Small violations of full correlation Bell inequalities for multipartite pure random states

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Abstract

We estimate the probability of random $N$-qudit pure states violating full-correlation Bell inequalities with two dichotomic observables per site. These inequalities can show violations that grow exponentially with $N$, but we prove this is not the typical case. For many-qubit states the probability to violate any of these inequalities by an amount that grows linearly with $N$ is vanishingly small. If each system’s Hilbert space dimension is larger than two, on the other hand, the probability of seeing any violation is already small. For the qubits case we discuss furthermore the consequences of this result for the probability of seeing arbitrary violations (i.e., of any order of magnitude) when experimental imperfections are considered.
I. INTRODUCTION

Bell inequality violations substantiate the claim that certain physical phenomena cannot be described by any local hidden-variables theory. As such, they constitute one of the most striking "nonlocal" features of quantum mechanics. Besides this foundational interest, Bell inequality violations also have operational applications in quantum and post quantum information processing. Example tasks include being able to assure the security of quantum cryptography [1] and reducing the communication complexity of certain protocols [2].

It is natural to inquire into the relationship between Bell inequality violations and entanglement, another nonlocal feature of quantum mechanics. It is known that every pure entangled state violates some Bell inequality [3, 4]. A different question is whether Bell inequality violations and entanglement are quantitatively related. It is known that, in this sense, these two features are genuinely different. For instance, maximally entangled states do not always achieve maximal violation of Bell’s inequalities (see, e.g., [5, 6] and references therein).

In this paper we contribute to this quantitative line of research by showing that there is an extreme mismatch between entanglement and Bell inequality violations for typical states of \(N \gg 1\) qudits. These states are known to be highly entangled both in the bipartite [7] and in the multipartite sense [8]. By contrast, we will show that the maximal degree of violation of full correlation Clauser-Horne-Shimony-Holt (CHSH)-type inequalities by a typical pure state is very small. By considering the whole set of these inequalities, which consist of correlation measurements of dichotomic observables in each system [9, 10], we find the following:

(1) If the individual systems are qubits, the maximal violation is typically upper bounded by a linear function of \(N\) (this will mean that we should not expect to see violations in the laboratory, as we will argue below).

(2) If the local dimension is strictly larger than 2, the typical degree of violation is actually zero.

Thus typical states are highly entangled but do not produce large violations of these Bell inequalities. This is in sharp contrast with, e.g., generalized Greenberger-Horne-Zeilinger (GHZ) states, which are not very entangled but achieve exponential violation of at least one of the proposed Bell inequalities. In other words, nearly maximal entanglement across many
partitions cannot guarantee strong violations of Bell inequalities, and vice versa.

Our result connects to other problems that have been recently studied. First, the family of inequalities we consider not only is of foundational interest, but also has an information-theoretic property. Reference [2] shows that a violation of one of these inequalities is a sufficient condition for decreasing the complexity of communication protocols.

Second, our paper adds to the large body of work that addresses how typical a given a physical property is, when one considers the set of all pure states in a quantum system. References [7, 8] establish that in high-dimensions, most pure states not only have entanglement, but almost the maximum of it. The same goes for entanglement dynamics, where most states follow almost the same trajectory, as far as entanglement is concerned [11]. The conditions for a quantum system to reach thermal equilibrium and the definition of a quantum analogue of the ergodic hypothesis have been approached recently from a typicality perspective, in the sense of showing that under certain conditions on the system’s Hamiltonian and Hilbert space most states equilibrate or satisfy the ergodic hypothesis [12, 13]. In the quantum computation field, it was demonstrated that most pure quantum states of several qubits are “useless” for one-way quantum computation, in the sense that the computation generated by it is equivalent to one performed by a classical computer assisted by random bits [14].

Finally, we will also argue that the linear violation of Bell inequalities we obtain for typical many qubits states is noise-sensitive. If we take into account experimental imperfections, namely, local noise, no matter how small, the majority of states can not violate any of the inequalities by a significant amount. This consequence of our result had been noted by Pitowsky [15], who conjectured that the maximal violation of a typical state was of the order $\sqrt{N \ln N}$ (his conclusion still holds for violations of the order $N$).

II. THE FULL CORRELATION BELL INEQUALITIES

In general a Bell inequality for $N$ parts can have $n$ distinct measurement settings per site with $m$ distinct outcomes per measurement. We shall deal with the case $n = m = 2$, i.e., two measurement settings per site, with two outcomes each. Moreover, we use only full-correlation measurements, where every inequality can be explicitly written [9, 10]. If $X = (x_1, ..., x_N) \in \{0, 1\}^N$, $A_0^j = \pm 1$ and $A_1^j = \pm 1$, $1 \leq j \leq N$ represents the deterministic
measurement results for the pair of measurements in site \( j \), then the inequalities read [9]:

\[
-1 \leq \sum_{X \in \{0,1\}^N} S(X) \prod_{j=1}^{N} \frac{(A^j_0 + (-1)^{x^j} A^j_1)}{2} \leq 1, \tag{1}
\]

where \( S : \{0,1\}^N \to \{1,-1\} \). The set of inequalities is given then by all the possible “sign” functions \( S \), which has a total number of \( 2^{2N} \). Some of these inequalities are equivalent, for instance, they are obtained from each other by changing the labels of the outcomes. Nevertheless, the number of nonequivalent inequalities still grows superexponentially [10].

