ABELIAN POINTS ON ALGEBRAIC CURVES

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Abstract. We study the question of whether algebraic curves of a given genus \( g \) defined over a field \( K \) must have points rational over the maximal abelian extension \( K^{ab} \) of \( K \). We give: (i) an explicit family of diagonal plane cubic curves without \( \mathbb{Q}^{ab} \)-points, (ii) for every number field \( K \), a genus one curve \( C/\mathbb{Q} \) with no \( K^{ab} \)-points, and (iii) for every \( g \geq 4 \) an algebraic curve \( C/\mathbb{Q} \) of genus \( g \) with no \( \mathbb{Q}^{ab} \)-points. In an appendix, we discuss varieties over \( \mathbb{Q}((t)) \), obtaining in particular a curve of genus 3 without \( \mathbb{Q}^{ab} \)-points.

Convention: All varieties over a field \( K \) are assumed to be nonsingular, projective and (as is especially important for what follows) geometrically irreducible.

1. Introduction

In [4], G. Frey demonstrated the existence of an algebraic variety \( V/\mathbb{Q} \) with no points rational over the maximal abelian extension \( \mathbb{Q}^{ab} \) of \( \mathbb{Q} \) (or “without abelian points.”) His argument uses a mixture of elliptic curve theory and valuation theory;\(^1\) from it, one can deduce the existence of an abelian variety \( A/\mathbb{Q} \) and a principal homogeneous space \( V/\mathbb{Q} \) for \( A \) such that \( V(\mathbb{Q}^{ab}) = \emptyset \). But one does not have any bound on the dimension of \( A \), nor any information about the curves lying on \( V \).\(^2\)

The purpose of this paper is to take a closer look at curves without abelian points. First, we shall give “optimally” concrete and simple examples of varieties without abelian points. For this, let us not delay the statement (and proof!) of the following

Theorem 1. Let \( p \equiv -1 \pmod{3} \) be a prime number, and let \( a, b, c \) be integers which are prime to \( p \). The curve
\[
aX^3 + bpY^3 + cp^2Z^3 = 0
\]
has no \( \mathbb{Q}^{ab}_p \)-points.

Proof: First observe that \( C \) has no \( \mathbb{Q}_p \)-rational points. For, if not, we would have a solution \((x, y, z) \in \mathbb{Z}_p^3\) with \( \min(\text{ord}_p(x), \text{ord}_p(y), \text{ord}_p(z)) = 0 \) and this is visibly impossible: looking at the equation we see that \( p \) must divide first \( x \), then \( y \), then finally \( z \). Indeed, the same argument works in any finite extension \( K/\mathbb{Q}_p \) in which the relative ramification degree \( e(K/\mathbb{Q}_p) \) is prime to 3.

But now suppose that there exists a solution in the ring of integers of \( \mathbb{Q}(\mu_N) \) for some positive integer \( N \). Write \( N = M \cdot p^t \) with \( (M, p) = 1 \), and let \( K \) be the completion of \( \mathbb{Q}(\mu_N) \) at some prime lying over \( p \). We have \( e(K/\mathbb{Q}_p) = \varphi(p^t) = p^{t-1}(p-1) \),

\(^1\)See also the “elliptic curve free” proof given in [3].
\(^2\)One does, of course, know that there exist curves on \( V \), but regarding the existence of an object as information about it seems to invite philosophical controversy.
which is, by our assumption on \( p \), prime to 3. Because the maximal abelian extensions of both \( \mathbb{Q} \) and \( \mathbb{Q}_p \) are those generated by all roots of unity,\(^3\) the proof is complete.

For “most” number fields \( K \), we can choose \( p \) such that the curves \( \textbf{1} \) fail to have \( K^{ab} \)-rational points. More precisely:

**Corollary 2.** Let \( K \) be a number field whose Galois closure does not contain \( \mathbb{Q}(\mu_3) \). Then there exists a prime \( p \) such that the curves \( \textbf{1} \) have no \( K^{ab} \)-rational points.

**Proof:** Thanks to our assumption on \( K \), Cebotarev’s density theorem guarantees the existence of infinitely many primes \( p \equiv -1 \pmod{3} \) such that \( p \) splits completely in \( K \). We then have an embedding \( K^{ab} \hookrightarrow \mathbb{Q}_p^{ab} \), and by Theorem \( \textbf{1} \) we conclude that \( aX^3 + pbY^3 + p^2cZ^2 = 0 \) has no \( K^{ab} \)-rational points.

In fact we will prove the following:

**Theorem 3.** For \( K \) a number field, there is a genus 1 curve \( C/\mathbb{Q} \) with \( C(K^{ab}) = \emptyset \).

Our second goal is to determine for which genera \( g \) there exists a genus \( g \) curve \( C/\mathbb{Q} \) with \( C(\mathbb{Q}^{ab}) = \emptyset \). Since quadratic extensions are abelian, one obvious sufficient condition for abelian points is for \( C/\mathbb{Q} \) to admit a degree 2 morphism to a curve \( Y \) with \( Y(\mathbb{Q}) \neq \emptyset \). Taking \( Y = \mathbb{P}^1 \) we see that there are abelian points on all hyperelliptic curves, and in particular on all curves of genus 0 or 2.

