ASYMPTOTIC LINKING OF VOLUME-PRESERVING ACTIONS
OF $\mathbb{R}^k$

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Abstract. We extend V. Arnold’s work on asymptotic linking for two volume preserving flows on a domain in $\mathbb{R}^3$ and $S^3$ to volume preserving actions of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ on certain domains in $\mathbb{R}^n$ and also to linking of a volume preserving action of $\mathbb{R}^k$ with a closed oriented singular $\ell$-dimensional submanifold in $\mathbb{R}^n$, where $n = k + \ell + 1$.

1. Introduction

V.I. Arnold, in his paper “The asymptotic Hopf invariant and its applications” published in 1986 (also see [2, 6, 15, 4]), considered a compact domain $\Omega$ in $\mathbb{R}^3$ or $S^3$ with a smooth boundary and trivial homology and two divergence free vector fields $X$ and $Y$ in $\Omega$ tangent to the boundary $\partial \Omega$. He defined an asymptotic linking invariant $\text{lk}(X, Y)$ that measures the average linking of trajectories of $X$ with those of $Y$, and another invariant $I(X, Y) = \int_{\Omega} \alpha \wedge d\beta$, where $d\alpha = i_X \omega$ and $d\beta = i_Y \omega$ (interior products with the volume form $\omega$ on $\Omega$), and showed that $\text{lk}(X, Y) = I(X, Y)$. We extend these results to volume-preserving actions $\Phi$ and $\Psi$ of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ on a compact convex domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, where $\Phi$ and $\Psi$ are tangent to $\partial \Omega$ and $k + \ell = n - 1$.

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Arnol’d defines the invariant \( \text{lk}(X,Y) \) as follows. For \( p \in \Omega \) and \( T > 0 \), let \( \vartheta_X(p,T) = \{ \varphi_X^t(p) | 0 \leq t \leq T \} \) be the segment of orbit beginning at \( p \) and continuing for a time \( T \), and let \( \vartheta_X(p,T) \) be this curve closed by adding a short path in \( \Omega \) from \( \varphi_X^T(p) \) to \( p \). Define \( \vartheta_Y(q,S) \) similarly. The asymptotic linking invariant of \( X \) and \( Y \) is

\[
\text{lk}(X,Y) = \int_{\Omega \times \Omega} \hat{\text{lk}}(p,q)
\]

where

\[
\hat{\text{lk}}(p,q) = \lim_{S,T \to \infty} \frac{1}{ST} \text{lk}(\vartheta_X(p,T), \vartheta_Y(q,S)).
\]

Then \( \text{lk}(X,Y) \) is well-defined, since \( \text{lk}(\vartheta_X(p,T), \vartheta_Y(q,S)) \) is defined and the limit exists for almost all \( (p,q) \in \Omega \times \Omega \), and furthermore the function \( \hat{\text{lk}}(p,q) \) is in \( L^1(\Omega \times \Omega) \) [13].

The way that Arnol’d closes up partial orbits with short curves was used earlier on by Schwartzman to define asymptotic cycles for a continuous flow \( \varphi \) on a compact polyhedron \( X \) [10]. Let \( \vartheta_\varphi(p,T) \) be the partial orbit from \( p \in X \) to \( \varphi_T(p) \), let \( \vartheta_{\varphi}(p,T) \) be a (possibly singular) loop formed by adding a short curve, and let \([\vartheta_{\varphi}(p,T)] \in H_1(X;\mathbb{R})\) be its first real homology class. Then the \( p \)-asymptotic cycle is the limit

\[
A_p = \lim_{t \to \infty} \frac{1}{t} [\vartheta_{\varphi}(p,T)] \in H_1(M;\mathbb{R})
\]

which exists for almost all points \( p \in X \), as described in a geometric interpretation ([10], p. 275). Schwartzman’s proof is quite different, since he uses homomorphisms from the cohomology to \( \mathbb{R} \) to define \( A_p \). If the short curves are chosen in a measurable fashion for a normalized invariant measure \( \mu \), then the \( \mu \)-asymptotic cycle is defined to be the integral \( A_\mu = \int_X A_p d\mu \in H_1(X;\mathbb{R}) \), the average of the cycles \( A_p \).

In [11], Schwartzman also defines asymptotic cycles for a smooth action of \( \mathbb{R}^k \) on a compact smooth manifold \( M^n \). This asymptotic cycle could also be defined by capping off the boundary of a partial orbit by a small (possibly singular) manifold, if that can be done in a measurable way, as in the present paper, though this is not carried out in [11].

In [12] we define an asymptotic linking invariant \( \text{lk}(\Phi, \Psi) \) which measures the degree of linking between orbits of the actions \( \Phi \) and \( \Psi \) and another invariant \( \mathcal{I}(\Phi, \Psi) \) defined in terms of differential forms. Our main result, Theorem 2 (proven in [11]), states that \( \text{lk}(\Phi, \Psi) = \mathcal{I}(\Phi, \Psi) \). Analogous results are given for the asymptotic linking of the action \( \Phi \) with a closed oriented \( \ell \)-dimensional submanifold \( N \) (Theorem 4 proven in [10]).

We use extensions of the gradient, curl, and divergence to multivectors in higher dimensions that are presented in [4] and in [7] an extension to higher dimensions of the classical Biot-Savart formula that gives an inverse for the curl of a divergence-free vector field on a compact domain \( \mathbb{R}^3 \). A version of the ergodic theorem due to Tempelman [13] that is used in the proofs is given in [4].

As an application, we show that our invariant gives a lower bound for the energy of an action in [12]. Examples in which the invariant is non-trivial are given in the last section, [13].

These results are taken from the doctoral thesis [9] of the first author, under the direction of the second author at the Pontifical Catholic University of Rio de Janeiro (PUC-Rio). Some similar results were obtained by García-Compéan and Santos-Silva in [5]. It would be interesting to extend these results to \( S^n \) and other
Riemannian manifolds and also to linking of \( \mathbb{R}^k \)-actions with leaves of foliations endowed with an invariant transverse volume form (see [7]).

2. Definitions and statements of results

Throughout the paper \( M \) is an oriented Riemannian \( n \)-dimensional manifold and \( \Omega \subset M \) is a compact convex domain with smooth boundary \( \partial \Omega \). In the main results of this paper, \( M \) will be \( \mathbb{R}^n \) with the standard metric, but many of the details are valid more generally. We consider a smooth \((C^\infty)\) action

\[
\Phi : \mathbb{R}^k \times \Omega \to \Omega,
\]

of the \( k \)-dimensional real vector space \( \mathbb{R}^k \) on \( \Omega \). Then \( \Phi \) is defined by \( k \) vector fields tangent to \( \partial \Omega \), \( X^1, X^2, \ldots, X^k \), whose corresponding flows \( \phi^1, \phi^2, \ldots, \phi^k \) commute with each other, so that for \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \) and \( x \in \Omega \),

\[
\Phi(t, x) = \phi^1(t_1, \phi^2(t_2, \ldots, \phi^k(t_k, x), \ldots)).
\]

In other words, if we set \( \Phi_t = \Phi(t, \cdot) \) and \( \phi^i_t = \phi^i(t, \cdot) \) for each \( i \), then \( \Phi_t = \phi^1_t \cdot \cdots \cdot \phi^k_t \). As usual, \( \phi^i \) is related to \( X^i \) by the identity \( \frac{d}{dt}\phi^i(t, x) = X^i(\phi^i(t, x)) \) and the commutation of \( \phi^i \) and \( \phi^j \) is equivalent to the vanishing of the Lie bracket \( [X^i, X^j] \).

**Definition 1.** A (smooth) action \( \Phi : \mathbb{R}^k \times \Omega \to \Omega \) on \( \Omega \) is conservative if it is volume-preserving (i.e., for each \( t \in \mathbb{R}^k \), \( \Phi_t : \Omega \to \Omega \) preserves the Riemannian volume form on \( M \)) and the generating vector fields \( X^i \) are tangent to the boundary \( \partial \Omega \).

Let \( \Phi : \mathbb{R}^k \times \Omega \to \Omega \) and \( \Psi : \mathbb{R}^\ell \times \Omega \to \Omega \) be conservative actions on \( \Omega \), \( k + \ell + 1 = n \). Let \( X = X^1 \wedge \cdots \wedge X^k \) and \( Y = Y^1 \wedge \cdots \wedge Y^\ell \) be the exterior products of the \( k \) vector fields that generate the action \( \Phi \) and the \( \ell \) vector fields that generate \( \Psi \), and let \( \omega \) be the volume form on \( \Omega \). Denote the differential forms of degree \( r \) on \( \Omega \) (resp., the forms that vanish on \( \partial \Omega \)) by \( E^r(\Omega) \) (resp., \( E^r(\Omega, \partial \Omega) \)). Since \( \Omega \) is convex, their deRham cohomology groups \( H^*(\Omega; \mathbb{R}) \) and \( H^*(\Omega, \partial \Omega; \mathbb{R}) \) vanish for \( 0 < r < n \). The differential forms \( i_X \omega \in E^{r+1}(\Omega, \partial \Omega) \) and \( i_Y \omega \in E^{k+1}(\Omega, \partial \Omega) \) given by the interior products with \( X \) and \( Y \) vanish on the boundary \( \partial \Omega \) since \( X \) and \( Y \) are tangent to the boundary, and these forms are closed since the actions are volume-preserving. Since \( \Omega \) is convex, they are exact, so there exist differential forms \( \alpha \in E^r(\Omega, \partial \Omega) \) and \( \beta \in E^k(\Omega, \partial \Omega) \) of degrees \( \ell \) and \( k \), respectively, such that \( d\alpha = i_X \omega \) and \( d\beta = i_Y \omega \). Then we define the invariant

\[
I(\Phi, \Psi) = \int_{\Omega} \alpha \wedge d\beta,
\]

which obviously does not depend on the choice of \( \beta \). Since \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^\ell \alpha \wedge d\beta \) and both \( \alpha \) and \( \beta \) vanish on \( \partial \Omega \), Stokes’ theorem gives the following result.

**Lemma 1.** This invariant satisfies

\[
I(\Phi, \Psi) = (-1)^{\ell+1} \int_{\Omega} d\alpha \wedge \beta = (-1)^{(\ell+1)(k+1)} I(\Psi, \Phi).
\]

Hence it depends only on the actions \( \Phi \) and \( \Psi \), and not on the choice of the differential forms \( \alpha \) and \( \beta \).
We shall define an asymptotic linking number \( \text{lk}(\Phi, \Psi) \) that measures the degree of linking between orbits of \( \Phi \) and \( \Psi \). For sets \( T \subset \mathbb{R}^k \) and \( Y \subset \Omega \) we set \( \Phi(T, Y) = \{ \Phi(t, y) \mid t \in T, y \in Y \} \). Let \( T_k \) be the set of \( k \)-rectangles
\[
T = [0, T_1] \times \cdots \times [0, T_k], \quad (T_1, \ldots, T_k) \in \mathbb{R}_+^k
\]
where \( \mathbb{R}_+^k \) is the space of \( k \)-tuples of non-negative real numbers, and fix a point \( \bar{p} \in \Omega \). Then we let \( \theta_\Phi(p, T) \) be the closed oriented singular \( k \)-manifold in the domain \( \Omega \)
\[
\theta_\Phi(p, T) = \Phi(T, p) \cup \sigma(p, T)
\]
where
\[
\sigma(p, T) = \Phi(\partial T, p) \ast \bar{p}
\]
is the cone composed of the geodesic segments joining each point of \( \Phi(\partial T, p) \) to \( \bar{p} \).

We construct the closed oriented singular \( \ell \)-manifold \( \theta_\Phi(q, S) = \Psi(S, q) \cup \sigma'(q, S) \) in like manner, replacing \( T \) by \( S = [0, S_1] \times \cdots \times [0, S_\ell] \) in \( T_k \) for some \( (S_1, \ldots, S_\ell) \in \mathbb{R}_+^\ell \), \( \Phi \) by \( \Psi \), and \( \bar{p} \) by another point \( \bar{q} \neq \bar{p} \).

For fixed \( T \) and \( S \), since the sum of the dimensions of \( \theta_\Phi(p, T) \) and \( \theta_\Phi(q, S) \) is \( n - 1 \), the following lemma holds. It will be proved in \( \S 3 \).

**Lemma 2.** Fix \( T \in T_k \) and \( S \in T_\ell \). Then for almost every pair \((p, q) \in \Omega \times \Omega \) the singular manifolds \( \theta_\Phi(p, T) \) and \( \theta_\Phi(q, S) \) are disjoint and therefore \( \text{lk}(\theta_\Phi(p, T), \theta_\Phi(q, S)) \) is defined.

The set \( D(\Phi, \Psi) = \{(p, T, q, S) \in \Omega \times T_k \times \Omega \times T_\ell \mid \theta_\Phi(p, T) \cap \theta_\Phi(q, S) = \emptyset \} \), where the compact sets \( \theta_\Phi(p, T) \) and \( \theta_\Phi(q, S) \) are disjoint, is clearly open, and since it has full measure, it must be dense, so we have:

**Corollary 1.** \( D(\Phi, \Psi) \) is an open dense set in \( \Omega \times T_k \times \Omega \times T_\ell \).

It follows from the Lemma that the function
\[
\text{lk}_{T,S}(p, q) := \frac{1}{\lambda_k(T)\lambda_\ell(S)} \text{lk}(\theta_\Phi(p, T), \theta_\Phi(q, S))
\]
is defined for almost all pairs \((p, q) \in \Omega \times \Omega \), where \( \lambda_k(T) = T_1 \cdots T_k \) and \( \lambda_\ell(S) = S_1 \cdots S_\ell \) are the Lebesgue measures on \( \mathbb{R}^k \) and \( \mathbb{R}^\ell \). The following theorem, proved in \( \S 4 \) affirms that this function is in \( L^1(\Omega \times \Omega) \) and permits us to define the linking index for the orbits of \( \Phi \) and \( \Psi \). We write \( T, S \to \infty \) to signify that \( \min\{T_1, \ldots, T_k, S_1, \ldots, S_\ell\} \to \infty \).

