Tachyon and Quintessence in Brane-Worlds

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Using tachyon or quintessence fields along with a barotropic fluid on the brane we examine the different cosmological stages in a Friedmann-Robertson-Walker (FRW) universe, from the first radiation scenario to the later era dominated by cosmic string networks. We introduce a new algorithm to generalize previous works on exact solutions and apply it to study tachyon and quintessence fields localized on the brane. We also explore the low and high energy regimes of the solutions. We show that the tachyon and quintessence fields are driven by an inverse power law potential. Finally, we find several simple exact solutions for tachyon and/or quintessence fields.

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I. INTRODUCTION

Recent progress in the superstring theory has shown that the strongly coupled $E_8 \times E_8$ heterotic string can be identified as the 11-dimensional limit of M-theory compactified on an $S^1/Z_2$ orbifold with a set of $E_8$ gauge fields at each ten-dimensional orbifold fixed plane $[1, 2]$. Furthermore, there exists a consistent compactification of this M-theory limit on a Calabi-Yau threefold, such that for energies below the unification scale there is a regime where the Universe appears five-dimensionally. This five-dimensional regime represents a new setting for early universe cosmology, which has been traditionally studied in the framework of the four-dimensional effective action. The theory is developed in a five-dimensional space which is a product of a smooth four-dimensional manifold times the orbifold $S^1/Z_2$. Therein the matter fields are confined to the four-dimensional spacetime while gravity can propagate in the full spacetime. This model and its noncompact analogs $[2, 4, 5, 6]$ (see Ref. $[3]$ for an account of earlier works) provide a novel setting for discussing various conceptual and phenomenological issues related to compactification of extra dimensions in models motivated by the M-theory.

Two interesting possibilities were suggested in $[1, 8]$. In the two-brane model $[8]$, the branes have tensions of opposite sign and the bulk cosmological constant is chosen in such a way that the classical solution describes five-dimensional space-time whose four-dimensional slices are flat.

It was shown that mass scales on the negative tension brane can be severely suppressed, leading to a solution of the hierarchy problem. Of course, this assumes that we live on the negative tension brane. It was shown by Shirouzu, Maeda, and Sasaki $[9]$ that the effective Einstein field equations on the negative tension brane involve a negative gravitational constant which means that gravity would be repulsive instead of attractive. However, they showed that one does recover the correct Einstein equations in the low energy limit on the positive tension brane. More recently, it has been shown $[10]$ that the problem with the negative tension brane may disappear if the extra dimension is stabilized by a radion field.

In the second scenario we live on the positive tension brane and the negative tension brane is moved off to infinity. Thus, in this scenario the extra dimension is infinite in extent and there is a single gravitational bound state confined to the brane that corresponds to the graviton. Even though the extra dimension is infinite, the effective gravitational interaction on the brane is that of a four-dimensional space-time with some very small corrections.

Randall and Sundrum have suggested $[8], [11]$ that four-dimensional gravity may be recovered in the presence of an infinite fifth dimension provided that we live on a domain wall embedded in anti-de Sitter space. Their linearized analysis showed that there is a massless bound state of the graviton associated with such a wall as well as a continuum of massive Kaluza-Klein modes. More recently, linearized analyses have examined the space-time produced by matter on the domain wall and concluded that it is in close agreement with four-dimensional Einstein gravity $[12]$. In Ref. $[4]$ a new static solution to the 5-D Einstein equations was presented in which space-time is flat on a 3-brane with positive tension provided that the bulk has an appropriate negative cosmological constant. Even if the fifth dimension is uncompactified, standard 4-D gravity (specifically, Newton’s force law) is reproduced on the brane. In contrast to the compactified case $[14]$, this follows because the near-brane geometry traps the massless graviton.

A natural extension of the Randall-Sundrum model is to include higher-order curvature invariant in the bulk action. This kind of terms arise in the anti-de Sitter/
conformal field theory correspondence as next to leading order corrections to the conformal field theory [15]. The Gauss-Bonnet combination of curvature invariants is quite relevant in five dimensions because it is the only invariant which leads to field equations of second order linear in the highest derivative thereby ensuring a unique solution[16]. The Einstein-Gauss-Bonnet equations projected onto the brane lead to a complicated Hubble equation in general (see [17, 13, 19, 20]). However, it reduces to a very simple equation $H^2 \approx \rho^\theta$ with $\theta = 1, 2, 2/3$ in limiting cases corresponding to general relativity, Randall-Sundrum and Gauss–Bonnet regimes, respectively.

