THE CAUCHY DUAL SUBNORMALITY PROBLEM VIA DE BRANGES-ROVNYAK SPACES

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Abstract. The Cauchy dual subnormality problem (for short, CDSP) asks whether the Cauchy dual of a 2-isometry is subnormal. In this paper, we address this problem for cyclic 2-isometries. In view of some recent developments in operator theory on function spaces (see [4, 22]), one may recast CDSP as the problem of subnormality of the Cauchy dual $M'_z$ of the multiplication operator $M_z$ acting on a de Branges-Rovnyak space $H(B)$, where $B$ is a vector-valued rational function. The main result of this paper characterizes the subnormality of $M'_z$ on $H(B)$ provided $B$ is a vector-valued rational function with simple poles. As an application, we provide affirmative solution to CDSP for the Dirichlet-type spaces $D(\mu)$ associated with measures $\mu$ supported on two antipodal points of the unit circle.

1. Cauchy dual subnormality problem for 2-isometries

The Cauchy dual subnormality problem (for short, CDSP) for 2-isometries can be seen as the manifestation of the rich interplay between positive definite and negative definite functions on abelian semigroups. Indeed, CDSP can be considered as the non-commutative variant of the fact from the harmonic analysis on semigroups that the reciprocal of a Bernstein function $f : [0, \infty) \to (0, \infty)$ is completely monotone (see [31, Theorem 3.6]). This fact turns out to be somewhat equivalent to the solution of CDSP for completely hyperexpansive weighted shifts (see [2] Proposition 6 for a generalization). Another early result towards the solution of CDSP asserts that the Cauchy dual of any concave operator is a hyponormal contraction (see [33, Equation (26)]). Later this fact was generalized in [13, Theorem 3.1] by deducing power hyponormality of the Cauchy dual of any concave operator. Around the same time CDSP was settled affirmatively for $\Delta_T$-regular 2-isometries in [S Theorem 3.4] and for 2-isometric operator-valued weighted shifts in [S Theorems 2.5 and 3.3] (see also [14 Corollary 6.2] for the solution for yet another subclass of 2-isometries). Further, it was shown in [S Examples 6.6 and 7.10] that there exist 2-isometric weighted shifts on directed trees (that include adjacency operators) whose Cauchy dual is not necessarily subnormal. Recently, a class of cyclic 2-isometric composition operators without subnormal Cauchy dual has been exhibited in [6, Theorem 4.4].

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For a complex Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of bounded linear operators on $\mathcal{H}$. A bounded linear operator $T$ in $\mathcal{B}(\mathcal{H})$ is cyclic if there exists a vector $f \in \mathcal{H}$ (to be referred to as the cyclic vector) such that $\{T^n f : n \geq 0\} = \mathcal{H}$. We say that $T$ is analytic if $\cap_{n \geq 0} T^n \mathcal{H} = \{0\}$. Following [32], the Cauchy dual $T'$ of a left-invertible $T \in \mathcal{B}(\mathcal{H})$ is defined by $T' = T(T^* T)^{-1}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is a 2-isometry if

$$I - 2T^* T + T^* T^2 = 0$$

(see [2] [26] for the basic properties of 2-isometries). By [26] Lemma 1], a 2-isometry $T$ is norm increasing (that is, $T^* T \geq I$), and hence the Cauchy dual of $T$ is well-defined. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if it has a normal extension, that is, there exists a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $N x = T x$ for every $x \in \mathcal{H}$ (the reader is referred to [10] for an excellent exposition on the theory of subnormal operators). With these notions, one can now state the Cauchy dual subnormality problem. This problem asks when the Cauchy dual $T'$ of a cyclic 2-isometry $T$ in $\mathcal{B}(\mathcal{H})$ is subnormal.

For a finite positive Borel measure $\mu$ on the unit circle $\mathbb{T}$, the Dirichlet-type space $\mathcal{D}(\mu)$ is defined by

$$\mathcal{D}(\mu) := \left\{ f \in \mathcal{O}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 P_{\mu}(z) \, dA(z) < \infty \right\},$$

where $P_{\mu}(z)$ denotes the Poisson integral $\int_{\mathbb{T}} \frac{1-|z|^2}{1-\overline{z}\zeta} d\mu(\zeta)$ of the measure $\mu$, $dA(z)$ denotes the normalized Lebesgue area measure on the open unit disc $\mathbb{D}$ and $\mathcal{O}(\mathbb{D})$ denotes the space of complex valued holomorphic functions on $\mathbb{D}$. It is well-known that any cyclic analytic 2-isometry is unitarily equivalent to the operator $\mathcal{M}_z$ of multiplication by the coordinate function $z$ on a Dirichlet-type space $\mathcal{D}(\mu)$ for some finite positive Borel measure $\mu$ on the unit circle $\mathbb{T}$ (see [26] Theorems 3.7 and 5.1]). In view of this model theorem and the fact that any 2-isometry decomposes into a direct sum of an analytic 2-isometry and a unitary (see [32] Proposition 3.4)), the foregoing problem is equivalent to the following variant of CDSP.

**Problem (CDSP).** Classify all finite positive Borel measures $\mu$ for which the Cauchy dual $\mathcal{M}'_z$ of the multiplication operator $\mathcal{M}_z$ on the Dirichlet-type space $\mathcal{D}(\mu)$ is subnormal.

The counter-example to CDSP, as provided in [3] Section 4], is a cyclic 2-isometry $T$ for which the range of $T^* T - I$ is of infinite dimension. This asks whether or not CDSP has affirmative answer for all cyclic 2-isometries for which $T^* T - I$ is of finite rank. It turns out that all such 2-isometries arise as the multiplication operator $\mathcal{M}_z$ acting on a Dirichlet-type space associated with a positive finitely supported measure (see Theorem 6.1). Since every analytic norm increasing operator is unitarily equivalent to the multiplication operator $\mathcal{M}_z$ acting on a de Branges-Rovnyak space (see [22] Theorem 4.6] and Lemma 3.4), the above problem for Dirichlet-type spaces associated with finitely supported measures can be addressed by examining de Branges-Rovnyak spaces of finite rank. This is the purpose of this paper.
2. Main Results

For complex separable Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$, the Schur class $\mathcal{S}(\mathcal{D}, \mathcal{E})$ is given by
\[ \mathcal{S}(\mathcal{D}, \mathcal{E}) = \{ B : \mathcal{D} \to \mathcal{B}(\mathcal{D}, \mathcal{E}) \mid B \text{ is holomorphic with } \sup_{z \in \mathcal{D}} \| B(z) \|_{\mathcal{B}(\mathcal{D}, \mathcal{E})} \leq 1 \}. \]

In case $\mathcal{D} = \mathbb{C} = \mathcal{E}$, the Schur class is equal to the closed unit ball of the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on the unit disc $\mathbb{D}$. For any $B \in \mathcal{S}(\mathcal{D}, \mathcal{E})$, the de Branges-Rovnyak space $\mathcal{H}(B)$ is the reproducing kernel Hilbert space associated with the $B(\mathcal{E})$-valued positive semi-definite kernel
\[ \kappa_B(z, w) = \frac{I_\mathcal{E} - B(z)B(w)^*}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}. \]

The kernel $\kappa_B$ is normalized if $\kappa_B(z, 0) = I_\mathcal{E}$ for every $z \in \mathcal{D}$. This is equivalent to $B(0) = 0$. If $\mathcal{D} = \mathcal{E} = \mathbb{C}$, then we refer to $\mathcal{H}(B)$, to be denoted by $\mathcal{H}(b)$, as the classical de Branges-Rovnyak space (refer to [29] and [19] for the basic theory of classical de Branges-Rovnyak spaces). Following [4], Definition 1.1], the spaces $\mathcal{H}(B)$ are referred to as finite rank de Branges-Rovnyak spaces provided $\mathcal{D}$ is finite dimensional and $\mathcal{E}$ is of dimension 1. More generally, for any $B \in \mathcal{S}(\mathcal{D}, \mathcal{E})$, the de Branges-Rovnyak space $\mathcal{H}(B)$ is said to be of rank $k$ if
\[ k = \inf \{ \dim \tilde{\mathcal{D}} : \tilde{B} \in \mathcal{S}(\tilde{\mathcal{D}}, \mathcal{E}) \text{ so that } \mathcal{H}(B) = \mathcal{H}(\tilde{B}) \text{ with equality of norms} \}. \]

Assume that $\mathcal{D}$ is finite dimensional and $\mathcal{E}$ is of dimension 1. Then $B = (b_1, \ldots, b_k)$, $z$ is a multiplier (that is, $zf \in \mathcal{H}(B)$ whenever $f \in \mathcal{H}(B)$) if and only if
\[ \int \log \left(1 - \sum_{j=1}^{k} |b_j|^2\right) d\theta > -\infty \]
(see [4] Theorem 5.2] and [19] Corollary 20.20]). If $z$ is a multiplier, then by the closed graph theorem, the operator $\mathcal{M}_z$ of multiplication by $z$ on $\mathcal{H}(B)$ is bounded. In this case, $\mathcal{M}_z$ turns out to be a norm increasing, that is,
\[ \| \mathcal{M}_z f \| \geq \| f \|, \quad f \in \mathcal{H}(B) \quad (2.1) \]
(see [3] Theorem 2.1] and [22] Pg 17]). Conversely, every analytic norm increasing operator is unitarily equivalent to the operator $\mathcal{M}_z$ of multiplication by $z$ acting on a de Branges-Rovnyak space $\mathcal{H}(B)$ for some Schur-class function $B$ (see [22] Theorem 4.6] and Lemma 3.4).

Here is the main result of this paper concerning the CDSP for a family of finite rank de Branges-Rovnyak spaces.

