Sharpness of Lenglart’s domination inequality and a sharp monotone version

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Abstract
We prove that the best so far known constant $c_p = \frac{p}{1-p}$, $p \in (0, 1)$ of a domination inequality, which originates to Lenglart, is sharp. In particular, we solve an open question posed by Revuz and Yor [13]. Motivated by the application to maximal inequalities, like e.g. the Burkholder-Davis-Gundy inequality, we also study the domination inequality under an additional monotonicity assumption. In this special case, a constant which stays bounded for $p$ near 1 was proven by Pratelli and Lenglart. We provide the sharp constant for this case.

Keywords: Lenglart’s domination inequality, Garsia’s Lemma, sharpness, monotone Lenglart’s inequality, BDG inequality

MSC2020 subject classifications: 60G44, 60G40, 60G42, 60J65

1 Introduction

In this note, we prove that the best so far known constant $c_p$ of a domination inequality, which originates to Lenglart [6, Corollaire II] (see Theorem 1.1), is sharp. In particular, we solve an open question posed by Revuz and Yor [13, Question IV.1, p.178]. Furthermore, motivated by the method of applying Lenglart’s inequality to extend maximal inequalities to small exponents, we study Lenglart’s domination inequality under an additional monotonicity assumption: A result by Pratelli [11] and Lenglart [6] implies (under the additional monotonicity assumption) a constant, which is bounded by 2, and hence considerably improves the constant of Lenglart’s inequality for $p$ near 1. We provide a sharp constant. The sharpness of our monotone version of Lenglart’s inequality is related to a result by Wang [17].

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a filtered probability space satisfying the usual conditions. The following lemma is [8, Lemma 2.2 (ii)]:

**Theorem 1.1** (Lenglart’s inequality). Let $X$ and $G$ be non-negative adapted right-continuous processes, and let $G$ be in addition non-decreasing and predictable such that $\mathbb{E}[X_\tau | \mathcal{F}_0] \leq \mathbb{E}[G_\tau | \mathcal{F}_0] \leq \infty$ for any bounded stopping time $\tau$. Then for all $p \in (0, 1),$

$$\mathbb{E}\left[\left(\sup_{t \geq 0} X_t\right)^p \mid \mathcal{F}_0\right] \leq c_p \mathbb{E}\left[\left(\sup_{t \geq 0} G_t\right)^p \mid \mathcal{F}_0\right]$$

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where $c_p := \frac{p}{1-p}$.

In the original work by Lenglart [6, Corollaire II], the inequality is proven for $c_p = \frac{2-p}{1-p}$, $p \in (0,1)$. The constant $c_p$ is improved to $\frac{p}{1-p}$ by Revuz and Yor in [13, Exercise IV.4.30] for continuous processes $X$ and $G$. This result is generalized to c\`adl\`ag processes by Ren and Shen in [12, Theorem 1] and is extended to a more general setting than [6, Corollaire II] by Mehri and Scheutzow [8, Lemma 2.2 (ii)]. Furthermore, the growth rate of the optimal constant $c_{opt}^{p}$ for c\`adl\`ag processes has been studied (see e.g. [13, Theorem IV.4.1]): It holds that $(c_{opt}^{p})^{1/q} = O(1/p)$ for $p \to 0^{+}$. We prove (see Theorem 2.1) that $\frac{p}{1-p}$ is sharp.

Lenglart’s inequality yields a very short proof of the Burkholder-Davis-Gundy inequality for continuous local martingales for small exponents (see e.g. [13, Theorem IV.4.1]): Let $(M_t)_{t \geq 0}$ be a continuous local martingale with $M_0 = 0$. To prove $\mathbb{E}[(M,M)_t^{q/2}] \lesssim \mathbb{E}[\sup_{s \leq t} |M_s|^q]$ for $q \in (0,2)$, take

$$X_t := \langle M, M \rangle_t, \quad G_t := \sup_{0 \leq s \leq t} |M_s|^2.$$ 

Using the BDG inequality for $q = 2$ we have $\mathbb{E}[X_\tau] \leq \mathbb{E}[G_\tau]$ for any bounded stopping time $\tau$. Applying Lenglart’s inequality with $p = q/2$, we obtain

