ON SECOND VARIATION OF WANG-YAU QUASI-LOCAL ENERGY

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Abstract. We study a functional on the boundary of a compact Riemannian 3-manifold of nonnegative scalar curvature. The functional arises as the second variation of the Wang-Yau quasi-local energy in general relativity. We prove that the functional is positive definite on large coordinate spheres, and more general on nearly round surfaces including large constant mean curvature spheres in asymptotically flat 3-manifolds with positive mass; it is also positive definite on small geodesics spheres, whose centers do not have vanishing curvature, in Riemannian 3-manifolds of nonnegative scalar curvature. We also give examples of functions $H$, which can be made arbitrarily close to 2, on the standard 2-sphere $(S^2, \sigma_0)$ such that the triple $(S^2, \sigma_0, H)$ has positive Brown-York mass while the associated functional is negative somewhere.

1. Introduction

In [10][11], Wang and Yau introduced a new quasilocal mass. Briefly speaking, its definition is as follows. Let $\Sigma$ be a closed 2-surface, in a spacetime $N$ satisfying the dominant energy condition, such that $\Sigma$ bounds a compact, spacelike hypersurface $\Omega$. Denote the induced Riemannian metric on $\Sigma$ by $\gamma$. Given a function $\tau$ on $\Sigma$ such that $\hat{\gamma} = \gamma + d\tau \otimes d\tau$ is a metric of positive Gaussian curvature, one considers the isometric embedding $X : (\Sigma, \gamma) \hookrightarrow \mathbb{R}^{3,1}$

where $X = (\hat{X}, \tau)$ and $\hat{X} = (\hat{X}_1, \hat{X}_2, \hat{X}_3)$ is an isometric embedding of $(\Sigma, \hat{\gamma})$ in $\mathbb{R}^3 = \{(x,0) \in \mathbb{R}^{3,1}\}$. Associated with each such a function $\tau$ or equivalently each such an isometric embedding $X$, Wang and Yau introduced a quantity, which we denote by $E_{WY}(\Sigma, \tau)$, called the quasi-local energy of $\Sigma$ in $N$ with respect to $\tau$. The Wang-Yau quasi-local

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mass of $\Sigma$ in $N$ is then defined by
\begin{equation}
(1.1) \quad m_{WY}(\Sigma) = \inf_{\tau} E_{WY}(\Sigma, \tau)
\end{equation}
where the infimum is taken over all admissible functions $\tau$ (see [11] for an exact formula of $E_{WY}(\Sigma, \tau)$ and the definition of admissibility). It was proved in [11] that $m_{WY}(\Sigma) \geq 0$ and $m_{WY}(\Sigma) = 0$ if the embedding $\Sigma \hookrightarrow N$ is isometric to $\mathbb{R}^3$, along $\Sigma$.

When $\Sigma$ bounds a time-symmetric $\Omega$ and $\gamma$ has positive Gaussian curvature, there is a well-known Brown-York quasi-local mass of $\Sigma$ ([1, 2]) given by
\begin{equation}
(1.2) \quad m_{BY}(\Sigma, \Omega) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) \, dv_\gamma
\end{equation}
where $H_0$ is the mean curvature of the isometric embedding of $(\Sigma, \gamma)$ in $\mathbb{R}^3$ and $H$ is the mean curvature of $\Sigma$ in $\Omega$. In this situation, one has $m_{BY}(\Sigma, \Omega) = E_{WY}(\Sigma, \tau_0)$, where $\tau_0 = 0$ is an admissible function and is also a critical point of $E_{WY}(\Sigma, \cdot)$ ([11]). The variational definition of $m_{WY}(\Sigma)$ suggests $m_{WY}(\Sigma) \leq m_{BY}(\Sigma, \Omega)$. A natural question is whether $m_{WY}(\Sigma) = m_{BY}(\Sigma, \Omega)$. (Results regarding the global minimization of $E_{WY}(\Sigma, \cdot)$ recently have been announced in [3].)

In this paper, we consider the local minimality of $m_{BY}(\Sigma, \Omega)$. A main corollary of our result for surfaces in an asymptotically flat manifold is:

**Theorem 1.1.** Let $(M, g)$ be an asymptotically flat 3-manifold. Let $S_r = \{x \in M \mid \lvert z \rvert = r\}$ be a coordinate sphere in an admissible coordinate chart $\{z_i\}$ on a given end. Suppose the ADM mass of the end is positive. For sufficiently large $r$, the Brown-York mass of $S_r$ is a strict local minimum of $E_{WY}(S_r, \cdot)$.

We will prove Theorem 1.1 by proving Theorem 4.1 in Section 4 for a larger class of “large surfaces”, namely nearly round surfaces near the infinity which were introduced in [9]. As mentioned in [9], besides large coordinate spheres, notable examples of nearly round surfaces in an asymptotically flat 3-manifold include the constant mean curvature surfaces constructed in [5] and [12].

For small geodesic spheres in a manifold of nonnegative scalar curvature, we have:

**Theorem 1.2.** Let $(M, g)$ be a Riemannian 3-manifold of nonnegative scalar curvature. Let $p \in M$ be a point. For $r > 0$, let $S_r$ be the geodesic sphere of radius $r$ centered at $p$ and $B_r$ be the corresponding
geodesic ball. If
\begin{equation}
\lim_{r \to 0} r^{-5} m_{BY}(S_r, B_r) > 0,
\end{equation}
then \( m_{BY}(S_r, B_r) \) is a strict local minimum of \( E_{WY}(S_r, \tau) \) for \( r > 0 \) sufficiently small.

We note that condition (1.3) in Theorem 1.2 is equivalent to
(i) \( R(p) > 0 \), or
(ii) \( R(p) = 0 \) and \( |\text{Ric}(p)| > 0 \), or
(iii) \( R(p) = 0 \), \( |\text{Ric}(p)| = 0 \), and \( (\Delta R)(p) > 0 \)
which follows from the asymptotic expansion of \( m_{BY}(S_r, B_r) \) in [4] and the assumption \( R \geq 0 \). Here \( R, \text{Ric} \) denote the scalar curvature, the Ricci curvature of \( g \).

For \( m_{BY}(\Sigma, \Omega) = E_{WY}(\Sigma, \tau_0) \) to locally minimize \( E_{WY}(\Sigma, \cdot) \), the second variation of \( E_{WY}(\Sigma, \cdot) \) at \( \tau_0 \) is necessarily nonnegative. We recall the following result from [6].

**Theorem 1.3** ([6]). Suppose \( m_{BY}(\Sigma, \Omega) \) is defined for a 2-surface \( \Sigma \) bounding a time-symmetric hypersurface \( \Omega \) in a spacetime \( N \). The second variation of \( E_{WY}(\Sigma, \cdot) \) at \( \tau_0 = 0 \) (up to multiplication by \( \frac{1}{8\pi} \)) is
\begin{equation}
F_{\gamma,H}(\eta) = \int_{\Sigma} \left[ \frac{(\Delta \eta)^2}{H} + (H_0 - H)|\nabla \eta|^2 - \mathbb{II}_0(\nabla \eta, \nabla \eta) \right] dv_{\gamma},
\end{equation}
where \( \mathbb{II}_0 \) is the second fundamental form of \( (\Sigma, \gamma) \) when it is isometrically embedded in \( \mathbb{R}^3 \). If there exists a constant \( \beta > 0 \) such that
\begin{equation}
F_{\gamma,H}(\eta) \geq \beta \int_{\Sigma} (\Delta \eta)^2 dv_{\gamma}, \quad \forall \eta \in W^{2,2}(\Sigma),
\end{equation}
then \( m_{BY}(\Sigma, \Omega) \) is a strict local minimum of \( E_{WY}(\Sigma, \cdot) \).

Therefore, to obtain the local minimality of \( m_{BY}(\Sigma, \Omega) \), it suffices to study the functional \( F_{\gamma,H}(\eta) \). The induced metric and the mean curvature function on the surfaces \( \{S_r\} \) in Theorems 1.1 and 1.2 (after rescaling) are close to the standard metric \( \sigma_0 \) on the unit sphere \( S^2 \) and the constant 2 respectively. Thus, one may ask whether \( F_{\gamma,H}(\eta) \) satisfies (1.5) if the pair \( (\gamma, H) \) is sufficiently close to \( (\sigma_0, 2) \).

Our first task in this paper is to derive some sufficient conditions on such a pair \( (\gamma, H) \) so that (1.5) is true. Applying these sufficient conditions, we can prove Theorem 4.1 and part of Theorem 1.2 which corresponds to cases (i) and (iii) above.

The other part of Theorem 1.2, which corresponds to case (ii), turns out to be more subtle. We will prove it using more refined estimation on \( (S^2, \sigma_0) \) (see Theorem 5.2 and Proposition 5.1). Motivated by our
proof of this part of Theorem 1.2 we also construct examples to show that on \((S^2, σ_0)\), there are functions \(H\) which can be arbitrarily close to 2, but \(F_{σ_0,H}(η) < 0\) for some \(η\).

We remark that the general validity of (1.5) is of significance in the study of boundary behaviors of compact manifolds of nonnegative scalar curvature. If (1.5) is always true, it will impose a necessary condition for a positive function \(H\) on \(Σ\) to arise as the mean curvature of \(Σ\) in some compact Riemannian 3-manifold of nonnegative scalar curvature, bounded by \((Σ, γ)\). So far, a major known necessary condition is \(\int_Σ (H_0 - H) dv_γ \geq 0\) by the result of [8]. It is worth to note that our examples of \(H\) above, with \(F_{σ_0,H}(η) < 0\) for some \(η\), also satisfies \(\int_{S^2} (2 - H) dv_{σ_0} > 0\). Thus, if the Brown-York mass always locally minimizes the Wang-Yau quasi-local energy in the time-symmetric situation, then (1.5) will constitute a new necessary condition.

This paper is organized as follows. In Section 2, we collect some lemmas which are to be used frequently in later sections. In Section 3, we obtain sufficient conditions for (1.5) to hold. In Section 4, we apply the derived sufficient conditions to prove Theorem 4.1 which implies Theorem 1.1. In Section 5, we establish the positivity of \(F_{γ,H}\) on small geodesic spheres in Theorem 5.1 which implies Theorem 1.2. There whether the scalar curvature vanishes at the center of a geodesic sphere makes an important difference in the proof. A main result related to the case of vanishing scalar curvature at the sphere center is Theorem 5.2 which we prove using a functional inequality on the standard sphere \((S^2, σ_0)\) (Proposition 5.1) which may have independent interest. In Section 6, we give examples of \(H\) on the standard unit sphere so that \(F_{σ_0,H}(η) < 0\) for some \(η\) while \(\int_{S^2} (2 - H) dv_{σ_0} > 0\). In the Appendix, we list some elementary computational results, which are needed in Section 5.

2. Preliminaries

Throughout this paper, \(Σ\) always denotes a closed 2-surface that is diffeomorphic to a 2-sphere. Given a metric \(γ\) of positive Gaussian curvature and a positive function \(H\) on \(Σ\), we let

\[
F_{γ,H}(η) = \int_Σ \left[ \frac{(Δη)^2}{H} + (H_0 - H)|∇η|^2 - \Pi_0(∇η, ∇η) \right] dv_γ
\]

for any \(η \in W^{2,2}(Σ)\). Here \(Δ\) and \(∇\) denote the Laplacian and the gradient on \((Σ, γ)\), \(H_0\) and \(Π_0\) are the mean curvature and the second fundamental form of \((Σ, γ)\) when it is isometrically embedded in \(\mathbb{R}^3\), and \(dv_γ\) is the volume form on \((Σ, γ)\). We also denote the symmetric
bilinear form associated to $F_{\gamma,H}$ by $Q_{\gamma,H}$. Namely
\begin{equation}
Q_{\gamma,H}(\eta_1,\eta_2) = \int_\Sigma \left[ \frac{\Delta \eta_1 \cdot \Delta \eta_2}{H} + (H_0 - H)\langle \nabla \eta_1, \nabla \eta_1 \rangle - \mathbb{II}_0(\nabla \eta_1, \nabla \eta_2) \right] d\gamma.
\end{equation}
All metrics on $\Sigma$ below will be assumed to be smooth for simplicity.

We recall some basic results from [6].

**Lemma 2.1.** Let $\gamma$ be a metric of positive Gaussian curvature on $\Sigma$. Let $X = (X_1, X_2, X_3) : (\Sigma, \gamma) \to \mathbb{R}^3$ be an isometric embedding of $(\Sigma, \gamma)$ in $\mathbb{R}^3$. The functional
\[
F_{\gamma,H_0}(\eta) = \int_\Sigma \left[ \frac{(\Delta \eta)^2}{H_0} - \mathbb{II}_0(\nabla \eta, \nabla \eta) \right] d\gamma
\]
satisfies:
(i) $F_{\gamma,H_0}(\eta) \geq 0, \forall \ \eta \in W^{2,2}(\Sigma)$ .
(ii) $F_{\gamma,H_0}(\eta) = 0$ if and only if $\eta \in \mathcal{L}(\gamma)$, where
\[
\mathcal{L}(\gamma) = \left\{ a_0 + \sum_{i=1}^3 a_i X^i \mid a_0, a_1, a_2, a_3 \text{ are arbitrary constants} \right\}.
\]
(iii) If $\eta \in \mathcal{L}(\gamma)$, then $Q_{\gamma,H_0}(\eta, \phi) = 0, \forall \ \phi \in W^{2,2}(\Sigma)$.

**Remark 2.1.** (i) and (ii) are proved in [9, Corollary 3.1]. (iii) is a direct consequence of (i) and (ii) by considering the first variation of $F_{\gamma,H_0}$.

**Remark 2.2.** Since any two isometric embeddings of $(\Sigma, \gamma)$ differ by a rigid motion in $\mathbb{R}^3$, the space $\mathcal{L}(\gamma)$ defined above is independent on the choice of $X$.

