A solvable model for octupole phonons

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Abstract. A solvable model is proposed for the description of octupole phonons in closed-shell nuclei, formulated in terms of shell-model particle–hole excitations. With some simple assumptions concerning single-particle energies and two-body interactions, closed expressions are derived for the energy and wave function of the octupole phonon. In particular, it is shown that the components of the octupole phonon are proportional to Wigner $3j$ coefficients. This analytic wave function is proven to be exactly valid in light nuclei, which have $LS$ shell closures that coincide with those of the three-dimensional harmonic oscillator, and to be valid to a good approximation in heavier nuclei, which have $jj$ shell closures due to the spin–orbit interaction. The properties of the solvable model are compared with the results of a realistic shell-model calculation for $^{208}$Pb.

1 Introduction

Nuclei with a closed-shell configuration for neutrons and/or protons frequently exhibit low-energy excitations with angular momentum $J = 3$ and negative parity. Such excitations are associated with nuclear shapes that break reflection symmetry and, in particular, with pear-like or octupole shapes \cite{1}. Given the closed-shell configuration of at least one type of nucleon, the nucleus is thought to have a spherical equilibrium shape in its ground state and to exhibit reflection asymmetric oscillations of the octupole type around that shape. Nuclei with neutrons and protons in the valence shell may acquire a permanent ground-state deformation and an open question is whether they can assume a permanent pear-like deformation. Interest in this question was rekindled in 2013 by observed indications of such static octupole deformation in the ground-state configuration of $^{224}$Ra \cite{2}.

By virtue of their supposed collective structure, octupole excitations are thought to exhibit phonon-like behaviour \cite{3}, which renders them of particular interest, being at the cross-roads of microscopic and collective descriptions of nuclei. Consequently, many models of nuclear octupole excitations have been considered in the past (for a review, see Ref. \cite{1}). From a shell-model point of view, an octupole vibration of a closed-shell nucleus corresponds to a coherent superposition of particle–hole excitations. This is the basis of the description of the $3^−$ octupole state in $^{208}$Pb proposed by Brown \cite{4}, leading to results in broad agreement with experimental findings. Such calculations are, however, challenging when extended to more complex structures such

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as multi-phonon states in terms of \( n \)-particle–\( n \)-hole excitations [4] or the coupling of particles or holes to an octupole phonon [5]. It is therefore of interest to construct an approximate but solvable model of octupole phonons in terms of particle–hole excitations. This is the purpose of the present contribution, which is structured as follows. In Sect. 2 some general notions on the description of octupole excitations in a shell-model framework are introduced. The solvable model for octupole excitations is described in Sect. 3, and subsequently applied to \(^{208}\text{Pb}\) and compared with the results of a realistic shell-model calculation in Sect. 4. The paper concludes with a summary and outlook in Sect. 5.

2 General considerations about octupole phonons

It is assumed that all excitations are with reference to a doubly-closed-shell nucleus, which is represented by \( |o\rangle \). The hole orbitals below the shell closure belong to a set \( \{ j_{o,k}, k = 1, 2, \ldots \} \) and the set of particle orbitals above the shell closure is \( \{ j_{p,k}, k' = 1, 2, \ldots \} \). Particle orbitals are consistently denoted with primed indices and hole orbitals with unprimed ones; both occur for neutrons (\( \rho = \nu \)) as well as for protons (\( \rho = \pi \)). A collective octupole phonon corresponds to a coherent superposition of particle–hole excitations with respect to \( |o\rangle \),

\[
|3^-_c \rangle \equiv \sum_{\rho \rho' k} \rho_{\rho' k} |j_{\rho k} j_{\rho' k}^{-1}; 3^- \rangle = \sum_{\rho \rho' k} \rho_{\rho' k} [a_{\rho k}^\dagger \times b_{\rho k}]^{(3)} |o\rangle,
\]

where \( \rho_{\rho' k} \) are coefficients. The operator \( a_{j_{\rho} k}^\dagger \) creates a nucleon particle in the orbital \( j' \) with projection \( m' \) while \( b_{j_{\rho} k}^\dagger \) creates a nucleon hole in the orbital \( j \) with projection \( m \). The single-particle and single-hole states are therefore

\[
|j' m' \rangle = a_{j' m'}^\dagger |o\rangle, \quad |j^{-1} m \rangle = b_{j m}^\dagger |o\rangle,
\]

and particle and hole operators are related through \( b_{j m}^\dagger = (-)^{j+m} a_{j m} \).

The coefficients \( \rho_{\rho' k} \) in Eq. (1) are expressed in the basis \( \{ j_{\rho k} j_{\rho' k}^{-1}; 3^- \} \), which includes neutron as well as proton particle–hole excitations. They result from the diagonalisation of the nuclear Hamiltonian, which generically can be written as

\[
\hat{H} = \sum_{\rho} \hat{H}_\rho + \hat{V}_{\nu\pi} = \sum_{\rho} \left( \sum_{k'} \epsilon_{\rho k} \hat{n}_{\rho k} - \sum_{k} \epsilon_{\rho k} \hat{n}_{\rho k} + \hat{V}_{\rho \rho} \right) + \hat{V}_{\nu\pi},
\]

where \( \epsilon_{\rho k} \) and \( \epsilon_{\rho k'} \) are the single-particle energies pertaining to the orbitals above and below the shell closure, respectively. The structure of the octupole phonon is therefore determined by the matrix elements

\[
\langle j_{\rho k} j_{\rho' k}^{-1}; 3^- | \hat{H} | j_{\rho' k'} j_{\rho k'}^{-1}; 3^- \rangle,
\]

which can be obtained by means of particle–hole conjugation. For the matrix element with \( \rho = \bar{\rho} \) one finds, up to an overall diagonal constant,

\[
\langle j_{\rho k} j_{\rho' k}^{-1}; j | \hat{H}_\rho | j_{\rho' k'} j_{\rho k'}^{-1}; j \rangle
\]

\[
= (\epsilon_{\rho k'} - \epsilon_{\rho k}) \delta_{k k'} \delta_{k' k'} - \sum_{R} (2R + 1) \left\{ j_{\rho k'} j_{\rho k} j_{\rho' k} j_{\rho k'} R \right\} \sqrt{\rho \rho R} \delta_{k' k'},
\]

where \( \rho \) is a doubly-closed-shell nucleus.
where the symbol between curly brackets is a Racah 6j coefficient \[6,7\] and \( V^{ppR}_{k'l'k} \) is a two-body matrix element of the like-nucleon interaction \( \hat{V}_{pp} \),

\[
V^{ppR}_{k'l'k} = \langle j_{k'} j_{k}; R | \hat{V}_{pp} | j_{k'} j_{k}; R \rangle.
\]  

