Relativistic $O(q^4)$ two-pion exchange nucleon-nucleon potential: parametrized version

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The chiral two-pion exchange nucleon-nucleon interaction has nowadays a rather firm conceptual basis, but depends on low-energy constants, which may be extracted from fits to data. In order to facilitate this kind of application, we present here a parametrized version of our relativistic expansion of this component of the force to $O(q^4)$, performed recently.

I. INTRODUCTION

Nuclear interactions are strongly dominated by the quarks $u$ and $d$ and can be accommodated into a two-flavor version of QCD. The masses of these quarks are small in the GeV scale and one is not far from the massless lagrangian limit, in which QCD is invariant under chiral $SU(2) \times SU(2)$ transformations. For this reason, low energy hadronic processes can be reliably described by means of effective lagrangians that are symmetric under the Poincaré and isospin groups and incorporate approximate chiral symmetry, realized in the Nambu-Goldstone mode.

The one-pion exchange $NN$ potential ($OPEP$) is simple, has been well established long ago, and dominates completely partial waves with orbital momentum $L \geq 5$. The two-pion exchange potential ($TPEP$), on the other hand, is rather complex and has become free of important ambiguities only in the 1990s, after the systematic use of chiral symmetry in its theoretical description [1–11].

Chiral perturbation theory is based on the existence of a characteristic scale $q$, set by both pion four-momenta and nucleon three-momenta, such that $q < 1$ GeV. Due to this technique, nowadays one understands rather well the internal hierarchies of the $NN$ potential in terms of chiral layers. Leading terms of the chiral $TPEP$ are of order $O(q^2)$ and expansions which go up to $O(q^4)$ are already available. One of them was produced recently by our group [10,11]. We departed from a relativistic lagrangian and evaluated the relevant Feynman diagrams covariantly, without resorting to heavy baryon approximations. The so obtained $\mathcal{T}$-matrix was then transformed into a potential, expressed in terms of covariant loop integrals and observable parameters. Without loss of generality, one may choose these parameters to be either the subthreshold coefficients extracted from $\pi N$ scattering [12] or the low-energy constants ($LECs$) present in the effective lagrangian. These two possibilities are formally equivalent, but suited to slightly different physical purposes. The former choice yields a closed prediction for the potential, whereas the latter gives rise to an open theoretical structure which may be used to fit $NN$ data.

Nowadays, a rather pressing issue is to determine the extent that nature backs this picture. The comparison of chiral predictions with empirical phase shifts is not straightforward, since existing theoretical potentials are not reliable for distances smaller than 1 fm [11]. Three complementary possibilities are available for overcoming this problem. The most direct one is to use peripheral waves and rely on those windows in angular momenta and energies for which the centrifugal barrier is effective in suppressing the interaction at short distances. When this happens, the Born approximation can be used and one does not need to know the potential close to the origin. However, this kind of test is not very stringent, since peripheral waves are small and uncertainties are large.
II. PARAMETRIZED POTENTIAL

The configuration space potential has the isospin structure

\[ V(r) = V^+(r) + \tau^{(1)} \cdot \tau^{(2)} V^-(r), \]

with

\[ V^\pm(r) = V^\pm_C + V^\pm_{LS} \Omega_{LS} + V^\pm_T \Omega_{T} + V^\pm_{SS} \Omega_{SS} + V^\pm_Q \Omega_{Q}, \]

and

\[ \Omega_{LS} = L(\sigma^{(1)} + \sigma^{(2)})/2, \quad \Omega_{T} = 3 \sigma^{(1)} \cdot r \sigma^{(2)} - \sigma^{(1)} \sigma^{(2)}, \quad \Omega_{SS} = \sigma^{(1)} \sigma^{(2)}. \]

The form of the operator \( \Omega_{Q} \) in configuration space is highly non-local and can be found in Ref. [14].

