WELL-POSEDNESS FOR THE CAUCHY PROBLEM OF THE KLEIN-GORDON-ZAKHAROV SYSTEM IN 2D

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Abstract. This paper is concerned with the Cauchy problem of the Klein-Gordon-Zakharov system with very low regularity initial data. We prove the bilinear estimates which are crucial to get the local in time well-posedness. The estimates are established by the Fourier restriction norm method. We utilize the nonlinear version of the classical Loomis-Whitney inequality.

1. Introduction

We consider the Cauchy problem of the Klein-Gordon-Zakharov system:

\[
\begin{aligned}
(\partial^2_t - \Delta + 1)u &= -nu, \\
(\partial^2_t - c^2\Delta)n &= \Delta|u|^2, \\
(u, \partial_t u, n, \partial_t n)|_{t=0} &= (u_0, u_1, n_0, n_1)
\end{aligned}
\]

\(t, x \in [-T, T] \times \mathbb{R}^d,\)

\(u, n\) are real valued functions, \(0 < c < 1\). As a physical model, \(\text{(1.1)}\) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma. The condition \(0 < c < 1\), which plays an important role in the paper, comes from a physical phenomenon. See Bellan [4], Masmoudi and Nakanishi [13]. There are some works on the Cauchy problem of \(\text{(1.1)}\) in low regularity Sobolev spaces. For 3D, Ozawa, Tsutaya and Tsutsumi [14] proved that \(\text{(1.1)}\) is globally well-posed in the energy space \(H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)\). In the case of \(c = 1\), \(\text{(1.1)}\) is very similar to the Cauchy problem of the following quadratic derivative nonlinear wave equation.

\[
\begin{aligned}
(\partial^2_t - \Delta)u &= uDu, \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1)
\end{aligned}
\]

\(t, x \in [-T, T] \times \mathbb{R}^3,\)

\(u \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)\),

where \(u, n\) are real valued functions, \(0 < c < 1\). As a physical model, \(\text{(1.1)}\) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma. The condition \(0 < c < 1\), which plays an important role in the paper, comes from a physical phenomenon. See Bellan [4], Masmoudi and Nakanishi [13]. There are some works on the Cauchy problem of \(\text{(1.1)}\) in low regularity Sobolev spaces. For 3D, Ozawa, Tsutaya and Tsutsumi [14] proved that \(\text{(1.1)}\) is globally well-posed in the energy space \(H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)\). In the case of \(c = 1\), \(\text{(1.1)}\) is very similar to the Cauchy problem of the following quadratic derivative nonlinear wave equation.

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(\partial^2_t - \Delta)u &= uDu, \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1)
\end{aligned}
\]

\(t, x \in [-T, T] \times \mathbb{R}^3,\)

\(u \in H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)\),

For \(s > 0\), the local well-posedness of \(\text{(1.2)}\) was obtained from the iteration argument by using the usual Strichartz estimates. As opposed to that, Lindblad showed that \(\text{(1.2)}\) is ill-posed for \(s \leq 0\), see [11]-[12]. In [14], Ozawa, Tsutaya and Tsutsumi showed that the difference between the propagation speeds of the two equations in \(\text{(1.1)}\) allows for a better result. That is, they applied the Fourier restriction norm method and obtained the local well-posedness of \(\text{(1.1)}\) in the energy space, and then by the energy conservation law, they extended solutions globally in time. By the similar argument, Tsugawa established that \(\text{(1.1)}\) is local well-posed in 2D for \(s \geq -1/2\). For 4 and higher dimensions, I. Kato [8] recently proved that \(\text{(1.1)}\) is locally well-posed at \(s = 1/4\) when \(d = 4\) and \(s = s_c + 1/(d+1)\) when \(d \geq 5\) where \(s_c = d/2 - 2\) is the critical exponent of \(\text{(1.1)}\). He also proved that if the initial data are radially symmetric then the small data globally well-posedness can be obtained at the scaling critical regularity for \(d \geq 4\). He utilized the \(U^2, V^2\) spaces introduced by Koch-Tataru [10]. We would like to emphasize that the above results hold under the condition \(0 < c < 1\). Our aim in this paper is to get the local well-posedness of \(\text{(1.1)}\) at very low regularity \(s\) in 2 dimensions. Hereafter we assume \(d = 2\).

Key words and phrases. well-posedness, Cauchy problem, low regularity, bilinear estimate, Strichartz estimate.
By the transformation $u_\pm := \omega_1 u \pm i \partial_t u$, $n_\pm := n \pm i (\omega_1)^{-1} \partial_t n$, $\omega_1 := (1 - \Delta)^{1/2}$, $\omega := (-\Delta)^{1/2}$, the well-posedness of (1.3) can be written as follows:

$$
\begin{align*}
(i\partial_t \mp \omega_1)u_\pm &= \pm (1/4)(n_+ + n_-)(\omega_1^{-1} u_+ + \omega_1^{-1} u_-), \\
(i\partial_t \mp \omega) n_\pm &= \pm (4c)^{-1} \omega^{-1} |\omega_1^{-1} u_+ + \omega_1^{-1} u_-|^2 + c(2\omega_1)^{-1}(n_+ + n_-), \\
(u_\pm, n_\pm)|_{t=0} &= (u_{\pm0}, n_{\pm0}) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2).
\end{align*}
$$

We state our main result.

**Theorem 1.1.** Let $0 < c < 1$ and $-3/4 < s < 0$. Then (1.3) is locally well-posed in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$.

**Remark 1.** We can show the local well-posedness of (1.1) in $H^{s+1} \times H^s \times \dot{H}^{s-1}$ instead of $H^{s+1} \times H^s \times H^{s-1}$. In that case, by the transformation $n_\pm := n \pm i (\omega_1)^{-1} \partial_t n$, (1.1) is written as follows:

$$
\begin{align*}
(i\partial_t \mp \omega_1)u_\pm &= \pm (1/4)(n_+ + n_-)(\omega_1^{-1} u_+ + \omega_1^{-1} u_-), \\
(i\partial_t \mp \omega) n_\pm &= \pm (4c)^{-1} \omega^{-1} |\omega_1^{-1} u_+ + \omega_1^{-1} u_-|^2, \\
(u_\pm, n_\pm)|_{t=0} &= (u_{\pm0}, n_{\pm0}) \in H^s(\mathbb{R}^2) \times \dot{H}^s(\mathbb{R}^2).
\end{align*}
$$

See Remark 2 below for the details.

We make a comment on Theorem 1.1. Applying the iteration argument by the usual Strichartz estimates, we get the local well-posedness of (1.3) for $-1/4 \leq s \leq 0$. This suggests that if $c = 1$ the minimal regularity such that the well-posedness of (1.3) holds seems to be $-1/4$. Tsugawa [17] found that if we utilize the condition $0 < c < 1$ in the same way as in [14] with minor modification, we can show that (1.3) is local well-posed only for $s \geq -1/2$. We can say that the known arguments is not enough to get the well-posedness for $s < -1/2$ which is the most difficult case. To overcome this, we employ a new estimate which was introduced in [5], [7] and applied to the Zakharov system in [1] and [2]. See Proposition 4.3 below. The Zakharov system consists of a wave equation and a Schrödinger equation:

$$
\begin{align*}
(i\partial_t \mp \Delta)u &= nu, & (t, x) &\in \mathbb{R} \times \mathbb{R}^d, \\
\partial_t^2 - \Delta n &= |\omega|^2, & (t, x) &\in \mathbb{R} \times \mathbb{R}^d.
\end{align*}
$$

Roughly speaking, the two systems (1.1) and (1.5) share two features:

(I) The two linear dispersive differential operators are different from each other.

(II) The nonlinear terms are all quadratic.

These similarities suggest that we might get the well-posedness of (1.3) for $s < -1/2$ in the same way as in [1] and [2].

We will prove Theorem 1.1 by the iteration argument in the spaces $X_{\mp}^{s,b}(\mathbb{R}^3) \times X_{\pm}^{s,b}(\mathbb{R}^3)$. This spaces are defined as follows:

Let $0 < c \leq 1$ and $N, L \geq 1$ be dyadic numbers. $\chi_\Omega$ denotes the characteristic function of a set $\Omega$. We define the dyadic decompositions of $\mathbb{R}^3$:

$$
K_{N,L}^{\pm,c} := \{ (\tau, \xi) \in \mathbb{R}^3 | N \leq |\xi| \leq 2N, L \leq |(\tau \pm c|\xi|) | \leq 2L \}.
$$

By using $K_{N,L}^{\pm,c}$, we introduce the solution spaces. Let $s, b \in \mathbb{R}$. We define $X_{\pm}^{s,b}(\mathbb{R}^3)$ as follows:

$$
X_{\pm}^{s,b}(\mathbb{R}^3) := \{ f \in S'(\mathbb{R}^3) \mid \| f \|_{X_{\pm}^{s,b}} < \infty \},
$$

$$
\| f \|_{X_{\pm}^{s,b}} := \left( \sum_{N,L} N^{2s} L^{2b} \| \chi_{K_{N,L}^{\pm,c}} f \|_{L^2_{\tau,\xi}}^2 \right)^{1/2}.
$$

Here $\hat{f}$ denotes the Fourier transform of $f$ in space and time. For convenience, we define $\hat{X}_{\pm}^{s,b}$ which is the Fourier transform of $X_{\pm}^{s,b}$:

$$
\hat{X}_{\pm}^{s,b}(\mathbb{R}^3) := \{ \hat{f} \in S'(\mathbb{R}^3) \mid \| \hat{f} \|_{\hat{X}_{\pm}^{s,b}} < \infty \},
$$

$$
\| \hat{f} \|_{\hat{X}_{\pm}^{s,b}} := \| f \|_{X_{\pm}^{s,b}}.
$$
We denote $K_{N,L}^{s,\pm}$ and $K_{N,L}^{s,0}$ by $K_{N,L}^{s,\pm}$ and $K_{N,L}^{s,0}$, the space $X_{s,1}^{\pm,1}$ by $X_{s,1}^{\pm,1}$ and its norm by $\| \cdot \|_{X_{s,1}^{\pm,1}}$, and also $X_{s,1}^{\pm,1}$ by $\hat{X}_{s,1}^{\pm,1}$ and its norm by $\| \cdot \|_{\hat{X}_{s,1}^{\pm,1}}$.

The key estimates to prove Theorem 1.2 are the following.

**Theorem 1.2.** Let $0 < c < 1$. For any $s \in (-3/4, 0)$, there exists $b \in (1/2, 1)$, $\varepsilon > 0$ and $C > 0$ which depend on $c$ such that

$$\| u(\omega_1^{-1}v) \|_{X_{s,2}^{\pm,1,s-1+\varepsilon}} \leq C \| u \|_{X_{s,0,c}^{\pm,1}} \| v \|_{X_{1}^{\pm,b}},$$

$$\| \omega_1((\omega_1^{-1}u)(\omega_1^{-1}v)) \|_{X_{s,0,c}^{\pm,1+s-1+\varepsilon}} \leq C \| u \|_{X_{s,1}^{\pm,1}} \| v \|_{X_{s,2}^{\pm,1}},$$

regardless of the choice of signs $\pm$.

**Remark 2.**

(1) The key bilinear estimates naturally derived from (1.3) in Remark 1 are slightly different from (1.6) and (1.7). They are described as follows:

$$\| u(\omega_1^{-1}v) \|_{X_{s,2}^{\pm,1,s-1+\varepsilon}} \leq C \| \omega_1 u \|_{X_{s,0,c}^{\pm,1}} \| v \|_{X_{1}^{\pm,b}},$$

$$\| \omega_1((\omega_1^{-1}u)(\omega_1^{-1}v)) \|_{X_{s,0,c}^{\pm,1+s-1+\varepsilon}} \leq C \| u \|_{X_{s,1}^{\pm,1}} \| v \|_{X_{s,2}^{\pm,1}},$$

Let $s \in (-3/4, 0)$. From the inequalities $\| u \|_{X_{s,2}^{\pm,1,b}} \leq \| \omega_1 u \|_{X_{s,0,c}^{\pm,1}}$ and

$$\| \omega_1((\omega_1^{-1}u)(\omega_1^{-1}v)) \|_{X_{s,0,c}^{\pm,1+s-1+\varepsilon}} \leq \| \omega_1((\omega_1^{-1}u)(\omega_1^{-1}v)) \|_{X_{s,2}^{\pm,1}},$$

it is clear that (1.6) and (1.7) imply (1.8) and (1.9), respectively.

