A parallel subgradient projection algorithm for quasiconvex equilibrium problems under the intersection of convex sets

Le Hai Yen\textsuperscript{a} and Le Dung Muu\textsuperscript{b}

\textsuperscript{a}Institute of Mathematics, VAST, Hanoi, Vietnam; \textsuperscript{b}TIMAS, Thang Long University and Institute of Mathematics, VAST, Hanoi, Vietnam

ABSTRACT

In this paper, we studied the equilibrium problem where the bi-function may be quasiconvex with respect to the second variable and the feasible set is the intersection of a finite number of convex sets. We propose a projection algorithm, where the projection can be computed independently onto each component set. The convergence of the algorithm is investigated and numerical examples for a variational inequality problem involving affine fractional operator are provided to demonstrate the behaviour of the algorithm.

1. Introduction

Let \( C \) be a nonempty closed convex set in \( \mathbb{R}^n \) and \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) be a given bifunction such that \( f(x, y) < +\infty \) for every \( x, y \in C \). We consider the problem

\[
\text{Find } x^* \in C : f(x^*, y) \geq 0 \quad \forall y \in C. \tag{EP}
\]

This inequality is often called the equilibrium problem. The interest of this problem is that it unifies many important problems such as the Kakutani fixed point, variational inequality, optimization and the Nash equilibrium problems \([1–4]\) in a convenient way. The inequality in (EP) first was used in \([5]\) by Nikaido and Isoda for a convex game model. The first result for solution existence of (EP) has been obtained by Fan \([6]\), where the bifunction \( f \) can be quasiconvex with respect to the second argument.

In what follows, as usual, we suppose that \( f(x, x) = 0 \) for all \( x \in C \). It should be noticed that when \( f(x, \cdot) \) is convex and subdifferentiable on \( C \), the equilibrium problem (EP) can be formulated as the following multivalued variational
inequality problem

\[
\text{Find } x^* \in C, \quad v^* \in F(x^*) : (v^*, x - x^*) \geq 0 \quad \forall x \in C, \quad (MVI)
\]

where \( F(x^*) = \partial_2 f(x^*, x^*) \) with \( \partial_2 f(x^*, x^*) \) being the subdifferential of the convex function \( f(x^*, \cdot) \) at \( x^* \). In the case \( f(x, \cdot) \) is semi-strictly quasiconvex rather than convex, Problem (EP) can take the form of (MVI) with \( F(x) := N^a_{f(x,x)} \setminus \{0\} \), where \( N^a_{f(x,x)} \) is the normal cone of the adjusted sublevel set of the function \( f(x, \cdot) \) at the level \( f(x, x) \), see [7]. More details about the links between equilibrium problems and variational inequalities can be found in [8].

One can easy to see that (EP) is equivalent to the fixed point problem

\[
\text{Find } x^* \in C : x^* \in S(x^*), \quad (FP)
\]

where the fixed point mapping \( S \) is defined by taking

\[
S(x) := \text{argmin} \{ f(x, y) : y \in C \}. \quad (P(x))
\]

This fact suggests the use of the iterative scheme \( x^{k+1} \in S(x^k) \) for the fixed point problem to solve inequality (EP). The first difficulty that we have to face with here is that the mapping \( S \) may not be singleton, even it is not defined at every point of \( C \), i.e. Problem \( P(x) \) is not solvable. To overcome this difficulty one can use the auxiliary problem principle that states that if \( f(x, \cdot) \) is convex, subdifferentiable on \( C \), then for any \( r > 0 \), Problem (EP) is equivalent to the following one:

\[
\text{Find } x^* \in C : f(x^*, y) + r\|y - x^*\|^2 \geq 0 \quad \forall y \in C, \quad (EP)
\]

in the sense that their solution-sets coincide. Then the corresponding fixed point mapping takes the form

\[
s(x) := \text{argmin}\{f(x, y) + r\|y - x\|^2 : y \in C \}. \quad (1)
\]

Since \( f(x, \cdot) \) is convex and \( \| \cdot - x \|^2 \) is strongly convex, the latter mathematical program is always uniquely solvable. However, in the case \( f(x, \cdot) \) is quasiconvex rather than convex, the auxiliary problem principle cannot be applied because of the fact that the mathematical programming problem defining the mapping \( s(\cdot) \) is nonlonger convex, even not quasiconvex, and therefore solving it is an extremely difficult task. Based upon the auxiliary problem principle, a lot of algorithms using techniques of mathematical programming methods have been developed for solving problem (EP), e.g. [2,9–18] and the references therein. However, to our best knowledge, there is only one algorithm that was developed in [19] for equilibrium problems with the bifunction being quasiconvex in its second variable and the feasible set is a polyhedron. About more on theory, methods and applications of equilibrium problem (EP) we refer the readers to the comprehensive monographs [2,20] (see also the survey papers [3,21]).
In our recent paper [22], we developed an algorithm for problem (EP), where the bifunction \( f \) may be quasiconvex in its second variable. In order to handle the quasiconvexity, we used the metric projection onto \( C \) along the direction defined by a star-subgradient of the quasiconvex function \( f(x, \cdot) \), rather than solving the mathematical programming problem (1). However, in general, the projection onto \( C \) is not easy to compute. In this paper, we continue our work by considering problem (EP) where the feasible set \( C \) is the intersection of a finite number of convex sets (often in practice), and we propose a projection algorithm, where the projection can be computed independently onto each component set.

