Non-integrability of open billiard flows  
and Dolgopyat type estimates

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Abstract. We consider open billiard flows in $\mathbb{R}^n$ and show that the standard symplectic form $d\alpha$ in $\mathbb{R}^n$ satisfies a specific non-integrability condition over their non-wandering sets $\Lambda$. This allows to use the main result in [St3] and obtain Dolgopyat type estimates for spectra of Ruelle transfer operators under simpler conditions. We also describe a class of open billiard flows in $\mathbb{R}^n$ ($n \geq 3$) satisfying a certain pinching condition, which in turn implies that the (un)stable laminations over the non-wandering set are $C^1$.

1 Introduction

It is well-known that hyperbolic billiard flows in compact domains (e.g. Sinai billiards in Euclidean spaces or on tori) are non-integrable, just like contact Anosov flows (see e.g. [KB] or Appendix B in [L]). However, when considering contact flows over basic sets $\Lambda$ the general non-degeneracy of the contact form (which implies the non-integrability) does not say much about the dynamics of the flow over $\Lambda$. It is much more natural, and it turns out to be important as well, to look at the restriction of the contact form over tangent vectors to $\Lambda$ (see Sect. 2 for the definition). This is what we do here for non-wandering sets of open billiard flows. The motivation to study this kind of non-integrability comes from [St3] which deals with spectral estimates of Ruelle transfer operators for flows on basic sets (see Sect. 6 below for some details).

Let $K$ be a subset of $\mathbb{R}^n$ ($n \geq 2$) of the form $K = K_1 \cup K_2 \cup \ldots \cup K_k$, where $K_i$ are compact strictly convex disjoint domains in $\mathbb{R}^n$ with $C^2$ boundaries $\Gamma_i = \partial K_i$ and $k_0 \geq 3$. Set $\Omega = \mathbb{R}^n \setminus K$ and $\Gamma = \partial K$. We assume that $K$ satisfies the following (no-eclipse) condition:

(H) \quad \{ \text{for every pair } K_i, K_j \text{ of different connected components of } K \text{ the convex hull of } K_i \cup K_j \text{ has no common points with any other connected component of } K. \}

With this condition, the billiard flow $\phi_t$ defined on the sphere bundle $S(\Omega)$ in the standard way is called an open billiard flow. It has singularities, however its restriction to the non-wandering set $\Lambda$ has only simple discontinuities at reflection points. Moreover, $\Lambda$ is compact, $\phi_t$ is hyperbolic and transitive on $\Lambda$, and it follows from [SU] that $\phi_t$ is non-lattice and therefore by a result of Bowen [B], it is topologically weak-mixing on $\Lambda$.

Our main aim in this paper is to show that the open billiard flow always satisfies a certain non-integrability condition on $\Lambda$. Let $d\alpha$ be the standard symplectic form on $T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$.

**Theorem 1.1.** There exist $z_0 \in \Lambda$ and $\mu > 0$ such that for any unit tangent vector $b \in E^u(z_0)$ to $\Lambda$ there exists a unit tangent vector $a \in E^s(z_0)$ to $\Lambda$ with $|d\alpha(a, b)| \geq \mu$.

If the map $\Lambda \ni x \mapsto E^u(x)$ is $C^1$, then the invariance of $d\alpha$ along the flow implies that the points $z_0 \in \Lambda$ with the above property form an open and dense subset of $\Lambda$. Theorem 1.1 is established by means of a certain pairing of points on the strong stable and unstable manifolds of an appropriately chosen point $z_0$ - see Sect. 3 and Lemma 3.1 there for details. As a consequence
of this and the main result in [SL3] one gets Dolgopyat type spectral estimates for pinched open billiard flows – see Sect. 6.

It is well-known that in general the maps \( \Lambda \ni x \mapsto E^u(x) \) (or \( E^s(x) \)) are only Hölder continuous (see e.g. [HPS] or [PSW]). The following pinching condition implies stronger regularity properties of these maps.

\((P)\): There exist constants \( C > 0 \) and \( 0 < \alpha \leq \beta \) such that for every \( x \in \Lambda \) we have

\[
\frac{1}{C} e^{\alpha x t} \| u \| \leq \| d\phi_t(x) \cdot u \| \leq C e^{\beta x t} \| u \| , \quad u \in E^u(x) , t > 0 ,
\]

for some constants \( \alpha_x, \beta_x > 0 \) depending on \( x \) but independent of \( u \) and \( t \) with \( \alpha \leq \alpha_x \leq \beta_x \leq \beta \) and \( 2\alpha_x - \beta_x \geq \alpha \) for all \( x \in \Lambda \).

For example in the case of contact flows \( \phi_t \), it follows from the results in [Ha2] (see also [Ha1]) that assuming \((P)\), the map \( \Lambda \ni x \mapsto E^u(x) \) is \( C^{1+\epsilon} \) with \( \epsilon = 2\alpha/\beta - 1 > 0 \) (in the sense that this map has a linearization at any \( x \in \Lambda \) that depends Hölder continuously on \( x \)). The same applies to the map \( \Lambda \ni x \mapsto E^s(x) \).

Notice that when \( n = 2 \) (then the local unstable manifolds are one-dimensional) this condition is always satisfied. It turns out that for \( n \geq 3 \) the condition \((P)\) is always satisfied when the minimal distance between distinct connected components of \( K \) is relatively large compared to the maximal sectional curvature of \( \partial K \) (see Proposition 1.2 below). An analogue of the latter for manifolds \( M \) of strictly negative curvature would be to require that the sectional curvature is between \(-K_0\) and \(-a K_0\) for some constants \( K_0 > 0 \) and \( a \in (0,1) \). It follows from the arguments in [HP] that when \( a = 1/4 \) the geodesic flow on \( M \) satisfies the pinching condition \((P)\).

Set \( d_{i,j} = \text{dist}(K_i, K_j) \) and \( d_0 = \min_{i \neq j} d_{i,j} \). Since every \( K_i \) is strictly convex, the operator \( L_x : T_x(\partial K) \to T_x(\partial K) \), \( L_x u = (\nabla_u \nu)(x) \), of the second fundamental form is positive definite with respect to the outward unit normal field \( \nu(y), y \in \partial K \). Then \( k(x,u) = \langle L_x u, u \rangle \) is the normal curvature of \( \partial K \) at \( x \) in the direction of \( u \in T_x(\partial K) \), \( \| u \| = 1 \).

Set

\[
\kappa_{\text{min}} = \min_{x \in \partial K} \min_{u \in T_x(\partial K), \| u \| = 1} \langle L_x(u), u \rangle , \quad \kappa_{\text{max}} = \max_{x \in \partial K} \max_{u \in T_x(\partial K), \| u \| = 1} \langle L_x(u), u \rangle ,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^p \).

Before continuing, notice that the condition \((H)\) implies the existence of a global constant \( \varphi_0 \in (0, \pi/2) \) such that for any \( x \in \Lambda \) and any reflection point \( q \) of the billiard trajectory \( \gamma(x) \) generated by \( x \) the angle \( \varphi \) between the reflected direction of \( \gamma(x) \) at \( q \) and the outward normal to \( \partial K \) at \( q \) satisfies \( \varphi \leq \varphi_0 \). Set \( \mu_0 = 2 \cos \varphi_0 \kappa_{\text{min}} \) and \( \lambda_0 = \frac{1}{d_0} + \frac{2\kappa_{\text{max}}}{\cos \varphi_0} \).

Let \( a > 0 \) be such that \( d_{i,j} \leq d_0 + a \) for all \( i, j = 1, \ldots, k_0, i \neq j \). Below we assume that \( d_0 \) is large compared to \( a \) and \( \kappa_{\text{max}} \), so that

\[
(1.1) \quad [1 + (d_0 + a) \lambda_0]^{d_0 + a} < (1 + d_0 \mu_0)^{2d_0} .
\]

(Notice that when \( a = r d_0 \), \( 0 < r < 1 \), then the above holds for all sufficiently large \( d_0 \), assuming \( \kappa_{\text{max}} \) and \( \kappa_{\text{min}} \) are uniformly bounded above and below, respectively, by positive constants.)

In Section 5 below we prove the following

**Proposition 1.2.** Assume that \((1.1)\) holds and the boundary \( \partial K \) is \( C^3 \). Then the open billiard flow \( \phi_t \) in the exterior of \( K \) satisfies the condition \((P)\) on its non-wandering set \( \Lambda \). Moreover, for any \( x \in \Lambda \) we can choose \( \alpha_x = \alpha_0 \) and \( \beta_x = \beta_0 \), where \( \alpha_0 = \frac{\ln(1 + d_0 \mu_0)}{d_0 + a} \) and \( \beta_0 = \frac{\ln(1 + (d_0 + a) \lambda_0)}{d_0} \).
This is relatively easy to derive from a formula for the growth of the differential of the flow on unstable manifolds (see Proposition 5.1). The latter can be proved using an argument similar to that in the Appendix in [St2] (dealing with the two-dimensional case), and also can be easily derived from more general facts about the evolution of unstable vectors for multidimensional dispersing billiards (see e.g. [BCST]).

Section 2 below contains some basic definitions and an example which concerns the geometry of the non-wandering set Λ. Sections 3 and 4 are devoted to the proof of Theorem 1.1. In Section 5 we use some well known formulae of Sinai for curvature operators related to unstable manifolds of dispersing billiards to prove Proposition 1.2. Section 6 deals with Dolgopayt type estimates for pinched open billiard flows – these are straightforward consequences of [St3] and the considerations in Sect. 3 below.