In Refs. [10,11] we have presented a \( O(q^4) \) relativistic expansion of the TPEP, which is reproduced, in an alternative form, in appendix A. The configuration space potential is written in terms of numerical coefficients which multiply dimensionless functions arising form the Fourier transforms of Feynman loop integrals. The former are combinations of external parameters representing the pion and nucleon masses, \( \mu \) and \( m \), respectively, the pion decay constant \( f_{\pi} \), the axial coupling constant \( g_{A} \), and the LECs \( c_{i} \) and \( d_{i} \). The latter are denoted by \( Z_{i} \) and depend on just \( \mu, m, f_{\pi} \), and \( x \equiv \mu r \). We keep the external quantities as free and parametrize the function \( Z_{i} \equiv (F_{i}, G_{i}) \) as

\[ Z_{i} = -\frac{\mu}{(4\pi)^{3/2}} \left( \frac{\mu}{f_{\pi}} \right)^{4} \sum_{n} \gamma_{i}^{n} x^{n} e^{-2x/x^{2}} \]

The coefficients \( \gamma_{i}^{n} \) corresponding to the various cases are given in the tables at the end of this section. This parametrization is more than 1% accurate in the range \( 0.8 \text{ fm} \leq r \leq 10 \text{ fm} \).

Using the definition \( \alpha \equiv \mu/m \), the profile functions are written as

\[ V^+_C = \frac{3g_{A}^{4}}{16} \left\{ 4m c_{1} \left[ 2I_{2} - I_{1} - 2\alpha(H_{1} - H_{2}) \right] + \frac{m c_{2}}{3} \alpha \left( 3H_{1} - 2H_{3} \right) - 2m c_{3} \left[ I_{1} - I_{3} + \alpha \left( 2H_{1} - 2H_{2} - H_{3} \right) \right] \right\} + \frac{3\alpha^{2}}{2} \left[ (4m c_{1})^{2} H_{1} + \frac{1}{5} (m c_{2})^{2} \left( 4H_{2} - H_{3} \right) + (2m c_{3})^{2} \left( H_{1} - H_{3} \right) \right] - \frac{16}{3} m^{2} c_{1} c_{2} H_{2} - 16m^{2} c_{1} c_{3} \left( 2H_{2} - H_{1} \right) + \frac{4}{3} m^{2} c_{2} c_{3} \left( 2H_{2} - H_{3} \right) + \frac{3g_{A}^{4} m^{2}}{16\pi f_{\pi}^{2}} \left( I_{1} - I_{3} \right) \]

\[ -\frac{3g_{A}^{4} \mu^{2}}{256\pi^{2} f_{\pi}^{2}} \left\{ 8 \left( I_{6} - I_{8} \right) - 7 \left( I_{5} - I_{7} \right) + 4\pi \left[ 4I_{3} + 6I_{2} - 7I_{1} \right] \right\}, \]

