Analytical approximation schemes for solving exact renormalization group equations. II
Conformal mappings.

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Abstract

We present a new efficient analytical approximation scheme to two-point boundary value problems of ordinary differential equations (ODEs) adapted to the study of the derivative expansion of the exact renormalization group equations. It is based on a compactification of the complex plane of the independent variable using a mapping of an angular sector onto a unit disc. We explicitly treat, for the scalar field, the local potential approximations of the Wegner-Houghton equation in the dimension $d = 3$ and of the Wilson-Polchinski equation for some values of $d \in [2, 3]$. We then consider, for $d = 3$, the coupled ODEs obtained by Morris at the second order of the derivative expansion. In both cases the fixed points and the eigenvalues attached to them are estimated. Comparisons of the results obtained are made with the shooting method and with the other analytical methods available. The best accuracy is reached with our new method which presents also the advantage of being very fast. Thus, it is well adapted to the study of more complicated systems of equations.

Key words: Exact renormalisation group, Derivative expansion, Critical exponents, Two-point boundary value problem
PACS: 02.30.Hq, 02.30.Mv, 02.60.Lj, 05.10.Cc, 11.10.Gh, 64.60.Fr

In a previous article [1] we presented two analytical approaches for studying the derivative expansion of the exact renormalization group equation (ERGE,
for reviews and recent pedagogical introductions see [2,3]). The two methods, based on the commonly used field expansion, were shown to be more efficient than the current approaches [4,5,6] which implicitly assumed that the simple field expansion converges in $[0, \infty]$ whereas it does not [6]. In the first method, introduced in [7], the infinite-boundary condition is explicitly accounted for via an auxiliary differential equation (ADE) whereas, in the second, the solutions looked for are approximated by generalized hypergeometric functions (HFA). Another method very similar to HFA has almost simultaneously been proposed in [8], it looks for the solutions under the form of Padé approximants. Both three methods work well (though the ADE method has a wider range of application) but they are rather heavy to implement (see table 1). In the present work we show that a simple conformal mapping onto the unit disc of a suitably chosen angular sector of the complex plane of the independent variable, compactifies the originally infinite integration domain so as to make the series of the field expansion in the new variable convergent on the whole disc of unit radius.

The paper is organized as follows. In section 1, the principle of the mapping method is introduced with the example of the Wegner-Houghton RG flow equation [9] in the local potential approximation (LPA). For the fixed point solution of this equation in three dimensions, one approximately knows the location of the closest singularity in the complex plane of the independent variable [10]. We show that the best convergence properties provided by the method correspond to the largest angular sector compatible with the analyticity of the solution in the original variable. The calculations of the eigenvalues with the mapping method is shown to be easy and we provide the best estimates ever obtained up to now of the fixed point solution and the eigenvalues. In section 2, we consider the Wilson-Polchinski RG flow equation [11,12] in the LPA. This equation allows us to illustrate the efficiency of the method for different values of the dimension $2 < d \leq 3$. Again, we provide the best results ever obtained up to now in three dimensions. We determine the locations of the critical and multicritical fixed points for $d = 3, 8/3$ and $5/2$ together with the associated eigenvalues for $d = 3$ and $8/3$ with an excellent accuracy. We pursue the determination of the critical fixed point for values of $d$ very close to 2. (At $d = 2$, the type of solutions we track disappears.) In section 3, we look at the second order of the derivative expansion $O(\partial^2)$ by considering explicitly the Morris RG equations [13] in three dimensions. These equations are much more difficult to treat than the previous ones, even in the LPA. Nevertheless, we are able to determine both the fixed point and the eigenvalues with an accuracy approaching that obtained with the shooting method [13,14]. An estimate of the subcritical “odd-exponent” is obtained for the first time $O(\partial^2)$. It, however, does not compare favourably with existing estimates [15,16]. We discuss the probable reasons of this disagreement. Finally we summarize and conclude.
1 The Wegner-Houghton flow equation in the LPA

As detailed in [1], to which article the reader is invited to refer for some basic definitions if necessary, the study of the existence of fixed points and of their stability in the derivative expansion of an ERGE, amounts to look for regular solutions in \( \phi \in [0, \infty] \) of coupled nonlinear ordinary differential equations (ODE). Here \( \phi \) is the (constant) scalar field. Hence the two boundaries associated to the ODEs under study are (see [1]):

1. \( \phi = 0 \) where the symmetry of interest is imposed to the solution
2. \( \phi = \infty \) where a specific behaviour in approaching this point is imposed to the solution

This problem of solving differential equations with boundary conditions may be numerically studied using a shooting or a relaxation method but these methods are not always efficient. Instead, one has often recourse to an analytical method based on an expansion of the solutions looked for in power series of \( \phi \) about a certain value (the origin \( \phi_0 = 0 \) [4], or the minimum of the potential [5,6]). One may also consider \( \phi_0 \) as an adjustable parameter [6] with a view to improve the convergence of the series. Unfortunately these series do not converge in the whole range \( \phi \in [0, \infty] \) [6,10] and the condition at the second boundary cannot be explicitly imposed.

At this stage, it is useful to consider a concrete example.

1.1 Fixed point equation

Let us take as a paradigm the Wegner-Houghton equation in the LPA [9]:

\[
\dot{U} = \ln [1 + U''] + dU + \left(1 - \frac{d}{2}\right) \phi U',
\]

in which \( d \) is the spatial dimension that we shall set equal to three in this section, \( U \) stands for the potential \( U(\phi, t) \), with \( t = -\ln (\Lambda/\Lambda_0) \) where \( \Lambda \) is the “running” momentum scale of reference compared to a fixed scale \( \Lambda_0 \), and \( \dot{U} = \partial U / \partial t, U' = \partial U / \partial \phi, U'' = \partial^2 U / \partial \phi^2 \).

