Effects of Delayed Feedback on a Parametrically Excited System with Quadratic and Cubic Nonlinearities

H L Shang, J Xu

School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 1239 Siping Road, China
Email: xujian@mail.tongji.edu.cn

Abstract. A delayed position feedback is considered to investigate the local and global dynamics induced by the delay in a parametrically excited system. A method is proposed to obtain the approximate resonant solution from which one can easily predict the delay-induced bifurcation branches. Also, the basin of attraction is obtained numerically when single equilibrium or three equilibriums occur in the system under consideration. It is found that the appropriate choice of the time delay and the feedback gain can improve the fractal of the boundary of the basin and stabilize the motion to the expected equilibrium state.

1. Introduction

Delayed systems are ubiquitous in areas such as neural networks[1-2], ecology[3-4], and controlled systems[5-11] due to finite signal transmission times, processing times in synapses, switching speeds and memory effects. Since the delay often gives rise to the instability of the controlled system, its effects on the performance of the system have been intensively studied for many decades. Maccari[8] studied the parametric resonance of a van der Pol oscillator under state feedback control with a time delay. Xu and Chung[9] investigated the mechanism for the action of time delay in a non-autonomous system and indicated that the time delay may be used as a simple but efficient “switch” to control motions of a system: either from order motion to chaos or from chaotic motion to order for different applications. The similar conclusion was also obtained by Jin and Hu[11].

With extension of our research, we tend to understand how the delayed feedbacks affect the dynamics of an unsymmetrical system. To this end, a parametrically excited system with quadratic and cubic nonlinearities is considered as our model. On the one hand, the parametrically excited system can model many engineering problems such that the related research for it has its significance. On the other hand, dynamics of the chosen model has been known without delayed feedbacks[12]. Thus, the dynamics can be easily compared when a delayed feedback is taken into account in the system.

The mathematical model in this paper is obtained by applying linear delayed position feedbacks to the original system in [12], given by

\[ \ddot{u} + 2\epsilon \mu \dot{u} + \omega_0^2 u + \epsilon \delta u^2 + \epsilon^2 \alpha u^3 + \epsilon g u \cos \Omega t = A(u_e - u), \quad (1) \]

1 To whom any correspondence should be addressed.
where $\varepsilon$ is a small dimensionless parameter, $\mu, \omega, \delta, \alpha, g$, and $\Omega$ are constants, $\tau$ represents the time delay, and $A$ the gain. When $\tau = 0$, the system (1) describes the response of a single-degree-of-freedom system to a principal parametric resonance [12] and becomes the symmetric one for $\delta = 0$.

The organization of the paper is as follows. In Section 2, the method of normal form (MNF) is used to investigate bifurcating periodic motions, and some discussions are done for the stability of the periodic motions. Section 3 shows the numerical results. Section 4 summarizes the results and discusses their significance and possible application.

2. Theoretical Analysis

The equilibrium positions of the undamped and unexcited system (1) without the delay can be easily obtained as

$$u = 0, \quad (2\varepsilon \alpha)^{-1}(-\delta \pm (\delta^2 - 4\alpha^2)^{1/2}).$$

The number and type of equilibrium positions as well as the shape of potential depend on values of $\alpha, \delta, \omega$, and $\varepsilon$. In this section, we concentrate on analyzing the motion in the neighborhood of the trivial equilibrium.

For the principal parametric resonance, i.e., $\Omega = 2\omega + \varepsilon \sigma$, we rescale the feedback gain in $A = \varepsilon A$, to observe effects of the gain on the principal parametric resonance, where $\sigma$ is called as a detuning parameter. According to paper [12], we assume the approximate solution of equation (1) in the form

$$u = r \cos\left(\frac{1}{2} \Omega t - \frac{\gamma}{2}\right) + \varepsilon (m_1 r^2 \cos(\Omega t - \gamma) + m_2 r^2 + m_3 r \cos\left(\frac{3}{2} \Omega t - \frac{\gamma}{2}\right)) + \cdots,$$

where $m_1$, $m_2$ and $m_3$ are coefficients to be determined, $r$ represents the amplitude, and $\gamma$ the phase.

