Continuous Maps on Aronszajn Trees*

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Abstract

Assuming ♠: Whenever $B$ is a totally imperfect set of real numbers, there is special Aronszajn tree with no continuous order preserving map into $B$.

1 Introduction

We use the following notation: If $\sqsubset$ is a relation on $T$ and $x \in T$, then $x\uparrow$ denotes $\{y \in T : x \sqsubset y\}$ and $x\downarrow$ denotes $\{y \in T : y \sqsubset x\}$. Then a tree is a set $T$ with a strict partial order $\sqsubset$ such that each $x\downarrow$ is well-ordered by $\sqsubset$. In a tree $T$, height($x$) is the order type of $x\downarrow$ and $L_\alpha = L_\alpha(T) = \{x \in T : \text{height}(x) = \alpha\}$. T is an $\omega_1$-tree iff $|T| = \aleph_1$, each $L_\alpha(T)$ is countable, and $L_{\omega_1}(T) = \emptyset$. An Aronszajn tree is an $\omega_1$-tree $T$ with no uncountable chains; then, $T$ is special iff $T$ is a countable union of antichains.

We give a tree $T$ its natural tree topology, in which $U \subseteq T$ is open iff for all $y \in U$ with height($y$) a limit ordinal, there is an $x \sqsubset y$ such that $x\uparrow \cap y\downarrow \subseteq U$. Then the elements whose heights are successor ordinals or 0 are isolated points. Note that $T$ need not be Hausdorff, although any tree that we construct explicitly will be Hausdorff (equivalently, $y\downarrow = z\downarrow \rightarrow y = z$).

Let $T$ be an $\omega_1$-tree. A map $\varphi : T \rightarrow \mathbb{R}$ is called order preserving iff $x \sqsubset y \rightarrow \varphi(x) < \varphi(y)$ for all $x, y \in T$. The existence of such a $\varphi$ clearly implies that $T$ is Aronszajn, but not necessarily special; there is a counter-example [2] under ♠. However, it is easy to see (first noted by Kurepa [3]) that $T$ is special iff there is an order preserving $\varphi : T \rightarrow \mathbb{Q}$.

Let $T$ be an Aronszajn tree. If there is an order preserving $\varphi : T \rightarrow \mathbb{R}$, then there is also a continuous order preserving $\psi : T \rightarrow \mathbb{R}$, where $\psi(y) = \varphi(y)$ unless height($y$) is a limit ordinal, in which case $\psi(y) = \sup\{\varphi(x) : x \sqsubset y\}$. If we assume $MA(\aleph_1)$, then every Aronszajn tree is special, as Baumgartner [1]

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proved by forcing with finite order preserving maps into $\mathbb{Q}$. Note that this same forcing also produces a continuous order preserving $\psi : T \to \mathbb{Q}$. We show here that this cannot be done in ZFC, since assuming $\Diamond$, there is an Aronszajn tree $T$ with an order preserving map into $\mathbb{Q}$ (so $T$ is special), but no continuous order preserving $\psi : T \to \mathbb{Q}$.\footnote{A continuous order preserving map $\psi$ from an Aronszajn tree $T$ into the rationals is a nice thing to have. Todorˇcevi´c [4, Remark 4.3.(d) on page 429] proved that a combination of such a map with his osc map can be used to color the 2-element chains of $T$ with countably many colors so that every chain of order type $\omega^\omega$ receives all the colors.}

This last result can be generalized somewhat. First, we can replace “order preserving” by the weaker requirement that each $\psi^{-1}\{q\}$ is discrete in the tree topology; observe that when $\psi$ is order preserving, each $\psi^{-1}\{q\}$ is an antichain, and hence closed and discrete. Then, we can replace $\mathbb{Q}$ by any metric space which has no Cantor subsets (that is, subsets homeomorphic to $2^{\omega}$):

**Theorem 1.1** Assume $\Diamond$, and fix a metric space $B$ with no Cantor subsets such that $|B| \leq \aleph_1$. Then there is a special Aronszajn tree $T$ which has no continuous map $\psi : T \to B$ such that each $\psi^{-1}\{b\}$ is discrete.

