Smooth approximation of Yang–Mills theory on \( \mathbb{R}^2 \): a rough path approach

May 9, 2017

Hideyasu Yamashita
Division of Liberal Arts and Sciences, Aichi-Gakuin University
Email: yamasita@dpc.aichi-gakuin.ac.jp

Abstract

In the context of rough path theory (RPT), the theories of Hairer (2014) and Gubinelli–Imkeller–Perkowski (2015) (GIP theory) gave new methods for construction of \( \Phi^4_3 \) model. Roughly, their results state that a quantum field in a \( \Phi^4_3 \) model can be smoothly approximated. Consider the following question: Can RPT be applied to quantum Yang–Mills (YM) gauge field theories to show that any YM theory can be smoothly approximated? In this paper we consider this problem in the simplest case of Euclidean YM theory, i.e. YM on \( \mathbb{R}^2 \) with the usual Euclidean metric, as a test case. We prove that a (quantum) \( SU(n) \) YM theory on \( \mathbb{R}^2 \) in axial gauge can be smoothly approximated for some class of Wilson loops. While our study is inspired by the theories of Hairer and GIP, instead we use the RPT framework of Friz–Victoir (2010) in proving the theorem.

Keywords: Yang–Mills theory, Rough path theory, Stochastic differential equation, White noise, Littlewood–Paley theory.

MSC2010: 60H10, 60H40, 81T13.

Contents

1 Introduction 2
2 Littlewood–Paley theory and Besov space 2
3 Lie algebra valued white noise 4
4 Classical gauge theory on \( \mathbb{R}^2 \) 6
5 Axial gauge 7
6 operator \( \mathcal{E} \) 9
7 Rough paths 11
8 Estimate for \( X^{(j)}_{s,t} \) 15
9 Estimate for \( X^j_{s,t} \) 18
10 Rough path convergence 27
11 Wilson loop 29
12 Open problems 31
1 Introduction

In the context of rough path theory (e.g. \cite{FV10, FH14}), the theory of regularity structure of Hairer \cite{Hai14}, and that of paracontrolled distributions of Gubinelli, Imkeller and Perkowski (GIP theory) \cite{GIP15} gave new methods of construction of models of quantum scalar fields, including the $\Phi^4_3$ model \cite{CC13, Hai14, Hai15, MW16, MWX16}. Their results are summarized very roughly in one sentence: A quantum field in a $\Phi^4_3$ model, which is represented by a distribution-valued random variable, can be approximated by smooth fields, which are $C^\infty$-valued random variables. Thus the following natural (and naive) questions arise: Can these methods be applied to quantum Yang–Mills (YM) gauge field theories to show that any YM theory can be smoothly approximated? More generally, can the notion of ‘rough gauge field’ be rigorously established?

In this paper we consider this problem in the simplest case of Euclidean YM theory, i.e. YM on $\mathbb{R}^2$ with the usual Euclidean metric, as a test case. Our main result (Theorem 11.5) states that a (quantum) $SU(n)$-YM theory on $\mathbb{R}^2$ in axial gauge can be smoothly approximated; More precisely, it is stated as follows: Let $g = \mathfrak{su}(n)$ be the Lie algebra of $G = SU(n)$, and $\Omega^1(\mathbb{R}^2, g)$ the space of smooth $g$-valued 1-forms on $\mathbb{R}^2$. For a curve $c : \mathbb{R} \to \mathbb{R}^2$ and a 1-form $A \in \Omega^1(\mathbb{R}^2, g)$, let $\mathcal{U}_{c, A}(t) \in G$ ($t \in \mathbb{R}$) denote the parallel transport along $c$. Suppose that a set of the curves $\{c^i : i \in \mathbb{N}\}$ satisfy some regularity conditions. Then there exists a probability space $(\Omega, \mathbb{P})$ and a sequence of $\Omega^1(\mathbb{R}^2, g)$-valued random variables $A^{(n)}$ such that

$$
\mathbb{P}\left[\forall i \in \mathbb{N}, \mathcal{U}_{c_i} := \lim_{n \to \infty} \mathcal{U}_{c_i, A^{(n)}} \text{ (uniform)} \in C([0,1], G)\right] = 1,
$$

and furthermore the set of the $G$-valued random variables $\{\mathcal{U}_{c_i}\}_{i \in \mathbb{N}}$ obeys the law the Wilson loops in the YM theory on $\mathbb{R}^2$. Note that this statement itself does not contain any term or notion specific to rough path theory (including the theories of Hairer and GIP). However, to prove the theorem, we shall make heavy use of rough path theory, as well as the Littlewood–Paley theory of Besov spaces, in this paper. While our study is inspired by the theories of GIP and regularity structure, we work in the framework of \cite{FV10}, without those theories.

While YM on $\mathbb{R}^2$ is called ‘trivial’ in the physical literature since this is a sort of free field theory in the sense that it does not describe any interaction, we find that this theory has highly ‘nontrivial’ aspects in the mathematical point of view; Although the above theorem can be viewed as a partial positive answer for the above questions, our result is yet too weak to establish the theory of ‘rough gauge fields.’ See Conjecture 12.1.

For the rigorous formulations of (Euclidean) quantum YM theories on a 2-dimensional Riemannian manifold, we refer to Driver \cite{Dri89}, Sengupta \cite{Sen92, Sen93, Sen97} and Lévy \cite{Lévy03}.

2 Littlewood–Paley theory and Besov space

For a general introduction to Besov spaces with the Littlewood–Paley theory, we refer to \cite{BCD11, Gra09} (see also Appendix of \cite{GIP15}), and for Besov (and Sobolev) spaces \textit{without} the Littlewood–Paley theory, we refer to \cite{Tar07}.

Let $F u = \hat{u}$ denotes the Fourier transform of $u$:
\[ F u(z) = \hat{u}(z) := \int_{\mathbb{R}^d} e^{-i(z \cdot x)} u(x) dx, \]

so that \( \hat{u}(z) := F^{-1} u(z) = (2\pi)^{-d} F u(-z) \). We consider only the case where \( d = 2 \).

Following Grafakos [Gra09], we fix a radial \( C^\infty \) function \( \rho = \rho_0 \) on \( \mathbb{R}^2 \) such that

\[ \rho_0 \geq 0, \text{ supp} \rho_0 \subset \left\{ \xi : 1 - \frac{1}{7} \leq |\xi| \leq 2 \right\} \]

\[ 1 \leq |\xi| \leq 2 - \frac{2}{7} \implies \rho_0(\xi) = 1 \]

\[ 1 \leq |\xi| \leq 4 - \frac{4}{7} \implies \rho_0(\xi) + \rho_0(\xi/2) = 1 \]

so that \( \sum_{j \in \mathbb{Z}} \rho_0(2^{-j} \xi) = 1 \) for \( \xi \in \mathbb{R}^2 \setminus \{0\} \). We also define \( \chi = \chi_0 \) so that

\[ \chi_0(\xi) := \sum_{j \leq -1} \rho_0(2^{-j} \xi) \text{ if } \xi \neq 0, \quad \chi_0(\xi) = 1 \text{ if } \xi = 0. \]

Set

\[ \rho_{-1} := \chi, \quad \rho_j := \rho_0(2^{-j} \cdot), \quad j \geq 0, \]

so that \( \sum_{j \geq -1} \rho_j = 1 \), and set

\[ \chi_j := \chi_0(2^{-j} \cdot) = \sum_{i=-1}^{j-1} \rho_i, \quad j \geq 0 \]

Define the Littlewood–Paley operators \( \Delta_j \) and \( S_j \) by

\[ \Delta_j u := F^{-1}(\rho_j F u) = \check{\rho}_j * u, \quad j \geq -1, \]

\[ S_j u := \sum_{i=-1}^{j-1} \Delta_i u = \check{\chi}_j * u. \]

For \( p, q \in [1, \infty] \) and \( s \in \mathbb{R} \), the Besov space \( B^s_{p,q} = B^s_{p,q}(\mathbb{R}^d, \mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) \) is defined by

\[ B^s_{p,q}(\mathbb{R}^d, \mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) : \|u\|_{B^s_{p,q}} := \left( \sum_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^p})^q \right)^{1/q} < \infty \right\}. \]

The Lipschitz space \( \text{Lip}^s = \text{Lip}^s(\mathbb{R}^d, \mathbb{R}^n) \) is defined by

\[ \text{Lip}^s(\mathbb{R}^d, \mathbb{R}^n) := B^s_{\infty, \infty}(\mathbb{R}^d, \mathbb{R}^n) \]

\[ = \left\{ u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) : \|u\|_{B^s_{\infty, \infty}} := \sup_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^\infty}) < \infty \right\} \]

The space \( B^s_{p,p}(\mathbb{R}^d, \mathbb{R}^n) \) is written as \( W^{s,p}(\mathbb{R}^d, \mathbb{R}^n) \), often called the Sobolev space.
For \( h \in \mathbb{R}^d \), let \( \tau_h \) denote the translation operator
\[
(\tau_h u)(x) := u(x + h)
\]
(2.1)

The following proposition will be used later.

**Proposition 2.1.** (e.g., [Tar07] Lemma 35.1) Let \( 0 < s < 1 \) and \( 1 \leq p \leq \infty \).

Define the seminorm \( \| \cdot \|_{B^s_p, \infty} \) and the norm \( \| \cdot \|_{B^s_p, \infty} \) by
\[
|u|'_{B^s_p, \infty} := \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{\|u - \tau_h u\|_{L^p}}{|h|^s}, \quad \|u\|_{B^s_p, \infty} := \|u\|_{L^p} + |u|'_{B^s_p, \infty}.
\]

Then \( u \in B^s_p((\mathbb{R}^d, \mathbb{R}^n)) \) if and only if \( \|u\|_{B^s_p, \infty} < \infty \). Moreover the norms \( \| \cdot \|_{B^s_p, \infty} \) and \( \| \cdot \|_{B^s_p, \infty} \) are equivalent.

## 3 Lie algebra valued white noise

Fix \( n_{\text{mat}} \in \mathbb{N} \) and let \( \text{Mat} := \text{Mat}(n_{\text{mat}}, \mathbb{C}) \cong \mathbb{R}^{2n_{\text{mat}}} \), equipped with the Hilbert–Schmidt inner product
\[
\langle X, Y \rangle = \langle X, Y \rangle_{\text{HS}} := \text{Tr} X^* Y, \quad X, Y \in \text{Mat},
\]
and the norm \( \|X\|_{\text{HS}} := \langle X, X \rangle_{\text{HS}}^{1/2} \). Let \( G := SU(n_{\text{mat}}) \subset \text{Mat} \), and \( g := \text{su}(n_{\text{mat}}) \subset \text{Mat} \), the Lie algebra of \( G \). We define the inner product \( \langle \cdot, \cdot \rangle_g \) on \( g \) by \( \langle X, Y \rangle_g := \langle X, Y \rangle_{\text{HS}} \). Note that \( \langle \cdot, \cdot \rangle_g \) is proportional to the Killing form on \( g = \text{su}(n_{\text{mat}}) \).

Let \( \mathcal{S}(\mathbb{R}^d, g) \) denote the space of functions of rapid decrease from \( \mathbb{R}^d \) to \( g \), and \( \mathcal{S}'(\mathbb{R}^d, g) \) denote its dual space, consisting of the continuous linear functionals from \( \mathcal{S}(\mathbb{R}^d, g) \) to \( \mathbb{R} \). This is discriminated from \( \mathcal{S}'(\mathbb{R}^d, g) \), the space of \( g \)-valued tempered distributions, which are continuous linear functionals from \( \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d, \mathbb{R}) \) to \( g \). However, for \( F \in \mathcal{S}'(\mathbb{R}^d, g) \), we can naturally define the corresponding \( g \)-valued distribution \( F^* \in \mathcal{S}'(\mathbb{R}^d, g) \) by
\[
\langle F^*(f), X \rangle_g = F(Xf), \quad X \in g, \ f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}),
\]
or more explicitly,
\[
F^*(f) := \sum_{k=1}^{\text{dim } g} F(e_k f)e_k, \quad f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}),
\]
where \( \{e_k : k = 1, \ldots, \text{dim } g\} \) is an orthonormal basis of \( g \). So we can identify \( \mathcal{S}'(\mathbb{R}^d, g) \) with \( \mathcal{S}'(\mathbb{R}^d, g) \) under some abuse of notation: If \( F \in \mathcal{S}'(\mathbb{R}^d, g) \) and \( f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \), let \( F(f) := F^*(f) \in g \). Conversely, if \( F^* \in \mathcal{S}'(\mathbb{R}^d, g) \) and \( f \in \mathcal{S}(\mathbb{R}^d, g) \), let \( F^*(f) := F(f) \in \mathbb{R} \).

Let \( (\Omega, \mathcal{F}) \) be a probability space. Let \( W \) be a \( g \)-valued white noise on \( \mathbb{R}^2 \), that is, an isometry from \( L^2(\mathbb{R}^2) \) to \( L^2((\Omega, \mathcal{F}), g) \). For the same reason as above, \( W \) can also be viewed as an isometry from \( L^2(\mathbb{R}^2, g) \) to \( L^2((\Omega, \mathcal{F}), \mathbb{R}) \). If we consider \( W : L^2(\mathbb{R}^2, g) \rightarrow L^2((\Omega, \mathcal{F}), \mathbb{R}) \), its covariance is expressed as
\[
\mathbb{E}(W(f)W(g)) = \langle f, g \rangle_{L^2(\mathbb{R}^2, g)}, \quad f, g \in L^2(\mathbb{R}^2, g),
\]
and if we consider $W : L^2(\mathbb{R}^2) \rightarrow L^2((\Omega, \mathcal{P}), \mathfrak{g})$, its covariance is expressed as
\[ E((W(f), W(g)) = \langle f, g \rangle_{L^2(\mathbb{R}^2)}, \quad f, g \in L^2(\mathbb{R}^2), \]
or more explicitly,
\[ E(W(f)_k W(g)_l) = \delta_{kl} \langle f, g \rangle_{L^2(\mathbb{R}^2)}, \quad f, g \in L^2(\mathbb{R}^2), \quad k, l = 1, \ldots, \dim \mathfrak{g}, \]
where $W(f)_k := (W(f), e_k)_\mathfrak{g}$. While these views are compatible, we mainly regard $W$ as $W : L^2(\mathbb{R}^2) \rightarrow L^2((\Omega, \mathcal{P}), \mathfrak{g})$ in this paper.

In the following we write $L^p(\mathbb{P}) := L^p((\Omega, \mathcal{P}), \mathbb{R})$ and $L^p(\mathbb{P}, \mathfrak{g}) := L^p((\Omega, \mathcal{P}), \mathfrak{g})$. $W$ is continuous on $\mathcal{S}(\mathbb{R}^2)$ a.s., that is,
\[ \mathbb{P}[(W \mid \mathcal{S}(\mathbb{R}^2)) \in \mathcal{S}(\mathbb{R}^2, \mathfrak{g})] = 1. \]

In the following we assume $(W(\omega) \mid \mathcal{S}(\mathbb{R}^2)) \in \mathcal{S}(\mathbb{R}^2, \mathfrak{g})$ for all $\omega \in \Omega$, and we simply write this as $W \in \mathcal{S}(\mathbb{R}^2, \mathfrak{g})$.

