \), define the operators \( E_t(\cdot) \) and \( G_t(\cdot) \) as
\[ E_t(t(Z)) := \int_0^l t(z) g(z)dz \]
\[ G_t(t(Z)) := \int_0^l t(z)g(z)dz \]

**Lemma A.1:** The roots of (11) are equivalent to the roots of the equation
\[ \eta G_t \left( \frac{Z}{l} - 1 \right) = G_t \left( \log \left( \frac{Z}{l} \right) \right) \]
(16)
with \( l \) then being given by (15).

**Proof:** Using the definitions of \( E_t(\cdot) \) and \( G_t(\cdot) \), (14) and (15) simplify to
\[ \Gamma(\pi) = \log(l) P(Z \leq l) - G_t(\log(Z)) \]
(17)
\[ \pi = l P(Z \leq l) - G_t(Z) \]
(18)
(18) provides the \( l \) (actually \( l(\pi) \)) to be substituted in (17). Substituting for \( \Gamma(\pi) \) (from (17)), and for \( l \) (from (18)), into (11), dividing across by \( P(Z \leq l) \), and using the definition of \( E_t(\cdot) \), we have,
\[ \log \left( \frac{\pi + G_t(Z)}{P(Z \leq l)} \right) - E_t(\log(Z)) - \frac{\eta \pi}{\pi + G_t(Z)} = 0 \]
\[ \log \left( \frac{\pi}{P(Z \leq l) + E_t(Z)} \right) - E_t(\log(Z)) - \frac{\eta \pi}{\pi + G_t(Z)} = 0 \]
\[ \log \left( \frac{\pi}{G_t(Z) + 1} E_t(Z) \right) + \log \left( e^{-E_t(\log(Z))} \right) - \frac{\eta \pi}{\pi + G_t(Z)} = 0 \]
Rearranging terms, we get,
\[ \log \left( \frac{\pi + G_t(Z)}{G_t(Z)} \right) + \log \left( E_t(Z) e^{-E_t(\log(Z))} \right) - \frac{\eta \pi}{\pi + G_t(Z)} = 0 \]
Denote \( b_l := \log \left( \frac{E_l(Z)e^{-E_l(\log(Z))}}{G_l(Z)} \right) \). Then, we have,
\[
\log \left( \frac{\pi + G_l(Z)}{G_l(Z)} \right) + b_l - \frac{\eta \pi}{\pi + G_l(Z)} = 0
\]
From (18), we have
\[
\frac{G_l(Z)}{\pi + G_l(Z)} = \frac{G_l(Z)}{lP(Z \leq l)} = \frac{E_l(Z)}{\pi}
\]
which, with the previous equation, yields
\[
\log \left( \frac{E_l(Z)}{\pi} \right) + b_l - \eta \left( 1 - \frac{E_l(Z)}{\pi} \right) = 0
\]
Recall that \( l \) is actually \( l(\pi) \). We now find that \( \pi \) appears in the equation only as \( l(\pi) \). Hence we can view this as an equation in the variable \( l(\pi) \). Rearranging terms, we get
\[
- \log \left( \frac{E_l(Z)}{\pi} \right) + \eta \frac{E_l(Z)}{\pi} = -(b_l - \eta)
\]
Exponentiating both sides, and substituting back for \( b_l \), yields
\[
\frac{E_l(Z)}{\pi} e^{-\eta \frac{E_l(Z)}{\pi}} = E_l(Z) e^{-E_l(\log(Z))} e^{-\eta}
\]
On cancelling \( E_l(Z) \), and transposing terms, we next obtain
\[
e^{-\eta \frac{E_l(Z)}{\pi} - 1} = E_l(\log(\pi))
\]
or,
\[
e^{-\eta \frac{E_l(Z)}{\pi}} = E_l(\log(\pi))
\]
Taking log on both sides, we have,
\[
\eta E_l \left( \frac{Z - l}{l} \right) = E_l \left( \log \left( \frac{Z}{l} \right) \right)
\]
In terms of \( G_l(\cdot) \), this is equivalent to
\[
\eta G_l \left( \frac{Z - l}{l} \right) = G_l \left( \log \left( \frac{Z}{l} \right) \right)
\]
which is the desired result.

We next address the question of a unique positive solution of (16). The following lemma guarantees the existence of a unique positive solution, when \( f(\cdot) \), the density of \( \frac{Z}{l} \), satisfies a certain monotonicity condition.

Lemma A.2: \( f(\cdot) \) has at most one positive solution if for any \( 0 < l_1 < l_2 \), \( \frac{f(\frac{Z}{l_1})}{f(\frac{Z}{l_2})} \) is a strictly monotone decreasing function of \( y \).

