Zakharov-Ito equation and Generalized Heisenberg ferromagnet-type equation: equivalence and related geometric curve flows

Zhanbala Umbetova,* Shynaray Myrzakul†, Kuralay Yesmakhanova‡, Tolkynay Myrzakul§, Gulgassyl Nugmanova¶ and Ratbay Myrzakulov∥
Eurasian International Center for Theoretical Physics and Department of General & Theoretical Physics, Eurasian National University, Nur-Sultan, 010008, Kazakhstan

Abstract
These results continue our studies of integrable generalized Heisenberg ferromagnet-type equations (GHFE) and their equivalent counterparts. We consider the GHFE which is the spin equivalent of the Zakharov-Ito equation (ZIE). We have established that these equations are gauge and geometrical equivalent to each other. The integrable motion of the space curve induced by the ZIE is constructed. The 1-soliton solution of the GHFE is obtained from the seed solution of the ZIE.

1 Introduction
The famous Korteweg-de Vries (KdV) equation
\[ u_t + 6uu_x + u_{xxx} = 0 \] (1)
is the first integrable equation in soliton theory. At present, there are exist several integrable and nonintegrable generalizations of the KdV equation in 1+1 and 2+1 dimensions. For example, the Zakharov-Ito equation (ZIE)
\[ u_t + 6uu_x + u_{xxx} + 0.5\rho_{xx} = 0, \] (2)
\[ \rho_t + 2(u\rho)_x = 0 \] (3)
is one of such integrable generalizations of the KdV equation in 1+1 dimension. Another interesting subclass of integrable systems is the Heisenberg ferromagnet type equations. The pioneering example of this subclass is the Heisenberg ferromagnet equation (HFE)
\[ iA_t + \frac{1}{2}[A, A_{xx}] = 0, \] (4)

*Email: zumurzakhova@gmail.com
†Email: srmyrzakul@gmail.com
‡Email: kryesmakhanova@gmail.com
§Email: kryesmakhanova@gmail.com
¶Email: gnnugmanova@gmail.com
∥Email: rmyrzakulov@gmail.com
where
\[ A = \begin{pmatrix} A_3 & A_- \\ A_+ & -A_3 \end{pmatrix}, \quad A^\pm = A_1 \pm iA_2, \quad A^2 = I, \quad A = (A_1, A_2, A_3). \] (5)

It is well-known that the HFE (4) is gauge/geometrical equivalent to the non-linear Schrödinger equation (NLSE) [1]-[2]
\[ iq_t + q_{xx} + 2|q|^2q = 0, \] (6)
where \( q(x, t) \) is a complex function. The purpose of this paper is to find and study the GHFE which is gauge/geometrical equivalent counterpart of the ZIE.

This paper is organized as follows. In Section 2, we give the GHFE, its Lax representation (LR) and a reduction. Geometric formulation of the ZIE in terms of space curves in 3-dimensional Euclidean space \( \mathbb{R}^3 \) is presented in Section 3. Some detail informations of ZIE we give in Section 4. In Section 5, we have established the gauge equivalence between the ZIE and the GHFE. The 1-soliton solutions of the GHFE is presented in Section 6. Last section is devoted to the conclusion.

2 The Kuralay-I equation

There are several integrable and nonintegrable GHFE (see, e.g., [3]-[42]). One of the representatives of such GHFE is the Kuralay-I equation (K-IE). In this section, we present some main informations of the K-IE.

2.1 Equation

The Kuralay-I equation (K-IE) has the form
\[ A_t + A_{xxx} + (BA)_x + \frac{1}{8\beta^2}[(\rho^2)_x Z]_x - 8i\beta u_x Z = 0, \] (7)
\[ Z_t = v'_{21} A - 2v'_{11} Z = 0, \] (8)
where
\[ A = \begin{pmatrix} A_3 & A_- \\ A_+ & -A_3 \end{pmatrix}, \quad A^2 = I, \quad A = (A_1, A_2, A_3), \quad A^2 = 1, \] (9)
\[ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & -z_{11} \end{pmatrix}, \quad Z^2 = 0, \quad Z^2_x = I, \quad Z^2_t = v'_{21} I, \] (10)
\[ B = A_x^2 - (4\beta^2 - 2\nu)I + 2i\beta A_{xx}. \] (11)

