QUANTIZATION OF POISSON ALGEBRAIC GROUPS AND POISSON HOMOGENEOUS SPACES

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Abstract

This paper consists of two parts. In the first part we show that any Poisson algebraic group over a field of characteristic zero and any Poisson Lie group admits a local quantization. This answers positively a question of Drinfeld and generalizes the results of [BFGP] and [BP]. In the second part we apply our techniques of quantization to obtain some nontrivial examples of quantization of Poisson homogeneous spaces.

1. Quantization of Poisson algebraic and Lie groups.

Below we will freely use the notation from our previous paper [EK].

1.1. Poisson algebras and manifolds.

Let $k$ be a field of characteristic 0, $B$ be a commutative algebra over $k$. A Poisson bracket on $B$ is a Lie bracket $\{,\} : B \otimes B \to B$ which satisfies the Leibnitz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$, $f, g, h \in B$. A commutative algebra $B$ equipped with a Poisson bracket is called a Poisson algebra.

Let $B, C$ be Poisson algebras. Then $B \otimes C$ has a natural structure of a Poisson algebra, defined by $\{b_1 \otimes c_1, b_2 \otimes c_2\} = b_1 b_2 \otimes \{c_1, c_2\} + \{b_1, b_2\} \otimes c_1 c_2$.

Let $X$ be an algebraic variety over $k$. Denote by $\mathcal{O}_X$ the structure sheaf of $X$. $X$ is called a Poisson algebraic variety if the sheaf $\mathcal{O}_X$ is equipped with the structure of a sheaf of Poisson algebras. Similarly one defines the notions of a smooth or complex analytic Poisson manifold.

If $X, Y$ are Poisson manifolds, then the product $X \times Y$ has a natural structure of a Poisson manifold. This is a consequence of the fact that the tensor product of Poisson algebras is a Poisson algebra.

1.2. Poisson groups. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over $k$. Then the ring $k[[\mathfrak{a}]]$ of formal power series on $\mathfrak{a}$ is a commutative topological Hopf algebra. We will denote by $A$ the “spectrum” of $k[[\mathfrak{a}]]$ We call $A$ the formal group associated
to \( a \), and say that \( k[[a]] \) is the structure ring of \( A \). Of course, the precise meaning of spectrum has to be clarified, but this is irrelevant for us, as we only use the structure ring \( k[[a]] \).

Let \( A \) denote either a formal or an affine algebraic group over \( k \). When we refer to both cases we will use the term “Poisson group”.

Let \( O_A \) be the structure ring of \( A \). Let \( \Delta : O_A \to O_{A \times A} \) be the standard coproduct. In the case of algebraic groups, this coproduct is given by the formula \( \Delta(f)(x, y) = f(xy) \), \( x, y \in A \).

Let \( \{,\} \) be a Poisson bracket on \( O_A \). This bracket defines a Poisson bracket on \( O_{A \times A} \), as explained in Section 1.1. We say that the Poisson bracket \( \{,\} \) is compatible with the group structure if the coproduct \( \Delta \) is a homomorphism of Poisson algebras. If \( A \) is equipped with a Poisson bracket compatible with the group structure, it is called a Poisson formal (respectively, algebraic) group.

Any Poisson algebraic group canonically defines a Poisson formal group. Indeed, let \( I \) be the ideal of all elements of \( O_A \) vanishing at the identity. It is clear that the Poisson bracket maps \( I^m \times I^n \) to \( I^{m+n-2} \), so it extends to the projective limit of \( O_A/I^n \), which is exactly \( k[[a]] \), where \( a \) is the Lie algebra of \( A \). So we obtain the structure of a Poisson formal group on \( k[[a]] \).

1.3. Poisson formal groups and Lie bialgebras.

**Theorem 1.1.** (Drinfeld) The category of Poisson formal groups over \( k \) is equivalent to the category of Lie bialgebras over \( k \).

