Exhausting pants graphs of punctured spheres by finite rigid sets

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Abstract

Let $S_{0,n}$ be an $n$-punctured sphere. For $n \geq 4$, we construct a sequence $(X_i)_{i \in \mathbb{N}}$ of finite rigid sets in the pants graph $P(S_{0,n})$ such that $X_1 \subset X_2 \subset \ldots \subset P(S_{0,n})$ and $\bigcup_{i \geq 1} X_i = P(S_{0,n})$.

1 Introduction

Let $S = S_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures and let $\text{Mod}^\pm(S) = \pi_0(\text{Homeo}(S))$ be the extended mapping class group. Ivanov [6], Korkmaz [7], and Luo [8] proved that, for most surfaces, the curve complexes $\mathcal{C}(S)$ is rigid, that is, $\text{Aut}(\mathcal{C}(S)) \cong \text{Mod}^\pm(S)$. In [2], Aramayona and Leininger proved that curve complexes contain finite rigid sets; meaning a finite subgraph such that every simplicial embedding is a restriction of an element of $\text{Mod}^\pm(S)$. Later in [3], they showed that there exists an exhaustion of the curve complexes by finite rigid sets.

For the pants graphs $P(S)$, the rigidity property was proved by Margalit [9] using the result of Ivanov, Korkmaz, and Lou. Aramayona [10] extended Margalit’s result to prove a stronger form of rigidity, that is, if $S$ and $S'$ are surfaces such that the complexity of $S$ is at least 2, then every injective simplicial map $\phi : P(S) \to P(S')$ is induced by a $\pi_1$-injective embedding $f : S \to S'$. In [10], we refined Aramayona’s result by showing that the pants graphs of punctured spheres are finitely rigid.

In this paper, we modify the tools Aramayona and Leininger built in [3], together with the finite rigid sets we constructed [10], to prove that we can exhaust the pants graphs of punctured spheres by finite rigid sets:
Theorem 1.1. Let $S_{0,n}$ be an $n$-punctured sphere. For $n \geq 4$, there exists a sequence of finite rigid sets $X_1 \subset X_2 \subset \ldots \subset \mathcal{P}(S_{0,n})$ such that $\bigcup_{i \geq 1} X_i = \mathcal{P}(S_{0,n})$.

Outline of the paper. Section 2 contains the relevant background and definitions. In Section 3 we describe the adjustments to the tools Aramayona and Leininger [3] developed to enlarge their rigid sets in the curve complex so we can use them in our setting. We use these tools to prove the main theorem in Section 4.

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2 Background and definitions

This section contains necessary definitions and background restricted to punctured spheres, for general settings see [1] and [9]. Let $S = S_{0,n}$ be an $n$-punctured sphere. A simple closed curve on $S$ is essential if it does not bound a disk or a once-punctured disk on $S$. Throughout this paper, a curve is a homotopy class of essential simple closed curves on $S$. Given two curves $\gamma$ and $\gamma'$, we denote their geometric intersection number by $i(\gamma, \gamma')$, which is the minimum number of transverse intersection points among the representatives of $\gamma$ and $\gamma'$. The two curves are disjoint if $i(\gamma, \gamma') = 0$.

A multicurve $Q$ is a set of pairwise distinct, disjoint curves on $S$. For a given multicurve $Q$, the nontrivial piece $(S - Q)_0$ of the complement of the curves in $Q$ is the union of the non thrice-punctured sphere components of the complement. We call a thrice-punctured sphere, a pair of pants.

A pants decomposition $P$ is a maximal multicurve: the complement in $S$ is a disjoint union of pairs of pants. A pants decomposition always contains $n - 3$ curves and we call this number the complexity $\kappa(S)$ of $S$. The deficiency of a multicurve $Q$ is the number $\kappa(S) - |Q|$. If $Q$ is a deficiency-1 multicurve then $(S - Q)_0$ is homeomorphic to $S_{0,4}$.

