Linear time determination of the scattering number for strictly chordal graphs

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Abstract

The scattering number of a graph $G$ was defined by Jung in 1978 as $sc(G) = \max\{\omega(G - S) - |S|, S \subseteq V, \omega(G - S) \neq 1\}$ where $\omega(G - S)$ is the number of connected components of the graph $G - S$. It is a measure of vulnerability of a graph and it has a direct relationship with the toughness of a graph. Strictly chordal graphs, also known as block duplicate graphs, are a subclass of chordal graphs that includes block and 3-leaf power graphs. In this paper we present a linear time solution for the determination of the scattering number and scattering set of strictly chordal graphs. We show that, although the knowledge of the toughness of the class is helpful, it is not sufficient to provide an immediate result for determining the scattering number.

Keywords: strictly chordal graph; scattering number; toughness; minimal vertex separator.

1 Introduction

Vulnerability in graphs is mainly related to the study of a graph when some of its elements are removed. There are many well known measures of vulnerability based on subsets of vertices. Some are more usual as connectivity, domination number, domatic number and independence number; another

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ones are more recent as toughness \[12\], binding number \[34\], scattering number \[21\], integrity \[3\], tenacity \[13\] and rupture degree \[27\]. Relationships between these invariants were studied in \[3\] and \[36\].

Toughness and scattering number remind vertex connectivity since both consider the cardinality of a separator but they also take into account the number of remaining connected components after its removal. The toughness of a graph was introduced by Chvátal in 1973 \[12\]. A graph \(G = (V, E)\) is \(t\)-tough if \(|S| \geq t \omega(G - S)\) for every subset \(S \subseteq V\) with \(\omega(G - S) > 1\). The toughness of \(G\), denoted \(\tau(G)\), is the maximum value of \(t\) for which \(G\) is \(t\)-tough, taking \(\tau(K_n) = \infty, n \geq 1\). Therefore, if \(G\) is not complete, 

\[
\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S)} \right\}
\]

where the minimum is taken over all separators \(S\) of vertices in \(G\). Subset \(S\) for which this minimum is attained is called a tough set. The scattering number of a graph \(G\) was defined by Jung in 1978 \[21\] as 

\[
sc(G) = \max \{ \omega(G - S) - |S|, S \subseteq V \text{ and } \omega(G - S) \neq 1 \}
\]

Subset \(S\) for which this maximum is attained is called a scattering set. In both definitions, \(\omega(G - S)\) is the number of connected components of the graph \(G - S\). Kratsch et al. \[23\] related toughness and scattering number as follows.

**Lemma 1** \[23\] For every graph \(G\) holds \(\tau(G) \geq 1\) if and only if \(sc(G) \leq 0\).

Furthermore, they presented an important result which provides an algorithmic approach to computing the scattering number.

**Theorem 1** \[23\] Let \(G = (V, E)\) be a graph which is not complete. Then

\[
sc(G) = \max_S \left\{ \sum_{i=1}^{k} \max \{ sc(G[C_i]), 1 \} - |S| \right\}
\]

where the maximum is taken over all minimal separators \(S\) of the graph \(G\) and \(C_1, \ldots, C_k\) are the connected components of \(G[V \setminus S]\).

For general graphs, the corresponding computational problems to determine these invariants are in \(NP\) \[5, 35\]. Few classes of graphs present polynomial results when dealing with the determination of the toughness: interval graphs and trapezoid graphs \[21\], cocomparability graph with \(\tau(G) \geq 1\) \[13\] and claw-free graphs, split graphs and \(2K_2\)-free graphs \[10\]. Markenzon and Waga \[32\] presented a linear time determination of the toughness of strictly chordal graphs. The scattering number of interval graphs \[11\], grid graphs and cartesian product of two complete graphs \[35\] and gear graphs \[2\] can be solved in linear time; however, trapezoid graphs maintain the polynomial result to the scattering number \[21\]. Observe that this invariant has some
variations: edge scattering number \([1]\) opts for \(S\) as a subset of edges of the graph; isolated scattering number \([33, 25]\) focus on components that are isolated vertices of \(G - S\) and weighted scattering number \([26]\) takes into account the importance of the vertices in the graph establishing weight for them.

The block duplicate graphs, a subclass of ptolemaic graphs, were introduced by Golumbic and Peled \([18]\). The class was also defined as strictly chordal graphs by Kennedy in \([22]\) based on hypergraph properties and it was proved to be gem-free and dart-free \([18, 22]\). Brandstädt and Wagner \([9]\) showed that the class is the same as the \((4, 6)\)-leaf power graphs. Strictly chordal graphs can also be characterized in terms of the structure of their separators \([30]\). Some known subclasses of strictly chordal graphs are: block graphs \([19]\), AC-graphs \([6]\), 3-leaf power graphs \([15, 8]\), strictly interval graphs \([31]\) and generalized core-satellite graphs \([16]\).

