In this talk I describe recent progress in investigating the high energy limit of perturbative QCD. I review some of the steps that have been done in the direction of constructing an effective field theory for this limit. I describe some of its building blocks and explain why we expect the effective theory to be a 2 + 1–dimensional conformal field theory.

1 Introduction

In this talk I would like to describe some recent developments in the field of small–x perturbative QCD. In particular, I will review the first steps that have been done in the direction of finding an effective theory for small–x QCD.

Regge physics has been around for thirty years now. The focus of Regge physics is the behavior of hadronic scattering amplitudes at very high energy $s$ and fixed momentum transfer $t$ (of the order of some hadronic mass scale),

$$s \gg t \simeq M_{\text{hadron}}^2.$$  \hspace{1cm} (1)

With the advent of the HERA machine, it has recently attracted new interest. Electron–proton collisions in the now available kinematic range of small Bjorken–x and large momentum transfer $Q^2$ allow for a new test of high energy QCD. The interesting subprocess here is the scattering of a highly virtual photon off the proton. From a theorists point of view it is particularly interesting, because the large photon virtuality $Q^2$ enables us to enter this region of high parton densities using perturbative methods.

Let us first consider hadron–hadron scattering. The optical theorem relates the total cross section to the imaginary part of the forward elastic scattering amplitude,

$$\sigma_{\text{tot}} = \frac{1}{s} \text{Im} A_{\text{el}}(s, t = 0).$$  \hspace{1cm} (2)
It is convenient to perform a Mellin transformation changing from energy $s$ to complex angular momentum $\omega$.

$$ A(s, t) = is \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{M^2} \right)^\omega A(\omega, t). \quad (3) $$

The high energy behavior of the total cross section is then determined by the singularities of $A(\omega, t)$ in the $\omega$–plane, the so–called Regge poles and Regge cuts. The rightmost singularity gives the leading contribution and is identified with the pomeron. As it describes an elastic amplitude (see (2)) it carries vacuum quantum numbers. The location of the Regge poles in hadron–hadron scattering cannot yet be calculated from first principles, but there are successful phenomenological models that describe the experimental data in this framework.

It follows from unitarity that the total cross section cannot grow infinitely fast at high energy. It has to satisfy the Froissart bound

$$ \sigma_{\text{tot}} \leq \text{const.} \log^2 (s). \quad (4) $$

In the remaining part of my talk I will concentrate on considerations applying to the processes of $\gamma^* p$ or $\gamma^* \gamma^*$ scattering. The high virtuality $Q^2$ of the photon allows a perturbative treatment and thus enables us to get a clear understanding of these processes. $\gamma^*$–proton scattering is the basic process in electron–proton collisions. The Bjorken–$x$ variable is at high energy given by $x \simeq \frac{Q^2}{s}$, high energy corresponds to small $x$.

At small $x$ the small value of the coupling constant $\alpha_s$ can be compensated by large logarithms of $x$. This leads us to the leading logarithmic approximation (LLA)

$$ \alpha_s \ll 1; \quad \alpha_s \log(1/x) \sim 1. \quad (5) $$

In this approximation, the infinite number of contributing diagrams can be resummed. The result is the BFKL equation. It describes the $t$–channel exchange of a bound state of two reggeized gluons. These reggeized gluons are collective excitations of the Yang–Mills field carrying gluon quantum numbers. They are the relevant degrees of freedom at small $x$.

The BFKL equation is an integral equation in two–dimensional transverse momentum space, because the longitudinal degrees of freedom decouple in the high energy limit. In detail, it has the form

$$ \omega \phi_\omega (k, q - k) = \phi_0 (k, q - k) + \int \frac{d^2k'}{(2\pi)^2} \frac{1}{k'^2(q - k')^2} \mathcal{K}(q, k, k') \phi_\omega (k', q - k'). \quad (6) $$
\( \mathbf{k}, \mathbf{q} - \mathbf{k} \) are the momenta of the two gluons, \( \phi^0 \) is an inhomogeneous term. The integral kernel \( K \) (the so–called Lipatov kernel) is given by

\[
K(\mathbf{k}, \mathbf{k'}, \mathbf{q}) = N_c g^2 \left[ -q^2 + \frac{k^2(q-k')^2}{(k-k')^2} + \frac{(q-k)^2k'^2}{(k-k')^2} \right] \\
- (2\pi)^3 k^2(q-k) \left[ \beta(k) + \beta(q-k) \right] \delta^2(k-k'). 
\]

(7)

The coupling constant is normalized to \( \alpha_s = \frac{g^2}{4\pi} \) and \( \alpha(1^2) = 1 + \beta(1^2) \) with

\[
\beta(1^2) = \frac{N_c}{2} g^2 \int \frac{d^2k'}{(2\pi)^3} \frac{l^2}{k'^2(l^2 - 1)^2}.
\]

(8)

is known as the gluon trajectory function. The variable \( \omega \) acts as an energy variable in the BFKL equation. It can be shown to be conjugate to rapidity. Without making use of the Mellin transformation we would have found the BFKL equation as an evolution equation in \( x \).

