Constant rank theorems for curvature problems via a viscosity approach

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Abstract
An important set of theorems in geometric analysis consists of constant rank theorems for a wide variety of curvature problems. In this paper, for geometric curvature problems in compact and non-compact settings, we provide new proofs which are both elementary and short. Moreover, we employ our method to obtain constant rank theorems for homogeneous and non-homogeneous curvature equations in new geometric settings. One of the essential ingredients for our method is a generalization of a differential inequality in a viscosity sense satisfied by the smallest eigenvalue of a linear map Brendle et al. (Acta Math 219:1–16, 2017) to the one for the subtrace. The viscosity approach provides a concise way to work around the well known technical hurdle that eigenvalues are only Lipschitz in general. This paves the way for a simple induction argument.

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1 Introduction

We introduce a viscosity approach to a broad class of constant rank theorems. Such theorems say that under suitable conditions a positive semi-definite bilinear form on a manifold, that satisfies a uniformly elliptic PDE, must have constant rank in the manifold. In this sense, constant rank theorems can be viewed as a strong maximum principle for tensors. The aim of

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this paper is two-fold. Firstly, we want to present a new approach to constant rank theorems. It is based on the idea that the subtraces of a linear map satisfy a linear differential inequality in a viscosity sense and the latter allows to use the strong maximum principle. This avoids the use of nonlinear test functions, as in [5], as well as the need for approximation by simple eigenvalues, as in [24]. Secondly, we show that the simplicity of this method allows us to obtain previously undiscovered constant rank theorems, in particular for non-homogeneous curvature type equations. To illustrate the idea, we give a new proof for the following full rank theorem for the Christoffel-Minkowski problem, a.k.a. the $\sigma_k$-equation.

**Theorem 1.1** [14, Theorem 1.2]

Let $(\mathbb{S}^n, g, \nabla)$ be the unit sphere with standard round metric and connection. Suppose $n \geq 2$, $1 \leq k \leq n - 1$ and $0 < s, \phi \in C^\infty(\mathbb{S}^n)$ satisfy

$$\nabla^2 \phi - \frac{1}{k} \phi - s g \geq 0, \quad 0 \leq r := \nabla^2 s + sg \in \Gamma_k, \quad \sigma_k(r) = \phi,$$

where $\sigma_k$ is $k$-th symmetric polynomial of eigenvalues of $r$ with respect to $g$ and $\Gamma_k$ is the $k$-th Garding cone. Then $r$ is positive definite.

**Proof** For convenience, we define

$$F = \sigma_k^{1/k}, \quad f = \phi^{1/k}.$$

Then $F = f$. Differentiate $F$ and use Codazzi, where a semi-colon stands for covariant derivatives and we use the summation convention:

$$F_{;ab} = F_{;ab} = F^{ij,kl} r_{ij;1} r_{kl;1} + F^{ij} r_{ij;ab} = F^{ij,kl} r_{ij;1} r_{kl;1} + F^{ij} r_{ab;i} - r_{ab} F^{ij} g_{ij} + F_{;ab}.$$

Hence the tensor $r$ satisfies the elliptic equation

$$F^{ij} r_{ab;i} = F^{ij} g_{ij} r_{ab} - F^{ij,kl} r_{ij;1} r_{kl;1} + F_{;ab}.$$

Now we deduce an inequality for the lowest eigenvalue of $r$, $\lambda_1$, in a viscosity sense. Let $\xi$ be a smooth lower support at $x_0 \in \mathbb{S}^n$ for $\lambda_1$ and let $D_1 \geq 1$ denote the multiplicity of $\lambda_1(x_0)$. Denote by $\Lambda$ the complement of the set $\{i, j, k, l > D_1\}$ in $\{1, \ldots, n\}^4$. We use a relation between the derivatives of $\xi$ and $r$, and the inverse concavity of $F$ (cf. [3, Lemma 5], [2]) to estimate in normal coordinates at $x_0$:

$$F^{ij} \xi_{;ij} \leq F^{ij} r_{11;ij} - 2 \sum_{j > D_1} \frac{F^{ij}}{\lambda_j} (r_{ij;1})^2$$

$$= - F^{ij,kl} r_{ij;1} r_{kl;1} - 2 \sum_{j > D_1} \frac{F^{ij}}{\lambda_j} (r_{ij;1})^2 + F^{ij} g_{ij} r_{11} - (f - f_{;1})$$

$$= - \sum_{i,j,k,l > D_1} F^{ij,kl} r_{ij;1} r_{kl;1} - 2 \sum_{j > D_1} \frac{F^{ij}}{\lambda_j} (r_{ij;1})^2 - (f - f_{;1})$$

$$- \sum_{(i,j,k,l) \in \Lambda} F^{ij,kl} r_{ij;1} r_{kl;1} + F^{ij} g_{ij} r_{11}$$

$$\leq - (f + 2f^{-1} f_{;1}^2 - f_{;1}) + c|\nabla \xi| + F^{ij} g_{ij} \xi$$

$$\leq F^{ij} g_{ij} \xi + c|\nabla \xi|.$$
Then the strong maximum principle for viscosity solutions (cf. [4]) implies that the set \( \{ \lambda_1 = 0 \} \) is open. Hence, if \( \lambda_1 \) was zero somewhere, it would be zero everywhere. However, we know it is positive somewhere, since at a minimum of \( s \) we have \( r > 0 \).

The proof may be summarized as follows: apply the viscosity differential inequality from [3, Lemma 5] for the minimum eigenvalue \( \lambda_1 \) of the spherical hessian of \( r \). Then the strong maximum principle shows that since there is a point at which \( \lambda_1 > 0 \) we must have \( \lambda_1 > 0 \) everywhere and hence the hessian has constant, full rank. A similar argument was employed in [19] for obtaining curvature estimates along a curvature flow.

Our main approach here is to generalize the viscosity inequality to the subtrace \( G_m = \lambda_1 + \cdots + \lambda_m \), the sum of the first \( m \) eigenvalues. See Lemma 3.2 below. Then by induction, we are able to show that if \( \lambda_1 = \cdots = \lambda_{m-1} \equiv 0 \), the strong maximum principle shows that either \( G_m > 0 \) or \( G_m \equiv 0 \) to conclude constant rank theorems (in short, CRT).

We say a symmetric 2-tensor \( \alpha \) is Codazzi, provided \( \nabla \alpha \) is totally symmetric. Here is a prototypical CRT:

**Theorem 1.2 (Homogeneous CRT) [10, Theorem 1.4]** Suppose \( \alpha \) is a Codazzi, non-negative, symmetric 2-tensor on a connected Riemannian manifold \((M, g, \nabla)\) satisfying \( \Psi(\alpha, g) = f > 0 \), where \( \Psi \) is one-homogeneous, inverse concave and strictly elliptic (see Definition 1.3 and Assumption 2.1), and we have \( \nabla^2 f^{-1} + \tau f^{-1} g \geq 0 \) with \( \tau(x) \) the minimum sectional curvature at \( x \). Then \( \alpha \) is of constant rank.

We state a more general version of CRT that allows the curvature function to be non-homogeneous and to explicitly depend on \( x \in M \) as well. To state the result, we need a few definitions.

**Definition 1.3** Let \( \Gamma \subset \mathbb{R}^n \) be an open, convex cone such that

\[ \Gamma_+ := \{ \lambda \in \mathbb{R}^n : \lambda_i > 0 \ \forall 1 \leq i \leq n \} \subset \Gamma. \]

Suppose \((M^n, g)\) is a smooth Riemannian manifold. A \( C^\infty \)-function

\[ F : \Gamma \times M \rightarrow \mathbb{R} \]

is said to be a pointwise curvature function, if for any \( x \in M \), the map \( F(\cdot, x) \) is symmetric under permutation of the \( \lambda_i \). Such a map generates another map (denoted by \( F \) again) given by

\[ F : \mathcal{U} \subset \mathbb{R}^{n \times n}_{\text{sym}} \times \mathbb{R}^{n \times n}_{\text{sym}} \times M \rightarrow \mathbb{R} \]

\[ (\alpha, g, x) \mapsto F(\alpha, g, x) = F(\lambda, x), \]

where \( \mathcal{U} \) is a suitable open set and \( \lambda = (\lambda_i)_{1 \leq i \leq n} \) are the eigenvalues of \( \alpha \) with respect to \( g \), or equivalently, the eigenvalues of the linear map \( \alpha^2 \) defined by \( g(\alpha^2(v), w) = \alpha(v, w) \). Note that \( F \) can be considered as a map on an open set of \( \mathbb{R}^{n \times n} \) via \( F(\alpha^2, x) = F(\alpha, g, x) \); see [23].