An important feature of the physics of octupole phonons is that, due to the neutron–proton interaction, a non-zero matrix element also exists between a neutron and a proton. Specifically, the relations can be obtained from general principles of particle–hole conjugation, as discussed by Bell \[9\], and summarised in Chapter 3 of the monograph \[10\]. Because considerations of anti-symmetry, it is worthwhile to give a careful derivation of the Pandya relation, following the formalism of Appendix 3B of Ref. \[11\]. Specifically, Eq. (3B-38) of Ref. \[11\], adapted to the present notation, relates reduced matrix elements as follows:

\[
\langle j_{k'} j_{k}^{-1}; J | \hat{V}_{\nu\pi} | j_{k}^{-1}; J \rangle = -\sum_{R} (2R + 1) \left\{ \frac{j_{k'} j_{k} J}{j_{\pi} j_{\pi} R} \right\} V^{\nu\pi R}_{k'l'k},
\]

where \( V^{\nu\pi R}_{k'l'k} \) is a two-body matrix element of the neutron–proton interaction \( \hat{V}_{\nu\pi} \),

\[
V^{\nu\pi R}_{k'l'k} = \langle j_{k'} j_{\pi}; R | \hat{V}_{\nu\pi} | j_{\pi}; R \rangle = -(-)^{j_{\nu} + j_{\nu} - R} \langle j_{k'} j_{\pi}; R | \hat{V}_{\nu\pi} | j_{k'} j_{\pi}; R \rangle.
\]  

The last term on the right-hand side of Eq. (5) and the right-hand side of Eq. (7) represent the Pandya transformation of the particle–particle two-body interaction \[3\]. These relations can be obtained from general principles of particle–hole conjugation, as discussed by Bell \[9\], and summarised in Chapter 3 of the monograph \[10\]. Because considerations of anti-symmetry, it is worthwhile to give a careful derivation of the Pandya relation, following the formalism of Appendix 3B of Ref. \[11\]. Specifically, Eq. (3B-38) of Ref. \[11\], adapted to the present notation, relates reduced matrix elements as follows:

\[
\langle j_{3}^{-1} j_{4}; J | \hat{V} | j_{1}^{-1} j_{2}; J \rangle = \sum_{R} \langle (j_{1} j_{2}) J, (j_{3} j_{4}) J; 0 | (j_{3} j_{4}) R, (j_{1} j_{2}) R; 0 \rangle \langle j_{1} j_{2}; R | \hat{V} | j_{3} j_{4}; R \rangle_{a},
\]

where the first symbol in angle brackets represents a re-coupling coefficient of four angular momenta and the subscript ‘a’ of the second symbol in angle brackets indicates that the reduced matrix element is taken between anti-symmetric two-particle states. This relation implies

\[
\langle j_{3}^{-1} j_{4}; J | \hat{V} | j_{1}^{-1} j_{2}; J \rangle = (-)^{j_{1} + j_{2} + j_{3} + j_{4}} \langle j_{2}^{-1}; J | \hat{V} | j_{1}^{-1} j_{3}; J \rangle
\]

\[
= -\sum_{R} (2R + 1) \left\{ \frac{j_{1} j_{2} J}{j_{3} j_{4} R} \right\} \langle j_{1} j_{2}; R | \hat{V} | j_{3} j_{4}; R \rangle_{a},
\]

\[
= -\sum_{R} (2R + 1) \left\{ \frac{j_{1} j_{2} J}{j_{3} j_{4} R} \right\} (-)^{j_{1} + j_{2} + j_{3} + j_{4}} \langle j_{4} j_{1}; R | \hat{V} | j_{2} j_{3}; R \rangle_{a},
\]

and therefore

\[
\langle j_{1} j_{2}^{-1}; J | \hat{V} | j_{3} j_{4}^{-1}; J \rangle = -\sum_{R} (2R + 1) \left\{ \frac{j_{1} j_{2} J}{j_{3} j_{4} R} \right\} \langle j_{1} j_{2}; R | \hat{V} | j_{3} j_{4}; R \rangle_{a}.
\]

The requirement of anti-symmetry implies that, if \( j_{1} = j_{4} \) or \( j_{2} = j_{3} \), the summation in Eq. (10) is restricted to even values of \( R \). The result (10) agrees with the transformation used in Eq. (5) because the summation in that case is unrestricted since particle-like (primed) and hole-like (unprimed) orbitals belong to different sets. The result (10) also agrees with transformation used in Eq. (7): in that case the orbitals may be the same but one orbital contains a neutron and the other a proton.
It is of interest to carry out an approximate diagonalisation of the matrix with elements (3) in two stages. First, the like-nucleon interaction is diagonalised for neutrons and protons separately with use of the expression (5), leading to a neutron and a proton octupole phonon,

\[ |3^-_{-\nu\pi} \rangle \equiv \sum_{k'k} \tilde{\rho}_{k'k} | j_{\nu k'} j_{\nu k}^{-1}; 3^- \rangle, \quad |3^-_{\nu\pi} \rangle \equiv \sum_{k'k} \tilde{\pi}_{k'k} | j_{\pi k'} j_{\pi k}^{-1}; 3^- \rangle, \]

where coefficients \( \tilde{\rho}_{k'k} \) (with tilde) are used to distinguish them from the coefficients \( \rho_{k'k} \) in Eq. (1), which result from the diagonalisation of the Hamiltonian in the complete particle–hole space. The eigenenergies of the diagonal matrix element \( E_{\nu\pi} \) where \( E \) octupole phonon is lowest in energy, \( \nu\pi \) combination of the neutron and proton octupole phonons.

