Fermionic Path Integrals and Two-Dimensional Ising Model with Quenched Site Disorder

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Abstract

The notion of the integral over the anticommuting Grassmann variables is applied to analyze the fermionic structure of the 2D Ising model with quenched site dilution. In the $N$-replica scheme, the model is explicitly reformulated as a theory of interacting fermions on a lattice. For weak dilution, the continuum-limit approximation implies the log-log singularity in the specific heat near $T_c$.

1 Introduction

The site dilution provides, probably, the simplest way to realize quenched disorder in real magnetic materials. In this case, some amount of the magnetic atoms in a sample, chosen at random, are replaced by nonmagnetic impurity atoms. Another sort of disorder is bond dilution, in which case some of the lattice bonds are assumed to be broken. The two-dimensional Ising model (2DIM) is a natural object to analyze the effects of disorder on ferromagnetic phase transition since the exact analytic solution is known for this model in the pure case. The disordered 2D Ising model already has been analyzed during the last decades both by theoretical tools and in the precise Monte-Carlo experiments. The theoretical studies were merely concerned with the case of bond dilution, making use of the fermionic interpretation of the problem. In this report, we apply a new anticommuting path integral technique to clarify the fermionic structure of the 2DIM with quenched site dilution. At the first stage, the partition function with fixed dilution is transformed into a fermionic Gaussian integral. The averaging over the disorder within the $N$-replica scheme then results a lattice fermionic theory with interaction. Unexpectedly, the interaction in the exact lattice theory appears to be of order $2N$ in fermions, where $N \to 0$ is the number of replicas. An effective four-fermion interaction arises, however, in a continuum-limit field theory responsible for the critical behaviour near $T_c$ for weak dilution due to the interplay of the short-wave and long-wave lattice fermionic modes as we pass to the continuum limit. In particular,
this implies the double-logarithmic singularity in the specific heat near $T_c$ for weak site dilution. The question about the effects of strong and moderated dilution, however, still remains to be an important open problem in the 2D Ising model, this holds true for both site and bond dilution.

2 The model

Let us start with a fixed distribution of the nonmagnetic sites over a lattice (fixed site dilution). The Ising spins, $\sigma_{mn} = \pm 1$, are located at lattice sites, $mn$, with $m,n = 1,2,\ldots,L$ running in the horizontal and vertical directions, respectively. Here $L$ is the lattice length, $L \to \infty$. To introduce site dilution, we accompany each Ising spin by a random variable $y_{mn} = 0,1$. The hamiltonian is

$$-\beta H \{y | \sigma\} = \sum_{mn} \left[ b_1 y_{mn} y_{m+1n} \sigma_{mn} \sigma_{m+1n} + b_2 y_{mn} y_{mn+1} \sigma_{mn} \sigma_{mn+1} \right],$$

where $b_{1,2} = \beta J_{1,2}$ are the bond coupling parameters, $J_{1,2}$ are the exchange energies, $\beta = 1/kT$ is the inverse temperature. We assume ferromagnetic case: $b_{1,2} > 0$. Noting the identity for a typical bond weight: $\exp (byy' \sigma \sigma') = \cosh(byy') + yy' \sigma \sigma' \sinh(byy')$, which readily follows from $\sigma \sigma' = \pm 1$ and $yy' = 0, 1$, the partition function can be written in the form: $Z \{y\} = R \{y\} Q \{y\}$, where $R \{y\}$ is a nonsingular spin-independent prefactor, to be ignored in what follows, and $Q \{y\}$ is the reduced partition function:

$$Q \{y\} = \text{Sp}_\sigma \left( \sigma \prod_{mn} (1 + t_1 y_{mn} y_{m+1n} \sigma_{mn} \sigma_{m+1n}) (1 + t_2 y_{mn} y_{mn+1} \sigma_{mn} \sigma_{mn+1}) \right),$$

where $t_{1,2} = \tanh b_{1,2}$ and a properly normalized spin averaging is assumed. Since we are interested in quenched disorder, we have to average the free energy rather than the partition function itself. The standard device to avoid the averaging of the logarithm is the replica trick:

