Homoclinic Points and Homoclinic Orbits for the Quadratic Family of Real Functions with Two Parameters

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Abstract

In this work, we study the homoclinic points and homoclinic orbits of the family of real functions with two parameters

$$H = \{ h_{a,b}(x) = ax^2 + b : a > 0, b \in \mathbb{R} \}.$$ We show that the function $h_{a,b} \in H$ has no homoclinic points for $b > \frac{2}{a}$, but has a homoclinic point for $b \leq \frac{-2}{a}$. Also, we prove that $h_{a,b} \in H$ has homoclinic orbits for $b \leq \frac{-2}{a}$.

1. Introduction

Recently, we witness that there is a clear attention in connecting orbits (homoclinic or heteroclinic orbits) in dynamical systems [1]. The notion of a homoclinic point was first introduced by Poincaré (1890) [2], in the study of a three-body problem. After about 70 years, Smale (1963, 1976) [3], presented the horseshoe notion to show the chaotic behavior of higher-dimensional systems with a transverse homoclinic point. Homoclinic points have been used to study the dynamics of two- or higher-dimensional dynamical systems. Devaney (1989) in [4], showed that the same result holds for a one-dimensional map of an interval into itself with a nondegenerate homoclinic point. Block and Coppel (1992) [5],
proved that for one-dimensional map \( f \) of an interval into itself, a homoclinic point leads to horseshoe for \( f^2 \). Recently, Li (2002) gave a simple proof of Block and Coppel result proof. The concept of homoclinic orbits and heteroclinic connections plays a central role in the studying the chaotic sets. In particular, it can be used for proof of the existence of chaos. In fact, it is proved in [6] [7] [8], the existence of a non-degenerate homoclinic orbit to an expanding fixed point of a smooth map \( f \) implies the existence of an invariant set in a neighborhood of the homoclinic orbit, on which \( f \) is chaotic. The same result is also true for non-degenerate heteroclinic connections. Homoclinic orbits and heteroclinic connections are relevant not only for the proof of the existence of chaos but also for the description of several bifurcations of chaotic attractors (crisis bifurcations) [9]. In this work, we find homoclinic points and homoclinic orbits for two-parameters family of real functions \( H = \{ h_{a,b} (x) = ax^2 + b : a > 0, b \in \mathbb{R} \} \). In Section 2, we study the fixed points of the family \( H = \{ h_{a,b} (x) = ax^2 + b : a > 0, b \in \mathbb{R} \} \) where \( h_{a,b} : \mathbb{R} \to \mathbb{R} \), and the nature of this fixed point for various values \( a \) and \( b \). In Section 3, we study the local unstable sets of the repelling fixed point \( P \) for the functions \( h_{a,b} (x) \in H \). We proved that \( w^u (P) = \left( \frac{1}{2a}, \infty \right) \). And in Section 4, we study the unstable sets of the repelling fixed points \( P \) for the functions \( h_{a,b} (x) \in H \) we proved that \( w^u (P) = \left( \frac{1}{a} - P_1, \infty \right) \). Finally, in Section 5, we study the homoclinic points and homoclinic orbits of the function \( h_{a,b} (x) \in H \) for the repelling fixed point \( P \). We show that \( h_{a,b} (x) \in H \) has a homoclinic point and orbit whenever \( b \leq \frac{-2}{a} \), and has no homoclinic point and orbit whenever \( b > \frac{-2}{a} \). We need some important definitions.

**Definition 1:** Let \( P \) be a repelling fixed point for a map \( f : I \to I \) on a compact interval \( I \subset R \). Then the unstable set of \( P \) is defined as \( w^u (P) = \{ x \in I : \lim_{n \to \infty} f^{-n}(x) = P \} \) [10], which equivalent to the expansivity \( \left| f(x) - f(P) \right| > \left| x - P \right|, \forall x \in I, x \neq P \) [4].

**Definition 2:** Let \( P \) be a repelling fixed point for a map \( f : I \to I \) on a compact interval \( I \subset R \) and \( U \) be an open interval near \( P \). Then, the local unstable set \( w^u_{loc} (P) \) of \( P \) is defined as \( w^u_{loc} (P) = \{ x \in U : \lim_{n \to \infty} f^{-n}(x) = P \} \) [10]. Which equivalent to the expansivity \( \left| f(x) - f(P) \right| > \left| x - P \right|, \forall x \in U, x \neq P \) [11], and equivalent to \( \left| \frac{df}{dx} \right| > 1 \) for any \( x \in U \) [6].

**Definition 3:** Let \( P \) be fixed point and \( \dot{f}(P) > 1 \) for a map \( f : I \to I \) on a compact interval \( I \subset R \). A point \( q \) is called homoclinic point to \( P \) if \( q \in w^u_{loc} (P) \) and there exists \( n > 0 \) such that \( f^n(q) = P \) [4].

Note that the sequence of images of a homoclinic point \( q \) and a suitable sequence of preimages of \( q \) consist of points which are also homoclinic, and both these sequences converge to \( P \). The union of these sequences
\[ O(P) = \{ \ldots, q_1, q_{i-1}, q_i, q_{i+1}, \ldots, q_m \} \quad \text{where} \quad q_0 = q, q_{i+1} = f(q_i) \quad \text{for} \quad i \leq m-1, \]
\[ q_m = P \quad \text{and} \quad \lim_{i \to \infty} q_i = P, \]
\( \text{is called the homoclinic orbit of} \ P \[12\].

2. The Fixed Points of the Function \( h_{a,b}(x) \in H \) and Their Nature

In this section we study the fixed points of the family
\[ H = \{ h_{a,b}(x) = ax^2 + b : a > 0, b \in \mathbb{R} \} \quad \text{where} \quad h_{a,b} : \mathbb{R} \to \mathbb{R}, \]
and the nature of these fixed points for various values \( a \) and \( b \). It is clear that the fixed points of \( h_{a,b}(x) \) are 
\[ P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a} \quad \text{and} \quad P_2 = \frac{1 - \sqrt{1 - 4ab}}{2a}. \]
The graphs show that the function \( h_{a,b}(x) \) has no fixed point for \( b > \frac{1}{4a} \), has a unique indifferent fixed point for \( b = \frac{1}{4a} \), and has two fixed points for \( b < \frac{1}{4a} \). See Figure 1.

The fixed point \( P_1 \) is always repelling for \( b < \frac{1}{4a} \). But the fixed point \( P_2 \) is attracting for \( \frac{-3}{4a} < b < \frac{1}{4a} \), indifferent for \( b = \frac{-3}{4a} \) and is repelling for \( b < \frac{-3}{4a} \). See Figure 2.

We need the following remarks in our work.

2.1. Remark

For \( b < \frac{1}{4a} \), \( h_{a,b}'(P_1) > 1 \) and for \( b < \frac{-3}{4a} \), \( h_{a,b}'(P_2) < -1 \).

