THE REPRESENTATIONS OF TEMPERLEY-LIEB-JONES ALGEBRAS

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Abstract

Representations of braid group obtained from rational conformal field theories can be used to obtain explicit representations of Temperley-Lieb-Jones algebras. The method is described in detail for SU(2)_k Wess-Zumino conformal field theories and its generalization to an arbitrary rational conformal field theory outlined. Explicit definition of an associated linear trace operation in terms of a certain matrix element in the space of conformal blocks of such a conformal theory is presented. Further for every primary field of a rational conformal field theory, there is a subfactor of hyperfinite II_1 factor with trivial relative commutant. The index of the subfactor is given in terms of identity - identity element of certain duality matrix for conformal blocks of four-point correlators. Jones formula for index ( < 4 ) for subfactors corresponds to spin \( \frac{1}{2} \) representation of SU(2)_k Wess-Zumino conformal field theory. Definition of the trace operation also provides a method of obtaining link invariants explicitly.

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1. Introduction

In recent years, a close relationship has emerged between the areas of physics such as two-dimensional conformal field theory (i.e., critical phenomena in two dimensions, completely integrable models) and those of mathematics such as infinite dimensional Lie algebras, von Neumann algebras, braid theory and topology of three dimensional manifolds with or without knots in them.

One such link between physics and mathematics is provided by topological quantum field theories. Indeed such a description of knot theory has proven powerful enough to yield many a deep mathematical result\(^1\)-\(^5\). The Chern-Simons functional integrals over three-manifolds with appropriate Wilson lines connecting points on the boundaries, provide representations of the groupoid of braids made of individual strands carrying arbitrary colours and orientations. These representations can be used to obtain a whole variety of invariants for multicoloured links\(^1, 3, 4\). Jones polynomial is the simplest invariant in this class, where all the component knots carry one colour characterized by spin \(\frac{1}{2}\) representation of \(SU(2)\). An explicit and complete method of computing expectation values of Wilson link operators in the \(SU(2)\) Chern-Simons theory on \(S^3\) has been presented in refs.(4). This thereby provides a way of obtaining the invariants for an arbitrary multicoloured link. Along with the theory of multi-coloured and oriented braids, two dimensional conformal field theories play an important role in these developments.

In his original method of obtaining now famous Jones link polynomials\(^6\), Jones constructed a new representation of the braid group. This representation is also related to a representation of a certain finite dimensional von Neumann (Temperley-Lieb-Jones) algebras which he had come across in his earlier studies of subfactors of a type \(\text{II}_1\) factor. In these studies, Jones introduced a notion of an index which in some sense measures the size of a subfactor in a hyperfinite \(\text{II}_1\) factor. This index takes values either as \(4\cos^2\frac{\pi}{\ell}, \ell \in \mathbb{N}, \ell \geq 3\) or in \([4, \infty)\). In the former case (index < 4), the subfactor always has a trivial relative commutant. This is not so in the case where the index is greater than four. All subfactors with
index $\geq 4$ studied by Jones have non-trivial commutants. It is of interest to find possible values of the index on the half-line $[4, \infty)$ for the case where subfactors have trivial relative commutants. Wenzl has obtained such values$^7$. Since Jones work many a study of subfactors and their index has appeared$^{8-11}$.

In this paper, we shall present a general and systematic method of obtaining the representations of Temperley-Lieb-Jones algebras. The techniques of conformal field theory will be exploited for this purpose. We shall study one such theory, namely $SU(2)_k$ Wess-Zumino field theory in detail. The method generalizes to any other compact semi-simple group $G$. Further we show that there is a subfactor for hyperfinite II$_1$ factor with trivial commutant for every primary field of the Wess-Zumino level $k$ conformal field theory based on a compact semi-simple Lie group $G$. The index of this subfactor is given by the square of $q$-dimension of the primary field, with $q = exp(2\pi i/(k + C_V))$, $C_V$ is the quadratic Casimir for the adjoint representation of the group $G$. Other rational conformal field theories, for example the minimal series (central charge $C < 1$) also provide solutions to the problem. This follows from an obvious adaptation of the method to such theories. In fact there is a subfactor (with trivial relative commutant) for every primary field of any rational conformal field theory. The index for such a subfactor is given in terms of certain element of the duality matrix for the four-point correlator of the primary field. The matrix element relevant here corresponds to the four-point conformal blocks where identity operator lives on the internal lines. The values of index so obtained agree with those in refs. 7 and 11.

The paper is organised as follows: In section 2, we present the required aspects of Temperley-Lieb-Jones algebras and discuss the representation obtained by Jones and the related representation of the braid group. In section 3, a class of representations of braid group will be presented. These are developed in the context of level $k$ $SU(2)$ Wess-Zumino conformal field theories. These representations then will be used to construct representations of Temperley-Lieb-Jones algebras along with an explicit presentation of a concept of trace in sections 4 and 5. All these correspond to subfactors of a hyperfinite II$_1$ factor with
trivial commutant. The generalization of the construction for a Wess-Zumino conformal field theory based on an arbitrary compact semi-simple group $G$ will then be discussed only briefly in section 6. In fact the procedure is valid for any rational conformal field theory. Some concluding remarks will be presented in the last section 7. In particular it will be indicated that the definition of trace developed in section 5 also yields link invariants.

2. Temperley-Lieb-Jones algebras

We shall be interested in the von Neumann algebras $A_{m-1}$ generated by $1$ and projectors $e_1, e_2, \ldots e_{m-1}$ which obey :

\begin{align}
(i) & \quad e_i^2 = e_i, \quad e_i^* = e_i \\
(ii) & \quad e_i e_{i+1} e_i = \tau e_i \\
(iii) & \quad e_i e_j = e_j e_i \quad |i - j| \geq 2
\end{align} \tag{2.1}

These algebras in addition admit of a trace, denoted by "tr" which is defined over $\bigcup_{m=1}^{\infty} A_{m-1}$ and determined by the normalization $tr 1 = 1$ and the following conditions :

\begin{align}
(iv) & \quad tr xy = tr yx, \quad x, y \in A_{m-1} \\
(v) & \quad tr xe_m = \tau trx, \quad x \in A_{m-1} \\
vii) & \quad tr x^* y > 0, \quad x, y \in A_{m-1} \tag{2.1'}
\end{align}

