Threshold Multiparticle Amplitudes
in $\phi^4$ Theories at Large $N$

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Abstract

I review some recent work on the problem of multiparticle production in a $\phi^4$-theory with an $O(N)$ symmetry. Threshold amplitudes with fixed number of produced particles are exactly calculated at large-$N$ to all loops and vanish on mass shell for $2\rightarrow n$ when $n>2$ due to a dynamical symmetry. I consider an extension to the cases when the $O(N)$ symmetry is softly broken by masses including spontaneous breaking of a remaining reflection symmetry. The exact solutions are obtained by the Gelfand–Dikiĭ technique of finding the diagonal resolvent of the Schrödinger operator which emerges due to factorization at large $N$. I report also some new results on the diagonal resolvent for a general Poschl–Teller potential, which could be useful for calculations of multiparticle amplitudes in the standard model.

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1 Introduction and résumé

1.1 $n!$

The history of the subject began in 1990 when Ringwald [1] and Espinosa [2] utilized $n!$, which is due to $n$ identical scalar particles in a final state, to get a large amplitude of $B+L$ violating processes in the standard model due to instantons in the dilute gas approximation. Soon after Cornwall [3] and Goldberg [4] applied a similar idea to a problem of perturbative multiparticle production in the $\phi^4$ theory with a small coupling constant $\lambda$. Some other early references include also [5, 6, 7].

In the $\phi^4$ theory one considers the process when a virtual particle with the 4-momentum $(nm, \vec{0})$ produces $n$ on-mass-shell particles of mass $m$ at rest with the 4-momentum $(m, \vec{0})$ each. Such a process may appear in a high energy collision of quark and antiquark when $n$ Higgs particles are produced. Since the amplitude of the process is proportional to $n!$:

$$a(n) \propto n!,$$  \hspace{1cm} (1.1)

a naive estimate of the cross-section in the tree approximation is

$$\sigma_{q\bar{q}\to nH} \sim \frac{1}{n!} |a(n)|^2 \sim n! \lambda^{n-1},$$  \hspace{1cm} (1.2)

which may become large for $n \gtrsim 1/\lambda$.

A special comment is needed about “triviality” of the $\phi^4$ theory. If the $\phi^4$ theory were a fundamental one, the renormalized value of $\lambda$ would vanish. A modern look at this problem is that it is no longer fundamental at the distances

$$r \sim m^{-1} e^{-8\pi^2/3\lambda}$$  \hspace{1cm} (1.3)

which are very small if (renormalized) $\lambda$ is small and play a role of a cutoff. Therefore, the estimate (1.2) makes sense for small enough $\lambda$ when $n \ll \exp(8\pi^2/3\lambda)$ can still be large.

1.2 The tree level

The explicit results for the tree amplitudes at threshold in the $\phi^4$ theory

$$a(n) = -(n^2 - 1) n! m^2 \left( \frac{\lambda}{cm^2} \right)^{(n-1)/2},$$  \hspace{1cm} (1.4)

where $c = 8$ [8] for an unbroken reflection symmetry $\phi \to -\phi$ and $c = 2$ [9] for a broken one, demonstrate the expected factorial growth. These results are obtained solving exact tree-level recurrence relations between the amplitudes $a(n)$ with different $n$, which is an alternative to semiclassical evaluations at large $n$. A bridge between the two approaches relies on a functional technique [10] which turns out to be very useful to go beyond the tree level.
1.3 Unitarity violation

While the threshold amplitudes have a vanishing phase volume, they are a lower bound for tree amplitudes with nonvanishing spatial momenta of produced particles [6]. This bound looks like the r.h.s. of Eq. (1.4) with $m$ substituted by the variable $\omega = E - (n - 1)m$ where $E$ is the total energy of produced particles in the rest frame ($\omega = m$ if all particles are produced at rest) and $c = 9$ or $c = 8/3$ [11, 12] for the unbroken or broken symmetry, respectively.

This behavior of the tree amplitudes leads to a violation of unitarity [6, 11, 12, 13]. Moreover, if one reconstructs the perturbative expansion by the unitarity relation, it means under some natural assumptions that the perturbation theory breaks down starting from some finite order. Therefore, loops should be taken into account.

1.4 The nullification

The one-loop calculation of the $1\to n$ amplitude by Voloshin [14] discovered a very interesting property of the tree $2\to n$ threshold amplitude which vanishes for $n>4$ when incoming particles are on mass shell. This nullification occurs for dynamical reasons because of the cancellation between diagrams. It has been extended [15–25] to more general models and is crucial for calculations of the amplitude $1\to n$ at the one-loop level. A dynamical symmetry which may be responsible for the nullification has been discussed in Ref. [26]. An interesting question is which properties of the tree and one-loop amplitudes could survive in the full theory which includes all loop diagrams.

1.5 $O(\infty)$ symmetry

The simplest exactly solvable to all loops model is the $O(N)$-symmetric $\phi^4$ theory in the large-$N$ limit when only bubble diagrams contribute if there is no multiparticle production. The amplitudes of multiparticle production at threshold are calculated for this model in the large-$N$ limit at fixed number of produced particles $n$ in Ref. [27]. The results, which are reviewed in Sect. 2 below, are quite similar to the tree amplitudes while the effect of loops is the renormalization of the coupling constant and mass. The form of the solution is due to the fact that the exact amplitude of the process $2\to n$ vanishes on mass shell for $n>2$ at large $N$ when averaged over the $O(N)$ indices of incoming particles. An extension [28] to the cases when the $O(N)$ symmetry is softly broken by masses including spontaneous breaking of a remaining reflection symmetry is considered in Sect. 3.
1.6 The rescattering problem

Since the large-\(N\) multiparticle amplitudes are similar to the tree-level ones, they are not very instructive from the viewpoint of unitarity restoration by loops (while agree with unitarity by themselves). More essential for this are loops associated with rescatterings of the produced particles which are controlled at large \(n\) by the parameter \(n^2\lambda\) \cite{14,15}. As \(\lambda \sim 1/N\) they give corrections to the large-\(N\) amplitudes which are controlled by \(n^2/N\).

The rescattering contribution is expected to be calculable by semiclassical methods \cite{18,29}. Nice explicit results showing how unitarity is restored by this mechanism are obtained \cite{30} in 2+1 dimensions by an alternative method of summing infrared logarithms with the aid of renormalization group. The conjecture about exponentiation of the \(n^2\lambda\) (or \(n^2/N\)) corrections, which is confirmed by various considerations \cite{31}, can help to solve the problem.

1.7 Yet other applications?

The techniques developed in studying multiparticle production might be useful for other problems (some of them are mentioned in Sect. 4). An idea is that for the given kinematics, which reduces the system at the tree and one-loop levels or at large \(N\) to an integrable lower-dimensional one, calculations could be simpler than, say, for all momenta to be of the same order.

