AN INDEX THEOREM FOR ANTI-SELF-DUAL ORBIFOLD-CONE METRICS

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Abstract. Recently, Atiyah and LeBrun proved versions of the Gauss-Bonnet and Hirzebruch signature theorems for metrics with edge-cone singularities in dimension four, which they applied to obtain an inequality of Hitchin-Thorpe type for Einstein edge-cone metrics. Interestingly, many natural examples of edge-cone metrics in dimension four are anti-self-dual (or self-dual depending upon choice of orientation). On such a space there is an important elliptic complex called the anti-self-dual deformation complex, whose index gives crucial information about the local structure of the moduli space of anti-self-dual metrics. In this paper, we compute the index of this complex in the orbifold case, and give several applications.

1. Introduction

We will be concerned with metrics with the following type of singularities.

Definition 1.1. Let $M$ be a smooth four-manifold with a smoothly embedded two-dimensional submanifold $\Sigma \subset M$. We will say that $g$ is an orbifold-cone metric on $(M, \Sigma)$ with cone angle $2\pi/p$, where $p \geq 1$ is an integer, if $g$ is a smooth metric on $M \setminus \Sigma$ and, near any point of $\Sigma$, the metric is locally the quotient of a smooth $\Gamma$-invariant metric on $\mathbb{R}^4$ around the origin for a cyclic group $\Gamma \subset U(2)$ with generator given by

$$(z_1, z_2) \mapsto (z_1, e^{i2\pi/p}z_2).$$

We will refer to $\Sigma$ as the singular set.

Around any point $q \in \Sigma$ there exists a neighborhood $U_q = \tilde{U}_q/\Gamma$, where $\tilde{U}_q$ is a neighborhood of the origin in $\mathbb{R}^4$. We can choose coordinates $(x_1, x_2, y_1, y_2)$ on $\tilde{U}_q$ so that $\Sigma$ is given by $y_1 = y_2 = 0$. Then, after changing coordinates to $(x_1, x_2, r, \theta)$ by setting $y_1 = r \cos(\theta)$ and $y_2 = r \sin(\theta)$, the metric on $U_q \cap (M \setminus \Sigma)$ can be expressed as

$$g = dr^2 + \left(\frac{r^2}{p^2}\right)d\theta^2 + f_{ij}(x)dx^i \otimes dx^j + r^2 h,$$

where the $f_{ij}(x)$ are smooth functions on $\Sigma$ that are symmetric in $i$ and $j$, and $h$ is a smooth symmetric two-tensor field.

Viewing orbifold-cone metrics in this way, it is clear that they form a subclass of edge-cone metrics, which are a generalization of Definition 1.1 allowing arbitrary cone
angle 2\pi \beta for any \beta \in \mathbb{R}, see [AL12] for the full definition. Edge-cone metrics have recently been of great interest in Kähler geometry, see for example [Bre11, Don11, JMR11]. For an edge-cone metric \(g\), define

\[
\chi_{\text{orb}}(M) = \frac{1}{8\pi^2} \int_M \left( |W|^2 - \frac{1}{2} |E|^2 + \frac{1}{24} R^2 \right) dV_g
\]

(1.2)

\[
\tau_{\text{orb}}(M) = \frac{1}{12\pi^2} \int_M \left( |W^+|^2 - |W^-|^2 \right) dV_g,
\]

(1.3)

where \(W\) is the Weyl tensor, \(E\) is the traceless Ricci tensor, \(R\) is the scalar curvature and \(W^\pm\) are the self-dual and anti-self-dual parts of the Weyl tensor defined below.

We let \([\Sigma]^2\) denote the self-intersection of \(\Sigma\) in \(M\), which is identified with the Euler class of the normal bundle of \(\Sigma\) paired with the fundamental class of \(\Sigma\). In the case that the singular set is non-orientable, we can understand this by pulling the Euler class of the normal bundle back to the orientable double cover, evaluating it on the corresponding fundamental class and then dividing by two.

In [AL12], Atiyah-LeBrun proved the following versions of the Gauss-Bonnet and Hirzebruch signature Theorems for edge-cone metrics:

**Theorem 1.2** (Atiyah-LeBrun [AL12]). Let \(g\) be an edge-cone metric on a smooth four-manifold \(M\) with singular set \(\Sigma \subset M\) and with cone angle \(2\pi \beta\). Then

\[
\chi_{\text{orb}}(M) = \chi(M) - (1 - \beta)\chi(\Sigma)
\]

\[
\tau_{\text{orb}}(M) = \tau(M) - \frac{1}{3}(1 - \beta^2)[\Sigma]^2.
\]

(1.4)

As an application, Atiyah-LeBrun proved a version of the Hitchin-Thorpe inequality for Einstein edge-cone metrics. They also discussed many examples of Einstein edge-cone metrics. Interestingly, all of the examples considered in that paper also happened to be self-dual or anti-self-dual metrics, which we now very briefly describe, and we refer the reader to [LV12] for more background and details.

It is well-known that on an oriented four-manifold \(M\), the Weyl tensor decomposes as \(W = W^+ + W^-\), where \(W^+\) and \(W^-\) are the self-dual and anti-self-dual parts of the Weyl tensor, respectively. A metric \(g\) is said to be anti-self-dual or self-dual if \(W^+ \equiv 0\) or \(W^- \equiv 0\), respectively. In the anti-self-dual case, local information of the moduli space of anti-self-dual metrics near \(g\) is contained in the elliptic complex

\[
\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S^2_0(T^*M)) \xrightarrow{\mathcal{D}^+} \Gamma(S^2_0(\Lambda^2_+)),
\]

(1.5)

where \(\mathcal{K}_g\) is the conformal Killing operator, \(S^2_0(T^*M)\) denotes traceless symmetric tensors, and \(\mathcal{D}^+ = (W^+)_g\) is the linearized self-dual Weyl curvature operator. In the self-dual case, the relevant complex is

\[
\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S^2_0(T^*M)) \xrightarrow{\mathcal{D}^-} \Gamma(S^2_0(\Lambda^2_-)),
\]

(1.6)

where \(\mathcal{D}^- = (W^-)_g\) is the linearized anti-self-dual Weyl curvature operator.
If \( g \) is a smooth anti-self-dual or self-dual Riemannian metric, from the Atiyah-Singer Index Theorem, the index of the complex (1.5) or (1.6) is given by

\[
\text{Ind}(M, g) = \dim(H^0) - \dim(H^1) + \dim(H^2) = \frac{1}{2}(15\chi(M) \pm 29\tau(M)),
\]

where \( \chi(M) \) is the Euler characteristic, \( \tau(M) \) is the signature of \( M \), and \( H^i \) is the \( i \)th cohomology of the complex (1.5) in the positive case, and the complex (1.6) in the negative case, for \( i = 0, 1, 2 \); see [KK92].

For an orbifold-cone metric, the index is computed by looking at smooth sections in the orbifold sense, see Section 4. In this setting, the formula given in (1.7) is not necessarily correct, and there are correction terms required arising from the singularities.

