ABSTRACT. Segre embedding was introduced by C. Segre (1863–1924) in his famous 1891 article [50]. The Segre embedding plays an important role in algebraic geometry as well as in differential geometry, mathematical physics, and coding theory. In this article, we survey main results on Segre embedding in differential geometry. Moreover, we also present recent differential geometric results on maps and immersions which are constructed in ways similar to Segre embedding.

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References

1. Introduction.

Throughout this article, we denote by $CP^n(c)$ the complex projective $n$-space endowed with the Fubini-Study metric with constant holomorphic sectional curvature $c$. Let $(z_0, \ldots, z_n)$ denote a homogeneous coordinate system on $CP^n(c)$.

There are two well-known examples of algebraic manifolds in complex projective spaces: the Veronese embedding $v_n : CP^n(2) \rightarrow CP^{n(n+3)/2}(4)$ and the Segre embedding $S_{hp} : CP^h(4) \times CP^p(4) \rightarrow CP^{h+p+hp}(4)$.

The Veronese embedding $v_n$ is a Kählerian embedding, that is a holomorphically isometric embedding, of $CP^n(2)$ into $CP^{n(n+3)/2}(4)$ given by homogeneous monomials of degree 2:

\begin{equation}
(1.1) \quad v_n : CP^n(2) \rightarrow CP^{n(n+3)/2}(4); \quad (z_0, \ldots, z_n) \mapsto \left( z_0^2, \sqrt{2} z_0 z_1, \ldots, \sqrt{2} \frac{z_0^{\alpha_i} z_1^{\alpha_j}}{\alpha_i \alpha_j}, \ldots, z_n^2 \right)
\end{equation}

with $\alpha_i + \alpha_j = 2$. For $n = 1$, this is nothing but the quadric curve

$$Q_1 = \left\{ (z_0, z_1, z_2) \in CP^2 : \sum_{j=0}^{2} z_j^2 = 0 \right\}$$

in $CP^2(4)$.

The Veronese embedding can be extended to $\alpha$-th Veronese embedding $v_n^\alpha$ with $\alpha \geq 2$:

\begin{equation}
(1.2) \quad v_n^\alpha : CP^n\left(\frac{4}{\alpha}\right) \rightarrow CP^{(n+\alpha)\alpha^{-1}}(4)
\end{equation}
defined by
\[(z_0, \ldots, z_n) \mapsto \left( \sqrt[\alpha_0]{z_0}, \sqrt[\alpha_1]{\alpha z_1}, \ldots, \sqrt[\alpha_1]{\alpha_{n-1}z_{n-1}}, \sqrt[\alpha_n]{z_n} \right) \]
with \(\alpha_0 + \cdots + \alpha_n = \alpha\).

On the other hand, the Segre embedding:
\[(1.4)\quad S_{hp}: \mathbb{CP}^h(4) \times \mathbb{CP}^p(4) \to \mathbb{CP}^{h+p}(4),\]
is defined by
\[(1.5)\quad S_{hp}(z_0, \ldots, z_h; w_0, \ldots, w_p) = (z_j w_t)_{0 \leq j \leq h, 0 \leq t \leq p},\]
where \((z_0, \ldots, z_h)\) and \((w_0, \ldots, w_p)\) are the homogeneous coordinates of \(\mathbb{CP}^h(4)\) and \(\mathbb{CP}^p(4)\), respectively. This embedding (1.4) was introduced by C. Segre in 1891 (see [50]). It is well-known that the Segre embedding \(S_{hp}\) is also a Kählerian embedding.

When \(h = p = 1\), the Segre embedding is nothing but the complex quadric surface, \(Q_2 = \mathbb{CP}^1 \times \mathbb{CP}^1\) in \(\mathbb{CP}^3\), defined by
\[(1.6)\quad Q_2 = \left\{ (z_0, z_1, z_2, z_3) \in \mathbb{CP}^3 : \sum_{j=0}^{3} z_j^2 = 0 \right\}.

The Segre embedding can also be naturally extended to product embeddings of arbitrary number of complex projective spaces as follows.

Let \((z_i^1, \ldots, z_i^{n_i})\) \((1 \leq i \leq s)\) denote the homogeneous coordinates of \(\mathbb{CP}^{n_i}\). Define a map:
\[(1.7)\quad S_{n_1, \ldots, n_s}: \mathbb{CP}^{n_1}(4) \times \cdots \times \mathbb{CP}^{n_s}(4) \to \mathbb{CP}^N(4),\]
where
\[N = \prod_{i=1}^{s} (n_i + 1) - 1,\]
which maps each point \(((z_i^1, \ldots, z_i^{n_i})), \ldots, (z_s^1, \ldots, z_s^{n_s}))\) in the product Kählerian manifold \(\mathbb{CP}^{n_1}(4) \times \cdots \times \mathbb{CP}^{n_s}(4)\) to the point \((z_1^1 \cdots z_s^1, z_1^2 \cdots z_s^2)_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_s \leq n_s}\) in \(\mathbb{CP}^N(4)\). This map \(S_{n_1, \ldots, n_s}\) is also a Kählerian embedding.

The Segre embedding is known to be the simplest Kählerian embedding from product algebraic manifolds into complex projective spaces. It is well-known that the Segre embedding plays an important role in algebraic geometry (see
The Segre embedding has also been applied to differential geometry as well as to coding theory (see, for instance, [36, 49, 52]) and to mathematical physics (see, for instance, [4, 51]).

The purpose of this article is to survey the main results on Segre embedding in differential geometry. Furthermore, we also present recent results in differential geometry concerning maps, immersions, and embedding which are constructed in ways similar to Segre embedding defined by (1.5).

2. Basic formulas and definitions.

Let $M$ be a Riemannian $n$-manifold with inner product $\langle \ , \ \rangle$ and let $e_1, \ldots, e_n$ be an orthonormal frame fields on $M$. For a differentiable function $\varphi$ on $M$, the gradient $\nabla \varphi$ and the Laplacian $\Delta \varphi$ of $\varphi$ are defined respectively by

\begin{equation}
\langle \nabla \varphi, X \rangle = X \varphi,
\end{equation}

\begin{equation}
\Delta \varphi = \sum_{j=1}^{n} \{ e_j e_j \varphi - (\nabla e_j) e_j \varphi \}
\end{equation}

for vector fields $X$ tangent to $M$, where $\nabla$ is the Levi-Civita connection on $M$.

If $M$ is isometrically immersed in a Riemannian manifold $\tilde{M}$, then the formulas of Gauss and Weingarten for $M$ in $\tilde{M}$ are given respectively by

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),
\end{equation}

\begin{equation}
\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi
\end{equation}

for vector fields $X, Y$ tangent to $N$ and $\xi$ normal to $M$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$, $\sigma$ the second fundamental form, $D$ the normal connection, and $A$ the shape operator of $M$ in $\tilde{M}$.

The second fundamental form and the shape operator are related by

\begin{equation}
\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle.
\end{equation}

The mean curvature vector $\vec{H}$ is defined by

\begin{equation}
\vec{H} = \frac{1}{n} \text{trace } \sigma = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),
\end{equation}
where \{e_1, \ldots, e_n\} is a local orthonormal frame of the tangent bundle \(TN\) of \(N\). The squared mean curvature is given by \(H^2 = \langle \vec{H}, \vec{H} \rangle\), where \(\langle \ , \ \rangle\) denotes the inner product. A submanifold \(N\) is called totally geodesic in \(\tilde{M}\) if the second fundamental form of \(N\) in \(\tilde{M}\) vanishes identically. And \(N\) is called minimal if its mean curvature vector vanishes identically.

The equation of Gauss is given by
\[
\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + \langle \sigma(X, Z), \sigma(Y, W) \rangle \\
- \langle \sigma(X, W), \sigma(Y, Z) \rangle,
\]
for \(X, Y, Z, W\) tangent to \(M\), where \(R\) and \(\tilde{R}\) denote the curvature tensors of \(M\) and \(\tilde{M}\), respectively.

For the second fundamental form \(\sigma\), we define its covariant derivative \(\bar{\nabla}\sigma\) with respect to the connection on \(TM \oplus T^\perp M\) by
\[
(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)
\]
for \(X, Y, Z\) tangent to \(M\). The equation of Codazzi is
\[
(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z),
\]
where \((\tilde{R}(X, Y)Z)^\perp\) denotes the normal component of \(\tilde{R}(X, Y)Z\).

A submanifold \(M\) in a Riemannian manifold \(\tilde{M}\) is said to have parallel second fundamental form if \(\bar{\nabla}\sigma = 0\) identically.

The Riemann curvature tensor of a complex space form \(\tilde{M}^{m}(4c)\) of constant holomorphic sectional curvature \(4c\) is given by
\[
\tilde{R}(X, Y; Z, W) = c \{ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle + \langle JX, W \rangle \langle JY, Z \rangle \\
- \langle JX, Z \rangle \langle JY, W \rangle + 2 \langle X, JY \rangle \langle JZ, W \rangle \}.
\]

If we define the \(k\)-th \((k \geq 1)\) covariant derivative of the second fundamental form \(\sigma\) by
\[
(\bar{\nabla}^k \sigma)(X_1, \ldots, X_{k+2}) = D_{X_{k+2}}((\nabla^{k-1} \sigma)(X_1, \ldots, X_{k+1})) \\
- \sum_{i=1}^{k+1}((\nabla^{k-1} \sigma)(X_1, \ldots, \nabla_{X_{k+2}} X_i, \ldots, X_{k+1}),
\]
then $\bar{\nabla}^k \sigma$ is a normal-bundle-valued tensor of type $(0, k + 2)$. Moreover, it can be proved that $\bar{\nabla}^k \sigma$ satisfies

\begin{equation}
(\bar{\nabla}^k \sigma)(X_1, X_2, X_3, \ldots, X_{k+2}) - (\bar{\nabla}^k \sigma)(X_2, X_1, X_3, \ldots, X_{k+2}) = R^D(X_1, X_2)((\bar{\nabla}^{k-2} \sigma)(X_3, \ldots, X_{k+2})) + \sum_{i=3}^{k+2}((\bar{\nabla}^{k-2} \sigma)(X_3, \ldots, R(X_1, X_2)X_i, \ldots, X_{k+2}),
\end{equation}

for $k \geq 2$. For simplicity, we put $\bar{\nabla}^0 \sigma = \sigma$.

3. Differential geometric characterizations of Segre embedding.

H. Nakagawa and R. Tagaki [42] classify complete Kählerian submanifold in complex projective spaces with parallel second fundamental form. In particular, they show that the Segre embedding are the only one which are reducible. More precisely, they prove the following.

**Theorem 3.1.** Let $M$ be a complete Kählerian submanifold of $n$ complex dimensions embedded in $CP^m(4)$. If $M$ is reducible and has parallel second fundamental form, then $M$ is congruent to $CP^{n_1}(4) \times CP^{n_2}(4)$ with $n = n_1 + n_2$ and the embedding is given by the Segre embedding.

