On generic supercuspidal representations of $Sp_{2n}$

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Abstract

We will construct a family of irreducible generic supercuspidal representations of the symplectic groups over non-archimedean local field $F$ of odd residual characteristic. The supercuspidal representations are compactly induced from irreducible representations of the hyperspecial compact subgroup which are inflated from irreducible representations of finite symplectic groups over the finite quotient ring of the integer ring of $F$ modulo high powers of the prime element.

1 Introduction

The supercuspidal representations of a classical group $G$ (that is a unitary, symplectic or special orthogonal group) over a non-archimedean local field $F$ are realized as compactly induced representations $\text{ind}_{J}^{G(F)}\delta$ where $\delta$ is an irreducible finite dimensional complex representation of an open compact subgroup $J$ of $G(F)$ (see [5]).

If $G$ is quasi-split and unramified over $F$, then $G$ has a smooth model over the integer ring $O_{F}$ of $F$. In this case, [1] shows that the supercuspidal representation $\text{ind}_{J}^{G(F)}\delta$ is generic if and only if $J$ is hyperspecial. We will consider the case $J = G(O_{F})$.

The irreducible complex representations of $G(O_{F})$ come from those of the finite group $G(O_{F}/p^{r})$ ($p$ is the maximal ideal of $O_{F}$) via the canonical surjection $G(O_{F}) \to G(O_{F}/p^{r})$. If $r = 1$, we have well-developed theories on the representation of the finite reductive group $G(\mathbb{F})$ ($\mathbb{F} = O_{F}/p$ is the residue class field of $F$). The understanding of the cases $r > 1$ is less complete. For example the representation theory of $GL_{n}(O_{F}/p^{r})$ with $r > 1$ is studied by [4] and then [2] to construct supercuspidal representations of $GL_{n}(F)$, however their descriptions of representations of $GL_{n}(O_{F}/p^{r})$ are rather complicated and not suitable for detailed treatment or for generalization to other classical groups.

Recently [6] gives a uniform and quite explicit description of the irreducible representations of $G(O_{F}/p^{r})$ ($r > 1$) associated with the regular adjoint orbit, under the assumption of triviality of certain Schur multiplier. The Schur multiplier is trivial if the characteristic polynomial of the adjoint orbit is irreducible modulo $p$. So starting from this explicit description of irreducible complex representations $\delta$ of $G(O_{F})$, we may be able to give a simple proof of the fact that the compactly induced representation $\text{ind}_{G(O_{F})}^{G(F)}\delta$ is a generic supercuspidal representation. It also gives a good foundation for the further detailed studies on the generic supercuspidal representations of classical groups.

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In this paper, we will study this procedure in detail for the case of the symplectic group. Our main results are presented as Theorem 2.2.1. The rest of this paper is devoted to the proofs.

2 Main results

2.1 Let $F$ be a non-dyadic non-archimedian local field. The integer ring of $F$ is denoted by $O_F$ with the maximal ideal $\mathfrak{p}$ generated by $\varpi$. The residue class field is denoted by $\mathbb{F} = O_F/\mathfrak{p}$. Let $\tau$ be a continuous unitary additive character of $F$ such that

$$\{x \in F \mid \tau(xO_F) = 1\} = O_F.$$ 

Let $G = Sp_{2n}$ be the symplectic group and $g = sp_{2n}$ its Lie algebra with respect to the symplectic form

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where $I_n = \begin{bmatrix} \ddots & 1 \\ & 1 \end{bmatrix}$.

that is to say $G$ (resp. $\mathfrak{g}$) is a smooth $O_F$-group scheme (resp. affine $O_F$-scheme) such that

$$G(L) = \{g \in GL_{2n}(L) \mid gJ_n^t g = J_n\}$$

(resp. $\mathfrak{g}(L) = \{X \in \mathfrak{gl}_{2n}(L) \mid XJ_n + J_n^t X = 0\}$)

for any commutative $O_F$-algebra $L$. Let us denote by

$$B : \mathfrak{g} \times \mathfrak{g} \to \text{Spec}(O_F[t])$$

the trace form, that is $B(X, Y) = \text{tr}(XY)$ for all $X, Y \in \mathfrak{g}(L)$ with any commutative $O_F$-algebra $L$.

For any $O_F$-group subscheme $H \subset G$ and for any integers $0 < r < l$, let us denote by $H(O_F/\mathfrak{p}^r)$ (resp. $H(\mathfrak{p}^r)$) the kernel of the canonical group homomorphism $H(O_F/\mathfrak{p}^r) \to H(O_F/\mathfrak{p}^l)$ (resp. $H(O_F) \to H(O_F/\mathfrak{p}^r)$). These group homomorphisms are surjective if $H$ is smooth over $O_F$.

2.2 Fix an integer $r > 1$ and put $l = l + l'$ with the smallest integer $l$ such that $0 < l' \leq l$, in other word

$$l' = \begin{cases} l & \text{if } r = 2l \text{ is even}, \\ l - 1 & \text{if } r = 2l - 1 \text{ is odd}. \end{cases}$$

Fix a $\beta \in \mathfrak{g}(O_F)$ such that the reduction modulo $\mathfrak{p}$ of the characteristic polynomial $\chi_\beta(t)$ of $\beta \in \mathfrak{gl}_{2n}(O_F)$ is irreducible over $\mathbb{F}$. Then the $F$-subalgebra $K = F[\beta]$ of the matrix algebra $M_{2n}(F)$ is an unramified field extension of $F$ of degree $2n$. The element $\tau \in \text{Gal}(K/F)$ of order 2 of the cyclic extension $K/F$ is given by $x^r = J_n \cdot xJ_n^{t-1}$. In particular $\beta^r = -\beta$. The integer ring and the residue class field of $K$ are denoted by $O_K$ and $\mathbb{K} = O_K/\varpi O_K$ respectively. On the other hand $\beta$ is a regular semisimple element of $\mathfrak{g}(F)$ and its centralizer $T = Z_G(\beta)$ in $G$ is a maximal torus of $G$ which splits over $K$. More precisely $T$ is a smooth $O_F$-group subscheme of $G$ such that

$$T(L) = \{g \in L[\mathfrak{g}]^\times \mid gJ_n^t gJ_n^{t-1} = 1_{2n}\}$$

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for any commutative $O_F$-algebra where $\beta \in g(L)$ is the image of $\beta \in g(O_F)$ by the canonical homomorphism $g(O_F) \to g(L)$. In particular

$$T(O_F/p^r) = \left\{ \pi \in (O_K/w^rO_K)^\times \mid \pi \in O_K^{s.t.} N_{K/k}(\pi) = 1 \right\}$$ (1)

where $K_+ \subset K$ is the fixed subfield of $\tau \in \text{Gal}(K/F)$.

A group isomorphism $g(O_F/p^r) \to G(p^r/p^r)$ is given by

$$X \pmod{p^r} \mapsto 1 + \omega^j X \pmod{p^r}.$$ Let us denote by $\psi_\beta$ the unitary character of $G(p^r/p^r)$ defined by

$$\psi_\beta(1 + \omega^j X \pmod{p^r}) = \tau \left( \omega^{-r} B(X, \beta) \right).$$

On $T(O_F/p^r) \cap G(p^r/p^r)$, the character $\psi_\beta$ is described by

$$\psi_\beta(1 + \omega^j x \pmod{p^r}) = \tau \left( \omega^{-r} T_{K/F}(x) \right)$$

with $x \in O_K = O_F[\beta]$ such that $x + x^\tau \equiv 0 \pmod{p^r}$.

