Abstract—In this paper, we present an algebraic construction of tail-biting trellises. The proposed method is based on the state space expressions, i.e., the state space is the image of the set of information sequences under the associated state matrix. Then combining with the homomorphism theorem, an algebraic trellis construction is obtained. We show that a tail-biting trellis constructed using the proposed method is isomorphic to the associated Koetter–Vardy (KV) trellis and tail-biting Bahl–Cocke–Jelinek–Raviv (BCJR) trellis. We also evaluate the complexity of the obtained tail-biting trellises. On the other hand, a matrix consisting of linearly independent rows of the characteristic matrix is regarded as a generalization of minimal-span generator matrices. Then we show that a KV trellis is constructed based on an extended minimal-span generator matrix. It is shown that this construction is a natural extension of the method proposed by McEliece (1996).

Index Terms—Block codes, algebraic construction, KV trellises, tail-biting BCJR trellises, tail-biting trellises.

I. INTRODUCTION

From the 1980s to 1990s, trellis representations of linear block codes were studied with a great interest [14], [16], [17], [20], [22], [38]. Subsequently, tail-biting trellises of linear block codes have received much attention. There have been many contributions to the subject [4], [9], [10], [11], [15], [18], [23], [25], [33], [35]. Given a linear block code, there exists a unique minimal conventional trellis. This trellis simultaneously minimizes all measures of trellis complexity. However, tail-biting trellises do not have such a property. That is, minimality of tail-biting trellises depends on the measure being used [15]. Despite these difficulties, tail-biting trellises have been studied with a great interest. This is due to the fact that the complexity of a tail-biting trellis may be much lower than that of the minimal conventional trellis. A remarkable advance has been made by Koetter and Vardy [15]. They showed that for a $k$-dimensional linear block code of length $n$ with full support, there exists a list of $n$ characteristic generators (i.e., a characteristic matrix [15]) from which all minimal tail-biting trellises can be obtained. A different method of producing tail-biting trellises was proposed by Nori and Shankar [23]. They used the Bahl-Cocke-Jelinek-Raviv (BCJR) construction [3] in order to obtain tail-biting trellises. In this construction, each path in the conventional trellis is displaced using a displacement matrix [23] which is defined based on the spans of a generator matrix of the given code. These works were further investigated by Gluesing-Luerssen and Weaver [9], [10]. They carefully examined the works by Koetter and Vardy and by Nori and Shankar. In particular, they noted the fact that the characteristic matrix associated with a given code is not unique in general. Taking account of this fact, they have refined and generalized the previous works.

This paper focuses on algebraic constructions of tail-biting trellises. In 1988, Forney [21], in an appendix to a paper on coset codes, proposed an algebraic characterization of conventional trellises, which resulted in a great interest in this subject. In connection with algebraic trellis constructions, Nori and Shankar [23] discussed a generalization of the Forney construction [21] to tail-biting trellises. On the other hand, the state and edge spaces of a tail-biting trellis have been characterized by Gluesing-Luerssen and Weaver [9], [10]. Let $M_i$ be the state matrix at level $i$ of a Koetter-Vardy (KV) trellis of a linear block code $C$. Then the state space is given by $V_i = \text{im}M_i$ (i.e., the image of $F^k$ under the linear mapping $M_i$), where $F^k$ denotes the set of information sequences of length $k$). Similarly, let $N_i$ be the state matrix at level $i$ of a tail-biting BCJR trellis of $C$. Then the state space is given by $V_i = \text{im}N_i$. From these expressions, we noticed that the homomorphism theorem can be applied to $V_i = \text{im}M_i$ ($V_i = \text{im}N_i$). That is, we have

$$V_i = \text{im}M_i \cong F^k / \text{ker}(M_i), \text{ for } i = 0, \ldots, n - 1$$

$$V_i = \text{im}N_i \cong F^k / \text{ker}(N_i), \text{ for } i = 0, \ldots, n - 1,$$

where $\text{ker}(M_i)$ and $\text{ker}(N_i)$ are the kernels of the linear mappings $M_i$ and $N_i$, respectively. These equations directly provide an algebraic construction of tail-biting trellises. In this paper, based on these fundamental relations, we propose an algebraic construction of tail-biting trellises for linear block codes. It is shown that a tail-biting trellis constructed using the proposed method is isomorphic to the associated KV trellis and tail-biting BCJR trellis. We also evaluate the complexity of the obtained tail-biting trellises.

On the other hand, note that characteristic generators may be regarded as a generalization of minimal-span generator matrices (MSGM’s) in the realm of conventional trellises [10]. Hence, it is reasonable to think that a tail-biting trellis can also be constructed based on a kind of MSGM. Suppose that $G$ consists of $k$ linearly independent rows of a characteristic matrix. Then $G$ is regarded as a generalization of MSGM. We call such a generator matrix an extended minimal-span generator matrix (e-MSGM). We show that a KV trellis is constructed based on an e-MSGM. We also discuss the

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relationship between the proposed algebraic construction and the construction based on e-MSGM’s.

The paper is organized as follows. The basic notions for tail-biting trellises are given in Section II. In Section III, we first review the algebraic trellis construction by Nori and Shankar. After that we will present an algebraic construction of tail-biting trellises. The proposed construction is based on the state-space expressions combined with the homomorphism theorem. The complexity of tail-biting trellises obtained using the proposed construction will be evaluated in Section IV. In Section V, we discuss the tail-biting trellis construction based on e-MSGM’s. Finally, conclusions are provided in Section VI.

II. PRELIMINARIES

In this section, we introduce the basic notions needed in this paper. We always assume that the underlying field is \( F = \text{GF}(2) \). Denote by \( C \) an \( (n, k) \) linear block code defined by a generator matrix \( G \) and a corresponding parity-check matrix \( H \). Let

\[
G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{pmatrix}, \quad (1)
\]

\[
H = \begin{pmatrix} \bar{g}_1 & \bar{g}_2 & \cdots & \bar{g}_n \\ \bar{h}_1 & \bar{h}_2 & \cdots & \bar{h}_n \end{pmatrix}. \quad (2)
\]

A tail-biting trellis \( T = (V, E) \) of depth \( n \) over the field \( F \) is a directed edge-labeled graph with the property that the vertex set \( V \) partitions into \( n \) disjoint sets \( V = V_0 \cup V_1 \cup \cdots \cup V_{n-1} \). Here every edge in \( T \) starts in \( V_{i-1} \) and ends in \( V_{i \mod n} \). An edge is a triple \((v, a, w) \in V_{i-1} \times F \times V_i \). We call \( V_i \) the state space of the trellis at level \( i \). Thus its elements are called the states at that level. A cycle in \( T \) is a closed path of length \( n \). We assume that the cycles start and end at the same state in \( V_0 \). If \(|V_0| = 1\), the trellis is called conventional.

In addition to the labeling of edges, each vertex in \( V_i \) can also be labeled. The resulting trellis is termed a labeled trellis. Then every cycle in a labeled tail-biting trellis \( T \) consists of the labels of edges and vertices in the cycle. Such a sequence is termed a label sequence in \( T \). The set of all the label sequences in a labeled tail-biting trellis \( T \) is called the label code of \( T \), denoted by \( S(T) \). We call a trellis reduced if every state and every edge appears in at least one cycle. A labeled trellis \( T \) is said to be linear, if \( T \) is reduced and \( S(T) \) is a linear code over \( F \). Linear trellises \( T = (V, E) \) and \( T' = (V', E') \) are called isomorphic if there exists a bijection \( \phi : V \rightarrow V' \) such that \( \phi \) is an isomorphism and \((v, a, w) \in E_i \iff (\phi(v), a, \phi(w)) \in E'_i \) for all \( i \) \( (1 \leq i \leq n) \).

