Measurement of the shear wave speed in a submerged plate

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Abstract. This paper develops an inverse method to estimate, in water, the shear wave speed in an isotropic, thick, elastomeric plate. The submerged plate is mechanically shaken and a scanning laser vibrometer is used to measure normal velocity on one surface. The temporal domain measurements are transformed into the frequency domain using a Fourier transform, then, the spatial domain measurements are transformed into the $k_x, k_y$ wavevector domain using two Fourier transforms. Once the data is in the wavevector-frequency domain, the propagation wavenumber of each specific wave type can be estimated by fitting a circle to each collection of spectral peaks. Using this measured estimate of the wavenumber corresponding to the propagating wave, the Newton-Raphson gradient method is applied (inserting the estimated wavenumber into to the theoretical dispersion curve equation for wave propagation in a fluid-loaded plate,) hence resulting in an estimate of the shear wave speed. An experiment is included to illustrate the method and statistical properties of the measurement are discussed.

1. Introduction
Inverse methods involving wave propagation are typically broken into two methods. The first is a natural frequency or resonant method [1-3], where natural frequencies of a structure are measured and equated to the corresponding analytical natural frequencies. The analytical natural frequencies are usually functions of material modulus and this resultant expression can be solved to produce a value for the modulus at each natural frequency. The second method is a transfer function method [2,4-6], which typically involve equating measured data with an analytical model of the system. This mathematical expression is rewritten so that the estimated parameters become functions of the data. Historically, most inverse methods have been developed in air (or in vacuo). The loading of underwater structures is more complicated than that of air loaded testing. Fluid forces apply a compressional load whose effects usually need to be incorporated into the model. Failure to properly include fluid loading will often produce inaccuracies.

Inverse methods do exist to measure the material properties of fluid-loaded plates. In one type of submerged inverse problem [7,8], the dilatational and shear wave speed and attenuation was determined by exploiting the phase angle measurement via an insertion loss test. In another, the complex dilatational wave speed was found by applying a system model to an echo reduction experiment [9]. These methods involve actively insonifying the plate, however, all acoustic test tanks have lower frequency limitations. Measurements below 15 kHz are typically not possible.

Elastic plate theory has been extensively developed [10-12]. Thick plates have not been used to measure material properties because they support multiple wave types propagating in the medium, and any measurement technique has to have the ability to discern between each wave type and its
contribution to the measurement. Transfer function methods that measure one output (at a single location) versus a fixed input do not have the capability to separate various wave types and their associated response levels. It is precisely this property of multiple waves that this paper exploits to measure the shear wave speed in a submerged thick plate. This is accomplished by shaking the plate at a fixed frequency while simultaneously measuring the normal velocity of the plate across its entire surface. These spatial domain measurements are transferred into a wavevector (two-wavenumber) domain by means of Fourier transforms. Individual waves are identified in this domain and the resulting free wave propagation wavenumbers are accurately estimated. Once they are known, they are inserted into a Newton-Raphson iterative solver applied to the theoretical dispersion equations for the propagation of plate waves. This results in estimates of the shear wave speed. An experiment is included to illustrate the technique when applied to an underwater elastomeric plate.

2. System equations

The theory of wave motion in a submerged, isotropic, elastic thick plate of infinite extent has been previously derived [12,13] and the transfer functions developed. The objective of this work is to estimate the shear wave speed in an underwater elastic plate using the underlying theoretical dispersion curve developed for the plate with fluid-loaded boundary conditions on the front and back. This dispersion curve is

\[ g(k_d,k_s,k_f) = p_1 \cos \left[ h(k_d^2 - k_f^2)^{1/2} \right] \cos \left[ h(k_s^2 - k_f^2)^{1/2} \right] + 2f_1 \sin \left[ h(k_d^2 - k_f^2)^{1/2} \right] \sin \left[ h(k_s^2 - k_f^2)^{1/2} \right] + 2f_2 \sin \left[ h(k_d^2 - k_f^2)^{1/2} \right] \cos \left[ h(k_s^2 - k_f^2)^{1/2} \right] + (p_2 + f_3) \sin \left[ h(k_d^2 - k_f^2)^{1/2} \right] \sin \left[ h(k_s^2 - k_f^2)^{1/2} \right] - p_1 = 0, \]

