1. Introduction.

This paper is concerned with a class of time-optimal control problems for the swing and the ski. We first consider the motion of a man standing on a swing. For simplicity, we neglect friction and air resistance and assume that the mass of the swinger is concentrated in his baricenter $B$. Let $\theta$ be the angle formed by the swing and the downward vertical direction and $r$ the radius of oscillation, i.e. the distance between the baricenter $B$ and the center of rotation $O$. Assuming that the mass is normalized to a unit, the corresponding Lagrangean function takes the form:

$$ L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + g r \cos(\theta) \quad (1.1) $$

where $g$ is the gravity acceleration.

We assume that, by bending his knees, the swinger can vary his radius of oscillation. This amounts to the addition of a constraint $r = u(t)$, implemented by forces acting on $B$, parallel to the vector $OB$. The function $u(t)$ can be regarded here as a control, whose values are chosen by the man riding on the swing, within certain physical bounds, say $u(t) \in [r_-, r_+]$ with $0 < r_- < r_+$.

Writing (1.1) as a first order system for $(\theta, \dot{\theta})$ we obtain an impulsive control system. On the other hand, using the coordinates $x_1 = \theta$, $x_2 = \dot{\theta}r^2$ (the angular momentum), following Alberto Bressan (1993), we obtain the nonimpulsive control system:

$$ \begin{cases} 
\dot{x}_1 = x_2/v^2 \\
\dot{x}_2 = -g \, v \, \sin(x_1) 
\end{cases} \quad (1.2) $$

For the ski, we use the model considered in Aldo Bressan (1991), Bressan and Motta (1994), with a special approximation of the skier.

Our main concern is the existence and the structure of time-optimal controls. As a first step, we introduce an auxiliary control system $\Sigma$ with control entering linearly and establish the relationships between this and the original system. We then perform a detailed study of the auxiliary system, using the geometric techniques developed in Sussmann (1987 a,b,c) and in Piccoli (1993 a,b). These geometric techniques permit us to know the structure of the time optimal trajectories in generic cases and possibly to solve explicitly some optimization problem. See section 4 for examples. In turn this provides accurate information on the time-optimal controls for the original system (1.2).
Since the set of admissible velocities for $\Sigma$ is always convex, a time optimal control exists for general boundary conditions. However, we prove that a time optimal control for (1.2) does exist only when the corresponding optimal control for $\Sigma$ is bang-bang. In this case the optimal trajectory is the same for the two control systems. We show that, for every control constraint $v \in [r_-, r_+]$ with $0 < r_- < r_+$, there exists some pair of points $x, \tilde{x} \in \mathbb{R}^2$ such that no control for (1.2) steers $x$ to $\tilde{x}$ in minimum time.

Then we consider the problem of reaching with the swing a given angle $\bar{\theta}$ in minimum time, having assigned initial condition $\theta, \dot{\theta} r^2$. This can be stated as a Mayer problem for (1.2). We show how to solve this problem under the assumptions $|\theta|, |\dot{\theta}| \leq \pi/2$. To solve numerically this problem, it suffices to find a suitable class of solutions to (1.2) with constant control $v \in \{r_-, r_+\}$, and then to solve the complementary linear system (2.6) for each trajectory. The optimal trajectory is characterized by the (final) transversality condition. We also show that, for a special class of boundary conditions and data $r_{\pm}$, no optimal solution exists.

For the ski model we obtain similar results under the assumption that the curvature of the ski trail is constant.

For an introduction to impulsive control systems we refer to Alberto Bressan (1993).

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2. Basic definitions.

By a curve in $\mathbb{R}^n$ we mean a continuous map $\gamma : I \mapsto \mathbb{R}^n$, where $I$ is some real interval. We use the symbol $\text{Dom}$ for the domain so that if $\gamma : I \mapsto \mathbb{R}^n$ then $\text{Dom}(\gamma) = I$. We use the symbol $\gamma \upharpoonright J$, where $J \subset \text{Dom}(\gamma)$ is an interval, to denote the restriction of $\gamma$ to $J$.

A $C^1$ vector field on $\mathbb{R}^2$ is a continuously differentiable map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It can be written in the form:

$$F = \alpha \partial_x + \beta \partial_y$$

where $\partial_x, \partial_y$ are the constant vector fields with components (1,0), (0,1) respectively. If we have the representation (2.1) for a vector field $F$ then we use the symbol $\nabla F$ to denote
the $2 \times 2$ Jacobian matrix:

$$\nabla F = \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix}. $$

The Lie-bracket of two vector fields $F, G$ is the vector field:

$$[F,G] = \nabla G \cdot F - \nabla F \cdot G. \quad (2.2)$$

We consider a two dimensional autonomous control system:

$$\dot{x} = h(x,u) \quad u \in U \quad (2.3)$$

where $U \subset \mathbb{R}^m$ is compact and $h \in C^1(\mathbb{R}^2 \times \mathbb{R}^m, \mathbb{R}^2)$. A control is a measurable function $u : [a, b] \mapsto U$ where $-\infty < a \leq b < +\infty$. As for the curves we use the symbol $\text{Dom}$ for the domain. A trajectory for a control $u$ is an absolutely continuous curve $\gamma : \text{Dom}(u) \mapsto \mathbb{R}^2$ which satisfies the equation:

$$\dot{\gamma}(t) = h(\gamma(t), u(t))$$

for almost every $t \in \text{Dom}(u)$.

If $\gamma : [a, b] \mapsto \mathbb{R}^2$ is a trajectory of (2.3) we use the symbol $\text{In}(\gamma)$ to denote its initial point $\gamma(a)$ and $\text{Term}(\gamma)$ to denote its terminal point $\gamma(b)$. We define:

$$T(\gamma) = b - a.$$ 

i.e. $T(\gamma)$ is the time along $\gamma$.

A trajectory $\gamma$ is said to be time optimal if for every trajectory $\gamma'$ with $\text{In}(\gamma') = \text{In}(\gamma)$ and $\text{Term}(\gamma') = \text{Term}(\gamma)$, we have that $T(\gamma') \geq T(\gamma)$.

If $u_1 : [a, b] \mapsto U$ and $u_2 : [b, c] \mapsto U$ are controls, we use $u_2 * u_1$ to denote the control defined by:

$$(u_2 * u_1)(t) = \begin{cases} u_1(t) & \text{for } t \in \text{Dom}(u_1) \\ u_2(t) & \text{for } t \in \text{Dom}(u_2) \end{cases}. $$

This control is called the concatenation of $u_1$ and $u_2$.

If $\gamma_1 : [a, b] \mapsto \mathbb{R}^2$, $\gamma_2 : [b, c] \mapsto \mathbb{R}^2$ are trajectories for $u_1$ and $u_2$ such that $\gamma_1(b) = \gamma_2(b)$, then the concatenation $\gamma_2 * \gamma_1$ is the trajectory:

$$(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in \text{Dom}(\gamma_1) \\ \gamma_2(t) & \text{for } t \in \text{Dom}(\gamma_2) \end{cases}. $$
Now we consider a control system with control entering linearly:

\[ \dot{x} = F(x) + u \, G(x) \quad |u| \leq 1 \quad (2.4) \]

where \( F, G \) are two \( C^1 \) vector fields on \( \mathbb{R}^2 \) and \( u \) is scalar. We define:

\[ X = F - G \quad Y = F + G. \]

An \( X \)-trajectory is a trajectory corresponding to a constant control \( u \) whose value is equal to \(-1\). We define \( Y \)-trajectories in a similar way, using the control \( u = +1 \) rather than \( u = -1 \). An \( X \ast Y \)-trajectory is concatenation of a \( Y \)-trajectory and an \( X \)-trajectory (the \( Y \)-trajectory comes first), and similarly is defined a \( Y \ast X \)-trajectory.

A bang-bang trajectory is a trajectory that is a concatenation of \( X \)- and \( Y \)-trajectories.

A time \( t \in \text{Dom}(\gamma) \) is called a switching time for \( \gamma \) if, for each \( \varepsilon > 0 \), \( \gamma \downarrow [t - \varepsilon, t + \varepsilon] \) is neither an \( X \)-trajectory nor a \( Y \)-trajectory. If \( t \) is a switching time for \( \gamma \) then we say that \( \gamma(t) \) is a switching point for \( \gamma \), or that \( \gamma \) has a switching at \( \gamma(t) \).

For the rest of this section we call \( \Sigma \) the system (2.4). An admissible pair for the system \( \Sigma \) is a couple \((u, \gamma)\) such that \( u \) is a control and \( \gamma \) is a trajectory corresponding to \( u \). We use the symbol \( \text{Adm}(\Sigma) \) to denote the set of admissible pairs and we say that \((u, \gamma) \in \text{Adm}(\Sigma)\) is optimal if \( \gamma \) is optimal.

A variational vector field along \((u, \gamma) \in \text{Adm}(\Sigma)\) is a vector-valued absolutely continuous function \( v : \text{Dom}(\gamma) \mapsto \mathbb{R}^2 \) that satisfies the equation:

\[ \dot{v}(t) = \left( (\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t)) \right) \cdot v(t) \quad (2.5) \]

for almost all \( t \in \text{Dom}(\gamma) \).

