HOMOLOGICAL STABILITY OF TOPOLOGICAL MODULI SPACES

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ABSTRACT. Given a graded $E_1$-module over an $E_2$-algebra in spaces, we construct an augmented semi-simplicial space up to higher coherent homotopy over it, called its canonical resolution, whose graded connectivity yields homological stability for the graded pieces of the module with respect to constant and abelian coefficients. We furthermore introduce a notion of coefficient systems of finite degree in this context and show that, without further assumptions, the corresponding twisted homology groups stabilize as well. This generalizes a framework of Randal-Williams and Wahl for families of discrete groups.

In many examples, the canonical resolution recovers geometric resolutions with known connectivity bounds. As a consequence, we derive new twisted homological stability results for e.g. moduli spaces of high-dimensional manifolds, unordered configuration spaces of manifolds with labels in a fibration, and moduli spaces of manifolds equipped with unordered embedded discs. This in turn implies representation stability for the ordered variants of the latter examples.

A sequence of spaces

\[ \cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots \]

is said to satisfy homological stability if the induced maps in homology are isomorphisms in degrees that are small relative to $n$. There is a well-established strategy for proving homological stability that traces back to an argument by Quillen for the classifying spaces of a sequence of inclusions of groups $G_n$. Given simplicial complexes whose connectivity increases with $n$ and on which the groups $G_n$ act simplicially, transitively on simplices and with stabilizers isomorphic to groups $G_{n-k}$ prior in the sequence, stability can often be derived by employing a spectral sequence relating the stabilizers. In [RW17], Randal-Williams and Wahl axiomatized this strategy of proof, resulting in a convenient categorical framework for proving homological stability for families of discrete groups that form a braided monoidal groupoid. This unifies and improves many classical stability results and has led to a number of applications since its introduction in many examples, the canonical resolution recovers geometric resolutions with known connectivity bounds. As a consequence, we derive new twisted homological stability results for e.g. moduli spaces of high-dimensional manifolds, unordered configuration spaces of manifolds with labels in a fibration, and moduli spaces of manifolds equipped with unordered embedded discs. This in turn implies representation stability for the ordered variants of the latter examples.

Instead of considering the single spaces $M_n$ and the maps $M_n \to M_{n+1}$ between them at a time, it is beneficial to treat them as a single space $M = \coprod_{n \geq 0} M_n$ together with a grading $g_M$: $M \to \mathbb{N}_0$ to the nonnegative integers, capturing the decomposition of $M$ into the pieces $M_n$ and a stabilization map $s: M \to M$ which restricts to the maps $M_n \to M_{n+1}$, so it increases the degree by one. From the perspective of homotopy theory, such $M$ that result from families $M_n$ that are known to stabilize homologically usually share the characteristic of forming a (graded) $E_1$-module over an $E_2$-algebra—the homotopy theoretical analogue of a module over a braided monoidal category. This observation is the driving force behind the work at hand.

Referring to Section 2.3 for a precise definition, we encourage the reader to think of a graded $E_1$-module $M$ over an $E_2$ algebra $A$ as a pair of spaces $(M, A)$ together with gradings $g_M$: $M \to \mathbb{N}_0$ and $g_A: A \to \mathbb{N}_0$, a homotopy-commutative multiplication $\otimes: A \times A \to A$ and a homotopy-associative action-map $\otimes: M \times A \to M$. These are required to satisfy various axioms, among them additivity with respect to $g_M$ and $g_A$ (see Definition 2.3). Given such $M$ and $A$, the choice of a stabilizing object $X \in A$, meaning an element of degree 1, results in a stabilization map

\[ s := (\cdot \otimes X): M \to M \]

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that increases the degree by 1 and hence gives rise to a sequence
\[ \ldots \rightarrow M_{n-1} \rightarrow \rightarrow M_n \rightarrow \rightarrow M_{n+1} \rightarrow \ldots \]
of the subspaces \( M_n = g^1_{M^n}(n) \) of a fixed degree. The sequences of spaces arising in this fashion are the ones whose homological stability behavior the present work is concerned with.

The key construction of this work is introduced in Section 2.2. We assign to \( M \) its canonical resolution
\[ R_n(M) \rightarrow M, \]
which is an augmented semi-simplicial space up to higher coherent homotopy—a notion made precise in Section 1.5, but which can be thought of as an augmented semi-simplicial space in the usual sense. The fiber \( W_n(A) \) of the canonical resolution at a point \( A \in M \) is an analogue of the simplicial complex in Quillen’s argument; it is a semi-simplicial space up to higher coherent homotopy whose space of \( p \)-simplices \( W_p(A) \) is the homotopy fiber at \( A \) of the \((p+1)\)st iterated stabilization map \( s^{p+1}: M \rightarrow M \). Thus \( W_n(A) \) should be thought of as the space of destabilizations of \( A \)—a terminology that suggests that the canonical resolution controls the stability behavior of \( M \), which is justified by Theorem A and C.

To state our main theorems we call the canonical resolution of \( M \) graded \((\phi \circ g_M)\)-connected in degrees \( \geq m \) for a function \( \phi: N_0 \rightarrow \mathbb{Q} \) and a number \( m \geq 0 \) if the restriction \( |R_n(M)|_{\phi} \rightarrow M_{\phi} \) of the geometric realization of \( \{ \} \) to the preimage of \( M_{\phi} \) is \((\phi(n))\)-connected in the usual sense for all \( n \geq m \). The first theorem, proven in Section 3, treats homological stability with constant and abelian coefficients, the latter being local systems on which the commutator subgroups of the fundamental groups at all basepoints act trivially.

**Theorem A.** Let \( M \) be a graded \( E_1 \)-module over an \( E_2 \)-algebra with stabilizing object and \( L \) a local system on \( M \). If the canonical resolution of \( M \) is graded \((\frac{g - 2 + k}{k})\)-connected in degrees \( \geq 1 \) for some \( k \geq 2 \), then
\[ s_i: H_i(M_n; s^k L) \rightarrow H_i(M_{n+k}; L) \]
(i) is an isomorphism for \( i \leq \frac{n-1}{k} \) and an epimorphism for \( i \leq \frac{n+2+k}{2k} \), if \( L \) is constant and
(ii) is an isomorphism for \( i \leq \frac{n+2+k}{2} \) and an epimorphism for \( i \leq \frac{n}{2} \), if \( L \) is abelian and \( k \geq 3 \).

In certain cases, discussed in Remark 2.24, the ranges of Theorem A can be improved marginally. By restricting to homological degree 0, the theorem has the following cancellation result as a consequence.

**Corollary B.** Let \( M \) be a graded \( E_1 \)-module over an \( E_2 \)-algebra with stabilizing object \( X \). If the connectivity assumption of Theorem A is satisfied, then the fundamental groupoid of \( M \) is \( X \)-cancellative for objects of positive degree, i.e. for objects \( A \) and \( A' \) of \( M \) of positive degree, \( A \oplus X \cong A' \oplus X \) in \( \Pi(M) \) implies \( A \cong A' \).

To cover more general coefficients, we note that the fundamental groupoid of an \( E_2 \)-algebra \( \mathcal{A} \) naturally carries the structure of a braided monoidal category \((\Pi(\mathcal{A}), \otimes, b, 0)\) and the fundamental groupoid of an \( E_1 \)-module \( M \) over \( \mathcal{A} \) becomes a right-module \((\Pi(M), \otimes)\) over it (see Section 2.1). In terms of this, we define in Section 4.1 a coefficient system \( F \) for \( M \) with stabilizing object \( X \) as an abelian group-valued functor \( F \) on \( \Pi(M) \), together with a natural transformation \( \sigma^F: F \rightarrow F(\otimes X) \), such that the image of the canonical morphism \( B_m \rightarrow \text{Aut}_{\mathcal{A}}(X^{\otimes m}) \) from the braid group on \( m \) strands acts trivially on the image of \( (\sigma^F)^m: F \rightarrow F(\otimes X^{\otimes m}) \) for all \( n \) and \( m \). Such a coefficient system enhances the stabilization map to a map of spaces with local systems
\[ (s; \sigma^F): (M_n; F) \rightarrow (M_{n+1}; F) \]
by restricting \( F \) to subspaces of homogenous degree. A coefficient system \( F \) induces a new one \( \Sigma F = F(\otimes X) \), called its suspension, which comes with a morphism \( F \rightarrow \Sigma F \) named the suspension map (see Definition 4.1). The coefficient system \( F \) is inductively said to be of degree \( r \) if the kernel of the suspension map vanishes and the cokernel has degree \( (r-1) \); the zero coefficient system having degree \( -1 \). In fact, we define a more general notion of being of \textit{(split) degree} \( r \) at \( N \) such that \( F \) is of degree \( r \) in the sense just described if it is of degree \( r \) at 0 (see Definition 4.1). This notion of a coefficient system of finite (split) degree generalizes the one introduced by Randal-Williams and Wahl [RW17] for braided monoidal groupoids which was itself inspired by work of Dwyer [Dwy80] and van der Kallen [Kal80] on general linear groups and of Ivanov [Iva93] on mapping class groups of surfaces.
Remark. There is an alternative point of view on coefficient systems for $\mathcal{M}$, namely as abelian-group valued functors on a category $\mathcal{C}(\mathcal{M})$ constructed from the action of $\Pi(\mathcal{A})$ on $\Pi(\mathcal{M})$ (see Remark 4.19).

Our second main theorem, demonstrated in Section 4.2, addresses homological stability of $\mathcal{M}$ with coefficients in a coefficient system of finite degree.

**Theorem C.** Let $\mathcal{M}$ be a graded $E_1$-module over an $E_2$-algebra with stabilizing object and $F$ a coefficient system for $\mathcal{M}$ of degree $r$ at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $(\frac{n}{2r+k})$-connected in degrees $\geq 1$ for $k \geq 2$, then the map induced by stabilization

$$(s;\sigma^F)_*: H_i(\mathcal{M}_n;F) \rightarrow H_i(\mathcal{M}_{n+1};F)$$

is an isomorphism for $i \leq \frac{n-k}{2}$ and an epimorphism for $i \leq \frac{n-r-k}{k}$, when $n > N$. If $F$ is of split degree $r$ at $N \geq 0$ then $(s;\sigma^F)_*$ is an isomorphism for $i \leq \frac{n-r-k}{k}$ and an epimorphism for $i \leq \frac{n-r}{k}$, when $n > N$.

As a proof of concept, we apply the developed theory to three main classes of examples to which we devote the remainder of this introduction.

**Configuration spaces.** The unordered configuration space $C^\ast_n(W)$ of a manifold $W$ with labels in a Serre fibration $\pi: E \rightarrow W$ is the quotient of the ordered configuration space

$$F^\ast_n(W) = \{(e_1,\ldots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W/\partial W\}$$

by the canonical action of the symmetric group. If $W$ is of dimension $d \geq 2$ and has nonempty boundary, then the union of its canonical resolution spaces $\mathcal{M} = \bigsqcup_{i \geq 2} C^\ast_n(W)$ forms an $E_1$-module over the $E_2$-algebra $\mathcal{A} = \bigsqcup_{i \geq 2} C_\ast(D^i)$ of configurations in a $d$-disc, graded by the number of points (see Lemma 5.1). In Section 5.1 we identify its canonical resolution with the resolution by arcs—an augmented semi-simplicial space of geometric nature which has already been considered in the context of homological stability (see e.g. [KM14, MW16]) and is known to be sufficiently connected to apply Theorems A and C (see Section 5.1).

**Theorem D.** Let $W$ be a connected manifold of dimension at least 2 with nonempty boundary and let $\pi: E \rightarrow W$ be a Serre fibration with path-connected fibers.

(i) For a local system $L$ on $C^\ast_n,(W)$, the stabilization map

$s_*: H_1(C^\ast_n(W);s^\ast L) \rightarrow H_1(C^\ast_{n+1}(W);L)$

is an isomorphism for $i \leq \frac{n-2}{2}$ and an epimorphism for $i \leq \frac{n}{2}$, if $L$ is constant. It is an isomorphism for $i \leq \frac{n-2}{2}$ and an epimorphism for $i \leq \frac{n}{2}$, if $L$ is abelian.

(ii) If $F$ is a coefficient system of degree $r$ at $N \geq 0$, then the stabilization map

$$(s;\sigma^F)_*: H_0(C^\ast_n(W);F) \rightarrow H_0(C^\ast_{n+1}(W);F)$$

is an isomorphism for $i \leq \frac{n-2r-2}{2}$ and an epimorphism for $i \leq \frac{n-2r}{2}$, when $n > N$. If $F$ is of split degree $r$ at $N \geq 0$, then it is an isomorphism for $i \leq \frac{n-2r-2}{2}$ and an epimorphism for $i \leq \frac{n-2r}{2}$, when $n > N$.

Configuration spaces have a longstanding history in the context of homological stability, starting with work of Arnold [Arm68] who established stability for $C_n(D^2)$ with constant coefficients. McDuff and Segal [McD75, Seg73, Seg79] observed that this behavior is not restricted to the 2-disc and proved stability for more general $C^\ast_n(W)$ with constant coefficients and $\pi = \text{id}_W$, which can be extended to general $\pi$ e.g. by adapting the proof for a trivial fibration presented in [Ran13] (see [KM14, CP15] for alternative proofs).

The stabilization map for configuration spaces is in fact split injective in homology with constant coefficients in all degrees—a phenomenon special to configuration spaces which is not captured by our general approach (see e.g. [Ran13] for a proof). However, we obtain a slightly better isomorphism range of $i \leq \frac{n}{2}$ than the one stated in Theorem D by employing the improvement of Remark 2.24.

For a trivial fibration, stability of $C^\ast_n(W)$ with respect to a nontrivial coefficient system $F$ was studied by Palmer [Pal13b], building on work of Betley [Bet02] on symmetric groups. The second part of Theorem D extends his result to nontrivial fibrations and a significantly larger class of coefficient systems, partly conjectured by Palmer [Pal13b, Rem. 1.13] (see Remark 5.14 for a more detailed comparison to his work). In the case of surfaces and a trivial fibration, a result similar to Theorem D but with respect to a slightly smaller class of coefficient systems, is contained in work by Randal-Williams and Wahl [RW17 Thm. D].
In Section 5.2 we provide a discussion of coefficient systems for configuration spaces by relating them e.g. to the theory of \( \mathcal{T} \)-modules as introduced by Church, Ellenberg and Farb [CEF13] or to coefficient systems studied in [RW17]. In particular we obtain a wealth of nontrivial coefficient systems \( F \) with respect to which the homology of \( C_n^\pi(W) \) stabilizes.

To our knowledge, stability with abelian coefficients for configuration systems of manifolds of dimensions greater than two has not been considered so far. We next discuss a direct consequence of stability with respect to this class of coefficients as the first item in a series of applications exploiting Theorem D.

**Oriented configuration spaces.** The oriented configuration space \( C_n^{\pi,+}(W) \) with labels in a Serre fibration \( \pi \) over \( W \) is the double cover of \( C_n^\pi(W) \) given as the quotient of the ordered configuration space \( F_n^\pi(W) \) by action of the alternating group \( A_n \), or equivalently the space of labelled configurations ordered up to even permutations. By the space version of Shapiro's lemma, the homology of \( C_n^{\pi,+}(W) \) is isomorphic to \( H_1(C_n^\pi(W); Z/[Z/2Z]) \), with the action of \( \pi_1(C_n^\pi(W)) \) on the ring \( Z/[Z/2Z] \) being induced by the composition of the sign homomorphism with the morphism \( \pi_1(C_n^\pi(W)) \to \Sigma_n \) obtained by choosing an ordering of a basepoint. These local systems are abelian and are preserved by pulling back along the stabilization map, hence homological stability for \( C_n^{\pi,+}(W) \) follows as a by-product of Theorem D.

**Corollary E.** Let \( W \) and \( \pi \) be as in Theorem D. The map induced by stabilization

\[
s_i: H_i(C_n^{\pi,+}(W); Z) \to H_i(C_n^{\pi,+}(W); Z)
\]

is an isomorphism for \( i \leq \frac{n-2}{2} \) and an epimorphism for \( i \leq \frac{n}{2} \).

Stability for oriented configuration spaces of connected orientable surfaces with nonempty boundary and without labels was proven by Guest, Kozlowski and Yamaguchi [GKY96] using computations due to Bödigheimer, Cohen, Taylor and Milgram [BCT89, BCM92]. Palmer [Pal13] extended this to manifolds of higher dimensions with nonempty boundary and labels in a trivial fibration. Corollary E gives an alternative proof of his result and enhances it by means of general labels and an improved stability range.

**Configuration spaces of embedded discs.** The configuration space \( C_n^k(W) \) of unordered \( k \)-discs in a manifold \( W \) of dimension \( d \) is the quotient by the action of the symmetric group of the configuration space of ordered \( k \)-discs

\[
F_n^k(W) = \text{Emb}(\coprod^n D^k, W \setminus \partial W),
\]

equipped with the \( C^\infty \)-topology. For \( k = d \) and oriented \( W \), there are variants \( F_n^d(W) \) and \( C_n^d(W) \) by restricting to orientation preserving embeddings. Mapping an embedding of a \( k \)-disc to its center point labelled with the \( k \)-frame induced by standard framing of \( D^k \) at the origin results in a map \( C_n^d(W) \to C_n^{\pi_d}(W) \), where \( \pi_d \) is the bundle of \( k \)-frames. This map can be seen to be a weak equivalence by choosing a metric and exponentiating frames. For \( k < d \), the fiber of \( \pi_k \) is path-connected, so the homological stability results of Theorem D carry over to \( C_n^k(W) \), comprising part of Corollary E below. The argument for \( C_n^{\pi,+}(W) \) is analogous using oriented framings instead.

The group of diffeomorphisms \( \text{Diff}_\partial(W) \) fixing a neighborhood of the boundary in the \( C^\infty \)-topology naturally acts on the configuration spaces \( F_n^k(W) \) and \( C_n^k(W) \) and the resulting homotopy quotients \( F_n^k(W)/\text{Diff}_\partial(W) \) and \( C_n^k(W)/\text{Diff}_\partial(W) \) model the classifying spaces of the subgroups

\[
P\text{Diff}_\partial(W) \subseteq \text{Diff}_\partial(W) \subseteq \text{Diff}(W),
\]

where \( P\text{Diff}_\partial(W) \) are the diffeomorphisms that fix \( n \) chosen embedded \( k \)-discs in \( W \) and \( \text{Diff}_\partial(W) \) are the ones permuting them (see Lemma 5.13). If \( W \) is orientable, the (sub)groups of orientation preserving diffeomorphisms are denoted with a (+) superscript. In Example 2.22 we explain how the canonical resolution of a graded \( E_1 \)-module \( M \) over an \( E_2 \)-algebra \( \mathcal{A} \) relates to that of the \( E_1 \)-module \( EG \times_C M \) over \( \mathcal{A} \) in the presence of a graded action of a group \( G \) on \( M \) that commutes with the action of \( \mathcal{A} \). An application of this consideration to the situation at hand implies the following, carried out in Section 5.3.1.

**Corollary F.** Let \( W \) be a \( d \)-dimensional manifold as in Theorem D and let \( 0 \leq k < d \).

(i) For a local system \( L \), the stabilization maps

\[
H_i(C_n^k(W); s^* L) \to H_i(C_n^{k+1}(W); L) \quad \text{and} \quad H_i(B \text{Diff}_\partial(W); s^* L) \to H_i(B \text{Diff}_\partial(W); L)
\]

are isomorphisms for \( i \leq \frac{k-2}{2} \) and epimorphisms for \( i \leq \frac{k}{2} \). The map on homotopy groups induced by the stabilization map is an isomorphism for \( i \leq \frac{k-3}{2} \) and an epimorphism for \( i \leq \frac{k-1}{2} \).
are isomorphisms for \( i \leq \frac{n-1}{2} \) and epimorphisms for \( i \leq \frac{n}{2} \), if \( L \) is constant. If \( L \) is abelian, then they are isomorphisms for \( i \leq \frac{n-2}{2} \) and epimorphisms for \( i \leq \frac{n}{2} \).

(ii) If \( F \) is a coefficient system of degree \( r \) at \( N \geq 0 \), then the maps induced by the stabilization \((s; \sigma^F)\)

\[
H_i(C_n^k(W); F) \rightarrow H_i(C_{n+r}^k(W); F)
\]

and

\[
H_i(B\text{Diff}^k_{\partial,n}(W); F) \rightarrow H_i(B\text{Diff}^k_{\partial,n+1}(W); F)
\]

are isomorphisms for \( i \leq \frac{n-2}{2} \) and epimorphisms for \( i \leq \frac{n-2}{2} \), when \( n > N \). If \( F \) is of split degree \( r \) at \( N \geq 0 \), then they are isomorphisms for \( i \leq \frac{n-2}{2} \) and epimorphisms for \( i \leq \frac{n-2}{2} \), when \( n > N \).

If \( W \) is oriented, the analogous statements hold for the variants \( C_n^{\pi,*}(W) \) and \( \text{Diff}^d_{\partial,n}(W) \).

The isomorphism range for constant coefficients can be improved to hold for \( i \leq \frac{n}{2} \) by virtue of Remark 2.2.4.

For compact \( W \), Tillmann \cite{Til16} has proven homological stability with constant coefficients for a variant of \( B\text{Diff}^0_{\partial,n}(W) \) and \( B\text{Diff}^d_{\partial,n}(W) \) involving diffeomorphisms that are only required to fix a disc in the boundary instead of the whole boundary. A Serre spectral sequence argument shows that stability for these variants follows from stability of the spaces \( B\text{Diff}^0_{\partial,n}(W) \) and \( B\text{Diff}^d_{\partial,n}(W) \). Hatcher and Wahl \cite{HW10 Prop. 1.5] have shown stability with constant coefficients for the mapping class groups \( \pi_0(\text{Diff}^0_{\partial,n}(W)) \), which can be seen to be equivalent to \( \text{Diff}^0_{\partial,n}(W) \) for compact 2-dimensional \( W \) as a result of the homotopy discreteness of the space of diffeomorphisms of a compact surface \cite{EE67, Gra73}. In this case, stability with respect to some of the twisted coefficients systems Corollary \cite{F} deals with is contained in work by Randal-Williams and Wahl \cite[Thm. 5.22]{RW17}.

Representation stability. The first rational homology group of the ordered configuration space of the 2-disc

\[
H_i(F_n(D^2); \mathbb{Q}) \cong \mathbb{Q}(i),
\]

as e.g. computed in \cite{Armo69}, exemplifies that—in contrast to unordered configuration spaces—the homology of the ordered variant does not stabilize. However, by incorporating the action of the symmetric groups \( \Sigma_n \), it does stabilize in a more refined, representation theoretic sense. To make this precise, recall the correspondence between irreducible representations of \( \Sigma_n \) and partitions of \( n \) \cite[Ch. 4.]{FH91}. We denote the irreducible \( \Sigma_{|\lambda|}\)-module corresponding to a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_k) \vdash |\lambda| \) of length \( |\lambda| \) by \( V_{\lambda} \) and define for \( n \geq |\lambda| + \lambda_1 \), the padded partition \( \lambda[n] = (n - l \geq \lambda_1 \geq \ldots \geq \lambda_k) \vdash n \). Using the Totaro spectral sequence \cite{Tot16} and the celebrated representation stability—by concept introduced by Church and Farb \cite{CF13}—this implies the existence of a constant \( N(i) \), depending solely on \( i \), such that the multiplicity of \( V_{\lambda[n]} \) in the \( \Sigma_n \)-module \( H^i(F_n(W); \mathbb{Q}) \) is independent of \( n \) for \( n \geq N(i) \). To date, Church’s result has been extended in several directions \cite{CEF15, MW16, Pet17, Tos16}.

A twisted Serre spectral sequence argument (see Lemma \cite[5.16]{5.16}) shows that the multiplicity of an irreducible \( \Sigma_n \)-module \( V_{\lambda} \) in \( H^i(F_n(W); \mathbb{Q}) \) equals the dimension of \( H_i(C_n^k(W); V_{\lambda}) \), where \( \pi_1(C_n^k(W)) \) acts on \( V_{\lambda} \) via the morphism \( \pi_1(C_n^k(W)) \rightarrow \Sigma_n \). This fact allows us to derive the stability of these multiplicities from Theorem \cite[5.22]{5.22} at least for all manifolds to which the latter theorem applies (see Section \cite[5.22]{5.22}).

**Corollary G.** Let \( W \) and \( \pi \) be as in Theorem \cite[5.22]{5.22} and \( Z_n \) one of the following sequences of \( \Sigma_n \)-spaces:

(i) \( F_n^\pi(W) \)

(ii) \( F_n^\pi(W) \) for \( 0 \leq k < d \)

(iii) \( F_n^{d,k}(W) \) if \( W \) is oriented

The \( \lambda[n] \)-multiplicity in \( H^i(Z_n; \mathbb{Q}) \) for a fixed partition \( \lambda \) is independent of \( n \) for \( n \) large relative to \( i \).

In Remark \cite[5.18]{5.18} we discuss explicit ranges for Corollary \cite[5.18]{5.18} and compare them to Church’s. Let’s at this juncture record that our approach leads to ranges which depend on \( |\lambda| \), so we do not recover uniform representation stability. On the other hand, in contrast to Church’s result, we neither require \( W \) to be orientable nor to have finite dimensional rational cohomology or \( \pi \) to be the identity.

Jiménez Rolland \cite[11]{11} has shown uniform representation stability for the cohomology groups \( H^i(\text{PD}^d_{\partial,n}(W); \mathbb{Q}) \) for compact orientable surfaces and for compact connected manifolds \( W \) of dimension \( d \geq 3 \), assuming that \( B\text{Diff}_{\partial}(W) \) has the homotopy type of a CW-complex of finite type. Furthermore, she
proved uniform representation stability for $\pi_0(\text{PDiff}^0_{\partial,n}(W))$ for compact orientable surfaces, as well as for higher-dimensional manifolds under some further assumptions.