**Proof.** Let \( k[[a]] \) be a Poisson formal group. This ring carries a formal series topology and is a commutative topological Hopf algebra. The space \( k[[a]]^* \) of continuous linear functionals on \( k[[a]] \) is the universal enveloping algebra \( U(a) \) of \( a \). Therefore, the adjoint map to \( \{,\} \) defines a skew-symmetric cobracket \( \delta : U(a) \to U(a) \otimes U(a) \) satisfying the Jacobi identity, the Leibnitz rule

\[
(\tilde{\Delta}_0 \otimes 1)(\delta(x)) = (1 \otimes \delta)(\tilde{\Delta}_0(x)) + s_{23}(\delta \otimes 1)(\tilde{\Delta}_0(x))
\]

(where \( s_{23} \) is the permutation of the second and the third component), and the compatibility relation \( \delta(xy) = \delta(x)\tilde{\Delta}_0(y) + \tilde{\Delta}_0(x)\delta(y) \). It is easy to show that \( \delta(a) \subset a \otimes a \), and that \( \delta \) defines a Lie bialgebra structure on \( a \). Conversely, if \( a \) is a Lie bialgebra, then the cobracket \( \delta \) on \( a \) can be uniquely extended to a map \( \delta : U(a) \to U(a) \otimes U(a) \) using the compatibility relation. This map is skew-symmetric and satisfies the Leibnitz rule and the Jacobi identity, so the adjoint of this map is a Poisson bracket on \( k[[a]] \) which defines the structure of a Poisson formal group on \( k[[a]] \). It is easy to check that this correspondence defines an equivalence of categories.

4. Quantization of Poisson groups.

**Definition 1.1.** Let \( A \) be a Poisson formal or algebraic group. A quantization of \( A \) is a product \( * : O_A[[h]] \otimes O_A[[h]] \to O_A[[h]] \) and coproduct \( \Delta : O_A[[h]] \to O_{A \times A}[[h]] \) on the topological \( k[[h]] \)-module \( O_A[[h]] \), such that \( O_A[[h]], *, \Delta \) is a topological Hopf algebra, and

\[
f * g = fg + \frac{h}{2} \{f, g\} + O(h^2), \Delta(f) = \Delta_0(f) + O(h^2), f, g \in O_A,
\]

where \( \Delta_0 \) is the undeformed coproduct in \( O_A \).
Proposition 1.2. Any Poisson formal group admits a quantization.

Proof. Let \( \mathfrak{a} \) be a Lie bialgebra. Let \( U_h(\mathfrak{a}) \) be the quantization of \( \mathfrak{a} \) introduced in [EK]. We identify it, as a \( k[[h]] \)-module, with \( U(\mathfrak{a})[[h]] \), by the map \( \mu : U(\mathfrak{a})[[h]] \to U_h(\mathfrak{a}) \) defined in Section 4.1 of [EK]. Consider the space \( H = U_h(\mathfrak{a})^* \) of all continuous \( k[[h]] \)-linear functionals \( U_h(\mathfrak{a}) \to k[[h]] \). This space carries a natural weak topology and is isomorphic to \( k[[\mathfrak{a}]][[h]] \) as a topological \( k[[h]] \)-module. Moreover, this space carries a natural structure of a topological Hopf algebra, dual to the structure of a topological Hopf algebra in \( U_h(\mathfrak{a}) \). It is clear from the definition of \( U_h(\mathfrak{a}) \) that \( H/hH \) is isomorphic to \( k[[\mathfrak{a}]] \) as a topological Hopf algebra, and for any \( f, g \in k[[\mathfrak{a}]] \subset H \) we have the expansions \( f \ast g = fg + \frac{1}{2} h \{ f, g \} + O(h^2), \Delta(f) = \Delta_0(f) + O(h^2) \), where \( \ast, \Delta \) are the product and coproduct in \( H \), and \( \Delta_0 \) is the coproduct in \( k[[\mathfrak{a}]] \). Thus, the algebra \( H \) is a quantization of the Poisson formal group \( k[[\mathfrak{a}]] \).

