Applications of the graphs to the Generalized Ornstein-Uhlenbeck process

Boubaker Smii\(^b\)\(^*\)

February 3, 2010

\(^b\) : King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
KFUPM Box 82, Dhahran 31261, Saudi Arabia
tel: +966530603701. Fax: +96638602340

Abstract

We consider the generalized Ornstein-Uhlenbeck equation \(\partial_t X = -m X_t + \eta\). In this paper we construct the Lévy noise \(\eta\). The generalized Ornstein-Uhlenbeck process \(X_t\) will be represented by a special types of graphs called rooted trees with two types of leaves.

**Key words:** Stochastic differential equations, Generalized Ornstein-Uhlenbeck processes, Lévy noise, Trees.

**MSC (2010):** 60H10, 05C05

1 Introduction

In this paper we study the generalized Ornstein-Uhlenbeck (OU) equation:

\[
\begin{align*}
\frac{\partial}{\partial t} X_t(x) &= -m X_t(x) + \eta(t) \\
X_0(x) &= \phi(x), \quad (t, \phi(x)) \in ]0, \infty[ \times \mathbb{R}^d
\end{align*}
\]

(1)

Where \(\eta = \eta(t)\) is a Lévy noise (more details will be given in section 2).

Generalized Ornstein-Uhlenbeck processes has been introduced by Barndorff-Nielsen (1998). However by replacing the continuous time \(t\) by a discrete one, we introduce the generalized random process, then by the use of the well known Bochner-Minlos theorem, see eg. [3], we construct the Lévy noise \(\eta\).

A graphical calculus will be applied to the generalized OU-process in this paper. In fact we will introduce a special types of graphs called rooted trees with two types of leaves, see.eg [4, 5] an analytic value will be given to each rooted trees and therefore a graphical representation of the OU-process will be recalled in this paper.

\(^*\)Electronic mail: boubaker@kfupm.edu.sa
The procedure of given each graph a numerical value is called “perturbation theory”. However our model of representing the solution of equation (1) in terms of rooted trees with two types of leaves is not restricted to such equation in fact one can generalize equation (1) and make it more complicated, e.g by introduction of non linear terms, e.g for force $F$ of gradient type, $F = \nabla V$ we obtain a non-linear SDE and one can ask the same questions as before, again expansion into graphs is possible. Generalizing further we pass from SDE’s driven by Lévy noise to SPDE’s.

Graphical representation of the OU process seems to be of a great importance in fact they can be used for modeling biological processes such as neuronal response, see e.g [7], also in Mathematical Finance, see e.g [6], the modeling of the dynamics of interest rates and volatilities of asset prices where each notions will have a specific graph representation.

The remainder of the article is organized as follows:
The next section will be reserved to the construction of Lévy noise, we will develop an interesting model which help the reader to understand the other sections of our paper.
In section 3 we recall a theorem which determine the solution of equation (1), the distribution of the generalized OU process will be given as well as a particular case when $\eta$ will be Gaussian noise.
In section 4 we introduce the rooted trees with two types of leaves. We will develop an algorithm which gives a numerical value to each rooted tree, in the last part of this section we will recall a graphical representation of the generalized OU process.

2 Construction of the Lévy noise

This section is devoted to the construction of the Lévy noise $\eta$, so far we introduce the Lévy characteristic $\psi$. By taking the time to be discrete the Fourier transform of the joint distribution of $\eta$ will converge to a function which depend of $\psi$, the generalized random process will be given as well. Hence by the use of the well known Bochner Minlos theorem we recall a theorem which gives the Lévy noise.

