FINITE ELEMENTS IN SOME VECTOR LATTICES OF NONLINEAR OPERATORS

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Abstract. We study the collection of finite elements $\Phi_1(U(E, F))$ in the vector lattice $U(E, F)$ of orthogonally additive, order bounded (called abstract Uryson) operators between two vector lattices $E$ and $F$, where $F$ is Dedekind complete. In particular, for an atomic vector lattice $E$ it is proved that for a finite element in $\varphi \in U(E, \mathbb{R})$ there is only a finite set of mutually disjoint atoms, where $\varphi$ does not vanish and, for an atomless vector lattice the zero-vector is the only finite element in the band of $\sigma$-laterally continuous abstract Uryson functionals. We also describe the ideal $\Phi_1(U(\mathbb{R}^n, \mathbb{R}^m))$ for $n, m \in \mathbb{N}$ and consider rank one operators to be finite elements in $U(E, F)$.

1. Introduction

The last time finite elements in vector lattices have been an object of an active investigation [5, 6, 7, 9, 17, 27]. This class of elements in Archimedean vector lattices was introduced as an abstract analogon of continuous functions (on a topological space) with compact support by Makarov and Weber in 1972, see [16]. Recently a systematic treatment of finite elements in vector lattices appeared in [28]. On the other hand the study of nonlinear maps between vector lattices is also a growing area of Functional analysis, where the background has to be found in the nonlinear integral operators, see e.g. [12]. The interesting class of nonlinear, order bounded, orthogonally additive operators, called abstract Uryson operators, was introduced and studied in 1990 by Mazón and Segura de León [18, 19], and then considered to be defined on lattice-normed spaces by Kusraev and Pliev in [14, 15, 21].

Now a theory of abstract Uryson operators is also a subject of intensive investigations [4, 8, 25]. In this paper the investigation of finite elements is extended to the vector lattice $U(E, F)$ of abstract Uryson operators from a vector lattice $E$ to a Dedekind complete vector lattice $F$.

2. Preliminaries

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices the reader can find in the books [1, 13, 28, 30].

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Definition 2.1. Let $E$ be an Archimedean vector lattice. An element $\varphi \in E$ called finite, if there is an element $z \in E$ satisfying the following condition: for any element $x \in E$ there exists a number $c_x > 0$ such that the following inequality holds
\[ |x| \land n|\varphi| \leq c_x z \quad \text{for all } n \in \mathbb{N}. \]

For a finite element $\varphi$ the (positive) element $z$ is called a majorant of $\varphi$. If a finite element $\varphi$ possesses a majorant which itself is a finite element then $\varphi$ is called totally finite. The collections of all finite and totally finite elements of a vector lattice $E$ are ideals in $E$ and will be denoted by $\Phi_1(E)$ and $\Phi_2(E)$, respectively. It is clear that 0 is always a finite element.

For our purpose we mention that the relations $\Phi_1(E) = E$ and $\Phi_1(E) = \{0\}$ are possible (for the complete list of the relations between $E$, $\Phi_1(E)$ and $\Phi_2(E)$ see [28], section 6.2). Finite elements in vector and Banach lattices have been studied in [5], in sublattices of vector lattices in [6], in $f$-algebras and product algebras [7, 9].

The relations between the finite elements in $E$ and the finite elements in vector sublattices of $E$ are manifold. The result we need later is the following.

Proposition 2.2 ([28], Theorem 3.28 and Corollary 3.29). Let $E_0$ be a projection band in the vector lattice $E$, and $p_0$ the band projection from $E$ onto $E_0$. Then $p_0(\Phi_1(E)) = \Phi_1(E) \cap E_0 = \Phi_1(E_0)$. If $E_1$ is another projection band in $E$ and $E = E_0 \oplus E_1$ then $\Phi_1(E) = \Phi_1(E_0) \oplus \Phi_1(E_1)$.

Recall that an element $z$ in a vector lattice $E$ is said to be a component or a fragment of $x$ if $z \perp (x - z)$, i.e. if $|z| \land |x - z| = 0$. The notations $x = y \cup z$ and $z \subseteq x$ mean that $x = y + z$ with $y \downarrow z$ and that $z$ is a fragment of $x$, respectively. The set of all fragments of the element $x \in E$ is denoted by $F_x$. Let be $x \in E$. A collection $(p_\xi)_{\xi \in \Xi}$ of elements in $E$ is called a partition of $x$ if $|p_\xi| \land |p_\eta| = 0$, whenever $\xi \neq \eta$ and $x = \sum_{\xi \in \Xi} p_\xi$.

Definition 2.3. Let $E$ be a vector lattice and let $X$ be a real vector space. An operator $T : E \to X$ is called orthogonally additive if $T(x + y) = T(x) + T(y)$ whenever $x, y \in E$ are disjoint elements, i.e. if $|x| \land |y| = 0$.

It follows from the definition that $T(0) = 0$. It is immediate that the set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

So, the orthogonal additivity of an operator $T$ will be expressed as $T(x \downarrow y) = T(x) + T(y)$.

Definition 2.4. Let $E$ and $F$ be vector lattices. An orthogonally additive operator $T : E \to F$ is called:

- positive if $Tx \geq 0$ holds in $F$ for all $x \in E$,
- order bounded if $T$ maps any order bounded subset of $E$ into an order bounded subset of $F$. 

An orthogonally additive order bounded operator \( T: E \to F \) is called an abstract Uryson operator.

The set of all abstract Uryson operators from \( E \) to \( F \) we denote by \( \mathcal{U}(E,F) \). If \( F = \mathbb{R} \) then an element \( f \in \mathcal{U}(E,\mathbb{R}) \) is called an abstract Uryson functional.

A positive linear order bounded operator \( A: E \to F \) defines a positive abstract Uryson operator by means of \( T(x) = A(|x|) \) for each \( x \in E \).

We will consider some examples. The most famous ones are the nonlinear integral Uryson operators which are well known and thoroughly studied e.g. in [12], chapt. 5.

Let \( (A, \Sigma, \mu) \) and \( (B, \Xi, \nu) \) be \( \sigma \)-finite complete measure spaces and denote the completion of their product measure space by \( (A \times B, \mu \times \nu) \). Let \( K: A \times B \times \mathbb{R} \to \mathbb{R} \) be a function which satisfies the following conditions\(^1\):

\begin{align*}
(C_0) & \quad K(s,t,0) = 0 \text{ for } \mu \times \nu\text{-almost all } (s,t) \in A \times B; \\
(C_1) & \quad K(\cdot,\cdot,r) \text{ is } \mu \times \nu\text{-measurable for all } r \in \mathbb{R}; \\
(C_2) & \quad K(s,t,\cdot) \text{ is continuous on } \mathbb{R} \text{ for } \mu \times \nu\text{-almost all } (s,t) \in A \times B.
\end{align*}

Denote by \( L_0(B,\Xi,\nu) \) or, shortly by \( L_0(\nu) \), the ordered vector space of all \( \nu \)-measurable and \( \nu \)-almost everywhere finite functions on \( B \) with the order \( f \leq g \) defined as \( f(t) \leq g(t) \) \( \nu \)-almost everywhere on \( B \). Then \( L_0(\nu) \) is a Dedekind complete vector lattice. Analogously, the space \( L_0(A,\Sigma,\mu) \), or shortly \( L_0(\mu) \), is defined.