The following results published in 1981 by B. Y. Chen [6] for $s = 2$ and by B. Y. Chen and W. E. Kuan [27, 28] for $s \geq 3$ can be regarded as “converse” to Segre embedding constructed in 1891 by C. Segre.

**Theorem 3.2 (6 [27, 28]).** Let $M_1^{n_1}, \ldots, M_s^{n_s}$ be Kählerian manifolds of dimensions $n_1, \ldots, n_s$, respectively. Then locally every Kählerian immersion

$$f : M_1^{n_1} \times \cdots \times M_s^{n_s} \to CP^N(4), \quad N = \prod_{i=1}^{s}(n_i + 1) - 1,$$

of $M_1^{n_1} \times \cdots \times M_s^{n_s}$ into $CP^N(4)$ is the Segre embedding, that is, $M_1^{n_1}, \ldots, M_s^{n_s}$ are open portions of $CP^{n_1}(4), \ldots, CP^{n_s}(4)$, respectively. Moreover, the Kählerian immersion $f$ is given by the Segre embedding.

Let $||\bar{\nabla}^k \sigma||^2$ denote the squared norm of the $k$-th covariant derivative of the second fundamental form. The Segre embedding can also be characterized by $||\bar{\nabla}^k \sigma||^2$ as given in the following.
Theorem 3.3 ([6, 27, 28]). Let $M^{n_1} \times \cdots \times M^{n_s}$ be a product Kählerian submanifold in $CP^m(4)$ of arbitrary codimension. Then we have

$$||\nabla^{k-2}\sigma||^2 \geq k! \cdot 2^k \sum_{i_1 < \cdots < i_k} n_1 \cdots n_k,$$

for $k = 2, 3, \cdots$.

The equality sign of (3.2) holds for some $k \geq 2$ if and only if $M^{n_1}, \ldots, M^{n_s}$ are open portions of $CP^{n_1}(4), \ldots, CP^{n_s}(4)$, respectively, and the Kählerian immersion is given by the Segre embedding.

When $k = 2$, Theorem 3.2 and Theorem 3.3 reduce to

Theorem 3.4 ([6]). Let $M^h_1$ and $M^p_2$ be two Kählerian manifolds of complex dimensions $h$ and $p$, respectively. Then every Kählerian immersion

$$f : M^h_1 \times M^p_2 \to CP^{h+p+hp}(4),$$

of $M^h_1 \times M^p_2$ into $CP^{h+p+hp}(4)$ is locally the Segre embedding, that is, $M^h_1$ and $M^p_2$ are open portions of $CP^h$ and $CP^p$, respectively, and moreover, the Kählerian immersion $f$ is given by the Segre embedding.

Theorem 3.5 ([6]). Let $M^h_1 \times M^p_2$ be a product Kählerian submanifold in $CP^m(4)$ of arbitrary codimension. Then we have

$$||\sigma||^2 \geq 8hp.$$

The equality sign of (3.2) holds if and only if $M^h_1$ and $M^p_2$ are open portions of $CP^h(4)$ and $CP^p(4)$, respectively, and the Kählerian immersion is given by the Segre embedding $S_{h,p}$.

Let $CP^n_s(4)$ denote the indefinite complex projective space of complex dimension $n$, index $2s$, and constant holomorphic sectional curvature 4 and let $S_{2s+1}^{2n+1}(1)$ be the $(2n+1)$-dimensional indefinite unit-sphere with index $2s$ and of constant sectional curvature 1. Thus a point of $CP^n_s(c)$ can be represented by $[(z, w)]$, where $z = (z_1, \ldots, z_s) \in C^n$, $w = (w_1, \ldots, w_{n-s-1}) \in C^{n-s+1}$, $(z, w) \in S_{2s+1}^{2n+1}(1) \subset C^{n+1}_s$ and $[(z, w)]$ is the equivalent class of the Hopf projection

$$\pi_H : S_{2s+1}^{2n+1}(1) \to CP^n_s(4).$$
Consider the map:
\[ \phi : CP^h_s(4) \times CP^p_t(4) \to CP^{N(h,p)}_{R(h,p,s,t)}(4) \]
with
\[ N(h,p) = h + p + hp, \]
\[ R(h,p,s,t) = s(p-t) + t(h-s) + s + t \]
given by
\[ \phi([(z,w)], [(x,y)]) = [(x_1 y_\alpha, w_k x_\alpha, z_j x_\beta, w_\ell y_\beta)] \]
for
\[ 1 \leq i, j \leq s; 1 \leq k, \ell \leq h - s + 1; 1 \leq a, b \leq t; 1 \leq \alpha, \beta \leq h - t + 1. \]

Then map \( \phi \) is a well-defined holomorphic isometric embedding, which is called the indefinite Segre embedding \( CP^h_s(4) \times CP^p_t(4) \) into \( CP^{N(h,p)}_{R(h,p,s,t)}(4) \).

T. Ikawa, H. Nakagawa and A. Romero [35] study indefinite version of Theorem 3.4 and obtain the following.

**Theorem 3.6.** Let \( M^n_s \) and \( M^m_t \) be two complete indefinite Kählerian manifolds with complex dimensions \( n \) and \( m \), and indices \( 2s \) and \( 2t \), respectively. If there exists a holomorphic isometric immersion from the product \( M^n_s \times M^m_t \) into an indefinite complex projective space of complex dimension \( N \) and index \( 2r \), then we have:

1. \( N \geq n + m + nm \) and \( r \geq s(m-t) + t(n-s) + s + t \).
2. If \( N = n + m + nm \), then the immersion is obtained by the indefinite Segre embedding.

**Remark 3.1.** The assumption of “Kähler immersion” in Theorems 3.2, 3.3 and 3.4 is necessary. In fact, let \( M_i \) be a projective nonsingular embedded variety of dimension \( n_i \geq 1 \) \((i = 1, 2, \cdots, r)\) and let
\[ M = M_1 \times \cdots \times M_r \subset CP^{n_1} \times \cdots \times CP^{n_r} \]
be the composition embedding from the product \( M_1 \times \cdots \times M_r \) into \( CP^N \) with \( N = \prod_{i=1}^{r}(n_i + 1) - 1 \) via the Segre embedding.

M. Dale considers in [31] the problem of finding the embedding dimension \( e \) such that \( M \) can be embedded (not necessary Kählerian embedded in general)
in $\mathbb{CP}^e$, but not in $\mathbb{CP}^{e-1}$, via a projection. Using an algebraic result of A. Holme, Dale characterizes $e$ in terms of the degree of the Segre classes of $M$, he proves that $e = 2(n_1 + \cdots + n_r) + 1$, unless $r = 2$, $X_1 = \mathbb{CP}^{n_1}$, $X_2 = \mathbb{CP}^{n_2}$, in which case $e = 2(n_1 + n_2) - 1$.

4. DEGREE OF KÄHLERIAN IMMERSIONS AND HOMOGENEOUS KÄHLERIAN SUBMANIFOLDS VIA SEGRE EMBEDDING.

By applying Segre embedding, R. Takagi and M. Takeuchi define in [54] the notion of tensor products of Kählerian immersions in complex projective spaces as follows:

Suppose that $f_i : M_i \to \mathbb{CP}^{N_i}(4)$, $i = 1, \ldots, s$, are full Kählerian embeddings of irreducible Hermitian symmetric spaces of compact type. Consider the composition given by

$$f_1 \boxtimes \cdots \boxtimes f_s : M_1 \times \cdots \times M_s \xrightarrow{f_1 \times \cdots \times f_s} \mathbb{CP}^{N_1} \times \cdots \times \mathbb{CP}^{N_s} \xrightarrow{\text{product embedding}} \mathbb{CP}^{N_1} \times \cdots \times \mathbb{CP}^{N_s} \xrightarrow{S_{N_1 \cdots N_s}} \mathbb{CP}^N(4),$$

with $N = \prod_{i=1}^s (N_i + 1) - 1$. This composition is a full Kählerian embedding, which is called the tensor product of $f_1, \ldots, f_s$.

H. Nakagawa and R. Tagaki in [42] and R. Tagaki and M. Takeuchi in [54] had obtained a close relation between the degree and the rank of a symmetric Kählerian submanifold in complex projective space; namely, they proved the following.

**Theorem 4.1.** Let $f_i : M_i \to \mathbb{CP}^{N_i}(4)$, $i = 1, \ldots, s$, are $p_i$-th full Kählerian embeddings of irreducible Hermitian symmetric spaces of compact type. Then the degree of the tensor product $f_1 \boxtimes \cdots \boxtimes f_s$ of $f_1, \ldots, f_s$ is given by $\sum_{i=1}^s r_i p_i$, where $r_i = \text{rank}(M_i)$.

Related with this theorem, we mention the following result by M. Takeuchi [55] for Kählerian immersions of homogeneous Kählerian manifolds.

**Theorem 4.2.** Let $f : M \to \mathbb{CP}^m(4)$ be a Kählerian immersion of a globally homogeneous Kählerian manifold $M$. Then
(1) $M$ is compact and simply-connected;  
(2) $f$ is an embedding; and  
(3) $M$ is the orbit in $CP^n(4)$ of the highest weight in an irreducible unitary representation of a compact semisimple Lie group.

The notion of the degree of Kählerian immersions in the sense of [54] is defined as follows: Let $V$ be a real vector space of dimension $2n$ with an almost complex structure $J$ and a Hermitian inner product $g$. Denote the complex linear extensions of $J$ and $g$ to the complexification $V^C$ of $V$ by the same $J$ and $g$, respectively. Let $V^+$ and $V^-$ be the eigensubspace of $J$ on $V_C$ with eigenvalue 1 and $-1$, respectively. Then $V^C = V^+ \oplus V^-$ is an orthogonal direct sum with respect to the inner product.

Let $E$ be a real vector bundle over a manifold $M$ with a Hermitian structure $(J,g)$ on fibres. The Hermitian structure induces a Hermitian inner product on $E^C$. We have subbundle $E^+$ and $E^-$ such that $E^C = E^+ \oplus E^-$. The map on the space of sections induced from the complex conjugation is denoted by $\Gamma(E^+) \to \Gamma(E^-)$. Let $(M,g,J)$ be a Kählerian manifold. Then we get a Hermitian inner product on the complexification $T(M)^C$ and subbundles $T(M)^\pm$ of $T(M)^C$ such that $T(M)^C = T(M)^+ \oplus T(M)^-$.  

Let $f : (M,g,J) \to (M',g',J')$ be a Kählerian immersion between Kählerian manifolds. The Levi-Civita connections of $M$ and $M'$ are denoted by $\nabla$ and $\nabla'$. The induced bundle $f^*T(M')$ has a Hermitian structure $(J',g')$ induced from the one on $M'$. Also it has a connection $\nabla'$ induced from $M'$.  