where

\[ 16 \left( 2I_{2} - I_{1} - 2\alpha(H_{1} - H_{2}) \right) = 8 \left( I_{6} - I_{8} \right) - 7 \left( I_{5} - I_{7} \right) + 4\pi \left[ 4I_{3} + 6I_{2} - 7I_{1} \right] \].
\[ V_{LS}^+ = \frac{3g^4_\alpha}{8} G_2 - 4g^4_\alpha \alpha^2 m c_2 H_5, \]  
\[
 V_T^+ = -\frac{g^4_\alpha}{16} G_3 + \frac{g^4_\alpha \alpha^2}{3} (m^2 \bar{d}_{14} - m^2 \bar{d}_{15}) (H_3 - 3 H_5) - \frac{g^4_\alpha \mu^2}{96 \pi^2 f_\pi} (H_3 - 3 H_5),
\]  
\[ V_{SS}^+ = \frac{g^4_\alpha}{8} G_4 - \frac{2g^4_\alpha \alpha^2}{3} (m^2 \bar{d}_{14} - m^2 \bar{d}_{15}) H_3 + \frac{g^4_\alpha \mu^2}{48 \pi^2 f_\pi} H_3, \]  
\[ V_C^- = \frac{g^4_\alpha}{8} G_5 + \frac{g^4_\alpha}{12} \left\{ 2 \left( 5 H_2 - 3 H_1 \right) - 3 \alpha \left( I_1 - I_3 \right) - 3 \alpha^2 \left( 2 H_1 - 2 H_2 - H_3 \right) \right\} 
+ \alpha^2 \left\{ 2 m c_4 \left( 5 H_3 - 12 H_1 + 12 H_2 \right) - 8 (m^2 d_1 + m^2 d_2) \left( 5 H_3 + 2 H_2 - 6 H_1 \right) 
- \frac{4 m^2 d_3}{5} \left( 7 H_3 - 8 H_2 \right) + 32 m^2 d_5 \left( 5 H_2 - 3 H_1 \right) \right\} 
+ \frac{1}{12} \left\{ H_2 + \alpha^2 \left\{ 2 m c_4 H_3 + 8 (m^2 d_1 + m^2 d_2) \left( 2 H_2 - H_3 \right) + \frac{12 m^2 d_3}{5} \left( 4 H_2 - H_3 \right) 
+ 32 m^2 d_5 H_2 \right\} \right\} 
- \frac{g^4_\alpha \mu^2}{288 \pi^2 f_\pi^2} \frac{1}{50} \left( 201 H_3 + 156 H_2 - 300 H_1 \right) 
- \frac{g^4_\alpha \mu^2}{144 \pi^2 f_\pi^2} \left\{ 5 H_2 - 3 H_7 - \frac{61}{20} H_3 + \frac{61}{5} H_2 - 3 H_1 \right\} 
- \frac{\mu^2}{288 \pi^2 f_\pi^2} \left[ H_8 - \frac{1}{10} \left( 14 H_3 - 76 H_2 \right) \right], \]
\[ V_{LS}^- = \frac{g^4_\alpha}{16} G_6 - \frac{g^4_\alpha \alpha}{12} \left[ 6 I_4 + \alpha \left( 3 - 40 m c_4 \right) H_5 + \alpha 24 m c_4 H_4 \right] 
+ \frac{\alpha^2}{24} \left( 3 + 16 m c_4 \right) H_5, \]  
\[ V_T^- = \frac{g^4_\alpha}{48} G_7 + \frac{g^4_\alpha \alpha}{144} \left( 1 + 4 m c_4 \right) \left\{ 6 \left( I_4 - 3 I_4 \right) + \alpha \left[ 8 H_3 - 12 H_1 + 12 H_2 \right] 
- 3 \left( 8 H_5 - 3 H_4 \right) \right\} 
- \frac{\alpha^2}{144} (1 + 4 m c_4)^2 \left( H_3 - 3 H_5 \right) - \frac{g^4_\alpha \mu^2}{96 \pi^2 f_\pi} (I_3 - 3 I_4) 
+ \frac{g^4_\alpha \mu^2}{84 \pi^2 f_\pi^2} \left[ (I_8 - 3 I_9) - 2 \pi (I_3 - 3 I_4) \right], \]  
\[ V_{SS}^- = \frac{g^4_\alpha}{24} G_8 - \frac{g^4_\alpha \alpha}{72} \left( 1 + 4 m c_4 \right) \left\{ 6 I_3 + \alpha \left( 8 H_3 - 12 H_1 + 12 H_2 \right) \right\} 
+ \frac{\alpha^2}{72} (1 + 4 m c_4)^2 \left( H_3 + \frac{g^4_\alpha \mu^2}{48 \pi^2 f_\pi} I_3 - \frac{g^4_\alpha \mu^2}{192 \pi^2 f_\pi^2} (I_8 - 2 \pi I_3) \right). \]
The parametrized profile functions given above depend explicitly on four well known quantities, namely $m$, $\mu$, $g_A$, $f_{\pi}$, and on the less known LECs $c_i$ and $d_i$. Therefore, the latter may be extracted from fits to data. When doing this, however, one has to bear in mind that, as discussed in Ref. [11], the influence of the LECs over the profile functions is rather uneven. Indeed, their influence over $V^+_{C\bar{C}}$, $V^+_{LS}$, $V^+_{T}$, and $V^+_{SS}$ is rather strong, but barely perceptible in $V^-_{C\bar{C}}$, $V^-_{LS}$, $V^-_{T}$ and $V^-_{SS}$. 