Proof:

It is clear that \( h_{a,b}'(x) = 2ax \), so for \( P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a} \),
\[ h_{a,b}'(P_1) = 1 + \sqrt{1 - 4ab}. \]
If \( b < \frac{1}{4a} \), then \( 1 - 4ab > 0 \), \( \sqrt{1 - 4ab} > 0 \), is defined and
\[ h_{a,b}'(P_1) = 1 + \sqrt{1 - 4ab} > 1. \]
Now for the fixed point \( P_2 = \frac{1 - \sqrt{1 - 4ab}}{2a} \),
\[ h_{a,b}'(P_2) = 1 - \sqrt{1 - 4ab}. \]
If \( b < \frac{-3}{4a} \), then \( 1 - 4ab > 2 \), thus \( \sqrt{1 - 4ab} < -2 \),
there for \( h_{a,b}'(P_2) = 1 - \sqrt{1 - 4ab} < -1 \).

According to the definition of the homoclinic points, we consider the repelling fixed point \( P_1 \) and we omit the repelling fixed point \( P_2 \).

2.2. Remark

For \( h_{a,b}(x) \in H \) with \( b < \frac{1}{4a} \), the fixed point \( P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a} > \frac{1}{2a} \).

Proof:

Let \( b < \frac{1}{4a} \). Then \( \sqrt{1 - 4ab} > 0 \) is defined and thus, \( 1 + \sqrt{1 - 4ab} > 1 \). There for \( P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a} > \frac{1}{2a} \).
3. The Local Unstable Sets

In this section, we study the local unstable sets of the repelling fixed point $P_1$ for the functions $h_{a,b}(x) \in H$.

**Proposition**

For $h_{a,b}(x) \in H$ the local unstable set of the fixed point $P_1$ is $w^u_{loc}(P_1) = \left(\frac{1}{2a}, \infty\right)$.

**Proof:**

To find the local unstable for the repelling fixed point $P_1$ we consider the inequality $|h'_{a,b}(P_1)| > 1$ [6]. Since $h_{a,b}$ is a continuous function, then there exists a neighborhood $U$ of the fixed point $P_1$ such that $|h'_{a,b}(x)| > 1, \forall x \in U$, i.e. $|2ax| > 1, \forall x \in U$. Thus $|x| > \frac{1}{2a} = \frac{1}{2a}$. Then either $x < -\frac{1}{2a}$, i.e. $x \in \left(-\infty, -\frac{1}{2a}\right)$, or $x > \frac{1}{2a}$, i.e. $x \in \left(\frac{1}{2a}, \infty\right)$. By remark (2.2) $P_1 > \frac{1}{2a}$, there for $w^u_{loc}(P_1) = \left(\frac{1}{2a}, \infty\right)$.

4. The Unstable Sets

The unstable sets of the repelling fixed points $P_1$ for the functions $h_{a,b}(x) \in H$ is calculated in the following proposition.

We need the following lemma in our studying.
4.1. Lemma

For \( h_{a,b}(x) \in H \), \( h_{a,b}^{-1}(P_1) = \frac{P_1 - b}{a} = \mp P_1 \) where \( P_1 > b \).

The proof is clear.

4.2. Proposition

For \( h_{a,b}(x) \in H \) the unstable set of the fixed point \( P_1 \) is \( w^u(P_1) = \left( \frac{1}{a} - P_1, \infty \right) \).

\[ \text{Proof:} \]

Consider the expansive inequality \[ |h_{a,b}(x) - P_1| > |x - P_1| \] where \( x \neq P_1 \).

Then for \( h_{a,b}(x) \in H \), we have \[ |ax^2 + b - P_1| > |x - P_1| \], thus

\[ a|x + \sqrt{(P_1 - b)/a}|x + \sqrt{(P_1 - b)/a}| |x - P_1| > |x - P_1|. \] By lemma (4.1). Thus we have

\[ a|x - P_1||x + P_1| > |x - P_1|. \] There for \( |x + P_1| > \frac{1}{a} \).

Either \( x + P_1 < \frac{1}{a} \) which implies \( x < \frac{1}{a} - P_1 \). Or \( x + P_1 > \frac{1}{a} \), which implies \( x > \frac{1}{a} - P_1 \). That is \( x \in \left(-\infty, \frac{1}{a} - P_1\right) \cup \left(\frac{1}{a} - P_1, \infty\right) \). To calculate \( w^u(P_1) \), by remark (2.2), \( P_1 > \frac{1}{2a} > 0 \). Now \( \left(-\infty, \frac{1}{a} - P_1\right) \subseteq \mathbb{R}^- \), which implies \( P_1 \notin \left(-\infty, \frac{1}{a} - P_1\right) \). So \( w^u(P_1) = \left(\frac{1}{a} - P_1, \infty\right) \).

4.3. Example

For \( h_{a,b}(x) = x^2 - 6 \), then \( w^u(P_1) = (-2, \infty) \).

\[ \text{Solution:} \]

It is clear that \( P_1 = 3 \). Since \[ |h_{a,b}(x) - P_1| > |x - P_1| \], where \( x \neq \mp P_1 \), implies that \[ |x^2 - 6 - 3| > |x - 3| \], then \[ |x^2 - 9| > |x - 3| \]. Thus \[ |(x - 3)(x + 3)| > |x - 3| \]. Since \( x \neq 3 \), then \( |x + 3| > 1 \), there for either \( x + 3 < -1 \), then \( x < -4 \), i.e. \( x \in (-\infty, -4) \).

Or \( x + 3 > 1 \), then \( x > -2 \), i.e. \( x \in (-2, \infty) \). Thus \( x \in (-\infty, -4) \cup (-2, \infty) \).

But \( P_1 = 3 \in (-2, \infty) \), then \( w^u(P_1) = \left(\frac{1}{3} - 3, \infty\right) \).

5. Homoclinic Points and Homoclinic Orbits for the Family \( H \)

In this section, we study the homoclinic points and homoclinic orbits of the function \( h_{a,b}(x) \in H \) for the repelling fixed point \( P_1 = \frac{1 + \sqrt{4ab - 4a}}{2a} \) where \( b < \frac{1}{4a} \), note that the other repelling fixed point \( P_2 = \frac{1 - \sqrt{4ab - 4a}}{2a} \). By remark (2.1), \( h'_{a,b}(P_2) < -1 \) where \( b < \frac{-3}{4a} \). Thus according to the definition of the homoclinic point, we don’t study the homoclinic points of \( P_2 \).
5.1. Homoclinic Points for the Functions $h_{a,b}(x) \in H$

To study the homoclinic points of the repelling fixed point $P_1$ of $h_{a,b}(x)$ we use the following technique concerned on the preimages of $h_{a,b}(x)$.

5.1.1. Remark

For $h_{a,b}(x) \in H$, $h_{a,b}^{-2}(P_1) = \mp \sqrt{-\frac{P_1 - b}{a}}$.

Proof:

By lemma (4.1), $h_{a,b}^{-1}(P_1) = \mp P_1$. Then $h_{a,b}^{-2}(P_1) = h_{a,b}^{-1}(\pm P_1) = \mp \sqrt{\frac{\pm P_1 - b}{a}} = \pm P_1$, so we have the same state, so is omitted.

For $h_{a,b}^{-2}(P_1) = h_{a,b}^{-1}(-P_1) = \mp \sqrt{-\frac{P_1 - b}{a}}$.

Now if $-P_1 \geq b$, in remark (5.1.1), then put $h_{a,b}^{-2}(P_1) = \mp \sqrt{-\frac{P_1 - b}{a}} = \mp q_{1,j}$ where $j \in N$. For $h_{a,b}^{-1}(q_{1,j}) = h_{a,b}^{-2}(P_1) = \mp \sqrt{\frac{\pm q_{1,j} - b}{a}} = \pm q_{2,j}$, where $\pm q_{1,j} > b$.