Given \( f \in L_0(\nu) \) the function \( |K(s,\cdot,f(\cdot))| \) is \( \nu \)-measurable for \( \mu \)-almost all \( s \in A \) and \( h_f(s) := \int_B |K(s,t,f(t))| \, d\nu(t) \) is a well defined \( \mu \)-measurable function. Since the function \( h_f \) can be infinite on a set of positive measure, we define

\[ \text{Dom}_B(K) := \{ f \in L_0(\nu): h_f \in L_0(\mu) \}. \]

**Example 2.5** (Uryson integral operator). Define an operator \( T: \text{Dom}_B(K) \to L_0(\mu) \)

by

\begin{equation}
(Tf)(s) = \int_B K(s,t,f(t)) \, d\nu(t) \quad \mu\text{-a.e.}
\end{equation}

Let \( E \) and \( F \) be order ideals in \( L_0(\nu) \) and \( L_0(\mu) \), respectively and \( K \) a function satisfying the conditions \((C_0) - (C_2)\). Then \( (2.1) \) is an orthogonally additive, in general, not order bounded, integral operator acting from \( E \) to \( F \) provided that \( E \subseteq \text{Dom}_B(K) \) and \( T(E) \subseteq F \). The operator \( T \) is called Uryson (integral) operator.

We consider the vector space \( \mathbb{R}^m \) for \( m \in \mathbb{N} \) as a vector lattice with the usual coordinate-wise order: for any \( x, y \in \mathbb{R}^m \) we set \( x \leq y \) provided \( e_i^*(x) \leq e_i^*(y) \) for all \( i = 1, \ldots, m \), where \( (e_i^*)_{m=1} \) are the coordinate functionals on \( \mathbb{R}^m \).

\(^1\)(C1) and (C2) are called the Carathéodory conditions.
Example 2.6. A map $T : \mathbb{R}^n \to \mathbb{R}^m$ belongs to $\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if there are real functions $T_{i,j} : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$ satisfying the condition $T_{i,j}(0) = 0$ and such that $T_{i,j}([a, b])$ is (order) bounded in $\mathbb{R}^m$ for each (order) interval $[a, b] \subset \mathbb{R}^n$, where the $i$-th component of the vector $T(x)$ is calculated by the usual matrix rule, i.e.

$$(T(x))_i = e_i^*(T(x_1, \ldots, x_n)) = \sum_{j=1}^n T_{i,j}(x_j), \quad i = 1, \ldots, m$$

In this case we write $T = (T_{i,j})$.

For more examples of abstract Uryson operators see [25].

In $\mathcal{U}(E, F)$ the order is introduced as follows: $S \leq T$ whenever $T - S$ is a positive operator. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If the vector lattice $F$ is Dedekind complete the following theorem is well known.

Theorem 2.7 ([18], Theorem 3.2.). Let $E$ and $F$ be vector lattices with $F$ Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover, for any $S, T \in \mathcal{U}(E, F)$ and $x \in E$ the following formulas hold

1. $(T \vee S)(x) = \sup\{T(y) + S(z) : x = y \sqcup z\}$.
2. $(T \wedge S)(x) = \inf\{T(y) + S(z) : x = y \sqcup z\}$.
3. $T^+(x) = \sup\{Ty : y \sqsubseteq x\}$.
4. $T^-(x) = -\inf\{Ty : y \sqsupseteq x\}$.
5. $T|_x = (T^+ \vee T^-)(x) = \sup\{T(y) - T(z) : x = y \sqcup z\}$
6. $|T|(x) \leq |T|(x)$.

The formulas (1) - (5) are generalizations of the well known Riesz-Kantorovich formulas for linear regular operators (see [1], Theorems 1.13 and 1.16).

We also need the following result which represents the lattice operations in $\mathcal{U}(E, F)$ in terms of directed systems.

Theorem 2.8 ([19], Lemma 3.2). Let $E$ and $F$ be vector lattices with $F$ Dedekind complete. Then for any $S, T \in \mathcal{U}(E, F)$ and $x \in E$ we have

1. $\left\{ \sum_{i=1}^n S(x_i) \wedge T(x_i) : x = \bigsqcup_{i=1}^n x_i, \ n \in \mathbb{N} \right\} \downarrow (S \wedge T)(x)$.
2. $\left\{ \sum_{i=1}^n S(x_i) \vee T(x_i) : x = \bigsqcup_{i=1}^n x_i, \ n \in \mathbb{N} \right\} \uparrow (S \vee T)(x)$.
3. $\left\{ \sum_{i=1}^n |T(x_i)| : x = \bigsqcup_{i=1}^n x_i, \ n \in \mathbb{N} \right\} \uparrow |T|(x)$.

3. Some properties of finite elements in the vector lattice $\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$

In the one-dimensional case, by definition, $\mathcal{U}(\mathbb{R}) := \mathcal{U}(\mathbb{R}, \mathbb{R})$ coincides with the set all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(0) = 0$ and for every (order) bounded set $A \subset \mathbb{R}$ its image $f(A)$ is also a bounded set.
The vector lattice of abstract Uryson operators has a lot of finite elements meaning that $\Phi_1(U(\mathbb{R}^n, \mathbb{R}^m)) \neq \{0\}$. The next proposition shows that $\{0\} \neq \Phi_1(U(\mathbb{R})) \neq U(\mathbb{R})$ in the one-dimensional case. For an arbitrary real function $f$ defined on $\mathbb{R}$ denote the set $\{x \in \mathbb{R}: f(x) \neq 0\}$ by $\text{supp}(f)$ and called it the support of $f$.

**Proposition 3.1.** The set of all finite elements $\Phi_1(U(\mathbb{R}))$ coincides with the set

$$\mathcal{F}(U(\mathbb{R})) = \{f \in U(\mathbb{R}): \text{supp}(f) \subset [a, b], a, b \in \mathbb{R}\}.$$ 

Moreover, $\Phi_2(U(\mathbb{R})) = \Phi_1(U(\mathbb{R}))$.

**Proof.** Fix an arbitrary element $f \in \mathcal{F}(U(\mathbb{R}))$. Then $\text{supp}(f) \subset [a, b]$ for some $a, b \in \mathbb{R}$. If $g \in U(\mathbb{R})$ then the number $c_g = \sup\{|g(x)|: x \in [a, b]\}$ belongs to $\mathbb{R}$. Define the function

$$z(x) = \begin{cases} 1, & \text{if } x \in \text{supp}(f) \\ 0, & \text{if } x \notin \text{supp}(f). \end{cases}$$

Then $z$ is a bounded function on $\mathbb{R}$. Due to $z(0) = 0$ and since in $\mathbb{R}$ any orthogonal representation of some element $x$ is trivial, i.e. consists of $x$ and some zeros, the function $z$ belongs to $U(\mathbb{R})$. For $x \in \text{supp}(f)$ we have

$$|g(x)| \leq c_g z(x).$$

The inequality trivially holds for $x \notin \text{supp}(f)$. Therefore $f$ is a finite element in $U(\mathbb{R})$ and $z$ one of its majorants. Hence $\mathcal{F}(U(\mathbb{R})) \subset \Phi_1(U(\mathbb{R}))$.