**Theorem 1.** Suppose that \( \Omega \) is a compact convex domain in \( \mathbb{R}^n \). Let \( \Phi : \mathbb{R}^k \times \Omega \to \Omega \) and \( \Psi : \mathbb{R}^\ell \times \Omega \to \Omega \) be conservative actions with \( k + \ell + 1 = n \). Then
1. The limit function \( \lim_{T, S \to \infty} \text{lk}_{T,S}(p, q) \) exists as a function in \( L^1(\Omega \times \Omega) \), i.e., there is an integrable function \( \text{lk}_{\Phi, \Psi} : \Omega \times \Omega \to \mathbb{R} \) defined almost everywhere such that
\[
\lim_{T, S \to \infty} \int_{\Omega} \int_{\Omega} |\text{lk}_{T,S}(p, q) - \text{lk}_{\Phi, \Psi}(p, q)| \, dp dq = 0.
\]
2. The integral \( \int_{\Omega} \int_{\Omega} \text{lk}_{\Phi, \Psi}(p, q) \, dp dq \) is independent of the choice of the distinct points \( \bar{p} \) and \( \bar{q} \).
Then the **asymptotic linking number** of $\Phi$ and $\Psi$ is defined to be

$$\text{lk}(\Phi, \Psi) := \int_{\Omega} \int_{\Omega} \tilde{\text{lk}}_{\Phi, \Psi}(p, q) dp dq$$

Our main theorem is the following.

**Theorem 2.** Under the hypotheses of Theorem 1, the asymptotic linking number and the invariant $I(\Phi, \Psi)$ coincide, i.e.,

$$\text{lk}(\Phi, \Psi) = I(\Phi, \Psi).$$

**Linking of an action with a submanifold.** There is a similar theory for asymptotic linking between a (smooth) conservative action $\Phi : \mathbb{R}^k \times \Omega \to \Omega$ and a closed oriented singular $\ell$-submanifold $N \subset \Omega$, where as above $\Omega$ is a compact convex domain in $n$-dimensional Euclidean space and $n = k + \ell + 1$. As before, let $\alpha$ be an $\ell$-form on $\Omega$ satisfying $d\alpha = i_{X} \omega$ where the vector fields $X^1, X^2, \ldots, X^k$ generate the action $\Phi$, $X = X^1 \wedge \cdots \wedge X^k$ and let $\omega$ be the volume form on $\Omega$. Then we define

$$I(\Phi, N) = \int_{N} \alpha. \quad (2)$$

By analogy to the previous case of two actions, we can also define an asymptotic linking number between the action $\Phi$ and $N$. As before, let $\theta_{\Phi}(p,T) = \Phi(T,p) \cup \sigma(p,T)$ with the apex of the cone at $\tilde{p} \in \Omega \setminus N$. The proof of the following Lemma is analogous to the proof of Lemma 2 and will also be given in §9.

**Lemma 3.** Fix $T \in T_k$ and let $N' \subset \Omega$, possibly with boundary. Then for almost every point $p \in \Omega$, $\theta_{\Phi}(p,T) \cap N' = \emptyset$.

Hence when $N' = N$, $\frac{1}{\lambda_k(T)} \text{lk}(\theta_{\Phi}(p,T), N)$ is defined for almost all $p \in \Omega$. Furthermore, the limit as $T \to \infty$ exists in $L^1(\Omega)$, and the integral is well-defined:

**Theorem 3.** Let $\Phi : \mathbb{R}^k \times \Omega \to \Omega$ be a conservative action on a compact convex domain $\Omega$ in $\mathbb{R}^n$ and let $N \subset \Omega$ be a smooth closed oriented $\ell$-manifold, with $k + \ell + 1 = n$. Then

1. The limit function $\text{lk}_{\Phi,N}(p) := \lim_{T \to \infty} \frac{1}{\lambda_k(T)} \text{lk}(\theta_{\Phi}(p,T), N)$ exists as a function in $L^1(\Omega)$, i.e., there is an integrable function $\tilde{\text{lk}}_{\Phi,N} : \Omega \to \mathbb{R}$ defined almost everywhere such that

$$\lim_{T \to \infty} \int_{\Omega} \left| \frac{1}{\lambda_k(T)} \text{lk}(\theta_{\Phi}(p,T), N) - \tilde{\text{lk}}_{\Phi,N}(p) \right| \, dp = 0.$$

2. The integral $\int_{\Omega} \tilde{\text{lk}}_{\Phi,N}(p) \, dp$ is independent of the choice of the point $\tilde{p}$.

Then we define the asymptotic linking number of $\Phi$ and $N$ to be

$$\text{lk}(\Phi, N) := \int_{\Omega} \tilde{\text{lk}}_{\Phi,N}(p) \, dp.$$
Theorem 4. Under the hypotheses if Theorem 3 the asymptotic linking number and the invariant \( I(\Phi, N) \) coincide, i.e.,
\[
\text{lk}(\Phi, N) = I(\Phi, N).
\]

Theorem 11 follows from Proposition 9 in §11, the proof of Theorem 2 is given at the end of §10. Theorem 8 follows from Proposition 7 in §10, and the proof of Theorem 3 is given in §10.

3. Higher Dimensional Vector Algebra

We recall vector algebra on an oriented Riemannian \( n \)-dimensional manifold with metric \( g \). Let \( E_{x,r} = \wedge_r T_x M \) be the \( r \)-th exterior power of the tangent space \( T_x M \) at \( x \in M \), with exterior multiplication \( \wedge : E_{x,r} \times E_{x,s} \to E_{x,r+s} \). The elements of \( E_{x,r} \) are called \( r \)-vectors or multivectors. Recall that the Hodge operator \( * : E_{x,r} \to E_{x,n-r} \) is defined for any positive orthonormal basis \( e_1, \ldots, e_n \) of \( \wedge_n T_x M = E_{x,1} \) by setting
\[
*(e_{i_1} \wedge \cdots \wedge e_{i_r}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-r}},
\]
if \( (i_1, \ldots, i_r, j_1, \ldots, j_{n-r}) \) is a positive permutation of \((1, \ldots, n)\), and extending over \( E_{x,r} \) by linearity and antisymmetry. Then
\[
(* \circ *) = (-1)^{r(n-r)} \text{id} : E_r \to E_r.
\]

The inner product given by the Riemannian metric \( \langle , \rangle \) on \( T_x M \) defines an inner product on \( E_{x,r} \); for decomposable multivectors \( u = u_1 \wedge \cdots \wedge u_r \) and \( v_1 \wedge \cdots \wedge v_r \), \( u \cdot v = \det(\langle u_i, v_j \rangle) \). This inner product extends to an \( \mathbb{R} \)-bilinear product
\[
: E_{x,r} \times E_{x,s} \to E_{x,n-r-s}, \quad (u, v) \mapsto u \cdot v = *(u \wedge v),
\]
and there is also a generalization to \( \mathbb{R}^n \) of the classical cross product on \( \mathbb{R}^3 \)
\[
\times : E_{x,r} \times E_{x,s} \to E_{x,n-r-s}, \quad (u, v) \mapsto u \times v = *(u \wedge v),
\]
In particular, \( u \cdot v = u \times v \).

Proposition 1. Let \( u \in E_{x,r}, v \in E_{x,s}, \) and \( w \in E_{x,m} \) be multivectors.

1. \( u \times (v \wedge w) = u \cdot (v \wedge w) \).
2. If \( r + s + m = n \), then
\[
(u \times v) \cdot w = *(u \wedge v \wedge w).
\]

Proof. 1. \( u \cdot (v \wedge w) = *(u \wedge *(v \wedge w)) = *(u \wedge (v \wedge w)) = u \times (v \wedge w) \).

2. \( u \cdot v \) and \( w \) are both in \( E_{x,m} \), so \( (u \times v) \cdot w \in E_{x,0} = \mathbb{R} \) and \( (u \times v) \cdot w = w \cdot (u \times v) \). Now \( w \cdot (u \times v) = *(w \wedge *(u \wedge v)) = (-1)^{m(r+s)}*(w \wedge u \wedge v) = *(u \wedge v \wedge w) \).

It follows from item 2 of the preceding Proposition that if the vectors \( u, v, \) and \( w \) are decomposable, say \( u = v_1 \wedge \cdots \wedge v_r, \quad v = v_{s+1} \wedge \cdots \wedge v_{r+s}, \) and \( w = v_{r+s+1} \wedge \cdots \wedge v_n \) with \( v_i = \sum_j a_{ij} e_j \) for a positive orthonormal basis \( e_1, \ldots, e_n \), then
\[
(u \times v) \cdot w = \det(a_{ij}).
\]

Example 1. As usual, a multi-index \( I \) is an ordered subset \( I = (i_1, \ldots, i_k) \) of \( \{1, \ldots, n\} \) with \( i_1 < i_2 < \cdots < i_k \), and we set \( e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \) and \( |I| = k \).
(1) For (ordered) multi-indices \( I \) and \( J \) we have
\[
e_I \times e_J = e_K
\]
if \( I \cap J = \emptyset \), \( K = \{1, \ldots, n\} \setminus (I \cup J) \), and the ordered union \( I \cup J \cup K \) is a positive permutation of \( (1, \ldots, n) \); but \( e_I \times e_J = 0 \) if \( I \cap J \neq \emptyset \).

(2) In addition,
\[
e_I \cdot e_J = (-1)^{|K|(|n-|J|)}e_K
\]
if \( I \subset J \), \( K = J \setminus I \), and the ordered union \( I \cup K \) is a positive permutation of \( J \); furthermore, \( e_I \cdot e_J \) vanishes if \( I \not\subset J \).

**Proposition 2.** For vectors \( u, v_1, \ldots, v_k \in T_x M^n \), we have
\[
u \cdot (v_1 \wedge \cdots \wedge v_k) = (-1)^{k-1} \sum_{i=1}^{k} (-1)^i (u \cdot v_i) \cdot v_1 \wedge \cdots \wedge \hat{v}_i \cdots \wedge v_k.
\]

**Proof.** Using (2) of Example 1 with \( e_I = e_{j_i} \) and \( e_J = e_{j_1} \wedge \cdots \wedge e_{j_k} \) with \( 1 \leq i \leq k \), we have \( |K| = k - 1 \) and so
\[
e_{j_1} \cdot (e_{j_1} \wedge \cdots \wedge e_{j_k}) = (-1)^{(k-1)(n-k)+i-1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \cdots \wedge e_{j_k}.
\]
Note that expanding \( u = \sum_{i=1}^{n} u_i e_i \) and \( v = v_1 \wedge \cdots \wedge v_k \) with \( v_i = \sum_{j_i=1}^{n} v_{j_i} e_{j_i} \), we obtain
\[
u \cdot v = \sum_{j_1, \ldots, j_k=1}^{n} \sum_{i=1}^{k} u_{j_i} (v_{j_1} \cdots v_{j_k})(e_{j_i} \cdot (e_{j_1} \wedge \cdots \wedge e_{j_k})).
\]
so the desired formula follows by substituting (7) and reassembling the terms \( u \) and \( v_1, \ldots, v_k \). \( \square \)

**Example 2.** For vectors \( u, v, w \) in \( \mathbb{R}^n \), by the definition of the product \( \times \) and Proposition 2, \( u \times (v \times w) = u \cdot (v \wedge w) = (-1)^n ((u \cdot v)w - (u \cdot w)v) \). In particular, in \( \mathbb{R}^3 \) we have the well-known formula \( u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \).

4. Extensions of Gradient, Curl, and Divergence.

Let \( E_k = E_k(M) \) be the space of smooth \( k \)-vector fields on a Riemannian manifold \( M \), and let \( E^k = E^k(M) \) be the dual space of differential \( k \)-forms. The inner product \( (U, V) \mapsto U \cdot V \) on \( E_k \) determines an isomorphism
\[
j : E_k \rightarrow E^k, \quad j(U)(V) = U \cdot V.
\]

The interior product \( i : E_k \times E^r \rightarrow E^{r-k}, (X, \alpha) \mapsto i_X \alpha \), is defined \( i_X \alpha(Y) = \alpha(X \wedge Y) \) for \( Y \in E_{r-k} \).

**Lemma 4.** Let \( \omega \) be the positive unit volume form on \( M \). Then \( i_X \omega = j(*X) \).

**Proof.** Consider \( X = e_I \) where \( I = (i_1, \ldots, i_k) \) and \( J = (j_1, \ldots, j_l) \) are ordered multi-indices such that \( (i_1, \ldots, i_k, j_1, \ldots, j_l) \) is a positive permutation of \( (1, \ldots, n) \), and let \( \eta_1, \ldots, \eta_n \) be the basis dual to a local positive orthonormal basis \( e_1, \ldots, e_n \). Then
\[
\eta_e \wedge \eta_{e_j} = \eta_{i_1} \wedge \cdots \wedge \eta_{i_k} = j(e_I) = j(*e_I)
\]
since \( e_J = *e_I \). The lemma follows since every \( X \in E_k \) is a linear combination of the elements \( e_I \). \( \square \)
The duality between $E_k$ and $E^k$ will be expressed using the isomorphism $j$. For example, the gradient operator $\nabla$, defined $\nabla f = j^{-1}(df)$ for a smooth function $f$ on $M$, can be extended to a linear operator $\nabla : E_k \to E_{k+1}$, $\nabla X = j^{-1}dj(X)$.

We can also extend the curl and divergence to operators $\text{rot} : E_k \to E_\ell$ and $\text{div} : E_k \to E_{k-1}$ by setting

\begin{equation}
\text{rot}(X) = (-1)^{(k+1)\ell} \ast (\nabla X)
\end{equation}

and

\begin{equation}
\text{div}(X) = (-1)^{(k+1)\ell} \ast \nabla(\ast X)
\end{equation}

where we always set $\ell = n - k - 1$. On $\mathbb{R}^3$ these definitions coincide with the classical definitions of curl and divergence for vector fields.

For the rest of this section we suppose that $M = \mathbb{R}^n$ with the canonical basis $\{e_1, \ldots, e_n\}$ and the dual basis $\{dx_1, \ldots dx_n\}$. For a $k$-vector field of the form $X = f e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $f$ is a smooth function it is easy to check that $j(f e_{i_1} \wedge \cdots \wedge e_{i_k}) = fdx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\nabla X = (\nabla f) \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}$, and

\begin{equation}
\text{div}(X) = (-1)^k \sum_{i=1}^k (-1)^i \frac{\partial f}{\partial x_{i_i}} e_{i_1} \wedge \cdots \wedge e_{i_k}.
\end{equation}

Recall that a vector field $U = \sum_{i=1}^n u_i e_i$ on $\mathbb{R}^n$ acts on a function $f$ by setting $U(f) = \langle U, \nabla f \rangle = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}$. The action of $U$ on a vector field $V = \sum_{i=1}^n v_i e_i$ is defined by setting

$$U(V) = \sum_{i=1}^n U(v_i) e_i = \sum_{i,j=1}^n u_j \frac{\partial v_i}{\partial x_j} e_i$$

so the Lie bracket can be written $[U, V] = U(V) - V(U)$.