Given a codeword \( x \in C \), a span of \( x \), denoted \([x]\), is a semiopen interval \([a, b)\) such that the corresponding closed interval \([a, b]\) contains all the nonzero positions of \( x \). We call the intervals \((a, b)\) and \([a, b)\) conventional if \( a \leq b \) and circular otherwise.

Remark 1: Note that \([x]\) does not contain the starting point \( a \) in the span. This is very convenient for the definition of elementary trellises \([15]\). On the other hand, a closed interval \([a, b)\] is adopted as a span in \([20]\) and \([23]\). Hence, we use the latter, if necessary. Also, take notice of the numbering of indices for a codeword. We assume that the index starts in 0 and ends in \( n - 1 \) for \( K \) trellises.

Let \( X = (F^{n \times n}) \) be a characteristic matrix \([15]\) of \( C \). The rows of \( X \) are called characteristic generators. Let \((a_1, b_1)\) be the span of a characteristic generator \( g_j \). Note that \( a_1, \ldots, a_n \) are distinct and \( b_1, \ldots, b_n \) are distinct. Let \( T_{g_j(a_1, b_1)} \) be the elementary trellis \([15]\) corresponding to a characteristic generator \( g_j \). A trellis of the form \( T_{g_1(a_1, b_1)} \times \cdots \times T_{g_k(a_k, b_k)} \), where \( g_1, \ldots, g_k \) are linearly independent rows of \( X \), is called a \( KV \) trellis \([9, 15]\) of \( C \).

Remark 2: The name \( KV \) trellises is used more generally for product trellises of the type \( T_{G, S} \), where \( G \) is a generator matrix and \( S \) is the corresponding span list.

Next, we consider a tail-biting BCJR trellis introduced by Nori and Shankar \([23]\). Denote by \( G \) and \( H \) the generator matrix and parity-check matrix of \( C \), respectively. Let \( S = \{|a_i, b_i|, 1 \leq i \leq k\} \) be a span list of \( G \). A displacement matrix \( \Theta \), defined by Nori and Shankar, is a design parameter for the construction of good trellises. In this paper, \( \Theta \) is defined based on \( S \) as follows (see \([9]\) or \([23]\)):

\[
\Theta = \begin{pmatrix} d_{g_1} \\ d_{g_2} \\ \vdots \\ d_{g_k} \end{pmatrix} \in F^{k \times (n-k)} \text{ with } d_i = \sum_{j=a_i}^n g_i h_j^T, \quad (4)
\]

where \( g_i = (g_{i1}, \ldots, g_{in}) \) is a generator in \( G \) with span \([a_i, b_i]\) \((T \text{ means transpose}) \). Note that if \( g_i \) has a conventional span, then \( d_i = 0 \). The displacement vector \( d_c \) for any codeword \( c \in C \) is defined as follows \([23]\):

\[
d_c = \sum_{i=1}^k \alpha_i d_{g_i}, \quad \text{where } c = \sum_{i=1}^k \alpha_i g_i, \quad \alpha_i \in F, g_i \in G. \quad (5)
\]

Denote by \( T_{(G, H, \Theta)} \) the resulting trellis. The trellis \( T_{(G, H, \Theta)} \) is called a tail-biting BCJR trellis \([9, 23]\).

III. ALGEBRAIC CONSTRUCTION OF TAIL-BITING TRELLISES

Nori and Shankar \([23]\) Section V] generalized the Forney construction \([7]\) for conventional trellises to tail-biting trellises. In this section, we will attempt to do the same thing in a somewhat different way.

A. Review of the tail-biting Forney trellis introduced by Nori and Shankar

Consider the \((7, 4)\) Hamming code \( C \) defined by the following generator matrix \( G \) and parity-check matrix \( H \) (cf. \([23]\) Examples 14 and 16)):

\[
H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad (6)
\]

\[
= \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}
\]
The resulting tail-biting BCJR trellis is given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} [1, 6] \\ [3, 7] \\ [6, 2] \\ [7, 4] \end{bmatrix}$$

(7)

First, we construct the tail-biting BCJR trellis introduced by Nori and Shankar [23]. Note that \(g_1\) and \(g_2\) have conventional spans, whereas \(g_3\) and \(g_4\) have circular spans [6, 2] and [7, 4], respectively. Hence, the corresponding displacement matrix \(\Theta\) is given by

$$\Theta = \begin{pmatrix} d_{g_1} & d_{g_2} & d_{g_3} & d_{g_4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$  

(8)

The resulting tail-biting BCJR trellis \(T_{(G, H, \Theta)}\) is shown in Fig.1.

On the other hand, \(G\) can be decomposed as follows:

$$G = \begin{pmatrix} G_0 \\ G_d \end{pmatrix},$$

(9)

where

$$G_0 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

(10)

$$G_d = \begin{pmatrix} g_3 \\ g_4 \end{pmatrix}.$$  

(11)

Here let \(C_0\) be the linear subcode generated by \(G_0\) and then consider the partition \(C/C_0 = \{C_0, C_1, C_2, C_3\}\). We have

\[
C_0 = \{000000, 0010111, 1000110, 1010001\}
\]

\[
C_1 = \{0100011, 0110100, 1100101, 1110010\}
\]

\[
C_2 = \{0111001, 0101110, 1111111, 1101000\}
\]

\[
C_3 = \{0011010, 0001101, 1011100, 1001011\}
\]

Remark 1: In this paper, we regard the first element of each coset as the representative.

We see that \(G_d\) is a generator matrix for the set of representatives of the cosets \(C_l\) \((0 \leq l \leq 3)\).

Next, for the cosets \(C_l\), \(0 \leq l \leq 3\), we define the following mappings:

\[
\pi_i(c) = c_h^T + \cdots + c_i h_i^T \\
\pi_i(c) = c_{11} h_1^T + \cdots + c_i h_i^T + d_{g_3} \\
\pi_i(c) = c_{11} h_1^T + \cdots + c_i h_i^T + d_{g_3} + d_{g_4} \\
\pi_i(c) = c_{11} h_1^T + \cdots + c_i h_i^T + d_{g_3} + d_{g_4} + d_{g_4}.
\]

(12)

(13)

(14)

(15)

where \(c = (c_1, c_2, \ldots, c_7)\) denotes any codeword in coset \(C_l\).

