State succinctness of two-way finite automata with quantum and classical states

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Abstract

Two-way quantum automata with quantum and classical states (2QCFA) were introduced by Ambainis and Watrous in 2002. In this paper we study state succinctness of 2QCFA. For any $m \in \mathbb{Z}^+$ and any $\epsilon < 1/2$, we show that:

1. there is a promise problem $A_{\text{eq}}(m)$ which can be solved by a 2QCFA with one-sided error $\epsilon$ in a polynomial expected running time with a constant number (that depends neither on $m$ nor on $\epsilon$) of quantum states and $O(\log \frac{1}{\epsilon})$ classical states, whereas the sizes of the corresponding deterministic finite automata (DFA), two-way nondeterministic finite automata (2NFA) and polynomial expected running time two-way probabilistic finite automata (2PFA) are at least $2m + 2$, $\sqrt{\log m}$, and $\sqrt[3]{(\log m)/\epsilon}$, respectively;

2. there exists a language $L_{\text{twin}}(m) = \{wcw|w \in \{a, b\}^*\}$ over the alphabet $\Sigma = \{a, b, c\}$ which can be recognized by a 2QCFA with one-sided error $\epsilon$ in an exponential expected running time with a constant number of quantum states and $O(\log \frac{1}{\epsilon})$ classical states,

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whereas the sizes of the corresponding DFA, 2NFA and polynomial expected running time 2PFA are at least $2^m$, $\sqrt{m}$, and $\sqrt[3]{m/b}$, respectively;

where $b$ is a constant.

Keywords: Computing models; Quantum finite automata; State complexity; Succinctness.

1 Introduction

An important way to get a deeper insight into the power of various quantum resources and features for information processing is to explore the power of various quantum variations of the basic models of classical automata. Of a special interest and importance is to do that for various quantum variations of classical finite automata because quantum resources are not cheap and quantum operations are not easy to implement. Attempts to find out how much one can do with very little of quantum resources and consequently with the simplest quantum variations of classical finite automata are therefore of a particular interest. This paper is an attempt to contribute to such line of research.

There are two basic approaches how to introduce quantum features to classical models of finite automata. The first one is to consider quantum variants of the classical one-way (deterministic) finite automata (1FA or 1DFA) and the second one is to consider quantum variants of the classical two-way finite automata (2FA or 2DFA). Already the very first attempts to introduce such models, by Moore and Crutchfields [23] and Kondacs and Watrous [16] demonstrated that in spite of the fact that in the classical case, 1FA and 2FA have the same recognition power, this is not so for their quantum variations (in case only unitary operations and projective measurements are considered as quantum operations). Moreover, already the first important model of two-way quantum finite automata (2QFA), namely that introduced by Kondacs and Watrous, demonstrated that very natural quantum variants of 2FA are much too powerful - they can recognize even some non-context free languages and are actually not really finite in a strong sense [16]. It started to be therefore of interest to introduce and explore some “less quantum” variations of 2FA and their power [1, 2, 3, 7, 20, 21, 22, 24, 26, 30, 31, 37, 38, 39].

A very natural “hybrid” quantum variations of 2FA, namely, two-way quantum automata with quantum and classical states (2QCFA) were introduced by Ambainis and Watrous [3]. Using this model they were able to show, in an elegant way, that an addition of a single qubit to a classical model can enormously increase the power of automata. A 2QCFA is essentially a classical 2FA augmented with a quantum memory of constant size (for states in a fixed Hilbert space) that does not depend on the size of the (classical) input. In spite of such a restriction, 2QCFA have been shown to be more powerful than two-way probabilistic finite automata (2PFA) [3].
State complexity and succinctness results are an important research area of classical automata theory, see [40], with a variety of applications. Once quantum versions of classical automata were introduced and explored, it started to be of large interest to find out, also through succinctness results, a relation between the power of classical and quantum automata models. This has turned out to be an area of surprising outcomes that again indicated that the relations between classical and corresponding quantum automata models are intriguing. For example, it has been shown, see [2, 4, 5, 6, 19], that for some languages 1QFA require exponentially less states than classical 1FA, but for some other languages it can be in an opposite way.

Because of the simplicity, elegance and interesting properties of the 2QCFA model, as well as its natural character, it seems to be both useful and interesting to explore state complexity and succinctness results of 2QCFA and this we will do in this paper.

In the first part of this paper, 2QCFA are recalled formally and some basic notations are given. Then we will prove state succinctness result of 2QCFA on an infinite family of promise problems. For any $m \in \mathbb{Z}^+$ let $A_{\text{eq yes}}(m) = \{w \in \{a, b\}^* | w = a^m b^m \}$ and $A_{\text{eq no}}(m) = \{w \in \{a, b\}^* | w \neq a^m b^m \text{ and } |w| \geq m \}$. For any $\epsilon < 1/2$ ($\epsilon$ is always a nonnegative number in this paper), we will prove that the promise problem $A^{\text{eq}}(m) = (A_{\text{eq yes}}(m), A_{\text{eq no}}(m))$ can be solved by a 2QCFA with one-sided error $\epsilon$ in a polynomial expected running time with a constant number of quantum states and $O(\log \frac{1}{\epsilon})$ (the base of logarithm is always 2 in this paper) classical states, whereas sizes of the corresponding DFA, 2DFA and 2NFA are at least $2m+2$, $\sqrt{\log m}$ and $\sqrt{\log m}$, respectively. We also show that for any $m \in \mathbb{Z}^+$, any 2PFA solves the promise problem $A^{\text{eq}}(m)$ with an error probability $\epsilon < 1/2$ and within polynomial expected running time has least $\sqrt{(\log m)/b}$ states, where $b > 0$ is a constant. Finally, we show a state succinctness result of 2QCFA on an infinite family of languages. For any $m \in \mathbb{Z}^+$ and any $\epsilon < 1/2$, there exists a 2QCFA that recognizes language $L^{\text{twin}}(m) = \{wcw | w \in \{a, b\}^* \}$ over the alphabet $\Sigma = \{a, b, c\}$ with one-sided error $\epsilon$ in an exponential expected running time with a constant number of quantum states and $O(\log \frac{1}{\epsilon})$ classical states. We use lower bound of communication complexity to prove that any DFA recognizing language $L^{\text{twin}}(m)$ has at least $2^m$ states. Next, we prove that the sizes of the corresponding 2DFA and 2NFA to recognize $L^{\text{twin}}(m)$ are at least $\sqrt{m}$. We also show that for any $m \in \mathbb{Z}^+$, any 2PFA recognizing $L^{\text{twin}}(m)$ with an error probability $\epsilon < 1/2$ and within polynomial expected running time has least $\sqrt{m}/b$ states, where $b > 0$ is a constant.

We now outline the remainder of this paper. Definition of 2QCFA and some auxiliary lemmas are recalled in Section 2. In Section 3 we prove a state succinctness result of 2QCFA on an infinite family of promise problems. Then we show a state succinctness result of 2QCFA on an infinite family of languages in Section 4. Finally, Section 5 contains a conclusion and some open problems.
2 Preliminaries

In the first part of this section we formally recall the model of 2QCFA we will use. Concerning the basics of quantum computation we refer the reader to [13, 25], and concerning the basic properties of automata models, we refer the reader to [13, 14, 15, 27, 29, 32].

2.1 2QCFA

2QCFA were first introduced by Ambainis and Watrous [3], and then studied by Qiu, Yakaryilmaz and etc. [28, 37, 41, 42]. Informally, we describe a 2QCFA as a 2DFA which has an access to a quantum memory of a constant size (dimension), upon which it performs quantum unitary transformations or projective measurement. Given a finite set of quantum states $Q$, we denote by $\mathcal{H}(Q)$ the Hilbert space spanned by $Q$. Let $\mathcal{U}(\mathcal{H}(Q))$ and $\mathcal{O}(\mathcal{H}(Q))$ denote the sets of unitary operators and projective measurements over $\mathcal{H}(Q)$, respectively.

Definition 1. A 2QCFA $A$ is specified by a 9-tuple

$$A = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$$

where:

1. $Q$ is a finite set of quantum states;
2. $S$ is a finite set of classical states;
3. $\Sigma$ is a finite set of input symbols; $\Sigma$ is then extended to the tape symbols set $\Gamma = \Sigma \cup \{ \phi, \$\}$, where $\phi \notin \Sigma$ is called the left end-marker and $\$ \notin \Sigma$ is called the right end-marker;
4. $q_0 \in Q$ is the initial quantum state;
5. $s_0 \in S$ is the initial classical state;
6. $S_{acc} \subset S$ and $S_{rej} \subset S$ satisfying $S_{acc} \cap S_{rej} = \emptyset$ are the sets of classical accepting and rejecting states, respectively.
7. $\Theta$ is the transition function of quantum states:

$$\Theta : S \setminus (S_{acc} \cup S_{rej}) \times \Gamma \to \mathcal{U}(\mathcal{H}(Q)) \cup \mathcal{O}(\mathcal{H}(Q)).$$

Thus, $\Theta(s, \gamma)$ is either a unitary transformation or a projective measurement.
8. $\delta$ is the transition function of classical states.
a) If Θ(s,γ) ∈ U(ℋ(Q)), then

\[ \delta : S \setminus (S_{\text{acc}} \cup S_{\text{rej}}) \times \Gamma \rightarrow S \times \{-1, 0, 1\}, \] (3)

which is similar to the transition function for 2DFA, \( \delta(s,\gamma) = (s',d) \) means that when the classical state \( s \in S \) scanning \( \gamma \in \Gamma \) is changed to state \( s' \), and the movement of the tape head is determined by \( d \) (moving right one cell if \( d = 1 \), left if \( d = -1 \), and being stationary if \( d = 0 \)).

b) If Θ(s,γ) ∈ O(ℋ(Q)), then we assume that Θ(s,γ) is a projective measurement with a set of possible eigenvalues \( R = \{r_1, \ldots, r_n\} \) and the projectors set \( \{P(r_i) : i = 1, \ldots, n\} \), where \( P(r_i) \) denotes the projector onto the eigenspace corresponding to \( r_i \). In such a case

\[ \delta : S \setminus (S_{\text{acc}} \cup S_{\text{rej}}) \times \Gamma \times R \rightarrow S \times \{-1, 0, 1\}, \] (4)

where \( \delta(s,\gamma)(r_i) = (s',d) \) means that when the projective measurement result is \( r_i \), the classical state \( s \in S \) is changed to \( s' \), and the movement of the tape head is determined by \( d \).

