Quantum Fisher information for reference frame alignment

Mear M. R. Koochakie, Vahid Jannesary, and Vahid Karimipour

Department of Physics, Sharif University of Technology, Tehran 14588, Iran

Abstract

Two coordinate systems in two remote labs are to be aligned by sending specific pre-determined states from one lab and measuring these states in the other. The data obtained from these measurements are used to align the two coordinate systems. What kind of states are the most suitable for this task? We find the suitable state by optimizing the relevant quantum Fisher information and hence minimizing the uncertainty in the final alignment. We do this both for the case where the two labs have one common axis and when they do not have any common axis.

Introduction

Almost any quantum communication protocol requires reference frames (phase, coordinates, time, etc) to be shared between the players. In the case of coordinate reference frames which is the subject of study in this paper, the two laboratories have to agree on a common conception of the $x, y$ and $z$ axis.
Such a Shared Reference Frame (or SRF for short) is almost an absolute necessity for the hallmark protocols of quantum communication, like quantum key distribution QKD [1], teleportation [2] and dense coding [2]. The same is true for testing the entanglement of states shared between the two labs by, for example, testing the Bell inequality violation [3]. This is just a short list of protocols which obviously need an SRF for their performance. It is true that one can bypass SRF and do for example QKD [4] or verify Bell inequality without a shared reference frame [5], but in both cases this will lead to consuming more resources for each protocol. For example, in reference frame independent QKD [4], one has to send three qubits each time instead of one qubit and the measurements should also be multi-qubit measurements. In Bell inequality violation also [5] the two labs have to measure correlations in a number of directions which is more than the standard tests of Bell inequality [3].

In the past few years, the subject of shared reference frames has been studied from different points of view in different works [6, 7, 8, 9]. In [10, 11, 12], a general theory on the asymmetry of quantum states has been set up within which this problem can also be addressed. In [13, 14, 15] general schemes for classical and quantum communication in the absence of shared reference frames have been analyzed. More specifically in [16] it has been shown how one can use shared entangled states for setting up a shared reference frame, in [17] and [18] it has been shown how relative parameters can be safely encoded into the relative coordinates (coordinates that are invariants of rotation) of product or entangled states respectively.

In this paper we want to study this problem from a different point of view. Naturally we expect that one of the labs, operated by Alice, sends quantum states to the other lab, operated by Bob, who measures these states in his own reference frame and then by analyzing the results of measurements aligns his coordinate system to conform to that of Bob. Like any other measurement, there is also an uncertainty in this measurement which is transferred to the uncertainty of the final alignment between the two frames. The uncertainty and its lower bound is measured by “Quantum Fisher Information” (QFI) which crucially depends on what kind of state Alice is sending to Bob. By calculating the QFI for different types of states, one can determine the optimum state for coordinate system alignment.

We shall now point the difference of our work with that of [19] and [7].
In [19] two inertial reference frames are partially aligned in the sense they are related to each other with a Lorentz transformation $g$ (a rotation in our case) with probability $P(g)$. Therefore any state $\rho$ which is sent by Alice is received by Bob as if it has undergone a twirling operation or a quantum channel given by

$$\mathcal{E}(\rho_\lambda) = \int dg P(g) U(g) |\psi_\lambda\rangle \langle \psi_\lambda| U^\dagger(g).$$  \hspace{1cm} (1)

In this setting the question that is investigated in [19] is the QFI of a state $\rho = |\psi_\lambda\rangle \langle \psi_\lambda|$ where $|\psi_\lambda\rangle = e^{-iK_\lambda} |\psi\rangle$ encodes a single parameter $\lambda$ and the goal of Bob is to estimate this single parameter. However in our case the two frames are connected by a fixed unknown rotation $\hat{R}$ and the goal of Bob is to estimate the three parameters (the Euler angles) of this rotation by repeatedly measuring a sequence of identical arbitrary-dimensional states sent by Alice and then aligning his reference frame with that of Alice. Our problem is to find the states which are best suited for this purpose in the sense that it leads to the least amount of uncertainty of alignment.

In [7] authors tackled the same problem as we are. The main distinction between their approach and ours is the difference in the figure of merit that is being used for optimization of the procedure; They tried to maximize the “fidelity” of reference frame transmission and we maximize QFI of the transmission. This enables us to find the optimum states analytically while they presented a numerical iterative method for the optimization.

