ON THE GHKS COMPACTIFICATION OF THE MODULI SPACE OF K3 SURFACES OF DEGREE TWO

KLAUS HULEK, CHRISTIAN LEHN, AND CARSTEN LIESE

Abstract. We investigate a toroidal compactification of the moduli space of K3 surfaces of degree 2 originating from the program formulated by Gross-Hacking-Keel-Siebert. This construction uses Dolgachev’s formulation of mirror symmetry and the birational geometry of the mirror family. Our main result is an analysis of the toric fan. For this we use the methods developed by two of us in a previous paper.

Contents

0. Introduction 1
1. Moduli spaces of K3 surfaces and their toroidal compactifications 4
2. Birational geometry of degenerations 10
3. The DNV family and the Mori fan 18
4. Cusp models 21
5. Construction of the fan 23
6. Cuspidal Cones 32
7. Models with dual intersection complex $\mathcal{P}$ 36
8. Models with dual intersection complex $\mathcal{\overline{P}}$ 42
9. Counting cones 45
References 47

0. Introduction

We investigate a new toroidal compactification of the moduli space of K3 surfaces of degree 2 inspired by mirror symmetry and examine its properties. This construction uses Dolgachev’s formulation of mirror symmetry [Dol96] and the birational geometry of the mirror family. It originates from the construction proposed by Gross-Hacking-Keel-Siebert [GHKS16] and our analysis relies on previous work by two of us [HL19].

Moduli spaces of polarized K3 surfaces have been a center of interest ever since. In [PŠS71], Pjatecki-Šapiro and Šafarevič have shown that the moduli functor of degree $2d$ polarized K3 surfaces is coarsely representable. Thanks to their global Torelli theorem, the corresponding

2020 Mathematics Subject Classification. 14J10, 14J28 (primary), 14D06, 14D20, 14E30, (secondary).

Key words and phrases. Moduli space, K3 surface, compactification, mirror symmetry, birational geometry, degeneration.
moduli space can be described as
\begin{equation}
\mathcal{F}_{2d} := \Gamma_{2d}\backslash \mathcal{D}_{2d}
\end{equation}
where \(\mathcal{D}_{2d}\) is the period domain and \(\Gamma_{2d}\) is an arithmetic group acting on it; we refer to Section 1 for more details. A cusp of the Baily–Borel compactification \(\mathcal{F}_{2d}^{\text{BB}}\) determines a certain cone and toroidal or semitoric compactifications of \(\mathcal{F}_{2d}\) are determined by (toric or semitoric) fans supported on these cones and equivariant for the action of \(\Gamma_{2d}\).

Let us assume that \(2d\) is square free. In this case there is a unique zero dimensional cusp in \(\mathcal{F}_{2d}^{\text{BB}}\) and there is a certain lattice \(M_{2d}\) of signature \((1,18)\) such that the cone in question is a connected component \(C_{2d}\) of the cone of positive vectors in \(M_{2d,\mathbb{R}} := M_{2d} \otimes \mathbb{R}\) or rather its rational closure \(C_{2d}^{\text{rc}} := \text{conv}(C_{2d} \cap M_{2d})\). Now a semitoric (respectively toroidal) compactification of \(\mathcal{F}_{2d}\) is determined by a semitoric (respectively toric) fan in \(M_{2d,\mathbb{R}}\) whose support is \(C_{2d}^{\text{rc}}\) and which is equivariant for the action of a certain subgroup \(\Gamma_{2d}^{+} \subset O(M_{2d})\).

There is a canonical such semitoric fan, the Coxeter fan \(\Sigma_{\text{Cox}}^{2d}\), whose maximal dimensional cones are the fundamental domains for the Weyl group action on \(C_{2d}^{\text{rc}}\), see [Vin75, Vin85, AET19]. The fan we consider in this paper is a refinement of the Coxeter fan obtained by the birational geometry of the mirror family to the moduli space \(\mathcal{F}_{2d}\), see Definition 5.19. We refer to it as the Gross–Siebert–Hacking–Keel fan or GHKS fan. Our first result is

**Theorem 0.1** (See Theorem 5.21). Let \(2d\) be an even square-free positive integer, let \(\Sigma_{\text{GHKS}}^{2d}\) be the GHKS fan in degree \(2d\), and let \(\mathcal{F}_{2d}^{\text{GHKS}}\) be the associated semitoric compactification of \(\mathcal{F}_{2d}\). Then \(\Sigma_{\text{GHKS}}^{2d}\) is a refinement of the Coxeter fan \(\Sigma_{\text{Cox}}^{2d}\) and there is a bimeromorphic morphism
\[
\mathcal{F}_{2d}^{\text{GHKS}} \to \mathcal{F}_{2d}^{\text{Cox}}.
\]

We build on and extend the investigation of the Dolgachev–Nikulin–Voisin family associated to a polarized K3 surface of degree 2 over the complex numbers that has been initiated in [HL19]. It turns out that for \(2d = 2\) the GHKS fan \(\Sigma_{2d}^{\text{GHKS}}\) is indeed a fan. By construction, it is a refinement of the Coxeter fan \(\Sigma_{2d}^{\text{Cox}}\) which in this case is also an honest toric fan. Alexeev, Engel, and Thompson constructed in [AET19] a semitoric coarsening of \(\Sigma_{2d}^{\text{Cox}}\). Altogether, this leads to the

**Corollary 0.2.** There is a sequence of morphisms
\[
\mathcal{F}_{2d}^{\text{GHKS}} \to \mathcal{F}_{2d}^{\text{Cox}} \to \mathcal{F}_{2d}^{\text{AET}}
\]
where \(\mathcal{F}_{2d}^{\text{GHKS}}\) and \(\mathcal{F}_{2d}^{\text{Cox}}\) are toroidal and \(\mathcal{F}_{2d}^{\text{AET}}\) is a semitoric compactification.

Our main result is

**Theorem 0.3** (See Corollary 5.25, Corollary 9.6, Corollary 9.7). The toric fan \(\Sigma_{2d}^{\text{GHKS}}\) has 31 maximal cones inside the fundamental domain of the Coxeter fan. The \(\Gamma_{2d}^{+}\)-action induces a residual \(S_3\)-action on the set of these cones with 17 orbits.

The ultimate goal is to examine modularity of the compactification \(\mathcal{F}_{2d}^{\text{GHKS}}\), in particular for degree 2 polarized K3 surfaces.
0.4. **Strategy of the proof and outline of the paper.** Let us outline the construction of the fan $\Sigma_{2d}^{GHKS}$. The first observation is that the lattice $M_{2d}$ is precisely the Picard group of a very general member in Dolgachev’s mirror moduli space $\mathcal{F}_{2d}$. This moduli space is one dimensional whereas $\mathcal{F}_{2d}$ is 19-dimensional. Observe that the Picard rank of the very general element of $\mathcal{F}_{2d}$ has rank 19 as opposed to 1 for the very general element of $\mathcal{F}_{2d}$. To be independent of any choices we consider the mirror family over a neighborhood of the unique cusp of $\mathcal{F}_{2d}$ — this is a uniquely determined projective K3 surface $\mathcal{S}$ over $\mathbb{C}(t)$ which, following GHKS, we call the Dolgachev–Nikulin–Voisin family or DNV family for short, see also [HL19, Remark 1.17 and Definition 1.18].

We can then realize $M_{2d}$ as the Picard group of $\mathcal{S}$ together with its intersection pairing. Next we consider certain degenerations $\mathcal{Y} \rightarrow S := \text{Spec} \mathbb{C}[[t]]$ with generic fiber $\mathcal{Y}_\eta = \mathcal{S}$, so-called models of the DNV family. Recall that in [HL19], the so-called Mori fan of $\mathcal{Y}$ was studied. Its cones are of the form $f^*\text{Nef}(\mathcal{Y}')$ where $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ are rational contractions of $\mathcal{Y}$ over $S$, see Definition 2.12, and $\text{Nef}$ denotes the effective nef cone, see (3.1). Let us denote by $\iota : \mathcal{Y}_\eta \rightarrow \mathcal{Y}$ the inclusion of the generic fiber. The fan $\Sigma_{2d}^{GHKS}$ is now roughly constructed in two steps:

- The Mori fan with support on the movable cone $\text{Mov}(\mathcal{Y})$ is pulled back to $\text{Nef}(\mathcal{Y}_\eta)$ via certain sections of the restriction $\iota^*$ constructed via birational geometry. This is done in Section 5.
- The thus obtained fan $\Sigma'$ with support on the nef cone of $\mathcal{Y}_\eta$ cone is shown to be equivariant under the action of the subgroup of $\Gamma_{2d}$ preserving the nef cone. It will be extended to a fan on the whole positive cone using the Weyl group action, see Definition 5.1.

In Section 1 we recall the basics on moduli spaces of K3 surfaces and their compactifications. Possibly only Section 1.10 is non-standard here and explains Looijenga’s construction of semitoric compactifications. In Section 2, we recall the theory of Friedman–Kulikov–Pinkham–Persson of degenerations of K3 surfaces and adapt it to our setting. In particular, we provide the necessary algebraization statements and analysis of the Picard groups. Section 3 is devoted to the study of the DNV family. We recall the Mori fan in Section 3.5. The central notion is that of a cusp model, introduced in Section 4. These are certain birational morphisms $\mathcal{Y} \rightarrow \mathcal{Y}'$ and are used to define the GHKS refinement of the Coxeter semitoric fan and hence a semitoric compactification of $\mathcal{F}_{2d}$ in Section 5. Sections 6 to 9 are devoted to the classification of cusp models. The idea is that the existence of a cusp model $\mathcal{Y} \rightarrow \mathcal{Y}'$ imposes strong restrictions on the Picard group of the components of the central fiber of $\mathcal{Y}$. While in Section 6 we recall the notion of a curve structure from [HL19] and obtain necessary conditions for the existence of cusp models, Sections 7 and 8 give a more detailed analysis depending on the dual intersection complex of the central fiber of $\mathcal{Y}$. The classification of cusp models and the counting of cones in degree 2 is carried out in Section 9 where our main result, Theorem 0.3, is proven.
0.5. **Previous work.** This article clearly relies on the Gross–Hacking–Keel–Siebert construction suggested in [GHKS16]. Even though the fan we consider is not exactly the same as theirs, see Remark 5.22, the construction is, of course, basically due to them. In the attempt to be self-contained, we prove some of the statements that one can also find in [GHKS16] and give a bit more details here and there, but no originality is claimed in the construction of the fan. We also decided to take a slightly different perspective: while GHKS use the construction to refine a given toric fan, we see it as a method to construct a semitoric fan in the sense of Looijenga [Loo03a, Loo03b]. This possibility has also been pointed out in [GHKS16, Remark 0.11] but was not further pursued there. The new contribution of the present paper is the detailed analysis of the degree 2 case where the GHKS-construction gives an actual toric fan and thus a honest toroidal compactification.

**Notation and terminology.** We will denote $S := \text{Spec} \mathbb{C}[[t]]$ and write $\eta$ for its generic point and $c$ for its closed point. A **lattice** will be a torsion-free $\mathbb{Z}$-module of finite rank together with an integral non-degenerate symmetric bilinear form.

**Acknowledgements.** We are grateful to Mark Gross, Paul Hacking, Sean Keel, and Bernd Siebert for sharing their unpublished manuscript [GHKS16]. The second named author would like to thank Ben Bakker, Philip Engel, and Luca Giovenzana for helpful discussions. We are grateful to Valery Gritsenko, Slava Nikulin, and Alessandra Sarti for answering our questions by email.

Klaus Hulek was partially supported by DFG grant Hu 337/7-1. Christian Lehn was supported by the DFG through the research grants Le 3093/2-2 and Le 3093/3-1.

1. **Moduli spaces of K3 surfaces and their toroidal compactifications**

In this section, we recall some basics about K3 surfaces, their moduli spaces, and compactifications. For K3 surfaces and their moduli spaces, we refer to Huybrecht’s textbook [Huy16], in particular its Sections 5 and 6, and references therein. For compactifications we refer to [AMRT10] and [Sca87].

1.1. **K3 surfaces and their moduli spaces.** The results stated in this section are well-known and we refer to [BHPdV04] or [Huy16] for proofs and additional references. Recall that a **K3 surface** is a smooth compact complex surface with trivial canonical bundle $\mathcal{O}_X \cong \omega_X$ and vanishing irregularity $h^1(X, \mathcal{O}_X)$. A **polarization** on a K3 surface $X$ is an ample divisor $H$ on $X$, a polarized K3 surface is a pair $(X, H)$ consisting of a K3 surface together with a polarization. If $H$ is only assumed big and nef, we refer to it as a **quasi-polarization** and to the pair $(X, H)$ as a quasi-polarized K3 surface. The **degree** of a polarization $H$ is the integer $H^2 \in 2\mathbb{Z}$. Sometimes we relax the smoothness hypothesis and ask $X$ to have at most ADE-singularities. We speak of **K3 surfaces with ADE-singularities** in this case. The notions of a polarization and a quasi-polarization extend to this setup.

Let $\mathcal{M}_{2d} : (\text{Sch}/\mathbb{C})^{\text{opp}} \to \text{Set}$ be the functor of degree $2d$ polarized K3 surfaces with ADE-singularities. Thanks to [PSS71], the moduli functor is coarsely representable. Pjateckii-Šapiro and Šafarevič use the Global Torelli Theorem to show this, see also [Vie90] for a
different approach in the spirit of GIT. We refer to [Huy16], especially Sections 5 and 6, for more details and references.

1.2. Periods of K3 surfaces. For any lattice $L$ we define

\begin{equation}
\mathcal{D}_L := \{ x \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0 \}.
\end{equation}

Then $\mathcal{D}_L$ is the period domain for weight 2 Hodge structures on the lattice $L$ with $h^{2,0} = 1$ such that the restriction of the pairing to the real space underlying $H^{2,0} \oplus H^{0,2}$ is positive definite and orthogonal to $H^{1,1}$. The geometry of the period domain, and in particular of the action of the orthogonal group $O(L)$ on it, is very sensitive to the signature of the lattice. If $L$ has signature $(3, n)$, which is the case relevant for K3 surfaces, $\mathcal{D}_L$ is connected and the group action has dense orbits and one does not have a reasonable quotient. It is worthwhile noting that precisely this ill-behaved group action can also be exploited to give a different proof of Torelli’s theorem, see e.g. [BL18, Theorem 1.1].

If $L$ has signature $(2, n)$, which is the case relevant for polarized K3 surfaces, then $\mathcal{D}_L$ is a hermitian symmetric domain of type IV and has two connected components. The quotient $O(L) \setminus \mathcal{D}_L$ is a quasiprojective variety, and we will briefly discuss the vast theory of compactifications of this quotient beginning with Section 1.6.

1.3. Marked K3 surfaces and the Torelli theorem. Let $X$ be a smooth K3 surface. Together with the intersection pairing, the group $H^2(X, \mathbb{Z})$ is known to be isomorphic to the so-called K3 lattice

\begin{equation}
\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}
\end{equation}

where $E_8(-1)$ stands for the negative definite root lattice of type $E_8$ and $U$ is the hyperbolic plane. The properties of the intersection pairing show that the weight 2 Hodge structure of $X$ lies in $\mathcal{D}_{H^2(X, \mathbb{Z})} \cong \mathcal{D}_\Lambda$. As the identification of the cohomology of the K3 surface with the K3 lattice $\Lambda$ is not canonical, one has to choose a marking, i.e. an isometry $\mu : H^2(X, \mathbb{Z}) \to \Lambda$. A pair $(X, \mu)$ consisting of a K3 surface and a marking is called a marked K3 surface. Let $\mathcal{M}_\Lambda$ be the marked moduli space, i.e. the space of all isomorphism classes of marked K3 surfaces $(X, \mu)$, where isomorphisms have to be compatible with the markings.

The Global Torelli Theorem was proven in [PSS71] for algebraic K3 surfaces and for Kähler K3 surfaces in [BR75]. After Verbitsky’s proof [Ver13] of the Global Torelli Theorem for irreducible symplectic manifolds, the following formulation has become popular:

**Theorem 1.4.** Consider the period map $\varphi : \mathcal{M}_\Lambda \to \mathcal{D}_\Lambda$ and let $\omega \in \mathcal{D}_\Lambda$. Then all $(X, \mu), (X', \mu') \in \varphi^{-1}(\omega)$ satisfy $X \cong X'$. Moreover, the marked moduli space $\mathcal{M}_\Lambda$ has two connected components and the restriction of the period map $\varphi$ to such a component $\mathcal{N}$ is surjective and injective over the complement of a countable union of hyperplanes.

We refer to [Huy16, Proposition 7.5.5] for the statement about the number of connected components.
1.5. Moduli spaces as quotients of period domains. For all hermitian symmetric domains \( \mathcal{D} \) together with an arithmetic group \( \Gamma \subset \text{Aut}(\mathcal{D}) \), the quotient \( \Gamma \backslash \mathcal{D} \) is a quasiprojective variety \([BB66]\). We continue to denote by \( \Lambda \) the K3 lattice from \([1.2]\). As explained in Section 1.2, the period domain \( \mathcal{D}_\Lambda \) for K3 surfaces is not hermitian symmetric, but the period domains for polarized K3 surfaces are. Let \((X,H,\mu)\) be a polarized marked K3 surface, i.e. a marked K3 surface \((X,\mu)\) together with a polarization \(H\). Then its period lies in the hyperplane

\[ \mathcal{D}_{v^+} = \mathcal{D}_\Lambda \cap \mathbb{P}(v^+) \quad \text{where} \quad v = \mu(H) \in \Lambda.\]

By \([Jam68\) Theorem] there is a unique \(O(\Lambda)\)-orbit of primitive vectors \(v\) with \(v^2 = 2d\). We can therefore choose \(v = e + df\) where \(e, f\) is a basis of a \(U\) summand in \([1.2]\) so that \(v^\perp\) can be identified with

\[ L_{2d} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus (-2d) .\]

Here \((m)\) stands for a rank one lattice with generator of square \(m\). As mentioned in Section 1.2, the period domain

\[ \mathcal{D}_{v^+} \cong \mathcal{D}_{L_{2d}} = \mathcal{D}_{2d} \cup \mathcal{D}'_{2d} \]

has two connected components and we choose one of them. Accordingly, we denote by

\[ O^+(L_{2d} \otimes \mathbb{R}) \subset O(L_{2d} \otimes \mathbb{R}) \quad \text{and} \quad O^+(L_{2d}) \subset O(L_{2d}) \]

the subgroups with real spinor norm 1, which are the subgroups preserving \(\mathcal{D}_{2d}\). Using the Torelli theorem one can show that the moduli functor \(\mathcal{M}_{2d}\) is represented by

\[ \mathcal{F}_{2d} := \Gamma_{2d} \backslash \mathcal{D}_{L_{2d}} = \Gamma_{2d}^+ \backslash \mathcal{D}_{2d} \]

where \(\Gamma_{2d} \subset O(\Lambda)\) is the subgroup fixing \(v = e + df\), see \([Huy16\) Corollary 6.4.3 and Remark 6.4.5], and \(\Gamma_{2d}^+ = O^+(\Lambda) \cap \Gamma_{2d}\). Here the action of \(\Gamma_{2d}\) on \(\mathcal{D}_{L_{2d}}\) is through the canonical restriction homomorphism \(\Gamma_{2d} \to O(L_{2d})\). It can be shown that the restriction yields an isomorphism \(\Gamma_{2d} \to \tilde{O}(L_{2d})\) with the finite index subgroup \(\tilde{O}(L_{2d}) \subset O(L_{2d})\) of elements acting trivially on the discriminant \(L_{2d}'/L_{2d}\). The group \(\tilde{O}(L_{2d})\) is called the stable orthogonal group. We will henceforth always refer to \(\mathcal{F}_{2d}\) as the moduli space of degree \(2d\) K3 surfaces. Keep in mind that it also parametrizes ADE-singular K3 surfaces as explained in Section 1.5.

1.6. Baily–Borel compactification. As all arithmetic quotients of hermitian symmetric domains, the moduli space \(\mathcal{F}_{2d}\) of polarized K3 surfaces has a canonical compactification, the Baily–Borel compactification \(\mathcal{F}^{BB}_{2d}\), see \([BB66]\). We also refer to Scattone’s book \([Sca87]\) for a general reference concerning \(\mathcal{F}^{BB}_{2d}\). We will give a rough idea of how it is constructed. Let us consider the quadric \(Q_{2d} \subset \mathbb{P}(L_{2d} \otimes \mathbb{C})\) cut out by the quadratic form and consider the closure of \(\mathcal{D}_{2d}\) inside \(Q_{2d}\). Boundary components are the maximal subsets of the topological boundary \(\partial \mathcal{D}_{2d}\) that are connected by analytic arcs. A boundary component \(F\) is called rational, if its stabilizer subgroup \(N(F) \subset O^+(L_{2d} \otimes \mathbb{R})\) is defined over \(\mathbb{Q}\). The rational closure \(\mathcal{D}^{rc}_{2d}\) is defined to be the union of \(\mathcal{D}_{2d}\) with all rational boundary components, endowed with the horocyclic topology. Then

\[ \mathcal{F}^{BB}_{2d} := \Gamma_{2d}^+ \backslash \mathcal{D}^{rc}_{2d}. \]
There is an alternative description as the Proj of a ring of modular forms, from which one immediately sees that it is projective. Moreover, \( F_{2d} \subset F_{2d}^{BB} \) is a Zariski open, dense subset.

The (rational) boundary components for type IV domains turn out to be either of dimension one respectively zero, and they correspond to (rational) isotropic planes respectively lines in \( L_{2d} \). Accordingly, also the complement \( F_{2d}^{BB} \setminus F_{2d} \) has a canonical stratification into so-called cusps which for type IV domains are either of dimension 1 or 0.

1.7. Toroidal compactifications. Apart from the Baily–Borel compactification, the moduli space \( F_{2d} \) of polarized K3 surfaces has plenty of toroidal compactifications, see [AMRT10]. These depend on certain collections \( \Sigma \) of fans similar to the fans in toric geometry and by construction map to the Baily–Borel compactification \( \beta : F_{2d}^{\Sigma} \to F_{2d}^{BB} \).

Let us be a bit more precise. For notational simplicity, we will abbreviate by \( \Gamma := \Gamma^{+}_{2d} \subset \text{O}^+(L_{2d}) \). For a (rational) boundary component \( F \), recall that \( N(F) \subset \text{O}^+(L_{2d} \otimes \mathbb{R}) \) was the stabilizer. By \( U(F) \subset N(F) \) we denote the center of its unipotent radical. Then \( U(F) \cong \mathbb{R}^k \) for some \( k \). Let us abbreviate \( N(F)_{\Gamma} := N(F) \cap \Gamma, U(F)_{\Gamma} := U(F) \cap \Gamma \), and

\[
\overline{\Gamma}_F := \text{Im} \left( N(F) \cap \Gamma \to \text{Aut}(U(F)) \right).
\]

For every boundary component \( F \), we consider \( F/N(F)_{\Gamma} \subset F_{2d}^{BB} \). Then the preimage \( \beta^{-1}(F) \) is the quotient of partial compactification of \( \Gamma \setminus \mathcal{D} \) inside a relative toric variety over a fiber bundle over \( F \). This toric variety is determined by a fan whose support is the rational closure \( C(F)^{rc} \) of a certain cone \( C(F) \) inside \( U(F) \). A toroidal compactification is then determined by what we call a \( \Gamma \)-admissible collection of fans.

\[
\Sigma := \{ \Sigma_F \mid F \text{ rational boundary component} \}
\]

where \( \Sigma_F \) is a \( \overline{\Gamma}_F \)-fan in \( U(F) \) with support the rational closure \( C(F)^{rc} \). This means that \( \Sigma \) has a certain equivariance for the group \( \Gamma \) and that the group \( \overline{\Gamma}_F \) acts on each \( \Sigma_F \) so that there are only finitely many cones up to the action of \( \overline{\Gamma}_F \).

