A SIMPLIFIED PROOF OF OPTIMAL $L^2$-EXTENSION THEOREM AND EXTENSIONS FROM NON-REDUCED SUBVARIETIES

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Abstract. We give a simplified proof of an optimal version of the Ohsawa-Takegoshi $L^2$-extension theorem. We follow the variational proof by Berndtsson-Lempert and use the method in the paper of McNeal-Varolin. As an application, we give an optimal estimate for extensions from possibly non-reduced subvarieties.

1. Introduction

The goals of this paper are to provide a simplified proof of an optimal version of the Ohsawa-Takegoshi $L^2$-extension theorem and to apply the same method to extensions from possibly non-reduced subvarieties to get an optimal estimate.

The Ohsawa-Takegoshi $L^2$-extension theorem [OT] is an extension theorem for holomorphic functions defined on a closed subvariety of a complex manifold with an $L^2$-estimate (For precise statement, see Theorem 2.1 below for example). This theorem is now widely used in complex and algebraic geometry. Recently, an optimal $L^2$-extension theorem was obtained by Blocki [Blo] and Guan and Zhou [GZ]. Their proofs are based on a version of Hörmander’s $L^2$-estimate for $\bar{\partial}$-equation and a careful choice of auxiliary functions. After that Berndtsson and Lempert obtained a new proof for the optimal $L^2$-extension theorem [BL] based on a variational result of Berndtsson [Ber].

In this paper, we will give a simplified proof of the $L^2$-extension theorem along with the proof in [BL]. We simplified a limiting argument in their proof (in particular, Lemma 3.5 in [BL]) by using a method by McNeal and Varolin [MV]. In [MV], the limiting argument is replaced by an existence result of an extension to a small neighborhood of the subvariety with a certain $L^2$-estimate (Theorem 4.5 in [MV]). Since they considered the jet extension, in [MV], $\bar{\partial}$-equation was solved to get an extension of jets to a small neighborhood. In our setting, it is enough to consider the restriction of a fixed arbitrary extension.

Moreover, by the same method, we will give an optimal $L^2$-estimate for extensions from non-reduced subvarieties in a formulation similar to [Dem]. Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain and $\phi \in PSH(\Omega) \cap C(\Omega)$ be a continuous plurisubharmonic function. Let $\psi$ be a negative plurisubharmonic function with neat analytic singularities (see Section...
3 for definitions). Let \(0 = m_0 < m_1 < \cdots < m_p < \cdots\) be the jumping numbers of \(\psi\). Then we have:

**Theorem 1.1.** Let \(f \in H^0(\Omega; \mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))\). Assume that there exists a holomorphic function \(F^\circ\) on \(\Omega\) such that \(F^\circ|_{\mathcal{O}/\mathcal{I}(m_p\psi)} = f\) and \(\int_{\Omega} |F^\circ|^2 e^{-\phi - m_{p-1}\psi} < +\infty\). Then the \(L^2\)-minimal extension \(F_0 \in A^2(\Omega, \phi + m_{p-1}\psi)\) of \(f\) satisfies

\[
\int_{\Omega} |F_0|^2 e^{-\phi - m_{p-1}\psi} \leq \limsup_{t \to -\infty} e^{-(m_p - m_{p-1})t} \int_{\psi < t} |F^\circ|^2 e^{-\phi - m_{p-1}\psi}.
\]

Note that the right-hand side is slightly different from the \(m_p\)-jet \(L^2\)-norm \(J^{m_p} f|_{\omega} = \int_{\Omega} |f|^2 dV_{\omega} Z^p_{\psi, \omega}[\psi]\), which was used in [Dem]. The relationship of these norms for jets are to be studied.

Very recently, Zhou and Zhu [ZZ] proved an optimal extension theorem in a more general setting. Their proof is based on a version of \(L^2\)-estimates for \(\overline{\partial}\)-equations. It may be interesting to know if variational methods can be applied to this general setting.

