Non-Abelian T-duality and the AdS/CFT correspondence: new $\mathcal{N} = 1$ backgrounds

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Abstract

We consider non-Abelian T-duality on $\mathcal{N} = 1$ supergravity backgrounds possessing well understood field theory duals. For the case of D3-branes at the tip of the conifold, we dualise along an $SU(2)$ isometry. The result is a type-IIA geometry whose lift to M-theory is of the type recently proposed by Bah et. al. as the dual to certain $\mathcal{N} = 1$ SCFT quivers produced by M5-branes wrapping a Riemann surface. In the non-conformal cases we find smooth duals in massive IIA supergravity with a Romans mass naturally quantised. We initiate the interpretation of these geometries in the context of AdS/CFT correspondence. We show that the central charge and the entanglement entropy are left invariant by this dualisation. The backgrounds suggest a form of Seiberg duality in the dual field theories which also exhibit domain walls and confinement in the infrared.

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1 Introduction

Within the context of gauge/string duality, solution generating techniques in supergravity are an extremely powerful tool. Prominent examples include the use of bosonic and fermionic T-dualities to show dual superconformal symmetry at strong coupling.
the T-s-T transformations that are the string analogue of \( \beta \)-deformations in gauge theory \[3\] and which can also be used to construct gravity duals for some non-relativistic field theories \[4, 5, 6\]; the use of \( G \) structure rotations to obtain solutions with/without back-reacted source branes in conifold related geometries \[7, 8, 9\]. Evidently some of these techniques, namely U-dualities, are understood to be symmetries of the underlying string theory. Fermionic T-duality, however, provides an example where the symmetry is only valid at tree-level in string perturbation theory but nonetheless has applications in AdS/CFT when considering just the planar limit.

Performing a T-duality with respect to a non-Abelian isometry group is also a solution generating technique of supergravity. Rather like the case of fermionic T-duality it is not expected to be a full symmetry of string perturbation theory. But it is nonetheless natural to ask what rôle it might have within the gauge/string correspondence. This study was initiated in \[10\] in which the dualisation of \( AdS_5 \times S^5 \) with respect to an \( SU(2) \) isometry group was carried out. The result was somewhat surprising; the dual was found to be a solution of type-IIA supergravity whose lift to M-theory captures some universal properties of the solutions found by Gaiotto and Maldacena in \[11\], as dual geometries to the generalised \( \mathcal{N} = 2 \) quiver SCFT’s proposed by Gaiotto in \[12\]. Further progress and works in studying non-Abelian T-duality in this context can be found \[13, 14, 15, 16, 17\] and a brief review of elementary aspects of non-Abelian T-duality in \[18\].

Motivated by this, in this paper we shall investigate non-Abelian T-duality applied to solutions with minimal supersymmetry whose field theory dual is well understood.

The Klebanov–Witten (KW) solution \[19\] provides the first such example; this solution represents the \( SU(N) \times SU(N) \) conformal field theory on D3-branes located at the tip of the conifold. We are also interested in gauge theories which are not conformal but rather, have non-trivial RG flows. The prototypical example is the Klebanov–Tseytlin (KT) solution \[20\] which incorporates fractional branes (D5-brans wrapped on the shrinking two-cycle of the conifold) and is a good model for the UV dynamics of a \( SU(N) \times SU(N + M) \) theory. As one flows towards the IR the theory under goes a sequence of Seiberg dualities to ever decreasing gauge group ranks. In the IR the solution of KT is singular, a fact which is remedied when \( M \) is a multiple of \( N \) (\( N = kM \)), as it occurs in the Klebanov–Strassler (KS) solution \[21\], wherein strong coupling effects take hold and remove the singularity by replacing the conifold with its
deformation. In the IR the theory exhibits R-symmetry breaking (or rather $\mathbb{Z}_{2n} \to \mathbb{Z}_2$), confinement, domain walls and other interesting phenomena. One level up in complexity is the construction of the gravity dual for the case in which the KS-field theory is exploring its baryonic branch; in this case there exists a one-parameter family of regular deformations [22] interpolating between the KS solution and the wrapped D5-brane solutions in [23, 24, 25].

All of these examples possess rich isometry groups containing at least an $SU(2)$ factor along which we will dualise. There is also a $U(1)$ isometry of the metric (at least in the KW and KT solutions) that may be understood in the dual field theory as the R-symmetry. The Killing spinors of the background are invariant under the $SU(2)$ action, in a sense which we shall explain. This corresponds to the fact the super symmetries are uncharged under the global (flavour-like) symmetries in the field theory and because of this performing the dualisation preserves supersymmetry. This should be contrasted with performing an Abelian T-duality in the internal space which would either destroy supersymmetry or result in a singular background. Since the $SU(2)$ isometry group has three generators one will arrive at solutions in (massive) IIA. Let us now summarise what happens in each case in ascending order of complexity.

We find that in the case of KW the dual geometry can be lifted to M-theory and can be directly matched to some solutions recently proposed by [26, 27], generalising the eleven-dimensional solutions of [28], as dual to the $\mathcal{N} = 1$ SCFTs obtained from M5-branes wrapped on a Riemann surface. Included in this class of SCFT are the so-called Sicilian quivers of [29]. This is a direct $\mathcal{N} = 1$ analogue of the dualisation of $AdS_5 \times S^5$ to Gaiotto–Maldacena-like geometries that was performed in [10]. Indeed, one can obtain the KW theory by considering the $\mathcal{N} = 2$ gauge theory dual to the orbifold $AdS_5 \times S^5/\mathbb{Z}_2$ adding a relevant deformation and flowing to the IR. We essentially find a T-dual complement of this relation.

For the dualisation of the KT solution one finds that the resultant geometry is a solution of massive IIA supergravity and the Romans mass is naturally quantised by the number of fractional branes. The reason for this can be understood intuitively by the fact that there is a component of the RR three-form with legs along all the $SU(2)$ directions. Upon dualisation, this then gets converted to a zero-form. Since this is a solution of massive IIA it has no lift to M-theory; the fractional branes of the type-IIB solution represent some obstruction to this.
To get a better handle on this novel background we perform a number of checks. The first is to look at the central charge before and after the dualisation, following the method of [30] and [31]. We find that, up to a subtlety that depends on the global properties of the geometry, the central charge before and after the non-Abelian duality, matches. As we will explain, this can be understood by the fact that the measure, \( \sqrt{\hat{g}} e^{-2\phi} \), is an invariant of the duality (just as it is for Abelian T-duality). The same invariance is present for the entanglement entropy. By using probe branes one can define a gauge coupling. A strange feature is that this suitably defined gauge coupling does not behave like those of a renormalisable 4d QFT where \( g^{-2} \sim \ln r \) (as in the KT case), instead going like \( g^{-2} \sim (\ln r)^{3/2} \) which hints at a rather unusual dual field theory (either that, or the coupling so defined does not represent the usual gauge interactions). Finally one can consider the Page and Maxwell charges after duality. Essentially what was D3-brane charge becomes D6-brane charge. The Maxwell charge of D6-branes changes logarithmically. As we will discuss, this is one among other similarities with the KS-cascade. We will discuss below, a form of Seiberg duality that appears after the duality.

To probe the low energy physics, one needs to look at the dual of the KS geometry. In this case things are rather more involved, but we verify that the IR signatures of confinement and domain walls are preserved after the dualisation. The same pattern shows-up if we start with the solution describing D5-branes wrapping SUSY two-cycles [33] and dualise it.

Interestingly, for the Type IIB solution describing the baryonic branch of the KS-field theory, something qualitatively different happens. After the dualisation, we find that the large radius asymptotics of the metric is no longer (logarithmically) approaching \( AdS_5 \). We provide two suggestions as to the field theoretic interpretation of this; either this is due to presence of an irrelevant operator in the dual QFT or more conservatively that this theory ceases to have a baryonic branch.

Let us present now a "road map" that summarises the points above and lay-out the general idea behind this long and technical paper.

1.1 General Idea and Road Map

We start with \( AdS_5 \times S^5 \) or better yet, with \( AdS_5 \times S^5 / Z_2 \), with \( N \) units of flux of the five form. The field theories associated are \( \mathcal{N} = 4 \) SYM or the \( \mathcal{N} = 2 \) version of the
two groups quiver $SU(N) \times SU(N)$ and adjoint matter. Following the paper [10] we can perform a non-Abelian T duality on the geometry to obtain a Type IIA/M-theory geometry of the form proposed by Maldacena and Gaiotto [11] with the following characteristics (see [10] for details):

- It contains a factor of $S^2$ instead of a hyperbolic plane $H^2$
- The resulting geometry is singular. The dilaton field diverges at a given angular position.
- In correspondence with the previous point, the ‘charge distribution’ in the language of [11] is $\lambda(\eta) = \eta$, which implies a quiver of the form

$$SU(2) \times SU(3) \times SU(4) \times \ldots \times SU(N) \times SU(N + 1) \times \ldots$$

A natural first step taken in this paper, is to apply the non-Abelian T-duality to examples preserving $\mathcal{N} = 1$ SUSY. We choose the Klebanov-Witten geometry [19] whose dual field theory is the mass deformation of the $N = 2$ field theory described above. The non-Abelian duality is performed in Section 3. Some interesting things are: that the solution is non-singular, preserves $\mathcal{N} = 1$ and falls within the class of geometries proposed in [29] (originally found in [32]). These geometries have been proposed to be dual to the mass deformation of the Maldacena-Gaiotto theories. Our geometry presented in Section 3 falls within this class, for the case in which we have an $S^2$ factor instead of an $H_2$. The dual field theory seems to be less understood in that case.
The following step is to continue with the known deformations of the Klebanov-Witten theory/geometry. We then study the case of the Klebanov-Tseytlin geometry, Klebanov-Strassler geometry, Baryonic Branch and the background of D5 branes wrapping a two cycle inside the resolved conifold. We obtain in this case new backgrounds in Massive IIA with a quantized mass parameter, proportional to the number of five branes $N_c$, the ‘deformation’ from the conformal point. We present arguments for the non-singular behavior of these new solutions (the transformed of the KT-solution is obviously singular as the seed solution is) and ‘define’ their field theory dual calculating observables with the background.

In more detail, the structure of the paper is the following: In Section 2 we develop the technology required to implement these non-Abelian duality transformations. In Section 3 we apply this to the KW background. In Sections 4, 5 and 6, we turn our attention to the dualisations and field theory analysis of the non-conformal backgrounds described above. We conclude in Section 7, presenting some open questions and future topics for research. We provide generous appendices describing our conventions and generic Buscher-like rules for dualisation.

2 Non-Abelian T-Duality Technology

In this section we give details of the dualisation procedure used. The hurried reader who simply wants to get the physical results should feel free to read the following "Non-Abelian Duality 101" and skip past the rest of the section returning when he wishes to know more of the technicalities.

2.1 Non-Abelian T-Duality 101

T-duality states equivalence between string theories propagating on two different target spacetimes containing some abelian isometries. In its simplest form, it is the equivalence between strings on circle of radius $R$ with those on a circle radius $1/R$. More generally T-duality provides a map, known as the Buscher rules, between one solution of supergravity and a second solution. A powerful approach to deriving these rules is the path integral approach (or Buscher procedure) [34]. This procedure is a three step recipe: one begins with the string sigma model for the first spacetime and gauges a $U(1)$ isometry of this spacetime; second, one invokes a flat connection for this gauge
field by means of a Lagrange multiplier; finally, one integrates by parts to yield an action with a non-propagating gauge field that can be eliminate by its equations of motion to produce the T-dual sigma model.

The Buscher procedure can be naturally generalised to the case of a target space equipped with a non-Abelian isometry group $G$. One follows exactly the same steps but in this case the gauge fields are valued in the algebra of $G$. Doing so produces a map between one solution of supergravity and another. It is in this spirit of solution generating that we employ non-Abelian T-duality in this paper.

Despite the fact the dualisation procedure is rather similar between the abelian and non-abelian cases there are some important differences. Generically the isometry (and potentially supersymmetry) enjoyed by the starting geometry is, at least partially, destroyed. However this lost isometry may be recovered as a non-local symmetry in the sigma model and the corresponding sigma models are canonically equivalent. A second point is the rather subtle effect of global issues that arise when performing the Buscher procedure on world sheets of arbitrary genera. These global concerns mean one should not view non-abelian duality as a full symmetry of string (genus) perturbation theory but just a tree-level symmetry. Nonetheless, if one’s focus is on supergravity (as it will be in this paper) or the planar limit then one may still harness its power as a solution generating symmetry.

Early work on non-Abelian duality can be found in [35, 36, 37, 38, 39, 40, 41] in context of purely Neveu-Schwarz backgrounds. This subject has had something of a revival following the work of [10], in which this procedure was extended to geometries contain RR fluxes. A particularly curious result came from performing a non-Abelian dualisation of an $SU(2)$ isometry group that acts within the sphere of $AdS_5 \times S^5$. After dualisation the resultant geometry was a solution of IIA whose lift to M-theory bore a very close resemblance to the Giaotto-Maldacena geometries that come from considering M5 branes wrapped on Riemann surfaces.

We close this section by giving an example to get the reader in the spirit. Consider the round metric on the $S^3$ which possess $SO(4) = SU(2)_L \times SU(2)_R$ isometry and may be written (in Euler angles) as

$$ds^2 = d\theta^2 + d\phi^2 + 2\cos \theta d\phi d\psi + d\psi^2 .$$

After performing the dualisation with respect to say the $SU(2)_L$ isometry one finds a
geometry that interpolates between $\mathbb{R} \times S^2$ and $\mathbb{R}^3$ with metric given by

$$\tilde{ds}^2 = dr^2 + \frac{r^2}{1 + r^2}(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.2)$$

In addition, in this example the dual geometry is supported by a NS two-form and dilaton given by

$$\tilde{B} = \frac{r^3}{1 + r^2} vol(S^2), \quad \tilde{\Phi} = -\frac{1}{2} \ln(1 + r^2). \quad (2.3)$$

This example serves to illustrates a two key features that we will encounter. Firstly the $SO(4) = SU(2)_L \times SU(2)_R$ isometry gets reduced to just $SU(2)$ that is reflected by the presence of the $S^2$ in the dual. Secondly there has been a serious topology change, indeed the dual geometry contains a non-compact direction. Whilst this example does not represent, evidently, a full solution of supergravity on its own, it may be embedded into true supergravity solutions and indeed it is prototypical of the dualisations that we will perform.

An important ingredient in this paper will be the incorporation of RR fields. Lets us illustrate how this works by supposing that in the example above the initial geometry is supported by a RR three-form

$$F_3 = vol(S^3). \quad (2.4)$$

To extract the dual fluxes one may use the following formula,

$$e^{\tilde{\Phi} \tilde{F}} = \tilde{F} \Omega^{-1}, \quad (2.5)$$

where the slashes indicate the RR poly form (sum of RR forms) contracted with gamma matrices to form a bispinor, i.e. $\tilde{F} = \Gamma^{123}$, and $\Omega$ is a matrix, the construction of which we describe in detail in the following section, given by in this case

$$\Omega^{-1} = \frac{1}{\sqrt{1 + r^2}}(-\Gamma^{123} + r \Gamma^r). \quad (2.6)$$

From this one ascertains that the dual geometry will contain a zero from and two form:

$$F_0 = 1, \quad F_2 = \frac{r^3}{1 + r^2} vol(S^2). \quad (2.7)$$

Notice that this would, when embedded into a true Type II supergravity background,
lead to a solution in massive type IIA (the $F_0$ is the Romans mass and comes when the $\Gamma_{123}$ in $\Omega$ annihilate the same factor in $F_3$). We shall see the same phenomenon happen in a number of the examples in this paper. The fact the type of the supergravity changed from IIB to IIA is due to the fact the isometry group dualised had an odd dimension (if it were to be even dimensional the type would have remained the same).

We now present details of how to technically compute the dualisation rules for the non-abelian duality.

### 2.2 Non-Abelian T-Duality; some nuts and bolts

We wish to consider backgrounds that support an $SU(2)$ isometry such that the metric can be cast as

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu L^i + g_{ij}(x)L^i L^j,$$

where $\mu = 1, 2, \ldots, 7$ and $L^i$ are the Maurer–Cartan forms. Our group theory conventions can be found in Appendix A. The NS sector comprises also the 2-form

$$B = B_{\mu\nu}(x) dx^\mu \wedge dx^\nu + B_{\mu i}(x) dx^\mu \wedge L^i + \frac{1}{2} b_{ij}(x) L^i \wedge L^j$$

and a dilaton

$$\Phi = \Phi(x).$$

Hence, all coordinate dependence on the $SU(2)$ Euler angles $\theta, \phi, \psi$ is contained in the Maurer–Cartan one-forms whilst the remaining data can all be dependent on the spectator fields $x^\mu$. Notice that we could have taken $b_{ij} = 0$, however, we will not do that since in the specific examples we will encounter it is necessary for a clear presentation of the various results.

In what follows it will be convenient to use a parametrisation of the frame fields given by

$$e^A = e_{\mu}^A dx^\mu, \quad e^a = \kappa^a_i L^i + \lambda^a_\mu dx^\mu,$$

To see that note that in (2.9) the relevant term becomes

$$B_{\mu i} dX^\mu \wedge L^i + \epsilon_{ijk} b_k L^i \wedge L^j = B_{\mu i} dX^\mu \wedge L^i + \sqrt{2} b_i dL^i$$

$$= (B_{\mu i} - \sqrt{2} \partial_{\mu} b_i) dX^\mu \wedge L^i + \sqrt{2} d(b_i L^i),$$

where in the second line we have performed a partial integration. Hence, the last term has no contribution to the field strength $dB$ and we may as well denote the first term by $B_{\mu i} dX^\mu \wedge L^i$. 

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where $A = 1, 2, \ldots, 7$ and $a = 1, 2, 3$. By demanding that

$$ds^2 = \eta_{AB} e^A e^B + e^a e^a ,$$

(2.13)

we obtain that

$$\lambda^a A^a = K_{\mu \nu} , \quad \eta_{AB} e^A e^B = G_{\mu \nu} - K_{\mu \nu} , \quad \kappa_i A^a = g_{ij} , \quad \kappa_i A^a = G_{\mu i} ,$$

(2.14)

where $\eta_{AB}$ is the seven-dimensional Minkowski metric. Note that $\kappa^a i$ and $\lambda^a \mu$ depend in general on the $x^\mu$'s.

### 2.2.1 The non-Abelian T-dual of the NS-sector

The corresponding Lagrangian density for the NS sector metric and antisymmetric fields is given by

$$\mathcal{L}_0 = Q_{\mu \nu} \partial_+ X^\mu \partial_- X^\nu + Q_{\mu i} \partial_+ X^\mu L^i_+ + Q_{i \mu} L^i_+ \partial_- X^\mu + E_{ij} L^i_+ L^j_+ ,$$

(2.15)

where, in accordance with (A.4), $L^i_\pm = -i \text{Tr}(t^i g^{-1} \partial_\pm g)$ and we have also defined

$$Q_{\mu \nu} = G_{\mu \nu} + B_{\mu \nu} , \quad Q_{\mu i} = G_{\mu i} + B_{\mu i} , \quad Q_{i \mu} = G_{i \mu} + B_{i \mu} , \quad E_{ij} = g_{ij} + b_{ij} .$$

(2.16)

To perform the non-Abelian T-duality we replace derivatives with covariant derivatives according to

$$\partial_\pm g \rightarrow D_\pm g = \partial_\pm g - A_\pm g ,$$

(2.17)

and add the Lagrange multiplier term

$$- i \text{Tr}(vF_\pm) , \quad F_\pm = \partial_+ A_+ - \partial_- A_+ - [A_+, A_-] .$$

(2.18)

Then the total action is invariant under

$$g \rightarrow h^{-1} g , \quad v \rightarrow h^{-1} vh , \quad A_\pm \rightarrow h^{-1} A_\pm h - h^{-1} \partial_\pm h ,$$

(2.19)

for a group element $h(\sigma^+, \sigma^-) \in SU(2)$. Under this transformation the fields $x^\mu$ stay inert and thus are called spectators. After some partial integrations the Lagrange multiplier term takes the form

$$\text{Tr}(i \partial_+ v A_- - i \partial_- v A_+ - A_+ f A_-) , \quad f_{ij} = f_{ij}^k v_k .$$

(2.20)
We now can integrate out the gauge fields to produce a dual theory that still depends on $\theta, \phi, \psi, v_i$ and the spectators. One must now gauge fix the $SU(2)$ isometry to remove three of these variables. The obvious way to proceed is to set

$$g = I,$$  \hspace{1cm} (2.21)

i.e. $\theta = \phi = \psi = 0$ in the notation of appendix A, which leaves an action in terms of just the $v_i$ and the spectators. There are other gauge fixing choices that may be more revealing by, for instance, making manifest some residual isometries. Different gauge fixing choices may be related, at least locally, through coordinate transformations as will demonstrate below in section 2.2.3. For the time being we proceed with the gauge fixing choice $g = I$. Integrating out the gauge fields gives

$$A^i_+ = iM^{-1}_{ji}(\partial_+ v_j + Q_{\mu j} \partial_+ X^\mu), \quad A^i_- = -iM^{-1}_{ij}(\partial_- v_j - Q_{\mu i} \partial_- X^\mu),$$  \hspace{1cm} (2.22)

where we have defined the matrix

$$M = E + f.$$  \hspace{1cm} (2.23)

Substituting back into the action gives the dual Lagrangian

$$\hat{\mathcal{L}} = Q_{\mu \nu} \partial_+ X^\mu \partial_+ X^\nu + (\partial_+ v_i + \partial_+ X^\mu Q_{\mu i}) M^{-1}_{ij}(\partial_- v_j - Q_{\mu j} \partial_- X^\mu).$$  \hspace{1cm} (2.24)

From this we read off the background fields of the NS-sector for the T-dual theory as

$$\hat{Q}_{\mu \nu} = Q_{\mu \nu} - Q_{\mu i} M^{-1}_{ji} Q_{\nu j}, \quad \hat{E}_{ij} = M^{-1}_{ij},$$

$$\hat{Q}_{\mu i} = Q_{\mu j} M^{-1}_{ji}, \quad \hat{Q}_{ij} = -M^{-1}_{ij} Q_{\mu i}.$$  \hspace{1cm} (2.25)

Additionally one finds that the dilaton receives a contribution at the quantum level just as in Abelian duality

$$\hat{\Phi}(x, v) = \Phi(x) - \frac{1}{2} \ln(\det M).$$  \hspace{1cm} (2.26)

It is clear from the above that the inverse of the matrix $M$ determines the dual geometry. Since we are working with $SU(2)$ isometries it is simple enough to evaluate this explicitly. In three-dimensions an antisymmetric matrix is dual to a vector, hence
we may write
\[ b_{ij} = \epsilon_{ijk} b_k . \] (2.27)
Rescaling also \( v_i \to v_i / \sqrt{2} \) we have to invert the matrix with elements
\[ M_{ij} = g_{ij} + \epsilon_{ijk} y_k , \quad y_i = b_i + v_i . \] (2.28)
To compute the inverse define an antisymmetric density and a vector as
\[ \tilde{\epsilon}_{ijk} = \sqrt{\det g} \epsilon_{ijk} , \quad z^i = \frac{y^i}{\sqrt{\det g}} = \frac{y^i}{\det \kappa} . \] (2.29)
In this way we may use the matrix \( g_{ij} \) to lower and raise indices in \( z^i \) since now the index has been transformed into a curved index. Then
\[ M_{ij} = g_{ij} + \tilde{\epsilon}_{ijk} z^k . \] (2.30)
Then the inverse is found to be
\[ (M^{-1})^{ij} = \frac{1}{1 + z^2} \left( g^{ij} + z^i z^j - \epsilon_{ijk} z^k \right) , \quad z^2 = z^i z^j g_{ij} = z^i z_i . \] (2.31)
Returning to our original variables
\[ (M^{-1})^{ij} = \frac{1}{\det g + y^2} \left( \det g g^{ij} + y^i y^j - \epsilon_{ijk} y_k \right) , \] (2.32)
where \( \bar{g}_i = g_{ij} y^j \) and \( y^2 = y^i y^j g_{ij} = \bar{g}_i \bar{g}_j g^{ij} \).

