Distillation of secret key from a class of compound memoryless quantum sources

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We consider secret-key distillation from tripartite compound classical-quantum-quantum (cqq) sources with free forward public communication under the strong security criterion. We design protocols which are universally reliable and secure in this scenario. These can be shown to asymptotically optimal as long as a certain regularity condition on the the set of cqq density matrices generating the source is fulfilled. We derive a multi-letter capacity formula for all compound cqq sources being regular in this sense. We also determine the forward secret-key distillation capacity in the situation, where the legitimate sending party has perfect knowledge of his/her marginal state deriving from the source. In this case regularity conditions can be dropped. Our results show, that the capacities with and without the mentioned kind of state knowledge are equal as long as the source is generated by a regular set of density matrices. We demonstrate, that regularity of cqq sources is not only a technical but also an operational issue. For this reason, we give an example of a source which has zero secret-key distillation capacity without sender knowledge, while achieving positive rates is possible if sender marginal knowledge is provided.

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I. INTRODUCTION

Common randomness shared by communication parties which is moreover uncorrelated to an eavesdropping third party is well known as a valuable resource in information theory. This fact becomes apparent e.g. when legitimate parties use a one-time pad coding [24] procedure to securely randomize codewords. In this way, they can enhance the security of messages sent over an insecure transmission line. A possible way to generate this resource is, to distill it from potentially noisy and insecure correlations preshared by the parties. Development of methods to obtain such secret-keys has been for long time a domain of cryptographic research [18]. The cryptographic approach is usually, to exploit assumed limited computational capabilities of eavesdropping parties. In such a setting, cooperation of legitimate parties can have substantial advantages in applying high-complexity protocols. This approach is even more powerful, when applied to quantum systems. Utilizing the structure of quantum theory led to a completely different type of protocols for distillation of secret-keys. An early example of such research was [4] which initiated extensive research on the theory of quantum key distribution. 

A disadvantage of the cryptographic approach is, that security is obtained under presumption of limitation of the eavesdroppers technical possibilities, which may be improved by technological progress. Even preassumptions deriving from the structure of quantum mechanics are subject to ongoing discussions [22].

This paper follows the so-called information-theoretic approach to security, where rather the principal limitations of the eavesdropping parties are utilized to obtain security. Initiated by works of Ahlswede and Csiszar [1] and Maurer [23], this direction was intensively studied in the past decades. Newer developments indicate, that integrating security on the physical layer of communication systems more and more heads towards technological application [23].

In this flavour, information-theoretic methods of secret-key distillation where also studied for quantum systems in [16], where the secret-key distillation of classical-quantum-quantum sources and completely quantum sources where studied.

However, to obtain these results, the sources under consideration where assumed to produce memoryless outputs with statistical properties perfectly known to the legitimate parties. Since these conditions are rather restrictive in real-world applications, we drop the second of the above conditions and assume presence of a compound memoryless quantum source. In this model, the source is memoryless, but the legitimate communication parties are assumed to have only limited information about the generating density matrix. Instead of knowing its identity, they are only provided with knowledge of a subset of states, containing the source states which possibly occur.

Consequently, the legitimate parties have to apply protocols which have the property to simultaneously generate secure common randomness for each of the possible states from the prescribed set.

The contributions of this paper are the following. We consider presence of a compound memoryless classical-quantum-quantum (cqq) source where one receiver is assumed to receive outputs of a classical source, while the remaining communication parties receive quantum systems. The classical systems-receiving party is also allowed for public communication of classical messages to support the distillation process. We consider a strong security criterion and design for given compound source universal protocols for secret-key distillation.

It turns out, that some compound cqq sources are of difficult structure for generating secret-keys. This fact leads us to introducing a regularity condition on sets of cqq density matrices. We demand, the possible sets of marginal states of the sender-legitimate receiver and sender-eavesdropper systems only to differ in a controllable amount when the derived marginals on the sender system dont differ much. For the class of density matrices fulfilling this property, the approximation methods we develop lead to a proof of achievability, whose optimal rates are also optimal in general, i.e. we obtain a full characterization of the forward secret-key capacity.

We also consider the case, where the sending party is equipped with perfect knowledge of the marginal distribution on his/her systems derived from the source. The capacities are shown to be equal for the cases with and without this kind of sender information. Moreover, the formula derived is shown to be also valid for all irregular sources.

The reader may ask for a general proof of validity of the mentioned capacity formula also for the case of no sender state information. Regarding this question we prove a disappointing negative result. The forward secret-key distillation capacities with and without sender state information differ substantially for some compound cqq sources.

Things get even harder. The counterexample we introduce shows, that the legitimate parties can reach positive capacity with zero error and zero correlation of the key in case of sender-state knowledge, while they are unable to achieve any positive rate without sender state knowledge. This sheds some light on the structure of compound cqq sources. Even if there may be weaker regularity conditions than the one
Presented here which lead to a general capacity formula, the notion of regularity bears an operational core. While perfect knowledge of the sender's marginal state does not help to achieve higher forward secret-key distillation rates for regular sources, irregularity of the source can split values of both capacities.

Related work

The information-theoretic approach to security was initiated in classical information theory by the works of Ahlswede and Csiszar [1], where among other results also considering channel models, the capacity for generation of secret keys from perfectly known tripartite classical memoryless sources with free one-way public communication where determined. The research area opened by the aforementioned work led to many relevant considerations. We mention [13] where the secret-key distillation capacity was also determined under constraints on the classical forward communication rate. An excellent overview of the research activities done on the field is Chapter 17 in [15].

The first results regarding information-theoretic security were obtained under a rather weak security criterion where the measures of security appeared as quantities regularized in blocklength. The notion of security was improved in [12], [24], where first results were proven using so-called strong security criteria. In this work, we define a quantum version of the strong security index known from classical information theory (see [15] for further information).

The spirit of [1] was injected to quantum information theory by Devetak and Winter where results from [1] were obtained under the assumption that the correlation used for secret-key generation are obtained from outputs of memoryless sources, while the statistics of the source are perfectly known to the communication parties. The compound source model was hardly considered including security constraints even in classical information theory. A first attempt to generalize some of the aforementioned results to the case of classical compound memoryless sources was pursued in [10], and [25] under collaboration of the first author of this paper. In the second of the mentioned papers, the forward secret-key distillation capacity of a classical compound source was determined for the case of a finite number of possible marginal states on the sender's systems. Also a lower bound on the capacity under restriction of the forward public communication was also derived therein.

Outline

In Section II we fix notation and introduce some conventions, we freely use in our considerations. We precisely state the relevant definitions and our main results in Section III. Therein, we also introduce a certain regularity condition for sets of cq density matrices which is defined in terms of Hausdorff continuity of the map connecting each possible marginal probability distribution on the sender's system with the cq density matrices from the compound source generating set being consistent with it.

In Sect. IV we provide a full proof of our main result, where we prove a multi-letter capacity formula for the forward secret-key distillation capacity for compound sources which fulfill the mentioned regularity conditions.

We may drop the regularity condition if we assume the sender to have perfect knowledge of his marginal distribution. In Section V we prove a full coding theorem to determine the secret-key distillation capacity of compound memoryless cq sources in case of sender marginal information (SMI). It turns out, that for regular sources, the secret-key distillation capacities with and without sender SMI are equal. However, the capacities do not match in general, when we are facing an irregular source. This claim is substantiated in Section VI where we provide an example of a compound cq source with a substantial gap between the forward secret-key distillation capacities with and without SMI. In Section VII we apply the general theory of set-valued maps to derive a weaker regularity condition. We show, that the definitions given in Section III implicitly allow to weaken the mentioned condition to lower semi-continuity.

The forward secret-key distillation capacity was only determined on a lower level of generality even in case of a fully classical source [23]. Therefore we indicate that our considerations also include the completely classical setting as a special case. We conclude the paper with some general remarks in Section VIII where we especially point out the relation of our results to the problem of one-way LOCC entanglement distillation from bipartite compound quantum sources.
II. NOTATION AND PRELIMINARY RESULTS

All Hilbert spaces appearing in this work are considered to be finite dimensional complex vector spaces. \( \mathcal{L}(\mathcal{H}) \) is the set of linear maps and \( \mathcal{S}(\mathcal{H}) \) the set of states (density matrices) on a Hilbert space \( \mathcal{H} \) in our notation. For a finite alphabet \( \mathcal{X} \) and a Hilbert space \( \mathcal{K} \), we denote by \( \mathcal{CQ}(\mathcal{X}, \mathcal{K}) \) the set of classical-quantum channels, i.e., maps from \( \mathcal{X} \) to \( \mathcal{S}(\mathcal{K}) \). The set of completely positive and trace preserving (c.p.t.p.) maps from \( \mathcal{L}(\mathcal{H}) \) to \( \mathcal{L}(\mathcal{K}) \) is denoted \( \mathcal{C}(\mathcal{H}, \mathcal{K}) \).

Regarding states on multiparty systems, we freely make use of the following convention for a system consisting of some parties \( X, Y, Z \), for instance, we denote \( \mathcal{H}_{XYZ} := \mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z \), while the marginals are labeled by indices assigned to the corresponding subsystems, i.e., \( \sigma_{XZ} := \text{tr}_{Y}(\sigma) \) for \( \sigma \in \mathcal{S}(\mathcal{H}_{XYZ}) \) and so on. The von Neumann entropy of a quantum state \( \rho \) is defined by

\[
S(\rho) := -\text{tr}(\rho \log \rho),
\]

where we denote by \( \log(\cdot) \) and \( \exp(\cdot) \) the base two logarithms and exponentials throughout this paper. Given a quantum state \( \rho \) on \( \mathcal{H}_{XY} \), we denote the conditional von Neumann entropy of \( \rho \) given \( Y \) by

\[
S(X|Y, \rho) := S(\rho) - S(\rho_Y),
\]

the quantum mutual information by

\[
I(X;Y, \rho) := S(\rho_X) + S(\rho_Y) - S(\rho).
\]

A convenient way of representing systems which have quantum as well as classical subsystems is by coherifying the classical systems. The density matrix

\[
\rho := \sum_{x \in \mathcal{X}} P_X(x) |x\rangle \langle x| \otimes \rho_x \in \mathcal{S}(\mathcal{H}_X \otimes \mathcal{K}_B)
\]

represents a density matrix of a source, where the statistics of a subsystem is driven by a classical random variable \( X \) with values in \( \mathcal{X} \) and \( \rho_x \in \mathcal{S}(\mathcal{K}_B) \) is a density matrix on \( \mathcal{K}_B \) for each \( x \in \mathcal{X} \). A quantum system with Hilbert space \( \mathcal{H}_X := \mathbb{C}^{3^{|X|}} \) was introduced where each \( x \in \mathcal{X} \) corresponds to the element \( |x\rangle \) of a once and for all fixed orthonormal basis (we may assume that this is for each system introduced the canonical basis). We set the convention to indicate the quantum systems belonging to coherified classical systems by the corresponding random variable. This convention extends to notation of entropic quantities. E.g.

\[
I(X;B, \rho) = H(X) + S(\rho_B) - S(\rho).
\]

corresponds to the quantum mutual information of the state \( \rho \) in (1). The conditional quantum mutual information of a density matrix \( \sigma_{ABX} \) is defined

\[
I(A;B|X, \sigma) := S(\sigma_{AX}) + S(\sigma_{BX}) - S(\sigma_{ABX}) - S(\sigma_X).
\]

If \( X \) belongs to a classical system i.e.

\[
\sigma = \sum_{x \in \mathcal{X}} P_X(x) |x\rangle \langle x| \otimes \sigma_{ABx}
\]

with \( \rho_{ABx} \) being a bipartite density matrix on the remaining systems Hilbert spaces, it holds

\[
I(A;B|X, \sigma) = \sum_{x \in \mathcal{X}} P_X(x) I(A;B, \rho_{ABx}).
\]

Whenever informational quantities are evaluated on classical systems, we feel free to express them in terms of the corresponding classical informational quantities evaluated on the corresponding probability distributions resp. random variables where we completely adopt the notation and calculational rules as presented in [15] if no further reference is given.

From [15], we also take the definition and properties of types and typical sequences. For given alphabet \( \mathcal{X} \) and \( n \in \mathbb{N} \) (which we always regard being finite) we denote the set of probability distributions on \( \mathcal{X} \) as \( \mathcal{P}(\mathcal{X}) \). We will use \( [N] \) as a shortcut for the set \( \{1, \ldots, N\} \) for each \( N \in \mathbb{N} \). The set of types (i.e. empirical distributions) on \( \mathcal{X}^n \) is denoted by \( \mathcal{T}(n, \mathcal{X}) \), it holds

\[
|\mathcal{T}(n, \mathcal{X})| \leq (n + 1)^{|\mathcal{X}|}
\]

\((n \in \mathbb{N})\).
For given type \( \lambda \in \mathcal{T}(n, \mathcal{X}) \), we denote the set of \( \lambda \)-typical words in \( \mathcal{X}^n \) by \( T^n_\lambda \). For each \( \delta > 0 \), \( p \in \mathcal{P}(\mathcal{X}) \), the set of \( \delta \)-typical sequences for \( p \) in \( \mathcal{X}^n \) is defined by

\[
T^n_{p, \delta} := \left\{ x^n \in \mathcal{X}^n : \forall a \in \mathcal{X} : \left| \frac{1}{n} N(a|x^n) - p(a) \right| \leq \delta \land p(a) = 0 \Rightarrow N(a|x^n) = 0 \right\},
\]

where \( N(a|x^n) \) is the number of occurrences of \( a \) in \( x^n \). Several kinds of bounds are known for these sets, we will explicitly employ the bound

\[
p^n\left( T^n_{p, \delta} \right)^c \leq 2^{-nc\delta^2}
\]

which holds with \( c := \frac{1}{2\ln 2} \) for each \( \delta > 0 \) and large enough \( n \).

For any two nonempty sets \( I, I' \) of states on a Hilbert space \( \mathcal{H} \), the Hausdorff distance (induced by the trace norm \( \| \cdot \|_1 \) ) is defined by

\[
d_H(I, I') := \max \left\{ \sup_{\sigma \in I} \inf_{\sigma' \in I'} \| \sigma - \sigma' \|_1, \sup_{\sigma' \in I'} \inf_{\sigma \in I} \| \sigma - \sigma' \|_1 \right\}
\]

\[
= \inf\{ \epsilon > 0 : I' \subset I_\epsilon \land I \subset I'_\epsilon \}
\]

where \( A_\epsilon \) denotes the \( \epsilon \)-blowup of \( A \) (with regard to \( \| \cdot \|_1 \) ) for each set \( A \). On the set of subsets of a bounded set, \( d_H \) has only finite values. If the set of compact subsets of such a bounded set are regarded, \( d_H \) becomes a full metric. Several properties of the Hausdorff distance are directly inherited from 1-norm on the underlying space. We will frequently use the triangle inequality

\[
d_H(A, C) \leq d_H(A, B) + d_H(B, C) \quad \quad \quad \quad (A, B, C \subset \mathcal{H})
\]

and monotonicity of \( d_H \) under c.p.t.p. maps, i.e. for each \( \mathcal{N} \in \mathcal{C}(\mathcal{H}, \mathcal{K}), A, B \in \mathcal{S}(\mathcal{H}) \), it holds

\[
d_H(A, B) \geq d_H(\mathcal{N}(A), \mathcal{N}(B)),
\]

where \( \mathcal{N}(C) \) is the image of each set \( C \subset \mathcal{L}(\mathcal{H}) \) under \( \mathcal{N} \).

### III. BASIC DEFINITIONS AND MAIN RESULT

In this section we give precise definitions of the secret-key distillations task and the corresponding capacities of compound memoryless cqq sources with and without assumption of SMI. We also state the two main results Theorem 8 and Theorem 9.

#### A. Source model

A compound memoryless quantum source generated by a set of density matrices \( I \subset \mathcal{S}(\mathcal{K}) \) on a Hilbert space \( \mathcal{K} \) is the source described by the set of possible output density matrices

\[
I^\otimes n := \{ \rho^\otimes n : \rho \in I \}
\]

for each blocklength \( n \in \mathbb{N} \). This source definition models a situation, where the source statistics is memoryless, but the generating density matrix is not known. It communication parties processing the source, only can be sure, that the output statistics is governed by memoryless extensions of a density matrix from \( I \). In this paper, the compound sources considered are generated by tripartite classical-quantum density matrix of the form

\[
\rho := \sum_{x \in \mathcal{X}} p(x) |x\rangle \langle x| \otimes \rho_{BE,x} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)
\]

which is the coherifed way to represent a statistics where \( A \) receives outputs of a classical source with distribution \( p \in \mathcal{P}(\mathcal{X}) \), while \( B \), and \( E \) receive quantum systems with joint state \( \rho_{BE,x} \in \mathcal{S}(\mathcal{H}_{BE}) \), dependent on the letter \( x \). If a system is classical, we regard the basis used for coherifying the systems as fixed once and for all (we fix it to be the canonical basis \( \{ |x\rangle \}_{x \in \mathcal{X}} \).
Definition 1. Let $\rho$ be a marginal density matrix for the channel $\mathcal{C}$, that is, $\rho = \sum_{x} p(x) |x\rangle\langle x| \otimes |\psi_x\rangle\langle \psi_x|$. We define $\mathcal{C}(\mathcal{H}_{AB})$ to be the set of all such density matrices.

To increase notational flexibility within our considerations, we define for each given set $I \in \mathcal{S}_{\text{cqq}}(\mathcal{H}_{AB})$

\[
P_I := \left\{ \rho \in \mathcal{P}(\mathcal{X}) : \exists \rho \in I \text{ with } \rho_A = \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x| \right\}
\]

and $I_p := \left\{ \rho \in I : \rho_A := \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x| \right\}$

for each $p \in P$. With these notations, $P_I$ is the set of marginal probability distributions which can occur at the sender’s site, while $I_p$ collects for each $p$ the set of possible occuring cqq density matrices under the constraint that $p$ generates the marginal distributions on the sender’s systems. For a more efficient notation of the capacity formulas appearing below, we also define the following sets of marginal distributions deriving from states on $I_p$ by

\[
I_p^{AB} := \{ \rho_{AB} : \rho \in I_p \}, \quad I_p^{AE} := \{ \rho_{AE} : \rho \in I_p \}
\]

for each $p \in P_I$.

B. Secret key generation from compound cqq sources: Definitions and results

For a cqq source with fixed density matrix $\rho \in \mathcal{S}_{\text{cqq}}(\mathcal{X}, \mathcal{H}_{AB})$, a secret key generation protocol for given blocklength $n$ is performed, informally speaking as follows. The $A$-party generates from his/her source output messages $l$ and $m$ where $m$ is the the key value for $A$ and $l$ is broadcasted to the remaining parties via a noiseless channel. The legitimate receiver subsequently determines a key value by a measurement, which can be chosen according to the received $l$. This results in a tuple $(K, K', A, X^n)$, where $K$ ($K'$) is the key random value of $A$ ($B$), $A$ the random variable representing the public transmission, and $X^n$ the classical random variable received by $A$ initially. The formal definition for the described type of protocol is as follows.

Definition 1. An $(n, M, L)$ (forward) secret-key distillation protocol for states on $\mathcal{S}_{\text{cqq}}(\mathcal{H}_{AB})$ is a pair $(T, D)$, with $T : \mathcal{X}^n \rightarrow \mathcal{P}([L] \times [M])$ being a stochastic matrix, and $D = \{ D_{lm} \}_{l \in [L], m \in [M]}$ being a set of matrices, $0 \leq D_{lm} \leq \mathbb{1}_{\mathcal{H}_B}^{\otimes n}$ such that

\[
\sum_{m=1}^{M} D_{lm} = \mathbb{1}_{\mathcal{H}_B}^{\otimes n}
\]

holds for each $l \in [L]$.

We will also consider the situation, where the sender has full knowledge of the statistics of his/her part of the source. If this is assumed, the sender can choose the stochastic matrix of a protocol according to this knowledge.
Definition 2. An \((n,M,L)\) (forward) secret-key distillation protocol with sender marginal information (SMI) for a set \(1 \in \mathcal{S}_{cq}(H_{ABE})\) is a family \((T_{p},D)_{p \in \mathcal{P}}\), with \((T_{p},D)\) being an \((n,M,L)\) forward secret key distillation protocol for states on \(\mathcal{S}_{cq}(H_{ABE})\).

In the next Definition, we will quantify the performance of \((n,L,M)\) forward secret-key distillation protocols with and without SMI performed in a compound source generated by a set \(1 := \{\rho_{s}\}_{s \in S} \subset \mathcal{S}_{cq}(H_{ABE})\).

For a protocol with SMI, \((T_{p},D)\) is performed, if the cqq density matrix is from \(1_{p}\). It is convenient, to express the aftermath in coherified manner by the state

\[
\rho_{\Lambda_{E}^{n},s} := \sum_{l=1}^{L} \sum_{m,m' = 1}^{M} \sum_{x \in X} p^{n}(x^{n}) T_{p}(l,m|x^{n}) |l\rangle \otimes |m\rangle \langle m| \otimes |m'\rangle \otimes \text{tr}_{\rho^{n}_{E}}((D_{lm'} \otimes 1^{\otimes n}) V^{\otimes n}(x^{n})).
\]

We are especially interested in the marginal state

\[
\rho_{\Lambda_{E}^{n},s} := \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{x \in X} p^{n}(x^{n}) T_{p}(l,m|x^{n}) \text{tr}(D_{lm} V^{\otimes n}(x^{n})) |l\rangle \otimes |m\rangle \otimes \rho_{E,x^{n}}.
\]

and the probability distribution \((K_{s},K'_{s})\) belonging to the key given by

\[
P_{K_{s},s}(m,m') = \langle m \otimes m', \rho_{K_{s},m \otimes m'} \rangle = \sum_{l=1}^{L} \sum_{m=m'=1}^{M} p^{n}(x^{n}) T_{p}(l,m|x^{n}) \text{tr}(D_{lm} \rho_{BE,x^{n}}) (m,m' \in [M]).
\]

The case of application of a protocol without sender marginal information can be regarded as the special case, where \(T_{p}\) does not depend on \(p\). The following definition quantifies the performance of each \((n,M,L)\) forward secret key distillation protocol with sender marginal information when performed on a set of cqq density matrices. The corresponding definition for the case without SMI is obtained by dropping all bracketed text from the next definition.