**Theorem 4.** For all \( g \geq 4 \), there exists \( C/\mathbb{Q} \) of genus \( g \) and such that \( C(\mathbb{Q}^{ab}) = \emptyset \).

Conspicuously missing is the case of \( g = 3 \). Equivalently, we wonder:

**Question 5.** Must a nonsingular plane quartic curve \( C/\mathbb{Q} \) have an abelian point?

The question remains of interest over \( \mathbb{Q}_p \) (and especially, over \( \mathbb{Q}_2 \)) and has prompted me to begin a more systematic study of rational points on curves over local fields.

The first step of the proofs has a similar flavor to the arguments of \( \textbf{1} \); namely, we begin in \( \S 2 \) by constructing genus one curves \( C_p \) defined over \( \mathbb{Q}_p \) (in fact, over arbitrary \( p \)-adic fields) without abelian points via Galois cohomological methods.

The second step is to “pull back” these \( C_p \)'s to curves \( C/\mathbb{Q} \) with \( C/\mathbb{Q}_p \cong C_p \). We thus construct curves \( C/\mathbb{Q} \) without \( \mathbb{Q}_p^{ab} \)-abelian points and a fortiori without \( \mathbb{Q}^{ab} \)-abelian points. That such pullbacks should exist (for suitably chosen \( C_p \)'s) seems unsurprising – it would be strange indeed if among members of a certain “class” of varieties there existed \( \mathbb{Q}_p \)-varieties without \( \mathbb{Q}_p^{ab} \)-rational points but no \( \mathbb{Q} \)-varieties without \( \mathbb{Q}^{ab} \)-rational points – but to prove their existence poses some technical challenges, as global Weil-Châtelet groups \( H^1(\mathbb{Q}, E) \) are agreeably large but unwieldy in structure. Moreover for our subsequent applications we need genus one curves \( C/\mathbb{Q} \) without abelian points and with further stringent conditions on the index, as described in Theorem \( \textbf{10} \). The proof of Theorem \( \textbf{10} \) revisits some ideas of \( \textbf{1} \). In particular, as in \( \textbf{1} \), use is made of elliptic curves \( E/\mathbb{Q} \) with rank zero and known (finite!) Shafarevich-Tate groups.\(^4\)

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\(^3\)The Kronecker-Weber theorem. In fact the elementary structure theory of finite extensions of local fields would suffice.

\(^4\)Whereas in \( \textbf{1} \) the \( E/\mathbb{Q} \) came courtesy of the work of Gross-Zagier and Kolyvagin, this time we find ourselves using an \( E/\mathbb{Q} \) coming from the (slightly earlier) work of Rubin and Mazur.
The curves of Theorem 4 are constructed as degree two coverings of the genus one curves of Theorem 10 following a suggestion of B. Poonen.

Obviously the present work is very far from being an authoritative treatment of abelian points on algebraic varieties. Many deeper questions remain, especially concerning varieties which have points everywhere locally and varieties “on the other side of the Calabi-Yau line,” e.g. hypersurfaces defined by a form of degree $d$ in more than $d$ variables. These two issues are related: for instance, it is unknown whether there exists a cubic surface $S_{/\mathbb{Q}}$ without $\mathbb{Q}_{ab}$-rational points, but by a theorem of Lang such an $S$ must have $\mathbb{Q}_{ab}^p$-rational points for all $p$. Further remarks on these and other related issues are made in §4.

After seeing an early draft of this note, B. Poonen commented that it might be of interest to investigate the existence of abelian points on varieties over arbitrary fields $F$, as is done in [14] for solvable points. In an appendix, we take $F = \mathbb{Q}(t)$, obtaining in particular curves of genus 3 and geometrically rational surfaces without abelian points.

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2. Local fields

We identify principal homogeneous spaces $V$ for an abelian variety $A_{/K}$ with Galois cohomology classes $\eta \in H^1(K, A)$. It is thus natural to speak of a field extension $L/K$ such that $V(L) \neq \emptyset$ as a splitting field for $V$ (or for the corresponding cohomology class).

For $\eta \in H^1(K, A)$, its period is the order of $\eta$ as an element of the torsion abelian group $H^1(K, A)$ and its index is the greatest common divisor of all $[L : K]$ for $L/K$ a splitting field for $\eta$. When $A = E$ has dimension one, the index of $\eta$ is equal to the least positive integer which is the degree of a $K$-rational divisor on the corresponding genus one curve $C$. To any $K$-rational divisor of degree $n \geq 3$ there corresponds a degree $n$ embedding of $C$ into $\mathbb{P}^{n-1}$ and conversely (up to linear equivalence of divisors and automorphisms of projective space); in particular, the curves of index 3 are precisely the plane cubic curves without rational points.

