QCD traveling waves at non-asymptotic energies

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Using consistent truncations of the BFKL kernel, we derive analytical traveling-wave solutions of the Balitsky-Kovchegov saturation equation for both fixed and running coupling. A universal parametrization of the “interior” of the wave front is obtained and compares well with numerical simulations of the original Balitsky-Kovchegov equation, even at non-asymptotic energies. Using this universal parametrization, we find evidence for a traveling-wave pattern of the dipole amplitude determined from the gluon distribution extracted from deep inelastic scattering data.

I. INTRODUCTION

In the past decade, there has been a large amount of work devoted to the description and understanding of quantum chromodynamics (QCD) in the high-energy/density limit corresponding to saturation. On the theoretical side, progress is made in obtaining non-linear QCD equations describing the evolution of scattering amplitudes towards saturation. On the phenomenological side, the discovery of geometric scaling [1] in deep inelastic scattering (DIS) at HERA was a stimulating observation. Indeed geometric scaling has a natural explanation [2] in terms of traveling-wave solutions of the Balitsky-Kovchegov (BK) non-linear equation [3, 4].

The BK equation is an equation for the evolution in rapidity \( Y \) of the scattering amplitude \( N(r, b, Y) \) of a dipole (a colorless \( q \bar{q} \) pair) off a given target. \( r \) is the transverse size of the dipole and \( b \) its impact parameter. The rapidity \( Y \) is the logarithm of the squared total center-of-mass energy of the collision. Assuming that the amplitude \( N \) is \( b \)-independent, the BK equation has been shown [2] to lie in the same universality class as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [5], extensively studied in statistical physics.

The FKPP equation is an evolution equation for the function \( u(x, t) \) where \( x \) is a space variable and \( t \) the time. It admits asymptotic solutions which are functions of the sole variable \( x-vt \), namely traveling-wave solutions \( u(x-vt) \) where \( v \) is the speed of the wave. It comes from a critical regime obtained by the competition of the exponential growth in the linear regime and the non-linear damping. This mechanism for the formation of traveling waves is more general [6] and applies to the BK equation for which time is replaced by rapidity and the logarithm of the dipole size plays the role of the position. It has even been extended beyond the \( b \)-independent case [8] to processes with non-zero momentum transfer.

However there are limitations for applying these solutions precisely because they are asymptotic, i.e. they require very large values of \( Y \) to appear; also analytical expressions are only known for relatively small values of \( r = |r| \). This leads to a limitation of analytical predictions to the tail of the wave \( (N \ll 1) \) which is essentially determined by the linear regime. For instance, the “interior” of the wave is not reproduced; it corresponds to the transition to saturation, an intermediate regime between the tail \( (N \ll 1) \) and the saturated region \( (N \sim 1) \). That moderates the phenomenological impact of those mathematically powerful results.

In order to circumvent these difficulties, it is necessary to extend the study to the non-asymptotic regime in a region of larger \( N \), which implies to use more information on the non-linear term of the equation than the previous approaches. For this sake, a new approach to study the BK equation has been developed in [9]. In short, the guiding idea is to try and find exact solutions of approximate QCD equations rather than approximate solutions of exact QCD equations.

The goal of this paper is to derive these analytical solutions and to develop their applications in various cases including the important running-coupling case. This allows to make a direct and model-independent comparison for saturation between QCD predictions and phenomenology.

The plan of the paper is as follows. In Section II, we define consistently truncated BK equations for which we derive exact traveling-wave solutions in the fixed-coupling case (IIA) and study the validity of the method by comparison with numerical simulations of the original BK equation (IIB). In Section III, we find solutions to the BK equation...
with running coupling. In Section IV, we apply the results to DIS phenomenology. Conclusions and outlook are given in Section V.

