NEUTRABELIAN ALGEBRAS

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Abstract. We introduce “neutrabelian algebras”, and prove that finite, heredi-
tarily neutrabelian algebras with a cube term are dualizable.

1. INTRODUCTION

This paper investigates the relationship between two commutator properties for
finite algebras in congruence modular varieties. The investigation is motivated by the
problem of determining which finite algebras in residually small congruence modular
varieties are dualizable in the sense of [2], i.e., which finite algebras in residually
small congruence modular varieties may serve as the character algebra for a natural
duality.

The first commutator property, new to this paper, is the property of being neutrabelian. This word is a portmanteau of neutral and abelian. Recall that if A is
any algebra in a congruence modular variety, then

\[ 0 \leq [\alpha, \beta] \leq \alpha \land \beta \]

for any two congruences \( \alpha \) and \( \beta \) on \( A \). The (universally quantified) commutator identity \((C4)\),

\[ [\alpha, \beta] = 0, \]

represents one extreme type of commutator behavior, where the left inequality in \((1.1)\) is equality for all \( \alpha, \beta \in \text{Con}(A) \). An algebra satisfying \((C4)\) is called abelian.

The other extreme behavior, where the right inequality in \((1.1)\) is equality for all
\( \alpha, \beta \in \text{Con}(A) \), is represented by the commutator identity \((C3)\),

\[ [\alpha, \beta] = \alpha \land \beta. \]

An algebra satisfying \((C3)\) is called neutral.

The neutrabelian property is a combination of \((C3)\) and \((C4)\).

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Definition 1.1. Let $A$ be a subdirectly irreducible algebra (= SI) in a congruence modular variety. Let $\mu$ be the monolith of $A$ and let $\nu = (0 : \mu)$ be the centralizer of the monolith. $A$ is a **neutrelabelian SI** if $\nu$ is comparable with all congruences of $A$, and the commutator operation on $\text{Con}(A)$ satisfies

"(C4)" $[\alpha, \beta] = 0$ if $\alpha$ and $\beta$ are both contained in $\nu$, and

"(C3)" $[\alpha, \beta] = \alpha \land \beta$ otherwise.

In words, this says that a neutrelabelian SI is a subdirectly irreducible algebra where $\nu$ is comparable to all other congruences and the commutator satisfies (C4) below $\rho$ and (C3) elsewhere. An SI algebra is abelian if and only if it is a neutrelabelian SI with $\nu = 1$, while an SI algebra is neutral if and only if it is a neutrelabelian SI with $\nu = 0$.

The shape of $\text{Con}(A)$, when $A$ is a neutrelabelian SI, is depicted in Figure 1. The lattice on the left indicates the situation where the monolith of $A$ is nonabelian, while the lattice on the right indicates the situation where the monolith is abelian.

![Figure 1. Con(A), µ nonabelian versus µ abelian](image-url)

In the left figure, the shape of $\text{Con}(A)$ is not restricted by the definition of “neutrelabelian SI”, except for the fact that neutral algebras in congruence modular varieties have distributive congruence lattices. However, in the right figure, where $\nu > 0$, the shape of $\text{Con}(A)$ is restricted. This lattice is articulated at $\nu$. We have indicated that $A$ is neutral above $\nu$ and abelian below $\nu$, but in fact more is true: if $\beta > \nu$ and $\alpha$ is arbitrary, we have $[\alpha, \beta] = \alpha \land \beta$.

The articulation at $\nu$ may seem to be a strange condition to put on $\text{Con}(A)$, but it is a shape that occurs in nature. The fact that any SI unital ring in a residually
small variety is a neutrabelian SI follows from [8]. The fact that any SI algebra in a finitely generated, finitely decidable variety is a neutrabelian SI follows from [5]. Any dualizable, endorigid, SI algebra in a congruence permutable variety is a neutrabelian SI, as can be shown using ideas in [6].

**Definition 1.2.** Let \( A \) be an algebra in a congruence modular variety. \( A \) is **neutrabelian** if every SI quotient of \( A \) is a neutrabelian SI according to Definition 1.1.

One must now ask whether this definition is consistent with Definition 1.1. Namely, is it true that if \( A \) is a neutrabelian SI according to Definition 1.1, then must \( A \) be neutrabelian according to Definition 1.2? To establish an affirmative answer to this question one must verify that any SI quotient \( A/\delta \) of a neutrabelian SI \( A \) is also a neutrabelian SI. Argue as follows: suppose that \( A \) is a neutrabelian SI and \( \nu \) is the centralizer of the monolith of \( A \). If \( \delta \geq \nu \), then \( A/\delta \) is neutral and SI, hence a neutrabelian SI. If \( \delta < \nu \), then it follows from conditions “(C4)” and “(C3)” of Definition 1.1 and properties of the modular commutator that \( \nu/\delta \) is the centralizer of the monolith of \( A/\delta \) and conditions “(C4)” and “(C3)” of Definition 1.1 hold for \( A/\delta \).