In this section the ground field $K$ is a finite extension of $\mathbb{Q}_p$, with valuation ring $R$ and residue field $\mathbb{F}_q$, $q = p^r$. Let $r$ be the cardinality of the (finite!) group of roots of unity in $K$, so that $q - 1 \mid r$ and $\frac{q - 1}{r}$ is a power of $p$.

Our point of departure is the following result:

**Theorem 6.** (Lang-Tate, [8]) Let $A_{/K}$ be an abelian variety with good reduction, and let $V$ be a principal homogeneous space for $A$ whose order, $n$, is prime to $p$. For a finite field extension $L/K$, the following are equivalent:
(i) $V(L) \neq \emptyset$.
(ii) The ramification index $e(L/K)$ is divisible by $n$. 
From this we deduce the following result, a sharpening of the example of [2] §2.3.

**Proposition 7.** Let $V$ be a principal homogeneous space of an abelian variety $A/K$ with good reduction, whose order $n$ is prime to $p$ and does not divide $r = \#\mu(K)$. Then $V(K^{ab}) = \emptyset$.

Proof: Suppose on the contrary that $L/K$ is an abelian extension with $V(L) \neq \emptyset$. We may decompose $L/K$ into a tower of extensions

$$K = F_0 \subset F_1 \subset F_2 \subset F_3 = L$$

such that $F_1/F$ is unramified, $F_2/F_1$ is totally ramified and of degree prime to $p$, and $F_3/F_2$ is totally ramified of degree a power of $p$ (e.g. [15] Prop. 11). By Theorem 5.1 we have that $n | d$, and by the known structure of totally tamely ramified Galois extensions of local fields [15], there exists a uniformizer $\pi$ of $F_1$ such that $F_2 = F_1[X]/(X^d - \pi)$. Since $L/K$ is abelian, so is $F_2/F_0$, which implies that $\mu_d \subset F_0$. In other words, $n | d | r$, a contradiction.

It is thus in our interest to give conditions for the existence of classes $\eta \in H^1(K, A)$ of a given order $n$ (prime to $p$ and not dividing $r$). To ease the notation in the proof of the next result, we write $B$ for the dual abelian variety $A^\vee$. (Of course $B \cong A$ for principally polarizable abelian varieties, and in particular for elliptic curves.)

**Lemma 8.** Let $\tilde{A}/\mathbb{F}_q$ be the reduction of $A$. If $\tilde{A}(\mathbb{F}_q)$ has an element of order $n$, with $(n, p) = 1$, then there exists $\eta \in H^1(K, B)$ of order $n$.

Proof: By a seminal theorem of Tate [18], the discrete abelian group $H^1(K, B)$ and the profinite abelian group $A(K)$ are in Pontryagin duality. It follows that the finite abelian groups $H^1(K, B)/[n]$ and $A(K)/nA(K)$ are in duality and hence are isomorphic. Reduction modulo the maximal ideal of $K$ gives an epimorphism $\mathcal{R} : A(K) \rightarrow \tilde{A}(\mathbb{F}_q)$, whose kernel $K$ is uniquely $n$-divisible. So $\mathcal{R}$ induces an isomorphism $A(K)/nA(K) \cong \tilde{A}(\mathbb{F}_q)/n\tilde{A}(\mathbb{F}_q)$, and the result follows.

**Theorem 9.** There is a genus one curve $C/K$ with $C(K^{ab}) = \emptyset$.

Proof: Assume for the moment the following

**Claim.** There exists a prime $\ell$ such that

- $\ell$ is prime to $q(q - 1)$.
- There is an elliptic curve $E_{/\mathbb{F}_q}$ with $\ell | \#E(\mathbb{F}_q)$.

Then: let $E$ be any lift of $E$ to an elliptic curve over $K$ (e.g., choose representatives in $R$ of the coefficients of a Weierstrass equation of $E$). By Lemma 5 there exists a class $\eta \in H^1(K, E)$ of order $\ell$, and since $\ell$ is prime to $(q - 1)q$ it does not divide $r$. So by Proposition 14 the corresponding genus one curve $C/K$ has $C(K^{ab}) = \emptyset$.

It remains to prove the claim. Once we refer the reader to the Deuring-Waterhouse classification of the integers $N$ which are $\#E(\mathbb{F}_q)$ for some elliptic curve $E_{/\mathbb{F}_q}$ [19] Theorem 4.1], she (resp. he) may prefer to work out the proof for herself (resp. himself). Our proof will use the fact that there exist such $N$ of the form $q + 1 - t$ when $t$ is an integer of absolute value at most $2\sqrt{q}$ satisfying either of the following additional hypotheses [19 loc. cit.]: (i) $(t, p) = 1$; or (ii) $p \neq 1$ mod 4 and $t = 0$.