Proof: Expanding \( G_l(\cdot) \), (16) becomes,
\[
\eta \int_0^l \left( \frac{z}{l} - 1 \right) g(z) dz - \int_0^l \log \left( \frac{z}{l} \right) g(z) dz = 0
\]
Rewriting the equation in terms of \( f(\cdot) \), we have,
\[
\int_0^l \left( \frac{z}{l} - 1 \right) - \log \left( \frac{z}{l} \right) \frac{1}{z^2} f \left( \frac{1}{z} \right) dz = 0
\]
Using a substitution \( y = \frac{1}{z} \) in the above equation, we get,
\[
\int_0^1 (\log(y) - \eta(y - 1)) \frac{1}{y^2 l^2} f \left( \frac{1}{y} \right) dy = 0 \tag{19}
\]
Define \( c(y) := (\log(y) - \eta(y - 1)) \frac{1}{y} \) and \( b_l(y) := f \left( \frac{1}{y} \right) \).

We are now interested in a positive \( l \) that solves
\[
\int_0^1 c(y) b_l(y) dy = 0
\]
Observe that \( \lim_{y \to 0} c(y) = -\infty \) and \( c(1) = 0 \). Further, there exists a unique \( y' \) such that \( c(y) \leq 0 \) for all \( 0 \leq y \leq y' \) and \( c(y) \geq 0 \) for all \( y' \leq y \leq 1 \). Since \( b_l(y) \geq 0 \) for all \( y \) and \( l \), we have \( c(y) b_l(y) \leq 0 \) for all \( 0 \leq y \leq y' \) and \( c(y) b_l(y) \geq 0 \) for all \( y' \leq y \leq 1 \). In particular,
\[
\int_0^{y'} c(y) b_l(y) dy \leq 0
\]
and
\[
\int_0^1 c(y) b_l(y) dy \geq 0
\]
Consider \( l_1, l_2 \) such that \( 0 < l_1 < l_2 \). By hypothesis, \( b_{l_2}(y) = b_{l_1}(y) \) is a strictly monotone decreasing function of \( y \).

Hence, \( \frac{c(y) b_{l_2}(y)}{c(y) b_{l_1}(y)} \) is also a strictly monotone decreasing function of \( y \). We then have,
\[
\int_0^{y'} c(y) b_{l_2}(y) dy = \int_0^{y'} c(y) b_{l_2}(y) b_{l_1}(y) dy > \int_0^{y'} c(y) b_{l_1}(y) dy
\]
And,
\[
\int_0^1 c(y) b_{l_2}(y) dy = \int_0^1 c(y) b_{l_2}(y) b_{l_1}(y) dy < \int_0^1 c(y) b_{l_1}(y) dy
\]
Hence,
\[
\int_0^{y'} c(y) b_{l_2}(y) dy > \int_0^1 c(y) b_{l_2}(y) dy > \int_0^{y'} c(y) b_{l_1}(y) dy
\]
Interchanging terms, we get,
\[
\int_0^{y'} c(y) b_{l_2}(y) dy > \int_0^{y'} c(y) b_{l_1}(y) dy
\]
i.e., the ratio of the negative area of the integral to the positive area of the integral is a strictly monotonic function of \( l \). Hence, as \( l \) increases, the integral (19) can cross 0 at most once, or, there exists at most one (non-trivial) solution for (19).
B. Proof of Theorem

In this section, we will use the variables and equations from the discussion in Appendix A.

Lemma B.3: \( d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right) \) is continuously differentiable with respect to \( d \).

Proof: Recall that \( \pi := \frac{\partial f}{\partial \pi^2} \), \( \Gamma(\pi) \) and \( \lambda(\pi) \) (equations (12) and (13)) are continuous function of \( \pi \), and \( \pi \) itself is a continuous function of \( d \). Hence, from (10), we see that \( d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right) \) is a continuously differentiable function of \( d \).

Lemma B.4: If \( H \) (or equivalently \( X := \frac{\partial f}{\partial \pi^2} \)) has a finite mean, then \( \lim_{d \to 0} d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right) = 0 \).

Proof: Consider (15)

\[
\int_0^l (l - z)g(z)dz = \pi
\]

where \( l \) is in fact \( l(\pi) \). Talking \( l \) outside the integral, we get,

\[
l \int_0^l \left( 1 - \frac{z}{l} \right) g(z)dz = \pi
\]

Rewriting the integral as an expectation, we have, \( l \mathbf{E}_z \left( 1 - \frac{z}{l} \right) = \pi \) or \( \mathbf{E}_z \left( 1 - \frac{z}{l} \right) = \pi \). Using Monotone Convergence Theorem, we get,

\[
\lim_{l \to \infty} \mathbf{E}_z \left( 1 - \frac{z}{l} \right) \uparrow 1
\]

or,

\[
\lim_{l \to \infty} \frac{\pi}{l} = 1
\]

From (15), we see that, \( l \to \infty \) as \( \pi \to \infty \) (\( d \to 0 \)). Hence, we have,

\[
\lim_{\pi \to \infty} l(\pi) = 1 \tag{20}
\]

Now, consider the following limit, \( \lim_{d \to 0} d \times \Gamma(\pi(d)) \), or equivalently, \( \lim_{\pi \to \infty} \pi^{-\frac{1}{2}} \Gamma(\pi) \). We know that,

\[
\pi^{-\frac{1}{2}} \Gamma(\pi) \geq 0
\]