The general form of the BFKL amplitude can be derived from the integral equation by iteration. Thus the high energy elastic scattering amplitude is in LLA described by a \( t \)–channel exchange of gluon ladders:

\[
s \rightarrow \infty \quad \Rightarrow \quad \sum_{\text{number of rungs}}.
\]

(9)

the rungs being Lipatov kernels. The BFKL equation has been solved analytically and leads to an increase of the amplitude

\[
A \sim x^{-(1+\omega_{\text{BFKL}})}; \quad \omega_{\text{BFKL}} = \frac{\alpha_s N_c}{\pi} 4 \ln 2 \approx 0.5
\]

(10)

for \( t = 0 \) in the limit \( x \rightarrow 0 \). It follows that the Froissart bound is violated at very small \( x \),

\[
\sigma_{\text{tot}}^{\gamma^*p} \sim x^{-\omega_{\text{BFKL}}} \ll \text{const. log}^2(1/x).
\]

(11)

This means that the BFKL pomeron violates unitarity at very small \( x \).

2 Unitarization of the BFKL-Pomeron

To restore unitarity we have to include nonleading (in \( \log(1/x) \)) corrections to the BFKL equation. The minimal set of corrections restoring unitarity can
be identified as the corrections with a larger number of reggeized gluons in the $t$–channel. Our goal is to find a consistent framework for the description of the infinite number of these terms. This framework will be a more general structure in which the BFKL pomeron appears as the first approximation.

Based on our knowledge of the BFKL equation we can already state some properties of this effective theory. Like in the BFKL case, the longitudinal degrees of freedom can be factored off and the dynamics will take place in two–dimensional transverse momentum space. Again, rapidity will act as the time–like coordinate. The appropriate degrees of freedom will be reggeized gluons.

As I will show, very important new elements are number changing vertices describing the creation and annihilation of $t$–channel gluons. Thus our aim to construct an effective theory of QCD at small $x$ turns out to be that of building a $2+1$–dimensional quantum field theory of reggeized gluons.

The program to be carried out in order to achieve this ambitious goal can be cut into the following pieces. Figure 1 tries to visualize the different elements that have to be extracted from QCD.

- **The leading order elements** are known: the BFKL equation and the quark loop.

- **The next-to-leading corrections.** The NLLA corrections to the BFKL equation are needed, for example, to understand which symmetries of the BFKL equation are only due to the LLA and how their breaking
occurs. These corrections are currently being calculated by Lipatov and Fadin and by Ciafaloni and Camici.

- The spectrum of the $n$–gluon state is a quantum mechanical problem formally described by the so–called BKP equations. Recently there has been a great leap forward in this longstanding problem. Lipatov, Faddeev and Korchemsky were able to prove that it is, in the large $N_c$ limit, equivalent to the integrable XXX Heisenberg model for noncompact spin zero. This connection opens the field for the application of some very powerful mathematical methods (Bethe ansatz etc.).

- Number changing vertices, which turn the problem into that of a quantum field theory. The vertex $2 \to 4$ gluons has been found recently, higher transition vertices, especially the $2 \to 6$ transition, are currently under investigation.

- Finding symmetries of the above elements is of course obligatory to understand the structure of the field theory. The most important symmetry of the known elements is their conformal invariance.

The number changing vertices, their emergence and the conformal invariance will be the topic of the following sections.

3 The Number Changing Vertices

The formal framework for our considerations is made up by amplitudes $D_n$ describing the creation of $n$ reggeized gluons in the $t$–channel from a quark loop at fixed $\omega$ (see fig. 1). These amplitudes will depend on the two–dimensional transverse momenta and on the color indices of the gluons. The amplitudes obey a tower of coupled integral equations that generalize the BFKL equation. For the purpose of this talk we state the equations only in a graphical form and only up to $n = 4$, a more detailed account can be found elsewhere.

\[
\omega \begin{array}{c} D_2 \\ D_3 \\ D_4 \end{array} = \begin{array}{c} D_{(2,0)} \\ D_{(3,0)} \\ D_{(4,0)} \end{array} + \begin{array}{c} D_2 \\ D_2 \\ D_2 \end{array} + \sum \begin{array}{c} D_3 \\ D_3 \\ D_4 \end{array} + \sum \begin{array}{c} D_4 \\ D_4 \end{array}
\]
The inhomogeneous terms $D_{(n,0)}$ stand for the perturbative coupling of $n$ gluons directly to the light quark loop. The first equation is the usual BFKL equation. The kernels appearing in the other equations can be calculated perturbatively and generalize the Lipatov kernel. They must not be confused with the number changing vertices we want to calculate. The latter emerge when we solve the integral equations which we can do at least partially, as I will explain. The summation symbols indicate that we have to include all possible pairwise interactions of $t$–channel gluons.