Lemma 2.1 shows $F_{\gamma,H}(\cdot)$ vanishes on $\mathcal{L}(\gamma)$ when $H = H_0$. For an arbitrary $H$, we have the following from (3.12) in [6, Proposition 3.2].

**Lemma 2.2.** Suppose $H > 0$. For any $\eta = a_0 + \sum_{i=1}^3 a_i X^i \in \mathcal{L}(\gamma)$,
\[
F_{\gamma,H}(\eta) = |a|^2 \int_\Sigma (H_0 - H)d\gamma + \int_\Sigma \langle a, \nu_0 \rangle^2 \frac{(H_0 - H)^2}{H} d\nu_\gamma,
\]
where $a = (a_1, a_2, a_3)$ and $\nu_0$ is the unit outward normal to $(\Sigma, \gamma)$ when it is isometrically embedded in $\mathbb{R}^3$.

If $\int_\Sigma (H_0 - H)d\gamma > 0$, then Lemma 2.2 implies
\[
F_{\gamma,H}(\eta) \geq \beta \int_\Sigma (\Delta \eta)^2 d\gamma
\]
for some $\beta > 0$ for all $\eta \in \mathcal{L}(\gamma)$.

Next we estimate $F_{\gamma,H_0}(\eta)$ for $\eta$ that are $\gamma$-$L^2$ orthogonal to $\mathcal{L}(\gamma)$. 
Lemma 2.3. Let $\sigma$ be a metric of positive Gaussian curvature on $\Sigma$. There exist positive constants $\delta$ and $\beta$ such that if $\gamma$ is a metric on $\Sigma$ satisfying $\|\gamma - \sigma\|_{C^{2,\alpha}(\Sigma,\sigma)} < \delta$, then

$$F_{\gamma,H_0}(\eta) \geq \beta \int_{\Sigma} (\Delta_\gamma \eta)^2 \, dv_\gamma$$

for all $\eta \in W^{2,2}(\Sigma)$ that is $\gamma$-$L^2$ orthogonal to $\mathcal{L}(\gamma)$. Here $\Delta_\gamma$ denotes the Laplacian on $(\Sigma, \gamma)$.

Proof. We argue by contradiction. Suppose it is not true, then there exists a sequence of metrics $\{\gamma_k\}$ on $\Sigma$ and a sequence of functions $\{\eta_k\} \subset W^{2,2}(\Sigma)$ such that

(2.3) $\lim_{k \to \infty} \|\gamma_k - \sigma\|_{C^{2,\alpha}(\Sigma,\sigma)} = 0$,

(2.4) $\int_{\Sigma} \eta_k \phi \, dv_{\gamma_k} = 0, \forall \phi \in \mathcal{L}(\gamma_k), \ k = 1, 2, 3, \ldots$,

and

(2.5) $F_{\gamma_k,H_0}(\eta_k) \leq \frac{1}{k} \int_{\Sigma} (\Delta_k \eta_k)^2 \, dv_{\gamma_k}$.

Here $H^k_0$ is the $H_0$ associated to $\gamma_k$ and $\Delta_k$ stands for $\Delta_{\gamma_k}$. We normalize $\eta_k$ such that

(2.6) $\int_{\Sigma} \eta_k^2 \, dv_{\gamma_k} = 1$.

Let $X : (\Sigma, \sigma) \to \mathbb{R}^3$ be an isometric embedding of $(\Sigma, \sigma)$. By (2.3) and the result of Nirenberg [7, p.353], for each large $k$, there exists an isometric embedding $X^k$ of $(\Sigma, \gamma_k)$ in $\mathbb{R}^3$ such that

(2.7) $\|X^k - X\|_{C^{2,\alpha}(\Sigma,\sigma)} \leq C \|\gamma_k - \sigma\|_{C^{2,\alpha}(\Sigma,\sigma)}$

where $C$ is some constant depending only on $\sigma$. Let $\mathbb{I}_0^k$, $\mathbb{I}_0$ be the second fundamental form of $X^k(\Sigma)$, $X(\Sigma)$ respectively. (2.7) implies that $\mathbb{I}_0^k$ and $\mathbb{I}_0$ (viewed as $(0,2)$ tensor fields on $\Sigma$ through the pullback by $X^k$ and $X$) satisfy

(2.8) $\lim_{k \to \infty} \|\mathbb{I}_0^k - \mathbb{I}_0\|_{C^{0,\alpha}(\Sigma,\sigma)} = 0$.

Consequently, $\{H^k_0\}$ converges to $H_0$ uniformly on $\Sigma$, where $H_0$ is the mean curvature of $X(\Sigma)$. 
It follows from (2.3) - (2.6), (2.8), the interpolation inequality for Sobolev spaces and the $L^2$-estimates that
\[
\int_{\Sigma} \left( \frac{(\Delta_k \eta_k)^2}{H_0^8} \right) dv_{\gamma_k} \leq \int_{\Sigma} II_0^k(\nabla_k \eta_k, \nabla_k \eta_k) dv_{\gamma_k} + \frac{1}{k} \int_{\Sigma} (\Delta_k \eta_k)^2 dv_{\gamma_k}
\]
\[
\leq \frac{1}{2} \int_{\Sigma} \left( \frac{(\Delta_k \eta_k)^2}{H_0^8} \right) dv_{\gamma_k} + C_1 + \frac{1}{k} \int_{\Sigma} (\Delta_k \eta_k)^2 dv_{\gamma_k}
\]
where $\nabla_k$ is the gradient on $(\Sigma, \gamma_k)$. Here and below, $\{C_i\}$ always denote positive constants that are independent on $k$. Now (2.9) shows
\[
\int_{\Sigma} (\Delta_k \eta_k)^2 dv_{\gamma_k} \leq C_2,
\]
which combined with (2.3), (2.6) and the $L^2$-estimates implies
\[
||\eta_k||_{W^{2,2}(\Sigma, \gamma_k)} \leq C_3.
\]
By (2.3), this in turn shows $||\eta_k||_{W^{2,2}(\Sigma, \sigma)} \leq C_4$. Hence $\exists \eta \in W^{2,2}(\Sigma)$ such that, passing to a subsequence, $\{\eta_k\}$ converges to $\eta$ weakly in $W^{2,2}(\Sigma, \sigma)$ and strongly in $W^{1,2}(\Sigma, \sigma)$.

We claim
\[
F_{\gamma, H_0}(\eta) = \int_{\Sigma} \frac{(\Delta \eta)^2}{H_0^8} - II_0(\nabla \eta, \nabla \eta) dv_{\sigma} \leq 0.
\]
If this is true, then we have a contradiction by Lemma 2.1.

To prove (2.12), we apply Lemma 2.1 to obtain $F_{\gamma_k, H_0^k}(\eta_k - \eta) \geq 0$. By (2.3), (2.5), (2.7), (2.8) and (2.10), we have
\[
\frac{C_2}{k} \geq F_{\gamma_k, H_0^k}(\eta_k)
\]
\[
\geq 2Q_{\gamma_k, H_0^k}(\eta_k, \eta) - F_{\gamma_k, H_0^k}(\eta)
\]
\[
= 2Q_{\gamma, H_0}(\eta, \eta) - F_{\gamma, H_0}(\eta) + o(1)
\]
as $k \to \infty$. Using the fact that $\eta_k$ converge to $\eta$ weakly in $W^{2,2}(\Sigma)$ and strongly in $W^{1,2}(\Sigma)$, we conclude from (2.13) that
\[
0 \geq 2Q_{\gamma, H_0}(\eta, \eta) - F_{\gamma, H_0}(\eta)
\]
\[
= F_{\gamma, H_0}(\eta)
\]
which verifies (2.12). The Lemma is proved.

Often we need estimates of $\int_{\Sigma} |\nabla \eta|^2 dv_{\gamma}$ by $\int_{\Sigma} (\Delta \eta)^2 dv_{\gamma}$ which depend explicitly on the metric $\gamma$. This can be given in terms of eigenvalues of $(\Sigma, \gamma)$. 

\[
\int_{\Sigma} \left( \frac{(\Delta_k \eta_k)^2}{H_0^8} \right) dv_{\gamma_k} \leq \int_{\Sigma} II_0^k(\nabla_k \eta_k, \nabla_k \eta_k) dv_{\gamma_k} + \frac{1}{k} \int_{\Sigma} (\Delta_k \eta_k)^2 dv_{\gamma_k}
\]
\[
\leq \frac{1}{2} \int_{\Sigma} \left( \frac{(\Delta_k \eta_k)^2}{H_0^8} \right) dv_{\gamma_k} + C_1 + \frac{1}{k} \int_{\Sigma} (\Delta_k \eta_k)^2 dv_{\gamma_k}
\]
Lemma 2.4. Let \((M, g)\) be a compact Riemannian manifold without boundary. Let \(E_k\) be the space of eigenfunctions with the \(k\)-th nonzero eigenvalue \(\mu_k\) of \((M, g)\). Let \(E_0\) be the space of constant functions. Suppose \(\phi \in W^{2,2}(M)\) is \(g\)-\(L^2\) orthogonal to \(E_0, E_1, \ldots, E_{k-1}\), then
\[
\mu_k \int_M |\nabla \phi|^2 dv_g \leq \int_M (\Delta \phi)^2 dv_g.
\]
In particular,
\[
\mu_1 \int_M |\nabla \phi|^2 dv_g \leq \int_M (\Delta \phi)^2 dv_g, \quad \forall \phi \in W^{2,2}(M).
\]

Proof. Since
\[
\int_M |\nabla \phi|^2 dv_g = - \int_\Sigma \phi \Delta \phi dv_g \leq \left( \int_M \phi^2 dv_g \right)^{\frac{1}{2}} \left( \int_\Sigma (\Delta \phi)^2 dv_g \right)^{\frac{1}{2}},
\]
we have
\[
\frac{\int_M |\nabla \phi|^2 dv_g}{\int_M \phi^2 dv_g} \leq \frac{\int_M (\Delta \phi)^2 dv_g}{\int_M |\nabla \phi|^2 dv_g},
\]
from which the claim follows. \(\square\)

3. Sufficient conditions for the positivity of \(F_{\gamma, H}\)

In this section, we provide some sufficient conditions guaranteeing the positivity of \(F_{\gamma, H}\). Note that Lemma 2.3 implies
\[
\inf_{\eta \in \mathcal{L}(\gamma)^{\perp}} \frac{F_{\gamma, H_0}(\eta)}{\int_\Sigma (\Delta \eta)^2 dv_\gamma} > 0
\]
where \(\mathcal{L}(\gamma)^{\perp}\) is the space of functions that are \(\gamma\)-\(L^2\) orthogonal to \(\mathcal{L}(\gamma)\).

Proposition 3.1. Let \(\gamma\) be a metric of positive Gaussian curvature on \(\Sigma\). Let \(\beta\) be a positive constant such that
\[
F_{\gamma, H_0}(\eta) \geq \beta \int_\Sigma (\Delta \eta)^2 dv_\gamma, \quad \forall \eta \in \mathcal{L}(\gamma)^{\perp}.
\]

Suppose the first nonzero eigenvalue of \((\Sigma, \gamma)\) is at least \(\lambda > 0\). Then \(\exists \delta > 0\), depending only on \(\beta, \lambda, H_0\) and a given constant \(\alpha > 0\), such that if \(H \geq \alpha\) is a function on \(\Sigma\) satisfying
\[
(a) \int_\Sigma (H_0 - H) \ dv_\gamma > 0
\]
\[
(b) \sup_\Sigma |(H_0 - H)_-| < \delta \quad \text{and} \quad \frac{\int_\Sigma |H_0 - H|^2 \ dv_\gamma}{\int_\Sigma (H_0 - H) \ dv_\gamma} < \delta,
\]

where \(H_0 = \int_\Sigma H \ dv_\gamma / \int_\Sigma \ dv_\gamma\).
where \((H_0 - H)_- = \min\{H_0 - H, 0\}\), then \(F_{\gamma,H}(\eta) \geq \tilde{\beta} \int_{\Sigma} (\Delta \eta)^2 \, dv_\gamma\), \(\forall \eta \in W^{2,2}(\Sigma)\), where \(\tilde{\beta} > 0\) is a constant depending only on \(H_0\) and \(\beta\).

**Proof.** Given any constant \(\alpha > 0\), let \(H \geq \alpha\) be a function on \(\Sigma\) satisfying (a) and (b) with \(\delta > 0\) to be chosen later.

Let \(X = (X_1, X_2, X_3)\) be an isometric embedding of \((\Sigma, \gamma)\) in \(\mathbb{R}^3\). Given any \(\eta \in W^{2,2}(\Sigma)\), decompose \(\eta = \eta_1 + \eta_2\) where \(\eta_1 = a_0 + \sum_{i=1}^3 a_i X_i \in \mathcal{L}(\gamma)\) and \(\eta_2 \in \mathcal{L}(\gamma)^\perp\). Let \(a = (a_1, a_2, a_3)\). If \(a = 0\), then

\[
F_{\gamma,H}(\eta) = F_{\gamma,H_0}(\eta_2) + \int_{\Sigma} (H_0 - H) \left( \frac{(\Delta \eta_2)^2}{H H_0} + |\nabla \eta_2|^2 \right) \, dv_\gamma
\]

\[
\geq \beta \int_{\Sigma} (\Delta \eta_2)^2 \, dv_\gamma - \delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) \int_{\Sigma} (\Delta \eta_2)^2 \, dv_\gamma
\]

\[
= \left[ \beta - \delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) \right] \int_{\Sigma} (\Delta \eta)^2 \, dv_\gamma
\]

where we have used (3.1) and Lemma 2.4.

