LINEAR COROTATION TORQUES IN NON-BAROTROPIC DISKS

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ABSTRACT

A fully analytic expression for the linear corotation torque to first order in eccentricity for planets in non-barotropic protoplanetary disks is derived, taking into account the effect of disk entropy gradients. This torque formula is applicable to both the co-orbital, corotation torques and the non-co-orbital, corotation torques—for planets in orbits with non-zero eccentricity—in disks where the thermal diffusivity and viscosity are sufficient to maintain the linearity of these interactions. While the co-orbital, corotation torque is important for migration of planets in Type I migration, the non-co-orbital, corotation torque plays an important role in the eccentricity evolution of giant planets that have opened gaps in the disk. The presence of an entropy gradient in the disk can significantly modify the corotation torque in both these cases.

Key words: hydrodynamics – planetary systems – planet–disk interactions – protoplanetary disks

1. INTRODUCTION

Planets are formed from protoplanetary disks surrounding young stars and are thought to migrate from their birthplaces into their final orbits. Planet–disk interactions tend to dominate the orbital evolution of a planet until the circumstellar disk dissipates after a few million years. These interactions occur mainly through the exchange of angular momentum and energy around resonances which occur at locations where the natural frequencies of the disk material can be excited by the periodic potential of the planetary perturber (Goldreich & Tremaine 1979).

Spiral density waves are launched at Lindblad resonances, located where the radial epicyclic frequency, \( \kappa \), matches the frequency of the perturbing potential felt by the disk material. Waves launched from the inner Lindblad resonances tend to drive outward migration of the planet, while waves launched from the outer Lindblad resonances tend to drive inward migration of the planet (Goldreich & Tremaine 1980). For small planets within typical protoplanetary disks, the outer Lindblad torque dominates over the inner Lindblad torque, and the net differential Lindblad torque drives planet migration inward, toward the star (Ward 1986), though wave reflection may alter the relative strength of the torques near a reflecting edge (Tsang 2011).

For circular orbits, the corotation resonance—where the perturbation frequency matches the orbital frequency of the disk material—occurs at the orbital radius of the planet. For eccentric orbits, non-co-orbital corotation resonances can occur both interior and exterior to the planet’s semi-major axis. These resonances also exchange energy and angular momentum with the planet. The sign of this exchange depends crucially on the disk parameters at the corotation point. In barotropic disks, the sign of the corotation torque depends on the gradient of the vortensity, \( \zeta \equiv \omega_z/\Sigma \), where \( \omega_z \equiv \hat{z} \cdot (\nabla \times \bm{u}) \) is the vorticity of the disk and \( \Sigma \) is the disk surface density. This can be understood in a qualitative fashion by considering the effect of the perturbing potential on collisionless particles (Goldreich & Sari 2003).

In barotropic disks the vortensity is conserved along streamlines. Material that is slightly outside the corotation point tends to be pushed inward by the planet’s potential, losing angular momentum, while the material slightly inside the corotation tends to be pushed outward, gaining angular momentum. The corotation torque depends on the relative difference between these two effects, which is determined by preserving the net vortensity at corotation and, thus, depends on the value of the vortensity gradient at the resonance.

Many previous simulations of disk torques have utilized a locally isothermal equation of state and ignore the entropy equation (see, e.g., Dunhill et al. 2013). In realistic disks, the barotropic assumption, that pressure depends only on density, does not necessarily hold. In such non-barotropic disks, vortensity is no longer conserved along streamlines and can be modified by a baroclinic term (see, e.g., Lovelace et al. 1999), which arises due to entropy gradients. This baroclinicity also modifies the corotation torque, as it changes the relative difference in the sign of the angular momentum transferred to the disk material in order to preserve the vortensity at the corotation (for a more thorough discussion see Section 2 of Tsang et al. 2014).

Previous works on the effect of non-barotropic equations of state on the corotation (Baruteau & Masset 2008; Paardekooper & Papaloizou 2008; Paardekooper et al. 2010, 2011) have primarily utilized numerical techniques to study the co-orbital corotation torques. While these works also consider the linear corotation torque, much more emphasis is appropriately placed on the role of nonlinear horseshoe torques that dominate the co-orbital region. These nonlinear horseshoe torques dominate the migration of small embedded planets, and for typical parameters, the co-orbital corotation torque was found to remain linear only for the order of a libration time (Paardekooper & Papaloizou 2009; Paardekooper et al. 2010) before becoming nonlinear, unless diffusivity and viscosity are sufficiently high. Detailed simulations have also been performed (e.g., Paardekooper & Mellema 2008; Kley & Crida 2008; Bitsch et al. 2013) including these effects for small planets in Type I migration, showing that indeed the effect of entropy gradients on the co-orbital corotation torques can halt or even reverse the migration of embedded planets.

For planets with small eccentricity, however, the non-co-orbital corotation torques are not as likely to become nonlinear, as they are located further from the planet and result from potential components that scale linearly with the eccentricity.

In the barotropic limit, Goldreich & Tremaine (1979) calculated the corotation torque for a “cold disk,” where the disk enthalpy response is negligible compared to the perturbing potential. Tanaka et al. (2002) revisited this calculation and numerically calculated the linear enthalpy response of the disk,
allowing a semi-analytic calculation of the corotation torque to be provided. Zhang & Lai (2006), in turn, derived the analytic disk enthalpy response and were able to provide a fully analytic form for the barotropic corotation torque.

For non-barotropic disks, Baruteau & Masset (2008) and Paardekooper & Papaloizou (2008) utilized a numerical evaluation of the enthalpy response, similar to Tanaka et al. (2002), to compute a semi-analytic linear corotation torque. Here, we will adopt the approach of Zhang & Lai (2006) and develop a fully analytic solution for the non-barotropic corotation torque.

Section 2 outlines the basic equations for a non-barotropic disk. Section 3 discusses the evaluation of the torque through the advective angular momentum flux. Section 4 analytically computes the entropy perturbation at corotation due to a planetary perturber, and in Section 5, this result is used to compute a fully analytic expression for the non-barotropic corotation torque. What follows is a discussion on the effects of thermal saturation in Section 6 and a calculation of the necessary thermal diffusivity needed to maintain linearity. Finally, there will be a summary of the results of this paper in Section 7.

In a companion paper to this work, Tsang et al. (2014), we discuss the effect of non-barotropic corotation torques on the eccentricity evolution for giant planets that can clear a gap in the disk. Utilizing results of the calculations in this paper, we show that stellar insolation of the gap (Varnière et al. 2006; Turner et al. 2012; Jang-Condell & Turner 2012) can result in entropy gradients and eccentricity excitation of giant planets, rather than the damping expected for a barotropic disk. We also suggest that the recently discovered “Eccentricity Valley” for low-metallicity exoplanetary systems (Dawson & Murray-Clay 2013) may be a signature of this effect.

