SIMPLE CONFORMALLY RECURRENT SPACE-TIMES ARE CONFORMALLY RECURRENT PP-WAVES

CARLO ALBERTO MANTICA AND LUCA GUIDO MOLINARI

Abstract. We show that in dimension $n \geq 4$ the class of simple conformally recurrent space-times coincides with the class of conformally recurrent pp-waves.

1. Introduction

Riemannian manifolds with recurrent Weyl tensor (conformally recurrent manifolds) were introduced by Adati and Miyazawa [1] and subsequently extended to pseudo-Riemannian manifolds by Derdziński [7], Roter [31, 32, 33, 34], Suh and Kwon [38], and to Lorentzian manifolds (spacetimes) by Mc Lenaghan et al. [24, 25], Hall [15, 16], De and Mantica [5]. In a recent study on conformally recurrent pseudo-Riemannian and Lorentzian manifolds in dimension $n \geq 5$, the authors showed that the Ricci tensor has at most two different eigenvalues, with restrictions on the metric structure, and gave an explicit representation of the Weyl tensor as a Kulkarni-Nomizu product [21, 22].

A pseudo-Riemannian manifold of dimension $n \geq 4$ is “conformally recurrent” if the Weyl curvature tensor is everywhere non-zero and satisfies the condition

$$\nabla_i C_{jklm} = \alpha_i C_{jklm}$$

where $\alpha_i$ is a non-zero 1-form named recurrence vector. The natural question arises whether the recurrent vector may be locally cancelled by a conformal transformation. This “simple” case was investigated by Roter [31, 34].

1 The local components of the Weyl tensor are [29]

$$C_{jklm} = R_{jklm} + \frac{1}{n-2} (\delta^m_j R_{kl} - \delta^m_k R_{jl} - g_{kl} R_{jm} - g_{jl} R_{km} - R \frac{\delta^m_k - \delta^m_j}{(n-1)(n-2)} g_{kl})$$

with Ricci tensor $R_{ij} = -R_{kij}^k$ and curvature scalar $R = R^k_k$. 

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Definition 1.1 (Roter). A conformally recurrent manifold \((\mathcal{M}, g)\) is “simple” if the metric is locally conformal to a conformally symmetric metric, i.e. at each point \(x \in \mathcal{M}\) there exist a neighbourhood \(U_x\) and a scalar function \(\sigma\) on \(U_x\) such that \(g'_{ij} = g_{ij}e^{2\sigma}\) and \(\nabla' C'_{jklm} = 0\) on \(U_x\).

The main result of this work is the statement that such spaces, with a Lorentzian metric, coincide with (conformally recurrent) pp-wave Lorentzian space-times.

pp-waves are important in general relativity (see for example [19] and references therein). They were studied in \(n > 4\) by Schimming [36], and appear in Kaluza-Klein theories and in string theory. In the literature (see for example [27]), pp-wave space-times are identified with Brinkmann-waves, i.e. space-times equipped with a null covariantly constant vector field, \(X^i X_i = 0, \nabla_j X_k = 0\) [4]. Here we conform to the more restrictive definition used in [36, 19, 17, 18]:

Definition 1.2. A pp-wave is a Brinkmann-wave whose Riemann tensor satisfies the condition \(R^p_q R_{pqlm} = 0\).

The layout of the paper is the following. In section 2 we present preliminary material about simple conformally recurrent pseudo-Riemannian manifolds and pp-wave space-times. In section 3 we prove that in dimension \(n \geq 4\) a conformally recurrent pp-wave space-time is a simple conformally recurrent Lorentzian manifold, and establish the converse implication. We obtain new curvature properties of conformally recurrent pp-waves, such as \(\nabla^m \nabla_m R_{ij} = 0\). In Section 4 we show that conformally recurrent pp-wave space-times are solutions to the extended theory of gravity with pure radiation source.

Throughout the paper, the manifolds are assumed to be connected, Hausdorff, with non-degenerate metric of arbitrary signature, i.e. \(n\)-dimensional pseudo-Riemannian manifolds. Some results are restricted to Lorentzian manifolds (space-times) i.e. to metrics of signature \(n - 2\).

2. Preliminary results

In this section we present preliminary material about conformally recurrent pseudo-Riemannian manifolds, simple conformally recurrent manifolds, and pp-wave space-times.