Let $P$ and $P'$ be pants decompositions of $S$. We say that $P$ and $P'$ differ by an elementary move if there are curves $\alpha, \alpha'$ on $S$ and a deficiency-1 multicurve $Q$ such that $P = \{\alpha\} \cup Q, P' = \{\alpha'\} \cup Q$ and $i(\alpha, \alpha') = 2$; see Figure 1 for an example of elementary moves.

The pants graph $\mathcal{P}(S)$ of $S$ is a graph with the set of vertices corresponding to pants decompositions. Two vertices are connected by an edge if the corresponding pants decompositions differ by an elementary move. The
pants graph $\mathcal{P}(S)$ is connected and the pants graph $\mathcal{P}(S_{0,4})$ of a 4-punctured sphere is isomorphic to a Farey graph, see [5].

A path in $\mathcal{P}(S)$ is an edge path determined by a sequence of distinct adjacent vertices of $\mathcal{P}(S)$. A cycle in $\mathcal{P}(S)$ is a subgraph homeomorphic to a circle. We call a cycle, a triangle, rectangle, or pentagon if it has 3, 4, or 5 vertices, respectively.

Let $X \subset \mathcal{P}(S_{0,n})$ and $\phi : X \to \mathcal{P}(S_{0,m})$ be an injective simplicial map. We say that a $\pi_1$-injective embedding $f : S_{0,n} \to S_{0,m}$ induces $\phi$ if there is a deficiency-$(n-3)$ multicurve $Q$ on $S_{0,m}$ such that $f(S_{0,n}) = (S_{0,m} - Q)_0$ and the simplicial map

$$f^Q : \mathcal{P}(S_{0,n}) \to \mathcal{P}(S_{0,m}),$$

defined by $f^Q(u) = f(u) \cup Q$ satisfies $f^Q(u) = \phi(u)$ for any vertex $u \in X$.

**Definition 2.1.** For $n \geq 4$, we say that $X \subset \mathcal{P}(S_{0,n})$ is rigid if for any punctured sphere $S_{0,m}$ and any injective simplicial map

$$\phi : X \to \mathcal{P}(S_{0,m}),$$

there exists a $\pi_1$-injective embedding $f : S_{0,n} \to S_{0,m}$ that induces $\phi$.

For $n = 4$, the isotopy class of $f$ is unique up to precomposing by an element $\sigma \in \text{Mod}(S_{0,4})$ inducing the identity on $\mathcal{P}(S_{0,4})$.

For $n \geq 5$, the isotopy class of $f$ is unique.

The following theorem is a refinement of Aramayona’s result [11] that we proved in [10].

**Theorem 2.2.** For $n \geq 4$, there exists a finite rigid subgraph $X_n \subset \mathcal{P}(S_{0,n})$. 

![Figure 1: Example of an elementary move.](image)
3 Tools for enlarging rigid sets

This section contains the definitions and theorems Aramayona and Leininger \cite{ar} developed to enlarge their rigid sets in the curve complexes. We make some necessary adjustments to them in order to enlarge rigid sets in the pants graphs.

Definition 3.1. Let $n \geq 5$. A set $X \subset \mathcal{P}(S_{0,n})$ is said to be **weakly rigid** if whenever $f_1, f_2 : S_{0,n} \to S_{0,m}$ are $\pi_1$-injective embeddings satisfy

$$f_1^{Q_1}|_X = f_2^{Q_2}|_X,$$

for some deficiency-$(n-3)$ multicurves $Q_1$ and $Q_2$ on $S_{0,m}$, then

$$Q_1 = Q_2$$

up to isotopy.

It is easy to see from the definition that a superset of a weakly rigid set is also weakly rigid.

Lemma 3.2. For $n \geq 5$, let $X_1, X_2 \subset \mathcal{P}(S_{0,n})$ be rigid sets. If $X_1 \cap X_2$ is weakly rigid then $X_1 \cup X_2$ is rigid.