2. BASIC EQUATIONS

We begin by limiting ourselves to the examination of two-dimensional perturbations and proceed with the vertically integrated disk variables \( \Sigma = \int \rho dz \) and \( P = \int p dz \), which are the surface density and vertically integrated pressure, respectively. We assume that the unperturbed axisymmetric disk state is given by the general \( \Sigma = \Sigma(r) \) and \( P = P(r) \) (as opposed to the strictly barotropic case \( P = P(\Sigma) \)), and the disk velocity profile is (nearly) Keplerian \( \mathbf{u} = r \Omega \hat{\mathbf{z}} \). We take the cylindrical coordinates \((r, \phi, z, t)\) to be centered at the central star.

The continuity and momentum equations are

\[
\partial_t \Sigma + \mathbf{u} \cdot \nabla \Sigma + \nabla \cdot (\Sigma \mathbf{u}) = 0, \tag{1}
\]

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\Sigma} \nabla P - \nabla \Phi, \tag{2}
\]

where \( \Phi \) is the gravitational potential, including both the star and the planetary perturber. Here we assume that the disk is pressure dominated and will work in the Cowling approximation, assuming the disk self-gravity to be negligible.

The star’s Newtonian potential is given by \( \Phi = -\frac{GM_p}{r} \) while the planet at location \( \mathbf{r}_p \) has a potential given by Goldreich & Tremaine (1980):

\[
\Phi_p = \frac{GM_p}{|\mathbf{r} - \mathbf{r}_p|} + M_p \Omega_p (r_p^2 \mathbf{r}_p / r), \tag{3}
\]

where we have defined \( M_p \) as the mass of the planet and \( M_* \) as the mass of the star. For an eccentric planetary perturber we have the semi-major axis \( a \) and eccentricity \( e \) such that \( \Omega_p = \Omega(a) \), and \( e \equiv (r_{\text{max}} - r_{\text{min}})/2a \). We take the eccentricity of the planet’s orbit \( e \ll 1 \). The radial epicyclic frequency of the planet is given by \( \kappa_p = \kappa(a) \), where in general \( \kappa(r) \equiv (2\Omega/r)(d^2r^2\Omega/dr^2) \).

The planet’s perturbing potential can be expanded in a Fourier series:

\[
\Phi_p = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} \Phi_{l,m} \cos [m\phi + (l - m)\kappa_p r], \tag{4}
\]

where to first order in \( e \) the non-zero Fourier components are given by Goldreich & Tremaine (1980) as

\[
\Phi_{l,m} = -\frac{GM_p}{2a} (2m\delta_{l,0}) \left[ \frac{1}{2} \pm \frac{m\Omega_p}{\kappa_p} + \frac{\beta d}{2 \partial \beta} b^{(l)}_{1/2}(\beta) \right], \tag{5}
\]

\[
\Phi_{m \pm 1,1} = -\frac{GM_p}{2a} e(2 - \delta_{m,0}) \left[ \frac{3}{2} - \frac{\kappa_p^2}{2\Omega_p^2} \mp \frac{\Omega_p}{\kappa_p} \delta_{m,1} \right], \tag{6}
\]

where \( \beta \equiv \Omega_p^2 a^3/(GM_p) \), \( \delta_{m,n} \) is the Kronecker delta function and

\[
b^{(l)}_{1/2}(\beta) = \frac{2}{\pi} \int_0^\infty \cos m\phi \frac{d\phi}{(1 - 2\beta \cos \phi + \beta^2)^{1/2}}, \tag{7}
\]

is the Laplace coefficient. Each of these components has a pattern frequency \( \omega_{l,m} = m\Omega_p + (l - m)\kappa_p \).

Assuming the linear perturbations have the form \( \delta \propto \exp(i\mathbf{m} \cdot \mathbf{r} - i\omega_{l,m} t) \), we can write the linear perturbations of the continuity and momentum equations:

\[
-i \dot{\omega} \delta \Sigma + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma r \delta u_r) + \frac{im}{r} \Sigma \delta u_\phi = 0, \tag{8}
\]

\[
-i \dot{\omega} \delta u_r - 2\Omega \delta u_\phi = \frac{1}{\Sigma} \frac{\partial}{\partial r} \delta P + \frac{\delta \Sigma}{\Sigma} \frac{\partial P}{\partial \Sigma} - \frac{\partial \Phi_{l,m}}{\partial r}, \tag{9}
\]

\[
-i \dot{\omega} \delta u_\phi + \frac{\kappa_p^2}{2\Omega_p^2} \delta u_r = -\frac{im}{r} \left( \frac{\delta P}{\Sigma} + \Phi_{l,m} \right), \tag{10}
\]

where \( \dot{\omega} \equiv \omega_{l,m} - m\Omega(r) \) is the perturbation frequency experienced in a frame corotating with the disk and \( \delta P \), \( \delta \Sigma \), and \( \delta \mathbf{u} \equiv \delta u_r \hat{\mathbf{r}} + \delta u_\phi \hat{\mathbf{\phi}} \) are the Eulerian perturbations of the pressure, density, and velocity, respectively. As in Tsang & Lai (2009), we examine adiabatic perturbations of non-barotropic disks, where the Lagrangian pressure and density perturbations are related by \( \Delta \Sigma = (1/c_s^2) \Delta P \) which relates the Eulerian perturbations as

\[
\delta \Sigma = \frac{1}{c_s^2} \delta P + \left( \frac{1}{c_s^2} \frac{dP}{dr} - \frac{d \Sigma}{dr} \right) \xi_r = \frac{1}{c_s^2} \delta P - \frac{\Sigma^2 N_r^2}{dP/dr} \frac{i \delta u_r}{\omega}, \tag{11}
\]

where \( c_s(r) \equiv (\partial P/\partial \Sigma)^{1/2} \) is the adiabatic sound speed and can be a general function of \( r \). \( \xi_r = i \delta u_r/\omega \) is the radial Lagrangian displacement, and \( N_r \) is the radial Brunt–Väisälä frequency defined by

\[
N_r^2 = \frac{1}{\Sigma^2} \left( \frac{dP}{dr} - \frac{1}{c_s^2} \frac{dP}{dr} - \frac{d \Sigma}{dr} \right), \tag{12}
\]

where we have corrected a typographical sign error in the definition of \( N_r^2 \) from Tsang & Lai (2009). Defining the enthalpy
perturbation \( \delta h \equiv \delta P/\Sigma \) and \( D_S \equiv k^2 - \tilde{\omega}^2 + N_r^2 \) and combining Equations (8)–(10) to eliminate \( \delta u_r \), we obtain

\[
\frac{\partial}{\partial r}(\delta h + \Phi_{l,m}) = \frac{2m\Omega}{\tilde{\omega}r}(\delta h + \Phi_{l,m}) - \frac{N_r^2}{dP/dr} \delta h - \frac{D_S}{\tilde{\omega}}i\delta u_r).
\]