1.5. Local quantization.

Definition 1.2. Let \( H = (O_A[[h]], \ast, \Delta) \) be a quantization of a Poisson group \( A \). Let \( D_A \) be the algebra of differential operators on \( A \). We say that \( H \) is a local quantization of \( A \) if each coefficient of the \( h \)-expansion of \( f \ast g \) is of the form \( \sum_{i=1}^m D_i f D_i' g \), where \( D_i, D_i' \in D_A \), and each coefficient of the \( h \)-expansion of \( \Delta(f) \) is of the form \( D \Delta_0(f) \), where \( D \in D_A \times A \).

Remark. The notion of a local quantization is important for the following reason. If we have a local quantization of a Poisson algebraic group \( A \), we can extend this quantization to the structure sheaf \( O_A \) of \( A \). Namely, for an open set \( U \subset A \) denote by \( O_A(U) \) the algebra of regular functions on \( U \). Then we have an associative product \( \ast : O_A(U)[[h]] \otimes O_A(U)[[h]] \to O_A(U)[[h]] \) and a coproduct \( \Delta : O_A(U)[[h]] \to O_A(A \times A)(m^{-1}(U))[[h]] \), where \( m : A \times A \to A \) is the product in \( A \). These operations are obtained by extension of the formulas for \( O_A \) to \( O_A(U) \), which is possible due to the locality of the quantization. In particular, we obtain a sheaf of associative algebras on \( A \) which is a quantization of the structure sheaf of \( A \) as a sheaf of Poisson algebras.

1.6. Existence of local quantization.
In this section we will prove the following theorem, which is the main result of Chapter 1:

Theorem 1.3. Any Poisson formal group \( G \) over \( k \) admits a local quantization. Any Poisson algebraic group \( G \) over \( k \) admits a local quantization.

To prove Theorem 1.3, we need the following result.

Proposition 1.4. (i) The quantization \( H \) of \( k[[\mathfrak{a}]] \) constructed in Section 1.4 is local.

(ii) If \( k[[\mathfrak{a}]] \) is the Poisson formal group associated to a Poisson algebraic group \( A \), then the coefficients of the quantization constructed in Section 1.4 are algebraic differential operators. More precisely, each coefficient of the \( h \)-expansion of \( f \ast g \) is of the form \( \sum_{i=1}^m D_i f D_i' g \), where \( D_i, D_i' \in D_A \), and each coefficient of the \( h \)-expansion of \( \Delta(f) \) is of the form \( D \Delta_0(f) \), where \( D \in D_A \times A \).

Proof. We first show the locality of the product \( \ast \). To be consistent with the notation of [EK], we denote the Lie bialgebra \( \mathfrak{a} \) by \( \mathfrak{g}_+ \), the dual Lie bialgebra to \( \mathfrak{g}_+ \) by \( \mathfrak{g}_- \), and the double \( \mathfrak{g}_+ \oplus \mathfrak{g}_- \) by \( \mathfrak{g} \).
In [EK], for every Drinfeld associator and any Lie bialgebra $\mathfrak{g}_+$ we defined the topological Hopf algebra $U_h(\mathfrak{g}_+)$ which is a quantization of $\mathfrak{g}_+$. The algebra $U_h(\mathfrak{g}_+)$ was defined as the space $\text{Hom}(M_+ \otimes M_-, M_-)$, where $M_+, M_-$ are the Verma modules over $\mathfrak{g}$, introduced in Section 2.3 of [EK], and $\text{Hom}$ is taken in the Drinfeld category $\mathcal{M}$ associated to $\mathfrak{g}$. Analogously to [EK], we identify $U_h(\mathfrak{g}_+)$ with $M_-[[h]]$ by the map $\nu_+ : U_h(\mathfrak{g}_+) \to M_-[[h]]$ given by $\nu_+(f) = f(1_+ \otimes 1_-)$. Then the coproduct in $U_h(\mathfrak{g}_+)$ is given by the following formula (formula (4.9) in [EK]):

$$\Delta(x) = J^{-1}i_-(x), x \in M_-,$$

$J \in U(\mathfrak{g}) \otimes U(\mathfrak{g})[[h]]$ is given by formula (3.1) of [EK]. Because $M_- = U(\mathfrak{g}_+)1_-$, and the map $i_- : M_- \to M_- \otimes M_-$ corresponds to the undeformed coproduct on $U(\mathfrak{g}_+)$, we obtain the following formula for the $\ast$-product on $k[[\mathfrak{g}_+]]$:

$$f \ast g = p((J^{-1})^\ast(f \otimes g)),$$

where $p : k[[\mathfrak{g}_+]] \otimes k[[\mathfrak{g}_+]] \to k[[\mathfrak{g}_+]]$ denotes the undeformed product. Thus, the locality property of the star-product is a consequence of the following lemma.

