Local resolution of singularities in foliated spaces

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Abstract

Let \( M \) be an analytic manifold over \( \mathbb{R} \) or \( \mathbb{C} \), \( \theta \) a monomial singular distribution and \( \mathcal{I} \) a coherent ideal sheaf defined on \( M \). We prove the existence of a local resolution of singularities of \( \mathcal{I} \) that preserves the monomiality of the singular distribution \( \theta \).

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1 Introduction

A foliated analytic manifold is the triple \((M, \theta, E)\) and a foliated ideal sheaf is a quadruple \((M, \theta, \mathcal{I}, E)\) where: \( M \) is a smooth analytic manifold of dimension \( n \) over a field \( \mathbb{K} \) (where \( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \)); \( \mathcal{I} \) is a coherent and everywhere non-zero ideal sheaf of \( M \); \( E \) is an ordered collection \( E = (E^{(1)}, \ldots, E^{(l)}) \), where each \( E^{(i)} \) is a smooth divisor on \( M \) such that \( \sum E^{(i)} \) is a reduced divisor with simple normal crossings; \( \theta \) is an involutive singular distribution defined over \( M \) and everywhere tangent to \( E \). In the same notation, a foliated analytic manifold is
the triple \((M, \theta, E)\).

The main objective of this work is to find a local resolution of singularities for \(I\) that preserves the class of singularities of \(\theta\). This is a generalization of the results in [Be]. In order to be precise and set notation, we briefly recall some basic notions of singular distributions and local resolution of singularities:

- **Singular distributions** (we follow [BaBo]): Let \(\text{Der}_M\) denote the sheaf of analytic vector fields over \(M\), i.e. the sheaf of analytic sections of \(TM\).

  An involutive singular distribution \(\theta\) is a coherent sub-sheaf of \(\text{Der}_M\) such that for each point \(p\) in \(M\), the stalk \(\theta_p := \theta_p \mathcal{O}_p\) is closed under the Lie bracket operation.

  Consider the quotient sheaf \(Q = \text{Der}_M / \theta\). The singular set of \(\theta\) is defined by the closed analytic subset \(S(\theta) = \{ p \in M : Q_p \text{ is not a free } \mathcal{O}_p \text{ module} \}\).

  A singular distribution \(\theta\) is called regular if \(S(\theta) = \emptyset\). On \(M \setminus S(\theta)\) there exists an unique analytic subbundle \(L\) of \(TM|_{M \setminus S(\theta)}\) such that \(\theta\) is the sheaf of analytic sections of \(L\). We assume that the dimension of the \(k\) vector space \(L_p\) is the same for all points \(p\) in \(M\) (this always holds if \(M\) is connected). It is called the leaf dimension of \(\theta\) and denoted by \(d\). In this case \(\theta\) is called an involutive \(d\)-singular distribution.

- **Local resolution of singularities**: An admissible local blowing-up \(\sigma : (\tilde{M}, \tilde{\theta}, \tilde{E}) \rightarrow (M, \theta, E)\) is the composition of an admissible blowing \(\sigma : (\bar{M}, \bar{\theta}, \bar{E}) \rightarrow (M, \theta, E)\) with an open immersion \(\text{Im} : (\tilde{M}, \tilde{\theta}, \tilde{E}) \rightarrow (\bar{M}, \bar{\theta}, \bar{E})\), i.e \(\tau = \sigma \circ \text{Im}\). In this case, the total transform of the ideal sheaf \(\mathcal{I}\) is the ideal sheaf \(\mathcal{I}^* := \mathcal{I} \mathcal{O}_{\tilde{M}}\).

Given an admissible local blowing-up \(\tau : (\tilde{M}, \tilde{\theta}, \tilde{E}) \rightarrow (M, \theta, E)\) we say that it has order one in respect to \(\mathcal{I}\) if the center \(\mathcal{C}\) of blowing-up is contained in the support of \(\mathcal{I}\) (see definition 3.65 of [K] for details). In this case, the controlled transform of the ideal sheaf \(\mathcal{I}\) is the ideal sheaf \(\mathcal{I}^c := \mathcal{O}(F)\mathcal{I}^*\), where \(F\) stands for the exceptional divisor of the blowing-up (see subsection 3.58 of [K]). An admissible local blowing-up of a foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) is the mapping:

\[
\tau : (\tilde{M}, \tilde{\theta}, \tilde{\mathcal{I}}, \tilde{E}) \rightarrow (M, \theta, \mathcal{I}, E)
\]

where the ideal sheaf \(\tilde{\mathcal{I}}\) is the controlled transform of \(\mathcal{I}\) whenever the blowing-up is of order one, and the total transform in every other case. We extend this notion to a sequence of local blowings-up in the obvious
manner, i.e. a sequence of local admissible blowings-up is a sequence:

\[(M_r, \theta_r, I_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_1} (M_0, \theta_0, I_0, E_0)\]

where each local blowing-up \(\tau_{i+1}\) is admissible in respect to \(E_i\). A local resolution of an ideal sheaf \(I\) over a point \(p\) of \(M\) is a finite collection of morphisms \(\tau_i : (M_i, \theta_i, I_i, E_i) \rightarrow (M, \theta, I, E)\) such that:

1. The morphism \(\tau_i\) is a finite composition of admissible local blowings-ups.
2. In each variety \(M_i\), there exists a compact set \(K_i \subset M_i\) such that the union of their images \(\bigcup \tau_i(K_i)\) is a compact neighborhood of \(p\);
3. The ideal sheaf \(I_i\) is the structural ring. In particular, this means that the total transform \(\tau_i^*I\) is a principal ideal sheaf with support contained in \(E_i\).

We remark that the existence of a local resolution of \(I\) is weaker than the notion of global resolution of singularities, which is first proved by Hironaka in [H] (modern proofs can be find in, e.g. [BRM], [K], [V1], [W]).

Now, we start our study by defining which class of singularities of \(\theta\) we want to preserve. Let us remark that equiresolution is a very special property which is not valid in general. Thus, the class of regular distributions is too restrictive for our objectives. This leads us to introduce the following class of singular distributions:

- \(R\)-monomial distributions are defined in section 2.1 and are deeply related with the notion of resolution in families and monomialization of maps (their leaves correspond to the level curves of a monomial map - see Example 2.4).