For each \(i\), \(\tilde{C}_i\) is \(\bigcup_{0 \leq l \leq 3}\{c \in C_l : \pi_i(c) = 0\}\) is computed as follows:

\[
i = 0 : \tilde{C}_0 = \{000000, 0010111, 1000110, 1010001\}
\]

\[
i = 1 : \tilde{C}_1 = \{000000, 0010111\}
\]

\[
i = 2 : \tilde{C}_2 = \{000000, 0010111, 0100011, 0110100\}
\]

\[
i = 3 : \tilde{C}_3 = \{000000, 0010001\}
\]

\[
i = 4 : \tilde{C}_4 = \{000000, 0100011, 0110100, 0111001\}
\]

\[
i = 5 : \tilde{C}_5 = \{000000, 0100011, 0110100, 0111001\}
\]

\[
i = 6 : \tilde{C}_6 = \{000000, 0001101, 0110010, 1111111\}
\]

Note that \(\tilde{C}_i\) is a linear subcode of \(C\) for each \(i\). Hence, we can think of the partition \(C/\tilde{C}_i\). We have the following:

\[
i = 0 : \tilde{C}_0 = \{000000, 0010111, 1000110, 1010001\}
\]

\[
i = 1 : \tilde{C}_1 = \{0111001, 0101110, 1111111, 1101000\}
\]

\[
i = 2 : \tilde{C}_2 = \{0100011, 0110100, 1100101, 1110010\}
\]

\[
i = 3 : \tilde{C}_3 = \{0011010, 0001101, 1011100, 1001011\}
\]

\[
i = 4 : \tilde{C}_4 = \{000000, 0010111\}
\]

\[
i = 5 : \tilde{C}_5 = \{1000110, 1010001\}
\]

\[
i = 6 : \tilde{C}_6 = \{1111111, 1101000\}
\]
Now a tail-biting trellis can be constructed based on the above results. That is, for each codeword $c \in C$, the corresponding path (i.e., cycle) is obtained by tracing the cosets $\tilde{C}_i$ which contain $c$. The obtained tail-biting trellis is shown in Fig.2. The states in Fig.2 are labeled by the representatives in the cosets $\tilde{C}_i$, $0 \leq i \leq 6$.

Also, a modified tail-biting trellis obtained using state permutations is shown in Fig.3. We observe that the trellis in Fig.3 is identical to the one in Fig.1.

B. Algebraic construction of a tail-biting BCJR trellis

The argument in the previous section, though it was presented in terms of a specific example, implies that a tail-biting BCJR trellis can be constructed using an algebraic method. Let $C$ be an $(n, k)$ linear block code defined by the generator matrix $G$ and parity-check matrix $H$. Denote by $\Theta$ the associated displacement matrix determined by $(G, H)$ and by the span list of $G$. Consider the tail-biting BCJR trellis $T_{(G,H,\Theta)}$ constructed based on $(G, H, \Theta)$. In the following, let $A_i$ denote the submatrix consisting of the first $i$ columns of $A$. The state matrix at level $i$, denoted by $N_i$, is given by

$$N_i = G_iH_i^T + \Theta, \quad \text{for } i = 0, \cdots , n - 1. \quad (16)$$

In this case \cite{9, 10}, the state space at level $i$ is expressed as

$$V_i = \text{im}N_i, \quad \text{for } i = 0, \cdots , n - 1. \quad (17)$$

Using the homomorphism theorem, the right-hand side becomes

$$\text{im}N_i \cong F^k/\ker(N_i), \quad \text{for } i = 0, \cdots , n - 1, \quad (18)$$

where $\ker(N_i)$ is the kernel of the linear mapping induced by $N_i$. Then we have

$$V_i = \text{im}N_i \cong F^k/\ker(N_i), \quad \text{for } i = 0, \cdots , n - 1. \quad (19)$$

Remark 2: Consider the example in the previous section. The kernels $\ker(N_i)$ $(0 \leq i \leq 6)$ are obtained as follows:

$$\begin{align*}
\ker(N_0) &= \{0000, 0100, 1000, 1100\} \\
\ker(N_1) &= \{0000, 0100\} \\
\ker(N_2) &= \{0000, 0010, 0100, 0110\} \\
\ker(N_3) &= \{0000, 0010\} \\
\ker(N_4) &= \{0000, 0001, 0010, 0011\} \\
\ker(N_5) &= \{0000, 0001, 0010, 0011\} \\
\ker(N_6) &= \{0000, 0001, 0010, 1001\}.
\end{align*}$$

Though the elements of $\ker(N_i)$ are $u$’s $\in F^4$, these elements can be identified with the corresponding codewords $c$’s $\in C$.\clearpage
We see that the resulting set of codewords coincides with \( \tilde{C}_i \), \( 0 \leq i \leq 6 \). In other words, \( \tilde{C}_i = \bigcup_{0 \leq l \leq 3} \{ c \in C_l : \pi_{i,l}(c) = 0 \} \) is equal to ker\( (N_i) \).

For \( i = 0, \ldots, n - 1 \), let

\[
F^k / \text{ker}(N_i) = \{ \tilde{C}_i, \tilde{C}_{i+1}, \ldots, \tilde{C}_{i,m(i) - 1} \}. \tag{20}
\]

Here it is assumed that the elements in \( \tilde{C}_i \) have been transformed into the corresponding codewords. In this case, a tail-biting trellis is constructed by tracing the cosets \( \tilde{C}_i \) which contain \( c \) for each codeword \( c \in C \). More precisely, there is an edge \( e \in E_i \) labeled \( c_i \) from a vertex \( v \in V_{i-1} \) to a vertex \( w \in V_i \), if and only if there exists a codeword \( c = (c_1, \ldots, c_n) \in C \) such that \( c \in v \cap w \).

Similarly, we have the following for the edge spaces \( E_i \):

\[
E_i = \text{im}(N_{i-1}, \bar{g}_i, N_i) \\ \cong F^k / \text{ker}(N_{i-1}, \bar{g}_i, N_i), \text{ for } i = 1, \ldots, n, \tag{21}
\]

where \( \bar{g}_i \) denotes the \( i \)th column of \( G \).

For the obtained tail-biting trellis, we have the following.

**Proposition 1:** A tail-biting trellis obtained using the proposed construction, denoted by \( T_{\text{alg}} \), is isomorphic to the associated tail-biting BCJR trellis \( T_{(G, H, \Theta)} \).

**Proof:** The proposed method is based on the isomorphism:

\[
\text{im} N_i \cong F^k / \text{ker}(N_i), \text{ for } n = 0, \ldots, n - 1.
\]

For \( u, u' \in F^k \), let

\[
c = (c_1, \ldots, c_i, \ldots, c_n) \\
c' = (c'_1, \ldots, c'_i, \ldots, c'_n)
\]

be the corresponding codewords. Here suppose that \( u - u' \in \text{ker}(N_i) \). This means that \( c \) and \( c' \) are contained in the same coset, i.e., go through the same state at level \( i \). On the other hand, \( u - u' \in \text{ker}(N_i) \) is equivalent to \( uN_i = u'N_i \). Hence, noting \( N_i = G_i H_i^T + \Theta \), we have

\[
c_i h_i^T + \cdots + c_i h_i^T + (u_1 d_{g_1} + \cdots + u_k d_{g_k}) \\= c'_i h_i^T + \cdots + c'_i h_i^T + (u'_1 d_{g_1} + \cdots + u'_k d_{g_k}).
\]

The last equation means that \( c \) and \( c' \) define the same state at level \( i \) in the tail-biting BCJR trellis.