From (14), we have,

\[
\pi^{-\frac{1}{2}} \Gamma(\pi) = \pi^{-\frac{1}{2}} \mathbf{E}_z \left( -\log \left( \frac{Z}{l(\pi)} \right) \right)
\]

Expanding the term inside the expectation, we have,

\[
= \pi^{-\frac{1}{2}} \mathbf{E}_z \left( \log \left( \frac{1}{Z} \right) + \log \frac{l(\pi)}{\pi} + \log(\pi) \right)
\]

Using the inequality \( \log \left( \frac{1}{z} \right) \leq \frac{1}{z} \) for \( z \geq 0 \) in the above inequality, we get,

\[
\leq \pi^{-\frac{1}{2}} \mathbf{E}_z \left( \frac{1}{Z} + \log \frac{l(\pi)}{\pi} + \log(\pi) \right)
\]

\( \mathbf{E}_z \left( \frac{1}{Z} \right) < \infty \) (follows from the definition \( Z := \frac{1}{\pi} \)) and the hypothesis on \( \mathbf{E}X \), \( \eta > 0 \) and from (20), we have the right hand side of the above expression \( \to 0 \) as \( \pi \to \infty \), which implies that \( \lim_{\pi \to \infty} \pi^{-\frac{1}{2}} \Gamma(\pi) = 0 \), or

\[
\lim_{d \to 0} d \times \Gamma(\pi(d)) = 0
\]

Lemma B.5: Let \( \eta \geq 2 \), \( \frac{1}{\eta} f(x) \) be continuously differentiable and \( \lim_{x \to 0} \frac{1}{\eta} f(x) = 0 \). Then \( \frac{\partial}{\partial \eta} (d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right)) \leq 0 \) as \( d \to \infty \).

Proof: From (10) and the discussion in the proof of Lemma A.1, we have,

\[
\frac{\partial}{\partial \eta} (d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right)) = \Gamma(\pi(d)) - \eta \pi(d) \lambda(\pi(d))
\]

\[
= \kappa \int_0^l \left( \eta \left( \frac{z}{l} - 1 \right) - \log \left( \frac{z}{l} \right) \right) \frac{1}{z^2} f \left( \frac{1}{z} \right) dz
\]

where \( \kappa \geq 0 \). Using a substitution \( y = \frac{z}{l} \), we get,

\[
\frac{\partial}{\partial \eta} (d \times \Gamma(\pi(d)))
\]

\[
= \kappa \int_0^l (\eta(y - 1) - \log(y)) \frac{1}{y^2} f \left( \frac{1}{y} \right) dy \tag{21}
\]

Define \( b(y) := \eta(y - 1) - \log(y) \). For \( \eta > 1 \), there exists a \( y' \) (depending on \( \eta \)) such that \( b(y) \geq 0 \) for \( 0 \leq y \leq y' \) and \( b(y) \leq 0 \) for \( y' \leq y \leq 1 \), also \( b(1) = 0 \). Then, in (21), we see that,

\[
\int_0^{y'} (\eta(y - 1) - \log(y)) \frac{1}{y^2} f \left( \frac{1}{y} \right) dy \geq 0
\]

\[
\int_0^1 (\eta(y - 1) - \log(y)) \frac{1}{y^2} f \left( \frac{1}{y} \right) dy \leq 0
\]

Further,

\[
\int_0^1 (\eta(y - 1) - \log(y)) dy = 1 - \frac{\eta}{2}
\]

For \( \eta \geq 2 \), the integral \( \int_0^1 b(y) dy \) is non-positive.

Let \( g(y) := \frac{1}{y} f \left( \frac{1}{y} \right) \). Then \( g(y) \) is continuously differentiable function and \( \lim_{y \to 0} g(y) = 0 \) (by hypothesis). Define \( y_0 \) as

\[ y_0 := \sup \{ y : g(z) = 0, 0 \leq z \leq y \} \]

If \( y_0 > 0 \), then, we see that for \( l \) sufficiently small,

\[
\int_0^{y'} (\eta(y - 1) - \log(y)) \frac{1}{y^2} f \left( \frac{1}{y} \right) dy = 0
\]

This is because for sufficiently small \( l \), \( \frac{1}{\eta} f \left( \frac{1}{y} \right) = 0 \) for \( 0 \leq y \leq y' \). Hence, \( \lim_{d \to \infty} d \times \Gamma \left( \frac{\partial f}{\partial \pi^2} \right) \leq 0 \).
If $y_0 = 0$, we then have $g'(y) \geq 0$ in a small neighbourhood of 0 (since $g$ is continuously differentiable by hypothesis). Hence, the function $g(y)$ is a monotonic increasing function in an $\epsilon$ neighbourhood of 0, i.e., $g(0) < g(y) \leq g(y') \leq g(\epsilon)$ for all $0 < y < y' < \epsilon$. Hence for all sufficiently small $l$, $\frac{1}{y^l}f\left(\frac{1}{y}\right)$ is a monotone increasing function of $y$ in $[0, 1]$. Hence, in (21), we have,

$$\int_0^y (\gamma(y-1) - \log(y)) \frac{1}{y^2} f\left(\frac{1}{y}\right) dy + \int_{y'}^1 (\gamma(y-1) - \log(y)) \frac{1}{y^2} f\left(\frac{1}{y}\right) dy$$

$$\leq \left(\frac{1}{y^l}\right)^2 f\left(\frac{1}{y^l}\right) \int_0^y (\gamma(y-1) - \log(y)) dy + \left(\frac{1}{y^l}\right)^2 f\left(\frac{1}{y^l}\right) \int_{y'}^1 (\gamma(y-1) - \log(y)) dy$$

$$= \left(\frac{1}{y^l}\right)^2 f\left(\frac{1}{y^l}\right) \left(1 - \frac{\eta}{l}\right)$$

