Explicit construction of the classical BRST charge for nonlinear algebras

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Received 03 July 2011; accepted 28 September 2011

Abstract: We give an explicit formula for the Becchi-Rouet-Stora-Tyutin (BRST) charge associated with Poisson superalgebras. To this end, we split the master equation for the BRST charge into a pair of equations such that one of them is equivalent to the original one and find a solution to this equation. The solution possesses a graphical representation in terms of diagrams.

PACS (2008): 11.15Kc
Keywords: BRST quantization

1. Introduction

The BRST symmetry [1, 2] plays an important role in quantization of gauge theories [3, 4]. It is generated by the BRST charge. If the quantum BRST charge exists it is essentially determined by the corresponding classical one.

The classical BRST charge is represented by a power series in ghosts. The first two terms of the series are well known in the general case. When constraints form a Lie algebra, these terms reproduce an exact BRST charge.

The fundamental equation for the BRST charge is equivalent to a system of recurrent equations. In the case of general polynomial Poisson algebras there exists an algorithm for the construction of a solution to these equations [5].

The classical BRST charge for some classes of quadratically nonlinear algebras was found in [6, 7]. Construction of the BRST charge for some boson Poisson algebras was investigated in [8]. In the case of general quadratically nonlinear algebras the expression for the third order contribution in the ghost fields to the BRST charge was found. In [9], the classical BRST charge for quadratically nonlinear superalgebras was discussed. For some classes of superalgebras the BRST charge was constructed up to the fourth order in the ghost fields.

In this paper we derive an explicit expression for the classical BRST charge in the case of general nonlinear superalgebras. We show that the system of equations for the classical BRST charge is equivalent to a smaller subsystem. Then we find the general solution to the subsystem. Expanding the solution in powers of the ghost fields one can find the BRST charge in an arbitrary order.

The paper is organized as follows. In Section 2, we introduce notations and represent the master equation for the BRST charge in the form which is convenient for our pur-
poses. In Section 3, we obtain an explicit expression for the classical BRST charge. We show that the expression possesses a graphical representation in terms of diagrams.

In what follows the Grassmann parity and ghost number of a function $A$ are denoted by $\epsilon(A)$ and $\text{gh}(A)$, respectively.

## 2. Structure of the master equation for the BRST charge

Let $G_\alpha, \alpha = 1, \ldots, J$, be the first class constraints which satisfy the following Poisson brackets

$$\{G_\alpha, G_\beta\} = F_{\alpha\beta}(G),$$

where $F_{\alpha\beta}(G)$ is a polynomial in the $G$'s such that $F_{\alpha\beta}(0) = 0$. The constraints are supposed to be independent and of definite Grassmann parity $\epsilon_\alpha, \epsilon(G_\alpha) = \epsilon_\alpha$.

Following the BRST method the ghost pair $(P_\alpha, e^\alpha)$ is introduced for each constraint $G_\alpha$ :

$$\{e^\alpha, P_\beta\} = \delta_\alpha^\beta,$$

$$\{e^\alpha, e^\beta\} = \{P_\alpha, P_\beta\} = \{G_\alpha, e^\beta\} = \{P_\alpha, G_\beta\} = 0,$$

$$\epsilon(P_\alpha) = \epsilon(e^\alpha) = \epsilon_\alpha + 1, \quad -\text{gh}(P_\alpha) = \text{gh}(e^\alpha) = 1.$$

Let $\mathcal{M}$ be the set of variables $G^\alpha, P_\alpha, e^\alpha$, and let $\mathcal{V} = R[[\mathcal{M}]]$ be the ring of formal power series in the variables $\mathcal{M}$.

The BRST charge $\Omega \in \mathcal{V}$ is defined as a solution to the equation

$$\{\Omega, \Omega\} = 0, \quad \epsilon(\Omega) = 1, \quad \text{gh}(\Omega) = 1, \quad (1)$$

and the boundary conditions

$$\frac{\partial \Omega}{\partial e^\alpha} \bigg|_{e=0} = G_\alpha. \quad (2)$$

These equations are consistent [10, 11]. One can write

$$\Omega = G_\alpha e^\alpha + M, \quad (3)$$

where

$$M = \sum_{\alpha = 2}^{\infty} \Omega^{(\alpha)}, \quad \Omega^{(\alpha)} \sim P^{\alpha-1} e^\alpha.$$