The organization of this paper is the following. The next section is preliminaries on the quasiconvex function on \( R^n \) and its star-subdifferential. A parallel algorithm for solving (EP) when \( C \) is the intersection of a finite number of convex sets is proposed, and its convergence analysis is studied in Section 3. In the last section, numerical experiences are provided to prove the efficiency of the algorithm for a class of equilibrium problems involving quasiconvex bifunctions.

2. Preliminaries on quasiconvex function and its subdifferentials

First of all, let us recall the well-known definitions of the quasiconvex function and its star-subdifferential that will be used in the algorithm.

**Definition 2.1 ([23,24]):** A function \( \varphi : R^n \to R \cup \{+\infty\} \) is called quasiconvex on a convex subset \( Y \) of \( R^n \) if and only if for every \( x, y \in Y \) and \( \lambda \in [0, 1] \), one has

\[
\varphi[(1 - \lambda)x + \lambda y] \leq \max[\varphi(x), \varphi(y)]. \tag{2}
\]

It is easy to see that \( \varphi \) is quasiconvex on a convex set \( Y \) if and only if the strict level set \( \{x \in Y : \varphi(x) < \alpha\} \) on \( Y \) of \( \varphi \) at \( x \) is convex for every \( \alpha \in R \).

We recall that a function \( \varphi : R^n \to R \) is said to be Lipschitz on \( Y \) at a point \( y \in Y \), if there exist a finite number \( L > 0 \) such that

\[
|\varphi(x) - \varphi(y)| \leq L\|x - y\| \quad \forall x \in Y.
\]

The star-subdifferential of \( \varphi \), see, e.g. [25] is defined as

\[
\partial^* \varphi(x) := \{g \in R^n : \langle g, y - x \rangle < 0 \ \forall \ y \in L_{\varphi}(x)\},
\]

where \( L_{\varphi}(x) := \{y \in R^n : \varphi(y) < \varphi(x)\} \) is the strict level set of \( \varphi \) at the level \( \varphi(x) \).

Clearly, if \( \partial^* \varphi(x) \) is nonempty and let us denote by \( \tilde{\partial}^* \varphi(x) \) the closure of \( \partial^* \varphi(x) \),
then
\[ \bar{\partial}^* \varphi(x) = \{ g \in \mathbb{R}^n; \langle g, y - x \rangle \leq 0 \ \forall \ y \in L_\varphi(x) \}. \]

Hence \( \partial^* \varphi(x) \equiv \mathbb{R}^n \) if \( x \) is a minimizer of \( \varphi \) over \( \mathbb{R}^n \), and if \( \varphi \) is continuous on \( \mathbb{R}^n \), then \( \partial^* \varphi(x) \cup \{0\} \) is the normal cone of \( L_\varphi(x) \) (see [26]), that is
\[ \partial^* \varphi(x) \cup \{0\} = \bar{\partial}^* \varphi(x) = N(L_\varphi(x), x) := \{ g \in \mathbb{R}^n; \langle g, y - x \rangle \leq 0 \ \forall \ y \in L_\varphi(x) \}. \]

Furthermore, \( \partial^* \varphi(x) \) contains nonzero vector [27,28]. This subdifferential thus is also called normal-subdifferential.

**Lemma 2.2 ([29, Lemma 3]):** Assume that \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) is continuous and quasiconvex. Then

\[ \partial^* \varphi(x) \neq \emptyset \ \forall \ x \in \mathbb{R}^n, \quad (3) \]

\[ 0 \in \partial^* \varphi(x) \iff x \in \mathrm{argmin}\{ \varphi(y) : y \in \mathbb{R}^n \}. \quad (4) \]

For simplicity of notation, let \( f_k(x) := f(x^k, x) \). For the star-subdifferential, we have the following results will be used in the sequel.

**Lemma 2.3 ([29, Lemma 6]):** If \( B(x, \epsilon) \subset L_{f_k}(x^k) \) for some \( x \in \mathbb{R}^n \), and \( \epsilon \geq 0 \), then for \( g_k \in \partial^* f_k(x_k) \) such that \( \| g_k \| = 1 \), it holds
\[ \langle g^k, x^k - x \rangle > \epsilon. \]

**Lemma 2.4:** Let \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) be a quasiconvex L-Lipschitz continuous function on \( \mathbb{R}^n \). Then for every \( y \in \bar{L}_h(z) \), \( g \in N(\bar{L}_h(z), z) \) such that \( \| g \| = 1 \), it holds
\[ L(g, y - z) \leq h(y) - h(z) \leq 0, \quad (5) \]