2.2.2 Computing the Lorentz transformation

By making use of (2.22) one can establish that the worldsheet derivatives transform under the non-Abelian T-duality as
\[ L^i_+ = - (M^{-1})_{ji} \left( \partial_+ v_j + Q_{j\mu} \partial_+ X^\mu \right) , \]
\[ L^i_- = M^{-1}_{ij} \left( \partial_- v_j - Q_{j\mu} \partial_- X^\mu \right) , \]
\[ \partial_\pm X^\mu \text{ is invariant} . \] (2.33)
These relations, in fact, provide a canonical transformation in phase space between pairs of T-dual sigma models [37, 40].
Crucial to us will be that by virtue of (2.33), left and right movers have different transformation rules and will define two different sets of frame fields. However, since these frame fields will describe the same geometry they must be related by a Lorentz transformation. Explicitly we find that the frames in (2.12) transforms, using the "plus" and the "minus" transformations (2.33), to the frames

\[ e \rightarrow \hat{e}^+ = -\kappa M^{-T}(dv + Q^T dX) + \lambda dX , \quad e \rightarrow \hat{e}^- = \kappa M^{-1}(dv - QdX) + \lambda dX , \]

(2.34)

where \((Q)_{i\mu} = Q_{i\mu}\). Writing

\[ \hat{e}^+ = \Lambda \hat{e}^- , \]

(2.35)

where \(\Lambda\) is the Lorentz transformation matrix to be computed. We find from equating the terms proportional to \(dv\) in (2.34) that

\[ \Lambda = -\kappa M^{-T}M^{-1} mass-1 = -\kappa^{-T}MM^{-T} \kappa^T . \]

(2.36)

The terms proportional to \(dX\) equate identically with no extra condition. To explicitly compute \(\Lambda\) note first that

\[ (\kappa^{-T}M\kappa^{-1})_{ab} = \delta_{ab} + \epsilon_{abc}\zeta^c , \]

(2.37)

where

\[ \zeta^a = \kappa^a iz^i , \]

(2.38)

is the flat index coordinate. Then

\[ (\kappa M^{-1} \kappa^T)^{ab} = \frac{1}{1 + \zeta^2} \left( \delta^{ab} + \zeta^a \zeta^b - \epsilon_{abc} \zeta^c \right) . \]

(2.39)

Then we compute that

\[ \Lambda^{ab} = \frac{\zeta^2 - 1}{\zeta^2 + 1} \delta_{ab} - \frac{2}{\zeta^2 + 1} (\zeta^a \zeta^b + \epsilon_{abc} \zeta^c ) , \]

(2.40)

where \(\zeta^2 = \zeta_a \zeta^a\). Note that this has exactly the same form as in [10] for the case of the PCM. Moreover, it is an \(O(3)\) rotation as it has \(\det \Lambda = -1\). The effect of the non-trivial extra couplings \(g_{ij}\) and \(b_{ij}\) is to "dress up" the original Lagrange multipliers and is hidden into the the definition of \(\zeta^a\).

This Lorentz transformation also induces an action on spinors given by a matrix \(\Omega\)
obtained by requiring that
\[ \Omega^{-1} \Gamma^a \Omega = \Lambda^a_{\ b} \Gamma^b. \] (2.41)

One finds that\(^2\)
\[ \Omega = \Gamma_{11} \frac{-\Gamma_{123} + \zeta_a \Gamma^a}{\sqrt{1 + \zeta^2}}, \] (2.42)
where \( \Gamma_{11} \) is the product of all ten Gamma matrices, that anticommutes with each one of them and for Minkowski spacetime, it squares to unity. Note also that \( \Omega \) leaves invariant the Gamma matrices \( \Gamma^A \) corresponding to the seven-dimensional spectator spacetime and it is of the same form as the corresponding matrix in [10].

### 2.2.3 General gauge fixing and coordinate transformations

As noted above, gauge choices different than (2.21) might be more convenient in certain applications. To expound this point let us consider a more general situation where the target admits an isometry group \( G \) and we dualise with respect to a subgroup \( H \) (for the case at hand we dualise the full \( SU(2) \) isometry and so \( \dim G = \dim H = 3 \)). Specific gauge choices among the original \( \dim G + \dim H \) variables are of the form
\[ f_i(g, v) = 0, \quad i = 1, 2, \ldots, \dim(H). \]
This leaves \( \dim G \) variables for the T-dual model. Nevertheless, one may show that the different gauge choices are related by coordinate transformations. This can be done by defining the "dressed" Lagrange multipliers as
\[ \hat{\vartheta}_i = D_{ji} v^j, \] (2.43)
where \( D_{ij} \) denotes the components of the matrix defined in (A.8), then the results we obtain for a general gauge fixing are given by just replacing in the previous expressions \( v_i \) by \( \hat{\vartheta}_i \). The details of the derivation are given in Appendix B. We also note that if the set of the \( v'_i \)'s is non-compact the same is true for the \( \hat{\vartheta}'_i \)'s since \( D \) is an orthogonal transformation.

In the present paper where the symmetry group is a freely acting \( SU(2) \), besides (2.21) these are two other natural choices for gauge fixing:

**Case 1:** One might choose to partially gauge fix the \( SU(2) \) group element, by setting two of the Euler angles \( \theta = \phi = 0 \), but leaving \( \psi \) as a variable in the dual. This choice is particularly motivated for backgrounds in which \( \partial_\psi \) is a Killing vector and corresponds, within the gauge/gravity correspondence, to the \( U(1)_R \) symmetry of the

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\(^2\)The general expression for \( \Omega \) for a freely acting group \( G \) can be found in [14].
\( \mathcal{N} = 1 \) supersymmetry in the field theory side. To fix the remaining gauge freedom one may fix one of the Lagrange multipliers \( v_2 = 0 \). The choice \( v_1 = 0 \) would work just as well, but \( v_3 = 0 \) is inadmissible since the fixing of \( g \) has already used up that particular gauge freedom. In this and similar manipulations one may use the transformations for the various variables given by (A.10) and (A.12). Then

\[
D = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
v = (v_1, 0, v_3) \implies \hat{v} = (\cos \psi v_1, \sin \psi v_1, v_3),
\]

(2.44)

\[
v = (0, v_2, v_3) \implies \hat{v} = (-\sin \psi v_2, \cos \psi v_2, v_3).
\]

Case 2: Another gauge choice is \( \phi = v_1 = v_2 = 0 \) leaving a dual depending on \( v_3, \theta, \psi \) as coordinates. In this case

\[
D = \begin{pmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
-\sin \psi & \cos \psi & 0 \\
\sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta
\end{pmatrix},
\]

\[
v = (0, 0, v_3) \implies \hat{v} = (\sin \theta \cos \psi v_3, \sin \theta \sin \psi v_3, \cos \theta v_3).
\]

(2.45)

For the above gauge choices the \( \hat{v}_i \)'s are obtained by a transformation that resembles the change of coordinates from Cartesian to either polar or spherical coordinates. However, this is only a formal analogy since in order to be precise one has to specify the range of variables. The above coordinate transformations imply that the dual backgrounds for different gauge choices are locally diffeomorphic and one must be rather careful about global properties. Indeed, the global properties of the dual coordinates should be established by demanding that the gauged and ungauged path integrals match. In Abelian T-duality one finds that the periodicity of the dual coordinates is determined by constraining the holonomies of the gauge connections to vanish. In the present non-Abelian case this is an open problem.

Finally we note that when we examine certain properties of the supergravity backgrounds produced by a non-Abelian duality we will need to perform certain integrations and thus will need the information about the global properties of the T-dual
coordinates. In doing so, we note the following relation
\[
e^{-2\Phi} \sqrt{|g|} \big|_{\text{original}} \times (\text{F.P.}) = e^{-2\Phi} \sqrt{|g|} \big|_{\text{final}} ,
\]
where the first factor in the left hand side is computed for the original background and for the specific gauge choice of the form \( f_a(g, \nu) = 0 \), \( a = 1, 2, 3 \), one has made. The second factor is the Faddeev–Popov determinant of the dim \( H \)-square matrix \( \Delta_{ij} \) arising in the variation \( \delta f_i = \Delta_{ij} \epsilon_j \) and in the specific gauge slice \( f_i = 0 \). Such a relation was first shown for gauged WZW models in [42], but it is valid the context of non-Abelian duality as well. This is not a surprise given their close relationship established in [38, 13].

2.2.4 Transformation of the RR flux fields

For the RR fields, the transformation rules were first realised for Abelian T-duality in [43] via the reduction and matching of type-IIA and type-IIB supergravities in nine dimensions. These rules were also obtained by considering how T-duality acts on spinors (or rather bispinors) and was detailed in [44, 45] from a space time perspective, in [46, 47] for the Green-Schwarz string and in [48, 49] for the pure spinor super string. In the democratic formalism [51] RR fields are combined with their Hodge duals to form a bispinor
\[
\text{IIB} : \ P = \frac{e^{\Phi}}{2} \sum_{n=0}^{4} \tilde{F}_{2n+1} , \quad \text{IIA} : \ \hat{P} = \frac{e^{\hat{\Phi}}}{2} \sum_{n=0}^{5} \hat{F}_{2n} ,
\]
where \( \tilde{F}_p = \frac{1}{p!} \Gamma_{\mu_1...\mu_p} \tilde{F}^{\mu_1\mu_2...m_p} \). The higher \( p \)-forms are related to the lower ones by
\[
F_p = (-1)^{\left[\frac{p}{2}\right]} \star F_{10-p} ,
\]
where our conventions for the Hodge dual are given by (C.1), assuming Minkowski signature spacetimes. The non-Abelian T-dual is simply obtained by multiplication with \( \Omega^{-1} \). If the transformation is from type-IIB to massive type-IIA the transformation rules for the RR-fluxes are given by comparing the two sides of [10]
\[
\hat{P} = P \cdot \Omega^{-1} .
\]
In the case of massive type-IIA to type-IIB the role of $P$ and $\hat{P}$ is interchanged. For a general ansatz one may read off the dual fluxes produced in this way and a systematic analysis is given in Appendix C.

2.2.5 A comment on singularities

One might wonder whether such a dualisation procedure can result in singular geometries starting with smooth geometries. To address this let us for a moment consider the case of abelian T-dualisation along a $U(1)$ isometry generated by a vector field, $\partial_{\theta}$, in adapted coordinates. The duality acts by inverting the component of the metric $g_{\theta\theta} \rightarrow \frac{1}{g_{\theta\theta}}$. It is clear that the dual may become singular at points for which $g_{\theta\theta}$ vanishes, in other words when the norm of the Killing vector about which we dualise vanishes. Indeed, this phenomenon occurs when dualising the polar angle of say $\mathbb{R}^2$ and in such cases non-perturbative effects would typically become important (an interesting related example of this in the context of mirror symmetry is found in [53] wherein the phase of chiral superfields are T-dualised and the dual superpotential receives vital instantonic corrections). More generally one anticipates singularities to be formed when the action of the isometry has fixed points.

In the examples considered in the remainder of the paper this is not the case; the norm of the Killing vectors can be seen to be nowhere vanishing and singularities are not created by the dualisation procedure.\footnote{However in our examples we will find some apparent singularities but these will be only coordinate in nature. In particular we will find bolt singularities that may be removed with an appropriate choice of ranges for dual coordinates.}

3 Dualisation of the Klebanov-Witten Background

The system of D3-branes placed at the tip of the conifold was studied within the AdS/CFT correspondence in [19]. In this section we work out the non-Abelian T-dual of this background and study various of its properties within the correspondence. The result of the dualisation and some of the of the properties of the background have first presented in [17].
3.1 The KW Background

The gauge theory on the branes for the background of [19] is an $N = 1$ superconformal field theory with product gauge group $SU(N) \times SU(N)$. There are two sets of bifundamental matter fields; $A_i$ in the $(N, \bar{N})$ forming a doublet of an $SU(2)$ global symmetry and $B^m$ in $(\bar{N}, N)$ forming a doublet of a second global $SU(2)$. The superpotential is given by

$$ W = \frac{\lambda}{2} \epsilon^{ij} \epsilon_{mn} \text{Tr}(A_i B^m A_j B^n) . \quad (3.1) $$

This gauge theory is dual to string theory on $AdS^5 \times T^{(1,1)}$ with $N$ units of RR flux on the $T^{(1,1)}$. The geometry and the 5-form self-dual flux form, are given by

$$ ds^2 = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2}{r^2} dr^2 + L^2 ds_{T^{(1,1)}}^2 , \quad F(5) = \frac{4}{g_s L} \left( \text{Vol}(AdS_5) - L^5 \text{Vol}(T^{(1,1)}) \right) . \quad (3.2) $$

Here $T^{(1,1)}$ is the homogenous space $(SU(2) \times SU(2))/U(1)$ with the diagonal embedding of the $U(1)$. It has an Einstein metric with $R_{ij} = 4g_{ij}$ given by [52, 55]

$$ ds_{T^{(1,1)}}^2 = \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{6} (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 . \quad (3.3) $$

Introducing the frame fields for the $S^2$

$$ \sigma_1 = \sin \theta_1 d\phi_1 , \quad \sigma_2 = d\theta_1 \quad (3.4) $$

and the invariant Maurer–Cartan forms for $S^3$, that up to an overall normalization factor coincide with (A.7),

$$ \sigma_1 = \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 , \quad \sigma_2 = \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 , \quad \sigma_3 = d\psi + \cos \theta_2 d\phi_2 , \quad (3.5) $$

allows one to recast the $T^{1,1}$ metric as

$$ ds_{T^{1,1}}^2 = \lambda_1^2 (\sigma_1^2 + \sigma_2^2) + \lambda_2^2 (\sigma_1^2 + \sigma_2^2) + \lambda^2 (\sigma_3 + \cos \theta_1 d\phi_1)^2 , \quad (3.6) $$

with $\lambda_1^2 = \lambda_2^2 = \frac{1}{6}$ and $\lambda^2 = \frac{1}{9}$. The $SU(2) \times SU(2) \times U(1)$ isometries of this metric correspond to Killing vectors. In particular, there are two commuting sets as in (A.11)
with \((\theta, \phi, \psi) \rightarrow (\theta_i, \phi_i, \psi)\), with \(i = 1, 2\). These can be labeled as \(k_a\) and \(k_{a+3}\), \(a = 1, 2, 3\). There is also \(k_0 = \partial_\psi\) corresponding to the \(U(1)_R\) symmetry in the dual field theory.

In the various computations below we will use the following frame

\[
e^\mu' = \frac{r}{L} dx^\mu, \quad \mu = 0, 1, 2, 3, \quad e^4 = \frac{L}{r} dr, \\
e^{\hat{1},\hat{2}} = \lambda_1 \sigma_{\hat{1},\hat{2}}, \quad e^{1,2} = \lambda_2 \sigma_{1,2}, \quad e^3 = \lambda (\sigma_3 + \cos \theta_1 d\phi_1).
\]

(3.7)

### 3.2 Action of \(SU(2)\) on Killing Spinors

The KW background has eight unbroken super symmetries, four of which correspond to Poicaré super symmetries and the other four corresponding to superconformal symmetries of the dual field theory. The Killing spinor equation coming from the gravitino variation in the \(T^{(1,1)}\) directions reduces to

\[
D_\mu \eta + \frac{i}{2L} \Gamma^{12\hat{1}\hat{2}} \Gamma_\mu \eta = 0, \quad \mu = 1, 2, \hat{1}, \hat{2}, \hat{3}.
\]

(3.8)

This is solved by a constant \(\eta\) obeying the projectors\(^4\)

\[
\Gamma_{12} \eta = i \eta, \quad \Gamma_{\hat{1}\hat{2}} \eta = -i \eta.
\]

(3.10)

We will dualise with respect one of the \(SU(2)\) isometries. It is natural to ask what portion of supersymmetry is preserved by this and what is the behaviour of the Killing spinors under the \(SU(2)\) action. In [10] (and further developed in [15]) it was shown that the criteria for whether Supersymmetry is preserved is provided by the spinor Lorentz-Lie/Kosmann derivative [56, 57, 58]. For a killing vector \(k\) this derivative is well defined and is given by

\[
\mathcal{L}_k \eta = k^\mu D_\mu \eta + \frac{1}{4} \nabla_\mu k^\nu \Gamma^{\mu\nu} \eta.
\]

(3.11)

\(^4\)Acting on the column vector \(\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix}\) where \(\eta = \eta_+ + i \eta_-\) we have the projection

\[
\Gamma_{12} \eta = - (i \sigma_2) \eta, \quad \Gamma_{45} \eta = + (i \sigma_2) \eta.
\]

(3.9)
Inserting the form of the Killing vectors we see that

\[ \mathcal{L}_{k(4)} \eta = 0 , \quad (3.12) \]

\[ \mathcal{L}_{k(5)} \eta = -\frac{1}{4} \sin(\phi_1) \csc(\theta_1) \left( \Gamma_{12} + \Gamma_{\hat{1}\hat{2}} \right) \eta , \quad (3.13) \]

\[ \mathcal{L}_{k(6)} \eta = \frac{1}{4} \cos(\phi_1) \csc(\theta_1) \left( \Gamma_{12} + \Gamma_{\hat{1}\hat{2}} \right) \eta . \quad (3.14) \]

Thus, one sees that the Killing spinor has vanishing Kosman derivative along the $SU(2)$. This corresponds to the statement that in the dual field theory the supersymmetry is not charged under the $SU(2)$ flavour symmetries. Hence we anticipate that supersymmetry is preserved after performing a T-duality along this $SU(2)$. Moreover we anticipate that the Killing spinor in the dual will have the form

\[ \hat{\eta} = \Omega \cdot \eta \quad (3.15) \]

where $\Omega$ is the spinorial representation of the Lorentz transformation between the left and right moving frames for the dual geometry.

Parallel to this discussion is the fact that the $U(1)_R$ symmetry commutes with the $SU(2)$ and hence one expects the corresponding isometry to be preserved after dualisation.

### 3.3 Dualisation of the NS sector and the Lorentz transformation

We will dualise with respect to the $SU(2)$ isometry group generated by \{k\textsuperscript{(4)}, k\textsuperscript{(5)}, k\textsuperscript{(6)}\} following the procedure outlined in section 2. Since the metric is block diagonal in the $AdS_5 \times T_{1,1}$ spacetime, it is sufficient to focus our attention on the $T^{(1,1)}$ factor alone. The $AdS_5$ just comes along for the ride as a spectator field\footnote{In what follows we have set $L = 1$; this can be restored by rescaling $\lambda_i$ and $\lambda$ by a factor of $L$ and by dividing the RR fields by $L$.} Our gauge choice will be given by the first of the choices in (2.44), i.e. $v_2 = 0$. Moreover we relabel $v_1 = 2x_1$ and $v_3 = 2x_2$.

Within $T_{1,1}$, two of the fields, i.e. $x^\mu = (\theta_1, \phi_1)$ are spectators. The various matrices we have introduced in (2.12) and directly enter in our expressions for the T-dual
background, using the frame \((3.7)\) take the form

\[
(\kappa)^a_i = \sqrt{2} \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\lambda)^a_\mu = \lambda \cos \theta_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

(3.16)

where \(\mu = 1, 2\) corresponds to \(\theta_1\) and \(\phi_1\), respectively.

The result of this procedure is a \(\sigma\)-model on a target space with NS fields given by

\[
d^{\hat{s}}^2 = ds^2_{AdS_5} + \lambda_1^2 (\sigma_1^2 + \sigma_2^2) + \lambda_2^2 \lambda^2 \Delta^{-1} x_1^2 \sigma_3^2
\]

\[
+ \frac{1}{\Delta} \left( (x_1^2 + \lambda_2^2 \sigma_3^2) dx_1^2 + (x_2^2 + \lambda_2^4) dx_2^2 + 2x_1 x_2 dx_1 dx_2 \right),
\]

\[
\tilde{B} = -\frac{\lambda^2}{\Delta} \left[ x_1 x_2 dx_1 + (x_2^2 + \lambda_2^2) dx_2 \right] \wedge \sigma_3,
\]

\[
e^{-2\hat{\Phi}} = \frac{8}{g^2} \Delta,
\]

(3.17)

where \(\sigma_3 = d\psi + \cos \theta_1 d\phi_1\) and

\[
\Delta \equiv \lambda_2^2 x_1^2 + \lambda^2 (x_2^2 + \lambda_2^4) .
\]

(3.18)

The metric, besides the symmetries of the \(AdS_5\) factor, evidently has a \(SU(2) \times U(1)_\psi\) isometry and for a fixed value of \((x_1, x_2)\) the remaining directions give a squashed three sphere. Although the geometry is regular and the dilaton never blows up we note that there is a removable bolt singularity (for a standard review see [54]). For small values of \(x_1\) and fixed \(x_2\) the metric on the internal space behaves as

\[
\lambda_1^2 (\sigma_1^2 + \sigma_2^2) + \frac{\lambda_2^2}{x_2^2 + \lambda_2^4} \left( dx_1^2 + x_1^2 \sigma_3^2 \right).
\]

(3.19)

For this to be removed we require the range of \(\psi\) to be \(2\pi\) (so that at fixed \(\theta, \phi\) the apparent singularity at \(x_1 = 0\) becomes just the coordinate singularity of \(\mathbb{R}^2\) written in polar coordinates)[6] Rather curiously, before dualisation the coordinate \(\psi\) had range \(4\pi\) (it was the coordinate of the fibre in \(T^{1,1}\) viewed as a \(U(1)\) bundle over \(S^2 \times S^2\)) so we see that the dualisation has effectively enforced a \(Z_2\) quotient on \(\psi\). This is illustrative of the point made earlier that the global properties of these geometries after dualisation may be rather subtle.

\[\text{This assumes that } x_1 \text{ takes values in the half-line, if it is allowed to range over the full real line the range of } \psi \text{ should be further restricted to } \pi.\]
From (2.34) one can obtain expressions for the "internal" frame fields $\hat{e}_{i\pm}$ for this metric and the corresponding Lorentz transformation relating the frames is given by (2.40). To explicitly compute that we take into account (2.38), (3.16) and the transformation (2.44), so that

$$\xi^a = \frac{1}{\lambda_2^2 \lambda} (\lambda_2 \cos \psi x_1, \lambda_2 \sin \psi x_1, \lambda x_2).$$

(3.20)

From this one can deduce the spinorial representation $\Omega$ of this Lorentz transformation by using (2.42). Explicitly, we find dependence on the gamma matrices $\Gamma_1$ and $\Gamma_2$ of the form $\cos \psi \Gamma_1 + \sin \psi \Gamma_2$. This will be used to obtain the RR fluxes as detailed below.