In his seminal paper, Jung stated that “the scattering number is in a certain sense the additive dual” for the concept of toughness”. Hence, as the toughness of a strictly chordal graph is already known, we study the determination of its scattering number exploring this relationship. However, although the determination of the toughness is quite simple, our proposed task was not straightforward. Firstly we show that, for \(\tau(G) \geq 1\), the determination of the scattering number is quite similar to the determination of the toughness; the scattering set is also composed by a sole minimal vertex separator of \(G\) but a tough set is seldom a scattering set. For \(\tau(G) < 1\), we need to establish a further partition of the set of graphs. After setting apart the graphs for which the scattering number is equal to one (we call them type A graphs) for the remaining graphs (type B graphs) a more algorithmic approach is required. For them, a scattering set must be build in order to determine their scattering number; this set can be composed by one or more minimal vertex separators of \(G\). In all cases our solution has linear time complexity.

2 Background

Basic concepts about chordal graphs (graphs possessing no chordless cycles) are assumed to be known and can be found in Blair and Peyton \([7]\) and Golumbic \([17]\). In this section, the most pertinent concepts are reviewed.

Let \(G = (V, E)\) be a connected graph, where \(|E| = m\) and \(|V| = n\). The neighborhood of a vertex \(v \in V\) is denoted by \(N(v) = \{w \in V; \{v, w\} \in E\}\) and its closed neighborhood by \(N[v] = N(v) \cup \{v\}\). Two vertices \(u\) and \(v\) are true twins in \(G\) if \(N[u] = N[v]\). A vertex \(v\) is said to be simplicial in \(G\) when \(N(v)\) is a clique in \(G\). For any \(H \subseteq V\), the subgraph of \(G\) induced by \(H\) is
Let $G = (V, E)$ be a chordal graph and $u, v \in V$. A subset $S \subset V$ is a separator of $G$ if at least two vertices in the same connected component of $G$ are in two distinct connected components of $G[V \setminus S]$; $S$ is a minimal separator of $G$ if $S$ is a separator and no proper subset of $S$ separates the graph. A subset $S \subset V$ is a vertex separator for non-adjacent vertices $u$ and $v$ (a $uv$-separator) if the removal of $S$ from the graph separates $u$ and $v$ into distinct connected components. If no proper subset of $S$ is a $uv$-separator then $S$ is a minimal $uv$-separator. If $S$ is a minimal $uv$-separator for some pair of vertices, it is called a minimal vertex separator (mvs). A minimal separator is always a minimal vertex separator but the converse is not true.

A clique-tree of $G$ is defined as a tree $T = (Q, E_T)$, where $Q$ is the set of maximal cliques of $G$ and for every two distinct maximal cliques $Q, Q' \in Q$ each clique in the path from $Q$ to $Q'$ in $T$ contains $Q \cap Q'$. Observe that a set $S \subset V$ is a minimal vertex separator of $G$ if and only if $S = Q \cap Q'$ for some edge $\{Q, Q'\} \in E_T$. Moreover, the multiset $\mathcal{M}$ of the minimal vertex separators of $G$ is the same for every clique-tree of $G$. The multiplicity of the minimal vertex separator $S$, denoted by $\mu(S)$, is the number of times that $S$ appears in $\mathcal{M}$. The set of minimal vertex separators of $G$ is denoted by $\mathcal{S}$. The determination of the minimal vertex separators and their multiplicities can be performed in linear time [29].

It is important to mention two types of cliques in a chordal graph $G$. A simplicial clique is a maximal clique containing at least one simplicial vertex. A simplicial clique $Q$ is called a boundary clique if there exists a maximal clique $Q'$ such that $Q \cap Q'$ is the set of non-simplicial vertices of $Q$.

A strictly chordal graph is a graph obtained by adding zero or more true twins to each vertex of a block graph $G$ [18]. The class was proved to be (gem,dart)-free [18, 22]. Strictly chordal graphs can also be characterized in terms of the structure of their minimal vertex separators as proved in Theorem 2.

**Theorem 2** [32] Let $G = (V, E)$ be a chordal graph and $\mathcal{S}$ be the set of minimal vertex separators of $G$. $G$ is a strictly chordal graph if and only if for any distinct $S, S' \in \mathcal{S}$, $S \cap S' = \emptyset$.

The clique-bipartite graph of $G$ is the bipartite graph $CB(G) = (\mathcal{S} \cup Q, F)$ in which there is an edge joining a maximal clique $Q \in Q$ and a minimal separator $S \in \mathcal{S}$ when $S \subset Q$ [28]. This structure is a generalization of the block-cut vertex graph defined by Harary [20]. Theorem 3 is an immediate consequence of results about clique-bipartite graphs presented in [28].
Theorem 3 Let $G$ be a strictly chordal graph. Then the clique-bipartite graph $CB(G)$ is a tree.

