On Fuzzy Regular Ternary Γ-Semi rings

K Revathi1, 2, D Madhusudhana Rao3, P Sundarayya4 and P Siva Prasad5

1, 4 Department of Mathematics, GITAM University, Visakhapatnam, A.P. India.
2 Department of Mathematics, Adikavi Nannaya University, Rajahmundry, A.P. India.
3 Department of Mathematics, VSR & NVR College, Tenali, A.P. India.
5 Department of BSH, VFSTR University, Vadlamudi, Guntur, A.P. India.

revathidintakurthi@gmail.com, dnrmaths@gmail.com, psundarayya@gmail.com, pusapatisivaprasad@gmail.com

Abstract: This paper divided into two main sections. In section three we study about fuzzy points in ternary Γ-semi ring and prove some properties of fuzz TΓ-ideals by using fuzzy points in ternary Γ-semi ring. In section four we introduce the concept of regular fuzzy TΓ-ideals in ternary Γ-semi rings and characteristic fuzzy regular ternary Γ-semi rings. Also we discuss the fuzzy simple ternary Γ-semi ring, fuzzy intra regular ternary Γ-semi rings.

1 Introduction

D. Madhusudhana Rao and G. Srinivasa Rao [6] characterized the ternary semi rings. The theory of fuzzy sets was first inspired by Zadeh [10]. Fuzzy ideals in rings were introduced by Liu [5] and it has been studied by several authors. Jun [2] and Kim and Park [4] have also studied fuzzy ideals in semi rings. In the year 2007, [7] we have introduced the notions of fuzzy ideals and fuzzy quasi-ideals in ternary semi rings. In the year 2015, Sajani Lavanya and MadhusudhanaRao[7, 8, 9] introduced the notion of ternary Γ-Semi rings.

2 Prerequisites

For preliminaries refer to the references and their references.

3 Fuzzy Points In Ternary Γ-Semi rings

Definition 3.1: Let T be a non-empty set and a ∈ T. Let τ ∈ (0, 1]. A fuzzy point aτ of T is a fuzzy subset of T, defined as

\[ a_\tau(x) = \begin{cases} 
\tau & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \]

, where x ∈ T. We denote FP(T) as the set of all fuzzy points of a ternary Γ-semi ring T.
Theorem 3.2: Let $T$ be a ternary $\Gamma$-semi ring and $FP(T)$ be the set of all fuzzy points of $T$. If $a_p, b_q, c_r \in FP(T)$ and $a, b, c \in T$, then $(a_p a b c) = (a_p a c b) = (a_p b c a) = (b_q a c) = (b_q c a) = (c_r a b) = (c_r b a)$.

Proof: Let $x \in T$.

Case I: If $x \neq u \forall v \forall w$, for any $u, v, w \in T$ and $p, q \in \Gamma$, Then $(a_p a b c(x)) = 0 = (a_p a b c(x))_{p \forall q \forall r}$.

Case II: If such exists, let $x = u \forall v \forall w$, for any $u, v, w \in T$ and $p, q \in \Gamma$.

Then we have $(a_p a b c(x)) = \bigvee_{x \neq y \neq z} \{a_p \land b_y \land c_z\}$.

(i) If $u = a, v = b, w = c$ and $x = y = z$, then $x = a \land b \land c$ and $a_p(a) = p, b_q(b) = q, c_r(c) = r$.

Therefore $(a_p a b c(x)) = a_p(a) \land b_q(b) \land c_r(c) = p \land q \land r = (a \land b \land c)_{p \forall q \forall r}$.

(ii) If either $u \neq a, v \neq b, w \neq c$ or $x \neq y \neq z$, then either $a_p(u) = 0$ or $b_q(v) = 0$ or $c_r(w) = 0$ and hence $(a \land b \land c)_{p \forall q \forall r} = 0$.

Therefore, from above, we find that $(a_p a b c(x)) = (a \land b \land c)_{p \forall q \forall r}$.

Theorem 3.3: Let $T$ be a ternary $\Gamma$-semi ring and $FP(T)$ be the set of all fuzzy points of $T$. Let $a \in FP(T)$. Then for any $x \in T$,

(i) $(T \Gamma T \Gamma a)(x) = \begin{cases} 1 & \text{if } x \in T \Gamma T \Gamma a, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $(a \Gamma T \Gamma T)(x) = \begin{cases} 1 & \text{if } x \in a \Gamma T \Gamma T, \\ 0 & \text{otherwise.} \end{cases}$

(iii) $(T \Gamma a \Gamma T)(x) = \begin{cases} 1 & \text{if } x \in T \Gamma a \Gamma T, \\ 0 & \text{otherwise.} \end{cases}$

Proof: Let $x \in T$.

