Sum of Hamiltonian manifolds

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Abstract. For any compact connected Lie group $G$, we study the Hamiltonian sum of two compact Hamiltonian group $G$-manifolds $(X^+, \omega^+, \mu^+)$ and $(X^-, \omega^-, \mu^-)$ with a common codimension 2 Hamiltonian submanifold $Z$ of the opposite equivariant Euler classes of the normal bundles. We establish that the symplectic reduction of the Hamiltonian sum agrees with the symplectic sum of the reduced symplectic manifolds. We also compare the equivariant first Chern class of the Hamiltonian sum with the equivariant first Chern classes of $X^\pm$.

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1 Introduction

Symplectic cut [L1] and symplectic sum [G, MW] are two of important operations on symplectic manifolds. These are effective methods of constructing new symplectic manifolds with specific properties, which can apply to study the problem how some important symplectic invariants, such as Gromov-Witten (GW) invariants, change under such symplectic operations ([H, LR, IP2, H, FZ]). Li and Ruan [LR] studied the degeneration formula of GW invariants under symplectic cut. Ionel and Parker [IP2] gave the symplectic sum formula of GW invariants under symplectic sum of two symplectic manifolds with a common codimension 2 symplectic submanifold. These methods were complemented and strengthened by Tehrani and Zinger [FZ].

In recent years, motivated by the Yang-Mills-Higgs theory in physics as well as some observations from the proof of Atiyah-Floer conjecture, some interesting invariants, called Hamiltonian or gauged GW invariants, were discovered for a symplectic manifold $(X, \omega)$ with a Hamiltonian action of a compact Lie group $G$ ([CGS, M, CGMS, MT, CWW]). A symplectic manifold $(X, \omega)$ with a Hamiltonian $G$-action is called a Hamiltonian $G$-manifold. The crucial notion for Hamiltonian manifolds is the moment map which is
a $G$-equivariant smooth map $\mu : X \to \mathfrak{g}^*$ generating the Hamiltonian $G$-action, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g} := \text{Lie}(G)$. We denote a Hamiltonian $G$-manifold by $(X, \omega, \mu)$. Hamiltonian GW invariants can be constructed by studying the equations of symplectic vortices derived from a principal $G$-bundle over a Riemann surface and $(X, \omega, \mu)$. In this paper, we study the sum operations on Hamiltonian $G$-manifolds. As a potential application, such sum operation would be useful for establishing the sum formula or degeneration formula of the Hamiltonian GW invariants.

Let $(X^+, \omega^+, \mu^+)$ and $(X^-, \omega^-, \mu^-)$ be two Hamiltonian $G$-manifolds of the same dimension, $Z = Z^+ \cong Z^-$ be their common codimension 2 Hamiltonian $G$-submanifold which has the same $G$-action induced from both sides, with restricted moment maps, denoted by $\mu_Z = \mu^+|Z = \mu^-|Z$. Suppose that the normal bundles $N_{X^+}Z$ and $N_{X^-}Z$ have opposite equivariant Euler classes $e_G(N_{X^+}Z) = -e_G(N_{X^-}Z)$. Then we can fix a $G$-equivariant isomorphism of the two $G$-trivial complex line bundles

$$\Phi : N_{X^+}Z \otimes_{\mathbb{C}} N_{X^-}Z \xrightarrow{\cong} Z \times \mathbb{C}.$$ (1.1)

The symplectic sum of $(X^+, \omega^+)$ and $(X^-, \omega^-)$ gives rises to a $(2n + 2)$-dimensional symplectic manifold $(S, \Omega)$, unique up to the (non-equivariant) homotopy class of $\Phi$, a smooth map $\pi : S \to D$ where $D$ is a sufficiently small neighbourhood of the origin in $\mathbb{C}$ such that

- $\pi$ is surjective and $S_0 = \pi^{-1}(0) = X^+ \cup_Z X^-,$
- $\pi$ is submersion away from $Z \subset S_0,$
- the restriction of $\Omega$ to $S_\lambda = \pi^{-1}(\lambda)$ is nondegenerate for every $\lambda \in D \setminus \{0\},$
- $\Omega|_{X^\pm} = \omega^\pm$ for $X^\pm \subset S_0$.

Our main result is the following Hamiltonian sum theorem which is in principle a combination of the symplectic sum construction in [IP2] [FZ] and a gluing construction of moment maps.

**Theorem 1** Let $G$ be a connected compact Lie group, $(X^+, \omega^+, \mu^+)$ and $(X^-, \omega^-, \mu^-)$ be two $2n$-dimensional Hamiltonian $G$-manifolds with a common codimension 2 Hamiltonian submanifold $Z = Z^+ \cong Z^-$ such that $\mu_Z = \mu^+|Z = \mu^-|Z$ and their respective normal bundles have opposite $G$-equivariant Euler classes. Then for each choice of homotopy class of equivariant isomorphisms [LJ], there exist a natural Hamiltonian structure on $(S, \Omega)$ with a moment map $\mu : S \to \mathfrak{g}^*$, such that for every $\lambda \in D \setminus \{0\}$, the fiber $(S_\lambda = \pi^{-1}(\lambda), \Omega|_{S_\lambda}, \mu_\lambda)$ is also a Hamiltonian $G$-manifold with the moment map $\mu_\lambda$ given by the restriction of $\mu$ to $S_\lambda$ and $\mu|_{X^\pm} = \mu^\pm$ for $X^\pm \subset S_0$. Moreover, if $0$ is a regular value of $\mu_Z$ and $G$-actions on $(\mu^\pm)^{-1}(0)$ are free, then for every $\lambda \in D \setminus \{0\}$

$$\mu_\lambda^{-1}(0)/G$$

is the symplectic sum of symplectic reductions $(\mu^+)^{-1}(0)/G$ and $(\mu^-)^{-1}(0)/G$ along their common codimension 2 symplectic submanifold $\mu_Z^{-1}(0)/G$. 


For every $\lambda \in D \setminus \{0\}$, the fiber $(\mathcal{S}_\lambda, \Omega|_{\mathcal{S}_\lambda}, \mu_\lambda)$ is called a Hamiltonian sum of $(X^+, \omega^+, \mu^+)$ and $(X^-, \omega^-, \mu^-)$, or simply denoted by $\mathcal{S}_\lambda = X^+ \#_{\mathcal{Z}, \lambda} X^-$. All $\mathcal{S}_\lambda$ are deformations, in the category of Hamiltonian manifolds, of the singular fibre $\mathcal{S}_0$. For $\lambda \neq 0$, these are isotopic to one another, as Hamiltonian $G$-manifolds.

The symplectic sum of symplectic orbifolds along a symplectic normalisable orbifolds was constructed in [Mu]. If $G$-actions on $(\mu^\pm)^{-1}(0)$ are locally free, symplectic reductions $(\mu^\pm)^{-1}(0)/G$ are symplectic orbifolds, and $\mu^\pm_0(0)/G$ is a symplectic normalisable suborbifold as in [Mu]. Then the principle that symplectic sum commutes with reduction holds in this Hamiltonian case.

The paper is organized as follows. In section 2, we review some definitions and properties of Hamiltonian manifolds and moment maps. We can generalize the moment map to a $G$-action on a smooth manifold with a $G$-invariant closed 2-form. In section 3, we review the construction of symplectic sum based on the paper [FZ]. In section 4, we show that the symplectic sum of two Hamiltonian $G$-manifolds has a natural Hamiltonian structure. In section 5, we compare the equivariant first Chern class of $(\mathcal{S}_\lambda, \Omega|_{\mathcal{S}_\lambda}, \mu_\lambda)$ with the equivariant first Chern classes of $(X^+, \omega^+, \mu^+)$ and $(X^-, \omega^-, \mu^-)$.

2 Hamiltonian manifolds, moment maps and generalization

In this section, we first review some basic notions and an example related to moment maps of Hamiltonian group $G$-action on symplectic manifolds ([MS]), then study its generalization to moment maps corresponding to any $G$-invariant closed 2-forms on general manifolds.

2.1 Moment maps for Hamiltonian $G$-manifolds

We first consider a symplectic manifold $(X, \omega)$. Let $G$ be a compact Lie group which acts covariantly on $(X, \omega)$ by symplectomorphisms. This means that there is a smooth group homomorphism from Lie group $G$ to the group of symplectomorphisms $\text{Symp}(X, \omega): g \mapsto \phi_g$. Denote the Lie algebra of $G$ by $\mathfrak{g} := \text{Lie}(G)$ as right invariant vector fields on $G$. The infinitesimal action determines a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of symplectic vector fields $\mathcal{X}(X, \omega): \xi \mapsto V_\xi$ defined by

$$V_\xi(x) := \frac{d}{dt} \bigg|_{t=0} \phi_{\exp(t\xi)} x, \quad \forall x \in X. \quad (2.1)$$

By calculation one can verify that

$$V_{\text{Ad}(g^{-1})\xi} = \phi_g^*V_\xi, \quad V_{[\xi, \eta]} = [V_\xi, V_\eta]. \quad (2.2)$$

So $V_\xi$ is $G$-equivariant. A vector field $V_\xi$ is Hamiltonian if there is a corresponding Hamiltonian function $H_\xi$ such that $\iota(V_\xi)\omega = dH_\xi$. The action of $G$ on $X$ is called Hamiltonian if the vector field $V_\xi$ is Hamiltonian for every $\xi \in \mathfrak{g}$ and the map

$$\begin{align*}
\mathfrak{g} & \to C^\infty(X, \mathbb{R}), \\
\xi & \mapsto H_\xi
\end{align*}$$
can be chosen to be \( G \)-equivariant with respect to the adjoint action of \( G \) on its Lie algebra \( \mathfrak{g} \), that is for \( \forall \, x \in X \),
\[
H_{\text{Ad}(g^{-1})}(x) = H_{g \cdot x}.
\]

A Hamiltonian \( G \)-manifold is a symplectic manifold \((X, \omega)\) with a Hamiltonian group \( G \)-action. By definition, the \( G \)-action is generated by the Hamiltonian vector fields \( V_\xi \) associated to the Hamiltonian functions \( H_\xi : X \to \mathbb{R} \) such that \( \xi \mapsto H_\xi \) is \( G \)-equivariant.

Then the (symplectic) moment map for the \( G \)-action on \((X, \omega)\) is a \( G \)-equivariant smooth map
\[
\mu : X \to \mathfrak{g}^* \cong \mathfrak{g}
\]
(isomorphism via an invariant inner product on the Lie algebra \( \mathfrak{g} \)), such that the Hamiltonian vector fields associated to the Hamiltonian functions \( H_\xi \)
\[
X \to \mathbb{R}
\]
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(isomorphism via an invariant inner product on the Lie algebra \( \mathfrak{g} \)), such that the Hamiltonian vector fields associated to the Hamiltonian functions defined by
\[
H_\xi(x) = \langle \mu(x), \xi \rangle
\] (2.3) generate the action, where \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( \mathfrak{g}^* \) and \( \mathfrak{g} \), or the invariant inner product on \( \mathfrak{g} \). In other words, \( \mu \) is a moment map for the \( G \)-action if and only if \( \mu \) satisfies the following

(a) \( G \)-equivariance condition
\[
\mu(g \cdot x) = \text{Ad}(g^{-1})^* \mu(x);
\] (2.4)

(b) Hamiltonian condition
\[
\langle d\mu(x) \tau, \xi \rangle = \omega(V_\xi(x), \tau)
\] (2.5)
for all \( g \in G, \xi \in \mathfrak{g}, x \in X, \tau \in T_x X \). We simply say that the Hamiltonian \( G \)-action is generated by the moment map \( \mu \), and denote a Hamiltonian \( G \)-manifold by the tuple \((X, \omega, G, \mu)\) or simply \((X, \omega, \mu)\) if the group \( G \) is prescribed. Here is an example of Hamiltonian manifolds.

