Introduction

The Vogt-Russell (VR) theorem states, crudely speaking, that a star’s mass and its chemical composition define uniquely its structure and hence its position in any suitably chosen characteristic diagram, such as the HR –, mass - radius –, central-density - temperature – diagram. The simple claim of the theorem seems, however, to be contradicted, at least by model stars. In any case, discussions of the VR theorem force astrophysicists to think thoroughly about the various approximations going into the modeling of stars, about limitations of numerical modeling, and the mathematical properties of the involved differential equations.

A first installment (Gautschy 2015) focused on the early years of the theorem and its reception to the end of the 1960s. The exposition stopped at about the time when the computational stellar evolution industry took off. The following second part of the historical discourse on the VR theorem focuses mainly on achievements during 1970s. Those years constituted the first phase of extensive stellar-evolutionary computations at various levels of abstraction. The advanced evolutionary stages of star models reached thereby revealed evidence of violations of the VR theorem. Parallel to all the computational work, more abstract mathematical methods were imported to study the solution properties of the stellar structure equations. During the early 1970s, a small number of researchers were attracted to the problem of the validity of the VR theorem. After the mid 1970s, however, that kind of formal stellar-physics research died out again. The quite formal and detached findings regarding the behavior of solutions of the stellar-evolution equations remained confined to a small group of researchers; but most importantly, the insights did not diffuse into the general literature and did not influence textbook opinions – with the self-explanatory exception of Kippenhahn & Weigert (1994). After the late 1970s, stellar-evolution research returned to and remained essentially a numerical-computation enterprise focusing on data-driven modeling.

Counterexamples and conjectures

Analyzing the results from his own stellar-evolution modeling and comparing them with data from the published literature,
Paczyński (1970) was confronted with annoying divergences in size and partly even the existence of blue loops traced out on the HR diagram by intermediate and massive stars (3, 5, 7, 10, and $15 M_\odot$) during their core helium-burning phase. After he failed to disclose any sensible correlations of model parameters with the blueest points on the HR plane reached during the blue loops of $5M_\odot$ stars, Paczyński conjectured that multiple solutions and associated thermal instabilities might be the source of the problem, at least for the more massive stars. The effect of the thermal instability was thought to be sensitive to the numerical treatment of the subphotospheric layers, therefore, Paczyński concluded that the different numerical treatments of the various authors as well as his own ones affected the eventual expression of the blue loops. Even though Paczyński did not mention the VR theorem explicitly, his suspicion of multiple solutions and the particular phrasing of the text must have meant to him that the VR theorem was apparently violated. Kozłowski (1971), a young researcher advised by Paczyński, looked closer into the blue-loop problem by constructing linear series of full-equilibrium (FE) models of $10M_\odot$ stars. In his fitted shooting models, he found that multiple solutions to the full-equilibrium stellar structure equations constructed for helium-core masses in the range of $2.43 < M_\odot < 2.53$. The multiple solutions of equal helium-core mass models mimicked indeed the blue loops on the HR plane of the full stellar-evolution models.

In a series of six papers published between 1972 and 1977, Paczyński and his various co-authors further explored the solution behavior of linear series of FE star models. The second paper, Paczyński & Kozłowski (1972), was devoted to homogeneous pure carbon stars, for which the total stellar mass was adopted as control parameter. At the very least, a stable main-sequence – like (non degenerate) solution family and a stable degenerate sequence, reminiscent of carbon white dwarfs were encountered. When neutrino losses were added to the computations, the authors reported hints of additional solution sequences, possibly thermally unstable ones, which at the time turned out to be very tricky to track numerically. These sequences were characterized by the number of regions with negative total luminosity therein; and this number remained invariant under change of the control parameter. Therefore, the authors conjectured that large number of sequences might lurk in solution space and hence, a correspondingly severe violation of the VR theorem might prevail. The solution branches with these multiple negative luminosity regions were, to the best of my knowledge, never followed up; so existence and relevance of such models remains obscure. Paper three of the series, Kozłowski & Paczyński (1973), gained some fame for its linear series of $10M_\odot$ star models with inconspicuous hydrogen/helium – envelopes that mimicked their core-helium burning phase. FE models with helium core masses in the range $0.37 – 0.375M_\odot$ sported nine simultaneous
equilibrium solutions.

Around 1970, Alfred Weigert started a small research group at Hamburg Observatory. Using, what was colloquially referred to as the Kippenhahn stellar-evolution code and to which Weigert was a founding contributor, the Hamburg group tackled various aspects of single- and binary-star evolution. Dietmar Lauterborn, then a young researcher in Weigert’s group, studied – independently of Paczyński’s quests – the physical cause of the blue loops of helium-burning, intermediate-mass (for $3M_\odot$ and mostly $5M_\odot$) stars (Lauterborn et al. 1971b) to grasp the bewildering variety of loci of such stars on the HR plane. The stellar models discussed in the paper relied on the generalized main-sequence approach of Giannone et al. (1968) where FE models were computed with the Kippenhahn code adopting prescribed composition profiles. The parameterized mass of the helium core mimicked thereby the temporal evolution of the model sequence.

The first paper of Lauterborn et al. (1971b) did not yet mention multiple solutions and secularly unstable model branches were not involved. Nonetheless, the paper left its mark and influenced the blue-loop literature by the analytical prescription of the H/He profile as a truncated ramp around the H-burning shell, this shape favors blue excursions of the respective model stars. The authors took advantage of the fact that the position on the HR plane of a centrally He-burning model star depended essentially on the macroscopic properties of the helium core. The detailed structure of the core is not important, only core mass, core radius, and the luminosity passing through the core’s surface were identified to matter. Specifying these core quantities as inner boundary conditions of envelope-only computations granted a great deal of freedom to construct envelope sequences for which one or several of the core properties (entering as inner boundary conditions) could be varied at discretion. In particular the physical properties of the bottoms of envelopes (BoE) computed in this artificial way provided crucial arguments on the issue of multiple solutions to the stellar structure equations. With this approach of splitting up a model star into a core and an attached envelope, the Henyey relaxation approach regained characteristics otherwise typical of two-sided shooting methods for boundary-value problems.

In the second paper (Lauterborn et al. 1971a), which addressed more massive stars at 7 and 9$M_\odot$, thermal instability and multiplicity of solutions entered the stage. Figure 2 sketches the effective temperatures as a function of helium core mass of the 9$M_\odot$ FE sequence of the complete star models from Lauterborn et al. (1971a). Two solution branches were encountered: As the helium core mass grows, a cool branch evolves at low effective temperatures, going through point $a$ to point $b$ where it terminates. Starting with the helium core mass of point $b$, a second, higher-temperature sequence exists that passes through point $d$. As the helium core grows, the star ‘evolves’ – in full equilibrium – to the upper right of the equilibrium solutions.

