SEP ARATING GEOMETRIC DATA WITH MINIMUM COST: TWO DISJOINT CONVEX HULLS

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ABSTRACT

In this study, a geometric version of an NP-hard problem ("Almost 2 − SAT" problem) is introduced which has potential applications in clustering, separation axis, binary sensor networks, shape separation, image processing, etc. Furthermore, it has been illustrated that the new problem known as "Two Disjoint Convex Hulls" can be solved in polynomial time due to some combinatorial aspects and geometric properties. For this purpose, an $O(n^2)$ algorithm has also been presented which employs the Separating Axis Theorem (SAT) and the duality of points/lines.

**Keywords** Convex Hull · Algorithm · Clustering · Binary Sensor Network · SAT · Separation Axis Theorem

1 Introduction

For some given point sets which constructs corners of the shapes reconstructing binary images with disjoint components is an important problem, for which there are several applications in digital/graphical games. Similar problems arise in the face of binary sensor networks. For all these applications, it is worth detecting the minimum number of noises which could be removed and caused construction of some separate components.

For instance, sensor networks are systems including simple atomic sensors over a site that senses events and reports them or tracks moving objects. Generally, these sensors are without much complexity, and different sensing aspects can be applied in these systems to various targets such as temperature, light, sound, etc. For some applications, the sensor generates some information as little as one bit at each time step in order to provide an inexpensive communication. These kinds of applications generate a binary model of sensor networks.

It is assumed that in the binary model, each node can determine one bit of information and broadcast it to the base station. These kinds of problems can be formulated as follows [Aslam et al., 2003]. Having a set of $n$ binary sensors $S = \{S_1, S_2, \ldots, S_m\}$ consider a single sample $s \in \{-1, +1\}^m$ of data, produced at time $t$. The following lemma illustrates an approximation for the location of the tracked object which is outside the convex hulls of both plus sensors and minus sensors.

**Lemma 1** Let $s \in \{-1, +1\}^m$ be a sample of the sensor values and $X(t)$ be the location of the target at the time $t$. Let $A = \{S_i | s_i = +1\}$, $B = \{S_j | s_j = -1\}$, and $CH(A)$ and $CH(B)$ their convex hulls. Then $CH(A) \cap CH(B) = \emptyset$. Furthermore, $X(t) \not\in CH(A) \cup CH(B)$ [Aslam et al., 2003].

This application provides the opportunity to define new geometry problems whose logical versions are $NP − hard$. Section 2 includes some subsections which represents results [Cygan et al., 2015] about these logical problems. In the next subsection, a new problem related to convex hull is introduced, which is applicable to shape modeling and sensor

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networks, and solvable in polynomial time. As a result, a naive $O(n^3)$ algorithm is also presented and the next part includes an $O(n^2)$ algorithm. Finally, Section 3 concludes this study and suggests some new open problems.

2 Two Disjoint Convex Hulls

As it is mentioned before, this problem is similar to a version of 2 – SAT problem which would be discussed in the following.

2.1 Variable Deletion Almost 2 – SAT

There are some basic hard problems in computational complexity which are $NP$ – hard. Although 2 – SAT problem belongs to class $P$, some versions of that like "Max 2 – SAT", "Almost 2 – SAT" and "Variable Deletion Almost 2 – SAT" are all $NP$ – hard. The last problem is known in theoretical computer science under the terms Almost$2$ – SAT, All – but – k 2 – SAT, 2 – CNF deletion, and 2 – SAT deletion [Razgon and O’Sullivan 2009].

In the "Almost 2 – SAT" problem, there are a 2 – CNF formula $\phi$, an integer $k$, and the question that whether one can delete at most $k$ clauses from $\phi$ to make it satisfiable. In "Variable Deletion Almost 2 – SAT" (variable-deletion
At first glance, it seems that the problem is NP-hard, and in this regard the following lemma had been proved [Cygan et al., 2015].

