Matrices resulting from the discretization of a kernel function, e.g., in the context of integral equations or sampling probability distributions, can frequently be approximated by interpolation. In order to improve the efficiency, a multi-level approach can be employed that involves interpolating the kernel functions and its approximations multiple times.

This article presents a new approach to analyze the error incurred by these iterated interpolation procedures that is considerably more elegant than its predecessors and allows us to treat not only the kernel function itself, but also its derivatives.

**Keywords:** Interpolation, variable-order interpolation, BEM, fast methods for boundary element matrices, $\mathcal{H}^2$-matrices

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1 Introduction

Let us consider a model problem from astrophysics: we have $n$ bodies with masses $m_1, \ldots, m_n$ at points $x_1, \ldots, x_n$ in space, and we want to evaluate the resulting gravitational forces $f_1, \ldots, f_n$ in $n$ points $y_1, \ldots, y_n$. Newton’s law yields

$$f_i = \sum_{j=1}^{n} m_j g(y_i, x_j), \quad g(y, x) := c \frac{x - y}{\|x - y\|^3}, \quad \text{for all } i \in [1 : n],$$

where $c$ is the gravitational constant, $\|z\|$ is the Euclidean norm, and $[1 : n] := \{1, \ldots, n\}$. Direct evaluation of all $f_i$ would require $\mathcal{O}(n^2)$ operations and is therefore unattractive or even practically impossible if $n$ is large. In order to evaluate the forces efficiently, we can approximate the function $g$ by sums of tensor products, i.e.,

$$g(y, x) \approx \sum_{\nu=1}^{k} a_{\nu}(y)b_{\nu}(x) \quad \text{for all } x \in \sigma, \ y \in \tau, \quad (1)$$
with suitable functions $a_1, \ldots, a_k$ on a domain $\tau$ and $b_1, \ldots, b_k$ on a domain $\sigma$. If all $y_i$ are in $\tau$ and all $x_j$ are in $\sigma$, we obtain

$$f_i \approx \sum_{j=1}^{n} m_j \sum_{\nu=1}^{k} a_\nu(y_i) b_\nu(x_j) = \sum_{\nu=1}^{k} a_\nu(y_i) \sum_{j=1}^{n} m_j b_\nu(x_j) =: \hat{x}_\nu$$

for all $i \in [1 : n]$, allowing us to compute $\hat{x}_\nu$ for all $\nu \in [1 : k]$ in $O(nk)$ operations and then evaluate $f_i$ for all $i \in [1 : n]$ in $O(nk)$ operations, i.e., we have a chance of reaching linear instead of quadratic complexity.

This leaves us with the challenge of finding approximations of the form eq. (1) that approximate $g$ sufficiently well. For gravitational and electrostatic forces, the fast multipole method [18, 11, 12] solves this task by using a special expansion optimized for this particular function $g$. For a significantly larger class of functions, standard polynomial approximations via Taylor expansion [14, 19] or interpolation [10, 3] can be employed to similar effect.

For the sake of simplicity, we focus here on interpolation of $x \mapsto g(y, x)$. Let $(\xi_{\sigma, \nu})_{\nu=1}^{k}$ be interpolation points in $\sigma$, and let $(\ell_{\sigma, \nu})_{\nu=1}^{k}$ be the corresponding Lagrange polynomials. Assuming that the interpolation error is under control, we have

$$g(y, x) \approx \sum_{\nu=1}^{k} g(y, \xi_{\sigma, \nu}) \ell_{\sigma, \nu}(x)$$

for all $x \in \sigma$, $y \in \tau$, and this is obviously an approximation of the required form eq. (1).

Except for very special cases, none of these techniques can give us a global approximation of $g$, i.e., an approximation that is valid for all $y_i$ and all $x_j$. This is not surprising since $g$ has a singularity at $x = y$ that cannot be resolved by an approximation of the form eq. (1). Instead we only get approximations on subdomains $\tau \times \sigma$ and have to use multiple subdomains to cover all combinations of points. The number of these subdomains can grow very large, frequently there are $O(n)$ subdomains, so we need an efficient approach to handling large numbers of subdomains.

A very successful strategy relies on a hierarchy of subdomains: assume that every subdomain $\sigma$ is either small, so that the few points it contains can be treated directly, or subdivided into two disjoint subdomains $\sigma_1$ and $\sigma_2$. If the vectors

$$\hat{x}_{\sigma_1, \nu} = \sum_{j=1 \atop x_j \in \sigma_1}^{n} m_j \ell_{\sigma_1, \nu}(x_j), \quad \hat{x}_{\sigma_2, \nu} = \sum_{j=1 \atop x_j \in \sigma_2}^{n} m_j \ell_{\sigma_2, \nu}(x_j)$$

have already been computed, we can re-interpolate the Lagrange polynomials $\ell_{\sigma, \nu}$ in the points in $\sigma_1$ and $\sigma_2$, respectively, to obtain

$$\ell_{\sigma, \nu} \approx \sum_{\nu_1=1}^{k} \ell_{\sigma, \nu}(\xi_{\sigma_1, \nu_1}) \ell_{\sigma_1, \nu_1}, \quad \ell_{\sigma, \nu} \approx \sum_{\nu_2=1}^{k} \ell_{\sigma, \nu}(\xi_{\sigma_2, \nu_2}) \ell_{\sigma_2, \nu_2},$$

for all $\nu \in [1 : k]$. This allows us to compute $\hat{x}_\nu$ for all $\nu \in [1 : k]$ in $O(nk)$ operations and then evaluate $f_i$ for all $i \in [1 : n]$ in $O(nk)$ operations, i.e., we have a chance of reaching linear instead of quadratic complexity.
and therefore

\[ \hat{x}_{\sigma,\nu} = \sum_{j=1}^{n} m_j \ell_{\sigma,\nu}(x_j) = \sum_{j=1}^{n} m_j \ell_{\sigma,\nu}(x_j) + \sum_{j=1}^{n} m_j \ell_{\sigma,\nu}(x_j) \]

\[ \approx \sum_{j=1}^{n} m_j \sum_{\nu_1=1}^{k} \ell_{\sigma,\nu}(\xi_{\sigma_1,\nu_1}) \ell_{\sigma_1,\nu_1}(x_j) + \sum_{j=1}^{n} m_j \sum_{\nu_2=1}^{k} \ell_{\sigma,\nu}(\xi_{\sigma_2,\nu_2}) \ell_{\sigma_2,\nu_2}(x_j) \]

\[ = \sum_{\nu_1=1}^{k} \ell_{\sigma,\nu}(\xi_{\sigma_1,\nu_1}) \hat{x}_{\sigma_1,\nu_1} + \sum_{\nu_2=1}^{k} \ell_{\sigma,\nu}(\xi_{\sigma_2,\nu_2}) \hat{x}_{\sigma_2,\nu_2}, \]

i.e., we can compute \( \hat{x}_{\sigma,\nu} \) by evaluating only \( 2k \) summands. This approach leads to fast multipole methods \cite{18,11} and \( H^2 \)-matrix representations \cite{13,2,1} that require only a total of \( O(nk) \) operations for all subdomains.

The resulting algorithm interpolates \( g \) on a domain \( \tau_0 \times \sigma_0 \), interpolates the result again on a smaller domain \( \tau_1 \times \sigma_1 \), which is then interpolated on an even smaller domain \( \tau_2 \times \sigma_2 \), until very small domains have been reached. The subject of this article is to investigate the cumulative effect of these iterated interpolation steps on the final error and the stability of the procedure.

Previous results rely either on Taylor expansion \cite{19} or Chebyshev expansions on Bernstein elliptic discs \cite{4,5}. Bernstein discs offer a very precise characterization of the convergence behaviour of interpolation on intervals \cite{8}, but they have so far only been used in intermediate steps.

The new approach presented in this article is based on a slight generalization (cf. theorem \cite{11} of a well-known result \cite[Theorem 7.8.1]{8}) that allows us to bound the error on an entire Bernstein disc instead of an interval.

Using this error estimate allows us to

- obtain a fairly general proof of convergence and stability for variable-order methods \cite{19,4}, cf. theorem \cite{9} and theorem \cite{10},
- prove that iterated interpolation is stable as long as the interpolation orders are not too small, cf. theorem \cite{12} and theorem \cite{13},
- prove that iterated interpolation can be used to approximate derivatives, e.g., to cover the double-layer and hypersingular operators in boundary element methods or to compute gradients of potentials, cf. theorem \cite{15} and
- prove that these results can be generalized to advanced approximation techniques for oscillatory kernel functions appearing, e.g., in boundary element methods for the high-frequency Helmholtz equation \cite{7,9,16,5}, cf. theorem \cite{17}.

To keep the presentation simple, this article focuses on the one-dimensional setting. Tensor methods can be used to extend the results to multi-dimensional interpolation.
Section 2 follows in the footsteps of \[8,\] Theorem 7.8.1] to prove the generalized best-approximation estimate theorem and the interpolation error estimate theorem. Section 3 investigates the relationship between Bernstein discs for nested intervals, with the key result of theorem 7 showing that if the intervals shrink uniformly, the transformed Bernstein discs grow uniformly. Section 4 takes advantage of this property to prove two error and stability estimates for iterated interpolation: theorem 9 is well-suited for variable-order interpolation, while theorem 12 allows us to handle derivatives of the interpolating polynomial. Section 5 covers a special class of interpolation operators tailored to the oscillatory kernel function of standard Helmholtz boundary element methods.

2 Interpolation on Bernstein discs

Before we discuss iterated interpolation, we briefly recall a few fundamental results concerning the approximation of holomorphic functions by interpolation. A key tool is the Joukowsky transformation \[15\] given by

\[\gamma: \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{z + 1/z}{2}.\]

For every \(w \in \mathbb{C}\), we can find a solution \(z \in \mathbb{C}\) of the quadratic equation \(z^2 - 2wz + 1 = 0\), this solution is non-zero, \(1/z\) is also a solution, and we have \(\gamma(z) = \gamma(1/z) = w\), so \(\gamma\) is surjective.

The Joukowsky transformation maps the unit circle \(S_1 = \{z \in \mathbb{C} : |z| = 1\}\) to the unit interval \([-1, 1]\) due to \(\gamma(z) = \Re(z)\) for all \(z \in S_1\).

