Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$

Rachidi Bolaji Salako and Wenxian Shen
Department of Mathematics and Statistics
Auburn University
Auburn University, AL 36849
U.S.A.

Abstract
In the current paper, we consider the following parabolic-elliptic semilinear Keller-Segel model on $\mathbb{R}^N$,

\[
\begin{aligned}
    u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + au - bu^2, \quad x \in \mathbb{R}^N, \quad t > 0, \\
    0 &= (\Delta - I)v + u, \quad x \in \mathbb{R}^N, \quad t > 0,
\end{aligned}
\]

where $\chi > 0$, $a \geq 0$, $b > 0$ are constant real numbers and $N$ is a positive integer. We first prove the local existence and uniqueness of classical solutions $(u(x, t; u_0), v(x, t; u_0))$ with $u(x, 0; u_0) = u_0(x)$ for various initial functions $u_0(x)$. Next, under some conditions on the constants $a, b, \chi$ and the dimension $N$, we prove the global existence and boundedness of classical solutions $(u(x, t; u_0), v(x, t; u_0))$ for given initial functions $u_0(x)$. Finally, we investigate the asymptotic behavior of the global solutions with strictly positive initial functions or nonnegative compactly supported initial functions. Under some conditions on the constants $a, b, \chi$ and the dimension $N$, we show that for every strictly positive initial function $u_0(\cdot)$,

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}^N} \left[ |u(x, t; u_0) - \frac{a}{b}| + |v(x, t; u_0) - \frac{a}{b}| \right] = 0,
\]

and that for every nonnegative initial function $u_0(\cdot)$ with non-empty and compact support $\text{supp}(u_0)$, there are $0 < c_{\text{low}}^*(u_0) \leq c_{\text{up}}^*(u_0) < \infty$ such that

\[
\lim_{t \to \infty} \sup_{|x| \leq ct} \left[ |u(x, t; u_0) - \frac{a}{b}| + |v(x, t; u_0) - \frac{a}{b}| \right] = 0 \quad \forall \ 0 < c < c_{\text{low}}^*(u_0)
\]

and

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} \left[ u(x, t; u_0) + v(x, t; u_0) \right] = 0 \quad \forall \ c > c_{\text{up}}^*(u_0).
\]

Key words. Parabolic-elliptic chemotaxis system, logistic source, classical solution, local existence, global existence, asymptotic behavior.

2010 Mathematics Subject Classification. 35B35, 35B40, 35K57, 35Q92, 92C17.
1 Introduction and the Statements of Main results

The movements of many mobile species are influenced by certain chemical substances. Such movements are referred to chemotaxis. The origin of chemotaxis models was introduced by Keller and Segel (see [18], [19]). The following is a general Keller-Segel model for the time evolution of both the density $u(x,t)$ of a mobile species and the density $v(x,t)$ of a chemoattractant,

$$\begin{align*}
    u_t &= \nabla \cdot (m(u) \nabla u - \chi(u,v) \nabla v) + f(u,v), \quad x \in \Omega, \quad t > 0 \\
    \tau v_t &= \Delta v + g(u,v), \quad x \in \Omega, \quad t > 0
\end{align*}$$

(1.1)

complemented with certain boundary condition on $\partial \Omega$ if $\Omega$ is bounded, where $\Omega \subset \mathbb{R}^N$ is an open domain, $\tau \geq 0$ is a non-negative constant linked to the speed of diffusion of the chemical, the function $\chi(u,v)$ represents the sensitivity with respect to chemotaxis, and the functions $f$ and $g$ model the growth of the mobile species and the chemoattractant, respectively.

In the last two decades, considerable progress has been made in the analysis of various particular cases of (1.1) on both bounded and unbounded domains. Among the central problems are the existence of nonnegative solutions of (1.1) which are globally defined in time or blow up at a finite time and the asymptotic behavior of time global solutions. The features of solutions of (1.1) depend on the geometric properties of the functions $m(u)$, $\chi(u,v)$, $f(u,v)$, and $g(u,v)$.

When $\tau > 0$, (1.1) is referred to as the parabolic-parabolic semilinear Keller-Segel model. In this case, when (1.1) is coupled with Neumann boundary condition on bounded domain, several results have been established for different choices of the functions $m(u)$, $\chi(u,v)$, $f(u,v)$, and $g(u,v)$. For example when $\tau = 1$, $m(u) = 1$, $\chi(u,v) = \chi u$, $g(u,v) = u - v$, $f(u,v) = u(a - bu)$, and $\frac{1}{\chi}$ is sufficiently large, it is shown in [48] that unique global classical solution exists for every nonnegative initial data $(u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega)$ and that the constant solution $(\frac{a}{b}, \frac{a}{b})$ is asymptotically stable. See also [31], [44] for the study of boundedness and global existence of classical solutions when $b$ is large. When $b$ is small, among others, Lankeit in [23] proved the existence of at least one global weak solution with given initial functions. The reader is referred to [2] for a recent survey.

In the present paper, we restrict ourselves to the case that $\tau = 0$, which is supposed to model the situation when the chemoattractant diffuses very quickly. System (1.1) with $\tau = 0$ reads as

$$\begin{align*}
    u_t &= \nabla \cdot (m(u) \nabla u - \chi(u,v) \nabla v) + f(u,v), \quad x \in \Omega, \quad t > 0 \\
    0 &= \Delta v + g(u,v), \quad x \in \Omega, \quad t > 0
\end{align*}$$

(1.2)

complemented with certain boundary condition on $\partial \Omega$ if $\Omega$ is bounded.

Global existence and asymptotic behavior of solutions of (1.2) on bounded domain $\Omega$ complemented with Neumann boundary conditions,

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{for} \quad x \in \partial \Omega,$$

(1.3)

has been studied in many papers. For example, in [38], the authors studied (1.2)+(1.3) with $m(u) \equiv 1$, $\chi(u,v) = \chi u$, $f(u,v) = au - bu^2$, which is referred to as the logistic source in literature, and $g(u,v) = u - v$, where $\chi$, $a$, and $b$ are positive constants. Among others, the following are proved in [38].
• If either $N \leq 2$ or $b > \frac{N-2}{2} \chi$, then for any initial data $u_0 \in C^{0,\alpha}(\bar{\Omega})$ ($\alpha \in (0,1)$) with $u_0(x) \geq 0$, (1.2)+(1.3) possesses a unique bounded global classical solution $(u(x,t;u_0),v(x,t;u_0))$ with $u(x,0;u_0) = u_0(x)$.

• If $b > 2^2 \chi$, then for any $u_0 \in C^{0,\alpha}(\bar{\Omega})$ with $u_0(x) \geq 0$ and $u_0(x) \neq 0$,

$$\lim_{t \to \infty} \left[ \left\| u(\cdot,t;u_0) - \frac{a}{b} \right\|_{L^\infty(\Omega)} + \left\| v(\cdot,t;u_0) - \frac{a}{b} \right\|_{L^\infty(\Omega)} \right] = 0.$$ 

It should be pointed out that, for the above choices of $m(u)$, $\chi(u,v)$, $f(u,v)$, and $g(u,v)$, when $N \geq 3$ and $b \leq \frac{N-2}{2} \chi$, it remains open whether for any given initial data $u_0 \in C^{0,\alpha}(\Omega)$, (1.2)+(1.3) possesses a global classical solution $(u(x,t;u_0),v(x,t;u_0))$ with $u(x,0;u_0) = u_0(x)$, or whether finite-time blow-up occurs for some initial data. The works [22], [46], [49] should be mentioned along this direction. It is shown in [22] that in presence of suitably weak logistic dampening (that is, small $b$) certain transient growth phenomena do occur for some initial data. It is shown in [46] that if we keep the choices of $m(u)$ and $\chi(u,v)$ as above and let $f(u,v) = au - bu^2$ with suitable $\kappa < 2$ (for instance, $\kappa = 3/2$) and $g(u,v) = u - \frac{1}{3\theta} \int_\Omega u(x)dx$, then finite-time blow-up is possible.

The reader is referred to [2], [7], [14], [40], [43], [45], [46], [47], [49], [50], [52], and references therein for other studies of (1.2) on bounded domain with Neumann or Dirichlet boundary conditions and with $f(u,v)$ being logistic type source function or 0 and $m(u)$, $\chi(u,v)$, and $g(u,v)$ being various kinds of functions.

There are also several studies of (1.2) when $\Omega$ is the whole space $\mathbb{R}^N$ and $f(u,v) = 0$ (see [5], [17], [28], [36], [35]). For example, in the case of $m(u) \equiv 1$, $\chi(u,v) = \chi u$, $f(u,v) = 0$, and $g(u,v) = u - v$, where $\chi$ is a positive constant, it is known that blow-up occurs if either $N=2$ and the total initial population mass is large enough, or $N \geq 3$ (see [2], [8], [28] and references therein). However, there is little study of (1.2) when $\Omega = \mathbb{R}^N$ and $f(u,v) \neq 0$.

The objective of this paper is to investigate the local/global existence and asymptotic behavior of positive solutions of (1.2) when $\Omega = \mathbb{R}^N$ and $f(u,v) = au - bu^2$ is a logistic source function, where $a$ and $b$ are positive constants. We further restrict ourselves to the choices $m(u) \equiv 1$, $\chi(u,v) = \chi u$, and $g(u,v) = u - v$, where $\chi$ is positive constant. System (1.2) with these choices on $\mathbb{R}^N$ reads as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + u(a - bu), & x \in \mathbb{R}^N, \ t > 0 \\ 0 = \Delta v + u - v, & x \in \mathbb{R}^N, \ t > 0. \end{cases}$$ (1.4)

We first investigate the local existence of solutions of (1.4) for various given initial functions $u_0(x)$. Note that, due to biological interpretations, only nonnegative initial functions will be of interest. We call $(u(x,t), v(x,t))$ a classical solution of (1.4) on $[0,T)$ if $u, v \in C(\mathbb{R}^N \times [0,T)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ and satisfies (1.4) for $(x,t) \in \mathbb{R}^N \times (0,T)$ in the classical sense. A classical solution $(u(x,t), v(x,t))$ of (1.4) on $[0,T)$ is called nonnegative if $u(x,t) \geq 0$ and $v(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times (0,T)$. A global classical solution of (1.4) is a classical solution on $[0,\infty)$.

Let

$$C^b_{\text{unif}}(\mathbb{R}^N) = \{ u \in C(\mathbb{R}^N) \mid u(x) \text{ is uniformly continuous in } x \in \mathbb{R}^N \text{ and } \sup_{x \in \mathbb{R}^N} |u(x)| < \infty \}$$ (1.5)

equipped with the norm $\|u\|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|$. For given $0 < \nu < 1$ and $0 < \theta < 1$, let

$$C^\nu_{\text{unif}}(\mathbb{R}^N) = \{ u \in C^b_{\text{unif}}(\mathbb{R}^N) \mid \sup_{x,y \in \mathbb{R}^N, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\nu} < \infty \}$$ (1.6)
equipped with the norm \( \|u\|_{\infty,\nu} = \sup_{x \in \mathbb{R}^N} |u(x)| + \sup_{x,y \in \mathbb{R}^N, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\nu} \), and

\[
C^\theta((t_1, t_2), C^\nu_{\text{unif}}(\mathbb{R}^N)) = \{ u(\cdot) \in C((t_1, t_2), C^\nu_{\text{unif}}(\mathbb{R}^N)) \mid u(t) \text{ is locally H"older continuous with exponent } \theta \}.
\]

We have the following result on the local existence and uniqueness of solution of (1.4) for initial data belonging to \( C^b_{\text{unif}}(\mathbb{R}^N) \).

**Theorem 1.1.** For any \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^N) \) with \( u_0 \geq 0 \), there exists \( T_{\text{max}}^\infty(u_0) \in (0, \infty] \) such that (1.4) has a unique nonnegative classical solution \((u(x, t; u_0), v(x, t; u_0))\) on \([0, T_{\text{max}}^\infty(u_0))\) satisfying that \( \lim_{t \to 0^+} u(\cdot, t; u_0) = u_0 \) in the \( C^b_{\text{unif}}(\mathbb{R}^N) \)-norm,

\[
u(\cdot, \cdot; u_0) \in C([0, T_{\text{max}}^\infty(u_0)), C^b_{\text{unif}}(\mathbb{R}^N)) \cap C^1((0, T_{\text{max}}^\infty(u_0)), C^b_{\text{unif}}(\mathbb{R}^N))
\]

and

\[
u(\cdot, \cdot; u_0), \partial_x u(\cdot, \cdot), \partial^2_{x_i x_j} u(\cdot, \cdot), \partial_t u(\cdot, \cdot; u_0) \in C^\theta((0, T_{\text{max}}^\infty(u_0)), C^\nu_{\text{unif}}(\mathbb{R}^N))
\]

for all \( i, j = 1, 2, \cdots, N, \ 0 < \theta \ll 1, \) and \( 0 < \nu \ll 1 \). Moreover, if \( T_{\text{max}}^\infty(u_0) < \infty \), then

\[
\limsup_{t \to T_{\text{max}}^\infty(u_0)} \|u(\cdot, t; u_0)\|_\infty = \infty.
\]

For given \( p \geq 1 \) and \( \alpha \in (0, 1) \), let \( X^\alpha \) be the fractional power space of \( I - \Delta \) on \( X = L^p(\mathbb{R}^N) \). We obtain the following results on the local existence and uniqueness of solutions of (1.4) for \( u_0 \in X^\alpha \).

**Theorem 1.2.** Assume that \( p > N \) and \( \alpha \in \left(\frac{1}{p}, 1\right) \). For every nonnegative \( u_0 \in X^\alpha \), there is a positive number \( T_{\text{max}}^\alpha(u_0) \in (0, \infty] \) such that (1.4) has a unique nonnegative classical solution \((u(x, t; u_0), v(x, t; u_0))\) on \( \mathbb{R}^N \times [0, T_{\text{max}}^\alpha(u_0)) \) satisfying that \( \lim_{t \to 0^+} u(\cdot, t; u_0) = u_0 \) in the \( X^\alpha \)-norm,

\[
u(\cdot, \cdot; u_0) \in C([0, T_{\text{max}}^\alpha(u_0)), X^\alpha) \cap C^1((0, T_{\text{max}}^\alpha(u_0)), L^p(\mathbb{R}^N)),
\]

and

\[
u(\cdot, \cdot; u_0), \partial_x u(\cdot, \cdot; u_0), \partial^2_{x_i x_j} u(\cdot, \cdot; u_0), \partial_t u(\cdot, \cdot; u_0) \in C^\theta((0, T_{\text{max}}^\alpha(u_0)), C^\nu_{\text{unif}}(\mathbb{R}^N))
\]

for all \( 0 \leq \beta < 1, \ i, j = 1, 2, \cdots, 0 < \theta \ll 1, \) and \( 0 < \nu \ll 1 \). Moreover, if \( T_{\text{max}}^\alpha(u_0) < +\infty \), then

\[
\lim_{t \to T_{\text{max}}^\alpha(u_0)} \|u(\cdot, t; u_0)\|_{X^\alpha} = \infty.
\]

Since \( X^\alpha \subset C^b_{\text{unif}}(\mathbb{R}^N) \) for \( p > N \) and \( \alpha \in \left(\frac{1}{p}, 1\right) \), the existence of local classical solution for initial data in \( X^\alpha \) is guaranteed by Theorem 1.1. However, \( u(\cdot, \cdot; u_0) \in C([0, T_{\text{max}}^\alpha(u_0)), X^\alpha) \cap C^1((0, T_{\text{max}}^\alpha(u_0)), L^p(\mathbb{R}^N)) \cap C^1((0, T_{\text{max}}^\alpha(u_0)), X^\beta) \) in Theorem 1.2, which is very important for later use, is not included in Theorem 1.1. The proof of Theorem 1.1 is based on the contraction mapping theorem and a technical result proved in Lemma 3.2 while the proof of Theorem 1.2 is based on the semigroup method. Theorem 1.2 is of particular interest because it helps to take advantage of the integration by parts theorems, thus, helps to get a weaker condition on the parameters \( \chi, b \) and \( N \) to ensure the global existence of classical solutions (see Theorem 1.6). Furthermore, Theorem 1.2 will be used to get some extension results for \( L^p \)-integrable initial data, which are not necessarily continuous, as stated in the next theorem and Theorem 1.7.
Since functions of $L^p(\mathbb{R}^N)$ are not always continuous, the definition of solution to (1.4) should be modified. For a nonnegative initial data $u_0 \in L^p(\mathbb{R}^N)$, by a solution of (1.4) on $[0,T)$ with initial data $u_0$ we mean nonnegative functions $u(x,t), v(x,t)$ satisfying that $u(\cdot, \cdot), v(\cdot, \cdot) \in C^{2,1}(\mathbb{R}^N \times (0,T))$, (1.4) holds for $(x,t) \in \mathbb{R}^N \times (0,T)$ in classical sense, and $\lim_{t \to 0^+} u(t) = u_0(t)$ in the $L^p(\mathbb{R}^N)$-norm.

**Theorem 1.3.** For every $p > N$ with $p \geq 2$ and $u_0 \in L^p(\mathbb{R}^N)$ with $u_0 \geq 0$, there is a positive number $T_{\text{max}}^p(u_0) \in (0, \infty]$ such that (1.4) has a unique non-negative solution $(u(x,t); u_0), v(x,t; u_0))$ on $[0,T_{\text{max}}^p(u_0))$ satisfying that $\lim_{t \to 0^+} u(t; u_0) = u_0(t)$ in the $L^p(\mathbb{R}^N)$-norm,

$$u(\cdot, \cdot; u_0) \in C([0,T_{\text{max}}^p(u_0)), L^p(\mathbb{R}^N)) \cap C^1((0,T_{\text{max}}^p(u_0)), L^p(\mathbb{R}^N)), \quad (1.12)$$

$$u(\cdot, \cdot; u_0) \in C((0,T_{\text{max}}^p(u_0)), X^\beta) \cap C^1((0,T_{\text{max}}^p(u_0)), C^a_{\text{unif}}(\mathbb{R}^N)), \quad (1.13)$$

and

$$u(\cdot, \cdot; u_0), \partial_x u(\cdot, \cdot), \partial_{x,x}^2 u(\cdot, \cdot; u_0), \partial_t u(\cdot, \cdot; u_0) \in C^b((0,T_{\text{max}}^p(u_0)), C^a_{\text{unif}}(\mathbb{R}^N)) \quad (1.14)$$

for all $0 \leq \beta < 1$, $i,j = 1, 2, \cdots, N$, $0 \leq \theta \ll 1$, and $0 < \nu \ll 1$. Moreover, if $T_{\text{max}}^p(u_0) < +\infty$, then $\lim_{t \to T_{\text{max}}^p(u_0)} \|u(\cdot, t; u_0)\|_{L^p(\mathbb{R}^N)} = \infty$.

The following theorem establishes the equality of the maximal existence intervals of the classical solutions of (1.4) in different phase spaces.

**Theorem 1.4.** Assume that $p > N$, $\alpha \in (\frac{2}{3},1)$, and $X^\alpha$ is the fractional power space of $I - \Delta$ on $X = L^p(\mathbb{R}^N)$. Then the following hold,

1. if $u_0 \in X^\alpha$ then $T_{\text{max}}^\infty(u_0) = T_{\text{max}}^\alpha(u_0)$;
2. if $u_0 \in X^\alpha$ and $p \geq 2$, then $T_{\text{max}}^\alpha(u_0) = T_{\text{max}}^p(u_0)$;
3. if $p \geq 2$ and $u_0 \in C_{\text{unif}}^b(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then $T_{\text{max}}^\infty(u_0) = T_{\text{max}}^p(u_0)$;
4. if $p \geq 2$ and $u_0 \in L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, then $u(\cdot, \cdot; u_0) \in C([0,T_{\text{max}}^p(u_0)), C^b_{\text{unif}}(\mathbb{R}^N))$,

where $T_{\text{max}}^\infty(u_0)$, $T_{\text{max}}^p(u_0)$, and $T_{\text{max}}^p(u_0)$ are as in Theorems 1.1, 1.2, and 1.3 respectively.

Next, we study the global existence of classical solutions of (1.4). The following are the main results on the global existence.

**Theorem 1.5.** Suppose that $\chi \leq b$. Then for every nonnegative $u_0 \in C_{\text{unif}}^b(\mathbb{R}^N)$, (1.4) has a unique global classical solution $(u(x,t); u_0), v(x,t; u_0))$ satisfying that $u(\cdot, \cdot; u_0) \in C([0,\infty), C_{\text{unif}}^b(\mathbb{R}^N))$. Furthermore if $\chi < b$, the solution is globally bounded.

**Theorem 1.6.** Suppose that $u_0 \in L^1(\mathbb{R}^N)$ satisfies the hypothesis of Theorem 1.2 and that

$$\frac{N}{2} < \frac{\chi}{(\chi - b)_+} \quad (1.15)$$

Then (1.4) has a unique global classical solution $(u(x,t; u_0), v(x,t; u_0))$ satisfying that $u(\cdot, \cdot; u_0) \in C([0,\infty), X^\alpha)$. Furthermore, it holds that

$$\|u(t)\|_{L^\infty} \leq C_1 t^{-\frac{N}{p}} e^{-t} \|u_0\|_{L^p} + C_2 \left[\|u_0\|_{L^p} + K_p \|u_0\|_1^{\frac{\lambda}{p}} \|u_0\|^{\frac{\tilde{\lambda}}{p}} \left(1 + \frac{1}{t} e^{(\lambda_p - 1)at}\right)\right] e^{at} \quad (1.16)$$

for any $t > 0$, where $K_p, \lambda_p, \tilde{\lambda}_p, \tilde{\lambda}, C_1$ and $C_2$ are positive constants depending on $N$, $p$, $a$, $b$, and $\chi$.  

---

5
Theorem 1.7. Let $N$ be a positive integer and $p$ be a positive real number with $p > N$ and $p \geq 2$. Suppose that (1.15) holds. Then for every nonnegative initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, (1.4) has a unique global classical solution $(u(x,t;u_0), v(x,t;u_0))$ satisfying that $u(\cdot,\cdot;u_0) \in C([0,\infty), L^p(\mathbb{R}^N)) \cap C([0,\infty), L^1(\mathbb{R}^N))$ and that (1.16) holds.