Next we describe the second commutator property that is the focus of this paper, the split centralizer condition. Let \( A \) be a finite algebra and let \( Q \) be a quasivariety containing \( A \). A **\( Q \)-congruence** on \( A \) is a congruence \( \kappa \) of \( A \) such that \( A/\kappa \in Q \). Let \( \delta \) be a completely meet irreducible congruence on \( A \) with upper cover \( \theta \), and let \( \nu = (\delta : \theta) \) be the centralizer of \( \theta \) modulo \( \delta \). A triple \( (\delta, \theta, \nu) \) defined in this way is called **relevant** if \( \theta \leq \nu \), that is, if \( \theta/\delta \) is abelian. The triple \( (\delta, \theta, \nu) \) is **split** by a triple of congruences \( (\alpha, \beta, \kappa) \) if

(i) \( \kappa \) is a \( Q \)-congruence on \( A \),
(ii) \( \beta \leq \delta \),
(iii) \( \alpha \wedge \beta = \kappa \),
(iv) \( \alpha \vee \beta = \nu \), and
(v) \( \alpha/\kappa \) is abelian.

The relationships between these congruences are depicted in Figure 2. We say that a relevant triple \( (\delta, \theta, \nu) \) is **split at 0** if it is split by a triple of the form \( (\alpha, \beta, \kappa) \) satisfying the above conditions with \( \kappa = 0 \).

In [7], the **split centralizer condition** for a finite algebra \( A \) was defined to be the condition that, for \( Q := \text{SP}(A) \) and for any subalgebra \( B \leq A \), each relevant triple \( (\delta, \theta, \nu) \) of \( B \) is split (relative to \( Q \)) by some triple \( (\alpha, \beta, \kappa) \). Here we shall consider a modified version of this condition, which allows us to ignore the role of the quasivariety \( Q \). Namely, we shall only consider the situation where relevant triples \( (\delta, \theta, \nu) \) are split at 0. Since 0 is a \( Q \)-congruence for any \( Q \) containing \( A \), we will be able to ignore the role of \( Q \). To make our assumptions explicit, we record this as:
Definition 1.3. Say that $A$ has centralizers split at 0 if every relevant triple $(\delta, \theta, \nu)$ of $A$ is split by a triple of the form $(\alpha, \beta, 0)$.

If $A_A$ is the expansion of $A$ by constants, then $A_A$ satisfies the split centralizer condition as defined in [7] if and only if $A_A$ has centralizers split at 0 as defined in Definition 1.3. But without expanding $A$ by constants we do not have equivalence. We have only that if $A$ and all subalgebras of $A$ have centralizers split at 0, then $A$ satisfies the split centralizer condition as defined in [7], but in the strong sense that the $\mathcal{Q}$-congruence $\kappa$ that arises in a splitting triple $(\alpha, \beta, \kappa)$ must be the zero congruence.

Our goal in this paper is to prove the following theorem.

Theorem 1.4. The following are equivalent for a finite algebra $A$ in a congruence modular variety.

1. $A$ is neutrelational.
2. $A$ has centralizers split at 0.

Property (2) of this theorem is a statement that certain congruences, namely centralizers of abelian prime quotients, have a special kind of “direct factorization”. Property (1) of this theorem is a statement about the commutator structure of the subdirect factors of $A$. When judging whether a theorem of this sort is interesting one might reflect on whether it is easier to verify if special direct factorizations exist or it is easier to verify if subdirect factors have given commutator properties. Our motivation for proving Theorem 1.4 was the feeling that it is hard to establish direct decompositions of congruences, and so Property (1) is more illuminating than Property (2).

The relevance of Theorem 1.4 for natural duality theory is the following corollary.
Corollary 1.5. If $A$ is a finite algebra with a cube term, and every subalgebra of $A$ is neutabelian, then $A$ is dualizable.

Proof. If every subalgebra of $A$ is neutabelian, then $A$ satisfies the split centralizer condition as defined in [7]. In Theorem 1.1 of [7] it is proved that any finite algebra with a cube term which satisfies the split centralizer condition is dualizable. □

This yields the first proof of the following fact.

Corollary 1.6. If $A$ is a finite algebra in a finitely decidable congruence modular variety, then $A$ is dualizable.

Proof. It follows from the Main Theorem of [5] that if $V$ is a finitely decidable congruence modular variety, then $V$ has a Maltsev term and every finite algebra in $V$ is neutabelian. Since Maltsev terms are special types of cube terms, this corollary follows from Corollary 1.5 above. □

2. The equivalence of two conditions

Recall that the commutator identity (C1) holds for $A$ exactly when, for every $\alpha, \beta \in \text{Con}(A)$, it is the case that

\[(C1) \quad [\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta].\]

Consider the following related condition for an SI, $S$, with monolith $\mu$: say that $S$ satisfies $(C1)^*$ if the centralizer of its monolith, $(0 : \mu)$, is abelian. It is proved in the opening paragraphs of Section 2 of [3] that if $A$ belongs to a congruence modular variety, then $A$ satisfies $(C1)$ if and only if every SI quotient of $A$ satisfies $(C1)^*$.

Recall also that the solvable radical of a congruence $\tau \in \text{Con}(A)$ is the largest congruence $\rho(\tau)$ for which the interval $I[\tau, \rho(\tau)]$ is solvable, if a largest such congruence exists. The solvable radical of the algebra $A$ is $\rho := \rho(0)$, the largest solvable congruence, if such exists. It follows from the properties of the modular commutator that if $\tau \leq \alpha, \beta$ and both $\alpha$ and $\beta$ are solvable over $\tau$, then $\alpha \lor \beta$ is solvable over $\tau$. Thus, if $\text{Con}(A)$ is finite, the join all congruences solvable over $\tau$ equals $\rho(\tau)$.