We will also use the fact that the only pairs of natural numbers $(s, t)$ such that
$2^a - 3^c = \pm 1$ are $(1, 0), (1, 1), (2, 1),$ and $(3, 2)$. This follows from the Catalan conjecture, which has recently been proved by P. Mihăilescu [11]. We consider several cases:

I. If $q \in \{2, 4, 16\}$, take $N = \ell = q + 1$.
II. If $a = 2^a$ for $a = 3$ or $a > 4$, take $N = q + 2 = 2^a + 2$. If $N$ were of the form $2^b \cdot 3^c$, then $b = 1$ and $2^a - 3^c = -1$, but as recalled above there is no such $c$. Thus $N$ is divisible by a prime $\ell \geq 5$, and any such prime will do.
III. If $q = 3$, take $N = \ell = 5$.
IV. If $q = 3^a$ for $a > 1$, take $N = q + 1$. Since $2^b - 3^c = 1$ has no solutions for $a > 1$, $N$ is divisible by a prime $\ell \geq 5$, and any such will do.
V. If $q = p^a$ for $p \geq 5$, take $N = q - 2$ and any prime $\ell | N$.

Remark: When $A$ has split purely toric reduction, every class $\eta \in H^1(K, A)$ has a unique minimal splitting extension $L_\eta$, which is abelian over $K$ [5, 2 §3.1].

3. NUMBER FIELDS

3.1. Curves of genus one. Let $E/Q$ be the Jacobian of Selmer’s cubic $3X^3 + 4Y^3 + 5Z^3 = 0$. Then $E(Q) = 0$, III($Q, E$) $\cong (\mathbb{Z}/3\mathbb{Z})^2$ [10]. Note that $j(E) = 0$, so that $E$ has CM by the maximal order in $Q(\sqrt{-3})$.

**Theorem 10.** Let $\ell$ be either 4 or an odd prime number. There exists a class $\eta \in H^1(Q, E)$ such that:
(i) $\eta$ has period and index equal to $\ell$.
(ii) $\eta$ does not have an abelian splitting field.

Proof: Suppose first that $\ell \geq 5$. Then by Poitou-Tate duality, a strong form of the local-global principle holds in $H^1(Q, E)[\ell^{\infty}]$, namely the natural map

\[ H^1(Q, E)[\ell^{\infty}] \to \bigoplus_p H^1(Q_p, E)[\ell^{\infty}] \]

is an isomorphism [12] 1.6.26(b). (Note that $H^1(Q, E)[\ell^{\infty}] = 0$ since $\ell$ is odd. In any case, $H^1(R, E) = 0$ for this $E$.) There exist infinitely many primes $p$ such that:

(i) $p > 3$;
(ii) $p \equiv -1 \pmod{3}$;
(iii) $p \equiv -1 \pmod{\ell}$.
(iv) $E$ has good reduction mod $p$.

Fix one such prime $p$. Condition (ii) means that $p$ is nonsplit in the CM field $Q(\sqrt{-3})$, so that by a well-known criterion of Deuring, $E$ has supersingular reduction modulo $p$ (or see [17] Example V.4.4), so that (using (i)), $\#E(F_p) = p + 1$ and hence, by (iii), $\ell | \#E(F_p)$. By Lemma 8 there exists $\eta_p \in H^1(Q_p, E)$ of order $\ell$. Because the map of [2] is an isomorphism, there exists a unique class $\eta \in H^1(Q, E)[\ell]$ restricting at $p$ to $\eta_p$ and having trivial restriction at all other primes. By [11] Prop. 6, we conclude that $\eta$ has period equals index equals $\ell$. Since $p \equiv 0 \pmod{\ell}$, evidently $\ell$ does not divide $p - 1 = \#\mu(Q_p)$, so by Proposition 4 $\eta_p$ (and a fortiori $\eta$ itself) has no abelian splitting field.

Next, note that the case $\ell = 3$ is covered by Theorem 11 (take $a = b = 1$, $c = 60$).
In fact it is not difficult to modify the above argument (taking into account that now the map of $E$ has a nontrivial, but still finite, kernel) in this case, and one gets essentially the “theoretical explanation” for the existence of the family of curves (11), since the curves $X^3 + pY^3 + 60p^2Z^3$ are indeed all principal homogeneous spaces of $E$.

For $\ell = 4$, take $p = 11$ (so that $4 \not| p - 1$); one checks easily that $\bar{E}(\mathbb{F}_{11}) \cong \mathbb{Z}/12\mathbb{Z}$.

3.2. Curves of genus $g \geq 4$. We will reduce to the case of curves of genus one via the following result, a version of which was suggested to me by B. Poonen en route to Sabino Canyon in 2003.\(^5\)

**Proposition 11.** Let $K$ be a field of characteristic different from 2, and let $Y/K$ be a genus one curve of index $n$. For any positive integer $k$ with $kn > 1$, there exists a curve $X/K$ of genus $nk + 1$ and a degree two covering $X \to Y$ defined over $K$.