**Lemma 1.5.** The algebra $U(\mathfrak{g})$ acts in $k[[\mathfrak{g}_+]] = M^\ast_-$ by differential operators. In addition, if $k[[\mathfrak{g}_+]]$ is the formal group corresponding to a Poisson algebraic group $G_+$, then these differential operators are in $D_{G_+}$.

**Proof of the Lemma.** It is enough to show that the Lie algebra $\mathfrak{g}$ acts in $k[[\mathfrak{g}_+]]$ by vector fields from $D_{G_+}$. This is obvious for the subalgebra $\mathfrak{g}_+ \subset \mathfrak{g}$, which acts by left-invariant vector fields. So, we have to prove the statement for the subalgebra $\mathfrak{g}_-$. To do this, we recall that $J = 1 + hr/2 + O(h^2)$, where $r \in \mathfrak{g}_+ \otimes \mathfrak{g}_-$ is the classical $r$-matrix of $\mathfrak{g}$. This shows that for any $f, g \in k[[\mathfrak{g}_+]]$ we have $\{g, f\} = p(r(f \otimes g)) = \sum_j x^+_j f \cdot x^-_j g$, where $x^+_j$ is a basis of $\mathfrak{g}_+$, $x^-_j$ is the dual basis of $\mathfrak{g}_-$. Thus, for any $g \in O_{G_+}$, the vector field $\sum_j (x^-_j g)x^+_j$ is from $D_{G_+}$. Since $\{x^+_j\}$ are a basis of the space of left-invariant vector fields on $G_+$, we see that the functions $x^-_j g$ belong to $O_{G_+}$ for all $j$, i.e. $x^-_j \in D_{G_+}$. The lemma is proved.

Now let us show the locality of the coproduct. Let $x, y \in M_-$. Let $X, Y \in \text{Hom}(M_+ \otimes M_-, M_-) = U_h(\mathfrak{g}_+)$ be such that $X(1_+ \otimes 1_-) = x, Y(1_+ \otimes 1_-) = y$. Let us compute the product $Z = XY$ in $U_h(\mathfrak{g}_+)$ and the vector $z = Z1_\ast \in M_-$. As follows from Chapter 4 of [EK], the element $z$ is given by the formula

$$z = X(1 \otimes Y)\Phi^{-1}(1_+ \otimes 1_+ \otimes 1_-),$$

where $\Phi$ is the associator. Therefore, it follows from Lemma 2.1 in [EK] that every coefficient of the $h$-expansion of $z$ is representable as a finite sum of terms of the form

$$X(1 \otimes Y)(a \otimes \Delta_0(b))(1_+ \otimes 1_+ \otimes 1_-), a, b \in U(\mathfrak{g}),$$

where $\Delta_0$ is the standard coproduct of $U(\mathfrak{g})$. Using the fact that $Y$ is an intertwiner, we can rewrite this expression as

$$X(a1_+ \otimes by).$$

Since $X$ is an intertwiner, we can further rewrite this expression as a linear combination of expressions of the form

$$cX(1_+ \otimes dy), c, d \in U(\mathfrak{g}).$$
Let $t \in U(g_+)$ be such that $t1_\pm = dy$. Then $cX(1_+ \otimes dy) = cX \tilde{\Delta}_0(t)(1_+ \otimes 1_-) = ctx$.