A variational covector field along \((u, \gamma) \in \text{Adm}(\Sigma)\) is an absolutely continuous function \( \lambda : \text{Dom}(\gamma) \mapsto \mathbb{R}^2_* \) that satisfies the equation:

\[ \dot{\lambda}(t) = -\lambda(t) \cdot \left( (\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t)) \right) \quad (2.6) \]

for almost all \( t \in \text{Dom}(\gamma) \). Here \( \mathbb{R}^2_* \) denotes the space of row vectors. In Sussmann (1987 a) it was proved:

**Lemma 2.1** Let \((u, \gamma) \in \text{Adm}(\Sigma)\) and let \( \lambda : \text{Dom}(\gamma) \mapsto \mathbb{R}^2_* \) be absolutely continuous. Then \( \lambda \) is a variational covector field along \((u, \gamma)\) if and only if the function \( t \mapsto \lambda(t) \cdot v(t) \) is constant for every variational vector field \( v \) along \((u, \gamma)\).
The Hamiltonian \( \mathcal{H} : \mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R} \) is defined as
\[
\mathcal{H}(\lambda, x, u) = \lambda \cdot (F(x) + uG(x)). \tag{2.7}
\]
If \( \lambda \) is a variational covector field along \((u, \gamma) \in \text{Adm}(\Sigma)\), we say that \( \lambda \) is maximizing if:
\[
\mathcal{H}(\lambda(t), \gamma(t), u(t)) = \max \{ \mathcal{H}(\lambda(t), \gamma(t), w) : |w| \leq 1 \} \tag{2.8}
\]
for almost all \( t \in \text{Dom}(\gamma) \).

The \textit{Pontryagin Maximum Principle} (PMP) states that, if \((u, \gamma) \in \text{Adm}(\Sigma)\) is time optimal, then there exists:

(PMP1) \textit{A non trivial maximizing variational covector field} \( \lambda \) \textit{along} \((u, \gamma)\)

(PMP2) \textit{A constant} \( \lambda_0 \leq 0 \) \textit{such that} \( \mathcal{H}(\lambda(t), \gamma(t), u(t)) + \lambda_0 = 0 \) for almost all \( t \in \text{Dom}(\gamma) \).

In this case \( \lambda \) is called an \textit{adjoint covector field along} \((u, \gamma)\) or simply an \textit{adjoint variable}, and we say that \((\gamma, \lambda)\) satisfies the PMP.

If \( \lambda \) is an adjoint covector field along \((u, \gamma) \in \text{Adm}(\Sigma)\), the corresponding \textit{switching function} is defined as:
\[
\phi_{\lambda}(t) = \lambda(t) \cdot G(\gamma(t)). \tag{2.9}
\]

From the above definition it follows:

\textbf{Lemma 2.2} \textit{Let} \((u, \gamma) \in \text{Adm}(\Sigma)\) \textit{be optimal and let} \( \lambda \) \textit{be an adjoint covector field along} \((u, \gamma)\). \textit{Then:}

a) \textit{The switching function} \( \phi_{\lambda} \) \textit{is continuous.}

b) \textit{If} \( \phi(t) > 0 \) \textit{for all} \( t \) \textit{in some interval} \( I \), \textit{then} \( u(t) \equiv 1 \) \textit{for almost all} \( t \in I \) \textit{and} \( \gamma \restriction I \) \textit{is a} \( Y \)-\textit{trajectory.}

c) \textit{If} \( \phi(t) < 0 \) \textit{for all} \( t \) \textit{in some interval} \( I \), \textit{then} \( u(t) \equiv -1 \) \textit{for almost all} \( t \in I \) \textit{and} \( \gamma \restriction I \) \textit{is an} \( X \)-\textit{trajectory.}

For each \( x \in \mathbb{R}^2 \), one can form the \( 2 \times 2 \) matrices whose columns are the vectors \( F \), \( G \), or \([F, G]\). As in Sussmann (1987 a), we shall use the following scalar functions on \( \mathbb{R}^2 \):
\[
\Delta_A(x) \doteq \det(F(x), G(x)) \quad \tag{2.10}
\]
\[
\Delta_B(x) \doteq \det(G(x), [F, G](x)) \quad \tag{2.11}
\]
where \( \det \) stands for determinant.
Consider \((u, \gamma) \in \text{Adm}(\Sigma), t_0 \in \text{Dom}(\gamma)\) and \(v_0 \in \mathbb{R}^2\). We write \(v(v_0, t_0; t)\) to denote the value at time \(t\) of the variational vector field along \((u, \gamma)\) satisfying (2.5) together with the boundary condition \(v(t_0) = v_0\). In Piccoli (1993 a) it was proved the following:

**Lemma 2.3.** Let \((u, \gamma) \in \text{Adm}(\Sigma), t_0 \in \text{Dom}(\gamma), \text{ and } v_0 \in \mathbb{R}^2, v_0 \neq 0\). For every \(t\) such that \(G(\gamma(t)) \neq 0\), define the angle:

\[
\alpha(t) = \arg \left( v_0, v(G(\gamma(t)), t_0) \right),
\]

(2.12)

Then, one has:

\[
\text{sgn}(\dot{\alpha}(t)) = \text{sgn} \left( \Delta_B(\gamma(t)) \right).
\]

(2.13)

A point \(x \in \mathbb{R}^2\) is called an ordinary point if

\[
\Delta_A(x) \cdot \Delta_B(x) \neq 0.
\]

On the set of ordinary points we define the scalar functions \(f, g\) as the coefficients of the linear combination

\[
[F, G](x) = f(x)F(x) + g(x)G(x).
\]

(2.14)

By direct calculations we have:

\[
f = -\frac{\Delta_B}{\Delta_A}.
\]

A point \(x\) at which \(\Delta_A(x)\Delta_B(x) = 0\) is called a nonordinary point. A nonordinary arc is a \(C^1\) one-dimensional connected embedded submanifold \(S\) of \(\mathbb{R}^2\), with the property that every \(x \in S\) is a nonordinary point. A nonordinary arc will be said isolated, and will be called an INOA, if there exists a set \(\Omega\) satisfying the following conditions:

(C1) \(\Omega\) is an open connected subset of \(\mathbb{R}^2\)
(C2) \(S\) is a relatively closed subset of \(\Omega\)
(C3) If \(x \in \Omega - S\) then \(x\) is an ordinary point
(C4) The set \(\Omega - S\) has exactly two connected components.

A turnpike is an isolated nonordinary arc that satisfies the following conditions:

(S1) For each \(x \in S\) the vectors \(X(x)\) and \(Y(x)\) are not tangent to \(S\) and point to opposite sides of \(S\)
(S2) For each \(x \in S\) one has \(\Delta_B(x) = 0\) and \(\Delta_A(x) \neq 0\)
(S3) Let $\Omega$ be an open set which satisfies (C1)-(C4) above. If $\Omega_X$ and $\Omega_Y$ are the connected components of $\Omega - S$ labelled in such a way that $X(x)$ points into $\Omega_X$ and $Y(x)$ points into $\Omega_Y$, then the function $f$ in (2.14) satisfies

$$f(x) > 0 \quad \text{on} \quad \Omega_Y$$

$$f(x) < 0 \quad \text{on} \quad \Omega_X.$$

Next, consider a turnpike $S$ and a point $x_0 \in S$. We wish to construct a trajectory $\gamma$ of (2.4) such that $\gamma(t_0) = x_0$ and $\gamma(t) \in S$ for each $t \in Dom(\gamma) \equiv [t_0, t_1]$. Clearly, one should have $\Delta_B(\gamma(t)) \equiv 0$ for all $t$. Since $\Delta_B(\gamma(t_0)) = 0$, it suffices to verify that:

$$\frac{d}{dt} \Delta_B(\gamma(t)) = (\nabla \Delta_B \cdot \dot{\gamma})(t) = 0.$$

The above holds provided that

$$(\nabla \Delta_B \cdot uG)(\gamma(t)) + (\nabla \Delta_B \cdot F)(\gamma(t)) = 0.$$

Assuming that

$$(\nabla \Delta_B \cdot G)(x) \neq 0 \quad \forall x \in S,$$  \hspace{1cm} (2.15)

the values of the control $u$ are thus uniquely determined by

$$u = \phi(x) \equiv -\frac{\nabla \Delta_B \cdot F(x)}{\nabla \Delta_B \cdot G(x)}.$$  \hspace{1cm} (2.16)

A turnpike is regular if for every $x \in S$ (2.15) holds true and $|\phi(x)| \leq 1$. A trajectory $\gamma$ is said to be a Z-trajectory if there exists a regular turnpike $S$ such that $\{\gamma(t) : t \in Dom(\gamma)\} \subset S$.