We note that the complex (1.5) yields local information about the structure of the moduli space of anti-self-dual metrics near \( g \). That is, there is a map, called the Kuranishi map

\[
\Psi : H^1 \to H^2
\]

which is equivariant under the action of \( H^0 \), such that the moduli space of anti-self-dual orbifold-cone metrics with singular set \( \Sigma \subset M \) and fixed cone angle \( \frac{2\pi}{p} \) near \( g, \mathcal{M}_g \), is locally isomorphic to \( \Psi^{-1}(0)/H^0 \). This is a standard fact in the setting of smooth manifolds, and the proof of existence of the Kuranishi map readily generalizes to the setting of orbifold-cone metrics. It is important to note that this map does not take into account deformations of the cone angle.

In a previous paper, the authors proved an extension of the index formula to anti-self-dual orbifold metrics with isolated cyclic quotient singularities [LV12]. In this paper, we prove an extension of the index formula (1.7) to anti-self-dual metrics with orbifold-cone singularities:

**Theorem 1.3.** Let \( g \) be an orbifold-cone metric on a smooth four-manifold \( M \) with singular set \( \Sigma \subset M \) and cone angle \( \frac{2\pi}{p} \). If \( g \) is anti-self-dual, then the index of the complex (1.5) is given by

\[
\text{Ind}^{\text{ASD}}(M, g) = \frac{1}{2}(15\chi(M) + 29\tau(M)) - 4\chi(\Sigma) - 4[\Sigma]^2,
\]

where \([\Sigma]^2\) denotes the self-intersection number of \( \Sigma \) in \( M \). If \( g \) is instead self-dual, then the index of the complex (1.6) is given by

\[
\text{Ind}^{\text{SD}}(M, g) = \frac{1}{2}(15\chi(M) - 29\tau(M)) - 4\chi(\Sigma) + 4[\Sigma]^2.
\]

We emphasize that the index is independent of the cone angle and only depends on the topologies of \( M \) and \( \Sigma \), and the embedding of \( \Sigma \) into \( M \). Our proof of this is an application of Kawasaki’s orbifold index theorem from [Kaw81].

**Remark 1.4.** There are many examples of continuous families of anti-self-dual edge-cone metrics arising from deforming the cone angle, see [AL12]. In all such examples the above index formula can be seen to hold for arbitrary real cone angles. Thus, it is likely that this formula holds in general for anti-self-dual edge-cone metrics.
with arbitrary cone angle $2\pi \beta$. However, in this case the index must be defined using appropriate weighted edge Hölder spaces to obtain Fredholm operators. This introduces considerable technical complications, and we plan to address this in a forthcoming paper.

It is useful to make the following definition.

**Definition 1.5.** An anti-self-dual (self-dual) orbifold-cone metric with $H^2 = \{0\}$ is called *unobstructed*.

It was conjectured by I.M. Singer in 1978 that a positive scalar curvature anti-self-dual metric is unobstructed. The evidence for this conjecture is very strong, but it has not yet been proven in full generality. However, the conjecture is certainly true in the Einstein case:

**Lemma 1.6.** Any anti-self-dual (self-dual) Einstein orbifold-cone metric with positive scalar curvature is unobstructed.

This was proved in the smooth case by [Ito95], and in Section 5 we will show that his proof extends to the orbifold-cone setting.

1.1. **Self-dual edge-cone metrics on $S^4$.** The first examples we consider were found by Hitchin in [Hit96]. They are a family of self-dual Einstein orbifold-cone metrics on $S^4$ with singular set an $\mathbb{RP}^2$ and cone angle $2\pi/(k-2)$, where $k \geq 3$ is an integer. These metrics have the 3-dimensional isometry group $\text{SO}(3)$. These metrics are self-dual, which determines an orientation on $S^4$. Everything we say below is with respect to this orientation.

The singular set is a Veronese $\mathbb{RP}^2 \subset S^4$. This arises by first looking at the representation of $S^2_0(\mathbb{R}^3)$ in $\mathbb{R}^5$, which yields an embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^4$, having two lifts into $S^4$ of self-intersection $\pm 2$ respectively. The singular set for Hitchin’s metrics is the $\mathbb{RP}^2$ with self-intersection $-2$. We have the following rigidity result for Hitchin’s metrics:

**Corollary 1.7.** For any $k \geq 3$, a Hitchin metric on $(S^4, \mathbb{RP}^2)$ is rigid as a self-dual orbifold-cone metric with cone angle $2\pi/(k-2)$.

Hitchin’s metrics all have singular set an embedded $\mathbb{RP}^2$. A natural question is whether there are self-dual Einstein orbifold-cone metrics on $S^4$ with other singular sets. Atiyah-LeBrun give the following family of examples on $(S^4, S^2)$ [AL12 page 21]: $S^4 \setminus S^2$ is conformally isometric to $\mathcal{H}^3 \times S^1$, with the product metric $h + d\theta^2$ where $\mathcal{H}^3$ is the 3-dimensional hyperbolic upper half plane with hyperbolic metric $h$. The metric $h + \beta^2 d\theta^2$ is then a constant curvature edge-cone metric with angle $2\pi \beta$, with singular set $S^2$.

**Corollary 1.8.** No unobstructed self-dual orbifold-cone metrics exist on $S^4$ with singular set diffeomorphic to a genus $j \geq 1$ orientable surface.

The natural question is then whether there are unobstructed self-dual orbifold-cone metrics on $S^4$ with singular set a smoothly embedded surface diffeomorphic to $j\# \mathbb{RP}^2$ when $j > 1$, which we denote here by $\Sigma^j$. 
For $\Sigma^j$ embedded in $S^4$, Whitney studied the possible values of the self-intersection number and proved that $[\Sigma^j]^2 \equiv 2\chi(\Sigma^j) \mod 4$. He also proposed a conjecture, which Massey later proved \cite{Mas69}, stating that $[\Sigma^j]^2$ could only take the following values:

\begin{align}
2\chi(\Sigma^j) - 4, \quad 2\chi(\Sigma^j), \quad 2\chi(\Sigma^j) + 4, \ldots, \quad 4 - 2\chi(\Sigma^j).
\end{align}

Since $\chi(\Sigma^j) = 2 - j$, this set of values can be written in terms of $j$ as:

\begin{align}
-2j, \quad -2j + 4, \quad -2j + 8, \ldots, \quad 2j.
\end{align}

Moreover, Massey also proved that any of these values can be obtained by an appropriate embedding of $\Sigma^j$ in $S^4$. The next result gives a restriction on the self-intersection number of the singular set for such a metric:

**Corollary 1.9.** If $g$ is an unobstructed self-dual orbifold-cone metric on $(S^4, \Sigma^j)$, where $\Sigma^j$ is diffeomorphic to $j \# \mathbb{RP}^2$ when $j \geq 1$, then we have the inequalities

\begin{align}
-2j \leq [\Sigma^j]^2 < -j.
\end{align}

It is an interesting problem to find examples of unobstructed self-dual orbifold-cone metrics on $(S^4, \Sigma^j)$ when $j > 1$.