In order to prove the converse assertion take an element $f \in U(\mathbb{R})$ such that there exist a sequence $(x_n)_{n=1}^\infty$ of real numbers $x_n \neq 0$ with the properties $\lim_{n \to \infty} x_n = \infty$ and $f(x_n) \neq 0$ for all $n \in \mathbb{N}$. Let $f \in \Phi_1(U(\mathbb{R}))$ and let $z$ be a majorant for $f$. Then we may assume $z(x_n) > 1$ for all $n \in \mathbb{N}$. By assumption, for every $g \in U(\mathbb{R})$ we have

$$\sup_n \{|g| \wedge n|f|\}(x_n) \leq c_g z \text{ for some } c_g \in \mathbb{R}_+,$$

where the existence of the supremum is guaranteed by the Dedekind completeness of $U(\mathbb{R})$ (see Theorem 2.7). In particular, for a function $g \in U(\mathbb{R})$ with

$$g(x) = \begin{cases} \exp(z(x)), & \text{if } x = x_n \text{ for } n = 1, 2, \ldots \\ 0, & \text{if } x \in \mathbb{R}, x \neq x_n \end{cases}$$

there exists $n_0 \in \mathbb{N}$, such that $g(x_n) > c_g z(x_n)$ for all $n \geq n_0$. Fix $m \in \mathbb{N}$ such that $m|f(x_{n_0})| > g(x_{n_0})$. Then

$$\sup_n \{|g| \wedge m|f|\}(x_{n_0}) \geq \left(|g| \wedge m|f|\right)(x_{n_0}) = g(x_{n_0}) > c_g z(x_{n_0}).$$

This contradicts to (3.1) and therefore, $\mathcal{F}(U(\mathbb{R})) \supset \Phi_1(U(\mathbb{R}))$.

Observe that the majorant $z \in U(\mathbb{R})$ of $f$, constructed in the first part of the proof, is such that $\text{supp}(z) = \text{supp}(f)$. So $\text{supp}(z) \subset [a, b]$, and by what has been proved one has $z \in \Phi_1(U(\mathbb{R}))$. Therefore $\mathcal{F}(U(\mathbb{R})) \subset \Phi_2(U(\mathbb{R}))$. 


Since $\Phi_2(\mathcal{U}(\mathbb{R})) \subseteq \mathfrak{F}(\mathcal{U}(\mathbb{R}))$ is clear from $\Phi_2(\mathcal{U}(\mathbb{R})) \subseteq \Phi_1(\mathcal{U}(\mathbb{R}))$, we conclude that any finite element in $\mathcal{U}(\mathbb{R})$ is even totally finite.

Guided by the previous proposition we describe the finite elements in the vector lattice $\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ (see Example 2.6).

**Proposition 3.2.** For the vector lattice $\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ the ideal $\Phi_1(\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m))$ coincides with the set of all operators $T = (T_{i,j}) \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ for which each function $T_{i,j}$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ satisfies the conditions $T_{i,j}(0) = 0$ and

\[
(3.2) \quad \text{supp}(T_{i,j}) \subset [a^{(ij)}, b^{(ij)}] \quad \text{for some real interval } [a^{(ij)}, b^{(ij)}].
\]

In this case $\Phi_1(\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)) = \Phi_2(\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m))$ also holds.

**Proof.** The assertion of this proposition is established by a similar argument as for Proposition 3.1. Consider an operator $T \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ such that its constituent functions $T_{i,j}$ have the properties $T_{i,j}(0) = 0$ and (3.2). Then for the interval $[a, b]$ with $a = \min_{ij} a^{(ij)}$ and $b = \max_{ij} b^{(ij)}$ one has $\text{supp}(T_{i,j}) \subset [a, b]$ for all $i$ and $j$. The set

\[
\text{supp}(T) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \exists i \in \{1, \ldots, m\} \text{ with } \sum_{j=1}^n T_{i,j}(x_j) \neq 0 \}
\]

will be called the support of the operator $T$. It is clear that $\text{supp}(T) = \{ x \in \mathbb{R}^n : T(x) \neq 0 \}$, $0 \notin \text{supp}(T)$ and $T(x) \neq 0$ for all $x \in \text{supp}(T)$. Define the map $Z = (Z_{i,j}) : \mathbb{R}^n \to \mathbb{R}^m$ by

\[
Z(x) = \left( \begin{array}{c}
Z_{1,1}(x_1) + \cdots + Z_{1,n}(x_n) \\
\vdots \\
Z_{m,1}(x_1) + \cdots + Z_{m,n}(x_n)
\end{array} \right),
\]

where $Z_{i,j}$ are real bounded functions with $Z_{i,j}(0) = 0$ for all $i, j$. Denote the set $\{1, 2, \ldots, n\}$ by $N$. For an arbitrary vector $w \in \mathbb{R}^n$ its support is the set

\[
\text{supp}(w) = \{ j \in N : w_j \neq 0 \}.
\]

In order to show the orthogonal additivity of $Z$ consider $x = u \uplus v$ with $u$ and $v$ being fragments of $x$. Then supp$(u) \cap$ supp$(v) = \emptyset$ and

\[
x_j = \begin{cases} u_j, & \text{if } j \in \text{supp}(u) \\ v_j, & \text{if } j \notin \text{supp}(u). \end{cases}
\]

Then

\[
Z(u \uplus v) = \left( \begin{array}{c}
\sum_{j \in \text{supp}(u)} Z_{1,j}(u_j) \\
\vdots \\
\sum_{j \in \text{supp}(u)} Z_{m,j}(u_j)
\end{array} \right) + \left( \begin{array}{c}
\sum_{j \notin \text{supp}(u)} Z_{1,j}(v_j) \\
\vdots \\
\sum_{j \notin \text{supp}(u)} Z_{m,j}(v_j)
\end{array} \right).
\]

By using that $Z_{i,j}(u_j) = 0$ for $j \notin \text{supp}(u)$ and $Z_{i,j}(v_j) = 0$ for $j \in \text{supp}(u)$ (for all $i = 1, \ldots, m$) the summation in each of the coordinates of the last
two vectors can be extended to the whole set $N$ and hence one obtains $Z(u \sqcup v) = Z(u) + Z(v)$ and, so $Z \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$.

For an arbitrary Uryson operator $S = (S_{i,j}) \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ define the number

$$c_S = \sup \{ \sum_{i,j=1}^{m,n} |S_{i,j}(x_j)| : x = (x_1, x_2, \ldots, x_n) \in [a, b] \}.$$ 

Then $c_S \in \mathbb{R}$ and for $x \in \text{supp}(T)$ one has

$$(|S| \land nT)(x) \leq |S|(x) \leq c_S Z(x).$$

For $x \notin \text{supp}(T)$ the inequality is also true due to $T(x) = 0$ in that case. So $T$ is a finite element in $\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ and $Z$ is one of its majorant.

