FROBENIUS CONDITION ON A PRETRIANGULATED CATEGORY, AND TRIANGULATION ON THE ASSOCIATED STABLE CATEGORY

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Abstract. As shown by Happel, from any Frobenius exact category, we can construct a triangulated category as a stable category. On the other hand, it was shown by Iyama and Yoshino that if a pair of subcategories $D \subseteq Z$ in a triangulated category satisfies certain conditions (i.e., $(Z, Z)$ is a $D$-mutation pair), then $Z/D$ becomes a triangulated category. In this article, we consider a simultaneous generalization of these two constructions.

1. Introduction and Preliminaries

Throughout this article, we fix an additive category $C$. Any subcategory of $C$ will be assumed to be full, additive and replete. A subcategory is called replete if it is closed under isomorphisms.

When we say $Z$ is an exact category, we only consider an extension-closed subcategory of an abelian category.

For any category $K$, we write abbreviately $K \in K$, to indicate that $K$ is an object of $K$. For any $K, L \in K$, let $K(K, L)$ denote the set of morphisms from $K$ to $L$. If $M, N$ are full subcategories of $K$, then $K(M, N) = 0$ means that $K(M, N) = 0$ for any $M \in M$ and $N \in N$. Similarly, $K(K, N) = 0$ means $K(K, N) = 0$ for any $N \in N$.

If $K$ is an additive category and $L$ is a full additive replete subcategory which is closed under finite direct summands, then $K/L$ denotes the quotient category of $K$ by the ideal generated by $L$. The image of $f \in K(X, Y)$ will be denoted by $f \in K/L(X, Y)$.

As shown by Happel [H], If we are given a Frobenius exact category $E$, then the stable category $E/I$, where $I$ is the full subcategory of injectives, carries a structure of a triangulated category.

On the other hand, it was shown by Iyama and Yoshino that if $D \subseteq Z$ is a pair of subcategories in a triangulated category $C$ such that $(Z, Z)$ is a $D$-mutation pair, then the quotient category $Z/D$ becomes a triangulated category. By definition, $(Z, Z)$ is a $D$-mutation pair if it satisfies

$(1)$ $C(Z, D[1]) = C(D, Z[1]) = 0,$

$(2)$ For any object $X \in Z$, there exists a distinguished triangle

$$X \to D \to Z \to \Sigma X$$

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with \( D \in \mathcal{D} \) and \( Z \in \mathcal{Z} \),

(3) For any object \( Z \in \mathcal{Z} \), there exists a distinguished triangle

\[
X \to D \to Z \to \Sigma X
\]

with \( X \in \mathcal{Z} \) and \( D \in \mathcal{D} \).

In this article, we make a simultaneous generalization of these two constructions, by using a slight modification of a \textit{pretriangulated category} in [BR]. To emphasize this modification, we call it a ‘pseudo-triangulated category. As in Definition 3.3, a \textit{pseudo-triangulated} category is an additive category \( \mathcal{C} \) with a \textit{pseudo-triangulation} \((\Sigma, \Omega, \triangleright, \langle, \psi\rangle)\).

As in Example 4.5, a pseudo-triangulated category \( \mathcal{C} \) is abelian if and only if \( \Sigma = \Omega = 0 \), and \( \mathcal{C} \) is triangulated if and only if \( \Sigma \cong \Omega^{-1} \). An \textit{extension} in \( \mathcal{C} \) is a simultaneous generalization of a short exact sequence in the abelian case, and a distinguished triangle in the triangulated case (Definition 4.1). For an extension-closed subcategory \( \mathcal{Z} \subseteq \mathcal{C} \), we define the \textit{Frobenius condition} on it (Definition 5.9). This is equivalent to the ordinary Frobenius condition in the case of \( \Sigma = \Omega = 0 \), and related to the existence of a mutation pair in the triangulated case (Example 5.10 and Corollary 5.16). As a main theorem, in Theorem 6.17, we show if \( \mathcal{Z} \) is Frobenius, then the associated stable category becomes a triangulated category. In the above two cases, this recovers the Happel’s and Iyama-Yoshino’s constructions, respectively.

| Pretriangulated | \( \Sigma = \Omega = 0 \) | \( \Sigma \cong \Omega^{-1} \) |
|-----------------|---------------------|---------------------|
| Extension       | short exact sequence | distinguished triangle |
| Frobenius condition | Frobenius condition | Corollary 5.16 |
| Theorem 6.17    | Happel’s construction | Iyama-Yoshino’s construction |

2. \textbf{ONE-SIDED TRIANGULATED CATEGORIES}

\textbf{Definition 2.1} (right triangulation cf. [BM], [BR]). Let \( \Sigma: \mathcal{C} \to \mathcal{C} \) be an additive endofunctor, and let \( \mathcal{R}T(\mathcal{C}, \Sigma) \) be the category of diagrams of the form

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.
\]

A morphism from \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) to \( A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A' \) is a triplet \((a, b, c)\) of morphisms \( a \in \mathcal{C}(A, A') \), \( b \in \mathcal{C}(B, B') \) and \( c \in \mathcal{C}(C, C') \), satisfying

\[
b \circ f = f' \circ a, \quad c \circ g = g' \circ b, \quad \Sigma a \circ h = h' \circ c.
\]

A pair \((\Sigma, \triangleright)\) of \( \Sigma \) and a full replete subcategory \( \triangleright \subseteq \mathcal{R}T(\mathcal{C}, \Sigma) \) is called a \textit{right triangulation} on \( \mathcal{C} \) if it satisfies the following conditions. Remark that \( \Sigma \) is not necessarily an equivalence.

(RTR1) For any \( A \in \mathcal{C}, 0 \to A \xrightarrow{id_A} A \to \Sigma 0 = 0 \) is in \( \triangleright \). For any morphism \( f \in \mathcal{C}(A, B) \), there exists an object \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) in \( \triangleright \).

(RTR2) If \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) is in \( \triangleright \), then \( B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma B \) is also in \( \triangleright \).

(RTR3) If we are given two objects \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) and \( A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A' \) in \( \triangleright \) and two morphisms \( a \in \mathcal{C}(A, A') \) and \( b \in \mathcal{C}(B, B') \) satisfying \( b \circ f = f' \circ a \), then there exists \( c \in \mathcal{C}(C, C') \) such that \((a, b, c)\) is a morphism in \( \triangleright \).
(RTR4) Let
\[
\begin{align*}
A & \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A, \\
A & \xrightarrow{\ell} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A, \\
A' & \xrightarrow{\ell'} M \xrightarrow{m'} B \xrightarrow{n'} \Sigma A'
\end{align*}
\]
be objects in $\triangleright$, satisfying $m' \circ \ell = f$.
Then there exist $g' \in C(B', C)$ and $h' \in C(C, \Sigma A')$ such that
\[
h' \circ g = n', \quad h \circ g' = n,
\]
\[
g' \circ m = g \circ m', \quad (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0,
\]
and
\[
A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'
\]
is an object in $\triangleright$. Here we put $f' = m \circ \ell'$.

If $(\Sigma, \triangleright)$ is a right triangulation on $\mathcal{C}$, we call $(\mathcal{C}, \Sigma, \triangleright)$ a right triangulated category.

**Caution 2.2.** Conditions (RTR4) is slightly different from that in [BM].

**Definition 2.3** (left triangulation). Let $\Omega: \mathcal{C} \to \mathcal{C}$ be an additive endofunctor, and let $\mathcal{LT}(\mathcal{C}, \Omega)$ be the category of diagrams of the form
\[
\begin{align*}
A & \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A, \\
A & \xrightarrow{\ell} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A, \\
A' & \xrightarrow{\ell'} M \xrightarrow{m'} B \xrightarrow{n'} \Sigma A'
\end{align*}
\]
A morphism in $\mathcal{LT}(\mathcal{C}, \Omega)$ is defined similarly as in Definition 2.1. A pair $(\Omega, \triangleleft)$ satisfying conditions (LTR1), (LTR2), (LTR3) and (LTR4) which are dual to (RTR1), (RTR2), (RTR3) and (RTR4) respectively, is called a left triangulation on $\mathcal{C}$, and $(\mathcal{C}, \Omega, \triangleleft)$ is called a left triangulated category.