**Proof:** By definition of the index, there exists a $K$-irreducible divisor $D'$ on $Y$ of degree $n$; put $D_0 = kD'$. If $n = 1$, then $D'$ consists of a single point, and since $k > 1$, $L((D_0))$ is basepoint free, so there exists a divisor $D_\infty$ linearly equivalent to $D_0$ and supported away from $D_0$ and a function $f \in K(Y)$ with $\text{div}(f) = D_0 - D_\infty$. If $n > 1$, then $L((D'))$ is already basepoint free, so there exists $E'$ linearly equivalent to $D'$ and with disjoint support, and hence a function $f \in K(Y)$ with $\text{div}(f) = k(D' - E')$. The extension of function fields $K(Y)(\sqrt{\chi})/K(Y)$ corresponds to a degree 2 cover $X \to Y$ with $2kn$ simple branch points. By the Riemann-Hurwitz theorem $X$ has genus $kn + 1$.

Let us now prove Theorem (4b). A positive integer $g \geq 4$ may be written as $k\ell + 1$ with $k \in \mathbb{Z}^+$ and $\ell$ either an odd prime or 4. By Theorem 10 there exists a genus one curve $Y/\mathbb{Q}$ of index $\ell$ without abelian points. Applying Proposition 11 with $n = \ell$, we get a curve $X$ of genus $g$ together with a degree two map $X \to Y$. Since $Y$ has no abelian points, neither does $X$.

3.3. The proof of Theorem 3. Let $n = [K : \mathbb{Q}]$. Let $\ell > 7$ and $p$ be distinct primes, each unramified in $K$, such that $\ell$ does not divide $p^a - 1$ for any $1 \leq a \leq n$; then no completion of $K$ at prime over $p$ has a rational $\ell$th root of unity. **Suppose** $E'/\mathbb{Q}_p$ is an elliptic curve with an $\mathbb{F}_p$-rational point of order $\ell$. Lift $E$ to an elliptic curve $E/\mathbb{Q}$. By a theorem of Ono-Skinner \([13]\), there exists an elliptic curve $E'/\mathbb{Q}$ such that:

(i) $E'/\mathbb{Q}_p \cong E/\mathbb{Q}_p$ (in particular $j(E) = j(E')$);

(ii) $E'$ has analytic rank zero.

(More precisely, $E'$ is the twist of $E$ by a quadratic Dirichlet character $\chi$ with $\chi(p) = 1$.) By the results of Gross-Zagier and Kolyvagin, it follows that $E'(\mathbb{Q})$ and $\text{III}(Q, E)$ are both finite. Moreover, since $\ell > 7$, Mazur’s theorem on rational torsion points on elliptic curves (10) gives $E'(\mathbb{Q}) \otimes \mathbb{Z}_{\ell} = 0$. Now the Poitou-Tate

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\(^5\)This same idea was later broached to Poonen’s student S. Sharif, whose 2006 Berkeley thesis employs it as the jumping-off point for a complete determination of the possible values of a period and index for a genus $g$ curve over a $p$-adic field. On the other hand, without further assumptions on $k$ the construction of Proposition 11 is “optimal” in a sense that we will discuss in a later work.
global duality theorem applies to show that the natural map
\[ H^1(Q, E')[\ell^\infty] \to \bigoplus_p H^1(Q_p, E')[\ell^\infty] \]
is a surjection. In particular, there exists a class \( \eta \in H^1(Q, E')[\ell^\infty] \) whose local restriction \( \eta_p \) has order \( \ell \). Since \( K/Q \) is unramified at \( p \), by Theorem 6 \( \eta|_K \) has exact order \( \ell \), and by the same arguments as above does not split over any abelian extension of the completion of \( K \) at any prime over \( p \).

It remains to show that, for some choices of \( \ell \) and \( p \) as above, there exists \( \hat{E}/\mathbb{F}_p \) with a point of order \( \ell \). For this, we consider primes \( p > n + 1 \). Another special case of the Deuring-Waterhouse classification [19, Theorem 4.1] gives that for any integer \( A \in (p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}) \), there exists \( E/\mathbb{F}_p \) with \( \#E(\mathbb{F}_p) = A \).

There are at most \( n \) primes \( \ell < \sqrt{p} \) whose order, as elements of \( \mathbb{F}_p^\times \), is at most \( n \), so when \( p \) is sufficiently large compared to \( n \) there are many primes \( \ell \) such that there exists \( E/\mathbb{F}_p \) with elements of order \( \ell \). This completes the proof of the theorem.

Remark: The proof we have given is a veritable showcase of the deepest results of \textit{fin de si\'ecle} elliptic curve theory. More care would probably lead to a more elementary (but perhaps less amusing) proof.

4. SOME FINAL REMARKS

With a single exception, this section contains not results but connections to other work, advertisements, conjectures, questions, and even “hearsay.”

