STANDARD MONOMIAL THEORY FOR WONDERFUL VARIETIES

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Abstract. A general setting for a multigraded standard monomial theory is introduced and applied to the Cox ring of a wonderful variety. This gives a degeneration result of the Cox ring to a multicone over a partial flag variety. Further, we deduce that the Cox ring has rational singularities.

1. Introduction

The first appearance of the idea of a standard monomial theory may be traced back to Hodge’s study of Grassmannians in [21], [22]. Then Doubilet, Rota and Stein found a similar theory for the coordinate rings of the space of matrices in [18]. This was reproved and generalized to the space of symmetric and antisymmetric matrices by De Concini and Procesi in [15].

A systematic program for the development of a standard monomial theory for quotients of reductive groups by parabolic subgroups was then started by Seshadri in [33] where the case of minuscule parabolics is considered. Further, in [26] Seshadri and Lakshmibai noticed that the above mentioned results could be obtained as specializations of their general theory.

This program was finally completed by Littelmann. Indeed, in [27], he found a combinatorial character formula for representations of symmetrizable Kac-Moody groups introducing the language of L-S paths. Moreover, he used L-S paths as an index set for the basis constructed in [29] and he proved that this basis defines a standard monomial theory for Schubert varieties of symmetrizable Kac-Moody groups. This theory has been developed in the context of LS algebras over posets with bonds in [8], [9] and [10].

We want now to briefly recall what a standard monomial theory is, the reader may see [11] for further details about this general setting. Let $A$ be a finite subset of an algebra $A$ and suppose we are given a transitive antisymmetric binary relation $\leftarrow$ on $A$. We define a formal monomial $a_1a_2\cdots a_N$ of elements of $A$ as standard if $a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_N$. If the set of standard monomials is a basis of the algebra $A$ as a vector space then we say that $(A, \leftarrow)$ is a standard monomial theory for $A$. Suppose, further, we have a monomial order $\leq_t$ on the monomials of elements of $A$. By the previous assumption, we may write any non-standard monomial $m'$ as a linear combination of standard monomials. If in such an expression only standard monomials $m$ with $m' \leq_t m$ appear, then we say that we have a straightening relation for $m'$. If we have a straightening relation for each non-standard monomial, then we say that $(A, \leftarrow, \leq_t)$ is a standard monomial theory with straightening relations.

Given a simply connected semisimple algebraic group $G$ over an algebraically closed field $k$ of characteristic 0, a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, let $\Lambda^+ \subset \Lambda$ be the monoid of dominant weights and the lattice of weights,
respectively. For a dominant weight \( \lambda \), let \( V_\lambda \) be the irreducible \( G \)-module of highest weight \( \lambda \). Let \( B \subset P \subset G \) be a parabolic subgroup of \( G \) stabilizing the line generated by a highest weight vector in \( V_\lambda \). Moreover, we denote by \( B_\lambda \) the set of L-S paths of shape \( \lambda \).

Littelmann’s construction provides a basis \( A_\lambda = \{ p_\pi | \pi \in B_\lambda \} \), indexed by L-S paths, for the module \( \Gamma(G/P, \mathcal{L}_\lambda) \cong V_\lambda^* \), where \( \mathcal{L}_\lambda \) is the line bundle over \( G/P \) associated with \( \lambda \). The ring of sections \( A_\lambda = \bigoplus_{n \geq 0} \Gamma(G/P, \mathcal{L}_{n\lambda}) \) is generated in degree one and it is the coordinate ring of the cone over the closed embedding \( G/P \hookrightarrow \mathbb{P}(V_\lambda) \) induced by \( \mathcal{L}_\lambda \). On the basis \( A_\lambda \) one may define a relation \( \preceq \) and a monomial order \( \preceq_t \) such that \( (A_\lambda, \preceq, \preceq_t) \) is a standard monomial theory with straightening relations for \( A_\lambda \).

In [12], the second and fourth named authors adapted Littelmann’s basis to the Cox ring (see below) of complete symmetric varieties; this class of varieties has been introduced by De Concini and Procesi in [16]. As a consequence, they proved the degeneration of the Cox ring to the coordinate ring of a suitable multicone over a flag variety. This degeneration allowed a new proof of the rational singularity property for the Cox ring of complete symmetric varieties.

The purpose of the present paper is a further extension of these results to the Cox ring of wonderful varieties. As a first step, we take the opportunity to introduce a general setting for a multigraded standard monomial theory modelled on the above recalled one. This setting may be briefly summarized as follows, see Section 2 below for details.

Let \( A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n \) be the union of disjoint finite subsets of an algebra \( A \). Suppose we have a binary relation \( \preceq \) on \( A \) such that \( \preceq \) restricted to \( A_i \) is transitive and antisymmetric for all \( i = 1, 2, \ldots, n \) and, further, suppose we have bijective maps \( \phi_{i,j} \), called swaps, from the set of comparable pairs \( a \preceq b \) of \( A_i \times A_j \) to the set of comparable pairs \( b' \preceq a' \) of \( A_j \times A_i \) satisfying some mild conditions. We define a formal monomial \( a_1a_2 \cdots a_N \) as weakly standard if \( a_1 \preceq a_2 \preceq \cdots \preceq a_N \), and we say it is standard if all monomials obtained by repeatedly swapping adjacent pairs in all possible ways, via the \( \phi_{i,j} \)'s, are weakly standard. We define the multigrade of a monomial \( a_1a_2 \cdots a_N \) as \( (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n \) where \( k_i \) is the number of elements of \( A_i \) in the monomial. If the set of standard monomials is a basis for \( A \) as a vector space and this basis induces a multigrading for \( A \), we say that \( (A, \preceq, \phi_{i,j}) \) is a multigraded standard monomial theory for \( A \). As above we introduce also the notions of monomial order and straightening relations for non-standard monomials.

We prove that the kernel of the natural map from the symmetric algebra over \( A \) to \( A \) is generated by the straightening relations of minimally non-standard monomials, that is by the straightening relations of those non-standard monomials which are not a product of non-standard monomials of smaller degree. In particular, we show that if any weakly standard monomial is standard then such kernel is generated in degree two.

We also show how, given a valuation map for monomials that is compatible with the total order \( \preceq_t \), one may construct a flat degeneration of \( A \).

As a motivating example for this setting one may see the multigraded standard monomial theory for the multicone over a flag variety constructed by the second named author in [10]. This is described in details in Section 3.

Now we recall which is the type of varieties we are interested in. A \( G \)-variety \( X \) is wonderful of rank \( r \) if it satisfies the following conditions:

- \( X \) is smooth and projective;
X possesses an open orbit whose complement is a union of r smooth prime
divisors, called the boundary divisors, with non-empty transversal intersec-
tions;

- any orbit closure in X equals the intersection of the prime divisors which
  contain it.

Examples of wonderful varieties are the flag varieties, which are the wonderful va-
rieties of rank zero, and the complete symmetric varieties. Wonderful varieties have
been considered in full generality by Luna in [30], [31] in the context of spherical
varieties. See [5] for a general introduction to wonderful varieties.

If X is a wonderful G-variety, then the Picard group Pic(X) is freely generated
by the classes of the B-stable prime divisors of X which are not G-stable. These
divisors are called the colors of X. Since X contains an open B-orbit, the colors
form a finite set Δ, so that Pic(X) is a free lattice of finite rank. Given D ∈ ZΔ,
we denote by L_D the corresponding line bundle.

The direct sum

$$C(X) = \bigoplus_{\mathcal{L} \in \text{Pic}(X)} \Gamma(X, \mathcal{L}),$$

has a ring structure (see Section 4 below) and it is called the Cox ring of X.