**Proof:** Let \( y \in \bar{L}_h(z) \). Thanks to the continuity of the function \( h \), we have that \( h(y) - h(z) \leq 0 \). Since \( z \not\in L_h(z) \), \( N(\bar{L}_h(z), z) \neq \emptyset \). Take \( g \in N(\bar{L}_h(z), z) \) such that \( \| g \| = 1 \), then
\[ \langle g, x - z \rangle \leq 0 \ \forall \ x \in \bar{L}_h(z). \]

Let \( H(g) \) be the supporting hyperplane of \( \bar{L}_h(z) \) at \( z \), that is, defined by
\[ H(g) = \{ x \in \mathbb{R}^n | \langle g, x - z \rangle = 0 \}. \]

Since \( L_h(z) \) is open, \( H(g) \cap L_h(z) = \emptyset \). If \( y \in H(g) \), then it is clear that \( (5) \) is true. If \( y \not\in H(g) \), let \( P(y) \) be the projection of \( y \) onto the hyperplane \( H(g) \). It is not difficult to see that \( h(P(y)) \geq h(z) \).
On the one hand,
\[ 0 \leq h(z) - h(y) \leq h(P(y)) - h(y) \]
\[ \leq L\|P(y) - y\|. \tag{6} \]

On the other hand,
\[ \|P(y) - y\| = \langle g, P(y) - y \rangle \]
\[ = \langle g, P(y) - z \rangle + \langle g, z - y \rangle \]
\[ = \langle g, z - y \rangle \text{ (because } P(y) \in H(g) \text{).} \tag{7} \]

From (6) and (7),
\[ L\langle g, y - z \rangle \leq h(y) - h(z) \leq 0. \]

We also need the following result to find the subdifferential in some important cases. Some calculus rules, optimality conditions and minimization methods concerning these subdifferentials have been studied in [25–27,29–33].

**Lemma 2.5 ([29, Lemma 4]):** Suppose \( \varphi(x) = \frac{a(x)}{b(x)} \) for all \( x \in \text{dom} \varphi \), where \( a \) is a convex function, \( b \) is finite and positive on \( \text{dom} \varphi \), \( \text{dom} \varphi \) is convex and one of the following conditions holds:

- (a) \( b \) is affine;
- (b) \( a \) is nonnegative on \( \text{dom} \varphi \) and \( b \) is concave;
- (c) \( a \) is nonpositive on \( \text{dom} \varphi \) and \( b \) is convex.

Then \( \varphi \) is quasiconvex and \( \partial(a - \alpha b)(x) \) is a subset of \( \partial^* \varphi(x) \) for \( \alpha = \frac{a(x)}{b(x)} \).

### 3. A parallel algorithm and its convergence

For presentation of the algorithm and its convergence, we make the following assumptions:

**Assumptions:**

- (A1) For every \( x \in C \), the function \( f(x,.) \) is continuous, quasiconvex on \( \mathbb{R}^n \), and \( f(.,.) \) is upper semicontinuous on an open set containing \( C \times C \).
- (A2) The bifunction \( f \) is pseudomonotone on \( C \), that is,
\[ f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0 \quad \forall x,y \in C, \]
and paramonotone on \( C \) with respect to the solution set \((EP)\), that is,
\[ x \in S(EP), \ y \in C \quad \text{and} \quad f(x,y) = f(y,x) = 0 \Rightarrow y \in S(EP). \]

The paramonotonicity of a bifunction is an extension of that for an operator, see, e.g. [32], which has been used in some papers, see, for example,
Algorithm 1 Subgradient-projection algorithm.
Take real sequences \( \{\alpha_k\} \) and \( \{\lambda_k\} \) and positive numbers \( \omega_1, \ldots, \omega_m \) satisfying the following conditions:

\[
\alpha_k > 0 \quad \forall k \in \mathbb{N}, \\
\sum_{k=1}^{\infty} \alpha_k = +\infty, \quad \sum_{k=1}^{\infty} \alpha_k^2 < +\infty, \\
0 < \lambda_k \leq \bar{\lambda} < \bar{\lambda} < 1 \quad \forall k \in \mathbb{N}, \\
0 < \omega_i < 1 \quad \forall i = 1, \ldots, m, \sum_{i=1}^{m} \omega_i = 1.
\]

Initial Step: choose \( x^0 \in \mathbb{R}^n \), let \( k = 0 \).

Step \( k (0,1,\ldots) \): Having \( x^k \in \mathbb{R}^n \), take \( g^k \in \partial^*_2 f(x^k, x^k) := \{ g \in \mathbb{R}^n : \langle g, y - x^k \rangle < 0 \quad \forall y \in L_{f_k}(x^k) \} \).

If \( g^k = 0 \) and \( x^k \in C \), stop: \( x^k \) is a solution.
(Note that in this case the set \( L_{f_k}(x^k) \) is empty.) If \( g^k \neq 0 \), normalize \( g^k \) to obtain \( \|g^k\| = 1 \). Compute

\[
x^{k+1} = (1 - \lambda_k) x^k + \lambda_k P_\omega (x^k - \alpha_k g^k). \tag{8}
\]

If \( x^{k+1} = x^k \) and \( x^k \in C \), then stop: \( x^k \) is a solution. Else update \( k \leftarrow k + 1 \).

