Disjoint hypercyclicity along filters

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Abstract

We extend a result of Bès, Martin, Peris and Shkarin by stating: $B_w$ is $\mathcal{F}$-weighted backward shift if and only if $(B_w, \ldots, B_r^w)$ is $d$-$\mathcal{F}$, for any $r \in \mathbb{N}$, where $\mathcal{F}$ runs along some filters containing strictly the family of cofinite sets, which are frequently used in Ramsey theory. On the other hand, we point out that this phenomenon does not occur beyond the weighted shift frame by showing a mixing linear operator $T$ on a Hilbert space such that the tuple $(T, T^2)$ is not $d$-syndetic.

We also investigate the relationship between reiteratively hypercyclic operators and $d$-$\mathcal{F}$ tuples, for filters $\mathcal{F}$ contained in the family of syndetic sets. Finally, we examine conditions to impose in order to get reiterative hypercyclicity from syndeticity in the weighted shift frame.

KEYWORDS: hypercyclic operators, disjoint hypercyclic operators, weighted backward shift

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1 Introduction

Let $X$ be a separable topological vector space, denote the set of bounded and linear operators on $X$ by $\mathcal{L}(X)$. An operator $T \in \mathcal{L}(X)$ is called hypercyclic provided there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geq 0\}$ is dense in $X$ and $x$ is called a hypercyclic vector for $T$. Hypercyclic operators are one of the most studied objects in Linear Dynamics, see [18] [3]

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for further information concerning concepts, results and a detailed account on this subject.

The aim of this paper is to analyze disjoint hypercyclicity along those filters mostly studied in Ramsey theory. The notion of disjoint hypercyclicity, a strengthening of hypercyclicity, concerning a tuple of linear operators, was introduced independently by Bernal [9] and by Bès and Peris [15] in 2007.

For any integer $N \geq 2$, the tuple $(T_1, \ldots, T_N)$ of hypercyclic operators, acting on the same topological vector space $X$ is said disjoint hypercyclic, or $d$-hypercyclic for short, provided there exists $(z, \ldots, z)$ in $X^N$ such that

$$\{(T_1^nz, \ldots, T_N^nz) : n \in \mathbb{Z}_+\}$$

is dense in $X^N$. Such a vector $z$ is called a $d$-hypercyclic vector for the tuple $(T_1, \ldots, T_N)$.

Bernal [9] gave sufficient conditions for $d$-hypercyclicity in the setting of $F$-spaces, which allowed him to provide examples of $d$-hypercyclic differential operators or composition operators on spaces of analytic functions. On the other hand, Bès and Peris [15] presented, among other results, a characterization of the so-called Hypercyclicity Criterion, which they call $d$-Hypercyclicity Criterion and proved its connection with hereditary $d$-hypercyclicity. Moreover, they provided a characterization of $d$-hypercyclic tuples of different powers of weighted shifts on $l_2$, showing that these tuples satisfy the $d$-Hypercyclicity Criterion.

Bès, Martin and Sanders [16] extended Salas’s characterization of hypercyclic shifts operators to the $d$-hypercyclicity setting. In fact, they gave a characterization of weighted backward shifts that are $d$-hypercyclic in terms of their weight sequences. As a consequence, they pointed out that weakly mixing property and the Blow-up/Collapse property are no longer equivalent, as happened in the case of a single hypercyclic operator. They showed that any $d$-hypercyclic tuple of weighted shift operators satisfies the Disjoint Blow-up/Collapse property but is never $d$-weakly mixing, and consequently it never meets the $d$-Hypercyclicity Criterion.

Bès, Martin, Peris and Shkarin [12] showed the following: if $T$ is linear and bounded operator on a topological vector space $X$ satisfying the Original Kitai Criterion, then the tuple $(T, \ldots, T^r)$ is $d$-mixing, for each $r \in \mathbb{N}$. As a consequence, any bilateral weighted shift $T$ on $l_p(\mathbb{Z}), (1 \leq p < \infty)$ or $c_0(\mathbb{Z})$ is mixing if and only if $(T, \ldots, T^r)$ is $d$-mixing, for each $r \in \mathbb{N}$. Nevertheless, they remarked that this phenomenon does not occur beyond the weighted
shift setting, by exhibiting an example of a mixing Hilbert space operator $T$ so that $(T, T^2)$ is not $d$-mixing.

The paper is structured as follows: In Section 2 we begin by studying conditions to impose on the weights in the search of a characterization of any tuple of different powers of a fixed weighted shift that is $d$-$\mathcal{F}$, with $\mathcal{F}$ a filter. As a consequence, and using a result of Ramsey theory about the preservation of certain notions of largeness in products we obtain the following: any weighted shift $T$ on $l_p$, $(1 \leq p < \infty)$ or $c_0$ is $d$-$\mathcal{F}$-operator if and only if $(T, \ldots, T^r)$ is $d$-$\mathcal{F}$, for each $r \in \mathbb{N}$, where $\mathcal{F}$ is a filter strictly containing the family of cofinite sets. Then, in Section 3, it can be seen that this fails beyond the setting of weighted shifts, since it exists a mixing linear operator $T$ on a Hilbert space such that the tuple $(T, T^2)$ is not $d$-syndetic. In Section 4, we investigate the interplay between reiteratively hypercyclic operators and $d$-$F$ tuples. Finally, it is known that reiteratively hypercyclic operators are syndetic, but the converse does not hold in general [13], then we examine conditions to impose in order to get reiteratively hypercyclicity from syndeticity in the weighted shift setting.

1.1 Preliminaries and main results

Special attention requires the study of sets of recurrence of the form $N(U, V) = \{n \in \mathbb{N} : T^{-n}(U) \cap V \neq \emptyset\}$ for any $U, V$ non-empty open sets of $X$ (open set for short). An operator $T$ is topologically transitive if $N(U, V) \neq \emptyset$ for any pair of open sets $U, V$. For second countable Fréchet spaces hypercyclicity coincides with topologically transitivity, by Birkhoff’s transitivity theorem, see theorem 1.2 [3].

Let $N$ sequences $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty \in \mathcal{L}(X)$, they are $d$-universal if $\{(T_{1,j}z, \ldots, T_{N,j}z) : j \in \mathbb{N}\}$ is dense in $X^N$ for some vector $z \in X$.

**Definition 1.1.** The $N$-tuple of operators $(T_1, \ldots, T_N)$ acting on $X$ is said disjoint transitive (d-transitive for short) if for any $N + 1$-tuple $(U_i)_{i=0}^N$ of open sets it holds

$$N_{T_1, \ldots, T_N}(U_1, \ldots, U_N; U_0) := \{n \in \mathbb{N} : T_1^{-n}U_1 \cap \cdots \cap T_N^{-n}U_N \cap U_0 \neq \emptyset\} \neq \emptyset.$$

If $T_i = T^i$, for $i = \{1, \ldots, N\}$, then we write $N_T$ instead of $N_{T_1, \ldots, T_N}$.

For recent results on disjoint hypercyclicity, see [10], [11], [12], [16], [25] and [28].
On the other hand, in [13] the authors introduce another refinement of the notion of hypercyclicity, relative to the set \( N(U, V) \) when belonging to a certain collection \( \mathcal{F} \) of subsets of \( \mathbb{N} \), namely an operator \( T \in \mathcal{L}(X) \) is called \( \mathcal{F} \)-operator if \( N(U, V) \in \mathcal{F} \), for any pair of open sets \( U, V \) in \( X \). In [13] can be found an analysis of the hierarchy established between \( \mathcal{F} \)-operators, whenever \( \mathcal{F} \) covers those families mostly studied in Ramsey theory.

Let \( \mathcal{F} \) be a set of subsets of \( \mathbb{N} \), we say that \( \mathcal{F} \) is a family provided (I.) \( |A| = \infty \) for any \( A \in \mathcal{F} \), where \( |A| \) stands for cardinality of \( A \) and (II.) \( A \subset B \) implies \( B \in \mathcal{F} \), for any \( A \in \mathcal{F} \).

A family \( \mathcal{F} \) is a filter if it is invariant by finite intersections, i.e. \( \mathcal{F} \) is a family such that for any \( A \in \mathcal{F}, B \in \mathcal{F} \) implies \( A \cap B \in \mathcal{F} \).

Now, in a natural way we can introduce the notion of \( \mathcal{F} \)-disjoint transitivity (or \( d-\mathcal{F} \) for short).

**Definition 1.2.** The tuple of sequence of operators \((T_{1,n_k}, \ldots, T_{N,n_k})_k\) is said to be \( d-\mathcal{F} \) if for any \( N+1 \)-tuple \((U_i)_{i=0}^N\) of open sets we have
\[
\{ k \in \mathbb{N} : T_{1,n_k}^{-1}(U_1) \cap \cdots \cap T_{N,n_k}^{-1}(U_N) \cap U_0 \neq \emptyset \} \in \mathcal{F}.
\]

In particular, when \( T_{i,n_k} = T_k^i \), for any \( k \in \mathbb{N}, 1 \leq i \leq N \) in the above definition, then the \( N+1 \)-tuple of operators \((T_1, \ldots, T_N)\) is said to be \( d-\mathcal{F} \).

Let us summarize some families commonly used in Ramsey theory. For a rich source on this subject, see [20]. Denote by \( \mathcal{F}^* = \{ A \subseteq \mathbb{N} : A \cap B \neq \emptyset, \forall B \in \mathcal{F} \} \) the dual family of \( \mathcal{F} \).

- \( J = \{ A \subseteq \mathbb{N} : |A| = \infty \} \)
- \( \Delta = \{ A \subseteq \mathbb{N} : B - B \subseteq A, \text{for some } B \} \)
- \( \mathcal{F}^\mathcal{P} = \{ A \subseteq \mathbb{N} : \exists (x_n)_n \subseteq \mathbb{N}, \sum_{n \in F} x_n \in A, \text{for any } F \in P_f(\mathbb{N}) \} \), where \( P_f(\mathbb{N}) = \{ A \subseteq \mathbb{N} : |A| < \infty \} \)
- \( A \) is syndetic set (\( A \in \mathcal{S} \) for short) if \( A \) has bounded gaps, i.e if \( A \) can be enumerated increasingly as \((x_n)_n = A\), then \( \max_n x_{n+1} - x_n < \infty \)
- \( A \) is thick set (\( A \in \mathcal{T} \) for short) if \( A \) contains arbitrarily long intervals, i.e \( \mathcal{T} = \{ A \subseteq \mathbb{N} : \forall L > 0, \exists n : \{ n, n+1, \ldots, n+L \} \subset A \} \)
- \( A \) is piecewise syndetic set (\( A \in \mathcal{PS} \) for short) if \( A \) can be written as the intersection of a thick and a syndetic set.
Now, $\mathcal{F}$ (family of cofinite sets), $\Delta^*$, $\mathcal{P}^*$ and $\mathcal{P}^*$ are filters. $\mathcal{I}^* \subseteq \Delta^* \subseteq \mathcal{P}^* \subseteq S$ and $\mathcal{I}^* \subseteq \mathcal{P}^* \subseteq S$. Concerning the hierarchy between $\mathcal{F}$-operators, see \cite{13}.

In linear dynamics recurrence properties have been frequently studied first in the context of weighted shifts.

Each bilateral bounded weight $w = (w_k)_{k \in \mathbb{Z}}$, induces a \textit{bilateral weighted backward shift} $B_w$ on $X = c_0(\mathbb{Z})$ or $l_p(\mathbb{Z})(1 \leq p < \infty)$, given by $B_w e_k := w_k e_{k-1}$, where $(e_k)_{k \in \mathbb{Z}}$ denotes the canonical basis of $X$ and $(e^*_k)_{k \in \mathbb{Z}}$ the associated sequence of coordinate functionals.

The following is the main result of Section 2, which has as main ingredients a characterization of any tuple of different powers of weighted shifts to be $d-$ $\mathcal{F}$ in term of their weight sequences, with $\mathcal{F}$ a filter, together with a result of Ramsey theory about the preservation of certain notions of largeness in products.