4.1. Conjectural strengthenings. I find it likely that the following stronger statements hold:

\textbf{Conjecture 12.} Let \( K \) be any number field and \( E/K \) any elliptic curve.
\begin{enumerate}
  \item There is a genus one curve \( C/K \), with Jacobian \( E \), and \( C(K^{ab}) = \emptyset \).
  \item For all \( d \geq 3 \), there is a degree \( d \) plane curve \( C/Q \) such that \( C(K^{ab}) = \emptyset \).
  \item For all \( g \geq 4 \), there is a curve \( C/Q \) of genus \( g \) such that \( C(K^{ab}) = \emptyset \).
\end{enumerate}

4.2. Higher-dimensional varieties. The following immediate generalization of Theorem 11 gives, for any odd prime \( \ell \), a Calabi-Yau \((\ell - 2)\)-fold \( V/Q \) without \( Q^{ab} \)-rational points.

\textbf{Proposition 13.} Let \( \ell \) be an odd prime and \( p \) a prime with \( \ell \nmid p(p-1) \). Then
\[ \sum_{i=0}^{\ell-1} p^i X_i^\ell = 0 \]
has no points over \( Q_p^{ab} \).

The proof does not go through for composite \( \ell \): in particular for \( \ell = 4 \) the construction does not yield examples of K3 surfaces without abelian points. Note that, given a quartic surface without abelian points, taking a general hyperplane section would give a negative answer to Question 5.
It would be of great interest to find a geometrically rational variety \( V/\mathbb{Q} \) without abelian points. Especially, it is a famous open question of Artin whether a Fano hypersurface (i.e., the zero locus of a degree \( d \) homogeneous polynomial in at least \( d+1 \) variables) defined over \( \mathbb{Q}^{ab} \) must have a rational point. As mentioned above, by a theorem of Lang \[7\], such hypersurfaces have points everywhere locally. By work of Brauer and Birch, the existence of quadratic points is known for “sufficiently Fano” hypersurfaces: for every fixed \( d \), there exists \( n = n(d) \) such that a degree \( d \) form in \( n \) variables over \( \mathbb{Q} \) has a nontrivial solution in (e.g.) \( \mathbb{Q}(\sqrt{-1}) \). Finally, work of Kanevsky \[6\] shows that if \( V/K \) is a cubic surface over a number field \( K \) such that for all finite extensions \( L/K \), the Brauer-Manin obstruction to the existence of \( L \)-rational points on \( V/L \) is the only one, then \( V(K^{ab}) \neq \emptyset \).

4.3. Varieties with abelian points. One can ask for nontrivial examples of varieties with abelian points, i.e., not coming from a quadratic covering of a variety with \( \mathbb{Q} \)-points. The best example I know is that of Severi-Brauer varieties (varieties \( V/\mathbb{Q} \) such that \( V/\mathbb{Q} \cong \mathbb{P}^N/\mathbb{Q} \)): there are always abelian points, but, in dimension at least two, usually not quadratic points. Indeed, the Brauer-Hasse-Noether theorem says that every element of the Brauer group of a number field is given by a cyclic algebra.

Somewhat distressingly, the following seems to be nontrivial:

**Problem 14.** For all positive integers \( n \) (or even for infinitely many \( n \)), exhibit a genus one curve \( C/\mathbb{Q} \) of index \( n \) and such that \( C(\mathbb{Q}^{ab}) \neq \emptyset \).

Perhaps the solution to Problem 14 will involve some Iwasawa theory.

4.4. Varieties with points everywhere locally. A motivation for this work came from a question of D. Jetchev, who asked whether Selmer’s curve \( 3X^3 + 4Y^3 + 5Z^3 = 0 \) has points in any abelian cubic field, or in any abelian number field. Theorem 1 gives negative answers for superficially similar curves: e.g. the curve \( 2X^3 + 4Y^3 + 5Z^3 = 0 \) fails to have abelian points. But Selmer’s curve has points everywhere locally, rendering useless the present approach.

Rephrasing the question slightly, we may ask:

**Question 15.** Fix a positive integer \( n \). Does every locally trivial genus one curve of index \( n \) defined over a number field have an abelian point?

I have been told that random matrix theory predicts a positive answer when \( n = 3 \).

4.5. Solvable points. We were also motivated by work in progress of M. Cipriani and A. Wiles, who study solvable points on curves of genus one. They are able to show (at least) that a genus one curve \( C/\mathbb{Q} \) which is locally trivial and with semistable Jacobian has a solvable point.

Using the solvability of the absolute Galois groups of \( \mathbb{Q}_p \) (and \( \mathbb{R} \)), it is easy to see that for every variety \( V \) over a number field \( K \), there is a solvable extension \( L/K \) such that \( V/L \) has points everywhere locally. Thus, an affirmative answer to
Question 15 for all \(n\) (which I must admit seems unlikely) would imply the existence of solvable points on all curves of genus one.

