PARALLEL RANDOM COORDINATE DESCENT METHOD FOR COMPOSITE MINIMIZATION

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Abstract. In this paper we consider a parallel version of a randomized (block) coordinate descent method for minimizing the sum of a partially separable smooth convex function and a fully separable non-smooth convex function. Under the assumption of Lipschitz continuity of the gradient of the smooth function, this method has a sublinear convergence rate. Linear convergence rate of the method is obtained for the newly introduced class of generalized error bound functions. We prove that the new class of generalized error bound functions encompasses both global/local error bound functions and smooth strongly convex functions. We also show that the theoretical estimates on the convergence rate depend on the number of blocks chosen randomly and a natural measure of separability of the objective function. Numerical simulations are also provided to confirm our theory.

Key words. Random coordinate descent method, parallel algorithm, partially separable objective function, convergence rate, Lipschitz gradient, error bound property.

1. Introduction. In recent years there has been an ever-increasing interest in the optimization community for algorithms suitable for solving convex optimization problems with a very large number of variables. These problems, known as big data problems, have arisen from more recent fields such as network control [1, 2], machine learning [3] and data mining [4]. An important property of these problems is that they are partially separable, which permits parallel and/or distributed computations in the optimization algorithms that are to be designed for them [1, 5]. This, together with the surge of multi-core machines or clustered parallel computing technology in the past decade, has led to the widespread focus on coordinate descent methods.

State of the art: Coordinate descent methods are methods in which a number of (block) coordinates updates of vector of variables are conducted at each iteration. The reasoning behind this is that coordinate updates for problems with a large number of variables are much simpler than computing a full update, requiring less memory and computational power, and that they can be done independently, making coordinate descent methods more scalable and suitable for distributed and parallel computing hardware. Coordinate descent methods can be divided into two main categories: deterministic and random. In deterministic coordinate descent methods, the (block) coordinates which are to be updated are chosen at each iteration are chosen in a cyclic fashion or based on some greedy strategy. For cyclic coordinate search, estimates on the rate of convergence were given recently in [6, 7], while for the greedy coordinate search the convergence rate is given e.g. in [8, 9]. On the other hand, in random coordinate descent methods, the (block) coordinates which are to be updated are chosen randomly based on some probability distribution. In [10], Nesterov presents a random coordinate descent method for smooth convex problems, in which only one coordinate is updated at each iteration. Under some assumption of Lipschitz gradient and strong convexity of the objective function, the algorithm in [10] was proved to have linear convergence in the expected values of the objective function. In [11, 12] a 2-block random coordinate descent method is proposed to solve linearly constrained smooth convex problems. The algorithm from [11, 12] was extended to linearly constrained composite convex minimization in [13]. The results in [10] and

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were combined in [15, 16], in which the authors propose a randomized coordinate descent method to solve composite convex problems. To our knowledge, the first results on the linear convergence of coordinate descent methods under more relaxed assumptions than smoothness and strong convexity were obtained e.g. in [8, 17]. In particular, linear convergence of these methods is proved under some local error bound property, which is more general than the assumption of Lipschitz gradient and strong convexity as required in [10, 11, 12, 13, 15]. However, the authors in [8, 17] were able to show linear convergence only locally. Finally, very few results were known in the literature on distributed and parallel implementations of coordinate descent methods. Recently, a more thorough investigation regarding the separability of the objective function and ways in which the convergence can be accelerated through parallelization was undertaken in [1, 12, 18, 19], where it is shown that speedup can be achieved through this approach. Several other papers on parallel coordinate descent methods have appeared around the time this paper was finalized [3, 11, 20, 21]. All of the limitations stated above motivate the work of this paper.

Motivation: Despite widespread use of coordinate descent methods for solving large convex problems, there are some aspects that have not been fully studied. In particular, in practical applications, the assumption of Lipschitz gradient and strong convexity is restrictive and the main interest is in finding larger classes of functions for which we can still prove linear convergence. We are also interested in providing schemes based on parallel and/or distributed computations. Finally, the convergence analysis has been almost exclusively limited to centralized stepsize rules and local convergence results. These represent the main issues that we pursue here.

Contribution: In this paper we consider a parallel version of random (block) coordinate gradient descent method for solving large optimization problems with a convex separable composite objective function, i.e. consisting of the sum of a partially separable smooth function and a fully separable non-smooth function. Our approach allows us to analyze in the same framework several methods: full gradient, serial random coordinate descent and any parallel random coordinate descent method in between. Analysis of coordinate descent methods based on updating in parallel more than one (block) component per iteration was given first in [1, 12, 20] and then further studied e.g. in [13, 19]. We provide a detailed rate analysis for our parallel coordinate descent algorithm under general assumptions on the objective function, e.g. error bound type property, and under more knowledge on the structure of the problem and we prove substantial improvement on the convergence rate w.r.t. the existing results from the literature. In particular, we show that this algorithm attains linear convergence for problems belonging to a general class, named generalized error bound problems. We establish that our class includes problems with global/local error bound objective functions and implicitly strongly convex functions with some Lipschitz continuity property on the gradient. We also show that the new class of problems that we define in this paper covers many applications in networks. Finally, we establish that the theoretical estimates on the convergence rate depend on the number of blocks chosen randomly and a natural measure of separability of the objective function. In summary, the contributions of this paper include:

(i) We employ a variant of parallel coordinate descent method in [18], and show that the algorithm still has sublinear convergence rate in the expected values of the objective function under different assumptions regarding the Lipschitz gradient of the objective function. For this algorithm, the iterate updates can be done independently and thus it is suitable for parallel and/or distributed computing architectures.
In Sections 3 and 4 we analyze the random coordinate descent algorithm:

(ii) We introduce a new class of generalized error bound problems for which we show that it encompasses both, problems with global/local error bound functions and smooth strongly convex functions and that it covers many practical applications.

(iii) Under the generalized error bound property, we prove that our parallel random coordinate descent algorithm has a global linear convergence rate.

(iv) We also perform a theoretical identification of which categories of problems and objective functions satisfy the generalized error bound property.

Paper Outline: In Section 2 we present our optimization model and discuss practical applications which can be posed in this framework. In Sections 3 and 4 we analyze the properties of a parallel random coordinate descent algorithm, in particular we establish sublinear convergence rate, under some Lipschitz continuity assumptions. In Section 5 we introduce the class of generalized error bound problems and we prove that the random coordinate descent algorithm has global linear convergence rate under this property. In Section 6 we investigate which classes of optimization problems have an objective function that satisfies the generalized error bound property. In Section 7 we discuss implementation details of the algorithm and compare it with other existing methods. Finally, in Section 8 we present some preliminary numerical tests on constrained lasso problem.

2. Problem formulation. In many big data applications arising from e.g. networks, control and data ranking, we have a system formed from several entities, with a communication graph which indicates the interconnections between entities (e.g. sources and links in network optimization [22], website pages in data ranking [3] or subsystems in control [2]). We denote this bipartite graph as $G = ([N] \times [\bar{N}], E)$, where $[N] = \{1, \ldots, N\}$, $[\bar{N}] = \{1, \ldots, \bar{N}\}$ and $E \in \{0, 1\}^{N \times \bar{N}}$ is an incidence matrix. We also introduce two sets of neighbors $N_j$ and $\bar{N}_i$ associated to the graph, defined as:

$$N_j = \{i \in [N]: E_{ij} = 1\} \quad \forall j \in [\bar{N}] \quad \text{and} \quad \bar{N}_i = \{j \in [\bar{N}]: E_{ij} = 1\} \quad \forall i \in [N].$$

The index sets $N_j$ and $\bar{N}_i$, which e.g. in the context of network optimization may represent the set of sources which share the link $j \in [\bar{N}]$ and the set of links which are used by the source $i \in [N]$, respectively, describe the local information flow in the graph. We denote the entire vector of variables for the graph as $x \in \mathbb{R}^n$. The vector $x$ can be partitioned accordingly in block components $x_i \in \mathbb{R}^{n_i}$, with $n = \sum_{i=1}^{N} n_i$. In order to easily extract subcomponents from the vector $x$, we consider a partition of the identity matrix $I_n = [U_1 \ldots U_N]$, with $U_i \in \mathbb{R}^{n \times n_i}$, such that $x_i = U_i^T x$ and matrices $U_{N_i} \in \mathbb{R}^{n \times n_i}$, such that $x_{N_i} = U_{N_i}^T x$, with $x_{N_i}$ being the vector containing all the components $x_j$ with $j \in N_i$. In this paper we address problems arising from such systems, where the objective function can be written in a general form as (for similar models and settings, see [1, 2, 18, 21, 22, 23]):

$$F^* = \min_{x \in \mathbb{R}^n} F(x) = \left( \sum_{j=1}^{\bar{N}} f_j(x_{N_j}) + \sum_{i=1}^{N} \psi_i(x_i) \right),$$

where $f_j : \mathbb{R}^{n_{N_j}} \to \mathbb{R}$ and $\psi_i : \mathbb{R}^{n_i} \to \mathbb{R}$. We denote $f(x) = \sum_{j=1}^{\bar{N}} f_j(x_{N_j})$ and $\Psi(x) = \sum_{i=1}^{N} \psi_i(x_i)$. The function $f(x)$ is a smooth partially separable convex function, while $\Psi(x)$ is fully separable convex non-smooth function. The local information structure imposed by the graph $G$ should be considered as part of the problem formulation. We
I. Necoara and D. Clipici consider the following natural measure of separability of the objective function $F$:

$$(\omega, \bar{\omega}) = (\max_{j \in [N]} |N_j|, \max_{i \in [\bar{N}]} |N_j|).$$

Note that $1 \leq \omega \leq N$, $1 \leq \bar{\omega} \leq \bar{N}$ and the definition of the measure of separability $(\omega, \bar{\omega})$ is more general than the one considered in [18] that is defined only in terms of $\omega$. It is important to note that coordinate gradient descent type methods for solving problem (2.1) are appropriate only in the case when $\bar{\omega}$ is relatively small, otherwise incremental type methods [22, 24] should be considered for solving (2.1). Indeed, difficulties may arise when $f$ is the sum of a large number of component functions and $\bar{\omega}$ is large, since in that case exact computation of the components of gradient (i.e. $\nabla f(x) = \sum_{j \in \mathcal{N}} \nabla f_j(x_{N_j})$) can be either very expensive or impossible due to noise. In conclusion, we assume that the algorithm is employed for problems (2.1), with $\bar{\omega}$ relatively small, i.e. $\bar{\omega}, \omega \ll n$.

Throughout this paper, by $x^*$ we denote an optimal solution of problem (2.1) and by $X^*$ the set of optimal solutions. We define the index indicator function as:

$$1_{N_j}(i) = \begin{cases} 1, & \text{if } i \in N_j \\ 0, & \text{otherwise,} \end{cases}$$

and the set indicator function as:

$$I_X(x) = \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{otherwise.} \end{cases}$$

Also, by $\| \cdot \|$ we denote the standard Euclidean norm and we introduce an additional norm $\|x\|_W = x^TWx$, where $W \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix. Considering these, we denote by $\Pi_X^W(x)$ the projection of a point $x$ onto a set $X$ in the norm $\| \cdot \|_W$:

$$\Pi_X^W(x) = \arg \min_{y \in X} \|y - x\|^2_W.$$ 

Furthermore, for simplicity of exposition, we denote by $\bar{x}$ the projection of a point $x$ on the optimal set $X^*$, i.e. $\bar{x} = \Pi_X^W(x)$. In this paper we consider that the smooth component $f_j(x_{N_j})$ of (2.1) satisfies the following assumption:

**Assumption 1.** We assume that the functions $f_j(x_{N_j})$ have Lipschitz continuous gradient with a constant $L_{N_j} > 0$:

$$(2.2) \quad \| \nabla f_j(x_{N_j}) - \nabla f_j(y_{N_j}) \| \leq L_{N_j} \|x_{N_j} - y_{N_j}\| \quad \forall x_{N_j}, y_{N_j} \in \mathbb{R}^{n_{N_j}}.$$ 

Note that our assumption is different from the ones in [1, 10, 13, 18], where the authors consider that the gradient of the function $f$ is coordinate-wise Lipschitz continuous, which states the following: if we define the partial gradient $\nabla_i f(x) = U_i^T \nabla f(x)$, then there exists some constants $L_i > 0$ such that:

$$(2.3) \quad \| \nabla_i f(x + U_i y_i) - \nabla_i f(x) \| \leq L_i \|y_i\| \quad \forall x \in \mathbb{R}^n, y_i \in \mathbb{R}^{n_i}.$$ 

As a consequence of Assumption [1] we have that [25]:

$$(2.4) \quad f_j(x_{N_j} + y_{N_j}) \leq f_j(x_{N_j}) + (\nabla f_j(x_{N_j}), y_{N_j}) + \frac{L_{N_j}}{2} \|y_{N_j}\|^2.$$
Based on Assumption \textbf{1}, we can show the following distributed variant of the descent lemma, which is central in our derivation of a parallel coordinate descent method and in proving the convergence rate for it.

**Lemma 2.1.** Under Assumption \textbf{1} the following inequality holds for the smooth part of the objective function $f(x) = \sum_{j=1}^{N} f_j(x_{N_j})$:

\begin{equation}
    f(x + y) \leq f(x) + \langle \nabla f(x), y \rangle + \frac{1}{2} \|y\|_W^2 \quad \forall x, y \in \mathbb{R}^n,
\end{equation}

where $W > 0$ is diagonal with its blocks $W_{ii} \in \mathbb{R}^{n_i \times n_i}$, $W_{ii} = \sum_{j \in \mathcal{N}_i} L_{N_j} I_{n_i}$, $i \in [N]$, and the remaining blocks are zero.

**Proof.** If we sum up (2.4) for $j \in [\bar{N}]$ and by the definition of $f$ we have that:

\begin{equation}
    f(x + y) \leq f(x) + \sum_{j=1}^{\bar{N}} \left( \langle \nabla f_j(x_{N_j}), y_{N_j} \rangle + \frac{L_{N_j}}{2} \|y_{N_j}\|^2 \right).
\end{equation}

Given matrices $U_{N_j}$, note that we can express the first term in the right hand side as:

\[
\sum_{j=1}^{\bar{N}} \langle \nabla f_j(x_{N_j}), y_{N_j} \rangle = \sum_{j=1}^{\bar{N}} \langle \nabla f_j(x_{N_j}), U_{N_j}^T y \rangle = \sum_{j=1}^{\bar{N}} \langle U_{N_j} \nabla f_j(x_{N_j}), y \rangle = \langle \nabla f(x), y \rangle.
\]

Note that since $W$ is a diagonal matrix we can express the norm $\| \cdot \|_W$ as:

\[
\|y\|_W^2 = \sum_{i=1}^{N} \left( \sum_{j \in \mathcal{N}_i} L_{N_j} \right) \|y_i\|^2.
\]

From the definition of $\mathcal{N}_j$ and $\bar{N}_i$, note that $1_{\mathcal{N}_j}(i)$ is equivalent to $1_{\bar{N}_i}(j)$. Thus, for the final term of the right hand side of (2.6) we have that:

\[
\frac{1}{2} \sum_{j=1}^{\bar{N}} L_{N_j} \|y_{N_j}\|^2 = \frac{1}{2} \sum_{j=1}^{\bar{N}} L_{N_j} \sum_{i \in \mathcal{N}_j} \|y_i\|^2 = \frac{1}{2} \sum_{j=1}^{\bar{N}} L_{N_j} \sum_{i=1}^{N} \|y_i\|^2 1_{\mathcal{N}_j}(i)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} \|y_i\|^2 \sum_{j=1}^{\bar{N}} L_{N_j} 1_{\mathcal{N}_j}(j) = \frac{1}{2} \sum_{i=1}^{N} \|y_i\|^2 \sum_{j \in \bar{N}_i} L_{N_j} = \frac{1}{2} \|y\|_W^2,
\]

and the proof is complete. \qed

Note that the convergence results of this paper hold for any descent lemma in the form (2.3) and thus the expression of the matrix $W$ above can be replaced with any other block-diagonal matrix $W > 0$ for which (2.5) is valid. Based on (2.3) a similar inequality as in (2.6) can be derived, but the matrix $W$ is replaced in this case with the matrix $\omega W' = \omega \text{diag}(L_i I_{n_i}; i \in [N])$ \cite{18}. These differences in the matrices will lead to different step sizes in the algorithms of our paper and of e.g. \cite{11,13}. Moreover, from the generalized descent lemma through the norm $\| \cdot \|_W$, the sparsity induced by the graph $G$ via the sets $\mathcal{N}_j$ and $\bar{N}_i$ and implicitly via the measure of separability $(\omega, \bar{\omega})$ will intervene in the estimates for the convergence rates of the algorithm. A detailed discussion on this issue can be found in Section \textbf{4}. The following lemma

establishes Lipschitz continuity for $\nabla f$ but in the norm $\| \cdot \|_W$, whose proof can be derived using similar arguments as in [25]:

**Lemma 2.2.** For a function $f$ satisfying Assumption [1] the following holds:

$$\| \nabla f(x) - \nabla f(y) \|_{W^{-1}} \leq \| x - y \|_W \quad \forall x, y \in \mathbb{R}^n.$$  

2.1. Motivating practical applications. We now present important applications from which the interest for problems of type (2.1) stems.