Jones has obtained a representation of this algebra and a corresponding representation of the braid group$^6$. For a braid generated by $b_i$, $i = 1, 2, \ldots$, we have the defining relations :

\begin{align}
b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\
b_i b_j &= b_j b_i \quad |i - j| \leq 2 \tag{2.2}
\end{align}

Clearly, if we define

\begin{align}
b_i &= q^{\frac{1}{2}} (q - (1 + q) e_i) \tag{2.3}
\end{align}

with

\begin{align}
b_i^2 - (q^{\frac{3}{2}} - q^{\frac{1}{2}}) b_i - q^2 &= 0 \tag{2.4}
\end{align}
the $e_i$’s satisfy relations (2.1) above with $\tau^{-1} = (q^{1/2} + q^{-1/2})^2$. Eqn.(2.4) corresponds to a representation of the braid generators where spin $\frac{1}{2}$ representation of $SU(2)$ is placed on the strands. It is this representation that finally led Jones to his knot invariants. Further $\tau^{-1}$ is the Jones index of the subfactor of $II_1$ factor. As stated earlier, Jones theorem restricts the values of this index from above representation to $\tau^{-1} = (q^{1/2} + q^{-1/2})^2$ with $q = \exp^{2\pi i/\ell}, \ell = 3, 4, \ldots$. We shall later see that these values correspond to the square of $q$-dimension of spin $\frac{1}{2}$ representation and is related to the duality matrix for the correlators of four spin $\frac{1}{2}$ primary fields in $SU(2)_k$ Wess-Zumino model (with the identification $k = \ell - 2$).

The algebra (2.1) written in terms of generators $U_i = \tau^{-1/2}e_i$ with $\tau^{-1} = 4\cos^2\lambda$ is known as Temperley-Lieb algebra in physics literature, and it first appeared in the context of exactly solvable statistical mechanical models. We shall refer to the algebras (2.1) with more general $\tau$ as Temperley-Lieb-Jones algebras.

In the following sections, we shall generalize the relation above between the representations of the braid group and those of von Neumann algebras (2.1). Using the von Neumann algebra generated by \{e_1,e_2,e_3,\ldots\} as the hyperfinite $II_1$ factor and that generated by \{e_2,e_3,\ldots\} as the subfactor, we shall obtain the index $\tau^{-1}$ for this subfactor.

3. A class of representations for braid generators

Now we shall present the necessary discussion of correlators of level $k$ $SU(2)$ Wess-Zumino conformal field theory on $S^2$. Their duality properties will be recapitulated. The representations of braid group will be given in terms of the monodromy properties of these correlators, which in turn will be used in sections 4 and 5.

$SU(2)_k$ Wess-Zumino conformal field theory admit of $k + 1$ primary fields (integrable representations) whose spins are given by $j = 0, 1/2, 1, \ldots, k/2$. The fusion rules are given by

$$(j_1) \otimes (j_2) = (|j_1 - j_2|) \oplus (|j_1 - j_2| + 1) \oplus \ldots \oplus \min(j_1 + j_2; k - j_1 - j_2)$$

(3.1)
The correlators of this theory can be given in terms of conformal blocks. There are more than one complete, but equivalent, sets of conformal blocks for a given correlator. These are related to each other by duality transformations. For example for the correlator for four primary fields in representations $j_1, j_2, j_3, j_4$, two sets of conformal blocks are represented by diagrams as shown in fig. 1. In this diagrammatic representation spins meeting at every trivalent point (e.g. here $(\ell_j j_2), (\ell_j j_3), (m_j j_3), (m_j j_4)$) satisfy fusion rules (3.1). The correlator is non-zero only if the product of the four spins $j_1, j_2, j_3, j_4$ contains a singlet. The duality matrix relating these two sets of conformal blocks is given in terms of Racah coefficients of $SU(2)_q$ with deformation parameter related to the level $k$ of the conformal field theory as $q = \exp 2\pi i/(k + 2)$:

$$a_{\ell m} \left[ j_1 j_2 j_3 j_4 \right] = (-)^{(j_1 + j_2 + j_3 + j_4)} \sqrt{[2\ell + 1][2m + 1]} \left( j_1 j_2 \ell \atop j_3 j_4 m \right) \times \sum_{z \geq 0} (-)^z [z + 1]! \left\{ [z - j_1 - j_2 - \ell]! [z - j_1 - j_4 - m]! \times [z - j_2 - j_3 - m]! [j_1 + j_2 + j_3 + j_4 - z]! \times [j_3 + j_4 + \ell + m - z]! [j_2 + j_4 + \ell + m - z]! \right\}^{-1}$$

where

$$\Delta(abc) = \sqrt{[-a + b + c]![a - b + c]![a + b - c]! \over [a + b + c + 1]!}$$

Here $[a]! = [a][a - 1] \ldots [3][2][1]$ and square brackets represent $q$-numbers:

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

These duality matrices satisfy the orthogonality condition:

$$\sum_\ell a_{\ell m} \left[ j_1 j_2 j_3 j_4 \right] a_{\ell m'} \left[ j_1 j_2 j_3 j_4 \right] = \delta_{mm'}$$

The duality properties of higher point correlators can be constructed in terms of these duality properties of four-point correlators. In particular, we shall be interested in the duality properties of $n$-point correlators as drawn in figs. 2 (a) and (b). Here we have the same spin
The eigenvalues are given in terms of the conformal weights of the representations $\rho$ where $b = \text{braid group}$. Consider a braid of $n$ strands, both carrying spin $j$ representation on all the external legs. The $SU(2)$ spins $(r_1, r_2, \ldots, r_{i-1}, r_i, r_{i+1} \ldots r_{n-3})$ and $(r_1, r_2, \ldots, r_{i-2}, \ell_i, r_i, r_{i+1} \ldots r_{n-3})$ respectively on the interval lines in these figures are in accordance with the fusion rules above. With this $n$-point correlator we associate a Hilbert space spanned by orthonormal basis vectors corresponding to these conformal blocks $|\phi_{r_1r_2 \ldots r_{n-3}}\rangle$ or equivalently $|\psi^{(i)}_{r_1r_2 \ldots r_{i-2}(\ell_i)\ell_i r_{i+1} \ldots r_{n-3}}\rangle$ referring to the two diagrams (a) and (b) in fig.2 respectively:

$$\langle \phi_{r_1r_2 \ldots r_{n-3}} | \phi_{r'_1r'_2 \ldots r'_{n-3}} \rangle = \prod_p \delta_{r_p r'_p} \tag{3.4}$$

$$\langle \psi^{(i)}_{r_1r_2 \ldots r_{i-2}(\ell_i)\ell_i r_{i+1} \ldots r_{n-3}} | \psi^{(i')}_{r_1' r_2' \ldots r'_{i-2}(\ell'_i)\ell'_i r'_{i+1} \ldots r'_{n-3}} \rangle = \delta_{\ell_i \ell'_i} \prod_p \delta_{r_p r'_p} \tag{3.5}$$