It is well known that an infinite divisible probability distribution $P$ is a probability distribution which satisfy the existence of a probability distribution $P_n$ such that $P = P_n \ast \cdots \ast P_n$ (n times). Let $P$ be an infinitely divisible probability distribution. By Lévy-Khinchine theorem (see e.g [3]), the Fourier transform, or characteristic function, of $P$ satisfies:

1. $C_P(t) = \int_\mathbb{R} e^{ixs}dP(s) = e^{\psi(t)}, t \in \mathbb{R}$, where $\psi$ is the Lévy characteristic of $P$ which is uniquely represented by , see. e.g Lukacs [12],

$$\psi(t) = iat - \frac{\sigma^2 t^2}{2} + z \int_{\mathbb{R} \setminus \{0\}} (e^{ist} - 1) \ dM(s), \ \forall \ t \in \mathbb{R}. \ (2)$$

with $M$ satisfies $\int_{\mathbb{R} \setminus \{0\}} \min(1, s^2)dM(s) < \infty$.

We can explain the meaning of (2) as follows:
If $\sigma^2 = M = 0$, then $\psi(t) = iat$ and the process $\{X(t), t \geq 0\}$ is simply deterministic motion, here $a$ is called the drift and it’s physical interpretation is the velocity of this motion.
Now if only $M = 0$ so that $\psi(t) = iat - \frac{\sigma^2 t^2}{2}$ and therefore $C_P(t) = e^{iat - \frac{\sigma^2 t^2}{2}}$ this is a characteristic function of a Gaussian random variable $X(t)$ having mean $ta$ and covariance $t\sigma$.

The process $\{X(t), t \geq 0\}$ in this case is known as the Brownian motion.
The last case is when $a \neq 0, \sigma \neq 0$ and $M \neq 0$, in this case the process $\{X(t), t \geq 0\}$ can be represented by $X(t) = bt + \sqrt{\sigma}B(t) + N(t)$ where $b$ is the drift, $B(t)$ the Brownian motion and
Generalized Ornstein-Uhlenbeck processes

$N(t)$ will be recognized as a poisson process with intensity $\lambda$ taking values in $\{nh \ n \in \mathbb{N}\}$.

In the following we denote by $F$ the probability distribution on $\mathbb{R}_+$ of a given stochastic process.

**Definition 2.1.** A Generalized stochastic process $\eta(t)$ is a stochastic process such that the following properties are satisfied:

- $F(\eta(t)) = F(\eta(s))$, $\forall \ t, s \geq 0$
- $\eta(t)$ is independent of $\eta(s)$, i.e $F(\eta(t)\eta(s)) = F(\eta(t)) \otimes F(\eta(s))$

From now on our stochastic process $\eta$ will be a generalized stochastic process and we would like to construct a generalized Lévy noise:

Let us start with a model, where the continuous time $t \in \mathbb{R}_+$ is replaced by discrete time $t \in \frac{1}{n}\mathbb{N}$. In this case one can thus model the noise $\{\eta(n)(t)\}_{t \in \frac{1}{n}\mathbb{N}}$ by a collection of i.i.d. random variables such that the "global random fluctuations" that the noise gives should not depend on the lattice scale $\frac{1}{n}$ fixed in our model, i.e. to model the noise independently of the lattice scale, we need that:

$$F(\eta(1)(1)) = F\left(\eta(n)\left(\frac{1}{n}\right) + \eta(n)\left(\frac{2}{n}\right) + \cdots + \eta(n)\left(\frac{n}{n}\right)\right) = \left(F(\eta(n)(1))\right)^n$$

Here $F(\eta(1)(1))$ is the random fluctuation of the unit interval.

**Definition 2.2.** The Fourier transform of $\eta(n)(t)$ is given by

$$\mathcal{F}(\eta(n))(f)(t) = e^{\psi(f(t))}n, f(t) \in \mathbb{R}^d$$

Where $\psi$ is the Lévy characteristic given in equation (2).

**Proposition 2.3.** The Fourier transform of the joint distribution of $\eta(n)(\frac{1}{n}), \eta(n)(\frac{2}{n}), \ldots$ converges to $e^{\int_{\mathbb{R}_+} \psi(f(t)) \ dt}$, as $n \rightarrow \infty$.