Denote by σ_1, . . . , σ_r the boundary divisors of X, and let s_i be the section of
O(σ_i) defining σ_i, for i = 1, . . . , r. As an algebra C(X) is generated by the sections
of the line bundles \mathcal{L}_D = O(D) with D ∈ Δ together with the sections s_1, . . . , s_r.

By definition, X contains a unique closed G-orbit Y ≃ G/P for a suitable parabolic
subgroup P, and given D ∈ NΔ we denote by λ_D the highest weight of the dual of
the simple G-module Γ(Y, \mathcal{L}_D|_Y), so that \mathcal{L}_D|_Y ≃ \mathcal{L}_{λ_D} corresponds
to the equivariant line bundle on G/P associated to the dominant weight λ_D. By
taking into account the description of Γ(X, \mathcal{L}_D) as a G-module, we lift Littelmann’s
basis of Γ(Y, \mathcal{L}_{λ_D}) to X, and we take as algebra generators for C(X) this set of lifts
together with the sections s_1, . . . , s_r.

Consider the coordinate ring C(Y) = \bigoplus_{E ∈ \mathbb{N} \Delta} \Gamma(Y, \mathcal{L}_E|_Y) of the multicone over
the flag variety Y associated to the dominant weights λ_D, with D ∈ Δ. In Section 4
we construct a multigraded standard monomial theory for C(X) by extending, in
a natural way, those of C(Y).

As a consequence of our standard monomial theory, we obtain a flat deformation
which degenerates C(X) to the product k[s_1, . . . , s_r] \oplus C(Y). Since multicones over
flag varieties have rational singularities by [23] and, since the property of having
rational singularities is stable under deformation by [20], it follows that the Cox
ring C(X) has rational singularities as well. From this it follows at once that,
given D ∈ ZΔ, also the ring C_D(X) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}_{nD}) has rational singularities.
Both C(X) and C_D(X), for any D ∈ ZΔ, can be seen as coordinate rings of affine
spherical varieties, see [7], Section 3.1], and the fact that affine spherical varieties
have rational singularities was already known, see [32] and [1].

2. Multigraded standard monomial theory

In this section, as a first step, we introduce the notion of a multigraded standard
monomial theory. This requires some technical machinery which we express in a
very abstract setting. In the next section we see the application to the multicone
over a flag variety and in Section 4 that to the Cox ring of a wonderful variety.

For further details the reader may see the various referenced papers as suggested
below. In particular, the multigraded standard monomial theory we are going
to introduce is modelled on the general definition of a standard monomial theory
given in [14]; see also [9] and [10] where such kind of standard monomial theory is
developed in the context of posets with bonds.
We begin with a field $k$ and a commutative $k$–algebra $A$. Let $A_1, A_2, \ldots, A_n$ be disjoint finite subsets of $A$, let $A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ and define the shape of $a \in A_i$ as the index $i$. We extend the notion of shape to formal monomials $\mathbb{m} = a_1 a_2 \cdots a_N$ of elements of $A$ by declaring that the shape of $\mathbb{m}$ is $(i_1, i_2, \ldots, i_N)$ if $a_h$ has shape $i_h$ for $h = 1, 2, \ldots, N$.

Suppose we have a binary relation $\leftarrow$ on $A$ that is antisymmetric and transitive when restricted to $A_i$ for all $i$. We say that a monomial $a_1 a_2 \cdots a_N$ of elements of $A$ is weakly standard if $a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_N$. Given $i, j$, let $\phi_{i,j}$ be a map from the set of weakly standard monomials of shape $(i, j)$ to the set of weakly standard monomials of shape $(j, i)$. If these maps verify $\phi_{i,i} = \text{Id}$ and $\phi_{i,j} \circ \phi_{j,i} = \text{Id}$, then we call them swap maps.

Now let $m = a_1 a_2 \cdots a_N$ be a weakly standard monomial. Since any pair $a_j a_{j'+1}$ is weakly standard, we may swap it and obtain a new monomial

$$m' = a_1 \cdots a_{j-1} a'_j a'_{j+1} a_{j+2} \cdots a_N,$$

where $a'_j a'_{j+1}$ is the swap of $a_j a_{j+1}$. If also $m'$ is weakly standard, then we may apply another swap, etc. If the shape of $m$ is a non decreasing sequence and if all monomials obtained from $m$ by swaps are weakly standard, then we say that $m$ is a standard monomial (notice that the number of swaps of $m$ is surely finite since $A$ is a finite set).

We say that the above datum $(A, \phi_{i,j}, \leftarrow)$ is a multigraded standard monomial theory for $A$ if the set of standard monomials in $A$, regarded as elements of $A$ via the natural map $S(A) \rightarrow A$ from the symmetric algebra over $A$ to $A$, is a basis of $A$ as a $k$–vector space and if this basis induces a multigrading on $A$ via the monomial shape: the multigrade of a monomial $m$ is $(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ if there are $k_i$ elements of shape $i$ in $m$.

A standard monomial theory usually has another feature, the straightening relations. In order to express them we introduce a monomial order $\leq_t$ on monomials in $A$ (see [10]), namely $\leq_t$ is a total order on the set of monomials in $A$ with the following properties:

i) if $m$, $m'$, $m''$ are monomials in $A$ and if $m' \leq_t m''$, then $mm' \leq_t mm''$,

ii) for every monomial $m$ in $A$ it holds $1 \leq_t m$.

Since the standard monomials are a $k$–basis of $A$, for every non-standard monomial $m'$ of elements of $A$ we have a relation

$$m' = \sum_m a_m m$$

expressing $m'$ as a linear combination of the standard monomials $m$. The basis of standard monomials induces a multigrading on $A$, so $m$ has the same shape of $m'$ whenever $a_m \neq 0$. If we have $m' \leq_t m$ whenever $a_m \neq 0$ we say that the above relation is a straightening relation for $m'$. If, further, we have straightening relations for all non-standard monomials then we say that $(A, \phi_{i,j}, \leftarrow, \leq_t)$ is a multigraded standard monomial theory with straightening relations for $A$.

A non-standard monomial $m$ is minimally non-standard if whenever $m = m_1 m_2$ with $m_1, m_2 \neq 1$ then $m_1$ and $m_2$ are standard. Notice that the straightening relations generate the kernel $\mathcal{R}$ of the map $S(A) \rightarrow A$, i.e. the ideal of relations in the generators $A$ for $A$. But fewer relations suffice as we see in the following theorem.

**Theorem 2.1.** If $(A, \phi_{i,j}, \leftarrow, \leq_t)$ is a multigraded standard monomial theory with straightening relations for $A$, then $\mathcal{R}$ is generated by the straightening relations of the minimally non-standard monomials.
Proof. Let \( \mathcal{I} \) be the ideal of \( S(\mathbb{A}) \) generated by the straightening relations SR(m) for minimally non-standard monomial \( m \). Since by definition \( \mathcal{I} \subseteq \mathcal{R} \) we have a surjective map \( S(\mathbb{A})/\mathcal{I} \to S(\mathbb{A})/\mathcal{R} \). Moreover the set of standard monomials is a \( \mathbb{k} \)-basis for \( S(\mathbb{A})/\mathcal{R} \cong A \), so if we show that the images of standard monomials are a basis also for \( S(\mathbb{A})/\mathcal{I} \) we have \( \mathcal{I} = \mathcal{R} \).