An interesting model that incorporates modification of gravitational laws at large distances was proposed by Dvali, Gabadadze and Porrati in [21]. The model describes a brane with four-dimensional worldvolume, embedded into flat five-dimensional bulk. Ordinary matter is supposed to be localized on the brane, while gravity can propagate in the bulk. A crucial ingredient of the model is the induced Einstein-Hilbert action on the brane, because it allows under a non trivial mechanism, to recover the four-dimensional Einstein gravity at moderate scales [22].

In recent times a great amount of work has been invested in studying the inflationary model with a tachyon field. The tachyon field associated with unstable D-branes might be responsible for cosmological inflation in the early evolution of the Universe, due to tachyon condensation near the top of the effective scalar potential [23], which could also add some new form of cosmological dark matter at late times [24]. Cosmological implications of this rolling tachyon were studied by Gibbons [25] who showed that it is quite natural to consider scenarios in which inflation is driven by the rolling tachyon. Later, an extended version of the tachyon Lagrangian [26] was found and it was shown how this new version could be related to the generalized Chaplygin gas. In addition, using a phenomenological approach was described which conditions the potential and the tachyon mass must fulfill in order to provide enough $e$-folds for inflation and to have density perturbations of the correct magnitude [27]. In addition, the structural stability of tachyonic inflation against changes in the shape of the potential was carefully explored in Ref. [28].

One of the main elements within the traditional brane worlds is a quadratic correction in the energy-momentum tensor projected onto the brane. Basically, this means that the Einstein's equations on the brane turn out to be a highly complex nonlinear system. Then, it is a very hard task to find exact solutions for brane worlds. In order to gain a better insight about the physics underlying in the brane models, some authors have found particular cosmological solutions. Recently, for a tachyon field two solution have been reported for the inflationary era on the brane. Basically, it was found that compared to scalar field inflation, tachyonic inflation is not smooth [24]. Also, the power law solution was obtained for inverse power potential in the Dvali, Gabadadze and Porrati model with tachyon source [20]. Others aspects concerning the inflation with tachyon source, in the brane cosmologies, were studied. For example, in Ref. [31] the Chaplygin equation of state is used to model the tachyonic matter on the brane.

Maartens et al [32] studied brane-world universe with massive scalar field. They focus on the effects of the chaotic inflationary models when the kinetic energy is much less than the potential energy ($V \gg \dot{\phi}^2$). Later, the opposite case ($V \ll \dot{\phi}^2$) was explored in Ref. [33]; this situation leads to the stiff matter equation of state ($p_c = \rho_c$) for the scalar field. Finally, we want to mention that Hawkins and Lidsey [34] proposed different algorithms to find exacts solutions on brane models.

The aim of this paper is to develop a general method to integrate brane–world cosmological theories containing scalar and tachyon fields with a self-interaction potential. We assume that the energy density of the matter and scalar (or tachyon) fields are functions of the scale factor. We have found that it is possible to reconstruct the scalar (or tachyon) field potential for simple cosmological solutions. We also examine the high and low energy limits of the solutions. Some of the solutions are used to describe the cosmological evolution in the inflationary, radiation, matter and cosmic string eras.

II. THE THEORY

Let us consider a 5D space-time with a brane world located at $\Phi(X^a) = 0$, where $X^a$, $a = 0, 1, 2, 3, 4$ are five-dimensional coordinates. The effective action in five-dimensional manifold is given by (see [4, 35, 36])

$$S = \int d^5 X \sqrt{-g_5} \left( \frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{\Phi=0} d^4 x \sqrt{-g_4} \left( \frac{1}{k_5^2} K_{ab}^\pm - \lambda + L_{\text{matter}} \right),$$

(1)

with $k_5^2 = 8\pi G_5$ is the five-dimensional gravitational coupling constant and where $x^a$ are the induced four-dimensional brane-world coordinates, so $\mu$ takes values $\{0, \ldots, 3\}$. $R_5$ is the 5D intrinsic curvature in the bulk, $K_{ab}^\pm$ is the intrinsic curvature on either side of the brane and $\lambda$ is the brane tension. Here, $L_{\text{matter}}$ corresponds to the Lagrangian for the matter fields.