| $\gamma_n$ | $-1/2$ | $-3/2$ | $-5/2$ | $-7/2$ | $-9/2$ | $-11/2$ |
|------------|--------|--------|--------|--------|--------|--------|
| $H_1$      | -1     | $-3/16$| $15/32$| $-105/8192$| $0.0069211$| $-0.002031054$|
| $H_2$      | -      | $3/2$  | $45/32$| $315/1024$  | $-0.0508797$| $0.0105639$|
| $H_3$      | -      | 6      | $165/8$| $8715/256$  | $27.45483$ | $5.43256$|
| $H_4$      | -      | 2      | $23/8$ | $153/256$   | $-0.0723934$| -     |
| $H_5$      | -      | -      | -      | $-129/16$   | $-3555/512$| $-1.33605$|
| $H_6$      | -      | $-3.89861$ | $4.23305$ | $-0.833136$| -     | -     |
| $H_7$      | -      | $5.78893$ | $-7.63019$ | $-2.69576$ | -     | -     |
| $H_8$      | -      | -      | -      | $-14.3654$  | $14.6375$ | $39.3909$ |

| $\gamma_n$ | 1 | 0 | $-1$ | 2 | $-3$ | 4 | 5 | 6 |
|------------|---|---|------|---|------|---|---|---|
| $G_1$      | - | 2.83823 | $-7.200711$ | 38.9637 | $-55.5164$ | 47.2443 | $-16.2395$ | - |
| $G_2$      | - | - | $-6.12315$ | $-28.1422$ | $-30.2813$ | 0.023458 | $-15.8996$ | 7.18869 |
| $G_3$      | - | 0.5579 | 17.1039 | 16.8038 | 9.94755 | 3.40171 | -2.7544 | - |
| $G_4$      | - | 0.569624 | 15.9429 | $-4.26031$ | 15.6445 | $-5.06641$ | - | - |
| $G_5$      | - | $-0.217221$ | $-9.98415$ | $-4.662$ | $-36.9761$ | 13.4087 | $-6.21047$ | - |
| $G_6$      | - | - | 7.90985 | 55.9568 | 86.3242 | 66.9540 | $-29.5680$ | 11.8985 |
| $G_7$      | - | 1.69219 | 25.5612 | 6.53589 | 160.459 | $-169.567$ | 120.612 | $-36.7881$ |
| $G_8$      | - | 1.7661 | 21.2122 | $-9.87710$ | 116.454 | $-144.344$ | 103.063 | $-30.9265$ |

| $\gamma_n$ | 2 | 1 | 0 | $-1$ | 2 | $-3$ | 4 |
|------------|---|---|---|------|---|------|---|
| $I_1$      | - | 0.000483761 | -0.0226386 | 1.53346 | 0.0595627 | $-0.0913580$ | 0.0291743 | - |
| $I_2$      | - | - | 0.000381934 | 0.0158372 | $-1.63009$ | $-0.660019$ | 0.0532419 | - |
| $I_3$      | - | - | - | 0.242214 | $-8.87827$ | $-6.47733$ | $-30.5206$ | - |
| $I_4$      | - | - | - | - | $-0.00147946$ | 2.99191 | 6.86185 | 1.82098 |

| $\gamma_n$ | 3/2 | 1/2 | $-1/2$ | $-3/2$ | $-5/2$ | $-7/2$ | $-9/2$ |
|------------|-----|-----|------|------|------|------|------|
| $I_5$      | 0.168731 | -0.00262 | $-2.18325$ | 1.79108 | - | - | - |
| $I_6$      | - | $-0.129344$ | 5.46315 | $-2.54740$ | 0.240122 | - | - |
| $I_7$      | - | $-0.464737$ | 20.2994 | 31.4762 | $-26.1396$ | - | - |
| $I_8$      | - | - | - | $-28.6452$ | $-101.827$ | 28.3771 | 99.2609 |
| $I_9$      | - | - | - | - | $-0.454235$ | 18.7535 | 24.4758 | $-31.2207$ |
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APPENDIX A: THEORETICAL POTENTIAL

The $O(q^4)$ relativistic expansion of the TPEP produced in Refs. [10,11] was based on the evaluation of three families of diagrams\(^1\). The first of them involves only pion and nucleon degrees of freedom into single loops and corresponds to the minimal realization of chiral symmetry [3]. It includes the subtraction of the iterated OPEP and yields the terms in the profile functions given below which are proportional to just $g^2_{\lambda}/f^2_N$, $g^2_\pi/f^2_N$ or $1/f^2_N$. Terms proportional to $1/f^6_N$, on the other hand, come from two-loop processes, either in the form of $t$-channel contributions from the second family or $s$ and $u$-channel terms embodied in the subthreshold coefficients of the third family. Finally, the third group of diagrams includes chiral corrections associated with other degrees of freedom, hidden within the LECs $c_i$ and $d_i$, and gives rise to contributions which are proportional to either $(LEC)/f^2_N$ or $(LEC)^2/f^4_N$. In Ref. [11] we have expressed this last class of results in terms of the $\pi N$ subthreshold coefficients. As these can be easily translated into LECs, in the present work we write the potential in terms of these constants, which appear directly into the effective lagrangians. The following expressions correspond to the updated version as described in Ref. [13].