2. **Simplest case: a simplification of Berndtsson-Lempert’s proof**

To make the idea clear, first we will explain the simplest case. In this case we obtain a simplified proof of an optimal \(L^2\)-extension theorem. The following version is in [BL]:

**Theorem 2.1** (Optimal \(L^2\) extension, [Blo], [GZ], [BL]). Let \(\Omega \subset \mathbb{C}^n\) be a bounded pseudoconvex domain and \(V \subset \Omega\) be a closed submanifold. Let \(\phi \in \text{PSH}(\Omega)\). Assume that there exists a Green-type function \(G\) on \(\Omega\) with poles along \(V\), i.e. \(G \in \text{PSH}(\Omega), G < 0\) on \(\Omega\), and for some continuous functions \(A\) and \(B\) on \(\Omega\),

\[
\log d^2(z, V) + A(z) \geq G(z) \geq \log d^2(z, V) - B(z).
\]

Then, for every holomorphic function \(f\) on \(V\) with \(\int_V |f|^2 e^{-\phi + kB} < +\infty\), there exists a holomorphic function \(F\) on \(\Omega\) such that \(F|_V = f\) and

\[
\int_{\Omega} |F|^2 e^{-\phi} \leq \sigma_k \int_V |f|^2 e^{-\phi + kB}.
\]

We will prove Theorem 2.1 following the proof in [BL]. The most important result used in the proof is the following:

**Theorem 2.2** ([Ber]). Let \(\Omega \subset \mathbb{C}^n\) be a bounded pseudoconvex domain and \(\Phi \in \text{PSH} \cap C^\infty(\overline{\Omega} \times \Delta)\) where \(\Delta = \{|t| < 1\}\) \(\subset \mathbb{C}\) is the unit disc. For each \(t \in \Delta\), we let \(\phi_t := \Phi(z, t)\), which is a plurisubharmonic function on \(\Omega\). Let \(\xi \in A^2(\Omega)\) be a bounded linear functional on the Hilbert space of \(L^2\) holomorphic functions on \(\Omega\) (note that \(A^2(\Omega, \phi_t) = A^2(\Omega)\) as vector spaces since by assumption \(\Phi\) is bounded). Then the function

\[
t \mapsto \log \|\xi\|_{(A^2_{\Omega, \phi_t})^*},
\]

is subharmonic.
Remark 2.3. It is interesting that, conversely, Theorem 2.2 can be proved by the optimal \(L^2\)-extension theorem [GZ]. Furthermore, in [DWZZ], Theorem 2.2 is obtained by the (non-optimal) \(L^2\)-extension theorem via an \(L^2\)-theoretic characterization of plurisubharmonic functions. According to these results, Theorem 2.2 also holds for singular weights.

Proof of Theorem 2.2. We will denote by \(A^2(\Omega, \phi)\) the Hilbert space of holomorphic functions \(F\) on \(\Omega\) with \(\int_{\Omega} |F|^2 e^{-\phi} < +\infty\). We can assume that \(\phi\) is continuous.

First take an arbitrary \(L^2\) extension \(F^0 \in A^2(\Omega, \phi)\) of \(f\) to \(\Omega\) by shrinking domain in advance and using the standard theory of Stein manifolds. (Thus we should consider a sequence of domains \(\Omega_j\) approximating \(\Omega\) from inside. We will prove the existence of an extension \(F_0^{(j)} \in \mathcal{O}(\Omega_j)\) of \(f|_{\Omega_j \cap V}\) with uniformly bounded \(L^2\)-norms. Thus we can extract a convergent subsequence. In the argument below, just for simplicity, we omit the subscript \(j\). We will need this approximation sequence of domains again in the last of this proof.)