Here
\[ A^2_x = 4(u - \frac{\beta^2}{16\beta^2})I, \quad Z^2_x = u'_{21} I, \] (12)
\[ u = 2\beta + 0.25tr(Z^2_t), \quad \rho^2 = 4\beta^2[8\beta + tr(Z^2_t) - tr(A^2_x)], \] (13)
\[ Z_x = u'_{21} A - 2u'_{11} Z, \] (14)
\[ v'_{11} = 2i\beta u - u_x, \] (15)
\[ v'_{21} = -4\beta^2 + 2u, \] (16)
\[ u'_{11} = -i\beta, \] (17)
\[ u'_{21} = -1. \] (18)
2.2 Lax representation

The K-IE (7)-(8) is integrable. Its LR is given by

\[ \Phi_x = U_1 \Phi, \quad (19) \]
\[ \Phi_t = V_1 \Phi. \quad (20) \]

Here

\[ U_1 = -i(\zeta - \beta)A - \frac{\rho^2}{16} (\zeta^{-2} - \beta^{-2})Z, \quad (21) \]
\[ V_1 = wA + (\zeta^2 - \beta^2) \{ A, A_x \} + \frac{\rho^2}{4\beta^2} Z + i(\zeta - \beta) \{ A_{xx}, A \} + \frac{(\rho^2)_x}{8\beta^2} Z + 
\]
\[ + 4[u - \frac{\rho^2}{16\beta^2}]A \} + \frac{u\rho^2}{8} (\zeta^{-2} - \beta^{-2})Z, \quad (22) \]

where

\[ w = -4i(\zeta^3 - \beta^3) + 2i(\zeta - \beta)u. \quad (23) \]

2.3 Reduction

Note that if \( \beta = 0 = \rho \), the K-IE equation admits the following reduction

\[ A_t + A_{xxx} + [(A_x^2 + 2uI)A]_x = 0 \quad (24) \]

or

\[ A_t + A_{xxx} + 1.5(A_x^2 A)_x = 0. \quad (25) \]

3 Integrable motion of space curves induced by the ZIE

In this section, we want to present the geometric formulation of the ZIE. To do this, let us consider a space curve in \( \mathbb{R}^3 \) with the position vector \( \gamma(x,t) \). Then \( \mathbf{v} = \frac{d\gamma}{dx} \) and \( \mathbf{a} = \frac{d\gamma}{dt} \) are the velocity and the acceleration. In this paper, we shall assume that \( x \) is the natural parameter, i.e. the length along the curve. Thus, \( x \) is the arc length of the curve at each time \( t \). In other words, the velocity has unit length: \( \mathbf{v}^2 = |\mathbf{v}|^2 = 1 \). Then the acceleration vector is orthogonal to the velocity vector \( \mathbf{v} \cdot \mathbf{a} = 0 \). The magnitude of the acceleration vector \( \kappa = |\mathbf{a}| \) is called the curvature of the curve. Let us introduce three vectors:

\[ \mathbf{e}_1 \equiv \mathbf{v}, \quad \mathbf{e}_2 = \frac{\mathbf{a}}{|\mathbf{a}|}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2. \]

From the differential geometry follows that such space curve \( \gamma(x,t) \) describes by the Frenet-Serret equation (FSE). In this paper, we consider the more general form of the FSE with two curvatures \( \kappa_1 \equiv \kappa \) and \( \kappa_2 \). Then the corresponding FSE and its temporal counterpart are given by

\[
\begin{pmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{pmatrix}_x = C
\begin{pmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{pmatrix},
\]
\[
\begin{pmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{pmatrix}_t = G
\begin{pmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{pmatrix},
\]

(26)
where \( e_1 = \gamma_x \) (the unit tangent vector), \( e_2 = \frac{\gamma_{xx}}{|\gamma_{xx}|} \) (principal normal vector) and \( e_3 = e_1 \wedge e_2 \) (the binormal vector), respectively. Here

\[
C = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}
\]

(27)

or

\[
C = -\tau L_1 + \kappa_2 L_2 - \kappa_1 L_3, \quad G = -\omega_1 L_1 + \omega_2 L_2 - \omega_3 L_3,
\]

(28)

where \( \tau, \kappa_1, \kappa_2 \) are the "torsion", "geodesic curvature" and "normal curvature" of the curve, respectively; \( \omega_j \) are some functions. The basis elements of \( so(3) \) are given by

\[
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(29)

They obey the commutation relations

\[
[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.
\]