An isolated nonordinary arc or INOA $S$ is a barrier in $\Omega$ (where $\Omega$ satisfies (C1)-(C4)) if it verifies:

(S1') For every $x \in S$, $X(x)$ and $Y(x)$ point to the same side of $S$

(S2') Each of the function $\Delta_A$, $\Delta_B$ is either identically zero on $S$ or nowhere zero on $S$.

A point $x \in \mathbb{R}^2$ is a near ordinary point if it is an ordinary point or belongs to an INOA that is either a turnpike or a barrier.

**Remark 2.1.**

The definition of near ordinary point given in Sussmann (1987 a) is more general, but for our purposes this simpler definition is sufficiently general.
3. Preliminary Theorems.

In this section we show three theorems on control systems with control appearing linearly as in (2.4) and prove a theorem relating the control systems for the swing and the ski with this type of control systems.

Consider the control system (2.4). For ordinary points we have the following:

**Theorem 3.1** Let \( \Omega \subset \mathbb{R}^2 \) be an open set such that each \( x \in \Omega \) is an ordinary point. Then all time optimal trajectories \( \gamma \) for the restriction of (2.4) to \( \Omega \) are bang-bang with at most one switching. Moreover if \( f > 0 \) throughout \( \Omega \) then \( \gamma \) is an \( X, Y \) or \( Y^*X \)-trajectory, if \( f < 0 \) throughout \( \Omega \) then \( \gamma \) is an \( X, Y \) or \( X^*Y \)-trajectory.

For the proof see Sussmann (1987 a, p.443). For near ordinary point we have a similar theorem on the local structure of optimal trajectories, see Sussmann (1987 a, p.459):

**Theorem 3.2** Let \( x \) be a near ordinary point. Then there exists a neighborhood \( \Omega \) of \( x \) such that every time optimal trajectory \( \gamma \) for the restriction of (2.4) to \( \Omega \) is concatenation of at most five trajectories each of which is an \( X-, Y- \) or \( Z- \)-trajectory.

Let consider now a control system as in (2.3):

\[
\dot{x} = h(x,u). \tag{3.1}
\]

If \( w_1, w_2 \in \mathbb{R}^2 \), define the triangle:

\[
C(w_1, w_2) = \{ w \in \mathbb{R}^2 : w = \lambda w_1 + \mu w_2; \lambda, \mu \geq 0; \lambda + \mu \leq 1 \}
\]

and consider the condition:

(P1) There exist two \( C^1 \) vector fields \( w^\pm(x) \) that for every \( x \) either are linearly independent or have the same versus, and such that:

\[
\{w^\pm(x)\} \subset \{h(x,u) : u \in U\}
\]

\[
\{h(x,u) : u \in U\} \subset \{x : x = \lambda w^+(x) + \mu w^-(x), \lambda, \mu \geq 0, 0 \leq \lambda + \mu < 1\} \cup \{w^\pm(x)\} \subset C(w^+(x), w^-(x)).
\]
Suppose that (3.1) verifies (P1). Then we can define a system of the form (2.4) choosing:

\[ F = \frac{w^+ + w^-}{2}, \quad G = \frac{w^+ - w^-}{2} \]  

thus \((F \pm G)(x) = w^\pm(x)\). We can also define the map:

\[ P : \mathbb{R}^2 \times U \to [-1, 1] \]

in the following way. If \(w^\pm(x)\) are independent then \(P(x,v) = u\) if and only if \(h(x,v)\) and \(F(x) + uG(x)\) are parallel. Otherwise, \(P(x,v)\) is constantly equal to +1 if \(F(x), G(x)\) have the same versus (that is if \(|w^+|\) is bigger than \(|w^-|\)) and to −1 if the opposite happens. From (P1) we have that this map is well defined.

Given two points \(x, \tilde{x} \in \mathbb{R}^2\) we consider the two endpoints problem of steering \(x\) to \(\tilde{x}\) in minimum time. Let define the value functions:

\[
V(x, \tilde{x}) = \inf \{T(\gamma) : \gamma \text{ trajectory of (2.4), (3.2), } In(\gamma) = x, Term(\gamma) = \tilde{x} \}
\]

\[
\bar{V}(x, \tilde{x}) = \inf \{T(\gamma) : \gamma \text{ trajectory of (3.1), } In(\gamma) = x, Term(\gamma) = \tilde{x} \}
\]

We have the following:

**Theorem 3.3** Assume that every point is near ordinary for (2.4),(3.2) and consider two points \(x, \tilde{x} \in \mathbb{R}^2\):

(i) there exists a trajectory \(\gamma\) of (2.4),(3.2) such that \(T(\gamma) = V(x, \tilde{x})\)

(ii) if \(\Gamma\) is the set of trajectories of (2.4),(3.2) corresponding to bang-bang controls and \(V'(x, \tilde{x}) = \inf \{T(\gamma) : \gamma \in \Gamma, In(\gamma) = x, Term(\gamma) = \tilde{x} \}\) then:

\[ V'(x, \tilde{x}) = V(x, \tilde{x}). \]

**Proof.** Assume that \(V(x, \tilde{x}) < +\infty\), otherwise there is nothing to prove. The statement (i) follows from the convexity of the set of velocities for (2.4),(3.2), see for example Lee and Markus (1967).

Let \(\gamma\) be a time optimal trajectory steering \(x\) to \(\tilde{x}\). From Theorem 3.1 and 3.2 we have that \(\gamma\) is concatenation of \(X,Y\) and \(Z\)–trajectories. If \(\gamma\) is bang-bang we are done. Assume now that \(\gamma \upharpoonright [t_0, t_1]\) is a \(Z\)–trajectory. From the definition of turnpike we have that for every \(t \in [t_0, t_1]\) the two vectors \(X(\gamma(t)), Y(\gamma(t))\) points to opposite sides of the image of
\[ \gamma. \text{ If } \gamma(t) \text{ is sufficiently near to } \gamma(t_0) \text{ then we can construct a bang-bang trajectory } \hat{\gamma} \text{ with one switching steering } \gamma(t_0) \text{ to } \gamma(t_1), \text{ in the following way. For } |t - t_0| \text{ sufficiently small the } Y-\text{trajectory } \gamma^+ \text{ passing through } \gamma(t_0) \text{ and the } X-\text{trajectory } \gamma^- \text{ passing through } \gamma(t) \text{ meet each other in at least one point. Let } \bar{x} \text{ be the first point in which } \gamma^+ \text{ intersect the image of } \gamma^-, \text{ after passing through } \gamma(t_0). \text{ We can construct } \hat{\gamma} \text{ following } \gamma^+ \text{ up to the point } \bar{x} \text{ and then } \gamma^- \text{ up to the point } \gamma(t). \text{ Divide now } [t_0, t_1] \text{ into } n \text{ equal subintervals inserting the points } k_i = t_0 + (i/n)(t_1 - t_0), i = 0, \ldots, n. \text{ If } n \text{ is sufficiently large we can construct, as above, a bang-bang trajectory steering } \gamma(k_i) \text{ to } \gamma(k_{i+1}), i = 0, \ldots, n - 1. \text{ Let } \hat{\gamma}_n \text{ be the concatenation of these trajectories. Let } K \text{ be a compact neighborhood of the image of } \gamma \text{ and let } M = 2 \max_K (|F| + |G|) < +\infty. \text{ If } n \text{ is sufficiently large then by construction } \hat{\gamma}_n \text{ lies in } K, \text{ hence:}

\[ |\gamma(t) - \hat{\gamma}_n(t)| \leq \frac{t_0 - t_1}{n} M \leq \frac{T(\gamma)}{n} M. \]

The subset \( J \) of Dom(\( \gamma \)) on which \( \gamma \) is a Z-\text{trajectory is a finite union of closed interval. Otherwise we can construct a sequence } t_n \text{ in } \partial J \text{ converging to a time } t \in \text{Dom}(\gamma) \text{ and then } \gamma(t) \text{ is not a near ordinary point. Repeating the same reasonings for every subinterval of } J \text{ we obtain a sequence of bang-bang trajectories converging uniformly to } \gamma. \]

Notice that every bang-bang trajectory for (2.4),(3.2) is a trajectory for (3.1), then \( \bar{V}(x, \bar{x}) \leq V'(x, \bar{x}). \) We can prove the following:

**Theorem 3.4** Assume that (P1) holds true and that every point is a near ordinary point for (2.4),(3.2). Consider two points \( x, \bar{x} \in \mathbb{R}^2 \) then

(i) if there exists a bang-bang time optimal control \( u \) for (2.4),(3.2) steering \( x \) to \( \bar{x} \), then there exists a time optimal control \( v \) for (3.1) corresponding to the same trajectory \( \gamma \) of \( u \), i.e. \( h(\gamma(t), v(t)) = F(\gamma(t)) + u(t)G(\gamma(t)) \in \{ w^\pm(\gamma(t)) \} \);