By $CH$ (which follows from $\Diamond$), $|B| \leq \aleph_1$ holds whenever $B$ is separable, as well as when $B$ has a dense subset of size $\aleph_1$.

Observe that if $T$ is special and $B \subseteq \mathbb{R}$ does have a Cantor subset $F$, then there must be a continuous order preserving $\psi : T \to B$. Just let $D \subseteq F$ be countable and order-isomorphic to $\mathbb{Q}$, let $\varphi : T \to D$ be order preserving, and then construct a continuous $\psi : T \to F$ as described above.

In Theorem 1.1, $T$ depends on $B$. There is no one tree which works for all $B$ by the following, which holds in ZFC (although it is trivial unless $CH$ is true):

**Theorem 1.2** Let $T$ be any special Aronszajn tree. Then there is a $B \subseteq \mathbb{R}$ with no Cantor subsets and a continuous order preserving map $\psi : T \to B$ such that for all $x, y \in T$, $\psi(x) \neq \psi(y)$ unless $x \downarrow = y \downarrow$.

So, $\psi$ is actually 1-1 if $T$ is Hausdorff. Theorem 1.1 is proved in Section 2, and Theorem 1.2 is proved in Section 3.

By Theorem 1.2, the “$|B| \leq \aleph_1$” cannot be removed in Theorem 1.1, since $B$ could be the direct sum of all totally imperfect subspaces of $\mathbb{R}$.

## 2 Killing Continuous Maps

Throughout, $T$ always denotes an $\omega_1$–tree and $B$ denotes a metric space. We begin with some remarks on pruning open $U \subseteq T$. In the special case when $U$ is a subtree (that is, $x \downarrow \subseteq U$ for all $x \in U$), the pruning reduces to the standard procedure of removing all $x \in U$ with $x \uparrow \cap U$ countable. For a general $U$, we replace “countable” by “non-stationary” (which is the same when $U$ is a subtree).
Definition 2.1 For \( U \subseteq T \): \( U \) is stationary iff \( \{ \text{height}(x) : x \in U \} \) is stationary, and \( U^p \) is the set of all \( x \in U \) such that \( x \uparrow \cap U \) is stationary.

Clearly \( U^p \subseteq U \). If \( U \) is open then \( U^p \) is open, since \( x \in U^p \rightarrow x \uparrow \cap U \subseteq U^p \).

Lemma 2.2 If \( U \subseteq T \) is open, then \( (U^p)^p = U^p \).

Proof. Fix \( a \in U^p \); so \( a \uparrow \cap U \) is stationary. We need to show: \( \{ x \in a \uparrow \cap U : x \uparrow \cap U \) is stationary} is stationary. So, we fix a club \( C \subseteq \omega_1 \), and we shall find an \( x \) such that \( \text{height}(x) \in C \) and \( a \subseteq x \) and \( x \in U \) and \( x \uparrow \cap U \) is stationary.

Since \( a \in U^p \), fix a stationary set \( S \) such that for all \( \beta \in S \): \( a \uparrow \cap U \cap L_{\beta}(T) \neq \emptyset \) and \( \beta \) is a limit point of \( C \). For each \( \beta \in S \): choose \( y_{\beta} \in a \uparrow \cap U \cap L_{\beta}(T) \); then, since \( U \) is open, choose \( x_{\beta} \subseteq y_{\beta} \) such that \( x_{\beta} \in a \uparrow \cap U \) and \( \text{height}(x_{\beta}) \in C \).

By the Pressing Down Lemma, fix \( x \) and a stationary set \( S' \subseteq S \) such that \( x_{\beta} = x \) for all \( x \in S' \). Then \( x \uparrow \cap U \) is stationary (since it contains \( \{ y_{\beta} : \beta \in S' \} \)) and \( \text{height}(x) \in C \) and \( a \subseteq x \) and \( x \in U \).

Lemma 2.3 If \( A \subseteq T \) is discrete in the tree topology and \( U \) is a stationary open set, then the set \( S := \{ \alpha : U \cap L_\alpha \neq \emptyset \land U \cap L_\alpha \subseteq A \} \) is non-stationary. Hence, \( U \setminus A \) is stationary.