We point out that if the domain is bounded, it allows us to take advantage of the fact that the domain has finite measure to obtain that the solution is globally bounded. However, in the present case where domain has infinite size, no such trick can be used. This makes the study of this problem on unbounded domain more complicated. We also point out that the global solution in Theorem 1.6 or Theorem 1.7 may not be bounded in $L^p(\mathbb{R}^N)$-norm (see the remarks after Theorem 1.9). We remark that Theorem 1.5 requires less assumption on the initial data and assumption on the parameters $(\chi, b, N)$ while Theorem 1.6 requires more assumption on the initial data and less assumption on the parameters. Theorem 1.7 generalizes the known results when (1.4) is studied on bounded domain with Neumann boundary condition. In the case of bounded domains, under (1.15), it follows from our results that (1.4) has a unique global classical solution with given initial function $u_0 \in L^p(\Omega)$. It is obvious that $C^0(\overline{\Omega}) \subset L^p(\Omega)$ for every $p \geq 1$ when $\Omega$ is bounded; then our results cover initial data in $C^0(\overline{\Omega})$.

Finally, we explore the asymptotic behavior of global classical solutions of (1.4) and obtain the following main results.

Theorem 1.8. Suppose that $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ with $\inf_{x \in \mathbb{R}^N} u_0(x) > 0$. If

$$b > 2\chi,$$

(1.17)

then the unique global classical solution $(u(x,t;u_0), v(x,t;u_0))$ of (1.4) with $u(x,0;u_0) = u_0(x)$ satisfies that

$$\|u(\cdot,t;u_0) - \frac{a}{b}\|_{\infty} + \|v(\cdot,t;u_0) - \frac{a}{b}\|_{\infty} \to 0 \text{ as } t \to \infty.$$  

(1.18)

Theorem 1.9. Assume that

$$\chi < \frac{2b}{3 + \sqrt{aN} + 1}.$$  

(1.19)

(1) Suppose that $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ is nonnegative and supp$(u_0)$ is non-empty. There is $c_{\text{low}}(u_0) > 0$ such that the unique global classical solution $(u(x,t;u_0), v(x,t;u_0))$ of (1.4) satisfies that

$$\lim_{t \to \infty} \left[ \sup_{|x| \leq ct} |u(x,t;u_0) - \frac{a}{b}| + \sup_{|x| \leq ct} |v(x,t;u_0) - \frac{a}{b}| \right] = 0 \quad \forall \ 0 \leq c < c_{\text{low}}^*.$$  

(1.20)

(2) Suppose that $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ is nonnegative and supp$(u_0)$ is non-empty and compact. There is $c_{\text{up}}(u_0) \geq c_{\text{low}}^*(u_0)$ such that the unique global classical solution $(u(x,t;u_0), v(x,t;u_0))$ of (1.4) satisfies that

$$\lim_{t \to \infty} \left[ \sup_{|x| \geq ct} u(x,t;u_0) + \sup_{|x| \geq ct} v(x,t;u_0) \right] = 0 \quad \forall \ c > c_{\text{up}}^*.$$  

(1.21)
We remark that \((\frac{\alpha}{b}, \frac{\alpha}{b})\) is usually called a trivial equilibrium solution of (1.4). By Theorem 1.8 when the logistic damping constant \(b\) is large, the trivial equilibrium \((\frac{\alpha}{b}, \frac{\alpha}{b})\) is globally stable with strictly positive initial data. As mentioned in the above, for (1.1) in the bounded domain with \(m(u) = 1, \chi(u, v) = \chi_u, f(u, v) = au - bu^2\), and \(g(u, v) = u - v\), the global stability of the trivial solution \((\frac{\alpha}{b}, \frac{\alpha}{b})\) has been obtained in [38] when \(\tau > 0\), and in [38] when \(\tau = 0\). It is worthwhile mentioning that, when \(b\) is not large, there may be lots of nontrivial equilibria - at least in bounded domains, quite a few have been detected (see [21], [38]).

We also remark that it is not required that \(\text{supp}(u_0)\) is compact in Theorem 1.9(1). Hence it applies to nonnegative \(u_0\) in Theorems 1.6 and 1.7. Then by (1.20), if \(p > N, p \geq 2, \chi < \frac{2b}{3+\sqrt{1+Na}}\) and \(u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \setminus \{0\}\), then
\[
\lim_{t \to \infty} \|u(\cdot, t; u_0)\|_{L^p(\mathbb{R}^N)} = \infty.
\]

The limit properties in (1.20) and (1.21) reflect the spreading feature of the mobile species. In the absence of the chemotaxis (i.e. \(\chi = 0\)), the first equation in (1.4) becomes the following scalar reaction diffusion equation,
\[
\frac{d}{dt}u_t = \Delta u + u(a - bu), \quad x \in \mathbb{R}^N, \quad t > 0,
\]
which is referred to as Fisher or KPP equations due to the pioneering works by Fisher [9] and Kolmogorov, Petrovsly, Piscunov [20] on the spreading properties of (1.23). It follows from the works [9], [20], and [11] that \(c^\ast_{\text{low}}(u_0)\) and \(c^\ast_{\text{up}}(u_0)\) in Theorem 1.9 can be chosen so that \(c^\ast_{\text{low}}(u_0) = c^\ast_{\text{up}}(u_0) = 2\sqrt{a}\) for any nonnegative \(u_0 \in C^0_\text{unif}(\mathbb{R}^N)\) with \(\text{supp}(u_0)\) being not empty and compact \((c^\ast := 2\sqrt{a}\) is called the spatial spreading speed of (1.23) in literature), and that (1.24) has traveling wave solutions \(u(t, x) = \phi(x - ct)\) connecting \(\frac{\alpha}{b}\) and 0 (i.e. \(\phi(-\infty) = \frac{\alpha}{b}, \phi(\infty) = 0\)) for all speeds \(c \geq c^\ast\) and has no such traveling wave solutions of slower speed. Since the pioneering works by Fisher [9] and Kolmogorov, Petrovsly, Piscunov [20], a huge amount research has been carried out toward the spreading properties of reaction diffusion equations of the form,
\[
\frac{d}{dt}u_t = \Delta u + uf(t, x, u), \quad x \in \mathbb{R}^N,
\]
where \(f(t, x, u) < 0\) for \(u \gg 1, \partial_u f(t, x, u) < 0\) for \(u \geq 0\) (see [3], [4], [10], [11], [25], [26], [29], [30], [33], [34], [13], [53], etc.).

When \(\chi > 0\), up to our best knowledge, the spreading properties of (1.4) is studied for the first time in this paper. It remains open whether \(c^\ast_{\text{low}}(u_0)\) and \(c^\ast_{\text{up}}(u_0)\) in (1.20) and (1.21) can be chosen so that \(c^\ast_{\text{low}}(u_0)\) and \(c^\ast_{\text{up}}(u_0)\) are independent of \(u_0\); whether \(c^\ast_{\text{low}}(u_0) = c^\ast_{\text{up}}(u_0)\); and what is the relation between \(c^\ast_{\text{low}}(u_0)\), \(c^\ast_{\text{up}}(u_0)\) and \(2\sqrt{a}\). These questions are very important in the understanding of the spreading feature of (1.4) because they are related to the issue whether the chemotaxis speeds up or slows down the spreading of the species. We plan to study these problems in our future works. Another interesting question about (1.4) is the existence of traveling wave solutions connecting \((\frac{\alpha}{b}, \frac{\alpha}{b})\) and \((0, 0)\). We also plan to study the existence of such solutions in our future works.

The rest of the paper is organized as follows. In section 2, we collect some important results from literature that will be needed in the proofs of our main results. Section 3 is devoted to the proofs of the local existence theorems (i.e. Theorems 1.1 to 1.4). In section 4, we prove the global existence theorems (i.e. Theorems 1.5 to 1.7). Finally in section 6, we present the asymptotic behavior of classical solutions and prove Theorems 1.8 and 1.9.
Acknowledgment. The authors would like to thank Professors J. Ignacio Tello and Michael Winkler for valuable discussions, suggestions, and references.

2 Preliminaries

In this section, we shall prepare several lemmas which will be used often in the next sections. We start by stating some standard definitions from semigroup theory. The reader is referred to [15], [32] for the details.

Let $X$ be a Banach space and $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ generated by $A$. It is well known that $A$ is closed and densely defined linear operator on $X$. Furthermore, there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{t\omega}$ for every $t \geq 0$ and $(\omega, \infty) \subset \rho(A)$ with

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{\lambda - \omega} \quad \forall \lambda > \omega,$$

where $\rho(A)$ denotes the resolvent set of $A$. Moreover for every $t > 0$ and every continuous function $w \in C([0, t] : X)$, the map

$$[0, t] \ni s \mapsto T(t - s)w(s) \in X$$

is continuous.

For our purpose, we will be concerned with the spaces $C^b_{\text{unif}}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$, and the analytic semigroup $T(t)$ generated by $A = \Delta - I$ on $X = C^b_{\text{unif}}(\mathbb{R}^N)$ or $X = L^p(\mathbb{R}^N)$. Observe that

$$(T(t)u)(x) = e^{-t}(G(\cdot, t) * u)(x) = \int_{\mathbb{R}^N} e^{-t}G(x - y, t)u(y)dy \quad (2.2)$$

for every $u \in X$, $t \geq 0$, and $x \in \mathbb{R}^N$, where $X = C^b_{\text{unif}}(\mathbb{R}^N)$ or $X = L^p(\mathbb{R}^N)$, and $G(x, t)$ is the heat kernel defined by

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}. \quad (2.3)$$

Let $X = C^b_{\text{unif}}(\mathbb{R}^N)$ or $X = L^p(\mathbb{R}^N)$ and $X^\alpha = \text{Dom}((I - \Delta)^\alpha)$ be the fractional power spaces of $I - \Delta$ on $X$ ($\alpha \in [0, \infty)$). Note that $X^0 = X$ and $X^1 = \text{Dom}(I - \Delta)$. It is well known that $\Delta$ generates a contraction $C_0$-semigroup defined by the heat kernel, $\{G(t)\}_{t \geq 0}$, on $X$ with spectrum $\sigma(\Delta) = (-\infty, 0]$ (see [15]). Thus, the Hille-Yosida theorem implies that the resolvent operator $R(\lambda)$ associated with $\Delta$ is the Laplace transform of $\{G(\cdot, t)\}_t$. Thus the operator $\Delta - I$ is invertible with

$$(I - \Delta)^{-1}u = \int_0^\infty e^{-t}G(\cdot, t) * udt = \int_0^\infty T(t)udt \quad (2.4)$$

for all $u \in X$. Furthermore the restriction operator $(\Delta - I)^{-1}|_{X^\alpha} : X^\alpha \to X^\alpha$ is a bounded linear map.

Our approach to prove Theorems [1.1], [1.2], and [1.3] is first to prove the existence of a mild solution with giving initial function $u_0$ and then to prove the mild solution is a classical solution, which will be partially achieved by the tools from semigroup theory. Hence it is necessary to collect some results that will be used from the semigroup theory. In this regards, we recall the following theorems on the existence of mild and classical solutions of

$$\begin{cases}
  u_t = (\Delta - 1)u + \tilde{F}(t, u), & t > t_0 \\
  u(t_0) = u_0.
\end{cases} \quad (2.5)$$
For given \( u_0 \in X^\alpha \), \( u(t) \) is called a mild solution of (2.5) on \([t_0, T]\) if \( u \in C([t_0, T), X^\alpha) \) and
\[
u(t) = T(t)u_0 + \int_{t_0}^t T(t-s)\tilde{F}(s,u(s))ds \quad \text{for} \quad s \in (t_0, T).
\]

**Theorem 2.1.** ([13 Theorem 3.3.3, Theorem 3.3.4, Lemma 3.5.1]) Assume that \( 0 \leq \alpha < 1 \), \( U \) an open subset of \( \mathbb{R} \times X^\alpha \), and \( F : U \to X \) is locally Hölder continuous and locally Lipschitz continuous in \( u \). Then for any \((t_0, u_0) \in U\) there exists \( T = T(u_0) > t_0 \) such that (2.6) has a unique mild solution \( u(t_0; u_0) \) on \([t_0, T)\) with initial value \( u(t_0; t_0, u_0) = u_0 \). Moreover, \( u(\cdot; t_0, u_0) \in C^1((t_0, T), X) \); \( u(t_0; t_0, u_0) \in \text{Dom}(\Delta - I) \) for \( t \in (t_0, T) \); (2.5) holds in \( X \) for \( t \in [t_0, T) \); and the mappings
\[
(t_0, T) \ni t \mapsto u(\cdot; t_0, u_0) \in X^\alpha, \quad (t_0, T) \ni t \mapsto \partial u(\cdot; t_0, u_0) \in X^\gamma
\]
are locally Hölder continuous for \( 0 < \gamma \ll 1 \).

**Theorem 2.2.** ([13 Theorem 3.3.4]) Assume that \( \tilde{F} \) is in as the previous theorem, and also assume that for every closed bounded set \( B \subset U \), the image \( \tilde{F}(B) \) is bounded in \( X \). If \( u(t_0; t_0, u_0) \) is a solution of (2.5) on \([t_0, t_1]\) and \( t_1 \) is maximal in the sense that there is no solution of (2.6) on \([t_0, t_2]\) if \( t_2 > t_1 \) (when \( t_1 < \infty \)), then either \( t_1 = +\infty \) or else there exists a sequence \( t_n \to t_1 \) as \( n \to +\infty \) such that \( u(t_n; t_0, u_0) \to \partial U \). (If \( U \) is unbounded, the point at infinity is included in \( \partial U \).)

**Theorem 2.3.** ([13 Theorem 7.1.3]) Assume that \( 0 \leq \alpha < 1 \), \( \tilde{F}(t, u) = B(t)u \), and \([t_0, T] \ni t \mapsto B(t) \in L(X^\alpha, X) \) is Hölder continuous. Then for any \( u_0 \in X \), there is a unique \( u(\cdot; t_0, u_0) \in C([t_0, T), X) \) such that \( u(t_0; t_0, u_0) = u_0 \); \( u(t; t_0, u_0) \in \text{Dom}(I - \Delta) \); \( u(\cdot; t_0, u_0) \in C^1((t_0, T], X) \); and \( u(t; t_0, u_0) \) satisfies (2.5) in \( X \) for \( t \in [t_0, T) \). Moreover, the mapping \((t_0, T) \ni t \mapsto u(t; t_0, u_0) \in X^\beta \) is locally Hölder continuous for any \( 0 \leq \beta < 1 \).

For given \( u_0 \in X^\alpha \) (0 ≤ \( \alpha < 1 \)) and \( T > 0 \), \([0, T] \ni t \mapsto (u(\cdot; t; u_0), v(\cdot; t; u_0)) \) \( \in X^\alpha \times X^\alpha \) is called a mild solution of (1.4) on \([0, T]\) with initial function \( u_0 \) if \( u(\cdot; \cdot; u_0), v(\cdot; \cdot; u_0) \in C([0, T], X^\alpha) \), \( v(\cdot; t; u_0) = -(\Delta - I)^{-1}(u(\cdot; t; u_0)) \), and \( u(t) := u(\cdot; t; u_0) \) satisfies
\[
u(t) = T(t)u_0 + \int_0^t T(t-s)(\chi \nabla \cdot (v(s)\nabla (\Delta - I)^{-1}(u(s))) + (a + 1)u(s) - bu^2(s))ds
\]
for \( t \in [0, T) \). Note that, if \((u_0, v_0) \) is a mild solution of (1.4) with initial function \( u_0 \in X^\alpha \), then \( u(t) := u(\cdot; t; u_0) \) is a mild solution of the Cauchy problem (initial value problem)
\[
\begin{align*}
\begin{cases}
u_t = (\Delta - I)u + F(u) & \text{in} \ (0, T) \\
u(0) = u_0 & \text{in} \ X^\alpha,
\end{cases}
\end{align*}
\]
(2.6)
where \( F(u) = \chi \nabla \cdot (u\nabla (\Delta - I)^{-1}u) + (a + 1)u - bu^2 \). Conversely, for given \( u_0 \in X^\alpha \), if \( u(t) \) is a mild solution of (2.6), then \((u(\cdot; t; u_0), v(\cdot; t; u_0)) \) is a mild solution of (1.4) with initial function \( u_0 \), where \( u(\cdot; t; u_0) = u(t) \) and \( v(\cdot; t; u_0) = -(\Delta - I)^{-1}(u(t)) \).

We next present some important embedding results on the fractional spaces in the case that \( X = L^p(\mathbb{R}^N) \) (see [15]). Let \( A = \Delta - I \). We have that \( \text{Dom}(A) = W^{2,p}(\mathbb{R}^N) \) (1 ≤ \( p < \infty \) and
the following continuous imbeddings

\[
X^\alpha \subset C^\nu \quad \text{if} \quad 0 \leq \nu < 2\alpha - \frac{N}{p}; \quad (2.7)
\]
\[
X^\alpha \subset W^{1,q}(\mathbb{R}^N) \quad \text{if} \quad \alpha > \frac{1}{2} \quad \text{and} \quad \frac{1}{q} > \frac{1}{p} - \frac{(2\alpha - 1)}{N}; \quad (2.8)
\]
\[
X^\alpha \subset L^q \quad \text{if} \quad \frac{1}{q} > \frac{2\alpha}{N}, \quad q \geq p. \quad (2.9)
\]

We have that \(X^\alpha = W^{1,p}(\mathbb{R}^N)\). Furthermore, there is a constant \(C\alpha\) such that

\[
\| (T(t) - I)u \|_{L^p} \leq C\alpha t^\alpha \| u \|_{X^\alpha} \quad \text{for all} \quad u \in X^\alpha. \quad (2.10)
\]

Using the \(L^p - L^q\) estimates for the convolution product, concretely,

\[
\| f * g \|_{L^q(\mathbb{R}^N)} \leq \| f \|_{L^p(\mathbb{R}^N)} \| g \|_{L^{p'}(\mathbb{R}^N)} \quad \forall 1 \leq p, q, r \leq +\infty \quad \text{and} \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \quad (2.11)
\]

we can easily show that the analytic semigroup \(\{T(t)\}_{t \geq 0}\) generated by \(\Delta - I\) on \(L^p(\mathbb{R}^N)\) enjoys the following \(L^p - L^q\) a priori estimates:

\[
\begin{cases}
\| T(t)u \|_{L^p(\mathbb{R}^N)} \leq Ct^\frac{\alpha - 1}{2} e^{-t} \| u \|_{L^p(\mathbb{R}^N)} \quad \text{for} \quad 1 \leq p \leq q \leq +\infty \\
\| (I - \Delta)\alpha T(t)u \|_{L^q} \leq C\alpha t^{-\alpha} \| u \|_{L^p(\mathbb{R}^N)} \quad \text{for} \quad 1 \leq p \leq q \leq +\infty
\end{cases} \quad (2.12)
\]

for every \(t > 0\) and \(\alpha \geq 0\), where \(C\) and \(C\alpha\) are constant depending only on \(p, q\) and \(N\). In fact the first inequality in (2.12) is a direct consequence of (2.11), while the second is a result of the combination of Theorem 1.4.3 in [15] and the first inequality.

We now state a result that will be needed in the proof of time global existence theorem. The result is a variant of Gagliardo-Nirenberg inequality.

**Lemma 2.4.** ([15] Lemma 2.4) Let \(N \geq 1\), \(a_0 > 2\), \(u \in L^{q_1}(\mathbb{R}^N)\) with \(q_1 \geq 1\) and \(u^\frac{r}{2} \in H^1(\mathbb{R}^N)\) with \(r > 1\). If \(q_1 \in [1, r], q_2 \in [\frac{r}{2}, a_0 \frac{r}{2}]\) and

\[
\begin{cases}
1 \leq q_1 \leq q_2 \leq \infty \quad \text{when} \quad N=1, \\
1 \leq q_1 \leq q_2 < \infty \quad \text{when} \quad N=2, \\
1 \leq q_1 \leq q_2 \leq \frac{Nq_1}{N-2} \quad \text{when} \quad N \geq 3,
\end{cases} \quad (2.13)
\]

then, it holds that

\[
\| u \|_{L^{q_2}(\mathbb{R}^N)} \leq C\frac{q_1}{q_2} \| u \|_{L^{q_1}(\mathbb{R}^N)}^{1-\theta} \| \nabla u^\frac{r}{2} \|_{L^2(\mathbb{R}^N)}^{2\theta} \quad (2.14)
\]

with

\[
\theta = r \left( \frac{1}{q_1} - \frac{1}{q_2} \right) \left( \frac{1}{N} - \frac{1}{2} + \frac{r}{2q_1} \right)^{-1},
\]

where

\[
\begin{cases}
C \text{ depends only on } N \text{ and } a_0 \text{ when } q_1 \geq \frac{r}{2}, \\
C = C_0^{\frac{r}{2}} \quad \text{with } C_0 \text{ depending only on } N \text{ and } a_0 \text{ when } 1 \leq q_1 < \frac{r}{2},
\end{cases}
\]

and

\[
\beta = \frac{r}{q_2 - q_1} \left[ \frac{2q_1}{r} + \left( 1 - \frac{2q_1}{r} \right) \frac{2N}{N+2} \right].
\]
We end this section by stating an important result that will be needed to complete the proof of the main theorem.

**Lemma 2.5. (Exercise 4*, page 190)** Assume that $a_1, a_2, \alpha, \beta$ are nonnegative constants, with $0 \leq \alpha, \beta < 1$, and $0 < T < \infty$. There exists a constant $M(a_2, \beta, T) < \infty$ so that for any integrable function $u : [0, T] \to \mathbb{R}$ satisfying that

$$0 \leq u(t) \leq a_1 t^{-\alpha} + a_2 \int_0^t (t-s)^{-\beta} u(s) ds$$

for a.e $t$ in $[0, T]$, we have

$$0 \leq u(t) \leq \frac{a_1 M}{1-\alpha} t^{-\alpha}, \text{ a.e. on } 0 < t < T.$$

### 3 Local existence of classical solutions

In this section, we investigate the local existence and uniqueness of classical solutions of (1.4) with various given initial functions and prove Theorems 1.1, 1.2, 1.3, and 1.4. We first establish some important lemmas.

**Lemma 3.1.** Let $p \in [1, \infty)$ and $\{T(t)\}_{t>0}$ be the semigroup in (2.2) generated by $\Delta - I$ on $L^p(\mathbb{R}^N)$. For every $t > 0$, the operator $T(t) \nabla \cdot$ has a unique bounded extension on $(L^p(\mathbb{R}^N))^N$ satisfying

$$\|T(t)\nabla \cdot u\|_{L^p(\mathbb{R}^N)} \leq C_1 t^{-\frac{1}{2}} e^{-t\|u\|_{L^p(\mathbb{R}^N)}} \quad \forall u \in (L^p(\mathbb{R}^N))^N, \forall t > 0,$$

(3.1)

where $C_1$ depends only on $p$ and $N$. Furthermore, for every $q \in [p, \infty]$, we have that $T(t)\nabla \cdot u \in L^q(\mathbb{R}^N)$ with

$$\|T(t)\nabla \cdot u\|_{L^q} \leq C_2 t^{-\frac{1}{2}} \frac{N}{2} e^{-t\|u\|_{L^p(\mathbb{R}^N)}} \quad \forall u \in (L^p(\mathbb{R}^N))^N, \forall t > 0,$$

(3.2)

where $C_2$ is constant depending only on $N$, $q$ and $p$.