If we denote the orthogonal complement of $f^*T(x(M))$ in $T_{f(x)}(M')$ by $T_{x}^\perp(M)$, then $T^\perp(M)$ is a subbundle of $f^*T(M')$. We have the orthogonal Whitney sum decompositions: $f^*T(M') = f^*T(M) \oplus T^\perp(M)$, $f^*T(M')^C = f^*T(M)^C \oplus T^\perp(M)^C$, and $f^*T(M')^\pm = f^*T(M)^\pm \oplus T^\perp(M)^\pm$.  

The orthogonal projection $f^*T(M') \to T^\perp(M)$ is denoted by $X \mapsto X^\perp$ and the induced projection $\Gamma(f^*T(M')) \to \Gamma(T^\perp(M))$ is denoted by $\xi \mapsto \xi^\perp$. The normal connection $D$ on $T^\perp(M)$ satisfies $D_X\xi = (\nabla_X\xi)^\perp$.  


Let $\sigma$ denote the second fundamental form of $f$. We have

(4.2) $\sigma(T_x(M)^+, T_x(M)^-) = \{0\}$, $\sigma(T_x(M)^\pm, T_x(M)^\mp) \subset T_x^\pm(M)^\pm$.

Put $\sigma_2 = \sigma$. For $k \geq 3$, we define $\sigma_k$ inductively just like (2.10) by

(4.3) $\sigma_{k+1}(X_1, \ldots, X_{k+1}) = D_{X_{k+1}}\sigma_k(X_1, \ldots, X_k)
- \sum_{i=2}^k \sigma_k(X_1, \ldots, \nabla_{X_{k+1}}X_i, \ldots, X_k),
$ for $X_i \in T_x(M)$.

Equations (4.2) and (4.3) imply that $\sigma_k(X_1, \ldots, X_k) \in T_x^+(M)^+$ for $X_1, X_2 \in T_x(M)^+$ and $X_3, \ldots, X_k \in T_x(M)^C$.

Let $H^k \in \Gamma(\Hom(\otimes^k T(M)^+, T^+(M)^+)) \ (k \geq 2)$ be defined by

$H^k(X_1, \ldots, X_k) = \sigma_k(X_1, \ldots, X_k), \ X_i \in T_x(M)^+.$

We put

$h =: \sum_{k \geq 2} \sigma_k \in \Gamma\left(\Hom\left(\sum_{k \geq 2} \otimes^k T(M)^+, T^+(M)^+\right)\right),
H =: \sum_{k \geq 2} H^k \in \Gamma\left(\Hom\left(\sum_{k \geq 2} \otimes^k T(M)^+, T^+(M)^+\right)\right).$

For an integer $k > 0$, we define a subspace $\mathcal{H}_x^k(M)$ of $T_{f(x)}(M)^+$ to be the subspace spanned by $T_x(M)^+$ and $H\left(\sum_{2 \leq j \leq k} \otimes^j T_x(M)^+\right)$. Then we get a series:

$\mathcal{H}_x^1(M) \subset \mathcal{H}_x^2(M) \subset \cdots \subset \mathcal{H}_x^k(M) \subset \mathcal{H}_x^{k+1}(M) \subset \cdots \subset T_{f(x)}(M)^+$

of increasing subspaces of $T_{f(x)}(M)^+$. Let $O_x^k(M)$ be the orthogonal complement of $\mathcal{H}_x^{k-1}(M)$ in $\mathcal{H}_x^k(M)$, where $O_x^0(M)$ is understood to be $\{0\}$. Then we have an orthogonal direct sum: $\mathcal{H}_x^k(M) = O_x^1(M) \oplus O_x^2(M) \oplus \cdots \oplus O_x^k(M)$.

Define $\mathcal{R}_1 = M$. For an integer $k > 1$, we define the set $\mathcal{R}_k$ of $k$-regular points of $M$ inductively by

$\mathcal{R}_k = \left\{u \in \mathcal{R}_{k-1} : \dim_C \mathcal{H}_x^k(M) = \max_{y \in \mathcal{R}_{k-1}} \dim_C \mathcal{H}_y^k(M)\right\}.$

Then we have the inclusions: $\mathcal{R}_1 \supset \mathcal{R}_2 \supset \cdots \supset \mathcal{R}_k \supset \mathcal{R}_{k+1} \supset \cdots$. Note that each $\mathcal{R}_k$ is an open nonempty subset of $M$ and, for each $k$, $\mathcal{H}_x^k(M) =$
\[ \cup_{x \in \mathcal{R}_k} \mathcal{H}_x^k(M) \] is a complex vector bundle over \( \mathcal{R}_k \) which is a subbundle of \( f^*T(M')^+|_{\mathcal{R}_k} \).

For an integer \( k \geq 1 \) and a point \( x \in \mathcal{R}_k \), we have

1. \( \nabla' X Y \in \mathcal{H}_x^{k+1}(M) \) for \( X \in T_x(M)^+ \) and local sections \( Y \) of \( \mathcal{H}_x^k(M) \);

2. \( O_x^{k+1}(M) = \{0\} \) if and only if, for each \( X \in T_x(M)^+ \) and each local section \( Y \) of \( \mathcal{H}_x^k(M) \), we have \( \nabla' X Y \in \mathcal{H}_x^k(M) \).

Thus, there is a unique integer \( d > 0 \) such that \( O_x^d(M) \neq \{0\} \) for some \( x \in \mathcal{R}_d \) and \( O_x^{d+1}(M) = \{0\} \) for each \( x \in \mathcal{R}_d \). The integer \( d \) is called the degree of the Kählerian immersion \( f : M \to \tilde{M} \).

5. CR-products and Segre embedding.

A submanifold \( N \) in a Kählerian manifold \( \tilde{M} \) is called a totally real submanifold [30] if the complex structure \( J \) of \( \tilde{M} \) carries each tangent space of \( N \) into its corresponding normal space, that is, \( JT_x N \subset T_x^\perp N, x \in N \). An \( n \)-dimensional totally real submanifold in a Kählerian manifold \( \tilde{M}^n \) with complex dimension \( n \) is called a Lagrangian submanifold. (For latest surveys on Lagrangian submanifolds form differential geometric point of view, see [14, 17]).

A submanifold \( N \) in a Kählerian manifold \( \tilde{M} \) is called a CR-submanifold [2] if there exists on \( N \) a holomorphic distribution \( \mathcal{D} \) whose orthogonal complement \( \mathcal{D}^\perp \) is a totally real distribution, that is, \( JD^\perp_x \subset T^\perp_x N \).

The notion of CR-products was introduced in [6] as follows: A CR-submanifold \( N \) of a Kählerian manifold \( \tilde{M} \) is called a CR-product if locally it is a Riemannian product of a Kählerian submanifold \( N_T \) and a totally real submanifold \( N_\perp \) of \( \tilde{M} \).

For a CR-submanifold \( N \) in a Kählerian manifold \( \tilde{M} \), we put

\[ JX = PX + FX, \quad X \in TN, \]

where \( PX \) and \( FX \) denote the tangential and the normal components of \( JX \), respectively.
It is proved in [6] that a submanifold $M$ of a Kählerian manifold is a CR-product if and only if $\nabla P = 0$ holds, that is, $P$ is parallel with respect to the Levi-Civita connection of $M$.

**Example 5.1.** Let $\psi_1 : N_T \to CP^{n_1}(4)$ be a Kählerian immersion of a Kählerian manifold $N_T$ into $CP^{n_1}(4)$ and let $\psi_2 : N_\parallel \to CP^{n_2}(4)$ be a totally real immersion of a Riemannian $p$-manifold $N_\parallel$ into $CP^{n_2}(4)$. Then the composition:

$$S_{n_1n_2} \circ (\psi_1, \psi_2) : N_T \times N_\parallel \xrightarrow{(\psi_1, \psi_2) \text{ product immersion}} CP^{n_1}(4) \times CP^{n_2}(4) \xrightarrow{S_{n_1n_2} \text{ Segre embedding}} CP^{n_1+n_2+n_1n_2}(4)$$

is isometric immersed as a CR-product in $CP^{n_1+n_2+n_1n_2}(4)$.

In particular, if $\iota : CP^h(4) \to CP^h(4)$ is the identity map of $CP^h(4)$ and $\varphi : N_\parallel \to CP^p(4)$ is a Lagrangian immersion of a Riemannian $p$-manifold $N_\parallel$ into $CP^p(4)$, then the composition:

$$S_{hp} \circ (\iota, \varphi) : CP^h(4) \times N_\parallel \xrightarrow{(\iota, \varphi) \text{ product immersion}} CP^h(4) \times CP^h(4) \xrightarrow{S_{hp} \text{ Segre embedding}} CP^{h+p+hp}(4)$$

is a CR-product in $CP^{h+p+hp}(4)$ which is called a standard CR-products [6].

For CR-products in complex space forms, the following results are known.

**Theorem 5.1 ([6]).** A CR-submanifold in the complex Euclidean $m$-space $C^m$ is a CR-product if and only if it is a direct sum of a Kählerian submanifold and a totally real submanifold of linear complex subspaces.

**Theorem 5.2 ([6]).** There do not exist CR-products in complex hyperbolic spaces other than Kählerian submanifolds and totally real submanifolds.

CR-products $N_T^h \times N_\perp^p$ in $CP^{h+p+hp}(4)$ are obtained from the Segre embedding as given in Example 5.1. More precisely, we have the following.

**Theorem 5.3 ([6]).** Let $N_T^h \times N_\perp^p$ a CR-product in $CP^m(4)$ with $\dim C N_T = h$ and $\dim R N_\perp = p$. Then we have:

$$m \geq h + p + hp$$
The equality sign of (5.3) holds if and only if the following statements hold:

(a) $N^h_T$ is an open portion of $CP^h(4)$.

(b) $N^p_\perp$ is a totally real submanifold.

(c) The immersion is the following composition:

$$N^h_T \times N^p_\perp \to CP^h(4) \times CP^p(4) \xrightarrow{S_{h,p}} CP^{h+p+h_p}(4).$$

**Theorem 5.4** ([10]). Let $N^h_T \times N^p_\perp$ be a CR-product in $CP^m(4)$. Then the squared norm of the second fundamental form satisfies

$$||\sigma||^2 \geq 4h_p.$$  

The equality sign of (5.4) holds if and only if the following statements hold:

(a) $N^h_T$ is an open portion of $CP^h(4)$.

(b) $N^p_\perp$ is a totally geodesic totally real submanifold.