Proof. Let $\phi : X_1 \cup X_2 \to \mathcal{P}(S_{0,m})$ be an injective simplicial map. Since $X_i$ is rigid, there exist a $\pi_1$-injective embedding $f_i : S_{0,n} \to S_{0,m}$ and a deficiency-$(n-3)$ multicurve $Q_i$ such that $f_i^{Q_i}|_{X_i} = \phi|_{X_i}$. Hence $f_1^{Q_1}|_{X_1 \cap X_2} = f_2^{Q_2}|_{X_1 \cap X_2}$. The weakly rigidity of $X_1 \cap X_2$ implies that $Q_1 = Q_2 = Q$ and $f_1 = f_2 = f$. Therefore $f$ is a $\pi_1$-injective embedding such that $f^{Q}|_{X_1 \cup X_2} = \phi$ which implies the rigidity of $X_1 \cup X_2$. \hfill $\square$

The following proposition is the key to enlarge rigid sets.

Proposition 3.3. For $n \geq 5$, let $\mathcal{X} \subset \mathcal{P}(S_{0,n})$ be a finite rigid set such that $\text{Mod}(S_{0,n}) \cdot \mathcal{X} = \mathcal{P}(S_{0,n})$. Suppose there exists a finite subset $C$ of curves on $S_{0,n}$ such that:

1. The set $\{T_{\alpha}^{i} \mid \alpha \in C\}$ generates $\text{Mod}(S_{0,n})$;
2. $\mathcal{X} \cap T_{\alpha}^{i}(\mathcal{X})$ is weakly rigid, for all $\alpha \in C$, and $i \in \{-1, 1\}$.

Then there exists a sequence $\mathcal{X} = \mathcal{X}_1 \subset \mathcal{X}_2 \subset ... \subset \mathcal{X}_n \subset ...$ such that each $\mathcal{X}_i$ is a finite rigid set, and

$$\bigcup_{i \in \mathbb{N}} \mathcal{X}_i = \mathcal{P}(S_{0,n}).$$
Proof. Since $\mathcal{X}$ is rigid and a half twist is a homeomorphism, $T^i_\alpha(\mathcal{X})$ is rigid for all $\alpha \in C$, and $i \in \{-\frac{1}{2}, \frac{1}{2}\}$. Given $\alpha, \beta \in C$ and $i, j \in \{-\frac{1}{2}, \frac{1}{2}\}$. By assumption (2) and by applying Lemma 3.2 we see that $\mathcal{X} \cup T^i_\alpha(\mathcal{X})$ is rigid. Recall that a superset of a weakly rigid set is also weakly rigid. Hence $(\mathcal{X} \cup T^i_\alpha(\mathcal{X})) \cap T^j_\beta(\mathcal{X})$, which contains $\mathcal{X} \cap T^j_\beta(\mathcal{X})$, is weakly rigid. Apply Lemma 3.2, we see that $\mathcal{X} \cup T^i_\alpha(\mathcal{X}) \cup T^j_\beta(\mathcal{X})$ is weakly rigid. By repeating above arguments, the set $\mathcal{X}_n := \mathcal{X} \cup \bigcup_{\alpha \in C} T^{\pm \frac{1}{2}}_\alpha(\mathcal{X}_n)$ is rigid. We define
\[
\mathcal{X}_{n+1} := \mathcal{X}_n \cup \bigcup_{\alpha \in C} T^{\pm \frac{1}{2}}_\alpha(\mathcal{X}_n),
\]
for $n \geq 2$. Since a weakly rigid set $\mathcal{X} \cap T^i_\alpha(\mathcal{X})$ is a subset of $\mathcal{X}_n \cap T^i_\alpha(\mathcal{X}_n)$, $\mathcal{X}_n \cap T^i_\alpha(\mathcal{X}_n)$ is weakly rigid. Again, by applying 3.2 repeatedly and use induction, we conclude that $\mathcal{X}_n$ is rigid for all $n$. Then the first claim is proved.