**Proposition 3.** Let $V = V^1 \wedge \cdots \wedge V^k$ be the exterior product of vector fields $V^1, \ldots, V^k$ on $\mathbb{R}^n$. Then

$$\text{div}(V) = (-1)^k \sum_{i=1}^k (-1)^i \text{div}(V^i) V^1 \wedge \cdots \wedge \hat{V}^i \cdots \wedge V^k$$

$$+ (-1)^k \sum_{1 \leq i < j \leq k} (-1)^{i+j} [V^i, V^j] \wedge V^1 \wedge \cdots \wedge \hat{V}^i \wedge \cdots \wedge \hat{V}^j \wedge \cdots \wedge V^k$$

where $[V^i, V^j]$ is the Lie bracket.

**Proof.** Note that this is a dual version of the well-known formula for the exterior derivative of a product of 1-forms evaluated on vector fields. Let

\begin{equation}
V^i = \sum_{\ell=1}^n v^i_\ell e_\ell
\end{equation}

for every $i$, so expanding $V$ we have

\begin{equation}
V = \sum_{\ell_1, \ldots, \ell_k=1}^n v^1_{\ell_1} \cdots v^k_{\ell_k} e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}.
\end{equation}
Then by (11)

\[ \text{div}(V) = (-1)^k \sum_{i=1}^{k} \sum_{\ell_1, \ldots, \ell_k=1}^{n} (-1)^i \frac{\partial (v^1_{\ell_i} \cdots v^k_{\ell_k})}{\partial x_{\ell_i}} e_{\ell_1} \wedge \cdots \wedge e_{\ell_k} \]

\[ = (-1)^k \sum_{i,j=1}^{k} \sum_{\ell_1, \ldots, \ell_k=1}^{n} (-1)^i \frac{\partial v^j_{\ell_i}}{\partial x_{\ell_i}} v^1_{\ell_j} \cdots v^k_{\ell_k} e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}. \]

Since \( \text{div}(V^i) = \sum_{\ell_i=1}^{n} \frac{\partial v^i}{\partial x_{\ell_i}} \), the terms with \( i = j \) give

\[ (-1)^k \sum_{i=1}^{k} (-1)^i \text{div}(V^i) V^1 \wedge \cdots \wedge V^k \]

while the remaining terms give the second sum in the proposition; in fact, if \( I_{ab} \) with \( a < b \) is the sum of the terms with \((i, j) = (a, b)\) and \((i, j) = (b, a)\), then

\[ I_{ab} = (-1)^k \sum_{\ell_a, \ell_b=1}^{n} (-1)^{a+b} \left( v^a_{\ell_a} \frac{\partial v^b}{\partial x_{\ell_a}} e_{\ell_b} - v^b_{\ell_b} \frac{\partial v^a}{\partial x_{\ell_b}} e_{\ell_a} \right) \wedge \]

\[ \wedge \left( v^1_{\ell_1} \cdots v^a_{\ell_a} \cdots v^b_{\ell_b} \cdots v^k_{\ell_k} \right) e_{\ell_1} \wedge \cdots \wedge e_{\ell_k} \]

\[ = (-1)^{k+a+b} [V^a, V^b] \wedge V^1 \wedge \cdots \wedge V^a \wedge V^b \cdots \wedge V^k \]

since

\[ \sum_{\ell_a, \ell_b=1}^{n} \left( v^a_{\ell_a} \frac{\partial v^b}{\partial x_{\ell_a}} e_{\ell_b} - v^b_{\ell_b} \frac{\partial v^a}{\partial x_{\ell_b}} e_{\ell_a} \right) = [V^a, V^b]. \]

\[ \square \]

**Example 3.** If \( U \) and \( V \) are vector fields in \( \mathbb{R}^n \), then by Proposition 3 and the definitions of \( \text{rot} \) and \( \times \),

\[ \text{rot}(U \times V) = \text{div}(U \wedge V) = (\text{div}(V))U - (\text{div}(U))V - [U, V]. \]

**Proposition 4.** Let \( \omega \) be the positive unit volume form. Given a \( k \)-vector field \( U \in E_k(\Omega) \) and a \( k \)-form \( \alpha \in E^k(\Omega) \) with \( 0 \leq k \leq n \), we have:

\[ \alpha(U) \omega = \alpha \wedge i_U \omega, \]

\[ dj(U) = i_{\text{rot}(U)} \omega. \]

**Proof.** If \( U = e_i \wedge \cdots \wedge e_{i_k} \) with \( i_1 < \cdots < i_k \) and \( \alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k} \) with \( j_1 < \cdots < j_k \), then \( \alpha(U) \neq 0 \) if and only if the sequences \((i_1, \ldots, i_k)\) and \((j_1, \ldots, j_k)\) coincide, and then \( \alpha(U) \omega = \omega = \alpha \wedge i_U \omega \). If the two sequences do not coincide, then both sides vanish. By expanding any \( U \) and \( \alpha \) and using linearity, we conclude that the equation (14) holds in general.

Next, \( dj(U) = j(\nabla U) = j((-1)^{(k+1)(n-k)} \ast * \nabla U) = j(*\text{rot}(U)) \) which is equal to \( i_{\text{rot}(U)} \omega \) by Lemma 4, thus proving (15). \[ \square \]
5. The Ergodic Theorem for actions of $\mathbb{R}^k$

In this section we present Theorem 5, a special case of Tempelman’s version of the Ergodic Theorem [13] (also see [14]), for volume-preserving actions of $\mathbb{R}^k$. This result is an essential step in showing that the asymptotic linking invariant is well-defined.

Let $M$ be a compact Riemannian manifold (possibly with boundary) with Riemannian volume form $\mu$ and let $\Phi : \mathbb{R}^k \times M \to M$ be a conservative action of $\mathbb{R}^k$ on $M$. Let $L^1(M)$ denote the space of measurable real functions $f : M \to \mathbb{R}$ such that $\int_M |f|d\mu < \infty$. Consider a sequence of $k$-rectangles

$$T_n := [0, T_n^1] \times \cdots \times [0, T_n^k], \quad n \in \mathbb{N}$$

with each $T_n^i > 0$, such that for each $i (1 \leq i \leq k)$ $\lim_{n \to \infty} T_n^i = \infty$. For a function $f \in L^1(M)$, define a sequence of means $f_n \in L^1(M), \quad n \in \mathbb{N}$, by setting

$$f_n(p) := \frac{1}{\lambda(T_n)} \int_{t \in T_n} f(\Phi_t(p))d\lambda(t)$$

$$= \frac{1}{T_n^1 T_n^2 \cdots T_n^k} \int_0^{T_n^1} \int_0^{T_n^2-1} \cdots \int_0^{T_n^k-1} f(\Phi(t_1, \ldots, t_k)(p))dt_1 dt_2 \cdots dt_k$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}^k$ and $t = (t_1, \ldots, t_k)$. The following theorem is a special case of Theorem 6.2 of Tempelman [13] and also of Theorem 3.3 of Lindenstrauss [8].

**Theorem 5.** (Ergodic Mean Theorem) There is a unique function $\tilde{f}$ in $L^1(M)$ to which the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges almost everywhere, i.e.,

$$\lim_{n \to \infty} \int_M |f_n - \tilde{f}| \, d\mu = 0.$$

Furthermore, $\tilde{f}$ is independent of the choice of the sequence $\{T_n\}_{n \in \mathbb{N}}$ and satisfies

$$\int_M \tilde{f} \, d\mu = \int_M f \, d\mu.$$

Of course, uniqueness of $\tilde{f}$ is understood in the sense of $L^1$, i.e., two such functions $\tilde{f}$ agree outside of a set of measure zero.

Lindenstrauss’ Theorem 3.3 implies this theorem since $\mathbb{R}^k$ is an amenable group and $\{T_n\}$ is a tempered Folner sequence.

**Outline of the Proof.** First we observe that for a fixed sequence $\{T_n\}$ of $k$-rectangles the set of $f \in L^1(M)$ for which the Theorem holds is a closed vector subspace of $L^1(M)$. Then the essential idea is Tempelman’s decomposition of $L^1(M)$ into invariant functions and functions with zero mean (Theorem 5.1 of [13]). Let $W$ be the vector subspace of $L^1(M)$ generated by functions $h - h \circ \Phi_t$ where $h = \chi_A$ is the characteristic function of a measurable set $A$ and $t \in \mathbb{R}^k$, and let $\overline{W}$ be its closure in $L^1(M)$. One shows that the conclusions of the Theorem hold for $f = h - h \circ \Phi_t$, if $h$ is the characteristic function of a measurable set $A$ in $\Omega$, and consequently for every $f \in W$. By approximation, the same is true for all $f \in \overline{W}$.

On the other hand, let $I \subset L_1(M)$ be the set of invariant functions where $f \in L_1(M)$ is invariant if there exists a measurable set $A$ with $\mu(M \setminus A) = 0$ such
that for every \( x \in A \) and \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \) we have \( f(\Phi_t(x)) = f(x) \). For every invariant function \( f \) it is clear that \( f_n = f \), so it is easy to see that the conclusions of the Theorem hold for every \( f \in I \) by setting \( \tilde{f} = f \). Since by Theorem 5.1 of \cite{13} every function \( f \in L^1(M) \) can be uniquely represented as a sum \( f = f_1 + f_2 \) with \( f_1 \in I \) and \( f_2 \in \overline{W} \), the Theorem holds for every \( f \in L^1(M) \). \( \square \)

6. The Generalized Gauss Divergence Theorem for a Multivector Field

In this section, \( \Omega \) is a compact domain with smooth boundary in \( \mathbb{R}^n \). We define the integral of a \( k \)-vector field \( X = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \in E_k(\Omega) \) to be the \( k \)-vector

\[
\int_{\Omega} X \omega := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \int_{\Omega} f_{i_1 \ldots i_k} \omega \right) e_{i_1} \wedge \cdots \wedge e_{i_k} \in E_k(\Omega)
\]

where \( \omega \) is the unit volume form. Using this definition of the integral, we can extend the Gauss divergence theorem to \( k \)-vector fields on \( \Omega \) with \( k > 1 \).

**Theorem 6.** (Generalized Gauss Divergence Theorem for a Multivector Field) If \( V \in E_k(\Omega) \), then

\[
\int_{\Omega} \operatorname{div}(V) \omega = (-1)^{(k+1)\ell} \int_{\partial \Omega} N \cdot V \ dA,
\]

where \( N \) is the unit normal vector field pointing outwards along \( \partial \Omega \), \( N \cdot V \) is the extended dot product \cite{3}. \( \omega \) and \( dA \) are the positive unit volume forms on \( \Omega \) and \( \partial \Omega \), and \( \ell = n - k - 1 \).

**Proof.** Since every element of \( E_k(\Omega) \) is a sum of decomposable ones, it suffices to prove the proposition for a decomposable \( k \)-vector \( V = V^1 \wedge \cdots \wedge V^k \) where \( V^i \) is given by \cite{12}. Then from \cite{13} and \cite{11} we get

\[
\operatorname{div}(V) = (-1)^k \sum_{i=1}^k \sum_{\ell_1, \ldots, \ell_k = 1}^n (-1)^i \operatorname{div}(v_{\ell_1}^1 \cdots v_{\ell_i}^i \cdots v_{\ell_k}^k) e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}
\]

since

\[
\operatorname{div}(v_{\ell_1}^1 \cdots v_{\ell_i}^i \cdots v_{\ell_k}^k V^i) = \sum_{\ell_i = 1}^n \frac{\partial (v_{\ell_1}^1 \cdots v_{\ell_k}^k)}{\partial x_{\ell_i}}.
\]

By Stokes’ Theorem we have

\[
\int_{\Omega} \operatorname{div}(v_{\ell_1}^1 \cdots v_{\ell_i}^i \cdots v_{\ell_k}^k V^i) \omega = \int_{\partial \Omega} v_{\ell_1}^1 \cdots v_{\ell_i}^i \cdots v_{\ell_k}^k < N, V^i > dA
\]

so

\[
\int_{\Omega} \operatorname{div}(V) \omega = (-1)^k \sum_{i=1}^k (-1)^i \int_{\partial \Omega} < N, V^i > V^1 \wedge \cdots \wedge V^k dA
\]

\[
= \int_{\partial \Omega} \left( (-1)^k \sum_{i=1}^k (-1)^i < N, V^i > V^1 \wedge \cdots \wedge V^k \right) dA
\]

\[
= (-1)^{(k+1)\ell} \int_{\partial \Omega} N \cdot (V^1 \wedge \cdots \wedge V^k) \ dA
\]

using \( V = \sum_{\ell_i = 1}^n v_{\ell_i}^i v_{\ell_i} \) and Proposition \cite{2} \( \square \)
Corollary 2. Set \( \Omega - x = \{ u - x \in \mathbb{R}^n \mid u \in \Omega \} \). For a \( k \)-vector field \( V(x,u) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) we have
\[
\text{div}_x \int_{\Omega-x} V(x,u)du = -(-1)^{k+1} \int_{\partial\Omega-x} N \cdot V(x,u) dA(u) + \int_{\Omega-x} \text{div}_x V(x,u)du.
\]

Proof. By the change of variables \( v = u + x \)
\[
\int_{\Omega-x} V(x,u)du = \int_{\Omega} V(x,v-x)dv,
\]
so
\[
\text{div}_x \int_{\Omega-x} V(x,u)du = \text{div}_{1,x} \int_{\Omega} V(x,v-x)dv + \text{div}_{2,x} \int_{\Omega} V(x,v-x)dv,
\]
where the notation indicates that the divergence is calculated with respect to the first or second occurrence of the variable \( x \). Now
\[
\text{div}_{1,x} \int_{\Omega} V(x,v-x)dv = \int_{\Omega} \text{div}_{1,x} V(x,v-x)dv
= \int_{\Omega-x} \text{div}_x V(x,u)du
\]
by reversing the change of variables. On the other hand, if we introduce a new variable \( z = x \) to separate the two arguments of \( V \),
\[
\text{div}_{2,x} \int_{\Omega} V(x,v-x)dv = \text{div}_z \int_{\Omega} V(z,v-x)dv
= -\text{div}_v \int_{\Omega} V(z,v-x)dv
= -(-1)^{(k+1)\ell} \int_{\partial\Omega} N \cdot V(z,v-x) dA(v)
= -(-1)^{(k+1)\ell} \int_{\partial\Omega-x} N \cdot V(x,u) dA(u)
\]
by Theorem 6 reversing the change of variables. Adding the last two expressions gives the desired result. \( \square \)

7. Extension of the Biot-Savart Formula

We now give an extension of the Biot-Savart formula to higher dimensions. For a smooth divergence-free vector field \( V \) that is tangent to the boundary on a bounded domain \( \Omega \) in \( \mathbb{R}^3 \), it is well known that the Biot-Savart formula
\[
BS(V)(x) = \frac{-1}{4\pi} \int \frac{(x - y) \times V(y)}{||x - y||^3} dy,
\]
gives a right inverse for the curl, i.e., \( \text{rot}(BS(V)) = V \) (e.g., see \[3\] §5). We generalize this result to \( \mathbb{R}^n \) since it will be used in our proofs.