These two sets of orthonormal vectors are related by the duality property

$$|\psi^{(i)}_{r_1r_2 \ldots r_{i-2}(\ell_i)\ell_i r_{i+1} \ldots r_{n-3}}\rangle = \sum_{r_{i-1}} a_{\ell_i} \left[ \begin{array}{cc} j_i & j \end{array} \right] |\phi_{r_1r_2 \ldots r_{n-3}}\rangle \tag{3.6}$$

where $a_{\ell i} = \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$ is the duality matrix above in (3.2).

These $n$-point conformal blocks provide a basis for a class of matrix representations of braid group. Consider a braid of $n$ strands where every strand carries a spin $j$ of $SU(2)$ and has same orientation. Such a braid is generated by generators $b_i, i = 1, 2, \ldots n - 1$ of fig.3. Now the generator $b_i$ is diagonal in the basis associated with conformal blocks drawn in fig.2(b):

$$b_i |\psi^{(i)}_{r_1r_2 \ldots r_{i-2}(\ell_i)\ell_i r_{i+1} \ldots r_{n-3}}\rangle = \lambda_{\ell_i} |\psi^{(i)}_{r_1r_2 \ldots r_{i-2}(\ell_i)\ell_i r_{i+1} \ldots r_{n-3}}\rangle \tag{3.7}$$

The eigenvalues are given in terms of the conformal weights of the representations $h_j = \frac{j(j+1)}{k+2}$. Consider a braiding by one unit of right-handed half-twist in two parallely oriented strands, both carrying spin $j$ representation. This is is represented by a braid matrix. Then for every spin $\ell = 0, 1, \ldots \min (2j, k - 2j)$ representation in the product $(j) \otimes (j) = \otimes (\ell)$ as dictated by fusion rules, we have an eigenvalue of this braid matrix given by:

$$\lambda_{\ell}(j, j) = (-)^{\ell} \exp \pi i (4h_j - h_\ell) = (-)^{\ell} q^{2j(j+1)-\ell(\ell+1)/2} \tag{3.8}$$
The braid generators \( b_i \) and \( b_{i \pm 1} \) are diagonal in the bases \( |\psi^{(i)}\rangle \) and \( |\psi^{(i \pm 1)}\rangle \) respectively. Using duality properties (3.6), it can easily be seen that these two bases, \( |\psi^{(i)}\rangle \) and \( |\psi^{(i \pm 1)}\rangle \), are related to each other by

\[
\langle \psi^{(i)}_{\ell_1 \ell_2 \ldots \ell_{i-2}} | \psi^{(i+1)}_{\ell_1 \ell_2 \ldots \ell_{i+1}} \rangle = A_{\ell_i \ell_{i+1}}
\]

\[
A_{\ell_i \ell_{i+1}} = \frac{\ell_i \ell_{i+1}}{r_i - j}
\]

(3.9)

Now equations (3.7) - (3.9) define representations of the braid generators. Using the properties of \( q \)-Racah coefficients of eqn.(3.2), one can verify that these representations do indeed satisfy the defining relations (2.2) of the braid group.

Alternatively, we may specify these representations of braid generators in terms of basis \( |\phi\rangle \) associated with conformal blocks of fig.(2a) instead of the basis \( |\psi\rangle \) (fig.2b) above. Then

\[
b_i |\phi_{\ell_1 \ell_2 \ldots \ell_{n-3}}\rangle = \sum_{\ell_{i-1} \ell_{i}} \lambda_{\ell_i} a_{r_{i-1} \ell_i} \left[ \begin{array}{cc} r_{i-2} & j \\ j & r_i \end{array} \right] a_{r_i \ell_{i+1}} \left[ \begin{array}{cc} r_{i-2} & j \\ j & r_{i+1} \end{array} \right] |\phi_{\ell_1 \ell_2 \ldots \ell_{n-3}}\rangle
\]

(3.10)

The representation of the braid group that Jones used in (2.3) and (2.4) to relate it to the representation of algebra (2.1) corresponds to spin \( j = 1/2 \) being placed on every strand of the braid. For this the eigenvalues of the braid generators from eqn.(3.8) are : \( \lambda_0 = q^{3/2} \) and \( \lambda_1 = -q^{1/2} \). Thus, these braid generators satisfy characteristic relation (2.4). Corresponding to other representations \( (j = 1, 3/2, \ldots) \) of the braid generators above, there exist new representations of algebras (2.1). Now we shall present these.

### 4. Representations of Temperley-Lieb-Jones algebras \( A_{m-1} \)

Consider an identity braid of \( 2m \) strands, each carrying \( SU(2) \) spin \( j \) with first \( m \) strands in one orientation and the rest in opposite orientation. On this we apply braid generator \( b_1, b_2, \ldots \) to obtain an arbitrary braid. Define projectors \( e_i, i = 1, 2, \ldots m - 1 \) as :
\[
e_i = \frac{(\lambda_n - b_i)(\lambda_{n-1} - b_i) \ldots (\lambda_1 - b_i)}{(\lambda_n - \lambda_0)(\lambda_{n-1} - \lambda_0) \ldots (\lambda_1 - \lambda_0)} \quad (4.1)
\]

where \( \lambda_\ell, \ \ell = 0, 1, \ldots n \equiv \min(2j, k - 2j) \) are the eigenvalues of the braid generators given by (3.8) and hence

\[
\prod_{\ell=0}^{n} (\lambda_\ell - b_i) = 0
\]