As mentioned before, the boundary components for type IV domains have dimension 1 or 0 and correspond either to an isotropic plane \( E \) or an isotropic line \( \ell \) in \( L_{2d} \). It turns out that \( U(E) \) is one dimensional, see e.g. [GHS07, Lemma 2.25], and there is the \( \Gamma \)-equivariance condition is automatic. On the other hand, \( U(\ell) \cong \ell^\perp/\ell \) which is a hyperbolic lattice of signature \((1,18)\).

1.8. The square free case. Let us assume from now on that \( 2d \) is square free. Then up to the action of \( \Gamma_{2d} \), there is a unique isotropic line \( \ell \subset L_{2d} \), see [Sca87, Theorem 4.0.1]. We
can therefore choose $\ell$ to be spanned by one of the basis vectors in one of the $U$ summands of $L_{2d}$ and identify $U(\ell) = \ell^\perp/\ell$ with the lattice
\begin{equation}
M_{2d} = E_8(-1)^{\oplus 2} \oplus U \oplus \langle -2d \rangle .
\end{equation}

The cone $C(\ell)$ is then realized as a connected component $C_{2d}$ of the cone of positive vectors in $M_{2d,\mathbb{R}} := M_{2d} \otimes \mathbb{R}$ and its rational closure is
\begin{equation}
C_{2d}^{rc} = \text{conv}(C_{2d} \cap M_{2d}) .
\end{equation}

Let us denote $\Gamma_{2d}^+ := (\Gamma_{2d}^+)\ell \subset O^+(M_{2d})$ the image of $N(\ell) \cap \Gamma_{2d}^+ \rightarrow O^+(M_{2d})$. Note that $\Gamma_{2d}^+$ preserves the cone $C_{2d}$. As explained in Section 1.7, the $\Gamma_{2d}^+$-compatibility with the fans $\Sigma_E$ corresponding to isotropic planes $E$ is automatic. Moreover, since there is only one orbit of isotropic lines, we may summarize our discussion as follows.

**Proposition 1.9.** A toroidal compactification of $\mathcal{F}_{2d}$ for square free $2d$ is determined by a $\Gamma_{2d}^+$-fan in $M_{2d,\mathbb{R}}$ whose support is $C_{2d}^{rc}$.

1.10. **Semitoric compactifications.** In [Loo03a], [Loo03b] Looijenga has introduced the notion of semitoric compactifications which generalizes both toroidal compactifications and the Baily–Borel compactification. Similarly to toroidal compactifications, these depend on an admissible collection of *semitoric fans* and are also constructed relatively over the Baily–Borel compactification. Here we shall recall the basic notions necessary for this construction. Since we shall only apply this in the case of square-free $d$ we will restrict to this case, as this allows us to keep the discussion considerably simpler.

We start with a lattice $L$ of signature $(2,n)$ (which will later become $L_{2d}$) and an arithmetic group $\Gamma^+ \subset O^+(L)$ (which will later become $\Gamma_{2d}^+$). We will make the assumption that $\Gamma^+$ operates transitively on the set of rational isotropic lines. Let $\ell$ be such an isotropic line and set $M := U(\ell) = \ell^\perp/\ell$. Then $M$ (which will later be $M_{2d}$) is a hyperbolic lattice of signature $(1, n-1)$ and we denote by $M_\mathbb{R}$ the associated real vector space. Let $C := C(\ell)$ with rational closure $C^{rc}$. The following definitions will be crucial. As described above the group $\Gamma^+ := (\Gamma^+)\ell \subset O^+(M)$ acts on $C^{rc}$.

**Definition 1.11.** By a rational cone we mean a rational polyhedral cone which is nondegenerate (i.e. does not contain an affine line). A rational cone system in $C^{rc}$ is a finite collection $\Sigma$ of rational cones in $C^{rc}$ such that

1. If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau \in \Sigma$
2. If $\sigma \in \Sigma$ and $\tau$ is a face of $\sigma$, then $\tau \in \Sigma$.

**Definition 1.12.** A locally rational cone in $C^{rc}$ is a cone $\sigma \subset C^{rc}$ such that the restriction of $\sigma$ to any rational polyhedral subcone of $C^{rc}$ is a rational cone.

**Definition 1.13.** A locally rational decomposition of $C^{rc}$ is a collection $\Sigma$ of convex cones with the following properties:

1. $\bigcup_{\sigma \in \Sigma} \sigma = C^{rc}$.
2. The restriction of $\Sigma$ to any rational subcone of $C^{rc}$ is a rational cone system.
If \( h \supset \ell \) is a rational isotropic plane, then \( J(h) := h/\ell \) is a rational isotropic line in \( M_\mathbb{R} \) (and every rational isotropic line in \( M_\mathbb{R} \) arises this way). We denote the half line of \( J(h) \) which belongs to \( C^{rc} \) by \( J^+(h) \).

**Definition 1.14.** A **semitoric fan** (for the group \( \Gamma^+ \)) is a collection \( \Sigma \) of cones which decompose the rational closure \( C^{rc} \) with the following properties:

1. \( \Sigma \) is \( \Gamma^+ \)-equivariant.
2. The origin \( \{0\} \) and the half lines \( J^+(h) \), where \( h \supset \ell \) is a rational isotropic plane are cones belonging to \( \Sigma \).
3. The cones of \( \Sigma \) define a locally rational decomposition of \( C^{rc} \).

This is Looijenga’s definition adapted to the special case where we have only one isotropic rational line up to the action of the group \( \Gamma^+ \). If there are several such lines, then one must choose a decomposition for all \( C^{rc}(\ell) \), or more intrinsically of the conical locus in the sense of Looijenga, and these must fulfill a (fairly complicated to formulate) compatibility condition. As we will not need this, we have omitted the details, which can be found in [Loo03b, Section 6].

By Looijenga’s theory every semitoric fan as above gives rise to a compactification of the quotient

\[
\mathcal{F} = \Gamma^+ \backslash D^+.
\]

We shall call the resulting compactification, which we will also denote by \( \mathcal{F}^\Sigma \) the **semitoric compactification** given by the semitoric fan \( \Sigma \). This is a normal compact complex space (possibly projective). Refinements of semitoric fans correspond to blow-ups of semitoric compactifications and all semitoric compactifications have a natural map to the Baily–Borel compactification \( \mathcal{F}^{BB} \) (as will become clear from the second example below). The following are the most important examples for us.

**Example 1.15.** Every fan for the group \( \Gamma^+ \) is also a semitoric fan for this group, in particular, every toroidal compactification is also a semitoric compactification.

The Baily–Borel compactification itself is also a semitoric compactification:

**Example 1.16.** The coarsest semitoric fan \( \Sigma \) consists simply of the cones \( \{0\} \), the half lines \( J^+(h) \) and \( C^{rc} \). This leads to the **Baily–Borel compactification** \( \mathcal{F}^{BB} \).

An important class of semitoric compactifications is defined via **hyperplane arrangements**, indeed, these were the motivating examples for Looijenga’s work.

**Example 1.17.** We consider a set \( \mathcal{H} \) of rational hyperplanes in \( L_\mathbb{R} \) of signature \( (2, n-1) \) which has the property that it consists of only finitely many \( \Gamma \)-orbits (and thus is locally finite). If \( H \in \mathcal{H} \) is a hyperplane containing our chosen isotropic line \( \ell \), then \( H/\ell \) defines a hyperplane in \( M_\mathbb{R} \). Intersecting the cone \( C^{rc} \) with these hyperplanes and taking the closures of the connected components defines a locally rational decomposition \( \Sigma = \Sigma(\mathcal{H}) \) of \( C^{rc} \). Special cases are:
(1) Again, we note that for $H = \emptyset$ we once again recover the Baily–Borel compactification.

(2) We can take the set of all roots $r \in L$ and the collection of hyperplanes $H_{\text{root}} := \bigcup_r H_r$ where $H_r = r^\perp$. We shall call the resulting semitoric fan the Coxeter semitoric fan.

At this point we would like to make a comment on terminology. In the literature one finds both names Coxeter chambers and Vinberg chambers. Indeed, both authors have contributed decisively to this theory. In order to avoid confusion in the literature we have decided to stay with the name Coxeter fan as in [AET19].

Applying the above discussion to the lattice $\Gamma_{2d}$ we obtain the semitoric fan analog of Proposition 1.9.

**Proposition 1.18.** A semitoric compactification of $F_{2d}$ for square free $2d$ is determined by a $\Gamma_{2d}^+$ semitoric fan in $M_{2d,\mathbb{R}}$ whose support is $C_{2d}^\circ$.

2. Birational geometry of degenerations

In this section we provide basic facts about the birational geometry of models. Much of this is standard and we claim no originality for this.

As before we denote $S = \text{Spec} \mathbb{C}[[t]] = \{c, \eta\}$. We will study degenerations of projective K3 surfaces over $S$. Before we get more specific, let us observe that such a degeneration always comes from an algebraic one.

**Proposition 2.1.** Let $X \rightarrow S$ be a flat, projective morphism of schemes. Then there exists a quasi-projective variety $C$, a projective morphism $X \rightarrow C$, and a cartesian diagram

\[
\begin{array}{ccc}
\widehat{X} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\widehat{S} & \longrightarrow & C
\end{array}
\]

in the category of formal schemes where $\widehat{X} \rightarrow \widehat{S} := \text{Spf} \mathbb{C}[[t]]$ denotes the formal completion of $X \rightarrow S$ along the central fiber.

**Proof.** We fix an ample line bundle $L$ on the central fiber $X := X_c$. By Grothendieck’s existence theorem [Gro95], see also [Ser96, Theorem 2.5.13], there is a deformation $\mathcal{X} \rightarrow \mathcal{I}$ of $X$ over a local noetherian scheme $(\mathcal{I}, c)$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that the pair $(\mathcal{X} \rightarrow \mathcal{I}, \mathcal{L})$ is versal for $(X, L)$ at $c$. By Artin’s result [Art69, Theorem 1.6], the formal completion of $\mathcal{X} \rightarrow \mathcal{I}$ at $c$ is induced from a projective scheme $\mathfrak{X} \rightarrow C$ as claimed. \(\square\)

Observe that in the statement of the proposition, the scheme theoretic image of $\varphi$ need not be a curve in general.

2.2. Kulikov models. A flat, projective morphism $\mathcal{Y} \rightarrow S$ from a normal scheme $\mathcal{Y}$ such that the generic fiber $\mathcal{Y}_\eta$ is a $K3$ surface will be called a degeneration of $K3$ surfaces. Such a degeneration is called $K$-trivial if the canonical sheaf is the trivial line bundle: $\omega_{\mathcal{Y}} \cong \mathcal{O}_X$.

**Definition 2.3.** A Kulikov model is a $K$-trivial degeneration of $K3$ surfaces $\mathcal{Y} \rightarrow S$ such that $\mathcal{Y}$ is a regular 3-fold and the central fiber $\mathcal{Y}_c$ is a reduced normal crossing divisor.
The central fibers of such degenerations are classified by [Per77, Kul77, PP81] into type I, II, and III, see also [Fri83a, §5] for more details and further references. Conversely, one may ask which normal crossing surfaces admit a smoothing to a K3 surface. For this purpose, Friedman introduced the notion of \textit{d-semistability}, which is defined as the triviality of the (intrinsically defined) infinitesimal normal bundle, see [Fri83a, Definition (1.13)]. It is easily seen to be satisfied by the central fiber of a Kulikov model and for the converse Friedman proved in [Fri83a, Theorem (5.10)] that \(d\)-semistable K3 surfaces always admit a smoothing.

As we are most interested in Kulikov models whose central fiber is of type III, we recall the following definition.

**Definition 2.4.** A \(d\)-semistable K3 surface of type III is a reduced, projective normal crossing surface \(Y\) such that

1. \(Y\) is \(d\)-semistable,
2. the dualizing sheaf \(\omega_Y\) is trivial,
3. the irreducible components are rational surfaces such that the preimage of the double curves form cycles of rational curves on the normalization, and
4. the dual intersection complex of \(Y\) is a triangulation of the 2-sphere \(S^2\).

As we will not consider type I or II degenerations in this article, we will usually use the term \textit{Kulikov models} as a synonym to Kulikov model of type III.

2.5. \textbf{Algebraization.} One parameter smoothings \(\mathcal{Y} \to S\) of a given \(d\)-semistable K3 surface are highly non-unique. One can obtain uniqueness (up to pullbacks along finite covers) by enforcing \(\mathcal{Y}\) to have a high rank Picard group which leads to a strong algebraicity property of so-called maximal Kulikov models, see Proposition 2.7.

**Definition 2.6.** Recall that \(\mathcal{Y} \to S\) is called \textit{maximal}, if the restriction homomorphism

\[
\text{Pic}(\mathcal{Y}) \to \text{Pic}(\mathcal{Y}_c)
\]

(2.1)

to the central fiber is an isomorphism.

Recall that a Kulikov model \(\mathcal{Y} \to S\) is maximal if and only if \(\mathcal{Y}_c\) is \textit{maximal} in the sense that its Carlson invariant is trivial. We refer to the discussion in [HL19, p. 11] for more details and further references. We will see in Proposition 2.15 that maximality is preserved under birational transformations. One can show that for a maximal Kulikov model one always has

\[
g(\mathcal{Y}) = g(\mathcal{Y}_c) = 18 + n,
\]

(2.2)

where \(n\) is the number of irreducible components of \(\mathcal{Y}_c\), see e.g. [Laz08, Section 3.1].

By deformation theory, we can always realize a \textit{maximal} Kulikov model as a base change from a finite type scheme, more precisely:

\footnote{To be precise, these references treat degenerations over a disk \(\Delta \subset \mathbb{C}\) in the analytic category. However, given \(\mathcal{Y} \to S\) as above, one can always reduce to this situation by Proposition 2.1.}
Proposition 2.7. Let $\mathcal{Y} \rightarrow S$ be a maximal Kulikov model of type III. Then there is a diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
S & \phi \rightarrow & C \\
\end{array}
\]

such that the following holds.

1. On the right, $\mathcal{Y}$ is a smooth 3-fold, and $C$ is a smooth, affine curve.
2. The morphism $\mathcal{Y} \rightarrow C$ is flat and projective, smooth over $C^* := C \setminus \{\varphi(c)\}$, and for $t \neq \varphi(c)$ the fiber $\mathcal{Y}_t$ is a K3 surface over $k(t)$.
3. The morphism $\varphi$ is étale at $c \in S$ and (2.3) is cartesian.
4. If $Y_c = Y_1 \cup \ldots \cup Y_n = \mathcal{Y}_{\varphi(c)}$ are the irreducible components of the central fiber and $\eta_C \in C$ denotes the generic point, then the diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \sum_{i=1}^m ZY_i & \longrightarrow & \text{Pic}(\mathcal{Y}) & / & \text{Pic}(C) & \longrightarrow & \text{Pic}(\mathcal{Y}_{\eta_C}) & \longrightarrow & 0 \\
0 & \longrightarrow & \sum_{i=1}^m ZY_i & \longrightarrow & \text{Pic}(\mathcal{Y}) & \longrightarrow & \text{Pic}(\mathcal{Y}_{\eta_C}) & \longrightarrow & 0 \\
\end{array}
\]

has exact rows and the vertical morphisms (obtained by restriction) are isomorphisms.
5. For $t \in C$ the restriction $\text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(\mathcal{Y}_t)$ induces a canonical injection

\[
\tilde{\rho}_t : \text{Pic}(\mathcal{Y}_{\eta}) \rightarrow \text{Pic}(\mathcal{Y}_{\eta_C}) \rightarrow \text{Pic}(\mathcal{Y}_{\eta_C}) / \text{Pic}(C) + \sum_{i=1}^m ZY_i \rightarrow \text{Pic}(\mathcal{Y}_t)
\]

which for very general $t \in C$ is an isomorphism.

Proof. This is a deformation theoretic argument using Grothendieck’s existence theorem and work of Friedman-Scattone [FSS80]. The proof is easily obtained from the stronger result [HL19, Proposition 1.13] where also more precise references can be found. Explicitly, the proof of the aforementioned result gives a cartesian diagram as in (2.3) which (after possibly shrinking $C$) satisfies (1) and (2) together such that for a choice of line bundles $L_1, \ldots, L_{19}$ on $\mathcal{Y}$ whose restrictions generate the Picard group of $\mathcal{Y}_c$, there are line bundles $L_1, \ldots, L_{19}$ on $\mathcal{Y}$ such that $(L_i)|_{\mathcal{Y}_c}$ is isomorphic to $(L_i)|_{Y_c}$ on $\mathcal{Y}_c = \mathcal{Y}_c$. This is where maximality is needed. Claim (3) follows by replacing $C$ with a finite cover and possibly shrinking it further, so let us prove the statements about the Picard group.

The horizontal morphisms in (2.4) form short exact sequences if there are no nontrivial divisors on $\mathcal{Y}$ respectively $\mathcal{Y}$ whose support is contained in a fiber different from the special fiber. This however holds, as the morphisms are smooth outside the special fibers and have connected fibers. By construction, $\alpha$ is surjective and by [Sta20, Lemma 0CC5] it is also injective, hence (4) follows. The existence of the morphisms in (2.5) is clear. Let us denote by $\mathcal{Y}^* \subset \mathcal{Y}$ the complement of the special fiber. By the localization sequence, $\text{Pic}(\mathcal{Y}^*) / \text{Pic}(C^*) \cong \text{Pic}(\mathcal{Y}_{\eta_C})$ and thus it follows from the sequence (4.11) in FSS0 that $\tilde{\rho}_t$ is injective and its image is saturated. As the $c_1(L_1), \ldots, c_1(L_{19})$ are linearly independent.
in \( H^2((\mathbb{C}^*)^{\text{an}}, \mathbb{Z}) \), the Picard rank of \( \mathcal{Y} \), is \( \geq 19 \) for every \( t \in C^* \). As \( \mathcal{Y} \to C \) is a type III degeneration, for every \( t \in C^* \) the classifying map \( (C, t) \to \text{Def}(\mathcal{Y}_t) \) is not constant, hence maps onto the Hodge locus. By the geometry of the period domain, a very general period in this Hodge locus has Picard number 19 so that by saturatedness of the image, \( \rho_t \) is also surjective for very general \( t \in C^* \). □

Note that this statement is much stronger than the standard algebraicity statement coming from Grothendieck’s existence theorem and Artin algebraicity result, cf. Proposition 2.1.

Remark 2.8. One could formulate the above statement equivalently in terms of the Néron–Severi group. In fact, the only non-trivial \( \text{Pic}^0 \) in (2.4) is the one of \( C \). This is because for a variety \( X \) the tangent space to \( \text{Pic}^0(X) \) is \( H^1(X, \mathcal{O}_X) \) which vanishes for all varieties above except \( C \) above by Lemma 2.14.

The exact sequence (2.4) crucially needs the regularity assumption (or actually factoriality). For a singular model we can however still compare the Picard group to the one of an algebraization. We use the following notation. A pointed scheme \((X, x)\) is a scheme \( X \) together with a distinguished closed point \( x \in X \). We write \( f: (X, x) \to (Y, y) \) for a morphism of schemes with \( f(x) = y \).

Proposition 2.9. Let \((C, c)\) be a smooth algebraic curve and let \( p: \mathcal{Y} \to \mathcal{Y}' \) be a birational \( C \)-morphism of normal schemes projective over \( C \) that is an isomorphism over \( C \setminus \{c\} \). Given a flat morphism \( \varphi: (S, c) \to (C, c) \), we consider the diagram of pullback squares

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\pi} & \mathcal{Y}' \\
j & & \downarrow j' \\
\mathcal{Y} & \xrightarrow{p} & \mathcal{Y}' \\
& & (C, c) \\
\end{array}
\]

Then the canonical morphism

\[
(2.6) \quad \text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}') \times_{\text{Pic}(\mathcal{Y})} \text{Pic}(\mathcal{Y})
\]

is an isomorphism. In particular, the canonical morphism \( \text{Pic}(\mathcal{Y}')/\text{Pic}(C) \to \text{Pic}(\mathcal{Y}') \) is an isomorphism in the situation of Proposition 2.7.

Proof. By normality of \( \mathcal{Y} \), the morphism (2.6) is injective because its composition with the projection to \( \text{Pic}(\mathcal{Y}) \) is. To prove surjectivity, let \( L' \) be a line bundle on \( \mathcal{Y}' \) and \( \mathcal{L} \) be a line bundle on \( \mathcal{Y} \) such that \( j^*\mathcal{L} = \pi^*L' \). As normality can be read off from the completion of a local ring, also \( \mathcal{Y} \) and \( \mathcal{Y}' \) are normal. Then \( j'^*p_*\mathcal{L} = \pi_*j^*\mathcal{L} = \pi_*\pi^*L' = L' \) by flat base change and normality. In particular, \( \mathcal{L}' := p_*\mathcal{L} \) is locally free of rank one along the central fiber \( \mathcal{Y}'_c \), and thus a line bundle satisfying \( j'^*\mathcal{L}' = L' \). Moreover, the canonical morphism \( p^*\mathcal{L}' \to \mathcal{L} \) is an isomorphism over \( C \setminus \{c\} \) and pulls back to an isomorphism under \( j \). Thus, \( p^*\mathcal{L}' = \mathcal{L} \) and (2.6) is surjective. The last claim is now immediate from Proposition 2.7. □

We continue with the following simple observation.
Lemma 2.10. Let \( Y \to S \) be a maximal Kulikov model and let \( Y \to C \) be an algebraization of \( Y \) as in Proposition 2.7. Then the following hold.

1. If \( p : Y \to Y' \) is a birational \( C \)-morphism of relative Picard rank \( k \), then the base change of \( p \) to \( S \) is a birational \( S \)-morphism of relative Picard rank \( k \).
2. Conversely, if \( \pi : Y \to Y' \) is a birational \( S \)-morphism and \( Y' \to S \) is projective, then after possibly shrinking \( C \) there is a birational \( C \)-morphism \( p : Y \to Y' \) where \( Y' \to C \) is projective and \( \pi \) is the base change of \( p \) to \( S \). Again, the relative Picard ranks of \( \pi \) and \( p \) coincide.

Proof. In both cases, the statement about the Picard rank follows from Proposition 2.9.

(1) It is clear that a \( C \)-morphism \( p : Y \to Y' \) induces an \( S \)-morphism \( \pi : Y \to Y' \) by base change and that \( \pi \) is birational if \( p \) is.

(2) Given \( \pi : Y \to Y' \), we chose a very ample line bundle on \( Y' \) and denote \( L \) its pullback to \( Y \). By (2.4) again, there is a line bundle \( \mathcal{L} \) on \( Y \) whose pull back to \( Y' \) is \( L \). The set of points \( t \in C \) where the restriction \( \mathcal{L}_t \) to the fiber \( Y_t \) is base point free is Zariski open and certainly contains the origin as \( \mathcal{L}_0 = L_c \) on \( Y_0 = Y_c \). Thus, after possibly shrinking \( C \) we get a \( C \)-morphism \( p : Y \to Y' \) whose base change is \( \pi \) and which therefore must be birational.

\[ \square \]

Remark 2.11. Proposition 2.7 and Lemma 2.10 allow us to treat maximal Kulikov models \( Y \to S \) as if they were finite type schemes. For example, we will make use of the MMP at various places without justifying it every time. This is possible as we may lift the Kulikov model to a morphism \( Y \to C \) as in the proposition, perform MMP there, and then base change to obtain a Kulikov model again.