Clearly, in the case of optimization when \( f(x, y) := g(y) - g(x) \), Assumption (A2) is satisfied. The next example satisfying assumption (A2) is given in Section 4, where the bifunction is affine fractional in its second variable. Another example where the bifunction is pseudomonotone–paramonotone and not monotone can be found in [35].

(A3) The solution set \( (EP) \) is nonempty.

Suppose that \( C = \bigcap_{i=1}^{m} C_i \), where \( C_i \) are closed convex sets in \( \mathbb{R}^n \). For simplicity of notation, for \( \omega^T = (\omega_1, \ldots, \omega_m) \) with \( \omega_i > 0 \) for every \( i \) and \( \sum_{i=1}^{m} \omega_i = 1 \), let us define the operator \( P_\omega \) as \( P_\omega(x) := \sum_{i=1}^{m} \omega_i P_{C_i}(x) \) for every \( x \). It is easy to see that \( P_\omega \) is nonexpansive for any \( \omega \). Since \( C \subseteq C_i \) for every \( i \), it follows that \( P_{C_i}(x) = P_\omega(x) = x \) for every \( x \in C \). For this operator, we have the following result.

Lemma 3.1 ([30, Fact 2.3], [36, Lemma 2.3]): If \( 0 < \omega_i < 1 \) for all \( i = 1, 2, \ldots, n \) and \( \bar{x} \) is a fixed point of \( P_\omega \), that is \( \bar{x} = P_\omega(\bar{x}) \), then \( \bar{x} \in C = \bigcap_{i=1}^{m} C_i \).

Now, we are in a position to describe an algorithm for solving equilibrium \( (EP) \), where the feasible domain \( C := \bigcap_{i=1}^{m} C_i \). The algorithm is a projection-subgradient one that takes the projection independently on each component set.
Remark 3.1: In the case of the minimization problem, the Subgradient-Projection algorithm is the subgradient algorithm considered in [29].

The sequence of the iterates generated by the algorithm has the following properties:

**Proposition 3.2:** For every $z \in C = \bigcap_{i=1}^{m} C_i$, and $k \in \mathbb{N}$, the following inequality holds

$$
\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 + 2\lambda_k \alpha_k \langle g^k, z - x^k \rangle + \lambda_k \alpha_k^2 - \lambda_k (1 - \lambda_k) \|x^k - P_\omega(x^k - \alpha_k g^k)\|^2. \quad (9)
$$

**Proof:** Let $z \in C$, by using the elementary equality

$$
\|(1 - t)a + tb\|^2 = (1 - t)\|a\|^2 + t\|b\|^2 - t(1 - t)\|a - b\|^2
$$

with $a = x^k - z$, $b = P_\omega(x^k - \alpha_k g^k) - z$, $t = \lambda_k$, we have

$$
\|x^{k+1} - z\|^2 = \|x^k + \lambda_k [P_\omega(x^k - \alpha_k g^k) - x^k] - z\|^2
\leq (1 - \lambda_k)\|x^k - z\|^2 + \lambda_k \|P_\omega(x^k - \alpha_k g^k) - z\|^2
- \lambda_k (1 - \lambda_k) \|x^k - P_\omega(x^k - \alpha_k g^k)\|^2. \quad (10)
$$

In addition,

$$
\|P_\omega(x^k - \alpha_k g^k) - z\|^2
= \left\| \sum_{i=1}^{m} \omega_i (P_{C_i}(x^k - \alpha_k g^k) - z) \right\|^2
\leq \sum_{i=1}^{m} \omega_i \|P_{C_i}(x^k - \alpha_k g^k) - z\|^2
\leq \sum_{i=1}^{m} \omega_i \|x^k - \alpha_k g^k - z\|^2
\leq \|x^k - \alpha_k g^k - z\|^2
\leq \|x^k - z\|^2 - 2\alpha_k \langle g^k, x^k - z \rangle + \alpha_k^2.
$$

From (10) and the last equality, it follows that

$$
\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 + 2\lambda_k \alpha_k \langle g^k, z - x^k \rangle + \lambda_k \alpha_k^2 - \lambda_k (1 - \lambda_k) \|x^k - P_\omega(x^k - \alpha_k g^k)\|^2. \quad \blacksquare
$$
Lemma 3.3:

\[ \liminf_{k \to +\infty} \langle g^k, x^k - z \rangle \leq 0 \quad \forall \, z \in C. \quad (11) \]

**Proof:** From Proposition 3.2 and \(0 < \lambda \leq \lambda_k \leq \bar{\lambda} < 1\), we obtain

\[ 2\alpha_k \langle g^k, x^k - z \rangle \leq \frac{1}{\lambda} (\|x^k - z\|^2 - \|x^{k+1} - z\|^2) + \alpha^2_k. \]

Applying this inequality for every \( k = 1, \ldots, \infty \) and summing up, we obtain

\[ \sum_{k=1}^{\infty} \alpha_k \langle g^k, x^k - z \rangle < +\infty, \]

which together with \( \sum_{k=1}^{\infty} \alpha_k = +\infty \), implies

\[ \liminf_{k \to +\infty} \langle g^k, x^k - z \rangle \leq 0. \]

\[ \square \]

We have the following convergence result.