In fact, there is a more convenient and simpler choice for the frame fields which can be obtained by performing an additional rotation in the 1–2 plane

$$\left( \begin{array}{c} \hat{e}'_{1\pm} \\ \hat{e}'_{2\pm} \end{array} \right) = \left( \begin{array}{cc} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{array} \right) \left( \begin{array}{c} \hat{e}_{1\pm} \\ \hat{e}_{2\pm} \end{array} \right).$$

(3.21)

Using the above and the relation between the world sheet derivatives as in (B.13) the frame fields are given by

$$\hat{e}'_{1\pm} = \pm \frac{\lambda_2}{\Delta} \left[ (x_1^2 + \lambda^2 x_2^2) dx_1 + x_1 x_2 (dx_2 \pm \lambda^2 \sigma_3) \right].$$

$$\hat{e}'_{2\pm} = \frac{\lambda_2}{\Delta} \left[ \lambda^2 x_2 dx_1 - \lambda^2 x_1 (dx_2 \pm \lambda^2 \sigma_3) \right],$$

(3.22)

$$\hat{e}_{3\pm} = \pm \frac{\lambda}{\Delta} \left[ x_1 x_2 dx_1 + (x_2^2 + \lambda^2_2) dx_2 \mp \lambda^2_2 x_2^2 \sigma_3 \right].$$

The Lorentz transformation $\Lambda$ relating the $\hat{e}'_{i\pm}$ frames is given by (2.40). For these rotated frames we have that $\Lambda' = D \Lambda D^T$. One finds that

$$\Lambda' = -I + \frac{2}{\Delta} \left( \begin{array}{cccc} \lambda^2 x_2^2 & -\lambda^2 x_2 \lambda_2 x_1 & -\lambda_2 \lambda x_1 x_2 \\ \lambda^2 \lambda_2^2 x_2 & \lambda^2 x_2^2 + \lambda^2 \lambda_2^2 x_1^2 & -\lambda_2^3 \lambda x_1 \\ -\lambda_2 \lambda x_1 x_2 & \lambda^3 \lambda x_1 & -\lambda_2^2 x_1^2 \end{array} \right).$$

(3.23)

### 3.4 RR Field Transformation

We may now use the rule for transforming the RR sector given in (2.49) in which the spinorial representation of the Lorentz transform acts on the right of the bispinor. A change for both frames induces a corresponding change in the bispinor with an extra
matrix $\Omega_{fr}$ which acts from the left as well as from the right. It reads

$$\hat{P} = \Omega_{fr} \cdot P \cdot \Omega^{-1} \cdot \Omega_{fr}^{-1} = P' \cdot \Omega'^{-1},$$

(3.24)

where

$$\Omega' = \Omega_{fr} \cdot \Omega \cdot \Omega_{fr}^{-1}, \quad P' = \Omega_{fr} \cdot P \cdot \Omega_{fr}^{-1}. \quad (3.25)$$

In our case using (2.41) with $\Lambda$ the two-dimensional $SO(2)$ matrix in (3.21) we easily find that

$$\Omega_{fr} = \cos \frac{\psi}{2} \mathbb{I} + \cos \frac{\psi}{2} \Gamma_{12}. \quad (3.26)$$

Then in $\Omega'$ the dependence on $\psi$ disappears and we have that

$$\Omega' = \frac{\Gamma_{11}}{\sqrt{\Delta}} \left( -\lambda_2^2 \lambda_1 \Gamma_{123} + \lambda_2 x_1 \Gamma_1 + \lambda x_2 \Gamma_3 \right), \quad (3.27)$$

which precisely corresponds to the Lorentz matrix (3.23). On the other hand $P' = P$ since the dependence on the frames $e^1$ and $e^2$ is of the form $e^1 \wedge e^2$. This amounts to the "naive" use of (2.49) without taking into account the effect of the rotation of the bispinor $P$, with just renaming the combination $\cos \phi \Gamma_1 + \sin \psi \Gamma_2$ by $\Gamma_1$. Using the general formulae of Appendix C, in particular (C.4), we see that the only non-vanishing form is $G_2^{(3)} = -\frac{4\lambda_2^2}{g_s} \sigma_1 \wedge \sigma_2$. Then using eqs.(C.11) and (C.12) we compute the forms supporting the type-IIA supergravity background as

$$\hat{F}_2 = \frac{8\sqrt{2}}{g_s} \lambda_1^2 \lambda \sigma_1 \wedge \sigma_2, \quad \hat{F}_4 = -\frac{8\sqrt{2}}{g_s} \lambda_1 \lambda_2^2 \lambda_2 x_1 \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge (\lambda_2^2 x_1 d x_2 - \lambda_2^2 x_2 d x_1). \quad (3.28)$$

These fluxes support the NS geometry as a solution of the supergravity equations of motion. Moreover the solution has $\mathcal{N} = 1$ supersymmetry as explained in detail in the Appendix E. Using eqs.(C.11) and (C.12) we also compute the higher forms

$$\hat{F}_6 = \frac{8\sqrt{2}}{g_s} \text{Vol}(AdS_5) \wedge (\lambda_2 x_1 \hat{e}^1 + \lambda x_2 \hat{e}^3)$$

\footnote{We use thought this paper the $e^a$ frame to display the results.}
\[
= -\frac{8\sqrt{2}}{g_s} \text{Vol(AdS}_5) \wedge (x_1 dx_1 + x_2 dx_2),
\]
\[
\hat{F}_8 = \frac{8\sqrt{2}}{g_s} \lambda_1^2 \lambda_2 \text{Vol(AdS}_5) \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 \tag{3.29}
\]
\[
= \frac{8\sqrt{2}}{g_s} \lambda_1^2 \lambda_2 \frac{x_1}{\Delta} dx_1 \wedge dx_2 \wedge \sigma_3.
\]

These obey \(\hat{F}_6 = -(\star \hat{F}_4)\) and \(\hat{F}_8 = \star \hat{F}_2\), in agreement with (2.48), as they should. In fact, we prove in Appendix C, that these consistency relations relating higher to lower rank RR flux forms are preserved by non-Abelian T-duality.

### 3.5 M-theory lift

We are interested in lifting and interpreting our solution to M-theory. To do that we first read off, using (D.5), the potentials corresponding to the fluxes \(\hat{F}_2\) and \(\hat{F}_4\) in (3.28).\(^8\) We find that

\[
C_1 = \frac{1}{27} \sigma_3, \quad C_3 = \frac{x_2}{27} \sigma_1 \wedge \sigma_2 \wedge \sigma_3. \tag{3.30}
\]

The lift to eleven dimensions (along the circle with coordinate \(x_\#\)) of the geometry we find after non-Abelian T-duality is given by,

\[
ds^2 = \Delta^{1/3} \left( ds^2_{\text{AdS}_5} + \lambda_1^2 \sigma_1^2 + \sigma_2^2 \right) + \Delta^{-2/3} \left[ (x_1^2 + \lambda_2^3) dx_1^2 \\
+ (x_2^3 + \lambda_4^3) dx_2^2 + 2x_1 x_2 dx_1 dx_2 + \lambda_2^3 x_1^2 \sigma_2^2 + (dx_2 + \sigma_3/27)^2 \right], \tag{3.31}
\]

where \(\Delta\) is given in (3.18). The four-form flux field is given by

\[
F_4 = d(C_3 + B \wedge dx_\#) = \frac{1}{27} dx_2 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + H \wedge dx_\#, \tag{3.32}
\]

where \(H = dB\) is computed using the expression for \(B\) in (3.17).

In eleven dimensions our solution preserves two commuting \(U(1)\) isometries and eight supercharges. Recently, a class of \(\mathcal{N} = 1\) (generically non-Lagrangian) SCFT’s, arising as the IR fixed point of the dynamics of M5-branes wrapped on a genus \(g\) surface \(\Sigma_g\), was engineered in [26, 27]. These field theories enjoy not only a \(U(1)_R\) global symmetry but also an additional \(U(1)\) global symmetry. The geometrical dual to these

\(^8\)In the rest of this section and in the next section we set besides \(L = 1\), the string coupling \(g_s = 2\sqrt{2}\) and \(\lambda_1 = \lambda_2 = \frac{1}{\sqrt{6}}, \lambda = \frac{1}{3}\), even if we keep the notation in some of the eqs.
theories fit within the ansatz of [32], but now specialised to the case that the internal six-dimensional manifold is a squashed \( S^4 \) fibration over \( \Sigma_g \). Our solution fits in this ansatz as we will soon demonstrate after a brief review of the solutions of [26, 27].

### 3.5.1 Brief review of the solution in [26, 27]

To give the metric and four-form flux in [26, 27], we first introduce the constant curvature metric on the Riemann surface \( \Sigma_g \)

\[
ds_2^2(\Sigma_g) = e^{2A(y_1,y_2)} (dy_1^2 + dy_2^2),
\]

with scalar curvature \( 2\kappa \), where \( \kappa = 1, 0, -1 \) for the two-sphere \( (g = 0 \) for which we will be particularly interested), for the torus and a hyperbolic surface, respectively. Hence the conformal factor obeys Liouville’s equation

\[
(\partial_{y_1}^2 + \partial_{y_2}^2)A + \kappa e^{2A} = 0 .
\]

It is also convenient to define the constants

\[
a_1 = \frac{2}{\kappa} - \frac{2g}{\kappa} e^{2\nu}, \quad a_2 = -12 \frac{2}{\kappa} \frac{2g}{\kappa} e^{2\nu} \left( 1 - \frac{\kappa}{6} e^{-2\nu} \right),
\]

\[
e^{2\nu} = \frac{1}{6} \left( -\kappa \pm \sqrt{\kappa^2 + 3\kappa^2 z^2} \right),
\]

the functions

\[
k(q) = \frac{a_2}{a_1} q^2 + q - \frac{1}{36}, \quad f(y) = f_0 + 6 \frac{a_2}{a_1} y^2, \quad e^{6\lambda(y,q)} = q f(y) + 4y^2,
\]

as well as the one forms \( V \) and \( \rho \)

\[
dV = \frac{\kappa}{2 - 2g} e^{2A(y_1,y_2)} dy_1 \wedge dy_2, \quad \int_{\Sigma_g} dV = 2\pi,
\]

\[
\rho = (2 - 2g)V - \frac{1}{2} \left( a_2 + \frac{a_1}{2q} \right) (d\chi + V).
\]

Then the metric takes the form

\[
ds_{11}^2 = e^{2\lambda(y,q)} \left[ ds_{AdS_5}^2 + e^{2\nu} ds_2^2(\Sigma_g) \right] + e^{-4\lambda(y,q)} ds_{M_4}^2,
\]
where $ds^2_{AdS_5}$ is the metric of $AdS_5$ with unit radius and

$$
\begin{align*}
\mathcal{M}_4^2 &= \left( 1 + \frac{4y^2}{qf(y)} \right) dy^2 + \frac{f(y)q}{k(q)} \left( dq + \frac{12yk(q)}{qf(y)} q \right)^2 \\
&\quad + \frac{a_f^2}{4} \frac{f(y)k(q)}{q} (d\chi + V)^2 + \frac{qf(y)}{9} (d\psi + \rho)^2 ,
\end{align*}
$$

(3.39)
in which the $U(1)$ isometry generated by $\partial_\psi$ corresponds to the R-symmetry and is supplemented by an additional $U(1)$ generated by $\partial_\chi$. Demanding that the metric is positive definite places bounds on the coordinates $y$ and $q$ by imposing

$$
qf(y) \geq 0 \,, \quad k(q) \geq 0 .
$$

(3.40)

To support the geometry of the ansatz (3.38) a flux is turned one with structure

$$
F_4 = e^{2A(y_1,y_2)+2\nu} dy_1 \wedge dy_2 \wedge G_2 + V \wedge G_3 + G_\perp .
$$

(3.41)

The various functions are given by

$$
\begin{align*}
G_2 &= \frac{(24q-1)(4y^2 + qf(y)) - 144y^2k(q)}{18q(4y^2 + qf(y))} dy \wedge d\psi \\
&\quad - \frac{(24q-1)(a_1 + 2a_2q)[4y^2 + qf(y)] - 36k(q) \left[ 4 (a_1 + 2(6a_1 + a_2)q) y^2 + a_1qf(y) \right]}{72q^2(4y^2 + qf(y))} dy \wedge d\chi \\
&\quad - \frac{1}{3} \frac{yf(y)}{4y^2 + qf(y)} dq \wedge [2d\psi - (6a_1 + a_2)d\chi]
\end{align*}
$$

(3.42)

and

$$
\begin{align*}
G_3 &= - \frac{a_1qf(y)}{36} \frac{96y^2 + f(y)(1 + 12q)}{(4y^2 + qf(y))^2} dq \wedge dy \wedge [2(g - 1)d\chi + d\psi] , \\
G_\perp &= - \frac{a_1qf(y)}{36} \frac{96y^2 + f(y)(1 + 12q)}{(4y^2 + qf(y))^2} dq \wedge dy \wedge d\chi \wedge d\psi .
\end{align*}
$$

(3.43)

\footnote{Some terms in the expression below seem to be missing in the corresponding expressions of the original literature.}
3.5.2 Explicit change of coordinates

Let’s consider the case where \( g = 0 \), i.e. \( \Sigma_{g=0} = S^2 \) and also take the numbers \( \kappa = z = 1 \). Then the various constants are given by

\[
e^{2\nu} = \frac{1}{6}, \quad a_1 = \frac{2}{3}, \quad a_2 = 0.
\] (3.44)

The remaining functions take the form

\[
f(y) = f_0, \quad k(q) = q - \frac{1}{36}, \quad e^{6\lambda(y,q)} = f_0 q + 4y^2.
\] (3.45)

The one-forms that determine the fibration are given by

\[
\rho = 2V - \frac{1}{6q}(d\chi + V), \quad dV = \frac{1}{2} e^{2A(y_1,y_2)} dy_1 \wedge dy_2.
\] (3.46)

The conformal factor for the metric on \( S^2 \) is given by

\[
e^{A(y_1,y_2)} = \frac{2}{1 + y_1^2 + y_2^2},
\] (3.47)

and the conversion to polar coordinates is given by

\[
y_1 = \cos \phi_1 \cot \frac{\theta_1}{2}, \quad y_2 = \sin \phi_1 \cot \frac{\theta_1}{2},
\] (3.48)

whence

\[
V = \frac{1}{2} \cos \theta_1 d\phi_1.
\] (3.49)

Then the metric (3.38) can be written exactly in the form (3.31) if we make the coordinate change

\[
q = \frac{1}{36} + \frac{3}{2} x_1^2, \quad y = \frac{x_2}{6}, \quad x_2 = \frac{1}{9} \chi - \frac{1}{18} \psi, \quad f_0 = \frac{1}{9}.
\] (3.50)

From the positivity requirement of the metric (and these coordinate redefinitions) we expect \textit{a priori} that \( q \in \left[ \frac{1}{36}, \infty \right] \) and \( y \in [-\infty, \infty] \). There may be further conditions from quantisation of fluxes.

For the forms in eqs.(3.42) and (3.43) we find that,

\[
G_{\perp} = -\frac{1}{54} \frac{(1 + 12q + 864y^2)}{(q + 36y^2)^2} dq \wedge dy \wedge d\chi \wedge d\psi,
\]
Then, with the above coordinate change the four-form flux \((3.41)\) becomes \((3.32)\).

### 3.6 Brief comments on the field theory

Next we present a very quick synopsis the work of [26] which interprets these geometries as coming from wrapped M5-branes. Before back-reaction one considers the M5-brane on a genus \(g\) curve \(C_g\) in a Calabi Yau threefold that is decomposable as

\[
\mathbb{C}^2 \rightarrow L_1 \oplus L_2, \\
\downarrow \quad C_g
\]

where \(L_i\) are line bundles, so that the space has a natural \(U(1)_1 \times U(1)_2\) symmetry acting as phase rotations in the respective line bundle. The Calabi–Yau condition restricts the Chern classes of the line bundles, \(c_1(L_1) = p, c_2(L_2) = q\), so that they cancel-off against the curvature coming from the curve \(C_g\) i.e. one requires \(p + q = 2g - 2\). In what follows it is helpful to encode the solution by the genus \(g\) and the twisting parameter \(z\) defined through

\[
p = (1 + z)(g - 1), \quad q = (1 - z)(g - 1).
\]

The parameter \(z\) is related to the warp factor \(e^\nu\) entering into the supergravity solutions above as in \((3.35)\). In our case recall that \(\kappa = z = 1\), which gives \((3.44)\). The construction in [26], allows one to calculate the four-dimensional central charges \(c, a\) using the anomaly polynomial of the six dimensional \(\mathcal{N} = (2, 0)\) theory (here specialised to the \(A_N\) case)

\[
a = \frac{(g - 1)(N - 1)\xi^3 + k\eta^3 - \kappa(1 + \eta)(9 + 21\eta + 9\eta^2)z^2}{48(1 + \eta)^2z^2},
\]

\[
c = \frac{(g - 1)(N - 1)\xi^3 + k\eta^3 - \kappa(1 + \eta)(6 - k\zeta + 17\eta + 9\eta^2)z^2}{48(1 + \eta)^2z^2},
\]

(3.54)
where \( \eta = N^2 + 1 \) and \( \zeta = (\eta^2 + (1 + 4\eta + 3\eta^2)z^2)^{1/2} \). Generically, in the large \( N \) limit the leading behaviour of these charges match and scale as \( N^3 \) reflecting the six dimensional M5-brane origin of the theories.

The corresponding field theories for \( g > 1 \) were constructed in [26], starting from the theories discussed by Gaiotto and Maldacena [11], integrating out the adjoint chiral multiplet inside the \( N = 2 \) vector multiplets and following a simple set of rules. However this construction does not seem to extend easily for genus zero and the field theory interpretation remains somewhat mysterious in this case. A surprising result is that for the particular geometry that we obtained by dualising KW we have \( g = 0, z^2 = 1 \). Then the anomaly polynomial result for central charges in (3.54) in the large \( N \) limit gives,

\[
a = \frac{3}{8}(N - 1) + \mathcal{O}\left(\frac{1}{N}\right), \quad c = \frac{1}{4}(N - 1) + \mathcal{O}\left(\frac{1}{N}\right).
\]  

(3.55)

The leading \( N^3 \) term cancelled out leaving just a linear dependence on \( N \). Since the leading term vanished one need not have that \( a \) and \( c \) match. A strange feature, however, is that the result has no \( N^2 \) dependent piece as one might have expected. As we shall see later this does not match the expectation from supergravity where we find the central charge is an invariant of the T-dualisation. The resolution of the puzzle probably lies on the fact that a-maximization, used in [26] is breaking-down in this case due to the presence of extra accidental symmetries.

These features point to the fact that the theory obtained when M5-branes wrap a two-sphere preserving conformality and minimal SUSY are apparently out-of-line with the general ideas of [11], [29] and [26].

On the other hand, the existence of a BPS operator corresponding to a wrapped \( M2 \)-brane seems guaranteed here. We will see below how to calculate its dimension. In line with the differences pointed above, we will also find qualitative differences regarding this operator and that defined in [11], [29] and [26].

### 3.6.1 An operator associated with M2-branes

Gaiotto and Maldacena [11] discussed the existence of an operator associated with the \( T_N \) theories. They called this operator \( O_{ijk} \) and its characteristics are described in [11]. It is uncharged under the \( U(1) \) symmetries of the field theory. The dimension of this
operator in the string dual is equated with the energy of an M2-brane extending in
time and wraps a two-cycle (in their case an \( H2 \)-space) which is not fibered with
the rest of the space. In other words, the calculation in the putative type-IIA would give a
D2-brane without any worldvolume gauge field.

We can proceed in analogy and define the dimension of an operator by the volume of
an M2-brane that extends in time and wraps a two-cycle. The main subtle difference is
that in our case, we will not be able to have this operator uncharged. When we place
our M2-brane on the manifold, utilizing our M-theory geometry \( (3.31) \), \( M_3 = [t, \theta_1, \phi_1] \)
at constant values of \( x_1, x_2, \psi, x_{11} \) and for the radial coordinate \( r = r_0 \), we can then
calculate the induced metric on this M2-brane to be

\[
ds^2_{\text{ind,M2}} = \Delta^{1/3} \left[ -r_0^2 dt^2 + \frac{1}{6} (d \theta_1^2 + \sin^2 \theta_1 d \phi_1^2) + \frac{f(x_1, x_2)}{6} \cos^2 \theta_1 d \phi_1^2 \right],
\]

where \( \Delta \) is given by \( (3.18) \). We see here that we will have non-zero \( U(1) \) charge,
as there seems to be no way of getting rid of the fibration represented by the term
\( \hat{c}_3 = d \psi + \cos \theta_1 d \phi_1 \). The energy of this M2-brane is given in terms of the complete
elliptic function of the second kind \( E \) as

\[
E = r_0 \Delta^{1/2} \frac{\pi}{6} \int_0^\pi d \theta_1 \sqrt{f(x_1, x_2) \cos^2 \theta_1 + \sin^2 \theta_1}
= r_0 \Delta^{1/2} \frac{\pi}{6} \left[ E \left( \sqrt{1 - f} \right) + \sqrt{f} \ E \left( \sqrt{1 - f^{-1}} \right) \right].
\]

After lengthy algebraic manipulations one can show that the minimum of the energy
as a function of \( x_1 \) and \( x_2 \) is at \( x_1 = x_2 = 0 \) and has the arithmetic value \( E_{\text{min}} \simeq 0.039r_0 \)
(note our choice for \( g_s \) in footnote 8).

In a very similar way, we can calculate the central charge at this conformal point,
by defining it as proportional to the volume of the internal six-manifold. We will
study the central charge in more detail in Section \( \text{4.2.4} \). Since the results there can be
specialised to the constant value obtained at the fixed point, we will postpone this
study until Section \( \text{4.2.4} \).
4 Dualisation of the Klebanov-Tseytlin Background

Let us now turn our attention to non-conformal backgrounds. One can start with the KW solution and break conformal invariance by adding $M$ fractional D3-branes i.e. D5-branes wrapping a contractible two cycle of $T^{(1,1)}$ as in $[59, 20, 21]$. This modifies the field theory to be $SU(N) \times SU(N + M)$, hence no longer conformal. In fact this theory has a rich RG dynamics undergoing a sequence of Seiberg dualities to lower rank gauge groups as one proceeds to the IR (see $[60]$ for a careful summary). In the IR, strong coupling dynamics takes hold giving rise to spontaneous $Z_{2M}$-symmetry breaking and confinement.

For the time being let us concentrate not on the full solution of Klebanov and Strassler $[21]$ but rather on the simpler case of Klebanov and Tseytlin (KT) that appeared earlier $[20]$. This background gives a good description of the UV of the duality cascade, but is singular in the IR (where the strong dynamics cures this pathology, replacing the singular conifold with the smooth deformed conifold).

One of the purposes of this section then is to develop our ‘intuitions’ on the effect of the non-Abelian duality on cascading geometries. All of our results will be trustable away from the singularity, placed at the origin of the radial coordinate (that will be labelled by $u$ below). The philosophy adopted here, will be that the generated background ‘defines’ a dual QFT, that we will start to understand with the calculations proposed in this section.

The geometry is given $[20]$ by$^{10}$

$$ds_{10}^2 = e^{-\frac{2}{3}(B+4C)} ds_5^2 + ds_5', \quad (4.1)$$

where

$$ds_5^2 = du^2 + e^{2A} dx_{1,3}^2, \quad (4.2)$$

is a deformation of $AdS_5$ and

$$ds_5'^2 = \frac{e^{2C}}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin \theta_i^2 d\phi_i^2) + \frac{e^{2B}}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2. \quad (4.3)$$

$^{10}$The dilaton is zero so that there is no difference between string and Einstein frame.
It is convenient to define the natural forms

\[ e^{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i, \quad e^{\phi_i} = \frac{1}{\sqrt{6}} \sin \theta_i d\phi_i, \quad i = 1, 2, \]

\[ e^\psi = \frac{1}{3} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2), \quad \]

\[ \omega_2 = \frac{1}{\sqrt{2}} (e^{\theta_1} \wedge e^{\phi_1} - e^{\theta_2} \wedge e^{\phi_2}). \]

Note that the above one-forms are not the frames defining the metric since they are missing the appropriate exponentials. The fluxes are given by

\[ B_2 = -T \omega_2, \quad F_3 = -P e^\psi \wedge \omega_2, \]

\[ F_5 = K e^\psi \wedge e^{\theta_1} \wedge e^{\phi_1} \wedge e^{\theta_2} \wedge e^{\phi_2} - K e^{4A - \frac{8}{3}(B + 4C)} du \wedge dx_{1,3} \]

and the dilaton is, as explained, vanishing. Since it is needed for the dualisation below, we also give, using (2.48), the expression for

\[ F_7 = -(\star F_3) = -P e^{4A + 4f - 4q} du \wedge dx_{1,3} \wedge \omega_2. \]

The functions \( A, B, C, K, T \) depend only on the radial direction \( u \), whereas \( P \) is a constant. Introducing the functions

\[ f = -\frac{1}{5} (B - C), \quad q = \frac{2}{15} (B + 4C). \]

one finds that the BPS conditions lead to

\[ K = Q - PT, \quad A = q + \frac{2}{5} f - \frac{1}{P} T, \]

as well as to a set of first order non-linear system of equations

\[ \frac{dT}{du} = -P e^{4f - 4q}, \]

\[ \frac{df}{du} = -\frac{3}{5} e^{4f - 4q} \left( 1 - e^{-10f} \right), \]

\[ \frac{dq}{du} = \frac{2}{15} e^{-4q + 4f} \left( 3 + 2e^{-10f} \right) - \frac{1}{6} (Q - PT) e^{-10q}. \]

One may check explicitly that the flux equations and Bianchi identities are satisfied

\[ \text{11 there is a sign convention choice } P_{\text{here}} = -P \text{ in } [20]. \]
on these BPS equations. The dilaton equation is also satisfied by virtue of the identity 
\[ H_3^2 = F_3^2. \]

We recall the special logarithmically running solution of KT. This is constructed by 
setting the function \( f = 0 \), which is consistent with the system (4.9) and changing 
variables as 
\[ e^{3q} = r^2 h^{1/2}, \quad du = e^{4q} \frac{dr}{r}. \]  
(4.10)

One finds the metric 
\[ ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2} \left( dr^2 + r^2 ds_{T_{1,1}}^2 \right), \quad h = b_0 + \frac{p^2}{4 r^4} \ln(r/r_\ast), \]  
\[ T = \tilde{T} - P \ln(r/r_\ast), \quad K = P^2 \ln(r/r_\ast) - \frac{p^2}{4}, \]  
(4.11)

where \( \tilde{T} \), \( r_\ast \), and \( b_0 \) are integration constants. The latter should be set to zero in order 
to decouple the dual QFT from gravity. Then clearly, the gravitational description 
brakes down at \( r \sim r_\ast \). However, already at a larger radius at which the function 
\( K(r) \) vanishes the gravitational force has changed sign. This indicates that one needs a 
non-singular completion of this solution towards the IR, which was achieved in [21]. 
In what follows we shall keep the solution completely general and given in terms of 
the functions entering into the BPS equations. Only final results may use the explicit 
expression in (4.11).