As early as 1964, blue loops of intermediate-mass stars had been sighted in evolutionary computations performed with the “Kippenhahn code” (Hofmeister et al. 1964), and also in results obtained with the “Iben code” (Iben 1964).

4 This means that hydrogen shell-burning drives the star’s evolution. The effect of the helium core-burning luminosity and the associated composition change can be neglected in this case.
the theorem that was none

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diagram. Again, starting at point d, the high-$T_{\text{eff}}$ solution branch of FE star models can be followed to point c upon lowering the helium-core mass where the equilibrium series terminates again. If the helium core mass is further reduced, no FE model star can be found in the direct neighborhood of point c. In the tiny range $0.1893 < M_{\text{He core}}/M_* < 0.1905$ the two sequences overlap, i.e. two equilibrium solutions exist per prescribed helium core mass. Actually, points b and c can be connected by an FE solution branch (dash-dotted line hinted at schematically in Fig. 2), which is secularly unstable.

For an actual star, the double- or actually triple-solution episode never causes a problem. Evolution and the appropriate physical processes resolve the di- or better trilemma: A star would approach point a as the helium core grows and it would – essentially in full equilibrium continue its evolution until the equilibrium sequence terminates at point b. Because the helium core of an actual star cannot only grow at this point, the star adapts to the circumstances by going thermally unstable and evolving – quickly compared with FE evolution – in the vicinity of line b-d to find new full equilibrium states at around epoch d. Thereafter, the star continues its evolution at higher effective temperature, again close to a full-equilibrium locus, hinted at by the dashed line to the upper right of the figure.6

After Lauterborn et al. (1971a), it was evident that blue loops of core helium-burning stars could develop also in full equilibrium models, at least for not too massive stars. With growing total stellar mass, FE tends to get lost and secular instability develops as the helium core grows, and the magnitude (measured in the effective-temperature range they swept on the HR plane) of the loops grows. Hence, the existence of blue loops does not depend on secular instability and therefore blue loops are not causally related to uniqueness issues of stellar models.

March 1972 must have been a busy period at Hamburg Observatory: Within 18 days, three papers on the same topic were submitted for publication by different members of Weigert’s research group (although Lauterborn apparently was on leave of absence at JILA in Boulder, most likely with John P. Cox). The editorial office of A&A received the Lauterborn (1972) manuscript on March 2nd, that of Kähler (1972) on March 3rd, and finally the Roth & Weigert (1972) one on March 20th.

Lauterborn (1972) set out to understand the BoE loci and those traced out by the surfaces of core solutions on the radius – pressure plane.7 The computations revealed that core and envelopes loci produced up to three intersections and therefore gave rise to triple degeneracies for some cases of equilibrium-structure solutions of blue-looping supergiants. Section 3 of the Lauterborn paper eventually addressed the violation of the VR theorem. In a terse paragraph, he kept the ball low when he emphasized that the correct interpretation of the classical VR theorem be a local, not a global one – i.e. only one solution may exist in a suitably chosen

Figure 2: Core-mass – effective temperature relation for a blue-looping $9 M_\odot$ model sequence after Lauterborn et al. (1971a).

6 Intermediate-mass and massive stars live through a comparable situation when they enter their Schönberg-Chandrasekhar instability phase.

7 The core was defined as the volume of the star where $X = 0$; the rest of the mantle with $X > 0$ was attributed to the envelope; i.e. the H-burning shell was considered as the bottom part of the stars’ envelopes.
neighborhood of a given solution. Lauterborn did not dive deeper into mathematical technicalities but focused on the particular class of stars at hand and he studied how the solutions depended on the numerous physical parameters that usually enter the modeling of these stars. A bump (also referred to as the hook in pertinent papers) in the BoE loci on the pressure – radius plane emerged as being the reason of potential multiplicity of the solutions to the FE stellar structure problem. The physical origin of the non-monotonicity was, however, not yet elaborated.

The \textit{Roth & Weigert (1972)} paper expanded on the multiple solutions of FE models with pure helium cores and hydrogen-rich envelopes. The authors reviewed earlier papers that dealt with generalized main-sequences and emphasized that the double solutions encountered in that context were caused by the different material properties of helium cores that could successfully match envelope solutions: One solution branch was made up by non-degenerate He-burning cores, the other consisted of degenerate, isothermal cores. \textit{Roth & Weigert (1972)} contrasted the \textit{Lauterborn et al. (1971b)} multiple solutions found for massive stars emphasizing that the multiple solutions were caused by the particular material properties of the envelopes rather than the cores. Roth et al. managed to compute much broader radius and pressure ranges on the core – envelope fitting plane. This allowed them to recover not only the “hook” localized by \textit{Lauterborn et al. (1971b)}; they also tracked down three additional solution possibilities of models with isothermal cores. Therefore, the triple solutions of Lauterborn et al. were actually sextuple solutions.

For solutions to FE model stars with isothermal helium cores, Roth & Weigert identified the roots that are associated with the Schönberg-Chandrasekhar instability. Most interestingly, the authors outlined concisely the topological behavior of the roots at the onset of the secular instability. Using the concept of the Henyey-determinant – which was elaborated on in \textit{Paczyński (1972)} and \textit{Kähler (1972)} – the authors pointed out that zeros of the determinant indicate the loss of FE at the onset of the triple solutions of the 9$M_\odot$ star studied previously by Lauterborn and collaborators. \textit{Murai (1974)} looked into the multiple-solution problem resorting to the methods developed earlier on by the Japanese stellar-astrophysics school around Hayashi, Hōshi, and Sugimoto. By performing the fitting of interior and exterior solutions in the plane of the homology invariants $U$ and $V$, the spread of the fitting quantities on the fitting plane could be significantly reduced.\footnote{The reason is that homologous solutions of the stellar structure equations are invariant on the $U,V$ plane, i.e. they trace the same locus thereon.} Murai found the reason of the hook to be tied to the particular opacity behavior in the deep interior of the envelope models. It is the functional form of the Kramers law that induces a spiraling motion on the $UV$-plane of envelope solutions. Effects of radiation pressure and of course of convection suppress this spiraling tendency. Consequently, stars that are sufficiently hot for their matter to be dominated by electron scattering or cool enough to have extended
convective envelopes lose their ability to allow for multiple solutions at a given total mass and a given chemical composition.

Additionally, Murai pointed out the coincidence of the occurrence of the hook in the fitting behavior of envelope models and the development of the Hertzsprung-gap. Even though earlier authors attributed the Schönberg-Chandrasekhar instability to the isothermal spiral of the inert helium core. Figure 7 of this paper illustrates that for the example of a $5M_\odot$ model star with pure electron-scattering opacity only (i.e. models with suppressed "hook") can be evolved through the Hertzsprung gap in FE, which is to say that they do not experience the SC instability anymore.