The final expression is non-positive for $\eta \geq 2$. Thus, $\frac{\partial}{\partial d} \left(d \times \Gamma\left(\frac{\bar{P}}{\alpha}\right)\right) \leq 0$ as $d \to \infty$. $\blacksquare$

**Lemma B.6:** Let $\eta \geq 2$ and $\frac{1}{x}f\left(\frac{1}{x}\right)$ be continuously differentiable. If for large $x$, $P(X > x) = O\left(\frac{1}{x^2}\right)$ (or equivalently for $H = \frac{x^2}{\alpha}$), then $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}}{\alpha}\right) = 0$.

**Proof:** Let $P(X > x) = O\left(\frac{1}{x^2}\right)$ for large $x$, i.e.,

$$\int_x^\infty f(x) dx = O\left(\frac{1}{x^2}\right)$$

Using a substitution $z = \frac{1}{x}$, we have,

$$\int_0^1 \frac{1}{z^2} f\left(\frac{1}{z}\right) dz = O(z^2)$$

Define $g(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right)$. Then,

$$\int_0^z g(z) dz = O(z^2)$$

(22)

Since $g(z) \geq 0$ and continuous (by hypothesis), we have, $g(0) = 0$. Suppose not, then, we have $g(z) \geq \epsilon$ for all $0 \leq z < \delta$ for some $\delta$. Then,

$$\int_0^z g(z) dz \geq \epsilon z$$

for all $z \leq \delta$, which is a contradiction to (22). Hence $\lim_{z \to 0} g(z) = 0$ or $\lim_{z \to 0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 0$.

We know from (10) that

$$\frac{\partial}{\partial d} (d \Gamma(\pi(d))) = \Gamma(\pi(d)) - \eta \pi(d) \lambda(\pi(d))$$

Now from Lemma B.5, we see that, for $\eta \geq 2$, and for $d \to \infty$,

$$\Gamma(\pi(d)) - \eta \pi(d) \lambda(\pi(d)) \leq 0$$

In other words,

$$\Gamma(\pi(d)) \leq \eta \pi(d) \lambda(\pi(d))$$

Multiplying by $d$ on both the sides, we have,

$$d \Gamma(\pi(d)) \leq \eta \pi(d) \lambda(\pi(d)) d = \eta \frac{\bar{P}}{d} \lambda(\pi(d))$$

(23)

Since $\frac{\partial}{\partial d} \left(d \Gamma\left(\frac{\bar{P}}{\alpha}\right)\right) \leq 0$ as $d \to \infty$, the function $d \Gamma(\pi(d))$ is monotonic decreasing for $d \to \infty$. Also $d \Gamma(d) \geq 0$. Suppose that, $\lim_{d \to \infty} d \Gamma(\pi(d)) \neq 0$, it implies that $\lim_{d \to \infty} d \Gamma(\pi(d)) \geq \epsilon > 0$, which, using (23), implies that $\frac{\lambda(\pi(d))}{d^{\eta - 1}} \geq \epsilon$ or as $d \to \infty$.

$$\lambda(\pi(d)) \geq \epsilon d^{\eta - 1}$$

(24)

From (13), we have,

$$\int_\lambda^\infty \left(\frac{1}{x} - \frac{1}{x}\right) f(x) dx = \frac{\bar{P}}{d^\eta}$$

ignoring the negative term, we have,

$$\frac{1}{\lambda} \int_\lambda^\infty f(x) dx \geq \frac{\bar{P}}{d^\eta}$$

or,

$$\int_\lambda^\infty f(x) dx \geq \frac{\bar{P}}{d^\eta} \lambda$$

Substituting from (24), we have,

$$\int_\lambda^\infty f(x) dx \geq \frac{\bar{P}}{d^\eta} \epsilon d^{\eta - 1} = \frac{\bar{P}}{d^\eta} \epsilon \frac{1}{d}$$

(25)

But we have

$$\int_\lambda^\infty f(x) dx = P(X > \lambda) = O\left(\frac{1}{\lambda^2}\right) \leq O\left(\frac{1}{d^{2\eta - 2}}\right)$$

(26)

where the last inequality follows from (24). For $\eta \geq 2$, (25) and (26) yields a contradiction. Hence, $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}}{\alpha}\right) = 0$. $\blacksquare$
C. Discrete Fading States