With the convention \( \alpha^i_j = g^{ik} \alpha_{kj} \), where \( (g^{kl}) \) is the inverse of \( (g_{kl}) \):

\[ F^i_j := \frac{\partial F}{\partial \alpha^i_j}, \quad F^{ij} := \frac{\partial F}{\partial \alpha_{ij}}, \quad F^{ij, kl} := \frac{\partial F}{\partial \alpha_{ij} \partial \alpha_{kl}}. \]

1 Note that in [10, Theorem 1.4] \( F := -\Psi^{-1} \).
Note that $F^{ij} = F^i_k g^{kj}$. Moreover, $F$ is said to be

(i) **Strictly elliptic**, if $F^{ij} \eta_i \eta_j > 0 \ \forall 0 \neq \eta \in \mathbb{R}^n$,
(ii) **One-homogeneous**, if for all $x \in M$, $F(\cdot, x)$ is homogeneous of degree one, and
(iii) **Inverse concave**, if the map $\tilde{F} \in C^\infty(\Gamma_+ \times M)$ defined by

$$\tilde{F}(\lambda_i, x) = -F(\lambda_i^{-1}, x)$$

is concave.

We use the convention for the Riemann tensor from [11]. For a Riemannian or Lorentzian manifold $(M, g, \nabla)$,

$$\text{Rm}(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y, X]}Z$$

and we lower the upper index to the first slot:

$$\text{Rm}(W, X, Y, Z) = g(W, \text{Rm}(X, Y)Z).$$

The respective local coordinate expressions are $(R_{ijkl}^m)$ and $(R_{iijkl})$.

**Definition 1.4**

(i) A pointwise curvature function $F \in C^\infty(\Gamma \times M)$ is $\Phi^1$-inverse concave for some $\Phi^1 \in C^\infty(\Gamma \times M, T^4, 0(M))$, provided at all $\beta > 0$ we have

$$F^{ij, kl} \eta_i \eta_j \eta_k \eta_l + 2F^{ik} \tilde{\beta}^{jl} \eta_i \eta_j \eta_k \eta_l \geq \Phi^{ij, kl} \eta_i \eta_j \eta_k \eta_l,$$

where $\tilde{\beta}^{ik} \beta_{kj} = \delta^i_j$.

(ii) For $\alpha \in \Gamma$ we define a curvature-adjusted modulus of $\Phi^1$-inverse concavity,

$$\omega_F(\alpha)(\eta, v) = \Phi^{ij, kl} \eta_i \eta_j \eta_k \eta_l + D^2_{xx} F(v, v) + 2D_{x^k} F^{ij} \eta_i \eta_j v^k$$

$$+ \text{tr}_g \text{Rm}(\alpha^x, v, D_{\alpha^x} F, v),$$

where $D$ denotes the product connection on $\mathbb{R}^{n \times n} \times M$. Here the curvature term denotes contracting the vector parts of the $(1, 1)$ tensors $\alpha^x = \alpha^i_j$, $D_{\alpha^x} F = F^k_l$ with the Riemann tensor and tracing the resulting bilinear form with respect to the metric so that

$$\text{tr}_g \text{Rm}(\alpha^x, e_m, D_{\alpha^x} F, e_m) = g^{jl} \alpha^i_j F^k_l R_{i j k m}.$$

**Remark 1.5** If $(A, x) \mapsto -F(A^{-1}, x)$ is concave (i.e., $F$ is inverse concave), then we take $\Phi = 0$ and for all $(\eta, v)$ we have

$$\omega_F(\eta, v) \geq \text{tr}_g \text{Rm}(\alpha^x, v, D_{\alpha^x} F, v).$$

On several occasions, where there is a homogeneity condition on $F$, we will be able to choose a good positive $\Phi$ that allows to relax assumptions on the other variables of the operator $F$; see Sect. 2.

We state the main result of the paper which contains Theorem 1.2 as a special case.

**Theorem 1.6** (Non-homogeneous CRT) Let $(M, g, \nabla)$ be a connected Riemannian manifold and $\Gamma$ an open, convex cone containing $\Gamma_+$. Suppose $F \in C^\infty(\Gamma \times M)$ is a $\Phi^1$-inverse concave, strictly elliptic pointwise curvature function. Let $\alpha$ be a Codazzi, non-negative, symmetric $2$-tensor with eigenvalues in $\Gamma$ and

$$F(\alpha^x, \cdot) = 0 \ \text{on} \ M.$$
Suppose for all $\Omega \subset M$ there exists a positive constant $c = c(\Omega)$, such that for all eigenvectors $v$ of $\alpha^2$ there holds
\[ \omega_F(\alpha)(\nabla_v \alpha, v) \geq -c(\alpha(v, v) + |\nabla \alpha(v, v)|). \]
Then $\alpha$ is of constant rank.

**Remark 1.7** It might seem more natural to replace the condition on $\omega_F$ with the condition
\[ \omega_F(\alpha)(\eta, v) \geq -c(\alpha(v, v) + |\nabla \alpha(v, v)|) \]
for every $\eta$ and all $v$. Indeed such a condition certainly leads to constant rank theorems since taking in particular $\eta = \nabla_v \alpha$, and $v$ and eigenvector, we may apply Theorem 1.6. However, the requirement holding for all $\eta, v$ is too restrictive for applications such as in Theorem 1.2. See the proof in Sect. 2 below where the required inequality is only proved to hold for $\eta = \nabla_v \alpha$ and $v$ an eigenvector.

An application of Theorem 1.6 to a non-homogeneous curvature problem is given in Theorem 2.4. Such a result was declared interesting in [16]. The full results are listed in Sect. 2.

CRT (also known as the microscopic convexity principle) was initially developed in [9] in two-dimensions for convex solutions of semi-linear equations, $\Delta u = f(u)$ using the maximum principle and the homotopy deformation lemma. The result was extended to higher dimensions in [20]. The continuity method combined with a CRT yields existence of strictly convex solutions to important curvature problems. For example, a CRT was an important ingredient in the study of prescribed curvature problems such as the Christoffel-Minkowski problem and prescribed Weingarten curvature problem [12, 14, 15]. Later, general theorems for fully nonlinear equations were obtained in [5, 10] under the assumption that $A \mapsto F(A^{-1})$ is locally convex. These approaches are based on the observation that a non-negative definite matrix valued function $A$ has constant rank if and only if there is an $\ell$ such that the elementary symmetric functions satisfy $\sigma_\ell \equiv 0$ and $\sigma_{\ell-1} > 0$. To apply this observation requires rather delicate, long computations and the introduction of clever auxiliary functions. The difficulties are at least in part due to the non-linearity of $\sigma_\ell$. An alternative approach was taken in [24, 25], using a linear combination of lowest $m$ eigenvalues, which provides a linearity advantage at the expense of losing regularity compared with $\sigma_\ell$. The authors get around this difficulty by perturbing $A$ so that the eigenvalues are distinct (thus restoring regularity) but then using an approximation argument. Our approach based on the viscosity inequality shows that $G_m$ enjoys sufficient regularity to apply the strong maximum principle and this suffices to obtain a self-contained proof of the CRT.