4.6. **Metabelian points.** Note that in Section 2, all our examples of principal homogeneous spaces over \(\mathbb{Q}_p\) without abelian points have points over the maximal abelian extension of \(\mathbb{Q}_p^{ab}\), i.e., over a metabelian extension of \(\mathbb{Q}_p\). It was suggested to me a few years ago by B. Mazur that every genus one curve over \(\mathbb{Q}\) should have metabelian points. As far as I know, this remains open even over \(\mathbb{Q}_p\), although special cases follow from results of Lang-Tate [8] and Lichtenbaum [9].

**Appendix: varieties over \(\mathbb{Q}(t)\)**

A somewhat different perspective would be to fix a “class” of algebraic varieties (e.g., curves of a given genus \(g\), or hypersurfaces of degree \(d\) in \(\mathbb{P}^N\)) and ask whether for any field \(F\),\(^7\) a variety \(V/F\) of this type must have points in the maximal abelian extension of \(F\). With “abelian” replaced by “solvable,” this is the setting of recent work of A. Pál [14]. In this appendix we will show that, with a suitable choice of \(F\), there are additional classes of \(F\)-varieties without \(F^{ab}\)-points.

It is convenient to work with \(F = \mathbb{Q}(t)\). The absolute Galois group \(\mathfrak{g}_F\) of \(F\) lies in a split exact sequence

\[
1 \to \hat{\mathbb{Z}} \to \mathfrak{g}_F \to \mathfrak{g}_\mathbb{Q} \to 1
\]

where the action of \(\mathfrak{g}_\mathbb{Q}\) on \(\hat{\mathbb{Z}}\) is by the cyclotomic character. It follows that the maximal abelian extension of \(F\) is generated by the roots of unity together with \(t^{1/2}\). The field \(F(\mu_\infty, t^{1/2})\) is Henselian with respect to the discrete valuation \(\text{ord}_{t^{1/2}}\).

We will work instead with its completion \(\mathbb{Q}_{\text{ab}}^{ab}(t^{1/2})\), which – by a small abuse of notation – we will denote by \(F^{ab}\). Since this field contains what is literally the maximal abelian extension of \(F\), finding varieties \(V/F\) with \(V(F^{ab}) = \emptyset\) is a priori a stronger result than showing that they do not have abelian points. (But in fact it is equivalent: a variety defined over a discretely valued field has points rational over the Henselization iff it has points rational over the completion.)

**Proposition 16.** Every Severi-Brauer variety \(V/F\) has an abelian point, but there exist Severi-Brauer varieties \(V/F^{ab}\) without rational points.

Proof: This may be viewed as a question about the Brauer groups \(\text{Br}(F)\) and \(\text{Br}(F^{ab})\) (e.g. [15, §X.6]). For any complete, discretely valued field \(K\) with perfect residue field \(K\), there is an exact sequence

\[
0 \to \text{Br} K \to \text{Br} K \xrightarrow{\sigma} X(\text{Br}_k) \to 0,
\]

where the last term is the character group of the Galois group of the residue field [15 Theorem X.3.2]. Consider first the case of \(K = \mathbb{Q}(t)\), \(k = \mathbb{Q}\). Then, for \(\alpha \in \text{Br} K\) we can split the character \(\chi(\alpha)\) via a unique unramified abelian extension \(L/K\) with (abelian) residue extension \(l/\mathbb{Q}\). By the exact sequence, \(\alpha|_L \in \text{Br}(l)\). Now every element of the Brauer group of a number field can be split by a cyclotomic extension, so overall we get that \(\text{Br}(\mathbb{Q}(t)) = \text{Br}(\mathbb{Q}^{ab}(t)/\mathbb{Q}(t))\), giving the first statement of the proposition.

On the other hand, \(F^{ab} = \mathbb{Q}^{ab}(t^{1/2}) \cong \mathbb{Q}^{ab}(t)\) is again a local field, whose

\(^7\)Just for simplicity, let us assume that \(F\) has characteristic 0.
Lemma 18. There exists a geometrically rational 4-fold $V/F$ with $V(F^{ab}) = \emptyset$.

Proof: The quaternion algebra $(\sqrt{2}, t^2)$ defined over $F' = F(\sqrt{2}, t^2)$ is non-split over $F^{ab}$; this corresponds to a conic $C_{F'}$ with $C(F^{ab}) = \emptyset$. Restriction of scalars (or “Weil restriction”) from $F'$ to $F$ gives a fourfold $V/F$ such that $V/F \cong (\mathbb{P}^1)^4$ and $V(F^{ab}) = C(F' \otimes_F F^{ab}) = C(\prod_{i=1}^4 F^{ab}) = C(F^{ab})^4 = \emptyset$.

Remark: A famous theorem of Merkurjev implies that over an arbitrary field $K$, Severi-Brauer varieties $V_{K}$ have metabelian points. Work of Wedderburn and Albert shows that every Severi-Brauer surface is split over a cyclic cubic extension and every Severi-Brauer threefold is split over a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-extension. I am not personally in possession of an example of a division algebra without an abelian splitting field, but I presume this is the generic situation when the index is divisible by a sufficiently high power of a prime.