For the fixed point equation with \( d = 3 \):

\[
\ln [1 + U''] + 3U - \frac{\phi}{2} U' = 0,
\]

the conditions at the boundaries for the solution \( U(\phi) \) are:
\[ U(0) = \gamma, \quad (3) \]
\[ U'(0) = 0, \quad (4) \]
\[ U(\phi) \to G \phi^6, \quad (5) \]

in which \( \gamma \) (or \( G \)) is the integration constant the value \( \gamma^* \) (or \( G^* \)) of which has to be determined. The connection parameter \( \gamma^* \) (or \( G^* \)) is often considered as a substitute to the solution \( U^*(\phi) \) because this latter is deduced from the knowledge of \( \gamma^* \) (or \( G^* \)) by a simple numerical integration of the ODE.

For convenience, instead of \( \gamma \), we shall deal with \( r = U''(0) \) from which, according to (2, 3, 4), one deduces \( \gamma \) via the relation:

\[ \gamma = -\frac{1}{3} \ln [1 + r]. \]

Let us expand \( U(\phi) \) about \( \phi = 0 \) up to a finite order \( 2M \). Accounting for the conditions (3, 4) at the first boundary (the origin \( \phi = 0 \)), it comes:

\[ U_M(\phi) = -\frac{1}{3} \ln [1 + r] + \frac{1}{2} rz + \sum_{n=2}^{M} a_n(r) z^n, \]
\[ z = \phi^2, \quad (6) \]

in which we have introduced \( z \) for convenience and the coefficients \( a_n(r) \) are determined as functions of \( r \) so that the fixed point equation (2) is satisfied order by order in powers of \( z \). For example, the two first coefficients read:

\[ a_2(r) = -\frac{r(1 + r)}{12}, \]
\[ a_3(r) = \frac{r(1 + r) (1 + 7r)}{360}. \]

It remains to determine the value \( r^* \) of \( r \) so that the condition (5) at the second boundary is satisfied. To impose this condition, we need an evaluation for a large value of \( z \) of the Taylor polynomial of degree \( M \) introduced in (6). This evaluation is possible if the radius of convergence \( R_c \) of the Taylor series in powers of \( z \) is infinite. However, in general, in the derivative expansion of the ERGE, this is not the case: \( R_c \) is always finite due to the presence of singularities in the complex plane of \( z \). For the unique nontrivial fixed point solution of (2), knowing the corresponding \( r^* \), the singularity the closest to the origin has been numerically located in the complex plane of \( z \) at the following point \( z_0 \):
\[ |z_0| = 9.7344, \quad \text{Arg} (z_0) = 0.514 \pi = \alpha_c \pi, \quad (8) \]
\[ \text{which implies that} \quad R_c = 9.7344. \quad (10) \]

From the known localization of the singularity the closest to the origin, we may assume that the solution \( U^* (\phi) \) we are looking for is analytical in the angular sector which symmetrically straddles the positive real axis, has its vertex located on the negative real axis at \( -|z_0| \) and its angle equal to \( \text{Arg} (z_0) \) (see figure 1). This assumption implies that none of the eventual supplementary singularities lies inside this sector.

The basis of our approach consists in exploiting the analyticity of the solution \( U^* \) looked for in the range \( z \in [0, \infty[ \). Actually, if this solution exists then there is an angular sector involving the whole positive real axis of the complex plane of \( z \), and including the origin, in which \( U^* \) is analytic. We thus introduce a conformal mapping of an angular sector which symmetrically straddles this positive real axis onto the unit disc centered at the origin of the complex plane of a new variable \( w \) defined by:

\[ w = (1 + z/R)^{1/\alpha} - 1 \quad (11) \]

in which \( R \) and \( \alpha \) characterize the position of the vertex and the angle of an angular sector of the complex plane of \( z \) as shown in figure 1.

The inverse transformation is:

\[ z = R \left[ \left( \frac{1 + w}{1 - w} \right)^\alpha - 1 \right]. \quad (12) \]

In particular, if we adjust the angular sector in such a way that the presently known closest-to-the-origin singularity [see equations (8, 9)] lies right on one of its edge i.e.:

\[ \alpha = \alpha_c = 0.514, \quad (13) \]
\[ R = R_c, \quad (14) \]

then the resulting Taylor series of the solution \( U^* \) expressed in powers of \( w \) should converge on the whole disc \( |w| < 1 \).

Consequently, to obtain an auxiliary condition for determining the value \( r^* \), we proceed as follows.
Fig. 1. When the interior of an angular sector involving the positive real axis of $z$ defines a region of analyticity for the function $U^*(\phi)$ (with $z = \phi^2$), it may be mapped onto the interior of a circle of unit radius for the complex variable $w$ defined by eq (11). In the present case the singularity $z_0$ of $U^*(\phi)$ the closest to the origin determines the analyticity domain provided the other singularities lay outside the angular sector containing the origin.

We deduce from (6, 12) the following Taylor polynomial:

$$\tilde{U}_M (w) = -\frac{1}{3} \ln [1 + r] + \sum_{n=1}^{M} c_n (r) w^n,$$

where $\tilde{U}_M (w) = U_M [\phi (w)]$ with $\phi (w)$ given by (7, 12).

Then, from the convergence of the resulting series in powers of $w$, we may:

(1) either, impose when $w \to 1$ (the counterpart of $z \to \infty$), that $\tilde{U}_M (w)$
satisfies the counterpart of (5), (2) or, in a less rigorous but often efficient manner, implicitly exploit the convergence of the Taylor series in powers of \(w\) on the whole unit disc by imposing, as proposed in [17], that the \(M\)th coefficient \(c_M(r)\) vanishes (this assumes that the remaining higher terms of the series do not contribute too much).

Choosing one of these two variants yields an auxiliary condition under the form of a polynomial equation for \(r\) the zeros of which are candidates for the solution \(r^*\) we are looking for.