**Proof.** Since $C^\infty_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, it is enough to prove that inequalities (3.1) and (3.2) hold on $(C^\infty_c(\mathbb{R}^N))^N$. For every $u = (u_1, u_2, \cdots, u_N) \in (C^\infty_c(\mathbb{R}^N))^N$ and $t > 0$, using integration by parts, we obtain that

$$T(t)\partial_{x_i} u_i = \frac{e^{-t}}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}} \partial_{x_i} u_i(x-z) dz = \frac{e^{-t}}{2t (4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} z_i e^{-\frac{|z|^2}{4t}} u_i(x-z) dz.$$

Using the $L^p - L^q$ estimates (2.11), we have that

$$\|T(t)\partial_{x_i} u_i\|_{L^p(\mathbb{R}^N)} \leq \frac{e^{-t}}{2t (4\pi t)^{\frac{N}{2}}} \|H_i(\cdot, t)\|_{L^1(\mathbb{R}^N)} \|u_i\|_{L^p(\mathbb{R}^N)}$$

(3.3)

and

$$\|T(t)\partial_{x_i} u_i\|_{L^q(\mathbb{R}^N)} \leq \frac{e^{-t}}{2t (4\pi t)^{\frac{N}{2}}} \|H_i(\cdot, t)\|_{L^r(\mathbb{R}^N)} \|u_i\|_{L^q(\mathbb{R}^N)},$$

(3.4)
where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{p}$ and

$$H_i(z, t) = z_i e^{-\frac{|z|^2}{4t}}. \quad (3.5)$$

Making change of variable $z = \sqrt{4t}y$, we obtain that

$$\|H_i(\cdot, t)\|_{L^r(\mathbb{R}^N)} = (4t)^{\frac{1}{2} + \frac{N}{r}} \|H_i(\cdot, 1)\|_{L^r(\mathbb{R}^N)} = (4t)^{\frac{1}{2} + \frac{N}{r} (1 + \frac{1}{p} - \frac{1}{q})} \|H_i(\cdot, 1)\|_{L^r(\mathbb{R}^N)}. \quad (3.6)$$

Combining the fact that

$$t^{\frac{1}{2} + \frac{N}{r}} = \left\{ \begin{array}{ll} t^{-\frac{1}{2} - \frac{N}{r} (\frac{1}{p} - \frac{1}{q})} & \text{if } q > p \\ t^{-\frac{1}{2}} & \text{if } q = p \end{array} \right. \quad (3.7)$$

with inequalities (3.8), (3.9) and (3.10) we obtain inequalities (3.1) and (3.2).

Considering the analytic semigroup $\{e^{-t\Delta}\}_{t \geq 0}$ generated by $\Delta$ on bounded domains coupled with Neumann boundary condition, a result in the style of inequality (3.1) was first established in [16] (Lemma 2.1) and inequality (3.2) was later obtained in [13] (Lemma 3.3). The authors in [16] and [13] used different methods to establish these results. It should be noted that the proof presented in [16] is difficult to be adapted for the whole space because it is based on the measure of the domain. Taking advantage of the explicit formula of the analytic semigroup $\{e^{-t\Delta}\}_{t \geq 0}$ generated by $\Delta$ on the whole space $\mathbb{R}^N$, our proof is simpler and yields the same results.

Note that $C_c^\infty(\mathbb{R}^N)$ is not a dense subset of $C^b_{\text{unif}}(\mathbb{R}^N)$. Hence the arguments used in the previous proof can not be applied directly on $C^b_{\text{unif}}(\mathbb{R}^N)$. This problem can be overcome by choosing an adequate dense subset. This leads to a version of this result on $C^b_{\text{unif}}(\mathbb{R}^N)$ that we formulate in the next lemma.

**Lemma 3.2.** Let $T(t)$ be the semigroup in (2.2) generated by $\Delta - I$ on $C^b_{\text{unif}}(\mathbb{R}^N)$. For every $t > 0$, the operator $T(t)\nabla \cdot$ has a unique bounded extension on $(C^b_{\text{unif}}(\mathbb{R}^N))^N$ satisfying

$$\|T(t)\nabla \cdot u\|_\infty \leq \frac{N}{\sqrt{\pi t}} t^{-\frac{1}{2}} e^{-\frac{t}{4}} \|u\|_\infty \quad \forall u \in (C^b_{\text{unif}}(\mathbb{R}^N))^N, \forall t > 0. \quad (3.8)$$

**Proof.** Let

$$C^{1,b}_{\text{unif}}(\mathbb{R}^N) = \{ u \in C^1(\mathbb{R}^N) | u(\cdot), \partial_x u(\cdot) \in C^b_{\text{unif}}(\mathbb{R}^N), i = 1, 2, \cdots, N \}.$$ 

Since $C^{1,b}_{\text{unif}}(\mathbb{R}^N)$ is dense in $C^b_{\text{unif}}(\mathbb{R}^N)$, it is enough to prove that inequalities (3.8) hold on $(C^{1,b}_{\text{unif}}(\mathbb{R}^N))^N$. For every $u = (u_1, u_2, \cdots, u_N) \in (C^{1,b}_{\text{unif}}(\mathbb{R}^N))^N$ and $t > 0$, we have

$$T(t)\partial_x u_i = \frac{e^{-t}}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-z|^2}{4t}} \partial_x u_i(x-z) dz = \lim_{R \to \infty} \left[ \frac{e^{-t}}{(4\pi t)^{\frac{N}{2}}} \int_{B(0,R)} e^{-\frac{|x|^2}{4t}} \partial_x u_i(x) dz \right]. \quad (3.9)$$

Next, for every $R > 0$ using integration by parts, we have

$$\int_{B(0,R)} e^{-\frac{|x|^2}{4t}} \partial_x u_i(x-z) dz = \frac{1}{2t} \int_{B(0,R)} z_i e^{-\frac{|x|^2}{4t}} u_i(x-z) dz - \int_{\partial B(0,R)} e^{-\frac{|x|^2}{4t}} u_i(x-z) \nu_i(z) ds(z)$$

$$= \frac{1}{2t} \int_{B(0,R)} z_i e^{-\frac{|x|^2}{4t}} u_i(x-z) dz - e^{-\frac{R^2}{4t}} \int_{\partial B(0,R)} u_i(x-z) \frac{z_i}{R} ds(z). \quad (3.10)$$
Since $u$ is uniformly bounded and the function $z \in \mathbb{R}^N \mapsto z_i e^{-\frac{|z|^2}{4R}}$ belongs to $L^1(\mathbb{R}^N)$, then
\[
\lim_{R \to \infty} \frac{1}{2t} \int_{B(0,R)} z_i e^{-\frac{|z|^2}{4R}} u_i(x - z)dz = \frac{1}{2t} \int_{\mathbb{R}^N} z_i e^{-\frac{|z|^2}{4R}} u_i(x - z)dz.
\] (3.11)

On the other hand we have
\[
\left| e^{-\frac{R^2}{4t}} \int_{\partial B(0,R)} u_i(x - z) \frac{\partial_i}{R} ds(z) \right| \leq \|u_i\|_{\infty} e^{-\frac{R^2}{4t}} \int_{\partial B(0,R)} ds(z) \to 0 \text{ as } R \to \infty.
\] (3.12)

Combining (3.9), (3.10), (3.11) and (3.12), we obtain that
\[
T(t)\partial_x u_i = \frac{e^{-t}}{2t \sqrt{(4\pi t)^N}} \int_{\mathbb{R}^N} z_i e^{-\frac{|z|^2}{4t}} u_i(x - z)dz = \frac{e^{-t}}{2t \sqrt{(4\pi t)^N}} \int_{\mathbb{R}^N} H_i(z,t)u_i(x - z)dz,
\]
where the function $H_i$ is defined by (3.5). Thus, using (3.6) and (3.7), we obtain that
\[
\|T(t)\partial_x u_i\|_{\infty} \leq \frac{t^{-\frac{1}{2}} e^{-t}}{\sqrt{\pi}} \|H_i(\cdot,1)\|_{L^1(\mathbb{R}^N)} \|u_i\|_{\infty}.
\]

Direct computations yield that
\[
\|T(t)\partial_x u_i\|_{\infty} \leq \frac{t^{-\frac{1}{2}} e^{-t}}{\sqrt{\pi}} \|u_i\|_{\infty}.
\] (3.13)

Inequality (3.8) easily follows from (3.13). \hfill \Box

In the next Lemma, we shall provide an explicit a priori estimate of the gradient of the solution $v(\cdot, \cdot)$ in the second equation of (1.4). This a priori estimate will be useful in the proof of existence theorem and the discussion on the asymptotic behavior of the solution.

**Lemma 3.3.** For every $u \in C^b_{unif}(\mathbb{R}^N)$, we have that
\[
\|\partial_{x_i} (\Delta - I)^{-1} u\|_{\infty} \leq \|u\|_{\infty}
\] (3.14)
for each $i = 1, \cdots, N$. Therefore we have
\[
\|\nabla (\Delta - I)^{-1} u\|_{\infty} \leq \sqrt{N} \|u\|_{\infty}
\] (3.15)
for every $u \in C^b_{unif}(\mathbb{R}^N)$.

**Proof.** Let $u \in C^b_{unif}(\mathbb{R}^N)$ and set $v = (I - \Delta)^{-1} u$. According to (2.14) it follows that
\[
v(x) = \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-s}}{(4\pi s)^N} e^{-\frac{|x-z|^2}{4s}} u(z)dzds
\]
for every $x \in \mathbb{R}^N$. Hence
\[
\partial_{x_i} v(x) = \int_0^\infty \int_{\mathbb{R}^N} \frac{(z_i - x_i)e^{-s}}{2s(4\pi s)^N} e^{-\frac{|x-z|^2}{4s}} u(z)dzds
\]
\[
= \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-s}}{\pi^{\frac{N}{2}} \sqrt{s}} H_i(z,1)u(x - z)dzds.
\]
Thus, using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\|H(\cdot, 1)\|_{L^1(\mathbb{R}^N)} = \pi^{\frac{N-1}{2}}$, we obtain that

$$|\partial_x v(x)| \leq \frac{1}{\pi^{\frac{1}{2}}} \left[ \int_0^\infty s^{-\frac{1}{2}} e^{-s} ds \right] ||H_i(\cdot, 1)||_{L^1(\mathbb{R}^N)} ||u||_\infty = ||u||_\infty.$$  

The lemma thus follows. $\square$

Next, we prove Theorems 1.1, 1.2, 1.3, and 1.4 in subsections 3.1, 3.2, 3.3, and 3.4, respectively. Throughout subsections 3.1, 3.2, 3.3, and 3.4, $C$ denotes a constant independent of the initial functions and the solutions under consideration, unless specified otherwise.

3.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. The main tools for the proof of this theorem are based on the contraction mapping theorem and the existence of classical solutions for linear parabolic equations with Hölder continuous coefficients. Throughout this subsection, $X = C^b_{unif}(\mathbb{R}^N)$ and $X^\alpha$ is the fractional power space of $I - \Delta$ on $X$ ($\alpha \in (0, 1)$).

Proof of Theorem 1.1  

(i) **Existence of mild solution.** We first prove the existence of a mild solution of (2.6) with given initial function $u_0 \in C^b_{unif}(\mathbb{R}^N)$, which will be done by proving five claims.

Fix $u_0 \in C^b_{unif}(\mathbb{R}^N)$. For every $T > 0$ and $R > 0$, let

$$S_{R,T} := \left\{ u \in C([0, T], C^b_{unif}(\mathbb{R}^N)) \mid ||u||_X \leq R \right\}.$$  

Note that $S_{R,T}$ is a closed subset of the Banach space $C([0, T], C^b_{unif}(\mathbb{R}^N))$ with the sup-norm.

**Claim 1.** For any $u \in S_{R,T}$ and $t \in [0, T]$, $(Gu)(t)$ is well defined, where

$$(Gu)(t) = T(t)u_0 + \chi \int_0^t T(t-s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))ds$$

$$+ (1 + a) \int_0^t T(t-s)u(s)ds - b \int_0^t T(t-s)u^2(s)ds,$$

and the integrals are taken in $C^b_{unif}(\mathbb{R}^N)$. Indeed, let $u \in S_{R,T}$ and $0 < t \leq T$ be fixed. Since the function

$$[0, t] \ni s \mapsto (a + 1)u(s) - bu^2(s) \in C^b_{unif}(\mathbb{R}^N)$$

is continuous, then the function $F_1 : [0, t] \rightarrow C^b_{unif}(\mathbb{R}^N)$ defined by

$$F_1(s) := (1 + a)T(t-s)u(s) - bT(t-s)u^2(s)$$

is continuous. Hence the Riemann integral $\int_0^t F_1(s)ds$ in $C^b_{unif}(\mathbb{R}^N)$ exists. Observe that for every $0 < \varepsilon < t$ and $s \in [0, t - \varepsilon]$, we have

$$F_{2,\varepsilon}(s) := T(t-s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s)) = T(t - \varepsilon - s)T(\varepsilon)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s)),$$
and the function \([0, t - \varepsilon] \ni s \mapsto T(\varepsilon)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s)) \in C^b_{\text{unif}}(\mathbb{R}^N)\) is continuous. Thus the function \(F_{2, \varepsilon} : [0, t - \varepsilon] \to C^b_{\text{unif}}(\mathbb{R}^N)\) is continuous for every \(0 < \varepsilon < t\). Thus, the function \(F_2 : [0, t) \to C^b_{\text{unif}}(\mathbb{R}^N)\) defined by

\[
F_2(s) := T(t - s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))
\]

is continuous. Moreover, using Lemma 3.2 and inequality (3.15), we have that

\[
\int_0^t \|F_2(s)\|_{\infty} ds \leq \chi \int_0^t \|T(t - s)\nabla \cdot \|u(s)\|_{\infty}\|\nabla (\Delta - I)^{-1}u(s)\|_{\infty} ds
\leq \frac{N\sqrt{N}}{\sqrt{\pi}} \chi \int_0^t (t - s)^{-\frac{1}{2}} e^{-\frac{1}{2}(t - s)} \|u(s)\|_{\infty}^2 ds
\leq \frac{NR^2\sqrt{N}}{\sqrt{\pi}} \chi \Gamma\left(\frac{1}{2}\right).
\]

Hence, the Riemann integral \(\int_0^t F_2(s) ds\) in \(C^b_{\text{unif}}(\mathbb{R}^N)\) exists. Note that \((Gu)(t) = T(t)u_0 + \int_0^t F_2(s) ds + \int_0^t F_1(s) ds\). Whence, Claim 1 follows.

**Claim 2.** For every \(u \in S_{R,T} \) and \(0 < \beta < \frac{1}{2}\), the function \((0, T] \ni t \mapsto (Gu)(t) \in X^\beta\) is locally Hölder continuous, and \(G\) maps \(S_{R,T}\) into \(C([0, T], C^b_{\text{unif}}(\mathbb{R}^N))\).

First, observe that

\[
(Gu)(t) = T(t)u_0 + \chi \int_0^t \underbrace{T(t - s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))}_{I_0(t)} ds + \underbrace{\int_0^t T(t - s)(\alpha + 1)u(s) - bu^2(s)) ds}_{I_1(t)}. \quad (3.16)
\]

For every \(t > 0\), it is clear that \(T(t)u_0 \in X^\beta\) because the semigroup \(\{T(t)\}_t\) is analytic. Furthermore, the divergence operator \(T(t)\nabla \cdot\) satisfy

\[
T(t)\nabla \cdot w = T(\frac{t}{2})T(\frac{t}{2})\nabla \cdot w \in \text{Dom}(\Delta) \subset X^\beta
\]

for all \(t > 0, w \in (C^b_{\text{unif}}(\mathbb{R}^N))^N\). Using Lemma 3.2 and inequality (3.15), we obtain that

\[
\int_0^t \|T(t - s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))\|_{X^\beta} ds = \int_0^t \|(\Delta - I)^{\beta}T(t - s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))\|_{\infty} ds
\leq C \int_0^t (t - s)^{-\frac{1}{2}\beta} e^{-\frac{1}{2}(t - s)} \|u(s)\nabla (\Delta - I)^{-1}u(s)\|_{\infty} ds
\leq CR^2 \int_0^t (t - s)^{-\frac{1}{2}\beta} e^{-\frac{1}{2}(t - s)} ds
\leq CR^2 \Gamma\left(\frac{1}{2} - \beta\right). \quad (3.17)
\]

Since the operator \((\Delta - I)^{\beta}\) is closed, we have that

\[
I_1(t) = \int_0^t T(t - s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s)) ds \in X^\beta
\]
for every $t > 0$. Similar arguments show that $I_2(t) \in X^\beta$ for every $0 < t \leq T$. Hence $u(t) \in X^\beta$ for every $t > 0$.

Next, let $t \in (0, T)$ and $h > 0$ such that $t + h \leq T$. We have

\[
\|I_0(t+h) - I_0(t)\|_\beta \leq \|(T(h) - I)T(t-\cdot)\|_{X^\beta} \leq Ch^\beta e^{-h}\|T(t)u_0\|_{X^\beta} \\
\leq Ch^\beta t^{-\beta}e^{-(h+t)}\|u_0\|_\infty \leq Ch^\beta t^{-\beta}\|u_0\|_\infty, \\
\quad (3.18)
\]

\[
\|I_1(t+h) - I_1(t)\|_{X^\beta} \leq \int_0^t \|(T(h) - I)T(t-s)\|_\infty \nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))\|_{X^\beta} ds \\
+ \int_t^{t+h} \|(T(t+h-s)\|_\infty \nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))\|_{X^\beta} ds \\
\leq Ch^\beta \int_0^t (t-s)^{-\frac{1}{2}}e^{-(h+t-s)}\|u(s)\nabla (\Delta - I)^{-1}u(s)\|_{X^\beta} ds \\
+ C \int_t^{t+h} (t+h-s)^{-\frac{1}{2}}e^{-(h+t-s)}\|u(s)\nabla (\Delta - I)^{-1}u(s)\|_{X^\beta} ds \\
\leq CR^2 h^\beta \int_0^t (t-s)^{-\frac{1}{2}}e^{-(h+t-s)} ds + CR^2 \int_t^{t+h} (t+h-s)^{-\frac{1}{2}}e^{-(h+t-s)} ds \\
\leq CR^2(h^\beta + h^{\frac{1}{2}-\beta}), \\
\quad (3.19)
\]

and

\[
\|I_2(t+h) - I_2(t)\|_{X^\beta} \leq \int_0^t \|(T(h) - I)T(t-s)((a+1)u(s) - bu^2(s))\|_{X^\beta} ds \\
+ \int_t^{t+h} \|(T(t+h-s)((a+1)u(s) - bu^2(s))\|_{X^\beta} ds \\
\leq Ch^\beta \int_0^t (t-s)^{-\beta}e^{-(t+h-s)}\|(a+1)u(s) - bu^2(s)\|_{X^\beta} ds \\
+ C \int_t^{t+h} (t+h-s)^{-\beta}e^{-(t+h-s)}\|(a+1)u(s) - bu^2(s)\|_{X^\beta} ds \\
\leq CR^2(h^\beta + h^{1-\beta}). \\
\quad (3.20)
\]

Combining $(3.16)$, $(3.18)$, $(3.19)$ and $(3.20)$, we deduce that the function $(0, T) \ni t \rightarrow (Gu(t)) \in X^\beta$ is locally Holder continuous.

Now it is clear that $t \rightarrow (Gu(t)) \in C^b_{\text{unif}}(\mathbb{R}^N)$ is continuous in $t$ at $t = 0$. The claim then follows.

**Claim 3.** For every $R > \|u_0\|_\infty$, there exists $T := T(R)$ such that $G$ maps $S_{R,T}$ into itself.
First, observe that for any $u \in S_{R,T}$, we have

\[
\|G(u)(t)\|_\infty \leq \|T(t)u_0\|_\infty + \chi \int_0^t \|T(t-s) \nabla \cdot (u(s) \nabla (\Delta - I)^{-1} u(s))\|_\infty ds + (1+a) \int_0^t \|T(t-s) u(s)\|_\infty ds + b \int_0^t \|T(t-s) u^2(s)\|_\infty ds
\]

\[
\leq e^{-t} \|u_0\|_\infty + \chi \int_0^t \|T(t-s) \nabla \cdot (u(s) \nabla (\Delta - I)^{-1} u(s))\|_\infty ds + (1+a) R \int_0^t e^{-(t-s)} ds + bR^2 \int_0^t e^{-(t-s)} ds
\]

\[
= e^{-t} \|u_0\|_\infty + R ((1 + a) + bR) (1 - e^{-t}) + \chi \int_0^t \|T(t-s) \nabla \cdot (u(s) \nabla (\Delta - I)^{-1} u(s))\|_\infty ds.
\]

(3.21)

Using Lemma 3.2 and inequality (3.15), the last inequality can be improved to

\[
\|G(u)(t)\|_\infty \leq e^{-t} \|u_0\|_\infty + R ((1 + a) + bR) (1 - e^{-t}) + C\chi \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|u(s) \nabla (\Delta - I)^{-1} u(s)\|_\infty ds
\]

\[
\leq e^{-t} \|u_0\|_\infty + R ((1 + a) + bR) (1 - e^{-t}) + C\chi R^2 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} ds
\]

\[
\leq e^{-t} \|u_0\|_\infty + R ((1 + a) + bR) (1 - e^{-t}) + 2C\chi R^2 t^\frac{1}{2}.
\]

(3.22)

Now, by (3.22), we can now chose $T > 0$ such that

\[
\|G(u)(t)\|_\infty \leq e^{-t} \|u_0\|_\infty + R ((1 + a) + bR) (1 - e^{-t}) + 2C\chi R^2 t^\frac{1}{2} < R \ \forall \ t \in [0,T].
\]

This together with Claim 2 implies Claim 3.