The Joukowsky transformation maps the real half-axis \(\mathbb{R}_{\geq 1}\) onto itself and is monotonically increasing, i.e., we have

\[x > y \iff \gamma(x) > \gamma(y)\]

for all \(x, y \in \mathbb{R}_{\geq 1}\). \hspace{1cm} (2)

Finally, for every \(\varrho \in \mathbb{R}_{>1}\), the Joukowsky transformation maps the (open and closed) annuli

\[A_\varrho := \{z \in \mathbb{C} : 1/\varrho < |z| < \varrho\}, \quad \bar{A}_\varrho := \{z \in \mathbb{C} : 1/\varrho \leq |z| \leq \varrho\}\]

to the (open and closed) Bernstein elliptic discs

\[D_\varrho := \{w \in \mathbb{C} : |w - 1| + |w + 1| < 2\gamma(\varrho)\}, \quad \bar{D}_\varrho := \{w \in \mathbb{C} : |w - 1| + |w + 1| \leq 2\gamma(\varrho)\},\]

this follows from eq. (2) and the identity \(|\gamma(z) - 1| + |\gamma(z) + 1| = 2\gamma(|z|)\) for all \(z \in \mathbb{C} \setminus \{0\}\).

This last property allows us to investigate the approximation of holomorphic functions by polynomials \([8, \S 7.8]\): let \(\varrho \in \mathbb{R}_{>1}\), and let \(f: D_\varrho \to \mathbb{C}\) be holomorphic. Then \(\hat{f} := f \circ \gamma\) is holomorphic in the annulus \(A_\varrho\) and therefore has a Laurent series expansion

\[\hat{f}(z) = \sum_{n=-\infty}^{\infty} a_n z^n\]

for all \(z \in A_\varrho\).
with coefficients

\[ a_n := \frac{1}{2\pi i} \int_{|z|=r} \frac{\hat{f}(z)}{z^{n+1}} \, dz \quad \text{for all } n \in \mathbb{Z}, \quad (3) \]

where any \( r \in (1/\varrho, \varrho) \) can be chosen due to Cauchy’s integral theorem. Since \( \gamma(1/z) = \gamma(z) \), we also have \( \hat{f}(1/z) = \hat{f}(z) \) and obtain \( a_{-n} = a_n \) for all \( n \in \mathbb{N} \), and therefore

\[ \hat{f}(z) = a_0 + 2 \sum_{n=1}^{\infty} a_n \frac{z^n + z^{-n}}{2} \quad \text{for all } z \in \mathcal{A}_\varrho. \]

It is easy to verify that the Chebyshev polynomials given by

\[ C_n(w) := \begin{cases} 1 & \text{if } n = 0, \\ w & \text{if } n = 1, \\ 2wC_{n-1}(w) - C_{n-2}(w) & \text{otherwise} \end{cases} \quad \text{for all } w \in \mathbb{C}, \ n \in \mathbb{N}_0 \]

satisfy the equation

\[ C_n(\gamma(z)) = \frac{z^n + z^{-n}}{2} \quad \text{for all } z \in \mathbb{C} \setminus \{0\}, \ n \in \mathbb{N}_0. \]

Let \( w \in \mathcal{D}_\varrho \). We have seen that we can find \( z \in \mathcal{A}_\varrho \) such that \( w = \gamma(z) \) and therefore

\[ f(w) = \hat{f}(z) = a_0 + 2 \sum_{n=1}^{\infty} a_n \frac{z^n + z^{-n}}{2} = a_0 + 2 \sum_{n=1}^{\infty} a_n C_n(w), \]

i.e., the Laurent series of \( \hat{f} \) corresponds to the Chebyshev expansion of \( f \). Truncating the Chebyshev expansion yields polynomial approximations of \( f \).

In order to estimate the approximation error, we require bounds for the coefficients \( a_n \) and the Chebyshev polynomials \( C_n \). We introduce the notation

\[ \|f\|_{\infty, \Omega} := \sup \{|f(w)| : w \in \Omega\} \]

for functions \( f : \Omega \to \mathbb{C} \) and sets \( \Omega \subseteq \mathbb{C} \). For the coefficients eq. \( (3) \) we have

\[ |a_n| \leq \lim_{r \to \varrho} \max_{r^n} \frac{\hat{f}(z) : |z|=r}{r^n} \leq \frac{\|\hat{f}\|_{\infty, \mathcal{D}_\varrho}}{\varrho^n} \quad \text{for all } n \in \mathbb{N}, \quad (4) \]

while for \( \varrho \in [1, \varrho] \) we have

\[ |C_n(w)| \leq \frac{|z^n + |z|^{-n}|}{2} \leq \varrho^n \quad \text{for all } w \in \mathcal{D}_\varrho, \quad (5) \]

where we choose \( z \in \mathcal{A}_\varrho \) with \( \gamma(z) = w \). Combining both estimates yields an error estimate.
Theorem 1 (Approximation error) Let $\varrho \in \mathbb{R}_{>1}$ and $\hat{\varrho} \in [1, \varrho)$. Let $f : D_{\varrho} \to \mathbb{C}$ be holomorphic. For any $m \in \mathbb{N}$ we can find an $m$-th order polynomial $p$ such that

$$\|f - p\|_{\infty, \bar{D}_{\hat{\varrho}}} \leq \frac{2}{\varrho/\hat{\varrho} - 1} \left(\frac{\hat{\varrho}}{\varrho}\right)^m \|f\|_{\infty, D_{\varrho}}.$$ 

Proof: The proof is a slight modification of [8, Theorem 7.8.1]. Let $m \in \mathbb{N}$ and $p := a_0 + 2 \sum_{n=1}^m a_n C_n$.

Combining eq. (4) and eq. (5), we obtain

$$|f(w) - p(w)| = 2 \left| \sum_{n=m+1}^{\infty} a_n C_n(w) \right| \leq 2 \sum_{n=m+1}^{\infty} |a_n| |C_n(w)| \leq 2 \sum_{n=m+1}^{\infty} \|f\|_{\infty, D_{\varrho}} \frac{\hat{\varrho}^n}{\varrho^{n+1}} \frac{\hat{\varrho}}{\varrho} = 2 \|f\|_{\infty, D_{\varrho}} \frac{\hat{\varrho}^{m+1}}{\varrho^m - \hat{\varrho}^m} \left| \frac{\hat{\varrho}}{\varrho} \right|$$

by using the geometric series equation. □

Due to $D_1 = [-1, 1]$, the special case $\hat{\varrho} = 1$ yields

$$\|f - p\|_{\infty, [-1, 1]} \leq \frac{2}{\varrho - 1} \varrho^{-m} \|f\|_{\infty, D_{\varrho}}.$$ 

While proving the existence of an approximating polynomial is reassuring, practical applications require us to actually construct such a polynomial. We will accomplish this task by interpolation: let $m \in \mathbb{N}$, and let $\xi_0, \ldots, \xi_m \in [-1, 1]$ be pairwise distinct interpolation points and $\ell_0, \ldots, \ell_m$ the corresponding $m$-th order Lagrange polynomials. We denote the corresponding interpolation operator by

$$\mathcal{I}_m[f] := \sum_{\nu=0}^m f(\xi_\nu) \ell_\nu$$

for all $f \in C[-1, 1]$. (6)

We have $\mathcal{I}_m[p] = p$ for any $m$-th order polynomial $p$ and

$$\|\mathcal{I}_m[f]\|_{\infty, [-1, 1]} \leq \Lambda_m \|f\|_{\infty, [-1, 1]}$$

for all $f \in C[-1, 1]$, (7)

where the Lebesgue constant $\Lambda_m$ is given by

$$\Lambda_m := \max \left\{ \sum_{\nu=0}^m |\ell_\nu(x)| : x \in [-1, 1] \right\}.$$ 

In order to extend this stability estimate from $[-1, 1]$ to a closed Bernstein disc $D_{\hat{\varrho}}$, we use the Bernstein inequality.
Lemma 2 (Bernstein inequality) Let \( p \) be an \( m \)-th order polynomial, let \( \hat{\varrho} \in \mathbb{R}_{\geq 1} \). We have
\[
\| p \|_{\infty, \hat{D}_\varrho} \leq \hat{\varrho}^m \| p \|_{\infty, [-1, 1]}.
\]

Proof: cf. [Theorem 4.2.2]

Corollary 3 (Interpolation error) Let \( \varrho \in \mathbb{R}_{>1} \) and \( \hat{\varrho} \in [1, \varrho) \). Let \( f : D_\varrho \to \mathbb{C} \) be holomorphic. We have
\[
\| f - \mathcal{I}_m[f] \|_{\infty, \hat{D}_\varrho} \leq \frac{2(1 + \Lambda_m)}{\varrho/\hat{\varrho} - 1} \left( \frac{\hat{\varrho}}{\varrho} \right)^m \| f \|_{\infty, D_\varrho}.
\]

Proof: Let \( p \) be the \( m \)-th order polynomial constructed in theorem [1]. Due to \( \mathcal{I}_m[p] = p \) and theorem [2] we have
\[
\| f - \mathcal{I}_m[f] \|_{\infty, \hat{D}_\varrho} = \| f - p + \mathcal{I}_m[p - f] \|_{\infty, \hat{D}_\varrho} \leq \| f - p \|_{\infty, \hat{D}_\varrho} + \| \mathcal{I}_m[f - p] \|_{\infty, \hat{D}_\varrho}
\]
\[
\leq \| f - p \|_{\infty, \hat{D}_\varrho} + \hat{\varrho}^m \| \mathcal{I}_m[f - p] \|_{\infty, [-1, 1]}
\]
\[
\leq \| f - p \|_{\infty, \hat{D}_\varrho} + \hat{\varrho}^m \Lambda_m \| f - p \|_{\infty, [-1, 1]}
\]
\[
\leq \frac{2}{\varrho/\hat{\varrho} - 1} \left( \frac{\hat{\varrho}}{\varrho} \right)^m \| f \|_{\infty, D_\varrho} + \hat{\varrho}^m \Lambda_m \frac{2}{\varrho/\hat{\varrho} - 1} \left( \frac{1}{\varrho} \right)^m \| f \|_{\infty, D_\varrho}
\]
due to \( 1 \leq \hat{\varrho} < \varrho \) and therefore \( \varrho \geq \varrho/\hat{\varrho} > 1 \).