This implies,

\[
b_i e_i = e_i b_i = \lambda_0 e_i \quad (4.2)
\]

and therefore \( e_i^2 = e_i \) which is condition (i) of (2.1). Notice each \( e_i \) has one eigenvalue 1 and \( n \) eigenvalues all zero:

\[
e_i |\psi_{r_1r_2...r_{i-2}(\ell_i)r_{i}...r_{2m-3}}^{(i)}\rangle = \delta_{\ell,0} |\psi_{r_1r_2...r_{i-2}(\ell_i)r_{i}...r_{2m-3}}^{(i)}\rangle \quad (4.3)
\]

This may be equivalently reexpressed in basis \( |\phi> \) as

\[
e_i |\phi_{r_1r_2...r_{2m-3}}\rangle = \sum_{r'_{i-1}} a_{r'_{i-1}0} \left[ \begin{array}{c} r_{i-2} \\ j \\ r_i \end{array} \right] \left[ \begin{array}{c} r_{i-2} \\ j \\ r_i \end{array} \right] \left[ \begin{array}{c} r_{i-2} \\ j \\ r_i \end{array} \right] |\phi_{r_1r_2...r_{i-2}r'_{i-1}r_{i}r_{2m-3}}\rangle \quad (4.4)
\]

Now since,

\[
a_{00} \left[ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j_4 \end{array} \right] = (-)^{j_1 + j_3 - \ell} \frac{\sqrt{[2\ell + 1]} \delta_{j_1j_4} \delta_{j_2j_3}}{\sqrt{[2j_1 + 1][2j_3 + 1]}}
\]

we have

\[
e_i |\phi_{r_1r_2...r_{2m-3}}\rangle = \sum_{r'_{i-1}} \delta_{r_{i-1}} (-)^{(2j+2r_i-r_{i-1}+r'_{i-1})} \frac{\sqrt{[2r_{i-1} + 1][2r'_{i-1} + 1]}}{[2j + 1][2r_{i-2} + 1]} |\phi_{r_1r_2...r_{i-2}r'_{i-1}r_{i}...r_{2m-3}}\rangle \quad (4.5)
\]

Now to check that the generators \( e_i \) so defined do indeed satisfy condition (ii) of (2.1), we use (4.4) or (4.5) to calculate \( e_i e_{i \pm 1} e_i \)

\[
e_i e_{i \pm 1} e_i |\phi_{r_1r_2...r_{2m-3}}\rangle = \left( a_{00} \left[ \begin{array}{c} j \\ j \end{array} \right] \right)^2 e_i |\phi_{r_1r_2...r_{2m-3}}\rangle
\]

The following identities are useful in proving this relation:
\[
\sum_{r'_{i-1}} a_{r'_{i-1}0} \begin{bmatrix} r_{i-2} & j \\ j & r_i \end{bmatrix} a_{r_i0} \begin{bmatrix} r'_{i-1} & j \\ j & r_i+1 \end{bmatrix} \\
\times a_{r'_{i-1}0} \begin{bmatrix} r'_{i-1} & j \\ j & r_i+1 \end{bmatrix} a_{r_{i-1}0} \begin{bmatrix} r_{i-2} & j \\ j & r'_i \end{bmatrix} = \delta_{r_ir'_i} \delta_{r_ir_{i-2}} \frac{1}{[2j+1]^2}
\]

\[
\sum_{r'_{i-1}} a_{r'_{i-1}0} \begin{bmatrix} r_{i-2} & j \\ j & r_i \end{bmatrix} a_{r_{i-1}0} \begin{bmatrix} r_{i-3} & j \\ j & r'_{i-1} \end{bmatrix} \\
\times a_{r_{i-1}0} \begin{bmatrix} r_{i-3} & j \\ j & r'_{i-1} \end{bmatrix} a_{r'_{i-1}0} \begin{bmatrix} r'_{i-2} & j \\ j & r_i \end{bmatrix} = \delta_{r_ir_{i-2}} \delta_{r_ir_{i-2}} \frac{1}{[2j+1]}
\]

Now since \(a_{00} \begin{bmatrix} j & j \\ j & j \end{bmatrix} = (-)^{2j} \left([2j+1]\right)^{-1}\) we, have

\[e_i e_{i\pm 1} e_i = \frac{1}{[2j+1]^2} e_i \quad (4.6)\]

The equations (4.3) or (4.4) or (4.5) define matrix representations of the Temperley-Lieb-Jones algebras (2.1) with

\[\tau(j) = \left(a_{00} \begin{bmatrix} j & j \\ j & j \end{bmatrix}\right)^2 = \frac{1}{[2j+1]^2} \quad (4.7)\]

5. Trace

The algebras \(A_{m-1}\) generated by \((1, e_1, e_2 \ldots e_{m-1})\) where \(e_i\)'s are as given above and satisfy the conditions (2.1) with \(\tau\) as in eqn.(4.7), also admit of a linear trace. This trace defined on \(\bigcup_{m=1}^{\infty} A_{m-1}\), satisfies the properties listed in (2.1'). The structure of the algebras \(A_{m-1}\) is essentially decided by this trace. We shall now present an explicit definition of this trace. To this purpose consider the conformal blocks for \(2m\) primary fields, all in spin \(j\) representation, diagrammatically represented by fig.4(a). We have labeled the first \(m\) primary fields by \(1, 2, 3, \ldots \) \(m\) and the rest are denoted by \(m', (m-1)', \ldots 2', 1'\) as indicated. The orthonormal basis vectors of the associated vector space corresponding to these conformal blocks will be denoted by \(|\chi_{\ell_1\ell_2\ldots\ell_{2m-3}}\rangle\). Fig.4(b) is a redrawing of fig.2(a) with this new convenient labeling of the lines. This corresponds to the basis vector \(|\phi_{r_1r_2\ldots r_{m-1}r_{m-2}\ldots s_2s_1}\rangle\) with
identification \(s_{m-1} \equiv r_{m-1}\). These two sets of vectors are related to each other by duality transformation which can be constructed in terms of the elementary duality transformation involving four representations at a time (fig.1). The result of this exercise is:

\[
|\chi_{\ell_1\ell_2...\ell_{2m-3}}\rangle = \sum_{r_1s_1} A(\ell_1\ell_2...\ell_{2m-3})(r_1r_2...r_{m-1}s_{m-2}...s_{2s_1}) |\phi_{r_1r_2...r_{m-1}s_{m-2}...s_{2s_1}}\rangle
\]

with

\[
A(\ell_1\ell_2...\ell_{2m-3})(r_1r_2...r_{m-1}s_{m-2}...s_{2s_1}) = \prod_{p=1}^{m-1} a_{r_p\ell_2p} \begin{bmatrix} r_{p-1} & j \\ \ell_{2p-1} & s_p \end{bmatrix} a_{s_p\ell_{2p+1}} \begin{bmatrix} r_{p-1} & \ell_{2p} \\ j & s_{p-1} \end{bmatrix}
\]

Here we identify \(\ell_{2m-2} \equiv j, \ell_{2m-1} \equiv 0, r_0 \equiv j, s_0 = j\) and \(s_{m-1} \equiv r_{m-1}\).