(ii) if every time optimal control \( u \) for (2.4),(3.2) is not bang-bang (i.e. if \( \gamma(t) \) belongs to a turnpike for some interval) then the time optimal control for (3.1) does not exist but we have:

\[ V(x, \bar{x}) = \bar{V}(x, \bar{x}) \]

and for each \( \varepsilon > 0 \) there exists a control \( v \), corresponding to a trajectory \( \eta \) steering \( x \) to \( \bar{x} \), such that \( h(\eta(t), v(t)) \in \{ w^\pm(\eta(t)) \} \) for each \( t \in \text{Dom}(\gamma) \) and \( T(\eta) \leq \bar{V}(x, \bar{x}) + \varepsilon. \)

**Proof.** Suppose first that there exists a bang-bang time optimal control \( u \) and, by contradiction, that there exists a control of (3.1), with corresponding trajectory \( \eta \) steering \( x \) to \( \bar{x} \), such that \( T(\eta) < T(\gamma) \). If \( h(\eta(t), v(t)) \in \{ w^\pm(\eta(t)) \} \) for almost every \( t \) then \( \eta \) is
a trajectory of (2.4), (3.2) contradicting the optimality of $\gamma$. If this is not true let define the feedback control $\bar{u}(t) = P(\eta(t), v(t))$, where $P$ is the map in (3.3). There exists a trajectory $\tilde{\gamma}$ corresponding to $\bar{u}$ that is a reparametrization of $\eta$ and is a trajectory of (2.4), (3.2). Finally $|F(\eta(t)) + \bar{u}(\eta(t))G(\eta(t))| \geq |h(\eta(t), v(t))|$, therefore:

$$T(\eta) \geq T(\tilde{\gamma}) \geq T(\gamma)$$

that gives a contradiction.

Suppose now that every time optimal control $u$ is not bang-bang. From Theorems 3.1 and 3.2 we have that there exists $I \subset Dom(\gamma)$ such that $\gamma(I)$ is contained in a turnpike. Consider a control $v$ for (3.1) corresponding to a trajectory $\eta$ steering $x$ to $\tilde{x}$. Let define $\bar{u}, \tilde{\gamma}$ as above and define $S = \{t : |h(\eta(t), v(t))| < |F(\eta(t)) + \bar{u}(\eta(t))G(\eta(t))|\}$. If $\text{meas}(S) = 0$ then $\eta$ is a bang-bang trajectory of (2.4), (3.2) and is not time optimal. If $\text{meas}(S) > 0$ then there exists $n$ and $\delta > 0$ such that

$$\text{meas} \left( \left\{ t : |h(\eta(t), v(t))| < |F(\eta(t)) + \bar{u}(\eta(t))G(\eta(t))| - \frac{1}{n} \right\} \right) \geq \delta > 0.$$

Therefore:

$$T(\eta) \geq T(\tilde{\gamma}) + \delta \frac{1}{n} > T(\gamma)$$

for every time optimal $\gamma$.

From Theorem 3.3 it follows the second part of the statement (ii).

Theorem 3.4 shows the relationships between the two control system (2.4), (3.2) and (3.1). If we are able to determine the time optimal control $u$ for a given problem for (2.4), (3.2) then we immediately know all about the same problem for (3.1). Indeed if the time optimal control $u$ is bang-bang then the trajectory $\gamma$ of $u$ is a trajectory also for (3.1), is optimal and corresponds to a control $v$ taking values in $\{r_-, r_+\}$. If every time optimal control is not bang-bang then the time optimal control for (3.1) does not exists.

Thank to Theorems 3.1 and 3.2 in most cases we are able to know the local structure of time optimal control for (2.4), (3.2) and then for (3.1). Using the same methods of Sussmann (1987 a,b,c) and Piccoli (1993 a,b), we are able to solve explicitly many optimization problems for the swing and the ski models. Some examples will be given in the following sections.

4. Time Optimal Control of the Swing.
In this section we treat the problem of time optimal control for the swing considering the minimum time problem and a Mayer type problem.

Recall equation (1.2). We have to verify that this control system satisfies the condition (P1) of section 3. If \( \sin(x_1) = 0 \) then \( \dot{x}_2 = 0 \) for every \( v \in [r_-, r_+] \) and the set of velocities is a segment. If \( \sin(x_1) \neq 0 \), from the second equation of (1.2) we obtain:

\[
v = -\frac{\dot{x}_2}{g \sin(x_1)}
\]

and replacing (4.1) in the first equation of (1.2):

\[
\dot{x}_1 = \frac{x_2 g^2 \sin^2(x_1)}{\dot{x}_2^2}.
\]

It is easy to check from (4.2) that the set of velocities for (1.2) at a given point lies on a branch of hyperbola. Defining \( w^\pm = h(x, r^\pm) \), (P1) holds. Notice that with this definition we have:

\[
(F + G)(x) = h(x, r_-) \quad (F - G)(x) = h(x, r_+).
\]

Therefore if \( u \) is a bang-bang control for the auxiliary system (2.4),(3.2) with corresponding trajectory \( \gamma \), then \( \gamma \) is also a trajectory for (3.1) and corresponds to the control:

\[
v(t) = r_- \quad \text{if} \quad u(t) = 1, \quad v(t) = r_+ \quad \text{if} \quad u(t) = -1.
\]

Hence we can compute the linear system (3.2) associated to this control system as in section 3, explicitly:

\[
F = \begin{pmatrix}
\frac{r_+^2 + r_-^2}{2} x_2 \\
-\frac{g}{2} (r_+ + r_-) \sin(x_1)
\end{pmatrix}, \quad
G = \begin{pmatrix}
\frac{r_+^2 - r_-^2}{2} x_2 \\
\frac{g}{2} (r_+ - r_-) \sin(x_1)
\end{pmatrix}.
\]

For simplicity we define:

\[
a \doteq (r_+ - r_-)^2 \quad b \doteq r_+ + r_- \quad c \doteq r_+ - r_- \quad d \doteq r_+^2 + r_-^2
\]

then:

\[
F = \begin{pmatrix}
\frac{d}{2a} x_2 \\
-\frac{g}{2b} \sin(x_1)
\end{pmatrix}, \quad
G = \begin{pmatrix}
\frac{bc}{2a} x_2 \\
\frac{g}{2c} \sin(x_1)
\end{pmatrix}.
\]

To investigate the local structure of time optimal trajectories we have to compute the functions \( \Delta_A, \Delta_B \), defined in (2.10,11):

\[
\Delta_A = \frac{g c (d + b^2)}{4a} x_2 \sin(x_1)
\]
\[ [F, G] = \left( \begin{array}{c} -\frac{g c (d+b^2)}{4a} \sin(x_1) \\ \frac{g c (d+b^2)}{4a} b x_2 \cos(x_1) \end{array} \right) \]

hence:
\[ \Delta_B = \frac{g c^2 (d+b^2)}{8a} \left( \frac{b}{a} x_2^2 \cos(x_1) + g \sin^2(x_1) \right). \] (4.6)

Every turnpike is subset of \( \Delta_B^{-1}(0) \), then we have to solve the equation \( \Delta_B(x) = 0 \) that is equivalent to:
\[ \frac{b}{a} x_2^2 \cos(x_1) + g \sin^2(x_1) = 0 \]

that gives:
\[ x_2^2 = -\frac{g a \sin^2(x_1)}{b \cos(x_1)}. \]

By periodicity we can restrict ourselves to the case \( x_1 \in [0, 2\pi] \). There is the isolated solution \((0, 0)\) and if \( x_1 \in ]\frac{\pi}{2}, \frac{3\pi}{2}[, \) we have the two solutions:
\[ x_2 = \pm \sqrt{-\frac{g a \sin^2(x_1)}{b \cos(x_1)}}. \] (4.7)

otherwise there is no solution.

The two branches of solutions form two curves, one contained in the first quadrant, the other in the forth quadrant. The two curves meet each other at the point \((\pi, 0)\) and \((0, 0), (\pi, 0)\) are the only points of \( \Delta_B^{-1}(0) \) that verify \( \nabla \Delta_B(x) = 0 \). For each \( x \in \Delta_B^{-1}(0) \setminus \{(0,0), (\pi,0)\} \) we can compute the control \( \phi(x) \) defined in (2.8):
\[ \phi(x) = \phi(x_1) = \frac{(2b^2 - d) \cos^2(x_1) - d}{b c (3 \cos^2(x_1) + 1)}. \] (4.8)

From (4.8) we have:
\[ \lim_{x_1 \to \frac{\pi}{2}} \phi(x_1) = \lim_{x_1 \to \frac{\pi}{2}} \phi(x_1) = -\frac{r_+^2 + r_-^2}{r_+^2 - r_-^2} < -1 \]
\[ \lim_{x_1 \to \pi} \phi(x_1) = \frac{2b^2 - 2d}{4b c} = \frac{r_+ r_-}{r_+^2 - r_-^2} > 0 \]
\[ \frac{d\phi(x_1)}{dx_1} = -\frac{4(b^2 + d) \sin(x_1) \cos(x_1)}{b c (3 \cos^2(x_1) + 1)^2} \]

then \( \phi \) is increasing for \( x_1 \in ]\pi/2, \pi[ \) and decreasing for \( x_1 \in ]\pi, 3\pi/2[ \). Therefore there exist \( \varepsilon_1 > 0, \varepsilon_2 \geq 0 \) such that \( |\phi(x_1)| \leq 1 \) for \( x_1 \in ]\pi/2 + \varepsilon_1, \pi - \varepsilon_2[ \cup ]\pi + \varepsilon_2, 3\pi/2 - \varepsilon_1[ \neq \emptyset \). We have that \( \varepsilon_2 = 0 \) if and only if:
\[ \frac{r_+ r_-}{r_+^2 - r_-^2} \leq 1 \]
i.e. if and only if:

\[ r_+ \geq r_- \frac{1 + \sqrt{5}}{2}. \]

Hence regular turnpikes do exist. We can choose two points \( x, \tilde{x} \in \mathbb{R}^2 \) such that the only time optimal trajectory for (2.4), (4.4) that steers \( x \) to \( \tilde{x} \) is not bang-bang: it is sufficient to take two points of the same turnpike, see Sussmann (1987 a). In this case the time optimal control for (1.2) does not exist and the second part of Theorem 3.4 applies.