$$-\beta f Q \{y\} = \ln Q \{y\} = \lim_{N \to 0} \frac{1}{N} (Q^N \{y\} - 1).$$

In this scheme, we take $N$ identical copies of the original partition function and average $Q^N \{y\}$. The formal limit $N \to 0$ to be performed at final stages. In what follows, we assume the simplest distribution of the impurities in the definition of the averaging:

$$W (y_{mn}) = p \delta (1 - y_{mn}) + (1 - p) \delta (y_{mn}),$$

where $\delta(\ )$ are the correspondent Kronecker’s symbols, $p$ is the probability that any given site, chosen at random, is occupied by the normal Ising spin, while $1 - p$ is the probability that the given site is dilute.

3 Fermionic interpretation

The partition function with fixed disorder can be transformed into a Gaussian fermionic integral following the method of the mirror-ordered factorization for the density ma-

\[\text{\footnote{The rules of the integration over the purely anticommuting (Grassmann) variables were first introduced by F.A. Berezin. In a close analogy with the bosonic case, a Gaussian fermionic integral of any kind can be expressed in terms of the determinant of the associated matrix. For a short comment about the Gaussian fermionic integrals also see [1, 2].}}\]
explicitly illuminates the Majorana-Dirac structures of 2DIM already at the lattice level of the reduced representation like (6) is that the corresponding action, in the pure case, integrals (5) and (6) are completely equivalent to each other and to (2). The advantage results the theory with interaction of the following kind \[ 10 \]:

\[ Q\{y\} = \int \prod_{mn} d\bar{c}_{mn} d\bar{c}_{mn} \exp \sum_{mn} \left[ y_{mn}^{-2} c_{mn} \bar{c}_{mn} + y_{mn}^{2} \left( t_{1}c_{m-1n} + t_{2}c_{mn-1} \right) - y_{mn}^{-2} t_{1}t_{2}c_{m-1n}c_{mn-1} \right] , \tag{6} \]

where \( c_{mn}, \bar{c}_{mn} \) are Grassmann variables, \( y_{mn}^{2} \exp \left( y_{mn}^{-2} c_{mn} \bar{c}_{mn} \right) = y_{mn}^{2} + c_{mn} \bar{c}_{mn} \). The integrals (5) and (6) are completely equivalent to each other and to (2). The advantage of the reduced representation like (6) is that the corresponding action, in the pure case, explicitly illuminates the Majorana-Dirac structures of 2DIM already at the lattice level (10).

The averaging of (6) over the disorder within the \( N \)-replica scheme, see (3) and (4), results the theory with interaction of the following kind (10):

\[ \overline{Q^{N}}\{y\} = \int \prod_{mn}^{N} \prod_{\alpha=1}^{N} d\bar{c}_{mn}^{(\alpha)} d\bar{c}_{mn}^{(\alpha)} \prod_{mn}^{N} \left[ p \prod_{\alpha=1}^{N} e^{c_{mn}^{(\alpha)}} + (1 - p) \prod_{\alpha=1}^{N} c_{mn}^{(\alpha)} \bar{c}_{mn}^{(\alpha)} \right] \tag{7} \]

\[ = pL^{2} \int \prod_{mn}^{N} \prod_{\alpha=1}^{N} d\bar{c}_{mn}^{(\alpha)} d\bar{c}_{mn}^{(\alpha)} \exp \sum_{mn}^{N} \left[ S_{mn}^{(\alpha)} + \frac{1 - p}{p} \prod_{\alpha=1}^{N} c_{mn}^{(\alpha)} \bar{c}_{mn}^{(\alpha)} e^{\Delta_{mn}^{(\alpha)}} \right] , \]