We remark here, that our method is capable of reproving the results in [5, 10], namely with the help of Theorem 3.4 it is possible to prove that any convex solution $u$ to
\[ H(\nabla^2 u, \nabla u, u, \cdot) = 0 \]
has constant rank under the assumption that
\[ (A, u, x) \mapsto -H(A^{-1}, p, u, x) \]
is concave for fixed $p$. This result does not follow from Theorem 1.6, but by using a suitably redefined $\omega_F$ in Theorem 3.4, this result follows in the same way as Theorem 1.6. Here we rather want to focus on geometric problems.

We proceed as follows: In Sect. 2 we collect and prove direct applications of Theorem 1.6. In Sect. 3 we prove the viscosity inequality satisfied by the subtrace, a result that is of interest by itself. After some further corollaries, we conclude with the proof of Theorem 1.6.
2 Applications

In this section, we collect a few applications of Theorem 1.6. We fix an assumption that we need on several occasions.

Assumption 2.1 Let $\Gamma$ be as in Definition 1.3.

(i) $\Psi \in C^\infty(\Gamma)$ is a positive, strictly elliptic, homogeneous function of degree one and normalized to $\Psi(1, \ldots, 1) = n$,

(ii) $\Psi$ is inverse concave.

Recall that such a function $\Psi$ at invertible arguments $\beta$ satisfies

$$\Psi^{ij,kl} \eta_{ij} \eta_{kl} + 2 \Psi^{ik} \tilde{p}^{jl} \eta_{ij} \eta_{kl} \geq \frac{2}{\Psi} (\Psi^{ij} \eta_{ij})^2 \quad (2.1)$$

for all symmetric $(\eta_{ij})$; see for example [2].

In order to facilitate notation, for covariant derivatives we use semi-colons, e.g., the components of the second derivative $\nabla^2 T$ of a tensor are denoted by $T_{;ij} = \nabla_i \nabla_j T - \nabla_j \nabla_i T$.

First, we illustrate how Theorem 1.2 follows from Theorem 1.6.

**Proof of Theorem 1.2** We define $F = \Psi - f$. In view of (2.1) and Definition 1.4, we have

$$\Phi^{ij,kl} \eta_{ij} \eta_{kl} = 2 \Psi^{-1} (\Psi^{ij} \eta_{ij})^2.$$

Let $x_0 \in M$ and $(e_i)_{1 \leq i \leq n}$ be an orthonormal basis of eigenvectors for $\alpha^\sharp(x_0)$. In the associated coordinates, we calculate

$$\omega_F(\alpha)(\nabla_{e_m} \alpha, e_m) \geq 2 f^{-1} f_m^2 - f_{;mm} + \tau \Psi^{kr} \eta_{kr} \eta_{mr}$$

$$\geq 2 f^{-1} f_m^2 - f_{;mm} + \tau f - c \alpha_{mm}$$

$$= f^2 \left( \left( f^{-1} \right)_{mm} + \tau f^{-1} \right) - c \alpha_{mm},$$

for some constant $c$. Hence the claim follows from Theorem 1.6. \hfill \Box

For a $C^2$ function $\zeta$ on a space $(M, g)$ of constant curvature $\tau_M$,

$$r_M[\zeta] := \tau_M \nabla^2 \zeta + g \zeta.$$  

The next theorem contains the full rank theorems from [14, 15, 17] as special cases.

**Theorem 2.2** ($L_p$-Christoffel-Minkowski Type Equations) Suppose $(M, g, \nabla)$ is either the hyperbolic space $\mathbb{H}^n$ or the sphere $\mathbb{S}^n$ equipped with their standard metrics and connections. Let $\Psi$ satisfy Assumption 2.1, $k \geq 1$, $p \neq 0$ and $0 < \phi, s \in C^\infty(M)$ satisfy

$$r_M[s] \geq 0, \quad s^{1-p} \Psi^k (r_M[s]) = \phi.$$  

If either

$$r_{\mathbb{H}^n}[\phi^{-\frac{1}{p+k-1}}] \geq 0, \quad p + k - 1 < 0,$$

or

$$r_{\mathbb{S}^n}[\phi^{-\frac{1}{p+k-1}}] \geq 0, \quad p \geq 1,$$

then $r_M[s]$ is of constant rank. In particular, if $M = \mathbb{S}^n$, then we have $r_{\mathbb{S}^n}[s] > 0$. 

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**Proof** Note that $\alpha = r_M[s]$ is a Codazzi tensor. We define

$$F = \Psi - (\phi s^{-p-1})^\frac{1}{k} = \Psi - f.$$  

For simplicity, we rewrite $f = us^{\frac{q}{2}}$, where $u = \phi^\frac{1}{k}$ and $q = \frac{p+k-1}{k}$.

As in the proof of Theorem 1.2, we have

$$\omega_F(\alpha)(\nabla_{e_m} \alpha, e_m) \geq 2f^{-1}f_{;m}^2 - f_{;mm} + \tau_M f - c\alpha_{mm}.$$  

Now we calculate

$$f_{;mm} - 2f^{-1}f_{;m}^2 - \tau_M f = -(\tau_M q u + \frac{q+1}{q}(u_m^2) - u_{;mm})s^{q-1}$$  

$$\quad - \frac{q-1}{q} \left( \frac{u_m}{u} + q \frac{s_m}{s} \right)^2 f$$  

$$\quad + \tau_M (q-1) f s^{-1} r_M[s]_{mm}.$$  

Therefore, if either $r_{\Omega^n}[u^{-\frac{q}{2}}] \geq 0$, $q < 0$ or $r_{\Omega^n}[u^{-\frac{q}{2}}] \geq 0$, $q \geq 1$, then

$$f_{;mm} - 2f^{-1}f_{;m}^2 - \tau_M f \leq c\alpha_{mm},$$

for some $c \geq 0$. The result follows from Theorem 1.6. Since $\mathbb{S}^n$ is compact, at some point $y$ we must have $r_{\Omega^n}[s](y) > 0$. Hence $r_{\Omega^n}[s] > 0$ on $M$.

**Remark 2.3** Let $M = x(\Omega), x : \Omega \hookrightarrow \mathbb{R}^{n+1}$ be a co-compact, convex, spacelike hypersurface. The support function of $M$, $s : \mathbb{R}^n \to \mathbb{R}$, is defined by $s(z) = \inf\{-\langle z, p \rangle; p \in M\}$, and $r_{\Omega^n}[s]$ is non-negative definite. Moreover, if $r > 0$, then the eigenvalues of $r$ with respect to $g$ are the principal radii of curvature; e.g., [1]. Therefore, the curvature problem stated in the previous theorem can be considered as an $L_p$-Christoffel-Minkowski type problem in the Minkowski space.

In [16] the authors asked the validity of CRT for non-homogeneous curvature problems. In this respect we have the following theorem. First we have to recall the definition of the Garding cones:

$$\Gamma_\ell = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_\ell(\lambda) > 0 \},$$

where $\sigma_k$ is the $k$-th elementary symmetric polynomial of the $\lambda_i$. In $\Gamma_\ell$, all $\sigma_k, 1 \leq k \leq \ell$, are strictly elliptic and the $\sigma_k^{\frac{1}{k}}$ are inverse concave, see [18]. For a cone $\Gamma \subset \mathbb{R}^n$, on a Riemannian manifold $(M, g)$ a bilinear form $\alpha$ is called $\Gamma$-admissible, if its eigenvalues with respect to $g$ are in $\Gamma$.