Similarly to the triangulated case, the following are satisfied.

**Proposition 2.4.** Let $\mathcal{C}$ be an additive category.

1. If $(\Sigma, \triangleright)$ is a right triangulation on $\mathcal{C}$, then for any object $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\triangleright$ and for any $E \in \mathcal{C}$, the induced sequence
\[
\begin{align*}
\mathcal{C}(A, E) & \xleftarrow{\cdot} \mathcal{C}(B, E) \xleftarrow{\cdot} \mathcal{C}(C, E) \xleftarrow{\cdot} \mathcal{C}(\Sigma A, E) \xleftarrow{\cdot} \mathcal{C}(\Sigma B, E) \xleftarrow{\cdots}
\end{align*}
\]
is exact.
2. Dually for a left triangulation.

**Proof.** Left to the reader.
3. PSEUDO-TRIANGULATED CATEGORY

In this section, we introduce a notion unifying triangulated categories and abelian categories. We make a slight modification of the pretriangulated category in [BR], for the sake of Example 4.5. We call it a ‘pseudo-’triangulated category, to make the reader beware of this modification. Roughly speaking, a pseudo-triangulated category is an additive category endowed with right and left triangulated triangulations, satisfying some gluing conditions (Definition 3.3).

**Definition 3.1.** Let $(\Sigma, \triangleright)$ be a right triangulation on $\mathcal{C}$, and let $f : A \to B$ be any morphism in $\mathcal{C}$.

1. $f$ is $\Sigma$-null if it factors through some object in $\Sigma \mathcal{C}$.
2. $f$ is $\Sigma$-epic if for any $B' \in \mathcal{C}$ and any $b \in \mathcal{C}(B, B')$, $b \circ f = 0$ implies $b$ is $\Sigma$-null.

For a left triangulation $(\Omega, \triangleleft)$, dually we define $\Omega$-null morphisms and $\Omega$-monic morphisms.

**Remark 3.2.** For any morphism $f \in \mathcal{C}(A, B)$, the following are equivalent.

1. $f$ is $\Sigma$-epic.
2. There exists an object in $\triangleright$

\[
A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A
\]

such that $g$ is $\Sigma$-null.
3. For any object in $\triangleright$

\[
A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A,
\]

$g$ becomes $\Sigma$-null.

Dually for $\Omega$-monics.

**Definition 3.3.** A pseudo-triangulation $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$ on $\mathcal{C}$ is a pair $(\Sigma, \triangleright)$ and $(\Omega, \triangleleft)$ of right and left triangulations, together with an adjoint natural isomorphism

\[
\psi_{A,B} : \mathcal{C}(\Omega A, B) \xrightarrow{\cong} \mathcal{C}(A, \Sigma B) \quad (A, B \in \mathcal{C}),
\]

which satisfies the following gluing conditions (G1) and (G2).

(G1) If $g \in \mathcal{C}(B, C)$ is $\Sigma$-epic, then for any objects

\[
\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \in \triangleleft,
\]

\[
A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A \in \triangleright,
\]

there exists an isomorphism $c \in \mathcal{C}(C', C)$ such that

\[
c \circ g' = g \quad \text{and} \quad -\psi(e) \circ c = h'.
\]

Roughly speaking, this means that any $\Sigma$-epic morphism agrees with the ‘cokernel’ of its ‘kernel’.
(G2) Dually, if \( f \in \mathcal{C}(A, B) \) is \( \Omega \)-monic, then for any objects

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \quad \in \triangleright,
\]

\[
\Omega C \xrightarrow{\epsilon} A' \xrightarrow{f'} B \xrightarrow{g} C \quad \in \triangleleft,
\]

there exists an isomorphism \( a \in \mathcal{C}(A, A') \) such that

\[
f' \circ a = f \quad \text{and} \quad -a \circ \psi^{-1}(h) = e'.
\]

If we are given a pseudo-triangulation \( (\Sigma, \Omega, \triangleright, \triangleleft, \psi) \) on \( \mathcal{C} \), then we call the 6-tuple \( (\mathcal{C}, \Sigma, \Omega, \triangleright, \triangleleft, \psi) \) a pseudo-triangulated category. We often represent a pseudo-triangulated category simply by \( \mathcal{C} \).

**Example 3.4.** Let \( (\mathcal{C}, \Sigma, \Omega, \triangleright, \triangleleft, \psi) \) be a pseudo-triangulated category.

1. \( \mathcal{C} \) is an abelian category if and only if \( \Sigma = \Omega = 0 \).
2. \( \mathcal{C} \) is a triangulated category if and only if \( \Sigma \) is the quasi-inverse of \( \Omega \) and \( \psi \) is the one induced from the isomorphism \( \Sigma \circ \Omega \cong \text{Id}_\mathcal{C} \).

**Proof.** (1) We only show that \( \Sigma = \Omega = 0 \) implies the abelianess of \( \mathcal{C} \). The converse is confirmed by a routine work. Since \( \Sigma = 0 \), Proposition 2.4 means \( g = \text{cok}(f) \) holds for any object

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A
\]

in \( \triangleright \).

Thus (RTR1) implies the existence of a cokernel for each morphism. Dually for the existence of \( \ker(f) \). Moreover, in this case \( f \) is \( \Sigma \)-null if and only if \( f = 0 \), and \( f \) is \( \Sigma \)-epic if and only if it is epimorphic. Thus (G1) means that any epimorphism \( g \) agrees with \( \text{cok}(\ker(g)) \). Dually for monomorphisms.

(2) In this case, any morphism is at the same time \( \Sigma \)-null and \( \Sigma \)-epic, and \( \Omega \)-null and \( \Omega \)-monic. Moreover, \( \triangleright \) and \( \triangleleft \) agree. We only show \( \triangleleft \subseteq \triangleright \).

By (LTR2), for any object

\[
(3.1) \quad \Omega C \xrightarrow{\epsilon} A \xrightarrow{f} B \xrightarrow{g} C
\]

in \( \triangleleft \), the shifted one

\[
\Omega B \xrightarrow{-\Omega g} \Omega C \xrightarrow{\epsilon} A \xrightarrow{f} B
\]

is also in \( \triangleleft \). By (G1), we obtain an object in \( \triangleright \)

\[
\Omega C \xrightarrow{\epsilon} A \xrightarrow{f} B \xrightarrow{\psi(\Omega g)} \Sigma \Omega C,
\]

which is isomorphic to (3.1). \( \square \)
4. Extensions

In this section, $\mathcal{C}$ is a pseudo-triangulated category with pseudo-triangulation $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$. We define the notion of an extension which generalizes a short exact sequence in an abelian category, and a distinguished triangle in a triangulated category.

**Definition 4.1.** A sequence in $\mathcal{C}$

$$\Omega \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is called an *extension* if it satisfies

$$(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A) \in \triangleright,$$

$$(\Omega \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C) \in \triangleleft,$$

$$h = -\psi_{\mathcal{C},A}(e).$$

Since $e$ and $h$ determines each other, we sometimes omit one of them.

A *morphism of extensions* from

$$\Omega \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

to

$$\Omega' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

is a triplet $(a, b, c)$ of $a \in \mathcal{C}(A, A')$, $b \in \mathcal{C}(B, B')$ and $c \in \mathcal{C}(C, C')$ satisfying

$$b \circ f = f' \circ a, \quad c \circ g = g' \circ b, \quad (\Sigma a) \circ h = h' \circ c.$$

Remark that $(\Sigma a) \circ h = h' \circ c$ is equivalent to $a \circ e = e' \circ (\Omega c)$. Thus, a morphism of extensions is essentially the same as a morphism in $\triangleright$ or $\triangleleft$.