**Example 1 (Nori and Shankar [23]):** Consider the \((7,4)\) Hamming code defined by the following generator matrix \( G \) and parity-check matrix \( H \):

\[
H = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix} \tag{22}
\]

\[
G = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{23}
\]

\[
= \begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix}
\]

\begin{align*}
\text{The tail-biting BCJR trellis } T_{(G, H, \Theta)} \text{ is shown in Fig. } 4. \\
\text{In order to construct a tail-biting trellis using the proposed method, we first compute the state matrices } N_i \,(0 \leq i \leq 6). \\
\text{They are given as follows:}
\end{align*}

\[
N_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix} \quad (\Theta)
\]

\[
N_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
N_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
N_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
N_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Next, $\ker(N_i)$ ($0 \leq i \leq 6$) are determined as follows:

$$
\ker(N_0) = \{000000, 001100, 011000, 011100, 001101, 011011, 110010, 111111\}
$$

$$
\ker(N_1) = \{000000, 001100, 100000, 101010\}
$$

$$
\ker(N_2) = \{000000, 001000, 100000, 101010\}
$$

$$
\ker(N_3) = \{000000, 001000, 100000, 011010\}
$$

$$
\ker(N_4) = \{000000, 001000, 011010, 011011\}
$$

$$
\ker(N_5) = \{000000, 001000, 011010, 011011\}
$$

$$
\ker(N_6) = \{000000, 001000, 011010, 011011\}
$$

These kernels can also be expressed in terms of the corresponding codewords:

$$
i = 0 : \tilde{C}_{00} = \{000000, 001100, 101000, 110010, 001101, 011011, 110010, 111111\}
$$

$$
i = 1 : \tilde{C}_{10} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

$$
i = 2 : \tilde{C}_{20} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

$$
i = 3 : \tilde{C}_{30} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

$$
i = 4 : \tilde{C}_{40} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

Then the partitions $C/\tilde{C}_{00}$ are given as follows:

$$
i = 0 : \tilde{C}_{00} = \{000000, 001100, 101000, 110010, 001101, 011011, 110100, 111111\}
$$

$$
i = 1 : \tilde{C}_{10} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

$$
i = 2 : \tilde{C}_{20} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

$$
i = 3 : \tilde{C}_{30} = \{000000, 001100, 000110, 001101, 0001101, 0011011\}
$$

The resulting tail-biting trellis is shown in Fig.5. A modified tail-biting trellis obtained using state permutations is shown in Fig.6. We observe that the trellis in Fig.6 is identical to the one in Fig.4.

C. Algebraic construction of a BCJR-dual trellis

In this section, we consider the BCJR-dual trellis $T^\perp = T_{(H,G,\theta)}$ corresponding to a tail-biting BCJR trellis $T_{(G,H,\theta)}$. Note that the state matrix $\hat{N}_i$ associated with $T^\perp$ is given by $\hat{N}_i = N_i^T$ [23]. Let $V_i$ be the state space of $T^\perp$ at level $i$. We have

$$
\hat{V}_i = \text{im}\hat{N}_i \cong \mathbb{F}^{n-k}/\ker(\hat{N}_i), \text{ for } i = 0, \ldots, n-1. \tag{25}
$$

Based on this equation, we can construct a tail-biting trellis which is isomorphic to the tail-biting BCJR-dual trellis.

Example 1 (Continued): Again, consider Example 1. We only need to compute $\ker(\hat{N}_i)$ after obtaining the state matrices.
For the obtained tail-biting trellis, we have the following.

**Proposition 2:** A tail-biting trellis obtained using the proposed construction, denoted by \( T_{alg} \), is isomorphic to the associated KV trellis \( T_{G,S} \). If the coset representatives (in terms of \( u \)) are placed in ascending order at each level \( l \), then \( T_{alg} \) is identical to \( T_{G,S} \).

**Proof:** The proposed method is based on the isomorphism:

\[
\text{im} M_i \cong F^k / \ker(M_i), \quad \text{for } i = 0, \ldots, n - 1.
\]

For \( u, u' \in F^k \), let \( c \) and \( c' \) be the corresponding codewords. Suppose that \( u - u' \in \ker(M_i) \). This means that \( c \) and \( c' \) are contained in the same coset, i.e., go through the same state in the trellis at level \( i \). On the other hand, \( u - u' \in \ker(M_i) \) is equivalent to \( uM_i = u'M_i \). We have two cases:

1) \( \mu_i^l = 1 \): In this case, since

\[
\begin{align*}
\text{u}M_i &= (\ldots, u_l, \ldots), \\
\text{u}'M_i &= (\ldots, u_l', \ldots),
\end{align*}
\]

it follows that \( u_l = u_l' \).

2) \( \mu_i^l = 0 \): In this case, we have

\[
\begin{align*}
\text{u}M_i &= (\ldots, 0, \ldots), \\
\text{u}'M_i &= (\ldots, 0, \ldots),
\end{align*}
\]

regardless of the values of \( u_l \) and \( u_l' \). Hence, only \( u_l \) such that \( \mu_i^l = 1 \) are effective. That is, \( u \) and \( u' \) whose components coincide at positions \( l \) such that \( \mu_i^l = 1 \) are contained in the same coset.

On the other hand, a KV trellis is obtained as the product of elementary trellises \( T_{g_i} \). Let \( i \) be any level in the trellis. If \( \mu_i^l = 1 \) holds, then \( T_{g_i} \) has “two” vertices at level \( i \) from the definition of an elementary trellis \([15]\). Hence, both 0 and 1 are allowed as the value of \( u_l \) in the product of elementary trellises. This is equivalent to considering only the components \( u_l \) such that \( \mu_i^l = 1 \).

Next, consider the transition from level \((i - 1)\) to level \( i \). Without loss of generality, suppose that \( \mu_{i-1}^l = 0 \), \( \mu_i^l = 1 \).

First, consider the “algebraic” construction. Since \( \mu_i^l = 1 \), \( u_l = 0 \) and \( u_l = 1 \) are distinguished at level \( i \). Hence, denote by \( U_i(u_l = 0) \) and \( U_i(u_l = 1) \) the cosets which contain \( u = (\ldots, u_l = 0, \ldots) \) and \( u' = (\ldots, u_l = 1, \ldots) \), respectively. On the other hand, since \( \mu_{i-1}^l = 0 \), \( u_l \) is not effective at level \( i - 1 \). That is, \( u = (\ldots, u_l = 0, \ldots) \) and \( u' = (\ldots, u_l = 1, \ldots) \) are contained in the same coset \( U_{i-1}(u_l = 0) \). Noting these facts, if \( u_l = 0 \) at level \( i \), then we regard the value of \( u_l \) at level \( i - 1 \) as 0, whereas if \( u_l = 1 \) at level \( i \), then we regard the value of \( u_l \) at level \( i - 1 \) as 1. This means that there exist transitions: \( U_{i-1}(u_l = 0) \to U_i(u_l = 0) \) and \( U_{i-1}(u_l = 0) \to U_i(u_l = 1) \).