### 1.2. LeBrun’s hyperbolic monopole metrics.

Next, we turn to LeBrun’s hyperbolic monopole metrics from \cite{LeB91}. These metrics are defined similarly to the Gibbon-Hawking multi-Eguchi-Hanson metrics \cite{GH78}, but with hyperbolic 3-space $\mathcal{H}^3$ replacing Euclidean 3-space. To define these metrics, first choose $n$ points $\{p_i\}$ in hyperbolic 3-space, and let

\begin{align}
V = 1 + \sum_{i=1}^n \Gamma_{p_i}
\end{align}

where $\Gamma_{p_i}$ is the hyperbolic Green’s function based at $p_i$ with normalization $\Delta \Gamma_{p_i} = -2\pi \delta_{p_i}$. Letting $P$ denote the collection of monopole points $p_i$, $\ast dV$ is a closed 2-form on $\mathcal{H}^3 \setminus P$, and $(1/2\pi)[\ast dV]$ is an integral class in $H^2(\mathcal{H}^3 \setminus P, \mathbb{Z})$. Let $\pi : X_0 \to \mathcal{H}^3 \setminus P$ be the unique principal $U(1)$-bundle determined by the the above integral class. By Chern-Weil theory, there is a connection form $\omega \in H^1(X_0, i\mathbb{R})$ with curvature form $i(\ast dV)$. LeBrun’s metric is defined by

\begin{align}
g_{LB} = V \cdot g_{\mathcal{H}^3} - V^{-1} \omega \circ \omega.
\end{align}

Next define a larger manifold $X$ by attaching points $\tilde{p}_j$ over each $p_j$, and by adding $\Sigma = S^2$ corresponding to the boundary of hyperbolic space. By an appropriate choice of conformal factor, the metric extends smoothly to this compactification, which is diffeomorphic to $n \# \mathbb{CP}^2$. All of these conformal classes admit an $S^1$-action.

In \cite[Section 5]{AL12} it was noted that by replacing $V$ with

\begin{align}
V = \beta^{-1} + \sum_{i=1}^n \Gamma_{p_i},
\end{align}

one obtains an self-dual edge-cone metric on $(n \# \mathbb{CP}^2, \Sigma)$, where $\Sigma = S^2$, with cone angle $2\pi\beta$. 

For $\beta = 1$ and $n \geq 3$, it is well-known that LeBrun’s metrics admit non-$S^1$-equivariant deformations, with the moduli space locally of dimension $7n - 15$ [LeB92]. However, for orbifold-cone metrics, assuming these metrics are unobstructed, then somewhat surprisingly this is no longer true:

**Corollary 1.10.** Let $g$ be an unobstructed LeBrun self-dual orbifold-cone metric on $(n\#\mathbb{CP}^2, \Sigma)$ with cone angle $2\pi/p$. Then any self-dual edge-cone deformation of $g$ with cone angle $2\pi/p$ also admits an $S^1$-action. Thus the moduli space of such metrics near $g$ is of dimension $3(n - 2)$ for $n \geq 3$.

Unobstructedness is true in the smooth case ($\beta = 1$), but the proof of this relies on tools which do not easily generalize to the orbifold-cone case. However, we do expect that these metrics are also unobstructed for $\beta \neq 1$, so this assumption is probably not necessary.

**1.3. Ricci-flat anti-self-dual metrics.** Finally, we consider the Ricci-flat case. There are many known example of such metrics with edge-cone singularities. For example, in [Bre11] examples of Kähler Ricci-flat metrics with edge-cone singularities were obtained. These are of relevance to this paper, since such a metric in dimension 4 is in necessarily anti-self-dual. The following result computes the dimension of the moduli space of anti-self-dual metrics in the more general Ricci-flat anti-self-dual case:

**Corollary 1.11.** Let $g$ be an anti-self-dual Ricci-flat orbifold-cone metric on $M$ with singular set $\Sigma$ and cone angle $2\pi/p$. Assume that there are no parallel vector fields, and also that there are no parallel sections of $S^0_0(\Lambda^2_+)$ with respect to $g$. Then the moduli space of anti-self-dual orbifold-cone metrics on $M$ with singular set $\Sigma$ and cone angle $2\pi/p$ near $g$ is a smooth manifold $\mathcal{M}_g$ of dimension

$$\dim(\mathcal{M}_g) = -\frac{1}{2}(15\chi(M) + 29\tau(M)) + 4\chi(\Sigma) + 4[\Sigma]^2. \quad (1.17)$$

Since there are many examples of Kahler-Ricci flat edge-cone metrics which are not orbifold-cone metrics, it is a very interesting problem to generalize Corollary 1.10 to the more general edge-cone case. We expect the same formula holds for any cone angle.

We end with a brief outline of the paper. In Section 2 we give the necessary background and set-up. In Section 3 we compute the required equivariant Chern characters. Theorem 1.3 is then proved in Section 4. Finally, in Section 5 the proofs of Corollaries 1.7-1.11 are given.

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**2. Analysis of $T^*M_{\Sigma}$ and $\Lambda^2_+|_{\Sigma}$**

To compute the index it is necessary to understand how the pullback of the complexified principle symbol of the complex (1.5), and the $K$-theoretic Thom class of the complexified normal bundle decompose into line bundles, and how the orbifold
structure group acts on these decompositions. This will be dealt with in Section 3. In this section, however, we will analyze $T^*M_{|\Sigma}$ and $\Lambda^2_{\pm|\Sigma}$ because these bundles play a crucial role in the decompositions in Section 3. For the rest of the paper we will denote the complexification of a real bundle, $E$, by $E_C$.

We begin by decomposing, in real coordinates,

\[ T^*M_{|\Sigma} = T^* \oplus N^*, \]

which are the tangent and normal bundles to the singular set respectively. In local orthonormal coordinates we can write

\[ T^* = \text{span}\{e^1, e^2\} \quad \text{and}, \]

\[ N^* = \text{span}\{e^3, e^4\}. \]

Using these coordinates we describe the orbifold structure group. We are considering orbifold-cone metrics with cone angle $2\pi/p$ so the orbifold structure group is the cyclic group $\Gamma$, of order $p$, consisting of elements $\gamma_j$, which can be locally written as

\[ \gamma_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi}{p}j\right) & -\sin\left(\frac{2\pi}{p}j\right) \\ 0 & 0 & \sin\left(\frac{2\pi}{p}j\right) & \cos\left(\frac{2\pi}{p}j\right) \end{pmatrix}. \]

We will often refer to a general element $\gamma \in \Gamma$ and denote the angle of the corresponding action by $\theta$.