(13)

Further eliminating \( \delta u_r \), we arrive at the second-order inhomogeneous differential equation

\[
\frac{\partial^2 \delta h}{\partial r^2} - \left[ \frac{\partial}{\partial r} \ln \left( \frac{D_S}{r\Sigma} \right) \right] \frac{\partial \delta h}{\partial r} - \left[ \frac{m^2}{r^2} + \frac{2m}{\tilde{\omega}r} + \frac{\tilde{\omega}}{c_s^2} \right] \frac{\partial \delta h}{\partial r} - \left[ \frac{1}{L_S^2} + \frac{\tilde{\omega}}{r\tilde{\omega}L_S} \right] \frac{\partial \Phi_{l,m}}{\partial r} + \frac{m^2 N_r^2}{r^2 \tilde{\omega}^2} \Phi_{l,m},
\]

(14)

where

\[
1 \equiv \frac{\Sigma N_r^2}{dP/dr}.
\]

(16)

is the inverse of the length scale related to the entropy variation in the disk background.

If the adiabatic index \( \gamma = \frac{c_s^2}{P} / \Sigma \) is assumed constant, then we can define the two-dimensional entropy \( S \equiv P/\Sigma \), such that

\[
N_r^2 = -\frac{1}{\gamma} \frac{dP}{dr} \ln S, \quad \text{and} \quad \frac{1}{L_S} = \frac{1}{\gamma} \frac{dS}{d\ln r}.
\]

(17)

we recover Equation (16) of Baruteau & Masset (2008). The left-hand side of the equation is the homogeneous Equation (8) from Tsang & Lai (2009). In the barotropic disk limit (\( N_r^2 \to 0 \)) Equation (13) from Goldreich & Tremaine (1979) is recovered. Equation (15) is our master equation describing the vertically integrated perturbations and response of a protoplanetary disk.

3. DISK TORQUE AND THE ADVECTIVE ANGULAR MOMENTUM FLUX

The torque acting on a disk due to the perturber can be evaluated as the time-averaged rate of change of the disk angular momentum (in the vertical direction), while the torque acting on the planet by the disk is equal and opposite to this:

\[
\Gamma_{\text{disk}} = -\Gamma_{p \to d} = -\left( \frac{dL_z}{dt} \right)_{\text{disk}},
\]

(18)

where \( \langle \ldots \rangle \) denotes the time average over a period and \( L_z \) is the total disk angular momentum in the vertical direction.

The vertical angular momentum area density \( l_z = dL_z/dA \) is given by

\[
l_z = \left( \Sigma + \delta \Sigma \right)(r \times u),
\]

\[
= r^2 \Sigma \delta \Sigma + r^2 \Sigma \delta u_r + r^2 \Sigma \delta u_\phi.
\]

(19)

The torque surface density of the planet acting on the disk is then given by the conservation of angular momentum:

\[
\gamma_{p \to d} = \left[ \frac{\partial l_z}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rl_z \delta u_r) + \frac{1}{r} \frac{\partial}{\partial \phi} [l_z (r \Omega + \delta u_\phi)] \right].
\]

(20)

We then have the total torque on the disk given by

\[
\Gamma_{p \to d} = \left( \int_0^{2\pi} d\phi \left( \frac{\partial}{\partial \phi} (rl_z) + \frac{\partial}{\partial r} (r \delta u_r) \right) \right. 
\]

\[
+ \left. \frac{\partial}{\partial \phi} [l_z (r \Omega + \delta u_\phi)] \right),
\]

(21)

Let us assume that the disk is in a steady state, relative to the orbital timescale, such that all transient perturbations have died away, and we can therefore take \( \omega \) to be real. We see that the first term in the integrand above has a zero net contribution when integrated over a period for the time average. Similarly, the last term also has no net contribution, after integration over the azimuthal angle \( \phi \) from 0 to \( 2\pi \). We thus only have a contribution from the second term, which corresponds to the angular momentum flux,

\[
\Delta F_l \equiv \left( \int_0^{2\pi} d\phi (r^2 \Sigma \delta u_\phi \delta u_r + r^3 \Omega \delta \Sigma \delta u_r + r^3 \Sigma \Omega \delta \delta u_\phi) \right)
\]

(22)

where \( \Delta \) denotes the complex part of all perturbations, \( r_+ \) and \( r_- \) are the cylindrical boundaries of the part of the disk in which we are interested, and we have kept only up to quadratic order in the perturbation. The last term above is linear in the perturbation and thus yields no contribution when integrated over \( \phi \) and averaged over the period.

Expressing the flux in terms of complex perturbations, we then have

\[
F_l = \left( \int_0^{2\pi} d\phi (r^2 \Sigma \Re \delta u_\phi \Re \delta u_r + r^3 \Omega \Re \delta \Sigma \Re \delta u_r) \right),
\]

(23)

\[
= r^2 \Sigma \Re \delta u_\phi \Re \delta u_r^* + r^3 \Omega \Re \delta \Sigma \Re \delta u_r^*,
\]

(24)

where \( \Re[z] \) is the real part of \( z \) and \( z^* \) denotes the complex conjugate of \( z \).

The second term above \( F_{mf} \) corresponds to the angular momentum transport due to total mass flux through a radius \( r \). However, the angular momentum content of this mass is not changing but is merely being transported outward. Therefore, when evaluated over the entire disk \( \Delta F_{mf} \) does not provide a contribution to the total torque on the planet. Thus, the only term that contributes to the torque of the planet is \( F_{adv} \equiv r^2 \Sigma \Re \delta u_\phi \Re \delta u_r^* \), the advective angular momentum flux (Lynden-Bell & Kalnajs 1972),

\[
F_{adv} = -\Delta F_{adv} = F_{adv} (r_+) - F_{adv} (r_-).
\]

(25)
Utilizing Equations (9)–(11), $F_{\text{adv}}$ above can be evaluated for a particular $l, m$ component, in terms of $\delta h$ and $\Phi_{l,m}$:

$$F_{\text{adv}} = \frac{m \pi \Sigma_r}{D_S} \Im [(\delta h + \Phi_{l,m}(\delta h' + \Phi_{l,m}'))] + \frac{1}{L_S} \frac{m \pi \Sigma_r}{D_S} \Im [\delta h \Phi_{l,m}^*]$$

\hspace{3cm} (26)

(Baruteau & Masset 2008), where $\Im[z]$ is the imaginary part of $z$ and $f'$ denotes $\delta f/\delta r$. In the barotropic limit, $L_S \rightarrow \infty$, and this reduces to the flux from Goldreich & Tremaine (1979).