Define the $j$th smooth approximation $W^{(j)} \in C^\infty(\mathbb{R}^2, \mathfrak{g})$ of $W$ by
\[ W^{(j)} := S_j W. \tag{3.1} \]
$W^{(j)}$ converges to $W$ in $\mathcal{S}(\mathbb{R}^2, \mathfrak{g})$.

4 Classical gauge theory on $\mathbb{R}^2$

Let $\mathcal{C} = \mathcal{C}_{[0, 1]}$ the set of smooth maps $\mathbb{R} \ni t \mapsto c(t) = (c_1(t), c_2(t)) \in \mathbb{R}^2$ such that $\text{supp} c \subset [0, 1]$ where $\dot{c}(t) := \frac{d}{dt} c(t)$, in other words, $c$ is constant on $(-\infty, 0]$ and $[1, \infty)$, respectively.

For $c \in \mathcal{C}$, define $\overline{c} \in \mathcal{C}$ by $\overline{c}(t) := c(1 - t)$. If two curves $c^{(1)}, c^{(2)} \in \mathcal{C}$ satisfy $c^{(1)}(1) = c^{(2)}(0)$, we define the concatenation $c^{(2)}c^{(1)} \in \mathcal{C}$ by
\[ c^{(2)}c^{(1)}(t) := \begin{cases} c^{(1)}(2t) & (t \in (-\infty, 1/2]) \\ c^{(2)}(2t - 1) & (t \in [1/2, \infty)) \end{cases}, \]
equivalently, $c^{(2)}c^{(1)}(t) := c^{(1)}(2t) + c^{(2)}(2t - 1) - c^{(2)}(0)$.

Fix $c \in \mathcal{C}_{[0, 1]}$. Additionally we assume that any $c \in \mathcal{C}$ satisfies $c_1(t) > 0$ for all $t$; this assumption is not essential, but this simplifies the calculations.

Let $\Omega^1 = \Omega^1(\mathbb{R}^2, \mathfrak{g})$ denote the space of $\mathfrak{g}$-valued smooth 1-forms on $\mathbb{R}^2$. An element $A \in \Omega^1$ is called a gauge field in the physical context. Let $A = A_1 dx_1 + A_2 dx_2 \in \Omega^1(A_1, A_2 \in C^\infty(\mathbb{R}^2, \mathfrak{g}))$. In the notation $A(\dot{c}(t))$, $\dot{c}(t)$ should be seen as a tangent vector in the tangent bundle $T_{c(t)} \mathbb{R}^2$; that is,
\[ A(\dot{c}(t)) = A \left( \sum_{k=1}^2 \dot{c}_k(t) \frac{\partial}{\partial x^k} \right) = \sum_{k=1}^2 A_k(c(t))\dot{c}_k(t). \]
The parallel transport $\Psi_{c,A}(t) \in G$ $(t \in \mathbb{R})$ along $c \in \mathcal{C}_{[0, 1]}$ is defined by the ODE
\[ \frac{d\Psi_{c,A}(t)}{dt} = A(\dot{c}(t)) \Psi_{c,A}(t) = \sum_{k=1}^2 A_k(c(t))\dot{c}_k(t)\Psi_{c,A}(t), \quad \Psi_{c,A}(0) = e \quad (4.1) \]
For \( t \geq 0 \), define \( X_t = X(t) \) to be the line integral of \( A \) along \( c \mid [0, t] \):

\[
X(t) = X_{c,A}(t) := \int_{c|[0,t]} A = \int_0^t A(\dot{c}(s))\,ds = \int_0^t \sum_{k=1}^2 A_k(\dot{c}(s))\dot{c}_k(s)\,ds. \tag{4.2}
\]

Let \( V : \text{Mat} \to L(\text{Mat}, \text{Mat}) \) be a bounded smooth map such that

\[
\forall (U)M = MU, \quad \forall U \in G, \forall M \in \text{Mat}.
\tag{4.3}
\]

(Recall \( G := SU(n_{\text{mat}}) \subset \text{Mat} \)). Then the ODE \( \text{E.11} \) is rewritten as a normal form

\[
d\mathcal{U}_c(t) = V(\mathcal{U}_c(t))dX_{c,A}(t). \tag{4.4}
\]

If \( c \) is a loop (i.e. \( c(0) = c(1) \)), we call \( \mathcal{U}_{c,A}(1) \in G \) the holonomy along \( c \). It is also called the Wilson loop, mainly when \( \mathcal{U}_{c,A}(1) \) is a \( G \)-valued random variable.

The most basic class of loops is that of the simple (Jordan) loops, i.e. loops \( c \) such that if \( s, t \in [0, 1] \) and \( c(s) = c(t) \) then \( s = t \). However, it is useful to consider a slightly broader class of loops, called lassos \( \text{[Dri89, Sen92]} \).

Let \( D \subset \mathbb{R}^2 \). Suppose \( c \in \mathcal{C} \), \( c(0) = c(1) \), \( c \) is simple. Let \( D \subset \mathbb{R}^2 \) be the closed domain enclosed by the arc \( c([0, 1]) \). \( c \) is called a lasso based on \( x \in \mathbb{R}^2 \) if there exists \( c^1, c^2 \in \mathcal{C} \) such that \( c^2 \) is a simple closed curve enclosing \( D \subset \mathbb{R}^2 \) anticlockwise, and that

\[
c^1(0) = x, \quad c^1(1) = c^2(0) = c^2(1), \quad c = c^1c^2c^1
\]

In this case, we write

\[
D(c) := D, \quad \gamma(c) := c^1
\]

A simple loop is also a lasso where \( c^1 \) is trivial (i.e. a constant map). The set of lassos based on \( x \in \mathbb{R}^2 \) is denoted by \( \text{Lasso}(x) \), and let \( \text{Lasso} := \bigcup_{x \in \mathbb{R}^2} \text{Lasso}(x) \).

Let \( \mathcal{D} \) be the set of subsets \( D \subset \mathbb{R}^2 \) such that there exists a simple loop \( c \in \mathcal{C} \) enclosing \( D \).

**Lemma 4.1.** Fix \( A \in \Omega^1 \). Let \( c \in \mathcal{C} \cap \text{Lasso}(x) \). Suppose \( D_1, \ldots, D_n \in \mathcal{D} \) satisfy

(i) \( D(c) = \bigcup_{k=1}^n D_k \),

(ii) \( D_k \cap D_l = \emptyset \) if \( k \neq l \), and

(iii) \( \bigcup_{1 \leq l \leq k} D_l \) is connected for all \( k = 1, \ldots, n \). Then there exists \( c^1, \ldots, c^n \in \mathcal{C} \cap \text{Lasso}(x) \) such that \( D(c^k) = D_k, \ k = 1, \ldots, n \), and

\[
\mathcal{U}_{c,A}(1) = \mathcal{U}_{c^1,A}(1) \cdots \mathcal{U}_{c^n,A}(1),
\]

Proof. Easily shown by induction for \( n \), using the relation \( \mathcal{U}_c^{-1} = \mathcal{U}_c^{-1} \). \( \square \)

From the definition of \( \mathcal{U} \), one can easily show the following:

**Lemma 4.2.** Fix \( x = (x_1, x_2) \in \mathbb{R}^2 \), and suppose that for each \( \epsilon_1, \epsilon_2 > 0 \), \( \epsilon_{\epsilon_1, \epsilon_2} \) is a lasso in \( \mathcal{C} \cap \text{Lasso} \) such that

\[
\epsilon_{\epsilon_1, \epsilon_2}(0) = \epsilon_{\epsilon_1, \epsilon_2}(1) = x, \quad D(\epsilon_{\epsilon_1, \epsilon_2}) = [x_1, x_1 + \epsilon_1] \times [x_2, x_2 + \epsilon_2]
\]

Then
Axial gauge

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\Omega_{\epsilon_1, \epsilon_2, A}(1) - 1}{\epsilon_1 \epsilon_2} = F_{12}(x),
\]

where \( F_{12}(x) := \partial_1 A_2(x) - \partial_2 A_1(x) + A_2(x) A_1(x) - A_1(x) A_2(x). \)

The above \( F_{12} = F_{12, A} \in C^\infty(\mathbb{R}^2, \mathfrak{g}) \) is called the field strength in physical terminology. The curvature 2-form \( F = F_A \in \Omega^2(\mathbb{R}^2, \mathfrak{g}) \) is defined by

\[
F(x) := F_{12}(x) dx_1 \wedge dx_2.
\]

We see \( F_A = dA + [A, A] \), more exactly,

\[
F_A(X, Y) = dA(X, Y) + [A(X), A(Y)], \quad X, Y \in T_x \mathbb{R}^2.
\]

However, in this paper we shall impose the axial gauge condition later, which implies \([A, A] = 0\). In this case the linear relation \( F = dA \) holds.

5 Axial gauge

For \( u \in C^\infty(\mathbb{R}^2, G) \), define the action \( \mathcal{G}_u \), called the gauge transformation, on \( A \) by

\[
\mathcal{G}_u A_k(x) = A^u_k(x) := u^{-1}(x) A_k(x) u(x) - (\partial_k u^{-1}(x)) u(x),
\]

so that

\[
\mathcal{G}_A \mathcal{G}_u A(t) = u(t)^{-1} \mathcal{G}_u A(t) u(t(0)).
\]

Note that if \( t(0) = t(1) \), the holonomies \( \mathcal{H}_u A(1) \) and \( \mathcal{H}_A \mathcal{G}_u A(1) \) are conjugate. Since

\[
F_{\mathcal{G}_u, A}(x) = u^{-1}(x) F(x) u(x),
\]

naturally we define the gauge transform of \( F \) by \( \mathcal{G}_u F = F^u := u^{-1} F u \).

Let \( e_\theta = (e_{\theta_1}, e_{\theta_2}) := (\cos \theta, \sin \theta) \in \mathbb{R}^2 \setminus \{0\} \) and \( e'_\theta = (e'_{\theta_1}, e'_{\theta_2}) := e_{\theta + \pi/2} \). If \( A = A_1 dx_1 + A_2 dx_2 \in \Omega^1 \) satisfies \( \sum_{k=1}^2 A_k e_{\theta_k} \equiv 0 \) for some \( \theta \in [0, 2\pi] \), then \( A \) is said to be in \( (\theta, \pi) \)-axial gauge. In this case we have \([A, A] = 0\), and hence \( F = dA \). This axial gauge fixing condition is not complete in that for a given \( F = F_{12} dx_1 \wedge dx_2 \in \Omega^2(\mathbb{R}^2, \mathfrak{g}) \), the 1-form \( A \in \Omega^1 \) in \( \theta \)-axial gauge satisfying \( F = dA \) is not unique. Instead if we assume two conditions

\[
\sum_{k=1}^2 A_k(x) e_{\theta_k} \equiv 0, \quad \sum_{k=1}^2 A_k(r e'_{\theta_k}) e'_{\theta_k} = 0, \quad \forall r \in \mathbb{R}
\]

we have a unique \( A \) for any \( F \). In this paper we say that \( A \) is in \( \theta \)-gauge if these conditions are satisfied. We see that any \( A \in \Omega^1 \) can be gauge-transformed to satisfy this condition. If \( \theta = 0 \), \( A \) in \( \theta \)-gauge is determined by \( F \) as follows:

\[
A_1(x) \equiv 0, \quad A_2(x) := \int_0^{x_1} F_{12}(\xi, x_2) d\xi, \quad x = (x_1, x_2) \in \mathbb{R}^2
\]

We assume [5.2] in the following. We see

\[
A(\hat{t}(t)) = \int_0^{\hat{t}(t)} F_{12}(x_1, c_2(t)) \xi_1 dx_1.
\]
and

\[ X_{c,t} \equiv X_c(t) = \int_0^t A(\dot{c}(t')) dt' = \int_0^t A_2(\dot{c}(t')) \dot{c}_2(t') dt' = \int_0^t \int_0^t F_{12}(x_1, c_2(t')) \dot{c}_2(t') dx_1 dt' \]  

(5.3)

(5.4)

Let \( \mathcal{R}_1 \) be the set of \( E \in \mathcal{D} \) such that \( E \) is convex w.r.t. \( x_1 \), i.e.

\[ \mathcal{R}_1 := \{ E \in \mathcal{D} : \text{if} \ (x_1, x_2), (x'_1, x_2) \in E \text{ and } x_1 \leq x''_1 \leq x'_1, \text{ then } (x''_1, x_2) \in E \}. \]  

(5.5)

Fix \( D \in \mathcal{R}_1 \). Let

\[ a := \inf \{ x_2 \in \mathbb{R} : \exists x_1 \in \mathbb{R}, (x_1, x_2) \in D \}, \]  

\[ b := \sup \{ x_2 \in \mathbb{R} : \exists x_1 \in \mathbb{R}, (x_1, x_2) \in D \}. \]

Then there exists \( c^1, c^2 \in \mathcal{C} \cap \text{Lasso} \) such that \( D(\overline{c^2}, \overline{c^1}) = D \), and that

\[ c^1(0) = c^2(0) = a, \quad c^1(1) = c^2(1) = b, \quad c^1(t) = c^2(t), \ \forall t \in [0,1]. \]

Then corresponding parallel transport \( \mathcal{U}_c \) is defined by (4.4):

\[ d\mathcal{U}_c(t) = \mathcal{V}(\mathcal{U}_c(t)) dX_{c,t}, \quad \mathcal{U}_c(0) = I. \]

For \( \tau \in [a,b] \), let

\[ D_\tau := D \cap (\mathbb{R} \times [a, c^2(\tau)]), \quad F^D_\tau := \int_{D_\tau} F_{12}(x) dx. \]

Let \( c^* \in \mathcal{C} \cap \text{Lasso} \) satisfy \( D(c) = D_\tau \) and \( c^*_1(0) = c^*_1(1) = a \). Let \( U(\tau) := \mathcal{U}_{c^*}(1) \), the holonomy of \( c^* \).

The following lemmas are easily shown from these definitions:

**Lemma 5.1.** For \( t \in [a,b] \), \( U(t) = \mathcal{U}_{c^1}(t)^{-1} \mathcal{U}_{c^2}(t) \) holds.

**Lemma 5.2.** For \( t \in [a,b] \),

\[ U(t)^{-1} \frac{d}{dt} U(t) = -\mathcal{U}_{c^1}(t)^{-1} \left( \int_{c^1(t)}^{c^2(t)} F_{12}(x_1, c^2(t)) dx_1 \right) \mathcal{U}_{c^1}(t) \]

holds. Equivalently,

\[ dU(t) = -U(t) \mathcal{U}_{c^1}(t)^{-1} dF^D_t \mathcal{U}_{c^1}(t) = -U(t) dB^D_t, \]  

(5.6)

where

\[ B^D_t := \int_a^t \mathcal{U}_{c^1}(s)^{-1} dF^D_s \mathcal{U}_{c^1}(s). \]
6 operator $\mathcal{E}$

Set $F_{1j} := W^{(j)}$, $j$th approximation of the $\mathfrak{g}$-valued white noise $W$ on $\mathbb{R}^2$ defined by $\mathbb{K}_{\mathfrak{g}}$, then a unique $\Omega^1$-valued random variable $A^{(j)}$ is determined by $\mathbb{K}_{\mathfrak{g}}$.