(30)

The basis elements of \( su(2) \) algebra read as

\[
e_1 = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2i} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(31)

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(32)

are the Pauli matrices. These elements obey the commutation relations:

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.
\]

(33)

Note that the Pauli matrices obey the commutation relations:

\[
[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2
\]

(34)

or

\[
[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.
\]

(35)

The isomorphism between the Lie algebras \( su(2) \) and \( so(3) \) induced the following correspondence between their basis elements \( L_j \leftrightarrow e_j \). The compatibility condition of the equations (26) has the form

\[
C_t - G_x + [C, G] = 0.
\]

(36)

This matrix equation in terms of elements takes the form

\[
\kappa_1 - \omega_3 \omega_2 - \kappa_2 \omega_1 + \tau \omega_2 = 0, \quad (37)
\]

\[
\kappa_2 - \omega_2 \omega_1 + \kappa_1 \omega_1 - \tau \omega_3 = 0, \quad (38)
\]

\[
\tau - \omega_1 \omega_2 - \kappa_1 \omega_2 + \kappa_2 \omega_3 = 0. \quad (39)
\]
Let us assume that
\[ \kappa_1 = -2iu_{11}, \quad \kappa_2 = -(u_{12} - u_{21}), \quad \tau = -i(u_{12} + u_{21}), \]  
\[ \omega_1 = -i(v_{12} + v_{21}), \quad \omega_2 = -(v_{12} - v_{21}), \quad \omega_3 = -2iv_{11}, \]
where
\[ u_{11} = -i\zeta, \quad u_{12} = u - \frac{\rho^2}{16\zeta^2}, \quad u_{21} = -1, \]  
\[ v_{11} = -(4i\zeta^3 - 2i\zeta u + u_x), \]  
\[ v_{12} = 4\zeta^2u + 2i\zeta u_x - (u_{xx} + 2u^2 + 0.25\rho^2) + \frac{u\rho^2}{8\zeta^2}, \]  
\[ v_{21} = -(4\zeta^2 - 2u). \]
Then Eqs. (37)–(39) give us the following equations for \( u, \rho \):
\[ u_t + 6uu_x + u_{xxx} + 0.5\rho\rho_x = 0, \]  
\[ \rho_t + 2(u\rho)_x = 0. \]
It is nothing but the ZIE. Thus we have constructed the integrable motion of the space curve induced by the ZIE. Our next aim is to find the integrable spin system, or in other words, the integrable generalized Heisenberg ferromagnet type equation which is geometrical equivalent counterpart of the ZIE. To construct this spin systems, we introduce two vectors as
\[ \mathbf{e}_3 = (A_1, A_2, A_3) \equiv \mathbf{A}, \quad \mathbf{Z} = (z_1, z_2, z_3) = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3, \]
where \( \mathbf{e}_3 \) and \( \mathbf{Z} \) are the vector equivalents of the matrix functions \( A \) and \( Z \), respectively. Here \( p_j = p_j(x, t) \) are some functions to be determined and
\[ z_1 = 0.5(z_{21} + z_{12}), \quad z_2 = -0.5i(z_{21} - z_{12}), \quad z_3 = z_{11}. \]
In terms of the functions \( g_j \), the components of the vectors \( \mathbf{e}_3 \) and \( \mathbf{Z} \) take the forms
\[ A_1 = \frac{g_1g_2 + g_1g_2}{\Delta}, \quad A_2 = \frac{g_1g_2 - g_1g_2}{i\Delta}, \quad A_3 = \frac{|g_1|^2 - |g_2|^2}{\Delta}, \]  
\[ z_1 = \frac{g_1^2 - g_2^2}{2\Delta}, \quad z_2 = \frac{g_1^2 + g_2^2}{2\Delta}, \quad z_3 = \frac{g_1g_2}{\Delta}. \]
We now ready to write the following set of equations
\[ \mathbf{e}_{3t} = -\omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2, \]  
\[ \mathbf{Z}_t = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3, \]
where
\[ b_1 = p_{1t} - \omega_3p_2 - \omega_2p_3, \]  
\[ b_2 = p_{2t} + \omega_3p_1 - \omega_1p_3, \]  
\[ b_3 = p_{3t} + \omega_2p_1 + \omega_1p_2, \]
are some real functions. In fact, this set of equations (52)–(53) is the K-IE written in the slightly other (vector) form. This vector K-IE is equivalent to the matrix K-IE (4)–(5). Thus we have proved the Lakshmanan (geometrical) equivalence between the ZIE and the K-IE.
4 Zakharov-Ito equation