$$\mathbb{E}[(M,M)^{q/2}_t] \leq c_{q/2} \mathbb{E}[\sup_{t \geq 0} |M_t|^q].$$

For $q = 1$, this implies $c_{BDG,1} = c_{q/2} = 2\sqrt{2} \approx 2.8284$. The optimal BDG constant can be computed numerically for this case (see Schachermayer and Stebegg [14]) and is $c_{BDG,1}^{(opt)} \approx 1.2727$. A better constant than $c_{q/2}$ can be achieved if we apply the following proposition due to Lenglart [6, Proposition I] and Pratelli [11, Proposition 1.2] instead:

**Proposition 1.2** (Lenglart, Pratelli). Let $F$ be a concave non-decreasing function with $F(0) = 0$ and let $c > 0$ be a constant. Let $Y$ and $G$ be adapted non-negative right-continuous processes starting in 0. Furthermore, let $G$ be non-decreasing and predictable. Assume that $\mathbb{E}[Y_\tau] \leq c \mathbb{E}[G_\tau]$ holds for all finite stopping times $\tau$. Then, for all finite stopping times $\tau$, we have

$$\mathbb{E}[F(Y_\tau)] \leq (1 + c) \mathbb{E}[F(G_\tau)].$$

Let $X$ and $G$ be as in Theorem 1.1. Assume in addition that both processes start in 0. Then Proposition 1.2 implies, choosing $F(x) = x^p$ for some $p \in (0,1)$ and optimizing over $c$, that

$$\mathbb{E}[X^p_t] \leq (1-p)^{-(1-p)}p^{-p} \mathbb{E}[G^p_t].$$

Hence, Proposition 1.2 gives $c_{BDG,1} = 2$. We show that the constant of Proposition 1.2 in the special case $F(x) := x^p, p \in (0,1)$ can be improved to $p^{-p}$ (see Theorem 2.2), which is sharp. In particular, by the argument described above we now achieve $c_{BDG,1} = \sqrt{2} \approx 1.4142$. For the right-hand side of the BDG inequality $\mathbb{E}[\sup_{t \geq 0} |M_t|^q] \lesssim \mathbb{E}[(M,M)^{q/2}_t]$, the sharp constant for $q = 1$ ($c_{1,BDG} = 1.4658$) was found by Osekowski [10]. Here, the monotone version of Lenglart’s inequality does not yield a sharper constant than the normal Lenglart’s inequality.
Lenglart’s inequality is frequently applied to extrapolate maximal inequalities to smaller exponents (see e.g. [2], [7], [13], [16] and [18]). Furthermore, Lenglart’s inequality is a useful tool for proving stochastic Gronwall inequalities (see e.g. [1] and [8]) and more generally studying SDEs (see e.g. [5] and [9]). In many of the application examples listed above, the additional assumption, that $X$ is non-decreasing is satisfied. Hence, instead, Theorem 2.2 could be applied, improving the constant considerably for $p$ near 1.

2 Main results

We assume, unless otherwise stated, that all processes are defined on an underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ which satisfies the usual conditions.

The following theorem answers the open question posed by Revuz and Yor [13, Question IV.1, p. 178].

**Theorem 2.1** (Sharpness of Lenglart’s inequality). For all $p \in (0, 1)$, there exist families of continuous processes $X^{(n)} = (X^{(n)}_t)_{t \geq 0}$ and $G^{(n)} = (G^{(n)}_t)_{t \geq 0}$ (depending on $p$) which satisfy the assumptions of Theorem 1.1 such that

$$\frac{p - p}{1 - p} = \lim_{n \to \infty} \frac{\mathbb{E} \left[ \left( \sup_{t \geq 0} X^{(n)}_t \right)^p \right]}{\mathbb{E} \left[ \left( \sup_{t \geq 0} G^{(n)}_t \right)^p \right]}. \quad (1)$$

In particular, the constant $c_p = \frac{p - p}{1 - p}$ in Theorem 1.1 is sharp.

