ENDPOINT STRICHARTZ ESTIMATE FOR THE DAMPED WAVE EQUATION AND ITS APPLICATION

TAKAHISA INUI AND YUTA WAKASUGI

Abstract. Recently, the Strichartz estimates for the damped wave equation was obtained by the first author [5] except for the wave endpoint case. In the present paper, we give the Strichartz estimate in the wave endpoint case. We apply the argument of Keel–Tao [6]. However, it does not follow directly from [6] since the solution map is not unitary unlike wave equation. Moreover, we apply the endpoint Strichartz estimate to the unconditional uniqueness for the energy critical nonlinear damped wave equation.

Contents

1. Introduction 2
  1.1. Background 2
  1.2. Main results 3
2. Endpoint Strichartz estimate 8
  2.1. Stationary phase method 8
  2.2. Bilinear estimate and the proof of the endpoint Strichartz estimates 9
3. Local well-posedness when $d \geq 6$ 13
  3.1. Function spaces 13
  3.2. Nonlinear estimates 14
  3.3. The proof of L.W.P 16
4. Unconditional uniqueness 17
  4.1. Paraproduct estimates 17
  4.2. Proof of the unconditional uniqueness 20
Appendix A. Besov type Strichartz estimates 26
Appendix B. Proof of Lemma 4.1 29
Appendix C. An estimate of Besov norm for Hölder continuous functions 30
Acknowledgment 31
References 31

Date: March 15, 2019.

2010 Mathematics Subject Classification. 35L71; 35A02.

Key words and phrases. damped wave equation, endpoint Strichartz estimates, unconditional uniqueness.
1. Introduction

1.1. Background. We consider the damped wave equation.

\[ \begin{align*}
\frac{\partial_t^2 \phi}{\Delta} - \Delta \phi + \partial_t \phi &= 0, & (t, x) &\in (0, \infty) \times \mathbb{R}^d, \\
(\phi(0), \partial_t \phi(0)) &= (\phi_0, \phi_1), & x &\in \mathbb{R}^d,
\end{align*} \]

where \( d \in \mathbb{N} \), \((\phi_0, \phi_1)\) is given, and \( \phi \) is an unknown complex valued function. Matsumura [9] applied the Fourier transform to (1.1) and obtained the formula

\[ \phi(t, x) = \mathcal{D}(t)(\phi_0 + \phi_1) + \partial_t \mathcal{D}(t)\phi_0, \]

where \( \mathcal{D}(t) \) is defined by

\[ \mathcal{D}(t) := e^{-\frac{t}{2}} \mathcal{F}^{-1} L(t, \xi) \mathcal{F} \]

with

\[ L(t, \xi) := \begin{cases} \frac{\sin(t \sqrt{1/4 - |\xi|^2})}{\sqrt{1/4 - |\xi|^2}} & \text{if } |\xi| \leq 1/2, \\ \frac{\sin(t \sqrt{|\xi|^2 - 1/4})}{\sqrt{|\xi|^2 - 1/4}} & \text{if } |\xi| > 1/2. \end{cases} \]

Matsumura [9] also obtained an \( L^p - L^q \) type estimate. After his work, many researchers showed \( L^p - L^q \) type estimates (see e.g. [11, 3, 10, 13, 4] and references therein).

In [14], Watanabe discussed the Strichartz estimates for (1.1) when \( d = 2, 3 \) and its application to nonlinear problem. Recently, the first author [5] obtained the following homogeneous and inhomogeneous Strichartz estimates.

**Proposition 1.1** (Homogeneous Strichartz estimates). Let \( d \geq 2 \), \( 2 \leq r < \infty \), and \( 2 \leq q \leq \infty \). Set \( \gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d}{2} \} \). Assume

\[ \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \geq \frac{1}{q}, \]

Then, we have

\[ \begin{align*}
\|\mathcal{D}(t) f\|_{L^q_t(L^r_x(\mathbb{R}^d))} &\lesssim \|\nabla \|^{\gamma - 1} f\|_{L^2}, \\
\|\partial_t \mathcal{D}(t) f\|_{L^q_t(L^r_x(\mathbb{R}^d))} &\lesssim \|\nabla \|^{\gamma} f\|_{L^2}, \\
\|\partial_t^2 \mathcal{D}(t) f\|_{L^q_t(L^r_x(\mathbb{R}^d))} &\lesssim \|\nabla \|^{\gamma + 1} f\|_{L^2}.
\end{align*} \]

**Proposition 1.2** (Inhomogeneous Strichartz estimates). Let \( d \geq 2 \), \( 2 \leq r, \tilde{r} < \infty \), and \( 2 \leq q, \tilde{q} \leq \infty \). We set \( \gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d}{2} \} \) and \( \tilde{\gamma} \) in the same manner. Assume that \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfies

\[ \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) + \frac{d}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) > \frac{1}{q} + \frac{1}{\tilde{q}}, \]

or

\[ (q, r) = (\tilde{q}, \tilde{r}) = (\infty, 2). \]
Moreover, we exclude the end-point case, that is, we assume \((q, r) \neq (2, 2(d - 1)/(d - 3))\) and \((\tilde{q}, \tilde{r}) \neq (2, 2(d - 1)/(d - 3))\) when \(d \geq 4\). Then, we have
\[
\left\| \int_0^t D(t - s)F(s)ds \right\|_{L^q_t(I;L^r_x(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta - 1} F \right\|_{L^{q'}_t(I;L^{r'}_x(\mathbb{R}^d))},
\]
\[
\left\| \int_0^t \partial_t D(t - s)F(s)ds \right\|_{L^q_t(I;L^r_x(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta} F \right\|_{L^{q'}_t(I;L^{r'}_x(\mathbb{R}^d))},
\]
where \(\delta = 0\) when \(\frac{1}{q} (1/2 - 1/r) = \frac{1}{\tilde{q}} (1/2 - 1/\tilde{r})\) and in the other cases \(\delta \geq 0\) is defined in the Table 1 below.

| \(\delta\) | \(\frac{1}{q} (\frac{1}{2} - \frac{1}{r}) < \frac{1}{\tilde{q}} (\frac{1}{2} - \frac{1}{\tilde{r}})\) | \(\frac{1}{q} (\frac{1}{2} - \frac{1}{r}) > \frac{1}{\tilde{q}} (\frac{1}{2} - \frac{1}{\tilde{r}})\) |
|---|---|---|
| \(d - \frac{1}{2} (\frac{1}{2} - \frac{1}{r}) \geq \frac{1}{q}\) | 0 | 0 |
| \(d - \frac{1}{2} (\frac{1}{2} - \frac{1}{r}) \geq \frac{1}{q}\) | \(\times\) | \(\frac{1}{q} \left\{ \frac{1}{q} - \frac{d - 1}{2} (\frac{1}{2} - \frac{1}{r}) \right\}\) |
| \(d - \frac{1}{2} (\frac{1}{2} - \frac{1}{r}) < \frac{1}{q}\) | \(\frac{1}{q} \left\{ \frac{1}{q} - \frac{d - 1}{2} (\frac{1}{2} - \frac{1}{r}) \right\}\) | \(\times\) |
| \(d - \frac{1}{2} (\frac{1}{2} - \frac{1}{r}) \geq \frac{1}{q}\) | \(\frac{1}{q} \left\{ \frac{1}{q} - \frac{d - 1}{2} (\frac{1}{2} - \frac{1}{r}) \right\}\) | \(\times\) |
| \(d - \frac{1}{2} (\frac{1}{2} - \frac{1}{r}) < \frac{1}{q}\) | \(\frac{1}{q} \left\{ \frac{1}{q} - \frac{d - 1}{2} (\frac{1}{2} - \frac{1}{r}) \right\}\) | \(\frac{1}{q} \left\{ \frac{1}{q} - \frac{d - 1}{2} (\frac{1}{2} - \frac{1}{r}) \right\}\) |

Table 1. The value of \(\delta\). (\(\times\) means that the case does not occur.)

Roughly speaking, the Strichartz estimates of the damped wave equation is a combination of those of heat and wave. Indeed, the assumption of the exponents \((q, r)\) and \((\tilde{q}, \tilde{r})\) comes from the low frequency part of the solution map \(D(t)\), which behaves like the heat equation. On the other hand, The derivative losses \(\gamma\) and \(\tilde{\gamma}\) come from the high frequency part, which behaves like the wave equation with exponential time decay. See the previous work [5].

We remark that the homogeneous Strichartz estimate holds in the wave endpoint case, i.e., \((q, r) = (2, 2(d - 1)/(d - 3))\) when \(d \geq 4\), since the high frequency part of the homogeneous term can be reduced to the wave solution map and its endpoint estimate is known by Keel–Tao [6]. However, it has not been clear whether the inhomogeneous Strichartz estimates hold or not in the wave endpoint case since its high frequency part of the inhomogeneous term can not be reduced to the inhomogeneous term of the wave equation.

In the present paper, we are interested in the endpoint Strichartz estimates and its application to a nonlinear problem.

### 1.2. Main results
First, we give the endpoint Strichartz estimate.
We need to treat the exponential term \( e^{2t} \) defined in the table 1 above. Proposition 1.1 and 1.4. These Besov type Strichartz estimates are useful to analyze nonlinear problems.

Theorem 1.3 (Endpoint Strichartz estimate). Let \( d \geq 4 \) and we assume \((q, r) = (2, (d-1)/(d-3))\). Then, we have

\[
\left\| \int_0^t \mathcal{D}(t-s) F(s) ds \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma+\delta-1} F \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))},
\]

\[
\left\| \int_0^t \partial_t \mathcal{D}(t-s) F(s) ds \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma+\delta} F \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))},
\]

where \( \gamma := d(1/2 - 1/r) - 1/q \).

The low frequency part of the inhomogeneous term can be treated easily since it behaves like the heat equation. The difficulty comes from the high frequency part, which corresponds to the wave equation with exponential term related to time. We apply the argument of Keel–Tao [6] to the high frequency part. However, it is not applicable directly since, roughly, we need to estimate \( \int_0^t e^{-(t-s)/2} e^{(t-s) \langle \nabla \rangle} F(s) ds \).

Combining this endpoint Strichartz estimate and the argument to prove Proposition 1.2, we obtain the following general Strichartz estimates containing the wave endpoint case.

Proposition 1.4. Let \( d \geq 2, \ 2 \leq r, \tilde{r} < \infty, \text{ and } 2 \leq q, \tilde{q} \leq \infty \). We set \( \gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\} \) and \( \tilde{\gamma} \) in the same manner. Assume that \((q, r) \) and \((\tilde{q}, \tilde{r}) \) satisfies

\[
\frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) + \frac{d}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) > \frac{1}{q} + \frac{1}{\tilde{q}},
\]

\[
\frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) + \frac{d}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } 1 < \tilde{q}' < q < \infty,
\]

or

\((q, r) = (\tilde{q}, \tilde{r}) = (\infty, 2)\).

(We may take \((q, r) = (2, 2(d-1)/(d-3))\) or \((\tilde{q}, \tilde{r}) = (2, 2(d-1)/(d-3))\) when \(d \geq 4 \).) Then, we have

\[
\left\| \int_0^t \mathcal{D}(t-s) F(s) ds \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma+\delta-1} F \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))},
\]

\[
\left\| \int_0^t \partial_t \mathcal{D}(t-s) F(s) ds \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma+\delta} F \right\|_{L_t^q(I; L_x^r(\mathbb{R}^d))},
\]

where \( \delta = 0 \) when \( \frac{1}{q}(1/2 - 1/r) = \frac{1}{\tilde{q}}(1/2 - 1/\tilde{r}) \) and in the other cases \( \delta \geq 0 \) is defined in the table 1 above.

By the Littlewood–Paley decomposition, we get the Besov type estimate from Proposition 1.1 and 1.4. These Besov type Strichartz estimates are useful to analyze nonlinear problems.
**Proposition 1.5** (Besov type homogeneous Strichartz estimates). Let $s \in \mathbb{R}$. Assume that $(q, r)$ satisfies the assumptions in Proposition 1.1. Let $\gamma$ be as in Proposition 1.1. Then we have the following Besov type homogeneous Strichartz estimates.

$$
\|D(t) f\|_{L^q(I; B^s_{r,2}(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma - 1} f \right\|_{B^s_{r,2}},
$$

$$
\|\partial_t D(t) f\|_{L^q(I; B^s_{r,2}(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma} f \right\|_{B^s_{r,2}},
$$

$$
\|\partial^2_t D(t) f\|_{L^q(I; B^s_{r,2}(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + 1} f \right\|_{B^s_{r,2}},
$$

where $I \subset \mathbb{R}$ is a time interval.

**Proposition 1.6** (Besov type inhomogeneous Strichartz estimates). Let $s \in \mathbb{R}$. Assume that $(q, r)$ and $(\tilde{q}, \tilde{r})$ satisfy the assumptions in Proposition 1.4. Let $\gamma$ and $\tilde{\gamma}$ be as in Proposition 1.4 and $\delta$ be defined in Table 1. Then we have the following Besov type inhomogeneous Strichartz estimates.

$$
\left\| \int_0^t D(t-s) F(s) ds \right\|_{L^q(I; B^s_{r,2}(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta - 1} F \right\|_{L^\tilde{q}(I; B^s_{\tilde{r},2}(\mathbb{R}^d))},
$$

$$
\left\| \int_0^t \partial_t D(t-s) F(s) ds \right\|_{L^q(I; B^s_{r,2}(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta} F \right\|_{L^\tilde{q}(I; B^s_{\tilde{r},2}(\mathbb{R}^d))},
$$

where $I \subset \mathbb{R}$ is a time interval.