Let $X^{(j)} = X_{t}^{(j)} = X_{t,A^{(j)}}$, i.e.

$$X^{(j)}(t) = X_{t}^{(j)}(t) := \int_{t[0,1]} A^{(j)}.$$  \hspace{1cm} (6.1)

For $H : \mathbb{R}^2 \to \mathbb{R}$ and $h \in L^\infty(\mathbb{R})$, let

$$\hat{\mathcal{E}}(H,h) := \int_{\mathbb{R}} \int_{t[0,1]} c_{1}(t) H(x,\epsilon_2(t)) h(t) \epsilon_2(t) dx dt.$$  \hspace{1cm} if the integral in the r.h.s. exists. Let

$$\|\hat{\mathcal{E}}\|_{2,h} := \sup \left\{ \|\hat{\mathcal{E}}(H,h)\| ; H \in L^2(\mathbb{R}^2), \|H\|_{L^2(\mathbb{R}^2)} \leq 1 \right\}.$$  \hspace{1cm} We shall see in Lemma 6.1 that $\|\hat{\mathcal{E}}\|_{2,h} < \infty$ for all $h \in L^\infty(\mathbb{R})$, and hence we can define the bounded linear operator $\mathcal{E} : L^\infty(\mathbb{R}) \to L^2(\mathbb{R}^2)$ as follows:

$$\langle H, \mathcal{E}h \rangle_{L^2(\mathbb{R}^2)} = \hat{\mathcal{E}}(H,h), \quad H \in L^2(\mathbb{R}^2), \quad h \in L^\infty(\mathbb{R}).$$  \hspace{1cm} Clearly $\text{supp}(\mathcal{E}h) \subset \mathbb{R}^2$ is compact. $W^{(j)}(\mathcal{E}h) \in \mathfrak{g}$ is naturally defined by

$$W^{(j)}(\mathcal{E}h) = \langle W^{(j)}, \mathcal{E}h \rangle := \int_{\mathbb{R}^2} W^{(j)}(x) \cdot (\mathcal{E}h)(x) dx.$$  \hspace{1cm} This integral is well-defined because $W^{(j)}$ is smooth, $\mathcal{E}h \in L^2(\mathbb{R}^2)$, and $\text{supp}\mathcal{E}h$ is compact. We see the following relations:

$$W^{(j)}(\mathcal{E}h) = W(S_j \mathcal{E}h) = \hat{\mathcal{E}}(W^{(j)}, h).$$  \hspace{1cm} We also see

$$X^{(j)}(t) = \hat{\mathcal{E}}(W^{(j)}, 1_{[0,1]}) = \langle W^{(j)}, \mathcal{E}1_{[0,1]} \rangle.$$  \hspace{1cm} Here define the $\mathfrak{g}$-valued random variable $X(t)$ by

$$X(t) = X_{t}(t) = W(\mathcal{E}1_{[0,1]}) = \langle W, \mathcal{E}1_{[0,1]} \rangle.$$  \hspace{1cm} while the last expression is useful but rather formal because it is neither a $L^2$ inner product, nor a pairing of $\mathcal{S}'$ and $\mathcal{S}$.

Hereafter we use the notations such as

$$\mathbb{R}^2_{<} := \left\{ (s,t) \in \mathbb{R}^2 : s < t \right\}, \quad [0,T]_{<}^2 := \left\{ (s,t) \in [0,T]^2 : s < t \right\}, \quad \text{etc.}$$  \hspace{1cm} Let

$$T_{\pm} = T_{\epsilon,\pm} := \{ t \in (0,1) ; \epsilon_2(t) \gtrless 0 \}, \quad T_0 := \{ t \in (0,1) ; \epsilon_2(t) = 0 \}$$  \hspace{1cm} then these are unions of countable disjoint open intervals:
Thus we have

\[ T_\pm = \bigcup_{i=1}^{N_\pm} I_{\pm,i}, \quad I_{\pm,i} = (t_{i,0}^\pm, t_{i,1}^\pm), \quad T_0 = \bigcup_{i=1}^{N_0} I_{0,i}, \quad N_\pm, N_0 \in \mathbb{N} \cup \{\infty\}, \]

Define \( \mathcal{E}_{c_i} h \in L^2(\mathbb{R}^2) \) as follows: for each \( x = (x_1, x_2) \in \mathbb{R}^2 \), let

\[
(\mathcal{E}_{c_i} h)(x) := \begin{cases} h(t) & \text{if } \exists t \in I_{\pm,i}, \ x_2 = c_2(t), \ 0 \leq x_1 \leq c_1(t), \\ 0 & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} h(c_2^{-1}(x_2; I_{\pm,i})) & \text{if } x_2 \in c_2(I_{\pm,i}), \ 0 \leq x_1 \leq c_1(c_2^{-1}(x_2; I_{\pm,i})) \\ 0 & \text{otherwise} \end{cases}
\]

where \( c_2^{-1}(x_2; I_{\pm,i}) \) is defined to be \( t \in I_{\pm,i} \) such that \( c_2(t) = x_2 \).

If \( \mathcal{E}_c h \in L^2(\mathbb{R}^2) \), we can check that \( \mathcal{E}_c h \) is explicitly expressed by

\[
\mathcal{E}_c h = \sum_{i=1}^{N_+} \mathcal{E}_{c_i}^+ h - \sum_{i=1}^{N_-} \mathcal{E}_{c_i}^- h \tag{6.4}
\]

**Lemma 6.1.** If we define \( \mathcal{E}_c h \) by (6.4), then \( \mathcal{E}_c h \in L^2(\mathbb{R}^2) \) for all \( h \in L^\infty(\mathbb{R}) \) and \( c \in \mathcal{C} \).

**Proof.** If \( N_+ < \infty \) or \( N_- < \infty \), this is clear. Suppose \( N_+ = N_- = \infty \).

Since \( \mathcal{E}_2(t_{i,0}^\pm) = \mathcal{E}_2(t_{i,1}^\pm) = 0 \) for all \( i \), and \( \sum_{i,\pm} (t_{i,1}^\pm - t_{i,0}^\pm) < \infty \), we have

\[
|c_2(t_{i,1}^\pm) - c_2(t_{i,0}^\pm)| = \left| \int_{t_{i,0}^\pm}^{t_{i,1}^\pm} \dot{c}_2(t) dt \right| \leq \int_{t_{i,0}^\pm}^{t_{i,1}^\pm} |\dot{c}_2(t)| dt
\]

\[
\leq \int_{t_{i,0}^\pm}^{t_{i,1}^\pm} \|\dot{c}_2\|_{L^\infty} (t - t_{i,0}^\pm) dt = \frac{1}{2} \|\dot{c}_2\|_{L^\infty} (t_{i,1}^\pm - t_{i,0}^\pm)^2
\]

Thus

\[
\|\mathcal{E}_{c_i}^\pm h\|_{L^2(\mathbb{R}^2)}^2 = \int_{c_2(t_{i,0}^\pm)}^{c_2(t_{i,1}^\pm)} dx_2 \int_0^{c_1 \left( c_2^{-1}(x_2; I_{\pm,i}) \right)} dx_1 \left| h(c_2^{-1}(x_2; I_{\pm,i})) \right|^2
\]

\[
\leq \|h\|_{L^\infty}^2 \int_{c_2(t_{i,0}^\pm)}^{c_2(t_{i,1}^\pm)} dx_2 \int_0^{c_1 \left( c_2^{-1}(x_2; I_{\pm,i}) \right)} dx_1
\]

\[
\leq \|h\|_{L^\infty}^2 \|c_1\|_{L^\infty} \|c_2(t_{i,1}^\pm) - c_2(t_{i,0}^\pm)\|
\]

\[
< \frac{1}{2} \|h\|_{L^\infty}^2 \|c_1\|_{L^\infty} \|\dot{c}_2\|_{L^\infty} (t_{i,1}^\pm - t_{i,0}^\pm)^2
\]

Therefore we have
\[ \| \mathcal{E}_t h \|_{L^2(\mathbb{R}^2)} \leq \sum_{\pm} \sum_{i=1}^{\infty} \left( \frac{1}{2} \| h \|^2_{L^\infty} \| \mathcal{E}_t h \|_{L^\infty} \| \mathcal{E}_t h \|_{L^\infty} \right)^{1/2} (t^\pm_{i+1} - t^\pm_{i}) < \infty \]

Define subsets \( \mathcal{C}_{\text{Rot}}, \mathcal{C}_\infty \) of \( \mathcal{C} \) by

\[
\mathcal{C}_{\text{Rot}} = \mathcal{C}_{[0,1],\text{Rot}} := \left\{ \varepsilon \in \mathcal{C}_{[0,1]} : \text{Rot}(\varepsilon) := \sup_{(s,t) \in \mathbb{R}^2} \| \mathcal{E}_s 1_{[s,t]} \|_{L^\infty} < \infty \right\},
\]

\[
\mathcal{C}_\infty = \mathcal{C}_{[0,1],\infty} := \left\{ \varepsilon \in \mathcal{C}_{[0,1]} : \| \mathcal{E}_\varepsilon \|_{L^\infty} < \infty \right\}.
\]

where \( \| \mathcal{E}_\varepsilon \|_{L^\infty} := \sup \{ \| \mathcal{E}_t h \|_{L^\infty} : h \in L^\infty(\mathbb{R}), \| h \|_{L^\infty} \leq 1 \} \).

Clearly we see \( \mathcal{C}_\infty \subset \mathcal{C}_{\text{Rot}} \). Roughly speaking, a curve \( \varepsilon \in \mathcal{C}_{[0,1]} \) is in \( \mathcal{C}_{[0,1],\text{Rot}} \) if \( \varepsilon \) does not rotate (clockwise or anti-clockwise) infinitely many times around any point in \( \mathbb{R}^2 \), and Rot(\varepsilon) is the maximum rotation number of \( \varepsilon \).

Note that in our definition of ‘smooth curve \( \varepsilon \),’ possibly \( \dot{\varepsilon}(t) = 0 \) holds for some \( t \in (0,1) \). Hence possibly the range \( \varepsilon(\mathbb{R}) = \varepsilon([0,1]) \subset \mathbb{R}^2 \) is not a smooth curve in the usual sense. For example, we see that any (finitely) piecewise linear curves are in \( \mathcal{C}_\infty \) (and \( \mathcal{C}_{\text{Rot}} \)).

By these definitions we easily find the following:

**Lemma 6.2.** If \( \varepsilon \in \mathcal{C}_{[0,1],\text{Rot}} \), then \( \mathcal{E}_\varepsilon 1_{[s,t]} \) is a finite (\( \leq 2 \text{Rot}(\varepsilon) \)) linear combination of characteristic functions; There exists disjoint subsets \( D_k \subset \mathbb{R}^2 \) (\( - \text{Rot}(\varepsilon) \leq k \leq \text{Rot}(\varepsilon) \)) such that

\[ \mathcal{E}_\varepsilon 1_{[s,t]} = \sum_{k \in -\text{Rot}(\varepsilon)} k 1_{D_k}. \]

7 Rough paths

For rough path theory, we refer to [PV10, FH14].

Let \( V \) be a finite-dimensional linear space, where \( V = \mathfrak{g} = \mathfrak{su}(n_{\text{mat}}) \) case is our main concern. Let

\[ T^{(2)}(V) := \mathbb{R} \oplus V \oplus (V \otimes V), \]

equipped with the truncated tensor product \( \otimes \), that is, if \( A = (a, b, c) \in T^{(2)}(V) \) and \( A' = (a', b', c') \in T^{(2)}(V) \), define \( A \otimes A' \) by

\[ A \otimes A' := (aa', ab' + a'b, ac' + a'c + b \otimes b'). \]

Let \( T^{(2)}_1(V) := \{ (a, b, c) \in T^{(2)}(V) \} \). Then naturally \( T^{(2)}_1(V) \) becomes a Lie group under \( \otimes \). We denote an element of \( T^{(2)}_1(V) \) as \( x = (1, x^{(1)}, x^{(2)}) \), or more readably, \( x = (1, x, \mathfrak{x}) \), etc.
If $x : [0, T] \to T^{(2)}_1(g)$, we write $x_{s,t} := x^{-1}_s \otimes x_t = (1, x_{s,t}, x_t - x_s \otimes x_{s,t})$, $x_{s,t} := x_t - x_s$, $s, t \in [0, T]$ If $x \in C^{1\text{-var}}([0, T], V)$, i.e. $x$ is a continuous path of bounded variation, define the truncated signature $\text{sig}(x) : [0, T] \to T^{(2)}_1(g)$ by $\text{sig}(x)_{s,t} := \left(1, x_{s,t}, \int_{s<u_1<u_2<t} dx_{u_1} \otimes dx_{u_2}\right) \in T^{(2)}_1(V)$. Note that if $x$ is smooth, $\int_{s<u_1<u_2<t} dx_{u_1} \otimes dx_{u_2} = \int_s^t x_{s,r} \otimes dx_r = \int_s^t \frac{dx_r}{dr} dr.$ When $x_0 = 0$ (i.e. $x_{0,t} = x_t$), the path $t \mapsto \text{lift}(x)_t := \text{sig}(x)_{0,t} = \left(1, x_t, \int_0^t x_r \otimes dx_r\right)$ is called the (step-2) lift of $x$.

**Theorem 7.1.** (Chen’s relation [FY10, Theorem 7.11, p.133]) For $x \in C^{1\text{-var}}([0, T], V)$ and $0 \leq s < t < u \leq T$, we have $\text{sig}(x)_{s,u} = \text{sig}(x)_{s,t} \otimes \text{sig}(x)_{t,u}$. Define the subgroup $G^{(2)}(V)$ of $T^{(2)}_1(V)$ by

$$G^{(2)}(V) := \left\{ \text{sig}(x)_{0,1} : x \in C^{1\text{-var}}([0, 1], V) \right\} \quad (7.1)$$

It is shown that $G^{(2)}(V)$ is expressed more explicitly as follows:

$$G^{(2)}(V) = \left\{ \left(1, x, \frac{1}{2} x \otimes x + y \otimes z - z \otimes y \right) : x, y, z \in V \right\} \quad (7.2)$$

$G^{(2)}(V)$ is given the Carnot-Caratheodory metric $d_{CC}[FY10,FH14]$. In this paper, the only information needed for $d_{CC}$ is the following:

$$d_{CC}(x, y) \simeq |y - x| + |y - x - x \otimes (y - x)|^{1/2}, \quad x, y \in G^{(2)}(V),$$

where $|\cdot|$ is the usual norm on the linear space $T^{(2)}(V)$. In particular, $d_{CC}(x, o) \simeq |x| + |x|^{1/2}$, where $o := 1_{G^{(2)}(V)} = (1, 0, 0) \in G^{(2)}(V)$.