**Example 1 (Induced Hamiltonian action on cotangent bundle)** In classical mechanics, any cotangent bundle is a symplectic manifold corresponding to the phase space of a Hamiltonian system. Let \( G \) be a Lie group, \( M \) a differential manifold with a \( G \)-action.

Then there exist associated natural vector bundle \( G \)-actions on the tangent bundle \( TM \) and the cotangent bundle \( T^* M \). The action on \( TM \) is given by \( g \cdot v = dg_x(v) \), for \( v \in T_x M \).

The action on the cotangent bundle \( T^* M \) is given by
\[
\langle g \cdot \eta, v \rangle_d = \langle \eta, g^{-1} \cdot v \rangle_d
\]
for \( \forall \, v \in T_x M \) and \( \eta \in T^*_x M \), where \( \langle \cdot, \cdot \rangle_d \) is the dual between \( T^* M \) and \( TM \). It is well-known that there exist a canonical 1-form \( \theta \) on \( T^* M \) defined as
\[
\theta_{(x, \alpha)}(v, \eta) = \langle \alpha, v \rangle,
\]
where \( (x, \alpha) \in T^*_x M \), \( (v, \eta) \in T_{(x, \alpha)}(T^* M) \). The canonical symplectic form on cotangent bundle \( T^* M \) is
\[
\omega_{\text{can}} = -d\theta.
\]
For such induced \( G \)-action on \( T^* M \), the moment map is given by the canonical 1-form
\[
\langle \mu_{\phi}(\cdot), \xi \rangle = -\theta(V_\xi(\cdot)),
\]
for \( \forall \, \xi \in \mathfrak{g} \). Condition (2.3) obviously holds. Condition (2.4) holds because \( V_\xi \) is \( G \)-equivariant. So \((T^* M, \omega_{\text{can}}, G, \mu_{\phi})\) is a Hamiltonian \( G \)-manifold. \( \square \)
2.2 Generalization of moment maps

In fact, for any manifold $M$ with a fixed closed 2-form $\omega$ (might not be non-degenerate), one can generalize the notion of moment maps for any group action on $M$ preserving the closed 2-form $\omega$. Now the pair $(M, \omega)$ might not be a symplectic manifold. Let $G$ be a compact Lie group which acts on the pair $(M, \omega)$ preserving the closed 2-form $\omega$. This means that there is a smooth group homomorphism from Lie group $G$ to the group of $\omega$-preserving diffeomorphisms $\text{Diff}(M, \omega)$.

For an element $\xi \in \mathfrak{g}$, the infinitesimal action determines a vector field $V_\xi$ defined as in (2.1). Since the closed 2-form $\omega$ is $G$-invariant, by Cartan formula

$$0 = \mathcal{L}_{V_\xi} \omega = d(\iota_{V_\xi} \omega) + \iota_{V_\xi} d\omega = d(\iota_{V_\xi} \omega),$$

we see that $\iota_{V_\xi} \omega$ is a closed 1-form. A smooth map

$$\mu_\omega : M \to \mathfrak{g}^*$$

satisfying (2.4) and (2.5) is called the $\omega$-moment map for the $G$-action on $(M, \omega)$. Equivalently, $\mu_\omega$ is $G$-equivariant and the following equality holds for all $\xi \in \mathfrak{g}$

$$d(\mu_\omega, \xi) = \iota_{V_\xi} \omega.$$  

(2.7)

In particular, if the closed 2-form $\omega$ is non-degenerate, an $\omega$-moment map is just the (symplectic) moment map for Hamiltonian $G$-action on a symplectic manifold.

If the $G$-invariant 2-form $\omega$ is the exterior differential of a $G$-invariant 1-form $\alpha$, that is $\omega = d\alpha$, then by Cartan formula

$$0 = \mathcal{L}_{V_\xi} \alpha = d(\iota_{V_\xi} \alpha) + \iota_{V_\xi} d\alpha = d(\iota_{V_\xi} \alpha) + \iota_{V_\xi} \omega.$$  

From (2.7), this means

$$d(\mu_\omega, \xi) = -d(\iota_{V_\xi} \alpha).$$

Then a $G$-equivariant smooth map

$$\mu_\alpha : M \to \mathfrak{g}^*$$

is called the $\alpha$-moment map for the $G$-action on $(M, \omega = d\alpha)$ if the following equality holds for all $\xi \in \mathfrak{g}$

$$\langle \mu_\alpha, \xi \rangle = -\iota_{V_\xi} \alpha.$$  

(2.9)

We remark that the $\alpha$-moment map was first defined by Lerman [L2] and further studied by Chiang and Karshon in [CK] for contact manifolds. Note that in this case the $d\alpha$-moment map is just the $\alpha$-moment map $\mu_{d\alpha} = \mu_\alpha$.

Let $f : M \to \mathbb{R}$ be a smooth $G$-invariant function. Then it is easy to get the following relation between the $\alpha$-moment map and $f\alpha$-moment map

$$\mu_{f\alpha} = f \mu_\alpha.$$  

(2.10)

In section 4, we will consider a Hamiltonian $G$-manifold $(X, \omega, \mu)$ such that the symplectic form can be expressed as $\omega = \omega_0 + d\alpha$, where $\omega_0$ is a $G$-invariant closed 2-form and $\alpha$ is a $G$-invariant 1-form on $X$. The following lemma is obvious.

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Lemma 2.1 Given a symplectic manifold \((X, \omega)\) with a \(G\)-action. Suppose that the symplectic form \(\omega = \omega_0 + d\alpha\) such that \(\omega_0\) is a \(G\)-invariant closed 2-form and \(\alpha\) is a \(G\)-invariant 1-form. Let \(\mu_{\omega_0}\) and \(\mu_{\alpha}\) be the \(\omega_0\)-moment map and \(\alpha\)-moment map on \(X\). Then the sum
\[
\mu := \mu_\omega = \mu_{\omega_0} + \mu_{\alpha}
\] (2.11)
is the moment map of the \(G\)-action on \(X\). So \((X, \omega, \mu)\) is a Hamiltonian \(G\)-manifold.

Given two manifolds \(M\) and \(N\) with \(G\)-action, we consider a smooth \(G\)-equivariant map \(\varphi : M \to N\). Let \(\omega\) and \(\alpha\) be a \(G\)-invariant closed 2-form and a \(G\)-invariant 1-form on \(N\), respectively. Then \(\varphi^*\omega\) and \(\varphi^*\alpha\) are \(G\)-invariant closed 2-form and \(G\)-invariant 1-form on \(M\). Furthermore, if \(\mu_{\omega}\) and \(\mu_{\alpha}\) are \(\omega\)-moment map and \(\alpha\)-moment map on \(N\), then the \(\varphi^*\omega\)-moment map and \(\varphi^*\alpha\)-moment map on \(M\) are
\[
\mu_{\varphi^*\omega} = \mu_\omega \circ \varphi, \quad \text{and} \quad \mu_{\varphi^*\alpha} = \mu_\alpha \circ \varphi;
\] (2.12)
respectively.

3 Review of symplectic sum construction

In this section, we review the construction of symplectic sum based on \([FZ]\) (see also \([G, IP2]\)). Let \((X^+, \omega^+)\) and \((X^-, \omega^-)\) be two compact symplectic manifolds of dimension \(2n\), \(Z = Z^+ \cong Z^-\) be their common codimension 2 symplectic submanifold. Take an almost complex structure \(J^\pm\) on \(X^\pm\) compatible with \(\omega^\pm\) such that \(J^+|_Z = J^-|_Z\). Since the normal bundle \(N^\pm Z := N_{X^\pm Z}\) inherits a symplectic structure from \(\omega^\pm\) and thus a complex structure up to homotopy, both \(N_+ Z\) and \(N_- Z\) are oriented. We assume that they have opposite Euler classes:
\[
e(N_+ Z) + e(N_- Z) = 0.\] (3.1)

There exists an isomorphism of complex line bundles
\[
\Phi : N_+ Z \otimes_C N_- Z \to Z \times \mathbb{C}.
\] (3.2)
These data determine a family of symplectic sums. The following theorem is taken from \([FZ]\) Proposition 3.1.

Theorem 2 (Symplectic connect sum) Given two compact \(2n\)-dimensional symplectic manifolds \((X^+, \omega^+)\) and \((X^-, \omega^-)\) with a common codimension 2 symplectic submanifold \(Z\) satisfying (3.1). Then for each choice of homotopy class of isomorphisms (3.2), there exist a \((2n + 2)\)-dimensional symplectic manifold \((S, \Omega)\), a smooth map \(\pi : S \to D\) over a sufficiently small neighbourhood \(D\) of the origin in \(\mathbb{C}\), and an \(\Omega\)-compatible almost complex structure \(J_S\) on \(S\) such that

1. \(\pi\) is surjective and is a submersion outside of \(Z \subset S_0\), and \(S_0 = X^+ \cup_Z X^-\);
2. the restriction \(\omega_\lambda\) of \(\Omega\) to \(S_\lambda = \pi^{-1}(\lambda)\) is nondegenerate for \(\forall \lambda \in D^* = D \setminus \{0\}\);
3. on the singular fibre \(S_\lambda\), \(\Omega|_{X^+} = \omega^+\), \(\Omega|_{X^-} = \omega^-\).
(4) $J_S$ preserves $T\mathcal{S}_\lambda$ for every $\lambda \in D^*$;

Then every $(\mathcal{S}_\lambda, \omega_\lambda)$, $\lambda \in D^*$, is a smooth compact symplectic manifold, called symplectic sum of $X^+$ and $X^-$ along $Z$ with gluing parameter $\lambda$, and simply denoted as $\mathcal{S}_\lambda = X^+ \#_{Z,\lambda} X^-$. They are symplectically isotopic to one another and can be regarded as deformations, in the symplectic category, of the singular fiber $\mathcal{S}_0$.