Consider a short exact sequence of Hilbert spaces:

\[ 0 \rightarrow A^2(\Omega, \phi) \cap \mathcal{I}_V \rightarrow A^2(\Omega, \phi) \rightarrow \frac{A^2(\Omega, \phi)}{A^2(\Omega, \phi) \cap \mathcal{I}_V} \rightarrow 0, \]

where \(A^2(\Omega, \phi) \cap \mathcal{I}_V\) denotes the space \(A^2(\Omega, \phi) \cap H^0(\Omega, \mathcal{I}_V)\). Let \(F_0\) be the \(L^2\)-minimum extensions of \(f\). The \(L^2\) norm \(\|F_0\|_{A^2(\Omega, \phi)}\) of \(F_0\) is equal to the quotient norm of \(F^0\) in the space \(A^2(\Omega, \phi)/A^2(\Omega, \phi) \cap \mathcal{I}_V\). Considering dual spaces, we can write the norm as

\[ \|F^0\|_{A^2/\mathcal{I}_V} = \sup \left\{ \frac{|\langle \xi, F^0 \rangle|}{\|\xi\|_{A^2(\Omega, \phi)^*}} : \xi \in A^2(\Omega, \phi), \langle \xi, h \rangle = 0 \text{ for every } h \in A^2(\Omega, \phi) \cap \mathcal{I}_V \right\}. \]

Let us consider a good class of \(\xi\). To do that, fix a smooth function \(g\) on \(V\) with compact support. Define a linear functional \(\xi_g\) on \(A^2(\Omega, \phi)\) by

\[ \langle \xi_g, h \rangle := \sigma_k \int_V h \eta e^{-\phi + kB}. \]

It can be shown that the set of such functionals \(\xi_g\) is a dense subspace of \((A^2(\Omega, \phi)/A^2(\Omega, \phi) \cap \mathcal{I}_V)^*\). Thus the supremum above can be written also as

\[ \sup_{g} \frac{|\langle \xi_g, F^0 \rangle|}{\|\xi_g\|_{A^2(\Omega, \phi)^*}}. \]

For \(p \geq 0\) and \(t \in \mathbb{C}\) with \(\text{Re} \, t \leq 0\), we let \(\phi_{t,p}(z) := \phi(z) + p \max(G(z) - \text{Re} \, t, 0)\). Define \(A^2_{t,p} := A^2(\Omega, \phi_{t,p})\). For fixed \(p \geq 0\), the function \(t, z \mapsto \phi_{t,p}(z)\) is plurisubharmonic. Applying Theorem 2.2 to the family \(\{A^2(\Omega, \phi_{t,p})\}_t\), we obtain that the function

\[ t \mapsto \|\xi_g\|_{(A^2(\Omega, \phi_{t,p}))^*} \]

is subharmonic. Since \(\phi_{t,p}\) depends only on the real part of \(t\), it is convex as a function of \(\text{Re} \, t\). From here we assume that \(t \in \mathbb{R}_{\leq 0}\). The following lemma describes the limiting behavior of \(\|\xi_g\|_{(A^2(\Omega, \phi_{t,p}))^*}\). We will write this norm by \(\|\xi_g\|_{t,p}\) for short.
Lemma 2.4 ([BL Lemma 3.2]). For fixed \( p > 0 \), it holds that

\[ \| \xi_g \|^2_{t,p} e^{kt} = O(1) \]

when \( t \to -\infty \). In particular, \( \log \| \xi_g \|^2_{t,p} + kt \) is convex in \( t \) and bounded from above when \( t \to -\infty \), thus increasing in \( t \).

The proof is the same as one in [BL] up to this point. From here we use the idea in [MV]. We will denote by \( F_{t,p} \) the \( L^2 \)-minimal extension of \( f \) in \( A^2_{t,p} \) := \( A^2(\Omega, \phi_{t,p}) \). By the same reason as before, we have

\[ \| F_{t,p} \| = \sup_g \frac{|\langle \xi_g, F^\circ \rangle|}{\| \xi_g \|_{t,p}}. \]

By the lemma before, we have that \( e^{-kt} \| F_{t,p} \|^2_{A^2_{t,p}} \) is decreasing in \( t \), and thus

\[ \| F_{0} \|^2_{A^2(\Omega)} \leq e^{-kt} \| F_{t,p} \|^2_{A^2_{t,p}}. \]

Since \( F_{t,p} \) is \( L^2 \)-minimal, we have

\[ e^{-kt} \| F_{t,p} \|^2_{A^2_{t,p}} \leq e^{-kt} \| F^\circ \|^2_{A^2_{t,p}}. \]