Similarly for $h_{a,b}^{-3}(\pm q_{2,j}) = h_{a,b}^{-4}(P_1) = \mp \sqrt{\frac{\pm q_{2,j} - b}{q}} = \mp q_{3,j}$, where $\pm q_{2,j} > b$.

And so on (see Figure 3). Thus we have sequences of preimage points of $h_{a,b}(x)$, $q_{i,j} \in \mathbb{R}$. If $\mp q_{i,j} \in w_{a,b}^\pm(P_1)$, for some $i,j \in N$, then these points are the homoclinic points of the repelling fixed point $P_1$.

In following propositions and examples we study the homoclinic points of the fixed point $P_1$ of $h_{a,b}(x) \in H$ for various values of $a$ and $b$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Tree of the homoclinic points of $h_{a,b}(x)$.}
\end{figure}
5.1.2. Proposition
For $h_{a,b}(x) \in H$. If $b = 0$ then the fixed point $P_1$ has no homoclinic points.

Proof:
It is clear that $h_{a,0}(x) = ax^2$ and $P_1 = \frac{1}{a}$. By proposition (3.1),
$$w^\infty_{\text{loc}}(P_1) = \left(\frac{1}{2a}, \infty\right).$$
Now the first preimage of $h_{a,0}(x)$ is $h_{a,0}^{-1}(x) = \frac{x}{\sqrt{a}}$ where $x > 0$. For $P_1 = \frac{1}{a}$ and by lemma (4.1), we have
$$h_{a,0}^{-1}(\frac{1}{a}) = \pm \sqrt{\frac{1}{a}} = \pm \frac{1}{a} = \mp P_1.$$ But $\mp P_1 = \frac{1}{a}$ is a fixed point, and
$$-P_1 = -\frac{1}{a} \not\in w^\infty_{\text{loc}}(P_1) = \left(\frac{1}{2a}, \infty\right).$$
Moreover, by proposition (5.1.1), we have
$$h_{a,0}^{-2}(P_1) = \mp \sqrt{-\frac{P_1}{a}} = \pm \sqrt{-\frac{1}{a}} \not\in \mathbb{R},$$
so for $n \geq 2$. Thus $h_{a,0}(x) = ax^2$ has no homoclinic points to the fixed point $P_1$.

5.1.3. Example
For $h_{0}(x) = x^2$ has no homoclinic points to the fixed point $P_1 = 1$.

Solution:
By proposition (3.1), $w^\infty_{\text{loc}}(P_1) = \left(\frac{1}{2}, \infty\right)$. The first preimage of $h_{0}(x)$ is
$$h_{0}^{-1}(x) = \mp \sqrt{x}$$
where $x > 0$. For $h_{0}^{-1}(1) = \mp \sqrt{1} = \mp P_1$. But $P_1 = 1$ is a fixed point of $h_{0}(x)$, and $-1 \not\in w^\infty_{\text{loc}}(P_1)$. Now for $h_{0}^{-1}(-1) = h_{0}^{-1}(1) = \mp \sqrt{1} \not\in \mathbb{R}$. So $h_{0}(x) = x^2$ has no homoclinic points to the fixed point $P_1$.

5.1.4. Proposition
For $h_{a,b}(x) \in H$, the first preimage of the fixed point $P_1$ cannot be a homoclinic point to the fixed point $P_1$.

Proof:
The first preimage of $h_{a,b}(x)$ is $h_{a,b}^{-1}(x) = \mp \sqrt{\frac{x-b}{a}}$ where $x > b$. For
$$P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a}$$
and by lemma (4.1) we have $h_{a,b}^{-1}(P_1) = \mp \sqrt{\frac{P_1 - b}{a}} = \mp P_1$ where $P_1 > b$. But $\mp P_1$ is a fixed point, by remark (2.2) $P_1 > \frac{1}{2a}$, we have $-P_1 < \frac{1}{2a}$ and
$$w^\infty_{\text{loc}}(P_1) = \left(\frac{1}{2a}, \infty\right).$$
So $-P_1 \not\in w^\infty_{\text{loc}}(P_1) = \left(\frac{1}{2a}, \infty\right)$. Thus the fixed point $P_1$ has no homoclinic points at the first preimage.

5.1.5. Example
$h_{1,-2}(x) = x^2 - 2$ has no homoclinic points to the fixed point $P_1$ in the first
preimage.

**Solution:**
The fixed point \( P_1 = 2 \), and the first preimage of \( h_{-2}(x) \) is
\[
\begin{align*}
    h_{-2}^{-1}(x) &= \pm \sqrt{x + 2}.
\end{align*}
\]
From proposition (3.1), \( w_{loc}^u(P_1) = \left\{ \frac{1}{2}, \infty \right\} \). For \( P_1 = 2 \), then \( h_{-2}^{-1}(2) = \pm \sqrt{2+2} = \pm 2 = \pm P_1 \). But \( P_1 = 2 \) is a fixed point of \( h_{-2}(x) \) and \(-2 \notin w_{loc}^u(P_1)\). So \( h_{-2}(x) = x^2 - 2 \) has no homoclinic points to the fixed point \( P_1 \) in the first preimage.

Following remarks assert that some inverse images of \( P_1 \), \( h^{-n}_{a,b}(P_1) \) are belong to the local unstable set of \( P_1 \) for \( b \leq \frac{-2}{a} \).

5.1.6. Remark
If \( b < -\left(\frac{5 + 2\sqrt{5}}{4a}\right) \) of \( h_{a,b} \in H \), then the second preimage of the fixed point \( P_1 \) belongs to the local unstable set of \( P_1 \).

**Proof:**
Let \( b < -\frac{5-2\sqrt{5}}{4a} \), it is clearly that \( b < -\frac{5+2\sqrt{5}}{4a} \). There for
\[
\begin{align*}
    ab + \frac{5+2\sqrt{5}}{4} < 0 \quad \text{and} \quad ab + \frac{5-2\sqrt{5}}{4} < 0.
\end{align*}
\]
Then \( 4a^2b^2 + 10ab + \frac{5}{4} > 0 \), thus \( 1 - 4ab < 4a^2b^2 + 6ab + \frac{9}{4} \), which implies
\[
\begin{align*}
    1 - 4ab < \left(2ab + \frac{3}{2}\right)^2. \quad \text{Since} \quad \left(2ab + \frac{3}{2}\right)^2 = \left(-2ab + \frac{3}{2}\right)^2, \text{there for}
\end{align*}
\]
\[
\begin{align*}
    1 - 4ab < \left(-2ab - \frac{3}{2}\right)^2, \quad \text{then} \quad \sqrt{1 - 4ab} < -2ab - \frac{3}{2}.
\end{align*}
\]
Then
\[
\begin{align*}
    \frac{1}{2a} + \sqrt{1 - 4ab} < -b - \frac{1}{4a}, \quad \text{i.e.} \quad P_1 < -b - \frac{1}{4a}. \quad \text{There for} \quad \sqrt{-\frac{P_1 - b}{a}} > \frac{1}{2a}. \quad \text{By remark (5.1.1)} \quad h^{-2}_{a,b}(P_1) = \sqrt{-\frac{P_1 - b}{a}}, \quad \text{So} \quad h^{-2}_{a,b}(P_1) > \frac{1}{2a}. \quad \text{Thus} \quad h^{-2}_{a,b}(P_1) \in w^u_{loc}(P_1). \quad \text{(See proposition (3.1)).}
\end{align*}
\]

5.1.7. Remark
If \( \frac{-(5+2\sqrt{5})}{4a} \leq b \leq \frac{-2}{a} \) of \( h_{a,b} \in H \), then the third preimage of the fixed point \( P_1 \) belongs to the local unstable set of \( P_1 \).

