We present a renormalization group analysis of the disorder effects on the low-energy behaviors of two-dimensional tilted Dirac-fermion systems, in which the fermions have two distinct orbitals unrelated by any symmetry. Four types of disordered potential, two interorbital and two intraorbital, are considered. If there is only one type of interorbital disorder, the fermion-disorder scattering induces logarithmic or power-law corrections to the fermion density of states and specific heat. In contrast, the intraorbital disorder can turn the system into a strongly disordered phase. In this disordered phase, calculations based on self-consistent Born approximation reveal that the Dirac point is destroyed and replaced by a bulk Fermi arc. We also study the interplay of four types of disorder, and find that the Dirac point can either remain intact or give place to a Fermi arc. We obtain the condition for the emergence of a Fermi arc in this case. Our results indicate that disorders can result in rich low-energy properties of tilted Dirac fermions.

PACS numbers: 71.10.Hf, 73.43.Nq, 74.62.En

I. INTRODUCTION

Tilted Dirac/Weyl semimetal (SM), characterized by the tilting of the conic spectrum of fermionic excitations[1–4] has attracted increasing theoretical and experimental interest. For sufficiently large tilt, the Fermi surface crossing the Dirac nodes becomes lines in experimental interest. For sufficiently large tilt, the excitations by the tilting of the conic spectrum of fermionic SM such a system is usually called type-II Dirac/Weyl SM[45]. The tilt-induced unusual Fermi surface is found to produce a variety of novel phenomena, including unconventional magnetic-optical response[9,15], magnetic breakdown[16], anomalous Hall effect[17,18] and anomalous Nernst and thermal Hall effect[19,20]. Meanwhile, several scenarios have been proposed to realize tilted Dirac/Weyl fermions in different regimes[14,46,19]. Recent angle-resolved photoemission spectroscopy experiment[20,24] have reported evidence of their existence.

In previous works, the two degenerate states at Dirac point usually refer to the spin components. In this case, the free tilted Dirac fermions respect at least one of the time-reversal, spatial inversion, and particle-hole symmetries, although the fundamental Lorentz symmetry is always broken by the tilt. However, it is in principle possible that the two degenerate states of fermions arise from two distinct degrees of freedom that are not related by any symmetry. For example, on the (001) surfaces of topological crystalline insulators SnTe and Pb1−xSnxTe, the two components of Dirac fermions are made out of the cation Sn/Pb orbital and the anion Te/Se orbital, respectively. Similar features occur in some heavy fermion SMs due to the hybridization of f- and d-bands[25,27].

Disorder plays different roles in tilted Dirac/Weyl fermion systems with and without symmetry constraints. If at least one of the time-reversal, spatial inversion, and particle-hole symmetries is respected, disorder can lead to several possible quantum phase transitions. For instance, a compressible diffusive metal (CDM) phase[22,15] in which the fermions acquire a finite zero-energy disorder scattering rate γ0 and a finite zero-energy density of states (DOS) ρ (0), is realized if some kind of disorder is coupled to a single tilted Weyl con[23,15]. Disorder may trigger a metal-insulator transition in tilted Weyl fermion system[11]. Moreover, disorder is predicted to drive a novel topological phase transition between type-I and type-II SM states by reducing the topological mass[11,12]. In case the Dirac/Weyl fermions have two components that are not related by any symmetry, the disorder effect is still not well studied. In Ref.13, the authors considered one special type of disorder that has never been studied before, and showed that such disorder can destroy the Dirac point and replace it with a bulk Fermi arc. Thus far, it remains unclear how other types of disorder influence the low-energy behavior of the system.

In this paper, we study the disorder effects on 2D tilted Dirac fermions that have two distinct orbitals. We only consider bilinear fermion-disorder couplings based on the requirement that any single type of disorder can exist alone without generating other types of disorder under the renormalization group (RG) flow. In particular, we will identify four types of disorder (see Appendix A for a detailed analysis), two interorbital and two intraorbital. After carrying out RG calculations, we find that the RG flow of one type of interorbital disorder behaves as the random gauge potential (RGP) widely studied in conventional 2D Dirac SM[44] and that the other one resembles random mass (RM). RGP is unrenormalized owing to a local time-independent gauge symmetry, and produces a stable non-Fermi liquid (NFL) state in which the density of states and specific heat receive power-law corrections. Different from RGP, RM is a marginally irrelevant perturbation, leading to logarithmic enhancement of DOS and specific heat. Neither RGP nor RM could generate the bulk Fermi arc[14]. The RG behavior for intraorbital disorder is analogous to the random scalar potential (RSP) of conventional 2D Dirac SM[41]. Both of the two types of intraorbital disorder are marginally relevant and turn the system into a strongly disordered phase[13]. We calculate the fermion self-energy in such a disordered phase by using the self-consistent Born approximation.
(SCBA), and obtain two different scattering rates, corresponding to two different orbitals. The difference in these two scattering rates gives rise to a bulk Fermi arc, consistent with previous work\textsuperscript{[53]}. We also consider the case in which all the four types of disorder coexist, and demonstrate that they all become marginally relevant at low energies due to their interplay, which invariably drives the system into a strongly disordered phase. We re-perform an SCBA calculation, and find that the system might exhibit either stable Dirac point or bulk Fermi arc, depending on the values of a number of model parameters. We obtain the conditions for the emergence of Fermi arc after computing the fermion self-energy functions. It is interesting that two distinct orbitals may have exactly the same disorder scattering rate, which prevents the emergence of Fermi arc.

The remainder of the paper is organized as follows. We present the model Hamiltonian and derive the RG equations in Sec. II. The impact of each single type of disorder and the interplay of all four types of disorder are analyzed in Sec. III. We analyze the conditions for Fermi arc or Dirac point to exist in Sec. IV. We summarize the results and highlight possible future works in Sec. V.

\section{Model and Flow Equations}

As the starting point, we consider non-interacting tilted Dirac fermions near one single Dirac cone described by the Hamiltonian:\textsuperscript{[53]}

\[ H_0(p) = \psi^\dagger(p) \left( v_x p_x \sigma_z + v_y p_y \sigma_x + w v_x p_x \sigma_0 \right) \psi(p), \]

where \( \psi^\dagger = \left( \psi_1^\dagger, \psi_2^\dagger \right) \) is a two-component fermion field, \( \sigma_0 \) is the 2 \times 2 identity matrix, and \( \sigma_i \) (\( i = x, y, z \)) are the three Pauli matrices. We use \( v_x \) and \( v_y \) to denote the fermion velocities along the \( x \)- and \( y \)-directions, respectively. Without loss of generality, we choose \( v_{x,0} > 0 \). A dimensionless tilting parameter \( w \) is introduced along the \( x \)-axis. For type-I Dirac fermions, the tilt is limited to the range of \( |w| < 1 \), whereas \( |w| > 1 \) for type-II. The point at which \( |w| = 1 \) is called Lifshitz transition point, which separates type-I from type-II Dirac fermions. The energy dispersion of Hamiltonian Eq. (1) is given by

\[ E_\pm(p) = w v_x p_x \pm \sqrt{v_x^2 p_x^2 + v_y^2 p_y^2}. \]

For type-II Dirac fermions, setting \( E_\pm = 0 \) produces two Fermi lines, described by the relation:

\[ v_y p_y = \pm v_x p_x \sqrt{w^2 - 1}. \]

Now the Fermi surface is not point-like. To perform RG analysis, one can either adapt the low-energy effective theory to model the fermionic excitations near the Fermi surface\textsuperscript{[35]}, or employ a parameter regularization scheme\textsuperscript{[60]} to ensure that the RG transformations are scaled to the Fermi surface. For our purpose, we restrict the tilt to the range of \( |w| < 1 \) to make the Fermi surface closed. It is worth mentioning that, such type-I tilted Dirac fermions can be realized in several materials, including the (001) surface state of SnT\textsubscript{2} and the organic conductor \( \alpha \)-(BEDT-TTF)\textsubscript{2}I\textsubscript{3}\textsuperscript{[14,15,16,17,18,19]}. We now incorporate various disordered potentials into the non-interacting Hamiltonian by adding the following fermion-disorder coupling term:\textsuperscript{[10,11,12,13,14,15,16,17,18,19]}

\[ H_{\text{dis}}(x) = \sum_\gamma A_\gamma(x) \psi^\dagger(x) \gamma \psi(x), \]

where \( A_\gamma(x) \) represents a given type of randomly distributed potential and \( \gamma \) is a 2 \times 2 matrix. Here, \( A_\gamma(x) \) is assumed to be a quenched, Gaussian white noise potential, characterized by two features

\[ \langle A_\gamma(x) \rangle = 0, \ \langle A_\gamma(x) A_{\gamma'}(x') \rangle = \delta_{\gamma,\gamma'} \Delta_\gamma \delta^2(x-x'). \]

Dimensionless variance \( \Delta_\gamma \) measures the strength of random potential.