Case I: If $x \neq p \forall q \forall r$ for any $p, q, r \in S$ and $p, q \in \Gamma$, then $(T \Gamma T \Gamma a)(x) = 0$.

Case II: If $x = p \forall q \forall r$ for any $p, q, r \in S$ and $p, q \in \Gamma$, then $(T \Gamma T \Gamma a)(x) = 0$.

So we find that $(T \Gamma T \Gamma a)(x) = 0$, when $x \in T \Gamma T \Gamma a$.

Sub case I: If $x \neq u \forall v \forall w$ for $u, v \in T$ and $a, b \in T$, then $(T \Gamma T \Gamma a)(x) = 0$.

We find that $x \in T \Gamma T \Gamma a$.

Sub case II: If such exists, let $x = u \forall v \forall w$, for some $u, v \in T$ and $a, b \in T$, then $x \in T \Gamma T \Gamma a$.

$(T \Gamma T \Gamma a)(x) = \bigvee_{x \neq y \neq z} \{T(p) \land T(q) \land a(r)\} = T(u) \land T(v) \land a(t) = 1 \land 1 \land t = t$.

Thus the result follows.

Corollary 3.4: Let $T$ be a ternary $\Gamma$-semi ring and $FP(T)$ be the set of all fuzzy points of $T$. Let $a \in FP(T)$. Then for any $x \in T$,

$(a, U(T \Gamma T \Gamma a))(x) = \begin{cases} 1 & \text{if } x = a \text{ or } x \in T \Gamma T \Gamma a, \\ 0 & \text{otherwise.} \end{cases}$

Proof: Clearly, $(a, U(T \Gamma T \Gamma a))$ is a fuzzy left $TT$-ideal of $T$ containing $a$. If $x = a$, then $a, U(T \Gamma T \Gamma a)(x) = a(a) \lor (T \Gamma T \Gamma a)(a) = a = 0 \lor t$. If $x = p \forall q \forall r \in T \Gamma T \Gamma a$ for some $p, q, r \in S$ and $p, q \in \Gamma$, then $a(U(T \Gamma T \Gamma a)(x) = a(a \lor p \forall q \forall r) \lor (T \Gamma T \Gamma a)(p \forall q \forall r) = 0 \lor t$ (by theorem 2.3.3) = t. Thus the result follows.
Definition 3.5: Let $\mu$ be a non-empty fuzzy subset and $x_i$ be a fuzzy point of a non-empty set $T$. If $\mu(x) \geq t$, then we say $x_i$ belongs to $\mu$ and it is denoted by $x_i \in \mu$.

Theorem 3.6: Let $T$ be a ternary $\Gamma$-semi ring, $F(T)$ be the set of all non-empty fuzzy subsets of $T$, and $FP(T)$ be the set of all fuzzy points of $T$. Then for any $\mu \in F(T)$ and $x_i, y_i \in FP(T)$,

(i) $x_i \in \mu$ if and only if $x_i \subseteq \mu$.

Proof: (i), (iii) and (iv) are obvious.

(iii) Suppose that $x_i$ is a fuzzy point such that $x_i \in \mu$. Since $x_i (a) = (a)$, for all $a \in T$, we have

$$\left( \bigcup_{x_i \in \mu} x_i \right) = \bigvee_{x_i \in \mu} x_i (a) \leq \bigcup_{x_i \in \mu} \mu(a) = \mu(a).$$

Thus $\bigcup_{x_i \in \mu} x_i \subseteq \mu$.

On the other hand, for each $x \in T$, put $x_i \in \mu$. In fact $a \neq x$, then $x_i (a) = 0 \leq \mu(a)$. If $a = x$, then $x_i (a) = t = \mu(x) \leq \mu(a)$. Thus $x_i \in \mu$, and $\mu(x) = x_i (x) = \bigcup_{x_i \in \mu} x_i (x)$, that is, $\mu \subseteq \bigcup_{x_i \in \mu} x_i$.

Definition 3.7: Let $T$ be a ternary $\Gamma$-semi ring and $a_i$ be a fuzzy point of $T$. Then the intersection of all fuzzy left $T\Gamma$-ideals of $T$ containing $a_i$ is a fuzzy left $T\Gamma$-ideal of $T$ containing $a_i$, denoted by $L(a_i)$ or $<a_i>_L$ and defined as $<a_i>_L = \bigcap_{a_i \in \mu} \mu$, where $FTTI_L(T)$ is the set of all fuzzy left $T\Gamma$-ideals of $T$.