The equivalence classes of the irreducible complex representations of the finite group $G(O_F/p^r)$ is denoted by $\text{Irr}(G(O_F/p^r))$ which is identified with the equivalence classes of the representations of the compact group $G(O_F)$ trivial on $G(p^r)$.

Let us denote by $\text{Irr}(G(O_F/p^r), \psi_\beta)$ the equivalence classes of the irreducible complex representations $\delta$ of $G(O_F/p^r)$ such that the restriction $\delta|_{G(p^r/p^r)}$ contains the character $\psi_\beta$.

The set of the group homomorphisms $\theta : T(O_F/p^r) \to \mathbb{C}^\times$ such that $\theta = \psi_\beta$ on $T(O_F/p) \cap G(p^r/p^r)$ is denoted by $\Theta(T(O_F/p^r), \psi_\beta)$.

Then a bijection $\theta \mapsto \delta_{\beta, \theta}$ of $\Theta(T(O_F/p^r), \psi_\beta)$ onto $\text{Irr}(G(O_F/p^r), \psi_\beta)$ is given as an application of the general theory developed by [6]. See section 3 for some details.

Now our main result is

**Theorem 2.2.1** For any $\theta \in \Theta(T(O_F/p^r), \psi_\beta)$, the compactly induced representation $\pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta}$ is

1) irreducible generic supercuspidal representation of $G(F)$,

2) with the formal degree

$$\dim \delta_{\beta, \theta} = q^n q^{-c}(1 - q^{-n}) \prod_{k=1}^{n-1} (1 - q^{-2k})$$

if the Haar measure of $G(F)$ is normalized so that the volume of $G(O_F)$ is one.

3) The irreducible representation $\delta_{\beta, \theta}$ is contained in $\pi_{\beta, \theta}|_{G(O_F)}$ with multiplicity one, and

4) if an irreducible complex representation $\delta$ of $G(O_F/p^r)$ is contained in $\pi_{\beta, \theta}|_{G(O_F)}$, then $\delta = \delta_{\beta, \theta}$.

The explicit description of $\delta_{\beta, \theta}$ will be given in the next section. The proof of Theorem 2.2.1 except the genericity of $\pi_{\beta, \theta}$ is given in Section 3. The genericity is proved in Section 5.
3 Representations of hyperspecial compact subgroup

In this section, we will recall the explicit descriptions of the bijection $\theta \mapsto \delta_{\beta,\theta}$ of $\Theta(T(O_F/p^r), \psi_\beta)$ onto $\text{Irr}(G(O_F/p^r), \psi_\beta)$ given by [3].

We will keep the notations of section 2.

3.1 Let us denote by $H(O_F/p^r)$ the subgroup of $g \in G(O_F/p^r)$ such that

$$\psi_{\beta}(ghg^{-1}) = \psi_{\beta}(h) \text{ for all } h \in G(p'/p^r).$$

In other word

$$H(O_F/p^r) = \left\{ g \pmod{p^r} \in G(O_F/p^r) \mid \text{Ad}(g)_{\beta} \equiv \beta \pmod{p^r} \right\}$$

$$= T(O_F/p^r) \cdot G(p'/p^r)$$

because $T$ is a smooth $O_F$-group scheme, and hence the canonical group homomorphism $T(O_F/p^r) \rightarrow T(O_F/p)$ is surjective.

Let us denote by $\text{Irr}(H(O_F/p^r), \psi_\beta)$ the set of the equivalence classes of the irreducible complex representations $\sigma$ of $H(O_F/p^r)$ such that the restriction $\sigma|_{G(p'/p^r)}$ contains the character $\psi_\beta$. Then Clifford’s theory says that

$$\sigma \mapsto \text{Ind}_{H(O_F/p^r)}^{G(O_F/p^r)} \sigma$$

gives a bijection of $\text{Irr}(H(O_F/p), \psi_\beta)$ onto $\text{Irr}(G(O_F/p^r), \psi_\beta)$. We have a bijection $\theta \mapsto \sigma_{\beta,\theta}$ of $\Theta(T(O_F/p^r), \psi_\beta)$ onto $\text{Irr}(H(O_F/p^r), \psi_\beta)$ described as in the following two subsections. Then

$$\theta \mapsto \delta_{\beta,\theta} = \text{Ind}_{H(O_F/p^r)}^{G(O_F/p^r)} \sigma_{\beta,\theta} \quad (G = G(O_F/p^r), H = H(O_F/p^r))$$

gives the required bijection of $\Theta(T(O_F/p^r), \psi_\beta)$ onto $\text{Irr}(G(O_F/p^r), \psi_\beta)$.

3.2 Suppose that $r = 2l$ is even. Then

$$H(O_F/p^r) = T(O_F/p^r) \cdot G(p'/p^r)$$

and $\theta \in \Theta(T(O_F/p^r), \psi_\beta)$ define a character $\sigma_{\beta,\theta}$ of $H(O_F/p^r)$ by

$$\sigma_{\beta,\theta}(gh) = \theta(g) \cdot \psi_\beta(h)$$

for $g \in T(O_F/p^r)$ and $h \in G(p'/p^r)$.

3.3 Suppose that $r = 2l - 1 \geq 3$ is odd. In this case we have

$$H(O_F/p^r) = T(O_F/p^r) \cdot G(p^{l-1}/p^r).$$

We will construct a representation of $G(p^{l-1}/p^r)$ by means of Schrödinger representation, and then we will extend it to a representation of $H(O_F/p^r)$ by means of Weil representation over the finite field $\mathbb{F}$.

Let $t = \text{Lie}(T)$ be the Lie algebra of the $O_F$-group scheme $T = Z_G(\beta)$. Then, for any commutative $O_F$-algebra $L$

$$t(L) = \{ X \in g(L) \mid [X, \beta] = 0 \}$$
is the centralizer of \( \mathfrak{F} \in \mathfrak{g}(L) \) (the image of \( \beta \in \mathfrak{g}(O_F) \) by the canonical homomorphism \( \mathfrak{g}(O_F) \to \mathfrak{g}(L) \)) in \( \mathfrak{g}(L) \).

Let us denote by \( Z(p^{l-1}/p^r) \) the inverse image of \( t(\mathbb{F}) \) under the surjective group homomorphism \( G(p^{l-1}/p^r) \to \mathfrak{g}(\mathbb{F}) \) defined by

\[
1_{2n} + \omega^{-1}X \pmod{p^r} \mapsto X \pmod{p}
\]

which is a normal subgroup of \( G(p^{l-1}/p^r) \) containing \( G(p^l/p^r) \).

Put \( \mathbb{V} = \mathfrak{g}(\mathbb{F})/t(\mathbb{F}) \) which is a symplectic \( \mathbb{F} \)-space with respect to the symplectic form \( (X,Y) = B([X,Y],\mathbb{F}) \) for \( X,Y \in \mathfrak{g}(\mathbb{F}) \) and \( \beta = \beta \pmod{p} \in \mathfrak{g}(\mathbb{F}) \).