**Theorem 1.3.** Let $\mathcal{F} = \Delta^*, \mathcal{P}^*, \mathcal{P}^*$. Then for any $r \in \mathbb{N}$, the following are equivalent:

(i) $B_w$ is $\mathcal{F}$-operator

(ii) for any $M > 0$, $j \in \mathbb{Z}$ and $t = 1, \ldots, r$

\[
\left\{ m \in \mathbb{N} : \prod_{i=j+1}^{j+tm} |w_i| > M \right\} \in \mathcal{F}, \quad \left\{ m \in \mathbb{N} : \frac{1}{\prod_{i=j-tm+1}^{j} |w_i|} > M \right\} \in \mathcal{F}
\]

(iii) $(B_w, \ldots, B^{r}_w)$ is $d-$ $\mathcal{F}$

(iv) $B_w \oplus \cdots \oplus B^{r}_w$ is $\mathcal{F}$-operator on $X^r$.

Recall that $\mathcal{I}^*$-operators are commonly known as mixing operators. Obviously, mixing operators are $\Delta^*$-operators, but the converse is not true as exhibited in \cite{13} and the example is a weighted shift. Therefore, the conclusion of theorem \cite{13} does not necessarily follows from the statement: $B_w$ is mixing if and only if $(B_w, \ldots, B^{r}_w)$ is $d$-mixing, for any $r \in \mathbb{N}$, shown in \cite{12}.

A similar result holds for unilateral weighted shifts. Each unilateral bounded weight $w = (w_n)_{n \in \mathbb{Z}_+}$ induces a \textit{unilateral weighted backward shift} $B_w$ on $X = c_0(\mathbb{Z}_+)$ or $l_p(\mathbb{Z}_+)(1 \leq p < \infty)$, given by $B_w e_n := w_n e_{n-1}, n \geq 1$ with $B_w e_0 := 0$, where $(e_n)_{n \in \mathbb{Z}_+}$ denotes the canonical basis of $X$ and $(e^*_n)_{n \in \mathbb{Z}_+}$ the associated sequence of coordinate functionals.

**Theorem 1.4.** Let $\mathcal{F} = \Delta^*, \mathcal{P}^*, \mathcal{P}^*$. Then for any $r \in \mathbb{N}$, the following are equivalent:
(i) $B_w$ is $\mathcal{F}$-operator
(ii) for any $M > 0$, $j \in \mathbb{Z}_+$, $t = 1, \ldots, r$
\[ \left\{ m \in \mathbb{N} : \prod_{i=j+1}^{j+tm} |w_i| > M \right\} \in \mathcal{F} \]
(iii) $(B_w, \ldots, B_r^w)$ is $d$-$\mathcal{F}$
(iv) $B_w \oplus \cdots \oplus B_r^w$ is $\mathcal{F}$-operator on $X^r$.

The family of syndetic sets is not a filter, consider $2\mathbb{N}$ and $2\mathbb{N} + 1$, they are syndetic sets but disjoint. However, $S$-operators behaves as filter operators, since syndetic operators coincide with $PS^*$-operators \cite{13}, then as expected $B_w$ is syndetic weighted backward shift if and only if $(B_w, \ldots, B_r^w)$ is $d$-$S$, for any $r \in \mathbb{N}$. For the remaining non-filter families such as $\Delta, \mathcal{I} \mathcal{P}, \mathcal{P} \mathcal{S}$ and $\mathcal{I}$, a result in the vein of theorem \cite{14} does not hold.

In \cite{12}, the authors show that there exists a mixing operator $T$ on a Hilbert space such that $(T, T^2)$ is not $d$-mixing. With a slight modification in the construction of the operator $T$, we show a little more, i.e.

Theorem 1.5. There exists $T \in \mathcal{L}(l_2)$ such that $T$ is mixing and the sequence of operators $(2T^n - T^{2n})_{n \in \mathbb{N}}$ is not $S$-transitive. In particular, $(T, T^2)$ is not $d$-syndetic. Consequently, $T$ is $\mathcal{F}$-operator but $(T, T^2)$ is not $d$-$\mathcal{F}$, where $\mathcal{F} = \Delta^*, \mathcal{I} \mathcal{P}^*, \mathcal{P} \mathcal{S}^*, S$.

Consequently, theorem \cite{13} (respectively, theorem \cite{14}) does not hold beyond the setting of weighted backward shifts, and this is the content of section 3.

Now, let us recall the different notions of density in $\mathbb{N}$. Let $A \subseteq \mathbb{N}$.

The asymptotic density: the upper and lower asymptotic density are defined respectively by
\[ \overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} \]
\[ \underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} \]
Set $\overline{\mathcal{D}} = \{A \subseteq \mathbb{N} : \overline{d}(A) > 0\}$ and $\underline{\mathcal{D}} = \{A \subseteq \mathbb{N} : \underline{d}(A) > 0\}$.
The Banach density: for every real number $s \geq 1$, define $\alpha^s = \limsup_{k \to \infty} |A \cap [k+1, k+s]|$ and $\alpha_s = \liminf_{k \to \infty} |A \cap [k+1, k+s]|$.

Now, the upper and lower Banach density are defined respectively by

$$\overline{Bd}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s},$$

$$\underline{Bd}(A) = \lim_{s \to \infty} \frac{\alpha_s}{s}.$$ 

Set $\overline{BD} = \{ A \subseteq \mathbb{N} : \overline{Bd}(A) > 0 \}$ and $\underline{BD} = \{ A \subseteq \mathbb{N} : \underline{Bd}(A) > 0 \}$.

Definition 1.6. Let $T \in \mathcal{L}(X)$ and suppose there exists $x \in X$ such that for every open set $U$, it holds

- $d(N(x, U)) > 0$, then it is said that $T$ is frequently hypercyclic
- $\overline{d}(N(x, U)) > 0$, then it is said that $T$ is $\mathcal{U}$-frequently hypercyclic
- $\overline{Bd}(N(x, U)) > 0$, then it is said that $T$ is reiteratively hypercyclic.

Reiteratively hypercyclic operators were introduced by Peris in [22], $\mathcal{U}$-frequently hypercyclic operators were introduced by Shkarin [29] while frequently hypercyclic operators were introduced by Bayart and Grivaux [2].

Obviously, frequently hypercyclic operators are $\mathcal{U}$-frequently hypercyclic, $\mathcal{U}$-frequently hypercyclic are reiteratively hypercyclic, and these in turn are syndetic operators. More on the hierarchy between these operators can be found in [3], [6], [14].

In section 4, we investigate the interplay between reiteratively hypercyclicity and $d$-$\mathcal{F}$ tuples. The main result of the section is the following:

Theorem 1.7. Let $T$ bounded and linear operator on a topological vector space $X$. If $T$ is reiteratively hypercyclic then $(T, \ldots, T^r)$ is $d$-syndetic or not $d$-transitive, for any $r \in \mathbb{N}$.

The main ingredient of the proof is a result of Bergelson and McCutcheon concerning essential idempotents of $\beta\mathbb{N}$ (the Stone-Čech compactification of $\mathbb{N}$), and Szemerédi’s theorem for generalized polynomials [8].
2 Tuple of powers of weighted shifts

In this section we characterize \( F \)-disjointness for tuples of power of a fixed weighted shift on \( c_0 \) or \( l_p, 1 \leq p < \infty \), where \( F \) is a filter on \( \mathbb{N} \) and prove theorem 1.3 (respectively, theorem 1.4).

Analogous to the \( d \)-Hypercyclicity Criterion for tuples of linear operators introduced in [15], we define the \( d \)-\( F \) Hypercyclicity Criterion, for \( F \) a family on \( \mathbb{N} \). Let \( X \) be a separable infinite dimensional Fréchet space. Let us recall the definition of limits along a family.

**Definition 2.1** (\( F \)-limit).

\[ F - \lim_n T^n(x) := y \]

if and only if for every open neighborhood \( V \) of \( y \),

\[ \{ n \in \mathbb{N} : T^n(x) \in V \} \in F. \]

**Definition 2.2** (\( d \)-\( F \) Hypercyclicity Criterion).

Let \((n_k)_k\) be a strictly increasing sequence of positive integers and \( F \) a family on \( \mathbb{N} \). We say that \((T_1, \ldots, T_N)\) a tuple of operators in \( \mathcal{L}(X) \) satisfies \( d \)-\( F \) Hypercyclicity Criterion with respect to \((n_k)_k\) provided there exists dense subsets \( X_0, \ldots, X_N \) of \( X \) and mappings \( S_{l,k} : X_l \rightarrow X, (1 \leq l \leq N, k \in \mathbb{N}) \) satisfying

(i) \( F - \lim_{k \rightarrow \infty} T_{l,k}^{n_k}(x) = 0, \) for any \( x \in X_0 \)

(ii) \( F - \lim_{k \rightarrow \infty} S_{l,k}(x) = 0, \) for any \( x \in X_l, (1 \leq l \leq N) \)

(iii) \( F - \lim_{k \rightarrow \infty} (T_{l,k}^{n_k} S_{l,k} - \delta_{i,l} I d_{X_l}) x = 0, \) for any \( x \in X_l \) \((1 \leq i, l \leq N)\).

We say that \((T_1, \ldots, T_N)\) satisfies the \( d \)-\( F \) Hypercyclicity Criterion if there exists some sequence \((n_k)_k\) for which (i)-(iii) is satisfied.

When \( X_s = X_l, \) for \( 0 \leq s, l \leq N \) and \((n_k) = (k)\) then we say that \((T_1, \ldots, T_N)\) satisfies the Original Kitai \( d \)-\( F \) Criterion.

Connection between \( d \)-\( F \) Hypercyclicity Criterion and \( d \)-\( F \) tuples is the content of the following proposition that follows exactly the same sketch of proof of proposition 2.6 [15].
 Proposition 2.3. Let $\mathcal{F}$ a filter on $\mathbb{N}$. If $(T_1, \ldots, T_N)$ satisfies the $d$-$\mathcal{F}$ Hypercyclicity Criterion with respect to $(n_k)_k$, then the sequence $(T_{n_k}^1, \ldots, T_{n_k}^N)_k$ is $d$-$\mathcal{F}$.

Proof. Let $(V_i)_{i=0}^N$ be a finite sequence of open subsets of $X$. Pick $y_l \in V_l \cap X_l, (0 \leq l \leq N)$ and $\epsilon > 0$ such that $B(y_l; (N + 1)\epsilon) \subset V_l$, where $B(x; r)$ denote the ball centered at $x$ with radius $r$. Set

$$A = \{ k \in \mathbb{N} : T_{n_k}^l y_0 \in B(0; \epsilon) \} \in \mathcal{F}$$

$$B_l = \{ k \in \mathbb{N} : S_{i,k}y_l \in B(0; \epsilon) \} \in \mathcal{F}$$

$$C_{i,l} = \{ k \in \mathbb{N} : T_{n_k}^l S_{i,k}y_i - \delta_{i,l}y_i \in B(0; \epsilon) \} \in \mathcal{F}, \quad (1 \leq i, l, \leq N).$$

But $\mathcal{F}$ is a filter, hence

$$D := A \cap \bigcap_{l=1}^N (B_l \cap \bigcap_{i=1}^N C_{i,l}) \in \mathcal{F}.$$ 

Then, for each $k \in D$ we have $z_k := y_0 + \sum_{i=1}^l S_{i,k}y_i \in V_0$ and

$$T_{n_k}^l z_k = T_{n_k}^l y_0 + T_{n_k}^l \left( \sum_{i=1}^N S_{i,k}y_i \right) = T_{n_k}^l y_0 + \sum_{i=1}^N T_{n_k}^l S_{i,k}y_i.$$ 

On the other hand,

$$\sum_{i=1}^N T_{n_k}^l S_{i,k}y_i = (T_{n_k}^l S_{i,k}y_l - y_l) + y_l + \sum_{i=1}^N T_{n_k}^l S_{i,k}y_i \in B(y_l; N \cdot \epsilon).$$ 

Hence, $T_{n_k}^l z_k \in B(y_l; (N + 1)\epsilon) \subset V_l \quad (1 \leq l \leq N)$. That is,

$$D \subset \{ k \in \mathbb{N} : V_0 \cap T_{-n_k}^1(V_1) \cap \cdots \cap T_{-n_k}^N(V_N) \neq \emptyset \}.$$ 

Hence, $(T_{n_k}^1, \ldots, T_{n_k}^N)$ is $d$-$\mathcal{F}$. \qed
2.1 Bilateral weighted shifts

Let \( X = c_0(\mathbb{Z}) \) or \( l_p(\mathbb{Z})(1 \leq p < \infty) \). For \( l = 1, \ldots, N \) consider \( w_l = (w_{l,j})_{j \in \mathbb{Z}} \) a bounded bilateral sequence of non-zero scalars and \( B_{w_l} \) be the associated bilateral weighted backward shift on \( X \).