In the present paper, we also discuss the application of the endpoint Strichartz estimates to a nonlinear problem. We consider the following energy critical nonlinear damped wave equation.

\[ (\text{NLDW}) \quad \begin{cases} 
\partial^2_t u - \Delta u + \partial_t u = |u|^4 u, & (t, x) \in [0, T) \times \mathbb{R}^d, \\
(u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d,
\end{cases} \]

where $d \geq 3$, $(u_0, u_1)$ is given, and $u$ is an unknown complex valued function. In the previous paper [5] (see also [14]), we show the local well-posedness of (NLDW) when $3 \leq d \leq 5$. In this paper, we will show the unconditional uniqueness of the solution to (NLDW) in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ when $d \geq 6$. In these local well-posedness, we needed auxiliary function spaces to prove the uniqueness. We will show the unconditional uniqueness of the solution to (NLDW) in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ when $d \geq 4$. Namely, we will remove those auxiliary spaces applying the endpoint Strichartz estimates. We give the precise definition of the solution to (NLDW).

**Definition 1.1** (solution). We say that $u$ is a solution to (NLDW) on a time interval $I$ with $0 \in I$ if $u$ satisfies $(u, \partial_t u) \in C(I : H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ and the Duhamel formula

$$
u(t, x) = D(t)(u_0 + u_1) + \partial_t D(t)u_0 + \int_0^t D(t-s)(|u(s)|^4 u(s))ds\]$$

in the sense of tempered distributions for every $t \in I$.

**Remark 1.1.** We emphasize that the solutions may not belong to the Strichartz spaces. This definition is different from Definition 1.1 in [5].

First, we show the local well-posedness when $d \geq 6$, which was not treated in [5].
Theorem 1.7 (L.W.P when \(d \geq 6\)). Let \(d \geq 6\) and \(T \in (0, \infty]\). Let \((u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) satisfy \(\|u_0, u_1\|_{H^1 \times L^2} \leq A\). Then, there exists \(\delta = \delta(A) > 0\) such that if

\[
\|D(t)(u_0 + u_1) + \partial_t D(t)u_0\|_{L_{t,x}^{2(d+1)/(d-2)}}([0,T]) \leq \delta,
\]

then there exists a solution \(u\) to (NLDW) with \(\|u\|_{L^{2(d+1)/(d-2)}([0,T])} \leq 2\delta\). Moreover, we have the standard blow-up criterion, that is, if the maximal existence time \(T_+ = T_+(u_0, u_1)\) is finite, then the solution satisfies \(\|u\|_{L^{2(d+1)/(d-2)}([0,T_+])} = \infty\).

We give the unconditional uniqueness of the solution to (NLDW) in the energy space \(H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\).

Theorem 1.8 (unconditional uniqueness). Let \(d \geq 4\). Let \(u, v\) be solutions (in the sense of Definition 1.1) to (NLDW) with the initial data \((u_0, u_1), (v_0, v_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\), respectively. If \((u_0, u_1) = (v_0, v_1)\), then we have \(u = v\).

Remark 1.2. The unconditional uniqueness when \(d = 3\) remains since its endpoint Strichartz estimate is an open problem.

For the corresponding energy critical nonlinear wave equation, the local well-posedness when \(d \geq 6\) and the unconditional uniqueness when \(d \geq 4\) was obtained by Bulut et.al. [2]. The proof of the local well-posedness when \(d \geq 6\) depends on exotic Strichartz estimates and the proof of the unconditional uniqueness relies on paraproduct estimates for the homogeneous Besov spaces. Though our argument is based on their argument, the homogeneous Besov spaces do not match the damped wave equation. We need the inhomogeneous Sobolev and Besov spaces and thus, in particular, we need to show paraproduct estimates for the inhomogeneous Besov spaces.

Notation. For the exponent \(p\), we denote the Hölder conjugate of \(p\) by \(p'\). The bracket \((\cdot)\) is Japanese bracket, i.e., \((a) := (1 + |a|^2)^{1/2}\).

We use \(A \lesssim B\) to denote the estimate \(A \leq CB\) with some constant \(C > 0\). The notation \(A \sim B\) stands for \(A \lesssim B\) and \(A \lesssim B\).

For a function \(f : \mathbb{R}^n \to \mathbb{C}\), we define the Fourier transform and the inverse Fourier transform by

\[
\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx,
\]

\[
\mathcal{F}^{-1}[f](x) = (2\pi)^{-n/2} \int_{\mathbb{R}^d} e^{ix\xi} \hat{f}(\xi) d\xi.
\]

For a measurable function \(m = m(\xi)\), we denote the Fourier multiplier \(m(\nabla)\) by

\[
m(\nabla)f(x) = \mathcal{F}^{-1} \left[ m(\xi) \hat{f}(\xi) \right](x).
\]

For \(s \in \mathbb{R}\) and \(1 \leq p \leq \infty\), we denote the usual Sobolev space by

\[
W^{s,p}(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : \|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p} < \infty \right\}.
\]

We write \(H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)\) for simplicity. Let \(\dot{W}^{s,p}(\mathbb{R}^d)\) and \(\dot{H}^s(\mathbb{R}^d)\) denote the corresponding homogeneous Sobolev spaces.
For a time interval $I$ and $F : I \times \mathbb{R}^d \to \mathbb{C}$, we set
\[ \|F\|_{L^q(I; L^r(\mathbb{R}^d))} := \left( \int_I \|F(t, \cdot)\|_{L^r(\mathbb{R}^d)}^q \, dt \right)^{1/q} \]
and $L^q_{t,x}(I) := L^q(I : L^r(\mathbb{R}^d))$. We sometimes use $L^p_x$ and $L^p_t$ to uncover time variables $s$ and $t$.

We define the Littlewood–Paley decomposition as follows. Let $\chi$ be a radial nonnegative $C^\infty$-function supported in $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{25}{24}\}$ with $\chi(\xi) = 1$ on the unit ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For $a > 0$, we define
\[ \chi_{\leq a}(\xi) := \chi \left( \frac{\xi}{a} \right) \quad \text{and} \quad \chi_{> a}(\xi) := 1 - \chi_{\leq a}(\xi). \]

For each integer $j \in \mathbb{Z}$, we set
\[ \widehat{\Delta_{\leq j}f}(\xi) := \chi_{\leq 2^j}(\xi) \hat{f}(\xi), \quad \widehat{\Delta_{> j}f}(\xi) := \chi_{> 2^j}(\xi) \hat{f}(\xi), \quad \widehat{\Delta_j f}(\xi) = \chi_{2^j}(\xi) \hat{f}(\xi) := (\chi_{\leq 2^j}(\xi) - \chi_{< 2^j}(\xi)) \hat{f}(\xi). \]

We also define
\[ \widehat{\Delta_{< j}f}(\xi) := \chi_{< 2^{j-1}} \hat{f}(\xi), \quad \widehat{\Delta_{> j}f}(\xi) := \chi_{> 2^{j-1}} \hat{f}(\xi), \quad \widehat{\Delta_j f}(\xi) := \widehat{\Delta_{\leq j}f}(\xi) - \widehat{\Delta_{< j}f}(\xi) = (\chi_{\leq 2^j}(\xi) - \chi_{< 2^j}(\xi)) \hat{f}(\xi). \]

for $j < l$. Moreover we assume that $\{\chi_j\}_{j \in \mathbb{Z}}$ give a dyadic partition of unity as follows.
\[ \Delta_{\leq 0} + \sum_{j=1}^{\infty} \Delta_j = \text{Id on } \mathcal{S}' \text{ and } \sum_{j \in \mathbb{Z}} \Delta_j = \text{Id on } \mathcal{S}' \cap \mathcal{S}''. \]

We also set $\hat{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$ and correspondingly, $\hat{\chi}_{2^j}(\xi) = \chi_{2^j}(\xi) + \chi_{2^j}(\xi) + \chi_{2^{j+1}}(\xi)$.

For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, we define inhomogeneous Besov norm by
\[ \|f\|_{B^s_{p,q}(\mathbb{R}^d)} := \|\Delta_{\leq 0} f\|_{L^p} + \|2^sj\|\Delta_j f\|_{L^p}\|_{L^q} \|\Delta_{\leq j} f\|_{L^q} \leq 1 \|_{L^q} \]
and inhomogeneous Besov space by
\[ B^s_{p,q}(\mathbb{R}^d) := \{ f \in \mathcal{S}' : \|f\|_{B^s_{p,q}(\mathbb{R}^d)} < \infty \}. \]

We denote the homogeneous space of a space-time function space $\mathcal{X}$ by $\dot{\mathcal{X}}$, that is, we replace the inhomogeneous Sobolev space or the inhomogeneous Besov spaces by their homogeneous spaces and we do not change the Lebesgue space.

This paper is structured as follows. Section 2 is devoted to show the endpoint Strichartz estimates. Especially, the $L^\infty - L^1$ estimate is proven in Section 2.1 and the bilinear estimates and the endpoint Strichartz estimate are given in Section 2.2. In Section 3, we prove the local well-posedness of (NLDW) when $d \geq 6$. In Section 4, we show the unconditional uniqueness of (NLDW). In Appendix, we collect some lemmas and we give the proof of the Besov type Strichartz estimates.
2. ENDPOINT STRICHARTZ ESTIMATE

We apply the method of Keel and Tao [6] to prove the endpoint Strichartz estimate. In this section we use $N \in 2^\mathbb{Z}$ to denote the dyadic numbers for spatial variables in order to distinguish dyadic numbers for time variable, which are denoted by $2^t$, and them. We set $P_N = \Delta_j$ for $N = 2^j$. $P_{>N}$, $P_{<N}$ e.t.c. are defined in the same way.

2.1. Stationary phase method. First, we give an $L^\infty$-$L^1$ estimate for the high frequency part of the linear solution.

**Lemma 2.1.** The following estimate is valid for $t > 0$.

\begin{equation}
\left\| e^{it\Delta} \right\|_{L^\infty} \lesssim (1 + tN)^{-\frac{d-1}{2}} N^d \| \partial_t \|_{L^1}.
\end{equation}

**Proof.** The proof is very similar to the $L^\infty$-$L^1$ estimate for the wave equation. See e.g. [7, Lemma 2.1]. However, we give a proof for the reader’s convenience. We only treat the case of $N \geq 1$ since $P_N P_{>N} = 0$ when $N \leq 1/2$.

**Case 1.** First, we assume “$d = 1$” or “$d \geq 2$ and $t \lesssim N^{-1}$”. By the Bernstein inequality, we obtain

\begin{equation}
\left\| e^{it\Delta} \right\|_{L^\infty} \lesssim N^{\frac{d}{2}} \| \partial_t \|_{L^2} \lesssim N^d \| \partial_t \|_{L^1}.
\end{equation}

When “$d = 1$” or “$d \geq 2$ and $|t| \leq N^{-1}$”, we have $(1 + tN)^{-\frac{d+1}{2}} \gtrsim 1$. Thus, we obtain the desired estimate.

**Case 2.** We assume that $d \geq 2$ and $t \gg N^{-1}$. Since $P_N \tilde{P}_N = 1$, it follows that

\begin{equation}
\left\| e^{it\Delta} \right\|_{L^\infty} \lesssim N^{\frac{d+1}{2}} \| \partial_t \|_{L^2} \lesssim N^d \| \partial_t \|_{L^1}.
\end{equation}

Therefore, it is enough to show

\begin{equation}
\left\| e^{it\Delta} \right\|_{L^\infty} \lesssim N^\frac{d+1}{2} \| \partial_t \|_{L^1}.
\end{equation}

by the Young inequality. We set

\begin{align}
I & := \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{it \sqrt{\xi^2 - 1/4}} \chi_{>1}(\xi) \tilde{\chi}_N(\xi) \, d\xi \\
& \lesssim N^d \int_{\mathbb{R}^d} \int_0^\infty \left| \int_{\mathbb{R}^d} e^{it \sqrt{(N\tau)^2 - 1/4}} \chi_{>1}(N\tau) \tilde{\chi}_1(r) \, d\tau \right| \, dr \sigma(\omega) \\
& = N^d \int_{\mathbb{R}^d} \int_0^\infty e^{it \sqrt{(N\tau)^2 - 1/4}} \sigma(N\tau) \chi_{>1}(N\tau) \tilde{\chi}_1(r) \, d\tau \, dr,
\end{align}

where $\sigma(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix \cdot \omega} \, d\omega$.