For our investigation, we need interpolation operators of different order on general domains. Let \( (\mathcal{I}_m)_{m=1}^\infty \) be a family of interpolation operators of the type eq. [5] on the reference interval \([-1, 1]\) with corresponding Lebesgue numbers \( (\Lambda_m)_{m=1}^\infty \). For an interval \([a, b] , a < b\), we use the simple transformation
\[
\Phi_{a,b} : [-1, 1] \to [a, b], \quad x \mapsto \frac{b + a}{2} + \frac{b - a}{2}x,
\]
to define the transformed interpolation operators \( \mathcal{I}_{[a,b],m} \) for all \( m \in \mathbb{N} \) by
\[
\mathcal{I}_{[a,b],m}[f] := \mathcal{I}_m[f \circ \Phi_{a,b}] \circ \Phi_{a,b}^{-1} \quad \text{for all } f \in C[a,b].
\]
For our error estimates, we introduce the transformed Bernstein elliptic discs
\[
D_{[a,b],\varrho} :\Phi_{a,b}(D_\varrho), \quad \bar{D}_{[a,b],\varrho} :\Phi_{a,b}(\bar{D}_\varrho)
\]
and the short notation
\[
\| f \|_{[a,b],\varrho} := \| f \|_{\infty, \bar{D}_{[a,b],\varrho}} = \max \{ |f(w)| : w \in D_{[a,b],\varrho} \} \quad \text{for all } f \in C(D_{[a,b],\varrho}).
\]
In a slight abuse of notation, we apply this norm also to functions with domains larger than \( D_{[a,b],\varrho} \). Theorem [3] takes the following form:

\[
\]
Corollary 4 (Interpolation error) Let \( g \in \mathbb{R}_{>1} \) and \( \hat{g} \in [1, g) \). Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( f : D_{[a,b],[\hat{a},\hat{b}]} \to \mathbb{C} \) be holomorphic. We have
\[
\| f - \mathcal{J}_{[a,b],m}[f] \|_{[a,b],[\hat{a},\hat{b}]} \leq \frac{2(1 + \Lambda_m)}{\hat{g}/g - 1} \left( \frac{\hat{g}}{g} \right)^m \| f \|_{[a,b],[\hat{a},\hat{b}]}.
\]

Proof: The function \( \hat{f} := f \circ \Phi_{a,b} \) is holomorphic in \( D_{\hat{g}} \). Theorem 3 yields
\[
\| f - \mathcal{J}_{[a,b],m}[f] \|_{[a,b],[\hat{a},\hat{b}]} = \| f \circ \Phi_{a,b} - \mathcal{J}_{m}[f \circ \Phi_{a,b}] \|_{\infty, D_{\hat{g}}} = \| \hat{f} - \mathcal{J}_{m}[\hat{f}] \|_{\infty, D_{\hat{g}}} \leq \frac{2(1 + \Lambda_m)}{\hat{g}/g - 1} \left( \frac{\hat{g}}{g} \right)^m \| \hat{f} \|_{[a,b],[\hat{a},\hat{b}]},
\]
where we have used \( \mathcal{J}_{[a,b],m}[f] \circ \Phi_{a,b} = \mathcal{J}_m[f \circ \Phi_{a,b}] \). \( \square \)

3 Bernstein discs for nested intervals

Since theorem 4 requires \( \hat{g} < g \), we can expect iterated interpolation to work only if the Bernstein disc \( D_{[a,b],[\hat{a},\hat{b}]} \) for an interval \([a, b] \subseteq [-1,1]\) is contained in the Bernstein disc \( D_{[-1,1],[\hat{a},\hat{b}]} \) with \( \hat{g} > g \). If we can ensure that the ratio between the lengths of \([a, b]\) and \([-1,1]\) is bounded and that \( \hat{g} \) is not too small, we can prove \( \hat{g} \geq \sigma g \) with \( \sigma > 1 \) and thus obtain an estimate for the rate of convergence.

Lemma 5 (Nested Bernstein discs) Let \( a, b \in \mathbb{R} \) with \(-1 \leq a < b \leq 1\), let \( \delta := \frac{b-a}{2} \). The function
\[
\gamma^\dagger : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}, \quad \varrho \mapsto \varrho + \sqrt{\varrho^2 - 1},
\]
satisfies \( \gamma(\gamma^\dagger(\varrho)) = \varrho \) for all \( \varrho \in \mathbb{R}_{>1} \). For \( \varrho \in \mathbb{R}_{>1} \) let
\[
\varrho_{a,b} := \gamma^\dagger \left( \frac{\gamma(\varrho) - 1}{\delta} + 1 \right).
\]
We have \( D_{[a,b],[\varrho_{a,b}]} \subseteq D_{\varrho} \).

Proof: Let \( w \in D_{[a,b],[\varrho_{a,b}]} \). By definition, this means that there is a \( \hat{w} \in D_{\varrho_{a,b}} \) with \( w = \Phi_{a,b}(\hat{w}) \). We observe
\[
|w - b| + |w - a| = \left| \frac{b + a}{2} + \frac{b - a}{2} \hat{w} - b \right| + \left| \frac{b + a}{2} + \frac{b - a}{2} \hat{w} - a \right|
= \left| \delta \hat{w} - \frac{b - a}{2} \right| + \left| \delta \hat{w} + \frac{b - a}{2} \right|
= \delta |\hat{w} - 1| + \delta |\hat{w} + 1| < 2\delta \gamma(\varrho_{a,b}).
\]
For all \( \varrho \in \mathbb{R}_{\geq 1} \), we have
\[
\gamma(\gamma^\dagger(\varrho)) = \frac{1}{2} \left( \varrho + \sqrt{\varrho^2 - 1} + \frac{1}{\varrho + \sqrt{\varrho^2 - 1}} \right) = \frac{(\varrho + \sqrt{\varrho^2 - 1})^2 + 1}{2(\varrho + \sqrt{\varrho^2 - 1})} = \varrho.
\]
Due to \( b \leq 1 \) and \(-1 \leq a\) and using the definition of \( \varrho_{a,b} \), we have
\[
|w - 1| + |w + 1| = |w - b + b - 1| + |w - a + a + 1| \\
\leq |w - b| + |b - 1| + |w - a| + |a + 1| \\
= |w - b| + 1 - b + |w - a| + a + 1 \\
< 2\delta\gamma(\varrho_{a,b}) + 2 - (b - a) = 2(\delta\gamma(\varrho_{a,b}) + 1 - \delta) \\
= 2\left(\delta\left(\frac{\gamma(\varrho) - 1}{\delta} + 1\right) + 1 - \delta\right) = 2\gamma(\varrho),
\]
and therefore \( w \in D_\varrho \). \( \square \)

If we want to interpolate a holomorphic function \( f \) given in \( D_\varrho \) on the subinterval \([a, b] \subseteq [-1, 1]\), we find that the function is holomorphic in \( D_{[a,b]}\varrho_{a,b} \) and that the error \( D_{[a,b]}\varrho_{a,b} \) will converge at a rate of \( \varrho/\varrho_{a,b} \). We have
\[
\gamma(\varrho_{a,b}) = \gamma(\varrho) - \frac{1}{\delta} + 1 = \gamma(\varrho) + \frac{1 - \delta}{\delta}(\gamma(\varrho) - 1) > \gamma(\varrho),
\]
and eq. (2) yields \( \varrho_{a,b} > \varrho \), i.e., we can expect exponential convergence. Finding a bound for the rate of convergence is a slightly more challenging task.

**Lemma 6 (Rate of convergence)** The function
\[
\hat{\sigma}: \mathbb{R}_{\geq 1} \times (0,1) \to \mathbb{R}_{\geq 1}, \quad (\varrho, \delta) \mapsto \frac{\gamma^{1}\left(\frac{\gamma(\varrho) - 1}{\delta} + 1\right)}{\varrho},
\]
is monotonically increasing in \( \varrho \) and monotonically decreasing in \( \delta \) with the limits
\[
\hat{\sigma}(1, \delta) = 1, \quad \lim_{\varrho \to \infty} \hat{\sigma}(\varrho, \delta) = 1/\delta \quad \text{for all } \delta \in (0, 1).
\]

**Proof:** Since \( \gamma^{1} \) is monotonically increasing, \( \hat{\sigma} \) is monotonically decreasing in \( \delta \).

To prove that \( \hat{\sigma} \) is monotonically increasing in \( \varrho \), we fix \( \delta \in (0, 1) \). Due to eq. (2), the identity \( \gamma(\gamma^{1}(\gamma(\varrho))) = \gamma(\varrho) \) implies \( \gamma^{1}(\gamma(\varrho)) = \varrho \) for all \( \varrho \in \mathbb{R}_{\geq 1} \), and we have
\[
\hat{\sigma}(\varrho, \delta) = \frac{\gamma^{1}\left(\frac{\gamma(\varrho) - 1}{\delta} + 1\right)}{\gamma^{1}(\gamma(\varrho))} \quad \text{for all } \varrho \in \mathbb{R}_{\geq 1}.
\]

We already know that \( \gamma \) is monotonically increasing, so it suffices to prove that
\[
g: \mathbb{R}_{\geq 1} \to \mathbb{R}, \quad x \mapsto \frac{\gamma^{1}\left(\frac{x - 1}{\delta} + 1\right)}{\gamma^{1}(x)},
\]
is monotonically increasing. Using
\[
\frac{\partial}{\partial \varrho} \gamma^{1}(\varrho) = 1 + \frac{2\varrho}{2\sqrt{\varrho^2 - 1}} = \frac{\gamma^{1}(\varrho)}{\sqrt{\varrho^2 - 1}} \quad \text{for all } \varrho \in \mathbb{R}_{\geq 1},
\]

9
the chain and quotient rules yield

\[
\frac{\partial g}{\partial x}(x) = \frac{\gamma^\dagger(x) - \gamma^\dagger \left( \frac{x-1}{\delta} + 1 \right) \frac{\gamma^\dagger(x)}{\sqrt{x^2-1}}}{\delta \sqrt{\left( \frac{x-1}{\delta} + 1 \right)^2 - 1}}
\]

\[
= \frac{1 - \delta \sqrt{\left( \frac{x-1}{\delta} + 1 \right)^2 - 1}}{\gamma^\dagger(x) \delta \sqrt{\left( \frac{x-1}{\delta} + 1 \right)^2 - 1}} \gamma^\dagger \left( \frac{x-1}{\delta} + 1 \right).
\]

Since \( \gamma^\dagger \) maps into \( \mathbb{R}_{\geq 1} \), it suffices to prove

\[
0 \leq 1 - \delta \sqrt{\left( \frac{x-1}{\delta} + 1 \right)^2 - 1} = 1 - \sqrt{\left( \frac{x-1 + \delta}{x^2} \right)^2 - \delta^2}.
\]

Due to \( \delta < 1 \), we have

\[
\frac{(x-1 + \delta)^2 - \delta^2}{x^2 - 1} = \frac{(x-1)^2 + 2\delta(x-1)}{(x+1)(x-1)} = \frac{x-1 + 2\delta}{x+1} < 1,
\]

i.e., \( g \) is monotonically increasing, and so is \( \varrho \mapsto \hat{\sigma}(\varrho, \delta) = g(\gamma(\varrho)) \).