The determination of the toughness of strictly chordal graphs, as seen in Theorem 4, can be performed in linear time complexity.

Theorem 4 Let $G$ be a non-complete strictly chordal graph and $\mathcal{S}$ be the set of minimal vertex separators of $G$. Then $\tau(G) = \min_{S \in \mathcal{S}} \left\{ \frac{|S|}{\mu(S)+1} \right\}$.

3 Scattering number of strictly chordal graphs

As already seen in Section 1, the scattering number of $G$ is

$$sc(G) = \max \{ \omega(G - S) - |S|; S \subseteq V, \omega(G - S) \neq 1 \}$$

where $\omega(G - S)$ is the number of connected components of the graph $G - S$ and a subset $S \subseteq V$ for which this maximum is attained is a scattering set of $G$.

Some basic properties of the separators of strictly chordal graphs will be fundamental for the determination of their scattering numbers. They can be stated:

Property 1 Let $G = (V, E)$ be a strictly chordal graph and $\mathcal{S}$ its set of minimal vertex separators.

a) for any distinct $S, S' \in \mathcal{S}$, $S \cap S' = \emptyset$.
b) $S \in \mathcal{S}$ is a minimal separator of $G$.
c) boundary cliques of $G$ contain only one mvs.
d) $\omega(G - S) = \mu(S) + 1$, for every $S \in \mathcal{S}$.
e) every separator $S$ of $G$ is the union of pairwise disjoint minimal vertex separators of $G$.

The maximal cliques that contain $S \in \mathcal{S}$ are called adjacent cliques of $S$; the cardinality of this set is $\mu(S) + 1$. The set of boundary cliques that contains the mvs $S \in \mathcal{S}$ is denoted by $B(S)$.

In Theorem 5 it is proved the first result relating the minimal vertex separators and the boundary cliques of $G$. It will be used for the development of an efficient algorithm to determine a scattering set.
**Theorem 5** Let $G$ be a strictly chordal graph with $|S| > 1$. Then there is a mvs $S \in \mathcal{S}$ of $G$ such that $|B(S)| = \mu(S)$.

*Proof.* As $G$ is a strictly chordal graph, every boundary clique in $G$ has only one mvs. Let $CB(G)$ be the clique-bipartite graph of $G$. The boundary cliques of $G$ are the leaves of $CB(G)$ and for every mvs $S \in \mathcal{S}$, the degree of the vertex that represents $S$ in $CB(G)$ is $\mu(S) + 1$.

Suppose that there is not a mvs $S$ of $G$ such that $|B(S)| = \mu(S)$, i.e., every mvs of $G$ is a subset of at least two maximal cliques that are not boundary cliques. However, the maximal cliques that are not boundary cliques have at least two minimal vertex separators. Since $G$ has a finite number of maximal cliques, $CB(G)$ must contain a cycle. Contradiction, by Theorem 3. Then there is a mvs $S$ of $G$ such that $|B(S)| = \mu(S)$.

Any mvs $S \in \mathcal{S}$ described in Theorem 5 is called a border minimal vertex separator of $G$.

From now on, the determination of the scattering number and the scattering set of strictly chordal graphs is addressed. If the graph has only one minimal vertex separator, the result is immediate.

**Theorem 6** Let $G$ be a non-complete strictly chordal graph. If $|S| = 1$ then $sc(G) = \mu(S) + 1 - |S|$ with $S \in \mathcal{S}$.

In the remaining of this section we will consider graphs with at least two minimal vertex separators and our approach will be the analysis of the graph according to its toughness.

### 3.1 graphs with $\tau(G) \geq 1$

**Theorem 7** Let $G$ be a strictly chordal graph with $|S| > 1$ and $\tau(G) \geq 1$. Then $sc(G) = \max_{S \in \mathcal{S}} \{\mu(S) + 1 - |S|\}$.

*Proof.* As $\tau(G) \geq 1$, for every mvs $S' \in \mathcal{S}$, $|S'| \geq \mu(S') + 1$ and $sc(G) \leq 0$ (Lemma 1).

Consider a mvs $S \in \mathcal{S}$ such that $\alpha = \mu(S) + 1 - |S|$ is the highest possible; $\alpha \leq 0$. Since the scattering number is defined as the maximum, let us analyse a separator of $G$ that is the union of $S$ and other mvs $S' \in \mathcal{S}$, $S' \neq S$. For every mvs $S' \in \mathcal{S}$, $\mu(S') + 1 - |S'| \leq \alpha \leq 0$. So, $\omega(G - \{S \cup S'\}) - |S \cup S'| < \omega(G - S) - |S| = \alpha$, i.e., any union presents a worse result than the result obtained by $S$. Then, $S$ is a scattering set and $sc(G) = \max_{S \in \mathcal{S}} \{\mu(S) + 1 - |S|\}$. \hfill \Box
Observe that, for a graph $G$, $sc(G) = 0$ if and only if $\tau(G) = 1$.