Now suppose \(a \neq 0\), we may normalize \(a\) so that \(|a| = 1\). Then for any \(\epsilon_1, \epsilon_2 > 0\),

\[
F_{\gamma,H}(\eta) - F_{\gamma,H_0}(\eta)
= \int_{\Sigma} \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) (\Delta \eta)^2 + (H_0 - H)|\nabla \eta|^2 \right] \, dv_\gamma
= \int_{\Sigma} \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) (\Delta \eta_1)^2 + (H_0 - H)|\nabla \eta_1|^2 \right] \, dv_\gamma
+ \int_{\Sigma} \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) (\Delta \eta_2)^2 + (H_0 - H)|\nabla \eta_2|^2 \right] \, dv_\gamma
+ 2 \int_{\Sigma} \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) \Delta \eta_1 : \Delta \eta_2 + (H_0 - H)(\nabla \eta_1, \nabla \eta_2) \right] \, dv_\gamma
\]

\[
\geq \int_{\Sigma} (H_0 - H)dv_\gamma - \delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) \int_{\Sigma} (\Delta \eta_2)^2 dv_\gamma
- \epsilon_1^{-1} \int_{\Sigma} \left( \frac{1}{H} - \frac{1}{H_0} \right)^2 (\Delta \eta_1)^2 dv_\gamma - \epsilon_1 \int_{\Sigma} (\Delta \eta_2)^2 dv_\gamma
- \epsilon_2^{-1} \int_{\Sigma} (H - H_0)^2 |\nabla \eta_1|^2 dv_\gamma - \epsilon_2 \int_{\Sigma} |\nabla \eta_2|^2 dv_\gamma
\]

\[
\geq (1 - \epsilon_1^{-1} \alpha^{-2} \delta - \epsilon_2^{-1} \delta) \int_{\Sigma} (H_0 - H)dv_\gamma
- \left( \delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) + \epsilon_1 + \lambda^{-1} \epsilon_2 \right) \int_{\Sigma} (\Delta \eta_2)^2 dv_\gamma
\]
where we have used Lemma 2.2, the fact \((\Delta \eta_1)^2 = \langle a, \nu_0 \rangle^2 H_0^2\) and \(|\nabla \eta_1|^2 = 1 - \langle a, \nu_0 \rangle^2\). On the other hand, it follows from Lemma 2.1 and (3.1) that

\[
F_{\gamma,H_0}(\eta) = F_{\gamma,H_0}(\eta_2) \geq \beta \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma},
\]

Hence

\[
F_{\gamma,H}(\eta) \geq (1 - \epsilon_1^{-1} \alpha^{-2} \delta - \epsilon_2^{-1} \delta) \int_{\Sigma} (H_0 - H) dv_{\gamma} + \left[ \beta - \delta \left( \frac{1}{\alpha_{\inf \Sigma} H_0} + \frac{1}{\lambda} \right) \right] \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma}.
\]

Let \(\epsilon_1 = \lambda^{-1} \epsilon_2 = \frac{\beta}{4}\) and choose \(\delta > 0\) such that \(1 - (\epsilon_1^{-1} \alpha^{-2} + \epsilon_2^{-1} \delta) \geq \frac{1}{2}\), and \(\delta \left( \frac{1}{\alpha_{\inf \Sigma} H_0} + \frac{1}{\lambda} \right) \leq \frac{\beta}{4}\), then

\[
F_{\gamma,H}(\eta) \geq \frac{1}{2} \int_{\Sigma} (H_0 - H) dv_{\gamma} + \frac{\beta}{4} \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma}.
\]

Combining this with (3.2) and the fact \((\Delta \eta_1)^2 \leq H_0^2\), we conclude that the Proposition is true.

In [6, Theorem 3.1], it was proved that \(F_{\gamma,H}\) is positive definite if \(H \leq H_0\) and \(\int_{\Sigma} (H_0 - H) dv_{\gamma} > 0\). By arguments similar to the proof of Proposition 3.1, it can be shown that \(F_{\gamma,H}\) remains positive definite if \(H\) is allowed to be slightly bigger than \(H_0\). First, we give a quantitative estimate of the case \(H \leq H_0\).

**Lemma 3.1.** Let \(\gamma\) be a metric of positive Gaussian curvature on \(\Sigma\). Let \(\beta > 0\) be a constant such that

\[
F_{\gamma,H_0}(\eta) \geq \beta \int_{\Sigma} (\Delta \eta)^2 dv_{\gamma}, \ \forall \ \eta \in \mathcal{L}(\gamma)^\perp.
\]

Let \(\lambda > 0\) be a lower bound for the first nonzero eigenvalue of \((\Sigma, \gamma)\). Given any positive function \(H\) on \(\Sigma\) with \(H \leq H_0\), let \(\alpha > 0\) be a lower bound of \(H\). Then

\[
F_{\gamma,H}(\eta_1 + \eta_2) \geq \frac{\beta}{\alpha^{-1} + \lambda^{-1} \sup_{\Sigma} H_0 + \frac{\alpha}{2}} \int_{\Sigma} (H_0 - H) dv_{\gamma} + \frac{\beta}{2} \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma}
\]

for any \(\eta_2 \in \mathcal{L}(\gamma)^\perp\) and \(\eta_1 = a_0 + \sum_{i=1}^{3} a_i X^i \in \mathcal{L}(\gamma)\) with \(a = (a_1, a_2, a_3)\) being a unit vector.
Proof. Similar to (3.3), using the fact $H_0 \geq H$, we have for any constant $0 < \epsilon < 1$ that

$$F_{\gamma,H}(\eta) - F_{\gamma,H_0}(\eta) \geq (1 - \epsilon) \int_\Sigma \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) (\Delta \eta_1)^2 + (H_0 - H)|\nabla \eta_1|^2 \right] d\gamma$$

$$+ (1 - \epsilon^{-1}) \int_\Sigma \left[ \left( \frac{1}{H} - \frac{1}{H_0} \right) (\Delta \eta_2)^2 + (H_0 - H)|\nabla \eta_2|^2 \right] d\gamma$$

$$\geq (1 - \epsilon) \int_\Sigma (H_0 - H) d\gamma + (1 - \epsilon^{-1}) \left( \alpha^{-1} + \lambda^{-1} \sup_\Sigma H_0 \right) \int_\Sigma (\Delta \eta_2)^2 d\gamma.$$

Hence

$$F_{\gamma,H}(\eta) \geq (1 - \epsilon) \int_\Sigma (H_0 - H) d\gamma$$

(3.4)

$$+ \left[ \beta + (1 - \epsilon^{-1}) \left( \alpha^{-1} + \lambda^{-1} \sup_\Sigma H_0 \right) \right] \int_\Sigma (\Delta \eta_2)^2 d\gamma$$

where we can choose $0 < \epsilon < 1$ such that

$$\beta + (1 - \epsilon^{-1}) \left( \alpha^{-1} + \lambda^{-1} \sup_\Sigma H_0 \right) = \frac{\beta}{2}.$$

The Lemma now follows from (3.4). \qed

**Proposition 3.2.** Let $\gamma$ be a metric of positive Gaussian curvature on $\Sigma$. Let $\beta > 0$ be a constant such that

$$F_{\gamma,H_0}(\eta) \geq \beta \int_\Sigma (\Delta \eta)^2 d\gamma, \forall \eta \in L^2(\Sigma).$$

Let $\lambda > 0$ be a lower bound for the first nonzero eigenvalue of $(\Sigma, \gamma)$. Given a positive function $H$ on $\Sigma$, let $(H_0 - H)_- = \min\{H_0 - H, 0\}$.

Let $\alpha > 0$ be any lower bound of $H$. Define $\alpha_1 = \min\{\alpha, \inf_\Sigma H_0\}$,

$$\theta = \frac{\beta}{\alpha_1^{-1} + \lambda^{-1} \sup_\Sigma H_0 + \frac{\beta}{2}} \quad \text{and} \quad \delta = \frac{\beta}{4} \left( \frac{1}{\alpha \inf_\Sigma H_0} + \frac{1}{\lambda} \right)^{-1}.$$

Suppose

(i) $\theta \int_\Sigma (H_0 - H) + 2 \int_\Sigma (H_0 - H)_- > 0$

(ii) $\sup_\Sigma [(H_0 - H)_-] < \delta$,

then $F_{\gamma,H}(\eta) \geq \tilde{\beta} \int_\Sigma (\Delta \eta)^2 d\gamma, \forall \eta \in W^{2,2}(\Sigma)$, for some $\tilde{\beta} > 0$. 


Proof. Suppose $H$ satisfies (i) and (ii). Given $\eta \in W^{2,2}(\Sigma)$, let $\eta = \eta_1 + \eta_2$ and $\eta_1 = a_0 + \sum_{i=1}^{3} a_i X^i$ be given as in the proof of Proposition 3.1. Let $a = (a_1, a_2, a_3)$. If $a = 0$, similar to (3.2), we have

$$F_{\gamma,H}(\eta) \geq \beta \int_{\Sigma} (\Delta \eta_2)^2 + \int_{\Sigma} (H_0 - H) \left( \frac{(\Delta \eta_2)^2}{H_0 H} + |\nabla \eta|^2 \right) dv_{\gamma} \geq \left[ \beta - \delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) \right] \int_{\Sigma} (\Delta \eta_2)^2.$$  

(3.5)

Next, suppose $a \neq 0$. We normalize $a$ so that $|a| = 1$. Define

$$H_1 = \min \{H_0, H\} \quad \text{and} \quad H_2 = \max \{H_0, H\}.$$  

Then $H_1 + H_2 = H + H_0$, $1/H_1 + 1/H_2 = 1/H + 1/H_0$ and $H_0 - H_2 = (H_0 - H)_{-}$. By Lemma 3.1, we have

$$F_{\gamma,H}(\eta) = F_{\gamma,H_1}(\eta) + \int_{\Sigma} (H_0 - H_2) \left( \frac{(\Delta \eta_1)^2}{H_2 H_0} + |\nabla \eta_1|^2 \right) dv_{\gamma} \geq \theta \int_{\Sigma} (H_0 - H_1) dv_{\gamma} + \frac{\beta}{2} \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma}$$

$$+ 2 \int_{\Sigma} (H_0 - H)_{-} \left( \frac{(\Delta \eta_1)^2}{H_2 H_0} + |\nabla \eta_1|^2 \right) dv_{\gamma}$$

$$+ 2 \int_{\Sigma} (H_0 - H)_{-} \left( \frac{(\Delta \eta_2)^2}{H_2 H_0} + |\nabla \eta_2|^2 \right) dv_{\gamma}$$

$$\geq \theta \int_{\Sigma} (H_0 - H) dv_{\gamma} + 2 \int_{\Sigma} (H_0 - H)_{-} dv_{\gamma} + \left[ \frac{\beta}{2} - 2\delta \left( \frac{1}{\alpha \inf_{\Sigma} H_0} + \frac{1}{\lambda} \right) \right] \int_{\Sigma} (\Delta \eta_2)^2 dv_{\gamma}.$$  

(3.6)

Proposition 3.2 now follows from (3.5) and (3.6). □

4. Nearly round surfaces in AF manifolds

In this section, we apply Lemma 2.3 and Proposition 3.1 to study the positivity of $F_{\gamma,H}$ on certain “large surfaces” near infinity in an asymptotically flat 3-manifold, based on existing results in [4, 9].

We adopt the following definition in [4] for an asymptotically flat 3-manifold and an admissible coordinate chart.

Definition 4.1. A Riemannian 3-manifold is called asymptotically flat (AF) of order $\tau$ (with one end) if there is a compact set $K$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$, where $B_R(0)$ is a coordinate
ball of radius $R > 0$ centered at the origin, such that in the standard coordinates $\{z_i\}$ on $\mathbb{R}^3$ the metric $g$ satisfies

$$g_{ij} = \delta_{ij} + h_{ij}$$

with

$$|h_{ij}| + |z||\partial h_{ij}| + |z|^2|\partial^2 h_{ij}| + |z|^3|\partial^3 h_{ij}| = O(|z|^{-\tau})$$

for some constant $\tau > \frac{1}{2}$. Here $|z|$ and $\partial$ denote the coordinate length of $z$ and the usual partial derivative operator on $\mathbb{R}^3$ respectively.

A coordinate chart $\{z_i\}$ on $M$ in which the metric $g$ satisfies the above conditions is called an admissible coordinate chart.

Large coordinate spheres in an admissible coordinate chart are examples of nearly round surfaces (see [9]). These surfaces are intrinsically defined as follows.

**Definition 4.2** ([9]). On an asymptotically flat 3-manifold $(M, g)$ of order $\tau > \frac{1}{2}$, let $r(x)$ be the $g$-distance from $x$ to a fixed point. A 1-parameter family of surfaces $\{\Sigma_r\}$, where $r = \min_{\Sigma_r} r(x)$ and $\Sigma_r$ is topologically a 2-sphere, is called nearly round as $r$ tends to infinity if

1. $|\hat{A}| + r|\nabla A| \leq Cr^{-1-\tau}$
2. $\max_{x \in \Sigma_r} r(x) \leq Cr$
3. $\text{diam}(\Sigma_r) \leq Cr$
4. $\text{Area}(\Sigma_r) \leq Cr^2$.

Here $\hat{A}$ is the traceless part of the second fundamental form of $\Sigma_r$, $\nabla$ and $|\cdot|$ denote the covariant derivative and the norm on $\Sigma_r$ with respect to the induced metric, $\text{diam}(\cdot)$ and $\text{Area}(\cdot)$ denote the diameter and the area of a surface, and $C$ is a constant independent of $r$.