II. THE FIXED-COUPLING CASE

Let us consider the Fourier transform of the dipole scattering amplitude defined as

\[ \mathcal{N}(k, Y) = \int_0^\infty \frac{dr}{r} J_0(kr) N(r, Y). \] (1)

The BK equation for \( \mathcal{N}(k, Y) \) then reads [4]

\[ \partial_Y \mathcal{N} = \chi(-\partial_L) \mathcal{N} - \mathcal{N}^2 \] (2)

where \( L = \log \left( \frac{k^2}{k_0^2} \right) \) and the rapidity \( Y \) is measured here in units of \( \bar{\alpha} \equiv \frac{\alpha_s N_c}{N_c} \) where the strong coupling constant \( \alpha_s \) is kept fixed.

\[ \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \] (3)

is the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel [10], \( k_0 \) is an arbitrary scale.

It has been proved [2] that the equation (2) admits asymptotic traveling-wave solutions \( \mathcal{N}(L-vY) \) where \( v \) is the speed of the wave. Starting with the initial condition \( \mathcal{N}(L, Y_0) \sim \exp(-\gamma L) \), traveling-wave solutions are formed during the \( Y \)-evolution with the following value for the speed:

- “pushed front”: \( 0 < \gamma_i < \gamma_c; \ v = \chi(\gamma_i)/\gamma_i \),
- “pulled front”: \( \gamma_i \geq \gamma_c; \ v = v_c \equiv \chi(\gamma_c)/\gamma_c \),

where \( \gamma_c \) is solution of the equation \( \chi(\gamma)/\gamma = \chi'(\gamma) \) and is called the critical anomalous dimension. Its value is \( \gamma_c = 0.6275 \) leading to the critical velocity \( v_c = 4.883 \). Note that [2] only the large-\( L \) behavior of the initial condition matters to determine whether one lies in a pushed or pulled-front case.

A. Exact traveling-wave solutions of truncated BK equations

In order to avoid mathematical difficulties related to the infinite-order differential equation [2], let us now formally consider a set of truncated approximations of the BK equation:

\[ \partial_Y \mathcal{N} = \chi_{P,\gamma_0}(-\partial_L) \mathcal{N} - \mathcal{N}^2 \] (4)

where each truncated kernel \( \chi_{P,\gamma_0} \) is given by the expansion of the original kernel \( \chi \) around \( \gamma_0 \) \((0 < \gamma_0 < 1)\) up to order \( P \geq 2 \):

\[ \chi_{P,\gamma_0}(-\partial_L) = \sum_{p=0}^{P} \frac{\chi^{(p)}(\gamma_0)}{p!} (-\partial_L - \gamma_0)^p = \sum_{p=0}^{P} (-1)^p A_p \partial_L^p. \] (5)

\( P \) and \( \gamma_0 \) are then two parameters which define a given approximation of the original kernel. Each particular kernel \( \chi_{P,\gamma_0} \) is polynomial in \( -\partial_L \) and its coefficients \( A_p \) can be fully computed from the original kernel \( \chi \): they are given by

\[ A_p = \sum_{i=0}^{P-p} (-1)^i \frac{\chi^{(i+p)}(\gamma_0)}{i! \ p!} \gamma_0^i. \] (6)

For the sake of simplicity, we do not write the subscript \( P,\gamma_0 \) of the coefficients \( A_p \). We want to consider kernels \( \chi_{P,\gamma_0} \) that are somewhat close to \( \chi \). To be more specific, we shall limit our study to kernels that are positive definite and feature \( A_0 > 0, A_1 < 0 \) and \( A_2 > 0 \). We also require that, just as the original BK equation [2], the equations [4] admit asymptotic traveling-wave solutions. In other words, for a kernel \( \chi_{P,\gamma_0} \) to be considered, the equation \( \chi_{P,\gamma_0}(\gamma)/\gamma = \chi_{P,\gamma_0}'(\gamma) > 0 \) should have a unique solution \( \gamma_c^{P,\gamma_0} \), which we shall call the critical anomalous dimension. We shall also denote the corresponding critical velocity \( v_c^{P,\gamma_0} \).
traveling waves. Indeed, consider the anzatz

\[ A_0 \mathcal{N} - \mathcal{N}^2 - \partial_Y \mathcal{N} - A_1 \partial_L \mathcal{N} + \sum_{p=2}^{P} (-1)^p A_p \partial_p^2 \mathcal{N} = 0 . \]