Definition 3. An \((n,M,L)\) forward secret-key distillation protocol (with SMI) for a set \(1 := \{\rho_{s}\}_{s \in S} \subset \mathcal{S}_{cq}(H_{ABE})\) is an \((n,M,L,\lambda)\) forward secret-key distribution protocol (with SMI) for \(1\) if the following two assertions are satisfied simultaneously for all \(s \in S\):

1. \(Pr(K_{s} \neq K'_{s}) \leq \lambda\).
2. \(\log M - H(K_{s}) + I(K_{s};E^{n}A_{s} \rho_{\Lambda_{E}^{n},s}) \leq \lambda\).

The first condition above is a bound on the probability, that key values mismatch. The second one guarantees for small \(\lambda\), that the key is approximately equidistributed and secure. The left hand side of the inequality in (8) of the above definition can be regarded as a quantum version of the so-called security index introduced in classical information theory [14]. For a pair \((K,Z)\) of classical random variables, the security index of \(K\) against \(Z\) is given by the expression

\[
S_{SID}(K|Z) := \log \text{supp}(p_{k}) - H(K) + I(K;Z).
\]

The security index is well-known as a useful criterion for quantifying equidistribution and the degree of decoupling from the eavesdropper (see e.g. [15] for more information). From the classical security index, also the above introduced quantum version stems its operational significance. If \((K,A,Z)\), is a tuple of random variables with \(K_{s}\) being the key random variable, \(A_{s}\) belonging to the public message of the protocol and \(Z_{s}\) the classical random variable obtained by measurement on the eavesdropper’s system, the second condition in (8) implies \(S_{SID}(K_{s},A_{s}|Z_{s}) \leq \lambda\), because

\[
S_{SID}(K_{s}|A_{s},Z_{s}) = \log M - H(K_{s}) + I(K_{s};Z_{s},A_{s}) \leq \log M - H(K_{s}) + I(K_{s};E^{n}A_{s} \rho_{\Lambda_{E}^{n},s}) \leq \lambda
\]

holds by the Holevo bound [21].
Remark 4. In their work [16], Devetak and Winter imposed a slightly stronger security criterion to be satisfied instead of the one used in Definition 3. The authors of the present paper feel, that in general the security criterion therein will hardly be satisfied by universal secret-key distillation protocols in general. The results in the subsequent sections applied on the case of a compound source \( I \) with \( |I| = 1 \) show, that imposing the weaker criteria in Definition 3 does not lead to higher capacities than in [16] in case of perfectly known source statistics.

Definition 5. A nonnegative number \( R \) is called an achievable secret-key distillation rate for \( I \) (with SMI), if for each \( \epsilon > 0, \delta > 0 \) exist numbers \( n_0 \) and \( 0 < R_\epsilon < \infty \) such for each possible marginal state \( \rho_A \) there is an \((n,M,L,\epsilon)\) secret-key distillation protocol for \( I \) (with SMI), such that

\[
M \geq \exp(n(R-\delta)), \text{ and } L \leq \exp(nR_\epsilon)
\]

for each \( n > n_0 \). We define the forward secret-key capacity of \( I \) with SMI by

\[
K_{\rightarrow,SMI}(I) := \sup \{ R \geq 0 : \text{ \( R \) is an achievable secret key distillation rate for \( I \) with SMI} \}.
\]

and the forward secret-key capacity of \( I \) without SMI by

\[
K_{\rightarrow}(I) := \sup \{ R \geq 0 : \text{ \( R \) is an achievable secret key distillation rate for \( I \) without SMI} \}.
\]

What we define next, is a regularity condition on generating sets of compound cqq sources.

Definition 6 (Regularity Condition). We call a set \( I \subset S_{cqq}(\mathcal{H}_{ABE}) \)

- \( \epsilon \)-regular, if there is a \( \delta > 0 \) such that the implication

\[
\| p - q \| < \delta \Rightarrow d_H(I^{AB}_p, I^{AB}_q) + d_H(I^{AE}_p, I^{AE}_q) < \epsilon
\]

holds for each pair \( p, q \in \mathcal{P}_I \), where \( d_H \) denotes the Hausdorff distance generated by the trace norm distance on the underlying matrix spaces.

- regular, if \( I \) is \( \epsilon \)-regular for each \( \epsilon > 0 \).

Remark 7. The regularity condition given above aims to cover an as large as possible class of reasonable sets of cqq density matrices under the condition that general protocol constructions are successful. The reader interested in detailed discussion of this condition is referred to Section [VI]. In Section [VI.B] we show using results from the theory of set-valued functions, that the above condition of regularity can be weakened somewhat to include even a larger class of cqq sources.

The following two theorems state the main results proven in this paper.

Theorem 8. Let \( I \) be a regular set of cqq density matrices on \( \mathcal{H}_{ABE} \). It holds

\[
K_{\rightarrow}(I) = \lim_{k \to \infty} \frac{1}{k} K^{(1)}_{\rightarrow}(I^A_k),
\]

where for a set \( \mathcal{A} := \{ \sum_{y \in \mathcal{Y}} p(y) |y\rangle \langle y| \otimes \sigma_y \} \) on some space,

\[
K^{(1)}_{\rightarrow}(\mathcal{A}) := \inf_{p \in \mathcal{P}_A} \sup_{T, \sigma_T} \left( \inf_{T \in \mathcal{T}} I(U; B | T, \sigma_T) - \sup_{\sigma \in \mathcal{A}_p} I(U; E | T, \sigma) \right).
\]

The supremum above is over all markov chains \( T \leftarrow U \leftarrow Y_p \) resulting from application of markov transition matrices \( P_{T | U}, P_{U | Y} \) on \( p \) for each \( p \in \mathcal{P}_p \), and

\[
\sigma_{T | U} = \sum_{y \in \mathcal{Y}} \sum_{t \in \mathcal{T}} \sum_{u \in \mathcal{U}} P_{T | U}(t | u) P_{U | Y}(u | y) p(y) | t \rangle \langle t| \otimes | u \rangle \langle u| \otimes \sigma_y
\]

for given transition matrices \( P_{T | U}, P_{U | Y} \) and

\[
\sigma = \sum_{y \in \mathcal{Y}} p(y) | y \rangle \langle y| \otimes \sigma_y.
\]
Theorem 9. Let $I$ be a set of cqq density matrices on $H_{ABE}$. It holds

$$K_{\to,\text{SMI}}(I) = \lim_{k\to\infty} \frac{1}{k} K^{(1)}(I^\otimes k),$$

(12)

where the function $K^{(1)}$ is defined in the preceding theorem.

Notice that the inequality

$$K_{\to}(I) \leq K_{\to,\text{SMI}}(I).$$

(13)

holds for each $I$, which can be directly observed by the definitions of achievable rates given above. The next section is devoted to giving a full argument which justifies the claim of Theorem 8. We now give short outline of the proof. In a sequence of Propositions with increasing level of approximation we prepare ourselves for proving the assertion

$$K_{\to}(I) \geq K^{(1)}(I)$$

(14)

in Proposition 15. For this reason, we first design suitable protocols of suboptimal rate for the special case of a source parameterized by a full cartesian product of probability distributions and cq channels. We improve this in Proposition 14, where we derive protocols suitable for the same type of source, but allowing preprocessing of the source by a fixed Markov chain for optimization. Finally, this result is combined with a fine-grained approximation of an arbitrary regular source by a number of sources of type subject to the mentioned propositions. The proof of achievability (i.e. the lower bound in (12)) follows almost immediately from (14), since we show, that regularity of $I$ implies, for each $k \in \mathbb{N}$, regularity of the set $I^\otimes k$ of all $k$-fold tensor extensions for states from $I$.

In Section IV we give a full proof of Theorem 9. The achievability part therein is derived also from the results gathered in Section V. The protocol construction used for proving achievability in Theorem 8 can be employed in case of SMI. To do so we use a certain type of finite covering in Hausdorff space to decompose a general set $I$ into a finite family of regular sets. Moreover, we provide a proof to the converse assertion.

The reader may ask, whether Theorem 8 may hold also without assumption of regularity. We give a negative answer to this question in Section VI, where an example of a cqq set of density matrices with $K_{\to}(I) < K_{\to,\text{SMI}}(I)$ is established.

IV. SECRET-KEY DISTILLATION WITHOUT STATE KNOWLEDGE

In this chapter, we give a detailed argument to prove Theorem 8. The first assertion, we prove is on a restricted type of cq density matrices. Assume $Q \subset \mathcal{P}(Y)$ be a set of probability distributions and $V \subset \mathcal{CQ}(Y,K_{BE})$ be a set of cq-channels. We define

$$\rho_{(p,V)} := \sum_{y \in Y} p(y) |y\rangle\langle y| \otimes V(y) \quad (p \in Q, V \in V),$$

and the set

$$J := \{\rho_{(p,V)}\}_{(p,V) \in Q \times V}. \quad (15)$$

We set for each $V \in \mathcal{V}$, $V_B = \text{tr}_{K_B} \circ V$, and $V_E = \text{tr}_{K_E} \circ V$.

Proposition 10. Let $J$ be the source defined in (15), and $\delta > 0$. There is a constant $C_1 > 0$ and a number $n_0$ such that for each $n > n_0$ there is an $(n,M,L,\mu)$ forward secret-key distillation protocol with

$$\frac{1}{n} \log M \geq \inf_{q \in Q} \left( \inf_{V \in V} \chi(q,V_B) \right) - \delta$$

$$\frac{1}{n} \log L \leq \sup_{p \in P_j} \left( \sup_{V \in V} \chi(p,V_B) \right) + \delta$$

$$\mu \leq 2^{-1/\psi c_1}.$$
Within the proof of Proposition 10, we will use some auxiliary results, we introduce first. The following lemma states, for given compound memoryless classical-quantum channel (DMcqC) existence of random codes being of constant composition (i.e. all codewords having the very same type) and equidistributed over the typical sets. The assertion is a direct consequence of coding results stated in Appendix A, where also the basic definitions regarding codes for message transmission over compound DMcq channels can be found.

**Lemma 11.** Let $V \subset CQ(\mathcal{X}, \mathcal{K})$ be a set of cq channels. For each $\gamma > 0$, there is a number $n_1(\gamma, V)$ such that for each $n > n_1(\gamma)$ and each type $\lambda \in T(n, \mathcal{X})$ the following assertion is true. It exists a random $(n, M_\lambda)$-code

$$C(U) := (U_m, D_m(U))_{m=1}^{M_\lambda}$$

fulfilling the following three properties

1. $U = (U_1, \ldots, U_{M_\lambda})$ is an i.i.d. sequence of random variables, such that $U_m$ is equidistributed on $T^n_\lambda$ for each $m \in [M_\lambda]$.

2. $M_\lambda \geq \exp\left(n \left(\inf_{V \in \mathcal{V}} \chi(\lambda, V) - \gamma\right)\right)$

3. $E \left[\sup_{V \in \mathcal{V}} T(C(U), V^\otimes n)\right] \leq 2^{-\lambda \psi_\gamma},$

with a constant $\ell(\gamma, V) > 0$ (independent of $\lambda$).

**Proof.** We need only consider types with

$$\inf_{V \in \mathcal{V}} \chi(\lambda, V) - \gamma > 0,$$  \hspace{1cm} (16)

since for all other types, the bounds in the assertion of the lemma can be satisfied by trivial coding. Setting $\delta := \frac{\gamma}{2}$ in Proposition 30 in Appendix A, ensures us, that for each large enough blocklength $n$ and each type $\lambda \in T(n, \mathcal{X})$ the hypothesis of Proposition 31 is fulfilled with an $M_\lambda$ which fulfills

$$M_\lambda \geq \exp\left(n \left(\inf_{V \in \mathcal{V}} \chi(\lambda, V) - \frac{\gamma}{2}\right)\right) > 2^{n \frac{\gamma}{4}},$$  \hspace{1cm} (17)

and $\mu \leq 2^{-\lambda \psi_\gamma(\frac{\gamma}{2})}$. Note, that the rightmost inequality in (17) is satisfied because we only consider types, which fulfill the condition in (16). Setting $\delta := \frac{\gamma}{2}$, we conclude with Proposition 31 that we find, for large enough $n \in \mathbb{N}$ and random $(n, M_\lambda)$ message transmission code fulfilling the properties demanded. \hfill \Box

The following matrix-concentration inequality results from the powerful matrix Chernoff bound \cite{2} and was proven in \cite{16}.

**Proposition 12** (\cite{16}, Prop. 2.4). Let $n \in \mathbb{N}$, $W \in CQ(\mathcal{X}, \mathcal{K})$, $\lambda \in T(k, \mathcal{X})$, $U := (U_1, \ldots, U_M)$ an i.i.d. sequence of random variables generically equidistributed on $T^n_\lambda$, and

$$\sigma_{n,\lambda}(W) := \frac{1}{|T^n_\lambda|} \sum_{x^n \in T^n_\lambda} W^\otimes n(x^n).$$

For each $\epsilon, \delta > 0$, there is a number $k := k(\epsilon, \delta)$, such that if $n > k$, then

$$\Pr\left(\left\|\frac{1}{M} \sum_{m=1}^{M} W^\otimes n(U_m) - \sigma_{n,\lambda}(W)\right\| \geq \epsilon\right) \leq 2 \cdot \dim K^n \cdot \exp\left(-M \Delta_n \cdot \epsilon\right)$$

holds with

$$\Delta_n := -\frac{1}{288 \ln 2} \cdot \exp\left(-n(\chi(\lambda, W) - \delta)\right).$$

The next assertion will help us to approximate the set $V$ assumed in Proposition 10 by a finite subset in a suitable way.
Lemma 13. Let \( \mathcal{V} \subset \mathcal{CQ}(\mathcal{X}, \mathcal{K}) \) be a set of classical quantum channels. For each \( \alpha \in (0, \frac{1}{e}) \) exists a subset \( \mathcal{V}_\alpha \subset \mathcal{V} \), which fulfills the following three conditions.

1. \( |\mathcal{V}_\alpha| \leq \left( \frac{2}{\alpha} \right)^{|X| \dim \mathcal{K}^2} \)

2. Given any \( n \in \mathbb{N} \), to each \( V \in \mathcal{V} \) exists a \( W \in \mathcal{V}_\alpha \), such that
   \[ \|V \otimes^n(x^n) - W \otimes^n(x^n)\|_1 \leq 2n\alpha \]
   holds for each \( x^n \in \mathcal{X}^n \).

3. For each \( p \in \mathcal{P}(\mathcal{X}) \), it holds
   \[ \left| \min_{W \in \mathcal{V}_\alpha} \chi(p, W) - \inf_{V \in \mathcal{V}} \chi(p, V) \right| \leq 2\alpha \log \frac{\dim \mathcal{K}}{2\pi}. \]

Proof of Proposition 10. Set
   \[ R := \inf_{q \in \mathcal{Q}} \left( \inf_{V \in \mathcal{V}} \chi(q, V_B) - \sup_{V \in \mathcal{V}} \chi(q, V_E) \right), \]
and let \( \delta > 0 \) be a number small enough for fulfilling \( R - \delta > 0 \), otherwise, there is nothing left to prove. Let \( \frac{1}{n} > \eta > 0 \), be fixed and small enough such that the inequality
   \[ 12\eta + \log \dim \mathcal{K}_{BE} + 4h(2\eta) \leq \frac{\delta}{16}. \]
   is valid. Let \( n \in \mathbb{N} \) be large enough to simultaneously satisfy
   \[ \frac{1}{n} \leq \frac{1}{16}\delta \text{ and } \frac{1}{n} \leq 2\eta. \]
Define
\[ \mathcal{T}_n := \mathcal{T}(n, Y) \cap \mathcal{Q}_\eta, \]
where \( \mathcal{Q}_\eta := \{ q \in \mathcal{P}(Y) : \exists p \in \mathcal{Q} : \|p - q\|_1 \leq \eta \} \) is the \( \eta \)-blowup of \( \mathcal{Q} \) regarding the variational distance. We set for each probability distribution \( q \in \mathcal{P}(Y) \)

\[ \chi_{B,q} := \inf_{V \in \mathcal{V}} \chi(q, V_B), \]
\[ \chi_{E,q} := \sup_{V \in \mathcal{V}} \chi(q, V_E), \]
\[ \chi_q := \chi_{B,q} - \chi_{E,q}, \]
\[ \chi_n := \min_{\lambda \in \mathcal{T}_n} \chi_\lambda. \]

Our choice of \( \eta \) and \( n \) implies
\[ d_H(\mathcal{T}_n, \mathcal{Q}) \leq d_H(\mathcal{T}_n, \mathcal{Q}_\eta) + d_H(\mathcal{Q}_\eta, \mathcal{Q}) \leq \frac{1}{2n} + \eta \leq 2\eta, \]
where the first inequality above is the triangle inequality for the Hausdorff distance, and the second is by (19). From (20), and (18) together with twofold application of Lemma 35 we infer
\[ |\chi_n - R| \leq \frac{\delta}{16}. \]
Set, for each \( \lambda \in \mathcal{T}_n \)

\[ L_\lambda := \left| \exp\left(n(H(\lambda) - \chi_{B,\lambda} + \frac{3}{4}\delta)\right) \right|, \]
\[ S_\lambda := \left| \exp\left(n(\chi_{E,\lambda} + \chi_\lambda - \chi_n + \frac{1}{4}\delta)\right) \right|, \]
and
\[ M := \left| \exp(n(R - \delta)) \right|. \]
The above definitions, together with (21) and the second inequality of (19) the bounds
\[
M \cdot S_\lambda \leq \exp\left(n(\log M - \frac{7}{36}\delta)\right),
\]
\[
M \cdot L_\lambda \leq \exp\left(n(H(\lambda) - \log M - \frac{7}{36}\delta)\right), \text{ and}
\]
\[
\Gamma_\lambda := \frac{L_\lambda \cdot S_\lambda \cdot M}{|T_\lambda^n|} \geq 2^{-\frac{3}{8}n}.
\]
are valid. The strategy for the rest of the proof is the following. We will in a first step, generate a class of one-way-secret key distribution protocols for \(J\), and then show, that with high probability, the protocols meet the properties demanded.

Define for each \(\lambda \in T_n\), a random matrix
\[
U^{(\lambda)} := \{U^{(\lambda)}_{lms}(1, n) \in [L_\lambda] \times [M] \times [S_\lambda]\}
\]
with all entries being independent and generically equidistributed on \(T_\lambda^n\). We collect the matrices defined above in an independent family
\[
U := \{U^{(\lambda)}\}_{\lambda \in T_n}.
\]

Define, for each \(y^n \in \mathcal{Y}_n\), \(\lambda \in T_n\) a random set
\[
A_\lambda(y^n, U) := \{\lambda, l, m, s : U_{lms}^{(\lambda)} = y^n\}.
\]

Obviously, the sets defined above fulfill for each outcome \(u\) of \(U, \lambda \in T_n\) the following relations
\[
A_\lambda(y^n, U) = \emptyset \quad (y^n \notin T_\lambda^n),
\]
\[
A_\lambda(y^n, U) \cap A_\lambda(z^n, U) = \emptyset \quad (y^n \neq z^n),
\]
and
\[
\bigcup_{y^n \in T_\lambda^n} A_\lambda(y^n, U) = [\lambda] \times [L_\lambda] \times [M] \times [S_\lambda].
\]
We regard, for each \(\lambda \in T_n\) and \(l \in [L_\lambda]\),
\[
U_{l,l} := \{U_{lms}^{(\lambda)}(1, n) \in [M] \times [S_\lambda]\}
\]
as a random i.i.d. constant composition codebook of size \(M \cdot S_\lambda\) with codewords equidistributed over \(T_\lambda^n\). Since we have the bound in (22), we know from Lemma [11], that there is a random \((n, M \cdot S_\lambda)\) constant composition code
\[
C_{\lambda,l}(U_{l,l}) := \{U_{lms}^{(\lambda)}, D_{lms}^{(\lambda)}(1, n) \in [M] \times [S_\lambda]\}
\]
for the compound DMCqC generated by \(\mathcal{V}_B := \{V_B : V \in \mathcal{V}\}\) which has expected average error bounded
\[
\mathbb{E}\left[\sup_{V \in \mathcal{V}_A} \tilde{\mathbb{P}}(C_{\lambda,l}(U_{l,l}), V^\otimes n)\right] \leq 2^{-\left(\frac{1}{8}\hat{\delta}\right)} \leq \beta_0
\]
(26)
with a strictly positive constant \(\hat{\delta}\) independent of \(\lambda\). Define, for \(\beta_3 > 0, \lambda \in T_n\) a random set
\[
B_\lambda(U, \beta_3) := \left\{l \in [L_\lambda] : \max_{V \in \mathcal{V}} \tilde{\mathbb{P}}(C_{\lambda,l}(U_{l,l}), V^\otimes n) < \beta_3\right\},
\]
which collects all indices \(l \in [L_\lambda]\), such that \(C_{\lambda,l}\) is \(\beta_3\)-good regarding the average error criterion. Define a random stochastic matrix
\[
T_U : \mathcal{X} \to \tilde{\mathbb{P}}(\mathcal{T}(n, \mathcal{Y}) \times [L_\lambda] \times [M] \times [S_\lambda])
\]
with entries
\[
T_U(\lambda, l, m, s | y^n) := \begin{cases} |A_\lambda(y^n, U)|^{-1} & \text{if } (\lambda, l, m, s) \in A_\lambda(y^n, U) \\ 0 & \text{otherwise} \end{cases}
\]
for each \( \lambda \in T_n \). The values of \( T_\ell(\lambda, l, m, s|y^n) \) with \( \lambda \) being not in \( T_n \) will be of no special interest for us, so they may be defined in any consistent way. Let, for each \( \lambda \in T_n \), \( V \in \mathcal{V} \)

\[
\sigma_\lambda(V) := \frac{1}{|T_\lambda|} \sum_{y^n \in T_\lambda} V_\ell^n(y^n).
\]