**Theorem 3.4:** Under the assumptions (A1)–(A3) it holds that

(i) If Algorithm 1 terminates at iteration \( k \), then \( x^k \) is a solution of \((EP)\).

(ii) If the algorithm does not terminate, then there exists a subsequence of \( \{x^k\} \) converging to a solution of \((EP)\) whenever \( \{x^k\} \) is bounded. In addition, if Problem \((EP)\) is uniquely solvable, in particular, if \( f \) is strongly pseudomonotone, the whole sequence \( \{x^k\} \) converges to the solution.

**Proof:** (i) Suppose that the algorithm terminates at iteration \( k \). Then, if \( 0 \in \partial f^*(x^k, x^k) \) and \( x^k \in C \), we have

\[ x^k \in \text{argmin}_{y \in \mathbb{R}^n} f(x^k, y). \]

Hence, \( f(x^k, y) \geq f(x^k, x^k) = 0 \) for every \( y \in \mathbb{R}^n \). Since \( x^k \in C \), it is a solution of \((EP)\).

If \( x^{k+1} = x^k \) and \( x^k \in C \), we have

\[ P_{\omega}(x^k - \alpha_k g^k) = x^k \]

\[ \iff \sum_{i=1}^{m} \omega_i P_{C_i}(x^k - \alpha_k g^k) = x^k. \]

For every \( y \in C \subseteq C_i \) and every \( i \in \{1, \ldots, m\} \), it holds that

\[ \langle x^k - \alpha_k g^k - P_{C_i}(x^k - \alpha_k g^k), y - P_{C_i}(x^k - \alpha_k g^k) \rangle \leq 0. \]
Equivalently
\[
\langle x^k - P_{C_i}(x^k - \alpha_k g^k), y - P_{C_i}(x^k - \alpha_k g^k) \rangle \leq \langle \alpha_k g^k, y - P_{C_i}(x^k - \alpha_k g^k) \rangle.
\]

Multiplying by \( \omega_i \) and summing up, we obtain
\[
\sum_{i=1}^m \omega_i \langle x^k - P_{C_i}(x^k - \alpha_k g^k), y - P_{C_i}(x^k - \alpha_k g^k) \rangle
\leq \sum_{i=1}^m \omega_i (\alpha_k g^k, y - P_{C_i}(x^k - \alpha_k g^k)).
\]

By a simple computation, using \( \sum_{i=1}^m \omega_i P_{C_i}(x^k - \alpha_k g^k) = x^k \) and \( \sum_{i=1}^m \omega_i = 1 \), we arrive at
\[
-\|x^k\|^2 + \|x^k\|^2 \leq \sum_{i=1}^m \omega_i \alpha_k (g^k, y - P_{C_i}(x^k - \alpha_k g^k)) = \alpha_k (g^k, y - x^k).
\]

Since \( g^k \in \partial^* f(x^k, x^k) \), the last inequality \( \langle g^k, y - x^k \rangle \geq 0, \forall y \in C \) implies \( f(x^k, y) \geq f(x^k, x^k) = 0 \) for every \( y \in C \), which means that \( x^k \) is a solution of (EP).

(ii) We consider two cases.

Case 1: There exists a solution \((EP)\) and an index \(k_0\) such that for \(k \geq k_0\),
\[
\|x^{k+1} - x^*\| \leq \|x^k - x^*\|.
\]

By the nonnegativity of \(\|x^k - x^*\|\), we conclude that the sequence \(\{\|x^k - x^*\|\}\) is convergent.

Moreover, from Proposition 3.2 it follows that
\[
\lambda_k (1 - \lambda_k) \|x^k - P_\omega (x^k - \alpha_k g^k)\|^2
\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - 2\lambda_k \alpha_k \langle g^k, x^k - x^* \rangle + \lambda_k \alpha_k^2.
\]

When \(k\) goes to infinity, the right-hand side of the above inequality goes to 0. Since \(0 < \underline{\lambda} \leq \lambda_k \leq \overline{\lambda} < 1\), we obtain
\[
\lim_{k \to \infty} \|x^k - P_\omega (x^k - \alpha_k g^k)\| = 0.
\]

We also have that
\[
\|x^k - P_\omega (x^k)\| \leq \|x^k - P_\omega (x^k - \alpha_k g^k)\| + \|P_\omega (x^k - \alpha_k g^k) - P_\omega (x^k)\|
\leq \|x^k - P_\omega (x^k - \alpha_k g^k)\| + \|x^k - \alpha_k g^k - x^k\|
\leq \|x^k - P_\omega (x^k - \alpha_k g^k)\| + \alpha_k.
\]

Since \(\lim_{k \to \infty} \alpha_k = 0\),
\[
\lim_{k \to \infty} \|x^k - P_\omega (x^k)\| = 0.
\]
From Lemma 3.3 follows
\[ \liminf_{k \to \infty} \langle g^k, x^k - x^* \rangle \leq 0. \tag{14} \]