**Claim 4.** $G$ is a contraction map for $T$ small and hence has a fixed point $u(\cdot) \in S_{R,T}$. Moreover, for every $0 < \beta < \frac{1}{2}$, $(0,T] \ni t \rightarrow u(t) \in X^\beta$ is locally Holder continuous.
For every $u, w \in \mathcal{S}_{R,T}$, using again Lemma 3.2, we have

$$
\|(G(u) - G(w))(t)\|_\infty \\
\leq \chi \int_0^t \|T(t - s)\nabla \cdot (u(s)\nabla(\Delta - I)^{-1}u(s) - w(s)\nabla(\Delta - I)^{-1}w(s))\|_\infty ds \\
+ (1 + a) \int_0^t \|T(t - s)(u(s) - w(s))\|_\infty ds + b \int_0^t \|T(t - s)(u^2(s) - w^2(s))\|_\infty ds \\
\leq C\chi \int_0^t (t - s)^{-\frac{1}{2}}e^{-(t-s)}\|(u(s)\nabla(\Delta - I)^{-1}u(s) - w(s)\nabla(\Delta - I)^{-1}w(s))\|_\infty ds \\
+ \left((1 + a) + 2Rb\right) \int_0^t e^{-(t-s)}\|(u(s) - w(s))\|_\infty ds
$$

Hence, choose $T$ small satisfying

$$
4CR\chi t^{\frac{1}{2}} + (1 + a + 2Rb)t < 1 \quad \forall \ t \in [0, T],
$$

we have that $G$ is a contraction map. Thus there is $T > 0$ and a unique function $u \in \mathcal{S}_{R,T}$ such that

$$
u(t) = T(t)u_0 + \chi \int_0^t T(t - s)\nabla \cdot (u(s)\nabla(\Delta - I)^{-1}u(s))ds + \int_0^t T(t - s)((a + 1)u(s) - bu^2(s))ds.
$$

Moreover, by Claim 2, for every $0 < \beta < \frac{1}{2}$, the function $t \in (0, T] \to u(t) \in X^\beta$ is locally H"older continuous. Clearly, $u(t)$ is a mild solution of (2.6) on $[0, T]$ with $\alpha = 0$ and $X^0 = C^b_{unif}(\mathbb{R}^N)$.

**Claim 5.** There is $T_{\text{max}} \in (0, \infty)$ such that (2.6) has a mild solution $u(\cdot)$ on $[0, T_{\text{max}})$ with $\alpha = 0$ and $X^0 = C^b_{unif}(\mathbb{R}^N)$. Moreover, for every $0 < \beta < \frac{1}{2}$, $(0, T_{\text{max}}) \ni t \mapsto u(\cdot) \in X^\beta$ is locally Hölder continuous. If $T_{\text{max}} < \infty$, then $\limsup_{t \to T_{\text{max}}} \|u(t)\|_\infty = \infty$.

This claim follows the regular extension arguments.

**(ii) Regularity and non-negativity.** We next show that the mild solution $u(\cdot)$ of (2.6) on $[0, T_{\text{max}})$ obtained in (i) is a nonnegative classical solution of (2.6) on $[0, T_{\text{max}})$ and satisfies (1.7), (1.8).

Let $0 < t_1 < T_{\text{max}}$ be fixed. It follows from claim 2 that for $0 < \nu \ll 1$, $u_1 := u(t_1) \in C^\nu_{unif}(\mathbb{R}^N)$, and the mappings

$$
t \to u(\cdot, t + t_1) := u(t + t_1)(\cdot) \in C^\nu_{unif}(\mathbb{R}^N), \ t \mapsto v(\cdot, t + t_1) \in C^\nu_{unif}(\mathbb{R}^N)
$$

18
are locally Hölder continuous in \( t \in (-t_1, T_{\text{max}} - t_1) \), where \( v(\cdot, t + t_1) := (I - \Delta)^{-1}u(\cdot, t + t_1) \) and \( i, j = 1, 2, \ldots, N \). Consider the initial value problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial}{\partial t} \tilde{u} = (\Delta - 1) \tilde{u} + \tilde{F}(t, \tilde{u}), & x \in \mathbb{R}^N, \; t > 0 \\
\tilde{u}(x, 0) = u_1(x), & x \in \mathbb{R}^N,
\end{cases}
\end{aligned}
\tag{3.23}
\]

where \( \tilde{F}(t, \tilde{u}) = -\chi \nabla v(\cdot, t + t_1) \nabla \tilde{u} + (a + 1 - \chi v(\cdot, t + t_1) - (b - \chi)u(\cdot, t + t_1))\tilde{u} \). Then by \[12\] Theorem 11 and Theorem 16 in Chapter 1, (3.23) has a unique classical solution \( \tilde{u}(x, t) \) on \( [0, T_{\text{max}} - t_1) \) with \( \lim_{t \to 0^+} \| \tilde{u}(\cdot, t) - u_1 \|_{\infty} = 0 \). In fact \( \tilde{u} \) is given by the equation

\[
\tilde{u}(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t, y, t_1) u_1(y) dy
\]

with the function \( \Gamma \) satisfying the inequalities

\[
|\Gamma(x, t, y, \tau)| \leq C e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}\text{ and } |\partial_{x_i}\Gamma(x, t, y, \tau)| \leq C e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}^\frac{N+1}{2}
\]

for every \( 0 < \lambda_0 < 1 \). By a priori interior estimates for parabolic equations (see \[12\] Theorem 5), we have that

\[
\tilde{u}(\cdot, \cdot) \in C^1((0, T_{\text{max}} - t_1), C_{\text{unif}}^{\nu}(\mathbb{R}^N),
\]

and the mappings

\[
t \mapsto \tilde{u}(\cdot, t) \in C_{\text{unif}}^{\nu}(\mathbb{R}^N), \quad t \mapsto \frac{\partial}{\partial x_i} \tilde{u}(\cdot, t) \in C_{\text{unif}}^{\nu}(\mathbb{R}^N),
\]

\[
t \mapsto \frac{\partial^2}{\partial x_i \partial x_j} \tilde{u}(\cdot, t) \in C_{\text{unif}}^{\nu}(\mathbb{R}^N), \quad t \mapsto \frac{\partial}{\partial t} \tilde{u}(\cdot, t) \in C_{\text{unif}}^{\nu}(\mathbb{R}^N)
\]

are locally Hölder continuous in \( t \in (0, T_{\text{max}} - t_1) \) for \( i, j = 1, 2, \ldots, N \) and \( 0 < \nu \ll 1 \). Hence, by \[15\] Lemma 3.3.2, \( \tilde{u}(t)(\cdot) = \tilde{u}(\cdot, t) \) is also a mild solution of (3.23) and then satisfies the following integral equation,

\[
\tilde{u}(t) = T(t)u_1 + \int_0^t T(t-s)(-\chi \nabla v(s + t_1) \nabla \tilde{u}(s) + (a + 1 - \chi v(s + t_1) - (b - \chi)u(s + t_1))\tilde{u}(s))ds
\]

for \( t \in [0, T_{\text{max}} - t_1) \).

Now, using the fact that \( \nabla \tilde{u} \nabla v(\cdot + t_1) = \nabla \cdot (\tilde{u} \nabla v(\cdot + t_1)) - (v(\cdot + t_1) - u(\cdot + t_1))\tilde{u} \), we have

\[
\tilde{u}(t) = T(t)u_1 - \chi \int_0^t T(t-s)(\nabla \cdot (\tilde{u}(s) \nabla v(s + t_1))ds + \chi \int_0^t T(t-s)(v(s + t_1) - u(s + t_1))\tilde{u}(s)ds
\]

\[
+ \int_0^t T(t-s)(a + 1 - \chi v - (b - \chi)u(s + t_1))\tilde{u}(s)ds
\]

\[
= T(t)u_1 - \chi \int_0^t T(t-s)\nabla \cdot (\tilde{u}(s) \nabla v(s + t_1))ds + \int_0^t T(t-s)(a + 1 - bu(s + t_1))\tilde{u}(s)ds.
\]

\tag{3.24}
On the other hand from equation (3.25), we have that
\[
    u(t + t_1) = T(t)u_1 - \chi \int_0^t T(t - s) \nabla \cdot (u(s + t_1) \nabla v(s + t_1))ds \\
    + \int_0^t T(t - s)(a + 1 - bu(s + t_1))u(s + t_1)ds.
\]
(3.25)

Taking the difference side by side of (3.24) and (3.25) and using Lemma 3.2, we obtain for any \( \epsilon > 0 \) and \( 0 < t < T_\epsilon < T_{\max} - t_1 - \epsilon \) that
\[
    \|\tilde{u}(t) - u(t + t_1)\|_{\infty} \leq \chi \int_0^t \|T(t - s)\nabla \cdot ((u(s + t_1) - \tilde{u}(s))\nabla v(s + t_1))\|_{\infty}ds \\
    + \int_0^t \|T(t - s)(a + 1 - bu(s + t_1))(u(s + t_1) - \tilde{u}(s))\|_{\infty}ds \\
    \leq C\chi \int_0^t (t - s)^{-\frac{1}{2}}e^{-(t-s)}\|u(s + t_1) - \tilde{u}(s)\|_{\infty}ds \\
    + \int_0^t e^{-(t-s)}\|(a + 1 - bu(s + t_1))(u(s + t_1) - \tilde{u}(s))\|_{\infty}ds \\
    \leq C\chi \sup_{s \in [0,T_\epsilon]} \|\nabla v(s + t_1)\|_{\infty} \int_0^t (t - s)^{-\frac{1}{2}}e^{-(t-s)}\|u(s + t_1) - \tilde{u}(s)\|_{\infty}ds \\
    + C(a + 1 + b) \sup_{s \in [0,T_\epsilon]} \|u(s + t_1)\|_{\infty} \int_0^t e^{-(t-s)}\|u(s + t_1) - \tilde{u}(s)\|_{\infty}ds.
\]

Combining this last inequality with Lemma 2.5, we conclude that
\[
    \tilde{u}(t) = u(t + t_1)
\]
for every \( t \in [0,T_\epsilon] \). We then have that \( u \) is a classical solution of (2.6) on \([0,T_{\max})\) and satisfies (1.7) and (1.8). Since \( u_0 \geq 0 \), by comparison principle for parabolic equations, we get \( u(x,t) \geq 0 \).

Let \( u(\cdot,t;u_0) = u(t)(\cdot) \) and \( v(\cdot,t;u_0) = (I - \Delta)^{-1}u(\cdot,t;u_0) \). We have that \( (u(\cdot,\cdot;u_0),v(\cdot,\cdot;u_0)) \) is a nonnegative classical solution of (1.4) on \([0,T_{\max})\) with initial function \( u_0 \) and \( u(\cdot,t;u_0) \) satisfies (1.7) and (1.8).

(iii) Uniqueness. We now prove that for given \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^N) \), (1.4) has a unique classical solution \( (u(\cdot,\cdot;u_0),v(\cdot,\cdot;u_0)) \) satisfying (1.7) and (1.8).

Any classical solution of (1.4) satisfying the properties of Theorem 1.1 clearly satisfies the integral equation (3.25). Suppose that for given \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^1) \) with \( u_0 \geq 0 \), \( (u_1(t,x),v_1(t,x)) \) and \( (u_2(t,x),v_2(t,x)) \) are two classical solutions of (1.4) on \( \mathbb{R}^N \times [0,T) \) satisfying the properties of Theorem 1.1. Let \( 0 < t_1 < T' < T \) be fixed. Thus \( \sup_{0 \leq t \leq T'} \|u_1(\cdot,t)\|_{\infty} + \|u_2(\cdot,t)\|_{\infty} < \infty \). Let \( u_i(t) = u_i(\cdot,t) \) and \( v_i(t) = (I - \Delta)^{-1}u_i(t) \) for every \( i = 1,2 \) and \( 0 \leq t < T \). For every \( t \in [t_1,T'] \), and \( i = 1,2 \) we have that
\[
    u_i(t) = T(t - t_1)u_i(t_1) + \chi \int_{t_1}^t T(t - s)\nabla \cdot (u_i(s)\nabla v_i(s))ds + \int_{t_1}^t T(t - s)(a + 1 - bu_i(s))u_i(s)ds.
\]
Hence for $t_1 \leq t \leq T'$,

$$
\|u_1(t) - u_2(t)\|_\infty \leq \|(u_1(t_1) - u_2(t_1))\|_\infty + C\chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_1(s)\|_\infty + C\chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_2(s)\|_\infty ds
+ \int_0^t e^{-(t-s)} \|u_1(s) - u_2(s)\|_\infty (a + b(\|u_1(s)\|_\infty + \|u_2(s)\|_\infty)) ds
\leq \|(u_1(t_1) - u_2(t_1))\|_\infty + C\chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_1(s) - u_2(s)\|_\infty \|\nabla v_1(s)\|_\infty ds
+ C\chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_2(s)\|_\infty \|\nabla (v_2(s) - v_1(s))\|_\infty ds
+ (a + b \sup_{0 \leq \tau \leq T'} (\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty)) \int_0^t e^{-(t-s)} \|u_1(s) - u_2(s)\|_\infty ds
\leq \|(u_1(t_1) - u_2(t_1))\|_\infty + C\sqrt{N} \chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_1(s) - u_2(s)\|_\infty \|u_1(s)\|_\infty ds
+ C\sqrt{N} \chi \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_2(s)\|_\infty \|u_2(s) - u_1(s)\|_\infty ds
+ (a + b \sup_{0 \leq \tau \leq T'} (\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty)) \int_0^t e^{-(t-s)} \|u_1(s) - u_2(s)\|_\infty ds
\leq \|(u_1(t_1) - u_2(t_1))\|_\infty + M \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_2(s)\|_\infty \|u_2(s) - u_1(s)\|_\infty ds,
$$

where $M = a + 1 + (C\sqrt{N} + b\sqrt{T'}) \sup_{0 \leq t' \leq T'} (\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty) < \infty$. Let $t_1 \to 0$, we have

$$
\|u_1(t) - u_2(t)\|_\infty \leq M \int_{t_1}^t (t-s) \frac{1}{2} e^{-(t-s)} \|u_2(s)\|_\infty \|u_2(s) - u_1(s)\|_\infty ds.
$$

By Lemma 2.5 again, we get $u_1(t) \equiv u_2(t)$ for all $0 \leq t \leq T'$. Since $T' < T$ was arbitrary chosen, then $u_1(t) \equiv u_2(t)$ for all $0 \leq t < T$. The theorem is thus proved. 

### 3.2 Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2. Throughout this section, we let $\alpha \in (\frac{1}{2}, 1), \delta \in [0, 2\alpha - 1)$ and $p > N$ such that $\frac{(2\alpha - 1 - \delta)p}{N} > 1$. Let $X = L^p(\mathbb{R}^N)$ and $X^\alpha$ be the fractional power space of $\Delta - I$ on $X$. By the inequalities in (2.7), we have the continuous inclusions

$$
X^\alpha \subset C^{1+\delta}(\mathbb{R}^N) \quad \text{and} \quad X^\alpha \subset W^{1,l,p} \quad \forall \ l \geq 1.
$$

**Proof of Theorem 1.2** We prove this theorem using semigroup method.

First, consider the functions $B : X^\alpha \times X^\alpha \to L^p(\mathbb{R}^N)$ and $F : X^\alpha \to L^p(\mathbb{R}^N)$ defined by

$$
B(u, v) = -\chi \nabla u \nabla (\Delta - I)^{-1} v - \chi u (\Delta - I)^{-1} v - (b - \chi) uv
$$

and

$$
F(u) = \chi \nabla u \nabla (\Delta - I)^{-1} u - \chi u (\Delta - I)^{-1} u - (b - \chi) u^2 + (a + 1) u
$$
for every $u, v \in X^\alpha$. Clearly, $B$ is a bilinear function and $F(u) = B(u, u) + (1 + a)u$ for every $u \in X^\alpha$. Since $X^\alpha$ is continuously embedded in $W^{1,p}(\mathbb{R}^N) \cap C^{1+\delta}(\mathbb{R}^N)$, there is a constant $C > 0$ such that
\[
\|w\|_{W^{1,p}(\mathbb{R}^N)} + \|w\|_{C^{1+\delta}} \leq C\|w\|_{X^\alpha} \quad \forall \, w \in X^\alpha.
\]
Combining this with regularity and a priori estimates for elliptic equations, we obtain that
\[
\|B(u, v)\|_{L^p} \leq \chi\|u\|_{C^1}(\Delta - I)^{-1}v\|_{W^{1,p}} + \chi\|u\|_\infty\|((\Delta - I)^{-1}v\|_{L^p} + (b - \chi)\|u\|_\infty\|v\|_{L^p}
\leq 2\chi\|u\|_{C^1}(\Delta - I)^{-1}v\|_{W^{1,p}} + (b - \chi)\|u\|_\infty\|v\|_{L^p}
\leq C\|u\|_{X^\alpha}\|((\Delta - I)^{-1}v\|_{W^{1,p}} + \|v\|_{X^\alpha})
\leq C\|u\|_{X^\alpha}\|v\|_{X^\alpha}.
\]
Thus $B$ is continuous. Hence the function $F$ is locally Lipschitz continuous and maps bounded sets to bounded sets. It follows from Theorem 2.1 and Theorem 2.2 that there exists a maximal time $T_{\text{max}} > 0$ and a unique $u \in C([0, T_{\text{max}}), X^\alpha)$ satisfying the integral equation
\[
u(t) = T(t)u_0 + \int_0^t T(t - s)(-\chi \nabla \cdot (u(s) \nabla v(s)) + (a + 1)u(s) - bu^2(s))ds,
\]
where $v(s) = (I - \Delta)^{-1}u(s)$. Moreover, $u \in C^1([0, T_{\text{max}}), L^p(\mathbb{R}^N))$ and if $T_{\text{max}} < \infty$, then
\[
\lim_{t \to T_{\text{max}}} \|u(t)\|_{X^\alpha} = \infty.
\]
Next, by Theorem 3.5.2, $u \in C([0, T_{\text{max}}), X^\beta)$ (3.26) for any $0 \leq \beta < 1$. Note that $X^\alpha$ is continuously embedded in $C^{1+\delta}(\mathbb{R}^N)$. Then by Theorem 1.1 and (3.26), we have that (1.10) and (1.11) hold.

Now, let $u(\cdot, t; u_0) = u(t)(\cdot)$ and $v(\cdot, t; u_0) = (I - \Delta)^{-1}u(\cdot, t; u_0)$. We have that $(u(\cdot, \cdot; u_0), v(\cdot, \cdot; u_0))$ is a nonnegative classical solution of (1.4) on $[0, T_{\text{max}})$ with initial function $u_0$ and $u(\cdot, t; u_0)$ satisfies (1.9), (1.10), and (1.11). The uniqueness of classical solutions of (1.4) follows from the similar arguments as in the proof (iii) of Theorem 1.1.

\[\square\]

3.3 Proof of Theorem 1.3

In this subsection, we prove Theorem 1.3.

Proof of Theorem 1.3. The proof of this theorem follows the similar arguments used in the proof of Theorem 1.1. Hence lengthy details will be avoided. In the following, we fix $u_0 \in L^p(\mathbb{R}^N)$.

Claim 1. There is $T_{\text{max}} \in (0, \infty)$ such that (2.6) has a mild solution $u(\cdot)$ on $[0, T_{\text{max}})$ with $\alpha = 0$ and $X^0 = L^p(\mathbb{R}^N)$; for every $0 < \beta < 1$, $(0, T_{\text{max}}) \ni t \mapsto u(\cdot) \in X^\beta$ is locally Hölder continuous; and if $T_{\text{max}} < \infty$, then $\lim_{t \to T_{\text{max}}} \|u(t)\|_{L^p(\mathbb{R}^N)} = \infty$.

For every $T > 0$ and $R > 0$, let us set
\[
S_{R,T} := \left\{ u \in C([0, T], L^p(\mathbb{R}^N)) \mid \|u\|_{L^p(\mathbb{R}^N)} \leq R \right\}.
\]
Note that $S_{R,T}$ is a closed subset of the Banach space $C([0, T] : L^p(\mathbb{R}^N))$ with the sup-norm.
Define \( G : S_{R,T} \to C([0, T], L^p(\mathbb{R}^N)) \) by

\[
(Gu)(t) = T(t)u_0 + \chi \int_0^t \frac{T(t-s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))}{F_2(s)} ds + (1 + a) \int_0^t T(t-s)u(s)ds - b \int_0^t \frac{T(t-s)u^2(s)}{F_1(s)} ds.
\]

It is clear that the Riemann integral \( \int_0^t T(t-s)u(s)ds \) in \( L^p(\mathbb{R}^N) \) exists for every \( u \in S_{R,T} \) and \( t \in [0, T] \). Let \( u \in S_{R,T} \) and \( 0 < s \leq T \) be given. For every \( s_1, s_2 \in [0, t] \), we have that

\[
\|u(s_1)\nabla (\Delta - I)^{-1}u(s_1) - u(s_2)\nabla (\Delta - I)^{-1}u(s_2)\|_{L^p(\mathbb{R}^N)} \\
\leq \|u(s_1) - u(s_2)\|_{L^p(\mathbb{R}^N)}\| (\Delta - I)^{-1} u(s_1) \|_{C^{1,b}_{\text{unif}}(\mathbb{R}^N)} \\
+ \|u(s_2)\|_{L^p(\mathbb{R}^N)}\| (\Delta - I)^{-1} (u(s_2) - u(s_1)) \|_{C^{1,b}_{\text{unif}}(\mathbb{R}^N)}
\]

Since \( p > N \), by regularity and a priori estimates for elliptic equations, there is a constant \( C > 0 \) such that

\[
\| (\Delta - I)^{-1} u \|_{C^{1,b}_{\text{unif}}(\mathbb{R}^N)} \leq C \|w\|_{L^p(\mathbb{R}^N)}, \quad \forall \ w \in L^p(\mathbb{R}^N).
\]

Thus we have that

\[
\|u(s_1)\nabla (\Delta - I)^{-1}u(s_1) - u(s_2)\nabla (\Delta - I)^{-1}u(s_2)\|_{L^p(\mathbb{R}^N)} \\
\leq C (\|u(s_1)\|_{L^p(\mathbb{R}^N)} + \|u(s_2)\|_{L^p(\mathbb{R}^N)}) \|u(s_1) - u(s_2)\|_{L^p(\mathbb{R}^N)} \\
\leq 2CR \|u(s_1) - u(s_2)\|_{L^p(\mathbb{R}^N)}, \quad \forall 0 \leq s_1, s_2 \leq t.
\]

Hence the function \([0, t] \ni s \mapsto u(s)\nabla (\Delta - I)^{-1}u(s) \in L^p(\mathbb{R}^N)\) is continuous. Similar arguments as Theorem 1.1 Claim 1 yield that the the function \( F_2 : [0, t] \to L^p(\mathbb{R}^N) \) is continuous and the Riemann integral \( \int_0^t F_2(s)ds \) in \( L^p(\mathbb{R}^N) \) exists. Next, for every \( 0 < \varepsilon < t \) and \( s \in [0, t - \varepsilon] \), we have

\[
F_1(s) = T(t - \varepsilon - s)T(\varepsilon)u^2(s)
\]

and by (2.12), (3.1) and \( p \geq 2 \), and Hölder’s inequality, we have

\[
\|T(\varepsilon)u^2(s_1) - T(\varepsilon)u^2(s_2)\|_{L^p(\mathbb{R}^N)} \leq C e^{-\frac{N}{2p}} e^{-\varepsilon} \|u^2(s_1) - u^2(s_2)\|_{L^\frac{p}{2}(\mathbb{R}^N)} \\
\leq C e^{-\frac{N}{2p}} e^{-\varepsilon} \|u(s_1) - u(s_2)\|_{L^p(\mathbb{R}^N)} \|u(s_1) + u(s_2)\|_{L^p(\mathbb{R}^N)} \\
\leq 2RC e^{-\frac{N}{2p}} e^{-\varepsilon} \|u(s_1) - u(s_2)\|_{L^p(\mathbb{R}^N)}, \quad \forall s_1, s_2 \in [0, t - \varepsilon].
\]

Thus the function \( F_1 : [0, t] \to L^p(\mathbb{R}^N) \) is continuous. Moreover, by (2.12), (3.1), \( p > N \) and \( p \geq 2 \), we have

\[
\int_0^t \|F_1(s)\|_{L^p(\mathbb{R}^N)} ds \leq C \int_0^t (t-s)^{-\frac{N}{2p}} e^{-(t-s)} \|u^2\|_{L^\frac{p}{2}(\mathbb{R}^N)} ds \\
\leq CR \int_0^t (t-s)^{-\frac{N}{2p}} e^{-(t-s)} ds \leq CR\Gamma(1 - \frac{N}{2p}).
\]
Hence the Riemann integral \( \int_0^t F_1(s) ds \) in \( L^p(\mathbb{R}^N) \) exists. Therefore \((Gu)(t)\) is well defined and the integral is taken in \( L^p(\mathbb{R}^N) \).