**Moduli spaces of manifolds.** The moduli space $M$ of compact $d$-dimensional smooth manifolds with a fixed boundary $P$ forms an $E_1$-module over the $E_2$-algebra $A$ given by the moduli space of compact $d$-manifolds with a sphere as boundary (see Lemma 6.1). The homotopy types of $M$ and $A$ are given by

$$M \cong \bigsqcup [W] B\text{Diff}_0(W) \text{ and } A \cong \bigsqcup [N] B\text{Diff}_0(N),$$

where $[W]$ runs over diffeomorphism classes relative to $P$ of compact $d$-manifolds with $P$-boundary and $[N]$ over the ones of compact $d$-manifolds with a sphere as boundary. Acting with a manifold $X \in A$ on $M$ corresponds to taking the connected sum with the manifold $X$ that is obtained from $X$ by gluing in a disc to close the sphere boundary, so the resulting stabilization map restricts on path components to

$$s : B\text{Diff}_0(W) \to B\text{Diff}_0(W_{\#}\bar{X})$$

which models the map on classifying spaces induced by extending diffeomorphisms by the identity.

As shown in Section 6.1 the canonical resolution of $M$ with respect to a choice of a stabilizing manifold $X$ is equivalent to the resolution by embeddings—an augmented semi-simplicial space of submanifolds $W \in M$ together with embeddings of $W$ with a fixed boundary. For specific manifolds $X$ and $W$, this resolution and its connectivity has been studied to prove homological stability of $\text{Diff}_0$ first by Galatius and Randal-Williams in their groundbreaking work [GR17a] for $X \equiv S^p \times S^q$ and simply-connected $(2p)$-dimensional $W$ with $p \geq 3$. Their results extend the classical homological stability result for mapping class groups of surfaces [Har85] to higher dimensions. As in Harer’s result, the known connectivity of the resolution by embeddings, and hence the resulting stability ranges, depend on the $X$-genus of $W$.

$$g^X(W) = \max\{k \geq 0 \mid \text{there exists } M \in M \text{ such that } M#X^k \cong W \text{ relative to } P\},$$

which incidentally provides a method of grading $E_1$-modules $M$ in general (see Section 2.3). Perlmutter [Per16a] succeeded in carrying out this strategy for $X \equiv S^p \times S^q$ with certain $p \neq q$ depending on which $W$ is required to satisfy a connectivity assumption. Recently, Friedrich [Fri16] extended the work of Galatius and Randal-Williams to manifolds $W$ without nontrivial fundamental group in terms of the unitary stable rank [KMo2 Def. 6.3] of the ring $\mathbb{Z}[\pi_1(W)]$. These connectivity results can be restated in our context as graded connectivity for the canonical resolution of $M$ with respect to different gradings (see Corollary 6.7), which allows us to apply Theorem A and C.

Employing the improvement of Remark 2.2, the ranges with constant and abelian coefficients obtained from Theorem A agree with the ones established in [Fri16a, GR17a, Per16a] (after extending [Per16a] to abelian coefficients by adapting the methods of [GR17a]). The cancellation result for connected sums of manifolds that we derive from Corollary B coincides with their cancellation results as well. Our main contribution lies in the application of Theorem C, i.e. homological stability of moduli spaces of manifolds with respect to a large class of nontrivial coefficient systems, which has not been considered in the context of moduli spaces of high-dimensional manifolds so far. On path-components, it reads as follows.

**Theorem H.** Let $W$ be a compact $(p + q)$-manifold without boundary and $F$ a coefficient system of degree $r$. Denote by $g(W)$ the $(S^p \times S^q)$-genus of $W$ and set $u$ to be 1 if $W$ is simply connected and to be the unitary stable rank of $\mathbb{Z}[\pi_1(W)]$ otherwise. The stabilization map

$$(s, \sigma^F) : H_i(\text{Diff}_0(W); F) \to H_i(\text{Diff}_0(W\#(S^p \times S^q)); F)$$

(i) is an isomorphism for $i \leq \frac{g(W) - 2r - u - 3}{2}$ and an epimorphism for $i \leq \frac{g(W) - 2r - u - 3}{2} + 1$, if $p = q \geq 3$ and

(ii) is an isomorphism for $i \leq \frac{g(W) - 2r - m - 4}{2}$ and an epimorphism for $i \leq \frac{g(W) - 2r - m - 2}{2} + 1$, if $W$ is $(q - p + 2)$-connected and $0 < p < q < 2p - 2$ with $m = \min\{i \in \mathbb{N} \mid \text{there exists an epimorphism } \mathbb{Z}^i \to \pi_q(S^p)\}$.

If $F$ is (split) of degree $r$ at a number $N \geq 0$, the ranges in the theoreme change as per Theorem C.

**Remark.** The unitary stable rank [KMo2 Def. 6.3] of a group ring $\mathbb{Z}[G]$ need not be finite. To provide a class of examples of finite unitary stable rank, recall that $G$ is called virtually polycyclic if there is a series $1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$ such that $G_i$ is normal in $G_{i+1}$ and the quotients $G_{i+1}/G_i$ are either finite or cyclic. Its Hirsch length $h(G)$ is the number of infinite cyclic factors. Crowley and Sixt [CS11 Thm. 7.3]
showed \( \text{usr}(Z[G]) \leq h(G) + 3 \) for virtually polycyclic groups \( G \). In particular, we have \( \text{usr}(Z[G]) \leq 3 \) for finite groups and \( \text{usr}(Z[G]) \leq \text{rank}(G) + 3 \) for finitely generated abelian groups.

In Remark 6.8 we briefly elaborate on how to include the case of orientable surfaces in this picture by utilizing high-connectivity of the complex of tethered chains, a result of Hatcher and Vogtmann [HV17]. For constant coefficients, this implies Harer’s classical stability theorem [Har85] with a better, but not optimal range [Bo12, Ran16]. For twisted coefficients, it extends a result by Ivanov [Iva93] to more general coefficient systems. However, in the case of surfaces, stability with respect to most of these coefficient systems was already known by [RW17].

In Section 6.2 we show that coefficient systems for \( M \) are equivalent to certain families of modules over the mapping class groups \( \pi_0(\text{Diff}_\partial(W)) \cong \pi_1(\text{Diff}_\partial(W)) \) and explain how the action of the mapping class groups on the homology of the manifolds gives rise to a coefficient system of degree 1 for \( M \), from which we get the following corollary.

**Corollary 1.** Let \( W \) be a compact \((p + q)\)-manifold with nonempty boundary and \( k \geq 0 \). The stabilization

\[
    H_i(B \text{Diff}_\partial(W); H_k(W)) \rightarrow H_i(B \text{Diff}_\partial(W^\#(S^p \times S^q)); H_k(W^\#(S^p \times S^q)))
\]

is an epi- and isomorphism for the same \( W \) as in Theorem A and with the same ranges after replacing \( r \) by 1.

Furthermore, in Section 6.3 we provide a short discussion of how our methods can be applied to incorporate the case of certain stably parallizable \((2n - 1)\)-connected \((4n + 1)\)-manifolds \( X \) and 2-connected \( W \). This extends stability results by Perlmutter [Per16b]. Similarly, we also enhance work of Kupers [Kup15] on homeomorphisms of topological manifolds and automorphisms of piecewise linear manifolds.

**Modules over braided monoidal categories.** We close in Section 7 by explaining applicability of our results to discrete situations, e.g., to groups or monoids, and by drawing a comparison to [RW17].

The classifying space \( BM \) of a graded module \( M \) over a braided monoidal category is a graded \( E_1 \)-module over an \( E_2 \)-algebra (see Lemma 7.2), so forms a suitable input for Theorem A and C. In Lemma 7.6 we identify the space of destabilization \( W^r(A) \) of \( A \in M \) with a semi-simplicial set \( W^r_{\text{RW}}(A) \) in the case of \( M \) being a groupoid satisfying an injectivity condition. This identification gives rise to a framework for homological stability for modules over braided monoidal categories, phrased entirely in terms of \( M \) and semi-simplicial sets instead of semi-simplicial spaces up to higher coherent homotopy (see Remark 7.8).

Using this, we observe in a forthcoming small note [Kra] that work of Hepworth on homological stability for Coxeter groups [Hep16] with constant coefficients implies their stability with respect to a large class of coefficient systems without further effort, as well as stability of their commutator subgroups.

In the special case of a braided monoidal groupoid acting on itself, the semi-simplicial sets \( W^r_{\text{RW}}(A) \) were introduced by Randal-Williams–Wahl in [RW17] as part of their stability results for the automorphisms of a braided monoidal groupoid, which this work enhances in various ways. We generalize from braided monoidal groupoids to modules over such, remove all hypotheses on the categories one starts with, improve the stability ranges in certain cases (see Remark 7.10) and enlarge the class of coefficients systems (see Remark 7.12). We refer to Section 7.2 for a more detailed comparison of our results in the discrete setting to [RW17] and an analysis of the assumptions that they impose on the groupoid one starts with.

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1. Preliminaries

This section is devoted to fix conventions and collect general techniques used in the body of the work at hand. We work in the category of compactly generated spaces, use Moore paths throughout and denote the length of a path $\mu$ by $l(\mu) \in [0, \infty)$.

1.1. Graded spaces and categories. We denote by $(\mathbb{N}, +)$ the discrete abelian monoid that is the extension of the non-negative integers $(\mathbb{N}_0, +)$ by an element $\infty$ satisfying $k + \infty = \infty$ for all $k \geq 0$.

A graded space is a space $X$ together with a continuous map $g_X : X \to \mathbb{N}$. A map of graded spaces is a map which preserves the grading and a map of degree $k$ between graded spaces for a number $k \geq 0$ is a map which increases the degree by $k$. The category of graded spaces is symmetric monoidal with the monoidal product of two graded spaces $(X, g_X)$ and $(Y, g_Y)$ given by $(X \times Y, g_{X \times Y})$.

The subspace of elements of degree $n \in \mathbb{N}$ is denoted by $X_n = g_X^{-1}(\{n\}) \subseteq X$. By restricting the grading, subspaces of graded spaces are implicitly considered as being graded. A graded space $(X, g_X)$ or a map $(Y, g_Y) \to (X, g_X)$ of graded spaces is $(\varphi \circ g_X)$-connected in degrees $\geq m$ for a function $\varphi : \mathbb{N} \to \mathbb{Q}$ and a number $m \geq 0$ if $X_n$, respectively $Y_n \to \mathbb{Z}_n$, is $[\varphi(n)]$-connected for all $m \leq n < \infty$ in the usual sense. Note that we do not require anything on $X_\infty$ respectively on $Y_\infty \to X_\infty$.

A graded set $X$ is a graded discrete space. A graded category $C$ is a category internal to graded sets, i.e. a category $C$ with a function $g_C : \text{ob} C \to \mathbb{N}$ that is constant on categorical path components. This is equivalent to a grading on the classifying space of $C$. A graded monoidal category is a monoid internal to graded categories with the monoidal product $(C, g_C) \times (D, g_D) = (C \times D, g_{C \otimes D})$, i.e. a monoidal category $(\mathcal{A}, \otimes, 0)$ together with a grading $g_{\mathcal{A}}$ on $\mathcal{A}$ satisfying $g_{\mathcal{A}}(0) = 0$ and $g_{\mathcal{A}}(X \otimes Y) = g_{\mathcal{A}}(X) + g_{\mathcal{A}}(Y)$. \\/
A graded right-module \((M, \oplus)\) over a graded monoidal category \((\mathcal{A}, \oplus, 0)\) is a graded category \(M\) together with a right-action of \((\mathcal{A}, \oplus, 0)\) on \(D\) internal to graded categories, i.e. a functor \(\oplus: M \times \mathcal{A} \to M\) which is unital and associative up to coherent isomorphisms and satisfies \(g_M(A \oplus X) = g_M(A) + g_M(X)\).

1.2. \(C\)-spaces and their rectification. We set up an ad-hoc theory of spaces parametrized by a topologically enriched category, which serves us as a convenient language in the body of this work.

We call a space-valued functor \(X_*\) on a topologically enriched category \(C\) a \(C\)-space. An augmentation \(f_*: X_* \to X_{-1}\) of a \(C\)-space \(X_*\) over a space \(X_{-1}\) is a lift of \(X_*\) to a functor on the overcategory \(\mathcal{Top}/X_{-1}\) and an augmented \(C\)-space is a \(C\)-space together with an augmentation. We denote the value of an augmented \(C\)-space \(f_*: X_* \to X_{-1}\) at an object \(C\) by \(f_C: X_C \to X_{-1}\). A morphism of augmented \(C\)-spaces is a natural transformation of functors \(C \to \mathcal{Top}/X_{-1}\) and it is called a weak equivalence if it is an objectwise weak equivalence.

A morphism between a \(C\) space \(X_*\) augmented over \(X_{-1}\) and one \(Y_*\) over \(Y_{-1}\) consists of a map \(h: X_{-1} \to Y_{-1}\) and a morphism \(h_*: h_*(X_*) \to Y_*\) of \(C\)-spaces augmented over \(Y_{-1}\) where \(h_*(X_*)\) denotes \(X_*\) when considered augmented over \(Y_{-1}\) using \(h\). An augmented \(C\)-space \(f_*: X_* \to X_{-1}\) is called \(f\)ibrant if all maps \(f_C\) are Serre fibrations.

Example 1.1. For \(C\) being the opposite of the semi-simplicial category \(\Delta_{	ext{inj}}\) (see Section 1.4), \(C\)-spaces are known as \(\text{semi-simplicial spaces}\). This is the example which motivates our choice of notation.

Definition 1.2. The \(f\)ibrant replacement of an augmented \(C\)-space \(X_* \to X_{-1}\) is the augmented \(C\)-space \(X_{\text{fib}}^{\text{fib}} \to X_{-1}\) obtained by applying the path-space construction objectwise,

\[
X_C^{\text{fib}} = \{(x, \mu) \in X_C \times \text{Path} X_{-1} \mid \mu(\ell(\mu)) = f_C(x)\},
\]

considered as a space over \(X_{-1}\) by evaluating paths at zero. It is fibrant and admits a canonical weak equivalence \(X_* \to X_{\text{fib}}^{\text{fib}}\) of augmented \(C\)-spaces given by mapping \(x \in X_C\) to \((x, \text{const}_{f_C(x)}) \in X_C^{\text{fib}}\).

The fiber \(X_{*, C}\) of an augmented \(C\)-space \(f_*: X_* \to X_{-1}\) at \(x \in X_{-1}\) is the \(C\)-space, which assigns to an object \(C\) the fiber \(X_C, C = f_C^{-1}(x)\). Its homotopy fiber \(\text{hofib}_X(x)\) at \(x\) is the fiber of \(X_{\text{fib}}^{\text{fib}} \to X_{-1}\) at \(x\). If \(X_* \to X_{-1}\) is fibrant, then the weak equivalence \(X_* \to X_{\text{fib}}^{\text{fib}}\) induces a weak equivalence \(X_{*, C} \to \text{hofib}_X(x)\).

Definition 1.3. Let \(C\) be a small topologically enriched category.

(i) The bar construction \(B(Y_*, C, X_*)\) of a pair of \(C\)-spaces \((X_*, Y_*)\), where \(X_*\) is co- and \(Y_*\) is contravariant, is the realization of the simplicial space \(B_*(Y_*, C, X_*)\) with \(p\)-simplices

\[
\coprod_{C_0, \ldots, C_p \text{co} C} X_{C_0} \times C(C_0, C_1) \times \cdots \times C(C_{p-1}, C_p) \times Y_{C_p}.
\]

An augmentation \(X_* \to X_{-1}\) naturally induces a map \(B(Y_*, C, X_*) \to X_{-1}\).

(ii) The homotopy colimit

\[
\text{hocolim}_C X_* \to X_{-1}
\]

of an augmented \(C\)-space \(X_* \to X_{-1}\) is the bar construction \(B(\ast, C, X_*) \to X_{-1}\).

Remark 1.4. We realize simplicial spaces by means of the fat geometric realization, which has the pleasant property that level-wise weak equivalences realize to weak equivalences [Seg74 App. A].

A \(C\)-space is \(k\)-connected for \(k \geq 0\) if its homotopy colimit is so. If the base \(X_{-1}\) of an augmented \(C\)-space \(X_* \to X_{-1}\) is graded, then its values \(X_C\) and its homotopy colimit inherit gradings by pulling back \(g_{X_{-1}}\) from \(X_{-1}\). It is graded \((\varphi \circ g_{X_{-1}})\)-connected in degrees \(\geq m\) for \(\varphi: \mathbb{N} \to \mathbb{Q}\) if \(\text{hocolim}_C X_* \to X_{-1}\) is.

Lemma 1.5. Let \(X_* \to X_{-1}\) be an augmented \(C\)-space and \(x \in X_{-1}\). The canonical map

\[
\text{hocolim}_C(\text{hofib}_x(X_* \to X_{-1})) \to \text{hofib}_x(\text{hocolim}_C X_* \to X_{-1})
\]

is a weak equivalence.

Proof. After replacing \(X_* \to X_{-1}\) up to weak equivalence with a \(C\)-space augmented over a weak Hausdorff space, the claim follows from the fact that the functor \(\text{ev}_1: \mathcal{Top}/X_{-1} \to \mathcal{Top}/(\text{Path}_x X_{-1})\) given by pulling back the path fibration \(\text{ev}_1: \text{Path}_x X_{-1} \to X\) is a left adjoint [MS06 Prop. 2.1.3], so preserves colimits.
A functor between topologically enriched categories is said to be a \textit{weak equivalence} if it induces weak equivalences on morphism spaces and a bijection on the set of objects. By virtue of Remark \ref{rem: augmentation is an equivalence of categories}, the map on bar constructions induced by a weak equivalence \((X_*, C, Y_*) \rightarrow (X'_*, C', Y'_*)\) of triples, defined in the appropriate sense, is a weak homotopy equivalence. In particular, by taking homotopy colimits, weak equivalences of \(C\)-spaces augmented over \(X_{-1}\) induce weak equivalences of spaces over \(X_{-1}\).

**Lemma 1.6.** Let \(p \colon C \rightarrow D\) be a weak equivalence of topologically enriched categories. There is a functor
\[
 p_* \colon (\text{Top}/X_{-1})^C \longrightarrow (\text{Top}/X_{-1})^D
\]
for each space \(X_{-1}\), which fits into a zig-zag of natural transformations between endofunctors on \((\text{Top}/X_{-1})^C\)
\[
p^*p_* \longrightarrow \longrightarrow \text{id}_{(\text{Top}/X_{-1})^C},
\]
where \(p^* : (\text{Top}/X_{-1})^D \rightarrow (\text{Top}/X_{-1})^C\) is given by precomposition with \(p\). When evaluated at an augmented \(C\)-space, the zig-zag consists of weak equivalences of augmented \(C\)-spaces.

**Proof.** The value \(p_\bullet X_*\) for \(X_* \in (\text{Top}/X_{-1})^C\) is given by the homotopy left Kan-extension of \(X_*\) along \(p\), i.e. by mapping \(D \in \text{ob} \ D\) to the bar construction \(B(D(p(\_), D), C, X_*)\). For each \(C\) in \(C\), we have a zig-zag
\[
B(D(p(\_), p(C)), C, X_*) \leftrightarrow B(C(\_\leftarrow C), C, X_*) \longrightarrow X_C
\]
of spaces over \(X_{-1}\), in which the left arrow is induced by \(p\) and the right one by the augmentation \(B_*(C(\_\leftarrow C), C, X_*) \rightarrow X_C\) given by composing and evaluating. The latter admits an extra degeneracy by inserting the identity, so realizes to a weak equivalence (see Section 4.5 of \[Rie/one/four\]). Since \(p\) is a weak equivalence, the left arrow is one as well and the claim follows by naturality. \(\Box\)

### 1.3. Homology with local coefficients

A \textit{local system} on a pair of spaces \((X, A)\) with \(A \subseteq X\) is a functor \(F\) from the fundamental groupoid \(\Pi(X)\) to the category of abelian groups. It is called \textit{constant} if it is constant as a functor. For path-connected spaces, local systems can equivalently be described as modules over the fundamental group. Subspaces of spaces with local systems are implicitly equipped with the local system obtained by restriction. When we write \((X, A)\) for a map \(A \rightarrow X\) which is not necessarily an inclusion, we implicitly replace \(X\) by the mapping cylinder of \(A \rightarrow X\). A \textit{morphism} \((f; \eta)\) \textit{between with local systems} \((X, A; F)\) and \((Y, B; G)\) is a map of pairs \(f : (X, A) \rightarrow (Y, B)\) with a natural transformation \(\eta : F \rightarrow f^*G\) of functors on \(\Pi(X)\). A \textit{homotopy} between \((f_0; \eta_0)\) and \((f_1; \eta_1)\) from \((X, A; F)\) to \((Y, B; G)\) consists of a homotopy \(H_t : (X, A) \rightarrow (Y, B)\) of pairs from \(f_0\) and \(f_1\), such that
\[
\begin{array}{ccc}
F(\_\leftarrow \_\rightarrow \_\leftarrow \_\rightarrow \_\) & \xrightarrow{\eta_0} & G(f_0(\_\leftarrow \_\rightarrow \_\leftarrow \_\rightarrow \_)) \\
\downarrow & & \downarrow \\
\eta_1 & & G(f_1(\_\leftarrow \_\rightarrow \_\leftarrow \_\rightarrow \_))
\end{array}
\]
commutes. Taking singular chains with coefficients in a local system provides a homotopy invariant functor \(C_\bullet\), from pairs with local systems to chain complexes. The homology \(H_\bullet(X, A; F)\) of \((X, A; F)\) is the \textit{homology of the pair \((X, A)\) with coefficients in the local system \(F)\). A grading on \(X\) results in an additional grading \(H_\bullet(X, A; F) = \bigoplus_{n \in \mathbb{N}} H_n(X, A; F)\) of the homology. For a morphism \((X, A; F) \rightarrow (Y, B; G)\), the homology of the algebraic mapping cone of \(C_\bullet(X, A; F) \rightarrow C_\bullet(Y, B; G)\) is denoted by \(H_\bullet((Y, B; G), (X, A; F))\). If \(X\) and \(Y\) are graded and the underlying map of spaces \(X \rightarrow Y\) is of degree \(k\), then \(H_\bullet((Y, B; G), (X, A; F))\) inherits an extra grading
\[
H_\bullet((Y, B; G), (X, A; F)) = \bigoplus_{n \in \mathbb{N}} H_n((Y_{n+k}, B_{n+k}; G), (X, A; F)).
\]
We refer to Chapter IV of \[Whi/78\] for more details on homology with local coefficients.

### 1.4. Augmented semi-simplicial spaces

Denoting by \([p]\) the linearly ordered set \(\{0, 1, \ldots, p\}\), the \textit{semi-simplicial category} is the category \(\Delta_{\text{simp}}\) with objects \([0], [1], \ldots\) and order-preserving injections between them. A \textit{semi-simplicial space} \(X_*\) is a functor \(\Delta_{\text{simp}}^{op} \rightarrow \text{Top}\), i.e. a contravariant \(\Delta_{\text{simp}}\)-space or equivalently a collection of spaces \(X_p\) for \(p \geq 0\) together with \textit{face maps} \(d_i : X_p \rightarrow X_{p-1}\) for \(0 \leq i \leq p\) satisfying the \textit{face relations} \(d_i d_j = d_{i-1} d_i\) for \(i < j\). An \textit{augmented semi-simplicial space} \(X_* \rightarrow X_{-1}\) is a contravariant augmented \(\Delta_{\text{simp}}\)-space. As for simplicial spaces, augmented semi-simplicial spaces \(X_* \rightarrow X_{-1}\) have a \textit{geometric realization}, which is a space over \(X_{-1}\) denoted by \([X_*] \rightarrow X_{-1}\) (see \[ER/17\] Sect. 1.2]).
Lemma 1.7. The homotopy colimit of an augmented semi-simplicial space \( X_\bullet \to X_{-1} \) and its realization are weakly equivalent when considered as spaces over \( X_{-1} \).

Proof. The classifying space of the overcategory \( \Delta_\text{aug} / [p] \) is isomorphic to the \( p \)-th topological standard simplex \( \Delta^p \) as the nerve of \( \Delta_\text{aug} / [p] \) is the barycentric subdivision of the \( p \)-th standard simplicial simplex. This extends to an isomorphism \( \Delta^* \cong B(\Delta_\text{aug} / -) \) of co-simplicial spaces from which \(^{[17]}\) implies that, given an augmented semi-simplicial space \( X_\bullet \to X_{-1} \), the thin realization (see \(^{[17]}\) Sect. 1.2) of \( B_\bullet(\ast, \Delta_\text{aug}, X_\bullet) \) is homeomorphic over \( X_{-1} \) to the realization of \( X_\bullet \). But for augmented \( C \)-spaces \( X_\bullet \to X_{-1} \) on a discrete category \( C \), the fat and the thin geometric realization of \( B_\bullet(\ast, C, X) \) are weakly equivalent as maps over \( X_{-1} \) since \( B_\bullet(\ast, C, X) \) is good in the sense of \(^{[7]}\) Prop. A.1.