Next, consider the “product” construction. From the definition of an elementary trellis, our assumption \( \mu_{i-1}^l = 0 \), \( \mu_i^l = 1 \) corresponds to the left end of the span \((a_l, b_l) \) (i.e., \( i - 1 = a_l \) and \( i = a_l + 1 \)). Then there are two transitions: \( u_l = 0 \ (i - 1) \to u_l = 0 \ (i) \) and \( u_l = 0 \ (i - 1) \to u_l = 1 \ (i) \). This is equivalent to the above. Moreover, the edge label from \((i - 1)\) to \( i \) in \( T_{g_i} \) is defined as \( \beta \cdot g_{i-1} \ [15] \) Section IV-C]. Since \( \mu_i^l = 1 \), \( \beta \) is equal to \( u_l \) from the definition of \( \beta \). Here...
The KV trellis follows:

\[ G = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}. \quad (33)

A characteristic matrix associated with \( C \) is given by

\[ X = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (34)

Next, \( \ker(M_i) \) \((0 \leq i \leq 6)\) are obtained as follows:

\[
\ker(M_0) = \{0000, 0010, 0100, 0110, 1000, 1010, 1100, 1110\}
\]
\[
\ker(M_1) = \{0000, 0010, 1000, 1010\}
\]
\[
\ker(M_2) = \{0000, 0010, 1000, 1010\}
\]
\[
\ker(M_3) = \{0000, 0001, 1000, 1001\}
\]
\[
\ker(M_4) = \{0000, 0001, 0100, 0101\}
\]
\[
\ker(M_5) = \{0000, 0001, 0100, 0101\}
\]
\[
\ker(M_6) = \{0000, 0001, 0010, 0011, 1000, 0101, 0110, 0111\}.
\]

**Remark 4:** Note that \( \ker(M_i) = \ker(N_i) \) \((0 \leq i \leq 6)\) holds, where \( \ker(N_i) \) are the kernels in Example 1. We can show this equality under a general condition.

Then we have the partitions \( F^3/\ker(M_i) \):

\[
i = 0 : \bar{U}_{00} = \{0000, 0010, 0100, 0110, 1000, 1010, 1100, 1110\}
\]
\[
i = 1 : \bar{U}_{10} = \{0000, 0010, 1000, 1010\}
\]
\[
i = 2 : \bar{U}_{20} = \{0000, 0010, 1000, 1010\}
\]
\[
i = 3 : \bar{U}_{30} = \{0000, 0001, 1000, 1001\}
\]
\[
i = 4 : \bar{U}_{40} = \{0000, 0001, 0100, 0101\}
\]
\[
i = 5 : \bar{U}_{50} = \{0000, 0001, 0100, 0101\}
\]

In order to apply the proposed construction, we first compute the state matrices \( M_i \) \((0 \leq i \leq 6)\). They are given as follows:

\[
M_0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
M_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
M_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
M_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Fig. 8. The KV trellis obtained using the product construction.
\[ \hat{U}_{52} = \{1000, 1001, 1100, 1101\} \]
\[ \hat{U}_{53} = \{1010, 1011, 1110, 1111\} \]
\[ i = 6 : \hat{U}_{60} = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\} \]
\[ \hat{U}_{61} = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\} \].

Remark 5: We observe that at each level \(i\), the representatives of the cosets coincide with the states in the trellis in Fig.8.

Since we have seen that \( \ker(M_i) = \ker(N_i) \) \((0 \leq i \leq 6)\), the same tail-biting trellis as that in Fig.8. is obtained. The resulting trellis is shown in Fig.9.

As is stated above, we have the following.

**Lemma 1:**
\[ \ker(M_i) = \ker(N_i) \text{, for } i = 0, \cdots, n - 1. \] (36)

**Proof:** Suppose that \( \rho_i^+ = 0 \), i.e., \( i \notin (a_i, b_i) \). That is, the \( l \)-th row of \( M_i \) is the all-zero vector. We first show that the \( l \)-th row of \( N_i \), denoted by \( N_{il} \), is also the all-zero vector. Without loss of generality, let \( (a_i, b_i) \) be a circular span. From the assumption that \( i \notin (a_i, b_i) \), it follows that \( b_i + 1 \leq i \leq a_i \).

In this case, \( N_{il} \) is expressed as
\[ N_{il} = g_{t,0}h_{0}^T + g_{t,1}h_{1}^T + \cdots + g_{n,b_i}h_{b_i}^T + d_{gi}. \]
Here note that
\[ d_{gi} = g_{t,a_i}h_{a_i}^T + \cdots + g_{n-1, n-1}h_{n-1}^T. \]
Then we have
\[ N_{il} = g_{t,0}h_{0}^T + g_{t,1}h_{1}^T + \cdots + g_{t,b_i}h_{b_i}^T \]
\[ + g_{t,a_i}h_{a_i}^T + \cdots + g_{t,n-1}h_{n-1}^T. \]

Since \( g_i \) is a codeword, \( N_{il} \) is the all-zero vector.

On the other hand, it has been shown that \( \text{rank} M_i = \text{rank} N_i \) (see [9]). This implies that any two non-zero rows of \( N_i \) are different. Then it follows that \( \ker(M_i) = \ker(N_i) \).

Suppose that a KV trellis \( T_{G,S} \) and the associated tail-biting BCJR trellis \( T_{G,H,\hat{e}} \) are constructed using the proposed algebraic method. Since \( \ker(M_i) = \ker(N_i) \) holds, the two resulting trellises are identical. Hence, we have shown the following.

**Proposition 3:** \( T_{G,S} \) and \( T_{G,H,\hat{e}} \) are isomorphic. Note that this is the result in [9] Theorem IV.11.

IV. **Complexity of Tail-Biting Trellises Obtained Using the Proposed Construction**

In this section, we discuss the complexity of the tail-biting trellises obtained using the proposed construction. We have the following.

**Proposition 4:**
\[ |V_i| = 2^k - \dim \ker(N_i), \text{ for } i = 0, \cdots, n - 1. \] (37)

**Proof:** From \( V_i \equiv F^k / \ker(N_i) \), we have
\[ \dim V_i = k - \dim \ker(N_i). \]

A similar result holds for the edge spaces \( E_i \).

**Proposition 5:**
\[ |E_i| = 2^{k - \dim \ker(N_{i-1}, \hat{g}, N_i)}, \text{ for } i = 1, \cdots, n. \] (38)

**Proof:** From \( E_i \equiv F^k / \ker(N_{i-1}, \hat{g}, N_i) \), we have
\[ \dim E_i = k - \dim \ker(N_{i-1}, \hat{g}, N_i). \]

**Corollary 1:** Denote by \( \rho_i^+ \) and \( \rho_i^- \) the out-degree and in-degree at level \( i \), respectively. Then we have
\[ \rho_i^+ = 2^{\dim \ker(N_i) - \dim \ker(N_{i-1}, \hat{g}, N_i)} \]
\[ \rho_i^- = 2^{\dim \ker(N_i) - \dim \ker(N_{i-1}, \hat{g}, N_i)}. \] (39) (40)

**Proof:** The edge space at level \( i + 1 \) is defined as
\[ E_{i+1} \equiv F^k / \ker(N_i, \hat{g}_{i+1}, N_{i+1}). \]
Let \( v \in V_i \) be any state (i.e., \( v \in F^k / \ker(N_i) \)). Let us denote the set of edges of \( E_{i+1} \) leaving \( v \) by \( E_{i+1}^v \). Then \( E_{i+1}^v \) is the set of cosets \( c \in F^k / \ker(N_i, \hat{g}_{i+1}, N_{i+1}) \) such that \( c \in v \). Hence, each of the sets \( E_{i+1}^v \) has the same size. Then the former result follows from the relation:
\[ \rho_i^+ = \frac{|E_{i+1}|}{|V_i|}. \]

The proof of the latter equality is similar.