For every \( R > \|u_0\|_p \), there exists \( T := T(R) \) such that \( G \) maps \( S_{R,T} \) into itself. Indeed, for every \( u \in S_{R,T} \), we have

\[
\|G(u)(t)\|_{L^p(\mathbb{R}^N)} \leq e^{-t}\|u_0\|_{L^p(\mathbb{R}^N)} + \chi \int_0^t \|T(t-s)\nabla \cdot (u(s)\nabla (\Delta - I)^{-1}u(s))\|_{L^p(\mathbb{R}^N)} ds \\
+ (1+a) \int_0^t e^{-(t-s)}\|u(s)\|_{L^p(\mathbb{R}^N)} ds + b \int_0^t |T(t-s)u^2(s)|_{L^p(\mathbb{R}^N)} ds.
\]

Now, by (2.12), (3.1) and \( p \geq 2 \), the last inequality can be improved to

\[
\|G(u)(t)\|_{L^p(\mathbb{R}^N)} \leq e^{-t}\|u_0\|_{L^p(\mathbb{R}^N)} + C \int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\|u(s)\|_{L^p(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} ds \\
+ (a+1) \int_0^t e^{-(t-s)}\|u(s)\|_{L^p(\mathbb{R}^N)} + C \int_0^t (t-s)^{-\frac{3}{2}}e^{-(t-s)}\|u^2(s)\|_{L^p(\mathbb{R}^N)} ds \\
\leq e^{-t}\|u_0\|_{L^p(\mathbb{R}^N)} + C \int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^p(\mathbb{R}^N)}\|\nabla (\Delta - I)^{-1}u(s)\|_{C^{1,\beta}(\mathbb{R}^N)} ds \\
+ (a+1)Rt + CR^2 \int_0^t (t-s)^{-\frac{3}{2}}ds.
\]

Since \( p > N \), by regularity and a priori estimates for elliptic equations, there is a constant \( C > 0 \) such that

\[
\|\nabla (\Delta - I)^{-1}w\|_{C^{1,\beta}(\mathbb{R}^N)} \leq C\|w\|_{L^p(\mathbb{R}^N)}, \quad \forall \ w \in L^p(\mathbb{R}^N).
\]

Combining this with (3.27), we obtain that

\[
\|G(u)(t)\|_{L^p(\mathbb{R}^N)} \leq e^{-t}\|u_0\|_{L^p(\mathbb{R}^N)} + CR^2 t^{\frac{3}{2}} + (a+1)Rt + CR^2 t^{1-\frac{N}{2p}}.
\]

Hence we can now choose \( T > 0 \) such that

\[
e^{-t}\|u_0\|_{L^p(\mathbb{R}^N)} + CR^2 t^{\frac{3}{2}} + (a+1)Rt + CR^2 t^{1-\frac{N}{2p}} < R \quad \forall \ t \in [0,T].
\]

This implies that \( G \) maps \( S_{R,T} \) into itself.

Using again inequalities (2.12) and inequality (3.1), by following the same ideas as in the proof of claim 3 in Theorem 1.1, we have that \( G \) is a contraction map for \( T \) sufficiently small. Thus \( G \) has a unique fixed point, say \( u(\cdot) \). Using again Lemma 3.1 similar arguments used in the proof of Claim 2 of Theorem 1.1 yields that the function \((0,T) \ni t \mapsto u(t) \in X^\beta\) is locally Hölder continuous for every \( 0 < \beta < \frac{1}{2} \). Clearly, \( u(\cdot) \) is a mild solution of (2.6) with \( \alpha = 0 \) and \( X^0 = L^p(\mathbb{R}^N) \). The claim then follows from regular extension arguments.

**Claim 2.** \( u(\cdot) \) obtained in Claim 1 is the unique classical solution of (2.6) on \([0, T_{\text{max}}]\) satisfying (1.12), (1.13), and (1.14).

By Claim 1, for any \( 0 < \beta < 1/2 \) and \( t_1 \in (0, T_{\text{max}}) \), \( u(t_1) \in X^\beta \). It then follows that \( u_1(\cdot) = u(t_1)(\cdot) \in C^b_{\text{unif}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \) for \( 0 < \nu \ll 1 \). Consider (3.23). By the similar arguments as those in the proof (ii) of Theorem 1.1, \( u(\cdot) \) is a classical solution of (2.6) on \([0, T_{\text{max}}]\). Moreover, \( u(\cdot) \in C^1((0, T_{\text{max}}), C^b_{\text{unif}}(\mathbb{R}^N)) \) and satisfies (1.14). By Theorem 2.3 we have \( u(\cdot) \in C((0, T_{\text{max}}), X^\beta) \).
for any $0 \leq \beta < 1$. Hence $u$ satisfies (1.13). By the similar arguments as those in the proof (iii) of Theorem 1.1, we can prove the uniqueness and then the claim follows.

Claim 3. The function $u(t)$ obtained in Claim 1 is nonnegative.

We have that the function $u$ satisfies the integral equation

$$u(t) = T(t)u_0 + \chi \int_0^t T(t-s)\nabla \cdot (u \nabla (\Delta - I)^{-1}u(s))ds + \int_0^t T(t-s)((a+1)u(s) - bu^2(s))ds.$$ 

Since $u_0 \geq 0$, there is a sequence of nonnegative functions $\{u_{0n}\}_n \in L^p(\mathbb{R}^N) \cap C^0_{\text{unif}}(\mathbb{R}^N)$ such $\|u_{0n} - u_0\|_{L^p(\mathbb{R}^N)} \to 0$ as $n \to \infty$. For $R$ large enough, since $\sup_n \|u_{0n}\|_{L^p(\mathbb{R}^N)} < \infty$, the time $T$ can be chosen to be independent of $n$, such for each $n$, there is a unique $u_n(\cdot) \in \mathcal{S}_{R,T}$ satisfying the integral equation

$$u_n(t) = T(t)u_{0n} + \chi \int_0^t T(t-s)\nabla \cdot (u_n \nabla (\Delta - I)^{-1}u_n(s))ds + \int_0^t T(t-s)((a+1)u_n(s) - bu_n^2(s))ds$$

for every $n$. Since the $u_n \geq 0$ and belongs to $C^0_{\text{unif}}(\mathbb{R}^N)$, for every $n$, Theorem 1.1 implies that $u_n(t) \geq 0$. Now, similar arguments used to establish (3.26), yield as similar result

$$\|u_n(t) - u(t)\|_{L^p(\mathbb{R}^N)} \leq \|T(t)(u_{0n} - u_0)\|_{L^p(\mathbb{R}^N)} + C \int_0^t (t-s)^{-\frac{N}{p}}\|u_n(s) - u(s)\|_{L^p(\mathbb{R}^N)}ds$$

$$\leq \|u_{0n} - u_0\|_{L^p(\mathbb{R}^N)} + C \int_0^t (t-s)^{-\frac{N}{p}}\|u_n(s) - u(s)\|_{L^p(\mathbb{R}^N)}ds.$$ 

Next, using Lemma 2.5 it follows from the last inequality that

$$\|u_n(t) - u(t)\|_{L^p(\mathbb{R}^N)} \leq C\|u_{0n} - u_0\|_{L^p(\mathbb{R}^N)}$$

for every $n \geq 1$, $0 < t < T$, where $C > 0$ is positive constant independent of $n$. Letting $n$ goes to infinity, we obtain that

$$\|u_n(t) - u(t)\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to 0.$$ 

Thus, for every $t > 0$, $u(t)(x) \geq 0$ for a. e. $x \in \mathbb{R}^N$. Since $u(t)(x)$ is continuous in $x \in \mathbb{R}^N$ for each $t > 0$, we conclude that $u(t)(x) \geq 0$ for every $x \in \mathbb{R}^N$, $t \in (0,T]$.

Let $u(\cdot,t;u_0) = u(t)(\cdot)$ and $v(\cdot,t;u_0) = (I - \Delta)^{-1}u(\cdot,t;u_0)$. We then have that $(u(\cdot,\cdot;u_0), v(\cdot,\cdot;u_0))$ is a unique nonnegative classical solution of (1.3) on $[0,T_{\text{max}})$ with initial function $u_0$ and $u(\cdot,t;u_0)$ satisfies (1.12), (1.13), and (1.14).

3.4 Proof of Theorem 1.4

In this section, we prove Theorem 1.4.

Proof of Theorem 1.4 (1) Let $u(t) = u(\cdot,t;u_0)$. It is clear that $T_{\text{max}}^\alpha(u_0) \leq T_{\text{max}}^\alpha(u_0)$. Assume that $T_{\text{max}}^\alpha(u_0) < T_{\text{max}}^\alpha(u_0)$. Then $T_{\text{max}}^\alpha(u_0) < \infty$. Recall that for every $t \in (0,T_{\text{max}}^\alpha(u_0))$, 

$$u(t) = T(t)u_0 - \chi \int_0^t \underbrace{T(t-s)(\nabla u \nabla v)ds}_{I_1} - \chi \int_0^t \underbrace{T(t-s)(uv)ds}_{I_2} + \int_0^t \underbrace{T(t-s)((a+1)u + (\chi - b)u^2)ds}_{I_3}.$$ 

(3.28)
We investigate the $X^\alpha$-norm of each term in the right hand side of (3.28) for $0 < t < T_{\max}^\alpha(u_0)$. It follows from inequalities (2.12) that
\[
\|T(t)u_0\|_{X^\alpha} \leq C_\alpha e^{-t\alpha}\|u_0\|_{L^p}.
\] (3.29)

Using inequalities (2.12), we have that
\[
\|I_1\|_{X^\alpha} \leq \int_0^t \|(I - \Delta)^\alpha T(t - s)(\nabla u \nabla v)\|_{L^p} ds \leq C_\alpha \int_0^t e^{-(t - s)\alpha}\|\nabla u \nabla v\|_{L^p} ds \leq C_\alpha \int_0^t e^{-(t - s)\alpha}\|u\|_{C^1(R^N)}\|\nabla v\|_{L^p} ds.
\]

Since $v = (\Delta - I)^{-1}u$, elliptic regularity implies that
\[
\|\nabla v\|_{L^p(R^N)} \leq C\|u\|_{X^\alpha}.
\]

We then have
\[
\|I_1\|_{X^\alpha} \leq C \sup_{0 \leq \tau \leq T_{\max}^\alpha(u_0)} \|u(\tau)\|_{C^1} \int_0^t e^{-(t - s)\alpha}\|u(s)\|_{X^\alpha} ds.
\]

Similar arguments applied to $I_2$ and $I_3$ yield that
\[
\|I_2\|_{X^\alpha} \leq C \sup_{0 \leq \tau \leq T_{\max}^\alpha(u_0)} \|u(\tau)\|_{C^1} \int_0^t e^{-(t - s)\alpha}\|u(s)\|_{X^\alpha} ds
\]
and
\[
\|I_3\|_{X^\alpha} \leq C(a + 1 + \sup_{0 \leq \tau \leq T_{\max}^\alpha(u_0)} \|u(\tau)\|_{C^1} \int_0^t e^{-(t - s)\alpha}\|u(s)\|_{X^\alpha} ds.
\]

We then have that for every $t \in (0, T_{\max}^\alpha(u_0))$
\[
\|u(t)\|_{X^\alpha} \leq M(u_0) \left[ t^{-\alpha} + \int_0^t (t - s)^{-\alpha}\|u(s)\|_{X^\alpha} ds \right],
\]
where
\[
M(u_0) = C(\|u_0\|_{L^p(R^N)} + a + 1 + \sup_{0 \leq \tau \leq T_{\max}^\alpha(u_0)} \|u(\tau)\|_{C^1} + \sup_{0 \leq \tau \leq T_{\max}^\alpha(u_0)} \|u(\tau)\|_{C^1}^\infty).
\]

Thus, it follows from Lemma 2.5 that
\[
\|u(t)\|_{X^\alpha} \leq CM(u_0)t^{-\alpha} \quad \forall t \in (0, T_{\max}^\alpha(u_0)).
\]

This implies that $\limsup_{t \to T_{\max}^\alpha(u_0)}\|u(t)\|_{X^\alpha} < \infty$, a contradiction. Therefore, $T_{\max}^\alpha(u_0) = T_{\max}^\infty(u_0)$.

(2) It is clear that $T_{\max}^\alpha(u_0) \leq T_{\max}^p(u_0)$. Assume that $T_{\max}^\alpha(u_0) < T_{\max}^p(u_0)$. Then $T_{\max}^\alpha(u_0) < \infty$. By (1.13), $\limsup_{t \to T_{\max}^\alpha(u_0)}\|u(\cdot, t; u_0)\|_{X^\alpha} < \infty$, a contradiction. (2) then follows.
(3) By the arguments in (1), we have \( T_{\max}^p(u_0) \geq T_{\max}^\infty(u_0) \). By \([1, 13]\), \( T_{\max}^\infty(u_0) \geq T_{\max}^p(u_0) \).

(4) Let \( u(\cdot, t; u_0) \) be as in Theorem \([1, 3]\). For given \( T > 0 \) and \( R > \|u_0\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^p(\mathbb{R}^N)} \), consider the set

\[
S'_{R,T} := \{ u \in C([0, T], L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)) \mid \|u\|_{L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)} \leq R \},
\]

where \( \|u\|_{L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)} = \|u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \). Using inequality \([2, 12]\), for \( u \in S'_{R,T} \), we have that

\[
\left\| \int_0^t T(t-s)u^2 ds \right\|_{L^1(\mathbb{R}^N)} \leq \int_0^t \left\| T(t-s)u^2(s) \right\|_{L^1(\mathbb{R}^N)} ds \leq C \int_0^t e^{-(t-s)} \|u^2(s)\|_{L^1} ds
\]

\[
= C \int_0^t e^{-(t-s)} \|u(s)\|^2_{L^2} ds.
\]

Now, since \( 1 < 2 \leq p \), Holder’s inequality implies that \( \|u(s)\|_{L^2} \leq \|u(s)\|_{L^1}^{\lambda} \|u(s)\|_{L^p}^{1-\lambda} \leq R \) with \( \lambda = (\frac{1}{2} - \frac{1}{p})/(1 - \frac{1}{p}) \). Thus, the last inequality becomes

\[
\left\| \int_0^t T(t-s)u^2 ds \right\|_{L^1(\mathbb{R}^N)} \leq CR^2 \int_0^t e^{-(t-s)} ds \leq CR^2 t.
\]

This together with the arguments in Claim 1 of Theorem \([1, 3]\) implies that

\[
G : S'_{R,T} \to C([0, T], L^p(\mathbb{R}^N)) \cap C([0, T], L^1(\mathbb{R}^N))
\]

is well defined, where

\[
(Gu)(t) = T(t)u_0 + \chi \int_0^t T(t-s)\nabla \cdot (u(s)\nabla(\Delta - I)^{-1}u(s)) ds + (1 + a) \int_0^t T(t-s)u(s) ds - b \int_0^t T(t-s)u^2(s) ds.
\]

By the arguments in Claim 1 of Theorem \([1, 3]\) \( u(\cdot, t; u_0) \in C([0, T], L^1(\mathbb{R}^N)) \) for \( 0 < T \ll 1 \).

Let

\[
T_{\max}^{p,1}(u_0) = \sup \{ \tau \in [0, T_{\max}^p(u_0)) \mid \sup_{0 \leq t < \tau} \|u(\cdot, t; u_0)\|_{L^1(\mathbb{R}^N)} < \infty \}.
\]

Assume that \( T_{\max}^{p,1}(u_0) < T_{\max}^p(u_0) \). Then \( \sup_{0 \leq t < T_{\max}^{p,1}(u_0)} \|u(\cdot, t; u_0)\|_{L^1(\mathbb{R}^N)} = \infty \). Fix any \( t_1 \in (0, T_{\max}^{p,1}(u_0)) \). By Theorem \([1, 3]\) \( u_1 = u(\cdot, t_1; u_0) \in L^p(\mathbb{R}^N) \cap C_{\text{unif}}^b(\mathbb{R}^N) \) and then

\[
u_1(\cdot, \cdot; u_0), \partial_x u_1(\cdot, \cdot; u_0) \in C([t_1, T_{\max}^{p,1}(u_0)], C_{\text{unif}}^b(\mathbb{R}^N), i = 1, 2, \cdots, N.
\]

Using the arguments in (1) with \( u_0 \) being replaced by \( u_1 \) and \( p = 1, \alpha = 0 \), we have

\[
\limsup_{t \to T_{\max}^{p,1} - t_1} (u_0) \|u(\cdot, t; u_1)\|_{L^1(\mathbb{R}^N)} < \infty.
\]

Note that \( u(\cdot, t + t_1; u_0) = u(\cdot, t; u_1) \). We then have

\[
\limsup_{t \to T_{\max}^{p,1}(u_0)} \|u(\cdot, t; u_0)\|_{L^1(\mathbb{R}^N)} < \infty,
\]

which is a contradiction. Therefore, \( T_{\max}^{p,1}(u_0) = T_{\max}^p(u_0) \). \( \square \)
4 Global existence of classical solutions

In this section, we discuss the existence of global in time solutions to (1.4) and prove Theorems 1.5, 1.6, and 1.7. Throughout this section, $C$ denotes a constant independent of the initial functions and the solutions under consideration, unless specified otherwise.

We first recall a well known lemma for a logistic ODE for convenience and then prove Theorems 1.5, 1.6, and 1.7 in subsections 4.1, 4.2, and 4.3, respectively.

Lemma 4.1. Consider the ODE
\[ \dot{u} = u(a_0 - b_0 u), \]
where $a_0, b_0$ are positive constants. Let $u(t; u_0)$ be the solution of (4.1) with $u(0; u_0) = u_0 \in \mathbb{R}$. Then for any $u_0 > 0$,
\[ \lim_{t \to \infty} u(t; u_0) = \frac{a_0}{b_0}. \]

4.1 Proof of Theorem 1.5

In this subsection, we prove Theorem 1.5.

Proof of Theorem 1.5 Let $(u, v)$ be the classical local nonnegative solution given by Theorem 1.1 defined on the maximal interval $[0, T_{\text{max}}(u_0))$. We have that
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla u \nabla v - (b - \chi) u^2 + au - \chi uv \\
    &\leq \Delta u - \chi \nabla u \nabla v - (b - \chi) u^2 + au.
\end{align*}
\]
Let $u(t, \|u_0\|_{\infty})$ be solution of the initial value problem,
\[
\begin{cases}
    u' = -(b - \chi) u^2 + au \\
    u(0) = \max u_0.
\end{cases}
\]
Since $b - \chi \geq 0$, then $u(t, \|u_0\|_{\infty})$ is globally defined in time. Since $u_0 \leq u(0, \|u_0\|_{\infty})$, by the comparison principle for parabolic equations we have that
\[ u(x, t) \leq u(t, \|u_0\|_{\infty}) \quad \text{(4.2)} \]
for all $x \in \mathbb{R}^N$ and $t \geq 0$. Hence $u(x, t)$ is globally defined in time. Furthermore, by Lemma 4.1 if $\chi < b$, then
\[ u(t, \|u_0\|_{\infty}) \to \frac{a}{b - \chi} \quad \text{as} \quad t \to \infty. \quad \text{(4.3)} \]
This completes the proof of the theorem.

4.2 Proof of Theorem 1.6

In this subsection, we prove Theorem 1.6. In order to do so, we first prove an important theorem and some technical lemmas. Throughout this section, we let $\alpha \in (\frac{1}{2}, 1), \delta \in [0, 2\alpha - 1)$ and $p > N$ such that $\frac{(2\alpha - 1 - \delta)p}{N} > 1$. Let $X = L^p(\mathbb{R}^N)$ and $X^\alpha$ be the fractional power space of $\Delta - I$ on $X$. 