Based on this we find that the optimum states for alignment of coordinate systems are states of the form

$$\frac{1}{\sqrt{2}} (|j, j\rangle + e^{i\delta} |j, -j\rangle)$$  \hspace{1cm} (2)

where we have used the angular momentum notation $|j, m\rangle$ and where the value of QFI increases with $j$. Except for the case $j = \frac{1}{2}$, these states are not the eigenstate of any spin operator of the form $\mathbf{S} \cdot \mathbf{n}$, that is the state does not point to a specific direction, rather, specially when the state is realized by a addition of a large number of spin-$\frac{1}{2}$ states, it is a macroscopic GHZ type superposition of qubits. The exclusion of the $j = \frac{1}{2}$ case is due to the rotational invariance of the state $|\psi_0\rangle$ which is a singlet in this case and hence the angle of rotations of these states around an axis cannot be estimated.
This is reflected in the vanishing of the determinant of the QFI matrix for \( j = 1/2 \) case, a fact which we have checked numerically, although it is not evident analytically in the form of the QFI matrix.

The structure of this paper is as follows: In section (1) and for ease of reference, we review the known facts about Fisher information; In section (2), we consider the simple case where the two labs have one axis in common; In section (3), we consider the general case where the two labs do not have any common axis. We end the paper with a conclusion.

1 Quantum Fisher information

For the sake of completeness, in this section we briefly explain the concept of Fisher information and QFI. Let a system be described by a statistical model characterized by one real parameter \( \theta \). To determine the parameter \( \theta \), we perform measurements on an observable \( X \) which yields a sequence of sample values \( x \), in subsequent measurements. The aim is to estimate the value of \( \theta \) from these sample values. The imprecise or indirect nature of measurements yields a probability distribution for obtaining values of \( x \) for a given fixed \( \theta \), which is denoted by \( p(x|\theta) \). Expectation value of any function \( f \) of samples \( x \) given \( \theta \) is defined as

\[
\mathbb{E}[f(X)|\theta] := \int_{\mathcal{X}} f(x)p(x|\theta) \, dx,
\]

where \( \mathcal{X} \) denotes the samples space and \( X \) is the random variable corresponding to \( x \).

An estimator \( \theta_e(x) \) is a function that estimates the actual parameter \( \theta \) from the sequence of samples \( x \). An estimator \( \theta_e \) is called unbiased if

\[
\mathbb{E}[\theta_e(X)|\theta] = \theta.
\]

A basic question is how much the estimate of \( \theta \) \( (\theta_e(x)) \) can vary from its actual value. The aim of any kind of measurement and estimator is to reduce this variance as much as possible. However the Cramér-Rao bound \([20]\), puts a lower bound to this variance, where the lower bound for mean square error (variance) of any unbiased estimation is

\[
\frac{1}{F(\theta)} \leq \mathbb{E}[(\theta_e(X) - \theta)^2|\theta],
\]
in which
\[ F(\theta) := \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \log p(X|\theta) \right)^2 \right] |\theta]. \tag{6} \]
is the so-called “Fisher information.” The important point is that Fisher information does not depend on the estimator. For the cases where \( p(x|\theta) = p(x) \), which means \( X \) is independent of \( \Theta \), (meaning that our measurements do not yield any information on the value of \( \theta \)), Eq. (6) naturally gives a zero Fisher information, i.e., we cannot estimate \( \theta \) from the samples.

