Weak Greedy Algorithms and the Equivalence Between Semi-greedy and Almost Greedy Markushevich Bases

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Abstract
We introduce and study the notion of weak semi-greedy systems—which is inspired in the concepts of semi-greedy and branch semi-greedy systems and weak thresholding sets-, and prove that in infinite dimensional Banach spaces, the notions of semi-greedy, branch semi-greedy, weak semi-greedy, and almost greedy Markushevich bases are all equivalent. This completes and extends some results from (Berná in J Math Anal Appl 470:218–225, 2019; Dilworth et al. in Studia Math 159:67–101, 2003; J Funct Anal 263:3900–3921, 2012). We also exhibit an example of a semi-greedy system that is neither almost greedy nor a Markushevich basis, showing that the Markushevich condition cannot be dropped from the equivalence result. In some cases, we obtain improved upper bounds for the corresponding constants of the systems.

Keywords Bases · Markushevich · Semi-greedy · Almost greedy

Mathematics Subject Classification Primary 41A65 · Secondary 46B15 · 46B20

1 Introduction
Let $X$ be a Banach space over the real or complex field $\mathbb{K}$, with dual space $X'$. A sequence $(x_i)_i$ in $X$ is fundamental if $X = \{x_i : i \in \mathbb{N}\}$, and it is minimal or a minimal...
system if there is a sequence of biorthogonal functionals \((x'_i)_i \subseteq X'\) (i.e., \(x'_i(x_j) = \delta_{ij}\) for every \(i, j\)); a sequence \((x'_i)_i\) in \(X'\) is total if \(x'_i(x) = 0\) for every \(i \in \mathbb{N}\) implies that \(x = 0\). A fundamental minimal system \((x_i)_i \subseteq X\) whose sequence of biorthogonal functionals is total is a Markushevich basis for \(X\). From now on, unless otherwise stated \((x_i)_i \subseteq X\) denotes a fundamental minimal system for a Banach space \(X\) with (unique) biorthogonal functionals \((x'_i)_i \subseteq X'\), and all of our Banach spaces are infinite dimensional. Given \(x \in X\), \(\text{supp}(x)\) denotes the support of \(x \in X\), that is the set \(\{i \in \mathbb{N}: x'_i(x) \neq 0\}\). A decreasing ordering for \(x\) is an injective function \(\varrho_x : \mathbb{N} \to \mathbb{N}\) such that \(\text{supp}(x) \subseteq \varrho_x(\mathbb{N})\), and for all \(i \leq j\)

\[|x'_{\varrho_x(i)}(x)| \geq |x'_{\varrho_x(j)}(x)|.\]

The set of all decreasing orderings for a fixed \(x \in X\) will be denoted by \(\text{D}(x)\). The greedy ordering for \(x\) is the decreasing ordering \(x\) with the property that if \(i < j\) and \(|x'_{\varrho_x(i)}(x)| = |x'_{\varrho_x(j)}(x)|\), then \(\varrho_x(i) < \varrho_x(j)\) - and which is bijective if \(x\) has finite support.

Note that if \((x_i)_i \subseteq X\) is a fundamental minimal system and \((x'_i)_i\) is a bounded sequence, then \((x_i)_i\) is bounded below (i.e., there is \(r > 0\) such that \(\|x_i\| \geq r\) for each \(i \in \mathbb{N}\)) and \((x'_i)_i\) is \(w^*-\)null, so the greedy ordering is well defined.

The thresholding greedy algorithm (TGA) for a fundamental minimal system \((x_i)_i \subseteq X\) with bounded coordinates, gives approximations to each \(x \in X\) in terms of the greedy ordering. For \(m \in \mathbb{N}\), the \(m\)-term greedy approximation to \(x\) is defined as follows:

\[G_m(x) := \sum_{i=1}^{m} x'_{\rho(x,i)}(x)\rho(x,i),\]

where \(\rho : X \times \mathbb{N} \to \mathbb{N}\) is the unique mapping such that for each \(x \in X\), the function \(\rho(x, \cdot)\) is the greedy ordering for \(x\). In this paper, \(\rho\) will always denote this function. Using the convention that the sum over the empty set is zero, \(G_0(x) = 0\). As usual, given a finite subset \(A\) of \(\mathbb{N}\), \(P_A\) denotes the projection with indices in \(A\), that is \(P_A(x) = \sum_{i \in A} x'_i(x) x_i\), and \(|A|\) denotes the cardinal of \(A\).

The TGA was introduced by Temlyakov [18], in the context of the trigonometric system, and extended by Konyagin and Temlyakov to general Banach spaces in [16], where the authors defined the concept of greedy Schauder bases (in the context of Schauder bases, “TGA” refers to any algorithm that gives approximations induced by a decreasing ordering; the greedy ordering is chosen for convenience to study general minimal systems; see [17]).

**Definition 1.1** A Schauder basis \((x_i)_i \subseteq X\) is greedy if there is \(M > 0\) such that for every \(x \in X\) and each \(m \in \mathbb{N}\),

\[\|x - G_m(x)\| \leq M\sigma_m(x),\]
where $\sigma_m(x)$ is the \textit{best m-term approximation error} given by

$$\sigma_m(x) = \inf \{ \|x - y\| : |\text{supp } y| \leq m \}.$$ 

Notice that, by a density argument, $(x_i)_i \subseteq X$ is a greedy basis if and only if there is $M > 0$ such that for every $x \in X$ and each $m \in \mathbb{N}$,

$$\|x - \sum_{i=1}^{m} x_{\rho_x(i)}(x)x_{\rho_x(i)}\| \leq M\sigma_m(x),$$

for some (or for all) $\rho_x \in D(x)$, which is the original definition in [16].

In [16], the authors also introduce the concept of \textit{quasi-greedy} Schauder bases, developed independently for fundamental biorthogonal systems and quasi-Banach spaces (though with somewhat different terminology) by Wojtaszczyk [20]. This concept was studied in several papers (see for example [1, 4, 10–12, 17]). We follow the definition from [17].

\textbf{Definition 1.2} A fundamental minimal system $(x_i)_i \subseteq X$ with bounded coordinates is \textit{quasi-greedy} if there is $M > 0$ such that for all $x$ and for each $m$,

$$\|G_m(x)\| \leq M\|x\|. \quad (1)$$

It is clear from the definition that a fundamental minimal system is quasi-greedy if and only if there is a constant $M > 0$ such that for all $x$ and each $m$,

$$\|x - G_m(x)\| \leq M||x||. \quad (2)$$

In the literature, the “quasi-greedy constant” of the system has been defined as the minimum $M$ for which (1) holds (see, e.g., [10]), the minimum $M$ for which (2) holds (see, e.g., [5]), or the minimum $M$ for which both hold (see, e.g., [1, 11, 17]). The differences in notation in the literature have been discussed in [3], where the minimum $M$ for which (2) holds is called the suppression quasi-greedy constant of the basis. We will refer to them as \textit{first quasi-greedy constant (1)} and the \textit{second quasi-greedy constant (2)} of the system, respectively.

A notion between being greedy and quasi-greedy is that of being almost greedy. This concept was introduced by Dilworth, Kalton, Kutzarova and Temlyakov for Schauder bases [11], and also studied in the context of Markushevich bases in several papers (see, among others, [3, 5, 9, 12]).

\textbf{Definition 1.3} A Markushevich basis $(x_i)_i \subseteq X$ with bounded coordinates is \textit{almost greedy} if there is a constant $M > 0$ such that for each $x \in X$ and $m \in \mathbb{N}_0$,

$$\|x - G_m(x)\| \leq M\tilde{\sigma}_m(x), \quad (3)$$

where

$$\tilde{\sigma}_m(x) = \inf \{ \|x - P_A(x)\| : |A| \leq m \}.$$
The minimum $M$ for which the above inequality holds is called the \textit{almost greedy constant} of the basis.

Another natural weakening of the greedy notion is the concept of semi-greedy Schauder bases that was introduced in [10]. This concept was later extended to Markushevich bases (see e.g., [5, 13]) and can be considered for fundamental minimal systems in general. To do so, we extend the definition of the $m$-term approximation error to such systems, as follows:

\[ \sigma_m(x) = \inf\{\|x - y\| : |\text{supp}(y)| \leq m \text{ and } y = P_{\text{supp}(y)}(y)\}. \]

\textbf{Definition 1.4} A fundamental minimal system \((x_i)_{i \in \mathbb{N}} \subseteq X\) with bounded coordinates is \textit{semi-greedy} if there exists \(M > 0\) such that for every \(x \in X\) and every \(m \in \mathbb{N}\), there is \(z \in [x_i : i \in \mathcal{G}S_m(x)]\) such that

\[ \|x - z\| \leq M \sigma_m(x), \]

where \(\mathcal{G}S_m(x) := \{\rho(x, i) : 1 \leq i \leq m\}\) is the \textit{greedy set} of \(x\) of cardinality \(m\). Under these conditions, \(z\) is called an \(m\)-\textit{term Chebyshev approximant} to \(x\), and the minimum \(M\) for which the inequality holds is called the \textit{semi-greedy constant} of the system.

In semi-greedy systems, the \textit{Chebyshev Greedy Algorithm} (CGA) is used instead of the TGA. The CGA gives generally better approximations than the TGA, since the approximations to \(x\) are not limited to the projections.

Weaker versions of the TGA and the CGA have been studied in several papers (see e.g. [7–9, 12–14, 17, 19]). These algorithms give approximations in terms of \textit{weak thresholding sets}, which are defined (according to [17]) as follows:

\textbf{Definition 1.5} Let \((x_i)_{i \in \mathbb{N}} \subseteq X\) be a fundamental minimal system with bounded coordinates and let \(0 < \tau \leq 1\). Given \(x \in X\) and \(m \in \mathbb{N}\), a set \(\mathcal{W}^\tau(x, m)\) of cardinality \(m\) is called an \(m\)-\textit{weak thresholding set} for \(x\) with \textit{weakness parameter} \(\tau\) if

\[ |x'_i(x)| \geq \tau |x'_j(x)|, \]

for all \(i \in \mathcal{W}^\tau(x, m)\) and all \(j \in \mathbb{N}\setminus\mathcal{W}^\tau(x, m)\). In this paper, \(\mathcal{W}^\tau(x, m)\) always denotes one of these sets. Note that as \((x'_i)_{i}\) is bounded, the greedy ordering assures that at least, for each \(\tau, m\) and \(x\), there is one of these sets.

Weak thresholding greedy algorithms give approximations to \(x \in X\) by projections \(P_{\mathcal{W}^\tau(x, m)}\), whereas weak Chebyshev greedy algorithms give instead the best approximation in terms of the vectors in \([x_i : i \in \mathcal{W}^\tau(x, m)]\). In this paper, we focus on the following two properties:

\textbf{Definition 1.6} A fundamental minimal system \((x_i)_{i \in \mathbb{N}} \subseteq X\) with bounded coordinates is \textit{weak almost greedy} with \textit{weakness parameter} \(0 < \tau \leq 1\) (WAG(\(\tau\))) and \textit{constant} \(M\) if for every \(x \in X\) and every \(m \in \mathbb{N}\), there is a weak thresholding set \(\mathcal{W}^\tau(x, m)\) such that

\[ \|x - P_{\mathcal{W}^\tau(x, m)}(x)\| \leq M \tilde{\sigma}_m(x). \]
Definition 1.7 A fundamental minimal system \((x_i)_i \subseteq X\) with bounded coordinates is **weak semi-greedy** with weakness parameter \(0 < \tau \leq 1\) and constant \(M\) if, for every \(x \in X\) and every \(m \in \mathbb{N}\), there is a weak thresholding set \(W^\tau(x, m)\) and \(z \in \{x_i : i \in W^\tau(x, m)\}\) such that
\[
\|x - z\| \leq M\sigma_m(x).
\]
In that case, \(z\) is called an \(m\)-term Chebyshev \(\tau\)-greedy approximant for \(x\).

Note that the WAG(\(\tau\)) concept is an extension of the almost greedy concept (Definition 1.3) that corresponds to \(\tau = 1\). Indeed, the greedy set \(G_m(x)\) is a weak thresholding set with parameter 1, and \(G_m(x) = P_{G_m(x)}\), so any almost greedy system is WAG(1). Reciprocally, if \((x_i)_i\) is WAG(1), given \(x \in X, m \in \mathbb{N}\), a set \(W^1(x, m)\) and \(\epsilon > 0\), one can choose \(y \in X\) with the property that \(|x'_i(y)| \neq |x'_j(y)|\) for all \(i \neq j\) so that \(\|x - y\| \leq \epsilon\) and \(W^1(x, m) = G_m(y)\) is the only \(m\)-weak thresholding set for \(y\). It easily follows from this that (4) holds for every \(x, m\), and every set \(W^1(x, m)\), so in particular (3) holds. Similarly, the notion of WSG(\(\tau\)) systems is an extension of that of semi-greedy systems, which also corresponds to the case \(\tau = 1\).

Before we continue, a word about Definitions 1.6 and 1.7 is in order. We require that there is at least one set \(W^\tau(x, m)\) for which the relevant approximation holds. An alternative choice would have been to stipulate that (4) and (5) hold for all sets of the form \(W^\tau(x, m)\). Our choice is motivated by the following considerations:

First, we follow the line of the definitions given in [12] for branch quasi-greedy, branch semi-greedy and branch almost greedy systems. This allows us to prove that branch semi-greedy Markushevich bases in infinite dimensional Banach spaces are almost greedy, which answers a question raised in [12], see Corollary 5.7.

Second, our proofs only require the existence of one “good” weak thresholding set \(\mathcal{W}^\tau(x, m)\) for each \(\tau, x, m\).

It is worth pointing out that a combination of Proposición 2.3 and [17, Theorem 2.2] shows that for WAG(\(\tau\)) systems (all of which are Markushevich bases), it is equivalent to consider one or any \(m\)-weak thresholding set. The same holds for WSG(\(\tau\)) Markushevich bases as a consequence of Corollary 4.3 and [12, Theorem 7.1]. However, we do not know if the Markushevich condition can be dropped for semi-greedy systems.

We are now in a position to describe the goal of this paper, which is twofold. First, we study the relations between almost greedy and semi-greedy systems, and their relation with approximations involving weak thresholding sets. Second, we focus our attention on some aspects that are significant on finite-dimensional spaces, in which the relations of their respective constants are of relevance. To be more precise we describe the structure of the paper below.