Recall the norm form $N(L/K)$ associated to a separable field extension of degree $d$: choosing a $K$-basis $\alpha_1, \ldots, \alpha_d$ of $L$, the map

$$N : K^d \to K, \ (x_1, \ldots, x_d) \mapsto N(\sum_{i=1}^d x_i \alpha_i)$$

is a degree $d$ polynomial. Let us say that a homogeneous $K$-form $f(x_1, \ldots, x_d)$ is isotropic (resp. anisotropic) if there exist $(x_1, \ldots, x_n)$, not all zero, with $f(x_1, \ldots, x_d) = 0$ (resp. there is only the zero solution).

If $M/K$ is a separable field extension, then $N(L/K)/M$ is the norm form for the extension of algebras $L \otimes_K M/M$, and is anisotropic if and only if $L \otimes_K M$ is a field. In other words:

Lemma 18. Let $M/K$ be a separable field extension. Then the norm form $N = N(L/K)$ is anisotropic over $M$ if and only if $M$ and $L$ are linearly disjoint over $K$. In particular, if $L/K$ is not abelian, then $N(L/K)$ is anisotropic over every abelian extension of $K$.

Note however that the norm form $N(L/K)$ of a nontrivial field extension is geometrically reducible. Indeed, over the Galois closure of $L/K$, with a suitable choice of basis $N$ becomes $N(X_1, \ldots, X_d) = X_1 \cdots X_d \ (d = [L : K])$. The corresponding closed subscheme is of dimension $d - 2$ and has as its singular locus a finite union of linear subspaces of dimension $d - 3$.

Proposition 19. Fix a positive integer $d \geq 3$, and let $K/Q$ be a degree $d$ number field with Galois group $S_d$. Let $N$ be the norm form of $K/Q$. Then

$$N(X_1, \ldots, X_d) = tZ^n$$

has no $F_{ab}$-rational points.

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8We are again encountering the phenomenon that $(Q^{ab})^{ab}$, the maximal metabelian extension of $Q$, is very much larger than $Q^{ab}$.
Proof: There are no solutions with $Z = 0$ since $K(T) = K\otimes\mathbb{Q}F/F$ is linearly disjoint from $F^{ab}$. A solution with $Z \neq 0$ exists if and only if $t$ is a norm from $M = K\otimes F^{ab}$ down to $L = F^{ab}$. But $M/L$ is a finite unramified extension of complete discretely valued fields, so the image of the norm map consists of elements whose valuation is divisible by $d$, whereas $v_L(t) = 2$.

**Corollary 20.** There is a geometrically rational surface $S/F$ without abelian points.

Proof: Taking $d = 3$, one gets a geometrically integral cubic surface $S/F$ without abelian points, but with finitely many (in fact 3) singular points. We can resolve the singularities by a birational $F$-morphism $\tilde{S} \to S$, and $S(F^{ab}) = \emptyset$ implies $\tilde{S}(F^{ab}) = \emptyset$.

**Corollary 21.** There is a plane curve of any degree $d \geq 3$ without abelian points.

Proof: Since the singular locus of $\mathfrak{S}$ is a finite union of codimension two affine subspaces, one sees easily that intersecting with a general 2-plane gives a nonsingular curve. In particular, taking $n = 4$ we get a genus 3 curve $C/F$ with $C(F^{ab}) = \emptyset$.

Following [14], we get another approach to curves of genus 3 without abelian points:

**Proposition 22.** Let $F$ be a complete, discretely valued field whose residue field $f$ contains an extension $m$ which is Galois with group isomorphic to $S_4$. Then there exists a genus 3 curve $C/F$ with $C(F^{ab}) = \emptyset$.

Proof: After choosing an isomorphism of $G = \text{Gal}(m/f)$ with $S_4$, we get an action of $\mathfrak{g}_f$ on the complete graph $K_4$ on 4 vertices. By [14] Prop. 4.6], there exists a stable curve $C_f$ with rational geometric components, whose corresponding dual graph is isomorphic, as a $\mathfrak{g}_f$-module, to $K_4$ with the chosen $G$-action. By [14] Cor. 4.4], there exists a stable curve over the valuation ring $R_F$ of $F$ whose generic fiber is an honest (i.e., nonsingular and geometrically integral) curve $C_f$ and whose special fiber is isomorphic to $C_f$. Since the stabilizer of any vertex or edge of $K_4$ is a non-normal subgroup of $S_4$, after making any abelian residue extension $f'/f$, there are no $\mathfrak{g}_{f'}$-fixed vertices or edges of the dual graph, so $C(f') = \emptyset$. Hence if $F'/F$ is any extension with abelian residue extension – so in particular if $F'/F$ is itself abelian – $C(f') = \emptyset$ implies $C(R_{F'}) = C(F') = \emptyset$.

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