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2 Preliminaries

2.1 Basic definitions

Let M be a $C^1$ complete Riemann manifold, and $\phi_t : M \rightarrow M$ ($t \in \mathbb{R}$) a $C^1$ flow on M. A $\phi_t$-invariant closed subset $\Lambda$ of $M$ is called hyperbolic if $\Lambda$ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ such that there exists a $d\phi_t$-invariant decomposition $T_x M = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of $T_x M$ ($x \in \Lambda$) into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at $x$, $\|d\phi_t(u)\| = C \lambda^t \|u\|$ for all $u \in E^s(x)$ and $t \geq 0$, and $\|d\phi_t(u)\| \leq C \lambda^{-t} \|u\|$ for all $u \in E^u(x)$ and $t \leq 0$.

A non-empty compact $\phi_t$-invariant hyperbolic subset $\Lambda$ of $M$ which is not a single closed orbit is called a basic set for $\phi_t$ if $\phi_t$ is transitive on $\Lambda$ and $\Lambda$ is locally maximal, i.e. there exists an open neighbourhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \cap_{t \in \mathbb{R}} \phi_t(V)$.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0 , d(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow \infty} 0 \} ,$$

$$W^u_\epsilon(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0 , d(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow -\infty} 0 \}$$

be the (strong) stable and unstable manifolds of size $\epsilon$. Then $E^u(x) = T_x W^u_\epsilon(x)$ and $E^s(x) = T_x W^s_\epsilon(x)$. Given $z \in \Lambda$, let $\exp_x^s : E^u(z) \rightarrow W^u_0(z)$ and $\exp_x^s : E^s(z) \rightarrow W^s_0(z)$ be the corresponding exponential maps. A vector $b \in E^u(z) \setminus \{0\}$ is called tangent to $\Lambda$ at $z$ if there exist infinite sequences $\{v^{(m)}\}_0 \subset E^u(z)$ and $\{t_m\}_0 \subset \mathbb{R} \setminus \{0\}$ such that $\exp_x^u(t_m, v^{(m)}) \in \Lambda \cap W^u_\epsilon(z)$ for all $m$, $v^{(m)} \rightarrow b$ and $t_m \rightarrow 0$ as $m \rightarrow \infty$. It is easy to see that a vector $b \in E^u(z) \setminus \{0\}$ is tangent to $\Lambda$ at $z$ if there exists a $C^1$ curve $z(t)$, $0 \leq t \leq a$, in $W^u_\epsilon(z)$ for some $a > 0$ with $z(0) = z$, $z(0) = b$, and $z(t_n) \in \Lambda$ for some sequence $\{t_n\}_{n=1}^\infty \subset (0, a]$ with $t_n \rightarrow 0$ as $n \rightarrow \infty$. Tangent vectors to $\Lambda$ in $E^s(z)$ are defined similarly. Denote by $\overline{E}^u(z)$ (resp. $\overline{E}^s(z)$) the set of all vectors $b \in E^u(z) \setminus \{0\}$ (resp. $b \in E^s(z) \setminus \{0\}$) tangent to $\Lambda$ at $z$.

Remark 1. Although we have not sought to construct particular examples, it appears that in general the set of unit tangent vectors to $\Lambda$ does not have to be closed in the bundle $E^u_\Lambda$ (or $E^s_\Lambda$). That is, there may exist a point $z \in \Lambda$, a sequence $\{z_m\} \subset W^u(z) \cap \Lambda$ and for each $m$ a unit vector $\xi_m$ tangent to $\Lambda$ at $z_m$ such that $z_m \rightarrow z$ and $\xi_m \rightarrow \xi$ as $m \rightarrow \infty$, however $\xi$ is not tangent to $\Lambda$ at $z$. 

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Next, assume that \( K \) and \( \Omega \) are as in Sect. 1. The non-wandering set \( \Lambda \) for the flow \( \phi_t \) is the set of those \( x \in S(\Omega) \) such that the trajectory \( \{\phi_t(x) : t \in \mathbb{R}\} \) is bounded. Notice that the natural projection of \( \phi_t \) on the quotient space \( S(\Omega)/\sim \), where \( \sim \) is the equivalence relation \( (q,v) \sim (p,w) \) iff \( q = p \) and \( v = w \) or \( q = p \in \partial K \) and \( v \) and \( w \) are symmetric with respect to \( T_\nu(\partial K) \), is continuous. Moreover whenever both \( x \) and \( \phi_t(x) \) are in the interior of \( S(\Omega) \) and sufficiently close to \( \Lambda \), the map \( y \mapsto \phi_t(y) \) is smooth on a neighbourhood of \( x \). It follows from results of Sinai (\cite{St1}, \cite{St2}) that \( \Lambda \) is a hyperbolic set for \( \phi_t \), and it is easily seen that \( \Lambda \) is the maximal compact \( \phi_t \)-invariant subset of \( S(\Omega) \). Moreover, it follows from the natural symbolic coding for the natural section of the flow (the so-called billiard ball map) that the periodic points are dense in \( \Lambda \), and \( \phi_t \) is transitive on \( \lambda \). Thus, \( \Lambda \) is a basic set for \( \phi_t \) and the classical theory of hyperbolic flows applies in the case under consideration (see e.g. Part 4 in \cite{KT3}).

2.2 An example

Here we briefly describe a (non-trivial) example from \cite{St4} which shows that in general for every \( z \in \Lambda \) the space \( \text{span}(E_A^u(z)) \) generated by the vectors in \( E_A^u(z) \) tangent to \( \Lambda \) could be a proper subspace of \( E^u(z) \).

Example 2.1. (\cite{St4}) Assume that \( n = 3 \) and there exists a plane \( \alpha \) such that each of the domains \( K_j \) is symmetric with respect to \( \alpha \). Setting \( K' = K \cap \alpha \) and \( \Omega' = \Omega \cap \alpha \), it is easy to observe that every billiard trajectory generated by a point in \( \Lambda \) is entirely contained in \( \alpha \). That is, \( \Lambda = \Lambda' \), where \( \Lambda' \) is the non-wandering set for the open billiard flow in \( \Omega' \). Thus, \( \dim(\text{span}(E_A^u(z))) = 1 < \dim(E^u(z)) = 2 \) for any \( z \in \Lambda \). This example is of course trivial, since \( \Lambda \) is contained in the flow-invariant submanifold \( S^*(\Omega') \) of \( S^*(\Omega) \).

However with a small local perturbation of the boundary \( \partial K \) of \( K \) we can get a non-trivial example. Choosing standard cartesian coordinates \( x,y,z \) in \( \mathbb{R}^3 \), we may assume that \( \alpha \) is given by the equation \( z = 0 \), i.e. \( \alpha = \mathbb{R}^2 \times \{0\} \). Let \( \text{pr}_1 : S^*(\mathbb{R}^3) \sim \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3 \) be the natural projection, and let \( C = \text{pr}_1(\Lambda) \). We may choose the coordinates \( x,y \) in the plane \( \alpha = \{z = 0\} \) so that the line \( y = 0 \) is tangent to \( K_1' \) and \( K_2' \) and \( K' \) is contained in the half-plane \( y \geq 0 \). Let \( q_1 \in K_1' \) and \( q_2 \in K_2' \) be such that \( [q_1,q_2] \) is the shortest segment connecting \( K_1' \) and \( K_2' \). Take a point \( q_1' \in \partial K_1' \) close to \( q_1 \) and such that the \( y \)-coordinate of \( q_1' \) is smaller than that of \( q_1 \).

Consider the open arc \( A \) on \( \partial K_1' \) connecting \( q_1 \) and \( q_1' \). It is clear that \( A \cap C = \emptyset \).

Let \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) be a \( C^1 \) (we can make it even \( C^\infty \)) diffeomorphism with \( f(x) = x \) for all \( x \) outside a small open set \( U \) such that \( q_1 \in \overline{U} \) and \( U \cap \partial K' \subset A \). Then for any \( q \in C \) the tangent planes \( T_q(\partial K) \) and \( T_q(\partial K) \) coincide. We can choose \( f \) so that \( K_i = f(K_i) = K_i \) for \( i > 1 \), \( K_1 = f(K_1) \) is strictly convex, and \( \nu(f(q)) \notin \alpha \) for \( q \in A \) arbitrarily close to \( q_1 \). Here \( \nu \) is the outward unit normal field to \( \partial K \).

One can then show that the non-wandering set \( \tilde{\Lambda} \) for the billiard flow \( \tilde{\phi}_t \) in the closure \( \tilde{\Omega} \) of the exterior of \( \tilde{K} \) in \( \mathbb{R}^3 \) coincides with \( \Lambda \) (\cite{St4}). Thus, \( \dim(\text{span}(E_A^u(z))) = 1 < \dim(E^u(z)) \) for any \( z \in \tilde{\Lambda} \). However, it is clear from the construction that \( S^*(\alpha \cap \Omega) \) is not invariant with respect to the billiard flow \( \phi_t \). Moreover, it is not difficult to see that there is no two-dimensional submanifold \( \tilde{\alpha} \) of \( \tilde{\Omega} \) such that \( S^*(\tilde{\alpha}) \) is \( d\tilde{\phi}_t \)-invariant and \( \Lambda \subset S^*(\tilde{\alpha}) \); see Section 4 in \cite{St4} for details.

