Causality and dissipation in relativistic polarizeable fluids

David Montenegro\textsuperscript{1}, Giorgio Torrieri\textsuperscript{2}
\textsuperscript{1} IFT Unesp, Sao Paulo, Brasil
\textsuperscript{2} IFGW, Unicamp, Campinas, Brasil

We analyze the breakdown of causality for the perfect fluid limit in a medium with polarizeability. We show that to restore causality a relaxation term linking vorticity and polarization, analogous to the Israel-Stewart term linking viscous forces and gradients, is required. This term provides a minimum amount of dissipation a locally thermalized relativistic medium with polarizeability must have, independently of its underlying degrees of freedom. For ferromagnetic materials an infrared acausal mode remains, which we interpret as a Banks-Casher mode signaling spontaneous magnetization. With these considerations, we propose a candidate for a fully causal Lagrangian of a relativistic polarizeable system near the perfect fluid limit.

The question of whether there exists a universal limit to viscosity and/or dissipation (parameterized, in relativistic systems by the viscosity over entropy ratio $\eta/s$) is both profound and difficult to handle. On a fundamental level, it is plausible to argue that quantum uncertainty gives rise to fluctuations which dissipate information. However, the unitarity of quantum theory is difficult to reconcile rigorously with dissipation. The usual procedure, given a microscopic theory, is to assume thermal equilibrium and then use correlators obtained from finite-temperature field theory to calculate viscosity \cite{1}. This allows us in principle to calculate transport coefficients given a thermally equilibrated microscopic theory which is also tractable. However, since relativistic systems with low viscosity are usually strongly coupled, this is a very blunt instrument for claiming “universal” limits.

Thus, a fundamental limit has been claimed decades ago by combining the uncertainty principle with Boltzmann’s derivation of viscosity \cite{2}, $\eta/s \sim O(0.1)$. While this is a plausible order-of-magnitude estimate, it was always clear that Boltzmann’s derivation should not generally apply to strongly coupled quantum fields. More recently, Gauge-gravity correspondence allowed us to conclude \cite{3} that theories with a classical gravity dual have $\eta/s = (4\pi)^{-1}$ in their strong-coupling limit. The universality of this limit is the black-hole no-hair theorem, and hence it critically depends on the existence of a classical gravity dual, namely a planar limit and a conformal strongly coupled fixed point. Counter examples have been argued for beyond this limit \cite{4}.

These difficulties illustrate that most likely one cannot get a lower limit from top-down arguments, where hydrodynamics appears as a limit of a known microscopic theory. A bottom-up constraint, based on effective field theory constraints such as low-energy unitarity, causality, and convergence of the gradient expansion are necessary. Attempts in this direction can be formulated in terms of a basic ambiguity within hydrodynamics: The fact that in the low viscosity limit thermal fluctuations will propagate as hydrodynamic modes, and the Kubo formula will need to be “renormalized” to take this into account \cite{5}. In \cite{6} the renormalization happens via an assumed microscopic cutoff, and hence cannot be used to get an a priori limit on viscosity if, as in quantum field theories, the number of degrees of freedom is infinite. The fact that one can constrain further the microscopic scale by causality arguments seems to emerge from energy condition arguments \cite{6}, but a conclusive result is still elusive.

That these considerations lead to a ”bottom-up limit” to hydrodynamics is clear from considering the Kolmogorov cascade \cite{7}: Generally a low-viscosity turbulent fluid in three dimensions redistributes energy from high-amplitude low frequency degrees of freedom to low-amplitude high frequency ones. Eventually, this process will reach a limit where the frequency of perturbations, in natural units, is comparable to the energy carried by them. Basic quantum mechanics should make the Kolmogorov cascade stop at that point, and the stopping of the cascade can be used to define a microscopic viscosity. That said, an analytical lower limit from these considerations is very difficult to get since vortex degrees of freedom in three dimensions appear strongly non-perturbative \cite{8-10}. Perhaps a lattice calculation including dissipative hydrodynamics \cite{11,12} and lattice techniques \cite{13} can make progress in this direction.

Help might arrive from an apparently unrelated direction, the study of relativistic hydrodynamics in the presence of polarization. This development has been motivated by the experimental observation of transfer of angular momentum degrees of freedom from hydrodynamic vorticity to polarization \cite{14}. However, a version of relativistic hydrodynamics which incorporates microscopic polarization is still in development \cite{15-20}, since quite a lot of characteristics we usually associate with the ideal fluid limit (vorticity conservation, isotropy, coarse-graining) applies in a very different way when collective angular momentum excitations can be transferred to microscopic spin degrees of freedom.