This fuzzy left $T\Gamma$-ideal is called fuzzy left $T\Gamma$-ideal generated by the fuzzy point $a_i$.

Note 3.8: Here $<a_i>_L$ is the smallest fuzzy left $T\Gamma$-ideal of $T$ containing the fuzzy point $a_i$.

Definition 3.9: Let $T$ be a ternary $\Gamma$-semi ring and $a_i$ be a fuzzy point of $T$. Then the intersection of all fuzzy lateral $T\Gamma$-ideals of $T$ containing $a_i$ is a fuzzy lateral $T\Gamma$-ideal of $T$ containing $a_i$, denoted by $M(a_i)$ or $<a_i>_M$ and defined as $<a_i>_M = \bigcap_{a_i \in \mu} \mu$, where $FTTI_M(T)$ is the set of all fuzzy lateral $T\Gamma$-ideals of $T$. This fuzzy lateral $T\Gamma$-ideal is called fuzzy lateral $T\Gamma$-ideal generated by the fuzzy point $a_i$.

Note 3.10: Here $<a_i>_M$ is the smallest fuzzy lateral $T\Gamma$-ideal of $T$ containing the fuzzy point $a_i$.

Definition 3.11: Let $T$ be a ternary $\Gamma$-semi ring and $a_i$ be a fuzzy point of $T$. Then the intersection of all fuzzy right $T\Gamma$-ideals of $T$ containing $a_i$ is a fuzzy right $T\Gamma$-ideal of $T$ containing $a_i$, denoted by $R(a_i)$ or $<a_i>_R$ and defined as $<a_i>_R = \bigcap_{a_i \in \mu} \mu$, where $FTTI_R(T)$ is the set of all fuzzy right $T\Gamma$-ideals of $T$. This fuzzy right $T\Gamma$-ideal is called fuzzy right $T\Gamma$-ideal generated by the fuzzy point $a_i$.

Note 3.12: Here $<a_i>_R$ is the smallest fuzzy right $T\Gamma$-ideal of $T$ containing the fuzzy point $a_i$.

Definition 3.1: Let $T$ be a ternary $\Gamma$-semi ring and $a_i$ be a fuzzy point of $T$. Then the intersection of all fuzzy $T\Gamma$-ideals of $T$ containing $a_i$ is a fuzzy $T\Gamma$-ideal of $T$ containing $a_i$, denoted by $J(a_i)$ or $<a_i>$ and defined as $<a_i> = \bigcap_{a_i \in \mu} \mu$, where $FTTI(T)$ is the set of all fuzzy $T\Gamma$-ideals of $T$. This fuzzy $T\Gamma$-ideal is called fuzzy $T\Gamma$-ideal generated by the fuzzy point $a_i$. 


Note 3.14: Here $<a_t>$ is the smallest fuzzy $T^+$-ideal of $T$ containing the fuzzy point $a_t$.

Theorem 3.15: Let $T$ be a ternary $\Gamma$-semi ring and $FP(T)$ be the set of all fuzzy points of $T$. Let $a_t \in FP(T)$. Then for any $x \in T$,

(i) $<a_t>_L(x) = \begin{cases} 1, & \text{if } x \in <a_t>_L \\ 0, & \text{otherwise} \end{cases}$

(ii) $<a_t>_M(x) = \begin{cases} 1, & \text{if } x \in <a_t>_M \\ 0, & \text{otherwise} \end{cases}$

(iii) $<a_t>_R(x) = \begin{cases} 1, & \text{if } x \in <a_t>_R \\ 0, & \text{otherwise} \end{cases}$

(iv) $<a_t>(x) = \begin{cases} 1, & \text{if } x \in <a_t> \\ 0, & \text{otherwise} \end{cases}$

Proof: We have $<a_t>_L = \bigcap_{a_t \mu \in FTTL(T)} \mu$. Then for any $x \in T$,

$<a_t>_L(x) = \left( \bigcap_{a_t \mu \in FTTL(T)} \mu \right)(x) = \bigwedge_{a_t \mu \in FTTL(T)} \{\mu(x)\}.$

If $x = a$, then $<a_t>_L(x) = <a_t>_L(a) = \bigwedge_{a_t \mu \in FTTL(T)} \{\mu(a)\} = t$ (since $a_t \mu \in \mu$, $\mu(a) \geq t$).