All these linear inequalities can be replaced though by a single nonlinear inequality [9, 10]:

\[
\sum_{X \in \{0,1\}^N} \left| \prod_{j=1}^{N} \frac{(A^j_0 + (-1)^{x^j} A^j_1)}{2} \right| \leq 1, \tag{2}
\]

and we shall use this inequality for the rest of the paper. For a quantum state \( |\psi\rangle \) of \( N \) \( d \)-dimensional systems and a choice of a pair of observables \( A^j_x \), for \( x_j = 0 \) or \( 1 \), representing measurements with results \( \pm 1 \), for each system \( j = 1, ..., N \), the nonlinear Bell inequality is evaluated then through the function:

\[
Q_{NL}(|\psi\rangle, Q) \equiv \sum_{X \in \{0,1\}^N} |\langle \psi | \prod_{j=1}^{N} \frac{(A^j_0 + (-1)^{x^j} A^j_1)}{2} |\psi\rangle|, \tag{3}
\]

where \( Q \) represents the choices of observables.

### III. BOUND FOR THE PROBABILITY OF MAXIMAL VIOLATION

It was shown that the so-called generalized GHZ \( N \)-qubit states \( \alpha |0\rangle^\otimes N + \beta |1\rangle^\otimes N \), for \( |\alpha| = |\beta| = \frac{1}{\sqrt{2}} \), violate one of these inequalities by an amount of the order \( 2^{N} \) [10, 16]. On the other hand, for \( N \) odd, and small enough but still greater than zero \( |\alpha| \) (or \( |\beta| \)), so that the state is still entangled, it can not violate any inequality of this form.

Considering that we have random pure states drawn according to the normalized uniform measure on the unit sphere of \( (\mathbb{C}^d)^\otimes N \), one can still ask: Is it true that for every \( N \geq 2 \) almost all pure states do violate some of these inequalities? Considering also that for large \( N \) most states have almost maximum entanglement (average of bipartite entanglement over all bipartitions), do they violate some inequality also by a great amount?

Our main theorem addresses these questions by estimating the probability of the event \( A_v = \{ |\psi\rangle : \sup_Q Q_{NL}(\psi, Q) > v \} \), which looks to the maximal violation, optimized over all possible observables, that each state can achieve.
Theorem 1. For $N \geq 2$, $d \geq 2$ integers, $|\psi\rangle \in (\mathbb{C}^d)^\otimes N$ a unit vector distributed according to the uniform measure in the sphere $S_{2dN-1}$ of $(\mathbb{C}^d)^\otimes N$, $\mathcal{A}_v = \{|\psi\rangle : \sup_Q Q_{NL}(\psi, Q) > v\}$, the following inequality holds true:

$$P(\mathcal{A}_v) \leq 2 \left( \frac{N^2d^2}{\delta} + 2 \right)^{2d^2N} e^{- \frac{(v-c_{d,N})^2}{8\pi^3} \left( \frac{d}{2} \right)^N} \quad \text{(4)}$$

for any $\delta > 0$, $v > c_{d,N} + \delta$, while $c_{d,N} = \left( \sqrt{\frac{2}{d}} \right)^N + \frac{d-2}{d}$.

An immediate consequence of this bound is that, for large $N$, most pure states do not get even close to exhibiting a violation of the order $2^N$, as generalized GHZ states do. For qubits we have $c_{2,N} = 1$ for every $N$, so as long as $v \geq cN$, with $c$ a positive suitable constant, we have $P(\mathcal{A}_v) \rightarrow 0$ as $N \rightarrow \infty$. That is, this is a result close to the conjecture presented by Pitowsky [15], although the author assumed that a violation already of the order $\sqrt{N \ln N}$ would have vanishing probability.

For $d \geq 3$ we have a more drastic scenario. Here $c_{d,N} \rightarrow \frac{d-2}{d} < 1$ with $N$ so it is possible to take appropriate $\delta > 0$ and $\delta + c_{d,N} < v < 1$ such that $P(\mathcal{A}_v)$ already goes to zero (super exponentially). That is, the majority of states do not violate any of these inequalities if $N$ is large enough.

Idea of the Proof—The basic idea is to use that $Q_{NL}(|\psi\rangle, Q)$ is “well-behaved”, being Lipschitz in its variation with $|\psi\rangle$, as well as $Q$. From this we perform two extra steps:

1. We may construct an $\epsilon$ net, i.e. a discretization for the space of choices for $Q$ such that all elements in this continuous space are approximated by an element of the net up to an error of $\epsilon$. The fact that $Q_{NL}(|\psi\rangle, Q)$ is Lipschitz means that it does not vary for more than $\delta$ in considering the discretized $Q$, if $\epsilon$ is chosen accordingly.