While polarization appears irrelevant to viscosity, all known physical realizations of strongly coupled fluids as well as most non-trivial interacting field theories contain particles with spin, and the strongly coupled dynamics of such systems must self-evidently include spin-orbit interactions. There are several reasons why such interactions could provide a source of bottom-up minimal dissipation. Polarization is in the same order, in the probability dis-
tribution of microscopic degrees of freedom, as thermal fluctuations examined in [5] (not to be confused with the Knudsen number expansion parameter [7]). As argued in [15], polarization could resolve the vortex instability noticed in [8] by giving vortices a “soft mass gap”, which would also stop the Kolmogorov cascade, as it is primarily vortex driven. That this effect could be similar to viscosity is apparent from a microscopic quasi-particle picture: A particle with spin and a finite de-Broglie wavelength, moving in a fluid with momentum flow, could have its helicity flipped by the gradient. By angular momentum conservation, the helicity flip will quench some of the gradient, and hence helicity-momentum interactions will have the same effect as a shear viscosity. Note that, in the strong limit of the spin-orbit coupling this effect does not go away, only for short de-Broglie wavelengths (i.e. high temperature and microscopic degeneracy) it does!

Furthermore, [16] demonstrated that the ideal fluid limit of a fluid with polarization, obtained by lagrangian effective field theory techniques, is incompatible with causality. This is a general principle, since the fluid vorticity is an acceleration, and any lagrangian including it is susceptible to Ostrogradski instabilities. Mathematically, the violation of causality occurs because a fluid with polarization will inevitably mix the sound wave and vortex perturbation (a compression sound wave will change local temperature, which induces change in the vortical susceptibility, which changes the vorticity/polarization balance), and the resulting dispersion relations become quartic and unbounded by causality limits [10].

In [16] we further conjectured that the solution to this issue parallels the Israel-Stewart resolution to the problem of causality of the Navier-Stokes equations [21][22]. In this work, we show that this is indeed correct. The relaxation time thereby obtained, however, unlike in the Israel-Stewart case, corrects a previously non-dissipative system and hence introduces a minimal amount of dissipation, dictated solely by causality and a non-zero polarizability. In other words, a bottom-up lower limit of thermalization of a relativistic fluid whose microscopic constituents have spin.

Let us briefly recap the approach of [15][16]: There, following the formalism of [8] we construct a lagrangian which contains the information of the equation of state, including an entropy term derived from the fluid Lagrangian coordinate degrees of freedom $\phi_I$

$$b = \left| \det \left[ \partial_\mu \phi_I \partial_\nu \phi_I \right] \right|^{1/2}$$

as well as the polarization tensor $y_{\mu\nu}$. For small polarizations, the equation of state will reduce to an equation of the form

$$\mathcal{L} = F(b, y) = F(b(1 - c y_{\mu\nu} y^{\mu\nu})), \quad (2)$$

c $> 0$ implies the material is a ferromagnet, with the potential to get spontaneously magnetized. c $< 0$ is an antiferromagnet, with the ground state resisting magnetization. Both cases are realized in nature, and could correspond to systems with ideal-fluid behavior. Note that since $c$ can depend on temperature, “antiferromagnetic” here could mean a magnet above the curie temperature.

For a well-defined ideal fluid limit, i.e. the absence of non-hydrodynamic microscopic “spin-wave” modes, we need vorticity and polarization to be parallel, in other words

$$y_{\mu\nu} = \chi(b, \omega_\mu \omega^{\mu\nu}) \omega_{\mu\nu}, \quad \omega_{\mu\nu} = \nabla_y [u u_\mu] \quad (3)$$

Then, the Lagrangian becomes a Legendre transform of the Energy density, analogously to the case with chemical potential [9]. The linearized dispersion relation derived from this lagrangian is however generally acausal [10].