It is worth remarking that \(\mathbb{Z}\)-monomial distributions are possibly the smallest class of distributions where we can resolve singularities of \(I\) preserving the class of the distribution (in dimension 1, this is proved in [Be]). In particular, this class of distributions is a sub-class of the Log-Canonical distributions introduced by McQuillan in [Mc]. Our main result provides a local resolution of singularities that preserves the \(R\)-monomiality of the singular distribution. More precisely, the main Theorem of this work is:

**Theorem 1.1 (Main Theorem).** Let \((M, \theta, I, E)\) be an analytic d-foliated ideal sheaf such that \(\theta\) is \(R\)-monomial. Then, for every point \(p\) in \(M\), there exists a finite collection of morphisms \(\tau_i : (M_i, \theta_i, I_i, E_i) \rightarrow (M, \theta, I, E)\) such that:

- The morphism \(\tau_i\) is a finite composition of admissible local blowings-ups.
- The singular distribution \(\theta_i\) is \(R\)-monomial for all \(i\);
- In each variety \(M_i\), there exists a compact set \(K_i \subset M_i\) such that the union of their images \(\bigcup \tau_i(K_i)\) is a compact neighborhood of \(p\);
- The ideal sheaf \(I_i\) is the structural ring. In particular, this means that the total transform \(\tau_i^*I\) is a principal ideal sheaf with support contained in \(E_i\).
Remark 1.2. In [Be], it is proved that there exists a global resolution of the ideal sheaf $I$ whenever the leaf dimension of $\theta$ is one and $\theta$ is $R$-monomial or Log-canonical. Furthermore, the above result is generalization of Theorem 6.1.1 of [Be], where a local resolution is proved under the additional hypothesis that $\theta$ has leaf dimension two.

The originality of this result comes from the fact that the searched resolution can not be given by the usual Hironaka’s algorithm. In particular, the usual invariant (the order of $I$) is not enough to control the singular distribution. So, we introduce an invariant which we call tg-order (abbreviation for tangency order - see section 3.1 for the precise definition). This invariant measures the order of tangency between an ideal sheaf $I$ and a singular distribution $\theta$, even if the objects are singular. Blowing-up the maximal order locus of this invariant guarantees some necessary “compatibility” conditions between each blowing-up and the transforms of the singular distribution. These “compatibility” conditions are formalized by the notion of $\theta$-admissible centers (see section 2.2 for more details).

Technically, the difficulty is that the tg-order has no surface of maximal contact associated to it. This comes as no surprise since we deal with foliations, where surfaces of maximal contacts are rare, and it explains the local nature of Theorem 1.1. Following an idea of [C2], we blow-up centers which are not necessarily contained in the singular locus of $I$ in order to emulate the existence of a surface of maximal contact. This allows us to obtain good formulas for the transforms of $I$ under the expense of choosing one special direction at each step. As a consequence, the algorithm is only local, although one could probably globalize it for low dimensions, just as in [C2] (when $dim M \leq 3$).

The manuscript is divided in four sections counting the introduction. In the second section we introduce the main objects used in the rest of the manuscript, including the definitions and techniques of [Be] that are going to be necessary. In the third section we introduce the tg-order and we present the main idea of the proof: in particular, we prove Theorem 1.1 assuming the technical Proposition 3.3. The forth and last section is completely devoted to prove Proposition 3.3.

Finally, it is worth remarking that this manuscript is not completely self-contained: Proposition 3.1, Lemma 2.8 and Theorem 2.8 are proved in [Be] and we do not reproduce their proofs in this first version. Furthermore, we are also going to leave the discussion of applications of this result (such as monomialization of first integrals) for a subsequent manuscript.

Remark 1.3. Although all proofs and results of this manuscript are set in the analytic category, they should also work for the algebraic category. The proofs should follow the same arguments, but with different local technicalities. We stress that we have not verified in detail these adaptations.
2 Main Objects

2.1 Monomial Distributions

The class of Monomial Singular Distributions is a sub-class of the Log-Canonical Singular Distributions (see [Mc] for a definition). Its motivation lies in the study of families of ideals or vector-fields and in the study of monomialization of maps.

Definition 2.1 (Monomial singular distribution). Given a foliated manifold \((M, \theta, E)\) and a ring \(R\) such that \(\mathbb{Z} \subset R \subset \mathbb{K}\), we say that the singular distribution \(\theta\) is \(R\)-monomial at a point \(p\) if there exists set of generators \(\{\partial_1, \ldots, \partial_d\}\) of \(\theta, \mathcal{O}_p\) and a coordinate system \((u, w) = (u_1, \ldots, u_k, w_{k+1}, \ldots w_m)\) centered at \(p\) such that:

i) The exceptional divisor \(E\) is locally equal to \(\{u_1 \cdots u_l = 0\}\) for some \(l \leq k\);

ii) The singular distribution \(\theta\) is everywhere tangent to \(E\), i.e., \(\theta \subset \text{Der}_M(-\log E)\);

iii) Apart from re-indexing, the vector-fields \(\partial_i\) are of the form:

\[
\partial_i = \partial w_{m+1-i} \text{ for } i \text{ from } 1 \text{ to } m - k, \quad \partial_i = \sum_{j=1}^{k} \alpha_{i,j} u_j \partial u_j \text{ (where } \alpha_{i,j} \in R) \text{ otherwise.}
\]

iv) If \(\omega \subset \text{Der}_M(-\log E)\) is a \(d\)-singular distribution such that \(\theta \subset \omega\) then \(\theta = \omega\).

In this case, we say that \((u, w)\) is a monomial coordinate system and that \(\{\partial_1, \ldots, \partial_d\}\) is a monomial basis of \(\theta, \mathcal{O}_p\).

Remark 2.2 (Geometrical Interpretation of (iv)). Assuming conditions [i – iii] above, it is clear that Property [iv] implies that the codimension one part of the singularity set of \(\theta\) is contained in \(E\).

Notation 2.3 (An special monomial coordinate system). In what follows, we sometimes need to emphasis one of the coordinates in \(w\). To that end, we will denote by \((u, v, w)\) a monomial coordinate system and the vector field \(\partial_v\) is contained in \(\theta, \mathcal{O}_p\).

The Example\Lemma below shows an important feature of \(R\)-monomial singular distribution:

Lemma 2.4 (Monomial First Integrals). Given a foliated manifold \((M, \theta, E)\), the singular distribution \(\theta\) is monomial if, and only if, for any monomial coordinate system \((u, w) = (u_1, \ldots, u_k, w_{k+1}, \ldots w_m)\) centered at \(p\) there exists \(m - d\) monomials \(u^B = (u^{\beta_1}, \ldots, u^{\beta_{m-d}})\), where \(B\) has maximal rank, such that

\[
\theta, \mathcal{O}_p = \{\partial \in \text{Der}_p(-\log E) ; \partial(u^{\beta_i}) \equiv 0 \text{ for all } i\}
\]

In this case, we call \(u^B\) a complete system of first integrals.
Remark 2.5. The proof of Example 2.4 can be found in [Be], Lemma 2.2. We do not reproduce its proof because the result is not going to be explicitly used in this manuscript.