As it was seen in Theorems 4 and 7, the determination of the toughness and the determination of the scattering number of a graph $G$ with $\tau(G) \geq 1$ are quite similar. Observe that, for $\tau(G) > 1$, a tough set is not always a scattering set; an example is shown in Figure 1. Graph $G$ has $\tau(G) = 2$, its tough set is composed by the white vertices, $sc(G) = -4$ and the scattering set is composed by the black vertices.

3.2 graphs with $\tau(G) < 1$

In Subsection 3.1 it was seen that a graph $G$ with $\tau(G) \geq 1$ have all their minimal vertex separators such that $|S| > \mu(S)$. For graphs with $\tau(G) < 1$, the cardinality of the minimal vertex separators can be greater, equal or lower than their multiplicity; however it is mandatory to have at least one mvs $S$ with $S \leq \mu(S)$. In order to analyse this case we need to establish a further partition of the set of graphs. Let $G$ be a strictly chordal graph with $\tau(G) < 1$ and $S$ its set of minimal vertex separators: graph $G$ is called a type A graph if $|S| \geq \mu(S)$, for every mvs $S \in S$; otherwise it is called a type B graph, i.e., there exists at least one mvs $S \in S$ such that $|S| < \mu(S)$.

The determination of the scattering number of type A graphs is quite simple as we can see in the following theorem.

Theorem 8 Let $G$ be a type A graph. Then $sc(G) = 1$.

Proof. Graph $G$ is a type A graph, $\tau(G) < 1 \because sc(G) > 0 \because sc(G) \geq 1$ (Lemma 1) and for every $S' \in S$, $|S'| \geq \mu(S')$. As $\tau(G) < 1$, there is at least a mvs $S$ such that $|S| = \mu(S)$ and $\tau(G) = \frac{|S|}{|S|+1}$. So, $\omega(G - S) - |S| = \mu(S) + 1 - |S| = 1$. In fact, for every $S' \in S$ such that $|S'| = \mu(S')$, $\mu(S') + 1 - |S'| = 1$.

Consider the separator $S = S' \cup S''$ with $S', S'' \in S$, $S' \neq S''$. By Theorem 2, $|S| = |S'| + |S''|$. Firstly let us determine the number of components of $G - S$. Figure 1: $\tau(G) \geq 1$
If \( S \) is a maximal clique of \( G \), \( \omega(G - S) = \omega(G - S') + \omega(G - S'') - 2 = \mu(S') + \mu(S'') \); otherwise, \( \omega(G - S) = \omega(G - S') + \omega(G - S'') - 1 = \mu(S') + \mu(S'') - 1 \). If \( |S'| > \mu(S') \) or \( |S''| > \mu(S'') \), it is immediate that \( \omega(G - S) - |S| \leq 0 \).

If \( |S'| = \mu(S') \) and \( |S''| = \mu(S'') \), \( \omega(G - \{S \cup S'\}) - |S| \leq 1 \). Consider \( S \) is a separator of \( G \) with at least three minimal vertex separators of \( S \) and such that for every \( S' \subseteq S \), \( |S'| = \mu(S') \). By the same reasoning, \( \omega(G - S) - |S| \leq 1 \). Then, \( sc(G) = 1 \).

**Corollary 8.1** Let \( G \) be a type A graph. Then every mvs \( S \in S \) such that \( |S| = \mu(S) \) is a scattering set.

In Figure 2, graphs \( G_1 \) and \( G_2 \) have toughness less than 1. Graph \( G_1 \) is a type A graph with \( \tau(G_1) = .5 \), \( sc(G_1) = 1 \) and three scattering sets \( \{c\} \), \( \{a, j\} \) and \( \{a, c, j\} \). Graph \( G_2 \) is a type B graph with \( \tau(G_2) = .25 \), the scattering set is \( \{\ell, m, n, o\} \) and \( sc(G_2) = 5 \).

In order to determine the scattering number of a type B graph, Theorems 9, 10 and 11 contain a detailed study on border minimal vertex separators. Theorem 9 provides a criterion to decide whether or not a mvs should be considered as part of a scattering set of strictly chordal graphs in general. Theorems 10 and 11 present results to type B graphs.

**Theorem 9** Let \( G \) be a strictly chordal graph with \( |S| > 1 \), \( SC \) be a scattering set and \( S \in S \) be a border mvs. If \( |S| < |B(S)| \) then \( SC \cup S \) is a scattering set of \( G \).