If $x = p\gamma q\delta a$ for some $p, q \in T$ and $\gamma, \delta \in \Gamma$, then for any fuzzy left $T^+$-ideal $\mu$ of $T$,

$\mu(p\gamma q\delta a) \geq \mu(a) \geq t$ and hence $<a_t>_L(x) = \bigwedge_{a_t \mu \in FTTL(T)} \{\mu(p\gamma q\delta a)\} = t$.

If $x \neq a$ and $x \neq p\gamma q\delta a$ for some $p, q \in T$ and $\gamma, \delta \in \Gamma$, then by using corollary 3.4, we find that $<a_t>_L(x) = 0$ and hence the result. Similarly, we can prove the remaining parts.

Theorem 3.16: Let $T$ be a ternary $\Gamma$-semi ring and $FP(T)$ be the set of all fuzzy points of $T$. Let $a_t \in FP(T)$. Then

(i) $<a_t>_L = a_t \cup TTTT a_t$

(ii) $<a_t>_M = a_t \cup T T T a_t \Gamma T T T a_t \Gamma T T T$

(iii) $<a_t>_R = a_t \cup U a_t \Gamma T T T$

(iv) $<a_t> = a_t \cup a_t \Gamma T T T U T T T a_t \cup T T T a_t \Gamma T T T a_t \Gamma T T T$

Proof: Let $a_t \in FP(T)$. Then $a_t \cup TToa_t$ is a fuzzy left $T^+$-ideal of $T$ containing $a_t$. Let $\mu$ be the fuzzy left $T^+$-ideal of $T$ containing $a_t$ such that $\mu \subseteq a_t \cup T T T a_t$. Since $a_t \mu \subseteq \mu$, we have $T T T a_t \subseteq T T T a_t \mu \subseteq \mu$ (since $\mu$ is the fuzzy left $T^+$-ideal of $T$). This implies that $a_t \cup T T T a_t \subseteq a_t \cup T T T a_t$ and hence $\mu \subseteq a_t \cup T T T a_t$. So we find that $a_t \cup T T T a_t$ is the smallest fuzzy left $T^+$-ideal of $T$ containing the fuzzy point $a_t$. But $<a_t>_L$ is the smallest fuzzy left $T^+$-ideal of $T$ containing $a_t$. Therefore, $<a_t>_L = a_t \cup T T T a_t$ Similarly, we can prove the remaining parts.

4. Fuzzy Regular Ternary $\Gamma$-Semi ring

Theorem 4.1: In a ternary $\Gamma$-semi ring $T$ the following are equivalent.
(i) $T$ is multiplicatively regular

(ii) $\mu_1 \Gamma \mu_2 \Gamma \mu_3 = \mu_4 \Gamma \mu_5 \Gamma \mu_6$ for every fuzzy left $T \Gamma$-ideal $\mu_1$ and every fuzzy lateral $T \Gamma$-ideal $\mu_2$ and every fuzzy right $T \Gamma$-ideal $\mu_3$ of $T$.

**Theorem 4.2**: The following conditions in a ternary $\Gamma$-semi ring $T$ are equivalent:

(i) $T$ is ternary multiplicatively regular

(ii) For any fuzzy right $T \Gamma$-ideal $\mu$, fuzzy lateral $T \Gamma$-ideal $\lambda$ and fuzzy left $T \Gamma$-ideal $\nu$ of $T$, $\mu \lambda \Gamma \nu = \mu \lambda \nu$

(iii) For $a, b, c \in T$ and $t \in (0, 1]$,

$\langle a_t \rangle_{\Gamma} b_t \Gamma c_t \rangle_{\Gamma} = \langle a_t \rangle_{\Gamma} b_t \cap c_t \rangle_{\Gamma}.$

For all $a \in S$ and $t \in (0, 1], \langle a_t \rangle_{\Gamma} b_t \Gamma c_t \rangle_{\Gamma} = \langle a_t \rangle_{\Gamma} b_t \cap c_t \rangle_{\Gamma}.$

Proof: (i) $\iff$ (ii): By theorem 4.1 we get (i) and (ii) are equivalent.

Clearly, (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i): Suppose that $a \in S$ and $t \in (0, 1], \langle a_t \rangle_{\Gamma} b_t \cap c_t \rangle_{\Gamma} = \langle a_t \rangle_{\Gamma} b_t \cap c_t \rangle_{\Gamma}.$

We have $a_t \in \langle a_t \rangle_{\Gamma} b_t \cap c_t \rangle_{\Gamma} = \langle a_t \rangle_{\Gamma} \langle b_t \cap c_t \rangle_{\Gamma}.$

Thus $a = a b c d$ for some $b, c \in T$ and $a, b, c \in \Gamma$ (by theorem 4.1). Therefore $T$ is multiplicatively regular.

