ON SELBERG’S THEOREM C IN THE THEORY OF THE
RIEMANN ZETA-FUNCTION

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Abstract. In this paper we obtain new theorems about classes of exceptional
sets for the Selberg’s theorem C (1942). Our theorems, as based on discrete
method, are not accessible for Karatsuba’s theory (1984) since this theory is a
continuous theory. This paper is English version of our paper [8], the results
of our paper [9] are added too.

1. Introduction

1.1. We use the following notions. Let

(a) \( \psi(t) \) be a positive increasing to infinity function such that
\[
\psi(t) \leq \sqrt{\ln t},
\]

(b) \( S \) be the set of values of \( t \)
\[
t \in \left[ T, T + T^{1/2+\epsilon} \right]
\]
for which there is at least one zero point of the function
\[
\zeta \left( \frac{1}{2} + it \right)
\]
within the interval
\[
\left( t, t + \frac{\psi(t)}{\ln t} \right), \quad S = S(T, \epsilon, \psi).
\]

Let us remind that the set of segments (1.1) for every small and fixed \( \epsilon > 0 \) is the
minimal set for the Selberg’s theory. It is the assertion of the Selberg’s C theorem
(see [10], p. 49) relative to segment (1.1)
\[
m(S) \sim T^{1/2+\epsilon}, \quad T \to \infty,
\]
that is, the measure of the set \( \tilde{S} \) of such values
\[
t \in \left[ T, T + T^{1/2+\epsilon} \right]
\]
for which there is no zero of the function (1.2) in the interval (1.3) is
\[
m(\tilde{S}) = o(T^{1/2+\epsilon}), \quad T \to \infty.
\]
1.2. Next, let 
\[ \{g_\nu\} \]

denote the sequence that is defined by the formula
\[ \vartheta_1(g_\nu) = \frac{\pi \nu}{2}, \quad \nu = 1, 2, \ldots \]
(see [3, 4], \[ l_\nu = g_\nu \], comp. [3, 4]), where
\[ \vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}. \]

**Remark 1.** Since the set
\[ (1.6) \quad W = \{g_\nu : g_\nu \in [T, T + T^{1/2+\epsilon}]\} \]
is the finite one, then
\[ (1.7) \quad m(W) = 0. \]
Consequently, we have the following: no information is contained in the Selberg’s theorem C (comp. (1.5), (1.7)) about the zeros of odd order of the function (1.2) in the intervals
\[ \left( g_\nu, g_\nu + \frac{\psi(g_\nu)}{\ln g_\nu} \right), \quad g_\nu \in W \]
that is the set W is the exceptional set for the Selberg’s theorem C.

1.3. Now, let us remind the following deep methods of the English mathematicians:
(a) continuous method of Hardy-Littlewood (see [11]),
(b) discrete method of E.C. Titchmarsh (see [11]).

In our papers [6, 7], we have constructed a discrete analogue of the Hardy-Littlewood continuous method (that is, some synthesis of (a) and (b)). Especially, we have obtained the following estimate (see [7])
\[ (1.8) \quad N_0(T + T^{5/12}\psi \ln^3 T) - N_0(T) > A(\psi)T^{5/12}\psi \ln^3 T, \]
where \( N_0(T) \) denotes the number of zeros of the function
\[ \zeta \left( \frac{1}{2} + it \right), \quad t \in (0, T], \]
and \( A(\psi) \) is the constant that depends on choice of \( \psi \), for example, if
\[ \psi = \ln \ln \ln T, \]
then
\[ A(\ln \ln \ln T) \]
is an absolute constant.

**Remark 2.** We notice explicitly that
(a) our improvement of the classical Hardy-Littlewood exponent \( \frac{1}{2} \)
\[ \frac{1}{2} \rightarrow \frac{5}{12} \]
is a 16.6% change after 61 years,
(b) the estimate (1.8) was the first step on a way to proof of the Selberg’s hypothesis (see [10], p. 5, comp. [2], pp. 37,39).

**Remark 3.** Let us notice that I have sent the manuscripts of my papers [6, 7] to A.A. Karatsuba in the beginning of 1981.
After this analysis, it is clear that our estimate \(1.8\) is in need of a corresponding analogue of the Selberg’s theorem C. Consequently, in this paper we shall prove an analogue of that theorem for finite set \(W_1\) (and also for others) of values

\[
g_\nu \in [T, T + T^{5/12} \psi \ln^3 T]; \quad m(W_1) = 0,
\]

i. e. for the exceptional set in the sense of the Selberg’s theorem.