**Proof.** If \( S \subseteq SC \) then it is immediate that \( SC \cup S \) is a scattering set.

Let \( S \not\subseteq SC \). Since \( SC \) is a scattering set of \( G \), \( \omega(G - SC) - |SC| = s \) is maximum. As \( S \) is a border mvs, \( |B(S)| = \mu(S) \). Since \( |S| < |B(S)| = \mu(S) \) then \( \tau(G) < 1 \) and \( sc(G) = s > 0 \).

Let \( G - SC \) be a graph with \( C_1, C_2, \ldots, C_{\omega(G - SC)} \) connected components. Two cases must be analysed.
• $B(S) = C_i, 1 \leq i \leq \omega(G - SC)$. Consider the separator $SC \cup S$. After the removal of this separator, the component $C_i$ does not exist anymore and there are $|B(S)|$ new components. The scattering number would become: $s' = \omega(G - \{SC \cup S\}) - |SC \cup S| = s + |B(S)| - |S| - 1$.

By hypothesis, $|S| < |B(S)|$; so, $|B(S)| - |S| - 1 \geq 0$. But $s$ is maximum. Then if $|S| = |B(S)| - 1$, $S$ can be added to $SC$; if $|S| < |B(S)| - 1$, $s$ would not be maximum and $S$ must be already in $SC$, contradiction.

• $B(S) \subset C_i, 1 \leq i \leq \omega(G - SC)$. Consider the separator $SC \cup S$. After the removal of this separator, the component $C_i$ does not exist anymore and there are $B(S) + 1$ new components. The scattering number would become: $s' = \omega(G - \{SC \cup S\}) - |SC \cup S| = s + |B(S)| - |S|$.

By hypothesis, $|S| < |B(S)|$. So, $s$ would not be maximum and $S$ must be already in $SC$. Contradiction.

Hence, in any case, $SC \cup S$ is a scattering set of $G$.

We notice that a strictly chordal graph that satisfies the hypothesis of Theorem 9 is a type B graph. Theorem 10 and 11 concludes the study of border minimal vertex separators for type B graphs.

**Theorem 10** Let $G$ be a type B graph with $|S| > 1$ and let $S \in \mathbb{S}$ be a border mvs with $|S| > |B(S)|$. Then $S$ is not a subset of any scattering set of $G$.

**Proof.** As $G$ is type B graph, $\tau(G) < 1$. By Lemma 11 $sc(G) \geq 1$ and, in addition, $sc(G) \neq 1$; $sc(G) \geq 2$.

By hypothesis, $|S| > |B(S)| = \mu(S)$. So, $|S| = \mu(S) + k, k \geq 1$. Consider $S$ a separator of $G$ with $\omega(G - S) - |S| \geq 2$ such that $S \not\subseteq S$. It is immediate that $\omega(G - (S \cup S)) - |S \cup S| < \omega(G - S) - |S|$. In particular, if $S$ is a scattering set, $S$ is not a subset of $S$.

**Theorem 11** Let $G$ be a type B graph with $|S| > 1$, $S \in \mathbb{S}$ be a border mvs with $|S| = |B(S)|$. Then $sc(G) = sc(G')$ where $G' = G - (B(S) \setminus S)$.

**Proof.** As $G$ is type B graph, by the definition, there is a mvs $S' \in \mathbb{S}$ such that $|S'| < \mu(S')$. As $S$ is a border mvs with $|S| = |B(S)|$, it follows that $|B(S)| = \mu(S)$ and $S \neq S'$.

The set $B(S) \setminus S$ consists of simplicial vertices of $G$. The graph $G'$ is obtained by removing these vertices. Therefore, $S$ is a set of simplicial vertices of $G'$, $S'$ is a mvs of $G'$ and $G'$ is also a type B graph with $sc(G') = s \geq 2$.

Consider $SC'$ a scattering set of $G'$. It is immediate that $S \not\subseteq SC'$. Let us analyse the separator $SC' \cup S$ of $G$. If there is a maximal clique $Q$ of $G$ such
that \( S \subset Q \) and \( Q \subset SC' \cup S \), \( \omega(G - \{SC' \cup S\}) - |SC' \cup S| = \omega(G - SC') - |SC'| + \omega(G - S) - |S| - 2 = s - 1 \). Otherwise, \( \omega(G - \{SC' \cup S\}) - |SC' \cup S| = \omega(G - SC') - |SC'| + \omega(G - S) - |S| - 1 = s \). As \( G' = G - (B(S) \setminus S) \), \( SC' \) is also a scattering set of \( G \). Then, \( sc(G) = sc(G') \). 

For the determination of the scattering number of a type B graph, firstly a scattering set of the graph must be determined; the algorithm that performs this task consists of the application of Theorems \( \mathfrak{9} \), \( \mathfrak{10} \) and \( \mathfrak{11} \). So, after the computation of this set, it must be removed from the original graph and the remaining components accounted for, i.e., the scattering number is determined.