The identity \( \hat{\sigma}(1, \delta) = 1 \) follows directly from \( \gamma(1) = 1 \) and \( \gamma^\dagger(1) = 1 \). Due to

\[
\lim_{x \to \infty} \frac{\gamma^\dagger(x)}{x} = 2,
\]

\[
\lim_{\varrho \to \infty} \frac{\gamma(\varrho) - 1}{\varrho} = \frac{1}{2\delta},
\]

we obtain

\[
\lim_{\varrho \to \infty} \hat{\sigma}(\varrho, \delta) = \lim_{\varrho \to \infty} \frac{\gamma^\dagger \left( \frac{\gamma(\varrho) - 1}{\delta} + 1 \right) \frac{\gamma(\varrho) - 1}{\varrho} + 1}{\delta} = \frac{2}{2\delta} = \frac{1}{\delta},
\]

where we have used that \( \varrho \mapsto \frac{\gamma(\varrho) - 1}{\delta} + 1 \) grows to infinity as \( \varrho \to \infty \).

Combining theorem 5 with theorem 6 allows us to estimate the size of Bernstein discs around an interval \([a, b]\) contained in a larger interval \([c, d]\).

**Corollary 7 (Nested discs)** Let \( \varrho_0 \in \mathbb{R}_{>1} \) and \( \delta_0 \in (0, 1) \). There is a \( \sigma \in \mathbb{R}_{>1} \) such that

\[
D_{[a,b],\sigma} \subseteq D_{[c,d],\sigma} \quad \text{for all} \ \varrho \geq \varrho_0, \ c \leq a < b \leq d \ \text{with} \ b - a \leq \delta_0(d - c).
\]

**Proof:** Using the function \( \hat{\sigma} \) introduced in theorem 6 we choose \( \sigma := \hat{\sigma}(\varrho_0, \delta_0) \).

Let \( \varrho \in \mathbb{R}_{\geq \varrho_0} \) and \( a, b, c, d \in \mathbb{R} \) with \( c \leq a < b \leq d \), and \( (b - a) \leq \delta_0(d - c) \). In order to apply theorem 6 we have to transform \([c, d]\) to the reference interval \([-1, 1]\). Due to \( a, b \in [c, d] \), we have

\[
\hat{a} := \frac{2}{d-c} \left( a - \frac{d+c}{2} \right) \in [-1, 1], \quad \hat{b} := \frac{2}{d-c} \left( b - \frac{d+c}{2} \right) \in [-1, 1]
\]
and \( \Phi_{c,d}(\hat{a}) = a \) as well as \( \Phi_{c,d}(\hat{b}) = b \). We let
\[
\delta := \frac{\hat{b} - \hat{a}}{2} = \frac{b - a}{d - c} \leq \delta_0
\]
and use theorem \( \Box \) to find \( \sigma = \hat{\sigma}(\varrho_0, \delta_0) \leq \hat{\sigma}(\varrho, \delta) \), so that theorem \( \Box \) yields \( D_{[\hat{a}, \hat{b}], \varrho_0} \subseteq D_{\varrho} \).

Applying \( \Phi_{c,d} \) gives us
\[
D_{[a,b], \varrho_0} = \Phi_{c,d}(D_{[\hat{a}, \hat{b}], \varrho_0}) \subseteq \Phi_{c,d}(D_{\varrho}) = D_{[c,d], \varrho},
\]
and the proof is complete. \( \Box \)

### 4 Iterated interpolation

Approximation schemes like variable-order \( H^2 \)-matrices \([19, 4, 6]\) and \( DH^2 \)-matrices for the high-frequency Helmholtz equation \([9, 5]\) rely on interpolation along a nested sequence
\[
[a_L, b_L] \subseteq [a_{L-1}, b_{L-1}] \subseteq \ldots \subseteq [a_1, b_1] \subseteq [a_0, b_0]
\]
of intervals: we first interpolate a given function \( f \) on the second-largest interval \([a_1, b_1]\), then interpolate the result again on the third-largest interval \([a_2, b_2]\), and repeat the process until we reach \([a_L, b_L]\). Our task is to prove that this sequence of interpolation steps leads to a reasonable approximation of the original function \( f \).

In order to investigate the interpolation error for different orders, we require the family of interpolation operators to be stable, i.e., we assume that there are constants \( \Lambda, \lambda \in \mathbb{R} > 0 \) such that
\[
\Lambda_m \leq \Lambda(1 + m)^\lambda \quad \text{for all } m \in \mathbb{N}.
\] (8)

Chebyshev interpolation satisfies this assumption with \( \Lambda = \lambda = 1 \) \([17]\). Using this assumption, we obtain a more convenient estimate for the interpolation error.

**Theorem 8 (Interpolation error)** Let \( \sigma \in \mathbb{R}_{>1} \) and \( q \in (1/\sigma, 1] \). There is a constant \( C_{\text{in}} \) depending only on eq. (8), \( \sigma \) and \( q \) such that for all \( \varrho \in \mathbb{R}_{\geq 1} \), all \( \tau \in \mathbb{R}_{\geq 1} \), all \( a, b \in \mathbb{R} \) with \( a < b \) and all holomorphic \( f: D_{[a,b], \varrho \tau} \to \mathbb{R} \) we have
\[
\|f - \mathcal{I}_{[a,b],m}[f]\|_{[a,b], \varrho \tau} \leq C_{\text{in}} q^m \tau^{-m} \|f\|_{[a,b], \varrho \tau} \quad \text{for all } m \in \mathbb{N}.
\]

**Proof:** Due to the stability condition eq. (8) the supremum
\[
C_{\text{in}} := \sup \left\{ \frac{2(1 + \Lambda_m)}{(\sigma - 1)} \left( \frac{1}{\sigma q} \right)^m : m \in \mathbb{N} \right\}
\] (9)
is finite, since \( \sigma q > 1 \) implies that the exponential \((\sigma q)^m\) grows faster than \( \Lambda_m \) as \( m \) increases.
Now let \( \rho \in \mathbb{R}_{\geq 0} \), \( \tau \in \mathbb{R}_{\geq 1} \), \( a, b \in \mathbb{R} \) with \( a < b \) and \( m \in \mathbb{N} \). Let \( f \) be a function that is holomorphic in \( \mathcal{D}_{[a,b],\sigma \tau \varrho} \). We apply theorem \[\text{[1]}\] and obtain
\[
\|f - J_{[a,b],m}[f]\|_{[a,b],\varrho} \leq \frac{2(1 + \Lambda_m)}{\sigma \tau - 1} \left( \frac{\varrho}{\sigma \tau \varrho} \right)^m \|f\|_{\mathcal{D}_{[a,b],\sigma \tau \varrho}}
\leq \frac{2(1 + \Lambda_m)}{\sigma - 1} \left( \frac{1}{\sigma \varrho} \right)^m q^m \tau^{-m} \|f\|_{\mathcal{D}_{[a,b],\sigma \tau \varrho}} \leq C \min_{q} q^m \tau^{-m} \|f\|_{\mathcal{D}_{[a,b],\sigma \tau \varrho}}.
\]

In order to be able to apply theorem \[\text{[8]}\] to sequences of intervals, we assume that there is a \( \delta_0 \in (0, 1) \) such that
\[
b_{\ell} - a_{\ell} \leq \delta_0 (b_{\ell-1} - a_{\ell-1}) \quad \text{for all } \ell \in [1 : L].
\]

Our goal is to analyze the iterated interpolation operators given by
\[
J_{j,i} := \begin{cases} f & \text{if } i = j, \\ J_{[a,b],m_1} \circ \cdots \circ J_{[a_{\ell+1},b_{\ell+1}],m_{\ell+1}} & \text{otherwise} \\
\end{cases} \quad \text{for all } i,j \in [0 : L], \ i \leq j,
\]
where \( m_1, \ldots, m_L \in \mathbb{N} \) are the orders of interpolation.

For the investigation of the stability and the error of nested interpolation, we can follow two different approaches: the “approximation first” approach relies on the telescoping sum
\[
f - J_{j,i}[f] = \sum_{\ell=i+1}^{j} J_{j,\ell}[f - J_{[a_{\ell},b_{\ell}],m_{\ell}}[f]].
\]
Treating the telescoping sum with the triangle inequality means that we need error estimates for \( f - J_{[a_{\ell},b_{\ell}],m_{\ell}}[f] \) and stability estimates for \( J_{j,\ell} \). For the error estimates, we can take advantage of theorem \[\text{[7]}\] in combination with theorem \[\text{[8]}\] for \( \tau = \sigma^{\ell-1} \) to obtain error estimates of the form
\[
\|f - J_{[a_{\ell},b_{\ell}],m_{\ell}}[f]\|_{[a_{\ell},b_{\ell}],\varrho} \leq C \min_{q} q^{m_{\ell}} \sigma^{-m_{\ell}(\ell-1)} \|f\|_{[a_{\ell},b_{\ell}],\sigma \rho} \leq C \min_{q} q^{m_{\ell}} \|f\|_{[a_0,b_0],\varrho},
\]
i.e., the rate of convergence increases with \( \ell \), clearly a very desirable property. On the other hand, obtaining stability estimates independent of \( L \) for \( \|J_{j,\ell}[f]\|_{[a_{\ell},b_{\ell}],\varrho} \) poses a challenge unless variable-order techniques are employed.