For the converse let $T \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ be such that there is a sequence $(x^{(k)})$ of vectors $0 \neq x^{(k)} \in \mathbb{R}^n$ with the properties that $T(x^{(k)}) \neq 0$ and $(x^{(k)})$ leaves any ball in $\mathbb{R}^n$. If $T$ would belong to $\Phi_1(\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m))$ and $Z$ is a fixed majorant of $T$ then for any operator $S \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ one has

$$(3.3) \quad |S| \land nT \leq c_S Z \text{ for all } n \in \mathbb{N} \text{ and some number } c_S > 0.$$ 

In particular, this holds for an operator $0 < S = (S_{i,j}) \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ with

$$S_{i,j}(x_j) = \begin{cases} \exp(Z_{i,j}(x_j)), & \text{if } x \in \text{supp}(T), \\ 0, & \text{if } x \notin \text{supp}(T) \end{cases}, \quad x = (x_1, \ldots, x_n).$$

Then

$$(S \land nT)(x^{(k)}) = S(x^{(k)}) = \left( \sum_{j=1}^{n} \exp(Z_{1,j}(x_j^{(k)})) \right) \cdots \left( \sum_{j=1}^{n} \exp(Z_{m,j}(x_j^{(k)})) \right).$$

It is clear that for sufficiently large $k$ the last vector is greater than $c_S Z(x^{(k)})$ what is in contradiction to (3.3). $\blacksquare$

**Remark 3.3.** Actually it is proved that

$$T = (T_{i,j}) \in \Phi_1(\mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)) \text{ if and only if } T_{i,j} \in \Phi_1(\mathbb{R})$$

for all $i = 1, \ldots, m; j = 1, \ldots, n$.

For a band $H \subset F$ we get a result for the abstract Uryson operators which is similar to Theorem 2 in [7] for (linear) regular operators.

**Proposition 3.4.** Let $E, F$ be vector lattices with $F$ Dedekind complete and let $H$ be a band in $F$. Then $\mathcal{U}(E, H)$ is a projection band $\mathcal{U}(E, F)$ and the following equation holds

$$\Phi_1(\mathcal{U}(E, H)) = \Phi_1(\mathcal{U}(E, F)) \cap \mathcal{U}(E, H).$$

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2 i.e. $|x^{(k)}| \to \infty$.

3 due to the Dedekind completeness of $F$ any band is a projection band.
Proof. Let \( \pi: F \to H \) be the order projection in \( F \) onto \( H \). It is clear that \( \mathcal{U}(E, H) \) is an order ideal in the \( \mathcal{U}(E, F) \). Fix a net \((T_\alpha)\) in \( \mathcal{U}_+(E, H) \), such that \( T_\alpha \uparrow T \) for some \( T \in \mathcal{U}(E, F) \). Then one has \( T_\alpha = \pi T_\alpha \uparrow \pi T \in \mathcal{U}_+(E, H) \). Therefore \( T = \pi T \), i.e. the order ideal \( \mathcal{U}(E, H) \) is a band and, due to the Dedekind completeness of \( \mathcal{U}(E, F) \), even a projection band. Let \( \pi^*: \mathcal{U}(E, F) \to \mathcal{U}(E, H) \) be the related order projection. Then \( \pi^*(T) = \pi T \) holds for every \( T \in \mathcal{U}(E, F) \). To finish the proof, refer to Theorem 2.11 from [6], saying that the finite elements in \( \Phi_1(\mathcal{U}(E, H)) \) are exactly those finite elements of \( \Phi_1(\mathcal{U}(E, F)) \), which belong to \( \mathcal{U}(E, H) \).

Remark 3.5. By the mentioned result from [6] there is proved even the equality \( \pi^*(\Phi_1(\mathcal{U}(E, F))) = \Phi_1(\mathcal{U}(E, H)) \).

4. Finite elements in \( \mathcal{U}(E, \mathbb{R}) \)

Definition 4.1. A non-zero element \( u \) of a vector lattice \( E \) is called an atom, whenever \( 0 \leq x \leq |u|, 0 \leq y \leq |u| \) and \( x \land y = 0 \) imply that either \( x = 0 \) or \( y = 0 \).

If \( u \) is an atom in \( E \) then \( F_u = \{0, u\} \). Note that a non-zero element \( u \) of a vector lattice \( E \) is called discrete, if the ideal \( I_u \) generated by \( u \) in \( E \) coincides with the vector subspace generated by \( u \) in \( E \), i.e. if \( 0 \leq x < u \) implies \( x = \lambda u \) for some \( \lambda \in \mathbb{R}_+ \). We need the following properties of atoms.

Proposition 4.2 ([29], Theorem 26.4). Let \( E \) be an Archimedean vector lattice. Then the following holds:

(i) Atoms and discrete elements are the same.
(ii) For any atom \( u \) the ideal \( I_u \) is a projection band.
(iii) For any two atoms \( u, v \) in \( E \), either \( u \perp v \), or \( v = \lambda u \) for some \( 0 \neq \lambda \in \mathbb{R} \).

Definition 4.3. An Archimedean vector lattice \( E \) is said to be atomic\(^4\) if for each \( 0 < x \in E \) there is an atom \( u \in E \) satisfying \( 0 < u \leq x \).

A vector lattice is said to be atomless provided it has no atoms.

Equivalently (see [2]), \( E \) is atomic, if and only if there is a collection \((u_i)_{i \in I}\) of atoms in \( E \), such that \( u_i \perp u_j \) for \( i \neq j \) and for every \( x \in E \) if \( |x| \land u_i = 0 \) for each \( i \in I \) then \( x = 0 \). Such a collection is called a generating disjoint collection of atoms.

By Proposition 4.2, a generating collection of atoms in an atomic vector lattice is unique, up to a permutation and nonzero multiples.

Let \( E \) be a vector lattice. Consider any maximal collection of atoms \((u_i)_{i \in I}\) in \( E \), the existence of which is guaranteed by Proposition 4.2 and by applying Zorn’s Lemma. Let \( E_0 \) be the minimal band containing \( u_i \) for all \( i \in I \). If \( E_0 \) is a projection band then \( E = E_0 \oplus E_1 \), where \( E_1 = E_0^d \) is the

\(^4\) or discrete.
disjoint complement to $E_0$ in $E$, which is an atomless sublattice of $E$. So, we obtain the following assertion.

**Proposition 4.4.** Any vector lattice $E$ with the projection property\(^\text{\textsuperscript{5}}\) has a decomposition into mutually complemented bands $E = E_0 \oplus E_1$, where $E_0$ is an atomic vector lattice and $E_1$ is an atomless vector lattice.

The following theorem is the first main results of this section and deals with finite elements in atomic vector lattices.

**Theorem 4.5.** Let $E$ be an atomic vector lattice and $\varphi \in \Phi_1(\mathcal{U}(E, \mathbb{R}))$. Then there exists only a finite set $\{e_1, \ldots, e_n\}$ of the mutually disjoint atoms in $E$, such that $\varphi(e_i) \neq 0$ for $i = 1, \ldots, n$.

**Proof.** If $E$ is a finite dimensional vector lattice then $E$ is isomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$ and $\Phi_1(\mathcal{U}(\mathbb{R}^k, \mathbb{R})) \neq \{0\}$ by Proposition 3.2. Then the coordinate vectors $e^{(i)} = (0, \ldots, 0, 1, 0, \ldots, 0)$, $i = 1, \ldots, k$ are mutually disjoint atoms in $\mathbb{R}^k$. Obviously, among them for each $0 \neq \varphi \in \Phi_1(\mathcal{U}(\mathbb{R}^k, \mathbb{R}))$ there are some vectors, on which the functional $\varphi$ does not vanish.