On the five-dimensional space-time (usually named as the bulk), with the negative vacuum energy $\Lambda_5$ and the brane energy momentum as the source of the gravitational field, the Einstein field equations are given by

$$G_{ab} = k_5^2 T_{ab}, \quad T_{ab} = -\Lambda_5 g_{ab} + \delta(Y) \left( -\lambda + T_{ab}^{\text{matter}} \right).$$

(2)

In this space-time the brane is a fixed point of the $Z_2$ symmetry. In the following Latin indices run from 0 to 4 while Greek indices take the values $\{0, \ldots, 3\}$. Assuming a metric of the form $ds^2 = (u_a u_b + h_{ab})dx^a dx^b$,
with \( n_a dx^a = d\chi \) the unit normal to the \( \chi = \text{cte} \) hypersurfaces and \( h_{ab} \), the induced metric on \( \chi = \text{cte} \), the effective four-dimensional gravitational equations on the brane (which can be deduced from the Gauss-Codazzi equations) take the form \([8]\):

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_2^2 T_{\mu\nu} + k_4^4 S_{\mu\nu} - E_{\mu\nu}. \tag{3}
\]

In the above equation \( S_{\mu\nu} \) is the local quadratic energy-momentum correction

\[
S_{\mu\nu} = \frac{1}{12} \eta T_{\mu\nu} - \frac{1}{4} T^a_{\mu} T_{\nu a} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2), \tag{4}
\]

and \( E_{\mu\nu} \) is the nonlocal effect from the bulk free gravitational field transmitted projection of the bulk Weyl tensor \( C_{abde} \).

\[
E_{ab} = C_{aebd} n^n, E_{ab} \to E_{\mu\nu} \delta_a^\mu \delta_b^\nu \quad \text{as} \quad \chi \to 0. \tag{5}
\]

The four-dimensional cosmological constant, \( \Lambda \), and the coupling constant \( k_4 \), are given by

\[
\Lambda = \frac{k_2^2}{2} \left( \Lambda_5 + \frac{k_5^2}{6} \right), \quad k_4^2 = \frac{k_5^4}{6}. \tag{6}
\]

The Einstein equation in the bulk (Codazzi equation), also implies the conservation of the energy-momentum tensor of the matter on the brane: \( D_a T^a_{\mu\nu} = 0 \). Moreover, the contracted Bianchi identities on the brane imply that the projected Weyl tensor should obey the following constraint: \( D_a E^a_{\mu\nu} = k_2^2 D_{\mu} S^a_{\nu} \). Finally, Eq. (3) and the latter above give the complete set of field equations for the brane gravitational field.

For any non dissipate matter field the general form of the brane energy-momentum tensor can be covariantly written as

\[
T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p h_{\mu\nu}. \tag{7}
\]

The decomposition is irreducible for any chosen four vector \( u^\mu \). Here \( \rho \) and \( p \) are the energy density and the isotropic pressure, and \( h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu} \) projects orthogonal to \( u^\mu \). The symmetric properties of \( E_{\mu\nu} \) imply that, in general, we can decompose it irreducibly with respect to a chosen four-velocity field \( u^\mu \) as

\[
E_{\mu\nu} = -k^4 \left[ U \left( u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) \right], \tag{8}
\]

where \( k = k_5/k_4 \). In Eq. (3) \( U \) is the effective nonlocal energy density on the brane arising from free gravitational field in the bulk.

III. BRANE WORLD-MODEL

We consider a brane universe with the induced metric given by the Friedmann-Roberston-Walker (FRW) spacetime. In the following, we shall explore the evolution of a cosmological brane filled with a tachyon field \( \varphi \) or a quintessence field \( \phi \) and a perfect fluid. These fields have an energy-momentum tensor on the brane given by \( T_{\mu\nu} = p_a u_{\mu} u_{\nu} + p_h h_{\mu\nu} \), where the label \( a \) denotes the tachyon or quintessence respectively. For a homogeneous tachyon field the energy density and pressure are given by

\[
\rho_\varphi = \frac{U(\varphi)}{\sqrt{1 - \varphi^2}}, \quad p_\varphi = -U(\varphi) \sqrt{1 - \varphi^2}. \tag{9}
\]

while for the quintessence the density energy and pressure take the following form:

\[
\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{10}
\]