Now consider the elements $\hat{x}, \hat{y} \in U(g_+)$ such that $\hat{x}1_- = x$, $\hat{y}1_- = y$. For any $u \in U(g)$, $s \in U(g_+)$ denote by $u(s)$ the element of $U(g_+)$ such that $u(s)1_- = us1_-$. We have shown that each coefficient of the $h$-expansion of $z$ is a linear combination of terms of the form $c(d(\hat{y})\hat{x})$, $c, d \in U(g)$. Therefore, using Lemma 1, we can conclude that the coefficients of $\Delta(f)$ for $f \in k[[g_+]]$ are given by differential operators from $D_{G_+ \times G_+}$ acting on $\Delta_0(f)$. The proposition is proved.

**Proof of Theorem 1.3.** Let $A$ be a Poisson algebraic group whose Lie algebra is $\mathfrak{a}$. Proposition 1.3 implies that the formulas which define the quantization of $k[[\mathfrak{a}]]$ described in Section 1.4 automatically define a local quantization of $A$.

**Theorem 1.6.** Let $G$ be a Poisson formal or Poisson algebraic group whose Lie algebra is a quasitriangular Lie bialgebra. Then $G$ admits a local formal quantization such that the comultiplication is the same as in the classical case.

**Proof.** The theorem is an immediate consequence of Theorem 1.3 and Theorem 6.1 of [EK], which says that a quasitriangular Lie bialgebra $g$ admits a quantization $U_h(g)$ which is isomorphic to $U(g)[[h]]$ as a topological algebra.

**Remark.** Deformations with preserved coproduct are called preferred deformations [BFGP]. Theorem 1.6 for Poisson Lie groups was proved in the reductive case in [BFGP], then in general by Bidegain and Pinczon [BP] and P.Cartier (private communication), using the existence result of [EK].

1.7. **Quantization of Poisson Lie groups.**

One may consider the case when $k = \mathbb{R}$ or $\mathbb{C}$, and $A$ is a Lie group over $k$. In this case $O_A$ is the algebra of smooth (respectively, holomorphic) functions on $A$. The notions of a Poisson structure on $A$ and its quantization is introduced analogously to the algebraic case, and we have a result analogous to Theorem 1.3.

**Theorem 1.7.** Any Poisson Lie group admits a local quantization.

The construction of this quantization is the same as in the algebraic case.

8. **Functoriality of quantization.**

The universality of the quantization $U_h(g_+)$, proved in Chapter 10 of [EK], implies that the quantization of Poisson formal, algebraic and Lie groups obtained above is functorial. In other words, we have the following result.

**Theorem 1.8.** Let $\phi : G_1 \to G_2$ be a morphism of Poisson groups. Then the induced map $\phi^* : O_{G_2}[[h]] \to O_{G_1}[[h]]$ commutes with the deformed product $\ast$ and deformed coproduct $\Delta$, i.e. defines a homomorphism of topological Hopf algebras.

2. **Quantization of Poisson homogeneous spaces.**

2.1. **Poisson $G$-manifolds.**

Let $G$ be a formal or algebraic group over $k$. Let $g$ be the Lie algebra of $G$. By a $G$-manifold we mean a formal (respectively, algebraic) variety $X$ with a left action of $G$. More specifically, in the formal case a $G$-manifold $X$ is a commutative topological algebra $O_X = k[[X]]$ isomorphic to the algebra of formal power series in $\dim X$ variables, together with a coaction $\Delta_0 : k[[X]] \to k[[g]] \otimes k[[X]]$ of the Hopf algebra $k[[g]]$ such that $\Delta_0$ is a morphism of algebras. In the algebraic case, a $G$-manifold is a variety $X$ with a sheaf of commutative algebras $O^X$ (the structure
sheaf of $X$) together with a morphism of sheaves of algebras $\Delta_0 : m^*O^X \to O^{G \times X}$, where $m$ is the action of $G$ on $X$, $m : G \times X \to X$.

Let $G$ be a Poisson group. Assume that a $G$-manifold $X$ is equipped with a Poisson bracket $\{,\}$. We say that $X$ equipped with $\{,\}$ is a Poisson $G$-manifold if $\Delta_0$ is a homomorphism of (sheaves of) Poisson algebras. In particular, if $G$ acts transitively on $X$, we say that $X$ is a Poisson homogeneous space.