Next, consider the BCJR-dual trellis \( T_{\perp} \) corresponding to a tail-biting BCJR trellis \( T \). Let \( \hat{V}_i \) and \( \hat{E}_i \) be the associated state and edge spaces of \( T_{\perp} \), respectively. As in the case of a tail-biting BCJR trellis,
\[ \hat{V}_i = \text{im} \hat{N}_i \]
\[ \cong F^{n-k} / \ker(N_i), \text{ for } i = 0, \cdots, n - 1 \]
\[ \hat{E}_i = \text{im} \hat{N}_{i-1}, \hat{h}_i, \hat{N}_i \]
\[ \cong F^{n-k} / \ker(N_{i-1}, \hat{h}_i, \hat{N}_i), \text{ for } i = 1, \cdots, n \]
hold. We have the following.

**Proposition 6:**
\[ |V_i| = |\hat{V}_i|, \text{ for } i = 0, \cdots, n - 1. \] (41)

**Proof:** Note that
\[ \dim V_i = \dim \text{im} N_i = \text{rank} N_i \]
\[ \dim \hat{V}_i = \dim \text{im} \hat{N}_i = \text{rank} \hat{N}_i. \]
Here \( \text{rank} N_i = \text{rank} N_i^T = \text{rank} \tilde{N}_i \) holds.

Next, consider the relation between \( \text{dim ker}(N_{i-1}, \bar{g}_i, N_i) \) and \( \text{dim ker}(\tilde{N}_{i-1}, h_i, \tilde{N}_i) \). For the purpose, we impose some restrictions.

In [10], Gluesing-Luerssen and Weaver showed that for each complete set of characteristic generators of a code, there exists a complete set of characteristic generators of the dual code such that their resulting KV trellises are dual to each other if paired suitably. Hence, if \( T \) is a KV trellis, then there exists a KV trellis \( \tilde{T} \) which is dual to \( T \).

Example 3: Again consider the \((7, 4)\) Hamming code \( C \) defined by

\[
G = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

A characteristic matrix \( Y \) associated with \( C^\perp \) is given by

\[
Y = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

(42)

By selecting 3 linearly independent rows of \( Y \), we set

\[
\tilde{H} = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

(43)

Note that \((G, \tilde{H})\) is a “dual selection” of \((X, Y)\) [10]. We see that the KV trellises \( T_{G,S} \) and \( T_{H,S} \) are dual to each other. This can also be stated in terms of BCJR representations. That is, \( T(\tilde{G}, \tilde{S}) = T_{G,S}^\perp \) and \( T_{H,S}^\perp \) holds [10].

Taking account of the above argument, we first show the following lemma.

Lemma 2: Suppose that a KV trellis \( T \) and its dual \( \tilde{T} \) are constructed based on extended minimal-span generator matrices (e-MSGM’s) (see Section V). Also, let \((\alpha_i, \beta_i)\) and \((\hat{\alpha}_i, \hat{\beta}_i)\) be the variables for \( T \) and \( \tilde{T} \), respectively. In this case, we have

\[
\alpha_i = -\hat{\alpha}_{i-1} + \hat{\beta}_i + 1, \quad i = 1, \ldots, n \quad (44)
\]

\[
\hat{\alpha}_i = -\alpha_i + \beta_{i-1} + \beta_i + 1, \quad i = 1, \ldots, n \quad (45)
\]

Proof: See Appendix.

Now suppose that KV trellises \( T_{G,S} \) and \( T_{H,S} \) are dual to each other. Let \( H \) be a parity-check matrix corresponding to \( G \). Then we have the following.

\[
T_{G,S} \cong T_{(G,H,S)} \quad (46)
\]

\[
T_{H,S} \cong T_{G,S}^\perp \quad (47)
\]

Here the latter equation is derived from the relations \( T(\tilde{G}, \tilde{S}) = T_{(G,H,S)}^\perp \) and \( T_{G,S}^\perp \cong T_{(G,H,S)}^\perp \). Let \( N_i \) and \( \tilde{N}_i \) \((= N_i^T)\) be the state matrices of \( T_{(G,H,S)} \) and \( T_{(G,H,S)}^\perp \), respectively. Then we have the following.

**Proposition 7:**

\[
k - \text{dim ker}(N_{i-1}, \bar{g}_i, N_i) = (n - k) + \text{dim ker}(\tilde{N}_{i-1}, h_i, \tilde{N}_i)
\]

\[
= (k + 1) + \text{dim ker}(N_{i-1}, \bar{g}_i, N_i) - \text{dim ker}(\tilde{N}_i) \quad (48)
\]

\[
(n - k) - \text{dim ker}(\tilde{N}_{i-1}, h_i, \tilde{N}_i)
\]

\[
= (k + 1) + \text{dim ker}(N_{i-1}, \bar{g}_i, N_i) - \text{dim ker}(\tilde{N}_i) \quad (49)
\]

Proof: The number of vertices at level \( i \) is \(|E_i| = 2^{n_i}\). Using Lemma 2, we have

\[
|E_i| = 2^{-\alpha_i + \hat{\alpha}_{i-1} + \hat{\beta}_i + 1}.
\]

Also (cf. Appendix), it follows that

\[
\hat{\alpha}_i = (n - k) - \text{dim ker}(\tilde{N}_{i-1}, h_i, \tilde{N}_i)
\]

\[
\hat{\beta}_i = (n - k) - \text{dim ker}(\tilde{N}_i) \quad (50)
\]

Then we have

\[
-\hat{\alpha}_i + \hat{\beta}_{i-1} + \hat{\beta}_i + 1
\]

\[
= -(n - k) + \text{dim ker}(\tilde{N}_{i-1}, h_i, \tilde{N}_i) + (n - k) - \text{dim ker}(\tilde{N}_i) + (n - k) - \text{dim ker}(\tilde{N}_i) + 1
\]

\[
= (k + 1) + \text{dim ker}(N_{i-1}, \bar{g}_i, N_i) - \text{dim ker}(\tilde{N}_i) \quad (51)
\]

On the other hand, we know that

\[
|E_i| = 2^{k - \text{dim ker}(N_{i-1}, \bar{g}_i, N_i)}.
\]

Then the result follows. The second equation is derived in a similar way.

Example 4: Consider the tail-biting BCJR trellis in Example 1 and its BCJR-dual trellis in Example 1 (Continued). Take notice of the transition \( i - 1 = 6 \rightarrow i = 7 \) in these trellises. From

\[
(N_6|\bar{g}_7|N_7) = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

we have

\[
\text{ker}(N_6) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}
\]

\[
\text{ker}(N_7) = \{0000, 0010, 0100, 0110, 1000, 1010, 1100, 1110\}
\]

\[
\text{ker}(N_6, \bar{g}_7, N_7) = \{0000, 0010, 0100, 0110\}.
\]

Then

\[
\text{dim ker}(N_6) = 3
\]

\[
\text{dim ker}(N_7) = 3
\]

\[
\text{dim ker}(N_6, \bar{g}_7, N_7) = 2.
\]
Similarly, from
\[(\hat{N}_6|\hat{h}_7|\hat{N}_7) = (N_6^T|h_7|N_7^T)\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
we have
\[
\ker(\hat{N}_6) = \{000, 010, 101, 111\} \\
\ker(\hat{N}_7) = \{000, 010, 101, 111\} \\
\ker(\hat{N}_6, \hat{h}_7, \hat{N}_7) = \{000, 010, 101, 111\}.
\]
Then
\[
\dim \ker(\hat{N}_6) = 2 \\
\dim \ker(\hat{N}_7) = 2 \\
\dim \ker(\hat{N}_6, \hat{h}_7, \hat{N}_7) = 2.
\]
Hence, we have
\[
k - \dim \ker(N_{i-1}, \tilde{g}_i, N_i) = 4 - 2 = 2
\]
\[
(n - k + 1) + \dim \ker(\tilde{N}_{i-1}, \tilde{h}_i, \tilde{N}_i) - \dim \ker(\hat{N}_i) = (7 - 4 + 1) + 2 - 2 - 2 = 2.
\]
The second equality is confirmed in a similar way.