3 Non-integrability of open billiard flows

In this section we prove Theorem 1.1.
Let $K \subset \mathbb{R}^n$ be as in Sect. 1. For any $x \in \Gamma = \partial K$ we will denote by $\nu(x)$ the outward unit normal to $\Gamma$ at $x$. Given $\delta > 0$ denote by $S_\delta(\Omega)$ the set of those $(x, u) \in S(\Omega)$ such that there exist $y \in \Gamma$ and $t \geq 0$ with $y + tu = x$, $y + su \in \mathbb{R}^n \setminus K$ for all $s \in (0, t)$ and $\langle u, \nu_t(y) \rangle \geq \delta$.

Remark 2. Notice that the condition (H) implies the existence of a constant $\delta_0 > 0$ depending only on the obstacle $K$ such that any $(x, u) \in S(\Omega)$ whose backward and forward billiard trajectories both have a common point with $\Gamma$ belongs to $S_{\delta_0}(\Omega)$.

For $\epsilon \in (0, d_0/2)$ set

$$D_\epsilon(\Omega) = \{x = (q, v) \in S(\Omega) : \text{dist}(q, \partial K) > \epsilon\} \quad \Lambda_\epsilon = \Lambda \cap D_\epsilon(\Omega).$$

In what follows in order to avoid ambiguity and unnecessary complications we will consider stable and unstable manifolds only for points $x$ in $D_\epsilon(\Omega)$ or $\Lambda_\epsilon$; this will be enough for our purposes.

Fix for a moment arbitrary $\epsilon, \delta$ and $\lambda$ so that

$$0 < \delta \leq \epsilon < \lambda < \frac{d_0}{2}.$$  \hfill (3.1)

We will see later how small these numbers need to be.

Consider an arbitrary point $\sigma_0 = (x^{(0)}, \xi^{(0)}) \in \Lambda_\epsilon$ such that $z^{(0)} = x^{(0)} + \lambda \xi^{(0)} \in \partial K$, $\xi^{(0)} = -\nu(z^{(0)})$ and $x^{(0)} + t \xi^{(0)} \in \mathbb{R}^n \setminus K$ for all $t \in [0, \lambda)$. I.e. the billiard trajectory generated by $\sigma_0$ is perpendicular to $\partial K$ at $z^{(0)}$ and so the reflected direction at $z^{(0)} = -\xi^{(0)}$. Notice that there exist such points $\xi^{(0)}$ e.g. we can take $x^{(0)}$ on the shortest segment between two connected components $K_i$ and $K_j$ ($i \neq j$) with $\xi^{(0)}$ parallel to that segment. The local submanifolds $U = W^u(\sigma_0)$ and $S = W^s(\sigma_0)$ have the form

$$U = \{(x, \nu_X(x)) : x \in X\} \quad S = \{(y, \nu_Y(y)) : y \in Y\}$$

for some smooth local $(n - 1)$-dimensional submanifolds $X$ and $Y$ in $\mathbb{R}^n$, where $\nu_X$ and $\nu_Y$ are continuous unit normal fields on $X$ and $Y$. Moreover the second fundamental form $L^{(X)}$ of $X$ with respect to $\nu_X$ (resp. $L^{(Y)}$ of $Y$ with respect to $\nu_Y$) is positive (resp. negative) definite. Finally, we have $x^{(0)} \in X \cap Y$, $X$ and $Y$ are tangent at $x^{(0)}$ and $\nu_X(x^{(0)}) = \nu_Y(x^{(0)}) = \xi^{(0)}$. Since the tangent planes to $X$ and $Y$ at $x^{(0)}$ are parallel to the tangent plane to $\partial K$ at $z^{(0)}$, we have

$$T_{x^{(0)}}X = T_{x^{(0)}}Y = T_{x^{(0)}}(\partial K).$$

Consider the inversion $i : S(\Omega) \rightarrow S(\Omega)$ defined by $i(x, \xi) = (x, -\xi)$. It follows from the general properties of stable (unstable) manifolds that for any $(x, \xi) \in S$ (or $U$) sufficiently close to $\sigma_0$ we have $i \circ \phi_{2\lambda}(x, \xi) \in U$ ($S$, respectively). In other words the shift along the billiard flow $\phi_t$ of the convex front $S$ along the normal field $\nu_Y(y)$ in $2\lambda$ units coincides locally with the inversion of the convex front $X$ near $x^{(0)}$. Using Sinai’s formula ([Si2], cf. also [SiCh]) in the particular situation considered here we have

$$L_{x^{(0)}}^{(X)}(u) = \frac{B_{x^{(0)}}(u)}{1 + \lambda B_{x^{(0)}}(u)},$$

where

$$B_{x^{(0)}}(u) = \frac{L_{x^{(0)}}^{(Y)}(u)}{1 + \lambda L_{x^{(0)}}^{(Y)}(u)} + 2L_{x^{(0)}}(u), \quad u \in T_{x^{(0)}}X.$$
It is well-known (see [Si2]) that the curvature operators of strong unstable manifolds of \( \phi_t \) are uniformly bounded, so there exists a global constant \( C > 0 \) such that \( 2L_{x(0)}(u) \leq B_{x(0)}(u) \leq C \) for all \( u \in T_{x(0)}X, \|u\| \leq 1 \). Therefore,

\[
(3.2) \quad C' \leq L_{x(0)}^{(x)}(u) \leq C, \quad u \in T_{x(0)}X, \|u\| \leq 1,
\]

for some other global constant \( C' > 0 \) (depending on \( K \) but not an \( \lambda \) and \( u \)).

Consider the map \( \Phi : U \rightarrow S \) near \( \sigma_0 = (x^{(0)}, \xi^{(0)}) \) defined by \( \Phi(x, \xi) = \iota \circ \phi_{2\lambda}(x, \xi) \). In fact by the same formula (see below for more details) one defines \( \Phi \) as a local smooth map \( \Phi : T(R^n) = R^n \times R^n \rightarrow T(R^n) \) near \( \sigma_0 \). Given \( \epsilon > 0 \), we will assume \( \delta \in (0, \epsilon] \) is chosen sufficiently small, so that \( \Phi \) is well-defined and \( \Phi(U) \subset S \). Moreover, \( \Phi(z) = z' \) implies \( \Phi(z') = z \) (whenever \( \Phi(z') \) is defined) and locally \( \Phi(W^u_{\epsilon}(z)) = W^s_{\epsilon}(z') \). Finally, it is important to remark that \( \Phi \) preserves the set \( \Lambda \). Indeed, \( \phi_{2\lambda}(\Lambda) = \Lambda \) and \( i(\Lambda) = \Lambda \), as well. So, in particular

\[
(3.3) \quad \Phi(U \cap \Lambda) \subset S \cap \Lambda.
\]

To write down a more explicit expression for \( \Phi \), let \( f \) be a defining function for \( \partial K \) in \( R^n \) so that \( \|\nabla f\| = 1 \) near \( z^{(0)} \) and \( \nabla f(z) = \nu(z) \) is the outward unit normal to \( \partial K \) at \( z \in \partial K \). Then \( \partial K = f^{-1}(0) \) (locally near \( z^{(0)} \)). Given \( (x, \xi) \in R^n \times R^n \) close to \( \sigma_0 \), there exist a unique \( z(x, \xi) \in \partial K \) and a unique minimal \( t(x, \xi) \in \mathbb{R}_+ \) with \( z(x, \xi) = x + t(x, \xi)\xi \in \partial K \), i.e. such that

\[
(3.4) \quad f(x + t(x, \xi)\xi) = 0.
\]

By \( \eta(x, \xi) \) we denote the reflection of \( \xi \) with respect to \( \nabla f(z(x, \xi)) \), i.e.

\[
\eta(x, \xi) = \xi - 2\langle \xi, \nabla f(z) \rangle \nabla f(z).
\]

Here and in what follows we denote for brevity \( z = z(x, \xi) \). We will also use the notation \( t = t(x, \xi) \) and \( \eta = \eta(x, \xi) \). We then have

\[
\Phi(x, \xi) = (g(x, \xi), -\eta(x, \xi)),
\]

where \( g(x, \xi) = z + (2\lambda - t)\eta \). Since \( \Phi(U) \subset S \) and \( \Phi \) is a local diffeomorphism between \( U \) and \( S \), we have \( d\Phi_\sigma(E^u(\sigma)) = E^s(\sigma) \) for every \( \sigma \in U \). Moreover, it is easy to see that \( d\Phi_\sigma \) preserves the sets of tangent vectors to \( \Lambda \), namely if \( \sigma \in U \cap \Lambda \) and \( \xi \in E^u(\sigma) \setminus \{0\} \) is tangent to \( \Lambda \) at \( \sigma \), then \( d\Phi_\sigma \cdot \xi \) is tangent to \( \Lambda \) at \( \Phi(\sigma) \).

It is well known that we can take the constant \( C > 0 \) so large that \( \|d\Phi_\sigma\| \leq C \) for any \( \sigma \in U \) and any choice of \( \sigma_0 \) (see e.g. [Si2], [Ch1] or [BCST]). (See also the proof of Lemma 3.1 in Sect. 4 for an explicit formula for \( d\Phi_{\sigma_0} \).)

Set \( L = L_{x(0)}^{(x)} \) and \( H = L_{z(0)} \), and for any \( u \in T_{z(0)}X \), consider the vectors

\[
v(u) = (u, Lu) \in E^u(\sigma_0), \quad w(u) = d\Phi(\sigma_0) \cdot v(u) \in E^s(\sigma_0).
\]

It is easy to see that

\[
(3.5) \quad \|v(u)\| \leq \sqrt{1 + C^2} \|u\|.
\]

The following lemma is the main technical ingredient in the proofs of Theorem 1.1 and Proposition 6.2. Its proof is given in Sect. 4 below.