A vector field belongs to the Olszak distribution [26] if

\[
X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = 0.
\]
The condition was extensively studied in the geometric literature on pseudo-Riemannian manifolds (see for example [6, 9, 10, 11]).

A contraction of (2.1) with \( g^{im} \) gives \( X^m C_{jklm} = 0 \). Then, a contraction with \( X^i \) gives \( X^iX_i C_{jklm} = 0 \). If the Weyl tensor is non-zero, the Olszak distribution consists of null vectors, \( X^iX_i = 0 \).

**Lemma 2.1.** On a \( n \)-dimensional Lorentzian manifold with everywhere non-zero Weyl curvature, the Olszak distribution is one-dimensional.

**Proof.** Suppose that, besides \( X_i \), there exists another covector \( Y_i \) satisfying (2.1). Then also \( X_i + Y_i \) belongs to the Olszak distribution, \( X_iX_i = 0 \), \( Y_iY_i = 0 \) and \( X_iY_i = 0 \). On a Lorentzian manifold, two null orthogonal vectors must be collinear, \( Y_i = \mu X_i \) [35]. \( \square \)

The following general identity holds for the Weyl tensor on a pseudo-Riemannian manifold (eq. 3.8 in [1]):

\[
\nabla_i C_{jklm} + \nabla_j C_{kil} + \nabla_k C_{ijl} = \frac{1}{n-3} \nabla_p \left[ \delta_{ij} C_{kil} + \delta_{jk} C_{iil} + \delta_{ki} C_{jil} - \frac{1}{n-3} \delta_{ij} C_{kil} \right]
\]

**Proposition 2.2** (Adati and Myazawa [1]). Let \( (\mathcal{M}, g) \) be a conformally recurrent pseudo-Riemannian manifold of dimension \( n \geq 4 \). Then:
1) \( \nabla_mC_{jklm} = 0 \) if and only if \( \alpha_i \) belongs to the Olszak distribution;
2) If \( \nabla_mC_{jklm} = 0 \), then the recurrent covector is null, \( \alpha_i\alpha_i = 0 \).

**Proof.** Obviously \( \nabla_mC_{jklm} = 0 \) if and only if \( \alpha_mC_{jklm} = 0 \). The vanishing of the right hand side of (2.2) and recurrence imply that \( \alpha_i \) belongs to the Olszak distribution. Conversely, we have \( \alpha\alpha_i C_{jklm} = 0 \), i.e. either \( \alpha\alpha_i = 0 \) or \( C_{jklm} = 0 \). \( \square \)

**Theorem 2.3** (Roter [31, 34]). A pseudo-Riemannian manifold \( \mathcal{M} \) of dimension \( n \geq 4 \) is simple conformally recurrent if and only if:
1) \( C_{jklm} \neq 0 \) everywhere on \( \mathcal{M} \),
2) \( \nabla_i C_{jklm} = \alpha_i C_{jklm} \), with recurrence 1-form \( \alpha_i \) that is locally a gradient,
3) the Ricci tensor is a Codazzi tensor, \( \nabla_k R_{jl} = \nabla_j R_{kl} \).

The contraction of the Codazzi property with the metric tensor gives \( \nabla_k R = 0 \), where \( R \) is the scalar curvature. Moreover, it was shown:

**Proposition 2.4** (Roter [31]). On a non-locally symmetric simple conformally recurrent manifold, \( R = 0 \).

**Proposition 2.5** (Roter [34], lemma 12). If the Ricci tensor of a non-locally symmetric simple conformally recurrent manifold has the form \( R_{ij} = \pm d_id_j \),
then the following equations hold:

\[(2.3) \quad d_i C_{jklm} + d_j C_{kilm} + d_k C_{ijlm} = 0,\]

\[(2.4) \quad d_i R_{jklm} + d_j R_{kilm} + d_k R_{ijlm} = 0.\]

**Proposition 2.6** (Roter [34], theorem 1). Every simple conformally recurrent Lorentzian manifold is Ricci-recurrent, i.e. there exists a non-zero covector field \(\omega_i\) such that \(\nabla_k R_{jl} = \omega_k R_{jl}\).