**Theorem 2.1.** Let $B = (b_1, \ldots, b_k) \in \mathcal{S}(\mathbb{C}^k, \mathbb{C})$ be such that $B(0) = 0$, where
\[ b_j(z) = \frac{p_j(z)}{\prod_{j=1}^{k}(z - \alpha_j)} \]
for polynomials $p_j$ of degree at most $k$ and distinct numbers $\alpha_1, \ldots, \alpha_k$ in $\mathbb{C} \setminus \mathbb{D}$. For $r = 1, \ldots, k$, let $a_r = \prod_{1 \leq t \neq r \leq k}(\alpha_r - \alpha_t)$. Assume that the operator $\mathcal{M}_z$ of multiplication by $z$ on the de Branges-Rovnyak space $\mathcal{H}(B)$ is bounded.
Then the Cauchy dual $\mathcal{M}'_z$ of $\mathcal{M}_z$ is subnormal if and only if the matrix
\[
\sum_{r,t=1}^{k} p_j(\alpha_r)p_j(\alpha_t) \left( 1 - \frac{1}{\alpha_r\alpha_t} \right)^l \left( \frac{1}{\alpha_r^{m+2\alpha_t^2}} \right) m,n \geq 0
\]
is formally positive semi-definite for every $l \geq 1$.

If the rational symbol $B$ admits a pole of multiplicity bigger than 1, then the results obtained in this paper can be used to characterize the subnormality of the Cauchy dual of $\mathcal{M}_z$ in $\mathcal{H}(B)$. However, the characterization in this case is not as neat as we get in the theorem above. As the first application of Theorem 2.1, we obtain a handy criterion for the subnormality of the Cauchy dual of the multiplication operators under consideration.

**Corollary 2.2.** Assume that the hypotheses of Theorem 2.1 hold. If
\[
\sum_{j=1}^{k} p_j(\alpha_r)p_j(\alpha_t) = 0, \quad 1 \leq r \neq t \leq k,
\]
then $\mathcal{M}'_z$ is subnormal.

The following provides affirmative solution to CDSP for the Dirichlet-type spaces $\mathcal{D}(\mu)$ associated with measures $\mu$ supported on two antipodal points of the unit circle.

**Theorem 2.3.** For $\zeta \in \mathbb{T}$ and nonnegative numbers $c_1, c_2$, let $\mathcal{D}(\mu)$ be the Dirichlet-type space associated with the measure $\mu = c_1\delta_{\zeta} + c_2\delta_{-\zeta}$ and let $\mathcal{M}_z$ be the multiplication operator on $\mathcal{D}(\mu)$. Then the Cauchy dual $\mathcal{M}'_z$ of $\mathcal{M}_z$ is a subnormal contraction.

Here is the layout of the paper. In Section 3, we consider the Cauchy dual subnormality problem in the set-up of functional Hilbert spaces and apply its solution to general de Branges-Rovnyak spaces (see Theorems 3.3 and 3.5). Section 4 is devoted to CDSP for de Branges-Rovnyak spaces of finite ranks. In particular, we prove Theorem 2.1 and Corollary 4.3. In Sections 5 and 6, we apply these results to provide a proof of Theorem 2.3. In this analysis, we arrive at a precise formula for the symbol $B$ of the rank 2 de Branges-Rovnyak space $\mathcal{H}(B)$, which coincides with the Dirichlet-type space associated with a measure supported at anti-podal points (see Section 7).

### 3. The Cauchy dual subnormality problem in de Branges-Rovnyak spaces

It is well-known that every analytic left-invertible operator $T$ is unitarily equivalent to the multiplication operator $\mathcal{M}_z$ on a reproducing kernel Hilbert space $\mathcal{H}$ of $E$-valued holomorphic functions of a some disc centred at the origin, where $E = \ker T^*$ (refer to [32]). In view of this model theorem, we confine the discussion to the setting of reproducing kernel Hilbert spaces.

Let $\mathcal{E}$ be an auxiliary complex separable Hilbert space. Let $\mathcal{H}_\kappa$ be a reproducing kernel Hilbert space of $E$-valued holomorphic functions defined on the unit disc and let $\kappa : \mathbb{D} \times \mathbb{D} \to B(\mathcal{E})$ be the reproducing kernel for $\mathcal{H}_\kappa$, that is, $\kappa(\cdot, w)x \in \mathcal{H}_\kappa$ and
\[
\langle f, \kappa(\cdot, w)x \rangle_{\mathcal{H}_\kappa} = \langle f(w), x \rangle_{\mathcal{E}}, \quad f \in \mathcal{H}_\kappa, \; w \in \mathbb{D}, \; x \in \mathcal{E}.
\]
In the remaining part of this section, we assume the following:

(A1) \( z \) is a multiplier for \( \mathcal{H}_\kappa \), that is, \( zf \in \mathcal{H}_\kappa \) for every \( f \in \mathcal{H}_\kappa \),

(A2) \( \kappa \) is normalized at the origin, that is, \( \kappa(z,0) = I_E \) for every \( z \in \mathbb{D} \),

(A3) Under the assumptions (A1) and (A2), the orthogonal complement of \( \{zf : f \in \mathcal{H}_\kappa\} \) is spanned by the space of \( E \)-valued constant functions.

(A4) Under the assumption (A1), \( \mathcal{M}_z \) is left-invertible.

We refer to the pair \((\mathcal{H}_\kappa, E)\) satisfying (A1)-(A4) as the functional Hilbert space.

Remark 3.1. By (A1) and the closed graph theorem, the operator \( \mathcal{M}_z \) of multiplication by \( z \) on \( \mathcal{H}_\kappa \) is bounded. Further, since \( \mathcal{H}_\kappa \) consists of \( E \)-valued holomorphic functions, \( \mathcal{M}_z \) is analytic. It is also clear from (A2) and (A3) that \( \ker \mathcal{M}_z^* \) is spanned by the space of \( E \)-valued constant functions and \( \|x\|_{\mathcal{H}_\kappa} = \|x\|_E \) for every \( x \in E \). The backward shift operator \( \mathcal{L}_z \) given by

\[
(\mathcal{L}_z f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathcal{H}_\kappa.
\]

is bounded on \( \mathcal{H}_\kappa \) and satisfies

\[
\mathcal{M}_z^* \mathcal{M}_z = \mathcal{I} = \mathcal{L}_z \mathcal{M}_z.
\]

(3.1)

Indeed, in view of the closed graph theorem, it suffices to check that \( \mathcal{L}_z f \) belongs to \( \mathcal{H}_\kappa \) for every \( f \in \mathcal{H}_\kappa \). Since \( \mathcal{M}_z \) is left-invertible, the range of \( \mathcal{M}_z \) is closed. By (A2) and (A3),

\[
\{zf : f \in \mathcal{H}_\kappa\} = \mathcal{H}_\kappa \ominus \{\kappa(\cdot,0) : x \in E\} = \{g \in \mathcal{H}_\kappa : g(0) = 0\}.
\]

This shows that \( \mathcal{L}_z \) is bounded. Moreover, \( \mathcal{M}_z^* \mathcal{M}_z = \mathcal{I} = \mathcal{L}_z \mathcal{M}_z \). This together with \( \mathcal{M}_z^* x = 0 = \mathcal{L}_z x, x \in E \), yields (3.1).

For a nonnegative integer \( n \), let \( e_n : \mathbb{D} \to B(E) \) be given by

\[
e_n(z)x = \frac{\partial^n \kappa(z,w)x}{n!} \bigg|_{w=0}, \quad z \in \mathbb{D}, \; x \in E,
\]

where \( \partial^n \kappa(z,w) \) denotes the \( n \)th partial derivative of \( \kappa(z,w) \) with respect to \( w \). After interchanging the roles of analytic and coanalytic functions, it is easy to deduce from [18, Lemma 4.1] that

\[
e_n(\cdot)x \in \mathcal{H}_\kappa \quad \text{and} \quad \langle f, e_n(\cdot)x \rangle = \left\langle \frac{\partial^n f(0)}{n!}, x \right\rangle, \quad f \in \mathcal{H}_\kappa, \; x \in E.
\]

(3.2)

For integers \( m, n \geq 0 \), we use the short notation \( \partial^m \partial^n \kappa(0,0) \) to denote \( \partial^m \partial^n \kappa(0,0) \). Although the following is well-known (see [18, Lemma 4.1] and [12, Remark 2]), we include a proof for the sake of completeness.

**Proposition 3.2.** Let \( \kappa : \mathbb{D} \times \mathbb{D} \to B(E) \) be a kernel function given by

\[
\kappa(z,w) = \sum_{m,n=0}^{\infty} A_{m,n}z^m w^n, \quad z, w \in \mathbb{D},
\]

where \( A_{m,n} \in B(E) \). Then \( \kappa \) is a positive semi-definite kernel if and only if the matrix \( A = (A_{m,n})_{m,n \geq 0} \) is formally positive semi-definite.
Proof. If $\kappa$ is positive semi-definite, then by [18, Lemma 4.1(c)], the matrix $B = \left(\left(\frac{\partial^m \partial^n \kappa(0,0)}{m! n!}\right)_{m,n \geq 0}\right)$ is formally positive semi-definite. If $C = \left(\left(\frac{\delta_{m,n}}{m! n!}\right)_{m,n \geq 0}\right)$, then $A = C^* BC$, and hence $A$ is also formally positive semi-definite. This yields the necessity part. To see the sufficiency part, note that $\kappa_l(z, w) := \sum_{m,n=0}^l A_{m,n} z^m w^n = X(z) AX(w)^*,$ $z, w \in \mathbb{D}, \ l \geq 0,$

where $X(z) = (I_E, zI_E, \ldots, z^lI_E, 0, \ldots) : \ell^2(E) \to E$. Thus if $A$ is formally positive semi-definite, then $\kappa_l$ is a positive semi-definite kernel for every integer $l \geq 0$. Since the pointwise limit of positive semi-definite kernels is again positive semi-definite, the proof is complete. \hfill $\square$

We need the following characterization of the subnormality of the Cauchy dual of multiplication operator $M_z$ acting on functional Hilbert spaces in the proof of Theorem 3.5.

**Theorem 3.3.** Let $(H_\kappa, E)$ be a functional Hilbert space and let $M_z$ be the operator of multiplication by $z$ on $H_\kappa$. Assume that $M_z$ is norm-increasing. Then the following statements are equivalent:

(i) The Cauchy dual $M'_z$ of $M_z$ is subnormal.

(ii) $\sum_{j=0}^k (-1)^j \binom{k}{j} \left(\frac{\partial^m + j \partial^n \kappa(0,0)}{(m+j)! (n+j)!}\right)_{m,n \geq 0}$ is formally positive semi-definite for every $k \geq 0$.