The proof is same of the above remark with some more complicated details.

5.1.8. Theorem
For the family \( H = \left\{ h_{a,b}(x) = ax^2 + b \right\} \), there exist homoclinic points to the fixed point \( P_1 \) whenever \( b \leq \frac{-2}{a} \).

**Proof:**
According to a proposition (5.1.4), we begin with the second preimage of the fixed point $P_1$. By remark (5.1.1), we have

$$h^{-2}_{a,b}(P_1) = \pm \sqrt{\frac{P_1 - b}{a}}.$$ Suppose that $\pm \frac{P_1 - b}{a} \in \mathbb{R}$, i.e. $-P_1 \geq b$, then

$$P_1 \leq -b.$$ Since $P_1 = \frac{1 + \sqrt{1-4ab}}{2a}$, where $b < \frac{1}{4a}$, then $\frac{1 + \sqrt{1-4ab}}{2a} \leq -b$, which implies

$$\sqrt{1-4ab} \leq -2ab - 1.$$ Let $c = -2ab - 1$. Since $b < \frac{1}{4a}$, then $\sqrt{1-4ab} > 0$, which implies $0 < \sqrt{1-4ab} \leq c$, i.e. $c \in (0, \infty)$ ....(*)

Then we have three cases for $b$

**Case 1:** If $b > 0$, then $ab > 0$, thus $-2ab < 0$, there for $-2ab - 1 < -1$, which implies $c < -1$ which is a contradiction with $c \in (0, \infty)$ in (*).

**Case 2:** If $b = 0$, then by proposition (5.1.2) there is no homoclinic points to the fixed point $P_1$.

**Case 3:** If $b < 0$, then $ab < 0$. Thus $-2ab > 0$, there for $-2ab - 1 > -1$, which implies $c > -1$, i.e. $c \in (-1, \infty)$. Since $c \in (0, \infty)$ in (*), then $c \in (-1, \infty) \cap (0, \infty) = (0, \infty)$. There for $\sqrt{1-4ab} \leq -2ab - 1$, implies that $1 - 4ab \geq 4a^2b^2 + 4ab + 1$, thus $4ab(ab + 2) \geq 0$ ....(**).

Since $a > 0$ and $b < 0$, then $4ab < 0$, so by (**), $ab + 2 \leq 0$, thus $ab \leq -2$.

It follows that $b \leq -\frac{2}{a}$. By remark (2.1), $h'_{a,b}(P_1) > 1$.

Now:

1) For $b < -\left(\frac{5 + 2\sqrt{5}}{4a}\right)$, let $h^{-2}_{a,b}(P_1) = q_{1,1}$, by remark (5.1.6), $q_{1,1} \in w_{\infty}(P_1)$ and it is clear that $h^2_{a,b}(q_{1,1}) = P_1$ (see Figure 3).

2) For $\frac{-5 + 2\sqrt{5}}{4a} \leq b \leq -\frac{2}{a}$, let $h^{-3}_{a,b}(P_1) = q_{2,1}$, by remark (5.1.7), $q_{2,1} \in w_{\infty}(P_1)$ and it is clear that $h^3_{a,b}(q_{2,1}) = P_1$ (see Figure 3).

There for $q_{1,2}$ for $\frac{-5 + 2\sqrt{5}}{4a} \leq b \leq -\frac{2}{a}$ and $q_{1,1}$ for $b < -\left(\frac{5 + 2\sqrt{5}}{4a}\right)$ are the first homoclinic points for the repelling fixed point $P_1$.

**5.1.9. Theorem**

If $b > -\frac{2}{a}$, then $h_{a,b}(x) = ax^2 + b$ has no homoclinic points to the fixed point $P_1$.

**Proof:**

According to a proposition (5.1.4), we begin with the second preimage of the fixed point $P_1$. By remark (5.1.1), we have

$$h^{-2}_{a,b}(P_1) = \pm \sqrt{\frac{P_1 - b}{a}}.$$ Since $P_1$ is a repelling fixed point for $b < \frac{1}{4a}$. So, with
our assumption, we have \(-\frac{2}{a} < b < \frac{1}{4a}\). Now we divide the proof into three cases:

**Case 1:** If \(-\frac{2}{a} < b < 0\), then \(-2 < ab < 0\). Thus \(0 < ab + 2 < 2\), and \(-8 < 4ab < 0\). So \(4ab(2b + 2) < 0\), which implies \(4a^2b^2 + 4ab + 1 < 1 - 4ab\). Thus \((2ab + 1)^2 < 1 - 4ab \quad \ldots \ldots \quad (*)\).

Since \((2ab + 1)^2 = \left(- (2ab + 1)\right)^2\) and \(-\frac{2}{a} < b < 0\), then \(1 - 4ab > 0\). Thus (*) will being \(-2ab - 1 < \sqrt{1 - 4ab}\), which implies, \(-b < \frac{1 + \sqrt{1 - 4ab}}{2a}\), i.e. \(-b < P_1\), which implies \(-\frac{P_1 - b}{a} < 0\). So \(h^{-2}_{a,b}(P_1) = \pm \sqrt{\frac{-P_1 - b}{a}} \not\in \mathbb{R}\). There for \(h^{-n}_{a,b}(P_1) \not\in \mathbb{R}, \forall n \in \mathbb{N}\). Then \(h_{a,b}(x)\) has no homoclinic points to the fixed point \(P_1\) for \(-\frac{2}{a} < b < 0\).

**Case 2:** If \(b = 0\), then by proposition (5.1.2) there is no homoclinic point to the fixed point \(P_1\).

**Case 3:** If \(0 < b < \frac{1}{4a}\), then \(0 < ab < \frac{1}{4}\). Thus \(2 < ab + 2 < \frac{9}{4}\), and \(0 < 4ab < 1\). So \(4ab(2b + 2) > 0\), which implies \(4a^2b^2 + 4ab + 1 > 1 - 4ab\), thus \((2ab + 1)^2 > 1 - 4ab \quad \ldots \ldots \quad (*)\), Since \((2ab + 1)^2 = \left(- (2ab + 1)\right)^2\) and \(0 < b < \frac{1}{4a}\), then \(1 - 4ab > 0\). Thus (*) will being \(-2ab - 1 > \sqrt{1 - 4ab}\), which implies, \(-b > \frac{1 + \sqrt{1 - 4ab}}{2a}\), i.e. \(-b > P_1\), which implies \(-\frac{P_1 - b}{a} > 0\). So \(h^{-2}_{a,b}(P_1) = \pm \sqrt{\frac{-P_1 - b}{a}} \not\in \mathbb{R}\). But by theorem (5.1.8), if \(\pm \sqrt{\frac{-P_1 - b}{a}} \in \mathbb{R}\), then \(b \leq \frac{-2}{a}\) which is a contradiction with \(0 < b < \frac{1}{4a}\). There for \(h_{a,b}(x)\) has no homoclinic points to the fixed point \(P_1\) for \(0 < b < \frac{1}{4a}\).