Given an augmented semi-simplicial space \( X_\bullet \to X_{-1} \) and a local system \( F \to X_{-1} \), we obtain local systems on the spaces of \( p \)-simplices \( X_p \) and on the realization \( [X_\bullet] \) by pulling back \( F \) along the augmentation. Filtering \( [X_\bullet] \) by skeleta induces a strongly convergent homologically graded spectral sequence

\[ E_{p,q}^1 \cong H_q(X_p; F) \Longrightarrow H_{p+q+1}(X_{-1}, [X_\bullet]; F) \]

defined for \( q \geq 0 \) and \( p \geq -1 \) (see \(^{[17]}\) Sect. 1.4) and \(^{[5]}\) Lem. 2.7). The differential \( d^1: H_q(X_p; F) \to H_q(X_{p-1}; F) \) is the alternating sum \( \sum_{i=0}^p (-1)^i(d_i) \) of the morphisms induced by the face maps for \( p \geq 0 \) and induced by the augmentation for \( p = 0 \). Given a morphism of augmented semi-simplicial spaces \( (f_\bullet, f_{-1}): (X_\bullet \to X_{-1}) \to (Y_\bullet \to Y_{-1}) \), local systems \( F \to X_{-1} \) and \( G \to Y_{-1} \) and a morphism of local systems \( F \to f_{-1}^*G \), we obtain a morphism of augmented semi-simplicial objects in spaces with local systems, resulting in a relative version of the spectral sequence

\[ E_{p,q}^1 \cong H_q(Y_p; G)(X_p; F) \Longrightarrow H_{p+q+1}(Y_{-1}, [Y_\bullet]; G)(X_{-1}, [X_\bullet]; F). \]

If \( X_{-1} \) is graded, all spaces \( X_p \) and \( [X_\bullet] \) inherit a grading by pulling back \( g_{X_{-1}} \) along the augmentation. This results in a third grading of the spectral sequence \( (3) \) but since the differentials preserve the additional grading, it is just a sum of spectral sequences, one for each \( n \in \mathbb{N} \). Analogously, if the map \( f_{-1} \) of \( (f_\bullet, f_{-1}): (X_\bullet \to X_{-1}) \to (Y_\bullet \to Y_{-1}) \) is a map of degree \( k \) for gradings on \( X_{-1} \) and \( Y_{-1} \), the spectral sequence \( (4) \) splits as a sum with the \( n \)-th summand of the \( E_1 \)-page being \( E_{p,q,n}^1 \cong H_q((Y_{p+n}; k), G)(X_{p;n}; F)) \).

1.5. Semi-simplicial spaces up to higher coherent homotopy. In the course of this work, a number of constructions which are key to the theory, require choices of contractible ambiguity. To deal with such, we are led to consider objects that are good as semi-simplicial spaces, but only in a homotopical sense. To model those, let us define an (augmented) semi-simplicial space up to higher coherent homotopy as an (augmented) \( \tilde{\Delta}_\text{aug} \)-space \( X_\bullet \), defined on any topologically enriched category \( \tilde{\Delta}_\text{aug} \) together with a weak equivalence \( \tilde{\Delta}_\text{aug} \to \Delta_{\text{aug}} \). Roughly speaking, \( \tilde{\Delta}_\text{aug} \) is a category with the same objects as \( \Delta_{\text{aug}} \) and a (weakly) contractible space of choices for all morphisms in \( \Delta_{\text{aug}} \). In particular, a \( \tilde{\Delta}_\text{aug} \)-space \( X_\bullet \) includes spaces \( X_p \) for \( p \geq 0 \) together with face maps \( d_i: X_p \to X_{p-1} \) which are unique up to homotopy.

By precomposing with \( \tilde{\Delta}_\text{aug} \to \Delta_{\text{aug}} \), every semi-simplicial space is a \( \tilde{\Delta}_\text{aug} \)-space and in fact, in the light of Lemma 1.6, every \( \tilde{\Delta}_\text{aug} \)-space is equivalent to one arising in this way. By virtue of this rectification result and Lemma 1.7, all homotopy invariant constructions for semi-simplicial spaces carry over to \( \tilde{\Delta}_\text{aug} \)-spaces, so in particular we have analogues of the spectral sequences \( (3) \) and \( (4) \) with the differentials being the alternating sum \( \sum_{i=0}^p (-1)^i(d_i) \) of morphisms induced by (weakly) contractible choices \( d_i \) of face maps.

A \( \tilde{\Delta}_\text{aug} \)-space \( X_\bullet \) induces a simplicial set \( \pi_0(X_\bullet) \) by taking path components, together with a morphism \( X_\bullet \to \pi_0(X_\bullet) \) of \( \tilde{\Delta}_\text{aug} \)-spaces which is a weak equivalence if and only if \( X_\bullet \) is homotopy discrete, i.e. takes values in homotopy discrete spaces.

To emphasize similarities and by abuse of notation justified by Lemma 1.7 we call the homotopy colimit \( \text{hocolim}_{\tilde{\Delta}_\text{aug}} X_\bullet \to X_{-1} \) of an augmented \( \tilde{\Delta}_\text{aug} \)-space \( X_\bullet \) its realization and denote it by \( [X_\bullet] \to X_{-1} \).

2. The canonical resolution of an \( E_1 \)-module over an \( E_2 \)-algebra

2.1. \( E_1 \)-modules over \( E_n \)-algebras and their fundamental groupoids. We recall the notion of an \( E_1 \)-module over an \( E_n \)-algebra and explain its relation to modules over monoidal categories.

By an operad, we mean a symmetric colored operad in spaces (see e.g. \(^{[8]}\)) For a subspace \( X \subseteq \mathbb{R}^n \), we let \( D^k(X) \) be the space of tuples of \( k \) disjoint embeddings of the closed unit disc \( D^k \) into \( X \) that are...
compositions of scalings and translations. Recall the one-colored operad $D^\ast(D^n)$ of little $n$-discs [BV73, May72] with $k$-operations $D^k(D^n)$ and operadic composition induced by composition of embeddings.

**Definition 2.1.** Let $SC_n$ be the colored operad with colors $m$ and $a$ whose space of operations $SC_n(m^k, a^l; m)$ is empty for $k \neq 1$ and for $k = 1$ is the space of pairs $(s, \phi) \in [0, \infty) \times D^l(R^n)$ such that $\phi \in D^l((0, s) \times (-1, 1)^{n-1})$. The space $SC_n(m^k, a^l; a)$ is empty for $k \neq 0$ and equals $D^l(D^n)$ otherwise. The composition restricted to the $a$-color is given by the composition in $D^\ast(D^n)$ and the composition

$$y: SC_n(m, a^l; m) \times \left( SC_n(m, a^k; m) \times SC_n(a^{l_1}; a) \times \ldots \times SC_n(a^{l_i}; a) \right) \rightarrow SC_n(m, a^{k+l}; m)$$

for $i = \sum_i i_j$ by mapping an element $((s, \phi), ((s', \psi), (\phi^1, \ldots, \phi^l)))$ in the codomain to $(s' + s, (\psi, \phi_1 \circ \phi^1, \ldots, \phi_l \circ \phi^l)) \in SC_n(m, a^{k+l}; m)$. In words, it is defined by adding the parameters, putting the discs of $SC_n(m, a^k; m)$ to the left of the ones of $SC_n(m, a^l; m)$ and composing the embeddings of discs of the $SC_n(a^{l_i}; a)$ factors with the ones of $SC_n(m, a^{l_i}; m)$ as in operad of little $n$-discs, illustrated by Figure 1.

![Figure 1. The operadic composition in $SC_n$](image)

The canonical embedding $D^\ast(D^n) \rightarrow D^\ast(D^{n+1})$ of little disc operads, as described e.g. in [Fre17, Sect. 4.1.5], extends to an embedding of two-colored operads $SC_n \rightarrow SC_{n+1}$ by taking products with $(-1, 1)$ from the right. As a consequence, any algebra over $SC_{n+1}$ is also one over $SC_n$.

We call two colored operads *weakly equivalent*, if there is a zig-zag between them which consists of morphisms of operads that are weak homotopy equivalences on all spaces of operations.

**Remark 2.2.** The operad $SC_n$ is weakly equivalent to a suboperad of the $n$-dimensional version of the Swiss-Cheese operad [Vor99].

**Definition 2.3.** An $E_{1,n}$-operad is an operad weakly equivalent to $SC_n$. A graded $E_1$-module $M$ over an $E_n$-algebra $\mathcal{A}$ is an algebra $(M, \mathcal{A})$ over an $E_{1,n}$-operad considered as an operad in graded spaces, where $M$ corresponds to the $m$- and $\mathcal{A}$ to the $a$-color.

**Remark 2.4.** In other words, a graded $E_1$-module $M$ over an $E_n$-algebra $\mathcal{A}$ is an algebra $(M, \mathcal{A})$ over an (ungraded) $E_{1,n}$-operad, together with gradings $g_M: M \to N$ and $g_{\mathcal{A}}: \mathcal{A} \to \bar{N}$ which are compatible with the algebra structure, i.e. the degree of a multiplication of points is the sum of their degrees.

The fundamental groupoid of an algebra over the little 2-discs operad has a braided monoidal groupoid structure with the multiplication induced by the choice of a 2-operation [Fre17, Ch. 5-6]. Similarly, for a graded algebra $(M, \mathcal{A})$ over an $E_{1,2}$-operad $\mathcal{O}$ and operations $c \in \mathcal{O}(m,a;m)$ and $d \in \mathcal{O}(a_2; a)$, the fundamental groupoid $\Pi(\mathcal{A})$ is a graded braided monoidal groupoid with multiplication induced by $d$ and $\Pi(M)$ becomes a graded right-module over $\Pi(\mathcal{A})$ with the action induced by $c$. In other words the functor $\otimes: \Pi(M) \times \Pi(\mathcal{A}) \rightarrow \Pi(M)$ induced by $c$ is associative and unital up to coherent natural isomorphisms and compatible with the grading on $\Pi(M)$ and $\Pi(\mathcal{A})$ induced by the grading on $M$ and $\mathcal{A}$.

**Remark 2.5.** Since the path-components of a space coincide with the categorical path components of its fundamental groupoid, a grading on an $E_1$-module over an $E_n$-algebra is in fact equivalent to a grading of the induced right-module $(\Pi(M), \otimes)$ over the braided monoidal groupoid $(\Pi(\mathcal{A}), \otimes, b, 0)$. 


2.2. **The canonical resolution.** Let $M$ be a graded $E_1$-module over an $E_2$-algebra $A$ with underlying $E_{1,2}$-operad $O$ and structure maps $\theta$. We call a point $X \in A$ of degree 1 a stabilizing object for $M$ and define the stabilization map with respect to a stabilizing point $X$

$$s: M \to M$$

as the multiplication $\theta(c, \ldots, X)$ by $X$ using an operation $c \in O(m, a, m)$ which we fix once and for all. As $X$ has degree 1, so does the stabilization map which hence restricts to maps $s: M_n \to M_{n+1}$ between the subspaces of consecutive degrees for all $n \geq 0$. It will be convenient to denote the stabilization map also by $(\cdot \oplus X): M \to M$ and we use the two notations interchangeably.

**Remark 2.6.** We chose to restrict to stabilizing objects of degree 1 to simplify the exposition. However, by keeping track of the gradings, the developed theory generalizes to stabilizing objects of arbitrary degree.

In the following, we assign to a graded $E_1$-module $M$ over an $E_2$-algebra with stabilizing object $X$ an augmented semi-simplicial space $R_*(M) \to M$ up to higher coherent homotopy, called the *canonical resolution*. It will be given as an augmented $\Lambda_{inj}$-space for a topologically enriched category $\Lambda_{inj}$ that is weakly equivalent to the semi-simplicial category. We begin by constructing $\Lambda_{inj}$ from the underlying $E_{1,2}$-operad $O$ for which we recall the braided analogue of the category of finite sets and injections as introduced in [RW17].

**Definition 2.7.** Define the category $UB$ having objects $[0], [1], \ldots$ as in $\Lambda_{inj}$, no morphisms from $[q]$ to $[p]$ for $q > p$ and $UB([q], [p])$ for $q \leq p$ given by the cosets $B_{p+1}/B_{p+1}$, where $B_i$ denotes the braid group on $i$ strands and $B_{p-q}$ acts on $B_{p+1}$ from the right as the first $(p-q)$ strands. The composition is defined as

$$UB([1], [q]) \times UB([q], [p]) \to UB([1], [p])$$

$$[b], [b'] \mapsto [b'(1^{p-q} \oplus b)]$$

where $1^{p-q} \oplus b$ is the braid obtained by inserting $(p-q)$ trivial strands next to $b$.

![Figure 2. The categorical composition in $UB$](image)

The category $UB$ admits a canonical functor to the category $FI$ of finite sets and injections by sending a class in $B_{p+1}/B_{p-q}$ to the injection obtained by following the last $(q+1)$ strands of a representing braid. Visualizing $UB$ as indicated by Figure 2, two braids represent the same morphism if and only if they differ by a braid of the $\circ$-ends. Following the braids of the upper $\bullet$-ends to the lower ends gives the induced injections.

On the subcategory $\Lambda_{inj} \subseteq FI$, the functor $UB \to FI$ admits a section: Considering $\coprod_{n \geq 0} B_n$ as the free braided monoidal category on one object $X$, the section is given by mapping the $i$th face map $d_i \in \Lambda_{inj}([p-1], [p])$ to the class of $b^{-1}_{X^{\otimes i}, X} \oplus X^{\otimes p-i-1} \in B_{p+1}$.

**Remark 2.8.** In the language of [RW17], the category $UB$ is the *free pre-braided monoidal category on one object* [RW17 Sect. 1.2]. Unwinding the definitions, the semi-simplicial sets $W_n(A, X)$ [RW17 Section 2], associated to objects $A$ and $X$ of a pre-braided monoidal category $D$, equals the composition

$$\Lambda_{inj}^{op} \to UB^{op} \to D^{op} \to Sets,$$

in which the first arrow is the described section, the second is induced by $X$ and the third is $D(\cdot, A \oplus X^{\otimes n})$.

In the following, we introduce topological analogues of $UB$ and $\Lambda_{inj}$ for any $E_{1,2}$-operad $O$. To that end, denote by $O(k)$ the space obtained from $O(m, a^k; m)$ by quotienting out the action of the symmetric group $\Sigma_k$ on the $a$-inputs. To simplify the construction, we assume that the quotient maps $O(m, a^k; m) \to O(k)$
are Serre fibrations, although this is not necessary (see Remark 2.23). As the operadic composition \( \gamma \) on \( O \) is equivariant, it induces composition maps \( \gamma(c_\ast, 1_k^+): O(k) \times O(l) \to O(k + l) \). The fixed operation \( c \in O(1) \) used to define the stabilization map yields iterated operations \( c_k \in O(k) \) by setting \( c_0 \) as the unit \( 1_m \) and \( c_{k+1} \) inductively as \( \gamma(c; c_k, 1_k) \).

**Definition 2.9.** Define a topologically enriched category \( UO = U(O, c) \) with objects \([0], [1], \ldots \) and

\[
UO([q], [p]) = \{ (d, \mu) \in O(p - q) \times \text{Path}_{c_{p+1}} O(p + 1) \mid \mu(\ell(\mu)) = \gamma(c_{q+1}; d, 1_q^{q+1}) \},
\]

where \( \text{Path}_{c_{p+1}} O(p + 1) \) is the path of spaces in \( O(p + 1) \) starting at \( c_{p+1} \). The composition is given by

\[
UO([l], [q]) \times UO([q], [p]) \to UO([l], [p])
\]

\[
((e, \xi), (d, \mu)) \mapsto (\gamma(e; d, 1_e^{q+1}), \mu \cdot \gamma(\xi; d, 1_q^{q+1}))
\]

as visualized by Figure 3. Using Moore paths, this is associative and unital by the respective properties of the operadic composition. The construction is homotopy invariantly functorial in \( (O, c) \) and independent of \( c \) up to weak equivalence.

---

**Remark 2.10.** Using Quillen’s bracket-construction for modules over monoidal categories [Gra76, p. 219], the category \( U\mathcal{B} \) is given by \( \langle \mathcal{B}, \mathcal{B} \rangle \), where \( \mathcal{B} = \bigsqcup_{n \geq 0} B_n \) is the free braided monoidal category acting on itself. Similarly, \( UO \) can be obtained via an analogue of Quillen’s construction for monoidal categories internal to spaces, applied to the path-category of the monoid \( \bigsqcup_{n \geq 0} O(n) \).

**Lemma 2.11.** The category \( UO \) is homotopy discrete and we have \( \pi_0(UO) \cong U\mathcal{B} \).

**Proof.** By the homotopy invariance of \( U(\_ , \_ ) \), it suffices to prove the claim for \( O = SC_2 \). Mapping embeddings of discs to their center yields a homotopy equivalence from the space of operations \( SC_2(n) \) to the unordered configuration space \( C_n(R^2) \) of the plane, which is an Eilenberg-MacLane space \( K(B_n, 1) \) for the braid group \( B_n \). On fundamental groups, the map \( \gamma(c_{q+1}; 1_q^{q+1}) : O(p - q) \to O(p + 1) \) is injective as it is given by including \( B_{p-q} \) in \( B_{p+1} \) as the first \( p - q \) strands. From this, one concludes that its homotopy fiber \( \text{hofib}_{c_{p+1}}(\gamma(c_{q+1}; 1_q^{q+1})) = UO([q], [p]) \) is homotopy discrete with path components \( B_{p+1}/B_{p-q} \) and that, via this equivalence, the composition coincides with the one of \( U\mathcal{B} \). \( \Box \)

**Definition 2.12.** The thickening of the semisimplicial category \( \Delta_{mj} \) associated to an \( E_{1,2} \)-operad \( O \) is the subcategory \( \Delta_{mj} \subseteq UO \) obtained by restricting \( UO \) to the path components hit by the section \( \Delta_{mj} \to U\mathcal{B} \cong \pi_0(UO) \). It comes with a weak equivalence to \( \Delta_{mj} \), induced by the functor \( UO \to \mathcal{F}1 \).

**Definition 2.13.** Let \( M \) be a graded \( E_1 \)-module over an \( E_2 \)-algebra with stabilizing object \( X \). Define the functor \( B_\ast(M): UO^{op} \to \mathcal{F}op \) on objects by sending \([p]\) to

\[
B_p(M) = \{(A, \xi) \in M \times \text{Path } M \mid \xi(\xi(\ell)) = s_{p+1}(A)\}
\]

and on morphisms by

\[
UO([q], [p]) \times B_p(M) \to B_q(M)
\]

\[
((d, \mu), (A, \xi)) \mapsto (\theta(d; A, X^{p-q}), \xi \cdot \theta(\mu; A, X^{p+1}))
\]

which is functorial by the associativity of the module-structure and the composition of Moore paths. Evaluating paths at zero defines an augmentation \( B_\ast(M) \to M \) of the \( UO \)-space \( B_\ast(M) \).

---

**Figure 3.** The categorical composition in \( UO \)**
Definition 2.14. Let $M$ be a graded $E_1$-module over an $E_2$-algebra with stabilizing object $X$.

(i) The canonical resolution of $M$ is the augmented $\Delta_{inj}$-space

$$R_*(M) \rightarrow M$$

obtained by restricting the augmented $UO$ space $B_*(M)$ to the semi-simplicial thickening $\Delta_{inj} \subseteq UO$.

(ii) The space of destabilizations of a point $A \in M$ is the $\Delta_{inj}$-space $W_*(A)$ defined as the fiber of the canonical resolution $R_*(M) \rightarrow M$ at $A$.

Remark 2.15. Unwrapping the definition, the space of $p$-simplices of the canonical resolution $R_*(M)$ is

$$R_p(M) = \{(A, \zeta) \in M \times \text{Path} M | \zeta(\ell(\zeta)) = s^{p+1}(A)\}$$

and the augmentation to $M$ is the evaluation at zero. Hence, the space $W_p(A)$ of $p$-simplices of the space of destabilizations of $A$ is given by the homotopy fiber $\text{hofib}_A(s^{p+1})$. There is a contractable space of $i$th face maps of the canonical resolution, but a possible choice is given by

$$\hat{d}_i: R_p(M) \rightarrow R_{p-1}(M)$$

where $\mu \in \Omega_{\epsilon_{p+1}}(O(p + 1))$ is any loop in $O(p + 1) = O(m, s^{p+1}; m)/\Sigma_{p+1}$ corresponding to the braid

$$b_{\epsilon_{p+1}}^A \otimes X^{\otimes p-1} \otimes B_{p+1} \simeq \Omega B(1_{p+1}) \approx \Omega \epsilon_{p+1} O(p + 1).$$

Remark 2.16. We borrowed the term space of destabilizations from [RW17] where it stands for certain semi-simplicial sets $W_*(A, X)$ associated to a braided monoidal groupoid. In Section 3.3 it is explained that these semi-simplicial sets are special cases of the spaces of destabilizations in our sense.

Remark 2.17. The map $R_*(M) \rightarrow M$ is the standard construction for replacing $s^{p+1}: M \rightarrow M$ by a fibration, so $R_*(M) \rightarrow M$ is fibrant in the sense of Section 1.2. Hence its fiber $W_*(A)$ at $A$ is equivalent to its homotopy fiber $\text{hofib}_A(R_*(M))$, so by virtue of Lemma 1.5 the homotopy fiber at $A$ of the realization $|R_*(M)| \rightarrow M$ is equivalent to $|W_*(A)|$. In particular, the canonical resolution of $M$ is graded $(p \circ g_M)$-connected in degree $\geq m$ for a function $\varphi: \mathbb{N} \rightarrow \mathbb{Q}$ and a number $m \geq 0$ if and only if the spaces of destabilizations $W_*(A)$ of $(\{g_M(A)\} - 1)$-connected for all points $A \in M$ with degree $m \leq g_M(A) < \infty$. As points in the same path component have equivalent homotopy fibers, it is sufficient to check one point in each path component.

Remark 2.18. A priori, the canonical resolution depends on the choice of $X$ within its path-component and the choices of the operation $c$ and the isomorphism $\pi_0(UO) \cong UB$, but its realization turns out to be independent of these choices, at least up to weak equivalence.

Example 2.19. Recall the free $E_2$-algebra on a point $A^{\text{free}} = \bigsqcup_{n \geq 0} O(c^n; a)/\Sigma_n$, graded in the evident way. It has the free $E_1$-module on a point $M^{\text{free}} = \bigsqcup_{n \geq 0} O(c^n; a)/\Sigma_n$ as graded $E_1$-module over it. Choosing the unit $1 \in O(c; a)$ as the stabilizing object, the space of destabilizations $W_*(c_{p+1})$ is the $\Delta_{inj}$-space obtained by restricting the $UO$-space $UO(c, [p])$ to $\Delta_{inj}$. As the category $UO$ is homotopy discrete with $\pi_0(UO) \cong UB$ by Lemma 2.11 the $\Delta_{inj}$-space $W_*(c_{p+1})$ is equivalent to the semi-simplicial set given as the composition of the section $\Delta_{inj}^{op} \rightarrow UB^{op}$ with $UB(\ast, [p])$.

Remark 2.20. The choice of a stabilizing object $X \in \mathcal{A}$ for a graded $E_1$-module $M$ over an $E_2$-algebra $\mathcal{A}$ induces a graded $E_1$-module structure on $M$ over $\mathcal{A}^{\text{free}}$. The two canonical resolutions of $M$ when considered as an module over $\mathcal{A}$ or over $\mathcal{A}^{\text{free}}$ are identical. In fact, all our constructions and results solely depend on the induced module structure of $M$ over $\mathcal{A}^{\text{free}}$ and are, in that sense, independent of the $E_2$-algebra $\mathcal{A}$.

Remark 2.21. Let $M$ be a graded $E_1$-module over an $E_2$-algebra with stabilizing object $X$ and consider $M$ as a graded $E_1$-module over $\mathcal{A}^{\text{free}}$ (see Remark 2.20). For a union of path-components $M' \subseteq M$, closed under the multiplication by $X$, we define a new grading on $M$ as an $E_1$-module over $\mathcal{A}^{\text{free}}$ by modifying the original grading on $M'$ by assigning the complement of $M'$ degree $\infty$ and leaving the grading on $M'$ unchanged. We call $M$ with this new grading the localization of $M'$. An example for such a subspace $M'$ is given by the objects stably isomorphic to an object $A \in M$ by which we mean the union of the path components of $A \otimes X^{\otimes n}$ for all $n \geq 0$. 
Example 2.22. Let $M$ be a graded $E_1$-module over an $E_2$-algebra $A$ and $G$ a group acting on $M$ preserving the grading. If the actions of $A$ and $G$ on $M$ commute, then the Borel construction $EG \times_G M$ inherits a graded $E_1$-module structure. The choice of a point in $EG$ induces morphism

$$
\begin{array}{c}
R_n(M) \rightarrow R_n(EG \times_G M) \\
\downarrow \quad \downarrow \\
M \rightarrow EG \times_G M
\end{array}
$$

of augmented $\tilde{\Lambda}_{m!}$-spaces which induces weak equivalences on homotopy fibers. An application of Lemma 1.5 implies that the respective canonical resolutions have the same connectivity.

Remark 2.23. Some constructions of this section work in greater generality. The category $U\mathcal{O}$ and the augmented $U\mathcal{O}$-space $B_\mathcal{O}(M)$ can be defined for any colored operad and $U\mathcal{O}$ still admits a functor to $\mathcal{I}$, but might not be homotopy discrete nor admit a section on $\Delta_{m!} \subseteq \mathcal{I}$. The point-set assumption on the action of $S_k$ on $O(m, a^k; m)$ can be avoided by constructing $U\mathcal{O}$ using $O(m, a^k; m)$ instead of $O(k)$ which involves taking care of permutations corresponding to preimages of the quotient map $O(m, a^k; m) \rightarrow O(k)$.