Note, that

\[
\sigma_\lambda(V) = \mathbb{E}\left[V_\ell^n(U^n_{lms})\right]
\]

holds for all \((l, m, s) \in [L_\lambda] \times [M] \times [S_\lambda]\). Define, for \( \lambda \in T_n \), \( \beta_1, \beta_2, \beta_3 > 0 \) and each outcome \( u \) of \( U \), the following sets.

\[
C^{(1)}_\lambda(\beta_1) := \left\{ u: \forall y^n \in T^n_\lambda : (1 - \beta_1) \Gamma_\lambda \leq |A_\lambda(y^n, u)| \leq (1 + \beta_1) \Gamma_\lambda \right\}
\]

\[
C^{(2)}_\lambda(\beta_2) := \left\{ u: \forall (l, m) \in [L_\lambda] \times [M], V \in \mathcal{V} : \left\| \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} V_\ell^n(u_{lms}) - \sigma_\lambda(V) \right\|_1 \leq \beta_2 \right\}
\]

\[
C^{(3)}_\lambda(\beta_3) := \left\{ u: |B_\lambda(u, \beta_3)| \geq (1 - 2\beta_3) \Gamma_\lambda \right\}
\]

\[
A := \bigcap_{\lambda \in T_n} \bigcap_{i=1}^{3} C^{(i)}_\lambda(\beta_i).
\]

Eventually, we will show, that if an outcome \( u \) of \( U \) is an element of \( A \), it generates a suitable protocol for our needs. First we make sure, that for the right choice of parameters and each large enough blocklength, \( A \) is actually nonempty, which we do by actually bounding the r.h.s. of

\[
\Pr(A^c) \leq \sum_{i=1}^{3} \sum_{\lambda \in T_n} \Pr(C^{(i)}_\lambda(\beta_i))^c
\]

away from one. In the following we separately derive bound on each of the summands on the right hand side of (28). Let \( \lambda \) be a type from \( T_n \). Note, that

\[
|A_\lambda(y^n, u)| = \sum_{l=1}^{L_\lambda} \sum_{m=1}^{M} \sum_{s=1}^{S_\lambda} \mathbb{P}_{y^n}(u_{lms})
\]

holds, where \( \mathbb{P} \) is the indicator function, therefore,

\[
\mathbb{E}[|A_\lambda(y^n, u)|] = \frac{L_\lambda M S_\lambda}{|T_\lambda|} = \Gamma_\lambda \geq 2^{-n \frac{1}{8} \delta}
\]

where the rightmost inequality above results from (24). We infer

\[
\Pr(C^{(1)}_\lambda(\beta_1))^c = \sum_{y^n} \Pr((u: |A_\lambda(y^n, u)| \notin ((1 - \beta_1) \Gamma_\lambda, (1 + \beta_1) \Gamma_\lambda))
\]

\[
\leq 2|Y|^n \cdot \exp\left(-\Gamma_\lambda \beta_1^2 / (2 \ln 2)\right)
\]

\[
\leq 2 \cdot \exp(-2^n \frac{\delta}{8})
\]

for large enough blocklength \( n \), where the first inequality above is by Chernov-bounding with Proposition 32 and the second is by (24) together with the choice

\[
\beta_1 = 2^{-n \frac{\delta}{8}}.
\]

To bound the summands with \( i = 2 \), we choose an approximating set \( \hat{V}_n \) for \( V \) according to Lemma 13 with parameter

\[
\alpha := 2^{-4\sqrt{n}/(16\delta)(\dim K_{\hat{V}_n})}.
\]
which is possible with cardinality
\[ |\mathcal{V}_n| \leq 2^{-\frac{1}{2} \Psi n^\frac{3}{4}} \]
as long as \( n \) is large enough. Let for given \( V \in \mathcal{V}, W \in \mathcal{V}_n \) be a channel, such that \( \|V(x) - W(x)\| \leq 2\alpha \). It holds for each \( \lambda \in T_n, l \in [L_\lambda], m \in [M] \)
\[
\| \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} V_E^{\otimes n} (y^n) - \sigma_\lambda (V) \|_1 \leq \| \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} W_E^{\otimes n} (y^n) - \sigma_\lambda (W) \|_1 + \| \sigma_\lambda (V) - \sigma_\lambda (W) \|_1
\]
\[
\leq \| \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} W_E^{\otimes n} (y^n) - \sigma_\lambda (W) \|_1 + 2\alpha. \tag{30}
\]
If we now choose
\[ \beta_2 = 4\alpha \tag{31} \]
We can bound
\[
\Pr \left( C^{(2)}_\lambda (\beta_2)^c \right) = \Pr \left( \exists (l,m), V \in \mathcal{V} : \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} V_E^{\otimes n} (u_{lms}^1) - \sigma_\lambda (V) \|_1 > \beta_2 \right) \leq \Pr \left( \exists (l,m), V \in \mathcal{V}_n : \frac{1}{S_\lambda} \sum_{s=1}^{S_\lambda} V_E^{\otimes n} (u_{lms}^1) - \sigma_\lambda (V) \|_1 > \frac{\beta_2}{2} \right) \leq 4 \cdot L_\lambda M |\mathcal{V}_n| \cdot (\dim \mathcal{K}_E)^n \exp \left(-S_\lambda \cdot 2^{-n\left(4 \cdot \frac{1}{2} \frac{\beta_2}{576 \ln 2} \right)} \right) \leq \exp \left(2^{-\frac{n}{4}} \right)
\]
where the first inequality is by (30), the second by application of Proposition 12, and the last inequality holds for each large enough blocklength. At last,
\[
\Pr \left( C^{(3)}_\lambda (\beta_3)^c \right) = \Pr \left( \frac{1}{L_\lambda} \sum_{l \in L_\lambda} \mathbb{I}_{B_3 (\beta_3)} (l) \geq 2\beta_3 \right) \leq \frac{\mathbb{E} \left[ \mathbb{I}_{B_3 (\beta_3)} (l) \right]}{2\beta_3} \leq \frac{\beta_0}{2\beta_3} \tag{32}
\]
The first inequality above is Markov’s inequality applied, the second can be justified as follows. It holds
\[
\mathbb{E} \left[ \mathbb{I}_{B_3 (\beta_3)} (l) \right] = \Pr \left( B_3 (\mathcal{U}, \beta_3) (l)^c \right) = \Pr \left( \max_{V \in \mathcal{V}} \mathcal{C}_{\lambda,l} (V_E^{\otimes n}) \geq \beta_3 \right) \leq \frac{\mathbb{E} \left[ \max_{V \in \mathcal{V}} \mathcal{C}_{\lambda,l} (V_E^{\otimes n}) \right]}{\beta_3} \leq \frac{\beta_0}{\beta_3} \tag{33}
\]
The first inequality above is again by Markov-bounding. The second is by (26). By combination of the estimate in (32), and the choice
\[ \beta_3 = 2^{-\frac{1}{4} \Psi n^c/4} \tag{34} \]
we yield, for large enough blocklength
\[ \Pr\left(C_\lambda^{(3)}(\beta_3)^c\right) \leq 2^{-\sqrt{\eta n}}. \] (35)

Combining all the bounds derived above with (28), choosing the blocklength large enough, we arrive with our choice of the parameters \(\beta_i, 1 \leq i \leq 3\) at
\[ \Pr(A^c) \leq |\mathcal{T}_n|2^{-\sqrt{\eta n}} \]
with a strictly positive constant \(c\) if \(n\) is large enough. Since \(|\mathcal{T}_n| \leq |\mathcal{T}(n, \mathcal{Y})| \leq (n+1)^3\), we have for each large enough blocklength
\[ \Pr(A^c) \leq \frac{1}{2}, \]
which implies, that \(A\) is nonempty. Define
\[ L := \max_{\lambda \in \mathcal{T}_n} L_\lambda, \quad \text{and} \quad S := \max_{\lambda \in \mathcal{T}_u} S_\lambda. \]
We choose any \(u \in A\) and define a stochastic map
\[ T : \mathcal{Y}^n \to \mathcal{T}(n, \mathcal{Y}) \times [L] \times [M], \]
\[ y^n \mapsto T(\lambda, l, m | y^n) := \sum_{s=1}^{\tilde{S}} T_u(\lambda, l, m, s | y^n). \]
and
\[ D := (D_{lm}^1)_{(\lambda, l, m) \in \mathcal{T}(n, \mathcal{Y}) \times [L] \times [M]}, \]
where
\[ D_{lm}^1 := \sum_{s=1}^{\tilde{S}} D_{lm}^s \quad (\lambda, l, m) \in \mathcal{T}(n, \mathcal{Y}) \times [L] \times [M] \]
with \(D_{lm}^s\) being the decoding set with index \((m, s)\) from the code \(C_{\lambda,l}(u)\). Note that some entries of \(T_u\) as well as some of the decoding matrices are have been not defined yet (e.g. \(L > L_\lambda\) may occur for some \(\lambda\)), we populate the undefined entries of \(T_u\) with zeros, and add zero matrices, and add arbitrary but consistently, where decoding matrices are undefined.

To fit the above objects to the definition of a one-way secret key distillation protocol, we consider each public message as a tuple \(I = (\lambda, l)\). With the above definitions, \(D := (T, D)\) is an \((n, M, L)\) secret key distillation protocol with
\[ \frac{1}{n} \log M \geq R - \delta \] (36)
by definition of \(M\). It remains, to show, that actually the bound on the performance \(\mu\) stated is fulfilled. We fix an arbitrary member
\[ \rho_l = \sum_{y \in \mathcal{Y}} p(y) |y\rangle \langle y| \otimes V(y) \]
from \(J\). We first show, that
\[ \Pr(K_l \neq K_l') \leq \mu. \] (37)
holds. By construction, we have \(T(n, \mathcal{Y}) \subset Q_{\mathcal{Y}}\), which together with well known type-bounds implies
\[ p^n \left( \bigcup_{\lambda \in \mathcal{T}(n, \mathcal{Y}) \setminus \mathcal{T}_n} T_\lambda^n \right) \leq 2^{-n c\eta^2}. \]
with a universal constant \( c > 0 \). It holds
\[
\Pr(K_i \neq K'_i) = \sum_{y^n \in \mathcal{Y}^n} p^n(y^n) \cdot \Pr(K_i \neq K'_i \mid Y^n_i = y^n)
\leq \sum_{\lambda \in \mathcal{T}_n} \sum_{y^n \in \mathcal{T}_n^\lambda} p^n(y^n) \cdot \Pr(K_i \neq K'_i \mid Y^n_i = y^n) + 2^{-nc\eta^2}.
\tag{38}
\]

We upper-bound for each \( \lambda \in \mathcal{T}_n \) the corresponding summand on the r.h.s. of (38). For each \( y^n \in \mathcal{T}_n^\lambda \), we have
\[
p^n(y^n) \cdot \Pr(K_i \neq K'_i \mid Y^n_i = y^n) = \sum_{m=1}^{M} \sum_{m' \neq m} p^n(y^n) \cdot \Pr(K_i = m, K'_i = m' \mid Y^n_i = y^n)
= \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{l=1}^{L} p^n(y^n) \cdot \Pr(K_i = m, K'_i = m', A_i = (\lambda, l) \mid Y^n_i = y^n)
= \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{l=1}^{L} \sum_{s=1}^{S} \text{tr}(D_{lm^s}^{\lambda} V_B^{\otimes n}(y^n)) \cdot p^n(y^n) \cdot T(\lambda, l, m | y^n)
= \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{l=1}^{L} \sum_{s=1}^{S} \text{tr}(D_{lm^s}^{\lambda} V_B^{\otimes n}(y^n)) \cdot p^n(y^n) \cdot T_\lambda(\lambda, l, m, s | y^n).
\tag{39}
\]

On the r.h.s. of (39), only the summands survive, where \((\lambda, l, m, s)\) is in \( A_\lambda(y^n, u) \) by definition of \( T_\lambda \). For the nonzero summands, we can estimate
\[
T_\lambda(\lambda, l, m, s | y^n) = \frac{1}{|A_\lambda(y^n, u)|} \leq (1 - \beta_3)^{-1} \leq \frac{2}{L_A \cdot M \cdot S_\lambda}.
\tag{40}
\]

The left of the above inequalities is because \( u \in A \), the right holds, if \( n \) is large enough. We define the abbreviation \( A_\lambda(y^n) := A_\lambda(y^n, u) \). Counting only the nonzero summands, and using the estimate in (40), we yield
\[
p^n(y^n) \cdot \Pr(K_i \neq K'_i \mid Y^n_i = y^n) \leq \sum_{(l, m, s): (\lambda, l, m, s) \in A_\lambda(y^n)} \sum_{m' \neq m} \sum_{s=1}^{S} \text{tr}(D_{lm^s}^{\lambda} V_B^{\otimes n}(u^1_{lm^s})) \frac{2|T_\lambda^n| \cdot p^n(u^1_{lm^s})}{L_A S_\lambda M}
\leq \frac{2}{L_A M_1 S_1} \sum_{(l, m, s): (\lambda, l, m, s) \in A_\lambda(y^n)} \sum_{m' \neq m} \sum_{s=1}^{S} \text{tr}(D_{lm^s}^{\lambda} V_B^{\otimes n}(u^1_{lm^s}))
\]
where in the last inequality, we noted, since \( u^1_{lm^s} \) is of type \( \lambda \), \( |T_\lambda^n| \cdot p^n(u^1_{lm^s}) = p^n(T_\lambda^n) \leq 1 \) holds. Since, for each type \( \lambda \in \mathcal{T}_n \)
\[
\bigcup_{y^n \in \mathcal{T}_n^\lambda} A_\lambda(y^n) = \{ \lambda \} \times [L_A] \times [M] \times [S_\lambda]
\]
holds by construction (see (23)), we have (with some rearrangements of terms)
\[
\sum_{y^n \in \mathcal{T}_n^\lambda} p^n(y^n) \cdot \Pr(K_i \neq K'_i \mid Y^n_i = y^n) = \frac{2}{L_A} \sum_{l=1}^{L_A} \frac{1}{S_1 M_1} \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{s=1}^{S} \text{tr}(D_{lm^s}^{\lambda} V_B^{\otimes n}(u^1_{lm^s}))
\leq \frac{2}{L_A} \sum_{l=1}^{L_A} \beta_3 (C_{A_\lambda(u_{A_\lambda})}, V_B^{\otimes n})
\leq \frac{2}{L_A} (L_A \cdot \beta_3 + L_A \cdot 2\beta_3) = 6\beta_3.
\tag{41}
\]
The last inequality holds, because we have chosen our protocol in a way, that for at least a fraction of $1 - 2\beta_3$, the code $C^{(3)}(u_{\lambda})$ is $\beta_3$-good regarding the average error criterion (i.e. $u$ is in $A \subset C^{(3)}_1(\beta_3)$).

Collecting inequalities, we arrive at

$$\Pr(K \neq K') \leq (n + 1)^{3/4} \cdot 6\beta_3 + 2^{-n\eta} \leq 2^{-n\eta/8}$$

(42)

for large enough $n$ by (34). Next, we show, that the key is almost equidistributed. For each $\lambda \in T(n, \mathcal{Y})$, we consider the probability distribution $P_{K_{t,\lambda}}$ on $[M]$, given by

$$P_{K_{t,\lambda}}(m) := \frac{\Pr(K_{t} = m|Y^n = y^n)}{|T^n_{\lambda}|}$$

$$= \sum_{y^n \in T^n_{\lambda}} \sum_{l=1}^{L_{\lambda}} \sum_{s=1}^{S_{\lambda}} \frac{T_{\lambda}(\lambda, l, m, s|y^n)}{|T^n_{\lambda}|}$$

$$= \sum_{l=1}^{L_{\lambda}} \sum_{s=1}^{S_{\lambda}} T_{\lambda}(\lambda, l, m, s|u_{\lambda}|)$$

(43)

Using the properties of the protocol constructed together with (43), we arrive at

$$\frac{1}{1 + \beta_1} \frac{1}{M} \leq P_{K_{t,\lambda}}(m) \leq \frac{1}{1 - \beta_1} \frac{1}{M}$$

for each $\lambda \in \mathcal{T}_n$, from which we infer, that

$$||P_{K_{t,\lambda}} - \pi_{[M]}||_1 \leq 2\beta_1.$$ 

is true for all $\lambda \in \mathcal{T}_n$. We conclude

$$||P_{K_{t,\lambda}} - \pi_{[M]}||_1 \leq \sum_{\lambda \in \mathcal{T}_n} p^n(T^n_{\lambda}) \ ||P_{K_{t,\lambda}} - \pi_{[M]}||_1 \leq 2\beta_1 + 2^{-n\eta} \leq 3\beta_1,$$

where the last inequality holds if $n$ is large enough. This implies

$$H(K_{t}) \geq \log M - 3\beta_1 \log M - \frac{M}{3\beta_1} \geq \log M - \frac{\mu}{2}$$

if $n$ is large enough. It remains to bound $I(K; E^n, \Lambda, \rho_{AKE^n,t})$. We will actually show, that $\rho_{AKE^n,t}$ is close to a state $\gamma_t$ whith $I(K; E^n, \Lambda, \gamma_t) = 0$. Define

$$\gamma_t := \sum_{\lambda \in \mathcal{T}(n, \mathcal{Y})} p^n(T^n_{\lambda})|\lambda\rangle \langle \lambda| \otimes \gamma_{t,\lambda}$$

where set for each $\lambda \in \mathcal{T}(n, \mathcal{Y})$,

$$\gamma_{t,\lambda} := \frac{1}{L_{\lambda}} \cdot \frac{1}{M} \sum_{l=1}^{L_{\lambda}} \sum_{m=1}^{M} |l \otimes m\rangle \langle l \otimes m| \otimes \sigma_{\lambda}(V).$$

We write $\rho_{AKE^n,t}$ in the form

$$\rho_{AKE^n,t} = \sum_{\lambda \in \mathcal{T}(n, \mathcal{Y})} p^n(T^n_{\lambda})|\lambda\rangle \langle \lambda| \otimes \rho_{t,\lambda},$$

where we defined

$$\rho_{t,\lambda} := \sum_{y^n \in T^n_{\lambda}} \sum_{l=1}^{L_{\lambda}} \sum_{m=1}^{S_{\lambda}} \sum_{s=1}^{S_{\lambda}} T_{\lambda}(\lambda, l, m, s|y^n) |l \otimes m\rangle \langle l \otimes m| \otimes V^n_E(y^n).$$
We first consider $\lambda \in T_n$. Note, that if $(\lambda, l, m, s)$ is a member of $A_\lambda(u, y^n)$,
\[
\frac{1}{1 + \beta_1} \Gamma_{\lambda}^{-1} \leq T_u(\lambda, l, m, s|y^n) \leq \frac{1}{1 - \beta_1} \Gamma_{\lambda}^{-1},
\]
while being zero otherwise, which implies
\[
\sum_{y^n \in T_{\lambda}^n} \sum_{l=1}^{L_\lambda} \sum_{m=1}^{M} \sum_{s=1}^{S_\lambda} \left| \frac{T_u(\lambda, l, m, s|y^n)}{|T_{\lambda}^n|} - \frac{1}{L_\lambda M |T_{\lambda}^n| S_\lambda} \right| \leq 2 \beta_1.
\]
Also, we know, that for each $l \in L_\lambda$, $s \in S_\lambda$
\[
\left\| \frac{1}{S_\lambda} \sum_{n=1}^{S_\lambda} \rho_{E}^{\otimes n}(y^n) \right\|_1 \leq 2 \beta_2.
\]
We define
\[
\tau := \sum_{y^n \in T_{\lambda}^n} \sum_{l=1}^{L_\lambda} \sum_{m=1}^{M} \sum_{s=1}^{S_\lambda} \frac{1}{L_\lambda M S_\lambda |T_{\lambda}^n|} \left| l \otimes m \right\rangle \left\langle l \otimes m \right| \otimes \rho_{E}^{\otimes n}(y^n),
\]
and obtain
\[
\| \hat{\rho}_{t,\lambda} - \gamma_{t,\lambda} \|_1 \leq \| \hat{\rho}_{t,\lambda} - \tau \|_1 + \| \tau - \gamma_{t,\lambda} \|_1
\]
\[
\leq \sum_{y^n \in T_{\lambda}^n} \sum_{l=1}^{L_\lambda} \sum_{m=1}^{M} \sum_{s=1}^{S_\lambda} \left| \frac{T_u(\lambda, l, m, s|y^n)}{|T_{\lambda}^n|} - \frac{1}{L_\lambda M |T_{\lambda}^n| S_\lambda} \right| + \sum_{y^n \in T_{\lambda}^n} \left\| \sum_{n=1}^{S_\lambda} \rho_{E}^{\otimes n}(y^n) - \sigma_{t}(V) \right\|_1
\]
\[
\leq 2(\beta_1 + \beta_2).
\]
Therefore,
\[
\| \hat{\rho}_{KAE^n,t} - \gamma_{t} \|_1 \leq \sum_{\lambda \in T_{(n, y)}} p^n(T_{\lambda}^n) \| \hat{\rho}_{t,\lambda} - \gamma_{t,\lambda} \|_1
\]
\[
\leq 2(\beta_1 + \beta_2) + 2^{-n \eta^2}
\]
\[
\leq \frac{\mu}{12}
\]
By using the well-known Alicki-Fannes type bound for the quantum mutual information, we infer
\[
I(K; \Lambda, E^n, \rho_{KAE^n,t}) \leq I(K; \Lambda, E^n, \gamma_{t}) + \frac{1}{2} \mu \log(L \cdot \dim K_{E}^{\otimes n}) + h(\frac{\mu}{12}) \leq \mu,
\]
where the last inequality is by the fact, that $\gamma_{t}$ is an uncorrelated state, together with a large enough choice of $n$.