**Definition 4.3**: A fuzzy $\Gamma$-ideal of $T$ is said to be $\Gamma$-idempotent if $\mu \Gamma \mu = \mu$.

**Theorem 4.4**: Every left $T \Gamma$-ideal of a multiplicatively regular ternary $\Gamma$-semi ring $S$ is a $\Gamma$-idempotent.

Proof: Let $\mu$ be a fuzzy left $T \Gamma$-ideal of a multiplicatively regular $\Gamma$-semi ring $T$. Then $\mu \Gamma \mu = \mu$.

Let $x \in T$. Then there exists an element $a, b \in T$ such that $x = a b x y b = x$ for some $a, b, c \in \Gamma$.

Therefore

$$\mu \Gamma \mu (x) = \bigvee_{p \in \Gamma} \mu (p) \cap \mu (q) \cap \mu (r) \geq \mu (x a b x y b \delta ) \cap \mu (x) \geq \mu (x) \cap \mu (x) \geq \mu (x)$$

This implies that $\mu \subseteq \mu \Gamma \mu$. Consequently, $\mu = \mu \Gamma \mu$ and hence $\mu$ is $\Gamma$-idempotent.

**Theorem 4.5**: A ternary $\Gamma$-semi ring $T$ is multiplicatively regular if and only if every fuzzy $T \Gamma$-ideal of $T$ is $\Gamma$-idempotent.

Proof: Let $T$ be a multiplicatively regular $\Gamma$-semi ring and consider $\mu$ to be a fuzzy $T \Gamma$-ideal of $T$. Then $\mu \Gamma \mu$ is a fuzzy left $T \Gamma$-ideal of $T$.

Conversely, suppose that every fuzzy $T \Gamma$-ideal of the ternary $\Gamma$-semi ring $T$ is $\Gamma$-idempotent. Let $\mu, \chi$ and $\nu$ be three fuzzy $T \Gamma$-ideals of $T$. Then $\mu \Gamma \chi \nu \subseteq \mu \Gamma \chi \nu$. But $\mu \Gamma \chi \nu$ is a fuzzy $T \Gamma$-ideal of $T$.

Therefore, by our assumption, $(\mu \Gamma \chi \nu) = (\mu \Gamma \chi \nu) \Gamma (\mu \Gamma \chi \nu) \Gamma (\mu \Gamma \chi \nu) \subseteq \mu \Gamma \chi \nu$ and hence $(\mu \Gamma \chi \nu) = \mu \Gamma \chi \nu$. By theorem 4.1, $T$ is regular.

**Definition 4.6**: A ternary $\Gamma$-semi ring is said to be **fuzzy left simple** if every fuzzy left $T \Gamma$-ideal of $T$ is constant.
Theorem 4.7: For a ternary $\Gamma$-semi ring $T$, the following conditions are equivalent:

(i) $T$ is a left simple ternary $\Gamma$-semi ring.

(ii) $T$ is fuzzy left simple ternary $\Gamma$-semi ring.

Proof: Let $T$ be left simple and let $\mu$ be a fuzzy left $\Gamma$-ideal of $T$. Consider $x, y \in T$. Then $\Gamma \Gamma \Gamma x$ and $\Gamma \Gamma \Gamma y$ are the left $\Gamma$-ideals of $T$. Since $T$ is left simple, we have $T = \Gamma \Gamma \Gamma x$ and $T = \Gamma \Gamma \Gamma y$. So we can write $y = paxb \Gamma x$ and $x = r \beta \gamma \gamma \gamma$ for some $p, q, r, s \in T$ and $\alpha, \beta, \gamma \in \Gamma$. Therefore $\mu(y) = \mu(paxb \Gamma x) = \mu(x) = \mu(y)$ for all $x, y \in T$. Hence $\mu$ denotes a constant function. Consequently, $T$ is fuzzy left simple.

Conversely, let $T$ be fuzzy left simple and let $A$ be a left $\Gamma$-ideal of $T$. Then by known theorem, $\chi_A$ denotes a fuzzy left $\Gamma$-ideal of $T$. Now by hypothesis, $\chi_A$ is a constant function. Since $A$ is a left $\Gamma$-ideal of $T$, it is non-empty. Let $x$ be any element of $T$. Since $A$ is non-empty, $\chi_A (x) = 1$. This implies that $x \in A$. So we find that $T \subseteq A$ and hence $A = T$. Therefore $T$ is left simple.