Since $\{T^{\pm \frac{1}{2}}_\alpha \mid \alpha \in C\}$ generates $\text{Mod}(S_{0,n})$ and $\text{Mod}(S_{0,n}) \cdot \mathcal{X} = P(S_{0,n})$,
\[
\bigcup_{i \in \mathbb{N}} \mathcal{X}_i = P(S_{0,n}).
\]

\[\Box\]

4 The proof of the main theorem

We note that for $n \leq 3$, the pants graphs $P(S_{0,3})$ is empty. We give a separate proof for $n = 4$ as follow.

Proof of Theorem 1.1 for $S = S_{0,4}$. The pants graph of $S_{0,4}$ is isomorphic to the Farey graph. Any triangle in $S_{0,4}$ is rigid as proved in \cite{10}. Then we let $\mathcal{X}_1$ to be a triangle. Each edge in a pants graph of any punctured sphere is contained in exactly two triangles which are both in the same Farey graph. Then we can define $\mathcal{X}_{n+1}$ inductively; let $\mathcal{X}_{n+1}$ be an enlargement of $\mathcal{X}_n$ obtained by attaching one more triangle to each edge of $\mathcal{X}_n$ contained in only one triangle. Hence $\mathcal{X}_{n+1}$ is rigid for all $n \geq 1$, and by the construction, $\bigcup_{i \in \mathbb{N}} (\mathcal{X}_i) = P(S_{0,4})$. We conclude that sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is an exhaustion of $P(S_{0,4})$. \[\Box\]
For \( n \geq 5 \), we begin by recalling the construction of finite rigid sets \( X_n \) in [10]. First we construct \( S_{0,n} \) with a set of curves, then define \( X_5 \), and finally, define \( X_n \) for \( n \geq 6 \).

Consider a regular \( n \)-gon with the \( n \) vertices removed and label the sides as \( 1, 2, \ldots, n \), cyclically. For each non-adjacent pair of sides of the \( n \)-gon, draw a straight line segment to connect the two sides. Then double the \( n \)-gon to obtain \( S_{0,n} \) and a set of curves \( \Gamma_n \), see Figure 2 for the case of \( S_{0,8} \) and Figure 3 for the case of \( S_{0,5} \). Let \( a_{i,j} \in \Gamma_n \) be the curve connecting the \( i \)th side to the \( j \)th side of \( S_n \). We call \( a_{i,j} \) such that \( i - j \equiv \pm 2 \mod n \), a chain curve. Compare to [2, Section 3].

Let \( Z_n \) be a subgraph of \( P(S_{0,n}) \) induced by the set of vertices corresponding to pants decompositions consisting of curves from \( \Gamma_n \).

For \( P(S_{0,5}) \), we defined

\[
X_5 = Z_5 \cup \bigcup_{c \in \Gamma_5} T_c^{\frac{1}{2}}(Z_5),
\]

where \( T_c^{\frac{1}{2}} \) is a simplicial map on \( P(S_{0,5}) \) induced by the half-twist around the curve \( c \).

See Figure 3 for a partial figure of \( X_5 \). The subgraph \( X_5 \) consists of the alternating pentagon \( Z_5 \) and 10 of its images under the twists. Those 10 images form 10 triangles attached to \( Z_5 \). In [10], we proved that \( X_5 \) is rigid.

For \( n \geq 6 \), we construct \( X_n \) as follows. Let \( W \subset \Gamma_n \) be a deficiency-2 multicurve such that \( (S_{0,n} - W)_0 \cong S_{0,5} \). Let \( \Gamma_5^W = \{ \alpha \in \Gamma_n \mid \alpha \text{ is disjoint from all curves in } W \} \). There is a natural homeomorphism \( h : S_{0,5} \to (S_{0,n} - W)_0 \) such that \( h(\Gamma_5) = \Gamma_5^W \), see [10, Lemma 3.1]. Let

\[
X_5^W = h^W(X_5) = \{ h(u) \cup W \mid u \in X_5 \},
\]

where \( h^W : P(S_{0,5}) \to P(S_{0,n}) \) is the induced map of \( h \) defined by \( h^W(u) = h(u) \cup W \). Then \( X_5^W \cong X_5 \). Finally we let

\[
X_n = Z_n \cup \bigcup_W X_5^W,
\]

where the union is taken over all deficiency-2 multicurves in \( \Gamma_n \) with a 5-punctured sphere component. In [10], we proved that \( X_n \) is rigid.