**Lemma 3.1.** For any \( u, u' \in T_{x(0)}X \) we have \( d\alpha(v(u), w(u')) = \langle u, P u' \rangle \), where the linear operator \( P \) is given by \( P = 2H + 2L + 2\lambda(HL + LH + L^2 + \lambda LHL) \). Consequently, if \( \kappa > 0 \) is the
minimal principal curvature at a point on $\partial K$ and $\epsilon$ and $\lambda$ are chosen sufficiently small, then $P$ is positive definite, $\langle u, Pu \rangle \geq \kappa \|u\|^2$, and therefore $|d\alpha(v(u), w(u))| \geq \kappa \|u\|^2$ for all $u \in T_x(\partial K)$.

Proof of Theorem 1.1. Take $\epsilon > 0$ sufficiently small and then $\lambda$ with (3.1) small enough so that the operator $P$ in Lemma 3.1 is positive definite, where $\sigma_0$ is chosen as above. (Notice that $H$ and $L$ are uniformly bounded from below and above regardless of the choice of the point $\sigma_0$.) More precisely, as stated in Lemma 3.1, if $\kappa > 0$ is the minimal principal curvature at a point on $\partial K$, we can choose $0 < \epsilon < \lambda$ so small that $\langle u, Pu \rangle \geq \kappa \|u\|^2$ for any $u \in T_x(\partial K)$.

Set $z_0 = \sigma_0$ and let $b \in E_0^\* (z_0)$, $\|b\| = 1$. Then $b = v(u)$ for some $u \in T_x(\partial K)$. Moreover, by (3.3), $w(u) = d\Phi(\sigma_0) \cdot v(u) \in E_0^\* (z_0)$, and so $a = w(u)/\|w(u)\|$ is a unit vector in $E_0^\* (z_0)$. By Lemma 3.1,

$$|d\alpha(a, b)| = \frac{1}{\|w(u)\|} |d\alpha(v(u), w(u))| \geq \frac{\kappa \|u\|^2}{\|w(u)\|} \geq \frac{\kappa \|u\|^2}{C \|v(u)\|} = \frac{\kappa \|u\|^2}{C}. $$

On the other hand (3.5) implies $1 = \|b\| = \|v(u)\| \leq \sqrt{1 + C^2} \|u\|$, so $|d\alpha(a, b)| \geq \frac{\kappa}{C (1 + C^2)}$.

As a consequence of Lemma 3.1 one can also derive the following which however we do not need in this paper.

**Proposition 3.2.** For every $z \in \Lambda$ and every $\delta > 0$ there exists $\tilde{z} \in \Lambda \cap W^\delta_r (z)$ such that for any non-zero tangent vector $b \in E^u(\tilde{z})$ to $\Lambda$ there exists a tangent vector $a \in E^s(\tilde{z})$ to $\Lambda$ with $d\alpha(a, b) = 0$.

### 4 Proof of Lemma 3.1

We will use the notation from Sect. 3. Recall that the standard symplectic form $d\alpha$ has the form

$$d\alpha((u, \tilde{u}), (p, \tilde{p})) = \langle u, \tilde{p} \rangle + \langle \tilde{u}, p \rangle,$$

where $(u, \tilde{u}), (p, \tilde{p}) \in T^s(\mathbb{R}^n)$. Given $u, u' \in T_x(\partial K)$, let $v(u) = (u, \tilde{u}) \in E^u(\sigma)$, $v(u') = (u', \tilde{u}') \in E^u(\sigma)$ and $w(u') = d\Phi(\sigma_0)(v(u')) = (p, \tilde{p})$. Then

$$\begin{pmatrix} p \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \partial_x g(\sigma) & \partial_x g(\sigma) \\ -\partial_x \eta(\sigma) & -\partial_x \eta(\sigma) \end{pmatrix} \begin{pmatrix} u' \\ \tilde{u}' \end{pmatrix} = \begin{pmatrix} \partial_x g(\sigma) u' + \partial_x g(\sigma) \tilde{u}' \\ -\partial_x \eta(\sigma) u' - \partial_x \eta(\sigma) \tilde{u}' \end{pmatrix},$$

and so

$$d\alpha(v(u), w(u')) = \langle u, \partial_x \eta(\sigma) u' \rangle + \langle u, \partial_x \eta(\sigma) \tilde{u}' \rangle + \langle \tilde{u}, \partial_x g(\sigma) u' \rangle + \langle \tilde{u}, \partial_x g(\sigma) \tilde{u}' \rangle. \tag{4.1}$$

One needs the derivatives of $g$ and $\eta$. Differentiating (3.4) gives

$$\nabla_x t(\sigma_0) = \nabla f(z^{(0)}) = -\xi^{(0)} , \quad \nabla_\xi t(\sigma_0) = \lambda \nabla f(z^{(0)}) = -\lambda \xi^{(0)}. \tag{4.2}$$

Moreover, $z = x + t\xi$ implies

$$\frac{\partial z_t}{\partial x_j}(x, \xi) = \delta_{jt} + \frac{\partial t}{\partial x_j}(x, \xi) \xi_t , \quad \frac{\partial z_t}{\partial \xi_j}(x, \xi) = t\delta_{jt} + \frac{\partial t}{\partial \xi_j}(x, \xi) \xi_t. \tag{4.3}$$
Next, we have \( g(x, \xi) = z + (2\lambda - t)\eta = x + 2\lambda \xi - 2(2\lambda - t)\langle \xi, \nabla f(z) \rangle \nabla f(z) \). Hence

\[
\frac{\partial g_i}{\partial x_j}(x, \xi) = \delta_{ij} + 2\frac{\partial t}{\partial x_j}(x, \xi) \langle \xi, \nabla f(z) \rangle \frac{\partial f}{\partial x_i}(z) - 2(2\lambda - t) \left( \sum_{k, \ell = 1}^n \xi_k \frac{\partial^2 f}{\partial x_k \partial x_\ell}(z) \frac{\partial z_\ell}{\partial x_j}(z) \right) \frac{\partial f}{\partial x_i}(z)
\]

\[-2(2\lambda - t) \langle \xi, \nabla f(z) \rangle \sum_{\ell = 1}^n \xi_\ell \frac{\partial^2 f}{\partial x_i \partial x_\ell}(z) \frac{\partial z_\ell}{\partial x_j}(z),
\]

Similarly,

\[
\frac{\partial g_i}{\partial \xi_j}(x, \xi) = 2\lambda \delta_{ij} + 2\frac{\partial t}{\partial \xi_j}(x, \xi) \langle \xi, \nabla f(z) \rangle \frac{\partial f}{\partial x_i}(z) - 2(2\lambda - t) \frac{\partial f}{\partial x_i}(z) \frac{\partial f}{\partial x_i}(z)
\]

\[-2(2\lambda - t) \langle \xi, \nabla f(z) \rangle \sum_{\ell = 1}^n \xi_\ell \frac{\partial^2 f}{\partial x_i \partial x_\ell}(z) \frac{\partial z_\ell}{\partial \xi_j}(z),
\]

Notice that for \( v = (u, \bar{u}) \in E^u(\sigma_0) \) we have \( u, \bar{u} \perp \xi_0 \). Therefore (4.2) imply \( \langle \nabla_x t(\sigma_0), u \rangle = (\nabla x t(\sigma_0), u) = 0 \), and the same holds with \( u \) replaced by \( \bar{u} \). This and (4.3) give

\[
\sum_{j = 1}^n \frac{\partial z_\ell}{\partial x_j}(\sigma_0) u_j = u_\ell, \quad \sum_{j = 1}^n \frac{\partial z_\ell}{\partial \xi_j}(\sigma_0) u_j = \lambda u_\ell
\]

for any \( u \in T_{x(0)}X \). Using these, \( t(\sigma_0) = \lambda \), the above formulae and the Hessian matrix

\[
H' = \left( \frac{\partial^2 f}{\partial x_k \partial x_\ell}(z(0)) \right)_{k, \ell = 1}^n
\]

one gets:

\[
\sum_{j = 1}^n \frac{\partial g_i}{\partial x_j}(\sigma_0) u'_j = u'_i - 2\lambda \left( \sum_{k, \ell = 1}^n \xi_k^{(0)} u_\ell \frac{\partial^2 f}{\partial x_k \partial x_\ell}(z(0)) \right) \frac{\partial f}{\partial x_i}(z(0)) + 2\lambda \sum_{\ell = 1}^n u_\ell \frac{\partial^2 f}{\partial x_i \partial x_\ell}(z(0))
\]

\[
= u'_i - 2\lambda (u' \cdot H' \xi^{(0)}) \frac{\partial f}{\partial x_i}(z(0)) + 2\lambda (H'u')_i = u'_i + 2\lambda (H'u')_i,
\]

where \((H'u')_i\) is the \(i\)th coordinate of the (column) vector \(H'u'\). Here we used the fact that \( \xi^{(0)} = -\nabla f(z(0)) \) and \( H' \nabla f(z(0)) = 0 \), since \( \|\nabla f\| = 1 \) near \( \partial K \). Similarly,

\[
\sum_{j = 1}^n \frac{\partial g_i}{\partial \xi_j}(\sigma_0) u''_j = 2\lambda u''_i - 2\lambda \left( \sum_{k, \ell = 1}^n \xi_k^{(0)} u_\ell' \frac{\partial^2 f}{\partial x_k \partial x_\ell}(z(0)) \right) \frac{\partial f}{\partial x_i}(z(0))
\]

\[+ 2\lambda \sum_{\ell = 1}^n u_\ell' \frac{\partial^2 f}{\partial x_i \partial x_\ell}(z(0)) = 2\lambda u''_i + 2\lambda^2 (H''u'')_i,
\]
The trajectory \( q \) \( \gamma \)
where \( d \) "outward" unit normal given by the direction \( \gamma \) consecutive reflections of the trajectory \( W \)
containing the point \( B \)
then
the boundary representation for the Jacobi fields along a billiard trajectory.