Following examples explain the cases for \(b < \frac{-2}{a}\), \(b = \frac{-2}{a}\) and \(b > \frac{-2}{a}\) respectively.

**5.1.10. Example**

For \(h_{2,-3}(x) = 2x^3 - 3\), has a homoclinic point to the fixed point \(P_1\) at (0.86602540378).

**Solution:**

The fixed point \(P_1 = \frac{3}{2} = 1.5\) and the first preimage of \(h_{2,-3}(x)\) is \(h^{-1}_{2,-3}(x) = \frac{3 + x}{2}\). From proposition (3.1), \(u_{loc}^{e}(P_1) = \left(\frac{1}{4}, \infty\right)\). Clearly
For \( h'_{1,2} > 1 \). For \( P_1 = 1.5 \), then \( h^{-1}_{2,3} (1.5) = \pm \sqrt{\frac{1.5+3}{2}} = \pm 1.5 = \pm P_1 \). But \( P_1 = 1.5 \) is a fixed point of \( h_{2,3} (x) \), and \(-1.5 \notin w_{\text{loc}}^{u} (P_1) \). Now for

\[
h^{-1}_{2,3} (-1.5) = h^{-1}_{2,3} (1.5) = \pm \sqrt{\frac{1.5+3}{2}} = \pm 0.86602540378,
\]

where

\( 0.86602540378 \in w_{\text{loc}}^{u} (P_1) \) and \(-0.86602540378 \notin w_{\text{loc}}^{u} (P_1) \). Moreover \( h_{2,3} (0.86602540378) = -1.5 \) and \( h^{-1}_{1,2} (0.86602540378) = 1.5 \) So \( (0.86602540378) \) is a homoclinic point to the fixed point \( P_1 \).

5.1.11. Remark

Here we consider \( 0.86602540378 \) (the first) homoclinic point for the fixed point \( 1.5 \). In fact, there are many points belong to the local unstable set of \( 1.5 \) (i.e. homoclinic points to \( P_1 = 1.5 \)). In fact

\[
h^{-1}_{2,3} (0.86602540378) = h^{-1}_{2,3} (1.5) = \pm \sqrt{0.86602540378 + \frac{3}{2}} = \pm 1.390328271.
\]

Now, \( 1.390328271 \in w_{\text{loc}}^{u} (P_1) \), and \(-1.390328271 \notin w_{\text{loc}}^{u} (P_1) \). For

\[
h^{-1}_{2,3} (-0.86602540378) = h^{-1}_{2,3} (1.5) = \pm \sqrt{-0.86602540378 + \frac{3}{2}} = \pm 1.032950772,
\]

and

\( 1.032950772 \in w_{\text{loc}}^{u} (P_1) \), and \(-1.032950772 \notin w_{\text{loc}}^{u} (P_1) \). If we continue with this way, we get a set \( \{0.86602540378, 1.390328271, 1.032950772, \ldots\} \). Every point in this set belongs to \( w_{\text{loc}}^{u} (P_1) \). Each point of this set is a homoclinic point to the fixed point \( P_1 = 1.5 \). See Figure 4.

5.1.12. Example

For \( h_{1,2} (x) = x^2 - 2 \), \( \sqrt{2} \) is a homoclinic point to the fixed point \( P_1 \).

Solution:

It is clear that \( P_1 = 2 \), and the first preimage of \( h_{1,2} (x) \) is

\[
h^{-1}_{1,2} (x) = \pm \sqrt{x + 2}.
\]

From proposition (3.1), \( w_{\text{loc}}^{u} (P_1) = \left( \frac{1}{2}, \infty \right) \). Clearly \( h'_{1,2} (2) > 1 \). For \( P_1 = 2 \), then \( h^{-1}_{1,2} (2) = \pm \sqrt{2 + 2} = \pm 2 = \pm P_1 \). But \( P_1 = 2 \) is a fixed point of \( h_{1,2} (x) \), and \(-2 \notin w_{\text{loc}}^{u} (P_1) \). Now for \( h^{-1}_{1,2} (-2) = h^{-1}_{1,2} (2) = \pm \sqrt{-2 + 2} = \pm 0 \in w_{\text{loc}}^{u} (P_1) \). Now for \( h^{-1}_{1,2} (0) = h^{-1}_{1,2} (2) = \pm \sqrt{0 + 2} = \pm \sqrt{2} \).

\( \sqrt{2} \in w_{\text{loc}}^{u} (P_1) \) and \(-\sqrt{2} \notin w_{\text{loc}}^{u} (P_1) \). Moreover \( h_{1,2} (\sqrt{2}) = 0 \), \( h_{1,2} (\sqrt{2}) = -2 \) and \( h_{1,2} (\sqrt{2}) = 2 \). So \( \sqrt{2} \) is a homoclinic point to the fixed point \( P_1 = 2 \).

In fact, any preimage point contained in \( w_{\text{loc}}^{u} (P_1) \) is a homoclinic point for \( P_1 \). See Figure 5.

5.1.13. Example

\( h_{-1,2} (x) = x^2 - 1 \), has no homoclinic points to the fixed point \( P_1 \).
Figure 4. Tree of the homoclinic points of $h_{-1}(x) = 2x^2 - 3$.

Figure 5. Tree of the homoclinic points of $h_{-1}(x) = x^2 - 2$.

Solution:

It is clear that $P_1 = \frac{1 + \sqrt{5}}{2} = 1.618033989$; and the first preimage of $h_{1,0}(x)$ is $h_{1,-1}^{-1}(x) = \mp \sqrt{x + 1}$.

From proposition (3.1), $w_{\ locus}^{in}(P_1) = \left[\frac{1}{2}, \infty\right]$. For $P_1 = 1.618033989$, then $h_{1,-1}^{-1}(1.618033989) = \mp \sqrt{1.618033989 + 1} = \pm 1.618033989 = \pm P_1$. But $+P_1 = 1.618033989$ is a fixed point of $h_{1,0}(x)$, and $-1.618033989 \not\in w_{\ locus}^{in}(P_1)$.

Now for $h_{1,-1}^{-1}(-1.618033989) = h_{1,-1}^{-1}(1.618033989) = \mp \sqrt{-1.618033989 + 1}$. So $h_{1,-1}(x) = x^2 - 1$ has no homoclinic points to the fixed point $P_1$.

5.2. Homoclinic Orbits for $h_{a,b}(x) \in H$

In this part we study the homoclinic orbits for the family

$H = \{h_{a,b}(x) = ax^2 + b : a > 0, b \in \mathbb{R}\}$.

5.2.1. Remark

For $b > -\frac{2}{a}$, we proved in theorem (5.1.9), $h_{a,b}(x)$ has no homoclinic points,
so \( h_{a,b}(x) \) has no homoclinic orbits for the repelling fixed point \( P_1 \).

To study the homoclinic orbits of \( h_{a,b}(x) \) for \( b \leq -\frac{2}{a} \), we introduce the following theorems and lemmas.