The Cramér-Rao bound can be generalized to multi-parameter models \[20\], in which \( \theta := (\theta_1, \theta_2, \ldots, \theta_m) \) is a \( m \)-tuple of real numbers and the Fisher information is a \( m \times m \) real matrix:
\[ F_{i,j}(\theta) := \mathbb{E} \left[ \frac{\partial}{\partial \theta_i} \log p(X|\theta) \frac{\partial}{\partial \theta_j} \log p(X|\theta) \right] |\theta]. \tag{7} \]
It is straightforward to show that Fisher information matrices are positive semi-definite. Assuming invertibility of \( F(\theta) \), the multi-parameter Cramér-Rao inequality (Ref. \[20\]) is given by the matrix inequality
\[ \mathcal{K}(\theta; \theta_e) \succeq F^{-1}(\theta) \tag{8} \]
where \( \mathcal{K} \) is the covariance matrix with elements:
\[ \mathcal{K}_{i,j}(\theta; \theta_e) := \mathbb{E} \left[ (\theta_{ei}(X) - \theta_i)(\theta_{ej}(X) - \theta_j) \right] |\theta]. \tag{9} \]
Suppose now that we have a prior knowledge about the distribution of the parameters \( \theta \) and let the probability density function \( z(\theta) \) indicates this prior knowledge. Using Cramér-Rao inequality (8) one finds the following matrix inequality
\[ \mathcal{K} := \int \mathcal{K}(\theta) z(\theta) \, d\theta \geq F^{-1}. \tag{10} \]
Consider now the following “cost function” of estimator \( \theta_e \):
\[ C(\theta; \theta_e) := \sum_{i=1}^{m} \mathbb{E} \left[ (\theta_{ei}(X) - \theta_i)^2 |\theta] = \text{Tr}(\mathcal{K}) \tag{11} \]
and its average over prior distribution \( z(\theta) \), denoted by \( \overline{C} \). Inequality (10) then yields
\[ \overline{C} = \text{Tr}(\mathcal{K}) \geq \text{Tr}(F^{-1}). \tag{12} \]
$F$ is a real-valued invertible positive matrix of dimension $m$. Using the convexity of the function $f(A) = \frac{1}{A}$ for positive definite matrices, we find from the last inequality that

$$C \geq \text{Tr}(F^{-1}).$$ (13)

Using the inequality $\text{Tr}(X^{-1}) \text{Tr}(X) \geq m^2$ for $m$-dimensional positive definite matrix $X$ (which is simply proved by diagonalizing $X$ and using $\lambda + \frac{1}{\lambda} \geq 2$ for positive $\lambda$), we find

$$C \geq \frac{m^2}{\text{Tr}F}.$$ (14)

This is the final result for the lower bound of the cost function defined in Eq. (11).

The concept of Fisher information and many of its properties are extended to the quantum domain, leading to QFI which has played a vital role in recent developments in quantum metrology. In quantum mechanics a statistical model is described by a parameterized density matrix $\rho(\theta)$ and samples are given by POVM measurements:

$$p(x|\theta) = \text{Tr} \Pi_x \rho(\otimes^n(\theta)), \quad (15)$$

where $n$ instances of the quantum ensemble has been measured and $x$ represents the samples. In Ref. [22, 23], it has been shown that the Fisher information $F$ pertaining to the statistics of measurement $\Pi$ satisfies the following matrix inequality

$$F(\theta; \Pi) \leq nQ(\theta),$$ (16)

where $Q(\theta)$ is so-called “quantum Fisher information” matrix, which is independent of measurement $\Pi$, with elements:

$$Q_{ij}(\theta) := \frac{1}{2} \text{Tr} (\rho(\theta) \{L_i(\theta), L_j(\theta)\}),$$ (17)

in which $L_i(\theta)$ is the symmetric logarithmic derivative (SLD) of $\rho(\theta)$ w.r.t. $\theta_i$, which is defined such that

$$\frac{1}{2} \{L_i(\theta), \rho(\theta)\} = \frac{\partial}{\partial \theta_i} \rho(\theta).$$ (18)
When the quantum model is described by the pure state $|\psi(\theta)\rangle$, it is easy to check that

$$L_i(\theta) = 2|\partial_i\psi(\theta)\rangle\langle\psi(\theta)| + 2|\psi(\theta)\rangle\langle\partial_i\psi(\theta)|$$

(19)

where $|\partial_i\psi(\theta)\rangle := \frac{\partial}{\partial \theta_i}|\psi(\theta)\rangle$. Hence the QFI matrix of the pure system can be written as

$$Q_{ij}(\theta) = 2\langle\partial_i\psi(\theta)|\partial_j\psi(\theta)\rangle + 2\langle\partial_j\psi(\theta)|\partial_i\psi(\theta)\rangle + 4\langle\psi(\theta)|\partial_i\psi(\theta)\rangle\langle\psi(\theta)|\partial_j\psi(\theta)\rangle.$$  

(20)

One can define $\overline{Q}$ as the average of $Q(\theta)$ over the a priori distribution $z(\theta)$ of the parameter $\theta$. All the analysis for the covariance of measurements of parameters now carry over without change to the quantum case, except that due to (16), we can replace $F$ with $n\overline{Q}$. Doing this substitution, Eq. (14) is replaced with

$$\overline{C} \geq \frac{m^2}{n \text{Tr}(\overline{Q})},$$

(21)

in which $m$ is the number of parameters in $\theta$.