Proof. In fact, \( S \) is discrete in the ordinal (= tree) topology on \( \omega_1 \). To see this, suppose that \( \alpha \in S \) is a limit ordinal. Then fix \( y \in U \cap L_\alpha \). Note that \( y \in A \) since \( U \cap L_\alpha \subseteq A \). Since \( U \) is open and \( A \) is discrete, we may fix \( x \subseteq y \) such that \( x \uparrow \cap y \subseteq U \) and \( x \uparrow \cap y \cap A = \emptyset \). Let \( \xi = \text{height}(x) \). Then \( \xi < \alpha \), and \( S \) contains no ordinals between \( \xi \) and \( \alpha \).

The next lemma has a much simpler proof when \( B \) is separable (then, each \( W_n \) can be a singleton). For \( b \in B \) and \( \varepsilon > 0 \), let \( N_\varepsilon(b) = \{ z \in B : d(b, z) < \varepsilon \} \) (where \( d \) is the metric on \( B \)).

Lemma 2.4 Suppose that \( U \subseteq T \) is a stationary open set, \( B \) is any metric space, and \( \psi : U \rightarrow B \) is continuous, with each \( \psi^{-1}\{ b \} \) discrete. Then there are infinitely many \( b \in B \) such that \( \psi^{-1}(N_\varepsilon(b)) \) is stationary for all \( \varepsilon > 0 \).

Proof. Since each \( U \setminus \psi^{-1}\{ b \} \) is also stationary open by Lemma 2.3, it is sufficient to prove that there is one such \( b \). If there are no such \( b \), then \( B \) is covered by the open sets \( W \) such that \( \psi^{-1}(W) \) is non-stationary. By paracompactness of \( B \), this cover has a \( \sigma \)-discrete open refinement, \( \{ W_n : n \in \omega \} \). So, each \( W_n \) is a discrete (and hence disjoint) family of open sets \( W \) such that \( \psi^{-1}(W) \) is non-stationary, and \( B = \bigcup_{n \in \omega} (\bigcup W_n) \).

Fix \( n \) such that \( \psi^{-1}(\bigcup W_n) \) is stationary. We may assume that \( |W_n| \geq \aleph_1 \), since \( |W_n| \leq \aleph_0 \) yields an obvious contradiction. Also, we may assume that \( |B| \leq \aleph_1 \) (replacing \( B \) by \( \psi(U) \)), so that \( |W_n| = \aleph_1 \). Let \( W_n = \{ W_\xi : \xi < \omega_1 \} \).

For each \( \xi \), let \( C_\xi \) be a club disjoint from \( \{ \text{height}(y) : y \in \psi^{-1}(W_\xi) \} \). Let \( D \) be the diagonal intersection; so \( D \) is club and \( \xi < \alpha \in D \rightarrow \alpha \in C_\xi \). Let \( S \) be the
set of limit $\alpha \in D$ such that $L_\alpha(T) \cap \psi^{-1}(\bigcup W_n) \neq \emptyset$; then $S$ is stationary. For $\alpha \in S$, choose $y_\alpha \in L_\alpha(T) \cap \psi^{-1}(\bigcup W_n)$. Then $y_\alpha \in \psi^{-1}(W_{\xi_\alpha})$ for some (unique) $\xi_\alpha$, and $\xi_\alpha \geq \alpha$ since $\alpha \in D$. Then fix $x_\alpha \subset y_\alpha$ with $x_\alpha \uparrow \cap y_\alpha \downarrow \subset \psi^{-1}(W_{\xi_\alpha})$.

