ON THE PERFECTNESS OF NONTRANSITIVE
GROUPS OF DIFFEOMORPHISMS

Stefan Haller  Tomasz Rybicki

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Abstract. It is proven that the identity component of the group preserving
the leaves of a generalized foliation is perfect. This shows that a well-known
simplicity theorem on the diffeomorphism group extends to the nontransitive
case.

1. Introduction

The goal of this paper is to show a perfectness theorem for a class of diffeomorphism
groups connected with foliations. Throughout by foliations we shall understand
generalized foliations in the sense of P.Stefan [12]. Given
a smooth foliation $\mathcal{F}$ on a manifold $M$, the symbol $\text{Diff}(T^n, \mathcal{F})_0$ stands for
the identity component of the group of all leaf preserving $C^\infty$-smooth diffeo-
morphisms of $(M, \mathcal{F})$ with compact support.

Theorem 1.1. Let $\mathcal{F}$ be a foliation on $M$ with no leaves of dimension 0.
Then the group $\text{Diff}(T^n, \mathcal{F})_0$ and its universal covering $\tilde{\text{Diff}}(T^n, \mathcal{F})_0$ are per-
fect.

Theorem 1.1 and its proof extend a well known theorem of W.P.Thurston
[15] stating that $\text{Diff}(M)_0$, the identity component of the group of all com-
 pactly supported diffeomorphisms of class $C^\infty$ on a manifold $M$, is simple. The proof generalizes the case, when $M$ is the torus (Theorem of M.R.Herman [6]), to any manifold by a reasoning involving the homology
theory (cf.[1]). Further modifications and completions of [10], where the
regular case has been considered, enables us to prove Theorem 1.1. These
refinements are necessary in the generating as well as in the general case.

Note that, in general, a nontransitive group is not simple for obvious
reasons. Note as well that the perfectness implies the simplicity in a large
class of transitive groups of homeomorphisms (cf.[4]).

Notice that $\text{Diff}(T^n, \mathcal{F})_0$ belongs to the class of so-called modular diffeo-
morphism groups, see Section 2. A clue and difficult part of the proof consist

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in showing the fragmentation and deformation properties (Sect.4). The proof of it does not appeal to foliations and holds in the whole class.

K.Fukui in [5] has shown that the abelianization of the group stabilizing a point is nontrivial. This simple example indicates the necessity of the assumption in the theorem.

It is fundamental that any "isotopically connected" diffeomorphism group determines uniquely a foliation (cf. Section 2). It follows that the identity component of the leaf preserving diffeomorphism group is the largest connected group defining the foliation in question. The problem of the perfection of its subgroups still determining this foliation is very difficult in particular cases. Of course, such subgroups, even if "classical", need not be perfect in general (see [1], or [11] in the nontransitive case). The regular case of symplectic geometry has been studied by one of us in [11].

Throughout all manifolds, diffeomorphisms, foliations etc. are assumed to be of the class $C^\infty$. Most of reasonings are neither true for the class $C^r$ $(r$-finite) nor for $C^\omega$.

2. Foliations and Modular Groups of Diffeomorphisms

Let us recall some concepts from [12]. A foliation of class $C^r$ is a partition $\mathcal{F}$ of $M$ into weakly imbedded submanifolds (see below), called leaves, such that the following condition holds. If $x$ belongs to a $k$-dimensional leaf, then there is a local chart $(U, \phi)$ with $\phi(x) = 0$, and $\phi(U) = V \times W$, where $V$ is open in $\mathbb{R}^k$, and $W$ is open in $\mathbb{R}^{n-k}$, such that if $L \in \mathcal{F}$ then $\phi(L|U) \cap (V \times W) = V \times l$, where $l = \{w \in W : \phi^{-1}(0, w) \in L\}$.

Let $L$ be a subset of a $C^r$-manifold $M$ endowed with a $C^r$-differentiable structure which makes it an imbedded submanifold. Then $L$ is weakly imbedded if for any locally connected topological space $N$ and a continuous map $f : N \to M$ satisfying $f(N) \subset L$, the map $f : N \to L$ is continuous as well. It follows that in this case such a differentiable structure is unique.

It has been first stated in [12] that orbits of any set of local $C^r$-diffeomorphisms, $1 \leq r \leq \omega$, form a foliation. More precisely, a smooth mapping $\phi$ of an open subset of $\mathbb{R} \times M$ into $M$ is said to be a $C^r$-arrow if (1) $\phi(t, .) = \phi_t$ is a local $C^r$-diffeomorphism for each $t$, possibly with empty domain, (2) $\phi_0 = id$ on its domain, and (3) $\text{dom}(\phi_s) \subset \text{dom}(\phi_t)$ whenever $0 \leq s < t$.

Given an arbitrary set of arrows $A$ let $A^*$ be the totality of local diffeomorphisms $\psi$ such that $\psi = \phi(t, .)$ for some $\phi \in A$, $t \in \mathbb{R}$. Next $\hat{A}^*$ denotes the set consisting of all local diffeomorphisms being finite compositions of elements from $A^*$ or $(A^*)^{-1} = \{\psi^{-1} : \psi \in A^*\}$, and of the identity. Then the orbits of $\hat{A}^*$ are called accessible sets of $A$.

For $x \in M$ we let $A(x)$, $\hat{A}(x)$ be the vector subspaces of $T_x M$ generated by

\[
\{\dot{\phi}(t, y) : \phi \in A, \phi_t(y) = x\}, \quad \{d_y \psi(v) : \psi \in \hat{A}^*, \psi(y) = x, v \in A(y)\},
\]

respectively.

**Theorem 2.1** [12]. Let $A$ be an arbitrary set of $C^r$-arrows on $M$. Then:

(i) Every accessible set of $A$ admits a (unique) $C^r$-differentiable structure of a connected weakly imbedded submanifold of $M$.  

(ii) The collection of accessible sets defines a foliation $\mathcal{F} = \mathcal{F}(A)$.

(iii) $\tilde{A}(x)$ is the tangent distribution of $\mathcal{F}(A)$.

The following ”splitting” theorem will be of use.

**Theorem 2.2** [3, Theorem 2.1]. Let $\mathcal{F}$ be a foliation on $M$ and let $x$ lie on a $k$-dimensional leaf. There exists a chart $(U, \phi)$ such that $\phi(x) = 0$, $\phi(U) = V \times W$, where $V$ (resp. $W$) is open in $\mathbb{R}^k$ (resp. $\mathbb{R}^{n-k}$), and the foliation $\mathcal{F}|U$ is sent to a foliation $V \times \mathcal{F}_2$, where $\mathcal{F}_2$ is a foliation on $W$ with a 0-dimensional leaf at 0.

Let $G(M) \subset Diff^\infty(M)$ be any diffeomorphism group. By a smooth path (or isotopy) in $G(M)$ we mean any family $\{f_t\}_{t \in \mathbb{R}}$ with $f_t \in G(M)$ such that the map $(t, x) \mapsto f_t(x)$ is smooth. Next, $G(M)_0$ denotes the subgroup of all $f \in G(M)$ such that there is a smooth path $\{f_t\}_{t \in \mathbb{R}}$ with $f_t = id$ for $t \leq 0$ and $f_t = f$ for $t \geq 1$, and such that each $f_t$ stabilizes outside a fixed compact set. Notice that $G(M)_0$ is the connected component of $id$ if $G(M)$ is locally arcwise connected and $M$ is compact.