Next, we prove an achievability result for the same simple kind of compound source as in the previous proposition, but the lower bound on the key rate derived including possible preprocessing of the source outputs for the sender by Markov chains. For each set $A$ of probability distributions on $\mathcal{Y}$, we denote its diameter (regarding the variational distance) by
\[
\text{diam}(A) := \sup \{ \| q - q' \|_1 : q, q' \in A \}
\]

**Proposition 14.** Let $\mathcal{P} \subseteq \mathcal{P}(\mathcal{Y})$ a set of probability distributions, $\text{diam}(\mathcal{P}) \leq \Delta$, $\mathcal{V} \subseteq \mathcal{CQ}(\mathcal{Y}, K_{BE})$, and $\mathcal{U}, T$ finite alphabets. Define
\[
\mathcal{I} := \left\{ \rho_{(p, \mathcal{V})} := \sum_{y \in \mathcal{Y}} p(y) |y\rangle \langle y| \otimes V(y) \right\}_{(p, \mathcal{V}) \in \mathcal{P} \times \mathcal{V}}.
\]
For each $P_{U|V} : U \to \mathcal{P}(T)$, $P_{U|Y} : Y \to \mathcal{P}(U)$ stochastic matrices, and $\delta > 0$, there is a number $n_0$, such that for each $n > n_0$ we find an $(n, M, L, \mu)$-secret-key distillation protocol $\hat{T}$ which fulfills

$$\mu \leq 2^{-\frac{1}{\sqrt{c_2}}},$$

$$\frac{1}{n} \log L \leq \sup_{p \in \mathcal{P}} \inf_{p, \rho \in \mathcal{P}} S(U|BT, \rho) + \log |T|,$$

$$\frac{1}{n} \log M \geq \inf_{p \in \mathcal{P}} \bigg( \inf_{p, \tilde{\rho} \in \mathcal{P}} I(U; B|T, \tilde{\rho}) - \sup_{p \in \mathcal{P}} I(U; E|T, \tilde{\rho}) \bigg) - \delta - 12 \Delta \log |U| - 4h(\Lambda), \quad (44)$$

where $h(x) = -x \log x - (1 - x) \log(1 - x)$, $(x \in (0, 1))$ is the binary entropy, and $c_2$ is a strictly positive constant.

We used the definition

$$\hat{\rho} := \sum_{t \in T} \sum_{w \in U} \sum_{y \in Y} P_{U|Y}(t|u)p_{U|Y}(u|y)p(y) \ |u\rangle \langle t| \otimes V(y)$$

for each state

$$\rho = \sum_{y \in Y} p(y) \ |y\rangle \langle y| \otimes V(y) \quad (p \in \mathcal{P}, V \in \mathcal{V}).$$

Proof. For the proof, we define a set of effective cqq density matrices $\hat{I}$ on which we apply Proposition 11 from which we derive existence of a certain forward secret-key distillation protocol for $\hat{I}$. Afterwards, we show, that this protocol can be modified to a forward secret-key distillation protocol for $I$ which has the stated properties. Define, for each $p \in \mathcal{P}$ a probability distribution $q_p \in \mathcal{P}(U)$ by

$$q_p(u) := \sum_{y \in Y} P_{U|Y}(u|y)p(y) \quad (u \in U),$$

a stochastic matrix $W_p : U \to \mathcal{P}(Y)$ by

$$W_p(y|u) := \begin{cases} \frac{p(y)p_{U|Y}(u|y)}{q_p(u)} & \text{if } q_p(u) > 0 \\ \frac{1}{q_p(u)} & \text{otherwise}, \end{cases}$$

and a classical-quantum channel $\hat{V}_p : U \to S(\mathcal{K}_{BE})$ by

$$\hat{V}_p(u) := \sum_{y \in Y} W_p(u|y)V(y) \quad (V \in \mathcal{V}).$$

Moreover, we define, introducing spaces $\mathcal{K}_{BE} = \mathcal{K}_E = \mathbb{C}^{|T|}$ a classical-quantum channel $\hat{V} : U \to S(\mathcal{K}_{BE} \otimes \mathcal{K}_{E'})$ by

$$\hat{V}(u) := \sum_{y \in Y} P_{U|Y}(t|u) \ |t\rangle \langle t| \otimes \hat{V}_p(u) \otimes V(u). \quad (45)$$

Define for each $(p, p', V) \in \mathcal{P} \times \mathcal{P} \times \mathcal{V}$ a state

$$\hat{\rho}_{(p, p', V)} := \sum_{u \in U} q_p(u) \ |u\rangle \langle u| \otimes \hat{V}_p(u) \otimes V(u).$$

We define a set of classical-quantum channels $\hat{V} := \{\hat{V}_p \otimes \hat{V} : p' \in \mathcal{P}, V \in \mathcal{V}\}$ and a set

$$\hat{I} := \{\hat{\rho}_{(p, p', V)} : p, p' \in \mathcal{P}, V \in \mathcal{V}\} \subset S_{cqq}(\mathbb{C}^{|T|} \otimes \mathcal{K}_{BE} \otimes \mathcal{K}_{E'})$$

of cqq density matrices. Note, that $\hat{I}$ meets the specifications of Proposition 11 (the states in $\hat{I}$ are parametrized by $\mathcal{P} \times \hat{V}$). We apply Proposition 11 on $\hat{I}$ and infer in case of sufficiently large blocklength existence of an $(n, M, L, \mu)$ secret key distillation protocol $\hat{D} = (\hat{T}, \hat{D})$ for $\hat{I}$ with a stochastic matrix

$$\hat{T} : U \to [M] \times [\hat{L}]$$
and
\[ D := \{ \hat{D}_{lm}(l,m) \mid (l,m) \in [L] \times [M] \} \subset \mathcal{L}(K_{BB}^{\otimes n}) \]
being a POVM such that the key rate is lower-bounded by
\[
\frac{1}{n} \log M \geq \inf_{p \in P} \left( \inf_{(p',p,V) \in [P] \times [P] \times [V]} \chi(q_p, \hat{V}_{B,p'} \otimes \hat{V}_B) - \sup_{(p',p,V) \in [P] \times [P] \times [V]} \chi(q_p, \hat{V}_{E,p'} \otimes \hat{V}_E) \right) - \delta
\]
holds, and for each \( s := (p,p',V) \) the inequalities
\[
\log M - H(\hat{K}_s) + I(\hat{K}; E^m \Lambda, \hat{\rho}_{K \Lambda} \otimes |E^m \rangle) \leq \mu
\]
being satisfied with \( \mu = 2^{-n \log c_2} \) with a constant \( c_2 > 0 \). Notice, that the cq-channel \( \hat{V}_B \) defined in (45) has classical structure in the sense, that all its output quantum states are diagonal in the orthogonal basis \( |t \rangle \). Consequently, we can assume, that for each \( l \in [L], m \in [M] \) the corresponding effect \( \hat{D}_{lm} \) has the form
\[
\hat{D}_{lm} = \sum_{i' \in T} D_{i'lm} \otimes |i' \rangle \langle i' |.
\]
We define the POVM
\[
D := \{ D_{i'tnm} \mid (l',m) \in [L] \times [T] \times [M] \}
\]
and the stochastic matrix \( T : \mathcal{Y} \rightarrow [L] \times [T] \times [M] \) by
\[
T(l,t^n,m|y^n) := \sum_{u \in U} P^n_{U|(l,t^n|u^n)} P^n_{U|Y}(u^n|y^n) = \langle l,t^n,m|y^n \rangle \in [L] \times [T] \times [M] \times \mathcal{Y}^m.
\]
With these definitions, \( D := (T,D) \) is an \((n,M,\hat{L},|T|^m)\) secret-key distillation protocol for I. It holds for each \( s := (p,V) \in P \times \mathcal{Y}, m,m' \in [M] \)
\[
P_{K,K',s}(m,m') = \sum_{t=1}^{L} \sum_{i' \in T} p^n_{U}(y^n) T(l,t^n,m|y^n) \text{tr}(D_{i'tnm} \otimes \hat{V}_B^{\otimes n}(y^n)) \tag{49}
\]
\[
= \sum_{u \in U} \sum_{l=1}^{L} \sum_{i' \in T} \sum_{y \in \mathcal{Y}} p^n_{U}(y^n) P^n_{U|Y}(u^n|y^n) \text{tr}(D_{i'tnm} \otimes \hat{V}_B^{\otimes n}(y^n)) \tag{50}
\]
\[
= \sum_{u \in U} \sum_{l=1}^{L} \sum_{i' \in T} \sum_{y \in \mathcal{Y}} q^n_p(u^n) W^n_p(y^n|u^n) \times \text{tr}(D_{i'tnm} \otimes \hat{V}_B^{\otimes n}(y^n)) \tag{51}
\]
\[
= \sum_{u \in U} \sum_{l=1}^{L} q^n_p(u^n) \text{tr}(D_{i'tnm} \otimes \hat{V}_B^{\otimes n}(u^n)) \tag{52}
\]
\[
= P_{K,K',s}(p,p,V)(m,m'). \tag{53}
\]
The equality in (49) holds by definition, (50) is valid by (48). The equality in (51) is justified by definition of \( q_p, W_p \), and the fact, that
\[
\text{tr}(|t \rangle \langle t| \hat{V}_B(u)) = P_{T|U}(t|u) \tag{t \in T, u \in U}
\]
holds by definition of \( \hat{V} \). From (53), we directly infer
\[
\text{Pr}(K_s \neq K'_s) = \text{Pr}(\hat{K}_{s,p,p,V} \neq \hat{K}'_{s,p,p,V}) \leq \mu \text{, and } \tag{54}
\]
\[
H(K_s) = H(\hat{K}_{s,p,p,V}) \tag{55}
\]
Notice, that by definition of \( \hat{\rho}_{(p,p,V)} \) is (up to unitaries permuting tensor factors) equal to \( \hat{\rho}_{(p,V)} \), i.e.

\[
\hat{\rho}_{(p,p,V)} = \sum_{u \in U} q_p(u) \left| u \right\rangle \left\langle u \right| \otimes \hat{V}_p(u) \otimes V(u)
\]

\[
= \sum_{i \in I} \sum_{u \in U} \sum_{y \in Y} P_{i|U}(t|u)p_{i|Y}(y) \left| u \right\rangle \left\langle u \right| \otimes V(y) \otimes |t\rangle \langle t|
\]

holds for each \( (p, V) \in \mathcal{P} \times \mathcal{V} \). Consequently, it follows:

\[
I(K; \hat{TE}, \hat{\rho}_{K^ATE^{V,s}}) = I(K; \hat{EE}' \hat{A}, \hat{\rho}_{\hat{K} \hat{A}E'E''}(p,p,V)).
\] (56)

The inequalities contained in (55) and (56) together with the one in (47) yield

\[
\log M - H(K_i) + I(K; \hat{TE}, \hat{\rho}_{K^ATE^{V,s}}) \leq \mu
\]

for each \( (p, V) \), which, together with (54) makes \((T, D)\) an \((n, \hat{L}, |T|^n, M, \mu)\) forward secret-key distillation protocol for \( \hat{I} \). At last, we have to show, that \( M \) indeed satisfies the bound stated in (44). We will therefore, lower-bound the right-hand side of (44). Because Markov-processing does never increase the variational distance, it holds

\[
\|q_p - q_{p'}\|_1 \leq \|p - p'\|_1 \leq \text{diam}(\mathcal{P}) \leq \Delta
\] (57)

for each \( p, p' \in \mathcal{P} \), where the rightmost inequality is by assumption. We obtain for each \( V \in \mathcal{V} \)

\[
\left| \chi(q_p, \hat{V}_{B,p} \otimes \hat{V}_B) - \chi(q_{p'}, \hat{V}_{B,p} \otimes \hat{V}_B') \right| \leq 6\|q_p - q_{p'}\|_1 \log |U| \leq 6\Delta \log |U| + 2h(\Delta).
\] (58)

The first equality above is by application of Lemma 44 which can be found in Appendix B the second by (57). The bound in (58) directly implies

\[
\inf_{(p', V)} \chi(q_p, \hat{V}_{B,p} \otimes \hat{V}_B) \geq \inf_{V} \chi(q_p, \hat{V}_{B,p} \otimes \hat{V}_B) - \Delta \log |U| + 2h(\Delta)
\]

\[
= \inf_{p \in \mathcal{P}} I(U, BB', \hat{\rho}) - \Delta \log |U| - 2h(\Delta).\] (59)

for each \( p \in \mathcal{P} \). The equality in (59) holds by the identity

\[
\chi(q_p, \hat{V}_{B,p} \otimes \hat{V}_B) = I(U, BB', \hat{\rho}_{(p,p,V)}).
\]

By similar reasoning, we also yield the bound

\[
\sup_{(p', V)} \chi(q_p, \hat{V}_{E,p'} \otimes \hat{V}_E) \leq \sup_{p \in \mathcal{P}} I(U, EE', \hat{\rho}) + 6\Delta \log |U| + 2h(\Delta).\] (60)

Combination of (59) and (60) for each \( p \in \mathcal{P} \) ensures us, that

\[
\inf_{(p', V) \in \mathcal{P}^2 \times \mathcal{Y}} \chi(q_p, \hat{V}_{B,p'} \otimes \hat{V}_B') - \sup_{(p', V) \in \mathcal{P}^2 \times \mathcal{Y}} \chi(q_p, \hat{V}_{E,p'} \otimes \hat{V}_E') \geq \inf_{p \in \mathcal{P}} \sup_{p' \in \mathcal{P}} I(U, BB', \hat{\rho}) - \inf_{p \in \mathcal{P}} \sup_{p' \in \mathcal{P}} I(U, EE', \hat{\rho}) - 12\Delta \log |U| - 4h(\Delta)
\]

holds. Note, that the identities

\[
S(\hat{\rho}_{BT}) = S(\hat{\rho}_{BB'}), \quad S(\hat{\rho}_{ET}) = S(\hat{\rho}_{EE'}),
\]

are valid. Moreover, for each \( p \in \mathcal{P} \) the equalities

\[
I(U; B|T, \hat{\rho}) = H(p_{UT}) + S(\hat{\rho}_{BT}) - S(\hat{\rho}_{UBT}) - H(P_T), \quad \text{and}
\]

\[
I(U; E|T, \hat{\rho}) = H(p_{UT}) + S(\hat{\rho}_{ET}) - S(\hat{\rho}_{UET}) - H(P_T).
\] (61) (62)
hold by definition, where \( P_T, P_{TU} \) are the distributions for the random variables \( T \) and \( TU \). It is important, to notice here, that the distributions \( P_J \) and \( P_{TU} \) depend only on the Markov chain on the senders systems. Therefore, we have for each \( p \in P \)

\[
\inf_{\rho \in \rho_p} I(U; B| T, \rho) - \sup_{\rho \in \rho_p} I(U; E| T, \rho) = \inf_{\rho \in \rho_p} (S(\hat{\rho}_{BT}) - S(\hat{\rho}_{UBT})) - \sup_{\rho \in \rho_p} (S(\hat{\rho}_{ET}) - S(\hat{\rho}_{UET}))
\]

\[
= \inf_{\rho \in \rho_p} (S(\hat{\rho}_{BB}) - S(\hat{\rho}_{UBB})) - \sup_{\rho \in \rho_p} (S(\hat{\rho}_{EE'}) - S(\hat{\rho}_{UUE'}))
\]

\[
= \inf_{\rho \in \rho_p} \left( H(q_p) + S(\hat{\rho}_{BB'}) - S(\hat{\rho}_{UBB'}) \right) - \sup_{\rho \in \rho_p} \left( H(q_p) + S(\hat{\rho}_{EE'}) - S(\hat{\rho}_{UUE'}) \right)
\]

\[
= \inf_{\rho \in \rho_p} I(U; B', \hat{\rho}) - \sup_{\rho \in \rho_p} I(U; E', \hat{\rho}).
\]

(63)

Collecting the bounds obtained, we can prove the desired lower bound on the key rate. It holds

\[
\frac{1}{n} \log M \geq \inf_{(p, p', \mathcal{V}) \in P^2 \times \mathcal{V}} \chi(q_p, \hat{\nu}_{pp'} \otimes \hat{\nu}_{pp'}) - \sup_{(p, p', \mathcal{V}) \in P^2 \times \mathcal{V}} \chi(q_p, \hat{\nu}_{pp'} \otimes \hat{\nu}_{pp'}) - \frac{\rho}{\log n}
\]

\[
= \inf_{p \in \mathcal{P}} \left( \inf_{p \in \mathcal{P}} I(U; B', \hat{\rho}) - \sup_{p \in \mathcal{P}} I(U; E', \hat{\rho}) \right) - \frac{\rho}{\log n}
\]

\[
= \inf_{p \in \mathcal{P}} \left( \inf_{p \in \mathcal{P}} I(U; B| T, \rho) - \sup_{p \in \mathcal{P}} I(U; E| T, \rho) \right) - \frac{\rho}{\log n}
\]

The first inequality above is by (60), the first inequality is the one from (59), while the last inequality is by (63). We are done.

**Proposition 15.** Let \( \Delta > 0 \), and \( J \subset \mathcal{S}_{\text{cq}}(\mathcal{V}, \mathcal{K}_{BE}) \) be a \( \Delta \)-regular set of cq density matrices on \( \mathcal{H}_{ABE} \). For all \( z, z' \in \mathbb{N} \), it holds

\[
K_{-}(J) \geq K_{-}^{(1)}(J, z, z) - \frac{\rho}{\log n} - f_{\text{reg}}(z, \Delta),
\]

with a function \( f_{\text{reg}} : \mathbb{N} \times [0, \infty) \to [0, \infty) \) such that \( f(r, \Delta) \to 0 \) (\( \Delta \to 0 \)). For a set \( \mathcal{A} := \{ p(y) | y \rangle \langle y | \otimes \sigma_y \} \) on some space, and \( z, z' \in \mathbb{N} \), the function \( K_{-}^{(1)}(J, z, z') \) is defined

\[
K_{-}^{(1)}(\mathcal{A}) := \inf_{\rho \in \mathcal{P}} \sup_{\mathcal{Y} \in \mathcal{P}} \left\{ \inf_{\sigma \in \mathcal{A}} I(U; B| T, \rho) - \sup_{\sigma \in \mathcal{A}} I(U; E| T, \rho) \right\}.
\]

The supremum above is over all Markov chains \( T \leftarrow U \leftarrow Y_p \) resulting from application of Markov transition matrices \( P_{TU} : \mathcal{U} \to T, P_{UY} : \mathcal{Y} \to U \) on \( p \) for each \( p \in \mathcal{P} \) with \( | \mathcal{U} | = z, | \mathcal{T} | = z' \), and

\[
\sigma_{TU} := \sum_{y \in \mathcal{Y}} \sum_{t \in \mathcal{T}} \sum_{u \in \mathcal{U}} P_{TU}(t|u)p_{UY}(u|y)p(y) \langle t | \otimes | u \rangle \langle u | \otimes \sigma_y
\]

for given transition matrices \( P_{TU}, P_{UY} \) and

\[
\sigma = \sum_{y \in \mathcal{Y}} p(y) | y \rangle \langle y | \otimes \sigma_y.
\]

**Proof.** Assume the set \( J \) is parameterized such that \( \mathcal{P} \subset \mathcal{P}(\mathcal{V}) \) is the set of possible margian distributions on the sender’s system, while to each \( p \in \mathcal{P} \) a set \( \mathcal{Y}_p \subset \mathcal{C}(\mathcal{V}, \mathcal{K}_{BE}) \) is associated, i.e.

\[
J = \left\{ \rho := \sum_{y \in \mathcal{Y}} q(y) | y \rangle \langle y | \otimes \mathcal{V}(y) : q \in \mathcal{P}, \mathcal{V} \in \mathcal{Y}_p \right\}
\]

Let \( \delta > 0 \), \( z, z' \in \mathbb{N} \) be arbitrary but fixed numbers. We show, that

\[
K_{-}^{(1)}(J, z, z') - \frac{\rho}{\log n} - f_{\text{reg}}(z, \Delta)
\]
with \( f_{\text{reg}}(z, \Delta) \) being defined
\[
f_{\text{reg}}(z, \Delta) := 32\Delta \log(z \cdot \dim \mathcal{K}_{BE}) + 24h(\Delta)
\]
is an achievable forward secret-key distillation rate for \( \lambda \). Note that the function defined above indeed has the properties claimed above. The strategy of proof will be as follows. We will equip \( \mathcal{P} \) with a regular nonintersecting covering, where we utilize the set of types for large enough blocklength to define such. With the right choice of parameters, we obtain a finite family of sources which approximate \( \lambda \) and have the addition property for fulfilling the hypotheses of Proposition 14. Combining the protocols obtained for each member of the family with an estimation on the first \( \sqrt{n} \) letters for blocklength \( n \) leads us to a universal protocol for \( \lambda \).