Given $x, y \in C([0, T], G^{(2)}(V))$, we define the homogeneous Hölder distance $C([0, T], G^{(2)}(V))$ by

$$d_{b,\text{Höld}}(x, y) \equiv d_{CC,b,\text{Höld}}([0, T], \text{sig}(x, y)) := \sup_{0 \leq s < t \leq T} \frac{d_{CC}(x_{s,t}, y_{s,t})}{|t - s|^b} \quad (7.3)$$

and let

$$C_{b,\text{Höld}}([0, 1], G^{(2)}(V)) := \left\{ x \in C([0, T], G^{(2)}(V)) : d_{CC,b,\text{Höld}}([0, T], x, o) < \infty \right\}$$
Proposition 7.2. \cite{FV10} Proposition 8.12, p.174] Suppose $1/3 < \alpha_1 \leq 1/2$, \( x \in C^{\alpha}_\text{Hölder}([0, T], \Omega^2(V)) \) and \( x_0 = 0 \). Then there exists a sequence \( (x^{(n)}) \subset C^{1,\text{var}}([0, T], V), n \in \mathbb{N} \), such that \( \text{lift}(x^{(n)}) \to x \) uniformly as \( n \to \infty \), i.e.

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} d_{CC}(x_t, \text{lift}(x^{(n)})) = 0.
\]

If $1/3 < \alpha_1 \leq 1/2$, \( C^{\alpha}_\text{Hölder}([0, 1], \Omega^2(V)) \) is called the space of weak geometric Hölder rough paths \cite{FH14}.

Theorem 7.3. (Existence and uniqueness of RDE solution; step-2 case of \cite{FV10} Theorem 10.14, p.222] with \cite{FV10} Theorem 10.26, p.233]) Let \( d, e \in \mathbb{N}, \alpha_1 \in (1/3, 1/2] \), and assume the following:

(i) \( V: \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e) \) is in Lip\(^1(\mathbb{R}^e) \), where \( \gamma > 1/\alpha_1 \).

(ii) \( (x^{(n)})_{n \in \mathbb{N}} \) is a sequence in \( C^{1,\text{var}}([0, T], \mathbb{R}^d) \), such that

\[
\sup_n d_{CC, \text{Hölder}}([0, T], (\text{lift}(x^{(n)}), x)) < \infty.
\]

(iii) \( x \in C^{\alpha}_\text{Hölder}([0, T], \Omega^2(\mathbb{R}^d)) \) satisfies

\[
\lim_{n \to \infty} d_{CC, \text{Hölder}}([0, T], (\text{lift}(x^{(n)}), x)) = 0.
\]

(iv) \( y^{(n)} \in \mathbb{R}^e \) is a sequence converging to some \( y_0 \).

(v) \( y^{(n)} \) is the solution of the ODE

\[
d y^{(n)}(t) = V(y^{(n)}(t)) dx^{(n)}(t), \quad y^{(n)}(0) = y^{(n)}_0
\]

Then, \( y^{(n)} \) converges in uniform topology to a unique limit \( y \) in \( C([0, T], \mathbb{R}^d) \), i.e., \( \lim_{n \to \infty} \|y^{(n)} - y\|_{L^\infty([0, T], \mathbb{R}^d)} = 0 \).

In \cite{FV10}, \( y \) in the above theorem is called the solution of the RDE (rough differential equation)

\[
d y(t) = V(y(t)) dx(t), \quad y(0) = y_0,
\]

and written \( y = \pi_{(y)}(0, y_0; x) \). Then we have the following stronger result.

Theorem 7.4. (Existence and uniqueness of full RDE solution; step-2 case of \cite{FV10} Theorem 10.36, p.242] with \cite{FV10} Theorem 10.38, p.246]) Let \( d, e \in \mathbb{N}, \alpha_1 \in (1/3, 1/2] \), and assume (i)-(iii) in Theorem 7.3, and that \( y^{(0)} = (1, y^{(0)}_0, y^{(0)}_n) \in C^{2}(\mathbb{R}^e) \) is a sequence converging to some \( y_0 \). Then, \( y^{(0)} \cap \text{lift}(\pi_{(y)}(0, y^{(0)}_n; x_n)) \) converges in uniform topology to a unique limit \( y \) in \( C([0, T], \mathbb{R}^d) \), i.e.,

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} d_{CC}(y^{(n)}(t), y(t)) = 0.
\]

In \cite{FV10}, \( y \) in the above theorem is called the solution of the full RDE

\[
d y(t) = V(y(t)) dx(t), \quad y(0) = y_0.
\]

and written \( y = \pi_{(y)}(0, y_0; x) \). \( \pi_{(y)} \) is called the Itô–Lyons map.
Theorem 7.5. Suppose \( b' \leq b \) and \( R > 0 \), and let \( \mathcal{V} : \mathbb{R}^c \to L(\mathbb{R}^d, \mathbb{R}^e) \) is in \( \text{Lip}^\gamma(\mathbb{R}^c) \), for \( \gamma > 1/b' \geq 1 \), and let

\[
C_{\leq R}^{b-\text{Hö}l} := \{ x \in C([0, T], G^{(2)}(\mathbb{R}^d)) : d_{C_{\leq R}^{b-\text{Hö}l}}(x, o) \leq R \}.
\]

Then, the map

\[
\mathbb{R}^c \times \left( C_{\leq R}^{b-\text{Hö}l}, d_{C_{\leq R}^{b-\text{Hö}l}} \right) \to \left( C_{\leq R}^{b-\text{Hö}l}(0, T], G^{(2)}(\mathbb{R}^d) \right), d_{C_{\leq R}^{b-\text{Hö}l}}
\]

\[ (y_0, x) \mapsto \pi_{(\mathcal{V})}(0, y_0; x) \]

is uniformly continuous.

Proof. Set \( p = 1/b, p' = 1/b' \) and \( \omega(s, t) = |s - t| \) in [FV10] Corollary 10.40, p.247.

Theorem 7.6. (\( N = 2 \) case of [FV10] Theorem A.12, p.583) Let \( 0 < b < a \), and \( (X_t : t \in [0, T]) \) be a continuous \( G^{(2)}(\mathcal{V}) \)-valued process. Then there exists \( q_0 = q_0(a, b) \) and \( C = C(a, b, T) \) such that the following holds: if

\[
\| d_{C_{\leq R}^{b-\text{Hö}l}}(x_s, x_t) \|_{L^q(\mathcal{F})} \leq M |t - s|^a, \quad \forall s, t \in [0, T]
\]

holds for some \( q \geq q_0 \), then we also have

\[
\| d_{C_{\leq R}^{b-\text{Hö}l}}(X_s, o) \|_{L^q(\mathcal{F})} \leq CM
\]

Theorem 7.7. (Kolmogorov \( L^q \) convergence condition for rough paths [FV10, Proposition A.15, p.587]) Let \( x^{(n)} = (1, x_t^{(n)}, x_t^{(n)}) \) (\( n \in \mathbb{N} \)) and \( x^{(\infty)} = (1, x_t^{(\infty)}, x_t^{(\infty)}) \) be continuous \( G^{(2)}(\mathbb{R}^d) \)-valued processes defined on \( [0, T] \). Let \( q \in [1, \infty) \) and assume that

\[
\lim_{n \to \infty} \| d_{C_{\leq R}^{b-\text{Hö}l}}(x_t^{(n)}, x_t^{(\infty)}) \|_{L^q(\mathcal{F})} = 0 \quad \forall t \in [0, T], \tag{7.6}
\]

\[
\sup_{1 \leq a \leq \infty} \| d_{C_{\leq R}^{a-\text{Hö}l}}(x_t^{(n)}, x_t^{(\infty)}) \|_{L^q(\mathcal{F})} < \infty, \tag{7.7}
\]

then for \( \alpha' \in (0, \alpha) \),

\[
\lim_{n \to \infty} \| d_{C_{\leq R}^{b-\text{Hö}l}}(x_t^{(n)}, x_t^{(\infty)}) \|_{L^q(\mathcal{F})} = 0.
\]

Note that (7.6) is equivalent to

\[
\lim_{n} \| x_t^{(n)} - x_t^{(\infty)} \|_{L^q(\mathcal{F}, \mathbb{R})} = \lim_{n} \| x_t^{(n)} - x_t^{(\infty)} \|_{L^q(\mathcal{F}, \mathbb{R} \otimes \mathbb{G})} = 0, \quad \forall t \in [0, T],
\]

and (7.7) is equivalent to

\[
\sup_{n} \left\| \| x_t^{(n)} \|_{C_{\leq R}^{b-\text{Hö}l}} \right\|_{L^q(\mathcal{F})} < \infty, \quad \sup_{n} \left\| \| x_t^{(\infty)} \|_{C_{\leq R}^{2a-\text{Hö}l}} \right\|_{L^q(\mathcal{F})} < \infty.
\]
8 Estimate for $X_{s,t}^{(j)}$

Recall the definitions of $X^{(j)}$ and $X$ (Eqs. (6.1), (6.2), (6.3)), and set

$$X_{s,t} := X_t - X_s, \quad X_{s,t}^{(j)} := X_t^{(j)} - X_s^{(j)}. \quad (8.1)$$

In this section we prove an estimate for $X_{s,t}^{(j)}$ (Prop. 8.5).

**Lemma 8.1.** For $D \subset \mathbb{R}^2$, let $1_D : \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of $D$. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $a := x_2 - x_1 > 0$, $b := y_2 - y_1 > 0$, and $f := 1_{[x_1, x_2] \times [y_1, y_2]}$. Suppose $p \in [1, \infty)$, $s > 0$ and $1 - sp > 0$ i.e. $s \in (0, 1/p)$. Then

$$\|f\|_{B_{p,\infty}(\mathbb{R}^2)} \simeq \|f\|_{B_{p,\infty}(\mathbb{R}^2)}' \leq (ab)^{1/p} \left(1 + 4^{1/p} \min \{a, b\}^{1-s}\right).$$

Especially if $a \leq b \wedge 1$,

$$\|f\|_{B_{p,\infty}(\mathbb{R}^2)} \simeq \|f\|_{B_{p,\infty}(\mathbb{R}^2)}' \leq 5a^{1/p - 1}b^{1/p}.$$

**Proof.** By Lemma 2.3 and some elementary (but rather lengthy) calculations. □

**Lemma 8.2.** Let $D \subset \mathbb{R}^2$ be a bounded domain s.t. the boundary $\partial D$ is a curve with a finite length $\text{ leng}(\partial D) \in (0, \infty)$. Then $1_D \in B_{2,\infty}^s(\mathbb{R}^2)$ for all $s \in (0, 1/2]$. More precisely,

$$\|1_D\|_{B_{2,\infty}^s(\mathbb{R}^2)} \leq \text{ diam}(\partial D)\text{ diam}(D)^{1/2-s} \quad (8.2)$$

where $\text{ diam}(D)$ is the diameter of $D$. Hence there exists $C = C(s) > 0$ such that

$$\|1_D\|_{B_{2,\infty}^s(\mathbb{R}^2)} \leq C(s) \left(\text{ diam}(D) + \text{ leng}(\partial D)\text{ diam}(D)^{1/2-s}\right) \quad (8.3)$$

**Proof.** Let $L = \text{ leng}(\partial D)$ and $\delta := \text{ diam}(D)$. Let $\text{ Leb}(A)$ denote the Lebesgue measure of $A \subset \mathbb{R}^2$. Then

$$\|1_D(\cdot + x) - 1_D\|_{L^2}^2 = \int_{\mathbb{R}^2} |1_D(y + x) - 1_D(y)|^2 \, dy$$

$$= \int_{D \triangle (D + x)} |1_D(y + x) - 1_D(y)|^2 \, dy \leq \text{ Leb}(D \triangle (D + x)) \quad (8.4)$$

If $|x| > \delta$, we see $\text{ Leb}(D \triangle (D + x)) = 2\text{ Leb}(D)$, $\text{ Leb}(D \cap (D + x)) = 0$, and if $|x| \leq \delta$, we have

$$\text{ Leb}(D \triangle (D + x)) \leq \text{ Leb}\left(\bigcup_{t \in [0, 1]} (\partial D + tx)\right) \leq L|x|$$

Hence we have

$$\|1_D(\cdot + x) - 1_D\|_{L^2}^2 \leq 2\text{ Leb}(D) \quad \text{ if } |x| > \delta$$

$$\|1_D(\cdot + x) - 1_D\|_{L^2}^2 \leq L|x| \quad \text{ if } |x| \leq \delta.$$
Estimate for $X_{s,t}^{(j)}$

Hence if $|x| \leq \delta$,

$$\sup_{x \in \mathbb{R}^2, |x| \leq \delta} \| 1_D(x + x) - 1_D \|_{L^2(\mathbb{R}^2)} |x|^{-s}$$

$$\leq \sup_{x \in \mathbb{R}^2, |x| \leq \delta} (L |x|)^{1/2} |x|^{-s}$$

$$= (L^2 \delta^{1-2s})^{1/2}$$

and if $|x| > \delta$,

$$\sup_{x \in \mathbb{R}^2, |x| > \delta} \| 1_D(x + x) - 1_D \|_{L^2(\mathbb{R}^2)} |x|^{-s}$$

$$\leq \sup_{x \in \mathbb{R}^2, |x| > \delta} (2 \text{Leb}(D))^{1/2} |x|^{-s}$$

$$= 2^{1/2} \text{Leb}(D)^{1/2} \delta^{-s}$$

$$\leq 2^{-1/2} \pi^{1/2} \delta^{1-s} \text{ (using Leb}(D) \leq \pi (\delta/2)^2)$$

$$\leq 2^{-1} \pi^{1/2} \delta^{1/2-s} L^{1/2} \text{ (using } \delta \leq 2^{-1} L)$$

$$\leq \delta^{1/2-s} L^{1/2} = (L^2 \delta^{1-2s})^{1/2}$$

Thus we have (8.2). Moreover, from

$$\| 1_D \|_{L^2} \leq \text{Leb}(D)^{1/2} \leq \pi^{1/2} (\delta/2)$$

we have (8.3). \hfill \Box

**Lemma 8.3.** Let $c \in C_{\text{Rot}}$, $s \in (0, 1/2)$, and $0 \leq s < t \leq 1$. Then, $\mathcal{E}_c 1_{[s,t]} \in L^2_{2,\infty}(\mathbb{R}^2)$. Moreover, when $s$ is sufficiently near to $t$,

$$\| \mathcal{E}_c 1_{[s,t]} \|_{L^{2,\infty}} \lesssim (t - s)^{1/2-s}.$$  

In other words,

$$\| \Delta_j \mathcal{E}_c 1_{[s,t]} \|_{L^{2,\infty}} \lesssim (t - s)^{1/2-s} 2^{-sj}$$

**Proof.** Suppose $c_2(s) \leq c_2(t)$. (The case where $c_2(s) \geq c_2(t)$ can be considered similarly.) Let

$$D_n := \{ x \in \mathbb{R}^2; (\mathcal{E}_c 1_{[s,t]}(x)) = n \} \subset \mathbb{R}^2, \quad n \in \mathbb{Z}$$

then by Lemma 6.2 we see

$$\mathcal{E}_c 1_{[s,t]} = \sum_{n=-\text{Rot}(c)}^{\text{Rot}(c)} n 1_{D_n}.$$  

Define the intervals

$$I_i := \left[ \inf_{\tau \in [s,t]} c_i(\tau), \sup_{\tau \in [s,t]} c_i(\tau) \right] \subset \mathbb{R}, \quad i = 1, 2$$

and the rectangles $R_1, R_2$ in $\mathbb{R}^2$ by
We also see that $\partial D$.