Disorders are classified by the definition of \( \gamma \). For a Dirac fermion system that respects at least one of time-reversal, spatial inversion, and particle-hole symmetries, \( \gamma \) can be the identity matrix or any of the three Pauli matrices\textsuperscript{[14]}. In particular, \( \gamma = \sigma_0 \) corresponds to RSP, \( \gamma = \sigma_z \) to RM, and \( \gamma = (\sigma_x, \sigma_y) \) to the two components of an RGP. However, for the tilted Dirac fermions under consideration, \( \gamma \) should be chosen in a different way. As demonstrated in Ref.\textsuperscript{[15,16,17,18,19]}, the disorder defined by \( \gamma = \sigma_0 \) and the one by \( \gamma = \sigma_z \) are not self-closed in the RG analysis, because their coupling to Dirac fermions inevitably generates different types of disorder. In this paper, we choose to define intraorbital disorder by the matrix \( \gamma = A \sigma_0 + B \sigma_z \), where \( A \) and \( B \) are two constants. According to Appendix A the self-closeness of the RG analysis allows us to choose \( (A, B) = (1/2, \pm 1/2) \). We identify these two types of disorder as RSPs. For interorbital disorders, the matrix can be formally written as \( \gamma = A' \sigma_x + B' \sigma_y \). To obtain self-closed RG equations, we are forced to define \( \gamma = \sigma_x \) and \( \gamma = \sigma_y \), which are called RGP and RM, respectively.

We now see that, there are only four possible types of disorder in the model considered in this paper. Any one of them can exist alone. However, to make our analysis more generic, we assume that they exist simultaneously\textsuperscript{[15,16,17,18,19]}, called RGP and RM, respectively. The random potential \( \gamma \) can be the identity matrix or any of the three Pauli matrices\textsuperscript{[14]}. The Keldysh formalism\textsuperscript{[50]} and the replica trick\textsuperscript{[24,25,26,27]} at the perturbation level, they are equivalent\textsuperscript{[17,18,19]}. Here, we employ the replica trick, which has been most widely used in the literature. To average over \( \gamma(x) \), we assume that the spatial distribution of \( A_\gamma(x) \) is Gaussian, described by \( P[A_\gamma] = \exp \left[ - \int d^2 x A_\gamma(x) / (2 \Delta_\gamma) \right] \). After...
applying the replica trick, we express the total effective action in the Euclidean space as follows

\[
S = \int d\tau d^2x \psi_m^\dagger \left[ \partial_\tau - i(\sigma_z + w\sigma_0)v_x \partial_x - iv_y \sigma_x \partial_y \right] \psi_m - \sum_i \frac{\Delta_i}{2} \int d\tau d\tau' d^2x \left( \psi_m^\dagger \sigma_i \psi_m \right)_x \left( \psi_m^\dagger \sigma_i \psi_m \right)_{x'}, \tag{6}
\]

where \(i = x, y, \pm\) are used to indicate the disorder type and the parameters \(\Delta_i\) represent the corresponding disorder coupling constants. Moreover, \(m\) and \(n\) are replica indices, which are summed over from 1 to \(R\) automatically. At the last step of the calculation, the replica limit \(R \rightarrow 0\) should be taken. Although the replica limit may give rise to unphysical results in some non-perturbative studies\cite{62-64}, it is well-defined in the perturbation RG analysis\cite{59,65}.

We have completed the RG analysis of the action given by Eq. (6). Weak coupling expansion is adopted in the perturbative calculation. All the relevant one-loop Feynman diagrams are presented in Fig. 1. After integrating out the fast modes defined within the momentum shell \(e^{-\Lambda} \ll |p| < \Lambda\), followed by RG transformations\cite{65}, we obtain a number of coupled RG equations:

\[
\frac{d\psi}{d\ell} = -2 \left[ \beta_+ \left( \frac{1-w}{1+w} \right) + \beta_- \left( \frac{1+w}{1-w} \right) \right], \tag{7}
\]

\[
\frac{d\psi_x}{d\ell} = -\left[ \beta_+ - \beta_- \right] \psi_x, \tag{8}
\]

\[
\frac{d\psi_y}{d\ell} = -2 \left( \beta_y - \beta_x \right) \psi_y + \beta_x \left( \frac{1-w}{1+w} \right) \psi_y + \beta_- \left( \frac{1+w}{1-w} \right) \psi_y, \tag{9}
\]

\[
\frac{d\beta_x}{d\ell} = \beta_y \left[ \beta_+ \left( \frac{1-w}{1+w} \right) + \beta_- \left( \frac{1+w}{1-w} \right) \right], \tag{10}
\]

\[
\frac{d\beta_y}{d\ell} = -2 \beta_x \beta_y + \beta_+ \left( \frac{1-w}{1+w} \right) \beta_+ \left( \frac{1+w}{1-w} \right) \beta_- \left( \frac{1+w}{1-w} \right), \tag{11}
\]

\[
\frac{d\beta_i}{d\ell} = \frac{-w\beta_i^\dagger \pm \beta_i}{1 \mp w} + \beta_+ \left[ \beta_+ \left( \frac{1-w}{1+w} \right) \beta_- \left( \frac{1+w}{1-w} \right) \beta_y \right] + \beta_- \left( \frac{1-w}{1+w} \right) \beta_+ \left( \frac{1+w}{1-w} \right) \beta_y, \tag{12}
\]

where

\[
\beta_i = \frac{\Delta_i}{2\pi v_x v_y \sqrt{1-w^2}} \tag{13}
\]

represent the effective strength of fermion-disorder interaction. The detailed RG calculational steps can be found in Appendix A. In the next section, we analyze the solutions to these RG equations and discuss the impact of various random potentials on tilted Dirac fermions. We will first consider each single type of disorder, and then their interplay.

### III. DISORDER EFFECTS

Our first aim in this section is to judge the relevance (or irrelevance) of each type of disorder. For a marginal or an irrelevant disorder, the perturbative RG is well under control in the weak-coupling regime. The disorder effects can be examined by directly computing the interaction corrections to some observable quantities. For a relevant disorder, the perturbative RG method breaks down as the disorder always flows to a strong coupling regime at low energies. In this case, we will employ SCBA to calculate disorder scattering rate and then to analyze the low-energy properties of the strongly disordered phase.

#### A. Random gauge potential

We first consider interorbital disorder scattering. When RGP exists alone in the system, one can set \(\beta_\alpha = \beta_\pm = 0\). The RG equation of RGP parameter is simply

\[
\frac{d\beta_x}{d\ell} = 0, \tag{14}
\]

which means that \(\beta_x\) is marginal at the one-loop level. Actually, this conclusion is valid up to any order of loop expansion due to the presence of a local gauge symmetry:

\[
\psi_x \rightarrow \psi_x e^{i\xi(y)}, \quad A(x) \rightarrow A(x) + \partial_y \xi(y). \tag{15}
\]

A detailed proof of this un-renormalization is provided in Appendix B. In a (2+1)D untitled Dirac fermion system, RGP is also marginal at any finite loop level, because the time-independent gauge transformation guarantees that RGP is un-renormalized\cite{61,62}. Such a similarity provides further clue for the identification of the disorder defined by \(\gamma = \sigma_x\) as RGP.
Since RGP is marginal, we will be able to obtain analytical solutions for the fermion velocities and the tilt parameter. First, we set
\[ \beta_x (E) = \beta_x (\Lambda), \]  
(16)
where \( \Lambda \) is the UV cutoff that is determined by the bandwidth. \( E = \Lambda e^{-\ell} \) is the low energy scale in which we are interested. The constant \( \beta_x (\Lambda) \) is the disorder strength defined at UV cutoff \( \Lambda \). Hereafter, unless otherwise stated, the physical quantities defined at \( \Lambda \) are taken as constants, but are regarded as variables if defined at the varying energy \( E \). Upon substituting Eq. (16) into Eq. (7), the tilt parameter has the form
\[ w (E) = w (\Lambda) \left( \frac{E}{\Lambda} \right)^{\beta_x (\Lambda)}, \]  
(17)
which vanishes as \( E \to 0 \). We then substitute Eqs. (16) and (17) into Eq. (5) and Eq. (6), and obtain the following \( E \)-dependent velocities:
\[ v_x (E) = \frac{v_x (\Lambda) \left( 1 - w^2 (\Lambda) \right) (E/\Lambda)^{\beta_x (\Lambda)}}{(E/\Lambda)^{2\beta_x (\Lambda)} - w^2 (\Lambda)}, \]  
(18)
\[ v_y (E) = \frac{v_y (\Lambda) \sqrt{1 - w^2 (\Lambda)}}{(E/\Lambda)^{2\beta_x (\Lambda)} - w^2 (\Lambda)}, \]  
(19)
In the low-energy region, we simplify these expressions as power functions of momentum \( k \), namely
\[ v_{x,y} (E) \big|_{E \to 0} \sim E^{\beta_x (\Lambda)} \propto k^{\eta}, \]  
(20)
where \( \eta = \beta_x (\Lambda) \). We can see that \( v_x \) and \( v_y \) acquire the same finite anomalous dimension \( \eta \). A similar positive anomalous dimension has previously been obtained in Ref.23 when studying the fermion velocity renormalization of Dirac fermions in finite-density QED. Moreover, this kind of fermion velocity renormalization is a special property of several Dirac fermion systems, such as graphene2526 and high-\( T_c \) superconductors2728. It may lead to a number of unusual spectral and thermodynamic properties of untitled Dirac fermions2928. Here we show that this phenomenon is also induced by RGP in the tilted Dirac fermion system.