Equations of this type appear in many problems with an emission of energy. In particular, the large-\(N\) Schwinger–Dyson equations considered in Subsect. 2.2 coincide with those \cite{32} describing a non-equilibrium time evolution of quantum systems.\footnote{I thank E. Mottola for this comment.} The difference is that the latter problem is in real rather than in imaginary time and, what is more essential, the boundary conditions are quite different.

2 Unbroken \(O(\infty)\) symmetry \cite{27}

2.1 The definitions

The \(O(N)\)-symmetric \(\phi^4\) theory is defined in the 4-dimensional Minkowskian space by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^b)(\partial^\mu \phi^b) - \frac{1}{2} m^2 (\phi^b \phi^b) - \frac{1}{4} \lambda (\phi^b \phi^b)^2,
\]

where \(b = 1, \ldots, N\) and the summation over repeated indices is implied.

The amplitudes \(a_{b_1 \ldots b_n}^b(n)\) of production of \(n\) on-mass-shell particles with the \(O(N)\) indices \(b_1 \ldots b_n\) at rest by a (virtual) particle with the \(O(N)\) index \(b\) and the energy \(nm\)
are given by the generating function \[9, 10\]

\[
\Phi^b(\tau) = \sum_{n \geq 1} \frac{1}{n!} \langle b_1 \ldots b_n | \phi^b(0, -i\tau) | 0 \rangle \xi^b_1 \ldots \xi^b_n
\]

\[= m \xi^b e^{m\tau} + \sum_{n \geq 3} \frac{a^b(n)}{n! (n^2 - 1)} e^{nm\tau} m^{n-2}, \quad (2.2)\]

where \(\tau = ix_0\) is the imaginary-time variable and

\[
a^b(n) = a^b_{b_1 \ldots b_n}(n) \xi^{b_1} \ldots \xi^{b_n}. \quad (2.3)
\]

The multiplication by the \(O(N)\) vectors \(\xi^b_i\) results in a symmetrization over the \(O(N)\) indices of the produced particles. The tree-level estimate gives

\[
a^b(n) \sim (\lambda \xi^2)^{n-1} \xi^b. \quad (2.4)
\]

Since \(\lambda \sim 1/N\) in the large-\(N\) limit, we choose \(\xi^2 \sim N\) for all the amplitudes \((2.4)\) to be of the same order in \(1/N\).

It is convenient to introduce the coherent state \([28]\)

\[
\langle \Xi | = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^b_1 \ldots \xi^b_n \langle b_1 \ldots b_n |. \quad (2.5)
\]

Then Eq. \((2.2)\) can be written as

\[
\Phi^b(\tau) = \langle \Xi | \phi^b(0, -i\tau) | 0 \rangle. \quad (2.6)
\]

The second quantity which appears in the set of the Schwinger–Dyson equations of the next subsection is the amplitude \(D^b_{a_1 \ldots a_n}(n;p)\) of the process when two particles \(a\) and \(b\) with the 4-momenta \(p + nq\) and \(-p\), respectively, produce \(n\) on-mass-shell particles with the \(O(N)\) indices \(b_1 \ldots b_n\) and the equal 4-momenta \(q = (m, 0)\) in the rest frame. We define again

\[
D^{ab}(n;p) = D^b_{a_1 \ldots a_n}(n;p) \xi^{b_1} \ldots \xi^{b_n} \quad (2.7)
\]

and introduce the generating function

\[
G_{\alpha}^{ab}(\tau, \tau') = \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p}\vec{x}} \langle \Xi | \phi^a(\vec{x}, -i\tau) \phi^b(0, -i\tau') | 0 \rangle - \delta^{(3)}(\vec{p}) \Phi^a(\tau) \Phi^b(\tau')
\]

\[= \delta^{ab} e^{-\omega|\tau-\tau'|} + \int \frac{d\epsilon}{2\pi} \sum_{n=0}^{\infty} e^{(\epsilon+nm)\tau-\epsilon\tau'} D^{ab}(n;p). \quad (2.8)
\]

Here

\[
\omega = \sqrt{\vec{p}^2 + m^2}, \quad (2.9)
\]

\(p = (\epsilon, \vec{p})\) and the subtraction cancels disconnected parts which are not included in the definition of \(D^{ab}(n;p)\). The meaning of \(G_{\alpha}^{ab}(\tau, \tau')\) is that of the Green function in the imaginary-time representation while \(D^{ab}(n;p)\) is the one in the energy representation.
2.2 Schwinger–Dyson equations

The Schwinger–Dyson equations can be extracted from the following identity

\[
\langle \Xi \mid \frac{\delta S[\phi]}{\delta \phi^a(x)} F[\phi] \mid 0 \rangle = i \langle \Xi \mid \frac{\delta F[\phi]}{\delta \phi^a(x)} \mid 0 \rangle \quad (2.10)
\]

where \( S \) is the action of the model and \( F[\phi] \) is an arbitrary functional of \( \phi \). Eq. (2.10) results from the invariance of the measure in the path integral under an arbitrary variation of \( \phi^a(x) \). Substituting \( F = 1 \), one gets from Eq. (2.10)

\[
\left\{ \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \vec{x}^2} - m^2 \right\} \langle \Xi \mid \phi^a(\vec{x}, -i\tau) | 0 \rangle - \lambda \langle \Xi \mid \phi^2(\vec{x}, -i\tau) \phi^a(\vec{x}, -i\tau) | 0 \rangle = 0. \quad (2.11)
\]

In the large-\( N \) limit one can split the matrix elements of the operator \( \phi^2 \phi^a \) as

\[
\langle b_1 \ldots b_n | \phi^2 \phi^a | 0 \rangle = \sum_{p,p'} \sum_{n_1+n_2=n} \frac{n!}{n_1!n_2!} \langle p(b_1) \ldots p(b_{n_1}) | \phi^2 | 0 \rangle \langle p(b_{n_1+1}) \ldots p(b_n) | \phi^a | 0 \rangle \quad (2.12)
\]

where \( p \) and \( p' \) stand for permutations. This formula holds for \( n \) finite as \( N \to \infty \) and extends the standard factorization of \( O(N) \)-singlet operators at large \( N \) to the case of multiparticle production. According to the definition (2.5), we rewrite Eq. (2.12) in the form

\[
\langle \Xi \mid \phi^2(\vec{x}, -i\tau) \phi^a(\vec{x}, -i\tau) | 0 \rangle = \langle \Xi \mid \phi^2(\vec{x}, -i\tau) | 0 \rangle \langle \Xi \mid \phi^a(\vec{x}, -i\tau) | 0 \rangle. \quad (2.13)
\]

Using Eqs. (2.6), (2.13) and translational invariance, Eq. (2.11) can be written as

\[
\left\{ \frac{d^2}{d\tau^2} - m^2 - v(\tau) \right\} \Phi^a(\tau) = 0, \quad (2.14)
\]

where we have denoted

\[
v(\tau) \equiv \lambda \langle \Xi | \phi^2(0, -i\tau) | 0 \rangle = \lambda \Phi^2(\tau) + \frac{\lambda}{2\pi^2} \int_{m^2} d\omega \omega \sqrt{\omega^2 - m^2} G^{aa}_\omega(\tau, \tau) \quad (2.15)
\]

and the last equality holds due to the definition (2.8).