Definition 4.8: A ternary $\Gamma$-semi ring $T$ is said to be fuzzy lateral simple provided every fuzzy right $\Gamma$-ideal of $T$ is constant.

Theorem 4.9: For a ternary $\Gamma$-semi ring $T$, the following conditions are equivalent:

(i) $T$ is a lateral simple ternary $\Gamma$-semi ring.

(ii) $T$ is fuzzy lateral simple ternary $\Gamma$-semi ring.

Proof: Similar to 4.6.

Definition 4.10: A ternary $\Gamma$-semi ring $T$ is said to be fuzzy right simple provided every fuzzy right $\Gamma$-ideal of $T$ is constant.

Theorem 4.11: For a ternary $\Gamma$-semi ring $T$, the following conditions are equivalent:

(i) $T$ is a right simple ternary $\Gamma$-semi ring.

(ii) $T$ is fuzzy right simple ternary $\Gamma$-semi ring.

Proof: Similar to 4.6.

Definition 4.12: A ternary $\Gamma$-semi ring $T$ is said to be fuzzy simple provided every fuzzy $\Gamma$-ideal of $T$ is constant.

Theorem 4.13: For a ternary $\Gamma$-semi ring $T$, the following conditions are equivalent:

(i) $T$ is a simple ternary $\Gamma$-semi ring.

(ii) $T$ is fuzzy simple ternary $\Gamma$-semi ring.

Proof: Similar to 4.6.

Theorem 4.14: If $T$ be an intra-regular ternary $\Gamma$-semi ring, then $\mu \cap \nu \cap \xi \subseteq \mu$, for every fuzzy left $\Gamma$-ideal $\mu$, fuzzy lateral $\Gamma$-ideal $\nu$ and fuzzy right $\Gamma$-ideal of $T$.

Proof: Let $T$ be an intra-regular ternary $\Gamma$-semi ring and $x \in T$. Then there exist $a, b \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x = aaxbxy \delta \delta b = aaxbxy(aaxbxy \delta \delta b) \delta \delta b = (aaxbxy)(aaxbxy)(x \delta \delta b \delta \delta b)$ and for any fuzzy left $\Gamma$-ideal $\mu$, fuzzy lateral $\Gamma$-ideal $\nu$ and fuzzy right $\Gamma$-ideal $\xi$ of $T$, we have

\[
(\mu \Gamma \nu \xi)(x) = \bigvee_{x = p \delta q \gamma \delta \delta b} (\mu(p) \wedge \nu(q) \wedge \xi(r)) \geq \mu(aaxbxy) \wedge \nu(aaxbxy) \wedge \xi(x \delta \delta b \delta \delta b)
\]

Thus $\mu \cap \nu \cap \xi \subseteq \mu \Gamma \nu \xi$.

Theorem 4.15: Every fuzzy lateral $\Gamma$-ideal of an intra-regular ternary $\Gamma$-semi ring $T$ is a fuzzy $\Gamma$-ideal of $T$.

Proof: Let $\mu$ be a fuzzy lateral $\Gamma$-ideal of an intra-regular ternary $\Gamma$-semi ring $T$ and let $x, y, z \in T$. Then there exist $a, b, c, d \in T$ and $\alpha, \beta, \gamma, \delta, \epsilon, \theta \in \Gamma$ such that $x = aaxbxy \delta \delta b$ and $z = c \delta z \epsilon z \theta z \delta d$.
\[\mu(x\lambda y\pi z) = \mu((aax\beta x\gamma x\delta b)\lambda y\pi z)\]

Now \[\mu(x\lambda y\pi z) = \mu((aax\beta x)\gamma x\delta b)\lambda y\pi z) \geq \mu(x)\]

and \[\mu(x\lambda y\pi z) = \mu(x\lambda y\pi(c\delta z\epsilon z\eta\rho d))\]

\[= \mu((x\lambda y\pi)\delta z\epsilon(z\theta z\rho d)) \geq \mu(z)\]

This shows that \(\mu\) is a fuzzy left TT-ideal of \(T\) and fuzzy right TT-ideal of \(T\). Since \(\mu\) is a fuzzy lateral TT-ideal of \(T\). Therefore \(\mu\) is a fuzzy TT-ideal of \(T\).

**Theorem 4.16:** In a ternary \(\Gamma\)-semi ring \(T\), the following conditions are equivalent:

(i) \(T\) is intra-regular

(ii) For every fuzzy lateral TT-ideal \(\mu\) of \(T\), \(\mu(x) = (x\gamma x \delta)\) for all \(x \in T\) and \(\gamma \in \Gamma\).