Now for \(x \in A_{m-1}\), we define the trace "tr" as the following matrix element:

\[
tr x = \langle \chi_{ojoj...ojo} | x | \chi_{ojoj...ojo} \rangle
\]

This is the matrix element between vectors \(|\chi_{\ell_1...\ell_{2m-3}}\rangle\) (fig.(4a)) where all the odd-indexed internal lines carry the identity representation, \(\ell_{2p+1} = 0\) and therefore by fusion rules all the even indexed lines carry spin \(j\) representations, \(\ell_{2p} = j\).

This definition of trace can as well be written in terms of \(|\phi\rangle\) basis above. For this notice from (5.2)

\[
A(o\ell_{20}...o\ell_{2m-40})(r_1r_2...r_{m-1}s_{m-2}...s_{2s_1}) = (-)^{mj-r_{m-1}} \frac{\sqrt{[2r_{m-1}+1]}}{[2j+1]^{m/2}} \prod_{p=1}^{m-2} \delta_{\ell_{2p}j} \delta_{r_{p}s_p}
\]

Next let us define the abbreviated notation for these vectors corresponding to conformal blocks shown in fig.5 as

\[
|\phi^{(m)}_{(r)}\rangle \equiv |\phi^{(m)}_{(r_1r_2...r_{m-1})}\rangle \equiv |\phi_{r_1r_2...r_{m-2}r_{m-1}r_{m-2}...r_1}\rangle
\]

Then the trace (5.3) can be rewritten as

\[
tr x = \frac{1}{[2j+1]^m} \sum_{r_1...r_{m-1}} (-)^{2 mj - r_{m-1} - r'_{m-1}} \sqrt{[2r_{m-1}+1][2r'_{m-1}+1]}
\]

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The generators $e_i, i = 1, 2, \ldots m - 1$ act on the vectors (5.5) as

$$e_i |\phi^{(m)}_{(r_1 r_2 \ldots r_{m-1})}\rangle = \sum_{r_{i-1}} \delta_{r_ir_{i-2}} (-)^{(2j + 2r_i - r_{i-1} - r'_{i-1})}$$

$$\times \sqrt{\frac{[2r_{i-1} + 1][2r'_{i-1} + 1]}{[2j + 1][2r_{i-2} + 1]}} |\phi^{(m)}_{(r_1 r_2 \ldots r_{i-2} r'_{i-1} r_i \ldots r_{m-1})}\rangle$$

(5.7)

Now it is just a straightforward calculation to check that our definition (5.3) or equivalently (5.6) of trace does indeed satisfy the properties (2.1'). For example notice from (5.6):

$$tr 1 = \frac{1}{[2j + 1]^m} \sum_{r_1 \ldots r_{m-1}} (-)^{(2mj - 2r_{m-1})} [2r_{m-1} + 1]$$

Since $(r_{p-1}, r_p, j), p = 1, 2, \ldots m - 2$ (with $r_o \equiv j$) satisfy fusion rules, we have

$$\sum_{r_p} (-)^{2r_p} [2r_p + 1] = (-)^{2j + 2r_{p-1}} [2j + 1][2r_{p-1} + 1]$$

and thus

$$\sum_{r_1 r_2 \ldots r_{m-1}} (-)^{(2mj - 2r_{m-1})} [2r_{m-1} + 1] = [2j + 1]^m$$

so that $tr 1 = 1$. This is also directly obvious from the definition as in eqn.(5.3).

Further explicit calculations using definition (5.6) of the trace yield

$$tr e_i = \frac{1}{[2j + 1]^{i+1}} \sum_{r_1 \ldots r_i} (-)^{2(i+1)j + 2r_i} [2r_i + 1] a_{r_i,0} \left \{ \begin{array}{ccc} r_i-2 & j & r_i \\ j & r_i & 0 \end{array} \right \} a_{r_i-1,0} \left \{ \begin{array}{ccc} r_i-2 & j & \dot{j} \\ \dot{j} & r_i & \end{array} \right \}$$

(5.8)

and

$$tr e_i e_j = tr e_j e_i$$

Thus from (5.8), using $r_{-1} = 0, r_0 = r'_0 = j$ and $a_{j0} \left \{ \begin{array}{ccc} j & j & \\ j & r_1 \end{array} \right \} = \delta_{r_10}$:

$$tr e_1 = \frac{1}{[2j + 1]^2} = \left ( a_{00} \left \{ \begin{array}{ccc} j & j & \\ j & j \end{array} \right \} \right )^2$$

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Now we could use inductive argument of Jones to prove that this trace does satisfy other
the conditions in (2.1′) with \( \tau = \left( a_{00} \begin{bmatrix} j & j & j \\ j & j & j \end{bmatrix} \right)^2 = [2j + 1]^{-2} \). Alternatively, we could
evaluate \( tr \) \( xe_m \) directly. It is convenient to use the definition (5.3) instead of (5.6) for this
purpose. Thus we evaluate the matrix element \( \langle \chi_{0j0j...j0} | xe_m | \chi_{0j0j...j0} \rangle \) where the vector \( |\chi\rangle \)
here corresponds to \( 2(m+1) \)-point correlator (fig.4(a) with \( 2(m+1) \) external legs instead of
\( 2m \)). To do so we first perform a four-point duality transformation involving external lines
\( m \) and \( m + 1 \) on this vector so that the transformed vector is an eigen- function of \( e_m \), and
then operate by \( e_m \). This leads to a factor \( a_{00} \begin{bmatrix} j & j & j \\ j & j & j \end{bmatrix} \). Now transform back to the \( |\chi_{0j0j...j0}\rangle \)
basis through another four-point duality transformation . This yields one more factor of
\( a_{00} \begin{bmatrix} j & j & j \\ j & j & j \end{bmatrix} \). Thus we have the desired result:

\[
tr \ x \ e_m = \left( a_{00} \begin{bmatrix} j & j & j \\ j & j & j \end{bmatrix} \right)^2 tr \ x = \frac{1}{[2j + 1]^2} tr \ x
\]