For our convenience, let us here one more present the ZIE (see, e.g., [9])

\[
\begin{align*}
    u_t + 6uu_x + u_{xxx} + 0.5\rho p_x &= 0, \\
    \rho_t + 2(u\rho)_x &= 0.
\end{align*}
\]

(55)

(56)

The ZIE is integrable. Its LR reads as

\[
\begin{align*}
    \psi_{2xx} &= -(\zeta^2 + u - \frac{\rho^2}{16\zeta^2})\psi_2, \\
    \psi_{2t} &= u_x\psi_2 + (4\zeta^2 - 2u)\psi_{2x}.
\end{align*}
\]

(57)

(58)

Note that as \( \rho = 0 \), the ZIE admits the following reduction

\[
    u_t + 6uu_x + u_{xxx} = 0,
\]

(59)

which is the famous KdV equation.

5 Gauge equivalence between the ZIE and the K-IE

In the section 3, we have proved the geometrical (Lakshmanan) equivalence between the K-IE and the ZIE. In this section we want to show that between the K-IE (7)-(8) and the ZIE (55)-(56) take place also the gauge equivalence. To prove this statement, let us consider the gauge transformation

\[
    \Psi = g\Phi,
\]

(60)

where

\[
    \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1 = i\lambda\psi_2 - \psi_{2x}, \quad g = \Psi|_{\lambda=\beta}.
\]

(61)

Then this transformation induces the new Lax pair \( U_3 - V_3 \) from known \( U_1 - V_1 \) (21)-(22) as

\[
    U_3 = gU_1g^{-1} + g_xg^{-1}, \quad V_3 = gV_1g^{-1} + g_xg^{-1}.
\]

(62)

where

\[
\begin{align*}
    U_3 &= \begin{pmatrix} -i\lambda & u - \frac{\rho^2}{16\lambda^2} \\ -1 & i\lambda \end{pmatrix} = -i\lambda\sigma_3 + Q, \\
    V_3 &= F_{-3}\lambda^3 + F_2\lambda^2 + F_1\lambda + F_0 + F_{-2}\lambda^{-2} = F - (4i\lambda^3 - 2i\lambda u + u_x)\sigma_3
\end{align*}
\]

(63)

(64)
Here
\[ F_3 = 4i\sigma_3, \quad F_2 = 4 \left( \begin{array}{cc} 0 & u \\ -1 & 0 \end{array} \right), \]
\[ F_1 = 2i u_3 + 2i \left( \begin{array}{cc} 0 & u_x \\ 0 & 0 \end{array} \right), \]
\[ F_0 = -u_x \sigma_3 + \left( \begin{array}{c} 0 \\ 2u \end{array} \right) - (u_{xx} + \frac{\rho^2}{4} + 2u^2), \]
\[ F_{-2} = \frac{u \rho^2 \Sigma}{8}, \quad \Sigma = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \]
\[ F = \left( \begin{array}{cc} 0 & 4\lambda^2 u + 2i \lambda u_x - (u_{xx} + \frac{\rho^2}{4} + 2u^2) + \frac{u \rho^2 \Sigma}{8} \\ -4\lambda^2 u - 2u \end{array} \right). \]

Note that
\[ V_3 = 4\lambda^2 U_3 + V_3' = 4\lambda^2 U_3 + F_2' \lambda^2 + F_1 \lambda + F_0' + F_{-2} \lambda^{-2}. \]

We now consider the linear equations
\[ \Psi_x = U_3 \Psi, \]
\[ \Psi_t = V_3 \Psi. \]

The compatibility condition of these linear equations
\[ U_{3t} - V_{3x} + [U_3, V_3] = 0 \]
gives the ZIE (55)-(56). In fact, we have
\[ \lambda^0 : \quad u_t + 6u u_x + u_{xxx} + 0.5 \rho \rho_x = 0, \]
\[ \lambda^{-2} : \quad \rho_t + 2(\rho u)_x = 0, \]
\[ \lambda^j : \quad 0 = 0 \quad (j = 1, 2, 3). \]