Fix \( t < 0 \) and let \( p \to +\infty \). Since \( e^{-\phi_{t,p}} \downarrow 1_{\Omega_t} \cdot e^{-\phi} \), Lebesgue’s dominated convergence theorem shows that

\[ e^{-kt} \| F^\circ \|^2_{A^2_{t,p}} \to e^{-kt} \| F^\circ \|^2_{A^2(\Omega_t, \phi_{t,t})}. \]

Next let \( t \to -\infty \). Here we need an approximation sequence of domains \( \Omega_j \subset \Omega \) again. We have proved that the \( L^2 \)-minimum extension \( F^{(j)}_0 \in A^2(\Omega_j, \phi) \) of \( f \) satisfies

\[ \| F^{(j)}_0 \|^2_{A^2(\Omega_j, \phi)} \leq e^{-kt} \| F^\circ \|^2_{A^2(\Omega_j \cap \{ G < t \}, \phi)}. \]

By [BL Lemma 3.3], the limsup of the right-hand side when \( t \to -\infty \) is bounded by

\[ \sigma_k \int_V |f|^2_{e^{-\phi + kB}}. \]

Therefore the \( L^2 \)-norms \( \| F^{(j)}_0 \|_{A^2(\Omega_j, \phi)} \) are uniformly bounded. After taking a subsequence we get a function \( F^{(\infty)}_0 \in A^2(\Omega, \phi) \) satisfying

\[ \| F^{(\infty)}_0 \|^2_{A^2(\Omega, \phi)} \leq \sigma_k \int_V |f|^2_{e^{-\phi + kB}}. \]

□
3. General case

In the same manner, we can prove an optimal $L^2$-estimate for extensions from non-reduced subvarieties in a formulation similar to [Dem].

The setting is as follows. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded pseudoconvex domain and $\phi \in PSH(\Omega) \cap C(\Omega)$ be a continuous plurisubharmonic function. Let $\psi$ be a negative plurisubharmonic function on $\Omega$ with neat analytic singularities, i.e., for each point $x \in \Omega$, there exist a neighborhood $U$ of $x$, a positive number $c > 0$, a finite number of holomorphic functions $g_1, \ldots, g_N \in \mathcal{O}(U)$, and a smooth function $u \in C^\infty(U)$ such that

$$\psi(z) = c \log(|g_1(z)|^2 + \cdots + |g_N(z)|^2) + u(z)$$

for $z \in U$. We will regard $\psi$ as a generalization of Green-type functions $G$ appeared in Section 2.

Let $0 = m_0 < m_1 < \cdots < m_p < \cdots$ be the jumping numbers of $\psi$. By definition, it holds that

$$\mathcal{I}(m_{p-1}\psi) = \mathcal{I}(m\psi) \supseteq \mathcal{I}(m_p\psi)$$

for every $m \in [m_{p-1}, m_p)$. In this setting, we have the following theorem:

**Theorem 3.1.** Let $f \in H^0(\Omega, \mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi)).$ Assume that there exists a holomorphic function $F^0 \in A^2(\Omega, \phi + m_{p-1}\psi)$ such that $F^0|_{\mathcal{O}/\mathcal{I}(m_p\psi)} = f$. Then the $L^2$-minimal extension $F_0 \in A^2(\Omega, \phi + m_{p-1}\psi)$ of $f$ satisfies

$$\int_{\Omega} |F_0|^2 e^{-\phi - m_{p-1}\psi} \leq \limsup_{t \to -\infty} e^{-(m_p - m_{p-1})t} \int_{\Omega_t} |F^0|^2 e^{-\phi - m_{p-1}\psi},$$

where $\Omega_t := \{ \psi < t \}$.

**Remark 3.2.** The assumption that $\psi$ has neat analytic singularities may be weakened. For example, let $G$ be the pluricomplex Green function of the polydisc $\Delta^n \subset \mathbb{C}^n$ with a pole at the origin. Then we have $G = \max(\log |z_1|^2, \ldots, \log |z_n|^2)$ and it does not seem to have neat analytic singularities, but we can apply Theorem 2.1 also to this case. Furthermore, by using the Azukawa indicatrix, we can obtain a sharper estimate than Theorem 2.1 (see [Hos2]).