**Case 2-1.** We consider the case of $|x| \gtrsim t$. Since $|\sigma(x)| \lesssim \langle x \rangle^{-\frac{d}{2}}$ (see [7, Lemma 2.2]), it follows from (2.3), $|x| \gtrsim t$, $\chi_{>1} \lesssim 1$, and $t \gg N^{-1}$ that

\begin{align}
|I| & \lesssim N^d \int_0^\infty e^{it \sqrt{(N\tau)^2 - 1/4}} \langle N\tau |x| \rangle^{-\frac{d}{2}} \chi_{>1}(N\tau) \tilde{\chi}_1(r) \, d\tau \\
& \lesssim N^\frac{d+1}{2} \langle x \rangle^{-\frac{d}{2}} \int_0^\infty \chi_{>1}(N\tau) \tilde{\chi}_1(r) \, d\tau \\
& \lesssim N^\frac{d+1}{2} \langle x \rangle^{-\frac{d}{2}} N^d.
\end{align}
Case 2.2. We consider $|x| \ll t$. We use the formula in (2.2). Let $\phi(r) := t\sqrt{(Nr)^2 - 1/4 + N\omega}$. Then, we have $\phi'(r) = tN^2r/\sqrt{(Nr)^2 - 1/4 + N\omega}$ and thus $|\phi'(r)| \geq Nt$ since $t \gg |x|$. When $N \geq 2^r$, by $\chi_{N}(Nr)x_1(x) = \overline{x_1}(r)$, $e^{i\phi(r)} = (i\phi'(r))^{-1}\partial_r e^{i\phi(r)}$ and the integration by parts $k$ times, we get
\[
|I| \lesssim N^d t^{-k} \lesssim N^{d+1} t^{-\frac{d+1}{r}},
\]
where we take $k = (d-1)/2$ if $d$ is odd and $k = d/2$ if $d$ is even and we use $t \gg N^{-1}$. When $N = 1, 2$, by the integration by parts $k$ times, in the same way as above, we obtain
\[
|I| \leq C_N t^{-\frac{d+1}{r}} \leq \max_{N=1,2} \{C_N\} t^{-\frac{d+1}{r}}.
\]
Therefore, for $N \geq 1$, we get
\[
|I| \lesssim N^{d+1} t^{-\frac{d+1}{r}} \lesssim (1 + tN)^{-\frac{d+1}{r}} N^d
\]
since $t \gg N^{-1}$. This completes the proof. \hfill \Box

Combining (2.1) and
\[
\left\| e^{it/\sqrt{-\Delta - 1/4}} P_{N^*} P_N f \right\|_{L^r} = \| P_N f \|_{L^r},
\]
we get the following $L^r$-$L^{r'}$ estimate by the interpolation.

Corollary 2.2. Let $2 \leq r \leq \infty$. We have
\[
\left\| e^{it/\sqrt{-\Delta - 1/4}} P_{N^*} P_N f \right\|_{L^r} \lesssim (1 + tN)^{-\frac{d-1}{2r}} N^{\frac{d}{r}} \| P_N f \|_{L^{r'}},
\]
for $t > 0$.

2.2. Bilinear estimate and the proof of the endpoint Strichartz estimates.

We estimate the space-time norm of the inhomogeneous term:
\[
J := \left\| \int_0^\infty e^{-\frac{t}{\sqrt{\Delta + 1/4}}} e^{(t-s)\sqrt{\Delta + 1/4}} P_{N^*} P_N F(s) ds \right\|_{L^q(1,1')}.
\]
If $N \leq 1/2$, $J = 0$ since $P_{N^*} P_N = 0$. Thus, we only consider the case of $N \geq 1$. First, we treat the case of $N \geq 2$. It follows from the duality that
\[
J = \left\| \int_0^\infty e^{-\frac{t}{\sqrt{\Delta + 1/4}}} e^{(t-s)\sqrt{\Delta + 1/4}} P_{N^*} P_N F(s) ds \right\|_{L^q(1,1')} = \sup_{\|G\|_{L^{q'}(1,1')}} \left| \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty e^{-\frac{t-s}{\sqrt{\Delta + 1/4}}} e^{(\tau-s)\sqrt{\Delta + 1/4}} P_{N^*} P_N F(s, x) G(\tau, x) ds d\tau dx \right|
\]
where we set
\[
T(F, G) := \int_0^\infty \int_0^\infty e^{-\frac{t-s}{\sqrt{\Delta + 1/4}}} \left\langle e^{-is\sqrt{-\Delta - 1/4}} P_{N} F(s), e^{-i\tau\sqrt{-\Delta - 1/4}} P_{N^*} P_N G(\tau) \right\rangle_{L^2} ds d\tau.
\]
Set $W(t) := e^{it/\sqrt{-\Delta - 1/4}} P_{N^*}$ for simplicity. Since we have $P_N = P_{N^*} P_N$ from $N \geq 2$, it follows that
\[
T(F, G) = \int_0^\infty \int_0^\infty e^{-\frac{t-s}{\sqrt{\Delta + 1/4}}} \left\langle W^*(s) P_N F(s), W^*(\tau) P_N G(\tau) \right\rangle_{L^2} ds d\tau,
\]
Then, we have
\[ I(\tau, s) := e^{-\frac{s-x}{\gamma}} \left\langle W^*(s) P_N F(s), W^*(\tau) \tilde{P}_N G(\tau) \right\rangle_{L^2_x}, \]
for \( \tau \geq s \).

**Lemma 2.3.** For \( r \in [2, \infty] \), we have
\[ |I(\tau, s)| \lesssim e^{-\frac{s-x}{\gamma}} \{1 + (\tau-s)N\}^{-\frac{d-1(r-2)}{2r}} N^{\frac{d(r-2)}{r}} \| P_N F(s) \|_{L^r} \left\| \tilde{P}_N G(\tau) \right\|_{L^{r'}}, \]
for \( \tau \geq s \).

**Proof.** By the Hölder inequality and Corollary 2.2, we have
\[ |I(\tau, s)| \lesssim e^{-\frac{s-x}{\gamma}} \|W(\tau-s) P_N F(s)\|_{L^r} \left\| \tilde{P}_N G(\tau) \right\|_{L^{r'}}, \]
\[ \lesssim e^{-\frac{s-x}{\gamma}} \{1 + (\tau-s)N\}^{-\frac{d-1(r-2)}{2r}} N^{\frac{d(r-2)}{r}} \| P_N F(s) \|_{L^r} \left\| \tilde{P}_N G(\tau) \right\|_{L^{r'}}, \]
for \( \tau \geq s \).

From this, we get the following bilinear estimate for non-endpoint admissible pairs.

**Lemma 2.4.** Let \((q, r) \neq (2, 2(d-1)/(d-3))\). We have
\[ |T(F, G)| \lesssim N^{2q} \|P_N F\|_{L^{q'}(I; L^r(\mathbb{R}^d))} \left\| \tilde{P}_N G \right\|_{L^{q'}(I; L^r(\mathbb{R}^d))} \]

**Proof.** From Lemma 2.3 and the Hölder inequality for \( \tau \), we obtain
\[ |T(F, G)| \]
\[ \leq \int_0^\infty \int_0^\tau |I(\tau, s)| ds d\tau \]
\[ \lesssim \int_0^\infty \int_0^\tau e^{-\frac{s-x}{\gamma}} \{1 + (\tau-s)N\}^{-\frac{d-1(r-2)}{2r}} N^{\frac{d(r-2)}{r}} \| P_N F(s) \|_{L^r} \left\| \tilde{P}_N G(\tau) \right\|_{L^{r'}} ds d\tau \]
\[ \lesssim N^{\frac{d(r-2)}{r'}} \int_0^\tau e^{-\frac{s-x}{\gamma}} \{1 + (\tau-s)N\}^{-\frac{d-1(r-2)}{2r}} \| P_N F(s) \|_{L^r} ds \left\| \tilde{P}_N G \right\|_{L^{q'} L^r L^{r'}}, \]
where we used the Young inequality or the Hardy–Littlewood–Sobolev inequality in the last. See [5, Lemma 2.6] for the detail.

For \( l \in \mathbb{Z} \), we set
\[ T_l(F, G) := \int_0^\infty \int_{[0, \tau] \cap [\tau - 2^l + 1, \tau - 2^l]} I(\tau, s) ds d\tau. \]
Then, we have
\[ \sum_{l \in \mathbb{Z}} T_l(F, G) = T(F, G). \]
For simplicity, we set \( K_l = K_l(\tau) := [0, \tau] \cap [\tau - 2^l + 1, \tau - 2^l] \). Let \( d \geq 4 \) and \((q^*, r^*) = (2, 2(d-1)/(d-3))\).
Lemma 2.5. Let $2 \leq a, b \leq \infty$ and $\beta(a, b) := -1 + \frac{d-1}{2}(1 - \frac{1}{a} - \frac{1}{b})$. For $(1/a, 1/b)$ near $(1/r^*, 1/r^*)$, the following is valid.

$$|T_l(F, G)| \lesssim 2^{-l\beta(a, b)} N^{\gamma(a) + \gamma(b)} ||P_N F||_{L^2_x L^{\gamma}_{t'}} ||\widetilde{P}_N G||_{L^2_x L^{\gamma}_{t'}},$$

where $\gamma(a) = d(1/2 - 1/a) - 1/q(a)$ and $1/q(a) = \frac{d-1}{2}(1/2 - 1/a)$.

Proof. By the interpolation, it is enough to show the inequality in the following cases.

1. $a = b = \infty$.
2. $2 \leq a < r^*$ and $b = 2$.
3. $2 \leq b < r^*$ and $a = 2$.

(1). By Lemma 2.3 as $r = \infty$, $\tau - s \approx 2l$ in $K_l$ and the Cauchy–Schwarz inequality, we get

$$|T_l(F, G)| \leq \int_0^\infty \int_{K_l} |I(\tau, s)| ds d\tau \lesssim \int_0^\infty \int_{K_l} \{(\tau - s)N\}^{-\frac{d+1}{2}} N^d ||P_N F(s)||_{L^1} ||\widetilde{P}_N G(\tau)||_{L^1} ds d\tau \lesssim 2^{-\frac{d+1}{4}l} N^{\frac{d+1}{4}} \int_0^\infty \int_{K_l} ||P_N F(s)||_{L^1} ds \cdot ||\widetilde{P}_N G(\tau)||_{L^1} d\tau \lesssim 2^{-\frac{d+1}{4}l} N^{2\gamma(\infty)} ||H||_{L^2} \cdot ||\widetilde{P}_N G||_{L^2_x L^1_{t'}},$$

where we set

$$H(\tau) := \int_{K_l(\tau)} ||P_N F(s)||_{L^1} ds.$$

Here, we have

$$||H||^2_{L^2} = \int_0^\infty \left( \int_{K_l(\tau)} ||P_N F(s)||_{L^1} ds \right)^2 d\tau \lesssim 2^l \int_0^\infty \int_{K_l(\tau)} ||P_N F(s)||_{L^1}^2 ds d\tau = 2^l \int_0^\infty \int_{[s, \infty] \cap [s+2^l, s+2^{l+1}]} d\tau ||P_N F(s)||_{L^1}^2 ds \lesssim 2^{2l} ||P_N F(s)||^2_{L^1_x L^1_{t'}}.$$

Therefore, we obtain

$$|T_l(F, G)| \lesssim 2^{-l\beta(1,1)} N^{2\gamma(\infty)} ||P_N F(s)||_{L^2_x L^1_{t'}} \cdot ||\widetilde{P}_N G||_{L^2_x L^1_{t'}}.$$
(2). We have

\[ |T_i(F, G)| = \int_0^\infty \int_0^\infty \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]}(s) I(\tau, s) dsd\tau \]

\[ \leq \int_0^\infty \int_0^\infty e^{-\frac{a}{\tau^2}} W^*(s) \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]}(s) P_N F(s) dsd\tau \]

\[ \leq \int_0^\infty \int_0^\infty e^{-\frac{a}{\tau^2}} W^*(s) \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]}(s) P_N F(s) d\tau \]

where we use the Cauchy–Schwarz inequality in the last inequality. Here, by the

Hence, we get

where \( (p, q) \) is the non-endpoint admissible pair and \( \gamma(a) := d(1/2 - 1/a) - 1/q(a) \)

(see [5, Lemma 6.1]). Thus, by the Cauchy–Schwarz inequality, we get

\[ |T_i(F, G)| \lesssim N^{\gamma(a)} \int \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]} P_N F(s) ds \]

\[ \lesssim N^{\gamma(a)} \left( \int \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]} P_N F(s) ds \right)^{1/2} \]

Let \( \alpha \) satisfy \( 1/2 + 1/\alpha = 1/q(a) \). Then, by the Hölder inequality, we obtain

\[ \left( \int \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]} P_N F(s) ds \right)^{1/2} \]

\[ \leq 2^{\frac{1}{\alpha}} \left( \int \mathbb{1}_{[\tau-2^{i+1}, \tau-2^i]} P_N F(s) ds \right)^{1/2} \]

\[ = 2^{\frac{1}{\alpha}} \left( \int \mathbb{1}_{[s+2^{i+1}, s+2^i]} P_N F(s) ds \right)^{1/2} \]

\[ \lesssim 2^{\frac{1}{\alpha}} \left( \int \mathbb{1}_{[s+2^{i+1}, s+2^i]} P_N F(s) ds \right)^{1/2} \]

\[ \leq 2^{\frac{1}{\alpha}} \left( \int \mathbb{1}_{[s+2^{i+1}, s+2^i]} P_N F(s) ds \right)^{1/2} \]

\[ \leq 2^{\frac{1}{\alpha}} \left( \int \mathbb{1}_{[s+2^{i+1}, s+2^i]} P_N F(s) ds \right)^{1/2} \]

Hence, we get

\[ |T_i(F, G)| \lesssim N^{\gamma(a) + \gamma(2)} 2^{-l_\beta(\alpha, 2)} \left\| P_N F \right\|^2_{L^2_t L^{q'(a)'}_{x'}} \left\| \tilde{P}_N G \right\|^2_{L^2_t L^q_{x'}}. \]

(3). By the symmetry, we get the statement in the same way as (2).}

By Lemma 2.5 and the interpolation lemma (see [1, p.76, Exercises 5. (b)] or Lemma 6.1 in [6]), we obtain the endpoint Strichartz estimates for \( N \geq 2 \). We
omit the details. See Keel–Tao [6] and Machihara–Nakanishi–Ozawa [8] for more precise argument. For \( N = 1 \), we only use the estimate in [5, Remark 10]. Since \( N \) is bounded, so we do not need above complicated calculations. Combining these estimates and using Littlewood–Paley decomposition, we get Theorem 1.3. Applying duality argument and the Bernstein inequality as in [5], we obtain Proposition 1.4.

