Stable Skyrmions in two-component Bose-Einstein condensates

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We show that stable Skyrmions exist in two-component atomic Bose-Einstein condensates, in the regime of phase separation. Using full three-dimensional simulations we find the stable Skyrmions with topological charges \( Q = 1 \) and \( 2 \), and compute their properties. With reference to these computations we suggest the salient features of an experimental setup in which they might realized.

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Experimental advances in the formation and control of ultra-cold atomic gases are allowing detailed studies of the properties of Bose–Einstein condensates (BECs) containing multiple components. Multi-component condensates have been formed by the simultaneous trapping and cooling of atoms in distinct hyperfine or spin levels \([1,2]\), and there are prospects of condensed mixtures of different bosonic atomic species \([3,4]\). The extra internal degrees of freedom introduced by the multiple components lead to a much richer phenomenology than in the single component case.

One key feature of multi-component BECs is that, in general, their condensed groundstates are described by spinor order parameters \([5,6]\). This raises the possibility that these systems could support novel topological defects and textures. The existence of topological vortices \([7,8]\) and monopoles \([9]\), in which the topology is imposed by boundary conditions, has been noted. However, no localised topological textures, whose structure and topology is fixed only by simple energetic stability, have been identified. In Ref. \([10]\) it was shown that a configuration with the topology of a “Skyrmion” \([1,2]\) (a topological soliton of the \( S^3 \rightarrow S^3 \) map \([13]\)) can be imprinted in a two-component Bose-Einstein condensate, using a carefully designed sequence of Rabi transitions. The subsequent time evolution indicated that this object is energetically unstable. In related work \([14]\), it was demonstrated that a spherically-symmetric Skyrmion in a 2- or 3-component ferromagnetic BEC is also unstable to collapse. In Refs. \([11,14]\) a particular choice of interaction coefficients was considered. However, the general multi-component system is described by all mutual two-body scattering lengths \([5]\). Given the range of parameter space available it is natural to ask: Are there any conditions under which Skyrmions are stable?

In this Letter, we present the results of extensive numerical studies of a fully 3-dimensional system that show that stable Skyrmions do exist in two-component BECs \([17]\). We find that stable Skyrmions exist under the condition that phase separation occurs. We therefore expect Skyrmions to be stable in the experimentally relevant \(^{87}\)Rb \( |F = 2, m_F = 1 \rangle \) \( \{1, -1\} \) system \([2]\). As we will show, the Skyrmion can be viewed as a quantised vortex ring in one component close to whose core is confined the second component carrying quantised circulation around the ring. As such, the configurations closely resemble the “cosmic vortons” \([14]\) which may have formed in the early universe from superconducting cosmic strings \([17]\) providing a further interesting link with between low-temperature laboratory experiments and cosmology (see, for example, Ref. \([18]\)).

We consider a two-component system, in which the number of atoms of each component is separately conserved on the timescale of interest, and assume that the many-body wavefunction can be written as a simple condensate \([5]\)

\[
\Psi(\{r_1, r_2, \ldots\}) = \prod_{i=1}^{N_1 + N_2} [\psi_1(r_i)|1\rangle + \psi_2(r_i)|2\rangle] ,
\]

where the states \( |\alpha = 1, 2\rangle \) are the eigenstates of the two components (for simplicity of presentation, we take these to be two hyperfine levels). We are interested in stable stationary states, which are minima of the total energy

\[
E = \int d^3r \left[ \sum_{\alpha} \frac{\hbar^2}{2m} |\nabla \psi_\alpha|^2 + V_\alpha(r)|\psi_\alpha|^2 \right. \\
\left. + \frac{1}{2} \sum_{\alpha, \beta} U_{\alpha \beta} |\psi_\alpha|^2 |\psi_\beta|^2 \right] ,
\]

where the matrix \( U_{\alpha \beta} \) is determined by all mutual s-wave scattering amplitudes \([8]\). Since the number of atoms of each component is conserved, the minimisation is performed with constraints on

\[
N_\alpha = \int d^3r |\psi_\alpha|^2
\]

that is, having separate chemical potentials \( \mu_1 \neq \mu_2 \).
The parameters entering (2) allow many different regimes. We will study solutions confined to a region close to the centre of a large trap that is loaded with component-\([1]\). We may then neglect the effects of the confining potentials, and consider an infinite uniform system with boundary condition

\[
\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \bigg|_{r \to \infty} = \left( \begin{array}{c} \sqrt{\rho_0} \\ 0 \end{array} \right),
\]

where \(\rho_0\) is the density at the centre of the trap. Furthermore, we concentrate on situations in which the interactions are repulsive, and only weakly dependent on species \(U_{11} \sim U_{12} \sim U_{22}\). In this case, the stationary solutions vary on scales much larger than the healing length \(\xi \equiv (2\pi \rho_0 \hbar^2)^{-1/2}\). Consequently, we can neglect variations in the total density \(\rho(\mathbf{r})\), and write

\[
\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \sqrt{\rho_0} \begin{pmatrix} \cos(\theta/2)e^{i\phi_1} \\ \sin(\theta/2)e^{i\phi_2} \end{pmatrix},
\]

in which we choose a parameterisation of the remaining freedom in terms of \(\theta \in [0, \pi]; \phi_1 \in [0, 2\pi]\). The field (3) and boundary condition (4) allow a topological classification of (non-singular) field configurations, in terms of the winding number for the map \(S^3 \to S^3\)

\[
Q = \frac{1}{8\pi^2} \int \sin \theta \nabla_i \theta \nabla_j \phi_1 \nabla_k \phi_2 \epsilon_{ijk} d^3r.
\]