Sice \( B|_{t(\mathbb{F}) \times t(\mathbb{F})} \) is non-degenerate, we have \( \mathfrak{g}(\mathbb{F}) = \mathbb{V} \oplus t(\mathbb{F}) \) with

\[
V = \{ X \in \mathfrak{g}(\mathbb{F}) \mid B(X,t(\mathbb{F})) = 0 \}.
\]

Let us denote by \( v \mapsto [v] \) the inverse \( \mathbb{F} \)-linear mapping of the canonical \( \mathbb{F} \)-linear isomorphism \( \mathbb{V} \to \mathbb{V} \).

Let \( H_\beta \) be the Heisenberg group associated with the symplectic space \( (\mathbb{V},(,)_\beta) \), that is \( H_\beta = \mathbb{V} \times \mathbb{C}^1 \) with group operation

\[
(u,s) \cdot (v,t) = (u + v, s \cdot t \cdot \tilde{\tau}(2^{-1}(u,v)_\beta))
\]

where \( \tilde{\tau}(x \pmod{p}) = \tau(x^{-1}) \) and \( \mathbb{C}^1 \) is the multiplicative group of complex numbers of absolute value one. Fix a polarization \( \mathbb{V} = \mathbb{W}' \oplus \mathbb{W} \). Then we have an irreducible unitary representation \( (\omega_\beta, L^2(\mathbb{W}')) \) of \( H_\beta \) called Schrödinger representation. Here \( L^2(\mathbb{W}') \) is the complex vector space of the complex-valued functions on \( \mathbb{W}' \), and

\[
(\omega_\beta(u,s)f)(w) = s \cdot \tilde{\tau}(2^{-1}(u_-,u_+)_\beta + (w,u_+)\beta) \cdot f(w + u_+)
\]

for \( f \in L^2(\mathbb{W}') \) and \( (u,s) \in H_\beta \) with \( u = u_- + u_+ \) \((u_-,u_+ \in \mathbb{W}',u_+ \in \mathbb{W})\). By a general theory of Weil representation over finite field \( \mathbb{F} \), there exists a group homomorphism \( \Omega : \text{Sp}(\mathbb{V}) \to GL_C(L^2(\mathbb{W}')) \) such that

\[
\Omega_\beta(\omega\sigma,s) = \Omega(\sigma)^{-1} \circ \omega_\beta(u,s) \circ \Omega(\sigma)
\]

for all \( \sigma \in \text{Sp}(\mathbb{V}) \) and \( (u,s) \in H_\beta \).

Fix additive character \( \rho \) of \( t(\mathbb{F}) \). Then an irreducible representation \( \omega_{\beta,\rho} \) of \( G(p^{l-1}/p^r) \) on \( L^2(\mathbb{W}') \) is defined as follows. Take an element \( h = 1_{2n} + \omega^{-1}T \pmod{p^r} \) of \( G(p^{l-1}/p^r) \) and hence \( T(\text{mod} \ p) \in \mathfrak{g}(\mathbb{F}) \). Put \( T(\text{mod} \ p) = [v] + Y \) with \( v \in \mathbb{V} \) and \( Y \in t(\mathbb{F}) \). Then \( \omega_{\beta,\rho}(h) \in GL_C(L^2(\mathbb{W}')) \) is defined by

\[
\omega_{\beta,\rho}(h) = \tau(\omega^{-1}B(T, \beta) - 2^{-1}B(T^2, \beta)) \cdot \rho(Y) \cdot \omega_\beta(v,1).
\]

If \( h \in Z(p^{l-1}/p^r) \), then \( \omega_{\beta,\rho}(h) \) is the homothety of

\[
\psi_{\beta,\rho}(h) = \tau(\omega^{-1}B(T, \beta) - 2^{-1}B(T^2, \beta)) \cdot \rho(T(\text{mod} \ p)),
\]

where \( \psi_{\beta,\rho} \) is a character of \( Z(p^{l-1}/p^r) \) which coincides with \( \psi_\beta \) on \( G(p^l/p^r) \), and all extensions of \( \psi_\beta \) to \( Z(p^{l-1}/p^r) \) are given in this way. Furthermore we have

\[
\text{Ind}_{Z(p^{l-1}/p^r)}^{G(p^{l-1}/p^r)} \psi_{\beta,\rho} = \bigoplus_{\omega_{\beta,\ rho}} \omega_{\beta,\ rho}
\]
where
\[ \dim \omega_{\beta, \rho} = q^{2 - 1(\dim} g(\mathbb{F}) - t(\mathbb{F})) = q^{n^2}. \]

Take a \( g \in T(O_F/p^r) \) and put \( \mathfrak{g} = g(\text{mod } \mathfrak{p}) \in T(\mathbb{F}) \). Then \( \sigma_{\mathfrak{g}} \in Sp(V) \) is defined by \( v \sigma_{\mathfrak{g}} = Ad(\mathfrak{g})^{-1}X(\text{mod } t(\mathbb{F})) \) for \( v = X(\text{mod } t(\mathbb{F})) \in V \). Then an irreducible representation \( (\sigma_{\beta, \rho}, L^2(W')) \) of \( H(O_F/p^r) = T(O_F/p^r) \cdot G(p^{l-1}/p^r) \) is defined by
\[ \sigma_{\beta, \rho}(gh) = \Omega(\sigma_{\mathfrak{g}}) \circ \omega_{\beta, \rho}(h) \]
for \( g \in T(O_F/p^r) \) and \( h \in G(p^{l-1}/p^r) \). Note that on
\[ T(O_F/p^r) \cap G(p^{l-1}/p^r) \subset Z(p^{l-1}/p^r), \]
\( \sigma_{\beta, \rho} \) is the homothety by \( \psi_{\beta, \rho} \).

Now take a \( \theta \in \Theta(T(O_F/p^r), \psi_{\beta}) \) and restrict \( \rho \) by the condition that \( \psi_{\beta, \rho} = \theta \) on \( T(O_F/p^r) \cap G(p^{l-1}/p^r) \). In other word \( \rho \) is given by
\[ \rho(Y \pmod{\mathfrak{p}}) = \tau (-\omega^{-1}B(Y, \beta)) \cdot \theta(g) \]
for \( Y \in t(O_F) \) and
\[ g = 1_{2n} + \omega^{l-1}Y + 2^{-1} \omega^{2l-2}Y^2 \pmod{\mathfrak{p}^r} \in T(O_F/p^r). \]

Note that the canonical morphism \( t(O_F) \to t(\mathbb{F}) \) is surjective. Then finally the representation \( \sigma_{\beta, \theta} \) of \( H(O_F/p^r) \) is defined by
\[ \sigma_{\beta, \theta}(gh) = \theta(g) \cdot \sigma_{\beta, \rho}(gh) \]
for \( g \in T(O_F/p^r) \) and \( h \in G(p^{l-1}/p^r) \).