As will be shown in Sect. 3, for a short-range, attractive neutron–proton interaction, \( \nu\pi \) symmetric in the neutron and proton octupole phonons, depending on the sign of \( V_{\nu\pi} \). Of particular interest is the strength of the transition from the collective octupole phonon to the ground state, which is indicated by the size of its \( B(E3) \) value

\[ B(E3; 3^- \rightarrow 0^+) = \frac{1}{4} \left( |\langle 0^+ | \hat{T}(E3) | 3^- \rangle|^2 \right) \]

Let us dwell a little longer on the solution of Eq. (13). Assume that the neutron octupole phonon is lowest in energy, \( E(3^-_{\nu\pi}) < E(3^-_{c\rho}) \); the opposite case is obtained by interchanging neutron and proton indices in the following. The eigenenergies of the matrix (13) are given by

\[ E(3^-_{c\rho}) = E(3^-_{c\nu}) + \Delta E \pm \sqrt{\Delta E^2 + V_{\nu\pi}^2}, \]

where \( \Delta E = |E(3^-_{\nu\pi}) - E(3^-_{c\nu})|/2 \), and the associated eigenfunctions can be written as

\[ |3^-_{c\pm} \rangle = \frac{1}{\sqrt{2f_v(f_v \pm 1)}} \left[ -\left( 1 \mp f_v \right) |3^-_{c\nu}\rangle + v |3^-_{c\pi}\rangle \right], \]

where \( v = V_{\nu\pi}/\Delta E \) and \( f_v = \sqrt{1 + \nu^2} \geq 1 \). One therefore predicts the occurrence of two collective \( 3^- \) states, which, in the limit of large \( |v| \), are the symmetric and the antisymmetric combinations of the neutron and proton octupole phonons, respectively, since

\[ |3^-_{c\pm}\rangle \xrightarrow{|v| \rightarrow \infty} \frac{1}{\sqrt{2|v|^2}} \left[ \pm |v| |3^-_{c\nu}\rangle + v |3^-_{c\pi}\rangle \right] = \frac{1}{\sqrt{2}} \left[ \pm |3^-_{c\nu}\rangle \pm \text{sign}(V_{\nu\pi}) |3^-_{c\pi}\rangle \right]. \]

The collective octupole phonon that is lowest in energy can be symmetric or antisymmetric in the neutron and proton octupole phonons, depending on the sign of \( V_{\nu\pi} \). As will be shown in Sect. 3 for a short-range, attractive neutron–proton interaction, \( V_{\nu\pi} \) is negative and in that case the low-energy octupole excitation is a symmetric combination of the neutron and proton octupole phonons.

Of particular interest is the strength of the transition from the collective octupole phonon to the ground state, which is indicated by the size of its \( B(E3) \) value

\[ B(E3; 3^- \rightarrow 0^+) = \frac{1}{4} \left( |\langle 0^+ | \hat{T}(E3) | 3^- \rangle|^2 \right) \]
\[ \langle 0^+_L \parallel \hat{T}(E\lambda) \parallel j'j^{-1}; \lambda \rangle = (-)^{l' + j - \lambda} \langle 0^+_L \parallel \hat{T}(E\lambda) \parallel j^{-1}j'; \lambda \rangle = (-)^{j' + j - \lambda} \langle j \parallel \hat{T}(E\lambda) \parallel j' \rangle, \]

where in the last line use is made of Eq. (3B-25) of Ref. [11]. The electric-transition operator of multipolarity \( \lambda \) is \( \hat{T}(E\lambda) = r^3 Y_{\lambda \mu} \), in which case the reduced matrix element in Eq. (20) is obtained from

\[ \langle n\ell j \parallel r^3 Y_{\lambda \mu} \parallel n'\ell' j' \rangle = (-)^{j' - 1/2} \sqrt{(2j + 1)(2\lambda + 1)(2j' + 1)} \left( \begin{array}{ccc} j & \lambda & j' \\ -1/2 & 0 & 1/2 \end{array} \right) b^\lambda j_{n\ell n'}^{(\lambda)}, \]

where the symbol between round brackets is a Wigner 3j coefficient \([6,7]\) and where it is assumed that \((-)^{\lambda + \ell + \ell'} = +1\). The length parameter \( b \) characterises the size of the harmonic-oscillator potential and \( I_{n\ell n'\ell'}^{(\lambda)} \) is the radial integral

\[ I_{n\ell n'\ell'}^{(\lambda)} = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \frac{(r^\lambda \rho_k \rho_{n'\ell'})(r)}{r^2} dr. \]

In the phase convention of radial wave functions \( \rho_k(r) \) that are positive for \( r \to \infty \), the radial integral \([22]\) is the positive quantity

\[ I_{n\ell n'\ell'}^{(\lambda)} = \sqrt{\frac{n! n'!}{4\pi \Gamma(n + \ell + 3/2) \Gamma(n' + \ell' + 3/2)}} I^{(p + 1)} \times \sum_k \left( \begin{array}{ccc} p - \ell - 1/2 \\ n - k \end{array} \right) \left( \begin{array}{ccc} p - \ell' - 1/2 \\ n' - k \end{array} \right) \left( \begin{array}{ccc} p + k \\ k \end{array} \right), \]

with \( p = (\ell + \ell' + \lambda + 1)/2 \).