This result shows in more detail the relation between monomial singular distributions and monomialization of maps (see [C] for a definition of monomial of maps). Indeed, if a holomorphic map is monomial, the foliation generated by its level curves is \(\mathbb{Z}\)-monomial. Furthermore, in the study of families, the notion of quasi-smoothness (see [V2] for a definition) is closely related to a \(\mathbb{Z}\)-monomial distributions. We now turn to an important property of \(R\)-monomial singular distributions:

**Lemma 2.6.** If \(\theta\) is \(R\)-monomial singular distribution at a point \(p\) in \(M\), then there exists an open neighborhood \(U\) of \(p\) such that \(\theta\) is \(R\)-monomial at every point \(q\) in \(U\).

Remark 2.7. The proof of Lemma 2.6 can be found in [Be], Lemma 2.1. We do not reproduce its proof in this version of the manuscript for shortness.

### 2.2 The analytic strict transform

Given an admissible blowing-up \(\sigma : (\tilde{M}, \tilde{E}) \to (M, E)\) (i.e. the center of blowings-up \(\mathcal{C}\) has SNC with \(E\)) let \(F\) be the exceptional divisor of the blowing-up. Consider the sheaf of \(\mathcal{O}_{\tilde{M}}\)-modules \(\text{BlDer}_{\tilde{M}} := \mathcal{O}(F) \otimes_{\mathcal{O}_{\tilde{M}}} \text{Der}_{\tilde{M}}\). Notice that the morphism \(\sigma : (\tilde{M}, \tilde{E}) \to (M, E)\) gives rise to an application:

\[ \sigma^* : \text{Der}_M \longrightarrow \text{BlDer}_{\tilde{M}} \]

Now, consider the coherent sub-sheaf \(\text{Der}_{\tilde{M}}(-\log \tilde{E})\) of \(\text{Der}_{\tilde{M}}\) composed by all the derivations which leave the exceptional divisor \(\tilde{E}\) invariant. We notice that there exists a natural immersion:

\[ \zeta : \text{Der}_{\tilde{M}}(-\log \tilde{E}) \longrightarrow \text{BlDer}_{\tilde{M}} \]

Finally, the analytic strict transform \(\tilde{\theta}\) of \(\theta\) is the singular distribution:

\[ \tilde{\theta} = \zeta^{-1} \sigma^*(\theta) \]

where \(\zeta^{-1}(\omega)\) stands for the coherent sub-sheaf of \(\text{Der}_{\tilde{M}}(-\log \tilde{E})\) generated by the pre-image of \(\omega\). For a relation between the analytic strict transform and the usual strict transform, see remark 2.10 below. With these notations, the transform of a foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) under blowing-up is uniquely defined.

### 2.3 \(\theta\)-Admissible Blowings-up

Given a foliated manifold \((M, \theta, \mathcal{I}, E)\), the generalized \(k\)-Fitting operation (for \(k \leq d\)) is a mapping \(\Gamma_{\theta,k}\) that associates to each coherent ideal sheaf \(\mathcal{I}\) over \(M\) the ideal sheaf \(\Gamma_{\theta,k}(\mathcal{I})\) whose stalk at each point \(p\) in \(M\) is given by:

\[ \Gamma_{\theta,k}(\mathcal{I})_p = \langle \{ \text{det}[X_i(f_j)]_{i,j \leq k} : X_i \in \theta_p, f_j \in \mathcal{I}_p, O_p \} \rangle \]
where \( < S > \) stands for the ideal generated by the subset \( S \subset O_p \).

Now, consider a regular analytic sub-manifold \( C \) of \( M \) and the reduced ideal sheaf \( I_C \) that generates \( C \). We say that \( C \) is a \( \theta \)-admissible center if:

- \( C \) is a regular closed sub-variety that has SNC with \( E \);
- There exists \( 0 \leq d_0 \leq d \) such that the \( k \)-generalized Fitting-ideal \( \Gamma_{\theta,k}(I_C) \) is equal to the structural ideal \( O_M \) for all \( k \leq d_0 \) and is contained in the ideal sheaf \( I_C \) otherwise.

A \( \theta \)-admissible blowing-up is a blowing-up with \( \theta \)-admissible center. The following Theorem enlightens the interest of \( \theta \)-admissible blowings-up:

Theorem 2.8. (See Theorem of [Be]) Let \( (M, \theta, E) \) be a \( R \)-monomial d-foliated manifold and:

\[
\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)
\]

a \( \theta \)-admissible blowing-up. Then \( \theta' \) is \( R \)-monomial.

Remark 2.9 (Proof of Theorem). In [Be], a singular distribution is called monomial if it satisfies Properties [i-iii] of the Definition 2.1. So, Theorem 4.1.1 proves that \( \tilde{\theta} \) also satisfies Properties [i – iii], but Claims nothing about Property [iv]. Nevertheless, by Remark 2.2 it is clear that if \( \theta \) satisfies condition [iv], then \( \tilde{\theta} \) also satisfies condition [iv], which gives rise to the formulation used on this work.

Remark 2.10 (Strict Transform of a monomial singular distribution \( \theta \)). As a consequence of Theorem 2.8, if \( \theta \) is monomial and \( \tau : (\tilde{M}, \tilde{\theta}, \tilde{E}) \rightarrow (M, \theta, E) \) is \( \theta \)-admissible, then the analytic strict transform \( \tilde{\theta} \) coincides with the intersection of the strict transform of \( \theta \) with \( \text{Der}_{\tilde{M}}(-\tilde{E}) \). In particular, they are equal in the non-dicritical case (i.e. if the strict transform is tangent to \( \tilde{E} \)).

Because of Theorem 2.8, the \( \theta \)-admissible blowings-up are going to be an essential tool in this work. So, let us present a couple of examples and an intuitive description of the Definition:

Example 1: If \( C \) is an admissible and \( \theta \)-invariant center (i.e all leafs of \( \theta \) that intersects \( C \) are contained in \( C \) ) it is \( \theta \)-admissible.

Example 2: If \( C \) is an admissible and \( \theta \)-totally transverse center (i.e all vector-fields in \( \theta \) are transverse to \( C \) ) it is \( \theta \)-admissible.

Example 3: Let \( (M, \theta, E) = (C^3, \{\partial_x, \partial_y\}, \emptyset) \) and \( C = \{x = 0\} \). Then \( C \) is a \( \theta \)-admissible center, but it is neither invariant nor totally transverse. Indeed, \( \Gamma_{\theta,1}(I_C) = O_M \) and \( \Gamma_{\theta,2}(I_C) \subset I_C \).

Example 4: Let \( (M, \theta, E) = (C^3, \{\partial_x, \partial_y\}, \emptyset) \) and \( C = \{x^2 - z = 0\} \). Then \( C \) is not a \( \theta \)-admissible center. Indeed, \( \Gamma_{\theta,1}(I_C) = (x, z) \).