2. Theorem 1

2.1. Let (see [6], (2.5))

\[
\omega = \frac{\pi}{\ln \frac{2}{\pi}} = \frac{\pi}{2 \ln P_0}, \quad U = T^{5/12} \psi \ln^3 T,
\]

\[
\ln T < M < \sqrt[3]{T} \ln T.
\]

Next, let

\[
\bar{\psi}(t)
\]

be the function of the same kind as \(\psi(t)\) and fulfilling the condition

\[
\frac{\bar{\psi}}{\sqrt{\psi}} = o(1), \quad T \to \infty,
\]

and let

\[
G(T, \psi, \bar{\psi})
\]

denote the number of such

\[
g_\nu \in [T, T + U]
\]

that the interval

\[
(g_\nu, g_\nu + \bar{\psi}(g_\nu))
\]

contains a zero of the odd order of the function

\[
\zeta \left( \frac{1}{2} + it \right), \quad t \in [T, T + U].
\]

The following theorem holds true.

**Theorem 1.**

\[
G(T, \psi, \bar{\psi}) \sim \frac{1}{\pi} U \ln T, \quad T \to \infty.
\]

**Remark 4.** Since (see [6], (8))

\[
\sum_{T \leq g_\nu \leq T + U} 1 \sim \frac{1}{\pi} U \ln T, \quad T \to \infty
\]

then we have by Theorem 1 that for *almost all*

\[
g_\nu \in [T, T + U]
\]

the interval \(2.3\) contains a zero of the odd order of the function

\[
\zeta \left( \frac{1}{2} + it \right).
\]
Remark 5. Let \( N(T) \) denote the number of zeros of the function
\[
\zeta(s), \ s = \sigma + it, \ \sigma \in (0,1), \ t \in (0,T).
\]
It is then true that (comp. [12], p. 181)
\[
(2.6) \quad N(T + U) - N(T) \sim \frac{1}{2\pi} U \ln T, \ T \to \infty.
\]
Of course, our formula (2.4) is not in a contradiction with the formula (2.6) since many of intervals (2.5) can intersect.

Remark 6. Let us notice explicitly that also for the theory of Karatsuba giving the estimate (comp. [2], p. 39)
\[
N_0(T + T^{27/82+\epsilon}) - N_0(T) > A(\epsilon) T^{27/82+\epsilon},
\]
we have that the set of values
\[
g_\nu \in [T, T + T^{27/82+\epsilon}]
\]
is the exceptional set, since the theory is continuous as well as the classical theories of Hardy-Littlewood and Selberg. Consequently, our Theorem 1 is not improvable also by Karatsuba’s theory.

2.2. Now we give the proof of Theorem 1. The basic point of the proof is the estimate (see [7], (3.16))
\[
(2.7) \quad R < A \frac{U \ln^2 T}{M}
\]
where \( R \) denotes the number of such
\[
g_\nu^* \in [T, T + U]
\]
for which the sequence
\[
\{Z(g_\nu^* + k\omega)\}_{k=1}^M
\]
preserves the sign (comp. [7], (3.9), (3.11)). Next,
\[
\bar{\psi}(g_\nu) \geq \bar{\psi}(T), \ g_\nu \in [T, T + U],
\]
and
\[
\frac{\bar{\psi}(T)}{\omega} \sim \frac{1}{\pi} \bar{\psi} \ln T > \frac{1}{2\pi} \bar{\psi} \ln T \geq \left[ \frac{1}{2\pi} \bar{\psi} \ln T \right] = M_1,
\]
of course,
\[
M_1 \in (\ln T, \sqrt{\bar{\psi} \ln T})
\]
(see (2.1) - inequalities for \( M \) and (2.2)). Putting \( M = M_1 \) in (2.7) one obtains
\[
R = o(U \ln T).
\]
Now, the formula (2.4) follows from the previous by (2.5).
3. Lemmas about translations $g_\nu \rightarrow g_\nu(\tau), \tau \in [-\pi, \pi]$

3.1. Let

$$\{g_\nu(\tau)\}$$

denote the infinite set of sequences which are defined (comp. [5]) by the formula

$$\vartheta_1[g_\nu(\tau)] = \frac{\pi}{2} \nu + \frac{\tau}{2}, \quad \nu = 1, 2, \ldots, \tau \in [-\pi, \pi],$$

where, of course,

$$g_\nu(0) = g_\nu.$$

Now, we shall study how the lemmas from the papers [6], [7] are sensitive with respect to the translations

$$g_\nu \rightarrow g_\nu(\tau), g_\nu \in [T, T + U], \quad \tau \in [-\pi, \pi].$$