**Lemma 2** Variable Deletion Almost $2 - SAT$ can be solved in $4^n n^{O(1)}$ time.

### 2.2 The Main Problem

Detecting and testing the intersection between geometric objects are among the most important applications of computational geometry. It is one of the main questions addressed in Shamos’ article that lays the groundwork for computational geometry [Shamos, 1975], the first application of the plane sweep technique [Shamos and Hoey, 1976]. It is hard to overstate the importance of finding efficient algorithms for intersection testing or collision detection as this class of problems has many applications in Robotics, Computer Graphics and medical applications of image processing (e.g. [Moradkhani and Bigham, 2017], Computer-Aided Design, VLSI design [Toth et al., 2017], [Jiménez et al., 2001], [Lin and Gottschalk, 1998], the clustering problems in data science [Bohlouli et al., 2020] and facility location problems [Hassani and Eskandari, 2020]. In the plane, Shamos [Shamos, 1975] presents an optimal linear algorithm to construct the intersection of a pair of convex polygons. Another linear time algorithm is later presented by O’Rourke et al. [O’Rourke et al., 1982].

For the testing version of the problem, Lemma 3 has been proved [Barba and Langerman, 2014].

**Lemma 3** Let $P$ and $Q$ be two convex polygons with $n$ and $m$ vertices, respectively. The 2D-algorithm determines if $P$ and $Q$ intersect in $O(\log n + \log m)$ time [Barba and Langerman, 2014].

As discussed earlier, the problems are applicable to binary sensor networks and target tracking. Assume that some of the sensors do not work properly in a binary sensor network; these sensors send incorrect signs. Clearly, if two convex hulls overlap, it means that some of the sensors are making mistakes (Fig. 1), but the reverse is not necessarily right. In some applications, the minimum sensors are ignored because they are prone to make a wrong sign at the moment. This problem is summarized as Problem 1.

**Problem 1** Minimum Sign to Remove: There are a plus set $P_1$ and a minus set $P_2$ of points on the plane. For an integer $K \leq \min\{|P_1|,|P_2|\}$, are there $K$ points in $P_1 \cup P_2$ whose removing yields two disjoint convex hulls for $P_1$ and $P_2$?

At first glance, it seems that the problem is NP-hard, and structurally very similar to Variable Deletion Almost $2 - SAT$ problem. For some geometric reasons, however, the problem can be solved in a polynomial time. The following observations can be considered easily about two disjoint convex hulls of given points in the 2D plane.

**Observation 1** For two given sets $A$ and $B$ of points in Problem 1, the minimum points which should be removed, might be located outside of $CH(A) \cap CH(B)$ (Fig. 2).

**Observation 2** Removing each pair of points on a convex hull forces to remove all the points between them (either clockwise or counterclockwise), which are on the convex hull (Fig. 3).

**Observation 3** If we know the regions containing only plus (minus) signs, the problem may become simpler (Fig. 2).

To decide whether two convex polygons are intersecting, the Separating Axis Theorem can be used [Boyd et al., 2004, Gilbert et al., 1988]. Clearly, for two not-intersecting convex polygons, there exists a line passing between them. Obviously, such a line exists if and only if an edge of one of the polygons is parallel to the other edge of the line. The closest point of this line to the other polygon is one of its corners which is closest to the first polygon (Fig. 5). This edge will then form a separating axis between the polygons. And also if two middle edges of two polygons are parallel, both of them are separating axes. These are summarized in Lemma 4 and the idea has been used in the algorithm of this study [Boyd et al., 2004, Eberly, 2001].

**Lemma 4** Hyperplane Separation Theorem: Let $A$ and $B$ be two disjoint nonempty convex subsets of $R^n$. Then there exist a nonzero vector $v$ and a real number $c$ such that $< x, v > \geq c$ and $< y, v > < c$ for all $x$ in $A$ and all $y$ in $B$; i.e., the hyperplane $<.,v> = c$, $v$ the normal vector, separates $A$ and $B$ [Boyd et al., 2004].

For every two arrays $A_{2 \times n}$ and $B_{2 \times m}$ of points in the plane, Algorithm 1 computes a cost for each line segments made by a pair of points. It chooses the pair with the minimum cost and returns the points that should make the separating axis. As a result, the minimum number of removing signs (plus or minus) of Problem 1 will be represented. So a glance at the algorithm reveals that Theorem 1 is clear.
Figure 3: Removing points $a$ and $d$ forces to remove either $\{e\}$ or $\{b, c\}$.