We are now prepared to make a detailed analysis of QFI for alignment of two reference frames in two remote labs. We first consider the simple case where the two labs agree on an axis and try to orient their frames in the two-dimensional plane perpendicular to this axis. We then consider the general case where the two labs do not have any common axis.

2 Frame alignment with one common axis

Let Alice and Bob share a common axis. This axis could have been for example fixed by classical means. Let us call this common axis the $z$-axis. To align the other two axes, Alice sends a series of $|\psi_0\rangle$ to Bob. Let $\gamma$ be the angle between Bob’s $x$-axis and Alice’s $x$-axis. In Bob’s frame the state is of the form

$$|\psi_\gamma\rangle = e^{-i\gamma S_z}|\psi_0\rangle.$$  

(22)

Bob measures the incoming states $|\psi_\gamma\rangle$ one-by-one and estimates the angle $\gamma$ given that the state $|\psi_0\rangle$ is a known state to Alice and himself. The aim of
this section is to find which states $|\psi_0\rangle$ are the most appropriate for this task. Intuitively, it is clear that the optimal states should be perpendicular to the $z$-axis, since they are the most sensitive states to such a rotation. Here we want to test this by calculating the QFI of estimating $\gamma$. Regarding Eq. (20), to calculate QFI, we have following inner products:

$$\langle \partial_\gamma \psi_\gamma | \partial_\gamma \psi_\gamma \rangle = \langle \psi_0 | S_z^2 | \psi_0 \rangle,$$

$$\langle \partial_\gamma \psi_\gamma | \psi_\gamma \rangle = i \langle \psi_0 | S_z | \psi_0 \rangle.$$  

Putting above inner products into Eq. (20), QFI of estimating $\gamma$ can be written as

$$Q(|\psi_0\rangle) = 4(\langle \psi_0 | S_z^2 | \psi_0 \rangle - \langle \psi_0 | S_z | \psi_0 \rangle^2),$$

which is interestingly independent of $\gamma$, but certainly depends on the state $|\psi_0\rangle$. To maximize the QFI, we have to find states that maximize variance of $S_z$. According to Popoviciu’s inequality [24], for any bounded probability distribution $P(x)$, defined on the interval $[x_{\text{min}}, x_{\text{max}}]$ one has

$$\sigma^2 \leq \frac{1}{4}(x_{\text{max}} - x_{\text{min}})^2,$$

where $\sigma^2$ is the variance. It is then easy to check that the equality holds if half of the probability is concentrated at the two ends of the interval $[x_{\text{min}}, x_{\text{max}}]$. Therefore the optimum states $|\psi_0\rangle$ that maximize variance of $S_z$ have the form of

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|j, j\rangle + e^{i\delta}|j, -j\rangle),$$

where $S_z|j, m\rangle = m|j, m\rangle$ assuming $|\psi_0\rangle$ are spin-$j$ states. Hence maximum of QFI is

$$Q_{\text{max}} = 4j^2.$$  

It is interesting to note that for $j \neq \frac{1}{2}$, the $|\psi_0\rangle$ is not a spin state pointing to any specific direction since it is not the eigenstate of any operator $\textbf{S} \cdot \textbf{n}$ (this is seen by direct calculation: The equation $(S_z + \alpha S_+ + \alpha^* S_-)|\psi_0\rangle \propto |\psi_0\rangle$ has no solution for $\alpha$.)

One can take $n$ spin-$j$ particles in a product state $|\psi_0\rangle^\otimes n$ for which the variance scales linearly with $n$ and hence

$$Q(|\psi_0\rangle^\otimes n) = 4nj^2.$$  

8
The total spin of such a state is not $nj$ and so its QFI is less than that of an entangled state whose total spin is $nj$ which scales quadratically and is equal to $4n^2j^2$.

