Semiclassical noise beyond the second cumulant

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ABSTRACT
We show how the semiclassical Langevin method can be extended to calculations of higher-than-second cumulants of noise. These cumulants are affected by indirect correlations between the fluctuations, which may be considered as "noise of noise." We formulate simple diagrammatic rules for calculating the higher cumulants and apply them to mesoscopic diffusive contacts and chaotic cavities. As one of the application of the method, we analyze the frequency dependence of the third cumulant of current in these systems and show that it contains additional peculiarities as compared to the second cumulant. The effects of environmental feedback in measurements of the third cumulant are also discussed in terms of this method.

Keywords: nonequilibrium noise, higher cumulants, Langevin equation

1. INTRODUCTION
Since the nonequilibrium noise in mesoscopic systems became a hot topic in late 80s, the semiclassical Boltzmann–Langevin approach has been successfully competing with quantum-mechanical methods in these studies. While restricted to certain class of systems and parameters, this approach is generally more simple and physically appealing. It can easily handle some of the problems that are extremely difficult to solve by directly using quantum-mechanical equations.

Higher-order correlations of current became a subject of interest for theorists since early nineties. This work was pioneered by Levitov and Lesovik, who discovered that the charge transmitted through a single-channel quantum contact obeys a binomial distribution. Based on their quantum-mechanical formulas, higher cumulants of current were calculated for a variety of multichannel phase-coherent systems. Meanwhile a wide class of these systems, like diffusive conductors and chaotic cavities, allows a semiclassical description of their average transport properties and second cumulant of noise. Hence it is of interest to have a fully semiclassical theory for higher cumulants of noise in these systems.

Below we present an extension of the Boltzmann–Langevin approach that allows one to calculate the statistical properties of noise beyond the second cumulant. This extension is based (a) on the time-scale separation between the correlation time of microscopic sources of noise acting upon the system ("noise generators") and the response of the system to them and (b) on the smallness of fluctuations. It has been initially proposed for the higher cumulants of current in mesoscopic diffusive wires, where the resulting terms could be directly mapped on the quantum-mechanical diagrams for these quantities. Shortly after this, the cascade expansion was postulated for a chaotic cavity with quantum contacts and tested against quantum-mechanical results for the third and fourth cumulants of current. Very recently, a general proof of this expansion was given for semiclassical cumulants of arbitrary order.

An application of the semiclassical method to calculations of the frequency dependence of the third cumulant of current noise in mesoscopic diffusive wires and chaotic cavities gave very unexpected results. Unlike the conventional noise and ac electric response that exhibit a dispersion only at the inverse RC time of the system, the third cumulant of current has also a dispersion at the inverse dwell time of an electron on the system. This dispersion is due to slow fluctuations of the distribution function that do not violate electroneutrality and are akin to fluctuations of local temperature. These fluctuations do not directly contribute to the current and therefore are not seen in conventional noise, but they modulate the intensity of noise sources and therefore

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manifest themselves in higher correlations of current. As the dwell time is generally much larger than the RC time, the resulting dispersion may be experimentally observed.

The cascaded expansion of higher cumulants is valid not only for systems described by Boltzmann equation but also for other systems that satisfy the conditions of time-scale separation and smallness of fluctuations. Recently, this method has been used for calculating the feedback of the external circuit in measurements of the third cumulant of an arbitrary mesoscopic resistor.\(^{15}\)

The paper is organized as follows. In Section 2 we describe the general cascaded formalism. We give a brief overview of the standard Boltzmann–Langevin method, then we derive the expression for the third cumulant using a generalized Fokker–Planck equation. After this, we formulate the general diagrammatic rules for calculating cumulants of arbitrary order. In Section 3 we calculate the frequency dependence of the third cumulant for a mesoscopic diffusive wire and a chaotic cavity. Finally, we address the effect of the external circuit on the measurements of the third cumulant. The main conclusions are summarized in Section 4.

2. GENERAL FORMALISM

The basis for the semiclassical description of kinetics is the existence of two well separated time scales, one of which describes a "slow" classical evolution of the system and the other describes "fast" quantum processes. For example, the collision integral in the Boltzmann equation may be written as local in time because quantum-mechanical scattering is assumed to be fast as compared to the evolution of the distribution function. For this reason, it is possible to describe the noise associated with electron scattering by Langevin sources that are \(\delta\)-correlated in time. On the other hand, using the semiclassical kinetic equation implies coarse-grain averaging over \(N = (\Delta p \Delta x/2\pi \hbar)^3\) states located in an elementary cell of phase space whose characteristic dimensions \(\Delta p\) and \(\Delta x\) are small as compared to the characteristic scales of the problem. This suggests that the cumulants of the distribution function \(f\) should decay with number as \(1/N^{n-1}\).