We have the following sections of $T^*_C$:

\[ \alpha_1 = e^1 + ie^2 \quad \text{and} \quad \bar{\alpha}_1 = e^1 - ie^2, \]

and the following sections of $N^*_C$:

\[ \alpha_2 = e^3 + ie^4 \quad \text{and} \quad \bar{\alpha}_2 = e^3 - ie^4. \]

Now consider the line bundles over $F$:

\[ \Theta_i = \text{span}\{\alpha_i\} \quad \text{and} \quad \bar{\Theta}_i = \text{span}\{\bar{\alpha}_i\}. \]

It is clear that we have the line bundle decompositions:

\[ T^*_C = \Theta_1 \oplus \bar{\Theta}_1 \quad \text{and} \]

\[ N^*_C = \Theta_2 \oplus \bar{\Theta}_2, \]

and in these coordinates for $N^*_C$ we have

\[ \gamma|_{N^*_C} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \]
Recall that \( \Lambda^2_{+|\Sigma} \) is generated by three sections, which can be locally written as
\[
\begin{align*}
\omega^+_{1c} &= e^1 \wedge e^2 + e^3 \wedge e^4 \\
\omega^+_{2c} &= e^1 \wedge e^3 + e^4 \wedge e^2 = \frac{1}{2}(\alpha_1 \wedge \alpha_2 + \bar{\alpha}_1 \wedge \bar{\alpha}_2) \\
\omega^+_{3c} &= e^1 \wedge e^4 + e^2 \wedge e^3 = -\frac{i}{2}(\alpha_1 \wedge \alpha_2 - \bar{\alpha}_1 \wedge \bar{\alpha}_2),
\end{align*}
\]
(2.9)

Since \( \omega^+_{1c} \) is a global non-zero section, it spans a trivial line bundle which we will denote by \( \mathbb{C}_+ \). Therefore, we can decompose \( \Lambda^2_{+|\Sigma} \) into line bundles as
\[
\Lambda^2_{+|\Sigma} = \mathbb{C}_+ \oplus (\Theta_1 \otimes \Theta_2) \oplus (\Theta_1 \otimes \Theta_2),
\]
(2.10)
which, with respect to this decomposition, admits the group action
\[
\gamma|_{\Lambda^2_{+|\Sigma}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}.
\]
(2.11)

Similarly, recall that \( \Lambda^2_{-|\Sigma} \) is generated by three sections, which can be locally written as
\[
\begin{align*}
\omega^-_{1c} &= e^1 \wedge e^2 - e^3 \wedge e^4 \\
\omega^-_{2c} &= e^1 \wedge e^3 - e^4 \wedge e^2 = \frac{1}{2}(\alpha_1 \wedge \bar{\alpha}_2 + \bar{\alpha}_1 \wedge \alpha_2) \\
\omega^-_{3c} &= e^1 \wedge e^4 - e^2 \wedge e^3 = -\frac{i}{2}(\alpha_1 \wedge \bar{\alpha}_2 + \bar{\alpha}_1 \wedge \alpha_2),
\end{align*}
\]
(2.12)

Here we will denote the trivial line bundle that is the span of \( \omega^-_{1c} \) as \( \mathbb{C}_- \). Therefore, we can decompose \( \Lambda^2_{-|\Sigma} \) into line bundles as
\[
\Lambda^2_{-|\Sigma} = \mathbb{C}_- \oplus (\bar{\Theta}_1 \otimes \Theta_2) \oplus (\Theta_1 \otimes \bar{\Theta}_2),
\]
(2.13)
which, with respect to this decomposition, admits the group action
\[
\gamma|_{\Lambda^2_{-|\Sigma}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}.
\]
(2.14)

3. Equivariant Chern characters

Throughout the rest of this paper, we will denote the Euler class of \( T^* \) by \( e \), and the orbifold Euler class of \( N^* \) by \( \hat{h} \). We will also denote their pairings with the fundamental class of \( \Sigma \), \( \langle e, [\Sigma] \rangle \) and \( \langle \hat{h}, [\Sigma] \rangle \), by \( \chi(\Sigma) \) and \( [\Sigma]^2 \) respectively. It is important to notice that if we consider \( \Sigma \) as a smoothly embedded submanifold with the regular Euler class of its normal bundle \( h \), then \( [\Sigma]^2 = \langle h, [\Sigma] \rangle = p[\Sigma]^2 \), the self-intersection number of \( \Sigma \) in \( M \), where \( p \) comes from the cone angle. Also, for the remainder of the construction of the index we will assume that \( \Sigma \) is orientable. This is a necessary assumption for the Index theorem. However, once we prove Theorem [13] for \( \Sigma \) orientable, it is very easy to show that it also holds for \( \Sigma \) non-orientable.
We will frequently make use of the equivariant Chern characters of the complex line bundles in decomposition (2.7). Since \(\gamma\) acts trivially on \(\Theta_1 \oplus \Theta_1\), and acts on \(\Theta_2 \oplus \Theta_2\), we see that

\[
\begin{align*}
    ch_\gamma(\Theta_1) &= ch(\Theta_1) = \sum_{j=0}^{\infty} \frac{e^j}{j!} \\
    ch_\gamma(\bar{\Theta}_1) &= ch(\bar{\Theta}_1) = \sum_{j=0}^{\infty} \frac{(-e)^j}{j!} \\
    ch_\gamma(\Theta_2) &= e^{i\theta} ch(\Theta_2) = e^{i\theta} \sum_{j=0}^{\infty} \frac{h^j}{j!} \\
    ch_\gamma(\bar{\Theta}_2) &= e^{-i\theta} ch(\bar{\Theta}_1) = e^{-i\theta} \sum_{j=0}^{\infty} \frac{(-\tilde{h})^j}{j!}.
\end{align*}
\]

(3.1)

To find the anti-self-dual index, we need to compute the equivariant Chern character on the pullback of the complexified principle symbol over \(\Sigma\), \(i^*\sigma\):

\[
i^*\sigma = i^*[T^*M_\mathbb{C}] - i^*[S_0^2T^*M_\mathbb{C}] + i^*[S_0^2\Lambda^2_{+\mathbb{C}|\Sigma}] = 0
\]

(3.2)

where \(i : \Sigma \rightarrow M\) is the inclusion of the singular set \(\Sigma\) into the orbifold \(M\). We will also need to compute the equivariant Chern character of the \(K\)-theoretic Thom class of the complexified normal bundle:

\[
\lambda_{-1} N_\mathbb{C}^* = [\Lambda^0 N_\mathbb{C}^*] - [\Lambda^1 N_\mathbb{C}^*] + [\Lambda^2 N_\mathbb{C}^*]
\]

(3.3)

We begin this section by computing \(ch_\gamma(\Lambda^2_{+\mathbb{C}|\Sigma})\), next we compute \(ch_\gamma(i^*\sigma)\) and finally we compute \(ch_\gamma(\lambda_{-1} N_\mathbb{C}^*)\).