We can also get the Besov type Strichartz estimates. See Appendix A for the proof of Besov type Strichartz estimates, Propositions 1.5 and 1.6.

### 3. Local well-posedness when \( d \geq 6 \)

We give the local well-posedness when \( d \geq 6 \) by using the exotic Strichartz estimates.

#### 3.1. Function spaces

For \( d \geq 6 \), we define the function spaces as follows.

\[
\|u\|_{S(I)} := \|u\|_{L^\infty_{t,x}}(I),
\]

\[
\|u\|_{X(I)} := \|u\|_{L^{d/2} W^{2} x}^{\frac{2d+1}{d-1}}(I),
\]

\[
\|u\|_{X'(I)} := \|u\|_{L^{d/2} W^{3} x}^{\frac{2d+1}{d+3}}(I),
\]

\[
\|u\|_{Y(I)} := \|u\|_{L^{\frac{2d^3-7d^2+9d}{d^3-11d^2+11d-6}} W^{\frac{d^2-4d+2}{d^3-11d^2+11d-6}} x}^\frac{d^3-14d^2+18d}{d^3-11d^2+11d-6}(I),
\]

\[
\|u\|_{W(I)} := \|u\|_{L_{t}^{\frac{d+1}{2}} B^{\frac{d+1}{d+3}}_{d+3}(I)},
\]

\[
\|u\|_{W'(I)} := \|u\|_{L_{t}^{\frac{d+1}{2}} B^{\frac{d+1}{d+3}}_{d+3}(I)},
\]

\[
\|u\|_{S^1(I)} := \max \left\{ \|u\|_{L_{t}^{\frac{2d^3-7d^2+9d}{d^3-11d^2+11d-6}} B^{\frac{d^2-4d+2}{d^3-11d^2+11d-6}} x}^\frac{d^3-14d^2+18d}{d^3-11d^2+11d-6}, \|u\|_{L_{t}^{\frac{2d^3-7d^2+9d}{d^3-11d^2+11d-6}} B^{\frac{d^2-4d+2}{d^3-11d^2+11d-6}} x}^\frac{d^3-14d^2+18d}{d^3-11d^2+11d-6}, \|u\|_{W(I)} \right\}.
\]

We remark that the norms in the definition of \( S^1(I) \) have the form \( L_{t}^{q} B_{r}^{1-\gamma(r)} \) whose \((q, r)\) satisfies \( 1/q = \frac{d+1}{2} (1/2 - 1/r) \) and \( \gamma(r) = \frac{d+1}{2} (1/2 - 1/r) \), i.e., \((q, r)\) is a wave admissible pair.

Since we have \( B_{p,2}^{s} \leftrightarrow F_{p,2}^{s} \approx W^{s,p} \) if \( p \geq 2 \), we get

\[
\|u\|_{Y(I)} \lesssim \|u\|_{S^1(I)}.
\]

Moreover, by the Sobolev embedding and the embedding \( B_{p,2}^{s} \hookrightarrow W^{s,p} \) for \( p \geq 2 \), we get

\[
\|u\|_{X(I)} \lesssim \|u\|_{L_{t}^{\frac{d+1}{d}} W^{\frac{d^2-4d+2}{d^3-3d^2}} x}^\frac{d^3-14d^2+18d}{d^3-3d^2} \lesssim \|u\|_{S^1(I)}.
\]

We have the following interpolation.

**Lemma 3.1.** Let \( d \geq 6 \) and \( I \subseteq \mathbb{R} \) be any time interval. Then, we have the following.
For the first term, we have the following estimate by \[2, \text{Lemma 2.10}\].

\[
\|u\|_{X(I)} \lesssim \|u\|_{S(I)}^{\theta_1} \|u\|_L^{1-\theta_1} W_x^{\frac{2d-4}{d^2-4d-4}} \lesssim \|u\|_{S(I)}^{\theta_1} \|u\|_{L^\infty H^1_x}^{1-\theta_1}. 
\]

(2). Let \( \theta_2 = \frac{d}{d^2-3d-4} \).

\[
\|u\|_{S(I)} \lesssim \|u\|_{X(I)}^{\theta_2} \|u\|_{Y(I)}^{1-\theta_2} \lesssim \|u\|_{X(I)}^{\theta_2} \|u\|_{W(I)}^{1-\theta_2}.
\]

(3). Let \( \theta_3 = \frac{1}{2(d-3)} \).

\[
\|u\|_{L^\infty}^{2(d+1)} \|u\|_{X(I)}^{\frac{2d+1}{2(d+1)}} \lesssim \|u\|_{X(I)}^{\theta_3} \|u\|_{Y(I)}^{1-\theta_3} \lesssim \|u\|_{X(I)}^{\theta_3} \|u\|_{S(I)}^{1-\theta_3}.
\]

Proof. The statement for the homogeneous spaces are obtained by \[2, \text{Lemma 10}\]. The desired statement can be obtained in the same way as for the homogeneous spaces.

3.2. Nonlinear estimates. We collect some nonlinear estimates.

Lemma 3.2 (Estimates for difference). Let \( 1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 \), \( 1 < p_i < \infty \) for \( i = 1, 2, 3, 4 \). Assume that a function \( F \in C^{1,\alpha}(\mathbb{R} : \mathbb{R}) \) for some \( 0 < \alpha < 1 \) and that \( F'(0) = 0 \). Then, we have

\[
\|F(u) - F(v)\|_{B^{\frac{1}{p_1},2}_{p_1}} \lesssim \|u - v\|_{B^{\frac{1}{p_1},2}_{p_1}} \|u\|_{L^{p_2}}^{\alpha} + \|u - v\|_{L^{p_3}}^{\alpha} \|v\|_{B^{\frac{1}{p_4},2}_{p_4}}^{\frac{1}{p_4}}.
\]

Proof. We have

\[
\|F(u) - F(v)\|_{B^{\frac{1}{p_1},2}_{p_1}} \approx \|F(u) - F(v)\|_{B^{\frac{1}{p_1},2}_{p_1}} + \|F(u) - F(v)\|_{L^p}.
\]

For the first term, we have the following estimate by \[2, \text{Lemma 10}\].

\[
\|F(u) - F(v)\|_{B^{\frac{1}{p_1},2}_{p_1}} \lesssim \|u - v\|_{B^{\frac{1}{p_1},2}_{p_1}} \|u\|_{L^{p_2}}^{\alpha} + \|u - v\|_{L^{p_3}}^{\alpha} \|v\|_{B^{\frac{1}{p_4},2}_{p_4}}^{\frac{1}{p_4}}.
\]

For the second term, we have the following from the Hölder inequality and the mean value theorem.

\[
\|F(u) - F(v)\|_{L^p} \lesssim \|u - v\|_{L^{p_1}} \|u\|_{L^{p_2}}^{\alpha} + \|u - v\|_{L^{p_3}}^{\alpha} \|v\|_{L^{p_4}}.
\]

Combining them, we get the desired statement.
Lemma 3.3 (Nonlinear estimates). Let $F(u) = \pm |u|^\frac{4}{d-2} u$. Then, the following are true.

\[\|F(u)\|_{W^r(I)} \lesssim \|u\|_{X(I)}^{\theta_2 \frac{4}{d-2}} \|u\|_{S^1(I)}^{(1-\theta_2) \frac{4}{d-2} + 1},\]

\[\|F(u)\|_{X'(I)} \lesssim \|u\|_{X(I)}^{\theta_2 \frac{4}{d-2} + 1} \|u\|_{S^1(I)}^{(1-\theta_2) \frac{4}{d-2}},\]

\[\left\|\nabla \frac{t}{2} F'(u)\right\|_{L_t^\frac{4}{d+2} L_x^{d^3+d^2+2d+2}(I)} \lesssim \|u\|_{X(I)}^{\frac{4}{d-2}} W_{\frac{2(d+1)}{d-2}}^{\frac{2(d+1)}{d-2} - \frac{2(d+1)}{d-2}},\]

\[\|F(u) - F(v)\|_{X'(I)} \lesssim \|u - v\|_{X(I)} \left(\|u - v\|_{S^1(I)} + \|v\|_{S^1(I)}\right)\|u - v\|_{S^1(I)}^{1-\theta_2} \|v\|_{S^1(I)}^{\theta_2} \|u - v\|_{S^1(I)}^{\frac{4}{d-2}},\]

\[\|F(u) - F(v)\|_{W^r(I)} \lesssim \|u - v\|_{W^r(I)} \left(\|u - v\|_{S^1(I)} + \|v\|_{S^1(I)}\right)\|u - v\|_{S^1(I)}^{1-\theta_2} \|v\|_{S^1(I)}^{\theta_2} \|u - v\|_{S^1(I)}^{\frac{4}{d-2}} + \|u - v\|_{S^1(I)}^{\frac{4}{d-2}} \|v\|_{S^1(I)}^{\frac{4}{d-2}},\]

Proof. By the embedding $W^{s,p} \approx F_{p,2} \hookrightarrow B_{p,2}$ for $p \leq 2$, the fractional chain rule, the embedding $B_{p,2} \hookrightarrow W^{s,p}$ for $p \geq 2$, and Lemma 3.1 (2), we know

\[\|F(u)\|_{W^r(I)} \lesssim \|F(u)\|_{L_t^\frac{4}{d+2} W_{\frac{2(d+1)}{d-2}}^{\frac{2(d+1)}{d-2} - \frac{2(d+1)}{d-2}}},\]

\[\|F(u)\|_{X'(I)} \lesssim \|u\|_{S^1(I)}^{\frac{4}{d-2}} \|u\|_{X(I)}^{1-\theta_2} \|u\|_{S^1(I)}^{\frac{4}{d-2}},\]

We have

\[\|F(u)\|_{X'(I)} \lesssim \|u\|_{S^1(I)}^{\frac{4}{d-2}} \|u\|_{X(I)}^{1-\theta_2} \|u\|_{S^1(I)}^{\frac{4}{d-2}},\]

where we used the fractional chain rule and Lemma 3.1 (2).

It is known by [2] that

\[\left\|\nabla \frac{t}{2} F'(u)\right\|_{L_t^\frac{4}{d+2} L_x^{d^3+d^2+2d+2}(I)} \lesssim \|u\|_{X(I)}^{\frac{4}{d-2}} W_{\frac{2(d+1)}{d-2}}^{\frac{2(d+1)}{d-2} - \frac{2(d+1)}{d-2}},\]

It is enough to show that

\[\|F'(u)\|_{L_t^\frac{4}{d+2} L_x^{d^3+d^2+2d+2}(I)} \lesssim \|u\|_{X(I)}^{\frac{4}{d-2}} W_{\frac{2(d+1)}{d-2}}^{\frac{2(d+1)}{d-2} - \frac{2(d+1)}{d-2}},\]

This follows from the Hölder inequality.
By [2, Lemma 2.11], we have
\[
\|F(u) - F(v)\|_{X'(I)} \lesssim \|u - v\|_{X(I)} \left( \|u - v\|_{S(I)}^{\frac{4}{5}} + \|v\|_{S(I)}^{\frac{4}{5}} \right)
\]
\[+ \|u - v\|_{X(I)} \left( \|u - v\|_{S(I)} + \|v\|_{S(I)} \right)^{\frac{4}{5}} \times \left( \|u - v\|_{S(I)}^{\frac{4}{5}} \right)^{\frac{4}{5}} \]
\[\cdot \left( \|u - v\|_{S(I)}^{1 - \theta_3} \right)^{\frac{4}{5}} \left( \|v\|_{S(I)}^{\theta_3} \right)^{\frac{4}{5}} \left( \|v\|_{Y(I)}^{1 - \theta_3} \right)^{\frac{4}{5}}.\]

We get these Lebesgue-type inequality by the Hölder inequality and thus we get the desired inequalities coming from Lemma 3.2.