In Sect. 2 we study weak almost greedy systems, proving that this apparently weaker condition is, in fact, equivalent to that of being almost greedy (Proposition 2.3). In Sect. 3, we introduce (Definition 3.1) and study a separation property which turns out to hold for every Markushevich basis. This property allows us to replace arguments involving the Schauder basis constant when working with Markushevich bases. Section 4 is devoted to the study of weak semi-greedy systems for which we obtain our main result, Theorem 4.2. In particular, we prove that for Markushevich bases, the con-
cepts of semi-greedy, almost greedy, WSG(τ) and WAG(τ) systems are all equivalent, extending results from [5, 10]. From this, we settle the equivalence between semi-greedy and almost greedy Markushevich bases (Corollary 4.3). To show the extent of this equivalence, we exhibit two examples. The first one, Example 4.4, shows that it is possible to construct almost greedy Markushevich bases in ℓ_1 with almost greedy constants tending to infinity while the semi-greedy constants are uniformly bounded. With Example 4.5, we show that the condition of being a Markushevich basis cannot be weakened to be a fundamental minimal system in Theorem 4.2.

Finally, in Sect. 5, we turn our attention to finite-dimensional spaces from a quantitative point of view. In this context, we extend [12, Theorem 7.7], and answer a question posed by the authors about branch semi-greedy systems (pp. 3915). Additionally, in the infinite dimensional case, we address a question from [12] (pp. 3903) and show that branch semi-greedy Markushevich bases are semi-greedy.

2 Weak Almost Greedy Systems

In this section, we prove that the notions of WAG(τ) and almost greedy systems are equivalent. Note that one of the implications is immediate by definition. Indeed, any weak thresholding set \( W^1(x, m) \) is also a weak thresholding set \( W^\tau(x, m) \) for all \( 0 < \tau \leq 1 \) and almost greedy systems are WAG(1) systems, so they are WAG(τ) for all τ. Moreover, for almost greedy systems, it was proven in [17, Theorem 2.2] that (4) holds for every set \( W^\tau(x, m) \), with \( M \) only depending on \( \tau \) and the almost greedy constant of the basis (while [17, Theorem 2.2] is stated for Schauder bases, the proof does not use the Schauder property).

To prove that every WAG(τ) system is almost greedy, we will use the known equivalence between almost greediness and quasi-greediness plus democracy or superdemocracy—two concepts we define next. We also define the concept of hyper-democracy, a natural extension of both democracy notions that has its roots in [10, 11, 20].

**Definition 2.1** A sequence \((x_i)_{i \in \mathbb{N}} \subseteq X\) is superdemocratic if there is \( K > 0 \) such that

\[
\| \sum_{i \in A} a_i x_i \| \leq K \| \sum_{j \in B} b_j x_j \|,
\]

for any pair of finite sets \( A, B \subseteq \mathbb{N} \) with \(|A| \leq |B|\), and any scalars \((a_i)_{i \in A}, (b_j)_{j \in B}\) such that \(|a_i| = |b_j|\) for every \( i \in A, j \in B \). The minimum \( K \) for which the above inequality holds is called the superdemocracy constant of \((x_i)_{i \in \mathbb{N}}\). When (6) holds with \( a_i = b_j = 1 \) for all \( i, j \), the sequence is democratic, and the corresponding constant is the democracy constant of \((x_i)_{i \in \mathbb{N}}\), whereas if (6) holds with \(|a_i| \leq |b_j|\) for all \( i, j \), we say that the sequence is hyperdemocratic, and that the corresponding minimum constant is the hyperdemocracy constant of \((x_i)_{i \in \mathbb{N}}\).

From [11, Theorem 3.3] and its proof, we extract the following result, which characterizes almost greedy Markushevich bases, valid for real and complex Banach spaces.

**Theorem 2.2** Let \((x_i)_{i \in \mathbb{N}} \subseteq X\) be a Markushevich basis.
(a) If \((x_i)_i\) is almost greedy with constant \(K_a\), it is quasi-greedy with second quasi-greedy constant \(K_{2q} \leq K_a\) and democratic with constant \(K_d \leq K_a\).

(b) If \((x_i)_i\) is quasi-greedy with first quasi-greedy constant \(K_{1q}\) and democratic with constant \(K_d\), then it is almost greedy with constant \(K_a \leq 32K_d(1 + K_{1q})^4\).

Almost greedy Markushevich bases can also be characterized as quasi-greedy and superdemocratic (see, e.g., [5, 7, 10], which considers complex spaces and improves the order of the bound for \(K_a\) in (b) if democracy is replaced with superdemocracy). Moreover, almost greedy Markushevich bases are equivalent to quasi-greedy and hyperdemocratic bases. In fact, that every quasi-greedy hyperdemocratic basis is almost greedy follows at once from Theorem 2.2(b), whereas a proof that an almost greedy system is hyperdemocratic can be obtained combining Theorem 2.2(a), [20, Proposition 2] and [11, Lemma 2.2], with minor modifications for complex scalars. This implication is also established in Proposition 2.3 below, taking \(\tau = 1\).

Also, note that in Theorem 2.2, the hypothesis that the minimal system \((x_i)_i\) is a Markushevich basis is not necessary, since an almost greedy system is clearly quasi-greedy, and a quasi-greedy system is a Markushevich basis. We give a simple proof of this fact that follows from [20, Theorem 1] (see also the proof of [4, Corollary 3.5]). If \((x_i)_i\) is quasi-greedy and \(x'_i(x) = 0\) for all \(i \in \mathbb{N}\), then \(\mathcal{G}_m(y) = \mathcal{G}_m(y - x)\) for every \(y \in X\) and every \(m \in \mathbb{N}\). Thus,

\[
\|x\| \leq \|y - x\| + \|y - \mathcal{G}_m(y)\| + \|\mathcal{G}_m(y - x)\| \leq \|y - \mathcal{G}_m(y)\| + (1 + M)\|y - x\|.
\]

Since the system is fundamental, for any \(\varepsilon > 0\) there is \(m \in \mathbb{N}\) and \(y \in \{x_i : 1 \leq i \leq m\}\) such that \(\|x - y\| \leq \varepsilon\). Given that \(y = \mathcal{G}_m(y)\), it follows that \(\|x\| \leq (1 + M)\varepsilon\). As \(\varepsilon\) is arbitrary, this entails that \(x = 0\).

Now we prove that every \(\text{WAG}(\tau)\) system is almost greedy. The proof is based on that of [12, Proposition 4.4], adapted for our purposes.

**Proposition 2.3** Let \(0 < \tau \leq 1\), and let \((x_i)_i \subseteq X\) be a \(\text{WAG}(\tau)\) system with constant \(M\). Then, \((x_i)_i\) is a quasi-greedy Markushevich basis with quasi-greedy constant \(K_{1q} \leq (1 + M)(1 + M^2\tau^{-4})\), and is hyperdemocratic with constant \(K_{hd} \leq M^2\tau^{-2}\). Hence, \((x_i)_i\) is almost greedy.

**Proof** To prove the hyperdemocracy condition, fix nonempty finite sets \(A, B \subseteq \mathbb{N}\) with \(|A| \leq |B|\), and \((a_i)_{i \in A}, (b_j)_{j \in B}\) such that \(|a_i| \leq |b_j|\) for every \(i, j\), and choose a set \(C \subseteq \mathbb{N}\) so that \(|C| = |B|\) and \(C \cap (A \cup B) = \emptyset\). Assume without loss of generality that \(a := \max_{i \in A} |a_i| > 0\). For every \(0 < \varepsilon < 1\) we have

\[
(1 - \varepsilon)a\tau \sum_{k \in C} x_k \leq \sum_{k \in C} (1 - \varepsilon)a\tau x_k + \sum_{j \in B} b_j x_j - \sum_{j \in B} b_j x_j \leq M\tilde{\sigma}|C| \left( \sum_{k \in C} (1 - \varepsilon)a\tau x_k + \sum_{j \in B} b_j x_j \right) \leq M\sum_{j \in B} b_j x_j,
\]

(7)
the first inequality resulting from the fact that $B$ is the only $|C|$-weak thresholding set for $\sum_{k \in C} (1 - \epsilon) a \tau x_k + \sum_{j \in B} b_j x_j$ with weakness parameter $\tau$. Similarly, $C$ is the only $|C|$-weak thresholding set with weakness parameter $\tau$ for $\sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k + \sum_{i \in A} a_i x_i$, thus

$$\| \sum_{i \in A} a_i x_i \| = \| \sum_{i \in A} a_i x_i + \sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k + \sum_{i \in A} a_i x_i \|$$

$$\leq M \tilde{\sigma}_{|C|} (\sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k + \sum_{i \in A} a_i x_i)$$

$$\leq (1 + \epsilon) a \tau^{-1} M \| \sum_{k \in C} x_k \|. \quad (8)$$

By letting $\epsilon \to 0$, it follows from (7) and (8) that

$$\| \sum_{i \in A} a_i x_i \| \leq M^2 \tau^{-2} \| \sum_{j \in B} b_j x_j \|.$$ 

Therefore, $(x_i)$ is hyperdemocratic with $K_{hd} \leq M^2 \tau^{-2}$.

To prove that $(x_i)_{i}$ is quasi-greedy, fix $x \in X$ and $m \in \mathbb{N}$. If $G_m(x) = 0$, there is nothing to prove. Else, let

$$n : = \max \{1 \leq i \leq m : x'_{\rho(x,i)}(x) \neq 0\}.$$ 

Given that $x'_i(x) \neq 0$ for all $i \in G\mathcal{S}_n(x)$ and $(x'_i)_{i}$ is $w^*$-null, there is $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$ and each $i \in G\mathcal{S}_n(x)$,

$$|x'_j(x)| < \tau |x'_i(x)|.$$ 

Thus, if $j \geq j_0$, every weak thresholding set $\mathcal{W}^T(x, j)$ contains $G\mathcal{S}_n(x)$. Hence, there is a minimum $m_1 \in \mathbb{N}$ for which there is a weak thresholding set $\mathcal{W}^T(x, m_1)$ containing $G\mathcal{S}_n(x)$ and such that (4) holds. If $G\mathcal{S}_n(x) = \mathcal{W}^T(x, m_1)$, then

$$\| G_m(x) \| = \| G_n(x) \| \leq \| x \| + \| x - G_n(x) \| = \| x \| + \| x - \sum_{i \in \mathcal{W}^T(x, m_1)} x'_i(x) x_i \|$$

$$\leq (1 + M) \| x \| \leq (1 + M) (1 + M^2 \tau^{-4}) \| x \|.$$ 

On the other hand, if $G\mathcal{S}_n(x) \subseteq \mathcal{W}^T(x, m_1)$, let $\mathcal{W}^T(x, m_1 - 1)$ be a weak thresholding set for which (4) holds. By the minimality of $m_1$ we get that $G\mathcal{S}_n(x) \not\subseteq \mathcal{W}^T(x, m_1 - 1)$, so for every $i \in \mathcal{W}^T(x, m_1 - 1)$ we have

$$|x'_i(x)| \geq \tau |x'_{\rho(x,n)}(x)|. \quad (9)$$
Thus, if there is $i_0 \in W^\tau (x, m_1 - 1) \setminus W^\tau (x, m_1)$, it follows from (9) and Definition 1.5 that for all $j \in W^\tau (x, m_1)$,

$$|x'_j(x)| \geq \tau |x'_i(x)| \geq \tau^2 |x'_{\rho(x,n)}(x)|.$$ 

On the other hand, if $W^\tau (x, m_1 - 1) \subseteq W^\tau (x, m_1)$, given that $\mathcal{G}S_n(x) \not\subseteq W^\tau (x, m_1 - 1)$ and $\mathcal{G}S_n(x) \subseteq W^\tau (x, m_1)$, it follows that there is $1 \leq i_1 \leq n$ such that

$$W^\tau (x, m_1) = W^\tau (x, m_1 - 1) \cup \{\rho(x, i_1)\},$$

which implies that (9) also holds for all $i \in W^\tau (x, m_1)$. Therefore, in any case, we have

$$|x'_j(x)| \geq \tau^2 |x'_{\rho(x,n)}(x)|,$$

for all $j \in W^\tau (x, m_1)$. In what follows, put $W = W^\tau (x, m_1)$. As for all $i \in N \setminus \mathcal{G}S_n(x)$,

$$|x'_{\rho(x,n)}(x)| \geq |x'_i(x)|,$$

we obtain

$$\max_{i \in W \setminus \mathcal{G}S_n(x)} |x'_i(x)| \leq \min_{j \in W} \tau^{-2} |x'_j(x)|.$$

Hence, using that $\mathcal{G}_m(x) = \mathcal{G}_n(x)$ and applying the hyperdemocracy condition, we get

$$\|\mathcal{G}_m(x)\| \leq \| \sum_{i \in W} x'_i(x)x_i \| + \| \sum_{i \in W} x'_i(x)x_i - \mathcal{G}_n(x) \|$$

$$= \| \sum_{i \in W} x'_i(x)x_i \| + \| \sum_{i \in W \setminus \mathcal{G}S_n(x)} x'_i(x)x_i \|$$

$$\leq \| \sum_{i \in W} x'_i(x)x_i \| + Khd \| \sum_{i \in W} \tau^{-2} x'_i(x)x_i \|$$

$$\leq (1 + M^2 \tau^{-4}) \| \sum_{i \in W} x'_i(x)x_i \|$$

$$\leq (1 + M^2 \tau^{-4})(\| x \| + \| x - \sum_{i \in W} x'_i(x)x_i \|)$$

$$\leq (1 + M^2 \tau^{-4})(\| x \| + M\sigma_{m_1}(x))$$

$$\leq (1 + M)(1 + M^2 \tau^{-4})\| x \|.$$
This proves that $(x_i)_i \subseteq X$ is quasi-greedy (with $K_{1q}$ as in the statement). Then $(x_i)_i$ is a Markushevich basis, and it is almost greedy by Theorem 2.2. \qed

\section{The Finite Dimensional Separation Property}

In this section, we introduce and study a separation property inspired by some of the proofs in [2]. We give upper bounds for a constant associated with this property. The constant plays a key role in some of our proofs involving Markushevich bases.