Then we can check the following:

$$\text{supp} (\mathcal{E}_\epsilon 1_{[s,t]}) \subset R_1 \cup R_2, \quad R_1 \subset D_1, \quad n \neq 1 \Rightarrow D_n \subset R_2.$$  

Suppose $n \neq 1$. Then we see

$$\text{diam} D_n \leq \text{diam} R_2 \leq t - s.$$  

We also see that $\partial D_n$ ($n \neq 1$) consists of curve segments of $c$ on $[s,t]$, i.e. $\partial D_n \subset c([s,t]) \subset \mathbb{R}^2$, and hence we have

$$\text{leng}(\partial D_n) \leq t - s.$$  

Hence by Lemma 8.2 we have

$$\|1_{D_n}\|_{B^s_{2,\infty}} \lesssim t - s + (t - s)(t - s)^{1/2 - s} \simeq t - s.$$  

On the other hand we see

$$\text{diam}(D_1 \cap R_2) \lesssim t - s, \quad \text{leng}(\partial(D_1 \cap R_2)) \lesssim t - s$$

hence again by Lemma 8.2 we have

$$\|1_{D_1 \cap R_2}\|_{B^s_{2,\infty}} \lesssim t - s.$$  

Thus by Lemma 8.1 with $a := c_2(t) - c_2(s) \lesssim t - s$, $b := \inf_{\tau \in [s,t]} c_1(\tau)$, we have

$$\|1_{R_1}\|_{B^s_{2,\infty}} \leq C_1(s) a^{1/2 - s} b^{1/2} \leq C_1(s) a^{1/2 - s} \left( \sup_{\tau \in [0,1]} c_1(\tau) \right)^{1/2} \leq C_2(s, c)(t - s)^{1/2 - s},$$

when $s \approx t$. Hence, since $D_4 = R_1 \cup (D_1 \cap R_2),

$$\|1_{D_4}\|_{B^s_{2,\infty}} \leq \|1_{R_1}\|_{B^s_{2,\infty}} + \|1_{D_1 \cap R_2}\|_{B^s_{2,\infty}} \lesssim (t - s)^{1/2 - s} + t - s \simeq (t - s)^{1/2 - s}.$$  

Thus we have

$$\|\mathcal{E}_\epsilon 1_{[s,t]}\|_{B^s_{2,\infty}} \leq \sum_{|n| \leq \text{Rot}(c)} \|n1_{D_n}\|_{B^s_{2,\infty}} = \|1_{D_1}\|_{B^s_{2,\infty}} + \sum_{|n| \leq \text{Rot}(c), n \neq 1} |n| \|1_{D_n}\|_{B^s_{2,\infty}}$$

$$\lesssim (t - s)^{1/2 - s} + \sum_{|n| \leq \text{Rot}(c), n \neq 1} |n|(t - s) \simeq (t - s)^{1/2 - s}$$

Recall the definitions of $X^{(i)}$, $X$ (Eqs. 6.1, 6.2, 6.3), and of $X_{s,t}$, $X_{s,t}^{(i)}$ (Eq. 8.1).
Proof. Since $c \in C_{\text{Rot}}$ and $s \in (0, 1/2)$. Then when $s$ and $t$ are sufficiently near,

$$
\left\| X_{s,t} - X_{s,t}^{(j)} \right\|_{L^2(\mathbb{P}, g)} \lesssim (t - s)^{1/2} 2^{-js}
$$

i.e. \( \mathbb{E} \left[ \left| X_{s,t} - X_{s,t}^{(j)} \right|^2 \right] \lesssim (t - s) 2^{-2js} \)

Proof. Since $X_{s,t}^{(j)} \in B_{2,\infty}^2(\mathbb{R}^2)$ and $\|\Delta_j u\|_{L^p(\mathbb{R}^2)} \leq 2^{-js} \|u\|_{B^s_{p,\infty}}$ we obtain from Lemma 8.3,

$$
\left\| X_{s,t} - X_{s,t}^{(j)} \right\|_{L^2(\mathbb{P}, g)} = \| (W, (I - S_j) E_t^c 1_{[s,t]} \|_{L^2(\mathbb{P})} = \| (I - S_j) E_t^c 1_{[s,t]} \|_{L^2(\mathbb{R}^2)} 
$$

$$
\lesssim \| E_t^c 1_{[s,t]} \|_{B^s_{2,\infty}} 2^{-js} \lesssim (t - s)^{1/2} 2^{-js}
$$

9 \hspace{1cm} \text{Estimate for } X_{s,t}^{j}

For $0 \leq s < t \leq 1$ and $j \geq -1$, the $g \otimes g$-valued random variable $X_{s,t}^{(j)}$ by

$$
X_{s,t}^{(j)} = X_{s,t}^{(j)} : = \int_s^t X_{s,r}^{(j)} \otimes dX_{r}^{(j)} = \int_s^t X_{s,r}^{(j)} \otimes \dot{X}_{s,r}^{(j)} \, dr
$$

$$
= \int_s^t \{W, S_j E_t^c 1_{[s,r]} \} \otimes d \{W, S_j E_t^c 1_{[0,r]} \}
$$

so that $X^{(j)} = X_t^{(j)} : = (1, X^{(j)}, X^{(j)}) = \text{sig}(X^{(j)} : [0, 1] \rightarrow G(\mathbb{R})$. Let $X_t^{(j)} := X_0^{(j)}$, then $(1, X_t^{(j)}, X_t^{(j)}) = \text{lift}(X_t^{(j)})$.

Fix an orthonormal basis $e_k$ ($k = 1, \ldots, \dim g$) of $g$, and set

$$
X_{s,t}^{(j)} = \sum_{k,l} x_{s,t}^{(j); k,l} e_k \otimes e_l, \quad x_{s,t}^{(j); k,l} \in \mathbb{R}.
$$
Let
\[ \kappa_j(x) := \langle \hat{\chi}_j(\cdot - x), \hat{\chi}_j(\cdot) \rangle_{L^2(\mathbb{R}^2)}, \]
then we see \( \kappa_j(x - y) = \langle \hat{\chi}_j(\cdot - x), \hat{\chi}_j(\cdot - y) \rangle_{L^2(\mathbb{R}^2)} \) and the following:

**Lemma 9.1.** For all \( j \geq -1 \),
\[ \int_{\mathbb{R}^2} \kappa_j(x)dx = \left( \int_{\mathbb{R}^2} \hat{\chi}_j(x)dx \right)^2 \quad \text{and} \quad \| \kappa_j \|_{L^1(\mathbb{R}^2)} \leq \| \hat{\chi}_0 \|^2_{L^2(\mathbb{R}^2)}. \]

Let
\[ f_t = f^j_{c,t} := \frac{d}{dt} S_j \mathcal{E}_t 1_{[s,t]} \in \mathcal{S}(\mathbb{R}^2). \]  
(9.3)

We see
\[ f^j_{c,t}(x_1, x_2) = \hat{\epsilon}_2(t) \int_0^{\epsilon_1(t)} \hat{\chi}_j(x_1 - \xi, x_2 - \epsilon_2(t))d\xi, \]
and
\[ \langle f^j_{c,t_1}, f^j_{c,t_2} \rangle = \hat{\epsilon}_2(t_1)\hat{\epsilon}_2(t_2) \int_0^{\epsilon_1(t_1)} d\xi \int_0^{\epsilon_1(t_2)} d\xi' \kappa_j(\xi - \xi', \epsilon_2(t_1) - \epsilon_2(t_2)). \]  
(9.4)

**Lemma 9.2.** For any \( \epsilon \in \mathcal{C}_{\text{box}} \), there exists \( C = C(\epsilon) > 0 \) such that for all \( j \geq -1 \) and \( r_1, r_2 \geq s \),

\[ \| \langle S_j \mathcal{E}_t 1_{[s,r_1]}, f^j_{c,r_2} \rangle \| \leq C, \]

and hence
\[ \left| \int_s^t dr \langle S_j \mathcal{E}_t 1_{[s,r]}, f^j_{c,r} \rangle \right| \leq (t - s)C. \]

**Proof.** Let
\[ D_{[t,t+\epsilon]} := \{ x \in \mathbb{R}^2; \tau \in [t, t + \epsilon], \epsilon_2(\tau) = x_2, \tau \in [t, t + \epsilon], 0 \leq x_1 \leq \epsilon_1(\tau) \} \]
then we see
\[ \text{Leb} (D_{[t,t+\epsilon]}) \approx \epsilon_1(t)\hat{\epsilon}_2(t)\epsilon \]
for \( \epsilon \approx 0 \). Since \( f_t = 0 \) if \( \hat{\epsilon}_2(t) = 0 \), we suppose \( \hat{\epsilon}_2(t) > 0 \) without loss of generality \( (\hat{\epsilon}_2(t) < 0 \text{ case is similar}) \). Then we see for sufficiently small \( \epsilon > 0 \),

\[ \mathcal{E}_c 1_{[t,t+\epsilon]} = 1_{D_{[t,t+\epsilon]}}. \]

Hence, using \( S_j u = \hat{\chi}_j * u \) and the inequality \( \| \phi * \psi \|_{L^q} \leq \| \phi \|_{L^1} \| \psi \|_{L^q} \) \( (q \in [1, \infty]) \), we have
\[ \| f_t \|_{L^1} = \left\| \lim_{\epsilon \to 0} \epsilon^{-1} S_j \mathcal{E}_c 1_{[t,t+\epsilon]} \right\|_{L^1} = \lim_{\epsilon \to 0} \epsilon^{-1} \| S_j \mathcal{E}_c 1_{[t,t+\epsilon]} \|_{L^1} \]
\[ \leq \lim_{\epsilon \to 0} \epsilon^{-1} \| \hat{\chi}_j \|_{L^1} \| \mathcal{E}_c 1_{[t,t+\epsilon]} \|_{L^1} = \| \hat{\chi}_j \|_{L^1} \lim_{\epsilon \to 0} \epsilon^{-1} \| 1_{D_{[t,t+\epsilon]}} \|_{L^1} \]
\[ = \| \hat{\chi}_j \|_{L^1} \lim_{\epsilon \to 0} \epsilon^{-1} \text{Leb} (D_{[t,t+\epsilon]}) = \| \hat{\chi}_0 \|_{L^1} \| \epsilon_1(t)\hat{\epsilon}_2(t) \| \]
\[ \leq C \| \hat{\chi}_0 \|_{L^1}, \quad C := \sup_{r \in [0,1]} |\epsilon_1(r)| \sup_{r \in [0,1]} |\epsilon_2(r)|. \]
Thus
\[ |\langle S_j \mathcal{E}_t 1_{[s, r]}, f_{r_2} \rangle| \leq \| S_j \mathcal{E}_t 1_{[s, r_2]} \|_{L^\infty} \| f_{r_2} \|_{L^1} \leq C \| \hat{X}_j \|_{L^2} \| \mathcal{E}_t 1_{[s, r_1]} \|_{L^\infty} \| \hat{X}_0 \|_{L^1} \]
\[ = C \| \hat{X}_0 \|_{L^1}^2 \| \mathcal{E}_t 1_{[s, r_1]} \|_{L^\infty} \leq C \| \hat{X}_0 \|_{L^1}^2 \text{Rot}(c). \]

\[ \square \]

**Proposition 9.3.** For any \( \epsilon \in \mathcal{C}_\infty \), there exists \( C = C(\epsilon) > 0 \) such that for all \( j \geq -1 \), and \( r_1 \in [0, 1] \),
\[ \int_0^1 |\langle f_{t, r_1}^j, f_{t, r_2}^j \rangle| \, dr_2 \leq C. \]

\[ \text{Proof. Let } H_{t, j, r_1}(x_1, x_2) = \int_{[0, t]} \kappa_j ((\xi_1, \xi_2(r_1)) - (x_1, x_2)) \, d\xi_1. \]

We easily check \( \| H_{t, j, r_1} \|_{L_j^1(\mathbb{R}^2)} = \nu_j(r_1) \| \kappa_j \|_{L_j^1(\mathbb{R}^2)} \), hence By Prop. 9.1, we have
\[ \| H_{t, j, r_1} \|_{L_j^1(\mathbb{R}^2)} \leq \nu_j(r_1) \| \hat{X}_0 \|_{L^1}. \] (9.5)

Let \( s_\epsilon(t) = \text{sgn}(\hat{\epsilon}_2(t)) \), i.e.
\[ s_\epsilon(t) := \hat{\epsilon}_2(t) / |\hat{\epsilon}_2(t)|, \quad 0 \leq t \leq 1 \]
where \( s_\epsilon(t) := 0 \) if \( \hat{\epsilon}_2(t) = 0 \). Then by (9.4) and (9.5) we have,
\[ \int_{\mathbb{R}} |\hat{\epsilon}_2(r_1) \hat{\epsilon}_2(r_2) \int_{0}^{\nu_j(r_1)} \kappa_j ((\xi_1, \xi_2(r_1)) - (\xi_2, \xi_2(r_2))) \, d\xi_2 \, d\xi_1 \, dr_2 \]
\[ \leq |\hat{\epsilon}_2(r_1)| \int_{\mathbb{R}} \hat{\epsilon}_2(r_2) s_\epsilon(r_2) \]
\[ \times \int_{0}^{\nu_j(r_1)} \int_{0}^{\nu_j(r_1)} |\kappa_j ((\xi_1, \xi_2(r_1)) - (\xi_2, \xi_2(r_2)))| \, d\xi_1 \, d\xi_2 \, dr_2 \]
\[ = |\hat{\epsilon}_2(r_1)| \| \hat{\epsilon}_2 \|_{L_j^1(\mathbb{R}^2)} = |\hat{\epsilon}_2(r_1)| \langle H_{t, j, r_1}, \mathcal{E}_s s_\epsilon \rangle \]
\[ \leq |\hat{\epsilon}_2(r_1)| \| H_{t, j, r_1} \|_{L_j^1} \| \mathcal{E}_s s_\epsilon \|_{L^\infty} \]
\[ \leq |\hat{\epsilon}_2(r_1)| \| H_{t, j, r_1} \|_{L_j^1} \| \mathcal{E}_s \|_{L^\infty} \]
\[ \leq C(\epsilon) \]

\[ \square \]

