CRITICAL AND MINIMAL CONNECTIVITY OF POWER GRAPHS OF FINITE GROUPS

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Abstract. The power graph of a group $G$ is the graph whose vertex set is $G$ and two distinct vertices are adjacent if one is a power of the other. In this article, we show that $G(\mathbb{Z}_n)$ is not critically vertex connected when $n$ is a product of two or three distinct primes. We show that $G(D_n)$ is not critically vertex (edge) connected. We prove that $G(Q_n)$ is not critically vertex connected and classify $Q_n$ such that $G(Q_n)$ is critically edge connected. We classify $p$-groups whose power graphs are critically vertex (edge) connected. We obtain a characterization for power graphs of finite groups that are critically edge connected. We supply some necessary criterion for minimally vertex connected graphs. Then we classify finite groups whose power graphs are minimally vertex connected. We classify finite groups of odd order whose power graphs are minimally edge connected. We show that $G(D_n)$ and $G(Q_n)$ are not minimally edge connected. Further, we classify abelian $p$-groups whose power graphs are minimally edge connected.

1. Introduction

The notion of directed power graph of a semigroup $S$ was introduced by Kelarev and Quinn [8, 9] as the directed graph $\Gamma(S)$ with vertex set $S$ and there is an arc from a vertex $u$ to another vertex $v$ if $v = u^\alpha$ for some positive integer $\alpha \in \mathbb{N}$. Following this, Chakrabarty et al. [2] defined (undirected) power graph $\mathcal{G}(S)$ of a semigroup $S$ by ignoring the direction in $\Gamma(S)$, that is, with vertex set $S$ and distinct vertices $u$ and $v$ are adjacent in $\mathcal{G}(S)$ if $v = u^\alpha$ for some $\alpha \in \mathbb{N}$ or $u = v^\beta$ for some $\beta \in \mathbb{N}$. This article is concerned with power graphs of finite groups.

All graphs considered in this article are finite and simple, and all groups considered have order at least two.

A separating set of a graph $\Gamma$ is a set of vertices whose removal increases the number of components of $\Gamma$. A separating set is minimal if none of its proper subsets separates $\Gamma$. A separating set of $\Gamma$ with least cardinality is called a minimum separating set of $\Gamma$. A disconnecting set of $\Gamma$ is a set of edges whose removal increases the number of components of $\Gamma$. A disconnecting set is minimal if none of its proper

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of disconnected graphs or the trivial graph are always 0. The vertex connectivity of a graph \( \Gamma \), denoted by \( \kappa(\Gamma) \), is the minimum number of vertices whose removal results in a disconnected or trivial graph. The edge connectivity of \( \Gamma \), denoted by \( \kappa'(\Gamma) \), is the minimum number of edges whose removal results in a disconnected or trivial graph. So, the connectivity and edge-connectivity of disconnected graphs or the trivial graph are always 0.

Graphs having the property that the deletion of a vertex or an edge decreases their vertex connectivity and/or edge connectivity have been objects of interest for researchers. To this end, we have the following definitions.

A graph \( \Gamma \) is said to be critically \( k \)-vertex connected if \( \kappa(\Gamma) = k \) and \( \kappa(\Gamma - v) = k - 1 \) for every vertex \( v \) of \( \Gamma \). A critically \( k \)-vertex connected graph is a graph which is critically \( k \)-vertex connected for some \( k \). A graph \( \Gamma \) is said to be critically \( k \)-edge connected if \( \kappa'(\Gamma) = k \) and \( \kappa'(\Gamma - e) = k - 1 \) for every edge \( e \) of \( \Gamma \). A critically \( k \)-edge connected graph is a graph which is critically \( k \)-edge connected for some \( k \).

Analogously, a graph \( \Gamma \) is said to be minimally \( k \)-vertex connected if \( \kappa(\Gamma) = k \) and \( \kappa(\Gamma - v) = k - 1 \) for every vertex \( v \) of \( \Gamma \). A minimally \( k \)-vertex connected graph is a graph which is minimally \( k \)-vertex connected for some \( k \). A graph \( \Gamma \) is said to be minimally \( k \)-edge connected if \( \kappa'(\Gamma) = k \) and \( \kappa'(\Gamma - e) = k - 1 \) for every edge \( e \) of \( \Gamma \). A minimally \( k \)-edge connected graph is a graph which is minimally \( k \)-edge connected for some \( k \).

Critically and minimally connected graphs have been in the focus for their applications in communication networks. Halin [7] proved that the minimum degree of a minimally \( k \)-vertex connected graph is \( k \). Lick [12] gave a necessary condition for a graph to be critically \( k \)-vertex connected in terms of upper bound of its minimum degree. Lick [13] further showed that the minimum degree of a minimally \( k \)-edge connected graph is \( k \). Some of these notions were further generalized in [1, 14]. For more interesting results on these graph parameters, we can refer to [3, 10, 11].

In this article, we first show that when \( n \) is a product of two or three primes, then \( \mathcal{G}(\mathbb{Z}_n) \) is not critically vertex connected. We then prove that \( \mathcal{G}(D_n) \) and \( \mathcal{G}(Q_n) \) are not critically vertex connected. We ascertain that \( \mathcal{G}(D_n) \) is not critically edge connected and classify \( Q_n \) such that \( \mathcal{G}(Q_n) \) is critically edge connected. Further, we classify \( p \)-groups whose power graphs are critically vertex (edge) connected. We obtain a characterization of power graphs of finite groups that are critically edge connected involving minimum degree. Moving forward, we supply some necessary criterion for minimally vertex connected graphs. We give a classification of finite groups whose power graphs are minimally vertex connected. We show that if the proper power graph of a finite group is regular, then its power graph is minimally edge connected. Furthermore, converse holds if the group have odd order. Applying this, we classify finite groups of odd order whose power graphs are minimally edge connected. We also show that \( \mathcal{G}(D_n) \) and \( \mathcal{G}(Q_n) \) are not minimally edge connected. We then classify abelian \( p \)-groups whose power graphs are minimally edge connected.