Next, we analyze the influence of marginal RGP on several important quantities. The quasiparticle residue is defined as
\[ Z_f = \frac{1}{1 - \frac{\partial}{\partial \omega} \text{Re} \Sigma^R (\omega) \big|_{\omega \to 0}}, \]  
(21)
where \( \Sigma^R \) is the retarded fermion self-energy. Since the electrons of \( \sigma_z = \pm 1 \) orbitals are not connected by any symmetry, their one-loop self-energy corrections might be different, as can be seen from Eq. (A4) in Appendix A. Accordingly, the residue could take different values for the two orbitals. It is also possible to compute the residue within the RG framework. Making use of the RG solutions, we express the residue for \( \sigma_z = \pm 1 \) orbitals in the following form:
\[ \frac{dZ^\pm}{d\ell} = - \frac{\beta_x}{1 \pm w} Z_f^\pm. \]  
(22)
By substituting Eqs. (16) and (17) into this equation, we find that:
\[ Z_f^\pm (E) = \frac{1 \pm w (\Lambda)}{(E/\Lambda)^{\beta_x (\Lambda)} \pm w (\Lambda)}. \]  
(23)
At low energies, the residue exhibits the same energy dependence, namely
\[ Z_f^\pm (\omega) \sim \omega^{\beta_x (\Lambda)}. \]  
(24)
Here, we have simply replaced \( E \) with \( \omega \). Combining this result with Eq. (21), we obtain
\[ \text{Re} \Sigma_{ii}^R (\omega) \sim \omega^{1-\beta_x (\Lambda)}. \]  
(25)
Using the Kramers-Kronig relation, we obtain the imaginary part of retarded fermion self-energy:
\[ \text{Im} \Sigma_{ii}^R (\omega) \sim \omega^{1-\beta_x (\Lambda)}. \]  
(26)
Because \( \beta_x (\Lambda) > 0 \), both of the two different orbitals display NFL-like low-energy behaviors. Moreover, the difference in the residues \( Z_f^\pm \) and \( Z_f \) vanishes at low energies.

According to the results of Appendix C, the DOS and specific heat of clean tilted Dirac fermions are given by
\[ \rho_0 (E) = \frac{E}{\pi v_x v_y (1 - w^2)^{3/2}}; \]  
(27)
\[ C_v (T) = \frac{18 \zeta (3) T^2}{\pi v_x v_y (1 - w^2)^{3/2}}, \]  
(28)
where \( \zeta (x) \) is the Riemann zeta function. Finite tilt tends to enhance DOS and specific heat. As \( |w| \to 1 \), the DOS formally diverges, indicating the instability of the system. The disorder effects on DOS and specific heat are embodied in the quantum corrections to the fermion velocities and the tilt parameter. According to Eqs. (7)–9, although the velocity renormalization is not dependent on the disorder type, the tilt parameter renormalization is type sensitive. As a result, different types of disorder result in different behaviors of DOS and specific heat.

When the RGP-induced corrections are taken into account, the fermion DOS becomes
\[ \rho (E) = \rho (\Lambda) \left( \frac{E}{\Lambda} \right)^{1-2\beta_x (\Lambda)} \sim E^{1-2\beta_x (\Lambda)}. \]  
(29)
Similarly, the specific heat is altered by RGP to take the form
\[ C_v (T) = C_v (T_\Lambda) \left( \frac{T}{T_\Lambda} \right)^{2-2\beta_x (\Lambda)} \sim T^{2-2\beta_x (\Lambda)}. \]  
(30)
where $T_A$ is certain fixed high temperature. Thus we see that RGP gives rise to power-law enhancement of DOS and specific heat, characterized by two tilt-independent exponents. This stems from the fact that RGP reduces the tilt down to zero, which implies that the tilt becomes irrelevant at low energies. In addition, the power-law enhancement of DOS and specific heat can be interpreted as the emergence of NFL-like behavior. The RGP-induced breakdown of FL theory has been studied in several SM materials, including 2D Dirac SM and multi-Weyl SM. Analogous unusual behavior also appears in Dirac SM with long-range correlated RM.

### B. Random mass

We now consider RM. According to Eq. (11), when RM exists along its flow equation is given by

$$\frac{d\beta_y}{d\ell} = -2\beta_y^2.$$  

This equation has the following solution

$$\beta_y(E) = \frac{\beta_y(\Lambda)}{1 + 2\beta_y(\Lambda)\ln(\Lambda/E)},$$  

which approaches to zero in the low-energy limit. Therefore, RM is a marginally irrelevant perturbation. Repeating the RG steps performed in Sec. III A, we find the following solutions for the tilt parameter and the fermion velocities:

$$w(E) = \frac{w(\Lambda)}{\sqrt{1 + 2\beta_y(\Lambda)\ln(\Lambda/E)}},$$  

$$v_x(E) = \frac{\sqrt{1 + 2\beta_y(\Lambda)\ln(\Lambda/E)}}{1 - w^2(\Lambda) + 2\beta_y(\Lambda)\ln(\Lambda/E)} \times v_x(\Lambda) \left[1 - w^2(\Lambda)\right],$$  

$$v_y(E) = \frac{v_y(\Lambda)\sqrt{1 - w^2(\Lambda)}}{\sqrt{1 - w^2(\Lambda) + 2\beta_y(\Lambda)\ln(\Lambda/E)}}.$$  

All of these three quantities go to zero logarithmically as $E \to 0$. Interestingly, no anomalous dimension is generated. It turns out that RM leads to weaker corrections to the properties of tilted Dirac fermions than RGP. To confirm this, we now calculate the quasiparticle residue. Based on Eqs. (21), (32), and (33), we obtain

$$Z^\pm_f(\omega) = \frac{1 \pm w(\Lambda)}{\sqrt{1 + 2\beta_y(\Lambda)\ln(\Lambda/\omega)} \pm w(\Lambda)}.$$  

By replacing $E$ with $\omega$, the energy dependence of $Z^\pm_f$ is roughly given by

$$Z^\pm_f(\omega) \big|_{\omega \to 0} \propto \left[\ln\left(\frac{\Lambda}{\omega}\right)\right]^{-1/2}$$  

as $\omega \to 0$. The real part of retarded fermion self-energy is

$$\text{Re}\Sigma_{ii}^R(\omega) \sim \omega \left[\ln\left(\frac{\Lambda}{\omega}\right)\right]^{1/2},$$  

and the imaginary part is

$$\text{Im}\Sigma_{ii}^R(\omega) \sim \omega \left[\ln\left(\frac{\Lambda}{\omega}\right)\right]^{-1/2}.$$  

Thus RM also leads to violation of FL theory, in a way similar to marginal Fermi liquid (MFL). Again, the difference in the residues of different orbitals becomes irrelevant at low energies.

After incorporating the corrections due to RM, the DOS and specific heat become

$$\rho(\omega) = \rho(\Lambda) \left[\frac{\omega}{\Lambda}\right] \left[1 + 2\beta_y(\Lambda)\ln\left(\frac{\Lambda}{\omega}\right)\right]$$  

$$\sim \omega \ln\omega,$$

$$C_v(T) = C_v(T_A) \left[\frac{T}{T_A}\right]^2 \left[1 + 2\beta_y(\Lambda)\ln\left(\frac{T_A}{T}\right)\right]$$  

$$\sim T^2 \ln T.$$  

The calculational details are shown in Appendix C. We conclude that the marginally irrelevant RM only causes logarithmic enhancement of DOS and specific heat. In analog to RGP, RM also suppress the tilt down to zero at low energies, thus the tilt does not play an important role. In the RG scheme, the influence of RM seems to be quite weak due to its irrelevance. However, its effect might not be limited to such logarithmic corrections. Rare region effects are believed to play an important role in the case of perturbatively irrelevant disorder. Previous studies suggest that when these effects are considered, the Dirac SM phase could exist only in the ultra-clean limit. Moreover, due to rare region effects, the quantum critical region between the SM and DM phases may become a sharp crossover, with the putative critical point entirely avoided. However, rare region effects are non-perturbative and thus cannot be studied by means of the perturbative RG approach. We leave the rare region effects of marginally irrelevant RM to future research.

### C. Random scalar potential

The results of the previous two subsections indicate that, although RGP and RM strongly modify the low-energy properties of tilted Dirac fermions, the system remains stable. In this subsection, we will show that the role played by intraorbital disorder is entirely different from interorbital disorder.

Suppose that only one electron orbital, either $\sigma_+ = 1$ or $\sigma_+ = -1$, is subjected to RSP. The RG equations of $w$
Appendix D. As explained in Appendix D, $\Sigma(\epsilon)$ corresponds to the $\Sigma$ term in Eq. (26) and the definition of $\langle \gamma \rangle$ is given by Eq. (39). We plot the RG flow diagrams for $(w, \beta_-)$ and $(w, \beta_+)$ separately in Fig. 2(a) and Fig. 2(b). From these two diagrams, we see that disorder in the $\sigma_\pm$ orbitals drives the tilt to 1, whereas disorder in the $\sigma_z$ orbital drives the tilt to $-1$. As a result, the DOS of disordered orbital becomes large, according to Eq. (27). The enhanced DOS in turn increases the disorder strength. At ultra low energies, the disorder strength formally flows to infinity. When $w \to \pm 1$, the divergence of DOS and disorder strength indicates that the system is no longer stable, but enters into a strongly disordered phase. The perturbative RG approach cannot be used to analyze the properties of this disordered phase. To obtain the disorder scattering rate generated in the disordered phase, we will employ the SCBA method to self-consistently calculate the fermion self-energy. The case of $w \to 1$ has already been analyzed by Papaj et al. Here, we focus on the case of $\sigma_\pm$ orbital with $w \to -1$.