### 4.1 Dualisation of the background

We proceed now to indicate the result of performing a non-Abelian T-duality on the 
geometry (4.1)-(4.5). We will use the general results of Section 2 and Appendix C. Our 
gauge choice will be given by the second of the choices in (2.44), i.e. \( v_1 = 0 \). We 
note that the result of this dualisation for the special solution (4.11) have been already 
presented in [17].

We implement the dualisation with the same gauge fixing as before using (2.44). The 
matrices \( \kappa \) and \( \lambda \) are the same as in (3.16), but 
\[ \zeta^a = \left( -3 \sqrt{3} e^{-B-C} \sin \psi v_2, 3 \sqrt{3} e^{-B-C} \sin \psi v_2, \frac{1}{\sqrt{2}} e^{-2C} \mathcal{V} \right), \quad \mathcal{V} = 6 v_3 - T. \]  
(4.12)
The procedure leads us to define a set of frame fields

\[ \hat{e}^{\mu} = e^{A - \frac{1}{2}(B+4C)} dx^\mu, \quad \mu = 0, 1, 2, 3, \quad \hat{e}^4 = e^{-\frac{1}{2}(B+4C)} du, \]
\[ \hat{e}^\theta = \frac{e^C}{\sqrt{6}} d\theta_1, \quad \hat{e}^\phi = \frac{e^C}{\sqrt{6}} \sin \theta_1 d\phi_1, \]

(4.13)

We also have that

\[ \hat{e}'_1^\pm = -\sqrt{\frac{6}{81W}} \left( e^{2B+C} \sqrt{6} d \theta_1 + e^{3C} \sqrt{6} d \phi_1 \right) \],
\[ \hat{e}'_2^\pm = \mp \sqrt{\frac{3}{81W}} e^C \left( 2(e^{2B+2C} + 27v_2^2) dv_2 + \sqrt{6} e^{2B} \sigma_3 \right) \],
\[ \hat{e}'_3^\pm = \mp \frac{1}{54W} e^B \left( \sqrt{2} (2e^{4C} + \sqrt{6} d \phi_2 + 6v_2 (\sqrt{2} e^{2B} + 2e^{2C} v_2 \sigma_3) \right) \].

(4.14)

where we have made a frame rotation as in (3.21). In the above we have defined the function

\[ W = \frac{1}{81} \left( 2e^{2B+4C} + 54 e^{2C} v_2^2 + e^{2B} \right), \]

(4.15)

such that the dual metric is given by

\[ d\hat{s}^2 = e^{-\frac{3}{2}(B+4C)} ds_5^2 + \frac{e^{2C}}{6} \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right) + \sum_{a=1}^{3} \hat{e}_\pm^a \hat{e}_\pm^a, \]
\[ (\hat{e}'_1^\pm)^2 + (\hat{e}'_2^\pm)^2 + (\hat{e}'_3^\pm)^2 = \frac{1}{54W} \left( 4(e^{2B+2C} + 27v_2^2) dv_2^2 + 3(2e^{4C} + \sqrt{6} d\phi_2 + 6v_2 (\sqrt{2} e^{2B} + 2e^{2C} v_2 \sigma_3) \right) \].

(4.16)

We can either choose the \( e_+ \) or the \( e_- \) triads. In the following we will prefer \( e_+ \) in accordance with footnote 7. Note that there is also a removable bolt singularity at \( v_2 = 0 \) provided that the range of \( \psi \) is restricted to \( 2\pi \) or \( \pi \) depending on whether \( v_2 \) takes values in the half or entire real line, respectively, i.e. footnote 6. This metric has, besides the obvious Poicaré symmetry of the \( dx_{1,3}^2 \) factor, an \( SU(2) \times U(1)_\psi \) isometry as in the case of the T-dualised Klebanov–Witten background.

There is also a NS antisymmetric tensor given by

\[ \hat{B}_2 = -\frac{T \sin \theta_1}{6\sqrt{2}} d\theta_1 \wedge d\phi_1 + \frac{e^B}{3\sqrt{6}v_2} \left( -\sqrt{2} e^C \hat{e}^1 \wedge \hat{e}^3 + e^{-C} \sqrt{6} \hat{e}^2 \wedge \hat{e}^3 \right) \]

(4.17)
and a dilaton
\[ e^{-2\Phi} = \mathcal{W}. \] (4.18)

The above background does not get any more singular than the original one. For instance the dilaton in (4.18) never blows up. However, it still has any singular behaviour inherited from the original background, e.g. when (4.11) is used.

The Lorentz rotation is given by
\[ \Lambda' = -1 + \frac{1}{81\mathcal{W}} \begin{pmatrix} 2(54e^{2C}v_2^2 + e^{2B}v^2) & -2\sqrt{2}e^{2B+2C}v & 12\sqrt{3}e^{B+3C}v_2 \\ 2\sqrt{2}e^{2B+2C}v & 2e^{2B}v^2 & -6\sqrt{6}e^{B+C}v_2 \\ -12\sqrt{3}e^{B+3C}v_2 & -6\sqrt{6}e^{B+C}v_2 & 108e^{2C}v_2^2 \end{pmatrix}. \] (4.19)

The spinorial counter part of this rotation is
\[ \Omega = \frac{1}{9\sqrt{\mathcal{W}}} \Gamma_{11} \left( -\sqrt{2}e^{B+2C}\Gamma_{123} + 3\sqrt{6}e^Cv_2\Gamma_2 + e^B\mathcal{V}\Gamma_3 \right). \] (4.20)

Using the diagonal combinations \( f, g \) defined in (4.7), the fluxes are (the Ramond fields are obtained by the right-action of \( \Omega \) in (4.20) on the flux bi-spinor),
\[
\begin{align*}
\hat{F}_0 & = \frac{P}{9}, \\
\hat{F}_2 & = \frac{e^{-3q-2f}}{9\sqrt{2}} \left( (2K - PV)e^{\theta_1} \wedge \bar{e}^{\theta_1} - PV\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_2} + 3\sqrt{6}Pe^{5f}v_2\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_3} \right), \\
\hat{F}_4 & = \frac{e^{-6q-4f}}{9} e^{\theta_1} \wedge \bar{e}^{\theta_1} \wedge \left[ - (e^{6q+4f}P + KV)e^{\theta_1} \wedge \bar{e}^{\theta_2} \\
& \quad + 3\sqrt{6}e^{5f}v_2(\sqrt{2}K\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_3} + e^{2f+3d}\bar{e}^{\theta_2} \wedge \bar{e}^{\theta_3}) \right]. \end{align*}
\] (4.21)

Provided the BPS differential conditions (4.8) and (4.9) are satisfied, these fluxes obey the Bianchi identities and ensure that the Einstein equations are obeyed. Notice that the mass \( F_0 \) is quantised naturally by \( P \) which measured the number of fractional branes prior to dualisation.

We also obtain the higher forms
\[
\begin{align*}
\hat{F}_6 & = \frac{e^{-6q-4f}}{9} \text{Vol(AdS}_5) \wedge \left( 3\sqrt{6}e^{5f}Kv_2\bar{e}^{\theta_2} - 3\sqrt{6}e^{3q+7f}Pv_2\bar{e}^{\theta_1} + (e^{6q+4f}P + KV)e^{\theta_3} \right), \\
\hat{F}_8 & = \frac{e^{-3q-2f}}{9\sqrt{2}} \text{Vol(AdS}_5) \wedge \left( -3\sqrt{6}e^{5f}Pv_2\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_1} \wedge \bar{e}^{\theta_2} - PV\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_2} \wedge \bar{e}^{\theta_3}, \\
& \quad +(2K - PV)\bar{e}^{\theta_1} \wedge \bar{e}^{\theta_2} \wedge \bar{e}^{\theta_3} \right). \end{align*}
\] (4.22)
\[ \hat{F}_{10} = -\frac{P}{9} \text{Vol}_{10}, \]

which turn out to be the related to the lower ones as dictated by (2.48). Not surprisingly, the whole structure is very similar to what we have already seen in the Klebanov–Witten case. In that respect we mention that the dualisation has not introduced any new singularities to the background in addition to those that might be initially present, e.g. for the solution (4.11). Indeed, notice that the function \( \mathcal{W} \) is nowhere vanishing and therefore the string coupling is clearly bounded.

Also useful for us will be the (rather pleasingly simple) expressions for the RR potentials which are found using (D.5). They read

\[
\begin{align*}
C_1 &= -\frac{1}{27\sqrt{2}} (Q - 3Pv_3) \sigma_3, \\
C_3 &= -\frac{1}{324} \left( (6Qv_3 + TQ - 3PTv_3) - 9P(v_2^2 + v_3^2) \right) \sigma_1 \wedge \sigma_2 \wedge \sigma_3. 
\end{align*}
\] (4.23)

### 4.2 Probing the dual geometry

In order to learn lessons about the new configuration described in (4.16)-(4.21), we will perform some calculations with its geometry and fluxes. In the Klebanov–Tseytlin case these calculation provide an understanding of how field theory features are encoded in the background. The goal here will be to gain a similar understanding of the behaviour of a dual field theory using the geometry and fluxes as a way of defining it.

We will observe that various quantities, when calculated in the transformed background present qualitatively similar (or the same) behavior as in the original KT-case. One may think then that one is capturing the result of a correlation function that is "uncharged" under the group of transformations used by the non-abelian T-duality. Ideas of this sort were used in the context of other background-generating techniques. See for example [61] and [4].

We will start by defining a couple of "geometric" cycles.

#### 4.2.1 Two and Three-Cycles

In the original KT geometry (4.1)-(4.5), there is a two-cycle defined by

\[
\begin{align*}
\theta_1 &= \theta_2, \\
\phi_1 &= 2\pi - \phi_2, \\
\psi &= \psi_0.
\end{align*}
\] (4.24)
Notice that the definition above is such that the $U(1)$-fibration coming from the term

$$d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2,$$

vanishes. We use this criterion (absence of fibration) to identify a two cycle in the T-dualised geometry. Let us consider the sub-manifold defined by

$$\Sigma_2 = [\theta_1, \phi_1], \quad v_2 = v_3 = \psi = 0. \quad (4.25)$$

We can check that the fibration term that appears in the vielbein $\hat{e}_3$ vanishes, together with any contribution coming from $\hat{e}_1$ and $\hat{e}_2$. Hence, after the T-duality, $\Sigma_2$ is a well defined two manifold (actually we only need for that $v_2 = 0$ and $v_3, \psi = \text{const.}$). The two-cycle of (4.25), will be used below.

Let us now define a three-cycle in the geometry. Consider the submanifold

$$\Sigma_3 = [\theta_1, \phi_1, \psi], \quad v_2, v_3, u = \text{const.} \quad (4.26)$$

The three vielbeins $\hat{e}_+,^i$, when projected to this submanifold read,

$$\hat{e}_+^1 = \frac{\sqrt{12} v_2 e^{2C+2B}}{81 W} \sigma_3, \quad \hat{e}_+^2 = -\frac{\sqrt{6} v_2 e^{2B+C}}{81 W} \sigma_3, \quad \hat{e}_+^3 = \frac{2 v_2^2 e^{B+2C}}{9 W} \sigma_3.$$

The induced metric and the NS antisymmetric tensor on the three cycle are

$$d\hat{s}_3^2 = \frac{e^{2C}}{6} \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right) + \frac{2 e^{2B+2C} v_2^2}{27 W} (d\psi + \cos \theta_1 d\phi_1)^2,$$

$$\hat{B}_2|_{\Sigma_3} = -\frac{T}{6} \sin \theta_1 d\theta_1 \wedge d\phi_1, \quad (4.27)$$

while the expression of the dilaton is given in [4.18]

We propose that the cycles above will play an important role in the study of this geometry and some of its field theoretical aspects. To make this claim more solid, we will study possible field theory observables, computed by brane probes that partially wrap the cycles above. We will start with domain walls.

### 4.2.2 Domain Walls

All the material in this section should be taken with an important caveat: domain walls are characteristic effects of the IR dynamics, while here we have a singular ge-
ometry in IR (the singular behaviour inherited from KT). We will derive expressions that in principle, should be evaluated at the origin of the radial coordinate, which would lead to ugly divergences. But that is not something of concern as the calculation should be performed in the backgrounds we obtain once we consider the non-singular geometries—see Section 5. The point we want to understand what object should be identified with a domain wall after the duality. We will observe that in the simplified KT-case of this section, it is easy to identify the probe whose interpretation in the dual QFT will be that of a domain wall.

A domain wall in the KT-background (4.1)-(4.5) is defined as a D5-brane extending along the manifold

$$\Sigma_6 = [R^{1,2}, \theta_2, \phi_2, \psi] .$$  \hspace{1cm} (4.28)

We calculate the corresponding Born–Infeld action for this D5-brane and obtain

$$ds^2_{\text{ind},D5} = e^{2A-\frac{2(B+4C)}{3}}dx_{1,2}^2 + e^{2C} \left( \sigma_1^2 + \sigma_2^2 \right) + \frac{e^{2B}}{9} \sigma_3^2 , \quad B_{\theta_2\phi_2} = \frac{T}{6\sqrt{2}} \sin \theta_2 ,$$

$$S_{\text{BI}} = -T_{\text{eff}} \int d^{2+1}x ,$$  \hspace{1cm} (4.29)

where the effective tension is

$$T_{\text{eff}} = T_{D5} \left. \frac{8\pi^2 e^{3A-4C}}{9\sqrt{2}} \sqrt{2e^{4C} + T^2} \right|_{u_0} .$$  \hspace{1cm} (4.30)

In the computation we kept $\theta_1, \phi_1, u = u_0$ as well as the extra Poincaré coordinate $x_3$ fixed.

After the non-Abelian T-duality transformation we need to specify what object will represent a domain wall. We may reason as follows: we started with a D5-brane and non-Abelian T-dualize in the three directions of the brane $(\theta_2, \phi_2, \psi)$, so we end up with a D2-brane. We then propose that the domain wall in the new geometry is represented by a D2-brane extended in $R^{1,2}$ and placed at a particular point in the internal space. We will set the rest of the coordinates

$$v_2 = v_3 = \theta_1 = \phi_1 = \psi = 0$$  \hspace{1cm} (4.31)

and keep the extra Poincaré coordinate $x_3$ and the holographic coordinate $u$ fixed. Let us calculate the induced metric and BI action for this putative probe D2-brane. For
this we will need that

$$\mathcal{W}|_{v_2=v_3=\psi=\phi_1=\theta_1=0} = \frac{e^{2B(2e^4C + T^2)}}{81}$$  \hspace{1cm} (4.32)$$

and to recall (4.18). We then calculate the corresponding BI action

$$ds_{\text{ind,D2}}^2 = e^{2A-\frac{2B+8C}{3}}dx_{1,2}^2, \quad S_{D2} = -T_{\text{eff}} \int d^{2+1}x,$$  \hspace{1cm} (4.33)$$

with

$$T_{\text{eff}} = T_{D2} \frac{e^{3A-4C}(2e^{4C} + T^2)^{1/2}}{9}.$$  \hspace{1cm} (4.34)$$

The two effective tensions in (4.30) and (4.34) are in agreement up to a constant factor! This suggests that when we deal with the whole non-singular KS/baryonic branch background–after transformed under non-Abelian T-duality, see Section 5– the D2-brane we studied here will be an actual domain wall in the QFT dual to the transformed non-Abelian T-dual geometry.

Another quantity that in a less subtle way will behave similarly (will be "uncharged" under the non-Abelian duality) is the Wilson loop. Let us comment briefly upon that.

### 4.2.3 Asymptotics of $R^{1,3} \times R$ and Wilson loops.

Let us study the asymptotic behavior of the Minkowski and radial part of the space by specializing the internal coordinates as in (4.31). Then the metric of the five-dimensional space i.e. (4.1)-(4.2) reads,

$$e^{-\frac{2}{3}(B+4C)}(du^2 + e^{2A}dx_{1,3}^2).$$  \hspace{1cm} (4.35)$$

There is no mixing term coming from the "internal" manifold (the $\hat{B}$ field does not induced a new term in the $g_{uu}$ component of the metric). This implies that asymptotically, the space will behave like the Klebanov–Tseylin one and that a simple rectangular Wilson loop [62], calculated as a string on the configuration

$$x = \sigma, \quad t = \tau, \quad u = u(\sigma),$$  \hspace{1cm} (4.36)$$

will proceed along the same lines as before the T-duality transformation is performed (for general formulas see [63, 64]). We could think that this particular Wilson loop is ‘uncharged’ under the duality. Hence, the short distance behavior of the quark-
antiquark potential will be the same in the transformed dual field theory, exhibiting a Coulombic behaviour with a logarithmical running charge.

Let us analyse the radial behavior of the dilaton. Using (4.31), the transformed dilaton (which—we remind—is constant in the KT case) goes like,

\[ e^{-2\Phi} = \mathcal{W} = \frac{e^{2\mathcal{B}(2e^{2\mathcal{C}} + T^2)}}{81} \sim \left[ \ln \left( \frac{r}{r_\ast} \right) \right]^{5/2}, \]

where we have indicated the behaviour for the specific solution (4.11) (with \(b_0 = 0\)) and similarly in the following two expressions.

Let us finally propose an object that calculates the 't Hooft line (and the potential between two magnetic monopoles). In the KT geometry, this is usually identified with the 'Wilson loop' for an effective string associated with a D3-brane probe that extends on \(R^{1,1}\) and wraps the two-cycle in (4.24). The presence of the NS B-field must be taken into account. We get an effective tension given by

\[ T_{\text{eff}} = \frac{4\pi}{3\sqrt{2}} T_{D3} e^{2A - \frac{2}{3}(B + 4C)} \sqrt{2e^{4C} + T^2} \sim r^2 \left[ \ln \left( \frac{r}{r_\ast} \right) \right]^{1/2}. \] (4.37)

The last behaviour is valid for large values of the radial coordinate, so our previous comment on domain walls applies here too. (we should actually be calculating these quantities in backgrounds without IR singularity, like those in Section 5).

After the non-Abelian T-duality, we propose that the same observable is computed by extending a D4-brane on \(R^{1,1}\) and wrapping it over the three cycle \(\Sigma_3\) in (4.26). The effective tension in this case is given by

\[ T_{\text{eff}} = \frac{2\pi^2 v^2}{9\sqrt{3}} T_{D4} e^{2A - \frac{2}{3}(B + 4C)} \sqrt{2e^{4C} + T^2} e^{B + C} \sim r^2 \ln \left( \frac{r}{r_\ast} \right), \] (4.38)

where, as discussed below (4.16), we have used that the range of the angle \(\psi\) is \([0, \pi]\) to avoid a bolt-singularity. In summary, for the 't Hooft loop (defined this way), we are not obtaining exactly the same functional form. It remains to study the functional form and values in the IR-smooth case, which we will do in Section 6.

We will now study two other quantities that when computed before and after the non-Abelian duality show the same qualitative behaviour in a quite interesting way.
To assess the central charge we follow the procedure explained in [30]. If it is implemented in full generality one should reduce to a one-dimensional action depending on the unknown functions entering into the solution. In turn this should then be recasted as a five-dimensional gauged supergravity from which the central charge function may be determined. Fortunately [31] spares us of this technically challenging reduction by giving some general results that after a small modification are applicable here as well. For a ten-dimensional metric of the form
\[ ds^2 = \alpha \, dx_{1,3}^2 + \alpha \beta \, du^2 + g_{ij} \, d\Theta^i d\Theta^j \]
and calling
\[ V_{\text{int}} = \int d\Theta \, e^{-2\Phi} \sqrt{g_{ij}}, \]
we can define the functions
\[ H = V_{\text{int}}^2 \, \alpha^3, \quad \kappa = H^{1/3}, \]
such that the Einstein frame five dimensional metric is
\[ ds_5^2 = \kappa (dx_{1,3}^2 + \beta \, du^2) \]
and this in turn, implies a central charge
\[ c = 27 \beta^{3/2} \frac{H^{7/2}}{(H')^3}. \]
Let us test this formula above for our KT-metric in (4.1). We have
\[ \alpha = e^{2A - 2/3(B+4C)}, \quad \beta = e^{-2A}, \quad V_{\text{int}}^2 = \frac{(4\pi)^6 e^{8C+2B}}{36^2 \times 9}, \]
\[ c = \frac{2\pi^3}{27A^{15}} \sim \frac{1}{A^{15}}. \quad (4.39) \]
Now, let us analyze things after the non-Abelian T-duality. The functions are
\[ \alpha = e^{2A - 2/3(B+4C)}, \quad \beta = e^{-2A}. \]
After a short calculation one finds that

\[ V_{\text{int}} = 6\sqrt{2}\pi^2 e^{\beta+4C} \int dv_3 \int dv_2 e^{-2\Phi} \frac{v_2}{81} \mathcal{W}, \]

\[ H = 24^2 \left( \frac{\pi^4}{2} \right) e^{6A} \left[ \int dv_3 \int dv_2 \frac{v_2}{81} \right]^2 \]

and this will in turn produce a central charge

\[ c \sim \left[ \int dv_3 \int dv_2 \frac{v_2}{81} \right] \frac{1}{A^3}. \]  

(4.40)

At this point one can immediately see that the central charges before and after duality match up to a single RG scale independent coefficient (the integral appearing in the result above). In fact this is not a coincidence. As discussed in (2.46), the "measure" \( e^{-2\Phi} \sqrt{g} \) is an invariant of the duality up to a factor arising from the Fadeev–Popov determinant. Indeed the integral in (4.40) can be understood as the space time integral of the FP determinant and is completely determined by the global properties of the dual coordinates. This being so, follows the invariant of the central charge in the manner displayed above.

In exactly the same way the central charge was analysed, we can study the Entanglement Entropy [65]. Following equations (7)-(10) of [31], we see that the integral defining the entangled entropy (for the case of the Klebanov–Tseytlin background) is

\[ \frac{S}{V_2} = \frac{1}{4G_{10}} \frac{(4\pi)^3 e^{-2\Phi_0}}{108} \int_{-l/2}^{l/2} e^{3A} \sqrt{1 + e^{-2A}(\partial_x u)^2} \, dx. \]  

(4.41)

While if we compute things after the non-Abelian T-duality, for the same reasons as explained around (2.46) we will have a nice cancellation of the transformed dilaton and the involved combination we called \( \mathcal{W} \). We will then obtain, for the entanglement entropy after the non-abelian T-duality,

\[ \frac{S}{V_2} = \frac{24\pi^2}{16G_{10}} \left[ \int dv_3 \int dv_2 \frac{v_2}{81} \right] \int_{-l/2}^{l/2} e^{3A} \sqrt{1 + e^{-2A}(\partial_x u)^2} \, dx. \]  

(4.42)

We see again, that like with many other quantities described above the non-Abelian T-duality preserves the dynamical content of the central charge and the entanglement entropy. These will behave equally in the original Klebanov-Tseytlin cascading theory and in its non-abelian T-dual field theory.
Let us close this sub-section giving another argument explaining why the central charge (and similarly the entanglement entropy) should be invariant under the non-abelian transformation. In the case of the flow between $AdS_5 \times S^5 / \mathbb{Z}_2$ and $AdS_5 \times T^{1,1}$ the corresponding central charges obey

$$\frac{c_{IR}}{c_{UV}} = \frac{27}{32} . \quad (4.43)$$

After non-abelian T-duality, the corresponding "UV" geometry will just be some $\mathbb{Z}_2$ quotient of the Gaiotto-Maldacena type geometry found in [10] and the "IR" geometry the result provide in the preceding section (the Klebanov-Witten transformed). One then can say that the ratio of $c_{IR}/c_{UV}$ in (4.43) will also be preserved. Indeed exactly this result was obtained directly in [26]. It is satisfying to see a different interpretation for it.

We will now focus on a quantity that will display a qualitatively different behavior before and after the non-abelian T duality. This is probably linked to the changing behavior of the dilaton (respect to the constant value in KT) obtained in (4.21). We will move on to study a possible definition of the gauge coupling in the dual QFT of the background of eqs. (4.16)-(4.21).

### 4.2.5 Definition of a 4-d gauge coupling

We want to give a possible definition of the gauge coupling of the field theory dual to the geometry of (4.16). We begin by reviewing how things work in the KT-background.