The advantage of this equation w.r.t. the original BK equation is that it can be exactly solved in terms of traveling waves. Indeed, consider the anzatz

\[ \mathcal{N}(L,Y) = A_0 \ U(s) , \]

with the scaling variable \( s \) given by

\[ s = \frac{\lambda}{c} L - \left( A_0 + \frac{\lambda}{c} A_1 \right) Y \]

where \( \lambda = \sqrt{A_0/A_2} \) and \( c \) is a parameter related to the speed of the wave. The equation for the traveling wave \( U \) then becomes the following ordinary differential equation

\[ U(1 - U) + U' + \frac{1}{c^2} U'' + \sum_{p=3}^{P} \frac{(-1)^p \lambda^p}{c^p} \frac{A_p}{A_0} U^{(p)} = 0 . \]

Let us define

\[ v(c) = A_1 + c \sqrt{A_0 A_2} \]

such that \( s = \lambda (L - v(c) Y)/c \) and that \( v(c) \) is the speed of the traveling wave. For a given kernel \( \chi_{P,\gamma_0} \), the values of \( c > -A_1/\sqrt{A_0 A_2} \) for which equation (10) has a solution define the possible values of the speed \( v(c) \) of the traveling-wave solution.

It is crucial to notice that, in this approach, \( c \) is a free and thus adjustable parameter. It means that in general, there is a continuum of solutions with different speed to the equation (10) or equivalently (7). For instance, if one wants to describe asymptotic solutions, we are led to choose the value of \( c \) such that

\[ \begin{cases} v(c) = v_c &= \chi_{P,\gamma_0}^{P,\gamma_0} \gamma_c^{P,\gamma_0} \text{ if } \gamma_i \geq \gamma_c \\ v(c) &= \chi_{P,\gamma_0}^{P,\gamma_0}(\gamma_i)/\gamma_i \text{ if } 0 < \gamma_i < \gamma_c . \end{cases} \]

Indeed, once the initial condition is fixed, the asymptotic speed is \( v_c^{P,\gamma_0} \) or \( \chi_{P,\gamma_0}^{P,\gamma_0}(\gamma_i)/\gamma_i \) depending on the initial condition, as confirmed by numerical simulations. We shall use the freedom on \( c \) to also examine non-asymptotic properties.

Equation (10) provides a \( 1/c^p \) expansion in with the \( 1/c \) term missing. The key point of the method is that \( 1/c \) is indeed a small parameter allowing to find an iterative solution

\[ h(s) = h_0(s) + \frac{1}{c^2} h_2(s) + \sum_{p \geq 3} \frac{1}{c^p} h_p(s) = \frac{1}{2} - U(s) . \]

Inserting it into (10) translates into a hierarchy of equations

\[ \begin{align*}
    h_0' + h_0^2 - 1/4 &= 0 \\
    h_2' + 2h_0 h_2 + h_0' &= 0 \\
    h_3' + 2h_0 h_3 - \lambda^3 A_3 h_0''/A_0 &= 0 \\
    h_4' + 2h_0 h_4 + h_2^2 + h_2'' + \lambda^4 A_4 h_0'''/A_0 &= 0 \\
    &\vdots
\end{align*} \]

where we have written the equations up to \( \mathcal{O}(1/c^3) \). An iterative solution can easily be found with initial conditions being appropriately chosen, i.e. \( h_0(\pm \infty) = \pm \frac{1}{4} \) and \( h_{i \neq 0}(\pm \infty) = h_i(0) = 0 \). Note that if \( h(s) \) is solution, also \( h(s + s_0) \)
FIG. 1: The function $U(s)$ solution of the equation $U(1-U)+U''/c^2=0$ for $c=5/\sqrt{6}$ with $U(-\infty)=1$ and $U(0)=1/2$. Full line: exact solution (18). Dashed line: solution (16) containing only the first order. Dotted line (hardly distinguishable from the full line): solution (16) containing only the first and second order.