Within SCBA scheme, the self-consistent equation for the fermion self-energy in the $\sigma_\pm$ orbital takes the following form

$$\Sigma(\epsilon) = \Delta_+ \int d^2k \frac{1}{(2\pi)^2} \frac{1}{\epsilon - H_0(k) - \Sigma(\epsilon)},$$

where the definition of $\int d^2k$ is given by Eq. (D2) in Appendix D. As explained in Appendix D, $\Sigma(\epsilon)$ is the self-consistent equation for $\Sigma_\pm(\epsilon)$ of the form:

$$\Sigma_\pm(\epsilon) = \frac{2e}{1-w} \mp 2i\Lambda \sqrt{\frac{1+w}{1-w}},$$

where the upper and lower signs represent retarded and advanced self-energy functions, respectively. The solution for $\Sigma_\pm(0)$ leads to the following constant

$$\Gamma_1 = 2\Lambda \sqrt{\frac{1+w}{1-w}} \exp \left( -\frac{2}{\beta_+} \frac{1+w}{1-w} \right),$$

which defines a low energy scale. As the energy decreases down to $\Gamma_1$, the tilt approaches to $-1$, which means the one-loop RG becomes invalid. For energies well beyond $\Gamma_1$, the self-energy can be calculated analytically. At sufficiently high energies of $|\epsilon| \gg \Gamma_1$, Eq. (43) can be solved by using the iterative method in powers of $\beta_+$. At low energies $|\epsilon| \ll \Gamma_1$, the solution can be obtained by taking a series expansion in powers of $\epsilon$. Based on the calculations presented in Appendix D, the self-energy is given by

$$\Sigma_\pm(\epsilon) = \begin{cases} 
-\frac{4(1+w)\epsilon}{(1-w)^2\beta_+} \mp i\Gamma_1 & (|\epsilon| \ll \Gamma_1), \\
-\frac{2\beta_+}{1+w} \left[ \pm i\pi |\epsilon|/2 + \epsilon \\
\times \ln \left( \Lambda \sqrt{1-\epsilon^2} \right) \right] & (|\epsilon| \gg \Gamma_1).
\end{cases}$$

This self-energy tells us that the electrons in the $\sigma_\pm$ orbital acquire a finite scattering rate at $\omega = 0$, i.e.,

$$\gamma_0 = |\text{Im} \Sigma_\pm(0)| = \Gamma_1.$$
D. Coexistence of all four types of disorder

In the last three subsections, each type of disorder is supposed to exist individually in the system. But it happens in many realistic materials that more than one types of disorder coexist. The interplay of different types of disorder could give rise to much richer low-energy behaviors. According to Eq. (12), the coexistence of an RGP and an RM generates an RSP flow even if the system originally does not contain RSP. A more complete analysis reveals that, the coexistence of any two types of disorder invariably generates the rest ones. As a result, the system are eventually driven to contain all the four types of disorder. To analyze the properties of this situation, we need to analyze the full set of RG equations presented in Eqs. (7) - (12). Because the velocities are always reduced down to zero in the low-energy limit, which can be inferred from Eqs. (8) and (9), we will pay special attention to the RG flows of $w$ and four disorder parameters.

By numerically solving the coupled RG equations, we obtain the running behavior of $w$ and show the result in Fig. 3(a) The $\ell$-dependence is sensitively determined by the sign of the initial value of $w$. For a positive $w(0)$, $w(\ell)$ approaches to unity at a constant energy scale of $\ell_c = \ln(\Lambda/E)$. However, $w(\ell)$ flows to $-1$ at the same energy scale if $w(0)$ is negative. The $\ell$-dependence of disorder parameters obtained by starting from positive and negative $w(0)$ is plotted in Fig. 3(b) The main figure presents the results for a positive $w(0)$, and the inset for negative $w(0)$. We observe that, the flowing behaviors of $\beta_x$ and $\beta_y$ are not affected by the sign reversal of $w(0)$, whereas the flowing behaviors of $\beta_+$ and $\beta_-$ are interchanged. As shown in Fig. 3(b) $\beta_+ > \beta_-$ when $w \to 1$ at a fixed scale, but $\beta_+ < \beta_-$ when $w \to -1$. This is different from the case in which only one single type of disorder exists. From the analysis of Sec. [III C] it appears that a larger $\beta_-$ is favored if $w \to 1$, and $w \to 1$ leads to a larger $\beta_+$. The flip in this behavior is caused by the interplay between different types of disorder.

Regardless of the subtle difference caused by different initial values of $w$, we find that all the disorder parameters become divergent at the same constant energy if $|w| \to 1$. Once again, the system is no longer stable and driven into a strongly disordered phase. The new feature is that now four types of disorder are present simultaneously. To gain further insight into the strongly disordered phase, we will again make a SCBA analysis. Now, the self-energy should be decomposed in the form

$$\Sigma(\epsilon) = \Sigma_{11}(\epsilon) \sigma_+ + \Sigma_{22}(\epsilon) \sigma_-.$$  

Repeating the same calculational steps, we find the following relation between $\Sigma_{11}(\epsilon)$ and $\Sigma_{22}(\epsilon)$:

$$\left(\frac{\beta_x + \beta_y}{1+w} + \frac{\beta_-}{1-w}\right)\Sigma_{11}(\epsilon) = \left(\frac{\beta_x + \beta_y}{1-w} + \frac{\beta_+}{1+w}\right)\Sigma_{22}(\epsilon).$$  

(47)

This is another important result caused by the interplay of different disorders. This condition will be utilized to determine under what circumstance a bulk Fermi arc is realized.

It is difficult to obtain exact analytical solutions of $\Sigma_{11}(\epsilon)$ and $\Sigma_{22}(\epsilon)$. We now take the zero-energy limit, and find that the two components of fermion self-energy are given by
Therefore, the retarded self-energy takes the form
\[
\Sigma^R (\varepsilon = 0) = \begin{bmatrix} -i\Gamma_+ & 0 \\ 0 & -i\Gamma_- \end{bmatrix},
\]
where \( \Gamma_+ \) and \( \Gamma_- \) are two constant energy scales defined by Eq. (48) and Eq. (49), respectively. It is trivial to check that Eq. (47) is satisfied by Eq. (48) and Eq. (49), and, therefore, by \( \Gamma_+ \) and \( \Gamma_- \). For illustration, we rewrite Eq. (47) explicitly in terms of \( \Gamma_+ \) and \( \Gamma_- \):
\[
\begin{pmatrix} \beta_+ + \beta'_x \\ 1 + w \end{pmatrix} \Gamma_+ = \begin{pmatrix} \beta_x + \beta_y \\ 1 - w \end{pmatrix} \Gamma_- \tag{52}
\]
This result tells us immediately that \( \Gamma_+ = \Gamma_- \) when the disorder parameters satisfy
\[
\beta_- (1 + w) - \beta_+ (1 - w) = 2w (\beta_x + \beta_y). \tag{53}
\]
In this case, the Dirac point is robust, and the strongly disordered phase can be identified as a well-defined CDMD.\textsuperscript{[25,28]} Once Eq. (53) is violated, we always have \( \Gamma_+ \neq \Gamma_- \). This indicates that electrons in two orbitals have different scattering rates, similar to the case only the \( \sigma_z = +1 \) orbital is disordered.

IV. FERMI ARC VERSUS DIRAC POINT

In this section, we discuss how the Fermi surface is influenced if the condition Eq. (53) is satisfied and violated.

If the condition Eq. (53) is violated, we know that \( \Gamma_+ \neq \Gamma_- \). Adding the self-energy to the free Dirac Hamiltonian, we obtain the total Hamiltonian \( H (\mathbf{p}, \omega) = H_0 (\mathbf{p}) + \Sigma (\omega) \). At zero energy, \( \omega = 0 \), the Hamiltonian
\[
H (\mathbf{p}) = (wv_x \sigma_x - i\eta_+ \sigma_0 + (v_x \sigma_x - i\eta_- \sigma_z + v_y \sigma_y) \sigma_z, \tag{54}
\]
where \( \eta_{\pm} = (\Gamma_+ \pm \Gamma_-) / 2. \) Solving the equation \( \det [\mathcal{E}' - H (\mathbf{p})] = 0 \) leads us to the following quasiparticle energy dispersion
\[
\mathcal{E}' (\pm \mathbf{p}) = (wv_x \sigma_x - i\eta_+) \pm \sqrt{(v_x \sigma_x - i\eta_-)^2 + v_y^2 \sigma_y^2}, \tag{55}
\]
Since \( \Gamma_+ \neq \Gamma_- \), both \( \eta_+ \) and \( \eta_- \) are nonzero. As a result, \( \mathcal{E}' (\pm \mathbf{p}) \) now has a complex value. The genuine quasiparticle energy-momentum relation\textsuperscript{[25,28,108]} corresponds to its real part \( \text{Re} [\mathcal{E}' (\pm \mathbf{p})] \), which is found to have the form
\[
\mathcal{E}' (\pm \mathbf{p}) = wv_x \sigma_x \pm \frac{1}{\sqrt{2}} \sqrt{\mathcal{E}_0^2 + 4\eta^2 v_y^2 p_y^2 + \mathcal{E}_0}, \tag{56}
\]
where \( \mathcal{E}_0 = v_x^2 p_x^2 + v_y^2 p_y^2 - \eta_+^2. \) This energy-momentum relation is plotted in Fig. 4(b), and for comparison we also plot the energy dispersion of Eq. (2) in Fig. 4(a). It is clear that the Dirac point located at \( p_x = p_y = 0 \) is converted by disorder into a line. Setting the quasiparticle energies in the conduction and valence bands to be

\[\Sigma_{11}(\varepsilon = 0) = \mp 2iA \sqrt{1 - w^2} [\beta_+ (1 - w) + (\beta_x + \beta_y) (1 + w)] \frac{1}{h} \exp \left[-2 (1 - w^2) / h\right] \equiv \mp i\Gamma_+, \tag{48}\]
\[\Sigma_{22}(\varepsilon = 0) = \mp 2iA \sqrt{1 - w^2} [\beta_- (1 + w) + (\beta_x + \beta_y) (1 - w)] \frac{1}{h} \exp \left[-2 (1 - w^2) / h\right] \equiv \mp i\Gamma_-, \tag{49}\]
where
\[
h = \beta_+ (1 - w)^2 + \beta_- (1 + w)^2 + 2 (\beta_x + \beta_y) (1 - w^2). \tag{50}\]
equal, we find that this line is described by
\[ p_x = 0, \quad -\frac{|\eta|-}{v_y} \leq p_y \leq \frac{|\eta|}{v_y}. \tag{57} \]

Two bands are degenerate along this line, which is the Fermi arc. The two end points of this arc are \((k_x, k_y) = (0, \pm |\eta|/v_y)\). These two points, commonly identified as exceptional points, can be regarded as the result of Dirac point splitting. Beyond these two points, \(E_{+}^{R} > E_{-}^{R}\), and a band gap \(E'_{y} = E_{+}^{R} - E_{-}^{R}\) is generated, such that
\[ E'_{y} = \sqrt{2} \sqrt{E_{0}^{2} + 4\eta^{2}v_{x}^{2}p_{x}^{2} + E_{0}}. \tag{58} \]

The increase of this gap with momentum can be clearly seen in Fig. 4(a).