Given $G(M)$ the totality of $f_t$ as above constitutes a set of arrows. This set determines uniquely a foliation. Likewise, the flow of a $C^\infty$ vector field is an arrow. Therefore any set of vector fields $\mathcal{X}(M)$ defines a foliation.

Let $G(M) \subset Diff^\infty(M)$. To any smooth path $f_t$ in $G(M)_0$ one can attach a family of vector fields

$$\dot{f}_t = \frac{df_t}{dt}(f_t^{-1}).$$

Then the time-dependent family $\dot{f}_t$ is a unique smooth path in the Lie algebra corresponding to $G(M)_0$ which satisfies the equality

$$(2.1) \quad \frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = id.$$  

Conversely, given a smooth family $X_t$ of compactly supported vector fields there exists a unique solution $f_t$ of (2.1). Specifically, $f_t$ is a flow if and only if the corresponding $X_t = X$ is time-independent.

Although only few diffeomorphisms are elements of some flow, the simplicity theorem of Thurston [15] states that actually $Diff(M)_0$ is generated by elements of flows.

**Definition.** A Lie algebra of vector fields is called modular if it is a $C^\infty(M)$ module which is $C^0$ closed.

A group of diffeomorphisms $G(M)$ is said to be modular if its Lie algebra (cf. [7]) $\mathfrak{g}$ is modular. Consequently, there is a one-to-one correspondence between isotopies in $G(M)$ and in $\mathfrak{g}$ given by (2.1).

**Lemma 2.3.** Let $V \subseteq \mathcal{X}_c(M)$ be a $C^0$ closed $C^\infty(M)$ submodule. For $x \in M$ set $E_x := \{X(x) : X \in V\}$. Then $V = \{X \in \mathcal{X}_c(M) : X(x) \in E_x\}$.

**Proof.** One inclusion ($\subseteq$) is trivial, we show the other one. So let $X \in \mathcal{X}_c(M)$ such that $X(x) \in E_x$ for all $x \in M$ and suppose conversely $X \notin V$. Since $V$ is $C^0$-closed there exists $\varepsilon \in C^\infty(M; \mathbb{R}^+)$ such that

$$Y \in \mathcal{X}_c(M) : \|Y(y) - X(y)\| \leq \varepsilon(y) \quad \forall y \in M \quad \Rightarrow \quad Y \notin V$$
The norm is with respect to some fixed Riemannian metric on \( M \). For \( x \in M \)
we choose \( Y_x \in V \) with \( X(x) = Y_x(x) \) and a neighborhood \( U_x \) of \( x \) such that
\[ \|Y_x(y) - X(y)\| \leq \varepsilon(y) \]
for all \( y \in U_x \). Since the support of \( X \) is compact we find \( x_1, \ldots, x_n \) with
\( U_{x_1} \cup \cdots \cup U_{x_n} \supseteq \text{supp}(X) \). Finally we choose a partition
of unity \( \lambda_0, \lambda_1, \ldots, \lambda_n \) subordinate to \( \{ M \setminus \text{supp}(X), U_{x_1}, \ldots, U_{x_n} \} \) (that is
\( \text{supp}(\lambda_i) \subseteq M \setminus \text{supp}(X), \text{supp}(\lambda_i) \subseteq U_{x_i} \)) and define \( Y := \sum_{i=1}^n \lambda_i Y_{x_i} \in V \).
Using \( \lambda_0 X = 0 \) we obtain
\[
\|Y(y) - X(y)\| = \|\sum_{i=1}^n \lambda_i(y)(Y_{x_i}(y) - X(y))\| \\
\leq \sum_{i=1}^n \lambda_i(y)\|Y_{x_i}(y) - X(y)\| \leq \varepsilon(y) \\
\leq \lambda_i(y)\varepsilon(y)
\]
for all \( y \in M \) and therefore \( Y \notin V \), a contradiction. \( \square \)

**Proposition 2.4.** Suppose \( G(M) \) is modular and let \( \{ U_i \} \) be a finite family
of open balls of \( M \). If \( f_t \) is an isotopy in \( G(M) \) such that \( \bigcup_t \text{supp}(f_t) \subset \bigcup U_i \)
then there are isotopies \( f_t^j \) supported in \( U_{i(j)} \) which satisfy \( f_t = f_t^0 \circ \cdots \circ f_t^s \).

**Proof.** Let \( f_t \) be as above and let \( X_t \) be the corresponding family in
\( X_G(M) \). By considering \( f_t, f_{t-1/m}, \ldots, f_{t-(p-1/m)} \), \( p = 1, \ldots, m \), instead of \( f_t \)
we may assume that \( f_t \) is close to the identity.

First we choose a new family of open balls, \( \{ V_j \}_{j=1}^s \), satisfying \( \text{supp}(f_t) \subset V_1 \cup \cdots \cup V_s \) for each \( t \) and which is starwise finer that \( \{ U_i \} \), that is
\[
(\forall j) \ (\exists i) \ \text{star}(V_j) \subset U_{i(j)}, \quad \text{where} \quad \text{star}(V_j) = \bigcup_{V_j \cap V_k \neq \emptyset} V_k.
\]

Let \( (\lambda_j)_{j=1}^s \) be a partition of unity subordinate to \( (V_j) \), and let \( Y_t^j = \lambda_j X_t \).
We set
\[
X_t^j = Y_t^1 + \cdots + Y_t^j, \quad j = 1, \ldots, s,
\]
and \( X_t^0 = 0 \). Each of the smooth families \( X_t^j \) integrates to an isotopy \( g_t^j \) with
support in \( V_1 \cup \cdots \cup V_j \). We get the partition
\[
f_t = g_t^s \circ \cdots \circ f_t^1,
\]
where \( f_t^j = g_t^j \circ (g_t^{j-1})^{-1} \), with the required inclusions \( \text{supp}(f_t^j) = \text{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \text{star}(V_j) \subset U_{i(j)} \) which hold if \( f_t \) is sufficiently small. \( \square \)

**Proposition 2.5.** Let \( G(M) \) be modular. Then \( G(M)_0 \) is locally contractible
in the \( C^\infty \)-topology.

Indeed, it follows by a classical argument involving vector fields.

3. The Case of \( Diff(T^n, F) \)

The concept of \( \mathcal{L} \)-category was introduced in [14]. Roughly speaking,
an object in this category is a quadruple \((E, B, \mathcal{N}, S)\), where \( E \) is a Fréchet
space, \( \mathcal{N} = (| \cdot |_i) \) is an increasing sequence of norms defining the topology of
\( E, S = (S_t), \ t > 0 \), is a one-parameter family of "approximation" operators
on \( E \), and \( B \) is an open subset with respect to some norm from \( \mathcal{N} \). Let \( E_i \)
denote the completion of $E$ with respect to the norm $| i$, and let $\rho_{ji} : E_j \rightarrow E_i$ be an extension if $i d E$, $j \geq i$. Then topologically $E = \lim_{\rightarrow}(E_i, \rho_{ji})$. An interpretation of the operators $S$ spaces of $C$ generalized smooth structure on them motivated the definition of $L$ that their group products and inverse mappings are $L$ can define also a notion of the diffeomorphism groups are here the main example. In obvious way one can define also a notion of $L$-action of an $L$-group on an $L$-manifold.