If \( (B_{w_1}^{r_1}, \ldots, B_{w_N}^{r_N}) \) satisfies the d-Hypercyclicity Criterion then the integers \( r_1, \ldots, r_N \) are necessarily distinct, see corollary 3.4 \[9\], then since we are interested in tuples of weighted shifts satisfying a kind of \( d \)-Hypercyclicity Criterion along filters, we then must consider distinct powers.

Let \( 1 \leq r_1 < r_2 < \cdots < r_N, M > 0, j \in \mathbb{Z} \) and \( 1 \leq s < l \leq N \). Let \( (n_k)_k \) a strictly increasing sequence of positive integers. Denote

\[
A_{M,j,l,(n_k)} = \left\{ k \in \mathbb{N} : \prod_{i=j+1}^{j+r_l n_k} |w_{l,i}| > M \right\}
\]

\[
\bar{A}_{M,j,l,(n_k)} = \left\{ k \in \mathbb{N} : \prod_{i=j-r_l n_k+1}^{j} |w_{l,i}| > M \right\}
\]

\[
A_{M,j;(s,l),(n_k)} = \left\{ k \in \mathbb{N} : \prod_{i=j+1}^{j+(r_l-r_s)n_k+1} |w_{l,i}| \prod_{i=j+(r_l-r_s)n_k+1}^{j+r_l n_k} |w_{s,i}| > M \right\}
\]

\[
\bar{A}_{M,j;(s,l),(n_k)} = \left\{ k \in \mathbb{N} : \prod_{i=j-(r_l-r_s)n_k+1}^{j+r_l n_k} |w_{l,i}| \prod_{i=j-(r_l-r_s)n_k+1}^{j-r_s n_k} |w_{s,i}| > M \right\}.
\]

If \( (n_k) = (k) \), set \( A_{M,j,l}, \bar{A}_{M,j,l}, A_{M,j;(s,l)}, \bar{A}_{M,j;(s,l)} \) respectively. We have the following:

**Proposition 2.4.** Let \( \mathcal{F} \) be a filter on \( \mathbb{N} \), then the following are equivalent:

(i) \( (B_{w_1}^{r_{n_1}}, \ldots, B_{w_N}^{r_{n_N}})_{k \in \mathbb{N}} \) is \( d \)-\( \mathcal{F} \)

(ii) for each \( M > 0, j \in \mathbb{Z}, 1 \leq s < l \leq N \)

\[
A_{M,j,l,(n_k)} \in \mathcal{F}, \quad \bar{A}_{M,j,l,(n_k)} \in \mathcal{F}
\]

\[
A_{M,j;(s,l),(n_k)} \in \mathcal{F}, \quad \bar{A}_{M,j;(s,l),(n_k)} \in \mathcal{F}
\]

(iii) \( (B_{w_1}^{r_{n_1}}, \ldots, B_{w_N}^{r_{n_N}}) \) satisfies the \( d \)-\( \mathcal{F} \) Hypercyclicity Criterion with respect to \( (n_k)_k \).
Proof. It is enough to consider $(n_k)_k = (k)$. The sketch of the proof is the same for any $(n_k)_k$.

(iii) implies (i) Follows by proposition 2.3

(i) implies (ii) Let $M > 0, j \in \mathbb{Z}$ and $R > 1$ such that $MR > 1$. Consider the open set

$$A_R = \{x \in X : |e_j^*(x)| > 1/R\} \cap \{x \in X : \|x\| < 1\}.$$ 

Let $V = \{x \in X : \|x - (M + 1)e_j\| < \frac{1}{MR}\}$ and $m \in N(V, \ldots, V; A_R)$. Pick $x \in A_R$, such that $B_{w_im}^m x \in V$, where $1 \leq l \leq N$, then

$$\prod_{i=j+1}^{j+r_m} w_{l,i} x_{r_m+j} - (M + 1) < \frac{1}{MR} < 1, \quad 1 \leq l \leq N \quad (2.1)$$

$$\prod_{i=t+1}^{t+r_m} |w_{l,i} x_{r_m+l}| < \frac{1}{MR}, \quad t \neq j, \quad 1 \leq l \leq N. \quad (2.2)$$

We get by (2.1)

$$\prod_{i=j+1}^{j+r_m} |w_{l,i}| > \prod_{i=j+1}^{j+r_m} |w_{l,i} x_{r_m+j}| > M, \quad 1 \leq l \leq N.$$ 

Now, by (2.2) we obtain $\prod_{i=j-r_m+1}^{j} |w_{l,i} x_j| < \frac{1}{MR}$, hence

$$\prod_{i=j-r_m+1}^{j} |w_{l,i}| \cdot \frac{1}{R} < \prod_{i=j-r_m+1}^{j} |w_{l,i} x_j| < \frac{1}{MR}.$$ 

Thus

$$\prod_{i=j-r_m+1}^{j} |w_{l,i}| < \frac{1}{M}.$$ 

Again by (2.1) and (2.2) we get

$$\prod_{i=j+r_{m1}}^{j+r_m} |w_{l,i}| \prod_{i=j+(r_m+j)x_{r_m+1}}^{j+r_m} |w_{s,i}| x_{r_m} > \prod_{i=j+1}^{j+r_m} |w_{l,i} x_{j+r_m}| > M, \quad 1 \leq s < l \leq N.$$ 

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Finally, for $1 \leq s < l \leq N$

\[
\frac{\prod_{i=j}^{j+r_{m}} |w_{s,i}|}{\prod_{i=j}^{j+r_{m}} |w_{l,i}|} = \frac{\prod_{i=j}^{j+r_{m}} |w_{s,i} - x_j|}{\prod_{i=j}^{j+r_{m}} |w_{l,i} - x_j|} > \frac{M}{1/|MR|} > M.
\]

(ii) implies (iii) Let $X_0 = \cdots = X_N$, the set of all finitely supported vectors in $X$ and $S_{l,k} : X_0 \to X$, $(1 \leq l \leq N)$ be defined as

\[S_{l,k}e_t = \frac{1}{\prod_{i=t+1}^{l+r_{t,k}} w_{l,i}} e_{t+r_{l,k}}, \quad k \in \mathbb{Z}.
\]

a) Let us verify that $\mathcal{F}$-lim$k\to\infty B_{w_l}^{r_{l,k}}(x) = 0$, for every $x \in X_0$.

Let $x \in X_0$, denote $F = \{ j \in \mathbb{Z} : x_j \neq 0 \}$. Recall that the $(j - r_{l,k})$-th coefficient of $B_{w_l}^{r_{l,k}} x$ is equal to $\prod_{i=j-r_{l,k}}^{l} w_{l,i} x_j$. Let $M > 0$, then

\[\bigcap_{j \in F} \left\{ k \in \mathbb{N} : \prod_{i=j-r_{l,k}}^{l} |w_{l,i}| < \frac{1}{M \|x\|} \right\} \subseteq \left\{ k \in \mathbb{N} : \|B_{w_l}^{r_{l,k}} x\| < \frac{1}{M} \right\},
\]

hence

\[\bigcap_{j \in F} \bar{A}_{M\|x\|,j,l} \subseteq \left\{ k \in \mathbb{N} : \|B_{w_l}^{r_{l,k}} x\| < \frac{1}{M} \right\}
\]

and by hypothesis $\bigcap_{j \in F} \bar{A}_{M\|x\|,j,l} \in \mathcal{F}$ for any $M > 0$. Obviously $B_{w_l}^{r_{l,k}} S_{l,k} = Id_{X_0}$, for any $k$.

b) Let us verify that $\mathcal{F}$-lim$k\to\infty S_{l,k}(x) = 0$ for any $x \in X_0$.

Let $x \in X_0$, $F = \{ j \in \mathbb{Z} : x_j \neq 0 \}$, $\epsilon > 0$. Set $M_j = \frac{1}{\epsilon} |F|^{1/p} |x_j|$ whenever $X = l_0(\mathbb{Z})$ or $M_j = |x_j|/\epsilon$ whenever $X = c_0(\mathbb{Z})$. Set $M = \max_{j \in F} |M_j|$, then for each $k \in \bigcap_{j \in F} A_{M,j,l}$ it results $\|S_{l,k}(x)\| < \epsilon$, i.e.

\[\{ k \in \mathbb{N} : \|S_{l,k}(x)\| < \epsilon \} \in \mathcal{F}.
\]

c) Let $1 \leq s < l \leq N$, then

\[B_{w_l}^{r_{l,k}} S_{s,k} e_t = \frac{1}{\prod_{i=t+1}^{l+r_{s,k}} w_{s,i}} B_{w_l}^{r_{l,k}} e_{t+r_{s,k}} = \frac{\prod_{i=t+r_{s,k}}^{t+(r_{s-k})} w_{l,i}}{\prod_{i=t+1}^{l+r_{s,k}} w_{s,i}} e_{t+(r_{s-k})}.
\]

Now, arguing as in b), the last equation implies that $\mathcal{F}$-lim$k\to\infty B_{w_l}^{r_{l,k}} S_{s,k} x = 0$, for every $x \in X_0$, since $\mathcal{F}$ is a filter and $\bar{A}_{M,l,(s,l)} \in \mathcal{F}$ for any $M > 0$, $t \in \mathbb{Z}$.
Let \(1 \leq s < l \leq N\), then
\[
B_{r_s}^{r_k}S_{l,k}e_t = \frac{1}{\prod_{i=t+1}^{t+r_k} w_{t,i}} B_{r_s}^{r_k}e_{t+r_k} = \frac{\prod_{i=t+1}^{t+(r_l-r_s)k+1} w_{s,i}}{\prod_{i=t+1}^{t+r_k} w_{t,i}} e_{t+(r_l-r_s)k}.
\]
Again, arguing as in \(b)\), the last equation implies that \(\mathcal{F}-\lim_{k \to \infty} B_{r_s}^{r_k}S_{l,k}x = 0\), for every \(x \in X_0\), since \(\mathcal{F}\) is a filter and \(A_{M,t,(s,l)} \in \mathcal{F}\) for any \(M > 0, t \in \mathbb{Z}\). We can conclude that \((B_{r_s}^{r_1}, \ldots, B_{r_s}^{r_N})\) satisfy the \(d-\mathcal{F}\) Hypercyclicity Criterion with respect to \((k)\).