where \( h \) is the thickness of the plate (m), \( k_s \) is the shear wavenumber (rad m\(^{-1}\)), \( k_d \) is the dilatational wavenumber (rad m\(^{-1}\)), \( k_f \) is the fluid wavenumber (rad m\(^{-1}\)), and \( k \) is the measured free propagation wavenumber (rad m\(^{-1}\)) corresponding to a specific wave traveling in the plate. (For the experiments in this paper, the measurements of \( k \) will either correspond to a F(0) flexural wave, L(0) longitudinal wave, or a FE(1) flexural wave.) The relationship between the shear wavenumber and the shear wave speed is

\[ k_s = \omega / c_s \]

where \( \omega \) is the angular frequency (rad s\(^{-1}\)) and \( c_s \) is the shear wave speed (m s\(^{-1}\)); the relationship between the dilatational wavenumber and the dilatational wave speed is

\[ k_d = \omega / c_d \]

where \( c_d \) is the dilatational wave speed (m s\(^{-1}\)); and the relationship between the fluid wavenumber and fluid wave speed is

\[ k_f = \omega / c_f \]

where \( c_f \) is the fluid compressional wave speed. In equation (1), the constants are given by

\[ p_1 = -8(k_d^2 - k_f^2)^{1/2}(k_s^2 - k_f^2)^{1/2}k_s^2 k_f^2, \]

\[ f_1 = i\rho_f \rho^{-1}(k_f^2 - k_d^2)^{1/2}(k_d^2 - k_f^2)^{1/2}(k_s^2 - k_f^2)^2 k_s^2, \]

\[ f_2 = 4i\rho_f \rho^{-1}(k_f^2 - k_d^2)^{1/2}(k_d^2 - k_f^2)^{1/2}(k_s^2 - k_f^2)^2 k_s^2 k_f^2, \]

\[ p_2 = (k_s^2 - 2k_f^2)^4 + 16(k_s^2 - k_f^2)(k_s^2 - k_f^2)k_f^4, \]

and

\[ f_3 = \rho_f^2 \rho^{-2}(k_f^2 - k_d^2)^{-1}(k_d^2 - k_f^2)k_s^8, \]
where \( \rho \) is the density of the plate \((\text{kg m}^{-3})\) and \( \rho_f \) is the density of the fluid \((\text{kg m}^{-3})\). Equations (1) thru (6) define the wavenumber-frequency dispersion curve of the plate with a fluid load on both sides. Using known values of \( h, k_d, k_f, k, \rho \), and \( \rho_f \), the value of \( k_s \) (and \( c_s \)) will be estimated. It is noted that in the absence of a fluid load, \( f_1 = f_2 = f_3 = 0 \) and the zeros of equation (1) are at the exact location of the zeros of the Rayleigh-Lamb equation for wave propagation in a plate.

To understand the estimation process, it is useful to display equation (1) as a surface with respect to the dilatational and shear wavenumber and examine its functional behavior. Figure 1 is a plot of the frequency equation for the propagation of waves in a fluid loaded elastic plate versus shear and dilatational wavenumber using a free propagation wavenumber of \( k = 115.6 \text{ rad m}^{-1} \). In this figure, the right plot is a plot of the function with respect to the dilatational wavenumber with the shear wavenumber fixed at 100 \( \text{rad m}^{-1} \) and the bottom plot is a plot of the function with respect to the shear wavenumber with the dilatational wavenumber fixed at 15 \( \text{rad m}^{-1} \). The plate thickness \( h \) was 0.0254 m, the density of the plate \( \rho \) was 1100 \( \text{kg m}^{-3} \), the density of the fluid \( \rho_f \) was 1000 \( \text{kg m}^{-3} \), the fluid wavenumber \( k_f \) was 12.84 \( \text{rad m}^{-1} \), and the frequency \( \omega \) was \((2\pi)3000 \text{ rad s}^{-1}\).

Note that the function is varying with respect to the shear wavenumber and essentially flat with respect to the dilatational wavenumber. This important feature reveals three pertinent facts with respect to the estimation process: (1) the shear wavenumber (and hence the shear wave speed) can be accurately estimated, (2) this estimation process may be multi-valued due to more than one relative minimum of the function, and (3) the dilatational wavenumber cannot be accurately estimated using this method. Furthermore, the estimation of the shear wavenumber is relatively invariant with respect to the dilatational wavenumber. A previous measure of the dilatational wave speed of this material will be used in the estimation process.