### 3 A solvable model for octupole phonons

Let us study the features of octupole phonons in a simplified model, which assumes degenerate single-particle energies below and above the shell closures, such that

\[ \epsilon_{\rho k} = \epsilon_{\rho k} = \Delta \rho, \quad \forall k, k', \]

as illustrated on the right-hand side of Fig. [1]. Furthermore, it is assumed that a surface delta interaction (SDI) acts between the nucleons, whose matrix elements in the isospin formalism are \([12]\)

\[ \langle j_1 j_2; JT | \hat{V}_{\text{SDI}} | j_3 j_4; JT \rangle \equiv \langle n_1 \ell_1 j_1, n_2 \ell_2 j_2; JT | \hat{V}_{\text{SDI}} | n_3 \ell_3 j_3, n_4 \ell_4 j_4; JT \rangle \]

\[ = a_T \delta \left( \begin{array}{ccc} j_1 & j_2 & J' \\ 1/2 & -1/2 & 0 \end{array} \right) - \delta_T \left( \begin{array}{ccc} j_1 & j_2 & J \\ 1/2 & -1/2 & -1 \end{array} \right), \]

\[ \delta_T \equiv \sum_{J, J'} \sum_{\epsilon_J} \sum_{\epsilon_{J'}} \left( \begin{array}{ccc} J_1 & J_2 & J' \\ 1/2 & 1/2 & 0 \end{array} \right) \left( \begin{array}{ccc} J_1 & J_2 & J \\ 1/2 & -1/2 & -1 \end{array} \right). \]
structure depends solely on the radial wave functions. Due to the different phase convention for the radial wave functions, the particle–hole matrix element (5) can be written as

$$\langle j_1, j_2; JT = 1 | \hat{V}_{SDI} | j_3, j_4; JT = 1 \rangle = a_1 F F' \left( \begin{array}{cc} j_1 & j_2 \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right) \left( \begin{array}{cc} j_3 & j_4 \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right).$$  \tag{27}

This matrix element must be introduced for the particle–hole matrix element, which requires the evaluation of sums summarised in Appendix A. As a result, for a state with natural parity, that is, with \((-)^{f_{nk} + f_{np} + J} = +1\), the particle–hole matrix element (5) can be written as

$$\langle j_{pk}, j_{pk}^{-1}; J | \hat{H}_p | j_{pl}, j_{pl}^{-1}; J \rangle = \Delta \epsilon \delta_{k'k} \delta_{l'l} + \langle j_{pk}, j_{pk}^{-1}; J | \hat{V}_{SDI} | j_{pl}, j_{pl}^{-1}; J \rangle$$

$$= \Delta \epsilon \delta_{k'k} \delta_{l'l} + a_1 \left( f_{k'k}^p f_{l'l}^p - g_{k'k}^p g_{l'l}^p \right),$$  \tag{28}

where

$$f_{k'k}^p = (-)^{f_{nk}} \sqrt{(2j_{nk} + 1)(2j_{nk'} + 1)} \left( \begin{array}{cc} j_{nk} & j_{nk'} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right),$$

$$g_{k'k}^p = (-)^{f_{nk}' - 1/2} \sqrt{(2j_{nk} + 1)(2j_{nk'} + 1)} \left( \begin{array}{cc} j_{nk'} & j_{nk} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right).$$  \tag{29}

Fig. 1. Single-particle energies as taken in a realistic shell-model calculation for $^{208}$Pb (left) and as assumed in the solvable model (right).
The Hamiltonian matrix [23] can thus be written as the sum of two separable matrices, leading to a two-dimensional subspace that decouples exactly from the complete particle–hole space and to a secular equation in terms of a $2 \times 2$ matrix of the form

$$\frac{\alpha_{1p}}{2} \left[ \sum_{k'k} (f_{k'k}^p)^2 + \sum_{k'k} f_{k'k}^p g_{k'k}^p - \sum_{k'k} (g_{k'k}^p)^2 \right].$$  \hspace{1cm} (30)

The eigenvalues of this matrix correspond exactly to the two non-zero eigenvalues in the complete space. Furthermore, in most applications the off-diagonal matrix element in Eq. (30) is small compared with the diagonal matrix elements because of a property of Wigner $3j$ coefficients (see Appendix B),

$$\sum_{k'k} f_{k'k}^p g_{k'k}^p \ll \sum_{k'k} (f_{k'k}^p)^2 + \sum_{k'k} (g_{k'k}^p)^2. \hspace{1cm} (31)$$

The following approximation can therefore be made for the energy and wave function of the neutron and proton octupole phonons:

$$E(3c_p) \approx \Delta \epsilon_p - \frac{\alpha_{1p}}{2} S_p = \Delta \epsilon_p - \frac{\alpha_{1p}}{2} \sum_{k'k} (2j_{pk} + 1)(2j_{pk'} + 1) \left( \frac{j_{pk'}}{2} - \frac{1}{2} 0 \right)^2, \hspace{1cm} (32)$$

and

$$\tilde{p}_{k'k} \approx \frac{g_{k'k}^p}{\sqrt{S_p}} = (-)j_{pk}^{-1/2} \sqrt{\frac{(2j_{pk} + 1)(2j_{pk'} + 1)}{S_p}} \left( \frac{j_{pk'}}{2} - \frac{1}{2} 0 \right), \hspace{1cm} (33)$$

where $S_p$ is the sum

$$S_p = \sum_{k'k} (g_{k'k}^p)^2 = \sum_{k'k} (2j_{pk} + 1)(2j_{pk'} + 1) \left( \frac{j_{pk'}}{2} - \frac{1}{2} 0 \right)^2. \hspace{1cm} (34)$$