Substituting (3) into (1), one obtains

$$\delta M + \frac{1}{2} F + A M + \frac{1}{2} \{M, M\} = 0, \quad (4)$$

where

$$\delta = G_\alpha \frac{\partial}{\partial P_\alpha}, \quad F = e^\alpha F_{\alpha\beta}(G)e^\beta, \quad A = e^\alpha \{J_\alpha, \ldots\}.$$

Let $N$ be the counting operator

$$N = G_\alpha \frac{\partial}{\partial G_\alpha} + P_\alpha \frac{\partial}{\partial P_\alpha}.$$

The ring $\mathcal{V}$ splits as

$$\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n$$

with $N X = n X$ for $X \in \mathcal{V}_n$. One easily verifies that

$$N = \delta \sigma + \sigma \delta, \quad \sigma = P_\alpha \frac{\partial}{\partial G_\alpha}.$$

We define $N^+ : \mathcal{V} \rightarrow \mathcal{V}$ by

$$N^+ X = \begin{cases} \frac{1}{n} X, & X \in \mathcal{V}_n, \quad n > 0; \\ 0, & X \in \mathcal{V}_0. \end{cases}$$

Then $\delta^+ = \sigma N^+$ is a generalized inverse of $\delta$ :

$$\delta \delta^+ \delta = \delta, \quad \delta^+ \delta \delta^+ = \delta^+.$$

Let $\langle \ldots \rangle : \mathcal{V}^2 \rightarrow \mathcal{V}$ be defined by

$$\langle X_1, X_2 \rangle = -\frac{1}{2} (l + \delta^+ A)^{-1} \delta^+ (\{X_1, X_2\} + \{X_2, X_1\}),$$

where $l$ is the identity map, and

$$(l + \delta^+ A)^{-1} = \sum_{m=0}^{\infty} (-1)^m (\delta^+ A)^m.$$

In accordance with the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1 = PV, \mathcal{V}_2 = (I - P)V, \ P = \delta \delta^+, \ eq. \ (4)$ splits as

$$\delta M + \delta \delta^+ D = 0, \quad (5)$$

$$\delta^+ D = 0, \quad (6)$$
where

\[ D = \frac{1}{2} F + AM + \frac{1}{2} \{ M, M \}. \]

From eq. (5) it follows that \( M = Y - \delta^+ D \), or equivalently

\[ M = M_0 + \frac{1}{2} \langle M, M \rangle. \tag{7} \]

where

\[ M_0 = (I + \delta^+ A)^{-1} \left( Y - \frac{1}{2} \delta^+ F \right). \tag{8} \]

and \( Y \in \mathcal{V} \) is an arbitrary cocycle, \( \delta Y = 0 \), subject only to the restrictions \( \epsilon(Y) = 1 \), \( \text{gh}(Y) = 1 \),

\[ Y = \sum_{n=0}^{\infty} Y^{(n)}, \quad Y^{(n)} \sim \mathcal{P}^{n-1} \epsilon^n. \]

One can write \( Y = \delta W, W \in \mathcal{V} \) [11].

Eq. (7) can be iteratively solved as:

\[ M = M_0 + \frac{1}{2} \langle M_0, M_0 \rangle + \ldots. \tag{9} \]

Using (9), we can write

\[ \Omega^{(n)} = \delta W^{(n)} + \tilde{\Omega}^{(n)} \left( Y^{(2)}, Y^{(3)}, \ldots, Y^{(n-1)} \right), \quad n \geq 2, \]

where \( \delta W^{(n)} = Y^{(n)} \). The arbitrariness of the solution for \( \Omega^{(n)} \) is described by the transformation [12]

\[ \Omega^{(n)} \rightarrow \Omega^{(n)} + \delta Z^{(n)}, \quad Z \in \mathcal{V}. \]

This transformation is absorbed into a transitive group transformation of the coboundary \( \delta W^{(n)} \):

\[ \delta W^{(n)} \rightarrow \delta W^{(n)} + \delta Z^{(n)}. \]

Therefore, \( M \) (9) is the general solution to eq. (4). It follows that eq. (4) is equivalent to eq. (7). Eq. (6) can be omitted.

In the next section we obtain an explicit solution to eq. (7).