### 3.4 The explicit construction of \( \beta_{\beta, \theta} \) being given, the dimension of it is

**Proposition 3.4.1** For any \( \theta \in \Theta(T(O_F/p^r), \psi_{\beta}) \), we have
\[ \dim \delta_{\beta, \theta} = q^{n^2}r(1 - q^{-n}) \prod_{k=1}^{n-1} (1 - q^{-2k}). \]

**[Proof]** To begin with
\[ \dim \delta_{\beta, \theta} = (G(O_F/p^r) : H(O_F/p^r)) \cdot \dim \sigma_{\beta, \theta} \]
and
\[ \dim \sigma_{\beta, \theta} = \begin{cases} 1 & : \text{if } r = 2l \text{ is even,} \\ q^{l^2} & : \text{if } r = 2l - 1 \text{ is odd.} \end{cases} \]

Since \( H(O_F/p^r) = T(O_F/p^r) \cdot G(p^r/p^{r'}), \) we have
\[ |H(O_F/p^r)| = \frac{|T(O_F/p^r)||G(p^r)|}{|T(O_F/p^r) \cap G(p^{l-1}/p^r)|} \]
\[ = |T(O_F/p^r)| \cdot \frac{|G(O_F/p^r)|}{G(O_F/p^{r'})}. \]
Now
\[ |G(O_F/p^r)| = q^{n(n+1)r} \prod_{k=1}^{n} (1 - q^{-2k}). \]

On the other hand we have an exact sequence
\[ 1 \to T(O_F/p^r) \to (O_K/\varpi^rO_K)^{\times} \xrightarrow{N_{k'/k}^\times} (O_{K_+}/\varpi^rO_{K_+})^{\times} \to 1 \]
because of (1), and hence \( |T(O_F/p^r)| = q^{nr}(1 + q^{-n}). \)

4 Construction of irreducible supercuspidal representations

In this section we will prove Theorem 2.2.1 except the genericity.

We will keep the notations of the preceding sections. Fix a \( \theta \in \Theta(T(O_F/p^r), \psi_\beta) \).

Let us denote by \( V_{\beta, \theta} \) the representation space of \( \delta_{\beta, \theta} \).

4.1 To begin with, we will fix standard parabolic subgroups of \( G = \text{Sp}_{2n} \).

\[ B = \begin{Bmatrix} \begin{bmatrix} a & b \\ 0 & \tau_a^{-1} \end{bmatrix} \in \text{Sp}_{2n} \end{Bmatrix} = L \times U \]
is a Borel subgroup where

\[ L = \begin{Bmatrix} \begin{bmatrix} a & 0 \\ 0 & \tau_a^{-1} \end{bmatrix} \mid a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \end{Bmatrix}, \]

\[ U = \begin{Bmatrix} \begin{bmatrix} a & b \\ 0 & \tau_a^{-1} \end{bmatrix} \in \text{Sp}_{2n} \mid a = \begin{bmatrix} 1 & * \\ \vdots & \ddots \end{bmatrix} \end{Bmatrix}. \]

We use the notation \( \tau_a = I_n^t a I_n \) for a square matrix \( a \) of size \( n \), that is the matrix transposed with respect to the second diagonal. The standard maximal parabolic subgroups \( P_i = L_i \times U_i \) \((1 \leq i \leq n)\) are given by

\[ L_i = \begin{Bmatrix} \begin{bmatrix} g & a & b \\ c & d & \tau_a^{-1} \end{bmatrix} \mid g \in \text{GL}_i, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Sp}_{2(n-i)} \end{Bmatrix}, \]

\[ U_i = \begin{Bmatrix} \begin{bmatrix} 1_i & * & * & * \\ 1_{n-i} & 0 & 0 & * \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \in \text{Sp}_{2n} \end{Bmatrix}. \]

They are all smooth \( O_F \)-group subscheme of \( G = \text{Sp}_{2n} \). The Lie algebra \( u_i \) and \( l_i \) of \( U_i \) and \( L_i \) are given by

\[ u_i = \begin{Bmatrix} \begin{bmatrix} 0 & A & B & C \\ 0 & 0 & \tau B & \tau C \\ 0 & -\tau A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid A, B \in M_{i,n-i}, C \in M_i, \tau C = C \end{Bmatrix}. \]
and

\[
l_i = \left\{ \begin{bmatrix} X & A & B \\ C & -\tau A \\ \tau X \end{bmatrix} \middle| X \in \mathfrak{gl}_n, \begin{bmatrix} A & B \\ C & -\tau A \end{bmatrix} \in \mathfrak{sp}(n \times n) \right\}
\]

respectively. The Lie algebra of \( P_i \) is \( \mathfrak{p}_i = l_i \oplus \mathfrak{u}_i \).

### 4.2 Proposition 4.2.1
For all \( 1 \leq i \leq n \), the space of \( U_i(\mathfrak{p}^{r-1}/\mathfrak{p}^r) \)-fixed vectors of \( V_{\beta,\theta} \) is trivial.

**Proof** \( G(\mathfrak{p}^l/\mathfrak{p}^r) \) is a normal subgroup of \( G(O_F/\mathfrak{p}^r) \) and \( \delta_{\beta,\theta} \mid G(\mathfrak{p}^l/\mathfrak{p}^r) \) contains the character \( \psi_{\beta} \) of \( G(\mathfrak{p}^l/\mathfrak{p}^r) \). Then

\[
\delta_{\beta,\theta} \mid G(\mathfrak{p}^l/\mathfrak{p}^r) = \bigoplus_{\beta,\theta} \psi_{\beta}^e \quad (0 < e \in \mathbb{Z})
\]

where \( \bigoplus \) is the direct sum over the representatives \( g \) of \( H(O_F/\mathfrak{p}^r)\backslash G(O_F/\mathfrak{p}^r) \) and

\[
\psi_{\beta}^e(h) = \psi_{\beta}(ghg^{-1}) = \psi_{\Ad(g^{-1})\beta}(h).
\]

Note that \( U_i(\mathfrak{p}^{r-1}/\mathfrak{p}^r) \subset G(\mathfrak{p}^{l-1}/\mathfrak{p}^r) \). If there exists a non-trivial \( U_i(\mathfrak{p}^{r-1}/\mathfrak{p}^r) \)-fixed vector in \( V_{\beta,\theta} \), then there exists a \( g \in G(O_F) \) such that \( \psi_{\Ad(g)\beta}(h) = 1 \) for all \( h \in U_i(\mathfrak{p}^{r-1}/\mathfrak{p}^r) \). This means that

\[
\tau(\varpi^{-1}B(X, \Ad(g)\beta)) = 1 \text{ that is } B(X, \Ad(g)\beta) \equiv 0 \pmod{\mathfrak{p}}
\]

for all \( X \in \mathfrak{u}_i(O_F) \). This implies that \( \Ad(g)\beta \pmod{\mathfrak{p}} \in \mathfrak{p}_i(\mathbb{F}) \), and this means that the characteristic polynomial \( \chi_\beta(t) \pmod{\mathfrak{p}} \) is reducible in \( \mathbb{F}[t] \), contradicting the assumption on \( \beta \).