Let \( \{x^{k_i}\} \) be a subsequence of \( \{x^k\} \) such that
\[ \lim_{i \to \infty} \langle g^{k_i}, x^{k_i} - x^* \rangle = \liminf_{k \to \infty} \langle g^k, x^k - z \rangle. \]

Since \( \{x^k\} \) is bounded, \( \{x^{k_i}\} \) is bounded too. Let \( \bar{x} \) be a limit point of \( \{x^{k_i}\} \), and, without loss of generality, we assume that
\[ \lim_{i \to \infty} x^{k_i} = \bar{x}. \tag{15} \]

It is clear that
\[
\|x^{k_i} - P_\omega(\bar{x})\| \leq \|x^{k_i} - P_\omega(x^{k_i})\| + \|P_\omega(\bar{x}) - P_\omega(x^{k_i})\|
\leq \|x^{k_i} - P_\omega(x^{k_i})\| + \|\bar{x} - x^{k_i}\|. \tag{16}
\]

Thanks to (13) and (15), we have
\[ \lim_{i \to \infty} x^{k_i} = P_\omega(\bar{x}). \tag{17} \]

Combining this with (15), we see that \( P_\omega(\bar{x}) = \bar{x} \), which together with \( \omega_i > 0 \) for every \( i \) implies \( \bar{x} \in C \).

In addition, since \( x^* \) is a solution, by pseudomonotonicity of \( f \) on \( C \), we have \( f(\bar{x}, x^*) \leq 0 \). We show that \( f(\bar{x}, x^*) = 0 \). In fact, by contradiction, we assume that there exists \( a > 0 \) such that
\[ f(\bar{x}, x^*) \leq -a. \]

Since \( f(\cdot, \cdot) \) is upper semicontinuous on an open set containing \( C \times C \), there exist positive numbers \( \epsilon_1, \epsilon_2 \) such that, for any \( x \in B(\bar{x}, \epsilon_1) \), \( y \in B(x^*, \epsilon_2) \) we have
\[ f(x, y) \leq -\frac{a}{2}. \]

On the other hand, \( \lim_{i \to \infty} x^{k_i} = \bar{x} \) implies that there exist \( i_0 \) such that for \( i \geq i_0 \), \( x^{k_i} \) belongs to \( B(\bar{x}, \epsilon_1) \). So, for \( i \geq i_0 \) and \( y \in B(x^*, \epsilon_2) \), we have
\[ f(x^{k_i}, y) \leq -\frac{a}{2}, \tag{18} \]
which implies that \( B(x^*, \epsilon_2) \subset L_{f_{k_i}} (x^{k_i}) \). In addition, \( g^{k_i} \neq 0 \), because if \( g^{k_i} = 0 \) then \( f(x^{k_i}, y) \geq f(x^{k_i}, x^{k_i}) = 0 \) for every \( y \in \mathbb{R}^n \), which contradicts to (18). By Lemma 2.3, for \( i \geq i_0 \), it holds that
\[ \langle g^{k_i}, x^{k_i} - x^* \rangle > \epsilon_2, \]
which contradicts to (14). Thus, \( \bar{x} \in C \) and \( f(\bar{x}, x^*) = 0 \). Again by pseudomonotonicity, we obtain \( f(x^*, \bar{x}) = 0 \). Then, from paramonotonicity of \( f \) it follows that
\( \bar{x} \) is a solution of \((EP)\). In addition, since \( \{\|x^k - x^*\|\} \) is convergent for every solution \( x^* \) of \((EP)\), the sequence \( \{x^k\} \) converges to \( \bar{x} \) which is a solution of \((EP)\).

**Case 2:** For any solution \( x^* \) of \((EP)\), there exists a subsequence \( \{x_{k_i}\} \) of \( \{x^k\} \) that satisfies
\[
\|x^{k_i} - x^*\| < \|x^{k_{i+1}} - x^*\|.
\]

Now again, by Proposition 3.2 applying with \( x^* \), we can write
\[
\begin{align*}
\lambda_{k_i} (1 - \lambda_{k_i}) \|x^{k_i} - P_\omega (x^{k_i} - \alpha_{k_i} g^{k_i})\|^2 & \leq \|x^{k_i} - x^*\|^2 - \|x^{k_{i+1}} - x^*\|^2 - 2\lambda_{k_i} \alpha_{k_i} (g^{k_i}, x^{k_i} - x^*) + \lambda_{k_i} \alpha_{k_i}^2, \\
& \leq -2\lambda_{k_i} \alpha_{k_i} (g^{k_i}, x^{k_i} - x^*) + \lambda_{k_i} \alpha_{k_i}^2. 
\end{align*}
\]
(19)

Since \( \{x^{k_i}\} \) is bounded, and \( 0 < \lambda_k < 1, \alpha_k \to 0 \), from the last inequality it follows that
\[
\lim_{i \to \infty} \|x^{k_i} - P_\omega (x^{k_i} - \alpha_{k_i} g^{k_i})\| = 0.
\]
(20)

Also from (19) follows
\[
(g^{k_i}, x^{k_i} - x^*) \leq \frac{\alpha_{k_i}}{2},
\]
\[
\lim_{i \to \infty} sup (g^{k_i}, x^{k_i} - x^*) \leq 0. 
\]
(21)

Now by the same argument as in Case 1, we see that every limit point of the sequence \( \{x_{k_i}\} \) belongs to the solution set \( S(EP) \).