The final result of this section is that the optimal states are of the form
$$\frac{1}{2}(|j,j⟩ + e^{i\delta}|j,−j⟩),$$
which depending on the resources that we have (a single spin-$j$ particle or a collection of spin-$\frac{1}{2}$ particles), can pertain to a single high spin particle or an entangled state of lower spin particles. In any case, it is important to note that for $j \neq \frac{1}{2}$ this state is not the eigenstate of any operator $\mathbf{S} \cdot \mathbf{n}$.

## 3 Frame alignment in the general case

We now consider the general case, where the two labs do not have any common axis. The state $|Ψ_0⟩$ that Alice now sends to Bob appears in his frame as $|Ψ(\hat{R})⟩ = U(\hat{R})|Ψ_0⟩$, where
$$U(\hat{R}) = e^{-i\alpha S_z}e^{-i\beta S_y}e^{-i\gamma S_z},$$
and $\alpha$, $\beta$ and $\gamma$ are the Euler angles which transform Alice’s frame to that of Bob, (see Fig. 1). This is the natural generalization of the simple case where Alice’s and Bob’s had one common axis, but is considerably more involved. As in the previous case, the expression of the state $|Ψ_0⟩$ in Alice’s frame is known to both of them and Bob makes suitable measurement to discern these three angles and align his frame with that of Alice. This protocol inevitably has some inherent uncertainty in the final results for these angles. An uncertainty which is bounded below as shown in Eq. (14). Our aim is to find the optimum state $|Ψ_0⟩$ which minimizes this lower bound by maximizing the trace of QFI on the average ($\text{Tr} \bar{Q}$). By the average we mean average over a uniform distribution of rotations of the two frames. Therefore the state $|Ψ_0⟩$ that we find may act badly for some cases, but on the average it is the best state that minimize the lower bound of the cost.

Here we have used the fact that QFI is a convex function of the parameterized quantum states, and hence takes its optimum value at the extreme points of the space of states, that is the pure states. We can now use Eq. (20) and calculate QFI matrix of $|ψ(\alpha, \beta, \gamma)⟩$. The details of calculations are found in
Figure 1: Euler rotation of Bob’s frame (blue) into Alice’s frame (red). The angle $\gamma$ is not shown.

the appendix [A]. The result is

$$Q_{\alpha\alpha} = 4\left(\langle S_{\beta\gamma}^2 \rangle - \langle S_{\beta\gamma} \rangle^2\right), \quad (31a)$$

$$Q_{\beta\beta} = 4\left(\langle S_{\gamma}^2 \rangle - \langle S_{\gamma} \rangle^2\right), \quad (31b)$$

$$Q_{\gamma\gamma} = 4\left(\langle S_z^2 \rangle - \langle S_z \rangle^2\right), \quad (31c)$$

in which

$$S_{\beta\gamma} := \cos \beta S_z - \sin \beta (\cos \gamma S_x - \sin \gamma S_y), \quad (32)$$

$$S_{\gamma} := \cos \gamma S_y + \sin \gamma S_x. \quad (33)$$

To calculate $\overline{Q}$, we need a prior distribution for the parameters $\alpha, \beta$ and $\gamma$. We take a uniform distribution (SO(3) Haar measure)

$$z(\alpha, \beta, \gamma) \, d\alpha \, d\beta \, d\gamma = \frac{1}{8\pi^3} \sin \beta \, d\alpha \, d\beta \, d\gamma. \quad (34)$$

Here we have assumed that the distribution of the $z$ axis is uniform over a sphere in Bob’s frame. This gives the factor $\frac{1}{4\pi} \sin \beta \, d\alpha \, d\beta$ in [34]. Once the
z or the x-y plane is fixed, the x axis can be uniformly distributed in this place, with a distribution \( \frac{1}{2\pi} \, \text{d}\gamma \), hence Eq. (34).