![Figure 1](image_url)

**Figure 1.** A plot of the dispersion equation for waves in a fluid-loaded elastic plate at 3 kHz.
A Newton-Raphson method can be applied to equation (1) for the estimation of the shear wavenumber. To eliminate the ambiguity of both positive and negative wavenumbers, the estimation process is applied to the square of the wavenumber, rather than the wavenumber itself. This produces

\[
\left[ k_s^2 \right]_{i+1} = \left[ k_s^2 \right]_i - \left[ \frac{\partial g(k_d, k_x, k_y)}{\partial (k_s^2)} \right] \left[ g(k_d, k_x, k_y) \right]_i,
\]

where \( j \) is the iteration number at every fixed measurement location. Equation (7) is evaluated using the estimated free propagation wavenumber \( k \), and the result is an estimate of the square of the shear wavenumber. Finally, the relationship \( \frac{k_s}{\omega} = \frac{c_s}{\omega} \) is used to find the shear wave speed.

3. Experiment
An experiment was undertaken to validate the measurement technique. This estimation process uses the following assumption: (1) The return energy from the reflections at the edge of the plate does not interfere with the measurement process, and (2) the particle motion of the plate and fluid are linear. A plate was molded using Cytech Industries EN-6, a two-part urethane that consists of a mixture of a prepolymer and a curing agent. The plate was 0.780 m by 0.755 m by 0.0254 m thick and had a mass of 16.6 kg. The dilatational wave speed was previously measured as 1421 m s\(^{-1}\). The plate was mounted on four corners with bungee cords and a Wilcoxon Model F3/Z602WA electromagnetic shaker was attached to the backside near the middle. When the shaker was turned on, the front side was interrogated with a scanning Polytec LDV PSV-200 Doppler laser vibrometer that measured the normal velocity of the plate. The entire experiment was submerged in the Acoustic Test Facility at the Naval Undersea Warfare Center in Newport, Rhode Island, USA. This tank measures approximately 18.3 m by 12.2 m by 10.2 m deep and holds 2.46 million liters of fresh water. The experiment was conducted at a water temperature of 16.1 degrees Celsius. This experimental setup is shown in figure 2.

![Experimental Setup](image)

**Figure 2.** Experimental Setup.

A square grid of 90 by 90 points with a point-to-point spacing of 0.0082 m was used to collect 8100 spatial domain data points per frequency bin. After the data set was collected it was transformed into the frequency domain using a fast Fourier transform. Next, it was zero padded and transformed into the \( k_x, k_y \) wavevector domain using a two-dimensional 512 by 512 point fast Fourier transform.
Once this was accomplished, three waves were identified based on their relative spectra maxima. Isotropic elastic plate theory predicts that every wave will be circular in the $k_x, k_y$ wavevector domain and centered at $k_x = k_y = 0$, thus a circle was fit to each set of relative maxima points using an ordinary least square estimate to the wavevector domain data to estimate the propagation wavenumber. This was done at 1 to 6 kHz in increments of 1 kHz. This circular data set allows statistical properties of the estimation process to be computed.

Figure 3 is a plot of the measured wave propagation locations in the $k_x, k_y$ wavevector domain at (a) 4 kHz (left) and (b) 6 kHz (right). The F(0) flexural wave data is denoted with an x marker, the L(0) longitudinal wave data is denoted with a + marker, the FE(1) flexural wave is denoted with a o marker, and the circles fit to the markers are denoted with solid lines. For clarity, the markers have been decimated by eighty nine percent. Once the propagation wavenumbers are known, the shear wave speed can be estimated using equation (1). At the frequency of 5 kHz, the flexural wave is beginning to become incoherent across the major dimensions of the plate. The results of this estimation procedure are shown in Table 1 for all of the waves measured during the test at all experimental frequencies. Additionally, the in air test data are included in this table for comparison. (The measured propagation wavenumber of the flexural wave at 3 kHz was used to construct figure 1.)

The average value for the shear wave speed estimate for the flexural wave was 246.9 ms\(^{-1}\) and the average shear wave speed estimate for the longitudinal wave was 211.6 ms\(^{-1}\). This indicates a dispersion of the shear wave speed with respect to wave type. This is shown in figure 4, where all measurements are compared and displayed with their first standard deviation using an error bar plot. The average shear wave speed for all underwater measurements was 229.2 ms\(^{-1}\). Using this average value and the value of the dilatational wave speed, the dispersion curve in the wavenumber-frequency plane can be calculated. This is displayed as figure 5 along with the each data point. The slight mismatch between theory and experiment that is observed is due to the variation of the shear wave speed.