In order to prove this claim we use induction of the total degree of a minimally non-standard monomial \( m \) and on the order \( \leq t \) to prove that \( m \) is a sum of standard monomials in \( S(\mathbb{A})/\mathcal{I} \).

If either \( m \) has total degree one or it is \( \leq t \)-maximal of its degree, then it is standard as follows by the definition of standardness and by the order requirement in a straightening relation, respectively.

Now suppose \( m = m_{1}m_{2} \) with \( m_{1}, m_{2} \neq 1 \) and \( m_{1} \) non-standard. Since \( m_{1} \) has total degree less than that of \( m \), it is sum of standard monomials in \( S(\mathbb{A})/\mathcal{I} \), in particular \( m_{1} = \sum a_{n}n \) and in this sum \( n \) runs over the standard monomials such that \( m_{1} \leq t \). So

\[
m = m_{1}m_{2} \equiv \sum a_{n}nm_{2} \quad (\text{mod} \mathcal{I})
\]

and \( m \leq t \) \( m_{2} \) for all \( n \). Using the inductive hypothesis on \( \leq t \), all \( nm_{2} \)'s are sums of standard monomials, hence also \( m \) is sum of standard monomials in \( S(\mathbb{A})/\mathcal{I} \). \( \square \)

As a corollary we have the following result.

**Corollary 2.2.** If all weakly standard monomials are standard, then the ideal \( \mathcal{R} \) of relations is generated in degree two.

**Proof.** We prove that the total degree of a minimally non-standard monomial is necessarily 2. Indeed let \( m = a_{1}a_{2} \cdots a_{N} \) be a minimally non-standard monomial with \( N \geq 3 \). Then, writing \( m = (a_{i}a_{i+1})m' \) we see that \( a_{i}a_{i+1} \) is standard for all \( i \). So \( a_{i} \leftarrow a_{i+1} \) for all \( i \), and \( m \) is weakly standard, hence it is standard. \( \square \)

In the last part of this section we see how a degeneration for \( A \) may be constructed using the straightening relations (see also [9] for further details about this kind of degeneration in the language of LS algebras). Suppose we have a valuation \( \mathcal{A} \) on \( A \) such that, when extended to monomials by \( \delta(mn) = \delta(m) + \delta(n) \) for all \( m, n \), we have \( \delta(m) \leq \delta(n) \) if \( m \leq n \).

For an integer \( n \), let \( K_{n} \) be the ideal of \( A \) generated by those monomials \( m \) such that \( \delta(m) \geq n \) and consider the Rees algebra

\[
A \leftarrow \cdots \oplus At^{2} \oplus At \oplus A \oplus K_{1}t^{-1} \oplus K_{2}t^{-2} \oplus \cdots
\]

as a subalgebra of \( \mathbb{k}[t, t^{-1}] \otimes A \). This algebra is a torsion-free \( \mathbb{k}[t] \)-module, hence it is a flat \( \mathbb{k}[t] \)-algebra. For \( a \in \mathbb{k} \) let \( A_{a} \cong A/(t - a) \) be the fiber over \( a \). Notice that we have an action of \( \mathbb{k}^{*} \) on \( A \) given by \( \lambda : t = \lambda t \), for all \( \lambda \in \mathbb{k}^{*} \); hence isomorphisms \( A_{a} \rightarrow A_{a' \lambda} \) between the fibers. In particular all generic fibers, i.e. \( A_{a} \) with \( a \neq 0 \), are isomorphic to \( A_{1} \cong A \). On the other hand the special fiber \( A_{0} = A/(t) \) is isomorphic to the associated graded algebra

\[
A/K_{1} \oplus K_{1}/K_{2} \oplus K_{2}/K_{3} \oplus \cdots
\]

We may now state our deformation result.

**Theorem 2.3.** There exists a flat \( \mathbb{k}^{*} \)-equivariant degeneration of \( A \) to \( A_{0} \) whose all generic fibers are isomorphic to \( A \) while the special fiber \( A_{0} \) is isomorphic to the quotient of the symmetric algebra \( S(\mathbb{A}) \) by the ideal generated by the relations

\[
m' - \sum_{\delta(m) = \delta(m')} a_{m}m
\]

where \( m' \) is a minimally non-standard monomial and \( m' - \sum_{m} a_{m}m \) is its straightening relation.
Proof. We have only to prove the last part about \( \mathcal{A}_0 \). Consider the symmetric algebra \( \mathbb{T} \cong \mathbb{S}(\mathbb{A}, t) \) with indeterminates the set of generators \( \mathbb{A} \) and the parameter \( t \). Let \( B \) be the quotient of \( \mathbb{T} \) by the ideal generated by the modified straightening relations
\[
m' = \sum_m a_m t^{l(m)-d(m')}
\]
for all \( m' \) minimally non-standard. We may define a map \( \mathbb{T} \to \mathbb{A} \) by \( \mathbb{A} \ni a \mapsto a t^{-\delta(a)} \) and by \( t' \mapsto t \). It is clear that this map is well defined also on \( B \). Its image is \( \mathbb{A} \) by definition of this last algebra. Moreover it is an injective map since \( \mathbb{A} \) has a multigraded standard monomial theory defined in terms of that of \( A \) with any monomial \( m \) replaced by \( mt^{-\delta(m)} \).

It is now clear that \( \mathcal{A}_0 \cong B|_{t=0} \) is as claimed in the statement of the theorem. \( \square \)

3. Standard monomial theory for multicones over flag varieties

In this section we apply the abstract construction of the previous section to the multicone over a flag variety; this is the motivating example for the above general setting of a multigraded standard monomial theory.

Let \( G \) be a simply connected semisimple algebraic group over an algebraically closed field \( \mathbb{k} \) of characteristic 0 and let \( T \subseteq B \subseteq G \) be a maximal torus and a Borel subgroup of \( G \), respectively. Denote by \( W \) the Weyl group and by \( \Lambda \supseteq \Lambda^+ \) the lattice of integral weights and the monoid of dominant weights associated to \( \mathfrak{g} \). For a dominant weight \( \lambda \) denote by \( W_\lambda \subseteq W \) its stabilizer and by \( W^\lambda \subseteq W \) the set of minimal representatives of the cosets \( W/W_\lambda \); denote, moreover, by \( \mathfrak{S} \) the Bruhat order on \( W \) and on \( W^\lambda \).

Now let \( P \supseteq B \) be a parabolic subgroup of \( G \) stabilizing the line generated by a highest weight vector \( v_\lambda \) in the irreducible \( G \)-module \( V_\lambda \) of highest weight \( \lambda \). We have a natural map \( G/P \ni g \mapsto g \cdot v_\lambda \in \mathbb{P}(V_\lambda) \) from the flag variety \( G/P \) to the projective space over \( V_\lambda \). We use this map to define the line bundle \( L_\lambda \) on \( G/P \) as the pull-back of \( \mathcal{O}(1) \) on \( \mathbb{P}(V_\lambda) \); we denote the space of its sections by \( \Gamma(G/P, L_\lambda) \). Notice that, as \( G \)-modules, we have \( \Gamma(G/P, L_\lambda) \cong V_\lambda^* \), the dual of \( V_\lambda \). In the sequel we denote by \( \lambda^* \) the unique dominant weight such that \( V_{\lambda^*} \cong V_\lambda^* \) as \( G \)-modules.