If, in addition, $E$ is one dimensional, then (1) and (2) are equivalent to

(iii) $\left\{\frac{\partial^m \partial^n \kappa(0,0)}{m! n!}\right\}_{m,n \geq 0}$ is a complex moment sequence.

Proof. Assume that $M_z$ is norm-increasing. To see the equivalence of (i) and (ii), note that $M'_z$ is a contraction, and hence by Agler’s criterion [11, Theorem 3.1], $M'_z$ is subnormal if and only if for every integer $k \geq 0$,

$$\sum_{j=0}^k (-1)^j \binom{k}{j} M'^{m+j}_z M'^{n+j}_z \geq 0.$$ 

One may now infer from [32, Theorem 2.15] that $M'_z$ is subnormal if and only if for every integer $k \geq 0$,

$$\sum_{j=0}^k (-1)^j \binom{k}{j} \sigma^{*j} \kappa(z, w)$$

is formally positive semi-definite, where $\sigma^{*j}$ is given by

$$\sigma^{*j} \kappa(z, w) = \sum_{m,n \geq 0} \frac{\partial^m + j \partial^n \kappa(0,0)}{(m+j)! (n+j)!} z^m w^n,$$ $j \geq 0$.

The equivalence of (i) and (ii) is now immediate from Proposition 3.2.
To see the remaining implications, let $\mathcal{L}_z$ be the backward shift operator (see Remark 3.1). We claim that
\[
\mathcal{M}^n_z(e_n(\cdot)x) = e_{n+1}(\cdot)x \quad \text{for every } n \geq 0 \text{ and } x \in \mathcal{E}. \tag{3.3}
\]
Note that for any $f \in \mathcal{H}_\kappa$, by (3.2) (applied twice),
\[
\langle f, \mathcal{L}_z^* e_n(\cdot)x \rangle_{\mathcal{H}_\kappa} = \langle \mathcal{L}_z(f), e_n(\cdot)x \rangle_{\mathcal{H}_\kappa} = \left( \frac{(\partial^n \mathcal{L}_z(f))(0)}{n!}, x \right)_\mathcal{E} = \langle f, e_{n+1}(\cdot)x \rangle_{\mathcal{H}_\kappa}.
\]
Since $f$ is arbitrary, an application of (3.1) yields (3.3).

To see the implication (i) $\Rightarrow$ (iii), note that by (3.3) and (3.2), for all integers $m, n \geq 0$ and $x, y \in \mathcal{E}$,
\[
\langle \mathcal{M}_z^m x, \mathcal{M}_z^m y \rangle_{\mathcal{H}_\kappa} = \langle e_n(\cdot)x, e_m(\cdot)y \rangle_{\mathcal{H}_\kappa} = \left( \frac{(\partial^m e_n(\cdot)x)(0)}{m!}, y \right)_\mathcal{E} = \left( \frac{(\partial^{m+n} \mathcal{L}_z^* \kappa)(0)}{m! n!}, x, y \right)_\mathcal{E}.
\]
This shows that
\[
P_{\mathcal{E}} \mathcal{M}_z^m \mathcal{M}_z^m |_{\mathcal{E}} = \frac{(\partial^m \mathcal{L}_z^* \kappa)(0, 0)}{m! n!}, \quad m, n \geq 0, \tag{3.4}
\]
where $P_{\mathcal{E}}$ denotes the orthogonal projection of $\mathcal{H}_\kappa$ onto the space $\mathcal{E}$ of constant functions in $\mathcal{H}_\kappa$. Assume now that $\mathcal{M}_z^*$ is subnormal. By Bram’s characterization of subnormality [16, 1.9 Theorem(a)(i)], there exists a semispectral measure $Q$ compactly supported in $\mathbb{C}$ such that
\[
\mathcal{M}_z^m \mathcal{M}_z^m = \int z^n \overline{z}^m dQ(z), \quad m, n \geq 0.
\]
It now follows from (3.4) that \( \left\{ \frac{(\partial^m \mathcal{L}_z^* \kappa)(0, 0)}{m! n!} \right\}_{m, n \geq 0} \) is a complex moment sequence with the representing measure $P_{\mathcal{E}} Q(\cdot)|_{\mathcal{E}}$. This gives the implication (1) $\Rightarrow$ (3) (here we do not need the assumption that $\mathcal{E}$ is one dimensional).

To see the implication (iii) $\Rightarrow$ (i), assume that the dimension of $\mathcal{E} = \ker \mathcal{M}_z^*$ is 1. By [32, Corollary 2.8], the Cauchy dual of $\mathcal{M}_z$ is cyclic with cyclic vector in $\mathcal{E}$, and hence one may apply [34, Theorem 35] to complete the proof of (iii) $\Rightarrow$ (i).

We already noted that every analytic norm increasing operator can be modeled as the operator of multiplication by $z$ on a de Branges-Rovnyak space (see [22, Theorem 4.6] for a generalization). In view of the Shimorin’s model theorem (see [32]), the foregoing fact can be obtained using the reproducing kernel space techniques (see [1]).

**Lemma 3.4.** Let $(\mathcal{H}(\kappa), \mathcal{E})$ be a functional Hilbert space and let $\mathcal{M}_z$ be the operator of multiplication by $z$ on $\mathcal{H}_\kappa$. If $\mathcal{M}_z$ is norm increasing, then there exist a complex Hilbert space $\mathcal{D}$ and $B \in S(\mathcal{D}, \mathcal{E})$ such that $\mathcal{H}_\kappa$ coincides with $\mathcal{H}(B)$ with equality of norms.
Proof. Assume that \( \mathcal{M}_z^* \mathcal{M}_z \geq I \) and let \( P_{\text{ran} \mathcal{M}_z} \) denote the orthogonal projection of \( \mathcal{M}_z \) onto the ran \( \mathcal{M}_z \). Note that
\[
\mathcal{M}_z \mathcal{M}_z^* - P_{\text{ran} \mathcal{M}_z} = \mathcal{M}_z \mathcal{M}_z^* - \mathcal{M}_z \mathcal{M}_z^* \mathcal{M}_z,
\]
which is positive. It follows that for any \( f \in \mathcal{H}_\kappa \),
\[
\| \mathcal{M}_z^* f \| ^2 \geq \| P_{\text{ran} \mathcal{M}_z} f \| ^2 = \| f - f(0) \| ^2 = \| f \| ^2 - \| f(0) \| ^2,
\]
where we used the fact that \( f - f(0) \) and \( f(0) \) are orthogonal in \( \mathcal{H}_\kappa \). Letting \( f = \sum_{j=1}^n \kappa(\omega_j) c_j \) for \( c_1, \ldots, c_n \in \mathcal{E} \) and \( \omega_1, \ldots, \omega_n \in \mathbb{D} \), note further that
\[
\sum_{i,j=1}^n \omega_i \omega_j \langle \kappa(\omega_j, \omega_i) c_i, c_j \rangle_{\mathcal{E}} \geq \sum_{i,j=1}^n \langle \kappa(\omega_j, \omega_i) c_i, c_j \rangle_{\mathcal{E}} - \sum_{i,j=1}^n \langle c_i, c_j \rangle_{\mathcal{E}} = \sum_{i,j=1}^n \langle \kappa(\omega_j, \omega_i) - I_{\mathcal{E}} \rangle c_i, c_j \rangle_{\mathcal{E}}.
\]
Thus \( \eta(z, w) := z \omega \kappa(z, w) - (\kappa(z, w) - I_{\mathcal{E}}) \), \( z, w \in \mathbb{D} \), is a positive semi-definite kernel. The existence of \( B : \mathbb{D} \rightarrow B(\mathcal{D}, \mathcal{E}) \) such that \( \eta(z, w) = B(z)B(w)^* \), \( z, w \in \mathbb{D} \), now follows from the factorization theorem for positive semi-definite kernels (see [3, Theorem 2.62]). Moreover, since \( \eta(z, z) \leq I_{\mathcal{E}} \) for every \( z \in \mathbb{D} \), \( B \in S(\mathcal{D}, \mathcal{E}) \). Finally, note that \( \kappa(z, w) = \frac{I_{\mathcal{E}} - B(z)B(w)^*}{1 + z \omega} \) for every \( z, w \in \mathbb{D} \).

In view of Lemma [4, 3] the Cauchy dual subnormality problem for analytic norm increasing operators reduces to the same problem for the operator of multiplication by \( z \) on de Branges-Rovnyak spaces. The main result of this section characterizes subnormality of the Cauchy dual of the multiplication operator \( \mathcal{M}_z \) on de Branges-Rovnyak spaces.

Theorem 3.5. Let \( B(z) = \sum_{j=1}^\infty B_j z^j \) belong to the Schur class \( S(\mathcal{D}, \mathcal{E}) \). Assume that the operator \( \mathcal{M}_z \) of multiplication by \( z \) on \( \mathcal{H}(B) \) is bounded and the orthogonal complement of \( \{ zf : f \in \mathcal{H}(B) \} \) is spanned by the space of \( \mathcal{E} \)-valued constant functions. Then the following statements are equivalent:

(i) The Cauchy dual \( \mathcal{M}_z \) of \( \mathcal{M}_z \) is subnormal.

(ii) The matrix \( \sum_{j=0}^k (-1)^j \binom{k}{j} (B_{m+1+j} B_n^* B_{n+1+j}^*)_{m,n \geq 0} \) is formally positive semi-definite for every \( k \geq 1 \).