2. Since $Q_{NL}(|\psi\rangle, Q)$ is Lipschitz w.r.t $|\psi\rangle$, we upper bound the probability of violating the inequality for each $Q$ in the $\epsilon$-net using Lévy’s Lemma of measure concentration in high-dimensional spheres [17].

Bounding the expected value of $Q_{NL}$. The first element we need for the proof is an estimate of the expected value of the function $Q_{NL}$ for a fixed $Q$, that is, a fixed choice of measurement observables. To do so we define, for $j = 1, ..., N, x_j = 0, 1, B_{j,x_j} \equiv \frac{1}{2}(A^j_0 + (-1)^{x_j}A^j_1)$, and denote by $\lambda_{i_j,x_j}$, $i_j = 1, ..., d$ the eigenvalues of $B_{j,x_j}$. Since we are concerned with the maximal violation exhibited by a state, it is enough to consider $A^j_0$ and $A^j_1$ unitary Hermitian operators [18], i.e., $A^j_1 = (A^j_1)^\dagger = (A^j_1)^{-1}$. Note this includes the
operators $I$ and $-I$. It is easy to show then:

$$\text{Tr}B_{j,0}^2 + \text{Tr}B_{j,1}^2 = d, \quad (5)$$

$$|\text{Tr}B_{j,0}| + |\text{Tr}B_{j,1}| \leq \max\{|\text{Tr}A_{0}^j|, |\text{Tr}A_{1}^j|\} \leq d. \quad (6)$$

On the other hand, in order to have a violation of these Bell inequalities we must perform, in at least one of the systems, a pair of “nondull” measurements, i.e., represented by observables with both eigenvalues $\pm 1$. For the pair of observables on this system one can strengthen inequality (5):

$$|\text{Tr}B_{j,0}| + |\text{Tr}B_{j,1}| \leq d - 2, \text{ for at least one system } j. \quad (7)$$

To sum up, it is enough to consider measurement settings $Q$ where each system observable is Hermitian and unitary and, in at least one of the systems, the pair of observables has both eigenvalues $\pm 1$.

For a fixed $X = (x_1, ..., x_N)$, we expand an arbitrary state in the product eigenbasis $|i_1\rangle \otimes ... \otimes |i_N\rangle$ of $\bigotimes_{j=1}^{N} B_{j,x_j}$ [omitting its dependence on $(x_1, ..., x_N)$ to simplify notation]. To simplify notation even further, we use capital letters to denote the array of indexes $(i_1, ..., i_N) = I$ and write $|i_1\rangle \otimes ... \otimes |i_N\rangle \equiv |I\rangle$ and $\lambda_{i_1}...\lambda_{i_N} \equiv \lambda_I$, while $\alpha_I$ denote the expansion coefficient of a state in this basis. We can write then

$$\mathbb{E}[|\langle \psi | \bigotimes_{j=1}^{N} B_{j,x_j} |\psi \rangle|^4] = \mathbb{E}[|\sum_{I} |\alpha_I|^2 \lambda_I|^2]$$

$$\leq \{\mathbb{E}[\sum_{I} |\alpha_I|^4 \lambda_I^2]\}^{1/2}. \quad (8b)$$

The inequality is just a particular instance of Jensen’s [19], using that the square root is a concave function. Noting that $\mathbb{E}[|\alpha_I|^4] = 2\mathbb{E}[|\alpha_I|^2|\alpha_{L}^2|] = 2/d^N(d^N + 1)$ for every $I$ and

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1 It can be indeed advantageous to include these operators, even though they represent somewhat “dull” measurements. Take, for instance, the three-qubit mixed state $\frac{I}{2} \otimes |\Phi\rangle \langle \Phi|$, where $|\Phi\rangle$ is any maximally entangled state of two qubits. If we only perform measurements represented by Pauli operators (i.e., with eigenvalues 1 and $-1$), no violation will be seen (all expectation values will be zero). But if “dull” measurements are performed on the first qubit, one can see violations due to the entangled state on the second and third qubits.
we have:

\[
E[\sum I |\alpha_I|^2]^{1/2} = \left\{ \sum I E[|\alpha_I|^4] \lambda_I^2 + \sum_{I \neq L} E[|\alpha_I|^2|\alpha_L|^2] \lambda_I \lambda_L \right\}^{1/2} \quad (9a)
\]

\[
= \frac{1}{d^{N/2}(d^N + 1)^{1/2}} \left\{ 2 \sum I \lambda_I^2 + \sum_{I \neq L} \lambda_I \lambda_L \right\}^{1/2} \quad (9b)
\]

\[
= \frac{1}{d^{N/2}(d^N + 1)^{1/2}} \left\{ \sum I \lambda_I^2 + \sum_{I \neq L} \lambda_I \lambda_L \right\}^{1/2} \quad (9c)
\]

\[
= \frac{1}{d^{N/2}(d^N + 1)^{1/2}} \left\{ \prod_{j=1}^N \text{Tr} B_{j,x_j}^2 + \prod_{j=1}^N (\text{Tr} B_{j,x_j})^2 \right\}^{1/2} \quad (9d)
\]

\[
< \frac{1}{d^{N/2}} \left\{ \prod_{j=1}^N \sqrt{\text{Tr} B_{j,x_j}^2} + \prod_{j=1}^N |\text{Tr} B_{j,x_j}| \right\} \quad (9e)
\]