We begin setting up the covering of \( \mathcal{P} \). Define for each \( k, l \in \mathbb{N}, \lambda \in \mathcal{T}(k, \mathcal{Y}) \) a set
\[
T_{\lambda, l} := \{ q \in \mathcal{P}(\mathcal{Y}) : \forall y \in \mathcal{Y} : \lambda(y) - \frac{l}{2^k} < q(y) \leq \lambda(y) + \frac{l}{2^k} \}.
\]
Notice, that the diameter of \( T_{\lambda, l} \) is bounded by
\[
diam(T_{\lambda, l}) \leq \frac{l \cdot |\mathcal{Y}|}{k},
\]
and the sets in the family being pairwise nonintersecting for \( l = 1 \). We fix \( k \) to be specified later, define \( \mathcal{P}_\lambda := T_{\lambda, 3} \cap \mathcal{P} \) for each \( \lambda \in \mathcal{T}(k, \mathcal{Y}) \), and denote by \( \hat{\mathcal{T}} \) the collection of all \( \lambda \) with \( \mathcal{P}_\lambda \) being nonempty. We construct sets of cqq density matrices, which fit the specifications demanded in Proposition 14. Combining the protocols obtained for each \( \lambda \), \( \hat{\mathcal{P}} \), we define
\[
\hat{\mathcal{V}}_\lambda := \bigcup_{q \in \mathcal{P}_\lambda} \mathcal{V}_q, \quad \text{and} \quad \hat{\mathcal{J}}_{p, \lambda} := \left\{ \sum_{y \in \mathcal{Y}} p(y) |y\rangle \langle y| \otimes V(y) : p \in \mathcal{P}_\lambda, V \in \hat{\mathcal{V}}_\lambda \right\}
\]
for each \( \lambda \in \hat{\mathcal{T}} \). Fix the number \( k \) large enough, to ensure us, that for each \( \lambda \in \hat{\mathcal{T}} \), \( p, p' \) are in \( \mathcal{P}_\lambda \) implies
\[
d_H(J^{AB}_p, J^{AB}_{p'}) + d_H(J^{AE}_{l, p}, J^{AE}_{l, p'}) \leq \Delta,
\]
and, in addition \( diam(\mathcal{P}_\lambda) \leq \Delta \). Notice, that this choice of \( k \) is indeed possible because we assumed \( \lambda \) to be \( \Delta \)-regular. We consider the family \( \{ \hat{\mathcal{J}}_\lambda \}_{\lambda \in \hat{\mathcal{T}}} \), where for each \( \lambda, \hat{\mathcal{J}}_\lambda \) is the set of density matrices defined by
\[
\hat{\mathcal{J}}_\lambda := \bigcup_{p \in \mathcal{P}_\lambda} \hat{\mathcal{J}}_{p, \lambda} = \left\{ \sum_{y \in \mathcal{Y}} p(y) |y\rangle \langle y| \otimes V(y) : p \in \mathcal{P}_\lambda, V \in \hat{\mathcal{V}}_\lambda \right\}.
\]
The rightmost of the above equalities holds by construction. Notice, that since \( \mathcal{P}_\lambda \) has diameter bounded, and \( \hat{\mathcal{J}}_\lambda \) is parameterized by the full cartesian product \( \mathcal{P}_\lambda \times \hat{\mathcal{V}}_\lambda \), Proposition 14 can be applied in each case. Choose for each \( \lambda \in \hat{\mathcal{T}} \), stochastic matrices \( P_{U, \lambda} : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{T}) \) and \( P_{U|Y, \lambda} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{U}) \) such that
\[
\inf_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; B|T_\lambda, \hat{\rho}) - \sup_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; E|T_\lambda, \hat{\rho}) \geq \sup_{T - U - Y} \left( \inf_{\rho \in \hat{\mathcal{J}}_\lambda} I(U; B|T, \hat{\rho}) - \sup_{\rho \in \hat{\mathcal{J}}_\lambda} I(U; E|T, \hat{\rho}) \right) - \frac{\delta}{2}
\]
is fulfilled. Resulting from the choices made, it also holds
\[
\inf_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; B|T_\lambda, \rho) - \sup_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; E|T_\lambda, \rho) \geq \inf_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; B|T_\lambda, \rho) - \sup_{\rho \in \hat{\mathcal{J}}_\lambda} I(U_\lambda; E|T_\lambda, \rho) - f_{\text{reg}}(\Delta, z, z')
\]
for each \( \lambda \in \hat{\mathcal{T}}, p \in \mathcal{P}_\lambda \). The inequality above is by continuity together with properties of our construction and definition of \( f_{\text{reg}} \). The full argument for justification can be found in Appendix C. Combining the
above estimates, we have for each \( \lambda \in \hat{\mathcal{T}} \)

\[
\inf_{p \in \mathcal{P}_\lambda} \left( \inf_{\rho \in \mathcal{P}_{\lambda}} I(U_A; B|T_A, \rho) - \sup_{\rho \in \mathcal{P}_{\lambda}} I(U_A; E|T_A, \rho) \right) - f_{\text{reg}}(\Delta, z, z')
\]

\[
\geq \inf_{p \in \mathcal{P}_\lambda} \left( \inf_{\rho \in \mathcal{P}_{\lambda}} I(U; B|T, \rho) - \sup_{\rho \in \mathcal{P}_{\lambda}} I(U; E|T, \rho) \right) - \frac{\delta}{2} - \frac{1}{2} f_{\text{reg}}(\Delta, z, z')
\]

\[
\geq \inf_{p \in \mathcal{P}_\lambda} \sup_{T \in \mathcal{U} - Y} \left( \inf_{\rho \in \mathcal{P}_{\lambda}} I(U; B|T, \rho) - \sup_{\rho \in \mathcal{P}_{\lambda}} I(U; E|T, \rho) \right) - \frac{\delta}{2} - \frac{1}{2} f_{\text{reg}}(\Delta, z, z')
\]

\[
= \tilde{K}^{(1)}_{\lambda}(J, z, z') - \frac{\delta}{2} - \frac{1}{2} f_{\text{reg}}(\Delta, z, z'). \quad (67)
\]

fulfilled. The first inequality above holds by \([66]\), the second is by \([65]\). Let the blocklength \( n \in \mathbb{N} \) be fixed. We set \( n = a_n + b_n \) with \( a_n := \lceil \sqrt{n} \rceil \), \( b_n := n - a_n \), and consider the decomposition

\[
\mathcal{Y}^n = \mathcal{Y}^{b_n} \times \mathcal{Y}^{b_n}.
\]

Applying Proposition \([14]\) to each of the sets \( \hat{J}_\lambda \), we infer for each large enough \( n \), \( \lambda \in \hat{T} \) existence of an \((b_n, M, L, \hat{\delta})\) secret-key distillation protocol \((\hat{T}_\lambda, \hat{D}_\lambda)\) for \( \hat{J}_\lambda \) with

\[
\hat{\delta} \leq 2^{-1/4\psi_n c} \leq 2^{-1/4\psi_n c}
\]

with a strictly positive constant \( c_\lambda \) and \( c := \min_{\lambda \in \mathcal{T}} c_\lambda \) and

\[
M = \left[ \exp \left( b_n \left( \tilde{K}^{(1)}_{\lambda}(J) - \frac{3\hat{\delta}}{4} - f_{\text{reg}}(\Delta, z, z') \right) \right) \right].
\]

(68)

Note that the combination of \( M \) and \( \theta \) is indeed possible is justified by combining the claim of Proposition \([14]\) and the bound in \([67]\). Next, we define a two-phase protocol, where the first \( a_n \) letters from the source observed by the sender are used to estimate \( \lambda \in \hat{T} \), while the protocol \((\hat{D}_\lambda, \hat{T}_\lambda)\) for the estimated parameter \( \lambda \) is applied on the remaining \( b_n \) outputs of the source. To formalize this strategy, we define a stochastic matrix

\[
T : \mathcal{Y}^n \to \mathcal{D}([L] \times \mathcal{T}(k, \mathcal{Y}) \times [M])
\]

with entries

\[
T(l, \theta, m|y^n) := \hat{T}_\theta(l, m|y^n) \delta_{\theta \xi(y^n)} \quad (l, m, \theta, y^n) \in [L] \times [M] \times \hat{T} \times \mathcal{Y}^n)
\]

for each \( \mu \in \hat{T} \), where we defined a function \( \xi : \mathcal{Y}^n \to \mathcal{T}(k, \mathcal{Y}) \) which maps each \( y^n \) to the unique member \( \lambda = \xi(y^n) \) such that \( T_{\lambda, \lambda} \) contains the type of \( y^n \). Notice, that some of the entries may are undefined, if \( \hat{T} \) does not contain all elements of \( \mathcal{T}(k, \mathcal{Y}) \). In this case, entries can be defined in any consistent way, because they will be of no further relevance. Moreover, we introduce matrices

\[
D^\theta_{lm} := 1_{H_B^{\otimes n}} \otimes D^\theta_{lm} \in \mathcal{L}(H_B^{\otimes n}),
\]

where \( D^\theta_{lm} \) is the corresponding effect from the POVM \( D^\theta \) associated to \( \theta \). With these definitions, it is clear, that \((T, D)\) is an \((n, M, L, |\hat{T}|, \delta)\) forward secret-key distillation protocol for \( J \), with a number \( \delta \) we will bound below. Let

\[
\rho := \sum_{y \in \mathcal{Y}} p(y) |y \rangle \langle y | \otimes V(y)
\]

be any fixed member of \( J \), and \( \lambda_0 \) the unique type in \( \hat{T} \) such that \( p \in T_{\lambda_0, 1} \). It is important to notice, that not only for \( \lambda_0 \), but also for each \( \theta \in \hat{T} \) with \( \theta \in T_{\lambda_0, 3} \), \( \rho \) is also a member of \( \hat{J}_\theta \). Assuming application
of the protocol to \( \rho \), we suppress indicating the chosen member in the following formulas. By definition, it holds
\[
P_{KK'}(m, m') = T_0(m, l) |y^a \rangle \cdot \delta_{0, l} |y^{a_1} \rangle \cdot \text{tr}(D_{lm}^{\rho} V^\otimes |y^b \rangle) = P_{KK'}(m, m', l, \theta) |y^b \rangle \cdot \delta_{0, l} |y^{a_1} \rangle
\]
with \( P_{KK'}(m, m', l, \theta) \) being the conditional distribution generated by \( (T_0, \hat{D}_0) \). We define the sets
\[
i_1 := \{ y^{a_n} : \xi(y^{a_n}) \in I_{1,1} \}, \quad n \in \{ y^{a_n} : \xi(y^{a_n}) \in I_{1,3} \}, \quad (n \in \{ y^{a_n} : \xi(y^{a_n}) \in I_{2,3} \})
\]
It holds
\[
P_{KK'}(m, m') = \sum_{\theta \in \Theta(k, l)} \sum_{y^a \in \mathcal{Y}} \sum_{l=1}^{L} P_{KK'}(m, m', l, |y^a \rangle \cdot p^\theta(y^a)
\]
\[
= \sum_{\theta \in \Theta(k, l)} \sum_{y^a \in \mathcal{Y}} p^\theta(y^a) \sum_{n=1}^{L} P_{KK'}(m, m', l, |y^a \rangle \cdot p^\theta(y^a)
\]
\[
= \sum_{\theta \in \Theta(k, l)} p^\theta(n_1) P_{KK'}(m, m')
\]
for each \( m, m' \in [M] \). We denote the key random variables produced by performing \( (T_0, \hat{D}_0) \) on \( \rho^\otimes \) by \( \hat{K}_0 \) and \( \hat{K}_0' \). We directly obtain
\[
\Pr(K \neq K') = \sum_{\theta \in \Theta(k, l)} p^\theta(n_1) \cdot \Pr(\hat{K}_0 \neq \hat{K}_0')
\]
\[
\leq p^\theta(n_1) \cdot \delta + p^\theta(n_1) \cdot \delta + 2^{-a_n \cdot \delta}
\]
\[
\leq 2 \sqrt{w} + 2^{-a_n \cdot \delta}
\]
(69)

The first inequality above is by the fact, that the protocol associated to each \( \theta \in \tilde{\Theta} \cap \tilde{I}_{1,3} \) is \( \delta \)-good for \( \rho \) by construction. The second inequality is by standard type bounds. Explicitly, we have by construction \( y^{a_n} \in \tilde{I}_{1,3} \) implying
\[
\left| \frac{1}{a_n} N(e|y^{a_n}) - p(e) \right| > \frac{1}{k}
\]
for all \( e \in \mathcal{Y} \), where \( N(e|y^{a_n}) \) is the number of occurrences of the letter \( e \) in \( y^{a_n} \). Consequently
\[
p^\theta(n_1) \leq p^\theta(n_1) \left( T_{n_1} \right)^c \leq 2^{-a_n \cdot \delta}
\]
(70)

where \( c \) is a universal, strictly positive constant, and the last inequality holds for large enough choice of \( n \). Also, it holds
\[
H(K) - I(K; \Lambda | \Theta) = H(K) - I(K; \Theta) - I(K; \Lambda | \Theta) \Theta_{\Lambda | \Theta} = H(K; \Theta) - I(K; \Lambda | \Theta) \Theta_{\Lambda | \Theta}
\]
\[
= \sum_{\theta \in \Theta} p^\theta(n_1) \left( H(K; \Theta = \theta) - I(K; \Lambda | \Theta = \theta) \Theta_{\Lambda | \Theta}
\right)
\]
\[
= \sum_{\theta \in \Theta} p^\theta(n_1) \left( H(\hat{K}_0) - I(\hat{K}_0; \Lambda | \Theta) \Theta_{\Lambda | \Theta}
\right),
\]
(71)
which determines the conditional quantities as generated from application of the protocol. Therefore, we obtain
\[
\log M - H(K) + I(K; \Lambda \Theta E^n, \rho_{K \Lambda \Theta E^n}) = \sum_{\theta \in T} p^n(\theta) \left( \log M - H(\hat{K}^0) + I(\hat{K}^0; \Lambda \Theta E^n, \rho_{K \Lambda \Theta E^n}) \right) \\
\leq p^n(\theta) \cdot \hat{\delta} + p^n(\theta) \cdot (2 \cdot \log M + \log L + b_n \cdot \log \dim K_{BE}) \\
\leq 2 \hat{\delta}
\]
(72)
where the equality above follows from (71), the first inequality is by the fact, that the protocol \((\hat{T}_\theta, \hat{D}_\theta)\) is \(\hat{\delta}\)-good for \(\rho\) whenever \(\theta\) is a direct grid point neighbour of \(\lambda_0\), i.e. \(T_{\lambda_0} \subset T_{\lambda_0,3}\). Moreover we applied the ultimate bound \(I(A;B,\rho) \leq 2 \log \dim H_A \otimes H_B\) which holds for each state \(\sigma\) on any Hilbert space \(H_A \otimes H_B\). The last inequality holds with a large enough choice of \(n\) by application of the bound in (70). The bounds obtained in (72) and (69) show us, that \((T, D)\) is actually an \((n, M, L, \delta)\) forward secret-key distillation protocol for \(J\), with \(\delta \leq 2 \hat{\delta}\), and, since \(b_n/n \to 1\) for \(n \to \infty\), it holds
\[
\frac{1}{n} \log M \geq \frac{b_n}{n} \hat{K}^{(1)}(J) - \frac{3 \delta}{4} - f_{\text{reg}}(\Delta, z, z') \\
\geq \hat{K}^{(1)}(J) - \delta - f_{\text{reg}}(\Delta, z, z')
\]
if \(n\) is large enough, where the first inequality is from (68).

To prove achievability of the multi-letter formula claimed in Theorem 8 we have to ensure ourselves, that regularity conditions do not break down when considering the set \(\mathbb{I}^{n, (1)} : [\rho^{(n)} : \rho \in \mathbb{I}]\) instead of a set \(\mathbb{I}\) of cqq density matrices. The following two basic lemmata will turn out to be sufficient for our needs.

**Lemma 16.** Let \(I, J \subset \mathcal{L}(K)\) be any two sets of density matrices. It holds for each \(n \in \mathbb{N}\)
\[
d_H(\mathbb{I}^{n, (1)}, \mathbb{J}^{n, (1)}) \leq n \cdot d_H(I, J),
\]
where \(d_H\) is the Hausdorff distance induced by the trace norm on the underlying space.

**Proof.** The inequality
\[
\|a^{(1)} - b^{(1)}\|_1 \leq n \cdot \|a - b\|_1
\]
valid for any two matrices \(a, b \in \mathcal{L}(K)\) inherits to the Hausdorff distance. It holds
\[
\sup_{a \in \mathbb{I}} \inf_{b \in \mathbb{J}} \|a^{(1)} - b^{(1)}\|_1 \leq n \cdot \sup_{a \in \mathbb{I}} \inf_{b \in \mathbb{J}} \|a - b\|_1.
\]
\(\square\)

**Lemma 17.** Let \(\mathbb{I}\) be a set of cqq density matrices. It holds
\[
\mathbb{I} \in \epsilon - \text{regular} \Rightarrow \mathbb{I}^{k, \epsilon} \in \epsilon - \text{regular}
\]
for each \(k \in \mathbb{N}\).

**Proof.** Is by direct application of Lemma 16 and the definition of regularity.
\(\square\)

We now obtained sufficient preparations to tackle the proof of achievability in Theorem 8. Before we head to the proof, we ensure ourselves, that the limit in (11) indeed exists.

**Lemma 18.** Let \(\mathbb{I}\) be a set of cqq density matrices on \(H_{ABE}\). It holds
\[
\sup_{k \in \mathbb{N}} \frac{1}{k} K^{(1)}(\mathbb{I}^{k, (1)}) = \lim_{k \to \infty} \frac{1}{k} K^{(1)}(\mathbb{I}^{(1)}).
\]

**Proof.** The assertion of the lemma follows from application of Fekete’s lemma on the sequence \(K^{(1)}(\mathbb{I}^{k})\). We check that the hypotheses of Fekete’s lemma are fulfilled. Clearly, the sequence is bounded. We show, that it is also superadditive, i.e. \(K^{(1)}(\mathbb{I}^{k+i}) \geq K^{(1)}(\mathbb{I}^{k}) + K^{(1)}(\mathbb{I}^{i})\) being valid for all \(k, l \in \mathbb{N}\). We can for each \(k\) write \(K^{(1)}(\mathbb{I}^{k})\) in the form
\[
K^{(1)}(\mathbb{I}^{k}) = \inf_{p \in P} \sup_{z, z' \in M} \hat{K}^{(1)}(\mathbb{I}^{k}, p, z, z')
\]
where we defined

\[
\hat{K}^{(1)}(I^{\otimes k}, p, z, z') := \sup_{T - U - X} \left( \inf_{\sigma \in \Pi} I(U; B|T, \sigma) - \sup_{\sigma \in \Pi} I(U; E|T, \sigma) \right),
\]

with the outer maximisation above being over all Markov chains generated by transition matrices \( P_{U|X} : X \to \mathcal{D}(X) \) and \( P_{T|U} : U \to \mathcal{D}(T) \) with alphabets of cardinalities \(|U| = z, |T| = z'\), and

\[
\hat{\sigma} := \sum_{t \in T} \sum_{u \in U} \sum_{x \in X} P_{T|U}(t|u) P_{U|X}(u|x) p(x|u) \langle t \otimes V(x) \rangle
\]

for

\[
\sigma = \sum_{x \in X} p(x) \langle x \otimes V(x) \rangle.
\]

Notice, that for each \( p \in \mathcal{P}_1 \), \( z, z' \in \mathbb{N} \)

\[
\hat{K}^{(1)}(I^{\otimes (k+1)}, p, z, z') \geq \hat{K}^{(1)}(I^{\otimes k}, p, z, z') + \hat{K}^{(1)}(I^{\otimes l}, p, z, z'),
\]

and moreover, for each \( z_1 \leq z_2, z_1' \leq z_2' \),

\[
\hat{K}^{(1)}(I^{\otimes k}, p, z_2, z_2') \geq \hat{K}^{(1)}(I^{\otimes k}, p, z_1, z_1')
\]

holds for each \( k \in \mathbb{N}, p \in \mathcal{P}_1 \). We obtain

\[
\hat{K}^{(1)}(I^{\otimes (k+l)}, p, z_2, z_2') \geq \hat{K}^{(1)}(I^{\otimes k}, p, z_2, z_2') + \hat{K}^{(1)}(I^{\otimes l}, p, z_2, z_2')
\]

\[
\geq \hat{K}^{(1)}(I^{\otimes k}, p, z_1, z_1') + \hat{K}^{(1)}(I^{\otimes l}, p, z_1, z_1')
\]

Consequently, it holds

\[
\sup_{z, z' \in \mathbb{N}} \hat{K}^{(1)}(I^{\otimes (k+l)}, p, z, z') \geq \sup_{z, z' \in \mathbb{N}} \hat{K}^{(1)}(I^{\otimes k}, p, z, z') + \sup_{z, z' \in \mathbb{N}} \hat{K}^{(1)}(I^{\otimes l}, p, z, z')
\]

for each \( p \in \mathcal{P}_1 \). We conclude

\[
K^{(1)}(I^{\otimes (k+l)}) \geq K^{(1)}(I^{\otimes k}) + K^{(1)}(I^{\otimes l})
\]

\( \square \)

**Proof of Theorem** We first prove achievability, i.e. validity of the inequality

\[
K_{\ast}(l) \geq \lim_{k \to \infty} \frac{1}{k} K^{(1)}_{\ast}(I^{\otimes k})
\]

Let \( z, z', k \in \mathbb{N} \) and \( \delta > 0 \) be arbitrary and fixed. We show, that

\[
\frac{1}{k} K^{(1)}_{\ast}(I^{\otimes k}) - \delta
\]

is an achievable forward secret key distillation rate. We apply Proposition 13 with \( J = I^{\otimes k} \), and conclude, that for each large enough blocklength \( g \in \mathbb{N} \), we find an \((l, M, L, \delta)\) forward secret-key distillation protocol for \( I^{\otimes k} \) with \( \delta \leq 2^{-\frac{1}{10} \log 3} \) with a constant \( c_3 > 0 \), and

\[
\frac{1}{g} \log M \geq \hat{K}^{(1)}(I^{\otimes k}, z, z') - \frac{2}{3} \delta
\]

(73)

where we chose \( \Delta \) small enough to satisfy \( f_{reg}(z, z', \Delta) \leq \frac{2}{3} \). Since an \((g, K, M, \delta)\) protocol for \( I^{\otimes k} \) is obviously an \((g \cdot K, M, \delta)\) protocol for \( I \), we obtained sufficient protocols for all large enough blocklengths being integer multiples of \( k \). We can achieve sufficient protocols also for the remaining blocklengths just by wasting resources. To be explicit, let \( n = k \cdot g + r \) with \( 0 < r < k \) and assume \((\hat{T}_{gk}, \hat{D}_{gk})\) being an \((g \cdot K, M, L, \mu)\) protocol for \( I \). Define a protocol \((T_n, D_n)\) for blocklength \( n \) by setting

\[
T_n(l, m|x^n) = \hat{T}_{gk}(l, m|x_{1\cdot\cdot\cdot x_{gk}})
\]

\[
(x^n = (x_1, \ldots, x_n) \in \mathcal{X}^n)
\]
and effects

\[ D_{n,l,m} := \hat{D}_{k,l,m} \otimes I_{\mathcal{H}_B} \]

for each \( l \in [L], m \in [M] \). It is clear, that \((T_n, D_n)\) is an \((n, M, L, \mu)\) forward secret-key distillation protocol for \(I\) with rate

\[
\frac{1}{n} \log M = \frac{1}{g \cdot k + r} \log M \geq \frac{1}{g \cdot k} \log M - \frac{\delta}{3}
\]

(74) if \( n \) is large enough. It follows from (73) and (74), that we actually achieve

\[
\frac{1}{k} \hat{K}^{(1)}(\mathcal{I}^{\otimes k}, z, z') - \delta
\]

Since \( \delta, z, z' \) where arbitrary, we are done. We do not give a detailed argument for the converse inequality here, since the assertion directly follows from (13) together with a converse proof for the case of a source with SMI given in the next section. 