For the values given by \([10, 13, 14]\), and the variant (1) of the method as example, the real zeros obtained are distributed almost similarly to those displayed in figure 1 of [1]. An important difference with [1] however, is that it is easier to follow the zero of interest as \(M\) grows. The estimation we obtain this way for \(r^*\) is excellent since we get 28 stabilized digits for \(M = 145\):

\[
r^* = -0.4615337201162071199657576484.
\] (16)

It appears that the values of \(R\) and \(\alpha\) which correspond to the location of the closest known singularity provide the best convergence for \(r^*\). Of course, this situation is particular because, thanks to the results of [10], we know the location of the closest singularity. In general this is not the case and, as we shall illustrate in section 2.1.1, the only information practically accessible is a rough estimation of \(R_c\) whereas \(\alpha_c\) is not known. It is thus justified to try different values of \(R\) and \(\alpha\).

Let us define a practical measure of the accuracy of the results as \(M\) grows by the number of stabilized digits of \(r^*\):

\[
N_{\text{acc}} = -\log \left| 1 - \frac{r^*_M - 1}{r^*_M} \right|.
\] (17)

Figure 2 illustrates well the fact that, if one chooses \(R \neq R_c\) and \(\alpha \neq \alpha_c\), then, in general, one gets a worse convergence towards the (hopefully true) value of \(r^*\).

But, there are also other values for the couple \(\{R, \alpha\}\) which provide essentially the same convergence, in particular when \(R\) is chosen greater than \(R_c\). Notice that, in such a case, \(\alpha\) must be smaller than \(\alpha_c\) otherwise the singularity located at \(z_0\) would have an image in the interior of the unit circle.

\footnote{Taking into account only the criterium of analyticity, one may choose any value of \(R\) and \(\alpha\) such that \(R < R_c\) and \(\alpha < \alpha_c\). But, it often appears that the simultaneous largest possible values of \(R\) and \(\alpha\) yield the best convergences.}
in the complex plane of $w$. These favourable cases corresponding to $R > R_c$ exist because the closest singularity is not located on the negative real axis of $z$. With the Wilson-Polchinski equation [11][12], studied in section [2], one encounters situations where the condition $R \leq R_c$ is necessary, indicating that the closest singularity is right on (or very close to) the negative real axis of $z$. 
### 1.1.1 Eigenvalues

A linearization about the solution $U^*(\phi)$ of (2), using $U(\phi, t) = U^*(\phi) + \epsilon e^{\lambda t}g(\phi)$ with $\epsilon$ a small parameter, provides the eigenvalue equation:

$$\frac{g''}{1+U''} - \frac{\phi}{2}g' + (3 - \lambda)g = 0,$$

which may be studied by expanding both $U^*$ and $g$ in powers of $\phi$ (or $z$) as done above in (6, 7) for the fixed point.

Accounting for the arbitrariness of the normalization of the eigenfunction, the conditions at the origin depend on the symmetry of the solution looked for:

- even case: $g(0) = 1$, $g'(0) = 0$,
- odd case: $g(0) = 0$, $g'(0) = 1$,

whereas the condition at infinity is:

$$g(\phi) \xrightarrow{\phi \to \infty} \phi^{6-2\lambda}. \quad (18)$$

Once $r^*$ is fixed to the value (16), the coefficients of the series for $g$ depend only on $\lambda$ which plays the role of the connection parameter in the preceding fixed point equation. We perform the conformal mapping $z \to w$ defined by (11) with $R$ and $\alpha$ fixed to the preceding values, namely $R = 9.7344$ and $\alpha = 0.514$ and, because the power of $\phi$ in (18) depends on $\lambda$ we use the variant (2) of the method, although we could use again the variant (1) with the condition $(6 - 2\lambda)g - \phi g' = 0$ imposed when $w \to 1$ on the transformed series. The real zeros of the resulting polynomial in $\lambda$ are displayed in figure 3 which shows an exceptionally clear distribution of the spectrum of the eigenvalues.

For simplifying the notation, we generically note $\nu$ the inverse of each positive eigenvalue $\lambda$ and $\omega$ the opposite of each negative one in the even case, respectively $\bar{\nu}$ and $\bar{\omega}$ in the odd case (eventually with an index when there are several eigenvalues of the same kind for each case, see [1] for more details):

$$\nu = \frac{1}{\lambda} > 0 \quad , \quad \omega = -\lambda > 0.$$

Using an expansion up to $M = 104$, we get an estimate of $\nu$ with six supplementary digits compared to the already excellent determination by Aoki et al [6]:

$$\nu = 0.68945905616213484062727.$$
Fig. 3. Eigenvalue spectrum (even case) of the Wegner-Houghton equation in the LPA \((d = 3)\) linearized about the Wilson-Fisher fixed point. The open circles are the real zeros (as function of the degree of the Taylor polynomial) of the auxiliary condition provided by the mapping method (presently using variant (2)). The horizontal lines represent the estimates of the eigenvalues (even case). This extremely clear picture illustrates well the facility for determining the eigenvalues.

At order \(M = 40\), the estimates of the eigenvalues for both the even and odd
cases are the following:

\[ \nu = 0.68945905616, \]
\[ \omega_1 = 0.59524, \quad \tilde{\omega}_1 = 1.691340, \]
\[ \omega_2 = 2.8385, \quad \tilde{\omega}_2 = 3.999, \]
\[ \omega_3 = 5.19, \quad \tilde{\omega}_3 = 6.40, \]
\[ \omega_4 = 7.7, \quad \tilde{\omega}_4 = 9.0. \]

The values of the subleading critical exponents are in agreement with the previous estimates [18]. As in the case of \( \nu \), the accuracy of each of them may be easily improved by considering higher values of \( M \).

2 The Wilson-Polchinski flow equation in the LPA for \( 2 < d < 4 \)

The Wilson-Polchinski flow equation [11,12] in the LPA reads:

\[
\dot{U} = U'' - (U')^2 + \left(1 - \frac{d}{2}\right) \phi U' + dU.
\]

By considering this equation we aim at illustrating other aspects of the method, in particular its efficiency compared to other procedures (when \( d = 3 \)) but also the way it works when several fixed points exist (when \( 2 < d < 3 \)). A recent study of this equation for generic \( d \) may be found in [19] and the estimates that we provide below compare favourably to those obtained in [20].

