Perturbative Aspects of $q$-Deformed Dynamics

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Abstract

Within the framework of the $q$-deformed Heisenberg algebra a dynamical equation of $q$-deformed quantum mechanics is discussed. The perturbative aspects of the $q$-deformed Schrödinger equation are analyzed. General representations of the additional momentum-dependent interaction originating from the $q$-deformed effects are presented in two approaches. As examples, such additional interactions related to the harmonic-oscillator potential and the Morse potential are demonstrated.

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Recently $q$-deformed quantum mechanics has attracted much attention as a possible modification of the ordinary quantum mechanics at short distances. According to present tests of quantum electrodynamics, quantum theories based on Heisenberg’s commutation relation are correct at least down to $10^{-18}$ cm. The question arises whether there is a possible generalization of Heisenberg’s commutation relation at shorter distances. In searching for such a possibility considerations of the space structure are a useful guide. If the space structure at such short distances exhibits a non-commutative property, and thus is governed by a quantum group symmetry, it has been shown that $q$-deformed quantum mechanics is a possible pre-quantum theory at short distances. In the literature different frameworks of $q$-deformed quantum mechanics were established [1–16].

The framework of the $q$-deformed Heisenberg algebra developed in Refs. [2, 4] shows clear physical content: its relation to the corresponding $q$-deformed boson commutation relations and the limiting process of the $q$-deformed harmonic oscillator to the undeformed one are clear. In this framework the $q$-deformed uncertainty relation shows an essential deviation from that of Heisenberg [14]: the ordinary minimal uncertainty relation shows an undercut. A non-perturbative feature of the $q$-deformed Schrödinger equation is that the energy spectrum exhibits an exponential structure [3, 4, 15]. The pattern of quark and lepton masses is qualitatively explained by such a $q$-deformed exponential spectrum [15].

In this paper we discuss perturbative aspects of the $q$-deformed Schrödinger equation in the above framework. The perturbative expansion of the $q$-deformed Hamiltonian possesses a complex structure, which amounts to some additional momentum-dependent interaction [2–4, 15]. There are two approaches to showing such $q$-deformed effects: One includes it in the kinetic energy term, the other includes it in the potential. General results are presented, and as examples, the harmonic-oscillator system and the Morse potential are discussed in some detail.

In the following, we first review the necessary background of $q$-deformed quantum mechanics. In terms of $q$-deformed phase space variables — the position operator $X$ and the momentum operator $P$, the following $q$-deformed Heisenberg algebra has been developed [4, 16]:

\[
q^{1/2}XP - q^{-1/2}PX = iU, \quad UX = q^{-1}XU, \quad UP = qPU,
\]  

(1)
where $X$ and $P$ are hermitian and $U$ is unitary: $X^\dagger = X$, $P^\dagger = P$, $U^\dagger = U^{-1}$. Compared to the Heisenberg algebra the operator $U$ is a new member, called the scaling operator. The necessity of introducing the operator $U$ is as follows.

The algebra (1) is based on the definition of the hermitian momentum operator $P$. However, if $X$ is assumed to be a hermitian operator in a Hilbert space the $q$-deformed derivative \[ \partial_X X = 1 + qX\partial_X, \] which codes the non-commutativity of space, shows that the usual quantization rule $P \rightarrow -i\partial_X$ does not yield a hermitian momentum operator. Ref. [4] showed that a hermitian momentum operator $P$ is related to $\partial_X$ and $X$ in a nonlinear way by introducing a scaling operator $U$

\[ U^{-1} \equiv q^{1/2}[1 + (q - 1)X\partial_X], \quad \bar{\partial}_X \equiv -q^{-1/2}U\partial_X, \quad P \equiv -\frac{i}{2}(\partial_X - \bar{\partial}_X), \] (3)

where $\bar{\partial}_X$ is the conjugate of $\partial_X$. The operator $U$ is introduced in the definition of the hermitian momentum, thus it closely relates to properties of the dynamics and plays an essential role in $q$-deformed quantum mechanics. The nontrivial properties of $U$ imply that the algebra (1) has a richer structure than the Heisenberg commutation relation. In (1) the parameter $q$ is a fixed real number. It is important to make distinctions for different realizations of the $q$-algebra by different ranges of $q$ values [18–20]. Following Refs. [2, 4] we only consider the case $q > 1$ in this paper. In the limit $q \rightarrow 1^+$ the scaling operator $U$ reduces to the unit operator, thus the algebra (1) reduces to the Heisenberg commutation relation.

Such defined hermitian momentum $P$ leads to $q$-deformation effects, which are exhibited by the dynamical equation. Eq. (3) shows that the momentum $P$ depends non-linearly on $X$ and $\partial_X$. Thus the $q$-deformed Schrödinger equation is difficult to treat. In this paper we demonstrate its perturbative aspects.