### 5.2.2. Lemma

For \( h_{a,b}(x) \in H \) with \( b \leq -\frac{2}{a} \), if \( q < c \) where \( c \in \mathbb{R} \) is a constant and \( q \) is a homoclinic point to \( P_1 \), then \( P_2 + \frac{1}{a} < c \).

**Proof:**

Let \( b \leq -\frac{2}{a} \), then \( \sqrt{1-4ab} \geq 3 \), which implies \( P_2 = \frac{1-\sqrt{1-4ab}}{2a} \leq -\frac{1}{a} \), thus \( P_2 + \frac{1}{a} \leq 0 \).

Now since \( q \in w_{\infty}(P_1) \), then \( q > \frac{1}{2a} > 0 \) (see proposition (3.1)). So it is clearly \( P_2 + \frac{1}{a} < q \), thus \( P_2 + \frac{1}{a} < c \).

### 5.2.3. Lemma

\[
(P_1 + P_2 + \frac{1}{a})c - P_1(P_2 + \frac{1}{a}) = \frac{2c - b - P_1}{a},
\]

where \( c \in \mathbb{R} \) is a constant.

**Proof:**

Since \( P_1 = \frac{1+\sqrt{1-4ab}}{2a} \) and \( P_2 = \frac{1-\sqrt{1-4ab}}{2a} \).

\[
(P_1 + P_2 + \frac{1}{a})c - P_1(P_2 + \frac{1}{a})
= \left(1+\frac{\sqrt{1-4ab}}{2a} + \frac{1-\sqrt{1-4ab}}{2a} + \frac{1}{a}\right)c - \left(1+\frac{\sqrt{1-4ab}}{2a}\right)\left(1-\frac{\sqrt{1-4ab}}{2a} + \frac{1}{a}\right)
= \left(\frac{2}{2a} + \frac{1}{a}\right)c - \left(\frac{1-(1-4ab)}{4a^2}\right) - \left(\frac{1+\sqrt{1-4ab}}{2a}\right)
= \left(\frac{2}{a}\right)c - \left(\frac{4ab}{4a^2}\right) - \left(\frac{P_1}{a}\right) = \left(\frac{2}{a}\right)c - \left(\frac{b}{a}\right) - \left(\frac{P_1}{a}\right) = \frac{2c - b - P_1}{a}
\]

### 5.2.4. Theorem

Let \( b \leq -\frac{2}{a} \) for \( h_{a,b} \in H \). If \( P_2 \leq x \leq P_1 \), then \( \left\{ h_{a,b}^{-n}(x) \right\} \) is an increasing sequence.

**Proof:**

The first preimage of \( h_{a,b}(x) \) is \( h_{a,b}^{-1}(x) = \sqrt{\frac{x - b}{a}} \).

Claim: \( h_{a,b}^{-1}(x) \) is increasing for \( P_2 \leq x \leq P_1 \). To show this, let \( h_{a,b}^{-1}(x) \geq x \),
that is \( \sqrt{\frac{x-b}{a}} \geq x \), thus \( ax^2 - x + b \leq 0 \), there for \( (x - P_1)(x - P_2) \leq 0 \). Then either \( x \leq P_1 \) and \( x \geq P_2 \), or \( x \geq P_1 \) and \( x \leq P_2 \) (omitted because there is no intersection). Hence \( h_{a,b}^{-1}(x) \) is increasing whenever \( P_2 \leq x \leq P_1 \). Now \( h_{a,b}^{-1}(h_{a,b}^{-1}(x)) \geq h_{a,b}^{-1}(x) \). Which implies \( h_{a,b}^{-1}(x) \geq h_{a,b}^{-1}(x) \), \( (h_{a,b}^{-1} \) is increasing). Thus, we have \( h_{a,b}^{-1}(a+1)(x) \geq h_{a,b}^{-1}(x) \), for any \( n \in N \).

5.2.5. Theorem

Let \( b \leq -\frac{2}{a} \) and \( h_{a,b}(x) \in H \). If \( x \geq P_1 \), then \( \{h_{a,b}^{-n}(x)\} \) is a decreasing sequence.

The proof is the same as the above theorem.

5.2.6. Theorem

For \( h_{a,b}(x) \in H \) with \( b \leq -\frac{2}{a} \), if \( P_2 \leq x \leq P_1 \) then the upper bound of the increasing sequence of preimages of \( x \), \( \{h_{a,b}^{-n}(x)\} \) is \( P_1 \).

Proof:

It is clear that \( h_{a,b}^{-1}(x) = -\sqrt{\frac{x-b}{a}} \). We will prove \( h_{a,b}^{-n}(x) \leq P_1 \) by induction

Since \( x \leq P_1 \), then \( \frac{x-b}{a} \leq \frac{P_1-b}{a} \). Since \( x \geq P_2 \) and \( P_2 > b \) (because \( h_{a,b}^{-1}(P_2) = -\sqrt{\frac{P_2-b}{a}} = \frac{P_2}{a} \) so \( h_{a,b}^{-1}(P_2) \) is undefined if \( P_2 < b \), then \( x > b \).

There for \( \sqrt{\frac{x-b}{a}} \leq \sqrt{\frac{P_1-b}{a}} \). So, by lemma (4.1) then \( h_{a,b}^{-1}(x) \leq P_1 \).……….(*).

Now since \( h_{a,b}(h_{a,b}^{-1}(x)) = h_{a,b}^{-1}(x) \), \( i.e. \ a(h_{a,b}^{-1}(x))^2 + b = h_{a,b}^{-1}(x) \). Then by (*) we have \( a(h_{a,b}^{-2}(x))^2 + b \leq P_1 \), thus \( h_{a,b}^{-2}(x) \leq \sqrt{P_1-b} \). Thus by lemma (4.1), then \( h_{a,b}^{-2}(x) \leq P_1 \).

Now assume that \( h_{a,b}^{-n}(x) \leq P_1 \) is true and we have to show \( h_{a,b}^{-n}(x) \leq P_1 \).

Since \( a(h_{a,b}^{-n}(x))^2 + b = h_{a,b}^{-n}(x) \). Then, with our assumption \( h_{a,b}^{-n}(x) \leq P_1 \), we get \( a(h_{a,b}^{-n}(x))^2 + b \leq P_1 \), thus \( h_{a,b}^{-n}(x) \leq \sqrt{\frac{P_1-b}{a}} \). So by lemma (4.1), \( h_{a,b}^{-n}(x) \leq P_1 \).

5.2.7. Theorem

For \( h_{a,b}(x) \in H \) with \( b \leq -\frac{2}{a} \), if \( x \geq P_1 \) then the lower bound of the decreasing sequence \( \{h_{a,b}^{-n}(x)\} \) is \( P_1 \).

The proof is the same as the above theorem.

5.2.8. Theorem

If \( b \leq -\frac{2}{a} \), \( h_{a,b}(x) \in H \) and, \( P_2 \leq x \leq P_1 \) then the supremum of the increasing
sequence \( \{h_{ab}^n(x)\} \) is \( P_1 \). The first preimage of \( h_{ab}(x) \) is \( h_{ab}^{-1}(x) = \frac{x-b}{a} \).

If \( \sup \{h_{ab}^n(x)\} \neq P_1 \). Let \( c < P_1 \) and \( \sup \{h_{ab}^n(x)\} = c \). There for \( h_{ab}^n(x) \rightarrow c \). By lemma (5.2.2) \( P_2 + \frac{1}{a} < c \), there for \( P_2 + \frac{1}{a} < c < P_1 \), which implies \( P_1 - c > 0 \).