Now, using proposition 2.4 and adapting theorem 2.5 \([26]\) to bilateral weighted backward shifts, we obtain the following:

**Corollary 2.5.** Let \(X = c_0(\mathbb{Z})\) or \(l_p(\mathbb{Z})(1 \leq p < \infty), w = (w_j)_{j \in \mathbb{Z}}\) a bounded bilateral weight sequence, \(\mathcal{F}\) a filter on \(\mathbb{N}\) and \(r_0 = 0 < r_1 < \cdots < r_N\), then the following are equivalent:

(i) \((B_{r_1}^{r_1}, \ldots, B_{r_N}^{r_N})\) is \(d-\mathcal{F}\)

(ii) \((B_{r_1}^{r_1}, \ldots, B_{r_N}^{r_N})\) satisfies the \(d-\mathcal{F}\) Hypercyclicity Criterion with respect to \((k)\)

(iii) for any \(M > 0, j \in \mathbb{Z}\) and \(0 \leq s < l \leq N\) holds
\[
\left\{ m \in \mathbb{N} : \prod_{i=j+1}^{j+(r_l-r_s)m} |w_i| > M \right\} \in \mathcal{F}
\]
\[
\left\{ m \in \mathbb{N} : \prod_{i=j-(r_l-r_s)m+1}^{j} |w_i| > M \right\} \in \mathcal{F}
\]

(iv) \(\oplus_{0 \leq s < l \leq N} B_{r_s}^{(r_l-r_s)}\) is \(\mathcal{F}\)-operator on \(X^{\frac{N(N+1)}{2}}\).

**Remark 2.6.** Observe that the conclusion of corollary 2.5 does not hold in general, for bounded linear operators. In fact, let \(r_i = \frac{i}{N}, 1 \leq i \leq N\). Bès, Martin, Peris and Shkarin \([12]\), have proved that \((I + B_{r_1}, \ldots, (I + B_{r_N})^r)\) is \(d-\mathcal{J}^r\) (i.e. \(d\)-mixing). On the other hand, by a result of Grivaux \([17]\), \(I + B_{r_N}\) does not satisfy the \(d-\mathcal{J}^r\)-Hypercyclicity Criterion with respect to \((k)\), for \(B_{r_N}\) weighted backward shift on \(l_p\) and \((w_n)\) a decreasing sequence of positive weights such that \(\lim_{n \to \infty} n(\prod_{i=1}^{n} w_i)^{1/n} = 0\). Hence, \((i)\) and \((ii)\) in corollary 2.5 are not equivalent beyond the weighted shift frame. Moreover, the same situation take place between \((i)\) and \((iv)\), since Shkarin has exhibited a bounded linear operator \(T\) such that \(T \oplus T^2\) is mixing but \((T, T^2)\) is not \(d\)-mixing.
Proof of theorem 1.3

Proof. (i) implies (ii) Denote $A_{M;j} = \{m \in \mathbb{N} : |\prod_{i=j+1}^{j+m} w_i| > M \}$ and $\bar{A}_{M;j} = \{m \in \mathbb{N} : \frac{1}{\prod_{i=j+1}^{j+m} |w_i|} > M \}$. If $B_w$ is $\mathcal{F}$-operator then $A_{M;j} \in \mathcal{F}$ and $\bar{A}_{M;j} \in \mathcal{F}$ for any $M > 0, j \in \mathbb{Z}$ by corollary 2.5 with $N = 1$ and $r_1 = 1$. Let $r \in \mathbb{N}$, then by corollary 2.3 [7] and corollary 2.7 [7] we have

$A = \{m, 2m, \ldots, rm : m \in \mathbb{N}\} \cap (A_{M;j} \times \cdots \times A_{M;j}) \in \mathcal{F}$

(2.3)

in $\{m, 2m, \ldots, rm : m \in \mathbb{N}\}$.

Denote $\prod_i$ the projection onto the $i$-th coordinate. It is not difficult to see that $\prod_1(A) \in \mathcal{F}$ and $\prod_1(\bar{A}) \in \mathcal{F}$ in $\mathbb{N}$. Then, (2.3) is equivalent to say

$\{m \in \mathbb{N} : tm \in A_{M;j}\} \in \mathcal{F}$

$\{m \in \mathbb{N} : tm \in \bar{A}_{M;j}\} \in \mathcal{F}$

for any $M > 0, j \in \mathbb{Z}$ and $t = 1, \ldots, r$.

(ii), (iii) and (iv) are equivalent by corollary 2.5.

(iii) implies (i) Obvious.

Notice that the family of syndetic sets is not a filter, since $2\mathbb{N}$ and $2\mathbb{N} + 1$ are syndetic sets but disjoint. Nevertheless, at the level of operators, we know that syndetic operators behave as filter-operators in virtue of their equivalence with $\mathcal{PS}^*$-operators [13]. As expected, theorem 1.3 is also valid for $\mathcal{F} = S$.

Corollary 2.7. Let $X = c_0(\mathbb{Z})$ or $l_p(\mathbb{Z})(1 \leq p < \infty)$, a bilateral bounded weight $w = (w_j)_{j \in \mathbb{Z}}$. Then for any $r \in \mathbb{N}$, the following are equivalent:

(i) $B_w$ is syndetic weighted backward shift

(ii) $(B_w, \ldots, B_w^r)$ is $d$-$S$

(ii) $(B_w, \ldots, B_w^r)$ is $d$-$\mathcal{PS}^*$

(iv) $B_w \oplus \cdots \oplus B_w^r$ is syndetic weighted backward shift on $X^r$.

Corollary 2.7 follows from theorem 1.3 and the equivalence between syndetic and $\mathcal{PS}^*$-operators [13].
2.2 Unilateral weighted shifts

Let \( X = c_0(\mathbb{Z}) \) or \( l_p(\mathbb{Z}) (1 \leq p < \infty) \). For \( l = 1, \ldots, N \) let \( w_l = (w_{l,n})_{n \in \mathbb{N}} \) a bounded unilateral weight sequence of non-zero scalars and \( B_{w_l} \) be the associated backward weighted shift on \( X \).

Consider \( 1 \leq r_1 < r_2 < \cdots < r_N \). The following is the unilateral version of proposition 2.4.

**Proposition 2.8.** Let \( \mathcal{F} \) be a filter on \( \mathbb{N} \), then the following are equivalent:

(i) \( (B_{r_1}^1, \ldots, B_{r_N}^N)_{k \in \mathbb{N}} \) is d-\( \mathcal{F} \)

(ii) for each \( M > 0, j \in \mathbb{Z}_+ \), \( 1 \leq s < l \leq N \) it holds
\[
A_{M,j,s,(r_k)} \in \mathcal{F} \quad A_{M,j,l,(r_k)} \in \mathcal{F}
\]

(iii) \( (B_{r_1}^1, \ldots, B_{r_N}^N) \) satisfies the d-\( \mathcal{F} \) Hypercyclicity Criterion with respect to \( (r_k) \).

**Proof.** Analogue to the proof of theorem 2.4. \( \square \)

Let \( A \subseteq \mathbb{N} \) and suppose \( (A + i) \cap \mathbb{N} \neq \emptyset \) for every \( i \in \mathbb{Z} \), then \( \mathcal{F} \) is said to be shift invariant if \( (A + i) \cap \mathbb{N} \in \mathcal{F} \) whenever \( A \in \mathcal{F} \), for every \( i \in \mathbb{Z} \). Denote \( P_f(\mathbb{Z}) = \{ A \subseteq \mathbb{Z}_+: |A| < \infty \} \).

**Corollary 2.9.** Let \( 0 = r_0 < 1 \leq r_1 < \cdots < r_n \), \( \mathcal{F} \) a filter on \( \mathbb{N} \) and \( w = (w_n)_{n \in \mathbb{Z}_+} \) a bounded unilateral weight, then the following are equivalent:

(i) \( (B_{r_1}^1, \ldots, B_{r_N}^N) \) is d-\( \mathcal{F} \)

(ii) \( (B_{r_1}^1, \ldots, B_{r_N}^N) \) satisfies the d-\( \mathcal{F} \) Hypercyclicity Criterion with respect to \( (r_k) \)

(iii) \( \oplus_{0 \leq s < l \leq N} B_{w}^{r_k-s} \) is \( \mathcal{F} \)-operator on \( X^{\mathbb{N}(N+1)} \)

(iv) for any \( M > 0, j \in \mathbb{Z}_+ \) and \( 0 \leq s < l \leq N \) it holds
\[
\left\{ m \in \mathbb{N}: \prod_{i=j+1}^{j+m(r_l-r_s)} |w_i| > M \right\} \in \mathcal{F}
\]

In addition,

- If \( \mathcal{F} \) is a shift invariant, then each condition (i) – (iv) is equivalent to

\[
(iv') \quad \left\{ m \in \mathbb{N}: \prod_{i=1}^{m(r_l-r_s)} |w_i| > M \right\} \in \mathcal{F}
\]

for any \( M > 0 \) and \( 0 \leq s < l \leq N \).
Proof. (i)—(iv) are equivalent, follows the same steps as corollary 2.5 adapted to the unilateral case. Moreover, suppose \( \mathcal{F} \) is a shift invariant filter, let us see that (iv') implies (iv). Let \( M > 0, j \in \mathbb{Z}_+, 0 \leq s < l \leq N \) and \( k = \max_{t \in \mathbb{N}} |w_t| \). It is enough to verify

\[
\{ n \in \mathbb{N} : \prod_{i=1}^{(r_l-r_s)n} |w_i| > Mk^{(r_l-r_s)n} \} \subseteq \{ n \in \mathbb{N} : \prod_{i=j+1}^{j+(r_l-r_s)n} |w_i| > M \}.
\]

In fact, let \( m \in \mathbb{N} \) such that \( \prod_{i=1}^{(r_l-r_s)m} |w_i| > Mk^{(r_l-r_s)m} \) then

\[
j + (r_l-r_s)(m-j) \prod_{i=j+1}^{j+(r_l-r_s)(m-j)} |w_i| = \frac{\prod_{i=1}^{(r_l-r_s)m} |w_i|}{\prod_{i=1}^{j} |w_i| \cdot \prod_{i=(r_l-r_s)(m-j)+j+1}^{r_l} |w_i|} > \frac{Mk^{(r_l-r_s)n}}{k^{(r_l-r_s)n}} = M.
\]

\( \square \)

**Proof of theorem 1.4**

Theorem 1.4 can be proven in a similar way as theorem 1.3 with the aid of corollary 2.9.

**Corollary 2.10.** Let \( X = c_0(\mathbb{Z}_+) \text{ or } l_p(\mathbb{Z}_+)(1 \leq p < \infty) \), an unilateral bounded weight \( w = (w_n)_{n \in \mathbb{Z}_+} \). Then for any \( r \in \mathbb{N} \), the following are equivalent:

(i) \( B_w \) is syndetic weighted backward shift

(ii) \( (B_w, \ldots, B^r_w) \) is d-S

(iii) \( (B_w, \ldots, B^r_w) \) is d-\( \mathcal{P} \mathcal{S}^* \)

(iv) \( B_w \oplus \cdots \oplus B^r_w \) is syndetic weighted backward shift on \( X^r \).

Corollary 2.10 follows from theorem 1.4 and the equivalence between syndetic and \( \mathcal{P} \mathcal{S}^* \)-operators [13].

As a consequence we can say a little more about the examples given in [13] concerning the existence of a \( \mathcal{P} \mathcal{S}^* \setminus \Delta^* \)-operator, \( S \setminus \mathcal{P} \mathcal{S}^* \)-operator and \( \Delta^* \) but not mixing operator.

**Proposition 2.11.** Let \( X = c_0(\mathbb{Z}_+) \text{ or } l_p(\mathbb{Z}_+)(1 \leq p < \infty) \). For any \( r \in \mathbb{N} \) the following holds

(i) There exists a weighted backward shift \( B_w \) on \( X \) such that \( (B_w, \ldots, B^r_w) \) is d-\( \mathcal{P} \mathcal{S}^* \) but \( B_w \) is not \( \Delta^* \)-operator

(ii) There exists a weighted backward shift \( B_w \) on \( X \) such that \( (B_w, \ldots, B^r_w) \) is d-\( \mathcal{P} \mathcal{S}^* \) but \( B_w \) is not \( \mathcal{P} \mathcal{S}^* \)-operator

(iii) There exists a weighted backward shift \( B_w \) on \( X \) such that \( (B_w, \ldots, B^r_w) \) is d-\( \Delta^* \) but \( B_w \) is not mixing operator.
Proof. The examples given in [13] are weighted backward shifts on $X$. We conclude by theorem 1.4 and corollary 2.10.