Remark 2.11 (Intuition of the Definition). (See Proposition 4.3.1 of [Be]) If a center \( C \) is \( \theta \)-admissible then, for each point \( p \) in \( C \), there exists two singular distributions germs \( \theta_{inv} \) and \( \theta_{tr} \) such that: \( \theta_p \) is generated by \( \{\theta_{inv}, \theta_{tr}\} \); \( C \) is \( \theta_{inv} \)-invariant; and \( C \) is \( \theta_{tr} \)-totally transverse.
3 Strategy of the proof

3.1 Main Invariant

We briefly recall the definition of the tangency order given in [Be]. A chain of ideal sheaves consists of a sequence \((\mathcal{I}_i)_{i \in \mathbb{N}}\) of ideal sheaves such that \(\mathcal{I}_i \subset \mathcal{I}_j\) if \(i \leq j\). The length of a chain of ideal sheaves at a point \(p\) of \(M\) is the minimal number \(\nu_p \in \mathbb{N}\) such that \(\mathcal{I}_i \mathcal{O}_p = \mathcal{I}_{\nu_p} \mathcal{O}_p\) for all \(i \geq \nu_p\). We distinguish two cases:

- if \(\mathcal{I}_{\nu_p} \mathcal{O}_p = \mathcal{O}_p\), then the chain is said to be of type 1 at \(p\);
- if \(\mathcal{I}_{\nu_p} \mathcal{O}_p \neq \mathcal{O}_p\), then the chain is said to be of type 2 at \(p\).

Given a chain of ideal sheaf \((\mathcal{I}_n)\), it is not difficult that the following functions are upper semi-continuous:

\[
\nu : M \rightarrow \mathbb{N} , \quad \text{type} : M \rightarrow \{1, 2\}
\]

\[
p \mapsto \nu_p \quad p \mapsto \text{type}_p = \text{type of } (\mathcal{I}_n) \text{ at } p
\]

So, given a foliated ideal sheaf \((\mathcal{M}, \theta, \mathcal{I}, E)\), the tangency chain of the pair \((\theta, \mathcal{I})\) is a chain of ideal sheaves \(\{H(\theta, \mathcal{I}, i); i \in \mathbb{N}\}\), where:

\[
\begin{cases}
H(\theta, \mathcal{I}, 0) := \mathcal{I} \\
H(\theta, \mathcal{I}, i + 1) := H(\theta, \mathcal{I}, i) + \theta[H(\theta, \mathcal{I}, i)]
\end{cases}
\]

At each \(p \in M\), the length of this chain is called the tangent order (or shortly, the \(tg\)-order) at \(p\), and is denoted by \(\nu_p(\theta, \mathcal{I})\). The type of the chain is denoted by \(\text{type}_p(\theta, \mathcal{I})\). The main invariant we consider is the pair:

\[
(\nu, t) := (\nu_p(\theta, \mathcal{I}), \text{type}_p(\theta, \mathcal{I}))
\]

ordered in the lexicographical order. The proof of the Theorem \([1]\) relies on making the pair of invariant droops. This is divided in two steps:

- First step: \((\nu, 2) \rightarrow (\nu, 1)\);
- Second step: \((\nu, 1) \rightarrow (\nu - 1, 2)\).

that can be iterated in order to make the invariant droop to \((0, 0)\), i.e. the transform of the ideal sheaf \(\mathcal{I}\) becomes the structural ring. The precise formulation of these steps is given by the following Propositions:

**Proposition 3.1 (First Step).** Let \((\mathcal{M}, \theta, \mathcal{I}, E)\) be a d-foliated ideal sheaf and \(p\) a point of \(M\) such that \(\text{type}_p(\theta, \mathcal{I}) = 2\). Then, there exists a neighborhood \(M_0\) of \(p\) and sequence of \(\theta\)-invariant admissible blowings-up of order one:

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that \(\nu_{M_r}(\theta_r, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I})\) and \(\text{type}_{M_r}(\theta_r, \mathcal{I}_r) = 1\).

**Remark 3.2 (Proof of Proposition 3.1).** The proof of the above Proposition can be found in [Be], Proposition 5.4.1 and we do not reproduce it here.
Proposition 3.3 (Second Step). Let $(M, \theta, \mathcal{I}, E)$ be an analytic d-foliated ideal-sheaf and $p$ a point of $M$ such that $\text{type}_p(\theta, \mathcal{I}) = 1$ and $\nu_p(\theta, \mathcal{I}) > 0$. Suppose that Theorem 1.1 is true any foliated ideal-sheaf $(N, \omega, \mathcal{J}, F)$ with $\dim N < \dim M$. Then, there exists a finite collection of morphisms $\tau_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \to (M, \theta, \mathcal{I}, E)$ such that:

- The morphism $\tau_i$ is a finite composition of $\theta$-admissible local blowing-ups;
- In each variety $M_i$, there exists a compact set $K_i \subset M_i$ such that the union of their images $\bigcup \tau_i(K_i)$ is a compact neighborhood of $p$;
- The invariant $\nu$ has drooped, i.e. at each point $q_i$ in the pre-image of $p$ by $\tau_i$, $\nu(q_i, (\theta_i, \mathcal{I}_i)) < \nu_p(\theta, \mathcal{I})$.

Remark 3.4. Notice that, if $\dim M = 1$, the inductive hypothesis “Theorem 1.1 is true any foliated ideal-sheaf $(N, \omega, \mathcal{J}, F)$ with $\dim N < \dim M$” is trivially true.

The proof of Proposition 3.3 is given in section 4. In the next subsection we show how these two Propositions imply the main Theorem 1.1.

3.2 Proof of the Main Theorem 1.1 (Assuming Proposition 3.3)

We suppose that we are in the hypothesis of Theorem 1.1. We prove the Theorem by strong induction on the dimension of $M$.

If $\dim M = 1$, then the Theorem is trivial. Indeed, all ideals $\mathcal{I} \subset \mathcal{O}_M$ are locally principal and the resolution of singularities of $\mathcal{I}$ blow-up only points. Since, for any singular distribution $\theta$, points are always $\theta$-admissible centers, by Theorem 2.8 we are done.

So, let us assume that Theorem 1.1 is true any foliated ideal-sheaf $(N, \omega, \mathcal{J}, F)$ with $\dim N < \dim M$. By a recursive use of Propositions 3.1 and 3.3 there exists a finite collection of morphisms $\tau_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \to (M, \theta, \mathcal{I}, E)$ such that:

- The morphism $\tau_i$ is a finite composition of $\theta$-admissible local blowing-ups.
- In particular, from Theorem 2.8 if $\theta$ is $R$-monomial, then so is $\theta_i$;
- In each variety $M_i$, there exists a compact set $K_i \subset M_i$ such that the union of their images $\bigcup \tau_i(K_i)$ is a compact neighborhood of $p$;
- In each point $q_i$ in the pre-image of $p$, the pair of invariant $(\nu(q_i, (\theta_i, \mathcal{I}_i)), \text{type}_q(\theta_i, \mathcal{I}_i))$ is equal to $(0, 0)$.