The radial components of the potential are expressed in terms of the following profile functions\(^2\)

$$V^{\pm}_{C}(r) = \tau^{\pm} U^{\pm}_{C}(x), \quad (A1)$$

$$V^{\pm}_{LS}(r) = \tau^{\pm} \frac{\mu^2}{m^2} \frac{1}{x} \frac{d}{dx} U^{\pm}_{LS}(x), \quad (A2)$$

$$V^{\pm}_{T}(r) = \tau^{\pm} \frac{\mu^2}{m^2} \left[ \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} \right] U^{\pm}_{T}(x), \quad (A3)$$

$$V^{\pm}_{SS}(r) = -\tau^{\pm} \frac{\mu^2}{m^2} \left[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} \right] U^{\pm}_{SS}(x), \quad (A4)$$

where $\tau^{+} = 3$ and $\tau^{-} = 2$.

Defining $t$ as the Laplacian operator acting on the variable $x = \mu r$ we write our profile functions as

$$-U^{\pm}_{C} = \frac{3g^2_{\lambda}\mu^5}{256\pi^4f^2_N} \left\{ \left[ 1 - (1 - \alpha^2/4)\hat{t} + \hat{t}^2/4 \right] S_{x} - \left[ 1 - (1 + \alpha^2/4)\hat{t} + \hat{t}^2/4 \right] S_{\hat{t}} \right\} - \frac{3g^2_{\lambda}\mu^5\alpha}{32\pi^2f^2_N} \left\{ 4m c_1 \left[ 2(1 - \hat{t}/4) - 1 \right] S_{\hat{t}} - \frac{\alpha}{2} \hat{t} S_{\hat{t}} \right\} + \frac{m c_2}{3} \alpha \left[ 3 - 2(1 - \hat{t}/4) \hat{t} \right] S_{\hat{t}} - 2m c_3 \left[ 1 - (1 - \hat{t}/4) \hat{t} \right] S_{\hat{t}} + \alpha \left[ \hat{t}/2 - (1 - \hat{t}/4) \hat{t} \right] S_{\hat{t}} \right\}$$

$$+ \frac{3m^4 c^2}{16} \left\{ (4m c_1)^2 S_{\hat{t}} + \frac{1}{5} (m c_2)^2 \left[ 4(1 - \hat{t}/4) - (1 - \hat{t}/4) \hat{t} \right] S_{\hat{t}} \right\} + (2m c_3)^2 \left[ 1 - (1 - \hat{t}/4) \hat{t} \right] S_{\hat{t}} - \frac{16}{3} m^2 c_1 c_2 (1 - \hat{t}/4) S_{\hat{t}} - 16m^2 c_1 c_3 \left[ 2(1 - \hat{t}/4) - 1 \right] S_{\hat{t}}$$

\(^1\)Please see section II of Ref. [11], for a detailed discussion of the meaning and dynamical contents of these families of diagrams, which are given in its Fig.2.

\(^2\)Please see Eqs. (3.4)-(3.8) of Ref. [11]. Note that in Eq. (3.8) a multiplication factor of $\mu^3$ is missing.
\[ + \frac{4}{3} m^2 c^2 c_3 \left[ 2(1 - \frac{i}{4}) - (1 - \frac{i}{4} \hat{t}) S_t \right] \left[ 1 - (1 - \frac{i}{4} \hat{t}) \right] S_t + \frac{3 g_A^3 \mu^7}{4096 \pi^4 f_\pi^2} \left\{ \left[ 8(1 - \frac{i}{4}) - 7 - \left[ 8(1 - \frac{i}{4})^2 - 7(1 - \frac{i}{4}) \right] \hat{t} \right] S_t \right\}, \]

