Gravity from CFT on $S^N(X)$: Symmetries and Interactions

Antal Jevicki, Mihail Mihailescu and Sanjaye Ramgoolam

Brown University
Providence, RI 02912
antal, mm, ramgosk@het.brown.edu

The orbifold CFT dual to string theory on $ADS_3 \times S^3$ allows a construction of gravitational actions based on collective field techniques. We describe a fundamental role played by a Lie algebra constructed from chiral primaries and their CFT conjugates. The leading terms in the algebra at large $N$ are derived from the computation of chiral primary correlation functions. The algebra is argued to determine the dynamics of the theory, its representations provide free and interacting hamiltonians for chiral primaries. This dynamics is seen to be given by an effective one plus one dimensional field theory. The structure of the algebra and its representations shows qualitatively new features associated with thresholds at $L_0 = N$, $L_0 = N/2$ and $L_0 = N/4$, which are related to the stringy exclusion principle and to black holes. We observe relations between fusion rules of $SU_q(2|1,1)$ for $q = e^{i\pi/4}$, and the correlation functions, which provide further evidence for a non-commutative spacetime.
1. Introduction

In the context of the ADS/CFT duality [1,2,3] we began in [4] (hereafter referred to as I) a construction of elements of supergravity on $\text{ADS}_3 \times S^3$ based on a simple orbifold conformal field theory with target space $S^N(T^4)$ or $S^N(K3)$. The novel feature of the emerging gravitational theory is a ‘stringy exclusion principle’ (3) which follows from the CFT.

We argued that this exclusion principle implies a role for a non-commutative space-time. Similar aspects of non-commutativity in closed string backgrounds were also found recently in [6]. The precise form of non-commutativity suggested in I was $SU(2)_q \times SU_q(1,1)$ with $q = e^{i\pi N}$. One way to arrive at the value of $q$ was geometrical. A deformation of the $S^3$ sphere to $SU_q(2)$ implies, using the special properties of quantum groups at roots of unity, that KK reduction on the quantum sphere leads to a cutoff in the spins of KK states. Using the match of KK states with generating chiral primaries [7], the cutoff on such chiral primaries in the CFT was analyzed and shown to be in agreement with the above $SU_q(2) \times SU_q(1,1)$ spacetime. Another way to arrive at the same value of $q$ was dynamical as in similar work in earlier matrix models [8]. An $SL_q(2)$ subalgebra was identified in the algebra generated by the chiral primaries and their conjugates. This makes it clear that the parameter $1/N$ governing the non-commutativity of space-time is identical to the parameter measuring the strength of interactions, so that the emerging model of non-commutative gravity here does not involve doing classical gravity on a non-commutative spacetime followed by quantum effects being turned as another parameter is being varied. A review of ideas in the general subject of non-commutative space-times in the gravitational context, with extensive references is given in [9].

The techniques of collective field theory allow, in general, the construction of spacetime actions starting from the algebra of $S^N$ invariant variables. Since the short representations related to chiral primaries have been matched with gravity on $\text{ADS}_3 \times S^3$, it is instructive to focus on the dynamics associated with these representations and their CFT conjugates. We describe a central role played by a Lie algebra of observables associated to the chiral primaries. The simplest representations of this Lie algebra involve the Fock space generated by a certain class of chiral primaries (those which are generators of the chiral ring). One wants to focus on these representations first because they include states created by the graviton, whose propagation and dynamics is of interest. We describe this dynamics in terms of two dimensional field theory. This represents the simplest representation of the
algebra. Other reps. will be of relevance when we go beyond chiral primaries and study more stringy states $[10,11,12]$.

The paper is organized as follows. In ch.2, we summarize the field content and basic symmetries of the CFT, and then describe the computation of some exact correlation functions involving twisted sector chiral primaries. In ch.3 concentrating on the untwisted sector we review the algebra of observables given by the creation-annihilation operators and their commutators. We study a Fock space representation of this algebra where a creation operator is associated to each generating chiral primary. We use a coherent state description for the Fock space and derive, using the correlation functions of the chiral primaries, a formula for a symplectic form (equivalently the kinetic term of a Lagrangian) on the space of coherent states. We also obtain a formula for the Hamiltonian in this coherent state representation. We outline how these exact formulae can be developed in a $\frac{1}{N}$ expansion into a realization of the algebra in terms of free oscillators. In section 4, we describe the algebra when twisted sector operators are taken into account. We describe the leading terms in a systematic $\frac{1}{N}$ free field representation of the algebra. We study the form of the Hamiltonian, and observe that free field realizations which differ in detailed form, lead to the possibility of a quadratic Hamiltonian and an interacting one. We write a formula for the interacting Hamiltonian and observe similarities with other gravity-gauge theory correspondences, which suggest that the dynamics of chiral primaries can be understood as a reduction of $ADS \times S$ backgrounds to a $1 + 1$ dimensional system having some universal features. In section 5 we turn to finite $N$ effects, like the stringy exclusion principle, where the free field realizations start to break down. We outline a non-trivial property of the finite $N$ Lie algebra which follows from independently known facts about the cohomology of the instanton moduli spaces. We return to the picture of a non-commutative spacetime emphasized in I. The q-deformed symmetry $SU_q(2|1,1)$ of the q-spacetime proposed in I, is seen to govern the structure of the fusion rules implicit in the correlation functions of section 2. While I focused on the quantum spacetime interpretation of the single-particle states, we begin a study of multi-particle states in this framework. Finally we study the properties of the algebra and look for the value of $L_0$ where deviations from free field behaviour first show up.

We conclude with a summary and outline some future directions.
2. Orbifold Conformal Field Theory. Field Content and Symmetries

In this section we will discuss the field content and the symmetries of the SCFT on symmetric product $S^N(X)$, where $N = Q_1Q_5$ and $X$ is either $T^4$ or $K3$. It is known that this SCFT has $(4,4)$ superconformal symmetry in both cases. We will work with $T^4$ for simplicity. Many of the results extend simply to $K3$.

The field content of the theory is: $4N$ real free bosons $X_I^a$ representing the coordinates of the torus and their superpartners $4N$ free fermions $\Psi_I^a$, where $I = 1, \ldots, N$, $\alpha, \dot{\alpha} = \pm$ are the spinorial $S^3$ indices, and $a, \dot{a} = 1, 2$ are the spinorial indices on $T^4$. Using the relation between the Fermi fields: $\Psi^a = \epsilon_{\alpha\beta}\epsilon_{\dot{a}\dot{b}} \Psi^{\beta\dot{b}}$, the field content of the theory is determined to be $4N$ real free bosons and $2N$ Dirac free fermions, giving a central charge $c = 6N$. The left moving superconformal symmetry is generated by the following currents

$$T(z) = -\frac{1}{2} \sum_I \partial X_I^a \partial X_I a - \frac{1}{2} \sum_I \psi_I^a \partial \psi_I a,$$

$$G^{\alpha\dot{a}}(z) = i \sum_I \psi_I^a \partial X_I^a,$$

$$J^{\alpha\beta}(z) = \frac{1}{2} \sum_I \psi_I^a \psi_I^b,$$

where $\psi^{\alpha\dot{a}}$ is the left moving component of the corresponding fermion, and the $\epsilon_{\alpha\beta}, \epsilon_{\dot{a}\dot{b}}$ are used to raise and lower the spinorial indices. The lowest modes of these currents $\{L_0, \pm 1, G^{\alpha\dot{a}}, J^{\alpha\beta}\}$ will generate together to their right counterparts the $SU(2|1)_L \times SU(2|1)_R$ symmetry which is mapped in the AdS/CFT correspondence to the supersymmetries of the $AdS_3 \times S^3$. In addition, it is possible to construct other symmetries commuting with the previous set and related to global $T^4$ rotations, and they are given by the following currents:

$$K^{\dot{a}\dot{b}}(z) = \frac{1}{2} \sum_I (\psi_I^{\alpha\dot{a}} \psi_I^{\dot{b}b} - X_I^{aa} \partial X_I^{bb}),$$

and similar expressions for the right-movers. This symmetry acts non-trivially on the space of chiral primaries.

Although the underlying CFT on $T^4$ is free, non-trivial $S_N$ invariant chiral primary operators can be constructed in correspondence with conjugacy classes of $S_N$ \cite{13,14,15}. The basic twist operators for $n$ free bosons: $X_I, I = 1..n$, are defined by the OPE:

$$\partial X_I(z) \sigma_{(1..n)}(0) = z^{\frac{1}{n} - 1} e^{-\frac{2\pi i}{n} I_{(1..n)}(0)} + ..$$
They impose the boundary conditions:

\[ X_I(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = X_{I+1}(z, \bar{z}), \quad I = 1..n - 1, \quad X_n(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = X_1(z, \bar{z}) \] (2.6)

The general twist operator for a general conjugacy class of \( S_N \) is obtained by its decomposition into cycles. We may define a twisted sector vacuum by \( |(1..n)\rangle = \sigma_{(1..n)}(0)|0\rangle \). Over this vacuum we can write the following mode expansion:

\[ \partial X_I(z) = -\frac{i}{n} \sum_m \alpha_m e^{-\frac{2\pi i}{n} l m} z^{-\frac{m}{n} - 1} \] (2.7)

where \([\alpha_m, \alpha_n] = m\delta_{m+n,0} \) and \( \alpha_m |(1..n)\rangle = 0 \) for \( m \geq 0 \). The expression above can be generalized to any other primary operator constructed from \( X_I \), the only difference being that in the exponent of \( z \), 1 is replaced by the corresponding conformal dimension. This shows that the \( S_N \) invariant operators have nonsingular OPE with the twist operator. The dimension of the twist operator can be calculated by computing the energy momentum tensor in the state \( |(1..n)\rangle \) and it is (for one boson, left-mover):

\[ \Delta_{(1..n)} = \frac{1}{24} \left( n - \frac{1}{n} \right) \] (2.8)