We can use $(C1)$ and the concept of the solvable radical to express the neutabelian property for SIs in a slightly simpler way.

Lemma 2.1. Let $A$ be a finite SI algebra in a congruence modular variety. $A$ is neutabelian if and only if

1. $A$ satisfies $(C1)$.
2. The solvable radical $\rho$ is comparable with all congruences of $A$.
3. $\rho$ is abelian.
4. $A/\rho$ is neutral.
Proof. For both directions of the argument assume that $A$ is a finite SI with monolith $\mu$ and centralizer $\nu = (0 : \mu)$.

Now assume that $A$ is neutrabelian. For any two congruences $\alpha, \beta \in \text{Con}(A)$ it is the case that $[\alpha \land \beta, \beta] \leq \alpha \land [\beta, \beta]$. If the inequality is strict, then $0 \neq [\beta, \beta]$, so $\nu \prec \beta$ by “(C4)”. But then, by “(C3)”, we get $[\alpha \land \beta, \beta] = (\alpha \land \beta) \land \beta$ and $\alpha \land [\beta, \beta] = \alpha \land \beta$, and these are equal, so the inequality cannot be strict after all. This means that (C1) holds.

In a neutrabelian SI, $\nu$ is abelian, and $A/\nu$ is neutral. This is enough to imply that $\nu = \rho = \text{the solvable radical of } A$. Hence (2), (3), (4) hold in a neutrabelian SI by Definition 1.1.

Now we prove the reverse direction. Assume that $A$ is SI and satisfies (1)–(4). Since $A$ satisfies (C1), it satisfies (C1)$^*$, so $\nu$ is abelian. This yields $\nu \leq \rho$. On the other hand, $\rho$ is abelian (as we assume in (3)), so $[\rho, \mu] = 0$, either because $\rho = 0$ or because $\mu \leq \rho$. Hence $\rho \leq \nu$. This yields $\rho = \nu$. Hence (2) and (3) imply that $\nu$ is abelian and is comparable with all congruences. What remains to show is that $[\alpha, \beta] = \alpha \land \beta$ if $\nu \prec \beta$.

If $\nu \prec \alpha \land \beta$, then we get $[\alpha, \beta] = \alpha \land \beta$ from the neutrality of $A/\nu = A/\rho$ (Item (4)). So the only situation left to check is when $\alpha \land \beta \leq \nu = \rho < \beta$. In this situation we have

\[(\dagger) \quad [\alpha \land \beta, \beta] \overset{(C1)}{=} \alpha \land [\beta, \beta] = \alpha \land \beta,\]

since $[\beta, \beta] = \beta$ when $\rho < \beta$. But then

\[(\ddagger) \quad [\alpha \land \beta, \beta] \leq [\alpha, \beta] \leq \alpha \land \beta \overset{(\dagger)}{=} [\alpha \land \beta, \beta],\]

so we have equality in the middle of $(\ddagger)$, $[\alpha, \beta] = \alpha \land \beta$. 

\[\square\]

The characterization given by Lemma 2.1 is very close to the definition of “neutrabelian SI”, so let us take a moment to emphasize the difference. In the definition, we specify that $[\alpha, \beta] = \alpha \land \beta$ if at least one of $\alpha, \beta$ is strictly above $(0 : \mu)$. Hence to verify the definition holds for some algebra we may have to examine values of $[\alpha, \beta]$ in some cases where $\alpha < (0 : \mu) < \beta$. The lemma allows us to avoid examining these values. Given (C1) and the comparability of $(0 : \mu)$ with all congruences, it suffices to check the values of $[\alpha, \beta]$ when both $\alpha, \beta$ lie below $(0 : \mu)$ or when both lie above $(0 : \mu)$.

**Lemma 2.2.** Let $A$ be a finite algebra in a congruence modular variety. Under either one of the hypotheses

(i) $A$ is neutrabelian, or

(ii) $A$ has centralizers split at 0,

the following conclusions hold:

(1) $A$ satisfies the commutator identity (C1).
(2) A satisfies the $\langle 3, 2 \rangle$ and $\langle 4, 2 \rangle$-transfer principles of tame congruence theory.

(3) The solvable radical $\rho$ is the largest abelian congruence of $A$, and $A/\rho$ is neutral.

(4) If $\tau$ is a congruence of $A$, then the solvable radical $\rho(\tau)$ of $\tau$ equals $\rho \vee \tau$, the interval $I[\tau, \rho(\tau)]$ is abelian, and the interval $I[\rho(\tau), 1]$ is neutral.

Proof. In the first part of the proof we assume (i), which is the assumption that $A$ is a neutrabelian algebra, and we argue that each of (1)–(4) hold.

An algebra is neutrabelian if and only if each SI quotient is, and an algebra satisfies (C1) if and only if each SI quotient does. Hence, to prove Item (1), it suffices to prove that any neutrabelian SI satisfies (C1), which we did in Lemma 2.1.