Brinkmann and pp-wave space-times arise in the presence of a null covariantly constant vector field. They are special cases of Lorentzian manifolds with a recurrent null vector field, which have been studied for a long time (see for example [39, 4, 36, 17, 18]). A characterization of the metric through a set of canonical coordinates was obtained by Walker [39]. We recall Proposition 1 of [18]:

**Proposition 2.7.** Let \((\mathcal{M}, g)\) be a Lorentzian manifold of dimension \(n = d + 2 > 2\) with a recurrent null vector field, \(X^k X_k = 0, \nabla_k X_j = p_k X_j\).

1) This is equivalent to the existence of coordinates \((u, x^1, \ldots, x^d, v)\), in which the metric has the local expression

\[
ds^2 = 2dudv + H(u, \bar{x}, v)du^2 + \sum_{\rho=1}^{d} a_\rho(u, \bar{x})du^\rho + \sum_{\mu,\nu=1}^{d} g^*_{\mu\nu}(u, \bar{x})dx^\mu dx^\nu,
\]

where \(g^*_{\mu\nu}\) and \(a_\rho\) are independent of the coordinate \(v\), and \(H\) is a smooth function. We refer to these coordinates as Walker coordinates.

2) \(\nabla_k X_j = 0\) if and only if \(H\) does not depend on \(v\). We refer to these coordinates as Brinkmann coordinates, and to the manifold as a Brinkmann-wave (space-time) [4].

A pp-wave is a Brinkmann-wave with some restrictions. Schimming [36] gave the following coordinate characterization (see also [19, 17, 18]):

**Proposition 2.8** (Schimming [36]). A Lorentzian manifold of dimension \(d + 2 > 2\) is a pp-wave if and only if there exist coordinates \((u, \bar{x}, v)\) in which the metric has the following local expression:

\[(2.5) \quad ds^2 = 2dudv + H(\bar{x}, u)du^2 + \sum_{\rho=1}^{d} dx^{\rho 2},\]

where \(H(\bar{x}, u)\) is a smooth function independent of \(v\), usually called the “potential function of the pp-wave”.

The expression of the metric yields the Ricci tensor of a pp-wave:

\[ R_{kl} = \psi(\vec{x}, u) X_k X_l, \quad \psi(\vec{x}, u) = -\frac{1}{2} \sum_{\rho=1}^{d} \frac{\partial^2 H}{\partial x^\rho^2} \]

where \( X_k = \nabla_k u \) is a covariantly constant null vector \( (X^k X_k = 0, \nabla_j X_k = 0) \) [28]. It follows that the scalar curvature is zero, \( R = 0 \).

**Remark 2.9.** A metric such that \( R_{kl} = \psi X_k X_l \) with a null recurrent covector is called a pure radiation metric with parallel rays, or aligned pure radiation metric [20].

**Proposition 2.10** (Schimming [36], see also [19, 17, 18]). A Lorentzian manifold of dimension \( d + 2 > 2 \) with covariantly constant null vector field (i.e. a Brinkmann wave) is a pp-wave if and only if one of the following conditions is satisfied:

\[ X_i R_{jklm} + X_j R_{kilm} + X_k R_{ijlm} = 0; \]
\[ R_{jklm} = X_j X_m D_{kl} - X_j X_l D_{mk} - X_k X_m D_{jl} + X_k X_l D_{jm}; \]
\[ R^p_{\ jk} g^{pq} R_{pqlm} = \chi X_j X_k X_l X_m; \]

\( D_{ij} \) is a symmetric tensor and \( \chi \) is a suitable scalar function.

A contraction recovers the form of the Ricci tensor of a pp-wave, \( R_{kl} = \psi X_k X_l \). Leistner and Nurowski showed that in dimension \( d + 2 = 4 \) the conditions are equivalent to \( R_{jk pq} R_{pqlm} = 0 \) [19].

**Remark 2.11.** It is worth noting that for a Ricci tensor of the form (2.6) we have

\[ X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = X_i R_{jklm} + X_j R_{kilm} + X_k R_{ijlm}. \]

A contraction with \( g^{im} \) and the equality \( R_{ij} = \psi X_i X_j \) imply \( X^m C_{jklm} = X^m R_{jklm} \).

For a pp-wave, by eq. (2.7), we have \( X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = 0 \), and so \( X^m C_{jklm} = X^m R_{jklm} = 0 \).