In particular, apart from taking smaller neighborhoods $M_i$ and compacts $K_i$, we can assume that he ideal sheaf $\mathcal{I}_i$ is equal to the structural ring, i.e. $\mathcal{I}_i = \mathcal{O}_{M_i}$. We conclude that the pull-back ideal sheaf $\mathcal{I}_i \mathcal{O}_{M_i}$ is principal and has support contained in the exceptional divisor $E_i$. This finishes the proof.
4 Dropping the invariant

4.1 Preparation

We start by giving a convenient Normal Form:

**Lemma 4.1.** Let $(M, \theta, \mathcal{I}, E)$ be an analytic $d$-foliated ideal-sheaf and $p$ a point in $M$ such that $\text{type}_p(\theta, \mathcal{I}) = 1$ and let $\nu = \nu_p(\theta, \mathcal{I}) > 0$. Then, there exists a coordinate system $(u, v, w)$ of $p$ and a set of generators $(g_1, \ldots, g_n)$ of $\mathcal{I}_p$ such that the vector-field $\partial_u$ belongs to $\theta, \mathcal{O}_p$ and:

\[
g_1 = \nu^\nu U + \sum_{j=0}^{\nu-2} \nu^j a_{1,j}(u, w) \quad \text{where } U \text{ is an unity, and}
\]

\[
g_i = \nu^\nu \tilde{g}_i + \sum_{j=0}^{\nu-1} \nu^j a_{i,j}(u, w)
\]

We call this a Basic Normal Form.

**Proof.** By the definition of type it is clear that there exists a coordinate system $(u, v, w)$ of $p$ such that the vector-field $\partial_u$ belongs to $\theta, \mathcal{O}_p$ and, for any set of generators $(g_1, \ldots, g_n)$ of $\mathcal{I}$, without loss of generality, the function $\partial_u^\nu g_1$ is a unit. Furthermore, by the implicit function Theorem, there is a change of coordinates $(\tilde{u}, \tilde{v}, \tilde{w}) = (u, V(u, v, w), w)$ such that $\partial_u^{-1} g_1(\tilde{u}, 0, \tilde{w}) \equiv 0$. Thus:

\[
g_1 = \nu^\nu U + \sum_{j=0}^{\nu-2} \nu^j a_{1,j}(\tilde{u}, \tilde{w}) \quad \text{where } U \text{ is an unity, and}
\]

\[
g_i = \nu^\nu \tilde{g}_i + \sum_{j=0}^{\nu-1} \nu^j a_{i,j}(\tilde{u}, \tilde{w})
\]

Finally, since $\tilde{u} = u$ and $\tilde{w} = w$, we have that $\partial_{\tilde{v}} = U \partial_v$ for some unit $U$. This clearly implies that $\partial_{\tilde{v}}$ is contained in $\theta, \mathcal{O}_p$, which proves the Lemma.

**Definition 4.2** (Prepared normal form). We say that an analytic $d$-foliated ideal-sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the Prepared Normal Form at a point $p$ of $M$ if $\text{type}_p(\theta, \mathcal{I}) = 1$ and there exists a coordinate system $(u, v, w)$ of $p$ and a set of generators $(g_1, \ldots, g_n)$ of $\mathcal{I}_p$ such that the vector-field $\partial_u$ belongs to $\theta, \mathcal{O}_p$ and:

\[
g_1 = \nu^\nu U + \sum_{j=0}^{\nu-2} \nu^j u^{i,j} b_{1,j}(u, w) \quad \text{where } U \text{ is an unity, and}
\]

\[
g_i = \nu^\nu \tilde{g}_i + \sum_{j=0}^{\nu-1} \nu^j u^{i,j} b_{i,j}(u, w)
\]

where $b_{i,j}$ is either a unit or zero, and whenever $b_{i,j} \neq 0$ then $r_{i,j} \neq 0$.

**Lemma 4.3** (Preparation). Let $(M, \theta, \mathcal{I}, E)$ be an analytic $d$-foliated ideal-sheaf and $p$ a point in $M$ such that $\text{type}_p(\theta, \mathcal{I}) = 1$ and $\nu = \nu_p(\theta, \mathcal{I}) > 0$. Suppose that Theorem 4.1 is true for any foliated ideal-sheaf $(N, \omega, \mathcal{F}, F)$ with $\dim N < \dim M$. Then, there exists a finite collection of morphisms $\tau_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \to (M, \theta, \mathcal{I}, E)$ such that:
• The morphism $\tau_i$ is a finite composition of $\theta$-admissible local blowing-ups;

• In each variety $M_i$, there exists a compact set $K_i \subset M_i$ such that the union of their images $\bigcup \tau_i(K_i)$ is a compact neighborhood of $p$;

• At every point $q_i$ in the pre-image of $p$, the $d$-foliated ideal-sheaf $(M_i, \theta_i, \mathcal{I}_i, E_i)$ satisfies the Prepared Normal Form at $q_i$ with $\nu_{q_i}(\theta_i, \mathcal{I}_i) \leq \nu$.

**Proof.** By Lemma 4.1, the analytic $d$-foliated ideal-sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the Basic Normal Form at $p$, i.e. there exists a coordinate system $(u, v, w)$ of $p$ and a set of generators $(g_1, \ldots, g_n)$ of $\mathcal{I} \mathcal{O}_p$ such that the vector-field $\partial_v$ belongs to $\theta \mathcal{O}_p$ and the functions $g_i$ are given by equation (4.1).

Let $\pi : M_0 \to N$ be the projection map given by $\pi(u, v, w) = (u, w)$, where $M_0$ is a small enough neighborhood of $p$, and let $\mathcal{J}$ be the principal ideal sheaf generated by the product of all non-zero $a_{i,j}$. Then, it is clear that there exist a $d-1$ foliated ideal sheaf $(N, \omega, \mathcal{J}, F)$ such that:

• The singular distribution $\theta$ is generated by the set $\{\partial_v, \pi^*\omega\}$;

• The inverse image of $F$ is equal to $E \cap M_0$.

Now, since $\dim N < \dim M$, there exists a finite collection of morphisms $\sigma_i : (N_i, \omega, \mathcal{J}_i, F_i) \to (N, \omega, \mathcal{J}, F)$ such that:

• The morphism $\sigma_i$ is a finite composition of $\omega$-admissible local blowing-ups;

• In each variety $N_i$, there exists a compact set $V_i \subset N_i$ such that the union of their images $\bigcup \tau_i(W_i)$ is a compact neighborhood of $p$;

• The ideal sheaf $\mathcal{J}_i$ is the structural ring. In particular, this means that the total transform $\sigma_i^* \mathcal{J}$ is a principal ideal sheaf with support contained in $F_i$.