**Application I:** One specific example is the sparse logistic regression problem. This type of problem is often found in data mining or machine learning, see e.g. [26, 27]. In a training set $\{a^j, b^j\}$, with $j \in [N]$, the vectors $a^j \in \mathbb{R}^n$ represent $N$ samples, and $b^j$ represent the binary class labels with $b^j \in \{-1, +1\}$. The likelihood function for these $N$ samples is:

$$\frac{1}{N} \sum_{j=1}^N \mathbb{P}(b^j | a^j),$$

where $\mathbb{P}(b|a)$ is the conditional probability and is expressed as:

$$\mathbb{P}(b | a) = \frac{1}{1 + \exp(-b \langle a, x \rangle)},$$

with $x \in \mathbb{R}^n$ being the weight vector. In some applications (see e.g. [27]), we require a bias term $c$ (also called as an intercept) in the loss function; therefore, $\langle a, x \rangle$ is replaced with $\langle a, x \rangle + c$. The equality $\langle a, x \rangle = 0$ defines a hyperplane in the feature space on which $\mathbb{P}(b | a) = 0.5$. Also, $\mathbb{P}(b | a) > 0.5$ if $\langle a, x \rangle > 0$ and $\mathbb{P}(b | a) < 0.5$ otherwise. Then, the sparse logistic regression can be formulated as the following convex problem:

$$\min_{x \in \mathbb{R}^n} f(x) + \lambda \|x\|_1,$$

where $\lambda > 0$ is some constant and $f(x)$ is the average logistic loss function:

$$f(x) = -\frac{1}{N} \sum_{i=1}^N \log \left( \mathbb{P}(b^i | a^i) \right) = -\frac{1}{N} \sum_{j=1}^N \log \left( 1 + \exp \left( -b^j \langle a^j, x \rangle \right) \right).$$

Note that $\Psi(x) = \lambda \|x\|_1$, where $\| \cdot \|_1$ denotes the 1-norm, is the separable non-smooth component which promotes the sparsity of the decision variable $x$. If we associate to this problem a bipartite graph $G$ where the incidence matrix $E$ is defined such that $E_{ij} = 1$ provided that $a_i^j \neq 0$, then the vectors $a^j$ have a certain sparsity according to this graph, i.e. they only have nonzero components in $a^j_{N_j}$. Therefore, $f(x)$ can be written as $f(x) = \sum_{j=1}^N f_j(x_{N_j})$, where each function $f_j$ is defined as:

$$f_j(x_{N_j}) = -\frac{1}{N} \log \left( 1 + \exp \left( -b^j \langle a^j_{N_j}, x_{N_j} \rangle \right) \right).$$

It can be easily proven that the objective function $f$ in this case satisfies [25] with $L_{N_j} = \sum_{i \in N_j} \|a_i^j\|^2 / 4$ and (2.3) with $L_i = \sum_{j \in N_i} \|a_i^j\|^2 / 4$. Furthermore, we have that $f$ satisfies [24] with matrix $W = \text{diag} \left( \sum_{j \in N_i} \|a^j_{N_j}\|^2 / 4; i \in [n] \right)$. 

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Application II: Another classic problem which implies functions $f_j$ with Lipschitz continuous gradient of type (2.2) is:

$$
(2.8) \quad \min_{x_i \in X_i, \xi \leq \mathbb{R}^n} F(x) = \frac{1}{2} \|Ax - b\|^2 + \sum_{i=1}^{N} \lambda_i \|x_i\|_1,
$$

where $A \in \mathbb{R}^{N \times n}$, the sets $X_i$ are convex, $n = \sum_{i=1}^{N} n_i$ and $\lambda > 0$. This problem is also known as the constrained lasso problem [28] and is widely used e.g. in signal processing, fused or generalized lasso and monotone curve estimation [28, 29] or distributed model predictive control [1].

For example, in image restoration, incorporating a priori information (such as box constraints on $x$) can lead to substantial improvements in the restoration and reconstruction process (see [29] for more details). Note that this problem is a special case of problem (2.11), with $\Psi(x) = \sum_{i=1}^{N} [\lambda_i \|x_i\|_1 + I_{X_i}(x_i)]$ being block separable and with the functions $f_j$ defined as:

$$
 f_j(x_{N_j}) = \frac{1}{2} (a_{N_j}^T x_{N_j} - b_j)^2,
$$

where $a_{N_j}$ are the nonzero components of row $j$ of $A$, corresponding to $N_j$. In this case, functions $f_j$ satisfy (2.2) with $L_{N_j} = \|a_{N_j}\|^2$. Given these constants, we find that $f$ in this case satisfies (2.5) with the matrix $W = \text{diag} \left( \sum_{j \in N_i} \|a_{N_j}\|^2 I_{n_i} \right)$. Also, note that functions of type (2.8) satisfy Lipschitz continuity (2.3) with $L_i = \|A_i\|^2$, where $A_i \in \mathbb{R}^{N \times n_i}$ denotes block column $i$ of the matrix $A$.

Application III: A third type of problem which falls under the same category is derived from the following primal formulation:

$$
(2.9) \quad f^* = \min_{u \in \mathbb{R}^n} \sum_{j=1}^{\tilde{N}} g_j(u_j),
$$

s.t. $Au \leq b$,

where $A \in \mathbb{R}^{n \times m}$, $u_j \in \mathbb{R}^{m_j}$ and the functions $g_j$ are strongly convex with convexity parameters $\sigma_j$. This type of problem is often found in network control [2], network optimization or utility maximization [22]. In all these applications matrix $A$ is very sparse, i.e. both $(\omega, \tilde{\omega})$ are small. We formulate the dual problem of (2.9) as:

$$
\max_{x \in \mathbb{R}^n} \left[ \min_{u \in \mathbb{R}^n} \sum_{j=1}^{\tilde{N}} g_j(u_j) + \langle x, Au - b \rangle \right] - \Psi(x),
$$

where $x$ denotes the Lagrange multiplier and $\Psi(x) = I_{\mathbb{R}^n_+}(x)$ is the indicator function for the nonnegative orthant $\mathbb{R}^n_+$. By denoting by $g_j^*(z)$ the convex conjugate of the function $g_j(u_j)$, the previous problem can be rewritten as:

$$
(2.10) \quad f^* = \max_{x \in \mathbb{R}^n} \left[ \sum_{j=1}^{\tilde{N}} \min_{u_j \in \mathbb{R}^{m_j}} g_j(u_j) - \langle A_j^T x, u_j \rangle \right] - \langle x, b \rangle - \Psi(x)
$$

$$
= \max_{x \in \mathbb{R}^n} \sum_{j=1}^{\tilde{N}} -g_j^*(-A_j^T x) - \langle x, b \rangle - \Psi(x),
$$
where $A_j \in \mathbb{R}^{n \times m_j}$ is the jth block column of $A$. Note that, given the strong convexity of $g_j(u_j)$, then the functions $g_j^*(z)$ have Lipschitz continuous gradient in $z$ of type (2.2) with constants $\frac{1}{\sigma_j}$. Now, if the matrix $A$ has some sparsity induced by a graph, i.e., the blocks $A_{ij} = 0$ if the corresponding incidence matrix has $E_{ij} = 0$, which in turn implies that the block columns $A_j$ are sparse according to some index set $\mathcal{N}_j$, then the matrix-vector products $A_j^T x$ depend only on $x_{\mathcal{N}_j}$, such that $f_j(x_{\mathcal{N}_j}) = g_j^*\left(-A_{\mathcal{N}_j}^T x_{\mathcal{N}_j}\right) - \langle x_{\mathcal{N}_j}, \hat{b}_{\mathcal{N}_j} \rangle$, with $\sum_j \langle x_{\mathcal{N}_j}, \hat{b}_{\mathcal{N}_j} \rangle = \langle x, b \rangle$. Then, $f_j$ has Lipschitz continuous gradient of type (2.2) with constants $L_{\mathcal{N}_j} = \frac{\|A_{\mathcal{N}_j}\|_2}{\sigma_j}$. For this problem we also have componentwise Lipschitz continuous gradient of type (2.3) with $L_i = \sum_{j \in \mathcal{N}_i} \frac{\|A_{ij}\|_2}{\sigma_j}$. Furthermore, we find that in this case $f$ satisfies (2.5) with $W = \text{diag} \left( \sum_{j \in \mathcal{N}_i} \frac{\|A_{ij}\|_2^2}{\sigma_j}; \; i \in [n] \right)$. Note that there are many applications in distributed control or network optimization where $\omega$ or $\bar{\omega}$ or both are small. E.g., one particular application that appears in the area of network optimization has the structure (2.9), where the matrix $A$ has column linked block angular form, i.e., the matrix $A$ has a block structure of the following form:

$$A = \begin{bmatrix}
A_{11} & A_{21} & A_{22} \\
A_{31} & A_{32} & A_{33} \\
A_{n-1,n-1} & A_{n-1,n} & A_{nn}
\end{bmatrix}.$$

One of the standard distributed algorithms to solve network problems is based on a dual decomposition as explained above. In this case, by denoting $g_j^*(z)$ the convex conjugate of the function $g_j(u_j)$, the corresponding problem can be rewritten as:

\begin{equation}
(2.11) \quad f^* = \max_{x \in \mathbb{R}^n} \left[ -g_1^*(-A_{11}^T x) - \sum_{j = 2}^{\bar{N}} -g_j^*(-A_{jj}^T x_j) - \langle x, b \rangle - \Psi(x) \right].
\end{equation}

If we consider the block columns of dimension 1, then $g_j^*(-A_{ij}^T x)$ depends on all the coordinates of $x$, i.e., $\omega = \bar{N}$. On the other hand, note that given the structure of $A$, we have that $\bar{\omega} = 2 \ll \omega = \bar{N}$. The reader can easily find many other examples of objective functions where $\bar{\omega} \ll \omega$.

3. Parallel random coordinate descent method. In this section we consider a parallel version of the random coordinate descent method [10,11,15], which we call P-RCD. Analysis of coordinate descent methods based on updating in parallel more than one (block) component per iteration was given first in [1,12,21] and then further studied e.g. in [18,19]. In particular, such a method and its convergence properties has been analyzed in [1,18], but under the coordinate-wise Lipschitz assumption (2.5). Before we discuss the method however, we first need to introduce some concepts. For a function $F(x)$ as defined in (2.1), we introduce the following mapping in norm $\| \cdot \|_W$:

\begin{equation}
(3.1) \quad t_{[\mathcal{N}]}(x, y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \| y - x \|_W^2 + \Psi(y).
\end{equation}

Note that the mapping $t_{[\mathcal{N}]}(x, y)$ is a fully separable and strongly convex in $y$ w.r.t. to the norm $\| \cdot \|_W$ with the constant 1. We denote by $T_{[\mathcal{N}]}(x)$ the proximal step for
function $F(x)$, which is the optimal point of the mapping $t_{[N]}(x,y)$, i.e.:

\[
T_{[N]}(x) = \arg \min_{y \in \mathbb{R}^n} t_{[N]}(x,y).
\]

The proximal step $T_{[N]}(x)$ can also be defined in another way. We define the proximal operator of function $\Psi$ as:

\[
\text{prox}_\Psi(x) = \arg \min_{u \in \mathbb{R}^n} \Psi(u) + \frac{1}{2\lambda} \|u - x\|^2_W.
\]

We recall an important property of the proximal operator [30]:

\[
\|\text{prox}_\Psi(x) - \text{prox}_\Psi(y)\|_W \leq \|x - y\|_W.
\]

Based on this operator, note that we can write:

\[
T_{[N]}(x) = \text{prox}_\Psi(x - W^{-1} \nabla f(x)).
\]

Given that $\Psi(x)$ is generally not differentiable, we denote by $\partial \Psi(x)$ a vector belonging to the set of subgradients of $\Psi(x)$. Evidently, in both definitions, the optimality conditions of the resulting problem from which we obtain $T_{[N]}(x)$ are the same, i.e.:

\[
0 \in \nabla f(x) + W(t_{[N]}(x) - x) + \partial \Psi(T_{[N]}(x)).
\]

It will become evident further on that the optimal solution $T_{[N]}(x)$ will play a crucial role in the parallel random coordinate descent method. We now establish some properties which involve the function $F(x)$, the mapping $t_{[N]}(x,y)$ and the proximal step $T_{[N]}(x)$. Given that $t_{[N]}(x,y)$ is strongly convex in $y$ and that $T_{[N]}(x)$ is an optimal point when minimizing over $y$, we have the following inequality:

\[
F(x) - t_{[N]}(x,T_{[N]}(x)) = t_{[N]}(x,x) - t_{[N]}(x,T_{[N]}(x)) \geq \frac{1}{2} \|x - T_{[N]}(x)\|^2_W.
\]

Further, since $f$ is convex and differentiable and by definition of $t_{[N]}(x,y)$ we get:

\[
t_{[N]}(x,T_{[N]}(x)) \leq \min_{y \in \mathbb{R}^n} f(y) + \Psi(y) + \frac{1}{2\lambda} \|y - x\|^2_W
\]

\[
= \min_{y \in \mathbb{R}^n} F(y) + \frac{1}{2\lambda} \|y - x\|^2_W.
\]

In the algorithm that we discuss, at a step $k$, the components of the iterate $x^k$ which are to be updated are dictated by a set of indices $J^k \subseteq [N]$ which is randomly chosen. Let us denote by $x_J \in \mathbb{R}^n$ the vector whose blocks $x_i$, with $i \in J \subseteq [N]$, are identical to those of $x$, while the remaining blocks are zeroed out, i.e. $x_J = \sum_{i \in J} U_i x_i$ or:

\[
x_J = \begin{cases} x_i, & i \in J \\ 0, & \text{otherwise.} \end{cases}
\]

Also, for the separable function $\Psi(x)$, we denote the partial sum $\Psi_J(x) = \sum_{i \in J} \Psi_i(x_i)$ and the vector $\partial_i \Psi(x) = [\partial \Psi(x)]_J \in \mathbb{R}^n$. A random variable $J$ is uniquely characterized by the probability density function:

\[
P_J = P(J = \hat{J}) \quad \text{where} \quad \hat{J} \subseteq [N].
\]
For the random variable $J$, we also define the probability with which a subcomponent $i \in [N]$ can be found in $J$ as:

$$p_i = \mathbb{P}(i \in J).$$

In our algorithm, we consider a uniform sampling of $\tau$ unique coordinates $i, 1 \leq \tau \leq N$ that make up $J$, i.e. $|J| = \tau$. For a random variable $J$ with $|J| = \tau$, we observe that we have a total number of $\binom{N}{\tau}$ possible values that $J$ can take, and with the uniform sampling we have that $P_J = \frac{1}{\binom{N}{\tau}}$. Given that $J$ is random, we can express $p_i$ as:

$$p_i = \sum_{J: i \in J} P_J.$$

For a single index $i$, note that we have a total number of $\binom{N-1}{\tau-1}$ possible sets that $J$ can take which will include $i$ and therefore the probability that this index is in $J$ is:

$$p_i = \frac{\binom{N-1}{\tau-1}}{\binom{N}{\tau}} = \frac{\tau}{N}. \tag{3.9}$$

**Remark 1.** We can also consider other ways in which $J$ can be chosen. For example, we can have partition sets $J^1, \ldots, J^q$ of $[N]$, i.e. $[N] = \bigcup_{i=1}^q J^i$, that are randomly shuffled. We can choose $J$ in a nearly independent manner, i.e. $J$ is chosen with a sufficient probability, or we can choose $J$ according to an irreducible and aperiodic Markov chain, see e.g. [3, 24]. However, if we employ these strategies for choosing $J$, the proofs for the convergence rate of our algorithm follow similar lines.

Having defined the proximal step as $T_{[N]}(x^k)$ in [8, 24], in the algorithm that follows we generate randomly at step $k$ an index set $J^k$ of cardinality $1 \leq \tau \leq N$. We denote the vector $T_{J^k}(x^k) = [T_{[N]}(x^k)]_{J^k}$ which will be used to update $x^{k+1}$, i.e. in the sense that $[x^{k+1}]_{J^k} = T_{J^k}(x^k)$. Also, by $\bar{J}^k$ we denote the complement set of $J^k$, i.e. $\bar{J}^k = \lbrace i \in [N] : i \notin J^k \rbrace$. Thus, the algorithm consists of the following steps (note that our algorithm is similar to the ones studied in [1, 18]. In particular, in [18] the uniform sampling has also been considered there under the name of $\tau$-uniform sampling, but with different stepsizes and different Lipschitz assumption on the gradient of the smooth component $f$):

**Parallel random coordinate descent method (P-RCD)**

1. Consider an initial point $x^0 \in \mathbb{R}^n$ and $1 \leq \tau \leq N$
2. For $k \geq 0$:
   1. Generate with uniform probability a random set of indices $J^k \subseteq [N]$, with $|J^k| = \tau$
   2. Compute:
      $$x_{J^k}^{k+1} = T_{J^k}(x^k) \text{ and } x_{\bar{J}^k}^{k+1} = x_{\bar{J}^k}^k.$$

Note that the iterate update of (P-RCD) method can also be expressed as:

$$
\begin{align*}
&x^{k+1} = x^k + T_{J^k}(x^k) - x_{J^k}^k \\
&x^{k+1} = \arg\min_{y \in \mathbb{R}^n} \langle \nabla_{J^k} f(x^k), y - x^k \rangle + \frac{1}{2} \|y - x^k\|_W^2 + \Psi_{J^k}(y) \\
&x^{k+1} = \text{prox}_{\Psi_{J^k}}(x^k - W^{-1}\nabla_{J^k} f(x^k)).
\end{align*}
$$

\[\tag{3.10}\]
Note that the right hand sides of the last two equalities contain the same optimization problem whose optimality conditions are:

\[
\begin{align*}
W[x^k - x^{k+1}]_{j^k} &\in \nabla f(x^k) + \partial \Psi_{j^k}(x^{k+1}) \\
[x^{k+1}]_{j^k} &= [x^k]_{j^k}.
\end{align*}
\]  

(3.11)

Clearly, the optimization problem from which we compute the iterate of (P-RCD) is fully separable. Then, it follows that for updating component \(i \in J^k\) of \(x^{k+1}\) we need the following data: \(\Psi_i(x_i^k), W_{ii}\) and \(\nabla_i f(x^k)\). However, the \(i\)th diagonal entry \(W_{ii} = \sum_{j \in \mathcal{N}_i} L_{ij}\), and \(i\)th block component of the gradient \(\nabla f = \sum_{j \in \mathcal{N}_i} \nabla_i f_j\) can be computed distributively according to the communication graph \(G\) imposed on the original optimization problem. Therefore, if algorithm (P-RCD) runs on a multi-core machine or as a multi-thread process, it can be observed that component updates can be done distributively and in parallel by each core/thread (see Section 7 for details).