### 4.3 In this subsection, we will prove the statements 1), 3) and 4) of Theorem 2.2.1 except the genericity.

To begin with

**Proposition 4.3.1** \( E = \Ind_{G(O_F)}^{G(F)} \delta_{\beta,\theta} \) is admissible \( G(F) \)-module.

**Proof** For any \( 0 < m \in \mathbb{Z} \), the space of the \( G(\mathfrak{p}^m) \)-fixed vectors is

\[
E^{G(\mathfrak{p}^m)} = \bigoplus_g V_{\beta,\theta}^{G(O_F)\cap g^{-1}G(\mathfrak{p}^m)g}
\]

where \( \bigoplus \) is the direct sum over the representatives \( g \) of the double cosets \( G(\mathfrak{p}^m)\backslash G(F)/G(\mathfrak{p}^m) \). Pick up one such representative \( g \) which should be of the form \( g = k \begin{bmatrix} a & 0 \\ 0 & \tau_a^{-1} \end{bmatrix} \) with \( k \in G(O_F) \) and

\[
a = \begin{bmatrix} \varpi^{e_1} & \cdots & \varpi^{e_n} \end{bmatrix}, \quad e_1 \geq \cdots \geq e_n \geq 0.
\]
Assume that

\[ \text{Max}\{e_k - e_{k+1} \mid 1 \leq k < n\} = e_i - e_{i+1} \geq m. \]

Then we have

\[ \begin{bmatrix} a & 0 \\ 0 & \tau_a^{-1} \end{bmatrix} U_i(O_F) \begin{bmatrix} a & 0 \\ 0 & \tau_a^{-1} \end{bmatrix}^{-1} \subset G(p^m) \]

and hence \( U_i(O_F) \subset G(O_F) \cap g^{-1}G(p^m)g \). Then

\[ V_{\beta,\theta}^{G(O_F)\cap g^{-1}G(p^m)g} \subset U_i(O_F) = \{0\} \]

by Proposition 4.2.1. Now the number of the elements \( k \begin{bmatrix} a & 0 \\ 0 & \tau_a^{-1} \end{bmatrix} \) where \( k \) is the representative of \( G(p^m)\backslash G(O_F) \) and \( a = \begin{bmatrix} e_1^a \\ \vdots \\ e_n^a \end{bmatrix} \) with \( e_1 \geq \cdots \geq e_n \geq 0, \text{Max}\{e_k - e_{k+1} \mid 1 \leq k < n\} \leq m \)

is finite. So \( E^{G(p^m)} \) is finite dimensional. ■

Next we will prove

**Proposition 4.3.2** As a \( G(O_F) \)-module

1) \( \text{ind}^{G(O_F)}_{G(O_F)\delta_{\beta,\theta}} \) contains \( \delta_{\beta,\theta} \) with multiplicity one,

2) if an irreducible representation \( \delta \) of \( G(O_F/p^r) \) is contained in \( \text{ind}^{G(O_F)}_{G(O_F)} \delta_{\beta,\theta} \), then \( \delta = \delta_{\beta,\theta} \).

**[Proof]** Take an irreducible representation \( \delta \) of \( G(O_F/p^r) \) which is identified with a representation of \( G(O_F) \) via the canonical surjection \( G(O_F) \to G(O_F/p^r) \). We have

\[
\left( \text{ind}^{G(O_F)}_{G(O_F)} \delta_{\beta,\theta} \right)|_{G(O_F)} = \bigoplus_g \text{ind}^{G(O_F)}_{G(O_F)\cap g^{-1}G(O_F)g} \delta_{\beta,\theta}^g ,
\]

where \( \bigoplus \) is the direct sum over the representatives \( g \) of the double cosets \( G(O_F)\backslash G(F)/G(O_F) \) and \( \delta_{\beta,\theta}^g(h) = \delta_{\beta,\theta}(ghg^{-1}) \). Then we have

\[
\text{Hom}_{G(O_F)} \left( \delta, \text{ind}^{G(O_F)}_{G(O_F)} \delta_{\beta,\theta} \right) = \bigoplus_g \text{Hom}_{G(O_F)} \left( \delta, \text{ind}^{G(O_F)}_{G(O_F)\cap g^{-1}G(O_F)g} \delta_{\beta,\theta}^g \right)
\]

\[
= \bigoplus_g \text{Hom}_{G(O_F)\cap g^{-1}G(O_F)g} \left( \delta, \delta_{\beta,\theta}^g \right)
\]

\[
= \bigoplus_g \text{Hom}_{g^{-1}G(O_F)g} \left( \delta^g, \delta_{\beta,\theta} \right)
\]
by Frobenius reciprocity. Take a representative $g$ of $G(O_F)/G(F)/G(O_F)$ such that $g \notin G(O_F)$. Then we can assume that

$$g = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a^{-1} \end{bmatrix} \text{ with } a = \begin{bmatrix} \omega^{e_1} & & & \\ & \ddots & & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}$$

and $e_1 \geq \cdots \geq e_i > 0$ $(1 \leq i \leq n)$. In this case

$$g^{-1}G(O_F)g \cap G(O_F) \supset g^{-1}U_i(O_F)g \cap U_i(O_F) = U_i(O_F) \supset U_i(p^{-1})$$

so that

$$\text{Hom}_{g^{-1}G(O_F)g \cap G(O_F)}(\delta^\beta, \delta_\beta, \delta) \subset \text{Hom}_{U_i(p^{-1})}(\delta^\beta, \delta_\beta, \delta).$$

Now we have $gU_i(p^{-1})g^{-1} \subset U_i(p^r)$ and $\delta$ factors through $G(O_F) \to G(O_F/p^r)$, hence

$$\text{Hom}_{g^{-1}G(O_F)g \cap G(O_F)}(\delta^\beta, \delta_\beta, \delta) = \{0\}$$

because the space of $U_i(p^{-1})$-invariant vectors in $V_{\beta, \theta}$ is trivial. Then we have

$$\text{Hom}_{G(O_F)}(\delta, \text{ind}_{G(O_F)}^G(F)\delta_\beta, \delta) = \text{Hom}_{G(O_F)}(\delta, \delta_\beta, \delta)$$

and this complete the proof. ■

Then we have

**Proposition 4.3.3** \(\text{ind}_{G(O_F)}^G(F)\delta_\beta, \delta\) is an irreducible representation of $G(F)$.

**Proof** Put $\delta = \delta_\beta, \delta$ for simplicity. Suppose that $E = \text{ind}_{G(O_F)}^G(F)\delta$ has a non-trivial $G(F)$-submodule $V$. Then we have

$$\text{Hom}_{G(F)}(V, \text{ind}_{G(O_F)}^G(F)\delta) \subset \text{Hom}_{G(F)}(V, \text{ind}_{G(O_F)}^G(F)\delta) = \text{Hom}_{G(O_F)}(V, \delta)$$

by Frobenius reciprocity so that $V$ contains $\delta$ as a $G(O_F)$-module. On the other hand $M = E/V$ is an admissible $G(F)$-module and we have

$$\text{Hom}_{G(F)}(\text{ind}_{G(O_F)}^G(F)\delta, M) = \text{Hom}_{G(F)}(M^\vee, (\text{ind}_{G(O_F)}^G(F)\delta)^\vee)$$

$$= \text{Hom}_{G(O_F)}(M^\vee, \text{Ind}_{G(O_F)}^G(F)\delta^\vee)$$

$$= \text{Hom}_{G(O_F)}(M^\vee, \delta^\vee)$$

$$= \text{Hom}_{G(O_F)}(\delta^\vee, (M^\vee|_{G(O_F)})^\vee)$$

$$= \text{Hom}_{G(O_F)}(\delta, M)$$

(here $\vee$ denote the contragredient representation) so that $M$ contains $\delta$ as a $G(O_F)$-module. Since $E$ is semisimple $G(O_F)$-module, this means that $E$ contains $\delta$ at least twice, contradicting to Proposition 4.3.2. ■

Now the compactly induced representation $\pi_{\beta, \theta} = \text{ind}_{G(O_F)}^G(F)\delta_{\beta, \theta}$ is an irreducible admissible representation of $G(F)$, it is well known that $\pi_{\beta, \theta}$ is supercuspidal.
4.4 In this subsection, we will prove the statement 2) of Theorem 2.2.1. Put $(\delta_{\beta,\theta}, V_{\beta,\theta}) = (\delta, V)$ for simplicity.