**Theorem 2.4** *(A non-homogeneous curvature problem)* Let $\phi > 0$ be a smooth function on $(\mathbb{S}^n, g, \nabla)$ with

$$\phi g - \nabla^2 \phi \geq 0,$$

$\psi_\ell \equiv 1$ and $0 < \psi_k \in C^\infty(\mathbb{S}^n)$ for $1 \leq k \leq \ell - 1$ satisfy \footnote{Note this forces $\psi_1$ to be constant.}

$$\nabla^2 \psi_k - \frac{k}{k+1} \frac{\nabla \psi_k \otimes \nabla \psi_k}{\psi_k} + (k-1)\psi_k \geq 0.$$
Let \( \alpha \) be a \( \Gamma_{\ell} \)-admissible, Codazzi, non-negative, symmetric 2-tensor, such that
\[
\sum_{k=1}^{\ell} \psi_k(x)\sigma_k(\alpha, g) = \phi(x).
\]

Then \( \alpha \) is of constant rank. In particular, when \( \alpha = r_{S^\ell}[s] \geq 0 \) for some positive function \( s \in C^\infty(S^n) \), then in fact we have \( \alpha > 0 \).

**Proof** The result follows quickly from Theorem 1.6. We define
\[
F(\alpha, g, x) = \sum_{k=1}^{\ell} \psi_k(x)\sigma_k(\alpha, g) - \phi(x).
\]

Since \( \sigma_k^{1/k} \) is inverse concave and 1-homogeneous, \( F \) is \( \Phi \)-inverse concave with
\[
\Phi^{pq, rs} \eta_{pq} \eta_{rs} := \sum_{k=1}^{\ell} \frac{k + 1}{k} \psi_k \frac{\sigma_k^{pq} \sigma_k^{rs}}{\sigma_k} \eta_{pq} \eta_{rs}.
\]

Let \( x_0 \in M \) and \( (e_i)_{1 \leq i \leq n} \) be an orthonormal basis of eigenvectors for \( \alpha^2(x_0) \). Now using
\[
F^{kr} \alpha^l_{kRlir} = F^{kr} \alpha^l_{k}(g_{lr}g_{ii} - g_{li}g_{ri}) = \sum_{k=1}^{\ell} k\psi_k\sigma_k - F^{ii} \alpha^{ii},
\]
we deduce
\[
\omega_F(\alpha)(\nabla e_i, \alpha, e_i) + \phi_{ii}
\]
\[
\geq \sum_{k=1}^{\ell} \left( \sigma_k \psi_{k;ii} + 2\psi_{k;i}\sigma_k + k + \frac{1}{k} \psi_k (\sigma_k^{k;i})^2 + k\psi_k \sigma_k \right) - c\alpha_{ii}
\]
\[
\geq \sum_{k=1}^{\ell} \left( \psi_{k;ii} - \frac{k}{k + 1} \frac{(\psi_{k;i})^2}{\psi_k} + (k - 1)\psi_k + \psi_k \right) \sigma_k - c\alpha_{ii}
\]
\[
= \sum_{k=1}^{\ell} \left( \psi_{k;ii} - \frac{k}{k + 1} \frac{(\psi_{k;i})^2}{\psi_k} + (k - 1)\psi_k \right) \sigma_k + \phi + (\ell - 1)\sigma_{\ell} - c\alpha_{ii}.
\]

Therefore, \( \omega_F(\alpha)(\nabla e_i, \alpha, e_i) + c\alpha_{ii} \) is non-negative for some constant \( c \). \( \square \)

Let \( (N, \tilde{g}, \tilde{D}) \) be a simply connected Riemannian or Lorentzian spaceform of constant sectional curvature \( \tau_N \). That is, \( N \) is either the Euclidean space \( \mathbb{R}^{n+1} \), the sphere \( S^{n+1} \), the hyperbolic space \( \mathbb{H}^{n+1} \) with respective sectional curvature \( 0, 1, -1 \) or the \( (n+1) \)-dimensional Lorentzian de Sitter space \( \mathbb{H}^{n,1} \) with sectional curvature \( 1 \).

Assume \( M = x(\Omega) \) given by \( x: \Omega \hookrightarrow N \) is a connected, spacelike, locally convex hypersurface of \( N \) and
\[
f \in C^\infty(M \times \mathbb{R}_+ \times \tilde{N}),
\]
where \( \tilde{N} \) denotes the dual manifold of \( N \), i.e.,
\[
\mathbb{R}^{n+1} = S^n, \quad \tilde{S}^{n+1} = S^{n+1}, \quad \tilde{\mathbb{H}}^{n+1} = S^{n,1}, \quad \tilde{\mathbb{H}}^{n,1} = \mathbb{H}^{n+1}.
\]

Here \( f \) is extended as a zero homogeneous function to the ambient space. We write \( v, h, s \) for the future directed (timelike) normal, the second fundamental form and the support function.
of $M$, respectively (cf. [7, 8]). The eigenvalues of $h$ with respect to the induced metric on $\Sigma$ are ordered as $\kappa_1 \leq \cdots \leq \kappa_n$ and we write in short

$$\kappa = (\kappa_1, \ldots, \kappa_n).$$

The Gauss equation (cf. [11, (1.1.37)]) relates extrinsic and intrinsic curvatures,

$$R_{ijkl} = \sigma(h_{ik}h_{jl} - h_{il}h_{jk}) + \bar{R}m(x_{i}, x_{j}, x_{k}, x_{l})$$

$$= \sigma(h_{ik}h_{jl} - h_{il}h_{jk}) + \tau_N(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}),$$

where $\sigma = \bar{g}(\nu, \nu)$ and the second fundamental form is defined by

$$\bar{D}X Y = \nabla_X Y - \sigma(h(X), Y).$$

**Theorem 2.5** Let $(N, \bar{g}, \bar{D})$ be one of the spaces above and let $\Psi$ satisfy Assumption 2.1. Let $M$ be a connected, spacelike, locally convex and $\Gamma$-admissible hypersurface such that

$$\Psi(\kappa) = f(x, s, \nu),$$

where $0 < f \in C^\infty(M \times \mathbb{R}_+ \times \bar{N})$ and

$$\bar{D}^2_{xx} f^{-1} + \tau_N f^{-1} \bar{g} \geq 0.$$

Then the second fundamental form of $M$ is of constant rank.

**Proof** Define $F(h, \bar{g}, x) = \Psi(h^2) - f(x, s(x), v(x))$. Let $x_0 \in M$ and $(e_i)_{1 \leq i \leq n}$ be an orthonormal basis of eigenvectors for $h^2(x_0)$. Now in view of Theorem 1.6, the claim follows from [8, p. 15] and a computation using the Gauss equation (2.2):

$$\omega_F(h)(\nabla_{e_m} h, e_m)$$

$$\geq 2\Psi^{-1}(\Psi, \bar{\kappa})^2 - \bar{D}^2_{xx} f(e_m, e_m) + F^i h^k R_{kmlm} - c(h_{mm} + |\nabla h_{mn}|)$$

$$\geq 2\Psi^{-1}(\Psi, \bar{\kappa})^2 - \bar{D}^2_{xx} f(e_m, e_m) + \Psi^i h^k R_{kmlm} - c(h_{mm} + |\nabla h_{mn}|)$$

$$\geq 2f^{-1}(\bar{D} x f(e_m))^2 - \bar{D}^2_{xx} f(e_m, e_m) + \tau_N \Psi^i h^k (g_{kl} - g_{km} g_{lm})$$

$$- c(h_{mm} + |\nabla h_{mn}|)$$

$$\geq 2f^{-1}(\bar{D} x f(e_m))^2 - \bar{D}^2_{xx} f(e_m, e_m) + \tau_N f - c(h_{mm} + |\nabla h_{mn}|)$$

$$\geq -c(h_{mm} + |\nabla h_{mn}|).$$

The following corollary contains the CRT from [12, 13] as special cases.