**Proof.** we have:

$$\mathcal{F}\left(\eta(n)\left(\frac{1}{n}\right), \eta(n)\left(\frac{2}{n}\right), \ldots \right) (f(\frac{1}{n}), f(\frac{2}{n}), \ldots) = \prod_{t \in \frac{1}{n}\mathbb{N}} e^{\frac{\psi(f(t))}{n}} = e^{\sum_{t \in \frac{1}{n}\mathbb{N}} \frac{\psi(f(t))}{n}}$$

It is now easy to see that the last expression converges to $e^{\int_{\mathbb{R}_+} \psi(f(t)) \ dt}$, as $n \rightarrow \infty$.

Now it turns out that, though the above limit exists, it does not give us the definition of $\eta(t)$ as a random function of $t$, as the limit:

$$F\left((\eta(n)(1))\right) = \delta_0$$

shows that the fluctuations of the random variable $\eta(n)(t)$ vanish for $n \rightarrow \infty$.

The solution of this apparent paradox lies in the fact, that $\eta(t)$ is not a stochastic process in the classical sense, but rather a distribution in $t$ then a function of $t$. 
Definition 2.4. A generalized random process or a random field is a mapping:
\[ \phi : (\Omega, \mathcal{B}, P) \to \mathcal{S}'(\mathbb{R}, \mathbb{R}^d) \]

such that \( \omega \to \langle \phi(\omega), f \rangle \) is measurable and for \( f_n \to f \) in \( \mathcal{S}(\mathbb{R}, \mathbb{R}^d) \), \( F(\phi(f_n)) \to F(\phi(f)) \) in low, i.e.

\[
\lim_{n \to \infty} \int_{\Omega} F(\phi(f_n)) \ dP = \lim_{n \to \infty} \int_{\mathbb{R}^d} F(x)F(\phi(f_n))(x) \ dx \\
= \int_{\mathbb{R}^d} F(x)F(\phi(f))(x) \ dx \\
= \int_{\Omega} F(\phi(f)) \ dP
\]

Theorem 2.5. (P. Lévy) Let \( X \) and \( X_n \), \( n \in \mathbb{N} \) be a stochastic processes then:
\( X_n \to X \) in low if and only if \( \mathcal{F}(X_n)(k) \to \mathcal{F}(X)(k) \), \( \forall k \in \mathbb{R}^d \).

Definition 2.6. Two random fields \( \phi_1 \) and \( \phi_2 \) are equivalent in low if all finite dimensional distributions of \( \phi_1 \) and \( \phi_2 \) coincide,
i.e. If \( \forall f_1, ..., f_n \in \mathcal{S}(\mathbb{R}, \mathbb{R}^d) \) and \( \forall A_1, ..., A_n \in \mathcal{B}(\mathbb{R}) \)

\[
P(\phi_1(f_1) \in A_1, ..., \phi_1(f_n) \in A_n) = P(\phi_2(f_1) \in A_1, ..., \phi_2(f_n) \in A_n)
\]

which means that:
\[
F(\phi_1(f_1) \cdots \phi_1(f_n)) = F(\phi_2(f_1) \cdots \phi_2(f_n)) \quad (6)
\]

But how can one construct such random field?
We note by \( \mathcal{B} \) the \( \sigma \)-algebra generated by the cylinder sets of \( \mathcal{S}'(\mathbb{R}^d) \). Then \( (\mathcal{S}'(\mathbb{R}^d), \mathcal{B}) \) is a measurable space.
We define a characteristic functional on \( \mathcal{S}(\mathbb{R}^d) \), as a functional \( C : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \) such that:

1. \( C(0) = 1 \).
2. \( C \) is continuous on \( \mathcal{S}(\mathbb{R}^d) \);
3. \( C \) is positive-definite, i.e. \( \forall z_1, ..., z_n \in \mathbb{C}, \forall n \in \mathbb{N}, f_1, ..., f_n \in \mathcal{S}(\mathbb{R}^d) \)

\[
\sum_{l, j=0}^{n} z_l \bar{z}_j C(f_l - f_j) \geq 0. \quad (7)
\]