3. Explicit expression for the BRST charge

To solve eq. (7), we introduce the functions

\[ \langle \ldots \rangle : \mathcal{V}^m \rightarrow \mathcal{V}, \quad m = 1, 2, \ldots, \]

which recursively defined by \( \langle X \rangle = X \),

\[ \langle X_1, \ldots, X_m \rangle = \frac{1}{2} \sum_{i_1 \leq \ldots \leq i_m} \langle \langle X_{i_1}, \ldots, X_{i_m} \rangle, \ldots, \rangle \]

if \( m = 2, 3, \ldots, \) where \( \tilde{X} \) means that \( X \) is omitted. One easily verifies by induction that \( \langle X_1, \ldots, X_m \rangle \) is an \( m \)-linear symmetric function.

For \( m \geq 2, 1 \leq i, j \leq m \), let

\[ P_{ij} : \mathcal{V}^m \rightarrow \mathcal{V}^{m-1} \]

be defined by

\[ P_{ij} \langle X_1, \ldots, X_m \rangle = \langle \langle X_i, X_j \rangle, X_1, \ldots, \tilde{X}_i, \ldots, \tilde{X}_j, \ldots, X_m \rangle. \]

If \( X \in \mathcal{V} \) is given by

\[ X = P_{12} P_{m-2m-2} \cdots P_{i_{m-2}j_{m-2}} \]

for some \( (i_1j_1), \ldots, (i_{m-2}j_{m-2}) \), we say that \( X \) is a descendant of \( (X_1, \ldots, X_m) \). A descendant of \( X \in \mathcal{V} \) is defined as \( X \).

Each descendant can be represented by a diagram. In this diagram an element of \( \mathcal{V} \) is represented by the line segment \( \quad \). A product \( \langle X_i, X_j \rangle \rightarrow \langle X_i, X_j \rangle \) is represented by the vertex joining the line segments for \( X_i, X_j \) and \( \langle X_i, X_j \rangle \). The graph for \( P_{ij} \langle X_1, \ldots, X_m \rangle \) is depicted in figure 1. Here the points labeled by \( 1, \ldots, m \) represent the ends of the lines \( X_1, \ldots, X_m \). Using this prescription, one can consecutively draw the diagrams for \( P_{i_{m-2}j_{m-2}} \langle X_1, \ldots, X_m \rangle, P_{i_{m-2}j_{m-2}} \langle X_1, \ldots, X_m \rangle, \ldots, X \) (11). The diagram for \( X \) has \( m-1 \) vertices and \( m+1 \) external lines.

Lemma 1. The function \( \langle X_1, \ldots, X_m \rangle \) equals the sum of all the descendants of \( (X_1, \ldots, X_m) \).

To prove the lemma, we observe that the statement holds for \( \langle X \rangle \) and \( \langle X_1, X_2 \rangle \). Assume that it is true for all \( k < m \). Each descendant of \( (X_1, \ldots, X_m) \) can be written as

\[ \langle d_1(X_1), d_2(X_2) \rangle, \tag{12} \]
where \( I = (i_1,\ldots,i_k) \) and \( J = (j_1,\ldots,j_l) \) are increasing multi-indexes\(^1\), \( I \cup J = (1,\ldots,m) \), \( d_1(X_I) \) and \( d_2(X_J) \) are some descendents of \( X_I = (X_{i_1},\ldots,X_{i_k}) \) and \( X_J = (X_{j_1},\ldots,X_{j_l}) \), respectively. Let us compute the sum of all the different functions (12). First we sum all the functions with a fixed partition \( I \cup J = (1,\ldots,m) \). Using the inductive assumption, we get

\[
\langle\langle X_I, X_J \rangle\rangle = \frac{1}{2} \sum_{i,j=1}^{m} \langle X_i, X_j \rangle.
\]

Then summing all the functions (13), we obtain

\[
\frac{1}{2} \sum_{i,j=1}^{m} \langle X_i, X_j \rangle.
\]

This is exactly the right-hand side of (10).

For example,

\[
\langle X_1, X_2, X_3 \rangle = \langle\langle X_1, X_2 \rangle, X_3 \rangle + \langle\langle X_1, X_3 \rangle, X_2 \rangle + \langle\langle X_2, X_3 \rangle, X_1 \rangle.
\]

In figure 2, we show the diagram for \( \langle\langle X_1, X_2 \rangle, X_3 \rangle \).