3.1 Normal form for a codimensional two symplectic submanifold

Let $Z$ be a smooth manifold and $\pi_N : (N, i_N) \to Z$ be a complex line bundle (a rank-2 real vector bundle with a complex structure $i_N$ on each fiber). Let $(g_N, \nabla^N)$ be a Hermitian structure on $(N, i_N)$, which consists of a Hermitian metric and a connection on $N$ such that for $\forall v, w \in N_z, z \in Z$ and any sections $\xi, \eta \in \Gamma(N)$, the following hold

\[ g_N(i_N v, w) = ig_N(v, w) = -g_N(v, i_N w), \]  
\[ \nabla^N(i_N \xi) = i_N \nabla^N \xi, \]  
\[ d(g_N(\xi, \eta)) = g_N(\nabla^N \xi, \eta) + g_N(\xi, \nabla^N \eta). \]  

Denote by

\[ \rho_N : N \to \mathbb{R}, \]  
\[ \rho_N(v) = g_N(v, v) = |v|^2 \]  

the square of the norm function. Let

\[ \pi_{SN} : SN \to Z \]  

be the circle bundle of $N$. The connection $\nabla^N$ induces a splitting of the following short exact sequence

\[ 0 \to T^{\text{vert}}(SN) \cong \ker(d\pi_{SN}) \to T(SN) \xrightarrow{d\pi_{SN}} \pi_{SN}^*TZ \to 0 \]  

of vector bundles over $SN$. Denote by $\alpha_{SN}$ the 1-form on $SN$ vanishing on the image of $\pi_{SN}^*TZ$ in $T(SN)$ corresponding to this splitting such that

\[ \alpha_{SN} \left( \frac{d}{d\theta} e^{i\theta} v \bigg|_{\theta=0} \right) = 1, \quad \forall \ v \in SN. \]  

It can be extended to a 1-form $\alpha_N$ on $N - Z$ via the radial retraction

\[ N - Z \to SN, \quad v \to \frac{v}{|v|}. \]  

Then the 1-form $\rho_N \alpha_N$ is well-defined and smooth on the total space of the line bundle $N$.

Assume now that $(Z, \omega_Z)$ is a symplectic manifold. For $\epsilon > 0$, define a 2-form on the total space of a complex line bundle $N$ over $Z$

\[ \omega_{N,Z}^\epsilon := \pi_N^* \omega_Z + \frac{\epsilon^2}{2} d(\rho_N \alpha_N). \]  

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We see that $\omega^\epsilon_{N,Z}$ is closed and the restriction to the zero section of $N$: $\omega^\epsilon_{N,Z}|_Z = \omega_Z$. If $Z$ is compact, since $d\rho_N \wedge \alpha_N$ is nondegenerate on each fiber, there exists $\epsilon_\zeta \in \mathbb{R}^+$ such that the restriction of $\omega^\epsilon_{N,Z}$ to $N(\delta) = \{ v \in N : |v| < \delta \}$ is non-degenerate whenever $\delta, \epsilon \in \mathbb{R}^+$ and $\delta \epsilon < \epsilon_\zeta$. So $(N(\delta), \omega^\epsilon_{N,Z})$ is a symplectic manifold if $\delta < \epsilon$. In particular, if $\delta < \epsilon$, then $(N(\delta), \omega^1_{N,Z})$ is a symplectic manifold.

Now consider a symplectic manifold $(X, \omega_X)$. Let $Z$ be a codimension 2 symplectic submanifold of $X$. Denote the symplectic normal bundle of $Z$ in $X$ by

$$N_X Z := \frac{T X|_Z}{T Z} \cong (T Z)^{\omega_X},$$

where

$$T Z^{\omega_X} = \{ v \in T_x X \mid x \in Z, \omega_X(v, w) = 0, \forall w \in T_x Z \}.$$

So under the identification \textbf{(3.10)} we have a symplectic orthogonal decomposition

$$T X|_Z = T Z \oplus N_X Z, \quad \omega_X = \omega_Z \oplus \omega_N,$$

where $\omega_N$ is a fiberwise symplectic structure on $N_X Z$ such that on each fiber $N_x Z, \omega_N = \omega_X|_N_x$. Let $\omega_X^N$ be the induced symplectic form on the normal bundle $N_X Z$. We can especially consider (fiberwise) $\omega_X^N$-\textbf{compatible} complex structures $i_X$ on $N_X Z$ and $\omega_X^N$-\textbf{compatible} Hermitian structures $(g_X, \nabla^N)$ on $(N_X Z, i_X)$, \textit{i.e.}

$$\omega_X^N(i_X v, i_X w) = \omega_X^N(v, w), \quad g_X(v, w) = \omega_X^N(v, i_X w)$$

for $\forall v, w \in N_X Z|_Z, z \in Z$. The spaces of (fiberwise) $\omega_X^N$-compatible complex structures on $N_X Z$ and of $\omega_X^N$-compatible Hermitian structures on $(N_X Z, i_X)$ are non-empty and contractible.

For example, when $\delta \epsilon < \epsilon_N$ is sufficiently small and as a zero section, $Z$ is a compact codimension 2 symplectic submanifold of $(N(\delta), \omega^\epsilon_{N,Z})$, the symplectic form $\omega_{N,Z}$ in \textbf{(3.9)} is just the form $\omega_X^N|_{N(\delta)}$ up to a symplectomorphism. Then we have

$$T Z^{\omega^\epsilon_{N,Z}} \cong N_{N(\delta)} Z,$$

$$(\omega^\epsilon_{N,Z})_N = \frac{\epsilon^2}{2} d(\rho_N \alpha_N)|_{N(\delta)}.$$

Note that when $\delta < \epsilon$ is sufficiently small, the 2-form $\omega^1_{N,Z}$ restricts to $\omega_X^N$ on $T(N_Z X)|_Z$ under the isomorphism as in \textbf{(3.10)}. Then by the Symplectic Neighborhood Theorem, a neighborhood of $Z$ is completely determined by the restriction of $\omega$ to $Z$ together with the isomorphism class of the symplectic normal bundle $N_X Z$. That is, there exist $\delta_Z > 0$, an smooth injective open map,

$$\Psi_X : (N_X Z(\delta_Z), Z) \longrightarrow (X, Z)$$

\textbf{(3.12)} such that

$$d_x \Psi_X = id, \forall x \in Z, \quad \Psi_X^* \omega_X = \omega^1_{N,Z}|_{N_X Z(\delta_Z)}.$$

Note that here we can choose $\delta_Z < \epsilon_Z$ so that $\omega^1_{N,Z}|_{N_X Z(\delta_Z)}$ is a symplectic form on $N_X Z(\delta_Z)$.
For any \( \epsilon > 0 \), define
\[
\Psi_{X;\epsilon} : (N_XZ(\epsilon^{-1}\delta_Z), Z) \rightarrow (X, Z)
\]  
(3.13)

\[
\Psi_{X;\epsilon}(p = (z, v)) = \Psi_X(z, \epsilon v), \quad z \in Z, \ v \in N_XZ(\epsilon^{-1}\delta_Z)|_z,
\]
then \( \Psi_{X;\epsilon} \) is a smooth injective open map, satisfies
\[
\Psi^*_{X;\epsilon}\omega_X = \omega_{N;Z}|_{N_XZ(\epsilon^{-1}\delta_Z)}
\]
and restricts to the identity on \( Z \).

3.2 Symplectic sum

Let \((X^+, \omega^+)(X^-, \omega^-)\) be two \(2n\)-dimensional compact symplectic manifolds and \(Z \subset X^+, X^-\) be a common symplectic codimension 2 submanifold satisfying \(\omega_Z = \omega^+|Z = \omega^-|Z\) so that (3.1) holds. Fix (fiberwise) complex structures \(i_+\) and \(i_-\) on the normal bundles \(\pi_{+,Z} : N_+Z \rightarrow Z\) and \(\pi_{-,Z} : N_-Z \rightarrow Z\) that are compatible with \(\omega^+_\pm\) and \(\omega^-\), respectively. Here \(\omega^+_\pm = \omega^\pm_X\) as in the last subsection. Fix an isomorphism \(\Phi\) of trivial complex line bundles (3.2), let
\[
\Phi_2 : N_+Z \otimes \mathbb{C} \cong Z \times \mathbb{C} \rightarrow \mathbb{C}
\]
be the composition of \(\Phi\) with the projection \(Z \times \mathbb{C} \rightarrow \mathbb{C}\).

**Definition 3.1** We say that an isomorphism \(\Phi\) is \((\omega^+_\pm, \omega^-)\)-compatible if
\[
|\Phi_2((z, v) \otimes (z, w))|^2 = |\Phi_2(v \otimes w)|^2 = \omega^+_\pm(v, i_+v) \cdot \omega^-_N(w, i_-w),
\]  
(3.14)

for \(\forall v \in N_+Z|_z, w \in N_-Z|_z, z \in Z\).

We choose an isomorphism \(\Phi\) which is \((\omega^+_\pm, \omega^-)\)-compatible. In fact, (3.14) can be achieved by scaling any given isomorphism \(\Phi\) in (3.2) and this does not change the homotopy class of \(\Phi\).

Choose Hermitian structures \((g_+, \nabla_+)\) on \((N_+Z, i_+)\) and \((g_-, \nabla_-)\) on \((N_-Z, i_-)\) that are compatible with \(\omega^+_\pm\) and \(\omega^-\), in the sense of (3.11), and compatible with \(\Phi_2\), in the following sense
\[
d(\Phi_2(\xi \otimes \eta)) = \Phi_2((\nabla_+\xi) \otimes \eta) + \Phi_2(\xi \otimes (\nabla_-\eta))
\]  
(3.16)

for \(\forall v \in N_+Z|_z, w \in N_-Z|_z, z \in Z\) and \(\xi \in \Gamma(Z; N_+Z), \eta \in \Gamma(Z; N_-Z)\). Here \(\rho_\pm\) is the fiberwise normal square functions on \(N_\pm Z\), and the equation (3.16) is an identity of differential 1-forms on \(Z\).

Denote by \(\alpha_+\) and \(\alpha_-\) the connection 1-forms on \(N_+Z - Z\) and \(N_-Z - Z\) corresponding to \((g_+, \nabla_+)\) and \((g_-, \nabla_-)\), respectively. For a sufficiently small \(\epsilon > 0\), as in (3.3), define
\[
\omega^+_{\epsilon, Z} := \pi^*_{+,Z}\omega_Z + \frac{\epsilon^2}{2}d(\rho_+\alpha_+), \quad \omega^-_{\epsilon, Z} := \pi^*_\omega_Z + \frac{\epsilon^2}{2}d(\rho_-\alpha_-).
\]  
(3.17)
As in (3.12), there exist $\delta_Z > 0$ and the smooth injective open maps
\[ \Psi_{\pm} : (N_{\pm}Z(\delta_Z), Z) \longrightarrow (X^{\pm}, Z) \]
such that
\[ d_x \Psi_{\pm} = id, \quad \forall \ x \in Z, \quad \Psi_{\pm}^* \omega_{X^{\pm}} = \omega_{\pm}^1|N_{\pm}Z(\delta_Z) . \]
Following (3.13), there exist smooth injective open maps
\[ \Psi_{+\epsilon} : (N_{+}Z(\epsilon^{-1}\delta_Z), Z) \longrightarrow (X^+, Z), \quad \Psi_{-\epsilon} : (N_{-}Z(\epsilon^{-1}\delta_Z), Z) \longrightarrow (X^-, Z). \quad (3.18) \]
satisfying
\[ \Psi_{+\epsilon}^* \omega_{+} = \omega_{+}^\epsilon|N_{+}Z(\epsilon^{-1}\delta_Z), \quad \Psi_{-\epsilon}^* \omega_{-} = \omega_{-}^\epsilon|N_{-}Z(\epsilon^{-1}\delta_Z) \quad (3.19) \]
and restricting to the identity on $Z$. Note that $\epsilon$ and $\delta$ are chosen, $\epsilon^{-1}\delta_Z > 2$, so that $N_{\pm}Z(2)$ is contained in the domain of $\Psi_{\pm\epsilon}$.