As shown in [9], other examples of nearly round surfaces include the constant mean curvature surfaces constructed in [5] and [12]. The main result in this section is

**Theorem 4.1.** Let $(M, g)$ be an asymptotically flat 3-manifold of order $\tau > \frac{1}{2}$. Let $\{\Sigma_r\}$ be a family of nearly round surfaces as $r$ tends to infinity. Suppose the ADM mass of $(M, g)$ is positive. Then there exist constants $R > 0$ and $C > 0$ such that

$$\int_{\Sigma_r} \left[ \frac{(\Delta \eta)^2}{H} + (H_0 - H)|\nabla \eta|^2 - \mathbb{I}_0(\nabla \eta, \nabla \eta) \right] dv_r > C r \int_{\Sigma_r} (\Delta \eta)^2 dv_r,$$

$\forall \eta \in W^{2,2}(\Sigma_r)$ and $\forall r > R$.

As a result, the Brown-York mass of $\Sigma_r$ is a strict local minimum of the Wang-Yau quasi-local energy of $\Sigma_r$ for sufficiently large $r$. 
Proof. Let $K$ be a compact set such that $M \setminus K$ carries an admissible coordinate chart. Let $\hat{g}$ be a background Euclidean metric on $M \setminus K$. Let $H_r$ and $\hat{H}_r$ be the mean curvature of $\Sigma_r$ in $(M \setminus K, g)$ and $(M \setminus K, \hat{g})$ respectively. By (2.13) in $[9]$, one has

\begin{equation}
H_r = \hat{H}_r + O(r^{-1-\tau}).
\end{equation}

By Proposition 3.2 and Theorem 4 in $[9]$, for each sufficiently large $r$, there exists a number $r_0 \in \mathbb{R}$ such that

\begin{equation}
C^{-1}r \leq r_0 \leq Cr,
\end{equation}

\begin{equation}
\hat{H}_r = \frac{2}{r_0} + O(r^{-1-\tau}),
\end{equation}

\begin{equation}
|H_{0r} - \frac{2}{r_0}| \leq Cr_0^{-1-\tau},
\end{equation}

and

\begin{equation}
K_r = \frac{1}{r_0^2} + O(r^{-2-\tau}), \quad |\nabla K| = O(r^{-3-\tau}).
\end{equation}

Here $C$ is a constant independent on $r$, $K_r$ is the Gaussian curvature of $\gamma_r$, where $\gamma_r$ is the induced metric on $\Sigma_r$ from $g$, and $H_{0r}$ is the mean curvature of $(\Sigma_r, \gamma_r)$ when it is isometrically embedded in $\mathbb{R}^3$.

Let $\sigma_r = r_0^{-2}\gamma_r$. It follows from (4.4), (4.7) and the proof of Theorem 3 in $[9]$ (in particular (3.1) in $[9]$) that, for each large $r$, there is a conformal map $\Phi_r$ from $(S^2, \sigma_0)$ to $(\Sigma_r, \sigma_r)$ such that

\begin{equation}
||\Phi_r^*(\sigma_r) - \sigma_0||_{C^{2,\alpha}(S^2)} = O(r^{-\tau}).
\end{equation}

Here $\sigma_0$ is the standard metric on $S^2$.

Let $H(r)$ be the mean curvature of $\Sigma_r$ in $(M, r_0^{-2}g)$ and $H_0(r)$ be the mean curvature of $(\Sigma_r, \sigma_r)$ when it is isometrically embedded in $\mathbb{R}^3$. It follows from (4.3) – (4.6) that

\begin{equation}
H_0(r) - H(r) = O(r^{-\tau}).
\end{equation}

On the other hand, by Theorem 5 in $[9]$, the Brown-York mass of $\Sigma_r$ in $(M, g)$ satisfies

\begin{equation}
\lim_{r \to \infty} \int_{\Sigma_r} (H_{0r} - H_r)dv_{\gamma_r} = 8\pi m(g)
\end{equation}

where $m(g) > 0$ is the ADM mass of $(M, g)$. This together with (4.4) implies $\exists R_0 > 0$ such that

\begin{equation}
\int_{\Sigma_r} (H_0(r) - H(r))dv_{\gamma_r} > C^{-1}r^{-1}\pi m(g), \quad \forall r > R_0.
\end{equation}
Now choose $\gamma = \sigma_r$ and $H = H(r)$ in Proposition 3.1. By (4.8) and Lemma 2.3 the constant $\beta$ in (3.1) and the lower bound $\lambda$ for the first nonzero eigenvalue can both be chosen to be independent of $r$. Moreover, the conditions (a) and (b) are satisfied for large $r$ by (4.9), (4.11) and the fact $\tau > \frac{1}{2}$ and $m(g) > 0$. Therefore, by Proposition 3.1 we conclude that $\exists \ R > 0$ and $\tilde{\beta} > 0$ such that

\begin{equation}
F_{\sigma_r,H(r)}(\eta) \geq \tilde{\beta} \int_{\Sigma_r} (\Delta \eta)^2 dv_{\sigma_r}, \ \forall \ \eta \in W^{2,2}(\Sigma_r), \ \forall \ r > R.
\end{equation}

Theorem 4.1 now follows from (4.4), (4.12), and Theorem 1.3. □

5. SMALL GEODESIC SPHERES

Let $(M, g)$ be an arbitrary Riemannian 3-manifold of nonnegative scalar curvature. Let $p \in M$ be any given point. For small $r > 0$, let $S_r$ be the geodesic sphere of radius $r$ centered at $p$. Let $\gamma_r$ be the induced metric on $S_r$ and $H(r)$ be the mean curvature of $S_r$ in $(M, g)$. Let $H_0(r)$ be the mean curvature of $(S_r, \gamma_r)$ when it is isometrically embedded in $\mathbb{R}^3$. By [4, Theorem 3.1], the Brown-York mass of $S_r$ in $(M, g)$ satisfies

\begin{equation}
\frac{1}{8\pi} \int_{S_r} (H_0(r) - H(r)) dv_{\gamma_r},
\end{equation}

\begin{equation}
= \frac{r^3}{12} R(p) + \frac{r^5}{1440} \left[ 24|\text{Ric}(p)|^2 - 13R(p)^2 + 12\Delta R(p) \right] + O(r^6)
\end{equation}

where $R$ and $\text{Ric}$ denote the scalar curvature and the Ricci curvature of $(M, g)$ respectively, and $\Delta$ is the Laplacian on $(M, g)$. It follows from (5.1) that the condition

\begin{equation}
\lim_{r \to 0} r^{-5} \int_{S_r} (H_0(r) - H(r)) dv_{\gamma_r} > 0
\end{equation}

is equivalent to the union of the following three conditions

(i) $R(p) > 0$

(ii) $R(p) = 0$ and $|\text{Ric}(p)|^2 > 0$

(iii) $R(p) = |\text{Ric}(p)| = 0$ and $\Delta R(p) > 0$.

Note that if $R(p) = 0$, then $\Delta R(p) \geq 0$ by the assumption $R \geq 0$.

Theorem 5.1. Under the above notations, if the condition (5.2) holds, then (1.5) is true on $S_r$ for small $r$. Precisely, we have

(a) If (i) or (iii) holds, then $\exists$ constants $r_0 > 0$ and $C > 0$ such that, $\forall \ \eta \in W^{2,2}(S_r)$ and $\forall \ \ r < r_0$,

\begin{equation}
\int_{S_r} \left[ \frac{(\Delta \eta)^2}{H} + (H_0 - H)|\nabla \eta|^2 - I_0(\nabla \eta, \nabla \eta) \right] dv_r > Cr \int_{S_r} (\Delta \eta)^2 dv_r.
\end{equation}
(b) If (ii) holds, then there exist constants \( r_0 > 0 \) and \( C > 0 \) such that, \( \forall \eta \in W^{2,2}(S_r) \) and \( \forall r < r_0 \),

\[
\int_{S_r} \left[ \frac{(\Delta \eta)^2}{H} + (H_0 - H)|\nabla \eta|^2 - \Pi_0(\nabla \eta, \nabla \eta) \right] dv_r > C r^5 \int_{S_r} (\Delta \eta)^2 dv_r.
\]

Proof of Theorem 5.1(a). Let \( \{z_i\} \) be a geodesic normal coordinate chart centered at \( p \). For small \( r \), \( S_r = \{|z| = r\} \). The metric \( g \) satisfies (5.3)

\[
g_{ij} = \delta_{ij} + h_{ij}
\]

where \( h_{ij} \) is a smooth function near 0 with \( h_{ij}(0) = 0 \) and \( \partial h_{ij}(0) = 0 \). For each fixed \( r > 0 \), define a new coordinate chart \( \{x_i\} \) near \( p \) by

\[
x_i = r^{-1} z_i.
\]

Let \( S^2 \) be the unit coordinate sphere in the \( x \)-space. We identify \( S_r \) with \( S^2 \) through the map \( z \mapsto \). Let \( \sigma_r = r^{-2} \gamma_r \) be the induced metric on \( S^2 \) from \((M, r^{-2}g)\) and \( H(r) \) be the mean curvature of \( S^2 \) in \((M, r^{-2}g)\). Let \( H_0(r) \) be the mean curvature of the isometric embedding of \((S^2, \sigma_r)\) in \( \mathbb{R}^3 \).

By (5.3) and the results in [4](Lemma 3.4 and Theorem 3.1), we have

\[
||\sigma_r - \sigma_0||_{C^3(S^2, \sigma_0)} = O(r^2),
\]

(5.5)

\[
H(r) = 2 - \frac{r^2}{3} R_{ij}(p)x^i x^j + O(r^3),
\]

(5.6)

\[
H_0(r) = 2 + r^2 \left( \frac{1}{2} R(p) - \frac{4}{3} R_{ij}(p)x^i x^j \right) + O(r^3),
\]

(5.7)

\[
\frac{1}{8\pi} \int_{S_r} (H_0(r) - H(r)) dv_{\sigma_r} = \frac{r^2}{12} R(p) + \frac{r^4}{1440} \left[ 24|\text{Ric}(p)|^2 - 13R(p)^2 + 12\Delta R(p) \right] + O(r^5).
\]

Here \( \sigma_0 \) is the standard metric on \( S^2 \) and \( R_{ij}(p) = \text{Ric}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})(p) \).

If \( R(p) > 0 \), it follows from (5.6) - (5.7) that

\[
|H_0(r) - H(r)| = O(r^2)
\]

and

\[
\int_{S^2} (H_0(r) - H(r)) dv_{\sigma_r} > \frac{r^2}{3} \pi R(p)
\]

for small \( r \). Take \( \gamma = \sigma_r \) and \( H = H(r) \) in Proposition 3.1. By (5.4) and Lemma 2.3, the constant \( \beta \) in (3.1) and the lower bound \( \lambda \) for the first nonzero eigenvalue can both be chosen to be independent on \( r \).

Moreover, the conditions (a) and (b) are satisfied for small \( r \) by (5.8)
and \((5.9)\). Therefore, Proposition \(3.1\) implies there exist \(r_0 > 0\) and \(\tilde{\beta} > 0\) such that if \(r < r_0\), then
\[
F_{\sigma, H(r)}(\eta) \geq \tilde{\beta} \int_{S^2} (\Delta \eta)^2 \, dv_{\sigma_r}, \quad \forall \, \eta \in W^{2,2}(S^2).
\]

If \(R(p) = |\text{Ric}(p)| = 0\) and \(\Delta R(p) > 0\), it follows from \((5.5) - (5.7)\) that
\[
|H_0(r) - H(r)| = O(r^3)
\]
and
\[
\int_{S^2} (H_0(r) - H(r)) \, dv_{\sigma_r} \geq \frac{\Delta R(p)}{240} r^4
\]
for small \(r\). Again, Lemma \(2.3\) and Proposition \(3.1\) can be applied to show that \((5.10)\) holds for some \(\tilde{\beta}\) and \(r_0\).

Theorem \(5.1(a)\) now follows \((5.10)\). \(\square\)

The case \(R(p) = 0\) and \(|\text{Ric}(p)| > 0\) is more subtle because in this case \(|H_0(r) - H(r)| = O(r^2)\) while \(\int_{S^2} (H_0(r) - H(r)) \, dv_{\sigma_r} = O(r^4)\). The sufficient conditions in Section \(3\) do not apply in this situation.

To prepare for the proof of Theorem \(5.1(b)\), we choose \(\{z_i\}\) to be a geodesic normal coordinate chart centered at \(p\) such that the Ricci curvature of \(g\) is diagonalized by \(\{\frac{\partial}{\partial z_i}\}\) at \(p\), i.e.
\[
\text{Ric} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)(p) = \delta_{ij} \lambda_i
\]
where \(\{\lambda_i\}\) are the eigenvalues of \(\text{Ric}(p)\). Then \(\{\lambda_i\}\) satisfy
\[
\lambda_1 + \lambda_2 + \lambda_3 = R(p) = 0
\]
and
\[
\sum_{i=1}^{3} \lambda_i^2 = |\text{Ric}(p)|^2 > 0.
\]
Let \(\{x_i\}, \sigma_r, \sigma_0, H(r), H_0(r)\) be defined as in the proof of Theorem \(5.1(a)\). For convenience, we record \((5.4) - (5.7)\) in the current setting:
\[
||\sigma_r - \sigma_0||_{C^3(S^2)} = O(r^2)
\]
\[
H(r) = 2 - \frac{r^2}{3} \sum_{i=1}^{3} \lambda_i x_i^2 + O(r^3)
\]
\[
H_0(r) = 2 - \frac{4r^2}{3} \sum_{i=1}^{3} \lambda_i x_i^2 + O(r^3)
\]
and
\[
(5.18) \quad \int_{S^2} (H_0(r) - H(r)) dv_{\sigma_r} = \frac{\pi r^4}{15} \left[ 2 \sum_{i=1}^{3} \lambda_i^2 + \Delta R(p) \right] + O(r^5).
\]

Given any constant \( b \), define
\[
(5.19) \quad H_{(b)}(r) = H(r) + br^4 \sum_{i=1}^{3} \lambda_i^2.
\]

It follows from (5.14), (5.15), (5.18) and (5.19) that
\[
(5.20) \quad \int_{S^2} (H_0(r) - H_{(b)}(r)) dv_{\sigma_r} = 4\pi r^4 \left[ \frac{1}{30} - \left( b - \frac{1}{60} \frac{\Delta R(p)}{|\text{Ric}(p)|^2} \right) \right] \sum_{i=1}^{3} \lambda_i^2 + O(r^5).
\]

We are in a position to state the main result in the remaining part of this paper — a classification theorem on the positivity of \( F_{\sigma_r, H_{(b)}(r)} \), from which Theorem 5.1(b) will follow as a corollary.