Then we set

$$\begin{cases}
v = 2\dot{u} / \Omega, \\
z = e^{\varepsilon u}.
\end{cases}$$

By equations (3) and (4), it follows that

$$u_r = u \cos \frac{\Omega \tau}{2} - v \sin \frac{\Omega \tau}{2} + \varepsilon (u^2 ((\cos \Omega \tau - \cos \frac{\Omega \tau}{2}) m_1 + (1 - \cos \frac{\Omega \tau}{2}) m_2) - 2mv(u \cos \Omega \tau - 2 \sin \frac{\Omega \tau}{2})$$

$$+v^2 ((\cos \frac{\Omega \tau}{2} - \cos \Omega \tau)m_1 + (1 - \cos \frac{\Omega \tau}{2}) m_2) + m_3 \cos \frac{3\Omega \tau}{2} - \cos \frac{\Omega \tau}{2}(u \frac{z + \overline{z}}{2} + v \frac{z - \overline{z}}{2i}) + m_3 (\sin \frac{3\Omega \tau}{2} - 3 \sin \frac{\Omega \tau}{2})(-v \frac{z + \overline{z}}{2} + u \frac{z - \overline{z}}{2i}) + O(\varepsilon^2).$$

Substituting equations (4) and (5) into equation (1) and using the MNF yield a second-order approximation solution of equation (1), given by

$$u = r \cos\left(\frac{1}{2} \Omega t - \frac{\gamma}{2}\right) - \frac{\varepsilon}{\Omega^2} \left(\frac{2 \delta}{3} r^2 \cos(\Omega t - \gamma) + 2 \delta r^2 - \frac{g}{4} r \cos\left(\frac{3}{2} \Omega t - \frac{\gamma}{2}\right)\right) + O(\varepsilon^2),$$

where
\[ \dot{r} = (-\frac{c_2}{\Omega} - \frac{e g c_1}{\Omega^3} \cos \gamma - \frac{\frac{c_2}{\Omega}}{2} \sin \gamma) r, \]

\[ r \dot{\gamma} = (\frac{2c_1}{\Omega} + \Omega - 2 \omega_0 + \frac{2(c_1^2 + c_2^2)}{\Omega^3} + \frac{e^2 g^2}{4 \Omega} + \frac{2(\Omega - 2 \omega_0)c_1}{\Omega^3} + \frac{(\Omega - 2 \omega_0)^2}{2}) \]

\[ + (\frac{2e g c_2}{\Omega^3} \sin \gamma - 2(\frac{e g c_1}{\Omega^3} + \frac{e g}{2 \Omega}) \cos \gamma)r + e^2 \frac{20 \delta^2}{3 \Omega^3} - \frac{3 \alpha}{2 \Omega} \gamma^3, \]

with

\[ c_1 = A \cos(\Omega \tau / 2) - A, c_2 = A \sin(\Omega \tau / 2) + e g \mu \Omega. \]

Obviously, the approximate solution is related not only to \( \mu, \omega_0, \delta, \alpha, \epsilon \) and \( \Omega \), but also to \( A \) and \( \tau \).

To analyze the stability of the trivial solution, it is necessary to express equations (6) and (7) in Cartesian form, given by

\[ u = p \cos \frac{\Omega t}{2} + q \sin \frac{\Omega t}{2} + \frac{2 \epsilon \delta}{3 \Omega^2} (q^2 - p^2) \cos \Omega t - \frac{4 \epsilon \delta \rho}{3 \Omega^2} \sin \Omega t - \frac{2 \epsilon \delta}{\Omega^2} (p^2 + q^2) \]

\[ + \frac{\epsilon g}{4 \Omega^2} (p \cos \frac{3 \Omega t}{2} + q \sin \frac{3 \Omega t}{2}) + O(\epsilon^3), \]