The optimization problem (4) for the discrete fading state case, simplifies to

\[
\begin{align*}
\max & \quad \sum_{h \in \mathcal{H}} a_h \ln \left( 1 + \left( \frac{\alpha h_i}{\sigma^2} \right)^{P(h)} \right) \\
\text{subject to } & \quad \sum_{h \in \mathcal{H}} a_h P(h) \leq \bar{P}_i' \tag{27}
\end{align*}
\]

For notational convenience, let us index the set of fading states, \( \mathcal{H} \), in descending order by the index \( i \), \( 1 \leq i \leq |\mathcal{H}| \), i.e., \( h_1 > h_2 > h_3 > \cdots \). Further, denote

\[
a_{h_i} = a_i, \quad x_i = \frac{\alpha h_i}{\sigma^2}, \quad \text{and } \xi_i = \frac{P(h_i)}{d_0}
\]

Also, denote

\[
\Pi = \frac{\bar{P}_i'}{d_0}
\]

We will later recall that, for each power constraint \( \bar{P}_i' \), \( \Pi \) is a function of \( d \). Using this new notation and change of variables, we obtain the problem

\[
\begin{align*}
\max & \quad \sum_i a_i \ln \left( 1 + x_i \xi_i \right) \\
\text{subject to } & \quad \sum_i a_i \xi_i \leq \Pi \tag{28}
\end{align*}
\]

We have the maximisation of a concave mapping from \( \mathbb{R}^{|\mathcal{H}|} \) to \( \mathbb{R} \) subject to a linear constraint. The KKT conditions are necessary and sufficient, and the following “water pouring” form of the optimal solution is well known. There exists \( \lambda(\Pi) > 0 \), such that, for \( 1 \leq i \leq |\mathcal{H}| \),

\[
\xi_i = \left( \frac{1}{\lambda(\Pi)} - \frac{1}{x_i} \right)^+
\]

with \( \lambda(\Pi) \) being given by

\[
\sum_{\{i: x_i > 1\}} a_i \left( \frac{1}{\lambda(\Pi)} - \frac{1}{x_i} \right) = \Pi
\]

Defining, for \( 1 \leq k \leq |\mathcal{H}| \),

\[
p_k = a_1 + a_2 + \cdots + a_k, \quad \text{and } \alpha_k = \sum_{i=1}^{k} \frac{a_i}{x_i}
\]

and \( \Pi_0 = 0, \Pi_{|\mathcal{H}|} = \infty \), the Lagrange multiplier, \( \lambda(\Pi) \), is given by

\[
\lambda(\Pi) = \left( \frac{1}{p_k} (\alpha_k + \Pi) \right)^{-1} \tag{29}
\]

for \( \Pi_{k-1} < \Pi \leq \Pi_k \) when \( 1 \leq k \leq |\mathcal{H}| - 1 \), and for \( \Pi_{|\mathcal{H}|-1} < \Pi < \infty \) when \( k = |\mathcal{H}| \). Here the break-points \( \Pi_k, 1 \leq k \leq |\mathcal{H}|-1 \), are obtained by equating the values of \( \lambda(\Pi) \) on either sides of the break-points, and are expressed as

\[
\Pi_k = \left( \frac{\alpha_{k+1}}{p_{k+1}} \frac{\alpha_k}{p_k} \right)
\]

The denominator of this expression is clearly \( > 0 \), and a little algebra shows that, since \( x_{k+1} > x_i, 1 \leq i \leq k \), the numerator is also \( > 0 \).

For each \( \Pi \), let us denote the optimal value of the problem defined by (28) by \( \Gamma(\Pi) \). We infer that

\[
\frac{\partial \Gamma}{\partial \Pi} = \lambda(\Pi)
\]

Now, fixing the power constraint \( \bar{P}_i' \), and reintroducing the dependence on \( d \), we recall that \( \Pi(d) = \frac{\bar{P}_i'}{d_0^\gamma} \), and hence conclude that

\[
\frac{d \Gamma}{d_0} = \lambda(\Pi(d)) \left( -\eta \bar{P}_i' \frac{p_k}{d_0^\gamma} \right)
\]

Define \( d_0 = \infty, d_{|\mathcal{H}|} = 0 \), and, for \( 1 \leq k \leq |\mathcal{H}| - 1 \), define

\[
d_0^\gamma = \bar{P}_i' \left( \frac{p_k}{p_k} \frac{p_{k+1}}{p_{k+1}} \right)
\]

Note that \( 0 = d_{|\mathcal{H}|} < d_{|\mathcal{H}|-1} < \cdots < d_2 < d_1 = d_0 = \infty \). Now, substituting for \( \lambda(\Pi(d)) \) from (29) and integrating, yields the following result.

**Theorem C.1:** For given \( \bar{P}_i' \), the optimal value \( \Gamma(d) \) of the problem defined by (27) has the following characterisation.