### 3.1. Equivariant Chern characters of \(\Lambda^2_{+\mathbb{C}|\Sigma}\).

Using the decomposition (2.10) of \(\Lambda^2_{+\mathbb{C}|\Sigma}\), we have that

\[
ch_\gamma(\Lambda^2_{+\mathbb{C}|\Sigma}) = ch_\gamma(\mathbb{C}) + ch_\gamma(\Theta_1 \otimes \Theta_2) + ch_\gamma(\bar{\Theta}_1 \otimes \bar{\Theta}_2).
\]

(3.4)

The first term on the right hand side is 1 because the \(\gamma\)-action on \(\mathbb{C}_+\) is trivial. We compute the second two terms on the right hand side as:

\[
\begin{align*}
    ch_\gamma(\Theta_1 \otimes \Theta_2) &= ch_\gamma(\Theta_1) \cdot ch_\gamma(\Theta_2) = (1 + e + \frac{e^2}{2} + \cdots) \cdot e^{i\theta} (1 + \hat{h} + \frac{\hat{h}^2}{2} + \cdots) \\
    &= e^{i\theta} (1 + e + \hat{h} + e\hat{h} + \frac{e^2}{2} + \frac{\hat{h}^2}{2} + \cdots), \text{ and} \\
    ch_\gamma(\bar{\Theta}_1 \otimes \bar{\Theta}_2) &= ch_\gamma(\bar{\Theta}_1) \cdot ch_\gamma(\bar{\Theta}_2) = (1 - e + \frac{e^2}{2} + \cdots) \cdot e^{-i\theta} (1 - h + \frac{h^2}{2} + \cdots) \\
    &= e^{-i\theta} (1 - e - \hat{h} + e\hat{h} + \frac{e^2}{2} + \frac{\hat{h}^2}{2} + \cdots).
\end{align*}
\]
Therefore, we can combine these terms to find

\[ \text{(3.5)} \quad \text{ch}_\gamma (\Lambda^2_{+\Cl_{12}}) = 1 + \cos(\theta)(2 + 2e\hat{h} + e^2 + \hat{h}^2 + \cdots) + i\sin(\theta)(2e + 2\hat{h} + \cdots). \]

Similarly, we find that

\[ \text{(3.6)} \quad \text{ch}_\gamma (\Lambda^2_{-\Cl_{12}}) = 1 + \cos(\theta)(2 - 2e\hat{h} + e^2 + \hat{h}^2 + \cdots) + i\sin(\theta)(-2e + 2\hat{h} + \cdots). \]

### 3.2. Equivariant Chern character of \( i^*\sigma \)

We will begin by computing the equivariant Chern characters of the individual \( K \)-theoretic classes that compose \( i^*\sigma \) and then sum them accordingly to find \( \text{ch}_\gamma (i^*\sigma) \).

First consider the bundle \( i^*(T^*M_C) = T_C^* \oplus N_C^* \). We have

\[ \text{ch}_\gamma (i^*[T^*M_C]) = \text{ch}_\gamma (T_C^* \oplus N_C^*) \]

\[ = (2 + e^2 + \cdots) + e^{i\theta}(1 + \hat{h} + \frac{\hat{h}^2}{2} + \cdots) + e^{-i\theta}(1 - \hat{h} + \frac{\hat{h}^2}{2} + \cdots) \]

\[ = (2 + e^2 + \cdots) + \cos(\theta)(2 + \hat{h}^2 + \cdots) + i\sin(\theta)(2\hat{h} + \cdots). \]

Next, consider the bundle \( i^*(S_0^2T^*M_C) \). Using the formulas (3.5) and (3.6), and the bundle isomorphism \( S_0^2T^*M = \Lambda^2_+ \otimes \Lambda^2_- \) we compute

\[ \text{ch}_\gamma (i^*[S_0^2T^*M_C]) = \text{ch}_\gamma (i^*\Lambda^2_{+\Cl} \otimes i^*\Lambda^2_{-\Cl}) = \text{ch}_\gamma (\Lambda^2_{+\Cl_{12}} \cdot \text{ch}_\gamma \Lambda^2_{-\Cl_{12}}) \]

\[ = [1 + 4\cos(\theta) + 4\cos^2(\theta)] + \hat{h} [i4\sin(\theta) + i8\sin(\theta)\cos(\theta)] \]

\[ + e^2[4 + 2\cos(\theta)] + \hat{h}^2[-4 + 2\cos(\theta) + 8\cos^2(\theta)] + \cdots. \]

Finally, consider the bundle \( i^*(S_0^2\Lambda^2_{+\Cl}) = S_0^2\Lambda^2_{+\Cl_{12}} \), which decomposes as

\[ S_0^2\Lambda^2_{+\Cl_{12}} = \{(\Theta_1 \otimes \Theta_2) \oplus (\Theta_1 \otimes \Theta_2)\} \oplus S_0^2((\Theta_1 \otimes \Theta_2) \oplus (\Theta_1 \otimes \Theta_2)) \oplus \text{C}_{tr} \]

where \( \text{C}_{tr} \) is a trivial line bundle, with trivial \( \gamma \)-action, corresponding to the trace term. Using this, we are able to compute

\[ \text{ch}_\gamma (i^*[S_0^2\Lambda^2_+]) = \text{ch}_\gamma (\Theta_1 \otimes \Theta_2 \oplus \Theta_1 \otimes \Theta_2) + \text{ch}_\gamma (S_0^2(\Theta_1 \otimes \Theta_2 \oplus \Theta_1 \otimes \Theta_2)) + 1 \]

\[ = [\cos(\theta)(2 + 2e\hat{h} + e^2 + \hat{h}^2 + \cdots) + i\sin(\theta)(2e + 2\hat{h} + \cdots)] \]

\[ + [(\cos(\theta)(2 + 2e\hat{h} + e^2 + \hat{h}^2 + \cdots) + i\sin(\theta)(2e + 2\hat{h} + \cdots))^2 - 2] + 1 \]

\[ = [-1 + 2\cos(\theta) + 4\cos^2(\theta)] + e[i2\sin(\theta) + i8\sin(\theta)\cos(\theta)] \]

\[ + \hat{h}[i2\sin(\theta) + i8\sin(\theta)\cos(\theta)] + e^2[-8 + 2\cos(\theta) + 16\cos^2(\theta)] \]

\[ + e^2[-4 + \cos(\theta) + 8\cos^2(\theta)] + \hat{h}^2[-4 + \cos(\theta) + 8\cos^2(\theta)] + \cdots. \]

Now, we are able to compute the \( \text{ch}_\gamma (i^*\sigma) \) by taking the appropriate sum of the above Chern characters:

\[ \text{ch}_\gamma (i^*\sigma) = \text{ch}_\gamma (i^*[T^*M_C]) - \text{ch}_\gamma (i^*[S_0^2T^*M_C]) + \text{ch}_\gamma (i^*[S_0^2\Lambda^2_+]) \]

\[ = e[2i\sin(\theta) + 8i\sin(\theta)\cos(\theta)] + e\hat{h}[-8 + 2\cos(\theta) + 16\cos^2(\theta)] \]

\[ + e^2[8\cos^2(\theta) - \cos(\theta) - 7] + \cdots. \]
3.3. Equivariant Chern character of $\lambda_{-1}N^*_C$. We begin by examining the bundles representing the $K$-theoretic classes that compose $\lambda_{-1}N^*_C$.