2. Some preliminaries

In this section, we recall various definitions and results from the literature. Throughout the article, \( n \) denotes a positive integer and \( p \) denotes a prime number. A \( p \)-group is a finite group whose order is power of \( p \). The exponent of any
finite group $G$, denoted by $\exp(G)$, is the least common multiple of orders of all its elements. An abelian group $G$ having exponent $p$ is called an elementary abelian group or more specifically elementary abelian $p$-group. The additive group of integers modulo $n$ is denoted by $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

For any integer $n \geq 3$, the dihedral group $D_n$ is a finite group of order $2n$ having presentation
\[ D_n = \langle a, b \mid a^n = b^2 = e, ab = ba^{-1} \rangle, \tag{1} \]
where $e$ is the identity element of $D_n$.

**Remark 2.1.** The group $D_n$ satisfies the following properties.

(i) For any $0 \leq i < n$, $(a^i b)^2 = e$, so that $\langle a^i b \rangle = \{e, a^i b\}$.

(ii) $D_n = \langle a \rangle \cup \bigcup_{i=0}^{n-1} \langle a^i b \rangle$.

(iii) $\langle a \rangle \cap \langle a^i b \rangle = \{e\}$ for all $0 \leq i \leq n - 1$.

For any integer $n \geq 2$, the dicyclic group $Q_n$ is a finite group of order $4n$ having presentation
\[ Q_n = \langle a, b \mid a^{2n} = e, a^n = b^2, ab = ba^{-1} \rangle, \tag{2} \]
where $e$ is the identity element of $Q_n$. When $n$ is a power of 2, $Q_n$ is called a generalized quaternion group of order $4n$.

**Remark 2.2.** The group $Q_n$ satisfies the following properties.

(i) $(a^i b)^2 = a^n$ for all $0 \leq i \leq 2n - 1$.

(ii) For any $0 \leq i \leq n - 1$, $(a^i b)^3 = a^n a^{i} b = a^{n+i} b$ and $(a^{n+i} b)^3 = a^n a^{n+i} b = a^{i} b$.

Hence $\langle a^i b \rangle = \langle a^{n+i} b \rangle = \{e, a^{i} b, a^n, a^{n+i} b\}$ for all $0 \leq i \leq n - 1$.

(iii) Any element of $Q_n - \langle a \rangle$ can be written as $a^i b$ for some $0 \leq i \leq 2n - 1$.

(iv) $Q_n = \langle a \rangle \cup \bigcup_{i=0}^{n-1} \langle a^i b \rangle$.

(v) $\langle a \rangle \cap \langle a^i b \rangle = \{e, a^n\}$ for all $0 \leq i \leq 2n - 1$.

Throughout the article, we follow (1) and (2) for presentation of dihedral and dicyclic groups, respectively.

Given a group $G$, the cyclic subgroup generated by $x \in G$ is denoted by $\langle x \rangle$. We define a relation $\approx$ on $G$ as $x \approx y$ if $\langle x \rangle = \langle y \rangle$. Notice that it is an equivalence relation. For $x \in G$, we denote the equivalence class of $x$ under $\approx$ by $[x]$.

**Theorem 2.3** ([6, Theorem 11.1]). Every finite abelian group is a direct product of cyclic groups of prime power order. Moreover, the number of groups in the product and the orders of the cyclic groups are uniquely determined by the group.

**Remark 2.4.** If a group $G$ has exponent 2, it can be observed that $G$ is abelian. Hence by **Theorem 2.3**, $G$ is a direct product of finite copies of $\mathbb{Z}_2$, i.e., $G$ is an elementary abelian 2-group.

The subgraph of a graph $\Gamma$ obtained by deleting a vertex set $U$ is denoted by $\Gamma - U$. For a vertex $v$ of $\Gamma$, the subgraph $\Gamma - \{u\}$ is simply denoted as $\Gamma - u$. The degree of vertex $v$ in $\Gamma$ is denoted by $\deg_{\Gamma}(v)$ or simply $\deg(v)$. The minimum degree of $\Gamma$, that is the minimum of degrees of all vertices of $\Gamma$, is denoted by $\delta(\Gamma)$.

For any group $G$, if $H$ is a subgroup of $G$, then $G(H)$ is an induced subgraph of $G(G)$ (cf. [2, Proposition 4.5]). In fact, for any $A \subseteq G$, we denote the subgraph of $G(G)$ induced by $A$ as $G(A)$. The subgraph $G^*(G)$ is obtained by removing the
identity vertex from $G$. We denote the set of identity element and generators of $\mathbb{Z}_n$ by $S(\mathbb{Z}_n)$. Further, we denote $\mathbb{Z}_n' = \mathbb{Z}_n - S(\mathbb{Z}_n)$ and $G'(\mathbb{Z}_n) = G(\mathbb{Z}_n) - S(\mathbb{Z}_n)$.

Remark 2.5. For any $n \in \mathbb{N}$, each element of $S(\mathbb{Z}_n)$ is adjacent to every other element of $G(\mathbb{Z}_n)$.