We now establish that method (P-RCD) is a descent method, i.e. \(F(x^{k+1}) \leq F(x^k)\) for all \(k \geq 0\). From the convexity of \(\Psi(\cdot)\) and (2.5) we obtain the following:

\[
F(x^{k+1}) \leq F(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \|x^{k+1} - x^k\|^2_W
\]

(3.12)

\[
= F(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \|x^{k+1} - x^k\|^2_W
\]

\[
= F(x^k) + \langle W[x^k - x^{k+1}]_{j^k}, [x^{k+1} - x^k]_{j^k} \rangle + \frac{1}{2} \|x^{k+1} - x^k\|^2_W
\]

(3.13)

With (P-RCD) being a descent method, we can now introduce the following term:

\[
R_W(x^0) = \max_{x: F(x) \leq F(x^0)} \min_{x^* \in X} \|x - x^*\|_W.
\]

and assume it to be bounded. We also define the random variable comprising the whole history of previous events as:

\[
\eta^k = \{J^0, \ldots, J^k\}.
\]

4. Sublinear convergence for smooth convex minimization. In this section we establish the sublinear convergence rate of method (P-RCD) for problems of type (2.1) with the objective function satisfying Assumption 2. Our analysis in this section combines the tools developed in [18] for parallel coordinate descent methods with the convergence analysis in [14] for random one block coordinate descent methods. The next lemma provides some property for the uniform sampling with \(|J| = \tau|J^0|\):

**Lemma 4.1.** [18, Lemma 3] Let there be some constants \(\theta_i\) with \(i = 1, \ldots, N\), and a sampling \(J\) chosen as described above and define the sum \(\sum_{i \in J} \theta_i\). Then, the expected value of the sum satisfies:

\[
\mathbb{E}\left[\sum_{i \in J} \theta_i\right] = \sum_{i = 1}^N p_i \theta_i.
\]  

(4.1)
For any vector $d \in \mathbb{R}^n$ we consider its counterpart $d_J$ for a sampling $J$ taken as described above. Given the previous lemma and by taking into account the separability of the inner product and of the squared norm $\| \cdot \|^2_W$, it follows immediately that:

\begin{align}
\mathbb{E}[\langle x, d_J \rangle] &= \frac{\tau}{N} \langle x, d \rangle \\
\mathbb{E}[\|d_J\|^2_W] &= \frac{\tau}{N} \|d\|^2_W.
\end{align}

Based on relations (4.2) and (4.3), the separability of the function $\Psi(x)$, and the properties of the expectation operator, the following inequalities can be derived (see e.g. \[18\]):

\begin{align}
\mathbb{E}[\Psi(x + d_J)] &= \mathbb{E}\left[\mathbb{E}[\Psi(x + d)]\right] = \frac{\tau}{N} \Psi(x + d) + \left(1 - \frac{\tau}{N}\right) \Psi(x) \\
\mathbb{E}[F(x + d_J)] &\leq \left(1 - \frac{\tau}{N}\right) F(x) + \frac{\tau}{N} J_N(x, y).
\end{align}

We can now formulate an important relation between the gradient mapping in a point $x$ and a point $y$. By the definition of $t_N(x, y)$ and the convexity of $f$ and $\Psi$ we have:

\begin{align}
t_N(x, T_N(x)) &= f(x) + \langle \nabla f(x), T_N(x) - x \rangle + \frac{1}{2} \|T_N(x) - x\|^2_W + \Psi(T_N(x)) \\
&\leq f(y) + \langle \nabla f(x), x - y \rangle + \langle \nabla f(x), T_N(x) - x \rangle + \frac{1}{2} \|T_N(x) - x\|^2_W + \Psi(T_N(x)) \\
&\leq f(y) + \langle \nabla f(x), x - y \rangle + \langle \nabla f(x), T_N(x) - x \rangle + \frac{1}{2} \|T_N(x) - x\|^2_W \\
&\quad + \Psi(y) + \langle \partial \Psi(T_N(x)), T_N(x) - y \rangle.
\end{align}

Furthermore, from the optimality conditions (3.13) we obtain:

\begin{align}
t_N(x, T_N(x)) &\leq f(y) + \langle \nabla f(x), x - y \rangle + \langle \nabla f(x), T_N(x) - x \rangle \\
&\quad + \frac{1}{2} \|T_N(x) - x\|^2_W + \Psi(y) + \langle -\nabla f(x) - W(T_N(x) - x), T_N(x) - y \rangle \\
&= F(y) + \langle \nabla f(x), x - y \rangle + \langle \nabla f(x), T_N(x) - x \rangle + \frac{1}{2} \|T_N(x) - x\|^2_W \\
&\quad + \langle -\nabla f(x) - W(T_N(x) - x), T_N(x) - x \rangle + \langle -\nabla f(x) - W(T_N(x) - x), x - y \rangle \\
&= F(y) - \langle W(T_N(x) - x), x - y \rangle - \frac{1}{2} \|T_N(x) - x\|^2_W.
\end{align}

This property will prove useful in the following theorem, which provides the sublinear convergence rate for method (P-RCD).

**Theorem 4.2.** If Assumption \[7\] holds and considering that $R_W(x^0)$ defined in (3.13) is bounded, then for the sequence $x^k$ generated by algorithm (P-RCD) we have:

\begin{equation}
\mathbb{E}[F(x^k)] - F^* \leq \frac{N (1/2 \langle R_W(x^0) \rangle^2 + F(x^0) - F^*)}{\tau k + N}.
\end{equation}

**Proof.** Our proof generalizes the proof of Theorem 1 in \[16\] from one (block) component update per iterate to the case of $\tau$ (block) component updates, based on uniform sampling and on Assumption \[11\] and consequently on a different descent lemma. Thus, by taking expectation in (3.12) w.r.t. $J^k$ conditioned on $\eta^{k-1}$ we get:

\begin{align}
\mathbb{E}[F(x^{k+1})] &\leq F(x^k) - \frac{1}{2} \mathbb{E} \|x^{k+1} - x^k\|^2_W \leq F(x^k).
\end{align}
Now, if we take \( x = x^k \) and \( y_j \) = \( T_{j,k}(x^k) - x^k_j \) in (4.3), we get:

\[
(4.9) \quad \mathbb{E}[F(x^{k+1})] \leq \left(1 - \frac{\tau}{N}\right) F(x^k) + \frac{\tau}{N} \|T_{[N]}(x^k, T_{[N]}(x^k))\|_W.
\]

From this and (4.10) we obtain:

\[
(4.10) \quad \frac{\tau}{N} F(y) + \frac{N - \tau}{N} F(x^k) \geq \mathbb{E}[F(x^{k+1})] + \frac{\tau}{N} \langle W (T_{[N]}(x^k) - x^k), x^k - y \rangle + \frac{\tau}{2N} \|T_{[N]}(x^k) - x^k\|_W^2.
\]

Denote \( r^k = \|x^k - x^*\|_W \). From the definition of \( x^{k+1} \) we have that:

\[
(r^{k+1})^2 = (r^k)^2 + \sum_{i \in J^k} [2W_i \langle T_i(x^k) - x^k_i, x_i^* - x_i^* \rangle + W_i \|T_i(x^k) - x_i^*\|^2].
\]

If we divide both sides of the above inequality by 2 and take expectation, we obtain:

\[
\mathbb{E}\left[\frac{1}{2}(r^{k+1})^2\right] = \frac{(r^k)^2}{2} + \frac{\tau}{N} \langle W (T_{[N]}(x^k) - x^k), x^k - x^* \rangle + \frac{\tau}{2N} \|T_{[N]}(x^k) - x^k\|_W^2.
\]

Through this inequality and (4.10) we arrive at:

\[
\mathbb{E}\left[\frac{1}{2}(r^{k+1})^2\right] \leq \frac{(r^k)^2}{2} + \frac{\tau}{N} F^* + \frac{N - \tau}{N} F(x^k) - \mathbb{E}[F(x^{k+1})].
\]

After some rearranging of terms we obtain the following inequality:

\[
\mathbb{E}\left[\frac{1}{2}(r^{k+1})^2 + F(x^{k+1}) - F^*\right] \leq \left(\frac{(r^0)^2}{2} + F(x^0) - F^*\right) - \frac{\tau}{N} (F(x^k) - F^*).
\]

By applying this inequality repeatedly, taking expectation over \( \eta^k \) and from the fact that \( \mathbb{E}[F(x^k)] \) is decreasing from (4.8), we obtain the following:

\[
\mathbb{E}[F(x^{k+1})] - F^* \leq \mathbb{E}\left[\frac{1}{2}(r^{k+1})^2 + F(x^{k+1}) - F^*\right] \leq \frac{(r^0)^2}{2} + F(x^0) - F^* - \frac{\tau}{N} \sum_{j=0}^{k} (\mathbb{E}[F(x^j)] - F^*) \leq \frac{(r^0)^2}{2} + F(x^0) - F^* - \frac{\tau(k + 1)}{N} (\mathbb{E}[F(x^{k+1})] - F^*).
\]

By rearranging some items and since \( (r^0)^2 \leq (R_W(x^0))^2 \), we arrive at (4.7). We notice that given the choice of \( \tau \) we get different results (see Section (7) for a detailed analysis). We also notice that the convergence rate depends on the choice of \( \tau = |J| \), so that if the algorithm is implemented on a multi-core machine or cluster, then \( \tau \) reflects the available number of cores.

Now, given a suboptimality level \( \epsilon \) and a confidence level \( 0 < \rho < 1 \), we can establish a total number of iterations \( k^\rho \) which will ensure an \( \epsilon \)-suboptimal solution with probability at least \( 1 - \rho \).
Corollary 4.3. Under Assumption 1 and with $R_W(x^0)$ defined in (3.13) bounded, consider a suboptimality level $\epsilon$ and a probability level $\rho$. Then, for the iterates generated by algorithm (P-RCD) and a $k_\rho^\epsilon$ that satisfies

\begin{equation}
 k_\rho^\epsilon \geq \frac{c}{\epsilon} \left( 1 + \log \left( \frac{N (R_W(x^0))^2 + 2 (F(x^0) - F^*)}{4c\rho} \right) \right) + 2 - N,
\end{equation}

with $c = \frac{2N}{\tau} \max \{(R_W(x^0))^2, F(x^0) - F^*\}$, we get the following result in probability:

\begin{equation}
 P \left( F \left( x^{k_\rho^\epsilon} \right) - F^* \leq \epsilon \right) \geq 1 - \rho.
\end{equation}

Proof. By denoting $\delta^k = F(x^k) - F^*$, from (4.8) we have that $\delta^{k+1} \leq \delta^k$ and then it can be proven that $\delta^k$ satisfies (see e.g. [18]):

\[ E[\delta^{k+1}] \leq \left( 1 - \frac{\delta^k}{c} \right) \delta^k, \]

with $c$ defined above. From this and (4.7), we can choose $\epsilon \leq \delta^0$ and following the proof of [16, Theorem 2], then for $k_\rho^\epsilon$ defined in (4.11) we obtain (4.12).

5. Linear convergence for error bound convex minimization. In this section we prove that, for certain minimization problems, the sublinear convergence rate of (P-RCD) from the previous section can be improved to a linear convergence rate. In particular, we prove that under additional assumptions on the objective function, which are often satisfied in practical applications (e.g. the dual of a linearly constrained smooth convex problem or control problem), we have a generalized error bound property for our problem. In these settings we analyze the convergence behavior of algorithm (P-RCD) for which we are able to provide global linear convergence rate, as opposed to the results in [8, 17] where only local linear convergence was derived for deterministic descent methods or the results in [31] where global linear convergence is proved for a gradient method but applied only to problems where $\Psi$ is the set indicator function of a polyhedron.

5.1. Linear convergence in the strongly convex case. We assume that the function $f$ is additionally strongly convex in the norm $\| \cdot \|_W$ with a constant $\sigma_W$, i.e.:

\begin{equation}
 f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\sigma_W}{2} \|y - x\|^2_W.
\end{equation}

and since the function $f$ has Lipschitz continuous gradient in the the norm $\| \cdot \|_W$ with constant 1, then it automatically holds that $\sigma_W \leq 1$. Since $F$ is strongly convex function, it follows that optimization problem (2.1) has a unique optimal point $x^*$. The following theorem provides the convergence rate of algorithm (P-RCD) when $f$ satisfies (5.1) and its proof follows the lines of the proof in [14].

Theorem 5.1. For optimization problem (2.1) with the objective function satisfying Assumption 1 and the strong convexity property (5.1), the algorithm (P-RCD) has global linear convergence rate for the expected values of the objective function:

\begin{equation}
 E \left[ F(x^k) - F^* \right] \leq \left( \theta_{sc} \right)^k (F(x^0) - F^*) \quad \forall k \geq 0,
\end{equation}

where $\theta_{sc} = \tau \sigma_W / N < 1$.
Proof. If we subtract $F^*$ from both sides of inequality (5.4) we have that:

$$
E[F(x^{k+1})] - F^* \leq \left(1 - \frac{T}{N}\right) \left(F(x^k) - F^*\right) + \frac{T}{N} \left(t_{[N]}(x^k, T_{[N]}(x^k)) - F^*\right).
$$

From the definition of $t_{[N]}(x, y)$ in (5.1), of $T_{[N]}(x)$ in (3.2) and from the strong convexity (5.1) we have that:

$$
t_{[N]}(x^k, T_{[N]}(x^k)) = \min_{y \in \mathbb{R}^n} f(x^k) + \langle \nabla f(x^k), y - x^k \rangle + \frac{1}{2} \|y - x^k\|^2_W + \Psi(y)
\leq \min_{y \in \mathbb{R}^n} F(y) + \frac{(1 - \sigma_W)}{2} \|y - x^k\|^2_W.
$$

We now consider $y$ of the form $\alpha x^* + (1 - \alpha)x^k$ for $\alpha \in [0, 1]$ and since the functions above are convex, then we have that:

$$
t_{[N]}(x^k, T_{[N]}(x^k)) \leq \min_{\alpha \in [0, 1]} F(\alpha x^* + (1 - \alpha)x^k) + \frac{\alpha^2(1 - \sigma_W)}{2} \|x^k - x^*\|^2_W.
$$

Now, since $f$ is strongly convex, it satisfies (5.1) and also the following inequality for any $x, y$ and $\alpha \in [0, 1]$ (see e.g. [22]):

$$
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) - \frac{\alpha(1 - \alpha)\sigma_W}{2} \|x - y\|^2_W.
$$

In this inequality, if we take $x = x^*$, $y = x^k$, and considering that $\Psi$ is also convex, we get that:

$$
F(\alpha x^* + (1 - \alpha)x^k) \leq \alpha F(x^*) + (1 - \alpha)F(x^k) - \frac{\alpha(1 - \alpha)\sigma_W}{2} \|x^k - x^*\|^2_W.
$$

By replacing this inequality in (5.4) we get the following:

$$
t_{[N]}(x^k, T_{[N]}(x^k)) \leq \min_{\alpha \in [0, 1]} \alpha F(x^*) + (1 - \alpha)F(x^k) + \frac{\alpha(1 - \alpha)\sigma_W}{2} \|x^k - x^*\|^2_W.
$$

Now, we can choose a feasible $\alpha_0 = \sigma_W < 1$, and therefore we get:

$$
t_{[N]}(x^k, T_{[N]}(x^k)) \leq \sigma_W F(x^*) + (1 - \sigma_W)F(x^k).
$$

From this and (5.3) we obtain:

$$
E[F(x^{k+1})] - F^* \leq \left(1 - \frac{T}{N}\right) \left(F(x^k) - F^*\right) + \frac{T}{N} \left(\sigma_W F(x^k) - F^*\right)
= \left(1 - \frac{T\sigma_W}{N}\right) \left(F(x^k) - F^*\right) = \theta_{sc} \left(F(x^k) - F^*\right),
$$

where we note that $\theta_{sc} = \tau \sigma_W / N < 1$. Now if we denote $\delta^k = F(x^k) - F(\bar{x}^*)$, then by taking expectation over $\eta^{k-1}$ in (5.5) we arrive at:

$$
E[\delta^k] \leq \theta_{sc} E[\delta^{k-1}] \leq \cdots \leq (\theta_{sc})^k E[\delta^0],
$$

and linear convergence is proved. \(\square\)
5.2. Linear convergence in the generalized error bound property case.

We introduce the proximal gradient mapping of function \( F(x) \):

\[(5.6) \quad \nabla^+ F(x) = x - \text{prox}_x \left( x - W^{-1} \nabla f(x) \right) .\]

Clearly, a point \( x^* \) is an optimal solution of problem (2.1) if and only if \( \nabla^+ F(x^*) = 0 \). In the next definition we introduce the Generalized Error Bounded Property (GEBP):

**Definition 5.2.** Problem (2.1) has the generalized error bound property w.r.t. the norm \( \| \cdot \|_W \) if it satisfies the following relation:

\[(5.7) \quad \| x - \bar{x} \|_W \leq (\kappa_1 + \kappa_2 \| x - \bar{x} \|^2_W) \| \nabla^+ F(x) \|_W \quad \forall x \in \mathbb{R}^n, \]

where \( \kappa_1 \) and \( \kappa_2 \) are two nonnegative constants and \( \bar{x} = \Pi^W_{X^*}(x) \).

Note that the class of functions introduced in (5.7) includes other known categories of functions. For example, functions \( F \) composed of a strongly convex function \( f \) with a convex constant \( \sigma_W \) w.r.t. the norm \( \| \cdot \|_W \) and a general convex function \( \Psi \) satisfy our definition (5.7) with \( \kappa_1 = \frac{\sigma_W}{\gamma_W} \) and \( \kappa_2 = 0 \), see Section 6 for more details.

Next, we prove that on optimization problems having the (GEBP) property (5.7) our algorithm (P-RCD) still has global linear convergence. Our analysis will employ ideas from the convergence proof of deterministic descent methods in [8]. However, the random nature of our method and the the nonsmooth property of the objective function requires a new approach. For example, the typical proof for linear convergence of gradient descent type methods for solving convex problems with an error bound like condition requires a new approach. For example, the typical proof for linear convergence of deterministic descent methods in [8] includes other known categories of functions. For example, functions \( F \) composed of a strongly convex function \( f \) with a convex constant \( \sigma_W \) w.r.t. the norm \( \| \cdot \|_W \) and a general convex function \( \Psi \) satisfy our definition (5.7) with \( \kappa_1 = \frac{\sigma_W}{\gamma_W} \) and \( \kappa_2 = 0 \), see Section 6 for more details.

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**Lemma 5.3.** If our problem (2.1) satisfies (GEBP) given in (5.7), then a point \( x^k \) generated by algorithm (P-RCD) and its projection onto \( X^* \), denoted \( \bar{x}^k \), satisfy the following:

\[(5.8) \quad \| x^k - \bar{x}^k \|_W \leq (\kappa_1 + \kappa_2 \| x^k - \bar{x}^k \|^2_W) \frac{N^2}{\tau} \mathbb{E} \left[ \| x^{k+1} - x^k \|^2_W \right] . \]

**Proof.** For the iteration defined by algorithm (P-RCD) we have:

\[
\mathbb{E} \left[ \| x^{k+1} - x^k \|^2_W \right] = \mathbb{E} \left[ \| x^k + T_{J^k}(x^k) - x^k \|^2_W \right] = \mathbb{E} \left[ \| x^k - T_{J^k}(x^k) \|^2_W \right] = \frac{\tau}{N} \| x^k - T_{J^k}(x^k) \|^2_W = \frac{\tau}{N} \| \nabla F(x^k) \|^2_W .
\]

Through this equality and (5.7) we have that:

\[(5.9) \quad \| x^k - \bar{x}^k \|_W \leq (\kappa_1 + \kappa_2 \| x^k - \bar{x}^k \|^2_W) \| \nabla^+ F(x^k) \|_W \leq (\kappa_1 + \kappa_2 \| x^k - \bar{x}^k \|^2_W) \frac{N}{\tau} \mathbb{E} \left[ \| x^{k+1} - x^k \|^2_W \right] , \]
and the proof is complete. \[\square\]

**Remark 2.** Note that if the iterates of an algorithm satisfy the following relation:
\[
\|x^k - x^*\| \leq \|x^0 - x^*\| \quad \forall k \geq 1,
\]

see e.g. the case of the full gradient method \([22]\), then we have:
\[
(5.10) \quad \|x^k - \bar{x}^k\|_W^2 \leq \frac{\bar{\kappa}(x^0)N}{\tau} \mathbb{E} \left[ \|x^{k+1} - x^k\|_W^2 \right] \quad \forall k \geq 0,
\]
where \(\bar{\kappa}(x^0) = (\kappa_1 + \kappa_2\|x^0 - x^*\|_W^2)^2\).