28
Lemma 4.2. Suppose that satisfies the hypothesis of Theorem 1.2 and \((u, v)\) is the solution of (1.4) as in Theorem 1.2. For every \(r \geq 1\) satisfying
\[
r \leq \frac{\chi}{(\chi - b)_+},
\]
we have that
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq \|u_0\|_{L^p(\mathbb{R}^N)} e^{\alpha t} \quad \forall \ t \in [0, \ T_{\max}^\alpha(u_0)).
\] (4.4)

Proof. Let \(1 \leq r \leq \chi/(\chi - b)_+\). If \(\|u_0\|_{L^p(\mathbb{R}^N)} = \infty\) there is nothing show. Hence we might suppose that \(\|u_0\|_{L^p(\mathbb{R}^N)} < \infty\). Let us set \(\delta_r := b - \frac{\chi(r - 1)}{r} \geq 0\). We multiply the first equation in (1.4) by \(u^{r-1}\), and integrating it, we obtain
\[
\frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}^N} u^r = - \int_{\mathbb{R}^N} \nabla u \nabla u^{r-1} + \frac{\chi(r - 1)}{r} \int_{\mathbb{R}^N} \nabla u^r \nabla v + \int_{\mathbb{R}^N} (au^r - bu^{r+1})
\]
\[
= -(r - 1) \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 - \frac{\chi(r - 1)}{r} \int_{\mathbb{R}^N} u^r \Delta v + \int_{\mathbb{R}^N} (a - bu) u^r
\]
\[
= -(r - 1) \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 - \frac{\chi(r - 1)}{r} \int_{\mathbb{R}^N} u^r v - (b - \frac{\chi(r - 1)}{r}) \int_{\mathbb{R}^N} u^{r+1} + \int_{\mathbb{R}^N} au^r
\]
\[
\leq a \int_{\mathbb{R}^N} u^r.
\]
(4.4) then follows. \(\square\)

A natural question that one could ask is under which condition on the expression \(\frac{\chi}{(\chi - b)_+}\), the \(L^r\)–a priori estimate in Lemma 4.2 can be extended to all \(r \geq 1\) or for at least for every \(r = p\). An obvious condition would be to require that \(p \leq \frac{\chi}{(\chi - b)_+}\) so that the hypothesis of Lemma 4.2 are satisfied. Hence Lemma 4.2 and Theorem 1.4 have a direct consequence that we formulate in the next result.

Corollary 4.3. Suppose that \(u_0\) satisfies the hypothesis of Theorem 1.2 and \(p \leq \frac{\chi}{(\chi - b)_+}\). Then the solution \((u, v)\) of (1.4) with initial data \(u_0\) is global in time.

Proof. We have that \(p \leq \frac{\chi}{(\chi - b)_+}\), hence according to Lemma 4.2 we have
\[
\|u(t)\|_{L^p(\mathbb{R}^N)} \leq \|u_0\|_{L^p(\mathbb{R}^N)} e^{\alpha t}, \quad \forall \ 0 \leq t < T_{\max}^\alpha(u_0).
\] (4.5)

By Theorem 1.4 we have that \(T_{\max}^\alpha(u_0) = T_{\max}^\rho(u_0) = +\infty\). \(\square\)

Note that \(p\) was chosen to be strictly greater than \(N\). Thus, it would be nice to find a relationship between \(N\) and the expression \(\frac{\chi}{(\chi - b)_+}\) that will guarantee the existence of a global solution. Theorem 1.6 provides such a sufficient condition to obtain a global in time solution. Note that this condition is weaker than the one giving by Corollary 4.3.

Proof of Theorem 1.6. We divide the proof into two steps. In the first part, we prove that \(L^r\)–norms of the \(u(t)\) can be bounded by continuous function as required in Theorem 1.4. We then conclude that \(T_{\max} := T_{\max}^\alpha(u_0) = +\infty\). The last part is the proof of inequality (1.16).
Note that $\frac{\chi}{(x-b)^+} > 1$, so we can choose $q_1 \in \left(\max\{1, \frac{N}{2}\}, \min\{p, \frac{\chi}{(x-b)^+}\}\right)$. We have $\delta_{q_1} = b - \frac{\chi(q_1-1)}{q_1} > 0$, and Lemma 4.2 implies that

$$\|u(\cdot, t)\|_{L^{q_1}(\mathbb{R}^N)} \leq \|u_0\|_{L^{q_1}(\mathbb{R}^N)} e^{\delta_{q_1} t} \quad \forall \ t \in [0, T_{\text{max}}).$$

**Step 1.** We claim that for all $r \geq q_1$,

$$\|u(t)\|_{L^r(\mathbb{R}^N)} \leq \left[\|u_0\|_{L^r(\mathbb{R}^N)} + K_r^\frac{1}{r} \|u_0\|_{L^r(\mathbb{R}^N)} e^{(\lambda_r - 1)at}\right] e^{at} \quad \forall \ t \in [0, T_{\text{max}}),$$

(4.6)

where $K_r$ and $\lambda_r$ are nonnegative real numbers depending on $a$, $b$, $\chi$, $r$ and $N$ with $\lambda_r > 1$.

We multiply the first equation in (4.4) by $u^{r-1}$, and after integrating it by part, we obtain

$$\begin{align*}
\frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}^N} u^r &\leq -(r-1) \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 \left(b - \frac{\chi(r-1)}{r}\right) \int_{\mathbb{R}^N} u^{r+1} + a \int_{\mathbb{R}^N} u^r \\
&= -\frac{4(r-1)}{r^2} \int_{\mathbb{R}^N} |\nabla u|^2 \left(b - \frac{\chi(r-1)}{r}\right) \int_{\mathbb{R}^N} u^{r+1} + a \int_{\mathbb{R}^N} u^r. \quad (4.7)
\end{align*}$$

From Lemma 2.4 it follows that

$$\|u(t)\|_{L^{r+1}(\mathbb{R}^N)} \leq C_0^\frac{2r}{r+1} \|u(t)\|_{L^q(\mathbb{R}^N)} \|\nabla u\|^\beta_{L^2(\mathbb{R}^N)},$$

where

$$\theta = \frac{r}{2} \left(\frac{1}{q_1} - \frac{1}{r+1}\right),$$

$$\beta = \begin{cases} 
\frac{r+1-q_1}{r+1} \left(\frac{2q_1}{r} + \frac{1-2q_1}{N+2}\right) & \text{if } r > 2q_1 \\
1 & \text{if } r \leq 2q_1,
\end{cases}$$

and $C_0$ depends only on $N$. Notice that we used $a = 3$ in Lemma 2.4. Hence

$$\int_{\mathbb{R}^N} u(t)^{r+1} \leq C_0^\frac{2(r+1)}{r^3} \|u(t)\|_{L^q(\mathbb{R}^N)}^{(1-\theta)(r+1)} \|\nabla u\|^\beta_{L^2(\mathbb{R}^N)}. \quad (4.8)$$

Observe from the choice of $q_1$ that

$$\frac{\theta(r+1)}{r} = \frac{\frac{r+1-q_1}{q_1} - 1}{\frac{r+1-q_1}{q_1} + \frac{2}{N} - 1} = \frac{\frac{r}{q_1} - 1 + \frac{1}{q_1}}{\frac{r}{q_1} - 1 + \frac{2}{N}} < 1.$$ 

Combining this with (4.8) and using Young’s inequality, for every $\epsilon > 0$, we have that

$$\int_{\mathbb{R}^N} u(t)^{r+1} \leq \epsilon \|\nabla u\|^\beta_{L^2(\mathbb{R}^N)} + \epsilon^{-\frac{\theta(r+1)}{r}} C_0^\frac{2(r+1)}{r^3} \|u(t)\|_{L^q(\mathbb{R}^N)}^{(1-\theta)(r+1)} \|\nabla u\|^\beta_{L^2(\mathbb{R}^N)},$$

which is equivalent to

$$-\|\nabla u\|^\beta_{L^2(\mathbb{R}^N)} \leq -\frac{1}{\epsilon} \int_{\mathbb{R}^N} u(t)^{r+1} + \epsilon^{\left(1+\frac{\theta(r+1)}{r-\theta(r+1)}\right)} C_0^\frac{2(r+1)}{r^3} \|u(t)\|_{L^q(\mathbb{R}^N)}^{(1-\theta)(r+1)} \|\nabla u\|^\beta_{L^2(\mathbb{R}^N)}. \quad (4.9)$$
Combining inequalities (4.7) and (4.9), we obtain that

\[
\frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}^N} u^r \leq - \left( \frac{4(r-1)}{r^2} + \beta \frac{(r-1)}{r} \right) \int_{\mathbb{R}^N} u^{r+1} + a \int_{\mathbb{R}^N} u^r + \frac{4(r-1)C_0}{r^2 \varepsilon_r} \frac{2(r+1)}{(1+\frac{\theta}{r-\theta(r+1)})} \|u(t)\|_{L^p(\mathbb{R}^N)}^{(1-\theta)(r+1)} \|u(t)\|_{L^{\tilde{p}}(\mathbb{R}^N)}^{\frac{r}{r-\theta(r+1)}}.
\]

If we choose \( \varepsilon > 0 \) such that \( \varepsilon_r \geq 0 \) (for example \( \varepsilon = \frac{4}{r} \chi \) yields \( \varepsilon_r = b \)), we obtain that

\[
\frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}^N} u^r \leq a \int_{\mathbb{R}^N} u^r + \frac{4(r-1)C_0}{r^2 \varepsilon_r} \frac{2(r+1)}{(1+\frac{\theta}{r-\theta(r+1)})} \|u(t)\|_{L^p(\mathbb{R}^N)}^{(1-\theta)(r+1)} \|u(t)\|_{L^{\tilde{p}}(\mathbb{R}^N)}^{\frac{r}{r-\theta(r+1)}}.
\]

It then follows by Gronwall’s inequality and the mean value theorem that

\[
\|u(t)\|_{L^r(\mathbb{R}^N)} \leq \left[ \|u_0\|_{L^r(\mathbb{R}^N)} + K_r \|u_0\|_{L^{\tilde{p}}(\mathbb{R}^N)}^\lambda \right] e^{\lambda_r t} \forall t \in [0, T_{\max}),
\]

with

\[
\lambda_r = \frac{(1-\theta)(r+1)}{r-\theta(r+1)} \quad \text{and} \quad K_r = \frac{4(r-1)C_0}{r^2 \varepsilon_r} \frac{2(r+1)}{(1+\frac{\theta}{r-\theta(r+1)})}.
\]

(4.10) then follows.

**Step 2.** It follows from Theorem 1.2, Theorem 1.4, Lemma 1.2 and Step 1 that (1.3) has a unique global classical solution \((u, v)\). To complete the proof of this theorem, we need to prove the following estimate.

\[
\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_1 t^{-\frac{N}{p}} e^{-t} \|u_0\|_{L^p(\mathbb{R}^N)} + C_2 \left[ \|u_0\|_{L^p(\mathbb{R}^N)} + K_p \|u_0\|_{L^1(\mathbb{R}^N)}^\lambda \|u_0\|_{L^{\tilde{p}}(\mathbb{R}^N)}^\frac{1}{\tilde{p}} e^{\lambda_{\tilde{p}} t} \right] e^{\lambda t},
\]

where \( \lambda_p, \tilde{\lambda}_p, \tilde{\lambda}, K_p, C_1 \) and \( C_2 \) are positive constants depending on \( N, p, a, b, \) and \( \chi \). Indeed, let us recall that

\[
u(t) = T(t)u_0 - \chi \int_0^t \left( T(t-s)\nabla(u \nabla v)(s) + (a+1) \int_0^s T(t-s)u(s)ds - b \int_0^s T(t-s)u^2(s)ds \right).\]

Note that

\[
\|v\|_{W^{2,p}(\mathbb{R}^N)} \leq C \|u\|_{L^p(\mathbb{R}^N)}.
\]

(4.10)
Using Lemma 3.1 and inequalities (1.6) and (4.10), we have that
\[
\| J_1 \|_{L^\infty} \leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2p}} e^{-(t-s)} \| u \nabla v(s) \|_{L^p(\mathbb{R}^N)} ds \\
\leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2p}} e^{-(t-s)} \| u \|_{L^p(\mathbb{R}^N)} \| v \|_{C^{1,\alpha}(\mathbb{R}^N)} ds \\
\leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2p}} e^{-(t-s)} \| u \|_{L^p(\mathbb{R}^N)}^2 ds \\
\leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2p}} e^{-(t-s)} \left[ \| u_0 \|_{L^p(\mathbb{R}^N)} + K_{\frac{1}{p}} \frac{1}{t^p} \| u_0 \|_{L^{\eta_1}(\mathbb{R}^N)} e^{(\lambda_{p-1})at} \right]^2 e^{2at} ds \\
\leq C \left[ \| u_0 \|_{L^p(\mathbb{R}^N)} + K_{\frac{1}{p}} \frac{1}{t^p} \| u_0 \|_{L^{\eta_1}(\mathbb{R}^N)} \right] e^{2at} \Gamma \left( \frac{1}{2} - \frac{N}{p} \right).
\]

Since \( \frac{1}{2} + \frac{N}{2p} \in (\frac{1}{2}, 1) \), we have that \( X^{\frac{1}{2} + \frac{N}{2p}} \) is continuously embedded in \( L^\infty(\mathbb{R}^N) \). Thus
\[
\| J_2 \|_{L^\infty} \leq C \int_0^t \| T(t-s)u(s) \|_{X^{\frac{1}{2} + \frac{N}{2p}}} ds \\
\leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2p}} e^{-(t-s)} \| u(s) \|_{L^p(\mathbb{R}^N)} ds \\
\leq C \left[ \| u_0 \|_{L^p(\mathbb{R}^N)} + K_{\frac{1}{p}} \| u_0 \|_{L^{\eta_1}(\mathbb{R}^N)} \right] \Gamma \left( \frac{1}{2} - \frac{N}{p} \right).
\]

By (2.12), we have
\[
\| T(t)u_0 \|_{L^\infty} \leq C t^{-\frac{N}{2p}} e^{-\frac{t}{p}} \| u_0 \|_{L^p(\mathbb{R}^N)}. \tag{4.11}
\]

Since \( u^2(s) \geq 0 \) for all \( s \geq 0 \), then \( J_3 \geq 0 \). Combining these with the fact that \( u(t) \geq 0 \), we obtain that
\[
\| u(t) \|_{L^\infty} \leq \| T(t)u_0 \|_{L^\infty} + \chi \| J_1(t) \|_{L^\infty} + (a + 1) \| J_2(t) \|_{L^\infty}
\]

Therefore we conclude that
\[
\| u(t) \|_{L^\infty} \leq C_1 t^{-\frac{N}{2p}} e^{-\frac{t}{p}} \| u_0 \|_{L^p(\mathbb{R}^N)} + C_2 \left[ \| u_0 \|_{L^p(\mathbb{R}^N)} + K_{\frac{1}{p}} \| u_0 \|_{L^{\eta_1}(\mathbb{R}^N)} \right] e^{2at},
\]

where \( C_1 \) and \( C_2 \) are positive constants depending on \( N \), \( p \), \( a \), \( b \), and \( \chi \). Now, since \( 1 < q_1 < p \), then \( \| u_0 \|_{q_1} \leq \| u_0 \|_{\frac{N}{2p} + \lambda} \| u_0 \|_{p}^{-\lambda} \) for \( \lambda = \frac{p-q_1}{q_1(p-1)} \). Thus the Theorem follows.

\[\square\]

**Remark 4.4.** We first point out that Theorem 1.6 does not extend Theorem 1.5 because it requires for \( \| u_0 \|_{L^1} + \| u_0 \|_{L^p} \) to be finite. Also, it should be noted that using (2.12), inequality (4.11) can be replaced by
\[
\| T(t)u_0 \|_{L^1(\mathbb{R}^N)} \leq C_1 t^{-\frac{N}{2p}} e^{-\frac{t}{p}} \| u_0 \|_{L^1(\mathbb{R}^N)}. \tag{4.12}
\]

On the other hand, under the hypothesis of Corollary 4.3, that is if \( p \leq \frac{\chi}{(\lambda-b)_+} \), by following the arguments used in the second part of the proof of Theorem 1.7 and making use of inequality (4.1) we obtain that
(i) There is a constant $C > 0$ depending on $a, b, \chi, N$ and $p$ such
\[
\|u(t)\|_{L^\infty} \leq C \left[ t^{-\frac{N}{2p}} e^{-(1+2a)t} + \|u_0\|_{L^p(\mathbb{R}^N)}^2 \right] e^{2at} \quad \forall \; t > 0.
\]
(ii) For every $\varepsilon > 0$, we have that
\[
\lim_{t \to \infty} e^{-(2a+\varepsilon)t} \|u(t) - T(t)u_0\|_{L^\infty} = 0
\]
(iii) If in addition $a = 0$ then
\[
\sup_{t \geq 0} \|u(t)\|_{L^\infty} < \infty.
\]

4.3 Proof of Theorem 1.7

In this subsection, we extend the results of the previous section to more initial data set and prove Theorem 1.7. Note that the choice of the initial data $u_0 \in X^\alpha$ in Theorem 1.6 depends on $N, p$ and $\alpha \in (\frac{1}{2}, 1)$. Since $X^\beta$ is continuously imbedded in $X^\alpha$ for $\beta \geq \alpha$, then Theorem 1.6 covers any nonnegative initial data in $X^\alpha$ with $\alpha \geq 1$.

Proof of Theorem 1.7. Let $\varphi$ be a nonnegative smooth mollifier function with $\|\varphi\|_{L^1} = 1$. For every $\varepsilon > 0$ let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^N} \varphi(\frac{1}{\varepsilon}x)$ for every $x \in \mathbb{R}^N$. Next, we define $u_{0\varepsilon} = \varphi_\varepsilon * u_0$ for every $n \geq 1$. We have that $u_{0\varepsilon} \in C^\infty(\mathbb{R}^N) \cap W^{k,q}(\mathbb{R}^N)$ for $n \in \mathbb{N}, k \geq 1$ and $q \geq 1$. Furthermore, we have that $u_{0\varepsilon} \geq 0$ for every $n$ with
\[
\|u_{0\varepsilon}\|_{L^p(\mathbb{R}^N)} \leq \|\varphi_n\|_{L^1(\mathbb{R}^N)}\|u_0\|_{L^p(\mathbb{R}^N)} = \|u_0\|_{L^p(\mathbb{R}^N)} \quad \forall \; p \geq q \geq 1 \; n \geq 1 \quad (4.13)
\]
and
\[
\lim_{n \to \infty} \|u_{0\varepsilon} - u_0\|_{L^q(\mathbb{R}^N)} = 0 \quad \text{for all} \; q \in [1, p].
\]
Let us choose $\alpha \in (\frac{1}{2}, 1)$ and $0 < \delta \leq 2\alpha - 1$ satisfying
\[
\frac{(2\alpha - 1 - \delta)p}{N} > \frac{1}{2} > \frac{1}{2p} = \frac{1}{p} - \frac{1}{2p}.
\]
Hence, $X^\alpha$ is continuously imbedded in $C^{1+\delta}$. We have that $u_{0\varepsilon} \in X^\alpha$ for all $n \geq 1$. Thus according to Theorem 1.6, for every $n \geq 1$, there is a global in time unique solution $(u_{m}(x,t), v_{m}(x,t)) = (u(x,t; u_{0\varepsilon}), v(x,t; u_{0\varepsilon}))$ of (1.4) with initial data $u_{0\varepsilon}$.

By the arguments of Theorem 1.3
\[
\lim_{m \to \infty} \left[ \|u(\cdot, t; u_{0\varepsilon}) - u(\cdot, t; u_0)\|_{L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)} + \|v(\cdot, t; u_{0\varepsilon}) - v(\cdot, t; u_0)\|_{L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)} \right] = 0
\]
for any $t$ in the maximal existence interval $[0, T_{\text{max}}(u_0))$ of $(u(x,t; u_0), v(x,t; u_0))$. Choose $q_1 \in \left( \max\{1, \frac{N}{2}\}, \min\{p, \frac{\chi}{(\chi - b) \varepsilon} \} \right)$. By Lemma 3.2 and (4.6), we have
\[
\|u(\cdot, t; u_{0\varepsilon})\|_{L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)} \leq \left[ \|u_{0\varepsilon}\|_{L^p(\mathbb{R}^N)} + R_{\varepsilon} \frac{1}{t^p} \|u_{0\varepsilon}\|_{L^p(\mathbb{R}^N)} e^{(\frac{\lambda}{p} - 1)at} \right] e^{at} \quad \forall \; t \in [0, T_{\text{max}}(u_0)).
\]
This implies that
\[ \|u(\cdot, t; u_{0m})\|_{L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)} \leq \left[ \|u_0\|_{L^p(\mathbb{R}^N)} + K_p^T \|u_0\|_{L^q(\mathbb{R}^N)} e^{(\lambda_p-1)at} \right] e^{at} \quad \forall \; t \in [0, T_{\text{max}}^0(u_0)) \]
and hence \( T_{\text{max}}^0(u_0) = \infty \). Since (1.16) holds for every \( u(\cdot, t; u_0m) \), letting \( m \to \infty \), we obtain that \( u(\cdot, t : u_0) \) also satisfies (1.16). This completes the proof.

As an immediate consequence of the Theorem 1.5 and Theorem 1.7 we have the following result.

**Corollary 4.5.** Suppose that assumptions of Theorem 1.7 hold and \( \chi < b \). Let \( u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \) and \( (u(\cdot, \cdot; u_0), v(\cdot, \cdot; u_0)) \) be the global classical solution of (1.4) given by Theorem 1.7. Then for every \( T > 0 \), we have that
\[ \sup_{t \geq T} \left[ \|u(\cdot, t, u_0)\|_{\infty} + \|v(\cdot, t, u_0)\|_{\infty} \right] < \infty. \]  
(4.14)

**Proof.** We have that \( u(\cdot, \cdot, u_0) \in C^{2,1}(\mathbb{R}^N \times (0, \infty)) \) and satisfies
\[ \partial_t u \leq \Delta u - \chi \nabla v(\cdot, \cdot; u_0) + (a - (b - \chi)u)u. \]  
(4.15)
Since \( u(\cdot, T, u_0) \in C^b_{\text{unif}}(\mathbb{R}^N) \), same arguments as in the proof of Theorem 1.5 imply that inequality (4.14) holds.

5 Asymptotic behavior of solutions

In this section, we discuss the asymptotic behaviors of global bounded classical solutions of (1.4) under the assumption that \( b > 2\chi \). This will be done in two subsections. The first subsection is devoted for strictly positive initial data. Hence its results apply only for \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^N) \). While in the second part we shall deal with initial data with compact supports. Whence there is no restriction on the space \( X \) in this case. Again, throughout this section, \( C \) denotes a constant independent of the initial functions and the solutions under consideration, unless specified otherwise.