2.1 Fixed point equation

The fixed point ODE reads:

\[
U'' - (U')^2 + \left(1 - \frac{d}{2}\right) \phi U' + dU = 0,
\]

the solutions of which we are looking for have the following properties\(^2\):

\(^2\) In [11] the parameter \( A \) was noted \( b \).
\[ U(0) = k, \quad \text{(20)} \]
\[ U'(0) = 0, \quad \text{(21)} \]
\[ U(\phi) \rightarrow \frac{1}{2}\phi^2 - \frac{1}{d} + A\phi^{2d/(d+2)} + O[\phi^{2(d-2)/(d+2)}], \quad \text{(22)} \]

where \( k \) (or \( A \)) has to be fixed to the value of the connection parameter \( k^* \) (or \( A^* \)).

As in the preceding section, to determine \( k^* \), we consider the following Taylor polynomial of degree \( M \) in \( z = \phi^2 \):

\[ U_M(\phi) = k + \sum_{n=0}^{M} b_n(k) z^n , \]
\[ z = \phi^2, \]

where the coefficients \( b_n(k) \) are determined as function of \( k \) such that the fixed point equation (19) is satisfied order by order in powers of \( z \). Then we perform the conformal mapping (11). Since we have a priori no information on the analyticity properties of \( U^* \) as function of \( z \), we consider \( R \) and \( \alpha \) as free parameters. We proceed by trial and error to find their best values: we look at the variation of the properties of convergence of the estimate \( k^*_M \) obtained at order \( M \) by changing the trial values of \( R \) and \( \alpha \). Let us illustrate this procedure in the case \( d = 3 \) where the value of \( k^* \) is already known with accuracy [21].

2.1.1 \( d = 3 \)

We use the variant (1) adapted to (22) by imposing the counterpart of \( dU^*/dz \rightarrow 1/2 \) when \( w \rightarrow 1 \). Setting arbitrarily \( R = 1 \) and \( \alpha = 1 \), we easily distinguish, among all the real zeros of the polynomial auxiliary condition obtained at order \( M \), a generic zero which presumably will converge to \( k^* \) when \( M \rightarrow \infty \). The observed convergence is not excellent yet, but it is sufficient to allow a rough estimate of the radius of convergence \( R_c \) for the expected \( k^* \). Following the d’Alembert or Cauchy rules, this estimate may be obtained by looking at:

\[ R_{1,n} = \left| \frac{b_n}{b_{n+1}} \right|, \]

or:

\[ R_{2,n} = \left| b_n \right|^{-1/n}, \]

for a given value of \( k \) and large values of \( n \).

In general, we have proceeded that way, by trial and error, to estimate \( R_c \). In the present case, however, the radius of convergence of the fixed point series
Table 1

Comparison between estimates of the connection parameter $k^*$, of the Wilson-Polchinski RG equation in the LPA ($d = 3$), obtained using different efficient analytical methods at order $M = 25$ of the Taylor polynomial [compare to (23)]. The “time” given in the third column is a CPU time (in seconds) corresponding to the calculation, on the same computer, using each method.

| method  | $k^*$               | time  |
|---------|---------------------|-------|
| ADE     | 0.076199400812365   | 1523.84 |
| Padé    | 0.07619940081205    | 1364.73 |
| HFA     | 0.076199400812340   | 138.58  |
| mapping | 0.0761994008160     | 2.00    |

is known to be $[1]$:

$$R_c = 5.72167.$$  

By setting straight out $R$ to this value (to save time), we get a better accuracy on the estimation of $k^*$ than with $R = 1$, and on varying $\alpha$, we observe that the accuracy is better and better when $\alpha$ grows. Finally, as shown in figure 4, the best result is obtained with $\alpha = 5/2$. This high value of $\alpha$ suggests that the singularity is located on the negative real axis and is unique. Indeed when we try values of $R$ larger than $R_c$, the zero of interest disappears contrary to the preceding Wegner-Houghton case for which the singularity the closest to the origin lies outside the negative real axis. The fact that we can choose $\alpha > 2$ (when $R = R_c$) without spoiling the property of convergence of the mapping method, suggests that this singularity is unique.

We finally get an accuracy of 49 digits for $M = 120$:

$$k^* = 0.07619940081234064145788536913234906280801814336214 \pm 6 \times 10^{-50}.$$  

(23)

Table 1 shows a comparison of efficiencies at $M = 25$ between the four methods ADE, HFA, Padé and the present mapping method. This latter method appears to be, by far, the most efficient.

The mapping method provides also a global explicit representation of the solution via the Taylor polynomial in powers of $w$ re-expressed in terms of $\phi$ (the HFA and Padé methods give also a global representation of the solution but the procedure is much more heavy). From this global representation, knowing $k^*$, it is not difficult to estimate the value of the connection parameter $A^*$ which appears in (22). With three more terms in the asymptotic behaviour of $U^*$ than in (22), we have been able to obtain the following result:

$$A^* \simeq -2.3184.$$
Fig. 4. Study of the fixed point of the Wilson-Polchinski equation in the LPA for $d = 3$: approximate number $N_{\text{acc}}$ of stabilized digits on $k^*_M$ [similar to (17)] as function of the degree $M$ of the Taylor polynomial in $w$, for different values of $\alpha$ and for $R = 5.72167$.

which is very close to the value we have determined with the shooting method [1]:

$$A^* \simeq -2.3183.$$ 

Once the fixed point has been determined, the eigenvalues are looked for with the same couple $(R = R_c, \alpha = 5/2)$ and, as in the preceding case, we easily get the eigenvalues with high accuracy. We may thus give the best estimates ever obtained up to now (see [1] for the definitions of both the eigenvalue equation and the exponents, and compare the following values with the estimates obtained in [21]):