Then larger classes of open billiard flows could be shown to satisfy the condition \( (P) \).

Using \( \tilde{d} \alpha \)

The last four formulae imply \( \langle u, \partial_x \eta (\sigma_0) u' \rangle = \sum_{j=1}^n u_j \frac{\partial n_j}{\partial x} (\sigma_0) = 2 \langle u, H' u' \rangle \), and similarly \( \langle u, \partial_x \eta (\sigma_0) \tilde{u}' \rangle = \langle u, \tilde{u}' \rangle + 2 \lambda \langle u, H' \tilde{u}' \rangle \), \( \langle \tilde{u}, \partial_x g(\sigma_0) u' \rangle = \langle \tilde{u}, u' \rangle + 2 \lambda \langle \tilde{u}, H' u' \rangle \), and \( \langle \tilde{u}, \partial_x g(\sigma_0) \tilde{u}' \rangle = 2 \lambda \langle \tilde{u}, \tilde{u}' \rangle + 2 \lambda^2 \langle \tilde{u}, H' \tilde{u}' \rangle \).

Combining these with (4.1), one gets

\[
\text{d}\alpha (v(u), w(u')) = 2 \langle u, H' u' \rangle + \langle u, \tilde{u}' \rangle + 2 \lambda \langle u, H' \tilde{u}' \rangle + \langle \tilde{u}, u' \rangle + 2 \lambda \langle \tilde{u}, H' u' \rangle + 2 \lambda \langle \tilde{u}, \tilde{u}' \rangle + 2 \lambda^2 \langle \tilde{u}, H' \tilde{u}' \rangle.
\]

Using \( \tilde{u} = L(u) \), \( \tilde{u}' = L(u') \) and the fact that \( H' u = H u \) for all \( u \in T_{\gamma(t)} X \), it now follows that

\[
\text{d}\alpha (v(u), w(u')) = 2 \langle u, H' u' \rangle + \langle u, Lu' \rangle + 2 \lambda \langle u, H Lu' \rangle + \langle Lu, u' \rangle + 2 \lambda \langle Lu, H u' \rangle + 2 \lambda \langle Lu, Lu' \rangle + 2 \lambda^2 \langle Lu, H Lu' \rangle = \langle u, P u' \rangle,
\]

where \( P = 2H + 2L + 2 \lambda (HL + LH + L^2 + \lambda LH) \).

5 Pinched open billiard flows

In this section we describe some open billiard flows in \( \mathbb{R}^n \) \((n \geq 3)\) that satisfy the pinching condition \( (P) \). Clearly open billiards in \( \mathbb{R}^2 \) always satisfy this condition. As one can see below, the estimates we use are rather crude, so one would expect that with more sophisticated methods larger classes of open billiard flows could be shown to satisfy the condition \( (P) \).

First, we derive a formula which is useful in getting estimates for \( \|d\phi_t(x) \cdot u\| \) \((u \in E^n(x), x \in \Lambda)\), both from above and below. From the arguments in this section one can also derive a representation for the Jacobi fields along a billiard trajectory.

In what follows we use the notation from the beginning of Sect. 3. Here we assume that the boundary \( \partial K \) is at least \( C^3 \) smooth.

Fix for a moment a point \( x_0 = (q_0, v_0) \in \Lambda_\epsilon \). If \( \epsilon > 0 \) is sufficiently small, then \( W^n_\epsilon (x_0) \) has the form (cf. [11, 12]) \( W^n_\epsilon (x_0) = \{ (x, \nu_X(x)) : x \in X \} \) for some smooth hypersurface \( X \) in \( \mathbb{R}^n \) containing the point \( q_0 \) such that \( X \) is strictly convex with respect to the unit normal field \( \nu_X \).

Denote by \( B_x : T_q X \rightarrow T_q X \) the curvature operator (second fundamental form) of \( X \) at \( q \in X \). Then \( B_x \) is positive definite with respect to the normal field \( \nu_X \) ([12]).

Given a point \( q \in X \), let \( \gamma(x) \) be the forward billiard trajectory generated by \( x = (q, \nu_X(q)) \). Let \( q_1(x), q_2(x), \ldots \) be the reflection points of this trajectory and let \( \xi_j(x) \in S^{n-1} \) be the reflected direction of \( \gamma(x) \) at \( q_j(x) \). Set \( q_0(x) = q \), \( t_0(x) = 0 \) and denote by \( t_1(x), t_2(x), \ldots \) the times of the successive reflections of the trajectory \( \gamma(x) \) at \( \partial K \). Then \( \xi_j(x) = d_j(x) + d_1(x) + \ldots + d_{j-1}(x) \), where \( d_j(x) = \|q_{j+1}(x) - q_j(x)\| , 1 \leq j \). Given \( t \geq 0 \), denote by \( u_t(q) \) the shift of \( q \) along the trajectory \( \gamma(x) \) after time \( t \). Set \( X_t = \{ u_t(q) : q \in X \} \). When \( u_t(q) \) is not a reflection point of \( \gamma(x) \), then locally near \( u_t(q) \), \( X_t \) is a smooth convex \((n - 1)\)-dimensional surface in \( \mathbb{R}^n \) with "outward" unit normal given by the direction \( v_t(q) \) of \( \gamma(x) \) at \( u_t(q) \) (cf. [12]).

Fix for a moment \( t > 0 \) such that \( t_m(x_0) < t < t_{m+1}(x_0) \) for some \( m \geq 1 \), and assume that \( q(s), 0 \leq s \leq a \), is a \( C^3 \) curve on \( X \) with \( q(0) = q_0 \) such that for every \( s \in [0, a] \) we have \( t_m(x(s)) < t < t_{m+1}(x(s)) \), where \( x(s) = (q(s), \nu_X(q(s))) \). Assume also that \( a > 0 \) is so small

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that for all \( j = 1, 2, \ldots, m \) the reflection points \( q_j(s) = q_j(x(s)) \) belong to the same boundary component \( \partial K_{i_j} \) for every \( s \in [0, a] \).

We will now estimate \( \|d\phi_t(x_0) \cdot \xi_0\| \), where \( \xi_0 = \hat{q}(0) \in T_{q_0} X \).

Clearly \( \phi_t(x(s)) = (p(s), v_0(x(s))) \), where \( p(s) \), \( 0 \leq s \leq a \), is a \( C^3 \) curve on \( X_t \). For brevity denote by \( \gamma(s) \) the forward billiard trajectory generated by \( (q(s), v(q(s))) \) and set \( q_0(s) = q(s) \). Let \( \xi_j(s) \in S_j^{\text{adj}} \) be the reflected direction of \( \gamma(s) \) at \( q_j(s) \) and let \( \varphi_j(s) \) be the angle between \( \xi_j(s) \) and the outward unit normal \( \nu(q_j(s)) \) of \( \partial K \) at \( q_j(s) \). Let \( \phi_t(x(s)) = (u_t(s), v_t(s)) \), and let \( \tau_j(s) = t_j(x(s)) \) be the times of the consecutive reflections of the trajectory \( \gamma(s) \) at \( \partial K \). Set \( d_j(s) = d_j(x(s)) = \|q_{j+1}(s) - q_j(s)\| \) (\( 0 \leq j \leq m-1 \), \( t_0(s) = 0 \), \( t_{m+1}(s) = t \) and \( d_m(s) = t - t_m(s) \).

Denote by \( k_t(s) \) the normal curvature of \( X_t \) at \( u_t(s) \) in the direction of \( \frac{\partial}{\partial s} u_t(s) \).

Next, let \( k_0(s) \) be the normal curvature of \( X \) at \( q(s) \) in the direction of \( \hat{q}(s) \), and for \( j > 0 \) let \( k_j(s) > 0 \) be the normal curvature of \( X_{t_j(s)} \) (\( \partial K \) at \( q_j(s) \)) \( \|\hat{u}_j(s)\| = 1 \) of \( \lim_{t \to t_j(s)} X_t  \( (\|\hat{u}_j(s)\| = 1 \) of \( \lim_{t \to t_j(s)} X_t \)) \( \|\hat{u}_j(s)\| \). For \( j \geq 0 \) let

\[
B_j(s) : T_{q_j(s)}(X_{t_j(s)}) \to T_{q_j(s)}(X_{t_j(s)})
\]

be the curvature operator (second fundamental form) of \( X_{t_j(s)} \) at \( q_j(s) \), and define \( \ell_j(s) > 0 \) by

\[
(5.1) \quad [1 + d_j(s)\ell_j(s)]^2 = 1 + 2d_j(s)k_j(s) + (d_j(s))^2 \|B_j(s)\hat{u}_j(s)\|^2. \]

Finally, set

\[
(5.2) \quad \delta_j(s) = \frac{1}{1 + d_j(s)\ell_j(s)}, \quad 0 \leq j \leq m. \]

**Proposition 5.1.** For all \( s \in [0, a] \) we have

\[
(5.3) \quad \|\hat{q}_0(s)\| = \|\hat{p}(s)\|\delta_0(s)\delta_1(s) \ldots \delta_m(s). \]

As we mentioned in the Introduction, the above formula can be easily derived from the more general study of the evolution of unstable fronts in multidimensional dispersing billiards in [BCST] (see Section 5 there). Apart from that, one could prove (5.3) by using a simple modification of the argument in the Appendix of [Si2] dealing with the two-dimensional case. We omit the details.