The equation for $D_3$ can be solved exactly and the solution is

$$D_3^{a_1 a_2 a_3}(k_1, k_2, k_3) = C_3 g_{a_1 a_2 a_3} \times
\times (D_2(k_1 + k_2, k_3) - D_2(k_1 + k_3, k_2) + D_2(k_1, k_2 + k_3)), \quad (15)$$

where $a_i$ are color indices, $k_i$ are the transverse momenta of the gluons and $C_3$ is a normalization constant. This result tells us that, although $D_3$ is formally defined as a 3–gluon amplitude, it is — according to its analytic properties — actually a superposition of 2–gluon (i.e. BFKL) amplitudes. There is no intermediate 3–gluon state! The details of the calculation show that this fact strongly depends on the inhomogeneous term in the integral equation, namely the coupling of the gluons to the loop containing light (effectively massless) quarks.

The equation (14) for the 4–gluon amplitude can be solved at least partially. ‘Partially’ because the solution involves the full 4–gluon state which is not yet known. Nevertheless, we can extract the structure of the solution from the equation. For simplicity suppressing all color and normalization factors, it has the following form:

$$D_4(k_1, k_2, k_3, k_4) = \sum \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \quad (16)$$

The first term is the sum of 2–gluon (BFKL) amplitudes $D_2$. The summation extends over the (seven) possibilities to combine the four momenta into two sums which are then the two arguments of the $D_2$ amplitudes ($C_{1,2}$ indicate color tensors, but again we will not go into the details of $su(3)$ algebra.):

$$D_4^R(k_1, k_2, k_3, k_4) = C_1 [D_2(k_1 + k_3 + k_4, k_2) + D_2(k_1 + k_2 + k_4, k_3)]$$
\[-D_2(k_1 + k_2, k_3 + k_4) - D_2(k_1 + k_3, k_2 + k_4)]
+ C_2[D_2(k_1 + k_2 + k_3, k_4) + D_2(k_1, k_2 + k_3 + k_1) - D_2(k_1 + k_4, k_2 + k_3)] \] (17)

This first part $D^R_4$ being isolated, the remaining terms in the equation define the transition vertex $V_{2\rightarrow4}$. (The full derivation of the vertex is rather lengthy and will not be given here.) We can write the second term as the convolution $G_4 \otimes V_{2\rightarrow4} \otimes D_2$ with $G_4$ being the propagator of the 4–gluon state. Starting from the quark loop, we first have a 2–gluon state, then there is a transition from two to four reggeized gluons and at the bottom we have the full, i.e. interacting, 4–gluon state. This teaches us the important lesson that in leading logarithmic order it is not possible to couple the interacting system of four reggeized gluons directly to the quark loop! This coupling always involves the vertex $V_{2\rightarrow4}$ and the 2–gluon state.

Let us now have a closer look at the vertex $V_{2\rightarrow4}$. It can be written as

\[
V_{2\rightarrow4}^{a_1a_2a_3a_4}(\{q_j\}, k_1, k_2, k_3, k_4) = \delta_{a_1a_2}\delta_{a_3a_4}V(\{q_j\}, k_1, k_2; k_3, k_4) \\
+ \delta_{a_1a_3}\delta_{a_2a_4}V(\{q_j\}, k_1, k_3; k_2, k_4) \\
+ \delta_{a_1a_4}\delta_{a_2a_3}V(\{q_j\}, k_1, k_4; k_2, k_3) \] (18)

Again, the indices $a_i$ are color labels, $k_i$ are the outgoing momenta and $\{q_j\}$ the incoming momenta. The explicit analytic form of the function $V$ can be found in the literature. The representation (18) nicely demonstrates the complete symmetry of the vertex under permutations of the outgoing gluons and displays its simple color structure.

Essentially in the same way, it is possible to extract information about the higher $n$–gluon amplitudes from the corresponding equations even without explicit knowledge of the interacting $n$–gluon systems.

A new result I can present here is the following: Like in the case of the 3–gluon amplitude, also the 5–gluon intermediate state is absent, the mechanism being similar to that shown in (15): the 5–gluon amplitude $D_5$ can be decomposed into a sum of 4–gluon amplitudes $D_4$. Using the same notation as in (16),

\[
D_5(k_1, k_2, k_3, k_4, k_5) = \sum \quad + \sum \\
V 
\]

(19)
The summation in the last term now refers to the possibilities to combine two of the five momenta into a sum.

This mechanism seems to apply to every odd number of intermediate gluons, and only even numbers of gluons appear as intermediate states. What we observe here is a generalization of the so-called reggeization of the gluon, a rather deep property of QCD in the Regge limit.