(c) The immersion is the following composition:

$$N^h_T \times N^p_\perp \xrightarrow{\text{totally geodesic}} CP^h(4) \times CP^p(4) \xrightarrow{S_{h,p}} CP^{h+p+h_p}(4).$$

6. **CR-warped products and partial Segre CR-immersions.**

Let $B$ and $F$ be two Riemannian manifolds of positive dimensions equipped with Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive function on $B$. Consider the product manifold $B \times F$ with its natural projections $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$||X||^2 = ||\pi_*(X)||^2 + f^2(\pi(x))||\eta_*(X)||^2$$

for any tangent vector $X \in T_xM$. Thus, we have $g = g_B + f^2 g_F$. The function $f$ is called the warping function of the warped product (cf. [44]).
It was proved in [16, 1] that there does not exist a CR-submanifold in a Kählerian manifold which is locally the warped product \( N_\perp \times f N_T \) of a totally real submanifold \( N_\perp \) and a holomorphic submanifold \( N_T \). It was also proved in [16, 1] that there do exist many CR-submanifold in complex space forms which are the warped product \( N_T \times f N_\perp \) of holomorphic submanifolds \( N_T \) and totally real submanifolds \( N_\perp \) with non-constant warping functions \( f \).

A CR-submanifold of a Kähler manifold \( \tilde{M} \) is called in [16] a CR-warped product if it is the warped product \( N_T \times f N_\perp \) of a holomorphic submanifold \( N_T \) and a totally real submanifold \( N_\perp \), where \( f \) denotes the warping function.

A CR-warped product is called a non-trivial CR-warped product if its warping function is non-constant.

**Example 6.1.** Let \( \mathbb{C}^m \) be the complex Euclidean \( m \)-space with a natural Euclidean complex coordinate system \( \{z_1, \ldots, z_m\} \). We put \( \mathbb{C}^m_* = \mathbb{C}^m - \{0\} \).

Let \( (w_0, \ldots, w_q) \) denote a Euclidean coordinate system on the Euclidean \( (q+1) \)-space \( \mathbb{E}^{q+1} \).

Suppose \( z : N_T \to \mathbb{C}^m_* \subset \mathbb{C}^m \) is a Kählerian immersion of a Kählerian manifold of complex dimension \( h \) into \( \mathbb{C}^m \) and \( w : N_\perp \to S^q(1) \subset \mathbb{E}^{q+1} \) is an isometric immersion of a Riemannian \( p \)-manifold into the unit hypersphere \( S^q(1) \) of \( \mathbb{E}^{q+1} \) centered at the origin.

For each natural number \( \alpha \leq h \), we define a map:

\[
C^\alpha_{hp} : N_T \times N_\perp \to \mathbb{C}^m \times S^q(1) \to \mathbb{C}^{m+\alpha q}
\]

by

\[
C^\alpha_{hp}(u, v) = (w_0(v)z_1(u), w_1(v)z_1(u), \ldots, w_q(v)z_1(u), \ldots, w_0(v)z_\alpha(u), w_1(v)z_\alpha(u), \ldots, w_q(v)z_\alpha(u), z_{\alpha+1}(u), \ldots, z_m(u))
\]

for \( u \in N_T \) and \( v \in N_\perp \). Then (6.2) induces an isometric immersion:

\[
\hat{C}^\alpha_{hp} : N_T \times f N_\perp \to \mathbb{C}^{m+\alpha q}
\]

from the warped product \( N_T \times f N_\perp \) with warping function \( f = \sqrt{\sum_{j=1}^\alpha |z_j(u)|^2} \) into \( \mathbb{C}^{m+\alpha q} \) as a CR-warped product.
We put
\[ \mathbb{C}^h_\alpha = \left\{ (z_1, \ldots, z_h) \in \mathbb{C}^h : \sum_{j=1}^{\alpha} |z_j|^2 \neq 0 \right\}. \]

When \( z : N_T = \mathbb{C}^h \hookrightarrow \mathbb{C}^h \) and \( w : S^p(1) \hookrightarrow \mathbb{E}^{p+1} \) are the inclusion maps, the map \( (6.4) \) induces a map:
\[ S^\alpha_{hp} : \mathbb{C}^h_\alpha \times_f S^p(1) \rightarrow \mathbb{C}^{h+\alpha p} \]
defined by
\[ S^\alpha_{hp}(z, w) = \left( w_0 z_1, w_1 z_1, \ldots, w_p z_1, \ldots, \\
  w_0 z_\alpha, w_1 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h \right) \]
for \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h \) and \( w = (w_0, \ldots, w_p) \in S^p(1) \) with \( \sum_{j=0}^{p} w_j^2 = 1 \).

The warping function of \( \mathbb{C}^h_\alpha \times_f S^p(1) \) is given by
\[ f = \left\{ \sum_{j=1}^{\alpha} |z_j|^2 \right\}^{1/2}. \]

The map \( S^\alpha_{hp} \) is an isometric CR-immersion which is a CR-warped product in \( \mathbb{C}^{h+\alpha p} \). We simply call such a CR-warped product in \( \mathbb{C}^{h+\alpha p} \) a standard partial Segre CR-product.

The standard partial Segre CR-immersion \( S^1_{hp} \) is characterized by the following theorem (see [16, I]).

**Theorem 6.1.** Let \( \phi : T \times_f N_\perp \rightarrow \mathbb{C}^m \) be a non-trivial CR-warped product in the complex Euclidean m-space \( \mathbb{C}^m \) with \( \dim_{\mathbb{C}} N_T = h \) and \( \dim_{\mathbb{R}} N_\perp = p \).
Then we have:

(a) The squared norm of the second fundamental form satisfies the inequality:
\[ ||\sigma||^2 \geq 2p ||\nabla (\ln f)||^2. \]

(b) The CR-warped product satisfies the equality \( ||\sigma||^2 = 2p ||\nabla (\ln f)||^2 \) if and only if the following statements holds:
(b.1) \( N_T \) is an open portion of \( \mathbb{C}^h_1 \).
(b.2) \( N_\perp \) is an open portion of the unit p-sphere \( S^p(1) \).
The warping function is given by $f = |z_1|$. 

Up to rigid motions of $\mathbb{C}^m$, $\phi$ is the standard partial Segre CR-immersion $S_{hp}^1$. More precisely, we have

\begin{equation}
\phi(z, w) = \left( S_{hp}^1(z, w), 0, \ldots, 0 \right) = \left( z_1w_0, z_1w_1, \ldots, z_1w_p, z_2, \ldots, z_h, 0, \ldots, 0 \right),
\end{equation}

for

\begin{equation}
z = (z_1, \ldots, z_h) \in \mathbb{C}_1^h, \quad w = (w_0, \ldots, w_p) \in S^p(1) \subset \mathbb{E}^{p+1}.
\end{equation}

When $\alpha$ is greater than one, the standard partial Segre CR-immersion $S_{hp}^\alpha$ is characterized by the following.

**Theorem 6.2.** Let $\phi : N_T \times_f N_\perp \to \mathbb{C}^m$ be a CR-warped product in complex Euclidean $m$-space $\mathbb{C}^m$. Then we have

1. The squared norm of the second fundamental form of $\phi$ satisfies

\begin{equation}
||\sigma||^2 \geq 2p\left\{||\nabla (\ln f)||^2 + \Delta (\ln f)\right\}.
\end{equation}

2. If the CR-warped product satisfies the equality case of (6.10), then we have

   2.i) $N_T$ is an open portion of $\mathbb{C}_a^h$.
   2.ii) $N_\perp$ is an open portion of $S^p(1)$.
   2.iii) There exists a natural number $\alpha \leq h$ and a complex coordinate system $\{z_1, \ldots, z_h\}$ on $\mathbb{C}^h$ such that the warping function $f$ is given by

   \[ f = \left( \sum_{j=1}^{\alpha} z_j \bar{z}_j \right)^{1/2}. \]

   2.iv) Up to rigid motions of $\mathbb{C}^m$, $\phi$ is the standard partial Segre CR-immersion $S_{hp}^\alpha$; namely, we have

\begin{equation}
\phi(z, w) = \left( S_{hp}^\alpha(z, w), 0, \ldots, 0 \right) = \left( w_0z_1, \ldots, w_pz_1, \ldots, w_0z_\alpha, \ldots, w_pz_\alpha, z_{a+1}, \ldots, z_h, 0, \ldots, 0 \right)
\end{equation}

for $z = (z_1, \ldots, z_h) \in \mathbb{C}_a^h$ and $w = (w_0, \ldots, w_p) \in S^p(1) \subset \mathbb{E}^{p+1}$. 


7. Real hypersurfaces as partial Segre embeddings.

A contact manifold is an odd-dimensional manifold $M^{2n+1}$ equipped with a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a Legendre curve if $\eta(\beta'(t)) = 0$ along $\beta$.

We put

$$S^{2n+1}(c) = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \langle z, z \rangle = \frac{1}{c} > 0 \right\}.$$  

Let $\xi$ be a unit normal vector of $S^{2n+1}(c)$ in $\mathbb{C}^{n+1}$. Then $S^{2n+1}(c)$ is a contact manifold endowed with a canonical contact structure given by the dual 1-form of $J\xi$, where $J$ is the complex structure on $\mathbb{C}^{n+1}$.

Legendre curves are known to play an important role in the study of contact manifolds. For instance, a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

There is a simple relationship between Legendre curves and a second order differential equation obtained in [12].

**Lemma 7.1.** Let $c$ be a positive number and $z = (z_1, z_2) : I \to S^3(c) \subset \mathbb{C}^2$ be a unit speed curve, where $I$ is either an open interval or a circle. If $z$ satisfies the following differential equation:

$$z''(t) - i\lambda \gamma(t)z'(t) + cz(t) = 0$$  

for some nonzero real-valued function $\lambda$ on $I$, then $z = z(t)$ is a Legendre curve in $S^3(c)$.

Conversely, if $z = z(t)$ is a Legendre curve in $S^3(c) \subset \mathbb{C}^2$, then it satisfies the differential equation (7.1) for some real-valued function $\lambda$.

For real hypersurfaces in complex Euclidean spaces, we have the following classification theorem.
Theorem 7.1. \([18]\) Let \(a\) be a positive number and \(\gamma(t) = (\Gamma_1(t), \Gamma_2(t))\) be a unit speed Legendre curve \(\gamma : I \rightarrow S^3(a^2) \subset \mathbb{C}^2\) defined on an open interval \(I\). Then the partial Segre immersion:

\[
x(z_1, \ldots, z_n, t) = (a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \ldots, z_n), \quad z_1 \neq 0
\]

defines a CR-warped product real hypersurface, \(C^*_n \times_{a|z_1|} I\), in \(\mathbb{C}^{n+1}\), where

\[
C^*_1 = \{(z_1, \ldots, z_n) : z_1 \neq 0\}.
\]

Conversely, up to rigid motions, every real hypersurface, which is the warped product \(N \times_{f} I\) of a complex hypersurface \(N\) and an open interval \(I\), in \(\mathbb{C}^{n+1}\) is either a partial Segre immersion defined by (7.2) or a product real hypersurface: \(\mathbb{C}^n \times C \subset \mathbb{C}^n \times C^1\) of \(\mathbb{C}^n\), where \(C\) is a real curve in \(\mathbb{C}\).