In [27] Littelmann associated to a fixed piece-wise linear path \( \pi : [0,1]_{\mathbb{Q}} \to \Lambda \otimes \mathbb{Q} \) completely contained in the dominant Weyl chamber and ending in \( \lambda = \pi(1) \in \Lambda^+ \), a set \( \mathbb{B}_\pi \) of piece-wise linear paths. The set \( \mathbb{B}_\pi \) gives the character of the irreducible module \( V_\lambda \):
\[
\text{char } V_\lambda = \sum_{\eta \in \mathbb{B}_\pi} e^{\eta(1)}
\]
In particular, if we start with the path \( \pi_\lambda : t \mapsto t\lambda \) we obtain the set \( \mathbb{B}_\lambda \) of L-S paths of shape \( \lambda \); they may be combinatorially described in the following way.

Given a pair \( \tau < s_\beta \tau \) of adjacent elements in \( W^\lambda \), where \( s_\beta \) is the symmetry with respect to the root \( \beta \), we define \( f_\lambda(\tau, s_\beta \tau) = \langle \tau(\lambda), \beta \rangle \). Further, we extend \( f_\lambda \) to generic comparable pairs \( \sigma < \tau \) in \( W^\lambda \) by choosing a chain \( \sigma = \tau_1 < \cdots < \tau_n = \tau \) of adjacent elements in \( W^\lambda \) and defining \( f_\lambda(\sigma, \tau) = \gcd(f_\lambda(\tau_1, \tau_2), \ldots, f_\lambda(\tau_{n-1}, \tau_n)) \); indeed such gcd is independent of the chain used to compute it (see [17]).

A pair \( \eta = (\tau_1, \tau_2, \ldots, \tau_r; a_0, a_1, \ldots, a_r) \), where \( \tau_1 < \cdots < \tau_r \) is a sequence of comparable elements of \( W^\lambda \) and \( 0 = a_0 < a_1 < \cdots < a_r = 1 \) are rational numbers, is an L-S path if the integral condition \( a_i f_\lambda(\tau_i, \tau_{i+1}) \in \mathbb{N} \) holds for all \( i = 1, 2, \ldots, r-1 \). The pair \( \eta \) is identified with the path
\[
\pi(a_0-a_1, a_1-a_2, a_2-a_3, \ldots, a_r-a_{r-1}) \tau_1(\lambda) * \pi(a_{r-1}-a_r) \tau_r(\lambda) * \cdots * \pi(a_1-a_0) \tau_1(\lambda)
\]
where we denote concatenation of paths by *. The set \( \text{supp}\, \eta \triangleq \{ \tau_1, \tau_2, \ldots, \tau_r \} \) is called the support of the path \( \eta \).

Let \( W \) be the set of words in the alphabet \( W \) and denote by \( N_\lambda \) the least common multiple of the image of \( f_\lambda \); we define the word \( w(\eta) \) of the L-S path \( \eta = (\tau_1, \tau_2, \ldots, \tau_r; a_0, a_1, \ldots, a_r) \) as \( w(\eta) \triangleq \tau_1^{N_\lambda(a_1-a_0)} \cdots \tau_r^{N_\lambda(a_r-a_{r-1})} \); this will be needed in the sequel to define a monomial order.

The set \( B_\lambda \) not only describes the character of the irreducible \( G \)-module \( V_\lambda \), but also, in [29], Littelmann associates a section \( p_\pi \in \Gamma(G/P, L_\lambda) \) to an L-S path \( \pi \in B_\lambda \). The set \( A_\lambda \triangleq \{ p_\pi \mid \pi \in B_\lambda \} \) of these sections may be used to construct a standard monomial theory as follows. For more details about the combinatorics of L-S paths and their application to the geometry of Schubert varieties one may see [8].

Given two dominant weights \( \lambda, \mu \) we lift the Bruhat order on \( W^\lambda \) and \( W^\mu \) to \( W^\lambda \sqcup W^\mu \) by defining \( W^\lambda \ni \sigma \leq \tau \in W^\mu \) if there exist \( \sigma', \tau' \in W \) such that \( \sigma'W_\lambda = \sigma W_\lambda \), \( \tau'W_\mu = \tau W_\mu \) and \( \sigma' \leq \tau' \) with respect the Bruhat order of \( W \). For details we refer to [10]. Notice that this lift is still the Bruhat order if \( \lambda \) and \( \mu \) have the same stabilizer in \( W \). We use this order to define a relation \( \leq_{\lambda, \mu} \) on pairs in \( B_\lambda \times B_\mu \) as follows: we set

\[
\pi \leq_{\lambda, \mu} \eta \quad \text{if} \quad \pi \in B_\lambda, \eta \in B_\mu, \quad \text{and max supp} \pi \leq \text{min supp} \eta
\]

Notice that if \( \lambda = \mu \), then \( \leq_{\lambda, \mu} \) is a transitive and antisymmetric relation.

Recall that the set of pairs \( (\pi, \eta) \in B_\lambda \times B_\mu \) such that \( \pi \leq_{\lambda, \mu} \eta \) is in natural bijection with the basis \( B_{\pi_\lambda, \pi_\mu} \) as proved by Littelmann in [28]. Further the two bases \( B_{\pi_\lambda, \pi_\mu} \) and \( B_{\eta_\lambda, \eta_\mu} \) of the module \( V_{\pi_\lambda, \pi_\mu} \) are in bijection by a unique isomorphism of crystal graphs (see [27]). So we have the diagram

\[
\begin{array}{c}
\{ (\pi \leq_{\lambda, \mu} \eta) \} \quad \longrightarrow \quad B_{\pi_\lambda, \pi_\mu} \\
\downarrow \\
\{ (\eta' \leq_{\mu, \lambda} \pi') \} \quad \longleftarrow \quad B_{\eta_\lambda, \eta_\mu}
\end{array}
\]

and, for \( B_\lambda \ni \pi \leq_{\lambda, \mu} \eta \in B_\mu \), we define \( \phi(\pi, \eta) \triangleq \pi \rho_{\eta} \rho_{\pi} \) if \( (\eta', \pi') \) corresponds to \( (\pi, \eta) \) for the composition of the above three bijections.

Finally let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be dominant weights stabilized by the parabolic subgroup \( P \). As seen before for a pair of dominant weights, we define an order \( \leq \) on \( W^\lambda_1 \sqcup W^\lambda_2 \sqcup \cdots \sqcup W^\lambda_n \) by: \( W^\lambda_i \ni \sigma \leq \tau \in W^\lambda_j \) if \( i < j \) and there exist \( \sigma', \tau' \in W \) such that \( \sigma'W^\lambda_i = \sigma W^\lambda_i \), \( \tau'W^\lambda_j = \tau W^\lambda_j \) and \( \sigma' \leq \tau' \) with respect the Bruhat order on \( W \). Further we refine this order to a total order \( \leq_{\lambda, \mu} \) such that \( W^\lambda_i \ni \sigma \leq_{\lambda, \mu} \tau \in W^\lambda_j \) only for \( i < j \).

On the formal monomials in \( A \triangleq A_{\lambda_1} \sqcup A_{\lambda_2} \sqcup \cdots \sqcup A_{\lambda_n} \) we define the following monomial order

\[
\rho_{\eta_1} \rho_{\eta_2} \cdots \rho_{\eta_n} \leq_{\lambda, \mu} \rho_{\eta_1} \rho_{\eta_2} \cdots \rho_{\eta_n} \quad \text{if} \quad u < v \quad \text{or} \quad u = v \quad \text{and} \quad w(\eta_1)w(\eta_2) \cdots w(\eta_n) \leq_{\lambda, \mu} w(\epsilon_1)w(\epsilon_2) \cdots w(\epsilon_n).
\]

Finally notice that we may define a relation \( \leq \) on \( A \) by declaring \( \pi \leq \eta \) if \( \pi \in B_{\lambda_1}, \eta \in B_{\lambda_1} \) and \( \pi \leq_{\lambda_1, \lambda_1} \eta \).