This is in sharp contrast to what happens with scalar multiples of powers of the unilateral unweighted backward shift $B$. In fact, $(\lambda_1 B^{r_1}, \ldots, \lambda_N B^{r_N})$ is $d$-hypercyclic on $l_2$ if and only if $(\lambda_1 B^{r_1}, \ldots, \lambda_N B^{r_N})$ is $d$-mixing on $l_2$ if and only if $1 \leq r_1 < r_2 < \cdots < r_N$ and $1 < |\lambda_1| < \cdots < |\lambda_N|$ as shown in corollary 4.7 [16].

Remark 2.12. Let $X = c_0(\mathbb{Z})$ or $l_p(\mathbb{Z})$ (or $c_0(\mathbb{Z}^+)$ or $l_p(\mathbb{Z}^+)$), theorem 1.3 (theorem 1.4) can be obtained in another way. In fact, in its proof we have used proposition 2.4 (proposition 2.8) who gives us conditions on the weight $w$ in such a way that the tuple $(B_w, \ldots, B_rw)$ is $d$-$\mathcal{F}$, with $r \in \mathbb{N}$ and $\mathcal{F}$ a filter on $\mathbb{N}$. But we can exploit another information coming from the proof of proposition 2.4 (proposition 2.8) and is the fact that any $\mathcal{F}$-weighted backward shift on $X$ satisfies the Original Kitai $\mathcal{F}$-criterion. Now, an adaptation to filters of theorem 3.4 [12] and a further application of corollary 2.3 [7] and corollary 2.7 [7] allows us to conclude as in theorem 1.3 (theorem 1.4), that is, $B_w$ is $\mathcal{F}$-weighted backward shift if and only if $(B_w, \ldots, B_rw)$ is $d$-$\mathcal{F}$.

Theorem 2.13. Let $T \in \mathcal{L}(X)$, where $X$ is a topological vector space, $\mathcal{F} = \Delta^*, \mathcal{J}^*, \mathcal{P}^*$. If $T$ satisfies the Original Kitai $\mathcal{F}$-criterion then $(T, \ldots, T^r)$ is $d$-$\mathcal{F}$, for any $r \in \mathbb{N}$.

Proof. Make the adaptation to filters of theorem 3.4 [12] and apply corollary 2.3 [7] and corollary 2.7 [7]. Details are left to the reader. □

2.2.1 The non-filter case

Everything changes when dealing with non-filter families, with the exception of the family of syndetic sets. A slight modification of example 4.5 [15] gives us an immediate answer.

Proposition 2.14. Let $X = c_0(\mathbb{Z}^+)$ or $l_p(\mathbb{Z}^+)$ $(1 \leq p < \infty)$, then there exists a thick backward weighted shift $B_w$ (i.e. weakly mixing backward weighted shift) such that

- $(B_w, B_rw^2)$ is not $d$-topologically transitive. In particular, $B_w$ is $\mathcal{F}$-operator but $(B_w, B_rw^2)$ is not $d$-$\mathcal{F}$, for $\mathcal{F} = \Delta, \mathcal{J}^*, \mathcal{P}^*$.
• \((B_w, B^2_w)\) satisfies the d-thick Hypercyclicity Criterion with respect to \((k)\).

Recall that any tuple of powers of a fixed backward weighted shift on \(c_0\) or \(l_p\) is \(d\)-transitive if and only if it is \(d\)-hypercyclic if and only if it satisfies the \(d\)-I* Hypercyclicity Criterion with respect to some \((n_k)\). This follows by proposition 2.3, theorem 2.7 and theorem 4.1, in [15]. Hence, in this case, \((B_w, B^2_w)\) does not satisfy the \(d\)-I* Hypercyclicity Criterion with respect to any \((n_k)\). In other words, \((B_w, B^2_w)\) does not satisfy the \(d\)-Hypercyclicity Criterion, though it satisfies the \(d\)-thick Hypercyclicity Criterion with respect to \((k)\).

**Proof.** Let \(w = (w_k)_{k \in \mathbb{Z}^+}\) defined as

\[
  w_k = \begin{cases} 
    2 & \text{if } k \in \bigcup_{n \in \mathbb{N}} \{2^{2n} + 1, \ldots, 2^{2n} + 2n\} \\
    1 & \text{if } k = 2^{2n} + 2n + 1 \text{ for some } n \in \mathbb{N} \\
    \frac{1}{2^{2n}} & \text{otherwise.}
  \end{cases}
\]

Denote \(A_M := \{m \in \mathbb{N} : \prod_{i=1}^m |w_i| > M\}\). Let \(M > 0\) and \(j \in \mathbb{N}\) such that \(2^{j-1} < M \leq 2^j\), then \(\bigcup_{n \geq j+1} \{2^{2n} + j + 1, \ldots, 2^{2n} + 2n\} \subseteq A_M\). Obviously \(B_w\) is weakly mixing since there exists an increasing sequence \((n_k)\) such that \(\lim_{k \to \infty} \prod_{i=1}^{n_k} w_i = \infty\), see chapter 4 [15]. On the other hand, by corollary 4.4 [15], \((B_w, B^2_w)\) is \(d\)-hypercyclic if and only if

\[
\bigcap_{t=1,2} \left\{m \in \mathbb{N} : tm \in A_M\right\} \in \mathcal{F}
\]

for any \(M > 0\). Now, take into consideration that \(d\)-hypercyclicity and \(d\)-topologically transitivity coincides for tuples of powers of a fixed unilateral weighted shift. Observe that \(A_1 = \bigcup_{n \in \mathbb{N}} \{2^{2n} + 1, \ldots, 2^{2n} + 2n\} \cap (A_1 \times A_1) = \emptyset\). Hence \((B_w, B^2_w)\) is not \(d\)-topologically transitive.

On the other hand, the proof of corollary 2.9 tell us that for any shift invariant family \(\mathcal{F}\) (not necessarily a filter), satisfying both conditions \(A_M \in \mathcal{F}\) and \(\{m \in \mathbb{N} : 2m \in A_M\} \in \mathcal{F}\), for any \(M > 0\) implies that the tuple \((B_w, B^2_w)\) satisfies the \(d\)-\(\mathcal{F}\) Hypercyclicity Criterion with respect to \((k)\). In our case, \(A_M\) is thick and

\[
\bigcup_{n \geq j+1} \left\{2^{2n-1} + \left\lfloor \frac{j+1}{2} \right\rfloor, 2^{2n-1} + \left\lfloor \frac{j+1}{2} \right\rfloor + 1, \ldots, 2^{2n-1} + n\right\} \subseteq \left\{m \in \mathbb{N} : 2m \in A_M\right\}
\]

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and clearly the left-hand side of the last inclusion is a thick set. □

3 An \( \mathcal{F} \)-operator \( T \) for which \( (T, T^2) \) is not \( d-\mathcal{F} \)

In theorem 3.8 \([12]\) the authors show an example of a mixing Hilbert space operator \( T \) such that \( (T, T^2) \) is not \( d \)-mixing. A slight modification in the proof allows us to exhibit an operator with more detailed features, i.e. a mixing Hilbert space operator \( T \) such that \( (T, T^2) \) is not \( d \)-syndetic, in particular it is not \( d \)-mixing. As a consequence we have that theorem 1.3 (theorem 1.4) fails beyond the weighted shift frame, in other words, there exists a Hilbert space \( \mathcal{F} \)-operator \( T \) such that \( (T, T^2) \) is not \( d-\mathcal{F} \), for \( \mathcal{F} = \Delta^*, J\mathcal{P}^*, PS^*, S \). We describe here all the details for the sake of completeness.

Let \( 1 \leq p < \infty, -\infty < a < b < +\infty \) and \( k \in \mathbb{N} \). Recall that the Sobolev space \( W^{k,p}[a,b] \) is the space of functions \( f \in C^{k-1}[a,b] \) such that \( f^{(k-1)} \) is absolutely continuous and \( f^{(k)} \in L^p[a,b] \). The space \( W^{k,p}[a,b] \) endowed with the norm

\[
\|f\|_{W^{k,p}[a,b]} = \left( \int_a^b \left( \sum_{j=0}^{k} |f^{(j)}(x)|^p \right) dx \right)^{1/p}
\]

is a Banach space isomorphic to \( L^p[0,1] \). Now, \( W^{k,2}[a,b] \) is a separable infinite-dimensional Hilbert space for each \( k \in \mathbb{N} \). The family of operators to be considered lives on separable complex Hilbert spaces and is built from a single operator. Let \( M \in \mathcal{L}(W^{2,2}[-\pi, \pi]) \) be defined by the formula

\[
M : W^{2,2}[-\pi, \pi] \rightarrow W^{2,2}[-\pi, \pi], \quad Mf(x) = \exp^{ix} f(x). \tag{3.1}
\]

Denote \( \mathcal{H} = W^{2,2}[-\pi, \pi] \) and \( M^* \) the dual operator. Then, \( M^* \in \mathcal{L}(\mathcal{H}^*) \).

For each \( t \in [-\pi, \pi] \), \( \delta_t \in \mathcal{H}^* \), where \( \delta_t : \mathcal{H} \rightarrow \mathbb{C}, \delta_t(f) = f(t) \). Furthermore, the map \( t \rightarrow \delta_t \) from \([-\pi, \pi]\) to \( \mathcal{H}^* \) is norm-continuous. For a non-empty compact subset \( K \) of \([-\pi, \pi]\), denote

\[
X_K = \overline{\text{span}} \{ \delta_t : t \in K \}
\]

where the closure of span \( \{ \delta_t : t \in K \} \) is taken with respect to the norm of \( \mathcal{H}^* \).

Now, the functionals \( \delta_t \) are linearly independent, \( X_K \) is always a separable Hilbert space and \( X_K \) is infinite dimensional if and only if \( K \) is infinite. The following condition holds

\[
M^* \delta_t = \exp^{it} \delta_t, \quad \text{for each } t \in [-\pi, \pi].
\]
Hence, each $X_K$ is an invariant subspace for $M^*$, which allows us to consider the operator

$$Q_K \in \mathcal{L}(X_K), \quad Q_K = M^*|_{X_K}.$$  

The following is taken from [12] and tells us when $Q_K$ is mixing or transitive, we omit the proof.

**Proposition 3.1.** proposition 3.9 [12]

Let $K$ a non-empty compact subset of $[-\pi, \pi]$. If $K$ has no isolated points, then $Q_K$ is mixing. If $K$ has an isolated point, the $Q_K$ is non-transitive.

Hence, in order to obtain a mixing operator $T$ such that $(T, T^2)$ is not $d$-syndetic, it will be enough to find a non-empty compact set $K \subset [-\pi, \pi]$ with no isolated points such that the sequence $(2Q_K^n - Q_K^{2n})_n$ is not a syndetic operator. As in the proof of theorem 3.8 [12] we need lemma A.3 [12].

**Lemma 3.2.** lemma A.3 [12]

There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of $2\pi$-periodic functions on $\mathbb{R}$ such that $f_n|_{[-\pi, \pi]} \in W^{2,2}[-\pi, \pi]$, the sequence $\left(\|f_n\|_{W^{2,2}[-\pi, \pi]}\right)_n$ is bounded and $f_n(x) = 2\exp^{inx} - \exp^{2inx}$ whenever $|x - \frac{2\pi m}{n}| \leq 2/n^5$, for some $m \in \mathbb{Z}$.

Now, consider the set

$$K = \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^{n} 2\pi \epsilon_{n,r} \cdot \frac{1}{2^r + r} : \epsilon \in \{0,1\}^{\mathbb{N} \times \mathbb{N}} \right\}. \quad (3.2)$$

Note that $K$ is a compact subset of $[-\pi, \pi]$ with no isolated points and by proposition 3.1 $Q_K \in \mathcal{L}(X_K)$ is a mixing operator. We adapt the proof of proposition 3.10 [12] to the compact set $K$ defined in (3.2), this is the content of the following

**Proposition 3.3.** Let $K$ be the compact subset of $[-\pi, \pi]$ defined in (3.2), then the sequence $(2Q_K^{2n,r} - Q_K^{kn,r})_{n \in \mathbb{N}, 0 \leq r \leq n}$ of continuous linear operators on $X_K$ is non-transitive, where $k_{n,r} = 2^{rn} + r$ with $0 \leq r \leq n, n \in \mathbb{N}$.