The study of real hypersurfaces in non-flat complex space forms has been an active field over the past three decades. Although these ambient spaces might be regarded as the simplest after the spaces of constant curvature, they impose significant restrictions on the geometry of their real hypersurfaces. For instance, they do not admit totally umbilical hypersurfaces and Einstein hypersurfaces in non-flat complex space forms.

Recently, B. Y. Chen and S. Maeda prove in \([29]\) the following general result for real hypersurfaces in non-flat complex space forms.

Theorem 7.2. \([29]\) Every real hypersurface in a complex projective space (or in a complex hyperbolic space) is locally an irreducible Riemannian manifold.

In other words, there do not exist real hypersurfaces in non-flat complex space forms which are the Riemannian products of two or more Riemannian manifolds of positive dimension.

On contrast, there do exist many real hypersurfaces in non-flat complex space forms which are warped products. For real hypersurfaces in complex projective spaces, we have the following classification theorem.
Theorem 7.3. ([13]) Suppose that \( a \) is a positive number and \( \gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \) is a unit speed Legendre curve \( \gamma : I \to S^3(a^2) \subset \mathbb{C}^2 \) defined on an open interval \( I \). Let \( x : S^{2n+1}_* \times I \to C^{n+2} \) be the map defined by the partial Segre immersion:

\[
x(z_0, \ldots, z_n, t) = (a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n), \quad \sum_{k=0}^n z_k\bar{z}_k = 1.
\]

Then

1. \( x \) induces an isometric immersion \( \psi : S^{2n+1}_* \times_{a|z_0|} I \to S^{2n+3} \).
2. The image \( \psi(S^{2n+1}_* \times_{a|z_0|} I) \) in \( S^{2n+3} \) is invariant under the action of \( U(1) \).
3. The projection \( \psi_\pi : \pi(S^{2n+1}_* \times_{a|z_0|} I) \to CP^{n+1}(4) \) of \( \psi \) via \( \pi \) is a warped product hypersurface \( CP^n_0 \times_{a|z_0|} I \) in \( CP^{n+1}(4) \).

Conversely, if a real hypersurface in \( CP^{n+1}(4) \) is a warped product \( N \times f \) \( I \) of a complex hypersurface \( N \) of \( CP^{n+1}(4) \) and an open interval \( I \), then, up to rigid motions, it is locally obtained in the way described above via a partial Segre immersion.

8. Complex extensors, Lagrangian submanifolds and Segre embedding.

When \( \alpha = h = 1 \), the partial Segre CR-immersion

\[
S^1_{lp} : \mathbb{C}^* \times S^p(1) \to C^{p+1}
\]

defined in Section 6 is given by

\[
S^1_{lp}(z, w) = (zw_0, zw_1, \ldots, zw_p)
\]

for \( z \in \mathbb{C}^* = \mathbb{C} - \{0\} \) and \( w = (w_0, \ldots, w_p) \in S^p(1) \subset E^{p+1} \).

In this section, we discuss the notion of complex extensors introduced in [11] which are constructed in a way similar to (8.2).

Complex extensors are defined in [11] as follows:
Let \( z = z(s) : I \rightarrow \mathbb{C}^* \subset \mathbb{C} \) be a unit speed curve in the punctured complex plane \( \mathbb{C}^* \) defined on an open interval \( I \). Suppose that

\[
x = (x_1, \ldots, x_m) : M^{n-1} \rightarrow S_0^{m-1}(1) \subset \mathbb{E}^m
\]

\[
u \mapsto (x_1(\nu), \ldots, x_m(\nu))
\]

is an isometric immersion of a Riemannian \((n - 1)\)-manifold \( M^{n-1} \) into \( \mathbb{E}^n \) whose image is contained in \( S_0^{m-1}(1) \).

The complex extensor of \( x : M^{n-1} \rightarrow \mathbb{E}^m \) via the unit speed curve \( z : I \rightarrow \mathbb{C} \) is defined to be the map:

\[
\tau : I \times M^{n-1} \rightarrow \mathbb{C}^m
\]

\[
(s, u) \mapsto (z(s)x_1(u), \ldots, z(s)x_m(u))
\]

for \( s \in I \) and \( u \in M^{n-1} \). It was proved in [11] that \( I \times M^{n-1} \) is isometrically immersed by \( \tau \) as a totally real submanifold in \( \mathbb{C}^m \).

The complex tensor of the unit hypersphere \( S^{n-1} \hookrightarrow \mathbb{E}^n \) via a unit speed curve in \( \mathbb{C} \) is a \( SO(n) \)-invariant Lagrangian submanifold in \( \mathbb{C}^n \). In this way, we can construct many \( S(n) \)-invariant Lagrangian submanifolds in \( \mathbb{C}^n \).

Now, we recall the definition of Lagrangian \( H \)-umbilical submanifolds introduced in [11] [12].

**Definition 8.1.** A Lagrangian \( H \)-umbilical submanifold of a Kählerian manifold is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following form:

\[
\sigma(e_1, e_1) = \lambda Je_1, \quad \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu Je_1,
\]

\[
\sigma(e_1, e_j) = \mu Je_j, \quad \sigma(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \ldots, n
\]

for some suitable functions \( \lambda \) and \( \mu \) with respect to some suitable orthonormal local frame field \( \{e_1, \ldots, e_n\} \).

The condition (8.5) is equivalent to the single condition:

\[
\sigma(X,Y) = \alpha \left( \langle JX, \vec{H} \rangle \langle JY, \vec{H} \rangle \vec{H} \right) + \beta \left( \langle \vec{H}, \vec{H} \rangle \{\langle X,Y \rangle \vec{H} + \langle JX, \vec{H} \rangle JY + \langle JY, \vec{H} \rangle JX \right)
\]
for vectors $X, Y$ tangent to $M$, where
$$\alpha = \frac{\lambda - 3\mu}{\gamma^3}, \quad \beta = \frac{\mu}{\gamma^3}, \quad \gamma = \frac{\lambda + (n - 1)\mu}{n}$$
when $\vec{H} \neq 0$.

It is easy to see that non-minimal Lagrangian $H$-umbilical submanifold satisfies the following two conditions:

(a) $J\vec{H}$ is an eigenvector of the shape operator $A\vec{H}$.

(b) The restriction of $A\vec{H}$ to $(J\vec{H})^\perp$ is proportional to the identity map.

On the other hand, since the second fundamental form of every Lagrangian submanifold satisfies (see [30])
$$\langle \sigma(X, Y), JZ \rangle = \langle \sigma(Y, Z), JX \rangle = \langle \sigma(Z, X), JY \rangle$$
for vectors $X, Y, Z$ tangent to $M$, we know that Lagrangian $H$-umbilical submanifolds are indeed the simplest Lagrangian submanifolds which satisfy both Conditions (a) and (b). Hence, we can regard Lagrangian $H$-umbilical submanifolds as the simplest Lagrangian submanifolds, next to the totally geodesic ones.

**Example 8.1. (Whitney’s sphere).** Let $w : S^n \to \mathbb{C}^n$ be the map defined by
$$w(y_0, y_1, \ldots, y_n) = \frac{1 + iy_0}{1 + y_0^2}(y_1, \ldots, y_n), \quad y_0^2 + y_1^2 + \ldots + y_n^2 = 1.$$  
Then $w$ is a (non-isometric) Lagrangian immersion of the unit $n$-sphere into $\mathbb{C}^n$ which is called the Whitney $n$-sphere.

The Whitney $n$-sphere is a complex extensor of the inclusion $\iota : S^{n-1} \to \mathbb{E}^n$ via the unit speed curve $z$ which is an arclength reparametrization of the curve $\varphi : I \to \mathbb{C}$ given by
$$f(\varphi) = \frac{\sin \varphi + i \sin \varphi \cos \varphi}{1 + \cos^2 \varphi}.$$  

Whitney’s $n$-sphere is a Lagrangian $H$-umbilical submanifold satisfies (8.5) with $\lambda = 3\mu$. In fact, up to dilations, Whitney’s $n$-sphere is the only Lagrangian $H$-umbilical submanifold in $\mathbb{C}^n$ satisfying $\lambda = 3\mu$ (see [3, 11, 46]).
Example 8.2. (Lagrangian pseudo-spheres). For a given real number $b > 0$, let $z : \mathbb{R} \to \mathbb{C}$ be the unit speed curve given by
\[
    z(s) = \frac{e^{2bsi} + 1}{2bi}.
\]
With respect to the induced metric, the complex extensor of $\iota : S^{n-1} \to \mathbb{E}^n$ via this unit speed curve is a Lagrangian isometric immersion of an open portion of $S^n(b^2)$ into $\mathbb{C}^n$. This Lagrangian submanifold is known as a Lagrangian pseudo-sphere [11].

A Lagrangian pseudo-sphere is a Lagrangian $H$-umbilical submanifold satisfying (8.5) with $\lambda = 2\mu$ (see [11]).

Lagrangian pseudo-sphere is characterized by the following.

Theorem 8.1. [11] Let $L : M \to \mathbb{C}^n$ be a Lagrangian isometric immersion. Then, up to rigid motions of $\mathbb{C}^n$, $L$ is a Lagrangian pseudo-sphere if and only if $L$ is a Lagrangian $H$-umbilical immersion satisfying
\[
    \sigma(e_1, e_1) = 2bJe_1, \quad \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = bJe_1,
\]
\[
    \sigma(e_1, e_j) = bJe_j, \quad \sigma(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \ldots, n,
\]
for some nontrivial function $b$ with respect to some suitable orthonormal local frame field.

Moreover, in this case, $b$ is a nonzero constant.

The following theorem classifies Lagrangian $H$-umbilical submanifold in $\mathbb{C}^n$ with $n \geq 3$.

Theorem 8.2. [11] Let $n \geq 3$ and $L : M \to \mathbb{C}^n$ be a Lagrangian $H$-umbilical isometric immersion. Then we have:

(1) If $M$ is of constant sectional curvature, then either $M$ is flat or, up to rigid motions of $\mathbb{C}^n$, $L$ is a Lagrangian pseudo-sphere.
(2) If $M$ contains no open subset of constant sectional curvature, then, up to rigid motions of $\mathbb{C}^n$, $L$ is a complex extensor of the unit hypersphere of $\mathbb{E}^n$ via a unit speed curve in $\mathbb{C}^n$. \

Remark 8.1. Flat Lagrangian $H$-umbilical submanifolds in $\mathbb{C}^n$ are not necessary complex extensors (see [11]). For the explicit representation formula of flat Lagrangian $H$-umbilical submanifolds in $\mathbb{C}^n$, see [13].

Remark 8.2. Complex extensors in an indefinite complex Euclidean space $\mathbb{C}^n_s$ are introduced and are investigated in [26]. For the relationship between complex extensors and Lagrangian submanifolds in indefinite complex Euclidean spaces and their applications, see [26].