First of all we have (comp. [6], (22) – (36)) the following

**Lemma $\tilde{A}$.**

$$g_{\nu_1 + p + 1}(\tau) = g_{\nu_1}(\tau) + \tilde{\omega}_0 p - \tilde{\omega}_0 D(p) + O\left(\frac{U^3}{T^2 \ln T}\right),$$

where (comp. [6], (11), (12))

$$\tilde{\omega}_0 = \frac{\pi}{\ln \frac{T}{2\pi}}, \quad \frac{\pi^2}{2} \frac{1}{T \ln^3 \frac{T}{2\pi}} - \frac{\pi}{T \ln^2 \frac{T}{2\pi}},$$

$$Q = Q(T) = \frac{\pi}{T \ln^2 \frac{T}{2\pi}},$$

$$D(p) = \sum_{q=1}^{p} (1 - (1 - Q)^q), 1 \leq p \leq N_1 - 1, \quad D(0) = 0,$$

$$g_{\nu_1}(\tau) = \min_{g_\nu(\tau) \in [T, T + U]} \{g_\nu(\tau)\},$$

$$g_{\nu_1 + N_1}(\tau) = \max_{g_\nu(\tau) \in [T, T + U]} \{g_\nu(\tau)\}, \quad \nu_1 = \nu_1(\tau), \quad N_1 = N_1(\tau),$$

and the $O$ is valid uniformly for $\tau \in [-\pi, \pi]$.

3.2. Next, since (see [6], (118))

$$(3.1) \quad \tilde{\vartheta}_{1,k} = \vartheta_1[g_\nu(\tau) + k\omega] = \frac{\pi}{2} \nu + \frac{\tau}{2} + k\omega \ln P_0 + O\left(\frac{MU}{T \ln T}\right),$$

then (comp. [6], (121))

$$(3.2) \quad Z[g_\nu(\tau) + k\omega] \cdot Z[g_\nu(\tau) + l\omega] =$$

$$= 2 \sum_{m,n < P_0} \frac{1}{\sqrt{mn}} \cos \{g_\nu(\tau) \ln \frac{n}{m} + k\omega \ln \frac{P_0}{n} - l\omega \frac{P_0}{m}\} +$$

$$+ 2 \sum_{m,n < P_0} \frac{(-1)^\nu}{\sqrt{mn}} \cos \{g_\nu(\tau) \ln (mn) - \tau - k\omega \ln \frac{P_0}{n} - l\omega \frac{P_0}{m}\} +$$

$$+ O\left(\frac{MU}{\sqrt{T \ln T}}\right) + O(T^{-1/12} \ln T),$$

and the $O$-estimates in $\text{(3.1)}$, $\text{(3.2)}$ are valid uniformly for $\tau \in [-\pi, \pi]$. 

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Now, we put (see [3.2], comp. [6], (16), (17))

\[ \tilde{S}_1(T, U, M, \tau) = \sum \sum_{m<n<P_0} \frac{1}{\sqrt{mn}} \sum_{T \leq g_{\nu}(\tau) \leq T + U} \cos \left\{ g_{\nu}(\tau) \ln \frac{n}{m} + \varphi_1 \right\}, \]

where

\[ \varphi_1 = k \omega \ln \frac{P_0}{m} - l \omega \ln \frac{P_0}{n}. \]

and also (comp. [6], (19), (20))

\[ \tilde{S}_2(T, U, M, \tau) = \sum \sum_{m<n<P_0} \frac{1}{\sqrt{mn}} \sum_{T \leq g_{\nu}(\tau) \leq T + U} (-1)^{\nu} \cos \left\{ g_{\nu}(\tau) \ln (mn) + \tilde{\varphi}_2 \right\}, \]

where

\[ \tilde{\varphi}_2 = -k \omega \ln \frac{P_0}{n} - l \omega \ln \frac{P_0}{m} - \tau = \varphi_2 - \tau. \]

Since

\[ T \leq g_{\nu}(\tau) \leq T + U, \]

and

\[ \tilde{S}_2(T, U, M, \tau) = \text{Re} \left\{ e^{-i\tau} \sum \sum_{m,n<P_0} \frac{1}{\sqrt{mn}} \times \sum_{T \leq g_{\nu}(\tau) \leq T + U} (-1)^{\nu} \exp \{ i [g_{\nu}(\tau) \ln (mn) + \tilde{\varphi}_2] \} \right\} \]

then the following estimates (comp. [6], (18), (21), (37) – (93)) hold true

**Lemma B.**

\[ \tilde{S}_1(T, U, M, \tau) = \mathcal{O}(MT^{5/12} \ln^3 T) \]

uniformly for \( \tau \in [-\pi, \pi] \).