**Remark 4.2.** Consider a diagram in $\mathcal{C}$

(4.1) $$\Omega \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

satisfying $h = -\psi(e)$. By (G1) and (G2) (and (RTR1) and (LTR1)), the following are equivalent.

1. $\Omega \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ belongs to $\triangleleft$ and $g$ is $\Sigma$-epic.
2. $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ belongs to $\triangleright$ and $f$ is $\Omega$-monic.
3. (4.1) is an extension.

**Corollary 4.3.**

1. $g \in \mathcal{C}(B, C)$ is $\Sigma$-epic if and only if there exists an extension (4.1), if and only if there exists an object $A \rightarrow B \xrightarrow{g} C \rightarrow \Sigma A$ in $\triangleright$.
2. $f \in \mathcal{C}(A, B)$ is $\Omega$-monic if and only if there exists an extension (4.1), if and only if there exists an object $\Omega \rightarrow A \xrightarrow{f} B \rightarrow C$ in $\triangleleft$. 
Proof. We show only (1). If there exists an object \( A \xrightarrow{g} C \xrightarrow{\Sigma} \Sigma A \in \triangleright \), then by (RTR2), we have an object in \( \triangleright \)

\[
B \xrightarrow{g} C \xrightarrow{\Sigma} \Sigma A \rightarrow \Sigma B.
\]

Obviously this implies \( g \) is \( \Sigma \)-epic.

Conversely if \( g \) is \( \Sigma \)-epic, then by (LTR1) and Remark 4.2, we obtain an extension (4.1).

\[\square\]

Lemma 4.4. Let \( f \in C(A,B) \), \( m \in C(A,M) \) and \( e \in C(M,B) \) be morphisms satisfying \( e \circ m = f \).

1. If \( f \) is \( \Sigma \)-epic, then so is \( e \).
2. If \( f \) is \( \Omega \)-monic, then so is \( m \).

Proof. (1) By (RTR1) and (RTR3), there exists a morphism in \( \triangleright \)

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} \Sigma A \\
M & \xrightarrow{m} & B & \xrightarrow{g'} & D & \xrightarrow{h'} \Sigma M
\end{array}
\]

Then since \( g \) is \( \Sigma \)-null, so is \( g' \). (2) is shown dually. \[\square\]

Example 4.5. The notion of an extension becomes as follows in the two cases of Example 3.4.

1. If \( \Sigma = \Omega = 0 \) and \( C \) is abelian, then an extension is nothing other than a short exact sequence.
2. If \( C \) is a triangulated category as in Example 3.4, then an extension is nothing other than a distinguished triangle.

Proposition 4.6. For any \( A,B \in C \),

\[
\Omega B \xrightarrow{u} A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A
\]

is an extension, where \( i_A \) and \( p_B \) are the injection and the projection, respectively.

Proof. Let \( p_A \): \( A \oplus B \rightarrow A \) be the projection, and \( i_B \): \( B \rightarrow A \oplus B \) be the inclusion. Since \( \text{id}_B \) is \( \Sigma \)-epic by (RTR1), so is \( p_B \) by Lemma 4.4. Thus by Corollary 4.3, there is an extension

\[
\Omega B \xrightarrow{u} A \xrightarrow{i_B} A \oplus B \xrightarrow{p_B} B \xrightarrow{w} \Sigma C
\]

with some morphisms \( u, v, w \). Since \( p_B \) is the projection and \( w \circ p_B = 0 \) by Proposition 2.4, we have \( w = 0 \), and thus \( u = 0 \). By \( p_B \circ i_A = 0 \), there exists \( r \in C(A,C) \) such that \( v \circ r = i_A \).

Then we have

\[
v \circ (\text{id}_C - r \circ (p_A \circ v)) = v - v \circ r \circ p_A \circ v
\]

\[
= (\text{id}_C - i_A \circ p_A) \circ v
\]

\[
= (i_B \circ p_B) \circ v = 0.
\]
Thus $\text{id}_C - r \circ p_A \circ v$ factors through $u = 0$, which means
\[ r \circ (p_A \circ v) = \text{id}_C. \]
Since $(p_A \circ v) \circ r = p_A \circ i_A = \text{id}_A$, this means $r$ is an isomorphism. \hfill \Box

**Proposition 4.7.** Let
\[
\begin{array}{cccc}
\Omega C & \to & A & \to B & \to C & \to \Sigma A, \\
\Omega B' & \to & A & \to M & \to B' & \to \Sigma A, \\
\Omega B & \to & A' & \to M & \to B & \to \Sigma A',
\end{array}
\]
be extensions, satisfying $m' \circ \ell = f$. Then there exist $g' \in C(B', C)$ and $h' \in C(C, \Sigma A')$ such that
\[
\begin{align*}
h' \circ g &= n' , & h \circ g' &= n , \\
g' \circ m &= g \circ m' , & (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' &= 0,
\end{align*}
\]
and
\[
\Omega C \to A' \xrightarrow{f'} B' \xrightarrow{g'} C \xrightarrow{h'} \Sigma A'
\]
is an extension. Here we put $f' = m \circ \ell'$. Remark if we put $e' = -\psi^{-1}(h')$, then $(\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0$ is equivalent to $\ell' \circ e' + \ell \circ e = 0$.

**Proof.** By (RTR4), there exist $g' \in C(B', C)$ and $h' \in C(C, \Sigma A')$ such that
\[
\begin{align*}
h' \circ g &= n' , & h \circ g' &= n , \\
g' \circ m &= g \circ m' , & (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' &= 0,
\end{align*}
\]
and
\[
\begin{array}{c}
A' \xrightarrow{f'} B' \xrightarrow{g'} C \xrightarrow{h'} \Sigma A'
\end{array}
\]
is an object in $\triangleright$. Thus by Remark 4.2, it suffices to show $f'$ is $\Omega$-monic. This follows from (LTR4). In fact, applying (LTR4) to objects in $\triangleleft$
\[
\begin{array}{cccc}
\Omega B & \rightleftharpoons \Omega C \xrightarrow{c} A & \xrightarrow{f} B, \\
\Omega B' & \xrightarrow{k} A & \xrightarrow{\ell} M & \xrightarrow{m} B', \\
\Omega B & \xrightarrow{k'} A' & \xrightarrow{\ell'} M & \xrightarrow{m'} B,
\end{array}
\]

Dual statement also holds.
we obtain an object in
\[ \Omega B' \rightarrow \Omega C \rightarrow A' \xrightarrow{f'} B' , \]
which means \( f' \) is \( \Omega \)-monic. \( \square \)

5. Frobenius condition

In this section, we define an extension-closed subcategory \( Z \subseteq C \), and the Frobenius condition on it. This condition generalizes simultaneously the usual Frobenius condition for an exact category, and the the existence of a subcategory \( D \) such that \( (Z, Z) \) is a \( D \)-mutation pair in the case of a triangulated category.

**Definition 5.1.** A subcategory \( Z \subseteq C \) is said to be **extension-closed** if it satisfies the following.

\[(\ast) \text{ For any extension in } C \]
\[ \Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X , \]
\[ X, Z \in Z \text{ implies } Y \in Z . \]

In the following, we fix an extension-closed subcategory \( Z \subseteq C \).