As explained in the introduction, the application to maximal inequalities motivates us to consider the following monotone version of Lenglart’s inequality. We assume in addition that $X$ is non-decreasing and obtain a considerably improved constant for $p$ near 1.

**Theorem 2.2** (Sharp monotone Lenglart’s inequality). Let $X$ and $G$ be non-decreasing non-negative adapted right-continuous processes, and let $G$ be in addition predictable such that $\mathbb{E}[X_\tau | \mathcal{F}_0] \leq \mathbb{E}[G_\tau | \mathcal{F}_0] \leq \infty$ for any bounded stopping time $\tau$. Then for all $p \in (0, 1)$,

$$\mathbb{E} \left[ \left( \sup_{t \geq 0} X_t \right)^p \bigg| \mathcal{F}_0 \right] \leq p^{-p} \mathbb{E} \left[ \left( \sup_{t \geq 0} G_t \right)^p \bigg| \mathcal{F}_0 \right]. \quad (2)$$

Furthermore, for all $p \in (0, 1)$ there exist continuous processes $\tilde{X} = (\tilde{X}_t)_{t \geq 0}$ and $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$, satisfying the assumptions above such that

$$p^{-p} = \lim_{n \to \infty} \frac{\mathbb{E} \left[ \left( \sup_{t \geq 0} \tilde{X}_{t \wedge n} \right)^p \right]}{\mathbb{E} \left[ \left( \sup_{t \geq 0} \tilde{G}_{t \wedge n} \right)^p \right]}.$$

In particular, the constant $p^{-p}$ is sharp.

**Remark 2.3.** Inequality (2) is a sharpened special case of Proposition 1.2, its proof is a modification of the proof of [11, Proposition 1.2]. The theorem generalizes a result by Garsia [4, Theorem III.4.4, page 113]. In [17, Theorem 2], Wang proved that [4, Theorem III.4.4, page 113] is sharp. Hence, by translating his result from discrete to continuous time proves sharpness of $p^{-p}$.
Theorem 1.1, Theorem 2.1, and Theorem 2.2 also hold in discrete time. Here, sharpness
found in [6, Remarque after Corollaire II].

exists no finite constant in inequality (2). An example which demonstrates this can be
adapted processes, and let

\( \tau \) for all \( t \geq 0 \) and noting that \( (\hat{X}_t)_{t \geq 0} \) and

\( (G_{t,\tau})_{t \geq 0} \) satisfy the assumptions of Theorem 2.2.

Remark 2.4. Theorem 2.2 can be also applied when \( X \) is not non-decreasing. In that
case, the theorem implies for any stopping time \( \tau \) the inequality \( \mathbb{E}[X_\tau] \leq p^{-p} \mathbb{E}[G_\tau] \). This
can be seen by defining \( \hat{X}_t := X_t 1_{(\tau,\infty)}(t) \) for all \( t \geq 0 \) and noting that \( (\hat{X}_t)_{t \geq 0} \) and

Remark 2.5. In Theorem 2.2, the assumption that \( G \) is right-continuous and predictable
can be replaced by the assumption that \( G \) is left-continuous.

Remark 2.6. A key part of the proof of Lenglart’s inequality is the inequality

\[ \mathbb{P}
\left(
\sup_{t \geq 0} X_t > c \mid F_0
\right)
\leq \frac{1}{c}
\mathbb{E}
\left(
\sup_{t \geq 0} G_t \wedge d \mid F_0
\right)
+ \mathbb{P}
\left(
\sup_{t \geq 0} G_t \geq d \mid F_0
\right),
\]

for all \( c, d > 0 \). If \( X \) is non-decreasing, this can be improved to

\[ \frac{1}{c}
\mathbb{E}
\left(
\sup_{t \geq 0} X_t \wedge c \mid F_0
\right)
\leq \frac{1}{c}
\mathbb{E}
\left(
\sup_{t \geq 0} G_t \wedge d \mid F_0
\right)
+ \mathbb{P}
\left(
\sup_{t \geq 0} G_t \geq d \mid F_0
\right),
\]

which is used to prove the monotone version of Lenglart’s inequality.