**Example.** Let $H \subset G$ be a subgroup, $\mathfrak{h}$ be the Lie algebra of $H$. Assume that $\mathfrak{h}$ is a coideal in $\mathfrak{g}$, i.e. $\delta(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{g} \otimes \mathfrak{h}$. In this (and only this) case the Poisson bracket on $G$ descends to the homogeneous space $G/H$. Namely, in the formal case we obtain a Poisson bracket $\{,\}$ on $k[[\mathfrak{g}/\mathfrak{h}]]$ which comes from the natural Lie coalgebra structure on $\mathfrak{g}/\mathfrak{h}$. In the algebraic case we obtain a Poisson bracket $\{,\}$ on the algebra $O_{G/H}(U)$ for any open set $U \subset G/H$. In both cases the map $\Delta_0$ is a homomorphism of Poisson algebras, so $G/H$ is a Poisson $G$-manifold. We call such a manifold a Poisson homogeneous space of group type.

**Remarks.** 1. Not every Poisson homogeneous space is of group type. For example, if a group $G$ with trivial Poisson structure acts transitively on a Poisson manifold $X$ by Poisson automorphisms, then $X$ is a Poisson homogeneous space, but the Poisson structure on $X$ is not obtained from the Poisson structure on $G$.

2. In the case of algebraic homogeneous spaces we use the language of sheaves, since the homogeneous space $G/H$ may not be affine even if $G, H$ are affine, and so the ring of globally defined functions on $G/H$ could contain only constants.

2.2. *Equivariant quantization of Poisson $G$-manifolds.*

Let $G$ be a Poisson group, and $O_{h,G}$ be a local quantization of the algebra $O_G$. In the algebraic case, let $O_{h,G}$ be a quantization of the sheaf of regular functions on $G$.

**Definition 2.1.** (i) By an $O_{h,G}$-equivariant quantization of a Poisson formal $G$-manifold $X$ we mean the $k[[\mathfrak{h}]]$-module $O_X[[\mathfrak{h}]]$ equipped with a new associative product $*$ and coaction $\Delta : O_X[[\mathfrak{h}]] \to O_{h,G} \otimes O_X[[\mathfrak{h}]]$, such that $\Delta$ is a homomorphism of algebras, and

$$f*g = fg + \frac{\hbar}{2}\{f,g\} + O(\hbar^2), \Delta(f) = \Delta_0(f) + O(\hbar^2), f, g \in O_X.$$ (1)

(ii) By an $O_{h,G}$-equivariant quantization of a Poisson algebraic $G$-manifold $X$ we mean the sheaf of $k[[\mathfrak{h}]]$-modules $O_X[[\mathfrak{h}]]$ equipped with a new associative product $*$ and homomorphism of sheaves of algebras on $G \times X$ (with new multiplication), $\Delta : m^*(O_X[[\mathfrak{h}]]) \to O_{G \times X}[[\mathfrak{h}]]$, such that equations (1) are satisfied.

(iii) We say that a quantization of a Poisson $G$-manifold $X$ is local if each coefficient of the $\hbar$-expansion of $f*g$ is of the form $\sum_{i=1}^m D_i f D'_i g$, $D_i, D'_i \in D_X$, and each coefficient of the $\hbar$-expansion of $\Delta(f)$ is of the form $D \Delta_0(f)$, $D \in D_{G \times X}$, where $D_X$ is the algebra (sheaf of algebras) of differential operators on $X$.

**Remark.** In the formal case, it is convenient to talk about quantization in terms of dual spaces to function algebras. Denote the space $O_X^*$ of continuous linear functionals on $O_X$ by $T$. Then the structure of a Poisson $G$-manifold on $X$ amounts to an action of $\mathfrak{g}$ on $T$ and a structure of a co-Poisson algebra on $T$ (i.e. a coproduct $\Delta_0$ and cobracket $\delta$), such that the map $U(\mathfrak{g}) \otimes T \to T$ defining the action of $\mathfrak{g}$ on $T$ is a homomorphism of co-Poisson algebras. Further, the problem
of quantization of $X$ reduces to finding an action of $U_h(g)$ on $T[[h]]$ and a $U_h(g)$-invariant coassociative coproduct $\tilde{\Delta} : T[[h]] \to T[[h]] \otimes T[[h]]$ which has the form $\tilde{\Delta} = \tilde{\Delta}_0 + \frac{1}{2} h \delta + O(h^2)$.