2.1. Boltzmann–Langevin Method

The second cumulants of noise are conveniently described by the Boltzmann–Langevin equation,\(^6\) which is obtained by linearizing the standard Boltzmann equation with respect to a fluctuation of the distribution function \(\delta f\). It reads

\[
\left[ \frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} + e \mathbf{E} \frac{\partial}{\partial \varepsilon} \right] \delta f(p, r, t) + \delta I = -e \delta \mathbf{E} \cdot \nabla f + J^{\text{ext}},
\]

where \(\delta I\) is the linearized collision integral and \(J^{\text{ext}}\) is the Langevin source that accounts for the randomness of electron scattering. This equation should be supplemented by a self-consistency equation for the electric field

\[
\nabla \delta \mathbf{E} = 4\pi \delta \rho, \quad \delta \rho(r, t) = e \sum_p \delta f(p, r, t).
\]

The key issue in the Boltzmann–Langevin method is a derivation of the correlation function of Langevin sources. Several decades ago, Kogan and Shulman\(^6\) proposed a physically appealing expression for this function. Because of a short duration of scattering events, the Langevin sources are \(\delta\)-correlated in time. Furthermore, since different scattering events are independent, the scattering between each pair of states \((p, r)\) and \((p', r)\) presents a Poissonian process whose cumulants of any order are proportional to its average rate

\[
J(p \rightarrow p') = W(p, p') f(p, r, t)[1 - f(p', r, t)].
\]

Hence the cumulants of extraneous sources of the corresponding order related with randomness of scattering may be written as the sums of incoming and outgoing scattering fluxes taken with appropriate signs. For example, the second cumulant is given by\(^6\)

\[
\langle (J^{\text{ext}}(p_1, r_1, t_1) J^{\text{ext}}(p_2, r_2, t_2)) \rangle = \delta(r_1 - r_2) \delta(t_1 - t_2) \left\{ \sum_{p_1, p_2} [J(p_1 \rightarrow p') + J(p' \rightarrow p_1)] \right\}
\]
The Langevin equation in the form

$$-J(p_1 \to p_2) - J(p_2 \to p_1).$$

To calculate, e.g., the correlator of current fluctuations \(\langle \delta j(r_1, t_1)\delta j(r_2, t_2) \rangle\), one has to take two formal solutions of Eq. (1) in \(\delta f\), calculate the two fluctuations of current via

$$\delta j(r, t) = e \sum_p v \delta f(p, r, t),$$
multiply them, and average the product using Eq. (4). As the fluxes (3) depend on the average distribution function, the resulting noise is also a functional of \(f\).

The Boltzmann–Langevin method has been very successful in predicting the 1/3 suppression of shot noise in mesoscopic diffusive contacts\(^{16}\) and analyzing the effects of electron-electron scattering on it.\(^{17}\)

### 2.2. Calculation of the Third Cumulant

One might think that all higher cumulants of order \(n\) could be treated similarly just by averaging the product of \(n\) formal solutions of Eq. (1) with the corresponding analog of Eq. (4). However this is not the case. The problem is that the correlator of Langevin sources (4) as well as higher-order correlators of \(\delta J^{ext}\) are themselves functions of \(f\), which fluctuates in time, and this results in additional correlations between \(\delta f\)'s. To elucidate this point, we consider a simple example of a Markov process where a random quantity \(x(t)\) is described by a Langevin equation

$$\dot{x} = -\frac{1}{\tau}(x - \bar{x}) + \xi(t)$$

and \(\xi(t)\) is a random extraneous force with zero mean \(\langle \xi(t) \rangle = 0\). Suppose that an \(n\)th cumulant of \(\xi\) is inversely proportional to a large number \(N^{n-1}\) and explicitly depends on \(x\):

$$\langle \xi(t_1)\xi(t_2) \rangle = \frac{1}{N} \delta(t_1 - t_2)Q(x), \quad \langle \xi(t_1)\xi(t_2)\xi(t_3) \rangle = \frac{1}{N^2} \delta(t_1 - t_2)\delta(t_2 - t_3)R(x).$$

This suggests that as \(\Delta t \to 0\), the cumulants of increment of \(x\) are proportional to \(\Delta t\) and equal

$$\langle x(t + \Delta t) - x(t) \rangle = -\frac{\Delta t}{\tau}\bar{x}, \quad \langle [x(t + \Delta t) - x(t)]^2 \rangle = \frac{\Delta t}{N}Q(x), \quad \langle [x(t + \Delta t) - x(t)]^3 \rangle = \frac{\Delta t}{N^2}R(x).$$

It's a textbook knowledge that the probability \(w(x, t)\) in this case obeys a generalized Fokker–Planck equation in the form

$$\frac{\partial w}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial x} [(x - \bar{x})w] + \frac{1}{2N} \frac{\partial^2 (Qw)}{\partial x^2} - \frac{1}{6N^2} \frac{\partial^3 (Rw)}{\partial x^3}.$$