Suppose now that the solution is unique, then the sequence \( \{x^k\} \) converges to the unique solution \( x^* \) of \((EP)\). In fact, without loss of generality, we may assume that for \( k \not\in \{k_j\} \). Then
\[
\|x^{k+1} - x^*\| \leq \|x^k - x^*\|.
\]

For any \( \epsilon > 0 \), there exist \( i_0 \) such that for \( i \geq i_0 \),
\[
\|x^{k} - x^*\| < \epsilon.
\]
Furthermore, for any \( k > k_{i_0} \) that does not belong to \( \{k_i\} \), there exists \( i_1 \geq i_0 \) such that \( k_{i_1} < k < k_{i_1 + 1} \). Then \( \|x^k - x^*\| \leq \|x^{k_{i_1}} - x^*\| < \epsilon \), which means that \( \|x^k - x^*\| < \epsilon \) for \( k \geq k_{i_0} \). Hence, the whole sequence \( \{x^k\} \) converges to the unique solution of \((EP)\).

**Remark 3.2:** The assumption on boundedness of the sequence \( \{x^k\} \) is ensured if either

(C1) The set \( C_\omega := \{x \in \mathbb{R}^n : x = \sum_{j=1}^m \omega_j x^j : x^j \in C_j \; \forall j \in \{1, \ldots, m\}\} \) is bounded, and \( x^0 \in C_\omega \), or
(C2) The bifunction \( f(x, y) \) is pseudomonotone on \( C_\omega \) and for each \( x \), the function \( f_x(\cdot) := f(x, \cdot) \) is Lipschitz continuous with constant \( L_x \), and

\[
C \cap S(C_\omega, f) \neq \emptyset,
\]

where \( S(C_\omega, f) \) stands for the solution-set of the equilibrium problem

\[
\text{find } \tilde{z} \in C_\omega : f(\tilde{z}, z) \geq 0 \quad \forall \ z \in C_\omega. \quad (EP_\omega)
\]

Indeed, according to the algorithm, \( x^{k+1} \) is a convex combination of elements of the convex set \( C_\omega \), we see that \( x^{k+1} \in C_\omega \) for every \( k = 0, 1, \ldots \), with implies that \( \{x^k\} \) is bounded.

To see the assertion for (C2), we apply Lemma 2.4 with \( h(x) := f(x^k, x) \), \( z := x^k \) and \( y = \tilde{z} \in C \cap S(C_\omega, f) \). Then \( f(y, x^k) \geq 0 \), which, by pseudomonotonicity, implies \( f(x^k, y) \leq 0 \) for all \( k \). Hence \( y \in \tilde{L}_h(x^k) \). Thus by Lemma 2.4, \( \langle g^k, \tilde{z} - x^k \rangle \leq 0 \). Then, by applying Proposition 3.2 with \( z = \tilde{z} \), we obtain

\[
\|x^{k+1} - \tilde{z}\|^2 \leq \|x^k - \tilde{z}\|^2 + \alpha_k^2
\]

Hence, \( \{\|x^k - \tilde{z}\|^2\} \) is convergence, and therefore \( \{x^k\} \) is bounded.

In the case of optimization problem, where \( f(x, y) := \varphi(y) - \varphi(x) \), Condition (C2) means that \( \varphi \) is Lipschitz continuous and there exists \( \tilde{z} \in C \) such that \( \tilde{z} \) is a minimizer of the function \( \varphi(y) \) on \( C_\omega \).

Condition (C2) is inspired by the paper [16], where the equilibrium problems of the form

\[
\text{Find } x^* \in P \cap Q : f(x^*, y) \geq 0 \quad \forall \ y \in Q,
\]

with \( P, Q \) being convex sets have been studied. This common solution problem has been attracted much attention of researchers in recent years.

### 4. Computational experience

As an application, let us consider the following inverse variational inequality:

\[
\text{Find } x^* \in C : \langle F(x^*), G(y) - G(x^*) \rangle \geq 0 \quad \forall \ y \in C,
\]

where \( F, G : \mathbb{R}^n \to \mathbb{R}^n \). Such a problem has been considered in [37–39]. In [38], He pointed out that the extended linear-quadratic programming that has been studied by Rockafellar and Wets [40] can be formulated in the form of the above inverse variational inequality.
Table 1. Algorithm with $\alpha_k = \frac{100}{k+1}$.

| $n$ | No. of prob. | CPU-times (s) | Error 1 | Error 2 |
|-----|--------------|---------------|---------|---------|
| 5   | 100          | 5.674775      | 0.000127| 0.188295|
| 10  | 100          | 6.990620      | 0.000092| 0.187407|
| 20  | 100          | 8.7481101     | 0.000085| 0.186929|
| 50  | 100          | 40.770369     | 0.000083| 0.186708|