where \( S_{mn}^{(\alpha)} \) is the replicated Gaussian action from (6) for the pure case, with \( y_{mn} \equiv 1 \) at all sites, and \( \Delta_{mn}^{(\alpha)} = -S_{mn}^{(\alpha)} \). Taking into account the nilpotent property of fermions in prefactor before the exponential, another possible choice is: \( \Delta_{mn}^{(\alpha)} = t_{1}t_{2}c_{m-1n}^{(\alpha)} \bar{c}_{mn-1}^{(\alpha)} \) (10). The continuum-limit field theory for weak dilution that follows from (7) is commented in the next section. A relatively simple form of interaction in (7) also provides grounds to try the cases of strong and moderated dilution, starting directly with the exact lattice integral (7). The analysis of this kind, however, have not yet been performed.

4 The Gross-Neveu model \((N \rightarrow 0)\)

To extract the effective continuum-limit field theory from (7), we have to distinguish explicitly the higher- and low-momentum lattice fermionic modes (Fourier-harmonics) in the exact lattice action (7). Integrating out the higher-momentum modes in the first...
order of perturbation theory (weak dilution), we obtain an effective action for the low-
momentum fields (large distances). The effective action appears, finally, in the form of
the \( N \)-colored Gross-Neveu model \((N \rightarrow 0)\) [10]:

\[
S_{G-N} = \int d^2 x \left\{ \sum_{\alpha=1}^{N} \left[ \overline{m}_N \psi_1^{(\alpha)} \psi_2^{(\alpha)} + \frac{1}{2} \psi_1^{(\alpha)} (\partial_1 + i \partial_2) \psi_1^{(\alpha)} \right] \right\},
\]

where \( \psi_1, \psi_2 \) are the anticommuting Majorana components, \( \overline{m}_N \) and \( g_N \) are the effective
mass and charge, respectively. Here \( \langle A \rangle \) and \( \langle B \rangle \) are some lattice fermionic averages
(short distances) explicitly calculated in [10]. The Gaussian part in (8) is the replicated
Majorana action, corresponding to the pure case, with the mass term modified by disorder.
The \( N = 0 \) G-N model similar to (8) already has been analyzed intensively by DD-SSL
as an effective theory near \( T_c \) for weak bond dilution [3, 4, 5, 13]. The predictions are
the double-logarithmic singularity in the specific heat and the logarithmic corrections in
other thermodynamic functions. For more details see [4, 5, 14, 15]. The present analysis
thus confirms the idea of the universality for small amount of impurities in the 2D Ising
ferromagnets, for weak dilution. The situation is less evident, however, for strong and
moderated site and/or bond dilution [6, 14, 15].

5 Added note

Let \( f(c_1, c_2, ..., c_N) \) be any function of Grassmann variables \( c_1, c_2, ..., c_N \). From the basic
rules of fermionic integration, it follows:

\[
\int dc_N ... dc_2 dc_1 f(\lambda_1 c_1, \lambda_2 c_2, ..., \lambda_N c_N) = \lambda_1 \lambda_2 ... \lambda_N \int dc_N ... dc_2 dc_1 f(c_1, c_2, ..., c_N).
\]

Infinite series of related identities also follows by differentiating (9) with respect to
the parameters \( \lambda_1, \lambda_2, ..., \lambda_N \). In particular, applying these ideas to the integral (7), we obtain
the identity:

\[
\frac{1}{L^2} \sum_{mn} \left\langle S_{mn}^{(\alpha)} + \frac{1-p}{p} e^{\Delta_{mn}^{(\alpha)}} \prod_{\beta=1}^{N} \epsilon_{mn}^{(\beta)} e^{\Delta_{mn}^{(\beta)}} \right\rangle = 1,
\]

where the averaging is assumed to be taken with respect to the non-Gaussian fermionic
measure from (6). This identity can be used to check the consistency of approximations
of any kind when dealing with lattice theory (7). In a sense, the equation (10) can be
viewed as the analog of \( \langle p^2/2m \rangle = \theta/2, \theta = kT \), for the kinetic energy of a particle in
classic Maxwell gas (one degree of freedom). In any case, the latter identity readily follows
by exactly the same method.
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