3.1. Principalization of analytic singularities. In this subsection, we will explain a technique to analyze multiplier ideals of plurisubharmonic functions with analytic singularities. We will follow the exposition in [Dem].

Let $\psi$ be a plurisubharmonic function on $\Omega$ with neat analytic singularities. As before, we assume that locally

$$\psi(z) = c \log(|g_1(z)|^2 + \cdots + |g_N(z)|^2) + u(z)$$

for some $c > 0$, $g_j \in \mathcal{O}(U)$, and $u \in C^\infty(U)$. By Hironaka’s theorem, there exists a modification $\mu : X \to \Omega$ such that the ideal on $X$ generated by functions $g_1 \circ \mu, \ldots, g_N \circ \mu$ is equal to
\( O_X(-\Delta) \), where \( \Delta \) is a simple normal crossing divisor on \( X \). Moreover, we can assume that the zero-divisor of the Jacobian of \( \mu \) is contained in \( \Delta \). Taking a suitable coordinate, we can write \((g_j \circ \mu) = (w^a)\) locally for some multi-index \( a \). Then \( f \in \mathcal{I}(m\psi) \) if and only if
\[
\int_{\mu^{-1}(V)} \frac{|f \circ \mu(w)|^2|\text{Jac}(\mu)|^2}{|g \circ \mu(w)|^{2mc}} d\lambda(w) < +\infty
\]
in every local coordinate \( w \). If we write \( \text{Jac}(\mu) = \beta(w) \cdot w^b \) for non-vanishing holomorphic function \( \beta \), this condition can be rewritten as
\[
\int_{\mu^{-1}(V)} \frac{|f \circ \mu(w)|^2|w^b|^2}{|w^a|^{2mc}} d\lambda(w) < +\infty.
\]
It is equivalent to the condition
\[
f \circ \mu \text{ can be divided by } w^s, \text{ where } s_k = [mca_k - b_k]_+.
\]
Thus, \( m > 0 \) is a jumping number of \( \psi \) if and only if
\[
m = \frac{b_k + M}{ca_k}
\]
for some \( k \) and \( M \in \mathbb{N} \). For the \( p \)-th jumping number \( m_p \), we will write \( s_k^{(p)} := [m_pca_k - b_k]_+ \).
This means that, for \( f \in \mathcal{I}(m_p\psi) \), \( \mu^*f \) must have zeros of order \( s_k^{(p)} \) along \( w_k = 0 \).

3.2. Functionals on the space of \( L^2 \)-holomorphic functions. In this subsection, we will explain the definition of a functional \( \xi_g \) on \( A^2(\Omega, \phi + m_{p-1}\psi) \) which will be used in the proof of Theorem 3.1.

We consider the modification \( \mu : X \to \Omega \) described in the previous subsection. Fix a local coordinate \( w \) on \( X \) such that \( \Delta = \{w^a = 0\} \) for a multi-index \( a \). Let \( w_k \) be one of the coordinate functions. We assume that, at \( m = m_p \), the jump of ideals \( \mathcal{I}(m\psi) \) occurs along \( w_k = 0 \), i.e. \( m_p = \frac{b_k + M}{ca_k} \). We will write the coordinate as \( w = (w', w_k) \).

Fix a smooth function \( g \in C^\infty_c((\Delta_{\text{reg}})_k) \), where \( (\Delta_{\text{reg}})_k \) means the intersection of the set \( \{w_k = 0\} \) and the regular locus of \( \Delta \). We assume that the support of \( g \) is contained in an open set where the coordinate \( w \) is defined.