□

3.3. The proof of L.W.P. We prove the local well-posedness when \(d \geq 6\).

Proof. By Lemma 3.1 (1), we have
\[
\|S(t)(u_0, u_1)\|_{X(I)} \lesssim \|S(t)(u_0, u_1)\|_{S(I)}^{\theta_1} \|S(t)(u_0, u_1)\|_{L^\infty(I; H^1)}^{1 - \theta_1}
\]
\[\lesssim \|S(t)(u_0, u_1)\|_{S(I)}^{\theta_1} A^{1 - \theta_1},\]
where \(S(t)(u_0, u_1) = D(t)(u_0 + u_1) + \partial_t D(t)u_0\). Let \(\delta := C \|S(t)(u_0, u_1)\|_{S(I)}^{\theta_1} A^{1 - \theta_1}\) and set

\[\Phi(u) = \Phi[u_0, u_1](u) := S(t)(u_0, u_1) + \int_0^t D(t - s)(|u(s)|^\frac{4}{5} u(s)) ds.\]

Take \(u \in \{ u : \|u\|_{X(I)} \leq a, \|u\|_{S(I)} \leq b \}\). Then, it follows from the Strichartz estimates and Lemma 3.3 that

\[
\|\Phi(u)\|_{X(I)} \leq \delta + \left\| \int_0^t D(t - s)(|u(s)|^\frac{4}{5} u(s)) ds \right\|_{X(I)}
\]
\[\leq \delta + C \left\| |u|^{\frac{2}{5}} u \right\|_{X(I)}
\]
\[\leq \delta + C \left\| |u|^{\frac{2}{5}} u \right\|_{X(I)}^{\frac{1}{1 - \theta_2}} \left\| u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}}
\]
\[\leq \delta + C \left\| |u|^{\frac{2}{5}} u \right\|_{X(I)}^{\frac{1}{1 - \theta_2}} b^{(1 - \theta_2)\frac{2}{5}}
\]
and

\[
\|\Phi(u)\|_{S(I)} \leq \|S(t)(u_0, u_1)\|_{S(I)} + \left\| \int_0^t D(t - s)(|u(s)|^\frac{4}{5} u(s)) ds \right\|_{S(I)}
\]
\[\leq CA + C \left\| |u|^{\frac{2}{5}} u \right\|_{W^d(I)}
\]
\[\leq CA + C \left\| |u|^{\frac{2}{5}} u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}} b^{(1 - \theta_2)\frac{2}{5} + 1}
\]
\[\leq CA + C \left\| |u|^{\frac{2}{5}} u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}} b^{(1 - \theta_2)\frac{2}{5} + 1}.
\]

Therefore, taking \(a = 2\delta\) and \(b = 2CA\) and choosing sufficiently small \(\delta\) such that \(CA \left\| |u|^{\frac{2}{5}} u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}} \leq 1/2\), it follows that

\[\|\Phi(u)\|_{X(I)} \leq \delta + C \left\| |u|^{\frac{2}{5}} u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}} b^{(1 - \theta_2)\frac{2}{5}} \leq a,
\]
and

\[\|\Phi(u)\|_{S(I)} \leq CA + C \left\| |u|^{\frac{2}{5}} u \right\|_{S(I)}^{\frac{1}{1 - \theta_2}} b^{(1 - \theta_2)\frac{2}{5} + 1} \leq b.\]
Thus, \( \Phi \) is a mapping on \( \{ u : \| u \|_{X(I)} \leq a, \| u \|_{S_1(I)} \leq b \} \). For \( u, v \in \{ u : \| u \|_{X(I)} \leq a, \| u \|_{S_1(I)} \leq b \} \), by the Strichartz estimate and Lemma 3.3, we have
\[
\| \Phi(u) - \Phi(v) \|_{X(I)} \lesssim \| F(u) - F(v) \|_{X(I)}
\]
\[
\quad \lesssim \| u - v \|_{X(I)} \left( \| u - v \|_{S_1(I)} + \| v \|_{S_1(I)} \right)
\]
\[
\quad + \| u - v \|_{X(I)} \left( \| u - v \|_{S_1(I)} + \| v \|_{S_1(I)} \right) \frac{2^\theta_3}{\theta_3} \int_1^{2^\theta_3} \frac{1}{\theta_3} \frac{\| \phi_3 \|_{Y(I)}}{\| v \|_{Y(I)}}
\]
\[
\quad =: I + II.
\]
It follows from Lemma 3.1 (2) that
\[
I \lesssim \| u - v \|_{X(I)} \left( \| u - v \|_{S_1(I)}^{(1-\theta_2)} \| v \|_{S_1(I)}^{1-\theta_2} \right)
\]
\[
\lesssim \| u - v \|_{X(I)} a^{\theta_2, b^{1-\theta_2}} (a^{\theta_3, b^{1-\theta_3}})^{\frac{1}{\theta_3}}.
\]
We also have
\[
II \lesssim \| u - v \|_{X(I)} (a^{\theta_2, b^{1-\theta_2}}) \frac{2^{\theta_3}}{\theta_3} (a^{\theta_3, b^{1-\theta_3}})^{\frac{1}{\theta_3}}.
\]
Therefore, if \( \delta \) (i.e. \( a \)) is small, then \( \Phi \) is a contraction map on \( \{ u : \| u \|_{X(I)} \leq a, \| u \|_{S_1(I)} \leq b \} \) with the distance \( d(u, v) := \| u - v \|_{X(I)} \). From the contraction mapping principle, we get the unique solution. \( \square \)

4. Unconditional uniqueness

4.1. Paraproduct estimates. We show paraproduct estimates for the inhomogeneous Besov spaces.

We have an equivalence of Besov norms. See Appendix B for the proof.

Lemma 4.1. Let \( 1 < p < \infty, 1 \leq q \leq \infty, \) and \( s > 0 \). Then it holds that
\[
\| f \|_{B^s_{p,q}} \sim J \| \Delta \leq f \|_{L^p} + \| 2^j \Delta \|_{L^p} \| \Delta \|_{L^q},
\]
\[
\| f \|_{B^{-s}_{p,q}} \sim J \| \Delta \leq f \|_{L^p} + \| 2^{-j} \Delta \|_{L^p} \| \Delta \|_{L^q},
\]
where the implicit constants may depend on \( J \in \mathbb{Z}_{\geq 0} \).

We decompose the product of two functions \( f \) and \( g \) in the following way.
\[
f g = \Delta_{\leq 0}(fg) + \sum_{j=1}^{\infty} \Delta_j(fg)
\]
\[
= \Delta_{\leq 0}((\Delta_{\leq 3} f) g) + \Delta_{\leq 0}((\Delta_{>3} f) g)
\]
\[
+ \sum_{j=1}^{\infty} \Delta_j((\Delta_{\leq j+3} f) g) + \sum_{j=1}^{\infty} \Delta_j((\Delta_{>j+3} f) g)
\]
\[
= \Delta_{\leq 0}((\Delta_{\leq 3} f) g) + \Delta_{\leq 0}((\Delta_{>3} f) \Delta_{>1} g)
\]
\[
+ \sum_{j=1}^{\infty} \Delta_j((\Delta_{\leq j+3} f) g) + \sum_{j=1}^{\infty} \Delta_j((\Delta_{>j+3} f) \Delta_{>j+1} g).\]
We set
\begin{equation}
G_1(f, g) := \Delta_{\leq 0}((\Delta_{\leq 3} f) g) + \sum_{j=1}^{\infty} \Delta_j((\Delta_{\leq j+3} f) g),
\end{equation}
\begin{equation}
G_2(f, g) := \Delta_{\leq 0}((\Delta_{> 3} f) \Delta_{> 1} g) + \sum_{j=1}^{\infty} \Delta_j((\Delta_{> j+3} f) \Delta_{> j+1} g).
\end{equation}

Namely, we have the identity \( f g = G_1(f, g) + G_2(f, g) \). We prepare the following paraproduct estimates.

**Lemma 4.2** (paraproduct estimates). Let \( 1 < p_i < \infty \), \( i = 1, 2, \cdots, 8 \), \( s > 0 \), and \( \sigma > 0 \). Then, we have
\begin{equation}
\|G_1(f, g)\|_{B_{p_1, p_2}^{-s}} \lesssim \|f\|_{B_{p_1, p_2}^{-s}} \|g\|_{L^{p_2}} \quad \text{if } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\end{equation}
\begin{equation}
\|G_2(f, g)\|_{B_{p_1, p_2}^{-s}} \lesssim \|f\|_{B_{p_1, p_2}^{-s}} \|g\|_{B_{p_3, p_4}^{-s}} \quad \text{if } s_1 > s \text{ and } \frac{1}{p} + \frac{s_1}{d} = \frac{1}{p_3} + \frac{1}{p_4},
\end{equation}
\begin{equation}
\|G_2(f, g)\|_{B_{p_1, p_2}^{-s}} \lesssim \|f\|_{B_{p_1, p_2}^{-s}} \|g\|_{B_{p_3, p_4}^{-s}} \quad \text{if } \frac{1}{p} = \frac{1}{p_5} + \frac{1}{p_6}.
\end{equation}

Moreover, when \( g = f \), we have
\begin{equation}
\|G_1(f, f)\|_{B_{p_1, p_2}^{-s}} \lesssim \|f\|_{B_{p_1, p_2}^{-s}} \|g\|_{B_{p_3, p_4}^{-s}} \quad \text{if } \frac{1}{p} = \frac{1}{p_7} + \frac{1}{p_8}.
\end{equation}

**Proof.** We first prove (4.3). By the equivalence of the Besov norm, we get
\begin{align*}
\|G_1(f, g)\|_{B_{p_1, p_2}^{-s}} &= \|\Delta_{\leq 0} G_1(f, g)\|_{L^p} + \|\{2^{-js} \Delta_j G_1(f, g)\|_{L^p}\|_{j=1}^{\infty} \|_{L^2} \\
&\lesssim \|\Delta_{\leq 0}^2 (\Delta_{\leq 3} f) g\|_{L^p} + \|\Delta_{\leq 0} \Delta_1 (\Delta_{\leq 4} f) g\|_{L^p} \\
&\quad + 2^{-s} \|\Delta_1 \Delta_{\leq 0} (\Delta_{\leq 3} f) g\|_{L^p} \\
&\quad + \left\| \left\{ 2^{-js} \Delta_j \sum_{k=j-1}^{j+1} \Delta_k (\Delta_{\leq k+3} f) g\right\|_{L^p}\right\|_{j=1}^{\infty} \|_{L^2} \\
&=: A + B + C + D.
\end{align*}

It follows from the \( L^p \)-boundedness of the projections \( \Delta_j \) and \( \Delta_{\leq j} \) and the Hölder inequality that
\begin{equation*}
A + B + C \lesssim \|\Delta_{\leq 4} f\|_{L^{p_1}} \|g\|_{L^{p_2}}.
\end{equation*}

Moreover, we have
\begin{align*}
D &= \left\| \left\{ 2^{-js} \Delta_j \sum_{k=j-1}^{j+1} \Delta_k (\Delta_{\leq k+3} f) g\right\|_{L^p}\right\|_{j=1}^{\infty} \|_{L^2} \\
&\lesssim \left\| \left\{ \sum_{k=j-1}^{j+1} 2^{-js} \|\Delta_{\leq k+3} f\|_{L^{p_1}}\right\|_{j=1}^{\infty} \|_{L^2} \|g\|_{L^{p_2}} \\
&\lesssim \left\| \left\{ 2^{-js} \|\Delta_{\leq j+4} f\|_{L^{p_1}}\right\|_{j=1}^{\infty} \|_{L^2} \|g\|_{L^{p_2}} \\
&\lesssim \left\| \left\{ 2^{-js} \|\Delta_{\leq j} f\|_{L^{p_1}}\right\|_{j=1}^{\infty} \|_{L^2} \|g\|_{L^{p_2}}.
\end{align*}
Combining the estimates of $A, B, C,$ and $D$, we obtain

$$
\|G_1(f, g)\|_{B_{p_2}^{−s}} \lesssim \left( \|\Delta_{\leq 4}f\|_{L^{p_1}} + \left\{ 2^{-j}\|\Delta_{\leq j}f\|_{L^{p_1}} \right\}_{j=4}^{\infty} \right) \|g\|_{L^{p_2}}
\sim \|f\|_{B_{p_2}^{−s}} \|g\|_{L^{p_2}}.
$$

Next, we show (4.4). In the same way as before, we have

$$
\|G_2(f, g)\|_{B_{p_2}^{−s}} \lesssim \|\Delta_{\leq 0}(\Delta_{>3}\Delta_{>1}g)\|_{L^{p}} + \|\Delta_{\leq 0}\Delta_1(\Delta_{>3}\Delta_{>1}g)\|_{L^{p}}
+ \|\Delta_{\leq 0}\Delta_1(\Delta_{>3}\Delta_{>2}g)\|_{L^{p}}
+ \left\{ 2^{-j}\Delta_1 \sum_{k-j=1}^{j+1} \Delta_k((\Delta_{>k+3}\Delta_{>k+1}g))_{j=1}^{\infty} \right\}_{j=1}^{\infty}
=: A + B + C + D.
$$

Noting the supports of $\Delta_{>k+3}f$ and $\Delta_{>k+1}g$, we have

$$
D = \left\{ 2^{-j}\sum_{k-j+1}^{j+1} \sum_{|l-l'|| \leq 2} \Delta_j \Delta_k((\Delta_l f)\Delta_{l'} g)\right\}_{j=1}^{\infty}
\lesssim \left\{ \sum_{l>j+1, |l-l'| \leq 2} 2^{-j}\sum_{s=1}^{j+1} \|((\Delta_l f)\Delta_{l'} g)\|_{L^{p}} \right\}_{j=1}^{\infty}
\lesssim \left\{ \sum_{l>j+1, |l-l'| \leq 2} 2^{-j}\sum_{s=1}^{j+1} \|\Delta_l f\|_{L^{p_1}} \|\Delta_{l'} g\|_{L^{p_2}} \right\}_{j=1}^{\infty}
\lesssim \left\{ \sum_{l>j+1, |l-l'| \leq 2} 2^{-j-l}(s_1-s) 2^{-l_s} \|\Delta_l f\|_{L^{p_3}} \|\Delta_{l'} g\|_{L^{p_4}} \right\}_{j=1}^{\infty}
\lesssim \left\{ \sum_{l>j+1, |l-l'| \leq 2} 2^{-l_s} \|\Delta_l f\|_{L^{p_3}} \right\}_{j=1}^{\infty}
\lesssim \|g\|_{B_{p_3}^1, \infty}.
$$

By noting $s_1 > s$ and the Young inequality, we get

$$
\left\{ \sum_{l>j+2} 2^{-j-l}(s_1-s) 2^{-l_s} \|\Delta_l f\|_{L^{p_3}} \right\}_{j=1}^{\infty} \lesssim \|f\|_{B_{p_3}^{−s}}.
$$

In the same way as above, we can calculate $A, B, C$ and have

$$
A + B + C \lesssim \|f\|_{B_{p_3}^{−s}} \|g\|_{B_{p_1}^{s}, \infty}.
$$

Therefore, we obtain (4.4). The inequality (4.5) can be obtained in a similar way.