\begin{definition}
Let $(u_i)_i \subseteq X$ be a sequence. We say that $(u_i)_i$ has the finite dimensional separation property (FDSP) if there is a positive constant $M$ such that for every separable subspace $Z \subseteq X$ and every $\epsilon > 0$, there is a basic subsequence $(u_{i_k})_k$ with basis constant no greater than $M + \epsilon$ satisfying the following: For every finite dimensional subspace $F \subseteq Z$ there is $j_F \in \mathbb{N}$ such that

$$\|x\| \leq (M + \epsilon)\|x + z\|,$$

for all $x \in F$ and all $z \in [u_{i_k} : k > j_F]$. We call any such subsequence a finite dimensional separating sequence for $(Z, M, \epsilon)$ (and for $(u_i)_i$, leaving that implicit when clear), and we call the minimum $M$ for which this property holds the finite dimensional separation constant $M_{fs}$ of $(u_i)_i$.
\end{definition}

\begin{remark}
Note that in order to check whether a subsequence $(u_{i_k})_k$ of $(u_i)_i$ is finite dimensional separating for $(Z, M, \epsilon)$, it is enough to check that (10) holds for $x$ with $\|x\| = 1$ and $z \in [u_{i_k} : k > j_F]$.
\end{remark}

The following lemma gives a basic characterization of sequences with the FDSP. The proof is immediate and is left to the reader.

\begin{lemma}
Let $(u_i)_i \subseteq X$ be a sequence. For any sequence $(a_i)_i \subseteq K$ with $a_i \neq 0$ for all $i$, and any bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$, $(u_i)_i$ has the finite dimensional separation property with constant $M_{fs}$ if and only if, for any $l \in \mathbb{N}$, $(a_i u_{\pi(i)})_{i \geq l}$ does.
\end{lemma}

For our next result, we need the following technical lemmas; the second one is a variant of [2, Theorem 1.5.2].

\begin{lemma}[2, Lemma 1.5.1]
Let $S \subseteq X'$ be a subset such that $S$ is bounded below and $0 \in \overline{S}_{w^*}$. Then, for every $\epsilon > 0$ and every finite dimensional subspace $F \subseteq X'$, there is $x' \in S$ such that for all $y' \in F$ and $b \in K$,

$$\|y'\| \leq (1 + \epsilon)\|y' + bx'\|.$$

\end{lemma}

\begin{lemma}[2, Lemma 1.5.1]
Let $X$ be a Banach space and $(u'_i)_i \subseteq X'$ a sequence such that $(u'_i)_i$ is bounded below and $0 \in \overline{\{u'_{i_k}\} : i \in \mathbb{N}^*}$. Then, for any separable subspace $Z \subseteq X'$ and $\epsilon > 0$ there is a basic subsequence $(u'_{i_k})_k$ with basis constant no greater than $(1 + \epsilon)$

\begin{verbatim}
\end{verbatim}

\documentclass{article}
\usepackage{amsmath, amsthm, amssymb}
\usepackage{hyperref}
\newtheorem{definition}{Definition}
\newtheorem{lemma}{Lemma}
\newtheorem{remark}{Remark}

\begin{document}
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\begin{lemma}[2, Lemma 1.5.1]
Let $X$ be a Banach space and $(u'_i)_i \subseteq X'$ a sequence such that $(u'_i)_i$ is bounded below and $0 \in \overline{\{u'_{i_k}\} : i \in \mathbb{N}^*}$. Then, for any separable subspace $Z \subseteq X'$ and $\epsilon > 0$ there is a basic subsequence $(u'_{i_k})_k$ with basis constant no greater than $(1 + \epsilon)$

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satisfying the following: For any finite dimensional subspace $F \subset Z$ and every $\xi > 0$, there is $j_{F,\xi} \in \mathbb{N}$ such that for all $y' \in F$ and $v' \in [u'_{i_k}: k > j_{F,\xi}]$, 

$$\|y'\| \leq (1 + \xi)\|y' + v'\|.$$

In particular, $(u_i')_i$ has the finite dimensional separation property with constant 1.

**Proof** Choose a dense sequence $(v_i')_i$ in $Z$ and a sequence of positive scalars $(\epsilon_i)_i$ so that $\prod_{i=1}^{\infty} (1 + \epsilon_i) \leq (1 + \epsilon)$. Since $0 \in \{u_i'\}_{i \in \mathbb{N}}$ and $u_i' \neq 0$ for every $i \in \mathbb{N}$, it follows that $0 \in \{u_i'\}_{i \geq l}$ for every $l \in \mathbb{N}$. Thus, by Lemma 3.4, we can choose $i_2 > i_1 := 1$ so that for all $y' \in [v_{i_1}', u_{i_1}']$ and all $b \in \mathbb{K}$,

$$\|y'\| \leq (1 + \epsilon_1)\|y' + bu'_{i_2}\|.$$

By an inductive argument, we obtain a strictly increasing sequence of positive integers $(i_k)_{k \in \mathbb{N}}$ such that for all $y' \in [v_s', u_s']: 1 \leq s \leq i_k$, $b \in \mathbb{K}$ and $k \in \mathbb{N}$,

$$\|y'\| \leq (1 + \epsilon_k)\|y' + bu'_{i_{k+1}}\|.$$

Then, for any positive integers $j < l$, any $y' \in [v_s', u_s']: 1 \leq s \leq i_j$ and any scalars $(a_k)_{j < k \leq l}$,

$$\|y'\| \leq \prod_{k=j}^{l-1} (1 + \epsilon_k)\|y' + \sum_{k=j+1}^{l} a_k u'_{i_k}\| \leq \prod_{k=j}^{\infty} (1 + \epsilon_k)\|y' + \sum_{k=j+1}^{l} a_k u'_{i_k}\|.$$

In particular, $(u_{i_k})_k$ is basic with basis constant no greater than $\prod_{k=1}^{\infty} (1 + \epsilon_k) \leq 1 + \epsilon$, and the result holds for $F \subset [v_i': 1 \leq i \leq k]$ for some $k \in \mathbb{N}$. Now, standard density arguments allow us to obtain the result for any finite dimensional subspace of $Z = [v_i': i \in \mathbb{N}]$.

**Remark 3.6** Suppose that there is a sequence $(u_i)_i \subseteq X$ such that $(u_i)_i$ is bounded below and $0 \in \overline{\{u_i\}_{i \in \mathbb{N}}}$, Via the canonical injection $X \hookrightarrow X''$, Lemma 3.5 remains valid for any separable subspace $Z \subseteq X$ where the basic sequence $(u_{i_k})_k$ is a subsequence of $(u_i)_i$.

Next, we consider the case in which 0 may not be a weak or a weak star accumulation point of the sequence. We use $\hat{x}$ to denote $x \in X$ as an element of the bidual space $X''$, and we use $\hat{X}$ to denote $X$ as a subspace of $X''$. Also, for a bounded sequence $(u_i)_i \subseteq X$, we will consider $\beta((u_i)) := \overline{\{u_i\}_{i \in \mathbb{N}}} \setminus \hat{X}$.
Lemma 3.7 Let $X$ be a Banach space and $(u_i)_{i \in \mathbb{N}} \subseteq X$ a bounded sequence such that $[u_i]_{i \in \mathbb{N}}^\wedge$ is not weakly compact. Then, $(u_i)_{i}$ has the finite dimensional separation property with constant $M_{fs} \leq M$, where

$$M := \left(2 + \inf_{x'' \in \beta((u_i))} \frac{\|x''\|}{\text{dist}(x'', \hat{X})}\right)^2.$$

Proof Since $[u_i]_{i \in \mathbb{N}}^\wedge$ is not weakly compact but $[\hat{u_i}]_{i \in \mathbb{N}}^{w*}$ is weak star compact, there is $x'' \in \beta((u_i)) = [\hat{u_i}]_{i \in \mathbb{N}}^{w*} \setminus \hat{X}$, so $M$ is well defined. Given $\epsilon > 0$ and $Z \subset X$ a separable subspace, choose $0 < \xi < 1$ and $x''_0 \in [\hat{u_i}]_{i \in \mathbb{N}}^{w*} \setminus \hat{X}$ so that

$$M + \xi + \xi^2 + 2\xi(M + \xi) \leq \frac{M + \epsilon}{1 + \xi}, \tag{11}$$

and

$$(2 + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})})^2 \leq M + \xi. \tag{12}$$

Let $Z_1 := \hat{Z} + [x''_0, \hat{u}_i : i \in \mathbb{N}]$, and consider in $Z_1$ the seminormalized sequence $(\hat{u}_i - x''_0)_i$. Then, there exists a basic subsequence $(\hat{u}_i - x''_0)_k$ with basic constant no greater that $(1 + \xi)$ satisfying the conclusions of Lemma 3.5. Since $x''_0 \notin \hat{X}$, there is a bounded linear functional $x'''_1$ on $\hat{X} \oplus [x''_0]$ such that for all $x \in X$ and all $b \in \mathbb{K}$,

$$x'''_1(\hat{x} + bx'') = b.$$  

Suppose that $\|\hat{x} + bx''\| = 1$ with $b \neq 0$. Then,

$$|x'''_1(\hat{x} + bx'')| = |b| = |b| \frac{\|b^{-1}\hat{x} + x''\|}{\|b^{-1}\hat{x} + x''\|} = \frac{1}{\|b^{-1}\hat{x} + x''\|} \leq \frac{1}{\text{dist}(x''_0, \hat{X})}. $$

It follows that

$$\|x'''_1\| \leq \frac{1}{\text{dist}(x''_0, \hat{X})}. \tag{13}$$

By the Hahn–Banach Theorem, there is a norm-preserving extension of $x'''_1$ to $X''$, which we also call $x'''_1$. Let $F_1 := [x''_1]$. By the choice of $(\hat{u}_i - x''_0)_k$, there exists $j_{F_1, \xi} \in \mathbb{N}$ such that for all $z'' \in [\hat{u}_i - x''_0 : k > j_{F_1, \xi}]$ we have

$$\|z''\| \leq (1 + \xi)\|x''_0 + z''\|. \tag{14}$$

In particular, given that $x''_0 \neq 0$ this implies that $x''_0 \notin [\hat{u}_i - x''_0 : k > j_{F_1, \xi}]$. Thus, there is a bounded linear functional $x'''_2$ on $[\hat{u}_i - x''_0 : k > j_{F_1, \xi}] \oplus [x''_0]$ defined, for

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all $z'' \in [\hat{u}_{ik} - x'''_0 : k > j_{F_1, \xi}]$ and all $b \in \mathbb{K}$, by

$$x'''_2 (z'' + bx''_0) = b.$$  

As before, for any $z'' \in [\hat{u}_{ik} - x'''_0 : k > j_{F_1, \xi}]$ and $b \neq 0$ such that $\|z'' + bx''_0\| = 1$, by (14), we have

$$|x'''_2 (z'' + bx''_0)| = |b| = \frac{1}{\|b^{-1}z'' + x''_0\|} \leq \frac{1 + \xi}{\|x''_0\|}.$$  

Thus,

$$\|x'''_2\| \leq \frac{1 + \xi}{\|x''_0\|}. \quad (15)$$

Again, by the Hahn–Banach Theorem, we may consider $x'''_2$ defined on $X''$. Now define, for $x'' \in X''$, the following bounded linear operators:

$$T(x'') := x'' + (x'''_2 - x'''_1)(x''_0);$$
$$L(x'') := x'' - (x'''_2 - x'''_1)(x''_0).$$

By (13) and (15) we get

$$\|T(x'')\| \leq \|x''\| + \|x'''_2\| \|x''_0\| \|x'''_1\| \|x''_0\| \leq (2 + \xi + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})}) \|x''\|.$$  

Hence, $\|T\| \leq 2 + \xi + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})}$ and the same bound holds for $L$.

From this, (11) and (12) we get

$$\|T\| \|L\| \leq (2 + \xi + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})})^2 = (2 + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})})^2 + \xi^2 + 2\xi (2 + \frac{\|x''_0\|}{\text{dist}(x''_0, \hat{X})}) \leq M + \xi + \xi^2 + 2\xi (M + \xi) \leq M + \epsilon,$$  

(16)

It is easy to check that $T$ and $L$ are inverses of each other. Then, since $T(\hat{u}_{ik} - x''_0) = \hat{u}_{ik}$ for all $k > j_{F_1, \xi}$ and $(\hat{u}_{ik} - x''_0)_k$ is a basic sequence with basis constant no greater than $(1 + \xi)$, it follows that $(u_{ik})_{k > j_{F_1, \xi}}$ is a basic sequence with basis constant

$$K_b((u_{ik})_{k > j_{F_1, \xi}}) \leq \|T\| \|L\|(1 + \xi) \leq M + \epsilon,$$

where the last inequality follows from (16). Now let $F \subset Z$ be a finite dimensional subspace, and let $j_F := \max \{j_{L(\hat{F}), \xi}, j_{F_1, \xi}\}$. For any $x \in F$ and scalars $(a_k)_{j_F < k \leq m}$,
by the choice of \((\hat{u}_{ik} - x_0'')\) we have

\[
\|x\| = \|\hat{x}\| \leq \|T\| \|L(\hat{x})\| \leq \|T\| (1 + \xi) \|L(\hat{x}) + \sum_{j_F < k \leq m} a_k (\hat{u}_{ik} - x_0'')\|
\]

\[
= \|T\| (1 + \xi) \|L(\hat{x}) + \sum_{j_F < k \leq m} a_k \hat{u}_{ik}\| \leq \|T\| \|L\| (1 + \xi) \|\hat{x} + \sum_{j_F < k \leq m} a_k \hat{u}_{ik}\|
\]

\[
\leq (M + \epsilon) \|x + \sum_{j_F < k \leq m} a_k u_{ik}\|,
\]

where we apply again (16) to obtain the last inequality. By a density argument, it follows that \((u_{ik})_{k > j_{F_1}, \xi}\) has the desired properties. \(\square\)

**Remark 3.8** Note that the upper bound for \(M_{fs}\) given by Lemma 3.7 remains unchanged if one replaces \((u_i)\) with \((a_i u_i)\), for any seminormalized sequence \((a_i)\).

**Corollary 3.9** Let \((u_i)\) be a seminormalized sequence. The following are equivalent:

(i) \((u_i)\) has a basic subsequence.

(ii) Either \(0 \in \overline{\{u_i\}}^w\), or \(\overline{\{u_i\}}^w\) is not weakly compact.

(iii) \((u_i)\) has the finite dimensional separation property.

**Proof** The equivalence (i) \(\iff\) (ii) was proven in [15] (see also [2, Theorem 1.5.6]).

By Remark 3.6 and Lemma 3.7 it follows that (ii) \(\implies\) (iii). Finally, (iii) \(\implies\) (i) is clear. \(\square\)

Next we study the finite dimensional separation property of Markushevich bases, and give upper bounds for its constant in this context. Recall that for \(0 < c \leq 1\), a subspace \(S \subset X'\) is said to be \(c\)-norming for \(X\) if

\[
c \|x\| \leq \sup_{x' \in S, \|x'\| = 1} |x'(x)|, \quad \forall x \in X.
\]

We will use the following result.