For each \( h \in A^2(\Omega, \phi + m_{p-1}\psi) \), we want to define \( \xi_g \) by
\[
\xi_g(h) := \lim_{t \to -\infty} \int_{t < \psi < t + 1} \mu^*h(w) \bar{g}(w) e^{-\phi - m_p\psi} \mu^*d\lambda_\Omega,
\]
where \( \bar{g}(w) = g(w') w_k^{s^{(p)}-1} \). Note that \( \mu^*h(w) \) has zeros of order \( s_k^{(p-1)} \) along \( w_k = 0 \). By definition of \( s_k^{(p-1)} \), we have that
\[
s_k^{(p-1)} \leq m_{p-1}ca_k - b_k < s_k^{(p-1)} + 1.
\]
Since we assumed that jump occurs at \( m_p \) along \( w_k = 0 \), we also have
\[
s_k^{(p)} = m_pca_k - b_k = s_k^{(p-1)} + 1.
\]
Properties of $\xi_g$ are summarized in the following proposition:

**Proposition 3.3.**  
(1) The limit (3.1) exists and $\xi_g$ is a bounded linear functional on $A^2(\Omega, \phi + m_{p-1}\psi)$.

(2) The set of finite sums of functionals $\xi_g$ is a dense subspace of the dual space of

$$
\frac{A^2(\Omega, \phi + m_{p-1}\psi)}{A^2(\Omega, \phi + m_{p-1}\psi) \cap I(m_p\psi)}.
$$

(3) Let $\phi_{s,q}(z) := \phi(z) + q \cdot \max(\psi(z) - s, 0)$ for $s \leq 0$ and $q > 0$. Then $e^{(m_p-m_{p-1})s} \|\xi_g\|^2_{A^2(\Omega, \phi_{s,q} + m_{p-1}\psi)} = O(1)$ as $s \to -\infty$.

**Proof.** (1) We will prove the existence of the limit by computing the limit in more explicit terms. What we have to compute is

$$
\lim_{t \to -\infty} \int_{t < \psi < t+1} \mu^* h(w) \left( g(w') w_k^{(p) - 1} \right) e^{-\phi - m_p \psi} \mu^* d\lambda_{\Omega}.
$$

We can write as $\mu^* h(w) = k(w) \cdot w_k^{(p) - 1}$, where $k(w)$ is a holomorphic function. We can also write as

$$
\sum_{j=1}^{N} |g_j \circ \mu(w)|^2 = \gamma(w) \cdot |w^a|^2,
$$

where $\gamma(w)$ is a positive continuous function (this is a square sum of the form $\sum_j |g_j \circ \mu(w)/w^a|^2$).

Then,

$$
\mu^* \psi = \mu^*(c \log |g|^2 + u) = c \log(\gamma \cdot |w^a|^2) + \mu^* u.
$$

In addition, we can write as $\mu^* d\lambda_{\Omega} = |\text{Jac} \mu|^2 d\lambda(w) = |\beta(w)|^2 |w^b|^2 d\lambda(w)$, where $\beta(w)$ is a non-vanishing holomorphic function. Using these, the integral can be written as

$$
= \int_{w' \in \text{supp}(g)} k(w') g(w') |w_k|^{2(s_k^{(p)} - 1)} e^{-u^* \phi \gamma(w) - m_p c |w^a|^2 - 2 m_p c e^{-m_p u(w)} |\beta(w)|^2 |w^b|^2} d\lambda(w)
$$

$$
= \int_{w' \in \text{supp}(g)} k(w') g(w') |w_k|^{2(s_k^{(p)} - 1)} e^{-u^* \phi \gamma(w) - m_p c e^{-m_p u(w)} |\beta(w)|^2} d\lambda(w).
$$

The degree of $w_k$ is

$$
(s_k^{(p)} - 1) - (m_p c a_k - b_k) = -1,
$$

thus the integral is like

$$
= \int_{w' \in \text{supp}(g)} k(w') g(w') |w_k|^{-2} L(w) d\lambda(w),
$$

where $L(w)$ is a continuous function. Consider integration in $w_k$, using Fubini’s theorem. Then the domain of integration is

$$
t < c \log(\gamma(w)|w^a|^2) + \mu^* u(w) < t + 1.
$$
Writing as \( w^a = (w')^a w_k^a \), we have that

\[
t - c \log(\gamma(w)|(w')^a|^2) - \mu^* u(w) < ca_k \log |w_k|^2 < t - c \log(\gamma(w)|(w')^a|^2) - \mu^* u(w) + 1.
\]