3. Simple conformally recurrent Lorentzian manifolds and pp-waves.

In this section we obtain two important results. First we show, in several steps, that a conformally recurrent pp-wave space-time is a simple conformally recurrent space-time. Next, we show that a simple conformally recurrent space-time is a pp-wave space-time. Finally, some curvature properties of conformally recurrent pp-wave space-times are presented, as well as some
Lemma 3.1. On a pp-wave space-time of dimension \( n \geq 4 \), the Ricci tensor: 1) is recurrent with a closed recurrence 1-form, 2) it satisfies \([\nabla_i, \nabla_j]R_{kl} = 0\), 3) it satisfies

\[
(\nabla^i \nabla^m C_{jklm} = -\frac{n-3}{n-2} \nabla^2 R_{kl}
\]

where \( \nabla^2 = \nabla_k \nabla^k \). In particular, \( \nabla^2 R_{kl} = 0 \) if and only if \( \nabla^j \nabla^m C_{jklm} = 0 \).

**Proof.** In a coordinate system with (2.5) one has \( \nabla_j R_{kl} = (\nabla_j \psi) X_k X_l = (\nabla_j \log |\psi|) R_{kl} \). From this we infer \( [\nabla_i, \nabla_j]R_{kl} = 0 \), i.e. a pp-wave space-time is Ricci semi-symmetric \[30\].

The divergence of the Weyl tensor has the general expression

\[
(\nabla^m C_{jklm} = -\frac{n-3}{n-2} \left[ \nabla^2 R_{kl} - \frac{g_{kl} \nabla^2 R}{2(n-1)} - [\nabla_j, \nabla_k]R^j \right] + \frac{n-3}{2(n-1)} \nabla_k \nabla_l R.
\]

Ricci semi-symmetry and the property \( R = 0 \) give (3.1). □

**Lemma 3.2** (see \[21\] and \[23\] theorem 6.1.). A pp-wave space-time of dimension \( d + 2 \geq 4 \) with the metric \((2.5)\) satisfies \( \nabla_m C_{jklm} = 0 \) if and only if \( \nabla_k \psi = \lambda X_k \).

**Proof.** For a pp-wave space-time we have \( \nabla_j R_{kl} = (\nabla_j \psi) X_k X_l \) and \( R = 0 \). Eq.(3.2) becomes:

\[
\nabla^j \nabla^m C_{jklm} = \frac{n-3}{n-2} \left[ \nabla^2 R_{kl} - \frac{g_{kl} \nabla^2 R}{2(n-1)} - [\nabla_j, \nabla_k]R^j \right] + \frac{n-3}{2(n-1)} \nabla_k \nabla_l R.
\]

It follows that \( \nabla_m C_{jklm} = 0 \) if and only if \( (\nabla_k \psi) X_j = (\nabla_j \psi) X_k \), which is equivalent to the condition \( \nabla_k \psi = \lambda X_k \) for some scalar function. □

**Remark 3.3.** For a pp-wave the condition \( \nabla_m C_{jklm} = 0 \) is equivalent to \( \nabla_k R_{jl} = \nabla_j R_{kl} \), i.e. to the Ricci tensor being a Codazzi tensor. This follows from \((3.2)\) and \( R = 0 \).

**Proposition 3.4.** Let \( (\mathcal{M}, g) \) be a \( d + 2 \geq 4 \) dimensional pp-wave space-time with the metric \((2.5)\). If \( \psi = -\frac{1}{2} \sum_{p=1}^{d} \partial^2 H/\partial x_\rho^2 \) only depends on \( u \), then \( \nabla^2 R_{kl} = 0 \).

**Proof.** If \( \psi = \psi(u) \) we have \( \nabla_k \psi = \partial_k \psi(u) = \psi'(u) X_k \) and \( \nabla_m C_{jklm} = 0 \) by the previous lemma. Therefore \( \nabla_k R_{ij} = \psi' X_k X_i X_j \) and \( \nabla^2 R_{ij} = \psi''(u) X^k X_k X_i X_j = 0 \). □
Galaev classified the indecomposable conformally recurrent Lorentzian manifolds [12]. He showed that either the manifold is conformally flat, or \( \nabla_i R_{jklm} = 0 \), or it is a pp-wave. Then he found all pp-wave potential functions \( H(\vec{x}, u) \) such that the Weyl tensor in the metric (2.5) is recurrent, \( \nabla_i C_{jklm} = \alpha_i C_{jklm} \). The potential solves the following system of \( d + 1 \) equations:

\[
\begin{align*}
(\alpha_\rho - \partial_\rho)\Omega_{\mu\nu} &= 0 \quad (\rho = 1 \ldots d) \\
(\alpha_u - \partial_u)\Omega_{\mu\nu} &= 0
\end{align*}
\]

where \( \Omega_{\mu\nu} = \partial_\mu \partial_\nu H - \frac{1}{n} \delta_{\mu\nu} \sum_\rho \partial_\rho^2 H \) (\( \mu, \nu = 1 \ldots d \)).

Let \( \Omega^2 = \sum_{\mu\nu}(\Omega_{\mu\nu})^2 \). In an open set \( \mathcal{O} \subset \mathcal{M} \) where \( \Omega^2 \neq 0 \) the equations give \( \alpha_\mu = \frac{1}{2} \partial_\mu \log \Omega^2 \) and \( \alpha_u = \frac{1}{2} \partial_u \log \Omega^2 \). The potential for which the metric (2.5) is conformally recurrent turns out to be

\[
H(\vec{x}, u) = \sum_{\rho=1}^{d} x_{\rho}^2 [a(u) + F(u)\lambda_{\rho}^2]
\]

with functions \( a(u), F(u) \) and real numbers \( \lambda_1 + \cdots + \lambda_d = 0 \). Correspondingly, \( \psi(\vec{x}, u) = -na(u) \) and, since it only depends on \( u \), we conclude that for conformally recurrent pp-waves the divergence of the conformal tensor vanishes:

**Proposition 3.5.** On a conformally recurrent pp-wave space-time with the metric (2.5), we have \( \nabla_m C_{jklm} = 0 \); moreover there exists a domain \( \mathcal{O} \) where the recurrence covector is a gradient.

The recurrence 1-form \( \alpha_i \) being closed, it follows that \( [\nabla_p, \nabla_q]C_{jklm} = 0 \), i.e. the manifold is Weyl semi-symmetric (see [5]). Moreover, as \( [\nabla_i, \nabla_j]R_{kl} = 0 \), the manifold is also semi-symmetric, \( [\nabla_p, \nabla_q]R_{jklm} = 0 \).

Now, from Proposition 3.5, Remark 3.3 and Theorem 2.3, we have:

**Theorem 3.6.** In \( n \geq 4 \) any conformally recurrent pp-wave space-time is a simple conformally recurrent Lorentzian manifold.

Some further properties of conformally recurrent pp-wave space-times are collected here:

**Proposition 3.7.** For a conformally recurrent pp-wave space-time:

1) the recurrence covector \( \alpha_i \) is null and collinear with the covariantly constant covector \( X_i = \nabla_i u \)

2) the recurrence covector is recurrent with closed recurrence 1-form.

**Proof.** From Propositions 3.5 and 2.2 we get \( \alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0 \) i.e. the recurrence covector of a conformally recurrent pp-wave space-time
belongs to the Olszak distribution. In view of Remark 2.11 for a pp-wave space-time also \( X_i \) belongs to the Olszak distribution; then, by Lemma 2.1 \( \alpha_i = \mu X_i \). Taking the covariant derivative, we see that \( \nabla_j \alpha_i = (\nabla_j \mu) X_i = (\nabla_j \log |\mu|) \alpha_i \equiv q_j \alpha_i \) (i.e. the recurrence vector is itself recurrent, with closed recurrence 1-form).

□

**Proposition 3.8.** Let \((\mathcal{M},g)\) be a \( d + 2 \geq 4 \) dimensional pp-wave space-time with the metric (2.5). Then there exists a coordinate domain \( \mathcal{O} \) in \( \mathcal{M} \) where \( \nabla^2 C_{jklm} = 0 \) and \( \nabla^2 R_{jklm} = 0 \).