In the case of the Klebanov-Tseytlin background, one defines two gauge couplings in terms of the quantity $b_0$, 

$$b_0 = \frac{1}{4\pi^2} \int_{S^2} B_2 ,$$

$$\frac{8\pi^2}{\mathcal{S}_1^2} = \pi e^{-\Phi} [1 + b_0] , \quad \frac{8\pi^2}{\mathcal{S}_2^2} = \pi e^{-\Phi} [1 - b_0] ,$$

$$\frac{4\pi^2}{\mathcal{S}_+^2} = \pi e^{-\Phi} = \pi , \quad \frac{4\pi^2}{\mathcal{S}_-^2} = \pi e^{-\Phi} b_0 ,$$

where we have defined also the diagonal combinations $\mathcal{S}_\pm^2$ in the usual way. These definitions arise when considering String Theory on the conifold (actually for strings on $AdS_5 \times S^5 / \mathbb{Z}_2$). In the context of the KT-background, there are more practical ways.
of getting the information encoded by the coupling $g^2$:

- To consider the Action of an instanton $e^{-S_{\text{inst}}} = e^{-\frac{8\pi^2}{g^2} + i\Theta}$ and equate it with the Action of an euclidean D1 brane wrapping the two-cycle of (4.24). The presence of the background $B$-field needs to be considered in the BIWZ Action.

- To consider a probe D5 brane that wraps the two cycle in (4.24) and that contains a gauge field in the Minkowski part of its worldvolume. It is again crucial to take into account the effect of the $B_2$-field with nonzero projection on the two-cycle and worldvolume of the brane.

In summary, the calculation gives (in both cases described above),

$$\frac{1}{g^2} \sim 2\pi T_{D5} h(r)^{1/2} e^{2C} \sqrt{1 + \frac{B_{\theta_2\phi_2}^2}{h(r)e^{4C}}} \sim \ln r . \quad (4.44)$$

The high energy/large radius logarithmic behavior is the expected one in a four dimensional QFT.

After the non-abelian T-duality have acted upon the KT-background, we would like to define the gauge coupling in the dual QFT. We find again two possible definitions (that as above will agree): one in terms of a D2 brane that wrapping our three-cycle of (4.26) will behave as an instanton, the other in terms of a D6 (with a gauge field in its Minkowski directions) that wraps the same three-cycle. In broad lines the calculations go as explained below. The first definition considers an instanton and uses that $e^{-S_{\text{inst}}} \sim e^{-\frac{8\pi^2}{g^2} + i\Theta}$. An instanton in our background is given by an euclidean D2 brane that wraps the three cycle described above. Its action will be

$$S_{D2} = -T_{D2} \int_{\Sigma_3} e^{-\hat{\Phi}} \sqrt{\det[g_{ab} + B_{ab}]} + T_{D2} \int_{\Sigma_3} C_3 . \quad (4.45)$$

So, we can associate

$$\frac{1}{g^2} \sim -T_{D2} \int_{\Sigma_3} e^{-\hat{\Phi}} \sqrt{\det[g_{ab} + B_{ab}]}, \quad \Theta \sim \int C_3 . \quad (4.46)$$

The induced metric and the $B_2$ field that are relevant for this calculation are

$$ds_3^2 = \frac{e^{2C}}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{2e^{2B+2C}v^2}{27W} (d\psi + \cos \theta_1 d\phi_1)^2 ,$$

$$B_2 = -\frac{T(u)}{6\sqrt{2}} \sin \theta_1 d\theta_1 \wedge d\phi_1 . \quad (4.47)$$
So, we can calculate
\[ \frac{1}{g^2} \sim T_{D2} \frac{\pi^2 e^{B+C} v^2}{9\sqrt{3}} \sqrt{2e^{4C} + T^2}. \] (4.48)

Notice that the units are correct. The tension of the D2 together with the volume of the three cycle cancel to give something dimensionless. Similarly, with the C3 field associated with \( \hat{F}_4 \)—see (4.23), we can calculate the anomaly associated with the changes in the theta-angle.

*The definition in terms of D6's:*

Suppose that we consider a D6 brane wrapping the three cycle \( \Sigma_3 \) extended along \( R^{1,3} \times (\theta_1, \phi_1, \psi) \). We will also turn on a gauge field on the Minkowski part of the D6 \(^{12}\). We will now calculate
\[
S_{D6} = -T_{D6} \int_M e^{-\Phi} \sqrt{\det g_{ab} - (2\pi\alpha') F_{ab} + B_{ab} + T_{D6} \int_{R^{1,3}} \Sigma_3 C_7 + C_3 \wedge F_2 \wedge F_2}. 
\] (4.49)

If the three-cycle is calibrated, then in an expansion for small values of \( \alpha' \), the first term coming from the square root and the one coming from the WZ term proportional to \( C_7 \) should cancel. It would be interesting to check if this is the case. The second term in the expansion should give the gauge coupling and the anomaly,
\[
\Theta \sim \int_{\Sigma_3} C_3.
\]

Focusing on the gauge coupling, the induced metric and action are
\[
\det [g_{ab} + B_{ab} + 2\pi\alpha' F_{ab}] = \delta^4_{xx} \left( 1 - \delta^{tt} g^{xx} F_{tx}^2 \right) \frac{e^{2B+2C} v^2}{972} \sin^2 \theta_1 [2e^{4C} + T^2].
\]

Now we can expand for small values of \( \alpha' \) or small electromagnetic fields to get
\[
S_{D6} = \frac{1}{g^2} \int d^{3+1} x F_{tx}^2, \tag{4.50}
\]
where the prefactor of the Maxwell term is identified with the gauge coupling which is found to be
\[
\frac{1}{g^2} \sim T_{D6} (4\pi^2) (\pi\alpha')^2 \frac{e^{B+C} v^2}{9\sqrt{3}} \sqrt{2e^{4C} + T(u)^2}. \tag{4.51}
\]

Notice that aside from the constant-factors we have the same expression as in (4.48)—using a different initial definition. As with the previous definition the units are

\(^{12}\)It is enough to consider just the \( F_{tx} \) component to see the argument working.
correct. To close this section, let us observe that if we fix $v_2$ to a constant and we use the leading asymptotics, we will have a behavior for the gauge coupling

$$\frac{1}{g^2} \sim (\ln r)^{3/2}. \quad (4.52)$$

This rather strange scaling perhaps suggests that the field theory might be different from a conventional field theory. Comparing with (4.44), it is clear that the new gauge coupling is not behaving as a typical coupling in a 4d-theory. Another possibility is that the quantity we have defined is not related to the four dimensional gauge coupling of the QFT.

In hand with the non-conventional beta-function derived from the above behaviour, we can study anomalies, associated with changes in the $\Theta$-angle. Indeed, with our definition of the gauge coupling, naturally comes a definition for the $\Theta$ angle, as we stressed in (4.46). To calculate the integral of the RR-three form on the three-cycle defined in (4.26), we use the expression in (4.23). We will focus in the case $Q = 0$—well motivated, as this is the case in the smooth geometries that IR-complete the KT-background—and we will also choose $v_3 = 0$. The result of the $\Theta$ angle is

$$\Theta \sim \int_{\Sigma_3} C_3 = \frac{\pi^2}{9} P v_2^2. \quad (4.53)$$

Hence, changes of $\Theta$ in integer multiples of $2\pi n$ imply a periodicity in the $v_2$ coordinate or a quantisation on the changes of $v_2$. Similar reasoning applied to the KT-background gave a result for a anomalous breaking of the R-symmetry. In this case, we emphasise that the symmetry is not the $U(1)_R$ R-symmetry (associated with translations in the angle $\psi$).

So, we have analysed different dynamical quantities with the goal of narrowing or defining the possible field theory dual to our background in eqs.(4.16)-(4.21). It is of interest to analyse the behavior of quantities that are either gauge invariant, conserved or quantized like Maxwell and Page charges. We turn to this now.
4.3 Maxwell and Page Charges

Before the non-abelian T-duality, we calculate the Maxwell and Page charges \[67\] of D3 and D5 branes,

\[
Q_{\text{Max},D3} = \frac{1}{(16\pi^4)} \int_{\theta_1,\phi_1,\theta_2,\phi_2,\psi} F_5 = \frac{K(r)}{27\pi} \sim N_c \ln r , \\
Q_{\text{Max},D5} = \frac{1}{4\pi^2} \int_{\theta_2,\phi_2,\psi} F_3 = \frac{\sqrt{2}}{9} P .
\] (4.54)

This shows the usual logarithmic growth of the D3 brane charge, linked with the logarithmic deviation of the geometry from the $AdS_5$ Klebanov-Witten fixed point. Also, the Maxwell charge of D5 branes is quantized. We can also compute the Page charges,

\[
Q_{\text{Page},D5} = \frac{1}{4\pi^2} \int_{\theta_2,\phi_2,\psi} F_3 - B_2 \wedge F_1 = \frac{\sqrt{2}}{9} P , \\
Q_{\text{Page},D3} = \frac{1}{(16\pi^4)} \int_{\theta_1,\phi_1,\theta_2,\phi_2,\psi} F_5 - B_2 \wedge F_3 = \frac{K + TP}{27\pi} = \frac{Q}{27\pi} .
\] (4.55)

The quantity $Q$ is usually taken to zero, it indicates the number of "free/mobile" D3 branes on the conifold. In the full solution with a good IR behavior (the Klebanov-Strassler background or its baryonic branch counterpart), one takes $Q = 0$, precisely to avoid singularities.

All this analysis is valid and standard before the non-abelian T-duality. Let us analyse things after the duality. We follow this logic: we have D3 branes to begin with. We will perform the non-abelian T-duality in the directions $[\theta_2,\phi_2,\psi]$. This will generate D6 branes. To calculate the Maxwell charge of D6 branes we should integrate the expression for $F_2$ specialized on the cycle in (4.25)\[^{13}\]

\[
Q_{\text{Max},D6} = \frac{1}{\sqrt{2}\pi^2} \int_{\theta_1,\phi_1} F_2 = \frac{K + Q}{27\pi} .
\] (4.56)

We should compare this with the Maxwell charge of D3 branes, in (4.54). Following the same logic in the case of the D5 branes, we can define

\[
Q_{\text{Max},D8} = \sqrt{2} \int F_0 = \frac{\sqrt{2}P}{9} ,
\]

that can be put in correspondence with $Q_{\text{Max},D5}$ in (4.54). Let us now calculate Page

\[^{13}\]The normalization factor is chosen arbitrarily to match with previous expressions
Notice that we can make a correspondence also between Page charges (before and after the duality), if we choose $Q = 0$ as above.

Using these Page charges, one can play a game similar to the one in [66], to get a hint on what is the fate of Seiberg duality after the non-abelian T-duality. To this we now turn.

### 4.4 Maxwell, Page and Seiberg

In this section we study the fate of Seiberg duality, after the non-abelian T-duality.

We will define the geometric version of "Seiberg duality" in the Klebanov-Tseytlin quiver (before the non abelian T-duality), to be the operation that changes in integer units, the Maxwell charges (after a given change in the radial coordinate) or equivalently, as the operation that changes the Page charge after suitable large gauge transformation in the NS-B field (at a fixed value of the radial coordinate). Both these equivalent definitions were introduced in [66]. We will follow that logic after the non-abelian T-duality and we will learn that in the generated background/dual QFT, there seems to be a Seiberg duality at work. The result suggests some ideas for the generated quiver field theory.

#### 4.4.1 Seiberg duality before dualisation

Let us start by summarising the approach to understanding the geometric realisation of Seiberg duality in the background prior to dualisation. Before the T-duality we have

\[ Q_{\text{Page,D6}} = \frac{1}{\sqrt{2\pi^2}} \int_{\phi_1, \phi_2} F_2 - F_0 B_2 = \frac{2Q}{27\pi} , \]

\[ Q_{\text{Page,D8}} = \sqrt{2P} \cdot \frac{9}{9} . \]

The Page charge of D4 branes is formally given by,

\[ Q_{\text{Page,D4}} = \frac{1}{\mu_{D4}} \int F_4 - B_2 \wedge F_2 + \frac{1}{2} B_2 \wedge B_2 \wedge F_0 ; \]

but its interpretation is not clear, as we did not find a well defined 4-cycle to integrate on.
the D3 charges given by

\[ Q_{\text{Max},D3} = \alpha \int_{X_5} F_5 = \frac{4^3 \pi^3 \alpha K(r)}{108}, \]

\[ Q_{\text{Page},D3} = \alpha \int_{X_5} F_5 - B_2 \wedge F_3 = \frac{(4\pi)^3 \alpha}{108} \left( K(r) + PT(r) \right). \]

We have left the coefficient \( \alpha \) undetermined. From the previous subsection we know that \( 16\pi^4 \alpha = 1 \) and \( K(r) = Q_{\text{free}} - PT(r) \). The restriction of the NS field on the two cycle of \((4.24)\) gives

\[ B_2|_{\Sigma} = \frac{T(r)}{3\sqrt{2}} \sin \theta_1 d\theta_1 \wedge d\phi_1. \]

From which we can define

\[ b_0 = \frac{1}{4\pi^2} \int_{\Sigma} B_2 = \frac{T(r)}{3\pi \sqrt{2}}. \]

The logic to follow will be this: we will consider two ways of changing \( b_0 \) in \( n \) (integer) units,

\[ b_0 \rightarrow b_0 \pm n \quad (4.57) \]

- by changing \( \Delta T(r) = \pm 3\pi \sqrt{2} n \)
- by changing

\[ B_2 \rightarrow B_2 \pm \frac{n\pi}{2} \left( \sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2 \right). \]

The change in \( T(r) \) occurs as the radial coordinate varies. We will apply this change to the Maxwell charge. The large gauge transformation is performed at fixed radial coordinate. We will apply this to the Page charge. The idea is that both changes are the equivalent of performing a Seiberg duality \([66]\).

Let us first apply \( \Delta T = \pm 3\pi \sqrt{2} n \) to the Maxwell charge. We obtain,

\[ \Delta Q_{\text{Max}} = \alpha \frac{(4\pi)^3}{108} \Delta K = \alpha \frac{(4\pi)^3}{108} (\Delta Q_{\text{free}} - P \Delta T) = -\alpha \frac{(4\pi)^3 P}{108} \Delta T, \]

\[ \Delta Q_{\text{Max},D3} = \pm \alpha P \frac{4^3 \pi^4}{18 \sqrt{2}} n. \]
Now, we calculate the change in the Page charge under a large gauge transformation in the $B_2$ field.

$$\Delta Q_{\text{Page,D3}} = -\alpha \int_{X_3} \Delta B_2 \wedge F_3 = \pm \alpha P n \frac{4^3 \pi^4}{18 \sqrt{2}}. \quad (4.58)$$

So, we see that under these two different changes, the change in the Page charge equals that of the Maxwell charge for D3 branes. Plugging specific values of $P, \alpha$ we see that this amount to a change in the charge of $\Delta Q = \pm n M$, this is the effect of $n$ Seiberg dualities on the Klebanov-Tseytlin-Strassler quiver field theory.

### 4.4.2 Calculations after the non-abelian T-duality

Now, we will review this calculation, after the non-abelian duality. The object of study are D6 branes whose Page and Maxwell charges we recall are given by

$$Q_{\text{Max,D6}} = \hat{\alpha} \int_{\Sigma_2} F_2 = \hat{\alpha} \frac{4\pi}{54 \sqrt{2}} (2K(r) + PT(r)).$$

$$Q_{\text{Page,D6}} = \hat{\alpha} \int_{\Sigma_2} F_2 - F_0 B_2 = \hat{\alpha} \frac{4\pi}{54 \sqrt{2}} (2K(r) + PT(r) + 9F_0 T). \quad (4.60)$$

After the non-abelian duality, we have that

$$b_0 = \frac{1}{4\pi^2} \int_{\Sigma_2} B_2 = -\frac{T(r)}{6\pi \sqrt{2}}.$$

For an integer $k$, the two possible ways of changing $\Delta b_0 = \pm k$ are by moving radially such that

$$\Delta T = \mp 6\pi \sqrt{2} k,$$

or by performing a large gauge transformation

$$\Delta B_2 = n \pi \sin \theta d\theta_1 \wedge d\phi_1.$$

We will now calculate, the change in the Maxwell charge, under a change in $\Delta T$ and the change in the Page charge, under a change in the $B_2$ field as specified. We have

$$\Delta Q_{\text{Max,D6}} = -\hat{\alpha} \frac{4\pi}{54 \sqrt{2}} P \Delta T(r) = \pm \hat{\alpha} \frac{4\pi^2}{9} P k \quad (4.61)$$

If we use

$$K = -PT, \quad P = 9F_0, \quad \hat{\alpha} \sqrt{2} = \alpha 16\pi^2. \quad (4.59)$$

we obtain that Maxwell and Page charges are invariants under the non-abelian T-duality, as we saw in the previous section.
We now calculate the change in the Page charge, under a large gauge transformation in $B_2$ and we get,

$$\Delta Q_{\text{Page},D6} = \pm \hat{\alpha} 4\pi^2 F_0 k = \pm \hat{\alpha} \frac{4\pi^2}{9} P k .$$  \hspace{1cm} (4.62)

We used eq.(4.59). We then see that

$$\Delta Q_{\text{Max},D6}|_{\Delta T(r)} = \Delta Q_{\text{Page},D6}|_{\Delta B_2} .$$

Using the values for $a, \hat{\alpha}$ in (4.59), we observe that whenever the Page charge of D3 changed in $2M$ units, the Page charge of D6 branes—after the non-abelian duality—changes by $M$ units. Indeed, comparing (4.58) with (4.62)

$$\Delta Q_{\text{Page},D3} = \pm a P n \frac{4^3 \pi^4}{18 \sqrt{2} } , \quad \Delta Q_{\text{Page},D6} = \pm \hat{\alpha} \frac{4\pi^2}{9} P k .$$  \hspace{1cm} (4.63)

Hence, since we associate changes in the Page charge by $M$ units as a Seiberg duality applied on the quiver—before the non-abelian T-duality. We see that a change in $M$ units in the KT-quiver reflects in a change in $2M$ units in the transformed QFT.

This may suggest some ideas for what is the quiver after the non-abelian T-duality.

### 4.5 A summary of this Section

Let us summarise the results of this long section: we have constructed the non-abelian T-dual of the Klebanov-Tseytlin background. We learnt about the QFT dual to our new background in eqs.(4.16)-(4.21) by performing different calculations using the geometrical description of this field theory. Among these, we learn about the (would-be) domain walls, Wilson and ’t Hooft loops, gauge coupling, central charge, entanglement entropy, conserved and gauge invariant charges. We also got a glimpse at the existence of an operation like Seiberg duality. Our transformed background is probably dual to a QFT that presents a cascade similar to the one of the Klebanov-Tseytlin-Strassler background.

The information we gather, together with the one in the following sections may help in narrowing or deciding for a given field theoretical description of the background.

We will now move to study the full non-perturbative dynamics of this putative
QFT. We will do so by analysing the non-abelian T dual of geometries, like Klebanov-Strassler, the baryonic branch of KS or the background produced by D5 branes compactified on the resolved conifold. These are smooth geometries that ‘IR-complete’ the KT-case analysed above. As expected, the resulting backgrounds will be quite involved (mostly due to the fact that we are loosing the $U(1)_R$ associated with the Killing vector $\partial_\psi$ at the level of the metric). The contents in this section will give an orientation about the interesting observables to study.

## 5 Dualisation of smooth geometries

The purpose of this section is to apply the non-Abelian T-duality technique to three trademark backgrounds in type-IIB string theory, conjectured to be dual to $N = 1$ SUSY QFT. The non-perturbative dynamics (confinement, symmetry breaking, etc) of the QFT is captured by the backgrounds and in this sense we will refer to them as "IR-complete" solutions of type-IIB. They are the smoothed out version of the KT background of the previous section.

In Section 6, we will study QFT aspects of the backgrounds spelled out below.

### 5.1 Dualisation of Wrapped D5 solutions

An important class of theories are the wrapped-brane models. In what follows we will be interested in the near brane geometry of D5 branes wrapping a two-sphere with a twist in the normal bundle to preserved $\mathcal{N} = 1$ supersymmetry [33]. In the very far IR the gauge theory on the brane, described in [33], reduces to pure (i.e. without matter) $\mathcal{N} = 1$ SYM. In fact this shares many similarities with the "IR completed" geometry of the Klebanov-Strassler theory which we turn to afterwards.

#### 5.1.1 The wrapped D5 background

First we give the vielbeins of this solution (as ever we are in string frame):

\[
e^i = e^{\Phi/2} dx^i, \quad e^\theta = e^{\Phi/2 + k} d\rho, \quad e^{\phi_1} = e^{\Phi/2 + h} d\theta_1, \quad e^{\phi_1} = e^{\Phi/2 + h} \sin \theta_1 d\phi_1, \nonumber
\]

\[
e^1 = \frac{1}{2} e^{\Phi/2 + \sigma}(\sigma_2 + ad\theta_1), \quad e^2 = \frac{1}{2} e^{\Phi/2 + \sigma}(\sigma_1 - a \sin \theta_1 d\phi_1),
\] (5.1)
\[ e^3 = \frac{1}{2} e^{\Phi/2+k}(\sigma_3 + \cos \theta_1 d\phi_1) , \]

in which the \(\sigma\)'s are the SU(2) left invariant one-forms given in (3.5). This background is supported by a RR three-form

\[
F_3 = e^{-3\Phi/2} \left( f_1 e^1 \wedge e^2 \wedge e^3 + f_2 e^{\theta_1} \wedge e^{\phi_1} \wedge e^3 + f_3 (e^{\theta_1} \wedge e^2 \wedge e^3 + e^{\phi_1} \wedge e^1 \wedge e^3) \\
+ f_4 (e^\rho \wedge e^1 \wedge e^{\phi_1} + e^\rho \wedge e^{\phi_1} \wedge e^2) \right). \tag{5.2}
\]

The functions \(f\)'s above are defined by

\[
f_1 = -2N_c e^{-k-2g}, \quad f_2 = \frac{N_c}{2} e^{-k-2h} (a^2 - 2ab + 1), \]

\[
f_3 = N_c e^{-k-h-g} (a - b), \quad f_4 = \frac{N_c}{2} e^{-k-h-g} b'. \tag{5.3}
\]

The dilaton \(\Phi\) and the other functions \(k, g, h, a, b\) depend on the coordinate \(\rho\) and obey BPS equations the details of which can be found for instance in the appendix of [68].