Furthermore, it is clear that we can extend $\sigma_i$ to blowings-up at $M_0$ by taking the product of the centers of $\tau_i$ by the $v$-coordinate:

$$\tau_i(M_i, \theta_i, \mathcal{I}_i, E_i) \to (M_0, \theta_0, \mathcal{I}_0, E_0)$$

where all centers have SNC with the exceptional divisor and are invariant by the $v$-coordinate i.e. all centers are $\partial_v$-invariant. Moreover, since all centers of $\sigma_i$ are $\omega$-admissible, we conclude that all centers of $\tau_i$ are $\theta$-admissible.

Now, notice that no center is contained in the support of the ideal sheaf $\mathcal{I}$, which implies that $\mathcal{I}_i$ stands for the total transform of $\mathcal{I}$. So consider $q_i$ a point in the pre-image of $p$ by $\tau_i$ and let $(x, y, z)$ be a coordinate system at $q_i$ such that $\tau_i^* v = y$. Since the pull-back $(\tau_i \circ \pi)^* \mathcal{J}$ is a principal ideal sheaf, we conclude that:

$$g_1 = y^\nu U + \sum_{j=0}^{\nu-2} y^j x^{\nu-1-j} \bar{a}_{1,j}(x, z)$$

$$g_i = y^\nu \bar{g}_i + \sum_{j=0}^{\nu-1} y^j x^{\nu-1-j} \bar{a}_{i,j}(x, z)$$
where the functions $a_{i,j}$ are either zero or units. Finally, since $\partial_y = \tau^*\partial_x$, we conclude that $\partial_y \in \theta_i \mathcal{O}_e$. In particular, this clearly implies that $\nu_q(\theta_i, I_i) \leq \nu$, which concludes the proof of the Proposition.

**4.2 Combinatorial blowings-up**

**Definition 4.4** (Sequence of combinatorial blowings-up). Given a divisor $E$ in $M$, we say that $\tau : \tilde{M} \rightarrow M$ is a sequence of combinatorial blowings-up (with respect to $E$) if $\tau$ is a composition of blowings-up with centers that are strata of the divisor $E$ and its total transforms.

Given a globally defined system of coordinates $(u, v, w)$, let us consider a sequence of combinatorial blowings-up, $\tau : (\tilde{M}, \tilde{E}) \rightarrow (M, E)$, with respect to the declared exceptional divisor $F = \{u_1 \cdots u_t \cdot v = 0\}$. We can cover $\tilde{M}$ by affine charts with a coordinate system $(x, w)$ satisfying:

\[
\begin{align*}
    u_j &= x_1^{a_{j,1}} \cdots x_t^{a_{j,l+1}} \\
    v &= x_1^{a_{1,l+1}} \cdots x_t^{a_{t,l+1}} \\
    w_i &= w_i
\end{align*}
\]

(4.3)

that we denote, to simplify notation, by:

\[(u, v, w) = (x^A, w)\]

where $A$ is a $(l + 1)$-square matrix such that $\det(A) = \pm 1$, given by:

\[
A = \begin{bmatrix}
    a_{1,1} & \cdots & a_{1,l+1} \\
    \vdots & \ddots & \vdots \\
    a_{l+1,1} & \cdots & a_{l+1,l+1}
\end{bmatrix}
\]

Now, let $q$ be another point in this affine chart contained in $\tilde{E}$. Then, apart from re-indexing,

\[(x_1, \ldots, x_t, y_{t+1}, \ldots, y_{l+1}, \gamma_{t+1}, \ldots, \gamma_{l+1}),\]

is a coordinate system centered at $q$, and $\gamma_j \neq 0$ for all the $\gamma_j$. We can also assume that $t \neq 0$, since otherwise $q$ would be outside the exceptional divisor.

Now, we consider a decomposition of the matrix $A$:

\[
A = \begin{bmatrix}
    A_1 & A_2 \\
    \alpha_1 & \alpha_2
\end{bmatrix}
\]

where $A_1$ is a $l \times t$ matrix, $A_2$ is a $l \times (l + 1 - t)$ matrix, $\alpha_2$ is a $1 \times (l - t + 1)$, and $\alpha_1$ is a $1 \times t$ matrix. We remark that, since $q$ is a point on the exceptional divisor $\tilde{E}$, there exists at least one $u_i$ such that $\tau^* u_i(q) = 0$, which clearly implies that $A_1$ has to be a non-zero matrix.

**Lemma 4.5** (Claim 1). Assume $A_1$ has maximal rank and let $\mathcal{X}$ be a singular distribution generated by $\partial_y$. If $q$ is a point in the pre-image of $p$, then there exists a system of coordinates $(x, y, z, w) = (x_1, \ldots, x_t, y_{t+1}, \ldots, y_{l+1}, z, w)$ centered at $q$ that satisfies the following properties:
- The analytic strict transform of $X$ is generated by a vector-field $\partial_z + X$, where $X$ is a singular vector-field at $q$;

- The relation between the coordinate systems are given by:

$$
\begin{align*}
  u &= x^{A_1}(y - \tilde{\gamma})^{Id} \\
  v &= x^{\alpha_1}(z - \tilde{\gamma}_{t+1}) \\
  w &= w
\end{align*}
$$

where $\tilde{\gamma}_j \neq 0$ for all $j$ and the matrix $Id$ is the identify.

**Lemma 4.6** (Claim 2). Assume that $A_1$ does not have maximal rank. If $q$ is a point in the pre-image of $p$, then there is a system of coordinates $(x, y, w) = (x_1, \ldots, x_t, y_{t+1}, \ldots, y_l, w)$ centered at $q$ where the relation between the coordinate systems is given by:

$$
\begin{align*}
  u &= x^{A_1}(y - \lambda)^{\Lambda} \\
  v &= x^{\alpha_1} \\
  w &= w
\end{align*}
$$

where $\tilde{\gamma}_j \neq 0$ for all $j$ and the matrix $\Lambda = (\lambda_{i,j})$ of exponents has maximal rank. Furthermore, $\alpha_1$ doesn’t belong to the span of the rows of $A_1$.

**Remark 4.7.** Furthermore, if $(M, \theta, E)$ is a $d$-foliated manifold such that $\partial_v$ is a vector-field in $\theta$, then $\tau : (\tilde{M}, \tilde{\theta}, \tilde{E}) \to (M, \theta, E)$ is a sequence of $\theta$-admissible blowings-up. Indeed, the singular distribution can be decomposed by $\{\omega, \partial_v\}$ where $\omega$ leaves the divisor $F$ invariant. This clearly implies that all blowings-up are $\omega$-invariant. To conclude, one only needs to notice that all blowings-up are either $\partial_v$ invariant or $\partial_v$ transverse.