V. CONSTRUCTING A KV TRELLIS FROM AN EXTENDED MINIMAL-SPAN GENERATOR MATRIX

A conventional BCJR trellis can be constructed using a minimal-span generator matrix (MSGM) [20]. In this section, we show that this construction is extended to tail-biting trellises. First, we show an example and then extend to a general case.

A. An example

Consider the generator matrix
\[
G = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \begin{array}{c}
[4, 7] \\
[1, 4] \\
[3, 6] \\
[7, 3]
\end{array}
\]

Remark 1: This generator matrix has been considered in Section III-B (Example 1) and Section III-D (Example 2). We also remark that the numbering of indices for a codeword is shifted by 1 compared to that in [15] (also in [9]).

Note that left/right end-points of the spans of \( G \) are distinct. Hence, \( G \) can be regarded as a kind of MSGM [20]. We call such an MSGM an extended minimal-span generator matrix (e-MSGM). According to McEliece [20], let us define \( A_i \) and \( B_i \) as follows:

\[
\begin{align*}
A_i &= \{j : i \in [a_j, b_j]\}, \text{ for } i = 1, \cdots, n \quad (50) \\
A_0 &= A_n \\
B_i &= A_i \cap A_{i+1}, \text{ for } i = 0, \cdots, n - 1 \quad (52) \\
B_n &= B_0, \quad (53)
\end{align*}
\]

where the periodicity of a tail-biting trellis is taking into account. Here we assume an additional condition:

(2) Let \([a_i, b_i]\) be the circular span of \( g_i \) such that \( a_i = i + 1 \) and \( b_i = i \). In this case, \( l \) is not contained in \( B_i \).

We denote the cardinalities of \( A_i \) and \( B_i \) by \( \alpha_i \) and \( \beta_i \), respectively. Table I gives the \( A_i \)'s, \( B_i \)'s, \( \alpha_i \)'s, and \( \beta_i \)'s. Based on this table, we can construct a tail-biting trellis using the method in [20]. Let \( u \) be a binary \( \alpha_i \)-tuple and define the variables \( \text{init}(u) \), \( \text{fin}(u) \), and \( \lambda(u) \) as follows:

\[
\begin{align*}
\text{init}(u) &= u \cap B_{i-1} \quad (54) \\
\text{fin}(u) &= u \cap B_i \quad (55) \\
\lambda(u) &= u \cdot \tilde{g}_i \quad (56)
\end{align*}
\]

where \( \tilde{g}_i = \tilde{g}_i \cap A_i \). The notation “\( u \cap B \)” represents the binary vector obtained by extracting the components of \( u \) corresponding to the elements of \( B \). Then for edge spaces \( E_i \) (\( 1 \leq i \leq 7 \)), we have the following tables. A tail-biting trellis is constructed based on these tables. The resulting tail-biting trellis is shown in Fig.10.
Fig. 10. The tail-biting trellis constructed based on the e-MSGM.

### B. Generalization

As is shown in [9] and [10], the state matrix $M_i$ of a KV-trellis is given as follows:

$$M_i = \left( \begin{array}{cccc} 
\mu_1^i & \cdots & \mu_k^i 
\end{array} \right)$$

$$\mu_i^j = \begin{cases} 
1, & \text{if } i \in (a_l, b_l] \\
0, & \text{if } i \notin (a_l, b_l] 
\end{cases}$$

#### Remark 2: In [15], the span of a generator $g_l$, denoted by $[g_l]$, is defined as a semipen interval $(a_l, b_l]$ such that the corresponding closed interval $[a_l, b_l]$ contains all the nonzero positions of $g_l$. Now, consider the condition $\mu_i^0 = 1$. This means that $i - 1 \in [a_l, b_l]$ and $i \in [a_l, b_l]$. Here take notice of the numbering of indices for a codeword in the two papers, i.e., one is by Koetter and Vardy (also by Gluesing-Luerssen and Weaver) and the other is by McEliece (also by Nori and Shankar). We see that the numbering of indices is shifted by 1 between them. Hence, in the notation of McEliece, the condition $\mu_i^0 = 1$ is equivalent to $i \in [a_l, b_l]$ and $i + 1 \in [a_l, b_l]$ (i.e., $g_l$ is “active” at levels $i$ and $i + 1$). Taking these facts into consideration, we follow McEliece in this section.

Thus the next lemma has been proved.

**Lemma 3:**

$$\mu_i^l = 1 \leftrightarrow l \in B_i.$$  \hspace{1cm} (57)

Here consider the product $u \cdot M_i$. When we compute

$$\begin{pmatrix} \mu_{i}^1 \\
\vdots \\
\mu_{i}^k 
\end{pmatrix}, \quad \begin{pmatrix} u_1, \ldots, u_l, \ldots, u_5 
\end{pmatrix}$$

the product has the form $(\ldots, u_l, \ldots)$ if $\mu_i^l = 1$. On the other hand, if $l \in B_i$, then $u_l$ is extracted in the operation of $u \cap B_i$. Hence, it follows that

$$u_l \in u \cap B_i \leftrightarrow l \in B_i \leftrightarrow \mu_i^l = 1 \leftrightarrow (\ldots, u_l, \ldots) \in imM_i = V_i.$$  \hspace{1cm} (58)

Accordingly, we have

- $u_l \in \init(u) \leftrightarrow (\ldots, u_l, \ldots) \in V_{i-1}$,
- $u_l \in \fin(u) \leftrightarrow (\ldots, u_l, \ldots) \in V_i$,
- $\lambda(u) = u \cdot \vec{g}_i = c_i$.

Here we need not consider all the components of $u$. From the definition, $B_{i-1} \subseteq A_i$ and $B_i \subseteq A_i$ hold. Thus we can restrict $u$ to $u \cap A_i$ (i.e., $u$ is a binary $\alpha_i$-tuple). Hence, in the calculation of $\lambda(u)$, $\vec{g}_i$ can be replaced by $\vec{g}_i = \vec{g}_i \cap A_i$. We see that these are equivalent to the tables for $E_i$.

Thus we have shown the following.

**Proposition 8:** Consider a KV trellis $T_{G,S}$. Denote by $T_{[G,S]}$ the corresponding tail-biting trellis constructed by regarding $(G,S)$ as an e-MSGM. Then $T_{[G,S]}$ is identical to $T_{G,S}$.

In fact, we observe that the tail-bitting trellis in Fig.10 is identical to the KV trellis in Fig.8.

Moreover, we have the following.