**Remark 5.2.** When \( C \) is an abelian category as in Example 4.5, then \( Z \) is an exact category.

**Definition 5.3.** Let \( Z \subseteq C \) be an extension-closed subcategory as above.

1. A **conflation** is an extension in \( C \)
\[ (5.1) \Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X , \]
\[ X, Y, Z \in Z \text{ implies } Y \in Z . \]

A morphism of conflations is a morphism of the extensions.
2. A morphism \( f : X \rightarrow Y \) in \( Z \) is an **inflation** if there exists a conflation (5.1).
3. A morphism \( g : Y \rightarrow Z \) in \( Z \) is a **deflation** if there exists a conflation (5.1).

In the following, we fix an extension-closed subcategory \( Z \subseteq C \). For a full additive replete subcategory \( D \subseteq Z \), we consider the following condition (DS).

**Condition 5.4.**

\( (DS) \) \( D \) is closed under finite direct summands in \( Z \), namely, for any \( Z_1, Z_2 \in Z \) and \( D \in D \), \( D \cong Z_1 \oplus Z_2 \) implies \( Z_1, Z_2 \in Z \).

**Definition 5.5.** Let \( D \subseteq Z \) be a full additive replete subcategory satisfying (DS).

1. An object \( I \) in \( D \) is **injective** if
\[ Z(Y, I) \xrightarrow{-of} Z(X, I) \rightarrow 0 \]
is exact for any inflation \( f : X \rightarrow Y \). We denote the full subcategory of injective objects by \( I_D \subseteq D \). In particular \( I_Z \) is denoted by \( I \).
2. An object \( P \) in \( D \) is **projective** if
\[ Z(P, Y) \xrightarrow{g\circ} Z(P, Z) \rightarrow 0 \]
is exact for any deflation \( g : Y \rightarrow Z \). We denote the full subcategory of projective objects by \( P_D \subseteq D \). In particular \( P_Z \) is denoted by \( P \).

**Example 5.6.**
(1) If $Z \subseteq C$ is an exact category where $C$ is an abelian category as in Example 4.5, then $I$ is equal to the full subcategory of injective objects, and $P$ is equal to the full subcategory of projective objects.

(2) If $C$ is a triangulated category, and if $D$ satisfies $C(\Omega Z, D) = C(D, \Sigma Z) = 0$, then we have $I_D = P_D = D$.

**Caution 5.7.** The definitions of injective and projective objects are different from those in [B].

**Remark 5.8.**

1. $I_D$ and $P_D$ are full additive replete subcategories, which are closed under finite direct summands in $Z$.
2. $I_D = I \cap D$.
3. $P_D = P \cap D$.

**Proof.** Left to the reader. □

**Definition 5.9.** Let $(C, Z, D)$ be a triplet as above.

1. $(C, Z, D)$ **has enough injectives** if for any $X \in Z$, there exists an inflation $\alpha: X \to I$ such that $I \in I_D$. When $D = Z$, we simply say “$Z$ has enough injectives”.
2. $(C, Z, D)$ **has enough projectives** if for any $Z \in Z$, there exists a deflation $\beta: P \to Z$ such that $P \in P_D$. When $D = Z$, we simply say “$Z$ has enough projectives”.
3. $(C, Z, D)$ is **Frobenius** if it has enough injectives and projectives, and moreover $I_D = P_D$. When $D = Z$, we simply say “$Z$ is Frobenius”.

**Example 5.10.**

1. If $Z \subseteq C$ is an exact category as in Example 5.6, then $Z$ is Frobenius if and only if $Z$ is Frobenius as an exact category. In this case the stable category $Z/I$ is triangulated [H].
2. If $C$ is a triangulated category and if $(Z, Z)$ is a $D$-mutation pair in $C$ (in the definition in [IY]), then $(C, Z, D)$ is Frobenius. In this case $Z/I_D = Z/D$ becomes a triangulated category by Theorem 4.2 in [IY].

| C       | Happel’s construction [H]      | Iyama and Yoshino’s construction [IY] |
|---------|--------------------------------|--------------------------------------|
| $Z$     | abelian category               | triangulated category                |
| $Z$     | exact subcategory              | extension-closed subcategory          |
| $D$     | $Z = D$                       | $(Z, Z): D$-mutation pair             |
| $I_D$   | injective objects             | $I_D = D$                            |
| $P_D$   | projective objects            | $P_D = D$                            |

In section 6, in a pseudo-triangulated category $C$ satisfying Condition 6.1, we show $Z/I_D$ becomes a triangulated category for any Frobenius triplet $(C, Z, D)$ (Theorem 6.17), which we call the **stable category** associated to $(C, Z, D)$. In particular, if $Z$ is Frobenius, then $Z/I$ becomes a triangulated category. We call $Z/I$ the stable category associated to $Z$.

Although we have defined the Frobenius condition on a triplet $(C, Z, D)$, it is essentially the same as the Frobenius condition on $Z$ as follows (Corollary 5.13).

**Proposition 5.11.** Let $D \subseteq D' \subseteq Z$ be full additive replete subcategories satisfying (DS). If $(C, Z, D)$ is Frobenius, so is $(C, Z, D')$. Moreover, we have $I_{D'} = I_D$. 
Proof. This immediately follows from the lemma below. \qed

Lemma 5.12. Let \( D \subseteq D' \subseteq Z \) be as in Proposition 5.11. If \((C, Z, D)\) has enough injectives, then we have \( I_{D'} = I_D \). Similarly for projectives.

Proof. Remark that \( I_D = I_D' \cap D \). Thus it suffices to show \( I_D' \subseteq D \).

Since \((C, Z, D)\) has enough injectives, for any \( I' \in I_{D'} \), there exists a conflation \( \Omega \xrightarrow{e} I' \xrightarrow{f} I \xrightarrow{g} Z \xrightarrow{h} \Sigma I' \), where \( Z \in Z \) and \( I \in I_D \). Since \( I' \in I_{D'} \), there exists \( p \in Z(I, I') \) such that \( p \circ f = \text{id}_{I'} \). By \( f \circ e = 0 \), we have \( e = p \circ f \circ e = 0 \), and thus \( h = 0 \). By \( (\text{id}_I - f \circ p) \circ f = 0 \), there exists \( s \in Z(Z, I) \) such that \( s \circ g = \text{id}_I - f \circ p \). Since \( (\text{id}_Z - g \circ s) \circ g = 0 \), \( \text{id}_Z - g \circ s \) factors through \( h = 0 \), namely, we have \( \text{id}_Z = g \circ s \). Thus we obtain \( I = I' \oplus Z \). Since \( D \) is closed under finite direct summands in \( Z \), it follows \( I' \in D \). \qed

Thus if \((C, Z, D)\) is a Frobenius triplet, then \( Z \) is Frobenius, and satisfies \( I = I_D \).
In particular, their stable categories are equivalent.

Corollary 5.13. For any extension-closed subcategory \( Z \subseteq C \), the following are equivalent.

1. \( Z \) is Frobenius.
2. There exists a full additive replete subcategory \( D \subseteq Z \) satisfying (DS) such that \((C, Z, D)\) is Frobenius.

Moreover, there exists the minimum one.

Corollary 5.14. If \( Z \) is Frobenius, there exists the minimum \( D \), which makes \((C, Z, D)\) Frobenius.

Proof. We show \( I \) satisfies the desired conditions. By Remark 5.8, \( I \subseteq Z \) is a full additive replete subcategory satisfying (DS). If \( Z \) is Frobenius, it immediately follows that
\[ I_I = I = P = P_I, \]
and \((C, Z, I)\) becomes Frobenius. Obviously \( I \) is the minimum one, since any Frobenius triplet \((C, Z, D)\) satisfies \( I = I_D \subseteq D \). \qed

When \( C \) is a triangulated category and if \( D \subseteq Z \) is a full additive replete subcategory satisfying (DS) and
\[ C(\Omega Z, D) = C(D, \Sigma Z) = 0, \]
then \((C, Z, D)\) is Frobenius if and only if \((Z, Z)\) is a \( D \)-mutation pair. (We also remark that if there exists one such \( D \), then it is unique and must agree with the full subcategory of \( Z \) consisting of those \( D \in Z \) satisfying \( C(\Omega Z, D) = C(D, \Sigma Z) = 0 \).)