Let be $E$ an infinite-dimensional atomic vector lattice $E$. Let be $\varphi \in \Phi_1(\mathcal{U}(E, \mathbb{R}))$, $\varphi > 0$ with a fixed positive majorant $\psi$. Assume that for $\varphi$ there exists an infinite set of mutually disjoint atoms $e_n \in E$, $n \in \mathbb{N}$ such that $\varphi(e_n) > 0$ for every $n \in \mathbb{N}$. Without restriction of generality\(^\text{\textsuperscript{6}}\) we may assume $\sum_{n=1}^{\infty} \psi(e_n) < \infty$. For arbitrary $T \in \mathcal{U}_+(E, \mathbb{R})$ there exists a number $c_T > 0$ such that $(T \wedge n\varphi)x \leq c_T \psi(x)$ for every $n \in \mathbb{N}$ and $x \in E$, what implies $(\pi_\varphi T)x \leq c_T \psi(x)$. By applying the formula

\begin{equation}
(\pi_\varphi T)x = \sup_{\varepsilon > 0} \inf_{y \in F_x} \{ Ty : \varphi(x - y) \leq \varepsilon \varphi(x) \}.
\end{equation}

(which was proved for any $x \in E$ in [26], Formula (3.8)) to the atom $e_n$ and by taking into account that, due to $\varphi(0) = 0$ and $F_{e_n} = \{0, e_n\}$, the element $y = e_n$ is the only feasible in formula (4.1) (applied to $e_n$) we get

\begin{equation}
(\pi_\varphi T)e_n = \sup_{\varepsilon > 0} \inf_{y \in F_{e_n}} \{ Ty : \varphi(e_n - y) \leq \varepsilon \varphi(e_n) \} = Te_n.
\end{equation}

Therefore

\begin{equation}
Te_n \leq c_T \psi(e_n) \text{ for every } n \in \mathbb{N}
\end{equation}

and so, $\sum_{n=1}^{\infty} Te_n < \infty$ for each $T \in \mathcal{U}(E, \mathbb{R})$. For every $n \in \mathbb{N}$ choose a natural number $k_n \in \mathbb{N}$ such that $\psi(e_{k_n}) < \frac{1}{(n+1)^\alpha}$ and define numbers $\beta_{k_n}$ satisfying

\begin{equation}
\text{then } E \text{ is Archimedean.}
\end{equation}

\begin{equation}
\text{Otherwise replace } \varphi \text{ by an element with appropriate smaller values for } \varphi(e_n).
\end{equation}
the condition $\frac{1}{(n+1)^3} < \beta_{k_n} < \frac{1}{(n+1)^{1/2}}$. Take now a functional $T \in \mathcal{U}_+(E, \mathbb{R})$ such that

$$T e_k = \begin{cases} \beta_{k_n}, & \text{if } k = k_n \\ \psi(e_k), & \text{if } k \neq k_n \end{cases}$$

for $k = 1, 2, \ldots$.

It is clear that $\sum_{k=1}^{\infty} T e_k < \infty$. Fix $n_0 \in \mathbb{N}$ with $c_T < n_0$, where $c_T$ is the constant number for the functional $T$ one has according to the finiteness of $\varphi$. Then

$$c_T \psi(e_{k_0}) < n_0 \psi(e_{k_0}) < \frac{1}{(n_0 + 1)^3} < \beta_{k_0} = T e_{k_0}.$$  

This is a contradiction to (4.2).

Our aim now is to establish that for an atomless vector lattice the band of $\sigma$-laterally continuous abstract Uryson functionals possesses only the trivial finite element.

**Definition 4.6.** A sequence $(x_n)_{n \in \mathbb{N}}$ in a vector lattice $E$ is said to be laterally converging to $x \in E$ if $x_n \subseteq x_m \subseteq x$ for all $n < m$ and $x_n \overset{(o)}{\rightarrow} x$. In this case we write $x_n \overset{\text{lat}}{\rightarrow} x$. For positive elements $x_n$ and $x$ the notion $x_n \overset{\text{lat}}{\rightarrow} x$ means that $x_n \in \mathcal{F}_x$, $x_n \uparrow x$ and $x_n \overset{\text{lat}}{\rightarrow} x$.

**Definition 4.7.** Let $E, F$ be vector lattices. An orthogonally additive operator $T: E \rightarrow F$ is called $\sigma$-laterally continuous if $x_n \overset{\text{lat}}{\rightarrow} x$ implies $Tx_n \overset{(o)}{\rightarrow} Tx$. The vector space of all $\sigma$-laterally continuous abstract Uryson operators from $E$ to $F$ is denoted by $\mathcal{U}_{\sigma c}(E, F)$.

It turns out that $\mathcal{U}_{\sigma c}(E, F)$ is a projection bands in $\mathcal{U}(E, F)$ ([18], Proposition 3.8). We need the following auxiliary lemma.

**Lemma 4.8.** Let $E$ be an atomless vector lattice, $\varphi \in \mathcal{U}_{\sigma c}(E, \mathbb{R})$ and $\varphi(x) > 0$ for some vector $x \in E$. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of mutually disjoint fragments of $x$, such that $\varphi(x_n) > 0$, for every $n \in \mathbb{N}$.

**Proof.** Assume that for every fragments $x', x''$ of $x$ with $x' \perp x''$, we have $\varphi(x') = 0$ and $\varphi(x'') = \varphi(x)$. Put $^7 x_1 = x'$ and consider in the next step the element $x''$. By repeating the procedure we construct a sequence $(x_n)_{n=1}^{\infty}$ of mutually disjoint fragments of $x$ such that $x = \bigcup_{n=1}^{\infty} x_n$ and $u_n = \bigcup_{i=1}^{n} x_i \overset{\text{lat}}{\rightarrow} x$.

However $\varphi(u_n) = 0$ for each $n \in \mathbb{N}$, what is a contradiction to the fact that $\varphi$ belongs to $\mathcal{U}_{\sigma c}(E, \mathbb{R})$.

---

$^7$ Since $E$ is atomless there are nontrivial (i.e. different from 0 and $x$) elements in $\mathcal{F}_x$.  

Now we deal with lateral ideals in vector lattices.
Definition 4.9. A subset $D$ of a vector lattice $E$ is called a lateral ideal if the following conditions hold:

(i) if $x \in D$ then $y \in D$ for every $y \in F_x$,
(ii) if $x,y \in D$, $x \perp y$ then $x + y \in D$.

Example 4.10. Let $E$ be a vector lattice. Every order ideal in $E$ is a lateral ideal.

Example 4.11. Let $E$ be a vector lattice and $x \in E$. Then $F_x$ is a lateral ideal (see [4], Lemma 3.5).

Lemma 4.12. Let $E$ be a vector lattice and $D = (D_n)_{n \in \mathbb{N}}$ a sequence of mutually disjoint lateral ideals in $E$. Then the set

$$L(D) := \{ \bigcup_{i=1}^{k} x_i : x_i \in D_n, 1 \leq i \leq k, k \in \mathbb{N} \}$$

is also a lateral ideal.