**Lemma 3.10** [2, Proposition 3.2.3] Let \((x_i)\) be a Schauder basis for \(X\) with basis constant \(K_b\). Then \(\overline{\{x'_i : i \in \mathbb{N}\}} \subset X'\) is \(K_b^{-1}\)-norming for \(X\).

Also recall that a sequence \((v_i)\) is a block basis of a Markushevich basis \((x_k)\) if there are sequences of positive integers \((n_i)\), \((m_i)\) with \(n_i \leq m_i < n_{i+1}\) for all \(i\) and scalars \((b_k)\) such that

\[
v_i = \sum_{k=n_i}^{m_i} b_k x_k,
\]

with at least one nonzero \(b_k\) for each \(i \in \mathbb{N}\). In particular, any subsequence of a Markushevich basis is a block basis of it.
Proposition 3.11 Let \((z_l)_l \subset X\) be a seminormalized block basis of a Markushevich basis \((y_k)_k\) for \(Y \subset X\) with biorthogonal functionals \((y'_k)_k\). The following hold:

(a) Either \(\overline{\{z_l\}_{l \in \mathbb{N}}}^{w}\) is not weakly compact, or \((z_l)_l\) is weakly null. Hence, \((z_l)_l\) has the finite dimensional separation property.

(b) If either \(0 \in \overline{\{z_l\}_{l \in \mathbb{N}}}^{w}\) or \(X\) is a dual space and \(0 \in \overline{\{z_l\}_{l \in \mathbb{N}}}^{w*}\), then \(M_{f_s} = 1\).

(c) If \(\overline{\{z_l\}_{l \in \mathbb{N}}}^{w}\) is not weakly compact, then

\[
M_{f_s} \leq \left(2 + \inf_{x'' \in \beta((z_l))} \left\{ \frac{\|x''\|}{\text{dist}(x'', X)} \right\} \right)^2.
\]

(d) If \(Y = X\) and \(\overline{\{y'_k\}_{k \in \mathbb{N}}}^{w}\) is c-norming, then \(M_{f_s} \leq c^{-1}\).

(e) If \(Y = X\) and \((y_k)_k\) is a Schauder basis for \(X\) with constant \(K_b\), then \(M_{f_s} \leq K_b\).

Proof To prove (a), suppose that \(\overline{\{z_l\}_{l \in \mathbb{N}}}^{w}\) is weakly compact. Then, since \(Y\) is weakly closed, given a subnet \((z_{l_k})\) there is a further subnet \((z_{l_{k_0}})\) and \(v_0 \in Y\) such that

\[
z_{l_{k_0}} \overset{w}{\rightarrow} v_0.
\]

Since \((z_l)_l\) is a block basis of \((y_k)_k\), it follows that \(y'_k(v_0) = 0\) for all \(k \in \mathbb{N}\), so \(v_0 = 0\). Thus, \((z_l)_l\) is weakly null. It follows by Corollary 3.9 that \((z_l)_l\) has the finite dimensional separation property.

Lemma 3.5 and Remark 3.6 imply (b), and (c) follows by Lemma 3.7.

To prove (d), note that by (a), \((z_l)_l\) has a basic subsequence \((z_{l_j})_j\). Let \(F \subset X\) be a finite dimensional subspace, and fix \(0 < \epsilon < 1\). Choose \(0 < \xi < 1\) so that

\[
0 < \frac{c^{-1}(1 - \xi)^{-1}}{1 - c^{-1}(1 - \xi)^{-1}\xi} \leq c^{-1} + \epsilon.
\]

Take \((u_j)_{1 \leq j \leq m_1}\) unit vectors in \(F\) that form a \(\xi\)-net of the unit sphere of \(F\).

As \(\overline{\{y'_k\}_{k \in \mathbb{N}}}^{w}\) is c-norming so is \(\overline{\{y'_k\}_{k \in \mathbb{N}}}^{w}\). Hence, there is \(m_2 \in \mathbb{N}\) and unit vectors \((u'_j)_{1 \leq j \leq m_1} \subset \overline{\{y'_k\}_{k \in \mathbb{N}}}^{w}\) such that \(|u'_j(u_j)| \geq c(1 - \xi)\) for all \(1 \leq j \leq m_1\).

Now fix \(x \in F\) with \(\|x\| = 1\) and \(v \in \overline{\{z_{l_j}\}_{l > m_2}}^{w}\), and choose \(1 \leq j \leq m_1\) so that \(\|x - u_j\| \leq \xi\). Note that \(v \in \overline{\{y_k\}_{k > m_2}}^{w}\), so \(y'_k(v) = 0\) for all \(1 \leq k \leq m_2\). Hence,

\[
1 \leq c^{-1}(1 - \xi)^{-1}|u'_j(u_j)| = c^{-1}(1 - \xi)^{-1}|u'_j(u_j + v)| \leq c^{-1}(1 - \xi)^{-1}\|u_j + v\|
\]

\[
\leq c^{-1}(1 - \xi)^{-1}\|x + v\| + c^{-1}(1 - \xi)^{-1}\|u_j - x\|
\]

\[
\leq c^{-1}(1 - \xi)^{-1}\|x + v\| + c^{-1}(1 - \xi)^{-1}\xi.
\]

Thus,

\[
\|x\| = 1 \leq \frac{c^{-1}(1 - \xi)^{-1}}{1 - c^{-1}(1 - \xi)^{-1}\xi}\|x + v\| \leq (c^{-1} + \epsilon)\|x + v\|.
\]
Finally, (e) follows by (d) and Lemma 3.10.

\[\square\]

**Remark 3.12** Note that if \((v_i)_i \subset X\) is any block basis of a Markushevich basis, we can normalize it and apply Proposition 3.11 and then Lemma 3.3, so it has the FDSP.

## 4 Weak Semi-greedy Systems

In this section, we prove our main results for weak semi-greedy minimal systems.

It was proven in [10, Theorem 3.2] that every almost greedy Schauder basis is semi-greedy with constant only depending on its democracy and quasi-greedy constants (and thus, by Theorem 2.2, only on its almost greedy constant), with a proof valid also for Markushevich bases (see also [5, Theorem 1.10] and [13, Corollary 4.2]). Moreover, it is known that if \((x_i)_i\) is an almost greedy Markushevich basis, then for each \(0 < \tau \leq 1\) there is a constant \(M\) depending only on the first quasi-greedy and the democracy constants of the basis and \(\tau\) such that the conditions of Definition 1.7 hold for all \(x \in X\), \(m \in \mathbb{N}\), and every weak thresholding set \(W^\tau(x, m)\). This fact was stated in [12, Theorem 7.1] for finite dimensional Banach spaces. A detailed proof in the infinite dimensional case can be found in [8, Theorem 1.2]. On the other hand, results in the opposite direction are not yet complete. In [10, Theorem 3.6], it is proved that every semi-greedy Schauder basis for a Banach space with finite cotype is almost greedy. In [5, Theorem 1.10], the cotype condition is removed and it is proved that every semi-greedy Schauder basis is almost greedy with quasi-greedy and superdemocracy constants depending only on the basis constant and the semi-greedy constant, leaving the question of whether the implication from semi-greedy to almost greedy holds for general Markushevich bases ([5, Question 1]). Recently, Berná extended [5, Theorem 1.10] to a certain class of Markushevich bases (known as \(\rho\)-admissible) [6, Theorem 5.3]. To our knowledge the general case remained open until now. In this section, we complete the proof of the implication from semi-greedy to almost greedy Markushevich bases, and extend the result to WSG(\(\tau\)) Markushevich bases. We also study the (weak) semi-greedy property for general minimal systems, without the Markushevich hypothesis. We begin with an auxiliary lemma.

**Lemma 4.1** Let \(0 < \tau \leq 1\), and let \((x_i)_i \subset X\) be a WSG(\(\tau\)) system with constant \(K\). Then both \((x_i)_i\) and \((x'_i)_i\) are seminormalized.

Moreover, \(\sup_i \|x_i\| \leq 2K \tau^{-1} \inf_j (1 + \|x'_j\|\|x_j\|)\|x_j\|\).

**Proof** Since \((x'_j)_j\) is bounded, \((x_i)_i\) is bounded below. Given \(i \neq j\), it follows from Definitions 1.5 and 1.7 that the only 1-weak thresholding set with weakness parameter \(\tau\) for \(x_i + 2\tau^{-1}x_j\) is

\[W^\tau(x_i + 2\tau^{-1}x_j, 1) = \{j\}\.]
Let $a x_j$ be a Chebyshev $\tau$-greedy approximant for $x_i + 2\tau^{-1}x_j$. We have

$$
\|x_i\| \leq \|x_i + 2\tau^{-1}x_j - ax_j\| + \|2\tau^{-1}x_j - ax_j\|
= \|x_i + 2\tau^{-1}x_j - ax_j\| + \|x'_j(x_i + 2\tau^{-1}x_j - ax_j)x_j\|
\leq (1 + \|x'_j\|\|x_j\|)\|x_i + 2\tau^{-1}x_j - ax_j\| \leq (1 + \|x'_j\|\|x_j\|)K\sigma_1(x_i + 2\tau^{-1}x_j)
\leq 2(1 + \|x'_j\|\|x_j\|)\tau^{-1}K\|x_j\|.
$$

Thus, $(x_i)_i$ is bounded, which implies that $(x'_i)_i$ is bounded below. Since the above inequality holds also for $i = j$, the bound in the statement follows by taking infimum over $j$ and supremum over $i$. \(\Box\)

Now we prove that WSG(\(\tau\)) Markushevich bases are almost greedy. The proof combines arguments from the proofs of [10, Proposition 3.3] and [5, Theorem 1.10 b] - which we adapt to weak thresholding and weak Chebyshev greedy algorithms - with arguments based on the finite dimensional separation property - which allows us to work in the context of general Markushevich bases.

**Theorem 4.2** Let $0 < \tau \leq 1$, and let $(x_i)_i \subseteq X$ be a WSG($\tau$) system with constant $K$. The following are equivalent:

(i) $(x_i)_i$ is almost greedy.

(ii) $(x_i)_i$ is quasi-greedy.

(iii) $(x_i)_i$ is a Markushevich basis.

(iv) $(x_i)_i$ has the finite dimensional separation property.

If any (and thus all) of these conditions holds, $(x_i)_i$ has second quasi-greedy constant $K_{2q} \leq M_{f_s}K + M_{f_s}(M_{f_s} + 1)K^2\tau^{-2}$ and hyperdemocracy constant $K_{hd} \leq M_{f_s}(M_{f_s} + 1)K^2\tau^{-2}$, where $M_{f_s}$ is the finite dimensional separation constant of $(x_i)_i$.

**Proof** The implication (i) $\Rightarrow$ (ii) is immediate. The comments after Theorem 2.2 give that (ii) $\Rightarrow$ (iii) and Proposition 3.11 (applied to $z_i = x_i$) gives that (iii) $\Rightarrow$ (iv). To show that (iv) $\Rightarrow$ (ii) fix $0 < \epsilon < 1$ and let $(x_i)_i$ be a finite dimensional separation subsequence for $(X, M_{f_s}, \epsilon)$.

We first show that $(x_i)_i$ is quasi-greedy. Fix $x \in X$ and $m \in \mathbb{N}$, assuming that $G_m(x) \neq 0$ (otherwise, $\|x - G_m(x)\| = \|x\|$ and there is nothing to prove). Let

$$
n := \max\{1 \leq j \leq m : x'_{\rho(x,j)}(x) \neq 0\},
$$

and note that $G_n(x) = G_m(x)$. Since $x'_{\rho(x,n)}(x) \neq 0$ and $x'_i(x) \xrightarrow{i \to \infty} 0$, there is $j_0 \in \mathbb{N}$ such that for all $i \geq j_0$,

$$
|x'_i(x)| \leq \frac{\tau\epsilon}{2}|x'_{\rho(x,n)}(x)|.
$$

(17)
Now take \( F_0 := \{ x_i : 1 \leq i \leq j_0 \} \), let \( \mathcal{W} := \{ i_k : jF_0 + 1 \leq k \leq jF_0 + n \} \) and set \( z \) as follows.

\[
z := x - \mathcal{G}_n(x) + (1 + \epsilon)\tau^{-1}|x'_\rho(x,n)(x)| \sum_{j \in \mathcal{W}} x_j - \sum_{j \in \mathcal{W}} x'_j(x) x_j.
\]  

(18)

Since \( i_{jF_0+1} \geq j_0 \), we deduce from (17) and the choice of \( z \) that for every \( j \in \mathcal{W} \) and every \( l \in \mathbb{N} \setminus \mathcal{W} \),

\[
\tau |x'_j(z)| = (1 + \epsilon)|x'_\rho(x,n)(x)| > |x'_j(x - \mathcal{G}_n(x))| = |x'_j(z)|.
\]

It follows from this and Definition 1.5 that the only \( n \)-weak thresholding set for \( z \) with weakness parameter \( \tau \) is \( \mathcal{W} \). Let \( u \in [x_j : j \in \mathcal{W}] \) be an \( n \)-term Chebyshev \( \tau \)-greedy approximant for \( z \). Notice that both \( x \) and \( \mathcal{G}_n(x) \) belong to \( F_0 \). Also, by Lemma 4.1, \( (\|x_i\|)_i \) is bounded, and also is \( (\|x'_i\|)_i \), say by \( N \). As

\[
\| \sum_{j \in \mathcal{W}} x'_j(x) x_j \| \leq \sum_{j \in \mathcal{W}} \epsilon |x'_\rho(x,n)(x)||x_j| \leq \epsilon n N^2\|x\|,
\]

from (17), (18) and the choice of our subsequence we deduce that

\[
\|x - \mathcal{G}_n(x)\| \leq (M_{fs} + \epsilon)\|z - u\| \leq (M_{fs} + \epsilon)\sigma_n(z)
\]

\[
\leq (M_{fs} + \epsilon)\|x + (1 + \epsilon)\tau^{-1}|x'_\rho(x,n)(x)| \sum_{j \in \mathcal{W}} x_j - \sum_{j \in \mathcal{W}} x'_j(x) x_j \|
\]

\[
\leq (M_{fs} + \epsilon)\|x + (1 + \epsilon N^2)\| + (M_{fs} + \epsilon)\|1 + \epsilon)\tau^{-1}|x'_\rho(x,n)(x)| \sum_{j \in \mathcal{W}} x_j \|.
\]  

(19)

Now, in order to estimate \( |x'_\rho(x,n)(x)|\| \sum_{j \in \mathcal{W}} x_j \| \) we set

\[
w := (1 - \epsilon)\tau |x'_\rho(x,n)(x)| \sum_{j \in \mathcal{W}} x_j,
\]

and let \( v \) be an \( n \)-term Chebyshev \( \tau \)-greedy approximant for \( x + w \). We claim that \( v \in F_0 \). To prove this, from (17) and the definition of \( F_0 \) we deduce that for all \( i \in \mathbb{N} \setminus j_0 \setminus \mathcal{W} \),

\[
|x'_i(x + w)| = |x'_i(x)| \leq \frac{\tau}{2} |x'_\rho(x,n)(x)| < \tau |x'_\rho(x,n)(x)|,
\]

whereas for \( i \in \mathcal{W} \),

\[
|x'_i(x + w)| \leq (1 - \epsilon)\tau |x'_\rho(x,n)(x)| + \frac{\tau \epsilon}{2} |x'_\rho(x,n)(x)| < \tau |x'_\rho(x,n)(x)|.
\]
Combining both inequalities above we get that
\[ \{ i \in \mathbb{N} : |x'_i(x + w)| \geq \tau |x'_{\rho(x,n)}(x)| \} \subseteq \{1, \ldots, j_0\}. \tag{20} \]

On the other hand, for all \(1 \leq i \leq n\), we have \(i_j + 1 > j_0 > \rho(x, i)\), so
\[ |x'_{\rho(x,i)}(x + w)| = |x'_{\rho(x,i)}(x)| \geq |x'_{\rho(x,n)}(x)|. \]

From this and Definitions 1.5 and 1.7 we deduce that
\[ |x'_i(x + w)| \geq \tau |x'_{\rho(x,n)}(x)|, \]
for all \(i \in \text{supp}(v)\) which, combined with (20), implies that \(\text{supp}(v) \subseteq \{1, \ldots, j_0\}\).