This completes our explicit construction of a class of representations of the algebra (2.1)
along with the trace operation of (2.1′) with the parameter \( \tau \) given by the square of the
identity-identity matrix element of the duality matrix for correlators of four spin \( j \) primary
fields : \( \tau = \left( a_{00} \begin{bmatrix} j & j & j \\ j & j & j \end{bmatrix} \right)^2 = [2j + 1]^{-2} \). Notice \( [2j + 1] \) is the \( q \)-dimension of the spin \( j \)
representation of \( SU(2)_q \).

6. Other representations

This method of explicit construction of representations of Temperley-Lieb-Jones algebras
(2.1) using level \( k SU(2) \) Wess-Zumino conformal field theory has an obvious generalization
to the case of Wess-Zumino model based on an arbitrary compact semi-simple Lie group \( G \).
Let \( \lambda \) be the weights of a representation \( R \) of \( G \) and \( \theta \) the highest root of the Lie algebra
normalized as \( (\theta, \theta) = 2 \). Then the primary fields of \( G_k \) Wess-Zumino model are all those
representations with \( (\lambda, \theta) \leq k \). For example for \( SU(N)_k \), \( (\lambda, \theta) \) being the number of boxes
in the first row of the Young tableaux, the primary fields (integrable representations) are
all those with number of boxes in the first row of their Young tableau less than or equal to \( k \).

The fusion rules for these representations are given in terms of their depths. For every weight \( \lambda \) in the highest weight representation \( L(\Lambda) \), depth \( (\lambda) \) is the biggest integer \( j \) for which \( \lambda - j\theta \in L(\Lambda) \). Then fusion rules are given\(^{11} \) by \( R_1 \otimes R_2 = \oplus R(\lambda) \), if the Clebsch-Gordon coefficient \( C^{R(\lambda)}_{R_1R_2} \neq 0 \) and also if depth \( R_1 + \text{depth } R_2 \leq k - (\lambda, \theta) \).

In order to construct representations of (2.1), we consider a \( 2m \)-strand braid where we put representation \( R \) (allowed integerable representation of \( G_k \) conformal field theory) and its conjugate \( \tilde{R} \) alternately on the \( 2m \) strands. Further let us put same orientations on the first \( m \) strands and opposite orientation on the other \( m \) strands. The eigen values of the braid generators \( b_i \) (fig.3) are again given in terms of conformal weights of the involved representations, \( h_R = C_R/(C_V + k) \), where \( C_R \) is the quadratic Casimir of representation \( R \) and \( C_V \) that for the adjoint representation. These eigenvalues for braid generators introducing right handed half twists in two parallelly oriented strands carrying representations \( R \) and \( \tilde{R} \) are given by\(^3 \).

\[
\lambda_{R_s}(R, \tilde{R}) = (-)^{\varepsilon_s} e^{\pi i (4h_R - h_{R_s})} \equiv (-)^{\varepsilon_s} q^{2C_R - C_{R_s}/2} \tag{6.1}
\]

corresponding to the irreducible representations \( R_s \) consistent with the fusion rules given above in the product \( R \otimes \tilde{R} = \oplus_{s=0}^{n_s} R_s \) with the identification that \( R_0 \) is the identity representation. Here \( (-)^{\varepsilon_s} = (-)^{(\lambda_{R_s}, \theta)/2} = \pm 1 \). The parameter \( q \) is related to the level \( k \) of the Wess-Zumino conformal field theory through

\[
q = e^{\pi i \frac{2}{k + C_V}} \tag{6.2}
\]

The braid generators \( b_i \) act on the vectors corresponding to \( 2m \)-point conformal blocks, introducing half-twist in the \( i \)th and \( (i+1) \)th strands. We define a braiding operation \( \hat{b}_i = \sigma_i b_i \), which introduces a half-twist followed by interchange (\( \sigma_i \) of \( i \)th and \( (i+1) \)th labels (representations) of the conformal blocks it acts on. The generators \( e_i, \ i = 1, 2, \ldots m - 1 \) of the Temberley-Lieb-Jones algebra (2.1) now can be constructed as in (4.1) in terms of the first \( m - 1 \) braid generators \( \hat{b}_i \) as:
The definition of trace (5.3) for $SU(2)_k$ also generalizes as the matrix element between states corresponding conformal blocks of the type shown in fig. 4(a), now with representations $R$ and $\tilde{R}$ alternately placed on the external legs and all the odd indexed internal lines carrying the identity representation $R_0$. The generators (6.3) with this definition of trace then satisfy all the conditions (2.1) and (2.1') with the index $\tau$ given as the square of identity - identity duality matrix element for the four point correlator for representations $R, \tilde{R}, R, \tilde{R}$ as shown in fig. 6. This duality matrix $a_{R,R'} \begin{bmatrix} R & \tilde{R} \\ R & \tilde{R} \end{bmatrix}$ here is again given in terms of $q$-Racah coefficients of quantum group $G_q$ with deformation $q$ as given in (6.2). Further identity-identity element of this duality matrix is inverse of the $q$-dimension of representation $R$. Thus we have

$$\tau(R) = \left( a_{R_0R_0} \begin{bmatrix} R & R \\ R & R \end{bmatrix} \right)^2 = (\text{dim}_q R)^{-2}$$ (6.4)

Now the $q$-dimension of a representation $R$ of $G_q$ is given by the "Weyl dimensionality formula":

$$\text{dim}_q R(\lambda) = \prod_{\alpha_+ \in \Delta_+} \frac{[\langle \lambda, \alpha_+ \rangle + \langle \delta, \alpha_+ \rangle]}{[\langle \delta, \alpha_+ \rangle]}$$

where $\lambda$ is the highest weight of the representation $R(\lambda)$, $\Delta_+$ is the set of positive roots $\{\alpha_+\}$ and $\delta = 1/2 \sum_{\alpha_+ \in \Delta_+} \alpha_+$ and square brackets indicate $q$-numbers with $q$ given by eqn.(6.2).