**Corollary 2.6** (Curvature Measures Type Equations) Suppose the curvature function $\Psi$ satisfies Assumption 2.1, $1 \leq k \leq n - 1$, $p \in \mathbb{R}$ and $0 < \phi \in C^\infty(S^n)$. Let $M$ be a $\Gamma$-admissible convex hypersurface of $\mathbb{R}^{n+1}$ which encloses the origin in its interior and suppose

$$\Psi(\kappa) = (x, \nu)^p |x|^{-\frac{n+1}{k}} \phi \left( \frac{x}{|x|} \right)^{\frac{1}{k}}.$$

If

$$|x|^{\frac{n+1}{k}} \phi \left( \frac{x}{|x|} \right)^{-\frac{1}{k}}$$

is convex on $\mathbb{R}^{n+1} \setminus \{0\}$, then $M$ is strictly convex.
3 A viscosity approach

The following lemma served as the main motivation for us to study the constant rank theorems with a viscosity approach. It shows that the smallest eigenvalue of a bilinear form satisfies a viscosity inequality. In the context of extrinsic curvature flows a similar approach was taken to prove preservation of convex cones; see [21, 22]. There it was shown that the distance of the vector of eigenvalues to the boundary of a convex cone satisfies a viscosity inequality.

**Lemma 3.1** [3, Lemma 5] Let the eigenvalues of a symmetric 2-tensor $\alpha$ with respect to a metric $(g, \nabla)$ at $x_0$ be ordered via

$$\lambda_1 = \cdots = \lambda_{D_1} < \lambda_{D_1+1} \leq \cdots \leq \lambda_n,$$

for some $D_1 \geq 1$. Let $\xi$ be a lower support for $\lambda_1$ at $x_0$. That is, $\xi$ is a smooth function such that in an open neighborhood of $x_0$,

$$\xi \leq \lambda_1$$

and $\xi(x_0) = \lambda_1(x_0)$. Choose an orthonormal frame for $T_{x_0}M$ such that

$$\alpha_{ij} = \delta_{ij}\lambda_i, \quad g_{ij} = \delta_{ij}.$$ 

Then at $x_0$ we have for $1 \leq k \leq n$,

(1) $$\alpha_{ij;k} = \delta_{ij}\xi_i ; k \quad 1 \leq i, j \leq D_1,$$

(2) $$\xi_{;kk} \leq \alpha_{11;kk} - 2 \sum_{j > D_1} (\alpha_{1j;k})^2 \frac{\lambda_j - \lambda_1}{\lambda_j^2}.$$ 

While the previous lemma is sufficient for full rank theorems (i.e., when the respective linear map is non-negative, and positive definite at least at one point), we need to generalize [3, Lemma 5] from the smallest eigenvalue to an arbitrary subtrace of a matrix to treat constant rank theorems.

To formulate the following lemma, we introduce some notation. For a symmetric 2-tensor $\alpha$ on a vector space $V$ with inner product $g$, let $\alpha^\sharp$ be the metric raised endomorphism defined by $g(\alpha^\sharp(X), Y) = \alpha(X, Y)$. Then $\alpha^\sharp$ is diagonalizable and we write

$$\lambda_1 \leq \cdots \leq \lambda_n$$

for the eigenvalues with distinct eigenspaces $E_k$ of dimension $d_k = \dim E_k, 1 \leq k \leq N$. For convenience, let $E_0 = \{0\}$ and $d_0 = 0$. Define

$$\tilde{E}_j = \bigoplus_{k=0}^{j} E_k, \quad \tilde{d}_j = \dim \tilde{E}_j$$

for $0 \leq j \leq N$ so that

$$\{0\} = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_N = V, \quad \tilde{E}_k = \tilde{E}_{k-1} \oplus E_k.$$ 

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Let \((e_j)_{1 \leq j \leq n}\) be an orthonormal basis of eigenvectors corresponding to the eigenvalues \((\lambda_j)_{1 \leq j \leq n}\) giving \(E_k = \text{span}\{e_{d_k-1}, \ldots, e_{d_k}\}\) and \(\tilde{E}_k = \text{span}\{e_1, \ldots, e_{d_k}\}\). For each \(1 \leq m \leq n\), there is a unique \(j(m)\) such that
\[
\tilde{E}_{j(m)-1} \subset V_m := \text{span}\{e_1, \ldots, e_m\} \subset \tilde{E}_{j(m)}.
\]
Then \(\tilde{d}_{j(m)-1} < m \leq \tilde{d}_{j(m)}\). For convenience, we write
\[
D_m := \tilde{d}_{j(m)}.
\]
Note that \(D_m\) is the largest number such that
\[
\lambda_1 \leq \cdots \leq \lambda_m = \cdots = \lambda_{D_m} < \lambda_{D_m + 1} \leq \cdots \leq \lambda_n,
\]
and hence
\[
\tilde{E}_{j(m)} = \text{span}\{e_1, \ldots, e_{D_m}\}.
\]
The subspace \(V_m\) is invariant under \(\alpha^\sharp\) and the trace of \(\alpha^\sharp\) restricted to \(V_m\) is the subtrace,
\[
G_m := \sum_{k=1}^{m} \lambda_k.
\]
This subtrace is characterized by Ky Fan’s maximum principle (cf. [6, Theorem 6.5]), taking the infimum with respect to all traces of \(\pi_P \circ \alpha^\sharp|_P\) over \(m\)-planes of the tangent spaces where \(\pi_P\) is orthogonal projection onto an \(m\)-plane \(P\):
\[
G_m = \inf_P \{ \text{tr} \pi_P \circ \alpha^\sharp|_P : P = m\text{-plane}\} = \inf_{(w_k)_{1 \leq k \leq m}} \left\{ \sum_{k,l=1}^{m} g^{kl} \alpha(w_k, w_l) : (g(w_k, w_l))_{1 \leq k,l \leq m} > 0 \right\},
\]
where \((g^{kl})\) is the inverse of \(g_{kl} = g(w_k, w_l)\). Now suppose \(\alpha\) is a bilinear form on a Riemannian manifold \((M, g)\), \(x_0 \in M\) and \((e_i)_{1 \leq i \leq n}\) is an orthonormal basis of eigenvectors at \(x_0\) with eigenvalues
\[
\lambda_1(x_0) \leq \cdots \leq \lambda_n(x_0).
\]
Letting \(w_i(x), 1 \leq i \leq m\), be any set of linearly independent local vector fields around \(x_0\) with \(w_i(x_0) = e_i\), then we have a smooth upper support function for \(G_m\) at \(x_0\):
\[
\Theta(x) := \sum_{k,l=1}^{m} g^{kl} \alpha_{kl} \geq G_m(x), \quad \Theta(x_0) = G_m(x_0),
\]
where \(\alpha_{kl} = \alpha(w_k(x), w_l(x))\). We make use of \(\Theta\) to prove the next lemma generalizing Lemma 3.1.

**Lemma 3.2** Let \((M, g)\) be a Riemannian manifold and let \(\alpha\) be a symmetric 2-tensor on \(TM\). Suppose \(1 \leq m \leq n\) and \(\xi\) is a (local) lower support at \(x_0\) for the subtrace \(G_m(\alpha^\sharp)\). Then at \(x_0\) we have
\[
(1) \qquad \xi_{i,j} = \text{tr}_{V_m} \alpha_{ij} = \sum_{k=1}^{m} \alpha_{kk;i}.
\]
where $V_m = \text{span}\{e_1(x_0), \ldots, e_m(x_0)\}$ for any choice of $m$ orthonormal eigenvectors $e_k$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$ satisfying

\[ \lambda_1 \leq \cdots \leq \lambda_m = \cdots = \lambda_D < \lambda_{D+1} \leq \cdots \leq \lambda_n. \]

**Proof** For this proof we use the summation convention for indices ranging between 1 and $m$. Let $\xi$ be a lower support for $G_m$ at $x_0$. Fix an index $1 \leq i \leq n$ and let $\gamma(s)$ be a geodesic with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = e_i(x_0)$. Let $(v_k)_{1 \leq k \leq m}$ be any basis (not necessarily orthonormal) for $V_m$ as in the statement of the lemma. As mentioned above, for any $m$ linearly independent vector fields $(w_k(s))_{1 \leq k \leq m}$ along $\gamma$ with $w_k(0) = v_k(x_0)$, $\alpha_{kl} = \alpha(w_k, w_l)$ and $(g^{kl}) = (g(w_k, w_l))^{-1}$, the function

\[ \Theta(s) := g^{kl}(\alpha_{kl} - \xi(\gamma(s))) \]

satisfies

\[ \Theta(s) \geq 0, \quad \Theta(0) = 0 \]

and hence

\[ \dot{\Theta}(0) = 0, \quad \ddot{\Theta}(0) \geq 0. \]

Since $V_m \subseteq \bar{E}_{j(m)}$, choosing $w_k$ such that $\dot{w}_k(0) \perp \bar{E}_{j(m)}(x_0)$ gives

\[ \ddot{g}_{kl}(0) = g(\dot{w}_k(0), v_l) + g(v_k, \dot{w}_l(0)) = 0 \]

and hence also

\[ g^{kl}(0) = -g^{ka}(0)\dot{g}_{ab}(0)g^{bl}(0) = 0. \]

Then we compute

\[ 0 = \ddot{\Theta}(0) = \left( g^{kl}(\alpha_{kl};i) - \xi_i \right)|_{x_0} \]

giving the first part.