- for $M = 75$ in the even case ($d = 3$):
\[ \nu = 0.649561773880648017614299724015827 \pm 2 \times 10^{-33}, \]
\[ \omega_1 = 0.6557459391933387407836879749684 \pm 2 \times 10^{-31}, \]
\[ \omega_2 = 3.180006512059167532314140242, \]
\[ \omega_3 = 5.912230612747701026351105, \]
\[ \omega_4 = 8.796092825413903643907, \]
\[ \omega_5 = 11.798087658336857239. \]

• for \( M = 69 \) in the odd case \( (d = 3) \):

\[ \tilde{\omega}_1 = 1.8867038380914203710417873172 \pm 5.3 \times 10^{-28}, \]
\[ \tilde{\omega}_2 = 4.52439073670772780436353, \]
\[ \tilde{\omega}_3 = 7.337650643354135387526, \]
\[ \tilde{\omega}_4 = 10.2839007240259581722, \]
\[ \tilde{\omega}_5 = 13.3361699643459431. \]

2.1.2 \( d = 8/3 \)

A new non-trivial fixed point emerges from the Gaussian fixed point when \( d \) takes on a value below each threshold [22]:

\[ d = 2 \frac{p + 1}{p}, \]

where \( p \) takes on the integer values \( 1, 2, 3, \cdots, \infty \). Each one of these dimensions corresponds to a monomial in powers of \( \phi \), in the formally expanded potential, becoming relevant with respect to the Gaussian fixed point. Hence for \( 3 \leq d < 4 \), there is only one nontrivial fixed point: the Wilson-Fisher fixed point [23] which controls the behaviour of any Ising-like critical point. Below \( d = 3 \) appears a new fixed point which controls the tri-critical behaviour. Down to \( d = 8/3 \) there are two fixed points.

To determine these two fixed points, we proceed similarly to the previous case \( d = 3 \) by trial and error in order to determine the best values for \( R \) and \( \alpha \). It appears that we easily determine the critical fixed point and get a rather good convergence for \( R = 3.5 \) and \( \alpha = 2 \). We finally obtain with \( M = 120 \) the following estimation:

\[ k_c^* = 0.16736641293800245119399231271370187 \pm 3 \times 10^{-35}. \]

It is worth indicating that the radius of convergence of the series for this value of \( k^* \) is:

\[ R_c = 3.539. \]

An estimation of this connection parameter by a shooting method gives the
following result:
\[ k_c^* = 0.16736641293. \]

For the same values of \( R \) and \( \alpha \), and the variant \( \text{(2)} \), we get the following estimates for the (critical) eigenvalues (without trying to optimize the accuracy):

- in the even case for \( M = 50 \):
  \[
  \begin{align*}
  \nu &= 0.76520486063609135 \pm 1.8 \times 10^{-16}, \\
  \omega_1 &= 0.8148967394644831495 \pm 7.4 \times 10^{-15}, \\
  \omega_2 &= 3.32048207983, \\
  \omega_3 &= 6.045184748, \\
  \omega_4 &= 8.9242446, \\
  \omega_5 &= 11.92109, 
  \end{align*}
  \]

- in the odd case for \( M = 60 \):
  \[
  \begin{align*}
  \tilde{\omega}_1 &= 2.032987010721739017 \pm 5 \times 10^{-18}, \\
  \tilde{\omega}_2 &= 4.66036178907299, \\
  \tilde{\omega}_3 &= 7.46810111086, \\
  \tilde{\omega}_4 &= 10.409630969, \\
  \tilde{\omega}_5 &= 13.4559703. 
  \end{align*}
  \]

The tri-critical fixed point is more difficult to determine than the critical one because a clear identification of the correct zero is only possible above \( M \simeq 20 \), a value which is greater than that observed in the case of the critical fixed point. Nevertheless, we get a good estimate for \( R = 19.5 \) et \( \alpha = 0.47 \) (variant \( \text{(2)}, \) and \( M = 120 \)):

\[ k_t^* = -0.0152088617493 \pm 9.2 \times 10^{-13}, \]

whereas by the shooting method we get:

\[ k_t^* = -0.015208861395, \]

The radius of convergence of the series in this case is:

\[ R_c \simeq 19.4. \]

For the same values of \( R \) and \( \alpha \), and the variant \( \text{(2)} \) we get the following estimates for the (tri-critical) eigenvalues (without trying to optimize the accuracy):

- in the even case:
\[ \nu_1 = 0.501489558, \]
\[ \nu_2 = 1.0479506, \]
\[ \omega_1 = 0.343226, \]
\[ \omega_2 = 1.83594, \]
\[ \omega_3 = 3.484, \]
\[ \omega_4 = 5.26, \]
\[ \omega_5 = 7.2, \]

- in the odd case:
\[ \tilde{\nu}_1 = 0.659916, \]
\[ \tilde{\omega}_1 = 1.0682, \]
\[ \tilde{\omega}_2 = 2.643, \]
\[ \tilde{\omega}_3 = 4.36, \]
\[ \tilde{\omega}_4 = 6.27, \]
\[ \tilde{\omega}_5 = 8.4. \]

As expected, compared to the critical fixed point, the tri-critical fixed point has one additional positive even eigenvalue \((\lambda = 1/\nu_2)\), and also one additional positive odd eigenvalue \((\lambda = 1/\tilde{\nu}_1)\).

### 2.1.3 Lower dimensions

The calculations are more and more difficult as \(d\) decreases. This is because the radius of convergence \(R_c(d)\) for a given kind of fixed point (critical or multi-critical) also decreases. Consequently, in approaching \(d = 2\), the determination of the critical fixed point itself is made more and more difficult. In fact, at \(d = 2\), the regular solution we are searching (which satisfies the condition (22) when \(\phi \to \infty\)) disappears. Indeed, the same reasoning as that followed by Morris in [24], shows that for \(d = 2\) the only solutions to the fixed point equation in the LPA are either singular at finite \(\phi\) or periodic (see also [25]).