(iii) For every fuzzy TT-ideal \(\mu\) of \(T\), \(\mu(x) = (x\gamma x \delta)\) for all \(x \in T\) and \(\gamma \notin \Gamma\).

**Proof:**

(i) \(\Rightarrow\) (ii): Let \(S\) be an intra-regular ternary \(\Gamma\)-semi ring and \(\mu\) be a fuzzy lateral TT-ideal of \(T\). Let \(x \in T\). Then there exist \(a, b, c\), \(\gamma, \delta \in \Gamma\) such that \(x = aax\beta x\gamma x\delta \epsilon b\). Therefore \(\mu(x) = (aax\beta x\gamma x\delta \epsilon b) \geq (x\gamma x) = ((aax\beta x)(\gamma x\delta \epsilon b)) \geq \mu(x)\). Therefore \((x\gamma x) = \mu(x)\).

(ii) \(\Rightarrow\) (i): Suppose for every fuzzy TT-ideal \(\mu\) of \(T\), \(\mu(x\gamma x) = \mu(x)\) for all \(x \in S\) and \(\gamma \in \Gamma\).

Then we have the following cases:

Case 1: If \(x = x\gamma x\), then \(x = x\gamma x = (x\gamma x)(x\gamma x)\gamma (x\gamma x) \in T\Gamma (x\gamma x) \Gamma T\). Let \(x \in T\Gamma (x\gamma x) \Gamma T\), then \(x \in T\Gamma (x\gamma x) \Gamma T\). Let \(x \in T\Gamma (x\gamma x) \Gamma T\), then \(x \in T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\).

Case 2: If \(x \in T\Gamma (x\gamma x) \Gamma T\), then \(x \in T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\).

Case 3: If \(x \in T\Gamma (x\gamma x) \Gamma T\), then \(x \in T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\Gamma (x\gamma x) \Gamma T\).

Therefore from the above cases, it follows that \(x \in T\Gamma (x\gamma x) \Gamma T\). Thus \(T\) is intra regular.

**Definition 4.17:** A ternary \(\Gamma\)-sub semi ring \(A\) of a ternary \(\Gamma\)-semi ring \(T\) is said to be an interior TT-ideal of \(T\) provided \(TTA \Gamma T\Gamma T \subseteq A\).

**Definition 4.18:** A fuzzy subset \(\mu\) is called a fuzzy interior TT-ideal of ternary \(\Gamma\)-semi ring \(T\) if

(i) \(\mu(\alpha x\beta x\gamma x \delta x e) \geq \mu(a) \land \mu(b) \land \mu(c)\)

(ii) \(\mu(aabbc\gamma x \delta x e) \geq \mu(b) \land \mu(d)\) for all \(a, b, c, d, e \in T\) and \(\alpha, \beta, \gamma, \delta \in \Gamma\).

**Theorem 4.19:** If \(A\) is an interior TT-ideal of a ternary \(\Gamma\)-semi ring \(T\), then \(\chi_{\mu}\) is a fuzzy interior TT-ideal of \(T\).

**Proof:** Since \(A\) is a non-empty subset of \(T\). Let \(a, b, c, d, e \in T\) and \(\alpha, \beta, \gamma, \delta \in \Gamma\). From the hypothesis, \(aabbc\gamma x \delta x e \in A\). Then we have the following cases:

(i) If \(b, d \in A\), then \(\chi_{\mu}(b) = \chi_{\mu}(d) = 1\). Thus \(\chi_{\mu}(aabbc\gamma x \delta x e) = 1 \geq \chi_{\mu}(b) \land \chi_{\mu}(d)\).

(ii) If \(b, e \in A\), then \(\chi_{\mu}(b) = \chi_{\mu}(e) = 0\). Thus \(\chi_{\mu}(aabbc\gamma x \delta x e) = 0 \geq \chi_{\mu}(b) \land \chi_{\mu}(d)\).

(iii) If \(b \notin A, d \in A\), then \(\chi_{\mu}(b) = 1, \chi_{\mu}(d) = 0\).

Thus \(\chi_{\mu}(aabbc\gamma x \delta x e) = 1 \geq \chi_{\mu}(b) \land \chi_{\mu}(d)\).

(iv) If \(d \notin A, b \notin A\), then \(\chi_{\mu}(b) = 0, \chi_{\mu}(d) = 1\).

Thus \(\chi_{\mu}(aabbc\gamma x \delta x e) = 0 \geq \chi_{\mu}(b) \land \chi_{\mu}(d)\).