is solution; this freedom corresponds to the possibility of arbitrarily redefining $k_0$. The system is iteratively but fully solvable. One first solves the only non-linear equation of the hierarchy, obtaining $h_0=\frac{1}{2}\tanh(\frac{s}{2})$. Using the property

$$\frac{d}{ds} h_n(s) + 2 h_0 h_n(s) = \frac{1}{\cosh^2(s/2)} \frac{d}{ds} \left[ \cosh^2(s/2) h_n(s) \right], \quad (15)$$

all the other linear equations reduce to simple integrations of functions defined recursively from the hierarchy. The solution of the first three terms of the expansion is

$$U(s) = \frac{1}{1+e^s} - \frac{1}{c^2} \frac{e^s}{(1+e^s)^2} \log \left[ \frac{(1+e^s)^2}{4e^s} \right] - \frac{\lambda^3 A_0}{c^3 A_1} e^s \frac{1}{(1+e^s)^2} \left[ 3 \frac{(1-e^s)}{(1+e^s)} + s \right] + O\left( \frac{1}{c^4} \right). \quad (16)$$

The first two terms of the expansion (16) (order $1/c^0$ and $1/c^2$) are universal: they do not depend neither on $P$ nor on $\gamma_0$. Indeed equation (4) with any kernel $\chi_{P,\gamma_0}$ admits solutions whose first two terms (in the $1/c$ expansion) are those of (16). The solutions for different kernels only differ through the definition of the scaling variable $s$ which depends on $P$ and $\gamma_0$. In this sense, we obtain a universal parametric solution.

Let us concentrate on the case $P=2$. The equation (11) reduces to the known FKPP equation and one obtains the corresponding traveling-wave equation:

$$U(1-U)+U' + \frac{1}{c^2} U'' = 0. \quad (17)$$

One has $\chi_{2,\gamma_0}(\gamma_i)/\gamma_i = A_0/\gamma_i + A_1 + A_2 \gamma_i$ and $\chi_{2,\gamma_0}' = A_1 + 2\sqrt{A_0}A_2$ and therefore, using (12) one should take $c=\lambda/\gamma_i+\gamma_i/\lambda$ in the “pushed front” case and $c=2$ in the “pulled front” case to reproduce the correct critical speed.

In order to check the efficiency of the iterative solution, we use the existence of an analytic solution (11) to equation (17), for the particular value of $c=5/\sqrt{6}\approx 2.04$ :

$$U(s) = \left[ 1 + (\sqrt{2} - 1) \exp \left( \frac{cs}{\sqrt{6}} \right) \right]^{-2}. \quad (18)$$

This exact solution allows one to see that limiting the expansion after the second order is a very good approximation: in Fig.1 we have plotted the exact solution (18) along with the solution given by the expansion (16) limited after first and second order with $c=5/\sqrt{6}$. It is very hard to distinguish the exact solution from the solution at second order.
In terms of physical variables, the solution (16) of equation (4) reads:

\[
\mathcal{N}(k, Y) = \frac{A_0}{1 + \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\lambda/c}} - \frac{A_0}{1 + \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\lambda/c}} \log \left( 1 + \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\lambda/c} \right) + O \left( \frac{1}{c^3} \right),
\]

where \(Q_s^2(Y) = k^2 \exp \left[ \bar{\alpha} v(c) \right] \) plays the role of the saturation scale and where we have restored the \( \bar{\alpha} \) dependence of \( Y \). In the following we fix \( \bar{\alpha} = 0.2 \). The \( A_p \) coefficients are given by (6), \( \lambda = \sqrt{A_0} / A_2 \) and \( c \) is chosen so that the traveling speed \( v(c) = A_1 + c\sqrt{A_0} \) in the “pulled front” case and \( \chi_{P, \gamma_0} (\gamma_i) / \gamma_i \) in the “pushed front” case, see (12). In the following, we shall only use the first two terms given in (19) which provide a good approximation of the full traveling-wave solution.

Let us compare numerical solutions of the BK equation with the solution (19) for \( P = 2 \). For the numerical simulations, the initial condition we chose is \( \mathcal{N}(L, Y_0) = e^{-\gamma_i L / (2\gamma_i)} \) if \( L \geq 0 \) and \( \mathcal{N}(L, Y_0) = (1 - \gamma_i L) / (2\gamma_i) \) if \( L < 0 \).