A second approach relies somewhat counter-intuitively — on error estimates not for \( f \) itself, but for its interpolating polynomial \( J_{\ell-1,i}[f] \). This “stability first” approach relies on the telescoping sum
\[
f - J_{j,i}[f] = \sum_{\ell=i+1}^{j} J_{\ell-1,i}[f] - J_{[a_{\ell},b_{\ell}],m_{\ell}}[J_{\ell-1,i}[f]].
\]
Replacing \( \sigma \) provided by theorem \[\text{[7]}\] by \( \sigma^\vartheta \) for \( \vartheta \in (0, 1] \) in theorem \[\text{[8]}\] with \( \sigma^{1-\varrho} \) \( q \) instead of \( q \), \( \sigma^\varrho \) instead of \( \sigma \) and \( \tau = 1 \) gives us estimates of the form
\[
\|f - J_{[a_{\ell},b_{\ell}],m_{\ell}}[f]\|_{[a_{\ell},b_{\ell}],\sigma^{1-\varrho}} \leq C \min_{q} q^{m_{\ell}} \|f\|_{[a_{\ell},b_{\ell}],\sigma \rho} \leq C \min_{q} q^{m_{\ell}} \|f\|_{[a_{\ell-1},b_{\ell-1}],\varrho},
\]
i.e., we sacrifice convergence speed to gain error estimates on larger Bernstein discs.
Theorem 9 ("approximation first" approach) Let \( g_0 \in \mathbb{R}_{>1} \), let \( \sigma \in \mathbb{R}_{>1} \) be as in theorem 7, let \( q \in (1/\sigma, 1] \) and \( C_{in} \) be chosen as in theorem 8. Let \( g \in \mathbb{R}_{\geq 0} \) and let \( f: \mathcal{D}_{[a_0, b_0]} \to \mathbb{C} \) be holomorphic. We have

\[
||\mathcal{J}_{j,i}[f]||_{[a_j, b_j],g} \leq \left( \prod_{\ell = i+1}^{j} (1 + C_{in}q^m) \right) ||f||_{[a_i, b_i],g},
\]

(13a)

\[
||f - \mathcal{J}_{j,i}[f]||_{[a_j, b_j],g} \leq \sum_{k=i+1}^{j} \left( \prod_{\ell = k+1}^{j} (1 + C_{in}q^m) \right) C_{in}q^{m(k-r)} ||f||_{[a_r, b_r],g}
\]

(13b)

for all \( j, i, r \in [0 : L] \) with \( r \leq i \leq j \).

Proof: We first prove the stability estimate eq. (13a). The triangle inequality, theorem 8 with \( \tau = 1 \), and theorem 7 give us

\[
||\mathcal{J}_{[a_r, b_r],m}[f]||_{[a_r, b_r],g} \leq ||f||_{[a_r, b_r],g} + ||f - \mathcal{J}_{[a_r, b_r],m}[f]||_{[a_r, b_r],g}
\]

\[
\leq ||f||_{[a_r, b_r],g} + C_{in}q^{m} ||f||_{[a_r, b_r],g} \leq (1 + C_{in}q^{m}) ||f||_{[a_{r-1}, b_{r-1}],g}
\]

for all \( \ell \in [1 : L] \). A simple induction yields eq. (13a).

We prove the error estimate eq. (13b) by induction over \( j - i \in \mathbb{N}_0 \). The case \( j = i \) is trivial.

Let now \( n \in \mathbb{N}_0 \) be such that eq. (13b) holds for all \( i, j \in [0 : L] \) with \( j - i = n \).

Let \( i, j, r \in [0 : L] \) with \( j - i = n + 1 \) and \( r \leq i \). Using the triangle inequality, the stability estimate eq. (13a), theorem 8 with \( \tau = \sigma^{i-r} \), and theorem 7 (applied \( i + 1 - r \) times), we obtain

\[
||f - \mathcal{J}_{j,i+1}[f]||_{[a_j, b_j],g} \leq ||f - \mathcal{J}_{j,i+1}[f]||_{[a_j, b_j],g} + ||\mathcal{J}_{j,i+1}[f - \mathcal{J}_{[a_{i+1}, b_{i+1}],m_{i+1}}[f]]||_{[a_j, b_j],g}
\]

\[
\leq ||f - \mathcal{J}_{j,i+1}[f]||_{[a_j, b_j],g}
\]

\[
\leq \sum_{k=i+1}^{j} \left( \prod_{\ell = k+1}^{j} (1 + C_{in}q^m) \right) C_{in}q^{m(k-r)} ||f||_{[a_r, b_r],g}
\]

\[
= \sum_{k=i+1}^{j} \left( \prod_{\ell = k+1}^{j} (1 + C_{in}q^m) \right) C_{in}q^{m(k-r)} ||f||_{[a_r, b_r],g},
\]

relying on the induction assumption in the last step. □
Example 10 (Variable-order interpolation) When using $\mathcal{H}^2$-matrix methods to approximate certain integral operators, variable-order interpolation schemes \cite{[19],[4],[6]} can be very efficient: in order to reduce the storage requirements, we choose the orders as $m_\ell = \alpha + \beta(L - \ell)$ with $\alpha, \beta \in \mathbb{N}$, i.e., we use large orders on large intervals and small orders on small intervals.

If we choose $q < 1$ in theorem \cite{[2]} we have

\[
\prod_{\ell=i+1}^{j} (1 + C_m q^{m_\ell}) \leq \prod_{\ell=i+1}^{j} \exp(C_m q^{m_\ell}) = \exp \left( C_m \sum_{\ell=i+1}^{j} q^{\alpha + \beta(L - \ell)} \right) \\
= \exp \left( C_m q^{\alpha + \beta(L - j)} \sum_{\ell=i+1}^{j} q^{\beta(j - \ell)} \right) \\
\leq \exp \left( C_m q^{\alpha + \beta(L - j)} \frac{1 - q^\beta}{1 - q^\beta} \right) \leq \exp \left( C_m \frac{q^\alpha}{1 - q^\beta} \right) =: C_{st},
\]

by the geometric sum formula, i.e., the variable-order interpolation is uniformly stable for all $L, i, j \in [0, L]$ with $i \leq j$.

In order to obtain an error estimate, we have to investigate the terms $q^{m_k(k-r)}$ for $k \in [i+1, j]$. For the sake of simplicity, we consider only the case $i = r = 0$, $j = L > 1$, and can use

\[
m_k = (\alpha + \beta(L - k))k = \alpha k + \beta(L - k) + \beta(L - k)(k - 1) \\
\geq \min\{\alpha, \beta\}L + \begin{cases} 
\beta[L/2](k - 1) & \text{if } k \leq \lfloor L/2 \rfloor, \\
\beta(L - k)[L/2] & \text{otherwise},
\end{cases}
\]

to find

\[
\|f - \mathcal{J}_{[a, b]} [f]\|_{[a, b]} \leq \sum_{k=1}^{L} C_{st} C_m q^{m_k(k-r)} \|f\|_{\{a_0, b_0\}, \varphi} \\
\leq C_{st} C_m q^{\min\{\alpha, \beta\}L} \left( \sum_{k=1}^{\lfloor L/2 \rfloor} q^{\beta[L/2](k - 1)} + \sum_{k=\lfloor L/2 \rfloor}^{L} q^{\beta[L/2](L - k)} \right) \|f\|_{\{a_0, b_0\}, \varphi} \\
\leq 2C_{st} C_m q^{\min\{\alpha, \beta\}L} \sum_{k=0}^{\infty} q^{\beta[L/2]k} \|f\|_{\{a_0, b_0\}, \varphi} \leq 2C_{st} C_m q^{\min\{\alpha, \beta\}L} \frac{q^{\min\{\alpha, \beta\}L}}{1 - q^{\beta[L/2]}} \|f\|_{\{a_0, b_0\}, \varphi}
\]

by the geometric summation formula. The term $q^{\min\{\alpha, \beta\}L}$ lets the accuracy grow exponentially as $L$ increases without the need to adjust the parameters $\alpha$ and $\beta$.

Remark 11 (Shrinking condition) The condition eq. \cite{[10]} can be weakened: if we have $[a_2, b_2] \subseteq [a_1, b_1] \subseteq [a_0, b_0]$, but only $(b_1 - a_1) > \delta_0(b_0 - a_0)$, we can apply theorem \cite{[3]} to $\sigma^{1/2}$ instead of $\sigma$ and obtain

\[
\|f - \mathcal{J}_{[a_2, b_2]} [f]\|_{[a_2, b_2]} \|f\|_{[a_2, b_2], \sigma^{1/2}} \leq C_m q^{m_2/2} \|f\|_{[a_2, b_2], \sigma^{1/2}} \leq C_m q^{m_3/2} \|f\|_{[a_1, b_1], \sigma^{1/2}}.
\]
\[ \|f - J_{[a,b]}[f]\|_{[a,b],\sigma} \leq C_m q_m^{1/2} \|f\|_{[a,b],\sigma} \leq C_m q_m^{1/2} \|f\|_{[a,b],\sigma}, \]

i.e., the convergence rate is worse, but the basic structure of the stability and convergence proofs can be preserved.

**Theorem 12 ("stability first" approach)** Let \( \theta_0 \in \mathbb{R}_{>1} \), let \( \theta_1, \theta_2 \in (0,1) \) with \( \theta_1 + \theta_2 = 1 \), let \( \sigma \in \mathbb{R}_{>1} \) be as in theorem 7 and let \( q_1 \in (\sigma^{-\theta_1}, 1) \) and \( q_2 := \sigma^{-\theta_2} \). There is a \( C_m \in \mathbb{R}_{>0} \) such that for all \( q \in \mathbb{R}_{\geq \theta_0} \) and all holomorphic \( f : D_{[a,b],\sigma} \rightarrow \mathbb{C} \) we have

\[
\|J_{a,i}[f]\|_{[a,b],\sigma^{\theta_2(i-\ell)}} \leq \prod_{\ell=i+1}^{j} (1 + C_m q_m^{\ell}) \|f\|_{[a,b],\sigma},
\]

or

\[
\|J_{a,i}[f]\|_{[a,b],\sigma^{\theta_2(i-\ell)}} \leq \sum_{k=i+1}^{j} q_2^{m(k-\ell)} q_1^{m_k} \left( \prod_{\ell=i+1}^{k-1} (1 + C_m q_1^{m_\ell}) \right) \|f\|_{[a,b],\sigma},
\]

for all \( i, j \in [0 : L] \) with \( j \geq i \).