In the following we choose \((\Omega, \mathcal{B}) = (\mathcal{S}', \mathcal{B}(\mathcal{S}')).\)
By the well known Bochner Minlos theorem, we know that for a given a Characteristic functional \( C \), there exist a unique (up to equivalence in low) random field \( \phi \) such that:
\[
C(f) = \mathbb{E}[e^{i\phi(f)}] = \int_{\Omega} e^{i\phi(f)(\omega)} \ dP(\omega), \forall f \in \mathcal{S}(\mathbb{R}, \mathbb{R}^d) \quad (8)
\]
Generalized Ornstein-Uhlenbeck processes

Theorem 2.7. Let \( \psi \) be a Lévy characteristic given by the representation (2) then there exist a unique measure \( \eta \) on \( S' \) such that:

\[
C(f) = \int_{S'} e^{i \langle \omega, f \rangle} d\eta(\omega) = \exp \left( \int_{\mathbb{R}^d} \psi(f(x)) \, dx \right), \quad x \in \mathbb{R}^d.
\] (9)

Proof. It is easy to see that the right hand side of (9) is a characteristic equation, now Bochner Minlos theorem concludes.

Definition 2.8. The measure \( \eta \) given by theorem (2.7) is called white measure with Lévy characteristic \( \psi \), and \( (S'(\mathbb{R}^d), \beta, \eta) \) will be the generalized white noise space associated with \( \psi \). The associated coordinate process \( \eta: S(\mathbb{R}^d) \times (S'(\Gamma), \beta, \eta) \rightarrow \mathbb{C}, \quad \eta(f, \xi) = \langle f, \xi \rangle \quad \forall f \in S(\mathbb{R}^d), \xi \in S'(\Gamma) \) (10)

is called a Lévy noise.

3 Generalized Ornstein-Uhlenbeck processes

In this section the generalized stochastic process \( \eta \) will be taken as a Lévy noise in the sense of the previous section. We recall a result which gives the distribution of the generalized OU process, this later is known as ”Mehler's Formula.” When there is no confusion we note \( X(t) \) by \( X_t \).

Definition 3.1. The stochastic process \( (X_t)_{t \geq 0} \) verifying equation (7), where \( \eta \) is a Lévy noise is called the generalized Ornstein-Uhlenbek (OU) process.

Theorem 3.2. The solution of the generalized OU- equation (7) is given by:

\[
X_t(x) = e^{-mt} \phi(x) + \int_0^t e^{-m(t-t')} \eta(t') \, dt'
\] (11)

Proof. The solution of the Homogeneous equation associated to equation (11) is given by \( X_{t,C} = C e^{-mt} \), now by the method of variation of parameters one thus get

\[
X_t(x) = \int_0^t e^{-m(t-t')} \eta(t') \, dt' + C e^{-mt}
\] (12)

The result now follows by taking \( X_0(x) = C = \phi(x) \).

The problem now is how to define the integral in theorem (3.2)?

Let \( e_j = (0, ..., 1, 0...) \in \mathbb{R}^d \), we put

\[
\chi_{m,t}^j(t') = e_j 1_{t' \leq t} e^{-m(t-t')}
\] (13)

and

\[
X^j(t) = e^{-mt} \phi^j + \eta(\chi_{m,t}^j)
\] (14)

This looks better, but \( \chi_{m,t}^j \) is not a test function in \( S(\mathbb{R}, \mathbb{R}^d) \).

Let

\[
\chi_{m,t,n}^j(t') = e_j \chi^n(t-t') e^{-m(t-t')}
\] (15)

Here \( \chi^n \) is a sequence of \( C^\infty \)-test functions that approximate the function from below.
Proposition 3.3. The sequences \( \{X^n_j\}_{n \geq 1} \) converges in \( L^2(\Omega, P) \), as \( n \to \infty \).