(2) It might be natural that we use $\langle \tau \pm c(\xi) \rangle$ instead of $\langle \tau \pm c(\xi) \rangle$ in the definition of $K_{N,L}^{s,\pm,c}$. As was seen in (1.4), these two weights are equivalent and therefore $X_{s,\pm,c}^{\pm}$ does not depend on the choice of them in the definition of $K_{N,L}^{s,\pm,c}$.

Once Theorem 1.2 is verified, we can obtain Theorem 1.1 by the iteration argument given in [6] and many other papers. For example, see [9], [16]. Therefore we focus on the proof of Theorem 1.2 in this paper.

Next, we show the negative result for $s < -3/4$.

**Theorem 1.3.** Let $d = 2$, $0 < c < 1$ and $s < -\frac{3}{4}$. Then for any $T > 0$, the data-to-solution map $(u_0, u_1, n_0, n_1) \mapsto (u, n)$ of (1.1), as a map from the unit ball in $H^{s+1} \times H^s \times H^s \times H^{s-1}$ to $C([0,T]; H^{s+1}) \cap C^1([0,T]; H^s) \times C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ fails to be $C^2$.

Theorem 1.3 implies that the iteration argument, which is applied to the proof of Theorem 1.1, is no longer available for the case $s < -3/4$.

The paper is organized as follows. In Section 2, we introduce some fundamental estimates and property of the solution spaces as preliminary. In Section 3, we show (1.6) and (1.7) with $\pm = \pm$ which is the easier case compared to $\pm \neq \pm$. In Section 4, we prove (1.6) and (1.7) with $\pm \neq \pm$, and complete the proof of Theorem 1.2. Lastly as Section 5, we show the negative result, Theorem 1.3.

2. Preliminaries

In this section, we introduce some estimates which will be utilized for the proof of Theorem 1.2. Throughout the paper, we use the following notations. $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. Also, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. It should be emphasized that the signs $\lesssim$ and $\sim$ depend on $1 - c$ in the paper. Thus, the necessary condition of a time ingredient $T$ in (1.1) to show Theorem 1.1 also depends on $1 - c$. Since the aim of the paper is to show the local well-posedness, here we are not concerned with how the necessary condition of $T$ for Theorem 1.1 changes as $c$ approaches to 1. Let $u = u(t, x)$. $F_t u$, $F_x u$ denote the Fourier transform of $u$ in time, space, respectively. $F_{t,x} u = \hat{u}$ denotes the Fourier transform of $u$ in time and space. We first observe that fundamental properties of $X_{s,1}^{\pm,1,c}$. A simple calculation gives the following:

(i) $X_{s,1}^{\pm,c} = X_{s,1}^{\pm,1,c}$,

(ii) $(X_{s,1}^{\pm,1,c})^* = X_{s,1}^{\pm,1-c}$. 


for $0 < c \leq 1$ and $b \in \mathbb{R}$. Next we define the angular decomposition of $\mathbb{R}^3$ in frequency. For a dyadic number $A \geq 64$ and an integer $j \in \{-A, A-1\}$, we define the sets $\{D_j^A\} \subset \mathbb{R}^3$ as follows: 

$$D_j^A = \left\{ (\tau, |\xi| \cos \theta, |\xi| \sin \theta) \in \mathbb{R} \times \mathbb{R}^2 \mid \theta \in \left[ \frac{\pi}{A} j, \frac{\pi}{A}(j+1) \right] \right\}.$$ 

For any function $u : \mathbb{R}^3 \to \mathbb{C}$, $\{D_j^A\}$ satisfy

$$\mathbb{R}^3 = \bigcup_{-A \leq j \leq A-1} D_j^A, \quad u = \sum_{j=-A}^{A-1} \chi_{D_j^A} u \text{ a.e.}$$

Lastly we introduce the useful two estimates which are called the bilinear Strichartz estimates. The first one holds true regardless of $c$. As opposed to that, the second one is given by using the condition $0 < c < 1$. The first estimate is obtained by the same argument as in the proof of Theorem 2.1 in [15]. We omit the proof.

**Proposition 2.1** (Theorem 2.1, [15]). Let $0 < c, c_1, c_2 \leq 1$. Then we have

$$
\| \chi_{K_{N_0}^+} \sum_{k} \left( \chi_{K_{N_1}^+} f \ast \chi_{K_{N_2}^+} g \right) \|_{L_t^2 L_x^\infty} \lesssim \left( N_0^{12} L_1^{12} \right)^{1/2} (N_0^{12} L_1^{12})^{1/4} \| f \|_{L_t^2 L_x^\infty} \| g \|_{L_t^2 L_x^\infty},
$$

(2.1)

regardless of the choice of signs $\pm$. Here $*$ denotes the convolution of $\mathbb{R}^3$, $N_0^{12} := \min(N_0, N_1, N_2)$, and $N_0^{12}$, $L_1^{12}$, $L_2^{12}$ are defined similarly.

**Proposition 2.2.** Let $0 < c < 1$. Then we have

$$
\| \chi_{K_{N_0}^+} \sum_{k} \left( \chi_{K_{N_1}^+} f \ast \chi_{K_{N_2}^+} g \right) \|_{L_t^2 L_x^\infty} \lesssim \left( N_0^{12} L_1^{12} \right)^{1/2} \| f \|_{L_t^2 L_x^\infty} \| g \|_{L_t^2 L_x^\infty},
$$

(2.2)

regardless of the choice of $\pm$.

**Proof.** Let $A = 2^{10}(1 - c)^{-1/2}$. It follows from the finiteness of $A$ and

$$g = \sum_{j=-A}^{A-1} \chi_{D_j^A} g \text{ a.e.}$$

that we can replace $g$ with $\chi_{D_j^A} g$ in (2.2) for fixed $j$. After applying rotation in space, we may assume that

$$j = 0. \text{ Also we can assume that there exists } \xi' \in \mathbb{R}^2 \text{ such that the support of } \chi_{D_j^A} g \text{ is contained in the cylinder}$$

$$C_{N_0^{12}}(\xi') \equiv \{(\tau, \xi) \in \mathbb{R}^3 \mid |\xi - \xi'| \leq N_0^{12} \}.$$ 

We sketch the validity of the above assumption roughly. See [15] for more details. If $N_2 \sim N_0^{12}$ the above assumption is harmless obviously. Therefore we may assume that $N_0 = N_0^{12} \ll N_2$ or $N_1 = N_0^{12} \ll N_2$. Since both are treated similarly, we here consider only the former case. Note that the condition $N_0 \ll N_2$ means $N_2/2 \leq N_1 \leq 2N_2$, otherwise the left-hand side of (2.2) vanishes. We can choose the two sets $\{C_{N_0^{12}}(\xi'_k)\}_k$ and $\{C_{N_0^{12}}(\xi''_\ell)\}_\ell$ such that

$$
\begin{align*}
\# k &\sim \left( \frac{N_1}{N_0} \right)^2, \quad \text{supp } \chi_{D_k^A} g \subset \bigcup_k C_{N_0^{12}}(\xi'_k), \quad |\xi'_k - \xi'_k| \geq N_0^{12} \text{ for any } k, k', \\
\# \ell &\sim \left( \frac{N_1}{N_0} \right)^2, \quad \text{supp } f \subset \bigcup_\ell C_{N_0^{12}}(\xi''_\ell), \quad |\xi''_\ell - \xi''_\ell| \geq N_0^{12} \text{ for any } \ell, \ell',
\end{align*}
$$

where $\# k$ and $\# \ell$ denote the numbers of $k$ and $\ell$, respectively. We see that for fixed $k$, independently of $N_0, N_1, N_2$, there is only a finite number of $\ell$ which satisfy

$$
\left\| \chi_{K_{N_0}^+} \sum_{k} \left( \chi_{K_{N_1}^+} \cap C_{N_0^{12}}(\xi'_k) f \ast \chi_{K_{N_2}^+} \cap C_{N_0^{12}}(\xi''_\ell) g \right) \right\|_{L_t^2 L_x^\infty} > 0,
$$

4
and vice versa. This means that $k$ and $\ell$ depend on each other. Once we obtain

$$
\left\| \chi_{K_{N_0,0}^{\pm 0}} \left( \left( \chi_{K_{N_1,1}^{\pm 1}} f \right) * \left( \chi_{K_{N_2,2}^{\pm 2}} \cap C_{N_{012}^{\text{min}}} (\xi) \right) \right) \right\|_{L_{\xi,\tau}^2} \lesssim (N_{\text{min}} L_1 L_2)^{1/2} \| f \|_{L_{\xi,\tau}^2} \| g \|_{L_{\xi,\tau}^2}
$$

for fixed $k$, from Minkowski inequality and $l^2$ almost orthogonality, we confirm

$$
\lesssim \sum_{k,\ell} \left\| \chi_{K_{N_0,0}^{\pm 0}} \left( \left( \chi_{K_{N_1,1}^{\pm 1}} \cap C_{N_{012}^{\text{min}}} (\xi) \right) f \right) * \left( \chi_{K_{N_2,2}^{\pm 2}} \cap C_{N_{012}^{\text{min}}} (\xi) \right) \right\|_{L_{\xi,\tau}^2} \lesssim (N_{\text{min}} L_1 L_2)^{1/2} \sum_{k,\ell} \| \chi_{C_{N_{012}^{\text{min}}} (\xi)} f \|_{L_{\xi,\tau}^2} \| \chi_{C_{N_{012}^{\text{min}}} (\xi)} g \|_{L_{\xi,\tau}^2}
$$

which verify the validity of the assumption. Hereafter, we call the above argument "$l^2$ almost orthogonality".

We turn to the proof of (2.2).

$$
\left\| \chi_{K_{N_0,0}^{\pm 0}} \left( \left( \chi_{K_{N_1,1}^{\pm 1}} f \right) * \left( \chi_{K_{N_2,2}^{\pm 2}} \cap C_{N_{012}^{\text{min}}} (\xi) \right)\right) \right\|_{L_{\xi,\tau}^2} \lesssim \| E(\tau, \xi) \|_{L_{\xi,\tau}^2}^{1/2} \| f \|_{L_{\xi,\tau}^2} \| g \|_{L_{\xi,\tau}^2}^{1/2}
$$

where

$$
E(\tau, \xi) := \{ (\tau_1, \xi_1) \in C_{N_{012}^{\text{min}}} (\xi') \cap D_0^A \mid \langle \tau - \tau_1, \xi - \xi_1 \rangle \sim L_1, \langle \tau_1, \xi_1 \rangle \sim L_2 \}.
$$

Thus it suffices to show that

$$
\sup_{\tau, \xi} |E(\tau, \xi)| \lesssim N_{\text{min}}^{1/2} L_1 L_2.
$$

(2.3)

From $\langle \tau - \tau_1, \xi - \xi_1 \rangle \sim L_1$ and $\langle \tau_1, \xi_1 \rangle \sim L_2$, for fixed $\xi_1$,

$$
|\{ \tau_1 \mid (\tau_1, \xi_1) \in E(\tau, \xi) \}| \lesssim L_{\text{min}}^{12},
$$

(2.4)

It follows from $(\tau_1, \xi_1) \in D_0^A$ that

$$
|\partial_1 (\tau \pm |\xi_1| \pm c|\xi - \xi_1|)| \geq \frac{(\xi_1)_1}{|\xi_1|} - c \geq \left( \frac{(\xi_1)_1}{|\xi_1|} \right)^2 - c
$$

$$
= 1 - c - \left( \frac{(\xi_1)_2}{|\xi_1|} \right)^2 \geq (1 - c)/2,
$$

(2.5)

where $(\xi_1)_1$ is the first component of $\xi_1$ and $\partial_1$ is the derivative with respect to $(\xi_1)_1$. Combining $|\tau \pm |\xi_1| \pm c|\xi - \xi_1| | \lesssim L_{\text{max}}^{12}$ with (2.3), for fixed $(\xi_1)_2$ we have

$$
|\{ (\xi_1)_1 \mid (\tau_1, \xi_1) \in E(\tau, \xi) \}| \lesssim L_{\text{max}}^{12},
$$

(2.6)

Collecting (2.4), (2.5) and $\xi_1 \in C_{N_{012}^{\text{min}}} (\xi')$, we get (2.3).
3. Proof of Theorem 1.2 for $\pm_1 = \pm_2$.