Proof. Take arbitrary elements $x,y \in L(D)$, such that $x \perp y$. Then

$$x = \bigcup_{i=1}^{k} x_i \text{ for } x_i \in D_n, \quad y = \bigcup_{j=1}^{m} y_j \text{ for } y_j \in D_n,$$

$$x_i \perp y_j \text{ for } 1 \leq i \leq k, 1 \leq j \leq m \quad \text{and}$$

$$x + y = \bigcup_{r=1}^{k+m} z_r, \text{ where } z_r = \begin{cases} x_r, & \text{if } 1 \leq r \leq k \\ y_{r-k}, & \text{if } k < r \leq k + m. \end{cases}$$

Hence the condition (ii) from Definition 4.9 is proved for $L(D)$. Now, let $x \in L(D)$ and $y \in F_x$. Then $x = \bigcup_{i=1}^{k} x_i$ with $x_i \in D_n$. By the Riesz decomposition property every $x_i$ has a decomposition $x_i = y_i \uplus z_i$, where $y = \bigcup_{i=1}^{k} y_i$ and $y_i$ belongs to $D_{n_i}$ due to $y_i \subseteq x_i \in D_{n_i}$ for $i = 1, \ldots, k$. So, the condition (i) from Definition 4.9 is also shown. \hfill \qed

The following extension property of positive orthogonally additive operators was proved in [8].

Theorem 4.13 ([8], Theorem 1). Let $E,F$ be vector lattices with $F$ Dedekind complete and $D$ a lateral ideal in $E$. Let $T : D \to F$ be a positive orthogonally additive operator such that the set $T(D)$ is order bounded in $F$. Then there exists an operator $\overline{T} \in U_+(E,F)$ with $Tx = \overline{T}Dx$ for every $x \in D$.

The operator $\overline{T}_D : E \to F$ (or, for simplicity, $\overline{T} : E \to F$) is defined by the formula

$$\overline{T}x = \sup\{Ty : y \in F_x \cap D\}. \hspace{1cm} (4.3)$$

that means, $D$ is saturated in the sense of (i) and (ii).

At least $0 \in D \cap F_x$ for any $x \in E$. 

Such an extension of $T$ is not unique. Due to the next lemma the operator $	ilde{T} \in \mathcal{U}_+(E,F)$ is called the minimal extension (with respect to $D$) of the positive, order bounded orthogonally additive operator $T : D \to F$.

**Lemma 4.14.** Let $E,F,D,T,T$ be as in Theorem 4.13 and let $R : E \to F$ be a positive abstract Uryson operator such that $Rx = Tx$ for every $x \in D$. Then $\tilde{T}x \leq Rx$ for every $x \in E$.

**Proof.** Take an arbitrary element $x \in E$ and $y \in F_x \cap D$. Then
\[
R(x) = R(x - y) + R(y) = R(x - y) + Ty \geq Ty \quad \text{and} \quad R(x) \geq \sup\{Ty : y \in F_x \cap D\} = \tilde{T}x.
\]

Now the second main result of this section can be provided.

**Theorem 4.15.** Let $E$ be an atomless vector lattice. Then $\Phi_1(\mathcal{U}_{sc}(E,\mathbb{R})) = \{0\}$.

**Proof.** Assume that there exists an element $\varphi \in \Phi_1(\mathcal{U}_{sc}(E,\mathbb{R}))$, $\varphi > 0$. Fix a positive laterally $\sigma$-continuous majorant $\psi$ for $\varphi$. Then for some $x \in E$, $x \neq 0$ one has $\varphi(x) > 0$. Since $E$ is atomless by Lemma 4.8 it can be deduced that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of mutually disjoint fragments of $x$ such that $\varphi(x_n) > 0$ for every $n \in \mathbb{N}$. Take now a positive functional $T \in \mathcal{U}_{sc}(E,\mathbb{R})$ with $T(x) > 0$ and $T(x_n) > 0$ for every $n \in \mathbb{N}$ (e.g. $T = \varphi$). Since $\varphi$ is a finite element there is some $c_T > 0$, such that $(\pi_T(x_n)) \leq c_T \psi(x_n)$, $n \in \mathbb{N}$. Consider the functional
\[
G_n : F_{x_n} \to \mathbb{R}_+
\]
defined on the lateral ideal $F_{x_n}$ by $G_n(y) = (\pi_T(y))$. Then $G_n$ is an orthogonally additive functional, the set $G_n(F_{x_n})$ is (order) bounded and $G_n(x_n) = (\pi_T(x_n))$, $n \in \mathbb{N}$. According to Theorem 4.13, $G_n$ can be extended to the functional $\tilde{G}_n \in \mathcal{U}_+(E,\mathbb{R})$ which, according to (4.3), is well defined on $E$ for every $n \in \mathbb{N}$. Since $(\pi_T(x_n)) \geq (T \wedge n\varphi)(x_n)$ one has $(\pi_T(x_n)) > 0$, $n \in \mathbb{N}$. By Lemma 4.14 the inequality $\tilde{G}_n(x) \leq (\pi_T(x))$ holds for every $x \in E$, i.e. $\tilde{G}_n \leq \pi_T$ and $\tilde{G}_n \in \{\varphi\}^{++}$, $n \in \mathbb{N}$. Moreover, $\tilde{G}_n(x_n) = (\pi_T(x_n)) > 0$. It is clear that $\tilde{G}_n \leq c_T \psi$, $n \in \mathbb{N}$. In view of the fact\(^{10}\) that $\sum_{n=1}^{\infty} \psi(x_n) = \psi(x) < \infty$, for every $k \in \mathbb{N}$ there exists an index $n_k$, such that $\psi(x_{n_k}) < \frac{1}{k^2}$. For every $k \in \mathbb{N}$ fix now numbers $\beta_{n_k}$ such that
\[
\frac{1}{k^3 \tilde{G}_{n_k}(x_{n_k})} < \beta_{n_k} < \frac{1}{k^2 \tilde{G}_{n_k}(x_{n_k})}.
\]
Then
\[
\sum_{k=1}^{\infty} \beta_{n_k} \tilde{G}_{n_k}(x_{n_k}) < \sum_{k=1}^{\infty} \frac{\tilde{G}_{n_k}(x_{n_k})}{k^2 \tilde{G}_{n_k}(x_{n_k})} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]
\(^{10}\)see footnote at page 9.
Observe that \( \mathcal{F} = (F_{x_{n_k}})_{k \in \mathbb{N}} \) is a sequence of mutually disjoint lateral ideals. Thus by Lemma 4.12 the set \( L(\mathcal{F}) \) is also a lateral ideal. Denote the operators \( G_{n_{k_i}} \) by \( R_i \) and define the operator \( R: L(\mathcal{F}) \to \mathbb{R}_+ \) by the formula

\[
R \left( \bigcup_{i=1}^{k} u_i \right) = \sum_{i=1}^{k} R_i(u_i).
\]

It will be shown that \( R \) is an orthogonally additive operator from the lateral ideal \( L(\mathcal{F}) \) to \( \mathbb{R} \). Indeed, take \( u, v \in L(\mathcal{F}) \) such that \( u \perp v \). Then

\[
R(u + v) = R \left( \bigcup_{i=1}^{k} u_i + \bigcup_{j=1}^{m} v_j \right) = R \left( \bigcup_{r=1}^{k+m} z_r \right) = \sum_{r=1}^{k+m} R_r(z_r)
\]

where \( z_r = \begin{cases} u_r, & \text{if } 1 \leq r \leq k, \\ v_{r-k}, & \text{if } k < r \leq k + m. \end{cases} \)