We begin with an Implicit Function Theorem in the case of $L$-actions (cf.[14]). Let $G, H$ be $L$-groups of class $C^r$ ($r \geq 2$) and $M$ be an $L$-manifold. Denote by $\alpha : G \times G \rightarrow G, \beta : H \times H \rightarrow H$ the group products and let $\Phi : G \times M \rightarrow M, \Psi : H \times M \rightarrow M$ be $L$-actions of class $C^r$. Next, let $\Delta : G \times H \times M \rightarrow M$, be an ”action” of $G \times H$, defined by

$$\Delta(g, h, x) = \Phi(g, \Psi(h, x))$$

for $g \in G, h \in H, x \in M$. By $d\Delta$ we denote the differential of $\Delta$ with respect to two first variables. By the chain rule one has

$$d\Delta(g, h, x, g', h') = d_1\Phi(g, \Psi(h, x), g') + d_2\Phi(g, \Psi(h, x), d_1\Psi(h, x, h')).$$

(Here we adopt the notation $g' \in T_g(G), \ h' \in T_h(M)$ and so on.) Let us fix $x_0 \in M$. By making use of the local triviality of the tangent bundle $TM$ one can identify $T_x(M)$ with $T = T_{x_0}(M)$ for $x$ being near $x_0$. Likewise, $T_g(G)$ is identified with $T_1 = T_e(G)$, whenever $g \in G$ is near $e$, and $T_h(H)$ is identified with $T_2 = T_e(H)$, whenever $h \in H$ is near $e$. Then by applying Implicit Function Theorem one has the following

**Theorem 3.1** [14, 4.2.5]. Suppose that there exists an $L$-morphism of class $C^\infty$, $L : U \times T \rightarrow T_1 \times T_2$, where $U$ is a neighborhood of $e$ in $H$, such that if $L(h, \hat{x}) = (\hat{g}, \hat{h})$, then

$$d\Delta(e, e, \Psi(h, x), \hat{g}, \hat{h}) = \hat{x}.$$ 

Then there exists a neighborhood $V$ of $x_0$ in $M$ and a weak $L$-morphism of class $C^\infty$ $s : V \rightarrow G \times H$ such that $\Delta(g, h, x_0) = x$ if $s(x) = (g, h)$.

Now let $T^n$ be the $n$-dimensional torus. Let $1 \leq k < n$ and let $F_k$ denote the trivial $k$-dimensional foliation on the torus $T^n$, i.e. $F_k = \{T^k \times \{pt\}\}$. We have the canonical inclusion $\alpha \in T^k \mapsto R_\alpha \in Diff^{\infty}(T^n, F_k)_0$, where

$$R_\alpha(z_1, \ldots, z_n) = (e^{2\pi i \alpha_1}z_1, \ldots, e^{2\pi i \alpha_k}z_k, z_{k+1}, \ldots, z_n).$$
Given a foliation $\mathcal{F}'$ on $T^{n-k}$ we set $\mathcal{F} = T^k \times \mathcal{F}'$, so that $\mathcal{F}_k$ is a subfoliation of $\mathcal{F}$. It is clear that

$$\text{(3.1)} \quad \text{Diff}^\infty (T^n, \mathcal{F}_k)_0 \subset \text{Diff}(T^n, \mathcal{F})_0.$$

Recall that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ satisfies the Diophantine condition if there are small $c > 0$ and large $N$ such that for any $(n+1)$-tuple of integers $(q_0, q_1, \ldots, q_n)$ with $(q_1, \ldots, q_n) \neq 0$ one has

$$|q_0 + q_1 \alpha_1 + \cdots + q_n \alpha_n| > c(|q_1| + \cdots + |q_n|)^{-N}.$$

Next, $\alpha \in T^n$ verifies this condition if so does its representant in $\mathbb{R}^n$. The set of all such $\alpha$ is dense in $T^n$.

**Theorem 3.2.** Let $(M, \mathcal{F})$ be such that $\mathcal{F} = T^k \times \mathcal{F}'$ and let $\alpha \in T^k$ be Diophantine. There exist a neighborhood $\mathcal{U}$ of $R_\alpha$ in $\text{Diff}(T^n, \mathcal{F})_0$ and a continuous map $s : \mathcal{U} \to \text{Diff}(T^n, \mathcal{F})_0 \times T^k$ such that $h = R_\beta g^{-1} R_\alpha g$ whenever $h \in \mathcal{U}$, and $s(h) = (g, \beta)$. Furthermore, if $h_t$, $t \in I$, is a smooth isotopy in $\mathcal{U}$ and $s(h_t) = (g_t, \beta_t)$ then $g_t$, $\beta_t$ depend smoothly on $t$.

**Proof.** The starting observation is that if $\alpha \in T^k$ is Diophantine then so is $\alpha' = (\alpha, 0) \in T^n$.

Let $\bar{\alpha} \in \mathbb{R}^k$ be a representant of $\alpha \in T^k$. The components of $\bar{\alpha}$ can be chosen linearly independent over $\mathbb{Q}$. It follows that $\alpha$ generates a dense subgroup of $T^k$.

Let $G = T^k, H = \text{Diff}(T^n, \mathcal{F})_0$ endowed with the structure opposite to the usual. Define actions of $G$ and $H$, respectively, on $H$ by

$$\Phi(\lambda, h) = R_\lambda h,$$

$$\Psi(g, h) = g^{-1} hg.$$

Let $\Delta : G \times H \times H \to H$ be the composition of these actions

$$\Delta(\lambda, g, h) = \Phi(\lambda, \Psi(g, h)) = R_\lambda g^{-1} hg.$$

We use Theorem 3.1. We have

$$d\Delta(e, e, h, \hat{\lambda}, \hat{g}) = \hat{\lambda} + dh \cdot \hat{g} - \hat{g} \cdot \Delta,$$

where $\hat{\lambda} \in \mathbb{R}^k = T_e(T^k), \hat{g} \in = T_{id}(\text{Diff}(T^n, \mathcal{F})_0)$. Consider the equation

$$\hat{\lambda} + d(g^{-1} R_\alpha g) \cdot \hat{g} - \hat{g} \cdot (g^{-1} R_\alpha g) = \hat{h}.$$

In view of Theorem 3.1 we have to solve this equality for given $g, h \in \text{Diff}(T^n, \mathcal{F})_0, \hat{h} \in T_{id}(\text{Diff}(T^n, \mathcal{F})_0)$, and with respect to the unknowns $\hat{\lambda}$, $\hat{g}$. Set $\hat{f} = dg \cdot \hat{g} \cdot g^{-1} \in \text{Diff}(T^n, \mathcal{F})_0$. Since $dR_\alpha = id$, we get

$$\text{(3.2)} \quad \hat{f} - \hat{f} \cdot R_\alpha = dg \cdot (\hat{h} - \hat{\lambda}) \cdot g^{-1}$$

If $m$ be the normalized Haar measure on $T^n$, we have

$$\text{(3.3)} \quad \int_{T^n} dg \cdot (\hat{h} - \hat{\lambda}) \cdot g^{-1} \ dm = 0.$$
The equality (3.3) determines uniquely $\hat{\lambda} \in \mathbb{R}^k$, provided $g$ is sufficiently near $id$ in $\text{Diff}(T^n, \mathcal{F})_0$.

Now by using the condition on $\alpha$ and expanding in Fourier series the both sides of (3.2) we get as in [14] or [6] the existence of $\hat{f} \in C^\infty(T^n, \mathbb{R}^n)$ satisfying (3.2). Observe that in case of class $C^r$, $r$ finite, one could not avoid the "loss of smoothness", i.e. $\hat{f}$ is of class $C^{r-\beta}$, $\beta$ depending on $\alpha$, cf. [8].