**Proof:** We apply theorem 8 to \( \sigma^{\theta_1} \), \( q_1 \), and \( \sigma^{\theta_2(\ell-i)} \|f\|_{\sigma^{\theta_2(\ell-i)}} \) in place of \( \sigma \), \( q \), and \( \rho \) to get a constant \( C_m \) depending only on \( \sigma \) and \( q_1 \). Using the triangle inequality and \( \sigma^{\theta_2(\ell-i)} \|f\|_{\sigma^{\theta_2(\ell-i)}} \geq \rho \), we obtain

\[
\|J_{a,i}[f]\|_{[a,b],\sigma^{\theta_2(\ell-i)}} \leq \|f\|_{[a,b],\sigma^{\theta_2(\ell-i)}} + \|J_{a,i}[f]\|_{[a,b],\sigma^{\theta_2(\ell-i)}} \\
\leq (1 + C_m q_1^{m_\ell}) \|f\|_{[a,b],\sigma^{\theta_2(\ell-i)}} + (1 + C_m q_1^{m_\ell}) \|f\|_{[a,b],\sigma^{\theta_2(\ell-i)}} \\
\leq (1 + C_m q_1^{m_\ell}) \|f\|_{[a,b],\sigma^{\theta_2(\ell-i)}}
\]

for all \( \ell \in [i+1 : L] \), where we use theorem 7 in the last step. A simple induction leads to eq. 14a.

We will prove eq. 14b again by induction over \( j - i \in \mathbb{N}_0 \). The case \( j = i \) is trivial. Let now \( n \in \mathbb{N}_0 \) be such that eq. 14b holds for all \( i, j \in [0 : L] \) with \( j - i = n \).

Let \( i, j \in [0 : L] \) with \( j - i = n + 1 \). We have

\[
\|f - J_{a,i}[f]\|_{[a,b],\sigma} \leq \|f - J_{a,i-1}[f]\|_{[a,b],\sigma} \\
+ \|J_{a,i-1}[f] - J_{a,i-1}f\|_{[a,b],\sigma}.
\]

The first term can be handled by the induction assumption due to \( j - i = n \). For the second term, we use theorem 8 with \( \tau = \sigma^{\theta_2(j-i)} \) to get

\[
\|J_{a,i-1}[f] - J_{a,i-1}[f]\|_{[a,b],\sigma^{\theta_2(j-i)}} \\
\leq C_m q_2^{m_j} \|J_{a,i-1}[f]\|_{[a,b],\sigma^{\theta_2(j-i)+\theta_1}} \\
= C_m q_2^{(j-i)m_j} q_1^{m_j} \|J_{a,i-1}[f]\|_{[a,b],\sigma^{\theta_2(j-i)+\theta_1}} \\
\leq C_m q_2^{(j-i)m_j} q_1^{m_j} \left( \prod_{\ell=i+1}^{j-1} (1 + C_m q_1^{m_\ell}) \right) \|f\|_{[a,b],\sigma},
\]

(15)
where we have used the stability estimate eq. (14a) in the last step. Combining this estimate with the induction assumption yields

\[ \| f - J_{j,i}[f] \|_{[a_j,b_j],q} \leq C_{\text{st}} \sum_{k=i+1}^{j} q_2^{m_k(k-1)} q_1^{m_k} \left( \prod_{\ell=i+1}^{k-1} (1 + C_{\text{in}} q_1^{m_\ell}) \right) \| f \|_{[a,b],q}, \]

completing the induction. □

**Corollary 13 (Stability)** Let \( q_0 \in \mathbb{R}_{>1} \), let \( \sigma \in \mathbb{R}_{>1} \) be as in theorem 7, let \( \theta_1, \theta_2 \in (0,1) \) with \( \theta_1 + \theta_2 = 1 \), let \( q_1 \in (\sigma^{-\theta_1}, 1) \) and \( q_2 = \sigma^{-\theta_2} \), and let \( C_{\text{st}} \) be as in theorem 12.

There are \( \alpha_0 \in \mathbb{N} \) and \( C_{\text{st}} \in \mathbb{R}_{\geq 1} \), \( C_{\text{ap}} \in \mathbb{R}_{>0} \) such that if \( \alpha := \min \{m_\ell : \ell \in [1 : L] \} \geq \alpha_0 \) holds, we have

\[ \| J_{j,i}[f] \|_{[a_j,b_j],q} \leq C_{\text{st}} \| f \|_{[a,b],q}, \]

\[ \| f - J_{j,i}[f] \|_{[a_j,b_j],q} \leq C_{\text{ap}} q_1^{\alpha} q_2^{i} \| f \|_{[a,b],q}, \]

for all \( i, j \in [0 : L] \) with \( i \leq j \), all \( q \in \mathbb{R}_{\geq q_0} \) and all holomorphic \( f : D_{[a_0,b_0],q} \to \mathbb{C} \).

**Proof:** Let \( i, j \in [0 : L] \) with \( i \leq j \). Let \( \alpha := \min \{m_\ell : \ell \in [1 : L] \} \). eq. (14b) yields

\[ \| f - J_{j,i}[f] \|_{[a_j,b_j],q} \leq \sum_{k=i+1}^{j} C_{\text{in}} q_2^{\alpha(k-1)} q_1^{\alpha} \left( \prod_{\ell=i+1}^{k-1} (1 + C_{\text{in}} q_1^{m_\ell}) \right) \| f \|_{[a,b],q}. \]

Now we choose \( \alpha_0 \in \mathbb{N} \) large enough to guarantee \((1 + C_{\text{in}} q_1^{m_\ell}) q_2^{\alpha} \leq 1/2, \) assume \( \alpha \geq \alpha_0 \), and use the geometric sum equation to conclude

\[ \| f - J_{j,i}[f] \|_{[a_j,b_j],q} \leq 2 C_{\text{in}} q_1^{\alpha} q_2^{i} \| f \|_{[a,b],q}. \]

Choosing \( C_{\text{ap}} := 2 C_{\text{in}} \) proves the error estimate, and the triangle inequality yields

\[ \| J_{j,i}[f] \|_{[a_j,b_j],q} \leq \| f \|_{[a_j,b_j],q} + \| f - J_{j,i}[f] \|_{[a_j,b_j],q} \leq (1 + C_{\text{ap}} q_1^{\alpha} q_2^{i}) \| f \|_{[a,b],q}, \]

so we get the stability estimate with \( C_{\text{st}} := 1 + C_{\text{ap}} q_1^{\alpha} q_2^{i} \). □

The “stability first” approach can be used to obtain error estimates for the derivatives of the interpolation error, allowing us to approximate the derivatives of a function by the derivatives of its interpolating polynomial. The key tool is Cauchy’s bound for the derivatives of holomorphic functions.

**Lemma 14 (Cauchy’s inequality)** Let \( q_0 \in \mathbb{R}_{>1} \). There is a constant \( C_{ca} \in \mathbb{R}_{>0} \) such that

\[ \| f' \|_{\infty,[a,b]} \leq \frac{C_{ca}}{b-a} \| f \|_{[a,b],q} \]  \hspace{1cm} (16)

holds for all \( q \in \mathbb{R}_{\geq q_0} \), \( a, b \in \mathbb{R} \) with \( a < b \) and all functions \( f \) holomorphic in \( D_{[a,b],q} \).
Proof: Let \( q \in \mathbb{R}_{\geq 0} \) and \( r := \frac{(q-1)^2}{2q} \). A straightforward computation reveals

\[
\{ w \in \mathbb{C} : |w - x| < r \} \subseteq D_{\varrho} \quad \text{for all } x \in [-1, 1].
\]

Let \( f \) be holomorphic in \( D_{\varrho} \). Using Cauchy’s inequality for derivatives, we find

\[
|f'(x)| \leq \max \left\{ \frac{|f(w)|}{r} : w \in \mathbb{C}, |w - x| = \tilde{r} \right\} \leq \frac{\|f\|_{[-1,1], \varrho}}{\tilde{r}}
\]

for all \( x \in [-1, 1] \) and \( \tilde{r} \in (0, r) \), and therefore

\[
\|f'\|_{\infty, [-1,1]} \leq \frac{2\varrho}{(q-1)^2}\|f\|_{[-1,1], \varrho} \leq \frac{C_{ca}}{2}\|f\|_{[-1,1], \varrho}
\]

with \( C_{ca} := \frac{4\varrho}{(q-1)^2} \geq \frac{4\varrho}{(q-1)^2} \). A straightforward scaling argument using \( \Phi'_{a,b} = \frac{b-a}{2} \) completes the proof.

In order to keep the denominator \( b - a \) in the estimate eq. (16) under control, we assume that there is a \( \delta_1 \in \mathbb{R}_{>0} \) with

\[
|q| - a_\ell \geq \delta_1(b_{\ell-1} - a_{\ell-1}) \quad \text{for all } \ell \in [1 : L], \tag{17}
\]

this is a counterpart of the “shrinking assumption” eq. (10).

Theorem 15 (Derivatives) Let \( q_0 \in \mathbb{R}_{>1} \), let \( \theta_1, \theta_2 \in (0, 1) \) with \( \theta_1 + \theta_2 = 1 \), let \( \sigma \in \mathbb{R}_{>1} \) be as in theorem 7 let \( q_1 \in (\sigma^{-\theta_1}, 1) \) and \( q_2 = \sigma^{-\theta_2} \), and let \( C_{in} \) be as in theorem 12 and \( C_{ca} \) as theorem 14.

There are \( \alpha_0 \in \mathbb{N} \) and \( C_{ap} \in \mathbb{R}_{>0} \) such that if \( \alpha := \min \{ m_\ell : \ell \in [1 : L] \} \geq \alpha_0 \) holds, we have

\[
\| (f - \mathfrak{J}_{j,i}[f])' \|_{[a_j, b_j], \varrho} \leq \frac{C_{ap}}{|a_\ell - a_i|} q_2^{m_j} \|f\|_{[a_i, b_i], \varrho}
\]

for all \( i, j \in [0 : L] \) with \( i \leq j \), all \( q \in \mathbb{R}_{\geq 0} \), and all holomorphic \( f : D_{[a_0, b_0], \varrho} \rightarrow \mathbb{C} \).