First, both $\Lambda^0N^*_C$ and $\Lambda^2N^*_C$ have non-vanishing global sections, so they are trivial, and clearly admit a trivial $\gamma$ action. Next, it is clear that $\Lambda^1N^*_C = N^*_C$. Therefore, we find that

$$
ch_{\gamma}(\lambda_{-1}N^*_C) = ch_{\gamma}(\Lambda^0N^*_C) - ch_{\gamma}(\Lambda^1N^*_C) + ch_{\gamma}(\Lambda^2N^*_C)
$$

(3.11)

$$
= 2 - ch_{\gamma}(N^*_C)
$$

$$
= 2 - \cos(\theta)(2 + \hat{h}^2) - i \sin(\theta)(2\hat{h}) + \cdots.
$$

4. The index

We begin this section with some remarks on the definition of the index in the orbifold case. As mentioned in the introduction, the index is computed by looking at smooth sections in the orbifold sense. To define this, we recall that an orbifold vector bundle is defined in terms of orbifold charts. Over a neighborhood $U_x$ away from $\Sigma$ it is defined as a vector bundle in the usual sense, and over a neighborhood $U_q = \tilde{U}_q/\Gamma$ around $q \in \Sigma$, where $\tilde{U}_q$ is a neighborhood of the origin in $\mathbb{R}^4$, it is identified with the quotient of a smooth $\Gamma$-equivariant vector bundle over $\tilde{U}_q$. On overlaps the obvious compatibility conditions are satisfied. Smooth sections of an orbifold vector bundle are globally defined sections on $M$. On a neighborhood $U_x$ away from $\Sigma$ it is smooth in the ordinary sense, and on a neighborhood $U_q$ of $q \in \Sigma$, it is identified with a smooth $\Gamma$-equivariant section of the corresponding $\Gamma$-equivariant bundle over $\tilde{U}_q$ defining the orbifold vector bundle in that neighborhood.

With this understanding of the index, from [Kaw81] and [LM89], recall that the anti-self dual index for $(M, g)$, where $g$ is an orbifold-cone metric with singular set $\Sigma$, is given by

$$
Ind^{ASD}(M, g) = \frac{1}{2}(15\chi_{\text{orb}}(M) + 29\tau_{\text{orb}}(M)) - \left\langle \frac{1}{|\Gamma|} \sum_{\gamma \neq Id} \frac{ch_{\gamma}(i^*\sigma)}{ch_{\gamma}(\lambda_{-1}N^*_C)e^{\hat{A}(\Sigma)^2}}, [\Sigma] \right\rangle.
$$

Note that Kawasaki’s formula is written in terms of evaluation on the orbifold tangent bundle of the singular set, but writing it in terms of evaluation on the fundamental class of $\Sigma$ introduces the Euler class in the denominator.

Next, using the formulas (1.2) and (1.3), we can rewrite

$$
Ind^{ASD}(M, g) = \frac{1}{2}(15\chi_{\text{top}}(M) + 29\tau_{\text{top}}(M)) - \frac{15}{2} \left( \frac{p-1}{p} \right) \chi(\Sigma) - \frac{29}{6} \left( \frac{p^2-1}{p} \right) |\hat{\Sigma}|^2
$$

$$
- \left\langle \frac{1}{p} \sum_{j=1}^{p-1} \frac{ch_{\gamma_j}(i^*\sigma)}{ch_{\gamma_j}(\lambda_{-1}N^*_C)e^{\hat{A}(\Sigma)^2}}, [\Sigma] \right\rangle.
$$
4.1. Computation of correction terms. Using the computation of the denominator, (3.11), we have

\[
ch\gamma(\lambda-1N_C)^{-1} = \left[2 - \cos(\theta)(2 + \hat{h}^2) - i \sin(\theta)(2\hat{h})\right]^{-1} = \left[(2 - 2 \cos(\theta))[1 - \frac{1}{2 - 2 \cos(\theta)}(\cos(\theta)(\hat{h}^2) + i \sin(\theta)(2\hat{h}))]\right]^{-1} = \left[(2 - 2 \cos(\theta))[1 - \mathbb{D}]\right]^{-1}
\]

Then, by using a geometric series, we see that

\[
(ch\gamma(\lambda-1N_C)^{-1} = \frac{1}{2 - 2 \cos(\theta)}[1 + \mathbb{D} + \mathbb{D}^2 + \cdots]. \tag{4.1}
\]

Now, we compute:

\[
\frac{ch\gamma(i^*\sigma)}{ch\gamma(\lambda-1N_C)e} = \frac{ch\gamma(i^*\sigma)}{(2 - 2 \cos(\theta))e[1 + \mathbb{D} + \mathbb{D}^2 + \cdots]}
\]

\[
= \frac{1}{2}e \left(\frac{8 \cos(\theta) + 7(\cos(\theta) - 1)}{1 - \cos(\theta)}\right) + \hat{h} \left[-\frac{4}{1 - \cos(\theta)} + \frac{\cos(\theta)}{1 - \cos(\theta)} + \frac{8 \cos^2(\theta)}{1 - \cos(\theta)}\right] + \frac{[2i \sin(\theta) + 8i \sin(\theta) \cos(\theta)]}{2 - 2 \cos(\theta)} \hat{h} \frac{2i \sin(\theta)}{2 - 2 \cos(\theta)} + \cdots
\]

\[
= -\frac{1}{2}e[8 \cos(\theta) + 7] + \hat{h} \left[-9 - 8 \cos(\theta) + \frac{5}{1 - \cos(\theta)}\right] + \hat{h} \left[-1 + 5 \cos(\theta) + 4 \cos^2(\theta)\right] + \cdots
\]

\[
= -\frac{1}{2}e[8 \cos(\theta) + 7] + \hat{h} \left[-4 \cos(\theta) - \frac{5}{1 - \cos(\theta)}\right] + \cdots.
\]

Finally, we find:

\[
\frac{1}{p} \sum_{j=1}^{p-1} \frac{ch\gamma(i^*\sigma)}{ch\gamma(\lambda-1N_C)e} = \frac{1}{p} \left[-\frac{1}{2}e(7p - 15) + \hat{h}(4 - \frac{5}{6}(p^2 - 1))\right]. \tag{4.2}
\]