Since $g$ is leaf preserving it remains to show that $\hat{f} \in T_{id}(\text{Diff}(T^n, \mathcal{F})_0)$, i.e. $\hat{f}$ is a vector field tangent to $\mathcal{F}$.

First fix $p' \in T^k$ and denote $C(\cdot) := \hat{f}(p', \cdot)$. Then fix $p = (p', p'') \in T^n$ and let $L_p$ (resp. $L_p^k$) be the leaf of $\mathcal{F}$ (resp. $\mathcal{F}_k$) passing through $p$. Choose $(x, \bar{x}, y) = (x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_{l(p)}, y_1, \ldots, y_{n-1(l(p))})$, a distinguished chart of $\mathcal{F}$ at $p$, where $l(p) = \text{dim}(L_p) \geq k$. Consequently, $L_p$ is determined by $y = 0$ in this chart. In addition, we may and do assume that $\bar{x} = y = 0$ describes $L_p^k$ in the domain of $(x, \bar{x}, y)$.

The equation (3.2) can be rewritten as

\[(3.4) \quad \hat{f}_i - \hat{f}_i \cdot R_\alpha = \hat{k}_i, \quad i = 1, \ldots, n.\]

Here $\hat{k}(\hat{k}_i) \in T_{id}(\text{Diff}(T^n, \mathcal{F})_0)$ denotes the r.h.s. of (3.2). Observe that $\hat{k}_i(q) = 0$ if $i > \text{dim}(L_p)$, where $q \in L_p$.

W introduce $\tilde{f} = (\hat{f}_i)$ by

$$\tilde{f} := \hat{f} - C,$$

where $C$ is as above. Hence $\tilde{f}(0, \bar{x}, y) = 0$, $\forall \bar{x}, y$, and (3.4) is satisfied with $\hat{f}_i$ instead of $\hat{f}_i$.

For $i > k(p)$ we have by (3.4) $\tilde{f}_i(\alpha, \bar{x}, 0) = 0$ in the chart $(x, \bar{x}, y)$, and inductively

$$\tilde{f}_i(m\alpha, \bar{x}, 0) = 0, \quad \forall \bar{x} \forall m \in \mathbb{Z}^k.$$

Therefore, by the assumption on $\alpha$, $\tilde{f}_i|_{L_p} = 0$. By repeating this argument to any leaf of $\mathcal{F}$ we get that $\tilde{f}$ is tangent to $\mathcal{F}$. Since $C$, and consequently $\tilde{f}$, are $C^\infty$, the required solution of (3.2) is $\tilde{f}$.

The second assertion follows from the fact that $s$ is a $C^\infty$ $\mathcal{L}$-morphism (Theorem 3.1), and it sends smooth curves to smooth curves, cf.[1].

4. TOPOLOGICAL BACKGROUND

Let us fix notation and recall some facts from the homology of groups (see e.g. [2]). Let $G$ be a connected topological group. We shall be concerned with $H_1(G)$ which is identified with the abelianization $G/[G, G]$.

By $\hat{G}$ we denote the universal covering of $G$. Provided $G$ is locally arcwise connected, $\hat{G}$ can be thought of as the set of pairs $(g, \{g_t\})$, where $g \in G$, $g_t (t \in I)$ is a path connecting $g$ with $e$, and $\{g_t\}$ is the homotopy class of $g_t$ rel. endpoints. Then $\hat{G}$ is given a group structure by the pointwise multiplication or, equivalently, by the juxtaposition. Clearly the perfection of $\hat{G}$ yields the perfection of $G$.

With any $G$ we can associate some simplicial set $B\hat{G} = \bigcup Bm\hat{G}$ where $Bm\hat{G}$ is identified with the set $(G, e)^{(\Delta^m, e_n)}$ of continuous mappings of the standard $m$-simplex $\Delta^m$ into $G$ sending $e_0$ (the first vertex) to $e$. For the detailed definition see [2] or [9].
We recall some properties of $B\tilde{G}$. It is a Kan complex and one can give a purely combinatorial definition of homotopy groups (cf.[9]). Namely the following equivalence relation is given on $B\tilde{G}$: for any 1-simplices $\sigma, \tau \in B\tilde{G}$

$$\sigma \sim \tau \iff \exists c \in B_2\tilde{G} : \partial_0 c = \sigma, \partial_1 c = \tau, \partial_2 c = e$$

where $e$ is the constant map. Then the first homotopy group of $B\tilde{G}$ is defined by $\pi_1(B\tilde{G}) = B\tilde{G}/\sim$. It follows that for any $\sigma \in B\tilde{G}$ the classes of $\sigma$ with respect to the relation $\sim$ and with respect to the homotopy rel. endpoints are the same, that is $\pi_1(B\tilde{G}) = B\tilde{G}/\sim = \tilde{G}$. One then has

$$H_1(B\tilde{G}, \mathbb{Z}) = H_1(\pi_1(B\tilde{G})) = H_1(\tilde{G})$$

since $H_1(B\tilde{G}, \mathbb{Z}) = \pi_1(B\tilde{G})/[\pi_1(B\tilde{G}), \pi_1(B\tilde{G})]$.

5. Fragmentation and Deformation for Modular Groups of Diffeomorphisms

Let $G = G(M)$ be a modular group and $\mathfrak{g}$ its Lie algebra.

Let $\Delta^p := \{(t_0, \ldots, t_p) : 0 \leq t_i \leq 1 ; \sum_{i=0}^{p} t_i = 1\} \subseteq \mathbb{R}^{p+1}$ denote the standard $p$-simplex and $e_0, \ldots, e_p$ the unit vectors in $\mathbb{R}^{p+1}$, that is the edges of $\Delta^p$. If $g : \Delta^p \to G$ is smooth then the right logarithmic derivative $\delta_r g \in \Omega^1(\Delta^p; \mathfrak{g})$ satisfies the Maurer Cartan equation

$$d(\delta_r g) - \frac{1}{2}[\delta_r g, \delta_r g] = 0$$

Conversely, if $\sigma \in \Omega^1(\Delta^p; \mathfrak{g})$ satisfies $d\sigma - \frac{1}{2}[\sigma, \sigma]$ then there exists $g : \Delta^p \to G$ with $\delta_r g = \sigma$. Moreover $g$ is unique if one assumes $g(e_0) = \text{id}$. (See [7]).

If $\sigma \in \Omega^1(\Delta^p; \mathfrak{g})$ then one obtains a distribution on $\Delta^p \times M$ by setting $E_{(t,x)} = \{(X_t, \sigma(X_t)(x)) : X_t \in T_t\Delta^p\} \subseteq T_{(t,x)}\Delta^p \times M$. This distribution has codimension $\dim(M)$ and it is transversal to $\{pt\} \times M$. Moreover it is integrable if and only if $d\sigma - \frac{1}{2}[\sigma, \sigma] = 0$.

So we obtain a one-to-one correspondence between $C^\infty(\Delta^p, \mathfrak{g})$, one forms $\sigma \in \Omega^1(\Delta^p; \mathfrak{g})$ satisfying $d\sigma - \frac{1}{2}[\sigma, \sigma] = 0$, and foliations on $\Delta^p \times M$ of codimension $\dim(M)$ that are transversal to $\{pt\} \times M$. We will denote the set of all such simplices by $S_p(\Delta^p; \mathfrak{g})$. If $\mathcal{U}$ is a set of open sets in $M$ then $S_{\mathcal{U}}(\Delta^p; \mathfrak{g})$ will denote the set of all simplices with support in one of the sets of $\mathcal{U}$. On the free Abelian group $C_*(\Delta^p; \mathbb{Z})$ generated by $S_*(\Delta^p; \mathfrak{g})$ we have a differential $\partial = \sum_{i=0}^{p} (-1)^i \partial_i$, where $\partial_i = \delta_i^*$ and $\delta_i : \Delta^p \to \Delta^{p+1}$ are the inclusion of the faces. Notice that $0 = \partial : C_1(\Delta^p; \mathbb{Z}) \to C_0(\Delta^p; \mathbb{Z}) \cong \mathbb{Z}$.