The torques above can be evaluated through the advective fluxes, either outside the Lindblad resonances, or on either side of the corotation resonance. Note that we do not find a torque contribution from the singularity due to entropy advection in Equation (11), contrary to Baruteau & Masset (2008), who added an extraneous contribution due to the singular entropy perturbation. This was later shown to be due to nonlinear effects (thermal saturation) on the corotation torque (Paardekooper & Papaloizou 2008), which we will discuss in Section 6.

The torque expression above requires the enthalpy perturbation as a function of the forcing potential, particularly near the corotation resonance. Baruteau & Masset (2008) and Tanaka et al. (2002) approach this semi-analytically, by leaving this enthalpy perturbation (and its derivative) in the expression and computing it numerically. Here, we will adopt the approach of Zhang & Lai (2006), where we will explicitly solve the disk enthalpy response to the forcing potential analytically near the resonances to find the torque.

4. COROTATION RESONANCE

Expanding Equation (15) near the corotation resonance, where $\omega(r_c) = 0$, and keeping only singular terms and those of order $\sim (H/r)^{-2} \equiv r^2 \Omega^2/c_s^2$, where $H$ is the disk scale height, we find

$$\frac{d^2w}{dr^2} - \left[\frac{D_S}{c_s^2} + \frac{2q}{q^2\Omega^2(r - r_c)^2}\right]w = -\left[\frac{D_S}{c_s^2} + \frac{2q\Omega r}{q^2\Omega^2 r - r_c}\right] \Phi_{l,m},$$

\hspace{3cm} (27)

where $w \equiv \delta h + \Phi_{l,m}$ and $q \equiv -(d \ln \Omega/d \ln r)|_{r_c}$, and we have assumed that $m^2/r^2 \ll 1/H^2$.

Utilizing the Landau prescription to avoid the singularities by taking $z \equiv x + i\epsilon$, for some small $\epsilon > 0$ (such that $-\pi \leq \text{Arg}(z) \leq 0$) where $x \equiv \int r_c 2kdr$, and defining $k^2 \equiv D_S/c_s^2$, and $\psi \equiv k^{1/2}w$, we can further simplify

$$\frac{d^2\psi}{dz^2} + \left[\frac{1}{4} + \frac{1}{z} + \frac{1}{4} - \frac{\mu^2}{z^2}\right]\psi = -\left[\frac{1}{4} + \frac{c_s}{q^2\Omega r}\right] \frac{1}{\sqrt{k}} \Phi_{l,m},$$

\hspace{3cm} (28)

where we have assumed that $k$ does not change quickly near the corotation such that $D_S/c_s^2 \gg k^{1/2} \partial^2_r (k^{-1/2})$, and defined the important parameters

$$v \equiv \frac{c_s}{q^2\Omega r} \left(\frac{d}{dr} \ln \zeta - \frac{2}{L_S}\right)|_{r_c}$$

and

$$\mu \equiv \frac{1}{2} \left(1 - \frac{4q^2\Omega^2}{q^2\Omega^2}\right)^{1/2} |_{r_c},$$

\hspace{3cm} (29)

where $\zeta \equiv D_S/(2\Omega^2)$ evaluated at the corotation is the vortensity for the barotropic case. We recognize the homogeneous version (when $\Phi_{l,m} = 0$ of Equation (28) as the Whittaker differential equation (Abramowitz & Stegun 1965; Erdélyi et al. 1953; Olver et al. 2010), which is solved, unsurprisingly, by the Whittaker function $W_{\nu,\mu}(z)$. We can choose the two linearly independent homogenous solutions to be

$$k^{1/2}w_1 = \psi_1 \equiv W_{\nu,\mu}(z)$$

and $k^{1/2}w_2 = \psi_2 \equiv W_{-\nu,\mu}(z e^{-i\pi})$

\hspace{3cm} (30)

such that the asymptotic forms are convenient for determining the boundary conditions.\footnote{For a more thorough discussion of asymptotic expansion.} These are given as

$$w_1 \sim \begin{cases} \frac{1}{\sqrt{|\nu|}} \exp \left(-\int r_c k_{\text{eff}} dr\right) & \text{for } r \gg r_c, \\ \frac{1}{\sqrt{|\nu|}} e^{-\nu \pi} \exp \left(+ \int r_c k_{\text{eff}} dr\right) + \frac{1}{\sqrt{|\nu|}} \frac{T_1}{2} e^{-\nu \pi} \exp \left(- \int r_c k_{\text{eff}} dr\right) & \text{for } r \ll r_c, \end{cases}$$

\hspace{3cm} (31)

$$w_2 \sim \begin{cases} \frac{1}{\sqrt{|\nu|}} e^{-\nu \pi} \exp \left(+ \int r_c k_{\text{eff}} dr\right) + \frac{1}{\sqrt{|\nu|}} \frac{T_1}{2} e^{-\nu \pi} \exp \left(- \int r_c k_{\text{eff}} dr\right) & \text{for } r \gg r_c, \\ \frac{1}{\sqrt{|\nu|}} \frac{T_2}{2} e^{-\nu \pi} \exp \left(- \int r_c k_{\text{eff}} dr\right) & \text{for } r \ll r_c, \end{cases}$$

\hspace{3cm} (32)

where the Stokes multipliers (Heading 1962) are given by the $\Gamma$ function,

$$T_0 = \frac{2\pi i}{\Gamma\left(\frac{1}{2} - \mu + \nu\right) \Gamma\left(\frac{1}{2} + \mu + \nu\right)},$$

$$T_1 = \frac{2\pi i e^{\nu \pi}}{\Gamma\left(\frac{1}{2} - \mu - \nu\right) \Gamma\left(\frac{1}{2} + \mu - \nu\right)},$$

\hspace{3cm} (33)

$\sim$ denotes an asymptotic expansion, and $\gg$ and $\ll$ are here taken to mean the range of validity for an asymptotic expansion of a local solution.

We see above that the solutions $w_1$ and $w_2$ decay exponentially away from the corotation for $r \gg r_c$ and $r \ll r_c$, respectively, while both are unbounded on the opposing sides of the corotation.