Now we move on to the second derivatives. For this we make the additional assumptions, $v_k = e_k$ and $\dot{w}_k(0) = 0$. We first calculate

\[ \ddot{g}^{kl}(0) = \frac{1}{g^{km}g^{ml}g^{ra}g^{sb}g^{bl}-g^{ka}g^{sb}g^{bl}+g^{ka}g^{sb}g^{bl}} \]

since $\dot{g}_{kl}(0) = 0$ and $g^{kl}(0) = g^{kl}$. Then from $\dot{w}_k(0) = 0$ we obtain

\[ \ddot{g}^{kl}(0) = -[g(\dot{w}_k, w_l) + g(w_k, \dot{w}_l) + 2g(\dot{w}_k, \dot{w}_l)](0) \]

\[ = -2g^{ka}(\dot{w}_a(0), \dot{w}_b(0))g^{bl}. \]
From the local minimum property, 
\[ 0 \leq \dot{\varphi}(0) \]
\[ = \ddot{g}^{kl}(0)\alpha_{kl} + \delta^{kl} \frac{d^2}{ds^2} |_{s=0} \alpha_{kl}(s) - \xi; ii(x_0) \]
\[ = -2g(\dot{w}_k(0), \dot{w}_l(0))\alpha^{kl} + \delta^{kl}\alpha_{kl;ii} \]
\[ + 4\delta^{kl}\nabla_i\alpha(\dot{w}_k(0), w_l(0)) + 2\delta^{kl}\alpha(\dot{w}_k(0), \dot{w}_l(0)) - \xi; ii(x_0) \]
\[ = \sum_{k=1}^{m} \alpha_{kk;ii} - \xi; ii(x_0) \]
\[ + 2 \sum_{k=1}^{m} (2\nabla_i\alpha(\dot{w}_k(0), e_k) + \alpha(\dot{w}_k(0), \dot{w}_k(0)) - g(\dot{w}_k(0), \dot{w}_k(0))\lambda_k) \].

From \( \dot{w}_k(0) \perp \tilde{E}_{j(m)} \), we may write \( \dot{w}_k(0) = \sum_{r > D_m} c^r_k e_r \) giving
\[ \xi; ii(x_0) - \sum_{k=1}^{m} \alpha_{kk;ii} \leq 2 \sum_{k=1}^{m} \sum_{r > D_m} (2c^r_k\alpha_{kr;i} + (c^r_k)^2\lambda_r - (c^r_k)^2\lambda_k) \]
\[ = 2 \sum_{k=1}^{m} \sum_{r > D_m} c^r_k (2\alpha_{kr;i} + c^r_k\lambda_r - \lambda_k) \].

Optimizing yields the specific choice
\[ \dot{w}_k(0) = - \sum_{r > D_m} \frac{\alpha_{kr;i}}{\lambda_r - \lambda_k} e_r. \]

From this we obtain
\[ \xi; ii(x_0) - \sum_{k=1}^{m} \alpha_{kk;ii} \leq - 2 \sum_{k=1}^{m} \sum_{r > D_m} \frac{\alpha_{kr;i}}{\lambda_r - \lambda_k} (2\alpha_{kr;i} - \alpha_{kr;i}) \]
\[ = - 2 \sum_{k=1}^{m} \sum_{r > D_m} \frac{(\alpha_{kr;i})^2}{\lambda_r - \lambda_k}. \]

\[ \square \]

**Corollary 3.3** Let \( \alpha \) be a non-negative, symmetric 2-tensor on \( TM \). Suppose for some \( 1 \leq m \leq n \) that \( \dim \ker \alpha^2 \geq m - 1 \) or equivalently that the eigenvalues of \( \alpha^2 \) satisfy \( \lambda_1 \equiv \cdots \equiv \lambda_{m-1} \equiv 0 \). Then for all \( x_0 \) and any lower support \( \xi \) for \( G_m \) at \( x_0 \) and all \( 1 \leq i \leq n \) we have

1. \( (\nabla_i \alpha(x_0))|_{\ker \alpha^2 \times \ker \alpha^2} = 0 \),
2. \( (\nabla_i \alpha(x_0))|_{\tilde{E}_{j(m)} \times \tilde{E}_{j(m)}} = g \nabla_i \xi(x_0), \) if \( \lambda_m(x_0) > 0 \).

**Proof** We use a basis \((e_i)\) as in Lemma 3.2. To prove (1) we may assume \( \lambda_1(x_0) = 0 \), and hence the zero function is a lower support for \( \lambda_1 \). By Lemma 3.1, we have \( \nabla \alpha_{kl} = 0 \) for all \( 1 \leq k, l \leq d_1 \) proving the first equation.

Now we prove (2). For \( m = 1 \) the claim follows from Lemma 3.2-(1). Suppose \( m > 1 \). If \( d_1 \geq m \) at \( x_0 \) then \( \lambda_m(x_0) = 0 \) which violates our assumption. Hence \( d_1 = m - 1 \) and
Taking any unit vector \( v \in E_2(x_0) = \text{span}\{e_m, \ldots, e_{D_m}\} \) and applying Lemma 3.2-(1) with \( V_m = \{e_1, \ldots, e_{m-1}, v\} \) gives
\[
\nabla_i \alpha(v, v) = \text{tr}_{V_m} \nabla_i \alpha = \nabla_i \xi \quad \forall 1 \leq i \leq n.
\]
Polarizing the quadratic form \( v \mapsto \nabla_i \alpha(v, v) \) over \( E_2(x_0) \) then shows
\[
\nabla_i \alpha_{kl} = \delta_{kl} \nabla_i \xi \quad \forall m \leq k, l \leq D_m.
\]
\( \square \)

Now we state the key outcome of the results in this section. We want to acknowledge that the following proof is inspired by the beautiful paper [24] and their sophisticated test function
\[
Q = \sum_{q=1}^m G_q.
\]

**Theorem 3.4** Under the assumptions of Theorem 1.6, if \( \dim \ker \alpha^z \geq m - 1 \), for all \( \Omega \in M \) there exists a constant \( c = c(\Omega) \), such that for all \( x_0 \in \Omega \) and any lower support function \( \xi \) for \( G_m(\alpha^z) \) at \( x_0 \) we have
\[
F^{ij} \xi_{;ij} \leq c(\xi + |\nabla \xi|).
\]

**Proof** In view of our assumption \( \lambda_{m-1} = 0 \). Hence the zero function is a smooth lower support at \( x_0 \) for every subtrace \( G_q \) with \( 1 \leq q \leq m - 1 \). Therefore by Lemma 3.2, for every \( 1 \leq q \leq m - 1 \) and every \( 1 \leq i \leq n \) we obtain
\[
0 \leq \sum_{k=1}^q \sum_{j=1}^D F^{ii} \alpha_{kk;ii} - 2 \sum_{k=1}^q \sum_{j>D_q} \frac{(\alpha_{kj;i})^2}{\lambda_j - \lambda_k}.
\]
(3.1)