In our case, the matrices \( A \) and \( Z \) have the following forms
\[ A = g^{-1} \sigma_3 g, \quad Z = g^1 \Sigma g, \quad \Sigma = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad g = \left( \begin{array}{cc} g_1 & -g_2 \\ g_2 & \bar{g}_1 \end{array} \right) \]
or
\[ A = \frac{1}{\Delta} \left( \begin{array}{cc} |g_1|^2 - |g_2|^2 & -2g_1 \bar{g}_2 \\ -2g_2 \bar{g}_1 & |\bar{g}_2|^2 - |g_1|^2 \end{array} \right), \quad Z = \frac{1}{\Delta} \left( \begin{array}{cc} \bar{g}_1 g_2 & \bar{g}_1 \bar{g}_2 \\ -g_2 \bar{g}_1 & -g_1 \bar{g}_2 \end{array} \right). \]

From the gauge equivalence between the ZIE and the K-IE follows the some important relations between the solutions of these equations. Some of them look like as
\[ Z^2 = 0, \quad Z_x^2 = I, \quad Z_t^2 = v^2_{x,t} I, \]
\[ A_x^2 = 4(u - \frac{\rho^2}{16\lambda}) I, \quad Z_x^2 = u^2_{x,t} I. \]
6 Soliton solutions of the K-IE

As the integrable equation, the K-IE has all ingredients of integrable systems like LR, conservation laws, Hamiltonian structures, soliton solutions and so on. Here let us present the 1-soliton solution of the K-IE. To construct this 1-soliton solution, we use the gauge equivalence between the K-IE and the ZIE. First recall that the matrices $A$ and $Z$ can be expressed as

$$A = g^{-1} \sigma_3 g = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad Z = g^{-1} \Sigma_3 g = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & -z_{11} \end{pmatrix},$$

(82)

where

$$g = \begin{pmatrix} g_1 & -\bar{g}_2 \\ g_2 & g_1 \end{pmatrix}, \quad g^{-1} = \frac{1}{\Delta} \begin{pmatrix} \bar{g}_1 & \bar{g}_2 \\ -g_2 & g_1 \end{pmatrix}, \quad \Delta = |g_1|^2 + |g_2|^2.$$  

(83)

As result we obtain the following expressions for the components of the matrices $A$ and $Z$:

$$A^+ = -\frac{2g_1g_2}{\Delta}, \quad A_3 = \frac{|g_1|^2 - |g_2|^2}{\Delta},$$

(84)

$$z_{11} = \frac{g_1g_2}{\Delta}, \quad z_{12} = \frac{g_2^2}{\Delta}, \quad z_{21} = -\frac{g_2^2}{\Delta}.$$  

(85)

To construct the 1-soliton solution of the K-IE, we consider the following seed solution of the ZIE

$$u = \rho = 0.$$  

(86)

Then we obtain the following equations for $g_j$:

$$g_{1x} = -i\beta g_1,$$  

(87)

$$g_{2x} = -g_1 + i\beta g_2,$$  

(88)

$$g_{1t} = -4i\beta^3 g_1,$$  

(89)

$$g_{2t} = -4\beta^2 g_1 + 4i\beta^3 g_2.$$  

(90)

These equations admit the following solution

$$g_1 = a_1 e^{-\theta}, \quad g_2 = a_2 e^{\theta + \delta},$$

(91)

where $a_j$ are complex constants and

$$\theta = i\beta x + 4i\beta^3 t, \quad \delta = -a_1a_2^{-1}(x + 4t).$$  

(92)

Thus the 1-soliton solution of the K-IE has the form

$$A^+ = -\frac{e^{\delta - \delta + (\delta_3 + \delta_3)}}{\cosh \chi}, \quad A_3 = \tanh \chi, \quad z_{11} = \frac{e^{(\theta - \beta) + 0.5(\delta - \delta + \delta_3 - \delta_3)}}{2 \cosh \chi},$$

(93)

$$z_{12} = \frac{e^{-2\theta - 0.5(\delta - \delta) - 2i\delta_1 - \delta_3}}{2 \cosh \chi}, \quad z_{21} = -\frac{e^{2(\theta + \delta) - 0.5(\delta + \delta) + 2i\delta_2 + \delta_3}}{2 \cosh \chi},$$

