Remarks on a financial inverse problem by means of Monte Carlo Methods

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Abstract. Estimating the price of a barrier option is a typical inverse problem. In this paper we present a numerical and statistical framework for a market with risk-free interest rate and a risk asset, described by a Geometric Brownian Motion (GBM). After approximating the risk asset with a numerical method, we find the final option price by following an approach based on sequential Monte Carlo methods. All theoretical results are applied to the case of an option whose underlying is a real stock.

1. Introduction
Recently, the growth of financial markets, especially due to the diffusion of financial instruments as derivatives, has forced analysts to create efficient mathematical models to estimate risk assets trends. Since a risk asset can be identified, formally, with a Stochastic process respect to a fixed probability space, this problem is generally an inverse problem and involves the solution of differential equations and the estimation of such parameters.
Here we deal with European double barrier options, a particular kind of exotic options, which differ from vanilla options for the complexity of their financial structure. More in detail, this concerns the definition of the underlying or payoff function. A double barrier option is characterized by the presence of two numerical values $L, U$, $(L < U)$, called barriers. Moreover, two cases are possible: (i) a Knock-In barrier option that comes to life when the value of the underlying arrives at one of the barrier values; (ii) a Knock-out Barrier Option that concludes its life when the value of the underlying arrives at one of the barrier values.
Generally, a barrier option is less expensive than the vanilla option with the same underlying, strike price and maturity. Conversely, respect to the last one, it has a higher riskiness degree: the former property implies that traders are attracted more by a barrier option, the last property implies that a barrier option is suitable to a vanilla one when the underlying value has a regular trend. The main reference for options, and in general for derivatives, is in [8]; in particular, for the problem of option pricing we also refer to [2] and [3].
From a mathematical point of view, the problem of barrier option pricing is addressed by Monte Carlo methods and in this paper we adopt a sequential Monte Carlo approach: for a complete description of this topic see [9], [4] and [5], for the implementation issues you can also see [6] and [7]. Finally, we propose a stochastic model to describe the market and a numerical procedure to give a barrier option price.
The paper is organized in the following way: in the section 2 we give some mathematical
preliminaries; the section 3 is devoted to the pricing procedure and a real real case study; in the last section we draw the conclusions.

2. Mathematical and Financial preliminaries

The option pricing can be interpreted as a classical example of inverse problem. From a formal point of view, we have two stochastic processes \( Y_t \) and \( X_t \), where \( Y_t \) represents a phenomenon and \( X_t \) its stochastic casual factors, and we suppose that a functional dependence between them, indicated with \( F_t \), exists:

\[
Y_t = F_t(X_t, \theta_t),
\]

where \( \theta_t \) is a time-depending vector. An inverse problem aims at the estimation of \( F_t \) and \( \theta_t \), given a set of observations of \( Y_t \).

In our context, the stochastic process \( Y_t \) is identified as the option price, the stochastic process \( X_t \) as the underlying, the parameter vector is the risk-free interest rate and the volatility of the underlying. In this paper we deal with the determination of the function \( F_t \) and, here, we assume that in our market the interest rate and the volatility are known.

In the following we fix a time interval \([0; T]\), a Brownian motion \( W_t \) and a probability space \((\Omega; F_t; \mathbb{P})\) for all stochastic processes; we denote the constant risk-free interest rate with \( r \). The lognormal risk asset \( S_t \) with constant volatility \( \sigma \) and drift \( \mu \) solves the following stochastic differential equation:

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW_t. \tag{1}
\]

The drift parameter is the difference between the interest rate \( r \) and a constant dividend \( q \). The no arbitrage vanilla option price \( P^{bs}_0 \) at the instant 0 is given by the Black-Scholes formula (the Black-Scholes model is described in details in [1]):

\[
P^{bs}_0 = \begin{cases} 
S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2) & \text{if } a) \\
Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1) & \text{if } b)
\end{cases} \tag{2}
\]

where \( a) \) refers to a Call, which gives the traders the right to buy a quantity of the underlying at the maturity by paying an amount \( K \) (named strike price), while \( b) \) refers to a Put, which gives the right to sell a quantity of the underlying at the maturity by receiving an amount \( K \) (as in the case of a Call, it is called strike price). Moreover, in the previous formula, it holds:

\[
d_1 = \frac{\log(S_0/K) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}, \\
d_2 = d_1 - \sigma \sqrt{T} \\
\Phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx.
\]