Since, by Definition 1.7, \(v\) is a linear combination of the \(x_i\)'s with \(i\) in its support, it follows that \(v \in F_0\) (and so \(x - v \in F_0\)). Hence, applying the separation property of \((x_k)\) we deduce that
\[
(1 - \epsilon)\tau |x'_{\rho(x,n)}(x)| \| \sum_{j \in W} x_j \| \leq \| w \| \leq \| x + w - v \| + \| x - v \|
\]
\[ \leq \| x + w - v \| + (M_{fs} + \epsilon)\| x + w - v \|
\leq (1 + M_{fs} + \epsilon)K\sigma_n(x + w)
\leq (1 + M_{fs} + \epsilon)K\| x \|. \]

Then
\[ |x'_{\rho(x,n)}(x)| \| \sum_{j \in W} x_j \| \leq (1 - \epsilon)^{-1}\tau^{-1}(1 + M_{fs} + \epsilon)K\| x \|. \]

This result and (19) entail that
\[ \| x - G_n(x) \| \leq (M_{fs} + \epsilon)K(1 + \epsilon n^2)\| x \| + (M_{fs} + \epsilon)(1 + M_{fs} + \epsilon)\frac{1 + \epsilon}{1 - \epsilon}\tau^{-2}K^2\| x \|. \]

As \(G_n(x) = G_m(x)\), letting \(\epsilon \to 0\), we get
\[ \| x - G_m(x) \| = \| x - G_n(x) \| \leq (M_{fs}K + M_{fs}(M_{fs} + 1)K^2\tau^{-2})\| x \|. \]

Since \(x\) and \(m\) were chosen arbitrarily, we see that \((x_i)_i\) is quasi-greedy with second quasi-greedy constant \(K_2 \leq M_{fs}K + M_{fs}(M_{fs} + 1)K^2\tau^{-2}\). Thus, it is a Markuševich basis.

Let us show the hyperdemocracy condition. Choose \(\epsilon > 0\) and \((x_k)_k\) as before, and take \(A\) and \(B\) finite subsets of \(\mathbb{N}\) such that \(|A| \leq |B|\), and \((a_i)_{i \in A}, (b_j)_{j \in B}\) scalars with \(|a_i| \leq |b_j|\) for all \(i \in A, j \in B\). Set
\[ F_1 := \{x_i : i \in A \cup B\} \text{ and } a_0 := \max_{i \in A} |a_i|, \]
assuming without loss of generality that \( a_0 \neq 0 \). Take \( W := \{i_k : j_{F_1} + 1 \leq k \leq j_{F_1} + |A|\} \), and define:

\[
z_1 := \tau^{-1}a_0 \sum_{l \in W} x_l, \quad z_2 := \sum_{i \in A} a_i x_i \quad \text{and} \quad z_3 := \sum_{j \in B} b_j x_j.
\]

Note that for all \( i \in A \) and all \( l \in W \),

\[
|x'_i(z_2 + (1 + \epsilon)z_1)| = |x'_i(z_2)| = |a_i| < a_0(1 + \epsilon) = \tau|x'_i(z_2 + (1 + \epsilon)z_1)|.
\]

By Definition 1.5, it follows that the only \( |A| \)-weak thresholding set for \( z_2 + (1 + \epsilon)z_1 \) with parameter \( \tau \) is \( W \). Let \( u \in [x_i : i \in W] \) be a \( |A| \)-term Chebyshev \( \tau \)-greedy approximant for \( z_2 + (1 + \epsilon)z_1 \). We have

\[
\| \sum_{i \in A} a_i x_i \| = \| z_2 \| \leq (M_{fs} + \epsilon)\| z_2 + (1 + \epsilon)z_1 - u \|
\]

\[
\leq (M_{fs} + \epsilon)K\sigma_{|A|}(z_2 + (1 + \epsilon)z_1) \quad (21)
\]

\[
\leq (1 + \epsilon)(M_{fs} + \epsilon)K\| z_1 \|.
\]

Similarly, since for all \( j \in B, (1 - \epsilon)\tau a_0 < \tau|b_j| \), the only \( |B| \)-weak thresholding set for \( z_3 + (1 - \epsilon)\tau^2z_1 \) with parameter \( \tau \) is \( B \). Thus, by the WSG(\( \tau \)) condition there is \( v \in [x_i : i \in B] \) such that

\[
\| z_3 + (1 - \epsilon)\tau^2z_1 - v \| \leq K\sigma_{|B|}(z_3 + (1 - \epsilon)\tau^2z_1) \leq K\| z_3 \| = K\| \sum_{j \in B} b_j x_j \|.
\]

Hence,

\[
(1 - \epsilon)\tau^2\| z_1 \| \leq \| z_3 + (1 - \epsilon)\tau^2z_1 - v \| + \| z_3 - v \|
\]

\[
\leq (1 + M_{fs} + \epsilon)\| z_3 + (1 - \epsilon)\tau^2z_1 - v \|
\]

\[
\leq (1 + M_{fs} + \epsilon)K\| \sum_{j \in B} b_j x_j \|.
\]

From this and (21) we obtain

\[
\| \sum_{i \in A} a_i x_i \| \leq \frac{(1 + \epsilon)(M_{fs} + \epsilon)(M_{fs} + 1 + \epsilon)}{(1 - \epsilon)}K^2\tau^{-2} \| \sum_{j \in B} b_j x_j \|.
\]

We complete the proof of the hyperdemocracy property by letting \( \epsilon \to 0 \). Finally, an application of Theorem 2.2 gives that \( (x_i)_i \) is almost greedy. \( \square \)

**Corollary 4.3** Let \( (x_i)_i \) be a Markushevich basis. The following are equivalent:

(i) For every \( 0 < \tau \leq 1 \), \( (x_i)_i \) is WSG(\( \tau \)).

(ii) There is \( 0 < \tau \leq 1 \) such that \( (x_i)_i \) is WSG(\( \tau \)).

\( \Box \)
(iii) For every $0 < \tau \leq 1$, $(x_i)_i$ is WAG($\tau$).
(iv) There is $0 < \tau \leq 1$ such that $(x_i)_i$ is WAG($\tau$).
(v) $(x_i)_i$ is semi-greedy.
(vi) $(x_i)_i$ is almost greedy.

Proof The implications (i) $\implies$ (ii) and (iii) $\implies$ (iv) are immediate.
Also, (v) $\implies$ (i) and (vi) $\implies$ (iii) follow at once from the definitions.
That (ii) $\implies$ (vi) and (v) $\implies$ (vi) follow by Theorem 4.2.
By Proposition 2.3, we see that (iv) $\implies$ (vi).
Finally, (vi) $\implies$ (v) follows by [10, Theorem 3.2] (see also [13, Corollary 4.2]). $\square$

When the conditions of Proposition 3.11(b) hold, we have $M_{f \xi} = 1$ for any Markushevich basis. Thus, in such cases Theorem 4.2 gives upper bounds for the second quasi-greedy and the hyperdemocracy constant depending only on $K$ and $\tau$ (cf. [5, Theorem 5.11] and [6, Proposition 5.2]). On the other hand, and unlike the implication from (weak) almost greedy to semi-greedy, in general there is no upper bound for the almost greedy constant of a WSG($\tau$) Markushevich basis depending only on the WSG($\tau$) constant and $\tau$. The following example illustrates that.

Example 4.4 Let $(e_j)_j$ and $(e'_j)_j$ be the unit vector basis of $\ell_1$ and its sequence of coordinate functionals respectively. Given $\alpha > 0$, define

\[ x_i := e_i + 2(\alpha + 1)(-1)^i e_1 \quad \text{for all } i \geq 2; \]
\[ X := [x_i : i \geq 2]; \]
\[ x'_i := e'_i|_X \text{ for all } i \geq 2. \]

The following statements hold:
(a) $(x_i)_{i \geq 2}$ is a basic sequence equivalent to the unit vector basis of $\ell_1$ with democracy constant $K_d \geq 2\alpha + 3$.
(b) For every $0 < \tau \leq 1$, $(x_i)_{i \geq 2}$ is WAG($\tau$). Moreover, if $M(\tau)$ is a WAG($\tau$) constant for $(x_i)_{i \geq 2}$, then

\[ M(\tau) \geq \tau(2\alpha + 3). \]

In particular, the almost greedy constant of $(x_i)_{i \geq 2}$ is no less than $2\alpha + 3$.
(c) The quasi-greedy constants of $(x_i)_{i \geq 2}$ are greater than $\alpha$.
(d) $(x_i)_{i \geq 2}$ is a semi-greedy Markushevich basis for $X$ with semi-greedy constant $K_s \leq 4$. Moreover, for every $x \in X$, $m \in \mathbb{N}$ and every set $\mathcal{W}^T(x, m)$, there are scalars $(b_i)_{i \in \mathcal{W}^T(x, m)}$ such that

\[ \|x - \sum_{i \in \mathcal{W}^T(x, m)} b_i x_i\| \leq 4\tau^{-1} \sigma_m(x). \]

Proof Let us show that (a) holds. For each $n \geq 2$, we have

\[ \sum_{i=2}^{n} |a_i| \leq \sum_{i=2}^{n} |a_i| + |\sum_{i=2}^{n} 2(\alpha + 1)(-1)^i a_i| = \| \sum_{i=2}^{n} a_i x_i\| \leq (3 + 2\alpha) \sum_{i=2}^{n} |a_i|. \]
We see that \((x_i)_{i \geq 2}\) is basic and equivalent to the unit vector basis of \(\ell_1\), so in particular, it is democratic. Since \(\|x_2 + x_3\| = \|e_2 + 2(\alpha + 1)e_1 + e_3 - 2(\alpha + 1)e_1\| = 2\) and \(\|x_2 + x_4\| = \|e_2 + e_4 + 4(\alpha + 1)e_1\| = 4\alpha + 6\), it follows that

\[
K_d \geq \frac{4\alpha + 6}{2} = 2\alpha + 3,
\]

which proves (a).

As \((x_i)_{i \geq 2}\) is equivalent to \((e_i)_{i \geq 2}\), it is clearly WAG(\(\tau\)) for all \(0 < \tau \leq 1\). To complete the proof of (b), choose \(0 < \epsilon < \tau\), and set

\[
x := (x_2 + x_3) + (\tau - \epsilon)(x_4 + x_6).
\]

Then the only 2-weak thresholding set for \(x\) with parameter \(\tau\) is \(\{2, 3\}\). Thus,

\[
(\tau - \epsilon)(4\alpha + 6) = \|((\tau - \epsilon)(x_4 + x_6))\| = \|x - P_{[2,3]}(x)\| \leq M(\tau)\|x - P_{[4,6]}(x)\| = 2M(\tau),
\]

so we obtain the result by letting \(\epsilon \to 0\).

Now let \(K_{1q}\) and \(K_{2q}\) be the first and second quasi-greedy constants of the basis, respectively. Since \(G_1(x_2 + x_3) = x_2\), we get that \(K_{1q} \geq \frac{2\alpha + 3}{2} > \alpha + 1\), so \(K_{2q} > \alpha\), and (c) holds.