**Proof of Lemma 4.6** After a re-indexing of the $u_j$’s we can write

$$
A_1 = \begin{bmatrix} A'_1 \\
  A''_1
\end{bmatrix}, \quad A_2 = \begin{bmatrix} A'_2 \\
  A''_2
\end{bmatrix},
$$

where $\det(A'_1) \neq 0$, and $A'_1$ and $A'_2$ have the same height.

We can write equations (4.3) in the compact form

$$
\begin{align*}
  u' &= x^{A'_1}(y - \gamma)^{A'_2} \\
  u'' &= x^{A''_1}(y - \gamma)^{A''_2} \\
  v &= x^{\alpha_1}(y - \gamma)^{\alpha_2}
\end{align*}
$$

Now, it is clear that there exists a coordinate system $(\bar{x}, \bar{y}, \bar{z})$ and constant $\bar{\gamma}$ such that:

$$
\begin{align*}
  u' &= \bar{x}^{A'_1} \\
  u'' &= \bar{x}^{A''_1}(\bar{y} - \bar{\gamma})^{Id} \\
  v &= \bar{x}^{\alpha_1}(\bar{z} - \bar{\gamma}_{t+1})
\end{align*}
$$

where $Id$ is the identity matrix. In this coordinate system, the pull-back of $v\partial_v$ is given by:

$$
v\partial_v = (\bar{z} - \bar{\gamma}_{t+1})\partial_z + \sum \alpha_{1,j}\bar{x}_j\partial_{\bar{x}_j}
$$

13
which implies that the analytic strict transform of \(X\) is locally generated by the vector-field:

\[
\frac{\partial_\tilde{z}}{\tilde{z} - \tilde{\gamma}_{l+1}} + \sum_{1,j} \alpha_{1,j} \partial_\tilde{x}_j
\]

So, taking \(X = \frac{\partial_\tilde{z}}{\tilde{z} - \tilde{\gamma}_{l+1}} + \sum_{1,j} \alpha_{1,j} \partial_\tilde{x}_j\) we have verified all conditions of the Lemma.

\[\Box\]

**Proof of Lemma 4.6.** Since \(A_1\) does not have maximal rank, but \(A\) does, we conclude that \(\alpha_i\) is not generated by the rows of \(A_1\). So, we can change coordinates so that the unit \((y - \gamma)^{\alpha_2}\) is absorbed into the monomial \(x^{\alpha_1}\).

**4.3 Dropping the invariant from Prepared Normal Form**

**Lemma 4.8.** Let \((M, \theta, \mathcal{I}, E)\) be an analytic d-foliated ideal-sheaf and \(p\) a point in \(M\) such that \((M, \theta, \mathcal{I}, E)\) satisfies the Prepared Normal Form at \(p\) with \(\nu = \nu_p(\theta, \mathcal{I}) > 0\). Then, for a small enough neighborhood \(M_0\) of \(p\), there exists a sequence of \(\theta\)-admissible blowings-up \(\tau : (M_r, \theta_r, \mathcal{I}_r, E_r) \to (M_0, \theta_0, \mathcal{I}_0, E_0)\) such that, for all point \(q\) in the pre-image of \(p\), the invariant \(\nu_q(\theta_r, \mathcal{I}_r)\) is strictly smaller than the initial invariant \(\nu_p(\theta, \mathcal{I})\).

**Proof.** By hypothesis, there exists a local coordinate system \((u, v, w)\) that satisfies the Prepared Normal Form at \(p\) with \(\nu = \nu_p(\theta, \mathcal{I})\), i.e. that satisfies equations (4.2). We consider the ideal \(J\) generated by:

\[
v^\nu, \quad \{v^j u^r i b_{i,j}\}_{0 \leq j < d}
\]

where we recall that all \(b_{i,j}\) are either units or zero. Let us consider a sequence of blowings-up:

\[
\tau : (M_r, \theta_r, \mathcal{I}_r, E_r) \to (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

that principalize \(J\), where \(M_0\) is any fixed open neighborhood of \(p\) where \(J\) is well-defined. Since \(J\) is generated by monomials in the variables \(u\) and \(v\), this sequence can be chosen to be combinatorial with respect to the divisor \(F := E \cup \{v = 0\}\) (see Definition 4.4). Furthermore, since \(\theta\) is adapted to \(E\) and the vector-field \(\partial_v\) is contained in \(\theta\), by Remark 4.7 the sequence \(\tau\) is \(\theta\)-admissible.

Now, let \(q\) be a point of \(M_r\) in the pre-image of \(p\). Since \(\tau\) is a sequence of combinatorial blowings-up in respect to the divisor \(F = E \cup \{v = 0\}\), the point \(q\) satisfies the hypothesis of either Lemma 4.5 or 4.6 (where we recall that \(\partial_v\) belongs in \(\theta_0\)). Thus, we have two cases to consider:

**Case 1:** We assume we are in conditions of Lemma 4.5. There exists a system of coordinates \((x, y, z, w) = (x_1, \ldots, x_t, y_{t+1}, \ldots, y_l, z, w)\) centered at \(q\) that satisfies the following properties:

- The analytic strict transform of \(X = (\partial_\gamma) \subset \theta_0\) is generated by a vector-field \(\partial_\gamma + X\), where \(X\) is a singular vector-field at \(q\). In particular, \(\partial_\gamma + X\) belongs to \(\theta_r, O_q\).
The relation between the coordinate systems are given by equation (4.4):

\[
\begin{align*}
    u &= x^{A_1}(y - \tilde{\gamma})^I \\
    v &= x^{\alpha_1}(z - \tilde{\gamma}_{l+1}) \\
    w &= w
\end{align*}
\]

where \( \tilde{\gamma}_j \neq 0 \) for all \( j \) and the matrix \( \text{Id} \) is the identify.