Now consider the \( \mathbb{k} \)-algebra

\[
A(\lambda_1, \lambda_2, \ldots, \lambda_n) \triangleq \bigoplus \Gamma(G/P, L_{m_1 \lambda_1 + m_2 \lambda_2 + \cdots + m_n \lambda_n})
\]

where the sum runs over all \( n \)-uples of positive integers \( m_1, m_2, \ldots, m_n \geq 0 \). This algebra is the coordinate ring of the multicone over the partial flag variety \( G/P \) mapped diagonally in \( \mathbb{P}(V_{\lambda_1}) \times \cdots \times \mathbb{P}(V_{\lambda_n}) \).

We finally have all we need to define a standard monomial theory.
**Theorem 3.1** ([10] Proposition 4.1, [23] Proposition 2). The set of generators $\Lambda$, the swap maps $\phi_{\lambda_1\lambda_2}$ and the relation $\leftarrow$ define a standard monomial theory for the algebra $A(\lambda_1, \lambda_2, \ldots , \lambda_n)$. With respect to the monomial order $\leq_{\mathfrak{t}}$, any non-standard monomial in the $p_{\lambda}$’s has a straightening relation. Moreover, the algebra $A(\lambda_1, \lambda_2, \ldots , \lambda_n)$ is isomorphic to the quotient of the symmetric algebra $S(k)$ by the ideal generated by the quadratic straightening relations.

**Remark 3.2.** We point out that in the proof of [10] Proposition 4.1] there is a slight inaccuracy: in that paper only the quadratic straightening relations are proved. Indeed, to our best knowledge, the higher degree straightening relations cannot, in general, be derived by the quadratic straightening relations. And this is true also if the ideal of relations in $A$ are generated by quadratic relations (see [23]). The point is: the order requirement in the higher degree straightening relations cannot be, in general, deduced by the same order requirement of the degree two straightening relations (this is the false argument used in [10]). However, for the multicone over a flag variety, the order requirement of the straightening relations of any degree may be proved by generalizing verbatim Proposition 7.3 and Corollary 7.4 in [25] to higher degrees and to products of sections $p_{\lambda}$’s of different shapes.

4. **Standard monomial theory for the Cox ring of a wonderful variety**

Let $X$ be a wonderful $G$–variety with (unique) closed $G$–orbit $Y$, and fix a parabolic subgroup $P \supseteq B$ such that $Y \simeq G/P$. By [30], $X$ is spherical, i.e. it possesses an open $B$–orbit, say $B \cdot x_0 \subseteq X$. Since $B \cdot x_0$ is affine, $G \cdot x_0 \sim B \cdot x_0$ is a union of finitely many $B$–stable divisors and we denote by $\Delta$ the set of their closures in $X$:

$$\Delta = \{D \subseteq X : D \text{ is a } B \text{–stable prime divisor, } D \cap G \cdot x_0 \neq \emptyset \}.$$ 

The elements of $\Delta$ are called the colors of $X$.

Denote by $B^-$ the opposite Borel subgroup of $B$ and let $y_0 \in Y$ be the unique $B^-$fixed point of $X$. The normal space of $Y$ in $X$ at $y_0$, $T_{y_0}X/T_{y_0}Y$, is a multiplicity-free $T$–module. The elements of the set

$$\Sigma = \{T$–weights of $T_{y_0}X/T_{y_0}Y \}$$

are called the spherical roots of $X$ and they naturally correspond to the local equations of the boundary divisors of $X$, which are $G$–stable. If $\sigma \in \Sigma$, we denote by $X^\sigma$ the associated boundary divisor of $X$ such that $T_{y_0}X/T_{y_0}X^\sigma$ is the one-dimensional $T$–module of weight $\sigma$. Notice that $Z\Sigma$ is a sublattice of $Z\Delta$.

Recall that every line bundle on $X$ or on $Y$ has a unique $G$–linearization. As a group, $\text{Pic}(X)$ is freely generated by the equivalence classes of line bundles $L_D \simeq \mathcal{O}(D)$, for $D \in \Delta$ (see [10] Proposition 2.2)). For all $E \in Z\Delta$, the associated line bundle $L_E \simeq \mathcal{O}(E)$ is globally generated, respectively ample, if and only if $E$ is a non-negative, respectively positive, combination of colors.

The restriction of line bundles to the closed orbit induces a map $\lambda : \text{Pic}(X) \longrightarrow \Lambda$; given $E \in Z\Delta$ we set $\lambda_E \simeq \lambda(L_E)$ in such a way that $\Gamma(Y, L_E|_{Y}) \simeq V_{X_E}^*$ and, moreover, we set $V_E \simeq V_{X_E}^*$ for short. (Hence $L_E|_{Y} \simeq \lambda_E$, where this last line bundle is defined in the previous section.) Moreover, in particular, $\Gamma(X, L_E)$ contains a copy of $V_E$ and, $X$ being spherical the decomposition of $\Gamma(X, L_E)$ is multiplicity-free.

If $\gamma = \sum a_{\sigma} \sigma \in N\Sigma$, we denote by $s_\gamma \in \Gamma(X, L_{X_\gamma})$ a section whose divisor is equal to $X_\gamma \simeq \sum a_{\sigma} X^\sigma$; notice that this section is $G$–invariant. If $E, F \in Z\Delta$ are such that $F - E \in N\Sigma$, then we write $E \leq_{\Sigma} F$. If $E \in N\Delta, F \in Z\Delta$ and $E \leq_{\Sigma} F$
the multiplication by \( s^{F-E} \) induces a \( G \)-equivariant map from the sections of \( \mathcal{L}_E \) to the sections of \( \mathcal{L}_F \), in particular we have \( s^{F-E} V_E \subseteq \Gamma(X, \mathcal{L}_F) \). Moreover

**Proposition 4.1** ([8] Proposition 2.4). Let \( F \in \mathbb{Z}\Delta \), then

\[
\Gamma(X, \mathcal{L}_F) = \bigoplus_{E \in \mathbb{N}\Delta, E \subseteq F} s^{F-E} V_E.
\]

Since \( \text{Pic}(X) \) is a free lattice, the space

\[
C(X) = \bigoplus_{D \in \mathbb{Z}\Delta} \Gamma(X, \mathcal{L}_D)
\]

is a ring; in analogy with the toric case \( C(X) \) is called the *Cox ring* of \( X \). The ring \( C(X) \) was studied in [12] and [11] in the case of a wonderful symmetric variety (where it is called respectively the *ring of sections* of \( X \) and the coordinate ring of \( X \)), and in [7] in the case of a wonderful variety (where it is called the *total coordinate ring* of \( X \)).