Figure 4: All the sensors in region $A$ ($B$) are plus (minus) and we can remove only from region $C$.

**Theorem 1** Algorithm 1 solves Two Disjoint Convex Hulls in $O(m + n)^3$.

### 2.3 An $O(m + n)^2$ algorithm

As mentioned earlier, in this section a new fast algorithm is presented for the problem that uses the duality of given points. As illustrated in Fig. 6, if the dual space is drawn, $(m + n)^2$ regions appear in which $m = |A|$ and $n = |B|$. In order to find the optimal line $l_{opt}$ in the primal space, an optimal cell $cell_{opt}$ and a point $v_{opt}$ in it should be found in dual space that $v_{opt}$ demonstrates the dual of $l_{opt}$.

For this purpose, firstly the dual line of all the given points in $A$ (plus signs) and $B$ (minus signs) is drawn in the dual space. Red and blue colors are assigned to dual lines of set $A$ and $B$, respectively (Lines 2-3). There appear $O(m + n)^2$ segments and half-lines in two colors in the dual space. The dual space is stored in a Doubly Connected Edge List (DCEL) in which the edge list contains the color of every edges as well. This process takes $O(m + n)^2$ time. At this time, a planar graph $G(V, E)$ (Lines 4-6) can be generated as follow. The starting cell which vertex $v_1$ is assigned to that, is the cell above the upper envelope (Fig. 7 (a)). A vertex $v_i$ is assigned to every other cell in any order. There is an edge between two vertices if those are adjacent, so $E$ contains all the edges between every two adjacent vertices (Line 6)(Fig. 7(b)). The resulted graph has $O(m + n)^2$ vertices and edges, each of which processes once.

The next step is computing a pair of numbers $w_1$ and $w_2$ as weights of the vertices (7-25). Firstly, set $w_1(v_1) = m$ and $w_2(v_1) = 0$, then set weights of the adjacent vertices according to the segment between them, so that if the segment
Figure 5: Dividing axis for two convex hulls.

Figure 6: Primal Plane, red points \( \{r_1, r_2, \ldots, r_7\} \) and blue points \( \{b_1, b_2, \ldots, b_8\} \)

is red decrease \( w_1 \) without any change in \( w_2 \) and if the segment is blue reduce \( w_1 \) and \( w - 1 \) stay without change. Therefore, for each vertex \( v \), \( w_1(v) \) and \( w_2(v) \) show the number of red lines under \( v \) and the number of blue lines above it respectively. Using a queue structure \( Q \), the weight for all the vertices (Lines 8-25) is computed. In each step, when an unweighted vertex appears by crossing a red (blue) segment, \( w_1 \) (\( w_2 \)) is decreased (increased) by one. Computing the weight for new vertex takes constant time; hence, in \( O((m + n)^2) \) time the weighted planar graph can be computed in which the weights \( w_1(v) \) and \( w_2(v) \) demonstrate the red lines below and blue lines above the cell containing vertex \( v \). In both sensor network application and finding two disjoint convex hulls, if the separation line is the dual line of vertex \( v \), then \( w_1(v) \) plus signs and \( w_2(v) \) minus signs should be removed. Algorithm 2 find the optimal vertex \( v_{opt} \) with \( \min\{w_1 + w_2\} \) and line \( l_{opt} \) in Lines (26-27) in \( O((m + n)^2) \) time. We can conclude all the results in Theorem 2.

**Theorem 2** Let \( A \) and \( B \) be two sets of points in the plane, \( |A| = m \) and \( |B| = n \). Then Algorithm 2 finds the minimum number of removal points in order to have two disjoint convex hulls in \( O(m + n)^2 \).

3 Conclusion

In this paper, a new problem entitled "Two Disjoint Convex Hulls" was introduced. A useful application of binary sensor networks along with some observations were also discussed in detail. Furthermore, a naive algorithm \( O(n^3) \) and a faster one \( O(n^2) \) were presented. By swapping the sign of minimum number of sensors, another interesting problem aroused which was solved using the same algorithm. As for the future work, the results promise applications in
Algorithm 1 A Naive Algorithm for Two Disjoint Convex Hulls

**Input:** \( n \) points with plus sign \( P = \{p_1, p_2, ..., p_n\} \) and \( m \) points with minus sign \( M = \{m_1, m_2, ..., m_m\} \) in the plane.