4.2. Computation of the index. Using formula (4.2), we begin to compute the index:

\[
Ind^{ASD}(M, g) = \frac{1}{2}(15\chi_{top}(M) + 29\tau_{top}(M))
\]

\[
- \frac{15}{2}(1 - p^{-1})\chi(\Sigma) - \frac{29}{6} \left(\frac{p^2 - 1}{p}\right) [\hat{\Sigma}]^2
\]

\[
- \left\langle \frac{1}{p} \left[-\frac{1}{2}e(7p - 15) + \hat{h}(4 - \frac{5}{6}(p^2 - 1))\right] \hat{A}(\Sigma)^2, [\Sigma]\right\rangle. \tag{4.3}
\]
For a real oriented plane bundle $E$, whose complexification decomposes into complex line bundles as $E_C = l \oplus \bar{l}$, we have that

$$\hat{A}(E)^2 = Td_C(E_C) = Td(l \oplus \bar{l}) = Td(l)Td(\bar{l})$$

(4.4)

$$= (1 + \frac{1}{2}c_1(l) + \frac{1}{12}c_1(l)^2 + \cdots)(1 - \frac{1}{2}c_1(l) + \frac{1}{12}c_1(l)^2 + \cdots)$$

$$= 1 - \frac{1}{12}c_1(l)^2 + \cdots.$$ 

In the fourth term on the right hand side of (4.3) we see that $\hat{A}^2(\Sigma)$ is only multiplied by terms containing Euler classes of the tangent and normal bundles of $\Sigma$. Since this product is paired with the fundamental class of a surface, it is clear that only the first term in $\hat{A}^2(\Sigma)$ contributes to the index. Therefore

$$Ind^{ASD}(M, g) = \frac{1}{2}(15\chi_{top}(M) + 29\tau_{top}(M)) + \frac{1}{p}\left[\left(\frac{7}{2}p - \frac{15}{2}\right) + \left(\frac{15}{2} - \frac{15}{2}p\right)\right] \chi(\Sigma)$$

$$+ \frac{1}{p}\left[4 + \frac{5}{6}(p^2 - 1) - \frac{29}{6}(p^2 - 1)\right] |\Sigma|^2$$

$$= \frac{1}{2}(15\chi_{top}(M) + 29\tau_{top}(M)) - 4\chi(\Sigma) - 4|\Sigma|^2.$$ 

Since $p|\Sigma|^2 = |\Sigma|^2$ we arrive at formula (1.9):

(4.5)  

$$Ind^{ASD}(M, g) = \frac{1}{2}(15\chi_{top}(M) + 29\tau_{top}(M)) - 4\chi(\Sigma) - 4|\Sigma|^2.$$ 

It is clear from examining the sign changes in the above computations that the formula for the self-dual complex is

(4.6)  

$$Ind^{SD}(M, g) = \frac{1}{2}(15\chi_{top}(M) + 29\tau_{top}(M)) - 4\chi(\Sigma) + 4|\Sigma|^2,$$

which is formula (1.10).

Finally, when $\Sigma$ is a non-orientable surface the formulas (1.9) and (1.10) still hold. This is proved by evaluating the pullbacks of the respective Euler classes to the orientable double cover, evaluating on that fundamental class and then dividing by 2.

5. PROOFS OF COROLLARIES

We begin this section with a proposition bounding $\dim(H^0)$ for an orbifold-cone metric on $(M, \Sigma)$, which will be very useful in the following proofs.

**Proposition 5.1.** Let $g$ be an orbifold-cone metric on $(M, \Sigma)$. Then

(5.1)  

$$\dim(H^0) \leq 11,$$
with equality possible only if $\Sigma = S^2$. Moreover

$$\dim(H^0) \leq \begin{cases} 
7 & \text{when } \Sigma = T^2 \\
5 & \text{when } \Sigma = j\#T^2 \text{ for } j > 1 \\
8 & \text{when } \Sigma = \mathbb{RP}^2 \\
7 & \text{when } \Sigma = \mathbb{RP}^2\#\mathbb{RP}^2 \\
5 & \text{when } \Sigma = j\#\mathbb{RP}^2 \text{ for } j > 2.
\end{cases} \tag{5.2}$$

\textbf{Proof.} We begin by proving bounds on the size of the conformal automorphism group. The proof follows the idea of Bagaev-Zhukova [BZ03], and we briefly recall their argument here:

We have the natural homomorphism

$$\phi : Conf(M, \Sigma) \rightarrow Conf(\Sigma), \tag{5.3}$$

where $Conf(M, \Sigma)$ and $Conf(\Sigma)$ are the conformal automorphism group of $(M, \Sigma)$ and $\Sigma$ respectively, and an exact sequence

$$1 \rightarrow Ker(\phi) \rightarrow Conf(M, \Sigma) \rightarrow Im(\phi) \rightarrow 1. \tag{5.4}$$

This implies that $Conf(M, \Sigma) = Ker(\phi) \rtimes Im(\phi)$.

Consider an element $f \in Ker(\phi)$ and its pushforward map $f_* : TM \rightarrow TM$. This induces the maps $f_*|_{T\Sigma} = Id$ since $f|_{\Sigma} = Id$, and $f_*|_{N\Sigma} \in O(2)$. So we get a homomorphism

$$\alpha : Ker(\phi) \rightarrow O(2) \tag{5.5}$$

by sending $f \mapsto f_*|_{N\Sigma} \in O(2)$. Therefore

$$\dim(Ker(\phi)) \leq \dim(O(2)) + \dim(Ker(\alpha)) \leq 5, \tag{5.6}$$

since $\dim(Ker(\alpha))$ is less than or equal to the dimension of the first prolongation of the Lie algebra of the $G$-structure group, which is 4. Therefore, $\dim(Conf(M, \Sigma)) \leq \dim(Conf(\Sigma)) + 5$.

Now, $\dim(Im(\phi)) \leq \dim(Conf(\Sigma)) \leq 6$, with equality if and only if $\Sigma = S^2$. We also know that

$$\dim(Conf(j\#T^2)) = \begin{cases} 2 & \text{for } j = 1 \\
0 & \text{for } j \geq 2 \end{cases} \quad \text{and} \quad \dim(Conf(j\#\mathbb{RP}^2)) = \begin{cases} 3 & \text{for } j = 1 \\
2 & \text{for } j = 2 \\
0 & \text{for } j \geq 3. \end{cases}$$

We complete the proof by showing that

$$\dim(H^0) = \dim(Conf(M, \Sigma)) \leq \dim(Conf(\Sigma)) + 5. \tag{5.7}$$

In the smooth case, a conformal Killing field is the derivative of a 1-parameter family of conformal transformations. Thus, the space of conformal Killing fields is identified with the Lie algebra of the conformal automorphism group. For the orbifold case take a neighborhood $U_q$ around $q \in \Sigma$ and lift it to a neighborhood $\tilde{U}_q$ around the origin in $\mathbb{R}^4$, on which the metric $g$ pulls back to $\tilde{g}$, a $\Gamma$-invariant metric. Any conformal
Killing field on $U_q$ lifts to a $\Gamma$-invariant conformal Killing field on $\tilde{U}_q$ since $\tilde{g}$ is $\Gamma$-invariant. The local 1-parameter families of diffeomorphisms on $\tilde{U}_q$ coming from the lifts of these conformal Killing fields must also be $\Gamma$-invariant since the flow is locally defined. From the uniqueness of the flow on each $\tilde{U}_q$, these patch together to give a globally defined 1-parameter group of conformal transformations on the orbifold. Therefore, in the orbifold case we also have, $\dim(H^0) = \dim(Conf(M, \Sigma))$. 

The fact that a self-dual positive scalar curvature Einstein orbifold-cone metric is unobstructed is crucial to the proof of Corollary 1.7, so we now prove Lemma 1.6:

**Proof of Lemma 1.6.** Let $g$ be an anti-self-dual Einstein orbifold-cone metric with positive scalar curvature on $M$ with singular set $\Sigma$. Let $Z \in Ker(D^+\ast)$. Then

$$D^+D^+\ast Z = 0,$$

using the Weitzenböck formula of Itoh in the case that $g$ is Einstein [Ito95], where $R$ is the scalar curvature.