We now want to study the birational geometry of models. We recall the definition of a birational contraction. Later we will also use the notion of a rational contraction due to Hu–Keel [KH00, Definition 1.1] which generalizes it.

Definition 2.12. Let \( T \) be a scheme and \( X, Y \) be normal projective \( T \)-schemes. A rational \( T \)-map \( f : X \dashrightarrow Y \) is called a birational contraction if its inverse does not contract any divisor. This can be reformulated as follows. If \( X \overset{p}{\leftarrow} W \overset{q}{\to} Y \) is a resolution of indeterminacy, i.e. \( p, q \) are proper and \( p \) is birational, then \( f \) is a birational contraction if and only if every \( p \)-exceptional divisor is \( q \)-exceptional. A rational \( T \)-map \( f \) is called a rational contraction if, with the same notation every \( p \) exceptional divisor is \( q \)-fixed, i.e. no effective Cartier divisor \( D \) whose support is \( p \)-exceptional is \( q \)-moving. This means that the natural map \( \mathcal{O}_Y \to q_* \mathcal{O}_W(D) \) is an isomorphism.
Lemma 2.13. Let $Y \to S$ be a maximal Kulikov model. If $\varphi : Y \dashrightarrow Y'$ is a birational contraction, then there is a maximal Kulikov model $X \to S$ and a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow \varphi & & \downarrow \psi \\
Y' & & 
\end{array}
$$

over $S$ such that $f$ is a composition of flops and $\psi$ is a regular contraction.

Proof. For an ample divisor $A$ on $Y'$, the pull back to $Y$ is contained in the effective movable cone $\text{Mov}(Y)$, see (3.2). Therefore, there is a marked minimal model $f : Y \dashrightarrow X$ over $S$ such that the pullback of $A$ to $X$ is nef, see [Kaw97, Theorem 2.3]. This implies that $X \dashrightarrow Y'$ is actually regular by Kawamata’s base point free theorem. Moreover, it follows from the proof of loc. cit. that $f$ is a composition of flops.

A priori $X$ only has $\mathbb{Q}$-factorial terminal singularities, but by [Kol89, Theorem 2.4] (see also [KM98, Theorem 6.15]) flops preserve regularity so that $X$ is smooth as well. Clearly, maximality is preserved under flops so that $X \to S$ is also a maximal Kulikov model. □

Lemma 2.14. Let $Y \to S$ be a maximal Kulikov model and let $Y \dashrightarrow Y'$ be a birational contraction. Then the following holds:

1. $K_{Y'} = 0$ and $Y'$ has canonical, hence rational singularities.
2. $Y'$ and the central fiber $Y'_c$ are Cohen Macaulay.
3. We have $h^i(Y'_c, \mathcal{O}_{Y'_c}) = \begin{cases} 1 & \text{for } i = 0, 2, \\ 0 & \text{for } i = 1. \end{cases}$
4. Base change holds, i.e. $H^i(Y', \mathcal{O}_{Y'}) \otimes_{\mathbb{C}[t]} \mathbb{C} \xrightarrow{\cong} H^i(Y'_c, \mathcal{O}_{Y'_c})$, and moreover

$$
H^i(Y', \mathcal{O}_{Y'}) = \begin{cases} \mathbb{C}[t] & \text{for } i = 0, 2, \\ 0 & \text{for } i = 1. \end{cases}
$$

Proof. By Lemma 2.13 we may assume $Y \to Y'$ to be regular. The morphism $Y \to Y'$ is an isomorphism over the regular locus of $Y'$, so $K_{Y'} = 0$. In particular, $K_{Y'}$ is Cartier. Consequently, $Y \to Y'$ is crepant and, in particular, $Y'$ has canonical singularities. Canonical singularities are rational by [Elk81] and rational singularities are Cohen Macaulay, see [KM98, Theorem 5.10]. Being an effective Cartier divisor in $Y'$ also $Y'_c$ is Cohen Macaulay. As it is connected and reduced, we have $H^0(\mathcal{O}_{Y'_c}) = \mathbb{C}$ and from Serre duality we infer $H^2(\mathcal{O}_{Y'_c}) = \mathbb{C}$. Finally, $H^1(\mathcal{O}_{Y'_c}) = 0$ follows as the generic fiber is a K3 surface and the Euler characteristic is constant in families. The last statement follows now from [3] and base change, see e.g. [Har77, Theorem III.12.11]. □

Let $Y \to S$ be a maximal Kulikov model and let $f : Y \dashrightarrow Y'$ be a birational contraction. Lemma 2.10 and Lemma 2.13 imply that there is an algebraization $\mathfrak{Y} \dashrightarrow \mathfrak{Y'} \to (C, c)$. We denote by $\Delta \subset C^\text{an}$ a small disk centered at $c$ and by $\mathfrak{Y}_{\Delta}^\text{an}$ the base change to $\Delta$ of the analytification.
Proposition 2.15. Let $\mathcal{Y} \to S$ be a maximal Kulikov model and let $f : \mathcal{Y} \to \mathcal{Y}'$ be a birational contraction. Then the following holds:

1. If $\mathcal{Y}' \to (C,c)$ is an algebraization, the morphism

\[
\text{Pic}(\mathcal{Y}')/\text{Pic}(C) \to \text{Pic}(\mathcal{Y}'\Delta), \quad L \mapsto L^\Delta \otimes O_C O_{\Delta}
\]

is an isomorphism. In particular, there is a canonical isomorphism $\text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}'\Delta)$.

2. The degeneration $\mathcal{Y}' \to S$ is maximal, i.e. the restriction

\[
\text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}_c')
\]

is an isomorphism.

3. If $f : \mathcal{Y} \to \mathcal{Y}'$ is a morphism, the pullbacks

\[
f^* : \text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}) \quad \text{and} \quad f^* : \text{Pic}(\mathcal{Y}_c') \to \text{Pic}(\mathcal{Y}_c)
\]

are injective of the same corank.

Proof. We will prove all statements at the same time. First we bound the size of $\text{Pic}(\mathcal{Y}'\Delta)$. The fibers $\mathcal{Y}'_t$ for $t \neq c$ are contractions of K3 surfaces and therefore satisfy $H^1(\mathcal{Y}'_t, O_{\mathcal{Y}'_t}) = 0$. Moreover, $H^1(\mathcal{Y}'_c, O_{\mathcal{Y}'_c}) = 0$ by Lemma 2.14. By GAGA, the same vanishing holds true after analytification and therefore $H^1(\mathcal{Y}'_{\Delta}^\text{an}, O_{\mathcal{Y}'_{\Delta}^\text{an}}) = H^0(\mathcal{Y}'_{\Delta}^\text{an}, R^1 h_* O_{\mathcal{Y}'_{\Delta}^\text{an}}) = 0$ by base change where $h : \mathcal{Y}'_{\Delta}^\text{an} \to \Delta$ is the projection. From the exponential sequence we thus obtain the diagram

\[
\begin{array}{ccc}
H^1(\mathcal{Y}'_{\Delta}^\text{an}, O_{\mathcal{Y}'_{\Delta}^\text{an}}^\times) & \to & H^2(\mathcal{Y}'_{\Delta}^\text{an}, \mathbb{Z}_{\mathcal{Y}'_{\Delta}^\text{an}}) \\
\downarrow & & \downarrow \iota^* \\
H^1(\mathcal{Y}'_{c}^\text{an}, O_{\mathcal{Y}'_{c}^\text{an}}^\times) & \to & H^2(\mathcal{Y}'_{c}^\text{an}, \mathbb{Z}_{\mathcal{Y}'_{c}^\text{an}})
\end{array}
\]

where the left horizontal maps are injections. The morphism $\iota^*$ is an isomorphism thanks to Lojasiewicz’s theorem [Loj64], see also [BHPVdV04, Theorem I.8.8]. This implies that the restriction

\[
\text{Pic}(\mathcal{Y}'_{\Delta}^\text{an}) = H^1(\mathcal{Y}'_{\Delta}^\text{an}, O_{\mathcal{Y}'_{\Delta}^\text{an}}^\times) \to H^1(\mathcal{Y}'_{c}^\text{an}, O_{\mathcal{Y}'_{c}^\text{an}}^\times) = \text{Pic}(\mathcal{Y}'_{c}^\text{an})
\]

is injective. The analogous statements hold for $\mathcal{Y}$. Thanks to Lemma 2.13 we may assume that $f : \mathcal{Y} \to \mathcal{Y}'$ is a morphism. Because of Proposition 2.7 we may replace $\text{Pic}(\mathcal{Y})$ by $\text{Pic}(\mathcal{Y})/\text{Pic}(C)$ and $\text{Pic}(\mathcal{Y}_c)$ by $\text{Pic}(\mathcal{Y}_c)$ everywhere.
and the same for \( Y' \) by Proposition 2.9. Let us consider the diagram

\[
\begin{array}{c}
\text{Pic}(\mathcal{Y}') / \text{Pic}(\mathcal{C}) \xrightarrow{\eta} \text{Pic}(\mathcal{Y}^\Delta_{\text{an}}) \\
\text{Pic}(\mathcal{Y}) / \text{Pic}(\mathcal{C}) \xrightarrow{\phi} \text{Pic}(\mathcal{Y}^\Delta_{\text{an}}) \\
\text{Pic}(\mathcal{Y}^\Delta_{c}) \xrightarrow{\alpha} \text{Pic}(\mathcal{Y}^\Delta_{c}) \\
\text{Pic}(\mathcal{Y}^\Delta_{c}) \xrightarrow{\beta} \text{Pic}(\mathcal{Y}^\Delta_{c}) \\
\text{Pic}(\mathcal{Y}) \xrightarrow{\delta} \text{Pic}(\mathcal{Y}^\Delta_{\text{an}}) \\
\text{Pic}(\mathcal{Y}^\Delta_{\text{an}}) \xrightarrow{\gamma} \text{Pic}(\mathcal{Y}^\Delta_{\text{an}})
\end{array}
\]

where all maps are given by pullback along the canonical morphisms of ringed spaces. The two lower horizontal maps are isomorphisms by GAGA, the two right vertical maps are injective by the observations made so far. By maximality, \( \alpha \) is an isomorphism and thus the front square consists of isomorphisms. We will see below that \( \beta \) and \( \gamma \) are injective. The map \( \delta \) is injective by normality of \( \mathcal{Y}' \) and injectivity of \( \varepsilon \) follows from the bottom square admitting that \( \gamma \) is injective. It follows from the diagram that \( \zeta \) and hence \( \eta \) are injective.

We will show next the surjectivity of \( \vartheta \) and injectivity of \( \beta \) and \( \gamma \). For this we consider the following diagram of cohomology groups induced by the exponential sequence.

\[
\begin{array}{c}
\text{Pic}(\mathcal{Y}^\Delta_{c}) \xrightarrow{\delta} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{Z}_{\mathcal{Y}^\Delta_{c}}) \xrightarrow{\vartheta} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{O}_{\mathcal{Y}^\Delta_{c}}) \\
\text{Pic}(\mathcal{Y}^\Delta_{c}) \xrightarrow{\gamma} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{Z}_{\mathcal{Y}^\Delta_{c}}) \xrightarrow{\beta} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{O}_{\mathcal{Y}^\Delta_{c}}) \\
\text{Pic}(\mathcal{Y}^\Delta_{c}) \xrightarrow{\varepsilon} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{Z}_{\mathcal{Y}^\Delta_{c}}) \xrightarrow{\zeta} H^2(\mathcal{Y}^\Delta_{c}, \mathcal{O}_{\mathcal{Y}^\Delta_{c}})
\end{array}
\]

As above, injectivity of the first column of horizontal morphisms follows from Lemma 2.14 and bijectivity of \( \iota^*, \iota'^* \) from Lojasiewicz’ theorem. Invoking Lemma 2.14 once more, we conclude that \( \mathcal{Y}' \) (and hence also \( \mathcal{Y}' \) and \( \mathcal{Y}^\Delta_{\text{an}} \)) has rational singularities. Now a standard argument, see e.g. [BL16, Lemma 2.1], shows that we have \( R^1 f_{\Delta*} \mathcal{Z}_{\mathcal{Y}^\Delta_{\text{an}}} = 0 \) and hence \( s \) is injective. This immediately implies injectivity of \( \beta \) and \( \gamma \). The kernel of \( u \) is identified with the kernel of \( v \) under \( \iota^* \). Rationality of singularities implies that \( t \) is an isomorphism and we obtain surjectivity of \( \vartheta \) by a diagram chase.

Now one shows literally as in Proposition 2.9 that the top square is cartesian, in particular, \( \eta \) is an isomorphism. This implies (1). From (2.9), we deduce that also \( \zeta \) is an isomorphism, so (2) follows. Putting everything together, we also obtain (3). \( \square \)
3. The DNV family and the Mori fan

In this section we introduce the DNV family in Definition 3.1 and review some of its basic properties. Afterwards in Section 3.5 we recall the construction of the Mori fan from [HL19], which will be crucial for the rest of the paper.

Let us briefly recall the notion of primitivity. Given a Kulikov model \( Y \to S \), we can always consider an analytic family \( Y \to \Delta \) with the same central fiber, e.g. by analytifying the algebraic deformation from Proposition 2.1. By [FS86, Theorem (0.5)], the monodromy action \( T \) on the cohomology \( H^2(Y_t, \mathbb{Z}) \) of a nearby fiber of \( Y \to \Delta \) of this degeneration only depends on the central fiber and the logarithm \( N := \log T \) is integral, that is, \( N \) is an endomorphism of \( H^2(Y_t, \mathbb{Z}) \). Following Friedman-Scattone, we call \( Y \to S \) primitive, if \( N \) is primitive as a vector in \( \text{End}_\mathbb{Z} (H^2(Y_t, \mathbb{Z})) \).

It follows from [HL19, Proposition 1.15] that the generic fiber of a primitive maximal Kulikov model \( Y \to S \) of a \( d \)-semistable K3 surface is uniquely defined up to isomorphism.

The existence statement of [HL19, Proposition 1.16] ensures that the following definition makes sense.

**Definition 3.1.** Fix a square free integer \( d > 0 \). The generic fiber \( Y_\eta \) of a primitive maximal Kulikov model \( Y \to S \) of a \( d \)-semistable K3 surface is called the Dolgachev–Nikulin–Voisin family (DNV family) of degree \( 2d \) if Pic(\( Y_\eta \)) is isomorphic to the lattice \( M_{2d} \) from (1.5). We refer to \( Y \to S \) as a model of the DNV family in this case.

The DNV family is thus a K3 surface \( Y_\eta \) over the field \( \mathbb{C}((t)) \). If we simply speak of the Dolgachev–Nikulin–Voisin family, it is understood that we mean the Dolgachev–Nikulin–Voisin family of degree \( 2d \) for some square free \( d \).

**Corollary 3.2.** If in Lemma 2.13 the morphism \( Y \to S \) is a semistable model of the DNV family, then \( X \to S \) is so as well.

**Proof.** Clearly, maximality and primitivity are preserved under flops so that \( X \to S \) is also a model of the DNV family by [HL19, Proposition 1.15]. \( \square \)

We recall the construction of these models, see [HL19 §1.2] for more details.

**Example 3.3.** Recall that there is a bijection between triangulations of the sphere \( S^2 \) such that no vertex has valency greater than 6 and maximal \( d \)-semistable K3 surfaces of type III in so-called \((-1)\)-form, see [Laz08 §5.1] and [FS86 §3.9]. Under this bijection, vertices correspond to irreducible components of the surface whose normalizations are weak del Pezzo surfaces of degree \( s \) equal to the valency of the vertex, and the double locus of the central fiber gives rise to an anticanonical cycle on the normalizations of the components. For \( s \leq 6 \) there is a unique such anti-canonical pair \((\mathcal{O}_s, D_s)\) which is a component of a \( d \)-semistable K3 surface with trivial Carlson map, see [HL19 Construction 1.23], and also [Laz08 Proposition 5.2 and Lemma 5.14].

There are precisely two different triangulations of \( S^2 \) with two triangles, and they correspond to dual intersection complexes of degenerations with central fibers having three components. We denote by \( \mathcal{P} \) the triangulation given by two triangles glued along the boundary.
Figure 1. The triangulations \( \mathcal{P} \) and \( \mathcal{T} \).

and by \( \mathcal{T} \) the one given by two triangles glued along one side to each other, see Figure 1. The corresponding surfaces \( Y_{\mathcal{P}} \) and \( Y_{\mathcal{T}} \) look like this: \( Y_{\mathcal{P}} \) is obtained by gluing three copies of \( \mathcal{Y}_2 \) by identifying components of the boundary cycle, and \( Y_{\mathcal{T}} \) is a copy of \( \mathcal{Y}_4 \), with two opposite components, say \( D_1 \) and \( D_3 \), of the anticanonical curve \( D \) identified, and two copies of \( \mathcal{Y}_1 \) glued to the (images) of \( D_2 \) and \( D_4 \). We refer to [HL19, Construction 1.23] for details.

The associated maximal smoothings \( Y_{\mathcal{P}} \to S \) and \( Y_{\mathcal{T}} \to S \) are models of the Dolgachev family in degree 2.

For later use we record

**Lemma 3.4.** Let \( \mathcal{Y} \to S \) be a regular model of the DNV family of degree \( 2d \). Then the number of irreducible components of the central fiber \( \mathcal{Y}_c \) is \( d + 2 \).

**Proof.** This is proven e.g. in [GHKS, Proposition 6.3]. We will give an independent argument. If \( n \) is the number of components of \( \mathcal{Y}_c \), then the number of triple points is \( t = 2n - 4 \). This follows from \( t - e + n = 2 \) where \( e \) is the number of edges in the dual complex and the obvious relation \( 3t = 2e \). Comparing the Picard lattices, it follows from [HL19, Proposition 1.13] that \( -2n + 4 = -2d \) so that \( n = d + 2 \). \( \square \)

**3.5. The Mori fan.** Let \( \mathcal{X} \to S = \text{Spec} \, \mathbb{C}[[t]] \) be a projective morphism of a normal threefold \( \mathcal{X} \). Let us first discuss the various cones of interest inside \( \text{NS}(\mathcal{X})_\mathbb{R} \). As the Picard group of \( S \) is trivial, the notion of relative and absolute ampleness for line bundles on \( \mathcal{X} \) coincide and similarly for nef line bundles. We denote by

\[
\text{Nef}(\mathcal{X}) \subset \text{NS}(\mathcal{X})_\mathbb{R}
\]

the nef cone of \( \mathcal{X} \), that is, the closure of the ample cone. We further denote by \( \text{Eff}(\mathcal{X}) \) the convex cone generated by all effective line bundles and by

\[
\text{Nef}^e(\mathcal{X}) := \text{Nef}(\mathcal{X}) \cap \text{Eff}(\mathcal{X}) \subset \text{NS}(\mathcal{X})_\mathbb{R}
\]

the effective nef cone. A line bundle \( L \) is movable if its base locus \( \text{Bs}(L) \subset \mathcal{X} \) has codimension at least 2. We denote by \( \overline{\text{Mov}}(\mathcal{X}) \) the closed movable cone defined as the closure of the convex cone generated by movable line bundles and by \( \text{Eff}(\mathcal{X}) \) the convex cone generated by all effective line bundles. The cone of interest to us will be

\[
\text{Mov}(\mathcal{X}) = \overline{\text{Mov}}(\mathcal{X}) \cap \text{Eff}(\mathcal{X}).
\]
to which we refer as the effective movable cone, sometimes also somewhat imprecisely just as the movable cone.

We can now define the Mori fan.

**Definition 3.6.** Let $\mathcal{Y} \to S$ be flat projective morphism from a $\mathbb{Q}$-factorial normal scheme $\mathcal{Y}$ with $\dim \mathcal{Y} = 3$ whose generic fiber is the DNV family and whose relative canonical sheaf is the trivial line bundle. For a rational map $f: \mathcal{Y} \dashrightarrow \mathcal{Y}'$, we put $C(f) := f^*\text{Nef}^e(\mathcal{Y}') \subset \text{NS}(\mathcal{Y})$. Then the Mori fan of $\mathcal{Y}$, denoted by $\text{MF}(\mathcal{Y})$, is defined as

$$\text{(3.3)} \quad \text{MF}(\mathcal{Y}) := \{ C(f) \mid f: \mathcal{Y} \dashrightarrow \mathcal{Y}' \text{ is a rational contraction} \}.$$ 

Note that if $f$ is the map to a point, then $C(f) = \{0\}$.

This fan was first considered by Hu and Keel for Mori dream spaces, see [KH00, 1.11 Proposition]. In our context, the definition is due to Gross–Hacking–Keel–Siebert, see [GHKS16, Section 6].

**Remark 3.7.** It follows readily from the definition of the Mori fan that its maximal cones are of the form $C(f)$ where $f: \mathcal{Y} \dashrightarrow \mathcal{Y}'$ is a small $\mathbb{Q}$-factorial modification. Recall that $f: \mathcal{Y} \dashrightarrow \mathcal{Y}'$ is called a small $\mathbb{Q}$-factorial modification (over $S$) if $\mathcal{Y}'$ is $\mathbb{Q}$-factorial and $f$ is a birational $S$-map and an isomorphism in codimension one.

**Remark 3.8.** We emphasize that the nef cone is in general strictly larger than its subcone generated by effective nef divisors. However, for K3 surfaces $\mathcal{X}$ containing curves of negative self intersection, the cones $\text{Nef}(\mathcal{X})$ and $\text{Nef}^e(\mathcal{X})$ coincide by [Kov94, Corollary 1].

From [GHKS16, Theorem 6.5], we infer:

**Proposition 3.9.** The support of $\text{MF}(\mathcal{Y})$ is $\text{Mov}(\mathcal{Y})$ and $\text{Mov}(\mathcal{Y})$ is the rational closure of its interior. □

**Remark 3.10.**

1. The fact that one can run an MMP for $\mathcal{Y} \to S$ implies that a small modification $f: \mathcal{Y} \dashrightarrow \mathcal{Y}'$ factors into flops. Hence, $\mathcal{Y}' \to S$ is also a model of the DNV family.
2. Similarly to [Kaw97, Lemma 1.5] one shows that if $f: \mathcal{Y} \dashrightarrow \mathcal{Y}'$ and $g: \mathcal{Y} \dashrightarrow \mathcal{Y}''$ are birational contractions such that $C(f) \cap C(g)$ has a point that is interior to both $C(f)$ and $C(g)$, then there is an isomorphism $\beta: \mathcal{Y}'' \to \mathcal{Y}'$ with $f = \beta \circ g$, and hence $C(f) = C(g)$.
3. Note that if $\mathcal{Y}' \to S$ is another regular model of the DNV family, then every birational map $\mathcal{Y} \dashrightarrow \mathcal{Y}'$ yields an identification of $\text{MF}(\mathcal{Y}')$ with $\text{MF}(\mathcal{Y})$. We refer to the discussion of Section 2.2 of [HL19] for the fact that $\text{MF}(\mathcal{Y})$ is indeed a fan. This follows from the fact that the lattice $M_2$ is a reflexive lattice, i.e. the quotient $O^+(M_2)/W_2$ of the orthogonal group of $M_2$ preserving the positive cone by the Weyl group is finite. This was shown by Nikulin in [Nik83].
4. One can show that nef line bundles on $\mathcal{Y}$ are semi-ample, see [HL19, Remark 2.6].
5. The cone $\text{Nef}(\mathcal{Y}_\eta)$ is finitely polyhedral if and only if $d = 1$, see e.g. [HL19, Remark 2.7].
4. Cusp models

In this short section, we introduce cusp models, prove their existence, and introduce cuspidal cones which will be central in the definition of the GHKS compactification in the next section.