Proceeding from the conclusion of [10], we consider Eq.[3] to be an asymptotic limit of a relaxation Maxwell-Cattaneo type equation [21],

$$\tau_y \partial_\tau \delta Y_{\mu\nu} + \delta Y_{\mu\nu} = y_{\mu\nu} = \chi(b, \omega^2) \omega_{\mu\nu} \quad (4)$$

Where non-equilibrium polarization $Y_{\mu\nu}$ evolves to vorticity. $Y_{\mu\nu}^{\mu\nu}$ is positive definite, has only dissipative terms and depends on initial conditions, and its introduction as a new purely dissipative degree of freedom should prevent Ostrogradski-type instabilities. This equation can be easily obtained from the Lagrangian formalism [12] via an Israel-Stewart type lagrangian, written in doubled coordinates [11] and employing non-equilibrium polarization degrees of freedom $Y_{\mu\nu}$

$$L = F(b(1 - c y_{\mu\nu} y^{\mu\nu})) + \mathcal{L}_{IS-vortex} \quad (5)$$

$$\mathcal{L}_{IS-vortex} = \frac{1}{2} \tau_y (Y_{\mu\nu}^{\mu\nu} u^\alpha \partial_\alpha Y_{\mu\nu} + Y_{\mu\nu}^{\mu\nu} u^\alpha \partial_\alpha Y_{\mu\nu} - Y_{\mu\nu}^{\mu\nu} u^\alpha \partial_\alpha Y_{\mu\nu}) + \frac{F'(\alpha)}{\sim y_{\mu\nu} y^{\mu\nu}} (6)$$

One could worry as to the extent of the universality of this choice, as opposed, for example, of writing a Lagrangian in terms of magnon/spinwave degrees of freedom. To answer this question, we point out that Eqs. [3] and [5] are equivalent to considering magnons, giving them a purely dissipative dynamics and coupling them only to collective degrees of freedom with angular momentum, i.e. vortices. The alternative (for example, adding a non-dissipative kinetic term for $Y_{\mu\nu}$ in the Lagrangian) would necessitate calculating transport properties for magnons from this Lagrangian, i.e. this would become a microscopic Lagrangian to be coarse-grained. If Eq. [3] leads to causal dynamics then, provided a general magnon lagrangian will lead to dynamics close enough to the ideal fluid limit, this is what it will coarse-grain to since additional terms would contain more derivatives. Causality what we aim to test for in this work.
Considering a system without further parameters, i.e. without chemical potential, shear and bulk viscosity will give us dissipative modes in $Y_{\mu \nu}$ and sound and vortex modes due to EoS. Following the prescription of [23], the field $\phi^I$ describes a fluid out-of-equilibrium an general expansion can be made from the hydrostatic coordinates $\phi^I = x^I$

$$\phi^I(x) = x^I + \pi^I + \frac{1}{2!} \pi \cdot \partial \pi^I + \frac{1}{3!} \partial(\pi \cdot \partial \pi) + \mathcal{O}(\pi^4) \quad (7)$$

The equation of motion to a general polarization from Euler Lagrange equation becomes

$$2\nu \partial_\mu \partial_\nu \left( \frac{Y_{r\sigma}}{Y_{\rho \sigma}} \frac{\partial Y_{r\sigma}}{\partial (\partial_\mu \partial_\nu \pi^I)} \right) = A \left( c_s^2 \partial_I [\partial \pi] - \pi^I \right) \quad (8)$$

with $c_s^2 = \frac{F'(b_o) + 2b_o F''(b_o)}{F'(b_o)}$, $A = b_o F'(b_o)$ and $\partial I = \partial_I \pi_J$, $[\partial \pi] = \partial_I \pi^I$ (using the notation in [10]).

To linear order Fluctuations of field could be written as

$$\bar{\pi} = \bar{\pi}_T + \bar{\pi}_L = \bar{\nabla} \Phi^I(x^I,t) + \bar{\nabla} \times \bar{\Omega}(x,t) \quad (9)$$

where $\pi_L$ usually parametrize a sound wave, a deformation of coordinates $\Phi_I$ parallel to the perturbation while $\pi_T$ is a vortex, in the direction perpendicular to propagation of sound. Because of sound-vortex mixing, $k \neq 0$ for $\pi_T$. Polarization terms $Y_{\mu \nu}$ once relaxation terms Eq. [4] are included, will propagate differently from sound and vortices. Thus, the sound potentials in Eq. [9] can be Fourier-expanded separately

$$\begin{pmatrix} \Phi \\ \Omega \\ Y \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Omega_0 \\ Y_0 \end{pmatrix} \exp \left[ i \left( w_{L,T,Y} t - \vec{k} \cdot \vec{x} \right) \right] \quad (10)$$