The solution to the inhomogeneous Equation (28) can then be given by the method of variation of parameters to be

$$w(x) = -W_{\nu,\mu}(z) \int_x^\infty \frac{W_{-\nu,\mu}(z)}{W} \left(\frac{1}{4} - \frac{c_s}{q^2\Omega r}\right) \Phi_{l,m} dx$$

and

$$-W_{-\nu,\mu}(z) \int_x^\infty \frac{W_{\nu,\mu}(z)}{W} \left(\frac{1}{4} - \frac{c_s}{q^2\Omega r}\right) \Phi_{l,m} dx,$$

\hspace{3cm} (34)

where $W \equiv \psi_1 \psi_2^* - \psi_1^* \psi_2 = e^{-i\pi \nu}$ is the Wronskian (Olver et al. 2010), and we have taken the limits of integration such that the solution is bounded on either side of the corotation.

For the calculation of the torque, we will need to evaluate Equation (34) at $x = 0$. To do this, we can utilize the Laplace
transform identity for Whittaker functions (Erdélyi et al. 1953; Olver et al. 2010),
\[
\int_0^\infty e^{-\mu t}t^{b-1}W_{\nu,\mu}(t)dt = \frac{\Gamma\left(\frac{3}{2} + \mu + b\right) \Gamma\left(\frac{1}{2} - \mu + b\right)}{\Gamma(1 - \nu + b)}
\times 2F_1\left(\frac{1}{2} - \mu + b; 1 - \nu + b; -\frac{s}{2}\right),
\]
for \(b = 1\) and \(s = 0\), and
\[
\int_0^\infty \frac{1}{x} W_{\nu,\mu}(x)dx = \frac{\Gamma\left(\frac{3}{2} + \mu\right) \Gamma\left(\frac{1}{2} - \mu\right)}{\Gamma(1 - \nu)}
\times 2F_1\left(\frac{1}{2} - \mu, \frac{3}{2} + \nu; 1; -\frac{1}{2}\right),
\]
for \(b = 0\) and \(s = 0\), where we note that the Laplace transform (Equation (35)) is valid to use since \(1/2 - \mu > 0\) for radically stable stratified disks,\(^2\) with \(N_z^2 > 0\).

This allows us to evaluate Equation (34) at \(x = rc_\epsilon\), assuming all disk properties and the forcing potential to be roughly constant in the vicinity of the corotation,
\[
w(rg) = \frac{\Phi_{l,m}(rc_\epsilon)e^{i\pi v}}{4}
\times \left[ W_{\nu,\mu}(i\epsilon) \frac{\Gamma\left(\frac{3}{2} + \mu\right) \Gamma\left(\frac{3}{2} - \mu\right)}{\Gamma(1 - \nu)}
\times 2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \nu; 2; -\frac{1}{2}\right)
\right.
\]
\[
\times W_{-\nu,\mu}(-i\epsilon) \frac{\Gamma\left(\frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2} - \mu\right)}{\Gamma(2 - \nu)}
\times 2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \nu; 2; -\frac{1}{2}\right)
\]
\[
- \frac{c_z\Phi_{l,m}(rc_\epsilon)e^{i\pi v}}{qKL_S}
\times \left[ W_{\nu,\mu}(i\epsilon) \frac{\Gamma\left(\frac{3}{2} + \mu\right) \Gamma\left(\frac{1}{2} - \mu\right)}{\Gamma(1 - \nu)}
\times 2F_1\left(\frac{1}{2} - \mu, \frac{3}{2} + \nu; 1; -\frac{1}{2}\right)
\right]
\times \left. W_{-\nu,\mu}(-i\epsilon) \frac{\Gamma\left(\frac{1}{2} + \mu\right) \Gamma\left(\frac{3}{2} - \mu\right)}{\Gamma(2 - \nu)}
\times 2F_1\left(\frac{1}{2} - \mu, \frac{3}{2} + \nu; 1; -\frac{1}{2}\right)\right].
\]

For general small imaginary component \(\epsilon > 0\) from the Landau prescription above, we can evaluate \(W_{\pm,\nu,\mu}(\pm i\epsilon)\) using
\[\epsilon > 0\] for which formally for \(\mu \neq 1/2\) as \(|z| \to 0\); then \(W_{\nu,\mu}(z) \to 0\), however, it does so only logarithmically slowly. Thus, as we show in the Appendix, for sufficient large (but still quite small) \(\epsilon\), we can approximate
\[
W_{\nu,\mu}(i\epsilon) \simeq W_{\nu,1/2}(0) = \frac{1}{\Gamma(1 - \nu)}.
\]
which gives the value for \(w = \delta h + \Phi_{l,m}\) evaluated at the corotation,
\[
w(rc_\epsilon) \simeq \frac{\Phi_{l,m}(rc_\epsilon)e^{i\pi v}}{4}
\times \left[ F(\mu, \nu) + F(\mu, -\nu) \right]
\times \frac{c_z\Phi_{l,m}(rc_\epsilon)e^{i\pi v}}{qKL_S}
\times \left[ G(\mu, +\nu) - G(\mu, -\nu) \right],
\]
where we have utilized the well-known recurrence relation, \(z\Gamma(z) = \Gamma(1 + z)\), and reflection formula, \(\Gamma(1 - z)\Gamma(z) = \pi / \sin(\pi z)\), for the gamma function, and we have defined
\[
F(\mu, \nu) = 2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2 + v; \frac{1}{2}\right), \quad \text{and} \quad G(\mu, \nu) = 2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 1 + v; \frac{1}{2}\right).
\]
In the barotropic limit, we have \((1/2 - \mu) \sim N_z^2 \to 0\) and \(L_z^{-1} \sim N_z^2 \to 0\), and noting that \(\mathcal{G}(1/2, \pm \nu) = 1\), we can recover Equation (107) from Zhang & Lai (2006, where their \(p/2q = v\)).