The comparison is shown in Fig. 2: the “pulled front” case is represented on the left plot for \( \gamma_i = 1 \) and with the kernel expanded around \( \gamma_0 = \gamma_c \), the critical value. The “pushed front” case is represented on the right plot for \( \gamma_0 = \gamma_i = 0.5 \). One sees that as soon as the asymptotic traveling-wave regime is reached, there is a good description of the “interior” of the wave, as defined in Ref. [7]. In both cases, one sees that equation (19) does not describe neither the saturated region (small \( k \)) nor the “leading-edge” (large \( k \)). Indeed, the exact scaling properties of traveling waves are mathematically expected only in the interior region.

FIG. 2: The traveling wave \( \mathcal{N}(L, Y) \) (fixed coupling) at large \( Y \) as a function of \( L \). The different values of \( Y \) are 0, 20, 40, ..., 100. The full lines are numerical solutions of the BK equation and the dashed line are the scaling solutions (19) for \( P = 2 \). Left plot, “pulled front” case \( (\gamma_0 = \gamma_c, \gamma_i = 1) \): the asymptotic traveling-wave regime is reached after some evolution with the critical speed \( v_c = 4.883 \). Right plot, “pushed front” case \( (\gamma_0 = \gamma_i = 0.5 < \gamma_c) \): the asymptotic regime is driven by the initial condition and the critical speed is \( \chi_{2, 1/2} (1/2) / (1/2) = 5.55 \).

For our purpose, it is useful to investigate the non-asymptotic regime since it is of interest for phenomenological applications. In Fig. 3, we compare the numerical solutions of the BK equation with the solution (19), in the pulled front case, where we adjusted the parameter \( c \) to describe the speed of the wave \( v \) at moderate values of the rapidity \( Y \) up to 20. Again the agreement in the interior region is satisfactory and it shows that the solution we obtained can be used also in the physical range of non-asymptotic rapidities.

1 This form ensures correct infrared and ultraviolet behaviors with an appropriate matching. The details of this matching are not relevant for the asymptotic properties.
Let us finally point out the role of the kernel truncation. First, the truncation up to a finite \( P \) is mathematically necessary to apply our method. Indeed, when considering the full analytic expansion to define the coefficients \( A_i \) (i.e. \( P \to \infty \)), they are infinite due to the \( \gamma = 0 \) singularity of the BFKL kernel. This explains why the solutions describe the interior region of the BK front but not the whole \( k \) range. For instance, one does not describe the saturated region [12]: formula (19) describes fronts which saturate at \( A_0 \) while the BK front behaves as \( \log 1/k \) when \( k \) is very small. A second remark is that we observed, e.g. by comparison of the results of the \( P = 2 \) and \( P = 4 \) truncations, that a good description of the interior region of the wave is already obtained for \( P = 2 \), i.e. the diffusive approximation of the BFKL kernel. In this paper, we shall thus stick to \( P = 2 \).

III. THE RUNNING-COUPLING CASE

Let us now perform the analysis in the running-coupling case with

\[
\frac{N_c}{\pi} \alpha_s(k^2) = \frac{1}{bL}, \quad b = \frac{11N_c - 2N_f}{12N_c}, \quad L = \log \left( \frac{k^2}{\Lambda_{QCD}^2} \right),
\]

and \( \Lambda_{QCD} = 200 \text{ MeV} \). We thus write the following form of the BK equation with running coupling constant and leading-order BFKL kernel [3]:

\[
bL \partial_Y N = \chi (-\partial_L) N - N^2.
\]

It has been proved [2] that this equation admits asymptotic traveling-wave solutions of the form \( N(L - \sqrt{(2v/b)Y}) \) where as before \( v = v_c \) in the “pulled front” case and \( v = \chi(\gamma_i)/\gamma_i \) in the “pushed front” case. Let us consider the same truncated kernels than in the previous section. That transforms the equation into

\[
A_0 N - N^2 - bL \partial_Y N - A_1 \partial_L N + \sum_{p=2}^P (-1)^p A_p \partial_L^p N = 0
\]

and we consider again the following form

\[
N(L,Y) = A_0 \, U(\tilde{s}).
\]