1. The derivative of \( \Gamma(d) \) w.r.t. \( d \) is given by

\[
\frac{d \Gamma}{d} = \frac{1}{d} \left( -\eta \bar{P}_i' \frac{p_k}{d_0^\gamma} + \frac{\bar{P}_i'}{d_0^\gamma} \right) \tag{30}
\]

for \( d_k \leq d < d_{k-1} \) when \( 1 \leq k \leq |\mathcal{H}| - 1 \), and for \( 0 < d < d_{|\mathcal{H}|-1} \) when \( k = |\mathcal{H}| \). 

2. \( \frac{d \Gamma}{d_0} \) is a negative, continuous and increasing function of \( d \). In particular \( \Gamma(d) \) is a decreasing, and convex function of \( d \).

3. The function \( \Gamma(d) \) is given by

\[
\Gamma(d) = p_k \ln \left( \alpha_k + \frac{\bar{P}_i'}{d_0^\gamma} \right) \gamma_k \tag{31}
\]

for \( d_k \leq d < d_{k-1} \) when \( 1 \leq k \leq |\mathcal{H}| - 1 \), and for \( 0 < d < d_{|\mathcal{H}|-1} \) when \( k = |\mathcal{H}| \), with the constants of integration \( \gamma_k \) being given as follows.

\[
\gamma_1 = \frac{1}{\alpha_1} = \frac{x_1}{a_1}
\]
and, for \(2 \leq k \leq \mathcal{H}\), \(\gamma_k\) is obtained recursively as

\[
\gamma_k = \left(\frac{\alpha_{k-1} + \frac{p_k}{\sigma_{k-1}^2}}{\alpha_k} \right) \gamma_{k-1} \frac{p_{k-1}}{p_k} \frac{\sigma_{k-1}^2}{\sigma_k^2}
\]

Proof: \((31)\) is obtained by integrating the derivative in \((30)\) over each segment of its definition. The integration constants \(\gamma_k\) are obtained by equating \(\Gamma(d)\) on either sides of the break-points of the argument \(d\).

1) Optimisation over \(d\): Using Theorem C.1, we conclude that we need to look at the stationary points of \(\Gamma(d)\). To this end, consider the solutions of

\[
\Gamma(d) + d \Gamma'(d) = 0
\]

Reintroducing the variable \(\Pi = \frac{\bar{b}}{\alpha}\), and canceling \(p_k\), we need the solutions of

\[
\ln \left(1 + \frac{\Pi}{\alpha_k}\right) \alpha_k \gamma_k - \frac{\eta \Pi}{\alpha_k + \Pi} = 0
\]

for \(\Pi_{k-1} < \Pi \leq \Pi_k\) when \(1 \leq k \leq |\mathcal{H}| - 1\), and for \(\Pi_{|\mathcal{H}|-1} < \Pi < \infty\) when \(k = |\mathcal{H}|\), with the break-points \(\Pi_k\), \(1 \leq k \leq |\mathcal{H}|\), as given earlier. Let us write \(\Pi_{k+1} = 1 - \frac{b_k}{\alpha_{k+1}}\), define \(b_k = \ln \alpha_k \gamma_k\) (observe that \(b_1 = 0\)), and, for given \(k\), use the new variable

\[
y = \frac{1}{1 + \frac{b_k}{\alpha_k}}
\]

Note that, for \(0 < \Pi < \infty\), \(1 > y > 0\). Define \(\delta_k = \frac{1}{1 + \frac{b_k}{\alpha_k}}\). Then we seek the solutions of

\[
\ln \frac{1}{y} + b_k - \eta (1 - y) = 0
\]

for \(\delta_k \leq y < \delta_{k-1}\), for each \(k\), \(1 \leq k \leq |\mathcal{H}|\); note that \(\delta_1 = 1\), and \(\delta_{|\mathcal{H}|} = 0\). The equations can be written more simply as

\[
\eta = ye^{-\eta y}
\]

and are depicted in Figure 3. At this point we can conclude the following

**Theorem C.2:** There are at most \(2|\mathcal{H}|-1\) stationary points of \(\Gamma(d)\) in \(0 < d < \infty\).

Proof: The result follows from the arguments just before the theorem statement, since each line \(e^{(b_k - \eta)}\), for \(2 \leq k \leq |\mathcal{H}|\), has at most two intersections with \(ye^{-\eta y}\), in \(0 < y < 1\), and \(e^{-\eta}\) has only one such intersection.

**D. Fixed Transmission Time vs Fixed Packet Size**

In this section, we will formally establish that fixed transmission time schemes are more throughput efficient compared to fixed packet size schemes, for a given average power constraint. We will prove this result in a general framework, without explicitly modelling the underlying MAC, the power control schemes used or the channel fading distribution.