Let \( \Omega \) be a bounded domain with smooth boundary \( \partial\Omega \) in \( \mathbb{R}^n \) and consider \( k \) commuting vector fields \( V_1, \ldots, V_k \) on \( \Omega \) that are divergence-free and tangent to \( \partial\Omega \), \( 1 \leq k < n \), with \( \ell = n - k - 1 \). They generate an action of \( \mathbb{R}^k \) on \( \Omega \). Let \( V = V_1 \wedge \cdots \wedge V_k \) be the exterior product of the vector fields \( V_i \).

Theorem 7. For \( x \in \Omega \), the \( \ell \)-vector field
\[
BS(V)(x) = \frac{(-1)^k}{a_n} \int_{\Omega} \frac{(x - y)}{||x - y||^n} \times V(y) dy,
\]
where \( a_n \) is the \((n-1)\)-volume of the unit sphere in \( \mathbb{R}^n \) and we use the standard Lebesgue measure \( dy \) on \( \mathbb{R}^n \), satisfies

\[
\text{rot}(BS(V))(x) = V(x).
\]

**Proof.** Note that the integral is well defined since the pole along the singular set has order \( n - 1 \). We prove the theorem for \( x \in \Omega \), to avoid the problem of a singularity of order \( n - 1 \) when we integrate along \( \partial \Omega \). It will follow by continuity that the theorem holds for every \( x \in \Omega \).

By the change of variables \( u = y - x \) on \( \Omega - x \), we have

\[
BS(V)(x) = \frac{(-1)^k + 1}{a_n} \int_{\Omega - x} \frac{u}{||u||^n} \times V(u + x) du.
\]

Since \( \text{rot}(u \times v) = \text{div}(u \wedge v) \), from Corollary 2 we get

\[
I := \text{rot}(BS(V))(x) = \frac{(-1)^k + 1}{a_n} \text{div}_x \int_{\Omega - x} \frac{u}{||u||^n} \wedge V(u + x) du = I_1 + I_2
\]

where

\[
I_1 = \frac{(-1)^k + 1}{a_n} \int_{\Omega - x} \text{div}_x \left( \frac{u}{||u||^n} \wedge V(u + x) \right) du
\]

and

\[
I_2 = -\frac{(-1)^{k+1 + k(\ell + 1)}}{a_n} \int_{\partial \Omega - x} N \cdot \left( \frac{u}{||u||^n} \wedge V(u + x) \right) dA(u).
\]

Applying Proposition 3 and the facts that \( \text{div}(V^i) = 0 \), \([V^i, V^j] = 0\), and \( \frac{u}{||u||^n} \) does not depend on the variable \( x \), we have

\[
I_1 = \frac{1}{a_n} \int_{\Omega - x} \sum_{i=1}^{k} (-1)^i \left( \frac{u}{||u||^n} \right)_{x} (V^i(u + x)) \left( V^1 \wedge \cdots \wedge V^k(u + x) \right) du,
\]

where \( \left( \frac{u}{||u||^n} \right)_{x} (V^i) \) is the action of the vector field \( \frac{u}{||u||^n} \) on \( V_i(u + x) \) with derivatives in the variable \( x \). Expanding the \( V_i \)'s by (12), using the definition of the integral (13), and avoiding the singularity at \( u = 0 \), we can write

\[
I_1 = -\frac{1}{a_n} \lim_{\epsilon \to 0} \sum_{j_1, \ldots, j_k = 1}^n \int_{\Omega'} \left\langle \frac{u}{||u||^n}, \nabla_x(v_{j_1}^{v_{j_k}}(u + x)) \right\rangle du e_{j_1} \wedge \cdots \wedge e_{j_k}
\]

where \( \Omega' = (\Omega - x) \setminus \{ ||u|| \leq \epsilon \} \) and \( e_{j_1} \wedge e_{j_2} \cdots e_{j_k} = (-1)^{i-1} e_{j_1} \cdots e_{j_k} \). Now

\[
\nabla_x(v_{j_1}^{v_{j_k}})(u + x) = \nabla_u(v_{j_1}^{v_{j_k}})(u + x),
\]

so, for \( \epsilon > 0 \) so small that \( \{ ||u|| \leq \epsilon \} \subset \hat{\Omega} \), the integral

\[
I(\epsilon) := \int_{\Omega'} \left\langle \frac{u}{||u||^n}, \nabla_x(v_{j_1}^{v_{j_k}})(u + x) \right\rangle du
\]

can be written as

[12]
\[
I(\epsilon) = \int_{\Omega'} \left( \frac{u}{||u||^n} \nabla u(v^1_{j_1} \ldots v^k_{j_k})(u+x) \right) \, du \\
= \int_{\Omega'} \left( \text{div}_u(v^1_{j_1} \ldots v^k_{j_k} \frac{u}{||u||^n}) - v^1_{j_1} \ldots v^k_{j_k} \text{div}_u\left( \frac{u}{||u||^n} \right) \right) \, du \\
= \int_{\Omega'} \text{div}_u(v^1_{j_1} \ldots v^k_{j_k} \frac{u}{||u||^n}) \, du \\
= \int_{\partial \Omega - x} <N, \frac{u}{||u||^n} > v^1_{j_1} \ldots v^k_{j_k} \, dA(u) \\
- \int \left( ||u|| = \epsilon \right) \frac{1}{\epsilon^{n-1}} v^1_{j_1} \ldots v^k_{j_k} \, dA(u)
\]

by Theorem \[\text{by Theorem } 6\] since \(\text{div}_u\left( \frac{u}{||u||^n} \right) = 0\) on \(\mathbb{R}^n\). Thus

\[
\lim_{\epsilon \to 0} I(\epsilon) = \int_{\partial \Omega - x} <N, \frac{u}{||u||^n} > v^1_{j_1} \ldots v^k_{j_k} \, dA(u) - a_n v^1_{j_1} \ldots v^k_{j_k} (x),
\]

so

(18) \(I_1 = -\frac{1}{a_n} \int_{\partial \Omega - x} <N, \frac{u}{||u||^n} > V^1 \ldots V^k(u+x) \, dA(u) + V^1 \wedge \ldots \wedge V^k(x).\)

Next, returning to (18), we get

\[
I_2 = \frac{(-1)^{k+1}}{a_n} \int_{\partial \Omega - x} N : \left( \frac{u}{||u||^n} \wedge V(u+x) \right) dA(u) \\
= \frac{1}{a_n} \int_{\partial \Omega - x} <N, \frac{u}{||u||^n} > V(u+x) dA(u).
\]

by Proposition \[\text{by Proposition } 2\] since the \(V_i\)’s are tangent to \(\partial \Omega - x\). Adding the last result to (18) we obtain the desired conclusion, \(I = \text{rot}(BS(V))(x) = I_1 + I_2 = V(x). \)

\[\text{Corollary 3. Let } \Omega \text{ be convex with unit volume form } \omega \text{ and let } V \in E_k(\Omega) \text{ be as above. Then}\]

(19) \(dj(BS(V)) = i_V \omega \) and

(20) \(I(\Phi, \Psi) = \int j(BS(X)) \wedge d\beta \)

**Proof.** By (15) and Theorem \[\text{by Theorem 7}\]

\(djBS(V) = i_{\text{rot}(BS(V))} \omega = i_V \omega \).

proving (19). By Lemma \[\text{by Lemma } 4\]

\(I(\Phi, \Psi) = \int_\Omega \alpha \wedge d\beta \)

is independent of \(\alpha\), provided that \(d\alpha = i_X \omega\). Then by (19) with \(V = X\)

\(I(\Phi, \Psi) = \int j(BS(X)) \wedge d\beta. \)

\[\text{□}\]
8. Linking of submanifolds

In order to study the asymptotic linking invariant we recall the linking of singular submanifolds in \( \mathbb{R}^n \). Let \( N \) and \( N' \) be closed, oriented, possibly singular, disjoint submanifolds of \( \mathbb{R}^n \) of dimensions \( k \) and \( \ell \), where we always suppose that \( n = k + \ell + 1 \). Then the linking number \( \text{lk}(N, N') \) of \( N \) and \( N' \) can be defined as follows. Let \( C \) be a compact oriented singular \( k + 1 \)-dimensional manifold in \( \mathbb{R}^n \) with \( \partial C = N \). By a small deformation of \( C \), if necessary, we may suppose that \( C \) is transverse to \( N' \) and only intersects it in non-singular points of \( N' \). Then the linking number of \( N \) and \( N' \) is defined to be

\[
\text{lk}(N, N') := \sum_{p} \varepsilon_p
\]

where the sum is taken over all points \( p \in C \cap N' \), with \( \varepsilon_p = +1 \) if the orientation of \( C \times N' \) coincides with that of \( \mathbb{R}^n \) or \(-1\) if the orientations are opposite. It is well known that this linking number is symmetric, does not depend on the choice of \( C \), and can also be calculated as

\[
\text{lk}(N, N') = \text{deg}(f : N \times N' \to S^{n-1})
\]

where

\[
f(p, q) := \frac{q - p}{\|q - p\|}
\]

is the normalized vector pointing from \( p \in N \) to \( q \in N' \) and \( \text{deg}(f) \) is the degree of the mapping \( f \) relative to the orientations of \( N \), \( N' \), and \( S^{n-1} \). If \( N \) and \( N' \) are disjoint images of smooth maps \( g : \bar{N} \to \mathbb{R}^n \) and \( g' : \bar{N'} \to \mathbb{R}^n \), then the linking number can be calculated by

\[
\text{lk}(N, N') = \frac{1}{a_n} \int_{\bar{N} \times \bar{N'}} \tilde{f}^*(\sigma)
\]

where \( \tilde{f} = f \circ (g \times g') \) and \( a_n = \int_{S^{n-1}} \sigma \) is the volume form on \( S^{n-1} \).

In order to prove the next proposition, we observe that if \((t_1, t_2, \ldots, t_k)\) are local coordinates in \( N \), then the volume form \( d\eta \) on \( N \) can be written in these coordinates as

\[
d\eta = \left| \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_k} \right| dt_1 dt_2 \cdots dt_k.
\]

and similarly for the volume form \( d\eta' \) on \( N' \) with local coordinates \( s_1, \ldots, s_\ell \).

**Proposition 5.** If \( N \) and \( N' \) are disjoint immersed closed oriented submanifolds in \( \mathbb{R}^n \), then the linking number \( \text{lk}(N, N') \) can be calculated by the formula

\[
\text{lk}(N, N') = \frac{(-1)^k}{a_n} \int_{p \in N} \int_{q \in N'} \frac{(q - p) \times U(p) \cdot U'(q)}{|q - p|^n} d\eta(p) d\eta'(q)
\]

where \( U(p) \) is a unit \( k \)-vector on \( N \) at \( p \) and \( U'(q) \) is a unit \( \ell \)-vector on \( N' \) at \( q \) and \( \eta \) and \( \eta' \) are the volume measures in \( N \) and \( N' \).

Furthermore, this formula holds if \( N \) and \( N' \) are the disjoint images of smooth manifolds \( \bar{N} \) and \( \bar{N'} \) under smooth singular maps \( q : \bar{N} \to \mathbb{R}^n \) and \( q' : \bar{N'} \to \mathbb{R}^n \), since the images of the singular sets (where \( U(p) = 0 \) or \( U'(q) = 0 \)) have measure zero on \( N \) and \( N' \), by Sard’s Theorem.
Proof. Note that the volume form \( \sigma = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \ldots \hat{dx}_i \ldots dx_n \) on \( S^{n-1} \) can be written \( \sigma = i_Y dx_1 \ldots dx_n \), where \( Y = \sum_{i=1}^{n} x_i e_i \) is the position vector in \( S^{n-1} \). Then, since \( dx_1 \ldots dx_n(Z) = *Z \) for any \( Z \in \Lambda_n(\mathbb{R}^n) \),

\[
\sigma(v_2 \wedge v_3 \wedge \cdots \wedge v_n) = i_Y dx_1 \ldots dx_n(v_2 \wedge \cdots \wedge v_n) \\
= dx_1 \ldots dx_n(Y \wedge v_2 \wedge \cdots \wedge v_n) \\
= *(Y \wedge v_2 \wedge \cdots \wedge v_n).
\]

(24)

On the other hand, using local coordinates \((t_1, \ldots, t_k, s_1, \ldots, s_\ell)\) in \( N \times N' \), since \( f(p, q) = \frac{q-p}{||q-p||} \) and \( \bar{f} = f \circ (g \times g') \), we have

\[
\frac{\partial \bar{f}}{\partial t_i}(p, q) = -\frac{1}{||q-p||} \frac{\partial}{\partial t_i} (p) + \left[ \frac{1}{||q-p||} \right]_{t_i} (q-p), \\
\frac{\partial \bar{f}}{\partial s_j}(p, q) = \frac{1}{||q-p||} \frac{\partial}{\partial s_j} (q) + \left[ \frac{1}{||q-p||} \right]_{s_j} (q-p).
\]

Setting \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_k} \) and \( \frac{\partial}{\partial s} = \frac{\partial}{\partial s_1} \wedge \cdots \wedge \frac{\partial}{\partial s_\ell} \) we get

\[
\frac{\partial \bar{f}}{\partial t} \wedge \frac{\partial \bar{f}}{\partial s} = \frac{\partial \bar{f}}{\partial t_1} \wedge \cdots \wedge \frac{\partial \bar{f}}{\partial t_k} \wedge \frac{\partial \bar{f}}{\partial s_1} \wedge \cdots \wedge \frac{\partial \bar{f}}{\partial s_\ell} \\
= \frac{(-1)^k}{||q-p||^{k+\ell}} \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s} + W \wedge (q-p)
\]