**Output:** cost \( c \) and Indexes \( i \) and \( j \) which minimizes \( \text{cost} [P, M] \).

1: \( c = \infty, a = p_1, b = m_1; \)
2: for \( i = 1, ..., n \) do
3:     for \( j = 1, ..., m \) do
4:         if \( p_i \) and \( m_j \) are the same points then
5:             \( \text{cost}[i, j] = \infty \)
6:         else
7:             \( \text{rightp}[i, j] = \text{Number of plus points in} \ P \text{ that are at the right hand side of line} \ ij; \)
8:             \( \text{leftp}[i, j] = \text{Number of plus points in} \ P \text{ that are at the left hand side of line} \ ij; \)
9:             \( \text{rightm}[i, j] = \text{Number of minus points in} \ M \text{ that are at the right hand side of line} \ ij; \)
10:            \( \text{leftm}[i, j] = \text{Number of minus points in} \ M \text{ that are at the left hand side of line} \ ij; \)
11:            \( \text{cost}[i, j] = \min\{ (\text{rightp}[i, j] + \text{leftm}[i, j]), (\text{rightm}[i, j] + \text{leftp}[i, j]) \}; \)
12:        end if
13:        if \( \text{cost}[i, j] \leq c \) then
14:            \( c = \text{cost}[i, j], a = i, b = j; \)
15:        end if
16:    end for
17: end for
18: Return \( c, a, b; \)

Figure 7: (a) Vertex \( V_1 \) is assigned to the starting cell in the upper envelope. (b) every other cell is labeled in any order and there is an edge between two vertices if and only if their corresponding cells are adjacent.

computer graphics and wireless sensor networks. Moreover, considering the mentioned problem in higher dimensions is another fascinating feasible problem for further studies.

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Figure 8: Dual plane of Fig 6, dual of points (lines) and a black point in a region that separates blue and red lines with minimum error (only $D(b_7)$ and $D(r_8)$ are at the wrong side of black point.

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Algorithm 2 Two Disjoint CHs by Dual Space

**Input:** Two set of $n$ plus and $m$ minus points

**Output:** Optimal separation line to have minimum number of removal points and two disjoint convex hulls.

1: Compute the dual line for every points in $A$ and $B$
2: Generate DCEL for the arrangement;
3: For each half-edges in DCEL assign label $r$ (red for Plus points) or $b$ (blue for minus points);
4: Find the upper envelope of arrangement;
5: Generate a planar graph $G(V, E)$;
6: $V = \{v_1, v_2, ..., v_{(m+n)z}\}$ in which $v_1$ stands for the cell above the upper envelope and $v_2, ..., v_{(m+n)^z}$ for other cells;
7: Make $E$ by connecting every two vertices which their cells are adjacent;
8: Set weight $w_1(v_1) = m$ and $w_2(v_1) = 0$, which $m$ is the number of red lines.
9: Add $v_1$ to empty queue $Q$;
10: $V' = \{v_1\}$;
11: for all item $u$ in queue $Q$ do
12:  for all vertex $v$ in $V$ which their cell are adjacent to $u$ do
13:    if $v$ is not in $V'$ and the segment between $u$ and $v$ is red then
14:      $w_1(v) = w_1(u) - 1$;
15:      $w_2(v) = w_2(u)$;
16:    else
17:      if $v$ is not in $V'$ and the segment between $u$ and $v$ is blue then
18:        $w_1(v) = w_1(u)$;
19:        $w_2(v) = w_2(u) + 1$;
20:      end if
21:    end if
22:  end for
23:  Add $v$ to $V'$;
24:  Add $v$ to queue $Q$;
25: end for
26: Remove $u$ from queue $Q$;
27: Find the vertex $v_{opt}$ in $G$ with minimum $w_1 + w_2$;
28: Return $l_{opt}=\text{dual}(v_{opt})$, the optimal separation line in the prime space;