If the iterates of an algorithm satisfy \((3.13)\) with \(R_W(x^0)\) bounded, see e.g. the case of our algorithm \((P-RCD)\) which is a descent method, as proven in \((3.12)\), then \((5.10)\) is satisfied with \(\bar{\kappa}(x^0) = (\kappa_1 + \kappa_2(R_W(x^0))^2)^2\).

Let us now note that given the separability of function \(\Psi : \mathbb{R}^n \to \mathbb{R}\), then for any vector \(d \in \mathbb{R}^n\) if we consider their counterparts \(\Psi_J\) and \(d_J\) for a sampling \(J\) taken as described above the expected value \(\mathbb{E} [\Psi_J(d_J)]\) satisfies:
\[
(5.11) \quad \mathbb{E}[\Psi_J(d_J)] = \sum_{J \subseteq |N|} \left( \sum_{i \in J} \Psi_i(d_i) \right) P_J = \sum_{J \subseteq |N|} \left( \sum_{i=1}^{N} \Psi_i(d_i) \mathbb{I}_J(i) \right) P_J
\]
\[
= \sum_{i=1}^{N} \Psi_i(d_i) \sum_{J \subseteq |N| : i \in J} P_J = \sum_{i=1}^{N} p_i \Psi_i(d_i) = \frac{\tau}{N} \sum_{i=1}^{N} \Psi_i(d_i) = \frac{\tau}{N} \Psi(d).
\]
Furthermore, considering that \(\bar{x}^k \in X^*\), then from \((5.8)\) we obtain:
\[
(5.12) \quad \|x^k - \bar{x}^k\|_W \leq c_s(\tau) \sqrt{\mathbb{E} [\|x^{k+1} - x^k\|^2]},
\]
where \(c_s(\tau) = (\kappa_1 + \kappa_2(R_W(x^0))^2)^2 \sqrt{\frac{N}{\tau}}\). We now need to express \(\mathbb{E}[\Psi(x^{k+1})]\) explicitly, where \(x^{k+1}\) is generated by \((P-RCD)\). Note that \(x_{j_k}^{k+1} = x_{j_k}^k\). As a result:
\[
\mathbb{E}[\Psi(x^{k+1})] = \mathbb{E} \left[ \sum_{i \in j_k} \Psi_i \left( [T_{j_k}(x^k)]_i \right) + \sum_{i \notin j_k} \Psi_i \left( [x^k]_i \right) \right] \quad \tag{5.13}
\]
\[
= \frac{\tau}{N} \Psi \left( T_{|N|}(x^k) \right) + \frac{N - \tau}{N} \Psi(x^k).
\]
The following lemma establishes an important upper bound for \(\mathbb{E}[F(x^{k+1}) - F(x^k)]\).

**Lemma 5.4.** If function \(F\) satisfies Assumption \([12]\) and the \((GEBF)\) property defined in \((5.7)\) holds, then the iterate \(x^k\) generated by \((P-RCD)\) method has the following property:
\[
(5.14) \quad \mathbb{E} \left[ F(x^{k+1}) - F(x^k) \right] \leq \mathbb{E}[\Lambda^k] \quad \forall k \geq 0,
\]
where
\[
\Lambda^k = \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \|x^{k+1} - x^k\|_W^2 + \Psi(x^{k+1}) - \Psi(x^k).
\]
Furthermore, we have that:

\[(5.15) \quad \frac{1}{2} \| x^{k+1} - x^k \|_W^2 \leq -\Lambda^k \quad \forall k \geq 0. \]

\[\text{Proof.} \quad \text{Taking } x = x^k \text{ and } y = x^{k+1} - x^k \text{ in } (2.5) \text{ we get:} \]

\[f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \| x^{k+1} - x^k \|_W^2. \]

By adding \(\Psi(x^{k+1})\) and subtracting \(\Psi(x^k)\) in both sides of this inequality and by taking expectation in both sides we obtain (5.14). Recall the iterate update (3.10): \[x^{k+1} \in \arg \min_{y \in \mathbb{R}^n} \langle \nabla J_{x^k}, f(x^k), y - x^k \rangle + \frac{1}{2} \| y - x^k \|_W^2 + \Psi(x^k)(y). \]

Given that \(x^{k+1}\) is optimal for the problem above and if we take a vector \(y = \alpha x^{k+1} + (1 - \alpha)x^k\), with \(\alpha \in [0, 1]\), we have that:

\[\langle \nabla J_{x^k} f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \| x^{k+1} - x^k \|_W^2 + \Psi(x^{k+1}) - \Psi(x^k) \leq (1 - \alpha) \left[ \langle \nabla J_{x^k} f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \| x^{k+1} - x^k \|_W^2 + \Psi(x^{k+1}) - \Psi(x^k) \right] \leq 0. \]

If we divide this inequality by \((1 - \alpha)\) and let \(\alpha \uparrow 1\) we have that:

\[\langle \nabla J_{x^k} f(x^k), x^{k+1} - x^k \rangle + (\Psi(x^{k+1}) - \Psi(x^k)) \leq -\| x^{k+1} - x^k \|_W^2. \]

By adding \(\frac{1}{2} \| x^{k+1} - x^k \|_W^2\) in both sides of this inequality and observing that:

\[\langle \nabla J_{x^k} f(x^k), x^{k+1} - x^k \rangle = \langle \nabla f(x^k), x^{k+1} - x^k \rangle \quad \text{and} \]

\[\Psi(x^{k+1}) - \Psi(x^k) = \Psi(x^{k+1}) - \Psi(x^k), \]

we obtain (5.15). \(\blacksquare\)

Additionally, note that by applying expectation in \(J_{x^k}\) to \(\Lambda^k\) we get:

\[\mathbb{E}[\Lambda^k] = \frac{\tau}{N} \mathbb{E} \left[ \langle \nabla f(x^k), T_N(x^k) - x^k \rangle + \frac{1}{2} \mathbb{E} \left[ \| x^{k+1} - x^k \|_W^2 \right] \right] \]

\[\mathbb{E} \left[ \Psi(x^{k+1}) - \Psi(x^k) \right] \]

\[\mathbb{E}[\Lambda^k] = \frac{\tau}{N} \mathbb{E} \left[ \langle \nabla f(x^k), T_N(x^k) - x^k \rangle + \frac{1}{2} \mathbb{E} \left[ \| x^{k+1} - x^k \|_W^2 \right] \right] \]

\[\mathbb{E} \left[ \Psi(x^{k+1}) - \Psi(x^k) \right]. \]

The following theorem, which is the main result of this section, proves the linear convergence rate for the algorithm \((P-\text{RCD})\) on optimization problems having the generalized error bound property \((P-\text{RCD})\).
Theorem 5.5. On optimization problems \( (P-RCD) \) with the objective function satisfying Assumption \( \{4.1\} \) and the generalized error bound property \( \{5.7\} \), the algorithm \( (P-RCD) \) has the following global linear convergence rate for the expected values of the objective function:

\[
\mathbb{E} \left[ F(x^k) - F^* \right] \leq \theta^k (F(x^0) - F^*) \quad \forall k \geq 0,
\]

where \( \theta < 1 \) is a constant depending on \( N, \tau, \kappa_1, \kappa_2 \) and \( R_W(x^0) \).

Proof. We first need to establish an upper bound for \( \mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \). By the definition of \( F \) and its convexity we have that:

\[
F(x^{k+1}) - F(\bar{x}^k) \\
= f(x^{k+1}) - f(\bar{x}^k) + \Psi(x^{k+1}) - \Psi(\bar{x}^k) \\
\leq \langle \nabla f(x^{k+1}), x^{k+1} - \bar{x}^k \rangle + \Psi(x^{k+1}) - \Psi(\bar{x}^k) \\
= \langle \nabla f(x^{k+1}) - \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle + \langle \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle + \Psi(x^{k+1}) - \Psi(\bar{x}^k) \\
\leq \|\nabla f(x^{k+1}) - \nabla f(x^k)\|_W - \|x^{k+1} - \bar{x}^k\|_W + \langle \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle + \Psi(x^{k+1}) - \Psi(\bar{x}^k) \\
\leq \|x^{k+1} - x^k\|_W + \|x^{k+1} - \bar{x}^k\|_W + \langle \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle + \Psi(x^{k+1}) - \Psi(\bar{x}^k).
\]

By taking expectation in both sides of the previous inequality we have:

\[
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq \mathbb{E}[\|x^{k+1} - x^k\|_W^2] + \mathbb{E}[\|x^{k+1} - x^k\|_W \|x^k - \bar{x}^k\|_W] \\
\leq \mathbb{E}[\|x^{k+1} - x^k\|_W^2] + \mathbb{E}[\|x^{k+1} - x^k\|_W] \mathbb{E}[\|x^k - \bar{x}^k\|_W] \\
\leq c_\alpha(\tau) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] \\
\leq c_\alpha(\tau) \mathbb{E}[\|x^{k+1} - x^k\|_W^2],
\]

where the last step comes from Jensen’s inequality. Thus, \( \{5.18\} \) becomes:

\[
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq c_1(\tau) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] + \mathbb{E}[\langle \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle] \\
+ \mathbb{E}[\Psi(x^{k+1})] - \Psi(\bar{x}^k),
\]

where \( c_1(\tau) = (1 + c_\alpha(\tau)) \). We now explicitly express the second term in the right hand side of the above inequality:

\[
\mathbb{E}[\langle \nabla f(x^k), x^{k+1} - \bar{x}^k \rangle] \equiv \mathbb{E}[\langle \nabla f(x^k), x^k + T_{jk}(x^k) - x^k_{jk} - \bar{x}^k \rangle] \\
= \langle \nabla f(x^k), x^k - \bar{x}^k \rangle + \mathbb{E}[\langle \nabla f(x^k), T_{jk}(x^k) - x^k_{jk} \rangle] \\
\leq \langle \nabla f(x^k), x^k - \bar{x}^k \rangle + \frac{T}{N} \langle \nabla f(x^k), T_{\{N\}}(x^k) - x^k \rangle \\
= \left(1 - \frac{T}{N}\right) \langle \nabla f(x^k), x^k - \bar{x}^k \rangle + \frac{T}{N} \langle \nabla f(x^k), T_{\{N\}}(x^k) - x^k \rangle.
\]
So, by replacing it in (5.19) and through (5.22) we get:

$$
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq c_1(\tau) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] + \frac{\tau}{N} \langle \nabla f(x^k), T_{[N]}(x^k) - \bar{x}^k \rangle
$$

\begin{align*}
&+ \left( 1 - \frac{\tau}{N} \right) \langle \nabla f(x^k), x^k - \bar{x}^k \rangle + \frac{\tau}{N} \Psi(T_{[N]}(x^k)) \\
&+ \left( 1 - \frac{\tau}{N} \right) \Psi(x^k) - \Psi(\bar{x}^k) .
\end{align*}

By taking $y = \bar{x}^k$ and $x = x^k$ in (5.21) we obtain:

$$
\tag{5.20}
f(\bar{x}^k) \leq f(x^k) + \langle \nabla f(x^k), \bar{x}^k - x^k \rangle + \frac{1}{2} \|\bar{x}^k - x^k\|_W^2 .
$$

By rearranging this inequality, we obtain:

$$
\langle \nabla f(x^k), x^k - \bar{x}^k \rangle \leq f(x^k) - f(\bar{x}^k) + \frac{1}{2} \|\bar{x}^k - x^k\|_W^2 .
$$

Through this and by rearranging terms in (5.20), we obtain:

$$
\tag{5.21}
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq c_1(\tau) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] + \left( 1 - \frac{\tau}{N} \right) \left( F(x^k) - F(\bar{x}^k) \right) \\
+ \frac{1}{2} \left( 1 - \frac{\tau}{N} \right) \|x^k - \bar{x}^k\|_W^2 \\
+ \frac{\tau}{N} \left( \Psi(T_{[N]}(x^k)) + \langle \nabla f(x^k), T_{[N]}(x^k) - \bar{x}^k \rangle - \Psi(\bar{x}^k) \right) .
$$

Furthermore, from (5.12) we obtain:

$$
\tag{5.22}
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq \left( c_1(\tau) + \frac{1}{2} \left( 1 - \frac{\tau}{N} \right) c_\kappa(\tau)^2 \right) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] \\
+ \left( 1 - \frac{\tau}{N} \right) \left( F(x^k) - F(\bar{x}^k) \right) \\
+ \frac{\tau}{N} \left( \Psi(T_{[N]}(x^k)) + \langle \nabla f(x^k), T_{[N]}(x^k) - \bar{x}^k \rangle - \Psi(\bar{x}^k) \right) .
$$

Through the convexity of $\Psi(x)$ we have:

$$
\Psi(\bar{x}^k) \geq \Psi(T_{[N]}(x^k)) + \langle \partial \Psi(T_{[N]}(x^k)), \bar{x}^k - T_{[N]}(x^k) \rangle
$$

and by rearranging it we obtain:

$$
\Psi(T_{[N]}(x^k)) - \Psi(\bar{x}^k) \leq \langle \partial \Psi(T_{[N]}(x^k)), T_{[N]}(x^k) - \bar{x}^k \rangle .
$$

From this and the optimality condition (5.5) and by replacing in (5.22) we obtain:

$$
\tag{5.23}
\mathbb{E}[F(x^{k+1})] - F(\bar{x}^k) \leq \left( c_1(\tau) + \frac{1}{2} \left( 1 - \frac{\tau}{N} \right) c_\kappa(\tau)^2 \right) \mathbb{E}[\|x^{k+1} - x^k\|_W^2] \\
+ \left( 1 - \frac{\tau}{N} \right) \left( F(x^k) - F(\bar{x}^k) \right) \\
+ \frac{\tau}{N} \left( -W(T_{[N]}(x^k) - \bar{x}^k) , T_{[N]}(x^k) - \bar{x}^k \right) .
$$
Furthermore, by rearranging some terms and through the Cauchy-Schwartz inequality we obtain:

\[
\langle -W(T_{[N]}(x^k) - x^k), T_{[N]}(x^k) - \bar{x}^k \rangle = \langle -W(T_{[N]}(x^k) - x^k), T_{[N]}(x^k) - x^k + x^k - \bar{x}^k \rangle \\
\leq \langle W(T_{[N]}(x^k) - x^k), x^k - x^k \rangle \\
\leq ||W(T_{[N]}(x^k) - x^k)||_W^{-1}||\bar{x}^k - x^k||_W \\
= ||T_{[N]}(x^k) - x^k||_W||\bar{x}^k - x^k||_W.
\]

Now, recall that:

\[
E[||x^{k+1} - x^k||^2_W] = \frac{\tau}{N}||x^k - T_{[N]}(x^k)||^2_W.
\]

Thus, from this and (5.12) we get:

\[
\frac{\tau}{N}||T_{[N]}(x^k) - x^k||_W||\bar{x}^k - x^k||_W \leq c_\kappa(\tau)\sqrt{\frac{\tau}{N}E[||x^{k+1} - x^k||^2_W]}.
\]

By replacing this in (5.23) we obtain:

\[
E[F(x^{k+1}) - F(\bar{x}^k)] \leq \left( c_1(\tau) + \frac{1}{2} \left( 1 - \frac{\tau}{N} \right) c_\kappa(\tau)^2 + c_\kappa(\tau)\sqrt{\frac{\tau}{N}} \right) E[||x^{k+1} - x^k||^2_W] \\
+ \left( 1 - \frac{\tau}{N} \right) (F(x^k) - F(\bar{x}^k))
\]

(5.24)

From (5.14) we have:

\[
E[||x^{k+1} - x^k||^2_W] \leq -2E[\Lambda^k].
\]

Now, through this and by rearranging some terms in (5.24) we obtain:

\[
\frac{\tau}{N} (E[F(x^{k+1}) - F(\bar{x}^k)]) \leq -2c_2(\tau)E[\Lambda^k] + \left( 1 - \frac{\tau}{N} \right) (F(x^k) - E[F(x^{k+1})]).
\]

Furthermore, from (5.13) we obtain:

\[
E[F(x^{k+1}) - F(\bar{x}^k)] \leq \frac{N}{\tau} \left( 2c_2(\tau) + \left( 1 - \frac{\tau}{N} \right) \right) (F(x^k) - E[F(x^{k+1})])
\]

(5.25)

By rearranging this inequality, we obtain:

\[
\frac{c_3(\tau)}{1 + c_3(\tau)} (F(x^k) - F(\bar{x}^k)).
\]

We denote \( \theta = \frac{c_3(\tau)}{1 + c_3(\tau)} < 1 \) and define \( \delta^k = F(x^{k+1}) - F(\bar{x}^k) \). By taking expectation over \( \eta^{k-1} \) in (5.26) we arrive at:

\[
E[\delta^k] \leq \theta E[\delta^{k-1}] \leq \cdots \leq \theta^k E[\delta^0],
\]
and linear convergence is proved. 

Note that we have obtained global linear convergence for our distributed random coordinate descent method on the general class of problems satisfying the generalized error bound property (GEBP) given in [5, 7], as opposed to the results in [8, 17] where the authors only show local linear convergence for deterministic coordinate descent methods applied to local error bound functions, i.e. for all \( k \geq k_0 > 1 \), where \( k_0 \) is an iterate after which some error bound condition of the form \( \| x^k - \bar{x} \| \leq \bar{k} \| \nabla F(x^k) \| \) is implicitly satisfied. In [31] global linear convergence is also proved for the full gradient method but applied only to problems having the error bound property where \( \Psi \) is the set indicator function of a polyhedron. Further, our results are more general than the ones in [1, 10, 11, 12, 13, 18], where the authors prove linear convergence for the more restricted class of problems having smooth and strongly convex objective function. Moreover, our proof for convergence is different from those in these papers.