**Proof.** For $f \in W^{2,2}[-\pi, \pi]$, consider $\phi_f \in X_K^*$ defined by $\phi_f(y) = y(f)$. Remark that $\|\phi_f\| \leq \|f\|_{W^{2,2}[-\pi, \pi]}$ and that $\phi_f = \phi_g$ if $f|_K = g|_K$, then

$$\|\phi_f\| \leq \inf \left\{ \|g\|_{W^{2,2}[-\pi, \pi]} : f|_K = g|_K \right\}. \quad (3.3)$$
Use the symbol $1$ to denote the constant $1$ function in $W^{2,2}[-\pi, \pi]$. The functional $\phi_1 \in X_K^*$ is non-zero, since $\phi_1(\delta_t) = 1$ for each $t \in K$. Denote $T_{n,r} = 2Q_{K,n,r}^* - Q_{K,n,r} \in L(X_K)$ and estimate $\|T_{n,r}^*\|\phi_1$. By definition of $Q_K$, it holds $Q_K^* \phi_f = \phi_{Mf}$, for each $f \in W^{2,2}[-\pi, \pi]$, where $M$ is the multiplication operator defined in (3.1). It follows that $T_{n,r}^* \phi_1 = \phi_{h_{k,n,r}}$, where $h_j(t) = 2 \exp^{ijt} - \exp^{2ijt}$. By lemma 3.2, there is a bounded sequence $(f_{k,n,r})_{n \in \mathbb{N}, 0 \leq r \leq n}$ in the Hilbert space $W^{2,2}[-\pi, \pi]$ such that $f_{k,n,r}(t) = h_{k,n,r}(t)$ whenever $|t - 2\pi m/k_{n,r}| \leq \frac{2}{k_{n,r}}$ for some $m \in \mathbb{Z}$.

Now, let $t \in K$, then $t = \sum_{j=1}^{\infty} \sum_{r=0}^{j} 2^{\pi \epsilon_{k,r}}/k_{j,r}$ for some $\epsilon \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$. For each $n \in \mathbb{N}$, it holds $t = y + u$, where $y = \sum_{j=1}^{n+1} \sum_{r=0}^{j} 2^{\pi \epsilon_{k,r}}/k_{j,r}$ and $u = \sum_{j=n+2}^{\infty} \sum_{r=0}^{j} 2^{\pi \epsilon_{k,r}}/k_{j,r}$. Obviously there exists $m \in \mathbb{N}$ such that $y \leq 2\pi m/k_{n,r}$ and

$$0 \leq u \leq 2\pi \sum_{j=n+2}^{\infty} \sum_{r=0}^{j} \frac{1}{k_{j,r}}$$

$$= 2\pi \left[ \frac{1}{2^{7n+2}} + \frac{1}{2^{7n+2} + 1} + \cdots + \frac{1}{2^{7n+2} + n + 2} \right]$$

$$+ \left( \frac{1}{2^{7n+3}} + \frac{1}{2^{7n+3} + 1} + \cdots + \frac{1}{2^{7n+3} + n + 3} \right) + \cdots$$

$$< 2\pi \left[ \frac{1}{2^{7n+1}} + \frac{1}{2^{7n+1} + 1} + \cdots + \frac{1}{2^{7n+1} + n + 2} \right]$$

$$+ \left( \frac{1}{2^{7n+2}} + \frac{1}{2^{7n+2} + 1} + \cdots + \frac{1}{2^{7n+2} + n + 3} \right) + \cdots$$

$$< 2\pi \sum_{j=7n+1}^{\infty} 2^{-j} < \frac{4\pi}{2^{7n+1}} = \frac{4\pi}{(2^{7n} + r)^2} < \frac{4\pi}{(2^{7n} + r)^6} < \frac{2}{(2^{7n} + r)^5} = 2k_{n,r}^{-5}.$$ 

Hence, $t = u + y \leq u + (2\pi m/k_{n,r})$, which implies $|t - (2\pi m/k_{n,r})| \leq 2k_{n,r}^{-5}$. Thus $f_{k,n,r}(t) = h_{k,n,r}(t)$ for each $t \in K$ and $n \in \mathbb{N}$. By (3.3), $\|\phi_{h_{k,n,r}}\| \leq \|f_{k,n,r}\|_{W^{2,2}[-\pi, \pi]}$ for $n \in \mathbb{N}, 0 \leq r \leq n$. Since the sequence $(f_{k,n,r})_{n,r}$ is bounded in $W^{2,2}[-\pi, \pi]$, the sequence $(\|\phi_{h_{k,n,r}}\|)_{n,r}$ is bounded, i.e. there exists $C > 0$ such that $\|\phi_{h_{k,n,r}}\| \leq C$ for each $n \in \mathbb{N}, 0 \leq r \leq n$. Since $T_{n,r}^* \phi_1 = \phi_{h_{k,n,r}}$, it follows that $|\phi_1(T_{n,r}(x))| = |T_{n,r}^* \phi_1(x)| \leq C\|x\|$ for each $x \in X_K$. Since $\phi_1$ is a non-zero continuous linear functional on $X_K$, by Hahn-Banach theorem the set $\{T_{n,r}x : n \in \mathbb{N}, 0 \leq r \leq n\}$ cannot be dense in $X_K$, for any given $x \in X_K$. In other words, $\{T_{n,r} : n \in \mathbb{N}, 0 \leq r \leq n\}$ is non-universal. □
Now, we can prove theorem 1.5.

**Proof of theorem 1.5.**

Proof. Let \( K \) the compact set defined in (3.2). By proposition 3.1 \( Q_K \) is a mixing operator on the separable infinite dimensional Hilbert space \( X_K \). On the other hand, by proposition 3.3 \((2Q_K^n - Q_K^{2n})_{n \in \mathbb{N}}\) is non-transitive for some thick set \( A \) written increasingly as \( A = (a_n)_n \). Hence, there exists open sets \( U, V \) in \( X_K \) such that \((2Q_K^n - Q_K^{2n})_n : (U) \cap V = \emptyset, \) for any \( n \in \mathbb{N} \). In other words, \( \{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})_n : (U) \cap V \neq \emptyset \} \cap A = \emptyset, \)

i.e., \((2Q_K^n - Q_K^{2n})_{n \in \mathbb{N}}\) is not a \( S \)-transitive sequence of operators. In particular, \((Q_K, Q_K^2)\) is not \( d \)-syndetic. In fact, let \( U, V \) open sets in \( X_K \) such that \( \{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})_n : (U) \cap V \neq \emptyset \} \) is not a syndetic set, and pick \( V_0 \) open set such that \( 2V_0 - V_0 \subseteq V \) (denote \( B(x; r) \) the open ball centered at \( x \) in \( X_K \) with radius \( r \)). Pick \( x \in X_K, r \in \mathbb{R}_+ \) such that \( B(x; r) \subset V, \) then set \( V_0 := B(x; r/3) \). Hence,

\[ \{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})_n : (U) \cap V \neq \emptyset \} \subseteq \{ n \in \mathbb{N} : 2Q_K^n - Q_K^{2n} : (U) \cap V \neq \emptyset \}. \]

Consequently, \( \{ n \in \mathbb{N} : U \cap Q_K^n(V_0) \cap Q_K^{2n}(V_0) \neq \emptyset \} \) can not be a syndetic set and then \((Q_K, Q_K^2)\) is not \( d \)-syndetic. Since all separable infinite dimensional Hilbert spaces are isomorphic to \( l_2 \), there is a mixing \( T \in L(l_2) \) such that the sequence \((2T^n - T^{2n})_{n \in \mathbb{N}}\) is not \( S \)-transitive, and then \((T, T^2)\) is not \( d \)-syndetic. \( \square \)

### 4 Reiteratively hypercyclicity vs. \( d \)-\( \mathcal{F} \) tuples

In this section we are interested in examine the connection between reiteratively hypercyclic operators and \( d \)-\( \mathcal{F} \) tuples. On one hand, there exists a \( \mathcal{F}^* \)-weighted backward shift which is not reiteratively hypercyclic on both \( c_0(\mathbb{Z}_+) \) and \( l_p(\mathbb{Z}_+), (1 \leq p < \infty) \). As a consequence, by corollary 2.10 we have the following:

**Proposition 4.1.** There exists a weighted backward shift \( B_w \) on both \( c_0(\mathbb{Z}_+) \) and \( l_p(\mathbb{Z}_+), (1 \leq p < \infty) \), such that the tuple \( (B_w, \ldots, B_w) \) is \( d \)-\( \mathcal{F}^* \) for any \( r \in \mathbb{N} \) and \( B_w \) is not reiteratively hypercyclic.
Conversely, in [13] it was shown that any reiteratively hypercyclic weighted backward shift on \( c_0 \) or \( l_p, (1 \leq p < \infty) \) is necessarily a \( \Delta^* \)-operator. On the other hand, in [23] it was proven that there exists a filter \( \mathcal{G} \) with \( \mathcal{F} \not\subset \mathcal{G} \not\subset \mathcal{S} \) such that any reiteratively hypercyclic weighted shift \( B_w \) on \( X = c_0 \) or \( l_p, (1 \leq p < \infty) \) is necessarily a \( \Delta^* \)-operator. On the other hand, in [23] it was proven that there exists a filter \( \mathcal{G} \) with \( \mathcal{F} \not\subset \mathcal{G} \not\subset \mathcal{S} \) such that any reiteratively hypercyclic weighted shift \( B_w \) on \( X = c_0 \) or \( l_p, (1 \leq p < \infty) \) is necessarily a \( \Delta^* \)-operator.

Now by corollary 2.5 (corollary 2.9) implies that \( (B_w, \ldots, B_r) \) is \( d-\mathcal{G} \), for any \( r \in \mathbb{N} \). Now by theorem 1.25 (theorem 1.4) we can unify all of these conclusions by stating:

**Theorem 4.2.** Let \( X = c_0(\mathbb{Z}) \) or \( l_p(\mathbb{Z}), (1 \leq p < \infty) \). If \( B_w \) is a reiteratively hypercyclic bilateral weighted backward shift on \( X \) then \( (B_w, \ldots, B_r) \) is \( d-\Delta^* \), for any \( r \in \mathbb{N} \).

Analogously,

**Theorem 4.3.** Let \( X = c_0(\mathbb{Z}_+) \) or \( l_p(\mathbb{Z}_+), (1 \leq p < \infty) \). If \( B_w \) is a reiteratively hypercyclic unilateral weighted backward shift on \( X \) then \( (B_w, \ldots, B_r) \) is \( d-\Delta^* \), for any \( r \in \mathbb{N} \).

Observe that the conclusion of proposition 4.3 is optimal since in [1] appears an example of a frequently hypercyclic, hence reiteratively hypercyclic weighted shift on \( c_0(\mathbb{Z}_+) \) which is not mixing.

In general, for linear operators, let us prove theorem 1.7.