We need the following lemmas to prove the main theorem for \( n \geq 5 \).

**Lemma 4.1.** \( \text{Mod}(S_{0,n}) \cdot X_n = P(S_{0,n}) \)
Proof. In the first part of this proof, we will show that, for a given vertex \( P \) in \( \mathcal{P}(S_{0,n}) \), there exist a vertex \( P' \) in \( X_n \) and \( f \in \text{Mod}(S_{0,n}) \) such that \( f(P') = P \). To do this, we obtain a pants decomposition \( P' \) from a dual graph of the pants decomposition \( P \). For the second part, we will show that there is a homeomorphism that send a given edge in \( \mathcal{P}(S_{0,n}) \) to an edge in \( Z_n \subset X_n \).

Given a vertex \( P \) in \( \mathcal{P}(S_{0,n}) \). Recall that we consider \( S_{0,n} \) as a double of a regular \( n \)-gon. Consider \( P \) as a pants decomposition on \( S_{0,n} \). The following construction of a dual graph of \( P \) was given in [5]. For each pair of pants component of \( (S_{0,n} - P) \), we mark a vertex on the interior of the component. We also mark the \( n \) punctures as \( n \) vertices. Two vertices are connected by an edge if (1) they are vertices on the interior of two pants components which have a common boundary, or (2) one of the vertices is on the interior of a pair of pants component and another vertex is a puncture of the same component. The result is a tree with \( 2n - 2 \) vertices; all puncture-vertices have degree 1, while the rest of the vertices have degree 3, see Figure 3.

Since a tree is planar, we can redraw this tree on the plane inside a regular \( n \)-gon so that all \( n \) puncture-vertices are the \( n \) vertices of the \( n \)-gon. We reconstruct a pants decomposition consisting of curves in \( \Gamma_n \) by drawing a curve connecting two sides of the regular \( n \)-gon whenever this curve can cross exactly one edge of the tree and both endpoints of this edge are not puncture-vertices. Double the regular \( n \)-gon. We now have a pants decomposition \( P' \) consisting of curves in \( \Gamma_n \), i.e., \( P' \) is a vertex in \( Z_n \subset X_n \).

The above construction of \( P' \) from \( P \) gives a one-to-one correspondence between the pants components \( S_{0,n} - P \) and the pants components \( S_{0,n} - P' \).
This correspondence describes a homeomorphism $f$ such that $f(P') = P$, as desired.

Next we show that if $P_1$ and $P_2$ are adjacent vertices in $\mathcal{P}(S_{0,n})$, then after applying some homeomorphisms on $S_{0,n}$ to $P_1$ and $P_2$, we get two vertices that are adjacent in $Z_n$.

Given adjacent vertices $P_1$ and $P_2$ in $\mathcal{P}(S_{0,n})$, then there exist curves $u_1, u_2$ on $S_{0,n}$ and a deficiency-1 multicurve $Q$ such that $P_1 = \{u_1\} \cup Q$ and $P_2 = \{u_2\} \cup Q$. By the first part of the proof, there is $f \in \text{Mod}(S_{0,n})$ such that $f(P_1)$ is a vertex in $Z_n$. If $f(P_2)$ is also in $Z_n$, then we are done.