We now establish the number of iterations \( k_\rho^\epsilon \) which will ensure an \( \epsilon \)-suboptimal solution with probability at least \( 1 - \rho \). In order to do so, we first recall that for constants \( \epsilon > 0 \) and \( \gamma \in (0, 1) \) such that \( \delta_0 > \epsilon > 0 \) and \( k \geq \gamma \log \left( \frac{\delta_0}{\epsilon} \right) \) we have:

\[
(5.26) \quad (1 - \gamma)^k \delta_0 = \left( 1 - \frac{1}{1/\gamma} \right)^{(1/\gamma)^k} \delta_0 \leq \exp(-\gamma k) \delta_0 \leq \exp\left( -\log(\delta_0/\epsilon) \right) \delta_0 = \epsilon.
\]

**Corollary 5.6.** For a function \( F \) satisfying Assumptions 1 and the generalized error bound property (5.7), consider a probability level \( \rho \in (0, 1) \), suboptimality \( 0 < \theta < \delta_0 \) and an iteration counter:

\[
k_\rho^\epsilon \geq \frac{1}{1 - \theta} \log \left( \frac{\delta_0}{\epsilon \rho} \right),
\]

where recall that \( \delta_0 = F(x^0) - F^* \) and \( \theta \) is defined in Theorem 5.5. Then, we have that the iterate \( x^{k_\rho^\epsilon} \) generated by (P-RCD) method satisfies:

\[
(5.27) \quad P(F(x^{k_\rho^\epsilon}) - F^* \leq \epsilon) \geq 1 - \rho.
\]

**Proof.** Under Theorem 5.5 we have that:

\[
E \left[ \delta^{k_\rho^\epsilon} \right] \leq \theta^{k_\rho^\epsilon} E \left[ \delta^0 \right] = (1 - (1 - \theta))^{k_\rho^\epsilon} E \left[ \delta^0 \right] = (1 - (1 - \theta))^{k_\rho^\epsilon} \delta_0.
\]

Through Markov’s inequality and (5.26) we have that:

\[
P(\delta^{k_\rho^\epsilon} > \epsilon) \leq \frac{E[\delta^{k_\rho^\epsilon}]}{\epsilon} \leq \frac{(1 - (1 - \theta))^{k_\rho^\epsilon}}{\epsilon} \delta_0 \leq \rho.
\]

and the proof is complete. 

6. Conditions for generalized error bound functions. In this section we investigate under which conditions a function \( F \) satisfying Assumption 1 has the generalized error bound property defined in [5, 7] (see Definition 5.2).

6.1. Case 1: \( f \) strongly convex and \( \Psi \) convex. We first show that if \( f \) satisfies Assumption 1 and additionally is also strongly convex, while \( \Psi \) is a general convex function, then \( F \) has the generalized error bound property defined in [5, 7]. Note that a similar result was proved in [8]. For completeness, we also give the proof. We consider \( f \) to be strongly convex with constant \( \sigma_W \) w.r.t. the norm \( \| \cdot \|_W \), i.e.:

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\sigma_W}{2} \| y - x \|_W^2.
\]
If $\Psi$ is strongly convex w.r.t. the norm $\| \cdot \|_W$, with convexity parameter $\sigma^\Psi_W$, then we can redefine $f \leftarrow f + \frac{\sigma^\Psi_W}{2} \| x - x^0 \|_W^2$ and $\Psi \leftarrow \Psi - \frac{\sigma^\Psi_W}{2} \| x - x^0 \|_W^2$, so that all the above assumptions hold for this new pair of functions.

By Fermat’s rule [30] we have that $T_{[N]}(x)$ is also the solution of the following problem:

$$T_{[N]}(x) = \arg \min_y \langle \nabla f(x) + W(T_{[N]}(x) - x), y - x \rangle + \Psi(y)$$

and since $T_{[N]}(x)$ is optimal we get:

$$\langle \nabla f(x) + W(T_{[N]}(x) - x), T_{[N]}(x) - x \rangle + \Psi(T_{[N]}(x)) \leq \langle \nabla f(x) + W(T_{[N]}(x) - x), y - x \rangle + \Psi(y) \quad \forall y.$$ 

Since $f$ is strongly convex, then $X^*$ is a singleton and by taking $y = \bar{x}$ we obtain:

$$\langle \nabla f(x) + W(T_{[N]}(x) - x), T_{[N]}(x) - x \rangle + \Psi(T_{[N]}(x)) \leq \langle \nabla f(x) + W(T_{[N]}(x) - x), \bar{x} - x \rangle + \Psi(\bar{x}).$$

On the other hand from the optimality conditions for $\bar{x}$ and convexity of $\Psi$ we get:

$$\Psi(\bar{x}) + \langle \nabla f(\bar{x}), \bar{x} \rangle \leq \Psi(T_{[N]}(x)) + \langle \nabla f(\bar{x}), T_{[N]}(x) \rangle.$$ 

By adding up the above two inequalities we obtain:

$$\| T_{[N]}(x) - x \|^2_W + \langle \nabla f(\bar{x}) - \nabla f(x), x - \bar{x} \rangle \leq \langle \nabla f(\bar{x}) - \nabla f(x), T_{[N]}(x) - x \rangle + \langle W(T_{[N]}(x) - x), \bar{x} - x \rangle.$$

Now, from strong convexity [6.1] and Lipschitz continuity [2.7] we get:

$$\| T_{[N]}(x) - x \|^2_W + \sigma_W \| x - \bar{x} \|^2_W \leq 2 \| x - \bar{x} \|_W \| T_{[N]}(x) - x \|_W.$$

Dividing now both sides of this inequality by $\| x - \bar{x} \|_W$, we obtain:

$$\| x - \bar{x} \|_W \leq \frac{2}{\sigma_W} \| \nabla F(x) \|_W,$$

i.e. $\kappa_1 = \frac{2}{\sigma_W}$ and $\kappa_2 = 0$ in the Definition 5.2 of generalized error bound functions.

6.2. Case 2: $\Psi$ indicator function of a polyhedral set. Another important category of problems [2.1] that we consider has the following objective function:

$$\min_{x \in \mathbb{R}^n} F(x) = \tilde{f}(Px) + c^T x + I_X(x),$$

where $f(x) = \tilde{f}(Px) + c^T x$ is a smooth convex function, $P \in \mathbb{R}^{p \times n}$ is a constant matrix upon which we make no assumptions and $\Psi(x) = I_X(x)$ is the indicator function of the polyhedral set $X$. Note that an objective function $F$ with the structure in the form (6.2) appears in many applications, see e.g. the dual problem [2.10] obtained from the primal formulation [2.9] given in Section 2.1. Now, for proving the generalized error bound property, we require that $f$ satisfies the following assumption:

Assumption 2. We consider that $f(x) = \tilde{f}(Px) + c^T x$ satisfies Assumption 4.

We also assume that $f(z)$ is strongly convex in $z$ with a constant $\sigma$ and the set of optimal solutions $X^*$ for problem (2.1) is bounded.
For problem (6.2), functions \( f \) under which the set \( X^* \) is bounded include e.g. continuously differentiable coercive functions \([32]\). Also, if (6.2) is a dual formulation of a primal problem (2.9) for which the Slater condition holds, then by Theorem 1 of \([33]\) we have that the set of optimal Lagrange multipliers, i.e. \( X^* \) in this case, is compact. Also, for \( \Psi(x) = I_X(x) \) we only assume that \( X \) is a polyhedron (possibly unbounded).

Our approach for proving the generalized error bound property is in a way similar to the one in \([8, 17, 31]\). However, our results are more general in the sense that they hold globally, while in \([8, 17]\) the authors prove their results only locally and in the sense that we allow the constraints set \( X \) to be an unbounded polyhedron as opposed to the recent results in \([31]\] where the authors show an error bound like property only for bounded polyhedra or for the entire space \( \mathbb{R}^n \). This extension is very important since it allows us e.g. to tackle the dual formulation of a primal problem (2.9) in which \( X = \mathbb{R}^n_+ \) is the nonnegative orthant and which appears in many practical applications.

Last but not least important is that our error bound definition and gradient mapping introduced in this paper is more general than the one used in the standard analysis of the classical error bound property (see e.g. \([8, 17, 31]\]).

By definition, given that \( \Psi(x) \) is a set indicator function, we observe that the gradient mapping of \( F \) can be expressed in this case as:

\[
\nabla^+ F(x) = x - \Pi^W_X(x - W^{-1}\nabla f(x)),
\]

and also note that \( x^* \) is an optimal solution of (6.2) and of (2.1) if and only if \( \nabla^+ F(x^*) = 0 \).

**Lemma 6.1.** For a function \( F \) whose smooth component satisfies Assumption \([7]\) we have that

\[
(6.3) \quad \|\nabla^+ F(x) - \nabla^+ F(y)\|_W \leq 3\|x - y\|_W \quad \forall x, y \in X.
\]

**Proof.** By definition of \( \nabla^+ F(x) \) we have that:

\[
\begin{align*}
\|\nabla^+ F(x) - \nabla^+ F(y)\|_W & = \|x - y + T_{[N]}(y) - T_{[N]}(x)\|_W \\
& \leq \|x - y\|_W + \|\text{prox}_\phi(x - W^{-1}\nabla f(x)) - \text{prox}_\phi(y - W^{-1}\nabla f(y))\|_W \\
& \leq \|x - y\|_W + \|x - y + W^{-1}(\nabla f(y) - \nabla f(x))\|_W \\
& \leq 2\|x - y\|_W + \|\nabla f(x) - \nabla f(y)\|_W^{-1} \\
& \leq 3\|x - y\|_W,
\end{align*}
\]

and the proof is complete. \( \square \)

The following lemma introduces an important property for the operator \( \Pi^W_X \).

**Lemma 6.2.** Given a convex set \( X \), its projection operator \( \Pi^W_X \) satisfies:

\[
(6.4) \quad \langle W(\Pi^W_X(x) - x), \Pi^W_X(x) - y \rangle \leq 0 \quad \forall y \in X.
\]

**Proof.** Following the definition of \( \Pi^W_X \), we have that:

\[
(6.5) \quad \|x - \Pi^W_X(x)\|_W^2 \leq \|x - d\|_W^2 \quad \forall d \in X.
\]
Since $X$ is a convex set, consider a point:

$$d = \alpha y + (1 - \alpha)\Pi^W_X(x) \in X \quad \forall y \in X, \alpha \in [0, 1],$$

and by (6.3) we obtain:

$$\|x - \Pi^W_X(x)\|_W^2 \leq \|x - (\alpha y + (1 - \alpha)\Pi^W_X(x))\|_W^2.$$  

If we elaborate the squared norms in the inequality above we arrive at:

$$0 \leq \alpha \langle W(\Pi^W_X(x) - x), y - \Pi^W_X(x) \rangle + \frac{1}{2} \alpha^2 \|y - \Pi^W_X(x)\|^2.$$  

If we divide both sides by $\alpha$ and let $\alpha \downarrow 0$, we get (6.4).

The following lemma establishes an important property between $\nabla f(x)$ and $\nabla^+ F(x)$.

**Lemma 6.3.** Given a function $f$ that satisfies (2.7) and a convex set $X$, then the following inequality holds:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq 2\|\nabla^+ F(x) - \nabla^+ F(y)\|_W \|x - y\|_W \quad \forall x, y \in X.$$  

**Proof.** Denote $z = x - W^{-1}\nabla f(x)$, then by replacing $x = z$ and $y = \Pi^W_X(y - W^{-1}\nabla f(y))$ in Lemma 6.2 we obtain the following inequality:

$$\langle W(\Pi^W_X(z) - x) + \nabla f(x), \Pi^W_X(z) - \Pi^W_X(y - W^{-1}\nabla f(y)) \rangle \leq 0.$$  

Through the definition of the projected gradient mapping, this inequality can be rewritten as:

$$\langle \nabla f(x) - W\nabla^+ F(x), x - \nabla^+ F(x) - y + \nabla^+ F(y) \rangle \leq 0.$$  

If we further elaborate the inner product we obtain:

$$\langle \nabla f(x), x - y \rangle \leq \langle W\nabla^+ F(x), x - y \rangle + \langle \nabla f(x), \nabla^+ F(x) - \nabla^+ F(y) \rangle - \langle W\nabla^+ F(x), \nabla^+ F(x) - \nabla^+ F(y) \rangle.$$  

By adding two copies of (6.6) with $x$ and $y$ interchanged we have:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq \langle W(\nabla^+ F(x) - \nabla^+ F(y)), x - y \rangle + \langle \nabla f(x) - \nabla f(y), \nabla^+ F(x) - \nabla^+ F(y) \rangle - \|\nabla^+ F(x) - \nabla^+ F(y)\|_W^2.$$  

$$\leq \langle W(\nabla^+ F(x) - \nabla^+ F(y)), x - y \rangle + \langle \nabla f(x) - \nabla f(y), \nabla^+ F(x) - \nabla^+ F(y) \rangle.$$  

From this inequality, through Cauchy-Schwartz and (2.7) we arrive at:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq \|\nabla^+ F(x) - \nabla^+ F(y)\|_W \|x - y\|_W + \|\nabla f(x) - \nabla f(y)\|_W^1$$  

$$\leq 2\|\nabla^+ F(x) - \nabla^+ F(y)\|_W \|x - y\|_W,$$

and the proof is complete.

We now introduce the following lemma regarding the optimal set $X^*$, see also [17, 31].

**Lemma 6.4.** Under Assumption [2] there exists a unique $z^*$ such that:

$$P_x z^* = z^* \quad \forall x^* \in X^*,$$
and furthermore:
\[ \nabla f(x) = P^T \nabla f(z^*) + c \]
is constant for all \( x \in Q = \{ y \in X : Py = z^* \} \).

**Proof.** Given that \( f(x) \) as defined in problem (6.2) is a convex function, then for any two optimal solutions \( x_1^*, x_2^* \in X^* \) we obtain:
\[ f \left( \frac{(x_1^* + x_2^*)}{2} \right) = \frac{1}{2} (f(x_1^*) + f(x_2^*)) , \]
which by the definition of \( f \) is equivalent to:
\[ \hat{f} \left( \frac{(P_{x_1^*} + P_{x_2^*})}{2} \right) + \frac{1}{2} c^T (x_1^* + x_2^*) = \frac{1}{2} \left( \hat{f}(P_{x_1^*}) + \hat{f}(P_{x_2^*}) + c^T (x_1^* + x_2^*) \right) . \]

If we substract \( c^T (x_1^* + x_2^*) \) in both sides and by the strong convexity of \( \hat{f} \) we have that \( P_{x_1^*} = P_{x_2^*} \). Thus, \( z^* = P_{x^*} \) is unique. From this, it is straightforward to see that \( \nabla f(x) = P^T \nabla f(z^*) + c \) is constant for all \( x \in Q \). □

Consider now a point \( x \in X \) and denote by \( q = \Pi^W_X (x) \) the projection of the point \( x \) onto the set \( Q = \{ y \in X : Py = z^* \} \), as defined in Lemma 6.4 and by \( \bar{q} \) its projection onto the optimal set \( X^* \), i.e. \( \bar{q} = \Pi^W_{X^*} (q) \). Given the set \( Q \), the distance to the optimal set can be decomposed as:
\[ \| x - \bar{x} \|_W \leq \| x - \bar{q} \|_W \leq \| x - q \|_W + \| q - \bar{q} \|_W . \]

Given this inequality, the outline for proving the generalized error bound property (GEBP) from (6.7) in this case is to obtain appropriate upper bounds for \( \| x - q \|_W \) and \( \| q - \bar{q} \|_W \). In the sequel we introduce lemmas for establishing bounds for these two terms.

**Lemma 6.5.** Under Assumption 2 there exists a constant \( \gamma_1 \) such that:
\[ \| x - q \|_W \leq \gamma_1 \frac{2}{\sigma} \| \nabla^+ F(x) \|_W \| x - \bar{x} \|_W \quad \forall x \in X. \]

**Proof.** Corollary 2.2 in [34] states that if we have the following two sets of constraints:

\begin{align*}
(6.7) & \quad Ay \leq b_1, \; Py = d_1 \\
(6.8) & \quad Ay \leq b_2, \; Py = d_2,
\end{align*}

then there exists a finite constant \( \gamma_1 \) such that for a point \( y_1 \) which satisfies the first set of constraints and a point \( y_2 \) which satisfies the second one we have:
\[ \| y_1 - y_2 \|_W \leq \gamma_1 \| \Pi_{R^+} (b_1 - b_2) \|_W . \]

Furthermore, \( \gamma_1 \) is only dependent on the matrices \( A \) and \( P \) (see [34] for more details). Given that \( X \) is polyhedral, we can express it as \( X = \{ x \in \mathbb{R}^n : Ax \leq b \} \). Thus, for \( x \in X \), we can take \( b_1 = b, \; d_1 = Px \) in (6.7), and \( b_2 = b, \; d_2 = z^* \) in (6.8) such that:

\begin{align*}
(6.10) & \quad Ay \leq b, \; Py = Px \\
(6.11) & \quad Ay \leq b, \; Py = z^*.
\end{align*}
Evidently, a point \( x \in X \) is feasible for (6.10). Consider now a point \( y_2 \) feasible for (6.11). Therefore, from (6.9) there exists a constant \( \gamma_1 \) such that:

\[
\| x - y_2 \|_W \leq \gamma_1 \| Px - z^* \|_W \quad \forall x \in X.
\]

Furthermore, from the definition of \( q \) we get:

\[
\| x - q \|_W^2 \leq \| x - y_2 \|_W^2 \leq \gamma_1^2 \| Px - z^* \|_W^2 \quad \forall x \in X.
\]

From the strong convexity of \( \tilde{f}(z) \) we have the following property:

\[
\sigma \| Px - z^* \|_W^2 \leq \langle \nabla \tilde{f}(Px) - \nabla \tilde{f}(P\bar{x}), Px - P\bar{x} \rangle = \langle \nabla f(x) - \nabla f(\bar{x}), x - \bar{x} \rangle
\]

for all \( \bar{x} \in X^* \). From this inequality and Lemma 6.3 we obtain:

\[
\sigma \| Px - z^* \|_W^2 \leq 2 \| \nabla^+ F(x) - \nabla^+ F(\bar{x}) \|_W \| x - \bar{x} \|_W.
\]

Since \( \bar{x} \in X^* \), it is well known that \( \nabla^+ F(\bar{x}) = 0 \). Thus, from this and (6.12) we get:

\[
\| x - q \|_W^2 \leq \frac{2}{\sigma} \| \nabla^+ F(x) \|_W \| x - \bar{x} \|_W
\]

and the proof is complete. \( \square \)

Note that, if in (6.2) we have \( c = 0 \), then by definition we have that \( Q = X^* \), and thus the term \( \| q - \bar{q} \|_W = 0 \). In such a case, also note that \( q = \bar{x} \) and through the previous lemma, in which we established an upper bound for \( \| x - q \|_W \), we can prove outright the error bound property (5.7) with \( \kappa_1 = \gamma_2^2 \sigma \) and \( \kappa_2 = 0 \). If \( c \neq 0 \), the following two lemmas are introduced to investigate the distance between a point and a solution set of a linear programming problem and then to establish a bound for \( \| q - \bar{q} \|_W \).