Finally, let us prove (d). Fix \(0 < \tau \leq 1\), \(x \in X\), \(m \in \mathbb{N}\), and a set \(W = W^\tau(x, m)\). Given \(A \subseteq \mathbb{N}_{\geq 1}\) with \(|A| = m\) and scalars \((a_i)_{i \in A}\), we proceed as follows: If \(A = W\), we choose \(b_i := a_i\) for each \(i\). Otherwise, fix \(\pi\) any bijection

\[
\pi : W \setminus A \to A \setminus W.
\]

For every \(j \in W\), we define

\[
b_j := \begin{cases} 
  a_j & \text{if } j \in A; \\
  (-1)^{j+\pi(j)}a_{\pi(j)} & \text{otherwise}.
\end{cases}
\]

Let us estimate the \(\ell_1\)-norm \(\|x - \sum_{j \in W} b_jx_j\|\) in terms of \(\|x - \sum_{i \in A} a_i x_i\|\). For the first coordinate we get

\[
e'_1(x - \sum_{j \in W} b_jx_j) = e'_1(x) - \sum_{i \in W \cap A} a_i e'_1(x_i) - \sum_{j \in W \setminus A} a_{\pi(j)}(-1)^{j+\pi(j)}e'_1(x_j)
\]

\[
e'_1(x) - \sum_{i \in W \cap A} a_i e'_1(x_i) - \sum_{j \in W \setminus A} 2a_{\pi(j)}(\alpha + 1)(-1)^{j+\pi(j)}(-1)^j
\]

\[
e'_1(x) - \sum_{i \in W \cap A} a_i e'_1(x_i) - \sum_{i \in A \setminus W} 2a_i(\alpha + 1)(-1)^j
\]

\[
e'_1(x - \sum_{i \in A} a_i x_i).
\]
Now, suppose that \( l > 1 \). For \( l \in \mathbb{N} \setminus A \cup \mathcal{W} \),

\[
e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_l(x) = e'_l(x - \sum_{i \in A} a_i x_i).
\]

Also, if \( l \in A \cap \mathcal{W} \) then

\[
e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_l(x) - b_l = e'_l(x) - a_l = e'_l(x - \sum_{i \in A} a_i x_i).
\]

On the other hand, if \( l \in \mathcal{W} \setminus A \) we compute the \( l \)- and the \( \pi(l) \)-coordinates components at the same time. From the fact that \( \pi(l) \notin \mathcal{W} \) we deduce that \( |e'_l(x)| = |x'_l(x)| \geq \tau |x'_{\pi(l)}(x)| = \tau |e'_{\pi(l)}(x)| \). Thus,

\[
|e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j)| + |e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_j x_j)| = |e'_l(x) - b_l| + |e'_{\pi(l)}(x)|
\]

\[
\leq \frac{|e'_l(x)|}{\tau} + |a_{\pi(l)}|
\]

\[
\leq 2 \max \left\{ \frac{|e'_l(x)|}{\tau}, |a_{\pi(l)}| \right\}
\]

\[
\leq 2 \max \left\{ \frac{|e'_l(x)|}{\tau}, 2|a_{\pi(l)}| - \frac{|e'_l(x)|}{\tau} \right\}
\]

\[
= \frac{4}{\tau} \max \left\{ \frac{|e'_l(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e'_l(x)|}{\tau} \right\}.
\]

Similarly,

\[
|e'_l(x - \sum_{i \in A} a_i x_i)| + |e'_{\pi(l)}(x - \sum_{i \in A} a_i x_i)| = |e'_l(x)| + |e'_{\pi(l)}(x) - a_{\pi(l)}|
\]

\[
\geq \max \left\{ |e'_l(x)|, |a_{\pi(l)}| - |e'_{\pi(l)}(x)| \right\}
\]

\[
\geq \max \left\{ |e'_l(x)|, \frac{|e'_l(x)|}{\tau} - \frac{|e'_l(x)|}{\tau} \right\}
\]

\[
\geq \tau \max \left\{ \frac{|e'_l(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e'_l(x)|}{\tau} \right\}.
\]

Comparing (22) and (23), we obtain

\[
|e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j)| + |e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_j x_j)| \leq \frac{4}{\tau} (|e'_l(x - \sum_{i \in A} a_i x_i)| + |e'_{\pi(l)}(x - \sum_{i \in A} a_i x_i)|).
\]
Combining the above estimates we get
\[ \|x - \sum_{j \in \mathcal{W}} b_j x_j\| \leq \frac{4}{\tau} \|x - \sum_{i \in A} a_i x_i\|. \]

Now, the left-hand side of the inequality is greater than or equal to the infimum over all scalars \((b_j)_{j \in \mathcal{W}}\), which in fact is a minimum since \(\mathcal{W}\) is finite. Then, taking the infimum over all \(A \subseteq \mathbb{N}\) with \(|A| = m\) and scalars \((a_i)_{i \in A}\) on the right-hand side, we conclude that
\[ \min_{(b_j)_{j \in \mathcal{W}} \subseteq \mathbb{K}} \|x - \sum_{j \in \mathcal{W}} b_j x_j\| \leq \frac{4}{\tau} \sigma_m(x). \]

Taking \(\tau = 1\), we conclude that \((x_i)_{i \geq 2}\) is semi-greedy, and we get the bound for \(K_s\). \(\Box\)

A natural question in this context is whether the implication from WSG(\(\tau\)) to almost greedy holds for all WSG(\(\tau\)) systems, or—equivalently in light of Theorem 4.2—whether every weak semi-greedy system is a Markushevich basis. The answer is negative. The following example shows a semi-greedy system that is neither quasi-greedy nor democratic.

**Example 4.5** Let \((e_i)_i\) be the unit vector basis of \(c_0\) and let \((e'_i)_i\) be the sequence of biorthogonal functionals. Set
\[
x_i := e_i + (-1)^i e_1 \text{ for all } i \geq 2;
x'_i := e'_i \text{ for all } i \geq 2.
\]

The following statements hold:

(a) \((x_i)_{i \geq 2}\) is a fundamental minimal system for \(c_0\), but not a Markushevich basis. Thus, it is not quasi-greedy.

(b) \((x_i)_{i \geq 2}\) is not democratic.

(c) \((x_i)_{i \geq 2}\) is a semi-greedy system for \(c_0\) with semi-greedy constant no greater than 3. Moreover, for any \(0 < \tau \leq 1\), \(x \in X\), \(m \in \mathbb{N}\) and every set \(\mathcal{W}^\tau(x, m)\), there are scalars \((b_i)_{i \in \mathcal{W}^\tau(x, m)}\) such that
\[ \|x - \sum_{i \in \mathcal{W}^\tau(x, m)} b_i x_i\| \leq 3^{-1} \sigma_m(x). \]

**Proof** To show that (a) holds, first note that for all \(n \in \mathbb{N}\),
\[ \|e_1 - \sum_{i=1}^n \frac{x_{2i}}{n} \| = \| \sum_{i=1}^n \frac{e_{2i}}{n} \| = \frac{1}{n}. \]
This entails that $e_1 \in [x_i : i \geq 2]$, so $(x_i)_{i \geq 2}$ is fundamental. Since $x_j'(e_1) = 0$ for every $j \geq 2$, $(x_i)_{i \geq 2}$ is not a Markushevich basis, and thus it is not quasi-greedy.

To see that $(x_i)_{i \geq 2}$ is not democratic, notice that for all $n \in \mathbb{N}$,

$$\| \sum_{i=2}^{2n+1} x_i \| = \| \sum_{i=2}^{2n+1} e_i \| = 1,$$

but

$$\| \sum_{i=1}^{2n} x_{2i} \| = \| 2ne_1 + \sum_{i=1}^{2n} e_i \| = 2n.$$

Hence, (b) holds. To prove (c), we proceed as in the proof of Example 4.4. Fix $0 < \tau \leq 1$, $x \in X$, $m \in \mathbb{N}$ and a set $\mathcal{W} = \mathcal{W}^\tau (x, m)$. Take a set $A \subseteq \mathbb{N}_{>1}$ with $|A| = m$ and $A \neq \mathcal{W}$, and scalars $(a_i)_{i \in A}$, and let

$$\pi : \mathcal{W} \setminus A \to A \setminus \mathcal{W}$$

be a bijection. For every $j \in \mathcal{W}$, define

$$b_j := \begin{cases} a_j & \text{if } j \in A; \\ (-1)^{j + \pi(j)}a_{\pi(j)} & \text{otherwise.} \end{cases}$$

Now, we estimate the supremum norm of $x - \sum_{j \in \mathcal{W}} b_j x_j$ in terms of that of $x - \sum_{i \in A} a_i x_i$.

First note that if $l > 1$ and $l \in \mathbb{N} \setminus A \cup \mathcal{W}$ or $l \in A \cap \mathcal{W}$, we have

$$e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_l(x - \sum_{i \in A} a_i x_i).$$

This equality also holds for $l = 1$, indeed

$$e'_1(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_1(x) - \sum_{i \in \mathcal{W} \cap A} a_i e'_1(x_i) - \sum_{j \in \mathcal{W} \setminus A} a_{\pi(j)}(-1)^{\pi(j)}$$

$$= e'_1(x) - \sum_{i \in \mathcal{W} \cap A} a_i e'_1(x_i) - \sum_{j \in \mathcal{W} \setminus A} a_{\pi(j)}e'_1(x_{\pi(j)})$$

$$= e'_1(x) - \sum_{i \in \mathcal{W} \cap A} a_i e'_1(x_i) - \sum_{i \in A \setminus \mathcal{W}} a_i e'_1(x_i)$$

$$= e'_1(x - \sum_{i \in A} a_i x_i).$$

For each $l \in \mathcal{W} \setminus A$, we have $|e'_l(x)| = |x'_l(x)| \geq \tau |x'_{\pi(l)}(x)| = \tau |e'_{\pi(l)}(x)|$. Hence, considering together the $l$- and the $\pi(l)$-th coordinates we have
\[
\max \left\{ \left| e'(x - \sum_{j \in \mathcal{W}} b_j x_j) \right|, \left| e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_j x_j) \right| \right\} = \max \left\{ \left| e'(x) - b_l \right|, \left| e'_{\pi(l)}(x) \right| \right\}
\]
\[
\leq \max \left\{ \left| e'(x) \right| + \left| a_{\pi(l)} \right| \right\}
\]
\[
\leq 3 \max \left\{ \left| e'(x) \right|, \left| a_{\pi(l)} \right| - \left| e'(x) \right| \right\}.
\]

Similarly, we obtain
\[
\max \left\{ \left| e'(x - \sum_{i \in \mathcal{A}} a_i x_i) \right|, \left| e'_{\pi(l)}(x - \sum_{i \in \mathcal{A}} a_i x_i) \right| \right\} \geq \tau \max \left\{ \left| e'(x) \right|, \left| a_{\pi(l)} \right| - \left| e'(x) \right| \right\}.
\]

From the inequalities given above,
\[
\| x - \sum_{j \in \mathcal{W}} b_j x_j \| \leq \frac{3}{\tau} \| x - \sum_{i \in \mathcal{A}} a_i x_i \|.
\]

The proof of (c) is completed by the same argument given in Example 4.4.

**Remark 4.6** The system of Example 4.5 can also be considered in \(\ell_p\) for all \(1 < p < \infty\). With only minor adjustments to the calculations given above, we obtain that \((x_i)_{i \geq 2} \subseteq \ell_p\) is semi-greedy (with constant no greater than \(3 \ast 2^{\frac{1}{p}}\)), but neither democratic nor quasi-greedy.

Our next proposition shows that from any \(\text{WSG}(\tau)\) system that is not a Markushevich basis, one can obtain an almost greedy Markushevich basis for the space, with superdemocracy and first quasi-greedy constants depending only on \(\tau\) and the \(\text{WSG}(\tau)\) constant of the system—and thus, by Theorem 2.2, with almost greedy constant also depending only on said constants. In order to prove our result, we need two technical lemmas. The notation used below is natural and according to the context.

**Lemma 4.7** Let \(\mathcal{B}_1 = (x_i)_{i \in \mathbb{N}}\) be a fundamental minimal system for \(Y\), and suppose that \(\mathcal{B}_2 := (x_0, x_i)_{i \in \mathbb{N}}\) is a fundamental minimal system for \(X := \{x_i : i \in \mathbb{N}_0\}\) with biorthogonal functionals \((x'_0, x'_i)_{i \in \mathbb{N}} \subseteq X'\) satisfying
\[
\|x_0\| \cdot \|x'_0\| = 1 \quad \text{and} \quad \|x_0\| = \sup_{i \in \mathbb{N}} \|x_i\|.
\]

The following hold:

(a) If \(\mathcal{B}_1\) is quasi-greedy with first quasi-greedy constant \(K_{1q}(\mathcal{B}_1)\), then \(\mathcal{B}_2\) is quasi-greedy with first quasi-greedy constant
\[
K_{1q}(\mathcal{B}_2) \leq 2K_{1q}(\mathcal{B}_1) + 1.
\]
(b) If $\mathcal{B}_1$ is superdemocratic with constant $K_{sd}(\mathcal{B}_1)$, then $\mathcal{B}_2$ is superdemocratic with constant

$$K_{sd}(\mathcal{B}_2) \leq 4K_{sd}(\mathcal{B}_1).$$

**Proof** To prove (a), fix $x \in X$ and $m \in \mathbb{N}$. Then, we have

$$G_{\mathcal{B}_2, m}(x) = \begin{cases} G_{\mathcal{B}_1, m-1}(x - x'_0(x)x_0) + x'_0(x)x_0 & \text{if } x \in GS_{\mathcal{B}_2, m}(x); \\ G_{\mathcal{B}_1, m}(x - x'_0(x)x_0) & \text{otherwise}. \end{cases}$$

Thus,

$$\|G_{\mathcal{B}_2, m}(x)\| \leq K_{1q}(\mathcal{B}_1)\|x - x'_0(x)x_0\| + \|x'_0(x)x_0\| \leq (2K_{1q}(\mathcal{B}_1) + 1)\|x\|. $$

It follows that $\mathcal{B}_2$ is quasi-greedy and $K_{1q}(\mathcal{B}_2) \leq 2K_{1q}(\mathcal{B}_1) + 1$.

To prove (b), suppose first that $D \subseteq \mathbb{N}_0$ is a finite nonempty set and take scalars $(a_k)_{k \in D}$ with $|a_k| = 1$ for each $k \in D$. If $0 \notin D$, then

$$\|x_0\| = \|x_0\||x'_0(\sum_{k \in D} a_k x_k)| \leq \sum_{k \in D} a_k x_k. \quad (24)$$

On the other hand, if $0 \notin D$ we have

$$\|x_0\| = \sup_{k \in \mathbb{N}} \|x_k\| \leq K_{sd}(\mathcal{B}_1)\|\sum_{k \in D} a_k x_k\|. \quad (25)$$

Now let $A, B \subseteq \mathbb{N}_0$ be finite nonempty sets with $|A| \leq |B|$, and take $(a_i)_{i \in A}, (b_j)_{j \in B}$ scalars such that $|a_i| = |b_j| = 1$ for all $i \in A, j \in B$. If $0 \notin A \cup B$, there is nothing to prove. If $0 \in A \setminus B$, by (25) we have

$$\|\sum_{i \in A} a_i x_i\| \leq \|x_0\| + \|\sum_{i \in A \setminus \{0\}} a_i x_i\| \leq \|x_0\| + K_{sd}(\mathcal{B}_1)\|\sum_{j \in B} b_j x_j\|$$

$$\leq 2K_{sd}(\mathcal{B}_1)\|\sum_{j \in B} b_j x_j\|.$$
\[(1 + 2K_{sd}(B_1)) \| \sum_{j \in B} b_j x_j \|.
\]

If \(0 \in B \setminus A\) and \(|B| > 1\), we choose \(i_0 \in A\) and using (24) we get that
\[
\| \sum_{i \in A} a_i x_i \| \leq \| x_{i_0} \| + \| \sum_{i \in A \setminus \{i_0\}} a_i x_i \| \leq 2K_{sd}(B_1) \| \sum_{j \in B \setminus \{0\}} b_j x_j \|
\]
\[
\leq 2K_{sd}(B_1)(\| \sum_{j \in B} b_j x_j \| + \| x_0 \|)
\]
\[
\leq 4K_{sd}(B_1) \| \sum_{j \in B} b_j x_j \|.
\]

The only case left is \(A \neq B = \{0\}\). Then \(A = \{i_0\}\) for some \(i_0 \in \mathbb{N}\) and
\[
\| \sum_{i \in A} a_i x_i \| = \| x_{i_0} \| \leq \sup_{k \in \mathbb{N}} \| x_k \| = \| x_0 \| = \| \sum_{j \in B} b_j x_j \|.
\]

From the above estimations, \(B_2\) is superdemocratic and \(K_{sd}(B_2) \leq 4K_{sd}(B_1)\).