By multiplying both parts of Eq. 8 by \(\delta x \equiv x(t) - \bar{x}, \delta x^2, \) and \(\delta x^3\) and integrating them over \(x\) by parts the corresponding number of times, one obtains equations of motion for the conditional cumulants of \(x\)

$$\left(\frac{\partial}{\partial x} + \frac{1}{\tau}\right) \langle \delta x(t) \rangle = 0, \quad \left(\frac{\partial}{\partial x} + \frac{2}{\tau}\right) \langle \delta x^2(t) \rangle = \frac{1}{N}Q(t),$$

and

$$\left(\frac{\partial}{\partial x} + \frac{3}{\tau}\right) \langle \delta x^3(t) \rangle = \frac{3}{N} \langle \delta x(t)Q(t) \rangle + \frac{1}{N^2}R(t),$$

where \(\ldots\) stand for any functions of \(x\) taken at previous instants of time, \(\langle Q(t) \rangle \equiv \langle Q(\bar{x} + \delta x(t)) \rangle\), and \(\langle R(t) \rangle \equiv \langle R(\bar{x} + \delta x(t)) \rangle\). By subsequently solving the equations for lower cumulants with senior cumulants as the initial conditions, one obtains the expressions for the two- and three-time correlation functions of \(\delta x\) for \(t_1 > t_2 > t_3\) in the form

$$\langle \delta x(t_1)\delta x(t_2) \rangle = \frac{1}{N} \int_{-\infty}^{t} dt' K(t_1 - t')K(t_2 - t')\langle Q(t') \rangle$$

\(\text{Eq. (11)}\)
and
\[ \langle \delta x(t_1) \delta x(t_2) \delta x(t_3) \rangle = \frac{1}{N} \int_{t_3}^{t_2} dt' K(t_1 - t') K(t_2 - t') \langle Q(t') \delta x(t_3) \rangle \]
\[ + \int_{-\infty}^{t_3} dt' K(t_1 - t') K(t_2 - t') K(t_3 - t') \left[ \frac{3}{N} \langle Q(t') \delta x(t') \rangle + \frac{1}{N^2} \langle R(t') \rangle \right], \quad (12) \]
where \( K(t - t') = \exp[\{-t - t'\}/\tau] \).

Expand now \( Q(t') \) and \( R(t') \) in powers of \( \delta x(t') \) about \( \bar{x} \). By substituting these expansions
\[ Q(t) = Q(\bar{x}) + \delta x(t') \frac{dQ}{dx} \bar{x} + \frac{1}{2} \delta x^2(t') \frac{d^2Q}{dx^2} \bar{x} + \ldots \]
\[ R(t) = R(\bar{x}) + \delta x(t') \frac{dQ}{dx} \bar{x} + \frac{1}{2} \delta x^2(t') \frac{d^2Q}{dx^2} \bar{x} + \ldots \]
to Eqs. (11) and (12), one obtains a system of equations where cumulants of all orders are coupled together and which cannot be solved in a general case. However, one can make use of the smallness of \( 1/N \) into Eqs. (11) and (12), one obtains a system of equations where cumulants of all orders are coupled together becomes just
\[ \langle \delta x(t_1) \delta x(t_2) \rangle = \frac{1}{N} \int_{-\infty}^{t_2} dt' K(t_1 - t') K(t_2 - t') Q(\bar{x}). \quad (13) \]

This is exactly what the standard Langevin approach with \( Q = Q(\bar{x}) \) gives.

The expression for the third cumulant appears to be more involved. Taking into account that \( K(t_i - t_i') \langle Q(t') \delta x(t') \rangle = \langle \delta x(t_i) Q(t_i') \rangle \) if \( t_i > t_i' \), it is easily brought to a form
\[ \langle \delta x(t_1) \delta x(t_2) \delta x(t_3) \rangle = \frac{1}{N} \int_{-\infty}^{t_3} dt' K(t_1 - t') K(t_2 - t') \langle Q(t') \delta x(t_3) \rangle \]
\[ + \frac{1}{N} \int_{-\infty}^{t_3} dt' K(t_2 - t') K(t_3 - t') \langle Q(t') \delta x(t_1) \rangle + \frac{1}{N} \int_{-\infty}^{t_2} dt' K(t_1 - t') K(t_3 - t') \langle Q(t') \delta x(t_2) \rangle \]
\[ + \frac{1}{N^2} \int_{-\infty}^{t_3} dt' K(t_1 - t') K(t_2 - t') K(t_3 - t') R(\bar{x}). \quad (14) \]

We note that the first three terms of Eq. (14) present convolutions of the functional derivatives of the second cumulant with another second cumulant, so that symbolically, this equation may be recast in a form
\[ \langle \delta x_1 \delta x_2 \delta x_3 \rangle = \langle \delta x_1 \delta x_2 \delta x_3 \rangle_{Q=0} + P_{123} \left\{ \frac{\delta \langle \delta x_1 \delta x_2 \rangle}{\delta \bar{x}} \langle \delta \bar{x} \delta x_3 \rangle \right\}, \quad (15) \]
where \( P_{123} \) denotes a summation over all inequivalent permutations of indices \((123)\) and \( \delta \langle \ldots \rangle / \delta \bar{x} \) denotes a functional derivative with respect to \( \bar{x} \). The first term in Eq. (15) presents a direct contribution to the third cumulant of \( x \) from the third cumulant of extraneous sources, and the second term presents the cascade corrections, which result from modulation of the second cumulant by fluctuations of \( x \).