**Lemma C.**

\[ \tilde{S}_2(T, U, M, \tau) = \mathcal{O}(T^{5/12} \ln^2 T) \]

uniformly for \( \tau \in [-\pi, \pi] \).

3.3. Let (see [6], (3))

\[ \tilde{J} = \tilde{J}(T, U, M, \tau) = \sum_{T \leq g_{\nu}(\tau) \leq T + U} \left\{ \sum_{k=0}^{M} Z[g_{\nu}(\tau) + k\omega] \right\}^2. \]

Now we obtain by method [6], (94) – (127) the following

**Lemma \( \tilde{\alpha} \).**

\[ \tilde{J} = AMU \ln^2 T + o(MU \ln^2 T), \]

\((A > 0 \text{ is an absolute constant})\) uniformly for \( \tau \in [-\pi, \pi] \).

Next, we have, instead of [7], (5.1), (5.2), the following

\[ 4 \cos \tilde{\vartheta}_k \cos \tilde{\vartheta}_l \cos (\tilde{\vartheta}_k - \tilde{\vartheta}_l) = \]

\[ = 1 + (-1)^{k+l} + (-1)^{\nu+k} \cos \tau + (-1)^{\nu+l} \cos \tau + \mathcal{O}\left( \frac{MU}{T \ln T} \right), \]

\[ - 4 \cos^2 \vartheta_k = -2 - 2(-1)^{\nu+k} \cos \tau + \mathcal{O}\left( \frac{MU}{T \ln T} \right). \]

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Now, putting (comp. [7], (3.4), (3.5))
\[
\bar{N} = \sum_{T \leq g_{\nu}(\tau) \leq T + U} |\bar{K}|^2, \\
\bar{K} = \sum_{k=0}^{M} \left\{ e^{-i\theta[|g_{\nu}(\tau) + k\omega|} Z[g_{\nu}(\tau) + k\omega] - 1 \right\},
\]
we obtain by method [7], (4.1) – (7.3) the following

**Lemma \(\bar{\beta}\).**
\[
\bar{N} = \mathcal{O}(MU \ln^2 T)
\]
uniformly for \(\tau \in [-\pi, \pi]\).

4. **Two theorems connected with translations** \(g_{\nu} \rightarrow g_{\nu}(\tau), \tau \in [-\pi, \pi]\)

4.1. First of all we give (comp. [7], (2.6) – (2.8)) the following

**Definition 1.** We shall call the segment
\[
[g_{\nu}(\tau) + k(\nu)\omega, g_{\nu}(\tau) + (k(\nu) + 1)\omega]
\]
where
\[
g_{\nu}(\tau) \in [T, T + U], \tau \in [-\pi, \pi], 0 \leq k(\nu) \leq M_2 = [\delta \ln T], \delta > 1,
\]
and \(k(\nu) \in \mathbb{N}_0\) as the good segment (comp. [7], [11]) if
\[
Z[g_{\nu}(\tau) + k(\nu)\omega] \cdot Z[g_{\nu}(\tau) + (k(\nu) + 1)\omega] < 0.
\]
Next, let
\[
G_1(T, U, \delta, \tau)
\]
denote the number of non-intersecting good segments within the interval \([T, T + U]\).
Then we obtain, similarly to [7], (3.7), (3.20), the following result

**Theorem 2.** There are
\[
\delta_0 > 1, A(\psi, \delta_0) > 0, T_0(\psi, \delta_0) > 0
\]
such that
\[
(4.1) \quad G_1(T, U, \delta_0, \tau) > A(\psi, \delta_0)U, \quad T \geq T_0(\psi, \delta_0)
\]
for all \(\tau \in [-\pi, \pi]\).

**Remark 7.** We notice explicitly that the estimate [7], (2.9) concerning the number of good segments (relatively to \(\{g_{\nu}\}\)) is invariant with respect to translations
\[
g_{\nu} \rightarrow g_{\nu}(\tau), \tau \in [-\pi, \pi], g_{\nu} \in [T, T + U].
\]
4.2. Above listed facts make clear that we have obtained a kind of generalization of our Theorem 1. Namely, let 
\[ G_2(T, \psi, \bar{\psi}, \tau) \]
stand for the number of values 
\[ g_\nu(\tau) \in [T, T + U] \]
such that the interval
\[ (g_\nu(\tau), g_\nu(\tau) + \bar{\psi}_\nu(\tau)) \]
contains a zero of the odd order of the function 
\[ \zeta \left( \frac{1}{2} + it \right) . \]

Then the following theorem holds true.