From the above analysis, we know that a bulk Fermi arc appears whenever two orbitals acquire different disorder scattering rates. However, once the condition Eq. (53) is met, two distinct orbitals have exactly the same disorder scattering rate, i.e., \(\Gamma_{-} = \Gamma_{+} = \Gamma_{0}\). As a result, the energy dispersion is the direct sum of tilted Dirac fermion energy and a constant damping rate, namely
\[ E'_{\pm} = w_{z}v_{x}p_{x} \pm \sqrt{v_{x}^{2}p_{x}^{2} + v_{y}^{2}p_{y}^{2} - i\Gamma_{0}} = E_{\pm} - i\Gamma_{0}. \tag{59} \]

In this case, there is no Fermi arc. This feature occurs in the strongly disordered phase of conventional Dirac fermion system, in which the symmetry between two orbitals ensures that the two scattering rates are equal. In the model considered in this paper, the two orbitals of tilted Dirac fermion are not related by any symmetry and physically distinct. It is the coexistence of different types of disorder that generates the same scattering rate for these two distinct orbitals.

We regard the condition Eq. (53) as a criterion for judging the presence or absence of bulk Fermi arc in tilted Dirac fermion system supporting independent orbitals. The system exhibits isolated Dirac point if Eq. (53) is satisfied, and bulk Fermi arc otherwise. This is the main result of our paper. The condition Eq. (53) corresponds to a four-dimensional subspace of a five-dimensional space spanned by the parameters \(\beta_{x}, \beta_{y}, \beta_{+}, \beta_{-}\), and \(w\). For a given sample, these five parameters take certain constant values. The Dirac point is stable only when these parameters lie in this subspace. In this respect, the appearance of a bulk Fermi arc is quite generic.

V. SUMMARY AND DISCUSSION

In summary, we have presented a RG analysis of four types of disorder allowed to exist by itself in 2D tilted Dirac fermion systems. Our results indicate that, when only one type of disorder exists, intraorbital disorder scattering can produce a bulk Fermi arc, consistent with previous work of Ref. [13]. Such an arc is generated as long as the Dirac fermions from two orbitals are not related by any symmetry. By contrast, in the case of interorbital disorder scattering, the Dirac cone remains intact, and no Fermi arc appears. Instead, interorbital disorder leads to logarithmic or power-law quantum corrections to such quantities as DOS and specific heat of tilted Dirac fermions. We have also examined the mutual influence of different disorders, and showed that the coexistence of two or more types of disorder dynamically generate the rest types of disorder. Consequently, unless there is strictly one single type of disorder, the system inevitably contains all the four types of disorder. The interplay of different disorders bring about physics not reported previously. Interestingly, the Fermi arc does not always replace the Dirac point, even if the fermions of distinct orbitals are unrelated. We obtain a condition for the emergence of a bulk Fermi arc. The tilted Dirac fermion system exhibits isolated Dirac point if this condition is satisfied, and a bulk Fermi arc is formed when this condition is violated.

We then remark on possible future research projects. The formation of a bulk Fermi arc due to different disorder scattering rates has been studied by several groups. The topological band theory of different scattering rates is put forward in Ref. [22]. The authors of Ref. [22] considered the possibility of inducing a Fermi arc by the electron-phonon interaction in 2D untitled Dirac fermion system. The disorder induced Fermi arc is predicted to emerge in 3D tilted Weyl fermion system[19] and a 2D tilted Dirac-fermion system with only the \(\sigma_{z} = -1\) orbital being disordered. Moreover, there is experimental evidence of a bulk Fermi arc produced by the splitting of one single Dirac point into a pair of exceptional points[101]. So far, it remains unclear whether such a Fermi arc can be realized in other Dirac/Weyl SMs, such as semi-Dirac SM[47,100].

This problem deserves further investigation.

Finally, we would like to address another interesting question: How should we identify the strongly disordered phase that features two different fermion scattering rates? In previous studies, a strongly disordered phase with one universal scattering rate is usually identified as a CDM phase[23,35]. There is no bulk Fermi arc in such a phase. For the Dirac fermion system considered in this paper, the strongly disordered phase cannot be regarded as CDM, although a finite zero-energy DOS is generated. The appearance of two distinct disorder scattering rates produces a Fermi arc, which makes such a strongly disordered phase quite different from the conventional one[23,35].

VI. ACKNOWLEDGMENTS

We would like to thank Jing-Rong Wang for helpful discussion, and acknowledge the support provided
Appendix A: Derivation of coupled RG equations

We first discuss how to determine the four types of bilinear fermion-disorder coupling, and then derive the RG equations given by Eqs. (8)-(11) by calculating the diagrams of Fig. 1(a)-Fig. 1(d).

The calculation will be based on the free fermion propagator of fermions:

\[
G_0 \left( ip_0, \mathbf{p} \right) = \frac{1}{-ip_0 \sigma_0 + (\sigma_z + w \sigma_0) v_x p_x + v_y \sigma_x p_y} = \frac{(ip_0 - wv_x p_x) \sigma_0 + v_x p_x \sigma_z + v_y p_y \sigma_x}{(p_0 + iwv_x p_x)^2 + v_x^2 p_x^2 + v_y^2 p_y^2}.
\]  (A1)

1. Determining the matrices for fermion-disorder coupling

Intraorbital disorder can be formally defined via the matrix \( \gamma = A \sigma_0 + B \sigma_z \), where \( A \) and \( B \) are both real to ensure that \( \gamma \) is hermitian. Now, we consider the loop correction of Fig. 1(b) with disorder vertex \( \gamma \), and find that

\[
\delta \Delta^{(b)}_\gamma = \Delta^2 \int \frac{d^2 p}{(2\pi)^2} \gamma G_0(0, \mathbf{p}) \gamma G_0(0, \mathbf{p}) \gamma
\]

\[
= \Delta^2 \frac{\Delta^2}{2\pi v_x v_z (1 - w^2)^{3/2}} \begin{pmatrix} (1 - w)(A - Bw)(A + B)^2 & 0 \\ 0 & (1 + w)(A - Bw)(A - B)^2 \end{pmatrix},
\]  (A2)

where

\[
\int d^2 p = \int_{\Lambda_{\kappa}^{-\ell}}^{\Lambda} \cosh{\theta} \ d \theta, \quad (p_x = |p| \cos{\theta}, p_y = |p| \sin{\theta}).
\]  (A3)

To make RG analysis self-consistent, the matrix appearing in Eq. (A2) must be proportional to \( \gamma = \begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix} \) for any \( w \in (-1, 1) \). It is easy to verify that this condition is fulfilled only when \( A = \pm B \). Therefore, we should choose the matrices as \( \gamma = A(\sigma_0 + \sigma_z) \equiv \sigma_+ \) and \( \gamma = A(\sigma_0 - \sigma_z) \equiv \sigma_- \). The constant \( A \) can be absorbed into the effective disorder parameter defined by Eq. (13). As a result, the value of \( A \) does not affect the RG equations, and we simply set \( A = 1/2 \).

For interorbital disorder denoted by \( \gamma = A' \sigma_x + B' \sigma_y \), similar analysis indicates that self-closed RG analysis is obtained only when \( A' = 0 \) or \( B' = 0 \). This implies that \( \gamma = \sigma_x \) or \( \gamma = \sigma_y \).

2. Interaction corrections and RG equations

After specifying the disorder vertices, we proceed to calculate the interaction corrections and to derive the coupled RG equations.

Fig. 1(a) is the fermion self-energy correction caused by disorder scattering. It can be written as

\[
\Sigma_{\text{dis}}(ip_0) = -\sum_i \Delta_i \int d^2 \mathbf{p} \frac{d^2 \mathbf{p}}{(2\pi)^2} \sigma_i G_0(0, \mathbf{p}) \sigma_i
\]

\[
= -ip_0 \ell \frac{1}{2\pi v_x v_y \sqrt{1 - w^2}} \left[ \left( \frac{\Delta_+}{1 + w} + \frac{\Delta_+ + \Delta_\phi}{1 - w} \right) \sigma_+ + \left( \frac{\Delta_-}{1 - w} + \frac{\Delta_- + \Delta_\phi}{1 + w} \right) \sigma_- \right].
\]  (A4)

Fig. 1(b)-Fig. 1(d) represent the corrections to fermion-disorder vertices. We find that

\[
\delta \Delta^{(b)}_i = \Delta_i \left( \psi^\dagger_m \sigma_i \psi_m \right) \psi_n^\dagger \sum_j \Delta_j \int d^2 \mathbf{p} \frac{d^2 \mathbf{p}}{(2\pi)^2} \sigma_j G_0(0, \mathbf{p}) \sigma_i G_0(0, \mathbf{p}) \sigma_j \psi_n.
\]  (A5)
Before going further, we re-define the disorder strength parameters: $\beta_i$.