Consider the projections
\[ \Pi_{\pm} : N_{\pm}Z \oplus N_{-}Z \longrightarrow Z, \]
\[ \Pi_+ : N_{+}Z \oplus N_{-}Z \longrightarrow N_{+}Z, \]
\[ \Pi_- : N_{+}Z \oplus N_{-}Z \longrightarrow N_{-}Z, \]
and the natural product map
\[ \mathcal{P} : N_{+}Z \oplus N_{-}Z \longrightarrow \mathbb{C}, \quad (z, v, w) \mapsto \Phi_2(v \otimes w). \quad (3.20) \]
Then the following identity holds
\[ \mathcal{P}^* \omega_{\mathbb{C}} = \frac{1}{2} d(\rho_+ \rho_- \mathcal{P}^* d\theta) = \frac{1}{2} d(\rho_+ \rho_- (\Pi_+^* \alpha_+ + \Pi_-^* \alpha_-)), \quad (3.21) \]
where $\omega_{\mathbb{C}} = rdr \wedge d\theta = \frac{1}{2}(r^2d\theta)$ is the standard symplectic form on $\mathbb{C}$, $\rho_+$ and $\rho_-$ also denote the extensions on $N_{X^+}Z \oplus N_{X^-}Z$ defined by $\rho_+(v, w) = |v|^2$, $\rho_-(v, w) = |w|^2$.

The following pieces are the basic building blocks in the symplectic sum construction
\begin{align*}
\mathcal{E}_+ & := (X^+ - \Psi_{+\epsilon}(N_{+}Z(1))) \times \mathbb{D}_\delta, \quad (3.22) \\
\mathcal{E}_- & := (X^- - \Psi_{-\epsilon}(N_{-}Z(1))) \times \mathbb{D}_\delta, \quad (3.23) \\
\mathcal{S}_Z & := \{(z, v, w) \in N_{+}Z \oplus N_{-}Z \mid |v|, |w| < 2, \epsilon|\mathcal{P}(v, w)| < \delta\}, \quad (3.24) \\
\mathcal{S}_{Z,+} & := \{(z, v, w) \in \mathcal{S}_Z : |v| > 1\}, \quad \mathcal{S}_{Z,-} := \{(z, v, w) \in \mathcal{S}_Z : |w| > 1\}. \quad (3.25) 
\end{align*}
where $\mathbb{D}_\delta = \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$, and $N_{\pm}Z(1)$ denote the closed unit disc bundles of $N_{\pm}Z$.

We first choose a sufficiently small $\epsilon > 0$, then assume that
\[ 2\epsilon < \delta_Z, \quad 2\delta < \epsilon. \quad (3.26) \]

Define the gluing maps to be the open maps
\[ gl_+ : \mathcal{S}_{Z,+} \longrightarrow \mathcal{E}_+, \quad (z, v, w) \mapsto (\Psi_{+\epsilon}(z, v), \epsilon \mathcal{P}(v, w)), \quad (3.27) \]

\footnote{This identity is just the formula (3.8) of [PZ].}
In particular, $\omega$ restriction of this closed 2-form
is the natural

\begin{align}
gl_- : S_{Z,-} \longrightarrow \mathcal{E}_-, \quad (z,v,w) \mapsto \left(\Psi_{-\epsilon}(z,w), \epsilon\mathcal{P}(v,w)\right). \tag{3.28}
\end{align}

From the assumptions in \[\text{(3.20)}\], $gl_+$ and $gl_-$ are well-defined diffeomorphisms between open subsets of their domains and targets. Let $S$ be the resulting smooth manifold from gluing $\mathcal{E}^+, \mathcal{E}^-$ and $S_Z$ by the maps $gl_+$

\begin{align}
S := \mathcal{E}_- \bigcup_{gl_-} S_Z \bigcup_{gl_+} \mathcal{E}_+. \tag{3.29}
\end{align}

The maps

\begin{align}
\pi_{\pm,C} : \mathcal{E}_\pm \longrightarrow \mathbb{D}_\delta, \quad : (x^\pm, \lambda) \mapsto \lambda,
\pi_{Z,C} : S_Z \longrightarrow \mathbb{D}_\delta, \quad (z, v, w) \mapsto \epsilon\mathcal{P}(v, w)
\end{align}

are intertwined by $gl_+$ and $gl_-$, so they induce a smooth map

\begin{align}
\pi_\epsilon : S \longrightarrow \mathbb{D}_\delta. \tag{3.30}
\end{align}

By the second assumption in \[\text{(3.26)}\], every fiber $S_\lambda = \pi_\epsilon^{-1}(\lambda)$ of $\pi_\epsilon$ is compact.

We now follow [FZ] to construct a symplectic form $\Omega_\epsilon$ on $S$. First, note that $\mathcal{E}_\pm$ are symplectic with the symplectic form given by $p_+^* \omega_+ + \pi_{+C}^* \omega_C$ where $p_\pm$ is the natural projection

\begin{align}
p_\pm : \mathcal{E}_\pm \longrightarrow X^\pm - \Psi_{\pm\epsilon}(N_{X_{\pm}Z}(1)), \quad : (x^\pm, \lambda) \mapsto x^\pm. \tag{3.31}
\end{align}

Take a cut-off function $\beta : \mathbb{R} \rightarrow [0, 1]$, which is a smooth function such that

\begin{align}
\beta(t) = \begin{cases} 0, & \text{if } t \leq \frac{1}{2}, \\ 1, & \text{if } t \geq 1. \end{cases} \tag{3.32}
\end{align}

Define a closed 2-form on $S_Z$

\begin{align}
\omega_{\mathcal{E}_Z}^\epsilon := \Pi_{-}^* \omega_Z + \frac{\epsilon^2}{2} d(\alpha_#), \tag{3.33}
\end{align}

where the 1-form

\begin{align}
\alpha_# := (1 - \beta \circ \rho_-) \Pi_+^* (\rho_+ \alpha_+) + (1 - \beta \circ \rho_+) \Pi_-^* (\rho_- \alpha_-) + (\beta \circ \rho_+ + \beta \circ \rho_-) \rho_+ - \rho_- (\Pi_+^* \alpha_+ + \Pi_-^* \alpha_-). \tag{3.34}
\end{align}

Since on $S_Z$, $|\mathcal{P}(v,w)| < \epsilon^{-1}\delta < \frac{1}{2}$, we have $\rho_+^\frac{1}{2} = |v| > 1$, $\rho_-^\frac{1}{2} = |w| <\frac{1}{2}$ on $S_{Z,+}$. In particular, $\beta \circ \rho_+ = 1$ and $\beta \circ \rho_- = 0$ on $S_{Z,+}$. Using \[\text{(3.19)}, \text{(3.21) and (3.27)}\], the restriction of this closed 2-form $\omega_{\mathcal{E}_Z}^\epsilon$ to $S_{Z,+}$ is

\begin{align}
\Pi_+^* \omega_Z + \frac{\epsilon^2}{2} d\left(\Pi_+^* (\rho_+ \alpha_+) + \rho_+ \rho_- (\Pi_+^* \alpha_+ + \Pi_-^* \alpha_-)\right) = \left(\Pi_+^* \omega_Z + \frac{\epsilon^2}{2} d\Pi_+^* (\rho_+ \alpha_+)\right) + \epsilon^2 \frac{1}{2} d(\rho_+ \rho_- (\Pi_+^* \alpha_+ + \Pi_-^* \alpha_-))
= gl_+^*(p_+^* \omega_+) + \epsilon^2 \pi_{+C}^* \omega_C = gl_+^*(p_+^* \omega_+ + \pi_{+C}^* \omega_C). \tag{3.35}
\end{align}

Similarly, the restriction of $\omega_{\mathcal{E}_Z}^\epsilon$ to $S_{Z,-}$ is $gl_-^*(p_-^* \omega_- + \pi_{-C}^* \omega_C)$. So along with the 2-forms

\begin{align*}
p_+^* \omega_+ + \pi_{+C}^* \omega_C \text{ on } \mathcal{E}_+ \quad \text{and} \quad p_-^* \omega_- + \pi_{-C}^* \omega_C \text{ on } \mathcal{E}_-,
\end{align*}

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we get a closed 2-form $\Omega_S^\epsilon$ on $S$ via the gluing construction (3.29). From the same calculation as the one of [FZ, Page 23], one can verify that the closed 2-form (3.33) is also nondegenerate if $\epsilon$ and $\delta$ is chosen to be small enough as in (3.26). Thus, we obtained a symplectic manifold $(S, \Omega = \Omega_S^\epsilon)$ of dimension $2n + 2$ such that

$$
\Omega_S = \begin{cases} 
p_{-}^* \omega_\cdot + \pi_{-,\cdot}^* \omega_{\cdot, \cdot}, & \text{on } E_{-}, \\
\omega_{S_Z}, & \text{on } S_Z, \\
p_{+}^* \omega_\cdot + \pi_{+,\cdot}^* \omega_{\cdot, \cdot}, & \text{on } E_{+},
\end{cases}
$$

and a fibration (see (3.30))

$$
\pi := \pi_\epsilon: S \longrightarrow \mathbb{D}_\delta.
$$

Note that

$$
gl_{+}^{-1} \left( \left( \Psi_{+,\epsilon}(N_+(2) - N_+(1)) \right) \times \{0\} \right) = \{(v,0) \in \mathbb{N}_+ \mathbb{Z} \oplus \{0\} \mid 1 < |v| < 2\},$$

$$
gl_{-}^{-1} \left( \left( \Psi_{-,\epsilon}(N_- (2) - N_- (1)) \right) \times \{0\} \right) = \{(0, w) \in \{0\} \oplus \mathbb{N}_- \mathbb{Z} \mid 1 < |w| < 2\},$$

and

$$
\pi_{Z,C}^{-1}(0) = \{(v,0) \in \mathbb{N}_+ \mathbb{Z} \oplus \{0\} \mid |v| < 2\} \bigcup \{(0,w) \in \{0\} \oplus \mathbb{N}_- \mathbb{Z} \mid |w| < 2\}.
$$

So

$$
S_0 = \pi^{-1}(0) := (X^+ - \Psi_{+,\epsilon}(N_+(1))) \times \{0\} \bigcup_{gl_+} \pi_{Z,C}^{-1}(0)
$$

$$
\bigcup_{gl_-} (X^- - \Psi_{-,\epsilon}(N_- (1))) \times \{0\}
$$

$$
\cong X^+ \bigcup_Z X^- \quad (3.38)
$$

is a singular fiber of (3.37). For $\lambda \in \mathbb{D}_\delta^* = \mathbb{D}_\delta \setminus \{0\}$, denote the fibre by $S_\lambda = \pi^{-1}(\lambda)$. To study the nondegeneracy of the restriction of $\Omega_S^\epsilon$ to the fibre $S_\lambda$ (the restriction of $\omega_{S_Z}^\epsilon$ to the fibre $S_\lambda \cap S_Z$), one needs to construct $\Omega_S^\epsilon$-tame and compatible almost complex structure $J_S$ on $S$ which preserves the tangent spaces to the fibers of the fibration (3.37). We refer the reader to [FZ] for the detailed construction and the proof the following proposition.