The meaning of the amplitude \( G^{aa} \) which is summed over the \( O(N) \) indices of incoming particles becomes clear after the obvious decomposition

\[
G^{ab}_\omega(\tau, \tau') = \left( \delta^{ab} - \frac{\xi a \xi b}{\xi^2} \right) G^{T}_\omega(\tau, \tau') + \frac{\xi a \xi b}{\xi^2} G^{S}_\omega(\tau, \tau'). \quad (2.16)
\]

At large \( N \) one gets

\[
\frac{1}{N} G^{aa}_\omega(\tau, \tau') = \left( 1 - \frac{1}{N} \right) G^{T}_\omega(\tau, \tau') + \frac{1}{N} G^{S}_\omega(\tau, \tau') \to G^{T}_\omega(\tau, \tau'). \quad (2.17)
\]

Only \( G^T \) enters the Schwinger–Dyson equations at large \( N \).
In order to close the set of equations, one derives an equation for the propagator
\[ G^T_\omega(\tau, \tau') \] which corresponds to \[ F = \phi^a(0, -i\tau') \] in Eq. (2.10). Using the large-\( N \) factorization and integrating over \( d^3\vec{x} \), one gets
\[
\left\{ \frac{d^2}{d\tau'^2} - \omega^2 - v(\tau) \right\} G^T_\omega(\tau, \tau') = -\delta(\tau - \tau').
\] (2.18)

The set of Eqs. (2.14), (2.15) and (2.18) is closed and unambiguously determined the solution presented in the next subsection.

### 2.3 The exact solution

Eqs. (2.14), (2.15) and (2.18) look like a quantum mechanical problem which can be solved \[ [27] \] applying the Gelfand–Dikiĭ technique \[ [33] \] which is described in Subsect. 4.1.

The solution, which satisfies proper boundary conditions as \( \tau \to -\infty \), reads
\[
G^T_\omega(\tau, \tau) = \frac{1}{2\omega} - \frac{\bar{\lambda} \Phi^2(\tau)}{4\omega(\omega^2 - m^2_R)} ,
\] (2.19)
\[
v_R(\tau) = \bar{\lambda} \Phi^2(\tau),
\] (2.20)
where
\[
\bar{\lambda} = \frac{\lambda_R}{1 + \frac{N}{8\pi^2}},
\] (2.21)
and
\[
\Phi^a(\tau) = \frac{\xi^a m_R e^{m_R \tau}}{1 - \frac{\lambda^2}{8} e^{2m_R \tau}}.
\] (2.22)

The solution is expressed via the renormalized mass \( m_R \) and coupling \( \lambda_R \) which are related to the bare ones \( m \) and \( \lambda \) by
\[
m^2 = m^2_R - \frac{\lambda N}{4\pi^2} \int_{m_R^{2}}^\infty d\omega \sqrt{\omega^2 - m^2_R},
\] (2.23)
and
\[
\frac{1}{\bar{\lambda}} = \frac{1}{\lambda_R} - \frac{N}{8\pi^2} \int_{m_R^{2}}^\infty d\omega \frac{\sqrt{\omega^2 - m^2_R}}{\omega^2},
\] (2.24)
which exactly coincide with the standard renormalizations of mass and the coupling constant at large \( N \). The renormalization of \( v(\tau) \) is given by
\[
v_R(\tau) = v(\tau) + m^2 - m^2_R.
\] (2.25)

Eqs. (2.19), (2.22) coincide with the large-\( N \) limit of the tree-level results for \( G^T_\omega(\tau, \tau) \) \[ [17] \] and \( \Phi^a(\tau) \) \[ [10] \] in \( O(N) \)-symmetric \( \phi^4 \) theory, while the only effect of loops, besides the renormalization, is the change of \( \lambda_R \) given by Eq. (2.21).
Figure 1: The tree diagrams for $D^T(4; p)$ at large $N$. The continuous lines are associated with propagation of $O(N)$ indices. The contributions of the diagrams a), b) and c) equal, respectively, $-1/4$, $1/8$ and $1/8$ for the given on-mass-shell kinematics when the momenta of the incoming particles (in the units of mass) are $(2, \sqrt{3}, 0, 0)$ and $(2, -\sqrt{3}, 0, 0)$, while that of each produced particle is $(1, 0, 0, 0)$. The sum of three diagrams vanishes which illustrates the nullification of $2\to 4$ on mass shell at the tree level.

The fact that (2.19) has a pole only at $\omega^2 = m_R^2$ means the nullification of the on-mass-shell amplitudes $D^T(n; p)$ for $n>2$. An analogous property holds [20] at $N = 1$ for the tree level amplitudes in the case of the sine-Gordon potential and for the $\phi^4$-theory with spontaneously broken symmetry [13]. It can be explicitly demonstrated by the cancellation of the tree diagrams for the process $2\to 4$ which are depicted in Fig. 1.

The large-$N$ amplitude (2.22) is real which is due to the nullification. This is different from the $N = 1$ one-loop result [14] while is similar to the case of spontaneously broken reflection symmetry where the $1\to n$ amplitude is real to all orders [18]. The fact that the amplitude $1\to 5$ is real at the one-loop level is illustrated by Fig. 2. While the sum of the imaginary parts vanishes, the real part is described by Eqs. (2.21), (2.22).

Analogously, the imaginary parts of the one-loop diagrams for the $2\to 4$ on-mass-shell amplitude $D^T(4; p)$ are mutually cancelled for the same reason while the real part is given by Eqs. (2.19), (2.21) and (2.22). This amplitude is also real to all loops. Therefore, both amplitudes are real due to the nullification which, in turn, is due to a hidden dynamical symmetry.

3 Softly broken $O(\infty)$ symmetry [28]
Figure 2: The one-loop diagrams for $a^b(5; p)$ at large $N$. The dashed lines represent unitary cuts over two-particle intermediate states. The imaginary part vanishes after summing over all three diagrams since the amplitude $D^T(4; p)$ vanish on-mass-shell at the tree level as is illustrated by Fig. [ ]

3.1 The $O(N_1) \times O(N_2)$ model

The model consists of an $O(N_1)$ multiplet $\chi^b$, $b = 1, ..., N_1$, coupled to a $O(N_2)$ scalar field $\phi^\beta$, $\beta = 1, ..., N_2$. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \chi^b)(\partial^\mu \chi^b) + \frac{1}{2}(\partial_\mu \phi^\beta)(\partial^\mu \phi^\beta) - \frac{1}{2}m_1^2(\chi^b \chi^b) - \frac{1}{2}m_2^2(\phi^\beta \phi^\beta) - \frac{1}{4}\lambda(\chi^b \chi^b + \phi^\beta \phi^\beta)^2, \quad (3.1)$$

where interaction term is $O(N_1+N_2)$ invariant while the mass term possesses the $O(N_1) \times O(N_2)$ symmetry for nonequal masses $m_1$ and $m_2$. All the formulas of this section are proper extensions of those of Sect. 2 to $m_1 \neq m_2$.