**Theorem 2.6** ([2]). Let $G$ be a finite group.

(i) The power graph $G$ is connected.

(ii) The power graph $G$ is complete if and only if $G$ is a cyclic group of order 1 or $p^a$, for some prime $p$ and $a \in \mathbb{N}$.

(iii) If $H$ is a subgroup of $G$, then $G(H)$ is an induced subgraph of $G(\mathbb{G})$.

The following results on equality of edge connectivity and minimum degree

**Theorem 2.7** ([19]). If $\Gamma$ is a graph with $\text{diam}(\Gamma) \leq 2$, then $\kappa'(\Gamma) = \delta(\Gamma)$.

**Theorem 2.8** ([18, Theorem 3.2]). If $G$ is a finite group, then $\kappa'(G) = \delta(G)$.

We recall the following results on minimum degree of power graphs of dihedral and dicyclic group.

**Theorem 2.9** ([18, Theorem 5.3]). For $n \geq 3$, $\delta(G(D_n)) = \deg(a^ib) = 1$ for all $0 \leq i \leq n - 1$.

**Theorem 2.10** ([18, Theorem 5.4]). For $n \geq 2$, $\delta(G(Q_n)) = \deg(a^ib) = 3$ for all $0 \leq i \leq 2n - 1$.

We require the following results on power graphs of $p$-groups.

**Lemma 2.11** ([15]). For any $p$-group $G$, $G \cong K_1 \vee nK_{p-1}$ for some $n$ if and only if $\exp(G) = p$.

**Lemma 2.12** ([16, Corollary 4.1]). Let $G$ be a $p$-group. Then $G^*(G)$ is connected if and only if $G$ is cyclic or generalized quaternion.

3. Critical connectivity

In this section, we examine whether (or not) power graphs of finite groups are critically vertex (edge) connected.

3.1. Vertex connectivity.

In this subsection, we show that $G(\mathbb{Z}_n)$ is not critically vertex connected when $n$ is a product of two or three distinct primes. We then show that $G(D_n)$ and $G(Q_n)$ are not critically vertex connected. Finally, we classify $p$-groups whose power graphs are critically vertex connected.

**Theorem 3.1.** If $G$ is a finite group such that $G(G)$ is critically vertex connected, then $\kappa(G(G)) > 1$.

**Proof.** Since $e$ is adjacent to all other vertices in $G(G)$, $\kappa(G(G) - g) \geq 1$ for all $g \in G^*$. Thus, if $\kappa(G(G)) = 1$, then $G(G)$ is not critically vertex connected. Hence the proof follows. □

**Theorem 3.2.** If $n$ is a product of two or three distinct primes, $G(\mathbb{Z}_n)$ is not critically vertex connected.
Proof. Let \( n = pq \), where \( p \) and \( q \) are distinct primes. Then by [4, Theorem 3(ii)], 
\[ \kappa(G(Z_n)) = \phi(n) + 1 \]
Thus in view of Remark 2.5, we have \( \kappa(G(Z_n) - \pi) = \phi(n) + 1 \) for every \( \pi \in \mathbb{Z}_n^\ast \). Consequently, \( G(Z_{pq}) \) is not critically vertex connected.

Now let \( n = pqr \) for primes \( p < q < r \). Let \( P = \mathbb{Z}_n^\ast - \pi \) and \( \approx \) be the restriction of \( \approx \) to \( P \). Then the equivalence classes of \( P \) under \( \approx \) are given by
\[
[p], \{p\}, [\pi], [\pi], [\pi r], [\pi r] \text{ and } [\pi r].
\] (3)

Observe that a minimal separating set of \( G(P) \) is precisely the union of any two \( \approx \)-classes \( C_1 \) and \( C_2 \) from (3) such that no element of \( C_1 \) is adjacent to any element of \( C_2 \) in \( G(P) \). Comparing the cardinalities of \( \approx \)-classes in (3), \( |[\pi]| \cup |[\pi]| \) is a minimum separating set of \( G(P) \). Moreover, it was shown in [18, Theorem 2.40] that \( |[\pi]| \cup |[\pi]| \) is also a minimum separating set of \( G'(Z_n) \). Hence \( \kappa(G'(Z_n) - \pi) = \kappa(G'(\D_n)) \). As a result, by Remark 2.5, \( G(Z_n) \) is not critically vertex connected. \( \square \)

**Theorem 3.3.** For any integer \( n \geq 3 \), \( G(D_n) \) is not critically vertex connected.

**Proof.** It was shown in [4, Theorem 5] that \( \kappa(G(D_n)) = 1 \). This together with Theorem 3.1 give the required proof. \( \square \)

**Theorem 3.4.** For any integer \( n \geq 2 \), \( G(Q_n) \) is not critically vertex connected.

**Proof.** Let \( \Gamma_b = G(Q_n) - b \). We deduce from Remark 2.2 that \( \Gamma_b - g \) is connected for all \( g \in G - b \), whereas \( \Gamma_b - \{e, a^n\} \) is disconnected. Thus \( \kappa(\Gamma_b) = 2 \). However, it was shown in [4, Theorem 7] that \( \kappa(G(Q_n)) = 2 \). Hence \( G(Q_n) \) is not critically vertex connected. \( \square \)

**Theorem 3.5.** If \( G \) is a p-group, then \( G(G) \) is critically vertex connected if and only if \( G \) is cyclic.