5.1.2 The Wrapped D5 dual geometry

As before, we perform a T-duality along the SU(2) isometry under which the \(\sigma\)'s are invariant. As in the previous section we will chose a gauge fixing that sets \(\theta_2 = \phi_2 = 0\) and introduces new coordinates \(v_2\) and \(v_3\) in the dual background. The duality transformation leaves the frames \(\{e^x, e^\rho, e^{\theta_1}, e^{\phi_1}\}\) invariant and acts by sending \(e^i \rightarrow \hat{e}^i\). To express compactly the dual geometry we find it convenient to rotate the frame fields in the 1-2 plane using the same rotation matrix as in (3.21). Also to express the results it is very helpful to recombine the dependence on the angular coordinates \(\theta_1, \phi_1, \psi\) into a new set of SU(2) left invariant forms which we denote by \(\omega^j\).\footnote{The \(\sigma\)'s here are related to the \(\tilde{\omega}\)'s of [68] in the following way: \(\sigma_1 = \tilde{\omega}_2, \sigma_2 = \tilde{\omega}_1\) and \(\sigma_3 = \tilde{\omega}_3\), while the angles \(\theta_1, \phi_1\) correspond to the angles \(\theta, \phi\) and also \(\theta_2, \phi_2\) here are the tilded angles of the previous paper .}

We then find that

\[
\hat{e}^{\rho} = \frac{e^{g+\frac{3}{2} \Phi}}{8\mathcal{W}} \left[ 4e^{2g} v_2 (dv_3 - v_2 a \omega_1) + \sqrt{2e^{2g+2k+\Phi}} (v_2 \omega_3 - a v_3 \omega_2) \right]
\]

\footnote{The \(\omega\)'s here are defined in the same way as the \(\sigma\)'s in (3.5) but here instead of the angles \(\theta_2, \phi_2\) we have \(\theta_1, \phi_1\).}
\[-4e^{2k}v_3 (dv_2 + a v_3 \omega_1) \],

\[
\hat{e}^2 = -\frac{e^{g + \frac{1}{2} \Phi}}{8 V} \left[ 4e^{2k+\Phi}v_3 (v_2 \omega_3 - a v_3 \omega_2) + \sqrt{2} e^{2g+2k+2 \Phi} (dv_2 + a v_3 \omega_1) \right. \\
\left. + 8\sqrt{2}v_2 (v_2 dv_2 + v_3 dv_3) \right],
\]

\[
\hat{e}^3 = -\frac{e^{g + \frac{1}{2} \Phi}}{8 V} \left[ \sqrt{2} e^{4g+2 \Phi} (dv_3 - v_2 a \omega_1) - 4e^{2g+\Phi}v_2 (v_2 \omega_3 - a v_3 \omega_2) \\
+ 8\sqrt{2}v_3 (v_2 dv_2 + v_3 dv_3) \right],
\]

and the T-dual metric is given by

\[
ds^2 = (e^x)^2 + (e^\theta)^2 + (e^\phi)^2 + (\hat{e}^i)^2.
\]

The NS-two form field is

\[
\hat{B}_2 = -\frac{e^{g+k+\Phi}}{2\sqrt{2}v_2} \hat{e}^1 \wedge \hat{e}^3 + \frac{e^{-g+k}v_3}{v_2} \hat{e}^2 \wedge \hat{e}^3 + \frac{e^{g+\Phi/2}}{2} \hat{e}^2 \wedge \omega_2 + \frac{e^{k+\Phi/2}}{2v_2} \hat{e}^3 \wedge \omega_2.
\]

The dual dilaton is given by

\[
\hat{\Phi} = \Phi - \frac{1}{2} \ln \mathcal{W},
\]

in which we defined

\[
\mathcal{W} = \det \hat{M} = \frac{1}{8} e^{4g+2k+3\Phi} + e^{2g+\Phi}v_2^2 + e^{2k+\Phi}v_3^2.
\]

This geometry is supported by a cornucopia of RR fluxes with \( \hat{F}_0, \hat{F}_2 \) and \( \hat{F}_4 \) all activated, given by

\[
\hat{F}_0 = \frac{N_c}{\sqrt{2}}
\]

\[
\hat{F}_2 = e^{g-\Phi}f_4 v_2 (\cos \psi e^\theta \wedge e^{\theta_1} + \sin \psi e^\theta \wedge e^{\phi_1}) - e^{k-\Phi} f_2 v_3 e^{\theta_1} \wedge e^{\phi_1}
\]

\[
- \frac{1}{4} e^{k-\Phi} f_3 (\sqrt{2} e^{2g+\Phi} \sin \psi + 4 \cos \psi v_3) (e^{\theta_1} \wedge e^{\theta_2} + e^{\phi_1} \wedge \hat{e}^2)
\]

\[
+ \frac{1}{4} e^{k-\Phi} f_3 (\sqrt{2} e^{2g+\Phi} \cos \psi - 4 \sin \psi v_3) (e^{\phi_1} \wedge e^{\theta_1} - e^{\theta_1} \wedge e^{\phi_2})
\]

\[
+ e^{g-\Phi} f_3 v_2 (\cos \psi e^{\theta_1} \wedge \hat{e}^3 - \sin \psi e^{\theta_1} \wedge e^3) + e^{k-\Phi} f_1 v_3 \hat{e}^{\theta_1} \wedge \hat{e}^2 - e^{g-\Phi} f_1 v_2 \hat{e}^{\theta_1} \wedge \hat{e}^3
\]
\[ \hat{F}_4 = e^{g-\Phi} f_4 v_2 e^\theta \wedge (\sin \psi e^{\theta_1} - \cos \psi e^{\theta_1}) \wedge \hat{e}^1 \wedge \hat{e}^2 \]  
\[ - \frac{1}{4} e^{k-\Phi} f_4 (\sqrt{2} e^{2g+\Phi} \cos \psi - 4 v_3 \sin \psi) e^\theta \wedge (e^{\theta_1} \wedge \hat{e}^1 + e^{\theta_1} \wedge \hat{e}^2) \wedge \hat{e}^3 \]  
\[ + \frac{1}{4} e^{k-\Phi} f_4 (\sqrt{2} e^{2g+\Phi} \sin \psi + 4 v_3 \cos \psi) e^\theta \wedge (e^{\theta_1} \wedge \hat{e}^2 - e^{\theta_1} \wedge \hat{e}^1) \wedge \hat{e}^3 \]  
\[ + e^{g-\Phi} f_2 v_2 e^{\theta_1} \wedge e^{\theta_1} \wedge \hat{e}^2 \wedge \hat{e}^3 - \frac{e^{2g+k}}{2\sqrt{2}} f_2 e^{\theta_1} \wedge e^{\theta_1} \wedge \hat{e}^1 \wedge \hat{e}^2 \]  
\[ + e^{g-\Phi} f_3 v_2 (\cos \psi e^{\theta_1} + \sin \psi e^{\theta_1}) \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 \]  

(5.9)

Using Mathematica we have verified that indeed the equations of motion for fluxes, Bianchi identities, dilaton and Einstein’s equations are all satisfied on the same set of BPS equations as the original geometry.

**5.2 Dualisation of Klebanov Strassler**

**5.2.1 The KS background**

We now consider the Klebanov Strassler solution [21] in which the conifold is replaced with its smooth deformation so that the metric is given by

\[ ds_{10}^2 = h^{-\frac{1}{2}}(\tau) dx_n dx_n + h^{\frac{1}{2}}(\tau) ds_6^2 \]  

(5.10)

where \( ds_6^2 \) is the metric of the deformed conifold. The details of supergravity solution can be found in section 5 of [21]. The radial coordinate we now denote by \( \tau \) in keeping with the notation of [21].

So as to write the metric in a form compatible with our ansatz we find it convenient to introduce some functions which depend on the radial coordinate. The dictionary between the functions we use here and the original paper is as follows

\[ h_1 = \varepsilon^4 h(\tau)^{\frac{1}{2}} K(\tau), \quad h_2 = g_s M(f(\tau) + k(\tau)), \quad h_3 = g_s M(f(\tau) - k(\tau)). \]  

(5.11)

The functions \( h(\tau), f(\tau), k(\tau) \) obey BPS equations given explicitly in [21] and \( K(\tau) \) is a function fixed by the deformed conifold metric. The appearance of the parameter \( \varepsilon \) which describes the deformation of the conifold can be thought of the supergravity dual of dimensional transmutation.

---

\[^{18}\text{At large } \tau \text{ the KS solution asymptotes to the KT solution with a the radial coordinates related by } r^3 \sim e^2 e^1.\]
As with the KT solution the geometry is supported by an NS two-form, an RR three form and a RR five form whose details may be found in [21]. Let us just remark that the RR three form interpolates between that of the KT solution at large $\tau$ and, to prevent the infinite charge density origin of the singularity, something with support only in the non-shrinking $S^3$ at $\tau = 0$.

As before we perform an non-abelian dualisation along the $SU(2)$ action that acts on the coordinates $\{\psi, \theta_2, \phi_2\}$ and we choose a gauge fixing choice in which $\theta_2 = \phi_2 = v_1 = 0$ such that the coordinates of the dual theory are $\{\tau, x^i, \psi, \theta_1, \phi_1, v_2, v_3\}$. We express our results more compactly by writing derivatives of the remaining Euler angles as the left invariant Maurer-Cartan forms $\omega_i[\psi, \theta_1, \phi_1]$ as we did in Section 5.1.2.

### 5.2.2 The KS dual geometry

To express compactly the dual geometry it is convenient to introduce the combinations

$$V = h_2 + \sqrt{2}v_3, \quad U = \cosh \tau h_3 - V,$$

$$W = \det M = \frac{h_1}{12K^3} \left( \cosh^2 \tau h_1^2 + 4V^2 + 12 \cosh \tau K^3 v_2^2 \right),$$

(5.12)

and the one-forms

$$\Lambda_1 = -U\omega_2 - \sqrt{2} \cosh \tau v_2 \omega_3, \quad \Lambda_2 = U\omega_1 - \sqrt{2} \cosh \tau dv_2, \quad \Lambda_3 = \cosh \tau dv_3 - v_2 \omega_1.$$  

(5.13)

Furthermore we perform a rotation of the dual frame in the 1-2 plane as we did in (3.21).

The frame fields in the dual are given by

$$e'^{1} = \frac{h_1^3}{12WK^3 \cosh^2 \tau} \left[ - h_1 \cosh \tau \Lambda_1 + 2V \Lambda_2 + 6 \cosh \tau K^3 v_2 \Lambda_3 \right],$$

$$e'^{2} = \frac{h_1^4}{12WK^3 \cosh^2 \tau} \left[ 2h_1 V \Lambda_1 + h_1^2 \cosh \tau \Lambda_2 + 12K^3 v_2 (v_2 \Lambda_2 - \Lambda_3) \right],$$

(5.14)

$$e'^{3} = -\frac{h_1^2}{4\sqrt{3}WK^2 \cosh \tau} \left[ 2h_1 v_2 \cosh \tau \Lambda_1 - 4V (v_2 \Lambda_2 - \Lambda_3) + h_1^2 \cosh^2 \tau \Lambda_3 \right].$$
so that the dual metric is given by

\[ ds^2 = h^{-\frac{1}{2}}(\tau) dx_n dx_n + \frac{h_1}{6K^3} d\tau^2 + \frac{h_1}{4} \sinh \tau \tanh \tau (\omega_1^2 + \omega_2^2) + \sum_{i=1...3} (\hat{e}^i)^2. \]  

(5.15)

The NS 2-form is

\[
\hat{B} = -\frac{h_1}{2\sqrt{3}K^3 v_2} \hat{e}^1 \wedge \hat{e}^3 + \frac{\cosh^{-\frac{1}{2}} \tau}{\sqrt{3}K^3 v_2} \hat{e}^2 \wedge \hat{e}^3 + \frac{h_1}{2} \cosh^{-\frac{1}{2}} \tau \hat{e}^2 \wedge \omega_2 \]  

\[ -\frac{h_1}{2\sqrt{3}K^3 v_2} \hat{e}^3 \wedge \omega_2 + \frac{h_3 \cosh^{-\frac{1}{2}} \tau}{h_1} \hat{e}^1 \wedge \omega_2 - \frac{h_2 \cosh \tau - h_3}{2 \cosh \tau} \omega_1 \wedge \omega_2 \]  

(5.16)

and the dilaton is

\[ \hat{\Phi} = -\frac{1}{2} \ln \mathcal{W} \]  

(5.17)

We will not quote the transformed RR fields.

### 5.3 Dualisation of the Klebanov-Strassler-baryonic branch

#### 5.3.1 The baryonic branch background

The geometry that describes the whole Baryonic branch of the KS field theory was constructed in [22]. This background is given by the frame fields\(^\text{19}\)

\[
e^x = e^{\frac{\Phi}{2}} \hat{h}^{-\frac{1}{4}} dx^i, \quad e^\rho = e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} d\rho, \quad e^{\theta_1} = e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} d\theta_1, \quad e^{\phi_1} = e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} \sin \theta_1 d\phi_1, \]

\[
e^1 = \frac{1}{2} e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} (\omega_2 + a \ d\theta_1), \quad e^2 = \frac{1}{2} e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} (\omega_1 - a \ \sin \theta_1 d\phi_1), \]

\[
e^3 = \frac{1}{2} e^{\frac{\Phi}{2}} + \hat{h}^{\frac{1}{4}} (\omega_3 + \cos \theta_1 d\phi_1). \]  

(5.18)

Notice the factor \( \hat{h} \) dressing up the frame fields in comparison to the wrapped D5 background in (5.1).

The metric, RR and NSNS fields are

\[ ds^2 = \sum_{i=1}^{10} (e^i)^2, \]

\[ F_3 = \frac{e^{-\frac{3}{2} \Phi}}{\hat{h}^{3/4}} \left[ f_1 e^{123} + f_2 e^{\theta_1 \phi_1 3} + f_3 (e^{\theta_1 23} + e^{\phi_1 13}) + f_4 (e^{\theta_1 1} + e^{\phi_1 2}) \right]. \]

\(^{19}\)Our conventions here are related to the conventions of [27]-[8] in the following way: \((\theta_1, \phi_1) = (\hat{\theta}, \hat{\phi}), (\theta_2, \phi_2) = (\tilde{\theta}, \tilde{\phi})\) and \(\sigma_1 = \hat{\omega}_2, \sigma_2 = \hat{\omega}_1, \sigma_3 = \omega_3\).
\[
B_2 = \kappa e^\Phi \frac{e^{\rho^3} - \cos \alpha (e^{\theta_1 \phi_1} + e^{12}) - \sin \alpha (e^{\theta_1} + e^{\phi_1})}{h^{1/2}} ,
\]
\[
H_3 = -\kappa e^{1/2} \left[ -f_1 e^{\theta_1 \phi_1} + f_2 e^{\phi_1} - f_3 (e^{\theta_1} + e^{\phi_1}) + f_4 (e^{\theta_1} + e^{\phi_1}) \right] ,
\]
\[
C_4 = -\kappa e^{2/3} \sqrt{h} d\rho \bigwedge dx^1 \bigwedge dx^2 \bigwedge dx^3 ,
\]
\[
F_5 = \kappa e^{-2} \Phi - k \frac{1}{\hat{h}} \partial_\rho \left( e^{2 \Phi} \right) \left[ e^{\theta_1 \phi_1} - e^{\theta_1 \phi_1} \right] .
\]

We have defined
\[
\cos \alpha = \frac{\cosh(2\rho) - a}{\sinh(2\rho)} , \quad \sin \alpha = -\frac{2e^{\rho - g}}{\sinh(2\rho)} , \quad \hat{h} = 1 - \kappa^2 e^{2\Phi} ,
\]

where \(\kappa\) is a constant that we will choose to be \(\kappa = e^{-\Phi(\infty)}\), requiring the dilaton to be bounded at large distances.

### 5.3.2 The Baryonic Branch dual geometry

We again perform the dualisation as outlined in section (5.1.2) however in this case the presence of the NS two-form field renders the resulting geometry rather complicated and in particular mixes the radial direction with the internal space.

To express the dual geometry it is expedient to introduce a few combinations,

\[
\mu_1 = ae^g \cos \alpha + 2e^h \sin \alpha , \quad \mu_2 = ae^g \cos \alpha + 4e^h \sin \alpha ,
\]
\[
\nu = 2\sqrt{2} v_3 + e^{2g + 2\Phi} \cos \alpha ,
\]
\[
\mathcal{U} = a\nu + e^{g + 2\Phi} \kappa \mu_1 ,
\]
\[
\mathcal{W} = \det M = \frac{1}{8} e^{\Phi} \hat{h}^{1/2} \left( e^{4g + 2k + 2\Phi} \hat{h} + e^{2k} \nu^2 + 8e^{2g} v_3^2 \right) ,
\]

and to introduce a one-form
\[
\Lambda = dv_3 - \kappa \frac{e^{2k + 2\Phi}}{\sqrt{2}} d\rho ,
\]

that will neatly encode the mixing of the radial and internal direction in the metric.
Then T-dual vielbeins, after the rotation described in section (5.1.2), are given by

\[
\hat{e}'_1 = \frac{e^{g+\frac{3}{2}\Phi}h^\frac{3}{4}}{16W} \left[ 8e^{2h}v_2(\Lambda - \alpha v_1) - e^{2k}a\nu^2\omega_1 - 2\sqrt{2}e^{2k}\nu dv_2 \\
- e^{2k+\Phi} \left( e^{\Phi}b\nu\kappa\mu_1\omega_1 + e^h(\nu\omega_2 - 2\sqrt{2}v_2\omega_3) \right) \right],
\]

\[
\hat{e}'_2 = -\frac{e^{g+\frac{1}{2}\Phi}h^\frac{1}{4}}{16W} \left[ e^{2g+2k+2\Phi}h(\nu\omega_1 + 2\sqrt{2}v_2\nu) - e^{2k+\Phi}h\nu(\nu\omega_2 - 2\sqrt{2}v_2\omega_3) \\
+ 8v_2(\nu\Lambda + e^{g+2\Phi}\nu\kappa\mu_1\nu\omega_1 + 2\sqrt{2}v_2\nu) \right],
\]

\[
\hat{e}'_3 = -\frac{e^{g+\frac{1}{2}\Phi}h^\frac{1}{4}}{8W} \left[ \sqrt{2}e^{2g+\Phi}hnu(\nu\omega_2 - 2\sqrt{2}v_2\omega_3) + \sqrt{2}v(\nu\Lambda + 2\sqrt{2}v_2\nu) \\
+ 8\sqrt{2}e^{g+2\Phi}(\nu\lambda\kappa\mu_1\nu\omega_1 + e^h(\Lambda - \alpha v_1)) \right].
\]

The metric is then given by

\[
ds^2 = (e^x)^2 + (e^y)^2 + (e^{\theta_1})^2 + (e^{\phi_1})^2 + (\hat{e}'_i)^2,
\]

the NS two-form by

\[
\hat{B} = \frac{e^{g+\Phi}h^\frac{1}{4}}{2\sqrt{2}v_2} \hat{e}'_1 \wedge \hat{e}'_3 + \frac{e^{g+k}\nu}{2\sqrt{2}v_2} \hat{e}'_2 \wedge \hat{e}'_3 + \frac{e^{g+\Phi/2}a\hat{h}^\frac{1}{4}}{2} \hat{e}'_2 \wedge \omega_2 + \frac{e^{k+\Phi/2}a\hat{h}^\frac{1}{4}}{4\sqrt{2}v_2} \hat{e}'_3 \wedge \omega_2 \\
- \frac{\kappa e^{2\Phi}}{4}(\alpha(\mu_1 + \mu_2)e^{g} - 4e^{2h}\cos\alpha)\omega_1 \wedge \omega_2 - \frac{\kappa e^{g+2\Phi}}{2} \omega_3 \wedge d\rho - \frac{\kappa\mu_1\nu^3/2}{2h^\frac{1}{4}} \hat{e}'_1 \wedge \omega_1
\]

and the dilaton by the usual formula

\[
\hat{\Phi} = \Phi - \frac{1}{2} \ln W.
\]

As a consistency check one can readily verify that these reduce to the results of the wrapped D5 system in the limit where \( \hat{h} \to 1 \). As before, this geometry is supported by fluxes \( F_0, F_2 \) and \( F_4 \) the exact details of which we do not require at this stage and so omit for concision.
6 Analysis of the dual field theories.

In this section, we will complete the analysis initiated in Section 4. As we repeatedly emphasized, the study of many of the observables considered in Section 4 was incomplete, due to the fact that we were analysing certain low-energy effects in backgrounds that do not contain the information of the IR dynamics (as it is the Klebanov-Tseytlin background and its non-abelian T-dual transformed). The logic was that we were defining the way to calculate these observables in the simpler setting of the KT-transformed background.

In this section we remedy this deficiency. We will study IR effects in backgrounds that are fully non-singular, containing all the information on the non-perturbative dynamics of the dual QFT. We organise this section by presenting different observables and quoting the result when calculated in the non-abelian transformed of the following:

- (wD5): The wrapped D5 background, that as explained in [8] could be thought of as the "low energy effective theory" of the KS/baryonic branch backgrounds
- (KS): The Klebanov-Strassler background. A specially symmetric case, with a four-dimensional UV dynamics.
- (KS+bb): The baryonic branch, interpolating between both previous cases.

To add clarity to the presentation, in what follows we mark the formulas by (wD5), (KS), (KS+bb) to indicate the case to which they refer.

We start by writing explicitly the induced metrics and IR behavior of the two-cycle and three-cycle we defined in eqs. (4.25) and (4.26), respectively. Then, we will go down our list of non-perturbative effects, drawing conclusions on the different characteristics of the associated dual field theories.

6.1 Two-cycle and Three-cycle

The two cycle $\Sigma_2 = [\theta_1, \phi_1]$ with $v_2 = v_3 = \psi = 0$ defined previously in the dual of KT, (4.25) is also well defined in each of the three cases that occupy us here. The induced
metric on the two cycle is given by

\[ (wD5) : \quad ds^2|_{\Sigma_2} = e^{2h+\Phi} \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right), \]

\[ (KS) : \quad ds^2|_{\Sigma_2} = \frac{h_1(-8h_2h_3 + \cosh^2 \tau h_1^2 + \cosh \tau(4h_2^2 + 4h_3^2 - h_1^2))}{4(4h_2^2 + \cosh^2 \tau h_1^2)} \times \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right), \]

\[ (KS + bb) : \quad ds^2|_{\Sigma_2} = \frac{e^{\Phi} h}{\hat{h} + e^{2\Phi} \kappa^2 \cos^2 \alpha} \left( e^{h}(e^{h} h + e^{h+2\Phi} \kappa^2 \sin^2 \alpha + a e^{2h} \kappa^2 \sin 2\alpha) + e^{2\Phi} (a^2 e^{2h} + e^{2h}) \kappa^2 \cos^2 \alpha \right) \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right). \]

The volume of the cycle follows immediately. The IR asymptotics are (making use of \[21\] and \[8\], \[20\]),

\[ (wD5) : \quad Vol[\Sigma_2] \sim \rho^2 + \ldots, \quad (KS) : \quad Vol[\Sigma_2] \sim \tau^2 + \ldots, \]

\[ (KS + bb) : \quad Vol[\Sigma_2] \sim \rho^2 + \ldots. \]

The two cycle vanishes in the origin of the radial coordinate.

Let us move to the study of the three cycle defined in (4.26). The three cycle is defined by \[\Sigma_3 = [\theta_1, \phi_1, \psi_1]\] at constant values of the other internal coordinates. The induced metric on this cycle is given by,

\[ (wD5) : \quad ds^2|_{\Sigma_3} = \left( e^{2h+\Phi} + \frac{a^2 e^{2h+2\Phi}}{4V} (e^{2h} v_2 + e^{2h} v_3) \right) \left( \omega_1^2 + \omega_2^2 + \frac{V_2 e^{2h+2k+2\Phi}}{4W} \omega_3^2 \right) \]

\[ - \left( \frac{a^2 v_2 e^{2h+2k+2\Phi}}{4W} \right) \omega_2 \omega_3 \]

\[ (KS) : \quad ds^2|_{\Sigma_3} = \left( \frac{h_1^2 U^2}{4 \cosh \tau K^3 W} \right) \left( \omega_1^2 + \omega_2^2 + \frac{\cosh \tau h_1^2 v_2^2}{6K^3 W} \omega_3^2 \right) \]

\[ + \left( \frac{4h_2^2 + h_1^2 v_2^2}{3W} \right) \omega_1^2 + \left( \frac{h_1^2 U v_2}{3 \sqrt{2} K^3 W} \right) \omega_2 \omega_3, \]

\[ (KS + bb) : \quad ds^2|_{\Sigma_3} = \left( e^{2h+\Phi} h_1^2 + \frac{e^{2h+2k+2\Phi} U^2}{32W} \right) \left( \omega_1^2 + \omega_2^2 + \frac{V_2 e^{2h+2k+2\Phi}}{4W} \omega_3^2 \right) \]

\[ ^{20}\text{The careful reader will find useful the following expansions (at } \tau \to 0 \text{ and } \rho \to 0) \]

\[ h_1 \sim K(\tau) \sim h(\tau) \sim 1, \quad h_2 \sim h_3 \sim \tau \]

\[ \hat{h} \sim e^h \sim e^\Phi \sim \sin \alpha \sim 1, \quad e^h \mu_1 \sim \mu_2 \sim \mu_3 \sim \cos \alpha \sim \sin 2\alpha \sim \rho. \]
\[ + \left( \frac{a^2 e^{4s + 2\Phi} h + e^{2s + 4\Phi} \kappa^2 \mu_1^2 v_2^2}{4W} \right) \omega_1^2 - \left( \frac{v_2 U e^{2s + 2k + 2\Phi}}{4\sqrt{2}W} \right) \omega_2 \omega_3. \]

The volume of these three cycle behaves in the far IR as

\[(wD5) : \text{Vol}[\Sigma_3] \sim \rho + ...., \quad (KS) : \text{Vol}[\Sigma_3] \sim \tau ...., \quad (KS + bb) : \text{Vol}[\Sigma_3] \sim \rho + .... .\]

and in all cases the functional dependence of \(v_i\) is shared in the IR limit with the volume behaving as

\[\sqrt{g_{\Sigma_3}} \sim \rho v_2 \frac{\sqrt{v_3^2 + v_2^2}}{\text{const} + v_3^2 + v_2^2} + \ldots \quad (6.5)\]

reflecting the underlying similarities of these geometries.