An intuitive algorithm can be drafted: we search graph \( G \) looking for a border \( mvs \) \( S \), that always exists (Theorem \( \mathfrak{5} \)). If \( |S| \geq |B(S)| \), \( S \) can be ignored: the vertices of \( B(S) \setminus S \) must be removed from \( G \) and the vertices of \( S \) become simplicial vertices in the updated graph \( G \). If \( |S| < |B(S)| \), the vertices of \( S \) must be included in the scattering set of \( G \) and the vertices of \( B(S) \) removed from the graph. This step is repeated until all minimal vertex separators are covered. For each border \( mvs \), we must go through the updated graph again. This algorithm has \( O(nm) \) time complexity since in a strictly chordal graph there are \( O(n) \) minimal vertex separators and, for each one, the entire graph must be searched. However, a more efficient implementation can be presented.

The algorithm \textit{Scattering-set determination}, presented here, relies on a depth-first search over the clique-bipartite graph \( CB(G) = (S \cup Q, F) \), which is a tree (Theorem \( \mathfrak{3} \)). It is immediate to see that, although the algorithm performs on the tree, it works at the same time with the structures that are represented by the vertices of the tree. The search must begin in a vertex that represents a \( mvs \) of \( G \); all leaves of the depth-first search tree are vertices representing maximal cliques.

At each step of the algorithm, a vertex representing a maximal clique or a \( mvs \) is considered. They take turns in the tree, since \( CB(G) \) is a bipartite graph. They are analysed at the moment that the vertex comes out from the recursion stack. Observe that at this point all the action over its descendants is already performed, that is, the subgraph induced by them is already updated.

The algorithm maintains labels for the vertices of \( CB(G) \):

- \textit{entry}(v): the order in which the vertex is visited;
- \textit{parent}(v): the parent of the vertex in the depth-first search tree;
- \textit{card}(v): the cardinality of the maximal clique or the \( mvs \) represented by \( v \);
- \( \mu(v) \): equal to \( \mu(S) \), being \( S \in S \) represented by \( v \);
Algorithm Scattering-set determination;
Input: $CB(G)$;
Output: $\text{scattering-set}(G)$;
begin
  Initialize arrays $\text{card}$, $\text{status}$, $\text{entry}$;
  $\text{SC}$, $\text{scattering-set}(G) \leftarrow \emptyset$;
  $\text{entry} \_ \text{order} \leftarrow 0$;
  $\text{root} \leftarrow v \in V$ such that $\text{status}(v) = \text{mvs}$; $\text{parent}($root$) = \text{NULL}$;
  $\text{dfs}($root$)$;
  if $\text{card}($root$) < \mu($root$)$ then
    $\text{SC} \leftarrow \text{SC} \cup \{v\}$;
  for $v \in \text{SC}$ do  \hspace{1em} \% let $S$ be the mvs represented by $v$
    $\text{scattering-set}(G) \leftarrow \text{scattering-set}(G) \cup S$;
procedure $\text{dfs}(v)$;
begin
  $\text{entry}(v) \leftarrow \text{entry} \_ \text{order} \leftarrow \text{entry} \_ \text{order} + 1$;
  for $w \in \text{Adj}(v)$ do
    if $\text{entry}(w) = 0$ then
      $\text{parent}(w) \leftarrow v$;
      $\text{dfs}(w)$;
    if $\text{status}(v) \neq \text{mvs}$ then \hspace{1em} (*)
      if $\text{status}(v) = \text{false} \_ \text{clique}$ then
        $\mu($parent$(v)) \leftarrow \mu($parent$(v)) - 1$;
      else if $v \neq \text{root}$ then
        if $\text{card}(v) < \mu(v)$ then
          $\text{SC} \leftarrow \text{SC} \cup \{v\}$;
          if $\text{card}($parent$(v)) = \text{card}(v) + \text{card}($parent($($parent$(v))$) then \hspace{1em} (**)\hspace{1em}
            $\text{status}($parent$(v)) \leftarrow \text{false} \_ \text{clique}$
          else $\text{card}($parent$(v)) \leftarrow \text{card}($parent$(v)) - \text{card}(v)$;
        end
      end
    end
end
end.

$- \text{status}(v) = \begin{cases} 
\text{mvs}, & v \text{ represents a mvs} \\
\text{true} \_ \text{clique}, & v \text{ represents an existing clique} \\
\text{false} \_ \text{clique}, & v \text{ represents a clique that does not exist anymore in the updated graph}
\end{cases}$

Let $v$ be the vertex to be analysed.

- If $v$ represents a maximal clique:
  If $v$ is a leaf of $CB(G)$, nothing happens. Otherwise, its status must be addressed; if it is $\text{false} \_ \text{clique}$ the number of adjacent cliques of $\text{parent}(v)$, which represents a mvs, $\mu($parent$(v))$ must be decreased.
Observe that this computation simulates that all the subgraph induced by \( v \) and its descendants, i.e, the subgraph represented by these vertices on the graph, does not belong anymore to the graph.

