Casimir force between Chern-Simons surfaces

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Abstract

We calculate the Casimir force between two parallel plates if the boundary conditions for the photons are modified due to presence of the Chern-Simons term. We show that this effect should be measurable within the present experimental technique.

1 Introduction

The Chern-Simons gauge theories [1, 2] attract much attention due to both their theoretical beauty and practical applications to certain condensed matter phenomena. Most notable is the generation of states with fractional statistics first observed by Wilczek [3]. Later this phenomenon was used to describe composite fermions [4] in the theory of the fractional quantum Hall effect (FQHE) (see, e.g., [5]).

The Casimir effect (see, e.g., the book [6]) is a nowadays also experimentally well established [7, 8] macroscopic quantum effect. Several applications are known, ranging from the bag model in QCD [9] to constraints on hypothetical long range interactions [10].

In the present paper we investigate another situation where the Casimir effect eventually may serve as a test for certain theoretical models in solid state theory, namely for models including a Chern-Simons term. We consider the situation when the Maxwell volume action (which is in (3+1) dimensions) is supplemented by a Chern-Simons surface term. We argue that such boundary term naturally leads to a modification of ordinary conductor and bag boundary conditions. Such a modification can also be considered on its own right as the only possible one.

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with a local gauge invariant $P$-odd term without any new dimensional coupling. The boundary condition and quantization are discussed in the next section.

In section 3 we calculate the ground state energy for the system of two parallel planes bearing different Chern-Simons charges. We show that the Casimir force depends substantially on these charges and can even change sign under certain conditions. Given recent advances in the Casimir force experiments \cite{8}, the effect could be measurable. Theoretical and phenomenological implications of our work are discussed in the last section.

2 Chern-Simons boundary conditions for ordinary and dual potentials

Consider a region $M$ in the space-time with a boundary $\partial M$. Let the Maxwell action on $M$ be supplemented by the Chern-Simons surface action:

$$
S = -\frac{1}{4} \int_M d^4 x \ g^{\frac{1}{2}} F_{\mu \nu} F^{\mu \nu} - \frac{a}{2} \int_{\partial M} d^3 x \ \varepsilon^{ijk} A_i \partial_j A_k,
$$

where $\varepsilon^{ijk}$ is the Levi-Civita tensor and the $x^j$ are coordinates on the boundary $\partial M$, $a$ being a real parameter. After integration by parts one gets

$$
S = -\frac{1}{2} \int_M d^4 x \ g^{\frac{1}{2}} A_\mu (-g^{\mu \nu} \Box + \nabla^\nu \nabla^\mu) A_\nu
- \frac{1}{2} \int_{\partial M} d^3 x (h \varepsilon^{ijk} (\partial_N A_i - \partial_i A_N) A^j + a \varepsilon^{ijk} A_i \partial_j A_k),
$$

where $\Box$ is the d’Alambertian, $N$ is the outward pointing normal vector, and $h$ is the determinant of the induced metric on $\partial M$. The volume term in (2) (with a suitable gauge choice) generates the wave equation for $A_\mu$, the surface term generates the boundary conditions.

There are two sets of local gauge-invariant boundary conditions which ensure the vanishing of the surface term in (2). The first one is called relative boundary conditions

$$
A_i|_{\partial M} = 0, \quad (\partial_N + k) A_N|_{\partial M} = 0.
$$

Here $k$ is the trace of the second fundamental form of the boundary. The second set reads \cite{11}:

$$
A_N|_{\partial M} = 0, \quad (\partial_N A_i + a \varepsilon^{jik} \partial_j A_k)|_{\partial M} = 0 \quad i = 0, 1, 2, .
$$

Note that variation of the surface term in (2) with respect to $A^i$ gives precisely the sum of the two conditions (4).

Let us suppose for simplicity that the boundary $\partial M$ consists of two parallel infinite plates located at $x^3 = \text{const}$. In this case $k = 0$. Consider the right plate,
The boundary conditions (3) and (4) can be rewritten in terms of the field strengths giving respectively

\[ H_3|_{\partial M} = 0, \quad E_\parallel|_{\partial M} = 0; \quad (E_3 + aH_3)|_{\partial M} = 0, \quad (H_\parallel - aE_\parallel)|_{\partial M} = 0 \]

When the coupling to charged particles is absent (as in the problem of the Casimir energy calculations) one can quantize the photons in terms of the dual potentials \( A_\mu^* \):

\[ *F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\rho\sigma} F_{\rho\sigma}, \quad *F_{\mu\nu} = \partial_\mu A_\nu^* - \partial_\nu A_\mu^* \]