Since \( X \) is irreducible, \( C(X) \) is generated as a \( k \)-algebra by the sections \( s^\sigma \) and by the modules \( V_D \subseteq \Gamma(X, \mathcal{L}_D) \) for \( D \in \Delta \). It follows that \( C(X) \) is a quotient of the symmetric algebra

\[
S(X) = \mathbb{k}[s_1, \ldots, s_r] \otimes \bigoplus_{D \in \Delta} V_D
\]

where we fix an ordering \( \Sigma = \{\sigma_1, \ldots, \sigma_r\} \) and we set \( s_i = s^{\sigma_i} \) for short. Further, notice that the quotient of \( C(X) \) by the ideal generated by the sections \( s_1, \ldots, s_r \) is isomorphic to the coordinate ring of a multicone over the flag variety \( Y \simeq G/P \), that is

\[
C(Y) \simeq A(\lambda_{D_1}, \lambda_{D_2}, \ldots, \lambda_{D_q}) = \bigoplus_{D \in \Delta} \Gamma(Y, \mathcal{L}_D|_Y) \simeq \bigoplus_{D \in \Delta} V_D.
\]

where \( \Delta = \{D_1, \ldots, D_q\} \) is any fixed ordering of \( \Delta \). Therefore we have surjective maps

\[
S(X) \twoheadrightarrow C(X) \twoheadrightarrow C(Y).
\]

The rings \( S(X) \), \( C(X) \) and \( C(Y) \) all have natural \( \mathbb{Z}\Delta \)-gradings, and the previous maps are morphisms of \( \mathbb{Z}\Delta \)-graded \( G \)-algebras.

By Theorem 3.1 we have a standard monomial theory with straightening relations for \( C(Y) \). Our aim is to extend it to a standard monomial theory for the Cox ring \( C(X) \), and deduce a degeneration result for such a ring. The ideal \( \mathcal{I}_X \) which defines \( C(X) \) in \( S(X) \) is generated by quadratic relations (see [8] Proposition 3.3.1]), in our main result we will give a description of \( \mathcal{I}_X \) in terms of our standard monomial theory, namely we will show that \( \mathcal{I}_X \) is generated by the quadratic straightening relations.

Given \( D \in \Delta \) we denote by \( B_D \) the set of \( L \)-\( S \) paths of shape \( \lambda_D \) for short. Given \( \pi \in B_D \), let \( x_\pi \in V_D \subseteq \Gamma(X, \mathcal{L}_D) \) be the unique section such that \( x_\pi|_Y = p_\pi \) and let

\[
\kappa_D = \{x_\pi : \pi \in B_D\} \subset V_D.
\]

Then \( \kappa_D \) is a basis of \( V_D \subseteq \Gamma(X, \mathcal{L}_D) \). Further, define

\[
\kappa_\Sigma = \{s^\sigma : \sigma \in \Sigma\}, \quad \kappa_\Delta = \bigsqcup_{D \in \Delta} \kappa_D \quad \text{and} \quad \kappa_X = \kappa_\Sigma \sqcup \kappa_\Delta.
\]

In particular, \( S(X) \) is the symmetric algebra in the indeterminates \( \kappa_X \). If \( x_\pi \in \kappa_\Delta \), its *shape* is the unique \( D \in \Delta \) such that \( x_\pi \in \kappa_D \).

Let \( B_\Delta \) and set

\[
\kappa_Y = \{p_\pi : \pi \in B_\Delta\},
\]

...
which is naturally identified with the subset $\mathbb{A}_D \subset \mathbb{A}_X$ via the bijection $x_\pi \mapsto p_\pi = x_\pi|_Y$.

By Theorem 3.1 we have a standard monomial theory for $C(Y)$. We denote by $M(Y) \subset S(\mathbb{A}_Y) \simeq S(\mathbb{A}_D)$ the set of monomials in the coordinates $\mathbb{A}_Y$, and by $SM(Y) \subset M(Y)$ the subset of standard monomials. In particular, using the bijections $\mathbb{A}_D \simeq \mathbb{A}_X$, for all $D, D' \in \Delta$ we have swap maps $\phi_{D, D'}$, a relation $\leftarrow$ on $\mathbb{A}_D$ and a monomial order $\leq_t$ on $M(Y)$ as defined in the previous section.

First we extend the relation $\leftarrow$ to $\mathbb{A}_X$ by declaring $s_1 \leftarrow s_2 \leftarrow \cdots \leftarrow s_r$ and $s_i \leftarrow x_\pi, x_\pi \leftarrow s_i$ for all $i = 1, 2, \ldots, r$ and all $\pi \in \mathbb{B}_\Delta$. Next we extend the swap maps by

\[
\phi_{D, \Sigma}(s_i, x_\pi) \doteq (x_\pi, s_i) \quad \phi_{D, \Sigma}(x_\pi, s_i) \doteq (s_i, x_\pi)
\]

for all $i = 1, 2, \ldots, r$ and all $\pi \in \mathbb{B}_\Delta$. Finally, we extend the monomial order $\leq_t$ to monomials in $\mathbb{A}_X$: if $m_1, m_2$ are two monomials in $S(\mathbb{A}_D) \simeq S(\mathbb{A}_Y)$, and if $\gamma_1, \gamma_2 \in \mathbb{N}\Sigma$, then we set $s^{\gamma_1}m_1 \leq_t s^{\gamma_2}m_2$ if either $\gamma_1 <_\Sigma \gamma_2$, or $\gamma_1 = \gamma_2$ and $m_1|_Y \leq_t m_2|_Y$. We denote by $M(X) \subset S(X)$ the set of the monomials in the indeterminates $\mathbb{A}_X$, endowed with the total order $\leq_t$.

Let $m = s^\gamma x_{\pi_1} \cdots x_{\pi_N}$ be a generic monomial in $M(X)$. Then $\nu(m) = \gamma$ is called the vanishing of $m$, and if $D_i \in \Delta$ is the shape of $x_{\pi_i} \in \mathbb{A}_D$, we define the shape of $m$ as

\[
\gamma + \sum_{i=1}^N D_i,
\]

namely the degree of $m$ with respect to the $\mathbb{Z}\Delta$-grading.

The inclusion $\mathbb{A}_Y \hookrightarrow \mathbb{A}_X$ defines a shape-preserving bijection between $M(Y)$ and the subset of $M(X)$ of the monomials $m$ such that $\nu(m) = 0$. Given $n \in M(Y)$ we denote by $\bar{n} \in M(X)$ the corresponding monomial, so we have $\bar{n}|_Y = n$ and, in particular, $\bar{p}_\pi = x_\pi$. Conversely, given a monomial $m \in M(X)$ we may define a monomial $\bar{m} \in M(Y)$ with $\nu(\bar{m}) = 0$ by setting $\bar{m} = s^{-\nu(m)}m$. Notice that $m \in M(X)$ is a standard monomial if and only if $\bar{m} \in M(Y)$ is a standard monomial, if and only if $\bar{m}|_Y \in M(Y)$ is a standard monomial. We denote by $SM(X) \subset M(X)$ the set of standard monomials in $\mathbb{A}_X$, and if $E \in \mathbb{Z}\Delta$ we denote by $SM_E(X) \subset M_E(X)$ the set of standard monomials and of all monomials of shape $E$, respectively.

Following [12, Theorem 3], we are now able to construct a standard monomial theory for the Cox ring $C(X)$ of a wonderful variety $X$.

**Theorem 4.2.**

i) Given $E \in \mathbb{Z}\Delta$, the images of the standard monomials of shape $E$ form a basis of $\Gamma(X, \mathcal{L}_E)$.

ii) Given a non-standard monomial $m'$ the equality

\[
m' = \sum_{m \in SM(X)} a_m m
\]

guaranteed by i) is a straightening relation in $C(X)$; that is, we have $m' \leq_t m$ whenever $a_m \neq 0$. Moreover,

\[
\bar{m}' = \sum_{m \in SM(X), \nu(m) = \nu(m')} a_m m
\]

is a straightening relation in $C(Y)$.

iii) The defining ideal $\mathcal{I}_X \subset S(X)$ is generated by the quadratic straightening relations, i.e. by the relations in ii) for the quadratic non-standard monomials.