We see that the volume of the three cycle also vanishes in the far IR. Though both the two and three cycle defined above vanish, this does not imply that we will get trivial results for all observables. Indeed, in most cases, as we will see, it is the "stringy volume", that is \(\text{det}[g + B]\) what plays a role in calculations.

We will move now into calculating observables of the dual QFT, using the three new geometries we have found.

### 6.2 Domain walls

In the original backgrounds, domain walls are defined as D5 branes that extend along \(R^{1,2}\) and \((\theta_2, \phi_2, \psi)\). These objects have finite tension when placed in the origin of the radial coordinate. This indicates the objects "exist" and are formed due to non-perturbative dynamics. From the induced Action one can calculate the tension of the domain wall (before the non-abelian T-duality) to be,

\[T_{DW} = T_{D5} 2\pi^2 e^{2\Phi + 2s + k} \sim 2\pi^2 e^{2\Phi(0)}. \quad (6.6)\]

We used the IR expansions of the functions, quoted, for example in Appendix A of [8]—see also our (6.2). In the T-dualised background we have defined the Domain walls as D2 branes extending along \(R^{1,1}\) with vanishing \((v_2, v_3, \theta_1, \phi_1, \psi)\). Such configurations have vanishing B field on their world-volume. The Born Infeld action which in this case is for the non-abelian transformed of the wrapped D5 system

\[(wD5) : \quad S = -T_{D2} e^{-\tilde{\Phi} + 2\Phi/2} \int d^{2+1}x \quad (6.7)\]
Using the expression for the dual dilaton we obtain for the effective tension,  

\[ (wD5) : \quad T_{\text{eff}} = T_D 2 \frac{e^{2g+k+2\Phi}}{2\sqrt{2}} \bigg|_{\rho=0} \]  

(6.8)

again, we observe a finite tension object. This indicates that the field theory dual to the transformed background contains different vacua, separated by these walls, formed non-perturbatively.

In the cases of the KS-Baryonic Branch non-abelian T dual backgrounds the expressions read,

\[ (KS) : \quad T_{\text{eff}} = T_D 2 \frac{e^{2/3h(\tau)^{-1/2}(4h_2^2 + \cosh^2 \tau K(\tau)^2 e^{-\Phi^2})^{1/2}}}{2\sqrt{3}K(\tau)} \bigg|_{\tau=0} \]  

(6.9)

Using the small radius expansions in [21] and [8], we find that the domain walls have constant tension in the three analysed cases.

### 6.3 The ‘t Hooft loop and gauge coupling

In Section 4 we defined the ‘t Hooft loop as the non-local operator calculated with a D4 brane extending in \( R^{1,1} \) and wrapping the three-cycle \( \Sigma_3 \) of (4.26). In the case of the background we obtain when transforming the wrapped D5 system— see Section 5.1.2 The induced metric on the probe D4 takes the form,

\[ ds^2 = e^\Phi dx_{1,1}^2 + \left( e^{2h+\Phi} + \frac{a^2 e^{2g+2\Phi}}{4W} (e^{2g}v_2^2 + e^{2k}v_3^2) \right) (\omega_1^2 + \omega_2^2) + \left( \frac{e^{2g+2k+2\Phi}}{4W} \right) \omega_3^2  

\[ - \left( \frac{a^2 e^{2g+2\Phi}}{4W} \right) \omega_2^2 - \left( \frac{av_2v_3 e^{2g+2k+2\Phi}}{2W} \right) \omega_2 \omega_3 . \]  

(6.10)

while the B field induced on the brane is,

\[ B = \frac{a e^{4g+2k+3\Phi}}{8 \sqrt{2} W} (v_2 \omega_1 \wedge \omega_3 - a v_3 \omega_1 \wedge \omega_2) \]  

(6.11)
Now one calculates the Born-Infeld action and the effective tension for the above configuration— for the non-abelian T-dual of the wrapped D5 background,

\[(wD5): \quad S = T_{\text{eff}} \int d^{1+1}x, \quad (6.12)\]

\[T_{\text{eff}} = T_{D4} 4\pi^2 e^{g+h+k+2\Phi} v_2 \sqrt{a^2 e^{2g} + 4 e^{2h}}.\]

Note that the above result is independent of \(v_3\). Also, as already observed, it is the stringy volume what plays a role in this calculation.

Using the small radius expansion from Appendix A of [8] or (6.2), we can see that this object becomes tensionless in the far IR as \(T_{\text{eff}} \sim \rho\). This indicates that monopoles are "screened", which is a typical signature of a confining field theory. Let us briefly analyse the analog result on the KS/Baryonic Branch non-abelian T-dual backgrounds where the effective tensions are

\[(KS): \quad T_{\text{eff}} = T_{D4} \frac{\pi^2 e^{\frac{4}{3}}}{2\sqrt{3}K(\tau)^{\frac{1}{2}}} v_2 \left(16g_4M((-1 + \cosh \tau)f(\tau)^2 + (1 + \cosh \tau)k(\tau)^2\right)\]

\[+ 2e^{\frac{8}{3}h(\tau)K(\tau)^2 \cosh \tau \sinh^2 \tau)^{\frac{1}{2}}),\]

\[(KS + bb): \quad T_{\text{eff}} = T_{D4} \pi^2 e^{g+h+k+2\Phi} h^{\frac{1}{2}} v_2 \sqrt{a^2 e^{2g} + 4 e^{2h}}.\quad (6.13)\]

Using the IR expansions of [21] and [8], we see that monopoles are screened in the three backgrounds/dual QFTs.

Finally, in Section 4, we defined the gauge coupling in terms of a D2 brane that wraps the three-cycle in (4.26). The induced NS B field in this case is the same as that we found in (6.11). The Born Infeld action for this euclidean "instantonic" D2 brane gives

\[(wD5): \quad S = T_{D2} \pi^2 e^{g+h+k+\Phi} v_2 \sqrt{a^2 e^{2g} + 4 e^{2h}}\quad (6.14)\]

Notice that we have again considered the range of \(\psi\) to be \([0, \pi]\), to avoid the bolt-singularity. Note also that this result is \(v_3\) independent. Using this, we obtain (in the case of the non-abelian T-dual to the wrapped D5 system), that a logic similar to the one spelled around (4.48) tells us that the inverse gauge coupling vanishes in the far IR as

\[(wD5): \quad \frac{1}{g_4^2} \sim \rho, \quad (6.15)\]

which is again a characteristic sign of a confining theory. The definition based on D6
branes is expected to give the same result.

Let us briefly analyse the analog result on the KS/Baryonic Branch non-abelian T-dual backgrounds. The euclidean D2 actions are

\[
(KS) : \quad S = T_{D2} \pi^2 \frac{h_1}{K^{3/2} 4\sqrt{6}} (-8h_2 h_3 + \cosh^3 \tau h_1^2 + \cosh \tau (4h_2^2 + 4h_3^2 - h_1^2))^\frac{1}{2}
\]

\[
(KS + bb) : \quad S = T_{D2} \pi^2 e^{\Phi + \Phi^2} h_1 \frac{1}{2} \cos^2 \alpha + 4e^{2h} (\mu_1 - 2ae^\alpha (\mu_1 + \mu_2) \cos \alpha) \frac{1}{2}. \quad (6.16)
\]

Correspondingly we obtain the behaviour in the deep IR (and at fixed \(v_2\)),

\[
(KS) : \quad \frac{1}{g_4^2} \sim \tau, \quad (KS + bb) : \quad \frac{1}{g_4^2} \sim \rho.
\]

Notice that the expression in (6.16) reduces to (6.14) in the un-dressing limit \(\kappa \to 0, \hat{h} \to 1\).

In conclusion, the gauge coupling defined as in Section 4.2.5, is such that in all three backgrounds/QFTs grows unbounded for small energies. As explained, this is in good correlate with confinement of quarks and the screening of monopoles discussed above.

### 6.4 Central charge and Entanglement Entropy

In Section 4, we found expressions that hinted at the fact that the central charge and the entanglement entropy were invariants of the non-abelian T-dual operation. indeed, we found that up-to a factor depending on the volume of the \(v_2, v_3\) space, expressions before and after the non-abelian duality were identical.

In the cases of the non-abelian T-dual of the wrapped D5 and the KS we can readily obtain the expressions for the central charge using the formalism of [31]. Since in these cases the space spanned by \(\mathbb{R}^{1,3}\) and the radial coordinate is unaltered it is sufficient just to look at the (dilaton adjusted) volume of the internal manifold which are given by

\[
(wD5) : \quad V_{int} = \int e^{-2\Phi} \sqrt{g} = \sqrt{2} \pi^2 e^{2h+k+\Phi/2} I,
\]
\begin{align*}
\text{(KS)} : \quad V_{\text{int}} &= \int e^{-2\Phi} \sqrt{g} = \frac{\pi^2}{2\sqrt{3}} \epsilon^{10} h(\tau)^{\frac{5}{2}} K(\tau) \sinh^2 \tau \mathcal{I}, \quad (6.17)
\end{align*}

where in both cases
\begin{align*}
\mathcal{I} &= \int dv_3 \int dv_2 v_2,
\end{align*}

is the RG scale independent integral left over from the Fadeev-Popov determinant. Up to this constant factor these precisely match the corresponding expressions before dualisation and hence the central charges agree. Interestingly, notice that \( V_{\text{int}} \) vanishes in the far IR. The situation is rather more subtle when it comes to the Baryonic branch; the ‘radial’ coordinate \( \rho \) mixes with the internal directions, this means a direct use of [31] is not viable. We will comment more on this unusual feature in the next section.

It may be interesting to push these calculations a bit more and analyse as the authors of [31] did the possible first order (quantum) phase transition observed when plotting the entanglement entropy in terms of the separation between regions \( l \). We leave this demanding numerical work for the future.

\subsection*{6.5 Wilson loops, Asymptotics of the Dilaton and five dimensional metric}

In Section [4.2.3], we considered the "radial" behavior of the dilaton and the metric of the \( R^{1,3} \times R \) of the metric for the non-abelian T-dual of the Klebanov-Tseytlin background. We observed that when fixing the internal directions as in (4.31), the metric of the five-space received no "contributions" from the internal manifold. The conclusion of this was the fact that the UV-asymptotics of the Wilson loop—computed with a string hanging from the far UV in the configuration of (4.36)—was not supposed to present changes. We can quickly verify which is situation in the non-abelian T dual transformed of our wrapped D5, KS and Baryonic branch backgrounds.

In the transformed solutions corresponding to the case in which the "seed" background are those of wrapped D5s and KS, we check that the \( R^{1,3} \times R \) remains the same before and after the non-abelian T-duality, in the case of the Baryonic Branch background something uncanny occurs. Indeed, the one form \( \Lambda \) defined in (5.22), generates a new term in the component of \( g_{\rho \rho} \) even when the internal directions are fixed to be constants.
In particular we find that the $R^{1,3} \times R$ part of the metric is given by

$$ds_{R^{1,3} \times R}^2 = e^{\Phi} h^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + e^{2k + \Phi} h^{1/2} d\rho^2$$

$$+ \frac{e^{4k + 4\Phi}}{8W} (e^{4\Phi} + V^2) d\rho^2. \quad (6.19)$$

In addition to this there is an off-diagonal mixing

$$g_{\rho a} dx^a = -\frac{e^{2k + 2\Phi} \kappa}{4W} \left[ 4V v_2 dv_2 + (e^{4\Phi} + V^2) dv_3 
- \sqrt{2} e^{4\Phi} v_2 (A e^{2\Phi} h - V \kappa \mu_1) \omega_1 \right]. \quad (6.20)$$

What distinguished the baryonic branch from the KS system and the D5 wrapped background (all considered before the non-abelian T-duality) is that in the baryonic branch case we have a new component in the NS-$B_2$ field, that is ultimately responsible for this mixing expressed by the one form $\Lambda$. Field theoretically, the baryonic branch contains a non-zero VEV for an operator of dimension two. The operator is called $\mathcal{U}$ in [69] and is roughly indicating the differences in VEVs of baryon and antibaryon fields. Here, it seems that what was a difference of VEV's before the non-abelian T-duality gets mapped in to an irrelevant operator that deviates the geometry away from the "logarithmically approaching AdS" characteristic of KS and the baryonic branch. This in indicated by the last term in (6.19). The numerology points to an irrelevant of dimension six. Another more conservative interpretation is that there simply does not exist a baryonic branch of the field theories obtained after T-dualisation; indeed since some global symmetries are lost it might not be possible to from an appropriate baryonic operator. We leave for a more dedicated study the understanding of this feature of the QFT.

Let us summarize the results of this section. We performed an analysis that complements that of Section 4, for the observables sensitive to the non-perturbative IR dynamics of the QFT. The information obtained here is key in deciding about the quiver/lagrangian of the dual QFT.

We have presented a set of three geometries in massive IIA String theory, that aside from being smooth geometries, are dual to a minimally SUSY field theory that confines, generates domain walls (hence different vacua), presents a mechanism similar to Seiberg duality (for the KS case ) when flowing down from an approximate fixed
Lots of observables that have been calculated with the trademark backgrounds of Type IIB, could be computed in our new geometries to learn more about these QFT.

7 Conclusions and Outlook

In this work we have demonstrated the utility of non-abelian T-duality as a solution generating technique of supergravity backgrounds, in particular in the context of the AdS/CFT correspondence. We began by considering dualisation of the Klebanov Witten geometry with respect to an SU(2) isometry group. The result of this procedure was to find an \( N = 1 \) supersymmetric background in type IIA supergravity whose lift to M-theory coincides with a special case of the geometries dual to certain \( N=1 \) quiver gauge theories described in [26]. However, in our situation the dual geometry contains a genus zero surface and the corresponding gauge theories are, perhaps surprisingly, less understood than those of higher genus. Understanding more precisely the theories of [29], [26] in the genus zero case and their connection to the geometry present here is an interesting question that we hope will be the subject of further study.

We also looked at dualisation of geometries dual to non-conformal theories, beginning with the Klebanov Tseytlin geometry and then considering its completion in the IR. Here we find an immediate departure to the conformal case; the backgrounds we generate are solutions of massive type IIA supergravity with the mass parameter naturally quantised by the number of fractional branes before dualisation. The action of T-duality has a natural consequence for the Page charges; the D5 and D3 charges get mapped on to D8 and D6 charges respectively. Under a large gauge transformation we saw that these charges display the signatures of a Seiberg duality cascade encoded in the geometry. A puzzling feature is that whereas a Seiberg duality before the T-duality changed charges (or the ranks of gauge groups) by \( M \) units, after duality it has the effect of changing charges by \( 2M \) units. A fundamental question to ask is then what is the would be cascading field theory corresponding to this solution.

We hope that by performing a number of probes of this geometry we have assembled some facts that will be of use in future attempts to answer this question. Many features that were present before the dualisation seem to persist. Features that are preserved (and which can thus be thought of as ‘neutral’ under the isometry group...
dualised) are the presence of well defined two and three cycles; the aforementioned Page charges and indications of duality cascade; the constant tension of domain walls and the fact that the $R^{1,3} \times R$ directions, logarithmically approach $AdS_5$. Perhaps the most compelling similarity is the fact that the central charge is preserved under the dualisation (up to a single RG scale independent integral that depends on the global properties of the geometry). There however are some notable changes after dualisation. Firstly the geometry is supported by a non-constant dilaton. Somewhat related to this is that a suitable gauge coupling, defined in terms of a brane probe wrapping the three cycle, has a rather peculiar asymptotic behaviour and not what is typical in 4d gauge theories. On the same vein, we find an associated anomaly, that is not the usual one breaking the $U(1)_R$ symmetry of the seed backgrounds.

In order to probe the IR physics we also addressed IR smooth $N=1$ dual geometries, namely the Klebanov Strassler, the wrapped five brane and the baryonic branch that interpolates between them. In the UV the story is qualitively similar to the Klebanov Tseytlin. In the deep IR we find that although the cycles are shrinking the presence of the dilaton and NS flux ensures finite tension for the domain walls. The ’t Hooft loop, defined as a probe D4 wrapping the three cycle, has an effective tension that vanishes in the deep IR indicating screening of monopoles, a signature of confinement. In alignment with this we find that the rectangular Wilson loop, uncharged under global symmetries, displays an area law. Also, the gauge coupling described above diverges at low energies. Again the central charges are preserved by the dualisation.

For the case of the baryonic branch the dualisation has rather more severe consequences; the $R^{1,3} \times R$ space actually mixes with the internal space (this is due to the activation of a certain component of the NS two form in baryonic branch background). This strongly deviates the geometry from the logarithmically approaching $AdS$. One interpretation of this could be that the dimension two operator that takes a VEV on the baryonic branch has been converted into an irrelevant operator in the dualisation.

Let us now describe some unresolved puzzles that we hope will be the topic of future study.

*What are the dual field theories?*

The dual of the conformal case seems to have at least superficial relation to the theories coming from wrapped M5 branes on Riemann surfaces [29], [26]. This corresponds well to the fact that a similar dualisation applied to $AdS_5 \times S^5$ gave a geometry
rather similar to that of Gaiotto and Maldacena [11] which are an $N = 2$ counter part to the Sicilian theories of [29]. However the case of genus zero that seems relevant to us is the least understood.

Moreover, it is puzzling that in the non-conformal case the geometries are in massive IIA which does not lift to M-theory in the conventional way. One is then led to ask how to modify the picture of wrapped M5 branes to induce a mass. One might wonder if there is some non-geometric compactification involved especially given the connection between non-abelian duality and gauged supergravity [71]. Equally it is know that certain supergravities, which don’t have a higher dimensional origin in the conventional geometric sense, can be obtained by reduction in the T/U-duality symmetric generalised geometry [72, 73]. It is interesting to ask whether these massive IIA theories can be obtained from an appropriate reduction of the generalised M-theory considered in [74].

**Clarifying and developing the underlying geometry**

At first sight, the solutions presented here, seem rather complicated. However certain similarities emerge due to the simplicity of the $SU(2)$ group dualised. One might hope to find a better language with which to describe the geometries presented within this work. In fact all of the initial geometries studied can be nicely phrased in terms of $SU(3)$-structures (i.e. by the data in a globally defined two form $J$ and holomorphic three form $\Omega_{hol}$). The action of abelian T-duality on the $SU(3)$-structure has been studied before and it seems very likely the situation would be similar for the non-abelian duality. It is then natural to conjecture that these geometries may be described by non-local $SU(2)$ structures [75]. With such technology one might even hope to find a more general class of cascading solutions in massive type IIA by using the solutions found here as the starting point for an intelligent ansatz.

**Compatibility with S-duality**

In this paper one system considered was the wrapped D5 brane whose geometry is supported by quantised RR 3-forms flux. In general one can consider a $p - q$ system of wrapped NS5 and D5 branes whose geometry has both NS and RR 3-forms. It is natural to ask what becomes of this system under the duality. A similar question stands for the D1-D5 system considered in [10].

**Developing the technology of non-abelian T-duality**
One of the challenges in non-abelian T-duality is the difficulty (at the very least) to demonstrate it to be an exact symmetry in string (genus) perturbation theory. One reason for this is that it is hard to understand how to constrain the periodicities and global properties of the Lagrange multipliers so that partition functions match exactly before and after duality. Although one may not be concerned about this difficulty when thinking about the dualisation as a solution generating technique of supergravity or in the context of large N AdS/CFT where string genus corrections are suppressed, it would be desirable to understand better the global properties of the backgrounds we have presented within.

Wider applications

It seems that here, and in recent works, we have only started seeing the utility of these duality transformations. In principle one could apply these procedures whenever a space-time admits some non-abelian isometry group. There are of course many examples of this and we hope that further study will prove fruitful.

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A Group theory conventions

We give details of the conventions used in performing dualisation with respect to $SU(2)$ isometry groups. The Pauli matrices are

$$
\begin{align*}
\tau_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \tau_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \tau_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
$$

(A.1)

and obey $\tau_i \tau_j = \delta_{ij} \mathbb{I} + 2 \epsilon_{ijk} \tau_k$. We define $SU(2)$ generators

$$
t^i = \frac{1}{\sqrt{2}} \tau_i,
$$

(A.2)

such that

$$
\text{Tr}(t^i t^j) = \delta^{ij}, \quad [t^i, t^j] = i f^{ikj} t^k = i \sqrt{2} \epsilon_{ijk} t^k.
$$

(A.3)

Left invariant one-forms are defined in general by

$$
L^i = -i \text{Tr}(t^i g^{-1} dg)
$$

(A.4)

and obey

$$
dL^i = \frac{1}{2} f^{ijk} L^j \wedge L^k.
$$

(A.5)

In the Euler parametrisation a group element is given by

$$
g = e^{\frac{i}{2} \phi \tau_3} \cdot e^{\frac{i}{2} \theta \tau_2} \cdot e^{\frac{i}{2} \psi \tau_3}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi
$$

(A.6)

and the left-invariant Maurer–Cartan forms by

$$
\begin{align*}
L_1 &= \frac{1}{\sqrt{2}} (-\sin \psi d\theta + \cos \psi \sin \theta d\phi), \\
L_2 &= \frac{1}{\sqrt{2}} (\cos \psi d\theta + \sin \psi \sin \theta d\phi), \\
L_3 &= \frac{1}{\sqrt{2}} (d\psi + \cos \theta d\phi).
\end{align*}
$$

(A.7)

A useful matrix is the adjoint action

$$
D^{ij} = \text{Tr} \left( t^i g t^j g^{-1} \right).
$$

(A.8)
which is an orthogonal matrix, i.e. it obeys
\[ D^{ij}D^{ik} = \delta^{jk} . \] (A.9)

Its explicit expression in terms of the Euler angles is not necessary for our purposes. Finally note that the \( SU(2) \) transformations on the group element have the following action on the Euler angles
\[
\begin{align*}
\delta \theta &= \epsilon_1 \sin \phi + \epsilon_2 \cos \phi , \\
\delta \phi &= \cot \theta (\epsilon_1 \cos \phi - \epsilon_2 \sin \phi) + \epsilon_3 , \\
\delta \psi &= \frac{1}{\sin \theta} (-\epsilon_1 \cos \phi + \epsilon_2 \sin \phi) .
\end{align*}
\] (A.10)
The corresponding Killing vectors are
\[
\begin{align*}
k_{(1)} &= -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi - \csc \theta \sin \phi \partial_\psi , \\
k_{(2)} &= -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi + \csc \theta \cos \phi \partial_\psi , \\
k_{(3)} &= -\partial_\phi ,
\end{align*}
\] (A.11)
and obey \([k_{(i)}, k_{(j)}] = -\epsilon_{ijk}k_{(k)}\), i.e. the \( su(2) \) algebra.

Similarly the Lagrange multipliers are infinitesimally transforming as, i.e.\[ (2.19) \]
\[ \delta v_a = \epsilon_{abc} \epsilon_b v_c . \] (A.12)

B General gauge fixing

In this appendix we consider the general case in which we gauge fix some variables of the group group element and use the residual symmetry to gauge fix in addition some of the Lagrange multipliers. Let \( g \) and \( v \) denote the set of \( \dim G \) variables left after gauge fixing \( \dim H \) among the original \( \dim G + \dim H \) ones. Then we have the replacement
\[ L_i^i \to L_i^i + iD_{ji}A_j^i . \] (B.1)
In addition we denote
\[ Q_{\mu i} \partial_\pm X^\mu = Q_{\pm i} , \quad Q_{ij} \partial_\pm X^\mu = Q_{i\pm} . \] (B.2)
Then we have the Lagrangian (we suppress group indices)

\[ \mathcal{L} = Q_\mu \partial_+ X^\mu \partial_- X^\nu + Q_+ L_- + L_+ Q_- + L_+ EL_- \]

\[ iA_+(DQ_- + DEL_- - \partial_- v) + i(Q_+ D^T + L_+ ED^T + \partial_+ v) A_- \]

\[ -A_+(DED^T + f) A_- . \] (B.3)

(B.4)

Integrating out the gauge fields we find that

\[ A_+ = i\hat{M}^{-T} \left( Q_+ D^T + L_+ ED^T + \partial_+ v \right), \]
\[ A_- = -i\hat{M}^{-1} \left( \partial_- v - DQ_- - DEL_- \right), \] (B.5)

where

\[ \hat{M} = DED^T + f . \] (B.6)

The dual action is

\[ \mathcal{L}_{\text{dual}} = \mathcal{L}_0 + (Q_+ D^T + L_+ ED^T + \partial_+ v) \hat{M}^{-1} (\partial_- v - DQ_- - DEL_-), \] (B.7)

where \( \mathcal{L}_0 \) the original action.