In the case of the electromagnetic field in a dielectric the quantization in terms of \( A_\mu^* \) even gives some technical advantages [12]. One can also consider the Chern-Simons term with the ordinary vector potential \( A_\mu \) replaced by the dual potential \( A_\mu^* \). The corresponding boundary conditions will be dual to the two sets (5) and (6):

\[ E_3|_{\partial M} = 0, \quad H_\parallel|_{\partial M} = 0; \quad (H_3 - aE_3)|_{\partial M} = 0, \quad (E_\parallel + aH_\parallel)|_{\partial M} = 0 \]

The conditions (3) correspond a conducting boundary. Their duals (8) are called bag boundary conditions. The conditions (4) and (9) generalize (8) and (5) respectively for the case of non-zero Chern-Simons coupling. The two sets (9) and (4) are related by \( a \to -1/a \). Therefore, it is enough to consider just one set (9). Different connected pieces of the boundary can correspond to different values of \( a \). Equal values of \( a \) everywhere will correspond to complete dual invariance.

At this point we must note that there is no exact equivalence between boundary conditions and a boundary action. Here we are considering a “mixed” problem when the \( a = 0 \) physics is described by rigid boundary conditions which are later modified by the presence of the Chern-Simons boundary term. A more rigorous way would be to describe the full interaction of photons with the boundary by a boundary action and treat it as a \( \delta \)-function potential (for the mathematical set-up see e.g. [13]). This would lead however to considerable technical complexities. We believe that our present approach gives correct (zeroth order) approximation to the real physics.

A suitable gauge choice is

\[ \partial_j A_j = 0. \]

The boundary conditions (4) are invariant under the gauge transformations \( A \to A + \partial \omega \) provided the gauge parameter \( \omega \) satisfies

\[ \partial_N \omega|_{\partial M} = 0. \]
Upon quantization (11) become boundary conditions for the ghost field. Propagators for the ghosts and for the $A_3$ does not depend on the Chern-Simons coupling $a$. Moreover, in the gauge (10) they does not contain $\partial_3$ and, hence, does not contribute to the Casimir force. Therefore, in the following we will be interested in the two polarizations of the field $A_i$ satisfying (10).

By performing the Fourier transformation in the $x^j$ directions we arrive at the problem of diagonalization of the operator $L_i^k = \varepsilon_{ijk}k_j$. With our sign conventions $\varepsilon_{012} = 1$. The following two polarizations solve the problem

$$A_i^\pm = \begin{pmatrix} k_1k_0 \pm k_2\sqrt{k_2^2 - k_0^2} \\ k_2k_0 \pm k_1\sqrt{k_2^2 - k_0^2} \end{pmatrix} b^\pm(x^3),$$

(12)

$i = \{0, a\}$, $a = 1, 2$. Both vectors $A_i^\pm$ satisfy the gauge condition (10). They correspond to the eigenvalues $\pm\sqrt{k_2^2 - k_0^2}$ of the operator $L_i^k$.

### 3 Casimir force

We define the ground state energy density per unit area of the plates in the usual way by

$$E_0 = \frac{1}{2} \sum_n \int \frac{dk_a}{(2\pi)^2} (k_a^2 + \omega_n^2)^{s/2} = \frac{1}{2} \sum_n \int \frac{dk_a}{(2\pi)^2} (k_a^2 + \omega_n^2)^{s/2},$$

(13)

where $\omega_n$ is discrete momentum in the third direction, $s > \frac{3}{2}$ is the (zeta-functional) regularization parameter with $s \to 0$ in the end.

Consider two planes located at $x^3 = 0, L$ with different Chern-Simons charges, $a(0)$ and $a(L)$. From the equation of motion we note $k_0^2 = k_a^2 + k_3^2$. The boundary conditions on the two planes for the modes $b^\pm$ become:

$$\left( \frac{\partial}{\partial x^3} \pm a(0,L)k_3 \right) b^\pm(x^3)|_{x^3=0,L} = 0$$

(14)

Note that $\partial_N|_{x^3=L} = -\partial_N|_{x^3=0} = \partial_3$. The sign before the Levi-Civita tensor should be also different on the two components of the boundary due to the reversed orientation. Consider the $b^+$ polarization for definiteness. The boundary condition at $x^3 = 0$ is solved by the following harmonics:

$$b^+(x^3) = -a(0) \sin k_3x^3 + \cos k_3x^3.$$  

(15)

Substituting (13) in the boundary condition at $x^3 = L$ one gets

$$(a(L) - a(0)) \cos k_3L - (1 + a(L)a(0)) \sin k_3L = 0,$$

(16)

whose solutions are $k_3 = \omega_n$. It is useful to rewrite condition (14) in the form

$$f(k) = \sin(kL + \delta)$$

(17)
where \( \delta = \arctan \left( \frac{(a(0) - a(L))/1}{(1 + a(L)a(0))} \right) \). Obviously, when \( \delta = 0 \) \((a(0) = a(L))\) the Chern-Simons surfaces interact with the same potential as two Dirichlet planes. In the case \( \delta = \frac{\pi}{2} \) interaction coincides with that of a Dirichlet with a Newmann plane.