By the Pressing Down Lemma, fix $x$ and a stationary $S' \subseteq S$ such that $x_\alpha = x$ for all $\alpha \in S'$. Then, using $\xi_\alpha \geq \alpha$, fix stationary $S'' \subseteq S'$ such that the $\xi_\alpha$, for $\alpha \in S''$, are all different. Then the sets $x \uparrow \cap y_\alpha \downarrow$, for $\alpha \in S''$ are pairwise disjoint, which is impossible because $L_{\sup(x\uparrow \cap T)}$ is countable. ☺

**Proof of Theorem 1.1.** Call $\psi : T \to B$ a DP map iff $\psi$ is continuous and each $\psi^{-1}\{b\}$ is discrete.

We build $T$, along with an order-preserving $\varphi : T \to \mathbb{Q}$, and use ♦ to defeat all DP maps $\psi : T \to B$.

As a set, $T$ will be the ordinal $\omega_1$, and the root will be 0. We shall define the tree order $\sqsubset$ so that $L_0(T) = \{0\}$, $L_1(T) = \omega \setminus \{0\}$, $L_{n+1}(T) = \{\omega \cdot n + k : k \in \omega\}$ for $0 < n < \omega$, and $L_\alpha(T) = \{\omega \cdot \alpha + k : k \in \omega\}$ when $\omega \leq \alpha < \omega_1$. As in the usual construction of a special Aronszajn tree, we construct $\varphi : T \to \mathbb{Q}$ and $\sqsubset$ recursively so that $\varphi(0) = 0$ and

$$\forall x \in T \forall \alpha < \omega_1 \forall q \in \mathbb{Q} [\alpha > \sup(x) \land q > \varphi(x) \to \exists y \in L_\alpha(x)[x \sqsubset y \land \varphi(y) = q]]. \quad (\ast)$$

This implies, in particular, that each node has $\aleph_0$ immediate successors.

Let $\langle \psi_\alpha : \alpha < \omega_1 \rangle$ be a ♦ sequence, where each $\psi_\alpha : \alpha \to B$. Such a sequence exists by ♦ because $|B| \leq \aleph_1$.

In the recursive construction of $\sqsubset$ and $\varphi$, do the usual thing in building each $L_\gamma(T)$ to preserve $(\ast)$. But in addition, whenever $\omega \cdot \gamma = \gamma > 0$ (so $T_\gamma = \gamma$ as a set, and $\psi_\gamma : T_\gamma \to B$): if $\psi_\gamma$ is a DP map, then if it is possible, extend $\sqsubset$ so that the node $\gamma \in L_\gamma(T)$ satisfies:

$$\sup(\varphi(x) : x \sqsubset \gamma) \leq 1 \text{ and } \langle \psi_\gamma(x) : x \sqsubset \gamma \rangle \text{ does not converge in } B. \quad (\dagger)$$

This implies that $\psi_\gamma$ could not extend to a continuous map into $B$. Use the nodes $\gamma + 1, \gamma + 2, \ldots$ to preserve $(\ast)$, so if $(\dagger)$ is possible, we may let $\varphi(\gamma) = 1$. If $(\dagger)$ is impossible, then ignore it and just preserve $(\ast)$. To ensure that the tree will be Hausdorff, make sure that if $j \neq k$ then $\gamma + j$ and $\gamma + k$ are limits of distinct branches.

**Lemma 2.5 (Main Lemma)** Suppose that $\psi : T \to B$ is a DP map. Then there is a club $C \subseteq \omega_1$ so that for all limit points $\gamma$ of $C$: $\omega \cdot \gamma = \gamma$, and if $\psi_\gamma = \psi|\gamma$, then $(\dagger)$ is possible at level $\gamma$.

The theorem follows immediately, since choosing such a $\gamma$ for which $\psi_\gamma = \psi|\gamma$, we see that $\psi$ cannot be continuous at node $\gamma \in L_\gamma(T)$.

So, we proceed to prove the Main Lemma. We use a standard definition of $C$ — namely, let $\langle M_\xi : \xi < \omega_1 \rangle$ be a continuous chain of countable elementary
submodels of \( H(\theta) \) (for a suitably large regular \( \theta \)), such that \( \varphi, \psi, \sqsubseteq, B \in M_0 \) and each \( M_\xi \in M_{\xi+1} \). Let \( C = \{ M_\xi \cap \omega_1 : \xi < \omega_1 \} \).