In the inequality we use that \((d^N + 1)^{1/2} > d^N\) and \((a^2 + b^2)^{1/2} \leq |a| + |b|\) for any \(a, b \in \mathbb{R}\).

Summing over all \(X \in \{0,1\}^N\), we have

\[
E[Q_{NL}(|\psi\rangle, \mathcal{Q})] < \frac{1}{d^N} \prod_{j=1}^N \left( \sqrt{\text{Tr} B_{j,0}^2} + \sqrt{\text{Tr} B_{j,1}^2} \right) + \prod_{j=1}^N (|\text{Tr} B_{j,0}| + |\text{Tr} B_{j,1}|) \quad (10)
\]

\[
\leq \frac{(\sqrt{2d})^N}{d^N} + \frac{(d-2)}{d} \equiv c_{d,N} \quad (11)
\]

while the last inequality is obtained using that \(|a| + |b| \leq \sqrt{2}(a^2 + b^2)^{1/2}\) for any \(a, b \in \mathbb{R}\) and Eqs. (5), (6), and (7).

**Bounding the Lipschitz constant of** \(Q_{NL}\). Regardless of the local dimension \(d\), the Bell operators \(B_{S,Q} = \sum_{X \in \{0,1\}^N} S(X) \otimes_{j=1}^N B_{j,x_j}\) satisfy \(||B_{S,Q}||_1 \leq \frac{2^{N-1}}{d}||B_{S,Q}||_\infty\) where \(||\cdot||_\infty\) denotes the usual operator norm. Now, given two arbitrary states \(|\psi\rangle\) and \(|\psi'\rangle\) we have

\[
|Q_{NL}(|\psi\rangle, \mathcal{Q}) - Q_{NL}(|\psi'\rangle, \mathcal{Q})| = \sum_{X \in \{0,1\}^N} ||\langle \psi| \otimes_{j=1}^N B_{j,x_j} |\psi'\rangle||
\]

\[
= \sum_{X \in \{0,1\}^N} ||\langle \psi| \otimes_{j=1}^N B_{j,x_j} |\psi'\rangle|| \quad (12)
\]

\[
\leq \sum_{X \in \{0,1\}^N} S^*(X) \left[ \langle \psi| \otimes_{j=1}^N B_{j,x_j} |\psi'\rangle - \langle \psi'| \otimes_{j=1}^N B_{j,x_j} |\psi'\rangle \right] \quad (13)
\]

\[
= |\text{Tr}[(\sum_{X} S^*(X) \otimes_{j=1}^N B_{j,x_j})(\langle \psi| \langle \psi| - |\psi'\rangle \langle \psi'|)|]. \quad (14)
\]

\[\text{In Ref. } 20 \text{ the author compute the expected value of any random variable of the form } (\sum_{I \in I} |\alpha_I|^2)^n \text{ for any non-negative integer } n \text{ and } I \subset \{0,1\}^N. \text{ From these it is straightforward to compute the expected values we use.}\]
where $S^*(X) = \langle \psi | \bigotimes_{j=1}^N B_{j,x_j} | \psi \rangle - \langle \psi' | \bigotimes_{j=1}^N B_{j,x_j} | \psi' \rangle / \langle \psi | \bigotimes_{j=1}^N B_{j,x_j} | \psi' \rangle$ if the expression in the parenthesis is not zero, and (say) +1 otherwise. From von Neumann’s trace inequality [21], followed by Hölder’s [19], the last expression can be bounded and we get

$$
|Q_{NL}(|\psi\rangle, Q) - Q_{NL}(|\psi'\rangle, Q)| \leq \| S^*(X) \bigotimes_{j=1}^N B_{j,x_j} \|_{\infty} \| |\psi\rangle - |\psi'\rangle \|_1 \leq 2^{\frac{d^2 N}{2}} \| |\psi\rangle - |\psi'\rangle \|_1,
$$

where $\| \cdot \|_1$ is the trace operator norm and using that $\| |\psi\rangle - |\psi'\rangle \|_1 \leq 2 \| |\psi\rangle - |\psi'\rangle \|$, $\| \cdot \|$ being just the Hilbert space norm.