For \( X_1 = \ldots = X_n = M_0 \) eq. (10) takes the form

\[
\langle M_0^n \rangle = \frac{1}{2} \sum_{i=1}^{m-1} \frac{1}{i!} \langle\langle M_0^i \rangle, M_0^{m-i-1} \rangle), \quad \langle M_0 \rangle = \langle M_0, M_0, \ldots, M_0 \rangle.
\]

We claim that a solution to eq. (7) is given by

\[
M = \langle e^{M_0} \rangle,
\]

where

\[
\langle e^{M_0} \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \langle M_0^m \rangle, \quad \langle M_0^0 \rangle = 0.
\]

Indeed, substituting (15) into (7), we get

\[
\frac{1}{2} m! \langle M_0^m \rangle = \sum_{r=1}^{m} \frac{1}{r!} \langle\langle M_0^r \rangle, M_0^{m-r} \rangle).
\]

To conclude the proof, it remains to use (14).

Our previous results lead to the following expression for the BRST charge

\[
\Omega = G_a c^a + \sum_{m=1}^{\infty} \frac{1}{m!} \langle M_0^m \rangle.
\]

From (8) it follows immediately that \( M_0 = O(c^2) \). Using (14) and the induction method, one can show that

\[
\langle M_0^m \rangle = O(c^{m+1}).
\]

Hence, in the case of bosonic constraints, \( \epsilon(G_a) = 0, \epsilon(c^a) = 1 \), eq. (16) takes the form

\[
\Omega = G_a c^a + \sum_{m=1}^{\infty} \frac{1}{m!} \langle M_0^m \rangle.
\]

For example, using lemma 1, we get

\[
\Omega = G_a c^a + \sum_{m=1}^{5} \frac{1}{m!} \langle M_0^m \rangle + O(c^3),
\]

where

\[
\frac{1}{3!} \langle M_0^3 \rangle = \frac{1}{2} \langle\langle M_0, M_0, M_0 \rangle, M_0 \rangle.
\]

\(^1\) The multi-index \( I = (i_1,\ldots,i_n) \) is said to be increasing if \( i_1 < \ldots < i_n \).
\[
\frac{1}{4!} \langle \langle M_6^4 \rangle \rangle = \frac{1}{2} \langle \langle \langle (M_0, M_0), M_0 \rangle \rangle + \frac{1}{8} \langle \langle (M_0, M_0), (M_0, M_0) \rangle \rangle.
\]

\[
\frac{1}{3!} \langle \langle M_6^3 \rangle \rangle = \frac{1}{2} \langle \langle \langle (M_0, M_0), M_0 \rangle \rangle + \frac{1}{4} \langle \langle (M_0, M_0), M_0 \rangle \rangle + \frac{1}{8} \langle \langle (M_0, M_0), (M_0, M_0) \rangle \rangle.
\]

4. Conclusions

An explicit construction providing the classical BRST charge for general Poisson superalgebra is given in this paper. The master equation for the BRST charge is rewritten in the form which enables us to apply a simple iterative method. The iterative solution is represented in an explicit form. It is shown that the solution possesses a graphical representation in terms of diagrams. A similar construction can be used in the BRST-anti-BRST quantization, deformation quantization and in the other models which are based on nonlinear equations with quadratic nonlinearity.

References

[1] C. Becchi, A. Rouet, R. Stora, Commun. Math. Phys. 42, 127 (1975)
[2] I. V. Tyutin, Preprint No. 39 (Lebedev Physics Institute, Moscow, Russia, 1975)
[3] D. M. Gitman, I. V. Tyutin, Quantization of fields with constraints (Springer-Verlag, Berlin Heidelberg, 1990)
[4] M. Henneaux, C. Teitelboim, Quantization of gauge systems (Princeton Univ. Press, Princeton, USA, 1992)
[5] E. S. Fradkin, T. E. Fradkina, Phys. Lett. 72B, 343 (1978)
[6] K. Schoutens, A. Sevrin, P. van Nieuwenhuizen, Commun. Math. Phys. 124, 87 (1989)
[7] A. Dresse, M. Henneaux, J. Math. Phys. 35, 1334 (1994)
[8] I. L. Buchbinder, P. M. Lavrov, J. Math. Phys. 48, 082306 (2007)
[9] M. Asorey, P. M. Lavrov, O. V. Radchenko, A. Sugamoto, Int. J. Mod. Phys. A 24, 5033 (2009)
[10] M. Henneaux, Phys. Rep. 126, 1 (1985)
[11] I. A. Batalin, P. M. Lavrov, I. V. Tyutin, J. Math. Phys. 31, 6 (1990)
[12] I. A. Batalin, I. V. Tyutin, Int. J. Mod. Phys. A 6, 3255 (1991)