At a meeting of the Astronomische Gesellschaft in 1975, Lauterborn (1976) reviewed the physical origin of the earlier found triple solutions to massive stars in their central helium-burning phase; he made extensive use of Murai’s UV-plane viewpoint and illustrated the robustness of Murai’s identification of the Kramers opacity effect as the origin of the multiple of solutions in the $7M_\odot$-star case.

Local uniqueness

Kähler, another young member of Weigert’s research group in Hamburg, chose to attack the problem of the VR theorem more formally without having specific evolutionary scenarios in mind. In the first paper of Kähler (1972), he focused on the behavior of FE models. The starting point was the set of the canonical stellar-evolution equations with thermodynamic basis $\{P, T\}$; for a particular stellar model, the set of parameters, $\varphi = \varphi(M_*, \vec{\chi})$, was assumed to be prescribed.

The solution of the FE structure problem was assumed to be obtained with some two-sided shooting method. In this case, the in-out integration, starting at the stellar center to the to some fitting mass, $m_F$, constitutes an initial-value problem, which starts at a regular singularity. The annoyance of this singularity is usually cushioned by starting the computation at some small distance off the center with the solution expanded into a power series. Other than that the right-hand sides of the differential equations are well behaved so that they can be considered locally Lipschitz. Therefore, the existence of a solution (of the IVP) is guaranteed. The same reasoning applies to the out-in integration, which starts at a suitable, physically motivated specification of a stellar surface; again, the integration extends to $m_F$. Figure 5 illustrates the two-sided integration strategy. Two trial solutions obtained from the in-out and the out-in integrations meet at $m_F$. Only if they exactly match, and if they have additionally the same derivatives at $m_F$ do they constitute a physically acceptable solution.

The distributed boundary conditions allow the two equations to be integrated from one side of the problem each; trial values are assumed for the remaining two boundary conditions: $P_c$ and $T_c$ in the center and say $L$ and $R$ at the surface. Usually, the magnitudes...
of the dependent variables to not match at the fitmass \( m_F \), cf. the red state vectors at the fitmass plane in Fig. 3. Iteratively improving the guesses of the yet unknown boundary values will eventually lead to a solution of the stellar structure problem via a series of IVP integrations.

Kähler (1972) chose to split up the problem. He specified surface values for \( L \) and \( R \) and integrated the mass and energy equations (eqs. 1.2 and 1.8) down to \( m_F \). Analogously, these equations can be integrated in-out from the center, whereas they depend implicitly on \( P_c \) and \( T_c \) via the quantities \( \rho \) and \( \varepsilon \). The parameters are iteratively updated until the solutions match at \( m_F \):

\[
\begin{align*}
    r_I(m_F; P_c, T_c) & = r_O(m_F; R, L), \\
    L_I(m_F; P_c, T_c) & = L_O(m_F; R, L).
\end{align*}
\]

Equations (1) and (2) suggest that the outer boundary conditions \( R \) and \( L \) can be related to \( P_c \) and \( T_c \). If these relations are monotonous in the physically relevant parameter space then relations \( R(P_c, T_c), L(P_c, T_c) \) can be derived. Eventually, the remaining fitting conditions at \( m_F \) boil down to two relations that depend on \( P_c \) and \( T_c \) only:

\[
\begin{align*}
    g_1 = & \ P_I(m_F; P_c, T_c) - P_O(m_F; R(P_c, T_c), L(P_c, T_c)) \\
    g_2 = & \ T_I(m_F; P_c, T_c) - T_O(m_F; R(P_c, T_c), L(P_c, T_c)).
\end{align*}
\]

The two relations, \( g_1 \) and \( g_2 \), illustrate the nature of the boundary-value problem. An intersection of \( g_1 = 0 \) and \( g_2 = 0 \) on the \( P_c - T_c \) plane means that the stellar-structure problem admits of a solution. The behavior of the two nonlinear equations \( g_1 = 0 \) and \( g_2 = 0 \) can be complicated with, in principle, wild twists and turns (e.g. Fig. 4). It is possible that \( g_1 = 0 \) and \( g_2 = 0 \) do not intersect at all; they may intersect once or multiple times for a fixed choice of \( \varphi \); this latter case will then be a realization of the infamous multiple-solution cases that have been encountered numerically. The multiple-solution cases can be manifold, Fig. 4 suggest two solutions with a finite distance on the \( P_c - T_c \) plane (\( g_1 = 0 \) and \( g_2 = 0 \)). If \( g_2^* = 0 \) should violently oscillate with an accumulation point,
ininitely many solutions might be possible. The more realistic and pertinent tangential case of $g_2^{**} = 0$ will be encountered again later and discussed then.

Because no a priori insight into the expected topologies of the $g_1 = 0$ and $g_2 = 0$ loci on the $P_c - T_c$ plane is possible, i.e. no direct answer to the question of global uniqueness appeared to be accessible, Kähler resolved to concentrate first on the simpler, *local uniqueness* question.

Starting out with assuming that an equilibrium solution, i.e. $g_1 = 0$ and $g_2 = 0$ exists, Kähler inquired the properties of the fitting conditions in the direct neighborhood of such an equilibrium solution. The linear approximation of the functions $g_1 = 0$ and $g_2 = 0$ at the found solution leads to the condition

$$J_g \cdot \left( \begin{array}{c} \delta P_c \\ \delta T_c \end{array} \right) = 0 \quad (5)$$

for an additional solution displaced by $(\delta P_c \ \delta T_c)^T$ on the $P_c - T_c$ plane with $\wp$ remaining unchanged and with

$$J_g = \left( \begin{array}{cc} \partial P_c g_1 & \partial T_c g_1 \\ \partial P_c g_2 & \partial T_c g_2 \end{array} \right),$$

being the Jacobian of the fitting conditions $g$. For $\det J_g \neq 0$, the equilibrium stellar-structure problem is *locally unique* because only $(\delta P_c \ \delta T_c)^T = \vec{0}$ obtains; i.e. no other FE star model with the same set of parameters $\wp$ can be found in the neighborhood of an already obtained one.

The case of $\det J_g = 0$, on the other hand, allows for a nonvanishing $(\delta P_c \ \delta T_c)^T$ vector, i.e. for another solution to the FE equations, in an essentially arbitrarily close neighborhood of an already existing one for the same choice of parameters $\wp$. Hence, $\det J_g = 0$ amounts to a violation of the classical VR theorem formulation (such as illustrated in intersections of $g_1 = 0$ with $g_2^{*} = 0$ or $g_2^{**} = 0$ in Fig. 4). An intersection case such as $g_1 = 0$ and $g_2^{*} = 0$, with
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its accumulation point, seems unlikely to be physically realized in naturalistic stellar astrophysics but mathematically it remains a viable option. Contacts such as encountered in $g_1 = 0$ and $g_2^* = 0$, with common tangents, are involved whenever a star changes from secular stability to instability or vice versa.