(25)

where \( W \) is a \((k+\ell-1)\)-vector. Thus, at the point \((p, q)\) in \( N \times N' \) that corresponds to the point \( \frac{q-p}{||q-p||} \in S^{n-1} \), using local coordinates and \( k + \ell + 1 = n \) we get

\[
\bar{f}^*(\sigma)(p, q) = \sigma \left( \frac{\partial \bar{f}}{\partial t} \wedge \frac{\partial \bar{f}}{\partial s} dt_1 \ldots dt_k ds_1 \ldots ds_\ell \right)
\]
Remark 1. (See, e.g., \[\alpha\] by arclength by well-known Gauss linking number formula)

\[\text{Corollary 4.} \quad \text{\textit{Lk}}(\mathbb{N}, \mathbb{N}') = \int_{\mathbb{N}} \int_{\mathbb{N}'} (q - p) \cdot U'(q) \, dq \, dq' \quad \text{by} \quad (3)\]

since \(U'(q)\) and \((q - p) \times U(p)\) are in the same dimension \(\ell\) so the dot product commutes. Thus by (21)

\[\text{lk}(N, N') = \frac{1}{a_n} \int_{P \in \mathbb{N}} \int_{q \in \mathbb{N}'} \tilde{f}^*(\sigma)(p, q)\]

\[= \frac{(-1)^k}{a_n} \int_{P \in \mathbb{N}} \int_{q \in \mathbb{N}'} \frac{((q - p) \times U(p)) \cdot U'(q)}{||q - p||^n} \, dq \, dq'. \quad \square\]

**Remark 1.** (See, e.g., [3]) In dimension 3, when \(N\) and \(N'\) are curves parametrized by arclength by \(\alpha : [0, t_0] \to \mathbb{N}\) and \(\alpha' : [0, s_0] \to \mathbb{N}'\), the formula (23) becomes the well-known Gauss linking number formula

\[\text{lk}(N, N') = \frac{-1}{4\pi} \int_0^{t_0} \int_0^{s_0} \frac{(\alpha'(s) - \alpha(t)) \times \hat{\alpha}(t) \cdot \hat{\alpha}'(s)}{||\alpha'(s) - \alpha(t)||^3} \, ds \, dt.\]

A double differential form \(L(x, y)\) on \(\mathbb{R}^n \times \mathbb{R}^n\) of bidegree \((k, \ell)\), \(k + \ell = n - 1\), is called a **linking form** if whenever \(N = g(\mathbb{N})\) and \(N' = g'(\mathbb{N}')\) are disjoint images of smooth singular maps \(g : \mathbb{N} \to \mathbb{R}^n\) and \(g' : \mathbb{N}' \to \mathbb{R}^n\), where \(\mathbb{N}\) and \(\mathbb{N}'\) are closed oriented manifolds of dimensions \(k\) and \(\ell\), then we have

\[\text{lk}(N, N') = \int_{\mathbb{N}} \int_{\mathbb{N}'} L(x, y).\]

**Corollary 4.**

\[L = L(x, y) = \frac{(-1)^k}{a_n} \frac{((y - x) \times U(x)) \cdot U'(y)}{||y - x||^n} \, dq(x) \, dq'(y) \quad \text{(26)}\]
is a linking form on $\mathbb{R}^n \times \mathbb{R}^n$, where $U(x)$ is a unit vector on $N$ at $x$, $U'(y)$ is a unit vector on $N'$ at $y$, and $\eta$ and $\eta'$ are the volume measures in $N$ and $N'$.

This is evident from Proposition 5.

9. Proofs of Lemmas 2 and 3

As in §2 consider two volume-preserving actions $\Phi : \mathbb{R}^k \times \Omega \rightarrow \Omega$ and $\Psi : \mathbb{R}^\ell \times \Omega \rightarrow \Omega$ on a compact convex domain $\Omega$ in a Riemannian $n$-manifold $M$ tangent to the (smooth) boundary $\partial \Omega$, $n = k + \ell + 1$. Recall that $\mathcal{T}_k$ is the set of $k$-rectangles $T = [0, T_1] \times \cdots \times [0, T_k] \subset \mathbb{R}^k$ for $(T_1, \ldots, T_k) \in \mathbb{R}_+^k$. Fix points $\tilde{p}, \tilde{q} \in \Omega$, $\tilde{p} \neq \tilde{q}$, and consider the geodesic cones $\sigma(p, T)$, $(p, T) \in \Omega \times \mathcal{T}_k$, and $\sigma'(q, S)$, $(q, S) \in \Omega \times \mathcal{T}_\ell$, with apices $\tilde{p}$ and $\tilde{q}$, as defined in (1). We now prove Lemma 2.

**Proof of Lemma 2** We must show that for every $T \in \mathcal{T}_k$ and $S \in \mathcal{T}_\ell$ the set

$$X = \{(p, q) \in \Omega \times \Omega \mid \theta_\Phi(p, T) \cap \theta_\Psi(q, S) \neq \emptyset\}$$

has measure zero in $\Omega \times \Omega$. Set

$$A_q = \Phi(-T, \Psi(S, q)), \quad B_q = \Phi(-T, \sigma'(q, S)),$$

$$B'_p = \Psi(-S, \sigma(p, T)), \quad \text{and} \quad C_p = \{q \in \Omega \mid \sigma(p, T) \cap \sigma'(q, S) \neq \emptyset\}.$$

Note that for any set $K \subset \Omega$ and $p \in \Omega$, $p \in \Phi(-T, K) \iff \Phi(T, p) \cap K \neq \emptyset$. Consequently

$$p \in A_q \iff \Phi(T, p) \cap \Psi(S, q) \neq \emptyset,$$

$$p \in B_q \iff \Phi(T, p) \cap \sigma'(q, S) \neq \emptyset, \text{and}$$

$$q \in B'_p \iff \Psi(S, q) \cap \sigma(p, T) \neq \sigma(p, T)\emptyset.$$

Since $\theta_\Phi(p, T) = \Phi(T, p) \cup \sigma(p, T)$ and similarly for $\theta_\Psi(q, S)$, it follows that

$$X = \bigcup_{q \in \Omega} ((A_q \cup B_q) \times \{q\}) \cup \bigcup_{p \in \Omega} (\{p\} \times (B'_q \cup C_p)).$$

Each of the sets $A_p, B_p$, and $B'_p$ is a singular compact $(n - 1)$-dimensional sub-manifold with open dense complements in $\Omega$, and therefore has measure zero in $\Omega$.

Next we shall show that if $p \neq \tilde{q}$ the set $C_p$ has measure zero in $\Omega$. Let $\tilde{N}$ be the cone consisting of straight segments beginning at $\tilde{q}$, passing through a point of $\sigma(p, T)$, and ending at a point of $\partial \Omega$. Let $N$ be the closure of the component of $\tilde{N} \setminus \sigma(p, T)$ that does not contain the point $\tilde{q}$. Now $\sigma'(q, S)$ meets $\sigma(p, T)$ if and only if $\Psi(\partial S, q)$ meets $N$. Thus $C_p = \Psi(-\partial S, N)$, which is a compact singular manifold (the product of the image of the union of the $2\ell$ faces of $S$ with $N$) of dimension $(\ell - 1) + (k + 1) = n - 1$, so it has measure zero.

Note that each of the sets

$$\cup_q (A_q \times \{q\}), \cup_q (B_q \times \{q\}), \cup_p (\{p\} \times B'_p), \text{ and } \cup_p (\{p\} \times C_p)$$

is closed and therefore measurable in $\Omega \times \Omega$. Hence the function $f : \Omega \times \Omega \rightarrow \{0, 1\}$, defined by setting $f(p, q) = 1$ if $p \in A_q$ and 0 otherwise, is measurable. Since $A_q$ has measure zero in $\Omega$ for almost all $q \in \Omega$, and therefore $\int_\Omega f(p, q)dp = 0$ for almost all $q$, Fubini’s theorem shows that $\int_\Omega \int_\Omega f(p, q)dqdq = 0$, which means that the set $\cup_q (A_q \times \{q\})$ has measure zero in $\Omega \times \Omega$. Parallel arguments show that the
clearly, Proposition 6. \( \sigma \) measure zero, \( h \) and \( \Omega \)

\[ \int_{\Omega} \Phi^\ast(\omega) = 0 \]

\( \Omega \) is a union of \( k \) rectangles \( T = [0, T_1] \times \cdots \times [0, T_k] \subset \mathbb{R}^k \) for \( (T_1, \ldots, T_k) \in \mathbb{R}_+^k \), \( p \in \Omega \setminus N \) is fixed, and \( \mathbf{X} = X_1 \wedge \cdots \wedge X_k \) generates \( \Phi \). According to Lemma 3, for every \( T \in \mathcal{T}_k \) the sets \( \sigma(p, T) \) defined in (1) are disjoint from \( N \) for almost all \( p \in \Omega \). The invariant \( I(\Phi, N) = \int_N \alpha \) with \( da = i_X \omega \) was defined in (2).

Lemma 5. This invariant satisfies \( I(\Phi, N) = \int_N jBS(X) \) and does not depend on the choice of \( \alpha \).

Proof. By (19) \( d jBS(X) = i_X \omega = da \) so \( d(\alpha - jBS(X)) = 0 \). Since \( \Omega \) is convex, \( \alpha - jBS(X) \) is exact and there exists a form \( \theta \) such that \( d \theta = \alpha - jBS(X) \). Then \( I(\Phi, N) - \int_N jBS(X) = \int_N \alpha - \int_N jBS(X) = \int_N d \theta = \int_{\partial N} \theta = 0 \) since \( \partial N = \emptyset \). Clearly \( \int_N jBS(X) \) does not depend on \( \alpha \).

Proposition 6. The following conditions are satisfied:

1. The sets \( \sigma(p, T) \) vary measurably in the sense that for every \( T \in \mathcal{T}_k \) there is a function \( h_T : \Omega \to \mathbb{R} \) defined by

\[ h_T(p) = \frac{1}{T_1 \cdots T_k} \int_{\sigma(p, T)} \int_{y \in N} L(x, y), \]

and \( h_T \in L^1(\Omega) \), i.e., \( \int_{\Omega} |h_T(p)| \, dh(p) < \infty \).
2. The family of functions \( \{h_T\} \) converges to zero in \( L^1(\Omega) \), i.e.,

\[ \lim_{T_1, \ldots, T_k \to \infty} \int_{\Omega} |h_T(p)| \, dh(p) = 0. \]

Proof. To prove (1), let \( Y_T := \{p \in \Omega \mid \sigma(p, T) \cap N = \emptyset\} \) and note that \( h_T(p) = (T_1 \cdots T_k)^{-1} \int_{\sigma(p, T)} \int_N L(x, y) \) is defined and varies continuously on the dense open set \( \Omega \setminus Y_T \), where the compact sets \( \sigma(p, T) \) and \( N \) are disjoint. Then since \( Y_T \) has measure zero, \( h_T \) is measurable in \( \Omega \).

To show that \( h_T \) is integrable and that the limit converges to zero, we parametrize \( \sigma(p, T) \) by setting

\[ T^i = [0, T_1] \times \cdots \times [0, T_i] \]

and

\[ \partial_\delta T = [0, T_1] \times \cdots \times \{t_\delta\} \times \cdots \times [0, T_k] \]
Lemma 6. There exists a constant $W_i > 0$ such that for all $t^i \in T^i$ and $y \in N$

$$\int_{r \in [0,1]} \int_{p \in \tilde{\Omega}} \bar{L}(r, t^i, y, p)drd\lambda(p) \leq W_i$$

where $d\lambda(p)$ is the euclidean measure on $\Omega$.

This lemma will be proven at the end of this section. We use it now to show that $h^{i\delta}_T \in L^1(\Omega)$. In fact, by (29),

$$\int_{p \in \tilde{\Omega}} |h^{i\delta}_T(p)| \leq \frac{1}{|T|} \int_{r \in [0,1]} \int_{p \in \tilde{\Omega}} \int_{t^i \in T^i} \int_{y \in N} \bar{L}(r, t^i, y, p)drd\lambda(p)$$

$$= \frac{1}{|T|} T \int_{t^i \in T^i} \int_{y \in N} \left[ \int_{r \in [0,1]} \int_{p \in \tilde{\Omega}} \bar{L}(r, t^i, y, p)drd\lambda(p) \right] dt^i d\eta(y)$$

$$\leq \frac{W_i d\lambda(N)}{|T|} T \int_{t^i \in T^i} dt^i \left[ \int_{y \in N} d\eta(y) \right] = \frac{W_i \text{Vol}(N) T \tilde{T}_1 \cdot \tilde{T}_i \cdot \tilde{T}_k}{T} = \frac{W_i \text{Vol}(N)}{T}.$$
so $h^{(i)}_T \in L^1(\Omega)$ and $\lim_{T \to \infty} \int_{p \in \Omega} |h_T(p)| d\lambda(p) = 0.$

\textbf{Proof of Lemma 6}. Using $\sigma_p(r, t^i)$ and its derivatives,

$$\frac{\partial \sigma_p}{\partial r} = \frac{\partial \sigma_p}{\partial r} \wedge \frac{\partial \sigma_p}{\partial t_1} \wedge \cdots \wedge \frac{\partial \sigma_p}{\partial t_k}$$

$\quad = (1 - r)^{k-1} [\Phi(t^{i^0}, p) - \tilde{p}] \wedge X_1(\Phi(t^{i^0}, p)) \wedge \cdots \wedge X_k(\Phi(t^{i^0}, p))$

$\quad = (1 - r)^{k-1} [\Phi(t^{i^0}, p) - \tilde{p}] \wedge X_i(\Phi(t^{i^0}, p))$

where $X_i = X_1 \wedge \cdots \wedge \hat{X}_i \cdots \wedge X_k$. Note that $(1 - r)^{k-1} \leq 1$, $|\Phi(t^{i^0}, p) - \tilde{p}|$ is less than or equal to the diameter $D$ of $\Omega$, there is a constant $B$ such that $|X_i(\Phi(t^{i^0}, p))| \leq B$ for all $p \in \Omega$, and $||U(y)|| = 1$, so by (30) we have

$$L(r, t^i, y, p) \leq \frac{||\frac{\partial \sigma_p}{\partial r}(r, t^i)|| ||U(y)||}{||\frac{\partial \sigma_p}{\partial r}(r, t^i) - y||^{n-1}}$$

$$\leq \frac{(1 - r)^{k-1} ||\Phi(t^{i^0}, p) - \tilde{p}|| ||X_i(\Phi(t^{i^0}, p))|| ||U(y)||}{||\frac{\partial \sigma_p}{\partial r}(r, t^i) - y||^{n-1}}$$

$$\leq \frac{DB}{||\Phi(t^{i^0}, p) - \tilde{p}||^{n-1}}.$$

Thus

$$\int_{r \in [0, 1]} \int_{p \in \Omega} L(r, t^i, y, p) d\lambda(p)$$

(31) $\leq \int_{r \in [0, 1]} \int_{p \in \Omega} \frac{DB}{||\Phi(t^{i^0}, p) - \tilde{p}||^{n-1}} d\lambda(p)$.