**Proposition 3.1** For every $\lambda \in \mathbb{D}_\delta^*$ with sufficiently small $\delta$, the fibre $S_\lambda$ is a compact symplectic submanifold of $(S, \Omega_S^\epsilon)$ with restricted symplectic form $\omega_\lambda := \Omega_S^\epsilon|_{S_\lambda}$, called the symplectic sum of $X^+$ and $X^-$ along $Z$, and symplectically isotopic to one another. Moreover, there exists a compatible almost complex structure $J_S$ on $S$, such that $S_\lambda$ is $J_S$-invariant.
4 Sum of Hamiltonian manifolds

In this section, we establish the main result of the paper on the operation of sum of two Hamiltonian manifolds along a common codimension 2 Hamiltonian submanifold. Fix a compact and connected Lie group $G$. Consider a Hamiltonian $G$-manifold $(X, \omega_X, \mu_X)$. Let $Z$ be a codimension 2 compact Hamiltonian submanifold of $X$. Under the identification \((3.10)\) we have a $G$-equivariant symplectic orthogonal decomposition
\[
TX|_Z = TZ \oplus N_XZ, \quad \omega_X = \omega_Z \oplus \omega_N,
\]
where $N_XZ = (TZ)^{\omega_X}$ is the $G$-equivariant symplectic normal bundle of $Z$ in $X$.

The induced symplectic form $\omega^N_X$ on the total space of the normal bundle $\pi: N_XZ \to Z$ is $G$-invariant. The spaces of (fiberwise) $\omega^N_X$-compatible complex structures on $N_XZ$ and of $\omega^N_X$-compatible Hermitian metrics on $(N_XZ, i_X)$ are non-empty and contractible. By averaging over $G$, we can choose a $G$-invariant $\omega^N_X$-compatible complex structure $i_X$ on $N_XZ$, a $G$-invariant $\omega^N_X$-compatible Hermitian metric $g_N$ and $G$-invariant Hermitian connection $\nabla^N$ on $(N_XZ, i_X)$. In particular, the square of the norm function $\rho_N: N_XZ \to \mathbb{R}$, $\rho_N(x, v) = g_N(v, v)$ $\forall v \in N_XZ|_x, x \in Z$ is $G$-invariant. The $G$-invariant connection $\nabla^N$ on $N_XZ$ induces a $G$-invariant connection 1-form on the unit circle bundle $SN_X$ of $N_XZ$, and can be extended to a $G$-invariant 1-form $\alpha$ on $N_XZ - Z$ via the radial retraction $N_XZ - Z \to SN_X, v \mapsto \frac{v}{|v|}$.

Therefore $\rho_N \alpha_N$ is a $G$-invariant 1-form on the total space of $N_XZ$. If $\epsilon \delta$ is small enough, then the restriction of $\omega^N_{X,Z} := \pi^*_N \omega_Z + \frac{\epsilon^2}{2} d(\rho_N \alpha_N)$ to $N_XZ(\delta) = \{v \in N : |v| < \delta\}$ is a $G$-invariant symplectic form. From the discussion in subsection 3.2 we know that the $G$-action on $N_XZ(\delta)$ is Hamiltonian with the moment map $\mu^\epsilon_{N,Z}: N_XZ(\delta) \to g^*$ given by
\[
\mu^\epsilon_{N,Z} = \mu_Z \circ \pi_N + \frac{\epsilon^2}{2} \mu_{\rho_N \alpha_N},
\]
where $\mu_{\rho_N \alpha_N}$ is the $\rho_N \alpha_N$-moment map defined in subsection 2.2.

Since $Z$ is $G$-invariant in both $N_XZ(\delta)$ and $X$, and $\omega^N_{X,Z}|_Z = \omega_X|_Z$, by an equivariant version of the Symplectic Neighborhood Theorem (a direct result from the Darboux-Weinstein Theorem in [GS]), there exist $\delta_Z > 0$, a tubular neighborhood $O_X(Z)$ of $Z$ in $X$ and a $G$-equivariant diffeomorphism $\Psi_X: (N_XZ(\delta_Z), Z) \to (O_X(Z), Z)$ such that $\Psi_X \omega_X = \omega^N_{X,Z}|_{N_XZ(\delta_Z)}, \quad \Psi_X|_Z = Id_Z$. 

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Moreover, we have
\[ \mu_X \circ \Psi_X = \mu_Z \circ \pi_N + \frac{1}{2} \rho_N \alpha_N. \]

For any \( \epsilon > 0 \), the smooth \( G \)-equivariant injective open map
\[ \Psi_{X,\epsilon} : (N_X Z(\epsilon^{-1} \delta_Z), Z) \longrightarrow (X, Z) \] (4.1)
satisfies
\[ \Psi_{X,\epsilon}^* \omega_X = \omega_{N,Z} |_{N_X Z(\epsilon^{-1} \delta_Z)} \]
and restricts to the identity on \( Z \). The corresponding moment maps satisfy
\[ \mu_X \circ \Psi_{X,\epsilon} = \mu_Z \circ \pi_N + \frac{\epsilon^2}{2} \rho_N \alpha_N. \] (4.2)

Let \((X^+, \omega^+, \mu^+)\) and \((X^-, \omega^-, \mu^-)\) be two Hamiltonian \( G \)-manifolds of the dimension \( 2n, Z = Z^+ \cong Z^- \) be their common codimension 2 Hamiltonian \( G \)-submanifold satisfying
\[ \omega_Z = \omega^+ |_Z = \omega^- |_Z, \quad \mu_Z = \mu^+ |_Z = \mu^- |_Z. \]

Take \( G \)-invariant almost complex structures \( J^\pm \) on \( X^\pm \) compatible with \( \omega^\pm \) such that \( J^+ |_{T Z} = J^- |_{T Z} \). The symplectic normal bundles
\[ \pi_{\pm,Z} : N_{\pm,Z} = (TZ)^{\omega^\pm} \longrightarrow Z \]
are \( G \)-equivariant and can be endowed with \( G \)-invariant fiberwise complex structures \( i_{\pm} \), compatible with the induced \( G \)-invariant symplectic form \( \omega_{\pm} = \omega^\pm |_{N_{\pm,Z}} \) as above. In particular, these determine \( G \)-invariant Hermitian metrics \( g_{\pm} \) on \((N_{\pm,Z}, i_{\pm})\) in the sense of \([8,11]\). We assume that \( N_+ Z \) and \( N_- Z \) have opposite \( G \)-equivariant Euler classes:
\[ e_G(N_+ Z) + e_G(N_- Z) = 0. \] (4.3)

By the classification theorem of equivariant complex line bundles (Theorem C.47 [GGK]), there is a \( G \)-equivariant isomorphism of complex line bundles
\[ \Phi : N_+ Z \otimes \mathbb{C} N_- Z \longrightarrow Z \times \mathbb{C}, \] (4.4)
where \( Z \times \mathbb{C} \) is a trivial \( G \)-equivariant line bundle in the sense that \( G \)-action on \( \mathbb{C} \) is trivial.

Fix a \( G \)-equivariant isomorphism \( \Phi \) which is compatible with \((\omega_+^N, \omega_-^N)\) (see \([8,14]\)). Choose \( G \)-invariant \( \omega_+^N \)-compatible connections \( \nabla_{\pm} \) on \( N_{\pm,Z} \) in the sense of \([3,13]\) and \([3,10]\). Then we have \( G \)-invariant norm square functions
\[ \rho_{\pm} : N_{\pm,Z} \longrightarrow \mathbb{R}, \quad \rho_{\pm}(v) = g_{\pm}(v, v). \]
Choose \( G \)-invariant connections \( \nabla_{\pm} \) on \( N_{\pm,Z} \) which in turn define \( G \)-invariant connection 1-forms \( \alpha_{\pm} \) on the unit circle \( SN_{\pm,Z} \) of \( N_{\pm,Z} \). Note that \( \alpha_{\pm} \) can be extended to \( G \)-invariant 1-forms on \( N_{\pm,Z} - Z \) via the radial retraction. Therefore we have \( G \)-invariant 1-forms \( \rho_{\pm} \alpha_{\pm} \) on the total spaces of the symplectic normal bundles \( N_{\pm,Z} \rightarrow Z \).
When \( \epsilon \delta \) is small enough,

\[
(N_{\pm Z}(\delta), \omega_{\pm Z}^\epsilon = \pi_{\pm Z}^* \omega_Z + \frac{\epsilon^2}{2} d(\rho_{\pm} \alpha_{\pm}))
\]

are Hamiltonian \( G \)-manifolds with the moment map \( \mu_{\pm Z}^\epsilon : N_{\pm Z}(\delta) \to g^* \) given by

\[
\mu_{\pm Z}^\epsilon = \mu_Z \circ \pi_{\pm Z} + \frac{\epsilon^2}{2} \mu_{\rho_{\pm} \alpha_{\pm}},
\]

where \( \mu_{\rho_{\pm} \alpha_{\pm}} \) are the \( \rho_{\pm} \alpha_{\pm} \)-moment maps defined in subsection 2.2.