Namely, we have the following.

Claim 5.15. Let \( D \subseteq Z \) be a full additive replete subcategory satisfying (DS). The following are equivalent.

1. \((C, Z, D)\) is Frobenius, and \( C(\Omega Z, D) = C(D, \Sigma Z) = 0 \).
2. \((Z, Z)\) is a \( D \)-mutation pair.

Regarding Corollary 5.13 and Corollary 5.14, we obtain the following.

Corollary 5.16. For any \( Z \), the following are equivalent.
6. Triangulation on the stable category

In this section, as a main theorem, we show give a triangulation on the stable category associated to an extension-closed subcategory of a pseu-dotriangulated category satisfying the following condition. Remark that this condition is trivially satisfied in the two cases of Example 3.4.

**Condition 6.1.** Let

\[
\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,
\]

be extensions.

(AC1) If \(e \in C(C, C')\) satisfies \(h \circ c = 0\) and \(c \circ g = 0\), then there exists \(e' \in C(C, B')\) such that \(g' \circ e' = c\).

(AC2) If \(a \in C(A, A')\) satisfies \(f' \circ a = 0\) and \(a \circ e = 0\), then there exists \(a' \in C(B, A')\) such that \(a' \circ f = a\).

**Remark 6.2.** If we impose the following conditions (1) and (2) on \(C\) (cf. [BR]), then Condition 6.1 is satisfied.

1. There exists an adjoint natural isomorphism

\[\varphi_{A,B} : C(\Sigma A, B) \xrightarrow{\cong} C(A, \Omega B) \quad (A, B \in C).\]

2. Let \(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A\) and \(\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C'\) be any object in \(\triangleright\) and \(<\), respectively.

   For any \(a \in C(A, \Omega C')\) and \(b \in C(B, A')\) satisfying \(b \circ f = e' \circ a\), there exists \(c \in C(C, B')\) such that \(c \circ g = f' \circ b\) and \(\varphi_{A,C'}^{-1}(a) \circ h = g' \circ c\).

   For any \(c \in C(C, B')\) and \(d \in C(\Sigma A, C')\) satisfying \(d \circ h = g' \circ c\), there exists \(b \in C(B, A')\) such that \(c \circ g = f' \circ b\) and \(b \circ f = e' \circ \varphi_{A,C'}(d)\).

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \\
\circ \downarrow \circ \downarrow \circ \downarrow \\
\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C'
\end{array}
\]

In the rest, \(C\) is assumed to satisfy Condition 5.14. First, we construct the shift functor.

**Lemma 6.3.** Let

\[
\begin{array}{cccc}
\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X, & \Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X, \\
\circ \downarrow \circ \downarrow \circ \downarrow \circ \downarrow \circ \downarrow \circ \downarrow & \circ \downarrow \circ \downarrow \circ \downarrow \circ \downarrow \\
\Omega S \xrightarrow{g} M \xrightarrow{\alpha} I \xrightarrow{\beta} S \xrightarrow{\gamma} \Sigma M, & \Omega S \xrightarrow{g} M \xrightarrow{\alpha} I \xrightarrow{\beta} S \xrightarrow{\gamma} \Sigma M,
\end{array}
\]

be morphisms of conflations, with \(I \in \mathcal{I}_D\). Then \(x = x'\) in \(Z/I_D\) implies \(z = z'\) in \(Z/I_D\).
Proof. Obviously, it suffices to show that $x = 0$ implies $x = 0$ in the first diagram. Since $x = 0$, there exist $I_0 \in \mathcal{I}_D$, $x_1 \in \mathcal{Z}(X, I_0)$ and $x_2 \in \mathcal{Z}(I_0, M)$ such that $x = x_2 \circ x_1$. Since $I_0 \in \mathcal{I}_D$ and $f$ is an inflation, there exists $x_3 \in \mathcal{Z}(Y, I_0)$ such that $x_3 \circ f = x_1$. Thus we have $x \circ e = x_2 \circ x_3 \circ f \circ e = 0$, which implies

$$((\Sigma x) \circ h = -((\Sigma x) \circ \psi(e)) = -\psi(x \circ e) = 0.$$ 

Put $\eta = y - \alpha \circ x_2 \circ x_3$. By $\eta \circ f = 0$, there exists $s \in \mathcal{Z}(Z, I)$ such that $s \circ g = \eta$. Thus we have

$$\gamma \circ (z - \beta \circ s) = \gamma \circ z = (\Sigma x) \circ h = 0,$$
$$z - \beta \circ s) \circ g = z \circ g - \beta \circ y = 0.$$

By (AC1), there exists $t \in \mathcal{Z}(Z, I)$ such that $z - \beta \circ s = \beta \circ t$, namely $z = \beta \circ (s + t)$. □

Construction 6.4. Assume $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough injectives. For any $X \in \mathcal{Z}$, take a conflation

$$\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$$

with $I_X \in \mathcal{I}_D$. Define $S(X) = SX$ to be the image of $S_X$ in $\mathcal{Z}/\mathcal{I}_D$.

For any morphism $f \in \mathcal{Z}(X, Y)$, take a conflation

$$\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y$$

similarly for $Y$. Since $\alpha_X$ is an inflation and $I_Y \in \mathcal{I}_D$, there exists $I_f \in \mathcal{Z}(I_X, I_Y)$ such that $I_f \circ \alpha_X = \alpha_Y \circ f$. By (RTR3), there exists $S_f \in \mathcal{Z}(S_X, S_Y)$ such that $(f, I_f, S_f)$ is a morphism of conflations.

For any $f \in \mathcal{Z}/\mathcal{I}_D(X, Y)$, define $S_f$ to be the image $S_f$ of $S_f$ in $\mathcal{Z}/\mathcal{I}_D$. This is well-defined by Lemma 6.3, and the following proposition holds.

Proposition 6.5. $S: \mathcal{Z}/\mathcal{I}_D \to \mathcal{Z}/\mathcal{I}_D$ gives an additive functor.

Proof. This immediately follows from Lemma 6.3. □

Remark 6.6. Dually, if $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough projectives, then we have an additive functor $S^*: \mathcal{Z}/\mathcal{P}_D \to \mathcal{Z}/\mathcal{P}_D$, defined by a conflation

$$\Omega X \to S^*X \to P_X \to X \to \Sigma S^*X$$

for any $X \in \mathcal{Z}$, where $P_X \in \mathcal{P}_D$.

Proposition 6.7. If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, then $S$ and $S^*$ are quasi-inverses.

Proof. This follows immediately from the definitions of $S$ and $S^*$. □

In the rest, $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is assumed to be Frobenius. Next, we define the class of distinguished triangles on $\mathcal{Z}/\mathcal{I}_D$. 

**Definition 6.8.** Let $\Omega \xrightarrow{\xi} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be any conflation, and take a conflation $\Omega S_X \xrightarrow{\delta} X \xrightarrow{\alpha_X} F_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$ where $I_X \in \mathcal{I}_D$.

If there exist $p \in \mathcal{Z}(Y, I_X)$ and $q \in \mathcal{Z}(Z, S_X)$ satisfying

$$p \circ f = \alpha_X, \quad q \circ g = \beta_X \circ p, \quad \gamma_X \circ q = h$$

(namely, $(id, p, q)$ is a morphism of conflations)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & \circ & \downarrow \circ \\
X & \xrightarrow{\alpha_X} & I_X \\
\end{array} \xrightarrow{g} \quad \begin{array}{ccc}
Z & \xrightarrow{h} & \Sigma X \\
\downarrow & \circ & \downarrow \circ \\
S_X & \xrightarrow{\gamma_X} & \Sigma X \\
\end{array}
\]

then we call the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} S X$$

a **standard triangle**. Remark that by (RTR3) and the injectivity of $I_X$, there exists at least one such pair of morphisms $(p, q)$. We define the class of distinguished triangles $\Delta$ to be the category of triangles (6.1)

$$X \to Y \to Z \to S Z$$

in $\mathcal{Z}/\mathcal{I}_D$, which are isomorphic to standard triangles.