V. SECRET-KEY DISTILLATION WITH SENDER MARGINAL INFORMATION (SMI)

In this section, we assume that the sender has perfect knowledge of his/her marginal distribution deriving from the source statistics. We will prove the achievability part of Theorem 9 by decomposing each compound cqq source into a finite collection of regular compound cqq sources. To obtain such an approximation, we need the following basic assertion. For a given set \( X \), we use the notation \( 2^X \) for the power set.

Lemma 19. Let \( d_H \) be the Hausdorff distance on \( 2^\mathbb{R}^n \) generated by the 1-norm distance on \( \mathbb{R}^n \). Let \( A \subset \mathbb{R}^n \) be a subset of \( \mathbb{R}^n \) with \( \text{diam}(A) \leq a < \infty \). For each \( \Delta > 0 \), there exists a family \( R_A := \{ \tilde{A}_\omega \}_{\omega=1}^\Omega \subset 2^\mathbb{R}^n \setminus \{ \emptyset \} \) with the following properties.

1. \( \Omega \leq \exp\left(\left(\frac{a}{\Delta}\right)^n\right) \).
2. For each \( B \subset A \) there exists \( \omega \in [\Omega] \), such that

\[
d_H(B, \tilde{A}_\omega) \leq \Delta, \quad B \subset \tilde{A}_k.
\]

Proof. Equip \( \mathbb{R}^n \) with the regular pairwise-disjoint covering, generated by the \( n \)-dimensional half-open cubes

\[
\left[ \left( \frac{k_1}{n}, \ldots, \frac{k_n}{n} \right), \left( \frac{k_1 + 1}{n}, \ldots, \frac{k_n + 1}{n} \right) \right) \text{ for } ((k_1, \ldots, k_n) \in \mathbb{Z}^n) \text{.}
\]

Since \( \text{diam}(A) \leq a \) is assumed, we do not need more than \( K := \left( \frac{a^2}{\Delta} \right)^n \) of these cubes to cover \( A \). Let \( \{G_k\}_{k=1}^K \) be any parameterization of the family of cubes intersecting with \( A \) by \([K] := \{1, \ldots, K\}\). Define, for each \( \omega \subset [K] \)

\[
\tilde{A}_\omega := \bigcup_{k \in \omega} G_k.
\]

We show, that

\[
\mathcal{R}_A := \{ \tilde{A}_\omega \}_{\omega=1}^\Omega
\]

indeed has the properties stated in the lemma. The first property is fulfilled by the bound on \( K \) and the fact, that there are not more, than \( 2^K \) different values for \( \omega \). The member

\[
\omega := \{ k \in [K] : G_k \cap A \neq \emptyset \}
\]

fulfills the properties demanded for the second property. 

\[ \square \]
Proof of Theorem 9 For proving achievability, we the following strategy is applied. We approximate \( I \) by a finite family \( \{I_\omega\}_{\omega \in \Omega} \) and apply Theorem 15 for each degree of regularity. Let \( \Phi = \{\rho_\lambda\}_{\lambda \in \mathbb{S}} \) be a parameterization of \( I \), and \( \{\rho_{\lambda, t}\}_{t \in T} \) be a parameterization of the set of marginal states on \( \mathcal{H}_\lambda \) which derive from members of \( I \). Fix an arbitrary \( \Delta > 0 \) and let \( \tilde{\omega}_{\lambda, t} \) be an approximation of \( I \) with the properties stated in Lemma 19 with parameter \( \lambda \). Note, that by identifying \( \mathbb{C} \) to \( \mathbb{R}^2 \) in the usual way, the approximation satisfies

\[
|\Omega| \leq \exp \left( \frac{4 \dim \mathcal{H}_{ABE}^2}{\Delta} \frac{4 \dim \mathcal{H}_{ABE}^2}{\Delta} \right) < \infty,
\]

where we only use the fact, the cardinality of \( \Omega \) is finite. Let, for each \( t \in T \), \( \omega(t) \) the element of \( \Omega \) as defined in (75) for \( I_t \). It holds

\[
I_t \subset \tilde{\omega}_{\omega(t)}, \quad d_{H}(\tilde{\omega}_{\omega(t)}, I_t) \leq \Delta.
\]

\[
\tilde{I}_a := \bigcup_{t: \omega(t) = a} I_t \quad (a \in \Omega).
\]

The family \( \{\tilde{I}_a\}_{a \in \Omega} \) is decomposition of \( I \) into a family of pairwise disjoint sets of cqq density matrices with the additional feature, that for each \( a \in \Omega \), \( \tilde{I}_a \) is \( 4\Delta \)-regular, which can be justified as follows. For each \( t, t' \) with \( \omega(t) = \omega(t') = \beta \), it holds

\[
d_{H}(I_t, I_{t'}) \leq d_{H}(I_t, \tilde{\omega}_{\beta}) + d_{H}(\tilde{\omega}_{\beta}, I_{t'}) \leq 2\Delta.
\]

The left of the above inequalities is the triangle inequality for the Hausdorff distance applied, the right hand inequality is by (76). Therefore, we infer, using monotonicity of the Hausdorff distance under taking partial traces,

\[
d_{H}(I_t^{AB}, I_{t'}^{AB}) + d_{H}(I_t^{AE}, I_{t'}^{AE}) \leq 2 \cdot d_{H}(I_t, I_{t'}) \leq 4\Delta.
\]

From applying Proposition 15 on each of the sets \( \tilde{I}_\beta \), \( t \in T \), we know, that for each given \( \delta, \mu > 0 \), there is a number \( k_0(\beta) \), such that we find for each \( n > k_0(\beta) \) an \( (n, M_{\beta}, L_{\beta}, \mu_{\beta}) \) forward secret-key distillation protocol \( (T^{(\beta)}, D^{(\beta)}) \) for \( \tilde{I}_\beta \), with

\[
\log M_{\beta} \geq \tilde{K}^{(1)}(I_{\beta}, z; z') - f_{reg, \beta}(z, \Delta) - \delta
\]

\[
\geq \tilde{K}^{(1)}(I, z, z') - f_{reg, \beta}(z, \Delta) - \delta
\]

for each \( z, z' \in \mathbb{N} \) with a function \( f_{reg, \beta} \) as stated in Proposition 15. Moreover, we have bounds

\[
\mu_{\beta} \leq 2^{-\frac{1}{4}n_{\beta \epsilon}}, \quad L_{\beta} \leq 2^{n_{R_{\epsilon, \beta}}}
\]

with constants \( \epsilon_{\beta} > 0 \) and \( R_{\epsilon, \beta} \in \mathbb{R}^+ \) for each \( \beta > 0 \). We define \( \epsilon := \min_{\beta \in \Omega} \epsilon_{\beta} \), \( R_{\epsilon} := \min_{\beta \in \Omega} R_{\epsilon, \beta} \), \( L = 2^{n_{R_{\epsilon}}} \), \( M := \min_{\beta \in \Omega} M_{\beta} \). If we define a stochastic matrix \( T_{\beta} \) with entries

\[
T_{\beta}(I, l, m|x^n) := T^{(\beta)}(I, l, m|x^n) \cdot \delta_{\rho_{\omega(t)}}
\]

and effects

\[
D_{\beta lm} := D^{(\beta)}_{lm} \quad (\beta \in \Omega, l \in [L], m \in [M]),
\]

Then \(((T_{\beta}, D_{\beta lm}))_{\beta \in T}) \) with \( D := \{D_{\beta lm}\}_{(\beta, l, m) \in \Omega \times [L] \times [M]} \) is an \( (n, M, |\Omega| \cdot L, \mu) \) secret-key distillation protocol for \( I \) with SMI, such that

\[
\log M \geq \tilde{K}^{(1)}(I, z, z') - f_{reg, \beta}(z, \Delta) - \delta
\]

holds by (78). Since \( \Delta > 0 \) was arbitrary,

\[
\tilde{K}^{(1)}(I, z, z') \geq 2\delta
\]
is achievable for each $z,z' \in \mathbb{N}$. Consequently, it holds

$$ K_{\rightarrow,SMI} \geq \sup_{z,z' \in \mathbb{N}} R^{(1)}(I, z, z') - 2\delta = R^{(1)}(I) - 2\delta $$

The same reasoning can be applied for $I^{\otimes k}, k \in \mathbb{N}$, which implies, that

$$ \frac{1}{k} R^{(1)}(I^{\otimes k}) $$

is achievable as well. It remains to prove the converse inequality. Assume $\mathcal{P} \subset \mathcal{D}(\lambda')$ to be the set of marginal probability distributions on the sender’s systems deriving from $I$. Define $V_p \subset CQ(V, H_{BE})$ to be the set of classical-quantum channels associated to each $p \in \mathcal{P}$. I.e. Fix $k \in \mathbb{N}$, and assume $(T,D_p)_{p \in \mathcal{P}}$ to be an $(k,M,L,\mu)$ forward secret-key distillation protocol for the set

$$ I_p := \left\{ \rho_{p,V} := \sum_{x \in \lambda} p(x) |x\rangle \langle x| \otimes V(x) : V \in V_p \right\} $$

of density matrices from $I$ having sender marginal distribution $p$. Fix any $p \in \mathcal{P}$ we suppress the index $p$ for the next lines. Denote by

$$ \rho_{A|K^{(E^n)}_{V,V}} $$

the state resulting from performing $(T_p,D)$ on $\rho_V$ according to [7] for each $V \in V_p$. Note that the resulting pair $(A,K)$ of random variables does not depend on the chosen state $\rho_V$ since all state in $I_p$ have same sender marginal distribution. Since $\log M - H(K)$ and $I(K;A|E^n,\rho_{A|K^{(E^n)}_{V,V}})$ are nonnegative by definition of the protocol and nonnegativity of the quantum mutual information, the inequalities

$$ \log M - H(K) \leq \mu, \quad \text{and} \quad \sup_{V} I(K;A|E^n,\rho_{A|K^{(E^n)}_{V,V}}) \leq \mu $$

(79)

are simultaneously fulfilled. Moreover, we have

$$ H(K) = I(K;K'_V) + H(K|K'_V) $$

$$ \leq I(K;K'_V) + \mu \log M + h(\mu) $$

$$ \leq I(K;K'_V|A) + \mu \log M + h(\mu) $$

$$ \leq I(K;B^nA,\rho_{K^{(E^n)}_{V,V}}) + \mu \log M + h(\mu), $$

(80)

where the first inequality is by Fano’s inequality together with the assumption $Pr(K \neq K'_V) \leq \mu$, while the last two inequalities follow from the data processing inequalities for the classical and quantum mutual information. We infer

$$ \log M \leq H(K) + \mu $$

$$ \leq \inf_{V \in \mathcal{V}} I(K;B^nA,\rho_{K^{(E^n)}_{V,V}}) + \mu + \mu \log M + h(\mu) $$

$$ \leq \inf_{V \in \mathcal{V}} I(K;B^nA,\rho_{K^{(E^n)}_{V,V}}) - \sup_{V \in \mathcal{V}} I(K;E^nA,\rho_{K^{(E^n)}_{V,V}}) + 2\mu + \mu \log M + h(\mu) $$

$$ \leq \inf_{V \in \mathcal{V}} I(K;B^n|A,\rho_{K^{(E^n)}_{V,V}}) - \sup_{V \in \mathcal{V}} I(K;E^n|A,\rho_{K^{(E^n)}_{V,V}}) + 2\mu + \mu \log M + h(\mu) $$

$$ \leq K^{(1)}(I^{\otimes k}_p) + 2\mu + \mu \log M + h(\mu), $$

(81)

The first and the third of the above inequalities are from [79], while the second is from [80]. The fourth is by definition of the quantum mutual information together with the fact, that the distribution $(K;A)$ does not depend on the chosen $V$. The last one results from observing, that $X \rightarrow (\Lambda,K) \rightarrow \Lambda$ is a Markov chain. The estimate in [81] is valid for each $p \in \mathcal{P}$. Minimization over all $p \in \mathcal{P}$ leads to

$$ \log M \leq \inf_{p \in \mathcal{P}} K^{(1)}(I^{\otimes k}_p) + 2\mu + \mu \log M + h(\mu) $$

$$ = K^{(1)}(I^{\otimes k}) + 2\mu + \mu \log M + h(\mu), $$

where the equality above is by definition of the function $K^{(1)}$. \qed
VI. DISCUSSION OF REGULARITY OF COMPOUND CQ Sources

This section is of twofold purpose. First, we point out that regularity issues have operational significance for forward secret-key distillation from tripartite compound sources. While for regular sources, there is no gap between the forward secret-key key distillation capacities with and without SMI, there may be serious differences in capacities, if the source is not regular. Second, we introduce a weaker notion of regularity than the one introduced in Definition 6, where we utilize notions from the theory of set-valued functions.

A. Operational significance of regularity conditions

We have seen in the previous section, that there is no difference between the forward secret key distillation capacities with and without SMI, as long as the source is regular in the sense of Definition 6. We admit, that there may be weaker notions of regularity which also exhibit this property (an example of such a condition is introduced in the next section). Regularity conditions seem somewhat technical on a first view. One can easily imagine large classes of sets of cq density matrices, which are notoriously easy to process even in the case without sender knowledge, while being irregular. This feature is shared in a trivial way by all irregular sources having zero forward secret key distillation capacity under sender knowledge. The following example depicts the fact, that also in nontrivial cases irregularities may be of no consequences for the behaviour of the source regarding forward secret-key distillation.

Example 20. Define for a finite alphabet \( X, A := \{ p \in \mathcal{D}(X) \colon \forall x \in X \colon p(x) \in Q \} \), \( V \in CQ(\lambda, \mathcal{H}_B \otimes \mathcal{H}_E) \), and let \( K_B = C^{\otimes 2} \) be the Hilbert space of an additional system assigned to the legitimate receiver. Define states \( \rho_a := \sum_{x \in X} a(x) |x\rangle \langle x| \otimes V_{BE}(x) \otimes |e_a\rangle \langle e_a| \)

with \( \{ e_1, e_2 \} \) being an orthonormal basis in \( K_B \). \( e_a := e_1 \) if \( a \in A \) and \( e_a = e_2 \) if \( a \in A^c \). The source defined by \( I := \{ \rho_a \}_{a \in \mathcal{D}(X)} \) is not regular, but can be easily converted to a regular one by just discarding the systems on \( K_B \).

Beside the mentioned facts, the question of regularity in principle, bears strong operational significance. The next Theorem shows, that for irregular sources, the capacities with and without sender marginal state knowledge may be substantially different.

Theorem 21. The equality

\[ K_{\rightarrow SMI}(I) = K_{\rightarrow}(I) \]

does not hold in general.

Proof. We construct an example of a set \( I \) with

\[ K_{\rightarrow SMI}(I) = 1 \quad \text{and} \quad K_{\rightarrow}(I) = 0. \]  \( \tag{82} \)

Let \( X = Y = \{ 0, 1 \} \), and \( \mathcal{H}_B = \mathcal{H}_E = C^2 \otimes C^2 \). We introduce classical-quantum channels \( W_1, W_2 : \{ 0, 1 \} \rightarrow \mathcal{S}(C^2 \otimes C^2) \) by

\[
W_1(x,y) = W_1(x) := |x\rangle \langle x| \otimes \Pi, \\
W_2(x,y) = W_2(y) := \Pi \otimes |y\rangle \langle y| \\
((x,y) \in X \times Y),
\]

where \( \Pi := \frac{I}{2} \) is the flat state on \( C^2 \). We set

\[
V_{1,B} = V_{2,E} = W_1, \quad V_{2,B} = V_{1,E} = W_2,
\]

and define states

\[
\rho_p \begin{cases} \sum_{x \in X} \frac{1}{2} |x\rangle \langle x| \otimes \Pi \otimes \Pi X_{1,B}(x) \otimes X_{1,E}(y) & \text{if } p = \pi \\
\sum_{x \in X} \frac{1}{2} p(y) |x\rangle \langle x| \otimes \Pi X_{2,B}(y) \otimes X_{2,E}(x) & \text{otherwise}, \end{cases}
\]

where \( \pi \) denotes the equidistribution on \( \{ 0, 1 \} \), i.e. \( p(0) = p(1) = \frac{1}{2} \). Consider the set \( I := \{ \rho_p : p \in \mathcal{D}(Y) \} \). We first show the left equality in (82). If we define stochastic matrices \( \rho^1_{U|XY}, \rho^2_{U|XY} : X \times Y \rightarrow \mathcal{U} := \{ 0, 1 \} \) with entries

\[
\rho^1_{U|XY}(u|x,y) := \delta_{xu}, \quad \text{and} \quad \rho^2_{U|XY}(u|x,y) := \delta_{yu} \quad (x \in X, y \in Y, u \in \mathcal{U}),
\]

and define states

\[
\rho_p \begin{cases} \sum_{x \in X} \frac{1}{2} |x\rangle \langle x| \otimes \Pi \otimes \Pi X_{1,B}(x) \otimes X_{1,E}(y) & \text{if } p = \pi \\
\sum_{x \in X} \frac{1}{2} p(y) |x\rangle \langle x| \otimes \Pi X_{2,B}(y) \otimes X_{2,E}(x) & \text{otherwise}, \end{cases}
\]

where \( \pi \) denotes the equidistribution on \( \{ 0, 1 \} \), i.e. \( p(0) = p(1) = \frac{1}{2} \). Consider the set \( I := \{ \rho_p : p \in \mathcal{D}(Y) \} \). We first show the left equality in (82). If we define stochastic matrices \( \rho^1_{U|XY}, \rho^2_{U|XY} : X \times Y \rightarrow \mathcal{U} := \{ 0, 1 \} \) with entries

\[
\rho^1_{U|XY}(u|x,y) := \delta_{xu}, \quad \text{and} \quad \rho^2_{U|XY}(u|x,y) := \delta_{yu} \quad (x \in X, y \in Y, u \in \mathcal{U}),
\]
and use the sender’s preprocessings $P^{(1)}_{U|X|Y}$ for $\rho_\pi$ and $P^{(2)}_{U|X|Y}$ for each $p \neq \pi$, we achieve the maximum in the capacity formula derived in Theorem 1. The corresponding states are

$$\rho_\pi := \sum_{u \in U} \sum_{x \in X} \sum_{y \in Y} P^{(1)}_{U|X|Y}(u|x,y) \frac{1}{4} |u\rangle \langle u| \otimes V_{1,\mu}(x) \otimes V_{1,E}(y)$$

$$= \sum_{u \in U} \frac{1}{2} |u\rangle \langle u| \otimes |u\rangle \otimes \Pi \otimes \Pi \otimes \Pi$$

$$\rho_p := \sum_{u \in U} \sum_{x \in X} \sum_{y \in Y} P^{(2)}_{U|X|Y}(u|x,y)p(x) \frac{1}{2} |u\rangle \langle u| \otimes V_{2,B}(y) \otimes V_{2,E}(x)$$

$$= \sum_{u \in U} \frac{1}{2} |u\rangle \langle u| \otimes \Pi \otimes |u\rangle \otimes \sigma_{p,E}$$

where $\sigma_{p,E} := \sum_{x \in X} p(x)V_{2,E}(x)$. Note, that both of the above states contain perfect common randomness between the legitimate users without sharing any correlations to the eavesdropper, which is the optimum they can achieve, as is easily observed. It therefore holds

$$K_{\rightarrow SMI}(I) = \log 2 = 1.$$  

The situation is completely different, if no SMI is present. Let $\mu > 0$ be fixed and $(T,D)$ an arbitrary $(n,M,L,\mu)$ forward secret key distillation protocol for $I$ without SMI. I.e. the inequalities

$$\Pr(K_p \neq K_\pi) \leq \mu$$  

and

$$\log M - H(K_p) + I(K;A|E^n,\rho_{KA|E^n,p}) \leq \mu$$  

are satisfied for each $p \in \mathcal{P}(\mathcal{Y})$. If we define the states

$$\rho_p := \sum_{x \in X} \sum_{y \in Y} p(x) \frac{1}{2} |x \otimes y\rangle \langle x \otimes y| \otimes V_{1,B}(x) \otimes V_{2,E}(x)$$

for each $p \in \mathcal{P}(\mathcal{Y}) \setminus \{\pi\}$, the identity

$$I(K;A|E^n,\rho_{KA|E^n,p}) = I(K;A^B^n,\rho_{KB|E^n,p})$$

is fulfilled by symmetry. Moreover, it holds

$$\|\rho_{KB|E^n,p} - \rho_{KB|E^n,p}\|_1 \leq \|\text{tr}_{E^n}|\rho_\pi|^{\otimes n} - \text{tr}_{E^n} \rho_{E^n}^{\otimes n} \rho_\pi^{\otimes n} \|_1 = \|p^n - \pi^n\|_1 \leq n\|p - \pi\|_1$$

where the first inequality is by ctp monotonicity of the trace norm distance, and the second is by construction. Combining and with Fannes’ inequality for the quantum mutual information, we yield

$$I(K;A|E^n,\rho_{KA|E^n,p}) \leq I(K;A^B^n,\rho_{KB|E^n,p})$$

$$\leq I(K;A^B^n,\rho_{KB|E^n,p}) + f(n\|p - \pi\|_1)$$

for each $p \in \mathcal{P}(\mathcal{Y}) \setminus \{\pi\}$, where $f$ is a function with $f(a) > 0$ for all $a > 0$, and $f(a) \to 0, (a \to 0)$. Therefore, we have for each $p \neq \pi$

$$\log M \leq H(K_p) - I(K;A|E^n,\rho_{KA|E^n,p}) + \mu$$

$$\leq H(K_p) - I(K;A^B^n,\rho_{KB|E^n,p}) + \mu + f(n\|p - \pi\|_1)$$

$$\leq H(K_p) - I(K_\pi;K'_\pi) + \mu + f(n\|p - \pi\|_1)$$

$$\leq H(K_p) - I(K_\pi) + H(K'_\pi|K'_\pi) + \mu + f(n\|p - \pi\|_1)$$

$$\leq H(K_p) - I(K_\pi) + \mu \log M + h(\mu) + \mu + f(n\|p - \pi\|_1)$$

$$\leq \mu \log M + h(\mu) + \mu + 2f(n\|p - \pi\|_1).$$

where the first inequality is by , the second is by , the third is by the quantum data processing inequality, the fifth by Fano’s inequality together with , and the last is by Fannes’ inequality. By taking the infimum over all $p$ in the above inequality arrive at

$$\log M \leq \mu \log M + h(\mu) + \mu.$$  

We conclude, that $R = 0$ is the only achievable forward secret key distillation rate.
Definition 23. Let \( f \) be a set-valued map. We define for each \( \omega \in \Omega \)
\[
    f^+(\omega) := \{ \theta \in \Theta : f(\theta) \subset \omega \}, \quad \text{and} \quad f^-(\omega) := \{ \theta \in \Theta : f(\theta) \cap \omega \neq \emptyset \}.
\]

Proposition 24. Let \( f : \Theta \to 2^\Omega \) be a set-valued map. We define for each \( E \subset \Omega \)
\[
    f^+(E) := \{ \theta \in \Theta : f(\theta) \subset E \}, \quad \text{and} \quad f^-(E) := \{ \theta \in \Theta : f(\theta) \cap E \neq \emptyset \}.
\]

Definition 23. We call a set-valued map \( f : \Theta \to 2^\Omega \)

1. upper hemi-continuous, if for each \( \theta \in \Theta \) the following is true. Whenever \( \theta \in f^+(E) \) for an open set \( E \), there is a neighbourhood \( U(\theta) \) of \( \theta \) with \( U(\theta) \subset f^+(E) \).