Proof: We modify the proof of theorem 12 by theorem 14 and eq. (15), we obtain

\[
\| (\mathfrak{J}_{j-i}[f] - \mathfrak{J}_{j-i}[f])' \|_{\infty, [a_j, b_j]} \leq \frac{C_{ca}}{|b_j - a_j|} \|\mathfrak{J}_{j-i}[f] - \mathfrak{J}_{j-i}[f]\|_{[a_j, b_j], \varrho} \leq \frac{C_{ca} q_2^{m_j} j-i}{|b_j - a_j|} \left( \prod_{\ell=i+1}^{j-1} (1 + C_{in} q_1^{m_\ell}) \right) \|f\|_{[a_i, b_i], \varrho} \leq \frac{C_{ca}}{|b_i - a_i|} C_{in} \left( \frac{q_2^{m_j}}{\delta_1} \right) \left( \prod_{\ell=i+1}^{j-1} (1 + C_{in} q_1^{m_\ell}) \right) \|f\|_{[a_i, b_i], \varrho}
\]

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for all \(i, j \in [0 : L]\) with \(i \leq j - 1\). Using the same strategy as in theorem 13, we choose \(\alpha_0 \in \mathbb{N}\) large enough to ensure
\[
\frac{q_2^{\alpha_0}}{\delta_1}(1 + C_{in}q_1^{\alpha_0}) \leq \frac{1}{2}.
\]
Assuming \(\alpha := \min\{m_\ell : \ell \in [1 : L]\} \geq \alpha_0\) yields
\[
\| (\mathcal{J}_{j-1,i}[f] - \mathcal{J}_{[a_j,b_j],m_j}[\mathcal{J}_{j-1,i}[f]])'\|_{\infty,[a_j,b_j]} \leq \frac{C_{ca}}{b_i - a_i}C_{in}q_1^{\alpha} \left( \frac{1}{2} \right)^{j-i} \| f \|_{[a_i,b_i],\delta}. \]
Using induction as in theorem 12 gives us
\[
\| (f - \mathcal{J}_{j,i}[f])'\|_{\infty,[a_j,b_j]} \leq \sum_{k=i+1}^{j} \frac{C_{ca}}{b_i - a_i}C_{in}q_1^{\alpha} \left( \frac{1}{2} \right)^{k-i} \| f \|_{[a_i,b_i],\delta} \]
\[
\leq \frac{2C_{ca}}{b_i - a_i}C_{in}q_1^{\alpha} \| f \|_{[a_i,b_i],\delta}.
\]
Setting \(C_{ap} := 2C_{ca}C_{in}\) completes the proof. \(\Box\)

5 Iterated interpolation of oscillatory functions

The kernel function
\[
g(x, y) = \frac{\exp(i\kappa\|x - y\|)}{4\pi\|x - y\|}
\]
of the three-dimensional Helmholtz operator oscillates quickly if the wave number \(\kappa\) is large. This means that standard interpolation is a poor fit for constructing fast methods for Helmholtz boundary element methods.

An effective solution is to split the kernel function into a plane wave and a locally smooth remainder that can be approximated [7, 9, 16, 5]: we choose a unit vector \(\hat{c} \in \mathbb{R}^3\) and apply interpolation to the modified kernel function
\[
g_c(x, y) = \frac{\exp(i\kappa\|x - y - \langle \hat{c}, x - y \rangle\|)}{4\pi\|x - y\|},
\]
then we can use
\[
g(x, y) = \exp(i\kappa\langle \hat{c}, x - y \rangle) g_c(x, y) = \exp(i\kappa\langle \hat{c}, x \rangle) \overline{\exp(i\kappa\langle \hat{c}, y \rangle)} g_c(x, y)
\]
to reconstruct the original kernel function. Multiplication by
\[
\exp(i\kappa\langle \hat{c}, x \rangle) = \exp(i\kappa\hat{c}_1 x_1) \exp(i\kappa\hat{c}_2 x_2) \exp(i\kappa\hat{c}_3 x_3)
\]
is a tensor operation, therefore we can restrict our analysis to the one-dimensional multiplication operators \(E_c\) given by
\[
E_c[f](x) := \exp(i\kappa x) f(x),
\]
where $c$ is the product of $\hat{c}$ and the wave number $\kappa$. Instead of interpolating $g$ directly, we divide by a plane wave, i.e., apply $\mathcal{E}_{-c}$, interpolate the result, and then multiply by the plane wave again, i.e., apply $\mathcal{E}_c$. Our task is to investigate the resulting “oscillatory interpolation operators”

$$J_{[a,b],m,c} := \mathcal{E}_c \circ J_{[a,b],m} \circ \mathcal{E}_{-c}.$$ 

In order to obtain efficient numerical schemes, we have to use iterated oscillatory interpolation, i.e., we again fix a sequence of nested intervals with corresponding “directions” $c_0, \ldots, c_L \in \mathbb{R}$ and interpolation orders $m_1, \ldots, m_L \in \mathbb{N}$. The iterated interpolation operators are now given by

$$J_{j,i} := \begin{cases} I & \text{if } i = j, \\ J_{[a_i,b_i],m_j,c_j} \circ J_{j-1,i} & \text{otherwise} \end{cases} \quad \text{for all } i, j \in [0 : L], \ i \leq j.$$ 

We again need the shrinking condition eq. (10), and we require the “directions” of neighbouring steps in the interpolation chain to be sufficiently close, i.e., we assume that there is a constant $\omega \in \mathbb{R}_{>0}$ such that

$$|c_{\ell} - c_{\ell-1}|(b_{\ell} - a_{\ell}) \leq \omega \quad \text{for all } \ell \in [1 : L].$$

**Lemma 16 (Bounded oscillations)** Let $g \in \mathbb{R}_{>1}$. There is a constant $C_{os} \in \mathbb{R}_{>0}$ such that for all $i, j \in [0 : L]$ with $i \leq j$ and all holomorphic $f : D_{[a_i,b_i],g} \to \mathbb{C}$ we have

$$\|\mathcal{E}_{c_i-c_j}[f]\|_{[a_i,b_i],g} = \|\mathcal{E}_{c_i-c_j}[f]\|_{[a_j,b_j],g} \leq C_{os}\|f\|_{[a_j,b_j],g}.$$ 

**Proof:** We first prove

$$|c_j - c_i| (b_j - a_j) \leq \frac{\omega}{1 - \delta_0} \quad \text{for all } i, j \in [0 : L], \ i \leq j \quad (18)$$

with $\delta_0$ from eq. (10) by induction over $j - i \in \mathbb{N}_0$. The case $j = i$ is trivial.

Let now $n \in \mathbb{N}_0$ be such that eq. (18) holds for all $i, j \in [0 : L]$ with $j - i = n$.

Let $i, j \in [0 : L]$ with $j - i = n + 1$. The triangle equality, eq. (10), and the induction assumption yield

$$|c_j - c_i| (b_j - a_j) \leq |c_j - c_{j-1}| (b_j - a_j) + |c_{j-1} - c_i| (b_j - a_j) \leq \omega + \delta_0|c_{j-1} - c_i| (b_{j-1} - a_{j-1}) \leq \omega + \delta_0 \frac{\omega}{1 - \delta_0} \leq \omega \frac{1 - \delta_0}{1 - \delta_0} \omega = \frac{\omega}{1 - \delta_0}.$$ 

Let now $i, j \in [0 : L]$ with $i \leq j$, let $w \in D_{[a_i,b_i],g}$, and let $\hat{w} \in D_{\hat{c}}$ with $\Phi_{a,b}(\hat{w}) = w$. We can find $\hat{z} = x + iy \in A_\hat{c}$ with $\gamma(\hat{z}) = \hat{w}$ and get

$$\hat{w} = \frac{1}{2} \left( x + iy + \frac{x - iy}{x^2 + y^2} \right) = \frac{x}{2} \left( 1 + \frac{1}{x^2 + y^2} \right) + \frac{y}{2} \left( 1 - \frac{1}{x^2 + y^2} \right),$$
which allows us to conclude $|\Im(\hat{w})| \leq \frac{e}{2}(1 - \frac{1}{\gamma}) = \frac{e - 1/\gamma}{2}$ and $|\Im(w)| \leq (b_j - a_j) \frac{e - 1/\gamma}{2}$.

We obtain

$$|\exp(\imath(c_j - c_i)w)| \leq \exp(|c_j - c_i||\Im(w)|) \leq \exp\left(|c_j - c_i|(b_j - a_j) \frac{e - 1/\gamma}{2}\right)$$

and this proves our claim, since $E_{c_j - c_i}$ is a simple multiplication operator. □

In this section’s setting, $f$ may be oscillatory, i.e., it may grow exponentially along the imaginary axis. Therefore we cannot expect the absolute value of the error to converge reasonably well in a Bernstein disc $D_{[a,b],\varrho}$. We can, however, investigate the “smoothed” error obtained via the operator $E_{-c}$ eliminating the exponential growth.

**Theorem 17 (Oscillatory interpolation)** Let $\varrho = \varrho_0 \in \mathbb{R}_{>1}$, let $\varrho \in \mathbb{R}_{>1}$ be as in theorem 7, let $q \in [1/\sigma, 1]$ and $C_{in}$ be chosen as in theorem 5. Let $C_{os}$ be the constant of theorem 17 with $\sigma_q$ instead of $\varrho$. Let $f : D_{[a_0,b_0],\varrho} \rightarrow \mathbb{C}$ be holomorphic. We have

$$\|E_{-c_i}J_{j,i}[f]\|\leq \prod_{\ell=1}^{j} (1 + C_{os}C_{in}q^{m_{\ell}})\|E_{-c_i}f\|_{[a_i,b_i],\varrho}, \quad (19a)$$

$$\|E_{-c_i}(f - J_{j,i}[f])\|\leq \sum_{k=i+1}^{j} \prod_{\ell=k}^{j-1} (1 + C_{os}C_{in}q^{m_{\ell}})$$

$$C_{os}C_{in}q^{m_k}\|E_{-c_i}f\|_{[a_i,b_i],\varrho} \quad (19b)$$

for all $i, j \in [0 : L]$ with $i \leq j$.

**Proof:** Let $i \in [0 : L]$. Using theorem 8, theorem 16 and theorem 7 we find

$$\|E_{-c_i}(f - J_{[a_i,b_i],m_{\ell},c_{\ell}}[f])\|\leq \|E_{-c_i}(E_{-c_{\ell}}f - J_{[a_i,b_i],m_{\ell}}[E_{-c_{\ell}}f])\|\leq C_{os}\|E_{-c_{\ell}}f - J_{[a_i,b_i],m_{\ell}}[E_{-c_{\ell}}f]\|\leq C_{os}C_{in}q^{m_{\ell}}\|E_{-c_{\ell}}f\|_{[a_i,b_i],\varrho} \leq C_{os}C_{in}q^{m_{\ell}}\|E_{-c_{\ell}}f\|_{[a_i,b_i],\varrho}$$

$$\leq C_{os}C_{in}q^{m_{\ell}}\|E_{-c_{\ell}}f\|_{[a_{\ell-1},b_{\ell-1}],\varrho} \quad (20)$$

for all $\ell \in [i+1 : L]$. The triangle inequality gives us

$$\|E_{-c_i}J_{[a_i,b_i],m_{\ell},c_{\ell}}[f]\|\leq \|E_{-c_i}f\|_{[a_i,b_i],\varrho} \leq C_{os}C_{in}q^{m_{\ell}}\|E_{-c_{\ell}}f\|_{[a_i,b_i],\varrho}$$

for all $\ell \in [i+1 : L]$, and a straightforward induction yields eq. (19a).