Suppose $f(P_2)$ is not in $Z_n$. Use Figure 5 as a reference for the rest of the proof. We note that $f(Q) \subset \Gamma_n$ and it has deficiency-1. The nontrivial component $(S_{0,n} - f(Q))_0 \cong S_{0,4}$ contains exactly two curves in $\Gamma_n$; one curve is $f(u_1)$ and we call the other curve $\alpha$. Then $i(f(u_2), \alpha) = 2n$ for some $n \in \mathbb{N}$. 

\[ A = \{\alpha, \beta\} \]
\[ B = \{\delta, \beta\} \]
\[ C = \{\delta, \epsilon\} \]
\[ D = \{\gamma, \epsilon\} \]
\[ E = \{\alpha, \gamma\} \]
Applying one full twist around \( f(u_1) \) in an appropriate direction reduces the intersection number by 4. \( f(P_i) \) is invariant under this full twist. So we can choose a new \( f \) (by composing the old one with some power of full twists) and assume that \( i(f(u_2), \alpha) = 0 \) or \( i(f(u_2), \alpha) = 2 \). If \( i(f(u_2), \alpha) = 0 \), then \( f(u_2) = \alpha \) and we are done.

Suppose \( i(f(u_2), \alpha) = 2 \). We compose \( f \) by an appropriate half twist \( T \) around \( f(u_1) \): here a half twist in \( f(u_1) \) is a homeomorphism on \( S_{0,n} \), whose square is the Dehn twist in Lemma 4.2. Around \( T \), we assume that \( f \) whose square is the Dehn twist in Lemma 4.2.

Let \( \alpha \) be a curve on \( S_{0,n} \). We defined \( P_\alpha(S_{0,n}) \) to be a subgraph of \( P(S_{0,n}) \) induced by vertices corresponding to pants decompositions containing \( \alpha \).

The following lemma is proved in [10] and we use this lemma to prove Lemma 4.3.

**Lemma 4.2.** For \( n \geq 6 \), let \( \alpha \) be a chain curve on \( S_{0,n} \) and let \( X_{n-1}^\alpha = X_n \cap P_\alpha(S_{0,n}) \).

Then \( X_{n-1}^\alpha \cong X_{n-1} \). Moreover, this isomorphism is induced by \( h : S_{0,n-1} \to (S_{0,n} - \alpha)_0 \) as \( h^\alpha(v) = h(v) \cup \{\alpha\} \in X_{n-1}^\alpha \).

**Lemma 4.3.** \( X_n \cap T_i^\alpha(X_n) \) is weakly rigid, for \( i \in \{-\frac{1}{2}, \frac{1}{2}\} \) and for all chain curves \( \alpha \) in \( S_{0,n} \).
Figure 5: Example of an edge \{f(P_1), f(P_2)\} and its images after composing with a power of full twist around the curve \(f(u_1)\) and a *half twist* around the same curve.

**Proof.** Let \(\alpha\) be a chain curve and \(i \in \{-\frac{1}{2}, \frac{1}{2}\}\). Suppose \(f_1, f_2 : S_{0,n} \to S_{0,m}\) are \(\pi_1\)-injective embeddings such that

\[
\left. f_1 \right|_{X_n \cap T_i^\alpha(X_n)} = \left. f_2 \right|_{X_n \cap T_i^\alpha(X_n)},
\]

for some deficiency-\((n-3)\) multicurves \(Q_1\) and \(Q_2\) on \(S_{0,m}\).

We first prove the case of \(n = 5\). Recall the definition of \(X_5\). By a direct calculation, we see that \(X_5 \cap T_i^\alpha(X_5)\) consists of two alternating pentagons which are \(Z_5 = T_i^\alpha(T_\alpha^{-i}(Z_5))\) and \(T_i^\alpha(Z_5)\). They share an edge together with four triangles as shown in Figure 5. Since \(Z_5\) is an alternating pentagon and \(\left. f_1^{Q_1} \right|_{Z_5} = \left. f_2^{Q_2} \right|_{Z_5}\), \(^\square\) Lemma 4.2 implies that \(Q_1 = Q_2\) and

\[
f_1 = f_2 \text{ or } f_1 = f_2 \circ e,
\]