For our system, we will bosonize first the fermions by introducing \( \Phi_I^{1,2}(z, \bar{z}) = \phi_I^{1,2}(z) + \bar{\phi}_I^{1,2}(\bar{z}) \), the final theory being one of \( 6N \) free bosons and we will construct the chiral primaries using these bosons:

\[ \psi_{\bar{I}}^+ a(z) = e^{i\phi^+_I(z)} \quad \psi_{\bar{I}}^- a(z) = \epsilon_{ab} e^{-i\phi^-_I(z)} \] (2.9)

for left-movers and:

\[ \bar{\psi}_{\bar{I}}^+ a(z) = e^{i\bar{\phi}^+_I(\bar{z})} \quad \bar{\psi}_{\bar{I}}^- a(z) = \epsilon_{ab} e^{-i\bar{\phi}^-_I(\bar{z})} \] (2.10)

for right-movers. Using these expressions, the \( SU(2)_L \) currents are given in the standard way:

\[ J^3 = \frac{i}{2} \sum_I (\partial \phi^+_I(z) + \partial \phi^-_I(z)) \],

\[ J^+ = \sum_I e^{i(\phi^+_I + \phi^-_I)(z)} \],

\[ J^- = \sum_I e^{-i(\phi^+_I + \phi^-_I)(z)} \] (2.11)

Let us construct the \( Z_n \) twist operators which can be used to build \( S_N \) invariant chiral primaries by averaging over \( S_N \). These twist operators play a distinguished role
in the chiral ring in the sense that they can be used to generate the rest by using the ring structure. In the correspondence with gravity in $ADS_3 \times S^3$ they are in one-one correspondence with single particle states. These vertex operators are written in terms of the 6 free scalar fields and the twist operators. Consider first the fields appearing in the untwisted sector, $n = 1$:

\[
\begin{align*}
O_{(1)}^{(0,0)} &= 1, \\
O_{(1)}^{a} &= \psi_1^{a}, \\
O_{(1)}^{\bar{a}} &= \bar{\psi}_1^{\bar{a}}, \\
O_{(1)}^{a,\bar{b}} &= \psi_1^{a} \bar{\psi}_1^{\bar{b}}, \\
O_{(1)}^{aa',\bar{b}} &= \psi_1^{a} \psi_1^{a'} \bar{\psi}_1^{\bar{b}}, \\
O_{(1)}^{(2,2)} &= \psi_1^{+1} \psi_1^{+2} \bar{\psi}_1^{+1} \bar{\psi}_1^{+2}.
\end{align*}
\]

where (1) represents the one cycle containing only $I = 1$. We will also construct vertex operators out of the simple twists which will correspond to the nontrivial chiral fields associated with the twist. For that we consider the cycle $(1..n)$ and we define the following $S_n$ invariant 6 dimensional vector observables (left and right):

\[
\begin{align*}
Y_L(z) &= \frac{1}{n} \sum_{I=1..n} (X^1_I L, X^2_I L, X^3_I L, X^4_I L, \phi^1_I, \phi^2_I)(z), \\
Y_R(\bar{z}) &= \frac{1}{n} \sum_{I=1..n} (X^1_I R, X^2_I R, X^3_I R, X^4_I R, \bar{\phi}^1_I, \bar{\phi}^2_I)(\bar{z}),
\end{align*}
\]

(2.13)

Let us consider now the following field which consists of the twist operator for all 6 fields and having momenta along the 2 extra $\Phi$ dimensions:

\[
O_{(1..n)}^{(0,0)}(z, \bar{z}) = e^i \sum_I \left( \frac{n-1}{2n} \phi^1_I + \frac{n-1}{2n} \phi^2_I(z) + \frac{n-1}{2n} \bar{\phi}^1_I(\bar{z}) + \frac{n-1}{2n} \bar{\phi}^2_I(\bar{z}) \right) \sigma_{(1..n)}(\Phi, X)(z, \bar{z})
\]

(2.14)

The dimensions of this field is $(\frac{n-1}{2}, \frac{n-1}{2})$ and its charges can be shown to be equal to the dimensions. In comparison with the formulae in I, we have extracted and made explicit the $U(1)$ charges by exhibiting exponentials of the bosons $\Phi$, leaving a twist operator for both the $\Phi$ and $X$. This field will be used to construct the $S_N$ invariant chiral primary $O_{n}^{(0,0)}(z, \bar{z})$. A more precise way to write which we will further generalize to the other cases is to write:

\[
O_{(1..n)}^{(0,0)}(z, \bar{z}) = e^{i(k_L Y_L(z) + k_R Y_R(\bar{z}))} \sigma_{(1..n)}(\Phi, X)(z, \bar{z})
\]

(2.15)
where \( k_L = (0, 0, 0, 0, \frac{n-1}{2}, \frac{n-1}{2}) \), \( k_R = (0, 0, 0, 0, \frac{n-1}{2}, \frac{n-1}{2}) \) represents the left and right momenta in 6 dimensions. The dimensions of a field of type (2.13) is given by the following general formula:

\[
\Delta_O = 6 \frac{1}{24} (n - \frac{1}{n}) + \frac{1}{2n} k_L^2, \\
\bar{\Delta}_O = 6 \frac{1}{24} (n - \frac{1}{n}) + \frac{1}{2n} k_R^2,
\]

and for our \( k \)'s we obtain the dimensions above. For the charge we can read it from the momenta on only the two extra dimensions, the twist being uncharged. In order to construct all the other chiral fields we will combine the construction in untwisted sector with the twist. We will focus then on the following construction:

\[
O^A_{(1..n)}(z, \bar{z}) \leftarrow O^A_{(1)}(z, \bar{z}) \ O^{(0,0)}_{(1..n)}(z, \bar{z})
\]  

where \( A \) index takes care of the spinorial indices which already fully appear at untwisted level. We give for these operators a construction similar to (2.15) where what makes the distinction between them is the value for the momenta. This construction will be further justified by looking to the OPE of chiral fields.

We will focus further on scalar chiral primaries, namely those fields coresponding to \( O_n^{(0,0)}(z, \bar{z}), O_n^{(1,1)}(z, \bar{z}), O_n^{(2,2)}(z, \bar{z}) \) only. We can characterize the fields as being a twist operator in 6 bosonic dimensions and having definite momenta on the bosonic fields coming from the bosonization \( \Phi^{1,2} \) (the dimension and the charge for this operators are equal):

\[
O^{(0,0)}_{(1..n)}(z, \bar{z}) \text{ corresponds to momenta}
\]

\[
k_L = (0, 0, 0, 0, \frac{n-1}{2}, \frac{n-1}{2}) \quad k_R = (0, 0, 0, 0, \frac{n-1}{2}, \frac{n-1}{2})
\]

and has dimension \( (\frac{n-1}{2}, \frac{n-1}{2}) \),

\[
O^{(1,1)}_{(1..n)}(z, \bar{z}) \text{ corresponds, for example, to momenta}
\]

\[
k_L = (0, 0, 0, 0, \frac{n+1}{2}, \frac{n-1}{2}) \quad k_R = (0, 0, 0, 0, \frac{n+1}{2}, \frac{n-1}{2})
\]

3 other combinations of left-right momenta in the case of \( T^4 \) are possible. The dimension is \( (\frac{n}{2}, \frac{n}{2}) \).

\[
O_n^{(12,12)}(z, \bar{z}) \text{ corresponds to momenta}
\]

\[
k_L = (0, 0, 0, 0, \frac{n+1}{2}, \frac{n+1}{2}) \quad k_R = (0, 0, 0, 0, \frac{n+1}{2}, \frac{n+1}{2})
\]
and has dimension \((\frac{n+1}{2}, \frac{n+1}{2})\).

We will define below the precise relation between the \(O_n\) and the corresponding \(O_{(1..n)}\) in a general context. For this let us consider the following basis of forms leaving on the target space \(X\) and spanning its \(H^{(1,1)}(X)\); we will denote them as \(\omega^r_{\bar{a}a}\) where \(r\) counts the forms (for example, \(r = 1..4\) for \(T^4\), and \(r = 1..20\) for \(K3\)) and \(a, \bar{a} = 1, 2\) and they are \(X\) indices. Using this forms and summing over all permutations, it is possible to describe the scalar chiral primaries up to a normalization constant which will be determined in the next section. We write here the full expression for this operators:

\[
O_n^{(0,0)}(z, \bar{z}) = \frac{1}{(N!(N-n)!n)!^2} \sum_{h \in S_N} O_{h(1..n)h^{-1}}(z, \bar{z}),
\]

\[
O_n^{(0,r)}(z, \bar{z}) = \frac{1}{(N!(N-n)!n)!^2} \sum_{h \in S_N} O_{h(1..n)h^{-1}}^{a,\bar{a}} \omega^r_{\bar{a}a}(z, \bar{z}),
\]

\[
O_n^{(2,2)}(z, \bar{z}) = \frac{1}{4} \frac{1}{(N!(N-n)!n)!^2} \sum_{h \in S_N} O_{h(1..n)h^{-1}}^{ab,\bar{a}\bar{b}} \epsilon_{ab} \epsilon_{\bar{a}\bar{b}}(z, \bar{z}),
\]

2.1. Correlation Functions

We will compute in this section the OPE of two chiral primary operators focusing only on the constant appearing in front of the resulting single-particle chiral primary operator. From this we are able to write their three-point correlation functions. Using the definition of the chiral primaries, we see that there are some restrictions coming from charge conservation, from the fact that except the twist operator, these operators are fermionic in nature and from \(S_N\) multiplication law. It is straightforward to observe that we will be interested from the permutation point of view in two kind of processes:

1) the joining of two cycles overlapping on only one of their components giving a longer permutation, which gives correlation functions between fields which have \((p, q)\) form indices as follows: i) \((0, 0) + (0, 0) \rightarrow (0, 0)\), ii) \((0, 0) + r \rightarrow r\), iii) \((0, 0) + (2, 2) \rightarrow (2, 2)\), iv) \(s + r \rightarrow (2, 2)\),

2) the joining of two cycles overlapping on two components giving a longer permutation, which gives correlation functions of the form: \((0, 0) + (0, 0) \rightarrow (2, 2)\).