where \( U(\varphi) \) and \( V(\phi) \) are the potentials. We suppose that the barotropic components, radiation, dust and cosmic string have equation of states, \( p_m = (\gamma_m - 1) \rho_m \) where the barotropic indices are \( \gamma_m = 4/3, 1 \) and 2/3 respectively. They exchange energy with tachyon or quintessence fields only gravitationally, so that, each one of the matter and field components satisfies separate equations of conservation. Under this scheme the dynamics on the brane is completely determined by the following set of equations

\[
3 \left( H^2 + \frac{k}{a^2} \right) = \Lambda + \rho + \frac{3}{\lambda^2} (\rho^2 + 12 U), \tag{11}
\]

\[
\dot{\rho}_a + 3 H (\rho_a + p_a) = 0, \tag{12}
\]

\[
\dot{\rho}_m + 3 H \rho_m \gamma_m = 0, \tag{13}
\]

\[
\frac{36}{\lambda^2} U = -\frac{1}{a^4}, \tag{14}
\]

where \( k = -1, 0, 1 \) is the spatial curvature, and \( \rho = \rho_m + \rho_a \) is the total energy density. In the absence of anisotropic stress, the nonlocal energy density \( U \), containing the effects from the bulk, takes the form of dark radiation with its behavior given by Eq. (11). As a final remark, one can see that the evolution of the early universe can be separated into eras. In the high energy regime, \( 3 \rho^2/\lambda^2 \gg \rho \), the quadratic density term in Eq. (11) becomes dominant and we get an unconventional expansion law for the universe \( H^2 \approx \rho^2/\lambda^2 \). However, in the low energy regime, \( 3 \rho^2/\lambda^2 \ll \rho \), the linear density term in Eq. (11) dominates and we recover the standard Friedmann equation in four dimension \( 3 H^2 \approx \rho \). Below, we develop a simple algorithm for tachyon and quintessence fields that allows us to describe the evolution of the Universe from radiation to cosmic string eras. In addition, it will be used to find exact solutions for the modified Friedmann equation on the brane.

A. Tachyon case (TC)

Because of the recent interest of considering a tachyon field component in cosmology we investigate a brane-world universe where the field localized on the brane is
given by a single tachyon [31, 37, 38]. We wish to extract information from the modified Einstein equations assuming that the energy density depends only on the scalar factor \( \rho_{\phi} = \rho_{\phi}(a) \). Inserting it in the conservation equation (12) we obtain

\[
3\dot{\phi}^2 = -a'\rho_{\phi}, \tag{15}
\]

where the prime denotes a differentiation with respect to the scale factor. Combining Eqs. (9), (11) and (15) we may write the potential and the tachyon field

\[
U(\phi) = \left[ \rho_{\phi}^2 + \frac{a}{3} \rho_{\phi}' \rho_{\phi} \right]^{1/2}, \tag{16}
\]

as functions of the scale factor. Integrating the last equation leads to \( \varphi(a) \) and inverting it gives \( a(\varphi) \). Finally, by using (16) it follows the potential \( U(a(\varphi)) \). Thus the above procedure induces a model where the tachyon field is driven by this potential with an exact scale factor produced by the desired energy density \( \rho_{\phi}(a) \).

**B. Quintessence case (QC)**

Following the same steps above we write the energy density of the quintessence scalar field as a function of scale factor \( \rho_{\phi} = \rho_{\phi}(a) \). Inserting it in the conservation equation (12), we obtain

\[
3\dot{\phi}^2 = -a'\rho_{\phi}. \tag{18}
\]

From this equation we are able to write the potential energy as a function of the scale factor,

\[
V(a) = \rho_{\phi} + \frac{a}{6} \rho_{\phi}' \rho_{\phi}. \tag{19}
\]

Employing Eqs. (11) and (18) we find the quintessence field \( \phi(a) \)

\[
\phi = \pm \int \left[ \frac{-\rho_{\phi}' \rho_{\phi}}{a(\Lambda - \frac{3}{4} \rho_{\phi} a^2 + \rho + \frac{1}{2} \dot{\phi}^2 - \frac{\phi^2}{a^2})} \right]^{1/2} da. \tag{20}
\]

Replacing \( \rho_{\phi}(a) \) in the last equation, it gives \( \phi(a) \). Inverting it gives \( a(\phi) \) and by using (19), it follows \( V(a(\phi)) \). Thus the procedure determines \( V(\phi) \) and defines a model with an exact solution on the brane with the desired energy density \( \rho_{\phi}(a) \). In the next section we shall use the algorithm for tachyon and quintessence field to depict different cosmological eras.