Remark 2.7. If \( G \) is not predictable and no further assumptions are made, then there
exists no finite constant in inequality (2). An example which demonstrates this can be
found in [6, Remarque after Corollaire II].

Theorem 1.1, Theorem 2.1, and Theorem 2.2 also hold in discrete time. Here, sharpness
of \( p^{-p} \) follows immediately from [17, Theorem 2].

Corollary 2.8 (Discrete Lenglart’s inequality). Let \((X_n)_{n \in \mathbb{N}_0}\) and \((G_n)_{n \in \mathbb{N}_0}\) be non-negative
adapted processes, and let \( G \) be in addition non-decreasing and predictable such that
\( \mathbb{E}[X_\tau \mid F_0] \leq \mathbb{E}[G_\tau \mid F_0] \leq \infty \) for any bounded stopping time \( \tau \). Then for all \( p \in (0,1) \),

\[ \mathbb{E}
\left(
\left(
\sup_{n \in \mathbb{N}_0} X_n
\right)^p \mid F_0
\right)
\leq c_p \mathbb{E}
\left(
\left(
\sup_{n \in \mathbb{N}_0} G_n
\right)^p \mid F_0
\right),
\]

where \( c_p := \frac{p^{-p}}{\mathbb{E}[G_\tau \mid F_0]} \) and the constant \( c_p \) is sharp.

If we assume in addition, that \((X_n)_{n \in \mathbb{N}_0}\) is non-decreasing, then we have

\[ \mathbb{E}
\left(
\left(
\sup_{n \in \mathbb{N}_0} X_n
\right)^p \mid F_0
\right)
\leq p^{-p} \mathbb{E}
\left(
\left(
\sup_{n \in \mathbb{N}_0} G_n
\right)^p \mid F_0
\right)
\]

and the constant \( p^{-p} \) is sharp.

3 Proof of Theorem 2.1

Proof of Theorem 2.1. Choose an arbitrary \( p \in (0,1) \) for the remainder of this proof.
First, we define non-decreasing processes \( \hat{X} = (\hat{X}_t)_{t \geq 0} \) and \( \hat{G} = (\hat{G}_t)_{t \geq 0} \) which satisfy the
assumptions of Theorem 1.1, such that

\[ p^{-p} = \lim_{n \to \infty} \frac{\mathbb{E}[(\sup_{t \geq 0} \hat{X}_{t \wedge n})^p]}{\mathbb{E}[(\sup_{t \geq 0} \hat{G}_{t \wedge n})^p]}. \]
To obtain the extra factor $(1 - p)^{-1}$, we modify $\tilde{X}$ and $\tilde{G}$ using an independent Brownian motion: This gives us the families $\{(X_t^{(n)})_{t \geq 0}, n \in \mathbb{N}\}$ and $\{(G_t^{(n)})_{t \geq 0}, n \in \mathbb{N}\}$.

Note that if we have non-negative random variables $X_{RV} := 1$ and $G_{RV}$ with $\mathbb{E}[X_{RV}] = \mathbb{E}[G_{RV}]$, then we obtain $\mathbb{E}[X_{RV}^p] >> \mathbb{E}[G_{RV}^p]$ for example by choosing $G_{RV}$ to be very large on a set with small probability and everywhere else 0. Keeping this in mind, we construct $\tilde{X}$ and $\tilde{G}$ as follows: Let $Z$ be an exponentially distributed random variable on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[Z] = 1$. Set
\[
A : [0, \infty) \to [0, \infty), \quad t \mapsto \exp(t/p).
\]
Define for all $t \geq 0$
\[
\tilde{X}_t := A(Z) \mathbb{1}_{[Z, \infty)}(t), \quad \tilde{G}_t := \int_0^{t \wedge Z} A(s) ds.
\]
Choose $\tilde{F}_t := \sigma(\{Z \leq r\} \mid 0 \leq r \leq t)$ for all $t \geq 0$. Observe that $\tilde{X}$ and $\tilde{G}$ are non-decreasing non-negative adapted right-continuous processes, and $\tilde{G}$ is in addition continuous, hence predictable. Furthermore, due to $Z$ being exponentially distributed, $\tilde{G}$ is the compensator of $\tilde{X}$, implying $\mathbb{E}[\tilde{X}_\tau] = \mathbb{E}[\tilde{G}_\tau]$ for all bounded $\tau$.