where \(e : S_{0,5} \to S_{0,5}\) is the involution exchanging the two pentagons (as we consider \(S_5\) as a double of a pentagon). The map \(e\) induces a simplicial map on \(P(S_{0,5})\) that fixes \(Z_5\) and exchanges two triangles on each side of \(Z_5\). But \(f_1\) and \(f_2\) also agree on the four triangles attached to \(Z_5\) so \(f_1 = f_2\). Hence the case of \(n = 5\) is proved.
Figure 6: Examples of half twist around the thick curves. Two pants decompositions in $\mathbb{Z}_{10}$ and $\mathbb{Z}_{11}$ are given to help visualize the homeomorphisms. Note that after a half twisting, we get a new pants decomposition that is still in $\mathbb{Z}_{10}$ or $\mathbb{Z}_{11}$.

Let $n \geq 6$ and let $\alpha$ be any chain curve. By Lemma 4.2, a subgraph $X_{n-1}^\alpha = X_n \cap P_\alpha(S_{0,n}) \cong X_{n-1}$. Since each vertex of $X_{n-1}^\alpha$ contains $\alpha$, $T_\alpha(X_{n-1}^\alpha) = X_{n-1}^\alpha$. Hence $X_n \cap T_\alpha(X_n)$ contains $X_{n-1}^\alpha \cong X_{n-1}$. Consider the restrictions of $f_1$ and $f_2$ on the subsurface $(S_{0,n} - \{\alpha\})_0$. Since $X_{n-1}^\alpha$ is rigid, so is $X_{n-1}^\alpha$, and the uniqueness part of Definition 2.1 implies that $f_1$ agrees with $f_2$ on $(S_{0,n} - \{\alpha\})_0$ and $Q_1 \cup \{f_1(\alpha)\} = Q_2 \cup \{f_1(\alpha)\}$.

We can see that $X_{n-1}^\alpha$ is a proper subgraph of $X_n \cap T_\alpha(X_n)$. For example, choose a vertex $P$ in $Z_n \cap P_\alpha(S_{0,n}) \subset X_{n-1}^\alpha$. Then change $P$ to $P'$ by the elementary move which replaces $\alpha$ by the other curve $\alpha'$ in $\Gamma_n$. The vertex $T_\alpha(P')$ is adjacent to $P$ and it is a vertex in $X_n \cap T_\alpha(X_n)$. Hence $f_1$ and $f_2$ agree on $T_\alpha(P')$. Since $Q_1$ and $Q_2$ are the intersections of all vertices in $f_1(X_n \cap T_\alpha(X_n))$ and $f_2(X_n \cap T_\alpha(X_n))$, respectively, and $\alpha \notin T_\alpha(P')$, it follows $f_1(\alpha) = f_2(\alpha)$ is not in the intersection. Therefore $Q_1 = Q_2$ and $f_1 = f_2$. 

**Proof of Theorem 1.1 for $S_{0,n}, n \geq 5$.** We are ready to prove the main theorem for $n \geq 5$. We check that all conditions in Proposition 3.3 are satisfied.
Let $\mathcal{X} = X_n$. Lemma 4.1 states that $\text{Mod}(S_{0,n}) \cdot \mathcal{X} = \mathcal{P}(S_{0,n})$. The set

$$C = \{T^{\pm \frac{1}{2}}(\alpha) \mid \alpha \text{ a chain curve}\}$$

generates $\text{Mod}(S_{0,n})$, see [4, Corollary 4.15]. And by Lemma 4.3, $X_n \cap T_{i}^\alpha(X_n)$ is weakly rigid, for $i \in \{-\frac{1}{2}, \frac{1}{2}\}$ and for all chain curves $\alpha$ in $S_{0,n}$. Therefore Proposition 3.3 gives us a sequence of finite rigid set $\mathcal{X} = \mathcal{X}_1 \subset \mathcal{X}_2 \subset \ldots \subset \mathcal{X}_m \subset \ldots$ such that $\bigcup_{i \in \mathbb{N}} \mathcal{X}_i = \mathcal{P}(S_{0,n})$, as desired.

\[\square\]

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