We can now use the rick analogous to that used in [22] to invert equation[4] The Left Hand Side of Eq. [4] becomes, in Fourier space ($\eta_{\mu \nu}$ are the metric components),

$$\frac{\partial \bar{Y}_{\mu \nu}}{\partial (k_\alpha k_\beta \pi^I)} = \frac{2}{(1 + i \omega_Y T_Y)} \chi(b_o, 0) \left\{ \eta_{\mu \rho} \eta_{\nu \sigma} \delta_\alpha^\rho \delta_\beta^\sigma \delta^I_L \right\} \quad (11)$$

note that, as conjectured in [16], equation [4] now only has gradients up to order two, in contrast to the equations of motion of a fluid where polarization and vorticity align automatically. Ostrogradski’s instabilities therefore should be absent.

Plugging these into Eq. [8] and replace we get separate dispersion relations for the transverse and longitudinal parts, because we take up to second order $\pi^I$. After a some algebra Eq. [8] decomposed in transverse and longitudinal parts gives us

$$\begin{cases}
\omega_T^2 + k^2 \omega_T^2 = \frac{4c^2 \chi^2(b_o, 0)}{b_o F'(b_o)(1 + i \omega_Y T_Y)^2} - \omega_T^2 = 0 \quad (12) \\
\omega_L^2 + k^2 \omega_L^2 = \frac{4c^2 \chi^2(b_o, 0)}{b_o F'(b_o)(1 + i \omega_Y T_Y)^2} - \omega_L^2 + c_s^2 k^2_L = 0 \quad (13)
\end{cases}$$

We can then express $w_Y(w_L, w_T)$ and solve these equations for the group velocity $v_g = dw_{T,L}/dk$ of the longitudinal and transverse modes.

the dispersion relations are shown in Fig. [1] where for brevity we defined with

$$B = 4c^2 \chi^2(b_o, 0), \quad A = b_o F'(b_o)$$

As can be seen, when $\gamma^2_{[T/A]} \simeq 3$ the group velocity is not credible and its asymptotic velocity goes to negative values as we can note in fig 2 for sound modes. In the large $k$ limit dispersion relations are monotonic. In this UV limit the group velocity is calculable analytically. As this is the limit where deviations from the EFT should manifest themselves, examining it in a bottom-up approach will tell us if the ideal hydrodynamic limit to an arbitrary
scale is well-defined. For the ferromagnetic $c > 0$ and
anti-ferromagnetic $c < 0$ cases we get
\[
\lim_{k \gg 1} \frac{d[\omega_T]}{dk} \bigg|_{c \leq 0} = \frac{1}{\sqrt{1 + \frac{\tau_Y^2}{(B/A)}}} \tag{14}
\]
The equivalent for the longitudinal case are
\[
\lim_{k \gg 1} \frac{d[\omega_L]}{dk} \bigg|_{c \leq 0} = \sqrt{\frac{c_T^2\tau_Y^2 \mp (B/A)}{\tau_Y^2 \pm (B/A)}} \tag{15}
\]
These are plotted in Fig. 2 again for the transverse and longitudinal modes for both ferromagnetic and anti-ferromagnetic materials.

As can be seen from Fig. 2 an antiferromagnetic material can be causal given a constraint on $\tau_Y^2$, given by
\[
\tau_Y^2 \geq \frac{8c_T^2(b_o, 0)}{(1 - c_T^2)b_o F'(b_o)} \tag{16}
\]
It relates the vortical susceptibility $\chi$ to non-vortical coefficients (speed of sound, enthalpy, hydrostatic entropy).

The denominator can be thought of as $\Delta \rho$ (it is the numerator of the speed of sound), and the numerator is proportional to vorticity’s absorption by angular momentum. Thus, it has exactly the form required of a coefficient describing an effective viscosity arising from spin. For an unpolarizable medium (where $\chi = 0$ by definition) the limit of $\tau_Y^2$ goes to zero, as expected. What this shows is that when polarization is present, taking the UV cutoff $\sim \tau_Y^{-1}$ of hydrodynamic applicability, with zero polarization susceptibility and finite entropy density $b_o$ and $F'(b_o)$, is incompatible with causality.