**IV. DESCRIPTION OF THE RADIATION, MATTER AND COSMIC STRING ERAS**

We investigate a spatially flat FRW universe with no cosmological constant and nonvanishing dark radiation energy density, filled with a tachyon or quintessence field and a barotropic component. In what follows, we shall examine the cosmological evolution of the Universe from the radiation to the cosmic string eras by using the model developed in the Sect. III. In this case the Friedmann equation reads

\[
3H^2 = \rho + \frac{3}{\Lambda} \rho_{\phi}^2 - \frac{1}{a^4} \tag{21}
\]

with \( \rho = \rho_m + \rho_a \). To solve the nonlinear Eqs. (12), (13) and (21) we assume that both the energy densities of the tachyon and quintessence fields have an inverse power law dependence with the scale factor \( \rho_{\phi} = \rho_{\phi}(a^{-q}) \), \( \rho_{\phi} = \rho_{\phi}(a^{-q}) \), and the barotropic index satisfies the following inequalities \( s, q < 3\gamma_m \). The nonlocal effects from the bulk will be important in the first radiation era and neglected in the remaining ones.

**A. Radiation dominated era**

In the TC and QC, we split the radiation dominated era, with \( \rho_m = \rho_r = \rho_r/a^4 \), into two stages corresponding to the high and low energy regimes. These regimes are separated by the critical energy density \( \rho_c = \lambda^2/3 \) where the linear and quadratic terms in the Friedmann equation are comparable. For the TC, in the higher energy regime, \( \rho \gg \rho_c \), the tachyon field and the potential have the following behaviors

\[
\varphi_r \approx \frac{\lambda}{4\rho_r^1/3} a^{4/3}, \quad U_r \approx U_{r0} \varphi^{-s/4}. \tag{22}
\]

For the QC, we find that the quintessence field is given by

\[
\phi_r \approx \frac{2\lambda}{(8-q)\rho_r^0} \left[ \frac{q\rho_{\phi}}{3} a^{-q/2} \right], \tag{23}
\]

and the potential reads

\[
V_r \approx V_{r0} \phi^{-2q/(8-q)}. \tag{24}
\]

In this way we recover the results obtained in Ref. [39], and the proposal of a tracking potential arises as a consequence of the domination assumptions. As long as the Universe expands the energy density decreases and there exists a critical cosmological time where the quadratic and linear terms become equal. After that the quadratic term becomes subdominant, it begins the low regime \( \rho < \rho_c \), and the conventional cosmology is recovered. In Ref. [40] it was shown that nucleosynthesis is not restricted to the conventional radiation domination era and could have begun earlier, during the late
brane radiation domination era. Then, assuming that $T_c > T_{NS} \approx 1$ MeV we recover the results obtained in Ref. [39, 41].

When the Universe enters in the second radiation era the effects of the brane becomes negligible. In this cosmological stage, for the TC the tachyon field and the potential become

$$\phi_r \approx \frac{1}{2} \sqrt{s \rho_0} a^2, \quad U_r \approx U_{r0} \phi^{-s/2},$$

(25)

whereas for the QC, the quintessence field and the potential energy exhibit the following behaviors:

$$\phi_r \approx \frac{2}{q - 4} \sqrt{\frac{q r a^{2-2q/2}}{r a^{2-q/2}}, \quad V_r \approx V_{r0} \phi^{-2q/(4-q)},}$$

(26)

where $r = \rho_0 / \rho_{00}$. Solving Eqs. (13) and (21) for radiation eras with $\gamma_n = 4/3$ and $\rho_m = \rho_r = \rho_0 / a^4$, we obtain the scale factor

$$a^4 \approx \frac{4(\rho_0 - \Gamma_2^2)}{3} t^2 + \frac{4 \rho_0}{\lambda} t.$$  

(27)