(iii) There exists a \( B(\ell^2(\mathcal{E})) \)-valued semi-spectral measure \( F \) supported in \([0, 1]\) such that
\[
(B_{m+1+j} B_n^*)_{m,n \geq 0} = \int_0^1 t^j F(dt), \quad j \geq 0.
\]

Proof. Let \( \kappa_B \) denote the reproducing kernel for \( \mathcal{H}(B) \). Note that
\[
\kappa_B(z, w) = \sum_{n=0}^\infty z^n \omega^n I_{\mathcal{E}} - B(z)B(w)^* \sum_{n=0}^\infty z^n \omega^n
\]
\[
= \sum_{n=0}^\infty z^n \omega^n I_{\mathcal{E}} - \left( \sum_{j,k=1}^\infty B_j B_k^* z^j \omega^k \right) \sum_{n=0}^\infty z^n \omega^n.
\]
Comparing the coefficients of $z^m \overline{w}^n$, $m, n \geq 0$, on the both sides, we obtain
\[
\frac{\partial^m \mathcal{F}^j \kappa_B(0, 0)}{m! n!} = \delta_{m,n} I_E - \sum_{j=1}^{m} \sum_{k=1}^{n} B_j B_k^* \delta_{m-j, n-k} \tag{3.5}
\]
where $\delta_{m,n}$ denotes the Kronecker delta and the convention that sum over the empty set is 0 is used. Set
\[
f_k(m, n) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial^{m+j} \mathcal{F}^{j+i} \kappa_B(0, 0)}{(m+j)!(n+j)!}, \quad k \geq 1, \ m, n \geq 0.
\]
A routine verification using \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \), \( n \geq k \), shows that
\[
f_{k+1}(m, n) = f_k(m, n) - f_k(m+1, n+1), \quad k \geq 1, \ m, n \geq 0. \tag{3.6}
\]
We verify by induction on $k \geq 1$ that
\[
f_k(m, n) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} B_{m+1+j} B_{n+1+j}^*, \quad m, n \geq 0. \tag{3.7}
\]
By (3.3), for integers $m, n \geq 0$,
\[
f_1(m, n) = \frac{\partial^m \mathcal{F} \kappa_B(0, 0)}{m! n!} - \frac{\partial^{m+1} \mathcal{F}^{1+i} \kappa_B(0, 0)}{(m+1)!(n+1)!} = B_{m+1} B_{n+1}^*.
\]
Thus (3.7) holds for $k = 1$. If (3.7) holds for $k \geq 1$, then by (3.6),
\[
f_{k+1}(m, n) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} B_{m+1+j} B_{n+1+j}^* - \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} B_{m+2+j} B_{n+2+j}^* = \sum_{j=0}^{k} (-1)^j \binom{k}{j} B_{m+1+j} B_{n+1+j}^*.
\]
This completes the verification of (3.7).

To complete the proof, recall that $\mathcal{M}$ is norm increasing (see (2.1)) or equivalently $\mathcal{M}_z$ is a contraction. One may now apply Theorem 3.3 together with (3.7) to obtain the equivalence of (i) and (ii). In view of the polarization technique (see [20, Proof of Theorem 4.2]), the equivalence of (ii) and (iii) follows from the solution of the Hausdorff moment problem (cf [10, Proposition 6.11, Chapter 4]) provided we show that $\{B_{m+1+j} B_{n+1+j}^*\}_{m,n \geq 0}$ defines a bounded linear operator on $l^2(\mathcal{E})$ for every integer $j \geq 0$. Since $B$ belongs to $S(\mathcal{D}, \mathcal{E})$, the function $B(\cdot)x$ belongs to $H^2_B(\mathcal{D})$. Thus the linear map $C : \mathcal{D} \rightarrow l^2(\mathcal{E})$ given by $C(x) = (B_k x)_{k \geq 1}$ is well-defined. The boundedness of $C$ now follows either from the assumption that $B \in S(\mathcal{D}, \mathcal{E})$ or from the closed graph theorem. Since $\{B_{m} B_{n}^*\}_{m,n \geq 1} = C C^*$, the desired equivalence is immediate. \qed
Corollary 3.6. For a positive integer $k$, let $B(z) = \sum_{j=1}^{\infty} B_j z^j$ belong to the Schur class $S(\mathbb{C}^k, \mathbb{C})$. Assume that the operator $\mathcal{M}_z$ of multiplication by $z$ on $\mathcal{H}(B)$ is bounded. Then the following statements are equivalent:

(i) The Cauchy dual $\mathcal{M}'_z$ of $\mathcal{M}_z$ is subnormal.

(ii) The matrix $\sum_{j=0}^{\infty} (\frac{k}{j}) \{B_m B_{m+1+j}^*\}_{m,n \geq 0}$ is formally positive semi-definite for every $k \geq 1$.

(iii) There exists a $\mathcal{B}(\ell^2(\mathbb{N}))$-valued semi-spectral measure $F$ supported in $[0,1]$ such that

\[
\{B_m B_{m+1+j}^*\}_{m,n \geq 0} = \int_{0}^{1} t^j F(dt), \quad j \geq 0.
\]

Proof. In view of Theorem 3.5, it suffices to check that the orthogonal complement of $\{zf : f \in \mathcal{H}(B)\}$ is spanned by the space of constant functions. This is immediate from [3, Theorem 1.3].

4. The proof of main theorem and its consequences

We now complete the proof of the main result stated in Section 2.

Proof of Theorem 4.2. Write $B(z) = \sum_{j=1}^{\infty} B_j z^j$, $|z| < 1$, and note that by [24, Proposition 2.1], for any polynomial $p$ of degree less than $k$,

\[
p(z) = \prod_{i=1}^{k} (z - \alpha_i) = \sum_{i=1}^{k} p(\alpha_i) \frac{1}{z - \alpha_i}.
\]

Applying this to $\frac{p(z)}{z}$ for every $j = 1, \ldots, k$, we obtain

\[
b_j(z) = \frac{p_j(z)}{\prod_{i=1}^{k} (z - \alpha_i)} = \sum_{i=1}^{k} p_j(\alpha_i) \frac{z}{\alpha_i a_i} z - \alpha_i = - \sum_{i=1}^{k} p_j(\alpha_i) \sum_{l=0}^{\infty} \frac{z^{l+1}}{\alpha_i^{l+1}},
\]

(4.1)

For $j = 1, \ldots, k$, let $b_j(z) = \sum_{m=1}^{\infty} b_{j,m} z^m$, $z \in \mathbb{D}$ and note that $B_m = (b_1, b_2, \ldots, b_k, m \geq 1$. By (4.1), for any $s \geq 0$ and $m \geq 0$,

\[
b_{j,m+1+s} = - \sum_{i=1}^{k} p_j(\alpha_i) \frac{1}{\alpha_i^{m+s+2}},
\]

and hence

\[
B_{m+1+s} B_{n+1+s}^* = \sum_{j=1}^{k} b_{j,m+1+s} b_{j,n+1+s}
\]

\[
= \sum_{j=1}^{k} \left( \sum_{r=1}^{k} \frac{p_j(\alpha_r)}{\alpha_r^{m+s+2}} \right) \left( \sum_{t=1}^{k} \frac{p_j(\alpha_t)}{\alpha_t^{n+s+2}} \right)
\]

\[
= \sum_{r,t=1}^{k} \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} \frac{p_j(\alpha_r) p_j(\alpha_t)}{\alpha_r^{m+2+s+2} \alpha_t^{n+2+s}}.
\]

(4.2)

Note that

\[
\sum_{s=0}^{l} (-1)^s \binom{l}{s} \frac{1}{(\alpha_r \alpha_t)^s} = (1 - \frac{1}{\alpha_r \alpha_t})^l, \quad l \geq 1,
\]
and hence by (4.2), for \( m, n \geq 0 \), we get
\[
\sum_{s=0}^{l} (-1)^s \binom{l}{s} B_{m+1+s} B^*_{n+1+s} \]
\[
= \sum_{r,t=1}^{k} \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} p_j(\alpha_r) p_j(\alpha_t) \left( 1 - \frac{1}{\alpha_r \alpha_t} \right)^l \left( \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m \cdot n \geq 0 \right).
\]
This together with (i)⇔(ii) of Theorem 3.5 completes the proof. □

In the remaining part of this section, we present some applications of Theorem 2.1.

**Proof of Corollary 2.2.** By (2.2),
\[
\sum_{r,t=1}^{k} \left( \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} p_j(\alpha_r) p_j(\alpha_t) \right) (1 - \frac{1}{\alpha_r \alpha_t})^l \left( \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m \cdot n \geq 0 \right).
\]
Since \(|\alpha_r| > 1\), in view of Theorem 2.1 it suffices to check that \( \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m \cdot n \geq 0 \) is formally positive semi-definite for every \( r = 1, \ldots, k \). Since this matrix is equal to \( V_r V_r^* \) with \( V_r \) denoting the column vector \( \left( \frac{1}{\alpha_r} \right)^m \cdot n \geq 0 \), the desired conclusion is immediate. □

**Corollary 4.1.** Assume that the hypotheses of Theorem 2.1 hold. If \( \mathcal{M}_z \) is subnormal, then
\[
\sum_{r,t=1}^{k} \left( \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} p_j(\alpha_r) p_j(\alpha_t) \right) (1 - \frac{1}{\alpha_r \alpha_t})^l \left( \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m \cdot n \geq 0 \right)
\]
is a positive measure supported in \([0,1]\).

**Proof.** Assume that \( \mathcal{M}_z \) is subnormal and let
\[
\gamma_m = \sum_{r,t=1}^{k} \left( \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} p_j(\alpha_r) p_j(\alpha_t) \right) \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m, \quad m \geq 0.
\]
By Theorem 2.1 and (1.2), the sequence \( \{\gamma_m\}_{m \geq 0} \) is completely monotone. Hence, by the solution of the Hausdorff moment problem (see [10]), there exists a finite positive Borel measure \( \nu \) on \([0,1]\) such that
\[
\gamma_m = \int_0^1 t^m d\nu(t), \quad m \geq 0.
\]
On the other hand, if \( K = [0,1] \cup \left\{ \frac{1}{\alpha_r \alpha_t} : 1 \leq r, t \leq k \right\} \) and
\[
\mu := \sum_{r,t=1}^{k} \left( \frac{1}{\alpha_r \alpha_t} \sum_{j=1}^{k} p_j(\alpha_r) p_j(\alpha_t) \right) \left( \frac{1}{\alpha_r^{m+2} \alpha_t^{n+2}} \right)^m, \quad m \geq 0.
\]
then we get
\[ \gamma_m = \int_K z^m d\mu, \quad m \geq 0. \]
It follows that
\[ \int_0^1 t^m d\nu(t) = \int_K z^m d\mu, \quad m \geq 0. \]
Since \( K \) is a compact set with connected complement in \( \mathbb{C} \), by Mergelyan’s Theorem (see [28, 20.5 Theorem]), any continuous function on \( K \) can be approximated uniformly by polynomials in \( z \). By the Riesz representation theorem (see [28, 6.19 Theorem]), \( \mu \) is necessarily supported in \([0, 1]\) and it coincides with \( \nu \). □

Remark 4.2. For \( 1 \leq r, t \leq k \), let \( A_{r,t} = \{ (u, v) : \alpha_u \alpha_v = \alpha_r \alpha_t \} \) and
\[ c_{r,t} := \sum_{(u,v) \in A_{r,t}} \left( \frac{1}{\alpha_u^2 \alpha_v^2} \sum_{j=1}^k p_j(\alpha_u) p_j(\alpha_v) \right). \]
If the conclusion of Corollary 4.1 holds, then the following possibilities occur:
(i) If \( \frac{1}{\alpha_r \alpha_t} \notin [0, 1] \), then \( c_{r,t} = 0 \).
(ii) If \( \frac{1}{\alpha_r \alpha_t} \in [0, 1] \), then \( c_{r,t} \geq 0 \).