**Lemma 6.6.** Consider an LP on a nonempty polyhedral set \( Y \):

\[
\min_{y \in Y} b^T y,
\]

and assume that the optimal set \( Y^* \subseteq Y \) is nonempty, convex and bounded. Let \( \bar{y} \) be the projection of a point \( y \in Y \) on the optimal set \( Y^* \). For this problem we have that:

\[
\| y - \bar{y} \|_W \leq \gamma_2 ( \| y - \bar{y} \|_W + \| b \|_W^{-1} ) \| y - \Pi^W_Z (y - W^{-1} b) \|_W \quad \forall y \in Y,
\]

where \( Z \) is any closed convex set satisfying \( Y \subseteq Z \) and \( \gamma_2 \) depends on \( Y \) and \( b \).

**Proof.** Because the solution set \( Y^* \) is nonempty, convex and bounded, then the linear program (6.13) is equivalent to the following problem:

\[
\min_{y \in Y^*} b^T y,
\]

and as a result, the linear program (6.13) is solvable. Now, by the duality theorem of linear programming, the dual problem of (6.13):

\[
\max_{\mu \in \mathbb{R}^m} l(\mu),
\]

is well defined, solvable and strong duality holds, where \( Y^* \subseteq \mathbb{R}^m \) is the dual feasible set. For any pair of primal-dual feasible points \( (y, \mu) \) for problems (6.13) and (6.15), we have a corresponding pair of optimal solutions \( (y^*, \mu^*) \). By the solvability of (6.13)
we have from Theorem 2 of [34], that there exists a constant $\gamma_2$ depending on $Y$ and $b$ such that we have the bound:

$$\left\| \frac{y - y^*}{\mu - \mu^*} \right\|_{\text{diag}(W, I_m)} \leq \gamma_2 |b^T y - l(\mu)|.$$ 

By strong duality, we have that $l(\mu^*) = b^T \bar{y}$. Thus, taking $\mu = \mu^*$ and through the optimality conditions of (6.13) we obtain:

$$\|y - y^*\|_W \leq \gamma_2 \langle b, y - \bar{y} \rangle.$$ 

From this inequality and $\|y - \bar{y}\|_W \leq \|y - y^*\|_W$ we arrive at:

(6.16)  

$$\|y - \bar{y}\|_W \leq \gamma_2 \langle b, y - \bar{y} \rangle.$$ 

By Lemma 6.2 we have that:

$$\langle W (\Pi_Z^W (y - W^{-1}b)) - (y - W^{-1}b), \Pi_Z^W (y - W^{-1}b) - \bar{y} \rangle \leq 0.$$ 

This inequality can be rewritten as:

$$\langle b, y - \bar{y} \rangle \leq \langle W (y - \Pi_Z^W (y - W^{-1}b)), y - \bar{y} + W^{-1}b + \Pi_Z^W (y - W^{-1}b) - \bar{y} \rangle$$

$$\leq \langle W (y - \Pi_Z^W (y - W^{-1}b)), y - \bar{y} + W^{-1}b \rangle$$

$$\leq \|y - \Pi_Z^W (y - W^{-1}b)\|_W (\|y - \bar{y}\|_W + \|b\|_W^{-1}) \langle b, y - \bar{y} \rangle.$$ 

From this inequality and (6.16) we obtain:

$$\|y - \bar{y}\|_W \leq \gamma_2 (\|y - \bar{y}\|_W + \|b\|_W^{-1}) (y - \Pi_Z^W (y - W^{-1}b) \|_W \text{ and the proof is complete.}$$

**Lemma 6.7.** If Assumption 2 holds for optimization problem (6.2), then there exists a constant $\gamma_2$ such that:

(6.17)  

$$\|q - \bar{q}\|_W \leq \gamma_2 (\|q - \bar{q}\|_W + \|\nabla f(\bar{x})\|_W^{-1}) \|\nabla F^+(q)\|_W \quad \forall x \in X.$$

**Proof.** By Lemma 6.3 we have that $Px = z^*$ for all $x \in Q$. As a result, the following optimization problem:

$$\min_{x \in Q} \hat{f}(z^*) + c^T x$$

has the same solution set as problem (6.2), due to the fact that $X^* \subseteq Q \subseteq X$. Since $z^*$ is a constant, then we can formulate the equivalent problem:

$$\min_{x \in Q} \nabla f(\bar{x})^T x \quad \left(= \nabla f(z^*)^T z^* + c^T x \right).$$

Note that $\nabla f(\bar{x}) = P^T \nabla \hat{f}(z^*) + c$ is constant and under Assumption 2 we have that $X^*$ is convex and bounded. Furthermore, since $\bar{x}, q \in Q$, then $\nabla f(\bar{x}) = \nabla f(q)$. Considering these details, and by taking $Y = Q$, $Z = X$, $y = q$ and $b = \nabla f(\bar{x})$ in Lemma 6.6 and applying it to the previous problem, we obtain (6.17).

The next theorem establishes the generalized error bound property for optimization problems in the form (6.2) having objective functions satisfying Assumption 2.
Theorem 6.8. Under Assumption \( \mathcal{A} \), the function \( F(x) = \hat{f}(Px) + c^T x + \mathbf{1}_X(x) \) satisfies the following global generalized error bound property:

\[
(6.18) \quad \|x - \bar{x}\|_W \leq (\kappa_1 + \kappa_2 \|x - \bar{x}\|_W^2) \|\nabla^+ F(x)\|_W \quad \forall x \in X,
\]

where \( \kappa_1 \) and \( \kappa_2 \) are two nonnegative constants.

Proof. Given that \( \bar{x} \in X^* \), it is well known that \( \nabla^+ F(\bar{x}) = 0 \) and by Lemma 6.1 we have:

\[
\|
abla^+ F(x)\|_W = \|
abla^+ F(x) - \nabla^+ F(\bar{x})\|_W \leq 3\|x - \bar{x}\|_W.
\]

From this inequality and by applying Lemma 6.1 we also have:

\[
\|
abla^+ F(q)\|_W^2 \leq (\|
abla^+ F(x)\|_W + \|
abla^+ F(q) - \nabla^+ F(x)\|_W)^2 \\
\leq 2\|
abla^+ F(x)\|_W^2 + 2\|
abla^+ F(q) - \nabla^+ F(x)\|_W^2 \\
\leq 6 (\|
abla^+ F(x)\|_W \|x - \bar{x}\|_W + 3\|q - \bar{x}\|^2).
\]

From this and Lemma 6.7, we arrive at the following:

\[
(6.19) \quad \|q - \bar{q}\|_W^2 \leq \gamma_2^2 (\|q - \bar{q}\|_W + \|\nabla F(\bar{x})\|_{W^{-1}})^2 \|
abla F^+(q)\|_W^2 \\
\leq 6\gamma_2^2 (\|q - \bar{q}\|_W + \|\nabla F(\bar{x})\|_{W^{-1}}) (\|
abla^+ F(x)\|_W \|x - \bar{x}\|_W + 3\|q - \bar{x}\|^2).
\]

Note that since \( X^* \) is a bounded set, then we can imply the following upper bound:

\[
\|
abla f(\bar{x})\|_{W^{-1}} \leq \beta = \max_{x^* \in X^*} \|
abla f(x^*)\|_{W^{-1}}.
\]

Furthermore, \( \bar{q} \in Q \) since \( X^* \subseteq Q \). From this and through the nonexpansive property of the projection operator we obtain:

\[
\|q - \bar{q}\|_W \leq \|q - \bar{x}\|_W + \|\bar{x} - \bar{q}\|_W \leq \|x - \bar{x}\|_W + \|x - q\|_W \\
\leq \|x - \bar{x}\|_W + \|\bar{x} - q\|_W \leq 3\|x - \bar{x}\|_W.
\]

From this and (6.19), we get the following bound:

\[
(6.20) \quad \|q - \bar{q}\|_W^2 \\
\leq 6\gamma_2^2 (3\|x - \bar{x}\|_W + \beta)^2 (\|
abla^+ F(x)\|_W \|x - \bar{x}\|_W + 3\|q - x\|_W^2) \\
\leq 6\gamma_2^2 (18\|x - \bar{x}\|_W^2 + 2\beta^2) (\|
abla^+ F(x)\|_W \|x - \bar{x}\|_W + 3\|q - x\|_W^2).
\]

Given the definition of \( \bar{x} \) we have that:

\[
\|x - \bar{x}\|_W^2 \leq \|x - \bar{q}\|_W^2 \leq (\|x - q\|_W + \|q - \bar{q}\|_W)^2 \leq 2\|x - q\|_W^2 + 2\|q - \bar{q}\|_W^2.
\]

From Lemma 6.2 and (6.20), we can establish an upper bound for the right hand side of the above inequality:

\[
(6.21) \quad \|x - \bar{x}\|_W \leq (\kappa_1 + \kappa_2 \|x - \bar{x}\|_W^2) \|\nabla^+ F(x)\|_W \|x - \bar{x}\|_W,
\]

where:

\[
\kappa_1 = 24\gamma_2^2 \beta^2 \left( 1 + \frac{6\gamma_2^2}{\sigma} \right) + \frac{4\gamma_1^2}{\sigma}, \\
\kappa_2 = 256\gamma_2^2 \left( 1 + \frac{6\gamma_1^2}{\sigma} \right).
\]

If we divide both sides of (6.21) by \( \|x - \bar{x}\|_W \), the proof is complete. \( \square \)
6.3. Case 3: \( \Psi \) polyhedral function. We now consider general optimization problems of the form:

\[
\min_{x \in \mathbb{R}^n} F(x) = \hat{f}(Px) + c^T x + \Psi(x),
\]

where \( \Psi(x) \) is a polyhedral function. A function \( \Psi : \mathbb{R}^n \to \mathbb{R} \) is polyhedral if its epigraph, \( \text{epi} \, \Psi = \{(x, \zeta) : \Psi(x) \leq \zeta\} \), is a polyhedral set. There are numerous functions \( \Psi \) which are polyhedral, e.g. \( \mathbb{I}_X(x) \) with \( X \) a polyhedral set, \( \|x\|_1 \), \( \|x\|_\infty \) or combinations of these functions. Note that an objective function with the structure (6.22) appears in many applications (see e.g. the constrained Lasso problem (2.8) in Section 2.1). Now, for proving the generalized error bound property, we require that \( F \) satisfies the following assumption.

**Assumption 3.** We consider that \( f(x) = \hat{f}(Px) + c^T x \) satisfies Assumption 1. Further, we assume that \( \hat{f}(z) \) is strongly convex in \( z \) with a constant \( \sigma \) and the optimal set \( X^* \) is bounded. We also assume that \( \Psi(x) \) is bounded above on its domain by a finite value \( \bar{\Psi} < \infty \), i.e. \( \Psi(x) \leq \bar{\Psi} \) for all \( x \in \text{dom} \, \Psi \), and is Lipschitz continuous w.r.t. norm \( \| \cdot \|_W \) with a constant \( L_\Psi \).

The proof of the generalized error bound property under Assumption 3 is similar to that of [8], but it requires new proof ideas and is done under different assumptions, e.g. that \( \Psi(x) \) is bounded above on its domain. Boundedness of \( \Psi \) is in practical applications usually not restrictive. Since \( \Psi(x) \leq \bar{\Psi} \) is satisfied for any \( x \in \text{dom} \, \Psi \), then problem (6.22) is equivalent to the following one:

\[
\min_{x \in \mathbb{R}^n} f(x) + \Psi(x)
\]

\[
\text{s.t.} \quad \Psi(x) \leq \bar{\Psi}.
\]

Consider now an additional variable \( \zeta \in \mathbb{R} \). Then, the previous problem is equivalent to the following problem:

\[
\min_{x \in \mathbb{R}^n, \zeta \in \mathbb{R}} f(x) + \zeta
\]

\[
\text{s.t.} \quad \Psi(x) \leq \zeta, \, \Psi(x) \leq \bar{\Psi}.
\]

Take an optimal pair \((x^*, \zeta^*)\) for problem (6.23). We now prove that \( \zeta^* = \Psi(x^*) \). Consider that \( (x^*, \zeta^*) \) is strictly feasible, i.e. \( \Psi(x^*) < \zeta^* \). Then, we can imply that \( (x^*, \Psi(x^*)) \) is feasible for (6.23) and the following inequality holds:

\[
f(x^*) + \Psi(x^*) < f(x^*) + \zeta^*,
\]

which contradicts the fact that \( (x^*, \zeta^*) \) is optimal. Thus, it remains that \( \Psi(x^*) = \zeta^* \).

The following lemma establishes an equivalence between (6.22) and another problem:

**Lemma 6.9.** Under Assumption 3, the following problem is equivalent to (6.23):

\[
\min_{x \in \mathbb{R}^n, \zeta \in \mathbb{R}} f(x) + \zeta
\]

\[
\text{s.t.} \quad \Psi(x) \leq \zeta, \, \zeta \leq \bar{\Psi}.
\]

**Proof.** The proof of this lemma consists of the following two stages: we prove that an optimal point of (6.23) is an optimal point of (6.24), and then we prove its converse. Consider now an optimal pair \((x^*, \zeta^*)\) for (6.23). Since \( (x^*, \zeta^*) \) is feasible
for (6.23), we have that \( \Psi(x^*) \leq \zeta^* \) and \( \Psi(x^*) \leq \bar{\Psi} \). Recall that \( \Psi(x^*) = \zeta^* \). Then, \( \zeta^* \leq \bar{\Psi} \) and thus \((x^*, \zeta^*)\) is feasible for (6.24). Assume now that \((x^*, \zeta^*)\) is not optimal for (6.24). Then, there exists an optimal pair \((\bar{x}^*, \bar{\zeta}^*)\) of (6.24) such that:

\[
(6.25) \quad f(\bar{x}^*) + \bar{\zeta}^* < f(x^*) + \zeta^*.
\]

Since \((\bar{x}^*, \bar{\zeta}^*)\) is feasible for (6.24), we have that \(\bar{x}^* \leq \bar{\zeta}^*\) and inherently \(\bar{\Psi}(\bar{x}^*) \leq \bar{\Psi}\). Thus, \((\bar{x}^*, \bar{\zeta}^*)\) is feasible and from (6.26) note that it is optimal for problem (6.24), which contradicts the fact that \((x^*, \zeta^*)\) is optimal for (6.24).

Consider now the converse. That is, there exists a pair \((\bar{x}^*, \bar{\zeta}^*)\) which is optimal for (6.24) and is not optimal for (6.23). Following the same lines as before, note that \((\bar{x}^*, \bar{\zeta}^*)\) is feasible for (6.23). Assume now that \((\bar{x}^*, \bar{\zeta}^*)\) is not optimal for (6.23). Then, there exists a pair \((x^*, \zeta^*)\) such that:

\[
(6.26) \quad f(x^*) + \zeta^* < f(\bar{x}^*) + \bar{\zeta}^*.
\]

Since \((x^*, \zeta^*)\) is feasible for (6.23), recall that it is also feasible for (6.24). Thus, \((x^*, \zeta^*)\) is feasible and optimal for (6.24), which contradicts the fact that \((\bar{x}^*, \bar{\zeta}^*)\) is optimal for (6.24).

Now, if we denote \(z = [x^T \zeta]^T\), then problem (6.24) can be rewritten as:

\[
(6.27) \quad \min_{z \in \mathbb{R}^{n+1}} \tilde{F}(z) = \hat{f}(\tilde{P}z) + \tilde{c}^Tz
\]

subject to \(z \in Z\),

where \(\tilde{P} = [P \ 0]\) and \(\tilde{c} = [c^T 1]^T\). The constraint set for this problem is:

\[
Z = \{z = [x^T \zeta]^T : z \in \text{epi } \Psi, \zeta \leq \bar{\Psi}\}.
\]

Recall that from Assumption 3, we have that \(\text{epi } \Psi\) is polyhedral, i.e. there exists a matrix \(C\) and a vector \(d\) such that we can express \(\text{epi } \Psi = \{(x, \zeta) : C[x^T \zeta]^T \leq d\}\). Thus, we can write the constraint set \(Z\) as:

\[
Z = \left\{ z = [x^T \zeta]^T : \left[\begin{array}{c} C \\ \epsilon_{n+1} \end{array}\right] z \leq \left[\begin{array}{c} d \\ \Psi \end{array}\right] \right\},
\]

i.e. \(Z\) is polyhedral. Denote by \(Z^*\) the set of optimal points of problem (6.24). Then, from \(X^*\) being bounded in accordance with Assumption 3 and the fact that \(\Psi(x^*) = \zeta^*\), with \(\Psi\) continuous function, it can be observed that \(Z^*\) is also bounded.

We now denote \(\bar{z} = \Pi_{Z^*}^W(z)\), where \(W = \text{diag}(W, 1)\). Since by Lemma 6.14, we have that problems (6.23) and (6.27) are equivalent, then we can apply the theory of the previous subsection to problem (6.27). That is, we can find two nonnegative constants \(\kappa_1\) and \(\kappa_2\) such that:

\[
(6.28) \quad \|z - \bar{z}\|_W \leq (\kappa_1 + \kappa_2) \|z - \bar{z}\|_{\bar{W}} \|\nabla^+ \tilde{F}(z)\|_{\bar{W}} \quad \forall z \in Z.
\]

The proximal gradient mapping in this case, \(\nabla^+ \tilde{F}(z)\) is defined as:

\[
\nabla^+ \tilde{F}(z) = z - \Pi_{Z^*}^W \left( z - \bar{W}^{-1} \nabla \tilde{F}(z) \right),
\]

where the projection operator \(\Pi_{Z^*}^W\) is defined in the same manner as \(\Pi_Z^W\). We now show that from the error bound inequality (6.28) we can derive an error bound inequality...
for problem \((6.22)\). From the definitions of \(z, \bar{z}\) and \(\bar{W}\), we derive the following lower bound for the term on the right-hand side:

\[
\|z - \bar{z}\|_{\bar{W}} = \left\| \frac{x - \bar{x}}{\zeta - \bar{\zeta}} \right\|_{\bar{W}} \geq \|x - \bar{x}\|_{\bar{W}}.
\]

(6.29)

Further, note that we can express:

\[
\|z - \bar{z}\|_{\bar{W}}^2 = \|x - \bar{x}\|_{\bar{W}}^2 + \|\zeta - \bar{\zeta}\|^2.
\]

(6.30)

Now, if \(\zeta \leq \bar{\zeta}\), then from \(\zeta = \Psi(\bar{x})\) and the Lipschitz continuity of \(\Psi\) we have that:

\[
|\zeta - \bar{\zeta}| = \bar{\zeta} - \zeta \leq \Psi(\bar{x}) - \Psi(x) \leq L_\Psi \|x - \bar{x}\|_{\bar{W}}.
\]

Otherwise, if \(\zeta > \bar{\zeta}\), we have that:

\[
|\zeta - \bar{\zeta}| = \zeta - \bar{\zeta} \leq |\bar{\Psi}| + |\bar{\zeta}| \leq |\bar{\Psi}| + |\bar{\zeta}| = \kappa_1'.
\]

From these two inequalities we derive the following inequality for \(|\zeta - \bar{\zeta}|^2\):

\[
|\zeta - \bar{\zeta}|^2 \leq (\kappa_1' + L_\Psi \|x - \bar{x}\|_{\bar{W}}) \leq 2\kappa_1'^2 + 2L_\Psi^2 \|x - \bar{x}\|_{\bar{W}}^2.
\]

Therefore, the following upper bound for \(|z - \bar{z}|^2_{\bar{W}}\) is established:

\[
|z - \bar{z}|^2_{\bar{W}} \leq 2\kappa_1'^2 + (2L_\Psi^2 + 1)\|x - \bar{x}\|_{\bar{W}}^2.
\]

(6.31)

We are now ready to present the main result of this section that shows the generalized error bound property for problems in the form \((6.22)\) under general polyhedral \(\Psi\):

**Theorem 6.10.** Under Assumption 3, the function \(F(x) = f(Px) + c^T x + \Psi(x)\) satisfies the following global generalized error bound property:

\[
\|x - \bar{x}\|_{\bar{W}} \leq (\kappa_1 + \kappa_2 \|x - \bar{x}\|_{\bar{W}}^2) \|\nabla^+ F(x)\|_{\bar{W}} \quad \forall x \in \text{dom} \Psi,
\]

where \(\kappa_1 = (\tilde{\kappa}_1 + 2\kappa_1'^2\tilde{\kappa}_2)(2L_\Psi + 1)\) and \(\kappa_2 = 2\kappa_2(2L_\Psi + 1)(2L_\Psi^2 + 1)\).