**Lemma 9.4.** For any \( \epsilon \in \mathcal{C}_\text{Rot} \), there exists \( C = C(\epsilon) > 0 \) such that for all \( j \geq -1 \) and \( r_1, r_2 \in [s, t] \),
\[ |\langle S_j \mathcal{E}_t 1_{[s, r_1]}, S_j \mathcal{E}_t 1_{[s, r_2]} \rangle| \leq C(t - s). \]
Proof. We see

\[ \| \xi_s 1_{[s,r_2]} \|_{L^1} \leq \text{Rot}(c) \text{Leb}(\text{supp} \xi_s 1_{[s,r_2]}) \]
\[ \leq \text{Rot}(c) \sup_{t_1, t_2 \in [s,t]} |c_2(t_1) - c_2(t_2)| \sup_{t_1 \in [s,t]} c_1(t_1). \]

Hence, using \( f \ast g \|_{L^2} \leq \| f \|_{L^2} \| g \|_{L^2} \) (\( q \in [0, \infty) \)) and \( S_j u = \tilde{X}_j \ast u \) we have

\[ |\langle S_j \xi_s 1_{[s,r_1]}, S_j \xi_s 1_{[s,r_2]} \rangle| \leq \| S_j \xi_s 1_{[s,r_1]} \|_{L^\infty} \| S_j \xi_s 1_{[s,r_2]} \|_{L^1} \]
\[ \leq \| S_j \xi_s 1_{[s,r_1]} \|_{L^\infty} \| \tilde{X}_j \|_{L^1} \| \xi_s 1_{[s,r_2]} \|_{L^1} \]
\[ \leq \| \tilde{X}_j \|_{L^1} \text{Rot}(c) \| \tilde{X}/L \|_{L^1} \| \xi_s 1_{[s,r_2]} \|_{L^1} \]
\[ \leq \| \tilde{X}_0 \|_{L^1} \text{Rot}(c) \sup_{t_1, t_2 \in [s,t]} |c_2(t_1) - c_2(t_2)| \sup_{t_1 \in [s,t]} c_1(t_1) \]
\[ \leq C_1 \sup_{t_1, t_2 \in [s,t]} |c_2(t_1) - c_2(t_2)| \]
\[ \leq C_1 \sup_{t_1 \in [s,t]} |c_2(t_1) - c_2(t_2)| \]
\[ \leq C_1 \sup_{t_1 \in [s,t]} (t_1 - t) \leq C_2 (t - s). \]

Proposition 9.5. For any \( \epsilon \in \mathfrak{C}_\infty \) and \( p \in [1, \infty) \), there exists \( C = C(\epsilon, p) > 0 \) such that for all \( j \geq -1 \) and \( 0 \leq s < t \leq 1 \),

\[ \| X_s^{(j)} \|_{L^p(\mathbb{P}, \mathcal{G})} \leq C(t - s). \]

Proof. Since \( X_s^{(j)} \) is Gaussian, all \( L^p \)-norms \((p \in [1, \infty))\) for \( X_s^{(j)} \) are equivalent by \[\text{Jan77}]\ Theorem 3.50 p.39]. Hence it is enough to show the bound for \( p = 2 \). Using the equation

\[ \mathbb{E}[ABCD] = \mathbb{E}[AB] \mathbb{E}[CD] + \mathbb{E}[AC] \mathbb{E}[BD] + \mathbb{E}[AD] \mathbb{E}[BC] \] (9.6)

for any Gaussian random variables \( A, B, C, D \), we have

\[ \mathbb{E} \left[ |X_s^{(j)}|^{2k} \right]^2 = \mathbb{E} \left[ \left| \int_s^t dr \langle W, S_j \xi_s 1_{[s,r_1]} \rangle^k \langle W, f_r \rangle^l \right|^2 \right] \]
\[ = \int_s^t \int_s^t \mathbb{E} \left[ \left| \langle W, S_j \xi_s 1_{[s,r_1]} \rangle^k \langle W, f_r \rangle^l \right|^2 \right] dr_1 dr_2 \]
\[ = \int_s^t \int_s^t \left( \delta_{kl} \langle S_j \xi_s 1_{[s,r_1]}, f_r \rangle \delta_{kl} \langle S_j \xi_s 1_{[s,r_2]}, f_r \rangle \right) dr_2 dr_1 \]
\[ + \langle S_j \xi_s 1_{[s,r_1]}, S_j \xi_s 1_{[s,r_2]} \rangle \langle f_{r_1}, f_{r_2} \rangle \]
\[ + \delta_{kl} \langle S_j \xi_s 1_{[s,r_1]}, f_r \rangle \delta_{kl} \langle f_{r_1}, S_j \xi_s 1_{[s,r_2]} \rangle \]
\[ = \delta_{kl} \left( \int_s^t dr \langle S_j \xi_s 1_{[s,r]}, f_r \rangle \right)^2 \]
\[ + \int_s^t dr_1 \int_s^t dr_2 \langle S_j \xi_s 1_{[s,r_1]}, S_j \xi_s 1_{[s,r_2]} \rangle \langle f_{r_1}, f_{r_2} \rangle \]
\[ + \delta_{kl} \int_s^t \int_s^t \langle S_j \xi_s 1_{[s,r_1]}, f_r \rangle \langle S_j \xi_s 1_{[s,r_2]}, f_{r_1} \rangle dr_2 dr_1 \]
\[ =: (I) + (II) + (III). \]
By Lemma 9.2 we find 

(I) \leq C_1 \delta_{kl}(t-s)^2

By Lemma 9.4 we find

(II) = \int_s^t \int_s^t \langle S_j \mathcal{E}_\xi 1_{[s,r]}, S_j \mathcal{E}_\xi 1_{[s,r]} \rangle \, dr \, dr_1

\leq \int_s^t \int_s^t C_2(t-s) \langle f_{r_1}, f_{r_2} \rangle \, dr \, dr_1

\leq \int_s^t \int_s^t C_2(t-s) \int_\mathbb{R} \langle f_{r_1}, f_{r_2} \rangle \, dr \, dr_1

Hence by Prop 9.3 we have

(II) \leq C_3(t-s)^2

By Lemma 9.2 we find

(III) \leq \delta_{kl} \int_s^t \int_s^t C_5 \, dr \, dr_1 = \delta_{kl} C_4(t-s)^2

Thus we have

\mathbb{E} \left[ \left| X_{s,t}^{(j),k,l} \right|^2 \right] = (I) + (II) + (III) \leq C_5(t-s)^2.

Notice the following properties of delta functions:

**Lemma 9.6.** Let \( \delta \in \mathcal{D}'(\mathbb{R}^2) \) denote the Dirac delta function, and suppose that \( D \subset \mathbb{R}^2 \) is bounded and measurable. Then

(i) If \( 0 \in D^c \), \( \lim_{j,j',r,\to\infty} \langle S_j 1_D, S_j' \delta \rangle = 1. \)

(ii) If \( 0 \in (\mathbb{R}^2 \setminus D)^c \), \( \lim_{j,j',r,\to\infty} \langle S_j 1_D, S_j' \delta \rangle = 0. \)

(iii) If \( 0 \in \partial D \) and \( \partial D \) is a smooth curve on some neighborhood of 0, \( \lim_{j,j',r,\to\infty} \langle S_j 1_D, S_j' \delta \rangle = \frac{1}{2}. \)

**Proposition 9.7.** For each \( \epsilon \in \mathcal{C}_\infty \) and \( s,t \in [0,1] \), \( (X_{c,s,t}^{(j)})_{j=-1}^p \) is Cauchy in \( L^p(\mathbb{P}, g) \) for any \( p \in [1, \infty] \), i.e.

\( \lim_{j,j',r,\to\infty} \left\| X_{c,s,t}^{(j)} - X_{c,s,t}^{(j')} \right\|_{L^p(\mathbb{P}, g)} = 0. \)

**Proof.** This result follows immediately from Lemmas 9.8 9.10 and 9.11 below.

**Lemma 9.8.** For any \( \epsilon \in \mathcal{C} \),

\[ \left\| X_{c,s,t}^{(j')} - X_{c,s,t}^{(j)} \right\|_{L^2(\mathbb{P})} \leq N_1 + N_2 \]

where

\[ N_1 := \left\| \int_s^t \langle W, S_j \mathcal{E}_\xi 1_{[s,r]} \rangle \, dr \right\|_{L^2(\mathbb{P})} \quad (9.7) \]

\[ N_2 := \left\| \int_s^t \langle W, f_{c,r} \rangle \, dr \right\|_{L^2(\mathbb{P})} \quad (9.8) \]
Estimate for $\mathcal{X}_{s,t}$

**Proof.** By the definition (9.1) of $\mathcal{X}^{(j)}$, we see

$$\mathcal{X}^{(j)}_{s,t} = \int_s^t \langle W, S_j \mathcal{E}_\xi 1_{[s,r]} \rangle \otimes \langle W, f^j_r \rangle \, dr,$$

and hence the bound easily follows from (9.6).

Let

$$\delta_{\varepsilon,t} := \frac{d}{dt} \mathcal{E}_\xi 1_{[0,t]} = \dot{c}_2(t) \int_0^{\varepsilon(t)} \delta_{\xi, \varepsilon(t)} d\xi \in \mathcal{S}'(\mathbb{R}^2). \quad (9.9)$$

where

$$\delta_x(y) := \delta(y - x), \quad x, y \in \mathbb{R}^2.$$

$$S_{j,j'} := S_j - S_j = \sum_{i=j}^{j-1} \Delta_i. \quad (9.10)$$

$$\chi_{j,j'} := \chi_{j'} - \chi_j = \sum_{i=j}^{j-1} \rho_i. \quad (9.11)$$

We see $f_t \equiv f^j_t = S_j \delta_{\varepsilon,t}$.

**Lemma 9.9.** For any $c \in \mathfrak{C}$,

$$N^2 = I^2_1 + I_2 + I_3$$

where

$$I_1 := \int_s^t \delta_{kl} \langle S_{j'} \mathcal{E}_\xi 1_{[s,r]}, S_{j,j'} \delta_{\varepsilon,r} \rangle \, dr.$$  \hspace{1cm} (9.12)

$$I_2 := \int_s^t \int_s^t \langle S_{j'} \mathcal{E}_\xi 1_{[s,r]}, S_{j,j'} \delta_{\varepsilon,r} \rangle \langle S_{j,j'} \delta_{\varepsilon,r}, S_{j,j'} \delta_{\varepsilon,r} \rangle \, dr' \, dr.$$  \hspace{1cm} (9.13)

$$I_3 := \delta_{kl} \int_s^t \int_s^t \langle S_{j'} \mathcal{E}_\xi 1_{[s,r]}, S_{j,j'} \delta_{\varepsilon,r} \rangle \langle S_{j,j'} \delta_{\varepsilon,r}, S_{j,j'} \delta_{\varepsilon,r} \rangle \, dr' \, dr.$$  \hspace{1cm} (9.14)

**Proof.** By a straightforward calculation, using (9.9).

**Lemma 9.10.** For any $c \in \mathfrak{C}_{\text{Rot}}$,

$$\lim_{j,j' \to \infty} \int_s^t \langle S_{j'} \mathcal{E}_\xi 1_{[s,r]}, S_{j,j'} \delta_{\varepsilon,r} \rangle \, dr = 0.$$  \hspace{1cm} (9.15)

Especially, $I_1 := \int_s^t \delta_{kl} \langle S_{j'} \mathcal{E}_\xi 1_{[s,r]}, S_{j,j'} \delta_{\varepsilon,r} \rangle \, dr$ is Cauchy in $j, j'$, i.e.

$$\lim_{j,j' \to \infty} I_1 = 0.$$
Estimate for $X^i_{s,t}$

Proof. By Lemma 9.6 and $\delta_{c,t} = \dot{\varepsilon}_2(t) \int_0^{\varepsilon_1(t)} \delta_{r,\varepsilon_2(t)} d\xi$, we see

$$\lim_{j', j''} \langle S_{j'} \mathcal{E}_1, S_{j''} \delta_{c,t} \rangle = \frac{1}{2} \dot{\varepsilon}_2(r) \epsilon_1(r).$$

Hence by Lemma 9.2 and the dominated convergence,

$$\lim_{j', j''} \int_0^t \langle S_{j'} \mathcal{E}_1, S_{j''} \delta_{c,t} \rangle dr = \frac{1}{2} \int_0^t \dot{\varepsilon}_2(r) \epsilon_1(r) dr$$

and hence (9.15) holds.

Lemma 9.11. Define $I_2$ by (9.14). Then for any $\epsilon \in \mathbb{C}$, $\lim_{j', j'' \to \infty} I_2 = 0$.

Proof. Suppose $j < j'$. Let

$$R_{j,j'}(x) := \langle \tilde{X}_{j,j'}, \tilde{X}_{j,j'} \rangle, \quad x \in \mathbb{R}^2.$$

Then we have

$$\langle S_{j,j'} \delta_{c,r}, S_{j,j'} \delta_{c,r'} \rangle$$

$$= \left\langle S_{j,j'} \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \delta_{r_1,\varepsilon_2(r)} dx_1, S_{j,j'} \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} \delta_{r_1',\varepsilon_2(r')} dx_1' \right\rangle$$

$$= \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \delta_{r_1,\varepsilon_2(r)} dx_1 \int_0^{\varepsilon_1(r')} \langle S_{j,j'} \delta_{r_1,\varepsilon_2(r)}, S_{j,j'} \delta_{r_1',\varepsilon_2(r')} \rangle dx_1 dx_1'$$

$$= \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \delta_{r_1,\varepsilon_2(r)} dx_1 \int_0^{\varepsilon_1(r')} \langle S_{j,j'} \delta_{r_1,\varepsilon_2(r)}, S_{j,j'} \delta_{r_1',\varepsilon_2(r')} \rangle dx_1 dx_1'$$

$$= \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \delta_{r_1,\varepsilon_2(r)} dx_1 \int_0^{\varepsilon_1(r')} \langle S_{j,j'} \delta_{r_1,\varepsilon_2(r)} - \langle x_1, \epsilon_2(r) \rangle + \langle x_1, \epsilon_2(r) \rangle \rangle$$

$$= \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \delta_{r_1,\varepsilon_2(r)} dx_1 R_{j,j'} \left( -\langle x_1', \epsilon_2(r') \rangle + \langle x_1, \epsilon_2(r) \rangle \right) dx_1 dx_1'$$

Let

$$F_{j,j',r'}(r) := \left\langle S_{j'} \mathcal{E}_1, S_{j'} \mathcal{E}_1 \right\rangle,$$

$$R_{j,j'}'(x) := R_{j,j'} \left( -\langle x_1', \epsilon_2(r') \rangle + \langle x_1, \epsilon_2(r) \rangle \right).$$

