In this article, we present a general methodology for stochastic control problems driven by the Brownian motion filtration including non-Markovian and non-semimartingale state processes controlled by mutually singular measures. The main result of this paper is the development of a concrete method for characterizing and computing near-optimal controls for controlled Wiener functionals via a finite-dimensional approximation procedure. The theory does not require a priori functional differentiability assumptions on the value process and ellipticity conditions on the diffusion components. Explicit rates of convergence are provided under rather weak conditions for distinct types of non-Markovian and non-semimartingale states. The analysis is carried out on suitable finite dimensional spaces and it is based on the weak differential structure introduced by [38, 39] jointly with measurable selection arguments. The theory is applied to stochastic control problems based on path-dependent SDEs and rough stochastic volatility models, where both drift and possibly degenerated diffusion components are controlled. Optimal control of drifts for path-dependent SDEs driven by fractional Brownian motion with exponent $H \in (0, 1)$ is also discussed.
1. Introduction

Let $C_{n,T}$ be the set of continuous functions from $[0,T]$ to $\mathbb{R}^n$, let $\xi : C_{n,T} \to \mathbb{R}$ be a Borel functional, let $F = (\mathcal{F}_t)_{t \geq 0}$ be a fixed filtration and let $U^T_0 : 0 \leq t \leq T$ be a family of admissible $\mathcal{F}$-adapted controls defined over $(t,T]$. The goal of this paper is to develop a systematic approach to solve a generic stochastic optimal control problem of the form

\begin{equation}
\sup_{\phi \in U^T_0} \mathbb{E}[\xi(X^\phi)],
\end{equation}

where $\{X^\phi; \phi \in U^T_0\}$ is a given family of $\mathcal{F}$-adapted controlled continuous processes. A common approach to such generic control problem (see e.g [13, 18]) is to consider for each control $u \in U^T_0$, the value process given by

\begin{equation}
V(t,u) = \text{ess sup}_{\phi; \phi = u \text{ on } [0,t]} \mathbb{E}[\xi(X^\phi)|\mathcal{F}_t]; 0 \leq t \leq T.
\end{equation}

Two fundamental questions in stochastic control theory rely on sound characterizations of value processes and the development of concrete methods to produce either exact optimal controls $u^* \in U^T_0$ (when exists)

\begin{equation}
\mathbb{E}[\xi(X^{u^*})] = \sup_{\phi \in U^T_0} \mathbb{E}[\xi(X^\phi)],
\end{equation}

or near-optimal controls (see e.g [53]) which realize

\begin{equation}
\mathbb{E}[\xi(X^{u^*})] > \sup_{\phi \in U^T_0} \mathbb{E}[\xi(X^\phi)] - \epsilon,
\end{equation}

for an arbitrary error bound $\epsilon > 0$. Exact optimal controls may fail to exist due to e.g lack of convexity, moreover they are very sensitive to perturbations and numerical rounding. In this context, the standard approach is to consider near-optimal controls which exist under minimal hypotheses and are sufficient in most applications.

Two major tools for studying stochastic controlled systems are Pontryagin’s maximum principle and Bellman’s dynamic programming. While these two methods are known to be very efficient for establishing some key properties (e.g existence of optimal controls, smoothness of the value functional, sufficiency of subclasses of controls, etc), the problem of solving explicitly or numerically a given stochastic control problem remains a critical issue in the field of control theory. Indeed, except for a very few specific cases, the determination of an optimal control (either exact or near) is a highly nontrivial problem to tackle. The present article aims to provide a systematic method to compute and characterize $u^*$ realizing (1.3) for a given stochastic control problem (1.2) driven by a generic controlled process $\{X^\phi; \phi \in U^T_0\}$.

In the Markovian case, a classical approach in solving stochastic control problems is given by the dynamic programming principle based on Hamilton-Jacobi-Bellman (HJB) equations. One popular approach is to employ verification arguments to check if a given solution of the HJB equation coincides with the value function at hand, and obtain as a byproduct the optimal control. Discretization methods also play an important role towards the resolution of the control problem. In this direction, several techniques based on Markov chain discretization schemes [35], Krylov’s regularization and shaking coefficient techniques (see e.g [53, 34]) and Barles-Souganidis-type monotone schemes [2] have been successfully implemented. We also refer the more recent probabilistic techniques on fully non-linear PDEs given by Fahim, Touzi and Warin [19] and the randomization approach of Kharroubi, Langrené and Pham [30, 31, 32].