An appendix of the [Kähler 1972] article was devoted to the proof that the vanishing of $\det J_g$ was equivalent to the vanishing of the Henyey-determinant; i.e. the determinant of the finite-difference system that constitutes the numerical representation of the model stars.\(^\text{11}\)

Variations of the parameter $\varphi$ is a method to compute linear series of stellar models in FE. Assuming again that an equilibrium stellar model had been found for a choice of $\varphi$, one asks if another equilibrium solution, displaced on the $P_c - T_c$ plane by $(\delta P_c, \delta T_c)^T$, can be found upon varying the parameter by $\delta \varphi$. The resulting linear system of equations becomes

$$J_g \cdot \begin{pmatrix} \delta P_c \\ \delta T_c \end{pmatrix} = -\delta \varphi \begin{pmatrix} \partial \varphi g_1 \\ \partial \varphi g_2 \end{pmatrix}, \quad (6)$$

which is just an inhomogeneous form of eq. (5). Equation (6) admits of an unique solution $(\delta P_c, \delta T_c)^T$ upon a parameter-change $\delta \varphi$ if $\det J_g \neq 0$. Hence, as long as an equilibrium model is unique, a unique model continuation can be constructed upon variation of the control parameter $\varphi$. Such a parameter-$\varphi$ viewpoint can be adopted in the study of properties of say hydrogen or helium main sequences with the stellar mass as the control parameter. A suitable parameterization of the chemical composition, which be prescribed in $\varphi$, can be thought of as an emulation of the evolution of stars in FE. Such equilibrium models remain locally unique as long as $\det J_g$ does not vanish. In the inhomogeneous case of eq. (6) this means that a unique solution obtains if the rank of the augmented system is equal to rank $J_g$. If, on the other hand, $\det J_g = 0$, the number of local solutions depends again on the rank of the augmented system. The case of no solution signifies that the linear series reached a termination point. Alternatively, a branching of solutions, such as at the onset of the Schönberg-Chandrasekhar instability or blue-looping core helium-burning intermediate-mass stars, as illustrated in Fig. 2 can be encountered.

Quasi-hydrostatic equilibrium (QHE) models were the next logical step to take in the study of local uniqueness properties of stellar-evolution models. The paper of [Kähler & Weigert 1974] followed quite closely the approach used in [Kähler 1972], but now incorporating the thermal imbalance term $D_{s} s$. To get a better handle of the equations, the thermodynamic basis defined by the total pressure and the specific entropy, $\{P, s\}$, was found to be more appropriate. The formal structure of the QHE stellar-structure

\(^{11}\) Henyey’s scheme to solve the quasi-static stellar evolution problem applies essentially the Thomas-Algorithm to solve a large block bi-diagonal matrix. The Henyey-determinant can be easily obtained from computing the determinant of the last block matrix of during forward elimination. N.B. The magnitude of the Henyey-determinant depends on the gridding of the star model; the sign of the determinant, however, is invariant under regridding.
equations becomes then:
\[
\begin{align*}
\partial_m r &= f_1 (r, P, s) \quad \text{(from I.2)}, \quad (7) \\
\partial_m P &= f_2 (m, r) \quad \text{(from I.6)}, \quad (8) \\
\partial_m s &= f_3 (m, r, L, P, s) \quad \text{(from I.10)}, \quad (9) \\
\partial_m L &= f_4 (P, s, \partial_t s) \quad \text{(from I.8)}. \quad (10)
\end{align*}
\]

The resulting system of equations is hence no longer a BVP of ordinary differential equation. The problem turns into an initial-value – boundary-value problem which requires the theory of partial differential equations. However, the mathematical theory of PDEs is not general enough to put forth helpful statements for the types of equations popping up in the description of stellar structure/evolution problems. The situation can be improved when adopting a prescribed entropy profile, \( s(m) \), for the stellar model about which the local (linear) analysis is to be performed. In this case the equations (7) – (9) fall back onto a BVP of ordinary differential equations and the same line of arguments applies as in the FE case earlier on.

Analogously to the FE case, in-out and out-in integrations are performed, which are matched at some fitting mass \( m_F \). With a prescribed entropy profile, equations (7) and (8), i.e. the star’s mechanical equations decouple from the thermo-energetic part (eqs. (9) and (10)). The interior, mechanical solutions can be parameterized via the central pressure, \( P_c \), alone. Once again, exterior solutions, i.e. out-in integrations are performed to \( m_F \) where the solutions are matched. In the QHE case, \( r_1(P_c) = r_\Omega(R) \), is used to find a relation \( R = R(P_c) \) so that, eventually, the fitting condition, analogous to eq. (3) in the FE case, can be determined:
\[
g(P_c) = P_\Omega(R(P_c)) - P_1(P_c).
\]

Note, that once the mechanical structure is determined, the thermal one follows: The prescribed entropy profile allows to compute the luminosity profile in eq. (9) and with its spatial derivative, the \( D_t \delta \) profile can be obtained via eq. (10):
\[
D_t \delta = \frac{1}{T} \left[ \varepsilon(P, s) - D_m L \right] , \quad (12)
\]
so that the temporal evolution of a star’s entropy distribution is determined.

Notice furthermore that the prescription of the entropy profile reduces the number of fit-equations to one. Hence, the fitting procedure, and the linear perturbation analysis reduces to the study of scalar equations rather than to matrix properties as in the FE case.

Linearizing about a solution, varying \( P_c \), keeping fixed the mass, the chemical profile, as well as \( s(m) \), i.e. with all of \( \varphi \) unchanged, yields
\[
dP_c \cdot \delta P_c = 0 . \quad (13)
\]
For \( \partial P_c g \neq 0 \), the only solution is \( \delta P_c = 0 \); i.e. no neighboring solution can be found so that the found QHE solution is \textit{locally unique}. On the other hand, as already seen in the FE case, for \( \partial P_c g = 0 \), the solution is not locally unique and possibly multiple solutions exist in an arbitrary neighborhood of the already obtained QHE solution.