9. Partial Segre $CR$-immersions in complex projective space.

Let $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mathbb{C}^{m+1}_s = \mathbb{C}^{m+1} - \{0\}$. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{m+1}_s$ defined by

$$\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m)$$

for $\lambda \in \mathbb{C}^*$, where $\{z_0, \ldots, z_h\}$ is a natural complex Euclidean coordinate system on $\mathbb{C}^{m+1}_s$. Let $\pi(z)$ denote the equivalent class contains $z$. Then we have a projection: $\pi : \mathbb{C}^{m+1}_s \to \mathbb{C}^{m+1}_s / \sim$. It is known that the set of equivalent classes under $\pi$ is the complex projective $m$-space $CP^m(4)$. The coordinate system $\{z_0, \ldots, z_m\}$ are the homogeneous coordinate system on $CP^m(4)$. Thus, we have the projection:

$$\pi : \mathbb{C}^{m+1}_s \to CP^m(4).$$

For each integer $\alpha$ with $0 \leq \alpha \leq h$. We put

$$\mathbb{C}^{h+1}_\alpha = \left\{ (z_0, \ldots, z_h) \in \mathbb{C}^{h+1} : \sum_{j=0}^{\alpha} |z_j|^2 \neq 0 \right\}.$$

Consider the map:

\begin{equation}
S^\alpha_{hp} : \mathbb{C}^{h+1}_\alpha \times S^p(1) \to \mathbb{C}^{h+p+\alpha p+1}_s
\end{equation}

defined by

\begin{equation}
S^\alpha_{hp}(z, w) = (w_0 z_0, w_1 z_1, \ldots, w_p z_0, \ldots, w_0 z_\alpha, w_1 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h)
\end{equation}

for $z = (z_1, \ldots, z_h) \in \mathbb{C}^{h}_\alpha$ and $w = (w_0, \ldots, w_p) \in S^p(1)$. 

Since the image of $S^\alpha_{hp}$ is invariant under the action of $C^*$, the composition:

$$
\begin{align*}
\pi \circ S^\alpha_{hp} &: C^{h+1}_\alpha \times S^p(1) \xrightarrow{\text{Segre embedding}} C^{h+p+\alpha p+1}_s \\
&\xrightarrow{\text{projection}} CP^{h+p+\alpha p}(4)
\end{align*}
$$

induces an isometric CR-immersion:

$$
S^\alpha_{hp} : CP^h_\alpha \times f S^p(1) \to CP^{h+p+\alpha p}(4)
$$

of the product manifold $CP^h_\alpha \times S^p(1)$ into $CP^{h+p+\alpha p}(4)$, where $CP^h_\alpha$ is the open subset of $CP^h(4)$ defined by

$$
CP^h_\alpha = \left\{ (z_0, \ldots, z_h) \in CP^h(4) : \sum_{j=0}^\alpha |z_j|^2 \neq 0 \right\}.
$$

The metric on $CP^h_\alpha \times S^p(1)$ induced via (9.3) is a warped product metric with warping function, say $f$. Clearly, $CP^h_\alpha$ is a non-compact manifold.

We simply called such a CR-warped product immersion $S^\alpha_{hp}$ in $CP^{h+p+\alpha p}(4)$ a standard partial Segre CR-immersion in $CP^{h+p+\alpha p}(4)$.

The standard partial Segre CR-immersion $\bar{S}^\alpha_{hp}$ is characterized by the following theorem.

**Theorem 9.1.** (\cite{16, II}) Let $\phi : N_T \times f N_\perp \to CP^m(4)$ be a CR-warped product, where $h = \dim_{\mathbb{C}} N_T$ and $p = \dim_{\mathbb{R}} N_\perp$. Then we have:

(a) The squared norm of the second fundamental form satisfies the inequality

$$
||\sigma||^2 \geq 2p||\nabla(\ln f)||^2.
$$

(b) The CR-warped product satisfies the equality case of (9.5) if and only if the following statements hold:

(b.1) $N_T$ is an open portion of complex projective $h$-space $CP^h(4)$.

(b.2) $N_\perp$ is an open portion of a unit $p$-sphere $Sp$.

(b.3) Up to rigid motions, $\phi$ is the composition $\pi \circ S^0_{hp}$, where $S^0_{hp}$ is the standard partial Segre CR-immersion, that is,

$$
\phi(z, w) = (S^0_{hp}(z, w), 0, \ldots, 0) \\
= (z_0w_0, \cdots, z_0w_p, z_1, \ldots, z_h, 0, \ldots, 0),
$$

where $S^0_{hp}$ is the standard partial Segre CR-immersion.
for \( z = (z_0, z_1, \ldots, z_h) \in \mathbb{C}^{h+1} \) and \( w = (w_0, \ldots, w_p) \in \mathbb{E}^{p+1} \), and \( \pi \) is the natural projection \( \pi: \mathbb{C}^{m+1}_c \to \mathbb{C}^m \).

The standard partial Segre \( CR \)-immersions \( \tilde{S}^\alpha_{hp} \) with \( \alpha > 0 \) are characterized by the following theorem (see [24]).

**Theorem 9.2.** Let \( \phi: N_T \times_f N_\perp \to \mathbb{C}^m \) be a \( CR \)-warped product with \( h = \dim_{\mathbb{C}} N_T \) and \( p = \dim_{\mathbb{R}} N_\perp \). Then we have:

1. The squared norm of the second fundamental form of \( \phi \) satisfies the inequality:
   
   \[
   ||\sigma||^2 \geq 2p\left\{ ||\nabla \ln f||^2 + \Delta \ln f \right\} + 4hp.
   
2. The \( CR \)-warped product satisfies the equality case of (9.7) if and only if the following statements hold:
   
   (2.a) \( N_T \) is an open portion of complex projective \( h \)-space \( CP^h(4) \).
   
   (2.b) \( N_\perp \) is an open portion of unit \( p \)-sphere \( SP^p \).
   
   (2.c) There exists a natural number \( \alpha \leq h \) such that, up to rigid motions, \( \phi \) is given by \( \pi \circ S^\alpha_{hp} \), where \( S^\alpha_{hp} \) is the standard partial Segre \( CR \)-immersion, that is,
   
   \[
   \phi(z, w) = S^\alpha_{hp}(z, w) = (w_0z_0, \ldots, w_pz_0, \ldots, w_0z_\alpha, \ldots, w_pz_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0)
   
   \]
   
   for \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1} \) and \( w = (w_0, \ldots, w_p) \in \mathbb{E}^{p+1} \).

It follows from Example 7.1 that there exist many \( CR \)-warped products \( N_T \times_f N_\perp \) with non-constant warping function in \( CP^{h+p} \) with \( h = \dim_{\mathbb{C}} N_T \) and \( p = \dim_{\mathbb{R}} N_\perp \).

On contrast, when \( N_T \) is compact, the following theorem shows that the dimension of the ambient space is at least as the dimension of the Segre embedding.

**Theorem 9.3.** ([25]) Let \( N_T \times_f N_\perp \) with \( h = \dim_{\mathbb{C}} N_T \) and \( p = \dim_{\mathbb{R}} N_\perp \) be a \( CR \)-warped product in the complex projective \( m \)-space \( CP^m(4) \). If \( N_T \) is compact, then we have

\[
(9.9) \quad m \geq h + p + hp.
\]
When the dimension of the ambient space $CP^m$ is $m = h + p + hp$ which is the smallest possible, we have the following.

**Theorem 9.4.** (25) Let $N_T \times_f N_\perp$ with $h = \dim_{\mathbb{C}} N_T$ and $p = \dim_{\mathbb{R}} N_\perp$ be a CR-warped product which is embedded in $CP^{h+p+hp}(4)$. If $N_T$ is compact, then $N_T$ is holomorphically isometric to $CP^h(4)$.

10. **Convolution of Riemannian manifolds.**

The notion of convolution of Riemannian manifolds was introduced in [23, 20]. This notion extends the notion of warped products in a natural way.

**Definition 10.1.** Let $(N_1,g_1)$ and $(N_2,g_2)$ be two Riemannian manifolds and let $f$ and $h$ be two positive differentiable functions on $N_1$ and $N_2$, respectively. Consider the symmetric tensor field $h g_1 \ast_f g_2$ of type $(0,2)$ on $N_1 \times N_2$ defined by

\[ h g_1 \ast_f g_2 = h^2 g_1 + f^2 g_2 + 2f h df \otimes dh. \]  

(10.1)

The symmetric tensor field $h g_1 \ast_f g_2$ is called the **convolution of $g_1$ and $g_2$** via $h$ and $f$. The product manifold $N_1 \times N_2$ together with $h g_1 \ast_f g_2$, denoted by $h N_1 \star_f N_2$, is called a **convolution manifold**.

If $h g_1 \ast_f g_2$ is a positive-definite symmetric tensor, it defines a Riemannian metric on $N_1 \times N_2$. In this case, $h g_1 \ast_f g_2$ is called a **convolution metric** and the convolution manifold $h N_1 \star_f N_2$ is called a **convolution Riemannian manifold**.

When $f, h$ are irrelevant, $h N_1 \star_f N_2$ and $h g_1 \ast_f g_2$ are simply denoted by $N_1 \star N_2$ and $g_1 \ast g_2$, respectively.

The following result shows that the notion of convolution manifolds arises very naturally.

**Theorem 10.1.** (23) Let $x : (N_1, g_1) \to E^n_\ast \subset E^n$ and $y : (N_2, g_2) \to E^m_\ast \subset E^m$ be isometric immersions of Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ into $E^n_\ast$ and $E^m_\ast$, respectively. Then the map

\[ \psi : N_1 \times N_2 \to E^n \otimes E^m = E^{nm}; \]

\[ (u, v) \mapsto x(u) \otimes y(v), \quad u \in N_1, \ v \in N_2, \]

(10.2)
gives rise to a convolution manifold \( N_1 \star N_2 \) equipped with
\[
(10.3) \quad \rho_2 g_1 * \rho_1 g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2 \rho_1 \rho_2 d\rho_1 \otimes d\rho_2,
\]
where
\[
\rho_1 = \left\{ \sum_{j=1}^{n} x_j^2 \right\}^{1/2} \quad \rho_2 = \left\{ \sum_{\alpha=1}^{m} y_\alpha^2 \right\}^{1/2}
\]
denote the distance functions of \( x \) and \( y \), and
\[
x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_m)
\]
are Euclidean coordinate systems of \( E^n \) and \( E^m \), respectively.

**Definition 10.2.** Let \( \psi : (N_1, g_1) \rightarrow (N_2, h g_2) \) be a map from a convolution manifold into a Riemannian manifold. Then the map is said to be *isometric* if \( h g_1 * f g_2 \) is induced from \( \tilde{g} \) via \( \psi \), that is, we have
\[
(10.4) \quad \psi^* \tilde{g} = h g_1 * f g_2.
\]

**Example 10.1.** Let \( x : (N_1, g_1) \rightarrow E^n_n \subset E^n \) be an isometric immersion. If \( y : (N_2, g_2) \rightarrow S^{m-1}(1) \subset E^m \) is an isometric immersion such that \( y(N_2) \) is contained in the unit hypersphere \( S^{m-1}(1) \) centered at the origin. Then the convolution \( g_1 * g_2 \) of \( g_1 \) and \( g_2 \) is nothing but the warped product metric:
\[
g = g_1 + |x|^2 g_2.
\]

**Definition 10.3.** A convolution \( h g_1 * f g_2 \) of two Riemannian metrics \( g_1 \) and \( g_2 \) is said to be *degenerate* if \( \det(h g_1 * f g_2) = 0 \) holds identically.