If \( x \) denotes a semiclassical distribution function \( f(p, r, t) \), one may roughly imagine these cascade corrections as fluctuations of Nyquist noise of a resistor caused by fluctuations of its temperature. It should be noted that the conditions for validity of the cascaded approach are the same as the conditions of validity of Boltzmann description itself and no additional assumptions are needed.
2.3. Diagrammatic Expansions for Higher Cumulants

Similar expressions are valid for higher cumulants. Apart from direct products of \( n \) formal solutions of the Langevin equation, \( n \)th cumulant also contains cascade corrections from an interplay between lower cumulants. The cascade corrections are conveniently presented in a diagrammatic form (see Figs. 1 and 2). All diagrams present graphs whose outer vertices correspond to different instances of \( x \) and whose inner vertices correspond either to cumulants of extraneous currents or their functional derivatives. The number of arrows outgoing from an inner vertex corresponds to the order of the cumulant and the number of incoming arrows corresponds to the order of a functional derivative. Apparently, the difference between the total order of cumulants involved and the total number of functional differentiations should be equal to the order of the cumulant being calculated. As there should be no backaction of higher cumulants on lower cumulants, all diagrams are singly connected. Therefore any diagram for the \( n \)th cumulant of the current may be obtained from a diagram of order \( m < n \) by combining it with a diagram of order \( n - m + 1 \), i.e., by inserting one of its outer vertices into one of the inner vertices of the latter. Hence the most convenient way to draw diagrams for a cumulant of a given order is to start with diagrams of lower order and to consider all their inequivalent combinations that give diagrams of the desired order. The analytical expressions corresponding to each diagram contain numerical prefactors equal to the numbers of inequivalent permutations of the outer vertices.

At first, these rules had been postulated, but very recently they were proved independently by Jordan et al. using the stochastic path integral approach and by Gutman et al. using the supersymmetry method.

3. Applications of the Method

We now concentrate on the two most important mesoscopic systems that allow a semiclassical treatment, i.e. diffusive wires and classical chaotic cavities. Both these types of multichannel systems have a complicated internal dynamics characterized by essentially different time scales. In the case of a diffusive wire, there is a large separation between the elastic scattering time and the time of diffusion of an electron across the wire. In the case of a chaotic cavity, there is a large separation between the time of flight through the cavity and the dwell time of the electron on it. The fluctuations in these systems are relatively small because of a large number of quantum channels in them, so the cascaded Langevin approach may be conveniently applied to them.

Of a primary interest to us will be the frequency dependence of the third cumulant of current

\[
P_3(\omega_1, \omega_2) = \int d(t_1 - t_2) \int d(t_2 - t_3) \exp[i\omega_1(t_1 - t_3) + i\omega_2(t_2 - t_3)] \langle \delta I(t_1) \delta I(t_2) \delta I(t_3) \rangle
\]

in these systems. The point is that in the case of a good conductor with a short screening length, the dwell time of an electron in the system is seen neither in its linear response nor in the second cumulant of current because the charge transport is primarily controlled by the screening effects whose characteristic time scale is
Figure 2. The contributions to the fourth cumulant of the current. Dashed lines correspond to fluctuations of the distribution function. Full circles, triangles and squares correspond to the second, third, and fourth cumulants of extraneous currents. The empty triangles and squares present their functional derivatives.

the RC time of the system. Typically, this time is so short that the resulting dispersion of noise cannot be experimentally observed. In contrast to this, the third cumulant of current is affected not only by fluctuations of the total electron number at a given point, but also by fluctuations of their distribution in energy, which may affect the intensity of noise sources. The latter fluctuations have a much longer relaxation time since they do not violate the electroneutrality and therefore may result in a dispersion at experimentally accessible frequencies.

3.1. Diffusive wires

Consider a quasi-one-dimensional diffusive wire of length $L$ and conductivity $\sigma$ with a constant voltage drop $V$ across it. All dimensions are assumed to be much larger than the elastic mean free path and the screening length in the metal. We also assume that dimensions are much smaller than the electron-phonon scattering length, but take into account electron-electron scattering. It is assumed that the contact is located close to an external gate, so it is described by a single capacitance. In the case of a diffusive metal, the distribution function $f(\varepsilon, \mathbf{r})$ and its fluctuations are almost isotropic in the momentum space, and the Boltzmann–Langevin equation reduces to a stochastic diffusion equation

$$
\left(\frac{\partial}{\partial t} - D \nabla^2\right) \delta f - \delta I_{ee} = -e \delta \phi \frac{\partial f}{\partial \varepsilon} - \nabla \delta F^{imp} - \delta F^{ee},
$$

where $D$ is the diffusion coefficient, $\delta I_{ee}$ is the linearized electron-electron collision integral, and $\delta F^{imp}$ and $\delta F^{ee}$ are random extraneous sources associated with electron-impurity and electron-electron scattering. The fluctuation of the electric potential $\delta \phi$ that appears in this equation should be calculated self-consistently from the Poisson equation

$$
\nabla^2 \delta \phi = -4\pi \delta \rho,
$$

where the fluctuation of charge density $\delta \rho$ is given by

$$
\delta \rho = eN_F \left( \int d\varepsilon \delta f(\varepsilon) + e \delta \phi \right),
$$

and where $N_F$ is the Fermi density of states. In the case of a quasi-one-dimensional contact, a solution of Eqs. (17) - (19) is of the form

$$
\delta \phi(x, \omega) = \frac{1}{S_0 \sigma} \left( \nabla^2 + i \omega RC/L^2 \right)^{-1} \frac{\partial}{\partial x} \int d^2 r_{\perp} \delta j_x^{ext}(r),
$$

where $S_0$ is the cross-sectional area of the wire. The contribution to the fourth cumulant of the current is given by the functional derivative of the electric potential with respect to the external source $\delta j_x^{ext}(r)$.
where \( x \) is the coordinate along the contact, \( S_0 \) is the cross section area of the contact, \( C \) and \( R \) are the capacitance and the resistance of the contact, and

\[
\delta j^{\text{ext}} = eN_f \int d\varepsilon \delta F^{\text{imp}}.
\]