**Proof.** From the closedness condition \( \nabla_j \alpha_i = \nabla_i \alpha_j \) we infer \( q_i \alpha_j = q_j \alpha_i \) so that \( q_j = \rho \alpha_i \) and the recurrence 1-form \( \alpha_i \) satisfies \( \nabla_j \alpha_i = \rho \alpha_i \alpha_j \) and \( \nabla^i \alpha_i = 0 \). Thus, since \( \alpha^i \alpha_i = 0 \), we have \( \alpha^i \nabla_i C_{jklm} = 0 \) and consequently \( \nabla^i \nabla_i C_{jklm} = (\nabla^i (\alpha_i C_{jklm})) = (\nabla^i \alpha_i) C_{jklm} + \alpha^i (\nabla_i C_{jklm}) = 0 \). Moreover we have \( \nabla^i \nabla_i R_{kl} = 0 \) so that \( \nabla^i \nabla_i R_{jklm} = 0 \). □

Now we consider a \( n \)-dimensional Lorentzian simple conformally recurrent manifold and show that it is a conformally recurrent pp-wave space-time.

**Theorem 3.9.** Let \((\mathcal{M},g)\) be a Lorentzian simple conformally recurrent manifold of dimension \( n \geq 4 \): then there exists a coordinate domain \( \mathcal{O} \) in \( \mathcal{M} \) where the manifold is a pp-wave space-time.

**Proof.** From Theorem 2.3 and Proposition 2.4 we have \( \nabla_m C_{jklm} = 0 \) so that \( \alpha^m C_{jklm} = 0 \) and \( \alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0 \). Taking the covariant derivative, we get \( (\nabla_p \alpha_i) C_{jklm} + (\nabla_p \alpha_j) C_{kilm} + (\nabla_p \alpha_k) C_{ijlm} = 0 \). Since in a Lorentzian manifold the Olszak distribution is one-dimensional (Lemma 2.1), this implies proportionality, \( \nabla_i \alpha_j = p_i \alpha_j \) for some 1-form \( p_j \).

Proposition 2.6 states that a simple conformally recurrent Lorentzian manifold is Ricci recurrent, i.e. \( \nabla_k R_{jl} = \omega_k R_{jl} \) for some 1-form \( \omega_k \). Furthermore, by Theorem 2.3 the Ricci tensor is Codazzi, \( \nabla_k R_{jl} = \nabla_j R_{kl} \). Then we have

\[
(3.7) \quad \omega_j R_{kl} = \omega_k R_{jl}.
\]

Contraction with \( g^{jl} \) gives \( \omega_l R_{kl} = 0 \) because \( R = 0 \) (theorem 2.3). Thus on multiplying (3.7) by \( \omega_k \) it follows that \( (\omega^k \omega_k) R_{jl} = 0 \), i.e. \( \omega_k \) is a null vector.

Let \( \theta^k \) be a vector such that \( \theta^k \omega_k = 1 \). Eq.(3.7) gives \( R_{jl} = \omega_j \theta^k R_{kl} \) and, by symmetry, \( \omega_j \theta^k R_{kl} = \omega_l \theta^k R_{kj} \). Thus \( \theta^i R_{jl} = \omega_l (\theta^k \theta^i R_{kj}) \) from which we finally get:

\[
(3.8) \quad R_{ij} = \psi \omega_i \omega_j
\]
where $\psi = \theta^m \theta^j R_{mj}$ is a scalar function.

Let $d = \psi / |\psi|$ and $d_j = \omega_j \sqrt{|\psi|}$. Then $R_{ij} = dd_i d_j$ with $d^k d_k = 0$. By Proposition 2.5, equations (2.3) and (2.4) are recovered. In this way the vector $d_j$ belongs to the one-dimensional Olszak distribution: thus $d_i = \epsilon \alpha_i$ and the Ricci tensor takes the form $R_{ij} = \phi \alpha_j \alpha_i$. Furthermore, from

\[(3.9) \quad \alpha_i R_{jklm} + \alpha_j R_{kilm} + \alpha_k R_{ijlm} = 0\]

Remark 2.11 gives: $\alpha^m R_{jklm} = 0$. Taking the covariant derivative of $\nabla_k \alpha_l = p_k \alpha_l$ and using skew symmetrisation, we obtain $\alpha^m R_{jklm} = (\nabla_j p_k - \nabla_k p_j) \alpha_l = 0$. Thus the covector $p_j$ is locally a gradient, i.e. there exists a coordinate domain $O$ of $(\mathcal{M}, g)$ where $p_j = \nabla_j \eta$. The covector $\alpha_j = \alpha_j e^{-\eta}$ is covariantly constant ($\nabla_k \alpha_j = 0$). Eq.(3.9) with $\bar{\alpha}_j$ is Schimming’s condition (2.7) for the manifold to be locally a pp-wave space-time.