**Proposition 9:** Suppose that a KV trellis $T_{G,S}$ is constructed using the proposed algebraic method. Then the resulting tail-biting trellis $T_{alg}$ is isomorphic to $T_{[G,S]}$. If the coset representatives (in terms of $u$) are placed in ascending order at each level $i$, then $T_{alg}$ is identical to $T_{[G,S]}$.

**Proof:** For the proposed algebraic construction,

$$E_i \equiv F^k/\ker(M_{i-1}, \vec{g}_i, M_i), \quad \text{for } i = 1, \ldots, n.$$
holds. For \( u, u' \in F^k \), let \( c \) and \( c' \) be the corresponding codewords. Also, suppose that \( u - u' \in \ker(M_{i-1}, \bar{g}_i, M_i) \).

This means the following (cf. Proposition 2):
- At level \( i-1 \), \( u_l = u'_l \) holds for \( l \) such that \( \mu_i^{l-1} = 1 \),
- At level \( i \), \( u_l = u'_l \) holds for \( l \) such that \( \mu_i^l = 1 \),
- \( c_i = c'_i \).

The above are further rephrased as follows:
- If \( u_l \in \text{init}(u), u'_l \in \text{init}(u') \), then \( u_l = u'_l \),
- If \( u_l \in \text{fin}(u), u'_l \in \text{fin}(u') \), then \( u_l = u'_l \),
- \( c_i = c'_i \).

Finally, we remark that if a generator matrix \( G \) has a form of e-MSGM, then the corresponding tail-biting BCJR trellis \( T_{(G, H, \phi)} \) can also be constructed according to the procedure described above. In fact, it has been shown that the trellises \( T_{G,S} \) and \( T_{(G, H, \phi)} \) are isomorphic [9] Theorem IV.11).

**VI. CONCLUSION**

We have presented an algebraic construction of tail-biting trellises. The proposed method is based on a quite simple idea. We took notice of the state space expressions of a tail-biting trellis, where the state space is the image of the set of information sequences under the associated state matrix. Then by applying the homomorphism theorem to these expressions, an algebraic trellis construction is obtained. We have shown that a tail-biting trellis constructed using the proposed method is isomorphic to the associated KV trellis and tail-biting BCJR trellis. Also, we have evaluated the complexity of the obtained tail-biting trellises. On the other hand, a matrix consisting of linearly independent rows of the characteristic matrix is regarded as a generalization of minimal-span generator matrices. Then we have shown that a KV trellis is constructed based on an extended minimal-span generator matrix. It is shown that this construction is a natural extension of the method proposed by McEliece [20] Section VII).

**APPENDIX A**

**PROOF OF LEMMA 2**

Consider an \((n, k)\) linear block code \( C \). Let \( i \) be any level. Forney [7] defined the past and future subcodes \( P_i \) and \( F_i \) as follows (cf. [20]):

\[
P_i = \{ c \in C : c_{i+1} = c_{i+2} = \cdots = c_n = 0 \} \quad (59)
\]

\[
F_i = \{ c \in C : c_1 = c_2 = \cdots = c_i = 0 \}. \quad (60)
\]

We denote their dimensions by \( p_i \) and \( f_i \), respectively:

\[
p_i = \dim P_i, \quad i = 0, \cdots, n-1 \quad (61)
\]

\[
f_i = \dim F_i, \quad i = 1, \cdots, n. \quad (62)
\]

Consider the associated conventional trellis (i.e., the Forney trellis). Then for the state and edge spaces \( V_i \) and \( E_i \) at level \( i \), we have

\[
|V_i| = 2^{k-p_i-f_i} \quad (63)
\]

\[
|E_i| = 2^{k-p_{i-1}-f_i} \quad (64)
\]

A similar result holds for the dual code \( C^\perp \) of \( C \). Let \( \hat{p}_i \) and \( \hat{f}_i \) be the corresponding variables. Then it follows that

\[
\hat{p}_i = f_i + i - k \quad (65)
\]

\[
\hat{f}_i = p_i - i + (n-k). \quad (66)
\]

On the other hand, as is shown in [20], the same conventional trellis (i.e., the BCJR trellis) is constructed based on the MSGM. In this case, we have

\[
|V_i| = 2^{\beta_i} \quad (67)
\]

\[
|E_i| = 2^{\alpha_i} \quad (68)
\]

where \( \alpha_i = |A_i| \) and \( \beta_i = |B_i| \) (see Section V). Then it follows that

\[
\beta_i = k - p_i - f_i \quad (69)
\]

\[
\alpha_i = k - p_{i-1} - f_i. \quad (70)
\]

Similarly, we have

\[
\hat{\beta}_i = (n-k) - \hat{p}_i - \hat{f}_i \quad (71)
\]

\[
\hat{\alpha}_i = (n-k) - \hat{p}_{i-1} - \hat{f}_i. \quad (72)
\]

Based on these equations, we first derive the relations between \((\alpha_i, \beta_i)\) and \((\hat{\alpha}_i, \hat{\beta}_i)\) for conventional trellises. At this point, the derived relations hold only for conventional trellises. However, note that the relations are expressed in terms of \( \alpha_i, \beta_i, \hat{\alpha}_i, \) and \( \hat{\beta}_i. \) That is, variables such as \( p_i \) and \( f_i \) are not contained in the relations. On the other hand, we already have shown that the trellis construction based on MSGM’s is extended to the construction based on e-MSGM’s. This implies that the relations between \((\alpha_i, \beta_i)\) and \((\hat{\alpha}_i, \hat{\beta}_i)\) still hold for “tail-biting trellises”. This is a basic idea for the proof.

Now we go back to the proof. First, note that

\[
\alpha_i = k - p_{i-1} - f_i.
\]

From the duality formulae, it follows that

\[
\hat{p}_i = f_i + i - k
\]

\[
\hat{f}_{i-1} = p_{i-1} - (i-1) + (n-k).
\]

Adding each side, we have

\[
\hat{p}_i + \hat{f}_{i-1} = (p_{i-1} + f_i) + 1 + n - 2k.
\]

Hence,

\[
\alpha_i + (\hat{p}_i + \hat{f}_{i-1}) = 1 + (n-k)
\]

holds. On the other hand, from

\[
\hat{\beta}_{i-1} = (n-k) - \hat{p}_{i-1} - \hat{f}_{i-1}
\]

\[
\hat{\alpha}_i = (n-k) - \hat{p}_{i-1} - \hat{f}_i,
\]

it follows that

\[
\hat{\beta}_{i-1} - \hat{\alpha}_i = -\hat{f}_{i-1} + \hat{f}_i.
\]

By combining this with

\[
\hat{\beta}_i = (n-k) - \hat{p}_i - \hat{f}_i,
\]

we have

\[
-\hat{\alpha}_i + \hat{\beta}_{i-1} + \hat{\beta}_i = (n-k) - (\hat{p}_i + \hat{f}_{i-1}).
\]
Moreover, by combining this with
\[ \alpha_i + (\hat{\beta}_i + \hat{f}_{i-1}) = 1 + (n - k), \]
we finally have
\[ \alpha_i = -\hat{\alpha}_i + \hat{\beta}_{i-1} + \hat{\beta}_i + 1. \]
The equality
\[ \hat{\alpha}_i = -\alpha_i + \beta_{i-1} + \beta_i + 1 \]
is derived in a similar way.

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