The transformation of the worldsheet derivatives can be written as

\[ L_+ \rightarrow L_+ + iD^T A_+ = -D^T \hat{M}^{-T} (DQ_+ + fDL_+ + \partial_+ v), \]
\[ L_- \rightarrow L_- + iD^T A_- = D^T \hat{M}^{-1} (\partial_- v + fDL_- - DQ_-), \] (B.8)

where we have used (B.1) and utilized (B.5). Next we perform the coordinate transformation (2.43) that defines the dressed Lagrange multipliers in (2.43). Also let that

\[ \hat{f}_{ij} = f_{ijk} \hat{v}^k, \quad M \equiv E + \hat{f} , \] (B.9)

where note that \( M \) above is defined slightly different than (2.23). Then using the identity arising from group theory considerations

\[ D^T fD = \hat{f} , \] (B.10)

we obtain

\[ \hat{M} = DMD^T . \] (B.11)
Using that and in addition the identity
\[ \partial_\pm D^T v = \hat{f} L_\pm , \] (B.12)
we find that
\[ L_+ \to -M^{-T}(\partial_+ \hat{v} + Q_+) , \quad L_- \to M^{-1}(\partial_- \hat{v} - Q_-) , \] (B.13)
which are the same expressions as those found with the gauge choice \( g = I \), but with the \( v \)'s replaced with the \( \hat{v} \)'s as in (2.43).

Similar manipulations give for the dual action (B.7) the form (2.24), i.e.
\[ \tilde{L}_{\text{dual}} = Q^\mu_\nu \partial_+ X^\mu \partial_- X^\nu + (\partial_+ \hat{v}_i + \partial_+ X^\mu Q_{\mu i}) M^{-1}_{ij} (\partial_- \hat{v}_j - Q_{j\mu} \partial_- X^\mu) , \] (B.14)
with \( M \) given by (B.9), plus the term
\[ L_+ \partial_- \hat{v} - \partial_+ \hat{v} L_- - L_+ \hat{f} L_- , \] (B.15)
which is however a total derivative.

Finally, we note that the dilaton is still given by (2.26), but with \( M \) given by (B.9).

C  The general form of the transformation on RR fields

Our conventions on Hodge duality are such that on a \( p \)-form in a \( D \)-dimensional spacetime is
\[ (\ast F_p)_{\mu_{p+1}\ldots \mu_D} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1\ldots \mu_D} F^\mu_{\mu_1\ldots \mu_p} , \] (C.1)
where \( \epsilon_{01\ldots 9} = 1 \). With this we have the useful identity \[ \ast \ast F_p = s (-1)^{p(D-p)} F_p , \] where \( s \) is the signature of spacetime which we take to be mostly plus.

We would like to see how one works out the details concerning (2.49) from which the transformation of the flux fields arises. Its inverse (which is actually what we need) will necessarily have the form
\[ \Omega^{-1} = (A_0 \Gamma^1 \Gamma^2 \Gamma^3 + A_\alpha \Gamma^\alpha ) \Gamma_{11} , \] (C.2)
where \( A_0 \) and \( A_\alpha \) are some coefficients that may depend on fields and \( \alpha = 1, 2, 3 \).
runs over the directions that use to be a, generally squashed, $S^3$ that has the $SU(2)$ symmetry w.r.t. which we performed the non-Abelian T-duality. These coefficients are given in our case by

$$A_0 = \frac{1}{\sqrt{1 + \zeta^2}}, \quad A^a = \frac{\zeta^a}{\sqrt{1 + \zeta^2}}. \quad (C.3)$$

We will denote by $e^a$ the frames "containing" the original $SU(2)$ directions.

In general we have the following decomposition for a $p$-form

$$F_p = \mathcal{G}_p^{(0)} + G^a_{p-1} \wedge e^a + \frac{1}{2} G^{ab}_{p-2} \wedge e^a \wedge e^b + G^{(3)}_{p-3} \wedge e^1 \wedge e^2 \wedge e^3, \quad (C.4)$$

where $G^{ab}_{p-1} = -G^{ba}_{p-1}$. Then

$$F_p = \mathcal{G}_p^{(0)} \mathbb{1} + G^a_{p-1} \Gamma^a + \frac{1}{2} G^{ab}_{p-2} \Gamma^{ab} + \mathcal{G}^{(3)}_{p-3} \Gamma^{123}, \quad (C.5)$$

where $\mathcal{G}_q, q = p - 1, \ldots, p - 3$ is defined with the Gamma matrices $\Gamma^A$ corresponding to the seven-dimensional spectator spacetime.

If there is more symmetry some of the above forms are actually zero. For instance, if there is an extra $SU(2)$ symmetry so that the symmetry group is actually $SO(4)$, then

$$G^a_{p-1} = G^{ab}_{p-2} = 0. \quad (C.6)$$

The following identities are needed

$$\Gamma^a \Gamma^{123} = \frac{1}{2} \epsilon^{abc} \Gamma^{bc}, \quad \Gamma^{ab} \Gamma^{123} = -\epsilon^{abc} \Gamma^c, \quad \Gamma^{123} \Gamma^{123} = -1,$$

$$\Gamma^a \Gamma^b = \delta^{ab} + \Gamma^{ab}, \quad \Gamma^{ab} \Gamma^c = \delta^{bc} \Gamma^a - \delta^{ac} \Gamma^b + \epsilon^{abc} \Gamma^{123}, \quad (C.7)$$

$$\Gamma^{123} \Gamma^a = \frac{1}{2} \epsilon^{abc} \Gamma^{bc}.$$

Then we have that

$$F_p \Gamma^a = \mathcal{G}_p^{(0)} \Gamma^a + \mathcal{G}^a_{p-1} + \mathcal{G}^b_{p-1} \epsilon^{ba} - \mathcal{G}^{ab}_{p-2} \Gamma^b + \frac{1}{2} \mathcal{G}^{bc}_{p-2} \epsilon^{abc} \Gamma^{123} + \frac{1}{2} \mathcal{G}^{(3)}_{p-3} \epsilon^{abc} \Gamma^{123},$$

$$F_p \Gamma^{123} = \mathcal{G}_p^{(0)} \Gamma^{123} + \frac{1}{2} \mathcal{G}^a_{p-1} \epsilon^{abc} \Gamma^{bc} - \frac{1}{2} \mathcal{G}^{ab}_{p-1} \epsilon^{abc} \Gamma^c - \mathcal{G}^{(3)}_{p-3} \mathbb{1}. \quad (C.8)$$
Putting these together, and sorting by form degree, we get

\[
\mathcal{F}_p \cdot \Omega^{-1} = \tilde{\mathcal{F}}_{p-3} + \tilde{\mathcal{F}}_{p-1} + \tilde{\mathcal{F}}_{p+1} + \tilde{\mathcal{F}}_{p+3},
\]

(C.9)

where

\[
\begin{align*}
\tilde{\mathcal{F}}_{p-3} &= -A_0 G^{(3)}_{p-3}, \\
\tilde{\mathcal{F}}_{p-1} &= A_d \tilde{G}^a_{p-1} - \frac{A_0}{2} \delta^{ab} e^{abc} \Gamma^c - A_d \tilde{G}^a_{p-2} \Gamma^b + \frac{A_d}{2} \bar{G}^{(3)}_{p-3} e^{abc} \Gamma^{bc}, \\
\tilde{\mathcal{F}}_{p+1} &= A_d \tilde{G}^a_p \Gamma^a + \frac{A_0}{2} \delta^{ab} e^{abc} \Gamma^{bc} - A_d \tilde{G}^a_{p-1} \Gamma^b + \frac{A_d}{2} \bar{G}^{bc}_{p-2} \Gamma_{123}, \\
\tilde{\mathcal{F}}_{p+3} &= A_0 \bar{G}^0_p \Gamma_{123}.
\end{align*}
\]

Then one reads off the expression for the T-dual forms which is similar to that in (C.4), i.e.

\[
\tilde{\mathcal{F}}_p = \tilde{\mathcal{G}}^{(0)}_p + \tilde{\mathcal{G}}^a_{p-1} \wedge \varepsilon^a + \frac{1}{2} \tilde{\mathcal{G}}^{ab}_{p-2} \wedge \varepsilon^a \wedge \varepsilon^b + \tilde{\mathcal{G}}^{(3)}_{p-3} \wedge \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3,
\]

(C.11)

where

\[
\begin{align*}
\tilde{\mathcal{G}}^{(0)}_p &= e^{\Phi} \Phi^{-1} \left( -A_0 G^{(3)}_p + A_d G^a_p \right), \\
\tilde{\mathcal{G}}^a_{p-1} &= e^{\Phi} \Phi^{-1} \left( -\frac{A_0}{2} \delta^{abc} G^{bc}_{p-1} + A_b G^a_{p-1} + A_d \bar{G}^{(0)}_{p-1} \right), \\
\tilde{\mathcal{G}}^{ab}_{p-2} &= e^{\Phi} \Phi^{-1} \left[ e^{abc} \left( A_c G^{(3)}_{p-2} + A_0 G^c_{p-2} \right) - (A_d G^b_{p-2} - A_b G^a_{p-2}) \right], \\
\tilde{\mathcal{G}}^{(3)}_{p-3} &= e^{\Phi} \Phi^{-1} \left( \frac{A_d}{2} e^{abc} G^{bc}_{p-3} + A_0 \bar{G}^{(0)}_{p-3} \right).
\end{align*}
\]

(C.12)

We would like to show that the non-Abelian T-duality transformation preserved the degrees of freedom associated to the fact that not all of the RR \(p\)-forms are independent, by rather those with rank higher than five are related to those with lower than five rank, as stated by (2.48). To proceed consider a \(p\)-form \(X_p\) with legs in the seven-dimensional manifold. Then note the identities

\[
\begin{align*}
\ast X_p &= \ast_7 X_p \wedge \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3, \\
\ast (X_p \wedge \varepsilon^a) &= \frac{(-1)^{p+1}}{2} e^{abc} \ast_7 X_p \wedge \varepsilon^b \wedge \varepsilon^c \\
\ast (X_p \wedge \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3) &= (-1)^{p+1} \ast_7 X_p, \\
\ast (X_p \wedge \varepsilon^a \wedge \varepsilon^b) &= e^{abc} \ast_7 X_p \wedge \varepsilon^c.
\end{align*}
\]

(C.13)
valid for a Minkowskian seven-dimensional signature spacetime. Then the condition (2.48) that relates higher and lower forms gives the conditions

\[
\begin{align*}
G^{(0)}_p &= (-1)^{p+\left\lfloor \frac{p}{2} \right\rfloor} \star_7 G^{(3)}_{7-p}, \\
G^{a}_{p-1} &= (-1)^{\left\lfloor \frac{p}{2} \right\rfloor} e^{abc} \star_7 G^{bc}_{8-p}, \\
G^{ab}_{p-2} &= (-1)^{p+\left\lfloor \frac{p}{2} \right\rfloor} e^{abc} \star_7 G^{c}_{9-p}, \\
G^{(3)}_{p-3} &= (-1)^{\left\lfloor \frac{p}{2} \right\rfloor} \star_7 G^{(0)}_{10-p},
\end{align*}
\]

which, as one may verify are self consistent. In particular, the first and the fourth as well as the second and third are related by seven-dimensional Hodge duality. Finally, using the above we can check that the T-dual forms (C.12) preserve (C.14), equivalently (2.48), as well. This is a non-trivial check of various relevant factors in the T-duality transformations of the RR flux fields.

D  SUGRA review and conventions

D.1 Brief review of IIB supergravity

\[
S_{IIB} = \frac{1}{2\kappa^2} \int_{M_{10}} \sqrt{g} \left[ e^{-2\Phi} \left( R + 4(\partial \Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left( F^2_1 + \frac{F^2_3}{3!} + \frac{1}{2} \frac{F^2_5}{5!} \right) \right] - \frac{1}{2} C_4 \wedge H \wedge dC_2,
\]

where the potentials are

\[
H = dB, \quad F_1 = dC_0, \quad F_3 = dC_2 - C_0 H, \quad F_5 = dC_4 - H \wedge C_2.
\]

The field strength for \( F_5 \) is self dual (imposed by hand here). The Bianchi identities

\[
dH = 0, \quad dF_1 = 0, \quad dF_3 = H \wedge F_1, \quad dF_5 = H \wedge F_3.
\]
D.2 Brief review of (massive) IIA supergravity

In the conventions of [51], the action of the massive type-IIA supergravity [76] is given by

\[ S_{\text{Massive IIA}} = \frac{1}{2\kappa^2} \int_{M_{10}} \sqrt{g} \left[ e^{-2\Phi} \left( R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left( m^2 + \frac{F_2^2}{2} + \frac{F_4^2}{4!} \right) \right] \]

\[ - \frac{1}{2} \left( d\mathcal{C}_3 \wedge d\mathcal{C}_3 \wedge B + \frac{m}{2} d\mathcal{C}_3 \wedge B^3 + \frac{m^2}{20} B^5 \right), \tag{D.4} \]

where the field strengths are defined as

\[ H = dB, \quad F_2 = d\mathcal{C}_1 + mB, \quad F_4 = d\mathcal{C}_3 - H \wedge \mathcal{C}_1 + \frac{m}{2} B \wedge B, \tag{D.5} \]

and where \( m \) is the mass parameter. Note that, the presence of the \( \frac{1}{2} m^2 \) term in the action reveals that \( m \) plays the rôle of a zero-form \( F_0 \). The field strengths are invariant under the gauge transformations

\[ \delta B = d\Lambda, \quad \delta \mathcal{C}_1 = -m\Lambda, \quad \delta \mathcal{C}_3 = -m\Lambda \wedge B, \tag{D.6} \]

where \( \Lambda \) is a one-form. The Bianchi identities are

\[ dH = 0, \quad dF_2 = mH, \quad dF_4 = H \wedge F_2. \tag{D.7} \]

The topological term in the action can be written as

\[ -\frac{1}{2} \int_{M_{10}} d\mathcal{C}_3 \wedge d\mathcal{C}_3 \wedge B + \frac{m}{3} d\mathcal{C}_3 \wedge B^3 + \frac{m^2}{20} B^5 = -\frac{1}{2} \int_{M_{11}} F_4 \wedge F_4 \wedge H, \tag{D.8} \]

where \( \partial M_{11} = M_{10} \), so that gauge invariance under (D.6) becomes manifest.

The equations of motions that follow from varying the metric are

\[ R_{\mu\nu} + 2D_\mu D_\nu \Phi - \frac{1}{4} H^2_{\mu\nu} = e^{2\Phi} \left[ \frac{1}{2} (F_2^2)_{\mu\nu} + \frac{1}{12} (F_4^2)_{\mu\nu} \right. \]

\[ \left. - \frac{1}{4} g_{\mu\nu} \left( \frac{1}{2} F_2^2 + \frac{1}{24} F_4^2 + m^2 \right) \right], \tag{D.9} \]
whereas the dilaton equation is

\[ R + 4D^2 \Phi - 4(\partial \Phi)^2 - \frac{1}{12}H^2 = 0 . \tag{D.10} \]

From varying the fluxes we obtain (after simplifying using Bianchi identities)

\[ d \left( e^{-2\Phi} \star H \right) - F_2 \wedge \star F_4 - \frac{1}{2} F_4 \wedge F_4 = m \star F_2 , \]
\[ d \star F_2 + H \wedge \star F_4 = 0 , \tag{D.11} \]
\[ d \star F_4 + H \wedge F_4 = 0 . \]

This set of equations is consistent with the Bianchi identities as it can be seen by applying to each one of them the exterior derivative. In particular, we note the necessity of the term proportional to \( m \) in the right hand side of the first of (D.11).

The Bianchi identities and equations of motion can be recast in the following way:

\[ 0 = d(F_2 - mB) , \]
\[ 0 = d(F_4 - B \wedge F_2 + \frac{1}{2}mB \wedge B) , \]
\[ 0 = d(F_6 - B \wedge F_4 + \frac{1}{2}B^2 \wedge F_2 - \frac{1}{6}mB^3) , \]
\[ 0 = d(F_8 - B \wedge F_6 + \frac{1}{2}B^2 \wedge F_4 - \frac{1}{6}B^3 \wedge F_2 + \frac{1}{24}mB^4) \tag{D.12} \]

in which we defined \( F_6 = - \star F_4 \) and \( F_8 = \star F_2 \).

### D.3 Supersymmetry

Our conventions for supersymmetry variations follow those of [44]. To package these variations we find it handy to introduce a Killing spinor comprising of real Majorana–Weyl spinors

\[ e = \left( \begin{array}{c} e_+ \\ e_- \end{array} \right) . \tag{D.13} \]

In type-IIB we have \( \Gamma^{11} e = e \), while in type-IIA the conventions are such that \( \Gamma^{11} e_\pm = \mp e_\pm \), that is:

\[ \text{IIB} : \quad \Gamma_{11} e = 1_2 e , \quad \text{IIA} : \quad \Gamma_{11} e = -\sigma_3 e . \tag{D.14} \]
Using Pauli matrices, the type-IIA Killing spinor equations can be written as

$$\delta \lambda = \frac{1}{2} \partial \Phi e - \frac{1}{24} H \sigma_3 e + \frac{1}{8} e^\Phi \left[ 5m\sigma^1 + \frac{3}{2} F_2 (i\sigma^2) + \frac{1}{24} F_4 \sigma^1 \right] e,$$

$$\delta \psi_\mu = D_\mu e - \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho} \sigma_3 e + \frac{e^\Phi}{8} \left[ m\sigma^1 + \frac{1}{2} F_2 (i\sigma^2) + \frac{1}{24} F_4 \sigma^1 \right] \Gamma_\mu e,$$

where $D_\mu e = \partial_\mu e + \frac{1}{4} \omega^a_{\mu} \Gamma_{ab} e$. The Killing spinor of type-IIB are

$$\delta \lambda = \frac{1}{2} \partial \Phi e - \frac{1}{24} H \sigma_3 e + \frac{1}{2} e^\Phi \left[ F_1 (i\sigma^2) + \frac{1}{12} F_3 \sigma^1 \right] e,$$

$$\delta \psi_\mu = D_\mu e - \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho} \sigma_3 e - \frac{e^\Phi}{8} \left[ F_1 (i\sigma^2) + \frac{1}{6} F_3 \sigma^1 + \frac{1}{24} \frac{5!}{5^5} F_5 (i\sigma^2) \right] \Gamma_\mu e,$$

where as always we are using the notation $F_n \equiv F_{i_1...i_n} \Gamma^{i_1...i_n}$.

For IIB it is helpful to combine the MW spinors into a complex $e = e_+ + ie_-$. For the simplest case where we only have $F_5$ turned on we have a trivial dilation variation and

$$\delta \psi_\mu = D_\mu e + \frac{ie^\Phi}{2 \times 8 \times 5!} F_{\mu_1...\mu_5} \Gamma^{\mu_1...\mu_5} \Gamma_\mu e = 0$$

(E.17)

### E  SUSY in the dual of KW

In this appendix we evaluate explicitly the supersymmetry in the T-dual geometry of Klebanov-Witten.

Let us begin with the type IIB killing spinors of $T^{(1,1)}$ which obey

$$\Gamma_{12} \eta_1 = -\eta_2, \quad \Gamma_{45} \eta_1 = \eta_2, \quad \Gamma_{11} \eta_i = \eta_i.$$  \hspace{1cm} (E.1)

Let us define

$$e_1 = \eta_1, \quad e_2 = \Omega \eta_2,$$

(E.2)

which have chiralities

$$\Gamma_{11} e_1 = e_1, \quad \Gamma_{11} e_2 = -e_2,$$

(E.3)

and thus these combine to give an ansatz for the Killing spinors of type IIA.
We begin with the dilatino equation

\[
\delta \lambda_1 = \frac{1}{2} \Phi \epsilon_1 - \frac{1}{24} \mathcal{H} \epsilon_1 + \frac{1}{8} e^{\Phi} \left[ \frac{3}{2} F_2 + \frac{1}{24} F_4 \right] \epsilon_2 ,
\]

(E.4)

Upon inserting the above ansatz we find this vanishes. To see this first note that

\[
\epsilon_2 = e^{\Phi} \left( \lambda \lambda_1^2 \Gamma_3 \eta_1 - x_2 \lambda \Gamma_3 \eta_2 - x_1 \lambda_1 \Gamma_1 \eta_1 \right) .
\]

(E.5)

One may then use the projection conditions on the \( \eta_i \) under \( \Gamma_{45} \) and \( \Gamma_{12} \) to simplify when contracting with the fluxes.

One finds

\[
\frac{1}{2} \Phi \epsilon_1 = \frac{e^{2\Phi}}{2} \left( -x_2 \lambda^3 \Gamma_3 \eta_1 + x_1 \lambda_1^3 \Gamma_1 \eta_2 - \frac{x_1 x_2}{\lambda_1} (\lambda_1^2 - \lambda^2) \Gamma_2 \eta_2 \right) ,
\]

(E.6)

\[
\frac{3}{8} e^{\Phi} F_2 \epsilon_2 = \frac{e^{2\Phi}}{2} \left( \lambda_1^2 \lambda_1^4 \Gamma_3 \eta_2 + x_2 \lambda_1^2 \lambda_1^2 \Gamma_3 \eta_1 - x_1 \lambda \lambda_1^3 \Gamma_1 \eta_2 \right) ,
\]

(E.7)

\[
\frac{1}{8} e^{\Phi} F_4 \epsilon_2 = \frac{e^{2\Phi}}{2} \left( (x_1^2 \lambda_1^2 + x_2^2 \lambda^2) \Gamma_3 \eta_2 - x_2 \lambda_1^2 \lambda_1^2 \Gamma_3 \eta_1 + x_1 \lambda \lambda_1^3 \Gamma_1 \eta_2 \right) ,
\]

(E.8)

\[
- \frac{1}{24} \mathcal{H} \epsilon_1 = \frac{e^{2\Phi}}{2} \left( -\frac{1}{2} x_1 \lambda \lambda_1 \Gamma_1 \eta_2 + \frac{1}{2 \lambda_1} x_1 x_2 \lambda^2 \Gamma_2 \eta_2 + \lambda \left( -\frac{1}{2} x_1^2 - x_2^2 + x_2^2 \lambda_1^2 \lambda_1^{-2} - \lambda_1^4 \right) \Gamma_3 \eta_2 \right) ,
\]

(E.9)

Combining terms we get

\[
2e^{-2\Phi} \delta \lambda_1 = x_2 \lambda^2 \left( -\lambda + 2 \lambda_1^2 \right) \Gamma_3 \eta_1 - \frac{1}{2} x_1 \lambda_1 (\lambda^2 - 2 \lambda_1^2 + 4 \lambda \lambda_1^2) \Gamma_1 \eta_2 + x_1 x_2 \left( \frac{3}{2} \lambda^2 \lambda_1^{-1} - \lambda_1 \right) \Gamma_2 \eta_2 \\
+ \left( x_1^2 \left( -\frac{1}{2} \lambda + \lambda_1^2 \right) + x_2^2 \lambda \lambda_1^{-2} (\lambda^2 - \lambda_1^2 + \lambda \lambda_1^2) + \lambda \lambda_1^4 (3 \lambda - 1) \right) \Gamma_3 \eta_2 ,
\]

(E.10)

which vanishes as claimed when evaluated at the values of \( \lambda^2 = \frac{1}{5} \) and \( \lambda_1^2 = \frac{1}{6} \). The calculation of \( \delta \lambda_2 \) proceeds in the same vein with the same result.

One may also show that the gravitino variations also vanish, however we need to also take into account the dependence on the AdS coordinates for the Killing spinors. The full Killing spinors are then given by:

\[
\epsilon_{1+} = r^2 \eta_1^+, \quad \epsilon_{2+} = r^2 \Omega \eta_2^+ ,
\]

\[
\epsilon_{1-} = r^{-1} \eta_1^- + r^2 \Gamma_r (y^\mu \Gamma_y^\nu) \epsilon_{1-} , \quad \epsilon_{2-} = r^{-1} \Omega \eta_2^- + r^2 \Omega \Gamma_r (y^\mu \Gamma_y^\nu) \epsilon_{2-}
\]

(E.11)
In which the $\eta_{i\pm}$ are constant Majorana-Weyl spinors obeying
\[ \Gamma_{(11)} \eta_{i\pm} = \eta_{i\pm}, \quad \Gamma_{y^0y^1y^2} \eta_{1\pm} = \pm \eta_{2\pm}, \quad \Gamma_{12} \eta_{1\pm} = -\eta_{2\pm}, \quad \Gamma_{45} \eta_{1\pm} = \eta_{2\pm}. \] (E.13)

We conclude that the dual preserves supersymmetry.

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