Given an input \( w \), a 2QCFA \( A = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{\text{acc}}, S_{\text{rej}}) \) proceeds as follows: at the beginning, the tape head is positioned on the left end-marker \( \_ \), the quantum initial state is \( |q_0\rangle \), the classical initial state is \( s_0 \). In the next steps if the current quantum state is \( |\psi\rangle \), the current classical state is \( s \in S \setminus (S_{\text{acc}} \cup S_{\text{rej}}) \) and the current scanning symbol is \( \sigma \in \Gamma \), then the quantum state \( |\psi\rangle \) and the classical state \( s \) will be changed according to \( \Theta(s,\sigma) \) as follows:

1. if \( \Theta(s,\sigma) \) is a unitary operator \( U \), then \( U \) is applied to the current quantum state \( |\psi\rangle \) changing it into \( U|\psi\rangle \), and \( \delta(s,\sigma) = (s',d) \in S \times \{-1, 0, 1\} \) makes the current classical state \( s \) to become \( s' \), together with the tape head moving in terms of \( d \). In case \( s' \in S_{\text{acc}} \), the input is accepted, and in case \( s' \in Q_{\text{rej}} \), the input rejected;

2. if \( \Theta(s,\sigma) \) is a projective measurement, then the current quantum state \( |\psi\rangle \) is changed to the quantum state \( P_j|\psi\rangle/\|P_j|\psi\rangle\| \) with probability \( \|P_j|\psi\rangle\|^2 \) in terms of the measurement, and in this case, \( \delta(s,\sigma) \) is a mapping from the set of all possible results of the measurement to \( S \times \{-1, 0, 1\} \). For instance, for the result \( r_j \) of the measurement, and \( \delta(s,\sigma)(r_j) = (s_j, d) \), we have

(a) if \( s_j \in S \setminus (S_{\text{acc}} \cup S_{\text{rej}}) \), new classical state is \( s_j \) and the head moves in the direction \( d \);  
(b) if \( s_j \in S_{\text{acc}} \), the machine accepts the input and the computation halts;  
(c) and similarly, if \( s_j \in S_{\text{rej}} \), the machine rejects the input and the computation halts.
It is seen that if the current all possible classical states are in \( S_{\text{acc}} \cup S_{\text{rej}} \), then the computation for the current input string ends.

The computation will end if classical state is in \( S_{\text{acc}} \cup S_{\text{rej}} \). Therefore, similar to the definition of accepting and rejecting probabilities for 2QFA [16], the accepting and rejecting probabilities \( Pr[A \text{ accepts } w] \) and \( Pr[A \text{ rejects } w] \) in \( A \) for input \( w \) are respectively the sums of all accepting probabilities and all rejecting probabilities before the end of the machine for computing input \( w \).

Let \( L \subseteq \Sigma^* \) and \( \epsilon < 1/2 \). A 2QCFA \( A \) recognizes \( L \) with one-sided error \( \epsilon \) if

1. \( \forall w \in L, \ Pr[A \text{ accepts } w] = 1 \), and
2. \( \forall w \notin L, \ Pr[A \text{ rejects } w] \geq 1 - \epsilon \).

### 2.2 Notations and auxiliary lemmas

In this subsection we review some additional notations related to 2QCFA [28]. For convenience, let \( 2QCF_A_\epsilon \) denote the classes of all languages recognized by 2QCFA with a given error probability \( \epsilon \) and \( 2QCF_A_\epsilon(\text{ptime}) \) denote the classes of languages recognized in polynomial expected time by 2QCFA with a given error probability \( \epsilon \). Moreover, let \( QS(A) \) and \( CS(A) \) denote the numbers of quantum states and classical states of a 2QCFA \( A \) and let \( T(A) \) denote the expected running time of 2QCFA \( A \). For a string \( w \), the length of \( w \) is denoted by \( |w| \).

**Lemma 1** ([3]). For any \( \epsilon < 1/2 \), there is a 2QCFA \( A(\epsilon) \) that accepts any \( w \in L^{eq} = \{a^m b^m | m \in \mathbb{N} \} \) with certainty, rejects any \( w \notin L^{eq} \) with probability at least \( 1 - \epsilon \) and halts in expected running time \( O(|w|^4) \), where \( w \) is the input.

**Remark 1.** According to the proof of Lemma 1 in [3], for the above 2QCFA \( A(\epsilon) \) we further have \( QS(A(\epsilon)) = 2 \), \( CS(A(\epsilon)) \in \mathcal{O}(\log \frac{1}{\epsilon}) \).

**Lemma 2** ([28]). If \( L_1 \in 2QCF_{A_{\epsilon_1}}(2QCF_{A_{\epsilon_1}}(\text{ptime})) \) and \( L_2 \in 2QCF_{A_{\epsilon_2}}(2QCF_{A_{\epsilon_2}}(\text{ptime})) \), then \( L_1 \cap L_2 \in 2QCF_{A_\epsilon}(2QCF_{A_\epsilon}(\text{ptime})) \), where \( \epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2 \).

**Remark 2.** According to the proof of Lemma 2 in [28], if 2QCFA \( A_1 \) recognizes \( L_1 \) with one-sided error \( \epsilon_1 \) (in polynomial expected time) and 2QCFA \( A_2 \) recognizes \( L_2 \) with one-sided error \( \epsilon_2 \) (in polynomial expected time), then there is a 2QCFA \( A \) recognizes \( L_1 \cap L_2 \) (in polynomial expected time), where \( QS(A) = QS(A_1) + QS(A_2) \) and \( CS(A) = CS(A_1) + CS(A_2) + QS(A_1) \).

**Lemma 3** ([33, 34]). Every \( n \)-state 2DFA can be simulated by a DFA with \( (n + 1)^{n+1} \) states.

**Lemma 4** ([8]). Every \( n \)-state 2NFA can be simulated by a DFA with \( 2^{(n-1)^2+n} \) states.
Definition 2. Let language $L \subset \Sigma^*$ and $\epsilon < 1/2$, then a 2PFA $A$ recognizes $L$ with error probability $\epsilon$ if

1. $\forall w \in L$, $Pr[A accepts w] \geq 1 - \epsilon$, and
2. $\forall w \notin L$, $Pr[A rejects w] \geq 1 - \epsilon$.

A 2PFA $A$ recognizes $L$ if there is an $\epsilon < 1/2$ such that $A$ recognizes $L$ with error probability $\epsilon$.

Definition 3. Let $A, B \in \Sigma^*$ with $A \cap B = \emptyset$, then a 2PFA $A$ separates $A$ and $B$ if there is some $\epsilon < 1/2$ such that

1. $\forall w \in A$, $Pr[A accepts w] \geq 1 - \epsilon$, and
2. $\forall w \in B$, $Pr[A rejects w] \geq 1 - \epsilon$.

Lemma 5 ([11]). For every $\epsilon < 1/2$, $a > 0$ and $d > 0$, there exists a constant $b > 0$ such that, for any $c$, if $L$ is recognized by a $c$-state 2PFA with an error probability $\epsilon$ and within time $an^d$, then $L$ is recognized by some DFA with at most $c^{b \epsilon^2}$ states, where $n = |w|$ is the length of the input.

Lemma 6 ([11]). Let $A, B \subseteq \Sigma^*$ with $A \cap B = \emptyset$. Suppose there is an infinite set $I$ of positive integers and, for each $m \in I$, a set $W_m \subseteq \Sigma^*$ such that

1. $|w| \leq m$ for all $w \in W_m$,
2. for every integer $k$, there is an $m_k$ such that $|W_m| \geq m^k$ for all $m \in I$ with $m \geq m_k$, and
3. for every $m \in I$ and every $w, w' \in W_m$ with $w \neq w'$, there are words $u, v \in \Sigma^*$ such that either $uwv \in A$ and $uw'v \in B$ or $uwv \in B$ and $uw'v \in A$.

Then no 2PFA separates $A$ and $B$.

We recall some basic notations of communication complexity, and we refer the reader to [17, 18, 36] for more details. It deals with the situation where there are only two communicating parties and it deals with very simple tasks of computing two argument functions where one argument is known to one party and the other argument is known to the other party. It completely ignores the computational resources needed by the parties and it focuses solely on the amount of communication exchanged between the parties.