For any element \( u = \bigcup_{i=1}^{k} u_i \in L(\mathcal{F}) \) with \( u_i \in F_{x_{n_{k_i}}} \), due to

\[
G_{n_{k_i}}(u_i) = (\pi_\varphi T)(u_i) \leq c_T \psi(u_i) \leq c_T \psi(x_{n_{k_i}}) \leq \frac{c_T}{(k_i)^4},
\]

one has

\[
R(u) = R \left( \bigcup_{i=1}^{k} u_i \right) = \sum_{i=1}^{k} R_i(u_i) < c_T \sum_{i=1}^{k} \frac{1}{(k_i)^4} < c_T \sum_{k=1}^{\infty} \frac{1}{k^4},
\]

and therefore, the operator \( R \) is order bounded. In view of Theorem 4.13 there exists the minimal extension \( \tilde{R} \) of \( R \), which is a positive abstract Uryson functional \( \tilde{R}: E \to \mathbb{R} \) such that \( \tilde{R}(x) = \sup \{ R(v) : v \in F_x \cap L(\mathcal{F}) \} \) for any \( x \in E \). Observe that \( \tilde{R}(x_{n_k}) = \tilde{G}_{n_k}(x_{n_k}) \) for every \( k \in \mathbb{N} \). Let be \( S: E \to \mathbb{R} \) an abstract, positive Uryson functional such that \( S \geq \tilde{G}_{n_k} \) for any \( k \in \mathbb{N} \) and fix an arbitrary element \( x \in E \). Then for every decomposition \( x = y + z \), where \( y \perp z \) and

\[
z \in L(\mathcal{F}), \quad \text{i.e. } z = \bigcup_{i=1}^{m} u_i \text{ with } u_i \in F_{x_{n_{k_i}}} \text{ for some } m,
\]

one has

\[
S(x) = S(y + z) \geq S(z) = S \left( \bigcup_{i=1}^{m} u_i \right) \geq \sum_{i=1}^{m} G_{n_{k_i}}(u_i) = \sum_{i=1}^{m} R_i(u_i).
\]

Passing to the supremum over all fragments \( z \in L(\mathcal{F}) \) we conclude that \( S(x) \geq \tilde{R}(x) \) for every \( x \in E \). Hence \( \tilde{R} = \sup \{ \tilde{G}_{n_{k_i}} \} \) in \( \mathcal{U}(E, F) \). Using the fact that \( \tilde{G}_{n_{k_i}} \in \{ \varphi \}^{\perp\perp} \) for every \( i \in \mathbb{N} \), we deduce that \( \tilde{R} \in \{ \varphi \}^{\perp\perp} \).
Therefore a number $c_R > 0$ exists with $\bar{R} \leq c_R \psi$. For any number $k \in \mathbb{N}$ such that $c_R \leq k$ one has 

$$c_R \psi(x_{nk}) \leq k \psi(x_{nk}) < \frac{1}{k^3} < \beta_{nk} G_{nk}(x_{nk}) = \beta_{nk} \bar{G}_{nk}(x_{nk}) = \bar{R}(x_{nk}),$$

what is a contradiction. 

Now we are ready to put together the Theorems 4.5 and 4.15.

**Theorem 4.16.** Let $E$ be a vector lattice with the projection property and $\varphi \in \Phi_1(\mathcal{U}_{ac}(E, \mathbb{R}))$. Then there exists a finite dimensional projection band $M$ generated by a (finite) number of mutually disjoint atoms in $E$, such that $\varphi(x) = 0$ for every $x \in M^\perp$.

**Proof.** By Proposition 4.4 there exists a decomposition into mutually complemented bands $E = E_0 \oplus E_1$, where $E_0$ is an atomic vector lattice and $E_1$ is an atomless vector lattice. For the finite elements in $\mathcal{U}_{ac}(E, \mathbb{R})$ there holds the equality

$$\Phi_1(\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})) = \Phi_1(\mathcal{U}_{ac}(E_0, \mathbb{R})) \oplus \Phi_1(\mathcal{U}_{ac}(E_1, \mathbb{R})).$$

For that it is proved first that

$$\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R}) = \mathcal{U}_{ac}(E_0, \mathbb{R}) \oplus \mathcal{U}_{ac}(E_1, \mathbb{R}).$$

Take $f_i \in \mathcal{U}_{ac}(E_i, \mathbb{R}), i = 0, 1$. Define the functional $f = f_0 \oplus f_1$ for each $x = (x_0, x_1) \in E$ by the formula $f(x_0, x_1) = f_0(x_0) + f_1(x_1)$, where $x_0 \in E_0, x_1 \in E_1$. The functional $f$ belongs to the set $\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})$. Indeed, take a sequence $(x_n)_{n \in \mathbb{N}}$ in $E_0 \oplus E_1$, such that $x_n \xrightarrow{\text{lat.}} x$, where $x_n = (x_{n0}, x_{n1})$ and $x = (x_0, x_1)$. Then

$$f(x_n) = f(x_{n0}, x_{n1}) = f_0(x_{n0}) + f_1(x_{n1}) \xrightarrow{(o)} f_0(x_0) + f_1(x_1) = f(x).$$

On the other hand, let $f \in \mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})$. Denote by $f_i$ the restriction of $f$ on $E_i, i = 0, 1$. Then $f = f_0 + f_1$, with $f_i \in \mathcal{U}_{ac}(E_i, \mathbb{R})$.

Now it will be shown that $\mathcal{U}_{ac}(E_0, \mathbb{R})^\perp = \mathcal{U}_{ac}(E_1, \mathbb{R})$ and, therefore $\mathcal{U}_{ac}(E_0, \mathbb{R})$ and $\mathcal{U}_{ac}(E_1, \mathbb{R})$ are mutually disjoint bands in $\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})$. Hence the equality (4.4) will be established. Let $0 \leq f_i \in \mathcal{U}_{ac}(E_i, \mathbb{R}), i = 0, 1$ and $x \in E$. Then $x = x_0 \cup x_1$ with $x_i \in E_i, i = 0, 1$ and

$$(f_0 \wedge f_1)(x) = \inf \{f_0(y) + f_1(z) : x = y \cup z \} \leq f_0(x_1) + f_1(x_0) = 0.$$ 

Since $\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})$ is Dedekind complete, Proposition 2.2 guarantees the required equality

$$\Phi_1(\mathcal{U}_{ac}(E_0 \oplus E_1, \mathbb{R})) = \Phi_1(\mathcal{U}_{ac}(E_0, \mathbb{R})) \oplus \Phi_1(\mathcal{U}_{ac}(E_1, \mathbb{R})).$$

For $\varphi \in \Phi_1(\mathcal{U}_{ac}(E, \mathbb{R}))$ one has now $\varphi = \varphi_0 + \varphi_1$, where $\varphi_i \in \Phi_1(\mathcal{U}_{ac}(E_i, \mathbb{R}))$ for $i = 0, 1$. Theorem 4.15 implies $\varphi_1 = 0$ and, by Theorem 4.5 there exist only finite many $e_1, \ldots, e_n$ of mutually disjoint atoms in $E_0$, such that $\varphi_0(e_i) \neq 0, i = 1, \ldots, n$. Denote by $M$ the band in $E_0$, generated by $e_1, \ldots, e_n$. In view of the assumption on $E$ the band $M$ is a projection band.
in $E_0$. Then every element $x \in M^\perp$ is a linear combination of atoms disjoint to $M$ and therefore, $\varphi_0(x) = 0$. Thus $\varphi(x) = 0$ for all $x \in M^\perp$.