We prove eq. (19b) by induction over $j-i \in \mathbb{N}_0$. The case $j=i$ is trivial.

Let now $n \in \mathbb{N}_0$ be such that eq. (19b) holds for all $i, j \in [0 : L]$ with $j-i=n$. 

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Let $i, j \in [0 : L]$ with $j - i = n + 1$. The triangle inequality, the error estimate eq. (20), and the stability estimate eq. (19a) yield

$$
\|e_{-c_i}(f - J_{j,i}[f])\|_{[a_j,b_j],q} \leq \|e_{-c_i}(f - J_{j-1,i}[f])\|_{[a_j,b_j],q} + \|e_{-c_i}(J_{j-1,i}[f] - J_{j-1,i}[f])\|_{[a_j,b_j],q} \\
\leq \|e_{-c_i}(f - J_{j-1,i}[f])\|_{[a_j,b_j],q} + C_m^2 C_{os}^2 q^m_j \|e_{-c_i}(J_{j-1,i}[f])\|_{[a_j,b_j],q} \\
\leq \|e_{-c_i}(f - J_{j-1,i}[f])\|_{[a_j,b_j],q} + C_m^2 C_{os}^2 q^m_j \left( \prod_{\ell=i+1}^{j-1} (1 + C_m^2 C_{os}^2 q^{m_\ell}) \right) \|e_{-c_i}f\|_{[a_i,b_i],q} \\
= \sum_{k=i+1}^{j} C_m^2 C_{os}^2 q^{m_k} \left( \prod_{\ell=i+1}^{k-1} (1 + C_m^2 C_{os}^2 q^{m_\ell}) \right) \|e_{-c_i}f\|_{[a_i,b_i],q},
$$

where we have used the induction assumption in the last step. □

**Remark 18 (Stability condition)** Unfortunately, we cannot use the approach of theorem [12] to obtain uniform stability estimates for oscillatory interpolation, since theorem [12] only holds on fixed Bernstein discs. To ensure stability, we need an additional assumption (cf. [6, Theorem 5.6]). Let $\alpha := \min\{m_\ell : \ell \in [1 : L]\}$ denote the minimal order of interpolation. In order to have a stable method, we have to keep

$$
\prod_{\ell=i+1}^{j} (1 + C_m^2 C_{os}^2 q^{m_\ell}) \leq \prod_{\ell=i+1}^{j} (1 + C_m^2 C_{os}^2 q^{m_\ell}) \leq \prod_{\ell=i+1}^{j} \exp(C_m^2 C_{os}^2 q^{\alpha}) \\
= \exp(C_m^2 C_{os}^2 (j - i) q^{\alpha})
$$

under control for all $i, j \in [0 : L]$ with $i \leq j$. To do so, we choose $p \in (q, 1]$ and require

$$
\alpha \geq \frac{\log(L)}{\log(p) - \log(q)} \iff \log(L) + \log \left( \frac{q}{p} \right) \alpha \leq 0 \iff L \left( \frac{q}{p} \right)^{\alpha} \leq 1.
$$

**Remark 19 (Error estimates)** Due to $|\exp(ic_t w)| = 1$ for all $w \in \mathbb{R}$, we have $\|f - J_{j,i}[f]\|_{[a_j,b_j]} \leq \|e_{-c_i}(f - J_{j,i}[f])\|_{[a_j,b_j],q}$ for all $q \in \mathbb{R}_{\geq 1}$.
Standard discretization schemes with a mesh width of $h$ usually satisfy $kh \lesssim 1$. This translates to $|c_L|(b_L - a_L) \lesssim 1$, i.e., to a bound for $c_{-1}$, and we find
\[
\|f - J_{i,1}[f]\|_{[a_L,b_L],\varrho} \lesssim \|c_{-1}(f - J_{i,1}[f])\|_{[a_L,b_L],\varrho} \\
\leq C_{\text{os}}\|c_i(f - J_{i,1}[f])\|_{[a_L,b_L],\varrho}.
\]

Using this estimate in combination with theorem 17, we can even apply theorem 14 to obtain estimates for the derivative of the error.

In order to use our approximation and stability results in higher-dimensional settings, it is frequently useful to have stability estimates that only require $f$ to be bounded on the interval $[a_0,b_0] = D_{[a_0,b_0],1}$ instead of on a Bernstein disc $D_{[a_0,b_0],\varrho}$ with $\varrho > 1$. Using the “stability first” approach, we can obtain estimates of this type, at least for constant-order interpolation.

**Corollary 20 (Stability)** Let $m_\ell = \alpha$ for all $\ell \in [1 : L]$. There are $\alpha_0 \in \mathbb{N}$ and $C_{\text{st}} \in \mathbb{R}_{\geq 1}$ such that if $\alpha \geq \alpha_0$ holds, we have
\[
\|J_{j,i}[f]\|_{\infty, [a_j,b_j]} \leq C_{\text{st}} \Lambda_{m_{i+1}}\|f\|_{\infty, [a_i,b_i]} \tag{22}
\]
for all $i, j \in [0 : L]$ with $i < j$ and all $f \in C[a_i, b_i]$.

**Proof:** Let $\varrho = \varrho_0 \in \mathbb{R}_{> 1}$, let $\sigma \in \mathbb{R}_{> 1}$ be as in theorem 7, let $q \in (1/\sigma, 1)$ and $C_{\text{in}}$ be chosen as in theorem 8. Let $C_{\text{os}}$ be the constant of theorem 16 with $\sigma \varrho$ instead of $\varrho$. Let $p \in (q, 1]$ and
\[
\alpha_0 := \left[\log(L) \left(\log(p) - \log(q)\right)\right],
\]
just as in theorem 18, and assume $\alpha \geq \alpha_0$.

Let $i \in [0 : L]$, let $f \in C[a_i, b_i]$ and $\pi := J_{[a_{i+1},b_{i+1}],m_{i+1}}[c_{-1}[f]]$. We have $\|\pi\|_{\infty, [a_{i+1},b_{i+1}]} \leq \Lambda_{m_{i+1}}\|f\|_{\infty, [a_{i+1},b_{i+1}]}$, and $\hat{\pi} := c_{[a_{i+1}]}[\pi]$ is holomorphic in the entire complex plane. Using eq. (21) and $\|c_{[a_i]}[f]\|_{[a_i,b_i]} \leq \|f\|_{[a_i,b_i]}$ for all $\ell \in [0 : L]$, $c \in \mathbb{R}$, $f \in C[a_i, b_i]$, we can apply Theorem 8 with $\tau$ equal to $\varrho$ and $\varrho$ equal to 1 to get
\[
\|\hat{\pi} - J_{j,i+1}[\hat{\pi}]\|_{\infty, [a_j,b_j]} = \|\hat{\pi} - J_{j,i+1}[\hat{\pi}]\|_{[a_j,b_j],1} \\
\leq \|\hat{\pi} - J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],1} + C_{\text{in}}q^\alpha \varrho^{-\alpha}\|c_{-1}J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],\sigma \varrho} \\
\leq \|\hat{\pi} - J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],1} + C_{\text{os}}C_{\text{in}}q^\alpha \varrho^{-\alpha}\|c_{-1}J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],\sigma \varrho} \\
\leq \|\hat{\pi} - J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],1} + C_{\text{os}}C_{\text{in}}q^\alpha \varrho^{-\alpha}\exp(C_{\text{os}}^2C_{\text{in}})\|c_{-1}\hat{\pi}\|_{[a_{i+1},b_{i+1}],\varrho} \\
\leq \|\hat{\pi} - J_{j-1,i+1}[\hat{\pi}]\|_{[a_j,b_j],1} + C_{\text{os}}C_{\text{in}}q^\alpha \varrho^{-\alpha}\exp(C_{\text{os}}^2C_{\text{in}})\|c_{-1}\hat{\pi}\|_{[a_{i+1},b_{i+1}],\varrho}
\]
for all $j \in [i + 2, L]$. A straightforward induction yields
\[
\|\hat{\pi} - J_{j,i+1}[\hat{\pi}]\|_{\infty, [a_j,b_j]} \leq C_{\text{os}}C_{\text{in}}\exp(C_{\text{os}}^2C_{\text{in}})(j - i - 1)q^\alpha \varrho^{-\alpha}\|\pi\|_{[a_{i+1},b_{i+1}],\varrho} \\
\leq C_{\text{os}}C_{\text{in}}\exp(C_{\text{os}}^2C_{\text{in}})p^\alpha \varrho^{-\alpha}\|\pi\|_{[a_{i+1},b_{i+1}],\varrho}
\]
for all \( j \in [i+1:L] \). With theorem \( \text{2} \) and eq. \( \text{7} \) we conclude

\[
\| \mathcal{J}_{j,i} [f] \|_{\infty,[a_j,b_j]} = \| \mathcal{J}_{j,i+1} [\hat{\pi}] \|_{\infty,[a_j,b_j]} \leq \| \hat{\pi} \|_{\infty,[a_j,b_j]} + \| \hat{\pi} - \mathcal{J}_{j,i+1} [\hat{\pi}] \|_{\infty,[a_j,b_j]} \\
\leq \| \pi \|_{\infty,[a_j,b_j]} + C_{os} C_{in} \exp(C^2_{os} C_{in}) p^\alpha \| \hat{\pi} \|_{[a_{i+1},b_{i+1}],\hat{\pi}} \\
\leq \| \pi \|_{\infty,[a_j,b_j]} + C_{os} C_{in} \exp(C^2_{os} C_{in}) p^\alpha \| \pi \|_{[a_{i+1},b_{i+1}]} \\
\leq (1 + C_{os} C_{in} \exp(C^2_{os} C_{in}) p^\alpha) \Lambda_{m+1} \| f \|_{\infty,[a_{i},b_{i}]}.
\]

Choosing \( C_{st} := 1 + C_{os} C_{in} \exp(C^2_{os} C_{in}) p^\alpha \) completes the proof. \( \square \)

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