Let $X, Y, Z$ be arbitrary finite sets. We consider a two-argument function $f : X \times Y \to Z$ and two communicating parties, Alice is given an input $x \in X$ and Bob is given an input $y \in Y$. They wish to compute $f(x, y)$.
The computation of the value \( f(x, y) \) is done using a communication protocol. During the execution of the protocol, the two parties alternate roles in sending messages. Each of these messages is a string of bits. The protocol, based on the communication so far, specifies whether the execution terminated (in which case it also specifies what is the output). If the execution has not terminated, the protocol specifies what message the sender (Alice or Bob) should send next, as a function of its input and of the communication so far. A communication protocol \( \mathcal{P} \) computes the function \( f \), if for every input pair \((x, y) \in A \times B\) the protocol terminates with the value \( f(x, y) \) as its output.

We define the deterministic communication complexity of \( \mathcal{P} \) as the worst case number of bits exchanged by the protocol. The deterministic communication complexity of a function \( f \) is the communication complexity of the best protocol that computes \( f \), denoted by \( D(f) \).

Lemma 7 ([17]). If Alice and Bob each holds an \( n \) length string, \( x, y \in \{a, b\}^n \) and the equality function, \( \text{EQ}(x, y) \), is defined to be 1 if \( x = y \) and 0 otherwise, then

\[
D(\text{EQ}) = n + 1. \tag{5}
\]

3 State succinctness of 2QCFA on promise problems

In this section, we will give an infinite family of promise problems which can be solved by 2QCFA with one-sided error \( \epsilon \) in a polynomial expected running time with a constant number of quantum states and \( O(\log \frac{1}{\epsilon}) \) classical states.

A promise problem is a pair \( A = (A_{\text{yes}}, A_{\text{no}}) \), where \( A_{\text{yes}}, A_{\text{no}} \subset \Sigma^* \) are disjoint sets of strings [35]. (Languages may be viewed as promise problems that obey the additional constraint \( A_{\text{yes}} \cup A_{\text{no}} = \Sigma^* \).) For an alphabet \( \Sigma = \{a, b\} \) and any \( m \in \mathbb{Z}^+ \), let \( A_{\text{yes}}^m(m) = \{a^nb^m\} \) and \( A_{\text{no}}^m(m) = \{w \in \{a, b\}^* | w \neq a^mb^m \text{ and } |w| \geq m\} \). For any \( \epsilon < 1/2 \), we will prove that promise problems \( A_{\text{eq}}(m) = (A_{\text{yes}}^\epsilon(m), A_{\text{no}}^\epsilon(m)) \) can be solved by a 2QCFA with one-sided error \( \epsilon \) in a polynomial expected running time with a constant number of quantum states and \( O(\log \frac{1}{\epsilon}) \) classical states, whereas the sizes of the corresponding DFA, 2DFA and 2PFA grow without a bound.

In order to prove that the promise problem \( A_{\text{eq}}(m) \) can be solved by 2QCFA, we first prove that a simpler promise problem can be solved by 2QCFA.

For an alphabet \( \Sigma \) and an \( m \in \mathbb{Z}^+ \), let \( A_{\text{yes}}(m) = \{w \in \Sigma^* | |w| = m\} \) and \( A_{\text{no}}(m) = \{w \in \Sigma^* | |w| \neq m \text{ and } |w| \geq m/2\} \). For any \( \epsilon < 1/2 \), we will prove that there is a

\[\text{A promise problem } A = (A_{\text{yes}}, A_{\text{no}}) \text{ is solved by a 2QCFA } \mathcal{A} \text{ with one-sided error } \epsilon < 1/2 \text{ if (1) } \forall w \in A_{\text{yes}}, \text{ Pr}[\mathcal{A} \text{ accepts } w] = 1, \text{ and (2) } \forall w \in A_{\text{no}}, \text{ Pr}[\mathcal{A} \text{ rejects } w] \geq 1 - \epsilon. \text{ A promise problem } A = (A_{\text{yes}}, A_{\text{no}}) \text{ is solved by a 2PFA } \mathcal{A} \text{ with error probability } \epsilon < 1/2 \text{ if (1) } \forall w \in A_{\text{yes}}, \text{ Pr}[\mathcal{A} \text{ accepts } w] \geq 1 - \epsilon, \text{ and (2) } \forall w \in A_{\text{no}}, \text{ Pr}[\mathcal{A} \text{ rejects } w] \geq 1 - \epsilon. \text{ A promise problem } A = (A_{\text{yes}}, A_{\text{no}}) \text{ is solved by a DFA (2DFA, 2NFA) } \mathcal{A} \text{ if (1) } \forall w \in A_{\text{yes}}, \text{ } \mathcal{A} \text{ accepts } w \text{ and (2) } \forall w \in A_{\text{no}}, \text{ } \mathcal{A} \text{ rejects } w. \]
2QCFAs that can solve promise problem $A(m) = (A_{yes}(m), A_{no}(m))$ with one-sided error $\epsilon$ in a polynomial expected running time with a constant number of quantum states and $O(\log \frac{1}{\epsilon})$ classical states. The language $L(m) = \{w \in \Sigma^* | |w| = m\}$ was showed to be recognized by a 7-state one way quantum finite automata with restart ($1QFA^\infty$) with one-sided error $\epsilon$ in an exponential expected time by Yakaryılmaz and Cem Say [37]. In the same paper, they mentioned that $1QFA^\infty$ can be simulated by 2QCFA easily. In following theorem we will prove in details that the promise problem $A(m)$ can be solved by a 2QCFA with one-sided error $\epsilon$ in a polynomial expected time.

**Theorem 8.** For any $m \in \mathbb{Z}^+$ and any $\epsilon < 1/2$, there exists a 2QCFA $A(m, \epsilon)$ which accepts any $w \in A_{yes}(m)$ with certainty, and rejects any $w \in A_{no}(m)$ with probability at least $1 - \epsilon$, where $QS(A(m, \epsilon))$ is a constant and $CS(A(m, \epsilon)) \in O(\log \frac{1}{\epsilon})$. Furthermore, we have $T(A(m, \epsilon)) \in O(|w|^4)$, where $w$ is the input.

**Proof.** The main idea is as follows: we consider a 2QCFA $A(m, \epsilon)$ with 2 quantum states $|q_0\rangle$ and $|q_1\rangle$. $A(m, \epsilon)$ starts with the quantum state $|q_0\rangle$. When $A(m, \epsilon)$ reads the left end-marker $\dagger$, the state is rotated by angle $\sqrt{2m\pi}$ and every time when $A(m, \epsilon)$ reads a symbol $\sigma \in \Sigma^*$, the state is rotated by angle $-\alpha = -\sqrt{2\pi}$ (notice that $\sqrt{2m\pi} = m\alpha$). When the right end-marker $\$ is reached, $A(m, \epsilon)$ measures the quantum state. If it is $|q_1\rangle$, the input string $w$ is rejected. Otherwise, the process is repeated.

We now complete the description of $A(m, \epsilon)$ as sketched in Figure 1. The states of the automaton will be over the orthogonal base $\{|q_0\rangle, |q_1\rangle\}$ and will use the following two unitary transformations

$$
\begin{align*}
U_{\dagger}|q_0\rangle &= \cos m\alpha |q_0\rangle + \sin m\alpha |q_1\rangle \\
U_{\dagger}|q_1\rangle &= -\sin m\alpha |q_0\rangle + \cos m\alpha |q_1\rangle
\end{align*}
$$

$$
\begin{align*}
U_{\alpha}|q_0\rangle &= \cos \alpha |q_0\rangle - \sin \alpha |q_1\rangle \\
U_{\alpha}|q_0\rangle &= \sin \alpha |q_0\rangle + \cos \alpha |q_1\rangle
\end{align*}
$$

**Lemma 9.** If the input $w \in A_{yes}(m)$, then the quantum state of $A(m, \epsilon)$ will evolve with certainty into $|q_0\rangle$ after the loop 2.

**Proof.** If $w \in A_{yes}(m)$, then $|w| = m$. The quantum state after the loop 2 can be described as follows:

$$
|q\rangle = (U_{\alpha})^m U_{\dagger}|q_0\rangle = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^m \begin{pmatrix} \cos m\alpha & -\sin m\alpha \\ \sin m\alpha & \cos m\alpha \end{pmatrix} |q_0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{pmatrix} |q_0\rangle = |q_0\rangle.
$$

**Lemma 10.** If $w \in A_{no}(m)$, $|w| = n$, then $A(m, \epsilon)$ rejects $w$ after the step 3 with a probability at least $1/(2(m - n)^2 + 1)$. 

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Repeat the following ad infinitum:
1. Set the quantum state to \( |q_0 \rangle \), read the left end-marker $\psi$, and perform \( U_\ell \) on \( |q_0 \rangle \).
2. Until the scanned symbol is the right end-marker $\$$, do the following:
   (2.1). Perform \( U_{-\alpha} \) on the current quantum state (\( U_\alpha \) is defined in the proof of Theorem 8).
   (2.2). Move the tape head one square to the right.
3. Measure the quantum state. If the result is not \( |q_0 \rangle \), reject.
4. Repeat the following subroutine two times:
   (4.1). Move the tape head to the first input symbol.
   (4.2). Move the tape head one square to the right.
   (4.3). While the currently scanned symbol is not $\psi$ or $\$$, do the following:
       Simulate a coin flip. If the result is “head”, move right. Otherwise, move left.
5. If both times the process ends at the right end-marker $\$$, do:
       Simulate \( k \) coin-flips and if all outcomes are “heads”, accept.