and

\[ \begin{cases} \dot{p} = (-\frac{c_2}{\Omega} - \frac{e g c_1}{\Omega^3}) p + (-\frac{e g c_1}{\Omega^3} - \frac{e g}{2 \Omega} - \frac{c_1}{\Omega^3} + \frac{(c_1^2 + c_2^2)}{\Omega^3} - \frac{e^2 g^2}{8 \Omega^3} - \frac{\Omega}{2} + \omega_0) \\ - (\frac{\Omega - 2 \omega_0}{\Omega^3}) c_1 - (\frac{\Omega - 2 \omega_0}{\Omega^3}) q - e^2 (\frac{10 \delta^2}{3 \Omega^3} - \frac{3 \alpha}{4 \Omega}) (p^2 + q^2) q, \end{cases} \]

\[ q = \left( -\frac{e g c_1}{\Omega^3} - \frac{e g}{2 \Omega} + \frac{c_1}{\Omega^3} + \frac{(c_1^2 + c_2^2)}{\Omega^3} + \frac{e^2 g^2}{8 \Omega^3} + \frac{\Omega}{2} - \omega_0 + \frac{(\Omega - 2 \omega_0)c_1}{\Omega^3} \right) \]

\[ + \frac{\Omega - 2 \omega_0}{4 \Omega^3} ) p + (-\frac{c_2}{\Omega} - \frac{e g c_2}{\Omega^3}) q + e^2 (\frac{10 \delta^2}{3 \Omega^3} - \frac{3 \alpha}{4 \Omega})(p^2 + q^2) p. \]

It can be seen that the steady-state solutions of the system (1) correspond to the fixed points of equation (8) obtained by \( \dot{r} = \dot{\gamma} = 0 \). Hence, there are two possibilities: either a trivial solution \( r = 0 \) or non-trivial solutions. The stability of the trivial solution is determined by the eigenvalues of the corresponding Jacobian matrix of equation (10)

\[ \lambda = -\frac{c_2}{\Omega} \pm \sqrt{\frac{c_2^2}{\Omega^2} - f}, \]

where

\[ f = \frac{c_2^2}{\Omega^2} - \frac{e^2 g^2 c_1^2}{\Omega^3} - \frac{e g c_1}{\Omega^3} - \frac{e g}{2 \Omega} + \frac{c_1}{\Omega^3} + \frac{c_1^2}{\Omega^3} + \frac{c_2^2}{\Omega^3} + \frac{e^2 g^2}{8 \Omega^3} - \frac{\Omega - 2 \omega_0}{2} + \frac{(\Omega - 2 \omega_0)c_1}{\Omega^3} + \frac{(\Omega - 2 \omega_0)^2}{4 \Omega}. \]

The eigenvalues for non-trivial solutions can be obtained from

\[ \lambda^2 + 4c_2^2 \Omega \lambda + e^2 (\frac{4 \Omega ^2}{\Omega ^2} - \frac{3 \alpha}{\Omega ^2}) (\frac{e g c_1}{\Omega ^3} + \frac{e g}{2 \Omega}) \cos \gamma - \frac{e g c_2}{\Omega ^2} \sin \gamma) r^2 = 0. \]

The amplitude of the periodic solution in function of the time delay is plotted in figure 1, where the solid line represents the stable solution and the dashing line the unstable one.
To investigate motions in the neighborhood of nontrivial equilibrium at \( u_0 \), one can set \( x = u - u_0 \) in equation (2), resulting in
\[
\ddot{x} + 2\epsilon \mu x + (\omega_0^2 + 2\epsilon \delta u_0 + 3\epsilon^2 \alpha u_0^2)x + \epsilon (\delta + 3\epsilon \alpha u_0)x^2 + \epsilon^2 \alpha x^3 + \epsilon g(u_0 + x)\cos \Omega t = A(x_\tau - x). \quad (14)
\]
It can be seen from equation (14) that the system is subject to both parametric and external excitations. Thus, the primary resonance and the principal parametric resonance for the system (14) can be discussed in a similar way.