Now consider Item (2). The $\langle i, j \rangle$-transfer principle for a finite algebra $A$ “is” (or is equivalent to) the statement that $\text{Con}(A)$ has no 3-element interval $\alpha \preceq \beta \prec \gamma$ with $\text{typ}(\alpha, \beta) = i$ and with $\text{typ}(\beta, \gamma) = j$. We need the following claim.

Claim 2.3. Assume that $A$ is a finite algebra in a congruence modular variety. If $\text{Con}(A)$ has a 3-element interval $\alpha \prec \beta \prec \gamma$ with $\text{typ}(\alpha, \beta) = i$ and with $\text{typ}(\beta, \gamma) = j$, then some SI quotient of $A$ has congruences $0 \prec \mu \prec \sigma$ with $\text{typ}(0, \mu) = i$ and with $\text{typ}(\mu, \sigma) = j$.

Proof of Claim. This claim is derivable from Theorem 3.13 of [1], where a similar claim is asserted in the more general context of congruence semimodular varieties. Nevertheless, we include a proof here. Assume that, for some $i$ and $j$, $\text{Con}(A)$ has some 3-element interval $I[\alpha, \gamma]$ with $\alpha \prec \beta \prec \gamma$. Choose a congruence $\delta \geq \alpha$ that is maximal for the property $\delta \not\geq \beta$. Necessarily $\delta$ is completely meet irreducible with upper cover $\delta \vee \beta$. Since $\delta \geq \alpha$ and $\delta \not\geq \beta$, we have $\delta \land \gamma \geq \alpha$ and $\delta \land \gamma \not\geq \beta$. Since the interval $I[\alpha, \gamma]$ contains only $\alpha, \beta, \gamma$, it follows that $\delta \land \gamma = \alpha = \delta \land \beta$. By modularity it follows that $\delta \lor \gamma \neq \delta \lor \beta$. By tame congruence theory we even get $\delta \prec \delta \lor \beta \prec \delta \lor \gamma$. In the SI quotient $A/\delta$, with monolith $\mu = (\delta \lor \beta)/\delta$ and upper cover $\sigma = (\delta \lor \gamma)/\delta$ this yields $0 \prec \mu \prec \sigma$.

Now we complete the proof of Item (2). The types that can appear in $\text{Con}(A)$ when $A$ belongs to congruence modular variety are $2, 3, 4$, and $\alpha \not\prec \beta$ if and only if $\beta$ is abelian over $\alpha$. In a neutrabelian SI, if the monolith is nonabelian (type 3 or 4), then the SI is neutral, hence all congruence coverings are nonabelian. This shows that $0 \not\prec \mu \not\prec \sigma$ and $0 \not\prec \mu \not\prec \sigma$ are forbidden in any finite neutrabelian SI. By Claim 2.3, 3-element intervals $\alpha \prec \beta \prec \gamma$ and $\alpha \prec \beta \prec \gamma$ are forbidden in any finite neutrabelian algebra. This is the statement that $A$ satisfies the $\langle 3, 2 \rangle$ and $\langle 4, 2 \rangle$-transfer principles.
Next we consider Item (3). The solvable radical is, by definition, the largest solvable congruence on $\mathbf{A}$. According to tame congruence theory, the solvable radical of $\mathbf{A}$ equals the largest congruence $\rho \in \text{Con}(\mathbf{A})$ such that the interval $I[0, \rho]$ contains only abelian types. Since $\mathbf{A}$ lies in a congruence modular variety, where the only abelian type is $2$, this means that $\rho$ is the largest congruence on $\mathbf{A}$ such that $\text{typ}\{0, \rho\} = \{2\}$. It follows from the $\langle 3, 2 \rangle$ and $\langle 4, 2 \rangle$-transfer principles (and the fact that $\text{typ}\{\mathbf{A}\} \subseteq \{2, 3, 4\}$) that $\text{typ}\{\rho, 1\} \subseteq \{3, 4\}$. This implies that $\text{Con}(\mathbf{A}/\rho)$ has no abelian types, which is equivalent to the statement that $\mathbf{A}/\rho$ is neutral.

Now we argue that $\rho$ is abelian. Let $\mathbf{A} \leq \prod \mathbf{A}_i$ be a representation of $\mathbf{A}$ as a subdirect product of neutrabelian SIs. If the solvable congruence $\rho$ were not abelian, then its projection onto some factor would have to be solvable but not abelian. Neutrabelian SIs have no solvable congruences that are not abelian, so we are forced to conclude that $\rho$ is abelian. This completes the proof of Item (3).

Now we prove Item (4). If $\mathbf{A}$ is neutrabelian, then any quotient $\mathbf{A}/\tau$ is neutrabelian, hence the radical $\rho^{\mathbf{A}/\tau} = \rho(\tau)/\tau$ is abelian and the quotient $\mathbf{A}/\rho^{\mathbf{A}/\tau}$ is neutral by Item (3). This implies that the interval $I[\tau, \rho(\tau)]$ in $\text{Con}(\mathbf{A})$ is abelian and the interval $I[\rho(\tau), 1]$ is neutral in $\text{Con}(\mathbf{A})$. It remains to prove that $\rho(\tau) = \rho \vee \tau$.