Using the above distribution, we show that (see Appendix B)

\[
\text{Tr} \, \overline{Q} = 16 \left( \frac{3}{3} \right) \left( \langle S_x^2 \rangle - \langle S_x \rangle^2 \right) + 10 \left( \frac{3}{3} \right) \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 \right) + 10 \left( \frac{3}{3} \right) \left( \langle S_z^2 \rangle - \langle S_z \rangle^2 \right)
\]

\[= 16 \left( \langle S_x^2 \rangle - \langle S_x \rangle^2 \right) + 10 \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 \right) + 10 \left( \langle S_z^2 \rangle - \langle S_z \rangle^2 \right) = 10 \left( \langle S_x^2 \rangle - \langle S_x \rangle^2 \right) + 2 \langle S_z^2 \rangle - \langle S_x \rangle^2 - \langle S_y \rangle^2 - \langle S_z \rangle^2
\]

Equation (35)

When \( j \neq \frac{1}{2} \), \( \text{Tr} (\overline{Q}) \) can be maximized by states for which \( \langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0 \) and \( \langle S_z^2 \rangle \) is the largest. These are states of the form

\[
|\Psi_0\rangle = \frac{|j, -j\rangle + e^{i\delta}|j, j\rangle}{\sqrt{2}},
\]

where \( |j, m\rangle \) are eigenstates of \( S_z \). These states maximize \( \text{Tr} (\overline{Q}) \) to the value

\[
16 \left( \frac{3}{3} \right) \frac{j^2}{3} \left( j + 1 \right) + 10 \left( \frac{3}{3} \right) \frac{j}{3} \quad \text{if} \quad j \neq \frac{1}{2}.
\]

Equation (36)

When \( j = \frac{1}{2} \), there is no state with the property \( \langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0 \). In this case the state \( |\Psi_0\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle + e^{i\delta}|\uparrow\rangle) \) which is again of the form Eq. (36) but maximize \( \text{Tr} (\overline{Q}) \) to the value \( \frac{13}{6} \).

Equation (37)

As can be seen, in \( \text{Tr} (\overline{Q}) \) the role of z direction differs from other directions. This is due to the way Euler rotations are parameterized. Except for the case \( j = \frac{1}{2} \) the state \( |\Psi_0\rangle \) is not the eigenstate of any spin operator \( S \cdot \mathbf{n} \) for any \( \mathbf{n} \), i.e. it is not pointing to any specific direction. It is not also a singlet state in the sense that \( S_{x,y,z} |\Psi_0\rangle = 0 \). However it has the property that \( \langle \Psi_0 | S_{x,y,z} |\Psi_0\rangle = 0 \). This state may not be optimal for a specific orientation of Bob’s frame with respect to that of Alice, however on the average (over all orientations) it is the best state that Alice can prepare and send to minimize the uncertainty of realignment. As it is well known the question of which specific measurement Bob should do to estimate the Euler’s angles is not relevant in investigations related to QFI which is only concerned with the lower bound of the variance of estimation.
References

[1] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).

[2] D. Bouwmeester, H. Weinfurter, A. Zeilinger, N. Gisin, J. G. Rarity, G. Weihs, J.-W. Pan, S. Bose, V. Vedral, P. L. Knight, Quantum Dense Coding and Quantum Teleportation.

[3] J. F. Clauser and A. Shimony, Bell’s theorem. Experimental tests and implications.

[4] F. Rezazadeh, A. Mani, V. Karimipour, Secure alignment of coordinate systems by using quantum correlation, Physical Review A 96 (2), 022310.

[5] Peter Shadbolt, Tamas Vertesi, Yeong-Cherng Liang, Cyril Branciard, Nicolas Brunner, and Jeremy L. OBrien, Guaranteed violation of a Bell inequality without aligned reference frames or calibrated devices, Scientific Reports 2, 470 (2012).

[6] Asher Peres and Petra F. Scudo, Entangled Quantum States as Direction Indicators, Phys. Rev. Lett. 86 (2001) 4160.

[7] Asher Peres and Petra F. Scudo, Transmission of a Cartesian Frame by a Quantum System, Phys. Rev. Lett. 87 (2001) 167901.

[8] Asher Peres, Petra F. Scudo, Unspeakable quantum information, arXiv:quant-ph/0201017

[9] Netanel H. Lindner, Asher Peres, and Daniel R. Terno, Elliptic Rydberg states as direction indicators, Phys. Rev. A 68, 042308 (2003).

[10] Iman Marvian, Robert W. Spekkens, The asymmetry properties of pure quantum states, Phys. Rev. A 90, 014102 (2014).