In case of a variation of parameters, the case of a modified entropy profile is now of particular interest (the rest, variation of mass and chemical profile is analogous to the FE situation). Assume that we write:

\[
s(m, \varphi) = s_0(m, \varphi_0) + f(m) \cdot \delta \varphi,
\]

with \( \delta \varphi \in \mathbb{R} \) and \( f(m) \) an arbitrary, continuous function of mass alone. Linearization about a solution for \( s_0(m, \varphi_0) \):

\[
\partial P_c g \cdot \delta P_c + \partial \varphi g \cdot \delta \varphi = 0. \tag{14}
\]

Starting from a locally unique QHE solution with \( s_0(m) \), eq. (14) yields one neighboring solution for a slightly changed entropy profile if \( \partial P_c g \neq 0 \) and \( \partial \varphi g \neq 0 \). This means that a model sequence with varying entropy profile, say, according to

\[
s(m, \varphi; t) = s_0(m, \varphi; t_0) + \Delta t \cdot D_t s|_{t_0},
\]

with \( D_t s \) computed from eq. (12), can be continued continuously as long as the QHE model is locally stable. If a model violates local uniqueness \( (\partial P_c g = 0) \) then \( \partial \varphi g \neq 0 \) suppresses any local solutions, but \( \partial \varphi g = 0 \) allows for multiple ones in the vicinity of \( s_0 \).

Stellar stability, at least in the linear approximation, can be related to local uniqueness, this aspect is also computationally interesting. In QHE modeling, with its thermal imbalance term (but neglected acceleration), assume a time independent FE solution with physical quantities \( \hat{y}_i \); perturb this solution by some \( \delta y_i \):

\[
y_i(m, t) = \hat{y}_i + \delta y_i, \quad \text{adopting} \quad \delta y_i = v_i(m) \cdot \exp(\sigma t).
\]

Together with appropriate boundary conditions, a boundary eigenvalue problem results for the perturbations of the equilibrium state, \( v_i \), become spatial eigenfunctions for a discrete set of eigenvalues \( \sigma \).\(^{12}\) For simplicity, assume first that the secular eigenvalues are all real. If an initially secularly stable model sequence is computed then all the associated eigenvalues \( \sigma < 0 \). If later, during the star’s evolution secular instability sets in, the eigenfrequency of the respective secular mode goes positive. Hence, an epoch is encountered when \( \sigma = 0 \):

\[
y_i(m, t) = \hat{y}_i + v_i(m),
\]

which is time independent, and which therefore violates local uniqueness with \( \det J_g = 0 \) at this particular epoch: It was mentioned further up that (Kähler 1972) pointed out the connection of

\(^{12}\) The physical nature of the time dependence entering the eigenvalue problem leads to it being referred to as thermal or secular stability problem. In contrast to the adiabatic pulsation problem, the secular eigenproblem is not Hermitian and lacks therefore any elegant properties of the eigensolutions. Most importantly, its eigenvalues \( \sigma \) can be complex.


\[
\det J_g = 0 \quad \text{and zeros of Henyey-determinant. At the very me time,}
\]
also Paczyński was active in the same field; his paper \textbf{[Paczyński, 1972]} reached the editorial office of Acta Astronomica on March 15th, 1972 (Kähler’s paper arrived at A&A’s editorial office on the 3rd of March, 1972). In the same manner as Kähler, Paczyński elaborated on the Henyey determinant, its zeros (which coincide with the zeros of the Schwarzschild determinant\textsuperscript{13}) and the connection of the roots with the secular stability of a stellar model. In particular, the passage of stars through the Schönberg-Chandrasekhar instability – if they are approximated by a linear series of full-equilibrium models – leads to a phase of three equilibrium solutions for the same set of stellar parameters, with one branch being secularly unstable.\textsuperscript{14} The same conclusion was also arrived at within the methodical framework of \textbf{Kähler (1972)}.

In the case of oscillatory secular modes, the Henyey-/Schwarzschild-determinant does not vanish anymore so that the onset of secular instabilities via oscillatory modes cannot be tracked by means of monitoring the sign of the Henyey determinant. However, for complex eigensolutions with not too large oscillation frequencies, their change of stability is frequently noticeable in the determinant by its dipping through a local minimum as the model sequence progresses.

Irrespective of the path to instability of \(\sigma\) on the complex plane, it became clear that sign \(\det J_g = +1\) is a necessary requirement for the model to be thermally stable; sign \(\det J_g = -1\), on the other hand, is a sufficient condition for instability.

\textbf{The global perspective}

Local analyses boil down to a linear expansions of the curves \(g_1 = 0\) and \(g_2 = 0\) about solutions \(s_0\); they are, however, only of limited use to understand the full solution topology. The approach breaks down if, for a chosen \(\mathcal{P}\), multiple solutions exist at finite distance of each other on the \(P_c - T_c\) plane or if a solution is not a regular one, i.e. if \(\det J_g = 0\). Applying heavier mathematical machinery than usual in astrophysics, \textbf{Kähler (1975)} drew from the field of algebraic geometry and found a way to tackle even such non-local problems. The resulting abstract, mathematically convoluted paper is though not for the formally fainthearted.\textsuperscript{15} To that effect, the paper had essentially no impact on the field (with only two citations from outside of the Hamburg group). Up to the 1994 edition of the Kippenhahn & Weigert textbook, its chapter 12 contained a digested synopsis of the contents of the \textbf{Kähler (1975)} study, attempting to make it more palatable to the students of the stars.

To retrace the path to Kähler’s findings, introduce the vector field \(g = (g_1, g_2)\) on the \(P_c - T_c\) plane.\textsuperscript{16} To study the nonlocal solution properties, consider first the loci \(g_1 = 0\) and \(g_2 = 0\) for a given set of \(\mathcal{P}\). In the neighborhood of a solution \(s_0\) of the stellar

\begin{footnotesize}
\begin{enumerate}
\item The Schwarzschild determinant is characterized by the fitting conditions of trial solutions as obtained from the IVP integrations once from the surface and once from the center to a prescribed fitting point, respectively.
\item The effect had been encountered and discussed in \textbf{Schwarzschild & Härm (1965)} and \textbf{Gabriel & Ledoux (1967)}.
\item Helmuth Kähler is the son of the eminent German mathematician Erich Kähler who is known, among other things, for contributions to algebraic geometry. Hence, it is well conceivable that young Kähler got good advice from within his family circle on mathematics that usually lies beyond the astrophysicists’ horizon.
\item The characteristic plane can also be defined by other suitable stellar-physical quantities, such as \(R_*\), \(L_*\), which then define a coordinate system homeomorphic to the HR plane.
\end{enumerate}
\end{footnotesize}
structure problem, \( g_1 \) and \( g_2 \) are both assumed to be representable by algebraic equations; this can be accomplished if \( g_1 \) and \( g_2 \) are expanded in higher-order Taylor series about \( s_0 \). Bézout’s theorem allows to count the number of intersections of algebraic curves on the basis of their degrees. Kähler referred to this intersection number as multiplicity, \( m \), of the joint roots of \( g_1 = 0 \) and \( g_2 = 0 \). The number \( m \) gives the maximum number of solutions to the stellar structure problem at \( P_c - T_c \). Regular solutions, with \( \det J_g \neq 0 \), have \( m = 1 \). In case of multiple solutions, i.e. \( \det J_g = 0 \) violating local uniqueness, \( m > 1 \). A double solution, for example, such as \( g_1 \) and \( g_2^{*^2} \) in Fig. 4, counts as \( m = 2 \).