**Variation of $Q_{NL}$ with $Q$.** Now we estimate how the function $Q_{NL}$ varies when we change the operators describing the pair of measurements in each site. This will be used to take a “representative” finite subset of the set of measurements (a $\epsilon$ net). First, observe that

$$
\| \langle \psi | \bigotimes_{j=1}^N B_{j,x_j} | \psi \rangle - \langle \psi | \bigotimes_{j=1}^N \tilde{B}_{j,x_j} | \psi \rangle \| \leq \| \langle \psi | \bigotimes_{j=1}^N B_{j,x_j} - \bigotimes_{j=1}^N \tilde{B}_{j,x_j} | \psi \rangle \| (17)
$$

$$
= | \langle \psi | [B_{1,x_1} \otimes \ldots \otimes B_{N-1,x_{N-1}} \otimes (B_{N,x_N} - \tilde{B}_{N,x_N}) + B_{1,x_1} \otimes \ldots \otimes B_{N-1,x_{N-1}} \otimes (B_{N-1,x_{N-1}} - \tilde{B}_{N-1,x_{N-1}}) \tilde{B}_{N,x_N} + \ldots + (B_{1,x_1} - \tilde{B}_{1,x_1}) \otimes \tilde{B}_{2,x_2} \otimes \ldots \otimes \tilde{B}_{N,x_N} ] |\psi\rangle | (18)
$$

$$
\leq N \sup_{j} \| B_{j,x_j} - \tilde{B}_{j,x_j} \|_{\infty}, (19)
$$

since $\| B_{j,x_j} \|_{\infty} \leq 1$. Furthermore,

$$
\| B_{j,x_j} - \tilde{B}_{j,x_j} \|_{\infty} = \frac{1}{2} \| A_{j}^0 - \tilde{A}_{j}^0 + (-1)^x(A_{j}^1 - \tilde{A}_{j}^1) \|_{\infty} \leq \sup_{i=0,1} \| A_{j}^i - \tilde{A}_{j}^i \|_{\infty}. (20)
$$

Defining $D(Q, \tilde{Q}) = \sup_{j=1, \ldots, N; i=0,1} \| A_{j}^i - \tilde{A}_{j}^i \|_{\infty}$, we have

$$
|Q_{NL}(|\psi\rangle, Q) - Q_{NL}(|\psi\rangle, \tilde{Q})| \leq N 2^N D(Q, \tilde{Q}), (21)
$$

where the factor $2^N$ comes from the sum in $X$.

We can cover the set of relevant hermitian operators $A$ in $C^d$ by a parametrization with $d^2$ real numbers $a_k$, corresponding to the real and imaginary parts of their matrix in a given orthonormal basis. For simplicity, we introduce a norm $\| A \|^\prime = \sup_{k=1, \ldots, d^2} |a_k|$ which satisfies, in particular,

$$
\| A \|_{\infty} \leq 2 d^2 \| A \|^\prime. (22)
$$
Defining the distance $D′(Q, \tilde{Q}) = \sup_{j=1,\ldots,N; i=0,1} |A^j_i - \tilde{A}^j_i|′$, the set of quantum measurements $Q$ can be seen as a subset of the hypercube $[0, 1]^{2d^2 N}$ endowed with the maximum or $\ell_\infty$ norm (the norm of a vector is the largest absolute value of its coordinates). One can check that for any $\epsilon$ there exists an $\epsilon$ net $N_\epsilon$ of the hypercube with
\[|N_\epsilon| = \# \text{ of elements in } N_\epsilon \leq \left(\frac{1}{\epsilon} + 2\right)^{2d^2 N}.\] (23)

Comparing $||A||_\infty$ and $||A||′$ and applying Eq. (21), we see that we may take $\epsilon = \delta / d^2 N 2^{N+1}$ in order to guarantee that two choices of $Q$ within distance $\epsilon$ have values of $Q_{NL}$ within distance $\delta$ of each other. This results in a net with size:
\[|N_\epsilon| \leq \left(\frac{N 2^{N+1} d^2}{\delta} + 2\right)^{2d^2 N}.\] (24)

**Proof of theorem 1.** Finally, with all these elements in hand, we can estimate the probability of $\mathcal{A}_\psi$:
\[
P(\sup_{Q} Q_{NL}(\langle \psi \rangle, Q) > v) \leq P(\sup_{Q \in N_\epsilon} Q_{NL}(\langle \psi \rangle, Q) > v - \delta)
\]
\[
\leq \sum_{Q \in N_\epsilon} P(Q_{NL}(\langle \psi \rangle, Q) > v - \delta)
\]
\[
\leq \sum_{Q \in N_\epsilon} P(Q_{NL}(\langle \psi \rangle, Q) - E[Q_{NL}(\langle \psi \rangle, Q)] > v - \delta - c_{d,N})
\]
\[
\leq 2|N_\epsilon| e^{-\frac{(v - \delta - c_{d,N})^2 (d/2)^N}{9\pi^3 \lambda^2}}.
\] (28)

The third inequality comes from Eq. (11), while the last one, assuming $v > \delta + c_{d,N}$, comes from [13, 17].