[11] Iman Marvian, Robert W. Spekkens, A no-broadcasting theorem for quantum asymmetry and coherence and a trade-off relation for approximate broadcasting, Phys. Rev. Lett. 123, 020404.

[12] Iman Marvian, Robert W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A 94, 052324 (2016).
[13] Stephen D. Bartlett, Terry Rudolph, R. W. Spekkens, *Classical and quantum communication without a shared reference frame*, Phys. Rev. Lett. 91, 027901 (2003).

[14] Stephen D. Bartlett, Terry Rudolph, Robert W. Spekkens, Peter S. Turner, *Quantum communication using a bounded-size quantum reference frame*, New J. Phys. 11, 063013 (2009).

[15] T. Rudolph, L. Grover, *Quantum communication complexity of establishing a shared reference frame*, Phys. Rev. Lett. 91, 217905 (2003).

[16] F Rezazadeh, A Mani, V Karimipour, *Secure alignment of coordinate systems using quantum correlation*, Phys. Rev. A 96, 022310 (2017).

[17] Ali Beheshti, Sadegh Raeisi, Vahid Karimipour, *Entanglement-assisted communication in the absence of shared reference frame*, Physical Review A 99 (4), 042330.

[18] Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens, *Optimal measurements for relative quantum information*, Phys. Rev. A 70, 032321.

[19] Mehdi Ahmadi, Alexander R. H. Smith, Andrzej Dragan, *Communication between inertial observers with partially correlated reference frames*, Phys. Rev. A 92, 062319 (2015).

[20] Harald Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946

[21] Kenneth Nordström, “Convexity of the inverse and Moore-Penrose inverse”, Linear Algebra and its Applications, Vol. 434 (6), pp. 1489-1512, 2011

[22] Carl W. Helstrom, *Quantum detection and estimation theory*, Academic Press, 1976

[23] Dénes Petz, *Quantum Information Theory and Quantum Statistics*, Theoretical and Mathematical Physics, Springer-Verlag Berlin Heidelberg, 2008

[24] Darij Grinberg, *Generalizations of Popoviciu’s inequality*, [arXiv:0803.2958](http://arxiv.org/abs/0803.2958) [math.FA].
A Deriving QFI matrix of Euler rotation

Consider QFI matrix of Eq. \([20]\). In our problem, \(|\psi(\theta)\rangle = |\Psi(\hat{R})\rangle\) is in the form of \(U(\theta)|\psi(0)\rangle = U(\hat{R})|\Psi_0\rangle\). To calculate the QFI matrix, we need to calculate following inner products:

\[
\langle \partial_\theta \psi(\theta) | \partial_\theta \psi(\theta) \rangle = \langle \psi(0) | T_i(\theta) T_j(\theta) | \psi(0) \rangle, \tag{38}
\]

\[
\langle \psi(\theta) | \partial_\theta \psi(\theta) \rangle = \langle \psi(0) | T_i(\theta) | \psi(0) \rangle, \tag{39}
\]

where

\[
T_i(\theta) = U^\dagger(\theta) \partial_\theta U(\theta). \tag{40}
\]

In our problem, \(\theta = (\theta_1, \theta_2, \theta_3) = (\alpha, \beta, \gamma)\) and \(U(\hat{R})\) is the unitary representing the rotation with Euler angles \((\alpha, \beta, \gamma)\) and we should calculate the above expression for \(U(\hat{R}) = U(\alpha, \beta, \gamma) = Z(\alpha)Y(\beta)Z(\gamma) = e^{-i\alpha S_z}e^{-i\beta S_y}e^{-i\gamma S_z}\). It is straightforward to check that

\[
T_\alpha(\theta) = -iZ^\dagger(\gamma)Y^\dagger(\beta)S_z Y(\beta)Z(\gamma), \tag{41}
\]

\[
T_\beta(\theta) = -iZ^\dagger(\gamma)S_y Z(\gamma), \tag{42}
\]

\[
T_\gamma(\theta) = -iS_z, \tag{43}
\]

which after straightforward calculations using angular momentum algebra, leads to

\[
T_\alpha = -iS_{\beta\gamma}, \quad T_\beta = -iS_\gamma, \quad T_\gamma = -iS_z, \tag{44}
\]

where

\[
S_{\beta\gamma} := \cos \beta S_z - \sin \beta (\cos \gamma S_x - \sin \gamma S_y), \tag{45}
\]

\[
S_\gamma := \cos \gamma S_y + \sin \gamma S_x. \tag{46}
\]