**Proof of theorem 1.7.**

**Proof.** Fix \( r \in \mathbb{N} \). Let \( T \) reiteratively hypercyclic, then there exists \( x \in X \) such that \( \overline{Bd}(N(x, U)) > 0 \), for any open set \( U \) in \( X \). First, let us see that there exists a filter \( \mathcal{G} \) on \( \mathbb{N} \) such that \( \mathcal{G} \not\subset \mathcal{S} \) and

\[
N_T(U, \ldots, U; U) \in \mathcal{G},
\]

for any open set \( U \) in \( X \). Let \( U \) open set, then

\[
A_U := \{ k \in \mathbb{N} : \overline{Bd}(N(x, U) \cap (N(x, U) - k) \cap \cdots \cap (N(x, U) - rk)) > 0 \} \subseteq \{ k \in \mathbb{N} : T^{-k}U \cap \cdots \cap T^{-rk}U \cap U \neq \emptyset \}.
\]

In fact, let \( k \in A_U \), then there exists a set \( A \) with positive upper Banach density such that for any \( n \in A \) it holds \( T^{n+ik}x \in U \), for any \( i \in \{0, \ldots, r \} \). Consequently, \( T^nx \in T^{-k}U \cap \cdots \cap T^{-rk}U \cap U \). Now, by theorem 1.25 [8] it
follows that, there exists a filter $\mathcal{G}$ on $\mathbb{N}$ such that $A_U \in \mathcal{G} \subseteq \mathcal{S}$. Then, \((4.1)\) holds.

Next, let $\left( U_j \right)_{j=0}^r$ a finite sequence of open sets in $X$. Now, suppose that $(T, \ldots, T^r)$ is $d$-transitive, we must show that $N_T(U_1, \ldots, U_r; U_0) \subseteq \mathcal{S}$. In fact, there exists $n \in \mathbb{N}$ such that

$$V_n := T^{-n}U_1 \cap \cdots \cap T^{-rn}U_r \cap U_0 \neq \emptyset.$$  

Thus, $V_n$ is open, then pick $O_1, O_2$ open sets such that $O_1 + O_2 \subseteq V_n$, then

$$T^{jn}(O_1 + O_2) \subset U_j,$$  

for any $j \in \{0, \ldots, r\}$. \(4.2\)

Now, by \((4.1)\) since $\mathcal{G}$ is a filter we have

$$A := N_T(O_1, \ldots, O_1; O_1) \cap N_T(O_2, \ldots, O_2; O_2) \in \mathcal{G} \subseteq \mathcal{S}.$$

Let us show that $A + n \subseteq N(U_1, \ldots, U_r; U_0)$, then we are done because $A + n \in \mathcal{S}$ since $\mathcal{S}$ is shift invariant.

In fact, let $t \in A + n$, then $t - n \in A$, which means

$$T^{-t}T^n(O_1) \cap \cdots \cap T^{-rt}T^{rn}(O_1) \cap O_1 \neq \emptyset,$$

$$T^{-t}T^n(O_2) \cap \cdots \cap T^{-rt}T^{rn}(O_2) \cap O_2 \neq \emptyset.$$  

By the linearity of $T$ we obtain

$$T^{-t}(T^n(O_1 + O_2)) \cap \cdots \cap T^{-rt}(T^{rn}(O_1 + O_2)) \cap (O_1 + O_2) \neq \emptyset.$$  

Then we conclude by \((4.2)\), i.e.

$$T^{-t}U_1 \cap \cdots \cap T^{-rt}U_r \cap U_0 \neq \emptyset.$$  

\[\square\]

**Corollary 4.4.** The operator $Q_K$ given by theorem 1.5, is a (non-weighted shift) mixing operator which is not reiteratively hypercyclic.

**Proof.** Follows by theorems 1.5 and 1.7. \[\square\]

Concerning the statement of theorem 1.7 we were not able to determine whether reiteratively hypercyclicity implies $d$-transitivity or not. On the other hand, in [12], the authors wonder about the existence of a mixing continuous linear operator $T$ such that $(T, T^2)$ is not $d$-transitive. Now, taking into account corollary 4.4, it makes sense to pose the following:
Question 4.5. Does there exist a reiteratively hypercyclic continuous linear operator $T$ on a separable Banach space such that $(T, T^2)$ is not $d$-transitive?

On the other hand, in the case the answer to the precedent question is positive, bearing in mind proposition 2.14, it makes sense the following:

Question 4.6. Does every reiteratively hypercyclic continuous linear operator $T$ on a separable Banach space is such that $(T, \ldots, T^r)$ satisfies the $d$-$\mathcal{F}$ Hypercyclicity Criterion for some family $\mathcal{F}$, and any $r \in \mathbb{N}$?

4.1 Moving from syndetic to reiteratively hypercyclic weighted shifts

As mentioned before, reiteratively hypercyclic operators are syndetic but the converse is not true, [13]. In this subsection, we will discuss under which conditions we can obtain reiterative hypercyclicity from syndeticity, for weighted backward shifts on $X = c_0(\mathbb{Z}_+)$ or $l^p(\mathbb{Z}_+), (1 \leq p < \infty)$.

Consider a sequence $(H_m)_m$ of pairwise disjoint subsets of $\mathbb{N}$ such that
\[
(H_m + [0, m]) \cap H_n = \emptyset \quad \forall m \neq n.
\]
Suppose that each $H_m$ is partitioned into a sequence of pairwise disjoint finite sets $(H_{k,m})_k$ such that for any $k_1 < k_2$
\[
s < t \quad \forall (s, t) \in H_{k_1, m} \times H_{k_2, m}.
\]
Denote $\mathcal{H} := \{(H_{k,m})_k : m \in \mathbb{N}\}$. Let $l \in H_m$, then there exists a unique $k$ such that $l \in H_{k,m}$. Set $[l]_{H_m} := H_{k,m}$. Fix $k_0, m_0$ and denote
\[
\tilde{H}^{(m)}_{k_0, m_0} := \left[ \min \left( (\max H_{k_0, m_0}, \infty) \cap H_m \right) \right]_{H_m}.
\]
Let $\mathcal{F}$ be a family in $\mathbb{N}$ and $w = (w_n)_n$ a bounded weight.

Definition 4.7. We say that a triple $\left( w, (H_m)_m, \mathcal{H} \right)$

- is called $\mathcal{F}$-triple if $H_m \in \mathcal{F}$ for any $m \in \mathbb{N}$.
- satisfies property $\mathcal{P}$ if

\[
w_{1+r} \cdots w_{(s+r)-t} > m \cdot n
\]
for any \((s, t) \in \bigcap_{n} H_n \times H_n, m \neq n, r \in \{0 \ldots m\}\), and

\[ w_1 + r \cdot w_{(s+r)-t} > m^2 \]

for any \((s, t) \in H_{k,m} \times H_{k,m}, k, m \in \mathbb{N}, s > t, r \in \{0 \ldots m\}\).

Let us recall a characterization of piecewise syndeticity for sets, see [20]. A set \(A \subseteq \mathbb{N}\) is piecewise syndetic (\(A \in \mathcal{PS}\)) if there exists a natural number \(G(A)\) such that \(\cup_{t=1}^{G(A)} (-t + A)\) is a thick set. Hence, \(A \in \mathcal{PS}\) is equivalent to say:

\[ \forall L > 0, \exists x : x, \ldots, x + L \in \cup_{t=1}^{G(A)} (-t + A) \]

i.e.

\[ \forall L > 0, \exists x : (x + i) + t_i \in A \]

for some \(1 \leq t_i \leq G(A)\) and any \(0 \leq i \leq L\). Recall that \(\mathcal{BD} = \{A \subseteq \mathbb{N} : Bd(A) > 0\}\).

**Proposition 4.8.** Let \(B_w\) a syndetic weighted backward shift, then there exists a \(\mathcal{BD}\)-triple associated to \(w\).

**Proof.** Recall that each central set is piecewise syndetic. On the other hand, as a consequence of theorem 2.12 [19] it holds that any central subset of \(\mathbb{N}\) can be partitioned into infinitely many pairwise disjoint central sets, see corollary 2.13 [19]. Hence, we can apply this theorem as many times as necessary, taking \(\mathbb{N}\) as the starting set, in order to obtain a sequence \((D_m)_m\) of pairwise disjoint \(\mathcal{PS}\)-sets satisfying

\[ (D_m + [0, m]) \cap D_n = \emptyset \]

for any \(m \neq n\).

We will define a sequence of natural numbers \((l_{j,k,m})_{m,k \in \mathbb{N}}\) and set

\[ H_m := \{l_{j,k,m} : k \in \mathbb{N}, 1 \leq j \leq k\} \]

and

\[ H_{k,m} := \{l_{j,k,m} : 1 \leq j \leq k\} \]

We begin by defining \(H_1 = \{l_{j,k,1} : k \in \mathbb{N}, 1 \leq j \leq k\}, H_2\) and so on as \(m\) increases, such that \(\min H_m \geq m + G(D_m)\), for any \(m \in \mathbb{N}\). Now, at each
level $m$, we will proceed by blocks $H_{k,m} = \{l_{j,k,m} : 1 \leq j \leq k\}$ indexed by $k$, as $k$ increases such that

$$l_{j,k,m} + m + t_j = l_{j+1,k,m}$$

for some $1 \leq t_j \leq G(D_m)$ and any $1 \leq j \leq k$ and

$$l_{k-1,k-1,m} < l_{1,k,m} < l_{k,k,m} < l_{1,k+1,m}$$

for any $k, m \in \mathbb{N}$.

Before beginning the construction of the sequence $(H_m)_m$, let us recall the following. For any $M > 0$, $j \in \mathbb{N}$, denote $A_{M;j} = \{n \in \mathbb{N} : \prod_{i=1}^{n} |w_{i+j}| > M\}$. By hypothesis $B_w$ is syndetic backward shift, which is equivalent to say $\cap_{j \in F} A_{M;j} \in \mathcal{PS}^*$ for any $M > 0$ and any finite subset of natural numbers $F$, according to [13]. Furthermore, recall that any $\mathcal{PS}^*$-set $A$ is thickly syndetic, i.e. for any $L > 0$, there exists a syndetic sequence $(x_n)_n$ such that $\{x_n, x_n + 1, \ldots, x_n + L : n \in \mathbb{N}\} \subseteq A$.

Now, how to proceed? Suppose we have defined $H_{\hat{m}}$ for any $\hat{m} < m$. Set $l_{1,1,m} = \max(m + G(D_m), \min_{l \in D_m} l)$. Let $k \in \mathbb{N}$ and suppose we have defined $\{l_{j,k,m} : \hat{k} < k, 1 \leq j \leq \hat{k}\}$. Let us define the block $H_{k,m} = \{l_{j,k,m} : 1 \leq j \leq k\}$.

By hypothesis, the set $A^{(k,m)} := \cap_{r=1}^{m} A_{m-l_{k-1,k-1,m},r}$ is thickly syndetic, hence there exists some syndetic sequence $(x_n^{(k,m)})_n$ such that

$$x_n^{(k,m)}, \ldots, x_n^{(k,m)} + l_{k-1,k-1,m}, \ldots, x_n^{(k,m)} + 3l_{k-1,k-1,m} \subseteq A^{(k,m)} \quad (4.4)$$

for any $n \in \mathbb{N}$. By commodity, set $D_m - l_{k-1,k-1,m}$ instead of $(D_m - l_{k-1,k-1,m}) \cap \mathbb{N}$, then because $\mathcal{PS}$ is shift invariant, we have $D_m - l_{k-1,k-1,m} \in \mathcal{PS}$ with $G(D_m) = G(D_m - l_{k-1,k-1,m})$. Let $S^{(k,m)}$ the gap of $(x_n^{(k,m)})_n$, then there exists $x \in \mathbb{N}$ such that

$$x + t_0 \in D_m - l_{k-1,k-1,m}$$

$$(x + 1) + t_1 \in D_m - l_{k-1,k-1,m}$$

$$\ldots$$

$$x + S^{(k,m)} + k(m + G(D_m)) + t S^{(k,m)} + k(m + G(D_m)) \in D_m - l_{k-1,k-1,m}$$

for some $1 \leq t_i \leq G(D_m)$ and any $0 \leq i \leq S^{(k,m)} + k(m + G(D_m))$.  