To proceed with the calculation of the ground state energy we first integrate in (13) over the momenta \( k \) and obtain

\[
\mathcal{E}_0 = -\frac{1}{12\pi} \frac{1}{1 - \frac{3}{2}s} \zeta_\text{cs}(s),
\]  

(18)

where we introduced the 'Chern-Simons' zeta function

\[
\zeta_\text{cs}(s) = \sum_n \omega_n^{3-2s}.
\]  

(19)

Now we proceed in the usual way by changing from the discrete sum to an integral. Following, e.g., the paper [14] we obtain

\[
\zeta_\text{cs}(s) = \int_\gamma \frac{dk}{2\pi i} k^{3-2s} \frac{\partial}{\partial k} \ln f(k),
\]  

(20)

where the integration path \( \gamma \) encircles the zeros of the function \( f(k) \). Having in mind that \( \delta = 0 \) corresponds to the well known case of Dirichlet boundary conditions on both planes we represent the ground state energy in the form

\[
\mathcal{E}_0 = -\frac{1}{L^3} \frac{\pi^2}{1440} h(\delta),
\]  

(21)

where the function \( h(\delta) \) describes the relative deviation of the Casimir energy for the Chern-Simons boundary conditions from that for two Dirichlet planes. From (18) and (20) we obtain

\[
h(\delta) = \frac{120}{\pi^2} \zeta(s \to 0)
\]  

(22)

where \( s \to 0 \) means the analytic continuation and we can put \( L = 1 \) in \( \zeta \).

In order to calculate the function \( h(\delta) \) we divide the integration path \( \gamma \) in (20) into an upper part \( \gamma_1 \) (with \( \Im k > 0 \)) and an lower part \( \gamma_2 \). On the upper part we represent

\[
\frac{\partial}{\partial k} \ln \sin(k + \delta) = \frac{\partial}{\partial k} \left( -i(k + \delta) + \ln \left( 1 - e^{2i(k+\delta)} \right) \right)
\]  

(23)

and on the lower part we choose

\[
\frac{\partial}{\partial k} \ln \sin(k + \delta) = \frac{\partial}{\partial k} \left( i(k + \delta) + \ln \left( e^{-2i(k+\delta)} - 1 \right) \right),
\]  

(24)

which is just the complex conjugate of (23). The first contributions on the r.h.s. of (23) and (24) do not depend on the Chern-Simons parameter \( a \) and represent
(divergent) energy density in the empty Minkowski space. We drop them therefore. Than we turn the integration contours, namely $k \rightarrow ik$ on $\gamma_1$ and $k \rightarrow -ik$ on $\gamma_2$. After that we can take the limit $s \rightarrow 0$ because of the now exponential convergence of the integral and obtain

$$
\zeta = \int_{0}^{\infty} \frac{dk}{2\pi} k^3 \frac{\partial}{\partial k} \ln \left( 1 + e^{-4k} - 2 \cos(2\delta) e^{-2k} \right).
$$

For the function $h(\delta)$ we obtain after some trivial transformations

$$
h(\delta) = -\frac{120}{\pi^3} \int_{0}^{\infty} dk \, k^3 \frac{e^{-2k} - \cos(2\delta)}{\cosh(2k) - \cos(2\delta)}.
$$

We note the special values $h(0) = 1$ which confirms the case of Dirichlet boundary conditions on both planes and

$$
h\left(\frac{\pi}{4}\right) = -\frac{7}{128}, \quad h\left(\frac{\pi}{2}\right) = -\frac{7}{8},
$$

where the last number reproduces the repulsive potential between Dirichlet and Neumann planes. This function can be easily plotted, the result is shown in Figure [1]. To obtain the complete interaction energy one should also take into account the contribution of the second polarization $b^-$ which results in an overall factor of 2 in the ground state energy (18). Obviously, this means that the full interaction energy of the Chern-Simons surfaces is obtained from the interaction energy of two conducting planes by multiplication by the same function $h(\delta)$.