Our strategy is to reduce the problem to the one we solved in the previous section through an appropriate change of variables. We require that the universal terms in the corresponding equation for \( U \) are the same as before: \( U(1-U)+U' \).
Postulating \( \tilde{s} = L \phi(Y/L^2) \) determines a solution for \( \phi \) and leads to the scaling variable

\[
\tilde{s} \equiv L \left( - \frac{A_0}{A_1} - \frac{1}{\tilde{c}} \sqrt{b - 2 A_1} \frac{Y}{L^2} \right)
\]  

(24)

where \( \tilde{c} \) is a free parameter. Putting this into (22) gives

\[
U(1 - U) + U' + \sum_{p=2}^{P} \left( \frac{A_0}{A_1} \right)^p \frac{A_p}{A_0} U^{(p)} + \mathcal{O} \left( \frac{1}{\tilde{c}} \right) = 0.
\]  

(25)

The leading-order terms in \( 1/\tilde{c} \) are of the same generic form as equation (11) and thus one can apply the previous method. In this running-coupling case, the \( \mathcal{O}(1/\tilde{c}) \) terms that we shall neglect contain scaling-violation corrections decreasing as \( 1/L \). Using the method described in the previous Section, one obtains the following solution

\[
U(\tilde{s}) = \frac{1}{1+e^{\tilde{s}}} - \left( \frac{A_0}{A_1} \right)^2 \frac{A_2}{A_0} \frac{e^{\tilde{s}}}{(1+e^{\tilde{s}})^2} \log \left[ \frac{(1+e^{\tilde{s}})^2}{4e^{\tilde{s}}} \right] + \mathcal{O} \left( \frac{A_3}{A_1^2} \right)
\]  

(26)

where the expansion parameter is \( A_0/A_1 \). As before, we shall only consider two first terms of the expansion. In order to describe solution to the BK equation (21), one has now to select the speed of the wave by fixing the parameter \( \tilde{c} \). Writing at large \( Y \)

\[
\tilde{s} \simeq - \frac{A_0}{A_1} \left( L + \frac{A_1}{A_0} \tilde{c} \sqrt{2A_1} Y \right)
\]  

(27)

and matching with the asymptotic form \( \mathcal{N}(L-\sqrt{(2v/b)} Y) \), one gets

\[
v(\tilde{c}) = - \frac{b A_3}{A_0^2 \tilde{c}^2}.
\]  

(28)

Adjusting \( \tilde{c} \) to obtain a given speed, we check that \( 1/\tilde{c} \) is indeed small. Defining \( \tilde{Q}_4(Y) = \Lambda_{QCD}^2 \exp(\sqrt{(2v(\tilde{c})/b)} Y) \) that plays the role of the scaling variable, the scaling variable reads in terms of physical variables

\[
\tilde{s} = - \frac{A_0}{A_1} \log \left( \frac{k^2}{\Lambda_{QCD}^2} \right) - \frac{1}{\tilde{c}} \sqrt{b \log^2 \left( \frac{k^2}{\Lambda_{QCD}^2} \right) + \frac{A_0^2}{A_1^2} \tilde{c}^2 \log^2 \left( \frac{\tilde{Q}_4(Y)}{\Lambda_{QCD}^2} \right)}.
\]  

(29)

Formula (29) along with

\[
\mathcal{N}(k,Y) = \frac{A_0}{1+e^{\tilde{s}}} - \frac{A_0^2 A_2}{A_1^4} \frac{e^{\tilde{s}}}{(1+e^{\tilde{s}})^2} \log \left[ \frac{(1+e^{\tilde{s}})^2}{4e^{\tilde{s}}} \right]
\]  

(30)

is the traveling-wave solution of the truncated BK equation with running coupling (21) at \( \mathcal{O}(1/\tilde{c}) \).

Let us compare numerical solutions of the full BK equation (21) with that parametrization. For the numerical simulations, the initial condition is the same as in the fixed-coupling case. The problem of the Landau pole of the running coupling is dealt with by introducing a regulator: \( \log(k^2/\Lambda_{QCD}^2) \rightarrow \log(\kappa+k^2/\Lambda_{QCD}^2) \) with \( \kappa = 3 \). As well as the details of the initial condition, this does not modify the traveling-wave pattern.