Cut out $N_\epsilon$, an $\epsilon$-tubular neighborhood of $\Sigma$ and denote the outer unit normal vector and induced volume form on $\partial(M \setminus N_\epsilon)$ by $n$ and $d\sigma$ respectively. Using Itoh’s Weitzenböck formula (5.8) and integrating by parts, we see that

$$\int_{M \setminus N_\epsilon} \langle D^+D^+\ast Z, Z \rangle dV = \int_{M \setminus N_\epsilon} \left[ \frac{1}{4} |\Delta Z|^2 + \frac{7}{24} R|\nabla Z|^2 + \frac{1}{12} R^2 |Z|^2 \right] dV$$

$$- \int_{\partial(M \setminus N_\epsilon)} \left[ \frac{7}{48} R\nabla_n |Z|^2 + \frac{1}{4} \langle \Delta Z, \nabla_n Z \rangle - \frac{1}{4} \langle \nabla_n \Delta Z, Z \rangle \right] d\sigma.$$  

Since $Z$ is a smooth section in the orbifold sense, $Z$ and its derivatives are bounded. Therefore, by dominated convergence, the solid integrals limit to the corresponding solid integrals on $M$ as $\epsilon \to 0$. For $\epsilon$ sufficiently small, $\partial(M \setminus N_\epsilon)$ is a smooth submanifold, and we have the estimate

$$\text{Area}(\partial(M \setminus N_\epsilon)) < C\epsilon,$$

for some constant $C$. Consequently the boundary integral limits to 0 as $\epsilon \to 0$, and we have that

$$\int_M \langle D^+D^+\ast Z, Z \rangle dV = \int_M \left[ \frac{1}{4} |\Delta Z|^2 + \frac{7}{24} R|\nabla Z|^2 + \frac{1}{12} R^2 |Z|^2 \right] dV \geq 0$$

with equality if and only if $Z = 0$, since $R > 0$. Therefore $H^2 = \{0\}$. The proof for self-dual metrics is analogous.

Finally, we prove the corollaries from the Introduction:
Proof of Corollary 1.7. For $k = 3$ this is the standard metric on $S^4$, which is rigid. Hitchin’s metrics, $\{g_k\}_{k \geq 4}$, all have singular set $\Sigma = \mathbb{R}P^2$ with self-intersection $-2$. Theorem 1.3 for self-dual metrics implies that

$$\text{Ind}^{SD}(S^4, g_k) = 15 - 4 \cdot 1 - 4 \cdot 2 = 3.$$  \hfill (5.12)

Since each $g_k$ is an unobstructed self-dual orbifold-cone metric with $\dim(H^0) = 3$, we have

$$3 - \dim(H^1) = 3,$$  \hfill (5.13)

which implies that $\dim(H^1) = 0$. Consequently, using the remarks about the Kuranishi map from the Introduction, these metrics are rigid. \hfill $\square$

Proof of Corollary 1.8. Let $g$ be an unobstructed self-dual orbifold-cone metric on $(S^4, \Sigma^j)$, where $\Sigma^j$ is diffeomorphic to an orientable surface of genus $j \geq 1$. We know that $\chi(\Sigma^j) = 2 - 2j$ and $[\Sigma^j]^2 = 0$. Therefore

$$\text{Ind}^{SD}(S^4, g) = \frac{1}{2}(15 - 29 - 2j) - 4(2 - 2j) + 4 \cdot 0 = 7 + 8j.$$  \hfill (5.14)

Since $g$ is unobstructed, using Proposition 5.1 we have the inequality

$$7 - \dim(H^1) \geq \dim(H^0) - \dim(H^1) = 7 + 8j.$$  \hfill (5.15)

However, since $j \geq 1$ this inequality cannot hold, which proves the second part of the corollary. \hfill $\square$

Proof of Corollary 1.9. Let $g$ be an unobstructed self-dual orbifold-cone metric on $S^4$ with singular set $\Sigma^j$, a smoothly embedded surface diffeomorphic to $j\mathbb{R}P^2$. We know that $\chi(j\mathbb{R}P^2) = 2 - j$. Therefore

$$\text{Ind}^{SD}(S^4, g) = \frac{1}{2}(15 - 29 - 0) - 4(2 - j) + 4[\Sigma^j]^2 = 7 + 4j + 4[\Sigma^j]^2.$$  \hfill (5.16)

Since $g$ is unobstructed we have the inequality

$$7 + 4j + 4[\Sigma^j]^2 \leq \dim(H^0).$$  \hfill (5.17)

Using the bounds on $\dim(H^0)$ in Proposition 5.1 we see that

$$[\Sigma^j]^2 \leq \begin{cases} -\frac{3}{4} & \text{when } j = 1 \\ -2 & \text{when } j = 2 \\ -\frac{1}{2} - j & \text{when } j \geq 3. \end{cases}$$  \hfill (5.18)

Combining these restrictions with the list of possible values of the self-intersection numbers (proved by Massey) (1.11), completes the proof. \hfill $\square$
Proof of Corollary 1.10. Let $g$ be an unobstructed LeBrun self-dual orbifold-cone metric on $n\#\mathbb{CP}^2$, with singular set $\Sigma = S^2$ and cone angle $2\pi/p$. Clearly, for $n \geq 3$, the moduli space of such metrics is of dimension $3n - 6$, which is obtained by counting the moduli space of monopole points modulo the action of the group of hyperbolic isometries. We know that

\begin{equation}
\chi(\Sigma) = (n + 2) \text{ and } [\Sigma]^2 = n.
\end{equation}

Therefore

\begin{equation}
\text{Ind}^{SD}(n\#\mathbb{CP}^2, g) = \frac{1}{2}(15(n + 2) - 29 \cdot n) - 4 \cdot 2 + 4 \cdot n = -3n + 7.
\end{equation}

Since $g$ is unobstructed and $\dim(H^0) = 1$ we have that

\begin{equation}
\dim(H^1) = 3n - 6.
\end{equation}

Since the dimension of the moduli space is greater than or equal to $3n - 6$, the action of $H^0$ on $H^1$ must be trivial. Therefore, the dimension of the moduli space is exactly $3n - 6$, so any sufficiently close self-dual deformation of $g$ must be $S^1$-equivariant. \[\square\]

Proof of Corollary 1.11. We can see that $H^2$ consists of parallel sections of $S^2_0(\Lambda^2_2)$ using the argument in the proof of Lemma 1.6 with $R = 0$, so $H^2 = \{0\}$ by assumption. Similarly, since $Ric = 0$, the standard Bochner argument works in the orbifold-cone setting to show that $H^0$ consists of parallel vector fields, and therefore $H^0 = \{0\}$, also by assumption. Consequently, using the facts about the Kuranishi map from the Introduction, the moduli space is smooth near $g$, and the dimension is computed by Theorem 1.3, yielding (1.17). \[\square\]

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