Let $d_{G(O_F)}(k)$ be the Haar measure on $G(O_F)$ such that the volume of $G(O_F)$ is one, and $d_{G(F)}(x)$ the Haar measure on $G(F)$ such that

\[ \int_{G(F)} \varphi(x) d_{G(F)}(x) = \sum_{x \in G(F)/G(O_F)} \int_{G(O_F)} \varphi(xk) d_{G(O_F)}(k) \]

for all compactly supported complex valued continuous functions $\varphi$. Then the volume of $G(O_F)$ with respect to $d_{G(F)}(x)$ is one.

Now $\text{Ind}_{G(O_F)}^{G(F)} \delta$ consists of the smooth function $\varphi : G(F) \to V$ such that

1) $\varphi(xk) = \delta(k)^{-1} \varphi(x)$ for all $k \in G(O_F)$ and

2) $\text{supp}(\varphi) \mod G(O_F)$ is compact in $G(F)/G(O_F)$.

On the other hand $\text{Ind}_{G(O_F)}^{G(F)} \delta^\vee$ consists of the smooth functions

\[ \psi : G(F) \to V^* = \text{Hom}_C(V, C) \]

such that $\psi(xk) = \psi(x) \circ \delta(k)$ for all $k \in G(O_F)$. A non-degenerate pairing

\[ \langle \cdot, \cdot \rangle : \text{Ind}_{G(O_F)}^{G(F)} \delta \times \text{Ind}_{G(O_F)}^{G(F)} \delta^\vee \to C \]

is defined by

\[ \langle \varphi, \psi \rangle = \sum_{x \in G(F)/G(O_F)} \langle \varphi(x), \psi(x) \rangle \]

where $\langle \cdot, \cdot \rangle : V \times V^* \to C$ is the canonical pairing.

Take any $v \in V$ and $\alpha \in V^*$ and define $\varphi_v \in \text{Ind}_{G(O_F)}^{G(F)} \delta$ and $\psi_\alpha \in \text{Ind}_{G(O_F)}^{G(F)} \delta^\vee$ by

\[ \varphi_v(x) = \begin{cases} \delta(x)^{-1} v & : x \in G(O_F), \\ 0 & : \text{otherwise}, \end{cases} \]

and by

\[ \psi_\alpha(x) = \begin{cases} \alpha \circ \delta(x) & : x \in G(O_F), \\ 0 & : \text{otherwise}, \end{cases} \]

respectively. Then we have

\[ \langle g \cdot \varphi_v, \psi_\alpha \rangle = \begin{cases} \langle \delta(g)v, \alpha \rangle & : g \in G(O_F), \\ 0 & : \text{otherwise} \end{cases} \]
for any $g \in G(F)$. Now
\[
\int_{G(F)} \langle g \cdot \varphi_v, \psi_\alpha \rangle \langle g^{-1} \cdot \varphi_v, \psi_\alpha \rangle d_{G(F)}(g)
= \sum_{g \in G(F)/G(O_F)} \int_{G(O_F)} \langle gk \cdot \varphi_v, \psi_\alpha \rangle \langle k^{-1} g^{-1} \cdot \varphi_v, \psi_\alpha \rangle d_{G(O_F)}(k)
= \sum_{g \in G(F)/G(O_F)} \int_{G(O_F)} \langle \varphi_\delta(k) v, \psi_\alpha \rangle \langle g \cdot \varphi_v, \psi_\delta(k) \alpha \rangle d_{G(O_F)}(k)
= \int_{G(O_F)} \langle \delta(k) v, \alpha \rangle \langle v, \delta'(k) \alpha \rangle d_{G(O_F)}(k)
= (\dim \delta)^{-1} (v, \alpha)^2 = (\dim \delta)^{-1} \langle \varphi_v, \psi_\alpha \rangle^2
\]
so that the formal degree of $\text{Ind}^{G(O_F)}_G(\delta)$ is
\[
\dim \delta = q^{n^2 + 1 - q^{-n}} \prod_{k=1}^{n-1} (1 - q^{-2k})
\]
by Proposition 3.4.1.

5 Genericity of supercuspidal representations

In this section we will show that the irreducible supercuspidal representation $\pi_{\beta, \theta} = \text{Ind}^{G(O_F)}_G(\delta_{\beta, \theta})$ constructed in the preceding section is generic, that is
\[
\text{Hom}_{G(F)} \left( \pi_{\beta, \theta}, \text{Ind}^{G(F)}_{U(F)}(\chi) \right) \neq 0
\]
for some generic character $\chi$ of $U(F)$.

5.1 The characters (that is the continuous group homomorphism to $\mathbb{C}^1$) of $U(F)$ are given by
\[
\chi_u(h) = \tau \left( \sum_{k=1}^{n-1} u_k a_{k,k+1} + u_n b_{n,1} \right) \text{ for } h = \begin{bmatrix} a & b \\ 0 & \tau a^{-1} \end{bmatrix} \in U(F)
\]
with $u = (u_1, \ldots, u_{n-1}, u_n) \in F^n$ where $a_{ij}$ and $b_{ij}$ denote the $(i,j)$-entry of the square matrices $a$ and $b$ respectively. Put $\chi_u^g(h) = \chi_u(g^{-1} hg)$ for any $g \in G(F)$.

If $g = \begin{bmatrix} t & 0 \\ 0 & \tau^{-1} \end{bmatrix} \in T(F)$, then $\chi_u^g = \chi_{u'}$ with
\[
u' = (t_1^{-1} t_2 u_1, \ldots, t_{n-1}^{-1} t_n u_{n-1}, t_n^{-2} u_n).
\]
So $\chi_u$ is generic, that is
\[
\{ g \in T(F) \mid \chi_u^g = \chi_u \} = Z(G(F)) = \{ \pm 12u \}
\]
if and only if $u_i \neq 0$ ($1 \leq i \leq n$). In this case
\[
\chi_u^\lambda = \chi_{(2,2,\ldots,2,\lambda)} \quad (\lambda \in F^\times)
\]
with suitable $g \in T(F)$. 