\[ - \frac{3 g_A^3 \mu^7}{4096 \pi^4 f_\pi^2} \left\{ \left[ 8(1 - \frac{i}{4}) - 7 - \left[ 8(1 - \frac{i}{4})^2 - 7(1 - \frac{i}{4}) \right] \hat{t} \right] S_t \right\}, +4 \pi \left[ 4(1 - \frac{i}{4}) \hat{t} + 6(1 - \frac{i}{4}) - 7 \right] S_t \}, \] (A5)

\[ -U_{L,S}^+ = \frac{3 m g_A^3 \mu^4}{128 \pi^4 f_\pi^2} \left\{ (1 - \frac{i}{2}) (\tilde{S}_b - S_t) - (3/2 - 5 \hat{t}/8) S_a + \frac{\alpha}{4} (1 + 2\hat{t} - \hat{t}^2/2) (S_x + S_b) \right\} + 2\alpha \hat{t} S_t \right\} - \frac{g_A^3 \mu^5}{4 \pi^2 f_\pi^2} m c_2 (1 - \frac{i}{4}) S_t \right\}, \] (A6)

\[ -U_T^+ = -U_{SS}^+ / 2 = -\frac{m^2 g_A^3 \mu^3}{256 \pi^4 f_\pi^2} \left\{ (1 - \frac{i}{4}) S_b + \frac{\alpha}{2} \left[ (1 - \frac{i}{2}) (S_t - \tilde{S}_b) + (1 - \frac{i}{4}) S_a \right] \right\} + \left[ (1 - \alpha^2/4) - (1 - \alpha^2) \hat{t}/4 - \alpha^2 \hat{t}^2/16 \right] S_x \right\} + \frac{g_A^3 \mu^5}{48 \pi^4 f_\pi^2} (m^2 d_{14} - m^2 \tilde{d}_{15}) (1 - \frac{i}{4}) S_t \right\}

\[ + \frac{m^2 g_A^3 \mu^5}{1536 \pi^4 f_\pi^2} (1 - \frac{i}{4}) S_t \right\} \] (A7)

\[ -U_S = \frac{g_A^3 \mu^5}{128 \pi^4 f_\pi^2} \left\{ \left[ 1 - (1 - \alpha^2/4) \hat{t} + (1 - \alpha^2) \hat{t}^2/4 + \alpha^2 \hat{t}^3/16 \right] S_x \right\} + \left[ 1 - (1 + \alpha^2/4) \hat{t} + (1 + \alpha^2) \hat{t}^2/4 - \alpha^2 \hat{t}^3/16 \right] S_b \right\} + \alpha \left[ (2 - 3\hat{t} + \hat{t}^2) S_t + (2 - \hat{t}) S_a \right] - [10/3 - (11/6 - \alpha^2) \hat{t} - \alpha^2 \hat{t}^2/2] S_t \right\}

\[ + \frac{g_A^3 \mu^5}{192 \pi^4 f_\pi^2} \left\{ 2 \left[ 5(1 - \frac{i}{4}) - 3 \right] S_t - 3 \alpha \left[ 1 - (1 - \frac{i}{4} \hat{t}) \right] S_t - 3 \alpha^2 \left[ \hat{t}/2 - (1 - \frac{i}{4} \hat{t}) \right] S_t \right\} + \alpha \left[ 2 m c_4 \left[ 5(1 - \frac{i}{4} \hat{t}) - 3 \right] S_t - 8(m^2 d_1 + m^2 d_2) \left[ 5(1 - \frac{i}{4} \hat{t}) + 2(1 - \frac{i}{4}) - 6 \right] S_t \right\}

\[ - \frac{4 m^2 d_3}{5} \left[ 7(1 - \frac{i}{4} \hat{t}) - 8(1 - \frac{i}{4}) \right] S_t + 32 m^2 d_5 \left[ 5(1 - \frac{i}{4} \hat{t}) - 3 \right] S_t \right\}

\[ + \frac{\mu^5}{192 \pi^4 f_\pi^2} \left\{ (1 - \frac{i}{4}) S_t + \alpha \left[ 2 m c_4 \left[ (1 - \frac{i}{4}) \right] S_t \right\} + 8(m^2 d_1 + m^2 d_2) \left[ 2(1 - \frac{i}{4}) - (1 - \frac{i}{4} \hat{t}) \right] S_t + \frac{12 m^2 d_3}{5} \left[ 4(1 - \frac{i}{4}) - (1 - \frac{i}{4} \hat{t}) \right] S_t \right\}