For given $\sigma_i$, the corrections to $\Delta_i$ are given by

$$
\delta \Delta_i^{(b)} = \begin{cases} 
0 & \sigma_i = \sigma_x, \\
\frac{\Delta_y (\Delta_x - \Delta_y) \ell}{2\pi v_x v_y \sqrt{1 - w^2}} (\psi_n^\dagger \sigma_y \psi_m) (\psi_n^\dagger \sigma_y \psi_n) & \sigma_i = \sigma_y, \\
\frac{\Delta_x \Delta_y \ell}{4\pi v_x v_y \sqrt{1 - w^2}} \left\{ \left[ \Delta_x + \Delta_y + \Delta_+ \left( \frac{1 - w}{1 + w} \right) \right] (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_+ \psi_n) \\
+ \left[ (\Delta_x + \Delta_y) \left( \frac{1 - w}{1 + w} \right) + \Delta_- \right] (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_- \psi_n) \right\} & \sigma_i = \sigma_+, \\
\frac{\Delta_x \Delta_y \ell}{4\pi v_x v_y \sqrt{1 - w^2}} \left\{ \left[ \Delta_+ + (\Delta_x + \Delta_y) \left( \frac{1 + w}{1 - w} \right) \right] (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_- \psi_n) \\
+ \left[ \Delta_x + \Delta_y + \Delta_- \left( \frac{1 + w}{1 - w} \right) \right] (\psi_n^\dagger \sigma_- \psi_m) (\psi_n^\dagger \sigma_- \psi_n) \right\} & \sigma_i = \sigma_-.
\end{cases}
$$

Here, notice that the terms proportional to $(\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_- \psi_n)$ are forbidden within the replica formalism. We simply discard these terms hereafter.

For the diagrams of Fig. 1(c) + Fig. 1(d), we get

$$
\delta \Delta^{(c)+(d)} = \sum_{ij} \Delta_i \Delta_j \int \frac{d^2 P}{(2\pi)^2} \psi_m^\dagger \left[ \sigma_j G_0(0, P) \sigma_j \right] \psi_m \psi_n^\dagger \left[ \sigma_i G_0(0, P) \sigma_i + \sigma_i G_0(-P) \sigma_j \right] \psi_n, 
$$

(A7)

One can see that $\delta \Delta^{(c)+(d)} = 0$ for $\sigma_i = \sigma_j$ because of the relation $G_0(0, -P) = -G_0(0, P)$. Thus, we only need to consider $\sigma_i \neq \sigma_j$, which contains six pairs as $(\sigma_i, \sigma_j) = \{(\sigma_x, \sigma_y), (\sigma_x, \sigma_+), (\sigma_x, \sigma_-), (\sigma_y, \sigma_+), (\sigma_y, \sigma_-), (\sigma_+, \sigma_-)\}$. After computing Eq. (A7) for all of these pairs, we eventually obtain

$$
\delta \Delta^{(c)+(d)} = \begin{cases} 
\frac{\Delta_y \Delta_y \ell}{2\pi v_x v_y \sqrt{1 - w^2}} \left( \frac{1 + w}{1 - w} \right) (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_+ \psi_n) & (\sigma_i, \sigma_j) = (\sigma_x, \sigma_y), \\
\frac{\Delta_y \Delta_y \ell}{2\pi v_x v_y \sqrt{1 - w^2}} \left( \frac{1 - w}{1 + w} \right) (\psi_n^\dagger \sigma_- \psi_m) (\psi_n^\dagger \sigma_- \psi_n) & (\sigma_i, \sigma_j) = (\sigma_x, \sigma_-), \\
\frac{\Delta_y \Delta_y \ell}{8\pi v_x v_y \sqrt{1 - w^2}} \left( \frac{1 - w}{1 + w} \right) (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_+ \psi_n) & (\sigma_i, \sigma_j) = (\sigma_y, \sigma_+), \\
\frac{\Delta_y \Delta_y \ell}{8\pi v_x v_y \sqrt{1 - w^2}} \left( \frac{1 + w}{1 - w} \right) (\psi_n^\dagger \sigma_+ \psi_m) (\psi_n^\dagger \sigma_+ \psi_n) & (\sigma_i, \sigma_j) = (\sigma_y, \sigma_-), \\
\frac{\Delta_y \Delta_y \ell}{8\pi v_x v_y \sqrt{1 - w^2}} \left( \frac{1 - w}{1 + w} \right) (\psi_n^\dagger \sigma_- \psi_m) (\psi_n^\dagger \sigma_- \psi_n) & (\sigma_i, \sigma_j) = (\sigma_+, \sigma_-).
\end{cases}
$$

(A8)

Before going further, we re-define the disorder strength parameters:

$$
\beta_i = \frac{\Delta_i}{2\pi v_x v_y \sqrt{1 - w^2}},
$$

(A9)
We insert all the one-loop corrections into the free action. After making the transformations

\[ \tilde{\omega} = e^{-\ell} \omega, \quad \tilde{k}_i = e^{-\ell} k_i, \quad \tilde{\psi}_1 (\tilde{p}) = \sqrt{Z_1} \psi_1 (p), \quad \tilde{\psi}_2 (\tilde{p}) = \sqrt{Z_2} \psi_2 (p), \quad v_x (1 - w) = Z_{w+} \tilde{v}_x (1 + \tilde{w}) \]

we can recast the renormalized action in its original form. Based on these manipulations, we derive the flow equations:

\[
\frac{d (uv_x)}{d \ell} = - \left[ \frac{\beta_x - \beta_\gamma}{2} - \frac{\beta_x + \beta_y}{2} \left( \frac{1}{w + 1} - \frac{1}{w - 1} \right) \right] v_x,
\]

\[
\frac{dv_x}{d \ell} = - \left[ \frac{\beta_x - \beta_\gamma}{2} + \frac{\beta_x + \beta_y}{2} \left( \frac{1 + w}{w - 1} - \frac{1}{w + 1} \right) \right] v_x,
\]

\[
\frac{dv_y}{d \ell} = - \frac{1}{2} \left[ (\beta_x + \beta_\gamma) + \frac{\beta_x + \beta_y}{1 + w} + (\beta_x + \beta_y) + \frac{\beta_x - \beta_y}{1 - w} \right] v_y,
\]

\[
\frac{d\beta_x}{d \ell} = \frac{\beta_y}{2} \left[ \frac{1 - w}{1 + w} + \frac{1}{1 - w} \right],
\]

\[
\frac{d\beta_y}{d \ell} = -2 (\beta_x - \beta_\gamma) \beta_y + \frac{\beta_x + \beta_y}{2} \left[ \frac{1}{w - 1} + \frac{1}{w + 1} \right] + \frac{\beta_x - \beta_y}{2},
\]

\[
\frac{d\beta_\gamma}{d \ell} = -\frac{\beta_x^2 w}{1 - w} + \frac{\beta_\gamma}{1 - w} \left[ 2 w (\beta_x + \beta_y) + \beta_x + 3 \beta_\gamma \right] + \frac{2 \beta_x \beta_y (1 + w)}{1 - w},
\]

\[
\frac{d\beta_-}{d \ell} = \frac{\beta_x^2 w}{1 - w} + \beta_- \left[ \frac{\beta_+}{1 - w} + \frac{2 w (\beta_x + \beta_y)}{1 - w^2} + \beta_x + 3 \beta_\gamma \right] + \frac{2 \beta_x \beta_y (1 - w)}{1 - w}.
\]

Eqs. (A12) - (A15) are just Eqs. (8) - (11) presented in the main text. Combining Eq. (A11) and Eq. (A12) leads to Eq. (B1) of the main text. Combining Eq. (A16) and Eq. (A17) leads to Eq. (B2) of the main text.

**Appendix B: Proof of the nonrenormalization of RGP**

According to Eq. (14), the disorder parameter of RGP is not renormalized and remains a marginal perturbation at one-loop level. The aim of this Appendix is to illustrate that this conclusion is valid at any order of loop expansion. To prove this, it is more convenient to first write the action in the imaginary-time formalism:

\[
S = \int d\tau d^2 x \tilde{\psi}^\dagger (x) [\sigma_0 \partial_x - i (\sigma_z + w \sigma_0) v_x \partial_x - i v_y \sigma_x \partial_y - v_d A(x) \sigma_x] \tilde{\psi} (x).
\]

Be using the replica method, we find that \( v_y \) is connected to the effective disorder coupling via the relation

\[
\beta_x = \frac{v_y^2}{2 \pi v_y v_x \sqrt{1 - w^2}}.
\]

Discarding the energy-independent constants, we only need to verify that

\[
\frac{d}{d \ell} \left( \frac{v_y^2}{v_y v_x \sqrt{1 - w^2}} \right) = 0.
\]

Making the local gauge transformation

\[
\tilde{\psi} \rightarrow \psi_i e^{i \xi (y)}, \quad A(x) \rightarrow A(x) + \partial_y \xi (y),
\]

we re-express the action as

\[
S = \int d\tau d^2 x \{ \tilde{\psi}^\dagger (x) [\sigma_0 \partial_x - i (\sigma_z + w \sigma_0) v_x \partial_x - i v_y \sigma_x \partial_y - v_d A(x) \sigma_x] \tilde{\psi} (x) + \psi^\dagger (x) \sigma_x \psi (x) \}
\times (v_y - v_d) \partial_y \xi (y).
\]