Remark 2.24. If $g_M$ is a grading of $M$, then so is $g_M + m$ for any fixed number $m \geq 0$. Consequently, if the canonical resolution of $M$ is graded $(\frac{g_M - m + k}{k})$-connected for a number $m \geq 2$, then we can apply Theorem A and C to $M$ with the grading $g_M + (m - 2)$, which results in a shift in the stability range. However, requiring more specific connectivity assumptions results in the following slight improvements of the stability ranges in Theorem A which can be obtained by adapting the ranges in its proof in Section 3 appropriately.

(i) If the canonical resolution is graded $(\frac{g_M - m + k}{k})$-connected for a number $m \geq 3$, the surjectivity range in Theorem A for constant coefficients can be improved from $i \leq \frac{n - m + k}{k}$ to $i \leq \frac{n - m + k + 1}{k}$ and the one for abelian coefficients from $i \leq \frac{n - m + k}{k}$ to $i \leq \frac{n - m + k}{k}$.

(ii) If the canonical resolution is graded $(g_M - 1)$-connected in degrees $\geq 1$, then the isomorphism range in Theorem A for constant coefficients can be improved from $i \leq \frac{n - 1}{2}$ to $i \leq \frac{n}{2}$, similar to the proof of [Ran15] (Thm. 5.1) for symmetric groups.

2.3. The stable genus. We extend the notion of the stable genus of a manifold as introduced in [GR17a] to our context, which provides us with a general way of grading modules over a braided monoid categories and, by Remark 2.25, also of grading $E_1$-modules over $E_2$-algebras.

Let $(\mathcal{A}, \otimes)$ be a right-module over a braided monoidal groupoid $(\mathcal{A}, \otimes, b, 0)$. Recall the free braided monoidal category in one object $\mathcal{B} = \bigoplus_{n \geq 0} B_n$, built from the braids groups. A choice of an object $X$ in $\mathcal{A}$ induces a functor $\mathcal{B} \rightarrow \mathcal{A}$ and hence a right-module structure on $M$ over $\mathcal{B}$. As a module over $\mathcal{B}$, a grading of $M$ which is compatible with the canonical grading on $\mathcal{B}$ is equivalent to a grading $g_M$ on $M$ as a category such that $g_M(A \otimes X) = g_M(A) + 1$ for all $A$ in $\mathcal{M}$.

Definition 2.25. Let $X$ be an object of $\mathcal{A}$.

(i) The $X$-genus of an object $A$ of $M$ is defined as

$$
g^X(A) = \text{sup}\{k \mid \text{there exists an object } B \in M \text{ with } B \otimes X^\otimes k \cong A\} \in \mathbb{N}.
$$

(ii) The stable $X$-genus of $A \in \text{ob } M$ is defined as

$$
g^X(A) = \text{sup}\{g^X(A \otimes X^\otimes k) - k \mid k \geq 0\} \in \mathbb{N}.
$$

As $g^X(A \otimes X) = g^X(A) + 1$ holds by definition, the stable $X$-genus provides a grading of $M$ when considered as a module over $\mathcal{B}$ via $X$. This stands in contrast with the (unstable) $X$-genus which does in general not define a grading since the inequality $g^X(A) + 1 \leq g^X(A \otimes X)$ might be strict.

For an $E_1$-module $M$ over an $E_2$-algebra $\mathcal{A}$, the choice of a point $X \in \mathcal{A}$ induces an $E_1$-module structure on $M$ over the free $E_2$-algebra on a point $\mathcal{A}^\text{free}$ (see Remark 2.20). After taking fundamental groupoids, this gives the module structure of $\Pi(M)$ over $\mathcal{B}$ discussed above, so the stable $X$-genus provides a grading for $M$ as an $E_1$-module over $\mathcal{A}^\text{free}$.

Remark 2.26. If the connectivity assumption of Theorem A is satisfied for an $E_1$-module $M$, graded with the stable $X$-genus, then the cancellation result Corollary B implies $g^X(A \otimes X) = g^X(A) + 1$ for objects $A$ of positive stable genus, which in turns implies that for such $A$, the genus and the stable genus coincide.
3. Stability with constant and abelian coefficients

Let $\mathcal{M}$ be a graded $E_1$-module over an $E_2$-algebra with stabilizing object $X$. In this section, we prove Theorem A via a spectral sequence constructed from the canonical resolution $R_\ast(\mathcal{M}) \to \mathcal{M}$. We consider all spaces $R_\ast(\mathcal{M})$ and the realization $[R_\ast(\mathcal{M})]$ as being graded by pulling back the grading from $\mathcal{M}$ along the augmentation.

3.1. The spectral sequence. Given a local system $L$ on $\mathcal{M}$, the canonical resolution gives rise to a tri-graded spectral sequence

$$E^1_{p,q,n} = \begin{cases} H_q(R_p(\mathcal{M})_n; L) & \text{if } p \geq 0 \\ H_p(\mathcal{M}_n; L) & \text{if } p = -1 \end{cases} \Rightarrow H_{p+q+1}(\mathcal{M}_n, [R_\ast(\mathcal{M})]_n; L),$$

with differential $d^1: E^1_{p,q,n} \to E^1_{p-1,q,n}$. Induced by the augmentation for $p = 0$ and the alternating sum $\sum_{i=0}^n (-1)^i(\delta_i)$, for $p > 0$, where $\delta_i$ is any choice of $i$th face map of $R_\ast(\mathcal{M})$ (see Section 1.4.[1.5]). As the differentials do not change the $n$-grading, it is just a sum of spectral sequences, one for each $n \in \mathbb{N}$. To identify the $E^3$-page, we recall from Section 2.1 that the fundamental groupoid $(\Pi(\mathcal{M}), \otimes)$ is a graded right-module over the graded braided monoidal groupoid $(\Pi(\mathcal{M}), \otimes, b, 0)$.

Lemma 3.1. We have $E^1_{p,q,n+1} \cong H_q(M_{n-p};(s^{p+1})^*L)$ and $d^1: E^1_{p,q,n+1} \to E^1_{p-1,q,n+1}$ identifies with

$$\sum_{i=0}^n (-1)^i(\delta_i)_*: H_q(M_{n-p};(s^{p+1})^*L) \to H_q(M_{n-p+1};(s^p)^*L),$$

where $\eta_i$ denotes the natural transformation

$$L(\_ \otimes b^{\otimes i}_{X, \_}) : L(\_ \otimes X^{\otimes p+1}) \to L(\_ \otimes X^{\otimes p+1}).$$

In particular, $d^1$ corresponds for $p = 0$ to the stabilization $(s; id)_*: H_q(M_n; s^*L) \to H_q(M_{n+1}; L)$. Thus if $L$ is constant, $d^1$ identifies with $s_*: H_q(M_{n-p}; L) \to H_q(M_{n-p+1}; L)$. If $L$ is constant, the transformations $\eta_i$ coincide for all $i$, so the terms in the alternating sum cancel out.

Proof. By definition of the $i$th face maps of the canonical resolution (see Remark 2.15), the square

$$\begin{array}{ccc}
(M_{n-p};(s^{p+1})^*L) & \xymatrix{ \ar[r]^{(s; id)_*} & (R_p(M)_{n+1}; L) } \\
(M_{n-p+1};(s^p)^*L) & \xymatrix{ \ar[r]^{(s; id)_*} & (R_{p-1}(M)_{n+1}; L) }
\end{array}$$

commutes up to homotopy of spaces with local systems for all $p \leq n$, where $i$ denotes the canonical inclusion mapping $A$ to $(A, \text{const}_{p+1}(A))$. Taking homology proves the claimed identification as the canonical inclusions are homotopy equivalences and hence induce isomorphisms. If $L$ is constant, the transformations $\eta_i$ coincide for all $i$, so the terms in the alternating sum cancel out.

Lemma 3.2. If the local system $L$ is abelian, then the following compositions are homotopic for all $0 \leq i \leq p$

$$(M; (s^{p+2})^*L) \xymatrix{ \ar[r]^{(s; id)_*} & (M; (s^{p+1})^*L) } \xymatrix{ \ar[r]^{(s; id)_*} & (M; (s^p)^*L) }.$$

The proof of Lemma 3.2 uses a self-homotopy of $s^2 : \mathcal{M} \to \mathcal{M}$ which turns out to be crucial for many other arguments. It is given by

$$[0, 1] \times \mathcal{M} \to \mathcal{M},
\quad (t, A) \mapsto \theta(\mu(t); A, X^{\otimes 2})$$

using the $E_1$-module structure $\theta$, where $\mu$ is a choice of loop at $c_2 \in O(2)$ such that $[1_n, \mu] \in \pi_0(UO([1], [1]))$ corresponds to the braid $b_{X, X}^{-1} \in U\mathcal{B}([1], [1])$ via the isomorphism $\pi_0(UO) \cong U\mathcal{B}$ we fixed in Section 2.2. Since $\mu$ is unique up to homotopy, the homotopy of $s^2$ is unique up to homotopy of homotopies.

Proof of Lemma 3.2. By the recollection of Section 1.3, the selfhomotopy of $s^2$ extends to a homotopy of maps of spaces with local systems between the $i$th and $(i + 1)$st composition in question if the triangle

$$\begin{array}{ccc}
L(\_ \otimes X^{\otimes p+2}) & \xymatrix{ \ar[r]^{L(\_ \otimes b_X X \otimes X^{\otimes p-1})} & L(\_ \otimes X^{\otimes p+2}) } \\
L(\_ \otimes X^{\otimes p+2}) & \xymatrix{ \ar[r]^{L(\_ \otimes b_X X \otimes X^{\otimes p-1})} & L(\_ \otimes X^{\otimes p+2}) }
\end{array}$$

commutes up to homotopy.
commutes. The braid relations give \((\_ \oplus b_{X,Y} \otimes X^{\otimes p+1})(\_ \otimes X \otimes b_{X,Y} \otimes X^{\otimes p+1}) = \_ \oplus b_{X,Y} \otimes X^{\otimes p+1}\), so the claim follows if we show that \([b_{X,Y}, X] = [X \otimes b_{X,Y}, X] \) holds in the abelianization. But the braid relation \((b_{X,Y} \otimes X)(X \otimes b_{X,Y}) = (X \otimes b_{X,Y})(b_{X,Y} \otimes X)\) abelianizes to \([b_{X,Y}, X] = [X \otimes b_{X,Y}, X] \) from which the claimed identity follows by induction on \(i\).

3.2. The proof of Theorem A

We prove Theorem A by induction on \(n\) using the spectral sequence \([3]\).

As \([R_n(M)]_{n+1} \rightarrow M_{n+1}\) is assumed to be \((\frac{n-1-k}{k})\)-connected for \(k \geq 2\) in the constant and \(k \geq 3\) in the abelian coefficients case, the summand of degree \((n+1)\) of the spectral sequence converges to zero in the range \(p + q \leq \frac{n-1}{k}\). The differential \(d^1 : E^1_{p,q,n+1} \rightarrow E^1_{p-1,q,n+1}\) identifies with the stabilization map \((s, id) : H_q(M_n; s^p \mathbf{L}) \rightarrow H_q(M_{n+1}; L)\) by Lemma 3.1. Since there are no differentials to \(E^2_{p,q,n+1}\) for \(k \geq 1\), the stabilization has to be surjective for \(i = 0\) if \(E^0_{p,q,n+1} = 0\) for all \(n \geq 0\). In particular, this implies the case \(n = 0\) for both, constant and abelian coefficients because the isomorphism claims are vacuous.

Constant coefficients. Assume the claim holds for constant coefficients holds in degrees smaller than \(n\). By Lemma 3.1, the differential \(d^1 : E^1_{p,q,n+1} \rightarrow E^1_{p-1,q,n+1}\) identifies with \(s \colon H_q(M_n; p) \rightarrow H_q(M_{n+1}; p)\) for even \(p\) and is zero for odd \(p\). From the induction assumption, we draw the conclusion that \(E^2_{p,q,n+1}\) vanishes for \((p, q)\) if \(p\) is even with \(0 < p \leq n\) and \(q \leq \frac{n-3-k}{k}\), and for \((p, q)\) if \(p\) is odd with \(0 < p < n\) and \(q \leq \frac{n-p-3-k}{k}\). So in particular, \(E^2_{p,q} = 0\) if \(p = 0\) and \(q \leq \frac{n-3-k}{k}\). As \(d^1 : E^1_{p-1,q,n+1} \rightarrow E^1_{p,q,n+1}\) is zero for all \(i\), injectivity of \(s\) : \(H_q(M_n; p) \rightarrow H_q(M_{n+1}; p)\) holds if \(p \leq n\) and \(E^2_{p,q,n+1} = 0\) for \(p + q = i + 1\) with \(q < i\). This is the case for \(i = \frac{n-1}{k}\) as claimed, which follows from the established vanishing ranges of \(E^0\) and \(E^2\). Similarly, the map in question is surjective in degree \(i\) if \(E^0_{p,q,n+1} = 0\) and \(E^2_{p,q,n+1} = 0\) for \(p + q = i\) with \(q < i\).

Abelian coefficients. Assume the statement holds in degrees smaller than \(n\). The differential \(d^1 : E^1_{p,q,n+1} \rightarrow E^1_{p-1,q,n+1}\) identifies with \(\sum (-1)^i(s, \eta) : H_q(M_n; s^p \mathbf{L}) \rightarrow H_q(M_{n+1}; s^p \mathbf{L})\) by Lemma 3.2. In the range \((s, id) : H_q(M_n; s^p \mathbf{L}) \rightarrow H_q(M_{n+1}; s^p \mathbf{L})\) is surjective, \(d^1\) identifies with the stabilization \((s, id) : H_q(M_n; p) \rightarrow H_q(M_{n+1}; p)\) for even and vanishes for odd \(p\). By induction, this happens for \((p, q)\) with \(0 \leq p \leq n - 1\) and \(q \leq \frac{n-p-3-k}{k}\), so by using the induction again, we conclude \(E^2_{p,q,n+1} = 0\) for \((p, q)\) if \(p\) is even satisfying \(0 < p \leq n - 1\) and \(q \leq \frac{n-p-3-k}{k}\). For odd \(p\) with \(q < n - 1\) satisfying \(0 < p < n - 1\) and \(q \leq \frac{n-p-3-k}{k}\), the rest of the argument proceeds as in the constant case, adapting the ranges and using that \(d^1 : E^1_{p,q,n+1} \rightarrow E^1_{p,q,n+1}\) is zero for \(i = \frac{n-1}{k}\).

4. Stability with twisted coefficients

We introduce a notion of twisted coefficient systems and prove Theorem C. Many ideas in this section are inspired by Section 4 of [RW17] which is itself a generalization of work by Dwyer, van der Kallen and Ivanov [Dwy80, Dwy80, Ivan80, Kal80]. We use similar notation to [RW17] to emphasize analogies.

4.1. Coefficient systems of finite degree. This section serves to define coefficient systems of finite degree for graded modules over graded braided monoidal categories, e.g. for fundamental groupoids of graded \(E_2\)-modules over \(E_2\)-algebras as described in Section 2.1. For the following definitions, let \((\mathcal{A}, \otimes, b, 0)\) be a graded right-module over a braided monoidal category \((\mathcal{A}, \otimes, b, 0)\) in the sense of Section 1.1. We fix a stabilizing object \(X\), i.e. an object of \(\mathcal{A}\) of degree 1, and recall the free braided monoidal category \(\mathcal{B} = \prod_{n \geq 0} B_n\) in one object, built of the braided groups \(B_n\). The choice of \(X \in \mathcal{A}\) defines a canonical functor \(\mathcal{B} \rightarrow \mathcal{A}\), so in particular homomorphisms \(B_n \rightarrow \text{Aut}_{\mathcal{A}}(X^{\otimes n})\) and a module-structure on \(M\) over \(\mathcal{B}\).

Definition 4.1. A coefficient system \(F\) for \(M\) is a functor

\[F : M \rightarrow \mathcal{A}b\]

to the category of abelian groups, together with a natural transformation

\[\sigma^F : F \rightarrow F(\_ \otimes X),\]
called the structure map of $F$, such the image of the canonical morphism $B_m \to \text{Aut}(X \otimes^m m)$ acts trivially on the image of $(\sigma F)^m : F \to F(\_ \otimes X \otimes^m m)$ for all $m \geq 0$. A morphism between coefficient systems $F$ and $G$ for $\mathcal{M}$ is a natural transformation $F \to G$ which commutes with the structure maps $\sigma F$ and $\sigma G$.

**Remark 4.2.** The triviality condition in the definition of a coefficient system for $\mathcal{M}$ is an requirement for all $m \geq 0$. However, it is equivalent to just demand it for $m = 2$ since the braid group is generated by elementary braids of consecutive strands.

**Remark 4.3.** The category of coefficient systems for $\mathcal{M}$ is abelian, so in particular has (co)kernels. In fact, it is isomorphic to a category of abelian group-valued functors on a category $C(\mathcal{M})$ (see Remark 4.10).

**Definition 4.4.** Define the suspension $\Sigma F$ of a coefficient system $F$ as

$$\Sigma F = F(\_ \otimes X)$$

together with the structure map $\sigma F : \Sigma F \to \Sigma F(\_ \otimes X)$ given by the composition

$$\Sigma F = F(\_ \otimes X) \xrightarrow{\sigma F(\_ \otimes X)} F(\_ \otimes X \otimes^2) \xrightarrow{F(\_ \otimes b_{X,X})} F(\_ \otimes X \otimes^2) = \Sigma F(\_ \otimes X),$$

where the braiding ensures the triviality condition. The structure map $\sigma F$ of $F$ induces a morphism $F \to \Sigma F$ called the suspension map, whose (co)kernel is the kernel ker($F$) respectively cokernel coker($F$) of $F$. We call $F$ split if the suspension map is split injective in the category of coefficient systems.

**Remark 4.5.** The suspension map in fact gives rise to a natural transformation $\text{id} \to \Sigma$ of endofunctors on the category of coefficient systems for $\mathcal{M}$.

For the remainder of the section, we fix a coefficient system $F$ for $\mathcal{M}$.

**Definition 4.6.** We denote by $F_n$ for $n \geq 0$ the restriction of $F$ to the full subcategory $\mathcal{M}_n \subseteq \mathcal{M}$ of objects of degree $n$ and define the degree and split degree of $F$ at an integer $N$ inductively by saying that $F$ has

(i) (split) degree $\leq -1$ at $N$ if $F_n = 0$ for $n \geq N$,

(ii) degree $r$ at $N$ for $r \geq 0$ if ker($F$) has degree $-1$ at $N$ and coker($F$) has degree $(r - 1)$ at $(N - 1)$ and

(iii) split degree $r$ at $N$ for $r \geq 0$ if $F$ is split and coker($F$) is of split degree $(r - 1)$ at $(N - 1)$.

**Remark 4.7.** Note that for all $N \leq 0$, $F$ is of (split) degree $r$ at 0 if and only if it is of (split) degree $r$ at $N$ and that the property of being of (split) degree $r$ at 0 is independent of the chosen grading. However, being of degree $r$ at a positive $N$ depends on the grading. If $g_M$ is a grading for $\mathcal{M}$, then so is $g_M + k$ for any $k \geq 0$ and by induction on $r$, one proves that for $k \geq 0$, $F$ is of (split) degree $r$ at $N$ with respect to a grading $g_M$ if and only if it is of (split) degree $r$ at $(N - k)$ with respect to the grading $g_M + k$.

**Lemma 4.8.** The iterated suspension $\Sigma^i F$ for $i \geq 0$ is given by $\Sigma^i F = F_{n+i}(\_ \otimes X^{\otimes i})$ with structure map

$$\Sigma^i F = F(\_ \otimes X^i) \xrightarrow{\sigma F(\_ \otimes X^{\otimes i})} F(\_ \otimes X^{\otimes i} \otimes X) \xrightarrow{F(\_ \otimes b_{X,X})} F(\_ \otimes X^{\otimes i}) = \Sigma^i F(\_ \otimes X).$$

**Proof.** This follows by induction on $i$ using the braid relation $(b_{X,X} \otimes X^i)(X \otimes b_{X,X}) = b_{X,X \otimes X,X}.$

**Lemma 4.9.** Let $F$ be a coefficient system for $\mathcal{M}$.

(i) For all $i \geq 0$, $\Sigma^i(\ker(F))$ is isomorphic to ker($\Sigma^i F$) and $\Sigma^i(\coker(F))$ to coker($\Sigma^i F$).

(ii) If $F$ is split, then $\Sigma^i F$ is split for all $i \geq 0$.

(iii) If $F$ is of (split) degree $r$ at $N$, then $\Sigma^i F$ is of (split) degree $r$ at $N - i$ for $i \geq 0$.

**Proof.** Using Lemma 4.8 the natural transformation

$$\Sigma^{i+1} F(\_ ) = F(\_ \otimes X^{\otimes i+1}) \xrightarrow{F(\_ \otimes b_{X,X}^{-1})} F(\_ \otimes X^{\otimes i+1}) = \Sigma^{i+1} F(\_ )$$

can be seen to commute with the structure map of $\Sigma^{i+1} F$, so defines an automorphism $\Phi$ of $\Sigma^{i+1} F$. Lemma 4.8 also implies the relation $\Sigma^i(\Phi) = \Phi\Sigma^i F$ and therefore $\Sigma^i((\text{co})\ker(\sigma F)) = (\text{co})\ker(\Phi\Sigma^i F)$. Hence, the coefficient systems in comparison are (co)kernels of morphisms differing by an automorphism, which proves the first claim. Given a splitting $s : \Sigma F \to F$ for $F$, the composition $\Sigma^i(s)\Phi$ splits $\Sigma^i F$, which shows the second. Finally, the third follows from the first by induction on $r$. □
Remark 4.10. The category of coefficient systems for \( M \) is isomorphic to the category of abelian group-valued functors on a category \( C(M) \). To construct \( C(M) \), recall Quillen’s bracket construction \( (\mathcal{E}, \mathcal{F}) \) of a monoidal category \( \mathcal{F} \) acting via \( \oplus : \mathcal{E} \times \mathcal{F} \to \mathcal{E} \) on a category \( \mathcal{E} \) [Gra76, p.219]. It has the same objects as \( \mathcal{E} \) and a morphism from \( C \) to \( C' \) is an equivalence class of pairs \( (D, f) \) with \( D \in \text{ob} \mathcal{F} \) and \( f \in \mathcal{E}(C \oplus D, C') \) where \( (D, f) \) and \( (D', f') \) are equivalent if there is an isomorphism \( g \in \mathcal{F}(D, D') \) satisfying \( f' = f(C \oplus g) \).

Using this construction, we obtain \( C(M) \) as \( (M, \mathcal{B}) \), where the free braided monoidal category in one object \( \mathcal{B} \) acts on \( M \) via the canonical functor \( \mathcal{B} \to \mathcal{A} \) induced by \( X \), followed by the action of \( \mathcal{A} \) on \( M \). The multiplication by \( X \) on \( M \) induces an endofunctor
\[
\Sigma : C(M) \to C(M)
\]
by mapping a morphism \( [D, f] : C \to C' \) to \( [D, (f \oplus X)(C \oplus b_{X,D})] : C \oplus X \to C' \oplus X \). It comes together with a natural transformation
\[
\sigma : \text{id} \to \Sigma
\]
given by \([X, \text{id}]\), such that the suspension of a coefficient system \( F \), seen as a functor on \( C(M) \), is the composition \( (F \circ \Sigma) \) and its suspension map is \( F(\sigma) : F \to (F \circ \Sigma) \).

Remark 4.11. If \( M \) is a groupoid such that all subcategories \( M_n \) are connected, then a coefficient system for \( M \) is equivalently given as a sequence of \( \text{Aut}(A \oplus X^{\otimes n-\delta(A)})\)-modules \( F_n \) for an element \( A \) of minimal degree \( \delta(A) \), together with \( (\_ \oplus X)\)-equivariant morphisms \( F_n \to F_{n+1} \) such that the image of \( B_m \) in \( \text{Aut}(X^{\otimes m}) \) acts via \( (A \oplus X^{\otimes n-\delta(A)}) \) trivially on the image of \( F_n \) in \( F_{n+m} \) for all \( n \) and \( m \).

4.2. Twisted stability of \( E_2 \)-modules over \( E_2 \)-algebras. Let \( M \) be a graded \( E_1 \)-module \( M \) over an \( E_2 \)-algebra with stabilizing object \( X \). Recall from Section 2.1 that its fundamental groupoid \( (\Pi(M), \oplus) \) is a graded right-module over the graded braided monoidal groupoid \( (\Pi(\mathcal{A}), \oplus, b, 0) \).

Definition 4.12. A coefficient system for \( M \) is a coefficient system for \( \Pi(M) \) in the sense of Definition 4.1.

The structure map of a coefficient system \( F \) for \( M \) induces stabilization maps of degree 1
\[
(s; \sigma^F) : (M; F) \to (M; F),
\]
which stabilize homologically according to Theorem C in the case of \( F \) being of finite degree and the canonical resolution sufficiently highly connected. This remainder of this chapter is devoted to its proof.

Remark 4.13. In the course of the proof, it will be convenient to have fixed a notion of a homotopy commutative square of spaces with local systems, by which we mean a square
\[
\begin{array}{ccc}
(X; F) & \longrightarrow & (Y; G) \\
\downarrow & & \downarrow \\
(X'; F') & \longrightarrow & (Y'; G')
\end{array}
\]

which depends on the homotopy. However, homotopies that are itself homotopic give homotopic morphisms on mapping cones hence they induce the same on homology. Horizontal composition of such squares including the homotopy induces the respective composition of \( F \). Even though \( F \) depends on the homotopy, the long exact sequences of the mapping cones still fit into a commutative ladder as in the strict case.