Now let \( \epsilon = P_1 - c \). Then \( \exists k \in N \) such that \( |h_{ab}^n(x) - c| < P_1 - c, \forall n > k \), thus \( \left( a \left( h_{ab}^{(n+1)}(x) \right)^2 + b \right) - c > P_1 - c \), which implies \( c - P_1 < \left( a \left( h_{ab}^{(n+1)}(x) \right)^2 + b \right) - c < P_1 - c \), thus

\[
2c - b - P_2 < h_{ab}^{(n+1)}(x) < \frac{P_1 - b}{a}. \]

By lemma (4.1) and lemma (5.2.3),

\[
\sqrt{P_1 + P_2 + \frac{1}{a}} c - P_1 \left( P_2 + \frac{1}{a} \right) < h_{ab}^{(n+1)}(x) < P_1 \ldots (*).
\]

But \( c > P_2 + \frac{1}{a} \) and \( c < P_1 \), which implies \( \left( c - \left( P_2 + \frac{1}{a} \right) \right)(c - P_1) < 0 \), thus \( c^2 - \left( P_1 + P_2 + \frac{1}{a} \right) c + P_1 \left( P_2 + \frac{1}{a} \right) < 0 \), hence \( c^2 < \left( P_1 + P_2 + \frac{1}{a} \right) c - P_1 \left( P_2 + \frac{1}{a} \right) \), there for \( c < \sqrt{P_1 + P_2 + \frac{1}{a}} c - P_1 \left( P_2 + \frac{1}{a} \right) \). Thus by (*) \( c < h_{ab}^{(n+1)}(x) < P_1 \) which is a contradiction with \( \sup \{h_{ab}^n(x)\} = c \). Thus \( \sup \{h_{ab}^n(x)\} = P_1 \).

5.2.9. Theorem

For \( b \leq -\frac{2}{a} \) of the functions \( h_{ab}(x) \in H \) and \( x \geq P_1 \) then the infimum of a decreasing sequence \( \{h_{ab}^n(x)\} \) is \( P_1 \). \( \inf \{h_{ab}^n(x) : n \geq 1\} = P_1 \).

The proof is the same as the above theorem.

Finally, we introduce the main theorem in this section.

5.2.10. Theorem

For \( h_{ab}(x) = ax^2 + b \) with \( b \leq -\frac{2}{a} \) and \( x \in \mathbb{R} \), the homoclinic orbit of the (first) homoclinic points \( q_{2,3} \) and \( q_{1,3} \) are \( O(q_{2,1}) = \{P_1, -P_1, q_{1,1}, q_{2,1}, \cdots, P_1\} \)

for \( \frac{-(5 + 2\sqrt{5})}{4a} \leq b \leq -\frac{2}{a} \) and \( O(q_{1,1}) = \{P_1, -P_1, q_{1,1}, \cdots, P_1\} \) for

\[
b < -\frac{(5 + 2\sqrt{5})}{4a}.
\]

The proof:

Since \( P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a} \). Then
1) By Remark (2.1), for \( b < \frac{1}{4a} \), \( h_{a,b}'(P_1) > 1 \).

2) By theorem (5.1.8), \( h_{a,b}^1(q_{2,1}) = P_1 \) for \( \frac{-(5+2\sqrt{5})}{4a} \leq b \leq \frac{-2}{a} \) and \( h_{a,b}^2(q_{1,1}) = P_1 \) for \( b < \frac{-(5+2\sqrt{5})}{4a} \) (see Figure 3). So \( h_{a,b}^n(q_{2,1}) = P_1 \) for \( n \geq 3 \), \( h_{a,b}^n(q_{1,1}) = P_1 \) for \( n \geq 2 \).

3) By theorems (5.2.4) and (5.2.8), (resp. (5.2.5) and (5.2.9)), \( h_{a,b}^n(q_{j,1}) \) where \( j = 1, 2 \) is an increasing with supremum (resp. decreasing with infimum) \( P_1 \).

Thus \( h_{a,b}^n(q_{j,1}) \rightarrow P_1 \). There for, 1, 2 and 3 show that \( q_{2,1}, q_{1,1} \) are the homoclinic points for \( P_1 \) with the homoclinic orbits

\[
O(q_{2,1}) = \{P_1, P_1, q_{1,1}, q_{2,1}, \ldots, P_1\} \quad \text{for} \quad \frac{-(5+2\sqrt{5})}{4a} \leq b \leq \frac{-2}{a} \quad \text{and}
\]

\[
O(q_{1,1}) = \{P_1, P_1, q_{1,1}, \ldots, P_1\} \quad \text{for} \quad b < \frac{-(5+2\sqrt{5})}{4a} \]. See Figure 3.

Following examples explain the cases for \( b = \frac{-2}{a} \) and \( b < \frac{-2}{a} \) respectively.

5.2.11. Example

For \( h_{-1,2}(x) = x^2 - 2 \), a homoclinic orbit of a homoclinic point \( \sqrt{2} \) is

\[
O(\sqrt{2}) = \{2, -2, 0, \sqrt{2}, \ldots, 2\}.
\]

Solution:

The forward orbit of \( \sqrt{2} \), \( h_{-1,2}(\sqrt{2}) = 0, h_{-1,2}(0) = -2, h_{-1,2}(-2) = 2 \), thus \( h_{-1,2}^1(\sqrt{2}) = 2 \). So \( h_{-1,2}^n(\sqrt{2}) = 2 \) for \( n \geq 3 \).

The backward orbit of \( \sqrt{2} \) is \( h_{-1,2}^{-n}(\sqrt{2}) = \{\sqrt{2}, \sqrt{2} + 2, \sqrt{2} + 2 + 2, \ldots\} \).

To prove \( h_{-1,2}^{-n}(\sqrt{2}) \rightarrow 2 \). Consider the sequence \( \{h_{-1,2}^{-n}(\sqrt{2})\} = \{a_n\} \).

\( \{a_n\} \) is an increasing sequence: It is easily shown that for \( -1 \leq x \leq 2 \) the function \( \sqrt{x+2} \) is an increasing function. Hence \( \cdots \geq h_{-1,2}^2(x) \geq h_{-1,2}^1(x) \geq x \), thus \( \{a_n\} \) is an increasing sequence.

Moreover for \( -1 \leq x \leq 2 \), by theorem (5.2.6) \( h_{-1,2}^{-n}(x) \leq 2, \forall n \) (i.e. 2 is an upper bound for \( \{a_n\} \)). Thus to show that \( \{h_{-1,2}^{-n}(x)\} \) converges to 2, it is enough prove that \( 2 = \sup \{\{a_n\}\} \). If \( 2 \neq \sup \{\{a_n\}\} \), let \( c < 2 \) and \( c = \sup \{\{a_n\}\} \). By lemma (5.2.2) then \( 0 < c < 2 \), since \( \{a_n\} \) is an increasing sequence and \( c = \sup \{a_n : n \in N\} \), then \( a_n \rightarrow c \). Now, since \( 0 < c < 2 \), then \( 2 - c > 0 \). Let \( \epsilon = 2 - c \).

Then \( \exists k \in N \) such that \( |a_k - c| < 2 - c, \forall n > k \).