We will first study the processes listed above on the components of the chiral primaries and then we will sum over all permutation to recover the \(S_N\) invariance. Using the notation of [13] we will denote the left-moving twist operator having momenta as \(O_{(1..n)}(k)\) where \(k\)
is the momenta and we assume that they are normalized in the sense that their two-point takes the following form:

$$\langle O_h(k_1) (\infty) O_g(k_2)(0) \rangle = \delta_{h,g} - \delta_{k_1,k_2}$$  \hspace{1cm} (2.22)

Using the same method, we will derive in appendix the OPE of a twist (1..n) having momenta $k_n$ and the twist (n n+1) having momenta $k_2$ involving only the twist (1..n+1) (corresponding to the first kind of process) the result being:

$$O_{(n n+1)}(k_2)(u) O_{(1..n)}(k_n)(0) = \frac{C(2, n|n+1; k_2, k_n)}{z^{\Delta_{n+1}} - \Delta_2 - \Delta_n} (O_{(1..n+1)}(k_2 + k_n)(0) +$$

$$+ O_{(1..n-1 n+1)}(k_2 + k_n)(0)),$$  \hspace{1cm} (2.23)

Since we are dealing with operators which are chiral primaries one has the exponent of $z$ being 0, and the $C(2, n|n+1; k_2, k_n)$ are determined in the appendix. We will write here the expression for $C(2, n|n+1; k_2, k_n)$ and we will see what they are in our processes (only left movers):

$$O_{(n n+1)}(u) O_{(1..n)}(0) = \frac{1}{2} \left( \frac{n+1}{n} \right)^{\frac{3}{2}} (O_{(1..n+1)}(0) + O_{(1..n+1 n)}(0)),$$

$$O_{(n n+1)}(u) O_{(1..n)}^1(0) = \frac{1}{2} (O_{(1..n+1)}^1(0) + O_{(1..n+1 n)}(0)),$$  \hspace{1cm} (2.24)

$$O_{(n n+1)}(u) O_{(1..n)}^{12}(0) = \frac{1}{2} \left( \frac{n}{n+1} \right)^{\frac{3}{2}} (O_{(1..n+1)}^{12}(0) + O_{(1..n+1 n)}^{12}(0)).$$

The method also gives us OPE for all the other cases involved in the list above case 1) for a 2 twist and an n twist overlapping over only one component. Knowing that the total twist is the product of the left and right moving twist and excluding out of the OPE those cases when the left and right twists do not coincide we see that the result of the OPE is essentially the square of what we have in the previous equations. We can also observe that an extrapolation to the $n = 1$ case, when the twist is nothing than the identity operator results in the following OPE:

$$O_{(12)}(u) \psi^{+a}(0) = O_{(12)}^a(0),$$

$$O_{(12)}(u) \psi_{(1..n)}^{+2}(0) = \frac{1}{\sqrt{2}} O_{(12)}^{12}(0).$$  \hspace{1cm} (2.25)

The equations above show that it is enough to figure out the OPE for the twist fields and then generalize to the other chiral primary fields. We will use from this point the CFT rules to compute the OPE for any twist operators overlapping on only one component of
the permutations. What we will do is to compute the OPE for 3 twist of type (1..n − 1),
(n − 1 n) and (n..n + k − 1) at different locations in the complex plane, and by making
different limits and an induction process we obtain the following results including now the
right moving twist:
\[ O_{(n..n+k-1)}(u, \bar{u}) O_{(1..n)}(0) = \frac{n+k-1}{2nk} (O_{(1..n+k-1)}(0) + ..), \]  
(2.26)
where the dots are for the field corresponding to the other permutation obtained by mul-
tiplying the two permutations. Using the equations above we derive the OPE for the
processes 1) and we list them here:
\[ O_{(n..n+k-1)}(u, \bar{u}) O_{(1..n)}^r(0) = \frac{1}{2k} (O_{(1..n+k-1)}^r(0) + ..), \]
\[ O_{(n..n+k-1)}(u, \bar{u}) O_{(1..n)}^{12,\bar{12}}(0) = \frac{n}{2k(n+k-1)} (O_{(1..n+k-1)}^{12,\bar{12}}(0) + ..), \]  
(2.27)
\[ O_{(n..n+k-1)}^r(u, \bar{u}) O_{(1..n)}^s(0) = \frac{-2}{n+k-1} \omega^r * \omega^s (O_{(1..n+k-1)}^{12,\bar{12}}(0) + ..). \]
Note that if we extrapolate the above expressions for the case \( n = 1 \) we have:
\[ O_{(1..n)}(u, \bar{u}) O_{(1)}^r(0) = \frac{1}{2k} (O_{(1..n)}^r(0) + ..), \]
\[ O_{(1..n)}(u, \bar{u}) O_{(1)}^{12,\bar{12}}(0) = \frac{1}{2k^2} (O_{(1..n)}^{12,\bar{12}}(0) + ..). \]  
(2.28)
where for the definition of untwisted operators we use (2.12).

In addition there is the 2) process which we will focus now on. Using the above rules
there is a single OPE which has to be figured out and this is the one involving only two
twists for the same 2 cycle (overlapping of 2), and then the result can be extended to the
other permutations using the known OPE from the previous list and an induction process.
Consider the OPE of 3 operators: the twists and the one formed from all 4 adjoints of
fermions and no twist. In the limit when we put together the twist and the fermions this
process is nothing else than the one used for normalization of the twist of length 2; in the
limit when one puts together the twists and then the fermions and ask the question on
what kind of operator would give this result one is led to consider the following OPE:
\[ O_{(12)}(u, \bar{u}) O_{(12)}(0) = \psi^1 \psi^2 \bar{\psi}^1 \bar{\psi}^2(0) \]  
(2.29)
The final needed OPE is then:
\[ O_{(1..n)}(u, \bar{u}) O_{(n..n+k-2)}(0) = \frac{1}{n k (n+k-3)} (O_{(1..\hat{n},..n+k-2)}^{12,\bar{12}}(0) + ..), \]  
(2.30)
where by \( \hat{n} \) we mean that \( n \) is missing in this permutation. We have at this moment the
2 and 3-point functions for all fields which we will use in constructing the chiral fields.
2.2. Correlation functions of Chiral primaries.

In this section we will put back the sums over permutations with appropriate normalization factors, which together with the 3 point functions deduced in the previous section will allow us to write the full 3-point functions for the chiral primaries. For $O_{n}^{(0,0)}$ we have:

$$O_{n}^{(0,0)}(z, \bar{z}) = \text{const.} \sum_{h \in S_{N}} O_{h(1..n)h^{-1}}(z, \bar{z}).$$  \hspace{1cm} (2.31)

The two point function for individual twists is given (2.22) with a constant to be determined.

$$\langle O_{n}^{(0,0)}(0) O_{n}^{(0,0)}(0) \rangle = \text{const.} \sum_{h_{1,2} \in S_{N}} \langle O_{h_{1}(1..n)h_{1}^{-1}}^{(0,0)}(\infty) O_{h_{2}(1..n)h_{2}^{-1}}^{(0,0)}(0) \rangle .$$  \hspace{1cm} (2.32)

Because the 2-point function for twists is normalized to 1, the sums will be rearranged in other two sums: a sum over all permutations and a sum on only those permutation which leave a cycle invariant. The sum over all permutation will give a factor of $N!$ whereas the second sum will give a factor of $(N - n)! n$ leading to the expressions already listed in equation (2.21).

For the three-point functions we will use the normalized chiral primaries operators and we will also show how one can determine them for only three $O^{(0,0)}$ and then list the full results for all the other cases. The conservation law for the R-symmetry suggest that the only possibility is having only the process listed in 1) namely a cycle having length $n + k - 1$, one having length $n$ and one having $k$ and the individual permutations have to overlap on one entry:

$$\langle O_{n+k-1}^{(0,0)}(\infty) O_{n}^{(0,0)}(1) O_{k}^{(0,0)}(0) \rangle = \text{const.} \sum_{h_{1,2,3} \in S_{N}} \langle O_{h_{1}(1..n+k-1)h_{1}^{-1}}^{(0,0)}(\infty) O_{h_{2}(n..n+k-1)h_{2}^{-1}}^{(0,0)}(1) O_{h_{3}(1..n) h_{3}^{-1}}(0) \rangle ,$$  \hspace{1cm} (2.33)

where the const. comes from the normalization of each chiral primary field. The individual terms which appear in the sum are nonzero only if:

$$h_{1}(1..n+k-1)^{-1}h_{1}^{-1}h_{2}(n..n+k-1)h_{2}^{-1}h_{3}(1..n)h_{3}^{-1} = 1$$

and in this case they are all equal to the expressions derived in the previous section. By rearranging the previous permutation equation it is possible to compute the sums, namely:

$$(1..n+k-1)^{-1}h_{1}^{-1}h_{2}(n..n+k-1)h_{2}^{-1}h_{3}(1..n)h_{3}^{-1}h_{2}^{-1}h_{1} = 1.$$
Then the result will be a $N!$ summing over $h_1$, a factor of $(N-k)!/(N-k-n+1)!$ coming from the possibilities of constructing a long permutation out of 2 small permutation, a factor of $(N-n)!n$ coming from the sum over $h_2^{-1}h_3$ which leave the $n$ cycle invariant and a factor of $(N-n-k+1)!(n+k-1)$ coming from $h_1^{-1}h_2$ which leave the full $n+k-1$ cycle invariant. In addition there is a factor of 2 coming from the two possibility of multiplying permutations in individual three-point functions. We will list below the final result gathering all factors and also the results for all nonzero nontrivial three-point functions of three chiral primaries:

\[
\begin{align*}
\langle O_{n+k-1}^{(0,0)}(\infty)O_k^{(0,0)}(1)O_n^{(0,0)}(0) \rangle &= \left( \frac{(N-n)! (N-k)! (n+k-1)^3}{(N-(n+k-1))! N! n k} \right)^{\frac{1}{2}}, \\
\langle O_{n+k-1}^{r+}(\infty)O_k^{(0,0)}(1)O_n^{s}(0) \rangle &= \left( \frac{(N-n)! (N-k)! n (n+k-1)}{(N-(n+k-1))! N! k (n+k-1)} \right)^{\frac{1}{2}} \delta^{rs}, \\
\langle O_{n+k-1}^{(2,2)}(\infty)O_k^{(0,0)}(1)O_n^{(2,2)}(0) \rangle &= \left( \frac{(N-n)! (N-k)! n^3}{(N-(n+k-1))! N! k (n+k-1)} \right)^{\frac{1}{2}}, \\
\langle O_{n+k-1}^{(2,2)}(\infty)O_k^{r}(1)O_n^{s}(0) \rangle &= -\left( \frac{(N-n)! (N-k)! n k}{(N-(n+k-1))! N! (n+k-1)} \right)^{\frac{1}{2}} \omega^r \ast \omega^s, \\
\langle O_{n+k-3}^{(2,2)}(\infty)O_k^{(0,0)}(1)O_n^{(0,0)}(0) \rangle &= 2 \left( \frac{(N-n)! (N-k)! (N-(n+k)+3)}{(N-(n+k-1))! N! (n+k+2) n k (n+k-3)} \right)^{\frac{1}{2}}.
\end{align*}
\]

Fixing the charges and taking $N \to \infty$:

\[
\begin{align*}
\langle O_{n+k-1}^{(0,0)}(\infty)O_k^{(0,0)}(1)O_n^{(0,0)}(0) \rangle &= \left( \frac{1}{N} \right)^{\frac{1}{2}} \left( \frac{n+k-1}{n} \right)^{\frac{3}{2}}, \\
\langle O_{n+k-1}^{r+}(\infty)O_k^{(0,0)}(1)O_n^{s}(0) \rangle &= \left( \frac{1}{N} \right)^{\frac{1}{2}} \left( \frac{n (n+k-1)}{k} \right)^{\frac{3}{2}} \delta^{rs}, \\
\langle O_{n+k-1}^{(2,2)}(\infty)O_k^{(0,0)}(1)O_n^{(2,2)}(0) \rangle &= \left( \frac{1}{N} \right)^{\frac{1}{2}} \left( \frac{n^3}{k (n+k-1)} \right)^{\frac{1}{2}}, \\
\langle O_{n+k-1}^{(2,2)}(\infty)O_k^{r}(1)O_n^{s}(0) \rangle &= -\left( \frac{1}{N} \right)^{\frac{1}{2}} \left( \frac{n k}{(n+k-1)} \right)^{\frac{3}{2}} \omega^r \ast \omega^s, \\
\langle O_{n+k-3}^{(2,2)}(\infty)O_k^{(0,0)}(1)O_n^{(0,0)}(0) \rangle &= 2 \left( \frac{1}{N} \right)^{\frac{1}{2}} \left( \frac{1}{n k (n+k-3)} \right)^{\frac{1}{2}}.
\end{align*}
\]
After the following rescalings
\[ O_n^{(0,0)} \rightarrow \frac{1}{n} O_n^{(0,0)}, \]
\[ O_n^{(0,0)\dagger} \rightarrow n O_n^{(0,0)\dagger}, \]
\[ O_n^r \rightarrow O_n^r, \]
\[ O_n^{\dagger r} \rightarrow O_n^{\dagger r}, \]
\[ O_n^{(2,2)} \rightarrow n O_n^{(2,2)}, \]
\[ O_n^{(0,0)\dagger} \rightarrow \frac{1}{n} O_n^{(0,0)\dagger}, \]

which preserve the two-point functions, we have three-point functions:

\[
\langle O_n^{(0,0)\dagger}(\infty) O_k^{(0,0)}(1) O_n^{(0,0)\dagger}(0) \rangle = \left( \frac{1}{N} \right)^{\frac{1}{2}} ((n + k - 1) n k)^{\frac{1}{2}}, \\
\langle O_n^{(2,2)\dagger}(\infty) O_k^{(0,0)}(1) O_n^{(2,2)\dagger}(0) \rangle = \left( \frac{1}{N} \right)^{\frac{1}{2}} ((n + k - 1) n k)^{\frac{1}{2}} \delta^r s, \\
\langle O_n^{(2,2)\dagger}(\infty) O_k^{(0,0)}(1) O_n^{(2,2)\dagger}(0) \rangle = -\left( \frac{1}{N} \right)^{\frac{1}{2}} ((n + k - 1) n k)^{\frac{1}{2}} \omega^r \ast \omega^s, \\
\langle O_n^{(2,2)\dagger}(\infty) O_k^{(0,0)}(1) O_n^{(0,0)\dagger}(0) \rangle = 2 \left( \frac{1}{N} \right)^{\frac{1}{2}} ((n + k - 3) n k)^{\frac{1}{2}}.
\]

We notice a certain degree of universality: all the three point functions exhibited above for the case of AdS3 have the same form factor \( \sqrt{n k (n + k)} \) as the three point functions of chiral primaries in other AdS examples for example \([16,17,18]\). We will have further comments on this universal behaviour and its meaning in subsequent discussion. In the future sections, we will denote as \( A_n^{(p,q)} \) the modes \( \sum O_{-\Delta(n,p,q)} \bar{O}_{-\Delta(n,p,q)} \) of \( O_n^{p,q}(z, \bar{z}) \) and by \( A_n^{p,q} \) the conjugates.

3. Algebra of Observables and the Lagrangian - Untwisted sector

At the heart of the discussion begun in I is consideration of the commutator algebra generated by the exact single particle creation and annihilation operators \( A_n^{p,q} \) and \( (A_n^{p,q})^\dagger \) (with the cutoffs on \( n \) as described in I). These operators represent exact eigenstates of the Hamiltonian and the full algebra is then by definition a spectrum generating one. We should emphasize the analogy of this with previous large algebras appearing in related contexts: the W-algebra of the matrix model \([19,20]\) the BPS algebras of \([21]\), and extended
algebras mentioned in [22]. An interesting example of a spectrum generating algebra in semiclassical AdS gravity is discussed in [23].

In this section we concentrate on the simplest, zero twist sector of the theory. It represents a reduction of CFT to a fermionic quantum mechanical system. We describe for this the algebra of chiral primaries and the manner in which this algebra defines the interacting Lagrangian. The collective [24] interacting theory emerges as a representation of the algebra in terms of invariants. The relevance of the algebra on the Poisson (phase space) structure of the theory will also be elaborated.

Let us begin from the operators representing the chiral primaries \( A_{-1}^{(p,q)} \) and their conjugates \( A_1^{p,q} \). The procedure to construct the associated super-Lie algebra is to consider the (graded) commutators of the conjugates with the chiral primaries. The terms appearing on the RHS of the commutators are then commuted with the chiral primaries and their conjugates, and with each other until closure is achieved. This clearly generates a finite dimensional super-Lie algebra of which the creation-annihilation operators and the hamiltonian are members. A set of mutually commuting hermitian operators in this algebra are

\[
\psi^\mu_I (\psi_\nu^\mu)^\dagger \\
(\psi^\mu_I \psi_\nu^\mu) (\psi^\nu_I \psi_\mu^\nu)^\dagger \\
(\psi^\mu_I \psi_\nu^\nu \psi_\sigma^\sigma) (\psi^\sigma_I \psi_\nu^\nu \psi_\mu^\sigma)^\dagger
\] (3.1)

The indices \( \mu, \nu, \sigma \) above run from 1 to 4 and we have 4 operators in the first line, 6 in the second line, and 4 in the third line. We should add to this an operator \( E \) which commutes with everything, and which appears, for example, in the commutator of a an operator \( A_{-1}^{(1,0)} \) and its conjugate. Note that the procedure of successive commutations of the \( A_{-1} \) operators do not generate terms involving a product of four fermions with their conjugates.

The other generators of the Lie algebra include of course the chiral primaries and conjugates, along with others generated by the commutations. We also have the following
operators and their conjugates.

\[
\begin{align*}
&\sum_I \psi_I^{\mu_1} (\psi_I^{\nu_1})^\dagger \\
&\sum_I \psi_I^{\mu_1} \psi_I^{\mu_2} (\psi_I^{\nu_1})^\dagger \\
&\sum_I \psi_I^{\mu_1} \psi_I^{\mu_2} (\psi_I^{\nu_1} \psi_I^{\nu_2})^\dagger \\
&\sum_I \psi_I^{\mu_1} \psi_I^{\mu_2} \psi_I^{\mu_3} (\psi_I^{\nu_1})^\dagger \\
&\sum_I \psi_I^{\mu_1} \psi_I^{\mu_2} \psi_I^{\mu_3} (\psi_I^{\nu_1} \psi_I^{\nu_2})^\dagger \\
&\sum_I \psi_I^{\mu_1} \psi_I^{\mu_2} \psi_I^{\mu_3} \psi_I^{\mu_4} (\psi_I^{\nu_1} \psi_I^{\nu_2} \psi_I^{\nu_3})^\dagger
\end{align*}
\]

(3.2)

Some of these operators have already been included in the description of the Cartan above. The operator of the form \((\psi)^4(\psi^\dagger)^4\) does not appear in the the commutators. Let us call this Lie super-algebra \(g_{\text{inv}}\), and the corresponding supergroup \(G_{\text{inv}}\). If we nevertheless include it we get a superlagebra which has rank 16 and is closely related to the clifford algebra which is in turn related to \(SU(16)\). This suggests that the Lie super-algebra is actually \(U(8|8)\).