**Proof.** Let \( G(G) \) be critically vertex connected. Then by Theorem 3.1, \( G^\ast(G) \) is connected. So either \( G \) is cyclic or generalized quaternion (cf. Lemma 2.12). Since it follows from Theorem 3.4 that \( G \) is not generalized quaternion, \( G \) is cyclic. Conversely, if \( G \) is cyclic then \( G(G) \) is complete (cf. Theorem 2.6(ii)), so that it is critically vertex connected. \( \square \)

**Corollary 3.6.** If \( n \in \mathbb{N} \) is a prime power, then \( G(Z_n) \) is critically vertex connected.

3.2. Edge connectivity.

In this subsection, we first supply a characterization of power graphs of finite groups that are critically edge connected involving minimum degree. We ascertain that \( G(D_n) \) is not critically edge connected. Then we classify \( Q_n \) and p-groups whose power graphs are critically edge connected.

**Theorem 3.7.** Let \( G \) be a finite group. Then \( G(G) \) is critically edge connected if and only if for any \( g \in G \), \( \deg(g) = \delta(G(G)) \) or \( g \) is adjacent to some \( h \in G \) with \( \deg(h) = \delta(G(G)) \).

**Proof.** Let \( G(G) \) be critically edge connected. Suppose \( g \in G \) is not adjacent to any \( h \in G \) with \( \deg(h) = \delta(G(G)) \). Then \( g \neq e \). If \( \deg(g) > \delta(G(G)) \), then using Theorem 2.7 and Theorem 2.8, we get \( \kappa'(G(G)) = \kappa'(G(G) - g) \). This is a contradiction. Hence \( \deg(g) = \delta(G(G)) \).
For converse, suppose $G(G)$ is not critically edge connected. Then by Theorem 2.7 and Theorem 2.8, there exists $g \in G^*$ such that $\delta(G(G)) = \delta(G(G) - g)$. Consequently, $\deg(g) > \delta(G(G))$ and $g$ is not adjacent to any $h \in G$ with $\deg(h) = \delta(G(G))$. The concludes the proof of the theorem.

**Theorem 3.8.** If $G$ is a finite group and $G(G)$ is critically edge connected, then $\kappa_1(G(G)) > 1$.

**Proof.** If possible, let $\kappa'(G(G)) = 1$. Suppose that there exists $g \in G$ such that $o(g) > 2$. Then $|[g]| > 2$ and, as a result, $G(G) - g$ is connected. Consequently, $G(G)$ is not critically edge connected. Now suppose $o(g) = 2$ for all $g \in G^*$. Then it follows from Lemma 2.11 that $G(G)$ remains connected even after deletion of any $g \in G^*$. This again implies that $G(G)$ is not critically edge connected. □

**Theorem 3.9.** For any integer $n \geq 3$, $G(D_n)$ is not critically edge connected.

**Proof.** Theorem 2.9 and Theorem 3.8 together give the required proof. □

**Theorem 3.10.** For any integer $n \geq 2$, $G(Q_n)$ is critically edge connected if and only if $n = 2$.

**Proof.** By Theorem 2.8 and Theorem 2.10, $\kappa'(G(Q_n)) = 3$. First let $n \geq 3$. For any $g \in \langle a \rangle$, if $o(g) \geq 3$, then $|[g]| > 2$. Then if $g \in \langle a \rangle$, $\deg(g) \geq 5$, otherwise $\deg(g) > \phi(2n) + 1 \geq 3$. On the other hand, if $o(g)$ is 1 or 2, then $g$ is $e$ or $a^p$, respectively. Hence $\deg(g) > 3$ for all $g \in \langle a \rangle$. Now let $h \in Q_n - \langle a \rangle$. By Remark 2.2, $a$ is not adjacent to $h$. Moreover, by following the proof of Theorem 2.10 and Remark 2.2, we have $\delta(G(Q_n)) = \deg(h) = 3$. Consequently, applying Theorem 3.7 we conclude that $G(Q_n)$ is not critically edge connected.

![Figure 1. $G(Q_2)$](image)

Now let $n = 2$. Following Remark 2.2, $G(Q_2)$ is depicted in Figure 1. Observe that $e$ and $a^2$ are adjacent to all other vertices of $G(Q_2)$. Hence for any $g \in Q_2$, we have $\text{diam}(G(Q_2) - g) = 2$ and $\delta(G(Q_2) - g) = 2$. Consequently, by Theorem 2.7, $G(Q_2)$ is critically edge connected. This proves the theorem. □
Theorem 3.11. If $G$ is a $p$-group, then $\mathcal{G}(G)$ is critically edge connected if and only if $G$ is cyclic or $G = Q_2$.

Proof. Let $\mathcal{G}(G)$ be critically edge connected. If possible, let $\mathcal{G}^*(G)$ be disconnected. That is, $\kappa'(\mathcal{G}(G) - e) = 0$. Thus as $\mathcal{G}(G)$ is critically edge connected, we have $\kappa'(\mathcal{G}(G)) = 1$. Since this contradicts Theorem 3.8, $\mathcal{G}^*(G)$ is connected. Hence by Lemma 2.12 and Theorem 3.10, $G$ is cyclic or $G = Q_2$.

Conversely, if $G$ is cyclic, then $\mathcal{G}(G)$ is complete (cf. Theorem 2.6(ii)). Hence $\mathcal{G}(G)$ be critically edge connected. Whereas, it follows from Theorem 3.10 that $\mathcal{G}(Q_2)$ is critically edge connected. □

An immediate corollary of Theorem 3.11 is the following.

Corollary 3.12. If $n \in \mathbb{N}$ is a prime power, then $\mathcal{G}(\mathbb{Z}_n)$ is critically edge connected.

4. Minimal connectivity

In this section, we examine whether (or not) power graphs of finite groups are minimally vertex (edge) connected.