**Definition 4.1.** Let \( f : \mathcal{Y} \to S \) be a model of the DNV family. A \textit{rational cusp model} for \( f \) is a flat projective morphism \( f' : \mathcal{Y}' \to S \) together with a birational contraction \( \varphi : \mathcal{Y} \dashrightarrow \mathcal{Y}' \) over \( S \) such that the following holds:

1. The scheme \( \mathcal{Y}' \) is normal with \( \mathbb{Q} \)-factorial singularities,
2. the map \( \varphi \) is an isomorphism over the generic point of \( S \), and
3. the central fiber \( \mathcal{Y}'_c \) of \( f' \) is irreducible.

A \textit{cusp model} is a rational cusp model such that \( \varphi : \mathcal{Y} \to \mathcal{Y}' \) is a morphism. In both cases, if \( Y \subset \mathcal{Y}_c \) is the unique component which is not contracted, we refer to \( \mathcal{Y}' \) as a (rational) cusp model for \( Y \) or a \( Y \)-cusp model.

Note that a cusp model is not a model of the DNV family in the terminology of Definition 3.1 since the total space of a model needs to be regular. Sometimes we will refer somewhat imprecisely to \( \mathcal{Y}' \) itself as a cusp model. Lemma 2.13 in particular applies to rational cusp models \( \mathcal{Y}' \).

**Lemma 4.2.** Let \( f : \mathcal{Y} \to S \) be a model of the DNV family. If \( \varphi : \mathcal{Y} \dashrightarrow \mathcal{Y}' \) is a rational cusp model, then \( \mathcal{Y}' \) has Picard number 19 and the restriction \( \text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}_\eta) \) to the generic fiber is an isomorphism. If \( \varphi \) is a morphism, also the restriction \( \text{Pic}(\mathcal{Y}') \to \text{Pic}(\mathcal{Y}'_c) \) to the special fiber is an isomorphism.

**Proof.** As \( \varphi \) is a contraction, the pullback \( \varphi^* \) on (rational) Picard groups is injective. The Picard rank of \( \mathcal{Y} \) is \( 19 + n \) if \( n + 1 \) is the number of irreducible components of the central fiber \( \mathcal{Y}_c \). Let \( D \) be a divisor on \( \mathcal{Y} \). As \( \mathcal{Y}' \) is \( \mathbb{Q} \)-factorial, \( \varphi^* \varphi_* D \) coincides with \( D \) up to exceptional divisors. The claim follows because by definition of a cusp model there are exactly \( n \) distinct exceptional divisors.

**Remark 4.3.** Notice that unlike in [GHKS], we ask cusp models to be \( \mathbb{Q} \)-factorial. This turned out to be convenient; it is e.g. crucial for Lemma 4.2.

Recall from Remark 2.11 that we have the MMP for Kulikov models at our disposal – just as for ordinary threefolds over a curve.

**Proposition 4.4.** Let \( \mathcal{Y} \to S \) be a model of the DNV family of degree \( 2d \) and let \( Y \) be an irreducible component of \( \mathcal{Y}_c \). Then there is a rational \( Y \)-cusp model \( \mathcal{Y} \dashrightarrow \mathcal{Y}' \).

**Proof.** We consider the \( \mathbb{R} \)-divisor \( \Delta := \varepsilon(\mathcal{Y}_c - Y) \) for a sufficiently small \( \varepsilon > 0 \) such that the pair \( (\mathcal{Y}, \Delta) \) is klt. Note that \( K_{\mathcal{Y}/S} + \Delta = K_{\mathcal{Y}} + \Delta = \Delta \). We run a \( (K_{\mathcal{Y}/S} + \Delta) \)-log MMP over \( S \) and obtain a sequence of rational \( S \)-maps

\[
\mathcal{Y} = \mathcal{X}_0 \to \mathcal{X}_1 \to \ldots \to \mathcal{X}_N =: \mathcal{Y}',
\]
with the following properties. All schemes $\mathcal{X}_i$ are $\mathbb{Q}$-factorial, and if we define $\Delta_0 := \Delta$ and inductively $\Delta_i := \phi_{i-1}^* \Delta_{i-1}$ for all $i$, then the pairs $(\mathcal{X}_i, \Delta_i)$ are klt. In addition, the divisor $\Delta_N$ is nef.

We claim that $K_{\mathcal{X}_i/S} = 0$ and that $\phi_{i+1}$ is either a $\Delta_i$-flip or a divisorial contraction whose exceptional locus is a divisor contained in the support of $\Delta_i$ for all $i = 1, \ldots, N$. Inductively, we may assume this to be the case for all $j < i$ where $i$ is fixed. By MMP, there is a $\Delta_{i-1}$-negative extremal ray $R$ and a contraction $c_R : X_{i-1} \rightarrow Z$ which contracts exactly those curves $C \subset X_{i-1}$ with $[C] \in R$ such that the following holds: either $c_R$ is a divisorial contraction or a Mori fiber space, $\mathcal{X}_i = Z$, and $\phi_i = c_R$ or $c_R$ is small and $\phi_i$ is the $\Delta_{i-1}$-flip. As the Kodaira dimension $\kappa(X_{i-1})$ is not $-\infty$, the contraction $c_R$ cannot be a Mori fiber space. In the other two cases, we still have $K_{\mathcal{X}_i/S} = 0$. Thus, the exceptional locus of $c_R$ is contained in $\text{supp} \Delta$ and the claim about $\phi_i$ follows. We deduce that the composition $\phi := \phi_N \circ \ldots \circ \phi_1$ is birational when restricted to $Y$. We claim that $\phi : Y \dashrightarrow Y'$ is a cusp model for $Y$.

Being the outcome of an MMP, $Y'$ is clearly $\mathbb{Q}$-factorial. It remains to show that $\Delta_N = 0$. Suppose this is not the case and write the central fiber as $Y'_c = Y' + Y''$ where $Y' \subset Y'_c$ is the irreducible component such that $\phi : Y \dashrightarrow Y'$ is birational. Then we have $\Delta_N = \varepsilon Y''$, i.e. $Y''$ is the support of $\Delta_N$. Choose a curve $C \subset Y''$ which is not contained in $Y'$ but has positive intersection with it: $C.Y' > 0$. Then

$$0 = \varepsilon C.Y'_c = C.\Delta_N + \varepsilon C.Y' > C.\Delta_N$$

and therefore $\Delta_N$ is not nef. We obtain a contradiction so that $\Delta_N = 0$ and $\phi : Y \dashrightarrow Y'$ is a cusp model for $Y$.

Recall the definition of $\text{MF}(\mathcal{Y})$ from Section 3.5.

**Definition 4.5.** A cone $\sigma$ of $\text{MF}(\mathcal{Y})$ is called *cuspidal* if there is a component $Y \subset \mathcal{Y}_c$ and a rational $Y$-cusp model $f : \mathcal{Y} \dashrightarrow \mathcal{Y}'$ such that $\sigma = C(f)$.

**Remark 4.6.**

1. As a consequence of Lemma 3.12, cuspidal cones are 19 dimensional.
2. Lemma 2.13 says that for every cuspidal cone $\sigma$ there is a marked minimal model $f : \mathcal{Y} \dashrightarrow \mathcal{X}$ and a cusp model $\pi : \mathcal{X} \rightarrow \mathcal{Y}'$ such that $\sigma = f^* \pi^* \text{Nef}(\mathcal{Y}')$. Cusp models are divisorial contractions of extremal rational faces of the nef cone of $\mathcal{X}$ that lie on the boundary of the movable cone. Therefore, cuspidal cones have to lie on the boundary of the movable cone.
3. The just mentioned fact that cuspidal cones correspond (not necessarily one-to-one) to marked minimal models, i.e. regular models of the DNV family, together with a divisorial contraction is what will allow us to classify cuspidal cones in Section 9. The task there will be to classify marked minimal models allowing for a divisorial contraction of relative Picard rank two that is an isomorphism on the generic fiber.

4. That is, intersections of $\text{Nef}(\mathcal{X})$ with a rational linear subspace not meeting the ample cone with non-empty relative interior.
5. Construction of the fan

Let \( Y \to S \) be a regular model of the DNV family of degree \( 2d \). We fix this model once and for all. Recall that the real Néron–Severi group \( \text{NS}(Y_\eta) \otimes \mathbb{R} \) of the generic fiber of our model is isomorphic to \( M_{2d, \mathbb{R}} \) where \( M_{2d} \) is the lattice from \( \text{(1.5)} \). We now fix an identification \( \text{NS}(Y_\eta) \cong M_{2d, \mathbb{R}} \) once and for all.

The goal of this section is twofold. First, we introduce the GHKS fan (see Definition 5.19), whose support is the rational closure \( C_{2d}^{rc} \subset M_{2d, \mathbb{R}} \) of the positive cone. In Theorem 5.21 we prove that the GHKS fan is a semitoric fan in the sense of Looijenga and thus gives rise to a semitoric compactification. This is implicit in [GHKS16], but there the emphasis lies more in using the GHKS fan to refine a given toric fan which then gives rise to another toroidal compactification. The techniques we use here are clearly based on [GHKS16].

The second goal of this section is the analysis of the GHKS fan in degree \( 2d = 2 \). This case has two important features: the construction is somewhat simpler due to extra symmetries and the resulting fan is an actual toric fan, see Propositions 5.24 and 5.26.

This GHKS fan will be the common refinement of certain fans \( \Sigma_Y \) coming from irreducible components \( Y \subset Y_c \) of the central fiber. We will first define their restriction to the nef cone.

**Definition 5.1.** Let \( Y \subset Y' \) be an irreducible component of the central fiber. We choose a rational cusp model \( Y' \to Y' \) for \( Y \) and denote by \( \iota : Y_\eta \to Y' \) the inclusion. Then we define \( \Sigma_Y^{\text{nef}} \) on \( \text{NS}(Y_\eta) \) to be the pull back of the Mori fan of the cusp model along \( \iota \), that is, the following collection of cones:

\[
\Sigma_Y^{\text{nef}} := \{ \iota^* (\sigma) \mid \sigma \in \text{MF}(Y') \}.
\]

**Lemma 5.2.** The collection of cones \( \Sigma_Y^{\text{nef}} \) from Definition 5.1 has support equal to \( \text{Nef}(Y_\eta) \). Moreover, the restriction of \( \Sigma_Y^{\text{nef}} \) to any rational subcone of \( C_{2d}^{rc} \) is a rational cone system.

**Proof.** The second statement follows from [GHKS16, Theorem 6.5], see also [HL19, Theorem 2.4]. In both references, the total space \( Y' \) is supposed to be regular, but this is not necessary. The main technical tools are Theorems 3 and 4 of [Kaw11] which are proven for \( \mathbb{Q} \)-factorial klt pairs. What is used is that an effective divisor \( B \) on \( Y' \) can always be scaled so as to make \( (Y', B) \) klt. Alternatively, we can deduce the first claim from the embedding of \( \text{MF}(Y') \) into \( \text{MF}(Y) \) as in Section 5.7 and argue for the Mori fan of the regular scheme \( Y' \).

Moreover, the isomorphism \( \iota^* : \text{NS}(Y') \to \text{NS}(Y_\eta) \) from Lemma 4.2 defines an identification \( \text{Mov}(Y') \cong \text{Nef}(Y_\eta) \), see also [GHKS16, Lemma 8.1]. As the support of \( \text{MF}(Y') \) is \( \text{Mov}(Y') \), the first claim follows.

The notation \( \Sigma_Y^{\text{nef}} \) is justified by the following lemma.

**Lemma 5.3.** Let \( Y \subset Y' \) be an irreducible component of the central fiber. Then the collection \( \Sigma_Y^{\text{nef}} \) is independent of the choice of a cusp model for \( Y \).

**Proof.** Let \( p_1 : Y \to Y_1, p_2 : Y \to Y_2 \) be cusp models for \( Y \). Then the induced map \( p = p_2 \circ p_1^{-1} : Y_1 \to Y_2 \) is an isomorphism in codimension one, thus a small \( \mathbb{Q} \)-factorial
modifications. The isomorphism \( p^* : \text{NS}(\mathcal{Y}_2) \to \text{NS}(\mathcal{Y}_1) \) maps \( \text{MF}(\mathcal{Y}_2) \) isomorphically to \( \text{MF}(\mathcal{Y}_1) \) and is compatible with restriction to the central fiber so the claim follows. □

Our goal is to construct a semitoric fan covering the rational closure of (a component of) the positive cone in \( M_{2d,\mathbb{R}} \). For this we need to understand the action of the group \( \Gamma_{2d}^+ \) on the cones of \( \Sigma^\text{eff}_Y \). We denote the set of roots in \( M_{2d,\mathbb{R}} \) by \( R_{2d} := \{ v \in M_{2d,\mathbb{R}} \mid v^2 = -2 \} \).

We choose a set of simple roots \( \Delta \subset R_{2d} \). It is well-known from the theory of reflection groups that the set \( V_{2d} := \{ v \in C_{2d}^{\text{rc}} \mid v^2 \geq 0, \alpha.v \geq 0 \text{ for all } \alpha \in \Delta \} \) is a fundamental domain for the action of \( W_{2d} \) on \( C_{2d}^{\text{rc}} \) and that the orthogonal group \( O(M_{2d}) \) decomposes as a semidirect product \( W_{2d} \ltimes P_{2d} \) where \( W_{2d} \) is the Weyl group (generated by reflections in roots in \( R_{2d} \)) and \( P_{2d} \) is the subgroup that fixes the fundamental domain \( V_{2d} \) (\textit{Hum90}, Theorem 12.2). Note that reflections on \( M_{2d} \) always extend to the K3 lattice so that \( W_{2d} \subset \Gamma_{2d}^+ \). Hence, the decomposition of the orthogonal group induces a decomposition
\[
\Gamma_{2d}^+ = W_{2d} \ltimes P_{2d}^+, \quad \text{where } P_{2d}^+ := P_{2d} \cap \Gamma_{2d}^+.
\]

**Lemma 5.4.** Under the identification \( M_{2d,\mathbb{R}} \cong \text{Pic}(\mathcal{Y}_\eta) \), a distinguished set of simple roots is given by the effective \((-2)\) classes. With this choice, the fundamental domain \( V_{2d} \) is isomorphic to the cone \( \text{Nef}(\mathcal{Y}_\eta) \).

**Proof.** From Riemann-Roch and Serre duality one deduces that effective \((-2)\) classes constitute a set of simple roots. It follows easily from the Nakai-Moishezon-Kleiman criterion, see e.g. \textit{Huy16}, 8, Theorem 1.2 that \( \text{Nef}(\mathcal{Y}_\eta) \) cut out by their orthogonals inside the positive cone. □

Recall that for complex K3 surfaces \( S \), the morphism \( \text{Aut}(S) \to O(H^2(S,\mathbb{Z})) \) is injective. With this in mind, we prove

**Proposition 5.5.** Under the identification \( \text{Pic}(\mathcal{Y}_\eta) \cong M_{2d} \), pullback via an automorphism induces a homomorphism \( G := \text{Aut}(\mathcal{Y}_\eta) \to O^+(M_{2d}) \), whose image contains \( P_{2d}^+ \). For \( d = 1 \) we have a short exact sequence
\[
0 \to \mathbb{Z}/2\mathbb{Z} \to G \to P_{2d}^+ \to 0
\]
where the kernel is given by a non-symplectic involution on \( \mathcal{Y}_\eta \).

**Proof.** We take an algebraization \( \mathcal{Y} \to C \) over a quasi-projective curve \( C \) such that \( \mathcal{Y} \to S \) is the base change under a morphism \( S \to C \), see Proposition 2.7. For \( t \in C \), there is a canonical morphism \( \text{Pic}(\mathcal{Y}_t) \to \text{Pic}(\mathcal{Y}_\eta) \) which is an isomorphism for the very general point \( t \), see (2.5). For the fibers over those points, also the nef cone is isomorphic to the fundamental domain \( V_{2d} \) by the same reasoning as in Lemma 5.4. We will show first that \( P_{2d}^+ \) is contained in the image of \( G \to O^+(M_{2d}) \). Let us observe that an \( s \in P_{2d}^+ \) lifts to an isometry \( \tilde{s} \in O(A) \) of the K3 lattice \( (1.2) \). By the effective Torelli theorem for complex K3 surfaces there is an
automorphism $\varphi_t$ of $\mathcal{Y}_t$ which induces $\tilde{s}$. Let $\Gamma_t \subset \mathcal{Y}_t \times \mathcal{Y}_t$ be the graph of $\varphi_t$. By countability of components of the relative Hilbert scheme of $\mathcal{Y} \times_C \mathcal{Y}$ over $C$, there is an irreducible cycle $\varGamma \subset \mathcal{Y} \times_C \mathcal{Y}$ whose fiber over uncountably many $t \in C$ equals $\Gamma_t$. Therefore, $\varGamma$ is the graph of a birational automorphism of $\mathcal{Y}$, thus an automorphism of $\mathcal{Y}_\eta$. By construction, it acts as $\tilde{s}$ on $\text{Nef}(\mathcal{Y}_\eta)$.

The following argument shows that the kernel of $G \to O^+(M_{2d})$ is at most $\mathbb{Z}/2\mathbb{Z}$. For a very general $t \in C$ the only Hodge isometries of the transcendental lattice are $\pm \text{id}$ which can be verified using a Mumford–Tate group argument. The Mumford–Tate group of the transcendental lattice $T \subset H^2(\mathcal{Y}_t, \mathbb{Z})$ for very general $t \in C$ is $SO(T \otimes \mathbb{Q})$, a Hodge isometry is invariant under the Mumford–Tate group in $\text{End}(T)$, and by elementary representation theory of the special orthogonal group the only such invariants are multiples of the identity. We refer to [CMSPL7] Chapter 15.2 for generalities on Mumford–Tate groups.

Let us now specialize to $d = 1$ and show that there is a non-trivial element in the kernel. For the very general $\mathcal{Y}_t$ as above, the transcendental lattice is given by $T = U \oplus (2d)$. We claim that there is an isometry of $\Lambda$ that restricts to $\text{id}$ on $M_{2d}$ and to $- \text{id}$ on $T$. Let us choose generators $U = \langle e, f \rangle$ and define an isometry $\alpha : U \to U$ by $e \mapsto -f$, $f \mapsto -e$. Putting $h := e + df$ and $h' = h - df$ we may write $T = U \oplus \langle h \rangle$ and $M_{2d} = U \oplus 2E_8(-1) \oplus \langle h' \rangle$ and the isometry

$$\text{id} \oplus \alpha \oplus \text{id} : U \oplus U \oplus (U \oplus 2E_8(-1)) \to U \oplus U \oplus (U \oplus 2E_8(-1))$$

restricts to $(- \text{id}_T) \oplus \text{id}_{M_{2d}}$ on the sublattice $T \oplus M_{2d} = U \oplus \langle h, h' \rangle \oplus (U \oplus 2E_8(-1)) \subset \Lambda$. This isometry is clearly Hodge and invoking Torelli once more we see as above that it is induced by an automorphism of $\mathcal{Y}_\eta$, which is a non-symplectic involution.

**Remark 5.6.** We recall that $\mathcal{P}_{2}^+ \cong \tilde{O}^+(M_2)/W_2 = O^+(M_2)/W_2 \cong S_3$. In this way we have recovered the well known result that the automorphism group of a $K3$ surface with $\text{NS}(X) \cong U \oplus 2E_8(-1) \oplus \langle -2 \rangle$ is given by $\text{Aut}(X) \cong S_3 \times \mathbb{Z}/2\mathbb{Z}$, see [Kon89]. Here $S_3$ is the group of symplectic automorphisms and the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is the anti-symplectic isomorphism constructed above.

### 5.7. Embedding into the Mori fan

Our next goal is to embed the collections $\Sigma^\text{nef}_Y$ into the Mori fan of $\mathcal{Y}$. This is a helpful feature as it allows to use the methods from [HL19] to analyze them. Most of this material can be found in [GHKS16], only here and there we give some more details or choose a different presentation.

Given a rational cusp model $p : \mathcal{Y} \dashrightarrow \mathcal{Y}_1$, we embed the Morifan $\text{MF}(\mathcal{Y}_1)$ into $\text{MF}(\mathcal{Y})$. Note however, that such an embedding is not given by pullback along $p$ as $p^*$ does not necessarily send cones to cones due to the lack of functoriality of pullbacks along rational maps. Indeed, given a cone $C(f) \in \text{MF}(\mathcal{Y}_1)$, say coming from a contraction $f : \mathcal{Y}_1 \dashrightarrow \mathcal{X}$, we have $C(f \circ p) \in \text{MF}(\mathcal{Y})$ and this cone is in general not equal to the set $p^*(C(f))$. While the map $p^*$ is linear, but does not preserve cones, the map we aim to construct respects cones, but will only be piecewise linear. The embedding $C(f) \mapsto C(f \circ p)$ leads to the following definition.
**Definition 5.8.** Let \( \mathcal{Y} \rightarrow S \) be a model of the DNV family, let \( Y \) be an irreducible component of the central fiber of \( \mathcal{Y} \rightarrow S \), and let \( p : \mathcal{Y} \rightarrow \mathcal{Y}_1 \) be a rational cusp model for \( Y \). Let \( q : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2 \) be a small \( \mathbb{Q} \)-factorial modification. Then on \( q^*\text{Nef}(\mathcal{Y}_2) \subset \text{Mov}(\mathcal{Y}_1) \) we define

\[
\begin{align*}
  s_{\mathcal{Y}_1} : \text{Mov}(\mathcal{Y}_1) & \rightarrow \text{Mov}(\mathcal{Y}), \\
  s_{\mathcal{Y}_1}|_{q^*\text{Nef}(\mathcal{Y}_2)} & := (q \circ p)^* \circ q_*.
\end{align*}
\]

As the following lemma shows, this does not depend on the choices made. More than that, the composition with the restriction to the generic fiber only depends on the component \( Y \).

**Lemma 5.9.** The map \( s_{\mathcal{Y}_1} : \text{Mov}(\mathcal{Y}_1) \rightarrow \text{Mov}(\mathcal{Y}) \) from Definition 5.8 is well defined, piece-wise linear, and continuous. If \( t_1 : \mathcal{Y}_1 \rightarrow \mathcal{Y}_1 \) is the inclusion, the composition \( s_{\mathcal{Y}_1} \circ (t_1)_*^{-1} \) is a section of the restriction \( \text{Mov}(\mathcal{Y}) \rightarrow \text{Mov}(\mathcal{Y}_1) = \text{Nef}(\mathcal{Y}_1) \) and depends only on \( Y \), but not on the rational cusp model.