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5.2 Put \( u = (2, 2, \cdots, 2, \varpi^e) \). Because of Iwasawa decomposition \( G(F) = G(O_F)T(F)U(F) \), we have
\[
\text{Hom}_{G(F)} \left( \text{Ind}_{G(O_F)}^G(\delta_{\beta, \theta}) \right) \cap \text{U}(F) \chi^g_u
\]
\[
= \prod_g \text{Hom}_{G(O_F) \cap gU(F) \beta^{-1}} (\delta_{\beta, \theta}, \chi^g_u)
\]
\[
= \prod_g \text{Hom}_{U(O_F)} (\delta_{\beta, \theta}, \chi^g_u)
\]
where the last \( \prod \) is the direct product over the representatives \( g \in T(F) \) of \( G(O_F) \setminus G(F)/U(F) \). We can put \( g = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{pmatrix} \) with
\[
\varpi^m = \begin{pmatrix} \varpi^{m_1} \\ \cdots \\ \varpi^{m_n} \end{pmatrix}, \quad m = (m_1, \cdots, m_n) \in \mathbb{Z}^n.
\]
Then we have

**Proposition 5.2.1** \( \text{Hom}_{U(O_F)} (\delta_{\beta, \theta}, \chi^g_u) \neq 0 \) for some \( g \) only if \( e \equiv r \pmod{2} \).

*In this case*
\[
\chi^g_u = \chi(2\varpi^{-r}, 2\varpi^{-r}, \cdots, 2\varpi^{-r}, \varpi^{-r}).
\]

**Proof** Put where Then \( \chi^g_u = \chi_{u'} \) with
\[
u' = (2\varpi^{m_2-m_1}, \cdots, 2\varpi^{m_n-m_{n-1}}, \varpi^{e-2m_n}).
\]
If \( \ell_{i+1} - \ell_i \geq -(r-1) \) for some \( 1 \leq i < n \) or \( e - 2m_n \geq -(r-1) \), then \( \chi_{u'}^g(h) = 1 \) for all \( h \in U_i(p^{r-1}) \) for some \( 1 \leq i \leq n \). In other word \( \delta_{\beta, \theta} \) contains non-trivial \( U_i(p^{r-1}) \)-invariant vectors which contradicts to Proposition [\ref{unity}]. So we have
\[
m_{i+1} - m_i \leq -r \quad (1 \leq i < n) \quad \text{and} \quad e - 2m_n \leq -r.
\]
On the other hand \( \delta_{\beta, \theta} \) is trivial on \( G(p^r) \), and hence \( \chi_{u'}^g(h) = 1 \) for all \( h \in U(O_F) \cap G(p^r) \). This means that
\[
m_{i+1} - m_i + r \geq 0 \quad (1 \leq i < n) \quad \text{and} \quad e - 2m_n + r \geq 0.
\]
Hence \( e + r = 2m_n \) and \( m_i - m_{i+1} = r \) (1 \( \leq i \leq n \)).

Put \( \chi_r = \chi(2\varpi^{-r}, \cdots, 2\varpi^{-r}, \varpi^{-r}) \) which is regarded as a character of \( U(O_F/p^r) \). On the other hand we have
\[
\delta_{\beta, \theta} = \text{Ind}_{H(O_F/p^r)}^{G(O_F/p^r)} \sigma_{\beta, \theta}
\]
which is considered as a representation of \( G(O_F) \) via the canonical surjection \( G(O_F) \to G(O_F/p^r) \). Then, putting \( U = U(O_F/p^r) \) and \( H = H(O_F/p^r) \) for the sake of simplicity, we have
\[
\text{Hom}_{U(O_F)} (\delta_{\beta, \theta}, \chi_r) = \text{Hom}_{U} \left( \text{Ind}_{H}^{G(O_F/p^r)} \sigma_{\beta, \theta}, \chi_r \right)
\]
\[
= \bigoplus_g \text{Hom}_{U} \left( \text{Ind}_{U \cap gH_g}^{U \cap gH_g-1} \sigma_{Ad(g)\beta, \theta}, \chi_r \right)
\]
\[
= \bigoplus_g \text{Hom}_{U \cap gH_g} \left( \sigma_{Ad(g)\beta, \theta}, \chi_r \right)
\]
where $\bigoplus$ is the direct sum over the representatives of $U \setminus G(O_F/p^r)/H$ and $\theta^g(h) = \theta(g^{-1}hg)$ ($h \in T(O_F/p^r)$). Note that

$$U(O_F/p^r) \cap gH(O_F/p^r)g^{-1} = U(p^{l'}/p^r)$$ for all $g \in G(O_F/p^r)$.

To show this fact, replacing $\text{Ad}(g)\beta$ with $\beta$, it is sufficient to consider the case $g = 1_{2n}$. In this case $U(O_F/p^r) \cap H(O_F/p^r)$ is the inverse image of $U(O_F/p^{l'}) \cap T(O_F/p^{l'})$ under the canonical surjection $U(O_F/p^r) \to U(O_F/p^{l'})$. Take a

$$h = 1_{2n} + X \pmod{p^{l'}} \in U(O_F/p^r) \cap T(O_F/p^{l'}) = U(O_F/p^{l'}) \cap (O_K/\pi^{l'}O_K)^{\times}.$$

Then $X \pmod{p^{l'}} \in O_K/\pi^{l'}O_K$ is nilpotent, and hence $X \equiv 0 \pmod{p^{l'}}$. So $U(O_F/p^{l'}) \cap T(O_F/p^{l'}) = \{1\}$ and we have

$$U(O_F/p^r) \cap H(O_F/p^r) = U(p^{l'}/p^r).$$

5.3 Suppose that $r = 2l$ is even, and hence $l' = l$. In this case the character $\sigma_{\beta,\theta}$ of $H(O_F/p^{l'}) = T(O_F/p^r)\Gamma(p^{l'}/p^r)$ is defined by

$$\sigma_{\beta,\theta}(gh) = \theta(g)\psi_{\beta}(h)$$ for $g \in T(O_F/p^r)$, $h \in G(p^{l'}/p^r)$.

Then we have

$$\text{Hom}_{\text{U}(p^{l'}/p^r)}(\sigma_{\beta,\theta}, \chi_r) = \text{Hom}_{\text{U}(p^{l'}/p^r)}(\psi_{\beta}, \chi_r).$$

Hence $\text{Hom}_{\text{U}(p^{l'}/p^r)}(\sigma_{\beta,\theta}, \chi_r) \neq 0$ if and only if $\psi_{\beta} = \chi_r$ on $U(p^{l'}/p^r)$.

5.4 Suppose that $r = 2l - 1$ is odd, and hence $l' = l - 1$. In this case the irreducible representation $\sigma_{\beta,\theta}$ of $H(O_F/p^{l'}) = T(O_F/p^r)\Gamma(p^{l'}/p^r)$ is defined by

$$\sigma_{\beta,\theta}(gh) = \theta(g) \cdot \Omega(\sigma_{\gamma}) \cdot \omega_{\beta,\rho}(h)$$ for $g \in T(O_F/p^r)$ and $h \in G(p^{l'-1}/p^r)$ with the notations of subsection 3.3.