The parameter $\tau^{-1}$ is the index (dimension) of the subfactors of hyperfinite II$_1$ factors with trivial relative commutant. For $SU(N)$, $N \geq 2$, the formula (6.4) agrees with that obtained by Wenzl$^7$:

$$\tau^{-1} = \prod_{1 \leq r < s \leq N} \frac{\sin^2((\lambda_r - \lambda_s + s - r)\pi/\ell)}{\sin^2((s - r)\pi/\ell)}$$ (6.5)

where $\lambda_1, \lambda_2, \ldots, \lambda_{N-1}$ are the number of boxes in the first, second, third, ..., $(N - 1)$th rows respectively in the Young tableaux (with $\lambda_1 \leq \ell - N$) of a representation of $SU(N)$. Here $\lambda_N$ is set to zero. The expression (6.5) is indeed square of the $q$-dimension of the representation,
with \( q = \exp\left(\frac{2\pi i}{k+N}\right) \) and identification \( \ell = k + N \). Formula (4.7) is the special case of (6.5) for \( N = 2 \). And formula (6.4) generalizes this index of subfactors of \( \text{II}_1 \) factors with trivial commutants for any arbitrary representation of a compact semi-simple Lie group.

The method of construction of representations of algebra (2.1) presented here in fact is valid not only for Wess-Zumino conformal field theories but also for any arbitrary rational conformal field theory; in particular, for example, for the minimal series. Any rational conformal field theory can be constructed\(^{15}\) as a coset construction \( G/H \) of Wess-Zumino conformal field theories, the Wess-Zumino theories discussed above corresponding to \( H \) being trivial. Using the properties of conformal blocks for correlators much like as we have done in section 3 for \( SU(2)_k \) Wess-Zumino model, representations of braid group corresponding to every primary field of a rational conformal field theory can be constructed. The relevant braids for our purpose are those which are obtained from identity braid with primary fields \( \phi \) and conjugate field \( \bar{\phi} \) live alternately on \( 2m \) strands. The first \( m \) strands are all oriented in one direction and the other \( m \) in opposite direction. The eigen values of the generators of such braids are given again in terms of the conformal weights in a similar fashion as in eqn.(6.1). The generator \( e_i \) of the von Neumann algebra (2.1) are given by a formula of the type (6.3). Generalization of the trace ”tr” of eqn.(5.3) is also possible. It is given in terms of the matrix element between the states depicted by fig.4(a) where now the external lines alternately carry primary fields \( \phi \) and \( \bar{\phi} \) of the conformal field theory under consideration and all the odd indexed internal lines carry the identity representation \( \phi_0 \). The index \( \tau^{-1} \) is then again related to identity-identity \( (\phi_0 - \phi_0) \) element of the duality matrix, \( a_{\phi_0\phi_0} \left[ \begin{array}{c} \phi \\ \bar{\phi} \\ \phi \\ \bar{\phi} \end{array} \right] \) for four point correlators for primary fields \( \phi, \bar{\phi}, \phi, \bar{\phi} \) as shown by a diagram of the type shown in fig.6. Thus we have a general result :

For every primary field \( \phi \) of a rational conformal field theory, there is a subfactor of hyperfinite \( \text{II}_1 \) factor with trivial relative commutant with index \( (\tau(\phi))^{-1} \), where
\[ \tau(\phi) = \left( a_{\phi_0 \phi_0} \left[ \begin{array}{c} \phi \\ \bar{\phi} \end{array} \right] \right)^2 \] (6.6)

On the other hand the calculations of ref.(11) give this index as \((S_{\phi_0 \phi}/S_{\phi_0 \phi_0})^2\) where \(S_{\phi \phi'}\) is the \(S\)-modular transformation \((\tau \rightarrow -1/\tau)\) , where this \(\tau\) is the modular parameter of a torus) matrix on the characters \(\chi_\phi\) of the conformal field theory. Thus our result is consistent with this result with the identification:

\[ \left( a_{\phi_0 \phi_0} \left[ \begin{array}{c} \phi \\ \bar{\phi} \end{array} \right] \right)^2 = \left( \frac{S_{\phi_0 \phi}}{S_{\phi_0 \phi_0}} \right)^2 \] (6.7)

These values of index so obtained also agree with those of ref.7.

Now let us study the minimal models with central charge \(c < 1\) as other examples. These models have a coset construction\(^{15}\) based on \(SU(2)_m \otimes SU(1)/SU(2)_{m+1}\). The central charge is given by

\[ c = 1 - \frac{6}{(m + 2)(m + 3)} \quad m = 1, 2, \ldots \]

and the conformal weights for the primary fields \(\phi_{(r,s)}\) \((1 \leq r \leq m + 1, \quad 1 \leq s \leq m + 2)\) are given by

\[ h_{(r,s)} = \frac{(m + 3)r - (m + 2)s)^2 - 1}{4(m + 2)(m + 3)} \]

The identity operator is \(\phi_{(1,1)}\). Due to the identification \(\phi_{(r,s)} \equiv \phi_{(m+2-r,m+3-s)}\) , we pick up the primary fields from the grid \(\phi_{(r,s)}\) by taking \(r + s\) to be even. Now consider the duality matrix corresponding to fig.1, where the external lines carry the primary fields \(\phi_{(r_1,s_1)}, \phi_{(r_2,s_2)}, \phi_{(r_3,s_3)}\) and \(\phi_{(r_4,s_4)}\). The internal lines carry the representations \(\phi_{(r,s)}\) and \(\phi_{(r',s')}\) respectively in the two diagrams of this figure. This duality matrix is given by the product of duality matrices for \(SU(2)_m\) and \(SU(2)_{m+1}\) theories :

\[ a_{\phi_{(r,s)} \phi_{(r',s')}} \left[ \begin{array}{c} \phi_{(r_1,s_1)} \phi_{(r_2,s_2)} \\ \phi_{(r_3,s_3)} \phi_{(r_4,s_4)} \end{array} \right] \]
\[
\begin{align*}
&= a_{2-1,2-1}^{(1)} \begin{bmatrix} r_1-1 & r_2-1 \\ r_3-1 & r_4-1 \end{bmatrix} a_{2-1,2-1}^{(2)} \begin{bmatrix} s_1-1 & s_2-1 \\ s_3-1 & s_4-1 \end{bmatrix}
\end{align*}
\]

The duality matrices \(a_{\ell m}^{(1)}\), \(a_{\ell m}^{(2)}\) on the right hand side are the same as (3.2) except for that the \(q\)-parameters are now given by \(q_1 = \exp \frac{2\pi i}{m+2}\) and \(q_2 = \exp \frac{2\pi i}{m+3}\) respectively for these two.