Corollary 4.3. Assume the hypotheses of Theorem 2.1. If \( \alpha_r \alpha_t \notin [1, \infty) \) for every \( 1 \leq r \neq t \leq k \), then \( \mathcal{M}'_z \) is subnormal if and only if
\[ \sum_{j=1}^k p_j(\alpha_r) p_j(\alpha_t) = 0, \quad 1 \leq r \neq t \leq k. \]
Proof. Combining Corollary 2.2 with the preceding remark yields the desired conclusion. □

5. CLASSICAL DE BRANGES-ROVNYAK SPACES

Let \( b \) be a non-extreme point of the closed unit ball of \( H^\infty(\mathbb{D}) \). It is well-known that there exists a unique outer function \( a \in H^\infty(\mathbb{D}) \) (that is, \( \sqrt{\{z^n a : n \geq 0\}} = H^2(\mathbb{D}) \)) such that \( |a|^2 + |b|^2 = 1 \) almost everywhere on unit circle and \( a(0) > 0 \) (see [19] Chapter 23, Section 1). We refer to \( a \) as the mate of \( b \). The following lemma provides a formula for the reproducing kernels of the so-called the Cauchy dual of classical de Branges-Rovnyak spaces.

Lemma 5.1. Let \( b \) be a nonextreme point of the closed unit ball of \( H^\infty \) such that \( b(0) = 0 \) and let \( a \) be a mate of \( b \). Let \( \phi = \frac{b}{a} \). Then the Cauchy dual \( \mathcal{M}'_z \) of \( \mathcal{M}_z \) is unitarily equivalent to the operator of multiplication by \( z \) on the reproducing kernel Hilbert space \( \mathcal{H}_{\kappa'_b} \), where \( \kappa'_b \) is given by
\[ \kappa'_b(z, w) = \frac{1 + \phi(z)\bar{\phi}(w)}{1 - zw}, \quad z, w \in \mathbb{D}. \]
Proof. Let \( \phi(z) = \sum_{j=0}^{\infty} c_j z^j \) for \( z \in \mathbb{D} \). By [15] Lemma 3.2,

\[
\langle z^m, z^n \rangle_{\mathcal{H}(b)} = \begin{cases} 
\delta_{m,n} + \sum_{k=0}^{n} \frac{\bar{c}_{m-k} c_k}{k!} & \text{if } m \geq n, \\
\sum_{j=0}^{m} c_j c_{n-j} & \text{if } m < n,
\end{cases}
\]  

(5.1)

Since \( \mathcal{M}_z \) is cyclic with cyclic vector 1, by [32] Corollary 2.8, \( \mathcal{M}_z' \) is analytic. One may now apply [32] Corollary 2.14 to \( T = \mathcal{M}_z' \) to conclude that \( \kappa'_b \) is given by

\[
\kappa'_b(z,w) = \sum_{m,n \geq 0} \langle z^m, z^n \rangle_{\mathcal{H}(b)} z^m \overline{w}^n, \quad z,w \in \mathbb{D}.
\]

It is now easy to see using (5.1) that \( \kappa'_b \) has the desired formula (see the proof of Theorem 3.5 for a similar argument).

The first main result of this section provides affirmative solution of the Cauchy dual subnormality problem for classical de Branges-Rovnyak spaces (cf. [8] Corollary 3.6 and [5] Theorem 3.3).

**Theorem 5.2.** Let \( b \) be a nonextreme point of the closed unit ball of \( H^\infty(\mathbb{D}) \) such that \( b(0) = 0 \). If \( \mathcal{M}_z \) on \( \mathcal{H}(b) \) is concave, then the Cauchy dual \( \mathcal{M}_z' \) of \( \mathcal{M}_z \) is a subnormal contraction. Moreover, the following hold:

(i) The sequence \( \{ \partial^m \overline{\partial}^n \kappa_b(0,0) \}_{m,n \geq 0} \) is a complex moment sequence with the representing measure \( \mu \) given by

\[
\left(1 - \nu \left(2 \Re \left( \frac{1}{1 - e^{-i\theta} \beta} \right) - 1 \right) \right) \frac{d\theta}{2\pi} + \nu \delta_0,
\]

where \( \nu = \frac{|\gamma|^2}{1 - |\beta|^2} \) for some scalars \( \gamma, \beta \in \mathbb{C} \) with \( |\beta| < 1 \).

(ii) \( \mathcal{M}_z' \) is unitarily equivalent to the operator of multiplication by \( z \) on the reproducing kernel Hilbert space \( \mathcal{H}_{\kappa'_b} \), where \( \kappa'_b \) is given by

\[
k'_b(z,w) = \frac{1 + \frac{|\gamma|^2 z^m}{(\rho - \sigma z)(\rho - \sigma \overline{w})}}{1 - z \overline{w}}, \quad z,w \in \mathbb{D}
\]

for some scalars \( \sigma, \rho \in \mathbb{C} \).

**Proof.** Assume that \( \mathcal{M}_z \) on \( \mathcal{H}(b) \) is concave. By [21] Theorem 1, there exist scalars \( \gamma, \beta \in \mathbb{C} \) such that \( |\beta| < 1 \) and

\[
b(z) = \frac{\gamma z}{1 - \beta z}, \quad z \in \mathbb{D},
\]

where we used \( b(0) = 0 \). It now follows from Corollary 2.2 that \( \mathcal{M}_z' \) is a subnormal contraction.

Note that \( b(z) = \sum_{n \geq 1} b_n z^n \), where \( b_n \) is given by

\[
b_n = \gamma \beta^{n-1}, \quad n \geq 1.
\]

(5.2)

To see (i), note that by Theorem 3.3 the representing measure of \( \mathcal{M}_z' \) coincides with the representing measure of \( \{ \partial^m \overline{\partial}^n \kappa_b(0,0) \}_{m,n \geq 0} \). It is easy to see using (3.3) and (5.2) that

\[
\frac{\partial^m \overline{\partial}^n \kappa_b(0,0)}{m!n!} = \begin{cases} 
\delta_{m,n} - |\gamma|^2 \beta^{m-n-1} \frac{1 - |\beta|^2}{1 - |\beta|^2} & \text{if } m \geq n, \\
\delta_{m,n} - |\gamma|^2 \beta^{-m-n-1} \frac{1 - |\beta|^2}{1 - |\beta|^2} & \text{if } m < n.
\end{cases}
\]
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Since for any integer \( \ell \geq 0 \),

\[
2 \int_T e^{i(m-n)\theta} \Re \left( \frac{e^{-i\theta}}{1 - e^{-i\beta}} \right) \frac{d\theta}{2\pi} = \begin{cases} 
\beta^{m-n-\ell} & \text{if } m > n, \\
2\delta_{0,\ell} & \text{if } m = n, \\
\beta^{n-m-\ell} & \text{if } m < n.
\end{cases}
\] (5.3)

it can be easily seen that \( \mu \), as given in part (i), is the representing measure for \( \mathcal{M}_z' \).

To see part (ii), note that by [21, Lemma 6], the mate of \( b \) is given by

\[
a(z) = \frac{\rho - \sigma z}{1 - \beta z}, \quad z \in \mathbb{D},
\]

for some \( \rho, \sigma \in \mathbb{C} \). The desired formula now follows from Lemma 5.1. □

Remark 5.3. The conclusion of Theorem 5.2 can be extended to Schur functions which do not necessarily vanish at the origin. Indeed, by [22, Lemma 4.7], \( \mathcal{M}_z \) on \( \mathcal{H}(b) \) is unitarily equivalent to the operator \( \mathcal{M}_z \) of multiplication by \( z \) on \( \mathcal{H}(\tilde{b}) \), where \( \tilde{b}(0) = 0 \). Thus the Cauchy dual \( \mathcal{M}_z' \) of any concave multiplication operator \( \mathcal{M}_z \) on \( \mathcal{H}(b) \) is a subnormal contraction. Moreover, by [21, Theorem 1],

\[
b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}
\]

for some \( c, \gamma, \beta \in \mathbb{C} \) with \( |\beta| < 1 \). One may now argue as above using (5.3) to see that the sequence \( \left\{ \frac{\partial^m\overline{\partial}^n(\alpha,0)}{m!n!} \right\}_{m,n \geq 0} \) is a complex moment sequence with the representing measure \( \mu \) given by

\[
\left( 1 - (\nu + |c|^2) \left( 2 \Re \left( \frac{1}{1 - e^{-i\theta}} \right) - 1 \right) - 2\gamma \Re \left( \frac{e^{-i\theta}}{1 - e^{-i\beta}} \right) \right) \frac{d\theta}{2\pi} + \nu \delta_\beta,
\]

where \( \nu = \frac{|c| \beta + \gamma}{1 - |\beta|^2} \) for some scalars \( c, \gamma, \beta \in \mathbb{C} \) with \( |\beta| < 1 \).

We conclude this section with an application to Dirichlet-type spaces associated with measures supported at a point.

Corollary 5.4. Let \( \lambda \) be a point in the unit circle \( \mathbb{T} \) and let \( \tau \) be a positive number. Then the Cauchy dual \( \mathcal{M}_z' \) of \( \mathcal{M}_z \) on the Dirichlet-type space \( \mathcal{D}(\tau\delta_\lambda) \) is a subnormal contraction.