In a next step, Kähler studied the character of solutions \( s \), making use of the nature of the singularities \( g = (0,0) \) of the vector field \( g \). For regular singularities, i.e. for locally unique solutions \( s \) with \( m = 1 \), he resorted to the sign \( (\det J_g) \) and baptized this quantity the charge of a model star. For an \( m = 1 \) solution, \( c \) is either +1 or −1. For degenerate solutions, \( c = 0 \) in case of even \( m \), and \( c = \pm 1 \) for odd \( m \). Setting up an admissible closed path \( B \) on the \( P_c - T_c \) plane (with no singularities on the locus) the Poincaré index \( C \) of \( g \) can be computed. This quantity \( C \) is made up of the sum over the charges, \( c_i \), of the \( B \)-enclosing singularities (solutions) \( s_i \):

\[
C = \sum_i c(s_i(\varphi)).
\]

By just evaluating a path integral, information can be gained about the charges captured by the closed path. Along the same line, the total number of solutions, \( N = \sum m_i \), is just the sum of the solutions \( s_i(\varphi) \) enclosed by \( B \), accounting correctly for respective multiplicities in the sense algebraic geometry.

If \( \varphi \) varies continuously (such as in linear series of stellar models), the total charge \( C \) is found to be conserved. The total number of solutions, on the other hand, is either constant or changes by even numbers. In other words, varying the control parameter of a linear series lets new solutions dis-/appear in pairs.

Since both, \( c \) and \( m \), attain integer numbers only and since both numbers admit of conservation properties or at least follow some well defined selection properties, Kähler considered them as the quantum numbers of a model star.

In terms of \( c \) and \( m \), Kähler concluded that a necessary condition for unique stable solutions to the stellar structure problem is: \( m = c = +1 \). Making use of the conservation properties of these stellar quantum numbers under \( \varphi \) variation, the claim is then that always at least one stable stellar model exists (amounting to an existence theorem); additional solutions might pop up as \( \varphi \) changes. These solutions appear in pairs, always with a stable one and one being thermally unstable (hence, uniqueness prevails even locally).

Figure 5 illustrates the variation of the quantum numbers \( c \) and \( m \), as well as of \( C \) and \( N \) for a model sequence as presented in Fig. 2 wherein the star’s core mass is the control parameter \( \varphi \) and \( Q \) measures the model star’s effective temperature. The model sequence
“evolves” from $\varphi_1$ to $\varphi_5$. Epochs at $\varphi_4$ and $\varphi_2$ are turning points. The solutions between the turning points constitute the branches of the model sequence. Evidently, according to the counting rules of Kähler, $C = \sum c$ is always $+1$, and $N = \sum m$ is 1 early on, for epochs before $\varphi_2$ and for epochs after $\varphi_4$. In between, $N = 3$, being unity plus an even number. This behavior is canonical for solution loci with turning points. If $Q$ of Fig. 5 would be suitably flipped upside down, the result would be reminiscent of a sufficiently massive star’s core radius as a function of its growing core mass when the respective equilibrium-star model passed through the Schönberg-Chandrasekhar instability.

Other solution topologies that show up in stellar modeling are termination points (as encountered if no model star is possible below/above some critical value of $\varphi$) and bifurcations that appear at the onset of the Schönberg-Chandrasekhar instability (at $M_{SC}$) in the total mass – core-radius diagram. The illustrative example of Fig. 6 is made up of equilibrium models with $m = 3$ and $c = +1$ (e.g. Paczyński 1972).

Rather than focusing on the behavior of trial integrations of the differential equations of stellar structure, the analysis of the properties of the differential operators constitutes an alternative approach to tackle the VR problem. This latter approach calls for tools and results in nonlinear functional analysis. At about the same time as Kähler (1975) dealt with algebraic geometry, Perdang (1975) traveled the path of functional analysis and published a rather formal mathematical treatise discussing existence and uniqueness of solutions. For that approach to be tractable at all, the model stars had to be sufficiently simplified: The stars were assumed to be in full equilibrium and completely radiative. First, Perdang recast the stellar structure equations into two higher-order ones, a mechanical and a thermal one. The direct coupling of the two components occurred via the matter density as well as the nuclear energy generation rate and the thermal conductivity, which are functions of pressure and temperature (i.e. the thermodynamic basis variables, which are also and the key variables of the mechanical and the thermal equations). At the surface, finite pressure and finite temperature were assumed to prevail. After transforming the
variables into dimensionless form and some heavier mathematical machinery of function spaces, norms, and products as it is customary in functional analysis, Perdang rewrote the two equations into integral form and morphed them into a nonlinear integral operator to which Perdang referred as the *stellar structure operator*. The stellar structure problem was thereby transformed into an eigenvalue problem for the stellar radius.

Under rather general restrictions (continuity with respect to the thermodynamical basis, single-valuedness, first and second partial derivatives do exist) imposed on the constitutive relations – equation of state, opacities, nuclear energy generation rates – Perdang found that at least one solution $R_*, P(r), T(r)$ exists if $R_*$ stayed smaller than some prescribed value $R$; such a solution can even be found to be unique if $R_*$ is smaller than a more restrictive bound $R'$. Perdang scrutinized his approach on simple model systems such as polytropes and isothermal, self-gravitating gas spheres. Because the radius plays the role of an eigenvalue in the integral-equation formulation, he used the stellar radius as a control parameter in linear model series. For the simple case of self-gravitating isothermal gas spheres, the properties of Bonnor-Ebert spheres, which are known to be unable to achieve an equilibrium state for too high external pressures, were recovered. From the point of view of linear series, this means that sequences of isothermal gas spheres, embedded in a pressure-exerting external medium, pass through a turning point so that above a critical radius no equilibrium configurations can persist anymore. Actually, Perdang pointed out that his existence and uniqueness arguments obtain only in case of radius being the control parameter of the series but not if mass were adopted.

Despite his efforts, also Perdang failed to generate momentum to motivate further work along his line to probe the fundamental properties of stars. The paper garnered only two citations through the years. Ultimately, the model stars that were accessible to Perdang’s approach are likely to have been too abstracted and the math itself too abstract to entice any students of the stars.