A fluctuation of the total current at the left end of the contact equals

\[
\delta I = \sigma \int d^2 r_\perp \frac{\partial \phi(x, \omega)}{\partial x} \bigg|_{x=-L/2}.
\]

Making use of the correlation function of extraneous sources

\[
\langle \delta F^{\text{imp}}_\alpha(\varepsilon, \mathbf{r}) \delta F^{\text{imp}}_\beta(\varepsilon', \mathbf{r'}) \rangle_\omega = 2\frac{D}{N_F} \delta(\mathbf{r} - \mathbf{r'}) \delta(\varepsilon - \varepsilon') \delta_{\alpha\beta} f(\varepsilon, \mathbf{r})[1 - f(\varepsilon, \mathbf{r})],
\]

one easily obtains the second-order correlation function for the fluctuations of the current as a functional of the distribution function \( f \). At frequencies much smaller than \( (RC)^{-1} \) it is of the form

\[
\langle \delta I(\omega_1) \delta I(\omega_2) \rangle = 4\pi\delta(\omega_1 + \omega_2)(RL)^{-1} \int dx \int d\varepsilon f(\varepsilon, x)[1 - f(\varepsilon, x)],
\]

To calculate the third cumulant of current in a diffusive wire, one may use the cascaded Langevin approach described in the previous section with \( f \) in place of \( x \). In the case of diffusive metal, the direct contributions from the higher-than-second cumulants of Langevin sources to the corresponding cumulants of current are negligibly small because these cumulants are proportional to the inverse elastic scattering time \( \tau^{-1} \) (see Section 2.1) and each solution of the Boltzmann–Langevin equation for the anisotropic part of \( \delta f \) gives an additional factor of \( \tau \), so that the \( n \)th cumulant of transport current would be proportional to \( \tau^{n-1} \) instead of \( \tau \). Hence all higher cumulants of current for diffusive contacts are dominated by diagrams constructed of the second cumulant of Langevin sources and its functional derivatives. As has been shown in Section 2.2, the third cumulant in this case is of the form

\[
\langle \delta I(t_1) \delta I(t_2) \delta I(t_3) \rangle = P_{123} \left\{ \int dt \int d\varepsilon \int d^3 r \frac{\delta(\delta I(t_1) \delta I(t_2)) \delta f(\varepsilon, \mathbf{r}, t)}{\delta f(\varepsilon, \mathbf{r}, t)} \langle \delta I(t_3) \rangle \right\},
\]

We restrict ourselves to the two limiting cases of absolutely elastic scattering and of a strong electron-electron scattering. In the case of purely elastic scattering, \( \delta I^{\text{ee}} \) and \( \delta I^{\text{ee}} \) in Eq. (17) may be dropped. Furthermore, we consider frequencies much smaller than \( (RC)^{-1} \), so the frequency-dependent term in Eq. (20) may be neglected and the solution of these equations is straightforward. Using the well-known expression for the average distribution function

\[
\bar{f}(\varepsilon, x) = \left( \frac{1}{2} + \frac{x}{L} \right) f_0(\varepsilon + eV/2) + \left( \frac{1}{2} - \frac{x}{L} \right) f_0(\varepsilon - eV/2),
\]

where \( f_0 \) is the equilibrium Fermi distribution, one obtains that in the frequency range of interest,

\[
P_3(\omega_1, \omega_2) = P(\omega_1) + P(\omega_2) + P(-\omega_1 - \omega_2),
\]

where

\[
P_{el}(\omega) = \frac{4 e^2 V}{3 R} \frac{q_{el} L (q_{el}^2 L^2 + 30) \sinh(q_{el} L) - 8 (q_{el}^2 L^2 + 6) \cosh(q_{el} L) + 2 q_{el}^2 L^2 + 48}{q_{el}^2 L^3 \sinh(q_{el} L)}
\]

at \( eV \gg T \) and

\[
P_{el}(\omega) = \frac{4 e^2 V}{3 R} \frac{2 \cosh(q_{el} L) - q_{el} L \sinh(q_{el} L) - 2}{q_{el}^2 L^3 \sinh(q_{el} L)}.
\]