Each of the measurements was statistically analyzed by calculating the standard deviation of the radius of the data points for each wave at every frequency. Once this was known, the shear wave speeds were estimated at plus and minus one standard deviation away from the mean. These results are shown for each individual wave in table 2. For the analysis using minus one standard deviation,
the average shear wave speed was estimated to be 237.4 m s\(^{-1}\), and for the analysis using plus one standard deviation, the average shear wave speed was estimated to be 221.8 m s\(^{-1}\). These estimates are off by the original estimate of 229.2 m s\(^{-1}\) by 3.6 and 3.2 percent respectively, which indicates that the estimation process is extremely stable and the propagation of errors thru the analysis are small.

**Figure 4.** Shear wave speed measurements showing one standard deviation.

**Figure 5.** Dispersion curve of fluid-loaded plate showing data (markers) and theory (solid lines).
Table 1. Estimated shear wave speeds for all measured waves.

| Wave Name | Frequency kHz | Measured $k$ rad m$^{-1}$ | Estimated $k_s$ rad m$^{-1}$ | Estimated $c_s$ (water) m s$^{-1}$ | Estimated $c_s$ (air) m s$^{-1}$ |
|-----------|---------------|---------------------------|-----------------------------|-----------------------------------|---------------------------------|
| F(0)      | 1             | 57.7                      | 25.1                        | 250.8                             | 229.9                           |
| F(0)      | 2             | 88.1                      | 52.0                        | 241.9                             | 237.3                           |
| F(0)      | 3             | 115.6                     | 78.0                        | 241.6                             | 233.6                           |
| F(0)      | 4             | 141.1                     | 102.4                       | 245.4                             | 234.2                           |
| F(0)      | 5             | 163.1                     | 123.3                       | 254.8                             | 243.2                           |
| L(0)      | 3             | 63.6                      | 90.5                        | 208.4                             | 201.3                           |
| L(0)      | 4             | 98.7                      | 120.0                       | 209.4                             | 207.1                           |
| L(0)      | 5             | 140.6                     | 147.8                       | 212.6                             | 208.2                           |
| L(0)      | 6             | 182.6                     | 174.5                       | 216.0                             | 209.5                           |
| FE(1)     | 6             | 91.4                      | 178.9                       | 210.7                             | 212.0                           |

Table 2. Statistical analysis of estimated shear wave speeds.

| Wave Name | Frequency kHz | Standard Deviation $k$ rad m$^{-1}$ | Estimated $c_s$ ($-1$ Std. Dev.) m s$^{-1}$ | Estimated $c_s$ (+1 Std. Dev.) m s$^{-1}$ |
|-----------|---------------|-------------------------------------|---------------------------------------------|-------------------------------------------|
| F(0)      | 1             | 3.5                                 | 283.2                                       | 224.4                                     |
| F(0)      | 2             | 3.2                                 | 257.0                                       | 228.5                                     |
| F(0)      | 3             | 2.9                                 | 250.6                                       | 233.1                                     |
| F(0)      | 4             | 2.2                                 | 250.6                                       | 240.4                                     |
| F(0)      | 5             | 3.0                                 | 260.9                                       | 249.0                                     |
| L(0)      | 3             | 1.9                                 | 212.9                                       | 204.2                                     |
| L(0)      | 4             | 1.6                                 | 211.4                                       | 207.3                                     |
| L(0)      | 5             | 1.6                                 | 214.0                                       | 211.1                                     |
| L(0)      | 6             | 2.8                                 | 218.3                                       | 213.7                                     |
| FE(1)     | 6             | 4.1                                 | 215.4                                       | 206.2                                     |

4. Conclusions
The shear wave speed of a submerged isotropic plate can be accurately estimated by the inverse method developed in this paper. This approach consists of shaking an underwater plate, measuring the normal velocity of one surface, transforming these spatial measurements into the $k_x$, $k_y$ wavevector domain, identifying the major propagation wave types, and then using these wavenumbers to estimate the shear wave speed via an inverse method applied to the theoretical dispersion curve equations for propagation of waves in a fluid-loaded plate. Experimental measurements show that the use of this method provides accurate estimates of the shear wave speed. Additionally, it was shown that the dilatational wave speed cannot be accurately estimated using this technique, as the dispersion curve equations of motion do not have well defined relative minima with respect to the dilatational wavenumber.
5. Acknowledgements
This work was funded by the Office of Naval Research. The authors wish to thank Program Manager Dr. David M. Drumheller for sponsoring this effort. Additionally, the authors wish to thank Walter Booher, Rene LaFleur, and Gorham Lau of the Naval Undersea Warfare Center Acoustic Test Facility for collecting the spatial domain data.

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