In (1.3)-(1.7), replacing $u$ and $n$ with its complex conjugates $\bar{u}$ and $\bar{n}$ respectively, we easily find that there is no difference between the case $(\pm_0, \pm_1, \pm_2)$ and $(\mp_0, \pm_1, \mp_2)$. Here $\pm_j$ denotes a different sign to $\pm_j$. Therefore we assume $\pm_1 = - \pm_1$ in (1.3)-(1.7) hereafter. By the dual argument and Plancherel theorem, we observe that

$$I_1 \iff \left| \int f(\omega_1^{-1} g_1) g_2 dt dx \right| \leq C \| f \|_{X_{\pm_0}^{s, b}} \| g_1 \|_{X_{\pm_0}^{s, b}} \| g_2 \|_{X_{\pm_0}^{-s, 1-b-e}}.$$

$$\iff \sum_{N_j, L_j (j=0,1,2)} N_j^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi \leq \left( \sum_{N_1} \sum_{N_0} + \sum_{N_2} \right) \sum_{N_0} N_0^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi,$$

where

$$I_1 := \sum_{N_j, L_j (j=0,1,2)} N_j^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi.$$

Similarly, (1.7) is verified by the following estimate.

$$\left( \sum_{N_1} \sum_{N_0} + \sum_{N_2} \right) \sum_{N_0} N_0^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi$$

where

$$I_2 := \sum_{N_1, L_1, N_0, L_0} N_1^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi.$$

We now try to establish (3.1) and (3.2). First we assume that $\pm_2 = -$. In this case, we can obtain (3.1) and (3.2) by using the bilinear estimates Propositions 2.1 and 2.2 and the following estimate:

Lemma 3.1. Let $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$ and $0 < c < 1$. Then we have

$$\max((\tau \pm c|\xi|), (\tau_1 - |\xi_1|), (\tau_2 - |\xi_2|)) \geq \max(|\xi_1|, |\xi_2|).$$

Proof.

$$\max((\tau \pm c|\xi|), (\tau_1 - |\xi_1|), (\tau_2 - |\xi_2|)) \geq \max(|\xi_1|, |\xi_2|)$$

$$\geq |\xi_1| + |\xi_2| - (|\xi_1| + |\xi_2|)$$

$$= (1 - c)(|\xi_1| + |\xi_2|)$$

$\square$

Theorem 3.2. Let $0 < c < 1$. For any $s \in (-3/4, 0)$, there exists $b \in (1/2, 1)$ such that for $f, g_1, g_2 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$, the following estimates hold:

$$\left( \sum_{N_1} \sum_{N_0} + \sum_{N_2} \right) \sum_{N_0} N_0^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi \leq \| f \|_{X_{\pm_0}^{s, b}} \| g_1 \|_{X_{\pm_0}^{s, b}} \| g_2 \|_{X_{\pm_0}^{-s, 1-b-e}},$$

$$\left( \sum_{N_1} \sum_{N_0} + \sum_{N_2} \right) \sum_{N_0} N_0^{-1} \int \left( \chi_{K_{N_0, L_0}}^{-} f \right) \left( \chi_{K_{N_1, L_1}}^{-} g_1 \right) \left( \chi_{K_{N_2, L_2}}^{+} g_2 \right) d\tau d\xi \leq \| f \|_{X_{\pm_0}^{-s, 1-b-e}} \| g_1 \|_{X_{\pm_0}^{b, b}} \| g_2 \|_{X_{\pm_0}^{b, b}},$$
where

\[
I_1^- := \sum_{L_j} \left| N_1^{-1} \int \left( \chi_{K_{N_0,L_0}^+, \ell_j} f \right) \left( \chi_{K_{N_1,L_1}^-} g_1 \right) \left( \chi_{K_{N_2,L_2}^-} g_2 \right) d\tau d\xi \right|
\]

\[
I_2^- := \sum_{L_j} \left| N_0 N_1^{-1} N_2^{-1} \int \left( \chi_{K_{N_0,L_0}^+, \ell_j} f \right) \left( \chi_{K_{N_1,L_1}^-} g_1 \right) \left( \chi_{K_{N_2,L_2}^-} g_2 \right) d\tau d\xi \right|
\]

**Proof.** For simplicity, we use \( f \pm c := \chi_{K_{N_0,L_0}^+, \ell_j} f, \ g_\pm := \chi_{K_{N_1,L_1}^-} g \) and \( (g^-_1 \cdot \cdot - \cdot) := g^-_1 \cdot \cdot - \cdot \) with \( k = 1, 2 \).

Since the proof of (3.3) is analogous to that of (3.2), we establish only (3.4). From Lemma 3.1 it holds that \( L_{\text{max}}^{12} \gtrsim N_{\text{max}}^{12} \). We decompose the proof into the three cases:

(I) \( 1 \leq N_0 \lesssim N_1 \sim N_2 \), (II) \( 1 \leq N_1 \lesssim N_0 \sim N_2 \), (III) \( 1 \leq N_2 \lesssim N_0 \sim N_1 \).

First we consider the case (I). Considering that \( L_{\text{max}}^{12} \gtrsim N_{\text{max}}^{12} \), we subdivide the cases further:

(Ia) \( N_1 \lesssim L_0 \). We deduce from Hölder inequality and Proposition 2.1 that

\[
\sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f_{\pm c} (g_1^- \ast g_2^-) d\tau d\xi \right|
\]

\[
\lesssim \sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \| f_{\pm c} \|_{L_2^2}, \| \chi_{K_{N_1,L_1}^-} (g_1^- \ast g_2^-) \|_{L_2^2} \right|
\]

\[
\lesssim \sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-3/4} N_0^{1/4} L_0^{1/2} L_2^{1/4} \| f_{\pm c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^- \|_{L_2^2},
\]

\[
\lesssim \| f \|_{L_2^2} \| g_1 \|_{L_2^2} \| g_2 \|_{L_2^2},
\]

(Ib) \( N_1 \lesssim L_1 \). Similarly, from Hölder inequality and Proposition 2.1 we get

\[
\sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f_{\pm c} (g_1^- \ast g_2^-) d\tau d\xi \right|
\]

\[
\lesssim \sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-3/4} N_0^{1/4} L_0^{1/2} L_2^{1/4} \| f_{\pm c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^- \|_{L_2^2},
\]

\[
\lesssim \| f \|_{L_2^2} \| g_1 \|_{L_2^2} \| g_2 \|_{L_2^2},
\]

(Ic) \( N_1 \lesssim L_2 \).

\[
\sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f_{\pm c} (g_1^- \ast g_2^-) d\tau d\xi \right|
\]

\[
\lesssim \sum_{N_1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-3/4} N_0^{1/4} L_0^{1/2} L_2^{1/4} \| f_{\pm c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^- \|_{L_2^2},
\]

\[
\lesssim \| f \|_{L_2^2} \| g_1 \|_{L_2^2} \| g_2 \|_{L_2^2},
\]

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For the case (II), we can show (3.4) in the same manner as above. We omit the proof. Lastly, we consider the case (III).

(IIIa) $N_0 \lesssim L_0$. We deduce from Hölder inequality and Proposition 2.1 that

$$\sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \left| N_0^{-1} \int f^{\pm,c}(g_1^- * g_2^+) \, d\tau d\xi \right| \lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| \chi_K \xi \| \| g_1^- * g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \| f \|_{\mathcal{F}_{\pm,1}^s} \| g_1 \|_{\mathcal{F}_{\pm,1}^s} \| g_2 \|_{\mathcal{F}_{\pm,1}^s}. $$

(IIIb) $N_0 \lesssim L_1$. Similarly,

$$\sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \left| N_0^{-1} \int f^{\pm,c}(g_1^- * g_2^+) \, d\tau d\xi \right| \lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| \chi_K \xi \| \| g_1^- * g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \| f \|_{\mathcal{F}_{\pm,1}^s} \| g_1 \|_{\mathcal{F}_{\pm,1}^s} \| g_2 \|_{\mathcal{F}_{\pm,1}^s}. $$

(IIIc) $N_0 \lesssim L_2$. In this case, we need to utilize Proposition 2.2 instead of Proposition 2.1

$$\sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \left| N_0^{-1} \int f^{\pm,c}(g_1^- * g_2^+) \, d\tau d\xi \right| \lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| \chi_K \xi \| \| g_1^- * g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \sum_{N_0} \sum_{1 \leq N_2 \leq N_0 \sim N_1} \sum_{L_j} \sum_{N_0} \| f^{\pm,c} \|_{L_2^2} \| g_1^- \|_{L_2^2} \| g_2^+ \|_{L_2^2},$$

$$\lesssim \| f \|_{\mathcal{F}_{\pm,1}^{s,1}} \| g_1 \|_{\mathcal{F}_{\pm,1}^{s,1}} \| g_2 \|_{\mathcal{F}_{\pm,1}^{s,1}}. $$

□

4. PROOF OF THEOREM 1.2 FOR $\pm_1 \neq \pm_2$.

In this section, we establish (3.1) and (3.2) with $\pm_2 = +$. Note that if one of the inequalities $|\xi_2| \leq \frac{1-c}{2(1+c)} |\xi_1|$ and $|\xi_1| \leq \frac{1-c}{2(1+c)} |\xi_2|$ holds, then we observe that for $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$,

$$\max(\langle \tau \pm_0 c |\xi| \rangle, \langle \tau_1 - |\xi_1| \rangle, \langle \tau_2 + |\xi_2| \rangle) \geq | \pm_0 c |\xi| - |\xi_1| + |\xi_2| | \geq |\xi_1| - |\xi_2| \neq c |\xi_1| - |\xi_2| \geq \frac{1-c}{2} \max(|\xi_1|, |\xi_2|)$$

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and we can verify \([3.1]\) and \([3.2]\) by the same proof as in the case \(\pm 2 = -\). To avoid redundancy, we omit the proof.

**Proposition 4.1.** Let \(0 < c < 1\). For any \(s \in (-3/4, 0)\), there exists \(b \in (1/2, 1)\) such that for \(f, g_1, g_2 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)\), the following estimates hold:

\[
\begin{align*}
&\left(\sum_{N_0} \sum_{1 \leq N_1 \ll N_0 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_2 \ll N_0 \sim N_1}\right) I_1^+ \lesssim \|f\| \|\tilde{X}_c^b\| \|g_1\| \|\tilde{X}_c^b\| g_2 \|\tilde{X}_{c, 1-b-c}, \\
&\left(\sum_{N_0} \sum_{1 \leq N_1 \ll N_0 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_2 \ll N_0 \sim N_1}\right) I_2^+ \lesssim \|f\| \|\tilde{X}_c^b\| \|g_1\| \|\tilde{X}_c^b\| g_2 \|\tilde{X}_{c, 1-b-c},
\end{align*}
\]

where

\[
I_1^+ := \sum_{L_j} \left|N_1^{-1} \int \left(\chi_{K^c_{N_1, L_1}} f \right) \left(\chi_{K^c_{N_1, L_1}} g_1 \right) \left(\chi_{K^c_{N_2, L_2}} g_2 \right) \right| d\tau d\xi, \\
I_2^+ := \sum_{L_j} \left|N_0 N_1^{-1} N_2^{-1} \int \left(\chi_{K^c_{N_1, L_1}} f \right) \left(\chi_{K^c_{N_1, L_1}} g_1 \right) \left(\chi_{K^c_{N_2, L_2}} g_2 \right) \right| d\tau d\xi.
\]