**Data Transmission Model:** In a fixed transmission time scheme, all data transmissions (with positive rate) are of a fixed amount of time \(T\), independent of the channel state \(h\) and the power used. Earlier, in our work (see Section III-B), we assumed that, when the channel fade is poor (and hence \(P(h) = 0\)), the channel is left idle for the next \(T\) seconds. Further, the optimal power control policy for such a system was found to be a non-randomized policy, where a node transmits with constant power \(P(h)\) every time the channel is in state \(h\) (see Section V-A). Here, we will allow the possibility of the channel being relinquished when bad with a fixed time overhead \(\leq T\). We consider a spatio-temporal fading process with successive transmitter-receiver pairs being selected by a distributed multiaccess contention mechanism. Hence, relinquishing the channel might improve throughput, as successive fade levels might have little correlation. The optimal policy for such a MAC could be a randomized policy. Hence, we will allow a randomized power control, i.e., for a channel state \(h\), the transmitter chooses a power \(P_h\) according to some distribution. In a fixed packet size scheme, all data transmissions (with positive rate) carry a fixed amount of data \(L\) independent of the channel state \(h\) and the power control used. Here as well, we will allow the possibility of a randomized power control and the possibility of relinquishing the channel with a fixed time overhead (when the channel fade is poor).

**Optimality Criterion:** The throughput optimality of a data transmission scheme is established either by comparing the energy required to send a certain amount of bits in a given time or by comparing the amount of bits sent with a given amount of energy in a given time. We will discuss more about this optimality criterion in Remark D.1. We study a data transmission scheme by considering two data transmissions of positive rates, in some arbitrary channel states with gains \(h_1\) and \(h_2\) and with applied powers
$P_{h_1}$ and $P_{h_2}$. We do not make any assumption on the probabilities of $h_1$ and $h_2$, and about the power control policy which yields the powers $P_{h_1}$ and $P_{h_2}$.

For a given power control scheme $(h, P_h)$, we will then assume that the transmission rate given by Shannon’s formula is achieved over the transmission burst; i.e., the transmission rate is given by

\[ C_h = W \log(1 + hP_h) \]

We have absorbed the factor \( \frac{\sigma_{d}}{\alpha_d} \) in to the term $h$ (since $d$ is fixed in this discussion). Hence, the time durations taken to transmit the $L$ bits during the channel states $h_1$ and $h_2$ (with the powers $P_{h_1}$ and $P_{h_2}$) are given by $T_{h_1} := \frac{L}{W \log(1 + h_1P_{h_1})}$ and $T_{h_2} := \frac{L}{W \log(1 + h_2P_{h_2})}$. Then, the total time occupied by these two transmissions is

\[ T_P = \frac{L}{W \log(1 + h_1P_{h_1})} + \frac{L}{W \log(1 + h_2P_{h_2})} \tag{32} \]

spending an amount of energy equal to

\[ E_p = \frac{LP_{h_1}}{W \log(1 + h_1P_{h_1})} + \frac{LP_{h_2}}{W \log(1 + h_2P_{h_2})} \tag{33} \]

Define $L_P := 2 \times L$ as the amount of bits sent in time $T_P$ using an energy $E_P$ in channel states $h_1$ and $h_2$.

Lemma D.7: Let $h_1 > h_2$. For a fixed packet size scheme, if $P_{h_1}$ and $P_{h_2}$ are applied powers during channel states $h_1$ and $h_2$, then having $h_1P_{h_1} \geq h_2P_{h_2}$ is throughput optimal.

Proof: Suppose that $h_1P_{h_1} < h_2P_{h_2}$. Then,

\[ \log(1 + h_1P_{h_1}) < \log(1 + h_2P_{h_2}) \]

Find power controls $\tilde{P}_{h_1}$ and $\tilde{P}_{h_2}$ such that

\[ \log(1 + h_1\tilde{P}_{h_1}) = \log(1 + h_2\tilde{P}_{h_2}) \tag{34} \]

\[ \log(1 + h_2\tilde{P}_{h_2}) = \log(1 + h_1\tilde{P}_{h_1}) \tag{35} \]

or, equivalently,

\[ h_1\tilde{P}_{h_1} = h_2\tilde{P}_{h_2} \tag{36} \]

\[ h_2\tilde{P}_{h_2} = h_1\tilde{P}_{h_1} \tag{37} \]

With the power control scheme $(h_1, \tilde{P}_{h_1}), (h_2, \tilde{P}_{h_2})$, the total time occupied in the transmissions of $2 \times L$ bits during the channel states $h_1$ and $h_2$ is,

\[ T_P = \frac{L}{W \log(1 + (h_1\tilde{P}_{h_1})} + \frac{L}{W \log(1 + h_2\tilde{P}_{h_2})} = T_P \tag{from 34 and 35} \]

Now, consider the energy spent to transmit these $2 \times L$ bits, i.e.,

\[ E_p = \frac{L\tilde{P}_{h_1}}{W \log(1 + h_1\tilde{P}_{h_1})} + \frac{L\tilde{P}_{h_2}}{W \log(1 + h_2\tilde{P}_{h_2})} \]

Substituting for $\tilde{P}_{h_1}$ and $\tilde{P}_{h_2}$ from (36) and (37), we have,

\[ E_p = \frac{1}{h_1} \frac{Lh_2P_{h_2}}{W \log(1 + h_2P_{h_2})} + \frac{1}{h_2} \frac{Lh_1P_{h_1}}{W \log(1 + h_1P_{h_1})} \]