4.1. Vertex connectivity.

In this subsection, we first present some results on effect of deletion of an edge on the vertex connectivity of a graph. Subsequently, we give some necessary conditions for minimally vertex connected graphs. We then apply these results to classify finite groups whose power graphs are minimally vertex connected.

Lemma 4.1 ([7]). For any minimally vertex connected graph, the vertex connectivity and minimum degree are equal.

Lemma 4.2. Let $\Gamma$ be a graph with an edge $\varepsilon$ such that $\Gamma - \varepsilon$ is connected. If $\kappa(\Gamma - \varepsilon) = \kappa(\Gamma) - 1$, then no minimum separating sets of $\Gamma - \varepsilon$ contains endpoints of $\varepsilon$.

Proof. Suppose $S$ is a minimum separating set of $\Gamma - \varepsilon$ and $\varepsilon$ is incident to $u$ and $v$. If possible, let $S$ contain at least one of $u$ or $v$. Then we get $(\Gamma - \varepsilon) - S = \Gamma - S$. So $S$ is a separating set of $\Gamma$. Thus $\kappa(\Gamma) \leq \kappa(\Gamma - \varepsilon)$, which contradicts the given condition. Hence no minimum separating set of $\Gamma - \varepsilon$ contains $u$ and $v$. □

The next result follows immediately from Lemma 4.2.

Lemma 4.3. If $\Gamma$ is a minimally vertex connected graph, then for any edge $\varepsilon$, no minimum separating set of $\Gamma - \varepsilon$ contains endpoints of $\varepsilon$.

Theorem 4.4. Let $\Gamma$ be a non-complete graph with $n$ vertices. If $\Gamma$ is minimally vertex connected, then it has at most one vertex of degree $n - 1$.

Proof. First observe that $n \geq 3$. If possible, let $\Gamma$ have two vertices of degree $n - 1$, say $u$ and $v$. Then $u$ and $v$ are adjacent. Let $\Gamma_\varepsilon = \Gamma - \varepsilon$, where $\varepsilon$ is the edge incident to $u$ and $v$, and $S$ be a minimum separating set of $\Gamma_\varepsilon$. From the fact that $\Gamma$ is non-complete and is minimally vertex connected, $\kappa(\Gamma_\varepsilon) < \kappa(\Gamma) \leq n - 2$. Thus $\Gamma_\varepsilon - S$ has at least one vertex not equal to $u$ and $v$. As both $u$ and $v$ have degree $n - 1$ in $\Gamma$, they are adjacent to every vertex in $\Gamma_\varepsilon - S$. This implies that $\Gamma_\varepsilon - S$ is connected; a contradiction. Consequently, the proof follows. □
Theorem 4.5. Let $\Gamma$ be a non-complete graph with an edge $\varepsilon$ such that $\kappa(\Gamma - \varepsilon) = \kappa(\Gamma) - 1$. If $\Gamma - \varepsilon$ is connected and $S$ is a minimum separating set of $\Gamma - \varepsilon$, then $S \cup \{u\}$ or $S \cup \{v\}$ is a minimum separating set of $\Gamma$, where $u$ and $v$ are endpoints of $\varepsilon$.

Proof. Let $\Gamma$ be a graph on $n$ vertices. Since $\Gamma$ is non-complete, $\kappa(\Gamma) \leq n - 2$, and hence $\kappa(\Gamma - \varepsilon) \leq n - 3$. Let $S$ be a minimum separating set of $\Gamma - \varepsilon$. Then $\Gamma - S$ is a connected graph with at least three vertices, and by Lemma 4.2, $u, v \notin S$. Hence in $\Gamma - S$ every vertex $w, w \neq u, v$, is connected by a path to exactly one of $u$ or $v$ that does contain edge $\varepsilon$. Hence $S \cup \{u\}$ or $S \cup \{v\}$ is a separating set of $\Gamma$. Since $\kappa(\Gamma - \varepsilon) = \kappa(\Gamma) - 1$, the proof follows. \qed

Theorem 4.6 ([17, Theorem 2.16]). If $T$ is a minimal separating set of $\mathcal{G}(G)$, then $T$ is a union of $\approx$-classes.

Proposition 4.7. Let $G$ be a finite group with distinct elements $g$ and $h$ such that $\langle g \rangle = \langle h \rangle$. If $\varepsilon$ is the edge incident to $g$ and $h$ in $\mathcal{G}(G)$, then $\kappa(\mathcal{G}(G) - \varepsilon) = \kappa(\mathcal{G}(G))$.

Proof. If possible, let $\kappa(\mathcal{G}(G) - \varepsilon) \neq \kappa(\mathcal{G}(G))$. Note that $\mathcal{G}(G) - \varepsilon$ is connected. Let $S$ be a minimum separating set of $\mathcal{G}(G) - \varepsilon$. Then it follows from Theorem 4.5 that $S \cup \{g\}$ or $S \cup \{h\}$ is a minimum separating set of $\mathcal{G}(G)$. By Theorem 4.6, $S \cup \{g\}$ is a union of $\approx$-classes. This implies $[g] \subset S \cup \{g\}$, so that $h \in S$. However, it follows from Lemma 4.2 that $h \notin S$. Hence $\kappa(\mathcal{G}(G) - \varepsilon) = \kappa(\mathcal{G}(G))$. \qed

We now apply the results obtained above to classify power graphs of finite groups that are minimally vertex connected.

Theorem 4.8. For a finite group $G$, $\mathcal{G}(G)$ is minimally vertex connected if and only if $G$ is a cyclic group of prime power order or $G$ is an elementary abelian 2-group.