**Theorem 5.2.** Under the above notations,

(i) if the constant
\[
\left( b - \frac{1}{60} \frac{\Delta R(p)}{|\text{Ric}(p)|^2} \right) < \frac{1}{90},
\]
then there exist constants \( r_0 > 0 \) and \( C > 0 \) such that for any \( 0 < r < r_0 \),
\[
F_{\sigma_r, H_{(b)}(r)}(\eta) \geq Cr^4 \int_{S^2} (\Delta_r \eta)^2 dv_{\sigma_r}, \quad \forall \eta \in W^{2,2}(S^2).
\]
Here \( \Delta_r \) denotes the Laplacian of the metric \( \sigma_r \).

(ii) if the constant
\[
\left( b - \frac{1}{60} \frac{\Delta R(p)}{|\text{Ric}(p)|^2} \right) > \frac{1}{90},
\]
then there exists a constant \( r_1 > 0 \) such that for any \( 0 < r < r_1 \), there exists a function \( \eta_r \in W^{2,2}(S^2) \) such that
\[
F_{\sigma_r, H_{(b)}(r)}(\eta_r) < 0.
\]

**Proof of Theorem 5.2(b).** Let \( b = 0 \), the result follows from Theorem 5.2(i). \( \square \)

The main ingredient in the proof of Theorem 5.2 is the following result on \((S^2, \sigma_0)\).
**Proposition 5.1.** Let $\sigma_0$ be the standard metric on $S^2 = \{ |x| = 1 \} \subset \mathbb{R}^3$. Given any three constants $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\sum_{i=1}^{3} \lambda_i = 0 \text{ and } \sum_{i=1}^{3} \lambda_i^2 > 0,$$

define $\phi = \sum_{i=1}^{3} \lambda_i x_i^2$. Consider the functional

$$G(\eta_1, \eta_2) = 4\pi \left( \frac{1}{30} - b \right) \sum_{i=1}^{3} \lambda_i^2 + \frac{1}{2} \int_{S^2} \eta_1^2 \phi^2 dv_{\sigma_0}$$

$$- 2 \int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2)}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] dv_{\sigma_0}$$

$$+ \int_{S^2} \left( \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right) dv_{\sigma_0}$$

where $b$ is a constant, $\Delta_0$ and $\nabla_0$ are the Laplacian and the gradient on $(S^2, \sigma_0)$, and $\eta_1, \eta_2 \in W^{2,2}(S^2)$ satisfy

- $\eta_1 = \sum_{i=1}^{3} a_i x_i$ for some vector $a = (a_1, a_2, a_3)$ with $|a| = 1$
- $\eta_2$ is $\sigma_0$-$L^2$ orthogonal to $L(\sigma_0)$, the space spanned by $\{1, x_1, x_2, x_3\}$.

Then

(i) if $b < \frac{1}{90}$, $G(\eta_1, \eta_2) > 0$ for any $\eta_1$ and $\eta_2$.

(ii) if $b > \frac{1}{90}$, given any $\eta_1$, $\exists$ an $\eta_2$ such that $G(\eta_1, \eta_2) < 0$.

**Proof.** Using the assumption $\sum_{i=1}^{3} \lambda_i = 0$ and $|a| = 1$, it is computed in Lemma [7.2] in the Appendix that

$$\int_{S^2} \eta_1^2 \phi^2 dv_{\sigma_0} = \int_{S^2} \left( \sum_{i=1}^{3} a_i x_i \right)^2 \left( \sum_{i=1}^{3} \lambda_i x_i^2 \right)^2 dv_{\sigma_0}$$

$$= 16\pi \left[ \frac{2 \sum_i a_i^2 \lambda_i^2}{3 \times 35} + \frac{\sum_i \lambda_i^2}{2 \times 3 \times 35} \right].$$

For simplicity, we write

$$A = 16\pi \left[ \frac{2 \sum_i a_i^2 \lambda_i^2}{3 \times 35} + \frac{\sum_i \lambda_i^2}{2 \times 3 \times 35} \right].$$

Next we define

$$B = \int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2)}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] dv_{\sigma_0}$$

and claim

$$B = 10 \int_{S^2} \phi \eta_1 \eta_2 dv_{\sigma_0}.$$
To see this, we note the following facts about $\phi$ and $\eta_1$:

\begin{equation}
\Delta_0 \phi = -6\phi
\end{equation}

where we used $\sum_{i=1}^{3} \lambda_i = 0$, and

\[
\langle \nabla_0 \phi, \nabla_0 \eta_1 \rangle = \langle \nabla \phi, \nabla \eta_1 \rangle - \langle \nabla \phi, X \rangle \langle \nabla \eta_1, X \rangle = 2 \sum_{i=1}^{3} \lambda_i a_i x_i - 2\phi \eta_1
\]

where $\nabla$ denotes the gradient on $\mathbb{R}^3$ and $X = (x_1, x_2, x_3)$. Now

\[
B = \int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2)}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] dv_{\sigma_0}
\]

\[
= \int_{S^2} \left[ -\frac{1}{2} \phi \eta_1 \Delta_0 \eta_2 + \langle \nabla_0 \eta_1, \nabla_0 (\phi \eta_2) \rangle - \eta_2 \langle \nabla_0 \eta_1, \nabla_0 \phi \rangle \right] dv_{\sigma_0}
\]

\[
= \int_{S^2} \left[ -\frac{1}{2} \Delta_0 (\phi \eta_1) \eta_2 - \phi \eta_2 \Delta_0 \eta_1 - \eta_2 \langle \nabla_0 \eta_1, \nabla_0 \phi \rangle \right] dv_{\sigma_0}
\]

\[
= \int_{S^2} [4\phi \eta_1 \eta_2 + 2\phi \eta_2 \eta_1 - 2\eta_2 \langle \nabla_0 \eta_1, \nabla_0 \phi \rangle] dv_{\sigma_0}
\]

\[
= 10 \int_{S^2} \phi \eta_1 \eta_2 dv_{\sigma_0}.
\]

To proceed, we let $\tau_2$ be the $L^2$ orthogonal projection of $\eta_2$ to the eigenspace of the second nonzero eigenvalue of $(S^2, \sigma_0)$ and let $\tau_3 = \eta_2 - \tau_2$. Then

\begin{equation}
\int_{S^2} \left( \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right) dv_{\sigma_0}
= \int_{S^2} \left( \frac{(\Delta_0 \tau_2)^2}{2} - |\nabla_0 \tau_2|^2 \right) dv_{\sigma_0} + \int_{S^2} \left( \frac{(\Delta_0 \tau_3)^2}{2} - |\nabla_0 \tau_3|^2 \right) dv_{\sigma_0}
\geq \frac{1}{3} \int_{S^2} (\Delta_0 \tau_2)^2 dv_{\sigma_0} + \frac{5}{12} \int_{S^2} (\Delta_0 \tau_3)^2 dv_{\sigma_0}
\end{equation}

where we have used the assumption that $\eta_2$ is $L^2$ orthogonal to $L(\sigma_0)$, Lemma 2.4, and the fact that the second and third nonzero eigenvalues of $(S^2, \sigma_0)$ are 6 and 12 respectively.

Note that $\tau_2$ is the restriction to $S^2$ of a homogeneous polynomial of degree two, hence $\phi \eta_1 \tau_2$ is the restriction to $S^2$ of a homogeneous
polynomial of degree five which implies $\int_{S^2} \phi \eta_1 \tau_2 dv_{\sigma_0} = 0$. Therefore

(5.25) \quad B = 10 \int_{S^2} \phi \eta_1 \tau_3 dv_{\sigma_0}.

Now it follows from (5.21), (5.24) and (5.25) that

(5.26) \quad G(\eta_1, \eta_2) \geq \left[ 4\pi \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 + \frac{A}{2} \right] - 2 \times 10 \int_{S^2} \phi \eta_1 \tau_3 dv_{\sigma_0}

\quad + \frac{5}{12} \int_{S^2} (\Delta_0 \tau_3)^2 dv_{\sigma_0} + \frac{1}{3} \int_{S^2} (\Delta_0 \tau_2)^2 dv_{\sigma_0}.

Next, we make use of the fact that $\tau_3$ is $L^2$ orthogonal to $E_1$, the subspace spanned by $\{x_1, x_2, x_3\}$. Therefore

\int_{S^2} \phi \eta_1 \tau_3 dv_{\sigma_0} = \int_{S^2} (\phi \eta_1 - \xi) \tau_3 dv_{\sigma_0}

where $\xi$ is the $L^2$ orthogonal projection of $\phi \eta_1$ to $E_1$. This implies

$$|B| = 10 \left| \int_{S^2} \phi \eta_1 \tau_3 dv_{\sigma_0} \right| \leq \frac{5}{6} \left( A - \int_{S^2} \xi^2 dv_{\sigma_0} \right)^{\frac{1}{2}} \left( \int_{S^2} (\Delta_0 \tau_3)^2 dv_{\sigma_0} \right)^{\frac{1}{2}}.$$  

To compute $\int_{S^2} \xi^2 dv_{\sigma_0}$, we have

$$\int_{S^2} \phi \eta_1 x_1 dv_{\sigma_0} = \int_{S^2} (\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2)(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_1 x_3) dv_{\sigma_0}$$

$$= \int_{S^2} a_1 x_1^2(\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2) dv_{\sigma_0}$$

$$= 4\pi a_1 \left( \frac{\lambda_1}{5} + \frac{\lambda_2}{15} + \frac{\lambda_3}{15} \right)$$

$$= \frac{8\pi a_1 \lambda_1}{15}.$$  

Similarly, for $i = 2, 3$, $\int_{S^2} x_i \eta_1 \phi dv_{\sigma_0} = \frac{8\pi b_i \lambda_i}{15}$. Hence, $\xi = \frac{2}{5} \sum_{i=1}^{3} a_i \lambda_i x_i$.

and

$$\int_{S^2} (\phi \eta_1 - \xi)^2 dv_{\sigma_0} = A - \frac{16\pi}{75} \sum_{i=1}^{3} a_i^2 \lambda_i^2.$$  

(In particular, this shows $\phi \eta_1 - \xi \neq 0$ by (5.21).) Therefore,

(5.27) \quad |B| \leq \frac{5}{6} \left( A - \frac{16\pi}{75} \sum_{i=1}^{3} a_i^2 \lambda_i^2 \right)^{\frac{1}{2}} \left( \int_{S^2} (\Delta_0 \tau_3)^2 dv_{\sigma_0} \right)^{\frac{1}{2}}.
By (5.26) and (5.27), we conclude that

\[ G(\eta_1, \eta_2) \geq (\alpha - 2\beta t + \gamma t^2) + \frac{1}{3} \int_{S^2} (\Delta_0 \tau_2)^2 d\nu_{\sigma_0} \]

where

\[ t = \left( \int_{S^2} (\Delta_0 \tau_3)^2 d\nu_{\sigma_0} \right)^{\frac{1}{2}}, \quad \gamma = \frac{5}{12}, \]

\[ \alpha = 4\pi \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 + \frac{A}{2}, \]

\[ \beta = \frac{5}{6} \left( A - \frac{16\pi}{15} \sum_{i=1}^{3} a_i^2 \lambda_i^2 \right)^{\frac{1}{2}}. \]

Direct calculation shows

\[ \beta^2 - \alpha \gamma = \frac{25}{36} \left( A - \frac{16\pi}{75} \sum_{i=1}^{3} a_i^2 \lambda_i^2 \right) - \frac{5}{12} \left[ \left( \frac{2\pi}{15} - 4\pi \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 + \frac{A}{2} \right] \]

\[ = \frac{35}{72} A - \frac{4\pi}{27} \sum_{i=1}^{3} a_i^2 \lambda_i^2 - \frac{\pi}{18} \sum_{i=1}^{3} \lambda_i^2 + \frac{5}{3} \pi \bar{b} \sum_{i=1}^{3} \lambda_i^2 \]

\[ = \frac{35}{72} \cdot 16\pi \left[ \frac{2 \sum_{i=1}^{3} a_i^2 \lambda_i^2}{3 \times 35} + \frac{\sum_{i=1}^{3} \lambda_i^2}{2 \times 3 \times 35} \right] - \frac{4\pi}{27} \sum_{i=1}^{3} a_i^2 \lambda_i^2 - \frac{\pi}{18} \sum_{i=1}^{3} \lambda_i^2 + \frac{5}{3} \pi \bar{b} \sum_{i=1}^{3} \lambda_i^2 \]

\[ = \frac{4\pi}{27} \sum_{i=1}^{3} a_i^2 \lambda_i^2 + \frac{\pi}{27} \sum_{i=1}^{3} \lambda_i^2 - \frac{4\pi}{27} \sum_{i=1}^{3} a_i^2 \lambda_i^2 - \frac{\pi}{18} \sum_{i=1}^{3} \lambda_i^2 + \frac{5}{3} \pi \bar{b} \sum_{i=1}^{3} \lambda_i^2 \]

\[ = - \left( \frac{1}{54} - \frac{5}{3} \bar{b} \right) \pi \sum_{i=1}^{3} \lambda_i^2. \]

Therefore, if \( \bar{b} < \frac{1}{90} \), by (5.28) and (5.29) we have

\[ G(\eta_1, \eta_2) \geq \frac{\alpha \gamma - \beta^2}{\gamma} = 4\pi \left( \frac{1}{90} - \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 > 0 \]

which proves (i).