We denote by $A_{\alpha_1...\alpha_n,\beta_1...\beta_k}(n, k)$ the amplitude of production of $n$ on-mass-shell particles $\chi$ with the $O(N_1)$-indices $\alpha_1, ..., \alpha_n$ and of $k$ on-mass-shell particles $\phi$ with the $O(N_2)$-indices $\beta_1, ..., \beta_k$ at rest by a (virtual) particle $\chi$ with the $O(N_1)$-index $\alpha_1$ and the energy $nm_1 + km_2$. We also denote by $B_{\beta_1...\beta_n,\alpha_1...\alpha_k}(n, k)$ the amplitude of production of $n$ on-mass-shell particles $\chi$ with the $O(N_1)$-indices $b_1, ..., b_n$ and of $k$ on-mass-shell particles $\phi$ with $O(N_2)$-indices $\beta_1, ..., \beta_k$ at rest by a (virtual) particle $\phi$ with the $O(N_2)$-index $\beta_1$ and the energy $nm_1 + km_2$. Analogues of Eq. (2.3) are

$$A_{\alpha_1...\alpha_n,\beta_1...\beta_k}(n, k) = A_{\alpha_1...\alpha_n,\beta_1...\beta_k}(n, k) \xi_{\alpha_1}^{\beta_1} \cdots \xi_{\alpha_n}^{\beta_n} \xi_{\beta_1}^{\alpha_1} \cdots \xi_{\beta_k}^{\alpha_k}, \quad (3.2)$$

$$B_{\beta_1...\beta_n,\alpha_1...\alpha_k}(n, k) = B_{\beta_1...\beta_n,\alpha_1...\alpha_k}(n, k) \xi_{\beta_1}^{\alpha_1} \cdots \xi_{\beta_n}^{\alpha_n} \xi_{\alpha_1}^{\beta_1} \cdots \xi_{\alpha_k}^{\beta_k}. \quad (3.3)$$

An analog of the coherent state (2.5) is

$$\langle \Xi | = \sum_{n,k=0}^{\infty} \frac{1}{n!k!} \xi_1^{b_1} \cdots \xi_n^{b_n} \xi_1^{\beta_1} \cdots \xi_k^{\beta_k} \langle b_1 \cdots b_n, \beta_1 \cdots \beta_k |, \quad (3.4)$$

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while the generating functions for the amplitudes (3.2), (3.3) are
\[
\Phi^a(\tau) = \langle \Xi|\chi^a(\vec{0}, -i\tau)|0\rangle = m_1\xi_1^a e^{m_1\tau} + \sum_{n,k \geq 3} \frac{A^a(n, k) e^{(nm_1 + km_2)\tau}}{n!k!((nm_1 + km_2)^2 - m_1^2)m_1^n m_2^k}, \quad (3.5)
\]
\[
\Psi^a(\tau) = \langle \Xi|\phi^a(\vec{0}, -i\tau)|0\rangle = m_2\xi_2^a e^{m_2\tau} + \sum_{n,k \geq 3} \frac{B^a(n, k) e^{(nm_1 + km_2)\tau}}{n!k!((nm_1 + km_2)^2 - m_2^2)m_2^n m_2^k}. \quad (3.6)
\]

The definition of the generating functions \(G_{\chi}^{ab}(\omega; \tau, \tau')\) and \(G_{\phi}^{ab}(\omega; \tau, \tau')\) which extend (2.7) and (2.8) is obvious.

The Schwinger–Dyson equations which extend Eqs. (2.14), (2.15) and (2.18) read
\[
\left\{ \frac{d^2}{d\tau^2} - m_1^2 - v(\tau) \right\} \Phi^a(\tau) = 0, \quad (3.7)
\]
\[
\left\{ \frac{d^2}{d\tau^2} - m_2^2 - v(\tau) \right\} \Psi^a(\tau) = 0, \quad (3.8)
\]
\[
\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) \right\} G_{\chi}^T(\omega; \tau, \tau') = -\delta(\tau - \tau'), \quad (3.9)
\]
\[
\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) \right\} G_{\phi}^T(\omega; \tau, \tau') = -\delta(\tau - \tau'), \quad (3.10)
\]

where
\[
v(\tau) = \lambda \langle \Xi|\chi^2(\vec{0}, -i\tau) + \phi^2(\vec{0}, -i\tau)|0\rangle = \lambda (\Phi^2(\tau) + \Psi^2(\tau))
\]
\[
+ \frac{\lambda N_1}{2\pi^2} \int_{m_1} d\omega \sqrt{\omega^2 - m_1^2} G_{\chi}^T(\omega; \tau, \tau) + \frac{\lambda N_2}{2\pi^2} \int_{m_2} d\omega \sqrt{\omega^2 - m_2^2} G_{\phi}^T(\omega; \tau, \tau). \quad (3.11)
\]

The derivation uses factorization at large \(N\).

### 3.2 Unbroken reflection symmetry

The solution to Eqs. (3.7)–(3.11) is very similar to that of Subsect. 2.3, while there are important differences. It reads
\[
G_{\chi}^T(\omega; \tau, \tau) = \frac{1}{2\omega} - \frac{\bar{\lambda}_1 \Phi^2}{4\omega(\omega^2 - m_{1R}^2)} - \frac{\bar{\lambda}_2 \Psi^2}{4\omega(\omega^2 - m_{2R}^2)}, \quad (3.12)
\]
where
\[
v_R = \bar{\lambda}_1 \Phi^2 + \bar{\lambda}_2 \Psi^2, \quad (3.13)
\]
\[
\bar{\lambda}_1 = \frac{\lambda_R}{1 + \frac{\lambda_R}{8\pi^2} I_1}, \quad \bar{\lambda}_2 = \frac{\lambda_R}{1 + \frac{\lambda_R}{8\pi^2} I_2}, \quad (3.14)
\]
and
\[
I_1 = N_1 + N_2 \int_{m_{2R}} d\omega \sqrt{\omega^2 - m_{2R}^2} \left[ \frac{1}{\omega^2 - m_{1R}^2} - \frac{1}{\omega^2} \right], \quad (3.15)
\]
Figure 3: The one-loop diagram for the process $\chi + \chi \rightarrow \phi + \phi$ which possesses an imaginary part for $m_2R > m_1R$ given by Eq. (3.21).