Proof. Let $\mathcal{G}(G)$ be minimally vertex connected. If $\mathcal{G}(G)$ is complete, then $G$ a cyclic group of prime power order (cf. Theorem 2.6(ii)). Now, suppose $\mathcal{G}(G)$ is non-complete. Let $g \in G^*$. By Proposition 4.7, if there exists $h \in [g] - g$, then $\kappa(\mathcal{G}(G) - \varepsilon) = \kappa(\mathcal{G}(G))$, where $\varepsilon$ is the edge incident to $g$ and $h$. However, this contradicts the fact that $\mathcal{G}(G)$ is minimally vertex connected. Thus $[g] = \{g\}$, and since $[g] = \phi(o(g))$, we have $o(g) = 2$. Hence in light of Remark 2.4, $G$ is an elementary abelian 2-group.

Conversely, if $G$ a cyclic group of prime power order, then $\mathcal{G}(G)$ is complete (cf. Theorem 2.6(ii)). Whereas, if $G$ is an elementary abelian 2-group, then it follows from Lemma 2.11 that $\kappa(\mathcal{G}(G)) = 1$ and $\mathcal{G}(G) - \varepsilon$ is disconnected for every edge $\varepsilon$ in $\mathcal{G}(G)$. Consequently, $\mathcal{G}(G)$ is minimally vertex connected. \qed

The following are some direct consequences of Theorem 4.8.

Corollary 4.9. For $n \geq 2$, $\mathcal{G}(\mathbb{Z}_n)$ is minimally vertex connected if and only if $n$ is a prime power.

Corollary 4.10. For any integer $n \geq 3$, $\mathcal{G}(D_n)$ is not critically vertex connected.

Corollary 4.11. For any integer $n \geq 2$, $\mathcal{G}(Q_n)$ is not critically vertex connected.

Corollary 4.12. For any $p$-group $G$, $\mathcal{G}(G)$ is minimally vertex connected if and only if $G$ is a cyclic or an elementary abelian 2-group.
4.2. Edge connectivity.

In this subsection, for power graph of a finite group that is minimally edge connected, we give a sufficient condition involving regularity. We then show that the converse holds for finite groups of odd order. Applying this, we classify finite groups of odd order whose power graphs are minimally edge connected. We then study this notion for \( G(D_n) \) and \( G(Q_n) \). Finally, we classify abelian \( p \)-groups whose power graphs are minimally edge connected.

**Theorem 4.13.** Let \( G \) be a finite group. Then \( \mathcal{G}(G) \) is minimally edge connected and \( \kappa'(\mathcal{G}(G)) = 1 \) if and only if \( G \) is an elementary abelian 2-group.

**Proof.** Suppose \( \mathcal{G}(G) \) is minimally edge connected and \( \kappa'(\mathcal{G}(G)) = 1 \). If possible, let there exist \( g \in G \) such that \( o(g) > 2 \) so that \( |[g]| \geq 2 \). Let \( \varepsilon \) be an edge with endpoints in \([g]\). Then \( \mathcal{G}(G) - \varepsilon \) is connected, contradicting the fact that \( \kappa'(\mathcal{G}(G)) = 1 \). Hence \( o(g) = 2 \) for all \( g \in G^{*} \). Consequently, by Remark 2.4, \( G \) is an elementary abelian 2-group.

For converse, let \( G \) be an elementary abelian 2-group. Then it follows from Lemma 2.11 that deleting any edge of \( \mathcal{G}(G) \) makes it disconnected. Hence \( \kappa'(\mathcal{G}(G)) = 1 \) and \( \mathcal{G}(G) \) is minimally edge connected. \( \Box \)

**Theorem 4.14.** Let \( G \) be a finite group. If \( \mathcal{G}^{*}(G) \) is regular, then \( \mathcal{G}(G) \) is minimally edge connected. The converse holds if \( G \) is of odd order.

**Proof.** Let \( \mathcal{G}^{*}(G) \) be regular. We set \( \kappa'(\mathcal{G}(G)) = k \). In view of Theorem 2.8, all non-identity elements have degree \( k \) in \( \mathcal{G}(G) \). Let \( \varepsilon \) be an edge in \( \mathcal{G}(G) \). Trivially \( \kappa'(\mathcal{G}(G) - \varepsilon) \geq k - 1 \). At least one endpoint of \( \varepsilon \) is a non-identity element, say \( g \). Since deletion all edges incident to \( g \) in \( \mathcal{G}(G) - \varepsilon \) (if any) will make it disconnected, we have \( \kappa'(\mathcal{G}(G) - \varepsilon) \leq k - 1 \). As a result, \( \kappa'(\mathcal{G}(G) - \varepsilon) = k - 1 \), so that \( \mathcal{G}(G) \) is minimally edge connected.