Figure 1: Description of the behaviour of 2QCFA \( A(m, \epsilon) \). The choice of \( k \) will depend on \( \epsilon \).

Proof. Starting with the state \( |q_0 \rangle \), \( A(m, \epsilon) \) changes its quantum state to \( |q \rangle = (U_{-\alpha})^n U_\ell |q_0 \rangle \) after the loop 2, the quantum state can be described as follows:

\[
|q \rangle = (U_{-\alpha})^n U_\ell |q_0 \rangle = \left( \begin{array}{cc}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{array} \right)^n \left( \begin{array}{cc}
\cos m\alpha & -\sin m\alpha \\
\sin m\alpha & \cos m\alpha
\end{array} \right) |q_0 \rangle
\]

\[
= \left( \begin{array}{cc}
\cos(n\alpha) & \sin(n\alpha) \\
-\sin(n\alpha) & \cos(n\alpha)
\end{array} \right) \left( \begin{array}{cc}
\cos m\alpha & -\sin m\alpha \\
\sin m\alpha & \cos m\alpha
\end{array} \right) |q_0 \rangle
\]

\[
= \left( \begin{array}{cc}
\cos((m + n)\alpha) & \sin((m - n)\alpha) \\
\sin((m - n)\alpha) & \cos((m + n)\alpha)
\end{array} \right) |q_0 \rangle
\]

The probability of observing \( |q_1 \rangle \) is \( \sin^2(\sqrt{2}(m - n)\pi) \) in the step 3. Without loss of generality, we assume that \( m - n > 0 \). Let \( l \) be the closest integer to \( \sqrt{2}(m - n) \). If \( \sqrt{2}(m - n) > l \), then \( 2(m - n)^2 > l^2 \). So we get \( 2(m - n)^2 - 1 \geq l^2 \) and \( l \leq \sqrt{2}(m - n)^2 - 1 \). We have

\[
\sqrt{2}(m - n) - l \geq \sqrt{2}(m - n) - \sqrt{2}(m - n)^2 - 1
\]

\[
= \frac{\sqrt{2}(m - n) - \sqrt{2}(m - n)^2 - 1}{\sqrt{2}(m - n) + \sqrt{2}(m - n)^2 - 1}
\]

\[
= \frac{1}{\sqrt{2}(m - n) + \sqrt{2}(m - n)^2 - 1} > \frac{1}{2\sqrt{2}(m - n)}.
\]

Because \( l \) is the closest integer to \( \sqrt{2}(m - n) \), we have \( 0 < \sqrt{2}(m - n) - l < 1/2 \). Let \( f(x) = \sin(x\pi) - 2x \). We have \( f''(x) = -\pi^2 \sin(x\pi) \leq 0 \) when \( x \in [0, 1/2] \). That is to say,
\( f(x) \) is concave in \([0, 1/2]\), and we have \( f(0) = f(1/2) = 0 \). So for any \( x \in [0, 1/2] \), it holds that \( f(x) \geq 0 \), that is, \( \sin(x\pi) \geq 2x \). Therefore, we have

\[
\sin^2(\sqrt{2}(m-n)\pi) = \sin^2((\sqrt{2}(m-n) - l)\pi)
\]

\[
\geq (2(\sqrt{2}(m-n) - l))^2 = 4(\sqrt{2}(m-n) - l)^2
\]

\[
> 4\left(\frac{1}{2\sqrt{2}(m-n)}\right)^2 = \frac{1}{2(m-n)^2} > \frac{1}{2(m-n)^2 + 1}.
\]

If \( \sqrt{2}(m-n) < l \), then \( 2(m-n)^2 < l^2 \). So we get \( 2(m-n)^2 + 1 \leq l^2 \) and \( l \geq \sqrt{2(m-n)^2 + 1} \). We have

\[
\sqrt{2}(m-n) - l \leq \sqrt{2}(m-n) - \sqrt{2(m-n)^2 + 1}
\]

\[
= \frac{(\sqrt{2}(m-n) - \sqrt{2(m-n)^2 + 1})(\sqrt{2}(m-n) + \sqrt{2(m-n)^2 + 1})}{\sqrt{2}(m-n) + \sqrt{2(m-n)^2 + 1}}
\]

\[
= \frac{1}{\sqrt{2}(m-n) + \sqrt{2(m-n)^2 + 1}} < \frac{-1}{2\sqrt{2(m-n)^2 + 1}}
\]

It follows that

\[
l - \sqrt{2}(m-n) > \frac{1}{2\sqrt{2(m-n)^2 + 1}}.
\]

Because \( l \) is the closest integer to \( \sqrt{2}(m-n) \), we have \( 0 < l - \sqrt{2}(m-n) < 1/2 \). Therefore, we have

\[
\sin^2(\sqrt{2}(m-n)\pi) = \sin^2((\sqrt{2}(m-n) - l)\pi)
\]

\[
= \sin^2((l - \sqrt{2}(m-n))\pi) \geq (2(l - \sqrt{2}(m-n)))^2
\]

\[
= 4(l - \sqrt{2}(m-n))^2 > 4\left(\frac{1}{2\sqrt{2(m-n)^2 + 1}}\right)^2 = \frac{1}{2(m-n)^2 + 1}.
\]

So the lemma has been proved.

Simulation of a coin flip in the steps 4 and 5 is a necessary component in the above algorithm. We will show that coin-flips can be simulated by a 2QCFA using two quantum states \(|q_0\rangle\) and \(|q_1\rangle\).

**Lemma 11.** A coin flip in the algorithm can be simulated by a 2QCFA \(A(m, \epsilon)\) using two quantum states \(|q_0\rangle\) and \(|q_1\rangle\).

**Proof.** Let us consider a projective measurement \(M = \{P_0, P_1\}\) defined by

\[
P_0 = |q_0\rangle\langle q_0|, P_1 = |q_1\rangle\langle q_1|,
\]

\(\square\)
whose classical outcomes will be denoted by 0 and 1, representing the “tail” and “head” of a coin flip, respectively. Hadamard unitary operator

\[ H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \] (25)

Hadamard operator changes basis states

\[ |q_0\rangle \rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}(|q_0\rangle + |q_1\rangle), \quad |q_1\rangle \rightarrow |\phi\rangle = \frac{1}{\sqrt{2}}(|q_0\rangle - |q_1\rangle). \] (26)

Suppose now that the machine starts with the state \( |q_0\rangle \), changes its quantum state by \( H \), and then measures the quantum state with \( M \). Then we will get the result 0 or 1 with probability \( \frac{1}{2} \). This is similar to a coin flip process.

**Lemma 12.** [3] If the length of the input string is \( n \), then every execution of the loops 4 and 5 leads to the acceptance with a probability \( 1/2^k(n+1)^2 \).

**Proof.** The loop 4 performs two times of random walk starting at location 1 and ending at location 0 (the left end-marker \( \$ \)) or at location \( n + 1 \) (the right end-marker \$). It is known from probability theory that the probability of reaching the location \( n + 1 \) is \( 1/(n+1) \) (see Chapter 14.2 in [12]). Repeating it twice and flipping \( k \) coins, we get the probability \( 1/2^k(n+1)^2 \).

If we take \( k = 1 + \lceil \log \frac{1}{\epsilon} \rceil \), then \( \epsilon \geq 1/2^k \). Assume also that \( |w| = n \). If \( w \in A_{yes}(m) \), the loop 2 always changes the quantum state \( |q_0\rangle \) to \( |q_0\rangle \), and \( A(m, \epsilon) \) never rejects after the measurement in the step 3. After the loops 4 and 5, the probability of \( A(m, \epsilon) \) accepting \( w \) is \( 1/2^k(n+1)^2 \). Repeating the loops 4 and 5 for \( cn^2 \) times, the accepting probability is

\[ Pr[A(m, \epsilon) \text{ accepts } w] = 1 - (1 - \frac{1}{2^k(n+1)^2})^{cn^2}, \] (27)

and this can be made arbitrarily close to 1 by selecting the constant \( c \) appropriately.

Otherwise, if \( |w| \in A_{no}(m) \), \( A(m, \epsilon) \) rejects the input after the steps 2 and 3 with probability

\[ P_r = \frac{1}{2(m-n)^2 + 1} \] (28)

according to Lemma 10. \( A(m, \epsilon) \) accepts the input after the loops 4 and 5 with probability

\[ P_a = 1/2^k(n+1)^2 \leq \epsilon/2(n+1)^2. \] (29)

If we repeat the whole algorithm indefinitely, the probability of \( A(m, \epsilon) \) rejecting input \( w \) is

\[ Pr[A(m, \epsilon) \text{ rejects } w] = \sum_{i \geq 0} (1 - P_a)^i (1 - P_r)^i P_r \] (30)
Let $f(x) = \frac{x}{e^{2x}+x} = 1 - \frac{x}{e^{2x}}$, then $f(x)$ is monotonous increasing in $(0, +\infty)$. By assumption, we have $n = \lvert w \rvert \geq m/2$. So we have $(n + 1^2)/(2(n - m)^2 + 1) > 1/2$. Therefore, we have

$$> \frac{1/2}{1/2 + \epsilon/2} = \frac{1}{1 + \epsilon} > 1 - \epsilon. \quad (34)$$

If we assume the input is $w$, then the step 1 takes $O(1)$ time, the loop 2 and the step 3 take $O(|w|)$ time, and the loops 4 and 5 take $O(|w|^2)$ time. The expected number of repeating the algorithm is $O(|w|^2)$.