For an open neighborhood $\mathcal{E}$ of $0 \in \mathfrak{g}$ we let

$$S_\mathcal{E}^p(\Delta^p) = \{ \sigma \in S_p(\Delta^p) : \sigma(D\Delta^p) \subseteq \mathcal{E} \}$$

where $D\Delta^p \subseteq T\Delta^p$ is the unit disk bundle of $\Delta^p$. Then $C_\mathcal{E}^p(\Delta^p; \mathbb{Z})$ is a subcomplex of $C_*(\Delta^p; \mathbb{Z})$. Moreover we will consider $S_{\mathcal{E},\mathcal{U}}^p := S_\mathcal{E}^p(\Delta^p) \cap S_{\mathcal{U}}^p(\Delta^p)$ and $C_{\mathcal{E},\mathcal{U}}^p(\Delta^p; \mathbb{Z})$. Using barycentric subdivision one can easily show...
Lemma 5.1. For any set $\mathcal{U}$ of open sets in $M$ and for every neighborhood $\mathcal{E}$ of $0 \in \mathfrak{g}$ the inclusion induces an isomorphism $H^E_{\ast \mathcal{U}}(BG; \mathbb{Z}) \cong H^\mathcal{U}_\ast(BG; \mathbb{Z})$.

Lemma 5.2. Let $\tau : \Delta^p \times M \to \Delta^q \times M$ be smooth with $pr_M \circ \tau = pr_M$ and let $G$ be modular. If $\sigma \in S_q(BG)$ such that the foliation corresponding to $\sigma$ is transversal to $\tau(t, \cdot) : M \to \Delta^q \times M$ for all $t \in \Delta^p$ then $\tau^* \sigma \in S_p(BG)$. Moreover we have $\text{supp}(\tau^* \sigma) \subseteq \text{supp}(\sigma)$.

Proof. Obviously $\tau^* \sigma$ is a foliation on $\Delta^p \times M$ with $\text{codim}(\tau^* \sigma) = \dim(M)$ which is transversal to $\{pt\} \times M$ and so we obtain at least $\tau^* \sigma \in S_p(B\text{Diff}_c^\infty(M), \sigma)$. If $Y \in T_i \Delta^p$ the defining equation for $\tau^* \sigma(Y)$ is

$$\sigma(T_{(t,x)}(pr_{\Delta^q} \circ \tau) \cdot (Y, \tau^* \sigma(Y)(x))(x)) = T_{(t,x)}(pr_M \circ \tau) \cdot (Y, \tau^* \sigma(Y)(x)) = (\tau^* \sigma)(Y)(x)$$

So we see that $(\tau^* \sigma)(Y)(x) \in E_x := \{X(x) : X \in \mathfrak{g}\}$ for all $x \in M$ hence by Lemma 2.3 we obtain $(\tau^* \sigma)(Y) \in \mathfrak{g}$ and thus $\tau^* \sigma \in S_p(BG)$. $\square$

Lemma 5.3. Let $\tau : M \to \Delta^p$ be smooth and define

$$\mathcal{E}_\tau := \{X \in \mathfrak{g} : \|T_x \tau \cdot X_x\| < 1 \ \forall x \in M\} \subseteq \mathfrak{g}.$$ 

Then $\mathcal{E}_\tau$ is a zero neighborhood in $\mathfrak{g}$ and for $\sigma \in S_p^{\mathcal{E}_\tau}(BG)$ the foliation on $\Delta^p \times M$ corresponding to $\sigma$ is transversal to $(\tau, \text{id}_M) : M \to \Delta^p \times M$.

Corollary 5.4. Let $\tau_i : M \to \Delta^p$ for $i = 1, \ldots, N$. Then there exists a zero neighborhood $\mathcal{E}$ in $\mathfrak{g}$ such that for $\sigma \in S_p^{\mathcal{E}}(BG)$ the foliation on $\Delta^p \times M$ corresponding to $\sigma$ is transversal to $(\mu, \text{id}_M)$, where $\mu := \sum_{i=1}^N t_i \tau_i$ is any convex combination of the $\tau_i$, that is $0 \leq t_i \leq 1$ and $\sum_{i=1}^N t_i = 1$.

For $N \in \mathbb{N}$ let $D^N_n := \{(m_0, \ldots, m_n) \in \mathbb{N}_0^{n+1} : \sum_{i=0}^n m_i = N\}$. If $\lambda \in \Delta^{N-1}$ and $m \in D^N_n$ we define $\tau^\lambda_m : \Delta^n$ by

$$\tau^\lambda_m := \left(\lambda_0 + \cdots + \lambda_{m_0-1}, \lambda_{m_0} + \cdots + \lambda_{m_0+m_1-1}, \ldots, \lambda_{m_n}\right)_{m_0}^{m_1}^{m_n}$$

Moreover we let

$$A^n_n := \{(m, \pi) \in D^n_N \times \mathfrak{S}_n : m + f_{\pi(1)} + \cdots + f_{\pi(n)} \in D^n_N \ \forall 0 \leq j \leq n\}$$

where $f_i := e_i - e_{i-1}$. If $(m, \pi) \in A^n_n$ we define $\tau^\lambda_{(m, \pi)} : \Delta^n \to \Delta^n$ by

$$\tau^\lambda_{(m, \pi)}(e_j) := \tau^\lambda_{m+f_{\pi(1)}+\cdots+f_{\pi(j)}}$$

for $0 \leq j \leq n$ and extend it affine.

If $\lambda \in C^\infty(M, \Delta^{N-1})$ is a finite partition of unity and $(m, \pi) \in A^n_n$ we define

$$\tau^\lambda_{(m, \pi)} : \Delta^n \times M \to \Delta^n \times M \qquad \tau^\lambda_{(m, \pi)}(t, x) = (\tau^\lambda_{(m, \pi)}(t), x)$$

From Corollary 5.4 we obtain a zero neighborhood $\mathcal{E}_\tau$ in such that $\tau^\lambda_{(m, \pi)}(t, \cdot) : M \to \Delta^p \times M$ is transversal to the foliation corresponding to $\sigma \in S_p^\mathcal{E}(BG)$ for all $p \leq n$ and $(m, \pi) \in A^n_n$. Moreover Lemma 5.2 yields

$$\sigma \in S_p^{\mathcal{E}}(BG) \Rightarrow (\tau^\lambda_{(m, \pi)})^* \sigma \in S^\mu_p(BG)$$

for all $p \leq n$ and $(m, \pi) \in A^n_n$. We are now in a position to give the following
**Definition 5.5 (Fragmentation mapping).** Let $G$ be a modular, $U$ a set of open sets in $M$, $N \in \mathbb{N}$ and $\lambda \in C^\infty (M, \Delta^{N-1})$. Then for $p \leq n$ we define

$$\varphi_p^\lambda : C^p_{\mathcal{E}^\lambda, \mathcal{M}} (BG; \mathbb{Z}) \to C^p_{\mathcal{M}} (BG; \mathbb{Z}) \quad \varphi^\lambda_p (\sigma) := \sum_{(m, \pi) \in A^p_N} \text{sgn}(\pi) (\tau^\lambda_{(m, \pi)})^* \sigma$$

where the simplex $\sigma$ is considered as foliation on $\Delta^p \times M$.