We denote by \( \text{Rel}(F) \) the relative groups \( H_n((M; F), (M; F)) \) with respect to the stabilization \( (s; \sigma^F) \), equipped with the additional grading \( \text{Rel}(F) = \bigoplus_{n \in \mathbb{N}} H_n((M_{n+1}; F), (M_n; F)) \). We consider the square
\[
\begin{array}{ccc}
(M; F) & \to & (M; F) \\
\downarrow \left( (s; \sigma^F) \right) & & \downarrow \left( (s; \sigma^F) \right) \\
(M; F) & \to & (M; F)
\end{array}
\]

(8)
as homotopy commutative via the homotopy \( [\] \) of Section \([\] \) By the triviality condition on coefficient systems, the homotopy extends to one of spaces with local systems, so we obtain a relative stabilization

\[
(s; \sigma^F)_*: \text{Rel}_*(F) \to \text{Rel}_*(F)
\]
of degree 1, where the superscript \( \sim \) indicates the twist by the homotopy. The homotopy commutative square \( [\) \) admits a factorization into a composition of homotopy commutative squares

\[
\begin{array}{ccc}
(M; F) & \xrightarrow{(\id, \sigma^F)} & (M; \Sigma F) \\
\downarrow^{(s, \sigma^F)} & \downarrow^{(s, \sigma^F)} & \downarrow^{(s, \sigma^F)} \\
(M; F) & \xrightarrow{(\id, \sigma^F)} & (M; \Sigma F)
\end{array}
\]
in which the square on the left strictly commutes because of the triviality condition and the one on the right up to homotopy using the same homotopy as for \( [\) \). This induces a factorization of the relative stabilization map as

\[
\text{Rel}_*(F) \xrightarrow{(\id, \sigma^F)_*} \text{Rel}_*(\Sigma F) \xrightarrow{(s, \id)_*} \text{Rel}_*(F),
\]
with the first map being of degree 0 and the second of degree 1.

**Lemma 4.14.** The composition \( \text{Rel}_*(F) \xrightarrow{(s, \sigma^F)_*} \text{Rel}_*(\Sigma F) \xrightarrow{(s, \id)_*} \text{Rel}_*(F) \) is trivial.

**Proof.** The mapping cones defining \( \text{Rel}_*(F) \) induce a commutative diagram of long exact sequences

\[
\begin{array}{ccccccccc}
\ldots & \to & H_*(M; F) & \to & H_*(M; F) & \to & \text{Rel}_*(F) & \xrightarrow{h_1} & H_{*-1}(M; F) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & H_*(M; F) & \to & H_*(M; F) & \to & \text{Rel}_*(F) & \xrightarrow{h_2} & H_{*-1}(M; F) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & H_*(M; F) & \xrightarrow{h_3} & H_*(M; F) & \xrightarrow{h_4} & \text{Rel}_*(F) & \xrightarrow{h_5} & H_{*-1}(M; F) & \to & \ldots
\end{array}
\]

As \( h_1 \) and \( h_2 \) both equal \( (s; \sigma^F)_* \), we get \( 0 = h_1h_3 = h_2h_4 = h_1h_5 \), so the image of \( h_4 \) is in the kernel of \( h_5 \) which is the image of \( h_6 \). Hence it is enough to show \( h_7h_6 = 0 \). Since \( h_8 = h_10 \) for the same reason as \( h_1 = h_2 \), we obtain the claim from \( h_7h_6 = h_7h_8 = h_8h_6 = 0 \). \( \square \)

**4.3. The relative spectral sequence.** Let \( M \) be a graded \( E_1 \)-module over an \( E_2 \)-algebra with stabilizing object \( X \). Recall from Section \([\] \) the category \( UO \) and the \( UO \)-spaces \( B_*(M) \). The endofunctor \( t: UO \to UO \) defined by

\[
\begin{align*}
UO([q], [p]) & \to UO([q + 1], [p + 1]) \\
(d, \mu) & \mapsto (d, \gamma(c; [\mu, 1])
\end{align*}
\]
gives rise to a map of augmented \( UO \)-spaces

\[
B_*(M) \to (B_*(M) \circ t)
\]

by mapping \( (A, \zeta) \in B_*(M) \) to \( (A, \theta(c; \zeta, X)) \in B_{p+1}(M) \). With respect to the isomorphism \( \pi_0(UO) \cong UB \) of Section \([\] \), the induced functor \( \pi_0(t): UB \to UB \) on path-components is given by

\[
\begin{align*}
UB([q], [p]) & \to UB([q + 1], [p + 1]) \\
[b] & \mapsto [b \oplus 1]
\end{align*}
\]

from which it follows that \( \pi_0(t) \) preserves the image of the section \( \Lambda_{\text{inj}} \to UB \) of Section \([\] \) whose path components defined the thickening \( \Lambda_{\text{inj}} \subseteq UO \) of the semi-simplicial category (see Definition \([\] \)). Consequently, the functor \( t \) restricts to an endofunctor on \( \Lambda_{\text{inj}} \) and we obtain a map of augmented \( \Lambda_{\text{inj}} \)-spaces.
by restricting to $\tilde{\Lambda}_{\text{inj}}$. The bottom map is the stabilization map, hence of degree 1. For coefficient systems $F$ for $M$, we get by the discussion in Section 4.4 [5.5] a tri-graded spectral sequence

$$E^1_{p,q,n} \cong \begin{cases} H_q((R_{p+1}(M)_{n+1}; F_{n+1})), & \text{if } p \geq 0 \\ H_q((M_{n+1}; F_{n+1})), & \text{if } p = -1 \end{cases}$$

$$\Rightarrow H_{p+q+n}(\mathcal{M}_{n+1}; (\mathcal{R}_{n+1}(M); (\mathcal{R}_{n}(M); |_{n}; F_{n})),$$

which is a sum of spectral sequences, one for each $n \in \mathbb{N}$. The selfhomotopy of $s^2$ used for the square (8) witnesses homotopy commutativity of

$$\begin{pmatrix} (\mathcal{M}; \Sigma^{p+1}F) & (\mathcal{M}; \Sigma^p F) \\ (\mathcal{M}; \Sigma^p F) & (\mathcal{M}; \Sigma^{p-1}F) \end{pmatrix}$$

which induces a morphism $\sum_{i=0}^p (-1)^i s^i_{p,q}$: $\text{Rel}_q(\Sigma^{p+1}F) \to \text{Rel}_q(\Sigma^p F)$ of degree 1. A relative version of the proof of Lemma [5.1] establishes the following identification of the spectral sequence.

**Lemma 4.15.** We have $E^1_{n,p,q,n} \cong \text{Rel}_q(\Sigma^{p+1}F)_{n-p}$ and the $E^1$-differential identifies with

$$\sum_{i=0}^p (-1)^i s^i_{n,p,q,n} : \text{Rel}_q(\Sigma^{p+1}F)_{n-p} \to \text{Rel}_q(\Sigma^p F)_{n-p+1}.$$

In particular, the differential $d^1: E^1_{n,n+1} \to E^2_{n+1,n+1}$ corresponds to the second map of [10] in degree $n$.

**Lemma 4.16.** The following composition is zero for $n \geq 1$,

$$\text{Rel}_n(F)_{n-1} \xrightarrow{(\text{id}, s^F)} \text{Rel}_n(\Sigma F)_{n-1} \xrightarrow{(\text{id}, s^F)} \text{Rel}_n(\Sigma^2 F)_{n-1} \cong E^1_{n,n+1} \xrightarrow{d^1} E^1_{n+1,n+1} \cong \text{Rel}_n(\Sigma F)_{n}$$

**Proof.** Using Lemma 4.15 the composition in question is the difference between the morphisms in degree $(n-1)$ induced by the composition of the two homotopy commutative squares

$$\begin{pmatrix} (\mathcal{M}; F) & (\mathcal{M}; \Sigma^2 F) \\ (\mathcal{M}; \Sigma^2 F) & (\mathcal{M}; \Sigma^{p-1}F) \end{pmatrix}$$

for $i = 0$ and $i = 1$, where the homotopy of the left square is trivial. The nontrivial homotopy of the right square becomes trivial after composing with the left square which follows from the triviality condition for coefficient systems, so the composition in question is the difference of the induced morphism induced by the two strictly commutative outer squares. But, again by the triviality condition, we have $F_{n+1}(- \oplus b_{X}X)_{p-1}^{\Sigma p} \sigma F = \Sigma^{p+1}F$, so the two outer squares coincide and the composition vanishes. \[\square\]

**4.4. The proof of Theorem C** By the long exact sequence

$$\begin{array}{cccccccccccc}
\ldots & \rightarrow & \text{Rel}_{i+1}(F) & \rightarrow & H_i(M; F) & \xrightarrow{(\sigma, F)} & H_i(M; F) & \rightarrow & \text{Rel}_i(F) & \rightarrow & H_{i-1}(M; F) & \rightarrow & \ldots \\
\end{array}$$

Theorem C follows from the next result.

**Theorem 4.17.** Let $F$ be a coefficient system for $M$ of degree $r$ at $N \geq 0$. If the canonical resolution of $M$ is graded $(\Sigma; -1)$-connected in degrees $\geq 1$ for a $k \geq 2$, then

(i) the group $\text{Rel}_r(F)_n$ vanishes for $n \geq \max(N+1, k(i+r))$ and

(ii) if $F$ is of degree split $r$ at $N \geq 0$, then $\text{Rel}_r(F)_n$ vanishes for $n \geq \max(N+1, k(i+r))$.

We prove Theorem 4.17 via a double induction on $r$ and $i \geq 0$ by considering the following statement.

$\mathbf{(H_{r,i})}$ The vanishing ranges of Theorem 4.17 hold for all $F$ of degree $< r$ at any $N \geq 0$ in all homological degrees $i$ and for all $F$ of degree $r$ at any $N \geq 0$ in homological degrees $< i$. \[\square\]
The claim \((H_{r,i})\) holds trivially if \(r < 0\) or if \((r, i) = (0, 0)\). If \((H_{r,i})\) holds for a fixed \(r\) and all \(i\), then \((H_{r+1,0})\) follows since there is no requirement on coefficient systems of degree \((r + 1)\). Hence, to prove the theorem, it is sufficient to show that \((H_{r,i})\) implies \((H_{r+1,i})\) for \(i, r \geq 0\). As

\[
\text{Rel}_i(F_n) \xrightarrow{(s, \sigma)} \text{Rel}_i(F_n)_{n+1} \xrightarrow{(s, \sigma)} \text{Rel}_i(F_n)_{n+2}
\]

is zero by Lemma 4.14, it is enough to show injectivity of both maps in the claimed range. Using the factorization (10), this is implied by the following.

**Lemma 4.18.** Let \(r \geq 0\) and \(i \geq 0\) satisfying \((H_{r,i})\) and let \(F\) be of degree \(r\) at some \(N \geq 0\).

(i) (\(id, \alpha_i\)): \(\text{Rel}_i(F_n) \to \text{Rel}_i(\Sigma F_n)\) is injective for \(n \geq \max(N, k(i + r))\) and surjective for \(n \geq \max(N, k(i + r - 1))\). If \(F\) is of split degree \(r\) at \(N \geq 0\), it is split injective for all \(n\) and surjective for \(n \geq \max(N, k(i + r - 1))\).

(ii) The map \((s, id)_0: \text{Rel}_i(\Sigma F)_n \to \text{Rel}_i(F)_{n+1}\) is injective in degrees \(n \geq \max(N + 1, k(i + r))\). If \(F\) is of split degree \(r\) at \(N \geq 0\), it is injective for \(n \geq \max(N + 1, k(i + r))\).

**Proof.** We first prove the first part of the statement. As \(\text{Rel}_i(\_\_\_)\) is functorial in the coefficient system, injectivity of the split case is clear. The remaining claims of the first statement follow from the long exact sequences in \(\text{Rel}_i(\_\_\_)\) induced by the short exact sequences

\[
0 \to \ker(\text{ker}(F) \to F \to \text{im}(F) \to \Sigma F) \to 0 \quad \text{and} \quad 0 \to \text{im}(F \to \Sigma F) \to \Sigma F \to \text{coker}(F) \to 0
\]

by applying \((H_{r,i})\) using that \(\ker(F)\) has degree \(-1\) at \(N\) and that \(\text{coker}(F)\) has (split) degree \((r - 1)\) at \((N - 1)\). To prove (ii), we use the spectral sequence \(11\) and Lemma 4.15. Since \(\text{Rel}_i(M)_{m} \to M_{m}\) is assumed to be \((m - \frac{r + 1}{k})\)-connected for \(m \geq 1\), the groups \(\text{Rel}_i(M)_{m} \to M_{m}\) vanish for \(r + q \leq \frac{m - 2}{k}\), from which we conclude \(E^2_{0,0,i+1} = 0\) for \(p + q \leq \frac{r - 1}{k}\). We claim that the differential \(E^2_{1,1,i,n+1} \to E^1_{0,1,i,n+1}\) vanishes for \(n \geq \max(N + 1, k(i + r))\) in the nonsplit case and for \(n \geq \max(N + 1, k(i + r))\) in the split one. By Lemma 4.16 this is the case if the maps \(\text{Rel}_i(F)_{n-1} \to \text{Rel}_i(\Sigma F)_{n-1} \to \text{Rel}_i(\Sigma^2 F)_{n-1}\) are surjective in that range, which holds by (i). Since the map we want to prove injectivity of identifies with the differential \(E^1_{0,0,i+1} \to E^1_{1,1,i,n+1}\) by Lemma 4.15, it is therefore enough to show that, in the ranges of the statement, \(E^1_{0,0,i+1} = 0\) and \(E^2_{p,q,i,n+1} = 0\) holds for \((p, q)\) with \(p + q = i + 1\) and \(q < i\). By the vanishing range of \(E^0\) noted above, we have \(E^2_{0,0,i+1} = 0\) in the required range. The claimed vanishing of \(E^2\) follows from the vanishing even on the \(E^1\)-page, which is proven by observing that, by \((H_{r,i})\) and Lemma 4.15, the groups \(E^1_{p,q,i,n+1} \cong \text{Rel}_q(\Sigma^p F)_{n-p}\) vanish for \((p, q)\) with \(q < i\) and \(n \geq \max(N - p, k(q + r))\) in the nonsplit and \(\Sigma F\) satisfying \(q < i\) and \(n \geq \max(N - p, k(q + r))\) in the split case since \(\Sigma^p F\) has (split) degree \(r\) at \((N - p - 1)\). \(\square\)

5. **Configuration spaces**

The **ordered configuration space** of a manifold \(W\) with labels in a Serre fibration \(\pi: E \to W\) is given by

\[
F^n_W(W) = \{(e_1, \ldots, e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W(\partial W)\}
\]

and the **unordered configuration space** as the quotient by the canonical action of the symmetric group

\[
C^n_W(W) = F^n_W(W)/\Sigma n.
\]

To establish an \(E_1\)-module structure on the unordered configuration spaces of \(W\), we assume that \(W\) has nonempty boundary, fix a collar \((-\infty, 0] \times \partial W \to W\) and attach an infinite cylinder to the boundary,

\[
\tilde{W} = W \cup [0, \infty) \times \partial W.
\]

Collar and cylinder assemble to an embedding \(\mathbb{R} \times \partial W \to \tilde{W}\) of which we make frequent use henceforth. We extend the fibration \(\pi\) over \(\tilde{W}\) by pulling it back along the projection \(\tilde{W} \to W\) and define the space

\[
\tilde{C}^n_W(W) = \{(s, e) \in [0, \infty) \times C^n_W(\tilde{W}) \mid \pi(e) \subseteq W \cup (-\infty, s) \times \partial W\},
\]

which is an equivalent model for \(C^n_W(W)\) since the inclusion in \(\tilde{C}^n_W(W)\) as the subspace with \(s = 0\) is an equivalence by choosing an isotopy of \(\tilde{W}\) which pushes \([0, \infty) \times \partial W\) into \((-\infty, 0) \times \partial W\). We furthermore fix an embedded cube \((-1, 1)^{d-1} \subseteq \partial W\) of codimension 0, together with a section \(l: (-1, 1)^{d-1} \to E\) of \(\pi\) which we extend canonically to a section \(l\) on \([0, \infty) \times (-1, 1)^{d-1} \subseteq \mathbb{R} \times \partial W\).
Lemma 5.1. Configurations $\bigcup_{n \geq 2} C_n(D^d)$ in a disc form a graded $E_d$-algebra with configurations $\bigcup_{n \geq 2} \tilde{C}_n(W)$ in a d-manifold $W$ with nonempty boundary as an $E_1$-module over it, graded by the number of points.

Proof. The operad $D^\bullet(D^d)$ of little $d$-discs acts on $\bigcup_{n \geq 2} C_n(D^d)$ by
\[
\theta: D^k(D^d) \times (\bigcup_{n \geq 2} C_n(D^d))^k \rightarrow \bigcup_{n \geq 2} C_n(D^d)
\]
and this action extends to one of $SC_d$ (see Definition 2.1) on the pair $(\bigcup_{n \geq 2} \tilde{C}_n(W), \bigcup_{n \geq 2} C_n(D^d))$ via
\[
\theta: SC_d(m, a^k; m) \times \bigcup_{n \geq 2} \tilde{C}_n(W) \times (\bigcup_{n \geq 2} C_n(D^d))^k \rightarrow \bigcup_{n \geq 2} \tilde{C}_n(W)
\]
using the section $l$ on $[0, \infty) \times (-1, 1)^{d-1} \subseteq \hat{W}$ and the translation $(s + s')$ by $s'$ in the $[0, \infty)$-coordinate (see Figure 3 for an example).

Figure 4. The $E_1$-module structure of unordered configuration spaces

5.1. The resolution by arcs. Let $W$ be a smooth connected manifold of dimension $d \geq 2$ with nonempty boundary and $\pi: E \rightarrow W$ a Serre fibration with path-connected fiber. By Lemma 5.1, the configuration spaces $\bigcup_{n \geq 2} \tilde{C}_n(W)$ form a graded $E_1$-module $M$ over $\bigcup_{n \geq 2} C_n(D^d)$, which we consider as an $E_2$-algebra via the canonical morphism $SC_2 \rightarrow SC_d$ (see Section 2.3). The stabilization map $s: M \rightarrow M$ with respect to a chosen stabilizing object $X \in C_1(D^d)$, restricted to the subspace of elements of degree $n$, has the form
\[
s: \tilde{C}_n(W) \rightarrow \tilde{C}_{n+1}(W).
\]

Remark 5.2. With respect to the described equivalence $C_n^0(W) = \tilde{C}_n^0(W)$, the stabilization map corresponds to the maps $C_n^0(W) \rightarrow C_{n+1}^0(W)$ that adds a point “near infinity” [McD75, Seg79].

We prove high-connectivity of the canonical resolution of $M$ (see Section 2.2) by identifying it with the resolution by arcs—an augmented semi-simplicial space of geometric nature, known to be highly connected.

Definition 5.3. The resolution by arcs is the augmented semi-simplicial space $R^\bullet_p(M) \rightarrow M$ with
\[
R_p^\bullet(M) \subseteq M \times (\text{Emb}([-1, 0], \hat{W}) \times \text{Maps}([-1, 0], E))^{p+1}
\]
consisting of tuples $((s, e), ((\phi_0, \eta_0), \ldots, (\phi_p, \eta_p)))$ such that
\begin{itemize}
  \item[(i)] the arcs $\phi_i$ are pairwise disjoint and connect points in the configuration $\phi_i(-1) \in \pi(e) \subseteq \hat{W}$ to points $\phi(0) \in \{s \times (-1, 1) \times \{0\}^{d-2} \subseteq [0, \infty) \times \partial W$ in the order $\phi_0(0) < \ldots < \phi_p(0)$,
  \item[(ii)] the interior of the arcs lie in $W \cup [0, s) \times \partial W$ and are disjoint from the configuration $\pi(e)$,
  \item[(iii)] the path of labels $\eta_i$ satisfies $(\pi \circ \eta_i) = \phi_i$ and connects the label of $\phi_i(-1) \in \pi(e)$ to $\eta_i(0) = l(\phi_i(0))$,
  \item[(iv)] there exists $e \in (0, 1)$ with $\phi_i(t) = (s + e \phi_i(1), 0, \ldots, 0) \in (-\infty, s] \times (-1, 1)^{d-2} \subseteq \hat{W}$ for $t \in (-\infty, 0]$.
\end{itemize}
The space $R^\bullet_p(M)$ is topologized using the compact-open topology on $\text{Maps}([-1, 0], E)$ and the $C^\infty$-topology on $\text{Emb}([-1, 0], W)$. The $i$th face map forgets $(\phi_i, \eta_i)$, as indicated by Figure 5.

Theorem 5.4. The resolution by arcs $R^\bullet_p(M) \rightarrow M$ is graded $(g_M - 1)$-connected.
Proof. Setting \( s = 0 \) in the definition of \( R_p^* (M) \) yields a sub-semi-simplicial space \( \tilde{R}_p^* (M) \subseteq R_p^* (M) \), augmented over \( \mathcal{M} = \bigcup_{n \geq 0} C_n^\ast (W) \). As the inclusion is a weak equivalence by the same argument as for \( C_n^\ast (W) \subseteq \bar{C}_n^\ast (W) \), the augmented semi-simplicial space \( \bar{R}_p^* (M) \to \mathcal{M} \) is as connected as \( R_p^* (M) \to \mathcal{M} \). The latter is the standard resolution by arcs for configurations of unordered points with labels in \( W \), which is known to have the claimed connectivity (see e.g. the proof of Theorem A.1 in [KM14]). \( \square \)

**Theorem 5.5.** The canonical resolution and the one by arcs are weakly equivalent as augmented \( \bar{A}_{\infty} \)-spaces.

Assuming Theorem 5.5, Theorem 5.4 ensures graded \((g - 1)\)-connectivity of the canonical resolution of \( M \) which in turn implies Theorem D by an application of Theorem A and C.

We prove Theorem 5.5 by means of a weak equivalence up to higher coherent homotopy between the canonical resolution and the fibrant replacement of the resolution by arcs (see Definition 2.2). For the proof, we fix the center \( X = \{ 0 \} \in C_1 (D^d) \) as the stabilizing object and start by defining an analogue of the resolution by arcs for the free graded \( E_1 \)-module \( M^\text{free} = \bigcup_{n \geq 0} SC_2 (n) / \Sigma_n \) (see Example 2.19).

**Definition 5.6.** Define the augmented semi-simplicial space \( \bar{R}_p^* (M^\text{free}) \to M^\text{free} \) with \( p \)-simplices

\[
R_p^* (M^\text{free}) \subseteq M^\text{free} \times \text{Emb}([-1, 0], (0, \infty) \times (-1, 1)^{d+1})
\]

consisting of tuples \((s, \{ \phi_i \}, \phi_0, \ldots, \phi_p)\) such that

(i) the arcs \( \phi_i \) are pairwise disjoint and connect center points \( \phi_i(-1) \in \{ \phi_j(0) \} \) of the discs to \( \phi_i(0) \in (s) \times (-1, 1) \) in the order \( \phi_0(0) < \ldots < \phi_p(0) \),

(ii) the interior of the arcs lie in \((0, s) \times (-1, 1)\) and are disjoint from the center points \( \{ \phi_0(0) \} \),

(iii) there exists an \( \epsilon \in (0, s) \), such that \( \phi_i(t) = (s + t, \phi_i(0)) \in (0, s) \times (-1, 1) \) holds for all \( t \in (-\epsilon, 0] \).

Multiplying elements \( d \) and \( e \) in \( M^\text{free} \) by \( \gamma(e; d, 1^g) \) turns \( M^\text{free} \) into a topological monoid whose multiplication map \( M^\text{free} \times M^\text{free} \to M^\text{free} \) is covered by a simplicial action of \( M^\text{free} \) on \( R_p^* (M^\text{free}) \),

\[
M^\text{free} \times R_p^* (M^\text{free}) \to R_p^* (M^\text{free})
\]

(12)

using the translation \( (\cdot, s_e) \) by the parameter of \( e \) in the \((0, \infty)\)-coordinate. The monoid \( M^\text{free} \) also acts on \( M \) acting with \( e \in M^\text{free} \) on \( A \in M \) via \( \theta(e; A, X^g(e)) \). This action is covered by a simplicial map

\[
M \times R_p^* (M^\text{free}) \to R_p^* (M)
\]

(13)

\[\Phi: (A, (e, \phi_i)) \mapsto (\theta(e; A, X^g(e)), \phi_i + s_e, l(\phi_i + s_e))\]

using the embedding \([0, \infty) \times (-1, 1) \times [0, \infty) \times (-1, 1)^d \to E \) and the section \( l: [0, \infty) \to (-1, 1)^d \to E \).

As a warm-up to the proof of Theorem 5.5, we show that the spaces of \( p \)-simplices of the two augmented \( \bar{A}_{\infty} \)-spaces in question are weakly equivalent for all \( p \).

**Lemma 5.7.** The spaces \( R_p (M) \) and \( R_p^\text{fib} (M) \) are weakly equivalent over \( M \).

**Proof.** For an element \((e, \phi_i) \in R_p (M^\text{free})\) of degree \((p + 1)\), consider the triangle

\[
\begin{array}{c}
M \\
\Phi_{(e, \phi_i)} \\
\end{array} \to \begin{array}{c} R_p^* (M) \\
\end{array} \to \begin{array}{c} M \\
\end{array}
\]

(14)
There is a map $R^\ast_p(\mathcal{M}) \to \mathcal{M}$ by forgetting the arcs and the points attached to them and by following configurations along the arcs, the horizontal map of the triangle is an equivalence. A choice of a path in $R_p(M^{\text{free}})$ between $(e, \varphi_i)$ and an element with $M^{\text{free}}$-coordinate $c_{\varphi i + 1}$ lets the triangle homotopy commute and hence induces an equivalence over $\mathcal{M}$ between the path-space constructions of the left and the right diagonal arrow, which are exactly $R_p(M)$ and $R^\ast_p(\mathcal{M})^{\text{fib}}$.  