There exist \( \delta_Z > 0 \), tubular neighborhoods \( O_X^{\pm}(Z) \) of \( Z \) in \( X^{\pm} \) and \( G \)-equivariant diffeomorphisms

\[
\Psi_{\pm} : (N_{\pm Z}(\delta_Z), Z) \rightarrow (O_X^{\pm}(Z), Z)
\]

such that

\[
\Psi_{\pm}^* \omega^{\pm} = \omega^1_{\pm Z}|_{N_{\pm Z}(\delta_Z)}, \quad \Psi_{\pm}|Z = Id_Z.
\]

Moreover, we have

\[
\mu_X \circ \Psi_X = \mu_Z \circ \pi_N + \frac{1}{2} \mu_{\rho_N \alpha_N}.
\]

and for any \( \epsilon > 0 \) the smooth \( G \)-equivariant injective open maps of Hamiltonian \( G \)-manifolds

\[
\Psi_{\pm,:} : N_{\pm Z}(\epsilon^{-1} \delta_Z) \rightarrow X^{\pm},
\]

\[
\Psi_{\pm,:}(z, v) = \Psi_{\pm}(z, \epsilon v), \quad z \in Z, \; v \in N_{\pm Z}(\epsilon^{-1} \delta_Z)|_z,
\]

satisfying

\[
\Psi_{\pm,:}^* \omega^{\pm} = \omega_{\pm Z}^\epsilon|_{N_{\pm Z}(\epsilon^{-1} \delta_Z)}
\]

and restricts to the identity on \( Z \). The corresponding moment maps satisfy

\[
\mu_X \circ \Psi_{X,:} = \mu_Z \circ \pi_N + \frac{\epsilon^2}{2} \mu_{\rho_N \alpha_N}.
\]

Recall the symplectic sum construction (3.29)

\[
S = \mathcal{E}_{-} \bigcup_{gl_-} S_Z \bigcup_{gl_+} \mathcal{E}_{+}.
\]

Here \( \mathcal{E}_{\pm} = (X^{\pm} - \Psi_{\pm,:}(N_{\pm Z}(1))) \times \mathbb{D}_{\delta} \) with symplectic form \( p_\pm^* \omega^\pm + \pi_{\pm,c}^* \omega_C \). Note that \( G \)-action on \( \mathbb{D}_{\delta} \) is trivial, the symplectic form \( p_\pm^* \omega^\pm + \pi_{\pm,c}^* \omega_C \) is \( G \)-invariant. Therefore, \( (\mathcal{E}_{\pm}, p_\pm^* \omega^\pm + \pi_{\pm,c}^* \omega_C) \) are Hamiltonian \( G \)-manifolds with the moment maps \( \mu_{\epsilon_{\pm}} : \mathcal{E}_{\pm} \rightarrow g^* \) given by

\[
\mu_{\epsilon_{\pm}} = \mu^\pm \circ p_\pm.
\]

The symplectic form \( \omega_{S_Z}^\epsilon = \Pi_Z^\epsilon \omega_Z + \frac{\epsilon^2}{2} d(\alpha_{\#}) \) on the middle part \( S_Z = \{(z, v, w) \in N_+ Z \oplus N_- Z \mid |v|, |w| < 2, \epsilon|P(v, w)| < \delta \} \) is \( G \)-invariant as \( \alpha_{\#} \), defined in (3.34). This implies that

\[
(S_Z, \omega_{S_Z}^\epsilon)
\]
is a Hamiltonian $G$-manifold with the associated moment map $\mu_{S_Z} : S_Z \to g^*$ given by

$$\mu_{S_Z} = \mu_Z \circ \Pi_Z + \frac{e^2}{2} \mu_{\alpha^\#}.$$

Under the gluing maps $gl_{\pm} : S_{Z,\pm} \to E_{\pm}$ defined in (3.27) and (3.28), we have

$$gl_{\pm}(p^*_{\pm}\omega^\pm + \pi^*_{\pm,\mathbb{C}}(\omega_C)) = \omega_{S_Z}|_{S_{Z,\pm}},$$

which implies

$$\mu_{S_Z}|_{S_{Z,\pm}} = \mu_{E_{\pm}} \circ gl_{\pm} = \mu_{\pm} \circ p_{\pm} \circ gl_{\pm}.$$

Therefore, the symplectic form $\Omega = \Omega^0_S$ on $S$

$$\Omega = \begin{cases} 
 p_{\pm}^*\omega^- + \pi_{\pm,\mathbb{C}}^*\omega^C, & \text{on } E_-, \\
 \omega_{S_Z}^s = \Pi_Z^*\omega_Z + \frac{e^2}{2}d(\alpha^\#), & \text{on } S_Z, \\
 p_{\pm}^*\omega^+ + \pi_{\pm,\mathbb{C}}^*\omega^C, & \text{on } E_+, 
\end{cases}$$

is $G$-invariant. That is, $(S, \Omega)$ is a Hamiltonian $G$-manifold with the associated moment map $\mu : S \to g^*$ given by

$$\mu = \begin{cases} 
 \mu^+ \circ p_+, & \text{on } E_-, \\
 \mu_Z \circ \Pi_Z + \frac{e^2}{2} \mu_{\alpha^\#}, & \text{on } S_Z, \\
 \mu^- \circ p_-, & \text{on } E_.
\end{cases} \quad (4.6)$$

Assume that 0 is a regular value of $\mu^\pm : X^\pm \to g^*$ and $G$ acts freely on $(\mu^\pm)^{-1}(0)$. Then the quotient manifold, the symplectic reductions of $(X^\pm, \omega^\pm, \mu^\pm)$, respectively,

$$X_0^\pm = (\mu^\pm)^{-1}(0)/G$$

has a unique symplectic structure $\omega_0^\pm$ such that

$$\pi_{\pm}^*(\omega_0^\pm) = \omega^\pm|_{(\mu^\pm)^{-1}(0)},$$

where $\pi_{\pm} : (\mu^\pm)^{-1}(0) \to X_0^\pm$ is the quotient map. These symplectic manifolds $(X_0^+, \omega_0^+)$ have a common symplectic submanifold

$$(Z_0 = \mu_Z^{-1}(0)/G, \omega_{Z_0}^Z)$$

with the opposite symplectic normal bundles. Here $\omega_{Z_0}^Z$ is the unique symplectic structure on $Z_0$ satisfying

$$(\pi_{Z_0})^*(\omega_{Z_0}^Z) = \omega_Z|_{\mu_Z^{-1}(0)},$$

for the quotient map $\pi_{Z_0} : \mu_Z^{-1}(0) \to \mu_Z^{-1}(0)/G$. The $G$-equivariant isomorphism (1.1) also descends to an isomorphism of complex line bundles

$$\Phi_0 : N_{X_0^+}Z_0 \otimes_{\mathbb{C}} N_{X_0^-}Z_0 \to Z_0 \times \mathbb{C}.$$
So we can apply the symplectic sum operation to \((X_0^+, \omega_0^+)\) and \((X_0^-, \omega_0^-)\) along \((Z_0, \omega_0^Z)\). Note that the symplectic reduction of \((S_Z, \omega^Z_{S_Z}, \mu_{S_Z})\) is
\[
( (\mu_{S_Z})^{-1}(0)/G, (\omega^Z_{S_Z})_0 )
\]
with the unique symplectic form \((\omega^Z_{S_Z})_0\) specified by
\[
(\pi_{S_Z,0})^*(\omega^Z_{S_Z})_0 = \omega^Z_{S_Z}(\mu_{S_Z})^{-1}(0)
\]
where \(\pi_{S_Z,0}\) is the quotient map \((\mu_{S_Z})^{-1}(0) \to (\mu_{S_Z})^{-1}(0)/G\).

From the symplectic sum construction of \((X^+, \omega^+)\) and \((X^-, \omega^-)\), we have
\[
\mu^{-1}(0) = (\mu_{S^-})^{-1}(0) \bigcup_{gl^-} \mu_{S^+}^{-1}(0) \bigcup_{gl^+} (\mu_{S^+})^{-1}(0),
\]
where the gluing maps \(gl^\pm : (\mu_{S^\pm})^{-1}(0) \cap S_{S^\pm} \to (\mu_{S^\pm})^{-1}(0)\) are well-defined open maps due to
\[
(\mu_{S^\pm})^{-1}(0) \cap S_{S^\pm} = (\mu_{S^\pm}|S_{S^\pm})^{-1}(0)
\]
and \(\mu_{S^\pm}|S_{S^\pm} = \mu_{S^\pm} \circ gl^\pm\). These gluing map descend to open maps
\[
\overline{gl}^\pm : (\mu_{S^\pm}|S_{S^\pm})^{-1}(0)/G \to (\mu_{S^\pm})^{-1}(0)/G
\]
of symplectic manifolds. Therefore,
\[
S \sslash G = \mu^{-1}(0)/G = (\mu_{S^-})^{-1}(0)/G \bigcup_{gl^-} (\mu_{S^+})^{-1}(0)/G \bigcup_{gl^+} (\mu_{S^+})^{-1}(0)/G,
\]
is a symplectic manifold with a unique symplectic structure \(\Omega_0\) given by
\[
\Omega_0 = \begin{cases} 
  p^+_\omega_0^- + \pi^*_{S^-|C} \omega_C, & \text{on } (\mu_{S^-})^{-1}(0)/G, \\
  (\omega^Z_{S_Z})_0, & \text{on } (\mu_{S_Z})^{-1}(0)/G, \\
  p^+_\omega_0^- + \pi^*_{S^+|C} \omega_C, & \text{on } (\mu_{S^+})^{-1}(0)/G,
\end{cases}
\]
Moreover, there is a smooth map \(\pi_0 : S \sslash G \to D\) where \(D\) is a sufficiently small neighbourhood of the origin in \(\mathbb{C}\) such that

- \(\pi_0\) is surjective and \(\pi_0^{-1}(0) = X_0^+ \cup Z_0 X_0^-\),
- \(\pi_0\) is submersion away from \(Z_0 \subset S \sslash G\),
- the restriction of \(\Omega_0\) to \(\pi_0^{-1}(\lambda)\) is nondegenerate for every \(\lambda \in D \setminus \{0\}\),
- \(\Omega_0|_{X_0^+} = \omega_0^+\) for \(X_0^+ \subset \pi_0^{-1}(0)\).

By the uniqueness of symplectic form for the symplectic reduction, for any \(\lambda \in D \setminus \{0\}\), the symplectic manifold
\[
(\pi_0^{-1}(\lambda), \Omega_0|_{\pi_0^{-1}(\lambda)})
\]
is the symplectic sum of \((X_0^+, \omega_0^+)\) and \((X_0^-, \omega_0^-)\) along \((Z_0, \omega_0^Z)\). This completes the proof of Theorem [4].
Remark 1 If 0 is a regular value of \( \mu^\pm : X^\pm \to g^* \) and \( \mu_Z : Z \to g^* \) but \( G \)-actions is not free, then \((X_0^+, \omega_0^+)\) are symplectic orbifolds with a common symplectic suborbifold \((Z_0, \omega_0^Z)\) which is normalizable in the sense of [Mu]. In this case, \((S // G, \Omega_0)\) is a symplectic orbifold with a smooth map \( \pi_0 : S // G \to D \) such that for any \( \lambda \in D - \{0\} \), the symplectic orbifold

\[
(\pi_0^{-1}(\lambda), \Omega_0|_{\pi_0^{-1}(\lambda)})
\]

is a symplectic orbifold sum of \((X_0^+, \omega_0^+)\) and \((X_0^-, \omega_0^-)\) along \((Z_0, \omega_0^Z)\) as defined in [Mu]. We leave the details of this construction to interested readers.