(94)

where $\delta_3 = \ln \frac{|a_2|}{|a_1|}, \quad a_j = |a_j| e^{i\delta_j}, \quad \chi = \theta + \bar{\theta} + 0.5(\delta + \delta).$ Finally we note that the presented 1-soliton solution satisfies the following conditions

$$|A^+|^2 + A_3^2 = 1, \quad z_{11}^2 + z_{12}z_{21} = 0,$$

(95)

which come from the following properties of the matrices $A$ and $Z$:

$$A^2 = I, \quad Z^2 = 0.$$  

(96)
7 Conclusion

In the paper, one of the integrable GHFE, namely, the K-IE is investigated. The LR of this equation and its reduction are given. The geometric formulation of the ZIE is presented. It is also shown that the ZIE is geometrical and gauge equivalent to the K-IE. The 1-soliton solution of the K-IE is obtained. Finally we note that it is interesting to investigate the surfaces induced by the K-IE and by the ZIE as well as integrable motion of curves in other geometries (see, e.g., [3]-[12]).

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References

[1] M. Lakshmanan, Phil. Trans. R. Soc. A, 369 1280-1300 (2011).
[2] M. Lakshmanan, Phys. Lett. A, 64, 53-54 (1977).
[3] Assem Mussatayeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Kuralay Yesmakhanova, Ratbay Myrzakulov. Integrable Motion of Curves, Spin Equation and Camassa-Holm Equation, arXiv:1907.10910
[4] Bayan Kutum, Gulgassyl Nugmanova, Tolkynay Myrzakul, Kuralay Yesmakhanova, Ratbay Myrzakulov. Integrable Deformation of Space Curves, Generalized Heisenberg Ferromagnet Equation and Two-Component Modified Camassa-Holm Equation, arXiv:1908.01371
[5] Gulmira Yergaliyeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Kuralay Yesmakhanova, Ratbay Myrzakulov. Camassa-Holm and M-CIV equations with self-consistent sources: geometry and peakon solutions, arXiv:1909.10606
[6] Aigul Taishiyeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Shynaray Myrzakul, Kuralay Yesmakhanova, Ratbay Myrzakulov. Geometric Flows of Curves, Two-Component Camassa-Holm Equation and Generalized Heisenberg Ferromagnet Equation, arXiv:1910.13281
[7] Julia Cen, Francisco Correa and Andreas Fring. Nonlocal gauge equivalence: Hirota versus extended continuous Heisenberg and Landau-Lifschitz equation, arXiv:1910.07272
[8] Z. Umurzakhova et al. Gauge equivalence between the Myrzakulov-CXI and modified Camassa-Holm equations [in preparation].
[9] Ivanov R.I. Two component integrable systems modelling shallow water waves: the constant vorticity case, arXiv:0906.0780
[10] G. Nugmanova, Z. Zhunussova, K. Yesmakhanova, G. Mamyrbekova, R. Myrzakulov. International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering, 9, N8, 328-331 (2015).
[11] U. Saleem, M. Hasan. J. Phys. A: Math. Theor., 43, 045204 (2010).
[12] R. Myrzakulov, S. Vijayalakshmi, G. Nugmanova, M. Lakshmanan Physics Letters A, 233, 14-6, 391-396 (1997).
[13] R. Myrzakulov, S. Vijayalakshmi, R. Syzdykova, M. Lakshmanan, J. Math. Phys., 39, 2122-2139 (1998).
[14] R. Myrzakulov, M. Lakshmanan, S. Vijayalakshmi, A. Danlybaeva, J. Math. Phys., 39, 3765-3771 (1998).
[15] Myrzakulov R, Danlybaeva A.K, Nugmanova G.N. Theoretical and Mathematical Physics, V.118, 13, P. 441-451 (1999).
[16] Myrzakulov R., Nugmanova G., Syzdykova R. Journal of Physics A: Mathematical & Theoretical, V.31, 147, P.9535-9545 (1998).
[17] Myrzakulov R., Daniel M., Amuda R. Physica A., V.234, 13-4, P.715-724 (1997).
[18] Myrzakulov R., Makhankov V.G., Pashaev O.?.. Letters in Mathematical Physics, V.16, N1, P.83-92 (1989)
[19] Myrzakulov R., Makhankov V.G., Makhankov A. Physica Scripta, V.35, N3, P. 233-237 (1987)
[20] Myrzakulov R., Pashaev O.?., Kholmurodov Kh. Physica Scripta, V.33, N4, P. 378-384 (1986)
[21] Anco S.C., Myrzakulov R. Journal of Geometry and Physics, v.60, 1576-1603 (2010)
[22] Myrzakulov R., Rahimov F.K., Myrzakul K., Serikbaev N.S. On the geometry of stationary Heisenberg ferromagnets. In: "Non-linear waves: Classical and Quantum Aspects", Kluwer Academic Publishers, Dordrecht, Netherlands, P. 543-549 (2004)
[23] Myrzakulov R., Serikbaev N.S., Myrzakul Kur., Rahimov F.K. On continuous limits of some generalized compressible Heisenberg spin chains. Journal of NATO Science Series II. Mathematics, Physics and Chemistry, V 153, P. 535-542 (2004)
[24] R.Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, M. Lakshmanan. Symmetry, 7(3), 1352-1375 (2015). [arXiv:1305.0098]
[25] R.Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, K. Yesmakhanova, M. Lakshmanan. Physics Letters A, 378, N30-31, 2118-2123 (2014). [arXiv:1304.2088]
[26] Myrzakulov R., Martina L., Kozhamkulov T.A., Myrzakul Kur. Integrable Heisenberg ferromagnets and soliton geometry of curves and surfaces. In book: "Nonlinear Physics: Theory and Experiment. II". World Scientific, London, P. 248-253 (2003)
[27] Myrzakulov R. Integrability of the Gauss-Codazzi-Mainardi equation in 2+1 dimensions. In “Mathematical Problems of Nonlinear Dynamics”, Proc. of the Int. Conf. "Progress in Nonlinear sciences", Nizhny Novgorod, Russia, July 2-6, 2001, V.1, P.314-319 (2001)