Hence \(\chi_{\mu}\) is a fuzzy interior TT-ideal of \(T\).
Theorem 4.20: Every fuzzy TT-ideal of a ternary Γ-semi ring T is a fuzzy interior TT-ideal of T.

Proof: Let μ be a fuzzy TT-ideal of a ternary Γ-semi ring S. Let x, y, z ∈ S, α, β ∈ Γ. Since μ is a fuzzy TT-idea ⇒ μ(aαbβcγdδe) ≥ μ(a) ∨ μ(b) ∨ μ(c).

Now μ be a fuzzy TT-idea ⇒ μ is fuzzy left TT-ideal, fuzzy lateral TT-ideal and fuzzy right TT-ideal.

Case 1: If μ is left TT-ideal, then μ(aαbβcγdδe) ≥ μ(bβcγdδe) ≥ μ(d).

Case 2: If μ is lateral TT-ideal, then μ(aαbβcγdδe) ≥ μ(aαbβcδe) ≥ μ(b).

Case 3: If μ is right TT-ideal, then μ(aαbβcγdδe) ≥ μ(bβcγdδe) ≥ μ(b).

Therefore, μ(aαbβcγdδe) ≥ μ(b) ∧ μ(d) and hence μ is fuzzy interior TT-ideal of T.

Note 4.21: A fuzzy interior TT-ideal μ of a ternary Γ-semi ring T need not be a fuzzy TT-ideal of T.

Theorem 4.22: If T is ternary multiplicatively regular ternary Γ-semi ring, then every fuzzy interior TT-ideal of T is a fuzzy TT-ideal of T.

Proof: Let μ be a fuzzy interior TT-ideal of T and x, y ∈ T. In this case, because of T is ternary multiplicatively regular, there exists a, b, c, d ∈ T and α, β, γ, δ, λ, μ ∈ Γ such that x = aαbβcγdδe and y = yλcμγeddδy. Thus μ(xaαbβcγdδe) = μ((yaαbβcγdδe)γeddδy) = μ((μ)(yλcμγeddδy)) ≥ μ(z).

Similarly, we can show that μ is a fuzzy right TT-ideal, lateral TT-ideal of T. This completes the proof.

Note 4.23: Combining theorems 4.20, and 4.22, if T is multiplicatively regular then fuzzy interior TT-ideals coincide.

References:
[1] Dutta and Chanda. T, Structures of Fuzzy Ideals of Γ-Ring; Bull. Malays. Math. Sci. Soc. (2) 28(1) (2005), 9-18.
[2] Jun, J. Noggers, H S Kim, On L-fuzzy ideals in semi rings I, Czechoslovak Math. Journal, 48 (123) (1998) 669-675.
[3] Kavikumar and Azme Bin Khamis, Fuzzy ideals and Fuzzy Quasi-ideals of Ternary Semi rings, IAENG International Journal of Applied Mathematics, 37: 2 (2007) 102-106.
[4] C B Kim and Mi-AePark, k-Fuzzy ideals in Semi rings, Fuzzy Sets and Systems 81 (1996) 281-286.
[5] W Liu, Fuzzy invariant subgroups and Fuzzy ideals, Fuzzy Sets and Systems, 8 (1982) 133-139.
[6] Madhusudhana Rao and G Srinivasa Rao, “Concepts on Ternary Semi rings”, International Journal of Modern Sciences and Engineering Technology, Vol.1, Issue 7, 2014, PP.105-110.
[7] M Sajani Lavanya, Dr. D Madhusudhana Rao, and V Syam Julius Rajendra; On Lateral Ternary Γ-Ideals of Ternary Γ-Semi rings, American International Journal of Research in Science, Technology, Engineering & Mathematics (AIJRSTM), 12(1), September-November, 2015, pp: 11-14.
[8] M Sajani Lavanya, Dr. D Madhusudhana Rao, and Prof. K Panduranga Rao; On simple Ternary Γ-Semi ring, International Journal of Engineering Research and Application, Vol. 7, Issue 2, (Part-5) February 2017, pp: 01-06.
[9] M Sajani Lavanya, Dr. D Madhusudhana Rao, and V B Subramanyeswar Rao Seetam Raju; On Right Ternary Γ-Ideals of Ternary Γ-Semi ring, International Journal of Research in Applied Natural and Social Sciences, Vol. 4, Issue 5, May 2016, 107-114.
[10] L A Zadeh, Fuzzy Sets, Information and Control, 8 (3) (1965) 338-353.