Equation (32) gives an approximate expression for the diagonal matrix elements in the $2 \times 2$ matrix (13). The remaining problem is to calculate the off-diagonal matrix element $V_{\nu \tau}$, originating from the neutron–proton interaction, which for the SDI has the following matrix elements

$$\langle j_{\nu k'}j_{\pi k}; J \mid \hat{V}_{\nu \tau}^{SDI} \mid j_{\nu k'}j_{\pi k}; J \rangle = \frac{1}{2^{F}} \left\{ b_1 \left( j_{\nu k'} j_{\pi k} J \frac{J}{2} - \frac{1}{2} 0 \right) \left( j_{\nu k'} j_{\pi k} J \frac{J}{2} - \frac{1}{2} 1 \right) + b_0 \left( j_{\nu k'} j_{\pi k} J \frac{J}{2} - \frac{1}{2} -1 \right) \left( j_{\nu k'} j_{\pi k} J \frac{J}{2} - \frac{1}{2} -1 \right) \right\}, \hspace{1cm} (35)$$

with

$$b_1 = (-)J_{\nu k'} + J_{\pi k} + \epsilon_{\nu k'} + \epsilon_{\pi k} \left[ \frac{a_1 + a_0}{2} + (-)^{J + \epsilon_{\nu k'} + \epsilon_{\pi k}} \frac{a_1 - a_0}{2} \right], \quad b_0 = -a_0. \hspace{1cm} (36)$$

If this expression is introduced in the particle–hole matrix element [7], the resulting sums can be evaluated (see Appendix A), leading to

$$\langle j_{\nu k'}j_{\pi k}; J \mid \hat{V}_{\nu \tau}^{SDI} \mid j_{\pi k'}j_{\nu k}; J \rangle = \frac{1}{2^{F}} \left\{ b'_1 \left( j_{\nu k'} j_{\pi k} J \frac{J}{2} - \frac{1}{2} 0 \right) \left( j_{\pi k'} j_{\nu k} J \frac{J}{2} - \frac{1}{2} 1 \right) + b'_0 \left( j_{\nu k'} j_{\nu k} J \frac{J}{2} - \frac{1}{2} -1 \right) \left( j_{\pi k'} j_{\nu k} J \frac{J}{2} - \frac{1}{2} -1 \right) \right\}, \hspace{1cm} (37)$$
with

\[ b'_1 = (-)^{j_{\nu'k}+j_{\nu'\ell}} \left[ (-)^{\ell_{\nu'k}+\ell_{\nu'\ell}}, \frac{a_1 + a_0}{2} + a_0 \right], \quad b'_0 = (-)^{j_{\nu'k}+j_{\nu'\ell}} \frac{a_1 - a_0}{2}. \]  \tag{38} \]

For a state with natural parity, \((-)^{j_{\nu'k}+j_{\nu'\ell}+J} = (-)^{\ell_{\nu'k}+\ell_{\nu'\ell}+J} = +1\), the matrix element reduces to

\[ \langle j_{\nu'k}, j_{\nu'\ell}; J | \hat{V}_{\nu'\pi}^{\text{SDI}} | j_{\nu\ell}, j_{\nu\ell}^{-1}; J \rangle = \frac{a_1 - a_0}{4} f_{k'k} f_{\nu'\ell}^{\nu} - \frac{a_1 + 3a_0}{4} g_{k'k}^{\nu} g_{\nu'\ell}^{\nu}. \]  \tag{39} \]

In a final step the particle–hole matrix element (39) is introduced in the expansion (14) with use of the coefficients (33), which are approximately valid for a SDI. The sums to be evaluated have the property (see Appendix [B])

\[ \sum_{k'k'\ell'\ell} f_{k'k}^{\nu} g_{\nu'\ell}^{\nu} g_{\nu'\ell'}^{\nu} \ll \sum_{k'k'\ell'\ell} g_{k'k}^{\nu} g_{\nu'\ell}^{\nu} g_{\nu'\ell'}^{\nu}, \]  \tag{40} \]

such that, to a good approximation, one obtains

\[ V_{\nu\pi} \approx -\frac{a_1 + 3a_0}{4} \sqrt{S_{\nu} S_{\pi}}. \]  \tag{41} \]

In summary, if one assumes degenerate single-particle energies below and above the shell closures and if a SDI among the nucleons is taken, the energies of the symmetric and anti-symmetric octupole phonons are, to a good approximation, obtained from the 2 \times 2 matrix

\[ \begin{bmatrix} \Delta \epsilon_{\nu} - \frac{a_1}{2} S_{\nu} & -\frac{a_1 + 3a_0}{4} \sqrt{S_{\nu} S_{\pi}} \\ -\frac{a_1 + 3a_0}{4} \sqrt{S_{\nu} S_{\pi}} & \Delta \epsilon_{\pi} - \frac{a_1}{2} S_{\pi} \end{bmatrix}. \]  \tag{42} \]

Since the isoscalar and isovector strengths \(a_0\) and \(a_1\) are positive, the off-diagonal element of this matrix is negative and therefore the low-energy collective 3− state is the (approximately) symmetric combination of the neutron and proton octupole phonons, which in general can be written as

\[ |3_{\gamma}^-\rangle = \alpha_{\nu} |3_{\nu}^-\rangle + \alpha_{\pi} |3_{\pi}^-\rangle, \]  \tag{43} \]

where \(\alpha_{\nu}\) and \(\alpha_{\pi}\) have the same sign.