Obviously, $QS(\mathcal{A}(m, \epsilon)) = 2$. We just need $O(k)$ classical states to simulate $k$ coin-flips and calculate the outcomes, therefore $CS(\mathcal{A}(m, \epsilon)) \in O(\log \frac{1}{\epsilon})$.

**Theorem 13.** For any $m \in \mathbb{Z}^+$ and any $\epsilon < 1/2$, there exists a 2QCFA $\mathcal{A}(m, \epsilon)$ which accepts any $w \in \mathcal{A}_{eq}^{\text{yes}}(m)$ with certainty, and rejects any $w \in \mathcal{A}_{eq}^{\text{no}}(m)$ with probability at least $1 - \epsilon$, where $QS(\mathcal{A}(m, \epsilon))$ is a constant and $CS(\mathcal{A}(m, \epsilon)) \in O(\log \frac{1}{\epsilon})$. Furthermore, we have $T(\mathcal{A}(m, \epsilon)) \in O(|w|^4)$ where $w$ is the input.

**Proof.** Let the alphabet $\Sigma = \{a, b\}$. Obviously, $\mathcal{A}_{eq}(m) = L_{eq} \cap \text{A}(2m)$. According to Lemma 1, for any $\epsilon_1 > 0$, there is a 2QCFA $\mathcal{A}_1(\epsilon_1)$ recognizes $L_{eq}$ with one-sided error $\epsilon_1$, and $QS(\mathcal{A}_1(\epsilon_1)) = 2$, $CS(\mathcal{A}_1(\epsilon_1)) \in O(\log \frac{1}{\epsilon_1})$ and $T(\mathcal{A}_1(\epsilon_1)) \in O(|w|^4)$. According to Theorem 8 for any $\epsilon_2 > 0$, there is a 2QCFA $\mathcal{A}_2(m, \epsilon_2)$ that solves the promise problem $\text{A}(2m)$ with one-sided error $\epsilon_2$, and $QS(\mathcal{A}_2(m, \epsilon_2)) = 2$, $CS(\mathcal{A}_2(m, \epsilon_2)) \in O(\log \frac{1}{\epsilon_2})$ and $T(\mathcal{A}_2(m, \epsilon_2)) \in O(|w|^4)$. For any $\epsilon < 1/2$, let $\epsilon_1 = \epsilon/2$ and $\epsilon_2 = \epsilon/2$.

According to Lemma 2, there is a 2QCFA $\mathcal{A}(m, \epsilon)$ solves the promise problem $L_{eq} \cap \text{A}(2m)$ with a one-sided error $\epsilon$, where $QS(\mathcal{A}(m, \epsilon)) = QS(\mathcal{A}_1(\epsilon_1)) + QS(\mathcal{A}_2(m, \epsilon_2)) = 4$, $CS(\mathcal{A}(m, \epsilon)) = CS(\mathcal{A}_1(\epsilon_1)) + CS(\mathcal{A}_2(m, \epsilon_2)) + QS(\mathcal{A}_1(\epsilon_1)) \in O(\log \frac{1}{\epsilon})$ and $T(\mathcal{A}(m, \epsilon)) = T(\mathcal{A}_1(\epsilon_1)) + T(\mathcal{A}_2(m, \epsilon_2)) \in O(|w|^4)$. Hence, the theorem has been proved.

**Remark 3.** Actually, $L_1$ and $L_2$ must be languages in Lemma 2. But in Theorem 13, we used a promise problem $\text{A}(2m)$. It is easy to show that Lemma 2 still holds for promise problem $\text{A}(2m)$ and language $L_{eq}$. We used Lemma 2 to prove Theorem 13 in this section. However, we can prove Theorem 13 directly.

Obviously, there exists a DFA depicted in Figure 2 that solves the promise problem $\mathcal{A}_{eq}(m)$ with $2m+2$ states.
Theorem 14. For any $m \in \mathbb{Z}^+$, any DFA solving the promise problem $A^{eq}(m)$ has at least $2m + 2$ states.

Proof. Let us consider the string set $W = \{a^0, a^1, \ldots, a^m, a^m b^1, a^m b^2, \ldots, a^m b^m\}$, where $a^0$ is the empty string. Obviously, for any two different strings $w_i, w_j \in W$, we have $|w_i| \neq |w_j|$, and if $|w_i| < |w_j|$, then $w_i$ is a prefix of $w_j$. For any string $x \in \Sigma^*$ and any $\sigma \in \Sigma$, let $\hat{\delta}(s, \sigma x) = \hat{\delta}(\hat{\delta}(s, \sigma), x)$; if $|x| = 0$, $\hat{\delta}(s, x) = s$. Assume that a $n$-state DFA $A(m)$ solves promise problem $A^{eq}(m)$. We show that $n$ cannot be less than $2m + 2$.

Assume that $s_0$ is the initial state of $A(m)$, and that there are two different strings $w_i, w_j \in W$ such that $\hat{\delta}(s_0, w_i) = \hat{\delta}(s_0, w_j)$. Without a loss of generality, we assume that $w_i$ is a prefix of $w_j$, so there is a string $x$ such that $w_j = w_i x$, where $|x| \neq 0$. Let $\hat{\delta}(s_0, w_i) = s$, we have $\hat{\delta}(s, x) = \hat{\delta}(s, x^y) = s$. Because $w_i$ is a prefix of $a^m b^m$, there exists a string $y$ satisfies that $\hat{\delta}(s_0, w_i y) = \hat{\delta}(s, y) = s_{acc}$, where $s_{acc}$ is an accepting state. It follow $\hat{\delta}(s_0, w_i x^k y) = s_{acc}$. Therefore, there is some $k \in \mathbb{Z}^+$ satisfy that $\hat{\delta}(s_0, w_i x^k y) = s_{acc}$ and $w_i x^k y \in A^{eq}_{\text{na}}(m)$, which is a contradiction. Hence, for any two different strings $w_i, w_j \in W$ satisfy that $\hat{\delta}(s_0, w_i) \neq \hat{\delta}(s_0, w_j)$.

For any $w_i \in W$, $\hat{\delta}(s_0, w_i)$ is a reachable state (i.e., there exists a string $z$ such that $\hat{\delta}(\hat{\delta}(s_0, w_i), z)$ is an accepting state). Therefore, there must be at least one state that is not reachable, for example, $\hat{\delta}(s_0, a^m b^m+1)$. There is $2m + 1$ elements in the set $W$ and at least one not reachable state. So any DFA solving the promise problem $A^{eq}(m)$ has at least $2m + 2$ states.

Theorem 15. For any $m \in \mathbb{Z}^+$, any 2DFA, 2NFA and any polynomial expected running time 2PFA solving the promise problem $A^{eq}(m)$ has at least $\sqrt{\log m}, \sqrt[3]{\log m}$ and $\sqrt{(\log m)/b}$ states, where $b$ is a constant.

Proof. Assume that an $n_1$-state 2DFA $A$ solves the promise problem $A^{eq}(m)$. It is easy to prove that $n_1 \geq 3$. According to Lemma 15, there is a DFA that solves the promise problem.
\( A^{eq}(m) \) with \((n_1 + 1)^{n_1 + 1}\) states. According to Theorem 14, we have

\[
(n_1 + 1)^{n_1 + 1} \geq 2m + 2 \Rightarrow (n_1 + 1) \log (n_1 + 1) > \log m + 1. \tag{35}
\]

Because \( n_1 \geq 3 \), we get

\[
n_1^2 > (n_1 + 1) \log (n_1 + 1) \Rightarrow n > \sqrt{\log m}. \tag{36}
\]

Assume that an \( n_2 \)-state 2NFA \( A \) solves the promise problem \( A^{eq}(m) \). According to Lemma 4, there is a DFA that solves the promise problem \( A^{eq}(m) \) with \( 2^{(n_2 - 1)^2 + n_2} \) states. According to Theorem 14, we have

\[
2^{(n_2 - 1)^2 + n_2} \geq 2m + 2 \Rightarrow (n_2 - 1)^2 + n_2 \geq \log m + 1 \Rightarrow n_2 > \sqrt{\log m}. \tag{37}
\]

Assume that an \( n_3 \)-state 2PFA \( A \) solves the promise problem \( A^{eq}(m) \) with the error probability \( \epsilon < 1/2 \) and within a polynomial expected running time. According to Lemma 5, there is a DFA that solves the promise problem \( A^{eq}(m) \) with \( n_3^{b n_3^2} \) states, where \( b > 0 \) is a constant. According to Theorem 14, we have

\[
n_3^{b n_3^2} \geq 2m + 2 \Rightarrow b n_3^2 \log n_3 \geq \log m \Rightarrow n_3 > \frac{\sqrt{\log m}}{b}. \tag{39}
\]

\[
n_3^3 \geq (\log m)/b \Rightarrow n_3 > \sqrt{\log m}/b. \tag{40}
\]

\( \square \)

4 State Succinctness of 2QCFA

For the alphabet \( \Sigma = \{a, b, c\} \) and any \( m \in \mathbb{Z}^+ \), let \( L^{twin}(m) = \{wcw|w \in \{a,b\}^*, |w| = m\} \). For any \( \epsilon < 1/2 \), we will prove that \( L^{twin}(m) \) can be recognized by a 2QCFA with one-sided error \( \epsilon \) in an exponential expected running time with a constant number of quantum states and \( \mathcal{O}(\log \frac{1}{\epsilon}) \) classical states. The language \( L^{twin} = \{wcw|w \in \{a,b\}^*\} \) over alphabet \( \Sigma = \{a, b, c\} \) was declared as being recognized by a 2QCFA by Yakaryılmaz and Cem Say [37]. However, they did not give details of such a 2QCFA. In the following, we will show such an automaton and its behavior in details.