To preserve the gauge invariance, one must demand that

\[
v_y (\ell) = v_d (\ell).
\]

The validity of this identity is loop independent. It is now obvious that

\[
\frac{dv_y}{d \ell} = \frac{dv_d}{d \ell}.
\]

In addition, a crucial character of static disorder is that the fermion self-energy is independent of momenta. Therefore, at any order of loop expansion, integrating out the fast modes defined within the shell \( (\Lambda e^{-\ell}, \Lambda) \) leads to an effective, renormalized action.
We re-scale the coordinates and field operators as follows: 
\[ \tau \rightarrow e^{\ell} \tau, \quad x \rightarrow e^{\ell}x, \quad \psi_i \rightarrow Z_i^{-1/2} \psi_i, \quad v_x (1-w) \rightarrow Z_v^{-1} v_x (1-w), \quad v_y (1+w) \rightarrow Z_v^{-1} v_y (1+w), \quad v_y \rightarrow Z_v^{-1} v_y. \]

Straightforward calculations give rise to 
\[ Z_w^+ = \frac{1}{1 + \Sigma_1(\ell)}, \quad Z_w^- = \frac{1}{1 + \Sigma_2(\ell)}, \]
\[ Z_v = \frac{1}{\sqrt{1 + \Sigma_1(\ell)} \sqrt{1 + \Sigma_2(\ell)}}. \]  
(B9)

Apparently, the following identity holds 
\[ Z^2_v = Z_w^+ (\ell) Z_w^- (\ell), \]  
(B10)

which directly leads to 
\[ \frac{d \ln v_y}{d \ell} = \frac{d Z_v}{d \ell} |_{\ell=0} \]
\[ = Z_v^+ (\ell) \frac{d Z_v^- (\ell)}{d \ell} |_{\ell=0} + Z_v^- (\ell) \frac{d Z_v^+ (\ell)}{d \ell} |_{\ell=0} \]
\[ = \frac{d Z_v^+ (\ell)}{d \ell} |_{\ell=0} + \frac{d Z_v^- (\ell)}{d \ell} |_{\ell=0} \]
\[ = d \ln v_x (1-w) + d \ln v_x (1+w) \]
\[ = d \ln v_x (1-w^2). \]  
(B11)

This can be further written as 
\[ \frac{d \ln v_y}{d \ell} = \frac{d \ln v_x \sqrt{1-w^2}}{d \ell}. \]  
(B12)

Based on Eq. (B7) and Eq. (B12), we find that 
\[ \frac{d}{d \ell} \ln \left( \frac{v_y^2}{v_x \sqrt{1-w^2}} \right) = 0. \]  
(B13)

Thus, we conclude that 
\[ \frac{d \beta_x}{d \ell} = 0 \]  
(B14)

is valid at any order of loop expansion.

**Appendix C: DOS and Specific heat**

Here we calculate the fermion DOS and specific heat, and then use the results to discuss the impact of interorbital disorder.

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1. **Low-energy DOS**

The DOS \( \rho(\omega) \) is defined as 
\[ \rho(\omega) = -N \int \frac{d^2 k}{(2\pi)^2} \text{Tr} \left[ \text{Im} G_R^R(\omega, k_x, k_y) \right], \]  
(C1)

where \( N = 2 \), represents the flavor of Dirac fermions. Carrying out analytic continuation \( ip_i \to \omega + i\gamma (\gamma \to 0) \) to Eq. (A1), we get 

\[ G_R^0 (\omega, k) = \left[ \frac{1}{k^2 v_x^2 + k_y^2 v_y^2 - \omega^2} - i\pi \text{sgn}(\omega') \right] \]
\[ \times \delta(\omega'^2 + k_x^2 v_x^2 + k_y^2 v_y^2) \]
\[ \times \omega' \sigma_0 + k_x v_x \sigma_2 + v_y k_y \sigma_3, \]  
(C2)

where \( \omega' = \omega - w v_x k_x \). It is now easy to get the spectral function 
\[ A_0 (\omega, k) = -\frac{1}{\pi} \text{Tr} \left[ \text{Im} G_0^R (\omega, k) \right] \]
\[ = 2 |\omega| \delta(\omega'^2 + k_x^2 v_x^2 + k_y^2 v_y^2). \]  
(C3)

The fermions DOS can be computed directly, i.e., 
\[ \rho_0(\omega) = N \int \frac{d^2 k}{(2\pi)^2} A_0 (\omega, k) \]
\[ = \frac{\pi |\omega|}{\pi v_y v_x (1-w^2)^{3/2}}, \]  
(C4)

which means that 
\[ \rho_0(E) \propto E \propto T, \]  
(C5)

where \( T \) is certain temperature.

The influence of disorder is usually embodied in the quantum corrections to the fermion velocities and the tilt parameter. To proceed, we appeal to the transformation \( \omega = \Lambda e^{-\ell} \), where \( \Lambda \) is the UV cutoff and \( \omega \) is certain lower energy scale. According to Eq. (C4), we have 
\[ \frac{d \ln \rho(E)}{d \ln E} = 1 - \frac{d \ln v_y (E)}{d \ln E} - \frac{d \ln v_x (E)}{d \ln E} \]
\[ = 1 - 2 \left[ \beta_x (E) + \beta_y (E) \right] - \frac{\beta_x (E) [1-w (E)]}{1+w (E)} \]
\[ - \frac{\beta_y (E) [1+w (E)]}{1-w (E)}. \]  
(C6)
When there is only RGP, \( \beta_x (E) = \beta_x (\Lambda) \), so we have
\[
\frac{d \ln \rho (E)}{d \ln E} = 1 - 2\beta_x (\Lambda), \tag{C7}
\]
which gives rise to
\[
\rho (\omega) = \rho (\Lambda) \left( \frac{E}{\Lambda} \right)^{1 - 2\beta_x (\Lambda)}, \tag{C8}
\]
where \( \rho (\Lambda) \) is a constant DOS defined at energy scale \( \Lambda \). This result implies that the marginal RGP leads to a power-law enhancement of low-energy DOS.

When there is only RM, the RG solution of the effective disorder parameter is
\[
\beta_y (E) = \frac{\beta_y (\Lambda)}{1 + 2\beta_y (\Lambda) \ln (\Lambda/E)}, \tag{C9}
\]
where \( \beta_y (\Lambda) \) can be taken as a constant. This then yields
\[
\frac{d \ln \rho (E)}{d \ln E} = 1 - 2\beta_y (E) = 1 - \frac{2\beta_y (\Lambda)}{1 + 2\beta_y (\Lambda) \ln (\Lambda/E)}. \tag{C10}
\]
This equation has the solution
\[
\rho (E) = \rho (\Lambda) \left( \frac{E}{\Lambda} \right) \left[ 1 + 2\beta_y (\Lambda) \ln (\Lambda/E) \right]
\sim E \ln E. \tag{C11}
\]
From this solution we conclude that the marginally irrelevant RM only gives rise to logarithmic enhancement to the low-energy DOS.

2. Specific heat

To calculate the specific heat, we need to first compute the free energy. After performing functional integration, we find that the free energy has the form
\[
F_f (T) = -2NT \sum_{\omega_n} \int \frac{d^2 p}{(2\pi)^2} \ln |\text{Det} [G^{-1} (\omega_n, p)]| \]
\[
= -T \sum_{\omega_n} \int \frac{d^2 p}{(2\pi)^2} \ln \left[ (\omega_n - iwp_x)^2 + v_{p_x}^2 p_y^2 \right. \\
+ \left. v_{p_y}^2 p_z^2 \right], \tag{C12}
\]
where \( \omega_n = (2n + 1)\pi T \) is the Matsubara imaginary frequency. Performing frequency summation yields
\[
F_f (T) = -2NT \int \frac{d^2 p}{(2\pi)^2} \left[ \ln (1 + e^{\beta E_-}) + \ln (1 + e^{\beta E_+}) \right], \tag{C13}
\]
where \( E_{\pm} = wp_x p_x \pm \sqrt{v_{p_x}^2 p_y^2 + v_{p_y}^2 p_z^2} \). This free energy is divergent. To regularize the integral, we need to replace \( F_f (T) \) by \( F_f (T) - F_f (0) \). After doing so, we obtain
\[
F_f (T) = -2NT \int \frac{d^2 p}{(2\pi)^2} \left[ \ln (1 + e^{\beta E_-}) + \ln (1 + e^{-\beta E_+}) \right] \\
= -\frac{3\zeta (3)}{\pi v_y v_x (1 - w^2)^{3/2}} T^3, \tag{C14}
\]
where \( \zeta (x) \) is the Riemann zeta function. The specific heat can be obtained by
\[
C_v (T) = -T \frac{\partial^2 F_f (T)}{\partial T^2} = \frac{18\zeta (3) T^2}{\pi v_y v_x (1 - w^2)^{3/2}}. \tag{C15}
\]

Applying the transformation \( T = T_\Lambda e^{-\ell} \), where \( T_\Lambda \) is the temperature corresponding to the UV cutoff \( \Lambda \), we find
\[
\frac{d \ln C_v (T)}{d \ln T} = 2 - \frac{d \ln v_x (T)}{d \ln T} - \frac{d \ln v_y (T)}{d \ln T} - \frac{3d \ln [1 - w^2 (T)]}{2d \ln T} \\
= 2 - 2 [\beta_\pm (T) + \beta_y (T)] - \frac{\beta_\pm (T) [1 - w (T)]}{1 + w (T)} \\
- \frac{\beta_y (T) [1 + w (T)]}{1 - w (T)}. \tag{C16}
\]

Now we can analyze the \( T \)-dependence of specific heat. When there is only RGP, one can show that
\[
C_v (T) = C_v (T_\Lambda) \left( \frac{T}{T_\Lambda} \right)^{2 - 2\beta_\pm (\Lambda)} \sim T^{2 - 2\beta_x (\Lambda)}. \tag{C17}
\]

When there is only RM, we get
\[
C_v (T) = C_v (T_\Lambda) \left( \frac{T}{T_\Lambda} \right)^2 \left[ 1 + 2\beta_y (\Lambda) \ln (T_\Lambda / T) \right] \\
\sim T^2 \ln T. \tag{C18}
\]

Again, RGP leads to power-law enhancement of specific heat, whereas RM results in a logarithmic enhancement.