We omit the detail.
Finally, we remark on the proof of (4.6). As before, we write
\[ G_1(f, f) = \|\Delta_{\leq 2}((\Delta_{\leq 3}f) f)\|_{L^p} + \|\Delta_{\leq 0}\Delta_1((\Delta_{\leq 4}f) f)\|_{L^p} + 2^\gamma\|\Delta_1\Delta_{\leq 0}((\Delta_{\leq 3}f) f)\|_{L^p} \]

\[ + \left\{ \begin{array}{l}
2^j \|\Delta_j \sum_{k=j-1}^{j+1} \Delta_k((\Delta_{\leq k+3}f) f)\|_{L^p} \end{array} \right\}_{j=1}^\infty =: A + B + C + D. \]

We immediately obtain
\[ A + B + C \lesssim \|\Delta_{\leq 3}f\|_{L^p} \|\Delta_{\leq 3}f\|_{L^p}. \]

For the term \( D \), noting the support of \( F(\Delta_j\Delta_k((\Delta_{\leq k+3}f) f)) \), we calculate
\[ D \lesssim \left\| \left\{ 2^j \|\Delta_j((\Delta_{\leq j+3}f) f)\|_{L^p} \right\}_{j=1}^\infty \right\|_{L^2} \]
\[ \lesssim \left\| \left\{ 2^{-j}\|\Delta_{j+3}f\|_{L^p} 2^{-(j+\sigma)}\|\Delta_jf\|_{L^{p_\infty}} \right\}_{j=1}^\infty \right\|_{L^2} \]
\[ \lesssim \left\| \left\{ 2^{-j}\|\Delta_{j+3}f\|_{L^p} 2^{-(j+\sigma)}\|\Delta_jf\|_{L^{p_\infty}} \right\}_{j=1}^\infty \right\|_{L^2} \]
\[ \lesssim \|f\|_{B_{p_\infty}^r, \infty} \|f\|_{B_{p_\infty}^r, \infty}. \]

Here, we have used Lemma 4.1 for the last inequality. \( \square \)

4.2. Proof of the unconditional uniqueness. We give the proof of Theorem 1.8. The proof is based on the Besov-type inhomogeneous endpoint Strichartz estimates and paraproduct estimates. The argument is almost the same as those of [2, Theorem 3.4] and [12, Proposition 2]. However, for readers' convenience, we give a complete proof.

Proof. We assume that \( u \) and \( v \) are solutions to (NLDW) on a time interval \( I \) with the same initial data \( (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) in the sense of Definition 1.1. By the proof of Theorem 1.7, we have constructed a local solution \( u \) satisfying
\[ \left\| u \right\|_{L^\infty_t L^\infty_{x} (I, B_{p_\infty}^r (\mathbb{R}^d))} \lesssim \left\| (u_0, u_1) \right\|_{H^1 \times L^2}, \]

because \( \left( \frac{2(d+1)}{2 \gamma - 2}, \frac{2(d-1)}{\gamma - 2 \delta d} \right) \) is admissible with \( 0 < \gamma = 1 - \frac{d}{2(d-1)} \) and \( \delta = 0 \). Therefore, we may assume this bound for \( u \) without loss of generality.

In the following, we divide the proof into the cases \( d \geq 5 \) and \( d = 4 \). First, we treat the case \( d \geq 5 \). Putting \( F(u) = |u|^\frac{4}{n+2} u \) and \( w = u - v \), we have
\[ w(t, x) = \int_0^t D(t-s) (F(u) - F(v)) \ ds. \]
To express the nonlinear term by \( u \) and \( w \), we calculate
\[
F(u) - F(v) = w \int_0^1 F'(\lambda u + (1 - \lambda)v) \, d\lambda
\]
\[
= w \int_0^1 (F'(u) - (1 - \lambda)F'(u)) \, d\lambda + w F'(u)
\]
\[
= w H(u, w) + w F'(u),
\]
where \( H := \int_0^1 (F'(u) - (1 - \lambda)w - F'(u)) \, d\lambda \). Since \( F'(u) = (1 + \frac{4}{d-2})|u|^{\frac{d-3}{2}} \) is \( \frac{4}{d-2} \)-Hölder continuous on \( \mathbb{R} \), we obtain
\[
|H(u, w)| \lesssim \int_0^1 |(1 - \lambda)w|^{\frac{d-3}{2}} \, d\lambda \lesssim |w|^{\frac{d-3}{2}}.
\]

Let \( I_0 \) be a small time interval such that \( 0 \in I_0 \) determined later. We decompose the product \( wH(u, w) \) into \( wH(u, w) = G_1(w, H(u, w)) + G_2(w, H(u, w)) \), where \( G_1 \) and \( G_2 \) are paraproducts defined by (4.1) and (4.2), respectively. In the same way, we also decompose \( wF'(u) \) into \( wF'(u) = G_1(w, F'(u)) + G_2(w, F'(u)) \). By applying Proposition 1.6 to (4.8) with \( s = -\frac{1}{d-1}, r = \frac{2(d-1)}{d-2} \), \( q = 2 \), we have
\[
\|w\|_{L^2(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))} \lesssim \|G_1(w, F'(u))\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))}
\]
\[
+ \|G_2(w, F'(u))\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))}
\]
\[
+ \|G_1(w, H(u, w))\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))}
\]
\[
+ \|G_2(w, H(u, w))\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))}.
\]

Here, we note that \( \gamma = \frac{d+1}{d-1} \), and in the first and second terms in RHS, we have taken \( q' = \frac{2(\delta+1)}{d+\delta} \), \( \rho' = \frac{2(\delta-1)}{d+2d-2} \) (in that case, \( \delta = 0 \) and \( \tilde{\gamma} = \frac{d-3}{2(d-1)} \), which give \( \gamma + \tilde{\gamma} + \delta - 1 = 0 \)). Also, in the third and forth terms in RHS, we have taken \( q' = 2 \), \( \rho' = \frac{2d-1}{d+1} \) (in that case, \( \delta = 0 \) and \( \tilde{\gamma} = \frac{d-1}{2(d-1)} \), which give \( \gamma + \tilde{\gamma} + \delta - 1 = 0 \)).

We first give the estimate for \( G_1(w, F'(u)) \) in (4.10). Applying (4.3) of Lemma 4.2 with \( s = -\frac{1}{d-1}, p = \frac{2d-1}{d+2d-2}, p_1 = \frac{2(\delta-1)}{d-2}, \) and \( p_2 = \frac{d+1}{d-1} \), and then using the Hölder inequality in time with \( \frac{1}{2} + \frac{2}{d+1} = \frac{d+3}{2(d+1)} \), we see that
\[
\|G_1(w, F'(u))\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))} \lesssim \|w\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))} \|F'(u)\|_{L^2_t(I_0; L^{\frac{d+1}{d-1}}(\mathbb{R}^d))}
\]
\[
\lesssim \|w\|_{L^2_t(I_0; B_{2(d+1)}^{\frac{d-3}{2}}(\mathbb{R}^d))} \frac{1}{L^{\frac{d-1}{d+1}}_t(I_0; L^{\frac{d+1}{d-1}}_x)}.
\]
Here, in the second and third lines we have used $|F'(u)| \lesssim |u|^\frac{3}{d-2}$ and the embedding $B^{\frac{d-1}{2(d-2)}}_{\infty, \infty}(\mathbb{R}^d) \subset B^{\frac{d}{2(d-2)}}_{\infty, \infty}(\mathbb{R}^d)$, respectively.

Secondly, we give the estimate for $G_2(w, F'(u))$ in (4.10). Applying (4.4) of Lemma 4.2 with $s = \frac{2}{d-1}$, $p = \frac{2(d^2-1)}{d^2+2d}$, $p_3 = \frac{2(d-1)}{d-3}$, $p_4 = \frac{d(d^2-1)}{2(d+1)}$, and $s_1 = \frac{2}{d-1}$, and then using the Hölder inequality in time with $\frac{1}{2} + \frac{2}{d+1} = \frac{d+5}{2(d+1)}$, we see that

$$\|G_2(w, F'(u))\|_{L_t^1(I_0; B^{\frac{1}{2(d-1)}}_{\frac{d}{d+2}, \frac{d}{2(d+1)}}(\mathbb{R}^d))} \lesssim \|w\|_{L_t^1(I_0; B^{\frac{1}{2(d-1)}}_{\frac{d}{d+2}, \frac{d}{2(d+1)}}(\mathbb{R}^d))} \|F'(u)\|_{L_t^1(I_0; B^{\frac{d}{2(d+1)}}_{\frac{d}{2(d+1)}, \infty}(\mathbb{R}^d))} \lesssim \|w\|_{L_t^1(I_0; B^{\frac{1}{2(d-1)}}_{\frac{d}{d+2}, \frac{d}{2(d+1)}}(\mathbb{R}^d))} \|F'(u)\|_{L_t^1(I_0; B^{\frac{d}{2(d+1)}}_{\frac{d}{2(d+1)}, \infty}(\mathbb{R}^d))}.$$ 

Here, in the second and third lines we have used $|F'(u)| \lesssim |u|^\frac{3}{d-2}$ and the embedding $B^{\frac{d-1}{2(d-2)}}_{\infty, \infty}(\mathbb{R}^d) \subset B^{\frac{d}{2(d-2)}}_{\infty, \infty}(\mathbb{R}^d)$, respectively.

Thirdly, we give the estimate for $G_2(w, H(u, w))$ in (4.10). Applying (4.5) of Lemma 4.2 with $\sigma = \frac{1}{d-1}$, $p = \frac{2(d-1)}{d+1}$, $p_3 = \frac{2(d-1)}{d-3}$, $p_4 = \frac{d(d^2-1)}{2(d+1)}$, and $s = \frac{1}{d-1}$, and then using the Hölder inequality in time with $\frac{1}{2} = \frac{1}{2} + \frac{1}{d+1}$, we see that

$$\|G_2(w, H(u, w))\|_{L_t^2(I_0; B^{\frac{1}{2(d-1)}}_{\frac{d}{d+2}, \frac{d}{2(d+1)}}(\mathbb{R}^d))} \lesssim \|w\|_{L_t^2(I_0; B^{\frac{1}{2(d-1)}}_{\frac{d}{d+2}, \frac{d}{2(d+1)}}(\mathbb{R}^d))} \|H(u, w)\|_{L_t^{\infty}(I_0; B^{\frac{d}{2(d-1)}}_{\frac{d}{2(d-1)}, \infty}(\mathbb{R}^d))}.$$ 

By the Gagliardo–Nirenberg interpolation inequality, the last term of RHS is further estimated as

$$\|H(u, w)\|_{L_t^{\infty}(I_0; B^{\frac{d}{d+1}}_{\frac{d}{d+1}, \infty}(\mathbb{R}^d))} \lesssim \|H(u, w)\|_{L_t^{\infty}(I_0; B^{\frac{d}{2(d-1)}}_{\frac{d}{2(d-1)}, \infty}(\mathbb{R}^d))}^{\frac{1}{2}} \|H(u, w)\|_{L_t^{\infty}(I_0; B^{\frac{1}{d-1}}_{\frac{1}{d-1}, \infty}(\mathbb{R}^d))}^{\frac{1}{2}}.$$ 

By (4.9) and Lemma C.1 in Appendix C with $f(z) = |z|^\frac{d}{d-2}$, we estimate

$$\|H(u, w)\|_{L_t^{\infty}(I_0; B^{\frac{d}{d+1}}_{\frac{d}{d+1}, \infty}(\mathbb{R}^d))} \lesssim \|w\|_{L_t^{\infty}(I_0; B^{\frac{d}{2(d-1)}}_{\frac{d}{2(d-1)}, \infty}(\mathbb{R}^d))} \lesssim \|w\|_{L_t^{\infty}(I_0; H^1(\mathbb{R}^d)).}$$
and
\[
\|H(u, w)\|_{L_t^\infty(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))}^{\frac{1}{p}} \\
\lesssim \left( \int_0^1 \|F'(u - (1 - \lambda)w) - F'(u)\|_{L_t^\infty(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))} \, d\lambda \right)^{\frac{1}{p}} \\
\lesssim \left( \int_0^1 \|u - (1 - \lambda)w\|_{L_t^\infty(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))}^{\frac{1}{p}} + \|u\|_{L_t^\infty(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))} \right)^{\frac{1}{p}} \\
\lesssim \|u\|_{L_t^\infty(I_0; H^1(\mathbb{R}^d))}^{\frac{1}{p}} + \|u\|_{L_t^\infty(I_0; H^1(\mathbb{R}^d))}^{\frac{1}{p}}.
\]