Hence $\sigma_{\beta,\theta} = \omega_{\beta,\rho}$ on $U(p^{l'-1}/p^r)$. On the other hand we have (2), hence

$$\text{Hom}_{\text{U}(p^{l'-1}/p^r)}(\sigma_{\beta,\theta}, \chi_r) \neq 0$$ if and only if

$$\text{Hom}_{\text{U}(p^{l'-1}/p^r)}(\text{Ind}_{\text{Z}(p^{l'-1}/p^r)}^G[p^{l'-1}/p^r])\psi_{\beta,\rho}, \chi_r) \neq 0.$$ (3)

Note that $U(p^{l'-1}/p^r) \cap Z(p^{l'-1}/p^r) = U(p^{l'}/p^r)$ because $1_{2n} + \pi^{l'-1}X \pmod{p^r} \in U(p^{l'-1}/p^r) \cap Z(p^{l'-1}/p^r)$ implies that $X \pmod{p}$ is nilpotent, that is $X \pmod{p} = 0$. Hence we have

$$\text{Hom}_{\text{U}(p^{l'-1}/p^r)}(\text{Ind}_{\text{Z}(p^{l'-1}/p^r)}^G[p^{l'-1}/p^r]\psi_{\beta,\rho}, \chi_r) = \bigoplus \text{Hom}_{U} \left(\text{Ind}^U_{\text{Z}(p^{l'-1}/p^r)}\psi_{\beta,\rho}, \chi_r\right)$$

$$= \bigoplus \text{Hom}_{\text{U}(p^{l'}/p^r)}(\psi_{\beta,\rho}, \chi_r)$$
where $U = U(p^{l-1}/p^r)$, $Z = Z(p^{l-1}/p^r)$ and $\bigoplus_g$ is the direct sum over the representatives $g$ of the double cosets $U \setminus G(p^{l-1}/p^r)/Z$. So (3) is equivalent to $$\psi_{\text{Ad}(g)\beta} = \chi_r$$ on $U(p^l/p^r)$ for some $g \in G(p^{l-1}/p^r)$.

5.5 The combination of the results of subsections 5.2, 5.3 and 5.4 implies that

$$\text{Hom}_{G(F)}(\pi_{\beta, \theta}, \text{Ind}_{U(F)}^G(\chi_u)) \neq 0$$

with $u = (2, 2, \cdots, 2, \varpi^e)$ ($e \equiv r \pmod{2}$) if and only if

$$\psi_{\text{Ad}(g)\beta} = \chi_{(2\varpi^{-r}, \cdots, 2\varpi^{-r}, \varpi^{-r})}$$ on $U(p^l/p^r)$

for some $g \in G(O_F)$. We will show that this is the case by proving the following proposition.

**Proposition 5.5.1** There exists a $g \in G(O_F)$ such that $\text{Ad}(g)\beta = \begin{bmatrix} A & \ast \\ C & -\tau A \end{bmatrix}$ with

$$A = \begin{bmatrix} * & \ast & \ast \\ 1 & \ast & \ast \\ 0 & \cdots & \cdot & \cdot & \cdot \\ 1 & \ast \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$ 

**Proof** $K$ is the splitting field of the characteristic polynomial $\chi_\beta(t)$ of $\beta \in g(O_F) \subset \mathfrak{g} l_{2n}(O_F)$, and we can put

$$\chi_\beta(t) = \prod_{i=1}^n (t^2 - \lambda_i^2) \quad (0 \neq \lambda_i \in K).$$

Let $\langle u, v \rangle = \langle uJ, v \rangle$ be the symplectic form on $K^{2n}$ and $V_\lambda \subset K^{2n}$ the eigen space of $\beta$ with eigen value $\lambda \in K$. Then $\langle V_\lambda, V_\mu \rangle \neq 0$ only if $\lambda + \mu = 0$, and we have

$$K^{2n} = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \oplus V_{-\lambda_n} \oplus \cdots \oplus V_{-\lambda_1}.$$ 

This means that there exists a symplectic $K$-basis $\{u_1, \cdots, u_n, v_n, \cdots, v_1\}$ of $K^{2n}$ such that the representation matrix of $\beta$ with respect to the basis is

$$\begin{bmatrix} \Lambda & 0 \\ 0 & -\tau \Lambda \end{bmatrix}$$ with $\Lambda = \begin{bmatrix} \lambda_1 & \cdots & \cdot \\ \cdot & \lambda_n \end{bmatrix}$.

In other word, there exists a $g \in G(K)$ such that $g\beta g^{-1} = \begin{bmatrix} \Lambda & 0 \\ 0 & -\tau \Lambda \end{bmatrix}$.

We can choose

$$\beta' = \begin{bmatrix} A & 0 \\ C & -\tau A \end{bmatrix} + \beta_0 \in g(O_F)$$

such that $\chi_{\beta'}(t) = \chi_\beta(t)$ with

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$
and the entries of \( \beta_0 \) are zero except the elements of the first row and of the last column. Then there exists a \( g \in G(K) \) such that \( g\beta g^{-1} = \beta \). Because \( T = Z_G(\beta) \) splits over \( K \), the first Galois cohomology group \( H^1(\text{Gal}(K/F), T(K)) \) is trivial. So there exists a \( g \in G(F) \) such that \( g\beta g^{-1} = \beta \). Due to Iwasawa decomposition \( G(F) = G(O_F)T(F)U(F) \), we can put \( g = ktu \) with \( k \in G(O_F) \), \( u \in U(F) \) and

\[
t = \begin{bmatrix}
\overline{\omega}^{-m} & 0 \\
0 & \tau \overline{\omega}^{-m}
\end{bmatrix} \in T(F) \text{ with } m = (m_1, \ldots, m_n) \in \mathbb{Z}^n.
\]

Then \( tu\beta u^{-1}t^{-1} = k^{-1}\beta k \in g(O_F) \) and

\[
tu\beta u^{-1}t^{-1} = \begin{bmatrix} X & * \\ Y & -\tau X \end{bmatrix}
\]

with

\[
X = \begin{bmatrix}
\overline{\omega}^{m_1-m_2} & * & \cdots & * \\
* & \cdots & \cdots & * \\
\cdots & \cdots & \cdots & \cdots \\
\overline{\omega}^{m_n-m_n-1} & \cdots & \cdots & \cdots \\
\end{bmatrix}, \quad Y = \begin{bmatrix}
0 & \cdots & 0 & \overline{\omega}^{-2m_n} \\
0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{bmatrix}.
\]

This implies that \( m_1 \leq m_2 \leq \cdots \leq m_n \leq 0 \). On the other hand

\[
\chi_{\beta'}(t) \pmod{p} = \chi_{\beta}(t) \pmod{p} \in \mathbb{F}[t]
\]

is irreducible, we have \( m_1 = m_2 = \cdots = m_n = 0 \). ■

5.6 We have proved the followings;

1) the irreducible supercuspidal representations \( \pi_{\beta, \theta} = \text{ind}_{G(0_F)}^{G(F)} \delta_{\beta, \theta} \) are generic,

2) \( \text{Hom}_{G(F)}(\pi_{\beta, \theta} \cdot \text{ind}^{G(F)}_{U(F)} \chi_u) \neq 0 \) with \( u = (2, 2, \cdots, 2, \overline{\omega}^e) \) if and only if \( e \equiv r \pmod{2} \).

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