The following result will allow us to handle the case \(\sigma_m(x) = 0\) in the proof of Proposition 4.9.

**Lemma 4.8** Let \((x_i)_i \subseteq X\) be a fundamental minimal system with both \((x_i)_i\) and \((x'_i)_i\) bounded. If \(x \in X\) is such that \(\sigma_m(x) = 0\) for some \(m \in \mathbb{N}\), then \(| \text{supp} (x) | \leq m\) and \(x = G_m(x) = P_{\text{supp}(x)}(x)\).

**Proof** Let \(B := \text{supp}(x)\). If \(|B| > m\), there is \(C \subset B\) with \(|C| = m + 1\). Thus, if \(A \subset \mathbb{N}\) and \(|A| \leq m\), there is \(j \in C \setminus A\). Then, for any scalars \((a_i)_{i \in A}\) it follows that
\[
\| x - \sum_{i \in A} a_i x_i \| \geq \frac{|x'_j(x - \sum_{i \in A} a_i x_i)|}{\| x'_j \|} = \frac{|x'_j(x)|}{\| x'_j \|} \geq \min_{i \in C} \| x'_i \| \geq \max_{i \in C} |x'_i(x)| > 0.
\]

Taking infimum over such sets and scalars, we get a contradiction to the hypothesis that \(\sigma_m(x) = 0\). Now let
\[
M := \sup_i \{ \| x_i \|, \| x'_i \| \}.
\]

Given that \(\sigma_m(x) = 0\) and \(|B| \leq m\), we have \(\sigma_{2m}(x - P_B(x)) = 0\). Fix \(\epsilon > 0\) and choose \(A \subset X\) with \(|A| = 2m\) and scalars \((a_i)_{i \in A}\) so that
\[
\| x - P_B(x) - \sum_{i \in A} a_i x_i \| \leq \epsilon.
\]
For each \( l \in A \), we have
\[
|a_l| = |x'_{l}(x - P_B(x) - \sum_{i \in A} a_i x_i)| \leq M\|x - P_B(x) - \sum_{i \in A} a_i x_i\| \leq M\varepsilon.
\]
Hence,
\[
\|x - P_B(x)\| \leq \varepsilon + \sum_{i \in A} M\|a_i x_i\| \leq \varepsilon + 2mM^2\varepsilon.
\]
Since \( \varepsilon \) is arbitrary and \( m, M \) are fixed, we get \( x = P_B(x) \), and thus \( x = G_m(x) \).  \( \square \)

Now we can show that a weak semi-greedy system that is not a Markushevich basis can be slightly modified to obtain a Markushevich basis (and therefore, by Theorem 4.2, an almost greedy system).

**Proposition 4.9** Let \( 0 < \tau \leq 1 \), and let \( \mathcal{B} := (x_i)_{i \in \mathbb{N}} \subseteq X \) be a WSG(\( \tau \)) system that is not a Markushevich basis, with constant \( K_{ws}(\tau, \mathcal{B}) \). There are \( x_0 \in X \) and \( x'_0 \in X' \) such that
\[
\{x_i\}_{i \in \mathbb{N}} \subseteq \{x_i\}_{i \in \mathbb{N}} \cup \{x_0\},
\]
and the system
\[
\mathcal{B}_1 := (x_0, x_i - x'_0(x_i)x_0)_{i \in \mathbb{N}}
\]
is an almost greedy Markushevich basis for \( X \) with biorthogonal functionals \( (x'_0, x'_i) \). In addition, \( \mathcal{B}_1 \) has first quasi-greedy constant
\[
K_{1q}(\mathcal{B}_1) \leq 3 + 4K_{ws}(\tau, \mathcal{B}) + 16K_{ws}(\tau, \mathcal{B})^2\tau^{-2},
\]
and superdemocracy constant
\[
K_{sd}(\mathcal{B}_1) \leq 32K_{ws}(\tau, \mathcal{B})^2\tau^{-2}.
\]

**Proof** By Theorem 4.2 and Corollary 3.9, the set \( \{x_i\}_{i \in \mathbb{N}} \) is weakly compact, and \( 0 \notin \{x_i\}_{i \in \mathbb{N}} \). Then, there is a subnet \( (x_{i_\lambda})_\lambda \) and \( u_0 \in X \setminus \{0\} \) such that
\[
x_{i_\lambda} \overset{w}{\rightharpoonup} u_0.
\]
By the Hahn–Banach Theorem, there is \( u'_0 \in X' \) such that
\[
\|u_0\|\|u'_0\| = 1 \quad \text{and} \quad u'_0(u_0) = 1.
\]
Now for \( x \in X \) define the linear operator \( P : X \to X \) by
\[
P(x) := x - u'_0(x)u_0.
\]
It is easy to check that $P$ is a projection, $\| P \| \leq 2$, $X = [u_0] \oplus P(X)$ and, as $x_i'(u_0) = 0$ for all $j \in \mathbb{N},$

$$B_2 := (x_i - u_0'(x_i)u_0)_{i \in \mathbb{N}}$$

is a fundamental minimal system for $P(X)$ with biorthogonal functionals $(x_i' \mid P(X))_{i \in \mathbb{N}}$. First, we show that $B_2$ is a WSG($\tau$) system for $P(X)$. Take $y \in P(X)$ and $m \in \mathbb{N}$, and fix $\epsilon > 0$. If $\sigma_{B_2,m}(y) \neq 0$ we choose a set $A \subseteq \mathbb{N}$ with $|A| = m$ and scalars $(a_i)_{i \in A}$ so that

$$\| y - \sum_{i \in A} a_i (x_i - u_0'(x_i)u_0) \| \leq (1 + \epsilon) \sigma_{B_2,m}(y),$$

and define

$$z := y + \sum_{i \in A} a_i u_0'(x_i)u_0.$$ 

By hypothesis, there is a set $\mathcal{W}_B^\tau(z, m)$ and scalars $(b_j)_{j \in \mathcal{W}_B^\tau(z, m)}$ such that

$$\| z - \sum_{j \in \mathcal{W}_B^\tau(z, m)} b_j x_j \| \leq K_{ws}(\tau, B) \sigma_{B_2,m}(z).$$

Then, it follows that

$$\| y - \sum_{j \in \mathcal{W}_B^\tau(z, m)} b_j (x_j - u_0'(x_j)u_0) \| = \| P(z - \sum_{j \in \mathcal{W}_B^\tau(z, m)} b_j x_j) \|$$

$$\leq 2 \| z - \sum_{j \in \mathcal{W}_B^\tau(z, m)} b_j x_j \|$$

$$\leq 2 K_{ws}(\tau, B) \| y - \sum_{i \in A} a_i (x_i - u_0'(x_i)u_0) \|$$

$$\leq 2(1 + \epsilon) K_{ws}(\tau, B) \sigma_{B_2,m}(y).$$

Since $x_i'(y) = x_i'(z)$ for every $i \in \mathbb{N}$, the set $\mathcal{W}_B^\tau(z, m)$ is also a weak thresholding set for $y$ with respect to $B_2$, and we obtain the estimate

$$\| y - \sum_{j \in \mathcal{W}_{B_2}^\tau(y, m)} b_j (x_j - u_0'(x_j)u_0) \| \leq 2(1 + \epsilon) K_{ws}(\tau, B) \sigma_{B_2,m}(y). \quad (26)$$

Now suppose that $\sigma_{B_2,m}(y) = 0$. By Lemma 4.1, both $(x_i)_i$ and $(x_i')_i$ are seminormalized. The former implies that $(x_i - u_0'(x_i)u_0)_i$ is bounded, so by Lemma 4.8, we have

$$y = \mathcal{G}_{B_2,m}(y) = \sum_{j \in \mathcal{G}_{\mathcal{S}_{B_2,m}(y)}} x_j'(y)(x_j - u_0'(x_j)u_0).$$
Hence, (26) holds also in this case taking \( \mathcal{W}_{B_2}^\tau (y, m) := Gs_{B_2, m}(y) \) and \( b_j := x'_j(y) \) for all \( j \in \mathcal{W}_{B_2}^\tau (y, m) \). Then we conclude that \( B_2 \) is a WSG(\( \tau \)) system for \( P(X) \) with constant

\[
K_{ws}(\tau, B_2) \leq 2(1 + \epsilon)K_{ws}(\tau, B).
\]

As \( x_i \xrightarrow{w} u_0 \) in the weak topology of \( X \), we get that \( x_i - u'_0(x_i)u_0 \xrightarrow{w} 0 \) in the weak topology of \( P(X) \). Then, by Corollary 3.9, Theorem 4.2 and Proposition 3.11(b), it follows that \( B_2 \) is an almost greedy Markushevich basis for \( P(X) \), with first quasi-greedy constant

\[
K_{1q}(B_2) \leq 1 + K_{ws}(\tau, B_2) + 2K_{ws}(\tau, B_2)^2\tau^{-2}
\]

\[
\leq 1 + 2(1 + \epsilon)K_{ws}(\tau, B) + 8(1 + \epsilon)^2K_{ws}(\tau, B)^2\tau^{-2}
\] (27)

and with superdemocracy constant

\[
K_{sd}(B_2) \leq 2K_{ws}(\tau, B_2)^2\tau^{-2} \leq 8(1 + \epsilon)^2K_{ws}(\tau, B)^2\tau^{-2}.
\] (28)

Now set

\[
a_0 := \frac{1}{\|u_0\|} \sup_{i \in \mathbb{N}} \|x_i - u'_0(x_i)u_0\|, \quad x_0 := a_0u_0, \quad \text{and} \quad x'_0 := a_0^{-1}u'_0.
\]

Since

\[
\|x_0\|\|x'_0\| = 1, \quad \|x_0\| = \sup_{i \in \mathbb{N}} \|x_i - x'_0(x_i)x_0\| \quad \text{and} \quad B_2 = (x_i - x'_0(x_i)x_0)_i,
\]

we may apply Lemma 4.7. Letting \( \epsilon \to 0 \) in (27) and (28), an application of the lemma gives that \( B_1 \) is a quasi-greedy Markushevich basis for \( X \), with first quasi-greedy constant

\[
K_{1q}(B_1) \leq 2K_{1q}(B_2) + 1 \leq 3 + 4K_{ws}(\tau, B) + 16K_{ws}(\tau, B)^2\tau^{-2},
\]

and it is superdemocratic with constant

\[
K_{sd}(B_1) \leq 4K_{sd}(B_2) \leq 32K_{ws}(\tau, B)^2\tau^{-2}.
\]

To finish the proof, let \( v \in X \) and suppose there is a subnet \((x_{i_{\gamma}})_{\gamma}\) such that

\[
x_{i_{\gamma}} \xrightarrow{w} v.
\]

It is immediate that \( x'_j(v) = 0 \) for all \( j \in \mathbb{N} \). Then, as \( B_1 \) is a Markushevich basis for \( X \) we get that \( v - x'_0(v)x_0 = 0 \). This proves that \( \{x_{i}\}_{i \in \mathbb{N}} \subseteq \{x_i\}_{i \in \mathbb{N}} \cup \{x_0\} \).

\[\square\]
5 Finite Dimensional Spaces and Branch Greedy Algorithms

In this section, we study the semi-greedy and almost greedy properties—and some weaker versions thereof—in finite dimensional Banach spaces, where each biorthogonal system is a greedy Schauder basis. Here, the questions concerning (weak) thresholding or Chebyshev greedy algorithms focus on the behavior and the relationships of their natural associated constants. We will consider branch semi-greedy and branch almost greedy bases, introduced and studied by Dilworth et al. [12], and extend some of their results.

Let us present the branch versions of the (weak) thresholding and Chebyshev greedy algorithms. While the original definitions have been given for weak parameters $0 < \tau < 1$, we allow $\tau = 1$, since the results in this section hold for this case as well. For a fixed weakness parameter $0 < \tau \leq 1$ and a Markushevich basis $(x_i)$ with seminormalized coordinates, the algorithm is defined as follows. First, set

$$A^\tau (x) := \{ i \in \mathbb{N} : |e'_i(x)| \geq \tau \max_{j \in \mathbb{N}} |e'_j(x)| \},$$

and let $G^\tau : X \setminus \{0\} \rightarrow \mathbb{N}$ be a function with the following properties:

(BG1) $G^\tau (x) \in A^\tau (x)$ for every $x \in X \setminus \{0\}$.

(BG2) $G^\tau (\lambda x) = G^\tau (x)$ for all $x \in X \setminus \{0\}$ and all $\lambda \in \mathbb{K} \setminus \{0\}$.

(BG3) If $A^\tau (x) = A^\tau (y)$ and $e'_i(x) = e'_i(y)$ for all $i \in A^\tau (x)$, then $G^\tau (x) = G^\tau (y)$.