For \( X \in T(N_1) \) we denote by \( |X|_1 \) the length of \( X \) with respect to metric \( g_1 \) on \( N_1 \). Similarly, we denote by \( |Z|_2 \) for \( Z \in T(N_2) \) with respect to metric \( g_2 \) on \( N_2 \).

**Proposition 10.1.** \((23)\) Let \( h N_1 \star f N_2 \) be the convolution of two Riemannian manifolds \( (N_1, g_1) \) and \( (N_2, g_2) \) via \( h \) and \( f \). Then \( h g_1 * f g_2 \) is degenerate if and only if we have:

1. The length \( |\text{grad } f|_1 \) of the gradient of \( f \) on \( (N_1, g_1) \) is a nonzero constant, say \( c \).

2. The length \( |\text{grad } h|_2 \) of the gradient of \( h \) on \( (N_2, g_2) \) is the constant given by \( c^{-1} \), that is, the reciprocal of \( c \).
The following result provides a criterion for a convolution $h_1 * f_1$ of two Riemannian metrics to be a Riemannian metric.

**Theorem 10.2.** ([23]) Let $h_1 \star f_2$ be the convolution of Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ via $h$ and $f$. Then $h_1 * f_2$ is a Riemannian metric on $h_1 \star f_2$ if and only if we have

\[ |\text{grad } f|_1 \cdot |\text{grad } h|_2 < 1. \]

11. Convolutions and Euclidean Segre maps.

Let $C^n_* = C^n - \{0\}$ and $E^m_* = E^m - \{0\}$. Assume that $(z_1, \ldots, z_n)$ is a complex Euclidean coordinate system of $C^n$ and $(x_1, \ldots, x_m)$ is a Euclidean coordinate system on $E^m$. Suppose that $z : C^h_* \to C^h$ and $x : E^p_* \to E^p$ are the inclusion maps.

Let $\psi$ be the map:

\[
\psi : C^h_* \times E^p_* \to C^h p
\]

defined by

\[
\psi(z, x) = (z_1 x_1, \ldots, z_1 x_p, \ldots, z_h x_1, \ldots, z_h x_p)
\]

for $z = (z_1, \ldots, z_h) \in C^h_*$ and $x = (x_1, \ldots, x_p) \in E^p_*$. The map (11.1) is called a Euclidean Segre map.

If we put $z_j = u_j + iv_j, i = \sqrt{-1}$, and

\[
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial u_j} - i \frac{\partial}{\partial v_j} \right)
\]

for $j = 1, \ldots, h$, then we obtain from (11.2) that

\[
d\psi \left( \sum_{j=1}^{h} z_j \frac{\partial}{\partial z_j} \right) = d\psi \left( \sum_{\alpha=1}^{h} x_\alpha \frac{\partial}{\partial x_\alpha} \right).
\]

Notice that the vector fields $\sum_{j=1}^{h} z_j \partial/\partial z_j$ and $\sum_{\alpha=1}^{h} x_\alpha \partial/\partial x_\alpha$ are nothing but the position vector fields of $C^h_*$ and $E^p_*$ in $C^h$ and $E^p$, respectively.

Equation (11.3) implies that the gradient of $|z| = \sqrt{\sum_{j=1}^{h} z_j \bar{z}_j}$ and the gradient of $|x| = \sqrt{\sum_{\alpha=1}^{p} x_\alpha^2}$ are mapped to the same vector field under $\psi$. 
From (11.2) and (11.3) it follows that $d\psi$ has constant rank $2h + p - 1$. Hence $\psi(C^h \times E^p_\ast)$ gives rise to a $(2h + p - 1)$-manifold, denoted by

\begin{equation}
C^h \otimes E^p_\ast,
\end{equation}

which equips a Riemannian metric induced from the canonical metric on $C^h \otimes E^p_\ast$ via $\psi$.

From (11.2) we can verify that $C^h \otimes E^p_\ast$ is isometric to the warped product $C^h_\ast \times S^{p-1}$ equipped with the warped product metric $g = g_1 + \rho_1 g_0$, where $\rho_1$ is the length of the position function of $C^h_\ast$ and $g_0$ is the metric of the unit hypersphere $S^{p-1}(1)$.

If we denote the vector field in (11.3) by $V$, then $V$ is a tangent vector field of $C^h_\ast \otimes E^p_\ast$ with length $|x| |z|$. The Riemannian metric on $C^h_\ast \otimes E^p_\ast$ is induced from the following convolution:

\begin{equation}
h g_1 * f g_2 = \mu^2 g_1 + \lambda^2 g_2 + 2\lambda \mu d\lambda \otimes d\mu, \quad \lambda = |z|, \quad \mu = |x|.
\end{equation}

**Definition 11.1.** An isometric map:

\begin{equation}
\phi : (C^h_\ast \times E^p_\ast, \rho_\ast g_1 \ast \rho_\ast g_2) \rightarrow (C^m, \tilde{g}_0)
\end{equation}

is called a CR-map if $\phi$ maps each complex slice $C^h_\ast \times \{v\}$ of $C^h_\ast \times E^p_\ast$ into a complex submanifold of $C^m$ and it maps each real slice $\{u\} \times E^p_\ast$ of $C^h_\ast \times E^p_\ast$ into a totally real submanifold of $C^m$.

The following two theorems characterize the Euclidean Segre maps in very simple ways. These two theorems can be regarded as the Euclidean versions of Theorem 3.4 with $s = 2$ and Theorem 3.5.

**Theorem 11.1.** (20) Let $\phi : (C^h_\ast \times E^p_\ast, \rho_\ast g_1 \ast \rho_\ast g_2) \rightarrow C^m$ be an isometric CR-map. Then we have:

1. $m \geq hp$.
2. If $m = hp$, then, up to rigid motions of $C^m$, $\phi$ is the Euclidean Segre map, that is,

\begin{equation}
\phi(z, x) = \psi_{z, x} = (z_1 x_1, \ldots, z_1 x_p, z_2 x_1, \ldots, z_2 x_p, \ldots, z_h x_p).
\end{equation}
Theorem 11.2. [20] Let $\phi : (C^h \times E^p_{\rho_1, \rho_2}, g_1 * g_2) \to C^m$ be an isometric CR-map. Then we have:

\[(11.8) \quad ||\sigma||^2 \geq \frac{(2h - 1)(p - 1)}{|x|^2 |z|^2}.
\]

The equality sign of (11.8) holds identically if and only if, up to rigid motions of $C^m$, $\phi$ is obtained from the Euclidean Segre map, that is, $\phi$ is given by

\[(11.9) \quad \phi(z, x) = (\psi_{z, x}, 0) = (z_1 x_1, \ldots, z_1 x_p, z_2 x_1, \ldots, z_2 x_p, \ldots, z_h x_p, 0, \ldots, 0).
\]

12. **Skew Segre embedding.**

Motivated from the Segre embedding and the Veronese embeddings, S. Maeda and Y. Shimizu define in [39] real analytic but not holomorphic embeddings:

\[(12.1) \quad f^n_\alpha : C P^n \left(\frac{2}{\alpha}\right) \to C P^{(n+\alpha)-1}(4)
\]

defined by

\[(z_0, \ldots, z_n) \mapsto \left(z_0 \alpha_0^\alpha \bar{z}_0, \ldots, \sqrt{\frac{\alpha}{\alpha_0! \cdots \alpha_n!}} z_0^{\alpha_0} \cdots z_n^{\alpha_n} \bar{z}_0^{\beta_0} \cdots \bar{z}_n^{\beta_n}ight)
\]

where $\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i = \alpha$, and $(z_0, \ldots, z_n)$ is a homogeneous coordinate system on $C P^n \left(\frac{2}{\alpha}\right)$.

If $\alpha = 1$, the embedding (12.1) reduces to the *skew-Segre embedding*:

\[(12.2) \quad f^n_1 : C P^n(2) \to C P^{(n+2)}(4);
\]

\[(z_0, \ldots, z_n) \mapsto (z_i \bar{z}_j)_{0 \leq i, j \leq n}.
\]

T. Maebashi and S. Maeda prove in [37] that the squared mean curvature function of $f^n_1$ is constant equal to $n^{-1}$.

Maebashi and Maeda also prove the following.
Theorem 12.1. (37) The skew-Segre embedding \( f^n_1 \) is equal to the following composition:

\[
CP^n(2) \xrightarrow{\text{minimal}} S^{n(n+2)-1} \left( \frac{n+1}{n} \right) \xrightarrow{\text{totally umbilical}} S^{n(n+2)}(1) \xrightarrow{\text{totally geodesic}} CP^{n(n+2)}(4).
\]

The first part, \( CP^n(2) \xrightarrow{\text{minimal}} S^{n(n+2)-1} \left( \frac{n+1}{n} \right) \), of the decomposition for the skew-Segre embedding has already been considered by G. Mannoury (1867–1956) in 1899 (see [40]).

The skew-Segre embedding \( f^n_1 : CP^n(2) \to CP^{n(n+2)}(4) \) is a totally real pseudo-umbilical embedding which has parallel mean curvature vector. Moreover, the squared norm of the second fundamental form of the skew-Segre embedding is constant.

For \( \alpha = 2 \), the embedding (12.1) reduces to

\[
(12.3) \quad f^n_2 : CP^n(1) \to CP^{(n+2)}(4);
\]

\[
(z_0, \ldots, z_n) \mapsto (\cdots, z_i^2 \bar{z}_k, \cdots, z_i^2 \bar{z}_k z_1, \cdots, z_i z_j^2 \bar{z}_k, \cdots, \sqrt{2} z_i z_j z_k \bar{z}_l, \cdots).
\]

S. Maeda and Y. Shimizu gave in [39] an analogous decomposition for this embedding.

Theorem 12.2. (39) The embedding \( f^n_2 : CP^n(1) \to CP^{(n+2)}(4) \) is equal to the following composition:

\[
CP^n(1) \xrightarrow{\text{minimal}} S^{m_1-1}(c_1) \times S^{m_2-1}(c_2) \xrightarrow{\text{Clifford embedding}} S^{m_1+m_2-1}(c) \xrightarrow{\text{totally umbilical}} S^{m_1+m_2}(1) \xrightarrow{\text{totally real}} CP^{(n+2)}(4),
\]

where

\[
m_1 = n(n + 2), \quad m_2 = \frac{1}{4} n(n + 1)^2(n + 4),
\]

\[
c_1 = \frac{(n + 1)(n + 3)}{4n}, \quad c_2 = \frac{(n + 2)(n + 3)}{n(n + 1)},
\]

\[
c = 1 + \frac{2}{n(n + 3)}.
\]
They observe that $f^2_2$ is a pseudo-umbilical totally real embedding and the mean curvature vector of $f^2_2$ is not parallel in the normal bundle although the mean curvature is constant.