For the E3 transition strength one finds under the same assumptions

\[ \langle 0^+_1 | \hat{T}_p (E3) | 3_{\gamma}^- \rangle = \sum_{k'/k} \rho_{k'k} \langle 0^+_1 | \hat{T}_p (E3) | j_{\nu'k}, j_{\nu'\ell}^{-1}; 3^- \rangle \]

\[ \approx \sum_{k'/k} \rho_{k'k} \langle 0^+_1 | \hat{T}_p (E3) | j_{\nu'k}, j_{\nu'\ell}^{-1}; 3^- \rangle \frac{S_{\nu}^{(3)}}{\sqrt{S_p}} b^3, \]  \tag{44} \]

where

\[ S_{\nu}^{(3)} \equiv \sqrt{2\lambda + 1} \sum_{k'/k} (2j_{\nu'k} + 1)(2j_{\nu'\ell} + 1) \left( j_{\nu'k} \frac{j_{\nu'\ell}}{2} \frac{\lambda}{2} 0 \right)^2 f_{\nu'k}^{(3)} f_{\nu'\ell}^{(3)} f_{\nu'k'}^{(3)} f_{\nu'\ell'}^{(3)}. \]  \tag{45} \]

The total E3 transition strength from the low-energy collective octupole excitation 3− to the ground state is therefore

\[ \langle 0^+_1 | \hat{T} (E3) | 3_{\gamma}^- \rangle = \left( e_{\nu} \alpha_{\nu} \frac{S_{\nu}^{(3)}}{\sqrt{S_p}} + e_{\pi} \alpha_{\pi} \frac{S_{\pi}^{(3)}}{\sqrt{S_p}} \right) b^3. \]  \tag{46} \]
All contributions in this expression add coherently. This is so for the terms appearing in the sum (45) because, as remarked earlier, the phase convention is such that all radial integrals $f_{n^k \nu^\prime}^{(3)}$ are positive. Furthermore, the neutron and proton contributions in Eq. (46) add coherently because the $3^-_\alpha$ state is the symmetric combination of the neutron and proton octupole phonons, implying that $\alpha_\rho$ and $\alpha_\pi$ have the same sign.

4 An application to $^{208}$Pb

Let us now investigate to what extent the simple properties of the collective octupole state in the solvable model of the previous section are found in a shell-model calculation with a realistic single-particle space and two-body interaction. As an example we consider the nucleus $^{208}$Pb.

The single-particle orbitals in the application presented in this section span two major oscillator shells. For the neutrons they include the orbitals below the $N = 126$ shell closure, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, $1f_{7/2}$, $0h_{9/2}$ and $0i_{13/2}$, and the ones above, $3s_{1/2}$, $2d_{3/2}$, $2d_{5/2}$, $1g_{7/2}$, $1g_{9/2}$, $0i_{11/2}$ and $0j_{15/2}$; for the protons they include the orbitals below the $Z = 82$ shell closure, $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$ and $0h_{11/2}$, and the ones above, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, $1f_{7/2}$, $0h_{9/2}$ and $0i_{13/2}$.

For this heavy nucleus the major shells obviously do not coincide with those of a harmonic oscillator and display intruding and extruding orbitals. As a result, the off-diagonal element in the matrix (30) is not exactly zero (see Appendix B) and a first question that arises is to what extent the analytic components (33) in terms of Wigner 3$j$ coefficients are valid. This is illustrated in Fig. 2, which compares the expression (33) with the results of a numerical calculation using the above single-particle space with degenerate single-particle energies and a SDI among the nucleons.

While the energy of $3^-_\alpha$ state depends on the splitting $\Delta \epsilon_\rho$ and the strength $\alpha_\rho$ of the SDI, its wave function is parameter free and determined by the matrix (30). The numerically calculated wave function therefore only depends on the choice of single-particle space. Furthermore, the correspondence between the wave function obtained numerically and the analytic components in terms of Wigner 3$j$ coefficients is seen to be excellent, which proves that the approximation (31) applies to a realistic choice of the single-particle space for $^{208}$Pb.

Nevertheless, actual single-particle energies are not degenerate nor do the SDI two-body matrix elements coincide with those taken in a realistic calculation. The single-particle energies appropriate for the $^{208}$Pb region have been deduced from the data by Rejmund et al. [13] and these are shown on the right-hand side of Fig. 1. There are about 35000 two-body matrix elements in this single-particle space, which can be obtained in a variety of ways, as described by Brown [4]. The set used in the present calculation is taken from Refs. [14,15]. Figure 3 compares the wave function components of the neutron and proton octupole states obtained in a realistic calculation with the analytic expression in terms of Wigner 3$j$ coefficients. The main conclusion from these results is that much of the collectivity vanishes from the lowest $3^-$ excitations since, in particular for the protons, the wave function is dominated by only a few components.

It must be emphasised, however, that the components of the neutron and proton octupole states as shown in Fig. 3 result from the neutron–neutron and proton–proton interactions only. They correspond to the coefficients $\rho_{k^\prime k}$ as obtained in the two-stage diagonalisation, which, as explained in Sect. 2, may be different from the coefficients $p_{k^\prime k}$, resulting from the diagonalisation of the Hamiltonian in the complete particle–hole space. The latter are shown in Fig. 4. The top panel compares the components, obtained from Eq. (43) together with the analytic expression (33), with those from a numerical calculation with degenerate single-particle energies and a SDI among
Fig. 2. The particle–hole components of the wave function of the low-energy collective $3_{c\rho}^{-}$ states for neutrons (top) and protons (bottom), calculated numerically (black) and analytically from Eq. (33) (red). The single-particle space is appropriate for $^{208}$Pb (see text). The numerical results are obtained with degenerate single-particle energies and with a SDI among the nucleons. On the $x$-axis are indicated all possible configurations in a simplified notation, e.g. $s_{1}f_{5/2}^{-1}$ stands for $3s_{1/2}1f_{5/2}^{-1}$.

the nucleons. It displays therefore the same information as in Fig. 2 but with each $3_{c\rho}^{-}$ component multiplied with $\alpha_{\rho}$ as obtained from the diagonalisation of the $2 \times 2$ matrix [42]. The bottom panel of Fig. 4 compares the same analytic expression with the components resulting from the realistic shell-model calculation. It is clear that some of the components are enhanced while others are suppressed as compared to the simple prescription in terms of a Wigner $3j$ coefficient. Nevertheless, it should be noted that the phases of the realistic particle–hole wave function, which constitute an important aspect of the collective octupole structure, are in complete agreement with those predicted by the Wigner $3j$ coefficient.
Fig. 3. The particle–hole components of the wave function of the low-energy collective $3_{cv}^-$ states for neutrons (top) and protons (bottom), calculated numerically (black) and analytically from Eq. (33) (red). The numerical results are obtained with realistic single-particle energies and two-body interaction appropriate for $^{208}$Pb (see text). On the $x$-axis are indicated all possible configurations in a simplified notation, e.g. $s_1 f_{5/2}^{-1}$ stands for $3s_{1/2} f_{5/2}^{-1}$.