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Hence, there exists $0 \leq S \leq S^{(k,m)}$ and $n \in \mathbb{N}$ such that $x_n^{(k,m)} = x + S$. Then set

\[ l_{1,k,m} := x_n^{(k,m)} + l_{k-1,k-1,m} + s_1 \in D_m \]
\[ l_{2,k,m} := l_{1,k,m} + m + s_2 \in D_m \]
\[ \ldots \]
\[ l_{k,k,m} := l_{k-1,k,m} + m + s_k \in D_m \]

for some $1 \leq s_i \leq G(D_m)$ and any $1 \leq i \leq k$. Observe that automatically it holds $l_{k-1,k-1,m} < l_{1,k,m} < l_{k,k,m}$. On the other hand, by (4.4) and since $k(m + G(D_m)) < 2l_{k-1,k-1,m}$, for any $k \geq 2$ (recall that $l_{1,1,m} \geq m + G(D_m)$), we have the following

\[ [l_{1,k,m}, l_{1,k,m} + m] \subseteq A^{(k,m)} \]
\[ [l_{2,k,m}, l_{2,k,m} + m] \subseteq A^{(k,m)} \]
\[ \ldots \]
\[ [l_{k,k,m}, l_{k,k,m} + m] \subseteq A^{(k,m)}. \] (4.5)

Again by (4.4)

\[ l_{j,k,m} - l \in A^{(k,m)}, \quad \forall l \leq l_{k-1,k-1,m}, \quad 1 \leq j \leq k. \] (4.6)

Note that the intervals $(l + [0, m])_{l \in H_m}$ are actually disjoint by condition (4.3). Finally let us see that $H_m$ has positive upper Banach density for any $m \in \mathbb{N}$. Let $m \in \mathbb{N}$, note that

\[ \alpha^n := \limsup_k |H_m \cap [l_{1,k,m}, l_{1,k,m} + n(m + G(D_m))]| \geq n + 1. \]

Hence,

\[ \overline{Bd}(H_m) \geq \lim_n \frac{\alpha^n}{n(m + G(D_m))} \geq \lim_n \frac{n + 1}{n(m + G(D_m))} = \frac{1}{m + G(D_m)} > 0. \]

So, we can associate to the weight of a syndetic weighted shift a $\overline{BD}$-triple with further properties enumerated in the above construction. \hfill \Box

Let $B_w$ a syndetic weighted backward shift, any $\overline{BD}$-triple with the characteristics described in the proof of proposition 4.8 will be called a $\overline{BD}$-triple associated to $w$. Clearly, by construction a $\overline{BD}$-triple associated to $w$ is not unique.
Theorem 4.9. \( B_w \) is reiteratively hypercyclic on \( c_0(\mathbb{Z}_+) \) if and only if \( B_w \) is syndetic on \( c_0(\mathbb{Z}_+) \) provided there exists a \( \mathcal{BD} \)-triple associated to \( w \) satisfying property \( \mathcal{P} \).

Proof. Set \( X = c_0(\mathbb{Z}_+) \), we need to show that \( B_w \) is reiteratively hypercyclic on \( X \) provided \( B_w \) is syndetic on \( X \) with a \( \mathcal{BD} \)-triple associated to \( w \) satisfying property \( \mathcal{P} \).

Without loss of generality, consider a dense sequence \( (z(m))_{m \in \mathbb{N}} \) on \( X \), where
\[
z(m) = (z(m)_0, ..., z(m)_m, 0, 0, ...) \]
and satisfying \( |z(m)_j| \leq m \), for every \( j = 0, \ldots, m \). Define
\[
U_m = \{ x \in X : \|x - z(m)\| < 1/m \}, m \in \mathbb{N}. 
\]
It suffices to find \( y \in X \) such that \( \mathcal{BD}(N(y, U_m)) > 0 \) for any \( m \in \mathbb{N} \). Take the pairwise disjoint intervals in \( \mathbb{N} \)
\[
I_{j,k,m} = [l_{j,k,m}; l_{j,k,m} + m], \tag{4.7}
\]
k, \( m \in \mathbb{N}, 1 \leq j \leq k \) and set,
\[
y := \sum_{k, m \in \mathbb{N}} \sum_{1 \leq j \leq k} y_{i} e_{i}. \tag{4.8}
\]

Then, for any \( m \in \mathbb{N} \), it suffices to show that \( B_{w_{l_{j,k,m}}}^{l_{j,k,m}} y \in U_m \), for any \( k \in \mathbb{N}, 1 \leq j \leq k \), since \( H_m \in \mathcal{BD} \). Note that
\[
B_{w_{l_{j,k,m}}}^{l_{j,k,m}} y = \left( \prod_{i=1}^{l_{j,k,m}} w_i y_{l_{j,k,m}}, \ldots, \prod_{i=1}^{l_{j,k,m}} w_{i+m} y_{l_{j,k,m}+m}, \prod_{i=1}^{l_{j,k,m}} w_{i+m+1} y_{l_{j,k,m}+m+1}, \ldots \right). 
\]
In order to guarantee \( B_{w_{l_{j,k,m}}}^{l_{j,k,m}} y \in U_m \), we set
\[
\prod_{i=1}^{l_{j,k,m}} w_{i+r} y_{l_{j,k,m}+r} = z(m)_r, \quad r = 0, \ldots, m; 
\]
which forces to define
\[
y_{l_{j,k,m}+r} := \frac{z(m)_r}{\prod_{i=1}^{l_{j,k,m}} w_{i+r}}, \quad r = 0, \ldots, m. \tag{4.9}
\]
At this point, we have all we need in order to verify condition
\[
\| B_{w}^{l_j,k,m} y - z(m) \| < 1/m \tag{4.10}
\]
for any \( k, m \in \mathbb{N}, 1 \leq j \leq k \).

Consider an arbitrary trio \( j_0, k_0, m_0 \).

\[
B_{w}^{l_{j_0},k_0,m_0} y - z(m_0) = \left( \sum_{i=1}^{l_{j_0},k_0,m_0} \prod_{i=1}^{l_{j_0},k_0,m_0} w_{i+m_0+1} y_{l_{j_0},k_0,m_0+t, \ldots} \right).
\]

If \( y_{l_{j_0},k_0,m_0+t} \neq 0, m_0 + 1 \leq t \) there exists \( k, m \) such that \( l_{j_0,k_0,m_0} + t = l_{j,k,m} + r \) with \( 1 \leq j \leq k, 0 \leq r \leq m \). Hence,
\[
y_{l_{j_0},k_0,m_0+t} = y_{l_j,k,m+r} = \frac{z(m)_r}{\prod_{i=1}^{l_{j,k,m}} w_i + r}
\]
and
\[
\prod_{i=1}^{l_{j_0},k_0,m_0} w_{i+t} y_{l_{j_0},k_0,m_0+t} = \frac{w_{1+l_{j,k,m}+r-l_{j_0},k_0,m_0} \cdots w_{l_{j,k,m}+r}}{w_{1+r} \cdots w_{l_{j,k,m}+r}} z(m)_r = \frac{z(m)_r}{w_{1+r} \cdots w_{l_{j,k,m}+r-l_{j_0},k_0,m_0}}.
\]

Let \( m \neq m_0 \), then there exists a unique number \( \tilde{k}(m) \) such that \( l_{\tilde{k}(m)-1,\tilde{k}(m)-1,m} < l_{j_0,k_0,m_0} < l_{1,\tilde{k}(m),m} \). Note that \( [l_{j_0,k_0,m_0}]_{H_{m_0}} = [l_{1,\tilde{k}(m),m}]_{H_m} \), then

\[
B_{w}^{l_{j_0},k_0,m_0} y - z(m_0) = \sum_{k,m \in \mathbb{N}} \sum_{i=1}^{l_{j_0},k_0,m_0 < l_{j,k,m}} \frac{z(m)_r}{w_{1+r} \cdots w_{l_{j,k,m}+r-l_{j_0},k_0,m_0}} c_{l_{j,k,m}+r} = \sum_{m \neq m_0} \left( \sum_{l \in H_{l}(m), m} \sum_{r=0}^{m} \frac{z(m)_r}{w_{1+r} \cdots w_{l+r-l_{j_0},k_0,m_0}} c_{l+r} + \sum_{k > \tilde{k}(m)} \sum_{l \in H_{l}(m), m} \sum_{r=0}^{m} \frac{z(m)_r}{w_{1+r} \cdots w_{l+r-l_{j_0},k_0,m_0}} c_{l+r} \right)
\]

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Now, let us estimate each summand. The fact that the $BD$-triple associated to $w$ satisfies property $\mathcal{P}$ implies

$$\sup_{r \in \{0, \ldots, m\} \atop l \in H_{k,m} \atop k > k(m) \atop m \neq m_0} \left| \frac{z(m)_r}{w_1 + \cdots + w_{l+r-l_{j_0,k_0,m_0}}} \right| < \frac{m}{m_0} = 1.$$

By (4.6) we have

$$\sup_{r \in \{0, \ldots, m\} \atop l \in H_{k,m} \atop k > k_0} \left| \frac{z(m)_r}{w_1 + \cdots + w_{l+r-l_{j_0,k_0,m_0}}} \right| < \frac{m}{m_0} \cdot \frac{1}{l_{j_0,k_0,m_0}} < \frac{1}{m_0}.$$

Concerning the last two summands we have the following estimations

$$\sup_{r \in \{0, \ldots, m\} \atop l \in H_{k_0,m_0} \atop l_{j_0,k_0,m_0} < l} \left| \frac{z(m_0)_r}{w_1 + \cdots + w_{l+r-l_{j_0,k_0,m_0}}} \right| < \frac{m_0}{m_0^2} = \frac{1}{m_0},$$

that follows because the $BD$-triple associated to $w$ satisfies property $\mathcal{P}$. Finally, again by (4.6)

$$\sup_{r \in \{0, \ldots, m_0\} \atop l \in H_{k,m_0} \atop k > k_0} \left| \frac{z(m_0)_r}{w_1 + \cdots + w_{l+r-l_{j_0,k_0,m_0}}} \right| < \frac{m_0}{m_0 \cdot l_{j_0-k_0-1,m_0}} < \frac{1}{m_0}.$$

Hence, it holds (4.10).

Evidently $H_m \subseteq N(y, U_m)$ by (4.10), which implies $BD(N(y, U_m)) > 0$. Note that $y \in X$ by condition (4.5). We conclude that $B_w$ is reiteratively hypercyclic on $c_0(\mathbb{Z}_+)$. □
It is not difficult to see that following the same steps of the proof of the characterization for frequently hypercyclic and $\mathcal{U}$-frequently hypercyclic weighted shifts on $c_0(\mathbb{Z}_+)$ given by Bayart and Ruzsa in [6], it is possible to obtain a characterization for reiteratively hypercyclic weighted shifts on $c_0(\mathbb{Z}_+)$. 

**Theorem 4.10.** Let $w = (w_n)_{n \in \mathbb{N}}$ be a bounded sequence of positive integers. Then $B_w$ is reiteratively hypercyclic on $c_0(\mathbb{Z}_+)$ if and only if there exists a sequence $(M(p))_{p \in \mathbb{N}}$ of positive real numbers tending to $+\infty$ and a sequence $(E(p))_{p \in \mathbb{N}}$ of subsets of $\mathbb{N}$ such that

i) for any $p \geq 1$, $\overline{Bd}(E_p) > 0$

ii) for any $p, q \geq 1, p \neq q$, $(E_p + [0, p]) \cap (E_q + [0, q]) = \emptyset$

iii) for any $p \geq 1$, $\lim_{n \to \infty, n \in E_p + [0, p]} w_1 \cdots w_n = +\infty$

iv) for any $p, q \geq 1$, for any $n \in E_p$ and any $m \in E_q$ with $m > n$, for any $t \in \{0 \ldots q\}$,

$$w_1 \cdots w_{m-n+t} \geq M(p)M(q).$$

**Remark 4.11.** Now, theorem 4.9 tells us that in order to move from syndeticity to reiterative hypercyclicity (for shifts) on $c_0(\mathbb{Z}_+)$ it suffices to verify condition iv) ”partially” in theorem 4.10. A similar statement in the vein of theorem 4.9 can be formulated for shifts on $l_p(\mathbb{Z}_+)$. 

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