Thanks to Proposition 4.1 we can assume that \(1 \leq N_0 \ll N_1 \sim N_2\). In this case, we no longer make use of the useful estimate such as \([3.3]\) and as we mentioned in the introduction, it appears that the bilinear estimates Propositions 2.1 and 2.2 are not enough to show \([3.1]\) and \([3.2]\). Thus we employ a new estimate developed by Bejenaru-Herr-Tataru [4] and applied to Zakharov system in [1]. To describe it precisely, we introduce the decomposition of \(\mathbb{R}^3 \times \mathbb{R}^3\) which was exploited in [1]. For dyadic numbers \(M_0, M_1\), to be chosen later, we decompose \(\mathbb{R}^3 \times \mathbb{R}^3\) into the sets \(\mathcal{D}_j\):

\[
\mathbb{R}^3 \times \mathbb{R}^3 = \left\{\angle(\xi_1, \xi_2) \leq \frac{16}{M_0} \pi \right\} \cup \bigcup_{64 \leq A \leq M_0} \left\{\frac{16}{A} \pi \leq \angle(\xi_1, \xi_2) \leq \frac{32}{A} \pi \right\}
\]

\[
\cup \left\{\pi - \frac{16}{M_1} \pi \leq \angle(\xi_1, \xi_2) \right\} \cup \bigcup_{64 \leq A \leq M_1} \left\{\pi - \frac{32}{A} \pi \leq \angle(\xi_1, \xi_2) \leq \pi - \frac{16}{A} \pi \right\}
\]

\[
= \bigcup_{-M_0 \leq j_1, j_2 \leq -M_0 \frac{1}{16}} \mathcal{D}^{M_0, j_1} \times \mathcal{D}^{M_0, j_2} \bigcup_{64 \leq A \leq M_0} \bigcup_{-A \leq j_1, j_2 \leq A - \frac{1}{16}} \bigcup_{16 \leq j_1, j_2 \leq 16} \mathcal{D}^A \times \mathcal{D}^A
\]

\[
\cup \bigcup_{-M_0 \leq j_1, j_2 \leq M_0 - \frac{1}{16}} \mathcal{D}^{M_1, j_1} \times \mathcal{D}^{M_1, j_2} \bigcup_{64 \leq A \leq M_1} \bigcup_{-A \leq j_1, j_2 \leq A - \frac{1}{16}} \bigcup_{16 \leq j_1, j_2 \leq 16} \mathcal{D}^A \times \mathcal{D}^A,
\]

where \(\angle(\xi_1, \xi_2) \in [0, \pi]\) is the smaller angle between \(\xi_1\) and \(\xi_2\).

First we assume that \(\pi/2 \leq \angle(\xi_1, \xi_2) \leq \pi\). We find that if \(\angle(\xi_1, \xi_2)\) is sufficiently close to \(\pi\), then the following helpful inequality holds true.

**Lemma 4.2.** Let \(\tau = \tau_1 + \tau_2\), \(\xi = \xi_1 + \xi_2\), \(0 < c < 1\) and \(M_1\) be the minimal dyadic number which satisfies

\[
M_1 \geq 2^7 (1 - c)\sqrt[3]{\frac{(|\xi_1| + |\xi_2|)}{|\xi|}},
\]

then for any \((\tau_1, \xi_1) \in \mathcal{D}^{M_1, j_1}, (\tau_2, \xi_2) \in \mathcal{D}^{M_1, j_2}\) where \(|j_1 - j_2| \leq 16\), the following inequality holds;

\[
\max(|\tau \pm c|\xi|), (\tau_1 - |\xi_1|), (\tau_2 - |\xi_2|) \gtrsim |\xi|
\]

**Proof.** After rotation, we may assume \(\xi_1 = (|\xi_1|, 0)\), and then \(|\xi_2 \pm M_1| \leq 16\). It follows from the inequality

\[
\max(|\tau \pm c|\xi|), (\tau_1 - |\xi_1|), (\tau_2 + |\xi_2|) \geq ||\xi_1| - |\xi_2| - c|\xi|, 
\]

it suffices to show \(||\xi_1| - |\xi_2|| > \sqrt{\frac{1+c}{2}|\xi|}\). Indeed,

\[
\sqrt{\frac{1+c}{2}} - c > \frac{1}{4}(1-c)(1+2c) > \frac{1-c}{4},
\]
From \(|j_2 \pm M_1| \leq 16\), we obtain
\[
|\xi|^2 = (|\xi_1| + |\xi_2| \cos(\angle(\xi_1, \xi_2)))^2 + (N_2 \sin(\angle(\xi_1, \xi_2)))^2 \\
< (|\xi_1| - |\xi_2|)^2 + 2|\xi_1||\xi_2|(1 + \cos(\angle(\xi_1, \xi_2))) \\
< (|\xi_1| - |\xi_2|)^2 + 4|\xi_1||\xi_2|(\angle(\xi_1, \xi_2))^2 \\
< (|\xi_1| - |\xi_2|)^2 + \frac{1-c}{2}|\xi|^2,
\]
which gives
\[
\frac{1+c}{2}|\xi|^2 < (|\xi_1| - |\xi_2|)^2.
\]
This completes the proof. \(\square\)

Next we consider the case \(64 \leq A \leq M_1\) and \(16 \leq |j_1 - j_2 \pm A| \leq 32\).

**Proposition 4.3.** Let \(\tau = \tau_1 + \tau_2, \xi = \xi_1 + \xi_2, 0 < c < 1\) and \(f, g_1, g_2 \in L^2\) be satisfy
\[
supp f \subset K_{N_0, L_0}^+, \quad supp g_1 \subset D_{j_1} \cap K_{N_1, L_1}, \quad supp g_2 \subset D_{j_2} \cap K_{N_2, L_2},
\]
and \(64 \leq N_0 \leq N_1 \leq N_2, 64 \leq A \leq M_1, 16 \leq |j_1 - j_2 \pm A| \leq 32\). Then the following estimate holds;
\[
\left| \int f(\tau, \xi)g_1(\tau_1, \xi_1)g_2(\tau_2, \xi_2)\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim \|f\|_{L^2_{\xi_2} \tau_2} \|g_1\|_{L^2_{\xi_1} \tau_1} \|g_2\|_{L^2_{\xi_2} \tau_2}.
\]

For the proof of the above proposition, we introduce the important estimate. See [2] for more general case.

**Proposition 4.4 ([3] Corollary 1.5).** Assume that the surface \(\tilde{S}^*_i (i = 1, 2, 3)\) is an open and bounded subset of \(\tilde{S}^*_i\) which satisfies the following conditions (Assumption 1.1 in [3]).
(i) \(\tilde{S}^*_i\) is defined as
\[
\tilde{S}^*_i = \{ \tilde{s} \in U_i \mid \Phi_i(\tilde{s}) = 0, \nabla \Phi_i \neq 0, \Phi_i \in C^{1,1}(U_i) \},
\]
for a convex \(U_i \subset \mathbb{R}^3\) such that \(\text{dist}(\tilde{S}_i, U_i) \geq \text{diam}(\tilde{S}_i)\);
(ii) the unit normal vector field \(\bar{n}_i\) on \(\tilde{S}^*_i\) satisfies the Hölder condition
\[
\sup_{\tilde{s}, \tilde{s}^\prime \in \tilde{S}_i} \frac{|\bar{n}_i(\tilde{s}) - \bar{n}_i(\tilde{s}^\prime)|}{|\tilde{s} - \tilde{s}^\prime|} + \frac{|\bar{n}_i(\tilde{s})(\tilde{s} - \tilde{s}^\prime)|}{|\tilde{s} - \tilde{s}^\prime|^2} \lesssim 1;
\]
(iii) the matrix \(\tilde{N}(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) = (\bar{n}_1(\tilde{s}_1), \bar{n}_2(\tilde{s}_2), \bar{n}_3(\tilde{s}_3))\) satisfies the transversality condition
\[
\frac{1}{2} \leq \det \tilde{N}(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) \leq 1
\]
for all \((\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) \in \tilde{S}_1^* \times \tilde{S}_2^* \times \tilde{S}_3^*\).

We also assume \(\text{diam}(\tilde{S}_i) \leq 1\). Let \(T : \mathbb{R}^3 \to \mathbb{R}^3\) be an invertible, linear map and \(S_i = T \tilde{S}_i\). Then for functions \(f \in L^2(S_1)\) and \(g \in L^2(S_2)\), the restriction of the convolution \(f \ast g\) to \(S_3\) is a well-defined \(L^2(S_3)\)-function which satisfies
\[
\|f \ast g\|_{L^2(S_3)} \lesssim \frac{1}{\sqrt{d}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)},
\]
where
\[
d = \inf_{\tilde{s} \in \tilde{S}_i} |\det N(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)|
\]
and \(N(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)\) is the matrix of the unit normals to \(S_i\) at \((\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)\).

**Remark 3.** As was mentioned in [3], the condition of \(S^*_i\) (i) is used only to ensure the existence of a global representation of \(S_i\) as a graph. In the proof of Proposition 4.3, the implicit function theorem and the other conditions may show the existence of such a graph. Thus we will not treat the condition (i) in the proof of Proposition 4.3.

By utilizing Proposition 4.4, we verify Proposition 4.3.
Proof of Proposition 4.3. Let $\theta_0^\pm \in (0, \pi)$ be defined as $\cos \theta_0^\pm = \pm c$. We divide the proof into the following two cases:

(I) $|\angle (\xi, \xi_1) - \theta_0^-| > 2^{10}(1-c)^{-1}A^{-3/4}$ and $|\angle (\xi, \xi_1) - \theta_0^+| > 2^{10}(1-c)^{-1}A^{-3/4}$,

(II) $|\angle (\xi, \xi_1) - \theta_0^-| \leq 2^{10}(1-c)^{-1}A^{-3/4}$ or $|\angle (\xi, \xi_1) - \theta_0^+| \leq 2^{10}(1-c)^{-1}A^{-3/4}$,

where $\angle (\xi, \xi_1) \in (0, \pi)$ is the smaller angle between $\xi$ and $\xi_1$. We here assume that $A > 2^{20}(1-c)^{-2}$. If $A \leq 2^{20}(1-c)^{-2}$, the proposition is verified by the almost same proof as that for the case (II) below.

We first consider the case (I). The proof is very similar to that for $\pm_1 = \pm_2$. We utilize the following two estimates.

Lemma 4.5. Let $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$, $0 < c < 1$, $2^{20}(1-c)^{-2} < A \leq M_1$ and $\angle (\xi, \xi_1)$ satisfies (I). Then the following inequality holds:

$$\max(|\tau + c|\xi|), \langle \tau_1 - |\xi_1|, \langle \tau_2 + |\xi_2| \rangle \rangle \geq A^{-3/4}|\xi|$$

Proof. After rotation, we may assume that $\xi_1 = (|\xi_1|, 0)$ and $\xi = (|\xi| \cos \theta, |\xi| \sin \theta)$ with $\theta := \angle (\xi, \xi_1) \in (0, \pi)$. By the simple calculation, we have

$$\max(|\tau + c|\xi|), \langle \tau_1 - |\xi_1|, \langle \tau_2 + |\xi_2| \rangle \rangle \geq |\tau + c|\xi| + |\tau_1 - |\xi_1|| \geq |c|\xi| + |\tau_1 - |\xi_1||$$

$$\geq |c|\xi| + \sqrt{|\tau_1| - |\xi_2|} - |\tau_1 - |\xi_1|| + |\tau_2 + |\xi_2||$$

$$\geq |c|\xi| + |\tau_1 - |\xi_1||$$

$$\geq |c|\xi| + |\tau_1 - |\xi_1||$$

$$\geq |c|\xi| + |\tau_1 - |\xi_1||$$

$$\geq |c|\xi| + |\tau_1 - |\xi_1||$$

From $\theta_0^\pm, \theta \in (0, \pi)$ and (I), we get

$$K_1 = |\xi| |\cos \theta_0^- - \cos \theta| \geq |\xi| \frac{\sqrt{1-c}}{4} |\theta_0^- - \theta|$$

$$\geq 2^5(1-c)^{-\frac{1}{2}}|\xi|A^{-\frac{3}{4}}.$$
Proof. By the same way as in the proof of Proposition 2.2, we observe that the desired estimate is proved by
\[
\sup_{\tau, \xi} |E(\tau, \xi)| \lesssim A N_0 L_1 L_2 \tag{4.2}
\]
where
\[
E(\tau, \xi) := \left\{ (\tau_1, \xi_1) \in D_0^A \cap C_N_0(\xi') \mid \langle \tau - \tau_1 - |\xi - \xi_1| \rangle \sim L_1, \langle \tau_1 + |\xi_1| \rangle \sim L_2 \right\}
\]
with \(16 \leq |j_2 \pm A| \leq 32\) and fixed \(\xi' \in \mathbb{R}^2\). From \(\langle \tau - \tau_1 - |\xi - \xi_1| \rangle \sim L_1\) and \(\langle \tau_1 + |\xi_1| \rangle \sim L_2\), for fixed \(\xi_1\),
\[
|\{\tau_1 \mid (\tau_1, \xi_1) \in E(\tau, \xi)\}| \lesssim L_{\min}^{12}. \tag{4.3}
\]
It follows from \((\tau_1, \xi_1) \in D_0^A\) and \((\tau - \tau_1, \xi - \xi_1) \in D_j^A\)
that
\[
|\partial_2 (\tau - |\xi_1| + |\xi - \xi_1|)| \geq \frac{|(\xi_1)_2|}{|\xi_1|} + \frac{|(\xi - \xi_1)_2|}{|\xi - \xi_1|} \gtrsim A^{-1}. \tag{4.4}
\]
Combining \(|\tau - |\xi_1| + |\xi - \xi_1|| \lesssim L_{\max}^{12}\) with (4.4), for fixed \((\xi_1)_1\) we have
\[
|\{\{\xi_1\}_2 \mid (\tau_1, \xi_1) \in E(\tau, \xi)\}| \lesssim AL_{\max}^{12}. \tag{4.5}
\]
Collecting (4.3), (4.5) and \(\xi_1 \in C_{N_0}(\xi')\), we get (4.2). \(\Box\)

We now prove Proposition 4.3 for the case (I). From Lemma 4.5 it holds that \(L_{\max}^{12} \gtrsim A^{-1} N_0\). We decompose the proof into the three cases:

(Ia) \(A^{-1} N_0 \lesssim L_0\), (Ib) \(A^{-1} N_0 \lesssim L_1\), (Ic) \(A^{-1} N_0 \lesssim L_2\).