5. THE NON-BAROTROPIC COROTATION TORQUE

We can arrive at an expression for the linear non-barotropic corotation torque by dividing Equation (27) by \(w\) (which we have shown in the Appendix to be non-zero at corotation for the small, but finite, imaginary component \(z = i\epsilon\) and then integrating over the singularity at corotation,
\[
\frac{1}{w(rc_\epsilon)} \frac{dw}{dr} = +i\pi \frac{2}{q} \left( \frac{d}{dr} \ln \zeta - \frac{2}{L_S} \right) + i\pi \frac{2}{qL_S} \Phi_{l,m}(rc_\epsilon),
\]
where we have evaluated across the residues assuming \(-\pi \leq \text{Arg}(r - rc_\epsilon) \leq 0\) consistent with the Landau prescription above. This gives the discontinuity at the corotation for the first derivative of \(w\),
\[
\Delta w(rc_\epsilon) = w'(rc_\epsilon) - w'(rc_\epsilon^-) = i\pi \frac{2}{q} \left( \frac{d}{dr} \ln \zeta - \frac{2}{L_S} \right) \delta h(rc_\epsilon)
\]
\[
+ i\pi \frac{2}{q} \left( \frac{d}{dr} \ln \zeta - \frac{1}{L_S} \right) \Phi_{l,m}(rc_\epsilon).
\]
Equation (26) can then be evaluated on either side of the corotation as
\[
\Delta F_{adv} = \frac{m\pi \Sigma_r}{DS} \text{Im} \left[ w(rc_\epsilon)\Delta w(rc_\epsilon) \right] + \frac{1}{L_S} \frac{m\pi \Sigma_r}{DS} \text{Im} \left[ \delta h\Phi_{l,m}^\tau \right].
\]
The second term above is clearly zero across corotation for the non-co-orbital corotation resonances \( l = m \pm 1 \), as at these distinct locations, the potential components \( \Phi_{m \pm 1, m} \) are continuous. For the co-orbital resonances \( (l = m) \), the potential components, \( \Phi_{m, m} \), diverge logarithmically at corotation; however, the inclusion of an arbitrarily small softening length will regularize the potential at corotation (Baruteau & Masset 2008), and we may take \( \Phi_{m, m} \) to be continuous and, thus, the second term above to again be zero.

Combining the above equations, we have

\[
\Gamma_{\text{disk}} = -\Delta F_{\text{adv}} = -\frac{m \pi^2 \Sigma r^2}{D S} \left\{ \frac{d}{dr} \ln \xi - \frac{2}{L_S} \right\} |w_{r, c}|^2 + \frac{1}{L_S} \Phi_{l, m} \text{Re}[w_{r, c}],
\]

where we have assumed a phase such that \( \Phi_{l, m} \) is purely real. Utilizing Equation (40), we then find an expression for the linear corotation torque on the planet due to the disk,

\[
\Gamma_{l, m}^{(\text{CR})} = -\left[ \frac{2m \pi^2 \Sigma \Phi_{l, m}^2}{(d \ln \Omega/dr) D S} \left\{ \frac{d}{dr} \ln \xi - \frac{2}{L_S} \right\} r_{c T} \right] \times \left[ \frac{\pi \left( \frac{1}{4} - \mu^2 \right) \sin \pi v}{\sin \pi \left( \frac{1}{2} - \mu \right) \pi v} 4 \left( \mathcal{F}(\mu, v) + \mathcal{F}(\mu, -v) \right) \right. \\
- \left( \frac{c_s}{q \sqrt{\frac{D S}{L_S}}} \right) r_{c T} \left. \frac{\pi}{\sin \pi \left( \frac{1}{2} - \mu \right) \pi v} \right] \left[ \mathcal{G}(\mu, v) - \mathcal{G}(\mu, -v) \right]^2 \\
- \left[ \frac{2m \pi^2 \Sigma \Phi_{l, m}^2}{(d \ln \Omega/dr) D S} \left\{ \frac{d}{dr} \ln \xi - \frac{2}{L_S} \right\} r_{c T} \right] \times \left[ \frac{\pi \left( \frac{1}{4} - \mu^2 \right) \sin \pi v}{\sin \pi \left( \frac{1}{2} - \mu \right) \pi v} 4 \left( \mathcal{F}(\mu, v) + \mathcal{F}(\mu, -v) \right) \right. \\
- \left( \frac{c_s}{q \sqrt{\frac{D S}{L_S}}} \right) r_{c T} \left. \frac{\pi}{\sin \pi \left( \frac{1}{2} - \mu \right) \pi v} \right] \left[ \mathcal{G}(\mu, v) - \mathcal{G}(\mu, -v) \right]^2.
\]

Taking the isentropic limit \( N_r^2 \rightarrow 0 \), such that \( 1/L_S \rightarrow 0 \), \( \mu \rightarrow 1/2 \), and \( \pi \left( (1/4) - \mu^2 \right) / \sin \pi \left( (1/2) - \mu \right) \rightarrow 1 \), recovers Equation (109) from Zhang & Lai (2006), and further taking the “cold-disk” limit of \( v \sim c_s / (r \Omega) \rightarrow 0 \) (and noting that \( \mathcal{F}[1/2, 0] = 0 \)), we obtain the classical barotropic cold-disk corotation torque from Goldreich & Tremaine (1979).

Taking the cold-disk limit, \( c_s / (r \Omega) \rightarrow 0 \), such that \( \sin \pi v / (\pi v) \rightarrow 1 \) and \( \mathcal{G}(\mu, v) - \mathcal{G}(\mu, -v) \rightarrow 0 \), but not the isentropic limit such that \( N_r^2 \rightarrow 0 \), we obtain

\[
\Gamma_{l, m}^{(\text{CR, cold})} = -\left[ \frac{2m \pi^2 \Sigma \Phi_{l, m}^2}{(d \ln \Omega/dr) D S} \left\{ \frac{d}{dr} \ln \xi - \frac{2}{L_S} \right\} r_{c T} \right] \times \left[ \frac{\pi \left( \frac{1}{4} - \mu^2 \right) \mathcal{F}(\mu, 0)}{\sin \pi \left( \frac{1}{2} - \mu \right) 2} \right]^2 + \frac{1}{L_S} \times \left[ \frac{\pi \left( \frac{1}{4} - \mu^2 \right) \mathcal{F}(\mu, 0)}{\sin \pi \left( \frac{1}{2} - \mu \right) 2} \right],
\]

where the second equality is true in the limit where \( |r/L_S| \sim 1/(d \ln \Sigma/d \ln r) < (H/r)^{-1} \). The error is in using this cold-disk limit in the above equation for small \( N_r^2 \geq 0 \) scales with \( \mathcal{O}(N_r^2) \). This simplified cold-disk limit for the non-barotropic torque is used in a companion paper to this work (Tsang et al. 2014), which discusses the impact of the non-barotropic torque on eccentricity evolution of giant planets. When \( L_S \sim 1/\langle d \ln \Sigma/dr \rangle \sim H_r \), then \( N_r^2 \sim L^2 \), and the full form of Equation (46) should be used.

### 6. Nonlinear Saturation and Thermal Diffusivity

Three separate nonlinear effects can play a role in the evolution of the corotation torque in non-barotropic disks. The first is due to the singular density and entropy perturbation at corotation (Paardekooper & Papaloizou 2008), which can cause the thermal effect to become saturated, reducing the corotation torque to that of the barotropic case. The second is the onset of the nonlinear horseshoe torque in the co-orbital region (Ward 1991), which dominates over the linear corotation torque after a libration timescale (Paardekooper & Papaloizou 2009; Paardekooper et al. 2010), though this does not occur for the non-co-orbital resonances. The third is the saturation of the corotation region, where the background quantities have been sufficiently modified by the corotation interaction to reduce or halt the corotation torque entirely (Paardekooper & Papaloizou 2009; Paardekooper et al. 2010). In all cases, sufficient viscosity and/or thermal diffusivity can prevent the nonlinear effects from arising, and the linear corotation torque we have derived will remain valid (Paardekooper et al. 2011).