So, after blowing-up we have the following expressions:

\[
\tau^* g_1 = U x^{S_v} (z - \tilde{\gamma}_{l+1})^\nu + \sum_{j=0}^{\nu-2} x^{S_{1,j}} (z - \tilde{\gamma}_{l+1})^j c_{1,j} \quad \text{where } U \text{ is an unity, and}
\]

\[
\tau^* g_i = \bar{g}_i x^{S_v} (z - \tilde{\gamma}_{l+1})^\nu + \sum_{j=0}^{\nu-1} x^{S_{i,j}} (z - \tilde{\gamma}_{l+1})^j c_{i,j}
\]

where the functions \( c_{i,j}(x, y, w) \) are either zero or unities, and the unity \( U \) can be written as \( \tilde{U}(x, y, w) + x^{\alpha_1} \Omega \) for some function \( \Omega \) (notice that \( \tilde{U} \) is independent of \( z \)). We consider two cases depending on which generator of \( \mathcal{J} \) pulls back to be a generator of the pull-back of \( \mathcal{J}^* \):

**Case 1.1:** The pull back of \( v^\nu \) generates \( \mathcal{J}^* \), i.e. \( S_v = \min\{S_{\nu, S_{i,j}}\} \) In this case we have:

\[
\tau^* g_1 = x^{S_v} [(\tilde{U}z + \bar{U} \tilde{\gamma}_{l+1}^\nu + x^{\alpha_1} \Omega_2)z^{\nu-1} + \text{ (terms where the exponent of } z \text{ is } < \nu - 1)]
\]

where \( \alpha_1 \) is a non-zero matrix and \( \Omega_2 = [z + \tilde{\gamma}_{l+1}^\nu] \Omega. \) Since \( \tilde{U}z + \bar{U} \tilde{\gamma}_{l+1}^\nu + x^{\alpha_1} \Omega_2 \) is a unit and the vector-field \( \partial_z + X \) belongs to \( \theta_r, \) it is clear that \( \nu_q(\theta_r, I_r) \leq \nu - 1 < \nu. \)

**Case 1.2:** There is a maximum \( 0 \leq j_1 < \nu \) such that the pull back of \( u^{i_1} v^j \) generates \( \mathcal{J}^* \) for some \( i_1, \) i.e \( S_{i_1,j_1} = \min\{S_{\nu, S_{i,j}}\}, S_{\nu} > S_{i_1,j_1}, \) and \( S_{i,j} > S_{i_1,j_1} \) for \( j > j_1 \) In this case we have:

\[
\tau^* g_{i_1} = x^{S_{i_1,j_1}} \left[ (z - \tilde{\gamma}_{l+1})^{j_1} c_{i_1,j_1} + \sum_{j=0}^{j_1-1} x^{S_{i_1,j} - S_{i_1,j_1}} (z - \tilde{\gamma}_{l+1})^j c_{i_1,j}(x, y, w) + \Omega(x, y, z, w) \right]
\]

where \( \Omega(0, y, z, w) \equiv 0. \) Since \( c_{i_1,j_1} \) is a unit and the vector-field \( \partial_z + X \) belongs to \( \theta_r, \) it is clear that \( \nu_q(\theta_r, I_r) = j_1 < \nu. \)

**Case 2:** We assume we are in conditions of Lemma 1.5. Thus, there is a system of coordinates \( (x, y, w) = (x_1, \ldots, x_t, y_{l+1}, \ldots, y_{l+1}, w) \) centered at \( q \) where the relation between the coordinate systems is given by equation (4.5):

\[
\begin{align*}
    u &= x^{A_1}(y - \gamma)^A \\
    v &= x^{\alpha_1} \\
    w &= w
\end{align*}
\]
where \( \tilde{\gamma}_j \neq 0 \) for all \( j \) and the matrix \( \Lambda = (\lambda_{i,j}) \) of exponents has maximal rank. Furthermore, \( \alpha_1 \) doesn’t belong to the span of the rows of \( A_1 \). So, after blowing-up we have the following expressions:

\[
\tau^* g_1 = U x^{S_\nu} + \sum_{j=0}^{\nu-2} x^{S_{1,j}} c_{1,j} \quad \text{where } U \text{ is an unity, and}
\]

\[
\tau^* g_i = \bar{g}_i x^{S_\nu} + \sum_{j=0}^{\nu-1} x^{S_{i,j}} c_{i,j}
\]

where the functions \( c_{i,j}(x,y,w) \) are either zero or unities. We remark that:

\[
S_\nu = \nu \alpha_1 \quad S_{i,j} = j \alpha_1 + r_{i,j} A_1, \text{ for } i, j = 0, \ldots, \nu - 1.
\]

So, for a fixed \( i \), each \( S_\nu \) and \( S_{i,j} \) is a sum of an element of the span of the rows of \( A_1 \) and a different multiple of the \( \alpha_1 \). Since \( \alpha_1 \) is linearly independent with the rows of \( A_1 \), this means that the exponents \( S_\nu \) and \( S_{i,j} \) are all distinct. Therefore, for each \( i \), the generator of \( J^* \) can only be one of the monomials \( x^{S_{i,j}} \) or \( x^{S_{1,j}} \). So, let \( (i_1, j_1) \) be an index such that \( S_{i_1,j_1} = \min\{S_\nu, S_{i,j}\} \) (where we consider, possibly, \( (i_1, j_1) = \nu \)). Then:

\[
\tau^* g_{i_1} = x^{S_{i_1,j_1}} U
\]

where \( U \) is a unit. In this case, it is clear that \( I_r \) is generated by \( x^{S_{i_1,j_1}} \). Therefore, it is locally principal and the invariant has dropped to zero.

4.4 Proof of Proposition 3.3

We suppose that we are in the hypothesis of Proposition 3.3. Then, by Lemma 4.3, there exists a finite collection of morphisms \( \sigma_i : (M_i, \theta_i, I_i, E_i) \to (M, \theta, I, E) \) such that:

- The morphism \( \sigma_i \) is a finite composition of \( \theta \)-admissible local blowing-ups;
- In each variety \( M_i \), there exists a compact set \( K_i \subset M_i \) such that the union of their images \( \bigcup \sigma_i(K_i) \) is a compact neighborhood of \( p \);
- At every point \( q_i \) in the pre-image of \( p \), the \( d \)-foliated ideal-sheaf \((M_i, \theta_i, I_i, E_i)\) satisfies the Prepared Normal Form at \( q_i \) with \( \nu_{q_i}(\theta_i, I_i) \leq \nu \).

Now, by Lemma 4.3 and compactness of \( \bigcup K_i \), there exists a finite collection of morphisms \( \sigma_{i,j} : (M_{i,j}, \theta_{i,j}, I_{i,j}, E_{i,j}) \to (M_i, \theta_i, I_i, E_i) \) such that:

- The morphism \( \sigma_{i,j} \) is a finite composition of \( \theta \)-admissible local blowing-ups;
- In each variety \( M_{i,j} \), there exists a compact set \( K_{i,j} \subset M_{i,j} \) such that the union of their images \( \bigcup \sigma_{i,j}(K_{i,j}) \) is a compact neighborhood of \( K_i \);
- At every point \( q_{i,j} \) in the pre-image of \( p \) by \( \sigma_i \circ \sigma_{i,j} \), the invariant \( \nu_{q_{i,j}}(\theta_{i,j}, I_{i,j}) \) is strictly smaller than the initial invariant \( \nu \).

Thus, taking the finite collection of morphisms \( \tau_{i,j} := \sigma_i \circ \sigma_{i,j} \), we obtain the necessary sequence of local blowings-up.
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