The comparison is represented in Fig.4 where the “pulled front” case is considered. The values of the coefficients \( A_0 = 9.55, A_1 = -25.6 \) and \( A_2 = 24.3 \) have been fixed by considering \( P = 2 \) and \( \gamma_0 = \gamma_c \). We use the free parameter \( \tilde{c} \) to adjust the speed of the wave. On the left plot, high values of the rapidity \( Y = 0, 10, 20, ..., 100 \) are considered while on the right plot, moderate values \( Y = 0, 2, 4, ..., 20 \) are represented. In both cases, in order to describe the numerical solutions, one has to adjust the actual speed of the wave \( v(\tilde{c}) \) to the value of \( 3.1 \) for the left plot and \( 2.3 \) for the right plot which are both significantly smaller than the critical value \( v_c \simeq 4.88 \). It confirms that the establishment of the asymptotic regime is much slower in the running-coupling case than in the fixed coupling case [2]. Indeed, the theoretical sub-asymptotic corrections to the critical speed have been computed [2] and predict such a behavior. There is a good description of the interior of the wave for the different ranges of rapidity, provided we adjust the speed of the wave. As expected, the parametrization does not describe the saturated and the leading-edge regions. Note that the scaling region at low values of \( Y \) is less extended in the running coupling case (Fig.3) than for fixed coupling (Fig.4), this could be due to the scaling-violation terms that we had to neglect in (25). However it is clear that an approximative traveling-wave pattern already emerges at moderate rapidities.
that the parameter \(A\) result of the fit, the obtained parameters being \(A\).

Finally, we use the relation
\[
\frac{\partial}{\partial k^2} xg(x, k^2)
\]
that \(f(Y, k^2)\) depends on the parameters \(A_0, A_1, \) and \(v\) the speed of the wave. We discarded the second term of \(30\) since we found that the parameter \(A_2\) stays undetermined by the fit in the kinematical region we considered. In Fig. 5, we show the result of the fit, the obtained parameters being \(A_0 = 17.1, A_1 = -15.8, \) and \(v = 1.76.\)

These results call for comments.

IV. APPLICATION TO DEEP INELASTIC SCATTERING

We shall now draw a link between traveling waves and phenomenology. The observation of geometric scaling asks the question whether it can be viewed as a consequence of QCD traveling waves. The known problem is that the range of rapidity for which asymptotic predictions of QCD non-linear evolution equations are available remains far from the moderate rapidity range accessible to experiments. Hence the phenomenological interest of our method is to open a way of confronting theory and experiment in some physical kinematic range.

The formula relating the dipole scattering amplitude in coordinate space \(N(r, b, Y)\) to the unintegrated gluon distribution of the target \(f(Y, k^2)\) is (see e.g. [13]):
\[
\int d^2b \ N(r, b, Y) = \frac{4\pi^2\alpha_s}{N_c} \int \frac{dk}{k} f(Y, k^2)(1 - J_0(k|r|)) \ . \tag{31}
\]
In the \(b\)–independent approximation, one writes \(\int d^2b \ N(r, b, Y) = \pi R_p^2 N(r, Y)\) where \(R_p\) is the radius of the proton target. Through formulae (11) and (31) a link is established between \(N(k, Y)\) and \(f(Y, k^2)\). One gets
\[
N(k, Y) = \frac{4\pi\alpha_s}{N_c R_p^2} \int_k^\infty \frac{dp}{p} f(Y, p^2) \log \left(\frac{p}{k}\right) \ . \tag{32}
\]
Finally, we use the relation
\[
f(Y, k^2) = \frac{\partial}{\partial k^2} xg(x, k^2) \ , \tag{33}
\]
where \(Y = \log(1/x)\) between the unintegrated gluon distribution \(f(Y, k^2)\) and the gluon distribution \(xg(x, k^2)\) to obtain
\[
N(k, Y) = \frac{\pi\alpha_s}{N_c R_p^2} \int_{k^2}^\infty \frac{dt}{t^2} xg(x, t) \log \left(\frac{t}{ck^2}\right) \ . \tag{34}
\]
Note that formula (33) is a well-known approximation which may only be valid at small values of \(x\). It is useful here as it makes the analysis simpler. For a more detailed study which is beyond the scope of this work, one could consider more advanced prescriptions [14].