It should be noted that while this is a relaxation time, its effect is very similar to a viscosity. This can be seen by evolving a small vortex with a finite dissipation time. If the system contains very little vorticity, Eq. 4 to-gether with a thermodynamically sensible form of $\chi$ (\(\chi(|\omega| \to 0) \to 0\), as do all its derivatives. A “third law of thermodynamics for vortices) should quench the vorticity and transform it into polarization. The role of relaxation should indeed break up turbulence cascades which make vortices unstable \[8\]. Interpreting this vortex dissipation scale in the same way as a viscosity \[24\], we would get
\[
\eta = \frac{c_T^2 b_o}{s T} \geq \sqrt{\frac{8c_T^2(b_o, 0)}{(1 - c_T^2)b_o F'(b_o)}} \tag{17}
\]
this equation again makes sense, since the right hand side $\sim 1/T$ while the left hand side $\sim \chi$. In a system with a large degeneracy (for example, the planar limit) for a finite amount of energy $cT \chi(b, 0) \to 0$, hence the limit of $\eta/s$ goes to zero. This is consistent with such fluctuation effects going away in the limit of “many degrees of freedom” per unit volume. In this limit (corresponding to the planar limit in Yang-Mills theories and the applicability of molecular chaos in transport equations), the amount of angular momentum redistributed in polarization DoFs also vanishes. We also wish to underline that this limit is “bottom-up”, dependent on the assumption of causality, local thermalization and symmetries and that, unlike Israel-Stewart, it corrects an ideal fluid limit rather than one which is already dissipative.

For the ferromagnetic ($c > 0$) material a causal mode remains, and, as one can see from Fig. 1 this mode is infrared (small $k$) rather than ultraviolet (large $k$). Hence, it is not part of any microscopic gradient expansion, but it is rather related to the thermodynamic vacuum instability of the system. Indeed, this mode appears to have the right characteristics to be the Banks-Casher \[25\] mode
\[
\langle y_{\mu \nu} \rangle \equiv \lim_{k \to 0} \rho(\omega_{T,L}(k)) \tag{18}
\]
where $\rho(\omega)$ is the spectral function. A soft vacuum instability in the infrared signals, exactly the appearance of spontaneous magnetization in ferromagnetic systems (Physically, what happens is that a low wavelength mode is indistinguishable from the formation of a magnetic condensate, and indeed below a critical temperature such a
formation is unavoidable. In this case Eq. 4 is not anymore a good effective theory, since the fluid degrees of freedom and the magnetic condensate will evolve and interact with their own equations of motion.

One can be puzzled by the fact that here the condensate appears for any \( c > 0 \), but here we are just examining the "bare" lagrangian of the theory. Thermal fluctuations should quench the condensate, and in this lagrangian the only non-coarse grained fluctuations are the ones of the fluid DoFs \( \phi \) and polarization DoFs \( y \). The effect of thermal fluctuations can be found by calculating the functional integral

\[
\ln Z = \int D[y, \phi, I] e^{\frac{T}{\hbar} \int L(y, \phi, I, c) d^4x} \simeq L_{\text{eff}} [y', \phi', c'] \big|_{T_0}
\]

in terms of the microscopic scale \( T_0 \) [1-13], using the lagrangian given here both fluid fluctuations and polarization will be treated on the same footing and could give the interplay between spontaneous magnetization, thermal fluctuations and hydrodynamic evolution which would manifest in a renormalization group flow of \( c \) between a magnetized and a demagnetized phase. 

This course a much more ambitious project, perhaps needing a numerical approach extending [13].

In conclusion, we find that the ideal limit of hydrodynamics with polarization is generally non-causal. In the ferromagnetic regime and infrared limit this non-causality is physical, signalling the instability of the fluid against spontaneous magnetization. In the ultra-violet limit this causality can be fixed by a relaxation type term, which signals that any material with a non-zero spin must have a minimum amount of dissipation. The fact that Lagrangian hydrodynamics can capture both textbook physics (spontaneous magnetization) and a widely expected but never quite proven lower limit on dissipation in strongly coupled systems certifies its status as a powerful theoretical tool to examine the behavior of relativistic fluids. The Lagrangian proposed in Eq. 5 can therefore be considered as a candidate for the Lagrangian of a polarizable medium close to the ideal fluid limit.

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References

[1] S. Jeon and L. G. Yaffe, Phys. Rev. D 53, 5799 (1996) doi:10.1103/PhysRevD.53.5799 [hep-ph/9512263].

[2] P. Danielewicz and M. Gyulassy, Phys. Rev. D 31, 53 (1985). doi:10.1103/PhysRevD.31.53.

[3] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87, 081601 (2001) arXiv:hep-th/0104066.