Write $\sim$ for the solvability relation of tame congruence theory. Since $0 \sim \rho$ in $\text{Con}(\mathbf{A})$ and $\sim$ is a congruence on $\text{Con}(\mathbf{A})$ we get $\tau = 0 \vee \tau \sim \rho \vee \tau$. This forces $\rho \vee \tau \leq \rho(\tau)$, since $\rho(\tau)$ is the largest congruence related to $\tau$ by $\sim$. In particular, $I[\rho \vee \tau, \rho(\tau)]$ is a solvable interval. However, $I[\rho \vee \tau, \rho(\tau)]$ is a subinterval of $I[\rho, 1]$, which by Item (3) is neutral. Since it is both solvable and neutral it is trivial. This forces $\rho(\tau) = \rho \vee \tau$.

We have proved the lemma under hypothesis (i). Now we prove it under hypothesis (ii), which is the assumption that $\mathbf{A}$ has centralizers split at 0.

For Item (1), it again suffices to prove that any SI quotient of $\mathbf{A}$ satisfies $(\text{C1})^\ast$. Let $\mathbf{A}/\delta$ be an SI quotient of $\mathbf{A}$, let $\theta, \nu \geq \delta$ in $\text{Con}(\mathbf{A})$ be such that $\theta/\delta$ is the monolith of $\mathbf{A}/\delta$ and $\nu/\delta = (\delta/\delta : \theta/\delta)$ is the centralizer of $\theta/\delta$. (This forces $\nu = (\delta : \theta)$.)

There is nothing to prove unless the monolith $\theta/\delta$ is abelian, so we may assume that $\delta \leq \theta \leq \nu$. In this situation $(\delta, \theta, \nu)$ is a relevant triple of $\mathbf{A}$, hence must be split at 0 by $(\alpha, \beta, 0)$ where $\beta \leq \delta$, $\alpha \wedge \beta = 0$, $\alpha \vee \beta = \nu$, and $\alpha$ is abelian. Now

$$[\nu, \nu] = [\alpha \vee \beta, \alpha \vee \beta] \leq [\alpha, \alpha] \vee \beta \leq 0 \vee \delta = \delta,$$

which proves that $\nu$ is abelian over $\delta$. From this it follows that the centralizer $(\delta/\delta : \theta/\delta) = \nu/\delta$ of the monolith $\theta/\delta$ in the quotient algebra $\mathbf{A}/\delta$ is abelian over $\delta/\delta$, i.e. that $(\text{C1})^\ast$ holds.

We temporarily skip Item (2) and prove Item (3). We will use the fact, proved in Lemma 4.16 of [9] (also appearing as Exercise 8.5 of [4]), that if $\mathbf{A}$ is an algebra in a congruence modular variety, and $\mathbf{A}$ satisfies $(\text{C1})$, then $\mathbf{A}$ has a largest abelian congruence, which we denote $\tilde{\alpha}$. 

Claim 2.4. $A/\hat{\alpha}$ is neutral.

Proof of Claim. If $A/\hat{\alpha}$ had any abelian prime quotient, then $A$ would have one, $\delta \prec \theta$, where $\delta$ is completely meet irreducible and $\hat{\alpha} \leq \delta \prec \theta$. If $\nu = (\delta : \theta)$, then necessarily

$$\hat{\alpha} \leq \delta \prec \theta \leq \nu$$

so $(\delta, \theta, \nu)$ would be a relevant triple. If this triple is split by $(\alpha, \beta, 0)$, then since $\alpha$ is abelian we must have $\alpha \leq \hat{\alpha}$. Since $(\alpha, \beta, 0)$ splits $(\delta, \theta, \nu)$ we get

$$\nu = \alpha \lor \beta \leq \hat{\alpha} \lor \delta = \delta < \nu,$$

which is impossible. This proves the claim.

Claim 2.4 implies that $\rho = \hat{\alpha}$ (since $\hat{\alpha}$ is abelian and $A/\hat{\alpha}$ is neutral). Thus Item (3) has been proved: $\rho$ is the largest abelian congruence of $A$, and $A/\rho$ is neutral.

We easily infer the truth of Item (4), as well. If $\tau$ is a congruence of $A$, then since $\rho$ is abelian over $0$ we get that $\rho \lor \tau$ is abelian over $0 \lor \tau = \tau$. Since $I[\rho \lor \tau, 1]$ is a subinterval of the neutral interval $I[\rho, 1]$, we get that $I[\rho \lor \tau, 1]$ is neutral. Thus the congruence 1 is neutral over $\rho \lor \tau$, while $\rho \lor \tau$ is abelian over $\tau$; this is enough to conclude that $\rho(\tau) = \rho \lor \tau$.

We conclude by proving Item (2). If $A$ fails the $(i, j)$-transfer principle for $i \in \{3, 4\}$ and $j = 2$, then by Claim 2.3 there is a completely meet irreducible congruence $\delta$ such that $\delta \prec \theta \prec \sigma$. Now we use the statement in Item (4) to get a contradiction. Since $\delta$ is completely meet irreducible with nonabelian upper cover we have $\delta = \rho(\delta) = \rho \lor \delta$. This implies that $\rho \leq \delta$. Since $\theta$ has an abelian cover $\sigma$ we have $\theta < \rho(\theta) = \rho \lor \theta$. This implies that $\rho \nleq \theta$. But the conclusions $\rho \leq \delta \leq \theta$ and $\rho \nleq \theta$ are contradictory. This proves Item (2). 