Now, Theorems 3.1 and 3.2 determine the local structure of the time optimal trajectories for the system (2.4), (4.4) and then for the swing model. Observe that \( \Delta_{A}^{-1}(0) = \{(x_1, x_2) : x_2 = 0 \text{ or } \sin(x_1) = 0\} \). Therefore it easy to check that \( \Delta_{A}^{-1}(0) \setminus \{(0, 0), (\pi, 0)\} \) is union of a finite number of INOAs that are barriers. Moreover, from the reasoning above we have that \( \Delta_{B}^{-1}(0) \setminus \{(0, 0), (\pi, 0)\} \) is union of a finite number of INOAs each of which is either a turnpike or a barrier. Hence every point of \( \Omega = \mathbb{R}^2 \setminus \{(0, 0), (\pi, 0)\} \) is near ordinary. Notice that if \( \gamma \) is a trajectory of (2.4), (4.4) and \( \gamma(t) \in \{(0, 0), (\pi, 0)\} \) for some time \( t \in \text{Dom}(\gamma) \), then \( \gamma \) is a constant trajectory. Indeed, in this case, the control has no effect being \( X(\gamma(t)) = Y(\gamma(t)) = 0 \). Moreover, \( (0, 0), (\pi, 0) \) are the only point in which either \( X \) or \( Y \) vanishes. Thus if \( \gamma \) is a trajectory of (2.4), (4.4) we have that either \( \gamma \) is constant or \( \gamma(t) \in \Omega \) for every \( t \in \text{Dom}(\gamma) \). In the latter case we can apply Theorems 3.1 and 3.2.

Given the expression of \( \Delta_{A}, \Delta_{B} \) we can calculate \( f \) of (2.14):

\[ f(x_1, x_2) \doteq \frac{c \left[ \frac{b}{a} x_2^2 \cos(x_1) + g \sin^2(x_1) \right]}{2 x_2 \sin(x_1)}. \]

Consider the set:

\[ Q \doteq \left\{ (x_1, x_2) : |x_1| \leq \frac{\pi}{2} \right\} \]

and a time optimal trajectory \( \gamma \) that verifies \( \gamma(t) \in Q \) for every \( t \in \text{Dom}(\gamma) \). We have, from Theorems 3.1 and 3.2, that \( \gamma \) is bang–bang and that \( \gamma \) can switches from control +1 to control −1 if \( x_2 \sin(x_1) > 0 \) and from control −1 to control +1 if \( x_2 \sin(x_1) < 0 \). Therefore \( Q \setminus \Delta_{A}^{-1}(0) \) is divided is four parts and on each parts only one kind of switching is permitted. This correspond to the fact that if the swing is raising his distance from the earth then the swinger can change only from control \( r_- \) to control \( r_+ \), instead if the swing is lowering his distance from the earth then the swinger can change only from \( r_+ \) to \( r_- \). In this case, the map \( P \) of (3.3) is bijective for almost every \( x \) and we can establish a bijective correspondence between the trajectories of (1.2) and of (2.4), (4.4). In particular
if $\gamma : [a,b] \to \mathbb{R}^2$ is a trajectory of (2.4),(4.4) then there exists a trajectory $\eta : [c,d] \to \mathbb{R}^2$ of (1.2) verifying $\eta(c) = \gamma(a), \eta(d) = \gamma(b)$ and $\eta(t) \in \gamma([a,b])$ for every $t \in [c,d]$. Therefore also some geometric properties of (2.4),(4.4) hold for (1.2). For example the reachable sets from a given point are the same for (2.4),(4.4) and for (1.2).

In Alberto Bressan (1993) it was considered the problem of raising the amplitude of the first half oscillation starting from a given point $x \in \mathbb{R}^2$. Notice that if $\sin(x_1) > 0$ then $F_2 - G_2 > F_2 + G_2 > 0$ and the opposite happens if $\sin(x_1) < 0$. Hence, comparing the vector $X,Y$, it is easily seen that if we want to reach the maximum amplitude of the first half oscillation we must choose the control $+1$ if $x_2 \sin(x_1) > 0$ and $-1$ if $x_2 \sin(x_1) < 0$. For the swing this means to choose $r_-$ if we are raising the distance from the earth and $r_+$ in the other case. Obviously this is also the optimal control to raise the amplitude after a given number of oscillation.

Using the Pontryagin Maximum Principle and Lemma 2.1,2.2 and 2.3 we can prove the following:

**Lemma 4.1.** Assume that $\eta : [a,b] \to \mathbb{R}^2$ is a bang–bang trajectory of (1.2), $\eta(t) \in Q$ for every $t \in [a,b]$, and that there exists $t_1,t_2 \in (a,b)$ such that either the first or the second component of $\eta(t_1)$ vanishes and the same hold for $\eta(t_2)$. If $\eta$ has no switching then $\eta$ can not be optimal.

**Proof.** Assume for example that $\eta$ corresponds to the constant control $+1$ and, by contradiction, that $\eta$ is optimal. Observe that $\eta$ is a trajectory of (2.4),(4.4), then if $\eta$ is optimal there exists an adjoint covector field $\lambda(t), t \in [a,b]$, along $\eta$. We have, from Lemma 2.2, that $\lambda(t_1) \cdot G(\eta(t_1)) \geq 0$ and, from $\Delta_A(\eta(t_1)) = 0$, that $G(\eta(t_1)), Y(\eta(t_1))$ are parallel. From Lemma 2.3 and (4.6) the function:

$$\alpha(t) = \arg \left( Y(\eta(t_1)), v(G(\eta(t)), t; t_1) \right)$$

is strictly increasing. It is easy to check from (2.5) and (4.4) that the function:

$$\psi(t) = \det \left( v(G(\eta(t_2)), t_2; t), v(Y(\eta(t_2)), t_2; t) \right)$$

is constant. Assume for example that $t_2 > t_1$. We have that $G(t_2) = v(G(\eta(t_2)), t_2; t_2)$ and $Y(t_2) = v(Y(\eta(t_2)), t_2; t_2)$ are parallel because $\Delta_A(\eta(t_2)) = 0$. Hence $v(G(\eta(t_2)), t_2; t_1)$ and $v(Y(\eta(t_2)), t_2; t_1)$ are parallel, but from (2.5) it follows $v(Y(\eta(t_2)), t_2; t_1) = Y(\eta(t_1))$. We conclude that $\alpha(t_2) = k \pi$ for some integer $k > 0$. 


Now, from Lemma 2.1 and 2.2:

\[ \lambda(t) \cdot G(\eta(t)) = \lambda(t_1) \cdot v(G(\eta(t)), t; t_1) \geq 0 \quad \forall t \in [a, b]. \]

But since \( \alpha(t_2) \geq \pi \) the vector \( v(G(\eta(t)), t; t_1) \) for \( t \in [a, b] \) makes a rotation of an angle strictly greater than \( \pi \) and then there is no vector \( \lambda(t_1) \) for which the above inequality can hold.

It is useful the following:

**Lemma 4.2.** Assume that \( \gamma \) is a time optimal trajectory of (2.4),(4.4), that \( \gamma \) has a switching at time \( t_1 \in \text{Dom}(\gamma) \) and that \( \Delta_A(\gamma(t_1)) = 0 \). Then \( \Delta_A(\gamma(t_2)) = 0, t_2 \in \text{Dom}(\gamma) \), if and only if \( t_2 \) is a switching time for \( \gamma \).