(Ia) From Hölder inequality and Lemma 4.6, we have
\[
\left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim \|f\|_{L_{\xi_1}^2} \|g_1 \ast g_2\|_{L_{\xi_2}^2} \lesssim A^{\frac{2}{3}} (L_0 L_1 L_2)^{\frac{2}{3}} \|f\|_{L_{\xi_1}^2} \|g_1\|_{L_{\xi_2}^2} \|g_2\|_{L_{\xi_2}^2}. \tag{4.6}
\]

(Ib) From Hölder inequality and Lemma 2.2, we have
\[
\left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim \|g_1\|_{L_{\xi_1}^2} \|f \ast g_2\|_{L_{\xi_2}^2} \lesssim A^{\frac{2}{3}} (L_0 L_1 L_2)^{\frac{2}{3}} \|f\|_{L_{\xi_1}^2} \|g_1\|_{L_{\xi_2}^2} \|g_2\|_{L_{\xi_2}^2}. \tag{4.6}
\]

(Ic) From Hölder inequality and Lemma 2.2, we have
\[
\left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim \|g_2\|_{L_{\xi_2}^2} \|f \ast g_1\|_{L_{\xi_1}^2} \lesssim A^{\frac{2}{3}} (L_0 L_1 L_2)^{\frac{2}{3}} \|f\|_{L_{\xi_1}^2} \|g_1\|_{L_{\xi_2}^2} \|g_2\|_{L_{\xi_2}^2}. \tag{4.6}
\]

Here \(g_{j,-}(\cdot) = g_j(-\cdot)\).

We next consider the case (II). We apply the same strategy as that of the proof of Proposition 4.4 in [1]. Applying the transformation \(\tau_1 = |\xi_1| + c_1\) and \(\tau_2 = -|\xi_2| + c_2\) and Fubini’s theorem, we find that it suffices to prove
\[
\left| \int f(\phi_{c_1}^+ (\xi_1) + \phi_{c_2}^- (\xi_2)) g_1(\phi_{c_1}^+ (\xi_1)) g_2(\phi_{c_2}^- (\xi_2)) d\xi_1 d\xi_2 \right| \lesssim A^{\frac{2}{3}} \|g_1 \circ \phi_{c_1}^+ \|_{L_{\xi_2}^2} \|g_2 \circ \phi_{c_2}^- \|_{L_{\xi_2}^2} \|f\|_{L_{\xi_1}^2}, \tag{4.6}
\]
where \(f(\tau, \xi)\) is supported in \(c_0 \leq \tau \leq c_0 + 2\) and \(\phi_{c_k}^\pm (\xi) = (\pm |\xi| + c_k, \xi)\) for \(k = 1, 2\).

First we decompose \(f\) by angular localization characteristic functions \(\chi_{D_j^A} f \chi_{D_j^A, -A_1}\) where \(A_1\) is the minimal dyadic number which satisfies \(A_1 \geq 2^{20(1-c)^{-2}} A\) and thickened circular localization characteristic functions
From the assumption (II), we see that the sum of \([4.10]\) is immediately established from \((\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 \mid \xi^0 \leq |\xi| \leq \xi^0 + \delta\) with \(\delta = 2^{-20}(1-c)^2 N_0 A^{-1/2}\) as follows:

\[
f = \sum_{k=-[\frac{N_0}{2\delta}]}^{[\frac{N_0}{2\delta}]+1} \sum_{j=-A_1}^{A_1+1} \chi_{S_{N_0 + k\delta}^0} \chi_{\mathcal{D}_j} f.
\]

for supp \(f \subset \mathcal{D}^{A_1} \cap S_{N_0 + k\delta}^0\) with fixed \(k \in [-[N_0/2\delta], [N_0/\delta] + 1], j \in [-A_1, A_1 + 1]\). We use the scaling \((\tau, \xi) \rightarrow (N_0 \tau, N_0 \xi)\) to define

\[
\tilde{f}(\tau, \xi) = f(N_0 \tau, N_0 \xi), \quad \tilde{g}_k(\tau_k, \xi_k) = g_k(N_0 \tau_k, N_0 \xi_k).
\]

If we set \(\tilde{c}_k = N_0^{-1} c_k\), inequality \((4.7)\) reduces to

\[
\int \tilde{f}(\phi^+_{\tilde{c}_k}(\xi_1) + \phi^-_{\tilde{c}_k}(\xi_2)) \tilde{g}_1(\phi^+_{\tilde{c}_k}(\xi_1)) \tilde{g}_2(\phi^-_{\tilde{c}_k}(\xi_2)) d\xi_1 d\xi_2 \lesssim A^\frac{5}{2} \| \tilde{g}_1 \circ \phi^+_{\tilde{c}_k} \|_{L^2_\xi} \| \tilde{g}_2 \circ \phi^-_{\tilde{c}_k} \|_{L^2_\xi} \| \tilde{f} \|_{L^2_\xi},
\]

for \(\tilde{g} \in \mathcal{D}^{A_1} \cap S_{N_0 + k\delta}^0\) with fixed \(k \in [-[N_0/2\delta], [N_0/\delta] + 1], j \in [-A_1, A_1 + 1]\). We use the scaling \((\tau, \xi) \rightarrow (N_0 \tau, N_0 \xi)\) to define

\[
\tilde{f}(\tau, \xi) = f(N_0 \tau, N_0 \xi), \quad \tilde{g}_k(\tau_k, \xi_k) = g_k(N_0 \tau_k, N_0 \xi_k).
\]

Note that \(\tilde{f}\) is supported in \(S^N_0(N_0)^{-1})\) where

\[
S^N_0(N_0)^{-1} = \{ (\tau, \xi) \in \mathcal{D}^{A_1} \cap S^N_0 \mid \tau = c|\xi| + \frac{\xi^0}{N_0} \leq \tau \leq \tau = c|\xi| + \frac{\xi^0 + 1}{N_0}\}
\]

with \(\tilde{\delta} = N_0^{-1} \delta\). Thus from the \(\ell^2\) almost orthogonality, we may assume that there exist \(\xi_1^0, \xi_2^0\) such that

\[
\frac{N_1}{2N_0} \leq |\xi_1^0| \leq 4 \frac{N_1}{N_0}, \quad \frac{N_2}{2N_0} \leq |\xi_2^0| \leq 4 \frac{N_2}{N_0}
\]

(4.9) such that space variables of supp \(\tilde{g}_1 \circ \phi^+_{\xi_1^0}\) and supp \(\tilde{g}_2 \circ \phi^-_{\xi_2^0}\) are contained in the balls \(B_\delta(\xi_1^0)\) and \(B_\delta(\xi_2^0)\), respectively. By density and duality it suffices to show for continuous \(\tilde{g}_1\) and \(\tilde{g}_2\) that

\[
\|\tilde{g}_1|_{S_1} \ast \tilde{g}_2|_{S_2}\|_{L^2(S^N_0(N_0)^{-1}))} \lesssim A^\frac{5}{2} N_0^{-\frac{5}{2}} \| \tilde{g}_1\|_{L^2(S_1)} \| \tilde{g}_2\|_{L^2(S_2)}
\]

(4.10) where \(S_1, S_2\) denote the following surfaces

\[
S_1 = \{ \phi^+_{\xi_1^0}(\xi_1) \in \mathbb{R}^3 \mid \xi_1 \in B_\delta(\xi_1^0) \},
\]

\[
S_2 = \{ \phi^-_{\xi_2^0}(\xi_2) \in \mathbb{R}^3 \mid \xi_2 \in B_\delta(\xi_2^0) \}.
\]

(4.10) is immediately established from

\[
\|\tilde{g}_1|_{S_1} \ast \tilde{g}_2|_{S_2}\|_{L^2(S^N_0(N_0)^{-1}))} \lesssim A^\frac{5}{2} \| \tilde{g}_1\|_{L^2(S_1)} \| \tilde{g}_2\|_{L^2(S_2)}
\]

(4.11) where

\[
S^N_0 = \{ (\psi^+ (\xi), \xi) \in \mathcal{D}^{A_1} \cap S^N_0 \mid \psi^+ (\xi) = \tau c|\xi| + \frac{\xi^0}{N_0}\}.
\]

For any \(\sigma_i \in S_i, i = 1, 2, 3\), there exist \(\xi_1, \xi_2, \xi\) such that

\[
\sigma_1 = \phi^+_{\xi_1^0}(\xi_1), \quad \sigma_2 = \phi^+_{\xi_2^0}(\xi_2), \quad \sigma_3 = (\psi(\xi), \xi),
\]

and the unit normals \(n_i\) on \(\sigma_i\) are written as

\[
n_1(\sigma_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1, & (\xi_1)_{1} \xi_1 | \xi_1^0 | \xi_1 | \xi_1 \end{pmatrix},
\]

\[
n_2(\sigma_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & (\xi_2)_{1} \xi_2 | \xi_2 | \xi_2 | \xi_2 \end{pmatrix},
\]

\[
n_3(\sigma_3) = \frac{1}{\sqrt{c^2 + 1}} \begin{pmatrix} \pm 1, & c(\xi)_1 \xi_1 | \xi | \xi \end{pmatrix}.
\]
We deduce from $1 \lesssim |\xi|$ and (4.9) that the surfaces $S_1, S_2, S_3^\mp$ satisfy the following Hölder condition.

\begin{align*}
\sup_{\sigma_i, \sigma_i' \in S_i} \frac{|n_1(\sigma_i) - n_1(\sigma_i')|}{|\sigma_i - \sigma_i'|} + \frac{|n_1(\sigma_i)(\sigma_i - \sigma_i')|}{|\sigma_i - \sigma_i'|^2} \lesssim 1, \\
\sup_{\sigma_j, \sigma_j' \in S_j^\mp} \frac{|n_3(\sigma_j) - n_3(\sigma_j')|}{|\sigma_j - \sigma_j'|} + \frac{|n_3(\sigma_j)(\sigma_j - \sigma_j')|}{|\sigma_j - \sigma_j'|^2} \lesssim 1.
\end{align*}

(4.12) (4.13)