We will now use Proposition 5.1 to prove Proposition 1.2.

In the notation above, let \( q_j = q_j(x) \) be the reflection points of the billiard trajectory \( \gamma(x) \) for some \( x = (q, \nu_x(q)) \), with \( q \in X \), and let \( t_j = t_j(x) \) and \( d_j = d_j(x) \). Consider the curvature operator \( B_j = B_{q_j} : \Pi_j = T_{q_j}(X_{t_j}) \to \Pi_j \) and let \( S_j : \mathbb{R}^n \to \mathbb{R}^n \) be the symmetry with respect to the tangent space \( T_j = T_{q_j}(\partial K) \); notice that \( S_j(\Pi_{j-1}) = \Pi_j \). Let \( N_j : T_j \to T_j \) be the curvature operator (second fundamental form) of \( \partial K \) at \( q_j \).

Notice that \( \Pi_j \) is the hyperplane in \( \mathbb{R}^n \) passing through \( q_j \) and orthogonal to \( \xi_j = \nu_{t_j(x)}(x) \); it will be identified with the \((n-1)\)-dimensional vector subspace of \( \mathbb{R}^n \) orthogonal to \( \xi_j \).

Before going on we need to recall the representation of the operator \( B_j \) due to Sinai [Si2] (cf. also Chernov [Ch1]). Introduce the linear maps \( V_j : \Pi_j \to T_j, V_j^* : T_j \to \Pi_j \) where \( V_j \) is (the restriction to \( \Pi_j \) of) the projection to \( T_j \) along the vector \( \xi_j \), while \( V_j^* \) is the projection to \( \Pi_j \) along the normal vector \( \nu_j = \nu(q_j) \). (Considering \( V_j : \mathbb{R}^n \to T_j \) and \( V_j^* : \mathbb{R}^n \to \Pi_j \), \( V_j^* \) is the self-adjoint of \( V_j \).) Let \( \varphi_j \) be the angle between \( \nu_j \) and \( \xi_j \). Then (Si2)

\[
(5.4) \quad B_j = S_j B_j^- S_j + 2 \cos \varphi_j V_j^* N_j V_j
\]

for \( 1 \leq j \leq m \), \( B_j^- = B_{j-1}(I + d_{j-1} B_{j-1})^{-1} \), and \( B_{m+1} = B_m(I + t' B_m)^{-1} \), where \( t' = t - t_m \geq \epsilon \).
Let \( \mu_j(x_0) \leq \lambda_j(x_0) \) be the minimal and the maximal eigenvalues of the operator \( B_j \). If \( \lambda \) is an eigenvalue of \( B_{j-1} \), then \( \lambda/(1 + d_{j-1}\lambda) \) is an eigenvalue of \( B_j \), and \( \frac{\lambda}{1 + \lambda d_{j-1}} < \frac{1}{d_{j-1}} \).

Next, a simple calculation shows that the spectrum of the operator \( V_j^* N_j V_j \) lies in the interval \([\kappa_{\min}, \frac{\kappa_{\max}}{\cos^2 \varphi_j}]\). Thus, using (5.4) we get

\[
(5.5) \quad \mu_0 \leq 2 \cos \varphi_j \kappa_{\min} \leq \mu_j(x_0) \leq \lambda_j(x_0) \leq \frac{1}{d_0} + \frac{2\kappa_{\max}}{\cos \varphi_j} \leq \lambda_0 ,
\]

where \( \mu_0 \) and \( \lambda_0 \) are as in Sect. 1.

**Proof of Proposition 1.2.** Before we continue, notice that there exist global constants \( 0 < c_1 < c_2 \) such that \( c_1 \| \xi \| \leq \| u \| \leq c_2 \| \xi \| \) for any \( u = (\xi, \eta) \in E^u(x), x \in \Lambda \) (see formula (3.5) above).

Assume that (1.1) holds. Fix an arbitrary \( x_0 = (q_0, v_0) \in \Lambda_\varepsilon \) and \( t > 0 \). We will now use the notation from the beginning of this section.

To estimate \( \|d\dot{\phi}_t(x_0) \cdot u\| \) for a given unit vector \( u = (\xi, \eta) \in E^u(x_0) \), consider a \( C^1 \) curve \( q(s) \), \( 0 \leq s \leq a \), on \( X = \text{pr}_1(W^u(x_0)) \) with \( q(0) = q_0 \) and \( \dot{q}(0) = \xi \), and define \( q_j(s), j = 1, \ldots, m \) and \( p(s) \) as in the beginning of this section. Then \( p(s) = \text{pr}_1(d\dot{\phi}_t(x(s))) \), so \( c_1 \| p(0) \| \leq \|d\dot{\phi}_t(x_0) \cdot u\| \leq c_2 \| p(0) \| \). Using this and Proposition 5.1, we get

\[
(5.6) \quad \frac{c_1 \| u \|}{c_2 \delta_1(0) \delta_2(0) \ldots \delta_m(0)} \leq \|d\dot{\phi}_t(x_0) \cdot u\| \leq \frac{c_2 \| u \|}{c_1 \delta_1(0) \delta_2(0) \ldots \delta_m(0)} .
\]

Recall that each \( \delta_j \) is given by (5.2) and (5.1), so if \( 0 < \mu_j(x_0) \leq \lambda_j(x_0) \) are the minimal and maximal eigenvalues of the operator \( B_j(0) \), then

\[
(1 + d_j(0)\ell_j(0))^2 \leq 1 + 2d_j(0)\lambda_j + d_j^2(0)\lambda_j^2 = (1 + d_j(0)\lambda_j(x_0))^2 ,
\]

so \( \ell_j(0) \leq \lambda_j(x_0) \). Similarly, \( \mu_j(x_0) \leq \ell_j(0) \). Moreover, it follows from (5.5) that \( \mu_0 \leq \mu_j(x_0) \) and \( \lambda_j(x_0) \leq \lambda_0 \) for all \( j \geq 1 \) and all \( x_0 \in \Lambda \).

Assuming \( \| u \| = 1 \) and recalling that \( d_j(x_0) = d_j(0) \) and \( t > d_1(x_0) + \ldots + d_m(x_0) \), (5.6) and (5.2) give

\[
\frac{1}{t} \ln \|d\dot{\phi}_t(x_0) \cdot u\| \leq \frac{\ln(c_2/c_1)}{t} + \frac{1}{t} \sum_{j=1}^{m} \ln(1 + d_j(x_0)\lambda_j(x_0)) \leq \frac{\ln(c_2/c_1)}{t} + \frac{\sum_{j=1}^{m} \ln(1 + (d_0 + a)\lambda_j)}{d_1(x_0) + \ldots + d_m(x_0)} \leq \frac{\ln(c_2/c_1)}{t} + \frac{m \ln(1 + (d_0 + a)\lambda_0)}{md_0} \leq \frac{\ln(c_2/c_1)}{t} + \frac{\ln(1 + (d_0 + a)\lambda_0)}{d_0} = \frac{\ln(c_2/c_1)}{t} + \beta_0 ,
\]

so \( \|d\dot{\phi}_t(x_0) \cdot u\| \leq (c_2/c_1) e^{t\beta_0} \) for all \( t > 0 \).

In a similar way, using (5.6) one derives that \( \|d\dot{\phi}_t(x_0) \cdot u\| \geq c'(c_1/c_2) e^{t\alpha_0} \) for \( t > 0 \), where \( \alpha_0 = \frac{\ln(1 + d_0 + \mu_0)}{d_0 + a} \) and \( c' > 0 \) is another global constant. Finally, notice that (1.1) implies \( 2\alpha_0 \geq \beta_0 + \alpha \) for some global constant \( \alpha > 0 \). Hence the condition (P) is satisfied. \( \square \)
6 Dolgopyat type estimates for pinched open billiard flows

Let \( \phi_t : M \to M \) be a \( C^1 \) flow on complete (not necessarily compact) Riemann manifold \( M \), and let \( \Lambda \) be a basic set for \( \phi_t \). It follows from the hyperbolicity of \( \Lambda \) that if \( \epsilon > 0 \) is sufficiently small, there exists \( \delta > 0 \) such that if \( x, y \in \Lambda \) and \( d(x, y) < \delta \), then \( W^s_\epsilon(x) \) and \( \phi_{[-\epsilon, \epsilon]}(W^u_\epsilon(y)) \) intersect at exactly one point \( [x, y] \in \Lambda \) (cf. [KH]). That is, there exists a unique \( t \in [-\epsilon, \epsilon] \) such that \( \phi_t(\langle x, y \rangle) \in W^u_\epsilon(y) \).

Let \( R = \{ R_i \}_{i=1}^k \) be a Markov family for \( \phi_t \) over \( \Lambda \) consisting of rectangles \( R_i = [U_i, S_i] \), where \( U_i \) (resp. \( S_i \)) are (admissible) subsets of \( W^s_\epsilon(z_i) \cap \Lambda \) (resp. \( W^u_\epsilon(z_i) \cap \Lambda \)) for some \( \epsilon > 0 \) and \( z_i \in \Lambda \) (cf. e.g. [PP] for details; see also [D]). The first return time function \( \tau : R = \bigcup_{i=1}^k R_i \to [0, \infty) \) is the projection along the leaves of local stable manifolds. Let \( \hat{\tau} \) be a translation in the definition of the vector \( \hat{\tau} \). Set \( U = \bigcup_{i=1}^k U_i \) and define the shift map \( \sigma : U \to U \) by \( \sigma = \rho \circ \hat{\tau} \), where \( \rho : R \to U \) is the projection along the leaves of local stable manifolds. Let \( \hat{U} \) be the set of all \( u \in U \) whose orbits do not have common points with the boundary of \( R \) (in \( \Lambda \)). Given a Lipschitz function (or map) on \( \hat{U} \), we will identify it with its (unique) Lipschitz extension to \( U \). Assuming that the local stable and unstable laminations over \( \Lambda \) are Lipschitz, the map \( \sigma \) is essentially Lipschitz on \( U \) in the sense that there exists a constant \( L > 0 \) such that if \( x, y \in U_i \cap \sigma^{-1}(U_j) \) for some \( i, j \), then \( d(\sigma(x), \sigma(y)) \leq L d(x, y) \). The same applies to \( \tau : U \to \mathbb{R} \).