In the following, we enhance the equivalence of Lemma 5.7 between $R_p(M)$ and $R^\ast_p(\mathcal{M})^{\text{fib}}$ to an equivalence up to higher coherent homotopy between the corresponding augmented $\Delta_{\mathcal{M}}$-spaces. To do so, we first define a thickening of $\Delta_{\mathcal{M}}^{op} \times [1]$ which we obtain as a subcategory $\Delta_{\mathcal{M}}^{op} \times [1] \subseteq U^\circ \mathcal{SC}_2$ of a category $U^\circ \mathcal{SC}_2$ constructed from the underlying operad $\mathcal{SC}_2$.

**Definition 5.8.** Define an enriched category $U^\circ \mathcal{SC}_2$ with objects $\text{ob} \Delta_{\mathcal{M}} \times \{0, 1, \Delta_{\mathcal{M}}\}$. The restriction to $\text{ob} \Delta_{\mathcal{M}} \times \{0\}$ is $\mathcal{SC}_2$ and the one to $\text{ob} \Delta_{\mathcal{M}} \times \{1\}$ are given as copies of $\Delta_{\mathcal{M}}$. We write $[q]$ for the object $(i, [p])$. The only morphisms between $U^\circ \mathcal{SC}_2$ and $\Delta_{\mathcal{M}}$ are from $[q']$ to $[p]$ for which are defined as

$$U^\circ \mathcal{SC}_2([q'], [p]) = \text{hofib}_{\varphi_i}(R^\ast_q(M^{\text{free}}) \to M^{\text{free}}).$$

The composition $U^\circ \mathcal{SC}_2([1], [q]) \times U^\circ \mathcal{SC}_2([q], [p]) \to U^\circ \mathcal{SC}_2([1], [p])$ is induced by the action (12) and the composition $\Delta_{\mathcal{M}}^{op}([1], [q]) \times U^\circ \mathcal{SC}_2([q'), [p]) \to U^\circ \mathcal{SC}_2([1], [p])$ by the simplicial structure of $R^\ast_p(\mathcal{M})$.

To determine the category of path-components of $U^\circ \mathcal{SC}_2$, we define a category $U^\circ \mathcal{SC}_2^{\times \mathbb{R}}$ with objects $\text{ob} \Delta_{\mathcal{M}} \times \{0\}$ and the one to $\text{ob} \Delta_{\mathcal{M}} \times \{1\}$ is $\Delta_{\mathcal{M}}$. The compositions are given by the composition in $U^\circ \mathcal{SC}_2$, after applying the section $\Delta_{\mathcal{M}} \to U^\circ \mathcal{SC}_2$ of Section 2.2 to define $\Delta_{\mathcal{M}}([1], [q]) \times U^\circ \mathcal{SC}_2([q], [p]) \to U^\circ \mathcal{SC}_2([1], [p])$.

**Remark 5.9.** A contravariant functor on $U^\circ \mathcal{SC}_2^{\times \mathbb{R}}$ corresponds to a contravariant functor $F$ on $U^\circ \mathcal{SC}_2$ and $G$ on $\Delta_{\mathcal{M}}^{op}$ together with a natural transformation from $F$ precomposed with $\Delta_{\mathcal{M}} \to U^\circ \mathcal{SC}_2$.

**Lemma 5.10.** The category $U^\circ \mathcal{SC}_2$ is homotopy discrete and $\pi_0(U^\circ \mathcal{SC}_2)$ is isomorphic to $U^\circ \mathcal{SC}_2^{\times \mathbb{R}}$.

**Proof.** Arguing as in Lemma 5.7 any choice of $(c_{\varphi i + 1}, \varphi_i) \in R^\ast_p(M^{\text{free}})$ yields an equivalence $U^\circ \mathcal{SC}_2(q, p) \to U^\circ \mathcal{SC}_2([q'], [p])$ using (12) which, combined with Lemma 2.11 and its proof, implies the claim.  

The simplicial map (13) induces an extension

$$B^\circ(M) : (U^\circ \mathcal{SC}_2)^{op} \to \text{Top}/M$$

of the functors $R^\ast_p(\mathcal{M})^{\text{fib}}$ on $\Delta_{\mathcal{M}}^{op}$ and $B_\ast(M)$ on $U^\circ \mathcal{SC}_2^{op}$ to a functor on $U^\circ \mathcal{SC}_2$.

**Proof of Theorem 5.5.** We have a functor $\Delta_{\mathcal{M}} \times [1]^{op} \to U^\circ \mathcal{SC}_2^{\times \mathbb{R}}$ which is the identity on $\Delta_{\mathcal{M}} \times \{0\}$ and agrees with the section $\Delta_{\mathcal{M}} \to U^\circ \mathcal{SC}_2$ elsewised. Restricting $U^\circ \mathcal{SC}_2$ to the path components it hits, we obtain a category $\Delta_{\mathcal{M}}^{op} \times [1]$ together with a weak equivalence $p : \Delta_{\mathcal{M}}^{op} \times [1] \to \Delta_{\mathcal{M}}^{op} \times [1]$. Restricting $B^\circ(M)$ to this subcategory results in a functor $B^\circ(M)$ which agrees with $R_p(M)$ on $\Delta_{\mathcal{M}}^{op}$ and with $R^\ast_p(\mathcal{M})^{\text{fib}}$ on $\Delta_{\mathcal{M}}^{op}$. Applying Lemma 1.6 we obtain a zig-zag $p, R^\circ(M) \leftrightarrow \ldots \to B^\circ(M)$ of weak equivalences. The functors $p, R^\circ(M)$ on $[1] \times \Delta_{\mathcal{M}}^{op}$ consists of augmented semi-simplicial spaces $p, R^\circ(M)^{\mathbb{R}}$ and $p, R^\circ(M)^{\mathbb{R}}$ together with a morphism $p, R^\circ(M)^{\mathbb{R}} \to p, R^\circ(M)^{\mathbb{R}}$. The argument of the proof of Lemma 5.7 implies that $R^\circ(M)$ maps all morphisms over $(id_p, 0 \to 1) \in (\Delta_{\mathcal{M}}^{op} \times [1])(p^0, p^1)$ to equivalences. That, together with the zig-zag implies that $p, R^\circ(M)^{\mathbb{R}} \to p, R^\circ(M)^{\mathbb{R}}$ is a weak equivalence. But, using the zig-zag again, $p, R^\circ(M)^{\mathbb{R}}$ is as a $\Delta_{\mathcal{M}}$-space weakly equivalent to $R_p(M)$ and $p, R^\circ(M)^{\mathbb{R}}$ to $R^\ast_p(\mathcal{M})^{\text{fib}}$. The latter is in turn equivalent to $R^\ast_p(\mathcal{M})$ (see Section 1.2), which proves the claim.  

**Remark 5.11.** The map (13) induces an equivalence of augmented $\Delta_{\mathcal{M}}$-spaces between the levelwise bar-construction $B(M, M^{\text{free}}, R^\ast_p(M^{\text{free}}))$ and $R^\ast_p(\mathcal{M})$, which leads to an alternative proof of Theorem 5.5.
5.2. **Coefficient systems for configuration spaces.** Let $W$ and $\pi$ be as in Theorem 4.11. Recall from Section 4.11 that the fundamental groupoid $\{\prod_{n \geq 0} \Pi(\check{C}_n^\pi(W)), \oplus\}$ is a module over $\{\prod_{n \geq 0} \Pi(\mathcal{C}_n(D^p)), \oplus, b, 0\}$ and hence, after fixing a stabilizing object $X \in \mathcal{C}_1(D^p)$, also one over the free braided monoidal category $B$ in one object (see Section 4.11). Denote by $A \in \check{C}_n^\pi(W)$ the empty configuration and note that, by Remark 4.11, a coefficient system for $\{\prod_{n \geq 0} \check{C}_n^\pi(W)\}$ is specified by

(i) a $\pi_1(\check{C}_n^\pi(W), A \oplus X^{\oplus n})$-module $M_n$ for each $n \geq 0$ together with

(ii) $(\underbrace{\oplus_X})$-equivariant morphisms $\sigma: M_n \to M_{n+1}$ such that $B_m$ acts via $(A \oplus X^{\oplus n} \oplus \_\_)$ trivially on the image of $\sigma^n: M_n \to M_{n+m}$.

Equivalently, a coefficient system is an abelian group-valued functor on Quillen’s construction

$$C^n(W) := \{\prod_{n \geq 0} \pi_1(\check{C}_n^\pi(W)), B\},$$

compare Remark 4.10. By making use of the canonical ordering of $A \oplus X^{\oplus n}$, a loop $\gamma$ in $\check{C}_n^\pi(W)$ induces a permutation in $\Pi$ letters, as well as $n$ ordered loops in $E$ by connecting the paths in $E$ forming $\gamma$ to a basepoint in $E$ via paths in the image of the section $I: [0, \infty) \times (-1, 1)^{d-1} \to E$. This induces a morphism

$$\pi_1(\check{C}_n^\pi(W)) \to \pi_1(E) \rtimes \Sigma_n$$

to the wreath product, which we use to relate $C^n(W)$ to other categories by means of a commutative diagram

$$\begin{array}{cccc}
\check{C}_n^\pi(W) & \to & \langle \pi_1(E) \rtimes \Sigma_n \rangle & \\
\downarrow & & \downarrow & \\
B^n(W) & \to & \mathcal{J}_1(\pi_1(E)) & \to \mathcal{J}_1 & \\
\cap & & \cap & \\
B^n(W)^{\#} & \to & \mathcal{J}_1^\#(\pi_1(E)) & \to \mathcal{J}_1^\# & \\
\end{array}$$

on which we elaborate in the following.

The category $\langle \pi_1(E) \rtimes \Sigma_n \rangle$ results from the action of the groupoid $\Sigma = \prod_{n \geq 0} \Sigma_n$ on $\pi_1(E) \rtimes \Sigma = \prod_{n \geq 0} \pi_1(\check{C}_n^\pi(W))$. It receives a functor from $C^n(W)$ which is induced by the morphisms (14). The category $\mathcal{J}_1(\pi_1(E))$ of finite sets and injective $\pi_1(\check{C}_n^\pi(W))$-maps $[\Sigma_{14}, \text{Cas}_{16}, \text{GL}_{15}, \text{Ram}_{16}]$ is isomorphic to $\langle \pi_1(E) \rtimes \Sigma, \pi_1(E) \rtimes \Sigma \rangle$. The functor from $\langle \pi_1(E) \rtimes \Sigma_n \rangle$ to $\mathcal{J}_1(\pi_1(E))$ is induced by the inclusion $\Sigma_1 \subseteq \pi_1(E) \rtimes \Sigma$. By forgetting $\pi_1(E)$, the category $\mathcal{J}_1^\#(\pi_1(E))$ maps to the category $\mathcal{J}_1^\#$ finite sets and injections. Functors on the category $\mathcal{J}_1^\#$ are extensively studied in the context of representation stability (see e.g. [CEF15, CEFN14]). Both categories, $\mathcal{J}_1$ and $\mathcal{J}_1(\pi_1(E))$ are subcategories of larger categories $\mathcal{J}_1^\#$ and $\mathcal{J}_1^\#(\pi_1(E))$ of partially defined $(\pi_1(E))$-injections $[\Sigma_{15}, \text{SS}_{14}]$.

The category of partial braids $B^n(W)^{\#}$ has the nonnegative integers as its objects. A morphism from $n$ to $m$ is a pair $(k, \mu)$ with $k \leq \min(n, m)$ and $\mu$ a morphism in $\Pi(\check{C}_k^\pi(W))$ from a subset of $A \oplus X^{\oplus k}$ to one of $A \oplus X^{\oplus m}$. For $\pi$ being trivial, the category $B^n(W)^{\#}$ has been studied by Palmer [Pal13b], who also introduced the subcategory $B^\#(W) \subseteq B^n(W)^{\#}$ of full braids which consists of morphisms $(k, \mu): n \to m$ with $k = n$. There is a functor $C^n(W) \to B^\#(W)$ which is the identity on objects and maps a morphism

$$[\gamma] \in C^n(W)(n, m) = \pi_1(\check{C}_n^\pi(W), A \oplus X^{\oplus n})/B_{m-n}$$

to the path in $\check{C}_n^\pi(W)$ which forms the first $n$ paths in $E$ of $\gamma$, i.e. the ones starting at $A \oplus X^{\oplus n} \subseteq A \oplus X^{\oplus m}$.

For $W = D^2$ and $\pi = \text{id}_{D^2}$, the category $B^n(W)$ has been considered by Schlichtkrull and Solberg [SS16].

**Remark 5.12.** If $W$ is of dimension $d \geq 3$, then the morphisms (14) are isomorphisms $[\text{Til}_{16}, \text{Lemma } 4.1]$, from which it follows that the three left horizontal functors in the diagram (15) are isomorphisms. If, in addition, $E$ is simply connected, then all functors except for the lower vertical inclusions are isomorphisms.

We call an abelian group valued functor on a category $C$ of the diagram a **coefficient systems on $C$**. There a notion of being of (split) degree $r$ at an integer $N$ for coefficient systems on any of the categories $C$, defined analogously to Definition 4.10 using an endofunctor $\Sigma$ on $C$, together with a natural transformation $\sigma: \text{id} \to \Sigma$, similar to $C^n(W)$ (see Remark 4.11). Most categories in the diagram are of the form $(N, G)$ for a braided monoidal groupoid $G$ acting on a category $N$ and for such, $\Sigma$ and $\sigma$ are defined as in Remark 4.10. For $B^n(W)^{\#}$, the functor $\Sigma$ maps a morphism $(k, \mu)$ to $(k + 1, s(\mu))$ using the stabilization and $\sigma$ consists of...
the constant paths at $A \oplus X^{\otimes n}$. For $\mathcal{B}^\sigma(W)$, we obtain $\Sigma$ and $\sigma$ by restriction from $\mathcal{B}^\sigma(W)^g$ and for $\mathcal{F}^g$ and $\mathcal{F}^g_{\Sigma \varphi(E)}$, the definition is analogous. Note that the transformations $\sigma$ of the categories with a $g$-superscript admit left-inverses, which results in all coefficient systems on them being split.

As all functors in the diagram are compatible with $\Sigma$ and $\sigma$, the property of being of (split) degree $r$ at $N$ is preserved by pulling back coefficient systems along them. In conclusion, by pulling back to $C^\sigma(W)$, all coefficient systems of finite degree on any of the categories in the diagram induce coefficient systems for which the homology of $C^n_\Sigma(W)$ stabilizes by Theorem D. The degree of coefficient systems on some of the categories has been examined before, providing us with a wealth of examples.

Example 5.13. (i) In [RW17] Def. 4.1, the (split) degree of coefficient systems on prebraided monoidal categories has been introduced. This includes $(\pi_1(E), \Sigma, \Sigma)$, $\mathcal{F}^g_{\Sigma \varphi(E)}$, $\mathcal{F}^g_{\Sigma \varphi(E)}$, $\mathcal{F}^g_{\Sigma \varphi(E)}$, and $\mathcal{F}^g_{\Sigma \varphi(E)}$.

(ii) A finitely generated coefficient system $F$ on $\mathcal{F}^g_{\Sigma \varphi(E)}$ in the sense of [SS14] is of finite degree, provided that $\pi_1(E)$ is finite [SS14 Prop. 3.4.2]. By [SS14 Rem. 3.4.3], this implication remains valid if $\pi_1(E)$ is virtually polycyclic (see the introduction for a definition) and even holds for arbitrary $\pi_1(E)$ in the case of $F$ being presented in finite degree or extending to $\mathcal{F}^g_{\Sigma \varphi(E)}$.

(iii) More quantitatively, coefficient systems on $\mathcal{F}^g$ which are generated in degree $\leq k$ and related in degree $\leq d$, as defined in [CE17] Def. 4.1, are of degree $k$ at $d + \min(k, d)$ by [RW17 Prop. 4.18].

(iv) The degree of a coefficient system on $\mathcal{B}^\sigma(W)^g$ has been studied by Palmer [Pal13b], who also provides examples of finite degree coefficient systems on $\mathcal{F}^g$ (see [Pal13b Sect.]). Note that the degree and the split degree of coefficient systems on these categories coincide.

(v) For $W = D^2$ and $\pi = \text{id}_{\mathbb{Z}_2}$, the category $C^n_\Sigma(W)$ is isomorphic to the category $UB$ as recalled in Definition 2.7. The Burau representation gives rise to an example of a coefficient system of degree 1 at 0 on $UB$ [RW17 Ex. 3.14]. On the basis of this example, Soulié [Sou17] has constructed coefficient systems on $UB$ of arbitrary degree by using the so called Long-Moody construction.

Remark 5.14. Inspired by work of Betley [Bet02], Palmer [Pal13b] proved homological stability for $C^n_\Sigma(W)$ in the case of $\pi$ being a trivial fibration for coefficient systems of finite degree on $\mathcal{B}^\sigma(W)^g$ with a surjectivity range agreeing with ours. However, his result includes split injectivity in all degrees—a phenomenon special to configuration spaces and not captured by our general approach. In Remark 1.13, Palmer suspects stability for coefficient systems of finite degree on $\mathcal{B}^\sigma(W)$. Theorem D confirms this and extends his result to a significantly larger class of coefficient systems and nontrivial labels.

5.3. Applications. We complete the proofs of Corollary F and C sketched in the introduction. Unless stated otherwise, $W$ denotes a manifold satisfying the assumptions of Theorem D.

5.3.1. Configuration spaces of embedded discs. Recall from the introduction the configuration spaces of (unordered) $k$-discs $C^n_\Sigma(W)$ and $F^n_\Sigma(W)$ of $W$, the related subgroups $\text{PDiff}^k_{\Sigma \varphi(W)} \subseteq \text{Diff}^k_{\Sigma \varphi(W)} \subseteq \text{Diff}^k(W)$ of diffeomorphisms fixing respectively permuting $n$ chosen $k$-discs in $W$ and the orientation-preserving variants denoted with a (+)-superscript for $k = d$ and $W$ being oriented. The action of $\text{Diff}^k(W)$ on $C^n_\Sigma(W)$ extends to one on $\bigsqcup_{\Sigma \varphi(W)} C^n_\Sigma(W)$ by extending diffeomorphisms of $W$ to $W$ via the identity. As the action commutes with the $E_d$-action of $\bigsqcup_{\Sigma \varphi(W)} C^n_\Sigma(D)$, the Borel construction $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} M$ inherits a graded $E_d$-module structure with a highly-connected canonical resolution by Example 2.22.

Consequently, Theorem A and C imply (twisted) stability for $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} C^n_\Sigma(W)$ for $k < d$ and, as the equivalence $C^n_\Sigma(W) \to C^n_\Sigma(W) \subseteq C^n_\Sigma(W)$ (see the introduction for the first map) is equivariant, also for $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} C^n_\Sigma(W)$. The same argument applies to $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} C^n_\Sigma(W)$.

As announced in the introduction, we identify these homotopy quotients with classifying spaces of diffeomorphism groups, which proves Corollary F.

Lemma 5.15. The Borel constructions $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} F^n_\Sigma(W)$ and $E \text{Diff}^k(W) \times_{\text{Diff}^k(W)} C^n_\Sigma(W)$ are models for the classifying spaces $B\text{PDiff}^k_{\Sigma \varphi(W)}$ respectively $B\text{Diff}^k_{\Sigma \varphi(W)}$ for $k < d$. For $k = d$ and $W$ being oriented, the analogue identifications for the variants with (+)-superscripts hold.

Proof. It suffices to show that $\text{Diff}^k(W)$ acts transitively on $F^n_\Sigma(W)$ and $C^n_\Sigma(W)$ as the stabilizers are precisely the subgroups $\text{PDiff}^k_{\Sigma \varphi(W)}$ respectively $\text{Diff}^k_{\Sigma \varphi(W)}$. This follows from the fact that the map $\text{Diff}^k(W) \to \text{Emb}(\bigsqcup_{\Sigma \varphi(W)} D^k, W \cup \partial W)$ given by acting on $n$ fixed disjoint parametrized $k$-discs is by [Pal06] a
fiber bundle $\text{Diff}_g(W) \to \text{Emb}(\prod^n D^k, W \setminus \partial W)$ with path-connected base $\text{Emb}(\prod^n D^k, W \setminus \partial W) \cong \mathcal{F}^n(W)$. This same argument applies to the variants $\mathcal{PDiff}^d_{g,n}(W)$ and $\mathcal{Diff}^d_{g,n}(W)$ using orientation preserving diffeomorphisms and embeddings.

5.3.2. Representation stability. We prove Corollary 5.15 using the notation of the introduction to which we also refer for a diminutive introduction to representation stability in the context of configuration spaces.

**Lemma 5.16.** Let $W$ and $\pi$ be as in Theorem 4 and $\lambda + n$ a partition. The $V_\lambda$-multiplicity in $H^i(F^\pi_n(W); \mathbb{Q})$ is the dimension of $H_\lambda(C_n^\pi(W); V_\lambda)$, where $\pi(C_n^\pi(W))$ acts on $V_\lambda$ via the morphism $\pi(C_n^\pi(W)) \to \Sigma_n$.

**Proof.** Delooping the covering space $\Sigma_n \to F^\pi_n(W) \to C_n^\pi(W)$ once results in a fibration sequence with base space $\Sigma_n$, which induces a twisted Serre spectral sequence using the local system $V_\lambda$ on $\Sigma_n$

$$E_2^{p,q} \cong H_p(\Sigma_n; H_q(F^\pi_n(W); V_\lambda)) \implies H_{p+q}(C_n^\pi(W); V_\lambda).$$

Since the action of $\pi(C_n^\pi(W))$ is trivial on $V_\lambda$, we remark

$$H_p(\Sigma_n; H_q(F^\pi_n(W); V_\lambda)) \cong H_p(\Sigma_n; H_q(F^\pi_n(W); \mathbb{Q}) \otimes V_\lambda),$$

which vanishes for $p \neq 0$ since $\Sigma_n$ has no rational cohomology in positive degree. Hence, the $E_2$-page is trivial except for the 0th column which is isomorphic to the cohomotopy $H_0(F^\pi_n(W); \mathbb{Q}) \otimes V_\lambda$, which are in turn isomorphic to the invariants $(H_0(F^\pi_n(W); \mathbb{Q}) \otimes V_\lambda)^{V_\lambda}$. As a result of this, the spectral sequence collapses and we can identify $H_q(C_n^\pi(W); V_\lambda)$ with $(H_q(F^\pi_n(W); \mathbb{Q}) \otimes V_\lambda)^{V_\lambda}$ which has dimension equal to its $V_\lambda$-multiplicity as $V_\mu \otimes V_\lambda$ for a partition $\mu + n$ only contains a trivial representation if $\mu = \lambda$ and in that case, it is 1-dimensional (see [HHR] Ex. 4.51). This proves the claim as the $V_\lambda$-multiplicity in $H^i(F^\pi_n(W); \mathbb{Q})$ equals the one in $H_\lambda(F^\pi_n(W); \mathbb{Q})$ by the universal coefficient theorem.

**Corollary 5.17.** For $W$ and $\pi$ as in Theorem 4, the $V_{[\lambda][n]}$-multiplicity in $H^i(F^\pi_n(W); \mathbb{Q})$ is independent of $n$ for $n$ large relative to $i$.

**Proof.** By [CEF15] Prop. 3.4.1, the $\Sigma_n$-representations $V_{[\lambda][n]}$ assemble into a finitely generated $\mathcal{F}$-module $V(\lambda)$ with $V(\lambda)_n \cong V_{[\lambda][n]}$ which pulls back along $C^\pi(W) \to \mathcal{F}$ of (15) to a coefficient system of finite degree for $\prod_{n \geq 0} C_n(M)$ by Example 5.13 ii). Combining Theorem 4 with Lemma 5.16 gives the claim.

**Proof of Corollary 5.17** setties the statement for $F^\pi_n(W)$. To conclude the claim about $F^\pi_n(W)$, observe that the equivalence $C_n^\pi(W) \cong C_n^{\pi,\ast}(W)$ discussed in the introduction is covered by a $\Sigma_n$-equivariant equivalence $F^\pi_n(W) \to F^{\pi,\ast}_n(W)$, so $H^i(F^\pi_n(W); \mathbb{Q}) \cong H^i(F^{\pi,\ast}_n(W); \mathbb{Q})$ as $\Sigma_n$-modules. The remaining part concerning $B \mathcal{PDiff}^d_{g,n}(W)$ is shown by using the model $B \mathcal{PDiff}^d_{g,n}(W) \cong E \mathcal{Diff}_g(W) \times_{\mathcal{Diff}_g(W)} F^d_n(W)$ provided by Lemma 5.15 and adapting the argument of Lemma 5.16 and 5.17 by replacing the covering space $\Sigma_n \to F^\pi_n(W) \to C_n^\pi(W)$ with

$$\Sigma_n \to E \mathcal{Diff}_g(W) \times_{\mathcal{Diff}_g(W)} F^\pi_n(W) \to E \mathcal{Diff}_g(W) \times_{\mathcal{Diff}_g(W)} C_n^\pi(W).$$

The statements about the variants $F^{d,+}_n(W)$ and $\mathcal{PDiff}^{d,+}_{g,n}(W)$ are proven in the same way.

The ranges in the following remark resulted from a discussion with Peter Patzt to whom we like to express our thanks.