4. SOME CONSEQUENCES FOR EXTENDED THEORIES OF GRAVITATION

We derive some consequences for extended theories of gravitation in dimensions $n \geq 4$. Since for a pp-wave the scalar curvature is zero, a pp-wave solves the Einstein’s field equations (in natural units)

\[R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}\]

with a pure radiation source, $T_{ij} = \Phi^2 k_i k_j$, where $k_i k^i = 0$ (see [37], eq. 5.8).

Generic theories of gravitation modify Einstein’s theory at short distances by expressing the action integral with zero source as a power series

\[(4.1) \quad I = \int d^n x \sqrt{-g} \left[ R - 2\Lambda_0 + \alpha R^2 + \beta R_{ij} R^{ij} \right.
\left. + \gamma (R_{jklm} R^{jklm} - 4 R_{jk} R^{jk} + R^2) + \sum_{p>2} C_p (\text{Riem}, \text{Ric}, \nabla\text{Riem}, \ldots) \right]\]

where $\alpha$, $\beta$, $\gamma$, $C_p$ are parameters provided by some microscopic theory such as string theory. The quadratic part represents “quadratic gravity” (Weyl-Eddington terms), and the terms of higher order are contracted products of the Riemann tensor and its derivatives. Such corrections scale as powers of $\ell_p / L$, where $\ell_p$ is Planck’s length and $L$ is a typical length for the variation of the metric [40], and could be significant in the early evolution of the universe.

The field equations of the full theory are complicated; nevertheless Gürses et al. [14] proved that for pp-wave metrics (2.5) the field equations with
cosmological constant $\Lambda_0 = 0$ may be written as:

\[
[a_0 + a_1 \nabla^2 + a_2 (\nabla^2)^2 + \ldots] R_{kl} = 0
\]

where $a_i$ are constants depending on $\alpha, \beta, \gamma, C_p$. If $\nabla^2 R_{kl} = 0$ the equation reduces to $a_0 R_{kl} = 0$, with the vacuum solution. In the same paper the authors noted that for pp-waves with $\nabla^2 R_{kl} = 0$ the field equations may include a pure radiation source:

\[
a_0 R_{kl} = T_{kl}
\]

It is thus interesting to find non-vacuum pp-wave solutions. According to Proposition 3.4, if $\psi = -\frac{1}{2} \sum_{\rho=1}^{d} \partial_\rho^2 H$ depends only on $u$, then $\nabla^2 R_{kl} = 0$.

**Proposition 4.1.** Let $(\mathcal{M}, g)$ be a $d+2 \geq 4$ dimensional pp-wave space-time with the metric (2.5): if $\sum_{\rho=1}^{d} \partial_\rho^2 H$ depends only on $u$, then the metric solves the field equations of the generic theory of gravitation with pure radiation source.

We give two examples where $\nabla^2 R_{kl} = 0$:

1) the pp-wave metric is conformally recurrent, with potential $H$ given by (3.6).

2) the pp-wave metric is two-symmetric i.e. $\nabla_i \nabla_k R_{jklm} = 0$. This occurs if and only if the potential function of the pp-wave has the form

\[
H(\vec{x}, u) = \sum_{\mu, \nu=1}^{d} (u a_{\mu} \delta_{\mu\nu} + b_{\mu\nu}) x^\mu x^\nu
\]

where $0 \leq a_1 \leq \cdots \leq a_d$ and $b_{\mu\nu} = b_{\nu\mu}$ are real numbers (see [2, 3, 13]). One evaluates $\psi(u) = -\sum_{\mu=1}^{d} (u a_{\mu} + b_{\mu\mu})$. Thus for two-symmetric pp-waves space-times the divergence of the conformal tensor vanishes and, as one may directly calculate, $\nabla^2 R_{kl} = 0$.

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I.I.S. Lagrange, Via L. Modignani 65, 20161, Milano, Italy
E-mail address: carloalberto.mantica@libero.it

Physics Department, Università degli Studi di Milano, and I.N.F.N. sez. Milano, Via Celoria 16, 20133 Milano, Italy
E-mail address: luca.molinari@unimi.it