Lemma 2.5. Let $A$ be a finite algebra in a congruence modular variety. Assume also that $A$ has centralizers split at $0$, and that a particular relevant triple $(\delta, \theta, \nu)$ is split by a particular triple $(\alpha, \beta, 0)$. If $\tau \in \text{Con}(A)$ is a completely meet irreducible congruence, then either

- $\tau \geq \alpha$, or
- $\tau \geq [\nu, \nu]$.

In the latter case, $\tau \lor \rho \geq \nu$.

Proof. Suppose that the upper cover of $\tau$ is $\mu$. Assume that the first bulleted item does not hold, that is that $\tau \nleq \alpha$. Then $\tau < \tau \lor \alpha$, so both $\mu$ and $\alpha$ are dominated by $\tau \lor \alpha$. Hence $\mu \lor \alpha \leq \tau \lor \alpha \leq \mu \lor \alpha$, yielding $\tau \lor \alpha = \mu \lor \alpha$. If we also had $\tau \land \alpha = \mu \land \alpha$, then $\{\alpha, \tau, \mu\}$ would generate a pentagon in $\text{Con}(A)$, contradicting modularity. Thus $\tau \land \alpha < \mu \land \alpha$, and by modularity we even have that the interval
$I[\tau, \mu]$ is perspective with $I[\tau \land \alpha, \mu \land \alpha]$. This implies that $(\tau : \mu) = (\tau \land \alpha : \mu \land \alpha)$. But $\alpha \leq (\tau \land \alpha : \mu \land \alpha)$, since 

$$[\alpha, \mu \land \alpha] \leq [\alpha, \alpha] = 0 \leq \tau \land \alpha,$$

and $\beta \leq (\tau \land \alpha : \mu \land \alpha)$, since 

$$[\beta, \mu \land \alpha] \leq [\beta, \alpha] \leq \beta \land \alpha = 0 \leq \tau \land \alpha.$$

Thus $\nu = \alpha \lor \beta \leq (\tau \land \alpha : \mu \land \alpha) = (\tau : \mu)$.

By (C1)* in $A/\tau$ we have the second $\leq$ in 

$$[\nu, \nu] \leq [(\tau : \mu), (\tau : \mu)] \leq \tau.$$

This shows that when $\tau \not\geq \alpha$ we have $\tau \geq [\nu, \nu]$, as claimed. Moreover, when $\tau \geq [\nu, \nu]$ we calculate that 

$$\tau \lor \rho \geq [\nu, \nu] \lor \rho = \rho([\nu, \nu]) \geq \nu,$$

where the equality holds by Lemma 2.2 (4). This justifies the last sentence of the lemma statement. □

Now we are prepared to prove one direction of Theorem 1.4.

**Theorem 2.6.** Let $A$ be a finite algebra in a congruence modular variety. If $A$ has centralizers split at 0, then $A$ is neutrabelian.

**Proof.** We must show that if $A$ has centralizers split at 0, then every SI quotient of $A$ is a neutrabelian SI. Let $\tau$ be a completely meet irreducible congruence of $A$ with upper cover $\mu$. We shall argue that $A/\tau$ is a neutrabelian SI.

As a first case, assume that $A/\tau$ has nonabelian monolith. That is, $\tau \prec^3 \mu$. It follows from Lemma 2.2 (2) (transfer principles hold) that $A/\tau$ is neutral, hence neutrabelian, and we are done in this case.

Next we tackle the case where $\tau \prec^2 \mu$. It is a consequence of Lemma 2.2 (1) and the fact that the class of algebras in congruence modular varieties that satisfy (C1) is closed under quotients that

**Claim 2.7.** $A/\tau$ satisfies (C1). ⊳

We also have that

**Claim 2.8.** $\rho(\tau)$ is comparable with all congruences in the interval $I[\tau, 1]$.

**Proof of Claim.** This proof will involve the introduction of several congruences. The order relationships between these congruences are indicated in Figure 3.

Assume there exists a congruence in $I[\tau, 1]$ that is not comparable with $\rho(\tau)$. Choose such a congruence $\psi \geq \tau$ that is minimal for the property that $\psi$ is not comparable with $\rho(\tau)$. Necessarily any lower cover $\psi_*$ of $\psi$ which lies in $I[\tau, 1]$ will be strictly below $\rho(\tau)$, hence $\psi_* := \psi \land \rho(\tau)$ must be the unique lower cover of $\psi$ that
lies in $I[\tau,1]$. Choose $\gamma$ so that $\psi_* < \gamma \leq \rho(\tau)$. Since $\tau \leq \psi_* < \gamma \leq \rho(\tau)$ is a solvable interval, $\psi_* \not< \gamma$.

Extend $\psi$ to a congruence $\delta$ maximal for $\delta \not< \gamma$. This congruence will be completely meet irreducible. Its upper cover will be $\theta := \delta \lor \gamma$. Since the interval $I[\delta,\theta]$ is perspective with $I[\psi_*,\gamma]$ we must have $\delta \not< \theta$. Let $\nu = (\delta : \theta)$. The triple $(\delta,\theta,\nu)$ is relevant, and $A$ has centralizers split at 0, so assume that $(\delta,\theta,\nu)$ is split by $(\alpha,\beta,0)$. That is, $\beta \leq \delta$, $\alpha$ is abelian, $\alpha \land \beta = 0$ and $\alpha \lor \beta = \nu$.