[4] A. Buchel, R. C. Myers and A. Simha, JHEP 0903, 084 (2009) doi:10.1088/1126-6708/2009/03/084 arXiv:0812.2521 [hep-th].

[5] P. Kovtun, G. D. Moore and P. Romatschke, Phys. Rev. D 84, 025006 (2011) doi:10.1103/PhysRevD.84.025006 arXiv:1104.1586 [hep-ph].

[6] L. V. Delacretaz, T. Hartman, S. A. Hartnoll and A. Lewkowycz, arXiv:1805.04194 [hep-th].

[7] G. Torrieri, Phys. Rev. D 85, 065006 (2012) doi:10.1103/PhysRevD.85.065006 arXiv:1112.4086 [hep-th].

[8] S. Endlich, A. Nicolis, R. Rattazzi and J. Wang, JHEP 1104, 102 (2011) arXiv:1011.6396 [hep-th].

[9] S. Dubovsky, L. Hui, A. Nicolis and D. T. Son, arXiv:1107.0731 [hep-th].

[10] B. Gripaios and D. Sutherland, Phys. Rev. Lett. 114, no. 7, 071601 (2015) doi:10.1103/PhysRevLett.114.071601 arXiv:1406.4422 [hep-th].

[11] S. Grodzanov and J. Polonyi, Phys. Rev. D 91, no. 10, 105031 (2015) doi:10.1103/PhysRevD.91.105031 arXiv:1305.3670 [hep-th].

[12] D. Montenegro and G. Torrieri, Phys. Rev. D 94, no. 6, 065042 (2016) doi:10.1103/PhysRevD.94.065042 arXiv:1604.05291 [hep-th].

[13] T. Burch and G. Torrieri, Phys. Rev. D 92, no. 1, 016009 (2015) doi:10.1103/PhysRevD.92.016009 arXiv:1502.05421 [hep-lat].

[14] L. Adamczyk et al. [STAR Collaboration], Nature 548, 62 (2017) doi:10.1038/nature23004 [arXiv:1701.06657 [nucl-ex]].

[15] D. Montenegro, L. Tinti and G. Torrieri, Phys. Rev. D 96, no. 5, 056012 (2017) doi:10.1103/PhysRevD.96.056012 arXiv:1701.08263 [hep-th].

[16] D. Montenegro, L. Tinti and G. Torrieri, Phys. Rev. D 96, no. 7, 076016 (2017) doi:10.1103/PhysRevD.96.076016 arXiv:1703.03079 [hep-th].

[17] W. Florkowski, A. Kumar and R. Ryblewski, arXiv:1806.02616 [hep-ph].

[18] W. Florkowski, B. Friman, A. Jaiswal and E. Speranza, Phys. Rev. C 97, no. 4, 041901 (2018) doi:10.1103/PhysRevC.97.041901 arXiv:1705.00587 [nucl-th].

[19] F. Becattini, V. Chandra, L. Del Zanna and E. Grossi, Annals Phys. 338, 32 (2013) doi:10.1016/j.aop.2013.07.004 arXiv:1303.3431 [nucl-th].

[20] F. Becattini and L. Tinti, Annals Phys. 325, 1566 (2010) arXiv:0911.0864 [gr-qc].

[21] W. Israel and J. M. Stewart, Annals Phys. 118, 341 (1979). doi:10.1016/0003-4916(79)90130-1.

[22] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP 0804, 100 (2008) doi:10.1088/1126-6708/2008/04/100 arXiv:0712.2451 [hep-th].

[23] S. Dubovsky, L. Hui, A. Nicolis and D. T. Son, Phys. Rev. D 85, 085029 (2012).

[24] J. Casalderrey-Solana, E. V. Shuryak and D. Teaney,
[25] T. Banks and A. Casher, Nucl. Phys. B 169, 103 (1980). doi:10.1016/0550-3213(80)90255-2

[26] Note that an “inverted” relaxation equation

\[ \tau \partial_t \delta \omega_{\mu\nu} + \delta \omega_{\mu\nu} = \chi (b, w^2)^{-1} y_{\mu\nu} \]

would, according to the reasoning in [15], be necessary to resolve the vortical instability noted in [8, 13]. However, non-equilibrium vorticity is ill-defined without viscous Israel-Stewart terms, hence we do not see a coherent way to define such an inverted equation in the ideal hydrodynamic limit. We shall therefore proceed with Eq. [4].