Remark 2 For a Hamiltonian \( G \)-manifold with a local Hamiltonian \( S^1 \)-action which commutes with \( G \)-action, the symplectic cut produces two Hamiltonian \( G \)-manifolds (Remark 1.2 in [LT]). For completeness, we give an explicit description of this construction. Given a Hamiltonian \( G \)-manifold \((X, \omega, \mu)\) with a local Hamiltonian \( S^1 \)-action which commutes with \( G \)-action. Let \( V \) be a \( G \)-invariant open subset of \( X \) with a \( G \)-commuting Hamiltonian \( S^1 \)-action and a \( G \)-invariant moment map

\[
H : V \to \mathbb{R}.
\]

Assume that there is a small interval \( I = (\delta, \delta) \) of regular values and \( Y = H^{-1}(0) \) is a separating compact hypersurface with a free \( S^1 \)-action. By the symplectic \( S^1 \)-reduction, there is a circle bundle \( \pi : Y \to Z = Y / S^1 \) and a symplectic structure \( \omega_Z \) on \( Z \) uniquely defined by

\[
\pi^* \omega_Z = \omega|_Y.
\]

As the \( G \)-action on \( Y \) commutes with the \( S^1 \)-action, \( \pi : Y \to Z = Y / S^1 \) is a \( G \)-equivariant circle bundle and \((Z, \omega_Z)\) is a Hamiltonian \( G \)-manifold. Choosing a \( G \)-invariant connection \( \alpha \) on \( Y \), there exists a \( G \)-invariant 1-form on \( Y \), also denoted by \( \alpha \), vanishing on horizontal vector fields defined by the connection and

\[
\alpha \left( \frac{d}{d\theta} e^{i\theta} y \bigg|_{\theta=0} \right) = 1, \quad \forall \ y \in Y.
\] (4.7)

For simplicity, identifying \( V = H^{-1}(I) \) with \( I \times Y \), the symplectic form \( \omega \) on \( I \times Y \) can be written as

\[
\omega = \pi^* \omega_Z + d(t \alpha)
\]

where \( t \) is the coordinate on \( I \). Moreover, let \( \mu_\alpha : Y \to g \) be the \( \alpha \)-moment map on \( Y \), then restricted to \( I \times Y \), we have

\[
\mu = \mu_Z \circ \pi + t \mu_\alpha,
\]

where \( \mu_Z : Z \to g \) is a moment map for the Hamiltonian \( G \)-action on \( Z \).

Let \( \omega_C \) denote the standard symplectic structure on \( \mathbb{C} \). Then \((V \times \mathbb{C}, \omega \oplus \omega_C)\) be a Hamiltonian \( G \times S^1 \)-manifold, where \( G \) acts trivially on \( \mathbb{C} \) and the \( S^1 \)-action on \( \mathbb{C} \) is given by the scalar multiplication \( e^{\pm i\theta} \). The moment maps for \( S^1 \)-actions are

\[
H_\pm : V \times \mathbb{C} \to \mathbb{R}, \quad H_\pm(x, z) = H(x) \mp \frac{1}{2} |z|^2.
\]
Let \((V_\pm = H_\pm^{-1}(0)/S^1, \omega_0^\pm)\) denote the \(S^1\)-symplectic reductions. Then

\[
(Z = (H_\pm^{-1}(0) \cap (V \times \{0\}))/S^1, \omega_Z)
\]
is a \(G\)-invariant symplectic submanifold of \(V_\pm\) with \(G\)-equivariant symplectic normal bundle

\[
N_\pm \cong Y \times_{\rho_\pm} \mathbb{C}.
\]

Here \(\pi_\pm : Y \times_{\rho_\pm} \mathbb{C} \to Z\) is the complex line bundle over \(Z\) associated to the representations \(e^{i\theta} \cdot z = e^{i\theta} z\) for \(z \in \mathbb{C}\).

The \(G\)-invariant connection \(\alpha\) on \(Y\) induces \(G\)-invariant connection \(\nabla^\pm\) on \(N_\pm\). Choose \(G\)-invariant Hermitian metrics \(g_\pm\) on \(N_\pm\) which are compatible with \(\nabla^\pm\). With respect to these metrics, we can identify the \(\delta\)-neighbourhood of \(Z\) in \(V_\pm\) for a sufficiently small \(\delta\), denoted by \(N_\pm^\delta\), as

\[
N_\pm^\delta \cong \{(y, t, z)||z|^2 = \pm 2t, |t| \leq \delta/2\} \subset V_\pm
\]
with the induced symplectic form and the associated moment map given by

\[
\omega^\delta_\pm = \pi^*_Z \omega_Z \pm d(\alpha t), \quad \mu^\delta_\pm = \mu_Z \circ \pi_\pm \pm t\mu_\alpha.
\]

The symplectic cut is a union of two symplectic manifolds obtained by gluing \(V_\pm\) and \(X - Y\) through the following symplectomorphisms

\[
\Psi_+: Y \times (0, \epsilon) \to N_{2+}^{2e} - Z, \quad \Psi_- : Y \times (-\epsilon, 0) \to N_{2-}^{2e} - Z
\]
defined by \(\Psi_+(y, t) = [y, t, \sqrt{\pm 2t}]\) for a sufficiently small \(\epsilon\). These result in two Hamiltonian \(G\)-manifolds

\[
X^+ := X^+_0 \cup_{\psi_+} V_+ \quad \text{and} \quad X^- := X^-_0 \cup_{\psi_-} V_-,
\]
where \(X^+_0\) and \(X^-_0\) are two components of \(X - Y\) with end modelled on \(Y \times (0, \epsilon)\) and \(Y \times (-\epsilon, 0)\) respectively. The normal bundles \(N_\pm\) of \(Z\) in \(X^\pm\) have opposite equivariant Euler classes.

## 5 Equivariant first Chern classes

In this section, we compare the equivariant first Chern class of the Hamiltonian sum \((S_\lambda, \Omega_\lambda, \mu_\lambda)\) with the equivariant first Chern classes of \((X^+, \omega^+, \mu^+)\) and \((X^-, \omega^-, \mu^-)\). Let \(J_\lambda\) be the \(G\)-invariant \(\Omega_\lambda\)-compatible almost complex structure on \(S_\lambda\) induced from the \(G\)-invariant \(\Omega\)-compatible almost complex structure on \(S\), and \(J^\pm\) be the induced \(G\)-invariant \(\omega^{\pm}\)-compatible almost complex structure on \(X^\pm\). Denote by

\[
e_1^G(TX^\pm) \in H^2_G(X^\pm_G, \mathbb{Z}), \quad e_1^G(TS_\lambda) \in H^2_G((S_\lambda)_G, \mathbb{Z})
\]
the first equivariant Chern classes of \(TX^\pm\) and \(TS_\lambda\). Here \(X^\pm_G = EG \times_G X^\pm\) and \((S_\lambda)_G = EG \times_G S_\lambda\) are the homotopy quotients of \(X^\pm\) and \(S_\lambda\) respectively.
We claim that $S_\lambda$ can be realised as $X^+ \#_{\varphi_\lambda} X^-$ obtained from gluing the complements of tubular neighbourhoods of $Z$ in $X^+$ and $X^-$ along their boundaries by an orientation-reversing $G$-equivariant diffeomorphism $\varphi_\lambda$ (defined by $\lambda$ and the $G$-equivariant Hermitian line bundle isomorphism (1.1)). To see this, recall that

$$S_\lambda = \left( X^- - \Psi_{-,\epsilon}(N_- Z(1)) \right) \cup_{gl_-} S_{Z,\lambda} \cup_{gl_+} \left( X^+ - \Psi_{+,\epsilon}(N_+ Z(1)) \right),$$  \hspace{1cm} (5.1)

where the middle part $S_{Z,\lambda} = \{(z, v, w) \in N_+ Z \oplus N_- Z \mid v, w < 2, \epsilon \mathcal{P}(v, w) = \lambda\}$ can be identified as the open annular bundle of $N_+ Z$ with radius from $\epsilon^{-1}|\lambda|/2$ to 2

$$(N_+ Z(2) - N_+ Z(\epsilon^{-1}|\lambda|/2)),$$

as we can solve $w$ in terms of $v$ from the equation $\epsilon \mathcal{P}(v, w) = \lambda$. From the construction of the symplectic sum, we have

$$\epsilon^{-1}|\lambda| < \epsilon^{-1} \delta < \frac{1}{2} \implies \epsilon^{-1}|\lambda|/2 < \frac{1}{4}.$$ 

Therefore $S_{Z,\lambda}$ consists of three parts:

- the domain of the gluing maps $gl_+$
  $$S_{Z,\lambda,+} = \{(z, v, w) \in N_+ Z \oplus N_- Z \mid 1 < |v| < 2, \epsilon \mathcal{P}(v, w) = \lambda\}$$
  $$\cong (N_+ Z(2) - N_+ Z(1)),$$
  under this identification, $gl_+ = \Psi_{+,\epsilon}$.

- the neck part $(N_+ Z(1) - N_+ Z(\epsilon^{-1}|\lambda|))$, which is a closed annular bundle of $N_+ Z$ with radius from $\epsilon^{-1}|\lambda|$ to 1. Note that the neck part is identified with $\Psi_{+,\epsilon}(\overline{N_+ Z(1)} - N_+ Z(\epsilon^{-1}|\lambda|))$.

- the domain of the gluing map $gl_-$
  $$S_{Z,\lambda,-} = \{(z, v, w) \in N_+ Z \oplus N_- Z \mid 1 < |w| < 2, \epsilon \mathcal{P}(v, w) = \lambda\}$$
  $$\cong (N_+ Z(\epsilon^{-1}|\lambda|) - N_+ Z(\epsilon^{-1}|\lambda|/2)).$$

Thus $S_\lambda$ as in (5.1) can be written as

$$S_\lambda = \left( X^- - \Psi_{-,\epsilon}(N_- Z(1)) \right) \cup_{\varphi_\lambda} \left( X^+ - \Psi_{+,\epsilon}(N_+ Z(\epsilon^{-1}|\lambda|)) \right),$$

where $\varphi_\lambda : \partial(X^+ - \Psi_{+,\epsilon}(N_+ Z(\epsilon^{-1}|\lambda|))) \to \partial(X^- - \Psi_{-,\epsilon}(N_- Z(1)))$ is the orientation-reversing $G$-equivariant diffeomorphism defined by the composition of the following three maps:

- the restriction of gluing map $(gl_+)^{-1} = (\Psi_{+,\epsilon})^{-1} : \partial(X^+ - \Psi_{+,\epsilon}(N_+ Z(\epsilon^{-1}|\lambda|))) \to SN_+ Z(\epsilon^{-1}|\lambda|)$ (the circle bundle of $N_+ Z$ of radius $\epsilon^{-1}|\lambda|$),
• an orientation-reversing $G$-equivariant diffeomorphism from $SN_+Z(\epsilon^{-1}|\lambda|)$ to $SN_-Z(1)$ (the unit circle bundle of $N_-Z$) by solving $w$ in terms of $v \in SN_+Z(\epsilon^{-1}|\lambda|)$ from the equation 

$$\epsilon P(v, w) = \lambda,$$

• the restriction of gluing map $gl_- : SN_-Z(1) \to \partial(X^- - \Psi_{\lambda}(N_-Z(1)))$.