5. Rank one operators as finite elements in $\mathcal{U}(E, F)$

Let $E, F$ be vector lattices. An operator $T \in \mathcal{U}(E, F)$ is called a finite rank operator, if $T = \sum_{i=1}^{n} \varphi_i \otimes u_i$ for some $n \in \mathbb{N}$, where $\varphi_i \in \mathcal{U}(E, \mathbb{R})$, $u_i \in F$ and, $(\varphi_i \otimes u_i)(x) = \varphi_i(x) u_i$, $x \in E$ for all $i = 1, \ldots, n$. Similarly to the case of linear rank one operators in the vector lattice of regular operators the modulus of a rank one abstract Uryson operator has a simple structure.

**Proposition 5.1.** Let $E, F$ be vector lattices, with $F$ Dedekind complete. Then the modulus of the operator $T = \varphi \otimes u \in \mathcal{U}(E, F)$ is the operator $\lvert T \rvert = \lvert \varphi \rvert \otimes \lvert u \rvert$.

**Proof.** Using the relation (3) of Theorem 2.8 one has

$$
\lvert T \rvert(x) = \lvert \varphi \otimes u \rvert(x) = \sup \left\{ \sum_{i=1}^{n} \lvert (\varphi \otimes u)(x_i) \rvert : x = \bigsqcup_{i=1}^{n} x_i, n \in \mathbb{N} \right\} 
$$

$$
= \sup \left\{ \sum_{i=1}^{n} \lvert \varphi(x_i) u \rvert : x = \bigsqcup_{i=1}^{n} x_i, n \in \mathbb{N} \right\} 
$$

$$
= \lvert u \rvert \sup \left\{ \sum_{i=1}^{n} \lvert \varphi(x_i) \rvert : x = \bigsqcup_{i=1}^{n} x_i, n \in \mathbb{N} \right\} = \lvert \varphi \rvert \lvert u \rvert.
$$

The following theorem tells us that the constituent parts of an abstract Uryson rank one operator $T$ are finite elements in the corresponding vector lattices, whenever $T$ is a finite element in $\mathcal{U}(E, F)$. Recall that the order dual of the vector lattice $F$ is denoted by $F^\sim$. The expression $F^\sim$ separates the points of $F$ means that for each $0 \neq y \in F$ there exists some $f \in F^\sim$ with $f(y) \neq 0$. The order dual separates the points of $F$, e.g., if $F$ is a Dedekind complete Banach lattice.

**Theorem 5.2.** Let $E, F$ be vector lattices with $F$ Dedekind complete and $F^\sim$ separates the points of $F$. Let $T \in \mathcal{U}(E, F)$ be a rank one operator, i.e. $T = \varphi \otimes u$ for some $\varphi \in \mathcal{U}(E, \mathbb{R})$ and $u \in F$. If $T \in \Phi_1(\mathcal{U}(E, F))$ then $\varphi \in \Phi_1(\mathcal{U}(E, \mathbb{R}))$ and $u \in \Phi_1(F)$.

**Proof.** By Proposition 5.1 it suffices to consider a positive abstract Uryson operator $T = \varphi \otimes u$, with $0 < \varphi \in \mathcal{U}(E, \mathbb{R})$, $0 < u \in F$. By assumption $T$ is a finite element and therefore an operator $Z \in \mathcal{U}_+(E, F)$ exists, such that for every $S \in \mathcal{U}(E, F)$ the inequality

$$
\lvert S \rvert \wedge nT \leq c_S Z
$$
holds for some \( c_S > 0 \) and every \( n \in \mathbb{N} \). Consider the operator \( S = \varphi \otimes h \) for \( h \in F \) and fix \( m \in \mathbb{N} \). Then for every \( x \in E \) we have

\[
\begin{align*}
c_S Z(x) & \geq (|S| \wedge mT)(x) \\
& = \inf \left\{ \sum_{i=1}^{n} |S|(x_i) \wedge mT(x_i) : x = \bigcup_{i=1}^{n} x_i, \ n \in \mathbb{N} \right\} \\
& = \inf \left\{ \sum_{i=1}^{n} |h| \varphi(x_i) \wedge mu \varphi(x_i) : x = \bigcup_{i=1}^{n} x_i, \ n \in \mathbb{N} \right\} \\
& = \inf \left\{ \sum_{i=1}^{n} \varphi(x_i)(|h| \wedge (mu)) : x = \bigcup_{i=1}^{n} x_i, \ n \in \mathbb{N} \right\} \\
& = \varphi(x)(|h| \wedge (mu)).
\end{align*}
\]

The abstract Uryson functional \( \varphi \) is a nonzero positive element in \( U(E, \mathbb{R}) \), hence there exists \( x_0 \in E \), such that \( \varphi(x_0) > 0 \). By means of the last estimation the inequality

\[
|h| \wedge (mu) \leq \frac{c_S}{\varphi(x_0)} Z(x_0) = \mu Z(x_0)
\]

holds with \( \mu = \frac{c_S}{\varphi(x_0)} \in \mathbb{R}_+ \) and arbitrary \( m \in \mathbb{N} \). Since \( h \) is an arbitrary element of \( F \) it is proved that \( u \in \Phi_1(F) \).

For proving the second assertion consider the rank one operator \( S = \theta \otimes u \), where \( \theta \in U(E, \mathbb{R}) \) is arbitrary. For every \( x \in E \) and \( n \in \mathbb{N} \) we may write

\[
(|\theta| \wedge n\varphi)(x)u \leq (|\theta|(x)u \wedge n\varphi(x)u) = (|S|(x)) \wedge (nT(x)).
\]

Then for every disjoint partition \( \{x_1, \ldots, x_n\} \) of \( x \), i.e. \( x = \bigcup_{i=1}^{n} x_i \), by using the orthogonal additivity of the abstract Uryson functional \( |\theta| \wedge n\varphi \) we have

\[
(|\theta| \wedge n\varphi)(x)u = \sum_{i=1}^{n} (|\theta| \wedge n\varphi)(x_i)u \leq \sum_{i=1}^{n} |S|(x_i) \wedge nT(x_i).
\]

After taking the infimum over all disjoint partitions of \( x \) on the right side of last formula, one has

\[
(|\theta| \wedge n\varphi)(x)u \leq (|S| \wedge nT)(x) \leq c_S Z(x)
\]

for every \( x \in E \) and \( n \in \mathbb{N} \). If now \( \tau \) is a positive linear functional on \( F \), such that \( \tau(u) = 1 \) then

\[
\tau((|\theta| \wedge n\varphi)(x)u) = (|\theta| \wedge n\varphi)(x) \leq \tau(c_S Z(x)) = c_S(\tau \circ Z)(x),
\]

where \( x \in E, \ n \in \mathbb{N} \), and \( \tau Z \in U_+(E, \mathbb{R}) \). Thus \( \varphi \in \Phi_1(U(E, \mathbb{R})) \) is proved and, \( \tau \circ Z \) is one of its majorants.

The converse statement still remains to be an open question.
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