[28] Chen Chi, Zhou Zi-Xiang. Darboux Transformation and Exact Solutions of the Myrzakulov-I Equations. Chin. Phys. Lett., 26, N8, 080504 (2009)

[29] Chen Hai, Zhou Zi-Xiang. Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation. Chin. Phys. Lett., 31, N12, 120504 (2014)

[30] Zhao-Wen Yan, Min-Ru Chen, Ke Wu, Wei-Zhong Zhao. J. Phys. Soc. Jpn., 81, 094006 (2012)

[31] Yan Zhao-Wen, Chen Min-Ru, Wu Ke, Zhao Wei-Zhong. Commun. Theor. Phys., 58, 463-468 (2012)

[32] K.R. Ysmakhanova, G.N. Nugmanova, Wei-Zhong Zhao, Ke Wu. Integrable inhomogeneous Lakshmanan-Myrzakulov equation, [nlin/0604034]

[33] Zhen-Huan Zhang, Ming Deng, Wei-Zhong Zhao, Ke Wu. On the integrable inhomogeneous Myrzakulov-I equation, [arXiv: nlin/0603069]

[34] Martina L, Myrzakul Kur., Myrzakulov R, Soliani G. Journal of Mathematical Physics, V.42, 13, P.1397-1417 (2001).

[35] Z.S. Yersultanova, M. Zhassybayeva, K. Yesmakhanova, G. Nugmanova, R. Myrzakulov. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1404.2270]

[36] Myrzakul Akbota and Myrzakulov Ratbay. Integrable Motion of Two Interacting Curves and Heisenberg Ferromagnetic Equations, Abstracts of XVIII-th Intern. Conference "Geometry, Integrability and Quantization", June 3-8, 2016, Bulgaria.

[37] Myrzakul Akbota and Myrzakulov Ratbay. Integrable motion of two interacting curves, spin systems and the Manakov system. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1606.06598]

[38] Myrzakul Akbota and Myrzakulov Ratbay. Darboux transformations and exact soliton solutions of integrable coupled spin systems related with the Manakov system. [arXiv:1607.08151]

[39] Myrzakul Akbota and Myrzakulov Ratbay. Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrodinger equation. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1608.08553]

[40] Myrzakulova Zh., Myrzakul A., Nugmanova G., MyrzakulovR. Notes on Integrable Motion of Two Interacting Curves and Two-layer Generalized Heisenberg Ferromagnet Equations, [arXiv:1811:12216]
[41] Hussien R.A., Mohamed S.G. *Generated Surfaces via Inextensible Flows of Curves in $\mathbb{R}^3$*. Journal of Applied Mathematics, v.2016, Article ID 6178961 (2016).

[42] C. Qu, J. Song and R. Yao. *Multi-Component Integrable Systems and Invariant Curve Flows in Certain Geometries*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 001, 19 pages

arXiv:1301.0180