In the rest, we show that $(\mathcal{Z}/\mathcal{I}_D, S, \Delta)$ is a triangulated category.

**Proposition 6.9.** $(\mathcal{Z}/\mathcal{I}_D, S, \Delta)$ satisfies (TR1).

**Proof.**

1. By definition, every diagram (6.1) isomorphic to an object in $\Delta$ also belongs to $\Delta$.
2. Let $f \in \mathcal{Z}(X, Y)$ be any morphism. Take a conflation

$$\Omega S_X \xrightarrow{\delta} X \xrightarrow{\alpha_X} F_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$$

with $I_X \in \mathcal{I}_D$, and put $f_X = (f, -\alpha_X)$. By Corollary 4.3, Lemma 4.4 and Proposition 4.7, $f_X : X \to Y \oplus I_X$ becomes an inflation. In fact, by Corollary 4.3 and Lemma 4.4, there exists an extension

$$\Omega C_f \to X \xrightarrow{f_X} Y \oplus I_X \xrightarrow{c_f} C_f \xrightarrow{\ell_f} \Sigma X,$$

and applying Proposition 4.7 to the following diagram (6.2) of extensions, we obtain an extension

$$\Omega S_X \to Y \xrightarrow{c} \Sigma X \to S Y,$$
and thus $C_f \in \mathcal{Z}$ by the extension-closedness of $\mathcal{Z}$.

Let $C(f)$ denote the image of $C_f$ in $\mathcal{Z}/\mathcal{I}D$. Then the above diagram means

$$X \xrightarrow{f} Y \oplus I_X \xrightarrow{c_f} C_f \xrightarrow{\ell_f} \Sigma X$$

is a standard triangle. If we put $g = c_f \circ i_Y$ where $i_Y : Y \to Y \oplus I_X$ is the inclusion, then $X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{\ell_f} SX$ becomes isomorphic to this standard triangle.

(3) By (RTR1), (RTR2) and (LTR1),

$$0 = \Omega 0 \to X \xrightarrow{id} X \to 0 \to \Sigma X$$

is a conflation, and it immediately follows that the triangle

$$X \xrightarrow{id} X \to 0 \to SX$$

belongs to $\triangle$.

\[\Box\]

**Proposition 6.10.** $(\mathcal{Z}/\mathcal{I}D, S, \triangle)$ satisfies (TR2).

**Proof.** It suffices to show, for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$$

arising from a morphism of conflations

$$\begin{array}{ccc}
\Omega Z & \xrightarrow{e} & X \\
\downarrow \Omega q & & \downarrow \Omega p \\
\Omega SX & \xrightarrow{\delta_X} & I_X
\end{array} \xrightarrow{\gamma_X} \begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow h \\
\Sigma X & & \end{array} \xrightarrow{\Sigma \gamma} \begin{array}{ccc}
\Sigma X & \xrightarrow{\gamma_X} & \Sigma X \\
\downarrow p & & \downarrow q \\
I_X & \xrightarrow{\beta_X} & \Sigma X
\end{array},

the shifted triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} SX \xrightarrow{-Sf} SY$$

also becomes a distinguished triangle.

We may replace $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ by the conflation $\Omega C_f \to X \xrightarrow{f_X} Y \oplus I_X \xrightarrow{c_f} C_f \xrightarrow{\ell_f} \Sigma X$ constructed in the proof of Proposition 6.9. Recall that $f_X = (f, -\alpha_X) = i_Y \circ f - i_{I_X} \circ \alpha_X$ where $i_Y$ and $i_{I_X}$ are the inclusions into $Y \oplus I_X$. 
Take conflations

\[
\begin{align*}
\Omega S_X & \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X, \\
\Omega I_X & \xrightarrow{0} Y \xrightarrow{\iota_Y} Y \oplus I_X \xrightarrow{-p I_X} I_X \xrightarrow{0} \Sigma Y.
\end{align*}
\]

By Proposition 4.7, there exists \(k \in C(\Omega S_X, Y)\) and \(\nu \in Z(C_f, S_X)\) such that

\[
\Omega S_X \xrightarrow{k} Y \xrightarrow{\mu} C_f \xrightarrow{\nu} S_X \rightarrow \Sigma Y
\]

is a conflation, where \(\mu = c_f \circ i_Y\), and

\[
\begin{align*}
\nu \circ c_f &= -\beta_X \circ p I_X, & \gamma_X \circ \nu &= \ell_f, \\
-\psi_{S_X,Y}(k) \circ \beta_X &= 0, & f_X \circ \delta_X + iv \circ k &= 0.
\end{align*}
\]

**Claim 6.11.** We have a morphism of conflations

\[
\begin{align*}
\Omega S_X & \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X, \\
\Omega I_X & \xrightarrow{-id} Y \xrightarrow{\iota_Y} I_X \xrightarrow{p} S_X \xrightarrow{\mu} C_f \xrightarrow{\nu} \Sigma Y.
\end{align*}
\]

**Proof of Claim 6.11.** This immediately follows from

\[
\begin{align*}
f \circ \delta_X &= p_Y \circ f_X \circ \delta_X = -p_Y \circ i_Y \circ k = -k, \\
c_f \circ i_{I_X} \circ \alpha_X &= c_f \circ i_{I_X} \circ (-p I_X) \circ f_X \\
&= c_f \circ (i_Y \circ p_Y \circ f_X - f_X) \\
&= c_f \circ i_Y \circ f, \\
\nu \circ c_f \circ i_{I_X} &= -\beta_X \circ p I_X \circ i_{I_X} = -\beta_X.
\end{align*}
\]

\(\square\)

If we take a conflation \(\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y\) where \(I_Y \in \mathcal{I}_D\), then there exist \(u \in Z(C_f, I_Y)\) and \(v \in Z(S_X, S_Y)\) such that \((id_Y, p, q)\) is a morphism
of conflations.

\[
\begin{align*}
&\Omega S_X \xrightarrow{k} Y \xrightarrow{\mu} C_f \xrightarrow{\nu} S_X \xrightarrow{\Sigma Y} \\
&\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y
\end{align*}
\]

(6.3)

By definition, we have a standard triangle in \(\Delta\)

\[
Y \xrightarrow{\mu} C(f) \xrightarrow{\nu} S_X \xrightarrow{\beta_Y} SY.
\]

Composing (6.3) with the morphism obtained in Claim 6.11, we obtain the following morphism of conflations, which means \(Sf = -v\).

\[
\begin{align*}
&\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X \\
&\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y
\end{align*}
\]

\(\square\)

**Lemma 6.12.** Let

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} SX
\]

and

\[
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{q'} S_X'
\]

be standard triangles in \(\mathbb{Z}/\mathcal{I}_D\) obtained from

\[
\begin{align*}
&\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \\
&\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X
\end{align*}
\]

and

\[
\begin{align*}
&\Omega Z' \xrightarrow{e'} X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X' \\
&\Omega S_{X'} \xrightarrow{\delta_{X'}} X' \xrightarrow{\alpha_{X'}} I_{X'} \xrightarrow{\beta_{X'}} S_{X'} \xrightarrow{\gamma_{X'}} \Sigma X'
\end{align*}
\]

If we are given a morphism of conflations

\[
\begin{align*}
&\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \\
&\Omega Z' \xrightarrow{e'} X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'
\end{align*}
\]

then we obtain the following morphism in \(\Delta\).