The $q$-deformed phase space variables $X$, $P$ and the scaling operator $U$ can be realized in terms of undeformed variables $\hat{x}$, $\hat{p}$ of the ordinary quantum mechanics, where $\hat{x}$, $\hat{p}$ satisfy: $[\hat{x},\hat{p}] = i$, $\hat{x} = \hat{x}^\dagger$, $\hat{p} = \hat{p}^\dagger$. The variables $X$, $P$ and the scaling operator $U$ are
related to $\hat{x}$, $\hat{p}$ by [4]:

$$ X = \frac{\hat{z} + \frac{1}{2}}{\hat{z} + \frac{1}{2}} \hat{x}, \quad P = \hat{p}, \quad U = q^{\hat{z}}, \quad (4) $$

where $\hat{z} = -\frac{i}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ and $[A]$ is the $q$-deformation of $A$, defined by $[A] = (q^A - q^{-A})/(q - q^{-1})$. Using (4) it is easy to check that $X$, $P$ and $U$ satisfy (4).

From (4) it follows that $X$ is represented as a function of $\hat{x}$ and $\hat{p}$ (note that $\hat{z} + \frac{1}{2} = -i\hat{x}\hat{p}$):

$$ X = i(q - q^{-1})^{-1}(q^{(\hat{z} + 1/2)} - q^{-(\hat{z} + 1/2)})\hat{p}^{-1}. \quad (5) $$

Using (5) it is convenient to discuss the perturbative expansion of $X$. Let $q = e^f = 1 + f$, with $0 < f \ll 1$. To the order $f^2$, $X$ reduces to

$$ X = \hat{x} + f^2g(\hat{x}, \hat{p}), \quad g(\hat{x}, \hat{p}) = -\frac{1}{6}(1 + \hat{x}\hat{p}\hat{x}\hat{p})\hat{x}. \quad (6) $$

The $q$-deformed phase space $(X, P)$ governed by the $q$-algebra is a $q$-deformation of the ordinary quantum mechanics phase space $(\hat{x}, \hat{p})$, thus all machinery of the ordinary quantum mechanics can be applied to the $q$-deformed quantum mechanics. By analogy, dynamical equations of the quantum system are the same for the undeformed phase space variables $\hat{x}$ and $\hat{p}$ and for the $q$-deformed phase space variables $X$ and $P$. Thus the starting point for establishing perturbative calculations of the $q$-deformed Schrödinger equation is as follows: first one uses $q$-deformed phase space variables $X$ and $P$ to write down the Hamiltonian of the system, then one uses (4) to express $X$ and $P$ by the undeformed phase space variables $\hat{x}$ and $\hat{p}$.

The $q$-deformed Hamiltonian with potential $V(X)$ is

$$ H(X, P) = \frac{1}{2\mu}P^2 + V(X). \quad (7) $$

For regular potentials $V(X)$, which are singularity free, to the order $f^2$ of the perturbative expansion, such potentials can be expressed by the undeformed variables $\hat{x}$ and $\hat{p}$ as

$$ V(X) = V(\hat{x}) + \hat{H}_f^{(q)}(\hat{x}, \hat{p}), \quad (8) $$
with the perturbation
\[
\hat{H}_I^{(q)}(\hat{x}, \hat{p}) = f^2 \sum_{k=1}^{\infty} \frac{V^{(k)}(0)}{k!} \left( \sum_{i=0}^{k-1} i (k-1)^{-i} g(\hat{x}, \hat{p}) \hat{x}^i \right),
\]
(9)
where \( V^{(k)}(0) \) is the \( k \)-th derivative of \( V(x) \) at \( x = 0 \) (\( x \) is the spectrum of \( \hat{x} \)). In (9) the ordering between the non-commutative quantities \( \hat{x} \) and \( g(\hat{x}, \hat{p}) \) is carefully considered.

Substituting for \( g(\hat{x}, \hat{p}) \) and summing over \( i \), the above result can be expressed as
\[
\hat{H}_I^{(q)}(\hat{x}, \hat{p}) = f^2 \sum_{k=1}^{\infty} \frac{V^{(k)}(0)}{k!} \hat{x}^k \left( k\hat{x}^2 \partial_x^2 + k(k+2)\hat{x} \partial_x + \frac{1}{6} k(k-1)(2k+5) \right).
\]
(10)
The remaining sum over \( k \) can be performed in terms of derivatives of the potential:
\[
\hat{H}_I^{(q)}(\hat{x}, \hat{p}) = \frac{f^2 \hat{x}^2}{6} \{ \hat{x} V'(\hat{x}) \partial_x^2 + [\hat{x} V''(\hat{x}) + 3V'(\hat{x})] \partial_x + \frac{1}{3} \hat{x} V'''(\hat{x}) + \frac{3}{2} V''(\hat{x}) \}.
\]
(11)