This is a modification of the A. Banyaga’s procedure used for the deformation for globally hamiltonian diffeomorphisms, see [1].

Next we subdivide $\Delta^p \times I$ into $p + 1$ simplexes in the usual way. For $1 \leq i \leq p + 1$ we define $s^{p+1}_i : \Delta^{p+1} \to \Delta^p \times I$ by

$$s^{p+1}_i (e_j) := \begin{cases} (e_j, 0) & 0 \leq j < i \\ (e_{j-1}, 1) & i \leq j \leq p + 1 \end{cases}$$

and extend it affine. For $(m, \pi) \in A^p_N$ we define $T^\lambda_{(m, \pi)} : \Delta^p \times I \times M \to \Delta^p \times M$ by

$$T^\lambda_{(m, \pi)} (t, 0, x) = \tau^\lambda_{(m, \pi)} (t, x) \quad \text{and} \quad T^\lambda_{(m, \pi)} (t, 1, x) = \tau^\lambda_{(m, \pi)} (t, x)$$

and extend it affine, where $\lambda_1 = (\frac{1}{N}, \ldots, \frac{1}{N}) \in C^\infty (M, \Delta^{N-1})$. For $p \leq n$ we define a homotopy $H_p : C^p_{\mathcal{E}^\lambda, \mathcal{M}} (BG; \mathbb{Z}) \to C^p_{\mathcal{M}} (BG; \mathbb{Z})$ on a $p$-simplex $\sigma$ by

$$H_p (\sigma) := \sum_{i=1}^{p+1} (-1)^i \sum_{(m, \pi) \in A^p_N} \text{sgn}(\pi) (s^{p+1}_i \times \text{id}_M)^* (T^\lambda_{(m, \pi)})^* \sigma$$

**Lemma 5.6.** In this situation ($n = 2$) we have $\partial \circ \varphi^\lambda_2 = \varphi^\lambda_1 \circ \partial$ and $(\varphi^\lambda_1)_* = j_* : H^p_{\mathcal{E}^\lambda, \mathcal{M}} (BG; \mathbb{Z}) \to H^p_{\mathcal{M}} (BG; \mathbb{Z})$, where $j$ is the inclusion.

**Proof.** In this low dimension it is easy to see that $\partial \circ \varphi^\lambda_2 = \varphi^\lambda_1 \circ \partial$ and $\varphi^\lambda_1 - \varphi^\lambda_1 = \partial H_1 + H_0 \partial$ (make a drawing!). So $(\varphi^\lambda_1)_* = (\varphi^\lambda_1)_*$, but the latter is the ordinary subdivision of the interval into $N$ subintervals and hence homotopic to the identity. So we obtain $(\varphi^\lambda_1)_* = (\varphi^\lambda_1)_* = j_*$. □

**Remark 5.7.** With some combinatorical difficulties one can show that this remains true for arbitrary $n$.

**Remark 5.8.** If $\lambda$ is subordinate to an open cover $U$, and $\mathcal{U}^{(n)} := \{ U_1 \cup \cdots \cup U_n : U_i \in \mathcal{U} \}$ one easily sees

$$\varphi^\lambda_p : C^p_{\mathcal{E}^\lambda, \mathcal{M}} (BG; \mathbb{Z}) \to C^p_{\mathcal{M}} (BG; \mathbb{Z}) \quad \forall p \leq n$$

and therefore the name fragmentation mapping.

**Remark 5.9.** If $\mathcal{V}$ is an open covering with the property

$$V_1, V_2 \in \mathcal{V}, V_1 \cap V_2 \neq \emptyset \Rightarrow V_1 \cup V_2 \subseteq U \in \mathcal{U}$$

then $\varphi^\lambda_p : C^p_{\mathcal{E}^\lambda, \mathcal{M}} (BG; \mathbb{Z}) \to C^p_{\mathcal{M}} (BG; \mathbb{Z})$ and therefore the name fragmentation mapping.
and $\lambda$ is subordinate to $\mathcal{V}$ then $\varphi_1^\lambda$ induces a mapping $(\varphi_1^\lambda)_* : H_1^{\mathcal{U},\tilde{\lambda}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z})$. One can see this as follows. If $\partial d = c \in C_1^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z})$ with $d \in C_2^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z})$ then Lemma 5.6 gives $\varphi_1^\lambda(c) = \partial \varphi_1^\lambda(d)$, but we only have $\varphi_2^\lambda(d) \in C_2^{\mathcal{U},2}(B\tilde{G};\mathbb{Z})$. If we write

$$\varphi_2^\lambda(d) = \sum_k \rho_k + \sum_i \kappa_i$$

with $\text{supp}(\rho_k) \subseteq V_i \cup V_j$ for some $i, j$ with $V_i \cap V_j \neq \emptyset$ and $\kappa_i$ such that there do not exist $V_i, V_j$ with this property then one can show that $\partial(\sum \kappa_i) = 0$ and hence $\varphi_2^\lambda(d) = \partial(\sum_k \rho_k)$ with $\rho_k \in S_2^{\mathcal{U}}(B\tilde{G})$ by the construction of $\mathcal{V}$.

**Theorem 5.10.** Let $G$ be modular, $\mathcal{U}$ be an open covering of $M$. Then the inclusion induces an isomorphism $i_* : H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z}) \cong H_1(B\tilde{G};\mathbb{Z})$.

**Proof.** It suffices to show this for simplices which have support in a fixed compact set $K$. Choose a covering $\mathcal{V}$ with the property

$$V_1, V_2 \in \mathcal{V}, V_1 \cap V_2 \neq \emptyset \Rightarrow V_1 \cup V_2 \subseteq U \in \mathcal{U}$$

and choose $V_1, \ldots, V_{N-1} \in \mathcal{V}$ that cover $K$. Let $\lambda \in C^\infty(M, \Delta^{N-1})$ be subordinated to $\{M \setminus K, V_1, \ldots, V_{N-1}\}$ and let $r_* : H_1(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z})$ be the inverse of the inclusion $j_*$ from Lemma 5.1. Then

$$(\varphi_1^\lambda)_* \circ r_* : H_1(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z})$$

is an inverse of $i_* : H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z}) \to H_1(B\tilde{G};\mathbb{Z})$. Indeed we have

$$j_* = i_* \circ (\varphi_1^\lambda)_* : H_1^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z}) \to H_1(B\tilde{G};\mathbb{Z})$$

by Lemma 5.6 with $\mathcal{U} = \{M\}$ and so $i_* \circ (\varphi_1^\lambda)_* \circ r_* = j_* \circ r_* = \text{id}$. On the other hand the two mappings

$$(\varphi_1^\lambda)_* \circ r_* \circ i_* : H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z}) \to H_1(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{E},\tilde{\lambda}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z})$$

and

$$(\varphi_1^\lambda)_* \circ r_* : H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U},\tilde{\lambda}}(B\tilde{G};\mathbb{Z}) \to H_1^{\mathcal{U}}(B\tilde{G};\mathbb{Z})$$

coincide and again by Lemma 5.6 $(\varphi_1^\lambda)_* \circ r_* \circ i_* = (\varphi_1^\lambda)_* \circ r_* = j_* \circ r_* = \text{id}$.

\[ \square \]

**Remark 5.11.** With some combinatorical difficulties one can even show that the inclusion induces isomorphisms $i_* : H_p^{\mathcal{U},n}(B\tilde{G};\mathbb{Z}) \cong H_p(B\tilde{G};\mathbb{Z})$ for all $p \leq n$, where $\mathcal{U}^{(n)} := \{U_1 \cup \cdots \cup U_n : U_i \in \mathcal{U}\}$.