**Lévy’s Lemma.** For every $\epsilon > 0$, $n \geq 1$ integer and $F : S_n \rightarrow \mathbb{R}$, a real-valued function with Lipschitz constant $\lambda$ (with respect to the Euclidean distance), the following inequality holds true:
\[
P(F - E[F] > \epsilon) \leq 2e^{-\frac{(n+1)\epsilon^2}{9\pi^3 \lambda^2}},
\] (29)
where $P$ denotes the uniform probability measure on the sphere $S_n$ and $E$ the corresponding expected value.

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3 Let $M$ be the largest integer smaller than $\frac{1}{\epsilon}$ and define $N_\epsilon = \{(n_1 M + 1, \ldots, n_{2d^2 N} M + 1) \in [0, 1]^{2d^2 N} : n_j = 0, 1, \ldots, M + 1 \text{ for } j = 1, \ldots, 2d^2 N\}$. Clearly, every point of the hypercube is at least $\frac{1}{M+1} \leq \epsilon$ close to a point of $N_\epsilon$ and we have $|N_\epsilon| = (M + 2)^{2d^2 N} \leq (\frac{1}{\epsilon} + 2)^{2d^2 N}$. 

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IV. SMALL VIOLATIONS BY ERROR-PRONE QUBITS

We have left open the question of whether most (or almost all) qubit states exhibit at least some violation. Even if that is indeed the case, however, we will show below that these violations would be extremely sensitive to experimental errors. This is intuitive, since if each qubit is subjected to some error and each term of the inequality is a product of measurements on each of them, the overall error will scale exponentially, while as we have seen, typically a state exhibits a violation that scales no more than linearly.

We prove this formally for a model of the inevitable local noise that each qubit is subjected to. Specifically, we consider the representative case of local white noise where each qubit is mapped to $\rho_2 \mapsto (1-\lambda)\rho_2 + \lambda \frac{I}{2}$, where $0 \leq \lambda \leq 1$. The local assumption is an adequate one, since measurements in a Bell-like experiment are especially interesting when the systems are space-like separated, so there must be a time interval where the systems cannot interact or communicate. The mapping for global pure states is then the following:

$$|\psi\rangle \langle \psi| \mapsto \rho|\psi\rangle \langle \psi|, \quad \lambda = \sum_{k=0}^{N} \lambda^k (1-\lambda)^{N-k} \sum_{P \subseteq \{1,\ldots,N\} : |P| = k} \text{Tr}_{P_c} |\psi\rangle \langle \psi| \otimes I_{2^{P_c}}$$

where $P_c$ denotes the complement of $P$. Now the function $Q^\lambda_{NL}(|\psi\rangle, Q)$, representing the degree of violation due to a pure state perturbed by this map is given by $Q^\lambda_{NL}(|\psi\rangle, Q) = Q_{NL}(\rho|\psi\rangle, \lambda, Q)$. To bound the chances of seeing a violation in this case we just follow the same steps of Theorem 1’s proof (see Appendix for details), with the following result:

**Theorem 2.** For $N \geq 2$, integer, $|\psi\rangle \in (\mathbb{C}^2)^\otimes N$ a unit vector distributed according to the uniform measure in the sphere $S_{2,2^{N-1}}$ of $(\mathbb{C}^2)^\otimes N$, $A_{\delta}^\lambda = \{|\psi\rangle : \sup Q^\lambda_{NL}(\psi, Q) > \delta\}$, the following inequality holds true:

$$P(A_{\delta}^\lambda) \leq 2 \left( \frac{\delta}{N2^{N+3}} + 2 \right)^{8N} e^{-\frac{(\delta-1)^2(2/\chi)^N}{9\pi^3}}$$

for any $\delta > 0$, $\nu > 1 + \delta$, while $\chi = [\lambda + (1-\lambda)\sqrt{2}]^2$.

The point is that, for $\lambda > 0$ we have $\chi < 2$. Therefore, for any fixed $\delta > 0$, the probability of having a violation larger than $1 + \delta$ is vanishingly small for large $N$ due to the super exponential factor $\exp \left[ -\frac{(\nu-\delta-1)^2(2/\chi)^N}{9\pi^3} \right]$. 


V. CONCLUSION

We proved that, for large $N$, most $N$-qubit pure states cannot violate full correlation Bell inequalities by a large amount, when the inequalities are restricted to two measurements settings per site and two outcomes per measurement. For $N$ $d$-dimensional systems, with $d \geq 3$, the result is even stronger, where most states do not show any violation. This constitutes another instance where there is no quantitative correspondence between Bell inequality violations and entanglement, since most pure states are highly entangled.

We have left open the question of whether (necessarily small) violations for qubits are typical. We argued, though, that even if they are, small experimental imperfections would make it impossible to actually probe these violations.

All these results reinforce the insights of Pitowsky in Refs. [15, 22], about why entanglement (and nonlocality, we might add) is never present in the classical world (this is in addition to the well-known argument based on decoherence). The author argues that most many-body quantum states and most observables are such that the quantum signatures they could reveal are small and henceforth hidden by experimental imperfections. He bases this line of reasoning on many-qubit states. We add to this by showing that if each of the subsystems is “large,” which is the case of classical systems, it gets even more difficult to see Bell inequality violations.