In particular, the set of generators $\mathbb{A}_X$ together with the above defined swap maps, relation and monomial order define a multigraded standard monomial theory with straightening relations for the Cox ring $C(X)$. 
Proof. We prove the first two statements together. Let \( \pi_1, \ldots, \pi_N \in B_\Delta \) be such that \( x_{\pi_1} \cdots x_{\pi_N} \) is not standard. In \( A(Y) \), by Theorem \ref{thm:main} we have a straightening relation

\[
p_{\pi_1} \cdots p_{\pi_N} = \sum_{n \in SM(Y)} a_n n,
\]

where \( p_{\pi_1} \cdots p_{\pi_N} \leq_t n \) for all \( n \) such that \( a_n \neq 0 \).

Since \( X - G \cdot x_0 \) is a normal crossing divisor with smooth irreducible components, a section in \( C(X) \) vanishes on the closed orbit \( Y \) if and only if it is in the ideal generated by the sections \( s_1, \ldots, s_r \). By construction the difference \( x_{\pi_1} \cdots x_{\pi_N} - \sum_n a_n n \) is homogeneous w.r.t. the \( \mathbb{Z}\Delta \)-grading, and it vanishes on \( Y \). Hence we have

\[
x_{\pi_1} \cdots x_{\pi_N} = \sum_{n \in SM(Y)} a_n n + \sum_{m \in M_E(X)} a_m m,
\]

where \( x_{\pi_1} \cdots x_{\pi_N} \leq_t n \) for all \( n \in SM(Y) \) with \( a_n \neq 0 \).

Proceeding inductively on the partial order \( \leq_\Sigma \), the previous equality implies that in \( C(X) \) the image of every monomial \( m' \in M_E(X) \) may be written as the image of a sum of standard monomials \( m \in SM_E(X) \) with \( m' \leq_t m \).

Therefore the image of the standard monomials of \( SM_E(X) \) in \( C(X) \) is a set of generators for \( \Gamma(X, L_E) \) as a vector space. On the other hand, by Theorem \ref{thm:main} the images of the standard monomials \( n \in SM(Y) \) form a basis for \( C(Y) \); hence for all \( F \in \mathbb{N}\Delta \) the images of the standard monomials \( n \in SM(Y) \) of shape \( F \) form a basis for the graded component \( C(Y)_F = V_F \). So using Proposition \ref{prop:main} we have

\[
\dim \Gamma(X, L_E) = \sum_{F \in \mathbb{N}\Delta : F \leq_\Sigma E} \dim V_F = \sum_{F \in \mathbb{N}\Delta : F \leq_\Sigma E} \left| SM_E(Y) \right|
\]

and this finishes the proof of i) and ii).

Now, in order to prove iii), let \( J \) be the ideal of \( S(X) \) generated by the quadratic straightening relations, i.e. by the above relations for non-standard monomials \( x_{\pi_1}, x_{\pi_2} \). Clearly \( J \subseteq I_X \) and we want to show that these two ideals are equal.

The quotient \( S(X)/J, s_1, \ldots, s_r \) is isomorphic to \( C(Y) \) since the relations for this last ring is generated by the quadratic straightening relations; indeed it is generated in degree 2 by \cite[Proposition 2]{23}. So, if \( m' \) is a non-standard monomial \( m' + \langle s_1, \ldots, s_r \rangle \) is a sum of standard monomials modulo \( J \). Hence in \( S(X)/J \) the monomial \( m' \) is a sum of standard monomials \( m \) with \( \nu(m) = 0 \) plus \( s_1 y_1 + \ldots + s_r y_r \) for some homogeneous elements \( y_1, \ldots, y_r \), whose shapes are \( \Sigma \)-stable of that of \( m' \).

Proceeding again by induction on the shape of a non-standard monomial we see that any straightening relation is an element of \( J \). So \( J = I_X \) and the last statement of the proposition is proved.

The standard monomial theory constructed in the previous theorem is compatible with the \( G \)-orbit closures in \( X \). Recall that the subsets \( I \subseteq \Sigma \) parametrize the \( G \)-orbits in \( X \); that is, for every \( x \in X \) there is a unique \( I \subseteq \Sigma \) such that \( \overline{G \cdot x} = \bigcap_{\sigma \in \Sigma \setminus I} X^\sigma \cong X_I \).

The \( G \)-stable subvariety \( X_I \) is again a wonderful variety; its set of spherical roots coincides with \( I \). Given \( I \subseteq \Sigma \), we say that a standard monomial \( m \in SM(X) \) is \( I \)-standard if \( \nu(m) \in (\Sigma \setminus I)_Z \). We denote by \( SM^I(X) \) the set of \( I \)-standard monomials, and by \( SM^I_E(X) \) the set of the \( I \)-standard monomials of shape \( E \in \mathbb{Z}\Delta \).

**Corollary 4.3.** Given \( E \in \mathbb{Z}\Delta \), the images of the \( I \)-standard monomials \( SM^I_E(X) \) are a basis for \( \Gamma(X_I, L_E|_{X_I}) \).
Proof. Let $J = \Sigma \setminus I$. Then the restriction of sections $\rho : \Gamma(X, \mathcal{L}) \rightarrow \Gamma(X_I, \mathcal{L}_E|_{X_I})$ is a surjective map, and we have
\[
\ker \rho = \bigoplus_{F \in \mathbb{N} \Delta, F \leq J} s^{E - F} V_F,
\]
where we write $F \leq J$ when $F$ is a degeneration of $E$ if and only if $E - F \in \mathbb{N}J$. It follows that the images of the $J$–standard monomials of shape $E$ give a basis for $\ker \rho$, whereas the restrictions of the images of the $I$–standard monomials of shape $E$ give a basis for $\Gamma(X_I, \mathcal{L}_E|_{X_I})$.

When $X$ is the wonderful compactification of a semisimple adjoint group regarded as a homogeneous $G \times G$–variety, the above constructed standard monomial theory is even compatible with the $B \times B$–orbit closures (see [2]).

5. Degeneration and rational singularities

Any straightening relation involves monomials with higher power of the sections $s_1, \ldots, s_r$. This allows us to degenerate Spec$C(X)$ to the product of the affine space $k'$ with a multicone over the flag variety $G/P \simeq Y$. Let us see the details for such a degeneration.

Corollary 5.1. There exists a flat $G \times k^*$–equivariant degeneration $C$ of $C(X)$ to the ring $k[s_1, \ldots, s_r] \otimes C(Y)$; further all generic fibers of $C$ are isomorphic to $C(X)$.

Proof. We define a map $\delta : A_X \rightarrow \mathbb{N}$ by $\delta(\lambda s) = 0$ for all $\pi \in \mathbb{B}_\Delta$ and $\delta(s_i) = 1$ for all $i = 1, 2, \ldots, r$. This map is a valuation for the standard monomial theory of $C(X)$ by Theorem 1.2. Hence we may apply Theorem 2.3. The special fiber is isomorphic to the ring in the statement of the theorem again by Theorem 1.2.

Moreover, for this valuation map the Rees algebra is
\[
C = \cdots \oplus C(X) t^2 \oplus C(X) t \oplus C(X) \oplus K t^{-1} \oplus K^2 t^{-1} \oplus \cdots
\]
with $K$ the ideal of $C(X)$ generated by the sections $s_1, s_2, \ldots, s_r$. So, being $K$ generated by $G$–invariants, the action of $G$ on $C(X)$ induces an action on $C$ by letting $G$ act trivially on $t$. In particular $G$ acts on each fiber and it is clear that this $G$–action commutes with the isomorphisms $C_a \rightarrow C_{\lambda^{-1} a}$ for any $a \in k$ and $\lambda \in k^*$. So the deformation is also $G$–equivariant.