3.1. Coherent State Representation

We derive in this section the action governing the dynamics of the chiral primaries for the untwisted sector and then comment on how it is extended to the full set of chiral primaries and beyond. Consider then the simplified model obtained by reducing the theory to a quantum mechanical version with the algebra described above. We deal then with the following set of operators as annihilation operators:

\[
\psi_I^{\pm \mu}; \quad \mu = 1..4, \quad I = 1..N,
\]

(3.3)

\footnote{1 We thank J. Gervais for this remark.}
where $\pm$ is the $SU(2)$ index, $\mu$ is the $T^4$ index and we also have the adjoints of these as creation operators. We also focus on the chiral primaries only meaning that we consider only highest weight states under the action of the $SU(2)$, namely $(\pm)$, and for now on we will also drop this index. The hamiltonian of the system (the one involving only highest weight under $SU(2)$) is considered to be the hamiltonian of a free system of fermions:

$$L_0 + \bar{L}_0 = \frac{1}{2} \sum_{I, \mu} \psi_{I, \mu}^\dagger \psi_{I, \mu}$$

(3.4)

The chiral primary operators in the untwisted sector are built as:

$$A_{-1}^\mu = \frac{1}{\sqrt{N}} \sum_I \psi_{I, \mu}^\dagger,$$

$$A_{-1}^{\mu\nu} = \frac{1}{\sqrt{N}} \sum_I \psi_{I, \mu}^\dagger \psi_{I, \nu}^\dagger,$$

$$A_{-1}^{\mu\nu\sigma} = \frac{1}{\sqrt{N}} \sum_I \psi_{I, \mu}^\dagger \psi_{I, \nu}^\dagger \psi_{I, \sigma}^\dagger,$$

$$A_{-1}^{\mu\nu\sigma\rho} = \frac{1}{\sqrt{N}} \sum_I \psi_{I, \mu}^\dagger \psi_{I, \nu}^\dagger \psi_{I, \sigma}^\dagger \psi_{I, \rho}^\dagger.$$

(3.5)

The idea of the construction is to build them using only the highest weight operators under $SU(2)$ and $S_N$ invariant combination of operators. The dimension and the charge of the operators are equal to the number of fermion operators.

Let us study now how we write an action involving only these special set of operators. For this we can use the technique of [25]. It consists in introducing a basis of coherent states using the $S_N$ invariant operators already derived and from the expression for the partition function we derive the hamiltonian and the symplectic form which governs the dynamics of the collective variables. The first step is to derive the measure which will be use together with the coherent basis $\mu(\xi_A, \bar{\xi}_A)$ as:

$$\mu(\xi_B, \xi_B^*) = \langle 0 | e^{\xi_B A_i^B} e^{\xi_B A_{-1}^B} | 0 \rangle$$

(3.6)

Here we have taken the index $B$ to run over all the 15 chiral primaries of the invariant sector. This can be computed using coherent state techniques to find :

$$\mu(\xi_A, \xi_A^*) = (Z(\xi, \xi^*))^N,$$

(3.7)
where
\[ Z[\xi, \xi^*] = 1 - \frac{1}{N} \tilde{\xi}^\mu \tilde{\xi}_\mu + \frac{1}{2N} (\tilde{\xi}^{\mu\nu})^* (\tilde{\xi})^{\mu\nu} - \frac{1}{3!N} (\tilde{\xi}^{\mu\nu\rho})^* (\tilde{\xi})^{\mu\nu\rho} + \frac{1}{4!N} (\tilde{\xi}^{\mu\nu\rho\sigma})^* (\tilde{\xi})^{\mu\nu\rho\sigma} \] (3.8)
is the answer for one species of fermions. Here the following variables are used:
\[ \tilde{\xi}^\mu = \xi^\mu, \]
\[ \tilde{\xi}^{\mu\nu} = \xi^{\mu\nu} - \frac{1}{\sqrt{N}} \xi^\mu \xi^\nu, \]
\[ \tilde{\xi}^{\mu\nu\sigma} = \xi^{\mu\nu\sigma} + \frac{3}{\sqrt{N}} \xi^\mu \xi^\nu \xi^\sigma, \]
(3.9)
\[ (\tilde{\xi}^{(2,2)}) = \xi^{(2,2)} + \frac{1}{3!\sqrt{N}} \epsilon^{\mu\nu\sigma\rho} \xi^{\mu\nu\sigma} \xi^\rho + \frac{1}{2!\sqrt{N}} \epsilon^{\mu\nu\sigma\rho} \xi^{\mu\nu} \xi^{\sigma\rho} + \ldots, \]
and the dots stand for terms of lower order in \( \frac{1}{N} \) terms.

For determining the representation of the Hamiltonian in \( \xi, \xi^* \) we evaluate its matrix elements in the coherent basis. In our case it is suitable to evaluate the following:
\[ Z[\beta, \xi, \xi^*] = <0|e^{\xi^B A^B} e^{\beta (L_0 + \bar{L}_0)} e^{A^B \xi_B} |0> \] (3.10)
Using the expression for the Hamiltonian (3.4) and coherent state techniques we compute the following expression:
\[ Z[\beta, \xi, \xi^*] = Z[(1 + \frac{1}{\beta}) \tilde{\xi}^\mu, (1 + \frac{1}{\beta}) \tilde{\xi}^{\mu\nu}; (1 + 1/2\beta)^2 \tilde{\xi}^{\mu\nu}, (1 + 1/2\beta)^2 \tilde{\xi}^{\mu\nu\nu}; \]
\[ (1 + 1/2\beta)^3 \tilde{\xi}^{\mu\nu\rho}; (1 + 1/2\beta)^3 \tilde{\xi}^{\mu\nu\rho\nu}; (1 + 1/2\beta)^4 \tilde{\xi}^{\mu\nu\rho\nu\nu}; (1 + 1/2\beta)^4 \xi^{\mu\nu\rho\nu\nu\nu}] \] (3.11)

We can write then the average for the Hamiltonian as:
\[ H(\xi, \xi^*) \equiv \frac{<\tilde{\xi}|(L_0 + \bar{L}_0)|\xi>}{\mu(\xi, \xi^*)} \]
\[ = \frac{1}{Z\beta} \frac{\partial}{\partial \beta} Z^\beta |_{\beta=0} \]
\[ = \frac{N}{Z} \frac{\partial Z}{\partial \beta} |_{\beta=0} \]
\[ = \frac{N}{2} \left( \frac{-\frac{1}{N} \tilde{\xi}^\mu \tilde{\xi}_\mu + \ldots}{1 - \frac{1}{N} \tilde{\xi}^\mu \tilde{\xi}_\mu + \ldots} \right) \] (3.12)
In the last line we wrote the first quadratic term, the other being also quadratic in \( \tilde{\xi} \) but for the remaining indices. Noting the expression for the tilde variables (eq.3.9) one has a sequence of cubic, quartic and higher terms which are explicitly determined by the above representation.

Using the equations above we can now derive the lagrangian governing the dynamics of the collective fields we introduced:

\[
L(\xi, \xi^*) = L_\omega(\xi, \xi^*) - H(\xi, \xi^*),
\]

(3.13)

\( L_\omega \) is determined from the above measure (3.7), (3.8) by

\[
L_\omega(\xi, \xi^*) = \left[ \frac{\partial \xi^\mu}{\partial t} \frac{\partial}{\partial \xi^\mu} + \frac{\partial \xi^{\mu\nu}}{\partial t} \frac{\partial}{\partial \xi^{\mu\nu}} + \frac{\partial \xi^{\mu\nu\alpha}}{\partial t} \frac{\partial}{\partial \xi^{\mu\nu\alpha}} b + \frac{\partial \xi^{\mu\nu\alpha\beta}}{\partial t} \frac{\partial}{\partial \xi^{\mu\nu\alpha\beta}} \right] \log \mu(\xi, \xi^*),
\]

(3.14)

and it is linear in time derivatives. Its expression gives the Poisson structure in the phase space of \( \xi, \xi^* \). For the other part of the lagrangian \( H \), we use the expression we computed in (3.12).

This coherent state technique can also be used to give a free field realization of the algebra of observables in a \( 1/N \) expansion which reproduces correlation functions involving a small number of operators compared to \( N \). To convert to free fields we have to find variables which convert the Lagrangian \( L(\xi, \xi^*) \) into \( L(a, a^*) = a \frac{\partial a^*}{\partial t} \). This can be done in a systematic large \( N \) expansion and reproduces correlators involving a small number of insertions compared to \( N \). When the number of insertions becomes comparable to \( N \), effects like the stringy exclusion principle become important. The properties of the measure \( \mu \) then show properties qualitatively different from free fields. For example

\[
\left( \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi^*} \right)^N = 1 \mu(\xi, \xi^*) = 0
\]

(3.15)

The coherent state technique offers a complementary insight into the exclusion principle. It is related to the fact that the symplectic manifold generated by the action of elements of \( G_{inv} \) on the Fock vacuum has a non-trivial symplectic form which cannot be globally brought to the form \( \omega = \sum_i dp^i \wedge dq^i \). This phenomenon was emphasized in simpler models of large \( N \) in [25]. We expect along the lines of I, that a transformation valid for finite \( N \) can be done using q-oscillators at roots of unity.
4. Algebra of observables - Twisted sector

The strategy for defining a Lie algebra associated with the single particle chiral primaries by taking successive commutators clearly works when we include the twisted sectors as well. Some elements were described in I. Here we will discuss further properties of the algebra, and its representations in terms of free fields. As we go from the zero twist to the twisted sector the the creation-annihilation operators acquire an additional index $n$, which is reflected in the commutator algebra. The commutators also become more complex, but are computable using CFT techniques. The general form of the terms are highly constrained by conservation of $L_0, J_0$, and the $S_N$ symmetry. The $N$ dependences can be obtained by $S_N$ combinatorics. The precise coefficients are related to computations of correlation functions of the kind done in section 2.