Figure 3.

According to Lemma 2.5, either $\tau \geq \alpha$ or $\tau \geq \nu$. In the former case we have $\alpha \leq \tau \leq \psi \leq \delta$. Since we also have $\beta \leq \delta$, we derive that $\theta \leq \nu = \alpha \lor \beta \leq \delta$, which is false. In the latter case we have $[\nu,\nu] \leq \tau \leq \nu$ and $\nu \leq \tau \lor \rho = \rho(\tau)$. But then $\psi \leq \delta \leq \nu \leq \rho(\tau)$, contrary to our assumption that $\psi$ is not comparable to $\rho(\tau)$. This proves the claim.

It is a consequence of Lemma 2.2 (4) that

Claim 2.9. $I[\tau,\rho(\tau)]$ is abelian and $I[\rho(\tau),1]$ is neutral.

According to Lemma 2.1, Claims 2.7, 2.8, and 2.9 establish that $A/\tau$ is neutrabelian. This completes the proof.

Finally, we prove the other direction of Theorem 1.4.

Theorem 2.10. Let $A$ be a finite algebra in a congruence modular variety. If $A$ is neutrabelian, then $A$ has centralizers split at 0.
Proof. Assume that $A$ is neutrabelian and that $(\delta, \theta, \nu)$ is a relevant triple of $A$. Since $\delta$ is a completely meet irreducible congruence of $A$, $A/\delta$ is a neutrabelian SI. Moreover, since $\theta$ is an abelian cover of $\theta$, $A/\delta$ has abelian monolith $\theta/\delta$.

By Lemma 2.2 (1), $A$ satisfies (C1) so the centralizer of $\theta/\delta$ is abelian. This congruence has the form $\nu/\delta$ for $\nu := (\delta : \theta)$. The fact that $\nu/\delta$ is abelian in $\text{Con}(A/\delta)$ translates exactly into the property that $[\nu, \nu] \leq \delta$ in $\text{Con}(A)$. Let $\beta = [\nu, \nu]$. The interval $I[\beta, \nu]$ is abelian (hence solvable), while the interval $I[\nu, 1]$ is neutral (since $A/\delta$ is neutrabelian and $\nu = (\delta : \theta)$), so $\rho(\beta) = \nu$.

By Lemma 2.2 (3) and (4), the solvable radical $\rho$ of $A$ is the largest abelian congruence of $A$, and $\rho \lor \beta = \rho(\beta) = \nu$. Our goal in the rest of this proof is to find $\alpha \leq \rho$ such that

1. $\alpha \lor \beta = \nu$, and
2. $\alpha \land \beta = 0$.

Such an $\alpha$ will be abelian (since $\alpha \leq \rho$) and then $(\alpha, \beta, 0)$ will satisfy all conditions necessary to be a triple that splits $(\delta, \theta, \nu)$ at zero.

We already have $\rho \lor \beta = \nu$. Shrink $\rho$ to an $\alpha \leq \rho$ still satisfying (1) $\alpha \lor \beta = \nu$ for which $\alpha \land \beta$ is minimal in $\text{Con}(A)$. We will argue that (2) must be satisfied by this congruence $\alpha$. To obtain a contradiction, assume instead that $\alpha \land \beta > 0$, and choose some $\sigma < \alpha \land \beta$. Since $\sigma < \alpha \land \beta \leq \alpha \leq \rho$ and $\rho$ is abelian, the covering $\sigma < \alpha \land \beta$ is abelian. Choose $\psi \geq \sigma$ maximal for $\psi \not\leq \alpha \land \beta$, and let $\psi^* = \psi \lor (\alpha \land \beta)$. The quotient $A/\psi$ is subdirectly irreducible with monolith $\psi^*/\psi$. The interval $I[\psi, \psi^*]$ is perspective with the abelian interval $I[\sigma, \alpha \land \beta]$, so the subdirectly irreducible quotient $A/\psi$ has abelian monolith.

We shall argue that, for $\alpha_* := \alpha \land \psi \leq \alpha$, we have (i) $\beta \land \alpha_* \prec \beta \land \alpha$ and (ii) $\beta \lor \alpha_* = \nu$, contradicting the minimality of $\beta \land \alpha$. We rewrite the necessary conditions as (i) $\beta \land (\alpha \land \psi) \prec \beta \land \alpha$ and (ii) $\beta \lor (\alpha \land \psi) = \nu$.

Claim 2.11. (i) $(\alpha \land \psi) \land \beta = \sigma \prec \alpha \land \beta$.

Proof of Claim. We chose $\sigma$ so that $\sigma \prec \alpha \land \beta$ and we chose $\psi$ so that $\sigma \leq \psi$ and $\alpha \land \beta \not\leq \psi$. This is enough to conclude that $(\alpha \land \beta) \land \psi = \sigma$. \hfill \diamond

Claim 2.12. (ii) $\beta \lor (\alpha \land \psi) = \nu$.