Appendix D: SCBA calculation

In this appendix, we derive the SCBA equation and then get its solution. To make a generic analysis, the derivation will be completed in the case that all the four types of disorder coexist. The self-consistent equation for the fermion self-energy is given by
\[ \Sigma(\epsilon) = \sum_i \Delta_i \int d^2k \frac{1}{(2\pi)^2} \frac{1}{\epsilon - H_0(k) - \Sigma(\epsilon)} \sigma_i \]

\[ = \int d^2k \frac{1}{(2\pi)^2} \frac{1}{(f_+ - f_+^2 + q_y^2 v_y^2)} \left[ \begin{array}{cc} (\Delta_x + \Delta_y)(f_- - f_+) - \Delta_+(f_- + f_+) & 0 \\ 0 & \Delta_-(f_- - f_+) - (\Delta_x + \Delta_y)(f_- + f_+) \end{array} \right] \]

where

\[ \int d^2k = \int_0^\Lambda |k| d|k| \int_0^{2\pi} \frac{1}{2\pi} d\theta, \quad f_- = \epsilon v_x k_x + \frac{\Sigma_{11}(\epsilon) - \Sigma_{22}(\epsilon)}{2}, \quad f_+ = \epsilon - w v_x k_x - \frac{\Sigma_{11}(\epsilon) + \Sigma_{22}(\epsilon)}{2}. \]

Eq. (D1) is decomposed into two coupled equations:

\[ \Sigma_{11}(\epsilon) = \int d^2k \frac{1}{(2\pi)^2} \frac{1}{f_+ - f_- + q_y^2 v_y^2} \left[ -\Delta_+(f_- + f_+) + (\Delta_x + \Delta_y)(f_- - f_+) \right], \]

\[ \Sigma_{22}(\epsilon) = \int d^2k \frac{1}{(2\pi)^2} \frac{1}{f_+ - f_-^2 + q_y^2 v_y^2} \left[ \Delta_-(f_- - f_+) - (\Delta_x + \Delta_y)(f_- + f_+) \right]. \]

Integrating over \( k \) in Eq. (D3) yields

\[ g = \pm 2i \Lambda \sqrt{1 - w^2} \exp \left[ \frac{-2\Sigma_{11}(\epsilon)}{g \left( \frac{\beta_+ + \beta_y}{1 + w} + \frac{\beta_+}{1 - w} \right)} \right], \quad \text{(D5)} \]

where \( g = (1 - w) \Sigma_{11}(\epsilon) + (1 + w) \Sigma_{22}(\epsilon) - 2\epsilon \) and +, − represent advanced and retarded self-energy functions respectively. After completing the integration of Eq. (D4), we obtain

\[ g = \pm 2i \Lambda \sqrt{1 - w^2} \exp \left[ \frac{-2\Sigma_{22}(\epsilon)}{g \left( \frac{\beta_+ + \beta_y}{1 + w} + \frac{\beta_+}{1 - w} \right)} \right]. \quad \text{(D6)} \]

It seems difficult to obtain the analytical solutions for \( \Sigma_{11}(\epsilon) \) and \( \Sigma_{22}(\epsilon) \). However, we observe from Eq. (D5) and Eq. (D6) that

\[ \left( \frac{\beta_+ + \beta_y}{1 + w} + \frac{\beta_+}{1 - w} \right) \Sigma_{11}(\epsilon) = \left( \frac{\beta_+ + \beta_y}{1 - w} + \frac{\beta_+}{1 + w} \right) \Sigma_{22}(\epsilon). \quad \text{(D7)} \]

If only the \( \sigma_z = -1 \) orbital is disordered, this relation is reduced to a trivial identity \( 0 = 0 \), which gives us no new information. However, when four types of disorder coexist, this relation provides a strong connection for the self-energy functions of two distinct orbitals. The physical implication of this constraint is discussed in greater detail in the main text.

Next, we consider the solution for zero energy, and we focus on the retarded self-energy. Substitute Eq. (D7) to Eq. (D5) and Eq. (D6) with \( \epsilon = 0 \) we get

\[ \Sigma_{11}^R(\epsilon = 0) = -2i \Lambda \sqrt{1 - w^2} \left[ \beta_+ (1 - w) + (\beta_+ + \beta_y) \times (1 + w) \right] \frac{1}{\hbar} \exp \left[ -2 (1 - w^2) / \hbar \right]. \quad \text{(D8)} \]

\[ \Sigma_{22}^R(\epsilon = 0) = -2i \Lambda \sqrt{1 - w^2} \left[ \beta_- (1 + w) + (\beta_+ + \beta_y) \times (1 - w) \right] \frac{1}{\hbar} \exp \left[ -2 (1 - w^2) / \hbar \right]. \quad \text{(D9)} \]

where

\[ h = \beta_+ (1 - w)^2 + \beta_- (1 + w)^2 + 2 (\beta_+ + \beta_y) (1 - w^2). \quad \text{(D10)} \]

Therefore,

\[ \Sigma^R(\epsilon = 0) = \left[ \begin{array}{cc} -i \Gamma_+ & 0 \\ 0 & -i \Gamma_- \end{array} \right], \quad \text{(D11)} \]

where \( \Gamma_+ \) and \( \Gamma_- \) correspond to two energy scales that can be obtained from Eq. (D8) and Eq. (D9). In the special case that only the \( \sigma_z = -1 \) orbital is disordered, one can easily get

\[ \Gamma_+ = 0, \quad \Gamma_- = 2 \Lambda \frac{1 - w}{1 + w} \exp \left( - \frac{2}{\beta_+} \frac{1 - w}{1 + w} \right). \quad \text{(D12)} \]

This result is consistent with that presented in Ref. 13. In fact, the self-energy \( \Sigma_{22}(\epsilon) \) can be solved by making an expansion in powers of \( \epsilon \), as previously showed in Ref. 13.

If only the \( \sigma_z = +1 \) orbital is disordered, the solutions are

\[ \Gamma_+ = 2 \Lambda \sqrt{1 - w} \exp \left( - \frac{2}{\beta_+} \frac{1 + w}{1 - w} \right), \Gamma_- = 0. \quad \text{(D13)} \]
This result is already analyzed in the main text. Here, $\Gamma_+$ represents a low energy scale below which the perturbative RG becomes invalid. Similarly, the self-energy can be solved by power expansion at small $\epsilon$ in two limits, namely $|\epsilon| \gg \Gamma_+$ and $|\epsilon| \ll \Gamma_+$. According to Eq. (D7), we find that $\Sigma_{22} = 0$ due to $\beta_- = \beta_x = \beta_y = 0$, and the left hand side can be obtained by setting $\Sigma_{22} = \beta_z = \beta_y = 0$ in Eq. (D5). Now we obtain

$$\Sigma_{11}(\epsilon) = \frac{2\epsilon}{1-w} \pm 2i\Lambda \sqrt{\frac{1+w}{1-w}} \exp \left\{ \frac{-2\Sigma_{11}(\epsilon) (1+w)}{\beta_+ (1-w) \Sigma_{11}(\epsilon) - 2\epsilon} \right\}.$$  \hspace{1cm} (D14)

In the small energy regime, one can show that

$$\Sigma_{11}^R = -\frac{4(1+w)\epsilon}{(1-w)^2 \beta_+} - i\Gamma_+, \quad (|\epsilon| \ll \Gamma_0).$$  \hspace{1cm} (D15)

In the limit of $|\epsilon| \gg \Gamma_+$, we know from Eq. (D14) that

$$\Sigma_{11}(\epsilon) = \frac{2\epsilon}{1-w} + \frac{\beta_+ (1-w)^2 \left[ \Sigma_{11}(\epsilon) - \frac{2\epsilon}{1-w} \right] \ln \left[ \frac{\Sigma_{11}(\epsilon) - \frac{2\epsilon}{1-w}}{\beta_+ (1-w)^2} \right] + 4\epsilon (1+w)}{\beta_+ (1-w)^2 \left[ \ln \left( \Lambda \sqrt{\frac{1+w}{1-w}} \right) - \frac{i\pi}{2} \right] - 2(1-w^2)}.$$  \hspace{1cm} (D16)

We then iterate the above equation once, and perform an expansion in powers of $\beta_+$, which leads to

$$\Sigma_{11}^R(\epsilon) = -\frac{2\beta_+}{1+w} \left[ \epsilon \ln \left( \frac{\Lambda \sqrt{1-w^2}}{\epsilon} \right) + \frac{i\pi |\epsilon|}{2} \right], \quad (|\epsilon| \gg \Gamma_0).$$  \hspace{1cm} (D17)

Making use of Eq. (D15) and Eq. (D17), we finally get Eq. (45) of the main text.

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