Consider the class of chiral primary operators $A_{-n}^{0,0}$ defined at the end of section 2, denoted here $A_{-n}$. By elementary commutator manipulations, followed by contour integrals, we derived in I an equation which reduces in the large $N$ limit to

$$[A_{-n}, A_{-n}^\dagger] = 1 + \frac{1}{N^2} \left[ \sum_{\sigma_n \in T_n} O^{\sigma_n} O^{\dagger\sigma_n} + \bar{O}^{\sigma_n} \bar{O}^{\dagger\sigma_n} \right]$$

(4.1)

We are keeping only the leading terms in the combinatoric factors, e.g $(N)_n$ has been approximated by $N^n$, to demonstrate the key features of the large $N$ counting. Each operator is normalized to 1.

We will now consider the form of commutators involving different chiral primaries $A_{-m_1}$ and $A_{-m_2}^\dagger$. For this we will need to discuss commutators of the chiral building blocks

$$[O^{\sigma_{m_1}}, O^{\sigma_{m_2} \dagger}] = C_{\sigma_3}^{\sigma_{m_1} \sigma_{m_2}} O_{\sigma_3}$$

(4.2)

In the above equation $\sigma_{m_i} \in T_{m_i}$ where $T_{m_i}$ denotes the set of permutations in $S_N$ which have one cycle of length $m_i$ and remaining cycles of length 1. We are not being specific about the order of magnitude of $C$ in the large $N$ expansion in the above equation, but we will cure that in a moment. We will assume that $m_2$ is larger than $m_1$. We can further decompose the sum over $\sigma_3$ by specifying the number of non-trivial cycles ( of length greater than 1 ) in the permutation $\sigma_3$. When $\sigma_3$ belongs to a permutation with $k$ non-trivial cycles of lengths $(n_1, n_2, \cdots n_k)$ we say $\sigma_3 \in T_{n_1, n_2, \cdots n_k}$, and we write $\sigma_3$ as
In the following equation we have normalized the operators to have 2-point functions equal to one. The $C$-factors are of order one. And the form of the leading $N$ dependences have been made explicit.

\[
\sum_{\sigma_{m_1} \in T_{1}, \sigma_{m_2} \in T_{2}} [O^{\sigma_{m_1}}, O^{\sigma_{m_2}}] =
\sum_{\sigma_3 \in T_{n_1}} N^{-\frac{1}{2}} (1 + O(\frac{1}{N})) \delta(m_2 - m_1 + 1, n_1) C_{\sigma_{m_1} \sigma_{m_2}} O^{\dagger \sigma_{n_1}} O^{\dagger \sigma_{n_2}}
+ \sum_{\sigma_3 \in T_{n_1}, n_2} N^{-1} (1 + O(\frac{1}{N})) \delta(m_2 - m_1 + 2, n_1 + n_2) C_{\sigma_{m_1} \sigma_{m_2}} O^{\dagger \sigma_{n_1}} O^{\dagger \sigma_{n_2}}
+ \ldots
+ \sum_{\sigma_3 \in T_{n_1}, n_2, \ldots, n_k} N^{-\frac{k}{2}} (1 + O(\frac{1}{N})) \delta(m_2 - m_1 + k, n_1 + n_2 + \ldots n_k) C_{\sigma_{m_1} \sigma_{m_2}} O^{\dagger \sigma_{n_1}} O^{\dagger \sigma_{n_2}} \ldots O^{\dagger \sigma_{n_k}}
\]

(4.3)

Note that we could have used a basis where the product operators are just products of the generating $S_N$ invariant generating chiral primaries. For example we could have written

\[
\sum_{\sigma_1 \in T_{n_1}, \sigma_2 \in T_{n_2}} O^{\sigma_1} O^{\sigma_2}
\]

(4.4)

rather than the sums associated with conjugacy classes as in (4.3). In the sum in (4.4) the terms $\sigma_1$ and $\sigma_2$ can involve elements which overlap while they do not involve overlapping elements in (4.3). The advantage of writing the algebra in the form above is that the leading coefficient can be read off directly from the 3-point functions that have been computed in section 2.

Another noteworthy point in (4.3) is that after correct normalization the terms involving permutations with higher numbers of non-trivial cycles are sub-leading in the large $N$ expansion. This means that by restricting attention to leading powers of $\frac{1}{N}$ we can define appropriate *contractions* of the Lie algebra of interest which can be simpler than the exact Lie algebra and may be investigated in connection with the qualitative properties such as the exclusion principle and the related properties of correlation functions which we discuss in section 5.

While it is easy to prove that the full set of operators appearing in the commutator in (4.1) is of the form given there, we have not proved that the above form of operators is the full set of terms that can appear in the commutator in (4.3). We have some evidence
that the form of the algebra, restricting attention to pure twists is of the above form.
For example we know that the RHS cannot contain a descendant of a chiral primary, which would not satisfy $L_0 = J_0$ as the LHS does. The $L_0$ and $J_0$ conservations alone do allow other forms of operators which can be ruled out by using $S_N$ selection rules. In particular, it appears that we do not need to include operators of the form $(OO^\dagger)O^\dagger$. The generalization of the above ansatz to include the chiral factors which come up in the $A_n^{p,q}$ is easily done at the cost of some extra notation.

We will be interested in a class of free field realizations of the algebra obtained by considering the full non-chiral operators. We note nevertheless that similar free oscillator realizations can be considered for the chiral half of the algebra and that has some similarities with the large $N$ counting relevant for the full operators. We can realize the algebra in terms of free oscillators, with $\alpha_{-n} \alpha_{-m} = \delta_{n,m}$. In the large $N$ expansion, for example, by

$$A_{-m} = \alpha_{-m} + \frac{1}{\sqrt{N}}(1 + O(\frac{1}{N})) \sum_{m_1,m_2} \alpha_{-m_1} \alpha_{-m_2} \delta(m_1 + m_2 - 1, m)$$

$$+ \cdots + \frac{1}{N^{k/2}}(1 + O(\frac{1}{N})) \sum_{m_1,m_2 \cdots m_k} \alpha_{-m_1} \alpha_{-m_2} \cdots \alpha_{-m_k} \delta(m_1 + m_2 + \cdots m_k - k + 1, m)$$

(4.5)

The coefficients are matched with the algebra by using a recursive procedure.

Now we move on to the combination of the left and right operators which give the full chiral primaries. Using the commutators of the chiral sectors of the form above, we can write down the commutators of the non-chiral operators which contain terms written below

$$\sum_{\sigma_{m_1} \in T_{m_1}, \sigma_{m_2} \in T_{m_2}} [O^{\sigma_{m_1}} O^{\sigma_{m_1}}, \bar{O}^{\dagger \sigma_{m_1}} \bar{O}^{\dagger \sigma_{m_2}}] =$$

$$\sum_{k} \sum_{\sigma_{m_1} \in T_{m_1}, n_2 \cdots n_k} N^{-\frac{k}{2}} \delta(m_2 - m_1 + k, n_1 + n_2 + \cdots n_k)$$

(4.6)

which have a similar structure to the terms in (1.3).

A free field representation for the algebra can be constructed where

$$A_{-n} = \alpha_{-n} + \frac{1}{\sqrt{N}}(1 + O(\frac{1}{N}))(\alpha \alpha + \cdots) + \cdots \frac{1}{N^{l/2}} (1 + O(1/N)) \alpha^{l+1}$$

(4.7)

The second term in the above expression is quadratic in the $\alpha$'s. The higher term weighted by $N^{-\frac{l}{2}}$ is a polynomial in the $\alpha, \alpha^\dagger$ of degree $l + 1$. It is noteworthy that, for
fixed degree of the polynomial, the expansion involves powers of $1/N$ and not $1/\sqrt{N}$. The odd powers of $\sqrt{N}$ can be removed by redefining the operators. This is in agreement with the idea that the algebra is closely related to the exclusion principle and to quantum groups, which both involve the parameter $N$, and not $\sqrt{N}$.

There are extra operators appearing on the RHS of the equation have the form

$$O_n\bar{O}_m(\bar{O}_{n-m})^\dagger = \alpha_n\alpha_m + \cdots$$

(4.8)

The composite operators have also been normalized to have unit two-point function. They are subleading in the $1/N$ expansion.