**Proof.** Let \( u(t), t \in \text{Dom}(\gamma) \), be the control corresponding to \( \gamma \). Since \( \gamma \) is optimal there exists an adjoint covector field \( \lambda \) along \((u, \gamma)\). Let \( t_2 \) be the first time that either is a switching time or that verifies \( \Delta_A(\gamma(t_2)) = 0 \). We have that \( u \uparrow [t_1, t_2] \) is constant, say \( u \equiv 1 \), and that \( \lambda(t_1) \cdot G(\gamma(t_1)) = 0 \). But \( G(\gamma(t_1)) \) and \( F(\gamma(t_1)) + G(\gamma(t_1)) \) are parallel, hence \( \lambda(t_1) \cdot [F(\gamma(t_1)) + G(\gamma(t_1))] = 0 \). From Lemma 2.1 have:

\[ \lambda(t_2) \cdot [F(\gamma(t_2)) + G(\gamma(t_2))] = \lambda(t_1) \cdot v(F(\gamma(t_2)) + G(\gamma(t_2)), t_2; t_1) = \]

\[ = \lambda(t_1) \cdot [F(\gamma(t_1)) + G(\gamma(t_1))] = 0. \]

Now if \( \Delta_A(\gamma(t_2)) = 0 \) we have that \( G(\gamma(t_2)), F(\gamma(t_2)) + G(\gamma(t_2)) \) are parallel then \( t_2 \) is a switching time. On the other hand, if \( t_2 \) is a switching time then \( \lambda(t_2) \cdot G(\gamma(t_2)) = 0 \), hence being \( \lambda(t_2) \neq 0 \) we have that \( G(\gamma(t_2)) \) and \( F(\gamma(t_2)) + G(\gamma(t_2)) \) are parallel. This means \( \Delta_A(\gamma(t_2)) = 0 \). We can argue in the same way for the other switching times.

Consider now the problem of reaching, with the swing, an angle \( \hat{\theta} \in (0, \pi/2) \) in minimum time with given initial condition. We argue about the corresponding problem for (2.4),(4.4) with initial point \( \bar{x} \in Q \). We can solve this problem constructing all the time optimal trajectories starting from \( \bar{x} \) and using the (final) transversality condition of the PMP, see for example Lee and Markus (1967), to select among these trajectories the optimal one. To construct all the time optimal trajectories we can proceed as in Piccoli (1993 a,b).

Assume, for example, that \( \bar{x} = (\bar{x}_1, \bar{x}_2), \bar{x}_1 > 0, \bar{x}_2 > 0 \), being similar the other cases. Let \( \gamma \) be a time optimal trajectory that verifies \( In(\gamma) = \gamma(0) = \bar{x} \). If \( \gamma \) lies on the first
Then if we want to construct all the time optimal trajectories it is enough to consider the trajectories $\gamma_s$, $\gamma_s(0) = \bar{x}$, that follow $\gamma^+$ for a given time $s \in [0, t^+]$ and then switch to control $-1$. Now if $\gamma_s$ is time optimal then there exists an adjoint covector field $\lambda_s$ along $\gamma_s$. Since $\lambda_s(s) \cdot G(\gamma_s(s)) = 0$ we can determine $\lambda_s$ up to the product by a positive scalar. Hence we can compute the switching function $\phi_s = \lambda_s \cdot G(\gamma_s)$ and, by Lemma 2.2, determine the behaviour of $\gamma_s$, that is its switching times. Using Lemma 4.2 we can see that, for every $s$, $\gamma_s$ has to switch on each quadrant. More precisely $\gamma_s$ will make a second switching before reaching the $x_2$-axis, then a third switching before the second time of intersection with the $x_1$-axis and so on. The set of switching points of $\gamma_s$ form some manifolds called switching curves, see Piccoli (1993 b) for the exact definition. Each switching curve lies on a quadrant. After a given number of oscillations some $\gamma_s$ reach the manifold $\mathcal{M} = \{(x_1, x_2) : x_1 = \bar{\theta}\}$. To select the optimal trajectory between these $\gamma_s$, we can use the transversality condition that, in this case, is $\lambda_s(t_s) \cdot (0, 1) = 0$ if $t_s$ is the first time of intersection of $\gamma_s$ with $\mathcal{M}$.

To solve numerically our problem, first we have to solve the equation for the swing (1.2), for constant control $r_+$ and $r_-$, and a class of initial data (that is to approximate some elliptic integrals). Indeed every $\gamma_s$ is a finite concatenation of such trajectories. Then we have to consider the complementary equation (2.6) for $\gamma_s$. This is a linear system and can be solved numerically by usual methods, determining, up to a scalar, the adjoint covector field $\lambda_s$ with initial condition $\lambda_s(s) \cdot G(\gamma_s(s)) = 0$. Finally the transversality condition determines a value $\bar{s}$, and then the corresponding trajectory $\gamma_{\bar{s}}$.

If $\bar{x} \notin Q$ and $|\bar{\theta}| > \pi/2$ then the construction of time optimal trajectories is more difficult. We can do it following Piccoli (1993 a,b), but in this case we have to take into account the turnpikes. It can happen that some time optimal trajectories are $Z$-trajectories for some time interval of positive measure, and the optimal control for the swing does not exist. An example of this situation is given in the following.
We now consider the Mayer problem:
\[
\begin{cases}
\dot{\gamma}(t) = F(\gamma(t)) + u(t) G(\gamma(t)) \\
\gamma(0) = \bar{x} \\
\min \{ t : \exists \gamma \text{ such that } \gamma(t) \in \mathcal{M} \}
\end{cases}
\] (4.9)
where \( \mathcal{M} \equiv \{(x_1, x_2) : x_1 = c\}, c \in \mathbb{R} \). We consider the case in which \( c = 3\pi/2 \), \( \bar{x} = (\bar{x}_1, \bar{x}_2) \) belongs to a turnpike and \( \bar{x}_1 \in ]\pi, 3\pi/2[, \bar{x}_2 > 0 \). Our aim is to show the existence of some values of \( r_\pm \) such that the problem (4.9) with the dynamics (1.2) does not have a solution.

If \( \gamma \) is a solution then in particular \( \gamma \) is time optimal for steering \( \bar{x} \) to \( \text{Term}(\gamma) \) and hence is a concatenation of \( X-, Y- \) and \( Z- \) trajectories. Following the algorithm in Piccoli (1993 a,b), we can cover a region of the plane with the time optimal trajectories starting from \( \bar{x} \). Let \( S \) be the turnpike to which \( \bar{x} \) belongs. The region of first quadrant below \( S \) is covered by \( Y- \) trajectories originating from \( S \) and the region over \( S \) is covered by \( X- \) trajectories. If \( x' \) is the endpoint of \( S \) that comes after \( \bar{x} \) for the orientation given by \( X, Y \), then the \( X- \) trajectories that cross \( \Delta_B^{-1}(0) \) over \( x' \) must switch changing the control to \( +1 \). The points in which these trajectories change control form a curve called switching curve, see Piccoli (1993 b). Therefore if \( \gamma \) is a time optimal trajectory we have two possibilities: either \( \gamma = \gamma_2 \ast \gamma_1 \) where \( \gamma_1 \) is a \( Z \)-trajectory possibly trivial and \( \gamma_2 \) is a \( Y \)-trajectory, or \( \gamma = \gamma_3 \ast \gamma_2 \ast \gamma_1 \) where \( \gamma_1 \) is a \( Z \)-trajectory possibly trivial, \( \gamma_2 \) is an \( X \)-trajectory and \( \gamma_3 \) is a \( Y \)-trajectory.

Our aim is to find some \( r_\pm \) such that the solution to (4.9) is of the first type with \( \gamma_1 \) not trivial, i.e. \( \text{Dom}(\gamma) \) is not a single point.

Let define \( \gamma_\pm \) as the \( Y- \) trajectory that verifies \( \text{In}(\gamma_\pm) = \gamma_\pm(0) = x \) and let \( \gamma^S_\pm \) be the \( Z- \) trajectory that satisfies \( \text{In}(\gamma^S_\pm) = \gamma^S_\pm(0) = x \). We have that \( \gamma_\pm \) satisfies:
\[
\begin{cases}
\dot{x}_1 = \frac{d+b \ c}{2 \ a} \ x_2 \\
\dot{x}_2 = \frac{g \ (c-b)}{2} \ \sin(x_1)
\end{cases}
\]
hence:
\[
\ddot{x}_1 = \left( \frac{d+b \ c}{2 \ a} \right) \ \dot{x}_2 = \left( \frac{d+b \ c}{2 \ a} \right) \ \left( \frac{g \ (c-b)}{2} \right) \ \sin(x_1).
\] (4.10)

We define:
\[
\alpha = \frac{d+b \ c}{2 \ a}, \quad \beta = \frac{g \ (c-b)}{2}
\] (4.11)
and \( \omega \) by:
\[
\omega^2 = -\alpha \beta.
\] (4.12)
We can solve (4.10) in $t$ using the first integral:

$$
\dot{x}_1^2 - 2 \omega^2 \cos(x_1)
$$

and obtaining:

$$
\dot{x}_1^2 = \alpha^2 \overline{x}_2^2 + 4 \omega^2 \left( \sin^2 \frac{x_1}{2} - \sin^2 \frac{x_1}{2} \right).
$$

(4.13)