5 Summary and outlook

The ingredients of the solvable model for octupole phonons in doubly-closed-shell nuclei are (i) degenerate single-particle energies above and below the shell closures and (ii) a surface delta interaction (SDI) among the nucleons. With these assumptions it follows that the two-dimensional space spanned by the collective neutron and proton octupole excitations to a very good approximation decouples from the full particle–hole space. The resulting neutron and proton octupole phonons are strongly coupled by the neutron–proton interaction, producing a symmetric combination at low energy and an anti-symmetric one (sometimes referred to as isovector) at higher energy.
Fig. 4. The particle–hole components of the wave function of the low-energy collective $3_c^-$ state, calculated numerically (black) and analytically from Eqs. (33) and (43) (red). The top panel compares the analytic expression with a numerical calculation with degenerate single-particle energies and a SDI among the nucleons. The bottom panel compares the same analytic expression with the results of a realistic shell-model calculation as described in the text.

The appealing property of this simplified analysis is that the wave-function components of the neutron and proton octupole phonons are essentially given by Wigner 3$j$ coefficients.

Confronted with the results of a shell-model calculation with realistic single-particle energies and two-body interaction matrix elements, differences with the above solvable model are revealed. In particular, the neutron–neutron and proton–proton interactions by themselves do not lead to collective neutron and proton octupole excitations since much of the collectivity disappears from the low-energy $3^-$ states mainly due to the non-degeneracy of the single-particle levels. Therefore, the decoupling property mentioned above is not confirmed by a realistic shell-model calculation. However, when all interactions are active, including the neutron–proton interaction, octupole collectivity is regained and the simple structure of the solvable model is approximately recovered. One can therefore paradoxically claim that the separate
neutron and proton octupole phonons exist by virtue of the neutron–proton interaction, which, besides generating their collective structure, also couples them.

The proposed solvable octupole model might be of use in the study of more complex systems or as a guidance to answer more intricate questions. An example is the analysis of odd-mass nuclei as regards the coupling of a particle or hole to an octupole phonon, as was recently discussed for $^{207}$Pb [5]. The issue of double octupole phonons may provide another example since the solvable octupole model might be amenable to an extension to two-particle–two-hole excitations. These problems are currently under study.

Acknowledgements

I wish to thank Emmanuel Clément, Antoine Lemasson and Maurycy Rejmund for raising my interest in this problem and for many fruitful discussions, and Maurycy Rejmund for providing the realistic shell-model single-particle energies and two-body matrix elements.

A Sums of $3j$ and $6j$ coefficients

In the treatment of the Pandya transformation of the SDI the following sums are needed:

$$S_0 = \sum_R (2R + 1) \left( \begin{array}{ccc} j_v & j_i & R \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) \left( \begin{array}{ccc} j_v & j_k & R \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) \left\{ j_v & j_k & J \right\} ,$$

$$\tilde{S}_0 = \sum_R (-)^R (2R + 1) \left( \begin{array}{ccc} j_v & j_i & R \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) \left( \begin{array}{ccc} j_v & j_k & R \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) \left\{ j_v & j_k & J \right\} ,$$

$$S_1 = \sum_R (2R + 1) \left( \begin{array}{ccc} j_v & j_i & R \\ \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right) \left( \begin{array}{ccc} j_v & j_k & R \\ \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right) \left\{ j_v & j_k & J \right\} ;$$

(47)

With use of Eq. (15.14) of Ref. [9] these sums reduce to

$$S_0 = S_1 = -(-)^{j_k + j_i} \left( \begin{array}{ccc} j_v & j_k & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right) \left( \begin{array}{ccc} j_v & j_i & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right) ,$$

$$\tilde{S}_0 = -(-)^{j_i} \left( \begin{array}{ccc} j_v & j_k & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) \left( \begin{array}{ccc} j_v & j_i & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) .$$

(48)

B Properties of the $2 \times 2$ matrix

The $2 \times 2$ matrix [30] in the model with degenerate single-particle energies and a SDI among the nucleons has the following elements:

$$\sum_{k'k} f_{k'k}^\rho g_{k'k}^\rho , \quad \sum_{k'k} (f_{k'k}^\rho )^2 , \quad \sum_{k'k} (g_{k'k}^\rho )^2 .$$

(49)

Such sums occur for the neutrons ($\rho = \nu$) as well as the protons ($\rho = \pi$). The summation indices run over the orbitals below the shell closure (unprimed $k$) and over the orbitals above the shell closure (primed $k'$), and the sums obviously depend on the orbitals included in the model space.
Let us first assume, as it happens in light nuclei (e.g., $^{16}\text{O}$ or $^{40}\text{Ca}$), that the shell closures coincide with those of the harmonic oscillator, in which case $j$ runs over all orbitals in a major oscillator shell, say $N, j = \frac{1}{2}, \frac{3}{2}, \ldots, N + \frac{1}{2}$, while $j'$ runs over all orbitals in the next oscillator shell $N + 1$. The first sum in Eq. (49) corresponds to

$$
\sum_{j,j'}(-)^{j+j'-1/2}(2j+1)(2j'+1)\left(\frac{j'}{2} \frac{j}{2} - 1 \frac{j'}{2} \frac{j}{2} - 1\right)\left(\frac{j'}{2} \frac{j}{2} - 1 \frac{j'}{2} \frac{j}{2} - 1\right),
$$