Thus we can write (3.2) as

\[
t - \Diamond(w') - \epsilon(w) < ca_k \log |w_k|^2 < t - \Diamond(w') + 1 + \epsilon'(w),
\]

where \( \Diamond(w') \) is a continuous function and \( \epsilon, \epsilon' \) is a small error term converging to 0 when \( t \to -\infty \). Then, by a straightforward computation, one can obtain that the limit exists and is

\[
\frac{\pi}{ca_k} \int_{w' \in \text{supp}(g)} k(w', 0)\overline{g(w')}L(w', 0)d\lambda(w').
\]

This shows the linearity of \( \xi_g \). Boundedness of \( \xi_g \) can be proved in the same way as in the proof of (3), so this part is postponed.

(2) What we need to prove is that, for each \( h \in A^2(\Omega, \phi + m_{p-1}\psi) \), if \( \xi_g(h) = 0 \) for every \( g \) then \( h \in H^0(\Omega, I(m_p\psi)) \). By the calculation in (1), \( \xi_g(h) = 0 \) for every \( g \) implies that \( h \) has zeros of order \( s_{k}^{(p)} \) along \( \{ w_k = 0 \} \) for every \( w_k \). This implies that \( h \in I(m_p\psi) \).

(3) The norm of \( \xi \) is

\[
\|\xi\|_{A^2(\Omega, \phi + m_{p-1}\psi)} = \sup_h \|h\|_{A^2(\Omega, \phi + m_{p-1}\psi)} \frac{|\langle \xi_g, h \rangle|^2}{|\langle \xi_g, h \rangle|}.
\]

By Cauchy-Schwarz inequality, the numerator can be estimated as

\[
\lim_{t \to -\infty} \left( \int_{w' \in \text{supp}(g)} \mu^* h \cdot \overline{g} \cdot e^{-\phi - m_p\psi} \mu^* d\lambda(w) \right)^2 \leq \lim_{t \to -\infty} \left( \int_{w' \in \text{supp}(g)} |\mu^* h|^2 e^{-\phi - m_p\psi} \mu^* d\lambda \right) \cdot \left( \int_{w' \in \text{supp}(g)} |g|^2 e^{-\phi - m_p\psi} \mu^* d\lambda \right).
\]

The second integral is independent of \( h \) and can be bounded by a constant, say \( C_g \). We want to prove the following estimate:

(3.3)

\[
\lim_{t \to -\infty} \int_{w' \in \text{supp}(g)} \left| \mu^* h \right|^2 e^{-\phi - m_p\psi} \mu^* d\lambda \leq C e^{-(m_p - m_{p-1})s} \int_{w' \in U} \psi(w', w_k < s) \left| \mu^* h \right|^2 e^{-\phi - m_{p-1}\psi} \mu^* d\lambda,
\]
where $U \subset \Delta_k$ is a neighborhood of $\text{supp}(g)$. Once it is proved, the right-hand side can be bounded as

$$(\text{RHS}) \leq Ce^{-(mp-m_p-1)s} \int_{\psi<s} |h|^2 e^{-\phi-m_p-1}\psi d\lambda(z)$$

$${\leq} Ce^{-(mp-m_p-1)s} \int_\Omega |h|^2 e^{-\phi_{s,q}-m_p-1}\psi.$$  

This gives the desired conclusion.

Let us prove (3.3). Using $\mu^*h = w_k^{(p)} k(w)$, $\mu^*\psi = c(\log(\gamma(w) \cdot |w^\alpha|^2)) + \mu^*u(w)$, and $\mu^*\lambda = \beta(w)w^h d\lambda(w)$, each integration can be rewritten as:

$$\int_{t<\psi<t+1} |w_k|^{2(s_k^{(p)}-1)} |k|^2 e^{-2\phi\gamma(w)-m_p c |w^\alpha|^2} m_p c e^{-m_p u} |\beta|^2 |w^h|^2 d\lambda(w),$$

$$\int_{\psi<s} |w_k|^{2(s_k^{(p)}-1)} |k|^2 e^{-2\phi\gamma(w)-m_p c |w^\alpha|^2} m_p c e^{-m_p u} |\beta|^2 |w^h|^2 d\lambda(w).$$