For the comparison, we evaluate $c_i^G(TX^\pm)$ on certain degree 2 equivariant homology classes of the form 

$$B_{\pm} = (u_{\pm})_*(\Sigma_{\pm}) \in H^2_{\mathbb{Z}}(X^\pm_G, \mathbb{Z})$$

coming from a principal $G$-bundle $P^\pm$ over oriented closed surfaces $\Sigma^\pm$ together with smooth sections $u_{\pm}$ of $P^\pm \times_G X^\pm$ as in the following diagram

$$
\begin{array}{ccc}
P^\pm \times_G X^\pm & \xrightarrow{u_{\pm}} & X^\pm_G \\
\Sigma^\pm & \xleftarrow{c_p} & BG,
\end{array}
$$

where $c_p : \Sigma_{\pm} \to BG$ is the classifying map of $P^\pm$.

We assume that sections $u_{\pm}$ in (5.2) satisfy the following conditions

$$(u_{\pm})^{-1}(P^\pm \times_G Z) = \{x_1^\pm, \ldots, x_k^\pm\} \subset \Sigma_{\pm},$$

$$u^+(x_i^+)_k = u^-(x_i^-) \in P \times_G Z, \quad \text{ord}_{x_i^\pm, Z}^-(u^+) = \text{ord}_{x_i^\pm, Z}^-(u^-),$$

for any $i = 1, \ldots, k$. Here $\text{ord}_{x_i^\pm, Z}^-(u^\pm)$ is the order of contact of $u^\pm$ with $Z$ at $x_i^\pm$ defined as follows. Take a local trivialization of $P^+$ in a small neighborhood $B_{x_i^+}$ of $x_i^+$ over which the section $u^+$ can be expressed as a smooth map from $B_{x_i^+}$ to $X^+$. Take a homotopy of $u^+$ on a small neighborhood of $u^+(x_i^+)$ without changing the intersection point $u^+(x_i^+)$ so that $du^+$ maps a small circle in $T_{x_i^+} \Sigma^+$ to a circle in $N_+Z | u^+(x_i^+)$. The degree of this map is the order of contact $\text{ord}_{x_i^+, Z}^+(u^+) \in \mathbb{Z}$. This degree is independent of the choice of local trivialization of $P^+$ near $x_i^+$ and the choice of the homotopy of $u^+$ on a small neighborhood of $x_i^+$. The order of contact of $u^-$ with $Z$ at $x_i^-$ is defined in the same way.

We can construct a smooth section $u^+ \# u^- \# P^-$ over a $\Sigma^+_\lambda \Sigma^-$ by the following steps

• removing small discs $D_{x_i^+}$ and $D_{x_i^-}$ around $x_i^\pm$ in $\Sigma^\pm$ for all $i$ to get surfaces $\widehat{\Sigma}^\pm$ with $k$ boundary circles. We can assume (by homotopying if necessary)

$$u_{\pm} \mid_{\partial D_{x_i^\pm}} : \partial D_i \longrightarrow \Psi_{\pm, \epsilon}(\partial N_{\pm}Z(1)_{u_{\pm}(x_i^\pm)}),$$

• gluing $\widehat{\Sigma}^+$ and $\widehat{\Sigma}^-$ along each of boundary circles by orientation-reversing diffeomorphisms $\{\varphi_i\}$ to form a smooth oriented surface $\Sigma^+_\lambda \Sigma^-$ with neck structure such that

$$\Sigma^+_\lambda \Sigma^- = \widehat{\Sigma}^- \cup_{\{\varphi_i\}} \left( \bigsqcup_{i=1}^k [\epsilon^{-1}|\lambda|, 1]_i \times S^1 \right) \cup \widehat{\Sigma}^+.$$
• gluing the principal bundles $P^+$ and $P^-$ along each boundary circle to form a principal bundle over $\Sigma^+ \#_\lambda \Sigma^-$, by using the trivializations $P^\pm|_{D_{x_i}^\pm} \cong D_{x_i}^\pm \times G$ and gluing parameters $\rho_i : P^+|_{x_i^+} \to P^-|_{x_i^-}$ ($G$-equivariant diffeomorphisms) for all $i$. The homotopying around $\partial D_i$ for each $i$ so that $u^- \circ \varphi_i = \varphi \circ u^+$ on $\partial D_i^+$ for all $i$. To be precise, choose a homotopy

$$u^+ = \{u^+_i\}_{i \in \{\pm\}} : \bigcup_{i=1}^k \bigcup_{i=1}^k [\epsilon^{-1} |\lambda|, 1]_i \times S^1 \to \bigcup_{i=1}^k gl_+ \left(N^+(\Sigma^\pm(1) - N^+ \lambda(\epsilon^{-1} |\lambda|))_{u^+(x_i^+)} \right)$$

such that for each $i$, $u^+_i|_{t=1} = u^+|_{t=1}$ and $u^-|_{t=1} \circ \varphi_i = \varphi \circ u^+|_{t=1}$. This can be achieved because $P^+ \times_G X^+$ is trivialized over $D_{x_i^+}$, so $u^+$ can be written as smooth maps $\partial D_i^+ \to X^+$ and the degrees of maps $\varphi \circ u^+ \circ \varphi_i^-$ and $u^-$ are the same.

• We remark that $S_\lambda$ contains a neck part $gl_+ \left(N^+(\Sigma^\pm(1) - N^+ \lambda(\epsilon^{-1} |\lambda|)) \right)$. Define

$$u^+ \#_{\varphi} u^- = \begin{cases} u^- & \text{on } \hat{\Sigma}^-, \\ u^+ & \text{on } \left[\bigcup_{i=1}^k \bigcup_{i=1}^k [\epsilon^{-1} |\lambda|, 1]_i \times S^1 \right] \end{cases} \quad (5.5)$$

Then $u^+ \#_{\varphi} u^-$ is a section of $(P^+ \#_\lambda P^-) \times_G S_\lambda$.

From the construction of $u^+ \#_{\varphi} u^-$ and the homotopy invariance of homology theory, $u^+$ and $u^-$ completely define the homology class of

$$(u^+ \#_{\varphi} u^-)_*([\Sigma^+ \#_\lambda \Sigma^-]) \in H_2(P \times_G S_\lambda, \mathbb{Z}),$$

thus an equivariant homology classes $((u^+ \#_{\varphi} u^-)_*) \in H^G_2(S_\lambda, \mathbb{Z})$.

The Hamiltonian sum construction gives rise to a Hamiltonian $G$-manifold of the form $X^+ \#_{\varphi} X^-$. We can compare equivariant first Chern class of the Hamiltonian sum with the equivariant first Chern classes of $X^+$ and $X^-$. 

**Proposition 5.1** Suppose that the sections $u^\pm$ in $[5.2]$ satisfy the conditions $[5.3]$ and $[5.4]$. Let $B_\pm$ be the equivariant homology classes $u^\pm_*([\Sigma^\pm]) \in H^G_2(X^\pm, \mathbb{Z})$ and $B_\pm \#_{\varphi} B_\pm$ be the equivariant homology class $H^G_2(S_\lambda, \mathbb{Z})$ defined by $u^+ \#_{\varphi} u^-$, then

$$\langle c_1^G(T(S_\lambda)), B_+ \#_{\varphi} B_- \rangle = \langle c_1^G(TX^+), B_+ \rangle + \langle c_1^G(TX^-), B_- \rangle - 2 \sum_{i=1}^k \ord_{x_i^+}^{\pm}(u^+).$$

and $\sum_{i=1}^k \ord_{x_i^+}^{\pm}(u^+) = \sum_{i=1}^k \ord_{x_i^-}^{\pm}(u^-)$ is the intersection number

$$(u^+_*)([\Sigma^+]) \cdot [P^+ \times_G \mathbb{Z}] = (u^-)_*[\Sigma^-]) \cdot [P^- \times_G \mathbb{Z}].$$
Proof: From the functoriality of the first Chern class, we have
\[ \langle c_1^G(TX^\pm), (u^\pm_0)_*([\Sigma^\pm]) \rangle = \langle c_1(P \times_G TX^\pm), (u^\pm)_*([\Sigma^\pm]) \rangle, \]
and
\[ \langle c_1^G(TS_\lambda), ((u^+ \#_\varphi u^-)_*)([\Sigma^+ \#_\lambda \Sigma^-]) \rangle = \langle c_1(P \times_G TS_\lambda), (u^\pm_0 \#_\varphi u^-_0)_*([\Sigma^+ \#_\lambda \Sigma^-]) \rangle, \]
In \( H_2^G(S, \mathbb{Z}) \), the equivariant homology class of \( B_+ \#_\varphi B_- \) is the sum of equivariant homology class of \( B_+ \) and \( B_- \). Note that the normal bundle of \( S_\lambda \) in \( S \) is trivial, we have
\[
\langle c_1(P \times_G TS_\lambda), (u^+ \#_\varphi u^-)_*([\Sigma^+] \#_\lambda [\Sigma^-]) \rangle = \langle c_1(P \times_G TS), (u^+)_*([\Sigma^+]) \rangle + \langle c_1(P \times_G TS), (u^-)_*([\Sigma^-]) \rangle.
\]
(5.6)
Note that \( u^\pm \) are sections of \( P \times_G X^\pm \subset P \times_G S_0 \). Along \( X^+ \subset S_0 \), we have the decomposition of \( G \)-equivariant vector bundle,
\[ TS|_{X^+} \cong TX^+ \oplus N_SX^+. \]
As a \( G \)-equivariant vector bundle, \( N_SX^+ \) is trivial on \( X^+ - \Psi_{\pm E}^{-1}(N_+ \mathbb{Z}(1)) \) and isomorphic to \( (\Psi_{\pm E}^{-1})^* \circ \pi_{\pm E}^* \)\( N_- \mathbb{Z} \) on the \( \Psi_{\pm E}(N_+ \mathbb{Z}(2)) \), where \( \pi_{\pm E} : N_+ \mathbb{Z}(2) \to \mathbb{Z} \) is the restriction of the bundle projection. As \( N_- \mathbb{Z} \) is complex conjugate to \( N_+ \mathbb{Z} \), we get
\[ c_1(P^+ \times_G N_SX^+) = -PD([P^+ \times_G Z]) \in H^2(P^+ \times_G X^+, \mathbb{Z}). \]
Thus
\[ \langle PD([P^+ \times_G Z]), (u^+)_*([\Sigma^+]) \rangle = (u^+)_*([\Sigma^+]) \cdot [P^+ \times_G Z] = \sum_{i=1}^k \text{ord}_{x_i^+,Z}^{P^+}(u^+). \]
Putting these together, we have
\[ \langle c_1(P \times_G TS), (u^+)_*([\Sigma^+]) \rangle = \langle c_1(P \times_G TX^+), (u^+)_*([\Sigma^+]) \rangle - \sum_{i=1}^k \text{ord}_{x_i^+,Z}^{P^+}(u^+). \]
Similarly, we have
\[ \langle c_1(P \times_G TS), (u^-)_*([\Sigma^-]) \rangle = \langle c_1(P \times_G TX^-), (u^-)_*([\Sigma^-]) \rangle - \sum_{i=1}^k \text{ord}_{x_i^-,Z}^{P^-}(u^-). \]
Together with (5.6), this completes the proof of the Proposition.

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