## 4 Conclusions

In this paper we have argued that the presence of the Chern-Simons boundary action naturally leads to a modification of the boundary conditions for the photons. More precisely, we show that one of the admissible sets of the boundary conditions, namely “bag” or “conductor” ones, receives a contribution from the Chern-Simons interaction (depending on which potentials, ordinary or dual, are used in the Chern-Simons term). Usually, the type of the potential does not matter because the Chern-Simons term originates from the dual invariant volume action $a \int_M * F^{\mu \nu} F_{\mu \nu}$. This case corresponds to equal Chern-Simons couplings on both surfaces and does not lead to modification of the ground state energy. In the present context the case of the broken duality ($a(0) \neq a(L)$) is more interesting, but less clear as far as a modification of the boundary conditions is concerned. To clarify this point one should consider the microscopic interaction of photons with the surface which is out of the scope of the present paper (see also a remark before the eq. (14)). In any case, we find it quite unnatural if the presence of a specific $P$-odd surface interaction will have no effect on the boundary conditions at all. The simplest relevant modification of the boundary conditions (local,
Figure 1: The function $h(\delta)$ showing the dependence of the ground state energy on the Chern-Simons parameter $\delta = \arctan \left( \frac{(a(0) - a(L))}{(1 + a(L) a(0))} \right)$.

gauge invariant, without new dimensional parameters) is just the one considered in the present paper.

For a given set of the Chern-Simons boundary conditions we have calculated rigorously the ground state energy for two parallel planes bearing different Chern-Simons charges. As can be seen from the fig. 1, the Casimir force exhibits a strong dependence on the Chern-Simons coupling and can even change its sign as compared to the case of two conducting planes. With the present experimental technique [7] maintaining a precision of about 1% even correspondingly small values of the Chern-Simons parameter $a$ should be measurable in an experiment with one conducting plane and another one presumably bearing the Chern-Simons interaction. Such a measurement would provide an important check for the theoretical models including $P$-odd interaction of electromagnetic field with a surface.

The most spectacular application would be the (FQHE). An exact relation between the mechanism of the fractional conductivity and the boundary conditions is not well established yet. As well, it is not clear which part of the virtual photons responsible for the Casimir force would “see” the FQHE. However, if any interaction exists and if it makes sense to idealize it by boundary conditions than it should be of the type considered here. Hence, the Casimir force between two surfaces might serve as a tool to study these effects. Even though an actual experimental realization of such a measurement is not going to be simple, it could result in independent and interesting results.
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References

[1] S. Deser, R. Jackiw, and S. Templeton. Three-dimensional massive gauge theories. *Phys. Rev. Lett.*, 48:975, 1982.

[2] J.F. Schonfeld. A mass term for three-dimensional gauge fields. *Nucl. Phys.*, B185:157, 1981.

[3] F. Wilczek. Magnetic flux, angular momentum, and statistics. *Phys. Rev. Lett.*, 48:1144, 1982.

[4] O. Heinonen. *Composite Fermions, A Unified View of the Quantum Hall Regime*. World Scientific, Singapore, 1998.

[5] Sankar Das Sarma and Aron Pinczuk, editors. *Perspectives in Quantum Hall Effects : Novel Quantum Liquids in Low-Dimensional Semiconductor Structures*. John Wiley & Sons, New York, 1996.

[6] V. M. Mostepanenko and N. N. Trunov. *The Casimir Effect and Its Applications*. Clarendon Press, Oxford, 1997.

[7] Anushree Roy, Chiung-Yuan Lin, and U. Mohideen. Improved precision measurement of the casimir force. *Phys. Rev.*, D60:111101, 1999.

[8] S. K. Lamoreaux. Demonstration of the casimir force in the 0.6 to 6 micrometers range. *Phys. Rev. Lett.*, 78:5–7, 1997.

[9] Peter Hasenfratz and Julius Kuti. The quark bag model. *Phys. Rept.*, 40:75–179, 1978.

[10] M. Bordag, B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko. Stronger constraints for nanometer scale yukawa-type hypothetical interactions from the new measurement of the casimir force. *Phys. Rev.*, D60:055004, 1999.

[11] E. Elizalde and D. V. Vassilevich. Heat kernel coefficients for chern-simons boundary conditions in qed. *Class. Quantum Grav.*, 16:813–823, 1999.

[12] M. Bordag, K. Kirsten, and D. Vassilevich. On the ground state energy for a penetrable sphere and for a dielectric ball. *Phys. Rev.*, D59:085011, 1999.
[13] M. Bordag and D. V. Vassilevich. Heat kernel expansion for semitransparent boundaries. *J. Phys. A*, 1999. to appear, hep-th/9907076.

[14] M. Bordag. Vacuum energy in smooth background fields. *J. Phys.*, A28:755–766, 1995.