We now apply this degeneration result to the study of the singularities of the algebra $C(X)$. A variety $X$ is said to have rational singularities if there exists a resolution of singularities $\pi : Y \rightarrow X$ of $X$ such that $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$ and $\pi_* \mathcal{O}_Y \simeq \mathcal{O}_X$. If such a property holds for a resolution then it holds for all resolutions. Finally a ring $A$ is said to have rational singularities if Spec$A$ has rational singularities.

We have the following properties:

(a) a multicone over a flag variety has rational singularities (see [23], Theorem 2);
(b) if $X$ is an affine $G$–variety with rational singularities and $G$ is a reductive group then $X//G$ has rational singularities (see [24]);
(c) if $(X, X) \rightarrow (S, s_0)$ is a flat deformation of a variety with rational singularities $X$ then there exists a neighborhood $U$ of $s_0$ such that for $s \in U$ the fiber over $s$ has also rational singularities (see [24], Théorème 4).

Given $D \in \mathbb{Z} \Delta$ consider the subalgebra of $C(X)$ defined as follows
\[
C_D(X) \doteq \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}_n D).
\]
This is the projective coordinate ring of a spherical variety, namely the image of $X$ in the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}_D)^*)$. It is known that these rings have rational singularities (see [32], or also [1] Remark 2.5 for another proof which is closer to the constructions of this paper).

**Proposition 5.2.** Let $X$ be a wonderful variety and let $D \in \mathbb{Z} \Delta$. Then $C(X)$ and $C_D(X)$ have rational singularities.

**Proof.** By Corollary 5.1 we have that $C(X)$ is a deformation of a multicone over a flag variety, which has rational singularities by (a), hence $C(X)$ has rational singularities as well by (c).

In order to show the second claim, let $\tilde{D} \in \mathbb{Z} \Delta$ be such that $\mathbb{Q} D \cap \mathbb{Z} \Delta = \mathbb{Z} \tilde{D}$. Then the inclusions $\mathbb{Z} \tilde{D} \subset \mathbb{Z} D \subset \mathbb{Z} \Delta$ define a torus $S \cong \text{Hom}(\mathbb{Z} \Delta / \mathbb{Z} \tilde{D})$ and a finite group $\Gamma \cong \text{Hom}(\mathbb{Z} \tilde{D} / \mathbb{Z} D)$. Moreover, we have natural actions of $S$ on $C(X)$ and of $\Gamma$ on $C(X)^S$, and $C_D(X) = (C(X)^S)^\Gamma$. Therefore, by (b) it follows that $C_D(X)$ has rational singularities as well. □

**References**

[1] V. Alexeev and M. Brion, Toric degenerations of spherical varieties, Selecta Math. (N.S.) 10 (2004), 453–478.
[2] K. Appel, Standard monomials for wonderful group compactifications, J. Algebra 310 (2007), 70–87.
[3] J.-F. Boutot, Singularités rationnelles et quotients par les groupes réductifs, Invent. Math. 88 (1987), no. 1, 65–68.
[4] P. Bravi, J. Gandini and A. Maffei, Projective normality of model varieties and related results, arXiv:1304.6352.
[5] P. Bravi and D. Luna, An introduction to wonderful varieties with many examples of type $F_4$, J. Algebra 329 (2011) 4–51.
[6] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphériques, Duke Math. J. 58 (1989), 397–424.
[7] M. Brion, The total coordinate ring of a wonderful variety, J. Algebra 313, (2007), 61-99.
[8] R. Chirivì, LS Algebras and Application to Schubert varieties, Transform. Groups 5 (2000), 245–264.
[9] R. Chirivì, LS algebras and Schubert varieties, PhD thesis, Scuola Normale Superiore (2000).
[10] R. Chirivì, Deformation and Cohen-Macaulayness of the multicone over the flag variety, Comment. Math. Helv. 76 (2001), 436–466.
[11] R. Chirivì, P. Littelmann and A. Maffei, Equations Defining Symmetric Varieties and Affine Grassmannians, Int. Math. Res. Not. 2009, 291–347.
[12] R. Chirivì and A. Maffei, The ring of sections of a complete symmetric variety, J. Algebra 261 (2003), 310–326.
[13] R. Chirivì and A. Maffei, Projective normality of complete symmetric varieties, Duke Math. J. 122 (2004), 93–123.
[14] R. Chirivì and A. Maffei, Plücker relations and spherical varieties: application to model varieties, Transform. Groups 19 (2014), 979–997.
[15] C. De Concini and C. Procesi, A characteristic free approach to invariant theory, Advances in Math., 21 no. 3 (1976), 330–354.
[16] C. De Concini and C. Procesi, Complete symmetric varieties, in: Invariant theory (Montecatini, 1982), Lecture Notes in Math. 996, Springer, Berlin, 1983, 1–44.
[17] R. Dehy, Combinatorial results on Demazure modules, J. Algebra 205 (1998), 505–524.
[18] P. Doubilet, G. Rota and J. Stein, On the foundations of the combinatorial theory, IX Combinatorial methods in invariant theory, Studies in Appl. Math., 53 (1974), 185–216.
[19] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry, Graduate Texts in Mathematics 150, Springer-Verlag, New York, 1995.
[20] R. Elkik, Singularités rationnelles et déformations, Invent. Math. 47 (1978), 139–147.
[21] W. V. D. Hodge, Some enumerative results in the theory of forms, Proc. Cambridge Philos. Soc., 39 (1943), 22–30.
[22] W. V. D. Hodge and D. Pedoe, Methods of algebraic geometry Vol. II, Book III: General theory of algebraic varieties in projective space, Book IV: Quadrics and Grassmann varieties, Cambridge University Press, Cambridge, 1994, Reprint of the 1952 original.
[23] G. R. Kempf and A. Ramanathan, *Multicones over Schubert varieties*, Invent. Math. **87** (1987), 353–363.
[24] F. Knop, H. Kraft and Th. Vust, *The Picard group of a G-variety*, in: Algebraische Transformationengruppen und Invariantentheorie, DMV Sem. **13**, Birkhäuser, Basel, 1989, pp. 77–87.
[25] V. Lakshmibai, P. Littelmann and P. Magyar, *Standard Monomial Theory and applications*, in: “Representation Theories and Algebraic Geometry” (A. Broer, ed.), Kluwer Academic Publishers (1998).
[26] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P-II, The work of De Concini and Procesi and the basic conjectures*, Indian Academy of Sciences: Proceedings Part A **87** (1978), 1–54.
[27] P. Littelmann, *Paths and root operators in representation theory*, Ann. of Math. **142**, (1995), 499–525.
[28] P. Littelmann, *A plactic algebra for semisimple Lie algebras*, Adv. Math. **124** (1996), 312–331.
[29] P. Littelmann, *Contracting Modules and Standard Monomial Theory for Symmetrizable Kac-Moody Algebras*, J. Amer. Math. Soc. **11** (1998), 551–567.
[30] D. Luna, *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), 249–258.
[31] D. Luna, *Variétés sphériques de type A*, Publ. Math. Inst. Hautes Études Sci. **94** (2001) 161–226.
[32] V. L. Popov, *Contraction of the actions of reductive algebraic groups*, Math. USSR-Sb. **58** (1987), 311–335.
[33] C. S. Seshadri *Geometry of G/P. I. Theory of standard monomials for minuscule representations*, in C. P. Ramanujam - A Tribute, 207–39, Tata Institute of Fundamental Research Studies in Mathematics 8, Berlin: Springer (1978).