\[
\begin{align*}
&X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} SX \\
&X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{q'} S_X'
\end{align*}
\]
Proof. It suffices to show \( q' \circ z = (S_x) \circ q \). By the definition of \( S_x \), we have
\[
(S_x) \circ q = S_x \circ q = (\gamma_X \circ g - \gamma_X \circ q' \circ z - \beta_X \circ s)
\]
there exists \( s \in \mathcal{Z}(Z, I_{X'}) \) such that \( s \circ g = I_x \circ p - p' \circ y \). If we put \( \zeta = S_z \circ q - q' \circ z - \beta_X \circ s \), then \( \zeta \) satisfies
\[
\gamma_{X'} \circ \zeta = (\Sigma) \circ q - \gamma_X \circ q - h' \circ s \circ g
\]
and
\[
\zeta \circ g = S_z \circ q \circ g - q' \circ z \circ g - \beta_X \circ s \circ g
\]
Thus by (AC1), there exists \( t \in \mathcal{Z}(Z, I_{X'}) \) such that \( \zeta = \beta_{X'} \circ t \), i.e.,
\[
S_z \circ q - q' \circ z = \beta_{X'} \circ (s + t).
\]

Proposition 6.13. \((\mathcal{Z}/\mathcal{I}_D, S, \triangle)\) satisfies (TR3).

Proof. Suppose we are given distinguished triangles
\[
\begin{align*}
X & \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX \\
X' & \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} SX'
\end{align*}
\]
and morphisms \( x \in \mathcal{Z}(X, X') \) and \( y \in \mathcal{Z}(Y, Y') \) satisfying \( y \circ f = f' \circ x \). We want to find \( z \in \mathcal{Z}(Z, Z') \) which satisfies \( z \circ q = g' \circ y \) and \( S_z \circ q = q' \circ z \).

We may assume these triangles are standard, arising from morphisms of conflations:
\[
\begin{align*}
\begin{array}{ccc}
\Omega Z & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\Omega S_Z & \xrightarrow{S_X} & SX
\end{array}
\quad
\begin{array}{ccc}
\Omega X & \xrightarrow{f} & \Omega Y \\
\downarrow & & \downarrow \\
\Omega S_X & \xrightarrow{S_Y} & SY
\end{array}
\end{align*}
\]

\qed
Since \( y \circ f = f' \circ x \), there exist \( I \in \mathcal{I}_\mathcal{D}, s_1 \in Z(X, I) \) and \( s_2 \in Z(Y, I') \) such that \( s_2 \circ s_1 = y \circ f - f' \circ x \). By the injectivity of \( I \), there exists \( s_3 \in Z(Y, I) \) such that \( s_3 \circ f = s_1 \). Then we have \( (y - s_2 \circ s_3) \circ f = f' \circ x \), and there exists \( z \in Z(Z, Z') \) such that \( z \circ g = g' \circ (y - s_2 \circ s_3) \) and \((\Sigma x) \circ h = h' \circ z \) by (RTR3). Thus Proposition 6.13 follows from Lemma 6.12.

\[ \begin{array}{c}
\Omega Z \xrightarrow{f} X' \xrightarrow{g'} Y' \xrightarrow{h'} Z' \rightarrow \Sigma X' \\
\end{array} \]

\[ \Omega S_X \xrightarrow{f} X' \xrightarrow{g'} Y' \xrightarrow{h'} Z' \rightarrow \Sigma X' \]

Proposition 6.14. \((\mathcal{Z}/\mathcal{I}_\mathcal{D}, S, \triangle)\) satisfies (TR4).

Proof. Let

\[ \begin{align*}
\text{(6.4)} & \quad X \xrightarrow{f} M \xrightarrow{m'} Y' \xrightarrow{q'} SX \\
\text{(6.5)} & \quad X' \xrightarrow{f'} M \xrightarrow{m'} Y \xrightarrow{q} SX' \\
\text{(6.6)} & \quad X \xrightarrow{f} Y \xrightarrow{q} Z \xrightarrow{g} SX,
\end{align*} \]

be distinguished triangles in \((\mathcal{Z}/\mathcal{I}_\mathcal{D})\) satisfying \( m' \circ f = f \). It suffices to show there exist \( g' \in Z(Y', Z) \) and \( g' \in Z(Z, S_X') \) such that \( X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{g} S_X' \) is a standard triangle, where \( f' = m \circ f' \), and satisfy

\[ \begin{align*}
& g' \circ m = q \circ m', \\
& g' \circ q = q', \\
& q \circ g' = q, \\
& S_X' \circ q' + S_X \circ q = 0.
\end{align*} \]

\[ \begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{q} Z \xrightarrow{g} S_X' \\
\text{\textcircled{}} \end{array} \]

\[ \begin{array}{c}
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{g} S_X' \\
\text{\textcircled{}} \end{array} \]

We may assume (6.4), (6.5), (6.6) are standard triangles, arising from the following morphisms of conflations.

\[ \begin{array}{c}
\Omega Y' \xrightarrow{\ell} X \xrightarrow{f} M \xrightarrow{m} Y' \xrightarrow{q} \Sigma X \\
\Omega S_X \xrightarrow{\ell} X \xrightarrow{f} M \xrightarrow{m} Y' \xrightarrow{q} \Sigma X' \\
\end{array} \]

\[ \begin{array}{c}
\Omega Y \xrightarrow{\ell} X \xrightarrow{f} M \xrightarrow{m} Y' \xrightarrow{q} \Sigma X \\
\Omega Y' \xrightarrow{\ell} X \xrightarrow{f} M \xrightarrow{m} Y' \xrightarrow{q} \Sigma X' \\
\end{array} \]
Claim 6.15. We may assume $m' \circ \ell = f$.

Proof of Claim 6.15. Since $m' \circ \ell = f$, there exist $I \in \mathcal{I}_D$, $f_1 \in Z(X,I)$ and $f_2 \in Z(I,Y)$ such that $f_2 \circ f_1 = f - m' \circ \ell$. Let $i_M : M \to M \oplus I$ and $p_M : M \oplus I \to M$ be the inclusion and the projection, respectively. By Corollary 4.3 and Lemma 4.4, we have extensions

$$
\begin{align*}
\Omega Q &\to X \xrightarrow{(f, f_1)} M \oplus I \to Q \to \Sigma X, \\
\Omega M &\to I \to M \oplus I \xrightarrow{p_M} M \to \Sigma I.
\end{align*}
$$

By Proposition 4.7, we obtain the following morphisms of extensions by Lemma 6.12.

Thus we have $Q \in Z$, and obtain an isomorphism of distinguished triangles:

$$
\begin{align*}
X \xrightarrow{(f, f_1)} M \oplus I &\to Q \to \Sigma X \\
\alpha \xleftarrow{\cong} \beta \xleftarrow{\cong} \gamma &\to \Sigma X.
\end{align*}
$$

Dually, there exist morphisms of extensions

$$
\begin{align*}
\Sigma R &\to Y \xrightarrow{m' + f_2} M \oplus I \xrightarrow{\exists R} \Omega X' \\
\Sigma X' \xrightarrow{-} \Sigma M &\to Y' \xrightarrow{\exists} \Omega X'.
\end{align*}
$$
which implies \( R \in \mathcal{Z} \) and yields an isomorphism of distinguished triangles

\[
\begin{array}{c}
X' \xrightarrow{\ell'} M \xrightarrow{m'} Y \xrightarrow{S} SX' \\
\cong \; \cong \; \cong \; \cong \; \cong \; \cong \\
R \xrightarrow{} M \oplus I \xrightarrow{m'+f_2} Y \xrightarrow{} SR
\end{array}
\]

Thus, replacing \( \ell \) by \((\ell, f_1)\) and \( m' \) by \( m' + f_2 \), we may assume \( m' \circ \ell = f \). \( \square \)

By Claim 6.15, assume \( m' \circ \ell = f \). Then by Proposition 4.7, there exist \( g' \in \mathcal{Z}(Y', Z) \) and \( h' \in \mathcal{C}(Z, \Sigma X') \) such that

\[
\Omega Z \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} \Sigma X'
\]

is a conflation, and make the following diagram commutative.