We search for topological solitons by minimising, within each topological subspace \(Q\), the energy functional

\[
E - E_0 = \int d^3r \left\{ \frac{\hbar^2 \rho_0}{2m} \left[ \frac{1}{4} \nabla \theta^2 + \cos^2(\theta/2)\nabla \phi_1^2 \right] \right. \\
+ \sin^2(\theta/2)\nabla \phi_2^2 + \Delta \sin \theta \right\},
\]

where \(\Delta \equiv \frac{1}{8\rho_0^2} [2U_{12} - U_{11} - U_{22}]\) with a constraint on

\[
N_2 = \frac{1}{2} \int d^3r (1 - \cos \theta).
\]

The constraint on \(N_1\) is implicit in the choice of \(\rho_0\). In \(Q\) the zero of the energy \(E_0\) is a constant which depends on \(U_{11}, U_{22}\) and the constrained numbers \(N_1\) and \(N_2\).

There are two important length-scales in the problem

\[
R_2 \equiv \left( \frac{N_2}{\rho_0} \right)^{1/3}; \quad \xi_\Delta \equiv \sqrt{\frac{\hbar^2 \rho_0}{2m \Delta}}.
\]

\(R_2\) may be interpreted as the typical size of a cloud of component-\([2]\); \(\xi_\Delta\) may be interpreted as the width of the transition region separating regions of component \([1]\) and \([2]\). Our neglect of the confining potential is justified provided \(\max(R_2, \xi_\Delta)\) is small compared to the overall size of the cloud; the restriction to constant density (3) is justified for \(\xi_\Delta \gg \xi_\Delta\), that is, \(|\Delta| \ll \bar{U}\). The dimensionless ratio \(\eta \equiv (R_2/\xi_\Delta)\), which is a measure of the amount of component-\([2]\), is the parameter that characterises the solutions of (5), with the energies taking the form

\[
E_Q(\Delta, N_2) = \left( \frac{\hbar^2 \rho_0 R_2}{m} \right) \mathcal{E}_Q(\eta),
\]

where \(\mathcal{E}_Q(\eta)\) is a dimensionless function.

Consider first non-topological solutions \((Q = 0)\). For the limit, \(\nabla \phi_1 = \nabla \phi_2 = 0\), the problem reduces to a 3D double sine-Gordon model, which is known to have soliton solutions for \(\Delta > 0\) and \(\eta > 1.92\) [10]. For large \(N_2\), the soliton can be viewed as a spherical domain of component-\([2]\) with radius \(\sim R_2\), separated from the region of component \([1]\), by a domain wall of width \(\sim \xi_\Delta\). Therefore, it describes a spherical inclusion of component-\([2]\) arising from phase-separation.
ing (8) using a Lagrange multiplier on a discretized grid with $100^3$ points. Starting from a wide range of different initial conditions without any particular symmetry, but with the same value of $Q$, the kinetic energy was periodically removed so as to prevent oscillatory motion and effectively simulate gradient flow. This technique had previously been applied successfully to Skyrmions in the original model of Skyrme [20] and also to the closely related Hopf solitons [21]. The solutions evolved quickly to the required value of $N_2$ and the subsequent relaxation was achieved after an acceptable period of time. As a simple test of the procedure we were able to reproduce quantitatively the results of Ref. [11] for $Q = 0$.

The results for $Q = 1, 2$ are presented in Fig. 1 for the phase separation regime, $\Delta > 0$, illustrating the existence of stable topological solitons. We have plotted 3D iso-surfaces of the topological charge density and the number density of species $2$ (the integrands of $\Phi$ and $\Theta$ respectively). Both illustrate the axial symmetry of the solutions: for this precise value of the threshold the topological charge density appears to be mostly localized around a central line in the shape of a “bolt”, although for a different threshold the figure would have appeared to be a vastly different since there is also some topological charge associated with the thin ($\sim \xi_{\Delta}$) axially symmetric domain wall. The number density is clearly in the form of an axially symmetric ring encircling the “bolt” of topological charge density.