**Proof.** By [Kaw97, Theorem 2.3], the nef cones of small \( \mathbb{Q} \)-factorial modifications cover \( \text{Mov}(\mathcal{Y}_1) \) so that it is sufficient to define \( s_{\mathcal{Y}_1} \) on these cones. Let us show that \( s_{\mathcal{Y}_1} \) does not depend on the choice of small \( \mathbb{Q} \)-factorial modification. Let \( q_2 : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2 \) and \( q_3 : \mathcal{Y}_1 \rightarrow \mathcal{Y}_3 \) be small \( \mathbb{Q} \)-factorial modifications such that \( q_2^*\text{Nef}(\mathcal{Y}_2) \cap q_3^*\text{Nef}(\mathcal{Y}_3) \neq \emptyset \). If the interiors of both cones intersect, then there is an isomorphism \( h : Y_2 \xrightarrow{\cong} Y_3 \) such that \( h \circ q_2 = q_3 \), see Lemma 5.10 below. In this case obviously \( (q_2 \circ p)^* \circ q_* = (q_3 \circ p)^* \circ q_* \).

Otherwise, the intersection has to be contained in the boundary of both cones. Suppose first that \( q_2^*\text{Nef}(\mathcal{Y}_2) \cap q_3^*\text{Nef}(\mathcal{Y}_3) \) intersects the interior of the big cone. By [Kaw88, Theorem 5.7] (see also [Kaw97, Theorem 1.9]), the nef cone is rational polyhedral inside the big cone so that there is a unique face \( F \) of \( q_2^*\text{Nef}(\mathcal{Y}_2), q_3^*\text{Nef}(\mathcal{Y}_3) \) such that \( q_2^*\text{Nef}(\mathcal{Y}_2) \cap q_3^*\text{Nef}(\mathcal{Y}_3) = F \) and a diagram

\[
(5.3) \quad \mathcal{Y}_2 \leftarrow \mathcal{Y}_3 \\
\quad \downarrow f_2 \quad \quad \downarrow f_3 \\
\quad \mathcal{Z}
\]

where \( f_2, f_3 \) are regular contractions associated to \( F \). In particular,

\[
F = q_2^* (f_2^* \text{Pic}(\mathcal{Z}) \cap \text{Nef}(\mathcal{Y}_2)) = q_3^* (f_3^* \text{Pic}(\mathcal{Z}) \cap \text{Nef}(\mathcal{Y}_3)).
\]

Take \( \alpha \in F \subset \text{Mov}(\mathcal{Y}_1) \) and write \( \alpha = q_2^* f_2^* \beta = q_3^* f_3^* \beta \) for some class \( \beta \) on \( \mathcal{Z} \). Taking a resolution \( W \) of indeterminacies of all the rational maps, and letting \( q := q_3 \circ q_2^{-1} \), we obtain a diagram

\[
\begin{array}{c}
W \\
\pi_0 \downarrow \pi_1 \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \uparrow \pi_2 \quad \uparrow \pi_3 \\
\mathcal{Y} \quad \rightarrow \quad \mathcal{Y}_1 \quad \rightarrow \quad \mathcal{Y}_2 \\
\downarrow p \quad \quad \quad \downarrow q_2 \quad \quad \quad \downarrow q \\
\mathcal{Y}_2 \quad \rightarrow \quad \mathcal{Y}_3 \\
\downarrow f_2 \quad \quad \quad \quad \downarrow f_3 \\
\mathcal{Z}
\end{array}
\]
and we have to show that \((q_2 \circ p)^* q_{2*, \alpha} = (q_3 \circ p)^* q_{3*, \alpha}\). Let us observe that \(\pi_2^* \circ f_2^* = \pi_2^* \circ f_2^*\) by commutativity of the diagram and therefore

\[
(q_2 \circ p)^* q_{2*, \alpha} = (q_2 \circ p)^* f_2^* \beta = \pi_0^* \pi_2^* f_2^* \beta = \pi_0^* \pi_3^* f_3^* \beta
\]

\[
= (q_3 \circ p)^* f_3^* \beta = (q_3 \circ p)^* q_{3*, \alpha}.
\]

Now suppose that \(q_2^* \text{Nef}^c(\mathcal{Y}_2) \cap q_3^* \text{Nef}^c(\mathcal{Y}_3)\) lies on the boundary of the big cone and let \(\alpha\) be contained in this intersection. As \(\text{Mov}(\mathcal{Y}_1)\) is the rational closure of its interior, it is sufficient treat the case where \(\alpha\) is rational, or even integral. Then \(\alpha\) gives rise to a fibration \(f_2 : \mathcal{Y}_2 \to B\) which is an elliptic fibration on the generic fiber over \(S\). Let \(A\) be the pullback along \(q\) of an ample prime divisor. As \(A\) is \(f_2\)-nef if and only if \(A + f_2^* L\) is nef for a sufficiently ample divisor \(L\) on \(B\) and clearly \(A + \pi^* L \in q^* \text{Amp}(\mathcal{Y}_3)\), we deduce that for \(A\) to be \(f_2\)-nef, we must have that \(q\) is an isomorphism by Lemma 5.10 below. Let us assume this is not the case. Then we run a log MMP for the pair \((\mathcal{Y}_2, \varepsilon A)\), where \(\varepsilon\) is small enough in order to make the pair klt, and obtain a sequence of flops over \(B\) connecting \(\mathcal{Y}_2\) to \(\mathcal{Y}_3\). Thus, we may reduce to the case where \(q\) is a flop over \(B\):

\[
\mathcal{Y}_2 \xrightarrow{q} \mathcal{Y}_3 \\
\mathcal{Y}_2 \xrightarrow{f_2} B \xleftarrow{f_3} \mathcal{Y}_3
\]

In this case, \(q_2^* \text{Nef}^c(\mathcal{Y}_2) \cap q_3^* \text{Nef}^c(\mathcal{Y}_3)\) is a facet (in both cones) and we are back in the previous case. We thus conclude that \(s_{\mathcal{Y}_1}\) is well-defined and continuous. It is piecewise linear by definition.

Next we show that \(s_{\mathcal{Y}_1} \circ (\iota_1^*)^{-1}\) is independent of the cusp model. Let \(p : \mathcal{Y} \to \mathcal{Y}_1\), \(p' : \mathcal{Y} \to \mathcal{Y}_1'\) be rational \(Y\)-cusp models and consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}_2 & \xrightarrow{q} & \mathcal{Y}_3 \\
\mathcal{Y}_2 \xrightarrow{f} \mathcal{Y}_1 \xleftarrow{f'} \mathcal{Y}_2 \\
\mathcal{Y} & \xrightarrow{p} & \mathcal{Y}_1 \\
\mathcal{Y} & \xrightarrow{p'} & \mathcal{Y}_1'
\end{array}
\]

where \(f, f'\) are small \(\mathbb{Q}\)-factorial modifications and \(r := p' \circ p^{-1}\). By \(\mathbb{Q}\)-factoriality of cusp models, also \(r\) is a small \(\mathbb{Q}\)-factorial modification. Showing that \(s_{\mathcal{Y}_1} \circ (\iota_1^*)^{-1} = s_{\mathcal{Y}_1'} \circ (\iota_1'^* )^{-1}\) is tantamount to showing \(s_{\mathcal{Y}_1} = s_{\mathcal{Y}_1} \circ r^*\). Let us consider these maps on the cone \(f^* \text{Nef}^c(\mathcal{Y}_2)\). We have \(f'_* = f_* \circ r^*\) as all these maps are small \(\mathbb{Q}\)-factorial modifications and thus

\[
s_{\mathcal{Y}_1} = (f' \circ p')^* \circ f'_* = (f \circ p)^* \circ f_* \circ r^* = s_{\mathcal{Y}_1} \circ r^*
\]

on \(f^* \text{Nef}^c(\mathcal{Y}_2)\). It is straightforward to see that \(s_{\mathcal{Y}_1} \circ (\iota_1^*)^{-1}\) is a section of the restriction and so we conclude the proof.
In the proof of the preceding lemma, we used the following well-known lemma. We record a proof for convenience.

**Lemma 5.10.** Let \( Y, Y', Y'' \) be projective \( S \)-schemes. Let \( f : Y \rightarrow Y' \) and \( h : Y \rightarrow Y'' \) be birational contractions such that the relative interiors of \( f^*\text{Nef}(Y') \) and \( h^*\text{Nef}(Y'') \) have a nonempty intersection. Then there exists an isomorphism \( g : Y'' \rightarrow Y' \) such that \( g \circ h = f \).

Note that we do not ask that \( Y' \) and \( Y'' \) be isomorphic in codimension one. This is a consequence.

**Proof.** Let us take a resolution of indeterminacy as in the following diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & Y \\
\downarrow b & & \downarrow f \\
Y & \xrightarrow{g} & Y'' \\
\end{array}
\]

We put \( g := f \circ h^{-1} \) and want to show that this is an isomorphism, in particular, a regular map. By assumption there are very ample Cartier divisors \( A' \) on \( Y' \) and \( A'' \) on \( Y'' \) such that \( h^*A'' = f^*A' \), i.e. \( \pi_*b^*A'' = \pi_*a^*A' \). We deduce that there are \( \pi \)-exceptional divisors \( E', E'' \geq 0 \) such that \( a^*A' + E' = b^*A'' + E'' \). But all \( \pi \)-exceptional divisors are also exceptional for \( a \) and \( b \). From the equality

\[
H^0(Y', \mathcal{O}_{Y'}(A')) = H^0(Z, \mathcal{O}_Z(a^*A' + E')) = H^0(Z, \mathcal{O}_Z(b^*A'' + E'')) = H^0(Y'', \mathcal{O}_{Y''}(A''))
\]

we see that the morphisms associated to the linear systems of \( A' \) and \( A'' \) fit into a commuting diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{a} & Y' \\
\downarrow b & & \downarrow g \\
Y'' & \rightarrow & \mathbb{P}^N_S, \\
\end{array}
\]

so that \( g \) has to be an isomorphism. \( \square \)

We can now finally construct embeddings of the collections \( \Sigma_Y^\text{nef} \) into the Mori fan \( \text{MF}(Y) \). This is obtained via the following

**Definition 5.11.** Let \( Y \subset Y_1 \) be an irreducible component and let \( Y \rightarrow Y_1 \) by a \( Y \)-cusp model. We denote by \( \iota_1 : Y_\eta \rightarrow Y_1 \) the inclusion and define the section

\[
(5.4) \quad s_Y := s_{Y_1} \circ (\iota_1)^{-1} : \text{Nef}(Y_\eta) \rightarrow \text{Mov}(Y).
\]

By Lemma 5.9 the section \( s \) does not depend on the \( Y \)-cusp model but only on \( Y \), which justifies the notation.
5.12. **The action of the birational automorphism group.** A semitoric fan on $\mathbb{C}^d$ is by definition endowed with an action of $\Gamma_+^d$. Our collection $\Sigma^\text{nef}_Y$ has support on $\text{Nef}(Y)$, so in view of Subsection 5.2 we will first describe the action of $\Gamma_+^d$ on $\Sigma^\text{nef}_Y$. In fact, by Proposition 5.5 we know that $\Gamma_+^d$ is contained in the image of the map $G = \text{Aut}(Y) = \text{Bir}(Y) \to \text{O}^+(M_2d)$, so it will be sufficient to describe an action of $G$. This is given by \( \gamma \times \text{MF}(Y) \to \text{MF}(Y), \quad (g, C(f)) \mapsto g^*C(f) \).

It is worthwhile to note that, unlike in our discussion at the beginning of Section 5.7, here we do have $g^*C(f) = C(f \circ g)$ because $g^*g = \text{id}$.

The automorphism group of $Y$ also acts on the irreducible components $Y_1, \ldots, Y_{d+2}$ of $Y_c$ by permutations. Let us denote by $\Pi_2 \subset S_{d+2}$ the image of $G \to S_{d+2}$. We infer from [HL19, Corollary 5.39] that $\Pi_2 = S_3$. Given a component $Y \subset Y_c$, we denote by $G_Y$ the stabilizer group of $Y$ under the action $G \to S_{d+2}$.

We record the following elementary lemma.

**Lemma 5.13.** Let $Y \subset Y_c$ be an irreducible component and let $\gamma \dashrightarrow Y'$ be a rational $Y$-cusp model. Then there is an action \( G_Y \times \text{MF}(Y') \to \text{MF}(Y'), \quad (g, C(f)) \mapsto g^*C(f), \)

which is compatible with the action \( \gamma \).

We deduce the following basic properties of cuspidal cones.

**Corollary 5.14.** Let $Y \subset Y_c$ be an irreducible component of the central fiber and denote by $\text{Cusp}_Y \subset \text{MF}(Y)$ the set of cuspidal cones associated to rational $Y$-cusp models. Then every automorphism $g \in G$ induces a bijection

\[ g^* : \text{Cusp}_Y \to \text{Cusp}_Y, \quad C \mapsto g^*C \]

and similarly for the Mori fan. Here, $g.Y$ denotes the permutation action of $g$ on the set of irreducible components of $Y_c$. In particular, the collection of cones $\Sigma_Y$ only depends on the $\Pi_2$-orbit of $Y$.

**Proof.** The map is well-defined, because for every rational $Y$-cusp model $f : Y \dashrightarrow Y'$ the composition $f \circ g : Y \dashrightarrow Y \dashrightarrow Y'$ is a rational $g.Y$-cusp model.

5.15. **The GHKS semitoric fan.** We will now define the GHKS semitoric fans. For this let $Y_1, \ldots, Y_{d+2}$ be the irreducible components of the central fiber of $Y \to S$, cf. Lemma 3.4.

**Definition 5.16.** We define the collection $\Sigma^\text{GHKS,nef}_{2d}$ to be the coarsest common refinement of the collections $\Sigma^\text{nef}_{Y_1}, \ldots, \Sigma^\text{nef}_{Y_{d+2}}$.

**Remark 5.17.** The coarsest common refinement is characterized by the property that every cone of $\Sigma^\text{GHKS,nef}_{2d}$ is a subcone of some cone of $\Sigma^\text{nef}_{Y_i}$ for all $i$ and if $\Sigma'$ is another collection with this property, then every cone in $\Sigma'$ is a subcone of some cone in $\Sigma^\text{GHKS,nef}_{2d}$. The usual proof, cf.
[AMRT10, Corollary I.4.9], that a finite set of fans has a coarsest common refinement, which is achieved by taking intersections of cones, carries over to this situation without problems.

**Lemma 5.18.** The group $\mathcal{P}^+_{2d}$ acts on each of the collections $\Sigma_{\text{nef}}^{Y_1}, \ldots, \Sigma_{\text{nef}}^{Y_{d+2}}$ and therefore also on their coarsest common refinement $\Sigma_{\text{GHKS}, \text{nef}}^{2d}$.

**Proof.** By Proposition 5.5 every element $h \in \mathcal{P}^+_{2d}$ is induced by an automorphism $g \in G$. Thus, $h$ clearly acts on $\text{NS}(Y_\eta)$. Given $\sigma \in \Sigma_{Y_i}$ for some $i$ we see that $h.\sigma \in \Sigma_{g.Y_i}$. But $\Sigma_{Y} = g.\Sigma_Y$ by Corollary 5.14. □

We are now ready to define the semitoric fans we are aiming at.

**Definition 5.19.** The Gross–Hacking–Keel–Siebert (GHKS) semitoric fan is defined by

\[
\Sigma_{\text{GHKS}}^{2d} := \{ s(\sigma) \mid \sigma \in \Sigma_{\text{GHKS}, \text{nef}}^{2d}, s \in W_{2d} \}.
\]

Similarly, given we put $\Sigma_Y^{\text{GHKS}} := \{ s(\sigma) \mid \sigma \in \Sigma_{\text{nef}}^{Y}, s \in W_{2d} \}$ for an irreducible component $Y \subset Y_c$.

We will usually refer to this semitoric fan simply as the GHKS fan, i.e. drop the word semitoric for simplicity.

In order to justify the definition retrospectively we state the

**Proposition 5.20.** The collections $\Sigma_{\text{GHKS}}^{2d}$, $\Sigma_Y^{\text{GHKS}}$ define semitoric fans for the group $\Gamma^+_{2d}$ and hence give rise to semitoric compactifications $\mathcal{F}_{\text{GHKS}}^{2d}$ and $\mathcal{F}_{Y}^{\text{GHKS}}$ of the moduli space of 2d-polarized $K3$ surfaces.

**Proof.** We must check that the conditions of Definition 1.14 are fulfilled. Condition (1), namely the equivariance with respect to $W_{2d}$ is true by construction, the equivariance by $\mathcal{P}^+_{2d}$ is true by Lemma 5.18 and thus the collections are $\Gamma^+_{2d}$ equivariant. The same is true for condition (3) as the support of $\Sigma_{\text{GHKS}, \text{nef}}^{2d}$ is $\text{Nef}(Y_\eta)$ and the orbits of $\text{Nef}(Y_\eta)$ under $W_{2d}$ cover $C_{2d}^{c}$. For the latter we refer to the geometric argument given in the proof of [GHKS16, Theorem 6.5]: a rational ray in $C_{2d}^{c}$ is spanned by a nef class after a finite number of reflections in $(-2)$-curves. The same argument also proves condition (3).

For (2) we need that every isotropic ray $R$ in $\text{Nef}(Y_\eta)$ is a cone given by some cusp model. Note that such a ray corresponds to an isotropic nef line bundle on the $K3$ surface $Y_\eta$ which is therefore isomorphic to the pull back of a very ample bundle along an elliptic fibration of $Y_\eta$.

We can thus take an arbitrary rational cusp model $Y_\eta \rightarrow Y_1$ and obtain an effective prime divisor $D$ on $Y_1$ such that the restriction $D_\eta$ lies in the ray $R$. If $D$ is not nef, we can run an MMP for the pair $(Y_1, \epsilon D)$ for $0 < \epsilon \ll 1$ which will terminate in a pair $(Y_2, \epsilon D')$ such that $D'$ is nef on the cusp model $Y_2$. Hence, it defines a cone in $\Sigma_{\text{GHKS}}^{2d}$. We have already remarked after Definition 3.6 that $\{0\}$ is also a cone in $\Sigma_{\text{GHKS}}^{2d}$. This proves (2). □

We have already discussed the Coxeter semitoric fan $\Sigma_{\text{Cox}}^{2d}$ in Example 1.17. By construction $\Sigma_{\text{GHKS}}^{2d}$ is a refinement of the Coxeter fan. Hence, we obtain the main result of this section:
Theorem 5.21. The GHKS semitoric fan $\Sigma^{GHKS}_{2d}$ defines a semitoric compactification $F_{2d}^{GHKS}$ of the moduli $F_{2d}$ of 2d-polarized K3 surfaces. This admits a semitoric morphism

$$F_{2d}^{GHKS} \rightarrow F_{2d}^{Cox}$$

to the semitoric compactification given by the Coxeter semitoric fan.

Remark 5.22. For their construction, Gross–Hacking–Keel–Siebert use an additional refinement of what we called the GHKS fan. They use an averaging process over the sections from Definition 5.11 resulting in a piecewise linear section $Nef(Y,\eta)_{\mathbb{R}} \rightarrow Mov(Y)$ which is linear on the cones of $\Sigma^{GHKS}_{2d}$. Then they pull back the Mori fan via these sections. But as seen in this section, already $\Sigma^{GHKS}_{2d}$ is a semitoric fan which we believe deserves further study.

5.23. The GHKS fan in degree two. The case $2d = 2$ is the most important for us because the GHKS construction yields a toroidal compactification in this case. Indeed, we first observe the following

Proposition 5.24. Let $Y \rightarrow S$ be a model of the DNV family of degree 2 and let $Y \subset Y$ be an irreducible component of the central fiber. Then the Coxeter semitoric fan $\Sigma^{Cox}_2$ and the GHKS semitoric fan $\Sigma^{GHKS}_{2d}$ are $\Gamma_2^+$-fans.

Proof. This is an immediate consequence of the fact that the nef cone $Nef(Y,\eta)$ is finitely polyhedral, which in turn follows from the fact that the lattice $M_2$ is reflexive, see also [HL19, Remark 2.7].

The following corollary is an immediate consequence of Proposition 5.24.

Corollary 5.25. Let $Y \rightarrow S$ be a model of the DNV family of degree 2. Then $F_{2d}^{GHKS}$ and $F_{2d}^{Cox}$ are toroidal compactifications.

Finally, we would like to point out that the refinement process described in Definition 5.16 is unnecessary in the case $d = 1$. Indeed, thanks to the section $s_Y: Nef(Y,\eta)_{\mathbb{R}} \rightarrow Mov(Y)$ from Definition 5.11 we can now interpret the collections $\Sigma^{nef}_{Y_i}$ from Definition 5.1 as living in the Mori fan of $Y$. Thus we can use the symmetry of the $2d = 2$ situation to show:

Proposition 5.26. Suppose $d = 1$ and denote by $Y_c = Y_1 \cup Y_2 \cup Y_3$ the decomposition of the central fiber into irreducible components. Then the collections from Definition 5.1 associated to these components coincide (as subsets of $Nef(Y,\eta)$):

$$\Sigma^{nef}_{Y_1} = \Sigma^{nef}_{Y_2} = \Sigma^{nef}_{Y_3} (= \Sigma^{GHKS,nef}_{2d}).$$

Proof. We can, without loss of generality, work with the model $Y = Y_\varphi$. By [HL19, Corollary 5.39] there is a subgroup $S_3$ of $Aut(Y_\varphi/S)$ which acts as group of permutations of the components of $Y_c$. If $\sigma$ is an element of this group and $p: Y \rightarrow Y'$ is a rational $Y_i$-cusp model for some $1 \leq i \leq 3$, it is clear that $p \circ \sigma$ is a $Y_{\sigma(i)}$-cusp model. The claim now follows from the independence of $\Sigma^{nef}_{Y_i}$ of the rational $Y_i$-cusp model, see Lemma 5.3. \qed
6. Cuspidal Cones

In order to prove our main result, Theorem 0.3, we need to better understand cusp models. The purpose of this section is to show that the existence of a cusp model \( Y \to Y' \) puts severe restrictions on the central fiber \( Y_c \). The idea is that as maximality of a degeneration is preserved by a birational contraction, the non-contracted component of \( Y_c \) has to account for most of its Picard group. The results of this section are important in the proof of Theorem 0.3 which is achieved in Section 9.

**Proposition 6.1.** Let \( Y \to S \) be a model of the DNV family of degree 2, and let \( f : Y \to Y' \) be a cusp model with \( Y \subset Y_c \) the non-contracted component of the central fiber. Let \( Y' \to Y \) be the normalization. Then the pullback map satisfies

\[
\text{rk} \left( \text{Pic}(Y') \to \text{Pic}(Y') \right) \geq 19.
\]

In particular, \( g(Y') \geq 19 \).

**Proof.** Let us denote by \( f_c : Y_c \to Y' \) the restriction of \( f \) to the central fiber. Observe that by Proposition 2.15 restriction gives an isomorphism \( \text{Pic}(Y') \cong \text{Pic}(Y'_c) \) and \( f^*_c : \text{Pic}(Y'_c) \to \text{Pic}(Y_c) \) is injective. By Lemma 4.2 we have \( g(Y'_c) = 19 \) and from [HL19, Lemma 1.10] we infer that the pullback \( \text{Pic}(Y'_c) \to \text{Pic}(Y_c) \) is injective where \( \nu : Y' \to Y_c \) is the normalization. Let \( K := \ker \left( \text{Pic}(Y'_c) \to \text{Pic}(Y_c) \right) \). It suffices to show that \( K \cap \im f^*_c = 0 \).