- If \( v \) represents a \emph{mvs} \( S \):

At the moment that vertex \( v \) is analysed, it corresponds to a \emph{mvs} of the updated graph \( G \). Its descendants in the depth search tree represent boundary cliques of \( G \) or a subgraph of \( G \) that does not contain any \emph{mvs} that must be in the scattering set. So, the \emph{mvs} can be considered as a border \emph{mvs} and Theorems 9, 10 and 11 can be applied. We must consider two cases. Firstly, if \(|S| \geq \mu(S)|\): the \emph{mvs} \( S \) does not need to be included in the scattering set of \( G \) and it is no more considered as a separator; nothing is done. Its vertices will be treated as simplicial vertices in the next iteration. In the second case, when \(|S| < \mu(S)|\), the vertices of \( S \) must be added to the scattering set of \( G \) and removed from the graph. The graph must be updated: the maximal clique \( Q \) that contains the \emph{mvs} and that remains in the graph is analysed: if \( Q \) is composed only by two minimal vertex separators, then \( Q \), as a maximal clique, will not exist anymore in the graph. This computation is implemented in the algorithm by testing \( \text{card}(\text{parent}(v)) \) and, if needed, updating the label \( \text{status}(\text{parent}(v)) \) (line (**)) of the algorithm.

The algorithm determines \( SC \), a set of vertices of \( CB(G) \), each one representing a \emph{mvs} of \( G \). The scattering set of \( G \) is the result of the union of these minimal vertex separators.

### 3.3 time complexity of the determination of the scattering number

**Theorem 12** The determination of the scattering number of a strictly chordal graph \( G \) has linear time complexity.

**Proof.** Let \( G = (V,E) \) be a strictly chordal graph. The set of maximal cliques, the set of minimal vertex separators of \( G \) and their multiplicities must be determined; this can be accomplished in linear time complexity \[29\].

The determination of the toughness of the graph depends on a traversal of \( S \) that has \( O(n + m) \) time complexity.

If \( \tau(G) \geq 1 \), by Theorem \[7\] we must determine a \emph{mvs} that satisfies \( \max_{S \in S_{mvs}} \{ \mu(S) + 1 - |S| \} \). As the minimal vertex separators and their multiplicities are already known, a new traversal of \( S \) is needed. It has \( O(n + m) \) time complexity.
If \( \tau(G) < 1 \) the partition of the set of graphs must be established. In order to perform it, the analysis of each \( mvs \) is, again, necessary. If \( G \) is a type A graph, the determination of the scattering number has constant time complexity, by Theorem 8. If \( G \) is a type B graph, the algorithm \textit{Scattering-set determination} must be performed. As seen, the algorithm relies on a depth-first search that has linear time complexity. As \( G \) is a strictly chordal graph, we know that for any distinct \( S, S' \in \mathbb{S} \), \( S \cap S' = \emptyset \) (Theorem 2). So, a vertex \( v \) is a simplicial vertex or it belongs exactly to one \( mvs \). By labeling the vertices of maximal cliques, it is possible to build \( CB(G) \) in linear time complexity. The vertex set of graph \( CB(G) \) is \( Q \cup S \). The sets \( Q \) and \( S \) have each one at most \( n \) elements; so, \( Q \cup S \) has also \( O(n) \) elements. As \( CB(G) \) is a tree it has at most \( 2n - 1 \) edges. The if statement marked (*) in the algorithm takes constant time for its execution. So, the building of \( CB(G) \) and the depth-first search have \( O(n + m) \) time complexity. Hence the theorem is proved.

References

[1] E. Aslan, A measure of graphs vulnerability: edge scattering number, \textit{Bull. Soc. Math. Banja Luka} 4 (2014) 53-60.

[2] E. Aslan, A. Kirlangic, Computing the scattering number and the toughness for gear graphs, \textit{Bulletin of International Mathematical Virtual Institute} 1 (2011) 1-11.

[3] C.A. Barefoot, R. Entringer, H.C. Swart, Vulnerability in graphs - A comparative survey, \textit{J. Combin. Math. Combin. Comput.} 1 (1987) 12-22.

[4] D. Bauer, H. Broersma, E. Schmeichel, Toughness in graphs - a survey, \textit{Graphs Combin.} 22 (1) (2006) 1-35.

[5] D. Bauer, S.L. Hakimi, E. Schmeichel, Recognizing tough graphs is NP-hard, \textit{Discrete Appl. Math.} 28 (1990) 191-195.

[6] J.R.S. Blair, The efficiency of AC graphs, \textit{Discrete Appl. Math.} 44 (1993) 119-138.

[7] J.R.S. Blair, B. Peyton, An introduction to chordal graphs and clique trees, in: J.A. George, J. R. Gilbert, J. W. H. Liu (Eds.), \textit{Graph theory and sparse matrix computation}. Springer Verlag, IMA 56, 1993, pp. 1-30.