5.1 Asymptotic behavior of solutions with strictly positive initial data

We shall assume that \( \inf u_0 > 0 \) for \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^N) \) with \( b > 2\chi \). Following the ideas given in [38] and [39], we consider the asymptotic behavior of the solution \( (u(x, t), v(x, t) := (u(x, t; u_0), v(x, t; u_0)) \) of (1.4) with \( u(x, 0; u_0) = u_0(x) \). Clearly, the sufficient conditions required for the existence of a unique bounded classical solution \((u, v)\) in Theorem 1.5 are satisfied and according to (4.2) and (4.3), it holds that
\[ 0 \leq u(x, t) \leq u(t, \|u_0\|_{\infty}) \quad \forall \; x \in \mathbb{R}^N, \; t \geq 0. \]
with
\[ \lim_{t \to \infty} u(t, \|u_0\|_{\infty}) = \frac{a}{b - \chi}. \]
where \( u(t, \|u_0\|_\infty) \) is the solution of (1.1). Define

\[
\overline{u} = \limsup_{t \to \infty} \left( \sup_{x \in \mathbb{R}^N} u(x, t) \right), \quad \underline{u} = \liminf_{t \to \infty} \left( \inf_{x \in \mathbb{R}^N} u(x, t) \right).
\]

Clearly \( 0 \leq \underline{u} \leq \overline{u} \leq \frac{a}{b-\chi} \). Our goal is to prove that \( \overline{u} = \underline{u} \). Observe that this will imply that \( \|u(t) - \overline{u}\|_{L^\infty} \to 0 \) as \( t \to \infty \). Note that if \( a = 0 \), then \( \underline{u} = \overline{u} = 0 \). Hence we shall suppose that \( a > 0 \) in this section.

By comparison principle for elliptic equations, we have that

\[
\inf_{x \in \mathbb{R}^N} u(\cdot, t) \leq v(x, t) \leq \sup_{x \in \mathbb{R}^N} u(\cdot, t) \quad \forall \ x \in \mathbb{R}^N, \forall \ t \geq 0.
\]

Hence, it follows that \( \sup_{x \in \mathbb{R}^N} v(x, t) < \infty \). Using definition of \( \limsup \) and \( \liminf \), for every \( \varepsilon > 0 \), there is \( t_\varepsilon > 0 \) such that

\[
\underline{u} - \varepsilon \leq u(x, t) \leq \overline{u} + \varepsilon \quad \forall \ x \in \mathbb{R}^N, \forall \ t \geq t_\varepsilon.
\]

Combining this with (5.2) we have

\[
\underline{u} - \varepsilon \leq v(x, t) \leq \overline{u} + \varepsilon \quad \forall \ x \in \mathbb{R}^N, \forall \ t \geq t_\varepsilon.
\]

Let us define

\[
Lu = \Delta u - \chi \nabla v \nabla u.
\]

Since \((u, v)\) solves (1.4), we have

\[
u_t - Lu = -\chi uv + u(a - (b - \chi)u) = u \left[ a - \chi v - (b - \chi)u \right].
\]

Note that \( 0 \leq u \). By (5.4) and (5.5), for \( t \geq t_\varepsilon \), we have

\[
u_t - Lu \leq u \left[ a - \chi (\underline{u} - \varepsilon) - (b - \chi)u \right]
\]

and

\[
u_t - Lu \geq u \left[ a - \chi (\overline{u} + \varepsilon) - (b - \chi)u \right].
\]

The following lemmas will be helpful in the proof of the main theorem of this section.

**Lemma 5.1.** Under the foregoing assumptions, we have that

\[
\inf_{x \in \mathbb{R}^N} u(x, t) > 0, \quad \forall \ t > 0 \quad \text{and} \quad a - \chi \overline{u} > 0.
\]

**Proof.** Let \( u(t, \inf_{x \in \mathbb{R}^N} u_0) \) be the solution of the following ordinary differential equation,

\[
\begin{cases}
U_t = -(b - \chi)U^2 + (a - \chi v_\infty)U \\
U_0 = \inf_{x \in \mathbb{R}^N} u_0.
\end{cases}
\]

where \( v_\infty := \sup_{x \in \mathbb{R}^N, t \geq 0} v(x, t) \). Since \( b - \chi > 0 \), then \( u(t, \inf_{x \in \mathbb{R}^N} u_0) \) is globally defined and bounded with \( 0 < u(t, \inf_{x \in \mathbb{R}^N} u_0) \) for all \( t \geq 0 \). Note that if \( a - \chi v_\infty \leq 0 \), then \( u(t, \inf_{x \in \mathbb{R}^N} u_0) \)
decreases to 0, while if \( a - \chi u_\infty > 0 \), by Lemma 4.1 we have \( u(t, \inf_{x \in \mathbb{R}^N} u_0) \to \frac{a - \chi u_\infty}{b - \chi} \). Since \( u_0 \geq u(0, \inf_{x \in \mathbb{R}^N} u_0) \), by the comparison principle for parabolic equations, we conclude that

\[
u(t, \inf_{x \in \mathbb{R}^N} u_0) \leq u(x, t)
\]

for all \( x \in \mathbb{R}^N \) and \( t \geq 0 \). Hence the first inequality in (5.8) follows. On the other hand, if we suppose by contradiction that \( a - \chi u \leq 0 \), then we would have that

\[
\frac{a}{\chi} \leq u \leq \frac{a}{b - \chi},
\]

which contradicts the fact that \( b > 2\chi \). Hence the second inequality in (5.8) holds.

Since \( u \leq u \), according to Lemma 5.1, we may suppose that \( 0 < a - \chi(u + \varepsilon) < a - \chi(u - \varepsilon) \) for \( \varepsilon \) very small.

**Lemma 5.2.** Under the forgoing assumptions, it holds that

\[
(b - \chi)\overline{u} \leq a - \chi u \quad \text{and} \quad a - \chi \overline{u} \leq (b - \chi)\underline{u}.
\]

**Proof.** Let \( \overline{w}(t) \) denote the solution of the initial value problem

\[
\begin{aligned}
\overline{w}_t &= \overline{w}[a - \chi(u - \varepsilon) - (b - \chi)\overline{w}] \quad \forall t > t_\varepsilon \\
\overline{w}(t_\varepsilon) &= \sup_{x \in \mathbb{R}^N} u(x, t_\varepsilon).
\end{aligned}
\]

By (5.6), (5.10), and the comparison principle for parabolic equations, we obtain that

\[
u(x, t) \leq \overline{w}(t) \quad \forall x \in \mathbb{R}^N, \quad t \geq t_\varepsilon.
\]

According to Lemma 5.1 we have that \( \sup_{x \in \mathbb{R}^N} u_0(x, t) > 0 \) for all \( t > 0 \). In particular we have that \( \overline{w}(t_\varepsilon) > 0 \). On the other hand, by Lemma 4.1

\[
\overline{w}(t) \to \frac{a - \chi(u - \varepsilon)}{b - \chi} \quad \text{as} \quad t \to \infty.
\]

Combining this with inequality (5.11), we obtain that

\[
\overline{u} \leq \frac{a - \chi(u - \varepsilon)}{b - \chi} \quad \forall \varepsilon > 0.
\]

By letting \( \varepsilon \to 0 \), we obtain the first inequality in (5.8).

Similarly, let \( \underline{w}(t) \) be solution of

\[
\begin{aligned}
\underline{w}_t &= \underline{w}[a - \chi(\overline{w} + \varepsilon) - (b - \chi)\overline{w}] \quad \forall t > t_\varepsilon \\
\underline{w}(t_\varepsilon) &= \inf_{x \in \mathbb{R}^N} u(x, t_\varepsilon).
\end{aligned}
\]

By (5.6), (5.12), and the comparison principle for parabolic equations, we have

\[
u(x, t) \geq \underline{w}(t) \quad \forall x \in \mathbb{R}^N, \quad t \geq t_\varepsilon.
\]

Same arguments as in above yield that \( \underline{w}(t_\varepsilon) > 0 \), and
Combining this with inequality \(5.3\), we obtain that
\[
\begin{align*}
\bar{u}(t) & \to \frac{a - \chi \overline{u}}{b - \chi} \text{ as } t \to \infty.
\end{align*}
\]

By letting \(\varepsilon \to 0\), we obtain the second inequality in \(5.3\). Lemma 5.2 is thus proved. \(\square\)

From these Lemmas, we can easily present the proof of Theorem 1.8.

**Proof of Theorem 1.8.** It follows from Lemma 5.2 that
\[
(b - 2\chi)\overline{u} = (b - \chi)\overline{u} - \chi \overline{u} = (b - \chi)\overline{u} + (a - \chi \overline{u}) - a \\
\leq a - \chi \overline{u} + (b - \chi)\overline{u} - a = (b - 2\chi)\overline{u}.
\]

Combining this with the fact that \(\underline{u} \leq \overline{u}\) and \((b - 2\chi) > 0\) we obtain that
\[
\underline{u} = \overline{u}. \tag{5.13}
\]

Equality (5.13) combining with Lemma 5.2 imply that
\[
(b - \chi)\overline{u} = a - \chi \overline{u}.
\]

Solving for \(\overline{u}\) in the last equality, we obtain that \(\underline{u} = \overline{u} = \frac{a}{b}\). The conclusion of the theorem follow ready from the last equality and \((5.1)\), and \((5.2)\). \(\square\)

When the initial data \(u_0\) is not bounded away from zero. The uniform convergence on \(\mathbb{R}^N\) of \(u(x, t)\) as \(t \to \infty\) to the constant steady solution \(\frac{a}{b}\) does not hold. However we have a uniform local convergence of \(u(x, t)\) as \(t \to \infty\) to the steady solution under an additional hypothesis. We establish these in the last subsection.

### 5.2 Asymptotic behavior of solutions with non-negative initial data

Throughout this section we suppose that \(u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)\) is nonnegative and not identically zero with \(2\chi < b\). We shall also denote by \((u(x, t), v(x, t))\), the global bounded classical solution of \((1.4)\) associated with initial data \(u_0\).

In order to study the asymptotic behavior of \(u(\cdot, t)\), we first need to get some estimate on \(\|\nabla v(x, t)\|\). Since \(\Delta v = v - \underline{u}\) and \(\|v(\cdot, t)\|_\infty \leq \|u(\cdot, t)\|_\infty\) for every \(t > 0\), it follows from Lemma 3.3 that
\[
\|\nabla v(x, t)\| \leq \sqrt{N}\|u(\cdot, t)\|_\infty \quad \forall \ x \in \mathbb{R}^N, \ t > 0. \tag{5.14}
\]

Let us define for \(U \in C^{2,1}(\mathbb{R}^N \times \mathbb{R})\)
\[
LU := \partial_t U - \Delta U - \chi \nabla v \nabla U. \tag{5.15}
\]

We have that
\[
Lu = \begin{cases} a - \chi v - (b - \chi)u, \quad x \in \mathbb{R}^N, \ t > 0. \end{cases}
\]

\[\tag{5.16}\]
Hence, since $v \geq 0$, it follows that

$$Lu \leq u(a - (b - \chi)u). \quad (5.17)$$

By the comparison principle for parabolic equations, we have that

$$u(x, t) \leq U(t, \|u_0\|_{\infty}) \quad \forall \ x \in \mathbb{R}^N, \ t \geq 0,$$  

where $U(t, \|u_0\|_{\infty})$ is the solution of the ODE

$$\begin{cases}
LU = F^2(U) \\
U(0) = \|u_0\|_{\infty}.
\end{cases} \quad (5.19)$$

By Lemma 4.1, $U(t, \|u_0\|_{\infty}) \to \frac{a}{b-\chi}$ as $t \to \infty$.

Next, we prove some lemmas.

**Lemma 5.3.** Assume that $0 < \chi < \frac{2b}{3 + \sqrt{1 + Na}}$. Then

$$\lim_{R \to \infty} \inf_{t \geq R, |x| > R} (4(a - \chi v(x, t)) - \chi^2 \|\nabla v(x, t)\|^2) > 0. \quad (5.20)$$

**Proof.** From (5.14), (5.18), and the fact that $U(t, \|u_0\|_{\infty}) \to \frac{a}{b-\chi}$ as $t$ goes to infinity, for (5.20) to hold, it is enough to have

$$4(a - \frac{\chi a}{b - \chi}) - \frac{Na \chi^2 a^2}{(b - \chi)^2} > 0. \quad (5.21)$$

Let $\mu = \frac{\chi}{b - \chi}$. (5.21) is equivalent to $4(1 - \mu) - Na \mu^2 > 0$. This implies that

$$0 < \mu = \frac{\chi}{b - \chi} < \frac{2}{1 + \sqrt{1 + Na}}$$

and then

$$0 < \chi < \frac{2b}{3 + \sqrt{1 + Na}}.$$

The lemma is thus proved. \hfill \square

**Lemma 5.4.** Let $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ be a nonnegative and non-zero function. Let $(u, v)$ be the classical bounded solution of (1.4) associated with $u_0$. If

$$\chi < \frac{2b}{3 + \sqrt{1 + Na}}, \quad (5.22)$$

then

$$\liminf_{t \to \infty} \inf_{|x| \leq ct} u(x, t) > 0 \quad (5.23)$$

for every $0 \leq c < c^*(\leq 2\sqrt{a})$, where

$$c^* = \lim_{R \to \infty} \inf_{|x| \geq R, t \geq R} (2\sqrt{a - \chi v(x, t)} - \chi \|\nabla v(x, t)\|). \quad (5.24)$$

38
Proof. First, we know that \( \|v(\cdot, t)\|_\infty \leq \|u(\cdot, t)\|_\infty \leq U(t, \|u_0\|_\infty) \) for all \( t \) with \( U(t, \|u_0\|_\infty) \to \frac{a}{b-\chi} \) as \( t \to \infty \). Hence for every \( \varepsilon > 0 \), there is \( T_\varepsilon > 0 \) such that

\[
\|v(\cdot, t)\|_\infty \leq \frac{a}{b-\chi} + \varepsilon, \quad \forall \ t \geq T_\varepsilon
\]  

(5.25)

and \( T_\varepsilon \to \infty \) as \( \varepsilon \to 0 \). Define \( \tilde{u}(x,t) = u(x,T_\varepsilon + t) \) and \( \tilde{v}(x,t) = v(x,T_\varepsilon + t) \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^N \). Then \( \tilde{u}(x,t) \) satisfies

\[
\tilde{u}_t = \Delta \tilde{u} - \chi \nabla \tilde{v} \nabla \tilde{u} + \tilde{F}(t,x,\tilde{u}),
\]

where \( \tilde{F}(t,x,u) = \tilde{u}(a - \chi \tilde{v} - (b - \chi)\tilde{u}) \). Note that

\[
4\partial_q \tilde{F}(x,t,0) - \|\chi \nabla \tilde{v}(x,t)\|^2 = 4(a - \chi \tilde{v}(x,t)) - \chi^2 \|\nabla v(x,t)\|^2.
\]

By Lemma 5.3,

\[
\lim_{R \to \infty} \inf_{R > |x| > R} (4\partial_q \tilde{F}(x,t,0) - \|\chi \nabla \tilde{v}(x,t)\|^2) > 0.
\]

Next, we introduce the linear operator

\[
\mathcal{L}w := \partial_t w - \Delta w + q(x,t) \cdot \nabla w - p(x,t)w
\]

for every \( w \in C^{2,1}(\mathbb{R}^N \times \mathbb{R}) \), where

\[
q(x,t) = \begin{cases} \chi \nabla \tilde{v}(x,t), & t \geq 0 \\ \chi \nabla \tilde{v}(x,0), & t < 0 \end{cases} \quad \text{and} \quad p(x,t) = \begin{cases} a - \chi \tilde{v}(x,t), & t \geq 0 \\ a - \chi \tilde{v}(x,0), & t < 0 \end{cases}.
\]

Following [3], the generalized principal eigenvalue associated to the operator \( \mathcal{L} \) is defined to be

\[
\lambda'_1 := \inf \{ \lambda \in \mathbb{R}^N : \exists \phi \in C^{2,1} \cap W^{1,\infty}(\mathbb{R}^N \times \mathbb{R}), \inf_{(x,t)} \phi > 0, \mathcal{L}\phi \leq \lambda \phi \}.
\]

We show that \( \lambda'_1 < 0 \) for small values of \( \varepsilon \). Indeed, for \( w(x,t) = 1 \), the constant function, using definition of \( \tilde{v} \) and inequality (5.25), we obtain that

\[
\mathcal{L}(w) = \begin{cases} -a + \chi \tilde{v}(x,t), & t \geq 0 \\ -a + \chi \tilde{v}(x,0), & t < 0 \end{cases} \leq -a + \frac{\lambda a}{b-\chi} + \chi \varepsilon = \chi \varepsilon - \frac{a(b-2\chi)}{b-\chi}.
\]

Hence \( \lambda'_1 < 0 \) whenever \( \varepsilon < \frac{a(b-2\chi)}{\chi(b-\chi)} \).

Now, by Theorem 1.5 in [3], it holds that

\[
\lim_{t \to \infty} \inf_{|x| \leq ct} \tilde{u}(x,t) > 0
\]

(5.26)

for every \( 0 \leq c < c^*_\varepsilon \) where

\[
c^*_\varepsilon = \lim_{|x| \to \infty} \inf_{t \geq T_\varepsilon} (2\sqrt{a - \chi v(x,t)} - \chi \|\nabla v(x,t)\|).
\]

39
By the definition of $\tilde{u}$ and (5.23), we deduce that
\[
\liminf_{t \to \infty} \inf_{|x| \leq ct} u(x, t + T_\varepsilon) > 0 \quad \forall \ 0 \leq c < c^*_\varepsilon. \quad (5.27)
\]

We claim that $\lim_{\varepsilon \to 0} c^*_\varepsilon = c^*$. In fact, recall that
\[
c^* = \lim_{R \to \infty} \inf_{|x| \geq R, t \geq R} \left(2\sqrt{a - \chi v(x, t)} - \chi \|\nabla v(x, t)\|\right),
\]
Using the fact that
\[
\inf_{|x| \geq R, t \geq R} f(x, t) = \inf_{|x| \geq R, t \geq R} f(x, t), \quad \forall \ R \geq T_\varepsilon,
\]
we have
\[
c^*_\varepsilon = \liminf_{|x| \to \infty} f(x, t) = \liminf_{R \to \infty} \inf_{|x| \geq R, t \geq R} f(x, t) \leq \liminf_{R \to \infty} \inf_{|x| \geq R, t \geq R} f(x, t) = c^*. \quad (5.28)
\]
Using the fact that for given $\delta > 0$, there is $R_\delta > 0$ such that
\[
c^* - \delta < f(x, t) \quad \forall \ |x|, t \geq R_\delta
\]
and that there is $\varepsilon_0$ such
\[
T_\varepsilon \geq R_\delta \quad \forall \ \varepsilon < \varepsilon_0,
\]
we have
\[
c^* - \delta \leq \inf_{|x| \geq R, t \geq T_\varepsilon} f(x, t) \quad \forall \ R \geq R_\delta, \ \forall \ \varepsilon < \varepsilon_0.
\]
Thus, for every $0 < \varepsilon < \varepsilon_0$, we have that
\[
c^* - \delta \leq \lim_{R \to \infty} \inf_{|x| \geq R, t \geq T_\varepsilon} f(x, t) = c^*_\varepsilon. \quad (5.29)
\]

By (5.28) and (5.29), we obtain
\[
\lim_{\varepsilon \to 0} c^*_\varepsilon = c^*.
\]
Finally, let $0 \leq c < c^*$ be fixed. There is some $\varepsilon > 0$ small enough such that $c < c^*_\varepsilon$. Choose $c^* \in (c, c^*_\varepsilon)$. Observe that
\[
ct = \hat{c}(t - T_\varepsilon) - (\hat{c} - c)(t - \frac{\hat{c}T_\varepsilon}{\hat{c} - c}) \leq \hat{c}(t - T_\varepsilon) \quad (5.30)
\]
whenever $t \geq \frac{\hat{c}T_\varepsilon}{\hat{c} - c}$. Hence, since $u(x, t) = u(x, t - T_\varepsilon + T_\varepsilon)$, we obtain that
\[
\inf_{\|x\| \leq ct} u(x, t) \geq \inf_{\|x\| \leq ct} u(x, t + T), \quad \forall \ t \geq \frac{\hat{c}T_\varepsilon}{\hat{c} - c}.
\]
Combining the last inequality with inequality (5.30), we conclude that inequality (5.23) hold. \qed

The next step to the proof of Theorem 1.9 is the following result. This result asserts that, under some conditions, the asymptotic behavior of the function $v(x, t)$ is quite similar to the one of the function $u(x, t)$.  

40
Lemma 5.5. (i) If there is a positive constant $c_{\text{low}}^*$ such that
\[
\lim_{t \to \infty} \sup_{|x| \leq ct} |u(x,t) - \frac{a}{b}| = 0 \quad \forall \ 0 \leq c < c_{\text{low}}^*,
\] (5.31)

then
\[
\lim_{t \to \infty} \sup_{|x| \leq ct} |v(x,t) - \frac{a}{b}| = 0 \quad \forall \ 0 \leq c < c_{\text{low}}^*.
\]

(ii) If there is a positive constant $c_{\text{up}}^*$ such that
\[
\lim_{t \to \infty} \sup_{|x| \geq ct} |u(x,t) - \frac{a}{b}| = 0 \quad \forall \ c > c_{\text{up}}^*,
\] (5.32)

then
\[
\lim_{t \to \infty} \sup_{|x| \geq ct} |v(x,t) - \frac{a}{b}| = 0 \quad \forall \ c > c_{\text{up}}^*.
\]

Proof. We first recall that
\[
v(x,t) = \int_0^\infty \int_{\mathbb{R}^N} e^{-s} \frac{|x-y|^2}{4\pi s} e^{-\frac{|x-u(t,y)|^2}{4s}} du \, dy ds = \frac{1}{\pi^\frac{N}{2}} \int_0^\infty \int_{\mathbb{R}^N} e^{-s} e^{-|z|^2} u(x-2\sqrt{s}z,t) dy \, ds. \tag{5.33}
\]

Let $\varepsilon > 0$ be fixed. Since
\[
\frac{1}{\pi^\frac{N}{2}} \int_0^\infty \int_{\mathbb{R}^N} e^{-s} e^{-|z|^2} dy \, ds = \left[ \int_0^\infty e^{-s} \, ds \right] \left[ \frac{1}{\pi^\frac{N}{2}} \int_{\mathbb{R}^N} e^{-|z|^2} \, dy \right] = 1
\]

and
\[
\sup_{t \geq 0} \|u(\cdot,t) + 1\|_\infty < \infty,
\]

there is $R > 0$ large enough such that
\[
\frac{1}{\pi^\frac{N}{2}} \int_{\{s \geq R \} \cup \{|z| \geq R\}} e^{-s} e^{-|z|^2} |u(x-2\sqrt{s}z,t) + 1| \, dy \, ds < \frac{\varepsilon}{2} \tag{5.34}
\]

for all $x \in \mathbb{R}^N$, $t \geq 0$.