Now we use the processes $\tilde{X}$ and $\tilde{G}$ to construct the families $\{(X_t^{(n)})_{t \geq 0}, n \in \mathbb{N}\}$ and $\{(G_t^{(n)})_{t \geq 0}, n \in \mathbb{N}\}$: Assume w.l.o.g. that there exists a Brownian motion $B$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the smallest filtration satisfying the usual conditions which contains $(\tilde{F}_t)_{t \geq 0}$ and w.r.t. which $B$ is a Brownian motion. Denote by $g_{n,n+1} : [0, \infty) \to [0, 1]$ a continuous non-decreasing function such that
\[
g_{n,n+1}(t) = 0 \quad \forall t \leq n, \quad \text{and} \quad g_{n,n+1}(t) = 1 \quad \forall t \geq n + 1.
\]
Define:
\[
\tau^{(n)} := \inf\{t \geq n + 1 \mid \tilde{X}_n + (B_t - B_{n+1}) \mathbb{1}_{\{t \geq n+1\}} = 0\},
\]
\[
X_t^{(n)} := g_{n,n+1}(t) \tilde{X}_n + (B_{t \wedge \tau^{(n)}} - B_{t \wedge (n+1)})
\]
\[
G_t^{(n)} := \tilde{G}_{t \wedge n}
\]
The stopping time $\tau^{(n)}$ ensures that $X_t^{(n)}$ is non-negative. By construction, we have for every bounded $(\mathcal{F}_t)_{t \geq 0}$ stopping time $\tau$
\[
\mathbb{E}[X_{\tau}^{(n)}] \leq \mathbb{E}[\tilde{X}_{\tau \wedge n} + B_{\tau \wedge \tau^{(n)}} - B_{\tau \wedge (n+1)}] = \mathbb{E}[\tilde{G}_{\tau \wedge n}] = \mathbb{E}[G_{\tau}^{(n)}].
\]
Hence, $(X_t^{(n)})_{t \geq 0}$ and $(G_t^{(n)})_{t \geq 0}$ are continuous processes that satisfy the assumptions of Theorem 1.

It remains to calculate $\mathbb{E}\left[(\sup_{t \geq 0} X_t^{(n)})^p\right]$ and $\mathbb{E}\left[(\sup_{t \geq 0} G_t^{(n)})^p\right]$, to show that equation (2) is satisfied. We have
\[
\mathbb{E}[\tilde{X}_t^p] = \int_0^\infty A(x)^p \mathbb{1}_{\{t \geq x\}} \exp(-x) dx = t,
\]
\[
\mathbb{E}[\tilde{G}_t^p] = \int_0^\infty \left( \int_0^{t \wedge x} A(s) ds \right)^p \exp(-x) dx \leq p^p (t + 1),
\]

(6)
which implies in particular that $\mathbb{E}[(\sup_{t \geq 0} G_t^{(n)})^p] \leq p^p(n + 1)$.

We calculate $\mathbb{E}[(\sup_{t \geq 0} X_t^{(n)})^p]$ using the independence of $Z$ and $B$. To this end, let $\tilde{B}$ be some Brownian motion and consider for all $0 \leq x < a^{1/p}$ the stopping times

$$\sigma_x := \inf\{t \geq 0 \mid \tilde{B}_t + x = 0\}, \quad \sigma_{x,a} := \inf\{t \geq 0 \mid \tilde{B}_t + x = a^{1/p}\}.$$ 

Define the family of random variables $Y_x := \sup_{t \geq 0} \tilde{B}_t \wedge \sigma_x + x, x \geq 0$. Then $\mathbb{E}[\tilde{B}_{\sigma_x} \wedge \sigma_{x,a}] = 0$ implies $\mathbb{P}[Y_x \geq a^{1/p}] = \mathbb{P}[\sigma_{x,a} < \sigma_x] = xa^{-1/p}$, and hence

$$\mathbb{E}[Y_x^p] = x^p + \int_{x^p}^{\infty} \mathbb{P}[Y_x \geq a^{1/p}] da = x^p + x^p \frac{p}{1 - p} = \frac{x^p}{1 - p}. \quad (7)$$