**Theorem 16.** For any \( \epsilon < 1/2 \), there exists a 2QCFA \( A(\epsilon) \) which accepts any \( w \in L^{twin} \) with certainty, rejects any \( w \not\in L^{twin} \) with probability at least \( 1 - \epsilon \), and halts in exponential expected time, where \( QS(A(\epsilon)) = 3 \) and \( CS(A(\epsilon)) \in \mathcal{O}(\log \frac{1}{\epsilon}) \).
Check whether the input is of the form $xcy$ ($x, y \in \{a, b\}^*$). If not, reject.

Otherwise, repeat the following ad infinitum:
1. Move the tape head to the first input symbol and set the quantum state to $|q_0\rangle$.
2. Until the currently scanned symbol $\sigma$ is $c$, do the following:
   (2.1). Perform $U_\sigma$ on the quantum state.
   (2.2). Move the tape head one square to the right.
3. Move the tape head to the last input symbol.
4. Until the currently scanned symbol $\sigma$ is $c$, do the following:
   (4.1). Perform $U^{-1}_\sigma$ on the quantum state.
   (4.2). Move the tape head one square to the left.
5. Measure the quantum state. If the result is not $|q_0\rangle$, reject.
6. Move the tape head to the last input symbol and set $b = 0$.
7. While the currently scanned symbol is not $c$, do the following:
   (7.1). Simulate $k$ coin-flips. Set $b = 1$ in case all results are not “heads”.
   (7.2). Move the tape head one square to the left.
8. If $b = 0$, accept.

Figure 3: Informal description of the actions of a 2QCFA for $L_{twin}$. The choice of $k$ will depend on $\epsilon$.

Proof. Let us consider $3 \times 3$ matrixes $U_a$ and $U_b$ defined as follows:

$$ A = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ -3 & 0 & 4 \end{pmatrix}. \quad (41) $$

We now describe formally a 2QCFA $A(\epsilon)$ that is described less formally in Figure 3 with 3 quantum states $\{|q_0\rangle, |q_1\rangle, |q_2\rangle\}$, with $|q_0\rangle$ being the initial state. $A(\epsilon)$ has two unitary operators $U_a = \frac{1}{\sqrt{5}}A$ and $U_b = \frac{1}{\sqrt{5}}B$ given in Eq. (41). They can also be described as follows:

| $U_a|q_0\rangle$ | $\frac{4}{\sqrt{5}}|q_0\rangle - \frac{3}{\sqrt{5}}|q_1\rangle$ | $U_b|q_0\rangle$ | $\frac{4}{\sqrt{5}}|q_0\rangle - \frac{3}{\sqrt{5}}|q_2\rangle$ |
| $U_a|q_1\rangle$ | $\frac{4}{\sqrt{5}}|q_0\rangle + \frac{4}{\sqrt{5}}|q_1\rangle$ | $U_b|q_1\rangle$ | $|q_1\rangle$ |
| $U_a|q_2\rangle$ | $|q_2\rangle$ | $U_b|q_2\rangle$ | $\frac{3}{\sqrt{5}}|q_0\rangle + \frac{4}{\sqrt{5}}|q_2\rangle$ |

We now summarize some concepts and results from [3] that we will use to prove the theorem. For $u \in \mathbb{Z}^3$, we use $u[i]$ ($i = 1, 2, 3$) to denote the $i$th entry of $u$. We define a function $f : \mathbb{Z}^3 \to \mathbb{Z}$ as

$$ f(u) = 4u[1] + 3u[2] + 3u[3] \quad (42) $$

for each $u \in \mathbb{Z}^3$, and we define a set $K \subseteq \mathbb{Z}^3$ as

$$ K = \{ u \in \mathbb{Z}^3 : u[1] \not\equiv 0(\text{mod } 5), f(u) \not\equiv 0(\text{mod } 5), \text{ and } u[2] \cdot u[3] \equiv 0(\text{mod } 5) \} \quad (43) $$
Lemma 17 (3). If \( u \in K \), then \( Au \in K \) and \( Bu \in K \).

Lemma 18 (3). If an \( u \in \mathbb{Z}^3 \) is such that \( u = Av = Bw \) for some \( v, w \in \mathbb{Z}^3 \), then \( u \notin K \).

Lemma 19. If \( u \in K \), there does not exist an \( l \in \mathbb{Z}^+ \) such that \( Xu = \pm 5^l (1, 0, 0)^T \), where \( X \in \{A, B\} \).

Proof. Suppose there is an \( l \in \mathbb{Z}^+ \) such that \( Xu = \pm 5^l (1, 0, 0)^T \). Assume that \( X = A \) (the proof for \( X = B \) is similar), then it holds

\[
Xu = Au = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} u[1] \\ u[2] \\ u[3] \end{pmatrix} = \begin{pmatrix} 4u[1] + 3u[2] \\ -3u[1] + 4u[2] \\ 5u[3] \end{pmatrix} = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} 5^l \tag{44}
\]

\[\Rightarrow \begin{pmatrix} u[1] \\ u[2] \\ u[3] \end{pmatrix} = \pm \begin{pmatrix} 4 \cdot 5^{l-2} \\ 3 \cdot 5^{l-2} \\ 0 \end{pmatrix}. \tag{45}\]

Since \( 4u[1] + 3u[2] + 3u[3] = \pm (16 \cdot 5^{l-2} + 9 \cdot 5^{l-2}) = \pm 5^l \), we conclude \( f(u) \equiv 0 \pmod{5} \). We get that \( u \notin K \), which contradicts the fact that \( u \in K \). Hence, the Lemma has been proved.

Corollary 20. Let

\[ u = X_k \cdots X_1 (1, 0, 0)^T, \tag{46}\]

where \( X_i \in \{A, B\} \). Then \( u = \pm 5^l (1, 0, 0)^T \) for no \( l \in \mathbb{Z}^+ \).

Proof. Clearly, \( (1, 0, 0)^T \in K \). According to Lemma 17, \( X_{k-1} \cdots X_1 (1, 0, 0)^T \in K \). According to Lemma 19, there does not exist \( l \in \mathbb{Z}^+ \) such that \( u = \pm 5^l (1, 0, 0)^T \).

Lemma 21. Let

\[ u = Y_{i-1} \cdots Y_k^{-1} (1, 0, 0)^T, \tag{47}\]

where \( Y_i \in \{A, B\} \). Then \( u = \pm \frac{1}{5^l} (1, 0, 0)^T \) for no \( l \in \mathbb{Z}^+ \).

Proof. Assume that there is an \( l \in \mathbb{Z}^+ \) satisfies that \( u = Y_{i-1} \cdots Y_k^{-1} (1, 0, 0)^T = \pm \frac{1}{5^l} (1, 0, 0)^T \), then we get \( Y_k \cdots Y_1 (1, 0, 0)^T = \pm 5^l (1, 0, 0)^T \). According to Corollary 20, such \( l \) does not exist.

Lemma 22. Let

\[ u = (5Y_1^{-1}) \cdots (5Y_{m-1}^{-1})(5^{-1}X_n) \cdots (5^{-1}X_1)(1, 0, 0)^T, \tag{48}\]

where \( X_j, Y_j \in \{A, B\} \). If \( m = n \) and \( X_j = Y_j \) for \( 1 \leq j \leq n \), then \( u[2]^2 + u[3]^2 = 0 \). Otherwise, \( u[2]^2 + u[3]^2 > 5^{-(n+m)} \).
Proof. If \( m = n \) and \( X_j = Y_j \) for \( 1 \leq j \leq n \), then we have
\[
u = Y_1^{-1} \cdots Y_n^{-1} X_n \cdots X_1(1, 0, 0)^T = (1, 0, 0)^T,
\]
and thus \( u[2]^2 + u[3]^2 = 0 \).

Otherwise, note that \( ||u|| = 1 \), since \( 5^{-1}X_j \) and \( 5Y_j^{-1} \) are unitary for each \( j \), and also note that \( 5^{(n+m)}u[i] \) \( (i = 1, 2, 3) \) is an integer. It therefore suffices to prove that \( u \neq \pm (1, 0, 0)^T \).

According to Corollary 20, for every \( Av \in \mathbb{R} \),
\[
\text{From that we conclude that } 5 \left( (\frac{1}{1}, i < m \right) \left( 1, 0, 0 \right) \right) = 5^{-m}X_{n-m} \cdots X_1(1, 0, 0)^T.
\]

We first prove the case that \( n \geq m \). If \( X_{n-j} = Y_{m-j} \) for \( 0 \leq j \leq m - 1 \), then
\[
u = (5Y_1^{-1}) \cdots (5Y_n^{-1}) (5^{-1}X_n) \cdots (5^{-1}X_1)(1, 0, 0)^T = 5^{-m}X_{n-m} \cdots X_1(1, 0, 0)^T.
\]

According to Corollary 20 for every \( l \in \mathbb{Z}^+ \),
\[
u = 5^{-m}X_{n-m} \cdots X_1(1, 0, 0)^T \neq \pm 5^{-m}5^l(1, 0, 0)^T.
\]

This implies that \( u \neq \pm (1, 0, 0)^T \) if \( l = n - m \).