Here, we have also used the embedding $H^1(\mathbb{R}^d) \subset B^{0, \frac{2d-2}{d+1}}_{\frac{2d}{d-1}}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d) \subset B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d)$. Consequently, we have

\begin{equation}
(4.11)
\end{equation}

Finally, we give the estimate for $G_1(w, H(u, w))$ in (4.10). We further decompose $G_1(w, H(u, w))$ into

\[
G_1(w, H(u, w)) = \sum_{j \in \mathbb{Z}} \Delta_j \left( (\Delta_{\leq j+3} w) H(u, w) \right) \\
= \sum_{j \in \mathbb{Z}} \Delta_j \left( (\Delta_{\leq j+3} w)(\Delta_{\geq j-3} H(u, w)) \right) \\
+ \sum_{j \in \mathbb{Z}} \Delta_j \left( (\Delta_{j-2 \leq : j+3} w)(\Delta_{< j-3} H(u, w)) \right) \\
=: G_{11}(w, H(u, w)) + G_{12}(w, H(u, w)).
\]

Here we remark that $2^{j-1} < |\xi| \leq 2^j$, $|\xi - \eta| \leq 2^{j+3}$, and $|\eta| \leq 2^{j-3}$ imply $2^{j-2} \leq |\xi - \eta| \leq 2^{j+3}$. We estimate $G_{11}(w, H(u, w))$ in the same way as (4.11) and have

\[
\|G_{11}(w, H(u, w))\|_{L_t^2(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))} \lesssim \|u\|_{L_t^2(I_0; B^{\frac{4-d}{2(d-1)}+\epsilon}_{\frac{2d-2}{d+1}}(\mathbb{R}^d))} \|u\|_{L_t^\infty(I_0; H^1(\mathbb{R}^d))}^{\frac{1}{p}} \\
\times \left( \|u\|_{L_t^\infty(I_0; H^1(\mathbb{R}^d))}^{\frac{1}{p}} + \|u\|_{L_t^\infty(I_0; H^1(\mathbb{R}^d))}^{\frac{1}{p}} \right).
\]
Next, we estimate $G_{12}(w, H(u, w))$. By the definition of the Besov space and the Hölder inequality with $\frac{d+1}{2(d-1)} = \frac{d^2 - 3d^2 + 6d - 2}{2(d^2 - d^2)} + \frac{2d - 1}{d^2}$, we deduce

$$\|G_{12}(w, H(u, w))\|_{L^2_t(J_0; B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d))} \leq \left( \sum_{j \in \mathbb{Z}} \left( 2^{\frac{1}{2} - 1} \|\Delta_j u \|_{L^2_x} \|\Delta_j - 3H(u, w)\|_{L^2_x} \|\Delta_j^2 H(u, w)\|_{L^2_x} \right)^2 \right)^{\frac{1}{2}}.$$ 

Here, for the second inequality we have used that $2^{\frac{1}{2} - 1} = 2^{\frac{1}{2} - 1} 2^{\frac{1}{2} j} 2^{\frac{1}{2} j}$ and the Hölder inequality for the variable $j$, and for the third inequality we have used Lemma 4.1 for the term $\|\{2^{-\frac{1}{2} j} \Delta_j u \|_{L^2_x} \} \|_{L_t^2(I_0)}$. Moreover, by the Sobolev embedding, we estimate

$$\|H(u, w)\|_{B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d)} \lesssim \|H(u, w)\|_{L^2_t(\mathbb{R}^d)}^{\frac{1}{4}} \lesssim \|w\|_{L^2_t(\mathbb{R}^d)}^{\frac{1}{4}}.$$

Plugging it into the previous inequality, we have

$$\|G_{12}(w, H(u, w))\|_{L^2_t(J_0; B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d))} \lesssim \left( \|w\|_{B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d)} \right)^2.$$ 

By the Gagliardo–Nirenberg interpolation inequality

$$\|w\|_{B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d)} \lesssim \|w\|_{H^1(\mathbb{R}^d)}^{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}} \|w\|_{H^1(\mathbb{R}^d)}^{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}.$$

we estimate

$$\|G_{12}(w, H(u, w))\|_{L^2_t(J_0; B_{\frac{1}{2(d-1)} \frac{2d - 1}{d^2}}(\mathbb{R}^d))} \lesssim \left( \|w\|_{H^1(\mathbb{R}^d)} \right)^2.$$
When $d \geq 5$, we see that $(1 + \frac{1}{\sigma^2} - \frac{3}{d}) \frac{d+2}{d \sigma^2} > 1$ holds, and hence, the Sobolev embedding $H^1(\mathbb{R}^d) \subset B^{\frac{d+1}{2(d-4)}}_{2,\frac{2(d-4)}{d-2}}(\mathbb{R}^d)$ implies

$$\|G_{12}(w, H(u, w))\|_{L^2_t(I_0; B^{\frac{d+1}{2(d-4)}}_{2,\frac{2(d-4)}{d-2}}(\mathbb{R}^d))} \lesssim \|w\|_{L^\infty_t(I_0; B^{\frac{d+1}{2(d-4)}}_{2,\frac{2(d-4)}{d-2}}(\mathbb{R}^d))} \|w\|_{L^\infty(I_0; H^1(\mathbb{R}^d))}.$$  

Taking all estimates into account, we have

$$\|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \lesssim \|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \approx \|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \times \left( \|w\|_{L^\infty(I_0; H^1(\mathbb{R}^d))} + \|w\|_{L^\infty(I_0; B^{\frac{d+1}{2}}_{2(d+1)}(\mathbb{R}^d))} \right).$$

Here, we note that $w \in C(I_0; H^1(\mathbb{R}^d))$ and $w(0) = 0$, which implies $\|w\|_{L^\infty(I_0; H^1(\mathbb{R}^d))} \to 0$ as $I_0$ becomes smaller. By (4.7), we also have

$$\|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \to 0$$

as $I_0$ becomes smaller. Therefore, taking $I_0$ sufficiently small, we see that $w = 0$ holds on $I_0$. Continuing this argument, we have $w = 0$ on the whole interval $I$. This completes the proof for the case $d \geq 5$.

For the case $d = 4$, we first note that

$$u^3 - v^3 = w^2(-3u + w) + 3wu^2.$$  

In the same way to (4.10), we have

$$\|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \lesssim \|3wu^2\|_{L^\infty(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} + \|w^2(-3u + w)\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))}.$$  

The term $\|3wu^2\|_{L^\infty(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))}$ can be estimated in the same manner as $wF'(u)$ in $d \geq 5$, because we did not use the condition $d \geq 5$ to handle it. Therefore, we have

$$\|3wu^2\|_{L^\infty(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \lesssim \|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))} \|w\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))}.$$  

For the term $\|w^2(-3u + w)\|_{L^2(I_0; B_6^{\frac{1}{2}}(\mathbb{R}^d))}$, by the fractional Leibniz rule and the Sobolev embedding, we have

$$\|w^2(-3u + w)\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \lesssim \|w^2\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)}.$$  

$$+ \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)}$$  

$$\lesssim \|w^2\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{B_6^{\frac{1}{2}}(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}.$$
Next, we show the product estimate

\[(4.15) \quad \|w^2\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \lesssim \|w\|_{H^1(\mathbb{R}^d)} \|w\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)}.\]

Indeed, applying the paraproduct \(w^2 = G_1(w, w) + G_2(w, w)\), we have

\[
\|G_j(w, w)\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \lesssim \|w\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \|w\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)}
\lesssim \|w\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}
\]

for \(j = 1, 2\). Here, we have used \((4.5)\) and \((4.6)\) in Lemma 4.1, respectively, and the Sobolev embedding. This proves \((4.15)\). Combining \((4.14)\) and \((4.15)\), we see

\[
\|w^2(-3u + w)\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \lesssim \|w\|_{\dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)} - 3u + w \|_{\dot{B}^{0,2}_{\infty,\infty}(\mathbb{R}^d)}.
\]

It follows from the H"older inequality that

\[(4.16) \quad \|w^2(-3u + w)\|_{L^2(I; \dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d))}
\lesssim \|w\|_{L^2(I; \dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d))} \|w\|_{L^\infty(I; H^1(\mathbb{R}^d))} - 3u + w \|_{L^\infty(I; \dot{B}^{0,2}_{\infty,\infty}(\mathbb{R}^d))}.
\]

Applying the estimates \((4.13)\) and \((4.16)\) to \((4.12)\), we conclude

\[
\|w\|_{L^2(I; \dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d))} \lesssim \|w\|_{L^2(I; \dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d))}
\times \left( \|w\|_{L^\infty(I; H^1(\mathbb{R}^d))} - 3u + w \|_{L^\infty(I; \dot{B}^{0,2}_{\infty,\infty}(\mathbb{R}^d))} + \|w\|^2_{L^2(I; \dot{B}^{\frac{1}{2},2}_{\infty,\infty}(\mathbb{R}^d))} \right)
\]

From this estimate, in the same way as in the case \(d \geq 5\), we conclude \(w = 0\) and complete the proof. \( \square \)

**APPENDIX A. BESOV TYPE STRICHARTZ ESTIMATES**

Using the following lemma, we get the Besov type Strichartz estimates, Proposition 1.5 and 1.6. We only give the proof of the lemma and omit the proof from the lemma to Proposition 1.5 and 1.6.

**Lemma A.1.** Let \(s \in \mathbb{R}\). Assume that \((q, r)\) satisfies the assumptions in Proposition 1.1. Let \(\gamma\) be as in Proposition 1.1. Then we have the following Besov type homogeneous Strichartz estimates.

\[
\|D(t)f\|_{L^q(I; \dot{B}^{\gamma-1}_{\infty,2}(\mathbb{R}^d))} \lesssim \left\| (\nabla)^{\gamma-1} f \right\|_{\dot{B}^{\frac{1}{2},2}_{2,2}(\mathbb{R}^d)} \approx \|f\|_{\dot{B}^{\gamma-1}_{\infty,2}(\mathbb{R}^d),}
\]

where \(I \subset \mathbb{R}\) is a time interval.
Proof. By the definition of Besov space, we have
\begin{equation}
\|D(t)f\|_{L^q(I;B^s_{p,r})} = \left\|D(t)\Delta_{\leq 0} f\right\|_{L^r} + \left\|\sum_{j \geq 1} \left(2^{sj} \|\Delta_j D(t)f\|_{L^2}\right)^2\right\|_{L^q(I)}^{\frac{1}{2}}
\end{equation}
\begin{equation}
\leq \left\|D(t)\Delta_{\leq 0} f\right\|_{L^r} \leq \left\|\sum_{j \geq 1} \left(2^{sj} \|\Delta_j D(t)f\|_{L^2}\right)^2\right\|_{L^q(I)}^{\frac{1}{2}}
\end{equation}

By the homogeneous Strichartz estimates in Proposition 1.1, the first term can be estimated as
\begin{equation}
\|D(t)\Delta_{\leq 0} f\|_{L^r} \lesssim \left\|\langle \nabla \rangle^{\gamma - 1} \Delta_{\leq 0} f\right\|_{L^2} = \left\|\Delta_{\leq 0} (\langle \nabla \rangle^{\gamma - 1} f)\right\|_{L^2}.
\end{equation}
The second term is estimated as
\begin{equation}
\left\|\sum_{j \geq 1} \left(2^{sj} \|\Delta_j D(t)f\|_{L^2}\right)^2\right\|_{L^q(I)}^{\frac{1}{2}} \leq \sum_{j \geq 1} \left(2^{sj} \|\Delta_j D(t)f\|_{L^2}\right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\leq \sum_{j \geq 1} \left(2^{sj} \|\Delta_j D(t)f\|_{L^2}\right)^{\frac{1}{2}} \leq \sum_{j \geq 1} \left(2^{sj} \|\Delta_j (\langle \nabla \rangle^{\gamma - 1} f)\|_{L^2}\right)^{\frac{1}{2}}
\end{equation}
Combining (A.2) and (A.3) with (A.1), we get
\begin{equation}
\|D(t)f\|_{L^q(I;B^s_{p,r})} \lesssim \left\|\Delta_{\leq 0} (\langle \nabla \rangle^{\gamma - 1} f)\right\|_{L^2} + \left\|\sum_{j \geq 1} \left(2^{sj} \|\Delta_j (\langle \nabla \rangle^{\gamma - 1} f)\|_{L^2}\right)^2\right\|_{L^q(I)}^{\frac{1}{2}}
\end{equation}
\begin{equation}
= \left\|\langle \nabla \rangle^{\gamma - 1} f\right\|_{B^s_{p,r}}.
\end{equation}
The last equivalency is a fundamental property of Besov spaces. \qed

The estimates for \(\partial_t D(t)\) and \(\partial_t^2 D(t)\) can be obtained in the same way as above. Thus, we get Proposition 1.5.