**Remark 5.18.** To obtain explicit ranges for Corollary 5.17, one can show that the $\mathcal{F}$-module $V(\lambda)$ used in the proof of Corollary 5.17 is generated in degree $|\lambda| + \lambda_1$ and related in degree $|\lambda| + \lambda_1 + 1$, so the corresponding coefficient system has degree $|\lambda| + 2\lambda_1 + 1$ by Example 5.13 i). Consequently, one deduces that the $V_{[\lambda][n]}$-multiplicities in the cohomology groups of Corollary 5.17 are constant for $i \leq \frac{d}{2} - (|\lambda| + \lambda_1 + 1)$. Note that our range is not uniform, i.e. depends on the partition. In contrast, the range for $H^i(F(W); \mathbb{Q})$ obtained by Church [Chu12] is $i \leq \frac{d}{2}$ if the dimension is $d \geq 3$ and $i \leq \frac{d}{4}$ for $d = 2$ for the manifolds $W$ for which result applies.
6. Moduli spaces of manifolds

Throughout the section, we fix a compact \((d-1)\)-manifold \(P\) without boundary, together with an embedding
\[
P \subseteq \mathbb{R}^{d-1} \times [0, \infty) \times \mathbb{R}^{\infty}
\]
which contains the open unit-cube \((-1, 1)^{d-1} \times [0) \subseteq \mathbb{R}^{d-1} \times [0, \infty) \times \mathbb{R}^{\infty}\).

We consider compact manifolds \(W\) with a specified identification \(\partial W = P\) and denote by \(\text{Diff}_d(W)\) the group of diffeomorphisms fixing a neighborhood of the boundary in the \(C^\infty\)-topology. To construct our preferred model of its classifying space, we choose a collar \(c:\((-\epsilon, 0) \times P \to W\) and denote by \(\text{Emb}_\epsilon(W, (-\infty, 0) \times \mathbb{R}^d \times \mathbb{R}^{\infty})\) for \(\epsilon > 0\) the space of embeddings \(\epsilon\) satisfying \((\epsilon \circ c)(t, x) = (t, x)\) for \(t \in (-\epsilon, 0]\), equipped with the \(C^\infty\)-topology.

We define the *moduli space of \(W\)-manifolds* \(\mathcal{M}(W)\) as the space of submanifolds
\[
W' \subseteq (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^{\infty}
\]
such that

(i) there is an \(\epsilon > 0\) with \(W' \cap (-\epsilon, 0] \times \mathbb{R}^d \times \mathbb{R}^{\infty} = (-\epsilon, 0] \times P\) and

(ii) there is a diffeomorphism \(\phi: W \to W'\) satisfying \(\phi \circ c|_{(-\epsilon, 0) \times P} = \text{inc}_{(-\epsilon, 0) \times P}\) for the inclusion \(\text{inc}\) ensured by (i). The space \(\mathcal{M}(W)\) is topologized as the quotient of
\[
\text{Emb}_\epsilon(W, (-\infty, 0) \times \mathbb{R}^d \times \mathbb{R}^{\infty}) = \text{colim}_{\epsilon \to 0} \text{Emb}_\epsilon(W, (-\infty, 0) \times \mathbb{R}^d \times \mathbb{R}^{\infty})
\]
by the action of \(\text{Diff}_d(W)\) via precomposition. The space \(\text{Emb}_\epsilon(W, (-\infty, 0) \times \mathbb{R}^d \times \mathbb{R}^{\infty})\) is weakly contractible due to Whitney’s embedding theorem and as the action of \(\text{Diff}_d(W)\) on it is free and admits slices \([BF81]\), the moduli space \(\mathcal{M}(W)\) provides a model for the classifying space \(B \text{Diff}_d(W)\).

In the case of \(P\) being the sphere \(S^{d-1}\), we define a weakly equivalent variant \(\mathcal{M}'(W)\) of \(\mathcal{M}(W)\), consisting of submanifolds
\[
W' \subseteq \mathbb{R}^d \times (-\infty, 0] \times \mathbb{R}^{\infty}
\]
such that

(i) the interior of \(W'\) lies in \((\mathbb{R}^d \setminus \partial \mathbb{R}^d) \times (-\infty, 0] \times \mathbb{R}^{\infty}\),

(ii) there exists an \(\epsilon > 0\) for which \(c': (-\epsilon, 0] \times S^{d-1} \to W'\) mapping \((t, x)\) to \((1 + t)x, 0)\) is a collar and

(iii) there is a diffeomorphism \(\phi: W \to W'\) satisfying \(\phi \circ c'|_{(-\epsilon, 0) \times P} = c'|_{(-\epsilon, 0) \times P}\).

We call \(\mathcal{M} = \bigcup_{[W]} \mathcal{M}(W)\) the *moduli space of manifolds with \(P\)-boundary*, the union taken over compact manifolds \(W\) with an identification \(\partial W = P\), one in each diffeomorphism class relative to \(P\). Analogously, the *moduli space of manifolds with sphere boundary* is \(\mathcal{A} = \bigcup_{[N]} \mathcal{M}'(N)\) for \(N\) with \(\partial N = S^{d-1}\).

**Lemma 6.1.** The moduli space \(\mathcal{A}\) of manifolds with sphere boundary forms an \(E_d\)-algebra with the moduli space \(\mathcal{M}\) of manifolds with \(P\)-boundary as an \(E_1\)-module over it.

**Proof.** The operad \(D^\bullet(D^d)\) of little \(d\)-discs acts on \(\bigcup_{[N]} \mathcal{M}'(N)\) by gluing manifolds along their sphere boundary into a disc, instructed by little \(d\)-discs. Formally, define
\[
\theta: D^k(D^d) \times \bigcup_{[N]} \mathcal{M}'(N) \to \bigcup_{[N]} \mathcal{M}'(N)
\]
\[
((\phi_1, \ldots, \phi_k), (N_1, \ldots, N_k)) \mapsto \bigcup_{i=1}^k (D^d \setminus \cup_{j=1}^k \text{im } \phi_j) \times \{0\}) \cup \bigcup_{i=1}^k r_i N_i + b_i
\]
where \(r_i\) is the radius and \(b_i\) the center of \(\phi_i: D^d \to D^d\) and \(r_i N_i + b_i\) is obtained from \(N_i\) by scaling by \(r_i\) and translating by \(b_i\), both in the \(D^d\)-coordinate. This extends to an action of \(\mathcal{S}_{C_d}\) (see Section 2.1) on the pair \((\bigcup_{[W]} \mathcal{M}(W), \bigcup_{[N]} \mathcal{M}'(N))\) via
\[
\theta: \mathcal{S}_{C_d}(m, a, m) \times \bigcup_{[W]} \mathcal{M}(W) \times \bigcup_{[N]} \mathcal{M}'(N) \to \bigcup_{[W]} \mathcal{M}(W),
\]
mapping \((s, \phi), (M, N_1, \ldots, N_k)\) to the submanifold obtained from
\[
M \cup (([0, s] \times P) \setminus \bigcup_{i=1}^k \text{im } \phi_i \times \{0\})) \cup \bigcup_{i=1}^k r_i N_i + b_i
\]
by translating in the first coordinate by \(s\) to the left. Loosely speaking, we attach a cylinder to the boundary of \(W\), glue in the \(N_i\) via the little \(d\)-discs and shift everything to the left, exemplified by Figure 6. □
6.1. The resolution by embeddings. In dimensions $d \geq 2$, the moduli space $\mathcal{M}$ of manifolds with $P$-boundary forms by virtue of Lemma 6.1 an $E_{1}$-module over the one of manifolds with sphere boundary $\mathcal{A}$, considered as an $E_{2}$-algebra via the embedding $\mathcal{S}_{d} \to \mathcal{S}_{d}$ of Section 2.1. For $A \in \mathcal{M}$ and $X \in \mathcal{A}$, the stabilized manifold $A \oplus X$ is a model for the connected sum $A \# X$ of $A$ with $X = X \cup_{S^{d-1}} D^{d}$. On path-components, the stabilization takes therefore the form $s: \mathcal{M}(A) \to \mathcal{M}(A \# X)$, modeling the map

$$s: B \text{Diff}_{P}(A) \to B \text{Diff}_{P}(A \# X)$$

induced by extending diffeomorphisms by the identity.

As we did for configuration spaces in Section 5.1, we identify the canonical resolution of $\mathcal{M}$ with an augmented semi-simplicial space $R_{\infty}^{\infty}(\mathcal{M})$ of geometric nature, which is a generalization of one introduced by Galatius and Randal-Williams [GR17a]. To that end, denote for $X \in \mathcal{A}$ by $H_{X}$ the manifold obtained from $X$ by gluing in $[-1,0] \times D^{d-1}$ along the embedding

$$\{1\} \times D^{d-1} \to \partial X = S^{d-1}$$

$$x \mapsto (\sqrt{1 - |x|}, x).$$

The resulting manifolds is, after smoothing corners, diffeomorphic to $X$, but contains a canonically embedded strip $[-1,0] \times D^{d-1} \subseteq H_{X}$. When considering embeddings of $H_{X}$ into a manifold with boundary, we always implicitly require that $\{0\} \times D^{d}$ is sent to the boundary and the rest of $H_{X}$ to the interior.

**Definition 6.2.** Let $W$ be a $d$-manifold, equipped with an embedding $e: (-\varepsilon, 0] \times R^{d-1} \to W$ for an $\varepsilon > 0$ satisfying $e^{-1}(\partial W) = \{0\} \times R^{d-1}$. Define a semi-simplicial space $K_{X}^{\infty}(W)$ with the space of $p$-simplices given by tuples $((\varphi_{0}, t_{0}), \ldots, (\varphi_{p}, t_{p})) \in \text{Emb}(H_{X}, W) \times R^{p+1}$ of embeddings with parameters such that

(i) the embeddings $\varphi_{i}$ are pairwise disjoint,

(ii) there exists an $\delta \in (0, \varepsilon)$ such that $\varphi_{i}(s, p + t_{i}e_{i})$ holds for $(s, p) \in (-\delta, 0] \times D^{d-1} \subseteq H_{X}$, where $e_{i} \in R^{d-1}$ is the first basis vector and

(iii) the parameters are ordered $t_{0} < \ldots < t_{p}$.

The embedding space is topologized in the $C^{\infty}$-topology and the ith face map is given by forgetting $(\varphi_{i}, t_{i})$.

For submanifolds $W \in \mathcal{M}$, we use the embedding $e: (-\varepsilon, 0] \times R^{d-1} \to W$ which is obtained from the canonically contained cube $(-\varepsilon, 0] \times (-1, 1)^{d-1} \subseteq (-\varepsilon, 0] \times P \subseteq W$ by use of the diffeomorphism

$$\begin{align*}
R & \to (-1, 1) \\
\begin{cases}
\frac{1}{\varepsilon} & \text{arctan}(x)
\end{cases}
\end{align*}$$

The group $\text{Diff}_{P}(W)$ acts simplicially on $K_{X}^{\infty}(W)$ by precomposition, so forming the level-wise Borel construction results an augmented semi-simplicial space

$$\text{Emb}_{P}(W, (-\infty, 0] \times R^{d} \times R^{\infty}) \times_{\text{Diff}_{P}(W)} K_{X}^{\infty}(W) \to \mathcal{M}(W)$$

in terms of which we define the resolution by embeddings as the augmented semi-simplicial space

$$R_{\infty}^{\infty}(\mathcal{M}) \to \mathcal{M}$$

obtained by taking coproducts of the $\text{Emb}_{P}(W, (-\infty, 0] \times R^{d} \times R^{\infty}) \times_{\text{Diff}_{P}(W)} K_{X}^{\infty}(W) \to \mathcal{M}(W)$ over compact manifolds $W$ with $P$-boundary, one in each diffeomorphism class relative $P$. This is the analogue of the resolution by arcs for configuration spaces. A point in $R_{\infty}^{\infty}(\mathcal{M})$ consists of a manifold $W \in \mathcal{M}$ and $(p + 1)$ embeddings of $H_{X}$ into $W$ that form an element of $K_{p}^{X}(W)$ (see the rightmost graphic of Figure 7).
for an example). Since the augmentation is a level-wise fiber bundle by construction, the fiber $X^A$ of the resolution by embeddings at $A \in \mathcal{M}$ is equivalent to its homotopy fiber.

**Theorem 6.3.** The canonical resolution and the resolution by embeddings are weakly equivalent as augmented $\Delta_{inj}$-spaces. In particular, $X^A$ for $A \in \mathcal{M}$ is weakly equivalent to the space of destabilizations $W(A)$ of $A$.

Following the proof of Theorem 5.5, for configuration spaces, we construct a weak equivalence up to higher coherent homotopy between the canonical resolution and the fibrant replacement of the resolution by embeddings. As a first step, we replace the semi-simplicial space $R^\infty_\bullet(M)$ (see Definition 5.6) with an equivalent variant $R^\infty_\bullet(M)$, which essentially includes a contractible choice of tubular neighborhoods of the arcs. Consider for $s > 0$ the simplicial space $K^D_\bullet((0, s) \times (-1, 1)^{d-1})$ using the embedding $e: (-s, 0) \times \mathbb{R}^{d-1} \to (0, s) \times (-1, 1)^{d-1}$ obtained from the diffeomorphism $(-s, 0) \times \mathbb{R}^{d-1}$ and the translation by $s$. Call a 0-simplex $(\varphi, t)$ therein a little $d$-disc with thickened tether, if the restriction $\varphi|_{D^d}: D^d \to (0, s) \times (-1, 1)^{d-1}$ is a composition of a scaling and a translation. The embedding $\varphi: H_d \to (0, s) \times (-1, 1)^{d-1}$ induces an arc $\varphi^t := \varphi|_{[-1, 0] \times \{0\}}: [-1, 0] \to (0, s) \times (-1, 1)^{d-1}$, called the tether of $\varphi$, which connects the little $d$-disc to the boundary. The embedding $\varphi$ furthermore induces a trivialization of the normal bundle of the tether, which we consider as a map $[-1, 0] \to V_{d-1}(\mathbb{R}^d)$ to the space of $(d - 1)$-frames in $\mathbb{R}^d$. We call a little $d$-disc with thickened tether $(\varphi, t)$ two-dimensional if

(i) the little $d$-disc $\varphi|_{D^d}$ is the image of a little 2-disc in $(0, s) \times (-1, 1)$ under $\mathcal{S}C_2 \to \mathcal{S}C_d$ (see Section 2.1),

(ii) the induced tether $\varphi^t$ lies in the slice $(0, s) \times (-1, 1) \times \{0\}$ and

(iii) the induced trivialization $[-1, 0] \to V_{d-1}(\mathbb{R}^d)$ equals, up to scaling by a smooth function $[-1, 0] \to (0, \infty)$, the parallel transport of the frame $(e_2, \ldots, e_d) \in V_{d-1}(\mathbb{R}^d)$ at $\varphi^t(0)$ along the tether $\varphi^t$, where $e_i \in \mathbb{R}^d$ denotes the $i$th basis vector.

**Definition 6.4.** Define the augmented semi-simplicial space $R^\infty_\bullet(M)$ with $p$-simplices

$$R^\infty_\bullet(M) \subseteq M^{free} \times (R \times \text{Emb}(H_d, (0, \infty) \times (-1, 1)))^{p+1}$$

consisting of $((s, (\varphi_i)), (\varphi_i, t_i))$ such that $(\varphi_i, t_i) \in K^D_\bullet((0, s) \times (-1, 1)^{d-1})$ and all $(\varphi_i, t_i)$ are two-dimensional little $d$-discs with thickened tethers whose induced little 2-disc is one of the $\varphi_i$.

As a two-dimensional little 2-disc with thickened tether is, up to a contractible choice of a thickening, determined by the associated little 2-disc and its tether, $R^\infty_\bullet(M)$ and $R^\infty_\bullet(M)$ are weakly equivalent. We define a category $U^{\mathcal{S}C_2}$ in the same way as the category with the identical name of Definition 5.5 by using $R^\infty_\bullet(M)$ instead of $R^\infty_\bullet(M)$ in the construction. Using the weak equivalence $R^\infty_\bullet(M) \simeq R^\infty_\bullet(M)$, the category $U^{\mathcal{S}C_2}$ is homotopy discrete with path-component category $U^{\mathcal{S}C_2}$ (see Lemma 5.10). To define an extension $B^p(M): (U^\mathcal{S}C_2)^p \to \text{Top}/M$ of $R_0(M)$ and $R^\infty_\bullet(M)$, we replace the simplicial map (13) we used in the proof for configuration spaces with

$$\Phi: M \times R_0^\infty(M) \to R_0^\infty(M),$$

mapping $(A, (e, (t, \varphi)))$ to the manifold $\theta(e; A, X^{p+1})$ equipped with the embeddings of $H_X$ obtained from the $\varphi_i$ by replacing $D^d$ by $X$ (see Figure 7 for an illustration).

![Figure 7](image)

**Figure 7.** The resolution by embeddings and the map $\Phi$

As in the proof of Theorem 5.5, $B^p(M)$ restricts to a functor $R^\infty(M)$ on a category $\Delta_{inj}^{op} \times [1]$ together with a weak equivalence to $\Delta_{inj}^{op} \times [1]$, which extends $R_0(M)$ and $R^\infty_\bullet(M)$\textsuperscript{fib}. The proof is completed
by the same argument we gave in the proof of Theorem 5.5 using that \( R^\theta(B) \) maps all morphisms over \((\text{id}, p), 0 \to 1 \in (\Delta^n_\text{op} \times \{1\})(p^0, p^1) \) for all \([p]\) to weak equivalences, which is implied by the following lemma.

**Lemma 6.5.** For all \( p \geq 0 \) and elements \((e, \phi_i) \in R^\infty_p(M)\) of degree \((p + 1)\), the simplicial map \( I \) induces a weak equivalence \( M \to R^\infty_p(M) \).

**Proof.** The line of argument given in [GR17a, Lem. 6.11] for \( X = S^p \times S^p \) generalizes verbatim. \( \square \)

Galatius and Randal-Williams [GR17a] proved high-connectivity of \( K^X(A) \) for simply connected manifolds \( A \) if \( X = X \cup_{S^2} D^2 \) is a product of two spheres of the same dimension. On the basis of this, Friedrich [Fri16] and Perlmutter [Per16a] proved connectivity results for more general manifolds \( A \) and \( X \). To state their results, recall the stable X-genus \( \tilde{g}^X \) (see Section 2.3) and denote by \( \text{usr}(Z[G]) \) for a group \( G \) the unitary stable rank \( [KMo9, Def. 6.3] \) of its group ring \( Z[G] \), considered as a ring with an anti-involution.

**Theorem 6.6.** The realization of \( K^X(A) \) for a connected manifold \( A \in M \) is

(i) \( \frac{1}{2}(\tilde{g}^X(A) - 4) \)-connected if \( X \cong S^p \times S^p \), \( p \geq 3 \) and \( A \) is simply connected,

(ii) \( \frac{1}{2}(\tilde{g}^X(A) - \text{usr}(Z[\pi(A)]) - 3) \)-connected if \( X \cong S^p \times S^p \) and \( p \geq 3 \) and

(iii) \( \frac{1}{2}(\tilde{g}^X(A) - 4 - m) \)-connected if \( X \cong S^p \times S^q \), \( 0 < p < q < 2p - 2 \) and of \( A \) is \((q - p + 2)\)-connected,

where \( m \) is the smallest number such that there exists an epimorphism of the form \( Z^i \to \pi_q(S^p) \).

**Proof.** The first two parts are [GR17a, Cor. 5.10] and [Fri16, Thm. 4.7]. Corollary 7.3.1 of [Per16a] proves the third claim for the genus \( \tilde{g}^X(B) \) instead of the stable variant \( \tilde{g}^X(B) \). However, the proof given therein goes through for \( \tilde{g}^X(B) \) if one replaces the relation between the genus of a manifold \( B \) satisfying the assumption in (ii) and the rank of its associated Wall form (see [Per16a, Prop. 6.1]) with the analogous statement relating the stable genus to the stable rank. \( \square \)

We denote by \( g^X_A \) for a manifold \( A \in M \) the grading of \( M \) obtained by localizing the stable \( X \)-genus at objects stably isomorphic to \( A \) (see Remark 6.21). Combining Theorem 6.3 with 6.6 implies the following.

**Corollary 6.7.** The canonical resolution \( R_\bullet(M) \to M \) is graded

(i) \( \frac{1}{2}(g^X_A - 3) \)-connected for \( X \cong S^p \times S^p \), \( p \geq 3 \) and any simply-connected \( A \in M \).

(ii) \( \frac{1}{2}(g^X_A - \text{usr}(Z[\pi(A)]) - 1) \)-connected for \( X \cong S^p \times S^p \), \( p \geq 3 \) and any connected \( A \in M \).

(iii) \( \frac{1}{2}(g^X_A - 2 - m) \)-connected for \( X \cong S^p \times S^q \), \( 0 < p < q < 2p - 2 \) and any \((q - p + 2)\)-connected \( A \in M \) with \( m \) defined as in Theorem 6.6.

**Remark 6.8.** In the case \( d = 2 \), one can use [HV17, Prop. 5.1] to show that \( K^X(A) \) is \( \frac{1}{2}(\tilde{g}^X(A) - 3) \)-connected for \( X = S^1 \times S^1 \) and \( A \in M \) being an orientable surface, which implies stability results for diffeomorphism groups of surfaces. However, their homotopy discreteness [Le67, Gra73] ensures their equivalence to their mapping class group for which stability has a longstanding history, going back to a breakthrough result by Harer [Har83], improved in manifold ways since then [Bol12, CMo9, Iva93, Ran16, RW17, Wao08].

By Remark 2.24, our main theorems A and C are applicable to \( M \), when graded by \( g^X_A + 2 \), by \( g^X_A + \text{usr}(Z[\pi(A)]) + 1 \) or by \( g^X_A + m + 2 \) for \( X \) and \( A \) depending on the respective three cases of Corollary 6.7. In particular, this implies Theorem 2, by restricting to path-components, noting that, in the relevant ranges, the genus and the stable genus agree. (See Remark 2.26).

### 6.2. Coefficient systems for moduli spaces of manifolds

Recall from Section 4.2 that coefficient systems for the moduli space of manifolds with \( P \)-boundary \( M \) are defined in terms of the module structure of the fundamental groupoid \( (\Pi(M), \oplus) \) over the braided monoidal category \( (\Pi(\mathcal{A}), \oplus, b, 0) \) induced by the \( E_1 \)-module structure of \( M \) over the moduli space of manifolds with sphere boundary \( \mathcal{A} \). In the following, we provide an alternative description for the fundamental groupoids \( (\Pi(M), \oplus) \) and \( (\Pi(\mathcal{A}), \oplus) \) which is more suitable to construct coefficient systems on \( M \).

Define the categories \( \text{mcg}(M) \) and \( \text{mcg}(\mathcal{A}) \) with the same objects as \( \Pi(M) \) respectively \( \Pi(\mathcal{A}) \) and the mapping class group \( \pi_0(\text{Diff}_\partial(M, N)) \) as morphisms between \( M \) and \( N \), where \( \text{Diff}_\partial(M, N) \) denotes the space of diffeomorphisms in the \( C^\infty \)-topology between \( M \) and \( N \) which preserve a germ of the canonical collars of \( M \) and \( N \). Composition of diffeomorphisms induces the composition in \( \text{mcg}(M) \) and \( \text{mcg}(\mathcal{A}) \).
Lemma 6.9. The categories $\text{mcg}(M)$ is canonically isomorphic to $\Pi(M)$ and $\text{mcg}(\mathcal{A})$ to $\Pi(M)$.

Proof. Recall the fiber bundle from the construction of $M(\mathcal{A})$ in the beginning of the chapter

$$\text{Diff}_\partial(A) \to \text{Emb}_\partial(A, (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty) \to M(A).$$

Lifting a path from $A$ to $B$ in $M$ to a path in the total space which starts at the inclusion $A \subseteq (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty$ gives a path of embeddings ending at an embedding with image $B$ and hence a diffeomorphism from $A$ to $B$ by restricting to the image. This provides a functor from $\text{mcg}(M)$ to $\Pi(M)$, which has an inverse induced by considering a diffeomorphism as an embedding, choosing a path in the contractible space $\text{Emb}_\partial(A, (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty)$ from the inclusion $A \subseteq (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty$ to the embedding obtained from the diffeomorphism and mapping this path to $M(A)$. The argument for $\text{mcg}(\mathcal{A}) \cong \Pi(\mathcal{A})$ is analogous. □

The module structure of $\Pi(M)$ over $\Pi(\mathcal{A})$ can be transported via the identification of the preceding lemma to one of $\text{mcg}(M)$ over $\text{mcg}(\mathcal{A})$, considered as a braided monoidal category by making use of the isomorphism $\text{mcg}(\mathcal{A}) \cong \Pi(\mathcal{A})$. More concretely, the monoidal structure on $\text{mcg}(\mathcal{A})$ is on objects given by the one of $\Pi(\mathcal{A})$ induced by the $E_2$-multiplication and on morphisms by multiplying $f \in \text{Diff}_\partial(A, B)$ and $g \in \text{Diff}_\partial(A, X)$ as $f \otimes g \in \text{Diff}_\partial(A \oplus A', B \oplus B')$ defined by extending $f$ and $g$ via the identity. The description of the module structure on $\text{mcg}(M)$ is analogous. Coefficient systems for $M$ are then given by coefficient systems for the module $\text{mcg}(M)$ over $\text{mcg}(\mathcal{A})$ in the sense of Definition 4.1.