**Example.** We say that a Poisson homogeneous space $G/H$ of group type is split if $h \subset g$ is a Lie subalgebra. Let us construct the quantization of a split homogeneous space. Let $U_h(g), U_h(h)$ be the quantizations of the Lie bialgebras $g, h$.

The functoriality of quantization implies that we have a Hopf algebra embedding $U_h(h) \to U_h(g)$. Let $T = \text{Ind}_{U_h(h)}^{U_h(g)} 1$ be a module over $U_h(g)$. This module is generated by a vector $1_T$ such that $a1_T = \epsilon(a)1_T$, $a \in U_h(h)$, where $\epsilon$ is the counit. It is clear that $T^*$ is naturally isomorphic to $k[[g/h]][[h]]$. Let $i_T : T \to T \otimes T$ be the intertwiner defined by the formula $i_T(1_T) = 1_T \otimes 1_T$. Define the product on $T^*$ by $f \ast g = i_T^*(f \otimes g)$, and the coproduct to be the dual map to the action $U_h(g) \otimes T \to T$. It is easy to check that this defines a local quantization of the formal homogeneous space $k[[g/h]]$. Moreover, if the space $G/H$ is algebraic, we automatically obtain its quantization as well, due to the locality property of the quantization.

**Remark.** Quantized homogeneous spaces for semisimple Lie groups are considered in [BFGP]. It is shown there that any preferred deformation of a compact group descends to a quotient of this group by a normal subgroup.

2.3. Quantization of homogeneous spaces arising from Manin quadruples.

**Definition 2.2.** A Manin quadruple is a quadruple $(g, g_+, g_-, h)$, where $(g, g_+, g_-)$ is a Manin triple, and $h$ is a Lagrangian Lie subalgebra in $g$.

**Example.** Let $(g, g_+, g_-)$ be a Manin triple, and $h_+ \subset g_+$ be a Lie subalgebra which is also a coideal, i.e. $\delta(h_+) \subset h_+ \otimes g_+ \oplus g_+ \otimes h_+$. Let $h_-$ be the orthogonal complement of $h_+$ in $g_-$. Then $h_-$ is also a Lie subalgebra and coideal in $g_-$, and the direct sum $h = h_+ \oplus h_-$ is an isotropic Lie subalgebra in $g$, so $(g, g_+, g_-, h)$ is a Manin quadruple.

Any Manin quadruple defines a formal Poisson homogeneous space, as follows. Let $G/H$ denote the formal groups corresponding to $g, h$. For any two functions $f, g$ on $G$ right invariant under $H$ set

$$\{f, g\} = \sum_i R_{x_i^+} f R_{x_i^-} g,$$

where $x_i^+$ is a basis of $g_+$, $x_i^-$ is the dual basis of $g_-$, and $R_x f$ denotes the action of the right invariant vector field corresponding to $x \in g$ on a function $f$.

**Lemma 2.1.** The operation $\{,\}$ is skew symmetric.

**Proof.** We can identify the space $k[[g/h]]^*$ with the left $U(g)$-module $T = U(g)/U(g)h$. Let $1_T$ denote the image of 1 under the natural projection $U(g) \to T$. Then we have

$$\{f, g\}(a1_T) = (f \otimes g)(r\tilde{\Delta}_0(a)(1_T \otimes 1_T)),$$

where $\tilde{\Delta}_0$ is the coproduct in $U(g)$. In particular,

$$\{(f, g) + (g, f)\}(a1_T) = (f \otimes g)((r + r^{op})\tilde{\Delta}_0(a)(1_T \otimes 1_T))$$

The element $\Omega = r + r^{op}$ is $g$-invariant, and $\Omega(1_T \otimes 1_T) = 0$ because the algebra $h$ is Lagrangian. Thus, $\{f, g\} + \{g, f\} = 0$. 