For the sake of shortness, we have simply tested that we can determine the critical fixed point (we have not considered the eigenvalues despite the fact that they are easy to determine) down to dimensions very close to \(d = 2\). Table 2 shows some of the results for \(k^*\). It shows also that the radius of convergence of the original series in powers of \(\phi\) decreases when the location of the minimum of the critical fixed point potential increases. Consequently, because this property is presumably general in the derivative expansion, the method of [56] based on an expansion about this minimum is presumably doomed to failure before reaching \(d = 2\).

Regarding the fixed points of higher criticality, we have not tried to determine them below \(d = 5/2\). For this value of \(d\) we get the locations of the fixed points
Table 2

| $d$   | $k^*$       | $R$  | $\alpha$ | $\phi_0$ | $r_c(\phi)$ |
|-------|-------------|------|-----------|-----------|-------------|
| 3     | 0.07619940081234 | 5.7  | 2.5       | 1.905202  | 2.39        |
| 8/3   | 0.16736641293800 | 3.5  | 2         | 1.991622  | 1.87        |
| 5/2   | 0.252995492797691 | 2.7  | 2         | 2.040021  | 1.64        |
| 12/5  | 0.33045938745626 | 2    | 2         | 2.077695  | 1.51        |
| 7/3   | 0.40022546744981 | 1.5  | 2         | 2.108868  | 1.41        |
| 21/10 | 0.96514877904597 | 0.75 | 2         | 2.214026  | 1           |
| 202/100 | 1.8411524065675 | 0.55 | 2         | 2.370782  | 0.77        |

Determination of the connection parameter $k^*$ of the critical-fixed-point solutions of (19) for decreasing values of the spatial dimension $d$. $\phi_0$ stands for the location of the minimum of the fixed point potential ($U^*(\phi_0) = 0$) as determined from the present mapping method. $r_c(\phi) = \sqrt{R_c(z)}$ is the radius of convergence of the original series in powers of $\phi$. Notice that $r_c(\phi)$ decreases when the location of the minimum of $U^*$ increases and becomes smaller than $|\phi_0|$. $R$ and $\alpha$ are the parameters of the conformal mapping defined in (11). displayed in table 3.

We show in figure 5 the global solutions we get from the Taylor polynomials in $w$ for the fixed points when $d = 5/2$. They have the right $n$-well potential form with $n$ growing with the number of directions of instability of the fixed point in agreement with [22].

3 The Morris equations (LPA and second order of the derivative expansion)

At second order of the derivative expansion [$O(\partial^2)$ in short], in addition to the potential $U(\phi, t)$, a new function appears: the coefficient $Z(\phi, t)$ of the kinetic term of the action $S[\phi]$. The flow equations for $U$ and $Z$ are coupled and the reparameterization invariance of the ERGE is broken [26,13] together with the so-called scheme invariance (invariance with respect to a change of the cut-off function which introduces the running momentum scale of reference $\Lambda$). This is, at least, the case for the Wilson-Polchinski ERGE. Actually the two invariances are not independent [28] and if, instead of the action $S[\phi]$, one considers the Legendre transformed or effective action $\Gamma[\varphi]$, then the breaking of the invariances may be reduced to a single one [2]. One may even go further and find an appropriate cut-off function so that the reparameterization invariance is effectively restored. Morris [13] has thus obtained the following
| type of fixed point | $k^*$ \{mapping shooting\} | min($M$) | $R$ \{R$_c$\} | $\alpha$ |
|---------------------|-----------------------------|---------|-------------|---------|
| critical            | $\begin{cases} 0.252995492976906647431 \\ 0.252995492980 \end{cases}$ | 15      | $2.71$ \{2.71\} | 2       |
| tri-critical        | $\begin{cases} -0.043027 \\ -0.043026994781 \end{cases}$ | 50      | $9$ \{9\}     | 0.6     |
| quadri-critical     | $\begin{cases} 0.0030001 \\ 0.003000121150 \end{cases}$ | 40      | $30$ \{30\}   | 0.5     |

Table 3

Determination of the values of the connection parameter $k^*$ for the three non-trivial fixed points of (19) when $d = 5/2$. min($M$) indicates the minimal value of the degree $M$ of the Taylor polynomial at which, in the mapping method, we can unambiguously identify a zero that one may easily track when $M$ grows. The estimates obtained by using the shooting method are shown on the second line within a bracket. $R$ and $\alpha$ are the parameters of the conformal mapping (11), the $R_c$ are estimated values of the radius of convergence of the solution in terms of the original independent variable $z = \phi^2$.

two equations (written below for $d = 3$ and keeping the original notations$^3$):

\[
\dot{V} = -\frac{1 - \eta/4}{\sqrt{K}\sqrt{V'^2 + 2K}} + 3V - \frac{1}{2}(1 + \eta)xV',
\]

\[
\dot{K} = (1 - \eta/4)\left\{\frac{1}{48} \frac{24KK'' - 19(K')^2}{K^{3/2}(V' + 2\sqrt{K})^{3/2}} - \frac{1}{48} \frac{58V'''K'\sqrt{K} + 57(K')^2 + (V''')^2 K}{K^{5/2}(V' + 2\sqrt{K})^{5/2}} + \frac{5}{12} \frac{(V''')^2 K + 2V'''K'\sqrt{K} + (K')^2}{\sqrt{K}(V' + 2\sqrt{K})^{7/2}}\right\}
- \frac{1}{2}(1 + \eta)xK' - \eta K,
\]

$^3$ Going from $S[\phi]$ to $\Gamma[\varphi]$ changes the couple $\{U, Z\}$ into $\{V, K\}$.
Fig. 5. The mapping method provides a precise global explicit information on the solutions looked for. Here are shown the $n$-well potentials as obtained from the Taylor polynomials in $w(\phi)$ of the three fixed-point solutions found for the Wilson-Polchinski RG equation in the LPA when $d = 5/2$.

where $x$ stands for $\varphi$ and $\eta$ is a parameter to be adjusted in order to find a fixed point.

Let us first consider the determination of the Wilson-Fisher fixed point.