Beyond the Markovian context, the value process (1.2) cannot be reduced to a deterministic PDE and the control problem (1.1) is much more delicate. Nutz [43] employs techniques from quasi-sure analysis to characterize one version of the value process as the solution of a second order backward
stochastic differential equation (2BSDE) (see \cite{50}) under a non-degeneracy condition on the diffusion component of a controlled non-Markovian stochastic differential equation driven by Brownian motion (henceforth abbreviated by CNM-SDE-BM). Nutz and Van Handel \cite{12} derive a dynamic programming principle in the context of model uncertainty and nonlinear expectations. Inspired by the work \cite{30}, under the weak formulation of the control problem, Fuhrman and Pham \cite{21} show a value process can be reformulated under a family of dominated measures on an enlarged filtered probability space where the CNM-SDE-BM might be degenerated. It is worth to mention that under a nondegeneracy condition on diffusion components of CNM-SDEs-BM, \cite{12} can also be viewed as a fully nonlinear path-dependent PDE in the sense of \cite{17} via its relation with 2BSDEs (see section 4.3 in \cite{17}). In this direction, Possamaï, Tan and Zhou \cite{51} derived a dynamic programming principle for a stochastic control problem w.r.t a class of nonlinear kernels. They obtained a well-posedness result for general 2BSDEs and established a link with path-dependent PDEs in possibly degenerated cases. We also drive attention to Qiu \cite{14} who characterizes \cite{12} driven by a CNM-SDE-BM as a solution of a suitable HJB-type equation and uniqueness is established under a non-degeneracy condition on the diffusion components. Under strong a priori functional differentiability conditions (in the sense of \cite{16} \cite{11}) imposed on \cite{12}, one can apply functional Itô’s formula to arrive at verification-type theorems. In this direction, we refer to e.g. Cont \cite{12} and Saporito \cite{49}.

Discrete-type schemes for the optimal value \cite{11} driven by CNM-SDEs-BM were studied in the context of G-expectations by Dolinsky \cite{15} and also by Zhang and Zhuo \cite{52}, Ren and Tan \cite{48} and Tan \cite{51}. In \cite{52} \cite{48}, the authors provide monotone schemes in the spirit of Barles-Souganidis for fully nonlinear path-dependent PDEs in the sense of \cite{17} and hence one may apply their results for the study of \cite{11}. Under ellipticity conditions, by employing weak convergence methods in the spirit of Kushner and Depuis, \cite{51} provides a discretization method for the optimal value \cite{11}. Convergence rates are available only under strong a priori regularity conditions on the value process \cite{12} (see \cite{52}) and in the state independent case \cite{16} \cite{51}.

1.1. Main setup and contributions. The main goal of this paper is to deepen the analysis of stochastic control problems. Rather than developing new representation results for \cite{12}, we aim to provide a systematic approach to construct stochastic near-optimal controls driven by possibly non-Markovian and non-semimartingale controlled states \{X^u; u \in U_T^A\} adapted to the Brownian motion filtration and parameterized by possibly mutually singular measures. Throughout this article, we call them as controlled Wiener functionals. Under a rather weak \textit{L}-type regularity condition imposed on controlled states, this article develops a concrete methodology to solve a given stochastic control problem of the form \cite{11}. None a priori differentiability condition on the value process \cite{12} is required and the theory is tailor-made for non-Markovian and / or non-semimartingale controlled states where current techniques do not apply. For instance, a controlled path-dependent degenerated SDE driven by a possibly non-smooth transformation of the Brownian motion (such as fractional Brownian motion in the rough regime \(0 < H < 1/2\)) is a typical non-trivial application of the theory. Our methodology can also be relevant in high-dimensional Markovian cases, when it is known that PDE methods are not efficient.

The methodology is based on a weak version of functional Itô calculus developed by Leão, Ohashi and Simas \cite{39} and inspired by Leão and Ohashi \cite{38}. A given Brownian motion structure is discretized which gives rise to differential operators acting on piecewise constant processes adapted to a jumping filtration in the sense of \cite{29} and generated by what we call a discrete-type skeleton \(\mathcal{S} = \{T, A^{k,j}; j = 1, \ldots, d, k \geq 1\}\) (see Definition 3.1). For a given controlled state process \(\{X^u; u \in U_T^A\}\), we construct a \textit{controlled imbedded discrete structure} \((\{V^k\}_{k \geq 0}, \mathcal{S})\) (see Definition 4.2 in Section 6) for the value process \cite{12}. This is a non-linear version of the imbedded discrete structures introduced by \cite{39} and it can be interpreted as a discrete version of \cite{12}.

In Proposition 4.3, by using measurable selection arguments, we aggregate the controlled imbedded discrete structure \((\{V^k\}_{k \geq 0}, \mathcal{S})\) into a single finite sequence of upper semianalytic value functions \(v^k_n: \mathbb{H}^{k,n} \rightarrow \mathbb{R}; n = 0, \ldots, \epsilon(k, T) - 1\). Here, \(\mathbb{H}^{k,n}\) is the \(n\)-fold cartesian product of \(A \times S_k\), where \(A\)
is the action space, $S_k$ is suitable finite-dimensional space which accommodates the dynamics of the structure $\mathcal{D}$ and $\epsilon(k, T)$ is a suitable number of periods to recover (1.1) over the entire period $[0, T]$ as the discretization level $k$ goes to infinity. This procedure allows us to derive a pathwise dynamic programming equation on $\mathbb{H}_k^{k,n}$ which is the building block to solve stochastic control problems much beyond the classical Markovian states, traditionally treated by Markov chain approximations and PDE methods.