**The state of the affair**

The paper of Kähler (1975) and the singular one by Perdang (1975) marked the endpoints of the few years of intensive research on existence and uniqueness of stellar models in general and on the VR theorem in particular.

Admittedly, the results on the VR theorem do not directly affect the practice of stellar modeling, in particular not if data-oriented computation is the focus. Even if multiple solutions to the stellar-structure/-evolution equations are possible, the history of a star imposes the ensuing direction of evolution. This starts with the condensation of the protostars out of the low-density, more or less homogeneous interstellar material, which governs the evolution...
from low to high densities and temperatures, and from simple to complex chemical composition and spatial structure. Structural possibilities that might also be possible, at least mathematically, are therefore automatically excluded.

All in all, star models are locally unique except at epochs where a model’s stability changes. At these critical points uniqueness is lost. Physically, the situation can be usually saved by then adopting a more comprising description of the problem (as going from FE to QHE, or going from QHE to dynamical evolution, for example). From the viewpoint of global properties of the solution manifold, the existence of at least one stable equilibrium solution is claimed to be guaranteed with reasonably weak restrictions only on the model properties (Kähler [1975]). When applying the result of the conservation of the charge of a model star upon continuously varying components of its $\mathcal{P}$ vector, it is reasonable to conclude that at least one stable model can be found on the HR plane (or equivalently on a plane homeomorphic to it). On the other hand, uniqueness is not ensured. As before, in the local context, a star’s history will nonetheless select a particular set of solutions, so that additional branches are not accessible to a model star and can therefore usually be neglected in the analyses.

Even though important results elucidating the connection between evolution and stellar stability were obtained in the 1970s, the model stars that were studied then had to be necessarily simple. Mathematical analyses of the more complex systems of equations that describe say dynamical stars are not yet in (see Appendix A). Substantial changes of the results as obtained up to now would, however, be surprising. Nonetheless, it would feel good to know that the foundations and the computations are on solid ground. Last but not least, the Universe itself has proven to be a wilder place than humans’ imagination. So we can – particularly now as the next Cosmic data deluge builds up – rest confident that if seemingly weird stellar configurations are realized somewhere in the Universe, sooner or later we are going to stumble over them and we will, as usual in astronomy, likely come to grips with them post festum by means of reverse engineering.

In any case, if stellar astrophysics aspires to be more than a branch of celestial engineering one could do worse than to sit back time and again and contemplate qualitatively fundamental questions framing the field of research. Here, these questions concerned the solution properties of the basic equations describing the macroscopic structure of the stars.

Appendix A: The nature of the equations - II

Evolving model stars that yield helpful results to astronomers is traditionally, for computational reasons, a one-dimensional enterprise. Resorting to suitable parameterizations to model deviations from radial symmetry due to rotation and/or tidal effects in close
 binaries even allow to continue along the one-dimensional path of simulating stars.

At full glory, though, stellar structure and evolution is a three-dimensional problem. The computational methods and the computer power are, however, far from ready to embark on full-scale stellar evolution simulations covering nuclear timescales in three dimensions in a foreseeable future. Nonetheless, the properties of the underlying equations can be studied, and this is where mathematical fluid dynamics enters the stage.

Two main avenues of mathematical modeling, independent of the dimensionality of the spatial description, are encountered:

1. Star models in FE constitute a boundary-value problem of an ODE system; evolution can be approximated by a prescribed chemical profile (see linear series) at the discretion of the modeller.

2. Star models with ‘built-in’ temporal evolution change the character of the mathematical problem to an initial – boundary-value problem of a PDE system not belonging to any standard class.

Mathematically speaking, the most elementary stellar structure problem (say in FE) is a two-sided boundary-value problem of Euler-Poisson type. If energy transport through the stellar matter is modeled via photon diffusion (eq. I9) or some sort of a mean-field convection ansatz, the problem acquires a diffusion-type (hyperbolic) contribution. Last but not least, nuclear energy sources responsible for the long-term energy supply of the star add further equations of reaction-diffusion type (eq. I11) to the problem.

The paper of Makino (1986) was an early contribution to the discussion of the existence of solutions of the three-dimensional Euler-Poisson system, adopting a barotropic equation of state and neglecting the effect of radiation. The compact support of the gas ball (i.e. the spatial confinement of the stellar matter) posed mathematical problems which Makino set out to solve. Eventually, he managed to formulate sufficient conditions on initial data for short-time existence of solutions to the Euler-Poisson system, which applied to polytropic indices \( n > 2 \).

Even though stars are usually considered low-viscosity fluid-dynamical environments, turbulent convection, winds, high radiation energy densities, shocks and other nonlinear fluid-dynamical processes require the Euler equation to be replaced by its Navier-Stokes brethren. Because the Millennium-Prize money for the incompressible multidimensional Navier-Stokes problem seems not to have been disbursed yet, the jury is still out on proofs of the existence of regular solutions given smooth initial data even on a simpler Navier-Stokes – type problem than what is needed for stars.

The mathematicians’ closest approach to an astrophysically pertinent radiating, self-gravitating gas blob was achieved in the
paper of [Ducomet (1996)]. That paper outlines the equations in their the most general form. In all approaches, though, the equation of state is that of an ideal gas with radiation and the thermal conductivity in the heat-diffusion equation is considered to be fixed. The resulting full set of equations had to be simplified drastically before local and global existence theorems could be proved.

For short times, Secchi (1991, 1990) proved existence and uniqueness theorems for Navier-Stokes – Poisson – Fourier systems. [Ducomet & Feireisl (2004)] returned to the problem stated by Ducomet in 1996 and put forth an existence theory for three-dimensional weak solutions that hold, in contrast to Secchi’s results, for arbitrary time intervals.

Much activity in mathematical fluid-dynamics went also in studying subproblems: For example, Deng & Yang (2006) studied stationary solutions of three-dimensional isentropic Euler-Poisson systems; Xie & Li (2012) extended the analysis to non-isentropic equations of state. The stationary velocity fields that were accounted for can be thought of as to deal with rotating stars. Uniqueness of solutions of simplified spherically symmetric Navier-Stokes systems with conduction could be shown by Umehara & Tani (2007).

In all the studies of the type mentioned just above, it is evident that the mathematical problems associated with the full set of equations that describe radiating, self-gravitating gas balls are so formidable that the equations need to be stripped down dramatically to be mathematically tractable. In the end, it is always difficult for outsiders to figure out if the resulting mathematical formulation is still germane to stellar physics.

Appendix B: Computational exercitia

To put some flesh on the dry, abstract bones contemplated up to now, pertinent properties of one-dimensional, canonical MESA-computed 5M⊙ and 15M⊙ star models are presented in the following.