**Proof.** From the previous discussion, it remains to show that we can find an appropriate upper bound for \(||\nabla^+ \bar{F}(z)\|_{\bar{W}}\). Given a point \(z = [x^T \zeta]^T\), it can be observed that the gradient of \(\bar{F}(z)\) is:

\[
\nabla \bar{F}(z) = \begin{bmatrix} P^T \nabla \tilde{f}(Px) + c \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla f(x) \\ 1 \end{bmatrix}.
\]

Now, denote \(z^+ = \Pi_{\bar{W}}^W \left( z - \bar{W}^{-1} \nabla \bar{F}(z) \right)\). Following the definitions of the projection operator and of \(\nabla^+ \bar{F}\), note that \(z^+\) is expressed as:

\[
z^+ = \arg \min_{y \in \mathbb{R}^n, \zeta' \in \mathbb{R}} \frac{1}{2} \left\| y - \left( x - W^{-1} \nabla f(x) \right) \right\|_{\bar{W}}^2
\]

\[
s.t. \quad \Psi(y) \leq \zeta', \quad \zeta' \leq \bar{\Psi}.
\]

Furthermore, from the definition of \(||\cdot||_{\bar{W}}\), note that we can also express \(z^+\) as:

\[
z^+ = \arg \min_{y \in \mathbb{R}^n, \zeta' \in \mathbb{R}} \langle \nabla f(x), y - x \rangle + \frac{1}{2} \| y - x \|_{\bar{W}}^2 + \frac{1}{2} (\zeta' - \zeta + 1)^2
\]

\[
s.t. \quad \Psi(y) \leq \zeta', \quad \zeta' \leq \bar{\Psi}.
\]
Also, given the structure of $z$, consider that $z^+ = [\tilde{T}_{[N]}(x)^T \zeta'']^T$. Now, by a simple change of variable, we can define a pair $(\tilde{T}_{[N]}(x), \tilde{\zeta})$ as follows:

(6.33) \[ (\tilde{T}_{[N]}(x), \tilde{\zeta}) = \arg \min_{y \in \mathbb{R}^n, \zeta' \in \mathbb{R}} \langle \nabla f(x), y - x \rangle + \frac{1}{2} \|y - x\|_{W^2}^2 + \frac{1}{2} (\zeta' + 1)^2 \]

s.t. \[ \Psi(y) - \tilde{\zeta} \leq \zeta', \; \tilde{\zeta} \leq \bar{\Psi} - \tilde{\zeta}. \]

Note that $\tilde{\zeta} = \zeta'' - \zeta$ and that we can express $z^+ = [\tilde{T}_{[N]}(x)^T \tilde{\zeta} + \zeta]^T$ and:

\[ \|\nabla^+ F(z)\|_{\tilde{W}} = \|x - \tilde{T}_{[N]}(x)\|_{\tilde{W}}. \]

From (6.34) and (6.36), we can write $\nabla^+ F(x) = x - T_{[N]}(x)$ and recall that $T_{[N]}(x)$ can be expressed as:

\[ T_{[N]}(x) = \arg \min_{y \in \mathbb{R}^n} \langle \nabla f(x), y - x \rangle + \frac{1}{2} \|y - x\|_{W^2}^2 + \Psi(y) - \Psi(x). \]

Thus, we can consider that $T_{[N]}(x)$ belongs to a pair $(T_{[N]}(x), \tilde{\zeta})$ which is the optimal solution of the following problem:

(6.34) \[ (T_{[N]}(x), \tilde{\zeta}) = \arg \min_{y \in \mathbb{R}^n, \zeta' \in \mathbb{R}} \langle \nabla f(x), y - x \rangle + \frac{1}{2} \|y - x\|_{W^2}^2 + \zeta'. \]

s.t. \[ \Psi(y) - \Psi(x) \leq \zeta'. \]

Following the same reasoning as in problem (6.23), note that $\tilde{\zeta} = \Psi(T_{[N]}(x)) - \Psi(x)$. Through Fermat’s rule [30] and problem (6.34), we establish that $(T_{[N]}(x), \tilde{\zeta})$ can also be expressed as:

(6.35) \[ (T_{[N]}(x), \tilde{\zeta}) = \arg \min_{y \in \mathbb{R}^n, \zeta' \in \mathbb{R}} \langle \nabla f(x) + W(T_{[N]}(x) - x), y - x \rangle + \zeta' \]

s.t. \[ \Psi(y) - \Psi(x) \leq \zeta'. \]

Therefore, since $(T_{[N]}(x), \tilde{\zeta})$ is optimal for the problem above, we establish the following inequality:

(6.36) \[ \langle \nabla f(x) + W(T_{[N]}(x) - x), T_{[N]}(x) - x \rangle + \tilde{\zeta} \]

\[ \leq \langle \nabla f(x) + W(T_{[N]}(x) - x), \tilde{T}_{[N]}(x) - x \rangle + \tilde{\zeta}. \]

Furthermore, since the pair $(\tilde{T}_{[N]}(x), \tilde{\zeta})$ is optimal for problem (6.33), we can derive a second inequality:

(6.37) \[ \langle \nabla f(x), \tilde{T}_{[N]}(x) - x \rangle + \frac{1}{2} \|\tilde{T}_{[N]}(x) - x\|_{W^2}^2 + \frac{1}{2} (\tilde{\zeta} + 1)^2 \]

\[ \leq \langle \nabla f(x), T_{[N]}(x) - x \rangle + \frac{1}{2} \|T_{[N]}(x) - x\|_{W^2}^2 + \frac{1}{2} (\zeta + 1)^2. \]

By adding up (6.36) and (6.37), we get the following relation:

\[ \|T_{[N]}(x) - x\|_{W^2} + \frac{1}{2} \|\tilde{T}_{[N]}(x) - x\|_{W^2} + \frac{1}{2} (\tilde{\zeta} + 1)^2 \]

\[ \leq \frac{1}{2} \|T_{[N]}(x) - x\|_{W^2} + (W(T_{[N]}(x) - x), \tilde{T}_{[N]}(x) - x) + \frac{1}{2} (\tilde{\zeta} + 1)^2 + \tilde{\zeta}. \]
If we further simplify this inequality we obtain:

\[
\frac{1}{2} \| T_{[N]}(x) - x \|_W^2 + \frac{1}{2} \| \tilde{T}_{[N]}(x) - x \|_W^2 \geq (W(T_{[N]}(x) - x), \tilde{T}_{[N]}(x) - x) + \frac{1}{2} \zeta^2 \leq \frac{1}{2} \hat{\zeta}^2.
\]

Combining the first three terms in the left hand side under the norm and if we multiply both sides by 2, the inequality becomes:

\[
\left\| (T_{[N]}(x) - x) - (\tilde{T}_{[N]}(x) - x) \right\|_W^2 + \hat{\zeta}^2 \leq \zeta^2.
\]

From this, we derive the following two inequalities:

\[
\hat{\zeta}^2 \leq \zeta^2 \text{ and } \left\| (T_{[N]}(x) - x) - (\tilde{T}_{[N]}(x) - x) \right\|_W^2 \leq \hat{\zeta}^2.
\]

If we take square root in both of these inequalities, and by applying the triangle inequality to the second, we obtain:

\[
(6.38) \quad |\hat{\zeta}| \leq |\zeta| \quad \text{and} \quad \left\| \tilde{T}_{[N]}(x) - x \right\|_W - \| T_{[N]}(x) - x \|_W \leq |\hat{\zeta}|.
\]

Recall that \( \hat{\zeta} = \Psi(T_{[N]}(x)) - \Psi(x) \), and through the Lipschitz continuity of \( \Psi \), we have from the first inequality of (6.38) that:

\[
|\hat{\zeta}| \leq |\zeta| = |\Psi(T_{[N]}(x)) - \Psi(x)| \leq L_\Psi \| T_{[N]}(x) - x \|_W.
\]

Furthermore, from the second inequality of (6.38) we obtain:

\[
\left\| \tilde{T}_{[N]}(x) - x \right\|_W \leq (L_\Psi + 1)\| T_{[N]}(x) - x \|_W.
\]

From these, we arrive at the following upper bound on \( \| \nabla^+ \tilde{F}(z) \| \):

\[
(6.39) \quad \| \nabla^+ \tilde{F}(z) \| = \left\| \frac{x - \tilde{T}_{[N]}(x)}{-\hat{\zeta}} \right\|_W \leq \left\| \tilde{T}_{[N]}(x) - x \right\|_W + |\hat{\zeta}|
\]

\[
\leq (2L_\Psi + 1)\| T_{[N]}(x) - x \|_W = (2L_\Psi + 1)\| \nabla^+ F(x) \|.
\]

Finally, from (6.28), (6.31) and (6.39) we obtain the following error bound property for problem (6.22):

\[
\| x - \bar{x} \|_W \leq (\kappa_1 + \kappa_2 \| x - \bar{x} \|^2) \| \nabla^+ F(x) \|,
\]

where \( \kappa_1 = (\hat{\kappa}_1 + 2\kappa_3^2 \hat{\kappa}_2)(2L_\Psi + 1) \) and \( \kappa_2 = 2\hat{\kappa}_3(2L_\Psi + 1)(2L_\Psi + 1) \).

**6.4. Case 4: dual formulation.** Consider now the following linearly constrained convex primal problem:

\[
(6.40) \quad \min_{u \in \mathbb{R}^m} \{ g(u) : Au \leq b \},
\]

where \( A \in \mathbb{R}^{n \times m} \). In many applications however, its dual formulation is used since the dual structure of the problem is easier, see e.g. applications such as network utility maximization [22] or network control [2]. Now, for proving the generalized error bound property, we require that \( g \) satisfies the following assumption:
ASSUMPTION 4. We consider that $g$ is strongly convex (with constant $\sigma_g$) and has Lipschitz continuous gradient (with constant $L_g$) w.r.t. the Euclidean norm and there exists $\hat{u}$ such that $A\hat{u} < b$.

Denoting by $g^*$ the convex conjugate of the function $g$, then from previous assumption it follows that $g^*$ is strongly convex with constant $\frac{1}{\sigma_g}$ and has Lipschitz gradient with constant $\frac{1}{\sigma_g}$ (see e.g. [30]). Moreover, from the condition $A\hat{u} < b$ it follows using Gauvin’s theorem that the set of optimal Lagrange multipliers is compact. In conclusion, the previous primal problem is equivalent to the following dual problem:

$$\max_{x \in \mathbb{R}^n} -g^*(-A^T x) - \langle x, b \rangle - \Psi(x), \quad (6.41)$$

where $\Psi(x) = \mathbb{I}_{\mathbb{R}^n_+}(x)$ is the set indicator function for the nonnegative orthant $\mathbb{R}^n_+$. From Section 6.2, for $P = -A^T$, it follows that the dual problem (6.41) satisfies our generalized error bound property defined in (5.7) (see Definition 5.2).

7. Convergence analysis under sparsity conditions. In this section we analyze the distributed implementation and the complexity of algorithm (P-RCD) w.r.t. the sparsity measure and compare it with other complexity estimates from literature.

7.1. Parallel and distributed implementation. Nowadays, many big data applications which appear in the context of networks can be posed as problems of the form (2.1). Due to the large dimension and the separable structure of these problems, distributed optimization methods have become an appropriate tool for solving such problems. From the iteration of our algorithm (P-RCD) it follows that we can efficiently perform parallel and/or distributed computations. E.g., in the case $\tau = N = \bar{N}$, we consider that each computer $i$ owns the (block) coordinate $x_i$ and the function $f_i$ (provided that it depends on $x_i$) and store them locally. Then, our iteration is defined as follows:

$$x_i^{k+1} = \arg \min_{y_i \in \mathbb{R}^{n_i}} \langle \nabla_i f(x^k), y_i - x_i^k \rangle + \frac{1}{2} \|y_i - x_i^k\|_{W_i}^2 + \Psi_i(y_i) \quad \forall i \in [N],$$

where the diagonal block components of the matrix $W = \text{diag}(W_{ii}; i \in [N])$ have the expression:

$$W_{ii} = \sum_{j \in \mathcal{N}_i} L_{ij} I_{n_i} \quad \forall i \in [N].$$

Clearly, for updating $x_i^{k+1}$ we need to compute distributively $\nabla_i f(x^k)$ and $W_{ii}$. However, $\nabla_i f(x)$ can be computed in a distributed fashion since

$$\nabla_i f(x) = \sum_{j \in \mathcal{N}_i} \nabla_i f_j(x_{\mathcal{N}_j}),$$

i.e. node $i$ needs to collect the partial gradient $\nabla_i f_j$ from all the functions $f_j$ which depend on the variable $x_i$ (see also [1] for more details on distributed implementation of such an algorithm in the context of network control). We can argue in a similar fashion for computing $W_{ii}$. Also, for the case where $\tau \leq N$, we can employ other distributed implementations for the algorithm such as the reduce-all approach presented in [3]: if we consider that we have a machine with $\tau$ available cores, then we can distribute the information regarding the functions and block-coordinates per cores,
i.e. each core will retain information regarding a multiple number of coordinates and functions $f_j$. Then, we will require an all-reduced strategy for computing the $\nabla_i f(x)$. Further, through the norm $\| \cdot \|_W$, which is inherent in $R_W(x^0)$, convergence rates from Theorems 4.2 and 5.3 depend also on the sparsity induced by the graph via the sets $\mathcal{N}_j$ and $\mathcal{N}_i$. As it can be observed, the size of the diagonal elements $W_{ii}$ depends on the values of the Lipschitz constants $L_{\mathcal{N}_j}$, with $j \in \mathcal{N}_i$. Clearly these constants $L_{\mathcal{N}_j}$ are influenced directly by the number $|\mathcal{N}_j|$ of variables that a function $f_j$ depends on. Moreover, $W_{ii}$ depends on the number $|\mathcal{N}_i|$ of individual functions $f_j$ in which block component $x_i$ is found as an argument. For example, let us consider the dual formulation (2.10) of the primal problem (2.9). In this case we have $L_{\mathcal{N}_i} = \frac{\|A_{\mathcal{N}_j}\|^2}{\sigma_j}$. Given that the matrix block $A_{\mathcal{N}_j}$ is composed of blocks $A_{ij}$, with $l \in \mathcal{N}_j$, and from the definition of $\bar{\omega}$ we have the following inequality:

$$L_{\mathcal{N}_j} = \frac{\|A_{\mathcal{N}_j}\|^2}{\sigma_j} \leq \sum_{l \in \mathcal{N}_j} \frac{\|A_{ij}\|^2}{\sigma_j} \leq \bar{\omega} \max_{l \in \mathcal{N}_j} \frac{\|A_{ij}\|^2}{\sigma_j} \quad \forall j.$$ 

Furthermore, from this inequality and definition of $\bar{\omega}$, the diagonal terms of the matrix $W$ can be expressed as:

$$W_{ii} = \sum_{j \in \mathcal{N}_i} L_{\mathcal{N}_j} \leq \bar{\omega} \max_{j \in \mathcal{N}_j} L_{\mathcal{N}_j} \leq \bar{\omega} \max_{l \in \mathcal{N}_j, j \in \mathcal{N}_i} \frac{\|A_{ij}\|^2}{\sigma_j} \quad \forall i.$$ 

Thus, from the previous inequalities we derive the following upper bound:

$$(R_W(x^0))^2 \leq \bar{\omega} \left( \max_{l \in \mathcal{N}_j, j \in \mathcal{N}_i} \frac{\|A_{ij}\|^2}{\sigma_j} \right) (R_{I_n}(x^0))^2.$$

In conclusion, our measure of separability ($\omega, \bar{\omega}$) for the original problem (2.1) appears implicitly in the estimates on the convergence rate for our algorithm (P-RCD). On the other hand, the estimate on the convergence rate in [18] depends on the maximum number of connections which a subsystem has, i.e. only on $\omega$. This shows that our approach is more general, more flexible and thus potentially less conservative, as we will also see in the next section.

### 7.2. Comparison with other approaches

In this section we compare our convergence rates with those from other existing methods under sparsity conditions. Recall that under Assumption 1 a function $f$ satisfies the lemma given in (2.6):

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \|y - x\|_W^2,$$

where the matrix $W' = \text{diag} \left( L_i I_{n_i}; i \in [N] \right)$, with $L_i$ being the Lipschitz constants such that $f$ satisfies (2.5). In [18], under an additional separability assumption on the
function $f$, Nesterov’s descent lemma was generalized as follows:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\omega}{2} \| y - x \|^2_W,$$

where $\omega$ is defined in Section 2. In order to be able to compare the convergence rates of our method with existing convergence results we assume below that $L_{N_j}$ and $L_i$ are of the same magnitude.