For potentials with singular term \( X^{-k} \), \( k = 1, 2, 3, \ldots \), we use the following operator equation to treat the perturbation expansion:
\[
\frac{1}{A + B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} - \frac{1}{A} B \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots,
\]
(12)
where the norms of the operators \( A \) and \( B \) satisfy \( ||B|| < ||A|| \). Thus to the order \( f^2 \) the perturbative expansion of \( 1/X \) reads:
\[
\frac{1}{X} = \frac{1}{\hat{x}} - f^2 \frac{1}{\hat{x}} g(\hat{x}, \hat{p}) \frac{1}{\hat{x}}.
\]
(13)

For the energy shift, in the state \( |n\rangle \), corresponding to Eq. (11), we may integrate by parts, and obtain
\[
\Delta \hat{E}_n^{(q)} = -\frac{f^2}{36} \int_{-\infty}^{\infty} dx \left\{ \psi_n^{(0)*}(x)V(x)[2x^3 \partial_x^3 + 9x^2 \partial_x^2 - 3]\psi_n^{(0)}(x) + \text{h.c.} \right\},
\]
(14)
where \( \psi_n^{(0)} \) is the unperturbed wave function. One can use the Schrödinger equation and rewrite this as
\[
\Delta \hat{E}_n^{(q)} = \frac{f^2}{6} \int_{-\infty}^{\infty} dx \psi_n^{(0)*}(x)(V(x)\{1 - 4\mu x^2[V(x) - E]\} - \frac{2}{3} \mu E x^3 V'(x))\psi_n^{(0)}(x),
\]
(15)
where \( E \) is the unperturbed energy.
There is another set of variables $\tilde{x}$ and $\tilde{p}$ of an undeformed algebra, which are obtained by a canonical transformation of $\hat{x}$ and $\hat{p}$ [4]:

$$\tilde{x} = \hat{x}F^{-1}(\hat{z}), \quad \tilde{p} = F(\hat{z})\hat{p}, \quad (16)$$

where (note that $\hat{z} - \frac{1}{2} = -i\hat{p}\hat{x}$)

$$F^{-1}(\hat{z}) = \frac{\hat{z} - \frac{1}{2}}{\hat{z} - \frac{1}{2}}, \quad (17)$$

Such defined variables $\tilde{x}$ and $\tilde{p}$ also satisfy the undeformed algebra: $[\tilde{x}, \tilde{p}] = i$, and $\tilde{x} = \tilde{x}^\dagger$, $\tilde{p} = \tilde{p}^\dagger$. Thus $\tilde{p} = -i\partial_{\tilde{z}}$. The $q$-deformed variables $X$, $P$ and the scaling operator $U$ are related to $\tilde{x}$ and $\tilde{p}$ as follows:

$$X = \tilde{x}, \quad P = F^{-1}(\tilde{z})\tilde{p}, \quad U = q^{\tilde{z}}, \quad \text{(18)}$$

where $\tilde{z} = -\frac{i}{2}(\tilde{x}\tilde{p} + \tilde{p}\tilde{x})$; and with $F^{-1}(\hat{z})$ defined by Eq. (17) for the variables $(\hat{x}, \hat{p})$. From Eqs. (16)–(18) it follows that $X$, $P$ and $U$ also satisfy (4), and Eq. (18) is equivalent to Eq. (4).

Using (18) to the order $f^2$ the perturbative expansions of $P$ and the kinetic energy $P^2/(2\mu)$ read

$$P = \tilde{p} + f^2h(\tilde{x}, \tilde{p}), \quad h(\tilde{x}, \tilde{p}) = -\frac{1}{6}(1 + \tilde{p}\tilde{x}\tilde{p}\tilde{x})\tilde{p}, \quad \text{(19)}$$

and

$$\frac{1}{2\mu}P^2 = \frac{1}{2\mu}\tilde{p}^2 + \tilde{H}^{(q)}_I(\tilde{x}, \tilde{p}), \quad \text{(20)}$$

with

$$\tilde{H}^{(q)}_I(\tilde{x}, \tilde{p}) = \frac{1}{2\mu}f^2\left[\tilde{p} h(\tilde{x}, \tilde{p}) + h(\tilde{x}, \tilde{p}) \tilde{p}\right]$$

$$= -\frac{1}{12\mu}f^2\left[2\tilde{x}^2\partial_{\tilde{z}}^4 + 8\tilde{x}\partial_{\tilde{z}}^3 + 3\partial_{\tilde{z}}^2\right] \quad \text{(21)}$$