Next suppose there exist an \( i < m \) such that \( X_{n-i} \neq Y_{m-i} \). Let \( k \) be the smallest integer such that \( X_{n-k} \neq Y_{m-k} \), and without loss of generality suppose \( X_{n-k} = A, Y_{m-k} = B \). Since \( X_{n-j} = Y_{m-j} \) for \( j < k \), we have
\[
u = (5Y_1^{-1}) \cdots (5Y_n^{-1}) (5^{-1}X_n) \cdots (5^{-1}X_1)(1, 0, 0)^T = 5^{-m}Y_1^{-1} \cdots Y_{m-k}X_{n-k} \cdots X_1(1, 0, 0)^T.
\]

For \( u = (1, 0, 0)^T \), we get
\[
u = 5^{-m}Y_1^{-1} \cdots Y_{m-k}X_{n-k} \cdots X_1(1, 0, 0)^T = (1, 0, 0)^T
\]
\[
\Rightarrow X_{n-k} \cdots X_1(1, 0, 0)^T = 5^{-m}Y_{m-k} \cdots Y_1(1, 0, 0)^T = Y_{m-k} \cdots Y_{15}^{-1}(1, 0, 0)^T
\]

Obviously, \( (1, 0, 0)^T \in K \) and \( 5^{-m}(1, 0, 0)^T \in K \). Let \( v = X_{n-k-1} \cdots X_1(1, 0, 0)^T \) and \( w = Y_{m-k-1} \cdots Y_{15}^{-1}(1, 0, 0)^T \), according to Lemma 17 we have \( v, w \in K, X_{n-k}v = Av \in K, \) and \( Y_{m-k}w = Bw \in K \). By Lemma 18 this implies \( Av \neq Bw \), which contradicts the Equation 55. From that we conclude \( u \neq (1, 0, 0)^T \). By similar reasoning we get that, \( u \neq -(1, 0, 0)^T \).

Now we deal with the case \( n < m \). If \( X_{n-j} = Y_{m-j} \) for \( 0 \leq j \leq n - 1 \), then
\[
u = (5Y_1^{-1}) \cdots (5Y_n^{-1})(5^{-1}X_n) \cdots (5^{-1}X_1)(1, 0, 0)^T = 5^{-m-n}Y_1^{-1} \cdots Y_{m-n}^{-1}(1, 0, 0)^T.
\]

According to Lemma 21 for every \( l \in \mathbb{Z}^+ \),
\[
u = 5^{-m-n}Y_1^{-1} \cdots Y_{m-n}^{-1}(1, 0, 0)^T \neq \pm 5^{-m-n}5^{-l}(1, 0, 0)^T.
\]
This implies that \( u \neq \pm (1, 0, 0)^T \) if \( l = m - n \).

Let us assume that there exist \( j < n \) such that \( X_{n-j} \neq Y_{m-j} \). Let \( k \) be the smallest index such that \( X_{n-k} \neq Y_{m-k} \). By similar reasoning as in the case \( n \geq m \), we get \( u \neq \pm (1, 0, 0)^T \).

If the input \( w \) is not of the form \( xcy \), \( A(\epsilon) \) rejects \( w \) immediately.

**Lemma 23.** If the input \( w = xcy \) and \( x = y \), then the quantum state of \( A(\epsilon) \) will evolve into \( |q_0 \rangle \) with certainty after the loop 4.

**Proof.** Let \( x = x_1 x_2 \ldots x_l = y_1 y_2 \ldots y_l \) for some \( l \). Starting with the state \( |q_0 \rangle \), \( A(\epsilon) \) changes its quantum state to \( |\psi \rangle \) after the loop 4, where

\[
|\psi \rangle = U_{y_1}^{-1} U_{y_2}^{-1} \cdots U_{y_l}^{-1} U_{x_1} \cdots U_{x_l} U_{x_1} U_{x_2} |q_0 \rangle = U_{x_1}^{-1} U_{x_2}^{-1} \cdots U_{x_l}^{-1} U_{x_1} \cdots U_{x_l} U_{x_1} |q_0 \rangle = |q_0 \rangle.
\]  

**Lemma 24.** If the input \( w = xcy \) and \( x \neq y \), then \( A(\epsilon) \) rejects \( w \) after the step 5 with the probability at least \( 5^{-(m+n)} \).

**Proof.** Let \( x = x_1 x_2 \cdots x_n, y = y_1 y_2 \cdots y_m \). Starting with state \( |q_0 \rangle \), \( A(\epsilon) \) changes its quantum state after the loop 4 to:

\[
|\psi \rangle = U_{y_1}^{-1} U_{y_2}^{-1} \cdots U_{y_m}^{-1} U_{x_n} \cdots U_{x_l} U_{x_1} |q_0 \rangle.
\]  

Let \( |\psi \rangle = \beta_0 |q_0 \rangle + \beta_1 |q_1 \rangle + \beta_2 |q_2 \rangle \). According to Lemma 22, \( \beta_1^2 + \beta_2^2 > 5^{-(n+m)} \). In the step 5, the quantum state \( |\psi \rangle \) is measured, \( A(\epsilon) \) then rejects \( w \) with the probability \( p_r = \beta_1^2 + \beta_2^2 > 5^{-(n+m)} \).

Every execution of the steps 6, 7 and 8 leads to an acceptance with the probability \( 2^{-k(n+m+1)} \).

Let \( k \geq \max \{ \log 5, \log \frac{1}{\epsilon} \} \). Assume that the input is of the form \( w = xcy \). If \( x = y \), 2QCFA \( A(\epsilon) \) always changes its quantum state to \( |q_0 \rangle \) after the loop 4, and \( A(\epsilon) \) never rejects the input after the measurement in the step 5. After the steps 6, 7 and 8, the probability of \( A(\epsilon) \) accepting \( w \) is \( 2^{-k(n+m+1)} \). Repeating the whole iteration for \( c2^{k(n+m+1)} \) times, the accepting probability is

\[
Pr[A(\epsilon) \text{ accepts } w] = 1 - (1 - 2^{-k(n+m+1)})^{c2^{k(n+m+1)}},
\]  

and this can be made arbitrarily close to 1 by selecting constant \( c \) appropriately.

Otherwise, if \( x \neq y \), then, according to Lemma 24, \( A(\epsilon) \) rejects the input after the step 5 with the probability

\[
P_r > 5^{-(m+n)}
\]
and, $A(\epsilon)$ accepts the input after the steps 6, 7 and 8 with the probability

$$P_a = 2^{-k(n+m+1)}. \quad (62)$$

If we repeat the whole iteration indefinitely, the probability of $A(\epsilon)$ rejecting input $w$ is

$$Pr[A(\epsilon) \text{ rejects } w] = \sum_{i \geq 0} (1 - P_a)^i (1 - P_r)^i P_r \quad (63)$$

$$= \frac{P_r}{P_a + P_r - P_aP_r} > \frac{P_r}{P_a + P_r} \quad (64)$$

$$> \frac{5^{-k(n+m+1)}}{2^{-k(n+m+1)} + 5^{-k(n+m+1)}} \quad (65)$$

$$> \frac{1}{1 + \epsilon} > 1 - \epsilon. \quad (66)$$

If the input is $w$, then the step 1 takes $O(1)$ time, the steps 2 and 3 take $O(|w|)$ time, the loops 4 and 5 take $O(|w|)$ time, the steps 6, 7 and 8 take $O(|w|)$ time. The expected number of iterations is $O(2^{|w|})$. Hence, the expected running time of $A(\epsilon)$ is $O(|w|2^{|w|})$. Obviously, the 2QCFA $A(\epsilon)$ has three quantum states. We just need $O(k)$ classical states to simulate $k$ coin-flips and calculate the outcomes, therefore $CS(A(\epsilon)) \in O(\log \frac{1}{\epsilon}).$

\[\square\]

In Theorem 16 we have proved that $L^{twin}$ can be recognized by 2QCFA. We will show that $L^{twin}$ cannot be recognized by 2PFA with error probability $\epsilon < 1/2$. Thus $L^{twin}$ is another witness of the fact that 2QCFA are more powerful than their classical counterparts 2PFA.

**Theorem 25.** There is no 2PFA recognizing $L^{twin}$ with error probability $\epsilon < 1/2$.

**Proof.** Let $A = L^{twin}$ and $B = \overline{L^{twin}} = \Sigma^* \setminus A$. Clearly, for each $m \in I$, there is a set $W_m \subseteq \Sigma^*$ satisfying conditions (1) and (2) of Lemma 6. For every $m \in I$ and every $w, w' \in W_m$ with $w \neq w'$, if we take $u = \lambda$ (the empty word) and $v = cw$, then $uwv = wcw \in A$ and $uw'v = w'cw \in B$. According to Lemma 3 there is no 2PFA separating $A$ and $B$. Thus, there is no 2PFA recognizing $L^{twin}$ and the Theorem has been proved. \[\square\]

For an alphabet $\Sigma$ and an $m \in \mathbb{Z}^+$, let $L(m) = \{w \mid |w| = m\}$.