\[ + 32 m^2 d_5 (1 - \frac{i}{4} \hat{t}) S_t \right\} \] - \frac{g_A^3 \mu^7}{4608 \pi^4 f_\pi^2} \frac{1}{50} \left[ 201(1 - \frac{i}{4} \hat{t}) + 156(1 - \frac{i}{4} \hat{t}) - 300 \right] S_t \right\} \] - \frac{g_A^3 \mu^7}{4608 \pi^4 f_\pi^2} \frac{1}{50} \left[ 1250(1 - \frac{i}{4})^2 - 1500(1 - \frac{i}{4}) + 450 \right] S_t \right\}

\[ + \left[ -346(1 - \frac{i}{4}) \hat{t} + 1084(1 - \frac{i}{4} \hat{t}) - 300 \right] S_t \right\} \]
\[ -\frac{g^2 \mu^7}{2304 \pi^4 f_0^2} \left\{ \left[ 5 \left( 1 - \frac{i}{4} \right)^2 - 3 \left( 1 - \frac{i}{4} \right) \right] S_{\ell \ell} + \left[ -\frac{61}{20} \left( 1 - \frac{i}{4} \right) + \frac{61}{5} \left( 1 - \frac{i}{4} \right) - 3 \right] S_{\ell} \right\} \]

\[ -\frac{\mu^7}{4608 \pi^4 f_0^2} \left\{ \left( 1 - \frac{i}{4} \right)^2 S_{\ell \ell} - \frac{1}{10} \left[ 14 \left( 1 - \frac{i}{4} \right) - 76 \left( 1 - \frac{i}{4} \right) \right] S_{\ell} \right\}, \quad (A8) \]

\[ U_{LS} = \frac{m g^2 \mu^4}{256 \pi^2 f_0^2} \left\{ (6 - 5 i/2) S_{\ell a} - (4 - 2 i) \left( S_{\ell b} \right) + 4 S_{\ell} \right\} + \alpha \left[ 2 \left( 1 - \frac{i}{4} \right) S_{\ell} + (1 - i/2)^2 \left( S_{\ell} - S_{\ell b} \right) \right] \]

\[ -\frac{m g^2 \mu^4}{192 \pi^2 f_0^2} \left\{ 6 \left( 1 - \frac{i}{4} \right) S_{\ell} + \alpha \left[ 3 - 40 m c_4 \right] \left( 1 - \frac{i}{4} \right) S_{\ell} + \alpha \left[ 24 m c_4 \right] S_{\ell} \right\} + \frac{\mu^5}{384 \pi^2 f_0^2} \left( 3 + 16 m c_4 \right) \left( 1 - \frac{i}{4} \right) S_{\ell}, \quad (A9) \]

\[ -U_T = -U_{SS}/2 = \frac{m g^2 \mu^4}{768 \pi^2 f_0^2} \left\{ \left( 1 - \frac{i}{4} \right) \left( S_{\ell b} + (1 - \frac{i}{4}) \left( S_{\ell a} - 2 S_{\ell} \right) \right) - \frac{\alpha}{12} \left( 16 - 7 i \right) S_{\ell} \right\} \]

\[ + \frac{m g^2 \mu^4}{2304 \pi^2 f_0^2} \left( 1 + 4 m c_4 \right) \left\{ 6 \left( 1 - \frac{i}{4} \right) S_{\ell} + \alpha \left[ 8 \left( 1 - \frac{i}{4} \right) - 3 \right] S_{\ell} \right\} \]

\[ -\frac{\mu^5}{2304 \pi^2 f_0^2} \left( 1 + 4 m c_4 \right)^2 \left( 1 - \frac{i}{4} \right) S_{\ell} - \frac{m^2 g^6 \mu^5}{1536 \pi^4 f_0^2} \left( 1 - \frac{i}{4} \right) S_{\ell} \]

\[ + \frac{m^2 g^6 \mu^5}{6144 \pi^4 f_0^2} \left( 1 - \frac{i}{4} \right)^2 S_{\ell \ell} - 2 \pi \left( 1 - \frac{i}{4} \right) S_{\ell}. \quad (A10) \]

The dimensionless functions $S_{\ell}(x)$ carry the spatial dependence of the potential and are given by Eqs. (3.16)-(3.23) of Ref. [11].