The fixed point equations correspond to:

\begin{align}
\dot{V} &= 0, \\
\dot{K} &= 0,
\end{align}

(26) (27)

where $\dot{V}$ and $\dot{K}$ stand for the rhs of (24, 25).
We look for solutions under the form of even functions of \(x\) which must satisfy the following conditions:

\[
\begin{align*}
V(0) &= k, \quad (28) \\
V'(0) &= 0, \quad (29) \\
K(0) &= K_0, \quad (30) \\
K'(0) &= 0, \quad (31) \\
V(x) &\sim G_1 x^{\frac{6}{1+\eta}}, \quad (32) \\
K(x) &\sim G_2 x^{-\frac{2\eta}{1+\eta}}. \quad (33)
\end{align*}
\]

and which have no singularity in the whole range \(x \in [0, \infty]\).

The two parameters \(k\) and \(\eta\) (or \(G_1\) and \(G_2\)) have to be adjusted to the two connection parameters \(k^*\) and \(\eta^*\) respectively (or \(G_1^*\) and \(G_2^*\)) which define the fixed point solution of interest\(^4\). By virtue of the reparameterization invariance, \(K_0\) may be set arbitrarily equal to one without changing the value \(\eta^*\) (and of the eigenvalues associated to the linearization of the flow equation about the fixed point).

As in the case of the LPA, we express the two functions as Taylor polynomials of degree \(M\) in \(z = x^2\):

\[
\begin{align*}
V_M(x) &= k + \sum_{n=0}^{M} b_n^{(1)}(k, \eta) z^n, \\
K_M(x) &= 1 + \sum_{n=0}^{M} b_n^{(2)}(k, \eta) z^n,
\end{align*}
\]

in which the coefficients \(b_n^{(1)}(k, \eta)\) and \(b_n^{(2)}(k, \eta)\) are determined so as to satisfy the fixed point equations (26, 27). To determine the values \(k^*\) and \(\eta^*\) using these two polynomials, we need two auxiliary conditions that we deduce from the conditions (32, 33) and the analyticity of the two functions \(V(x)\) and \(K(x)\) in an angular sector of the complex plane of the independent variable \(z = x^2\).

We thus introduce the same conformal mapping \(z \rightarrow w\) as defined by (11) and we adopt one of the two variants defined in section 1.1 to get the two auxiliary conditions either by directly imposing on the Taylor polynomials the counterpart of the asymptotic conditions (32, 33) when \(w \rightarrow 1\) or, less rigorously,

\[^4\] The parameters \(k^*\) and \(\eta^*\) are related to the parameter \(\sigma\) and \(\eta\) used in [13] by \(\eta^* = \eta\) and \(k^* = \frac{(1-\eta/4)}{3\sqrt{2+\sigma}}\).
by assuming the simultaneous vanishing of the two coefficients $b^{(1)}_M (k, \eta)$ and $b^{(2)}_M (k, \eta)$.

### 3.1 LPA

Let us first study the LPA of (24, 25) which then reduce to the following unique equation:

$$
\dot{V} = -\frac{1}{\sqrt{V''} + 2} + 3V - \frac{x}{2}V'.
$$

The numerical study of this flow equation is much more complicated than that of the Wegner-Houghton or Wilson-Polchinski cases. The largest degree of the Taylor polynomial reasonably accessible is, in general, smaller than in these preceding cases. Nevertheless, using the mapping method, we have correctly estimated the fixed point location and the eigenvalues with a better accuracy than the results available in the literature [13,14] (obtained by the shooting method, see table 4). For a better comparison, we have improved those latter estimates (see table 4).

For the determination of the fixed point in the LPA ($d = 3$), the maximum degree of the Taylor polynomial that we have employed is $M = 82$. The best estimation of $k^*$, given in table 4 is obtained with $R = 2$ and $\alpha = 1/2$. The radius of convergence $R_c$ of the original series in powers of $z$ is roughly equal to 2. These facts suggest that the singularity the closest to the origin lies close to the imaginary axis in the complex plane of $z$.

The process of determining the eigenvalues is similar to the cases already described in the preceding sections.

In the LPA, the eigenvalues have been estimated with shorter series than the fixed point, $M = 40$ and $M = 60$ for the even and odd cases, respectively, whereas $R$ and $\alpha$ remained unchanged. The clear distribution of the eigenvalues shown in figure 3 is preserved at the LPA and $O(\partial^2)$. This particularity of the mapping method is very interesting since it is not always easy to clearly determine the spectrum of the eigenvalues (using the shooting method for example, see table 4). In the LPA the mapping method provides essentially the same accuracy than the shooting method.

### 3.2 Second order

At the next order $O(\partial^2)$, the main difference with LPA is the number of equations: two instead of one. For the sake of shortness we shall not describe
| $k^*$ | 0.2753644064810282 | 0.258126 |
|-------|------------------|---------|
|       | 0.2753644064810124 | 0.25821491 |
|       | 0.275364406 | 0.2582144 |
| $\eta^*$ | 0 | 0.053941 |
| | | (0.0539320839) |
| | | (0.05393208) |
| $\nu$ | 0.660389431 | 0.618063 |
| | 0.660389431331 | 0.61806 |
| | 0.660389 | 0.6181 |
| $\omega_1$ | 0.6285575 | 0.8964 |
| | 0.6285575035 | 0.89727 |
| | 0.6285 | 0.8975 |
| $\omega_2$ | 3.04801 | 1.71 |
| | 3.048005033 | 1.7 |
| | – | – |
| $\omega_1$ | 1.8124863608 | 0.86562 |
| | 1.812486361 | 0.865569 |
| | – | – |
| $\omega_2$ | 4.32251050 | 2.8 |
| | 4.3225104975 | – |
| | – | – |

Table 4
Estimates of the critical and subcritical exponents of the Morris equations \[ \frac{24}{25} \] at $d = 3$ in the LPA, and at the second order of the derivative expansion. On the right of each bracket are given the results obtained by the mapping (first line) and the shooting (second line) methods used in this work and, when they exist, previous estimates obtained in \[ \cite{13,14} \].
again the different steps of the calculations.