**Lemma 26** (*37*). For any $\epsilon < 1/2$, there exists a 7-state 1QFA $^\oplus A(m, \epsilon)$ which accepts any $w \in L(m)$ with certainty, and rejects any $w \notin L(m)$ with probability at least $1 - \epsilon$. Moreover, the expected runtime of the $A(m, \epsilon)$ on $w$ is $O(2^{|w|}|w|)$. 

20
Lemma 27. For any 1QFA $A_1$ with $n$ quantum states and expected running time $t(|w|)$, there exists a 2QCFA $A_2$ with $n$ quantum states, $O(n)$ classical states, and expected running time $O(t(|w|))$, such that $A_2$ accepts every input string $w$ with the same probability that $A_1$ accepts $w$.

Theorem 28. For any $\epsilon < 1/2$, there exists a 2QFA $A(m, \epsilon)$ which accepts any $w \in \mathcal{L}(m)$ with certainty, and rejects any $w \notin \mathcal{L}(m)$ with probability at least $1 - \epsilon$. Moreover, $\mathcal{Q}(A(m, \epsilon)) = 7$, $CS(A(m, \epsilon))$ is a constant, and the expected runtime of the $A(m, \epsilon)$ on $w$ is $O(|w|/|w|)$.

Proof. It follows from Lemma 26 and Lemma 27.

Theorem 29. For any $\epsilon < 1/2$, there exists a 2QFA $A(m, \epsilon)$ which accepts any $w \in \mathcal{L}(m)$ with certainty, and rejects any $w \notin \mathcal{L}(m)$ with probability at least $1 - \epsilon$. Moreover, $\mathcal{Q}(A(m, \epsilon))$ is a constant, $CS(A(m, \epsilon)) \in O(\log 1/\epsilon)$, and the expected running time of $A(m, \epsilon)$ on $w$ is $O(|w|2^{k|w|})$.

Proof. Let the alphabet $\Sigma = \{a, b, c\}$. Obviously, $\mathcal{L}(m) = \mathcal{L}(m) \cap L(2m + 1)$. According to Theorem 28 for any $\epsilon_1 < 1/2$, there is a 2QCFA $A_1(\epsilon_1)$ recognizes $\mathcal{L}(m)$ with one-sided error $\epsilon_1$, and $\mathcal{Q}(A_1(\epsilon_1)) = 3$, $CS(A_1(\epsilon_1)) \in O(\log 1/\epsilon)$ and $T(A_1(\epsilon_1)) \in O(|w|2^{k|w|})$ where $k$ is a constant. According to Theorem 28 for any $\epsilon_2 < 1/2$, there is a 2QCFA $A_2(m, \epsilon)$ recognizes $L(2m + 1)$ with one-sided error $\epsilon_2$, and $\mathcal{Q}(A_2(m, \epsilon)) = 7$, $CS(A_2(m, \epsilon))$ is a constant and $T(A_2(m, \epsilon)) \in O(2^{k|w|})$. For any $\epsilon < 1/2$, let $\epsilon_1 = \epsilon/2$ and $\epsilon_2 = \epsilon/2$. According to Lemma 2 there is a 2QCFA $A(m, \epsilon)$ recognizes $\mathcal{L}(m) \cap L(2m + 1)$ with one-sided error $\epsilon$, where $\mathcal{Q}(A(m, \epsilon)) = \mathcal{Q}(A_1(\epsilon_1)) + \mathcal{Q}(A_2(m, \epsilon)) = 10$, $CS(A(m, \epsilon)) = CS(A_1(\epsilon_1)) + CS(A_2(m, \epsilon)) + CS(A_1(\epsilon_1)) \in O(\log 1/\epsilon)$ and $T(A(m, \epsilon)) = T(A_1(\epsilon_1)) + T(A_2(m, \epsilon)) \in O(|w|2^{k|w|})$. Hence, the theorem has been proved.

For a fix $m \in \mathbb{Z}^+$, $\mathcal{L}(m)$ is finite, and thus there exists a DFA accepting the language $\mathcal{L}(m)$. In the following we use methods and results of communication complexity to derive a lower bound on the number of states of finite automata accepting the language $\mathcal{L}(m)$.

Theorem 30. For any $m \in \mathbb{Z}^+$, any DFA recognizing $\mathcal{L}(m)$ has at least $2^m$ states.

Proof. Assume that a DFA $\mathcal{A}$ recognizes $\mathcal{L}(m)$. For an input string $x|y$ of $\mathcal{L}(m)$ let us consider the following communication protocol between Alice and Bob with Alice having $x$ at the beginning and Bob having $y$ at the beginning. A protocol can be derived for $EQ(x, y)$ as follows: Alice first simulates the path taken by DFA $\mathcal{A}$ on her input $x$. She then sends the name of the last state $s$ in this path to Bob, which needs $\log(|S|)$ bits, where $S$ is the set of states in DFA $\mathcal{A}$. Afterwards, Bob simulates DFA $\mathcal{A}$, starting from the state $s$, on input $cy$. At last, Bob sends the result to Alice, if $w$ is accepted, bob sends 1, otherwise 0. All
together, they get a simulation of DFA $A$ on the input $w = xcy$. By assumption, if $w = xcy$ is accepted by DFA $A$ then $E(Q(x, y) = 1$ while if $w$ is rejected then $E(Q(x, y) = 0$. Therefore, we have $D(EQ) \leq \log(|S|) + 1$. According to Lemma 7 we have

$$D(EQ) = m + 1 \leq \log(|S|) + 1$$

$$\Rightarrow m \leq \log(|S|) \Rightarrow |S| \geq 2^m.$$ (67)

(68)

**Theorem 31.** For any $m \in \mathbb{Z}^+$, any 2DFA, 2NFA and polynomial expected running time 2PFA recognizing $L_{twin}(m)$ have at least $\sqrt{m}$, $\sqrt{m}$ and $3\sqrt{m/b}$ states, where $b$ is a constant.

**Proof.** Assume that an $n_1$-state 2DFA $A$ recognizes $L_{twin}(m)$. It is easy to prove that $n_1 \geq 3$. According to Lemma 8 there is a DFA recognizes $L_{twin}(m)$ with $(n_1 + 1)^{n_1 + 1}$ states. According to Theorem 30 we have

$$(n_1 + 1)^{n_1 + 1} \geq 2^m \Rightarrow (n_1 + 1) \log(n_1 + 1) \geq m.$$ (69)

Because $n \geq 3$, we get

$$n_1^2 > (n_1 + 1) \log(n_1 + 1) > m \Rightarrow n_1 > \sqrt{m}.$$ (70)

Assume that an $n_2$-state 2NFA $A$ recognizes $L_{twin}(m)$. According to Lemma 4 there is a DFA recognizes $L_{twin}(m)$ with $2^{(n_2 - 1)^2 + n_2}$ states. According to Theorem 30 we have

$$2^{(n_2 - 1)^2 + n_2} \geq 2^m \Rightarrow (n_2 - 1)^2 + n_2 \geq m$$

$$\Rightarrow n_2^2 > m \Rightarrow n_2 > \sqrt{m}.$$ (71)

(72)

Assume that an $n_3$-state 2PFA $A$ recognizes $L_{twin}(m)$ with an error probability $\epsilon < 1/2$ and within a polynomial expected running time. According to Lemma 5 there is a DFA recognizes $L_{twin}(m)$ with $n_3^{bn_3^2}$ states, where $b > 0$ is a constant. According to Theorem 30 we have

$$n_3^{bn_3^2} \geq 2^m \Rightarrow bn_3^2 \log n_3 \geq m$$

$$\Rightarrow n_3^3 > m/b \Rightarrow n_3 > \sqrt[3]{m/b}.$$ (73)

(74)
5 Concluding remarks

2QCFA were introduced by Ambainis and Watrous [3]. In this paper, we investigated state succinctness of 2QCFA. We have showed that 2QCFA can be more space-efficient than their classical counterparts DFA, 2DFA, 2NFA and polynomial expected running time 2PFA, where the superiority cannot be bounded. For any \( m \in \mathbb{Z}^+ \) and any \( \epsilon < 1/2 \), we have proved that there is a promise problem \( A^{eq}(m) \) that can be solved by a 2QCFA with one-sided error \( \epsilon \) in a polynomial expected running time with a constant number of quantum states and \( O(\log \frac{1}{\epsilon}) \) classical states, whereas the sizes of the corresponding DFA, 2DFA, 2NFA and polynomial expected running time 2PFA are at least \( 2m + 2, \sqrt{\log m}, \sqrt{\log m} \) and \( \sqrt[3]{\log m}/b \). For any \( m \in \mathbb{Z}^+ \) and any \( \epsilon < 1/2 \), we have also showed that there exists a 2QCFA recognizing the language \( L^{twin}(m) \) with one-sided error \( \epsilon \) in an exponential expected running time with a constant number of quantum states and \( O(\log \frac{1}{\epsilon}) \) classical states, whereas the sizes of the corresponding DFA, 2DFA, 2NFA and polynomial expected running time 2PFA are at least \( 2^m, \sqrt{m}, \sqrt{m} \) and \( \sqrt[3]{m/b} \).

To conclude, we formulate some open problems:

1. Can the result related to a promise problem \( A^{eq}(m) \) be improved to deal with a language?

2. In Theorem [31], we gave a bound on a polynomial expected running time 2PFA. What is the bound when the expected running time is exponential?

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