Eqs. (20) and (21) show that in the $(\tilde{x}, \tilde{p})$ system the perturbative contribution comes from the kinetic-energy term.
Similar to Eqs. (14)–(15) (using the Schrödinger equation and integrating by parts), one can write the energy shift corresponding to Eq. (21) as

$$\Delta \tilde{E}_n^{(q)} = \frac{f^2}{6} \int_{-\infty}^{\infty} dx \psi_n^{(0)*}(x)[V(x) - E]\{1 - 4\mu x^2[V(x) - E]\}\psi_n^{(0)}(x).$$  (22)

The two expressions for the energy shift, Eqs. (15) and (22) are in fact equal, since the difference is given by

$$\frac{f^2}{6} E \int_{-\infty}^{\infty} dx \psi_n^{(0)*}(x)\{1 - 4\mu x^2[V(x) - E] - \frac{2}{5} x^3 \mu V'(x)\}\psi_n^{(0)}(x) = 0.$$  (23)

From this last form, Eq. (22), it is easy to see that the energy shift is negative since $\langle n|V|n \rangle < E$. Thus,

$$\Delta E_n^{(q)} < 0.$$  (24)

As a first application we consider the $q$-deformed “harmonic” system described by the Hamiltonian

$$H(X, P) = \frac{1}{2\mu} P^2 + \frac{1}{2} \mu \omega^2 X^2,$$  (25)

First we calculate $\Delta \tilde{E}_n^{(q)}$ in the $(\tilde{x}, \tilde{p})$ system. From Eq. (21) or (22) it follows that the shifts of the energy levels are

$$\Delta \tilde{E}_n^{(q)} = -\frac{f^2 \omega}{48} (4n^3 + 6n^2 + 20n + 9).$$  (26)

In the $(\tilde{x}, \tilde{p})$ system the only non-zero term in (3) is $V^{(2)}(0) = \mu \omega^2$, thus (3) reduces to:

$$\hat{H}_I^{(q)}(\tilde{x}, \tilde{p}) = -\frac{1}{12} f^2 \mu \omega^2 [\tilde{x}(1 + \tilde{x}\tilde{p}\tilde{x})\tilde{x} + (1 + \tilde{x}\tilde{p}\tilde{x})\tilde{x}^2]$$

$$= \frac{1}{12} f^2 \mu \omega^2 [2\tilde{x}^4 \partial_{\tilde{x}}^2 + 8\tilde{x}^3 \partial_{\tilde{x}} + 3\tilde{x}^2].$$  (27)

The corresponding energy shift, which can also be obtained from Eq. (15), is easily seen to be identical to that of Eq. (26).

As noted above, the shift in Eq. (26) is negative, and it increases with $n$, leading eventually to a breakdown of perturbation theory for $n \sim (12/f^2)^{1/3}$. The tendency exhibited by Eq. (26) agrees with the observation that for the $q$-deformed harmonic oscillator the spectrum has an upper bound [4].
In the limiting case \( q \to 1^+ \) we have \( H(X, P) \to H_{\text{un}}(\hat{x}, \hat{p}) = \frac{1}{2\mu} \hat{p}^2 + \frac{1}{2} \mu \omega^2 \hat{x}^2 \). Only in this sense \( H(X, P) \) defined in Eq. (25) is called the \( q \)-deformed “harmonic” system.

As another example, we study the Morse potential [21] in its “supersymmetric” form [22], where the ground state energy vanishes. It is given by the potential

\[
V(x) = A^2 + B^2 e^{-2\alpha x} - 2B \left( A + \frac{\alpha}{2\sqrt{2\mu}} \right) e^{-\alpha x}.
\]  
(28)

The corresponding energy shift can be obtained from either Eq. (15) or Eq. (22), the result is shown in Fig. 1 for \( \alpha = 1, \mu = 1, \) and some range of \( A \) and \( B \). For the harmonic oscillator, we saw that the shift increased in magnitude with the unperturbed energy. This is not the case for the Morse potential, where the shift may increase or decrease with the unperturbed energy, depending on the parameters.

![Figure 1: Energy shift for the Morse potential, \( \Delta E/f^2 \), vs. \( A \) and \( B \), for \( \alpha = 1 \).](image)

It should be emphasized again that \( \hat{H}_I^{(q)}(\hat{x}, \hat{p}) \) originates from the kinetic term, whereas \( \hat{H}_I^{(q)}(\hat{x}, \hat{p}) \) originates from the potential. At the level of operators, these two Hamiltonians are different. However, they differ only by a quantity whose expectation value vanishes.

At short distances, where \( q \)-deformation might be relevant, one also expects quantum mechanics to break down and have to be replaced by some kind of field theory. Some
progress is being made in this area. In a more realistic theory along such lines, some features of $q$-deformed quantum mechanics may survive. It is therefore hoped that studies of $q$-deformed dynamics at the level of quantum mechanics will give some clue for the further development.

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