Fix $w'$ and consider integrations in $w_k$. Then essentially it is enough to prove that

$$\lim_{t \to -\infty} \int_{t<\psi<t+1} \frac{|w_k|^{2(s_k^{(p)}-1)} |k|^2 e^{-2\phi\gamma(w)-m_p c |w^\alpha|^2} m_p c e^{-m_p u} |\beta|^2 |w^h|^2 d\lambda(w)}{c a_k}$$

$$\leq C(e^{-(mp-mp-1)s')} \int_{\log |w_k|^2 < \frac{a_k}{ca_k}} |w_k|^{2(s_k^{(p)}-1-m_p c a_k+b_k)} d\lambda(w_k).$$

Since $s_k^{(p)} - m_p c a_k + b_k = 0$, it is equivalent to

$$\lim_{t \to -\infty} \int_{t<\psi<t+1} \frac{|w_k|^{-2} d\lambda(w_k)}{c a_k}$$

$$\leq C(e^{-(mp-mp-1)s')} \int_{\log |w_k|^2 < \frac{a_k}{ca_k}} |w_k|^{2((mp-mp-1)c a_k-1)} d\lambda(w_k).$$

This can be proved by a simple computation. 

\[\square\]

3.3. **Proof of Theorem 3.1.** As in the proof of Theorem 2.1 the minimum $L^2$-norm of $L^2$-extension is equal to the quotient norm of $F^0$ in the space $A^2(\Omega, \phi + m_p \psi) \cap I(m_p \psi)$. By considering the dual, this equals to

$$\sup_g \frac{\langle \xi_g, F^0 \rangle}{\|\xi_g\|_{A^2(\Omega, \phi + m_p \psi)^*}},$$

where $\xi_g$ is the functional defined in the previous subsection.

For $s < 0$ and $q > 0$, let $\phi_{s,q}(z) := \phi(z) + q \max(G(z) - s, 0)$. Denote by $F_{s,q}$ the $L^2$-minimal extension of $f$ in the space $A^2(\Omega, \phi_{s,q} + m_p \psi)$. Then, by the same reason,

$$\|F_{s,q}\|_{A^2(\Omega, \phi_{s,q} + m_p \psi)} = \sup_g \frac{\langle \xi_g, F^0 \rangle}{\|\xi_g\|_{A^2(\Omega, \phi_{s,q} + m_p \psi)^*}}.$$
By a singular version of Theorem 2.2, we have that \( \log \| \xi_g \|_{A^2(\Omega, \phi_{s,q} + m_{p-1} \psi)} \) is convex in \( s \). (Here we have to consider singular weights in Theorem 2.2. In this situation, we should prove Theorem 2.2 in advance by the optimal \( L^2 \)-extension theorem (Theorem 2.1) as in [GZ]. Also see [Hos1].) By Proposition 3.3 (3), we also have that \( (m_p - m_{p-1})s + \log \| \xi_g \|_{A^2(\Omega, \phi_{s,q} + m_{p-1} \psi)} \) is bounded from above when \( s \to -\infty \), and thus increasing in \( s \).

Therefore, \( e^{-(m_p - m_{p-1})s} \| F_s \|_{A^2(\Omega, \phi_{s,q} + m_{p-1} \psi)} \) is decreasing in \( s \). Then we have the following estimates:

\[
\| F_0 \|_{A^2(\Omega, \phi + m_{p-1} \psi)} \leq e^{-(m_p - m_{p-1})s} \| F_s \|_{A^2(\Omega, \phi_{s,q} + m_{p-1} \psi)} \leq e^{-(m_p - m_{p-1})s} \| F^0 \|_{A^2(\Omega, \phi_{s,q} + m_{p-1} \psi)}
\]

\[
\lim_{q \rightarrow +\infty} e^{-(m_p - m_{p-1})s} \| F^0 \|_{A^2(\Omega_s, \phi + m_{p-1} \psi)}
\]

This completes the proof.

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