Hence, we have by (6), (7) and independence of $(B_t - B_{n+1})_{t \geq n+1}$ and $\mathcal{F}_{n+1}$:

$$\mathbb{E}[(\sup_{t \geq 0} X_t^{(n)})^p] = \mathbb{E}[\mathbb{E}[(\sup_{t \geq 0} X_t^{(n)})^p \mid \mathcal{F}_{n+1}]]$$

$$= \mathbb{E}\left[\frac{1}{1 - p} (\tilde{X}_n)^p\right]$$

$$= \frac{n}{1 - p}.$$ 

Therefore, we have:

$$c_p \geq \frac{\mathbb{E}[\sup_{t \geq 0} X_t^{(n)p}]}{\mathbb{E}[\sup_{t \geq 0} G_t^{(n)p}]} \geq \frac{n}{1 - p} \frac{p^{-p}}{n + 1},$$

which implies (1). \qed

4 Proof of Theorem 2.2

Remark 4.1. The following proof of inequality (2) is a modification of the proof of [11, Proposition 1.2]. Sharpness of the constant can be proven using [17, Theorem 2].

Proof of Theorem 2.2. We first show that $p^{-p}$ is the optimal constant. Sharpness of $p^{-p}$ can be proven by translating [17, Theorem 2] into continuous time. Alternatively, one can use the processes $\tilde{X}$ and $\tilde{G}$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ from the proof of Theorem 2.1. Equation (6) implies, that

$$p^{-p} = \lim_{n \to \infty} \frac{\mathbb{E}\left[\left(\sup_{t \geq 0} \tilde{X}_{t/n}\right)^p\right]}{\mathbb{E}\left[\left(\sup_{t \geq 0} \tilde{G}_{t/n}\right)^p\right]},$$

and therefore that $p^{-p}$ is sharp.

Now we prove that inequality (2) holds true. We may assume w.l.o.g. that $(G_t)_{t \geq 0}$ is bounded (because it is predictable). This implies $\mathbb{E}[\sup_{t \geq 0} X_t] < \infty$. To shorten notation, we define

$$X_\infty := \sup_{t \geq 0} X_t, \quad G_\infty := \sup_{t \geq 0} G_t.$$  

(8)
We use the following formulas for positive random variables $Z$ (equation (10) is a direct consequence of [9], alternatively see also [3] Theorem 20.1, p. 38-39):

\[
E[Z^p \mid \mathcal{F}_0] = \int_0^\infty P[Z \geq u^{1/p} \mid \mathcal{F}_0] \, du,
\]

(9)

\[
E[Z^p \mid \mathcal{F}_0] = p(1-p) \int_0^\infty E[Z \wedge u \mid \mathcal{F}_0] \, u^{p-2} \, du.
\]

(10)

We will apply (10) to $X$. Because (9), alternatively see also [3, Theorem 20.1, p. 38-39]):

\[
E[X_\tau^p \mid \mathcal{F}_0] = \int_0^\infty P[X_\tau \geq u^{1/p} \mid \mathcal{F}_0] \, du,
\]

(11)

\[
P[X_\tau \geq u \mid \mathcal{F}_0] \leq \lambda E[G_\infty \wedge \lambda t \mid \mathcal{F}_0] = \lambda \int_0^\infty P[G_\infty \geq \lambda u \mid \mathcal{F}_0] \, du.
\]

(12)

On $\{\tau = \infty\} \cup \{G_0 > \lambda t\}$ we have $\lim_{n \to \infty} X_{\tau^{(n)}} \wedge \lambda t = X_\infty \wedge \lambda t$, which implies:

\[
E[X_\infty \wedge t - X_{\tau^\wedge} \wedge t \mid \mathcal{F}_0] \leq t E[1_{(\tau<\infty) \cap \{G_0 \leq \lambda t\}} \mid \mathcal{F}_0].
\]

(13)