Now let $\tilde{x} = \gamma_{\tilde{x}}(T)$ where $T$ is the first time at which $\gamma_{\tilde{x}}$ intersects $\mathcal{M}$ and define:

$$
k \equiv \sin \frac{x_1}{2} \quad a_0^2 \equiv \frac{4 \omega^2}{\alpha^2 \overline{x}_2^2 + 4 \omega^2 k^2}.
$$

(4.14)

After straightforward calculations from (4.13,14) we obtain:

$$
T = \frac{1}{\sqrt{\alpha^2 \overline{x}_2^2 + 4 \omega^2 k^2}} \int_{\tilde{x}_1}^{\overline{x}_1} \frac{dx_1}{\sqrt{1 - a_0^2 \sin^2 \frac{x_1}{2}}}
$$

and, using the substitution $\theta = \frac{x_1}{2}$:

$$
T = \frac{2}{\sqrt{\alpha^2 \overline{x}_2^2 + 4 \omega^2 k^2}} \int_{\tilde{x}_1}^{\overline{x}_1} \frac{d\theta}{\sqrt{1 - a_0^2 \sin^2 \theta}} = \frac{2}{\sqrt{\alpha^2 \overline{x}_2^2 + 4 \omega^2 k^2}} \left[ E\left(a_0, \frac{\tilde{x}_1}{2}\right) - E\left(a_0, \frac{x_1}{2}\right) \right]
$$

(4.15)

where:

$$
E(l, \bar{\theta}) = \int_{0}^{\bar{\theta}} \frac{d\theta}{\sqrt{1 - l^2 \sin^2 \theta}}
$$

(4.16)

is the elliptic integral of the second type. Now we want to compute the time along the trajectory $\gamma_s$ that satisfies $\gamma_s \mid [0, s] = \gamma_{\tilde{x}}^R \mid [0, s]$ and that is a $Y$-trajectory after the time $s$. Define $T(s)$ to be the first time in which $\gamma_s$ intersects $\mathcal{M}$. We have:

$$
T(s) = s + \frac{2}{\sqrt{\alpha^2 \overline{x}_2^2(s) + 4 \omega^2 k_s^2}} \left[ E\left(a_s, \frac{3 \pi}{2}\right) - E\left(a_s, \frac{x_1(s)}{2}\right) \right]
$$

(4.17)

where:

$$
x(s) \equiv \gamma_{\tilde{x}}^R(s) \quad k_s \equiv \sin \left( \frac{x_1(s)}{2} \right) \quad a_s^2 \equiv \frac{4 \omega^2}{\alpha^2 \overline{x}_2^2(s) + 4 \omega^2 k_s^2}.
$$

(4.18)

For any $s \geq 0$ we can calculate the difference between $T$ and $T(s)$. It is clear that if:

$$
\left. \frac{d}{ds} T(s) \right|_{s=0}
$$

(4.19)
is negative then $\gamma_+^{x}$ is not optimal for (4.9). In computing (4.19) we will use the approximations:

$$x_1(s) = \bar{x}_1 + (F_1 + \phi G_1)(\bar{x}) s + o(s) \quad x_2(s) = \bar{x}_2 + (F_2 + \phi G_2)(\bar{x}) s + o(s).$$

After some calculations from (4.15,16,17,18) it follows:

$$\frac{d}{ds} T(s) \bigg|_{s=0} =$$

$$1 + (\alpha^2 \bar{x}_2^2 + 4\omega^2 k^2)^{-\frac{3}{2}} \left[ - \int_{\bar{x}_1}^{\bar{x}_2} (1 - a_0^2 \sin^2 \theta)^{\frac{1}{2}} + (1 - a_0^2 \sin^2 \theta)^{-\frac{3}{2}} a_0^2 \sin^2 \theta d\theta \right] +$$

$$\frac{-d + \phi(\bar{x}_1)}{2a} \bar{x}_2 \left[ (\alpha^2 \bar{x}_2^2 + 4\omega^2 k^2) \left( 1 - a_0^2 k^2 \right) \right]^{-\frac{1}{2}}$$

(4.20)

where:

$$A = \alpha^2 g \left( \phi(\bar{x}_1) c - b \right) \bar{x}_2 \sin \bar{x}_1 + 2\omega^2 \bar{x}_2 \sin \bar{x}_1 \cos \frac{\bar{x}_1}{2} \cos \frac{\bar{x}_1}{2} \left( \frac{d + \phi(\bar{x}_1) b c}{a} \right) =$$

$$= \frac{\alpha g c}{2a} \left( d + b^2 \right) \left( \phi(\bar{x}_1) - 1 \right) \bar{x}_2 \sin \bar{x}_1.$$  

(4.21)

Notice that the second and the third terms on the righthand side of (4.20) are negative. If $r_-$ tends to zero then the second term tends to:

$$- \frac{\cos(\bar{x}_1)}{\sin(\bar{x}_1)} \frac{3 \cos^2(\bar{x}_1) - \cos(\bar{x}_1) + 2}{3 \cos^2(\bar{x}_1) + 1} \left( \frac{\bar{x}_1 - \bar{x}_1}{2} \right).$$

(4.22)

Now if we choose $r_+$ sufficiently large then $|\phi(\bar{x}_1)| \leq 1$ for $\bar{x}_1 \in [\pi, \pi + \varepsilon[$ and some $\varepsilon > 0$, hence we can take $\bar{x}_1$ arbitrarily near to $\pi$. But as $\bar{x}_1$ tends to $\pi$, the expression in (4.22) tends to minus infinity. In particular we can choose $r_-, r_+$ and $\bar{x}_1$ in such a way that:

$$\frac{d}{ds} T(s) \bigg|_{s=0} < 0.$$  

(4.23)

Now it is easy to see that, choosing $r_+, \bar{x}_1$ as above, the $Y \ast X$–trajectory leaving from $\bar{x}$ can not be optimal for (4.9), in fact $X(\bar{x})$ tends to zero. On the other hand for (4.23) the $Y$–trajectory starting from $\bar{x}$ can not be optimal. Hence the time optimal trajectory for (4.9) contains a non trivial $Z$–trajectory. Using Theorem 3.4 we obtain that the problem (4.9) with the dynamics (1.2) has no optimal solution.
Given the time optimal control $u$ for (4.9), corresponding to the trajectory $\gamma$, we can consider the feedback control $v$ such that, see (3.3):

$$P(\gamma(t), v(\gamma(t))) = u(t).$$

Indeed, in this case, the function $P$, defined in the third section, is bijective outside the coordinate axes and the line $\{x_1 = \pi\}$. We have:

$$v(\gamma(t)) = \sqrt[3]{-a \frac{u(t) c - b}{u(t) b c + d}}.$$  \hspace{1cm} (4.24)

The trajectory $\eta$ corresponding to the control $v$ is a reparametrization of $\gamma$.

Since $\gamma_1(t), \eta_1(t) > 0$ for every $t$ we can consider the inverse functions $t_\gamma(x_1), t_\eta(x_1)$ and define $\gamma(x_1) = \gamma(t_\gamma(x_1))$, $\eta(x_1) = \eta(t_\eta(x_1))$. If $\bar{x} = \text{Term}(\gamma)$ then:

$$T(\gamma) = \int_{\bar{x}_1}^{\bar{x}_1} dx_1 \frac{dx_1}{F_1(\gamma(x_1)) + u(t_\gamma(x_1)) G_1(\gamma(x_1))}$$

$$T(\eta) = \int_{\bar{x}_1}^{\bar{x}_1} \frac{v^2(\eta(x_1))dx_1}{\eta_2(x_1)}.$$ \hspace{1cm}

Then we can calculate the difference of times $T(\gamma) - T(\eta)$ computing the difference of the two integrals. We can obtain a better performance using a bang-bang control but it is not easy to compute explicitly one such control and its total variation tends to infinity as its time tends to the minimum. The control $v$ has the advantage of being defined by the explicit formula (4.24) and of having a fixed variation.

\section{Time Optimal Control for the Ski.}

In this section we deal with time optimal control problems for the ski. We consider a skier on a one-dimensional trail described by a curve $(x(s), y(s))$ in the plane. The skier can choose the height $u$ of his baricentre, that is the distance from the point of contact with the trail, within certain physical bounds. We refer to Aldo Bressan (1991) for assumptions and notations. The dynamics is given by the system:

$$\begin{cases}
\dot{s} = \frac{p}{I(s, u)} \\
\dot{p} = \frac{t}{2 M} p^2 - M g (1 - c u) y'(s)
\end{cases} \hspace{1cm} u \in [r_-, r_+] \hspace{1cm} (5.1)$$
where \( s \) is the arclength of the trail, \( p \) its conjugate momentum, \( c(s) \) the curvature of the trail, \( \mathcal{I} \) is \( c^2 \) times the inertial moment of the pair ski-skier with respect to the centre of curvature, \( \mathcal{I}_s \) its partial derivative with respect to \( s \), \( g \) the gravity acceleration and \( M \) is the total mass of the pair ski-skier. We approximate the body of the skier with a system of \( n \) equal masses with height \( i h/n, \ i = 1, \ldots, n \), where \( h \) is the height of the top of the skier. We have:

\[
h = b u, \quad b = 2 \frac{m + m_s}{m} \frac{n}{n + 1}
\]

and, see Bressan (1991):

\[
\mathcal{I}(s, u) = \alpha c^2(s) + m_s + \frac{m}{n} \sum_{i=1}^{n} \left( 1 - c(s) \frac{b}{n} \frac{i}{u} \right)^2
\]  

(5.2)

where \( \alpha \) is the inertial moment of the ski with respect to its baricentre, \( m_s \) its mass and \( m \) the mass of the skier, so that \( M = m + m_s \). In Bressan (1991) it was assumed that:

\[
u c \leq 1 \quad u \in [r_-, r_+]
\]  

(5.3)

\[
\text{sgn} \left( \frac{\partial \mathcal{I}}{\partial u} \right) = -\text{sgn}(c)
\]  

(5.4)

where, by definition, \( \text{sgn}(0) = 0 \). It is easy to verify that (5.3,4) hold under the hypotheses (5.2).