Then from (9.14) we have

$$I_2 = \int_s^t \int_s^t F_{j,j',r'}(r) \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} R_{j,j'}'(x_1, \epsilon_2(r)) dx_1 dx_1' dr' dr$$

$$= \int_s^t \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} \int_s^t \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} F_{j,j',r'}(r) R_{j,j'}'(x_1, \epsilon_2(r)) dx_1 dx_1' dr' dr$$

$$= \int_s^t \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} \dot{\varepsilon}_2(r) \int_0^{\varepsilon_1(r)} F_{j,j',r'}(r) R_{j,j'}'(x_1, \epsilon_2(r)) dx_1 dx_1' dr'$$

$$= \int_s^t \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} \bigg\langle R_{j,j'}, \mathcal{E}_4 F_{j,j',r'} \bigg\rangle dx_1' dr'$$

$$= \left\langle R_{j,j'}, \int_s^t \dot{\varepsilon}_2(r') \int_0^{\varepsilon_1(r')} \bigg\langle \mathcal{E}_4 F_{j,j',r'} \bigg\rangle dx_1' dr' \right\rangle$$
where

\[(\tau_y f)(y) := f(y + x).\]

Notice the fact that for any function \(G \in C(\mathbb{R}^2)\) with compact support, \(\lim_{j,j' \to \infty} \langle R_{j,j'}, G \rangle = 0\) holds. We see that the function

\[\mathbb{R}^2 \ni x \mapsto \int_s^t \hat{c}_2(r') \int_0^{c_1(r')} (\tau_{\xi_1', \xi_2(r')} \mathcal{E}_x F_{j,j'}, r') (x) dx' dr'\]

is continuous, and its support is compact. Thus we have

\[\lim_{j,j' \to \infty} I_2 = 0.\]

\[\Box\]

**Lemma 9.12.** Define \(I_3\) by (9.14). Then for any \(\epsilon \in C_{\text{Rot}}, \lim_{j,j' \to \infty} I_3 = 0.\)

**Proof.** By Lemma 9.2 with \(f^j_{1,r} = S_{j_1} \delta_{c,r}, \) there exists \(C = C(\epsilon) > 0\) such that for all \(j, j' \geq 1\) and \(r, r' \in [s, t],\)

\[|\langle S_{j_1} \mathcal{E}_x 1_{[s,r]}, S_{j_2} \delta_{c, r'} \rangle| \leq \left| \langle S_{j_1} \mathcal{E}_x 1_{[s,r]}, S_{j_2} \delta_{c, r'} \rangle \right| < C.\]

By Lemmas 6.2 and 9.6 we find that for almost all \(r, r' \in [s, t]\) and \(x_1, x_1' \in \mathbb{R},\)

\[\lim_{j', j' \to \infty} \left\langle S_{j_1} \mathcal{E}_x 1_{[s,r]}, S_{j_2} \delta_{c(r'), r'} \right\rangle = 0.\]

Thus, by \(\delta_{c,t} = \hat{c}_2(t) \int_0^{c_1(t)} \delta_{c(t)} d\xi\) and the dominated convergence, we have

\[\lim_{j,j' \to \infty} I_3 = 0.\]

\[\Box\]

**Lemma 9.13.** Define \(N_1\) by (9.7). Then for any \(\epsilon \in C_{\text{Rot}}, \lim_{j,j' \to \infty} N_1 = 0.\)

**Proof.** Follows from Lemmas 9.2, 9.3, 9.11 and 9.12.

\[\Box\]

**Lemma 9.14.** For any \(\epsilon \in C,\)

\[N_2^2 = J_1^2 + J_2 + J_3\]

where

\[J_1 := \delta_{kl} \int_s^t \langle S_{j} \delta_{c,r'}, S_{j,j'} \mathcal{E}_x 1_{[s,r]} \rangle dr,\]

\[J_2 := \int_s^t \int_s^t \langle S_{j} \delta_{c,r}, S_{j,j'} \mathcal{E}_x 1_{[s,r']} \rangle \langle S_{j,j'} \mathcal{E}_x 1_{[s,r]}, S_{j,j'} \mathcal{E}_x 1_{[s,r']} \rangle dr' dr,\]

\[J_3 := \delta_{kl} \int_s^t \int_s^t \langle S_{j} \delta_{c,r}, S_{j,j'} \mathcal{E}_x 1_{[s,r']} \rangle \langle S_{j,j'} \mathcal{E}_x 1_{[s,r]}, S_{j,j'} \delta_{c,r'} \rangle dr' dr.\]
Proof. By a straightforward calculation.

\[ \text{Lemma 9.15.} \text{ For any } \epsilon \in \mathcal{C}_\infty, \text{ there exists } C = C(\epsilon) \text{ such that for all } j \text{ and } 0 \leq s < t \leq 1, \]
\[ \left\| \int_s^t |S_j \delta_{c,r}| \, dr \right\|_{L^2(\mathbb{R}^2)} \leq C. \]

Proof. Let
\[ H_{j,y}(x) := |(S_j \delta_{c}) (y)| = |(S_j \delta) (y - x)|, \quad x, y \in \mathbb{R}^2. \]
\[ s_{[s,t]}(r) := \text{sgn} (\hat{c}_2(r)) 1_{[s,t]}(r) \]

Then we have
\[ \int_s^t |(S_j \delta_{c,r}) (y)| \, dr \leq \int_s^t |\hat{c}_2(r)| \int_0^1 |(S_j \delta_{x_1, \epsilon_2(r)}) (y)| \, dx_1 \, dr 
= \int_0^1 \hat{c}_2(r) \int_0^1 |(S_j \delta_{x_1, \epsilon_2(r)}) (y)| \, dx_1 \, dr 
= \mathcal{E}_c (H_{j,y}, s_{[s,t]}) = \langle H_{j,y}, \mathcal{E}_c s_{[s,t]} \rangle = (H_{j,0} \ast \mathcal{E}_c s_{[s,t]})(y) \]

On the other hand we find
\[ ||\mathcal{E}_c s_{[s,t]}||_{L^2} \leq \text{Leb} (\text{supp} \mathcal{E}_c s_{[s,t]})^{1/2} ||\mathcal{E}_c s_{[s,t]}||_{L^\infty} \]
\[ \leq \text{Leb} (\text{supp} \mathcal{E}_c s_{[s,t]})^{1/2} ||\mathcal{E}_c||_{L^\infty} \leq C_1(\epsilon). \]

Thus
\[ \left\| \int_s^t dr |S_j \delta_{c,r}| \right\|_{L^2} \leq ||H_{j,0} \ast \mathcal{E}_c s_{[s,t]}||_{L^2} \]
\[ = ||(S_j \delta) \ast \mathcal{E}_c s_{[s,t]}||_{L^2} \leq ||S_j \delta||_{L^1} ||\mathcal{E}_c s_{[s,t]}||_{L^2} \]
\[ = ||S_0 \delta||_{L^1} ||\mathcal{E}_c s_{[s,t]}||_{L^2} \leq ||S_0 \delta||_{L^1} C_1(\epsilon) \leq C_2(\epsilon). \]

\[ \text{Lemma 9.16.} \text{ Define } N_2 \text{ by (66). Then for any } \epsilon \in \mathcal{C}_\infty, \text{ lim}_{j,j' \to \infty} N_2 = 0. \]

Proof. By Lemma \textbf{9.14}, it suffices to show that
\[ \lim_{j,j'} J_i = 0, \quad i = 1, 2, 3, \]
The proof of \( \lim_{j,j'} J_1 = 0 \) is similar to that of \( \lim_{j,j'} I_1 = 0 \). The proof of \( \lim_{j,j'} J_3 = 0 \) is similar to \( \lim_{j,j'} I_3 = 0 \). We will show \( \lim_{j,j' \to \infty} J_2 = 0 \). By Lemmas \textbf{S2} and \textbf{S3} for any \( s \in (0, 1/2) \) we have
\[ \mathcal{N}_c := \sup_{r \in [s,t]} ||\mathcal{E}_c 1_{[s,t]}||_{B_{2,\infty}(\mathbb{R}^2)} < \infty \]

Thus we have
and hence we find that if \( j \leq j' \),

\[
\| S_{j,j'} E_1 \|_{L^2(\mathbb{R}^2)} \leq \sum_{i=j}^{j'-1} \| \Delta_i E_1 \|_{L^2(\mathbb{R}^2)} \leq \frac{N_c \varepsilon 2^{-s}}{1 - 2^{-s}}
\]

and so

\[
|\langle S_{j,j'} E_1, S_{j,j'} E_1' \rangle | \leq \| S_{j,j'} E_1 \|_{L^2(\mathbb{R}^2)} \| S_{j,j'} E_1' \|_{L^2(\mathbb{R}^2)} \leq C 2^{-2sj}.
\]

Thus we have

\[
|J_2| = \left| \int_s^t \int_s^t \langle S_j \delta_{c,r}, S_j \delta_{c',r'} \rangle \langle S_{j,j'} E_1, S_{j,j'} E_1' \rangle \, dr' \, dr \right|
\leq C 2^{-2sj} \int_s^t \int_s^t \| \langle S_j \delta_{c,r}, S_j \delta_{c',r'} \rangle \| \, dr' \, dr
\leq C 2^{-2sj} \int_s^t \int_s^t \| S_j \delta_{c,r} \| \, dr' \, dr
\leq C 2^{-2sj} \left( \int_s^t \| S_j \delta_{c,r} \| \, dr \right)^2
\leq C 2^{-2sj} \left( \int_s^t \| S_j \delta_{c,r} \| \, dr \right)^2
\]

where the last inequality is by Lemma 10.1. Thus we have \( \lim_{j,j' \to \infty} J_2 = 0 \). This completes the proof.

### 10 Rough path convergence

**Lemma 10.1** (Uniform rough path bounds in \( L^p \)). Let \( c \in C_\infty \), \( q \in [1, \infty) \) and \( \alpha \in (1/3, 1/2) \). Then

\[
\sup_j \| d_{CC, \alpha, \text{Hol}(0,1)}(X^{(j)}, 0) \|_{L^q(P)} < \infty.
\]

**Proof.** Notice that \( d_{CC}(X^{(j)}_t, X^{(j)}_s) \approx |X^{(j)}_t - X^{(j)}_s| + |X^{(j)}_t - X^{(j)}_s - X^{(j)}_t \otimes (X^{(j)}_t - X^{(j)}_s)|^{1/2} \). Because \( (1, X^{(j)}, X^{(j)}) = \text{sig}(X^{(j)}) \) and \( X^{(j)}_0 = 0 \), it follows from Chen’s relation (Theorem 7.1) that \( X^{(j)}_{a,t} = X^{(j)}_t - X^{(j)}_s - X^{(j)}_a \otimes (X^{(j)}_t - X^{(j)}_s) \). Thus we see \( d_{CC}(X^{(j)}_t, X^{(j)}_s) \approx |X^{(j)}_{a,t}|^{1/2} \), and hence
By Prop. 8.5 and Prop. 9.5, we have for all \( j \) that
\[
\|X_{s,t}^{(j)}\|_{L^q(P)} \leq C_1 |t-s|^\beta, \quad \|X_{s,t}^{(j)}\|_{L^q(P)} \leq C_2 |t-s|^{2\beta},
\]
Hence there exists \( C_3 \) such that
\[
\|d_{cc}(X_{s,t}^{(j)}, X_{s,t}^{(j)})\|_{L^p(P)} \leq C_3 |t-s|^\beta \quad \forall j \geq -1, \beta \in (0,1/2), \, q \in [1,\infty),
\]
For \( 0 \leq b < a \), let \( C(a,b,T) \) be of Theorem 7.6 with \( M = C_3 \). Then we see
\[
\|d_{cc,\alpha-Hö}(0,T) (X^{(j)}, 0)\|_{L^p(P)} \leq C(\beta, \alpha, 1) C_3, \quad \forall j \geq -1, \alpha \in (0,\beta).
\]
This completes the proof.

**Lemma 10.2** (pointwise \( L^p \) convergence). For each \( p \in (1,\infty) \) and \( 0 \leq s < t \leq 1 \), \( X_{s,t}^{(j)} = (1, X_{s,t}^{(j)}, X_{s,t}^{(j)}) \) converges to an element \( X_{s,t} = (1, X_{s,t}, X_{s,t}) \) in \( L^p \), that is,
\[
\lim_{j} \|X_{s,t} - X_{s,t}^{(j)}\|_{L^p(P,\mathfrak{g})} = 0.
\]
hold. Equivalently,
\[
\lim_{j \to \infty} \|d_{cc}(X_{s,t}^{(j)}, X_{s,t})\|_{L^p(P)} = 0.
\]

**Proof.** The convergence of \( \lim_{j} X_{s,t}^{(j)} \) in \( L^p(P,\mathfrak{g}) \) follows from Prop. 8.5. The convergence of \( \lim_{j} X_{s,t}^{(j)} \) in \( L^p(P,\mathfrak{g} \otimes \mathfrak{g}) \) follows from Prop. 9.7.

**Theorem 10.3** (rough path convergence in \( L^p \)). Suppose \( \epsilon \in \mathcal{C}_\infty \), \( \eta \in (1/3,1/2) \), and \( p \geq 1 \). Let \( X_{s,t} = \lim_{j} X_{s,t}^{(j)} \) be given by Lemma 10.2 and \( X_t := X_{0,t} = (1, X_t, X_t) \). Then \( X \) is a weak geometric \( \eta \)-Hölder rough path, i.e. \( X \in C^{\eta}\text{-Hö}(\{0,1\}, C^2(\mathfrak{g})) \), and \( X^{(j)} \to X \) in \( C^{\eta}\text{-Hö}(\{0,1\}, C^2(\mathfrak{g})) \) and \( L^p(P) \), i.e.
\[
\lim_{j \to \infty} \|d_{cc,\eta\text{-Hö}}(0,1) (X, X^{(j)})\|_{L^p(P)} = 0.
\]

**Proof.** This immediately follows from Prop. 10.1, Prop. 10.2 and Theorem 7.7.

**Corollary 10.4.** Suppose \( \epsilon \in \mathcal{C}_\infty \), \( \eta \in (1/3,1/2) \). Then if \( n : \mathbb{N} \to \mathbb{N} \) increases rapidly enough,
\[
P \left[ \lim_{k \to \infty} d_{cc,\eta\text{-Hö}}(0,1) (X, X^{(n(k))}) = 0 \right] = 1.
\]
Now the ODE (44) for the $j$th approximate holonomy $\mathcal{W}^{(j)}_{t,A}$ associated with $W^{(j)}$ is written as
\[
d\mathcal{W}^{(j)}_{t,A} = \mathcal{V}(\mathcal{W}^{(j)}_{t,A})dX^{(j)}, \quad \mathcal{W}^{(j)}_{t,A}(0) = 1_G \in G.
\]
Recall that $X^{(j)}$ is expressed by $W^{(j)}$ by (52).