Let \( C \subset Y \) be an irreducible component of the double locus and denote by \( C' \subset Y'_c \) its image under \( f_c \). Suppose that \( L \) is a line bundle on \( Y'_c \) such that \( f^*_c L \in K \). We have to show that the pullback of \( L \) to every component of \( Y'_c \) is trivial. By assumption, its pullback to \( Y' \) is trivial and hence so is the pullback to \( C \). We conclude that the restriction \( L|_{C'} \) is numerically trivial. For any component \( X \subset Y'_c \) different from \( Y'_c \), the induced map \( h : X \to Y'_c \) factors through \( C' \) for some double curve \( C \) as above. Thus, \( h^* L \) is numerically trivial as well and therefore trivial as \( X \) is a rational surface. This concludes the proof. \( \square \)

We will improve upon this result in Proposition 6.7 below, this will however need the finer analysis. For this purpose we need so-called curve structures as developed in [HL19].

6.2. Curve structures. As in Example 3.3, we denote by \( \mathcal{P} \) and \( \mathcal{T} \) the two possible triangulations of the 2-sphere with two triangles, see Figure 1, and by \( Y_{\mathcal{P}} \) and \( Y_{\mathcal{T}} \) the corresponding \( d \)-semistable K3 surface of type III in \((-1)\)-form with trivial Carlson map. Whenever \( Y \to S \) is a model of the DNV family in degree 2, the central fiber \( Y_c \) fits into a sequence of elementary modifications of type I

\[
Y_{\mathcal{G}} \to Y_1 \to \cdots \to Y_i \to \cdots \to Y_n = Y_c
\]

where \( \mathcal{G} \in \{ \mathcal{P}, \mathcal{T} \} \). Recall that the components of \( Y_{\mathcal{P}} \) or \( Y_{\mathcal{T}} \) are isomorphic to \( \mathcal{Y}_1, \mathcal{Y}_2, \) or \( \mathcal{Y}_4 \), respectively, as introduced in Example 3.3. For every irreducible component \( Y \subset Y_c \), its curve structure \( \Gamma_Y \) is a combinatorial object that remembers how \( Y \) the sequence of type I flops was built from some \( \mathcal{Y}_i \) for \( i \in \{1, 2, 4\} \). The definition is by induction over the length of (6.1).
Denoting by $D$ the anticanonical divisor of $Y_i$, we set

$$C(Y_i) := \{ C \subset Y_i \mid C \text{ is an integral curve with } C^2 < 0, C \not\subset D \}.$$ 

For $Y$ as above, (6.1) induces a sequence

$$Y_i =: W_0 \rightarrow \ldots \rightarrow W_{n-1} \rightarrow W_n =: Y^\nu$$

of birational maps and we suppose that $C(Y_{n-1})$ has already been defined. Then the set of curves $C(W_n)$ on $W_n = Y^\nu$ is given by images of non-contracted curves in $C(W_{n-1})$ if $W_{n-1} \rightarrow W_n$ is a blow up or by either strict transforms of curves in $C(W_{n-1})$ or the exceptional curve if $W_{n-1} \rightarrow W_n$ is the inverse of a blow up.

**Definition 6.3.** The curve structure $\Gamma_Y$ is the dual graph of $C(Y)$ whose vertices are labelled with the self-intersection numbers. We say that $\Gamma_Y$ has type $d_i$ if $Y$ maps to $Y_i$ under a sequence of type I flops.

Let $Y^\nu \rightarrow Y$ be the normalization and $D = \sum D_j$ the anticanonical divisor of $Y^\nu$. The augmented curve structure $\Gamma^a_Y$ is obtained by appending a vertex $v_{D_j}$ to $\Gamma_Y$ for each $j$ and an edge between $v_{D_j}$ and $v \in \Gamma_Y$ if and only if $D_j.C_v \neq 0$ for the curve $C_v$ corresponding to $v$.

Let us review some basic properties of curve structures.

1. The curve structure is well defined, i.e. independent of the chosen sequence (6.1) by [HL19, Lemma 3.8].
2. Given $\mathcal{Y}_c$, the type of $\Gamma_Y$ is well-defined as type I flops do not change the number of components of the anticanonical divisor.
3. If $\Gamma_Y$ has type $d_i$ with $i \in \{1, 2, 4\}$, then $\{ C_v \mid v \in \Gamma_Y \}$ is a $\mathbb{Q}$-basis of Pic$(Y^\nu)$, see [HL19, Proposition 3.16].
4. If $C_v \subset Y^\nu$ denotes the curve corresponding to a vertex $v \in \Gamma_Y$, the intersection number $C_v.C_w$ for $v, w \in \Gamma_Y$ is either 1 or 0, see the paragraph after [HL19, Remark 3.11].

Next we recall the definition of an exceptional vertex, see [HL19, Definition 3.12].

**Definition 6.4.** A vertex $v \in \Gamma_Y$ is called exceptional if

1. $v^2 = -1$,
2. there is a unique vertex in $\Gamma_Y \setminus \{v\}$ intersecting $v$ nontrivially, and
3. if $D_0$ denotes the unique (by [HL19, Proposition 3.2]) component of the anticanonical cycle met by $v$, no other vertex in $\Gamma_Y \setminus \{v\}$ meets $D_0$.

The leg of an exceptional vertex $v$ is defined as the maximal connected (full) subgraph $L := L(v)$ of (the graph underlying) $\Gamma_Y$ containing $v$ such that

1. all vertices are adjacent to at most two vertices of $L$,
2. all vertices except possibly the end $e$ of $L$ (i.e. the unique vertex different from $v$ that is adjacent to exactly one other vertex in $L$) are adjacent to at most two vertices of $\Gamma_Y$, and
3. no vertex in $L \setminus \{v, e\}$ meets the anticanonical divisor.
Informally speaking, the leg of an exceptional vertex is obtained by joining adjacent vertices until one reaches a fork or a component of the anticanonical cycle.

**Definition 6.5.** A curve structure $\Gamma_Y$ is called degenerate if there is no exceptional vertex, or if for some exceptional vertex $v_e$ the leg $L(v_e)$ ends in a vertex $v$ with $v.D_1 \geq 1$ for some smooth irreducible double curve $D_1 \subset Y'$ which maps to a smooth curve $Y$. Curve structures that are not degenerate will be called non-degenerate.

The following notion refines the division into degenerate and non-degenerate curve structures. It has been defined in [HL19, Definition 3.17] for curve structures of type $d_2$ only. We drop this assumption here because it is not needed even though we mainly use it for type $d_2$.

**Definition 6.6.** We say that a curve structure $\Gamma_Y$ is regular if $|\Gamma_Y| > 1$ and no leg $L(e)$ of an exceptional vertex $e$ ends in a vertex $v$ with $v^2 = 0$. A curve structure which is not regular is called very degenerate and a curve structure which is regular and degenerate is called tamely degenerate.

We have the implications “very degenerate” $\Rightarrow$ “degenerate” $\Rightarrow$ “tamely degenerate”. Indeed, the second implication holds by definition. If a curve structure is very degenerate then either $|\Gamma_Y| = 1$ in which case there is no exceptional vertex and there is an exceptional vertex whose leg ends in a vertex $v$ with $v^2 = 0$. As $C_v$ is a rational curve, it must intersect $D$ with $C_v.D = 2$ by [HL19, Proposition 3.2], hence the curve structure is degenerate. In conclusion, the set of degenerate curve structures is divided into very degenerate and tamely degenerate ones.

**Proposition 6.7.** Let $Y \rightarrow S$ be a model of the DNV family of degree 2 and $f : Y \rightarrow Y'$ be a cusp model. Let $Y \subset Y_c$ be the non-contracted component of the central fiber with normalization $Y' \rightarrow Y$. If $\Gamma_Y$ is of type $d_2$ or $d_4$, then $g(Y') \geq 20$.

**Proof.** In view of Proposition 6.1 to prove the claim it is sufficient to exhibit a line bundle on $Y'$ that is not a pullback from $Y_c$.

Let us first treat the case where the type of $\Gamma_Y$ is $d_2$. In this case, $Y' = Y$ is smooth. Let $D_1 + D_2$ be the anticanonical divisor on $Y$. If a component $D_i$ is contracted to a point under $f$, the corresponding line bundle $\mathcal{O}_Y(D_i)$ is not a pullback from $Y_c$. If no $D_i$ is contracted, then $D_1$ and $D_2$ are glued under $f$. In case $\Gamma_Y$ has an exceptional vertex, by definition there is a curve $C$ in $\Gamma_Y$ that intersects $D_1$ but not $D_2$, so $\mathcal{O}_Y(C)$ cannot be a pullback. If $\Gamma_Y$ has no exceptional vertex, it must be degenerate. The degenerate curve structures of type $d_2$ with no exceptional vertex are classified, see [HL19, Figure 15], and necessitate $g(Y) \leq 2$, which is impossible.

If $\Gamma_Y$ is of type $d_4$, the claim follows similarly from [HL19, Proposition 4.1].

**Corollary 6.8.** Let $Y \rightarrow S$ be a model of the DNV family of degree 2 and let $f : Y \rightarrow Y'$ be a cusp model. Let $Y \subset Y_c$ be a component that is contracted. Then $|\Gamma_Y| \leq 4$ and $|\Gamma_Y| = 4$ only if the non-contracted component is of type $d_1$ and $\Gamma_Y$ is of type $d_4$. If $\Gamma_Y$ is of type $d_2$, then $\Gamma_Y$ is degenerate.
Proof. Write \( Y_c = Y_1 \cup Y_2 \cup Y_3 \) with \( Y = Y_1 \) and suppose \( Y_3 \) is not contracted. We have \( \sum |\Gamma_{Y_i}| = 24 \) as \( \Gamma_Y \) is a basis for \( \text{Pic}(Y_1)_Q \). So from Proposition 6.7, we have \( 1 \leq |\Gamma_Y| \leq 4 \) and even \( |\Gamma_Y| \leq 3 \) if \( Y_3 \) is of type \( d_2 \) or \( d_4 \). So if \( |\Gamma_Y| = 4 \), then \( Y_3 \) must be of type \( d_1 \) and \( Y_2 \) must have Picard number one, so it cannot be the type \( d_4 \) component. If \( \Gamma_Y \) is of type \( d_2 \) non-degenerate, we necessarily have \( |\Gamma_Y| \geq 4 \) as there are 2 legs that end on a fork by [HL19 Example 3.15].

For the enumeration of cusp models, we have to provide a deeper analysis of the possible models. For this purpose, we examine which curves cannot be contracted in a cusp model.

**Proposition 6.9.** Let \( \mathcal{Y} \) be a model of the DNV family of degree 2 and write \( \mathcal{Y}_c = Y_1 \cup Y_2 \cup Y_3 \). Let \( f : \mathcal{Y} \to \mathcal{Y}' \) be a \( Y_3 \)-cusp model. Let \( C \subset Y_1' \) be a \((-2)\)-curve on the normalization \( Y_1' \to Y_1 \). Then \( C \) is not contracted by \( f \).

**Proof.** Recall that \( C \) does not intersect the double locus by [HL19 Proposition 3.2]. Thus, the line bundle \( L := \mathcal{O}_{Y_1'}(C) \) has degree zero on the double locus by [HL19 Proposition 3.2]. Hence, by [HL19 Lemma 1.10] there is a line bundle \( L_c \) on \( \mathcal{Y}_c \) that pulls back to \( L \) under the canonical map. By maximality of the DNV family, one obtains a line bundle \( L \) on \( \mathcal{Y}_c \) whose restriction to \( \mathcal{Y}_c \) is \( L_c \). For the restriction \( L_{\eta} \), we have \( L_{\eta}^2 = -2 \), so being effective on the central fiber \( L_{\eta} \) is effective and hence \( L \) is effective. Let us write \( L = \mathcal{O}_{\mathcal{Y}}(C) \) for an effective divisor \( C \subset \mathcal{Y} \). Let us write \( f = \phi|_{B} \) for some divisor \( B \) on \( \mathcal{Y} \). Then \( f \) contracts \( C \) if and only if \( B.C = 0 \). The latter is equivalent to \( B.C_{\eta} = 0 \) which would say that \( f \) contracts a divisor on the generic fiber which is a contradiction. \( \square \)

The following construction is needed to rule out certain constellations.

**Construction 6.10.** Let \( \mathcal{Y} \) be a model of the DNV family of degree 2, let \( Y \subset \mathcal{Y}_c \) be an irreducible component of the central fiber, and let \( f : \mathcal{Y} \to \mathcal{Y}' \) be a \( Y \)-cusp model. Suppose \( C \subset \mathcal{Y}_c \) is a curve contracted by \( f \) and that one of the following holds:

1. \( C \) is an interior \((-1)\)-curve, and if \( \mathcal{Y}_c \to \mathcal{Y}_c^+ \) denotes the elementary modification of type \( I \) in \( C \), then \( \mathcal{Y}_c^+ \) is projective.
2. \( C \) is a component of the double locus and \( \nu^{-1}(C) = D_{ij} \cup D_{ji} \) and \( D_{ij}^2 = D_{ji}^2 = -1 \), where \( \nu : \mathcal{Y}_c^+ \to \mathcal{Y}_c \) denotes the normalization.

By [HL19 Corollaries 5.7, 5.8], there is a flopping contraction \( \phi : \mathcal{Y} \to \mathcal{Z} \) contracting exactly \( C \). Hence, \( f \) factors as \( \mathcal{Y} \to \mathcal{Z} \to \mathcal{Y}' \), and there is model of the DNV family \( \mathcal{Y}^+ \) and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\phi} & \mathcal{Y}^+ \\
\downarrow{f} & & \downarrow{f} \\
\mathcal{Z} & \to & \mathcal{Y}'
\end{array}
\]

\[5\text{Here, \textit{interior} means it is not part of the anticanonical cycle.}\]
Moreover, in the first case the central fiber of $\mathcal{Y}^+$ is exactly $\mathcal{Y}^+_c = Y^+_c$. By construction $\mathcal{Y}^+ \to \mathcal{Y}'$ is also a cusp model.

We record some consequences of Construction 6.10 in the next proposition.

**Proposition 6.11.** Let $\mathcal{Y}$ be a model of the DNV family of degree 2. Write $\mathcal{Y}_c = Y_1 \cup Y_2 \cup Y_3$. Let $f : \mathcal{Y} \to \mathcal{Y}'$ be a $Y_3$-cusp model. Let $C \subset \mathcal{Y}_c$ be a curve such that one of the following assertions holds:

1. There is $i = 1$ or 2 such that $C \subset Y_i$ is an interior $(-1)$-curve meeting $D_{i3}$, and if $\mathcal{Y}_c \dashrightarrow Y^+_c$ denotes the elementary modification of type $I$ in $C$, then $Y^+_c$ is projective.
2. $\nu^{-1}(C) = D_{21} \cup D_{12}$ and $D^2_{12} = -1$, where $\nu : \mathcal{Y}' \to \mathcal{Y}_c$ denotes the normalization.

Then $C$ is not contracted by $f$.

**Proof.** Consider the situation in Construction 6.10. In case (1) the flopped curve $C^+$ of $C$ is an interior curve on $Y^+_3$. It is contracted by $\mathcal{Y}^+ \to \mathcal{Y}'$, a contradiction to the exceptional locus being purely of codimension one, see e.g. [Ko96, Theorem VI.1.5]. Case (2) is proven similarly, using the description of type II flops, see e.g. [FM83]. □

7. Models with dual intersection complex $\mathcal{P}$

In this section, we characterize those models of the DNV family with dual intersection complex $\mathcal{P}$ that are total spaces of cusp models.

**Proposition 7.1.** Let $\mathcal{Y}$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{P}$ and let $f : \mathcal{Y} \to \mathcal{Y}'$ be a cusp model. If $Y \subset \mathcal{Y}_c$ is a contracted component, then $|\Gamma_Y| \leq 2$.

**Proof.** We know that $|\Gamma_Y| \leq 3$ and that $\Gamma_Y$ is degenerate by Corollary 6.8, so it is enough to exclude $|\Gamma_Y| = 3$. Let us write $\mathcal{Y}_c = Y_1 \cup Y_2 \cup Y_3$ and assume $Y_1 = Y$. The restriction of $f$ to $Y$ can be written as the map $\pi := \pi|_L : Y \to K$ associated to a linear system for some base point free $L \in \text{Pic}(Y)$. We will distinguish the cases $\Gamma_Y$ very degenerate or tamely degenerate.

Suppose $\Gamma_Y$ is very degenerate and $|\Gamma_Y| = 3$. Then $\Gamma_Y$ has an exceptional vertex $v_1$ and is equal to its leg $L(v_1) = (v_1, v_2, v_3)$. By Proposition 6.9, the $(-2)$-curve $v_2$ is not contracted so that $K \cong \mathbb{P}^1$. We will produce a contradiction by showing that $L^2 \neq 0$. Let us write $D_{12} + D_{13}$ for the double curve on $Y$ and assume that $D_{12}, v_1 = 1$. By [HL19, Proposition 6.3], the curve structure determines the surface uniquely, so using $|\Gamma_{Y_1}| = 3$ and that $\Gamma_{Y_1}$ is very degenerate, one deduces $D^2_{13} = 4$ and $D^2_{12} = -1$. Thus, [HL19, Corollary 3.7] implies

$$\text{NE}(Y_1) = \langle v_2, v_1, D_{12} \rangle.$$

Let us write $L = av_1 + bv_2 + cD_{12}$ with $a, b, c \in \mathbb{N}_0$. Then

$$0 = L^2 = a(L, v_1) + b(L, v_2) + c(L, D_{12}).$$

For each generator of $\text{NE}(Y_1)$ that is not contracted by $\pi$, the corresponding coefficient must vanish. It follows that $b = 0$. If $Y_2$ is not contracted by $f$, the curve $v_1$ is not contracted by
Proposition 6.11. Else, by the same proposition, $D_{12}$ is not contracted. In any event, $F = dC$ for some $d \in \mathbb{N}$ and $C \in \{v_1, D_{12}\}$ and the claim follows.

Next we suppose $\Gamma_Y$ is tamely degenerate and $|\Gamma_Y| = 3$. Then $\Gamma_Y$ has an exceptional vertex $v_1$ and is equal to the leg $(v_1, v_2)$ generated by $v_1$ together with a vertex $v_3$ meeting only $v_2$. Let us write $D_{12} + D_{13}$ for the double curve on $Y = Y_1$ and assume that $D_{12}, v_1 = 1$. As the leg of $v_1$ ends in a vertex (namely $v_2$) that meets $D_{13}$, coming from $\mathfrak{g}_2$ we must have contracted at least two $(-1)$-curves meeting birational transforms of $D_{13}$ and thus we have $D_{13}^2 > 0$. The curves with negative self-intersection are contained in the set $\{v_1, v_2, v_3, D_{12}\}$.

By [HL19, Corollary 3.7], we have

$$\text{NE}(Y) = \langle C \in \{v_1, v_2, v_3, D_{12}\} : C^2 < 0 \rangle.$$

Note that by Corollary 6.8 and [HL19, Proposition 4.2], exactly one of the components of $\mathcal{Y}_c$ has a non-degenerate curve structure (namely the one not contracted by $f$).

Suppose $\Gamma_{Y_2}$ is non-degenerate. Then $|\Gamma_{Y_3}| = 1$ by Proposition 6.7 and hence $Y_3 \cong \mathbb{P}^2$ by [HL19, Corollary 3.7]. As $D_{31}^2 < 0$, this is a contradiction. So $\Gamma_{Y_3}$ is non-degenerate. Then the elementary modification in $v_2$, call it $\mathcal{Y}_c \to Y_{v_2}$, defines a projective surface $Y_{v_2}$ by [HL19, Proposition 4.4]. Indeed, one checks that $D_{31}^2 < 0$ and $D_{32}^2 < 0$ so that the assumptions of loc. cit. are satisfied. Hence, $v_2$ is not contracted by $\pi$ by Proposition 6.11. Also, $Y_2 \cong \mathbb{P}^2$.

As $D_{12}^2 + D_{31}^2 = -2$, it follows that $D_{12}^2 \leq -2$.

Suppose $\pi$ contracts $v_1$. Then $\pi$ factors as $Y_1 \to Y' \to K$ where $Y_1 \to Y'$ denotes the contraction of the $(-1)$-curve $v_1$. Consequently, $Y'$ is a smooth surface of Picard number $\rho(Y') = 2$. We calculate $v_2'^2 = 0$ and $D_{12}^2 \leq -1$ for the strict transforms $v_2'$ and $D_{12}'$ of $v_2$ and $D_{12}$. So

$$\text{NE}(Y') = \langle v_2', D_{12}' \rangle.$$

Note that as $v_2$ is not contracted by $\pi$, also $Y' \to K$ does not contract $v_2'$. Thus, the fiber of $Y' \to K$ is a multiple of $D_{12}'$, hence square negative, a contradiction. $\Box$

The above proposition helps to characterize the set of possible total spaces of cusp models with dual intersection complex of the central fiber given by $\mathcal{P}$. In the remainder of this section, we show that any of these models of the DNV family admits a cusp model.

**Lemma 7.2.** Let $f : X \to X'$ be a projective birational morphism between complex algebraic varieties with rational singularities, and suppose that $\text{Exc}(f) = E_1 \cup \ldots \cup E_m$ for prime divisors $E_i$ where $m = \rho(X/X')$ is the relative Picard rank. If $X$ is $\mathbb{Q}$-factorial, then so is $X'$.

**Proof.** Let $\pi : Y \to X$ be a resolution of singularities, denote $\pi' := f \circ \pi$, and let $Y \subset \overline{Y}$ be a smooth compactification. We denote by $F_1, \ldots, F_n$ the prime exceptional divisors of $\pi$. By [KM92, (12.1.6) Proposition], we have to show that

$$\text{im} \big( H^2(Y, \mathbb{Q}) \to H^0(X', R^2\pi'(\mathbb{Q})) \big) = \text{im} \big( \mathbb{Q} \langle E_i, F_j \mid i, j \rangle \to H^0(X', R^2\pi'(\mathbb{Q})) \big),$$

given that the corresponding equality holds for $X$ instead of $X'$ (and hence only the span of the $F_j$ on the right-hand side). The inclusion $\supseteq$ is clear, for the other direction one uses the
Figure 2. The augmented regular curve structures $\Gamma_Y$ with $|\Gamma_Y| = 2$ for $n > -2$. For $n = -2$ the vertex of self intersection $-2$ does not meet the black nodes.

semi-simplicity of the category of polarizable Hodge structures as in the proof of loc. cit. to reduce showing that the right-hand side is equal to the image of $\text{Pic}(Y)_\mathbb{Q} \to H^0(X', R^2\pi'_\mathbb{Q})$ which in turn is equal to the image of $\text{Pic}(Y)_\mathbb{Q} \to H^0(X', R^2\pi'_\mathbb{Q})$. Let us observe that

$$\text{Pic}(Y)_\mathbb{Q} = \pi^* \text{Pic}(X)_\mathbb{Q} \oplus \mathbb{Q} \langle F_j | j \rangle = (\pi'_*)^* \text{Pic}(X')_\mathbb{Q} \oplus \mathbb{Q} \langle E_i, F_j | i, j \rangle$$

where the first equality comes from $\mathbb{Q}$-factoriality and the second one holds by assumption on the relative Picard rank. Clearly, the pullback of $\text{Pic}(X')$ maps to zero under the right-hand side of (7.1) and so the claim follows. \hfill \Box

Remark 7.3. In the proof of the preceding lemma, we tacitly used that the classes of the $E_i$ are linearly independent in the (rational) Picard group. This can be seen by reducing to the surface case using hyperplane sections on $X'$. For surfaces, it is obvious e.g. from (the easy direction of) Grauert’s criterion.