(i) Let $0 \leq c < c_{\text{low}}^*$ be fixed. Choose $c < \tilde{c} < c_{\text{low}}^*$. From (5.31), there is $t_0 > 0$ such that
\[
|u(x,t) - \frac{a}{b}| \leq \frac{\varepsilon}{2} \tag{5.35}
\]

for every $|x| \leq \tilde{c}t$, $t \geq t_0$. Using (5.34), for every $x \in \mathbb{R}^N$ and $t > 0$, we have
\[
|v(x,t) - \frac{a}{b}| \leq \frac{1}{\pi^\frac{N}{2}} \int_0^\infty \int_{\mathbb{R}^N} e^{-s} e^{-|z|^2} |u(x-2\sqrt{s}z,t) - \frac{a}{b}| \, dy \, ds
\]
\[
\leq \frac{1}{\pi^\frac{N}{2}} \int_{\{s \leq R \} \cup \{|z| \leq R\}} e^{-s} e^{-|z|^2} |u(x-2\sqrt{s}z,t) - \frac{a}{b}| \, dy \, ds
\]
\[
+ \frac{1}{\pi^\frac{N}{2}} \int_{\{s > R \} \cup \{|z| > R\}} e^{-s} e^{-|z|^2} |u(x-2\sqrt{s}z,t) + \frac{a}{b}| \, dy \, ds
\]
\[
\leq \frac{1}{\pi^\frac{N}{2}} \int_{\{s \leq R \} \cup \{|z| \leq R\}} e^{-s} e^{-|z|^2} |u(x-2\sqrt{s}z,t) - \frac{a}{b}| \, dy \, ds + \frac{\varepsilon}{2}. \tag{5.36}
\]
On the other hand we have that
\[ |x - 2\sqrt{s}z| \leq |x| + 2\sqrt{s}|z| \leq ct + 2R^2 = \tilde{c}t - (\tilde{c} - c)(t - \frac{2R^2}{\tilde{c} - c}) \leq \tilde{c}t \quad (5.37) \]
whenever \(|x| \leq ct, s \leq R, |z| \leq R, \) and \(t \geq \frac{2R^2}{\tilde{c} - c}\). Hence combining inequalities (5.35), (5.36) and (5.37), we obtain that
\[ \sup_{|x| \leq \infty} |u(x, t) - \frac{a}{b}| \leq \varepsilon \]
whenever \(t \geq \max\{t_0, \frac{2R^2}{\tilde{c} - c}\}\). This complete the proof of (i).

(ii) Let \(c > c_{up}\) be fixed. Choose \(c_{up} < \tilde{c} < c\). Since \(\tilde{c} > c_{up}\), according to (5.32), there is \(t_1 > 0\) such that
\[ \sup_{|y| \geq \tilde{c}t} u(y, t) < \frac{\varepsilon}{2}, \quad \forall \ t \geq t_1. \quad (5.38) \]
Using Triangle inequality, for every \((z, s) \in B(0, R) \times [0, R]\) and \(|x| \geq ct\), it hold that
\[ ct \leq |x| \leq |x - 2\sqrt{s}z| + 2\sqrt{s}|z| \leq |x - 2\sqrt{s}z| + 2R^2. \]
Which implies that
\[ ct - 2R^2 \leq |x - 2\sqrt{s}z| \quad (5.39) \]
for every \((z, s) \in B(0, R) \times [0, R]\) and \(|x| \geq ct\). But
\[ \tilde{c}t \leq ct - 2R^2 \iff t \geq \frac{2R^2}{\tilde{c} - c}. \quad (5.40) \]
Hence combining inequalities (5.38), (5.39) and (5.40) we have that
\[ \sup_{(z,s) \in B(0,R)\times[0,R]} u(x - 2\sqrt{s}z, t) \leq \varepsilon \quad (5.41) \]
for \(|x| \geq ct\) and \(t \geq \max\{t_1, \frac{2R^2}{\tilde{c} - c}\}\). This implies that
\[ \frac{1}{\pi^2} \int \int_{\{s \leq R, |z| \leq R\}} e^{-s}e^{-|z|^2} u(x - 2\sqrt{s}z, t)dzds < \frac{\varepsilon}{2} \quad (5.42) \]
for \(|x| \geq ct\) and \(t \geq \max\{t_1, \frac{2R^2}{\tilde{c} - c}\}\). From (5.33) and inequalities (5.34) and (5.42), it follows that
\[ \lim_{t \to \infty} \sup_{|x| \geq ct} v(x, t) = 0 \]
for every \(c > c_{up}\). \hfill \(\Box\)

We now prove Theorem 1.9.

Proof of Theorem 1.9. (1) From Lemma 5.4 we know that
\[ \lim_{t \to \infty} \inf_{|x| \leq ct} u(x, t) > 0 \]
for all \(0 \leq c < c^*\). We first prove that (1.20) holds for \(0 \leq c < c^*\), where \(c^*\) is as in Lemma 5.4.
Assume that there are constants $0 \leq c < c^*$, $\delta > 0$ and a sequence $\{(x_n, t_n)\}_{n \in \mathbb{N}}$ such $t_n \to \infty$, $\|x_n\| \leq ct_n$ and

$$|u(x_n, t_n) - \frac{a}{b}| \geq \delta, \quad \forall n \geq 1. \quad (5.43)$$

For every $n \geq 1$, let us define

$$u_n(x, t) = u(x + x_n, t + t_n), \quad \text{and} \quad v_n(x, t) = v(x + x_n, t + t_n)$$

for every $x \in \mathbb{R}^N$, $t \geq -t_n$. Choose $0 < \alpha < \frac{1}{2}$. Using the facts that

$$u(\cdot, t) = T(t)u_0 - \chi \int_0^t T(t-s)\nabla(u(s)\nabla v(s))ds + \int_0^t T(t-s)(au(s) - bu^2(s))ds.$$

For every $n \geq 1$, we have

$$\|u_{n0}\|_{X^\alpha} \leq \|T(t_n)u_0(\cdot + x_n)\|_{X^\alpha} + \chi \int_0^{t_n} \|T(t_n-s)\nabla(u(s)\nabla v(s))\|_{X^\alpha}ds$$

$$+ \int_0^{t_n} \|T(t_n-s)(au(s) - bu^2(s))\|_{X^\alpha}ds$$

$$\leq C \alpha t_n^{-\alpha} \|u_0\|_{\infty} + \chi \int_0^{t_n} \|T(t_n-s)\nabla(u(s)\nabla v(s))\|_{X^\alpha}ds$$

$$+ C \alpha \int_0^{t_n} e^{-\alpha(t_n-s)}(t_n-s)^{-\alpha} \|au(s) - bu^2(s)\|_{\infty}ds. \quad (5.44)$$

Next, using Lemma [3.2] the last inequality can be improved as

$$\|u_{n0}\|_{X^\alpha} \leq C \alpha t_n^{-\alpha} \|u_0\|_{\infty} + C \alpha \chi \int_0^{t_n} e^{-\alpha(t_n-s)}(t_n-s)^{-\frac{1}{2}-\alpha} \|u(s)\nabla v(s)\|_{\infty}ds$$

$$+ C \alpha \int_0^{t_n} e^{-\alpha(t_n-s)}(t_n-s)^{-\alpha} \|au(s) - bu^2(s)\|_{\infty}ds. \quad (5.44)$$

Combining inequality (5.44) and the fact that $\sup_t \|u(\cdot, t)\|_{\infty} < \infty$, inequality (5.44) becomes,

$$\|u_{n0}\|_{X^\alpha} \leq C \alpha t_n^{-\alpha} \|u_0\|_{\infty} + C \left( \int_0^\infty e^{-s \frac{1}{2} - \alpha} ds + \int_0^\infty e^{-s \alpha} ds \right).$$

$$= C \alpha t_n^{-\alpha} \|u_0\|_{\infty} + C \quad (5.45)$$

Since $t_n \to \infty$ as $n \to \infty$, then $\sup_n \|u_{n0}\|_{X^\alpha} < \infty$. Furthermore, similar arguments as in the proof of Theorem [13] show that the functions $u_n : [-T, T] \to X^\alpha$ are equicontinuous for every $T > 0$. Hence, Arzela-Ascoli’s Theorem and Theorem 15 (page 80 of [1]) imply that there is a function $(\tilde{u}, \tilde{v}) \in C^{2,1}(\mathbb{R}^N \times \mathbb{R})$ and a subsequence $\{(u_{n'}, v_{n'})\}_n$ of $\{(u_n, v_n)\}_n$ such that $(u_{n'}, v_{n'}) \to (\tilde{u}, \tilde{v})$ in $C^{1+\delta', \delta'}_l(\mathbb{R}^N \times (-\infty, \infty))$ for some $\delta' > 0$. Moreover, $\tilde{v} = (I - \Delta)^{-1} \tilde{u}$ and $(\tilde{u}, \tilde{v})$ solves (1.4) in classical sense. Note that

$$\tilde{u}(x, t) = \lim_{n \to \infty} u(x + x_{n'}, t + t_{n'}).$$
for every \( x \in \mathbb{R}^N, \ t \in \mathbb{R} \). Next, choose \( c \in (c, c^*) \). For every \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \), we have
\[
\|x + x_n\| \leq \|x\| + \|x_n\| \leq \|x\| + ct_n' = \tilde{c}(t_n' + t) - (\tilde{c} - c)(t_n' - \frac{\|x\| - \tilde{c}t}{\tilde{c} - c}) \leq \tilde{c}(t_n' + t)
\]
whenever \( t_n' \geq \frac{\|x\| + \tilde{c}t}{\tilde{c} - c} \). Thus, it follows that
\[
\tilde{u}(x, t) = \lim_{n \to \infty} u(x + x_n', t + t_n') \geq \liminf_{s \to \infty} \inf_{\|y\| \leq \tilde{c}s} u(y, s) > 0
\]
for every \((x, t) \in \mathbb{R}^N \times \mathbb{R} \). Hence \( \inf_{(x, t) \in \mathbb{R}^N \times \mathbb{R}} \tilde{u}(x, t) > 0 \). Using Theorem 1.8 we must have
\[
\lim_{t \to \infty} \|\tilde{u}(\cdot) - \frac{a}{b}\|_{\infty} = 0. \tag{5.46}
\]

**Claim.** \( \tilde{u}(x, t) = \frac{a}{b} \) for every \((x, t) \in \mathbb{R}^{N+1} \).

The proof of this claim is inspired from the ideas used to prove Theorem 1.8. Let us set \( \tilde{u}_0 = \inf_{(x, t) \in \mathbb{R}^{N+1}} \tilde{u} \) and \( \overline{u}_0 = \sup_{(x, t) \in \mathbb{R}^{N+1}} \tilde{u} \). Since \((\tilde{u}, \tilde{v})\) solves \( \{1.4\} \) in the classical sense and \( b > 2\chi \), it follows from Lemma 5.1 that
\[
a - \chi \tilde{u}_0 \geq a - \chi \overline{u}_0 > 0. \tag{5.47}
\]
For every \( t_0 \in \mathbb{R} \), let \( u(\cdot, t_0) \) and \( \overline{u}(\cdot, t_0) \) be the solutions of
\[
\begin{cases}
\frac{d}{dt} \overline{u} = \overline{u}(a - \chi \tilde{u}_0 - (b - \chi)\overline{u}), & t > t_0 \\
\overline{u}(t_0, t_0) = \overline{u}_0 
\end{cases}
\]
and
\[
\begin{cases}
\frac{d}{dt} u = u(a - \chi \overline{u}_0 - (b - \chi)u), & t > t_0 \\
u(t_0, t_0) = u_0,
\end{cases}
\]
respectively. Since \( 0 < \tilde{u}_0 \leq \tilde{v}(x, t) \leq \overline{u}_0 \) for every \((x, t) \in \mathbb{R}^{N+1} \), following the same arguments used to prove Lemma 5.2 we obtain that
\[
u(t - t_0, 0) = u(t, t_0) \leq \tilde{u}(x, t) \leq \overline{u}(t, t_0) = \overline{u}(t - t_0, 0) \quad \forall \ x \in \mathbb{R}^N, \ t \geq t_0. \tag{5.48}
\]
By Lemma 4.1 we have
\[
\lim_{t_0 \to -\infty} \overline{u}(t - t_0, 0) = \frac{a - \chi \tilde{u}_0}{b - \chi} \quad \text{and} \quad \lim_{t_0 \to -\infty} u(t - t_0, 0) = \frac{a - \chi \overline{u}_0}{b - \chi}. \tag{5.49}
\]
Combining \( \{5.48\} \) and \( \{5.49\} \) we obtain that
\[
\frac{a - \chi \tilde{u}_0}{b - \chi} \leq \tilde{u}(x, t) \leq \frac{a - \chi \overline{u}_0}{b - \chi}, \quad \forall \ (x, t) \in \mathbb{R}^{N+1}.
\]
This together with \( u_0 \leq \tilde{u}(x, t) \leq \overline{u}_0 \) implies that
\[
a - \chi \overline{u}_0 \leq (b - \chi)u_0 \quad \text{and} \quad (b - \chi)\overline{u}_0 \leq a - \chi \tilde{u}_0.
\]
These last inequalities are exactly the ones established in Lemma 5.2. Therefore, following the arguments of Theorem 1.8 we obtain that $u_0 = \pi_0 = \frac{a}{b}$. This complete the proof of the claim. It follows from above that $\tilde{u}(0,0) = \frac{a}{b}$. But by (5.43),

$$|\tilde{u}(0,0) - \frac{a}{b}| \geq \delta,$$

which is a contradiction. Thus

$$\lim_{t \to \infty} \sup_{|x| \leq ct} |u(x,t) - \frac{a}{b}| = 0$$

for all $0 \leq c < c^*$. This together with Lemma 5.5 (i) implies (1.20) with any $0 < c^*_{low} \leq c^*$.

(2) We prove (1.21). Let us denote by $U(x,t)$ the classical solution of the Initial Value Problem

$$\begin{cases}
L \bar{U} = \bar{F}^2(U) & x \in \mathbb{R}^N, \ t > 0 \\
U(x,0) = u_0(x) & x \in \mathbb{R}^N
\end{cases}$$

where $L \bar{U}$ is given by (5.15) and $\bar{F}^2(u) = u(a + d - (b - \chi)u)$ where $d \gg 1$ is chosen such that $\|u_0\|_\infty < \frac{a + d}{b - \chi}$. By the comparison principle for parabolic equations,

$$u(x,t) \leq \bar{U}(x,t) \leq \bar{U}(t, \|u_0\|_\infty)$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$.

It follows from (5.20) and the fact that $v \geq 0$ that

$$\lim_{R \to \infty} \sup_{t > R, |x| > R} (4\partial_u \bar{F}^2(0) - \chi^2 \|
abla v(x,t)\|^2) \geq \lim_{R \to \infty} \sup_{t > R, |x| > R} (4\partial_u \bar{F}^1(x,t,0) - \chi^2 \|
abla v(x,t)\|^2) > 0.$$

Since $\bar{F}^2$ is of KPP type with $\bar{F}^2(0) = \bar{F}^2(\frac{a + d}{b - \chi}) = 0$, by Theorem 1 in [6], there exist two compact sets $\bar{S} \subset \mathbb{R}^N$ with non-empty interiors such that

$$\begin{cases}
\text{for all compact set } K \subset \text{int}(\bar{S}), \lim_{t \to \infty} \sup_{x \in tK} |\bar{U}(x,t) - \frac{a + d}{b - \chi}| = 0, \\
\text{for all closed set } F \subset \mathbb{R}^N \setminus \bar{S}, \lim_{t \to \infty} \sup_{x \in tF} |\bar{U}(x,t)| = 0.
\end{cases}$$

Take $c^*_{up}$ to be the diameter of $\bar{S}$. For every $c > c^*_{up}$ we have that $F := \{ x : |x| \geq c \} \subset \mathbb{R}^N \setminus \bar{S}$ and closed. Hence

$$\lim_{t \to \infty} \sup_{|x| > ct} u(x,t) = 0$$

whenever $c > c^*_{up}$. This together with Lemma 5.5 implies (1.21). By (1.20) and (1.21), it is clear that $c^*_{up}(u_0) \geq c^*_{low}(u_0)$. 

References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, Vol. I. Abstract Linear Theory, Monographs in Mathematics, 89, Birkhuser Boston, Inc., Boston, MA, 1995.

[2] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663-1763.
[3] H. Berestycki, F. Hamel and G. Nadin, Asymptotic spreading in heterogeneous diffusive excitation media, *Journal of Functional Analysis*, **255** (2008), 2146-2189.

[4] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, I - Periodic framework, *J. Eur. Math. Soc.*, **7** (2005), 172-213.

[5] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, II - General domains, *J. Amer. Math. Soc.*, **23** (2010), no. 1, 1-34.

[6] H. Berestycki and G. Nadin, Asymptotic spreading for general heterogeneous Fisher-KPP type, preprint.

[7] J.I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis, *Advances in Mathematical Sciences and Applications*, **5** (1995), 659-680.

[8] J.I. Diaz, T. Nagai, J.-M. Rakotoson, Symmetrization Techniques on Unbounded Domains: Application to a Chemotaxis System on $\mathbb{R}^N$, *J. Differential Equations*, **145** (1998), 156-183.

[9] R. Fisher, The wave of advance of advantageous genes, *Ann. of Eugenics*, **7** (1937), 355-369.

[10] M. Freidlin, On wave front propagation in periodic media. In: *Stochastic analysis and applications, ed. M. Pinsky, Advances in probability and related topics*, 7:147-166, 1984.

[11] M. Freidlin and J. Gärtner, On the propagation of concentration waves in periodic and random media, *Soviet Math. Dokl.*, **20** (1979), 1282-1286.

[12] A. Friedman, Partial Differential Equation of Parabolic Type, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[13] K. Fujie, A. Ito, M. Winkler, and T. Yokota, Stabilization in a chemotaxis model for tumor invasion, *Discrete Contin. Dyn. Syst.*, **36** (2016), 151-169.

[14] E. Galakhov, O. Salieva and J. I. Tello, On a Parabolic-Elliptic system with Chemotaxis and logistic type growth, preprint.

[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag Berlin Heidelberg New York, 1981.

[16] D. Horstmann and M. Winkler, Boundedness vs. blow up in a chemotaxis system, *J. Differential Equations*, **215** (2005), 52-107.

[17] Kyungkeun Kanga, Angela Steven Blowup and global solutions in a chemotaxis-growth system, *Nonlinear Analysis*, **135** (2016), 57-72.

[18] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399-415.

[19] E.F. Keller and L.A. Segel, A Model for chemotaxis, *J. Theoret. Biol.*, **30** (1971), 225-234.

[20] A. Kolmogorov, I. Petrowsky, and N.Piscunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem, *Bjul. Moskovskogo Gos. Univ.*, **1** (1937), 1-26.
[21] K. Kuto, K. Osaki, T. Sakurai, and T. Tsujikawa, Spatial pattern formation in a chemotaxis-diffusion-growth model, *Physica D*, **241** (2012), 1629-1639.

[22] J. Lankeit, Chemotaxis can prevent thresholds on population density, *Discr. Cont. Dyn. Syst. B*, **20** (2015), 1499-1527.

[23] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Differential Eq.*, **258** (2015), 1158-1191.

[24] O.A Ladyzenskaja, V.A Solonnikov, N.N Urral’ceva, Linear and Quasi-linear Equation of Parabolic Type, Amer. Math. Soc. Tranl., vol 23, Amer. Math. Soc., Providence, RI, 1968.

[25] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), no. 1, 1-40.

[26] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *Journal of Functional Analysis*, **259** (2010), 857-903.

[27] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Anal.*, **92** (2009), 232-262.

[28] T. Nagai, T. Senba and K, Yoshida, Application of the Trudinger-Moser Inequality to a Parabolic System of Chemotaxis, *Funkcialaj Ekvacioj*, **40** (1997), 411-433.

[29] J. Nolen, M. Rudd, and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dynamics of PDE*, **2** (2005), 1-24.

[30] J. Nolen and J. Xin, Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle, *Discrete and Continuous Dynamical Systems*, **13** (2005), 1217-1234.

[31] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, *Nonlinear Analysis*, **51** (2002), 119-144.

[32] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[33] W. Shen, Variational principle for spatial spreading speeds and generalized propagating speeds in time almost and space periodic KPP models, *Trans. Amer. Math. Soc.*, **362** (2010), 5125-5168.

[34] W. Shen, Existence of generalized traveling waves in time recurrent and space periodic monostable equations, *J. Appl. Anal. Comput.*, **1** (2011), 69-93.

[35] Y. Sugiyama, Global existence in sub-critical cases and finite time blow up in super critical cases to degenerate Keller-Segel systems, *Differential Integral Equations*, **19** (2006), no. 8, 841-876.

[36] Y. Sugiyama and H. Kunii, Global Existence and decay properties for a degenerate keller-Segel model with a power factor in drift term, *J. Differential Equations*, **227** (2006), 333-364.
[37] Y. Tao and M. Winkler, Persistence of mass in a chemotaxis system with logistic source, *J. Differential Eq.*, **259** (2015), 6142-6161.

[38] J. I. Tello and M. Winkler, A Chemotaxis System with Logistic Source, *Communications in Partial Differential Equations*, **32** (2007), 849-877.

[39] Yilong Wang, A quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type with logistic source, *Journal of Mathematical Analysis and Applications*, **441** (2016), 259-292.

[40] L. Wang, C. Mu, and P. Zheng, On a quasilinear parabolic-elliptic chemotaxis system with logistic source, *J. Differential Equations*, **256** (2014), 1847-1872.

[41] H. F. Weinberger, Long-time behavior of a class of biology models, *SIAM J. Math. Anal.*, **13** (1982), 353-396.

[42] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Biol.*, **45** (2002), 511-548.

[43] M. Winkler, Chemotaxis with logistic source: Very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.*, **348** (2008), 708-729.

[44] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Part. Differential Eq.*, **35** (2010), 1516-1537.

[45] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *Journal of Differential Equations*, **248** (2010), 2889-2905.

[46] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, *Journal of Mathematical Analysis and Applications*, **384** (2011), 261-272.

[47] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, **100** (2013), 748-767, arXiv:1112.4156v1.

[48] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Differential Eq.*, **257** (2014), 1056-1077.

[49] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? *J. Nonlinear Sci.*, **24** (2014), 809-855.

[50] T. Yokota and N. Yoshino, Existence of solutions to chemotaxis dynamics with logistic source, Discrete Contin. Dyn. Syst. 2015, Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl., 1125-1133.

[51] J. Zheng, Boundedness of solutions to a quasilinear parabolic-elliptic Keller-Segel system with logistic source, *J. Differential Equations*, **259** (2015), 120-140.

[52] P. Zheng, C. Mu, X. Hu, and Y. Tian, Boundedness of solutions in a chemotaxis system with nonlinear sensitivity and logistic source, *J. Math. Anal. Appl.*, **424** (2015), 509-522.

[53] A. Zlatoš, Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations, *J. Math. Pures Appl.* (9) **98** (2012), no. 1, 89-102.