Due to the Ricci identity, we have the commutation formula
\[
\alpha_{ij;kl} = \alpha_{ki;jl} = \alpha_{ki;lj} + R^p_{kjl} \alpha_{pi} + R^p_{ijl} \alpha_{pk} = \alpha_{kl;ij} + R^p_{kjl} \alpha_{pi} + R^p_{ijl} \alpha_{pk}.
\]

Taking into account Lemma 3.2 and adding the inequalities (3.1) for \( 1 \leq q \leq m - 1 \), we have at \( x_0 \),
\[
F^{ij} \xi_{;ij} \leq \sum_{q=1}^m \sum_{k=1}^q F^{ij} \alpha_{kk;ij} - 2 \sum_{q=1}^m \sum_{k=1}^q \sum_{j>D_q} F^{ii} (\alpha_{kj;i})^2 \frac{1}{\lambda_j - \lambda_k}
\]
\[
\leq \sum_{q=1}^m \sum_{k=1}^q F^{ij} \left( \alpha_{ij;kk} - R^p_{kjk} \alpha_{pi} - R^p_{ijk} \alpha_{pk} \right)
\]
\[
- 2 \sum_{q=1}^m \sum_{k=1}^q \sum_{j>D_q} F^{ii} (\alpha_{kj;i})^2 \frac{1}{\lambda_j - \lambda_k}.
\]

Now differentiating the equation \( F(\alpha^z, x) = 0 \) yields
\[
0 = F^{ij} \alpha_{ij;k} + D_x F^{ij},
\]
\[
0 = F^{ij,rs} \alpha_{ij;k} \alpha_{rs;l} + D_x F^{ij} \alpha_{ij;k} + F^{ij} \alpha_{ij;kl} + D_x F^{rs} \alpha_{rs;l} + D_x^2 F.
\]
Then substituting above gives

\[
F^{ij} \xi_{ij} \leq -2 \sum_{q=1}^{m} \sum_{j>D_m} \sum_{k \geq D_m+1} \frac{F^{ii}(\alpha_{jk};i)}{\lambda_j - \lambda_k} - \sum_{q=1}^{m} \sum_{k \geq D_m+1} F^{ij}_{rs} \alpha_{ij;k} \alpha_{rs;k} \lambda_j
\]

\[
- \sum_{q=1}^{m} \sum_{j>D_m} \left( D_{x^k x^k}^2 F + 2 D_{x^k} F^{ij} \alpha_{ij;k} + F^{ij} \left( R_{kjk}^p \alpha_{pi} + R_{ijk}^p \alpha_{pk} \right) \right) + c \xi
\]

\[
\leq -2 \sum_{q=1}^{m} \sum_{j>D_m} \sum_{k \geq D_m+1} \frac{F^{ii}(\alpha_{jk};i)}{\lambda_j} - \sum_{q=1}^{m} \sum_{k \geq D_m+1} F^{ij}_{rs} \alpha_{ij;k} \alpha_{rs;k} \lambda_j
\]

\[
\sum_{q=1}^{m} \sum_{j>D_m} \left( D_{x^k x^k}^2 F + 2 D_{x^k} F^{ij} \alpha_{ij;k} + F^{ij} R_{kjk}^p \alpha_{pi} \right) + c \xi
\]

\[
+ C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk};i| - 2 \sum_{q=1}^{m} \sum_{j=D_m+1} \sum_{j>D_m} \sum_{k \geq D_m+1} \frac{F^{ii}(\alpha_{jk};i)}{\lambda_j}
\]

where we have used that \( \alpha \) is Codazzi and the fact that \( 1 \leq k \leq m \leq D_m \) in splitting the sum involving \( F^{ij}_{rs} \) into terms where at least two indices are at most \( D_m \) and the remaining indices \( i, j, r, s > D_m \). We have also used \( \lambda_j - \lambda_k \geq \lambda_j \), and that for some constant \( c \),

\[
F^{ij} R_{ijm}^p \alpha_{pm} \geq -c \xi.
\]

Now for every \( 1 \leq k \leq m \) define

\[
\eta_k = (\eta_{ijk}) = \begin{cases} 
\alpha_{ij;k}, & i, j > D_m \\
0, & i \leq D_m \text{ or } j \leq D_m.
\end{cases}
\]

Then

\[
F^{ij} \xi_{ij} \leq -2 \sum_{q=1}^{m} \sum_{j>D_m} \sum_{k \geq D_m+1} \frac{F^{ii}(\eta_{ijk})}{\lambda_j} - \sum_{q=1}^{m} \sum_{k \geq D_m+1} F^{ij}_{rs} \eta_{ijk} \eta_{rsk}
\]

\[
- \sum_{q=1}^{m} \sum_{j>D_m} D_{x^k x^k}^2 F - 2 \sum_{q=1}^{m} \sum_{k \geq D_m+1} D_{x^k} F^{ij} \eta_{ijk} - \sum_{q=1}^{m} \sum_{k \geq D_m+1} F^{ij} R_{kjk}^p \alpha_{pi}
\]

\[
+ C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk};i| - 2 \sum_{q=1}^{m} \sum_{j=D_m+1} \sum_{j>D_m} \sum_{k \geq D_m+1} \frac{F^{ii}(\alpha_{jk};i)}{\lambda_j} + c \xi.
\]

In addition we define \( \alpha^\varepsilon = \alpha^\sharp + \varepsilon \text{id} \), which has positive eigenvalues for \( \varepsilon > 0 \). In the sequel, a subscript \( \varepsilon \) denotes evaluation of a quantity at \( \alpha^\sharp \), e.g., we put \( F^{ij}_\varepsilon = F^{ij}(\alpha^\sharp) \). We have
In view of Definition 1.4, and the definition of \( \omega_F \),

\[
F^{ij} \xi_{ji} \leq \lim_{\epsilon \to 0} \left( -2 \sum_{q=1}^{m} \sum_{k=1}^{n} \frac{F_{ij}^{q} (\xi_{ijk})^2}{\lambda_j + \epsilon} \right) - \sum_{k=1}^{q} (D^2_{x^k x^k} F)_{ij} - 2 \sum_{k=1}^{q} (D_{x^k} F^{ij})_{ij} \xi_{ijk} - \sum_{k=1}^{q} F_{ij}^{p} R_{ijk}^p (\alpha_{e})_{pi} \\
+ C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk;i}| - 2 \sum_{q=1}^{m} \sum_{k=1}^{n} \sum_{j=D_q+1}^{D_m} F_{ij}^{q} (\alpha_{ij;k})^2 \xi_{ji} + c \xi,
\]

Adding and subtracting some terms gives

\[
F^{ij} \xi_{ji} \leq \lim_{\epsilon \to 0} \left( -2 \sum_{q=1}^{m} \sum_{k=1}^{n} \frac{\Phi^{ij,rs}_{e} \xi_{ij}}{\lambda_j + \epsilon} \right) - \sum_{k=1}^{q} (D^2_{x^k x^k} F)_{ij} - 2 \sum_{k=1}^{q} (D_{x^k} F^{ij})_{ij} \xi_{ijk} - \sum_{k=1}^{q} F_{ij}^{p} R_{ijk}^p (\alpha_{e})_{pi} \\
+ C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk;i}| - 2 \sum_{q=1}^{m} \sum_{k=1}^{n} \sum_{j=D_q+1}^{D_m} F_{ij}^{q} (\alpha_{ij;k})^2 \xi_{ji} + c \xi.
\]