$$I_2 = N_2 + N_1 \int_{m_{1R}} d\omega \sqrt{\omega^2 - m_1^2 R} \left[ \frac{1}{\omega^2 - m_2^2 R} - \frac{1}{\omega^2} \right],$$

while $\Phi^a$ and $\Psi^\alpha$ are similar to those of Ref. [24]:

$$\Phi^a = z_1^a \left( 1 - 2\bar{\lambda}_1 \frac{\kappa}{m_{2R}^2} z_2^2 \right) \left( 1 - \frac{2\bar{\lambda}_1}{m_{1R}^2} z_1^2 - \frac{2\bar{\lambda}_2}{m_{2R}^2} z_2^2 + \bar{\lambda}_1 \bar{\lambda}_2 \frac{\kappa^2}{m_{1R}^2 m_{2R}^2} z_1^2 z_2^2 \right)^{-1},$$

$$\Psi^\beta = z_2^\beta \left( 1 + 2\bar{\lambda}_1 \frac{\kappa}{m_{1R}^2} z_1^2 \right) \left( 1 - \frac{2\bar{\lambda}_1}{m_{1R}^2} z_1^2 - \frac{2\bar{\lambda}_2}{m_{2R}^2} z_2^2 + \bar{\lambda}_1 \bar{\lambda}_2 \frac{\kappa^2}{m_{1R}^2 m_{2R}^2} z_1^2 z_2^2 \right)^{-1},$$

with

$$\kappa = \frac{m_{1R} - m_{2R}}{m_{1R} + m_{2R}}$$

and

$$z_1^a = \xi_1^a e^{m_{1R} \tau}, \quad z_2^a = \xi_2^a e^{m_{2R} \tau}.$$  

The constants $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are complex in general. Assuming for definiteness that $m_{2R} \geq m_{1R}$, one can easily see that the constant $\bar{\lambda}_1$ is real while the imaginary part of $\bar{\lambda}_2$ is given by

$$\text{Im} \frac{1}{\lambda_2} = -\frac{N_1}{8\pi} \sqrt{\frac{m_{2R}^2 - m_{1R}^2}{2m_{2R}}}. $$

The appearance of the imaginary part is because the masses of the $O(N_1)$ and $O(N_2)$ particles are not equal. For $m_{2R}^2 \geq m_{1R}^2$ the imaginary part is associated with an inelastic process $\chi + \chi \rightarrow \phi + \phi$ which is given to lowest order by the diagram depicted Fig. 3.

### 3.3 Broken reflection symmetry

For $N_2 = 1$ when $\phi$ is a singlet field, the symmetry of the Lagrangian (3.1) under the reflection $\phi \rightarrow -\phi$ is spontaneously broken if $m_2^2 < 0$. The remaining $O(N_1) = O(N-1)$ symmetry is unbroken for $m_1^2 \geq -|m_2^2|$ (for $m_1^2 = m_2^2 < 0$ the whole $O(N)$ symmetry were spontaneously broken down to $O(N-1)$ and the associated massless Goldstone bosons would appear).
The vacuum expectation value of the field $\phi$ and the physical mass of the field $\chi$ are
\[
\eta_R = \frac{|m_{2R}|}{\sqrt{\lambda_R}}, \quad m_\chi^2 = m_{1R}^2 - m_{2R}^2 = m_1^2 - m_2^2, \tag{3.22}
\]
while the physical mass of the shifted field $\phi' = \phi - \eta_R$ is determined self-consistently by
\[
m_{\phi}^2 = 2\bar{\lambda}_2\eta_R^2 \tag{3.23}
\]
with $\bar{\lambda}_1, \bar{\lambda}_2$ given by Eq. (3.14) and
\[
I_1 = N \int_{m_x} d\omega \sqrt{\omega^2 - m_\chi^2} \left[ \frac{\omega^2}{(\omega^2 - m_\phi^2/4)(\omega^2 - m_\chi^2)} - \frac{1}{\omega^2} \right], \tag{3.24}
\]
\[
I_2 = N \int_{m_x} d\omega \sqrt{\omega^2 - m_\chi^2} \left[ \frac{1}{\omega^2 - m_\phi^2/4} - \frac{1}{\omega^2} \right]. \tag{3.25}
\]
The parameters $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are real in contrast to the case without the spontaneous breaking of the reflection symmetry.

The solution reads
\[
G_T^\chi(\omega; \tau, \tau) = \frac{\omega}{2(\omega^2 - m_\phi^2/4)} - \frac{\bar{\lambda}_1\omega\Phi^2}{4(\omega^2 - m_\phi^2/4)(\omega^2 - m_\chi^2)} - \frac{\bar{\lambda}_2\Psi^2}{4\omega(\omega^2 - m_\phi^2/4)}, \tag{3.26}
\]
\[
v_R = \bar{\lambda}_1\Phi^2 + \bar{\lambda}_2\Psi^2 - \frac{m_\phi^2}{2}, \tag{3.27}
\]
where
\[
\Phi^a = z_1^a - \frac{1 - (2m_\chi - m_\phi)}{(2m_\chi + m_\phi)\eta_R} \frac{z_1^2}{2\eta_R}, \quad z_1^a = m_\chi \xi_1^a e^{m_\chi \tau},
\]
\[
\Psi = \eta_R + \eta_R \frac{z_1^a}{2\eta_R} - \frac{8\lambda_1}{4m_\chi^2 - m_\phi^2} z_1^2 + \frac{16\lambda_1}{\eta_R (2m_\chi + m_\phi)^2} z_2^2, \tag{3.28}
\]
\[
z_2^a = m_\phi \xi_2 e^{m_\phi \tau}, \quad z_2 = m_\phi \xi_2 e^{m_\phi \tau}. \tag{3.29}
\]
This solution is similar to the tree level one \[24\] and is real.

The poles of (3.26) at $\omega = m_\chi$ and $\omega = m_\phi/2$ are associated with production of two $\chi$-particles or a single $\phi$ particle, respectively. There is no pole at $\omega = m_\phi$ which signals that the amplitude for the production of two $\phi$ particles vanishes in contrast to the cases without the spontaneous breaking and $N = 1$ with broken reflection symmetry \[13\]. It is easy to see shifting $\Psi = \eta_R + \Psi'$ that (3.26) and (3.27) recover the free values $1/2\omega$ and $0$, respectively, to zeroth order of perturbation theory in $\lambda_R$ around the shifted vacuum.
4 Generalizations

4.1 Gelfand–Dikiĭ technique for Pöschl–Teller potential

The results presented in Sects. 2, 3 are obtained by the Gelfand–Dikiĭ technique \[33\] which represents the diagonal resolvent of the Schrödinger operator in Eq. (2.18) as

\[
C^T_\omega(\tau, \tau) = R_\omega[v] \equiv \sum_{l=0}^{\infty} \frac{R_l[v]}{\omega^{2l+1}} \tag{4.1}
\]

where the differential polynomials \(R_l[v]\) are determined recurrently by

\[
R_{l+1}[v] = \frac{1}{2} \left( \frac{1}{2} D^2 - v - D^{-1}vD \right) R_l[v], \quad R_0 = \frac{1}{2} \tag{4.2}
\]

and the inverse operator is

\[
D^{-1}f(\tau) = \int_{-\infty}^{\tau} dx \ f(x). \tag{4.3}
\]

Eq. (4.2) stems from the fact that \(R_\omega[v]\) obeys the third order linear differential equation

\[
\frac{1}{2} \left( \frac{1}{2} D^3 - Dv - vD \right) R_\omega[v] = \omega^2 DR_\omega[v]. \tag{4.4}
\]

The potential (2.20), (2.22) is a particular case of a more general Pöschl–Teller potential

\[
v(\tau) = m^2 \left\{ \frac{s(s+1)}{\sinh^2(m(\tau-\tau_0))} - \frac{k(k+1)}{\cosh^2(m(\tau-\tau_0))} \right\} \tag{4.5}
\]

with \(s = 1\) and \(k = 0\). The simple answer (2.19) for the diagonal resolvent is because

\[
R_l[v] = m^{2l-2}R_1[v] = -\frac{1}{4} m^{2l-2}v \quad \text{for} \quad s = 1, k = 0, \tag{4.6}
\]

i.e. \(R_l[v]\) with \(l > 1\) reduces to \(R_1[v]\).