(50)

and can be rewritten as follows:

$$
\sum_{j}(-)^{j+j'-1/2}(2j+1)\sum_{j'=|j-j'|}^{j+j} (2j'+1)\left(\frac{j}{2} - \frac{1}{2} \frac{j'}{2} - \frac{1}{2}\right)\left(\frac{j}{2} - \frac{1}{2} \frac{j}{2} - \frac{1}{2}\right)
- \sum_{j=N-J+5/2}^{N+1/2}(-)^{j+j'-1/2}(2j+1)\sum_{j'=N+5/2}^{N+J+3/2} (2j'+1)\left(\frac{j'}{2} - \frac{1}{2} \frac{j}{2} - \frac{1}{2}\right)\left(\frac{j'}{2} - \frac{1}{2} \frac{j}{2} - \frac{1}{2}\right).
$$

(51)

Since the sum over $j'$ in the first term of Eq. (51) is unrestricted, orthogonality of the Wigner 3j coefficients implies that it vanishes. The second term corrects for the over-counting in the first sum and, for $J = 3$, it equals

$$
-4(N+1)(N+4)\left(\frac{N}{2} \frac{N}{2} + \frac{1}{2} \frac{3}{2} \frac{1}{2} + 1\right)\left(\frac{N}{2} \frac{N}{2} + \frac{1}{2} \frac{3}{2} \frac{1}{2} + 0\right)
+4(N+2)(N+4)\left(\frac{N}{2} \frac{N}{2} + \frac{3}{2} \frac{3}{2} \frac{1}{2} + 1\right)\left(\frac{N}{2} \frac{N}{2} + \frac{3}{2} \frac{3}{2} \frac{1}{2} + 0\right)
+4(N+2)(N+5)\left(\frac{N}{2} \frac{N}{2} + \frac{3}{2} \frac{3}{2} \frac{1}{2} + 1\right)\left(\frac{N}{2} \frac{N}{2} + \frac{3}{2} \frac{3}{2} \frac{1}{2} + 0\right),
$$

(52)

of which it can be shown that it also vanishes. It follows therefore that

$$
\sum_{k'k} f_{k'k} g_{k'k} = 0.
$$

(53)

The second and third sum in Eq. (49) can be worked out in a similar fashion, leading to

$$
\sum_{k'k} (f_{k'k})^2 = \sum_{k'k} (g_{k'k})^2 = \frac{N(N+1)(N+2)(4N+11)}{(2N+3)(2N+5)}.
$$

(54)

The conclusion is therefore that, in light nuclei with harmonic-oscillator shell closures for neutrons and protons, the matrix (50) is diagonal. Furthermore, the energy of the lowest octupole excitation is given by

$$
E(3c^-) = \Delta\epsilon_{\rho} - a_{1\rho} \frac{N_{\rho}(N_{\rho} + 1)(N_{\rho} + 2)(4N_{\rho} + 11)}{2(2N_{\rho} + 3)(2N_{\rho} + 5)}
\approx \Delta\epsilon_{\rho} - \frac{a_{1\rho}}{2} \left(\frac{7}{4} N_{\rho} - \frac{1}{2} + \cdots\right),
$$

(55)

where $N_{\rho}$ is the major oscillator quantum number associated with the hole orbitals below the shell closure for the neutrons ($\rho = \nu$) and for the protons ($\rho = \pi$).

In heavier nuclei, due to the spin–orbit splitting, unnatural orbitals intrude into the harmonic-oscillator shells. If $N$ denotes again the oscillator quantum number of the orbitals below the shell closure, then the single-particle angular momentum of
the intruding orbital below the shell closure is $N + \frac{3}{2}$ while it is $N + \frac{5}{2}$ above the shell closure. Because of parity, only one additional term is needed to each of the sums (49), corresponding to a particle–hole excitation between the intruder orbitals. The corrections to the first, second and third sum of Eq. (49) are, respectively,

$$4(N + 2)(N + 3) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right),$$

$$4(N + 2)(N + 3) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right),$$

$$4(N + 2)(N + 3) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{3}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} N + \frac{5}{2} \\ \frac{1}{2} \end{array} \right).$$

Insertion of the expressions for the Wigner $3 \! j$ coefficients leads to the following results:

$$\sum_{k'} k f_{k' k} g_{k' k} = \sqrt{3}(N + 2)(N + 3)(N^2 + 5N + 4),$$

$$\sum_{k'} (f_{k' k})^2 = \frac{(N - 1)N(N + 1)(4N + 7)}{(2N + 1)(2N + 3)} + \frac{(N + 1)(N + 2)(N + 3)(N + 4)}{4(2N + 3)(2N + 5)(2N + 7)},$$

$$\sum_{k'} (g_{k' k})^2 = \frac{(N - 1)N(N + 1)(4N + 7)}{(2N + 1)(2N + 3)} + \frac{3(N + 1)(N + 2)(N + 3)(N + 4)}{(2N + 3)(2N + 5)(2N + 7)}.$$

The conclusion is therefore that, in heavier nuclei with a spin–orbit shell closure for the $\rho$ nucleon, the matrix (30) to leading orders in $N_{\rho}$ can be written as

$$\frac{a_{1 \rho}}{2} \left[ \begin{array}{c} N_{\rho}^2 - \frac{7}{32}N_{\rho} - \frac{75}{64} + \cdots \\ \frac{1}{16}N_{\rho} + \frac{5}{32} + \cdots \end{array} \right].$$

This shows that the off-diagonal matrix element is small compared to the difference between the diagonal matrix elements. It follows that the energy of the lowest octupole excitation is approximately

$$E(3_{1\rho}) \approx \Delta \epsilon_{\rho} - \frac{a_{1 \rho}}{2} \left( N_{\rho}^2 + \frac{1}{8}N_{\rho} - \frac{163}{512} + \cdots \right),$$

which gives the first few terms in a $1/N_{\rho}$ expansion.

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