Agmon-Type Estimates for a Class of Difference Operators

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Abstract. We analyze a general class of self-adjoint difference operators $H_\varepsilon = T_\varepsilon + V_\varepsilon$ on $\ell^2((\varepsilon\mathbb{Z})^d)$, where $V_\varepsilon$ is a one-well potential and $\varepsilon$ is a small parameter. We construct a Finslerian distance $d$ induced by $H_\varepsilon$ and show that short integral curves are geodesics. Then we show that Dirichlet eigenfunctions decay exponentially with a rate controlled by the Finsler distance to the well. This is analogous to semiclassical Agmon estimates for Schrödinger operators.

1. Introduction

The central topic of this paper is the investigation of a rather general class of families of self-adjoint difference operators $H_\varepsilon$ on the Hilbert space $\ell^2((\varepsilon\mathbb{Z})^d)$, as the small parameter $\varepsilon > 0$ tends to zero.

The operator $H_\varepsilon$ is given by

$$H_\varepsilon = (T_\varepsilon + V_\varepsilon), \quad \text{where} \quad T_\varepsilon = \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} a_\gamma \tau_\gamma,$$

$$\tau_\gamma u(x) = u(x + \gamma) \quad \text{and} \quad (a_\gamma u)(x) := a_\gamma(x, \varepsilon)u(x) \quad \text{for} \quad x, \gamma \in (\varepsilon\mathbb{Z})^d \quad (1.1)$$

where $V_\varepsilon$ is a multiplication operator, which in leading order is given by $V_0 \in C_0^\infty(\mathbb{R}^d)$.

We remark that the limit $\varepsilon \to 0$ is analogous to the semiclassical limit $\hbar \to 0$ for the Schrödinger operator $-\hbar^2\Delta + V$. This paper is the first in a series of papers; the aim is to develop an analytic approach to the semiclassical eigenvalue problem and tunneling for $H_\varepsilon$ which is comparable in detail and precision to the well-known analysis for the Schrödinger operator (see Simon [24, 25] and Helffer–Sjöstrand [17]). Our motivation comes from stochastic problems (see Bovier–Eckhoff–Gayrard–Klein [8,9]). A large class of discrete Markov chains analyzed in [9] with probabilistic techniques falls into the framework of difference operators treated in this article.
We recall that sharp semiclassical Agmon estimates describing the exponential decay of eigenfunctions of appropriate Dirichlet realizations of the Schrödinger operator are crucial to analyze tunneling for the Schrödinger operator. We further recall that the original work of Agmon on the decay of eigenfunctions for second order differential operators is not in the semiclassical limit. It treats the limit $|x| \to \infty$ (in a non bounded domain of $\mathbb{R}^n$). Agmon realized in [3] that for a large class of such operators the exponential rate at which eigenfunctions decay is given by the geodesic distance in the Agmon metric. This is the Riemannian metric from Jacobi’s theorem in classical mechanics: For a Hamilton function whose kinetic energy is a positive definite quadratic form in the momenta, the projection to configuration space of an integral curve of the Hamiltonian vector field is a geodesic in the Agmon (Jacobi) metric.

This paper contains analog results for the class of operators $H_\varepsilon$, including a generalization of Jacobi’s theorem. It is essential that we consider these operators as semiclassical quantizations of suitable Hamilton functions and investigate the relation of these Hamilton functions to Finsler geometry. In this generality our results are new. We recall, however, that various examples extending the original framework of the semiclassical analysis in the work of Simon [24] and Helffer–Sjöstrand [17] have been analyzed: The operator $\cosh D_x + \cos x$ in Harper’s equation (see e.g. Helffer–Sjöstrand [18]), the Schrödinger operator with magnetic field (Helffer–Mohamed [15]), the Dirac and Klein–Gordon operator (see e.g. Helffer–Parisse [16], Servat [23]) and the Kac operator (Helffer [14]).

If $T^d := \mathbb{R}^d/(2\pi)\mathbb{Z}^d$ denotes the $d$-dimensional torus and $b \in \mathcal{C}^\infty(\mathbb{R}^d \times T^d \times (0,1])$, a pseudo-differential operator $\text{Op}_T^d(b) : \mathcal{K}'((\varepsilon \mathbb{Z})^d) \to \mathcal{K}'((\varepsilon \mathbb{Z})^d)$ is defined by

$$\text{Op}_T^d(b)v(x) := (2\pi)^{-d} \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi,\pi]^d} e^{\frac{i}{\varepsilon}(y-x)\gamma} b(x,\gamma;\varepsilon)v(y) d\gamma,$$

(1.3)

where

$$\mathcal{K}'((\varepsilon \mathbb{Z})^d) := \{u : (\varepsilon \mathbb{Z})^d \to \mathbb{C} \mid u \text{ has compact support}\}$$

(1.4)

and $\mathcal{K}'((\varepsilon \mathbb{Z})^d)$ is dual to $\mathcal{K}((\varepsilon \mathbb{Z})^d)$ by use of the scalar product $\langle u, v \rangle_{\ell^2} := \sum u(x)v(x)$.

We remark that under certain assumptions on the $a_\gamma$ defining $T_\varepsilon$ in (1.1), one has $T_\varepsilon = \text{Op}_T^d(t(.,.;\varepsilon))$, where $t \in \mathcal{C}^\infty([\mathbb{R}^d \times T^d \times (0,1])$ is given by

$$t(x,\gamma;\varepsilon) = \sum_{\gamma \in \mathbb{Z}^d} a_\gamma(x,\varepsilon) \exp \left( - \frac{i}{\varepsilon} \gamma \cdot \xi \right).$$

(1.5)

Here $t(x,\xi;\varepsilon)$ is considered as a function on $\mathbb{R}^{2d} \times (0,1]$, which is $2\pi$-periodic with respect to $\xi$. 

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**References**

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