3. Delay Induced Global Dynamics

As mentioned above, the delayed feedback affects the resonant solutions in the neighborhood of the equilibriums. To understand such effects globally, the numerical simulation is employed to obtain the basin of attractors for both single and multiple equilibriums to occur in the system (1). In simulation, the initial conditions are given by \( u(0) = \dot{u}(0) = 0.0 \) for \(-\tau \leq t < 0\). The values of \( \mu, \omega_0, \alpha, g, \epsilon \) and \( A \) are the same as those in figure 1. Two values of \( \delta \), i.e., \( \delta = 3.1 \) and \( \delta = 5 \), are chosen to show the cases of single equilibrium and three equilibriums respectively.

In figure 1, multiple attractions coexist when the delay varies from 4.09 to 5.25. Since different initial values may induce different solutions so that there will be different attraction basins, it is necessary to classify the basins of attraction. In this section, the region of attractors is drawn in the sufficiently large space region defined as \(-10 \leq u(0) \leq 20\), \(-20 \leq \dot{u}(0) \leq 20\) by generating 120699 points of starting conditions, the light grey region in the figures represents the basin of the trivial or non-trivial equilibrium and the black region the basin of the resonant solution.

3.1. The system with single equilibrium

For \( \delta = 3.1 \), there is only a trivial equilibrium in the system (1). The principal parametric resonance leads to the resonant solution in the neighborhood of the equilibrium point. Figure 2 shows the evolution of the basin of the trivial equilibrium and the resonant solution as the delay increases. It follows from figure 2 that the basin of the resonant solution becomes larger and larger by fractal from the basin of the trivial attractor with the delay increasing. The range of the delay where the two attractions coexist is in great agreement with that in figure 1, which verifies the validity of the analytical prediction qualitatively. Figure 3 shows that a change in the delay may lead to the quasi-periodic motion.
3.2. The system with three equilibriums

For $\delta = 5$, there are three equilibriums in the system (1). It follows from equation (14) that either the principal parametric resonance or the primary resonance can lead to the resonant solution in the neighborhood at the non-trivial equilibriums. In computation, $\Omega = 1.0$ and $\Omega = 2.0$ are chosen to correspond these two cases respectively.

When $\tau = 0$, there will be a stable equilibrium and resonant solution in the system (1) [12]. Figures 4 and 5 show the evolution of the basin of attraction of the trivial equilibrium and the resonant solution with the delay increasing for $\Omega = 1.0$ and $\Omega = 2.0$ respectively. It is obvious that the delayed position feedback control can not change the dynamics of the system qualitatively when the delay is small. However, the nature of the basin of the trivial equilibrium is changed. It can be seen from figures 4 and 5 that the boundary of the basin of the trivial equilibrium is fractal at first, then becomes smooth and finally fractal as the delay grows. Figures 6 and 7 show the complex dynamics the large value of the delay causes such as the quasi-periodic motion for $\Omega = 1.0$ and quasi-periodic and the chaotic motion for $\Omega = 2.0$, respectively.
4. Conclusions
The paper focuses on considering effects of the delayed feedback on a parametrically excited system with quadratic and cubic nonlinearities. A method is proposed to obtain the approximate solution induced by the time delay. The basin of attraction for the system under consideration is drawn by the numerical computation. As a result, the delayed feedback can change the stability of the equilibrium, transform the unstable motions to stable ones, control the fractal boundary of the basin of attraction,
and generate complex dynamical motions. These provide some potential applications for the study of stabilization of controlled systems and chaotic motions.

Figure 7. Poincaré maps of numerical solutions of equation (1) for $\delta = 5$ and $\Omega = 2$ : (a) $\tau = 4.88$; (b) $\tau = 5.3$.

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