Here we will consider in detail the effect of the singular entropy perturbation, as this can quickly saturate the effects of the entropy gradient, returning the corotation torque to its barotropic value. This occurs at both the co-orbital \( (l = m) \) and non-co-orbital \( (l = m \pm 1) \) corotation resonances. However, following the discussion of Paardekooper & Papaloizou (2008), we will show that sufficient thermal diffusivity restores the linear non-barotropic torque.

The linear entropy perturbation can be defined as

\[
\frac{\delta S}{S} = \delta \frac{P}{P} - \gamma \frac{\delta \Sigma}{\Sigma},
\]

which, when combined with Equation (11), yields

\[
\delta S = -\theta \frac{\delta S}{S} \frac{\delta \mu}{\delta r} \frac{\delta \Omega}{\delta r},
\]

which corresponds to the singular density perturbation at the same location. In the non-dissipative, non-diffusive limit in which we have performed our linear calculations, these singular density and entropy perturbations result in nonlinear effects that arise as the perturbation amplitude grows large (Paardekooper & Papaloizou 2008). When \( \theta \delta S \sim \delta S / \delta r \), the gradient of the entropy due to the perturbation is comparable to the background entropy gradient, and the corotation can thermally saturate,
reducing the torque to the barotropic value. This effect, seen in simulations, was misattributed to a component of the linear torque due to the singular density perturbation by Baruteau & Masset (2008); however, this singular component does not contribute directly to the linear corotation torque.

Including the effect of thermal diffusion, the above equation can be rewritten

\[ -i\omega \delta S + \delta u_r \frac{dS}{dr} = \frac{K}{\rho C_p} \nabla^2 \delta S, \]  

(50)

where \( \rho \) is the volume density, \( K \) is the thermal conductivity, and \( C_p \) is the specific heat at constant pressure, and where we have ignored the effect of the pressure perturbation which is well behaved at the corotation.

Expanding around the corotation, Paardekooper & Papaloizou (2008) showed that the equation above can be rewritten in terms of the inhomogeneous Airy differential equation, which can be solved (assuming bounded behavior away from the corotation) by the inhomogeneous Airy function \( Hi(z) \) (Abramowitz & Stegun 1965; Olver et al. 2010), such that

\[ \delta S = -\left( \frac{3^{1/3} \pi F}{\lambda^{2/3}} \right) Hi(\xi), \]  

(51)

where \( \lambda \equiv 3m/(2\pi^2 D_e) \), \( D_e \equiv (2H K)/(\Sigma C_p r_{\Omega}^2 \Omega_e) = (2e_c K)/(\Sigma C_p r_{\Omega}^2 \Omega_e^2) \) is the dimensionless diffusivity, \( F \equiv \delta u_r(dS/dr)/(D_{r,\Omega} r_{\Omega}^3 \Omega_e) \), \( \Omega_e \equiv \Omega(r_e) \) and

\[ \xi = \frac{2(i\omega - m^2 D_e \Omega_e)(9m/2D_e)^{1/3}}{3m \Omega_e}. \]  

(52)

This manifests as an entropy peak at the corotation, with the perturbation amplitude given by \( [\delta S] \sim 3^{1/3} \pi F/\lambda^{2/3} \), and the length scale of entropy perturbation at corotation is then given by \( \Delta S \sim \lambda^{-1/3} \), which implies that the corotation becomes thermally saturated when the background entropy gradient is comparable to that of the perturbation,

\[ \frac{dS}{dr} \sim \frac{[\delta S(r_e)]}{\Delta S} \sim \frac{dS}{dr} \frac{1}{D_e \delta u_r(r_e)} \left( \frac{r_d D_e}{m} \right)^{1/3} |\delta u_r|, \]  

(53)

thus non-barotropic linearity is preserved when the dimensionless diffusion constant is greater than

\[ D_e \gg \frac{1}{m^{1/2}} \left( \frac{\delta u(r_e)}{r_{\Omega}} \right)^{3/2} \sim m \left( \frac{2|w(r_e)|}{r_{\Omega}^{3/2} \kappa_p^2} \right)^{3/2}, \]  

(54)

where we have utilized Equation (13) to evaluate at the corotation. From Equation (40), we see that we can estimate \( |w(r_e)| \sim \Phi_{1,m} \) for small \( v \) and \( 1/2 - \mu \), which allows us to estimate the thermal diffusivity necessary to maintain linearity for the non-co-orbital corotation resonance,

\[ D_e \gg m \left( \frac{\Phi_{1,m}}{\kappa_p^2} \right)^{3/2} \sim m \left( \frac{\mu}{\kappa_p} \right)^{3/2} \left( \frac{M_p}{M_{e}} \right)^{3/2}, \]  

(55)

while for the co-orbital corotation the diffusivity required depends on the amount of softening used for the potential, as well as the perturbations caused by other azimuthal components of the potential. Paardekooper & Papaloizou (2008) numerically found that \( D_e > 10^{-6} \) was sufficient to prevent thermal saturation for the co-orbital torques in their simulations of smaller Earth-mass planets. Explicitly evaluating the above expression for a planet located at 5 AU (the example discussed in Tsang et al. 2014; Turner et al. 2012), in a Keplerian disk we see that the \( m = 3 \) non-co-orbital corotation resonance located at \( \approx 6.6 \text{ AU} \) remains thermally unsaturated if \( D_e > 6.3 \times 10^{-8} \text{ } (e/10^{-2})^{3/2} \left( M_p/(10^{-3} M_{e}) \right) \). Including thermal diffusion with \( D_e \) obeying these conditions is equivalent to adding a small positive \( \epsilon \) in the Landau prescription.

7. DISCUSSION AND CONCLUSION

We have examined the linear torque due to planet–disk interaction at the corotation resonance in non-barotropic disks. While other works have previously provided semi-analytic expressions that required numerical evaluation and a numerical fitting formula (Baruteau & Masset 2008; Paardekooper & Papaloizou 2008; Paardekooper et al. 2010) to the co-orbital corotation torque, we have developed a fully analytic expression for the linear corotation torque in a non-barotropic disk, for both the co-orbital and non-co-orbital corotation resonances. The main result of this work is Equation (46), which generalizes the corotation torque expressions of Zhang & Lai (2006) and Goldreich & Tremaine (1979).