Because of the contribution of the nonlocal dark radiation $-\Gamma_2^2 / a^4$ the coefficient of the quadratic term has not definite sign showing the relevance of the bulk. Let us examine the possible behaviors of the brane-world universe dominated by radiation. For $\rho_0 > \Gamma_2^2$ the singular solution [21] represents a universe which interpolates between a $t^{1/4}$ in the high energy regime and $a \propto t^{1/2}$ in the low energy regime. However, in the case $\rho_0 < \Gamma_2^2$ the nonlocal dark radiation component makes it possible for the Universe to collapse in a big crunch at $t_c = 3 \rho_0 / \lambda (\Gamma_2^2 - \rho_0)$. The Universe has a finite time span and the scale factor has a maximum at $a_{max} = 3 \rho_0 / \lambda^2 (\Gamma_2^2 - \rho_0)$, where the total energy density vanishes and the acceleration $a$ is negative. In the particular case $\rho_0 = \Gamma_2^2$, the scale factor $a \approx (\frac{3}{4 \lambda})^{1/4} t^{1/4}$ has an initial singularity.

**B. Cold dark mater (CDM) dominated era**

As soon as the radiation thermalizes and equilibrates with the baryonic matter as well as with the nonrelativistic nonbaryonic cold dark matter (whose energy-momentum tensor is dustlike in the first approximation $0 < \rho_{DM} \ll \rho_{CDM}$) the Universe is dominated by all these contributions. Actually, we have included both baryonic and dark matter energy densities in the same term, $\rho_{CDM} = \rho_{CDM0} / a^3$. Assuming that $s, q < 3$, we get in the TC, the following expressions for the tachyon field and its potential energy

$$\phi_{CDM} \approx \frac{2}{3} \sqrt{s \rho_{CDM0}} a^{3/2}, \quad U_{CDM} \approx U_{CDM0} \phi^{-2s/3},$$

(28)

while for the QC, we have

$$\phi_{CDM} \approx \frac{2}{q - 3} \sqrt{\frac{q r a^{3-q/2}}{r a^{2-q/2}}, \quad V_{CDM} \approx V_{CDM0} \phi^{-2q/(3-q)},}$$

(29)

with $r = \rho_{CDM0} / \rho_{00}$.

Typically, in this CDM era the effects of the brane diminish and the contribution of the nonlocal dark radiation term $-\Gamma_2^2 / a^4$ may be neglected. Hence, by solving Eqs. (13) one finds that the approximate scale factor is given by $a \approx (3 \rho_{CDM0} / 4)^{1/3} t^{2/3}$ and the Universe behaves as if it was dominated by matter.

**C. Cosmic string dominated era**

When the dark matter dominated era ends and before the Universe begins to feel the dark energy component, which is the probable responsible for its currents acceleration, the Universe is dominated by cosmic string networks. This intermediate regime, that is, the transition between the CDM regime with $\rho_{CDM} = \rho_{CDM0} a^{-3}$ (non accelerated stage) and the dark energy era (accelerated phase), is properly described by the cosmological cosmic string energy density $\rho_{st} = \rho_{st0} a^{-2}$. Then, in the cosmic string era, the tachyon field and potential in the TC take the following form

$$\phi_{st} \approx \sqrt{s \rho_{st0}}, \quad U_{st} \approx U_{s0} \phi^{-s},$$

(30)

and

$$\phi_{st} \approx \frac{2}{q - 2} \sqrt{\frac{q r a^{2-q}}{r a^{2-q}}}, \quad V_{st} \approx V_{s0} \phi^{-2q/(2-q)},$$

(31)

for the QC with $r = \rho_{st0} / \rho_{00}$.

In this era, discarding the effects of the brane and the contribution of the nonlocal dark radiation term, the cosmic string networks density makes the scale factor go linear with the cosmic time and has the form $a \approx (\rho_{st0} / 3)^{1/2} t$.

In conclusion, we have found that in every stage of the Universe, the corresponding tachyon and quintessence fields are driven by inverse power law potentials. It is important to remark that these kinds of potentials are very interesting for several reasons. For instance, they arise in supersymmetric condensate models of QCD [42] and can in principle act as a source of quintessence for a brane-world [39].