at \( T \gg eV \). Here \( q_{el} = (i\omega/D)^{1/2} \). At \( \omega = 0 \) these expressions give \( -(1/45)e^2 V/R \) and \( -(1/9)e^2 V/R \), which corresponds to \( P_3(0,0) = -(1/15)e^2 V/R \) and \( P_3(0,0) = -(1/3)e^2 V/R \). These zero-frequency results are in
Figure 3. The real and imaginary parts of the ratio \( P(\omega)/(e^2 I) \) versus normalized frequency \( \omega \tau_D \) for purely elastic scattering at \( eV \gg T \) (solid lines), hot-electron regime at \( eV \gg T \) (dashed lines), purely elastic scattering at \( eV \ll T \) (dotted lines), and hot-electron regime at \( eV \ll T \) (dash-dotted lines). Inset: charged and uncharged fluctuations of the distribution function \( f \). Charged fluctuations have a short relaxation time and contribute to fluctuations of current \( \delta I \). Uncharged fluctuations do not contribute to \( \delta I \) directly but affect the intensity of noise sources. They decay only via slow diffusion and result in the low-frequency dispersion of the third cumulant.

agreement with the results of Gutman and Gefen.\(^9\) At finite frequency, Eqs. (28) and (29) become complex-valued and tend to zero as \( i/\omega \) at \( \omega \to \infty \).

In the opposite limit of a strong electron–electron interaction, the distribution function may be assumed to have a Fermi shape with a coordinate-dependent temperature \( T_e(x) \) and electric potential \( \phi(x) \)

\[
f(\varepsilon, x) = \left[ 1 + \exp \left( \frac{\varepsilon - e\phi(x)}{T_e(x)} \right) \right]^{-1}
\]

and a fluctuation \( \delta f \) can be expressed in terms of fluctuations of these quantities

\[
\delta f(\varepsilon, r, \omega) = \frac{\partial f(\varepsilon, r)}{\partial \phi} \delta \phi + \frac{\partial f(\varepsilon, r)}{\partial T_e} \delta T_e
\]

A substitution of Eq. (31) into Eq. (25) and integration over the energy readily gives

\[
P_{\text{hot}}(\omega) = \frac{2}{RL} \int_{-L/2}^{L/2} dx \langle \delta T_e(x) \delta I \rangle_{\omega}. \tag{32}
\]

To calculate the correlator in Eq. (32), we have to obtain a Langevin-type equation for \( \delta T_e \). To this end, we multiply Eq. (38) by \( \varepsilon \) and integrate it over \( \varepsilon \), like it was done when deriving the equation of heat balance.\(^{17,19}\) This gives

\[
\left( \frac{\partial}{\partial t} - D \nabla^2 \right) \left( \frac{\pi^3}{3} T_e \delta T_e \right) - D \nabla^2 (e^2 \phi \delta \phi) = - \int d\varepsilon \varepsilon \nabla \delta F^{\text{imp}}. \tag{33}
\]

By solving this equation together with Eq. (20) and making use of the mean effective temperature\(^{17}\)

\[
\bar{T}_e(x) = \left[ T^2 + \frac{3}{\pi^2} (eV)^2 \left( \frac{1}{4} - \frac{x^2}{L^2} \right) \right]^{1/2}, \tag{34}
\]
one obtains

\[ P_{\text{hot}}(\omega) = -\frac{12e^2V}{\pi R} \frac{1}{q^2_\omega L^2} \left[ 1 - \frac{2}{q_\omega L} \tanh \left( \frac{q_\omega L}{2} \right) \right]. \tag{35} \]

at high temperatures \( T \gg eV \) and

\[ P_{\text{hot}}(\omega) = \frac{12e^2V}{\pi R} \sum_{k=0}^{\infty} \frac{J_0(\pi k + \pi/2)[J_1(\pi k + \pi/2) - (-1)^k]}{(2k + 1)[\pi^2(2k + 1)^2 + i\omega L^2/D]}. \tag{36} \]

at low temperatures \( eV \gg T \), where \( J_0 \) and \( J_1 \) are Bessel functions of order 0 and 1. The corresponding limiting values \( P_3(0,0) = -(3/\pi^2)e^2V/R \) at \( eV \ll T \) and \( P_3(0,0) = -(8/\pi^2 - 9/16)e^2V/R \) at \( eV \gg T \) coincide with the results of Gutman and Gefen.\(^9\)

Figure 3 shows that both real and imaginary parts of \( P \) have the most pronounced dispersion in the case of a high temperature or for strong electron-electron scattering, i.e. when the local distribution function has a nearly Fermian shape. This unexpected result is in a sharp contrast with the dispersion of quantum noise,\(^21\) which results from sharp singularities in the energy dependence of the distribution function. This difference comes from the different symmetry of the second-order correlation functions appearing in Eq. (25) in the elastic case at low temperatures and in the hot-electron regime. In the former case, the correlation function \( \langle \delta f(\epsilon, x) \delta I \rangle_\omega \) is an odd function of the coordinate, but in the latter case \( \langle \delta T(x) \delta I \rangle_\omega \) is an even function of \( x \).