It is easy to extend Eqs. (2.19), (2.20) to case \(s > 1, k = 0\) (or \(k > 1, s = 0\)) when

\[
R_\omega[v] = \sum_{l=0}^{s} \frac{C_l v^l}{\omega(\omega^2 - m^2) \cdots (\omega^2 - l^2m^2)} \tag{4.7}
\]

and the coefficients \(C_l\) are determined recurrently by

\[
C_{l+1} = \frac{l(l+1) - s(s+1)}{(l+1)s(s+1)} (l + \frac{1}{2}) C_l. \tag{4.8}
\]

These formulas can easily be proven substituting into Eq. (1.4) and noticing that

\[
\frac{1}{2} \left( \frac{1}{2} D^2 - v - D^{-1}vD \right) v^l = l^2 m^2 v^l + \frac{l(l+1) - s(s+1)}{(l+1)s(s+1)} (l + \frac{1}{2}) v^{l+1}. \tag{4.9}
\]

It is evident from Eq. (1.9) that \(v^s\), which is the highest power of \(v\) in (4.7), is an eigenvector of the operator entering Eq. (1.2) and, therefore, \(R_l[v]\) for \(l > s\) reduces to \(R_1[v], \ldots, R_s[v]\).
The simple form (4.7) of the diagonal resolvent for the potential (4.5) with \( k = 0 \) is due to the fact that the spectrum is very simple: there are \( s \) poles at \( \omega = 1, \ldots, s \) (in the units of \( m \)). For nonvanishing both \( s \) and \( k \), the spectrum is slightly more complicated: there are \( M = \max\{s, k\} \) poles which go from 1 up to \( |s - k| \) with the step 1 and then from \( |s - k| + 2 \) up to \( s + k \) with the step 2. Say, the spectrum for \( s = 2 \) and \( k = 5 \) is \( 1, 2, 3, 5, 7 \). This spectrum can be obtained from the one for \( k = 0 \) (or \( s = 0 \)) by a supersymmetry transformation \¶ which relates \((s, k)\) with \((s + 1, k + 1)\). It was explicitly done for \( k = 1 \) and any \( s \) by Smith [25].

For the potential (4.5) \( R_l[v] \) with \( l > M = \max\{s, k\} \) reduces to \( R_l[v], \ldots, R_M[v] \). This is based on the fact that

\[
\frac{1}{2} \left( \frac{1}{2} D^2 - v - D^{-1} v D \right) \sinh^{-2s} \tau \cosh^{-2k} \tau = (s + k)^2 \sinh^{-2s} \tau \cosh^{-2k} \tau \quad (4.10)
\]

for the potential (4.5) (I put \( m = 1 \) and \( \tau_0 = 0 \) for brevity), \( i.e. \) there exists an eigenvector of the operator entering the recurrence relation (4.2). As is shown in Ref. [27], if \( \Phi(\tau) \) is an eigenvector of the Schrödinger operator on the l.h.s. of Eq. (2.14) with an eigenvalue \( E \) and \( \Phi(-\infty) = 0 \), then \( \Phi^2(\tau) \) will be an eigenvector of the operator on the l.h.s. of Eq. (4.10) with the same eigenvalue. Therefore, spectra of the two operators coincide.

Since the spectrum is known, a modification of (4.7) for \( k \neq 0 \) is obvious while awkward. An analog of Eq. (4.6) can be obtained as follows. For the given spectrum \( m_1, \ldots, m_M \) let us calculate

\[
D^{(s,k)}(\omega) \equiv \prod_{i=1}^{M} (\omega^2 - m_i^2) = \omega^{2M} - \sum_{i=1}^{M} D_i^{(s,k)} \omega^{2(M-i)}
\]

(4.11)

to determine the coefficients \( D_i^{(s,k)} \). Then, the reduction of \( R_{M+1}[v] \) reads

\[
R_{M+1} = \sum_{i=1}^{M} D_i^{(s,k)} R_{M-i+1}.
\]

(4.12)

The reduction of \( R_l[v] \) for \( l > M+1 \) can be obtained from Eq. (4.12) recurrently using Eq. (4.2).

Let us illustrate Eq. (4.12) which relies on Eq. (4.10) by an example of \( (s=2, k=5) \) with the spectrum mentioned above. One gets

\[
D^{(2,5)}(\omega) = (\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)(\omega^2 - 25)(\omega^2 - 49) =
\]

\[
\omega^{10} - 88\omega^8 + 2310\omega^6 - 20812\omega^4 + 62689\omega^2 - 44100
\]

(4.13)

and

\[
R_6 = 88 R_5 - 2310 R_4 + 20812 R_3 - 62689 R_2 + 44100 R_1,
\]

(4.14)

\footnote{The potential (4.5) with \( k = 1 \) and arbitrary \( s \) (which is related to the ratio of fermion to vector-boson masses) appears in the standard model for longitudinally polarized vector bosons [23, 24].}

\footnote{I am grateful to E. Argyres and A. Johansen for discussions of this point.}
The diagonal resolvent is

\[
R_\omega[v] = \frac{1}{2\omega} + \frac{R_1}{\omega(\omega^2 - 1)} + \frac{R_2 - R_1}{\omega(\omega^2 - 1)(\omega^2 - 4)} + \frac{R_1 - 5R_2 + 4R_3}{\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)}
\]

\[
+ \frac{R_3 - 14R_4 + 49R_2 - 36R_1}{\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)(\omega^2 - 25)} + \frac{R_5 - 39R_4 + 399R_3 - 1261R_2 + 900R_1}{\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)(\omega^2 - 25)(\omega^2 - 49)}.
\]