Now for $\tilde{p} \in N$ there exists $\epsilon > 0$ such that for all $y \in N$ and $r \in [1 - \epsilon, 1]$

$$||\Phi(t^{i^0}, p) - \tilde{p}|| \geq d/2,$$

where $d$ is the distance from $\tilde{p}$ to $N$. Then

$$\int_{r \in [1 - \epsilon, 1]} \int_{p \in \Omega} \frac{DB}{||\Phi(t^{i^0}, p) - \tilde{p}||^{n-1}} d\lambda(p)$$

$$\leq \int_{r \in [1 - \epsilon, 1]} \int_{p \in \Omega} \frac{2^{n-1}DB}{d^{n-1}} d\lambda(p) = \frac{

On the other hand, for $r \in [0, 1 - \epsilon]$, $\Phi(t^{i^0}, \cdot) = \Phi_{t^{i^0}}$ is a volume-preserving diffeomorphism of $\Omega$, so we can make the substitution $p' = \Phi(t^{i^0}, p)$ and get

$$\int_{r \in [0, 1 - \epsilon]} \int_{p' \in \Omega} \frac{DB}{||\Phi(t^{i^0}, p') - \tilde{p}||^{n-1}} d\lambda(p')$$

$$= \int_{r \in [0, 1 - \epsilon]} \int_{p' \in \Omega} \frac{DB}{||(1 - r)p' + \tilde{p} - y||^{n-1}} d\lambda(p').$$

Now for each $r$ we let $p_r = (1 - r)p' + r\tilde{p}$. Then $d\lambda(p_r) = (1 - r)^n d\lambda(p')$ and $\Omega$ is replaced by $\Omega_r \subset \Omega$ (a contraction moving towards $\tilde{p}$), so

$$\int_{r \in [0, 1 - \epsilon]} \int_{p' \in \Omega} \frac{DB}{||(1 - r)p' + \tilde{p} - y||^{n-1}} d\lambda(p')$$

$$\leq \int_{r \in [0, 1 - \epsilon]} \int_{p' \in \Omega} \frac{DB}{||(1 - r)p' + \tilde{p} - y||^{n-1}} d\lambda(p').$$
since $1-r \geq \epsilon$ and $\Omega_r \subset \Omega$, by the following lemma, which holds since the singularity at $q$ has order $n-1$, and that is less than the dimension $n$.

**Lemma 7.** There is a constant $\Gamma$ such that the function

$$g(q) = \int_{\Omega \setminus \{q\}} \frac{1}{||p-q||^{n-1}} d\lambda(p)$$

satisfies $|g(q)| \leq \Gamma$ for all $q \in \Omega$.

□

Combining the last two results with (31), we get

$$\int_{r \in [0,1-\epsilon]} \int_{p \in \Omega} \tilde{L}(r, t^i, y, p) dr d\lambda(p) \leq 2^{n-1} DB \epsilon d^{n-1} + \frac{DB \Gamma (1-\epsilon)}{\epsilon^n} =: W_i.$$

□

Since $\theta_{\Phi}(p, T)$ and $N$ are disjoint for almost all $(p, T) \in \Omega \times T_k$, the linking number $\text{lk}(\theta_{\Phi}(p, T), N)$ is defined on an open dense set. Then we have

**Proposition 7.** The limit

$$\bar{\text{lk}}_{\Phi, N}(p) = \lim_{T_1, \ldots, T_k \to \infty} \frac{1}{T_1 \cdots T_k} \text{lk}(\theta_{\Phi}(p, T), N)$$

exists as an integrable $L^1$-function on $\Omega$ and does not depend on the choice of the point $\tilde{p} \in \Omega \setminus N$.

**Proof.**

(32) \hspace{1cm} \text{lk}(\theta_{\Phi}(p, T), N) = \int_{\Phi(T, p)} \int_{N} L + \int_{\sigma(p, T)} \int_{N} L.

By Proposition 6

(33) \hspace{1cm} \lim_{T \to \infty} \frac{1}{T_1 \cdots T_k} \int_{\sigma(p, T)} \int_{N} L = 0.

Let

$$g(p) = \frac{(-1)^k}{a_n} \int_{y \in N} \frac{(y-p) \cdot X(p) \cdot U(y)}{||y-p||^n} d\eta(y)$$

where $U$ is the positive unit $\ell$-form on $N$. The function $g$ is smooth on $\Omega \setminus N$. Then

$$|g(p)| \leq \frac{1}{a_n} \int_{y \in N} \frac{||y-p|| \cdot ||X(p)|| \cdot ||U(y)||}{||y-p||^n} d\eta(y).$$

Let $K$ be an upper bound for $||X(p)||$, $p \in \Omega$. Since $||U(y)|| = 1$,

$$|g(p)| \leq \frac{K}{a_n} \int_{y \in N} \frac{1}{||y-p||^{n-1}} d\eta(y).$$
By Fubini’s Theorem
\[ \int_{p \in \Omega} |g(p)| \, d\lambda(p) \leq \frac{K}{a_n} \int_{y \in N} \frac{1}{\|y-p\|^{n-1}} \, d\lambda(p) \, d\eta(y) \]
\[ \leq \frac{K\Gamma}{a_n} \int_N \, d\eta = \frac{K\Gamma \text{Vol}(N)}{a_n} \]
so \( g \in L^1(\Omega) \). On the other hand, note that
\[ \int_{x \in \Phi(T,p)} \int_{y \in N} L(x,y) = \]
\[ = \int_0^{T_1} \cdots \int_0^{T_k} \int_{y \in N} \frac{(y - \Phi(t,p)) \times X(\Phi(t,p)) \cdot U(y)}{\|y - \Phi(t,p)\|^n} \, d\eta(y) \, dt \]
\[ = \int_0^{T_1} \cdots \int_0^{T_k} g(\Phi(t,p)) \, dt. \]
Thus, by (32), (33), and the Ergodic Theorem, since \( g \in L^1(\Omega) \), the limit
\[ \lim_{T \to \infty} \frac{1}{T_1 \cdots T_k} \text{lk}(\theta(\Phi, T), N) = \lim_{T \to \infty} \frac{1}{T_1 \cdots T_k} \int_0^{T_1} \cdots \int_0^{T_k} g(\Phi(t,p)) \, dt \]
exists and defines an \( L^1 \) function \( \overline{\text{lk}}_{\Phi,N}(p) \) on \( \Omega \) that satisfies
\[ \int_{p \in \Omega} \overline{\text{lk}}_{\Phi,N}(p) \, d\lambda(p) = \int_{p \in \Omega} g(p) \, d\lambda(p) \]
and does not depend on the choice of \( \overline{\theta} \). \( \square \)

Then we define the asymptotic linking invariant to be \( \text{lk}(\Phi, N) = \int_{\Omega} \overline{\text{lk}}_{\Phi,N}(p) \, d\eta \) and prove Theorem 4, which states that \( \text{lk}(\Phi, N) = I(\Phi, N) \).

**Proof of Theorem 4**

\[ \text{lk}(\Phi, N) = \int_{p \in \Omega} \overline{\text{lk}}_{\Phi,N}(p) \, d\lambda(p) = \int_{p \in \Omega} g(p) \, d\lambda(p) \]
\[ = \int_{p \in \Omega} \frac{(-1)^k}{a_n} \int_{y \in N} \frac{(y - p) \times X(p) \cdot U(y)}{\|y - p\|^n} \, d\eta(y) \, d\lambda(p) \]
\[ = \int_{y \in N} \left[ \frac{(-1)^k}{a_n} \int_{p \in \Omega} \frac{(y - p) \times X(p)}{\|y - p\|^n} \, d\lambda(p) \right] \cdot U(y) \, d\eta(y) \]
by Fubini’s Theorem, so by (17) and the definition of the isomorphism \( j \)
\[ \text{lk}(\Phi, N) = \int_N BS(X) \cdot U \, d\eta = \int_N jBS(X)(U) \, d\eta. \]
Then since \( U \) is a unit \( \ell \)-vector and \( d\eta \) is a unit \( \ell \)-form, Lemma 5 shows that
\[ \text{lk}(\Phi, N) = \int_N jBS(X) = I(\Phi, N). \] \( \square \)
11. Asymptotic Linking of Two Actions

In this section, we assume that $M = \mathbb{R}^n$, so $\Omega$ is a compact convex region with smooth boundary in $\mathbb{R}^n$ and consider volume-preserving actions $\Phi$ and $\Psi$ of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ that are tangent to the boundary on $\Omega$, $k + \ell = n - 1$, as in \cite{22}. Recall that $D(\Phi, \Psi) \subset \Omega \times T_k \times \Omega \times T_\ell$ is the dense open set of points $(p, T, q, S)$ for which $\theta_\Phi(p, T)$ and $\theta_\Psi(q, S)$ are disjoint.

**Proposition 8.** The following conditions are satisfied:

1. The functions $(p, T) \mapsto \theta_\Phi(p, T)$ and $(q, S) \mapsto \theta_\Psi(q, S)$ are continuous functions on $\Omega \times \mathbb{R}^k$. Furthermore, the function $\int_{\theta_\Phi(p, T)} \int_{\theta_\Psi(q, S)} L(p, q)$ is continuous on $D(\Phi, \Psi)$ and therefore measurable.

2. The limits

$$
\lim_{T,S \to \infty} \frac{1}{\lambda_k(T)\lambda_\ell(S)} \int_{p \in \Omega} \int_{q \in \Omega} \{ \int_{A_p} \int_{B_q} L(p, q) \} \, dp \, dq = 0,
$$

where we set $(A_p, B_q)$ equal to $(\Phi(T, p), \sigma'(q, S)), (\sigma(p, T), \Psi(S, q))$, and $(\sigma(p, T), \sigma'(q, S))$, exist, and all three limits are zero.

**Proof.** (1) Since the actions are continuous and line segments depend continuously on their extremities, it is clear that the functions $(p, T) \mapsto \theta_\Phi(p, T)$ and $(q, S) \mapsto \theta_\Psi(q, S)$ are continuous, and so the function $\int_{\theta_\Phi(p, T)} \int_{\theta_\Psi(q, S)} L(p, q)$ is continuous and measurable on the dense open set $D(\Phi, \Psi)$.

Proof of (2). As before, $T, S \to \infty$ means that $\min(T_1, \ldots, T_k, S_1, \ldots, S_l) \to \infty$. When the compact sets $A_p$ and $B_q$ are disjoint, it is clear that the integral $\int_{A_p} \int_{B_q} L(p, q)$ converges, but it is not evident that the integral in (34) converges, although the integrand is measurable. First, consider $A_p = \Phi(T, p)$ and $B_q = \sigma'(q, S)$. We decompose $\sigma'(q, S) = \cup \sigma_j'(q, S)$ analogous to the decomposition of $\sigma(p, T)$ with the parametrization in \cite{22}. Let $s_{j_0} = 0$ and $s_{j_1} = S_j$ be the extremities of the interval $[0, S_j]$. Note that $\Psi(\partial S, q)$ is the union of $2\ell$ sets, $\Psi(\partial S, q) = \bigcup_{j=1}^{\ell} \bigcup_{z=0}^{1} \Psi(\partial S, q, z) \in \{0, 1\}$, where

$$
\partial_{\sigma' j} S = [0, S_j], x \cdots \{ s_{j_0} \} \times \cdots \times [0, S_j],
$$

so the singular submanifold

$$
\sigma'(q, S) = \bigcup_{j=1}^{\ell} \bigcup_{z=0}^{1} \sigma_{jz}'(q, S),
$$

where $\sigma_{jz}'(q, S)$ is the cone joining $\Psi(\partial_{\sigma' j} S, q)$ to the vertex $q$. We shall prove the Proposition for $B = \sigma_{jz}'(q, S)$ instead of $\sigma'(q, S)$; then the same proof works for the other components of $\sigma'(q, S)$.

Let $S^j = [0, S_1], \ldots, [0, S_j] \cdots \times [0, S_\ell]$. To each point

$$
s^j = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_\ell) \in S^j
$$

we naturally associate the point $s^{jz} = (s_1, \ldots, s_{j-1}, s_{jz}, s_{j+1}, \ldots, s_\ell) \in \partial_{\sigma' j} S$. We use the parametrizations $x_p(t) = \Phi(t, p), t \in T$, of $\Phi(T, p)$ and $y_q(u, s^j) = (1 - u)\Psi(s^{jz}, q) + u\bar{q}, (u, s^j) \in [0, 1] \times S^j,$

of $\sigma_{jz}'(q, S)$. Note that $\frac{\partial x_p}{\partial t} = X_i, \frac{\partial y_q}{\partial u} = (1 - u)Y_j$ and $\frac{\partial y_q}{\partial w} = \bar{q} - \Psi(s^{jz}, q)$. Since $\Omega$ is compact, there is a constant $C$ that is a common upper bound for $||X(p)|| = \cdots$
\[ ||X_1 \wedge \cdots \wedge X_k(p)||, ||Y_j(q)|| = ||Y_1 \wedge \cdots \wedge Y_l(q)|| \text{ and for } |\tilde{q} - \Psi(s^{j\xi}, q)|, p, q \in \Omega. \]

Recall that for multivectors \[ ||(u \times v) \cdot w|| \leq ||u|| ||v|| ||w||. \]

Then
\[
\left| \int_{A_p} \int_B L(p, q) \right| = \left| \int_{\Psi(T, p)} \Phi_{\gamma}(q, S) L \right| \\
\leq \int_{\Psi(T, p)} \Phi_{\gamma}(q, S) \left| L \right|,
\]

but using \[ \Phi_{\gamma}(x) = d\lambda(t), \text{ and } d\eta(t) = (\tilde{q} - \Psi(s^{j\xi}, q))d\lambda(s^j)du, \]

\[
\left| L \right| \leq \frac{1}{a_n} \frac{||x_p(t) - y_q(u, s^j)||}{||x_p(t) - y_q(u, s^j)||} ||X(\Phi(t, p))|| ||Y^j(y_q(u, s^j))|| d\eta(t) \]

\[
\leq C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t),
\]

where \[ C' = 3C/a_n, \] so

\[
\left| \int_{A_p} \int_B L \right| \leq C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t).
\]

Integrating \[ \left| \int_{A_p} \int_B L \right| \text{ on } \Omega \times \Omega \] we have

\[
\int_{p \in \Omega} \int_{q \in \Omega} \int_{A_p} \int_B L \lambda(p) \lambda(q)
\]

\[
\leq C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t) \lambda(p) \lambda(q)
\]

\[
\leq C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t) \lambda(p) \lambda(q)
\]

by Fubini’s Theorem, since we shall see that the last integral converges.