[8] A. Brandstädt, V.B. Le, Structure and linear time recognition of 3-leaf powers, \textit{Inform. Process. Lett.} 98 (2006) 133-138.
[9] A. Brandstädt, P. Wagner, Characterising \((k, \ell)-leaf\) powers, *Discrete Appl. Math.* 158 (2010) 110-122.

[10] H. Broersma, How tough is toughness? *Bull. Eur. Assoc. Theor. Comput. Sci.* EATCS 117 (2015).

[11] H. Broersma, F. Fiala, P.A. Golovach, T. Kaiser, P. Paulusma, P. Proskurowski, Linear-time algorithms for scattering number and hamilton-connectivity of interval graphs, *J. Graph Theory* 79 (4) (2015) 282-299.

[12] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* 5 (1973) 215-228.

[13] M. Cozzens, D. Moazzami, S. Stueckle, The tenacity of a graph, in: *Proceedings of 7th International Conference on the Theory and Applications of Graphs*, Wiley, New York, 1995, pp. 1111-1122.

[14] J.S. Deogun, C.D. Kratsch, G. Steiner, 1-Tough cocomparability graphs are Hamiltonian, *Discrete Math.* 170 (1997) 99-106.

[15] M. Dom, J. Guo, F. Hüffner, R. Niedermeier, Error compensation in leaf root problems, *Lecture Notes in Comput. Sci.* 3341(2004) 389-401.

[16] E. Estrada, M. Benzi, Core-satellite graphs: clustering, assortativity and spectral properties, *Linear Algebra Appl.* 517 (2017) 30-52.

[17] M.C. Golumbic, *Algorithmic graph theory and perfect graphs*, second edition, Academic Press, New York, 2004.

[18] M.C. Golumbic, U.N. Peled, Block duplicate graphs and a hierarchy of chordal graphs, *Discrete Appl. Math.* 124 (2002) 67-71.

[19] F. Harary, A characterization of block graphs, *Canad. Math. Bull.*, 6(1) (1963) 1-6.

[20] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.

[21] H.A. Jung, On a class of posets and the corresponding comparability graphs, *J. Combin. Theory Ser. B* 24 (1978) 125-133.

[22] W. Kennedy, Strictly chordal graphs and phylogenetic roots, Master Thesis, University of Alberta, 2005.

[23] D. Kratsch, T. Klots, H. Müller. Computing the toughness and the scattering number for interval and other graphs, INRIA, Rapport de recherche 2237 (1994) 1-22.

[24] D. Kratsch, T. Klots, H. Müller. Measuring the vulnerability for classes of intersection graphs, *Discrete Appl. Math.* 77 (1997) 259-270.
[25] F. Li, Q. Ye, Y. Sun, The isolated scattering number can be computed in polynomial time for interval graphs, *Anziam J.* 58(E) (2017) E81-E97.

[26] F. Li, X. Zhang, H. Broersma, A polynomial algorithm for weighted scattering number in interval graphs, *Discrete Appl. Math.* 264 (2019) 118-124.

[27] Y. Li, S. Zhang, X. Li, Rupture degree of graphs, *Int. J. Comput. Math.* 82(7) (2005) 793-803.

[28] L. Markenzon, Non-inclusion and other subclasses of chordal graphs, *Discrete Appl. Math.* 272 (2020) 43-47.

[29] L. Markenzon, P.R.C. Pereira, One-phase algorithm for the determination of minimal vertex separators of chordal graphs, *Int. Trans. Oper. Res.* 17 (2010) 683-690.

[30] L. Markenzon, C.F.E.M. Waga, New results on ptolemaic graphs, *Discrete Appl. Math.* 196 (2015) 135-140.

[31] L. Markenzon, C.F.E.M. Waga, Strictly interval graphs: characterization and linear time recognition, *Electron. Notes Discrete Math.* 52 (2016) 181-188.

[32] L. Markenzon, C.F.E.M. Waga, Toughness and Hamiltonicity of strictly chordal graphs, *Int. Trans. Oper. Res.* 26(2) (2019) 725-731.

[33] S.Y. Wang, Y.X. Yang, S.W. Lin, J. Li, Z.M. Hu, The isolated scattering number of graphs, *Acta Mathematica Sinica* 54(5) (2011) 861-874.

[34] D.R. Woodall, The binding number of a graph and its Anderson number, *J. Combin. Theory Ser. B* 15 (1973) 225-255.

[35] S. Zhang, X. Li, X. Han, Computing the scattering number of graphs, *Int. J. Comput. Math.* 79(2) (2002) 179-187.

[36] S. Zhang, S. Peng, Relationships between scattering number and other vulnerability parameters, *Int. J. Comput. Math.* 81(3) (2004) 291-298.