We may assume that there exist $\xi_1', \xi_2', \xi_3' \in \mathbb{R}^2$ such that

$$\xi_1' + \xi_2' = \xi', \quad \phi_{\xi_1}^+(\xi_1') \in S_1, \quad \phi_{\xi_2}^-(\xi_2') \in S_2, \quad (\psi^+(\xi'), \xi') \in S_3^\mp,$$

otherwise the left-hand side of (4.10) vanishes. Let $\sigma_1' = \phi_{\xi_1}^+(\xi_1')$, $\sigma_2' = \phi_{\xi_2}^-(\xi_2')$, $\sigma_3' = (\psi^+(\xi'), \xi')$. For any $\sigma_1 = \phi_{\xi_1}^+(\xi_1) \in S_1$, we deduce from $\xi_1, \xi_1' \in B_3(\xi_0)$ and $A \leq M_1 \leq 2^{10}(1 - c)^{-1}N_1/N_0$ that

$$|n_1(\sigma_1) - n_1(\sigma_1')| \leq 2^{-18}\frac{N_0}{N_1}(1 - c)^2A^{-\frac{3}{2}} \leq 2^{-8}(1 - c)A^{-\frac{3}{2}}. \quad (4.14)$$

Similarly, for any $\sigma_2 = \phi_{\xi_2}^-(\xi_2) \in S_2$ we have

$$|n_2(\sigma_2) - n_2(\sigma_2')| \leq 2^{-18}\frac{N_0}{N_2}(1 - c)^2A^{-\frac{3}{2}} \leq 2^{-8}(1 - c)A^{-\frac{3}{2}}. \quad (4.15)$$

For any $\sigma_3 \in S_3^\mp$, it follows from $S_3^\mp \subset \mathcal{D}^{A_1}$ that

$$|n_3(\sigma_3) - n_3(\sigma_3')| \leq 2^{-10}(1 - c)A^{-1}. \quad (4.16)$$

It is obvious that $|\sigma_1 - \sigma_1'|, |\sigma_2 - \sigma_2'| \leq 2\delta \leq 2^{-10}(1 - c)^2A^{-1/2}$ holds, then we get from (4.14) and (4.15) that

\begin{align*}
|(|\sigma_1 - \sigma_1'| \cdot n_1(\sigma_1')| & \leq 2^{-15}(1 - c)^2A^{-2}, \\
|(|\sigma_2 - \sigma_2'| \cdot n_2(\sigma_2')| & \leq 2^{-15}(1 - c)^2A^{-2}.
\end{align*}

(4.17) (4.18)

Similarly, we deduce from $|\sigma_3 - \frac{|\sigma_3|}{|\sigma_3'|}{\sigma_3'}| \leq 2^{-10}(1 - c)^2A^{-1}$ and (4.10) that

$$|(|\sigma_3 - \sigma_3'| \cdot n_3(\sigma_3')| = \left| \left( \sigma_3 - \frac{|\sigma_3|}{|\sigma_3'|}{\sigma_3'} \right) \cdot n_3(\sigma_3') \right| \leq 2^{-15}(1 - c)^2A^{-2}. \quad (4.19)$$

(4.17) means that $S_1$ is contained in an slab of thickness $2^{-15}(1 - c)^2A^{-2}$ with respect to the $n_1(\sigma_1')$ direction. From $\ell^2$ orthogonality, we may assume that $S_2$ and $S_3$ are also contained in similar $2^{-15}(1 - c)^2A^{-2}$ thick slabs;

\begin{align*}
|(|\sigma_2 \cdot n_2(\sigma_2')| & \leq 2^{-15}(1 - c)^2A^{-2}, \\
|(|\sigma_3 \cdot n_3(\sigma_3')| & \leq 2^{-15}(1 - c)^2A^{-2}.
\end{align*}

(4.17)

Similarly, we may assume that surfaces $S_1, S_2$ are contained in slabs of thickness $2^{-15}(1 - c)^2A^{-2}$ with respect to the $n_2(\sigma_2')$ direction and the surfaces $S_1, S_2$ are contained in slabs of thickness $2^{-15}(1 - c)^2A^{-2}$ with respect to the $n_3(\sigma_3')$ direction. Collection the above assumptions, for $i, j = 1, 2, 3$,

$$|(|\sigma_i \cdot n_j(\sigma_j')| \leq 2^{-15}(1 - c)^2A^{-2}. \quad (4.20)$$

We define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$T = 2^{-10}(1 - c)^2A^{-2}(N^\top)^{-1}, \quad N = N(\sigma_1', \sigma_2', \sigma_3').$$
If the following conditions are established, we immediately obtain the desired estimate (4.11) by applying Proposition 4.13 with \( T \) and \( \tilde{S}_i := T^{-1}S_i \) \((i = 1, 2, 3)\).

(I) \( \frac{1 - c}{2} A^{-1} \leq |\det N(\sigma_1, \sigma_2, \sigma_3)| \) for any \( \sigma_i \in S_i \).

(II) \( \text{diam}(\tilde{S}_i) < 1 \).

(III) \( \frac{1}{2} \leq \det(\tilde{n}_1(\tilde{\sigma}_i), \tilde{n}_2(\tilde{\sigma}_i), \tilde{n}_3(\tilde{\sigma}_i)) \leq 1 \) for any \( \tilde{\sigma}_i \in \tilde{S}_i \).

(IV) \( \sup_{\tilde{\sigma}_i, \tilde{\sigma}_i' \in \tilde{S}_i} \frac{|\tilde{n}_i(\tilde{\sigma}_i) - \tilde{n}_i(\tilde{\sigma}_i')|}{|\tilde{\sigma}_i - \tilde{\sigma}_i'|} + \frac{|\tilde{n}_i(\tilde{\sigma}_i') \cdot (\tilde{\sigma}_i - \tilde{\sigma}_i')|}{|\tilde{\sigma}_i - \tilde{\sigma}_i'|^2} \leq 1 \) for the unit normals \( \tilde{n}_i \) on \( \tilde{S}_i \).

We first show (I). From (4.14), (4.16) it suffices to show

\[
(1 - c)A^{-1} \leq |\det N(\sigma_1', \sigma_2', \sigma_3')|.
\]

Seeing that \( \sigma_1' = \phi_1^+(\xi_1'), \sigma_2' = \phi_2^-(\xi_2'), \sigma_3' = (\psi^+(\xi'), \xi') \) and \( \xi_1' + \xi_2' = \xi' \), we get

\[
|\det N(\sigma_1', \sigma_2', \sigma_3')| \geq \frac{1}{4} \left| \begin{array}{cc} 1 & \pm 1 \\ -1 & 1 \\ \end{array} \right| \left| \begin{array}{cc} -1 & 1 \\ \xi_1' & \xi_2' \\ \end{array} \right| \left| \begin{array}{cc} \xi_1' & \xi_2' \\ -1 & 1 \\ \end{array} \right| \left( 1 - c \right) \left| \begin{array}{cc} \xi_1' & \xi_2' \\ \xi_1' & \xi_2' \\ \end{array} \right|
\]

\[
\geq (1 - c)A^{-1}.
\]

(II) is established from (4.20).

\[
|T^{-1}(\sigma_i - \sigma_i)| = 2^{10} (1 - c)^{-2} A^2 \left| \begin{array}{ccc} n_1(\sigma_1') \cdot (\sigma_i - \sigma_i') \\ n_2(\sigma_2') \cdot (\sigma_i - \sigma_i') \\ n_3(\sigma_3') \cdot (\sigma_i - \sigma_i') \end{array} \right| \leq 2^{-3} < \frac{1}{2}.
\]

Next we show (III). Note that the unit normals \( \tilde{n}_i \) on \( \tilde{S}_i \) are written as follows.

\[
\tilde{n}_i(\tilde{\sigma}_i) = \frac{(T^{-1})^T n_i(T\tilde{\sigma}_i)}{|(T^{-1})^T n_i(T\tilde{\sigma}_i)|} = \frac{N^{-1}n_i(T\tilde{\sigma}_i)}{|N^{-1}n_i(T\tilde{\sigma}_i)|}.
\]

In particular, the unit normals on \( T^{-1}\sigma_i' \) are the unit vectors \( e_i \);

\[
\tilde{n}_i(T^{-1}\sigma_i') = N^{-1}n_i(\sigma_i') = e_i.
\]

From (4.22), we get

\[
\|N^{-1}\| = \|N^T\|^{-1} \leq 2|\det N^T|^{-1} \|N^T\|^2 \leq 12(1 - c)^{-1} A.
\]

Thus we obtain

\[
\|T\| \leq 2^{-6}(1 - c)A^{-1}.
\]

We deduce from (4.14), (4.16), (4.22), (4.24) that

\[
|N^{-1}n_i(T\tilde{\sigma}_i) - e_i| = |N^{-1}(n_i(T\tilde{\sigma}_i) - n_i(\sigma_i'))| \leq 2^{-7}.
\]

This gives \( |\tilde{n}_i(\tilde{\sigma}_i) - e_i| \leq 2^{-5} \) and (III) is now obtained. Finally we show (IV). It follows from (4.23) that

\[
\frac{|\tilde{n}_i(\tilde{\sigma}_i) - \tilde{n}_i(\tilde{\sigma}_i')|}{|\tilde{\sigma}_i - \tilde{\sigma}_i'|} \leq 3 \frac{|N^{-1}(n_i(T\tilde{\sigma}_i) - n_i(T\tilde{\sigma}_i'))|}{|\tilde{\sigma}_i - \tilde{\sigma}_i'|} \leq 3 \|N^{-1}\| \frac{|n_i(T\tilde{\sigma}_i) - n_i(T\tilde{\sigma}_i')|}{|T\tilde{\sigma}_i - T\tilde{\sigma}_i'|} \leq 1.
\]
This completes (IV).

We now consider $0 \leq \angle(\xi_1, \xi_2) \leq \pi/2$. First we show the estimate which is similar to Proposition 4.3 for $64 \leq A \leq N_0^\frac{1}{2}$ and $16 \leq |j_1 - j_2| \leq 32$. In this case, thanks to $0 \leq \angle(\xi_1, \xi_2) \leq \pi/2$, $N_0 \sim N_1 \sim N_2$ always holds true and we can obtain the better estimates compared to Proposition 4.3.

**Proposition 4.7.** Let $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$, $0 < c < 1$ and $f$, $g_1, g_2 \in L^2$ be satisfy

$$\text{supp } f \subset K_{N_0, L_0}^+, \quad \text{supp } g_1 \subset D_{j_1}^A \cap K_{N_1, L_1}^-, \quad \text{supp } g_2 \subset D_{j_2}^A \cap K_{N_2, L_2}^+,$$

and $N_0 \sim N_1 \sim N_2$, $64 \leq A \leq N_0^\frac{1}{2}$, $16 \leq |j_1 - j_2| \leq 32$. Then the following estimate holds:

$$\left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1d\tau_2d\xi_1d\xi_2 \right| \lesssim A^\frac{2}{3}(L_0L_1L_2)^{\frac{1}{3}} \|f\|_{L^2_2} \|g_1\|_{L^2_2} \|g_2\|_{L^2_2}. \quad (4.27)$$

**Proof.** The proof is almost analogous to that of Proposition 4.3. Difference between them is a step of decomposition. Precisely, in the proof of Proposition 4.3 we decomposed $f$ into $\sim A^\frac{2}{5}$ pieces. We here decompose functions into finitely many pieces. From supp $g_1 \subset D_{j_1}^A$, supp $g_2 \subset D_{j_2}^A$ and $16 \leq |j_1 - j_2| \leq 32$, after suitable and harmless decomposition, we can assume that there exists $j$ such that $16 \leq |j_1 - j| \leq 32$ and supp $f \subset D_{j}^A$. Furthermore we decompose $f, g_1, g_2$ into finitely many pieces as follows:

$$f = \sum_{j' = j_1}^{j_1 + k'} \chi_{D_{j'}^A} f, \quad g_1 = \sum_{j_1' = j_1}^{j_1' + k_1} \chi_{D_{j_1'}^A} g_1, \quad g_2 = \sum_{j_2' = j_2}^{j_2 + k_2} \chi_{D_{j_2'}^A} g_2$$

where $k$ is the minimal dyadic number which satisfies $k \geq 2^{20}(1-c)^{-2}$, $A_1 := kA$ and $j_0, j_0', j_0''$ satisfy

$$\bigcup_{j_0' \leq j' \leq j_0 + k} D_{j'}^A = D_j^A,$$

with fixed $j' \in [j_0, j_0 + k]$, $j_1' \in [j_1, j_1' + k]$, $j_2' \in [j_2, j_2 + k]$.