Given a Lipschitz real-valued function \( f \) on \( \hat{U} \), set \( g = g_f = f - \hat{\tau} \), where \( \rho = \rho_f \in \mathbb{R} \) is the unique number such that the topological pressure \( \rho_f(g) \) of \( g \) with respect to \( \sigma \) is zero (cf. e.g. [PP]). For \( a, b \in \mathbb{R} \), one defines the Ruelle operator \( L_{g-(a+b)} : C^{\text{Lip}}(\hat{U}) \to C^{\text{Lip}}(\hat{U}) \) in the usual way (cf. e.g. [PP] or [D]), where \( C^{\text{Lip}}(\hat{U}) \) is the space of Lipschitz functions \( g : \hat{U} \to \mathbb{C} \). By \( \text{Lip}(g) \) we denote the Lipschitz constant of \( g \) and by \( ||g||_0 \) the standard sup norm of \( g \) on \( \hat{U} \).

We will say that the Ruelle transfer operators related to the function \( f \) on \( U \) are eventually contracting if for every \( \epsilon > 0 \) there exist constants \( \rho < 1, a_0 > 0 \) and \( C > 0 \) such that if \( a, b \in \mathbb{R} \) are such that \( |a| \leq a_0 \) and \( |b| \geq 1/a_0 \), then for every integer \( m > 0 \) and every \( h \in C^{\text{Lip}}(\hat{U}) \) we have

\[
||L_{g-(a+b)}^m h||_{\text{Lip},b} \leq C \rho^m |b|^m ||h||_{\text{Lip},b},
\]

where the norm \( ||.||_{\text{Lip},b} \) on \( C^{\text{Lip}}(\hat{U}) \) is defined by \( ||h||_{\text{Lip},b} = ||h||_0 + \frac{\text{Lip}(h)}{|b|} \). This implies in particular that the spectral radius of \( L_{g-(a+b)}^\tau \) in \( C^{\text{Lip}}(\hat{U}) \) does not exceed \( \rho \).

Next, assume that \( \phi_t \) is a \( C^2 \) contact flow on \( M \) with a \( C^2 \) invariant contact form \( \omega \). The following condition says that \( d\omega \) is in some sense non-degenerate on \( \Lambda \) near some of its points:

(ND): There exist \( z_0 \in \Lambda, \delta_0 > 0 \) and \( \mu_0 > 0 \) such that for any \( \delta \in (0, \delta_0] \), any \( z \in \Lambda \cap W^u_\delta(z_0) \) and any unit vector \( b \in E^u(z) \), there exist \( \tilde{z} \in \Lambda \cap W^u_\delta(z) \) and a unit vector \( a \in E^s(\tilde{y}) \) tangent to \( \Lambda \) at \( \tilde{y} \) with

\[
|d\omega_z(a_z, b_z)| \geq \mu_0
\]

where \( b_z \) is the parallel translate of \( b \) along the geodesic in \( W^u_\delta(z) \) from \( \tilde{z} \) to \( \tilde{z} \), while \( a_z \) is the parallel translate of \( a \) along the geodesic in \( W^s_\delta(z) \) from \( \tilde{y} \) to \( \tilde{z} \).

Remark 3. In fact, it is clear from the proof of Proposition 6.1 in [ST3] that in (ND) the ‘parallel translation’ in the definition of the vector \( b_z \) can be replaced by any other uniformly continuous (linear) operator \( P_z : E^u(z) \to E^u(z) \). E.g. using a local coordinate system to ‘identify’ \( E^u(z) \) and \( E^u(\tilde{z}) \) would be good enough. The same applies to the ‘parallel translation’ in the definition of the vector \( a_z \). In the case of the open billiard considered in this paper, (6.1) can be replaced simply by \( |da_z(a, b)| \geq \mu_0 \).
As an immediate consequence of the main result in \cite{SE3} (see also Sect. 6 there) one gets the following:

**Theorem 6.1.** (\cite{SE3}) Let \( \phi_t : M \rightarrow M \) be a \( C^2 \) contact flow on a \( C^2 \) Riemann manifold \( M \) and let \( \Lambda \) be a basic set for \( \phi_t \) such that the conditions (P) and (ND) are satisfied for the restriction of the flow on \( \Lambda \). Then for any Lipschitz real-valued function \( f \) on \( \hat{U} \) the Ruelle transfer operators related to \( f \) are eventually contracting.

Notice that for open billiard flows both \( W^s(x) \cap \Lambda \) and \( W^u(x) \cap \Lambda \) are Cantor sets, i.e. they are infinite compact totally disconnected sets without isolated points. In this particular case we can always choose the Markov family \( \mathcal{R} = \{ R_t \}_{t=1}^k \) so that the boundary (in \( \Lambda \)) of each rectangle \( R_t \) is empty and therefore \( U = \hat{U} \).

Next, assume that \( K \) is as in Sect. 1. Let \( \phi_t \) be the open billiard flow in the exterior of \( K \) and let \( \Lambda \) be its non-wandering set.

The following consequence of Lemma 3.1 shows that under some regularity condition, the billiard flow satisfies the condition (ND) on \( \Lambda \).

**Proposition 6.2.** Assume that the map \( \Lambda \ni x \mapsto E^u(x) \) is \( C^1 \). Then there exist \( z_0 \in \Lambda, \delta_0 > 0 \) and \( \mu_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), any \( z \in \Lambda \cap W^s_\delta(z_0) \) and any unit vector \( b \in E^u(z) \) tangent to \( \Lambda \) at \( z \) there exist \( y \in W^s_\delta(z) \) and a unit vector \( a \in E^s(y) \) tangent to \( \Lambda \) at \( y \) with \( |da(a, z, b)| \geq \mu_0 \), where \( a_z \) is the parallel translate of \( a \) along the geodesic in \( W^s_\delta(s) \) from \( y \) to \( z \).

**Proof of Proposition 6.2.** Notice that the standard symplectic form \( d\alpha \) in \( \mathbb{R}^{2n} \) satisfies

\[
|d\alpha(\xi, \eta)| \leq \|\xi\| \|\eta\|, \quad \xi, \eta \in \mathbb{R}^{2n},
\]

where we use the standard norm \( \| \cdot \| \) in \( \mathbb{R}^{2n} \).

Assume that the map \( \Lambda \ni x \mapsto E^u(x) \) is \( C^1 \); then the map \( \Lambda \ni x \mapsto E^s(x) \) is \( C^1 \), as well. Fix \( \sigma_0 \) as in Sect. 3. set \( z_0 = \sigma_0 \), and choosing \( \epsilon_0 > 0 \) and \( \delta'_0 \in (0, \epsilon_0] \) sufficiently small, define the map

\[
\Phi : W^u_\delta(z_0) \rightarrow W^s_{\delta'_0}(z_0)
\]
as in Sect. 3. It follows from Lemma 3.1 that there exists a constant \( \mu_1 > 0 \) (e.g. take \( \mu_1 = \kappa/ \sqrt{1 + C^2} \)) such that \( |d\alpha(d\Phi(z_0) \cdot b, b)| \geq \mu_1 \) for all unit vectors \( b \in E^u(z_0) \). Take \( \delta'_0 > 0 \) so small that

\[
|d\alpha(d\Phi(z) \cdot b, b)| \geq \frac{\mu_1}{2}, \quad z \in W^u_{\delta'_0}(z_0), \quad b \in E^u(z), \quad |b| = 1.
\]

Further restrictions on \( \delta'_0 \) will be imposed later.

Next, assuming \( \delta'_0 \in (0, \epsilon_0) \) is sufficiently small, for any \( x \in \Lambda \) and \( y \in \Lambda \cap W^u_{\delta'_0}(x) \) the local holonomy map \( H^u_{x,y} : \Lambda \cap W^u_{\delta'_0}(x) \rightarrow \Lambda \cap W^s_{\epsilon_0}(y) \) along unstable laminations is well-defined and uniformly Hölder continuous (see e.g. \cite{HPS} or \cite{PSW}). Recall that the map \( H^u_{x,y} \) is defined as follows. Given \( x' \in \Lambda \cap W^s_{\delta'_0}(x) \), there exist a unique \( y' \in W^u_{\epsilon_0}(y) \) such that \( \phi_t(y') \in W^u_{\epsilon_0}(x') \) for some \( t \in \mathbb{R}, \ |t| \leq \epsilon_0 \). Then we set \( H^u_{x,y}(x') = y' \). Under the additional condition that the unstable laminations are \( C^1 \), the maps \( H^u_{x,y} \) are \( C^1 \) as well (see e.g. Fact (2) on p. 647 in \cite{Ha1}).