Proof. By [15, Theorem 3.1], \( \mathcal{D}(\tau\delta_\lambda) \) coincides with the de Branges-Rovnyak space \( \mathcal{H}(b) \), where \( b \) is given by

\[
b(z) = \frac{\sqrt{\eta} \alpha \overline{\lambda} z}{1 - \eta \overline{\lambda} z}, \quad z \in \mathbb{D},
\]

where \( \alpha \in \mathbb{C} \) is such that \( |\alpha|^2 = \tau \) and \( \eta \in (0,1) \) is such that \( \eta + 1/\eta = 2 + \tau \). Since \( \mathcal{M}_z \) acting on \( \mathcal{D}(\tau\delta_\lambda) \) is a 2-isometry (see [20, Theorem 3.7]), by Theorem 5.2 the Cauchy dual of \( \mathcal{M}_z \) is a subnormal contraction. □
6. Dirichlet-type spaces associated with finitely supported measures

In this section, we discuss cyclic analytic 2-isometries $T$ with finite rank defect operators $\Delta_T := T^*T - I$ and their relationship with Dirichlet-type spaces associated with finitely supported measures. In particular, we classify cyclic analytic 2-isometries with finite rank defect operators $\Delta_T$ and also $\Delta_T$-regular cyclic analytic 2-isometries. Recall that for any 2-isometry, $\Delta_T$ is a positive operator (see [26, Lemma 1]). Following [23], we say that a norm increasing operator $T$ is $\Delta_T$-regular if $\Delta_T T = \Delta_T^{1/2} T \Delta_T^{1/2}$.

**Theorem 6.1.** Let $T \in B(\mathcal{H})$ be a cyclic analytic 2-isometry and let $\Delta_T := T^*T - I$. Then the following statements are true:

(i) Let $k$ be a positive integer. The rank of $\Delta_T$ is $k$ if and only if there exist distinct points $\zeta_1, \ldots, \zeta_k$ on the unit circle and positive numbers $c_1, \ldots, c_k$ such that $T$ is unitarily equivalent to the multiplication operator $M_z$ on $\mathcal{D}(\mu)$, where $\mu = \sum_{j=1}^k c_j \delta_{\zeta_j}$.

(ii) The operator $T$ is $\Delta_T$-regular if and only if $T$ is unitarily equivalent to the multiplication operator $M_z$ on $\mathcal{D}(c \delta_\zeta)$ for some scalar $c \geq 0$ and $\zeta \in \mathbb{T}$.

**Proof.** In view of a model theorem of Richter for cyclic analytic 2-isometries (see [26, Theorem 5.1]), we may assume that $T = M_z$ acting on a Dirichlet-type space $\mathcal{D}(\mu)$ for some finite positive Borel measure $\mu$ on the unit circle $\mathbb{T}$. One may argue as in the proof of [23, Theorem 1.26] to see that with respect to the decomposition $\mathcal{D}(\mu) = \ker \Delta_{M_z} \oplus \text{ran} \Delta_{M_z}$, $M_z$ and $\Delta_{M_z}$ decompose as follows:

\[
M_z = \begin{pmatrix} S & E \\ 0 & W \end{pmatrix}, \quad \Delta_{M_z} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \tag{6.1}
\]

where $S$ is an isometry, $S^*E = 0$ and $D$ is an injective positive operator such that $W^*DW = D$.

(i) Assume that $\mu = \sum_{j=1}^k c_j \delta_{\zeta_j}$. It is well known that the point evaluation at $\zeta_j$, $j = 1, \ldots, k$, is bounded on $\mathcal{D}(\mu)$ (see [27, Corollary 2.3]). Thus there exist linearly independent vectors $\kappa_{\zeta_1}, \ldots, \kappa_{\zeta_k} \in \mathcal{D}(\mu)$ such that $\langle f, \kappa_{\zeta_j} \rangle = f(\zeta_j)$, $j = 1, \ldots, k$. By [27, Corollary 2.3], for any $f \in \mathcal{D}(\mu)$,

\[
\langle \Delta_{M_z} f, f \rangle = \|zf\|^2 - \|f\|^2 = \int_{\mathbb{T}} |f(\zeta)|^2 d\mu(\zeta) = \sum_{j=1}^k c_j |\langle f, \kappa_{\zeta_j} \rangle|^2.
\]

It follows that

the range of $\Delta_{M_z}$ is spanned by $\{ \kappa_{\zeta_j} : j = 1, \ldots, k \}$. \tag{6.2}

To see the converse, assume that the range of $\Delta_{M_z}$ is $k$-dimensional. Note that $D$ is an invertible operator on the finite dimensional Hilbert space ran $\Delta_{M_z}$. Hence, $D^{1/2}WD^{-1/2}$ is a unitary operator on ran $\Delta_{M_z}$. Consider the orthonormal basis of ran $\Delta_{M_z}$ consisting of eigenvectors $v_j$ of $D^{1/2}WD^{-1/2}$ corresponding to the eigenvalue $\zeta_j \in \mathbb{T}$, $j = 1, \ldots, k$. By [27, Corollary 2.3] and the polarization identity,

\[
\langle \Delta_{M_z} M_z^n 1, 1 \rangle = \langle z^{n+1}, z \rangle - \langle z^n, 1 \rangle = \int_{\mathbb{T}} \zeta^n d\mu(\zeta), \quad n \geq 0. \tag{6.3}
\]
Let $P$ denote the orthogonal projection of $\mathcal{D}(\mu)$ onto $\text{ran}\Delta_{\mathcal{H}_z}$. Since the set \( \{ D^{-1/2}v_j : j = 1, \ldots, k \} \) forms a basis for $\text{ran}\Delta_{\mathcal{H}_z}$, there exists scalars $\lambda_j \in \mathbb{C}$ such that $P1 = \sum_{j=1}^k \lambda_j D_{-1/2}v_j$. It now follows from (6.3) and (6.1) that for any integer $n \geq 0$,

\[
\int_T \zeta^n d\mu(\zeta) = \langle DW^n P1, P1 \rangle = \sum_{i,j=1}^k \lambda_i \lambda_j \zeta^n \langle v_i, v_j \rangle = \sum_{i=1}^k |\lambda_i|^2 \zeta^n.
\]

By the uniqueness of the trigonometric moment problem (which in turn follows from the density of trigonometric polynomials and the Riesz representation theorem [28]), $\mu$ is equal to $\sum_{j=1}^k |\lambda_j|^2 \delta_{\zeta_j}$. Since the range of $\Delta_{\mathcal{H}_z}$ is $k$-dimensional, by the first half, $\zeta_1, \ldots, \zeta_k$ are all distinct. This completes the proof of (i).

(ii) If $\mathcal{M}_z$ is an isometry, then the equivalence holds with $c = 0$. Suppose that $\mu = c \delta_{\zeta}$ for some scalar $c > 0$ and $\zeta \in \mathbb{T}$. By (i), $\Delta_{\mathcal{H}_z}$ is one-dimensional, and hence by [3 Corollary 3.6], $\mathcal{M}_z$ is $\Delta_{\mathcal{H}_z}$-regular. This yields the sufficiency part. To see the converse, assume now that $\Delta_{\mathcal{H}_z}$ is nonzero and $\mathcal{M}_z$ is $\Delta_{\mathcal{H}_z}$-regular. Hence, by [23 Proposition 5.1], $E$ as appearing in (6.1) is a one-to-one map from $\text{ran}\Delta_{\mathcal{H}_z}$ into $\ker\Delta_{\mathcal{H}_z}$ such that $E^*E = 0$. However, by [21 Theorem 3.2], $\ker\mathcal{S}^*$ is of dimension 1. It follows that the range of $E$ is at most of dimension 1. Since $E$ is injective, $\text{ran}\Delta_{\mathcal{H}_z}$ is one dimensional. The conclusion now follows from (i).

Remark 6.2. We note the following:

(1) It is worth mentioning that part (ii) above is applicable to the Brownian shift $B_{\sigma,e^{i\theta}}$ of covariance $\sigma > 0$ and angle $\theta$, as introduced and studied in [2] (the reader is referred to [2, Section 5] for the definition of the Brownian shift).

(2) Let $\mu$ be a finite Borel positive measure. If $\mathcal{D}(\mu)$ coincides with a de Branges-Rovnyak space $\mathcal{H}(B)$ of rank $k$, with equality of norms, then there exist $c_1, \ldots, c_k > 0$ and $\zeta_1, \ldots, \zeta_k \in \mathbb{T}$ such that $\mu = \sum_{j=1}^k c_j \delta_{\zeta_j}$ (cf. [30 Proposition 2] and [15 Theorem 3.1]).

In operator theoretic terms, the following fact recovers a special case of [3 Corollary 3.6] (without the cyclicity assumption).

Corollary 6.3. Let $T$ be a cyclic concave operator on $\mathcal{H}$. If the range of $T^*T - I$ is at most one-dimensional, then the Cauchy dual operator $T'$ of $T$ is a subnormal contraction.

The second result of this section describes the symbol of de Branges-Rovnyak model space of the Dirichlet-type spaces associated with finitely
supported measures. We capitalize below on the algorithm for computing the reproducing kernel for Dirichlet-type spaces associated with finitely supported measures, as presented in [17].

**Theorem 6.4.** For positive scalars \( c_1, \ldots, c_k \) and distinct points \( \zeta_1, \ldots, \zeta_k \) on the unit circle \( \mathbb{T} \), consider the positive Borel measure \( \mu = \sum_{j=1}^{k} c_j \delta_{\zeta_j} \), where \( \delta_{\zeta_j} \) denotes the Dirac delta measure supported at \( \zeta_j \). Let \( X(z) = (z, \ldots, z^k)^T \) and \( \{e_j\}_{j=1}^{k} \) denote the standard basis of \( \mathbb{C}^k \). Then there exist \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \) such that \( \alpha \) is an analytic norm increasing operator (see [26, Theorem 3.6]) and the reproducing kernel for \( D \) is given by

\[
\kappa(z, w) = \overline{(\alpha(z) - \alpha(w))}, \quad z, w \in \mathbb{T}.
\]

Moreover, \( \alpha_1, \ldots, \alpha_k \) are governed by

\[
\begin{align*}
\prod_{j=1}^{k} |z - \zeta_j|^2 + \sum_{j=1}^{k} c_j \prod_{l=1, l \neq j}^{k} |z - \zeta_l|^2 = \gamma \prod_{j=1}^{k} |z - \alpha_j|^2, & \quad z \in \mathbb{T} \quad (6.4)
\end{align*}
\]

for some \( \gamma > 0 \).