B.1. A 5 M⊙ star from ZAMS to core He-burning

Murai (1974) pointed to the functional behavior of the opacity in the deep envelopes as the reason for multiple solutions, i.e. of secular instabilities in intermediate-mass model stars. Hence, in contrast to models with canonical (ρ, T)-dependent microphysics, stellar models resorting to constant electron-scattering – only opacity should therefore lack the hook in the fitting curves on the log P – log T plane and hence multiple solutions should thereby be suppressed.

Figure 7 displays some evolutionary tracks of simple 5M⊙, X = 0.7, Z = 0.02 model stars on the HR plane. The continuous black and red lines trace the evolution from the ZAMS to the base
of the first-giant branch making use of realistic microphysics as supplied by MESA. In particular, we made use of opacities from OPAL tables. The full black line marks the part of the early evolution when the QHE models are secularly stable. The red line, covers the secularly unstable phase of evolution to the base of the giant branch. The full grey line traces the evolution of the same model star if $D_s \equiv 0$ is enforced, i.e. FE models. The latter constraint gives numerical solutions only if the Henyey determinant does not vanish, which happens if a star turns thermally unstable via a monotonic secular eigenmode. At the epoch of the first zero of the Henyey determinant, convergence fails as expected; this point is highlighted as turning point of the sequence on the HR plane of Fig. 7.

The dashed lines illustrate the effect of evolving $5M_\odot$, $X = 0.7$, $Z = 0.02$ model stars with $\kappa(m) = \kappa_{e^-}(m)$ only; the dashed black line traces the respective QHE evolution. The grey dashed line on the other hand traces the evolution of the FE case. QHE evolution to low effective temperatures stops and reverses at the onset of helium core-burning. The FE evolution, in contrast, continues to very low temperatures, failing to ignite helium core-burning. The lack of $H^-$ opacity in the $\kappa$ prescription lets the model star miss the Hayashi line so that it continues to cool, always in full FE, until the code fails to converge due to numerical problems in the equation of state. Most importantly though, none of the computed FE $e^-$-scattering models did encounter a zero in their Henyey determinant, which means that secular instability was indeed suppressed. This is compatible with Murai’s conjecture that the spatial $\kappa$ structure due to Kramers opacity contribution is responsible for the onset of multiple stellar models in equilibrium, or of secularly unstable QHE models, which also applies to the Schönberg-Chandrasekhar instability.

To determine the range of secularly stable QHE models (the full black line in Fig. 7) we computed secular eigensolutions of the respective stellar evolution models. For a $5M_\odot$ QHE model at the
gate to the Hertzsprung-gap, at \( \log T_{\text{eff}} = 4.15, \log L/L_\odot = 3.04 \), a pertinent region of the complex frequency, \( \sigma = (\sigma_R, \sigma_I) \), plane was scanned to visualize the distribution of the lowest-order secular eigenmodes thereon. Figure 8 shows the frequency plane with \( \sigma \) measured in units of the stars’ free-fall frequency. The magnitude of the determinant of the secular eigenproblem (computed with the Riccati method, cf. Gautschy & Althaus 2007) is plotted as a grey-scale map. The larger the magnitude of the determinant the brighter the color. Superimposed on the colormap are the loci of the zeroes of the real part (blue) and of the imaginary part (red) of the determinant function. Local minima of this determinant function, indicating eigenfrequencies, are highlighted with small white circles. Evidently, oscillatory and monotonous modes coexist in the chosen region of the frequency plane. The three oscillatory modes appear twice on the complex plane shown in Fig. 8 because the secular problem is symmetric about \( \sigma_I = 0 \). As mentioned further up, secular eigenvalues can be complex and frequently are so, constituting oscillatory eigenmodes; these have been known since Schwarzschild & Härm (1968). Aizenman & Perdang (1971) showed then that oscillatory secular modes occur even in rather simple model stars close to the main sequence.

B.2. A 15 \( M_\odot \) star: FE versus NNE evolution

In the FE approximation, \( Dt_s \equiv 0 \) is inflicted upon a star’s evolution. In the other extreme, the no-nuclear-evolution case (NNE), \( Dt_s \) is fully accounted for but any nuclear evolution is neglected.23 Nuclear energy generation works as usual but it has no compositional consequences for the star in the NNE approximation. From Fig. 1 we learn that the 15\( M_\odot \) model displayed there embarks on an extensive blue loop during helium core- and double shell-burning. Therefore, this 15\( M_\odot \) model sequence is used to illustrate how conservative
QHE evolution can be roughly pieced together as a succession of FE and NNE phases, at least during the H- and He-burning stages. Figure 9 replots the 15\(M_\odot\) QHE track of Fig. 1 as the continuous black line.

Figure 9: Evolutionary tracks of a 15\(M_\odot\) star with \(X = 0.7, Z = 0.02\). Superimposed on the thin black line of QHE evolution are phases of FE (red) and NNE (indigo). The right panel is a zoom-in to more easily identify the different phases in the red-giant region.

The first FE episode extends from the ZAMS to the end of the S-bend (close to epoch A). The second FE phase starts at the tip of the first-giant branch and it extends to the local luminosity minimum when the model stars embark on their blue loops (from epoch \(A'\) to \(B\)). The third FE stretch starts at the blue end of the loop (at epoch \(B'\)) where the model star starts to evolve back towards the red-giant region. FE evolution terminates at about \(\log T_{\text{eff}} = 3.85\) when the model goes secularly unstable again. Compared with the QHE track, the first and the third FE phase are partially overluminous, this happens whenever QHE evolution experiences significant \(D_t\) contributions, which is ignored in the FE case.

The NNE computations could never be started using the last converging FE model. It always took a QHE model with sufficient \(D_t\) contributions in the neighborhood of a terminal FE model for an NNE sequence to launch successfully. Therefore, the starting epochs and also their respective positions on the HR plane of the starts of NNE sequences are afflicted with some uncertainty. All the NNE phases, \(A - A', B - B',\) and \(C - C',\) are lived through in a few \(10^4\) years; i.e. on thermal timescales of the model stars. Notice that initial and final NNE epochs \(A, A', B, B',\) and \(C, C',\) respectively, are pairs of models with the same chemical compositions but different internal structures; they are hence examples of models violating the global VR theorem; a point made already by Kahler (1978). In other words, double solutions are regularly encountered in canonical stellar-evolution computations under unspectacular circumstances. Once the final new equilibrium stratifications were established, the time steps adopted by the NNE computations grew...
rapidly because the structure of the model stars did not change anymore. Much in contrast to what was reported at the beginning of this paragraph for terminal FE models, the final models of the NNE sequences served as reliable starting points for ensuing FE evolution (cf. epochs $A'$, $B'$, $C'$); usually lying nearby the respective QHE model on the HR plane.

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