4.1. Hamiltonian

The free field realizations of the algebra in the large $N$ limit provide representations of operators of the theory and of the Hamiltonian. Based on studying the free field reps of sub-algebras of the full algebra (in particular the global case of the previous section) we expect that there exists first a free representation where the Hamiltonian is quadratic:

$$H^{(f)} = \sum_{m,p,q} (m + \frac{p+q}{2})\alpha_{-m}^p\alpha_m^p$$

(4.9)

In this picture the interactions are contained in the nonlinear forms of the other (in particular the creation-anihilation ) generators. We expect another representation where the Hamiltonian is nonlinear and exibits interactions in $1/N$. For the chiral primaries represented by $(0,0)$, $(1,1)$ and $(2,2)$ forms this takes the form

$$H^I = \sum_{m,p,q} (m + \frac{p+q}{2})\alpha_{-m}^p\alpha_m^p$$

$$+ \sum_{n,k} \frac{1}{\sqrt{N}} v(k,n)\left[\alpha_{n-k+1} \alpha_k \alpha_n + \alpha_{n-k} \alpha_k \alpha_{n+k-1} + \alpha_{n-k+1} \alpha_k \alpha_n + h.c.\right]$$

$$+ \left[\alpha_{n-k+1} \alpha_k \alpha_n + \alpha_{n-k} \alpha_k \alpha_{n+k-1} + h.c.\right]$$

$$+ \left[\alpha_{n-k+1} \alpha_k \alpha_n + \alpha_{n-k} \alpha_k \alpha_{n+k-1} + h.c.\right]$$

(4.10)

with the form factor

$$v(k,n) = \sqrt{(n+k)nk}$$

(4.11)
and with appropriate numerical coefficients. The above form would represent the leading term with higher interactions in powers of \( N \). Concerning this Hamiltonian and the emerging 1 + 1 dimensional field theory one has the following comments. It effectively summarizes the dynamics of chiral primaries with their correlation functions. The dimension conjugate to the twist \( n \) corresponds to a coordinate of AdS obtained after chiral primary reduction. The structure of the 1+1 dimensional hamiltonian and its form factor \( v(k,n) \) is identical in form to the collective field theory \[26\] of 2d strings. The fact that there is an analogy between the 2d noncritical string and radial dynamics in the AdS/CFT correspondence has been observed earlier, for example \[27\]. We would also like to stress that the dynamics outlined above is universal. Chiral primaries in other AdS theories (\( AdS_5 \times S^5 \) or \( AdS_4 \times S^7 \)) are also described by an analogous one plus one dimensional hamiltonian. This follows by the fact that the form factor \( v(k,n) \) seems to be the same in all these theories. In the present description it reflects the fact that behind the AdS/CFT correspondence there is an underlying non-critical string theory describable by coll. field theory.

The above structure (of the algebra) and the hamiltonian is operational on the space of chiral primary operators. We should mention the extension to a more general class of states as follows: using the SUSY algebra would provide couplings to other fields of the full short multiplet, likewise the lowering operations of both \( SL(2) \times SL(2) \) and \( SU(2) \times SU(2) \) would specify couplings to the corresponding descendants spanning the full \( AdS_3 \times S^3 \) space-time. At leading order this extensions are direct, at higher order in \( N \) they remain a challenge. The non-linear realizations of some of the basic symmetries \[28\] might play an important role in these extensions.

5. Exclusion Principle and the Lie Algebra of observables.

The properties of the Lie algebra associated with the chiral primaries are closely related to the Exclusion principle. To clarify this we will need to review a few facts about the chiral ring of the \( S^N(X) \) theory, all of them related to the finiteness properties, which are referred to as the exclusion principle.

The simplest fact under this heading is that there an arbitrary element of the ring generated by the \( A \) operators of section 2 has its left and right \( SU(2) \) quantum numbers bounded as \( 2j_L \leq 2N \) and \( 2j_R \leq 2N \). This follows from the unitarity constraints.
of the $N = 2$ superconformal sub-algebras of the $N = (4,4)$ symmetry \cite{29}. For example $(A_{-2}^{(0,0)})^{2N+1}$ necessarily vanishes by this argument. Another aspect of the exclusion principle, emphasized in I, refers to the properties of single particle states, which can be defined in the CFT as a linearly independent set of chiral primary operators which cannot be written entirely in terms of products of other chiral primaries. They are the analogs of single trace operators in Yang Mills. It was found in I that these generators are also cutoff by their $SU_L(2)$ and $SU_R(2)$ quantum numbers, and this was an important piece of evidence in favour of a non-commutative spacetime. Another property of the vector space of chiral primaries is that it has a basis defined in terms of Fock space creation operators $B_{-n}^{\omega_{p,q}}$, which are related but not identical to the $A_{-n}^{\omega_{p,q}}$. We can describe a vector space $H_B$ as a truncated Fock space. Its states are in 1-1 correspondence with

$$B_{-n_1}^{\omega_1} B_{-n_2}^{\omega_2} \cdots B_{-n_k}^{\omega_k} |0>$$

with the restriction $\sum n_i = N$. The relation between the space of chiral primaries defined as generated by $A_{-n}^{\omega}$ subject to relations between them which follow from the OPE’s is as follows. Let these relations take the form

$$R_i(A) = 0$$

Consider the ring of polynomials in the $A_{-n}^{\omega}$ quotiented by the relations above, and let this space be $H/R$ where $R$ is the ideal generated by the $R_i$. We have an isomorphism between $H/R$ and $H_B$.

The relations $R_i$ are closely related to the structure of the Lie algebra we constructed from the $A$ and $A^\dagger$. Setting these elements to zero has to be consistent with the algebra. In other words

$$[L_a, R_i] = C_{aijb} L_b R_j$$

where $L_a$ and $L_b$ are elements in the Universal Enveloping algebra of the Lie algebra. An example of this kind of relation in the case of a simple $sl(2)$ subalgebra as given in I. In fact the unitarity conditions in the algebra were used to derive these relations.
5.1. \textit{q-deformed super-algebra structure of fusion rules}

For simplicity we restrict the discussion here to couplings involving only the pure-twist operators $A_{(0,0)}^{(0,0)}$. For $n \leq N$ these are generating elements of the chiral ring. For small values of $n$ we have fusion of the form

$$A_{-n}^{(0,0)} A_{-m}^{(0,0)} \sim A_{-n-m+1}^{(0,0)} \quad (5.4)$$

There can also appear on the RHS products of $A$’s but we can restrict attention to the terms on the RHS which are relevant for studying the 3-point functions of generators of the chiral ring. On the left we have operators of $SU(2)_L$ spins $2j_1 = n - 1$ and $2j_2 = m - 1$. On the right we have $2j = 2j_1 + 2j_2$. This is consistent with $SU(2)$ fusion rules but not all $SU(2)$ reps. allowed by $SU(2)$ tensor products appear on the RHS. Rather a good model for the fusion is given by the tensor products of short reps. of $SU_L (2|1,1)$ (since the $SU_R (2|1,1)$ quantum numbers are identical to the left moving ones it suffices to focus on the left moving symmetry). The spin $j$ then labels the largest $SU(2)$ spin present in the decomposition of the short $SU(2|1,1)$ rep. into reps. of its $SU(2)$ subalgebra.

A qualitatively new feature appears when $n$ comes close to $N$, and $n + k \geq N$. Then

$$A_{-n}^{(0,0)} A_{-k}^{(0,0)} \sim 0 \quad (5.5)$$

We can see this explicitly from the formulae for correlation functions we wrote down in (2.34). We rewrite the equation relevant for this feature here:

$$\langle O_{n+k-1}^{(0,0)}(\infty) O_k^{(0,0)}(1) O_n^{(0,0)}(0) \rangle = \frac{(N - n)! (N - k)! (n + k - 1)^3}{(N - (n + k - 1))! N! n k} \frac{1}{3} \quad (5.6)$$

We see that when $n + k - 1$ exceeds $N$ the denominator contains a $\Gamma$ function with a negative integer argument which causes the expression to vanish. This feature can be explained using fusion rules of $SU_q (2|1,1)$ where $q$ is, as in I, given by $q = e^{\frac{i\pi}{N+1}}$. The $q$-deformed superalgebra will have a family of unitary short (atypical) reps. which will include unitary irreps. of the $SU_q (2)$. Since all the reps. of the $SU_q (2)$ appearing in the decomposition from $SU_q (2|1,1)$ to $SU_q (2)$ have to be unitary, there will be a bound on these short reps. $2j \leq N - 1$. 

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5.2. \( SU_q(2|1,1) \) and multiparticle states

We have explained the qualitative features of the couplings between this family of generators of the chiral ring, using \( SU_q(2|1,1) \). It will be very interesting to find a more detailed comparison between the fusions of the chiral ring and the q-deformed super-algebra. Some novel features, unfamiliar from ordinary rational CFT, but that have appeared in studies of WZW on supergroups [30] have to be taken into account. For example it has been found that one needs, in general, to take into account the indecomposable reps. of the superalgebra as well. It is very plausible, that in this case, we also need to include reps. of \( SU_q(2|1,1) \) which contain indecomposables of the \( SU_q(2) \) sub-algebra. While the \( SU_q(2|1,1) \) with the \( q \) given above, nicely describes the cutoffs on generating \( A^{(p,q)}_{-n} \), it also seems to have enough structure to account for some other cutoffs in this theory. For example, \( SU_q(2) \) has, in addition to the standard irreps, with a cutoff \( 2j \leq N_1 - 1 \), a family of irreps \( I_p^z \) in the notation of [31]. The detailed form of these reps. is given in [32]. Usually one drops these reps. in studying standard connections between \( SU_q(2) \) and the \( SU(2) \) WZW, but in this model the chiral primaries transform in more complicated reps. of the \( SU(2) \) current algebra than the standard integrable reps. [33] [34] usually considered in \( SU(2) \) WZW of level \( k = N \). We may expect a larger class of \( SU_q(2) \) reps. to appear in the corresponding quantum group model.

We mention a very suggestive numerical observation in favour of the above line of argument. The family \( I^p_0 \) has a cutoff at \( 2j \leq 2N \). The \( A_{-n} \) operators coming from twisted sectors can be raised to powers which allow them to exceed the bound at \( N_1 - 1 \). It turns out, however that that there is a cutoff on the highest weight \( 2j \leq 2N \) which works for any chiral primary, as explained in the previous section. This means that if we associate reps containing these indecomposables to some of the chiral primaries which are products of \( A^{(p,q)}_{-n} \)'s, we can explain both the cutoff \( 2j \leq N_1 - 1 \) for the generators, and the cutoff \( 2j \leq 2N \) for arbitrary chiral primaries.