Combining inequalities (11) and (12) gives:

\[
E[X_\infty \wedge t \mid \mathcal{F}_0] \leq \lim_{n \to \infty} E[X_{\tau^{(n)}} \mid \mathcal{F}_0] 1_{\{G_0 \leq \lambda t\}} + \lim_{n \to \infty} E[X_\infty \wedge t - X_{\tau^\wedge} \wedge t \mid \mathcal{F}_0] \leq \lambda \int_0^\infty P[G_\infty \geq \lambda u \mid \mathcal{F}_0] \, du.
\]

(14)

Applying (10) to $X_\infty$ and inserting (13) gives:

\[
E[X_\infty^p \mid \mathcal{F}_0] \leq \lambda p(1-p) \int_0^\infty E[(G_\infty \wedge \lambda u) \mid \mathcal{F}_0] \, u^{p-2} \, du + p(1-p) \int_0^\infty P[G_\infty \geq \lambda u \mid \mathcal{F}_0] \, u^{p-1} \, du.
\]

(15)

Applying (9) and (10) to $G_\infty$ in the previous inequality implies:

\[
E[X_\infty^p \mid \mathcal{F}_0] \leq \lambda^{1-p} E[G_\infty^p \mid \mathcal{F}_0] + (1-p) \int_0^\infty P[G_\infty \geq \lambda y^{1/p} \mid \mathcal{F}_0] \, dy \leq \lambda^{-p}(\lambda + 1-p) E[G_\infty^p \mid \mathcal{F}_0].
\]

Choosing $\lambda = p$ implies the assertion of the theorem. \hfill \Box

5 Proof of Corollary 2.8

Proof of Corollary 2.8. We first prove inequalities (3) and (4): We turn the processes $(X_n)_{n \in \mathbb{N}_0}$ and $(G_n)_{n \in \mathbb{N}_0}$ into càdlàg processes in continuous time as follows: Set for all $n \in \mathbb{N}_0, t \in [n, n+1)$:

\[
X_t := X_n, \quad G_t := G_n, \quad \mathcal{F}_t := \mathcal{F}_n.
\]
As we can approximate \((G_t)_{t \geq 0}\) by left-continuous adapted processes, it is predictable. Now Theorem 1.1 and Theorem 2.2 immediately imply inequalities (3) and (4).

Sharpness of \(p-p\) follows from [17, Theorem 2]. We show that \(p- p\) is sharp. Let \(X^{(n)}, G^{(n)}, A, (F_t)_{t \geq 0}\) be as in proof of Theorem 2.1. Fix some arbitrary \(N \in \mathbb{N}\). Set for all \(k, n \in \mathbb{N}\)

\[
X^{(n,N)}_0 := X^{(n)}_0, \quad X^{(n,N)}_k := X^{(n)}_{k+2N},
\]

\[
G^{(n,N)}_0 := G^{(n)}_0, \quad G^{(n,N)}_k := G^{(n)}_{k+2N} + \int_{(k-1)2^{-N} \wedge n}^{k2^{-N} \wedge n} A(s)ds,
\]

\[
F^{(n,N)}_0 := F_0, \quad F^{(n,N)}_k := F_{k+2N}.
\]

The processes \((X^{(n,N)}_k)_{k \in \mathbb{N}_0}\) and \((G^{(n,N)}_k)_{k \in \mathbb{N}_0}\) are non-negative and adapted, \((G^{(n,N)}_k)_{k \in \mathbb{N}_0}\) is in addition non-decreasing and predictable. Since \(G^{(n)}_{k+2N} \leq G^{(n,N)}_k\), the processes satisfy the Lenglart domination assumption.

Hence, noting that

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left( \sup_{k \in \mathbb{N}_0} X^{(n,N)}_k \right)^p \right] = \mathbb{E} \left[ \left( \sup_{t \geq 0} X^{(n)}_t \right)^p \right],
\]

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left( \sup_{k \in \mathbb{N}_0} G^{(n,N)}_k \right)^p \right] = \mathbb{E} \left[ \left( \sup_{t \geq 0} G^{(n)}_t \right)^p \right],
\]

implies the assertion of the corollary. 

\[\square\]

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