**Sublinear convergence case:** Recall that the sublinear convergence rate of our algorithm (P-RCD), that holds under Assumption 1, is (see Theorem 4.2):

$$E[F(x^k)] - F^* \leq \frac{2Nc}{\tau k + N} \forall k \geq 0,$$

where $c = \max \left\{ 1/2(R_W(x^0))^2, F(x^0) - F^* \right\}$. For $\tau = 1$ we obtain a similar convergence rate to that of the random coordinate descent method in [10], i.e. of order $O(Nc/k)$, while for $\tau = N$ we get a similar convergence rate to that of the full composite gradient method of [14]. However, the distances are measured in different norms in these papers. For example, when $\tau = N$ the comparison of convergence rate in our paper and [14] is reduced to comparing the quantities $L_f R(x^0)^2$ of [14] with our amount $R_W(x^0)^2$, where $L_f$ is the Lipschitz constant of the smooth component of the objective function, i.e. of $f$, while $R(x^0)$ is defined in a similar fashion as our $R_W(x^0)$ but in the Euclidean norm, instead of the norm $\| \cdot \|_W$. Let us consider the two extreme cases. First, consider the smooth component of the objective function:

$$f(x) = \sum_{j=1}^{\tilde{N}} f_j(x_j),$$

i.e. smooth part is fully separable. Recall that we assume that each individual function $f_j$ is Lipschitz continuous with a constant $L_{N_j}$, as stated in Assumption 1. In this case, it can be easily proven that the Lipschitz constant of $f$ is $L_f = \max_{j \in [\tilde{N}]} L_{N_j}$. Thus considering the definition of the matrix $W$ in Lemma 2.1 we have that:

$$L_f \| x^0 - x^* \|^2 \geq \| x^0 - x^* \|^2_W,$$

i.e. $L_f R(x^0)^2 \leq R_W(x^0)^2$ and our convergence rate is usually better. On the other hand, if we have $f$ defined as follows:

$$f(x) = \sum_{j=1}^{\tilde{N}} f_j(x),$$

then it can be easily proven that $L_f = \sum_{j \in \tilde{N}} L_{N_j}$ and the quantities $L_f R(x^0)^2$ and $R_W(x^0)^2$ would be the same. Thus, we get better rates of convergence when $\tilde{\omega} < \tilde{N}$. Finally, we notice that our results are also similar with those of [18], but are obtained under further knowledge regarding the objective function and with a modified analysis. In particular, in [18] two algorithms are proposed: algorithms (PCDM1) and (PCDM2) which explicitly enforce monotonicity. However, in practical large scale applications algorithm (PCDM2) cannot be implemented due to the very large cost per iteration, as the authors also state in their paper. Thus, using a similar reasoning as in [10], we can argue, based on our analysis, that the expected value type of convergence rate given in (4.7) is better than the one in [18] under certain separability.
properties as described below. First, the convergence rate of the algorithm in the sublinear case, apart from essentially being of order $O\left(\frac{1}{k}\right)$, depends on the quantity $c$, i.e. implicitly on the stepsizes involved when computing the next iterate $x^{k+1}$, see (3.10). Thus, in essence, the requirement is to find the smallest values for the diagonal elements of matrix $W$ such that Lemma (2.1) is still valid. To this purpose, we consider the smooth component in (2.8) in the form $f(x) = \frac{1}{2}\|Ax - b\|^2$. For this problem, consider basic block coordinate, i.e. $n_i = 1$. Under these considerations, we observe from the table below that our stepsizes are better than those in $[18]$ as $\tau$ increases and $\bar{\bar{\omega}} \ll \omega$. Thus, for certain cases, our analysis can show improvement over the stepsizes in $[18]$ under the same sampling.

| Paper       | $W_{ii}$                                                                 |
|-------------|---------------------------------------------------------------------------|
| This paper  | $\sum_{j:i \in \mathcal{N}_i} \sum_{i=1}^n A_{ji}^2$                   |
| $[18]$      | PCDM1: $\sum_{j=1}^N \min(\omega, \tau) A_{ji}^2$ or PCDM2: $\sum_{j=1}^N \left(1 + \frac{(\omega - 1)(\tau - 1)}{\max(1, n - 1)}\right) A_{ji}^2$ |

Finally, we proceed to compare the convergence of our algorithm with the algorithms in $[18]$. Since in practical large scale applications algorithm (PCDM2) in $[18]$ cannot be implemented due to the very large cost per iteration, as the authors also state in their paper, in the sequel we consider algorithm (PCDM1) in $[18]$ under the $\tau$-uniform sampling, which is similar with our sampling strategy, and for which the authors of $[18]$ were able to derive rate of convergence. In this case, both algorithms ($\textbf{P-RCD}$) and (PCDM1) have similar costs per iteration. In this setting, (PCDM1) has the following sublinear convergence:

$$
\mathbb{E}[F(x^k)] - F^* \leq \frac{2Nc'}{\tau k + 2Nc'(F(x^0) - F^*)^{-1}} \forall k \geq 0,
$$

where we define $\beta = \min(\omega, \tau)$ and

$$
c' = \max\left\{\beta(R_W(x^0))^2, F(x^0) - F^*\right\}.
$$

$$
R_W(x^0) = \max_{x: F(x) \leq F(x^0)} \min_{x^* \in X^*} \|x - x^*\|_W.
$$

Consider that in both algorithms we have that $F(x^0) - F^*$ is the smallest term in the two maximums, i.e. $1/2(R_W(x^0))^2 \geq F(x^0) - F^*$ and $\beta(R_W(x^0))^2 \geq F(x^0) - F^*$. Let us make a comparison between the two convergence rates, (7.4) and (7.5) and note that this comparison comes down to the comparison between the norms $\|\cdot\|_W$ and the quantity $\beta\|\cdot\|_W$. From the definitions on the norms we can express:

$$
\|x\|^2_W = \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i} L_{N_j}\right) \|x_i\|^2
$$

$$
\beta\|x\|^2_W = \sum_{i=1}^N \min(\omega, \tau) L_i \|x_i\|^2.
$$

In conclusion, the sublinear convergence rate of algorithm ($\textbf{P-RCD}$) is improved under our assumptions, for similarly sized Lipschitz constants and for $\bar{\bar{\omega}} \ll \omega \leq \tau$, i.e. for problems where we have at least a function $f_j$ that depends on a large
number of variables (i.e. $\omega$ relatively large) but each variable does not appear in many functions $f_j$ (i.e. $\tilde{\omega}$ relatively small). Note that this scenario was also considered at the beginning of the paper since coordinate gradient descent type methods for solving problem (7.7) make sense only in the case when $\tilde{\omega}$ is small, otherwise incremental type methods [24] should be considered for solving (2.1).

**Linear convergence case:** The authors of [1, 10, 11, 12, 13, 16, 18] also provide linear convergence rate for their algorithms. A straightforward comparison between the convergence rates in this paper and of those in the papers mentioned above cannot be done, due to the fact the linear convergence in all these papers is proved under the more conservative assumption of strong convexity, while the convergence rate of our algorithm (P-RCD) is obtained under the more relaxed assumption of generalized error bound property (5.7) given in Definition 5.2. However, we can also consider $f$ to be strongly convex with a constant $\sigma_w$ r.w. the norm $\| \cdot \|_W$. From Section 6.1 it follows that strongly convex functions are included in our class of generalized error bound functions (5.7) with $\kappa_1 = \frac{2}{\sigma_w}$ and $\kappa_2 = 0$. In this case we can easily prove that we have the following linear convergence of (P-RCD):

$$
\mathbb{E} \left[ F(x^{k+1}) - F^* \right] \leq (1 - \gamma_{eb}^k) \left( F(x^0) - F^* \right),
$$

where $\gamma_{eb}^k = \frac{\gamma_{sc}}{N}$. Note that from (2.8) it follows that the Lipschitz constant of the gradient is equal to 1 and then combining (2.8) with (6.1) we get $\sigma_W \leq 1$ r.w. the norm $\| \cdot \|_W$. In this case, we notice that, given the choice of $\tau$, we obtain different linear convergence results of order $O(\theta^k)$. E.g., for $\tau = 1$ we obtain a similar linear convergence rate to that of the random coordinate descent method in [11, 12, 16, 18], i.e. $\gamma_{eb} = O(\sigma_W / N)$, while for $\tau = N$ we get a similar convergence rate to that of the full composite gradient method of [14], i.e. $\gamma_{eb} = O(\sigma_W)$. Finally, if we consider $f$ to be strongly convex in the norm $\omega\| \cdot \|_W$ with a constant $\sigma_w$, and under the same sampling strategy as considered here, then for example the algorithm (PCDM1) in [18] has a convergence rate:

$$
\mathbb{E} \left[ F(x^{k+1}) - F^* \right] \leq (1 - \gamma_{sc})^k \left( F(x^0) - F^* \right),
$$

where $\gamma_{sc} = \frac{\sigma_w N}{\gamma_{sc} + \sigma_W}$. Thus, the comparison of convergence rates in this case reduces to the comparison of $\gamma_{eb}$ and $\gamma_{sc}$. Then, if $\omega$ is sufficiently small and $\omega$ is sufficiently large, we get that $\gamma_{eb} \geq \gamma_{sc}$ and the linear convergence rate (P-RCD) is an improvement over that of [18]. Thus, we found again that under some degree of sparsity (i.e. $\tilde{\omega} \ll \omega$) our results are superior to those in [18].

Our convergence results are also more general than the ones in [10, 11, 13, 16, 18], in the sense that we can show linear convergence of algorithm (P-RCD) for larger classes of problems. For example, up to our knowledge the best global convergence rate results known for gradient type methods for solving optimization problems of the form dual formulation of a linearly constrained convex problem (2.10) or general constrained lasso (2.8) are of the sublinear form $O\left( \left( \frac{1}{k} \right)^{\frac{1}{2}} \right)$ or (2.10). In this paper we show for the first time global linear convergence rate for random coordinate gradient descent methods for solving this type of problems (2.8) or (2.10). Note that for the particular case of least-square problems $\min_x \| Ax - b \|^2$ the authors in [30], using also an error bound like property, were able to show linear convergence for a random coordinate gradient descent method. Our results can be viewed as a generalization of the results from [30] to more general optimization problems. Further, our approach allows us to analyze in the same framework several methods: full gradient, serial coordinate descent and any parallel coordinate descent method in between.
8. Numerical simulations. In this section we present some preliminary numerical results on solving constrained lasso problems in the form (2.3). The individual constraint sets $X_i \subseteq \mathbb{R}^{n_i}$ are box constraints, i.e. $X_i = \{x_i : lb_i \leq x_i \leq ub_i\}$. The regularization parameters $\lambda_i$ were chosen uniform for all components, i.e. $\lambda_i = \lambda$ for all $i$. The numerical experiments were done for two instances of the regularization parameter $\lambda = 1$ and $\lambda = 10$. The numerical tests were conducted on a machine with 2 Intel(R) Xeon(R) E5410 quad core CPUs @ 2.33GHz and 8GB of RAM. The matrices $A \in \mathbb{R}^{m \times n}$ were randomly generated in Matlab and have a high degree of sparsity (i.e. the measures of partial separability $\omega, \bar{\omega} \ll n$).

In the first experiment we solve a single randomly generated constrained lasso problem with matrix $A$ of dimension $m = \bar{N} = 0.99 \times 10^6$ and $n = N = 10^6$. In this case the two measure of separability have the values: $\omega = 37$ and $\bar{\omega} = 35$. The problem was solved on $\tau = 1, 2, 4$ and 7 cores in parallel using MPI for $\lambda = 10$. From Fig. 8.1 and 8.2 we can observe that for each $\tau$ our algorithm needs almost the same number of coordinate updates $\tau k_n$ to solve the problem. On the other hand increasing the number of cores reduces substantially the number of iterations $k_n$.

Fig. 8.1: Evolution of $F(x^k) - F^*$ along coordinate updates normalized $\frac{\tau k_n}{n}$.

Fig. 8.2: Evolution of $F(x^k) - F^*$ along iterations normalized $\frac{k_n}{n}$.

Then, in Fig. 8.3 for each 10 randomly generated problems with varying sparsity ranging from $10^{-3}$ to $6 \times 10^{-3}$ and dimension $m = \bar{N} = 0.9 \times 10^4$ and $n = N = 10^4$ we plot the average number of iterations. We consider $\tau = 50$ and $\lambda = 10$. We observe that the number of iterations increases once the sparsity decreases.

In the second set of experiments, provided in Table 8.1 the dimension of matrix $A$ ranges as follows: $m = \bar{N}$ from $0.9 \times 10^4$ to $1.1 \times 10^6$ and $n = N$ from $10^4$ to $10^6$. For the resulting problem our objective function satisfies the generalized error bound property (5.7) given in Definition 5.2 and in some cases it is even strongly convex. This series of numerical tests were undertaken in order to compare the full number of iterations of the algorithm under the original assumptions considered in [18] and the ones considered for the algorithm in this paper.

In these simulations the algorithms were implemented in a centralized manner, i.e. there is no inter-core transmission of data, with the number of updates per iteration of $\tau = 100$ in each case. In both cases the algorithms were allowed to reach the same optimal value $F^*$ which is presented in the last column and was computed with the serial ($\tau = 1$) random coordinate descent method. The second and third column of the table represent the dimensions of matrix $A$. The fourth column represents the degree
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Fig. 8.3: Average number of iterations for alg. (P-RCD) for each 10 randomly generated problems with varying sparsity.

| $\lambda$ | n   | m   | sparse | $\omega$ | $\omega$ | $\tau_{k_{\text{P-RCD}}} / n$ | $\tau_{k_{\text{BFGS}}} / n$ | $F^*$  |
|-----------|-----|-----|--------|----------|----------|------------------|------------------|-------|
| $10^4$    | 0.90 x 10$^4$ | 2 x 10$^{-3}$ | 95 | 38 | 117 | 279 | 2420.167 |
|           | 0.98 x 10$^4$ | 3 x 10$^{-3}$ | 55 | 52 | 274 | 432 | 2379.622 |
|           | 0.94 x 10$^4$ | 4 x 10$^{-3}$ | 64 | 63 | 418 | 605 | 1985.261 |
|           | 1 x 10$^4$   | 4 x 10$^{-3}$ | 71 | 66 | 364 | 556 | 2472.455 |
|           | 1.03 x 10$^4$ | 4 x 10$^{-3}$ | 68 | 69 | 397 | 635 | 2307.750 |
| $10^5$    | 0.97 x 10$^5$ | 1.3 x 10$^{-3}$ | 31 | 32 | 111 | 193 | 27768.840 |
|           | 0.91 x 10$^5$ | 1.5 x 10$^{-3}$ | 34 | 36 | 128 | 238 | 25918.885 |
|           | 0.93 x 10$^5$ | 2 x 10$^{-3}$ | 43 | 41 | 167 | 285 | 25860.573 |
|           | 1 x 10$^5$   | 2 x 10$^{-3}$ | 41 | 42 | 162 | 280 | 26894.849 |
|           | 1.046 x 10$^5$ | 2 x 10$^{-3}$ | 42 | 42 | 161 | 270 | 28405.369 |
| $10^6$    | 0.98 x 10$^6$ | 1.5 x 10$^{-3}$ | 40 | 38 | 119 | 207 | 287390.02 |
|           | 0.91 x 10$^6$ | 1.7 x 10$^{-3}$ | 34 | 36 | 144 | 235 | 251255.96 |
|           | 0.99 x 10$^6$ | 2 x 10$^{-3}$ | 43 | 44 | 109 | 229 | 227031.21 |
|           | 1 x 10$^6$   | 2 x 10$^{-3}$ | 46 | 43 | 101 | 187 | 273215.09 |
|           | 1.046 x 10$^6$ | 2 x 10$^{-3}$ | 51 | 53 | 99 | 182 | 239189.71 |
| 10        | 0.98 x 10$^4$ | 2 x 10$^{-3}$ | 39 | 42 | 24 | 38 | 4884.610 |
|           | 0.96 x 10$^4$ | 3 x 10$^{-3}$ | 51 | 52 | 38 | 62 | 4762.226 |
|           | 0.92 x 10$^4$ | 4 x 10$^{-3}$ | 64 | 70 | 52 | 85 | 4477.707 |
|           | 1 x 10$^4$   | 4 x 10$^{-3}$ | 65 | 65 | 57 | 81 | 4909.406 |
|           | 1.02 x 10$^4$ | 4 x 10$^{-3}$ | 68 | 68 | 51 | 84 | 4922.320 |
| 10        | 0.92 x 10$^5$ | 1.3 x 10$^{-3}$ | 34 | 31 | 13 | 28 | 46066.411 |
|           | 0.95 x 10$^5$ | 1.5 x 10$^{-3}$ | 32 | 35 | 16 | 33 | 47770.23 |
|           | 0.91 x 10$^5$ | 2 x 10$^{-3}$ | 40 | 46 | 23 | 43 | 45550.275 |
|           | 1 x 10$^5$   | 2 x 10$^{-3}$ | 42 | 43 | 23 | 43 | 49808.196 |
|           | 1.09 x 10$^5$ | 2 x 10$^{-3}$ | 46 | 43 | 22 | 41 | 54370.699 |
| 10        | 0.9 x 10$^6$ | 1.5 x 10$^{-3}$ | 35 | 37 | 14 | 26 | 449548.04 |
|           | 0.91 x 10$^6$ | 1.7 x 10$^{-3}$ | 41 | 40 | 18 | 33 | 467529.31 |
|           | 0.97 x 10$^6$ | 2 x 10$^{-3}$ | 42 | 43 | 22 | 42 | 452739.23 |
|           | 1.1 x 10$^6$ | 2 x 10$^{-3}$ | 43 | 44 | 17 | 36 | 426963.31 |
|           | 1.3 x 10$^6$ | 2 x 10$^{-3}$ | 48 | 43 | 19 | 39 | 412936.02 |

Table 8.1: Comparison of algorithms (P-RCD) and (PCDM1) of [18].
of sparsity which dictates that the total number of nonzero elements in the matrix $A$ is less than or equal to $n \times m \times \text{sparse}$. The fifth and sixth columns denote the degrees of partial separability $\bar{\omega}$ and $\omega$, while the seventh and eighth columns represent the total number of coordinate updates normalized that the algorithms completed. As it can be observed from Table 8.1, algorithm (P-RCD) outperforms (PCDM1) of [18] even in the case where $\bar{\omega}$ and $\omega$ are of similar size or equal. Moreover, note that between the problems where $m$ is slightly larger than $n$, i.e. where the resulting objective function $F(x)$ is strongly convex, and the problems where $m$ is slightly smaller than $n$, i.e. where $F(x)$ is not strongly convex but satisfies our generalized error bound property, the number of iterations of algorithm (P-RCD) is comparable. In conclusion, given that the constrained lasso problems of the form (2.8) satisfy the generalized error bound property (5.7), the theoretical result that linear convergence of algorithm (P-RCD) is attained under the generalized error bound property is confirmed also in practice.

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