**Theorem 10.5.** For any countable subset $\Gamma \subset \mathcal{C}_\infty$, and $n : \mathbb{N} \to \mathbb{N}$ increasing rapidly enough,
\[
P\left( \forall \gamma \in \Gamma, \mathcal{W}_t^{(\infty)} := \lim_{k \to \infty} \mathcal{W}^{(n(k))}_t (\text{uniform}) \in C([0, 1], G) \right) = 1.
\]
Moreover, for $h \in (1/3, 1/2)$, lift$(\mathcal{W}^{(n(k))}_t)$ converges to $\hat{\mathcal{W}}^{(\infty)}_t = (1, \mathcal{W}^{(\infty)[1]}_t, \mathcal{W}^{(\infty)[2]}_t) \in C^2_{\text{Hö}}([0, 1], G^2(\text{Mat}))$ a.s., where $\mathcal{W}^{(\infty)[1]}_t = \mathcal{W}^{(\infty)}_t$.

That is,
\[
P\left( \forall \gamma \in \Gamma, \lim_{k \to \infty} d_{CC, b, \text{Hö}}([0, 1], \left( \mathcal{W}^{(\infty)}_t, \text{lift}(\mathcal{W}^{(n(k))}_t) \right)) = 0 \right) = 1.
\]

**Proof.** Note that if we let $n_i : \mathbb{N} \to \mathbb{N}$ be increasing for each $i \in \mathbb{N}$, then $n(k) := \max_{1 \leq i \leq n} n_i(k)$ $(k \in \mathbb{N})$ increases more rapidly than each $n_i$. Thus the theorem follows from Theorems 10.3, 10.4, 10.5 and Corollary 10.4. \qed

We call $\mathcal{W}^{(\infty)}_t(1)$ the holonomy-valued random variable (or simply the holonomy variable) along $\gamma \in \text{Lasso} \cap \mathcal{C}_\infty$.

## 11 Wilson loop

The law of Wilson loops in the YM theory on $\mathbb{R}^2$ (with the usual Euclidean metric) is described as follows (e.g., [Lee03]): Let $\mathcal{L}$ be a set of lassos with some regularity condition. Then

(i) The Wilson loop $\mathcal{W}_t(1)$ is independent of $\mathcal{W}_t(1)$ if $\gamma, \gamma' \in \mathcal{L}$ and $D(\gamma) \cap D(\gamma') = \emptyset$.

(ii) The density $\rho$ of the Wilson loop $\mathcal{W}_t(1)$ on $G$ with respect to Haar measure $dg$ is given by $\rho(g) = Q_{t, \text{Leb}(D(\gamma))}(g)$, where $Q_t(x)$ $(t \geq 0)$ denotes the fundamental solution to the heat equation on the group $G$.

In this section we show that holonomy variables $\mathcal{W}^{(\infty)}_t$ given by Theorem 10.5 obey the law the Wilson loops in the YM theory on $\mathbb{R}^2$.

Recall that $\mathcal{D}$ is the set of subsets $D \subset \mathbb{R}^2$ such that there exists a simple loop $\gamma \in \mathcal{C}$ enclosing $D$, and that $\mathcal{R}_1$ is the set of $E \in \mathcal{D}$ such that $E$ is convex w.r.t. $x_1$ (see [Lee03]).

We use the following lemma in the proof of Theorem 11.2.

**Lemma 11.1.** [Sen92, Lemma 3.2.3] Let $M : \Omega \to \mathfrak{g}$ be a random variable, $\Sigma$ a $\sigma$-algebra of measurable subsets of $\Omega$, and $g : \Omega \to G$ a random variable which is measurable with respect to $\Sigma$. Assume that $M$ is independent of $\Sigma$ and that the distribution of $M$ is the same as that of $xMx^{-1}$ for every $x \in G$. Then the $\mathfrak{g}$-valued random variable $gMg^{-1}$ is independent of $\Sigma$ and has the same distribution as $M$.

If $E$ is a measurable subset of $\mathbb{R}^2$ then $\tau(E)$ will denote the $\sigma$-algebra generated by all the random variables $W(E')$ as $E'$ runs over the measurable subsets of $E$. 


Theorem 11.2. Let $\epsilon \in C_\infty \cap \text{Lasso}(x)$ satisfy $D(\epsilon) \in \mathcal{R}_1$. Then

(i) The $G$-valued random variable $\mathcal{F}_\mathcal{C}^{(\infty)}(1)$ is independent of the $\sigma$-algebra $\tau(\mathbb{R}^2 \setminus D(\epsilon))$.

(ii) The density $p$ of the Wilson loop $\mathcal{W}_\mathcal{C}^{(\infty)}(1)$ on $G$ with respect to Haar measure $dg$ is given by $p(g) = Q_{\text{Leb}(D(\epsilon))}(g)$. In other words,

$$\mathbb{E} \left[ f(\mathcal{F}_\mathcal{C}^{(\infty)}(1)) \right] = \int_G f(g) Q_{\text{Leb}(D(\epsilon))}(g) dg.$$

for every bounded Borel function $f$ on $G$.

Proof. The proof of (i) is similar to that of [Sen92, Lemma 3.2.6], and the proof of (ii) is to that of [Sen92, Theorem 3.2.10] (see also [Dri89]), and so we will give only a sketch.

(i) In the settings of Sec. 5 let $F_{12} = W^{(j)}$, and denote the corresponding $F^{(j)}$, $B^{(j)}$ and $U$ by $F_t^{D,(1)}$, $B_t^{D,(1)}$, and $U^{(j)}$, respectively. Let

$$F_t^{D,(\infty)} := \lim_{j \to \infty} F_t^{D,(j)} = W(D_t), \quad B_t^{D,(\infty)} := \lim_{j \to \infty} B_t^{D,(j)} \quad (11.1)$$

Let us write $B_t^{D,(\infty)}$ as a formal integral

$$B_t^{D,(\infty)} = \int_0^t \mathcal{H}_c^\infty(s)^{-1} dF_s^{D,(\infty)} \mathcal{H}_c^\infty(s). \quad (11.2)$$

We see that $F_t^{D,(\infty)}$ is a $t$-reparametrization of a standard $g$-valued Brownian motion such that

$$\mathbb{E} \left[ \|F_t^{D,(\infty)}\|_{\text{HS}}^2 \right] = \text{Leb}(D_t).$$

Hence the formal integral (11.2) can be justified as a rough integral for Brownian rough paths [FH14], and also as a stochastic integral in the Stratonovich sense. Thus we see that $B_t^{D,(\infty)}$ is also $t$-reparametrization of a standard $g$-valued Brownian motion with $\mathbb{E} \left[ \|B_t^{D,(\infty)}\|_{\text{HS}}^2 \right] = \text{Leb}(D_t)$. By Theorem 10.3 we see $B_t^{D,(n(k)))}$ converges to $B_t^{D,(\infty)}$ as $k \to \infty$ uniformly a.s., if $n : \mathbb{N} \to \mathbb{N}$ increases rapidly enough; Moreover we find that $\text{lift}(B_t^{D,(n(k)))})$ converges to $B^{D,(\infty)} = (1, B_t^{D,(\infty)} \mathcal{H}_c^{D,(\infty)})$ in $C^{b,\text{Holo}}([0,1], G^{C^2}(\text{Mat})).$

The ODE (11.1) is now written as

$$dU^{(j)}(t) = -U^{(j)}(t) dB_t^{D,(j)}.$$  

(11.3)

By Theorem 10.2 and 11.1 we find that $U^{(\infty)} := \pi(0, I; -B^{D,(\infty)})$ is well-defined, that is, the solution of the RDE

$$dU^{(\infty)}(t) = -U^{(\infty)}(t) dB_t^{D,(\infty)}, \quad (11.4)$$

uniquely exists. Since $F_t^{D,(\infty)}$ is independent of $\tau(\mathbb{R}^2 \setminus D(\epsilon))$, we see from (11.1) and Lemma 11.1 that $B_t^{D,(\infty)}$ is independent of $\tau(\mathbb{R}^2 \setminus D(\epsilon))$, and so is $B^{D,(\infty)}$.

Hence $U^{(\infty)}(t)$, especially $\mathcal{W}(1) = U^{(\infty)}(1)$, is also independent of $\tau(\mathbb{R}^2 \setminus D(\epsilon))$.

(ii) Since $B_t^{D,(\infty)}$ is a reparametrization of a standard $g$-valued Brownian motion with $E[\|B_t^{D,(\infty)}\|_{\text{HS}}^2] = \text{Leb}(D_t)$, Eq. (11.4) leads to the Stratonovich SDE

$$dU^{(\infty)}(t) = -U^{(\infty)}(t) \circ dB_t^{D,(\infty)},$$

(11.5)
which implies that \( U^{(\infty)}(t) \) is a \( t \)-reparametrization of a \( G \)-valued Brownian motion with density \( Q_{\text{Leb}}(D_1) \). Thus the Wilson loop \( \Psi^\infty(1) = U^{(\infty)}(1) \) has the density \( Q_{\text{Leb}}(D_1) = Q_{\text{Leb}}(D(1)) \).

Let \( \mathcal{R}_{1, \text{fin}} \) be the family of the finite unions of sets in \( \mathcal{R}_1 \) which is : \( \mathcal{R}_{1, \text{fin}} := \{ \bigcup_{k=1}^n D_k : D_k \in \mathcal{R}_1, 1 \leq k \leq n \in \mathbb{N} \} \).

**Corollary 11.3.** Let \( c \in \mathcal{C}_\infty \cap \text{Lasso}(x) \) satisfy \( D(c) \in \mathcal{R}_{1, \text{fin}} \). Then (i) and (ii) in Theorem 11.2 hold.

**Proof.** Follows from Lemma 11.1 □

**Corollary 11.4.** Let \( c', c^2, \ldots \in \mathcal{C}_\infty \cap \text{Lasso} \), and suppose that \( D(c') \in \mathcal{R}_{1, \text{fin}} \) for all \( k \in \mathbb{N} \), and \( D(c^k) \cap D(c^l) = \emptyset \) for \( k \neq l \). Then the Wilson loop \( \Psi^\infty(c^1) \) is independent of \( \Psi^\infty(c^1) \) if \( k \neq l \), and has the density \( Q_{\text{Leb}}(D(c^k)) \).

Our results are summarized as follows:

**Theorem 11.5.** Let \( c', c^2, \ldots \in \mathcal{C}_\infty \cap \text{Lasso} \), and suppose that \( D(c^k) \in \mathcal{R}_{1, \text{fin}} \) for all \( k \in \mathbb{N} \). Then there exists a probability space \((\Omega, \mathbb{P})\) and a sequence of \( \Omega^1(\mathbb{R}^2, g) \)-valued random variables \( \mathcal{A}_{(n)} \) such that

\[
\mathbb{P} \left[ \forall i \in \mathbb{N}, \mathcal{A}_i := \lim_{n \to \infty} \mathcal{A}_{\mathbf{c}^{(i)}(\infty)} \text{ (uniform)} \in C([0,1], G) \right] = 1,
\]

and the set of the G-valued random variables \( \{ \mathcal{A}_i \}_{i \in \mathbb{N}} \) obeys the law the Wilson loops in the YM theory on \( \mathbb{R}^2 \).

**12 Open problems**

**Conjecture 12.1.** Let \( \mathcal{C}_* \) denote one of \( \mathcal{C}_\infty \), \( \mathcal{C}_\text{Rot} \), \( \mathcal{C} \) and \( \mathcal{C}^{1\text{-var}} \) (continuous curves of bounded variation). There exists a probability space \((\Omega, \mathbb{P})\) and a sequence of \( \Omega^1(\mathbb{R}^2, g) \)-valued random variables \( \mathcal{A}_{(n)} \) such that

\[
\mathbb{P} \left[ \forall \mathbf{c} \in \mathcal{C}_*, \mathcal{A}_i := \lim_{n \to \infty} \mathcal{A}_{\mathbf{c}^{(i)}(\infty)} \text{ (uniform)} \in C([0,1], G) \right] = 1,
\]

and the set of the holonomy variables \( \{ \mathcal{A}_i(1) : \mathbf{c} \in \mathcal{C}_* \cap \text{Lasso} \} \) obeys the law the Wilson loops in the YM theory on \( \mathbb{R}^2 \).

This conjecture seems plausible for \( \mathcal{C}_* = \mathcal{C}_\infty, \mathcal{C}_\text{Rot} \), but the plausibility is obscurer for \( \mathcal{C}_* = \mathcal{C}, \mathcal{C}^{1\text{-var}} \). If the conjecture is the case, the following question will arise:

**Problem 12.2.** Does the mapping \( \mathcal{C}_* \ni \mathbf{c} \mapsto \mathcal{A}_i \) given in Conj. 12.1 have any continuity in the sense of rough paths?

This continuity is desirable to establish the notion of ‘rough gauge fields.’ However, thus far, we have no positive evidence of this continuity.

The method of [Dri89] [Sen92] [Sen93] [Sen97] strongly depend on special gauge fixing (axial gauge in [Dri89], radial gauge in [Sen92] [Sen93] [Sen97]), and seem difficult to be generalized to other gauges; Generally, the notions of gauge transformation and gauge symmetry are usually defined on the classical level (in terms of differential geometry), and the rigorous treatment of those notions is more difficult in the quantum level. Although in this paper we confined ourselves to the case of axial gauge, we conjecture that our method can be generalized to other gauges, simply because a quantum gauge field can be approximated by a classical (smooth) gauge fields in our method.
Acknowledgement

The author thanks Professor Yuzuru Inahama of Kyushu University for valuable advices.

References

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin. Fourier analysis and nonlinear partial differential equations. Springer, Berlin, 2011.

[CC13] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. arXiv:1310.0869, 2013.

[Dri89] B. K. Driver. YM$_2$: Continuum expectations, lattice convergence, and lassos. Commun. Math. Phys., 123:575–616, 1989.

[FH14] P. Friz and M. Hairer. A Course on Rough Paths. Springer, Berlin, 2014.

[FV10] P. Friz and N. Victoir. Multidimensional Stochastic Processes as Rough Paths. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. Forum of Mathematics, Pi, 3(6):1–75, 2015.

[CC13] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. arXiv:1310.0869, 2013.

[Lév03] T. Lévy. The Yang-Mills measure for compact surfaces, volume 166 of Memoirs Amer. Math. Soc. American Mathematical Society, Providence, 2003.

[MW16] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic $\Phi^4_3$ model on the torus. arXiv:1601.01234, 2016.

[MWX16] J.-C. Mourrat, H. Weber, and W. Xu. Construction of $\Phi^4_3$ diagrams for pedestrians. arXiv:1610.08897, 2016.

[Sen92] A. Sengupta. The Yang–Mills measure for $S^2$. J. Funct. Anal., 108:231–273, 1992.

[Sen93] A. Sengupta. Quantum gauge theory on compact surfaces. Ann. Phys. (NY), 221:17–52, 1993.

[Sen97] A. Sengupta. Gauge Theory on Compact Surfaces, volume 126 of Memoirs of the Amer. Math. Soc. American Mathematical Society, Providence, 1997.

[Tar07] L. Tartar. An Introduction to Sobolev Spaces and Interpolation Spaces. Springer, Berlin, 2007.