**V. GENERATING EXACTS BRANE-WORLDS SOLUTIONS**

**A. Quintessence solutions**

Let us conceive the energy density of the quintessence field $\rho_{\phi}$ as a mixture of vacuum energy density $\rho_v$, energy
with \( \alpha > 0 \), plus a cosmic string network density interpreted as a nonrelativistic matter whose barotropic index is \( \gamma = 2/3 \), \( \rho_s = \rho_{\phi 0}/a^2 \) \( \{3 \} \), with \( \rho_{\phi 0} > 0 \), so

\[
\rho_{\phi} = \alpha + \frac{\rho_{\phi 0}}{a^2}.
\]

With the idea of relating our finding with the results obtained in \( \{44\} \), we shall be restricted from now on to the case of \( \rho_m = 0 \).

1. \( \Gamma^2 = \frac{3\nu^2}{1 + \nu^2} \), \( \rho_{\phi 0} \left( 1 + \frac{6\alpha}{\lambda^2} \right) > 3k \), and \( \Lambda > -\alpha \left( 1 + \frac{6\alpha}{\lambda^2} \right) \).

Integrating Eq. \( \{11\} \), the scale factor becomes

\[
a(t) = \frac{\sqrt{\nu}}{\sqrt{3} \omega} \sinh \omega t,
\]

where \( \nu = \rho_{\phi 0} \left( 1 + \frac{6\alpha}{\lambda^2} \right) - 3k \). It represents an expanding singular universe with a final de Sitter scenario. By following Ref. \( \{44\} \) we work with the constants

\[
\omega^2 = \frac{1}{3} \left[ \Lambda + \alpha \left( 1 + 3 \frac{\alpha}{\lambda^2} \right) \right], \quad \Gamma^2 = \frac{27}{p^2} \left( k + \frac{\nu}{3} \right)^2, \quad p^2 = \lambda^2 + 36 \left( \omega^2 - \frac{\Lambda}{3} \right).
\]

By combining Eqs. \( \{18\}, \{32\}, \{33\} \) and integrating, we have the field as a function of the cosmic time

\[
\Delta \phi = \pm \sqrt{\frac{2\lambda}{p}} \left( 1 + \frac{3k}{\nu} \right) \ln \left| \tanh \frac{\omega t}{2} \right|.
\]

Now, by making the composition of the scale factor \( \{33\} \) with the scalar field \( \{36\} \), we obtain \( a = a(\phi) \). Then, from Eqs. \( \{19\} \) and \( \{32\} \), we get the potential

\[
V(\phi) = \frac{\lambda^2}{6} \left( \frac{p}{\lambda} - 1 \right) + \frac{2\lambda\omega^2}{p} \left( 1 + \frac{3k}{\nu} \right) \times \sinh^2 \left( \sqrt{\frac{p}{2\lambda(1 + \frac{k}{\nu})}} \Delta \phi \right).
\]

At the initial singularity the scalar field and the potential diverge but in the limit \( t \to \infty \) the scalar field vanishes and the potential reaches its minimum value. This means that the exact solution is stable. For \( k = \Lambda = 0 \) and \( \nu = 3\omega^2 \) the solution mentioned above coincides with the solution reported in Ref. \( \{44\} \).

2. \( \Gamma^2 = \frac{3\nu^2}{1 + \nu^2} \), \( \rho_{\phi 0} \left( 1 + \frac{6\alpha}{\lambda^2} \right) > 3k \), and \( \Lambda < -\alpha \left( 1 + \frac{6\alpha}{\lambda^2} \right) \).

Now, the solution of Eq. \( \{11\} \) represents a singular universe with a finite time span

\[
a(t) = \frac{\sqrt{\nu}}{\sqrt{3} \omega} \sin \omega t,
\]

which begins at \( t = 0 \) and ends in a “big-crunch” at \( t_{BC} = \pi/\omega \). Also, from \( \{18\}, \{22\} \) and \( \{33\} \), we get the dependence of the scalar field with the cosmic time and the potential \( \{11\} \) as a function of the scalar field, namely,

\[
\Delta \phi = \pm \sqrt{\frac{2\lambda}{p}} \left( 1 + \frac{3k}{\nu} \right) \ln |\tanh \frac{\omega t}{2}|, \quad (39)
\]

\[
V(\phi) = \frac{\lambda^2}{6} \left( \frac{p}{\lambda} - 1 \right) + \frac{2\lambda\omega^2}{p} \left( 1 + \frac{3k}{\nu} \right) \times \sinh^2 \left( \sqrt{\frac{p}{2\lambda(1 + \frac{k}{\nu})}} \Delta \phi \right). \quad (40)
\]

Note that the potential takes its minimum value at the maximum value of the scale factor and diverges at the initial and final singularities.