We utilize the same notations as in the proof of Proposition 4.3. By the same argument as of the proof of Proposition 4.3 we only need to verify the following estimate:

$$\|\tilde{g}_1|_{S_1} \ast \tilde{g}_2|_{S_2}\|_{L^2(S_1)} \lesssim A^\frac{2}{3} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)} \quad (4.28)$$

where

$$S_1 = \left\{ \phi_{\xi_1}^+(\xi) \in D_{j_1}^A : \frac{1-c}{4} \leq |\xi_1| \leq 2 \right\}, \quad S_2 = \left\{ \phi_{\xi_2}^-(\xi) \in D_{j_2}^A : \frac{1-c}{4} \leq |\xi_2| \leq 2 \right\},$$

$$S_3 = \left\{ (\psi^\top(\xi), \xi) \in D_{j_1}^A : \frac{1}{2} \leq |\xi| \leq 4, \psi^\top(\xi) = \mp c|\xi| + \frac{c_0}{N_0} \right\}.$$

We recall that the unit norms on $\sigma_i$ $\in S_i$ $(i = 1, 2, 3)$ are written as:

$$n_1(\sigma_1) = \frac{1}{\sqrt{2}} \left( -1, \frac{(\xi_1)_1}{|\xi_1|}, \frac{(\xi_1)_2}{|\xi_1|} \right), \quad n_2(\sigma_2) = \frac{1}{\sqrt{2}} \left( 1, \frac{(\xi_2)_1}{|\xi_2|}, \frac{(\xi_2)_2}{|\xi_2|} \right), \quad n_3(\sigma_3) = \frac{1}{\sqrt{2}(\pm 1, c(\xi_1)_1, c(\xi_2)_2)}.$$
where
\[ \sigma_1 = \phi_1^+ (\xi_1), \quad \sigma_2 = \phi_2^+ (\xi_2), \quad \sigma_3 = (\psi (\xi), \xi). \]
We may assume that there exist \( \xi_1', \xi_2', \xi_3' \in \mathbb{R}^2 \) such that
\[ \xi_1' + \xi_2' = \xi', \quad (\sigma_1' := \phi_1^+ (\xi_1')) \in S_1, \quad (\sigma_2' := \phi_2^+ (\xi_2')) \in S_2, \quad (\sigma_3' := (\psi (\xi'), \xi')) \in S_3. \]
From \( S_1 \subset \mathcal{D}_{j_1}^{A_1}, \quad S_2 \subset \mathcal{D}_{j_2}^{A_1} \) and \( S_3 \subset \mathcal{D}_{j_3}^{A_1} \), we easily observe
\begin{align*}
|n_1(\sigma_1) - n_1(\sigma_1')| &\leq 2^{-10} (1-c)A^{-1}, \\
|n_2(\sigma_2) - n_2(\sigma_2')| &\leq 2^{-10} (1-c)A^{-1}, \\
|n_3(\sigma_3) - n_3(\sigma_3')| &\leq 2^{-10} (1-c)A^{-1}.
\end{align*}
(4.29) (4.30) (4.31)
The above estimates (4.29)-(4.31) give
\begin{align*}
| (\sigma_1 - \sigma_1') \cdot n_1(\sigma_1') | &\leq 2^{-20} (1-c)^2 A^{-2}, \\
| (\sigma_2 - \sigma_2') \cdot n_2(\sigma_2') | &\leq 2^{-20} (1-c)^2 A^{-2}, \\
| (\sigma_3 - \sigma_3') \cdot n_3(\sigma_3') | &\leq 2^{-20} (1-c)^2 A^{-2}.
\end{align*}
By the same argument as in the proof of Proposition 4.3, we can assume
\[ |(\sigma_i - \sigma_i') \cdot n_j(\sigma_j')| \leq 2^{-20} (1-c)^2 A^{-2} \quad \text{for any } i, j = 1, 2, 3. \]
(4.32)
The remaining part is only to prove (I)-(IV) in Proposition 4.3 with
\[ T = 2^{-10} (1-c)^2 A^{-2} (N^2)^{-1}, \quad N = N(\sigma_1', \sigma_2', \sigma_3') \]
and \( \tilde{S}_i := T^{-1} S_i \quad (i = 1, 2, 3) \). (I)-(IV) are verified from (4.29)-(4.32) as we proved in the proof of Proposition 4.3. To avoid redundancy, we omit the proof of them.

Lastly, we consider the case of sufficiently small \( \langle \xi_1, \xi_2 \rangle \).

\textbf{Proposition 4.8.} Let \( \tau = \tau_1 + \tau_2, \quad \xi = \xi_1 + \xi_2, \quad 0 < c < 1 \) and \( M_0 \) is the minimal dyadic number which satisfies \( N_0^{-\frac{2}{3}} \leq M_0 \). We assume that \( f, g_1, g_2 \in L^2 \) satisfy
\[ \text{supp } f \subset K_{N_0, L_0}^{\pm, c}, \quad \text{supp } g_1 \subset \mathcal{D}_{j_1}^{M_0} \cap K_{N_1, L_1}, \quad \text{supp } g_2 \subset \mathcal{D}_{j_2}^{M_0} \cap K_{N_2, L_2}, \]
with \( N_0 \sim N_1 \sim N_2, \quad |j_1 - j_2| \leq 16 \). Then the following estimate holds;
\[ \left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim N_0^{\frac{4}{3}} (L_0 L_{\min}^{12})^{\frac{1}{3}} + \|f\|_{L_{\xi, r}^2} + \|g_1\|_{L_{\xi, r}^2} + \|g_2\|_{L_{\xi, r}^2}. \]
(4.33)

\textbf{Proof.} We can assume \( L_1 \leq L_2 \) by symmetry. By Hölder inequality, (4.33) is established if we show
\[ \| \chi_{K_{N_2, L_2}} \left( \chi_{K_{N_0, L_0}} f \right) \ast \left( \chi_{K_{N_1, L_1}} g \right) \|_{L_{\xi, r}^2} \lesssim \left( N_0^{\frac{4}{3}} L_0 \right)^{1/2} \|f\|_{L_{\xi, r}^2} + \|g\|_{L_{\xi, r}^2}. \]
(4.34)
regardless of the choice of the signs \( \pm_0, \pm_1 \). It is easily confirmed that (4.34) can be verified by the proof of Proposition 2.2 with minor modification. Indeed, same as in the proof of Proposition 2.2, we find that the desired estimate (4.34) is shown by
\[ \sup_{\tau, \xi} |E(\tau, \xi)| \lesssim N_0^{\frac{4}{3}} L_0 L_1 \]
(4.35)
where \( E(\tau, \xi) := \{ (\tau_1, \xi_1) \in \mathcal{D}_{0}^{M_0} \mid \langle \tau - \tau_1 \pm_0 c (\xi - \xi_1) \rangle \sim L_0, \langle \tau_1 \pm_1 |\xi_1| \rangle \sim L_1 \} \). Applying the same proof as in Proposition 2.2, we immediately obtain (4.35) thanks to \( N_1 M_0^{-1} \sim N_0^{\frac{4}{3}} \). \( \square \)

We now prove the crucial estimates (4.31) and (4.32) with \( \pm_2 = + \) and \( N_0 \lesssim N_1 \sim N_2 \).
where

\[ I_1^- := N_1^{-1} \int \left( \chi_{K_{N_1}^-, l_1} \cdot f \right) \left( \chi_{K_{N_1}^+, l_1} \cdot g_1 \right) d\tau d\xi, \]

\[ I_2^- := N_0 N_1^{-1} N_2^{-1} \int \left( \chi_{K_{N_0}^-, l_0} \cdot f \right) \left( \chi_{K_{N_0}^+, l_0} \cdot g_1 \right) d\tau d\xi. \]

**Proof.** We first note that if \( N_1 \lesssim L_{\max}^{012} \) then (4.36) and (4.37) are obtained by the same proof as that of Theorem 3.2. Therefore we can assume \( L_{\max}^{012} \lesssim N_1 \). We can also assume that \( 1 \ll N_0 \). Indeed, if \( N_0 \sim 1 \) then (4.36) and (4.37) are immediately obtained by using Proposition 2.1 as \( N_0 \sim 1 \).

If \( s \in (-3/4, -1/2) \), considering \( N_0 \lesssim N_1 \sim N_2 \), we observe that the latter estimate (4.37) is difficult to show compared with the former one. Clearly, the proof of (4.36) and (4.37) become easier as \( s \) gets greater. Therefore, we here focus our attention on proving (4.37) for \( s \in (-3/4, -1/2) \).

(4.37) is equivalent to

\[ \sum_{N_1} \sum_{N_0 \leq N_1 \sim N_2} \sum_{L_i \leq N_1} N_0 N_1^{-2} \left| \int f^{\pm, c}(\tau_1 + \tau_2, \xi_1 + \xi_2)g_1^-(\tau_1, \xi_1)g_2^+(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim \| f \|_{\tilde{H}_{-s, b}^{1, b}} \| g_1 \|_{\tilde{H}_{-s, b}^{1, b}} \| g_2 \|_{\tilde{H}_{-s, b}^{1, b}}. \]

Here we utilized the denotations \( f^{\pm, c} := \chi_{K_{N_0}^-, l_0} \cdot f, \ g_1^- := \chi_{K_{N_1}^-, l_1} \cdot g_1, \ g_2^+ := \chi_{K_{N_2}^+, l_2} \cdot g_2 \). For simplicity, we use

\[ I(f, g, h) := N_0 N_1^{-2} \left| \int f(\tau, \xi)g(\tau_1, \xi_1)h(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \]

where \( \tau = \tau_1 + \tau_2 \) and \( \xi = \xi_1 + \xi_2 \). By the decomposition of \( \mathbb{R}^3 \times \mathbb{R}^3 = \bigcup_{M_0, M_1} \mathcal{D}_{j_1}^{M_0} \times \mathcal{D}_{j_2}^{M_0} \cup \bigcup_{M_0, M_1} \mathcal{D}_{j_1}^{M_0} \times \mathcal{D}_{j_2}^{M_1} \cup \bigcup_{M_0, M_1} \mathcal{D}_{j_1}^{M_1} \times \mathcal{D}_{j_2}^{M_1} \cup \bigcup_{M_0, M_1} \mathcal{D}_{j_1}^{M_0} \times \mathcal{D}_{j_2}^{M_1} \),

we only need to show

(I) \[ \sum_{N_0 \sim N_1 \sim N_2 \sim N_1} \sum_{L_i \leq N_1} \sum_{M_0, M_1, j_1, j_2} I(f^{\pm, c}, g_1^{-\cdot M_0, j_1}, g_2^{+\cdot M_0, j_2}) \lesssim \| f \|_{\tilde{H}_{-s, b}^{1, b}} \| g_1 \|_{\tilde{H}_{-s, b}^{1, b}} \| g_2 \|_{\tilde{H}_{-s, b}^{1, b}}, \]

(II) \[ \sum_{N_0 \sim N_1 \sim N_2 \sim N_1} \sum_{L_i \leq N_1} \sum_{M_0, M_1, j_1, j_2} I(f^{\pm, c}, g_1^{-\cdot A, j_1}, g_2^{+\cdot A, j_2}) \lesssim \| f \|_{\tilde{H}_{-s, b}^{1, b}} \| g_1 \|_{\tilde{H}_{-s, b}^{1, b}} \| g_2 \|_{\tilde{H}_{-s, b}^{1, b}}, \]

(III) \[ \sum_{N_1 \sim N_0 \sim N_1 \sim N_2 \sim N_1} \sum_{M_0, M_1, j_1, j_2} I(f^{\pm, c}, g_1^{-\cdot M_1, j_1}, g_2^{+\cdot M_1, j_2}) \lesssim \| f \|_{\tilde{H}_{-s, b}^{1, b}} \| g_1 \|_{\tilde{H}_{-s, b}^{1, b}} \| g_2 \|_{\tilde{H}_{-s, b}^{1, b}}, \]