For converse, suppose \( G \) is of odd order and \( \mathcal{G}^{*}(G) \) is not regular. Then there exists \( g \in G \), \( g \neq e \) with \( \deg_{\mathcal{G}(G)}(g) > \delta(\mathcal{G}(G)) \). Moreover, \(|[g]| \geq 2\), so that there exists \( h \in [g] \), \( h \neq g \). Note that \( \deg_{\mathcal{G}^{*}(G)}(g) = \deg_{\mathcal{G}(G)}(h) \) (cf. [18, Lemma 4.1]). Let \( \varepsilon \) be the edge incident to \( g \) and \( h \) and \( \Gamma_{\varepsilon} := \mathcal{G}(G) - \varepsilon \). Then \( \deg_{\Gamma_{\varepsilon}}(g) = \deg_{\mathcal{G}(G)}(g) - 1 \) and \( \deg_{\Gamma_{\varepsilon}}(g_{1}) = \deg_{\mathcal{G}(G)}(g_{1}) \) for any \( g_{1} \in G - \{g, h\} \). Hence, as \( \deg_{\mathcal{G}(G)}(g) > \delta(\mathcal{G}(G)) \), we have \( \delta(\Gamma_{\varepsilon}) = \delta(\mathcal{G}(G)) \). Moreover, since \( \text{diam}(\Gamma_{\varepsilon}) \leq 2 \), we have \( \kappa'(\Gamma_{\varepsilon}) = \kappa'(\mathcal{G}(G)) \) (cf. Theorem 2.7). These facts along with Theorem 2.8 yield \( \kappa'(\Gamma_{\varepsilon}) = \kappa'(\mathcal{G}(G)) \). Hence \( \mathcal{G}(G) \) is not minimally edge connected. \( \Box \)

**Theorem 4.15** ([5, Theorem 4]). Let \( G \) be a finite group. Then \( \mathcal{G}^{*}(G) \) is regular if and only if \( G \) is cyclic \( p \)-group or \( \exp(G) = p \).

By applying Theorem 4.14 and Theorem 4.15, we have the following theorem.

**Theorem 4.16.** Let \( G \) be a finite group of odd order. Then \( \mathcal{G}(G) \) is minimally edge connected if and only if \( G \) is a cyclic group of prime power order or \( \exp(G) \) is a prime.

The following are two immediate corollaries of Theorem 4.16.

**Corollary 4.17.** For any odd integer \( n > 0 \), \( \mathcal{G}(\mathbb{Z}_{n}) \) is minimally edge connected if and only if \( n \) is a prime power.

**Corollary 4.18.** Let \( G \) be a \( p \)-group and \( p \geq 3 \). Then \( \mathcal{G}(G) \) is minimally edge connected if and only if \( G \) is cyclic or \( \exp(G) = p \).
We next show that power graphs of $D_n$ and $Q_n$ are not minimally edge connected.

**Theorem 4.19.** For any integer $n \geq 3$, $G(D_n)$ is not minimally edge connected.

**Proof.** Since $\exp(D_n) = n$, the proof follows from Theorem 2.9 and Theorem 4.13. □

**Theorem 4.20.** For any integer $n \geq 2$, $G(Q_n)$ is not minimally edge connected.

**Proof.** In view of Theorem 2.8 and Theorem 10, we note that $\kappa'(G(Q_n)) = 3$.

First let $n \geq 3$. Suppose $\varepsilon_1$ is an edge with endpoints in $[a]$. We denote $\Gamma_1 = G(Q_n) - \varepsilon_1$. Then $\deg_{\Gamma_1}(g) \geq 2n - 2 \geq 4$ for all $g \in [a]$. Whereas, all other vertices of $\Gamma_1$ and $G(Q_n)$ have same degree. In particular, $\deg_{\Gamma_1}(h) = \deg_{G(Q_n)}(h) = 3$ for all $h \in Q_n - \langle a \rangle$. In fact, by following the proof of Theorem 2.10, we have $\delta(\Gamma_1) = 3$.

Hence, as $\diam(\Gamma_1) = 2$, we have $\kappa'(\Gamma_1) = 3$ (cf. Theorem 2.7). We thus conclude that $G(Q_n)$ is not minimally edge connected.

Now let $n = 2$. Suppose $\varepsilon_2$ is the edge incident to $e$ and $a^2$. We denote $\Gamma_2 = G(Q_2) - \varepsilon_2$. In view of Figure 1, $\delta(\Gamma_2) = 3$ and $\diam(\Gamma_2) = 2$. Hence application Theorem 2.7 yields $\kappa'(\Gamma_2) = 3$. Consequently, $G(Q_2)$ is not minimally edge connected. □

**Lemma 4.21** ([17, Proposition 3.2]). For any $p$-group $G$, each component of $G^*(G)$ has exactly $p - 1$ elements of order $p$.

**Lemma 4.22.** Consider positive integers $r$ and $a_i$ for all $1 \leq i \leq r$. For any $1 \leq i \leq r$, if $u_i \in \mathbb{Z}_{p^a_1} \times \mathbb{Z}_{p^a_2} \times \ldots \times \mathbb{Z}_{p^a_r}$ with $i^{th}$ co-ordinate $\overline{\mathbf{u}}$ and all other co-ordinates are $\overline{0}$, then $\deg(u_i) = p^{a_i} - 1$.

**Proof.** Note that $o(u_i) = p^{a_i}$. If possible, let there exist $v$ adjacent to $u_i$ and $v \notin \langle u_i \rangle$. Then $u_i \in \langle v \rangle$, so that $u_i = cp^k v$ for some $k \in \mathbb{N}$ and integer $c$, $(c, p) = 1$. Now comparing the $i^{th}$ co-ordinates, $p | (cp^k a - 1)$, where $\overline{a}$ is the $i^{th}$ co-ordinate of $v$. This implies that $p | 1$, which is not possible. Hence we have $\deg(u_i) = |\langle u_i \rangle| - 1 = p^{a_i} - 1$. □

The following theorem gives the minimum degree of power graphs of abelian $p$-groups.

**Lemma 4.23** ([18, Theorem 5.1]). Let $G$ be a group isomorphic to direct product of finitely many cyclic $p$-groups. If the smallest group in this product has order $p^a$, then $\delta(G(G)) = p^a - 1$.