That is, for each \( x' \in \Lambda \cap W^s_{\delta'_0}(x) \) the map \( H^u_{x,y} \) has a linearization \( L^u_{x,y}(x') : E^s(x') \rightarrow E^u(y') \) at \( x' \in \Lambda \cap W^s_{\delta'_0}(x) \) and \( \|L^u_{x,y}(x')\| \leq C_1 \) for some constant \( C_1 > 0 \) independent of \( x, y \) and \( x' \). Notice that \( L^u_{x,y}(x') \) preserves the sets of tangent vectors to \( \Lambda \), namely if \( \xi \in E^s(x') \setminus \{0\} \) is tangent to \( \Lambda \) at \( x' \), then \( L^u_{x,y}(x') \cdot \xi \) is tangent to \( \Lambda \) at \( y' \).

\(^2\)In fact, the main result in \cite{SE3} is much more general, however we are not going to discuss it here.
Since the map $L^u_{z_0,z}(x)$ depends continuously on $z \in W^u_{\delta_0}(z_0) \cap \Lambda$ and $x \in W^s_{\delta_0}(z_0) \cap \Lambda$ and $L^u_{z_0,z}(z_0) = I$ (the identity operator), we can take $\delta_0 > 0$ so small that

$$
\|L^u_{z_0,z}(x) - I\| \leq \frac{\mu_1}{4}
$$

for all $z \in \Lambda \cap W^u_{\delta_0}(z_0)$ and $x \in \Lambda \cap W^s_{\delta_0}(z_0)$. We will assume $\delta_0 > 0$ is chosen so small that for all $z \in W^u_{\delta_0}(z_0) \cap \Lambda$, $y \in W^s_{\delta_0}(z) \cap \Lambda$ and unit vectors $a \in E^u(y)$ we have $\|a - a_z\| \leq \frac{\mu_1}{8CC_1}$, where $a_z$ is the parallel translate of $a$ along the geodesic on $W^s_{\delta_0}(y)$ from $y$ to $z$.

Finally, take $\delta_0 \in (0, \delta_0]$ so small that for any $z \in W^u_{\delta_0}(z_0) \cap \Lambda$ we have $d(\Phi(z), z_0) < \delta_0$ and $d(z, H^u_{z_0,z}(\Phi(z))) < \delta_0$, where $d$ is the standard distance in $T(\mathbb{R}^n) = \mathbb{R}^{2n}$.

Now consider an arbitrary $\delta \in (0, \delta_0)$ and an arbitrary $z \in W^u_{\delta}(z_0) \cap \Lambda$. Let $b \in E^u(z)$ be a unit vector tangent to $\Lambda$ at $z$. Set $x = \Phi(z)$, $a' = d\Phi(z) \cdot b$ and $y = H^u_{z_0,z}(x)$ Then $x \in W^s_{\delta}(z_0) \cap \Lambda$, $a' \in E^s(x)$ is a tangent vector to $\Lambda$ at $x$ (see Sect. 3) and $y \in W^s_{\delta}(z) \cap \Lambda$. Moreover, since $\|b\| = 1$, it follows from $\|d\Phi\| \leq C$ (see Sect. 3) that $\|a'\| \leq C$.

Next, the vector $\tilde{a} = L^u_{z_0,z}(x) \cdot a' \in E^s(y)$ is tangent to $\Lambda$ at $y$ and by (6.3), $\|\tilde{a} - a'\| \leq \frac{\mu_1}{4}$. Moreover, $\|\tilde{a}\| \leq C_1 \|a'\| \leq CC_1$. Hence $a = \frac{\tilde{a}}{\|\tilde{a}\|} \in E^s(y)$ is a unit vector tangent to $\Lambda$ at $y$, and using (6.2) and (6.3), we get

$$
|d\omega(a, b)| \geq |d\omega(a, b) - |d\omega(a - a_z, b)| \geq \frac{1}{\|a\|} |d\omega(a, b) - \frac{1}{8CC_1} \mu_1\|
$$

$$
\geq \frac{1}{CC_1} \left[|d\omega(a', b)| - |d\omega(a - a', b)|\right] - \frac{\mu_1}{8CC_1}
$$

$$
\geq \frac{1}{CC_1} \left[|d\omega(d\Phi(z) \cdot b, b)| - \frac{\mu_1}{4} \right] - \frac{\mu_1}{8CC_1} \geq \frac{\mu_1}{4CC_1} - \frac{\mu_1}{8CC_1} = \mu_0 ,
$$

where $\mu_0 = \frac{\mu_1}{8CC_1}$. This proves the assertion. ■

From Theorem 6.1 and Proposition 6.2 one derives the following.

**Theorem 6.3.** Assume that the billiard flow $\phi_t : \Lambda \rightarrow \Lambda$ satisfies the condition (P) on its non-wandering set $\Lambda$. Then for any Lipschitz real-valued function $f$ on $U$ the Ruelle transfer operators related to $f$ are eventually contracting.

**Proof of Theorem 6.3.** As mentioned in Sect. 1, the condition (P) implies that the map $\Lambda \ni x \mapsto E^u(x)$ is $C^1$. Then by Proposition 6.2, $\phi_t$ satisfies the condition (ND) on $\Lambda$. Now applying Theorem 6.1 proves the assertion. ■

Results of this kind were first established by Dolgopyat ([D]) for some Anosov flows (i.e. $\Lambda = M$, a compact Riemann manifold). His results apply to geodesic flows on any compact surface (for any $f$), and to transitive Anosov flows on compact Riemann manifolds with $C^1$ jointly non-integrable local stable and unstable foliations for the Sinai-Bowen-Ruelle potential $f = \log \det(d\phi_t)_{E^u}$.

As one can see, Theorems 6.1 and 6.3 work for any potential. Theorem 6.3 generalizes the result in [St2] which deals with open billiard flows in the plane.

It should be mentioned that Dolgopyat type estimates for pinched open billiard flows have already been used in [PS1], [PS2] and [PS3] to obtain some rather non-trivial results. The main result in [PS1] provides existence of an analytic continuations of the cut-off resolvent of the Dirichlet Laplacian in $\mathbb{R}^2 \setminus K$ in a horizontal strip above the level of absolute convergence and
polynomial estimates for the norm of the cut-off resolvent in such a domain. The Dolgopyat type estimates for the open billiard flow in $\mathbb{R}^n \setminus K$ play a significant role in the proof. These estimates are also essential for the proof of the main result in [PS2] which deals with estimates of correlations for pairs of closed billiard trajectories for open billiards. Previous results of this kind were established in [PoS2] for geodesic flows on surfaces of negative curvature. Finally, in a very recent preprint [PS3], using Theorem 6.3 a fine asymptotic was obtained for the number of closed billiard trajectories in $\Lambda$ with primitive periods lying in exponentially shrinking intervals $(x - e^{-\delta x}, x + e^{-\delta x})$, $\delta > 0$, $x \to +\infty$.

As in [SL3], using Theorem 6.3 and an argument of Pollicott and Sharp [PoS1], we get some rather significant consequences about the Ruelle zeta function $\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$. Here $\gamma$ runs over the set of primitive closed orbits of $\phi_t : \Lambda \to \Lambda$ and $\ell(\gamma)$ is the least period of $\gamma$. Let $h_T$ be the topological entropy of $\phi_t$ on $\Lambda$.

**Corollary 6.4.** Under the assumptions in Theorem 6.3, the zeta function $\zeta(s)$ of the flow $\phi_t : \Lambda \to \Lambda$ has an analytic and non-vanishing continuation in a half-plane $\Re(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, there exists $c \in (0, h_T)$ such that

$$\pi(\lambda) = \#\{ \gamma : \ell(\gamma) \leq \lambda \} = \text{li}(e^{h_T \lambda}) + O(e^{c\lambda})$$

as $\lambda \to \infty$, where $\text{li}(x) = \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}$ as $x \to \infty$.

As another consequence of Theorem 6.3 and the procedure described in [D] one gets exponential decay of correlations for the flow $\phi_t : \Lambda \to \Lambda$.

Given $\alpha > 0$ denote by $F_\alpha(\Lambda)$ the set of Hölder continuous functions with Hölder exponent $\alpha$ and by $\|h\|_\alpha$ the Hölder constant of $h \in F_\alpha(\Lambda)$.

**Corollary 6.5.** Under the assumptions in Theorem 6.3, let $F$ be a Hölder continuous function on $\Lambda$ and let $\nu_F$ be the Gibbs measure determined by $F$ on $\Lambda$. Assume in addition that the boundary of $K$ is at least $C^0$. Then for every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that

$$\left| \int_\Lambda A(x)B(\phi_t(x)) \, d\nu_F(x) - \left( \int_\Lambda A(x) \, d\nu_F(x) \right) \left( \int_\Lambda B(x) \, d\nu_F(x) \right) \right| \leq C e^{-ct\|A\|_{\alpha} \|B\|_{\alpha}}$$

for any two functions $A, B \in F_\alpha(\Lambda)$.

One would expect that much stronger results could be established by using the techniques recently developed in [BKL], [L], [BG], [GL], [T] (see the references there, as well). Still, there are not very many results of this kind. In fact, for dimensions higher than two the author is not aware of any other results of this kind concerning billiard flows. What concerns billiards in general, bounds of correlation decay known so far concern mostly the corresponding discrete dynamical system (generated by the billiard ball map from boundary to boundary) – see [BSC], [Y] and [Ch2]. See also [ChZ] and the references there for some related results. Recently, a sub-exponential decay of correlations for Sinai billiards in the plane was established by Chernov ([Ch3]). For open billiard flows in the plane exponential decay of correlations was proved in [SL2] (as a consequence of the Dolgopyat type estimates established there).

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