Hence,

\[
\langle \partial_\alpha \psi | \partial_\alpha \psi \rangle = \langle S_{\beta\gamma}^2 \rangle, \quad \langle \partial_\beta \psi | \partial_\beta \psi \rangle = \langle S_\gamma^2 \rangle, \quad \langle \partial_\gamma \psi | \partial_\gamma \psi \rangle = \langle S_z^2 \rangle, \tag{47}
\]

and

\[
\langle \psi | \partial_\alpha \psi \rangle = -i\langle S_{\beta\gamma} \rangle, \quad \langle \psi | \partial_\beta \psi \rangle = -i\langle S_\gamma \rangle, \quad \langle \psi | \partial_\gamma \psi \rangle = -i\langle S_z \rangle, \tag{48}
\]

where \(\psi = \psi(\alpha, \beta, \gamma)\) and \(\langle \rangle = \langle \psi(0) | \circ | \psi(0) \rangle\). Putting above relations into Eq. \([20]\), it is straightforward to check that the QFI matrix of Euler rotation can be written as Eqs. \([31]\).
B Deriving Tr($\overline{Q}$) of Euler rotation

To calculate the mean trace QFI Tr($\overline{Q}$) for Euler rotation w.r.t. the distribution Eq. (34), we should calculate following integral:

$$\text{Tr}(Q) = \frac{1}{8\pi^2} \int_{\gamma=0}^{2\pi} \int_{\beta=0}^{\pi} \int_{\alpha=0}^{2\pi} (Q_{\alpha\alpha} + Q_{\beta\beta} + Q_{\gamma\gamma}) \sin \beta \, d\alpha \, d\beta \, d\gamma,$$

where the quantum Fisher matrix, Q, is defined in Eqs. (31).

We need the following:

$$\int_{\gamma=0}^{2\pi} \int_{\beta=0}^{\pi} \left( \langle S_{\gamma\gamma}^2 \rangle - \langle S_{\beta\gamma} \rangle^2 \right) \sin \beta \, d\beta \, d\gamma =$$

$$= 2\pi \int_{\beta=0}^{\pi} \cos^2 \beta \sin \beta \, d\beta \left( \langle S_x^2 \rangle - \langle S_z \rangle^2 \right)$$

$$+ \int_{\beta=0}^{\pi} \sin^3 \beta \, d\beta \int_{\gamma=0}^{2\pi} \cos^2 \gamma \, d\gamma \left( \langle S_x^2 \rangle - \langle S_z \rangle^2 \right)$$

$$+ \int_{\beta=0}^{\pi} \sin^3 \beta \, d\beta \int_{\gamma=0}^{2\pi} \sin^2 \gamma \, d\gamma \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 \right)$$

$$= \frac{4\pi}{3} \left( \langle S_x^2 \rangle - \langle S_y \rangle^2 + \langle S_y^2 \rangle - \langle S_y \rangle^2 + \langle S_z^2 \rangle - \langle S_z \rangle^2 \right),$$

(51)

and

$$\int_{\gamma=0}^{2\pi} \left( \langle S_{\gamma}^2 \rangle - \langle S_{\gamma} \rangle^2 \right) \, d\gamma =$$

$$\int_{\gamma=0}^{2\pi} \cos^2 \gamma \, d\gamma \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 \right) + \int_{\gamma=0}^{2\pi} \sin^2 \gamma \, d\gamma \left( \langle S_x^2 \rangle - \langle S_x \rangle^2 \right)$$

$$= \pi \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 + \langle S_x^2 \rangle - \langle S_x \rangle^2 \right).$$

(52)

Putting everything together the final expression for Tr($\overline{Q}$) is simplified to

$$\text{Tr}(\overline{Q}) = \frac{10}{3} \left( \langle S_x^2 \rangle - \langle S_x \rangle^2 \right) + \frac{10}{3} \left( \langle S_y^2 \rangle - \langle S_y \rangle^2 \right) + \frac{16}{3} \left( \langle S_z^2 \rangle - \langle S_z \rangle^2 \right).$$

(53)