### B. Tachyon solutions

Using the method introduced in Sec. III.A for the energy density \( \rho_\phi = \rho_{\phi 0} a^{-2} \), we shall find new solutions for the tachyonic field localized on the brane.

1. \( \Gamma^2 = \frac{3\nu^2}{1 + \nu^2} \), \( \rho_{\phi 0} > 3k \), and \( \Lambda > 0 \).

By solving Eqs. \( \{11\} \) and \( \{17\} \), we obtain the scale factor and the tachyon field

\[
a(t) = \sqrt{\frac{\nu}{\Lambda}} \sinh \sqrt{\frac{\lambda}{3}} t, \quad (41)
\]

\[
\Delta \varphi = \sqrt{\frac{2}{\Lambda}} \arcsinh \sqrt{\frac{\Lambda}{\nu}} \varphi_0. \quad (42)
\]

where \( \nu = \rho_{\phi 0} - 3k \). By considering \( a = a(\varphi) \) and \( \rho_\phi \) in Eq. \( \{10\} \), we find the potential \( U(\varphi) \) for the tachyon field

\[
U(\varphi) = \frac{\Lambda \rho_{\phi 0}}{\sqrt{3} \nu \sinh^2 m \Delta \varphi} \quad (43)
\]

where \( m = (\Lambda/2)^{1/2} \) and \( \Delta \varphi = \varphi - \varphi_0 \). Let us examine the behavior of the scale factor and the potential energy. At the initial singularity, the potential energy of the tachyon field blows up but for large cosmological time, the scale factor has a final de Sitter stage \( a \propto e^{\sqrt{\frac{\lambda}{3}} t} \) and \( U(\varphi) \) becomes an exponential potential \( U \approx 4\Lambda e^{-\sqrt{\lambda \Delta \varphi}/\sqrt{3} \nu} \) so the solution is stable.

2. \( \Gamma^2 = \frac{3\nu^2}{1 + \nu^2} \), \( \rho_{\phi 0} > 3k \), and \( \Lambda < 0 \).

After using Eqs. \( \{11\} \) and \( \{17\} \), the scale factor and tachyon field are given by
By inserting Eq. (45) into (16), we get the potential for the tachyon field

$$U(\varphi) = \frac{-\Lambda \rho_0 \varphi}{\sqrt{3} \nu \sin^2(-\lambda/2)^{1/2} \Delta \varphi}.$$  (46)

In this case, the Universe has a finite time span and the tachyon field vanishes at the initial and final singularities while the potential diverges as $U \approx a^{-2}$. With the help of Eq. (12) and the tachyon energy $\rho = \rho_\varphi a^{-2}$ we obtain the relation $\rho + \rho_\varphi = \frac{\Lambda}{\nu} \rho_0 \geq 0$ which shows that the tachyon solutions satisfy the weak energy condition ($\rho \geq 0$, $\rho + p \geq 0$) and null energy condition ($\rho + p \geq 0$) conditions. As a final remark, we desire to stress that a very large set of solutions can be found extending the previous proposal on the energy density. In a forthcoming paper, starting from the energy density $\rho(a) = \alpha + \rho_0 a^{-n}$ with $n > 0$, we will find several types of solutions of physical interest.

VI. Conclusion

To sum up, we have presented a new algorithm for solving the brane-world field equations on FRW backgrounds with different types of sources. Basically, we work with a single tachyon, a classical minimally coupled scalar field and barotropic matter which are confined to the brane, embedded in five-dimensional Einstein gravity. Furthermore, we find it is possible to mimic with a tachyon (or quintessence) field the different cosmological stages, starting from a first radiation era to a universe dominated by a cosmic string networks era. In general, we have found that the potential for the tachyon (or quintessence) field is given by an inverse power law $U = U_0 \varphi^{-n}$. Also, our procedure seems to match quite well with the phenomenology of brane-world quintessence analyzed by Maeda [39] and Huey and Lidsey [41]. Additionally, we have found that due to the nonlocal dark radiation term $-\Gamma/\alpha^4$ the Universe could exhibit a big crunch. Besides, with this algorithm we reproduce and generalize the solutions found in [44], [45]. Later, we extend the previous analysis and report exact solutions of the full brane field equations with tachyon and/or quintessence sources.

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