Thus then for every representation \(\phi_{(r,s)}\) we have a subfactor whose index can be written as

\[
(\tau(\phi_{(r,s)}))^{-1} = \left( a_{\phi(1,1)\phi(1,1)} \begin{bmatrix} \phi_{(r,s)} & \phi_{(r,s)} \\ \phi_{(r,s)} & \phi_{(r,s)} \end{bmatrix} \right)^{-2} = ([r]_1[s]_2)^2 \tag{6.8}
\]

where square brackets with subscripts 1,2 represent \(q\)-numbers with parameters given respectively by \(q_1\) and \(q_2\) above.

We can go through a similar discussion with respect to other discrete series such as superconformal theories, which have a coset construction based on \(SU(2)_m \times SU(2)_{m+2}\) or even more general case based on the coset \(SU(2)_m \times SU(2)_{\ell}/SU(2)_{m+\ell}\).

7. Concluding remarks

We have exploited the properties of rational conformal field theories to obtain representations of Temperley-Lieb-Jones algebras(2.1). Correlator conformal blocks of these theories provide a basis for developing a class of representation of the braid group. These in turn yield representations of the algebras (2.1) through (6.3) and its generalization to arbitrary rational conformal field theory. The eigen value of the braid generator in the direction of identity operator (for definiteness \(\lambda_{R_0}\) in eqn.(6.3)) plays a special role in this construction. Replacing this eigen value by any other eigen value does not provide a representation of the algebra (2.1). Further a trace with requisite properties (2.1') is given in terms of a certain specific matrix element in the space of conformal blocks. Thus we have demonstrated that for every primary field \(\phi\) of an arbitrary rational conformal field theory on \(S^2\), there is a subfactor of \(\Pi_1\) factor with trivial relative commutant. The index of this subfactor is given by inverse of the square of identity-identity element of the four point duality matrix for
primary fields $\phi$, $\bar{\phi}$, $\phi$, $\bar{\phi}$. The discrete series with index $4\cos^2\left(\frac{\pi}{4}\right)$, $\ell = 3, 4, \ldots$ obtained by Jones corresponds to the spin 1/2 representation of $SU(2)_k$ conformal field theory with the identification $\ell = k + 2$.

The explicit definition of trace as developed in section 5 for $SU(2)_k$ conformal field theory and its obvious generalization for an arbitrary rational conformal field theory can also be used to obtain link invariants. For this purpose consider a link represented as the closure of an $m$-strand braid $B_m$ written as a word in terms of generators $b_1, b_2 \ldots b_{m-1}$. We think of this closure in terms of $2m$-strand braid $\hat{B}_m$ where all the braiding (i.e. $B_m$) lives in first $m$ parallelly oriented strands, all carrying representation $\phi$ of the conformal field theory. The rest of the $m$ strands have no braiding but are oppositely oriented and all carry conjugate representation $\bar{\phi}$. Then closure of the braid $B_m$ constitutes in connecting the first strand to the last, the second to the second last and so on from above and below in the so constructed $2m$-strand braid $\hat{B}_m$. The invariant for such a link is given by

$$V[L] = \left(a_{\phi_0\bar{\phi}_0} \left[ \begin{array}{c} \phi \\ \phi \\ \bar{\phi} \\ \bar{\phi} \end{array} \right] \right)^{-m} \text{tr} \hat{B}_m$$

(7.1)

The trace here is the matrix element between vectors representing conformal blocks for correlators of $2m$ primary fields, first $m$ being all $\phi$ and the other $m$ all conjugate $\bar{\phi}$ (of the type as shown in fig.4(a)). All the odd indexed internal lines are supposed to carry identity representation. Eqn.(7.1) represents a link invariant because as can be shown this trace has the property $\text{tr} \hat{B}_m b_m^{\pm 1} = a_{\phi_0\bar{\phi}_0} \left[ \begin{array}{c} \phi \\ \phi \\ \bar{\phi} \\ \bar{\phi} \end{array} \right] \text{tr} \hat{B}_m$ where $\hat{B}_m b_m$ is a $2(m+1)$ strand braid with weaving pattern in the first $m + 1$ strands. This property ensures that $V[L]$ above remains unaltered under Markov moves. Notice invariant for an unknot is $\left(a_{\phi_0\bar{\phi}_0} \left[ \begin{array}{c} \phi \\ \phi \\ \bar{\phi} \\ \bar{\phi} \end{array} \right] \right)^{-1}$. The concept of trace used above in (7.1) provides an explicit presentation of the formal Markov trace used in ref(11).

The statement (7.1) gives us link invariant for monocromatic links. This is a version of Theorem 6 of second of refs.(4) which gives the link invariant for a multicoloured link obtained as closure of a braid in an $SU(2)$ Chern-Simons theory.

We shall discuss the link invariants obtained through (7.1) and their generalization for
multi-colour links for the minimal series of conformal field theories as well as for super conformal field theories in more detail elsewhere\textsuperscript{18}.
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Figure captions

Fig.1. Duality of four-point correlators in $SU(2)_k$ Wess-Zumino theory.

Fig.2. Two sets of conformal blocks for correlator of $n$ spin $j$ primary fields.

Fig.3. Generator of $n$-braids.

Fig.4. Conformal blocks for $2m$-point correlators.

Fig.5. Conformal block relevant for the definition of trace.

Fig. 6. Duality transformation for four-point correlator in $G_k$ Wess-Zumino conformal field theory.
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