5.3. Black Holes and Correlation functions of Chiral primaries

We saw that the Exclusion Principle is one manifestation of the breakdown of the free-field representation of the algebra we have been considering. This algebra, as we have emphasized, is closely related to the correlation functions. We saw at the beginning of this section in the discussion of (5.5) and (5.6) that the exclusion principle shows up in a qualitative change in behaviour of the correlation functions as a function of a parameter
approaching $N$. We also saw in the discussion of the untwisted sector algebra (see in particular (3.15)) that correlators with a large number of insertions start to show non-free field behaviour.

We would like to use the properties of the orbifold CFT to find the lowest conformal weight where we may expect a divergence from free field behaviour. Recall the discussion of the map to free fields. We had for example

$$A_{-2}^{(0,0)} = \alpha_{-2} + \cdots$$

$$A_{-3}^{(0,0)} = \alpha_{-3} + \cdots$$

(5.7)

However we do not have

$$(A_{-2}^{(0,0)})^2 = \alpha_{-2}^2 + \cdots$$

(5.8)

This would contradict the fact that the correlation function $\langle A_3 A_{-2} A_{-3} \rangle$ is non-zero at order $\frac{1}{\sqrt{N}}$. Rather the object which behaves more like $\alpha_{-2}^2$ is the twist operator associated with the conjugacy class with two cycles of length 2, which is one of the terms in (4.6) for example. While $A_{-2}^l$ is not, for small $l$, a free oscillator raised to the $l$'th power, its behaviour in correlation functions should not be qualitatively different from a power of a free field because it is a linear combination including an operator which behaves like $\alpha_{-2}^l$. When $l$ hits $N/2$ the corresponding operator ceases to exist because we cannot have a permutation with more than $\frac{N}{2}$ cycles of length 2. We observe that this happens at $L_0 = \frac{N}{4}$. If we try the same thing with $A_{-3}^l$ we get a threshold of $l = \frac{N}{3}$ where we may expect a qualitative change in behaviour. If we consider a general operator of the form

$$(A_{-2})^{n_2} (A_{-3})^{n_3} \cdots A_{-n_k}^{k}$$

(5.9)

the corresponding free operator ceases to exist when $2n_2 + 3n_3 + \cdots kn_k = N$. It has

$$L_0 = \frac{1}{2}(n_2 + 2n_3 + \cdots kn_k) = \frac{1}{2}(N - (n_2 + n_3 + \cdots + n_k) = \frac{1}{2}\left(\frac{N}{2} + \frac{1}{2}n_3 + \frac{2}{2}n_3 + \cdots + \frac{(k-1)}{2}n_k\right)$$

(5.10)

It is now clear that the lowest threshold we get is $\frac{N}{4}$. Considering operators of type $A_{-n}^{p,q}$ with $p, q \neq 0$ only increases the threshold. Precisely this value $L_0 = N/4$ was obtained by \[7\] \[35\] as the threshold where black holes start to be relevant. We have argued that the same threshold appears if we ask for the lowest value of $L_0$ where operators in the chiral ring start to display behaviour qualitatively different from free fields. It will be very interesting to characterize the corresponding change of behaviour in the correlation functions and compare with expectations from black hole physics.
6. Conclusions

We have studied in this paper a Lie algebra associated with the chiral primaries of $S^N(T^4)$ CFT and their CFT conjugates. The structure constants of this Lie algebra are simply related to the correlation functions after an appropriate choice of basis. We obtained by CFT computations some of these structure constants, those which are determined by the 3-point functions of bosonic chiral primaries. This leads to a dynamics described by an effective 1+1 dimensional field theory which corresponds to the simplest representation of the algebra.

Connections between the Lie algebra and the stringy exclusion principle were studied in I, where the relation with q-algebras and non-commutative spacetimes was emphasized. These aspects were elaborated 5.1-5.3. In 5.4 we focused on a characterization of the exclusion principle as a deviation of the chiral primaries from free field behaviour in correlation functions, and we were lead to look for the lowest threshold where such deviation is expected. We found that the lowest threshold is at $L_0 = \frac{N}{4}$ which has been argued to be relevant to black holes in $ADS_3$. Developing further the relation between these correlation functions and black holes will be very interesting.

The Lie algebra is a much richer structure than the truncated Fock space of the chiral primaries themselves. The latter is only one representation of the Lie algebra. This simplest representation was studied in connection with actions associated to the correlation functions of chiral primaries. This was done in detail in section 3 for the untwisted sector. A coherent state basis was useful there, and we explained how the parameters in the coherent states are related to the multi-oscillator Heisenberg algebra at large $N$. In section 4, we pursued the study of the large $N$ map between the Lie algebra and the multi-oscillator Heisenberg algebra (free fields).

Our discussion naturally raises the question of other representations of the algebra which are not directly related to the Fock space of the chiral primaries. These will be important in studying the stringy states, which are certainly important since the free orbifold CFT is expected to be dual to a gravitational background where the graviton as well as other stringy degrees of freedom are important, as emphasized in [10,11,12]. Studying the stringy states as representations of this algebra derived from chiral primaries and their conjugates is particularly interesting, since the chiral primaries and their truncations show clear evidence of a non-commutative spacetime. This point of view on the stringy states should clarify the relevance of the non-commutative space-time to the dynamics of the full set of stringy degrees of freedom.
The precise role of the non-commutative spacetime in determining the dynamics of this gravitational theory remains to be further studied, but the simplest possibility is that, in analogy to the case of non-commutative Yang Mills [36] there is ‘simple action’ on a non-commutative spacetime which is related to a ‘complicated action’ on a commutative spacetime. The fact that the techniques of collective field theory allow the construction of spacetime actions starting from the CFT should allow one to explore the above possibility. Other approaches for studying the non-commutativity of spacetime coordinates in the context of string theory on ADS-type backgrounds have been suggested in [37][38].

Acknowledgements: We are happy to acknowledge enjoyable and instructive discussions with Miriam Cvetic, Sumit Das, Pei Ming Ho, Robert de Mello Koch, Vipul Periwal, Radu Tatar, T. Yoneya, E. Witten. M.M. would like to thank the organizers of TASI99 for hospitality while part of the work was being done. This research was supported by DOE grant DE-FG02/19ER40688-(Task A).

7. Appendix

Let us consider first the OPE we used which can be read from [15] where it is determined by a geometric construction. We will also neglect the fact that in our case the theory is a $T^4$ orbifold, whereas in [15], it was a noncompact orbifold. The result for the constant is taken from equation (5.24) in [15] with the following redefinitions:

\[ N = n + 1, \quad n_0 = n, \quad D = 6, \quad p_1 = k_n, \quad p_2 = 0, \quad p = k_2; \tag{7.1} \]

\[ C(2, n|n + 1; k_2, k_n) = 2^a n^b (n + 1)^c, \tag{7.2} \]

where:

\[ a = -\frac{9}{8} + \frac{k_2^2}{4}, \]
\[ b = -\frac{3}{8} - \frac{1}{n + 1} \left( \frac{1}{4} n^2 - \frac{1}{2} k_n^2 - k_2 k_n - \frac{n - 1}{4} k_2^2 \right), \tag{7.3} \]
\[ c = \frac{1}{8} + \frac{1}{4} \left( n + \frac{1}{n} \right) - \frac{k_2^2}{4} - \frac{k_n^2}{2 n}. \]

We outline here the derivation for the equations used in the text only for the twist field which does not involve any $T^4$ indices, $O_{(1, n)}(u, \bar{u})$. In this case, in the equations (7.2) we take $k_2 = (0, 0, 0, 0, 0, 1)$, $k_n = (0, 0, 0, 0, \frac{n-1}{2}, \frac{n-1}{2})$ which give $a = -1, \quad b = -\frac{1}{2}, \quad c = \frac{1}{2}$.
and we have the first equation in (2.24). Combining this result with an identical one for the right movers we obtain:

\[ O_{(n+1)}(u, \bar{u}) O_{(1..n)}(0) = \frac{1}{4} \frac{n+1}{n} (O_{(1..n+1)}(0) + O_{(1..n+1)}(0)) \]  \hspace{1cm} (7.4)

This result has a clear interpretation for the case when the length of the cycle at 0 is set to 1, namely \( n = 1 \) case. We used this extrapolation for the other cases to arrive at equations (2.25).

Starting from equation (7.4) we can use now the conformal field theory rules in order to determine more involved correlators. We look to an OPE involving three twists:

\[ O_{(n+k-1..n+k)}(u, \bar{u}) O_{(n..n+k-1)}(z, \bar{z}) O_{(1..n)}(0) = const. \ (O_{(1..n+k)}(0) + ...) \]  \hspace{1cm} (7.5)

where \( ... \) stand for the other 4 possible terms, and the right hand side is in the limit \( u \to 0; \ z \to 0 \). For consistency, the limit should not depend on the order, and from this we derive the following recurrence equation for the constant which appear in the OPE of a twist of length \( n \) and of one of length \( k \), denoted in this appendix as \( C(n,k) \):

\[ C(n, k + 1) = C(n, k) \frac{(n + k) k}{(n + k - 1) (k + 1)}, \]  \hspace{1cm} (7.6)

\[ C(n, 2) = \frac{(n + 1)}{4 n}. \]

These equations are solved by :

\[ C(n, k) = \frac{n + k - 1}{2 n k} \]  \hspace{1cm} (7.7)

It is interesting to observe that in this derivation of the constants appearing in the OPE of the twist operators it was helpful to use both the geometric construction \[15\] and the associativity of the OPE in conformal field theory.

Using a similar derivation, we are able to obtain also the OPE for the twist operators overlapping over a two cycle.
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