Slices through the energy density (the integrand of (8)) and the number density in the plane perpendicular to the ring illustrate the precise details of the solutions — in particular the two length-scales $R_2$ and $\xi_{\Delta}$. The energy density, whose morphology is very similar to that of the topological charge density, is largely localized along the line through the centre of the vortex ring. However, one can clearly see that, emanating from this line, there is a shell of energy associated with the domain wall. The corresponding slice through the number density reaffirms this: it is almost zero away from the ring and rises sharply to close to one in the interior. An alternative view of the solution is a slice perpendicular to the line through the centre of the ring.

In each case, the configuration can be viewed as a (unit) quantised vortex ring in component-1, close to whose core is confined a circulating ring of component-2 with quantised circulation (1 or 2 units, for $Q = 1, 2$). The kinetic energy of the vortex ring and the surface tension of the domain wall separating the two components energetically favour shrinking the ring to zero size. This is balanced by the kinetic energy of the circulating core, allowing for stable configurations of non-zero radius. In this respect our solutions have similar qualitative form to the unstable configurations discussed in Ref. [1] for $\Delta = 0$. Clearly, it is phase separation, induced when $\Delta > 0$ that allows stabilization of these solutions.

Our 3D numerical studies indicate that the stable Skyrmions with $Q = 1, 2$ are cylindrically symmetric. To determine accurate values for the energies of these configurations, we have performed simulations using a cylindrical ansatz consistent with the 3D results (and consistent with the variational equation for $\phi_2$): $\theta(\rho, z), \phi_1(\rho, z), \phi_2 = m \chi$ ([$(\rho, \chi, z)$ are cylindrical polar co-ordinates]). When $m \neq 0$, the remaining 2D problem for $\theta(\rho, z), \phi_1(\rho, z)$ has finite energy solutions provided $\theta = 0$ on $\rho = 0$ and at $\rho^2 + z^2 \to \infty$; this allows the configurations to be characterised by the winding number, $n$, of $\phi_1$ around the half-plane. Thus, there is a topological classification of the cylindrical ansatz in terms of the pair of integers $(m, n)$, which describe the circulation of the vortex ring in component-1 ($n$) and the circulation of component-2 in the core ($m$). The winding number for the $S^3 \to S^3$ map (8) is $Q = nm$.

We have determined the soliton energies from numerical minimisations using a standard conjugant gradient routine. The results agree, to within numerical accuracy, with those of the 3D simulations. For $m = 0$ we recover the non-topological soliton. We find stable topological solitons for all $m > 0$, but only for $|n| = 1$. The energies of the solitons $E_{m,n=1}(\eta)$ are plotted in Fig. 2. For $Q \geq 3$ (up to the largest we have studied) we find stable topological solitons within the cylindrical ansatz. These solutions may not be stable to non-axisymmetric perturbations which is currently under investigation.

![Fig. 2. Scaled energies $E_{m,n=1}$ of Skyrmions as a function of $\eta$. The circulation of the vortex ring is $n = 1$; the circulation of the core is $m = 1 \ldots 7$ (from bottom to top), such that the topological charge is $Q = m$. The lines terminate below some critical value of $\eta$ at which the solitons become unstable. Results are from calculations using the cylindrical ansatz on a $100 \times 200$ grid with $R_2 = 80$ lattice spacings.](image)

As described above, the Skyrmions can be viewed as generalisations of quantised vortex rings (for example, in superfluid $^4$He) to a two-component condensate.
One distinction we wish to emphasise is that the stable Skyrmions of Figs. 1, 2 do not move in space, whereas conventional vortex rings move at constant (non-zero) velocity. We can find Skyrmion configurations that do propagate, by adding a constraint to fix the impulse

$$P_i = \frac{\hbar}{2\pi} \int d^3r [j_i \nabla_i \bar{\psi}_\alpha \nabla_j \psi_\alpha - r_j \nabla_j \bar{\psi}_\alpha \nabla_i \psi_\alpha]$$

(11)

which is conserved by the time-dependent Gross-Pitaevskii equation. Under these (dissipationless) dynamics, one can show that the configuration that minimizes the energy at fixed $P$ translates in space at a constant velocity $v = \partial E/\partial P$, where $E$ is its energy. Dimensional analysis allows the energy of a state with constraints on $Q$, $N_2$ and $P$ to be written as $E_Q(\eta, p)$ in Eq. (11), where the new dimensionless parameter is $p = |P|/(\hbar v_0^{1/3} N_2^{2/3})$.

Figure 1 (c,d) shows the distribution of component-[-2] that would be observed experimentally for these configurations. Clearly, the essential condition for the formation of stable Skyrmions is that there is phase-separation between the two components. We therefore anticipate stable Skyrmions to exist for small deviations from the limit $\Delta \ll \bar{U}$ studied. Phase separation occurs provided $U_{12}^2 > U_{11} U_{22}$ (equivalent to $\Delta > 0$ for $\Delta \ll \bar{U}$). The observation of phase separation in the system $^{87}$Rb $\{F = 2, m_F = 1\} \{1, -1\} [3]$, makes this a candidate system in which the stable Skyrmions may be realised.

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