Using the usual algebraization arguments, we immediately deduce from Lemma 7.2 the following

**Corollary 7.4.** Let $\mathcal{Y}$ be a model of the DNV family. Let $\text{contr}_F: \mathcal{Y} \to \mathcal{Y}'$ be a contraction of an extremal face $F$ with $\dim F = 2$. Suppose $\text{Exc}(\text{contr}_F) = Y_1 \cup Y_2$ where $Y_1$ and $Y_2$ are components of $\mathcal{Y}_c$. Then $\mathcal{Y}'$ is $\mathbb{Q}$-factorial. \hfill \Box

7.5. $\Gamma_{Y_i}$ regular, $|\Gamma_{Y_i}| = 2$ for $i = 1, 2$. Let $Y$ be a component of a central fiber $\mathcal{Y}_c$ of a model of the DNV family $\mathcal{Y}$ of degree 2. Suppose the dual intersection complex of $\mathcal{Y}$ is $\mathcal{P}$. In particular, $Y$ is smooth. Let $D = D_0 + D_1$ be the anticanonical divisor given by the restriction of $\text{Sing}(\mathcal{Y}_c)$ to $Y$.

**Lemma 7.6.** Suppose $\Gamma_Y$ is regular and $|\Gamma_Y| = 2$. Then $\Gamma_Y$ consists of two vertices $v_1, v_0$ such that $D_0.v_1 = n + 2$, $D_1.v_1 = 0$, $D_1.v_0 = D_0.v_0 = 1$, $v_0^2 = 0$, $v_1^2 = n$, $D_0^2 = 4 + n$, and $D_1^2 = -n$ for some $n \geq -2$.

**Proof.** Given that $\Gamma_Y$ is regular, it is uniquely determined by $D_0^2$ and $D_1^2$ by [HL19, Proposition 3.18]. As $|\Gamma_Y| = 2$, we must have $D_1^2 + D_0^2 = 4$. Thus one easily classifies the unique such curve structure with $D_0^2 = 4 + n$, and $D_1^2 = -n$. As $v_1$ is a smooth rational curve, the adjunction formula gives the intersection number with $D_0$. The result is depicted in Figure 2 for $n \neq -2$. For $n = -2$ one obtains the same graph except that $D_0.v_1 = 0$. From $v_0^2 = 0$ and [HL19, Proposition 3.2] one deduces that $D_1.v_0 = D_0.v_0 = 1$. \hfill \Box
By [Fri83b, Lemma 3], $v_0$ is smooth rational of genus 0, thus isomorphic to $\mathbb{P}^1$. It is a consequence of [HL19, Proposition 3.5] that $Y$ is a Mori Dream space, so the nef bundle $\mathcal{O}(v_0)$ is semiample, inducing a fibration $Y \to K$ with fiber $\mathbb{P}^1$. Hence, $Y$ is the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$.

We have the following lemma.

**Lemma 7.7.** Let $\mathcal{Y} \to S$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{P}$ such that there are two components $Y_1, Y_2$ with $|\Gamma_{Y_1}| = 2$ and $\Gamma_{Y_1}$ regular. Then there is a cusp model $f: \mathcal{Y} \to \mathcal{Y}'$ contracting $Y_1$ and $Y_2$.

**Proof.** Let us write $\Gamma_{Y_1} = \{v_0, v_1\}$ and $\Gamma_{Y_2} = \{w_0, w_1\}$ with $v_0^2 = 0$, $v_1^2 = m$, $w_0^2 = 0$ and $w_1^2 = n$ such that $D_{12}.v_0 = D_{13}.v_0 = 1 = D_{21}.w_0 = D_{23}.w_0$, see Lemma 7.6. For a positive integer $p$ let $G_1$ be the divisor defined by $p v_0$ and $G_2$ the divisor defined by $p w_0$. If $p$ is suitably chosen, there is an ample divisor $G_3$ on $Y_3$ such that there is a $G \in \text{Pic}(\mathcal{Y})$ restricting to $G_i$ on $Y_i$ by [HL19, Proposition 3.20]. The divisors $G_1$ and $G_2$ are nef: if $C_1$ is a curve on $Y_1$, then $G_1.C_1 = p(v_0.C_1) \geq 0$. So replacing $G$ by some multiple, we obtain a projective morphism $\phi := \phi_{|G|}: \mathcal{Y} \to \mathcal{Z}$ by abundance.

If $C_1$ is a curve on $Y_1$ contracted by $\phi$, then $G_1.C_1 = v_0.C_1 = 0$. Write $C_1 = a v_0 + b v_1$ with $a, b \in \mathbb{Z}, c \in \mathbb{N}$ using [HL19, Proposition 3.12]. Then $b = 0$ and thus $C_1 \in \mathbb{R}_+[v_0]$. Similarly, one shows that for a curve $C_2 \subset Y_2$ contracted by $\phi$ we have $C_2 \in \mathbb{R}_+[w_0]$.

The morphism $\phi$ is also an isomorphism on $\mathcal{Y}_0$: Suppose there is a curve $E \subset \mathcal{Y}_0$ contracted by $\phi_{|\mathcal{Y}}$. Take the closure of $E$. This yields an effective Cartier divisor $\tilde{E}$ on $\mathcal{Y}$. The divisor $\tilde{E}$ is 2 dimensional and irreducible, so it does not contain any component of the central fiber. Let $E_i$ denote the restriction of $\tilde{E}$ to $Y_i$. Then $G_i.E_i = 0$ as $E$ is contracted and hence $E_1 = q v_0$, $E_2 = r w_0$. But $G_3$ is ample so $E_3 = 0$ and hence from the gluing conditions for divisors [HL19, Lemma 1.10], one has $q = r = 0$. Thus, $E = 0$ as well and the curves contracted by $\phi$ are precisely the curves in the cone $F$ generated by $C_{v_0}$ and $C_{w_0}$. It follows now from Corollary 7.4 that $\phi$ is the sought-for cusp model. \hfill $\square$

**Proposition 7.8.** In the situation of Lemma 7.7 the following holds:

1. If neither $Y_1$ nor $Y_2$ is $\mathbb{P}^1 \times \mathbb{P}^1$, the cone $C(f)$ is the only cuspidal cone contained in $\text{Nef}(\mathcal{Y})$.
2. If $Y_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then there are two distinct cusp models $f_j: \mathcal{Y} \to \mathcal{Y}_j$, $j = 1, 2$ contracting precisely $Y_1 \cup Y_2$. The cones $C(f_1)$ and $C(f_2)$ are the only cuspidal cones contained in $\text{Nef}(\mathcal{Y})$.

**Proof.** In the first case, the existence of the cusp model follows from Lemma 7.7 and Corollary 7.4. Let the curve structure of $Y_1$ be given by $v_1, v_0$, with $v_0^2 = 0$ and $D_{12}.v_0 = D_{13}.v_0 = 1$. If $Y_1$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, either $v_1^2 = -1$ or $D_{13}^2 < 0$ or $D_{12}^2 < 0$. Write $C$ for this curve. Then $\text{NE}(Y_1) = \langle C, v_0 \rangle$. It follows that any cusp model $f': \mathcal{Y} \to \mathcal{Y}'$ contracts $v_0$. The same reasoning applies to $Y_2$. Hence, any such model contracts the same curves as the one constructed in Lemma 7.7 and the uniqueness claim follows.
In the second case, let us write $\Gamma Y_1 = \{v_0, v_1\}$ and $\Gamma Y_2 = \{w_0, w_1\}$ with indexing as above. For $f_1$, we can take the cusp model constructed in the proof of Lemma 7.7. As we have seen, the corresponding extremal face is spanned by $R_1 = \mathbb{R}_+ w_0$ and $R_2 = \mathbb{R}_+ v_0$.

As $Y_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we have $v_1^2 = v_0^2 = 0$. Suppose first that $D_{12} \cdot v_1 = 2$, see Lemma 7.6. There is an ample divisor $A$ with $A.D_{31} = A.D_{32} = 2q$ for an appropriate $q \in \mathbb{N}$. Note that $D_{31}^2 = -2$. So $(A + qD_{31}).D_{31} = 0$ and $(A + qD_{31}).D_{32} = 4q$. Write $L_3 := A + qD_{31}$ and note that $L_3$ is nef and vanishes only on $D_{31}$. Consider the divisors $L_1 := 2qv_1$ on $Y_1$ and $L_2 := 4qv_0$ on $Y_2$. We have

$D_{12} \cdot L_1 = 4q = D_{21} \cdot L_2, \quad D_{13} \cdot L_1 = 0 = D_{31} \cdot L_3, \quad D_{23} \cdot L_2 = 4q = D_{32} \cdot L_3.$

Hence, there is a nef divisor $L$ on $\mathcal{Y}$ restricting to $L_i$ on $Y_i$. A suitable multiple of $L$ defines a contraction $g_1 = \pi_{|rL|}: \mathcal{Y} \to Z$.

Suppose now that $D_{12} \cdot v_1 = 0$. Then $D_{32}^2 = -8$. Let $A$ be an ample divisor on $Y_3$ and set $L_3 = 8A + (A.D_{32})D_{32}$. Then $L_3$ is nef and $L_3.C = 0$ for a curve $C$ on $Y_3$ if and only if $C$ is a multiple of $D_{32}$. Also, $L_3.D_{31} = q$ for some $q \in \mathbb{N}$ which is divisible by 2. Let $L_1$ be the divisor defined by $\frac{1}{2}qv_1$ and write $L_2 = O_{Y_2}$. We have

$L_1 \cdot D_{12} = 0 = L_2 \cdot D_{21}, \quad L_1 \cdot D_{13} = q = L_3 \cdot D_{31}, \quad L_2 \cdot D_{23} = 0 = L_3 \cdot D_{32}.$

Thus there is a divisor $L \in \text{Pic}(\mathcal{Y})$ restricting to $L_i$ on $Y_i$. Let $g_2 = \pi_{|rL|}$ for $r \gg 0$. Then $g_2: \mathcal{Y} \to Z$ is a contraction.

Now, let $f_2 = g_1$ or $g_2$, depending on the case. By a similar argument as in the proof of Lemma 7.7, one sees that $\text{Exc}(g_2) = Y_1 \cup Y_2$ and that $f_2$ is the contraction of the extremal face $\text{NE}(f_2) = \langle w_0, v_1 \rangle$. Hence, it follows from Corollary 7.4 that $f_2: \mathcal{Y} \to Z$ is a cusp model.

Note that in any case, $\text{NE}(Y_1) = \langle v_0, v_1 \rangle$ and $\text{NE}(Y_2) = \langle w_0, D_{21} \rangle$ and as above, one sees that $f_1$ and $f_2$ are the only two cusp models.

**Remark 7.9.** It follows from the analysis of the curve structure, that in the situation of Proposition 7.8 that we cannot have $Y_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for both $i = 1, 2$.

**7.10.** $\Gamma Y_1$ **regular**, $\Gamma Y_2$ **very degenerate and** $|\Gamma Y_i| = 2$ **for** $i = 1, 2$. Let $Y$ be a component of a central fiber $\mathcal{Y}_c$ of a model of the DNV family $\mathcal{Y}$ of degree 2. Suppose the dual intersection complex of $\mathcal{Y}$ is $\mathcal{P}$, so $Y$ is smooth. Let $D = D_0 + D_1$ be the anticanonical divisor given by the restriction of $\text{Sing}(\mathcal{Y}_c)$ to $Y$.

**Lemma 7.11.** Suppose $\Gamma Y$ is very degenerate and $|\Gamma Y| = 2$. Then $\Gamma Y$ consists of two vertices $v_1, v_0$ with $v_0^2 = 0$, $v_1^2 = -1$ and such that $D_0 \cdot v_0 = 2$, $D_1 \cdot v_1 = 1$, $D_0 \cdot v_1 = D_1 \cdot v_0 = 0$, $D_1^2 = 0$, and $D_2^2 = 4$, see Figure 5.

**Proof.** As $\Gamma Y$ is very degenerate, the self intersection numbers of the $D_i$ are uniquely determined by [HL19 Proposition 3.18(iii)] and the requirement that $D_1^2 + D_2^2 = 6 - |\Gamma Y|$. By [HL19 Proposition 3.18(i)] there is a unique such $Y$. Constructing it from $\mathcal{Y}_2$ by suitable blow-downs one sees that the curve structure has to be the one depicted in Figure 5. The claims about the intersection numbers of the vertices with the boundary now follow from [HL19 Proposition 3.2].
Proposition 7.12. Let $\mathcal{Y} \to S$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{P}$ such that there are two components $Y_1, Y_2$ such that $|\Gamma_{Y_1}| = 2$, $\Gamma_{Y_2}$ is very degenerate, and $\Gamma_{Y_2}$ is regular. Then there is a cusp model $f : \mathcal{Y} \to \mathcal{Y}'$ contracting precisely $Y_1$ and $Y_2$. The cone $C(f)$ is the only cuspidal cone contained in $\text{Nef}(\mathcal{Y})$.

Proof. Write $\Gamma_{Y_1} = \{v_0, v_1\}$ and $\Gamma_{Y_2} = \{w_0, w_1\}$. By Lemmas 7.8 and 7.11 and the triple point formula, we necessarily have $D_{21}^2 \in \{-2, -6\}$. Assume first that $D_{21}^2 = -2$. Let $L$ be an ample divisor on $Y_3$. Again by the triple point formula we have $D_{32}^2 = -8$ so $L_3 = 8A + (A.D_{32})D_{32}$ defines a nef divisor on $Y_3$. We have $L_3C = 0$ if and only if $C = D_{32}$ and $L_3.D_{31} = 2q$ for some $q \in \mathbb{N}$ because $D_{31}.D_{32} = 2$. Let $L_1$ be the divisor in $\text{Pic}(Y_1)$ defined by $qv_0$ and set $L_2 = \mathcal{O}_{Y_2}$. Then

$$L_1.D_{12} = 0 = L_2.D_{21}, \quad L_1.D_{13} = 2q = L_3.D_{31}, \quad L_2.D_{23} = 0 = L_3.D_{32}.$$ 

Thus there is a divisor $L \in \text{Pic}(\mathcal{Y})$ restricting to $L_i$ on $Y_i$. Let $f = \pi_{r,L_i}$ for suitable $r \gg 0$. Then $f : \mathcal{Y} \to Z$ is a contraction. Suppose $f_{\mathcal{Y}_i} : \mathcal{Y}_i \to Z_i$ contracts a curve $C_i$. Taking the closure of $C_i$ we obtain a divisor $C \in \text{Pic}(\mathcal{Y})$ with none of the $Y_i$ in the support of $C$. Write $C_i$ as the effective cone of $\mathcal{Y}_i$ for the rest of restriction of $C$ to $Y_i$. As the $C_i$ are contracted, one has that $C_1 = av_0$ and $C_3 = a'D_{32}$ with $a, a' \geq 0$. From $C_1.D_{13} = C_3.D_{31}$ one deduces that in fact $a = a'$. So $C_3.D_{32} = -8a$. The effective cone of $Y_2$ is given by $\text{Eff}(Y_2) = \langle w_0, D_{21} \rangle$. Writing $C_2 = bD_{21} + cw_0$ with $b, c \geq 0$ we have $-8a = C_2.D_{23} = 2b + c$. It follows that $C$ is trivial, a contradiction. So $f$ is an isomorphism on $\mathcal{Y}_i$. It is immediate to check that $\text{Exc}(f) = Y_1 \cup Y_2$ and the extremal cone of $f$ is $\text{NE}(f) = \langle v_0, w_0 \rangle$. It now follows from Corollary 7.4 that $f : \mathcal{Y} \to Z$ is a cusp model.

Now suppose $D_{21}^2 = -6$. Pick an ample divisor $A$ on $Y_3$. As $D_{31}^2 = -2$, $L_3 = 2A + (A.D_{31})D_{31}$ defines a nef divisor such that $L_3C = 0$ for a curve $C$ on $Y_3$ if and only if $C = D_{31}$. Set $q = \frac{1}{2}(L_3.D_{32}) \in \mathbb{N}$. Let $L_1$ be the divisor defined by $qv_0$ and $L_2$ the divisor defined by $2qw_0$. As before, one checks that there is a divisor $L \in \text{Pic}(\mathcal{Y})$ restricting to $L_i$ on $Y_i$, inducing a contraction $f : \mathcal{Y} \to Z$. If $C$ is a curve contracted by $f$, then $C$ is in the span of $w_0$ and $v_0$. By the same arguments as before, we conclude that $f$ is a cusp model.

In both cases, $v_0$ and $w_0$, interpreted as curves on $\mathcal{Y}$, generate extremal rays that are contracted by $f$. We have $\text{NE}(Y_1) = \langle v_0, v_1 \rangle$ and $\text{NE}(Y_2) = \langle w_0, C \rangle$ where $C$ is a curve with $C^2 < 0$. So if $g$ is a cusp model, it necessarily contracts $w_0$ and $v_0$, so $g$ factors through $f$ and thus $g = f$. \qed

7.13. $|\Gamma_{Y_1}| = 1$, $|\Gamma_{Y_2}| = 2$. In this case, we write $\Gamma_{Y_1} = \{v_0\}$ with $v_0^2 = 1$. We also have $D_{1j}^2 = 4$ and $D_{ii}^2 = 1$, with $i, j \in \{2, 3\}$, $i \neq j$, see Figure 4. In particular, $Y_1 \cong \mathbb{P}^2$. Note that in this case, $\Gamma_{Y_2}$ is automatically regular.

Proposition 7.14. Let $\mathcal{Y} \to S$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{P}$ such that there is a component $Y_1 \cong \mathbb{P}^2$ and a component $Y_2$ with $|\Gamma_{Y_2}| = 2$.
Then there is a cusp model \( f : \mathcal{Y} \to \mathcal{Y}' \) contracting precisely \( Y_1 \) and \( Y_2 \). Moreover, \( C(f) \) is the only cuspidal cone contained in \( \text{Nef}(\mathcal{Y}) \).

**Proof.** Either we have \( D_{12}^2 = 1 \) or \( D_{12}^2 = 4 \). In the first case we have \( D_{31}^2 = -6 \) and \( D_{32}^2 = -9 \), in the second case \( D_{31}^2 = -3 \) and \( D_{32}^2 = -12 \). Let \( A \) be ample on \( Y_3 \), then

\[
L_3 = 50A + (6(A.D_{32} + 2(A.D_{31})))D_{32} + (9(A.D_{31}) + 2(A.D_{32}))D_{31}
\]

in the first case and

\[
L_3 = 32A + (3(A.D_{32} + 2(A.D_{31})))D_{32} + (12(A.D_{31}) + 2(A.D_{32}))D_{31}
\]

in the second case are nef on \( Y_3 \) and \( L_3.C = 0 \) for an effective curve if and only if \( C = aD_{31} + bD_{32} \) with \( a, b \geq 0 \). Setting \( L_1 = \mathcal{O}_{Y_1} \) and \( L_2 = \mathcal{O}_{Y_2} \) we obtain a unique semi-ample \( L \) as before such that \( L|_{Y_i} = L_i \). The rest of the proof is similar to the previous cases. \( \square \)

**Remark 7.15.** Note that in all cases considered so far any cusp model \( f : \mathcal{Y} \to \mathcal{Y}' \) corresponds to a cone \( C(f) \) such that \( C(f) \) is not contained in a facet \( \sigma \) of \( \text{Nef}(\mathcal{Y}) \) corresponding to a small contraction.

### 8. Models with dual intersection complex \( \mathcal{T} \)

Let \( \mathcal{Y} \to S \) be a model of the DNV family of class \( \mathcal{T} \). There is a unique component \( Y_\omega \subset Y_c \) such that \( \Gamma_{Y_\omega} \) is of type \( d_4 \). We denote by \( \nu : Y_\omega' \to Y_\omega \) the normalization and by \( D_\omega \subset Y_\omega \) the unique component of \( Y_\text{c}^{\text{sing}} \cap Y_\omega \) such that \( \nu^{-1}(D_\omega) \) is reducible.

We have the following result.

**Lemma 8.1.** Let \( \mathcal{Y} \) be a model of the DNV family of class \( \mathcal{T} \). Let \( \mathcal{Y} \to \mathcal{Y}' \) be a cusp model contracting \( Y_\omega \). Then \( D_\omega \) is not contracted.

**Proof.** By projectivity of \( \mathcal{Y} \), the preimage \( \nu^{-1}(D_\omega) \) consists of two \((-1)\) curves by [HL19, Proposition 4.1]. The elementary modification in \( D_\omega \) defines a flop \( \phi : \mathcal{Y} \to \mathcal{Y}^+ \) such that \( \mathcal{Y}^+ \) has dual intersection complex \( \mathcal{P} \). If \( D_\omega \) is contracted by \( f \), then it follows from Construction 6.10 that there is a cusp model \( f^+ : \mathcal{Y} \to \mathcal{Y}' \) and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\phi} & \mathcal{Y}^+ \\
\downarrow f & & \downarrow f^+ \\
\mathcal{Y}' & & \\
\end{array}
\]

It follows that the cusp model \( \mathcal{Y}^+ \to \mathcal{Y}' \) factors through the flopping contraction associated to \( \phi \), a contradiction, cf. Remark 7.15. \( \square \)

There is only one curve structure of type \( d_1 \) with \( |\Gamma_Y| = 3 \), namely the one depicted in Figure 6.
Proposition 8.2. Let $\mathcal{Y}$ be a model of the DNV family of class $\mathcal{T}$. Let $f : \mathcal{Y} \to \mathcal{Y}'$ be a cusp model. Then $Y_\omega$ is not contracted.

Proof. We will write $Y_c = Y_1 \cup Y_2 \cup Y_3$ for the irreducible components and assume that the special component is $Y_\omega = Y_1$. Then $Y_2$ and $Y_3$ are the smooth components. Let us assume $Y_1$ and $Y_2$ are contracted by $f$. We will obtain a contradiction by a detailed analysis of the curve structure on $Y_1$.

Note that by the very nature of curve structures of type $d_4$, we always have $|\Gamma_{Y_1}| \geq 3$. By Proposition 6.1, we must have $|\Gamma_{Y_1}| + |\Gamma_{Y_2}| \leq 5$ and a lengthy but straightforward analysis of the curve structures shows that in this case $|\Gamma_{Y_1}| = 4$, the curve structure is unique and depicted in Figure 5. Moreover, we must have $D_{12}^2 = -9$ and $D_{13}^2 = 9$. Let $C_1, \ldots, C_4$ be the curves corresponding to the vertices in $\Gamma_{Y_1}$ such that $C_2^2 = 0$ and $C_3^2 = 7$ and let $D_{\omega_1}, D_{\omega_2}$ be the preimages of the curve $D_\omega = Y_1^{\text{sing}}$ under the normalization. From [HL19, Corollary 3.7] we infer that $\text{NE}(Y_1) = \langle D_{12}, D_{\omega_1}, D_{\omega_2}, C_2, C_4 \rangle$.

By Lemma 8.1, the special component $Y_1$ is not contracted to a point. The pullback of a very ample line bundle from $\mathcal{Y}'$ restricts to a nef line bundle $L$ on $Y_1$ whose linear system gives the restriction $\varphi_L : Y_1 \to \mathbb{P}^1$ of $f$. In particular, $L^2 = 0$. As $\Gamma_{Y_2} = 1$ (and thus $Y_2 \cong \mathbb{P}^2$), the curve $D_{12}$ is contracted by $\varphi_L$. Hence, if we write $L = aC_1 + bC_2 + cC_3 + dC_4$ we must have $a = 0$. Testing nefness against $C_2, C_4, D_{\omega_1}, D_{\omega_2}$ gives the conditions

$$c \geq b, \quad c \geq d, \quad b \geq 0, \quad d \geq 0,$$

so we can estimate

$$0 = L^2 = (2bc - b^2) + (2cd - d^2) + 7c^2 \geq b^2 + d^2 + 7c^2 > 0$$

where the right-hand side is strictly positive because $L$ is not the trivial bundle. This is a contradiction.