To prove (ii), we claim that the function \( \phi \eta_1 - \xi \) above is indeed an eigenfunction of the third nonzero eigenvalue 12. To verify this, we
compute
\[
\Delta_0(\phi \eta_1) = (\Delta_0 \phi) \eta_1 + \phi (\Delta_0 \eta_1) + 2 \langle \nabla_0 \phi, \nabla \eta_1 \rangle
\]
\[
= (-6) \phi \eta_1 + (-2) \phi \eta_1 + 2 \left[ \sum_{i=1}^{3} a_i \lambda_i x_i - 2 \phi \eta_1 \right]
\]
\[
= (-12) \phi \eta_1 + 4 \sum_{i=1}^{3} a_i \lambda_i x_i.
\]

Therefore,
\[
\Delta_0(\phi \eta_1 - \xi) = (-12) \phi \eta_1 + 4 \sum_{i=1}^{3} a_i \lambda_i x_i + \frac{4}{5} \sum_{i=1}^{3} a_i \lambda_i x_i
\]
\[
= (-12) (\phi \eta_1 - \xi).
\]

Now we fix an \(a\) (hence \(\eta_1\) is fixed), and let \(\eta_2 = k (\phi \eta_1 - \xi)\) where \(\xi\) is the defined above and \(k\) is an arbitrary constant. Then
\[
G(\eta_1, \eta_2) = \left[ 4 \pi \left( \frac{1}{30} - b \right) \sum_i \lambda_i^2 + \frac{A}{2} \right] - 20 k \int_{S^2} (\phi \eta_1 - \xi)^2 d\nu_0
\]
\[
+ \frac{5}{12} k^2 \int_{S^2} [\Delta_0 (\phi \eta_1 - \xi)]^2 d\nu_0
\]
\[
= \alpha - 2 \beta t + \gamma t^2
\]
where \(\alpha, \beta\) and \(\gamma\) are defined as same as before and
\[
t = 12 \left( A - \frac{16 \pi}{75} \sum_i a_i^2 \lambda_i^2 \right)^{\frac{1}{2}} k.
\]
Suppose \(b > \frac{1}{90}\), it follows from (5.29) that the above quadratic form of \(t\) has two distinctive roots. In particular, if \(k\) is chosen such that
\[
12 \left( A - \frac{16 \pi}{75} \sum_i a_i^2 \lambda_i^2 \right)^{\frac{1}{2}} k = \frac{\beta}{\gamma},
\]
then
\[
G(\eta_1, k(\phi \eta_1 - \xi)) = \frac{\alpha \gamma - \beta^2}{\gamma} = 4 \pi \left( \frac{1}{90} - b \right) \sum_{i=1}^{3} \lambda_i^2 < 0.
\]
This completes the proof. \(\square\)

We are now ready to prove Theorem 5.2. We first prove part (i):
Proof. Suppose (i) of Theorem (5.2) is not true, then there exist two sequences of positive numbers \( \{ r_k \} \), \( \{ \epsilon_k \} \) and a sequence of functions \( \{ \eta^{(k)} \} \subset W^{2,2}(S^2) \) such that

\[
r_k \to 0, \quad \epsilon_k \to 0,
\]

\[
F_{\sigma_k, H_0(r_k)}(\eta^{(k)}) < \epsilon_k r_k^4 \int_{S^2} (\Delta_{r_k} \eta^{(k)})^2 d\nu_{\sigma_k}
\]

and

\[
\int_{S^2} \eta^{(k)} d\nu_{\sigma_k} = 0.
\]

In the following, we denote \( \Delta_{r_k}, \sigma_{r_k} \) by \( \Delta_k, \sigma_k \) respectively. We also let \( \nabla_k \) denote the gradient on \((S^2, \sigma_k)\).

On \( S^2 \), recall that \( \{ x_i \} \) are the restriction of the standard coordinate functions in \( \mathbb{R}^3 \). Hence \( X_0 = (x_1, x_2, x_3) \) is an isometric embedding of \((S^2, \sigma_0)\). By (5.15) and the result of Nirenberg (page 353 in [7]), for each large \( k \), there exists an isometric embedding

\[
X_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}): (S^2, \sigma_k) \to \mathbb{R}^3
\]

satisfying

\[
||x_i^{(k)} - x_i||_{C^{2,\alpha}(S^2, \sigma_0)} = O(r_k^2), \quad \forall \ i = 1, 2, 3.
\]

Here, given an integer \( m \), we use the notation \( O(r_k^m) \) to denote some quantity \( \psi \) satisfying \( |\psi| \leq Cr_k^m \) for a constant \( C \) independent on \( k \). Given such an \( X_k \), we let \( \nu^{(k)}_0 \) be the unit outward normal vector to \( X_k(S^2) \) and \( \Pi^{(k)}_0 \) be the second fundamental form of \( X_k(S^2) \) in \( \mathbb{R}^3 \). It follows from (5.33) that

\[
||\nu^{(k)}_0 - X_0||_{C^{0,\alpha}(S^2, \sigma_0)} = O(r_k^2)
\]

and

\[
||\Pi^{(k)}_0 - \sigma_0||_{C^{0,\alpha}(S^2, \sigma_0)} = O(r_k^2).
\]

As before, we let \( \mathcal{L}(\sigma_0) \) and \( \mathcal{L}(\sigma_k) \) be the subspaces of \( W^{2,2}(S^2) \) which are spanned by \( \{ 1, x_1, x_2, x_3 \} \) and \( \{ 1, x_1^{(k)}, x_2^{(k)}, x_3^{(k)} \} \) respectively. For each \( k \), we decompose \( \eta^{(k)} = \eta_1^{(k)} + \eta_2^{(k)} \), where

\[
\eta_1^{(k)} = a_0^{(k)} + \sum_{i=1}^3 a_i^{(k)} x_i^{(k)} \in \mathcal{L}(\sigma_k)
\]

and

\[
\eta_2^{(k)} = \sum_{i=1}^3 a_i^{(k)} x_i^{(k)} \in \mathcal{L}(\sigma_k)
\]
and $\eta_2^{(k)}$ is $\sigma_k$-$L^2$ orthogonal to $\mathcal{L}(\sigma_k)$. Let $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)})$. Then

$$F_{\sigma_k, H(b)(r_k)}(\eta^{(k)}) = F_{\sigma_k, H(b)(r_k)}(\eta_1^{(k)}) + 2Q_{\sigma_k, H(b)(r_k)}(\eta_1^{(k)}, \eta_2^{(k)}) + F_{\sigma_k, H(b)(r_k)}(\eta_2^{(k)})$$

where

$$F_{\sigma_k, H(b)(r_k)}(\eta_1^{(k)}) = |a^{(k)}|^2 \int_{S^2} (H_0(r_k) - H(b)(r_k)) \, dv_{\sigma_k}$$

$$+ \int_{S^2} \langle a^{(k)}, \nu_0^{(k)} \rangle^2 \frac{(H_0(r_k) - H(b)(r_k))^2}{H(b)(r_k)} \, dv_{\sigma_k},$$

$$Q_{\sigma_k, H(b)(r_k)}(\eta_1^{(k)}, \eta_2^{(k)}) = \int_{S^2} (H_0(r_k) - H(b)(r_k)) \left[ \frac{(\Delta_k \eta_1^{(k)})(\Delta_k \eta_2^{(k)})}{H_0(r_k)H(b)(r_k)} + \langle \nabla_k \eta_1^{(k)}, \nabla_k \eta_2^{(k)} \rangle_{\sigma_k} \right] \, dv_{\sigma_k},$$

and

$$F_{\sigma_k, H(b)(r_k)}(\eta_2^{(k)}) = \int_{S^2} \left[ \frac{\Delta_k \eta_2^{(k)}}{H_0(r_k)} - \Pi_0^{(k)}(\nabla_k \eta_2^{(k)}, \nabla_k \eta_2^{(k)}) \right] \, dv_{\sigma_k}$$

$$+ \int_{S^2} (H_0(r_k) - H(b)(r_k)) \left[ \frac{(\Delta_k \eta_2^{(k)})^2}{H_0(r_k)H(b)(r_k)} + |\nabla_k \eta_2^{(k)}|^2_{\sigma_k} \right] \, dv_{\sigma_k}.$$

By Lemma 2.3, Lemma 2.4, (5.15) – (5.17) and (5.19), we have

$$F_{\sigma_k, H(b)(r_k)}(\eta_2^{(k)}) \geq \beta + O(r_k^2) \int_{S^2} (\Delta_k \eta_2^{(k)})^2 \, dv_{\sigma_k},$$

$$Q_{\sigma_k, H(b)(r_k)}(\eta_1^{(k)}, \eta_2^{(k)}) \leq C_1 r_k^2 \left( \int_{S^2} (\Delta_k \eta_1^{(k)})^2 \, dv_{\sigma_k} \right)^{\frac{1}{2}} \left( \int_{S^2} (\Delta_k \eta_2^{(k)})^2 \, dv_{\sigma_k} \right)^{\frac{1}{2}}.$$

Here $C_1$ and $\beta$ are some positive constant independent on $k$.

We normalize $\eta^{(k)}$ such that

$$\int_{S^2} \left[ (\Delta_k \eta_1^{(k)})^2 + (\Delta_k \eta_2^{(k)})^2 \right] \, dv_{\sigma_k} = 1.$$

Recall

$$\frac{(\Delta_k \eta_1^{(k)})^2}{H_0(r_k)^2} + |\nabla_k \eta_1^{(k)}|^2_{\sigma_k} = |a^{(k)}|^2.$$
Thus \((5.42)\) and \((5.43)\), together with Lemma 2.4 \((5.15)\) and \((5.17)\), imply that there is a constant \(C_2\) independent on \(k\) such that

\[(5.44)\]
\[|a^{(k)}| \leq C_2.\]

It follows from \((5.15)\) - \((5.17)\), \((5.19)\), \((5.20)\), \((5.33)\), \((5.37)\) and \((5.44)\) that

\[(5.45)\]
\[|F_{\sigma_k,H_b(r_k)}(\eta_1^{(k)})| \leq C_3 r_k^4\]

for some positive constants \(C_3\) independent on \(k\). By \((5.31)\), \((5.36)\), \((5.41)\), \((5.42)\) and \((5.45)\), we then have

\[(5.48)\]
\[\int_{S^2} \left[ \beta + O(r_k^2) \right] (\Delta_k \eta_2^{(k)})^2 dv_{\sigma_k} < 2\epsilon_k r_k^4 + C_3 r_k^4 + 2C_1 r_k^2\]

which shows

\[
\lim_{k \to \infty} \int_{S^2} (\Delta_k \eta_2^{(k)})^2 dv_{\sigma_k} = 0,
\]

and consequently

\[
\lim_{k \to \infty} \int_{S^2} (\Delta_k \eta_1^{(k)})^2 dv_{\sigma_k} = 1
\]

by \((5.42)\). Therefore, for large \(k\), by \((5.43)\) we have

\[(5.46)\]
\[|a^{(k)}| \geq C_4\]

for some positive constant \(C_4\) independent on \(k\).

Now we renormalize \(\eta^{(k)}\) such that \(|a^{(k)}| = 1\). By \((5.42)\) and \((5.46)\),

\[(5.47)\]
\[\int_{S^2} \left[ (\Delta_k \eta_1^{(k)})^2 + (\Delta_k \eta_2^{(k)})^2 \right] dv_{\sigma_k} \leq C_5\]

for some positive constant \(C_5\) independent of \(k\). Define

\[\xi_k = \frac{\eta_2^{(k)}}{r_k^2}.\]

It follows from \((5.15)\) - \((5.17)\), \((5.19)\), \((5.20)\), \((5.31)\), \((5.33)\) - \((5.34)\) and \((5.36)\) - \((5.37)\) that

\[(5.48)\]
\[\epsilon_k \int_{S^2} (\Delta_k \eta^{(k)})^2 dv_{\sigma_k} \geq 4\pi \left[ \left( \frac{1}{30} - \bar{\theta} \right) \sum_i \lambda_i^2 \right] + \frac{1}{2} A_k + O(r_k) + 2r_k^{-2} Q_{\sigma_k,H_b(r_k)}(\eta_1^{(k)}, \xi_k) + F_{\sigma_k,H_b(k)}(\xi_k)\]

where

\[\bar{\theta} = b - \frac{1}{60} \frac{(\Delta_g R)(p)}{|\text{Ric}(p)|^2}\]
On second variation of Wang-Yau quasi-local energy

\[ A_k = \int_{S^2} \left( \sum_{i=1}^{3} a_i^{(k)} x_i \right)^2 \left( \sum_{i=1}^{3} \lambda_i x_i^2 \right)^2 d\sigma_0 \]
\[ = 16\pi \left[ 2 \sum_{i=1}^{3} \frac{(a_i^{(k)})^2 \lambda_i^2}{3 \times 35} + \sum_{i} \frac{\lambda_i^2}{2 \times 3 \times 35} \right] \]

(see the definition of \( A \) in Proposition 5.1). Moreover, by (5.40) and (5.41), we have

\[ \left| r_k^{-2} Q_{\sigma_k, H_{(b)}(r_k)}(\eta_1^{(k)}, \xi_k) \right| \]
\[ \leq C_1 \left( \int_{S^2} (\Delta_k \eta_1^{(k)})^2 d\sigma_0 \right)^{\frac{1}{2}} \left( \int_{S^2} (\Delta_k \xi_k)^2 d\sigma_0 \right)^{\frac{1}{2}} \]

and

\[ F_{\sigma_k, H_{(b)}(r_k)}(\xi_k) \geq \left[ \beta + O(r_k^2) \right] \int_{S^2} (\Delta_k \xi_k)^2 d\sigma_0. \]