We now consider the case in which the curvature \( c \) is constant. Then \( \mathcal{I}_s = 0 \) and (5.1) reduces to:

\[
\begin{cases}
\dot{s} = p \left( \alpha c^2(s) + m_s + m + m c^2 b^2 u^2 \left( \frac{(2n+1)(n+1)}{6n^2} \right) - m c b u \left( \frac{(n+1)}{n} \right) \right)^{-1} \\
\dot{p} = -M g (1 - c u) y'(s)
\end{cases}
\]  

(5.5)

Now we want to verify that, under suitable conditions on the curvature, the control system (5.5) satisfies the hypothesis (P1) of the third section. If \( c = 0 \) then the control disappear and we obtain a dynamical system. If \( p = 0 \) or \( y'(s) = 0 \) the set of velocities is a segment and we are done. Hence from now on we assume that \( c, p, y'(s) \neq 0 \). From the second equation of (5.5) we can express \( u \) as a function of \( \dot{p} \) and substitute its value in the first equation obtaining:

\[
\dot{s} = \frac{p}{P(\dot{p})}
\]

where:

\[
P(\dot{p}) = \dot{p}^2 \left( \frac{m b^2 (2n + 1) (n + 1)}{M^2 g^2 [y'(s)]^2 6n^2} \right) + \dot{p} \left( \frac{m b [b(2n + 1) - 3n] (n + 1)}{M g y'(s) 3n^2} \right)
\]
\[ + \frac{6 \alpha c^2 n^2 + m b^2 (2n^2 + 3n + 1) - 6 m b n (n + 1) + 6 (m + m_s) n^2}{6 n^2}. \]

The discriminant of the polynomial \( P \) is:

\[ \text{disc}(P) = - \frac{m b^2 (n + 1) [2 \alpha c^2 (2n + 1) + m (n + 1) + 2 m_s (2n + 1)]}{3 M^2 g^2 [y'(s)]^2 n^2} \]

then \( P(\dot{p}) \) has no zeros.

The zeros of the second derivative of \( \dot{s} \) with respect to \( \dot{p} \) are:

\[ \dot{p}_\pm = \frac{g M y'(s) [3n - b (2n + 1)]}{b (2n + 1)} \pm \frac{g M y'(s) n}{\sqrt{mb (2n + 1) (n + 1)}} \sqrt{2 (\alpha c^2 + m_s) (2n^2 + 3n + 1) + m (n^2 - 1)}. \]

Then we compute \( u_\pm \in \mathbb{R} \) in such a way that \( \dot{p}(u_\pm) = \dot{p}_\pm \):

\[ u_\pm = n \left[ 3 \sqrt{m} (n + 1) \pm \sqrt{2 (\alpha c^2 + m_s) (2n^2 + 3n + 1) + m (n^2 - 1)} \right] \frac{b c \sqrt{m} (2n + 1) (n + 1)}{b c \sqrt{m} (2n + 1) (n + 1)}. \]

If \( c < 0 \) then \( u_+ < 0 \) and the condition \( u_- < 0 \) gives:

\[ c^2 < \frac{m (4 n + 5)}{\alpha (2 n + 1)} - \frac{m_s}{\alpha}. \quad (5.6) \]

If \( c > 0 \) then (5.6) gives \( u_- > 0 \) but if we assume:

\[ \frac{3 n}{b c (2n + 1)} - r_+ > 0 \quad (5.7) \]

then the condition \( u_- > r_+ \) gives:

\[ a_2 c^2 + a_1 c + a_0 > 0 \quad (5.8) \]

where:

\[ a_2 = (2n + 1) (n + 1) [m b^2 r_+ (2n + 1) (n + 1) - 2 \alpha n^2] \quad (5.9) \]

\[ a_1 = -6 m b r_+ n (2n + 1) (n + 1)^2 \quad a_0 = 2 n^2 (n + 1) [m (4n + 5) - m_s (2n + 1)]. \]

For \( n \) sufficiently large and for standard data, the conditions (5.7,8,9) are similar to the condition (5.3) only slightly more restrictive. If we assume that (5.6,7,8,9) are verified then \( \dot{s}(\dot{p}(u), u \in [r_+, r_-] \) is convex or concave, depending on the signes of \( p, c \). Moreover
we have that \( \dot{s}(\dot{p}) \) tends to zero as \( |\dot{p}| \) tends to infinity and it is easy to check that (P1) is verified.

The assumptions (5.6,7,8,9) are natural in fact in the application we have some bounds on the curvature.

We can now compute the associated system with control appearing linearly, as in section 3, to obtain information about the time optimal controls. If \( c = 0 \) the control does not appear and we have a dynamical system as observed above. Hence from now on we assume that \( c \neq 0 \). We have:

\[
F = \left( -\frac{mg}{2} y'(s) \left( 2 - c (r_+ + r_-) \right) \right) \quad \quad \quad G = \left( -\frac{mg}{2} y'(s) c (r_+ - r_-) \right)
\]

where \( I_{\pm} = I(r_{\pm}) \). After straightforward calculations it follows:

\[
[F, G] = \frac{mg}{2} L \begin{pmatrix} y'(s) \\ -p y''(s) \end{pmatrix}
\]

where:

\[
H = \frac{I_+ + I_-}{2 \cdot I_+ I_-} \quad J = \frac{I_+ - I_-}{2 \cdot I_+ I_-} \quad L = c \left( r_+ - r_- \right) - J \left( 2 - c (r_+ + r_-) \right)
\]

and:

\[
\Delta_B = \frac{mg}{4} \left( -2 J L p^2 y''(s) + m g L c (y'(s))^2 \left( r_+ - r_- \right) \right).
\]

Therefore the equation for turnpikes is:

\[
p^2 = \frac{mg}{2} \frac{c (r_+ - r_-) (y'(s))^2}{J y''(s)}. \tag{5.11}
\]

Observe that the first factor of (5.11) is negative in fact \( sgn(J) = -sgn(c) \) (from (5.4)).

For example if we consider the curve:

\[
\left( \frac{\cos(|c| s)}{|c|}, \frac{\sin(|c| s)}{|c|} \right), \quad s \in [0, 2\pi/|c|] \tag{5.12}
\]

we have:

\[
p = \pm \sqrt{\frac{mg}{2 |J|} \frac{c (r_+ - r_-) \cos^2(|c| s)}{\sin(|c| s)}}.
\]

We can argue as in section 4 for time optimal control problems. Given a minimum time problem, if every time optimal trajectory for the system (2.4),(5.10) contains a
$Z$-trajectory then the time optimal control for the ski does not exist. If the opposite happens then the time optimal control exists and is bang-bang. The geometric structure of the system (2.4),(5.10) is very similar to that one of the system (2.4),(4.4). The only difference is the sign of the function $f$ of (2.14) that can be different. Consider again the curve (5.12). Since we are considering a skier, (5.12) makes sense only if we consider the restriction $s \in [0, \pi/|c|]$ if $c < 0$ and $s \in [\pi/|c|, 2\pi/|c|]$ if $c > 0$.

For the case $c > 0$ we can repeat the reasoning made for the swing, that is for the system (2.4),(4.4). Indeed, with the above restriction, the system (2.4),(5.10) is similar to the system (2.4),(4.4) restricted to the set $Q$ of section 4. In particular we have that the time optimal control to reach a given position $\bar{s}$, with fixed initial data, in minimum time does exist and is bang–bang. The natural hypothesis is that the skier has at least one choice that permit him to reach $\bar{s}$ without turning back, this means $\dot{s}(t) > 0$ for every $t$. Hence we have that the optimal trajectory lies in the part of the plane $\{(s, p) : p > 0\}$. Therefore the time optimal control has at most two switchings and can be determined as in section 4.

If $c < 0$ we are in the opposite case because the natural restriction force us to stay in the part of the plane in which there are turnpikes. The time optimal control does not necessarily exist. Again we can repeat the reasoning of section 4.

For Mayer problem (4.9) we have obtained the nonexistence assuming some conditions on $r_{\pm}$, but it can happen that these conditions are not physically acceptable for the ski model.
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