### 3.2. Chaotic Cavity

The cascaded Langevin approach may be successfully applied not only to systems described by Langevin equations with Poissonian statistics of extraneous sources, which takes place in the case of a Boltzmann kinetics. It can be equally well applied to systems with arbitrary statistics of noise sources provided that the conditions of time-scale separation and smallness of the fluctuations are observed. An example of a semiclassical system with non-Poissonian sources of noise is a classical chaotic cavity, i.e. a metallic island of irregular shape connected to the electrodes \( L, R \) via two quantum point contacts of conductances \( G_{L,R} \gg e^2/h \) and arbitrary transparencies \( \Gamma_{L,R} \) (see Fig. 4). As the dwell time of an electron in the cavity \( \tau_D = e^2N_F/(G_L + G_R) \) is much larger than the time of flight through the cavity, the electrons in the cavity lose memory of their initial phase and are described by an energy-dependent distribution function \( f(\epsilon, t) \). Hence the low-frequency dynamics of \( f \) is described by a semiclassical equation, while quantum contact serve as generators of noise with essentially non-Poissonian statistics. The fluctuations of the electric current in the left and right contacts are given by equations

\[ \delta I_{L,R} = \int d\epsilon \left[ (\tilde{I}_{L,R})\epsilon + \frac{1}{e} G_{L,R} \delta f(\epsilon) \right]. \tag{37} \]

where \((\tilde{I}_L)_\epsilon\) and \((\tilde{I}_R)_\epsilon\) are the energy-resolved random extraneous currents generated by the left and right contacts. The fluctuation of the distribution function \( \delta f(\epsilon) \) obeys a kinetic equation

\[ \left( \frac{\partial}{\partial t} + \frac{1}{\tau_D} \right) \delta f(\epsilon, t) = -e U \frac{\partial f}{\partial \epsilon} - \frac{1}{eN_F} \delta f(\epsilon, t) = \frac{1}{eN_F} \left[ (\tilde{I}_L)_\epsilon + (\tilde{I}_R)_\epsilon \right], \tag{38} \]
Figure 5. A 3D plot of \( \text{Re } P_3(\omega_1, \omega_2) \) for \( G_L/G_R = 1/2, \Gamma_L = \Gamma_R = 3/4, \tau_Q = 1/3, \) and \( \tau_D = 10 \) (dimensionless units).

where \( \tau_D = e^2 N_F / (G_L + G_R) \) is the dwell time of an electron in the cavity, \( N_F \) is the density of states in it, and \( \delta U \) is a fluctuation of the electric potential of the cavity. This fluctuation is obtained from the charge-conservation law

\[
\frac{\partial \delta U}{\partial t} = \frac{1}{C} \frac{\partial \delta Q}{\partial t} = -\frac{1}{C} \int d\varepsilon [(\tilde{I}_L)_{\varepsilon} + (\tilde{I}_R)_{\varepsilon}],
\]

where \( C \) is the electrostatic capacitance of the cavity. Equations (37) - (39) suggest that different parts of \( \delta f \) are described by different time scales. The relaxation of electrically neutral fluctuations is described by the characteristic time \( \tau_D \), whereas the fluctuations of charge are described by \( \tau_Q = [(G_L + G_R)/C + 1/\tau_D]^{-1} \), which is much shorter than \( \tau_D \) for good conductors.

If the distribution function of electrons in the cavity \( f(\varepsilon, t) \) were not allowed to fluctuate, the contacts would be independent generators of current noise whose zero-frequency energy-resolved cumulants \( \langle \langle \tilde{I}_{L,R}^{n} \rangle \rangle_{\varepsilon} \) could be obtained from a quantum-mechanical formula

\[
\langle \langle \tilde{I}_{L,R}^{n} \rangle \rangle_{\varepsilon} = \frac{G_{L,R}}{\Gamma_{L,R}} \frac{\partial^{n}}{\partial \chi^{n}} \ln \{ 1 + \Gamma_{L,R} f_{L,R}(\varepsilon)[1 - f(\varepsilon)](e^{-e\chi} - 1) 
+ \Gamma_{L,R} f(\varepsilon)[1 - f_{L,R}(\varepsilon)](e^{e\chi} - 1) \} \mid_{\chi=0}.
\]

If the voltage is high enough, the noise of isolated contacts can be considered as white at frequencies at which the distribution function \( f \) fluctuates. This allows us to consider the contacts as independent generators of white noise, whose intensity is determined by the instantaneous distribution function of electrons in the cavity. Based on this time-scale separation, we perform a recursive expansion of higher cumulants of current in terms of its lower cumulants as described in Section 2. In the low-frequency limit, the expressions for the third and fourth cumulants coincide with those obtained by quantum-mechanical methods for arbitrary ratio of conductances \( G_L/G_R \) and transparencies \( \Gamma_{L,R} \).10 Very recently, similar equations for the zero-frequency limit were obtained as a saddle-point expansion of a stochastic path integral.20 Consider now the frequency dependence of the third cumulant. We will be interested in the case of a good conductor where the charge-relaxation time \( \tau_Q \) is much shorter than the dwell time \( \tau_D \). Unlike the second cumulant of current, the third cumulant \( P_3(\omega_1, \omega_2) \) in general exhibits a strong dispersion at \( \omega_{1,2} \sim 1/\tau_D \).14 In this limit,

\[
P_3(\omega_1, \omega_2) = e^2 I \left\{ 3G_L G_R \left[ (1 - \Gamma_R) G_L^2 - (1 - \Gamma_L) G_R^2 \right]^2 / (G_L + G_R)^6 - 2 \Gamma_R^2 G_L^2 + \Gamma_L^2 G_R^2 \right\} + \frac{3 \Gamma_R G_L^4 + \Gamma_L G_R^4}{(G_L + G_R)^4} - \frac{G_L^3 + G_R^3}{(G_L + G_R)^3}.
\]
$$P_3(\omega, 0) = -\frac{1}{32} \frac{e^2 I}{(e^2 I)^2} \frac{1 + 2\tau_D^2 \omega^2 + \tau_D^2 \tau_Q^2 \omega^4}{(1 + \omega^2 \tau_D^2)(1 + \omega^2 \tau_Q^2)^2}.$$  