**Proof.** It has been observed in [17] that there exist \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \setminus \mathbb{D} \) and \( \gamma > 0 \) such that (6.4) holds. Let \( p(z) = \frac{z^n}{\sqrt[n]{\prod_{j=1}^{k} (z - \zeta_j)}} \), where \( \theta \in \mathbb{R} \) is chosen such that \( \frac{d}{d \theta} \prod_{j=1}^{k} (z - \zeta_j) > 0 \). Since the multiplication operator \( M_z \) on \( \mathcal{D}(\mu) \) is an analytic norm increasing operator (see [28, Theorem 3.6]) and the reproducing kernel for \( \mathcal{D}(\mu) \) is normalized, by Lemma 3.3 there exists a positive semi-definite kernel \( \eta : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \) such that

\[
\eta(z, 0) = 0, \quad \kappa(z, w) = \frac{1 - \eta(z, w)}{1 - \overline{z}w}, \quad z, w \in \mathbb{D}. \quad (6.5)
\]

By [17] Theorems 5.1 and 4.4, the reproducing kernel \( \kappa(z, w) \) for \( \mathcal{D}(\mu) \) is given by

\[
\kappa(z, w) = \frac{O(z)}{1 - \overline{z}w} + \sum_{j=1}^{k} f_j(z) \beta_j(w), \quad z, w \in \mathbb{D}, \quad (6.6)
\]

where \( O, f_j, \beta_j \) are given by

\[
O(z) = \frac{p(z)}{q(z)}, \quad f_j(z) = \frac{O(z)}{O'(\zeta_j)(z - \zeta_j)},
\]

\[
\begin{pmatrix}
\beta_1(w) \\
\vdots \\
\beta_k(w)
\end{pmatrix} = \left( (\langle f_i, f_j \rangle)_{1 \leq i, j \leq k} \right)^{-1} \begin{pmatrix}
f_1(w) \\
\vdots \\
f_k(w)
\end{pmatrix}.
\]

Letting \( (b_{ij})_{1 \leq i, j \leq k} \) is the inverse of \( (\langle f_i, f_j \rangle)_{1 \leq i, j \leq k} \), we obtain

\[
\beta_j(w) = \sum_{i=1}^{k} b_{ji} f_i(w), \quad j = 1, \ldots, k.
\]
This combined with (6.5) and (6.6)

\[
\frac{1 - \eta(z, w)}{1 - \bar{w}w} = \frac{1}{1 - \bar{w}w} \frac{p(z)p(w)}{q(z)q(w)} + \sum_{i,j=1}^{k} \frac{b_{ji}f_j(z)f_i(w)}{1 - \bar{w}w} + \sum_{i,j=1}^{k} \frac{b_{ji}}{O'(\zeta_j)O'(\zeta_i)} \frac{1}{(z - \zeta_j)(\bar{w} - \zeta_i)}
\]

This yields

\[
q(z) \eta(z, w) q(w) = q(z)q(w) - p(z)p(w) \left( 1 + (1 - \bar{w}w) \sum_{i,j=1}^{k} \frac{b_{ji}}{O'(\zeta_j)O'(\zeta_i)} \frac{1}{(z - \zeta_j)(\bar{w} - \zeta_i)} \right).
\]

Since the expression on the right hand side is a polynomial in \(z\) and \(\bar{w}\), there exists a matrix \(\tilde{A} = (a_{ij})_{0 \leq i,j \leq k}\) such that

\[
q(z) \eta(z, w) q(w) = \sum_{i,j=0}^{k} a_{ij}z^i\bar{w}^j, \quad z, w \in \mathbb{D}.
\]

As \(\eta(z, 0) = 0\) for \(z \in \mathbb{D}\),

\[
q(z) \eta(z, w) q(w) = \sum_{i,j=1}^{k} a_{ij}z^i\bar{w}^j = \langle AX(z), X(w) \rangle, \quad z, w \in \mathbb{D},
\]

where \(A\) is the \(k \times k\) matrix obtained from \(\tilde{A}\) by removing first row and first column. Further, since \(q(z) \eta(z, w) q(w)\) is a positive semi-definite kernel, the matrix \(A\) is positive semi-definite (see Proposition 3.2). By Cholesky’s decomposition (see [11, Pg 2]), there exists a \(k \times k\) upper triangular matrix \(P\) such that \(A = P^*P\). It follows that

\[
q(z) \eta(z, w) q(w) = \langle AX(z), X(w) \rangle = \langle PX(z), PX(w) \rangle = \sum_{j=1}^{k} \langle PX(z), e_j \rangle \langle PX(w), e_j \rangle = \sum_{j=1}^{k} p_j(z)p_j(w).
\]

This shows that

\[
\eta(z, w) = \sum_{j=1}^{k} \frac{p_j(z)p_j(w)}{q(z)q(w)}, \quad z, w \in \mathbb{D}.
\]

This completes the proof. \(\square\)
Remark 6.5. As noticed in the above proof,

\[
\sum_{j=1}^{k} p_j(z)p_j(w) = q(z)q(w) - p(z)p(w)
\]

\[
\left(1 + \sum_{i,j=1}^{k} \overline{b}_{ij} \overline{\phi}(\overline{\phi}'(z - \zeta_j)(w - \zeta_i))\right),
\]

where the matrix \((b_{ij})_{1\leq i,j\leq k}\) is the inverse of \((\langle f_i, f_j \rangle_{D(\mu)})_{1\leq i,j\leq k}\) and \(f_j(z) = \frac{O(z)}{O'(\zeta_j)(z - \zeta_j)}, \ j = 1, \ldots, k\). Thus, for \(1 \leq r \neq t \leq k\), we have

\[
\sum_{j=1}^{k} p_j(\alpha_r)p_j(\alpha_t) = 0 \text{ if and only if } \sum_{i,j=1}^{k} \overline{b}_{ij} \overline{\phi}(\overline{\phi}'(\zeta_j)(\zeta_i - \zeta_j)) = 1.
\]

Note further that by [17, Lemmas 4.2 and 4.3], \(\langle f_i, f_j \rangle\) is given by

\[
\langle f_i, f_j \rangle = \begin{cases} c_i \zeta_i f'_i(\zeta_i) & \text{if } i = j, \\ \frac{1}{O'(\zeta_j)O'(\zeta_j)(1 - \zeta_j \overline{\zeta_i})} & \text{otherwise}. \end{cases}
\]

7. Dirichlet-type spaces associated with measures supported at antipodal points

In this section, we apply the results obtained in the previous sections to solve affirmatively the Cauchy dual subnormality problem for Dirichlet-type spaces associated with measures supported at antipodal points. We begin with a fact which allows us to reduce the above problem to the case in which the antipodal points are \(1\) and \(-1\).

Proposition 7.1. Let \(\mu\) be a finite positive Borel measure on the unit circle \(\mathbb{T}\). For \(\zeta \in \mathbb{T}\), let \(\mu_\zeta\) be the finite positive Borel measure defined by \(\mu_\zeta(\Delta) = \mu(\zeta \Delta)\) for every Borel subset \(\Delta\) of \(\mathbb{T}\). Let \(M_{z,\zeta}\) denote the operator of multiplication by \(z\) on the Dirichlet-type space \(D(\mu_\zeta)\). Then \(M_{z,\zeta}\) is unitarily equivalent to \(\overline{\zeta}M_{z,1}\). In particular, we have the following:

(i) The Cauchy dual operator \(M'_{z,\zeta}\) of \(M_{z,\zeta}\) is subnormal if and only if so is \(M'_{z,1}\).

(ii) If \(\kappa\) is the reproducing kernel for \(D(\mu)\), then the reproducing kernel \(\kappa_{\zeta}\) for \(D(\mu_\zeta)\) is given by

\[
\kappa_{\zeta}(z, w) = \kappa(\zeta z, \zeta w), \quad z, w \in \mathbb{D}.
\]

Proof. Note that for any \(z \in \mathbb{D}\), by the \(\mathbb{T}\)-invariance of the Poisson kernel \(P(z, \lambda) = \frac{1 - |z|^2}{|z - \lambda|^2}, \ z \in \mathbb{D}, \ \lambda \in \mathbb{T}\), we get

\[
P_{\mu_\zeta}(z) = \int_{\mathbb{T}} P(z, \lambda) d\mu_\zeta(\lambda) = \int_{\mathbb{T}} P(\zeta z, \zeta \lambda) d\mu(\zeta \lambda) = P_{\mu}(\zeta z).
\]
It now follows from the rotation invariance of the area measure that for any $f \in \mathcal{D}(\mu_\zeta)$,

$$\|f\|^2_{\mathcal{D}(\mu_\zeta)} = \|f\|^2_{H^2(\mathbb{D})} + \int_D |f'(z)|^2 P_\mu(\zeta z) dA(z)$$

$$= \|f\|^2_{H^2(\mathbb{D})} + \int_D |f'(\zeta z)|^2 P_\mu(z) dA(z)$$

$$= \|f\|^2_{\mathcal{D}(\mu)},$$

where $f_\zeta(z) = f(\zeta z), z \in \mathbb{D}$. This yields the unitary $U_\zeta : f \mapsto f_\zeta$ from $\mathcal{D}(\mu_\zeta)$ onto $\mathcal{D}(\mu)$. It is easy to verify that $U_\zeta \mathcal{M}_z \mathcal{M}_{\zeta} = \mathcal{M}_z \mathcal{M}_{\zeta} U_\zeta$. Part (i) now follows from the definition of the Cauchy dual operator and the fact that a scalar multiple of a subnormal operator is again subnormal. Since $U_\zeta$ is a unitary, $\kappa(\zeta z, \zeta w), z, w \in \mathbb{D}$, is easily seen to be a reproducing kernel for $\mathcal{D}(\mu_\zeta)$, and hence by the uniqueness of the reproducing kernel, part (ii) follows. □

To complete the proof of Theorem 2.3 in view of Proposition 7.1 one may focus on the antipodal points 1 and $-1$. This is discussed in the following example.