### 6. Proof of Theorem 1.1

We begin with some consequences of Theorem 3.2.
Proposition 6.1. Let $h_t$ be an isotopy in $\text{Diff}(T^n, \mathcal{F})_0$. Then $h_t$ can be written as $h_t = h_t^s \cdots h_t^1$ where

$$h_t^i = R_{\beta_t^i}^{-1} R_{\alpha_t^i}^{-1} (g_t^i)^{-1} R_{\alpha_t^i} g_t^i \quad \forall t$$

for some $\alpha \in T^k$ and some isotopies $g_t^i \in \text{Diff}(T^n, \mathcal{F})_0$ and $\beta_t^i \in T^k$.

Proof. Observe first that $h_t$ can be written as the product of

$$h_{(p/m)t} h_{(p-1/m)t}^{-1}, \quad p = 1, \ldots, m,$$

for $m$ sufficiently large.

Then we may assume that $h_t \in \mathcal{V}$, where $\mathcal{V}$ is a neighborhood of $\text{id}$ such that $R_\alpha \mathcal{V} \subset \mathcal{U}$, where $\mathcal{U}, \alpha$ are as in Theorem 3.2, and $\alpha$ is so small that $R_\alpha$ is in a contractible neighborhood of $\text{id}$. Thanks to Theorem 3.2 we have $R_\alpha h_t = R_{\beta_t^i} g_t^{-1} R_{\alpha_t^i} g_t$ as required. \qed

Proposition 6.2. $\{R_{\beta_t}\} = 0$ in $H_1(\text{Diff}(T^n, \mathcal{F})_0)$.

Indeed, it follows from Theorem 4 [10] and (3.1).

As a corollary we have

Theorem 6.3. $H_1(\text{Diff}(T^n, \mathcal{F})_0) = 0$, that is $\text{Diff}(T^n, \mathcal{F})_0$ and, consequently, $\text{Diff}(T^n, \mathcal{F})_0$ are perfect.

For the general case we need a more refined version of Theorem 6.3 which can be formulated as follows.

Proposition 6.4. Let $U_2, V_2$ be open balls in $T^{n-k}$ such that $\bar{U}_2 \subset V_2$ and let $V = T^k \times V_2$, $U = T^k \times U_2$. Suppose $\mathcal{F}|T^n - U$ is trivial, i.e. $L_y = T^k \times \{pt\}$ if $y \notin U$. If $h_t$ is an isotopy in $\text{Diff}_V(T^n, \mathcal{F})_0$ then $\{h_t\} = 0$ in $H_1(\text{Diff}_V(T^n, \mathcal{F})_0)$. In other words, the map

$$\iota_* : H_1(\text{Diff}_U(T^n, \mathcal{F})_0) \rightarrow H_1(\text{Diff}_V(T^n, \mathcal{F})_0)$$

is trivial, where $\iota : \text{Diff}_U(T^n, \mathcal{F})_0 \rightarrow \text{Diff}_V(T^n, \mathcal{F})_0$ is the canonical inclusion.

Proof. Let $h_t \in \text{Diff}_U(T^n, \mathcal{F})_0$. In view of Theorem 6.3 we have

$$h_t \sim [g_t^1 \cdot k_t^1] \cdots [g_t^r \cdot k_t^r]$$

where $\sim$ denotes the homotopy rel. endpoints, and $g_t^i, k_t^i \in \text{Diff}(T^n, \mathcal{F})_0$. We choose a smooth bump function $\mu : T^{n-k} \rightarrow [0, 1]$ such that $\text{supp} \mu \subset V_2$ and $\mu = 1$ on $U_2$. Then we let

$$\tilde{g}_t^i(x, y) = g_{\mu(y)}^i(x, y),$$

$$\tilde{k}_t^i(x, y) = k_{\mu(y)}^i(x, y),$$

where $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ is the standard chart for $(T^n, \mathcal{F}_k)$. Observe that since $g_t^i, k_t^i$ are leaf preserving diffeomorphisms then so are $\tilde{g}_t^i,$
that we have 

\[ h_t \sim [\bar{g}_t^1, \bar{k}_t^1] \ldots [\bar{g}_t^n, \bar{k}_t^n]. \]

This is a consequence of the fact that the initial homotopy is leafwise so that it can be modified in an obvious manner. \( \square \)

Now for technical reasons we introduce the following sequence of open sets in \( T^n \):

\[ U = U_1 \times U_2 \]
\[ U' = U_1 \times W_2 \]
\[ V = T^k \times V_2 \]
\[ W = T^k \times W_2 \]
\[ W' = T^k \times W'_2, \]

where \( U_1 \) is an open ball in \( T^k \), and \( U_2, V_2, W_2, W'_2 \) are open balls in \( T^{n-k} \) satisfying \( \bar{U}_2 \subset V_2 \subset \bar{V}_2 \subset W_2 \subset \bar{W}_2 \subset W'_2 \). We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Diff}_U(T^n, \mathcal{F})_0 & \xrightarrow{\iota_1} & \text{Diff}_{U'}(T^n, \mathcal{F})_0 \\
\iota_3 \downarrow & & \iota_2 \downarrow \\
\text{Diff}_V(T^n, \mathcal{F})_0 & \xrightarrow{\iota_4} & \text{Diff}_W(T^n, \mathcal{F})_0,
\end{array}
\]

where \( \iota_1 : \text{Diff}_U(T^n, \mathcal{F})_0 \to \text{Diff}_{U'}(T^n, \mathcal{F})_0 \) is the canonical inclusion and so on. The commutativity follows by the definition of the universal covering and by the fact that \( \iota_j \) are inclusions. This diagram descends to

\[
\begin{array}{ccc}
H_1(\text{Diff}_U(T^n, \mathcal{F})_0) & \xrightarrow{\iota_{1*}} & H_1(\text{Diff}_{U'}(T^n, \mathcal{F})_0) \\
\iota_{3*} \downarrow & & \iota_{2*} \downarrow \\
H_1(\text{Diff}_V(T^n, \mathcal{F})_0) & \xrightarrow{\iota_{4*}} & H_1(\text{Diff}_W(T^n, \mathcal{F})_0).
\end{array}
\]

Now let \( \mathcal{F}' \) be a foliation on \( M \) and let \( h_t \) be an isotopy in \( \text{Diff}^\infty(M, \mathcal{F}')_0 \). By Proposition 2.4 one can assume that \( h_t \in \text{Diff}_U(T^n, \mathcal{F})_0 \) where \( \mathcal{F} \) is some foliation on \( T^n \) such that \( \mathcal{F}|U = \mathcal{F}'|U \). In addition, we assume that \( (U', \phi) \) is a chart at \( x \) chosen as in Theorem 2.2 and such that \( U, U' \) are as above. This means that \( \phi(x) = 0, \dim(L_x) = k, \phi(U') = U_1 \times W_2 \), and \( \phi(\mathcal{F}'|U') = U_1 \times \mathcal{F}'_2 \) where \( \mathcal{F}'_2 \) is a foliation on \( W_2 \) with \( L_x = \{pt\} \).