Next we estimate the last two lines of (3.2). We have

\[
\sum_{q=1}^{m} \sum_{k=1}^{n} \omega_F (\alpha) (\nabla_{e_k} \alpha, e_k) - \sum_{q=1}^{m} \sum_{k=1}^{n} \omega_F (\alpha) (\eta_k, e_k) \leq C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk;i}|,
\]

\[
C \sum_{i=1}^{n} \sum_{j,k \leq D_m} |\alpha_{jk;i}| \leq C \sum_{i=1}^{n} \sum_{k=1}^{D_i} \sum_{j=D_q+1}^{D_m} |\alpha_{jk;i}| + c |\nabla \xi|,
\]
where for the last inequality we used Corollary 3.3. Let us define
\[ R = C \sum_{i=1}^{n} \sum_{k=1}^{D_{i}} \sum_{j=D_{i}+1}^{D_{m}} |\alpha_{jk;i}| - 2 \sum_{q=1}^{m} \sum_{k=1}^{D_{q}} \sum_{j=D_{q}+1}^{D_{m}} \frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_{j} - \lambda_{k}}. \]

Note that if \( \lambda_{m}(x_{0}) = 0 \), then \( D_{q} = D_{m} \) for all \( q \leq m \) and hence \( R = 0 \). If \( \lambda_{m}(x_{0}) > 0 \), then we have \( D_{q} = m - 1 \) for all \( q \leq m - 1 \) and
\[ R = C \sum_{i=1}^{n} \sum_{k=1}^{m-1} \sum_{j=m}^{D_{m}} |\alpha_{jk;i}| - 2 \sum_{k=1}^{m-1} (m-k) \sum_{j=m}^{D_{m}} \frac{F^{ii}(\alpha_{ij;k})^2}{\lambda_{j} - \lambda_{k}}. \]

Therefore, due to uniform ellipticity, we can use
\[ C \sum_{i=1}^{n} |\alpha_{jk;i}| \leq 2(m-k) \frac{F^{ii}(\alpha_{jk;i})^2}{\lambda_{j} - \lambda_{k}} + c\xi \]

to show that \( R \leq c\xi \). Then by the assumptions on \( \omega_{F} \), the right hand side of (3.2) is bounded by \( c(\xi + |\nabla \xi|) \) completing the proof. \( \square \)

**Remark 3.5** Here we crucially used that \( F \) is \( \Phi \)-inverse concave, then we took the limit \( \varepsilon \to 0 \) and finally swapped \( \eta_{\varepsilon} \) with \( \nabla_{\varepsilon} \alpha \) absorbing the extra terms. If on the other hand we tried to swap first without using \( \Phi \)-inverse concavity, the extra terms would involve
\[ \sum_{r=1}^{n} \frac{F^{ii}(\nabla_{\varepsilon}(\alpha_{ij;i}))^2}{\lambda_{r} + \varepsilon}. \]
Since \( \lambda_{r} = 0 \) for \( 1 \leq r \leq m - 1 \) this blows up in the limit \( \varepsilon \to 0 \) and cannot be absorbed.

**Proof of Theorem 1.6** Let \( k := \max_{x \in M} \dim \ker \alpha^{\varepsilon}(x) \). If \( k = 0 \), we are done. By induction we show that for all \( 1 \leq m \leq k \) we have \( \lambda_{m} \equiv 0 \). For \( m = 1 \), clearly we have \( \dim \ker \alpha^{\varepsilon} \geq m - 1 \) and hence by Theorem 3.4 a lower support \( \xi \) for \( G_{1} = \lambda_{1} \) locally satisfies
\[ F^{ij}\xi_{;ij} \leq c(\xi + |\nabla \xi|). \]

By the strong maximum principle [4], \( \lambda_{1} \equiv 0 \).

Now suppose the claim holds true for \( m - 1 \), i.e.,
\[ \lambda_{1} \equiv \cdots \equiv \lambda_{m-1} \equiv 0. \]

Then a lower support \( \xi \) for \( G_{m} \) satisfies
\[ F^{ij}\xi_{;ij} \leq c(\xi + |\nabla \xi|). \]
Hence \( G_{m} \equiv 0 \) for all \( m \leq k \). Since \( k \) indicates the maximum dimension of the kernel, we must have \( \lambda_{k+1} > 0 \) and the rank is always \( n-k \). \( \square \)

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References

1. Andrews, B., Chen, X., Fang, H., McCoy, J.: Expansion of Co-compact convex spacelike hypersurfaces in Minkowski space by their curvature. Indiana Univ. Math. J. 64(2), 635–662 (2015)
2. Andrews, B.: Pinching estimates and motion of hypersurfaces by curvature functions. J. für die Reine und Angewandte Math. 608, 17–33 (2007)
3. Brendle, S., Choi, K., Daskalopoulos, P.: Asymptotic behavior of flows by powers of the Gaussian curvature. Acta Math. 219(1), 1–16 (2017)
4. Bardi, M., Da Lio, F.: On the strong maximum principle for fully nonlinear degenerate elliptic equations. Arch. Math. 73(4), 276–285 (1999)
5. Bian, B., Guan, P.: A microscopic convexity principle for nonlinear partial differential equations. Invent. Math. 177(2), 307–335 (2009)
6. Bhatia, R.: Perturbation bounds for matrix eigenvalues, Classics in Applied Mathematics, vol. 53, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2007), Reprint of the 1987 original
7. Bryan, P., Ivaki, M.N., Scheuer, J.: Harnack inequalities for curvature flows in Riemannian and Lorentzian manifolds. J. für die Reine und Angewandte Math. 2020(764), 71–109 (2019)
8. Bryan, P., Ivaki, M.N., Scheuer, J.: Parabolic approaches to curvature equations. Nonlinear Analysis 203, 112174 (2021)
9. Luis, A.: Caffarelli and Avner Friedman, Convexity of solutions of semilinear elliptic equations. Duke Math. J. 52(2), 431–456 (1985)
10. Caffarelli, L., Guan, P., Ma, X.-N.: A constant rank theorem for solutions of fully nonlinear elliptic equations. Commun. Pure Appl. Math. 60(12), 1769–1791 (2007)
11. Gerhardt, C.: Curvature problems, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., Sommerville (2006)
12. Guan, P., Lin, C., Ma, X.-N.: The Christoffel-Minkowski problem II: Weingarten curvature equations. Chin. Ann Math. Series B 27(6), 595–614 (2006)
13. Guan, P., Lin, C., Ma, X.-N.: The existence of convex body with prescribed curvature measures. Int. Math. Res. Not. 2009(11), 1947–1975 (2009)
14. Guan, P., Ma, X.-N.: The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation. Invent. Math. 151(3), 553–577 (2003)
15. Guan, P., Ma, X.-N., Zhou, F.: The Christoffel-Minkowski problem III: existence and convexity of admissible solutions. Commun. Pure Appl. Math. 59(9), 1352–1376 (2006)
16. Guan, P., Zhang, X.: A class of curvature type equations. Pure Appl. Math. Quarterly 17(3), 865–907 (2021)
17. Changqing, H., Ma, X.-N., Shen, C.: On the Christoffel-Minkowski problem of Firey’s p-sum. Calc. Var. Partial. Differ. Equ. 21(2), 137–155 (2004)
18. Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. 183(1), 45–70 (1999)
19. Ivaki, M.N.: Deforming a hypersurface by principal radii of curvature and support function. Calculus of Vari. Partial Diff. Equ. 58(1), 1–18 (2019)
20. Korevaar, N.J., Lewis, J.L.: Convex solutions of certain elliptic equations have constant rank Hessians. Arch. Rational Mech. Anal. 97(1), 19–32 (1987)
21. Langford, M.: Motion of hypersurfaces by curvature. Australian National University, Australia (2014)
22. Langford, M.: A general pinching principle for mean curvature flow and applications. Calculus of Variat. Partial Diff. Equ. 56(4), 107 (2017)
23. Scheuer, J.: Isotropic functions revisited. Arch. Math. 110(6), 591–604 (2018)
24. Székelyhidi, G., Weinkove, B.: On a constant rank theorem for nonlinear elliptic PDEs. Discrete Contin. Dyn. Syst. Series A 36(11), 6523–6532 (2016)
25. Székelyhidi, G., Weinkove, B.: Weak Harnack inequalities for eigenvalues and constant rank theorems. Comm. Partial Diff. Equ. 46(8), 1585–1600 (2021)

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