Since the action \[ \Phi_t \] preserves the volume, if we set \[ \Phi(t, p) = p' \], the measure \[ d\lambda(p') \] coincides with \[ d\lambda(p) \], and the last integral becomes

\[ (36) \]

\[
C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t) \lambda(p) \lambda(q)
\]

\[
\Gamma d\lambda(q) d\lambda(s^j) d\lambda(t) \leq \Gamma C' \overline{Vol}(\Omega) \overline{Vol}(T) \overline{Vol}([0, 1]) \overline{Vol}(S^j)
\]

\[
eq C' \overline{Vol}(\Omega) T_1 \cdots T_k S_1 \cdots S_j \cdots S_l.
\]

Lemma \[ \ref{lemma7} \] shows that this integral converges. Then, working backwards, it follows that all the previous integrals in this proof also converge. The integral \[ (36) \] is less than or equal to

\[
C' \int_{T, \infty} \int_{S_j} \int_{T, \infty} \left| x_p(t) - y_q(u, s^j) \right|^n d\lambda(s^j)du \lambda(t) \lambda(p) \lambda(q)
\]

In the limit we have

\[
0 \leq \lim_{T, S \to \infty} \int_{T_1 \cdots T_k S_1 \cdots S_l} \int_{p \in \Omega} \int_{q \in \Omega} \left| \int_{A_p} \int_B L \right| d\lambda(p) \lambda(q)
\]

\[
= \lim_{T_1, \ldots, T_k, S_1, \ldots, S_l \to \infty} \frac{C' \overline{Vol}(\Omega)}{S_j} = 0,
\]
so (34) holds for $A_p = \Phi(T, p)$ and $B_q = B = \sigma^I_{\delta}(q, S)$. Thus the limit vanishes for $\Phi(T, p)$ and $\sigma^I(q, S)$ and similarly for the case $A_p = \sigma(p, T)$ and $B_q = \Psi(S, q)$.

For the case when $A_p = \sigma(p, T)$ and $B_q = \sigma^I(q, S)$, we use the decompositions (24) of $\sigma(p, T)$ and (35) of $\sigma^I(q, S)$ and the parametrizations

$$x_p = \sigma_p(r, t^i) = (1 - r)\Phi(t^i, p) + r\bar{p}, \quad (r, t^i) \in [0, 1] \times T,$$

and

$$y_q = \sigma^I_q(u, s^j) = (1 - u)\Phi(t^j, q) + u\bar{q}, \quad (u, s^j) \in [0, 1] \times T,$$

of $\sigma_\delta(p, T)$ and $\sigma^I_{\delta}(q, S)$, with $t^i, t^j, s^j$ and $s^j$ as before. Then we have

$$|L(x_p, y_q)| \leq \frac{1}{a_n} \frac{||x_p - y_q|| ||X(x_p)|| ||Y^j(y_q)||}{||x_p - y_q||^n} d\eta(y_q)d\eta(x_p) \leq C ||x_p - y_q||^{n-1}d\lambda(s^j)d\mu(t)d\nu,$$

where $C_{a_n}$ is an upper bound for $|X(p)| |Y(q)|$.

It suffices to show that the limit of

$$L = \frac{1}{|S| |T|} \int_{p \in \Omega} \int_{q \in \Omega} \int_{x_p \in \sigma_\delta(p, T)} \int_{y_q \in \sigma^I_{\delta}(q, S)} \frac{C}{||x_p - y_q||^{n-1}}d\lambda(s^j)d\mu(t)d\nu$$

converges to zero as $S, T \to \infty$. We shall do this in three cases.

Case 1. $r, u \in [1 - \epsilon, 1]$, where $\epsilon > 0$ is such that $||y_q - x_p|| \geq d/2$ when $u, r \in [1 - \epsilon, 1]$ and $d$ is the distance from $\bar{p}$ to $\bar{q}$. Such an $\epsilon$ exists since $x_p \to \bar{p}$ and $y_q \to \bar{q}$ as $r, u \to 1$. In this case

$$C \leq \left(\frac{2}{d}\right)^{n-1},$$

the volume $D$ of $\Omega$ is finite, $|T|^{-1}\text{Vol}(\sigma_\delta(p, T)) \leq 1/T$, and $|S|^{-1}\text{Vol}(\sigma^I_{\delta}(q, S)) \leq 1/S$ so the limit of $L$ is zero.

Case 2. $r \in [0, 1 - \epsilon]$.

$$L' = \int_{p \in \Omega} \int_{q \in \Omega} \int_{y_q \in \sigma^I_{\delta}(q, S)} \int_{x_p \in \sigma_\delta(p, T)} \frac{1}{||x_p - y_q||^{n-1}}d\lambda(s^j)d\mu(t)d\nu$$

Then $\Phi(t^i, \cdot) = \Phi_{t^i}$ is a volume-preserving diffeomorphism of $\Omega$, so we can make the substitution $p' = \Phi(t^i, p)$ and get

$$L' = \int \int \int_{r \in [0, 1 - \epsilon]} \int_{p' \in \Omega} \frac{1}{||(1 - r)p' + \bar{p}||^{n-1}}d\lambda(p')dqdrdt$$

For each $r$ we let $p_r = (1 - r)p' + \bar{p}$. Then $d\lambda(p_r) = (1 - r)^n d\lambda(p')$ and $\Omega$ is replaced by $\Omega_r \subset \tilde{\Omega} (a \text{ contraction moving towards } \bar{p})$, so

$$L' = \int \int \int_{r \in [0, 1 - \epsilon]} \frac{1}{(1 - r)^n} \int_{p_r \in \Omega_r} \frac{1}{||p_r - y||^{n-1}}d\lambda(p_r).$$
Proposition 9. The limit

\[
\leq \int_{r \in [0,1-e]} \frac{1}{e^n} \int_{p \in \Omega} \frac{1}{||p_r - y||^{n-1}} dr d\lambda(p_r)
\]

\[
\leq \int_{r \in [0,1-e]} \frac{1}{e^n} \frac{\Gamma(1-\epsilon)}{\epsilon^n}
\]

by Lemma 7, since \(1 - r \geq \epsilon\) and \(\Omega_r \subset \Omega\). Now the volume of \(\Omega\) is finite, \(\text{Vol}(\sigma_{is}(p,T)) \leq |T|/T^3\), and \(\text{Vol}(\sigma_{jT}^e(q,S)) \leq |S|/S^3\), so it follows that \(\lim_{S,T \to \infty} L = 0\).

Case 3. \(u \in [0,1-\epsilon]\). This case is exactly parallel to Case 2, with \(p\) and \(q\) interchanged, so it is omitted. There is an overlap in the three cases, but all values of \((r,u) \in [0,1] \times [0,1]\) are covered. \(\square\)

Then for almost all \((p,T) \in \Omega \times \mathcal{T}_k\) and \((q,S) \in \Omega \times \mathcal{T}_\ell\), \(\theta(p,T)\) and \(\theta'(q,S)\) are disjoint and the linking number \(\text{lk}((\theta(p,T), \theta'(q,S))\) is defined.

**Proposition 9.** The limit

\[(37) \quad \text{lk}(p,q) = \lim_{T_1, \ldots, T_k, S_1, \ldots, S_\ell \to \infty} \frac{1}{T_1 \ldots T_k 1 S_1 \ldots 1 S_\ell} \text{lk}(\theta(p,T), \theta'(q,S))\]

exists as an integrable \(L^1\)-function on \(\Omega \times \Omega\) and does not depend on the choice of the points \(\hat{p}\) and \(\hat{q}\).

**Proof.** Calculating the linking number using the linking form \((29)\), it suffices to integrate over the sets \(\Phi(p,T)\) and \(\Psi(q,S)\), since by Proposition 8 the limits of the integrals over the other three sets vanish, i.e.,

\[
\lim_{T,S \to \infty} \frac{1}{\lambda_k(T) \lambda_\ell(S)} \text{lk}(\theta(p,T), \theta'(q,S)) = \frac{1}{\lambda_k(T) \lambda_\ell(S)} \int_{\Phi(T,p)} \int_{\Psi(S,q)} L.
\]

As before, \(X = X_1 \wedge \cdots \wedge X_k\) and \(Y = Y_1 \wedge \cdots \wedge Y_\ell\) are the exterior products of the vector fields that generate the actions of \(\Phi\) and \(\Psi\), respectively. Define the function \(f : \Omega \times \Omega \to \mathbb{R}\) by

\[(39) \quad f(p,q) := (-1)^k [(q - p) \times X(p)] \cdot Y(q), \quad a_n ||q - p||^n
\]

For every \((p,q)\) we have

\[(40) \quad |f(p,q)| \leq \frac{||X(p)|| ||Y(q)||}{a_n ||q - p||^{n-1}} \leq \frac{K}{a_n ||q - p||^{n-1}}
\]

where \(K\) is an upper bound for \(||X(p)|| ||Y(q)||, p, q \in \Omega\). Now, by Lemma 8 \(g(p,q) = 1/||q - p||^{n-1}\) is an integrable function in \(\Omega \times \Omega\), since

\[
\int \int_{(p,q) \in \Omega \times \Omega} g(p,q) d\lambda(p) d\lambda(q) = \int_{p \in \Omega} \left[ \int_{q \in \Omega} \frac{1}{||q - p||^{n-1}} d\lambda(q) \right] d\lambda(p)
\]

\[
\leq \int_{p \in \Omega} \Gamma d\lambda(p) = \Gamma \text{Vol}(\Omega).
\]

Then by \((40)\) we get
\[
\int \int_{(p,q) \in \Omega \times \Omega} |f(p,q)| d\lambda(p) d\lambda(q) \leq \frac{\Gamma K \text{Vol}(\Omega)}{a_n}
\]
so \( f \in L^1(\Omega \times \Omega) \).

To calculate \( \int_{\Phi(T,p)} \int_{\Psi(S,q)} L \) we use the natural parametrizations \( \bar{p} = x_p(t) = \Phi_t(p) = \Phi(t,p) \) and \( \bar{q} = y_q(s) = \Psi_s(q) = \Psi(s,q) \) induced by the actions \( \Phi \) and \( \Psi \) on \( \Phi(T,p) \) and \( \Psi(S,q) \). Then

\[
\frac{\partial x_p}{\partial t}(t) = X_1(\Phi_t(p)), i = 1, \ldots, k, \text{ and } \frac{\partial y_q}{\partial s}(s) = Y_j(\Psi_s(q)), j = 1, \ldots, \ell.
\]

Let \( \frac{\partial x_p}{\partial t}(t) = X_1 \wedge \cdots \wedge X_k(\Phi_t(p)) = X(\Phi_t(p)) \) and

\[
\frac{\partial y_q}{\partial s}(s) = Y_1 \wedge \cdots \wedge Y_\ell(\Psi_s(q)) = Y(\Psi_s(q)).
\]

Let \( U(\bar{p}) \) and \( U'(\bar{q}) \) denote the unit \( k \)- and \( \ell \)-vectors at \( \bar{p} \in \Phi(T,p) \) and \( \bar{q} \in \Psi(S,q) \), respectively. Then by \( [20] \)

\[
(-1)^k a_n \int_{\Phi(T,p)} \int_{\Psi(S,q)} L = \int_{\bar{p} \in \Phi(T,p)} \int_{\bar{q} \in \Psi(S,q)} \frac{[\bar{q} - \bar{p}] \cdot U'(\bar{q})}{||\bar{q} - \bar{p}||^n} d\eta(p) d\eta(q)
\]

\[
= \int_{t \in T} \int_{s \in S} \frac{[(y_q(s) - x_p(t)) \times U(x_p(t))] \cdot U'(y_q(s))}{||y_q(s) - x_p(t)||^n} \left( \frac{\partial y_q}{\partial s}(s)|ds| \right) \left( \frac{\partial x_p}{\partial t}(t)|dt| \right)
\]

\[
= \int_{T} \int_{S} \frac{[(\Psi_s(q) - \Phi_t(p)) \times (X(\Phi_t(p)) \cdot Y(\Psi_s(q)))]}{||\Psi_s(q) - \Phi_t(p)||^n} d\eta(p) d\eta(q)
\]

\[
= \int_{t \in T} \int_{s \in S} \frac{f(\Phi_t(p), \Psi_s(q))}{||\Psi_s(q) - \Phi_t(p)||^n} d\eta(p) d\eta(q)
\]

\[
= \int_{t \in T} \int_{s \in S} f(\Theta(t,s)(p,q)) ds_1 \ldots ds_\ell dt_1 \ldots dt_k
\]

\[
= \lim_{T_1, \ldots, T_\ell, S_1, \ldots, S_\ell \to \infty} \frac{1}{T_1 \ldots T_\ell S_1 \ldots S_\ell} \int_{\Phi(T,p)} \int_{\Psi(S,q)} L
\]

\[
= \lim_{T,S \to \infty} \frac{1}{\lambda(T) \lambda(S)} \int_{0}^{T_1} \int_{0}^{T_2} \int_{0}^{S_1} \int_{0}^{S_2} \int_{0}^{S_3} \int_{0}^{S_k} f(\Theta(t,s)(p,q)) dt_1 \ldots dt_k ds_1 \ldots ds_\ell
\]
so we get

$$\int \int_{(p,q) \in \Omega \times \Omega} \tilde{l}_k(p,q) dp \times dq = \int \int_{(p,q) \in \Omega \times \Omega} f(p,q) dp \times dq.$$ 

Then (38) shows that this function satisfies (37). Clearly it does not depend on the choices of $\tilde{p}$ and $\tilde{q}$. □

As a consequence of this Proposition, we can define the asymptotic linking invariant to be

$$\text{lk}(\Phi, \Psi) = \int_{p \in \Omega} \int_{q \in \Omega} \tilde{l}_k(p,q) d\eta(p) d\eta(q),$$

and then Theorem 2 states that $\text{lk}(\Phi, \Psi) = I(\Phi, \Psi)$.