**Proposition 4.24.** Let $G$ be a finite group isomorphic to direct product of finite copies of $\mathbb{Z}_4$. Then $G(G)$ is minimally edge connected if and only if $G$ is isomorphic to $\mathbb{Z}_4$.

**Proof.** Let $G$ be isomorphic to $H := \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ ($t$ times) for some integer $t \geq 2$. We show that $G(H)$ is not minimally edge connected. We denote $\mathbf{0} = (\overline{0}, \ldots, \overline{0})$. It is known that order of an element in a direct product of finite groups is the least common multiple of orders each of its co-ordinates (cf. [6, Theorem 8.1]). Hence all non-identity elements in $H$ have order either two or four. Moreover, by Lemma 4.21, no two elements of order two are adjacent in $G(H)$.

We first claim that if $w$ is an element of order two in $H$, then $w$ is adjacent to exactly $2t$ elements of order four in $G(H)$. Clearly every co-ordinate of $w$ is either $\overline{0}$ or $\overline{1}$. We construct an element $z \in H$ as follows. If the $i^{th}$ co-ordinate of $w$ is $\overline{0}$, set
Let \( G \) be an abelian \( p \)-group. Then \( G(G) \) is minimally edge connected if and only if \( G \) is a cyclic or \( \exp(G) = p \).

**Proof.** Let \( G(G) \) be minimally edge connected. If \( p \geq 3 \), then it follows from **corollary 4.18** that \( G \) is a cyclic or \( \exp(G) = p \).

Now suppose \( p = 2 \) and \( G \) is not cyclic. In view of **Theorem 2.3**, \( G \) is isomorphic to \( H := \mathbb{Z}_{2^\alpha_1} \times \mathbb{Z}_{2^\alpha_2} \times \ldots \times \mathbb{Z}_{2^\alpha_t} \) for some positive integers \( t \geq 2 \) and \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_t \). Since isomorphic groups have isomorphic power graphs, \( G(H) \) is minimally edge connected. Further, it is enough to show that \( \exp(H) = 2 \).

For every \( 1 \leq i \leq t \), consider \( z_i \in H \) with \( i^{th} \) co-ordinate \( \overline{1} \) and all other co-ordinates are \( \overline{0} \). Then by **Lemma 4.22**, \( \deg_H(z_i) = 2^{\alpha_i} \) for all \( 1 \leq i \leq r \). Moreover, by **Lemma 4.23**, \( \delta(G(H)) = 2^{\alpha_i} \).

If possible, let \( \alpha_i > \alpha_1 \). Then \( o(z_i) \geq 4 \). Let \( \varepsilon \) be an edge with endpoints in \( [z_i] \) and \( \Gamma_\varepsilon := G(H) - \varepsilon \). Then \( \deg_{\Gamma_\varepsilon}(u) \geq \deg_{G(H)}(u) - 1 \) for all \( u \in [z_i] \) and \( \deg_{\Gamma_\varepsilon}(v) = \deg_{G(H)}(v) \) for all \( v \in H - [z_i] \). Hence we get \( \delta(G(H)) = \delta(\Gamma_\varepsilon) \).

Since \( \text{diam}(\Gamma_\varepsilon) \leq 2 \), in view of **Theorem 2.7** and **Theorem 2.8**, we have \( \kappa'(G(H)) = \kappa'(\Gamma_\varepsilon) \). This contradicts the fact that \( G(H) \) is minimally edge connected. So we have \( \alpha_1 = \alpha_2 = \ldots = \alpha_t \) (= \( \alpha \) say).
Let take \( \alpha \geq 2 \). Since \( \langle z_1 \rangle \) is a clique in \( G(H) \), \( p z_1 \) is adjacent to all other vertices in \( \langle z_1 \rangle \). Consider an element
\[
w = (a_1 p^{\alpha_1 - 1} + 1, a_2 p^{\alpha_2 - 1}, \ldots, a_t p^{\alpha_t - 1}),
\]
where \( 0 \leq a_i \leq 2 - 1 \) for all \( 1 \leq i \leq t \). We notice that \( w \) is adjacent to \( p z_1 = (x, y, \ldots, z) \), but if \( a_i \neq 0 \) for any \( 2 \leq i \leq t \), then \( w \notin \langle z_1 \rangle \). Additionally, from the proof of Lemma 4.22, \( \text{deg}(z_1) = |\langle z_1 \rangle| - 1 \). Hence
\[
\text{deg}_{G(H)}(g) > \text{deg}_{G(H)}(z_1)
\]
for all \( g \in [p z_1] \). We now have the following cases.

Case 1: \( \alpha \geq 3 \). By an argument similar to that of (4), we have
\[
\text{deg}_{G(H)}(4z_1) > \text{deg}_{G(H)}(z_1).
\]
Let \( \varepsilon_1 \) be the edge between \( p z_1 \) and \( 4z_1 \). Then by (4) and (5), \( \delta(G(H)) = \delta(G(H) - \varepsilon_1) \). Since \( \text{diam}(G(H) - \varepsilon_1) \leq 2 \), it follows from Theorem 2.7 and Theorem 2.8 that \( G(H) \) is not minimally edge connected.

Case 2: \( \alpha = 2 \). Then it follows from Proposition 4.24 that \( G(H) \) is not minimally edge connected.

We thus conclude that \( \alpha = 1 \). Hence \( \exp(G) = 2 \). Conversely, if \( G \) is cyclic or \( \exp(G) = p \), then by Lemma 2.11 and Theorem 4.14, \( G(H) \) is minimally edge connected. \( \Box \)

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