(IV) \[ \sum_{N_1 \sim N_0 \sim N_1 \sim N_2 \sim N_1} \sum_{M_0, M_1, j_1, j_2} I(f^{\pm, c}, g_1^{-\cdot A, j_1}, g_2^{+\cdot A, j_2}) \lesssim \| f \|_{\tilde{H}_{-s, b}^{1, b}} \| g_1 \|_{\tilde{H}_{-s, b}^{1, b}} \| g_2 \|_{\tilde{H}_{-s, b}^{1, b}}, \]
where \( g_1^{-, A, j_1} := g_1^1 |_{D^A} \) and \( g_2^{+, A, j_2} := g_2^2 |_{D^A} \). We further simplify (I)-(IV). From \( \ell^2 \) Cauchy-Schwarz inequality and \( L^{12}_{\text{max}} \lesssim N_1 \), it suffices to show that there exists \( 0 < \varepsilon' < 1 \) such that the following estimates hold:

\[
(I') \quad \sum_{-M_0 \leq j_1, j_2 \leq M_0-1} I(f^{\pm, c}, g_1^{-, M_0, j_1}, g_2^{+, M_0, j_2}) \lesssim N_0^{-1} \varepsilon' (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}},
\]

\[
(II') \quad \sum_{64 \leq A \leq M_0} \sum_{-A \leq j_1, j_2 \leq A-1} I(f^{\pm, c}, g_1^{-, A, j_1}, g_2^{+, A, j_2}) \lesssim N_0^{-\frac{1}{2}} N_1^{1/2} (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}},
\]

\[
(III') \quad \sum_{-M_1 \leq j, \leq M_1} I(f^{\pm, c}, g_1^{-, M_1, j}, g_2^{+, M_1, j}) \lesssim N_0^{-1} N_1^{1/2} (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}},
\]

\[
(IV') \quad \sum_{64 \leq A \leq M_1} \sum_{-A \leq j \leq A-1} I(f^{\pm, c}, g_1^{-, A, j}, g_2^{+, A, j}) \lesssim N_0^{-1} N_1^{1/2} (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}}.
\]

If \(-3/4 < s\), (I') is immediately established by using Proposition 1.7

\[
\sum_{-M_0 \leq j_1, j_2 \leq M_0-1} I(f^{\pm, c}, g_1^{-, M_0, j_1}, g_2^{+, M_0, j_2}) \\
\sim \sum_{-M_0 \leq j_1, j_2 \leq M_0-1} \frac{N_0^{-1}}{16} \left| \int f^{\pm, c}(\tau, \xi) g_1^{-, M_0, j_1} (\tau_1, \xi_1) g_2^{+, M_0, j_2} (\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \\
\lesssim N_0^{-\frac{1}{2}} (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}},
\]

Next we prove (II'). It follows from Proposition 4.7 that

\[
\sum_{64 \leq A \leq M_0} \sum_{-A \leq j_1, j_2 \leq A-1} I(f^{\pm, c}, g_1^{-, A, j_1}, g_2^{+, A, j_2}) \\
\sim \sum_{64 \leq A \leq M_0} \sum_{-A \leq j_1, j_2 \leq A-1} \frac{N_0^{-1}}{16} \left| \int f^{\pm, c}(\tau, \xi) g_1^{-, A, j_1} (\tau_1, \xi_1) g_2^{+, A, j_2} (\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \\
\lesssim \sum_{64 \leq A \leq M_0} N_0^{-\frac{1}{2}} A^{\frac{1}{2}} (L_0 L_1 L_2)^{\frac{1}{2}} \| f^{\pm, c} \|_{L^2_{\xi, \tau}}, \| g_1^{-} \|_{L^2_{\xi, \tau}}, \| g_2^{+} \|_{L^2_{\xi, \tau}}.
\]

(III') is verified as follows. By Lemma 4.2 we have \( N_0 \lesssim L^{12}_{\text{max}} \). For the sake of simplicity, we here consider the case of \( N_0 \lesssim L_0 \). The other cases can be proved similarly. We deduce from Proposition 2.1 and Hölder
inequality that
\[
\sum_{-M_1 \leq l_2, j_2 \leq M_1 -1} I(f^{\pm, c}, g_1^{-, A_1, j_1}, g_2^{+, A_1, j_2})
\]
\[
\sim N_0 N_1^{-2} \sum_{-M_1 \leq l_2, j_2 \leq M_1 -1} \left| \int f^{\pm, c}(\tau, \xi) g_1^{-, A_1, j_1}(\tau_1, \xi_1) g_2^{+, A_1, j_2}(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\]
\[
\leq N_0 N_1^{-2} \sum_{-M_1 \leq l_2, j_2 \leq M_1 -1} \sum_{|j_1| \geq 2} \left\| \chi_{K_{N_0 L_0}}(g_1^{-, A_1, j_1} \ast g_2^{+, A_1, j_2}) \right\|_{L^2_{\xi, \tau}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0 N_1^{-2} N_1^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_0^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \sum_{-M_1 \leq l_2, j_2 \leq M_1 -1} \sum_{|j_1| \geq 2} \left\| g_1^{-, A_1, j_1} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+, A_1, j_2} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0^{-s} N_1^{2-2}\varepsilon' (L_0 L_1 L_2)^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+} \right\|_{L^2_{\xi, \tau}}
\]
Lastly, we prove (IV)'. We use the two estimates depending on the relation between \(N_0\) and \(N_1\). Precisely, we utilize Proposition 2.2 if \(N_0^3 \lesssim N_1^2\), and if not so, we employ Proposition 4.3. We first assume \(N_0^3 \lesssim N_1^2\).
\[
\sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} I(f^{\pm, c}, g_1^{-, A_1, j_1}, g_2^{+, A_1, j_2})
\]
\[
\sim N_0 N_1^{-2} \sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} \left| \int f^{\pm, c}(\tau, \xi) g_1^{-, A_1, j_1}(\tau_1, \xi_1) g_2^{+, A_1, j_2}(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\]
\[
\leq N_0 N_1^{-2} \sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} \left\| \chi_{K_{N_1 L_1}}(f^{\pm, c}) \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-, A_1, j_1} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+, A_1, j_2} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0^{-s} N_1^{2-2}\varepsilon' (L_0 L_1 L_2)^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+} \right\|_{L^2_{\xi, \tau}}
\]
If \(0 < \varepsilon' \leq \frac{3}{2} (s + \frac{3}{4})\), this completes (IV)'. We next assume \(N_0^3 \gtrsim N_1^2\). From Proposition 4.3 and \(M_1 \sim N_1 / N_0\), we observe that
\[
\sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} I(f^{\pm, c}, g_1^{-, A_1, j_1}, g_2^{+, A_1, j_2})
\]
\[
\sim N_0 N_1^{-2} \sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} \left| \int f^{\pm, c}(\tau, \xi) g_1^{-, A_1, j_1}(\tau_1, \xi_1) g_2^{+, A_1, j_2}(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\]
\[
\lesssim N_0 N_1^{-2} \sum_{64 \leq A \leq M_1} \sum_{-A \leq j_1, j_2 \leq A -1} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} \left\| A_{\xi, \tau} (L_0 L_1 L_2)^{\frac{1}{2}} \right\|_{L^2_{\xi, \tau}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \sum_{-A \leq j_1, j_2, A \leq 16} \sum_{|j_1| \geq 2} \sum_{|j_2| \geq 2} \sum_{|A| \leq 16} \left\| g_1^{-, A_1, j_1} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+, A_1, j_2} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0 N_1^{-2} N_1^{\frac{1}{2}} N_0^{\frac{1}{2}} (L_0 L_1 L_2)^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0^{s+3} N_1^{3-\frac{3}{4}} (\varepsilon' + \frac{3}{4}) (L_0 L_1 L_2)^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+} \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim N_0^{s} N_1^{2-\varepsilon'} (L_0 L_1 L_2)^{\frac{1}{2}} \left\| f^{\pm, c} \right\|_{L^2_{\xi, \tau}} \left\| g_1^{-} \right\|_{L^2_{\xi, \tau}} \left\| g_2^{+} \right\|_{L^2_{\xi, \tau}}
\]
This completes the proof of (IV). □

5. Negative result

In this section, we establish Theorem [1]. For convenience, we restate the Theorem [1].

**Theorem 5.1.** Let $d = 2$, $0 < c < 1$ and $s < -\frac{1}{2}$. Then for any $T > 0$, the data-to-solution map $(u_0, u_1, n_0, n_1) \mapsto (u, n)$ of (1), as a map from the unit ball in $H^{s+1} \times H^s \times H^s \times H^{s-1}$ to $C([0, T]; H^{s+1}) \cap C^1([0, T]; H^s) \times C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ fails to be $C^2$.

**Proof.** By the same argument as in the proof of Theorem 1.4 in [7], it suffices to prove that for every $C > 0$, there exist real-valued functions $u_0 \in H^{s+1}$ and $n_0 \in H^s$ such that

$$
\sup_{0 < t \leq T} \left\| \int_0^t \frac{\sin((t-t')(\nabla))}{\langle \nabla \rangle} ((\cos(t'(\nabla))u_0)(\cos(t'|\nabla|)n_0))dt' \right\|_{H^{s+1}} 
\geq C \|u_0\|_{H^{s+1}} \|n_0\|_{H^s}.
$$

(5.1)

Let $N \gg 1$, we define the sets $D_1, D_2, D_3 \subset \mathbb{R}^2$ as

$$
D_1 := [N, N + 1] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}] \cup [-N - 1, -N] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}],
$$

$$
D_2 := \left[ \frac{2N}{1-c}, \frac{2N}{1-c} + 1 \right] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}] \cup \left[ \frac{2N}{1-c} - 1, \frac{2N}{1-c} \right] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}],
$$

$$
D_3 := \left[ \frac{1+c}{1-c} N - 1, \frac{1+c}{1-c} N + 1 \right] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}] \cup \left[ \frac{1+c}{1-c} N - 1, \frac{1+c}{1-c} N + 1 \right] \times [-N^{\frac{1}{2}}, N^{\frac{1}{2}}].
$$

We set $(u_{N,0}, n_{N,0})$ as

$$(F_x u_{N,0})(\xi) := N^{-s-\frac{1}{2}} \chi_{D_1}(\xi), \quad (F_x n_{N,0})(\xi) := N^{-s-\frac{1}{2}} \chi_{D_2}(\xi).$$

We easily verify that $u_{N,0}, n_{N,0}$ are real-valued and $\|u_{N,0}\|_{H^{s+1}} \sim 1, \|n_{N,0}\|_{H^s} \sim 1$. A simple calculation gives

$$
\mathcal{F}_x \left( \int_0^t \frac{\sin((t-t')(\nabla))}{\langle \nabla \rangle} ((\cos(t'(\nabla))u_{N,0})(\cos(t'|\nabla|)n_{N,0}))dt' \right) \chi_{D_3}(\xi)
\sim N^{-2s-\frac{1}{2}} \int_0^t \int_{\mathbb{R}^2} \left( e^{-it(\xi)} - e^{it(\xi)} \right) \chi_{D_3}(\xi) \times
\left( ((e^{-it'\xi}) - e^{it'\xi}) \chi_{D_3}(\xi) (e^{-it'|\xi|\xi_1} - e^{it'|\xi|\xi_1}) \chi_{D_2}(\xi) \right) d\xi dt'
\sim N^{-2s-2} \left( e^{-\frac{1+c}{1-c}N} - e^{\frac{1+c}{1-c}N} \right) t \chi_{D_3}(\xi) + N^{-2s-2}O(t^2) \chi_{D_3}(\xi),
$$

(5.2)

for $N^{-1} \leq t \ll 1$. For any sufficient small $t > 0$, we can choose $N \gg 1$ such that

$$
N^{-1} \leq t \ll 1 \quad \text{and} \quad \left| e^{-it\frac{1+c}{1-c}N} - e^{it\frac{1+c}{1-c}N} \right| \geq \frac{1}{2}.
$$

Thus we can choose $0 < t < T$, $N \gg 1$ such that the first term of (5.2) dominates the second term.

$$
\int_0^t \sin((t-t')(\nabla)) \left( ((\cos(t'(\nabla))u_{N,0})(\cos(t'|\nabla|)n_{N,0})) \right) dt' \bigg\|_{H^{s+1}}
\geq \left\| \chi_{D_3} \int_0^t \sin((t-t')(\nabla)) \left( ((\cos(t'(\nabla))u_{N,0})(\cos(t'|\nabla|)n_{N,0})) \right) dt' \right\|_{L^2}
\sim N^{-s-\frac{1}{2}} \chi_{D_3} L^2 \sim N^{-s-\frac{1}{2}}.
$$

This completes the proof of (5.1). □

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