Now, fix a limit point \( \gamma \) of \( C \), with \( \psi_\gamma = \psi | \gamma \). Let \( \alpha_n \uparrow \gamma \), with all \( \alpha_n \in C \). We shall build a Cantor tree of candidates for the path satisfying \((\dagger)\), and then prove that one of these works by using the fact that \( B \) does not have a Cantor subset. For \( s \in 2^{<\omega} \), construct \( W_s, U_s, x_s \) with the following properties; here, \(|s|\) denotes the length of \( s \).

1. \( W_s \subseteq B \) is open and non-empty, and \( \operatorname{diam}(W_s) \leq 1/|s| \).
2. \( W_0 = B \).
3. \( \overline{W_{s-0}, W_{s-1}} \subseteq W_s \) and \( \overline{W_{s-0}} \cap \overline{W_{s-1}} = \emptyset \).
4. \( U_s \) is a stationary open subset of \( T \), with \((U_s)^p = U_s \).
5. \( U_0 = \{ x \in T : \varphi(x) < 1 \} \).
6. \( U_{s-0}, U_{s-1} \subseteq U_s \) and \( U_s \subseteq \psi^{-1}(W_s) \).
7. \( x_s \in U_s \) and \( U_{s-i} \subseteq x_s \) for \( i = 0, 1 \).
8. \( x_0 = 0 \), the root node of \( T \).
9. For \( n = |s| \): \( \operatorname{height}(x_s) < \alpha_n \) and, when \( n > 0 \), \( \operatorname{height}(x_s) \geq \alpha_{n-1} \).
10. For \( n = |s| \) and \( \alpha_n = M_{\xi_n} \cap \omega_1 : W_s, U_s, x_s \in M_{\xi_n} \).

For each \( f \in 2^\omega \), conditions \((7)\) and \((9)\) guarantee that \( P_f := \bigcup \{ x_f \downarrow n : n \in \omega \} \) is a cofinal path through \( T_\gamma \). Now, fix \( f \) so that \( \bigcap_{n \in \omega} W_f \downarrow n = \emptyset \). There is such an \( f \) because otherwise, by conditions \((1)(3), \bigcup \{ \bigcap_{n \in \omega} W_f \downarrow n : f \in 2^\omega \} \) would be a Cantor subset of \( B \). Then, \((\dagger)\) will hold if we place node \( \gamma \) above the path \( P_f \); note that condition \((5)\) guarantees that \( \sup \{ \varphi(x) : x \sqsubseteq \gamma \} \leq 1 \), and every limit point of \( \langle \psi_\gamma(x) : x \sqsubseteq \gamma \rangle \) must lie in \( \bigcap_{n \in \omega} W_{f \downarrow n} \), which is empty.

Of course, we need to verify that the \( W_s, U_s, x_s \) can be constructed. Fix \( s \), with \( n = |s| \), and assume that we have \( W_s, U_s, x_s \). Note that \( U_s \cap x_s \) is stationary by \( (U_s)^p = U_s \). Applying Lemma 2.4 (to \( \psi^{-1}(U_s \cap x_s) : (U_s \cap x_s) \rightarrow W_s \)), there exist \( b_0 \neq b_1 \) in \( W_s \) such that \( \psi^{-1}(N_\varepsilon(b_1)) \cap U_s \cap x_s \) is stationary for all \( \varepsilon > 0 \); applying condition \((10)\), choose such \( b_0, b_1 \in M_{\xi_n} \). Then fix \( \varepsilon \) to be the smallest of \( 1/(n+1) \), \( d(b_0, b_1)/3 \), \( d(b_0, B \setminus W_s)/2 \), and \( d(b_1, B \setminus W_s)/2 \). Let \( W_{s-i} = N_\varepsilon(b_1) \) and \( U_{s-i} = (\psi^{-1}(W_{s-i}) \cap U_s \cap x_s)^p \).