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Lemma 2.2. The operation \{,\} is a Poisson bracket on \(G/H\) which makes \(G/H\) into a Poisson homogeneous space (not necessarily of group type).

Proof. The skew-symmetry follows from Lemma 2.1. The Leibnitz rule is obvious. The Jacobi identity follows from the classical Yang-Baxter equation for \(r\). Consistency with the Poisson bracket on \(G\) is checked directly.

Theorem 2.3. The Poisson formal homogeneous space \(G/H\) admits a local quantization. Moreover, if \(G',H'\) are Poisson algebraic groups with Lie algebras \(\mathfrak{g},\mathfrak{h}\) then the quantization of \(G/H\) automatically determines a quantization of \(G'/H'\).

Proof. Let \(T = \text{Ind}_{\mathfrak{h}}^G \mathfrak{g}\) be a \(U(\mathfrak{g})\) module. It is generated by a vector \(1_T\) such that \(\mathfrak{h}1_T = 0\). We regard \(T\) as an object of category \(\mathcal{M}\) defined in Section 1.4 of [EK]. Let \(i_T : T \to T \otimes T\) be a morphism in \(\mathcal{M}\) defined by the equation \(i_T(1_T) = 1_T \otimes 1_T\).

Lemma 2.4. The morphism \(i_T\) is coassociative in \(\mathcal{M}\), i.e. \((i_T \otimes 1)i_T = (1 \otimes i_T)i_T\).

Proof. This statement is proved analogously to Lemma 2.3 in [EK], using the identity \(\Phi(1_T \otimes 1_T \otimes 1_T) = 1_T \otimes 1_T \otimes 1_T\), which follows from the fact that \(\mathfrak{h}\) is a Lagrangian subalgebra.

Denote by \(F\) the tensor functor from the category \(\mathcal{M}\) and the category of representations of \(U_h(\mathfrak{g})\), constructed Chapter 2 of [EK]. Applying this functor to the morphism \(i_T\), we obtain a coassociative coproduct \(F(i_T) : F(T) \to F(T) \otimes F(T)\), which is an intertwiner for \(U_h(\mathfrak{g})\).

Now we can define a quantization of the formal homogeneous space \(G/H\). By definition, the space of functions on this quantized homogeneous space is the \(k[[\mathfrak{h}]]\)-module \(F(T)^*\). It is clear that \(F(T)^*\) is isomorphic to \(O_{G/H}[[\mathfrak{h}]]\) as a topological \(k[[\mathfrak{h}]]\)-module. Denote by \(\ast\) the map \(F(T)^* \otimes F(T)^* \to F(T)^*\) dual to the map \(F(i_T)\), and by \(\Delta\) the map \(F(T)^* \to O_G[[\mathfrak{h}]] \otimes F(T)^*\) dual to the action of \(U_h(\mathfrak{g})\) on \(F(T)\). Since the product \(\ast\) is associative and \(U_h(\mathfrak{g})\)-equivariant, the triple \((F(T)^*,\ast,\Delta)\) is a quantized homogeneous space. It is easy to check, analogously to Section 3.4 of [EK], that \(F(T)^*\) is in fact a quantization of the Poisson homogeneous space \(G/H\) defined above. This proves Theorem 2.3 in the formal case. The algebraic case follows immediately because of the locality of the quantization. The theorem is proved.

Remarks.
1. The results of this Chapter trivially generalize to smooth and complex analytic homogeneous spaces.

2. Not every Poisson homogeneous space can be quantized. For example, let \(X = \mathbb{R}^2\) with the standard symplectic structure, and \(G\) be the group of symplectic diffeomorphisms of \(X\) with the zero Poisson-Lie structure. It is clear that \(G\) acts transitively on \(X\), and \(X\) is a Poisson homogeneous space. Since the Poisson structure on \(G\) is zero, the quantization of \(X\) as a Poisson homogeneous space is the same as a \(G\)-equivariant quantization of \(X\) as a symplectic manifold. It is well known that such a quantization does not exist.

It is an interesting question which Poisson homogeneous spaces can be quantized. At present we are unable to answer this question.

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