The foliation \( \mathcal{F} \) on \( T^n \) can be defined as follows. \( \mathcal{F}'_2 \) being a smooth foliation on \( W_2 \), one has that its tangent distribution \( T\mathcal{F}'_2 \) is determined by a family of smooth vector fields, say \( S \) (cf.[13, p.545]). Then every vector field of \( S \) respects \( T\mathcal{F}'_2 \), and \( S \) is integrable [13, Cor.1]. Choose a bump function \( \mu : T^{n-k} \to [0,1] \) with \( \mu = 1 \) on \( U_2 \) and \( \text{supp} \mu \subset V_2 \), where \( V_2 \) is as above. The family of vector fields \( \mu S \) is still smooth and by the above reasoning it determines a foliation \( \mathcal{F}_2 \) on \( T^{n-k} \). We let \( \mathcal{F} = T^k \times \mathcal{F}_2 \). Note that \( \mathcal{F} = \mathcal{F}_k \) in a neighborhood of \( T^n - V \), and \( \mathcal{F} = \mathcal{F}' \) on \( U \).

Observe that Proposition 6.4 applies to \( \mathcal{F} \) and we have that \( \{h_t\} = 0 \) in \( H_1(\text{Diff}_V(T^n, \mathcal{F})_0) \). Our purpose is to show that \( \{h_t\} = 0 \) in the group...
$H_1(Diff_U^r(T^n,F)_0)$. This will be a consequence of the above diagram and Proposition 6.8 below.

First we need the following two lemmata.

**Lemma 6.5.** With the above notation, there exist a finite family of open balls $\{W_i\}_{i=1}^s$ such that $W = \bigcup W_i$ and a related family of isotopies $\{\phi_t^i\}_{i=1}^s$ in $Diff_{W_i}^r(T^n,F)_0$ such that $\phi_1^i(W_i) \subset U'$ and

$$\phi_1^i|W^i \cap W_j = \phi_1^i \circ \phi_1^j|W^i \cap W^j,$$

where $\phi_1^i$ is an isotopy in $Diff_{W_i}^r(T^n,F)_0$, for each $(i, j)$ such that $W^i \cap W^j \neq \emptyset$. Moreover, we may have that $W^i \cap W^j$, whenever nonempty, is a ball.

**Proof.** There exists a covering $\{U_i\}_{i=1}^s$ of $T^k$ by open balls such that $U_1^i \cap U_2^i$ is a ball whenever nonempty. Further, there exist isotopies $\psi_t^i$ in $Diff_{W_i}^\infty(T^k)_0$, and $\psi_t^ij$ in $Diff_{W_i}^r(T^k)$ such that

$$\psi_1^i|U_1^i \cap U_1^j = \psi_1^j|U_1^i \cap U_1^j$$

(see, e.g., 1]). Then we let $W_i = U_1^i \times W_2$ and $\phi_t^i = \psi_t^i(y) = \psi_t^i(y) \times id$, where $\mu : T^n \to [0, 1]$ is a bump function such that $\text{supp} \mu \subset W_2'$ and $\mu = 1$ on $W_2$. \[\square\]

Let us recall that $c \in B_n \bar{G}$ has its support in $U$ if and only if $\forall x, y \in \Delta^n$ the diffeomorphism $c(x)c(y)^{-1}$ is supported in $U$.

The fragmentation property (Theorem 5.10) can be specified to our situation as follows.

**Lemma 6.6.** Let $U, W, \{W_i = U_1^i \times W_2\}_{i=1}^r$ be as above. If a 1-chain $\sigma \in B\text{Diff}_U(T^n,F)_0$ is a boundary of a 2-chain $c = \sum c_\alpha \in B\text{Diff}_V(T^n,F)_0$ then $\sigma$ is a boundary of a 2-chain $C = \sum C_\alpha$ such that the supports of $C_\alpha$ are subordinate to $\{W_i\}$.

For any $g \in G$, $G$ being a topological group, we denote by $I_g$ the inner automorphism of $G$ induced by $g$. Then we have (cf. [2] or [1])

**Lemma 6.7.** If $\sigma$, $\tau$ are any 1-simplices in $G$, then $\sigma$, $I_{\tau(1)}\sigma$ are homological.

Now we are in a position to prove

**Proposition 6.8.** If $\sigma \in B_1\text{Diff}_U(T^n,F)_0$ satisfies $\iota_2 \iota_1^* \{\sigma\} = 0$ then $\iota_1^* \{\sigma\} = 0$.

**Proof.** Let $\sigma \in B_1\text{Diff}_U(T^n,F)_0$. Due to Proposition 6.4 $\sigma = \partial c$, where $c = \sum c_\alpha \in B_2\text{Diff}_V(T^n,F)_0$. Next, in light of Lemma 6.6 one has that support of each $c_\alpha$ is contained in some $W_i$.

Under the notation of Lemma 6.5 we assume the convention:

(i) $\text{supp}(\partial_j c_\alpha) \subset W_i^{i(j, \alpha)}$ and by $\phi_t^{i(j, \alpha)}$ we denote the corresponding isotopy;

(ii) we assume that $W_i^{i(j, \alpha)} = U'$ and $\phi_t^{i(i, \alpha)} = id$, if $\text{supp}(\partial_j c_\alpha) \subset U'$;

(iii) if $\partial_j c_\alpha = \pm \partial_i c_\beta$ then $W_i^{i(j, \alpha)} = W_i^{i(i, \beta)}$ and $\phi_t^{i(j, \alpha)} = \phi_t^{i(i, \beta)} ;$
We have the following equality:

$$\sigma = \sum_{\alpha} \sum_{j=0}^{2} (-1)^j \partial_j c_\alpha = \sum_{\alpha} \sum_{j=0}^{2} (-1)^j I_{\phi_1^{i(\alpha)}} (\partial_j c_\alpha).$$

If fact, if support of the edge $\partial_j c_\alpha$ is in $U'$ then, because of (ii), nothing changes in the r.h.s. Otherwise, this edge must be reduced in the sum on the l.h.s., and by (iii) so must be on the r.h.s.

Due to Lemmata 6.5 and 6.7 we get

$$I_{\phi_1^{i(\alpha)}} (\partial_j c_\alpha) = I_{\phi_1^{i(\alpha)}} (\partial_j c_\alpha),$$

where $\sim$ stands for the homology relation. Combining this with (6.1) we have

$$\sigma \sim \sum_{\alpha} \sum_{j=0}^{2} (-1)^j \partial_j I_{\phi_1^{i(\alpha)}} (c_\alpha) = \partial \sum_{\alpha} I_{\phi_1^{i(\alpha)}} (c_\alpha).$$

In view of (iv), $\text{supp}(\sum_{\alpha} I_{\phi_1^{i(\alpha)}} (c_\alpha)) \subset U'$. Thus $\sigma$ is a coboundary in $B\text{Diff}_{U'}(T^n, F)_0$, i.e. $\iota_1 \ast \{\sigma\} = 0$. $\square$

By combining Propositions 6.4 and 6.8 it is visible that Theorem 1.1 holds.

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(T.R.)
Institute of Mathematics
Pedagogical University
ul.Rejtana 16 A
35-310 Rzeszów, POLAND
e-mail: rybicki@im.uj.edu.pl

(St.H.)
Institute of Mathematics
University of Vienna
Strudlhofgasse 4
1090 Wien, Austria
e-mail: stefan@nelly.mat.univie.ac.at