In order to compare (24) to our predictions, we use the running-coupling case which is known to lead to traveling waves with a speed compatible with phenomenology [13].

We performed a fit of the parametrization (29) and (30) on the amplitude (34) obtained from the Martin-Roberts-Stirling-Thorne (MRST) gluon distribution [16]. The domain we considered is \(x < 10^{-3}\) and \(\frac{\alpha_s}{\pi}R_p^2 N(k, Y) > 0.01\) which is the region where traveling-wave patterns can be tested. Note that we considered only the first term of (30) which depends on the parameters \(A_0, A_1, \) and \(v\) the speed of the wave. We discarded the second term of (30) since we found that the parameter \(A_2\) stays undetermined by the fit in the kinematical region we considered. In Fig. 5, we show the result of the fit, the obtained parameters being \(A_0 = 17.1, A_1 = -15.8, \) and \(v = 1.76.\)
i) The dipole amplitude obtained from the MRST distribution (full lines on Fig.5) displays a structure compatible with an evolution towards traveling waves, namely a steeper slope at small rapidities which evolves toward a less steep and regular pattern at higher rapidities.

ii) Fig.5 shows a compatibility with the parametric form ((29), (30)) at high values of $Y$ and moderate values of $k^2$.

iii) On the theoretical ground, it is worth comparing the fitted parameters to various predictions. For the truncated BK kernel with $P = 2$ and $\gamma_0 = \gamma_c$, one finds $A_0 = 10$, $A_1 = -25$, and the speed $2.3 < v < 3$ when one includes pre-asymptotic corrections [2]. With $P = 4$ and $\gamma_0 = 1/2$, one finds $A_0 = 11$, $A_1 = -49$. The obtained parameters are qualitatively closer to the diffusive approximation ($P = 2$) of the BK equation but they are quantitatively different. This seems to suggest the presence of next-leading or higher-order corrections to the QCD kernel. This deserves further study.

iv) Note that we obtain $|A_0/A_1| \sim 1$ for the observed parameters which could invalidate the expansion (26). We checked both numerically and from the analytical form that the second and higher order terms stay sufficiently small to be still neglected.

V. CONCLUSION

Let us summarize the main results of our study.

- We found iterative traveling-wave solutions to non-linear BK evolution equations obtained by finite truncation of the BFKL kernel for both fixed and running coupling constant.

- These solutions exhibit universality properties, the first two dominant terms of the iteration have a parametric form independent of the truncation of the kernel and of the fixed or running coupling cases.

- The scaling variable is found to be a combination of $L = \log(k^2)$ and rapidity $Y$ which is given by formula (24) (resp. (23)) for fixed (resp. running) coupling.

- The obtained traveling-wave solutions match with the asymptotic solutions of the BK equation. This was verified by analytical and numerical checks. The remarkable new property is that they also match with the non-asymptotic behavior of the interior of the wave, provided an adjustment of the speed of the wave which is a free parameter in the iterative approach.

- As an application of the method and its validity at non-asymptotic energies, we considered the dipole amplitude in momentum space $N(k,Y)$ obtained from the MRST parametrization of the gluon distribution. We found evidence for an evolution pattern compatible with the formation of traveling waves. The obtained dipole amplitude is well described by the universal parametric form (30). The obtained parameters seem to point towards traveling-wave solutions for a BFKL kernel modified by higher orders and/or non-perturbative corrections.

Our results help establishing a stronger link between the theory of saturation and the phenomenological observation...
of geometric scaling [1]. Moreover, simple analytical parametrizations of the solutions of the BK equation in momentum space could help extending the present phenomenology [17] to a broader range of kinematics or observables. It would be interesting to investigate where one could distinguish between a DGLAP structure and traveling waves. On a more theoretical level, it seems feasible to extend the method to QCD evolution equations beyond the “mean field” BK equation [2]. In particular the extension to the stochastic versions of QCD evolution equations [18] would help our understanding of QCD in the high-energy limit.

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