The next step will be the calculation of the 6–gluon amplitude. From that we will learn more about the elements of the field theory. The first question to answer is whether there is a new $2 \rightarrow 6$ transition vertex. Further, we will be able to understand the transition from the 4–gluon to the 6–gluon system. This should involve the known vertex $V_{2 \rightarrow 4}$, but also its generalization to the case in which the 2 gluons above the vertex are not in a color singlet, as has been the case in the 4–gluon amplitude.

4 Conformal Invariance

The Lipatov kernel and the transition vertex $V_{2 \rightarrow 4}$ exhibit a high degree of symmetry in that they are conformally invariant. To explain the meaning of this conformal invariance we go by means of Fourier transformation from transverse momentum space (momenta $\{k\}$) to impact parameter space (vectors $\{\vec{\rho}\}$). We then introduce the complex coordinates $\rho = \rho_x + i\rho_y$ and $\rho^* = \rho_x - i\rho_y$. This is done for all arguments of the Lipatov kernel and of the vertex.

Möbius (or conformal) transformations are defined by

$$\rho \rightarrow \rho' = \frac{a\rho + b}{c\rho + d}; \quad ad - bc = 1.$$  \hspace{1cm} (20)

These transformations are characterized by the group

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})/\mathbb{Z}_2,$$  \hspace{1cm} (21)

i.e. the group of projective conformal transformations. The generators of this group form the subalgebra $sl(2, \mathbb{C})$ of the well-known Virasoro algebra. It was known for some time\textsuperscript{[4]} that the Lipatov kernel is invariant under the transformations (20). Recently, also the vertex $V_{2 \rightarrow 4}$ was shown to be symmetric under conformal transformations\textsuperscript{[11]} We expect that this will be true also for higher vertices.

The observation that the Lipatov kernel as well as the number changing vertex are conformally invariant naturally leads us to a further property of the effective theory: it will be a conformal field theory. Conformal field theories in two dimensions have been subject to very intense and fruitful investigation.
in the past years. The applications range from statistical physics to string theory. It turned out that conformal symmetry is an extremely powerful tool. One example which might possibly apply to our considerations is the fact that the \( n \)-point functions in a conformal field theory are highly restricted. The 3-point function, for instance, is fixed up to a constant. The connection with conformal field theory might be very useful for a deeper understanding of small-\( x \) QCD.

It is important to mention that the conformal symmetry described above is not an exact symmetry of Nature even in the Regge limit. It is known that the running of the gauge coupling \( \alpha_s \) (the gauge coupling is fixed in LLA) and possibly other next-to-leading corrections will break conformal invariance. Of course, it will be important to understand in detail how the symmetry breaking takes place.

5 Outlook

The unitarization of the BFKL pomeron is urgently needed to get a deeper understanding of the small-\( x \) behavior of structure functions. It is believed that the unitarization of the pomeron will lead us to an effective theory of QCD in the Regge limit.

We are still at the very beginning of constructing such an effective theory of small-\( x \) QCD. We have compelling evidence that this theory will be a 2 + 1 dimensional conformal field theory. Reggeized gluons have been identified as the correct degrees of freedom. Some of the elements of the effective theory are already known. Other very important elements are still missing, but there has been considerable progress in the last years. So far, the emerging picture is very encouraging.

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References

1. E. A. Kuraev, L. N. Lipatov, V. S. Fadin, Sov. Phys. JETP 45, 199 (1977); Ya. Ya. Balitskii, L. N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978)
2. J. Bartels, Nucl. Phys. B 175, 365 (1980)
3. V. S. Fadin, L. N. Lipatov, Nucl. Phys. B 477, 767 (1996) and references therein
4. G. Camici, M. Ciafaloni, *Nucl. Phys. B* 496, 305 (1997) and in preparation
5. J. Kwieciński, M. Praszałowicz, *Phys. Lett. B* 94, 413 (1980)
6. L. N. Lipatov, JETP Lett. 59, 596 (1994)
7. L. D. Faddeev, G. P. Korchemsky, *Phys. Lett. B* 342, 311 (1995)
8. J. Bartels, *Z. Phys. C* 60, 471 (1993), J. Bartels, M. Wüsthoff, *Z. Phys. C* 66, 157 (1995)
9. J. Bartels, C. Ewerz, in preparation
10. L. N. Lipatov, *Sov. Phys. JETP* 63, 904 (1986)
11. J. Bartels, L. N. Lipatov, M. Wüsthoff, *Nucl. Phys. B* 464, 298 (1996)
12. A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, *Nucl. Phys. B* 241, 333 (1984); P. Ginsparg in *Les Houches, Session XLIX, 1988*, eds. E. Brézin and J. Zinn–Justin (Elsevier Science Publishers B. V., 1989)