Proof.

\[
\mathbb{E} \left[ (X^{j}_{m,n}(t) - X^{j}_{m,n'}(t))^2 \right] = \mathbb{E} \left[ \eta \left( \chi^{j}_{m,t,n} - \chi^{j}_{m,t,n'} \right)^2 \right] = \alpha \int_0^\infty (\chi^{j}_{m,t,n} - \chi^{j}_{m,t,n'})^2 dt - \beta \left( \int_0^\infty (\chi^{j}_{m,t,n} - \chi^{j}_{m,t,n'}) dt \right)^2
\]

where \( \alpha, \beta \) are constants, now by the use of the dominated convergence theorem it is clear that the last expression goes to 0 as \( n \) goes to infinity.

Let \( F \) be the distribution of the generalized OU process, then the following result holds:

Theorem 3.4. The distribution \( F \) of the generalized OU process is given by:

\[
F(X_t) = F(e^{-mt}X_0) \ast F(\eta(\chi_{m,t}))
\]

(16)

Proof. The Fourier transform of \( X_t \) is given by:

\[
\mathcal{F}(X_t)(p) = \exp \left( i \langle X_0, p \rangle \exp^{-mt} + \int_0^t \psi(e^{-m(t-t')}p)dt' \right)
\]

\[
= \exp \left( i \langle X_0, p \rangle \exp^{-mt} + \int_0^t \psi(e^{-m(t-t')}p)dt' \right)
\]

\[
= \exp \left( i \langle X_0, p \rangle \exp^{-mt} + \int_0^t \psi(e^{-m(t-t')}p)dt' \right)
\]

Corollary 3.5. 1)- Let \( \eta \) be a Gaussian noise, then the distribution of the OU process is given by:

\[
F(X_t)(x) = (\det A)^{-\frac{1}{2}} \left( \frac{2\pi m}{1 - e^{-2mt}} \right)^{-\frac{d}{2}} \exp \left\{ -\frac{1}{2} \frac{m}{1 - e^{-2mt}} \langle (x - e^{-mt}X_0), D^{-1}(x - e^{-mt}X_0) \rangle \right\} dx
\]

where \( A = \left( \frac{D}{m}(1 - e^{-2mt}) \right)^{-1} \).

2)- The distribution of the Brownian motion is given by:

\[
F(B_t)(x) = (\det A)^{-\frac{1}{2}} (4\pi)^{-\frac{d}{2}} \exp \left\{ -\frac{1}{4t} \langle (x - X_0), D^{-1}(x - X_0) \rangle \right\} dx
\]

(17)

Proof. 1)- If the noise \( \eta \) is taken to be Gaussian noise, then \( \psi(k) = -\langle k, Dk \rangle \), thus:

\[
\int_0^t \psi(e^{-m(t-t')}k)dt' = -\langle k, Dk \rangle \int_0^t e^{-2m(t-t')}k dt'
\]

\[
= -\langle k, Dk \rangle \frac{1}{2m}(1 - e^{-2mt})
\]
Hence by applying the Fourier integral, one thus obtains:

\[ F(X_t)(x) = \frac{\sqrt{\det A}}{(2\pi)^{d/2}} \exp \left\{ -\frac{1}{2} \langle (x - e^{-mt} X_0), A(x - e^{-mt} X_0) \rangle \right\} dx \]

\[ = (\det A)^{-\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \exp \left\{ -\frac{m}{2} \langle (x - e^{-mt} X_0), D^{-1}(x - e^{-mt} X_0) \rangle \right\} dx \]

Here \( A = (\frac{2m}{m}(1 - e^{-2mt}))^{-1} \).

2) If \( m \to 0 \) the right hand side of equation (17) converges to

\[ (\det A)^{-\frac{1}{2}} (4\pi)^{-\frac{d}{2}} \exp \left\{ -\frac{m}{4} \langle (x - X_0), D^{-1}(x - X_0) \rangle \right\} dx \]

which the distribution of the Brownian motion.