Lemma A.2. Let \(s \in \mathbb{R}\). Assume that \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfy the assumptions in Proposition 1.4. Let \(\gamma\) and \(\tilde{\gamma}\) be as in Proposition 1.4 and \(\delta\) be defined in Table 1. Then we have the following Besov type inhomogeneous Strichartz estimates.
\begin{equation}
\left\|\int_0^t D(t - s)F(s)ds\right\|_{L^q(I;B^s_{p,r})} \lesssim \left\|\langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta - 1} F\right\|_{L^{\tilde{q}}(I;B^s_{\tilde{p},\tilde{r}})}
\end{equation}
where \(I \subset \mathbb{R}\) is a time interval.
Proof. For simplicity, we set $I = \int_0^t \mathcal{D}(t-s) F(s) ds$ and $G = (\nabla)^{\gamma+\gamma'-\delta-1} F$. By the definition of Besov spaces and $(a^2)^{1/2} + (\sum_j b_j^2)^{1/2} \leq \sqrt{2} (a^2 + \sum_j b_j^2)^{1/2}$, we have

$$(\text{L.H.S}) = \left\| \Delta_{\leq 0} I \right\|_{L^r} + \left\{ \sum_{j \geq 1} (2^{sj} \left\| \Delta_j I \right\|_{L^r})^2 \right\}^{1/2}_{L^q(t)} \lesssim \left\{ \left\| \Delta_{\leq 0} I \right\|^2_{L^r} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j I \right\|_{L^r})^2 \right\}^{1/2}_{L^q(t)}.$$ 

Since $q \geq 2$, the right hand side is estimated as

$$(\text{R.H.S}) \lesssim \left\{ \left\| \Delta_{\leq 0} I \right\|^2_{L^r} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j I \right\|_{L^r})^2 \right\}^{1/2}_{L^q(t)}.$$ 

By using the inhomogeneous Strichartz estimates, Proposition 1.4, we get

$$\left\{ \left\| \Delta_{\leq 0} I \right\|^2_{L^r} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j I \right\|_{L^r})^2 \right\}^{1/2} \lesssim \left\{ \left\| \Delta_{\leq 0} G \right\|^2_{L^{q'}_r L^{r'}} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j G \right\|_{L^{q'}_r L^{r'}})^2 \right\}^{1/2} = \left\{ \left\| \Delta_{\leq 0} G \right\|^2_{L^{q'}_r} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j G \right\|_{L^{r'}})^2 \right\}^{1/2}_{L^{q'}_r(t)}.$$ 

Since $q' \leq 2$, this right hand side is bounded as

$$(\text{R.H.S}) \lesssim \left\{ \left\| \Delta_{\leq 0} G \right\|^2_{L^{r'}} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j G \right\|_{L^{r'}})^2 \right\}^{1/2}_{L^{q'}_r(t)} = \left\{ \left\| \Delta_{\leq 0} G \right\|^2_{L^{r'}} + \sum_{j \geq 1} (2^{sj} \left\| \Delta_j G \right\|_{L^{r'}})^2 \right\}^{1/2}_{L^{q'}_r(t)} \lesssim \left\| \Delta_{\leq 0} G \right\|_{L^{r'}} + \left\{ \sum_{j \geq 1} (2^{sj} \left\| \Delta_j G \right\|_{L^{r'}})^2 \right\}^{1/2}_{L^{q'}_r(t)} = \left\| G \right\|_{L^{q'}_r(t; B^{r'}_{r',2})}.$$ 

This completes the proof. \qed

We can estimate $\int_0^t \mathcal{D}(t-s) F(s) ds$ in the same way. Therefore, we get Proposition 1.6.
Appendix B. Proof of Lemma 4.1

Proof of Lemma 4.1. First, we consider the first equivalence. We show \( \geq \). It is true that \( \|f\|_{B^s_{p,q}} \sim \|f\|_{L^p} + \|f\|_{B^s_{p,q}} \) for \( s > 0 \). By [2, Lemma 4.1], we have

\[
\|f\|_{B^s_{p,q}} \sim \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j \in \mathbb{Z}}\|_{l^q}.
\]

Hence, it follows that

\[
\|\Delta \leq f\|_{L^p} + \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq \|\Delta \leq f\|_{L^p} + \sum_{j=1}^{J-1} \|\Delta_j f\|_{L^p} + \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq \|\Delta \leq f\|_{L^p} + \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\sim \|\Delta \leq f\|_{L^p} + \|\Delta \geq f\|_{B^s_{p,q}} \\
\sim \|f\|_{B^s_{p,q}}.
\]

We prove \( \leq \). We have

\[
\|f\|_{B^s_{p,q}} = \|\Delta \leq f\|_{L^p} + \|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq J \|\Delta \leq f\|_{L^p} + \sum_{j=1}^{J-1} \|\Delta_j f\|_{L^p} + \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq J \|\Delta \leq f\|_{L^p} + \|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq J \|\Delta \leq f\|_{L^p} + \|\{2^{js} \|\Delta \geq f\|_{L^p}\}_{j=1}^\infty\|_{l^q}.
\]

Next, we show the second equivalence. We first prove \( \preceq \).

\[
\|f\|_{B^{-s}_{p,q}} = \|\Delta \leq f\|_{L^p} + \|\{2^{-js} \|\Delta_j f\|_{L^p}\}_{j=1}^\infty\|_{l^q} \\
\leq \|\Delta \leq f\|_{L^p} + \|\{2^{-js} \|\Delta \leq f\|_{L^p}\}_{j=0}^\infty\|_{l^q} \\
\leq J \|\Delta \leq f\|_{L^p} + \|\{2^{-js} \|\Delta \leq f\|_{L^p}\}_{j=1}^\infty\|_{l^q}
\]

where we use \( \Delta_j = \Delta \leq_j - \Delta \leq_{j-1} \) and the triangle inequality in the first inequality. We show \( \succeq \). Now, by the Young inequality, we have

\[
\|\{2^{-js} \|\Delta \leq f\|_{L^p}\}_{j=1}^\infty\|_{l^q} = \left\| \left\{ 2^{-js} \sum_{l \leq j} \Delta_l f \|_{L^p} \right\}_{j=1}^\infty \right\|_{l^q} \\
\leq \left\| \left\{ \sum_{l \leq j} 2^{-js} \|\Delta_l f\|_{L^p} \right\}_{j=1}^\infty \right\|_{l^q} \\
\leq \left\| \left\{ \sum_{l \leq j} 2^{-j(s-l)} \|\Delta_l f\|_{L^p} \right\}_{j=1}^\infty \right\|_{l^q} \\
\leq \left\| \left\{ 2^{-j(s)} \|\Delta_j f\|_{L^p} \right\}_{j=1}^\infty \right\|_{l^q} \\
\leq \left\| \left\{ 2^{-j(s)} \|\Delta_j f\|_{L^p} \right\}_{j=1}^\infty \right\|_{l^q}.
\]
Therefore, it follows that
\[
\| \Delta_{\leq j} f \|_{L^p} + \| \{2^{-j} \| \Delta_{\leq j} f \|_{L^p} \}_{j=1}^{\infty} \|_q \\
\lesssim_j \| \Delta_{\leq 0} f \|_{L^p} + \sum_{j=1}^J \| \Delta_j f \|_{L^p} + \| \{2^{-j} \| \Delta_j f \|_{L^p} \}_{j=1}^{\infty} \|_q \\
\lesssim_j \| \Delta_{\leq 0} f \|_{L^p} + \| \{2^{-j} \| \Delta_j f \|_{L^p} \}_{j=1}^{\infty} \|_q \\
\lesssim_j \| f \|_{B_{p,q}^{-s}}.
\]
This completes the proof. \(\square\)

**Appendix C. An estimate of Besov norm for Hölder continuous functions**

For the proof of unconditional uniqueness (in Section 4), we prepare the following estimate of Besov norm for Hölder continuous functions. The corresponding Besov norms has already given in [2, Lemma 2.1].

**Lemma C.1.** Let \(0 < s < 1\), \(0 < \alpha \leq 1\) and \(1 < p \leq \infty\) be given such that \(1 < p\alpha \leq \infty\). Let \(f(z)\) be a Hölder continuous function of order \(\alpha\) satisfying \(f(0) = 0\). Then, we have
\[
\| f(u) \|_{B_{p,q}^s} \lesssim \| u \|_{B_{p,q}^{s,\alpha}}.
\]

**Proof.** We recall the equivalence of Besov norm (see [1, p.162])
\[
\| u \|_{B_{p,q}^s(\mathbb{R}^d)} \sim \| u \|_{L_p(\mathbb{R}^d)} + \left( \int_{\mathbb{R}^d} \| u(\cdot + h) - u(\cdot) \|_{L_p(\mathbb{R}^d)} \frac{1}{|h|^{d+sq}} \, dh \right)^{\frac{1}{q}}
\]
for \(q < \infty\), and
\[
\| u \|_{B_{p,q}^s(\mathbb{R}^d)} \sim \| u \|_{L_p(\mathbb{R}^d)} + \sup_{h \in \mathbb{R}^d} |h|^{-s} \| u(\cdot + h) - u(\cdot) \|_{L_p(\mathbb{R}^d)}
\]
for \(q = \infty\). Since \(f(z)\) is Hölder continuous of order \(\alpha\) and satisfies \(f(0) = 0\), we deduce \(|f(u)| \lesssim |u|^{\alpha}\) and \(|f(u(x+h)) - f(u(x))| \lesssim |u(x+h) - u(x)|^{\alpha}\). Therefore, we calculate
\[
\| f(u) \|_{B_{p,q}^s} \lesssim \| f(u) \|_{L_p(\mathbb{R}^d)} + \sup_{h \in \mathbb{R}^d} \left[ |h|^{-s} \left( \int_{\mathbb{R}^d} |f(u(x+h)) - f(u(x))|^{p\alpha} \, dx \right)^{\frac{1}{p\alpha}} \right] \\
\lesssim \| u \|_{L_{p\alpha}^p(\mathbb{R}^d)} + \sup_{h \in \mathbb{R}^d} \left[ |h|^{-s} \left( \int_{\mathbb{R}^d} |u(x+h) - u(x)|^{p\alpha} \, dx \right)^{\frac{1}{p\alpha}} \right] \\
\lesssim \| u \|_{L_{p\alpha}^p(\mathbb{R}^d)} + \left( \sup_{h \in \mathbb{R}^d} |h|^{-\frac{\alpha}{p\alpha}} |u(\cdot + h) - u(\cdot)|_{L_{p\alpha}^p(\mathbb{R}^d)} \right)^{\alpha} \\
\lesssim \left( \| u \|_{L_{p\alpha}^p(\mathbb{R}^d)} + \sup_{h \in \mathbb{R}^d} |h|^{-\frac{\alpha}{p\alpha}} |u(\cdot + h) - u(\cdot)|_{L_{p\alpha}^p(\mathbb{R}^d)} \right)^{\alpha} \\
\sim \| u \|_{B_{p,q}^{s,\alpha}(\mathbb{R}^d)},
\]
which gives the desired estimate. \(\square\)
Acknowledgment

This work was supported by JSPS KAKENHI Grant Numbers JP16K17625, 18K13444.

References

[1] Jöran Bergh, Jörgen Löfström, Interpolation spaces. An introduction, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976. x+207 pp.

[2] Aynur Bulut, Magdalena Czubak, Dong Li, Nataša Pavlović, Xiaoyi Zhang, Stability and unconditional uniqueness of solutions for energy critical wave equations in high dimensions, Comm. Partial Differential Equations 38 (2013), 575–607.

[3] Takafumi Hosono, Takayoshi Ogawa, Large time behavior and $L^p$-$L^q$ estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004), no. 1, 82–118.

[4] Masahiro Ikeda, Takahisa Inui, Mamoru Okamoto, Yuta Wakasugi, $L^p$-$L^q$ estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data, Commun. Pure Appl. Anal. 18 (2019), no. 4, 1967–2008. doi: 10.3934/cpaa.2019090

[5] Takahisa Inui, The Strichartz estimates for the damped wave equation and the behavior of solutions for the energy critical nonlinear equation, preprint.

[6] Markus Keel, Terence Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), no. 5, 955–980.

[7] Herbert Koch, Daniel Tataru, Monica Vişan, Dispersive equations and nonlinear waves. Generalized Korteweg-de Vries, nonlinear Schrödinger, wave and Schrödinger maps, Oberwolfach Seminars, 45 Birkhäuser/Springer, Basel, 2014. xii+312 pp.

[8] Shuji Machihara, Kenji Nakanishi, Tohru Ozawa, Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation, Rev. Mat. Iberoamericana 19 (2003), no. 1, 179–194.

[9] Akitaka Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. 12 (1976/77), no. 1, 169–189.

[10] Takashi Narazaki, $L^p$-$L^q$ estimates for damped wave equations with odd initial data, Electron. J. Differential Equations 2005, No. 74, 17 pp.

[11] Kenji Nishihara, $L^p$-$L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003), no. 3, 631–649.

[12] Fabrice Planchon, On uniqueness for semilinear wave equations, Math. Z. 244 (2003), 587–599.

[13] Shigehiro Sakata, Yuta Wakasugi, Movement of time-delayed hot spots in Euclidean space, Math. Z. 285 (2017), no. 3-4, 1007–1040.

[14] Tomonari Watanabe, Strichartz type estimates for the damped wave equation and their application, RIMS Kôkyûroku Bessatsu, B63 (2017), 77–101.

(T. Inui) Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: inui@math.sci.osaka-u.ac.jp

(Y. Wakasugi) Department of Engineering for Production and Environment, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho, Matsuyama, Ehime, 790-8577, Japan
E-mail address: wakasugi.yuta.vi@ehime-u.ac.jp