**Example 7.2.** Consider the Dirichlet-type space $\mathcal{D}(\mu)$ associated with the measure $\mu = c_1 \delta_1 + c_2 \delta_{-1},$ where $c_1, c_2$ are positive numbers. We already noted that the multiplication operator $\mathcal{M}_z$ is a cyclic 2-isometry. Moreover, the range of $\mathcal{M}_z \mathcal{M}_{\zeta} - I$ is of dimension 2 (see Lemma 6.4(i) and (6.2)). By Theorem 6.4, there exist $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma > 0$ such that

$$|z - 1|^2 |z + 1|^2 + c_1|z + 1|^2 + c_2|z - 1|^2 = \gamma|z - \alpha_1|^2|z - \alpha_2|^2, \ z \in \mathbb{T}. \ (7.1)$$

It can be seen by a routine verification that the above values of $\gamma, \alpha_1, \alpha_2$ satisfy (7.1).\footnote{To see this, one may try to solve the above identity for real numbers $\alpha_1, \alpha_2$ and $\gamma$ assuming that $\alpha_1 > 1$ and $\alpha_2 < -1$, and evaluating (7.1) at $z = \pm 1, i$.}

$$\gamma = \frac{(\sqrt{4 + c_2^2} - c_+)^2}{4} = -\frac{1}{\alpha_1 \alpha_2},$$

$$\alpha_1 = \frac{(c_- + \sqrt{4 + c_2^2})(c_+ + \sqrt{4 + c_2^2})}{4},$$

$$\alpha_2 = \frac{(c_- - \sqrt{4 + c_2^2})(c_+ + \sqrt{4 + c_2^2})}{4}, \ (7.2)$$

where $c_+ = \sqrt{c_1} + \sqrt{c_2}$ and $c_- = \sqrt{c_1} - \sqrt{c_2}$. By Proposition 6.4 $D(\mu)$ coincides with the de Branges-Rovnyak space $\mathcal{H}(B)$ with $B = (b_1, b_2)$ and $b_j = \frac{p_j}{(z - \alpha_1)(z - \alpha_2)}$ for some degree 2 polynomials $p_1$ and $p_2$. We contend that

$$\sum_{j=1}^2 p_j(\alpha_1)\overline{p_j(\alpha_2)} = 0. \ (7.3)$$
In view of Remark 6.5, it suffices to check
\[\sum_{i,j=1}^{2} \frac{b_{ji}}{O'(\zeta_i)O'(\zeta_i)} \frac{1}{(\alpha_1 - \zeta_j)(\alpha_2 - \zeta_i)} = \frac{1}{\alpha_1 \alpha_2 - 1},\]  
(7.4)
where \(\zeta_1 = 1, \zeta_2 = -1\), and, after some routine calculations using Remark 6.5,
\[O(z) = \frac{1}{\sqrt{1 - (z - \alpha_1)(z - \alpha_2)}}, \quad O'(-1) = \frac{1}{\sqrt{\epsilon_1}}, \quad O'(1) = -\frac{1}{\sqrt{\epsilon_2}},
\]
\[f_1(z) = -\frac{\sqrt{\epsilon_1}}{\sqrt{1 - (z - \alpha_1)(z - \alpha_2)}}, \quad f_2(z) = \frac{\sqrt{\epsilon_2}}{\sqrt{1 - (z - \alpha_1)(z - \alpha_2)}},
\]
\[f_1'(1) = \frac{-3 + \alpha_1 + \alpha_2 + \alpha_1 \alpha_2}{2(1 + \alpha_1)(1 - \alpha_2)}, \quad f_2'(1) = \frac{3 + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2}{2(1 + \alpha_1)(1 - \alpha_2)},
\]
\[(b_{ji})_{1 \leq i,j \leq 2} = \frac{1}{-c_1 c_2 (f_2'(1) + \frac{1}{4})} \left( -c_2 f_2'(1) \right)
\]
To verify (7.4), note that
\[\sum_{i,j=1}^{2} \frac{b_{ji}}{O'(\zeta_i)O'(\zeta_i)} \frac{1}{(\alpha_1 - \zeta_j)(\alpha_2 - \zeta_i)} = \frac{1}{-c_1 c_2 (f_2'(1) + \frac{1}{4})} \left( -c_2 f_2'(1) \right)
\]
Thus the claim stands verified. This completes the verification of (7.3). Hence, by Corollary 2.2, the Cauchy dual \(\mathcal{M}'\) of \(\mathcal{M}\) is subnormal.

We are now in a position to complete the proof of Theorem 2.3.

Proof of Theorem 2.3 If \(c_1 = 0\) and \(c_2 = 0\), then \(\mathcal{M}\) is an isometry and hence it is a subnormal contraction. If either \(c_1\) or \(c_2\) is 0, then the conclusion follows from Corollary 5.3. The case in which \(c_1\) and \(c_2\) are positive follows from Proposition 7.1 and Example 7.2.
We conclude the paper with a modest generalization of [30, Proposition 2] (cf. [15, Theorem 3.1] and [22, Example 11.1]).

**Proposition 7.3.** For positive scalars $c_1, c_2$, consider the positive Borel measure $\mu = c_2 \delta_1 + c_2 \delta_{-1}$. Then the Dirichlet-type space $D(\mu)$ coincides with the de Branges-Rovnyak space $\mathcal{H}(B)$ with $B = (b_1, b_2)$ and $b_j = p_j/q_j$, where

$$p_1(z) = \gamma_1 z + \gamma_2 z^2, \quad p_2(z) = \gamma_3 z^2, \quad q(z) = (z - \alpha_1)(z - \alpha_2), \quad z \in \mathbb{D}$$

for scalars $\alpha_i$ as given by (7.2), and and $\gamma_j$ are given by

$$\gamma_1 = \sqrt{(\alpha_1 + \alpha_2)^2 + \frac{\alpha_1 \alpha_2 (1 - \alpha_1^2)(1 - \alpha_2^2)}{(-1 + \alpha_1 \alpha_2)}}$$

$$\gamma_2 = -\sqrt{\frac{1}{\gamma_1}(\alpha_1 + \alpha_2)(1 - \frac{2}{\gamma_1}(\frac{-\alpha_1 \alpha_2}{\gamma_1} + \frac{2}{\gamma_1}(1) + 1))}$$

$$\gamma_3 = \sqrt{1 - \frac{\alpha_1 \alpha_2 (3 - \alpha_1 \alpha_2 - \alpha_1^2 - \alpha_2^2)}{-1 + \alpha_1 \alpha_2} - |\gamma_2|^2}$$

**Proof.** By Proposition 6.4 one may choose $p_1$ and $p_2$ of the form

$$p_1(z) = \gamma_1 z + \gamma_2 z^2, \quad p_2(z) = \gamma_3 z^2,$$

which satisfy (6.7). We now compute $\gamma_1, \gamma_2, \gamma_3$. By (6.7), we obtain

$$\gamma_1^2 \gamma_1^2 w + \gamma_2 \gamma_1^2 z^2 w + \gamma_1 \gamma_2 z^2 w^2 + (|\gamma_2|^2 + |\gamma_3|^2) z^2 w^2$$

$$= \sum_{j=1}^{2} p_j(z) p_j(w)$$

$$= q(z) q(w) - p(z) p(w) \left(1 + \sum_{i,j=1}^{2} \frac{b_{ij}}{O'(\zeta_i) O'(\zeta_i)} \frac{1 - z w}{z - \zeta_i} \right)$$

$$= (z - \alpha_1)(z - \alpha_2)(\overline{w} - \alpha_1)(\overline{w} - \alpha_2) - \frac{1}{\gamma}(z^2 - 1)(z^2 - 1)$$

$$- \frac{1}{\gamma} \left( \frac{1}{f'_2(-1)f'_1(1) + 1/4} \right) \left( f'_2(-1)(z + 1)(w + 1) - f'_1(1)(z - 1)(w - 1) \right)$$

$$+ \frac{1}{\gamma} (z - 1)(w + 1) + \frac{1}{\gamma} (z + 1)(w - 1)$$

Comparing the coefficients on both sides, we obtain

$$|\gamma_1|^2 = (\alpha_1 + \alpha_2)^2 - \frac{2}{\gamma} \frac{1}{f'_2(-1)f'_1(1) + 1/4}$$

$$\gamma_2 \gamma_1 = -\alpha_1 - \alpha_2 + \frac{1}{\gamma} \frac{1}{f'_2(-1)f'_1(1) + 1/4} \left( f'_2(-1) + f'_1(1) \right)$$

$$|\gamma_2|^2 + |\gamma_3|^2 = 1 - \frac{1}{\gamma} + \frac{1}{\gamma} \frac{1}{f'_2(-1)f'_1(1) + 1/4} \left( f'_2(-1) - f'_1(1) + 1 \right)$$

It follows from computations of Example 7.2 that

$$f'_1(1)f'_2(-1) + 1/4 = \frac{2(\alpha_1 \alpha_2 - 1)}{(1 - \alpha_1^2)(1 - \alpha_2^2)}.$$
\[ f_1'(1) + f_2'(-1) = \frac{2(\alpha_1 \alpha_2 - 1)(\alpha_1 + \alpha_2)}{(1 - \alpha_1^2)(1 - \alpha_2^2)} \]
\[ f_2'(-1) - f_1'(1) + 1 = \frac{2(2 - \alpha_1^2 - \alpha_2^2)}{(1 - \alpha_1^2)(1 - \alpha_2^2)}. \]

Since \( \gamma = -\frac{1}{\alpha_1 \alpha_2} \) (see (7.2)), it is easy to see that (7.5) holds. This gives the coefficients of \( p_1 \) and \( p_2 \).

It is worth noticing that if \( c_1 = c_2 \), then \( \alpha_1 + \alpha_2 = 0 \) (see (7.2)), and hence \( p_1, p_2 \) and \( q \) takes the form
\[ p_1(z) = \gamma_1 z, \quad p_2(z) = \gamma_3 z^2, \quad q(z) = z^2 - \alpha_1^2, \quad z \in \mathbb{D}. \]

A particular case of this has been pointed out in [22, Example 11.1].

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