The pay-off function \( H \) of an option is defined in the following way:

\[
H(S_t) = \begin{cases} 
\max\{S_T - K, 0\} & \text{if Call} \\
\max\{K - S_T, 0\} & \text{if Put}
\end{cases}
\]

In the following we indicate the expected value of a random variable \( X \) with \( E[X] \) and the indicator function of an interval \([a; b]\) with \( 1_{[a,b]}(x) \).

The prices of a knock-in and knock-out barrier option at 0, with underlying \( S_t \) and pay-off function \( H(S_t) \), are expressed in the following way:
We deal with the resolution of the described problems in the next section.

3. Numerical pricing procedure

We describe a numerical procedure to apply the model defined in the previous section. We decompose the time interval \([0; T]\) with \(N\) equidistant points \(0 = t_0 < ... < t_{N-1} = T\) and we indicate the value of the underlying at the instant \(t_n\) and the discretization step respectively with the symbol \(S_n\) and \(h\); for every \(n\), the variables \(H(S_N)\) and \(1_{[L,U]}(S_n)\) are uncorrelated, the following approximations of the knock-out and knock-in barrier option prices are valid:

\[
\begin{align*}
P^{\text{out}}_0 &= e^{-rT} E \left( H(S_T) \prod_{t \in [0; T]} 1_{[L,U]}(S_t) \right) \\
\end{align*}
\]

At every step \(n\), our numerical procedure consists in approximating the value \(S_n\) of the previous system by the mean of \(M\) particles and in estimating the price values in \((4)\) by a Monte Carlo approach. More in detail, our schema is composed of the following steps: i) generation of a number \(M\) of the particles \(S^{(m)}_n\), with \(1 \leq m \leq M\) and \(0 \leq n \leq N - 1\); ii) rejection of the underlying values (the particles) with the lowest probability to stay in the interval \([L, U]\); iii) determination of the values in \((4)\).

We start to analyze the point i). We set \(X_t := \log S_t\) and we transform the system in \((1)\) into the following one:

\[
\frac{dX_t}{dt} = \mu + \sigma dW_t.
\]

The numerical solutions \(X^{(m)}_t\) of this equation, which represent the logarithms of the particles \(S^{(m)}_n\), are found by applying a first-order numerical method, as it follows:

\[
X^{(m)}_{n+1} = \mu^n_m + \sqrt{h} \sigma^{(m)} \tilde{z}^{(m)}_{n+1} \sim N(0; 1) \quad \mu^n_m = X^{(m)}_n + h \mu
\]

As it concerns the point ii), we re-sample these just obtained particles. This procedure is made by two suitable functions \(g\) and \(G\) defined in the following way:

\[
g(X^{(m)}_{n+1}; X^{(m)}_n) := \mathbb{P}(X^{(m)}_{n+1} \in [L; U] | X^{(m)}_n) = \frac{1}{\sigma \sqrt{2 h \pi}} \int_{\log L}^{\log U} e^{-\frac{(x-\mu^{(m)}_n)^2}{2h^2 \sigma^2}} dx.
\]

\[
G(X^{(m)}_{n+1}, X^{(m)}_n) = \begin{cases} 
 g(X^{(m)}_{n+1}; X^{(m)}_n), & \text{if } X^{(m)}_{n+1} \in [\log L; \log U[ \\
 0, & \text{otherwise}
\end{cases}
\]

We briefly explain the re-sampling technique. We only choose the particles with the highest values of the function \(G\). At every step, we summarize the values of the function \(G\) until this sum does not become greater than a drawn uniform number \(u_k\), with \(k\) an integer, and we indicate the number of the addends with \(s_k\); at this point, we set all the previous particles equal to \(s_k\)-th particle. For every time step, this procedure stops when the sum of these indexes \(s_k\) is equal to \(M\), in order to have always a number \(M\) of particles.
For the point iii), estimators for the knock-out and knock-in price $P_{\text{out}}$ and $P_{\text{in}}$ have been determined by making use of the law of large numbers:

\[
\begin{aligned}
P_{\text{out}}^0 &= e^{-rT} \left( \frac{1}{M} \sum_{m=1}^{M} H(s_{N-1}^{(m)}) \prod_{n=0}^{N-1} \frac{1}{M} \sum_{m=1}^{M} 1_{[L,U]}(S_n^{(m)}) \right) \\
P_{\text{in}}^0 &= P_{0}^{\text{bs}} - P_{\text{out}}^0.
\end{aligned}
\]