13. Tensor product immersions and Segre embedding

The map (1.5) which defines the Segre embedding can be regarded as a tensor product map. Here, we recall the notions of tensor product maps and direct sum maps (see [10, 32] for details).

Let $V$ and $W$ be two vector spaces over the field of real or complex numbers. Denote by $V \otimes W$ and $V \oplus W$ the tensor product and the direct sum of $V$ and $W$, respectively. Let $\langle \ , \rangle_V$ and $\langle \ , \rangle_W$ denote the inner products on $V$ and $W$ respectively. Then $V \otimes W$ and $V \oplus W$ are inner product spaces with the inner products defined respectively by

\begin{align*}
(13.1) & \quad \langle v \otimes w, x \otimes y \rangle = \langle v, x \rangle_V \cdot \langle w, y \rangle_W, \quad v \otimes w, x \otimes y \in V \otimes W, \\
(13.2) & \quad \langle v \oplus w, x \oplus y \rangle = \langle v, x \rangle_V + \langle w, y \rangle_W, \quad v \oplus w, x \oplus y \in V \oplus W
\end{align*}

By applying these algebraic notions, we have the notion of tensor product maps and direct sum maps:

\begin{align*}
(13.3) & \quad f_1 \otimes f_2 : M \to V \otimes W \\
(13.4) & \quad f_1 \oplus f_2 : M \to V \oplus W
\end{align*}

associated with two given maps $f_1 : M \to V$ and $f_2 : M \to W$ of a Riemannian manifold $(M, g)$. These maps are defined by

\begin{align*}
(13.5) & \quad (f_1 \otimes f_2)(u) = f_1(u) \otimes f_2(u) \in V \otimes W, \\
(13.6) & \quad (f_1 \oplus f_2)(u) = f_1(u) \oplus f_2(u) \in V \oplus W, \quad u \in M.
\end{align*}

Similarly, if $f : M \to V$ and $f : N \to W$ are maps from two Riemannian manifolds $M$ and $N$ into $V$ and $W$, respectively. Then we have the box-tensor product map and the box-direct sum map:

\begin{align*}
(13.7) & \quad (f \boxtimes h)(u, v) = f(u) \otimes h(v), \\
(13.8) & \quad (f \boxplus h)(u, v) = f(u) \oplus h(v) \quad u \in M, \quad v \in N.
\end{align*}
The Segre embedding, the partial Segre immersions, complex extensors, Euclidean Segre maps, convolutions, as well as skew-Segre immersions given above can all be expressed in terms of (box) tensor product maps and (box) direct sum maps.

**Example 13.1.** Let \( \iota_1 : C^h \to C^h \) and \( \iota_2 : C^p \to C^p \) be the inclusion maps. Then the map \( S_{hp} \) defined by (1.5) is nothing but the box tensor product \( \iota_1 \boxtimes \iota_2 \) of \( \iota_1 \) and \( \iota_2 \).

**Example 13.2.** Let \( z = (z_1, \ldots, z_m) : N_T \to C^m \subset C^n \) be a Kählerian immersion of a Kählerian \( h \)-manifold into \( C^m \) and
\[
w = (w_0, \ldots, w_p) : N_\perp \to S^q(1) \subset E^{q+1}
\]
be an isometric immersion from a Riemannian \( p \)-manifold into the unit hypersphere \( S^q(1) \). Then the partial Segre CR-immersion \( C^\alpha_{hp} \) defined by (6.3) is nothing but the map:
\[
C^\alpha_{hp} = (z^\alpha \boxtimes w) \boxplus z^\alpha_\perp : N_T \times N_\perp \to C^{m+\alpha q}
(u, v) \mapsto (z^\alpha(u) \odot w(v)) \oplus z^\alpha_\perp(u),
\]
where
\[
z^\alpha = (z_1, \ldots, z_\alpha) : N_T \to C^\alpha \subset C^n
\]
and
\[
z^\alpha_\perp = (z_{\alpha+1}, \ldots, z_m) : N_T \to C^\alpha \subset C^{m-\alpha},
\]

**Example 13.3.** Let \( z = z(s) : I \to C^* \subset C \) be a unit speed curve in the punctured complex plane \( C^* \) defined on an open interval \( I \) and let
\[
x = (x_1, \ldots, x_m) : M^{n-1} \to S^{m-1}_0 \subset E^n; u \mapsto (x_1(u), \ldots, x_m(u))
\]
be an isometric immersion of a Riemannian \( (n - 1) \)-manifold \( M^{n-1} \) into \( E^n \) whose image is contained in the unit hypersphere. Then the complex extensor \( \tau \) of \( x \) via the unit speed curve \( z \) is nothing but the box tensor product \( z \boxtimes x \).

**Example 13.4.** Let \( z = (z_1, \ldots, z_n) : C^m_n \to C^n \) be the inclusion map of \( C^m_n \) and let
\[
\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) : C^m_n \to C^n
\]
be the conjugation of \( z : C^m_n \to C^n \). Then the map \( (12.1) \) which defines the skew-Segre embedding is nothing but the box tensor product \( z \boxtimes \bar{z} \).
Example 13.5. Let $\iota_n : S^n(1) \to E^{n+1}$ be the inclusion map. Then, up to dilations, the tensor product immersion:

$$\iota_n \otimes \iota_n : S^n(1) \to E^{(n+1)^2}$$

is nothing but the first standard immersion of $S^n(1)$ (see [9]).

14. Conclusion.

From the previous sections we know that maps and immersions constructed in ways similar to the Segre embedding provide us many nice examples for various important classes of submanifolds. Moreover, we also see from the previous sections that such examples have many nice properties.

Example 14.1. For the inclusion maps $z : C^h \to C$ and $w : C^p \to C$, the Segre map:

$$S_{h^p} = z \boxtimes w = (z_j w_t)_{0 \leq j \leq h, 0 \leq t \leq p}$$

gives rise to a Kählerian immersion of the product Kählerian manifold $CP^h(4) \times CP^p(4)$ into $CP^{h+p^h+p}(4)$.

Example 14.2. For a given Kählerian immersion $z : N \to C^m \to C^m$, a given Riemannian immersion $w : N_\perp \to S^q(1) \to E^{q+1}$, and a natural number $\alpha \leq h$ with $h = \dim_C N_T$, the partial Segre map:

$$C_{h^p} : N_T \times N_\perp \to C^m \times S^q(1) \to C^{m+\alpha q}$$

defined by

$$C_{h^p} = (z^\alpha \boxtimes w) \boxplus z_\perp^\alpha$$

gives rise to a CR-submanifold. Such construction provide us many nice examples of CR-warped products in complex Euclidean spaces.

Example 14.3. Let $z : C^h \to C$ and $w : S^p(1) \to E^{p+1}$ be the inclusion maps. Then the partial Segre map:

$$C_{h^p} : N_T \times N_\perp \to C^m \times S^q(1) \to C^{m+\alpha q}$$

gives rise to the standard partial Segre CR-immersion in a complex projective space. Such standard partial Segre CR-immersions satisfy the equality case of the general inequality:

$$||\sigma||^2 \geq 2p ||\nabla (\ln f) ||^2.$$
Example 14.4. For a unit speed curve $z = z(s) : I \to \mathbb{C}^n \hookrightarrow \mathbb{C}$ and a spherical isometric immersion $x : M^{n-1} \to S^{m-1}_0 \hookrightarrow \mathbb{E}^m$, the complex extensor $\tau$ of $x$ via $z$ is nothing but the box tensor product $z \boxtimes x$. Such box tensor product immersions provide us many nice examples of Lagrangian submanifolds in complex Euclidean spaces.

Example 14.5. Let $a$ be a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ be a unit speed Legendre curve $\gamma : I \to S^3(a^2) \hookrightarrow \mathbb{C}^2$ defined on an open interval $I$. Then the partial Segre immersion defined by

$$ (a\gamma \boxtimes z^1) \boxplus z_1^n : C^1 \times M \to C^{n+1}, $$

(14.6)

defines a warped product real hypersurface in $C^{n+1}$.

Conversely, up to rigid motions, every real hypersurface in $C^{n+1}$ which is the warped product $N \times I$ of a complex hypersurface $N$ and an open interval $I$ is either the partial Segre immersion given by (14.6) or the product real hypersurface: $C^n \times C$ in $C^{n+1}$ over a curve $C$ in the complex plane.

Example 14.6. Let

$$ x : (N_1, g_1) \to \mathbb{E}^n_\ast \hookrightarrow \mathbb{E}^n, \quad y : (N_2, g_2) \to \mathbb{E}^m_\ast \hookrightarrow \mathbb{E}^m $$

be two isometric immersions of Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ into $E^n_\ast$ and $E^m_\ast$, respectively. Then the Euclidean Segre map:

$$ (u, v) \mapsto x(u) \otimes y(v), \quad u \in N_1, \ v \in N_2, $$

(14.7)

gives rise to the convolution manifold $N_1 \boxtimes N_2$ equipped with the convolution:

$$ \rho_2 g_1 \ast\rho_1 g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2 \rho_1 \rho_2 d\rho_1 \otimes d\rho_2, $$

(14.8)

where $\rho_1 = |x|$ and $\rho_2 = |y|$ are the distance functions of $x$ and $y$, respectively.

Example 14.7. Let $z : C^n_\ast \hookrightarrow \mathbb{C}^n$ be the inclusion map of $C^n_\ast$ and let $\bar{z}$ be the conjugation of $z : C^n_\ast \to \mathbb{C}^n$. Then the skew-Segre map $z \boxtimes \bar{z}$ gives rise to a totally real isometric immersion of $C \mathbb{P}^n(2)$ in $C \mathbb{P}^{n+2}(4)$.

Example 14.8. Let $\psi : M \to C \mathbb{P}^m(2)$ be a Kählerian immersion from a Kählerian manifold $M$ into $C \mathbb{P}^m(2)$ and let $\bar{\psi}$ be the conjugation of $\psi : M \to C \mathbb{P}^m(2)$. Then the map:

$$ \psi \boxtimes \bar{\psi} : M \to C \mathbb{P}^{m+n+2}(4), $$

(14.9)
defined by

\[(\psi \boxtimes \bar{\psi})(u) = (z_i(u)\bar{z}_j(u))_{0 \leq i,j \leq m}\]  

is a totally real immersion from $M$ into $CP^{m(m+2)}$.

Such box tensor product immersions $\psi \boxtimes \bar{\psi}$ provides us a way to construct many examples of totally real submanifolds in complex projective spaces.

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