Of course, the inequalities we consider form a very restricted set. However, our proof method seems quite robust. We believe it should be applicable to more general families of measurements, whenever the total number of degrees of freedom for the measurements is much smaller than the Hilbert space dimension. The main difficulty for such extensions is a characterization of maximum violation inequalities in the spirit of Refs. [9, 10].

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APPENDIX

Here we show the proof of Theorem 2, which has the same idea and structure as that of Theorem 1.

Bounding the expected value of $Q_{NL}^\lambda$. Again, first we need a bound for $\mathbb{E}[Q_{NL}^\lambda(|\psi\rangle, \mathcal{Q})]$. 

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To do that we first examine the contribution of each term in the sum describing $\rho_{|ψ⟩,λ}$. For any $P \subseteq \{1, ..., N\}$, we have:

$$|\text{Tr}[\bigotimes_{j=1}^{N} B_{j,x_j}(\text{Tr}_{P^c} |ψ⟩ \langle ψ|) \otimes \frac{I_{P^c}}{2|P^c|}]|$$

$$= |\text{Tr}_{P}[\bigotimes_{j \in P} B_{j,x_j}(\text{Tr}_{P^c} |ψ⟩ \langle ψ|)] \prod_{j \in P^c} \frac{\text{Tr}B_{j,x_j}}{2}|$$  \hspace{1cm} (A1)

$$= |\prod_{j \in P^c} \frac{\text{Tr}B_{j,x_j}}{2} \text{Tr}_P[\bigotimes_{j \in P} B_{j,x_j} \otimes I_{P^c} |ψ⟩ \langle ψ|] |$$  \hspace{1cm} (A2)

$$= |⟨ψ| \bigotimes_{j \in P} B_{j,x_j} \bigotimes_{j \in P^c} \left(\frac{\text{Tr}B_{j,x_j}}{2} I_2\right) |ψ⟩|. \hspace{1cm} (A3)$$

The operators $\frac{\text{Tr}B_{j,x_j}}{2} I_2$ appearing on the tensor product over $P^c$ are either zero or $\pm I_2$. Therefore, by the same method we used on the proof of Theorem 1 proof we can now bound the expected value:

$$\sum_X E[|\text{Tr}[\bigotimes_{j=1}^{N} B_{j,x_j}(\text{Tr}_{P^c} |ψ⟩ \langle ψ|) \otimes \frac{I_{P^c}}{2|P^c|}]|] \leq \sum_X E[|⟨ψ| \bigotimes_{j \in P} B_{j,x_j} \bigotimes_{j \in P^c} \left(\frac{\text{Tr}B_{j,x_j}}{2} I_2\right) |ψ⟩|] \leq 1. \hspace{1cm} (A4)$$

We have then, using the triangle inequality and the above bound,

$$E[Q^\lambda_{NL}(|ψ⟩, Q)] = \sum_X E[|\text{Tr}\bigotimes_{j=1}^{N} B_{j,x_j}ρ_{|ψ⟩,λ}|] \hspace{1cm} (A5)$$

$$\leq \sum_{k=0}^{N} \lambda^k (1 - \lambda)^{N-k} \sum_{P^c \subseteq \{1, ..., N\}: |P^c| = k} \sum_X E[|\text{Tr}[\bigotimes_{j=1}^{N} B_{j,x_j}(\text{Tr}_{P^c} |ψ⟩ \langle ψ|) \otimes \frac{I_{P^c}}{2|P^c|}]|] \hspace{1cm} (A6)$$

$$\leq \sum_{k=0}^{N} \lambda^k (1 - \lambda)^{N-k} \binom{N}{k} \hspace{1cm} (A7)$$

$$= 1. \hspace{1cm} (A8)$$

**Bounding the Lipschitz constant of $Q^\lambda_{NL}$.** The Lipschitz constant of this function has a smaller bound than the one we get for $Q_{NL}(|ψ⟩, Q)$ if $\lambda > 0$. This is basically the reason why we get a stronger bound for the probabilities of seeing small violations in this case.

First we get:

$$|Q^\lambda_{NL}(|ψ⟩, Q) - Q^\lambda_{NL}(|ψ'⟩, Q)| \leq |\text{Tr}[(\sum_X S^r(X) \bigotimes_{j=1}^{N} B_{j,x_j}) (ρ_{|ψ⟩,λ} - ρ_{|ψ'⟩,λ})]|, \hspace{1cm} (A9)$$
where, similarly, $S^*(X) = (\text{Tr} \bigotimes_{j=1}^{N} B_{j,x_j} \rho_{\psi})/\lambda - \text{Tr} \bigotimes_{j=1}^{N} B_{j,x_j} \rho_{\psi})/\lambda - \text{Tr} \bigotimes_{j=1}^{N} B_{j,x_j} \rho_{\psi})$ if the expression in the parenthesis is not zero, and (say) +1 otherwise.