**Remark 3.6.**
1. The expression of \( F(X_t) \) given by equation (17) is known as "Mehler’s Formula".

2. For the Gaussian case, i.e, \( \psi(k) = -\langle k, Dk \rangle \) one thus obtain the Heat equation:

\[ \begin{cases} \frac{\partial P_t(x)}{\partial t} = \Delta P_t(x) \\ P_0(x) = f(x) \end{cases} \]

3. The formula given by equation (18) is the semigroup of the Brownian motion. Such a formula for the non Gaussian case is missing!

If now \( \psi(k) \) is taken to be of the more general type, i.e, \( \psi(k) \) is a Lévy-Khinchine function, see e.g [3], one thus get:

\[ \frac{\partial}{\partial t} P_t(x) = -\sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} P_t(x) + \sum_{j,l=1}^{d} D_{jl} \frac{\partial^2}{\partial x_j \partial x_l} P_t(x) \\
+ z \int_{\mathbb{R}^d \setminus \{0\}} [P_t(x+y) - P_t(x)] dr(y). \]

Here \( a \in \mathbb{R}^d \), \( D \) is a real and positive semidefinite \( d \times d \) matrix, \( z \geq 0 \) and \( r \) is a probability measure on \( \mathbb{R}^d \setminus \{0\} \) s.t. its Fourier transform is entire analytic. The well-known interpretation of \( a \) is the drift vector, \( D \) determines the diffusion part, whereas \( z \) and \( r \) give the frequency and distribution of jumps, respectively.

\( \psi \) is called the generator of the (distribution of the) Lévy process and the function \( k \to \psi(ik) \) is called the symbol of \( \psi \).

## 4 graph expansion of the generalized OU process

This section will be reserved to the graph applications to equations of type (1). We recall a graphical representation of the generalized OU process by a special type of graphs called rooted
trees. Let us start with some useful tools which help the reader to understand our formalism. Let \( G(t, x) \) be the Green function which satisfies:

\[
\begin{align*}
\frac{\partial G(t, x)}{\partial t} &= -mG(t, x) + \delta(x), \\
G(t, x) &= 0, \quad t < 0
\end{align*}
\]  

(22)

Here \( \delta(x) \) is the Dirac distribution.

**Proposition 4.1.** The analytic solution of the generalized OU equation (1) is given by:

\[ X_t = G \ast \eta(x) + G \ast f(x) \]  

(23)

**Proof.** From the generalized OU equation we obtain

\[ (\frac{\partial}{\partial t} + m) X_t(x) = \eta(x) \]  

(24)

Now the result hold by applying the Green function given by equation (22).

---

**Definition 4.2.** A tree is a graph in which any two vertices are connected by exactly one simple path. In other words, any connected graph without cycles is a tree. A tree is called a rooted tree if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root.

Each vertex of a tree \( T \) connected with only one edge is called leaf.

In our model we consider a rooted tree \( T \) with two types of leaves. The set of all trees with two types of leaves and \( i \) inner vertices will be denoted by \( \mathcal{T}(i) \).

Now it’s of a great importance to ask about a method which gives an analytic value to each rooted tree \( T \in \mathcal{T}(i) \). The following algorithm named as "Feynman rule" gives a numerical value to each rooted tree with two types of leaves:

**Definition 4.3.** Let \( R(T, \eta) \) be the analytic value of a rooted tree \( T \in \mathcal{T}(i) \), then \( R \) is obtained as follows:

- Assign \( r = (t, x) \) to the root of the tree \( T \).
- For each edge of the tree \( T \) multiply by \( G \), where \( G \) is the green function given by equation (22).
- For each leaf of type one multiply by the noise \( \eta \).
- For each leaf of type two multiply by the initial condition \( f \).
- Integrate with respect to the Lebesgue measure \( dx \).