**Theorem 3.**

\[ G_2(T, \psi, \bar{\psi}, \tau) \sim \frac{1}{\pi} U \ln T, \quad T \to \infty, \quad \tau \in [-\pi, \pi]. \]

5. **Remarks on Selberg’s theorems about zeros of function \( \zeta \left( \frac{1}{2} + it \right) \)**

We have given a discrete commentary to fundamental Selberg’s memoir [10] in our paper [9]. Here we put two results from our paper [9].

5.1. Let
\[ H_1 \in [a_1, a_2 \sqrt{\ln P_0}], \]
where
\[ a_1 = \frac{10}{\pi \epsilon}, \quad a_2 = a_1 \sqrt{\frac{2}{\pi}}, \quad H_1 \in \mathbb{N}. \]
The origin of (5.1) is as follows: we put
\[ \omega = \frac{\pi}{2 \ln P_0}, \quad H_1 \omega = H, \quad \xi = \left( \frac{T}{2\pi} \right)^{\epsilon/10} = P_0^{\epsilon/5}, \quad \epsilon \leq \frac{1}{10}, \]
(see [9], (9)), and further, we assume that
\[ \frac{1}{\ln \xi} \leq H \leq \frac{1}{\sqrt{\ln \xi}}. \]
(see [9], (10)).

**Definition 2.** We shall call the segment
\[ [g_\nu(\tau) + (k(\nu, \tau) - 1)\omega, g_\nu(\tau) + k(\nu, \tau)\omega], \]
where
\[ g_\nu(\tau) \in [T, T + U], \quad 1 \leq k(\nu, \tau) \leq N_1, \]
and \( k(\nu, \tau) \in \mathbb{N} \) as the **good segment** (see [9], p. 113) if
\[ Z[g_\nu(\tau) + (k(\nu, \tau) - 1)\omega] \cdot Z[g_\nu(\tau) + k(\nu, \tau)\omega] < 0. \]
Next, let
\[ G_3(T, U, H_1, \tau) \]
denote the number of non-intersecting good segments within the interval \([T, T + U]\). Then the following theorem holds true.
Theorem 4. There are 
\[ H_1 \in [a_1, a_2 \sqrt{\ln P_0}], \quad A(\epsilon) > 0, \quad T_0(\epsilon) > 0 \]
such that
\begin{equation}
G_3(T, U, \tau) > A(\epsilon) U \ln T, \quad T \geq T_0(\epsilon),
\end{equation}
where, of course,
\[ G_3(T, U, \tau) = G_3(T, U, H_1, \tau) \]

Remark 8. Since
\begin{equation}
G_3(T, U, \tau) < AU \ln T,
\end{equation}
then the order of \( G_3 \) is \( U \ln T \) for every fixed \( \tau \in [-\pi, \pi] \) (comp. (5.2), (5.3)). We shall call this property as generalized Gram’s law for the set of sequences \( \{g_{\nu}(\tau)\} \).

Now, we obtain the following from our Theorem 4.

Corollary.
\[ N_0(T + T^{1/2+\epsilon}) - N_0(T) > A(\epsilon) T^{1/2+\epsilon} \ln T, \]
i.e. the Selberg’s theorem A (see [10], p. 46, \( a = 1/2 + \epsilon \)).

Remark 9. Consequently, the generalized Gram’s law is the discrete basis of fundamental Selberg’s theorem A.

Remark 10. Our Theorem 4 is also not improvable by the Karatsuba theory.

Remark 11. Finally, we mention that our Theorem 3 is valid also for the intervals of the following form
\[ \left( g_{\nu}(\tau), g_{\nu}(\tau) + \frac{\psi[g_{\nu}(\tau)]}{\ln g_{\nu}(\tau)} \right), \quad g_{\nu}(\tau) \in [T, T + T^{1/2+\epsilon} \ln T], \]
(comp. [9], p. 112). i.e. for our system of exceptional sets, where
\[ a_1 \leq \left[ \frac{\psi(\tau)}{2\pi} \right] \leq a_2 \sqrt{\ln P_0}, \]
as a discrete analogue of the Selberg’s theorem C.

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