The coefficients of the numerators in the last term on the r.h.s. are just \(D^{(1,4)}\) associated with the spectrum 1, 2, 3, 5 for \((s=1, k=4)\) as is prescribed by the supersymmetry. Analogously, the numerators of the 5th, 4th and 3rd terms on the r.h.s. correspond to the \((s=0, k=3)\), \((s=0, k=2)\) and \((s=0, k=1)\) spectra, respectively. Substituting (4.15) and using Mathematica, (4.16) can be written explicitly as

\[
R_\omega[v] = \frac{1}{2\omega} + \frac{3\sinh^{-2}\tau(4 + \cosh^{-2}\tau)}{2\omega(\omega^2 - 1)} + \frac{3\sinh^{-4}\tau(21 - 55 \cosh^{-2}\tau + 35 \cosh^{-4}\tau)}{2\omega(\omega^2 - 1)(\omega^2 - 4)}
\]

\[
+ \frac{3^2\sinh^{-4}\tau \cosh^{-2}\tau(10 - 35 \cosh^{-2}\tau + 28 \cosh^{-4}\tau)}{2\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)}
\]

\[
+ \frac{3^3 \sinh^{-4}\tau \cosh^{-6}\tau}{2\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)(\omega^2 - 25)} + \frac{3^6 \sinh^{-4}\tau \cosh^{-10}\tau}{2\omega(\omega^2 - 1)(\omega^2 - 4)(\omega^2 - 9)(\omega^2 - 25)(\omega^2 - 49)}.
\]

The highest vector is in agreement with Eq. (4.10).

### 4.2 Symmetrized amplitude and renormalons

The symmetrized amplitude

\[
D^S(n; p) = D^{ab}(n; p)\frac{\xi^a\xi^b}{\xi^2}
\]

reveals, on the contrary, a surprisingly nontrivial behavior even at large \(N\). Some of the diagrams which contribute to \(D^S(4; p)\) are depicted in Fig. [4]. The combinatorics is now different from that of the diagrams of Fig. [1] so there is no cancellation of tree diagrams.
Figure 5: The diagrammatic representation of some diagrams for $D^S(4; p)$ at large $N$ integrated over $d^4p$. The diagrams b) and c) are of the renormalon type.

for $D^S(4; p)$ on mass shell which happens for $n \geq 6$ according to the explicit result [17] for $D^S(4; p)$ at the tree level. Each of the vertices can be “dressed” with bubble chains and each of the lines of outgoing particles can be substituted by the exact amplitude $a^b(n)$ to obtain a diagram with more produced particles.

One can integrate $D^S(4; p)$ over the 4-momentum $p$ to obtain diagrams of the vertex type. Some of them are depicted in Fig. 5. The most interesting are the diagrams of Fig. 5b), c) which are of the type of renomalons and should behave to $k$-th order of perturbation theory as $k!$.

In order to calculate $D^S(n; p)$ for the unbroken $O(N)$ symmetry at large $N$, we first derive the proper Schwinger–Dyson equation substituting $F = \phi^b(\vec{0}, -i\tau')$ in Eq. (2.10) and symmetrizing according to Eq. (4.17). The resulting equation can be written as

$$\begin{align*}
\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) - 2\lambda \Phi^2(\tau) \right\} G^S(\tau, \tau') & - 2\lambda \Phi^a(\tau) \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} e^{i(p_1 + p_2) \vec{x}} \langle \Xi | \phi^2(\vec{x}, -i\tau) \phi^a(\vec{0}, -i\tau) | 0 \rangle_{\text{conn}} = -\delta(\tau - \tau') .
\end{align*}$$

This equation involves a new amplitude

$$D^a(\tau; p_1, p_2) = \frac{1}{N} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} e^{i(p_1 x + p_2 y - (\epsilon_1 + \epsilon_2)\tau)} \langle \Xi | \phi^b(x) \phi^b(y) \phi^a(\vec{0}, -i\tau) | 0 \rangle_{\text{conn}},$$

with three external lines having nonvanishing spatial momenta (which is a generating function for $D^a(n; p_1, p_2)$), where $a$ is the $O(N)$ index of an outgoing particle with the spatial momentum $\vec{p}_1 + \vec{p}_2$ while the averaging over the $O(N)$ indices of two incoming particles with the spatial momenta $\vec{p}_1$ and $\vec{p}_2$ is performed.
Its solution for $D^a(p_1, p_2)$, the circles with two lines represent $D^T(\tau, p)$ or $D^S(\tau, p)$ and the tadpole represents $D^a(\tau)$. It is depicted graphically in Fig. 6 and reads analytically as

\[
D^a(\tau, p_1, p_2) = \lambda ND^T(\tau, p_1)D^T(\tau, p_2)\times \left[\Phi^a D^S(\tau, p_1 + p_2) + \int \frac{d^4k}{(2\pi)^4} D^a(\tau, k, p_1 + p_2 - k)\right].
\]

(4.20)

Its solution for $D^a(\tau, p_1, p_2)$ is given by

\[
D^a(\tau, p_1, p_2) = \frac{\lambda ND^T(\tau, p_1)D^T(\tau, p_2)\Phi^a(\tau)D^S(\tau, p_1 + p_2)}{1 - \lambda N \int \frac{d^4k}{(2\pi)^4} D^T(\tau, k)D^T(\tau, p_1 + p_2 - k)}.
\]

(4.21)

The propagator $D^T(\tau, p)$ is determined by Eq. (2.18) and reads explicitly

\[
D^T(\tau, p) = \int \frac{d\tau'}{\tau'} e^{i(\tau' - \tau)} G_\omega(\tau, \tau') = \frac{i}{\epsilon^2 - \omega^2} \left[1 + \frac{2m(m + \epsilon)}{(\epsilon - 2m)^2 - \omega^2}\right] \sum_{n=0}^{\infty} \left(\frac{\lambda^2}{8} e^{-2m\tau}\right)^n \left(\frac{2m^2}{(\epsilon - 2m)^2 - \omega^2}\right)
\]

where the integral over $d\tau$ is understood as an analytic continuation from imaginary $\tau$ (cf. (2.8)). The expression (1.22) can be simplified using the formula

\[
\sum_{n=0}^{\infty} \frac{z^n}{n + u} = u^{-1} \, _2F_1(1, u; 1 + u; z).
\]

(4.23)

The calculation of the renormalon contribution would consist of three steps.

1. Calculate the integral $\int \frac{d^4k}{(2\pi)^4} D^T(\tau, k)D^T(\tau, p - k)$.

2. Substitute the (proper Fourier transformed) expression (4.21) into Eq. (1.18) and solve it for the diagonal resolvent $G^S_\omega(\tau, \tau)$.

3. Integrate $\int_{m^2}^\infty d\omega \sqrt{\omega^2 - m^2} G^S_\omega(\tau, \tau)$.

This equation differs at the tree level from the one of Ref. [21] since we use exact propagators which are “dressed” by emitting on-mass-shell particles.
Figure 7: The tree diagrams for a $2 \rightarrow 3$ amplitude in matrix $\phi^3$ theory at large $N$. The continuous lines are associated with propagation of matrix indices. The contributions of the diagrams a), b) and c) equal, respectively, $1/4$, $-1/6$ and $1/24$ for the given on-mass-shell kinematics when the momenta of the incoming particles (in the units of mass) are $(3/2, \sqrt{5}/2, 0, 0)$ and $(3/2, -\sqrt{5}/2, 0, 0)$, while that of the each produced particle is $(1, 0, 0, 0)$. The diagrams b) and c) have an extra combinatorial factor 2 each. The sum of three diagrams vanishes which illustrates the nullification of $2 \rightarrow 3$ on mass shell at the tree level.