2. lower hemi-continuous, if for each \( \theta \in \Theta \) the following is true. Whenever \( \theta \in f^-(E) \) for an open set \( E \), there is a neighbourhood \( U(\theta) \) of \( \theta \) with \( U(\theta) \subset f^-(E) \).

3. continuous, if \( f \) is both upper and lower hemi-continuous.

We will always regard \( \Theta \) and \( \Omega \) being finite-dimensional. In this case, we obtain sequential characterizations of upper and lower hemi-continuity, if we assume the set-valued function to have only compact values.

Proposition 24. Let \( f : \Theta \to 2^\Omega \) be a set-valued map with \( \Theta \subset \mathbb{R}^m \), \( \Omega \subset \mathbb{R}^k \), and \( f(\theta) \) compact for each \( \theta \in \Theta \).

It holds

1. \( f \) is upper hemi-continuous if and only if for each \( \theta \in \Theta \), every sequence \( (\theta_n)_{n \in \mathbb{N}} \) with \( \theta_n \to \theta \) \( (n \to \infty) \) and \( \omega_n \in f(\theta_n) \), \( n \in \mathbb{N} \) there is a subsequence \( (\omega_{n_k})_{k \in \mathbb{N}} \) with \( \lim_{k \to \infty} \omega_{n_k} \in f(\theta) \).

2. lower hemi-continuous, if for each \( \theta \in \Theta \) and sequence \( (\theta_n)_{n \in \mathbb{N}} \subset \Theta \), and \( \omega \in f(\theta) \) from \( \lim_{n \to \infty} \theta_n = \theta \) it follows, that there is a sequence \( (\omega_n)_{n \in \mathbb{N}} \) with \( \omega_n \in f(\theta_n) \), \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \omega_n = \omega \).

Proof. See [11], Proposition 11.11. \( \square \)

For our considerations the closed-graph characterization of upper hemi-continuity will be of utility.

Theorem 25. Let \( \Theta \subset \mathbb{R}^m \), \( \Omega \subset \mathbb{R}^k \), \( f : \Theta \to 2^\Omega \) be a set-valued map with \( \Omega \) being compact. If the graph of \( f \), i.e. the set

\[
    \text{Gr}_f := \{ (\theta, \omega) \in \Theta \times \Omega : \omega \in f(\theta) \}
\]

is closed, then \( f \) is upper hemi-continuous.

Proof. See for example Proposition 11.9 in [11] \( \square \)

We need the following basic Lemma.

Lemma 26. If a set-valued function is lower hemi-continuous, then its closure \( \overline{f} \) (i.e. the function defined by closing the graph of \( f \)) is lower hemi-continuous as well.

Proof. Assume, that there is a sequence \( (\theta_n)_{n \in \mathbb{N}} \) with \( \theta_n \to \theta \) and \( \omega \in \overline{f}(\theta) \), such that no sequence \( (\omega_n)_{n \in \mathbb{N}} \) exists with \( \omega_n \in f(\theta_n) \) for all \( n \in \mathbb{N} \) and \( \omega_n \to \omega \). If \( \omega \) is in \( f(\theta) \), such a sequence always exists by lower hemi-continuity of \( f \). If \( \omega \) is in \( f(\theta) \setminus f(\theta) \) the hypothesis is only true if \( \omega \) is no point of accumulation of \( f(\theta) \), which contradicts the definition of \( f \). \( \square \)
Definition 27. We call a set $I \subset S_{\text{cqq}}(\mathcal{H}_{AB})$ weakly regular, if the set-valued map

$$f_{AX} : P_1 \rightarrow 2^S_{\text{cqq}(\mathcal{H}_{AX})}$$

$$p \mapsto \hat{f}_p^{AX}$$

is lower hemi-continuous for $X = B, E$.

Proposition 28. Let $I \subset S_{\text{cqq}}(\mathcal{H}_{ABE})$ be a weakly regular set of cqq density matrices. It exists a regular set $\hat{I} \subset S_{\text{cqq}}(\mathcal{H}_{ABE})$ with

1. $I \subset \hat{I}$
2. $K_{\rightarrow}(I) \leq K_{\rightarrow}(\hat{I})$.
3. $\hat{I}$ is regular.

Proof. Assume $I$ being parameterized as

$$I := \left\{ \rho_{(p,V)} : \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x| \otimes V(x) \right\}_{(p,V) \in S}$$

with

$$S := \bigcup_{p \in P_1} \{ p \} \times V_p$$

with sets $P_1 \in \mathcal{P}(\mathcal{X}), V_p \subset \text{CQ}(\mathcal{Y}, \mathcal{H}_{BE}), p \in P_1$. We define $\hat{I}$ as the closure of $I$. Obviously, the first condition $I \subset \hat{I}$ stated in the proposition is fulfilled. We show that the two remaining conditions are also fulfilled.

Assume, that $(T, D)$ is an $(n, M, L, \mu)$ forward secret-key distillation protocol for $I$. Since the performance and security criteria in Definition 27 are defined in terms of functions being continuously dependent on the cqq density matrix, it is clear, that $(T, D)$ is an $(n, M, L, \mu)$ forward secret-key distillation protocol also for $\hat{I}$, which directly implies, that also the second claim of the proposition is satisfied. For validating the third claim we notice, that since $\hat{I}$ is closed, the corresponding set-valued functions $f_{AB}$ and $f_{AE}$ have closed graphs. Therefore both maps are upper hemi-continuous by Theorem 25. The hypothesis of $I$ being weakly regular, together with Lemma 26 ensures us, that $F_{AB}$ and $F_{AE}$ are also lower hemi-continuous. Therefore, they are continuous. Since the set of sender marginal distributions $P_1$ deriving from $\hat{I}$ is a compact set, we infer, that $f_{AB}, f_{AE}$ are uniformly continuous, which implies, that for each $\epsilon > 0$ we find a $\delta > 0$, such that the implication

$$||p - q|| < \delta \Rightarrow d_H(f_{AB}(p), f_{AB}(q)) + d_H(f_{AE}(p), f_{AE}(q)) < \epsilon$$

for each $p, q \in P_1$. Since

$$d_H(\hat{f}_p^{AB}, \hat{f}_q^{AB}) + d_H(\hat{f}_p^{AE}, \hat{f}_q^{AE}) = d_H(f_{AB}(p), f_{AB}(q)) + d_H(f_{AE}(p), f_{AE}(q))$$

holds by definition, $\hat{I}$ is regular.

Theorem 29. Let $I$ be a weakly regular set of cqq density matrices on $\mathcal{H}_{ABE}$. It holds

$$K_{\rightarrow}(I) = \lim_{k \to \infty} \frac{1}{k} K^{(1)}_{\rightarrow}(I^{\otimes k})$$

Proof. We approximate $I$ by the set $\hat{I}$ as defined in the proof of Proposition 28. The first and second property of $\hat{I}$ in Proposition 28 together imply that

$$K_{\rightarrow}(I) = K_{\rightarrow}(\hat{I}) = \lim_{k \to \infty} \frac{1}{k} K^{(1)}_{\rightarrow}(\hat{I}^{\otimes k})$$

holds. The rightmost of the above inequalities is by application of Theorem 8 on $\hat{I}$, which is possible, because $\hat{I}$ is regular by Proposition 28. In fact, $d_H(I, \hat{I}) = 0$, and consequently $d_H(I^{\otimes k}, \hat{I}^{\otimes k}) = 0$ holds for each $k \in \mathbb{N}$. Therefore

$$K^{(1)}_{\rightarrow}(I^{\otimes k}) = K^{(1)}_{\rightarrow}(\hat{I}^{\otimes k})$$

holds for each $k \in \mathbb{N}$ by continuity of $K^{(1)}$. We are done.
VII. SPECIAL CASE: FORWARD SECRET-KEY DISTILLATION CAPACITY OF A CLASSICAL TRIPARTITE COMPOUND SOURCES

Our results also cover the case of a completely classical tripartite source. Let \((X, Y, Z)\) be a triple of classical random variables with distribution \(P_{X,Y,Z} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})\). The state of this classical system coherified to a Hilbert space \(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z\) is represented by the density matrix

\[
\rho := \sum_{(x,y,z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}} P_{X,Y,Z}(x,y,z) |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes |z\rangle\langle z|.
\]  

(101)

Forward secret-key distillation for this kind of classical compound memoryless source was considered in previous work \([25]\) done under collaboration of one of the authors of this paper. Among other results obtained therein, it was derived a capacity formula for the case without sender marginal knowledge in case, that the set of sender marginal distributions deriving from the source is finite. Our results extend the capacity description also to the case of an arbitrary regular tripartite classical source. By coherifying each set \((X_s, Y_s, Z_s)\) of triples into a mutually commuting set of density matrices as in (101), Theorem 8 directly leads to a version of Theorem 2 in \([25]\) where instead of assuming finite cardinality of the set of marginal states the assumption is that the compound source is regular. Theorem 9 in the same way provides a capacity formula for the case where the sender party perfectly knows the distribution of his/her part of the source. The reader may reply, that the definition of a (deterministic) classical decoding procedure is more restrictive than that of a POVM in quantum theory, since the decoding sets are demanded to be pairwise nonintersecting. Our reasoning is not affected by this fact, because in case of pairwise commuting density operators, optimal decoding can always be achieved by using projection valued measures. Alternatively, one could replace Lemma 11 by a completely classical version.

We point out, that the need for a regularity condition as well as differences between the capacities with and without sender’s knowledge of the marginal state are not effects of the quantum nature of the sources considered in this paper. The reader may note, that our counterexample given to prove Theorem [21] is essentially classical, since all density matrices involved pairwise commute.

Since there are stronger results known in classical information theory (especially regarding error exponents for coding of classical compound channels), a classical method of proof will lead to faster decrease of error with a potentially simplified proof.

VIII. CONCLUSION

We have considered the the task of secret key distillation under free forward classical communication for compound memoryless sources with classical legitimate sender and quantum legal receiver and eavesdropper outputs. We derived a capacity formula for all sources of this class which additionally exhibit a certain regularity condition.

We also discussed the situation, where the legitimate sender has perfect knowledge of the probability distribution governing his/her outputs. In this case, we were able to derive a capacity formula which equals the one given for the case without sender marginal information for regular sources, and moreover does hold for all nonregular sources.

As we have also seen, the capacities with and without sender marginal information differ at least for some nonregular sources. We admit, that the regularity conditions assumed in this paper may be somewhat weakened to determine the forward secret-key generation capacity without sender knowledge for a larger class of sources. We provided a further step in this direction by applying the general theory of set-valued maps to derive a slightly broader class of compound cqq sources with a general capacity description.

We leave open the more general case of proving a capacity theorem for forward secret-key distillation from compound sources where the generating set of density matrices may contain members being not in the class of cqq density matrices. In [16] such a theorem were proven in case of a perfectly known source without restriction on the legitimate sender to be classical.

The strategy therein to prove a coding theorem was, to combine an achievability result for cqq sources with an optimization over instruments dephasing the sender’s system to a classical one. Notice, that such a strategy in general does not apply to compound sources in a direct way as it did in [16] (at least in case that the sender does not have perfect marginal knowledge). In general, there is no control wether or not a dephasing operation leads to a nonregular compound cqq source. However, approximation techniques presented here may lead to a better understanding of the secret-key distillation task even for tripartite quantum compound sources.
Another astonishing result in [16] is the close correspondence between the forward secret-key distillation and entanglement distillation tasks. It was demonstrated, that modifying forward secret-key distillations leads to one-way local operations and classical communications protocols suitable for proving the so-called hashing bound eventually determining the entanglement distillation capacity of bipartite memoryless quantum sources.

The authors of this paper are of the opinion, that following a similar strategy to derive the entanglement distillation capacity of bipartite compound memoryless quantum sources may be not successful in the same way as it is with perfect knowledge of the source. A closer look to the corresponding considerations in [16] may underpin this opinion. Therein, an important part of the coding strategy was to apply a non-destructive measurement on the sender’s marginal of the bipartite quantum state subject to entanglement distillation – a strongly state-dependent task, which can hardly be performed without sender marginal knowledge. For a generalization to the case of bipartite compound quantum case under assumption of SMI, the strategy may be feasible. We did not pursue this path, because the one-way entanglement distillation capacity of compound quantum sources is already known from [7] and [8].

This parallels a similar observation made in [9] for channel coding. Therein, it was argued, that the ingenious way to derive entanglement generation codes for quantum channels from codes for secret message transmission over classical-quantum wiretap channels used in [17] for proving the quantum coding theorem leads to suboptimal results for compound channels if no sender state knowledge is assumed.

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Appendix A: Universal random constant composition codes for compound DMCQ Channels

In this section, we state and prove some results on compound DMcq channels we need within the proof of Lemma 30. For convenience of the reader, we first provide definitions, necessary to understand the subsequent arguments. Let $V \subset CQ(X,K)$ be a set of cq channels mapping a finite alphabet $X$ to the set of density matrices on a Hilbert space $K$. The compound discrete memoryless classical quantum channel generated by $V$ (the DMcq $V$ for short) is given by the family $\{V^\otimes n : V \in V\}_{n \in \mathbb{N}}$ of possible outputs. To catch up with the notation in [6], we sometimes write $V = \{V^\otimes n : V \in V\}$ assuming a suitable parameterization of $V$ by an index set $S$. For given blocklength $n \in \mathbb{N}$, $M \in \mathbb{N}$, an $(n,M)$-code for transmission of classical messages over $V$ is a family $\mathcal{C} := \{u_m,D_m\}_{m=1}^M$ with $u_m \in X^n$, $D_m \in L(K^\otimes n)$ for each $m \in [M]$, with the additional property, that for all $m \in [M]$,

$$0 \leq D_m \leq \mathbb{1}^n_K, \quad \text{and} \quad \sum_{m=1}^M D_m = \mathbb{1}^n_K$$

holds. For given $(n,M)$-code $\mathcal{C}$, $s \in S$, we define the average error of transmission by

$$\mathcal{E}(\mathcal{C},V^\otimes n) := \frac{1}{M} \sum_{m=1}^M \text{tr}(D_m^\perp V^\otimes n(u_m)),$$

where we allow ourselves to define $A^\perp := \mathbb{1}^n_K - A$ (even if $A$ is not a projector). The following two Propositions combined, immediately imply the proof of Lemma 30 stated in the text. The claim of the first one follows by careful modification of the proof of Theorem 5.10 in [5] and delivers for each large enough blocklength and each type of sequences a suitable random compound transmission code having superpolynomially decrease of error, universal regarding the channels in the compound set, as well as the appearing types.

**Proposition 30.** Let $W \subset CQ(X,K)$ be an arbitrary set of cq channels. For each $\delta > 0$, there is a number $n_2 := n_2(\delta)$ such that for each $n > n_2$ and each type $\lambda \in \mathcal{T}(n,X)$ there exists a random $(n,M_1)$-code $\mathcal{C}_\lambda(U) := (U_m,D_m(U))_{m=1}^{M_1}$ for $W$ with the following properties

1. $U := (U_1,...,U_{M_1})$ is an i.i.d. sequence of random variables, each with values in $X^n$ and distribution $\lambda^\otimes n$. 
2. \( M_0 \geq \exp \left\{ n \left( \inf_{W \in \mathcal{W}} \chi(\lambda, W) - \delta \right) \right\} . \)

3. \( \mathbb{E} \left( \max_{W \in \mathcal{W}} C_{\lambda}(W^*W) \right) \leq 2^{-\frac{1}{n} \psi(n)} \)

with a constant \( c(\delta) > 0. \)

**Proof.** The assertion is basically contained in the proof of Theorem 5.10 in \([5]\) and follows by minor modifications of the argument given there. We assume the reader’s familiarity with the arguments presented in \([5]\) and restrict ourselves to indicate the steps of modification necessary to justify our claim.

We write \( \mathcal{W} := \{ W_t \}_{t \in T} \) with a suitable index set \( T \). Let \( n \in \mathbb{N} \), and consider a type \( \lambda \in T(n, \mathcal{X}) \). Assume, that

\[
\inf_{t \in T} \chi(\lambda, W_t) - \delta > 0, \tag{A1}
\]

holds, since otherwise the claim is trivially fulfilled for \( \lambda \). Choose an approximating set \( \mathcal{W}_n := \{ W_t' \}_{t \in T_n} \) for \( \mathcal{W} \) as used in \([5]\). We will execute the suggestions given in Remark 5.11 in \([5]\), and therefore choose the diameter of the partition of output states used to define \( \mathcal{W}_n \) to be \( 2^{-\frac{1}{n} \psi(n)} \) instead of \( \frac{1}{n^2} \). Define

\[
\Omega_{\lambda,n} := \{ \rho_{t,n} := \sum_{x \in \mathcal{X}} \lambda(x) |x\rangle \langle x| \otimes W_t(x) : t \in T_n \},
\]

\[
\lambda := \sum_{x \in \mathcal{X}} \lambda(x) |x\rangle \langle x|, \quad \text{and} \quad \sigma_{t,n} := \sum_{x \in \mathcal{X}} \lambda(x) \sigma_t(x) (t \in T_n).
\]

Note, that the properties of the set \( \mathcal{W}_n \) in \([5]\) (see Lemma 5.6 therein) together with the above definitions, implies the bound

\[
\lambda_{\min}(\lambda \otimes \sigma_{t,n}') \geq \min_{x \in \text{supp}(\lambda)} \lambda(x) \cdot \frac{1}{d} \cdot 2^{-\frac{1}{n} \psi(n)}
\]

\[
\geq \frac{2^{-\frac{1}{n} \psi(n)}}{n \cdot d} \tag{A2}
\]

on the minimal eigenvalue \( \lambda_{\min}(\lambda \otimes \sigma_{t,n}') \) of \( \lambda \otimes \sigma_{t,n}' \). The last estimate above follows from the fact, that \( \lambda \) is a type of sequences in \( \mathcal{X}^n \). Closely following the argument given in \([5]\), we are ensured, that choosing \( l_n = \lceil \sqrt{n} \rceil \), we find \( a_n, b_n \in \mathbb{N}, \) \( 0 \leq b_n < l_n \), with

\[
n = a_n l_n + b_n,
\]

and a PVM \( M_{l_n,\lambda} := \{ P_{l,n} \}, P_{l,n} \) \( t \in T_n \), such that for all \( t, s \in T_n \)

\[
S_{M_{l_n,\lambda}}(\rho_{t,n}^\lambda, \sigma_{s,n}'^\lambda) \geq l_n (\min_{t \in T_n} \chi(\lambda, W_t) - \xi_{l_n}(\lambda \otimes \sigma_{l_n,s}'^\lambda)) \tag{A3}
\]

holds. Careful investigation of the function \( \xi_{l_n} \) in \([5]\) using the type-independent bound in \([A2]\) shows that

\[
\lim_{n \to \infty} \max_{l \in T(n, \mathcal{X})} \xi_{l_n}(\lambda \otimes \sigma_{l_n,s}') = 0 \tag{A4}
\]

holds. Introducing the refinement \( \mathcal{Q}_{l_n,\lambda} \) of the PVM \( M_{l_n,\lambda} \), and the stochastic matrices \( V_{l_n,\lambda}, t \in T_n \) generated by \( Q_{l_n,\lambda} \) as in \([5]\) (note, that these, also may depend on the chosen type \( \lambda \)), it holds

\[
\frac{1}{l_n} \min_{t \in T_n} F(\lambda \otimes \sigma_{l_n,s}', V_{l_n,\lambda}) \geq \inf_{t \in T} \chi(\lambda, W_t) - n \cdot 2^{-\frac{1}{n} \psi(n)} C(d) - \xi_{l_n,\max},
\]

where we used \( \xi_{l_n,\max} \) defined by

\[
\xi_{l_n,\max} := \max_{\lambda \in T(n, \mathcal{X})} \max_{t \in T_n} \xi_{l_n}(\lambda \otimes \sigma_{l_n,s}').
\]
Since (A3) holds, we find for each $0 < \eta < \delta$ a number $n_2(\eta) \in \mathbb{N}$ (independent of $\lambda$) such that

$$\frac{1}{T_n} \min_{t \in T_n} I(\lambda^{\otimes n}, V_{t,\lambda}) \geq \inf_{t \in T} \chi(\lambda, W_t) - \eta > 0 \quad \text{(A5)}$$

is fulfilled for all $n > n_2$. Note, that the last inequality holds by (A1). The bound above differs a bit from the one given in Eq. (30) in [3]. However, it will be sufficient for the following argument. Let

$$\Theta := \left\{ \theta \in \mathbb{R} : 0 < \theta < \frac{\eta}{4} \right\}, \quad \text{(A6)}$$

and

$$I_{n,\lambda} := \min_{t \in T_n} I(\lambda^{\otimes n}, V_{t,\lambda}).$$

Following the lines of [3] (always respecting dependencies on $\lambda$), we yield the bound

$$\Pr(i_{t,\lambda}^{\alpha_n} \leq I_{n,\lambda} - 2a_n \theta) \leq \frac{1}{|T_n|} \sum_{t \in T_n} \Pr(i_{t,\lambda}^{\alpha_n} \leq I_n - I_n \theta) + |T_n| 2^{-a_n I_n \theta}. \quad \text{(A7)}$$