Proof of Claim. We shall derive this claim through an examination of the sublattice of $\text{Con}(A)$ that is generated by $X = \{\alpha, \beta, \psi\}$. This is a 3-generated modular lattice, so it has at most 28 elements, and it can easily be drawn. Some elements of this sublattice have already been named above, for example $\alpha \lor \beta = \nu$ and $\alpha \land \beta \land \psi = \sigma$ (the least element of the sublattice).

The sublattice of $\text{Con}(A)$ that is generated by $X = \{\alpha, \beta, \psi\}$ is a quotient of the modular lattice freely generated by the set $X$, which is drawn in Figure 4. But the sublattice of our lattice $\text{Con}(A)$ that is generated by $X$ is not freely generated by
Figure 4.

$X$, it is a proper quotient of the lattice in Figure 4. In particular, the sublattice of $\text{Con}(A)$ generated by $X$ must satisfy the following relations.

Subclaim 2.13.

- $\beta \vee \psi = \alpha \vee \beta \vee \psi$, and
- $\alpha \vee \psi = (\alpha \wedge \beta) \vee \psi$.

To prove the first bulleted item note that, since $\beta = [\nu, \nu]$, the interval from $[\beta, \beta]$ to $\nu$ is 2-step solvable. Hence $\nu \leq \rho([\beta, \beta]) = [\beta, \beta] \vee \rho$, implying that the interval from $[\beta, \beta]$ to $\nu$ is abelian. Therefore $\beta = [\nu, \nu] \leq [\beta, \beta] \leq \beta$, or just $\beta = [\beta, \beta]$. From this we derive from (C1) that if $x \leq \beta$, then

$$x = x \wedge \beta = x \wedge [\beta, \beta] = [x \wedge \beta, \beta] = [x, \beta].$$

In particular, when $x = \alpha \wedge \beta$ we get that $[\alpha \wedge \beta, \beta] = \alpha \wedge \beta \not\leq \sigma$, which we can express as $\beta \not\leq (\sigma : \alpha \wedge \beta)$. But the interval $I[\sigma : \alpha \wedge \beta]$ is perspective with $I[\psi, \psi^r]$, so $(\sigma : \alpha \wedge \beta) = (\psi : \psi^r)$, and we derive that $\beta \not\leq (\psi : \psi^r)$.

The SI quotient $A/\psi$ is neutrabelian, and $(\beta \vee \psi)/\psi \not\leq (0 : \psi^r/\psi)$, so $\beta \vee \psi > \rho(\psi)$. Since $\nu \geq \beta$ and $\nu \sim \beta$ we have $(\nu \wedge \psi)/\psi \geq (\beta \vee \psi)/\psi$ and $(\nu \vee \psi)/\psi \sim (\beta \vee \psi)/\psi$. But solvably related congruences above the radical of a neutrabelian SI are equal, so $(\nu \vee \psi)/\psi = (\beta \vee \psi)/\psi$. Hence $\nu \vee \psi = \beta \vee \psi > \rho(\psi) = \rho \vee \psi$. Now $\nu = \beta \vee \alpha$, so we get $\nu \vee \psi = \nu \vee \psi = \beta \vee \psi$, completing the proof of the first bulleted item of the subclaim.

For the second bulleted item of the subclaim, we carry along our earlier conclusions that $\alpha \vee \beta \vee \psi = \beta \vee \psi > \rho(\psi)$. This strict inequality means that $(\beta \vee \psi)/\psi$ is strictly above the radical of the neutrabelian SI quotient $A/\psi$. From the way the commutator
behaves on a neuterabelian algebra, $(\beta \lor \psi)/\psi$ must fail to centralize any proper interval in $\text{Con}(A/\psi)$ that lies below $(\beta \lor \psi)/\psi$. In particular, if $(\alpha \land \beta) \lor \psi < \alpha \lor \psi$, then we must have
$$[(\beta \lor \psi)/\psi, (\alpha \lor \psi)/\psi] \not\leq ((\alpha \land \beta) \lor \psi)/\psi.$$  
But this is in contradiction with basic properties of the commutator (additivity, submultiplicativity, homomorphism property), as we see in the calculation
$$[\beta \lor \psi, \alpha \lor \psi] \lor \psi = [\beta, \alpha] \lor \psi \leq (\alpha \land \beta) \lor \psi.$$
This proves that second bulleted item of the subclaim.

From Subclaim 2.13 we know that the sublattice of $\text{Con}(A)$ generated by $X = \{\alpha, \beta, \psi\}$ is a homomorphic image of the modular lattice presented by
$$\langle \alpha, \beta, \psi \mid \beta \lor \psi = \alpha \lor \beta \lor \psi, \alpha \lor \psi = (\alpha \land \beta) \lor \psi \rangle.$$  
This lattice can be constructed as a quotient of the lattice in Figure 4 and it is drawn in Figure 5. Observe that $\beta \lor (\alpha \land \psi) = \nu$ in this lattice, so this relation will hold in any quotient, such as the sublattice $\text{Con}(A)$ generated by $X = \{\alpha, \beta, \psi\}$. This ends the proof of Claim 2.12.

Claims 2.11 and 2.12 contradict the minimality of $\alpha \land \beta$ in the situation where $\alpha \land \beta > 0$, so we conclude that $\alpha \land \beta = 0$. This shows that the relevant triple $(\delta, \theta, \nu)$ is split at 0 by $(\alpha, \beta, 0)$.

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