4.3 Matrix Higgs field

Another interesting model to investigate is the case of the matrix Higgs field which is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \tr (\partial_{\mu} \phi)^2 - \frac{m^2}{2} \tr \phi^2 - \frac{\lambda_3}{3} \tr \phi^3 - \frac{\lambda_4}{4} \tr \phi^4,$$

where $\phi^{ij}(x)$ is generically $N \times N$ Hermitean matrix. The model greatly simplifies as $N \rightarrow \infty$ at fixed $\lambda_3^2 N$ or $\lambda_4 N$ when only the planar diagrams survive similar to the 't Hooft large-$N$ limit of QCD.

The nullification of on-mass-shell amplitudes holds in the case of the matrix cubic interaction for $2 \rightarrow 3$ at large $N$ at the tree level. The cancellation of diagrams is illustrated by Fig. [4]. This looks quite similar to the $O(N)$-vector case.

The large-$N$ limit of the matrix $\phi^4$ theory while being simpler than the $N = 1$ case to all loops is, however, quite nontrivial like the large-$N$ limit of QCD. Some of the rescattering diagrams are now included in the large-$N$ amplitudes so they should not look like the tree-level ones. The crucial problem is to find a simple set of observables like $\tr \phi^4$ for ordinary matrix models to close the Schwinger–Dyson equations for the given kinematics at large $N$. 

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5 Conclusion and some further problems

- Quantum field theory is not yet completely investigated for large multiplicities of identical particles \( n \sim 1/\lambda \).
- Both conventional perturbation theory and the \( 1/N \) expansion break down at \( n \sim 1/\lambda \) or \( n \sim N \).
- While for \( n \sim 1/\lambda \) or \( n \sim N \) the problem becomes semiclassical, there are subtleties in application of semiclassical technique. Alternative methods could be useful.
- The mechanism of unitarity restoration is not yet completely work out (while there is no problems with unitarity at large \( N \)). An exponentiation of the corrections in \( n^2\lambda \) or \( n^2/N \) looks promising.
- Cross-sections of multiparticle production may become large at very high energies \( E \sim m/\lambda \) while smaller than the unitary bound.
- The nullification survive at large \( N \) to all loops (factorization is crucial for the reduction to a quantum mechanical problem at the tree and one-loop levels or at large \( N \)).
- Which dynamical symmetry is behind the nullification is not yet found out and higher conservation laws which restrict multiparticle production are not yet constructed.
- Drastic simplifications occur for the given kinematics (in contrast to the case when all momenta are off mass shell). The threshold amplitudes are described by an integrable lower dimensional problem.
- One might try to solve more complicated \( D = 4 \) models imposing this kinematics.
- The developed technique might be applicable to some other problems.
- The nullification could have some phenomenological consequences for the standard model restricting the ratio of masses.
- An extension to nonvanishing spatial momenta would be very interesting, in particular, for matching the threshold behavior of multiparticle amplitudes with the standard multiperipheral picture of high energy collisions.

References

[1] A. Ringwald, Nucl. Phys. B330 (1990) 1.
[2] O. Espinosa, Nucl. Phys. B343 (1990) 310.
[3] J.M. Cornwall, Phys. Lett. B243 (1990) 271.
[4] H. Goldberg, Phys. Lett. B246 (1990) 445.
[5] M.B. Voloshin, Phys. Rev. D43 (1991) 1726.
[6] J.M. Cornwall and G. Tiktopoulos, Phys. Rev. D45 (1992) 2105.
[7] H. Goldberg and M.T. Vaughn, Phys. Rev. Lett. 66 (1991) 1267.
[8] M.B. Voloshin, Nucl. Phys. B383 (1992) 233.
[9] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Nucl. Phys. B391 (1993) 42.
[10] L.S. Brown, Phys. Rev. D46 (1992) R4125.
[11] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Nucl. Phys. B391 (1993) 57.
[12] M.B. Voloshin, Phys. Lett. B293 (1992) 389.
[13] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Phys. Lett. B296 (1992) 139.
[14] M.B. Voloshin, Phys. Rev. D47 (1993) R357; 1712.
[15] B.H. Smith, Phys. Rev. D47 (1993) 3518.
[16] M.B. Voloshin, Phys. Rev. D47 (1993) 2573.
[17] B.H. Smith, Phys. Rev. D47 (1993) 3521.
[18] M.B. Voloshin, Phys. Rev. D47 (1993) 3525.
[19] L.S. Brown and C. Zhai, Phys. Rev. D47 (1993) 5526.
[20] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Phys. Lett. B302 (1993) 70.
[21] B.H. Smith, Phys. Rev. D47 (1993) 5531.
[22] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Phys. Lett. B308 (1993) 292.
[23] E.N. Argyres, R. Kleiss and C.G. Papadopoulos, Phys. Lett. B308 (1993) 315.
[24] M.V. Libanov, V.A. Rubakov and S.V. Troitsky, Nucl. Phys. B412 (1994) 607.
[25] B.H. Smith, Phys. Rev. D49 (1994) 1081.
[26] M.V. Libanov, V.A. Rubakov and S.V. Troitsky, Phys. Lett. B318 (1993) 134.
[27] Yu. Makeenko, Exact multiparticle amplitudes at threshold in large-N component $\phi^4$ theory, NBI-HE-94-25 (April 1994), hep-ph/9404312.
[28] M. Axenides, A. Johansen and Yu. Makeenko, Exact multiparticle amplitudes at threshold in $\phi^4$ theories with softly broken $O(\infty)$ symmetry, NBI-HE-94-36 (July 1994), hep-ph/9407323.
[29] A.S. Gorsky and M.B. Voloshin, Phys. Rev. D48 (1993) 3843.
[30] D.T. Son, Talk at this Seminar;
   V.A. Rubakov and D.T. Son, Renormalization group for multiparticle production in (2+1) dimensions around the threshold, INR preprint (July 1994), hep-ph/9406362.
[31] S.V. Troitsky, Talk at this Seminar;
   M.V. Libanov, V.A. Rubakov, D.T. Son and S.V. Troitsky, Exponentiation of multiparticle amplitudes in scalar theories, INR preprint (July 1994), hep-ph/9407381.
[32] E. Mottola, Talk at this Seminar;
   F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. Paz and P. Anderson, Non-equilibrium quantum fields in the large N expansion, LA-UR-94-783 (February 1994), hep-ph/9403352.
[33] I.M. Gelfand and L.A. Diki˘ı, Russ. Math. Surv. 30 (1975) 77.
[34] J.M. Cornwall and D.A. Morris, Unifying logarithmic and factorial behavior in high energy scattering, UCLA94 /TEP/31 (August 1994), hep-ph/9408274.