For small planets, which have not cleared a gap in the disk, co-orbital corotation torque is likely to be linear only for roughly a libration time (Paardekooper & Papaloizou 2009; Paardekooper et al. 2010), and the nonlinear horseshoe torque will likely develop. However, the linearity of the system is more easily preserved for the non-co-orbital corotation resonance, as the potential varies much more smoothly far from the location of the planet, and the perturbing potential is additionally reduced by a factor of the eccentricity \( e \). For higher \( m \), the non-co-orbital corotation resonances are more closely bunched, and interactions between resonances can become important depending on the disk properties. For larger planets that have sufficiently clean gaps, the higher-\( m \) resonances located within the gap have their torques suppressed due to the low surface density. We also note that the torque formula we have derived is not valid for \( m \lesssim r/H \), above which the torque is cutoff due to the high resonance order and finite thickness effects (Ward 1989).

We have shown that entropy gradients in the disk, whether they arise from disk heating, opacity changes, or stellar illumination, can significantly modify the corotation torque acting on planets. Additionally, we have also calculated the minimum thermal diffusivity required to prevent thermal saturation, below which the torque returns to its barotropic value.

We have used a general prescription for adiabatic perturbations and have not assumed a particular form for the equation of state or energy equation, expressing both the radial Brunt–Väisälä frequency, \( N_r \), and the characteristic entropy length scale \( L_S \) in terms of density and pressure gradients. For simplified models, these key quantities can also be expressed in terms of the entropy \( S \) and the adiabatic index \( \gamma \) for particular equations of state. In taking the perturbations to be adiabatic, we have implicitly assumed that the characteristic timescale for energy transport in the disk is slower than the perturbation (orbital) timescale, as is the case for radiatively inefficient disks.

The expression for the corotation torque in Equation (46) above should be applicable to both Keplerian disks, where
the pressure gradients are negligible, and to disks that are significantly non-Keplerian, due to either the effects of general relativity, or strong pressure gradients (see, e.g., Goldreich & Sari 2003). We note that for truly sharp density or pressure transitions, such as gaps that have edge transitions of an order less than the scale height, fully three-dimensional calculations should be used, such as in Zhang & Lai (2006).

In Tsang et al. (2014), a companion paper to this work, we utilize the cold-disk limit of the results above to explore the effect of stellar illumination on a gap opened by a giant planet. We show that disk entropy gradients are sufficiently modified in many cases to allow eccentricity excitation of the planets to occur by reducing the effectiveness of the eccentricity damping corotation resonances. We also suggest that a signature of this process may be evident in the deficit of low-metallicity eccentric planets in the “Eccentricity Valley,” between \(~0.1\) and \(~1\) AU from their host stars (Dawson & Murray-Clay 2013), as this corresponds to the region that would be shadowed by an inflated dust rim in a low-metallicity disk (Dullemond et al. 2001; Muzerolle et al. 2003).

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APPENDIX

APPROXIMATING THE WHITTAKER FUNCTION AT ZERO

Above we have stated that we can approximate \( W_{\nu,\mu}(i\epsilon) \approx W_{\nu,1/2}(0) = 1/\Gamma(1-\nu) \), for \( \epsilon \) only fairly small. This is not immediately obvious as formally when \( \epsilon \to 0 \), \( W_{\nu,\mu}(i\epsilon) \to 0 \); however, it does so only logarithmically slowly, thus only a small imaginary component is sufficient to prevent significant variation from \( W_{\nu,1/2}(0) \). This can be seen by first expressing the Whittaker function in terms of the Kummer Function \( M(a, b, z) \) (Abramowitz & Stegun 1965):

\[
W_{\nu,\mu}(z) = \frac{\Gamma(-\mu)}{\Gamma(\frac{1}{2} - \mu - v)} \frac{e^{z/2} z^{\frac{1}{2} + \nu} M\left(\frac{1}{2} + \mu - v, 1 + 2\mu, z\right)}{\Gamma(\frac{1}{2} + \mu - v)} e^{\frac{-z}{2} - \mu} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - v)} e^{\frac{-z}{2} - \mu} \times M\left(\frac{1}{2} - \mu - v, 1 + 2\mu, z\right). \tag{A1}
\]

As \( |z| \to 0 \), \( M(a, b, z) \equiv 1 \) for \( b \in \mathbb{Z} \). We also have that \( 0 < \nu < 1/2 \), and thus, the second term above dominates as \( |z| \to 0 \). Taking \( z = i\epsilon \), we have

\[
\lim_{\epsilon \to 0} W_{\nu,\mu}(i\epsilon) = \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - v)} e^{i\epsilon/2 + \pi(\frac{1}{2} - \mu)/2} e^{\frac{1}{2} - \mu}. \tag{A2}
\]

As \( \epsilon \to 0 \), \( \Arg[W_{\nu,\mu}(i\epsilon)] = \pi((1/2) - \mu)/2 + \epsilon/2 \), while

\[
|W_{\nu,\mu}(i\epsilon)| = \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - v)} e^{\frac{1}{2} - \mu}. \tag{A3}
\]

Thus, for small \( \epsilon \), we find, expanding in terms of \( 1/2 - \mu \),

\[
|W_{\nu,\mu}(i\epsilon)| = \left(\frac{1}{2} - \mu\right) \Gamma(1 - \nu) \left(1 + 2\gamma_\nu + \psi(1 - \nu) + \ln \epsilon\right) + O\left(\left(\frac{1}{2} - \mu\right)^2\right), \tag{A4}
\]

where \( \gamma_\nu \equiv 0.57722 \) is the Euler–Mascheroni constant, and \( \psi(z) \equiv \Gamma'(z)/\Gamma(z) \) is the digamma function. Defining \( \Delta \equiv 1 - |W_{\nu,\mu}(i\epsilon)/W_{\nu,1/2}(0)| \ll 1 \) to be the fractional error, we have from taking \( W_{\nu,\mu}(i\epsilon) \approx W_{\nu,1/2}(0) \), we then find that the imaginary component, \( \epsilon \), required to have \( \Delta \) a less fractional error is given by

\[
\ln \epsilon \gtrsim -\frac{\Delta - \epsilon_0}{\frac{1}{2} - \mu - \gamma_\nu - \psi(1 - \nu)} + O(\frac{1}{2} - \mu); \tag{A5}
\]

therefore, the minimum \( \epsilon \) required to assume \( W_{\nu,\mu}(0) \approx W_{\nu,1/2}(0) \) scales exponentially as \( \sim \exp[-\Delta/(1/(2) - \mu)] \), such that only \( \epsilon \gtrsim 10^{-4.3} \) is required to have fractional error \( \Delta \lesssim 10\% \), with \( \nu = 0.1 \) and \( (1/2) - \mu = 10^{-3} \).

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