If we take a morphism of conflations

\[
\Omega Z \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} \Sigma X'
\]

then by Lemma 6.12, we obtain morphisms of standard triangles

\[
\begin{array}{c}
X' \xrightarrow{\ell'} M \xrightarrow{m'} Y \xrightarrow{\alpha} SX' \\
\cong \; \cong \; \cong \\
X' \xrightarrow{f'} Y' \xrightarrow{\beta} Z \xrightarrow{\gamma} SX'
\end{array}
\]

and

\[
\begin{array}{c}
X \xrightarrow{f} M \xrightarrow{m} Y \xrightarrow{s} SX \\
\cong \; \cong \; \cong \\
X \xrightarrow{} Y \xrightarrow{q} Z \xrightarrow{q'} SX
\end{array}
\]

Thus it remains to show \( St' \circ g' + S \ell \circ q = 0 \).

**Claim 6.16.** There exist morphisms of conflations

\[(6.8)\]

\[
\begin{array}{c}
\Omega Z \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \\
\cong \; \cong \; \cong \; \cong \; \cong \\
\Omega S_M \rightarrow M \xrightarrow{\alpha_M} I_M \xrightarrow{\beta_M} S_M \xrightarrow{\gamma_M} \Sigma M
\end{array}
\]

\[(6.9)\]

\[
\begin{array}{c}
\Omega Z \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} \Sigma X' \\
\cong \; \cong \; \cong \; \cong \; \cong \\
\Omega S_M \rightarrow M \xrightarrow{\alpha_M} I_M \xrightarrow{\beta_M} S_M \xrightarrow{\gamma_M} \Sigma M
\end{array}
\]
such that\[ r \circ m' + r' \circ m = \alpha_M.\]

Moreover, \(s\) and \(s'\) satisfy
\[
(6.10) \quad s = S\ell \circ q \quad \text{and} \quad s' = S\ell' \circ q'.
\]

Suppose Claim 6.16 is shown. Then by
\[
(s + s') \circ g \circ m' = s \circ g \circ m' + s' \circ g' \circ m = \beta_M \circ r \circ m' + \beta_M \circ r' \circ m = \beta_M \circ \alpha_M = 0,
\]
there exists \(w' \in C(\Sigma X', S_M)\) such that \(w' \circ n' = (s + s') \circ g\). Thus by \(((s + s') - w' \circ h') \circ g = 0\), there exists \(w \in C(\Sigma X, S_M)\) such that \(w \circ h = s + s' - w' \circ h'\), namely
\[ s + s' = w \circ h + w' \circ h'. \]

Take a conflation
\[ \Omega Z \to X_0 \to I_0 \xrightarrow{\beta_0} Z \xrightarrow{\gamma_0} \Sigma X_0 \]
with \(I_0 \in \mathcal{I}_D\). We have morphisms of conflations
\[
\begin{align*}
\Omega Z & \to X_0 \to I_0 \xrightarrow{\beta_0} Z \xrightarrow{\gamma_0} \Sigma X_0, \\
\Omega Z & \to X' \to Y' \to Z \xrightarrow{h} \Sigma X.
\end{align*}
\]
and thus obtain
\[ s + s' = (w \circ \xi + w' \circ \xi') \circ \gamma_0. \]
Since \(\gamma_M \circ (s + s') = (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0\), we can conclude that \(s + s'\) factors through \(I_M\) by (AC1).

\[
\Omega Z \to X_0 \to I_0 \xrightarrow{\beta_0} Z \xrightarrow{\gamma_0} \Sigma X_0
\]
\[
\Omega S_M \to M \xrightarrow{\alpha_M} I_M \xrightarrow{\beta_M} S_M \xrightarrow{\gamma_M} \Sigma M
\]
By (6.10), this means \(S\ell' \circ q' + S\ell \circ q = 0\), and Proposition 6.14 can be shown. Thus it suffices to show Claim 6.16.

Proof of Claim 6.16. By \(I_M \in \mathcal{I}_D\), there exists \(r \in \mathcal{Z}(Z, I_M)\) such that \(r \circ f = \alpha_M \circ \ell\). By \((\alpha_M - r \circ m') \circ \ell = 0\), there exists \(r' \in \mathcal{Z}(Y', I_M)\) such that \(r' \circ m = \alpha_M - r \circ m'\).

By (RTR3), there exist \(s', s' \in \mathcal{Z}(Z, S_M)\) such that (6.8) and (6.9) are morphisms of conflations.

By definition, \(S_I\) is a morphism which gives a morphism of conflations as follows.

\[
\begin{align*}
\Omega S_X & \to X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X, \\
\Omega S_M & \to M \xrightarrow{\alpha_M} I_M \xrightarrow{\beta_M} S_M \xrightarrow{\gamma_M} \Sigma M.
\end{align*}
\]
Composing with (6.7), we obtain a morphism of conflations

\[
\begin{array}{cccccc}
\Omega Z & \longrightarrow & X & \overset{f}{\longrightarrow} & Y & \overset{g}{\longrightarrow} & Z \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\Omega S & \longrightarrow & M & \overset{\alpha}{\longrightarrow} & I & \overset{\beta}{\longrightarrow} & S & \overset{\gamma}{\longrightarrow} & \Sigma M \\
\end{array}
\]

Thus, comparing with (6.8), we obtain \( s = S\ell \circ q \) by Lemma 6.3. Similarly for \( s' \).

By the above arguments, we obtain the following.

**Theorem 6.17.** Let \( C \) be a pseudo-triangulated category satisfying Condition 6.1, and let \( Z \subseteq C \) be an extension-closed subcategory, and let \( D \subseteq Z \) be a full additive replete subcategory closed under finite direct summands in \( Z \). If \((C, Z, D)\) is Frobenius, then \( Z/\mathcal{I}D \) becomes a triangulated category.

In particular, if \( Z \) is Frobenius, then the stable category \( Z/\mathcal{I} \) becomes a triangulated category.

7. Possibility of further generalizations

In [B], for any triangulated category \( C \), Beligiannis showed that if we are given a proper class of triangles \( \mathcal{E} \) on \( C \) satisfying some conditions similar to the Frobenius condition discussed in section 5, then \( C/P(\mathcal{E}) \) becomes triangulated (Theorem 7.2 in [B]). Here, \( P(\mathcal{E}) \) is the subcategory of ‘projectives’, defined in a similar, but different manner (Definition 4.1 in [B]). With that definition, \( P(\mathcal{E}) \) becomes closed under \( \Sigma \), but this conflicts with Iyama-Yoshino’s construction, in which the factoring category \( D \) satisfies \( C(D, \Sigma D) = 0 \). We wonder if there exists a general construction unifying the construction in [B] and that in section 6.

We also remark that there is another very general construction of a triangulated stable category. In [BM], Beligiannis and Marmaridis constructed a left triangulated category (in the sense of [B] or [BM]) from a pair \((C, \mathcal{X})\) of an additive category \( C \) and a contravariantly finite subcategory \( \mathcal{X} \) assuming some existence condition on kernels (Theorem 2.12 in [BM]). Therefore if \( \mathcal{X} \) is functorially finite and satisfies some nice properties, it is expected that this resulting category becomes triangulated. In fact, Happel’s construction is one of these cases (Remark 2.14 in [BM]). Although this existence condition is not satisfied by a triangulated category \( C \) unless we replace it by some ‘pseudo’ one, we hope some unifying construction will be possible.

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