Rearranging the terms, we have,

\[ E_p = \frac{1}{h_2} \frac{Lh_1P_{h_1}}{W \log(1 + h_1P_{h_1})} + \frac{1}{h_1} \frac{Lh_2P_{h_2}}{W \log(1 + h_2P_{h_2})} \]

\[ < \frac{1}{h_1} \frac{Lh_2P_{h_2}}{W \log(1 + h_2P_{h_2})} + \frac{1}{h_2} \frac{Lh_1P_{h_1}}{W \log(1 + h_1P_{h_1})} \]

\[ = \frac{1}{h_1} \frac{Lh_2P_{h_2}}{W \log(1 + h_2P_{h_2})} + \frac{1}{h_2} \frac{Lh_1P_{h_1}}{W \log(1 + h_1P_{h_1})} \]

\[ = E_p \]

where the inequality follows from the fact that

\[ \frac{Lh_1P_{h_1}}{W \log(1 + h_1P_{h_1})} \left( \frac{1}{h_2} - \frac{1}{h_1} \right) \]

since $h_1 > h_2$ and $h_1P_{h_1} < h_2P_{h_2}$ (by assumption) and the fact that $\log(1+x)$ is strictly monotone increasing.

It follows that an optimal power control must have $h_1P_{h_1} \geq h_2P_{h_2}$.

Remark: From Lemma D.7 we see that, when $h_1 > h_2$, $C_{h_1} := W \log(1 + h_1P_{h_1}) \geq W \log(1 + h_2P_{h_2}) =: C_{h_2}$, or equivalently, $T_{h_1} \leq T_{h_2}$.

We will now provide a comparison of the fixed packet size scheme with a fixed transmission time scheme and show the optimality of the fixed transmission time schemes. The comparison is done under the following assumption.

- The channel has the same marginal fading distribution, whenever sampled by a transmitter, for either schemes. This is a reasonable assumption as we consider spatio-temporal fading, with successive transmissions from possibly different source-destination pairs chosen by the distributed multiaccess contention scheme.

For the fixed packet size scheme, $L_P := 2 \times L$ bits were transmitted in $T_P = T_{h_1} + T_{h_2}$ time (see (32)) with an amount of energy equal to $E_p$ (see (33)), in two channel samples $h_1$ and $h_2$. A reasonable comparison would be to find the throughput of a fixed transmission time scheme for a total duration of $T_P$ seconds involving two data transmissions with channel samples $h_1$ and $h_2$ of equal duration $T = \frac{T_P}{2}$ and a total energy of $E_p$. We will assume that $P_{h_1}$ and $P_{h_2}$, the power used for the fixed packet size scheme are such that $T_{h_1} \leq T_{h_2}$ (see Lemma D.7). Hence, we have $T_{h_1} \leq T \leq T_{h_2}$, or, the fixed transmission time scheme spends relatively more time on a better channel. Clearly, its throughput is better than the fixed packet size scheme for the same energy constraint, as seen below.
Let $P_{th_1}$ and $P_{th_2}$ be the optimal power control for the fixed transmission time strategy such that

$$E_T := P_{th_1}T + P_{th_2}T = P_{th_1}T_{h_1} + P_{th_2}T_{h_2} = E_P$$

We have,

$$L_P = 2L = T_{h_1}W \log(1 + h_1 P_{h_1}) + T_{h_2}W \log(1 + h_2 P_{h_2})$$

Expanding the left hand side, we have,

$$2L = T_{h_1}W \log(1 + h_1 P_{h_1}) + (T_{h_2} - T)W \log(1 + h_2 P_{h_2})$$
$$+ TW \log(1 + h_2 P_{h_2})$$

Using $h_1 > h_2$, we get,

$$2L \leq T_{h_1} \log(1 + h_1 P_{h_1}) + (T_{h_2} - T) \log(1 + h_1 P_{h_1})$$
$$+ T \log(1 + h_2 P_{h_2})$$
$$\leq T \log(1 + h_1 P_{h_1}) + T \log(1 + h_2 P_{h_2})$$
$$=: L_T$$

where the last inequality follows from the fact that $(h_1, P_{h_1})$ and $(h_2, P_{h_2})$ is the optimal power control scheme for the fixed transmission time scheme with time $T_P (= 2 \times T)$ and energy $E_T (= E_P)$.

Remarks D.1: For $L(t)$ defined as the amount of bits sent up to time $t$, and $E(t)$ defined as the total energy spent up to time $t$, the average throughput ($\Theta$) and the average power ($\bar{P}$) of the system are, in general, defined as

$$\Theta := \lim_{t \to \infty} \inf \frac{L(t)}{t}$$
$$\bar{P} := \lim_{t \to \infty} \sup \frac{E(t)}{t}$$

Under additional assumptions on the fading process and the power control scheme used, the expressions are simplified as an ensemble average (for example, see (1) and (2) for a fixed transmission time scheme). In this section, the optimality of the schemes have been shown directly, by comparing the amount of bits transmitted for a particular sample of channel for a given amount of time and energy, or by comparing the amount of energy used to transmit a given amount of bits for a particular sample of channel in a given amount of time. For example, the argument provided here directly translates to an argument with the ensemble average for the discrete fading case. This approach is not only straightforward, but also is very general.