Then choose \( x_{s-i} \in U_{s-i} \) with \( \alpha_n \leq \operatorname{height}(x_{s-i}) \); such an \( x_{s-i} \) exists by \((U_{s-i})^p = U_{s-i} \). Also, make sure that \( x_{s-i} \in M_{\xi_{n+1}} \) (using \( M_{\xi_{n+1}} \prec H(\theta) \)), which guarantees that \( \operatorname{height}(x_{s-i}) < \alpha_{n+1} \) and that condition \((10)\) will continue to hold. ☺

3 Constructing Continuous Maps

**Proof of Theorem 1.2.** Let \( H = \{ 1, 4, 16, \ldots \} = \{ 2^n : i \in \omega \} \) and \( K = \{ 2, 8, 32, \ldots \} = \{ 2^{n+1} : i \in \omega \} \). Observe that \( H \cap K = \emptyset \) and

\[
\forall n_1, n_2 \in H \forall j_1, j_2 \in K \left[ n_1 + j_1 = n_2 + j_2 \rightarrow n_1 = n_2 \land j_1 = j_2 \right].
\]
Then define \( j \) values preserving.

To see this, let \( z \) be the range of \( \sum_{n \in H} z_n2^{-n} \), where each \( z_n \in P \). Then \( S \) is compact, since it is the range of the continuous map \( \Gamma : P^H \to \mathbb{R} \) defined by \( \Gamma(z) = \sum_{n \in H} z_n2^{-n} \). Also, \( \Gamma \) is 1-1; that is,

\[
\sum_{n \in H} z_n2^{-n} = \sum_{n \in H} w_n2^{-n} \Rightarrow \forall n \in H [z_n = w_n] \quad \text{(all } z_n, w_n \in P) . \quad (\star)
\]

To see this, let \( z_n = \sum_{j \in K} \varepsilon_{j,n}2^{-j} \) and \( w_n = \sum_{j \in K} \delta_{j,n}2^{-j} \). We then have

\[
\sum \{\varepsilon_{j,n}2^{-(j+n)} : j \in K \land n \in H\} = \sum \{\delta_{j,n}2^{-(j+n)} : j \in K \land n \in H\} .
\]

Since the values \( j + n \) are all different, each \( \varepsilon_{j,n} = \delta_{j,n} \).

For \( n \in H \), define the “coordinate projection” \( \pi_n : S \to P \) so that we have \( \pi_n(\sum_{n \in H} z_n2^{-n}) = z_n \). So, \( \pi_n = \hat{\pi}_n \circ \Gamma^{-1} \), where \( \hat{\pi}_n : P^H \to P \) is the usual coordinate projection.

Since \( T \) is special, fix \( a : T \to H \) such that each \( A_n := a^{-1}\{n\} \) is antichain. Also, fix a 1-1 function \( \zeta : T \to P \setminus \{0\} \) such that \( \zeta(T) \) has no perfect subsets. Then, define

\[
\psi(x) = \sum \{\zeta(t) \cdot 2^{-a(t)} : t \in x\} .
\]

Let \( B \) be the range of \( \psi \); then \( \psi : T \to B \) is clearly continuous and order preserving.

Note that \( \psi(x) = \sum_{n \in H} z_n2^{-n} \), where \( z_n = \zeta(t) \) if \( t \in A_n \cap x \downarrow \), and \( z_n = 0 \) if \( A_n \cap x \downarrow = \emptyset \). Then, \( x \downarrow \neq y \downarrow \Rightarrow \psi(x) \neq \psi(y) \) follows from (\( \star \)) and the fact that \( \zeta \) is 1-1.

Suppose that \( C \subseteq B \) is a Cantor set. Then each \( \pi_n(C) \) is a compact subset of \( \text{ran}(\zeta) \cup \{0\} \), and is hence countable. There is then a countable \( \alpha \) such that \( \pi_n(C) \subseteq \zeta(T_\alpha) \cup \{0\} \) for all \( n \in H \). So, fix \( x \in T \) with \( \psi(x) \in C \) and \( \text{height}(x) > \alpha \), let \( x \downarrow \cap L_\alpha(T) = \{t\} \), and let \( n = a(t) \). Then \( \zeta(t) = \pi_n(\psi(x)) \in \pi_n(C) \) and \( \zeta(t) \notin \zeta(T_\alpha) \cup \{0\} \), a contradiction.

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