The $P_3(\omega, 0)$ curve shows a clear minimum at $\omega \sim (\tau_D \tau_Q)^{-1/2}$ and the amplitude of its variation tends to $P_3(0, 0)$ as $\tau_Q/\tau_D \to 0$.

### 3.3. Effects of External Circuit on Measurements of the Third Cumulant

So far we applied the cascaded method to systems where the noise was governed by a microscopic distribution function of electrons. Now we give an example where the noise in a system is determined by a macroscopic quantity.

Recently, Levitov and Reznikov\textsuperscript{22} predicted that the third cumulant of noise in a tunnel contact should present a linear function of voltage independent of temperature. However when Reulet et al.\textsuperscript{23} tested these predictions experimentally, they found very poor agreement of their data with theory. The reason for this description has been explained by Beenakker et al.\textsuperscript{15} by the feedback of the external circuit using the cascade approach.

In experiments of Reulet and co-workers, the actual measurable quantity was the voltage drop across a macroscopic resistor $R_S$ connected in series with a mesoscopic contact with a resistance $R_0$, which produced a
non-Gaussian noise (see Fig. 7). Naively, one could expect that the third cumulant of voltage drop would be just proportional to the third cumulant of current generated by the contact, but this is not the case.

Indeed, consider the macroscopic resistor and the contact as independent generators of noise currents $\delta I_S$ and $\delta I_0$. The noise of the macroscopic resistor is Gaussian with the only nonzero second cumulant $\langle\langle I_S^2 \rangle\rangle = 2T/R_S$ representing its equilibrium noise, and the noise of the contact has nonzero second and third cumulants $\langle\langle I_0^2 \rangle\rangle$ and $\langle\langle I_0^3 \rangle\rangle$, which depend on the voltage across it. The Kirchhoff’s law for the fluctuation of the total current in the circuit may be written in a form

$$\delta I = -\frac{\delta \phi}{R_S} + \delta I_S = \frac{\delta \phi}{R_0} + \delta I_0,$$

where $\delta \phi$ is the fluctuation of electric potential to be measured. Hence

$$\delta \phi = \frac{R_0 R_S}{R_0 + R_S} (\delta I_S - \delta I_0).$$

One obtains the second cumulant of voltage just by multiplying and averaging two solutions (44), which gives

$$\langle\langle \phi^2 \rangle\rangle = \left( \frac{R_0 R_S}{R_0 + R_S} \right)^2 \left( \frac{\langle\langle I_0^2 \rangle\rangle}{\langle\langle I_0^2 \rangle\rangle} + 2 \frac{T}{R_S} \right),$$

Calculations of the third cumulant of voltage according to Eq. (15) give

$$\langle\langle \phi^3 \rangle\rangle = -\left( \frac{R_0 R_S}{R_0 + R_S} \right)^3 \langle\langle I_0^3 \rangle\rangle + 3 \left( \frac{R_0 R_S}{R_0 + R_S} \right)^4 \left( 2 \frac{T}{R_S} + \frac{\langle\langle I_0^2 \rangle\rangle}{\langle\langle I_0^2 \rangle\rangle} \right) \frac{d\langle\langle I_0^2 \rangle\rangle}{d\phi}. $$

The first term in this expression is the naively expected result, and the second term presents the cascade correction that results from modulations of the second cumulant of current generated by the mesoscopic contact by fluctuations of voltage across it. Equation (46) suggests that the feedback of the external circuit is very essential even at $R_S \ll R_0$. For a tunnel contact where $\langle\langle I_0^2 \rangle\rangle = (e\phi/R_0) \coth(e\phi/2T)$ and $\langle\langle I_0^3 \rangle\rangle = e^2 \phi/R_0$, it results in a change of slope of $\langle\langle \phi^3 \rangle\rangle$ versus $V$ curve from negative at $eV \ll T$ to positive at $eV \gg T$. This is exactly what is observed in experiment.

**4. SUMMARY**

In summary, we demonstrated a semiclassical method for calculating higher cumulants of noise in systems where a large separation exists between the characteristic times describing the correlations of microscopic sources of noise and the characteristic times describing the dynamics of averages. The calculations of the frequency-dependent third cumulants in a number of mesoscopic systems show that their dispersion may be very different from that of conventional noise and is very sensitive to fine properties of these systems that almost do not manifest themselves otherwise. As the conditions for validity of the method are rather general, it may be applied to macroscopic systems, too.
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