Table 2. Algorithm with $\alpha_k = \frac{100}{k+1}$.

| $n$ | No. of prob. | CPU-times (s) | Error 1 | Error 2 |
|-----|--------------|---------------|---------|---------|
| 5   | 100          | 11.339168     | 0.000224| 0.383520|
| 10  | 100          | 11.638716     | 0.000182| 0.383298|
| 20  | 100          | 13.612659     | 0.000170| 0.386529|
| 50  | 100          | 42.999559     | 0.000165| 0.388552|

Table 3. Algorithm with $\alpha_k = \frac{100}{k+1}$.

| $n$ | No. of prob. | CPU-times (s) | Error 1 | Error 2 | Error 3 |
|-----|--------------|---------------|---------|---------|---------|
| 5   | 100          | 10.218638     | 0.000210| 0.199647| 0.058665|
| 10  | 100          | 10.811854     | 0.000275| 0.200397| 0.051829|
| 20  | 100          | 12.970668     | 0.000498| 0.200389| 0.074603|
| 50  | 100          | 45.202791     | 0.000731| 0.200382| 0.086886|

To test the algorithm, we consider the equilibrium problem

Find $x^* \in C$ such that $f(x^*, y) \geq 0 \quad \forall \ y \in C,$ \hspace{1cm} (EP)

where the bifunction $f(x, y)$ is defined by

$$f(x, y) = \left( Ax + b, \frac{A_1 y + b_1}{c^T y + d} - \frac{A_1 x + b_1}{c^T x + d} \right),$$ \hspace{1cm} (22)

with $A, A_1 \in \mathbb{R}^{n \times n},$ $b, b_1, c \in \mathbb{R}^n,$ $d \in \mathbb{R}$ and $C \subset \{ x | c^T x + d > 0 \}$.

By applying Proposition 4.1 in [32] to the differentiable function $g(x) := A^T (\frac{A_1 x + b_1}{c^T x + d})$ one can easily check that $f$ is monotone on $C$ if and only if the matrix

$$\hat{A}_1(x) = A^T [A_1 c^T x - A_1 x c^T] + A^T [A_1 d - b_1 c^T]$$

is positive semidefinite and paramonotone if $\hat{A}_1(x)$ is symmetric for any $x \in C$.

We test the algorithm with the following three examples, where the projection onto each component set has a closed form. For each problem we chose $\lambda_k = 1/2$ for every $k$, and $\omega_i = 1/m$ for all $i$. We stop the computation at iteration $k$ if $g^k = 0$ or $err_1 = \|x^k - x^{k+1}\| < 10^{-4}$ and $err_2 = \|x^k - P_{C_1}(x^k)\| + \cdots + \|x^k - P_{C_m}(x^k)\| < 10^{-1}$, or the number of iteration exceeds 1000.

The average time and average errors for each size are reported in Tables 1–3 with different sizes, a hundred of problems have been tested for each size.
Example 4.1: In this example, we take
\[ C = C_1 \cap C_2, \]
where \( C_1 = [1, 3]^n \) and \( C_2 = \{ x \in \mathbb{R}^n \mid \| x \| \leq 3 \} \). Each entries of \( A, A_1, b, b_1, c, d \) is uniformly generated in the interval \([0, 1]\).

Example 4.2: In this example, we take
\[ C = C_1 \cap C_2 \cap C_3, \]
where \( C_1 = [1, 3]^n \), \( C_2 = \{ x \in \mathbb{R}^n \mid \| x \| \leq 3 \} \) and \( C_3 = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \geq N + 1 \} \). Each entries of \( A, A_1, b, b_1, c, d \) is uniformly generated in the interval \([0, 1]\).

Example 4.3: In this example, we take
\[ C = C_1 \cap C_2, \]
where \( C_1 = [1, 3]^n \) and \( C_2 = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{3} x_i \geq 3 \} \). Each entries of \( A, A_1, b, b_1, c, d \) is uniformly generated in the interval \([0, 1]\).

Note that \( x^* \in C \) is a solution of the equilibrium problem (EP) if and only if \( x^* \) belongs to the solution set of the following affine fractional programming problem:
\[ \min_{y \in C} g(x^*, y), \]
where \( g(x, y) = \langle Ax + b, \frac{A_1 y + b_1}{c^T y + d} \rangle \). Thus we can use linear programming algorithms to compute
\[ err_3 = \frac{\min_{y \in C} g(P_C(x^k), y) + g(P_C(x^k), P_C(x^k))}{g(P_C(x^k), P_C(x^k))}. \]
and use it as a stopping criterion.

5. Conclusion

We have proposed an iterative star-subgradient projection algorithm for solving a class of equilibrium problems over the intersection of closed, convex sets, where the bifunction is quasiconvex in its second variable. The search direction at each iteration is defined by a star-subgradient at the current iterate, and the projection is executed independently on each component of the intersection sets. Convergence of the algorithm has been shown, and some illustrative examples have been solved.

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