To illustrate how the identification can be used to construct coefficient systems on $M$, we discuss one example in detail. Consider for $i \geq 0$ the functor $H_i : \text{mcg}(M) \to \mathcal{Ab}$ which assigns a manifold $A \in M$ its $i$th singular homology group $H_i(A)$. The inclusions $A \subseteq A \oplus X$ induce a natural transformation $\sigma^1 : H_i(\_ \oplus X) \to H_i(\_ \oplus X)$ satisfying the triviality condition for coefficient systems (see Definition 4.1). To calculate the degree of $H_i$, consider the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & H_i(A) & \to & H_i(A \oplus X) & \to & \tilde{H}_i(X) & \to & 0 \\
\downarrow{\sigma^1} & & \downarrow{\sigma^1} & & \downarrow{d} & & \downarrow{d} & & \downarrow{d} \\
0 & \to & H_i(A \oplus X) & \to & H_i(A \oplus X \oplus X) & \to & \tilde{H}_i(X) & \to & 0
\end{array}
\]

induced by the long exact sequence of pairs together with the equivalence $(A \oplus X^{\oplus i}, A \oplus X^{\oplus i-1}) \simeq (X, \ast)$ of pairs obtained by collapsing the complement of the last $X$-summand. The leftmost vertical map is induced by the inclusion and the second one by the inclusion followed by $A \oplus X_X$. Naturality of the diagram in $A$ implies triviality of the kernel of $H_i$ and that its cokernel is constant, so of degree 0 if $H_i(X) \neq 0$ and of degree $-1$ else wise. Hence, $H_i$ is of degree 1 at 0 if $\tilde{H}_i(X) \neq 0$ and of degree 0 at 0 if $\tilde{H}_i(X) = 0$, from which Corollary 1 is implied by an application of Theorem 1.

6.3. Extensions.

6.3.1. Stabilization by $(2n-1)$-connected $(4n+1)$-manifolds. In [Per16b], Perlmutter showed high-connectivity of the semi-simplicial spaces $K^2_{\mathfrak{A}}(A)$ for 2-connected manifolds $A$ of dimension $(4n+1)$ with $n \geq 2$ and certain specific $(2n - 1)$-connected stably-parallelizable manifolds $X$ with finite $H_{2n}(X; \mathbb{Z})$ and trivial $H_{2n}^2(X, \mathbb{Z}/2\mathbb{Z})$. From this result, he derived homological stability with constant coefficients of

\[
B \text{Diff}_\partial(A) \to B \text{Diff}_\partial(A[X])
\]

for these $A$ and $X$. By using classification results of closed $(2n - 1)$-connected stably parallelizable $(4n + 1)$-manifolds by Wall [Wal67] and De Sapio [De70], he furthermore concluded that (18) stabilizes in fact for all $X$ with $X$ having the properties described above and not just the specific ones considered before. The methods of this section imply an extension of his homological stability result to abelian coefficients and coefficient systems of finite degree.

6.3.2. Automorphisms of topological and piecewise linear manifolds. In [Kup13], Kupers explains how one can adapt the methods of Galatius and Randal-Williams [GR17a] to prove high-connectivity of the relevant semi-simplicial spaces of locally flat embeddings to prove homological stability for classifying spaces of homeomorphisms of topological manifolds and PL-automorphisms of piecewise linear manifolds. Our framework applies also to these examples using the ideas of this section, which results in an extension of Kupers’ stability results to coefficients systems of finite degree.
7. Homological stability for modules over braided monoidal categories

We explain the applicability of our framework to modules over braided monoidal categories and make a comparison to the theory for braided monoidal groupoids developed by Randal-Williams–Wahl in [RW17].

7.1. \(E_1\)-modules over \(E_2\)-algebras from modules over braided monoidal categories. Recall the categorical (one-colored) operad of colored braids \(\text{CoB} \) [Fre17 Ch. 5]. The category of \(n\)-operations is the groupoid \(\text{CoB}(n)\) with linear orderings of \(\{1, \ldots, n\}\) as its objects and braids connecting the spots as prescribed by the orderings as its morphisms. The operadic composition is given by “cabling”. Algebras over \(\text{CoB}\) are the same as strict braided monoidal categories and that the topological operad obtained by taking classifying spaces is \(E_2\) (see [Fre17 Thm. 5.2.12] or [FSV13 Ch. 8]). Extending this, we construct a two-colored operad whose algebras are modules over braided monoidal categories and whose classifying space operad is \(E_1\) (see Section 2.1).

**Definition 7.1.** Define a categorical operad \(\text{CoBM}\) with colors \(m\) and \(a\) whose operations \(\text{CoBM}(m^k, a^l; m)\) are empty for \(k \neq 1\) and are \(\text{CoB}(l)\) otherwise. The operations \(\text{CoBM}(m^k, a^l; a)\) are empty for \(k \neq 0\) and equal \(\text{CoB}(l)\) else. Restricted to the \(a\)-color, \(\text{CoBM}\) is defined as \(\text{CoB}\). Requiring commutativity of

\[
\text{CoBM}(m, a^l; m) \times (\text{CoBM}(m, a^k; m) \times \text{CoBM}(a^i; a) \times \ldots \times \text{CoBM}(a^t; a)) \xrightarrow{Y_{\text{CoBM}}} \text{CoBM}(m, a^{k+l}; m)
\]

defines the remaining composition \(Y_{\text{CoBM}}\), where \(\tau\) interchanges the first two factors, \(Y_{\text{CoB}}\) is the composition of \(\text{CoB}\) and \(\oplus\) is \(Y_{\text{CoB}}(\text{id}_{[1 < 2]}; \omega \cdot \omega)\) in \(\text{CoB}\), i.e. puts braids next to each other (see Figure 8 for an example).

\[
\begin{align*}
\text{CoBM}(m, a^5; m) & \quad \text{CoBM}(m, a^3; m) \\
\text{CoBM}(a^5; a) & \quad \text{CoBM}(a^3; a) \\
\text{CoBM}(m, a^6; m) & \quad \text{CoBM}(m, a^4; m)
\end{align*}
\]

**Figure 8.** The operadic composition in \(\text{CoBM}\)

Recall the notion of a right-module \((M, \oplus)\) over a monoidal category \((\mathcal{A}, \oplus, 0)\) being a category \(M\) together with a functor \(\oplus: M \times \mathcal{A} \to M\) which is unital and associative up to coherent isomorphisms.

**Lemma 7.2.** The structure of a (graded) \(\text{CoBM}\)-algebra on a pair of categories \((M, \mathcal{A})\) is equivalent to a strict (graded) braided monoidal structure on \(\mathcal{A}\) and a strict (graded) right-module structure on \(M\) over it. Furthermore, the topological operad obtained from \(\text{CoBM}\) by taking level-wise classifying spaces is \(E_{1,2}\).

**Proof.** The demonstration of the corresponding statements concerning \(\text{CoB}\) in Chapter 5 of [Fre17] carries over to \(\text{CoBM}\) mutatis mutandis. \(\square\)

As a consequence, the classifying space of a graded module over a braided monoidal category is a graded \(E_1\)-module over an \(E_2\)-algebra.

**Remark 7.3.** The operad of parenthesized colored braids encodes non-strict braided monoidal categories and its classifying space operad is \(E_2\) as well [Fre17 Ch. 6]. By considering a parenthesized version of \(\text{CoBM}\), this extends in a similar fashion to non-strict right-modules over non-strict braided monoidal categories whose classifying spaces hence also give \(E_1\)-modules over \(E_2\)-algebras.
7.2. Homological stability for groups and monoids. Let \((\mathcal{A}, \oplus)\) be a graded right-module over a braided monoidal category \((\mathcal{A}, \oplus, b, 0)\) with a stabilizing object \(X\), i.e., an object of \(\mathcal{A}\) of degree 1. Taking classifying spaces results by Lemma 7.2 in a graded \(E_l\)-module \(BM\) over the \(E_l\)-algebra \(B\mathcal{A}\) with a stabilizing object \(X \in B\mathcal{A}\), hence gives a suitable input for Theorem \(\text{[A]}\) and \(\text{[B]}\). In the following, we introduce a condition on \(M\) which simplifies the canonical resolution of \(BM\).

**Definition 7.4.** The module \((M, \oplus)\) is called injective at \(A\) for an object \(A\) of \(M\) if the stabilization
\[
(\_ \oplus X^{\oplus p+1}): \text{Aut}(B) \to \text{Aut}(B \oplus X^{\oplus p})
\]
is injective for all \(p \geq 0\) and all objects \(B\) for which \(B \oplus X^{\oplus p+1}\) is isomorphic to \(A\).

**Definition 7.5.** Define for an object \(A\) of \(M\) a semi-simplicial set \(W^\mathcal{A}_p(A)\) with \(p\)-simplices given as equivalence classes of pairs \((B, f)\) of object \(B\) of \(M\) and a morphism \(\tilde{f} \in M(B \oplus X^{\oplus p+1}, A)\) with \((B, f)\) and \((B', f')\) being equivalent if there is an isomorphism \(g \in M(B, B')\) satisfying \(f' \circ (g \oplus X^{\oplus p+1}) = f\). The \(i\)th face of a \(p\)-simplex \([B, f]\) is defined as \([B \oplus X, f \circ (B \oplus b^{-1}_X \oplus X^{\oplus p-i})]\).

Recall the spaces of destabilizations \(W_*(A)\), i.e., the fibers of the canonical resolution (see Definition 2.14).

**Lemma 7.6.** If \(M\) is a groupoid, then the semi-simplicial set \(W^\mathcal{A}_p(A)\) for an object \(A\) of \(M\) is isomorphic to \(W^\mathcal{A}_p(A)\) and \(W_*(A)\) is homotopy discrete if and only if \(M\) is injective at \(A\).

**Proof.** The inclusion of the 0-simplices ob \(M \subseteq BM\) together with the natural map mor \(M \to \text{Path } M\) induces a preferred bijection \(W^\mathcal{A}_p(B) \to W_0(W^\mathcal{A}_p(A))\) for all \(p \geq 0\) since every path in \(BM\) between 0-simplices is homotopic relative to its endpoints to a one simplex, i.e., to a path in the image of mor \(M \to \text{Path } M\).

By the definition of the respective face maps, these bijections assemble to an isomorphism of semi-simplicial sets, which proves the first claim. The homotopy fiber \(W_p(B)\) at the base of the map \(B(\_ \oplus X^{\oplus p+1}): BM \to BM\) are homotopy discrete if and only if the induced morphisms on \(\pi_1\) based at objects \(B\) with \(B \oplus X^{\oplus p+1} \cong A\) for \(p \geq 0\) are injective, which is clearly equivalent to \(M\) being locally injective at \(A\).

**Remark 7.7.** If \(M\) and \(\mathcal{A}\) are groupoids, then \(M = \Pi(B\mathcal{A})\) holds naturally as a module over \(\mathcal{A} = \Pi(B\mathcal{A})\), so coefficient systems for \(BM\) (see Definition 4.12) agree with coefficient systems for \(M\) as in Section 4.1.

**Remark 7.8.** The combination of Lemma 7.6 and 7.7 implies a version of Theorem \(\text{[A]}\) and \(\text{[B]}\) that is phrased entirely in terms of discrete categories and semi-simplicial sets, since the connectivity of the canonical resolution can be tested on the spaces of destabilizations \(W_*(A)\) (see Remark 2.17). This provides a simplified toolkit for proving homological stability for graded modules over braided monoidal categories with a stabilizing object \(X\) for which the multiplication \((\_ \oplus X): \text{Aut}(B) \to \text{Aut}(B \oplus X)\) is injective for all objects \(B\) of finite degree.

7.3. Comparison with the work of Randal-Williams and Wahl. Let \((\mathcal{G}, \circ, b, 0)\) be a braided monoidal groupoid. In \([\text{RW17}]\), it is shown that, for objects \(A\) and \(X\) in \(\mathcal{G}\), the maps
\[
B(\_ \circ X): B \text{Aut}_\mathcal{G}(A \circ X^{\oplus n}) \longrightarrow B \text{Aut}_\mathcal{G}(A \circ X^{\oplus n+1})
\]
satisfy homological stability with constant, abelian and a class of coefficient systems of finite degree if a family of associated semi-simplicial sets \(W_n(A, X)\) is sufficiently connected, provided that \(\mathcal{G}\) satisfies

(i) injectivity of the stabilization map (\(\circ X\)): \(\text{Aut}_\mathcal{G}(A \circ X^{\oplus n}) \to \text{Aut}_\mathcal{G}(A \circ X^{\oplus n+1})\) for all \(n \geq 0\),

(ii) local cancellation at \((A, X)\), i.e., \(Y \circ X = A \circ X^{\oplus n}\) for \(Y \in \text{ob } \mathcal{G}\) and \(1 \leq m \leq n\) implies \(Y = A \circ X^{\oplus m-n}\),

(iii) no zero-divisors, i.e. \(U \circ V = 0\) implies \(U \circ 0 = 0\) and \(V \circ 0 = 0\),

(iv) the unit 0 has no nontrivial automorphisms.

As indicated by our choice of notation, the simplicial set \(W_n(A, X)\) equals \(W^\mathcal{G}_*(A \circ X^{\oplus n})\) when considering \(\mathcal{G}\) as a module over itself. Define the module \(\mathcal{G}^-\mathcal{A}_X = \bigcup_{n \geq 0} \text{Aut}_\mathcal{G}(A \circ X^{\oplus n})\) over the braided monoidal category \(\mathcal{G}^-\mathcal{A}_X = \bigcup_{n \geq 0} \text{Aut}_\mathcal{G}(A \circ X^{\oplus n})\), both graded in the evident way. By Theorem \(\text{[A]}\) and \(\text{[B]}\), the maps (19) stabilize homologically—without assumptions on \(\mathcal{G}\)—if the canonical resolution of \(B\mathcal{G}^-\mathcal{A}_X\) is sufficiently connected, or equivalently if the spaces of destabilizations \(W_*(A \circ X^{\oplus n})\) are.

The semi-simplicial sets \(W_n(A, X)\) of \([\text{RW17}]\) are equivalent to the spaces of destabilizations \(W_n(A \circ X^{\oplus n})\) of \(\mathcal{G}^-\mathcal{A}_X\) if conditions (i) and (ii) hold. Indeed, assumption (ii) implies that \(W_0(A, X)\) agrees with
$W_{\mathcal{R}W}(A \oplus X^n)$, i.e. with $\pi_0(W_{\mathcal{R}}(A \oplus X^n))$ by Lemma 7.6. The first condition imposes injectivity of $\mathcal{G}_{A,X}$ at all objects $A \oplus X^n$, which is by Lemma 7.6 equivalent to homotopy discreteness of $W_{\mathcal{R}}(A \oplus X^n)$.

Hence, if one prefers to work in a discrete setting as in [RW17], i.e. with semi-simplicial sets, condition (i) is necessary. Condition (ii) makes sure that the semi-simplicial sets of [RW17] agree with the spaces of destabilization $W_{\mathcal{R}}(A \oplus X^n)$, whose high-connectivity always imply stability, as demonstrated by Theorem [A] and [C]. The last two conditions are redundant, i.e. imposing (i) and (ii) already implies (twisted) homological stability of [19] under the connectivity assumptions of [RW17]. The presence of these additional assumptions in [RW17] is due to their usage of Quillen’s construction $(G, \mathcal{G})$ since the conditions (iii) and (iv) guarantee that the automorphism groups $\text{Aut}_{(G, \mathcal{G})}(A \oplus X^n)$ and $\text{Aut}_{\mathcal{G}}(A \oplus X^n)$ coincide. If (i)–(iii) are satisfied and $W_{\mathcal{R}}(A, X)$ are highly-connected, then [RW17] implies stability for $\text{Aut}_{(G, \mathcal{G})}(A \oplus X^n)$. Hence, in this case, high-connectivity of $W_{\mathcal{R}}(A, X)$ shows stability for both, $\text{Aut}_{\mathcal{G}}(A \oplus X^n)$ and $\text{Aut}_{(G, \mathcal{G})}(A \oplus X^n)$. The reason for this is that, although these automorphism groups might differ, their quotients $\text{Aut}(A \oplus X^n)/\text{Aut}(A \oplus X^n-p^{-1}) \cong W_{\mathcal{R}P}(A \oplus X^n)$ forming the corresponding semi-simplicial sets agree.

Remark 7.9. The coefficient systems [RW17] deals with are functors of finite degree on the subcategory $C_{A,X} \subseteq (G, \mathcal{G})$ generated by the objects $A \oplus X^n$. In contrast to that, Theorem [C] is applicable to functors of finite degree on $(\mathcal{G}_{A,X}, \mathcal{B})$ (see Remark 7.10 and 7.7). As the canonical functors $\mathcal{G}_{A,X} \to \mathcal{G}$ and $\mathcal{B} \to \mathcal{G}$ induce a functor $(\mathcal{G}_{A,X}, \mathcal{B}) \to C_{A,X}$, every coefficient system in the sense of [RW17] gives one in ours.

Remark 7.10. The ranges for coefficient systems of finite degree provided by Theorem [C] agree with the ones of [RW17] in the situations in which their work is applicable. The ranges for abelian coefficients of Theorem [A] improve the ones of [RW17] marginally, and so does the surjectivity range for constant coefficients in the case $k > 2$. Note that, as discussed in Remark 2.24, these ranges can in some cases be further improved.

**References**

[Arn68] Vladimir I. Arnol’d. “Braids of algebraic functions and cohomologies of swallowtails”. Uspehi Mat. Nauk 23.4 (142) (1968), pp. 247–248.

[Arn69] Vladimir I. Arnol’d. “The cohomology ring of the group of dyed braids”. Mat. Zametki 5 (1969), pp. 227–231.

[BCM93] Carl-Friedrich Bödigheimer, Frederick R. Cohen, and Richard J. Milgram. “Truncated symmetric products and configuration spaces”. Math. Z. 214.2 (1993), pp. 179–216.

[BCT89] Carl-Friedrich Bödigheimer, Frederick R. Cohen, and Laurence Taylor. “On the homology of configuration spaces”. Topology 28.1 (1989), pp. 111–123.

[Beto2] Stanislaw Betley. “Twisted homology of symmetric groups”. Proc. Amer. Math. Soc. 130.12 (2002), pp. 3439–3445.

[BF81] Ernst Binz and Hans R. Fischer. “The manifold of embeddings of a closed manifold”. Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978). Vol. 139. Lecture Notes in Phys. With an appendix by P. Michor. Springer, Berlin-New York, 1981, pp. 310–320.

[BM07] Clemens Berger and Ieke Moerdijk. “Resolution of coloured operads and rectification of homotopy algebras”. Categories in algebra, geometry and mathematical physics. Vol. 331. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 31–58.

[Bol12] Soren K. Boldsen. “Improved homological stability for the mapping class group with integral or twisted coefficients”. Math. Z. 270.1-2 (2012), pp. 297–329.

[BV73] J. Michael Boardman and Rainer M. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973, pp. x+257.

[Cas16] Kevin Casto. “$\mathcal{F}_I$-modules, orbit configuration spaces, and complex reflection groups”. ArXiv e-prints (Aug. 2016). arXiv:1608.06317

[CE17] Thomas Church and Jordan S. Ellenberg. “Homology of FI-modules”. Geom. Topol. 21.4 (2017), pp. 2375–2418.

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. “FI-modules and stability for representations of symmetric groups”. Duke Math. J. 164.9 (2015), pp. 1833–1910.
REFERENCES

[CEFN14] Thomas Church, Jordan S. Ellenberg, Benson Farb, and Rohit Nagpal. "FI-modules over Noetherian rings". Geom. Topol. 18.5 (2014), pp. 2951–2984.

[CF13] Thomas Church and Benson Farb. "Representation theory and homological stability". Adv. Math. 245 (2013), pp. 250–314.

[Chu12] Thomas Church. "Homological stability for configuration spaces of manifolds". Invent. Math. 188.2 (2012), pp. 465–504.

[CM09] Ralph L. Cohen and Ib Madsen. "Surfaces in a background space and the homology of mapping class groups". Algebraic geometry—Seattle 2005. Part 1. Vol. 80. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2009, pp. 43–76.

[CP15] Federico Cantero and Martin Palmer. "On homological stability for configuration spaces on closed background manifolds". Doc. Math. 20 (2015), pp. 753–805.

[CS11] Diarmuid Crowley and Jörg Sixt. "Stably diffeomorphic manifolds and $l_{2q+1}(\mathbb{Z}/\pi)$". Forum Math. 23.3 (2011), pp. 485–538.

[De 70] Rodolfo De Sapio. "On $(k-1)$-connected $(2k+1)$-manifolds". Math. Scand. 25 (1970), 181–189 (1970).

[Dwy80] William J. Dwyer. "Twisted homological stability for general linear groups". Ann. of Math. (2) 112.1 (1980), pp. 230–251.

[EE67] Clifford J. Earle and James Eells. "The diffeomorphism group of a compact Riemann surface". Bull. Amer. Math. Soc. 73 (1967), pp. 557–559.

[ER17] Johannes Ebert and Oscar Randal-Williams. "Semi-simplicial spaces". ArXiv e-prints (May 2017). arXiv:[1705.03774]

[FH91] William Fulton and Joe Harris. Representation theory. Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xvi+551.

[Fre17] Benoît Fresse. Homotopy of operads and Grothendieck–Teichmüller groups. Part I. Vol. 217. Mathematical Surveys and Monographs. The algebraic theory and its topological background. American Mathematical Society, Providence, RI, 2017, pp. xvi+532.

[Fri16] Nina Friedl. "Homological Stability of automorphism groups of quadratic modules and manifolds". ArXiv e-prints (Dec. 2016). arXiv:[1612.04584]

[FSV13] Z. Fiedorowicz, M. Stelzer, and R. M. Vogt. "Homotopy colimits of algebras over Cat-operads and iterated loop spaces". Adv. Math. 248 (2013), pp. 1089–1155.

[GKY96] Martin A. Guest, Andrzej Kozlowski, and Kohhei Yamaguchi. "Homological stability of oriented configuration spaces". J. Math. Kyoto Univ. 36.4 (1996), pp. 809–814.

[GL15] Wei Liang Gan and Leping Li. "Coinduction functor in representation stability theory". J. Lond. Math. Soc. (2) 92.3 (2015), pp. 689–711.

[GR17a] Søren Galatius and Oscar Randal-Williams. "Homological stability for moduli spaces of high dimensional manifolds. I". J. Amer. Math. Soc. (2017). to appear.

[GR17b] Søren Galatius and Oscar Randal-Williams. "Homological stability for moduli spaces of high dimensional manifolds. II". Ann. Math. (2) 186.1 (2017), pp. 127–204.

[Gra73] André Gramain. "Le type d’homotopie du groupe des difféomorphismes d’une surface compacte". Ann. Sci. École Norm. Sup. (4) 6 (1973), pp. 53–66.

[Gra76] Daniel Grayson. "Higher algebraic K-theory. II (after Daniel Quillen)". Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976). Springer, Berlin, 1976, 217–240. Lecture Notes in Math., Vol. 551.

[GW16] Giovanni Gandini and Nathalie Wahl. "Homological stability for automorphism groups of RAAGs". Algebr. Geom. Topol. 16.4 (2016), pp. 2421–2441.

[Har85] John L. Harer. "Stability of the homology of the mapping class groups of orientable surfaces". Ann. of Math. (2) 121.2 (1985), pp. 215–249.

[Hep16] Richard Hepworth. "Homological stability for families of Coxeter groups". Algebr. Geom. Topol. 16.5 (2016), pp. 2779–2811.

[HV17] Allen Hatcher and Karen Vogtmann. "Tethers and homology stability for surfaces". Algebr. Geom. Topol. 17.3 (2017), pp. 1871–1916.

[HW10] Allen Hatcher and Nathalie Wahl. "Stabilization for mapping class groups of 3-manifolds". Duke Math. J. 155.2 (2010), pp. 205–269.
REFERENCES

[Seg73] Graeme Segal. "Configuration-spaces and iterated loop-spaces". Invent. Math. 21 (1973), pp. 213–221.
[Seg74] Graeme Segal. "Categories and cohomology theories". Topology 13 (1974), pp. 293–312.
[Seg79] Graeme Segal. "The topology of spaces of rational functions". Acta Math. 143.1-2 (1979), pp. 39–72.
[Sou17] Arthur Soulié. “The Long-Moody construction and polynomial functors”. ArXiv e-prints (Feb. 2017). arXiv:1702.08279
[SS14] Steven V Sam and Andrew Snowden. "Representations of categories of G-maps". ArXiv e-prints (Oct. 2014). arXiv:1410.6054
[SS16] Christian Schlichtkrull and Mirjam Solberg. "Braided injections and double loop spaces". Trans. Amer. Math. Soc. 368.10 (2016), pp. 7305–7338.
[SW14] Markus Szymik and Nathalie Wahl. "The homology of the Higman-Thompson groups". ArXiv e-prints (Nov. 2014). arXiv:1411.5035
[Til16] Ulrike Tillmann. “Homology stability for symmetric diffeomorphism and mapping class groups”. Math. Proc. Cambridge Philos. Soc. 160.1 (2016), pp. 121–139.
[Tos16] Philip Tosteson. "Lattice Spectral Sequences and Cohomology of Configuration Spaces". ArXiv e-prints (Dec. 2016). arXiv:1612.06034
[Tot96] Burt Totaro. "Configuration spaces of algebraic varieties". Topology 35.4 (1996), pp. 1057–1067.
[Vor99] Alexander A. Voronov. “The Swiss-cheese operad". Homotopy invariant algebraic structures (Baltimore, MD, 1998). Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 365–373.
[Waho08] Nathalie Wahl. "Homological stability for the mapping class groups of non-orientable surfaces". Invent. Math. 171.2 (2008), pp. 389–424.
[Wal67] Charles T. C. Wall. "Classification problems in differential topology. VI. Classification of (s − 1)-connected (2s + 1)-manifolds". Topology 6 (1967), pp. 273–296.
[Whi78] George W. Whitehead. Elements of homotopy theory. Vol. 61. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1978, pp. xxi+744.

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