For each $x \neq 0$, this defines a function $\rho^\tau_x : \{1, \ldots, |\text{supp}(x)|\} \rightarrow \mathbb{N}$ if $|\text{supp}(x)| < \infty$ or $\rho^\tau_x : \mathbb{N} \rightarrow \mathbb{N}$ otherwise, given by $\rho^\tau_x (1) := G^\tau (x)$, and for $2 \leq i \leq |\text{supp}(x)|$,

$$\rho^\tau_x (i) := G^\tau (x - \sum_{j=1}^{i-1} x'_{\rho^\tau_x (j)}(x)x_{\rho^\tau_x (j)}).$$

Similarly, for every $x \in X \setminus \{0\}$ and $m \in \mathbb{N}$, the $m$-term branch greedy approximation to $x$ (with regard to a fixed branch $G^\tau$) is defined as

$$G^\tau_m (x) := \sum_{i=1}^{m} x'_{\rho^\tau_x (i)}(x)x_{\rho^\tau_x (i)}.$$

setting $x'_{\rho^\tau_x (i)}(x)x_{\rho^\tau_x (i)} := 0$ if $i > \max (\text{supp}(x))$, and $G^\tau_0 (x) := 0$. The idea of choosing a branch associated to a weakness parameter $\tau$ is applied in different contexts. It is useful, for instance, to better understand some structures, as can be seen in [14], where it is shown that the Haar system in $L^1(0, 1)$ is branch quasi-greedy for all $0 < \tau < 1$, but it is not quasi-greedy in the usual sense.

**Definition 5.1** [12, Definition 6.1] Let $N \in \mathbb{N}$, and let $E$ be a $N$-dimensional Banach space with a fundamental minimal system $(x_i)_{1 \leq i \leq N} \subseteq E$. The system is called branch almost-greedy with weakness parameter $0 < \tau \leq 1$ (BAG($\tau$)) and constant $M$ if, for
every $x \in E$ and every $0 \leq m \leq N$, we have

$$\|x - G_m^\tau(x)\| \leq M\tilde{\sigma}_m(x).$$

**Definition 5.2** [12, Definition 7.3] Let $N \in \mathbb{N}$, and let $E$ be a $N$-dimensional Banach space with fundamental minimal system $(x_i)_{1 \leq i \leq N} \subseteq E$. The system is called *branch semi-greedy* with weakness parameter $0 < \tau \leq 1$ (BSG$(\tau)$) and constant $M$ if, for every $x \in E$ and every $1 \leq m \leq N$, there are scalars $(a_i)_{1 \leq i \leq m}$ such that

$$\|x - \sum_{i=1}^m a_i x_{\nu_{\tau}^i(x)}\| \leq M\sigma_m(x).$$

**Remark 5.3** Note that if we consider the definition of WAG$(\tau)$ systems in the finite dimensional context, it is immediate that every BAG$(\tau)$ system with constant $M$ is also WAG$(\tau)$ with constant no greater than $M$. The same relation exists between WSG$(\tau)$ and BSG$(\tau)$ systems.

**Remark 5.4** Also, note that the greedy ordering provides a branch greedy algorithm with parameter $\tau$ for every $0 < \tau \leq 1$. We can simply define

$$G^\tau(x) := \rho(x, 1)$$

for all $x \in X$. It is easy to check that $G^\tau$ satisfies (BG1), (BG2), and (BG3) for all $0 < \tau \leq 1$.

Every BAG$(\tau)$ system with constant $M$ has quasi-greedy, democratic and almost greedy constants depending only on $M$ and $\tau$ [12, Theorem 6.4, Corollary 6.5]. Also, for an almost greedy system, the conditions of Definition 1.7 hold for all $x \in X$, $m \in \mathbb{N}$, and every weak thresholding set $W^\tau(x, m)$, with $M$ depending only on the first quasi-greedy constant and $\tau$, [12, Theorem 7.1]. This implies immediately that it is BSG$(\tau)$, and that every branch of the algorithm satisfies the BSG condition.

Going in the opposite direction, that is from the BSG$(\tau)$ to the almost greedy (or, equivalently, the BAG$(\tau)$) property, it was proved in [12, Theorem 7.7] that the almost greedy constant can be controlled by the BSG$(\tau)$ constant, the basis constant, the cotype constant of the space and $\tau$ [12, Theorem 7.7]. In the same paper, the authors left open the question of whether the BSG$(\tau)$ property implies in general the BAG$(\tau)$ property, that is if the constant of the latter can be controlled by that of the former (see the question below [12, Definition 7.3]). Now we are in a position to answer that question and extend [12, Theorem 7.7].

First, note that in Example 4.4, we did not use that the space is infinite dimensional to prove any of the bounds for the constants of the system. Indeed, the bounds for the quasi-greedy constants hold if we replace $\ell_1$ with $\ell_1^n$ for any $n \geq 3$. To get the bound for the democracy constant, we used $n \geq 4$, but $n \geq 3$ is enough to get $K_d \geq \alpha + 1$ by comparing $x_2 + x_3$ with $\|x_2\|$ instead of $\|x_2 + x_4\|$. The system in Example 4.4 has semi-greedy constant no greater than $4$, and in fact, by (d), all branches of the algorithm satisfy the BSG$(\tau)$ condition with constant no greater than $4\tau^{-1}$. Hence, (a)
and (c) show that there is no upper bound for the democracy, quasi-greedy or almost greedy constant that depends only on the semi-greedy or the BSG($\tau$) constant and $\tau$. Thus, by [12, Theorem 6.4] and [12, Corollary 6.5], it follows that there is no such upper bound for the BAG($\tau$) constant, either.

Second, it is possible to remove the cotype condition from [12, Theorem 7.7], and also extend the result to any WSG($\tau$) system. To do so, next we provide a bound for the second quasi-greedy constant of such systems. For the proof we combine ideas from the proofs of [5, Theorem 1.10] and Theorem 4.2 with further arguments that allow us to handle the finite dimensional case.

**Theorem 5.5** Let $N \in \mathbb{N}_{>1}$ and $E$ be a $N$-dimensional Banach space with a WSG($\tau$) basis $(x_i)_{1 \leq i \leq N}$, $0 < \tau \leq 1$. If $(x_i)_{1 \leq i \leq N}$ has WSG($\tau$) constant $K_{ws}(\tau)$ and basis constant $K_b$, then $(x_i)_{1 \leq i \leq N}$ is quasi-greedy with second quasi-greedy constant

$$K_{2q} \leq 5K_b^2K_{ws}(\tau) + 6K_b^3K_{ws}(\tau)^2\tau^{-2}.$$

**Proof** Let $N_1 := \left\lfloor \frac{N+1}{2} \right\rfloor$, and consider the finite sets $A_1 := \{j \in \mathbb{N} : 1 \leq j \leq N_1\}$ and $A_2 := \{j \in \mathbb{N} : N_1 < j \leq N\}$. Now, for all $x \in E$ and $i = 1, 2$ define the projection operators

$$P_i(x) := \sum_{j \in A_i} x_j(x) x_j.$$

Fix $x \in E$ and $1 \leq m \leq N$, assuming without loss of generality that $G_m(x) \neq x$ (else, there is nothing to prove). Set

$$m_1 := |A_1 \cap GS_m(x)| \quad \text{and} \quad m_2 := |A_2 \cap GS_m(x)|.$$

Note that

$$G_m(x) = G_{m_1}(P_1(x)) + G_{m_2}(P_2(x)).$$

Thus,

$$\|x - G_m(x)\| \leq \|P_1(x) - G_{m_1}(P_1(x))\| + \|P_2(x) - G_{m_2}(P_2(x))\|. \quad (29)$$

Let us consider first the case in which $m_1 \neq 0$ and $m_2 \neq 0$. Since $x \neq G_m(x)$, it follows that $x_{\rho(P_i(x),m_i)}(P_i(x)) \neq 0$ for $1 \leq i \leq 2$. Fix $0 < \xi < 1$, and let

$$y_1 := \tau^{-1}(1 + \xi)|x_{\rho(P_1(x),m_1)}(P_1(x))| \sum_{j=N_1+1}^{N_1+m_1} x_j;$$

$$y_2 := \tau^{-1}(1 + \xi)|x_{\rho(P_2(x),m_2)}(P_2(x))| \sum_{j=1}^{m_2} x_j.$$
Note that for any $N_1 < j \leq (N_1 + m_1)$ and $1 \leq i \leq N_1$ or $(N_1 + m_1) < i \leq N$, we have

$$\tau x_j'(y_1) = (1 + \xi)|x_j'(P_{(1,x,m_1)}(P_1(x))| > |x_j'(P_1(x) - G_{m_1}(P_1(x)))|.$$  

Hence, the only $m_1$-weak thresholding set for

$$P_1(x) - G_{m_1}(P_1(x)) + y_1$$

with weakness parameter $\tau$ is the set $\{ j : N_1 < j \leq N_1 + m_1 \}$. Similarly, the only $m_2$-weak thresholding set for

$$P_2(x) - G_{m_2}(P_2(x)) + y_2$$

with weakness parameter $\tau$ is the set $\{ j : 1 \leq j \leq m_2 \}$. Let $w_1$ and $w_2$ be an $m_1$-term and an $m_2$-term Chebyshev $\tau$-greedy approximant for $P_1(x) - G_{m_1}(P_1(x)) + y_1$ and $P_2(x) - G_{m_2}(P_2(x)) + y_2$, respectively. Considering that $\| P_1 \| \leq K_b$ and $\| P_2 \| \leq 1 + K_b$, we deduce that

$$\| P_1(x) - G_{m_1}(P_1(x)) \| \leq K_b \| P_1(x) - G_{m_1}(P_1(x)) + y_1 - w_1 \|$$

$$\leq K_b K_{ws}(\tau) \sigma_{m_1}(P_1(x) - G_{m_1}(P_1(x)) + y_1)$$

$$\leq K_b K_{ws}(\tau) \| P_1(x) \| + K_b K_{ws}(\tau) \| y_1 \|$$

$$\leq K_b^2 K_{ws}(\tau) \| x \| + K_b K_{ws}(\tau) \| y_1 \|.$$

Analogously, we get that

$$\| P_2(x) - G_{m_2}(P_2(x)) \| \leq (1 + K_b)^2 K_{ws}(\tau) \| x \| + (1 + K_b) K_{ws}(\tau) \| y_2 \|.$$

Reasoning as before, we see that any weak thresholding set of cardinality $m_1$ for

$$P_1(x) + \tau^2(1 - \xi)(1 + \xi)^{-1} y_1$$

is contained in $\{ 1 \leq j \leq N_1 \}$. So taking $u_1$ an $m_1$-term Chebyshev $\tau$-greedy approximant for $P_1(x) + \tau^2(1 - \xi)(1 + \xi)^{-1} y_1$, we deduce that

$$\| y_1 \| = \tau^{-2}(1 - \xi)^{-1}(1 + \xi)^{-1} \| \tau^2(1 - \xi)(1 + \xi)^{-1} y_1 \|$$

$$\leq \tau^{-2}(1 - \xi)^{-1}(1 + \xi)(1 + K_b) \| P_1(x) - u_1 + \tau^2(1 - \xi)(1 + \xi)^{-1} y_1 \|$$

$$\leq \tau^{-2}(1 - \xi)^{-1}(1 + \xi)(1 + K_b) K_{ws}(\tau) \sigma_{m_1}(P_1(x) + \tau^2(1 - \xi)(1 + \xi)^{-1} y_1)$$

$$\leq \tau^{-2}(1 - \xi)^{-1}(1 + \xi)(1 + K_b) K_{ws}(\tau) \| P_1(x) \|$$

$$\leq \tau^{-2}(1 - \xi)^{-1}(1 + \xi)(1 + K_b) K_{ws}(\tau) \| x \|.$$

Similarly, we obtain

$$\| y_2 \| \leq \tau^{-2}(1 - \xi)^{-1}(1 + \xi)(1 + K_b) K_{ws}(\tau) \| x \|.$$
From the above estimations, and letting $\xi \to 0$, we deduce that

$$\| P_1(x) - G_{m_1}(P_1(x)) \| \leq (K_b^2 K_{wS}(\tau) + (1 + K_b) K_b^2 K_{wS}^2(\tau) \tau^{-2})\| x \|; \quad (30)$$

$$\| P_2(x) - G_{m_2}(P_2(x)) \| \leq ((1 + K_b)^2 K_{wS}(\tau) + (1 + K_b)^2 K_b K_{wS}^2(\tau) \tau^{-2})\| x \|. \quad (31)$$

Now suppose that $m_1 = 0$. Then $G_{m_1}(P_1(x)) = 0$, so (30) is clear, and we can obtain (31) by the same argument as before because $m_2 \leq N_1$. Finally, assume $m_2 = 0$. Then, (31) is clear. Now, if $m_1 < N_1$, we apply the same argument as before to obtain (30). On the other hand, if $m_1 = N_1$, then $G_{m_1}(P_1(x)) = P_1(x)$, so (30) is immediate.

To finish the proof, from (29), (30) and (31) we infer that

$$\| x - G_m(x) \| \leq (5K_b^2 K_{wS}(\tau) + 6K_b^3 K_{wS}(\tau)^2 \tau^{-2})\| x \|,$$

from where the upper bound for $K_{2q}$ is obtained. \qed

Note that in [12, Theorem 7.4] it was proved that any system $(x_i)_{1 \leq i \leq N}$ that is BSG($\tau$) is also superdemocratic with constant depending only on the basis constant, the BSG($\tau$) constant and $\tau$. A careful look at the proof shows that it is also valid for WSG($\tau$) systems. Also, the bounds in Theorem 2.2 are extracted from the proof of [11, Theorem 3.3] (with minor modifications for complex scalars), which is valid for finite dimensional spaces. Combining these results with Theorem 5.5, we obtain the following extension of [12, Theorem 7.7].

**Theorem 5.6** Let $N \in \mathbb{N}$ and let $E$ be an $N$-dimensional Banach space. Let $(x_i)_{1 \leq i \leq N} \subset E$ be a WSG($\tau$) system for $E$, $0 < \tau \leq 1$, with constant $K_{wS}$ and basis constant $K_b$. Then, $(x_i)_{1 \leq i \leq N} \subset E$ has almost greedy constant depending only on $K_{wS}$, $K_b$ and $\tau$.

Finally, we note that the branch thresholding algorithm can be and has been considered in infinite dimensional spaces as well. Indeed, in [12], the authors do so and prove that every weakly null semi-normalized branch quasi-greedy basic sequence has a quasi-greedy subsequence. If we extend the definitions of branch semi-greedy and branch almost greedy systems to the infinite dimensional context in the natural manner, it is immediate from the definitions that every semi-greedy system is branch semi-greedy, every branch semi-greedy system is weak semi-greedy, and the corresponding implications hold for the almost greedy case.

Thus, by Corollary 4.3, we obtain the following result, which answers a question raised in [12, page 3903].

**Corollary 5.7** Every branch semi-greedy Markushevich basis in an infinite dimensional Banach space is almost greedy and therefore, it is also semi-greedy.

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