Since the iteration of this sequence is \( a_n = a_{n+1}^2 - 2 \), thus

\[
a_{n+1}^2 - 2 - c < 2 - c, \quad \text{which implies} \quad 2c < a_{n+1}^2 < 4, \text{ then} \]

\[
\sqrt{2c} < a_{n+1} < 2 \quad \text{………(*).}
\]

But \( c > 0 \) and \( c < 2 \) which implies \( c(c-2) < 0 \), therefore \( c < \sqrt{2c} \). Thus by (*), \( c < a_{n+1} < 2 \) which is a contradiction with \( c = \sup \{A(a_n)\} \). Thus
\( 2 = \sup \{ A(a_n) \} \), and \( a_n \to 2 \). So \( O(\sqrt{2}) = \{ 2, -2, 0, \sqrt{2}, \cdots, 2 \} \) is a homoclinic orbit of a homoclinic point \( \sqrt{2} \) for \( h_{\text{c}}(x) \).

### 5.2.12. Example

For \( h_{\text{c}}(x) = x^3 - 6 \), a homoclinic orbit of a homoclinic point \( \sqrt{3} \) is \( O(\sqrt{3}) = \{ 3, -3, \sqrt{3}, \cdots, 3 \} \).

**Solution:**

The forward orbit of \( \sqrt{3} \), \( h_{\text{c}}(\sqrt{3}) = -3 \), \( h_{\text{c}}^{-1}(-3) = h_{\text{c}}^{-1}(\sqrt{3}) = 3 \). So \( h_{\text{c}}^{-1}(\sqrt{3}) = 3 \) for \( n \geq 2 \).

The backward orbit of \( \sqrt{3} \) is \( h_{\text{c}}^{-n}(\sqrt{3}) = \{ \sqrt{3}, \sqrt{3} + 6, \sqrt{3} + 6 + 6, \cdots \} \).

To prove \( h_{\text{c}}^{-n}(\sqrt{3}) \to 3 \). Consider the sequence \( \{ h_{\text{c}}^{-n}(\sqrt{3}) \} = \{ a_n \} \).

\( \{ a_n \} \) is an increasing sequence: It is easily shown that for \( -2 \leq x \leq 3 \) the function \( \sqrt{x + 6} \) is an increasing function. Hence \( \cdots \geq h_{\text{c}}^{-2}(x) \geq h_{\text{c}}^{-1}(x) \geq x \), thus \( \{ a_n \} \) is an increasing sequence.

Moreover, for \( -2 \leq x \leq 3 \), by theorem (5.2.6) \( h_{\text{c}}^{-n}(x) \leq 3 \), \( \forall n \) (i.e. 3 is an upper bound for \( \{ a_n \} \)). Thus to show that \( \{ h_{\text{c}}^{-n}(x) \} \) converges to 3, it is enough prove that \( 3 = \sup \{ \{ a_n \} \} \). If \( 3 \neq \sup \{ \{ a_n \} \} \), let \( c < 3 \) and \( c = \sup \{ \{ a_n \} \} \). By lemma (5.2.2) then \( -1 < c < 3 \), since \( \{ a_n \} \) is an increasing sequence and \( c = \sup \{ a_n : n \in N \} \), then \( a_n \to c \). Now, since \( -1 < c < 3 \), then \( 3 - c > 0 \). Let \( \epsilon = 3 - c \).

Then \( \exists k \in N \) such that \( |a_n - c| < 3 - c, \forall n > k \).

Since the iteration of this sequence is \( a_n = a_{n+1}^2 - 6 \), thus

\[
\left( a_{n+1}^2 - 6 \right) - c < 3 - c, \text{ which implies } 2c + 3 < a_{n+1}^2 < 9, \text{ then } \\
\sqrt{2c + 3} < a_{n+1} < 3 \quad \text{ (\*).}
\]

But \( c > -1 \) and \( c < 3 \) which implies \( (c + 1)(c - 3) < 0 \), there for \( c < \sqrt{2c + 3} \). Thus by (\*), \( c < a_{n+1} < 3 \) which is a contradiction with \( c = \sup \{ \{ a_n \} \} \). Thus \( 3 = \sup \{ \{ a_n \} \} \), and \( a_n \to 3 \). So \( O(\sqrt{3}) = \{ 3, -3, \sqrt{3}, \cdots, 3 \} \) is a homoclinic orbit of a homoclinic point \( \sqrt{3} \) for \( h_{\text{c}}(x) \).

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

### References

[1] Lord, G.J., Champneys, A.R. and Hunt, G.W. (1999) Computation of Homoclinic or-Bits in Partial Differential Equations: An Application to Cylindrical Shell Buckling. *SIAM Journal on Scientific Computing*, **21**, 591-619. [https://doi.org/10.1137/S1064827597321647](https://doi.org/10.1137/S1064827597321647)

[2] Poincaré, H. (2017) The Three-Body Problem and the Equations of Dynamics: Poincaré’s Foundational Work on Dynamical Systems Theory, Volume 443. Springer, Berlin. [https://doi.org/10.1007/978-3-319-52899-1](https://doi.org/10.1007/978-3-319-52899-1)

[3] Smale, S. (1980) The Mathematics of Time. Springer, Berlin. [https://doi.org/10.1007/978-1-4613-8101-3](https://doi.org/10.1007/978-1-4613-8101-3)
[4] Devaney, R. (2018) An Introduction to Chaotic Dynamical Systems. CRC Press, London. https://doi.org/10.4324/9780429502309

[5] Block, L.S. and Coppel, W.A. (2006) Dynamics in One Dimension. Springer, Berlin. https://www.springer.com/gp/book/9783540553090

[6] Gardini, L. (1994) Homoclinic Bifurcations in n-Dimensional Endomorphisms, Due to Expanding Periodic Points. Nonlinear Analysis: Theory, Methods and Applications, 23, 1039-1089. https://doi.org/10.1016/0362-546X(94)90198-8

[7] Marotto, F.R. and Fr, M. (1978) Snap-Back Repellers Imply Chaos in RN. https://doi.org/10.1016/0022-247X(78)90115-4

[8] Marotto, F.R. (2005) On Redefining a Snap-Back Repeller. Chaos, Solitons & Fractals, 25, 25-28. https://doi.org/10.1016/j.chaos.2004.10.003

[9] Avrutin, V., Schenke, B. and Gardini, L. (2015) Calculation of Homoclinic and Heteroclinic Orbits in 1D Maps. Communications in Nonlinear Science and Numerical Simulation, 22, 1201-1214. https://doi.org/10.1016/j.cnsns.2014.07.008

[10] Onozaki, T. (2018) Nonlinearity, Bounded Rationality, and Heterogeneity. https://doi.org/10.1007/978-4-431-54971-0

[11] Chen, G. and Huang, Y. (2011) Chaotic Maps: Dynamics, Fractals, and Rapid Fluctuations. Synthesis Lectures on Mathematical Statistics, 4, 1-241. https://doi.org/10.2200/S00373ED1V01Y201107MAS011

[12] Laura, G., Viktor, A., Iryna, S. and Fabio, T. (2019) Continuous and Discontinuous Piecewise-Smooth One-Dimensional Maps: Invariant Sets and Bifurcation Structures, Volume 95. World Scientific, Singapore. https://lccn.loc.gov/2019017217