Definitions (4.2) and (4.3) can be summarized by the following table and graph:

**Theorem 4.4.** Let \( T_j \in \mathcal{T}(i), j \in \mathbb{N} \). The solution of the generalized OU equation (1) is given by a sum over all rooted trees that are evaluated according to the rule given in definition 4.3, i.e.

\[ X_t = \sum_{T_j \in \mathcal{T}(i)} R(T_j, \eta) \]  

(25)

---

1The various kinds of trees used as data structures in computer science are not really trees in this sense, but rather, types of ordered directed trees.
Table 1: Different types of vertices.

| root | inner vertices | leaf 1 | leaf 2 | edge |
|------|----------------|--------|--------|------|
| ×    | ●             | ○      | ⊗      | →    |

Figure 1: Construction of a rooted tree with two types of leaves and one inner vertex.

Proof. From proposition (4.1) one thus get:

\[
X_t = G \ast \eta(x) + G \ast f(x) \\
= \times - \otimes + \times - \diamond \\
= \mathcal{R}(T_1, \eta) + \mathcal{R}(T_2, \eta)
\]  

(26)

The graphical representations introduced previously can be applied for more complicated OU equations, e.g. by replacing the right hand side of equation (1) by \(-m X^p + \eta\), in this case our graph formalism still true and the rooted trees will contain more inner vertices, hence in the rule given by definition (4.3) we will integrate over all vertices.

References

[1] S. Albeverio, J-L. Wu and T. S. Zhang: *Parabolic SPDEs driven by Poisson white noise*. Stoch. Proc. Appl. 74 (1998), 21-36.

[2] D. Applebaum and J.L. Wu: *Stochastic partial differential equations driven by Lévy space time white noise*. Random Ops. and Stochastic equations. 8, 245-61 (2000).

[3] D. Applebaum: *Lévy Processes and stochastic Calculus*. Cambridge University Press 2004.

[4] S. Boubaker, *Application of the generalized Feynman graphs to stochastic equations driven by Lévy noise*, Phd Thesis, University of Bonn, Germany, 2006.

[5] S. Boubaker, *Graphical representation of the grand canonical partition function*, http://arxiv.org/math-ph/1001.1042 (2010).

[6] O.E Brandorff-Nielsen and N. Shephard, *Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics*. J. R. Stat. Soc. Ser. B Stat. Methodol. 63, no. 2, 167-241.

[7] K. Fukushima, *Cognitron: A Self-Organizing Multilayered Neural Network*, Biological Cybernetics 20: 121-136 (1975). doi:10.1007/BF00342633
[8] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions 4. Some applications of Harmonic Analysis*, Academic Press, 1964.

[9] H. Gottschalk, Boubaker, S and H. Thaler, *The Feynman graph representation of general convolution semigroups and its applications to Lévy statistics*. Journal of Bernoulli Society, 14(2), 322-351 (2008).

[10] H. Gottschalk and Boubaker, S, *How to determine the law of the solution to a SPDE driven by a Lévy space-time noise*, Journal of Mathematical Physics v.43. 1-22 (2007).

[11] H. Gottschalk, Boubaker, S and H. Ouerdiane, *Convolution calculus on white noise spaces and Feynman graph representation of generalized renormalization flows*. 101-111 pp. icmarp (2005).

[12] E. Lukacs, *Characteristic Functions, 2nd ed*, Griffin, 1970.

[13] Oystein Ore, *Theory of graphs*. American Mathematical Society, 190 Hope Street, Providence, Rhode Island 1962

[14] K.-I. Sato, *Lévy processes and infinite divisibility*. Cambridge University Press, 1999.

[15] E. Taufer and N. Leonenko, *Simulation of Lévy-driven Ornstein-Uhlenbeck process with given marginal distribution*. Com. Stat.and Data An. Elsevier, V. 53(6), 2427-2437 (2009).