Remark 8.3. One implication of Proposition 8.2 together with the observation in Remark 7.15 is that the situation described in Construction 6.10 does not occur. Cuspidal cones are not contained in interior facets.

Proposition 8.4. Let $\mathcal{Y} \to S$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{T}$. Let $f : \mathcal{Y} \to \mathcal{Y}'$ be a cusp model. Let $Y_1$ be a component of $\mathcal{Y}_c$ contracted by $f$. Then $|\Gamma_{Y_1}| \leq 2$. 

Proof. By the previous proposition, $Y_1$ is a smooth component. We only need to show that $|\Gamma_{Y_1}| = 3$ is impossible. There is only one curve structure of type $d_1$ with $|\Gamma_{Y}| = 3$, depicted in Figure 6. As the special component $Y_\omega$ is not contracted, it follows from Proposition 6.9 and Proposition 6.11 that none of the curves in $\Gamma_{Y_1}$ is contracted. As these generate $\text{NE}(Y_1)$, this is impossible.

We now determine which models of the DNV family with $Y_c$ of dual intersection complex $\mathcal{T}$ admit a cusp model. Note that if $Y$ is a smooth component of $Y_c$ with $|\Gamma_{Y}| = 2$, then $\Gamma$ is one of the two graphs in Figure 7.

**Proposition 8.5.** Let $\mathcal{Y} \to S$ be a model of the DNV family of degree 2 with dual intersection complex $\mathcal{T}$ such that for any smooth component $Y$ we have $|\Gamma_{Y}| \leq 2$. Then there is a cusp model $f : \mathcal{Y} \to \mathcal{Y}'$ contracting precisely the smooth components. Moreover, $C(f)$ is the only cuspidal cone contained in $\text{Nef}(\mathcal{Y})$.

**Proof.** Write $\mathcal{Y}_c = Y_1 \cup Y_2 \cup Y_3$ and assume $Y_2$ is the special component. So $Y_1$ and $Y_3$ are the smooth components. Consider first the case where $|\Gamma_{Y_1}| = |\Gamma_{Y_3}| = 2$. Write $\Gamma_{Y_1} = \{v_0, v_1\}$ and $\Gamma_{Y_3} = \{w_0, w_1\}$ with $w_0^2 = v_0^2 = 0$. Let $A$ be an ample divisor on $Y_2$ and denote by $A'$ its pullback to the normalization $Y_2' \to Y_2$. For suitable $n, k, m \in \mathbb{N}$, the triple of divisors $L_1 = mw_0$, $kA'$, and $L_3 = nw_0$, interpreted as a divisor on the normalization $\mathcal{Y}_c'$ of $\mathcal{Y}_c$, descends to a divisor on $\mathcal{Y}_c$ which then by maximality induces a divisor on $\mathcal{Y}$. This divisor is semiample because this is true for its restriction to the central fiber.

Let $f : \mathcal{Y} \to Z$ be the morphism induced by $|IL|$ for $l \gg 0$. By construction, $Y_1 \cup Y_2 \subset \text{Exc}(f)$. Note that if $C$ is a curve on $Y_1$ (respectively $Y_3$) contracted by $f$, then $C$ is a multiple of $v_0$ (respectively $w_0$). As $f$ does not contract any curve on $Y_2$, it follows that $f$ does not contract any curve on $\mathcal{Y}_\eta$. Hence, we have $Y_1 \cup Y_2 = \text{Exc}(f)$. It follows from Corollary 7.4.
that \( Z \) is \( \mathbb{Q} \)-factorial. So \( f \) is a cusp model and \( C(f) \) a cuspidal cone. As before one shows that \( C(f) \) is the unique cuspidal cone contained in \( \text{Nef}(Y) \).

The next case \( \Gamma_{Y_{1}} = \{ v_{0} \} \), \( \Gamma_{Y_{3}} = \{ w_{0} \} \) is similar: as \( D_{21}^{2} = D_{23}^{2} < 0 \) it is straight forward to construct a nef divisor \( L_{2} \) on \( Y_{2} \) such that \( L_{2}.C = 0 \) if and only if \( C \in \{ D_{21}, D_{23} \} \). Extending \( L_{2} \) by the trivial divisors on \( Y_{1} \) and \( Y_{3} \), as above one obtains a morphism \( f: Y \to Z \). By construction, \( Y_{1} \cup Y_{3} = \text{Exc}(f) \) and one concludes that \( f \) is a cusp model and \( C(f) \) a cuspidal cone. A similar proof with \( L_{2} \) chosen to be a nef divisor such that \( L_{2}.C = 0 \) if and only if \( C = D_{21} \) shows the remaining case, where \( |\Gamma_{Y_{1}}| = 1, |\Gamma_{Y_{3}}| = 2 \).

\[ \square \]

9. Counting cones

So far, we have examined curve structures of models of the DNV family in degree 2 that admit cusp models. Building on this, we will in the present section classify the models and use the classification to count cuspidal cones.

**Proposition 9.1.** Let \( Y \to Y' \) be a cusp model such that \( Y_c = Y_1 \cup Y_2 \cup Y_3 \) has dual intersection complex \( \mathcal{P} \). Suppose \( |\Gamma_{Y_{i}}| = 2 \) for \( i = 2, 3 \) (and hence that \( Y_2, Y_3 \) are contracted under \( Y \to Y' \)). Then \( Y \) is uniquely determined by the regularity type of the \( Y_i \) and \( n := D_{12}^{1} \).

More precisely:

1. If \( \Gamma_{Y_{i}} \) is regular for all \( i = 1, 2, 3 \), then \( Y \) is uniquely determined by \( n \) and there is one such model precisely for every \( n \in [-7, 1] \).
2. If \( \Gamma_{Y_{i}} \) is very degenerate for some \( i \), then \( Y \) is uniquely determined by \( n \) and there is one such model precisely for every \( n \in \{-2, -6\} \).

**Definition 9.2.** In the situation of the proposition, we write \( Y_{R}(n) := Y \) if all components \( Y_i \) are regular and \( Y_{VD}(n) := Y \) if there is a very degenerate \( Y_i \).

**Proof of Proposition 9.1** As \( |\Gamma_{Y_{i}}| = 2 \) for \( i = 2, 3 \), the components \( Y_2, Y_3 \) are degenerate by Corollary 6.8. From projectivity of \( Y_c \) and [HL19, Proposition 4.2] we deduce that \( \Gamma_{Y_{1}} \) is non-degenerate. Write \( n_1 = D_{2}^{2} \) and \( n_2 = D_{13}^{2} \). Using non-degeneracy, we deduce \( n_1 \leq 1 \) from [HL19, Proposition 3.18]. Let \( v_{i}, i = 1, 2 \) be the exceptional vertices, with \( v_{1}.D_{12} = 1 \).

Write \( l_{i} = |L(v_{i})| - 4 \) where \( L(v_{i}) \) is the leg of \( v_{i} \). From the condition \( |\Gamma_{Y_{2}}| = |\Gamma_{Y_{3}}| = 2 \) we get \( |\Gamma_{Y_{1}}| = 20 \). By the definition of curve structure of type \( d_{2} \), we have \( |L(v_{1})| + |L(v_{2})| = |\Gamma_{Y_{1}}| - 2 \).

So \( l_{1} + l_{2} = 12 \). As a leg of any exceptional vertex has at least 2 vertices, we have \( l_{i} \in [-2, 14] \).

Also, we have \( n_i = -1 - l_i \). So we have \( n_2 = -14 - n_1 \), and \( n_1 \in [-15, 1] \). Note that \( n_1 \) determines all self-intersection numbers \( D_{ij} \). Hence, as soon as we know the regularity of \( \Gamma_{Y_{2}} \) and \( \Gamma_{Y_{3}} \), we have a complete description of \( Y \) thanks to [HL19, Proposition 3.18]. Assuming all curve structures are regular, after possibly changing the indexing of \( Y_2 \) and \( Y_3 \) of \( Y_c \) we find \( Y \cong Y_{R}(n) \) for \( n \in [-7, 1] \), where we have taken into account that we count unordered pairs \( \{n_1, n_2\} \) with \( n_1 + n_2 = -14 \).

By the general yoga of curve structures one shows that if \( n_1 \notin \{-2, -6, -8, -12\} \), all \( \Gamma_{Y_{i}} \) are regular. If \( n_1 \in \{-2, -6, -8, -12\} \), there can be very degenerate curve structures. Note that by changing the indexing, we can reduce to the case \( n_1 \in \{-2, -6\} \). So assume \( \Gamma_{Y_{2}} \) is very degenerate. Then, \( D_{23}^{2} \in \{4, -6\} \) so \( \Gamma_{Y_{3}} \) is regular.
The existence is straightforward to show by constructing a suitable sequence of elementary modifications of type I starting with $Y_{\emptyset}$.

Let us now turn to models with dual intersection complex $\mathcal{C}$. It is also straightforward to show that for $(n_1, n_2) \in \{(-8, -8), (-8, -9), (-9, -9)\}$ there is a unique model $\mathcal{Y}(n_1, n_2)$ of the DNV family of degree 2 such that the central fiber $\mathcal{Y}(n_1, n_2)_c$ has dual intersection graph $\mathcal{C}$ and, if $Y_3$ is the special component, $D^2_{Y_3} = n_i$. These are precisely the models such that $|\Gamma_Y| \leq 2$ for any smooth component. We state this as a proposition.

**Proposition 9.3.** Let $\mathcal{Y} \to \mathcal{Y}'$ be a cusp model such that $\mathcal{Y}_c$ has dual intersection complex $\mathcal{C}$. Then $\mathcal{Y} \cong \mathcal{Y}(n_1, n_2)$ for $(n_1, n_2) \in \{(-8, -8), (-8, -9), (-9, -9)\}$.

As a last ingredient for counting cones, we show that in degree 2, every cuspidal cone is contained in a unique maximal cone, cf. Remark 4.6.

**Lemma 9.4.** Assume $2d = 2$. Let $\pi : \mathcal{Y} \to \mathcal{Y}'$ be a cusp model. Then $\mathcal{Y}'$ determines the model $\mathcal{Y}$ of the DNV family uniquely.

**Proof.** Suppose that $\mathcal{Y} \to \mathcal{Y}'$ and $\mathcal{X} \to \mathcal{Y}'$ were two such models of the DNV family. Let $B$ be the pullback along $\mathcal{Y} \to \mathcal{X}$ of an ample prime divisor on $\mathcal{X}$. As $B$ is $\pi$-nef if and only if $B + \pi^*L$ is nef for a sufficiently ample divisor $L$ on $\mathcal{Y}'$ and clearly $B + \pi^*L \in f^*\text{Amp}(\mathcal{X}')$, we deduce that for $B$ to be $\pi$-nef, we must have that $\mathcal{Y} \to \mathcal{X}$ is an isomorphism by Lemma 5.10.

Let us assume this is not the case. Then we run a log MMP for the pair $(\mathcal{Y}, \varepsilon B)$, where $\varepsilon$ is small enough in order to make the pair klt, and obtain a sequence of flops over $\mathcal{Y}'$ connecting $\mathcal{Y}$ to $\mathcal{X}$. Thus, we may reduce to the case where $\mathcal{Y} \to \mathcal{X}$ is a flop. Now the lemma follows from Remark 8.3.

Thus it makes sense to speak of a cusp model $\mathcal{Y}'$ of class $\mathcal{G} \in \{\mathcal{P}, \mathcal{C}\}$, meaning that the unique model $\mathcal{Y}$ of the DNV family admitting a regular contraction $\mathcal{Y} \to \mathcal{Y}'$ is of class $\mathcal{G}$.

**Theorem 9.5.** Let $\mathcal{Y}$ be a model of the DNV family of degree 2. There are 93 cuspidal cones in $\text{MF}(\mathcal{Y})$. Moreover, 81 of these correspond to cusp models of class $\mathcal{P}$ and 12 to cusp models of class $\mathcal{C}$.

**Proof.** Note first by Lemma 9.4 each cuspidal cone is contained in exactly one maximal cone of $\text{MF}(\mathcal{Y})$ so we are reduced to counting marked models of the DNV family which admit a cusp model. For a model $\mathcal{X} \to S$ of the DNV family, we denote by $\ell_X$ the orbit length of $\text{Nef}(\mathcal{X})$ under the action of the group $\text{Bir}(\mathcal{Y})$ of birational automorphisms and by $c_X$ the number of cuspidal cones in $\text{Nef}(\mathcal{X})$. The number of cuspidal cones in $\text{MF}(\mathcal{Y})$ is then given by $\sum_X c_X \cdot \ell_X$ where the sum runs over all (non-marked) models $\mathcal{X}$ of the DNV family.

Let us first count the models of class $\mathcal{P}$. Among those, models with $|\Gamma_Y| = 2$ for every contracted component $Y$ are precisely given by the models $\mathcal{Y}_R(n)$ for $-7 \leq n \leq 1$ and $\mathcal{Y}_{\text{VD}}(n)$ for $n = -2$ or $-6$ by Proposition 9.1. By [HL19, Proposition 5.43], the orbit length of $\mathcal{Y}_R(n)$ is 6 if $n \neq -7$ and 3 if $n = -7$ and it is 6 for the models $\mathcal{Y}_{\text{VD}}(n)$. Also, one reads off from the curve structures that $\mathcal{Y}_c$ has a component isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\mathcal{Y} \cong \mathcal{Y}_R(n)$ for $n \in \{-6, -2\}$. So by Propositions 7.8 and 7.12 the nef cones of these last models contain 2
cuspidal cones while all the other models contain 1 such cone. There is also the unique model characterized by having a component isomorphic to \( P^2 \), as in Proposition 7.14. It has orbit length 6 and a unique cuspidal cone inside its nef cone. Similarly, the two models \( \mathcal{Y}_{VD}(n) \), \( n \in \{-2,-6\} \) have orbits of length 6 and their nef cones contain a unique cuspidal cone. This gives another 12 cuspidal cones. So we have

\[
6 \cdot 6 + 6 \cdot 2 \cdot 2 + 3 \cdot 1 + 6 \cdot 1 + 6 \cdot 2 = 81
\]
cuspedal cones defined by cusp models \( \mathcal{Y} \to \mathcal{Y}' \) with \( \mathcal{Y}_c \) having dual intersection complex \( \mathcal{P} \).

By Proposition 9.3 there are 3 models \( \mathcal{Y}(n_1, n_2) \) of the DNV family with \( \mathcal{Y}_c \) having dual intersection complex \( \mathcal{T} \). Two of these are symmetric, i.e. two components of \( \mathcal{Y}_c \) are isomorphic (see [HL19, Definition 5.40]), while \( \mathcal{Y}(-8,-9) \) is not. Each cone \( \text{Nef}(\mathcal{Y}(n_1, n_2)) \) contains a unique cuspidal cone by Proposition 8.5. So there are 6 \( \cdot 1 + 3 \cdot 2 = 12 \) cuspidal cones defined by cusp models \( \mathcal{Y} \to \mathcal{Y}' \) with \( \mathcal{Y}_c \) having dual intersection complex \( \mathcal{T} \). Summing up, there are 93 = 81 + 12 cuspidal cones.

By the \( S_3 \)-symmetry of the set of cuspidal cones, we deduce our main result, Theorem 0.3.

**Corollary 9.6.** Let \( \mathcal{Y}' \to S \) be a cusp model. Then \( \text{MF}(\mathcal{Y}') \) has 31 maximal cones, 27 of these correspond to cusp models of class \( \mathcal{P} \) and 4 to cusp models of class \( \mathcal{T} \). In particular, the toric fan \( \Sigma_{GHKS}^2 \) has 31 maximal cones inside the fundamental domain of the Coxeter fan.

**Proof.** For every irreducible component \( Y_i \subset \mathcal{Y}_i \) we choose a rational \( Y_i \)-cusp model \( \mathcal{Y} \to Y_i \) for \( i = 1, 2, 3 \). By Corollary 5.14 the Mori fans \( \text{MF}(\mathcal{Y}_i) \) are in bijective correspondence via the \( S_3 \) action. Moreover, thanks to Proposition 6.1 a given model \( \mathcal{X} \to S \) of the DNV family in degree 2 cannot admit cusp models for different components of the central fiber. Thus, cuspidal cones for different components do not intersect: \( \text{Cusp}_{Y_i} \cap \text{Cusp}_{Y_j} = \emptyset \) for \( i \neq j \). The claim now follows from Theorem 9.5. \( \square \)

**Corollary 9.7.** The \( \Gamma_2^+ \)-action induces a residual \( S_3 \)-action on the set of maximal cones of the toric fan \( \Sigma_{GHKS}^2 \) inside the fundamental domain of the Coxeter fan with 17 = 14 + 3 orbits of class \( \mathcal{P} \) and \( \mathcal{T} \), respectively.

**Proof.** This follows from the counting arguments given in the proof of Theorem 9.5. Taking the orbit length into account, we obtain 6 + 2 \cdot 2 + 1 + 1 + 2 = 14 maximal cones of class \( \mathcal{P} \) and 1 + 2 = 3 maximal cones of class \( \mathcal{T} \). \( \square \)

**References**

[AET19] Valery Alexeev, Philip Engel, and Alan Thompson, *Stable pair compactification of moduli of K3 surfaces of degree 2*, 2019. arXiv:1903.09742

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties. With the collaboration of Peter Scholze. 2nd ed*, Cambridge University Press, 2010 (English).

[Art69] Michael Artin, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[BB66] Walter L. Baily, Jr. and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.
[BHPV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces, 2nd edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 4, Springer-Verlag, Berlin, 2004.

[BL16] Benjamin Bakker and Christian Lehn, *A global Torelli theorem for singular symplectic varieties*, 2016. [arXiv:1612.07894]

[BL18] ———, *The global moduli theory of symplectic varieties*, 2018. [arXiv:1812.09748]

[BR75] Dan Burns, Jr. and Michael Rapoport, *On the Torelli problem for Kählerian $K3$ surfaces*, Ann. Sci. École Norm. Sup. (4) 8 (1975), no. 2, 235–273.

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017, Second edition.

[Dol96] Igor Dolgachev, *Mirror symmetry for lattice polarised K3 surfaces*, J. Math. Sci. 81 (1996), 2599–2630.

[Elk81] Renée Elkik, *Rationalité des singularités canoniques*, Invent. Math. 64 (1981), no. 1, 1–6.

[Fri83a] Robert Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) 118 (1983), no. 1, 75–114.

[Fri83b] ———, *Linear systems on anticanonical pairs*, The birational geometry of degenerations (Cambridge, Mass., 1981), Progr. Math., vol. 29, Birkhäuser, Boston, Mass., 1983, pp. 162–171.

[FS86] Robert Friedman and Francesco Scattone, *Type III degenerations of K3 surfaces*, Invent. Math. 83 (1986), 1–39.

[GHKS16] Mark Gross, Paul Hacking, Sean Keel, and Bernd Siebert, *Theta functions on K3 surfaces*, in preparation.

[GHKS07] Valeri A. Gritsenko, Klaus Hulek, and Gregory Kumar Sankaran, *The Kodaira dimension of the moduli of K3 surfaces*, Invent. Math. 169 (2007), no. 3, 519–567.

[Gro95] Alexander Grothendieck, *Géométrie formelle et géométrie algébrique*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, pp. Exp. No. 182, 193–220, errata p. 390.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer Verlag, 1977, Graduate Texts in Mathematics, No. 52.

[HL19] Klaus Hulek and Carsten Liese, *The Mori fan of the Dolgachev-Nikulin-Voisin family in genus 2*, 2019. [arXiv:1911.06862]

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, vol. 29, Cambridge etc.: Cambridge University Press, 1990.

[Huy16] Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, 2016.

[Jam68] Donald Gordon James, *On Witt’s theorem for unimodular quadratic forms*, Pacific J. Math. 26 (1968), 303–316.

[Kaw88] Yujiro Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) 127 (1988), no. 1, 93–163.

[Kaw97] ———, *On the cone of divisors of Calabi-Yau fiber spaces*, Internat. J. Math. 8 (1997), no. 5, 665–687.

[Kaw11] ———, *Remarks on the cone of divisors*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 317–325.

[KH00] Sean Keel and Yi Hu, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331–348.
János Kollár and Shigefumi Mori, Classification of three-dimensional flips, J. Amer. Math. Soc. 5 (1992), no. 3, 533–703.

János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

János Kollár, Flops, Nagoya Math. J. 113 (1989), 15–36.

János Kollár, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.

Shigeyuki Kondō, Algebraic K3 surfaces with finite automorphism groups, Nagoya Math. J. 116 (1989), 1–15.

Sándor J. Kovács, The cone of curves of a K3 surface, Math. Ann. 300 (1994), no. 4, 681–691.

Viktor Stepanovich Kulikov, Degenerations of K3 surfaces and Enriques surfaces, Mathematics of the USSR-Izvestiya 11 (1977), no. 5, 957–989.

Radu Laza, Triangulations of the sphere and degenerations of K3 surfaces, 2008. arXiv:0809.0987

Stanisław Lojasiewicz, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 18 (1964), 449–474.

Eduard Looijenga, Compactifications defined by arrangements. I: The ball quotient case., Duke Math. J. 118 (2003), no. 1, 151–187.

Eduard Looijenga, Compactifications defined by arrangements. II: Locally symmetric varieties of type IV., Duke Math. J. 119 (2003), no. 3, 527–588.

Viacheslav V. Nikulin, Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebro-geometric applications, J. Sov. Math. 22 (1983), 1401–1475.

Ulf Persson, On degenerations of algebraic surfaces, Mem. Amer. Math. Soc. 11 (1977), no. 189, xv+144.

Ulf Persson and Henry Pinkham, Degeneration of surfaces with trivial canonical bundle, Ann. of Math. (2) 113 (1981), no. 1, 45–66.

Ilya I. Pjatecki˘ı- ˇSapiro and Igor’ RostislavovichˇSafareviˇc, Torelli’s theorem for algebraic surfaces of type K3, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572.

Francesco Scattone, On the compactification of moduli spaces for algebraic K3 surfaces, Mem. Amer. Math. Soc. 70 (1987), no. 374, x+86.

Edoardo Sernesi, Deformations of algebraic schemes, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer Verlag, 2006.

The Stacks project authors, The stacks project, https://stacks.math.columbia.edu 2020.

Misha Verbitsky, Mapping class group and a global Torelli theorem for hyperkähler manifolds, Duke Math. J. 162 (2013), no. 15, 2929–2986, Appendix A by Eyal Markman.

Eckart Viehweg, Weak positivity and the stability of certain Hilbert points. III, Invent. Math. 101 (1990), no. 3, 521–543.

Ernest Borisovich Vinberg, Some arithmetical discrete groups in Lobachevskij spaces, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), 1975, pp. 323–348.

Ernest Borisovich Vinberg, Hyperbolic groups of reflections, Uspekhi Mat. Nauk 40 (1985), no. 1(241), 29–66, 255.

Klaus Hulek, Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

Email address: hulek@math.uni-hannover.de
Christian Lehn, Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Strasse 39, 09126 Chemnitz, Germany
Email address: christian.lehn@mathematik.tu-chemnitz.de

Carsten Liese, Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
Email address: liese@math.uni-hannover.de