(8)

The algorithm which implements our schema is summarized in the following. The input variables are: i) the upper bound $T$ of the time interval; ii) the number $N$ of time steps; iii) the number $M$ of particles at every time step; iv) the barriers $L$ and $U$; v) the strike price $K$; vi) the risk-free interest rate $r$; vii) the volatility $\sigma$; vii) the current value of the underlying $S_0$; viii) the theoretical Black-Scholes vanilla price $P_{0}^{\text{bs}}$.

The output variables are: i) the knock-out barrier option price $P_{\text{out}}^0$; ii) the knock-in barrier option price $P_{\text{in}}^0$.

We conclude the section with the application of the proposed framework to a real case. More in detail, we have studied the pricing problem of a knock in down put (i.e. in our model we have set $L = -\infty$ and $U = 0$), with Volkswagen company stocks as underlying and 6M EURIBOR as the risk-free interest rate. We have applied our procedure and compared the values and the ones of a standard Monte Carlo method (as presented in [10]) with the real price $P = 0.36 \, \text{€}$ for different values of $N$ and $M$ between 10 and 100. In the following we list all the values of the input variables:

$$(r, q, S_0, B, K, \sigma, T, P_{0}^{\text{bs}}) = (0.0056, 0.00005, 75, 100, 0.09, 1 \, \text{year}, 78.46 \, \text{€})$$

We have inserted the results into the following table, constructed in the following way: i) the first and the second column contain respectively the values of $N$ and $M$; ii) the third, the fourth and the fifth columns contain respectively the estimations of the price by the algorithm (indicated with Sequential), the corresponding absolute error values (indicated with Error Seq) and the average time to complete a simulation (indicated with Time Seq); iii) the sixth, the seventh and the eighth columns contain respectively the approximations of the price by the standard Monte Carlo (indicated with Standard method), the corresponding absolute error values (indicated with Error Stan) and the average time to complete a simulation (indicated with Time Stan). All the numerical values have been expressed in Euro.

| N  | M  | Sequential | Error Seq | Time Seq | Standard | Error Stan | Time Stan |
|----|----|------------|-----------|----------|----------|------------|-----------|
| 200| 200| 0.68       | 0.31      | 1.11     | 0.99     | 0.63       | 0.02      |
| 200| 150| 0.44       | 0.08      | 0.83     | 0.79     | 0.43       | 0.02      |
| 150| 150| 0.60       | 0.24      | 0.64     | 0.33     | 0.03       | 0.13      |
| 100| 150| 0.42       | 0.07      | 0.47     | 1.06     | 0.70       | 0.03      |
| 100| 100| 0.60       | 0.24      | 0.43     | 0.69     | 0.33       | 0.01      |
| 80 | 80 | 0.55       | 0.19      | 0.18     | 0.73     | 0.37       | 0.01      |
| 80 | 20 | 0.35       | 0.01      | 0.06     | 0.51     | 0.15       | 0.003     |
| 60 | 50 | 0.16       | 0.26      | 0.09     | 0.77     | 0.41       | 0.004     |
| 60 | 30 | 0.37       | 0.01      | 0.06     | 1.43     | 1.07       | 0.003     |
| 40 | 50 | 0.04       | 0.33      | 0.07     | 1.23     | 0.87       | 0.003     |
| 20 | 50 | 0.43       | 0.07      | 0.07     | 0.34     | 0.02       | 0.01      |
| 20 | 20 | 0.84       | 0.48      | 0.03     | 0.85     | 0.49       | 0.003     |
| 10 | 20 | 0.78       | 0.42      | 0.01     | 1.49     | 1.13       | 0.003     |