Since $i_{t,\lambda}^{\alpha_n}$ is a sum of i.i.d. random variables each with values in the interval

$$[-I_n d \sqrt{\frac{n}{n}}, I_n d \sqrt{\frac{n}{n}}],$$

and

$$I_{n,\lambda} \leq \mathbb{E}_{t,\lambda}(i_{t,\lambda}^{\alpha_n})$$

holds for each $t \in T_n$, our counterpart to eq. (34) in [3],

$$\Pr(i_{t,\lambda}^{\alpha_n} \leq I_n - I_n \theta) \leq e^{16 \frac{a_n \theta^2}{|\Theta|}}$$

is valid. By closely following the lines of [3] (having in mind our bounds) together with the choice $\theta = \frac{\eta}{4}$, we know, that there is a projection $P_{n,\lambda,\theta}$ with

$$\text{tr}(\rho^{(n)}_{\lambda} P_{n,\lambda,\theta}) \geq 1 - e^{16 \frac{a_n \eta^2}{|\Theta|}} - |T_n| 2^{-a_n I_n \theta} \quad \text{(A8)}$$

and

$$\text{tr}(\chi(\lambda^{\otimes n} \otimes \sigma^{(n)}_{\lambda}) P_{n,\lambda,\theta}) \leq 2^{-a_n(I_n - 2a_n \theta)} \leq 2^{-a_n(I_n \inf_{t \in T} \chi(\lambda, W_t) - \frac{3}{4} \eta)}, \quad \text{(A9)}$$

where the last of the above inequalities above holds by (A5). Since $b < \sqrt{n}$,

$$\text{tr}(\chi(\lambda^{\otimes n} \otimes \sigma^{(n)}_{\lambda}) P_{n,\lambda,\theta}) \leq 2^{-n(I_n \inf_{t \in T} \chi(\lambda, W_t) - 2\eta)} \quad \text{(A8)}$$

holds for each $n > n_2$, if $n_2$ is chosen large enough. Setting $\eta := \frac{\delta}{4}$, and using the bounds in (A7) and (A8) together with Theorem 1.1. stated in the Appendix of [3] (and leaving out the derandomization step leading to a deterministic code in the Proof of Theorem 5.10 in [3]), we conclude that there is an $(n, M_1)$ random code $C_1(U) := (U_m, D_m(U))_{m=1}^{M_1}$ for the average channel $W^{\otimes n} := \frac{1}{|T_n|} \sum_{t \in T_n} W_t^{\otimes n}$ with $U = (U_1, \ldots, U_M)$ being a sequence of i.i.d. random variables each with values in $\chi^{\otimes n}$ and generic distribution $\lambda^{\otimes n}$, such that

1. $M \geq \left\lceil \exp \left\{ n \left( \inf_{t \in T} \chi(\lambda, W_t) - \frac{\delta}{2} \right) \right\} \right\rceil.$

2. $\mathbb{E}[\mathcal{C}_1, W^{\otimes n}] \leq 2^{-\frac{1}{4} \sqrt{n} \delta}$
with a positive constant \( c(\delta) > 0 \). From this, we can conclude, that there is a number \( n_0(\delta) \), such that for each \( n > n_0(\delta) \) (independent of \( \lambda \)) \( \mathcal{C}_\lambda \) is an \((n, M)\) random code of the above stated properties, with 2. above replaced by

\[
\mathbb{E} \left[ \sup_{W \in \mathcal{W}} \mathfrak{t}(\mathcal{C}_\lambda, W^\otimes n) \right] \leq 2^{-\frac{1}{2}c(\delta)},
\]

where \( c(\delta) := \frac{1}{2}c(\delta) \).

The following proposition is a compound version of Proposition 2.5 in [14]. It is proven by exactly the same strategy replacing the Holevo-Schumacher-Westmoreland codes for DMcqC with perfectly known generic cq channel by the codes constructed in [5] together with the modifications done in Proposition 30 above.

**Proposition 31** (cf. [14], Prop. 2.5). Let \( \mathcal{W} \subset CQ(\mathcal{X}, \mathcal{K}) \) be an arbitrary set of cq channels, \( n \in \mathbb{N} \) and \( \lambda \in T(n, \mathcal{X}) \) a type of sequences in \( \mathcal{X}^n \). If there exists a random random \((n, M')\)-classical message transmission code

\[
C'(U) = (U_m, D_{m}(U))_{m=1}^{M'}
\]

for the DMcqC \( \mathcal{W} \) which has the properties

1. \( U := (U_1, \ldots, U_{M'}) \) is an i.i.d. sequence of random variables with values in \( \mathcal{X}^n \) with generic distribution \( \lambda^n \)
2. \( \mathbb{E} \left[ \sup_{W \in \mathcal{W}} \mathfrak{t}(C, W^\otimes n) \right] \leq \mu \)

with \( \mu \in (0, 1) \), then there exists for each given \( \delta \in (0, 1) \) a random \((n, M)\)-message transmission code

\[
C(V) := (V_m, D_{m}(V))_{m=1}^{M}
\]

having the properties

1. \( V := (V_1, \ldots, V_{M}) \) is an i.i.d. sequence of random variables, each equidistributed on \( T_\lambda^n \).
2. \( M = [\delta \cdot (n + 1)^{1/|X|} \cdot M'] \).
3. and

\[
\mathbb{E} \left[ \sup_{W \in \mathcal{W}} \mathfrak{t}(C', W^\otimes n) \right] \leq \frac{2}{\delta} (n + 1)^{1/|X|} \mu + 2^{-M'(1-\delta^2)(n+1)^{-1}\ln 2}.\]

For proving the above assertion, we will make use of the following variant of the Chernov bound

**Proposition 32**. Let \( n \in \mathbb{N} \), \( \delta > 0 \) and \( X = (X_1, \ldots, X_n) \) be an i.i.d. sequence of random variables with \( 0 \leq X_1, \ldots, X_n \leq 1 \) and \( \mathbb{E}[X_i] = E \), \( i \in [n] \), then

\[
\Pr \left( \frac{1}{n} \sum_{i=1}^{n} X_i \leq (1 - \delta)E \right) \leq 2^{-n\delta^2E^2/\ln 2}.
\]

**Proof of Proposition 32** Define the event, that at least \( M \) codewords of a codebook \( u \) are \( \lambda \)-typical sequences by

\[
A(M) := \{ u = (u_1, \ldots, u_{M'}) \in \mathcal{X}^{nM'} : |\{ m : u_m \in T_{\lambda}^n \}| \geq M \}.
\]

For an i.i.d. sequence \( U = (U_1, \ldots, U_{M'}) \) as in the hypotheses of the proposition, it holds

\[
\Pr (A(M)^c) = \Pr \left( \sum_{i=1}^{M'} \mathbb{I}_{T_{\lambda}^n}(U_m) < M \right).
\]

Notice, that

\[
\mathbb{E} [ \mathbb{I}_{T_{\lambda}^n}(U_m) ] = \lambda^n(T_{\lambda}^n) \geq (n + 1)^{-1/|X|}.
\]
Then, define a function $\varphi : \mathcal{X}^{nM'} \to (T^a_n)^M$ by
\[
\varphi(u) := \begin{cases} (v_1, \ldots, v_M) = \nu & \text{if } u \in A(M) \\
(v, \ldots, v) & \text{otherwise}, \end{cases}
\]
where $v$ is any word from $T^a_n$. By symmetry, it holds
\[
\lambda^{nM'}(q^{-1}(\nu)) = \frac{\lambda^{nM'}(A(M'))}{|T^a_n|},
\]
i.e. the push forward measure $\lambda^{nM'} \circ q^{-1}$ of $\lambda^{nM'}$ under $q$ is nearly equidistributed. Explicitly,
\[
\left\| P_{T^a_n}^{\otimes M} - \lambda^{nM'} \circ q^{-1} \right\|_1 = |1 - \lambda^{nM'}(A(M))| = \lambda^{nM'}(A(M)^c) < \tau.
\]
For each outcome $\nu = (v_1, \ldots, v_M) \in T^a_n$, we define an $(n, M)$-message transmission code $C(\nu) := (v_m, D_{m'})_{m=1}^M$ as follows. Let
\[
u_v := \arg \min_{u \in q^{-1}(\nu)} \mathcal{E}(C'(u), W^\otimes_t),
\]
and $D_{m'} := D_m(u_v)$ for the respective $m$ (i.e. $(u_v, D_m(u_v))$ is a pair of codeword and decoding set in $C'(u_v)$). Then
\[
\sup_{t \in T} \mathcal{E}(C(\nu), W^\otimes_t) = \sup_{t \in T} \frac{1}{M} \sum_{m=1}^M \text{tr} \left( D_m(\nu) W^\otimes_t(v_m) \right)
\]
\[
\leq \sup_{t \in T} \frac{1}{M} \sum_{m=1}^{M'} \text{tr} \left( D_m(u_v) W^\otimes_t(u_{v,m}) \right)
\]
\[
\leq \frac{M'}{M} \sup_{t \in T} \mathcal{E}(C'(u_v), W^\otimes_t)
\]
\[
= \frac{M'}{M} \min_{u \in q^{-1}(\nu)} \sup_{t \in T} \mathcal{E}(C'(u_v), W^\otimes_t).
\]
The last equality above is by our code definition from Eq. (A10). To each $u \in A(M)^c$, we assign some valid $(n, M)$-code $C(\varphi(u)) := (v_{0,m}, D_m)_{m=1}^M$ being of no further interest. Let $\hat{V} = \varphi(U)$ (which is not i.i.d. so far!),

Except the one in Eq. (A9), which is by application of the bound in Proposition 32 all of the above estimates are by the preceding definitions and bounds.

Next, define a function $\varphi : \mathcal{X}^{nM'} \to (T^a_n)^M$ by
\[
\varphi(u) := \begin{cases} (v_1, \ldots, v_M) = \nu & \text{if } u \in A(M) \\
(v, \ldots, v) & \text{otherwise}, \end{cases}
\]
where $v$ is any word from $T^a_n$. By symmetry, it holds
\[
\lambda^{nM'}(q^{-1}(\nu)) = \frac{\lambda^{nM'}(A(M'))}{|T^a_n|},
\]
i.e. the push forward measure $\lambda^{nM'} \circ q^{-1}$ of $\lambda^{nM'}$ under $q$ is nearly equidistributed. Explicitly,
\[
\left\| P_{T^a_n}^{\otimes M} - \lambda^{nM'} \circ q^{-1} \right\|_1 = |1 - \lambda^{nM'}(A(M))| = \lambda^{nM'}(A(M)^c) < \tau.
\]
For each outcome $\nu = (v_1, \ldots, v_M) \in T^a_n$, we define an $(n, M)$-message transmission code $C(\nu) := (v_m, D_{m'})_{m=1}^M$ as follows. Let
\[
u_v := \arg \min_{u \in q^{-1}(\nu)} \mathcal{E}(C'(u), W^\otimes_t),
\]
and $D_{m'} := D_m(u_v)$ for the respective $m$ (i.e. $(u_v, D_m(u_v))$ is a pair of codeword and decoding set in $C'(u_v)$). Then
\[
\sup_{t \in T} \mathcal{E}(C(\nu), W^\otimes_t) = \sup_{t \in T} \frac{1}{M} \sum_{m=1}^M \text{tr} \left( D_m(\nu) W^\otimes_t(v_m) \right)
\]
\[
\leq \sup_{t \in T} \frac{1}{M} \sum_{m=1}^{M'} \text{tr} \left( D_m(u_v) W^\otimes_t(u_{v,m}) \right)
\]
\[
\leq \frac{M'}{M} \sup_{t \in T} \mathcal{E}(C'(u_v), W^\otimes_t)
\]
\[
= \frac{M'}{M} \min_{u \in q^{-1}(\nu)} \sup_{t \in T} \mathcal{E}(C'(u_v), W^\otimes_t).
\]
The last equality above is by our code definition from Eq. (A10). To each $u \in A(M)^c$, we assign some valid $(n, M)$-code $C(\varphi(u)) := (v_{0,m}, D_m)_{m=1}^M$ being of no further interest. Let $\hat{V} = \varphi(U)$ (which is not i.i.d. so far!),
then $\mathcal{C}(\hat{V}_m, D_m(\hat{V}))_{m=1}^M$ is a random constant composition $(n, M)$-code with

$$
\mathbb{E}\left[\sup_{t \in T} \mathcal{C}(C(V), W_t^{\otimes n})\right] = \sum_{v \in \varphi(A(M))} \lambda^{M'}(q^{-1}(v)) \sup_{t \in T} \mathcal{C}(C(v), W_t^{\otimes n}) + \sum_{v \in \varphi(A(M'))} \lambda^{M'}(q^{-1}(v)) \sup_{t \in T} \mathcal{C}(C(v), W_t^{\otimes n})
$$

\begin{align*}
&< \sum_{v \in \varphi(A(M))} \lambda^{M'}(q^{-1}(v)) \sup_{t \in T} \mathcal{C}(C(v), W_t^{\otimes n}) + \tau \\
&\leq \frac{M'}{M} \sum_{u \in A(M)} \lambda^{M'}(u) \sup_{t \in T} \mathcal{C}(C'(u), W_t^{\otimes n}) + \tau \\
&\leq \frac{M'}{M} \mathbb{E}\left[\sup_{t \in T} \mathcal{C}(C'(u), W_t^{\otimes n})\right] + \tau \\
&\leq \frac{M'}{M} \mu + 2\tau.
\end{align*}

Now, let $V = (V_1, \ldots, V_M)$ be a sequence of i.i.d. random variables each equidistributed on $T^n$. And $\mathcal{C}(V) := (V_m, D_m(V))_{m=1}^M$. It holds

$$
\frac{M'}{M} \mathbb{E}\left[\sup_{t \in T} \mathcal{C}(C'(u), W_t^{\otimes n})\right] \leq \frac{M'}{M} \mathbb{E}\left[\sup_{t \in T} \mathcal{C}(C'(u), W_t^{\otimes n})\right]
$$

\begin{align*}
&+ \left\| \chi_{T^n}^{\otimes M} - \lambda^{M'} \circ q^{-1} \right\|_1 \\
&< \frac{M'}{M} \mu + 2\tau.
\end{align*}

Since

$$
\frac{M'}{M} \leq \frac{2}{\delta}(n + 1)^{|Y|}
$$

we are done. \square

**Appendix B: Continuity bounds**

For convenience of the reader, we state and prove several continuity properties of entropic quantities. Most of them follow straightforwardly from the Alicki-Fannes bound \[3\] for von Neumann entropies.

**Theorem 33 (3).** Let $\rho, \sigma \in S(K_A \otimes K_B)$ be states on $K_A \otimes K_B$ with $\|\rho - \sigma\| \leq \epsilon$. It holds

$$
|S(\rho) - S(\sigma)| \leq 4\epsilon \log \dim K_A + 2h(\epsilon),
$$

where $h(x) := -x \log x - (1 - x) \log (1 - x)$ is the binary Shannon entropy of $(x, 1 - x)$.

The following bound is easily derived from Theorem 33

**Lemma 34.** Let $p, q \in \mathcal{P}(Y)$ be probability distributions with $\|p - q\|_1 \leq \epsilon$. For each cq-channel $V \in \mathcal{C}Q(Y, K)$, it holds

$$
|\chi(p, V) - \chi(q, V)| \leq 6\epsilon \log \dim K + 2h(\epsilon).
$$

**Lemma 35.** Let $Q, Q' \subset \mathcal{D}(Y)$ be probability distributions with $d_H(Q, Q') \leq \epsilon$. For each set $V \subset \mathcal{C}Q(Y, K_B \otimes K_E)$

$$
\left| \inf_{q \in Q} \left( \inf_{V \in V} \chi(p, V_B) - \sup_{V \in V} \chi(q, V_E) \right) - \inf_{q \in Q'} \left( \inf_{V \in V} \chi(p, V_B) - \sup_{V \in V} \chi(q, V_E) \right) \right| \leq 6\epsilon \log \dim K_{BE} + 4h(\epsilon).
$$
Lemma 36. Let $I, J \subset S_{cqq}(Y, K_X)$ of cqq density matrices and $d_H(I, J) \leq \Gamma$, and with stochastic matrices $P_{U|Y} : Y \rightarrow P(U)$ and $P_{T|U} : U \rightarrow P(T)$

$$\hat{\rho} := \sum_{t \in T} \sum_{u \in U} P_{T|U}(t|u)P_{U|Y}(u|y)p(y)|u\rangle\langle t| \otimes V(y)$$

if

$$\rho := \sum_{y \in Y} p(y)|y\rangle\langle y| \otimes V(y).$$

Then, the inequalities

$$\inf_{\rho \in I} I(U; X|T, \rho) \geq \inf_{\rho \in J} I(U; X|T, \rho) - 8\Delta \log(|U| \cdot \dim K) - 6h(\Delta)$$

$$\sup_{\rho \in I} I(U; X|T, \rho) \leq \sup_{\rho \in J} I(U; X|T, \rho) + 8\Delta \log(|U| \cdot \dim K) + 6h(\Delta)$$

are valid.

Proof. It holds for any two states $\rho, \sigma \in S_{cqq}(Y, K_X)$

$$\|\rho - \sigma\|_1 \leq \|\rho - \sigma\|_1. \quad (B1)$$

Note, that for each cqq density matrix $\rho$, it holds

$$I(U; X|T, \rho) = S(U|T, \rho) + S(X|T, \rho) - S(UX|\rho).$$

by definition of the quantum mutual information. If $\rho, \sigma$ fulfill $\|\rho - \sigma\|_1 \leq \delta$, then

$$|I(U; X|T, \rho) - I(U; X|T, \delta)| \leq 8\delta \log(|U| \cdot \dim K) + 6h(\delta) \quad (B2)$$

holds by (B1) and application of Lemma 33. From (B2) and the assumptions, we directly infer the claims.

Appendix C: Proof of Eq. (66)

Let $\lambda \in \hat{T}$ and $p \in \mathcal{P}_\lambda$. We define for each $q \in \mathcal{P}_q$ the set

$$\mathcal{I}_{(p, q)}^B := \left\{ \sum_{y \in Y} p(y)|y\rangle\langle y| \otimes V_B : V \in \mathcal{V}_q \right\}.$$

Observe the relations

$$\hat{I}_{p, \lambda} = \bigcup_{q \in \mathcal{P}_q} \mathcal{I}_{(p, q)}^B \quad \text{and} \quad \mathcal{I}_p^B = \mathcal{I}_{(p, p)}^B.$$

assume $V \in \mathcal{V}_p$, $V' \in \mathcal{V}_q$. It holds

$$\left\| \sum_{y \in Y} p(y)|y\rangle\langle y| \otimes V(y) - \sum_{y \in Y} p(y)|y\rangle\langle y| \otimes V'(y) \right\|_1 \leq \sum_{y \in Y} \|p(y)V(y) - p(y)V'(y)\|_1 \leq \sum_{y \in Y} \|p(y)V(y) - q(y)V'(y)\|_1 + \|p - q\|_1,$$

from which we directly infer

$$d_H(\mathcal{I}_p^B, \mathcal{I}_{(p, q)}^B) \leq d_H(\mathcal{I}_p^B, \mathcal{I}_q^B) + \|p - q\|_1.$$
Using these facts together with the first claim of Lemma 36, we obtain
\[
\inf_{\rho \in \tilde{I}_{p,\lambda}} I(U_A;B|T_A,\rho) = \inf_{\sigma \in \tilde{I}^p_{\rho,\lambda}} I(U_A;B|T_A,\sigma) \\
= \inf_{\sigma \in \tilde{I}^p_{\rho,\lambda}} I(U_A;B|T_A,\sigma) \\
\geq \inf_{\sigma \in \tilde{I}^p_{\rho,\lambda}} I(U_J;B|T_J,\sigma) - 8\Delta \log(|U| \cdot \dim K_B) - 6h(\Lambda) \\
= \inf_{\rho \in \tilde{I}_{p,\lambda}} I(U_A;B|T_A,\rho) - 8\Delta \log(|U| \cdot \dim K_B) - 6h(\Lambda) \quad (C1)
\]
Applying a similar reasoning leads us to
\[
\sup_{\rho \in \tilde{I}_{p,\lambda}} I(U_A;E|T_A,\rho) \leq \sup_{\rho \in \tilde{I}_{p,\lambda}} I(U_A;E|T_A,\rho) - 8\Delta \log(|U| \cdot \dim K_E) - 6h(\Lambda). \quad (C2)
\]
Combination of (C1) and (C2) for all \(p \in \mathcal{P}_\lambda\) yields the desired bound
\[
\inf_{\rho \in \tilde{I}_{p,\lambda}} I(U_J;B|T_J,\rho) - \sup_{\rho \in \tilde{I}_{p,\lambda}} I(U_J;E|T_J,\rho) \geq \inf_{\rho \in \tilde{I}_{p,\lambda}} I(U_J;B|T_J,\rho) - \sup_{\rho \in \tilde{I}_{p,\lambda}} I(U_J;E|T_J,\rho) - 16\Delta \log(|U| \cdot \dim K_{BE}) - 12h(\Lambda).
\]

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