It follows from (5.47) - (5.50) that there exists a positive constant \( C_6 \) independent on \( k \) such that

\[ \int_{S^2} (\Delta_k \xi_k)^2 d\sigma_0 \leq C_6. \]

On the other hand, we still have \( \xi_k = r_k^{-2} \eta_2^{(k)} \in \mathcal{L}(\sigma_k) \). Hence,

\[ \int_{S^2} \xi_k d\sigma_k = 0 \]

and

\[ \int_{S^2} \xi_k x_i^{(k)} d\sigma_k = 0, \ \forall \ i = 1, 2, 3. \]

By Lemma 2.4, (5.15), (5.51), (5.52) and the \( L^2 \)-estimates, we know

\[ \| \xi_k \|_{W^{2,2}(S^2, \sigma_k)} \leq C_7 \]

for some positive constant \( C_7 \) independent on \( k \). This combined with (5.15) in turn shows

\[ \| \xi_k \|_{W^{2,2}(S^2, \sigma_0)} \leq C_8 \]

for some positive constant \( C_8 \) independent on \( k \). Therefore, there exists some \( \xi \in W^{2,2}(S^2) \) such that, passing to a subsequence, \( \{ \xi_k \} \) converges to \( \xi \) weakly in \( W^{2,2}(S^2, \sigma_0) \) and strongly in \( W^{1,2}(S^2, \sigma_0) \). Furthermore, it follows from (5.15), (5.33), (5.52) and (5.53) that \( \xi \in \mathcal{L}(\sigma_0) \).
We will take limit in (5.48). By (5.15) - (5.17), (5.19), (5.33), (5.35), (5.54) and (5.55), we have

\begin{equation}
F_{\sigma_k, H(b)(r_k)}(\xi_k)
= \int_{S^2} \left[ \frac{(\Delta_k \xi_k)^2}{H(b)(r_k)} + (H_0(k) - H(b)(r_k))|\nabla_k \xi_k|^2 - \Pi_{0}^{(k)}(\xi_k, \nabla_k \xi_k) \right] dv_{\sigma_0} + O(r_k^2)
\end{equation}

\begin{equation}
= \int_{S^2} \left[ \frac{(\Delta_0 \xi_k)^2}{2} - |\nabla_0 \xi_k|^2 \right] dv_{\sigma_0} + O(r_k^2).
\end{equation}

Since \{\xi_k\} converges to \xi strongly in \(W^{1,2}(S^2, \sigma_0)\) and \{\Delta_0 \xi_k\} converges to \(\Delta_0 \xi\) weakly in \(L^2(S^2, \sigma_0)\), (5.56) implies

\begin{equation}
\liminf_{k \to \infty} F_{\sigma_k, H(b)(r_k)}(\xi_k) \geq \int_{S^2} \left[ \frac{(\Delta_0 \xi)^2}{2} - |\nabla_0 \xi|^2 \right] dv_{\sigma_0}.
\end{equation}

To take the limit of \(r_k^{-2}Q_{\sigma_k, H(b)(r_k)}(\eta_{1(k)}, \xi_k)\), we can assume that \{\eta_{1(k)}\} converges to some \(a = (a_1, a_2, a_3) \in S^2\) because \(|a^{(k)}| = 1\). By (5.16) (5.17) and (5.19), we have \(H_0(r_k) - H(b)(r_k) = -r_k^2 \phi + O(r_k^3)\), where \(\phi = \sum_{i=1}^{3} \lambda_i x_i^2\). Similar to (5.50), we now have

\begin{equation}
r_k^{-2}Q_{\sigma_k, H(b)(r_k)}(\eta_{1(k)}^{(k)}, \xi_k)
= -\int_{S^2} \phi \left[ \frac{(\Delta_k \eta_{1(k)}^{(k)}) (\Delta_k \xi_k)}{4} + \langle \nabla_k \eta_{1(k)}^{(k)}, \nabla_k \xi_k \rangle_{\sigma_k} \right] dv_{\sigma_0} + O(r_k^2)
\end{equation}

\begin{equation}
= -\int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_{1(k)}^{(k)}) (\Delta_0 \xi_k)}{4} + \langle \nabla_0 \eta_{1(k)}^{(k)}, \nabla_0 \xi_k \rangle \right] dv_{\sigma_0} + O(r_k^2)
\end{equation}

\begin{equation}
= -\int_{S^2} \phi \left[ \frac{(\Delta_0 (\sum_{i=1}^{3} a_i^{(k)} x_i)) (\Delta_0 \xi_k)}{4} + \langle \nabla_0 \left( \sum_{i=1}^{3} a_i^{(k)} x_i \right), \nabla_0 \xi_k \rangle \right] dv_{\sigma_0} + O(r_k^2)
\end{equation}

\begin{equation}
= -\int_{S^2} \phi \left[ -\frac{\langle \nabla_0 \left( \phi \Delta_0 \left( \sum_{i=1}^{3} a_i^{(k)} x_i \right) \right), \nabla_0 \xi_k \rangle}{4} + \phi \langle \nabla_0 \left( \sum_{i=1}^{3} a_i^{(k)} x_i \right), \nabla_0 \xi_k \rangle \right] dv_{\sigma_0} + O(r_k^2)
\end{equation}

\begin{equation}
\to -\int_{S^2} \phi \left[ \frac{(\Delta_0 (\sum_{i=1}^{3} a_i x_i)) (\Delta_0 \xi_k)}{4} + \langle \nabla_0 \left( \sum_{i=1}^{3} a_i x_i \right), \nabla_0 \xi_k \rangle \right] dv_{\sigma_0}, \text{ as } k \to \infty
\end{equation}

since \{a^{(k)}\} converges to \(a\) and \{\xi_k\} converges to \(\xi\) strongly in \(W^{1,2}(S^2, \sigma_0)\).
Combining (5.47) - (5.48) and (5.57) - (5.58), we conclude that
(5.59)
$$0 \geq 4\pi \left[ \left( \frac{1}{30} - \bar{b} \right) \sum \lambda_i^2 + \frac{1}{2} \int_{S^2} \left( \sum a_i x_i \right)^2 \phi^2 dv_{\sigma_0} \right. \right.$$\n$$\left. - 2 \int_{S^2} \phi \left[ \frac{(\Delta_0 \left( \sum a_i x_i \right))(\Delta_0 \xi_k)}{4} + \langle \nabla_0 \left( \sum a_i x_i \right), \nabla_0 \xi \rangle \right] dv_{\sigma_0} \right. \right.$$\n$$\left. + \int_{S^2} \left[ \frac{(\Delta_0 \xi)^2}{2} - |\nabla_0 \xi|^2 \right] dv_{\sigma_0}. \right.$$\n
Since
$$|a| = 1, \quad \xi \in \mathcal{L}(\sigma_0), \quad \text{and} \quad \bar{b} < \frac{1}{90},$$
this leads to a contradiction with (i) of Proposition 5.1. Therefore, (i) of Theorem 5.2 is proved. □

Next, we prove (ii) of Theorem 5.2.

Proof. Let \( \bar{b} = b - \frac{1}{60} \frac{(\Delta g R)(p)}{|\text{Ric}(p)|^2} \). Then \( \bar{b} > \frac{1}{90} \). By (ii) of Proposition 5.1, given any \( a = (a_1, a_2, a_3) \) with \( |a| = 1 \), there exists an \( \eta_2 \in W^{2,2}(S^2) \) such that \( \eta_2 \) is \( \sigma_0 - L^2 \) orthogonal to \( \mathcal{L}(\sigma_0) \) and
(5.60)
$$G(\eta_1, \eta_2) = 4\pi \left( \frac{1}{30} - \bar{b} \right) \sum \lambda_i^2 + \frac{1}{2} \int_{S^2} \eta_1^2 \phi^2 dv_{\sigma_0}$$\n$$- 2 \int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2)}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] dv_{\sigma_0}$$\n$$+ \int_{S^2} \left[ \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right] dv_{\sigma_0} < 0.$$\n
Here \( \eta_1 = \sum_{i=1}^3 a_i x_i \) and \( \phi = \sum_{i=1}^3 \lambda_i x_i^2 \).

Let \( X_0 = (x_1, x_2, x_3) \). For each small \( r \), let
$$X_r = (x_1^{(r)}, x_2^{(r)}, x_3^{(r)}) : S^2 \rightarrow \mathbb{R}^3$$
be an isometric embedding of \((S^2, \sigma_r)\) satisfying
(5.61) \( ||x_i^{(r)} - x_i||_{C^{2,\alpha}(\mathbb{S}^2, \sigma_0)} = O(r^2), \ \forall \ i = 1, 2, 3. \)

Let \( \nu_0^{(r)} \) be the unit outward normal vector to \( X_r(S^2) \) and \( \Pi_0^{(r)} \) be the second fundamental form of \( X_r(S^2) \) in \( \mathbb{R}^3 \). It follows from (5.61) that
(5.62) \( ||\nu_0^{(r)} - X_0||_{C^{0,\alpha}(\mathbb{S}^2, \sigma_0)} = O(r^2) \)
and

\begin{equation}
\|\mathcal{P}_0^{(r)} - \sigma_0\|_{C^{0,\alpha}(S^2, \sigma_0)} = O(r^2).
\end{equation}

With \(a\) and \(\eta_2\) fixed, we define

\[ \eta_1^{(r)} = \sum_{i=1}^{3} a_i x_i^{(r)} \quad \text{and} \quad \eta^{(r)} = \eta_1^{(r)} + r^2 \eta_2 \]

for each small \(r\). Then

\begin{equation}
F_{\sigma, H(b)(r)}(\eta^{(r)}) = F_{\sigma, H(b)(r)}(\eta_1^{(r)}) + 2r^2 Q_{\sigma, H(b)(r)}(\eta_1^{(r)}, \eta_2) + r^4 F_{\sigma, H(b)(r)}(\eta_2).
\end{equation}

We compare each term in (5.64) with the corresponding term in (5.60). First,

\begin{equation}
F_{\sigma, H(b)(r)}(\eta_1^{(r)}) = \int_{S^2} (H_0(r) - H(b)(r)) dv_{\sigma_r} + \int_{S^2} \langle a, \nu_0^{(r)} \rangle^2 \frac{(H_0(r) - H(b)(r))^2}{H(b)(r)} dv_{\sigma_r}
\end{equation}

\[ = 4\pi r^4 \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 + O(r^5) + \frac{1}{2} r^4 \int_{S^2} \eta_1^2 \phi^2 dv_{\sigma_0} + O(r^5) \]

where we have used (5.15) - (5.17), (5.19) - (5.20) and (5.62).

Next, let \(\Delta_r\) and \(\nabla_r\) be the Laplacian and the gradient on \((S^2, \sigma_r)\) respectively. Then

\begin{equation}
Q_{\sigma, H(b)(r)}(\eta_1^{(r)}, \eta_2) = \int_{S^2} (H_0(r) - H(b)(r)) \left[ \frac{(\Delta_r \eta_1^{(r)})(\Delta_r \eta_2)}{H_0(r) H(b)(r)} + \langle \nabla_r \eta_1^{(r)}, \nabla_r \eta_2 \rangle_{\sigma_r} \right] dv_{\sigma_r}
\end{equation}

\[ = \int_{S^2} (-\phi r^2 + O(r^3)) \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2) + O(r^2)}{4 + O(r^2)} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle + O(r^2) \right] (1 + O(r^2)) dv_{\sigma_0}
\]

\[ = - r^2 \int_{S^2} \phi \left[ \frac{\Delta_0 \eta_1 \Delta_0 \eta_2}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] dv_{\sigma_0} + O(r^3) \]

where we have used (5.15) - (5.17), (5.19) and (5.61).
Finally,
\[
F_{\sigma,H(b)}(\eta_0) = \int_{S^2} \left[ \frac{(\Delta \eta_0)^2}{H_0(r)} - \mathbb{R}_{\sigma}(\nabla_r \eta_0, \nabla_r \eta_0) \right] \, dv_{\sigma} \\
+ \int_{S^2} \left[ (H_0(r) - H(b)(r)) \left( \frac{(\Delta \eta_0)^2}{H_0(r)H(b)(r)} + |\nabla_r \eta_0|^2 \right) \right] \, dv_{\sigma},
\]
(5.67) where we have used (5.15) - (5.17), (5.19) and (5.63).

It follows from (5.64) - (5.67) that
\[
F_{\sigma,H(b)}(\eta_i(r)) = r^4 G(\eta_1, \eta_2) + O(r^5).
\]
Since \(G(\eta_1, \eta_2) < 0\), we conclude that there exists small \(r_1 > 0\) such that \(F_{\sigma,H(b)}(\eta_i(r)) < 0\) for any \(0 < r < r_1\). This completes the proof of (ii) of Theorem 5.2. \(\square\)

6. Examples

We end this paper by giving examples of positive functions \(H\) on \((S^2, \sigma_0)\) such that
(a) \(\int_{S^2} (2 - H) \, dv_{\sigma_0} > 0\)
(b) \(F_{\sigma_0,H}(\eta) < 0\) for some \(\eta_0\).
(c) \(||H - 2||_{C^k(S^2, \sigma_0)} < \epsilon\) for any given \(\epsilon > 0\) and \(k \geq 2\).

Such a function \(H\) can be taken as one of \(H_b(r)\) in the following.