**Proof of Theorem 2.** With the volume forms $\omega, d\eta(p)$, and $d\eta(q)$ on $\Omega$, we have

$$\text{lk}(\Phi, \Psi) = \int_{\Omega \times \Omega} f(p,q) dq(p) dq(q)$$

$$= \int_{\Omega \times \Omega} \frac{(-1)^k}{an} \int_{p \in \Omega} \int_{q \in \Omega} \left[ \frac{q - p}{||q - p||^n} \times X(p) \right] \cdot Y(q) dq(p) dq(q) \quad \text{by (39)}$$

$$= \int_{\Omega} \left[ \frac{(-1)^k}{an} \int_{p \in \Omega} \frac{q - p}{||q - p||^n} \times X(p) dq(p) \right] \cdot Y(q) dq(q)$$

by Fubini’s Theorem, and then, by the Biot-Savart formula (17), the definition of $j$, (14), and Corollary 3, this is equal to

$$\int_{\Omega} (BS(X) \cdot Y) \omega = \int_{\Omega} jBS(X)(Y) \omega$$

$$= \int_{\Omega} jBS(X) \wedge i_Y \omega = \int_{\Omega} jBS(X) \wedge d\beta = I(\Phi, \Psi).$$

□

12. A LOWER BOUND FOR THE ENERGY OF AN ACTION

We remark that in the case when $\Phi = \Psi$ and $n = 2k + 1$, the invariant $\text{lk}(\Phi, \Phi) = I(\Phi, \Phi)$ is a lower bound for the energy of the generating $k$-vector $X$.

**Definition 2.** Let $\Phi$ be a conservative $k$-action on $\Omega$ and let $X$ be the $k$-vector field that generates $\Phi$. The energy of the $k$-action $\Phi$ is defined to be the value of the integral

$$E(\Phi) = ||X||^2 = \int_{p \in \Omega} X(p) \cdot X(p) d\lambda(p) = \int_{p \in \Omega} ||X(p)||^2 d\lambda(p).$$

Note that we can decrease the energy of $\Phi$ by conjugating $\Phi$ by volume-preserving diffeomorphisms. Can we make it arbitrarily close to zero? The following result gives a negative answer to this question.

**Theorem 8.** There exists a constant $C > 0$ depending only on $\Omega$ such that

$$C^{-1} ||\text{lk}(\Phi, \Phi)|| \leq E(\Phi).$$
Proof. By Corollary 2, (7), and the definition of $j$, 

$$
\text{lk}(\Phi, \Phi) = \int_\Omega jBS(X) \wedge d\alpha = \int_\Omega jBS(X) \wedge i_X d\lambda \\
= \int_\Omega jBS(X) i_X d\alpha = \int_\Omega BS(X) \cdot X d\lambda.
$$

By the Cauchy-Schwarz inequality

$$
|\text{lk}(\Phi, \Phi)| = |<BS(X), X>| \leq ||BS(X)|| ||X||.
$$

Furthermore

$$
BS(X)(p) = \int_{q \in \Omega} \frac{(p - q) \times X(q)}{||p - q||^{2k+1}} d\lambda(q)
$$

so

$$
||BS(X)(p)|| \leq \int_{q \in \Omega} \frac{||p - q|| X(q)||}{||p - q||^{2k+1}} d\lambda(q) \\
\leq \int_{q \in \Omega} \frac{||X(q)||}{||p - q||^{2k}} d\lambda(q) \\
= \int_{q \in \Omega} \left[ \frac{||X(q)||}{||p - q||^{k}} \right] \left[ \frac{1}{||p - q||^{2k}} \right] d\lambda(q) \\
\leq \left[ \int_{q \in \Omega} \frac{||X(q)||^{2}}{||p - q||^{2k}} d\lambda(q) \right]^{1/2} \left[ \int_{q \in \Omega} \frac{1}{||p - q||^{2k}} \lambda(q) \right]^{1/2}
$$

by the Holder inequality. Then by Lemma 7 with $n = 2k + 1$

$$
||BS(X)(p)|| \leq \Gamma^{1/2} \int_{q \in \Omega} \left[ \frac{||X(q)||^{2}}{||p - q||^{2k}} d\lambda(q) \right]^{1/2}.
$$

Therefore

$$
||BS(X)||^2 = \int_{p \in \Omega} BS(X)(p) \cdot BS(X)(p) d\lambda(p) \\
= \int_{p \in \Omega} ||BS(X)(p)||^2 d\lambda(p) \\
\leq \Gamma \int_{p \in \Omega} \left[ \int_{q \in \Omega} \frac{||X(q)||^{2}}{||p - q||^{2k}} d\lambda(q) \right] d\lambda(p) \quad \text{by (42)} \\
= \Gamma \int_{q \in \Omega} ||X(q)||^{2} \left[ \int_{p \in \Omega} \frac{1}{||p - q||^{2k}} d\lambda(p) \right] d\lambda(q) \quad \text{by Fubini’s Theorem} \\
\leq \Gamma^2 \int_{q \in \Omega} ||X(q)||^{2} d\lambda(q)
$$

by Lemma 7. Thus

$$
||BS(X)|| \leq (\Gamma^2)^{1/2} \int_{q \in \Omega} ||X(q)||^{2} d\lambda(q) \right)^{1/2} = \Gamma ||X||.
$$

Substituting this inequality in (41) we get

$$
|\text{lk}(\Phi, \Phi)| \leq ||BS(X)|| ||X|| \leq \Gamma ||X||^2 = \Gamma E(\Phi).
$$
We can decrease the energy of $\Phi$ by volume-preserving diffeomorphisms, but these diffeomorphisms do not change the value of the asymptotic linking number $lk(\Phi, \Phi)$, so by $\Gamma^{-1}lk(\Phi, \Phi)$ is the desired lower bound for the energy of $\Phi$. □

13. Examples

Example 4. For every pair of integers $k, \ell \geq 1$, $k + \ell + 1 = n$, and every $t \in \mathbb{R}$, there exist conservative actions $\Phi$ of $\mathbb{R}^k$ and $\Psi$ of $\mathbb{R}^\ell$ on the unit closed ball $D^n \subset \mathbb{R}^n$ such that $\ell k(\Phi, \Psi) = I(\Phi, \Psi) = t$.

The construction uses several lemmas.

Lemma 8. Given disjoint smooth embeddings of closed oriented manifolds $M, N$, of dimensions $k$ and $\ell$ in $\mathbb{R}^n$, there exist disjoint smooth embeddings $M \times S^1, N \subset \mathbb{R}^{n+1}$ such that $\ell k(M \times S^1, N) = \ell k(M, N)$. The same holds if $N$ is an affine $\ell$-space disjoint from $M$.

Proof. Given $M$ and $N$, by a translation we may assume that their images lie in the positive half space $x_1 > 0 \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Let $P$ be the $(n-1)$-plane in $\mathbb{R}^n$ perpendicular to the $x_1$-axis, and rotate $M$ around $P$ to get $M \times S^1 \subset \mathbb{R}^{n+1}$. Clearly $M \times S^1$ is disjoint from $N$. If we let $\Sigma \subset \mathbb{R}^2$ be a compact singular $(k+1)$-manifold transverse to $N$ such that $\partial \Sigma = M$, then $\ell k(M, N) = \ell k(\Sigma, N)$. By rotating $\Sigma$ around $P$ we obtain $\Sigma \times S^1$, whose boundary is $M \times S^1$. Then $\ell k(\Sigma \times S^1, N) = \ell k(\Sigma, N)$, and therefore the linking number is the same.

In case $N$ is an affine $\ell$-plane a similar argument works, taking $P$ to be an affine plane parallel to $N$. □

Lemma 9. There exist disjoint embeddings of $T_k \times D^{k+1}$ and $T_\ell \times D^{\ell+1}$ in $D^n$, where $T_k$ and $T_\ell$ are tori of dimensions $k$ and $\ell$, such that $\ell k(T_k \times 0, T_\ell \times 0) = 1$.

Proof. Begin with disjoint smooth embeddings of two circles $M$ and $N$ in $\mathbb{R}^3$ such that $\ell k(M, N) = 1$. Applying Lemma 8 repeatedly, switching the roles of $M$ and $N$, gives disjoint embeddings of $T_k$ and $T_\ell$ in $\mathbb{R}^n$ with intersection number 1. Since the normal bundles are trivial we can extend the embeddings to disjoint embeddings of $T_k \times D^{k+1}$ and $T_\ell \times k+1$. Then a homothety will move these sets into $D^n$. □

Lemma 10. Let $T^r$ act on $T^r \times D^s$ by the product action on the first factor and identity on the second factor. For any smooth volume form $\omega$ on $T^r \times D^s$ there is a smooth isotopy $h_t$ of $T^r \times D^s$ taking each factor $T^r \times \{y\}$ to itself such that $h_0 = \text{id}$ and $h_t^*\omega$ is $T^r$-invariant.

Proof. Here we need a slightly modified form of Moser’s Theorem 12 acting on each orbit. We use the standard coordinates $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ and the standard Euclidean volume form $\omega^* = dx \wedge dy$ on $T^r \times D^s$ to simplify the notation. Let $f_0 : T^r \times D^s \to \mathbb{R}$ be the (unique) non-vanishing smooth function such that $\omega = f_0^*\omega^*$ and define $f_1 : D^s \to \mathbb{R}$ by setting $f_1(y) = \int_{T^r \times \{y\}} f_0(x, y)dx$. Note that $\alpha^*_y = f_0(y)dx$ is $T^k$-invariant. Now the volume forms $\alpha_y = f_1(x, y)dx$ and $\alpha^*_y$ have the same integral $\int_{T^r \times \{y\}} \alpha_y dx = \int_{T^r} \alpha^*_y dx$, so there exists a smooth function $f : T^r \times D^s \to \mathbb{R}$ such that $\alpha = f \alpha^*$ and we can apply Moser’s proof 12 on each factor $T^k \times \{y\}$. Following Moser, we may suppose that there is a positive $\epsilon$ such that $|f(y) - 1| < \epsilon$ for every $y$ by expressing any positive function $f$ as a sum of functions close to 1. We use the same cover of $T^k$ by open cubes $U_0, U_1, \ldots, U_m$ and the same functions $\eta_i(k = 1, \ldots, m)$, independent of $y$. Then it is straightforward to
check that Moser’s isotopies of each $T^k \times \{y\}$ fit together to give a smooth isotopy of $T^r \times D^s$ transforming each $\alpha_y$ into $\alpha_y^*$. This isotopy also transforms $\omega$ into $f_1 \omega$, which is invariant under the action of $T^k$. □

Construction of the Example. Take $W = T^k \times D^{\ell+1} \cup T^\ell \times D^{k+1}$ embedded in $D^n$ by Lemma 10, where $k + \ell + 1 = n$. The compact Lie groups $T^k$ and $T^\ell$ act on $W$, $T^k$ acting on $T^k \times D^{\ell+1}$ by multiplication on the first factor and trivially on $T^\ell \times D^{k+1}$, and analogously for the action of $T^\ell$.

By Lemma 10 we may conjugate the action of $T^k$ on $T^k \times D^{\ell+1}$ by a diffeomorphism isotopic to the identity so that it preserves the Euclidean volume form, and similarly for the action of $T^\ell$. Lift the actions of $T^k$ and $T^\ell$ to volume preserving actions $\phi : \mathbb{R}^k \times (T^k \times D^{\ell+1}) \to T^k \times D^{\ell+1}$ and $\psi : \mathbb{R}^\ell \times (T^\ell \times D^{k+1}) \to T^\ell \times D^{k+1}$.

Let $W_\epsilon = T^k \times D^{\ell+1}_0 \cup T^\ell \times D^{k+1}_0$ be a smaller invariant neighborhood of $T^k \cup T^\ell$, and let $\lambda : W \to [0,1]$ be constant on the orbits with the values 1 on $W_\epsilon$ and 0 on $D^n \setminus W$. Then let $\Phi(t,z) = \phi(\lambda(z)t,z)$ for $z$ in the $\epsilon$-neighborhood of $T^k \times D^{\ell+1}$ and $\Psi(t,z) = \psi(\lambda(z)t,z)$ on the $\epsilon$-neighborhood of $T^\ell \times D^{k+1}$ and identity elsewhere. Thus $\Phi$ and $\Psi$ are commuting conservative actions of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ on $D^n$. The linking number of the orbits $T^k \times \{y\}$ and $T^\ell \times \{z\}$ are $lk(T^k \times y, T^\ell \times z) = 1$ for $y \in D^{\ell+1}_0$ and $z \in D^{k+1}_0$.

Now it is easy to check that the linking number $lk(\Phi,\Psi) > 0$ since for points $p \in D^{\ell+1}_0$ and $q \in D^{k+1}_0$ and for $T = [0,2r\pi] \times S = [0,2s\pi]$, $lk(\theta_\Phi(p,T),\theta_\Psi(q,S)) = r^k s^\ell$ since for these rectangles $T$ and $S$ the cones $\sigma(p,T)$ and $\sigma'(q,S)$ are empty. When we normalize by dividing by $(2r\pi)^k \cdot (2s\pi)^\ell$ we get the constant $(2\pi)^{(k+\ell)}$, which is therefore the value of the limit for orbits in $W_\epsilon$ as $r,s \to \infty$. Other points $p,q$ contribute positively, so we get $lk(\Phi,\Psi) > 0$. To get a negative value it suffices to change one of the orientations. Finally by multiplying $t \in \mathbb{R}^k$ by $s$ we multiply the asymptotic linking number by $s^k$ and thus we can obtain all real numbers as values of $lk(\Phi,\Psi)$. □

Example 5. Given a closed connected oriented submanifold $N^\ell$ embedded in $D^n$ and a real number $t$, by a similar construction we can find a conservative action $\Phi$ of $\mathbb{R}^k$ on $D^n$, $k = n - \ell - 1$, such that $lk(\Phi, N) = t$.

Here the construction is similar to the previous example. By applying Lemma 8 repeatedly we can obtain $T^k \subset \mathbb{R}^n \setminus P$, where $P$ is an affine $\ell$-plane, such that the linking number is $lk(T^k,P) = 1$. Now locally the smooth embedding of $N$ in $D^n$ is diffeomorphic to the embedding of $P$ in $\mathbb{R}^n$, so we can find a small torus $T^k \subset D^k \setminus N$ such that $lk(T^k,N) = 1$. The rest of the construction proceeds as in Example 4.

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ASYMPTOTIC LINKING OF VOLUME-PRESERVING ACTIONS OF $\mathbb{R}^k$

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