For the fixed point, the maximum degree of the Taylor polynomial we have attained is $M = 17$. Moreover, for small values of $M$, it has been difficult to locate the correct solution of the fixed point equation because the auxiliary polynomial conditions have a large number of real zeros. Nevertheless we have been able to distinguish the correct zeros. The values of $k^*$ and $\eta^*$ displayed in table 4 have been obtained with $R = 2.5$ and $\alpha = 2$. We have estimated the common radius of convergence to be $R_c(z) \simeq 2.5$. For other values of $R$ and $\alpha$ we have observed different kinds of convergence towards a unique couple of values. So, we are able to provide a rough estimate of the error on the evaluation of the $O(\partial^2)$-fixed point by the mapping method. For $M = 17$, it comes:

\[
\begin{align*}
  k^* &= 0.258204 \pm 0.000023, \\
  \eta^* &= 0.05388 \pm 0.00011.
\end{align*}
\]

Regarding the eigenvalues, the largest degree of the Taylor polynomials considered is $M = 25$. The parameters $R$ and $\alpha$ were not necessarily fixed to the values mentioned above for the fixed point. The eigenvalues displayed in table 4 have been obtained with $R = 2$ and $\alpha = 2$. In addition to those eigenvalues, we have observed in the even case the presence of a very small eigenvalue ($\simeq 1.5 \times 10^{-6}$). It corresponds to the expected zero eigenvalue associated with the reparameterization invariance. Its absolute value may be seen as a rough measure of the accuracy of the calculations.

It is worth indicating that our $O(\partial^2)$-results are better than the estimates:

\[
\begin{align*}
  \eta &= 0.05425, \\
  \nu &= 0.617476,
\end{align*}
\]

obtained in [6] using an expansion about an adjustable non-zero value of the field.

### 3.3 Remark

Let us emphasize that table 4 displays for the first time, for the Morris equations (24,25), values of the eigenvalues in the odd case ($\tilde{\omega}_1, \tilde{\omega}_2$) together with the first estimate of $\omega_2$. The exponent $\tilde{\omega}_1$ is of particular interest since it is actually the first time that it is evaluated at the second order of the derivative expansion of an ERGE. Moreover, this quantity is not well determined.
since only two estimations are available: from the Wilson ERGE [11] using the scaling field method in [15]:

\[ \tilde{\omega}_1 = 2.4 \pm 0.4 , \]

and from the \( \epsilon \)-expansion up to the order \( \epsilon^3 \) in [16] from which we have extracted the value:

\[ \tilde{\omega}_1 = 2.34 \pm 0.49 . \]

These two results are close to each other and clearly compatible. This is not the case of our present evaluation from the Morris equations (24, 25). Indeed, assuming that the derivative expansion converges, we may roughly give an error bar on the estimate of \( \tilde{\omega}_1 \). Using the two values of table 4 it comes:

\[ \tilde{\omega}_1 = 1.34 \pm 0.5 , \]

a result which is incompatible with the two preceding estimates. It is thus likely that the present \( O(\partial^2) \)-calculations do not provide the correct value of \( \tilde{\omega}_1 \), either because the derivative expansion itself does not converge or simply because the Morris equations are not adequate to determine correctly the values of the critical exponents (at least at low orders of the derivative expansion). We think that the second possibility is the right explanation. This is because one has already observed in the results of [29] a bad convergence on \( \omega_1 \) and also on \( \eta^* \) as functions of the number \( N \) of components of the field in the \( O(N) \)-symmetry whereas another choice of cut-off function, as done in [30], seems to be better adapted (see [31] for example). It is very likely that the fundamental reason why the Morris equation is not efficient must be looked for in the constraint imposed to the \( O(\partial^2) \)-equations with a view to satisfy explicitly the reparameterization invariance. Perhaps this constraint is too strong at this order.

4 Summary and conclusion

We have presented a new efficient analytical method for solving two point boundary value problems when one of the two boundaries is located at infinity. The method is based on the analyticity of the solution looked for in an angular sector which contains the positive real axis including the origin. This angular sector is then conformally mapped onto the unit disc so that an expansion in powers of the new independent variable \( w \) of the solution looked for yields a convergent series on the whole disc. The conditions at the infinite boundary may then be safely imposed on an approximate solution written as a Taylor polynomial in \( w \) estimated when \( w \to 1 \).
We have illustrated and tested the method on three different kinds of ODEs that one encounters in the derivative expansion of the ERGE: the Wegner-Houghton and the Wilson-Polchinski equations in the LPA and, at second order of the derivative expansion, the Morris equations. For the first and second examples (LPA) we provide the best estimates of both the fixed point connection parameter and of the critical exponents ever produced up to now from the ODEs considered. For the second example we are able to follow the fixed point solution from the dimension $d = 3$ down to 2 where the kind of solutions we track disappears. For the intermediate dimensions $d = 8/3$ and $d = 5/2$, we are able to locate the existing multi-critical points and the critical exponents attached to them with great accuracy. The difficulties increase with the multiplicity of the fixed point and with the decreasing of $d$. The study of the Morris equations is more demanding since it involves two ODEs (second order of the derivative expansion) which are particularly difficult. We show that the method works again well. We are able to estimate subcritical exponents that were not calculated previously using the derivative expansion. The estimation of one of these subcritical exponents does not compare favourably with values coming from the $\epsilon$-expansion and the scaling field method for studying the ERGEs. We give some possible explanations why the two equations considered are presumably not the best choice for calculating the critical exponents.

Comparisons with other analytical methods show that the mapping method is actually very efficient. It may be useful in the study of more complicated systems of equations obtained from the derivative expansion.

Obviously, it can also be employed efficiently to study analogous kinds of ODEs that appear in other fields when one of the boundaries is at infinity.

5 Acknowledgements

We thank B. Delamotte for indicating us the difficulty of studying the LPA using a field expansion when the space dimension approaches the value two.

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