**Theorem 6.1.** Let \(\sigma_0\) be the standard metric on \(S^2\). Let \(\lambda_1, \lambda_2, \lambda_3\) be three constants satisfying \(\sum_{i=1}^{3} \lambda_i = 0\) and \(\sum_{i=1}^{3} \lambda_i^2 > 0\). Define a 2-parameter family of functions \(\{H_b(r)\}\) on \(S^2\) by
\[
H_b(r) = 2 + r^2 \sum_{i=1}^{3} \lambda_i x_i^2 - \left( \frac{1}{30} - \hat{b} \right) r^4 \sum_{i=1}^{3} \lambda_i^2,
\]
where \(\hat{b} \in \left( \frac{1}{90}, \frac{1}{30} \right)\) and \(r \in (0, \hat{r})\). Here \(\hat{r} > 0\) is any fixed constant such that \(H_b(r)\) is everywhere positive. (For instance, \(\hat{r}\) can be chosen such that \(2 - \hat{r}^2 \sum_{i=1}^{3} |\lambda_i| - \frac{1}{35} \hat{r}^4 \sum_{i=1}^{3} \lambda_i^2 > 0\).)

Then
(i) \(\int_{S^2} [2 - H_b(r)] \, dv_{\sigma_0} > 0\), \(\forall \, \hat{b} \in \left( \frac{1}{90}, \frac{1}{30} \right)\) and \(\forall \, r \in (0, \hat{r})\).
(ii) \( \exists \tilde{C} > 0 \) independent on \( \bar{b} \) such that \( F_{\sigma_0, H_b(r)}(\eta) < 0 \) for some \( \eta \) whenever \( \bar{b} \in \left( \frac{1}{90}, \frac{1}{30} \right] \) and \( 0 < r < C \left( \frac{1}{90} - \bar{b} \right) \sum_{i=1}^3 \lambda_i^2 \).

**Proof.** Since \( \sum_{i=1}^3 \lambda_i = 0, \sum_{i=1}^3 \lambda_i^2 > 0 \) and \( \bar{b} < \frac{1}{30} \), we have

\[
\int_{S^2} [2 - H_b(r)] d\nu_{\sigma_0} = 4\pi r^4 \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^3 \lambda_i^2 > 0
\]

which proves (i).

Since \( \bar{b} > \frac{1}{90} \), by (5.30) we know for any \( a = (a_1, a_2, a_3) \) with \( |a| = 1 \), there exists an \( \eta_2 \in W^{2,2}(S^2) \) such that \( \eta_2 \) is \( \sigma_0 \)-L^2 orthogonal to \( L(\sigma_0) \) and

\[
G(\eta_1, \eta_2) = 4\pi \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^3 \lambda_i^2 + \frac{1}{2} \int_{S^2} \eta_1^2 \phi^2 d\nu_{\sigma_0}
\]

\[
- 2 \int_{S^2} \phi \left[ \frac{(\Delta_0 \eta_1)(\Delta_0 \eta_2)}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] d\nu_{\sigma_0}
\]

\[
+ \int_{S^2} \left( \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right) d\nu_{\sigma_0}
\]

\[
= 4\pi \left( \frac{1}{90} - \bar{b} \right) \sum_{i=1}^3 \lambda_i^2 < 0
\]

where \( \eta_1 = \sum_{i=1}^3 a_i x_i \) and \( \phi = \sum_{i=1}^3 \lambda_i x_i^2 \).

With such \( \eta_1 \) and \( \eta_2 \) fixed, for each small \( r \), define

\[
\eta^{(r)} = \eta_1 + r^2 \eta_2.
\]

Similar to the proof of (ii) of Theorem [5.2], we have

\[
F_{\sigma_0, H_b(r)}(\eta^{(r)}) = F_{\sigma_0, H_b(r)}(\eta_1) + 2r^2 Q_{\sigma_0, H_b(r)}(\eta_1, \eta_2) + r^4 F_{\sigma_0, H_b(r)}(\eta_2)
\]

where

\[
F_{\sigma_0, H_b(r)}(\eta_1)
\]

\[
= \int_{S^2} [2 - H_b(r)] d\nu_{\sigma_0} + \int_{S^2} \eta_1^2 \frac{[2 - H_b(r)]^2}{H_b(r)} d\nu_{\sigma_0}
\]

\[
= 4\pi r^4 \left( \frac{1}{30} - \bar{b} \right) \sum_{i=1}^3 \lambda_i^2 + \frac{1}{2} r^4 \int_{S^2} \eta_1^2 \phi^2 d\nu_{\sigma_0} + O(r^5),
\]
On second variation of Wang-Yau quasi-local energy

\( Q_{\sigma_0, H_b(r)}(\eta_1, \eta_2) \)

\[
(6.4) \quad Q_{\sigma_0, H_b(r)}(\eta_1, \eta_2) = \int_{S^2} [2 - H_b(r)] \left[ \frac{\Delta_0 \eta_1 (\Delta_0 \eta_2)}{2H_b(r)} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle_{\sigma_0} \right] d\sigma_0 \\
= -r^2 \int_{S^2} \phi \left[ \frac{\Delta_0 \eta_1 \Delta_0 \eta_2}{4} + \langle \nabla_0 \eta_1, \nabla_0 \eta_2 \rangle \right] d\sigma_0 + O(r^3)
\]

and

\[
(6.5) \quad F_{\sigma_0, H_b(r)}(\eta_2) = \int_{S^2} \left[ \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right] d\sigma_0 \\
+ \int_{S^2} [2 - H_b(r)] \left[ \frac{(\Delta_0 \eta_2)^2}{2H_b(r)} + |\nabla_0 \eta_2|^2_{\sigma_0} \right] d\sigma_0 \\
= \int_{S^2} \left[ \frac{(\Delta_0 \eta_2)^2}{2} - |\nabla_0 \eta_2|^2 \right] d\sigma_0 + O(r^2).
\]

Here it is important to note that \( O(r^k) \) denotes a quantity \( f \) that satisfies \( |f| \leq Cr^k \) for some constant \( C \) independent on \( b \in (\frac{1}{90}, \frac{1}{30}) \).

It follows from (6.1) - (6.5) that

\[
F_{\sigma_0, H_b(r)}(\eta^{(r)}) = r^4 G(\eta_1, \eta_2) + O(r^5)
\]

\[
(6.6) \quad \leq r^4 \left[ 4\pi \left( \frac{1}{90} - \bar{b} \right) \sum_{i=1}^{3} \lambda_i^2 + Cr \right]
\]

for some constant \( C > 0 \) independent on \( \bar{b} \). Therefore,

\[
F_{\sigma_0, H_b(r)}(\eta^{(r)}) < 0
\]

whenever \( 0 < r < C4\pi \left( \bar{b} - \frac{1}{90} \right) \sum_{i=1}^{3} \lambda_i^2 \). This proves (ii). \( \square \)

7. Appendix

Lemma 7.1. Let \( \sigma_0 \) be the standard metric on \( S^2 = \{|x| = 1\} \) in \( \mathbb{R}^3 \).

Then

\[
(7.1) \quad \int_{S^2} x_1^{2k} d\sigma_0 = \frac{4\pi}{2k + 1}; \quad \int_{S^2} x_1^2 x_2^2 d\sigma_0 = \frac{4\pi}{15}; \quad \int_{S^2} x_1^4 x_2^2 d\sigma_0 = \frac{4\pi}{35}; \quad \int_{S^2} x_1^2 x_2 x_3^2 d\sigma_0 = \frac{4\pi}{3 \times 35}.
\]
Proof. The first integral follows directly from integration using polar coordinates. To verify the second and the third integral, one has
\[
\int_{S^2} x_1^2 x_2^2 \, dv_{\sigma_0} = \frac{1}{2} \int_{S^2} x_1^2 (x_2^2 + x_3^2) \, dv_{\sigma_0}
\]
\[
= \frac{1}{2} \int_{S^2} x_1^2 (1 - x_1^2) \, dv_{\sigma_0}
\]
\[
= \frac{4\pi}{15}
\]
and
\[
\int_{S^2} x_1^4 x_2^2 \, dv_{\sigma_0} = \frac{1}{2} \int_{S^2} x_1^4 (x_2^2 + x_3^2) \, dv_{\sigma_0}
\]
\[
= \frac{1}{2} \int_{S^2} x_1^4 (1 - x_1^2) \, dv_{\sigma_0}
\]
\[
= \frac{4\pi}{35}
\]
using the first integral. To check the fourth integral, one notes that
\[
\int_{S^2} x_1^4 x_2^2 x_3^2 \, dv_{\sigma_0} = \frac{4\pi}{3}
\]
\[
= \int_{S^2} x_1^2 \, dv_{\sigma_0}
\]
\[
= \int_{S^2} x_1^2 (x_1^2 + x_2^2 + x_3^2)^2 \, dv_{\sigma_0}
\]
\[
= \int_{S^2} x_1^2 (x_1^4 + x_2^4 + x_3^4 + 2x_1^2 x_2^2 + 2x_1^2 x_3^2 + 2x_2^2 x_3^2) \, dv_{\sigma_0}
\]
\[
= 4\pi \left( \frac{1}{7} + \frac{1}{35} + \frac{1}{35} + \frac{2}{35} + \frac{2}{35} \right) + 2 \int_{S^2} x_1^2 x_2^2 x_3^2 \, dv_{\sigma_0}
\]
So
\[
2 \int_{S^2} x_1^2 x_2^2 x_3^2 \, dv_{\sigma_0} = 4\pi \left( \frac{1}{3} - \frac{11}{35} \right) = \frac{8\pi}{3 \times 35}.
\]

\[\square\]

Lemma 7.2. Let \( \sigma_0 \) be the standard metric on \( S^2 = \{ |x| = 1 \} \) in \( \mathbb{R}^3 \). Let \( a_1, a_2, a_3 \) be three constants satisfying \( \sum_{i=1}^{3} a_i^2 = 1 \) and \( \lambda_1, \lambda_2, \lambda_3 \) be three constants satisfying \( \sum_{i=1}^{3} \lambda_i = 0 \). Then
\[
\int_{S^2} \left( \sum_{i=1}^{3} a_i x_i \right)^2 \left( \sum_{i=1}^{3} \lambda_i x_i^2 \right)^2 \, dv_{\sigma_0} = 16\pi \left[ \frac{2 \sum_{i=1}^{3} a_i^2 \lambda_i^2}{3 \times 35} + \frac{\sum_{i=1}^{3} \lambda_i^2}{2 \times 3 \times 35} \right].
\]
Proof. Let \( A = \int_{S^2} \left( \sum_{i=1}^{3} a_i x_i \right)^2 \left( \sum_{i=1}^{3} \lambda_i x_i^2 \right)^2 d\sigma_0 \). By Lemma 7.1,

\[
\int_{S^2} x_i^2 \left( \sum_{i} \lambda_i x_i^2 \right)^2 d\sigma_0 = \int_{S^2} x_i^2 \left( \sum_{i} \lambda_i^2 x_i^4 + 2\lambda_1 \lambda_2 x_1^2 x_2^2 + 2\lambda_1 \lambda_3 x_1^2 x_3^2 + 2\lambda_2 \lambda_3 x_2^2 x_3^2 \right) d\sigma_0
\]

\[
= 4\pi \left( \frac{\lambda_1^2}{7} + \frac{\lambda_2^2}{35} + \frac{\lambda_3^2}{35} + \frac{2\lambda_1 \lambda_2}{35} + \frac{2\lambda_1 \lambda_3}{35} + \frac{2\lambda_2 \lambda_3}{3 \times 35} \right)
\]

\[
= 4\pi \left( \frac{\lambda_1^2}{7} - \frac{2\lambda_1^2}{35} + \frac{1}{35}(\lambda_2 + \lambda_3)^2 - \frac{4\lambda_2 \lambda_3}{3 \times 35} \right)
\]

\[
= 16\pi \left( \frac{\lambda_1^2}{35} - \frac{\lambda_2 \lambda_3}{3 \times 35} \right)
\]

\[
= 16\pi \left( \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2 \times 3 \times 35} \right)
\]

where one uses the fact \( 2\lambda_2 \lambda_3 = (\lambda_2 + \lambda_3)^2 - \lambda_2^2 - \lambda_3^2 = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \).

Similarly,

\[
\int_{S^2} x_2^2 \left( \sum_{i} \lambda_i x_i^2 \right)^2 d\sigma_0 = 16\pi \left( \frac{\lambda_2^2}{35} - \frac{\lambda_3^2 - \lambda_1^2 - \lambda_2^2}{2 \times 3 \times 35} \right),
\]

\[
\int_{S^2} x_3^2 \left( \sum_{i} \lambda_i x_i^2 \right)^2 d\sigma_0 = 16\pi \left( \frac{\lambda_3^2}{35} - \frac{\lambda_2^2 - \lambda_1^2 - \lambda_2^2}{2 \times 3 \times 35} \right).
\]

On the other hand, \( \int_{S^2} x_i x_j \left( \sum_{i} \lambda_i x_i^2 \right)^2 d\sigma_0 = 0 \) for \( \forall i \neq j \). Hence, using the fact \( \sum_{i=1}^{3} a_i^2 = 1 \), one concludes

\[
A = \frac{16\pi}{35} \left[ \sum_{i=1}^{3} a_i^2 \lambda_i^2 + \frac{1}{2 \times 3} \left( -\sum_{i=1}^{3} a_i^2 \lambda_1^2 + a_1^2(\lambda_2^2 + \lambda_3^2) + a_2^2(\lambda_3^2 + \lambda_1^2) + a_3^2(\lambda_1^2 + \lambda_2^2) \right) \right]
\]

\[
= \frac{16\pi}{35} \left[ \sum_{i=1}^{3} a_i^2 \lambda_i^2 + \frac{1}{2 \times 3} \left( -2 \sum_{i=1}^{3} a_i^2 \lambda_1^2 + \sum_{i=1}^{3} \lambda_i^2 \right) \right]
\]

\[
= 16\pi \left[ \frac{2 \sum_{i=1}^{3} a_i^2 \lambda_i^2}{3 \times 35} + \frac{\sum_{i=1}^{3} \lambda_i^2}{2 \times 3 \times 35} \right].
\]

\( \square \)
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