Table 1. Simulations of barrier option prices for different values of $N$ and $M$. 
**Algorithm 1** Algorithm for Barrier Option pricing

**Require:** $T, M, N, L, U, K, r, \sigma, S_0, P_{bs}$

**Ensure:** $P_{0}^{\text{(out)}}$, $P_{0}^{\text{(in)}}$

\begin{align*}
X_1^{(m)} &= \log S_0^{(m)}; \quad h = \frac{T}{N} \quad m = 1, ..., M. \quad \text{//Inizialization}

\text{for } n = 1, N - 1 \text{ do} \\
\quad \text{for } m = 1, M \text{ do} \\
\quad \quad X_n^{(m)}_{n+1} &= X_n^{(m)} + hr + \sqrt{h} \sigma z_n^{(m)}; \quad z_n^{(m)} \sim N(0; 1).
\quad \quad \mu_n^{(m)} &= X_n^{(m)} + hr. \quad \text{//Generation of the values } X_n^{(m)}
\quad \quad g(X_n^{(m)}; X_n^{(m)}) &= \frac{1}{\sigma \sqrt{2\pi}} \int_{\ln L}^{\ln U} e^{-\frac{(\ln x - \mu_n^{(m)})^2}{2\sigma^2}} \, dx.

\text{if } X_{n+1}^{(m)} \in \{L; U\} \text{ then}
\quad G_1(X_{n+1}^{(m)}; X_n^{(m)}) &= g(X_{n+1}^{(m)}; X_n^{(m)}) \quad G(X_{n+1}^{(m)}; X_n^{(m)}) = \frac{G_1(X_{n+1}^{(m)})}{\sum_{k=1}^{M} G_1(X_{n+1}^{(m)}; X_n^{(m)})}.
\text{else } G(X_{n+1}^{(m)}; X_n^{(m)}) = 0. \quad \text{//Computing of the functions } g \text{ and } G
\text{end if}
\text{end for}
\text{end for}
\text{for } m = 1, M \text{ do} \quad (G = 0 \quad j = 1 \quad u^{(m)} \sim Uniform(0, 1))
\quad \text{while } G < u^{(m)} \text{ and } j \leq M \text{ do} \quad \text{//Re-sampling of the particles.}
\quad \quad G = \sum_{k=1}^{j} G(X_{n+1}^{(m)}; X_n^{(m)}) \quad j \leftarrow j + 1.
\text{end while}
\quad X_{n+1}^{(k)} = X_{n+1}^{(j)}; \quad G(X_{n+1}^{(k)}; X_n^{(k)}) = G(X_{n+1}^{(j)}; X_n^{(j)}) \quad k = 1, ..., j - 1.
\text{end for}
\text{end for}

S_n^{(m)} &= \exp(X_n^{(m)});
\quad P_{0}^{\text{(out)}} = e^{-rT} \left( \frac{1}{M} \sum_{m=1}^{M} H(S_{N-1}^{(m)}) \right) \prod_{n=0}^{N-1} \frac{1}{M} \sum_{m=1}^{M} 1_{L; U}(S_n^{(m)})
\quad P_{0}^{\text{(in)}} = P_{bs}^{0} - P_{0}^{\text{(out)}}. \quad \text{//Option pricing}
Our results suggest that:

- for high values of $N$ and $M$ both the sequential and the standard Monte Carlo methods have similar results in mean;
- for low values of $N$ and $M$ the sequential Monte Carlo method gives better results than a standard Monte Carlo procedure.

In conclusion, our method is more suitable than a standard Monte Carlo one respect to low values of $N$ and $M$: this is due in particular to three factors: i) the discretization of the underlying; ii) the re-sampling technique; iii) the chosen price estimator.

4. Conclusions

The procedure of pricing the financial options is an inverse problem that involves several numerical and statistical issues, as for example the Monte Carlo methods and differential equation methods. We have proposed a numerical procedure to estimate barrier options under the assumptions of the Black-Scholes model: (i) a risk asset described by a Geometric Brownian motion; (ii) arbitrages are avoided; (iii) short sellings are allowed.

In a future work, we will extend the proposed approach to the case of a non constant interest rate and volatility and we will develop a model for their estimation.

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