Next we look at the contribution that each term in $\rho_{\psi}/\lambda$ [Eq. (30)] gives to the right-hand side of Eq. (A9):

$$| \sum_X S^*(X) \text{Tr}\left[ \bigotimes_{j=1}^{N} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \otimes \frac{I}{2^{|P_{c}|}} \right] |
$$

$$= | \sum_{x_j \in P} \text{Tr}_{P_{c}}\left[ \bigotimes_{j \in P} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \right] \sum_{x_j \in P_{c}} S(X) \prod_{j \in P_{c}} \frac{\text{Tr}B_{j,x_j}}{2} | (A10)$$

$$= | \sum_{x_j \in P} c_{x_{j_1},...,x_{j_{N-k}}} \text{Tr}_{P_{c}}\left[ \bigotimes_{j \in P} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \right] | (A11)$$

where $c_{x_{j_1},...,x_{j_{N-k}}} = \sum_{x_j \in P} S(X) \prod_{j \in P_{c}} \frac{\text{Tr}B_{j,x_j}}{2}$, and $j_i \in P$ for $i = 1, ..., N - k$, is such that $-1 \leq c_{x_{j_1},...,x_{j_{N-k}}} \leq 1$, since these numbers can be seen as expected values of full-correlation Bell operators for $k$ qubits on the maximally mixed state and, as such, must satisfy the Bell inequality. Therefore, there must exist a sign function $S^{**} : P \to \{0, 1\}^{N-k}$ such that

$$| \sum_{x_j \in P} c_{x_{j_1},...,x_{j_{N-k}}} \text{Tr}_{P_{c}}\left[ \bigotimes_{j \in P} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \right] |
$$

$$\leq | \sum_{x_j \in P} S^{**}(x_{j_1},...,x_{j_{N-k}}) \text{Tr}_{P_{c}}\left[ \bigotimes_{j \in P} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \right] |, (A12)$$

Hence, we have

$$| \sum_X S^*(X) \text{Tr}\left[ \bigotimes_{j=1}^{N} B_{j,x_j} \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \otimes \frac{I}{2^{|P_{c}|}} \right] |
$$

$$\leq | \text{Tr}_{P_{c}}\left[ (\sum_{x_j \in P} S^{**}(x_{j_1},...,x_{j_{N-k}}) \bigotimes_{j \in P} B_{j,x_j}) \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) \right] | (A13)$$

$$\leq | \sum_{x_j \in P} S^{**}(x_{j_1},...,x_{j_{N-k}}) \bigotimes_{j \in P} B_{j,x_j}|_{\infty} | \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) ||_{1} (A14)$$

$$\leq 2^{(N-k-1)/2} | \text{Tr}_{P_{c}}(|\psi\rangle \langle\psi| - |\psi'\rangle \langle\psi'|) ||_{1} (A15)$$

$$\leq 2^{(N-k-1)/2} | |\psi\rangle \langle\psi'|-|\psi'\rangle \langle\psi||_{1} (A16)$$

$$\leq 2^{N-k+1/2} | |\psi\rangle - |\psi'\rangle |. (A17)$$

In the second inequality we use again von Neumann and Hölder inequalities. Realizing that $\sum_{x_j \in P} S^{**}(x_{j_1},...,x_{j_{N-k}}) \bigotimes_{j \in P} B_{j,x_j}$ is a full-correlation Bell operator for $N - k$ qubits gives us the third. The fourth is due to the monotone behavior of the trace distance with respect
to completely positive trace-preserving maps (the partial trace being a particular instance of them).

Finally, using the bound (A17) and expression (30) for $\rho_{\psi,\lambda}$ we can compute:

$$|Q_{NL}^\lambda(\psi,\mathcal{Q}) - Q_{NL}^\lambda(\psi',\mathcal{Q})|$$

$$\leq \sqrt{2} \sum_{k=0}^{N} \lambda^k (1-\lambda)^{N-k} \binom{N}{k} \sqrt{2}^{N-k} ||\psi - |\psi'||||$$

(A18)

$$= \sqrt{2} (\lambda + (1-\lambda)\sqrt{2})^N ||\psi - |\psi'|||.$$  

(A19)

**Variation of $Q_{NL}^\lambda$ with $\mathcal{Q}$.** The last element we need is an appropriate $\epsilon$–net to replace the continuous set of measurements by a discrete subset of it. But this can be obtained in the exact same way as before and the same bound we get for its number of elements, for $d=2$, apply here as well.

Finally, following the same line of reasoning of Eqs. (25)–(28) we get the bound:

$$P(\sup_{\mathcal{Q}} Q_{NL}^\lambda(\psi,\mathcal{Q}) > v) \leq 2^N \frac{2^{N+3}}{\delta} + 2^{8N} e^{-\frac{(v-\delta)^2}{9\pi^2}}.$$  

(A20)

where $\chi = (\sqrt{2} - \lambda(\sqrt{2} - 1))^2$.

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