The idea is to select candidates to near-optimal controls for (1.3) by means of a feasible backward maximization procedure based on integral functionals

\begin{equation}
\arg \max_{\phi \in U_0^k} \int_{S_k} \gamma_{k+1}^k (\phi_{k+1}^k, \gamma_{k+1}) \nu_k^k (d\gamma_{k+1}^k) \nabla^k (d\gamma_{k+1}^k); \quad n = \epsilon(k, T) - 1, \ldots, 0,
\end{equation}

starting with a terminal condition

\[ V_{\epsilon(k, T)}^k (\phi_{\epsilon(k, T)}^k) = \xi (\gamma_{\epsilon(k, T)}^k (\phi_{\epsilon(k, T)}^k)), \]

where $\gamma_{\epsilon(k, T)}^k$ is a pathwise representation of the discretized controlled state one has to build case by case, $\nu_k^k$ is a probability kernel of $\mathcal{D}$ acting on $S_k$ and $\gamma_k^k \in \mathbb{H}_k^{k,n}$ is the history of the imbedded discrete system. Since the terminal condition is explicitly given and the transition probability admits a closed-form expression (see Proposition 9.1), then our dynamic programming principle can be effectively implemented whenever the underlying controlled state is adapted to the Brownian filtration. We refer the reader to Section 11 for a simple numerical example which illustrates how one can make use of (1.4) in a stochastic control problem.

The connection between (1.4) and the original control problem (1.1) is made via a strong robustness property found in a wide class of Wiener functionals as described by 39 in the linear expectation case. In this work, we take one step ahead towards the fully non-linear case. Proposition 5.2 reveals that an arbitrary nonlinear expectation (see e.g 44) driven by a strongly controlled Wiener functional $(X, \mathcal{D})$ associated with a controlled imbedded discrete structure $\mathcal{X} = ((X_k)_{k \geq 1}, \mathcal{D})$ (see Definition 3.3 in Section 3) is represented by

\[ \sup_{\phi \in U_0^k} \mathbb{E} [\xi (X^\phi)] = \lim_{k \to +\infty} \sup_{\phi \in U_0^k, \epsilon(k, T)} \mathbb{E} [\xi (X^\phi)], \]

over a suitable set $U_0^k, \epsilon(k, T)$ of stepwise constant $\mathcal{D}$-adapted processes. The concept of strongly controlled Wiener functional covers a wide class of stochastic systems which cannot be reduced to Markovian states without adding infinitely many degrees of freedom (see Sections 1.2 and 5). More importantly, by reducing the analysis to $\mathcal{X} = ((X_k)_{k \geq 1}, \mathcal{D})$, Proposition 5.3 reveals that one can drastically reduce the dimensionality of the control problem and compute a near optimal stochastic control for (1.3) via (1.4). As a by-product, we are able to provide a concrete description of near-optimal controls for a generic optimal control problem (1.1) based on a given controlled state $\phi \mapsto X^\phi$. This gives in particular an original method to solve a large class of stochastic control problems (including non-Markovian and non-semimartingale states). The regularity conditions of the theory boils down to a mild $L^2$-type continuity hypothesis on the controlled state (see Assumption (B1)) combined with a Hölder modulus of continuity on the payoff functional $\xi$ (see Assumption (A1)).

In contrast to previous works, we remark that our approach does not rely on a given representation of the value process (1.2) in terms of path-dependent PDE or 2BSDE. We develop a fully pathwise structure $\phi^k_{j+1} = 0, \ldots, \epsilon(k, T) - 1$ which allows us to make use the classical theory of analytic sets to construct path wisely the near-optimal controls for (1.3) by means of a list of analytically measurable functions $C_{\gamma^k_{j+1}}: \mathbb{H}_k^{k,j} \to A; j = 0, \ldots, \epsilon(k, T) - 1$ which can be calculated from (1.4) for a given error bound $\epsilon \geq 0$. By composing those functions with the skeleton $\mathcal{D}$, we are able to construct pure jump $\mathcal{D}$-predictable near optimal (open-loop) controls

\[ \phi^{*, k, \epsilon} = (\phi_0^k, \ldots, \phi_{\epsilon(k, T) - 1}^k) \]
realizing (1.3) for \( k \) sufficiently large, where the near-optimal control at the \( j \)-th step depends on previous near-optimal controls \( n = 0, \ldots, j - 1 \) by concatenating \( \phi_{0,\epsilon}^k \otimes \cdots \otimes \phi_{j-1,\epsilon}^k \) for \( j = 1, \ldots, e(k,T) \).

For a given error bound \( \epsilon \geq 0 \), the rate of convergence of our approximation scheme is proportional to

\[
\sup_{u \in U_0^k} \mathbb{E}[\xi(X^u)] - \mathbb{E}[\xi(X^\phi^k)] = O(h_k + r_k + \epsilon) \tag{1.5}
\]

and

\[
\sup_{\phi \in U_0^{k,\epsilon}(k,T)} \mathbb{E}[\xi(X^{k,\phi})] - \mathbb{E}[\xi(X^\phi)] = O(h_k + r_k), \tag{1.6}
\]
as \( k \to +\infty \). The rate \( (h_k)_{k \geq 1} \) in (1.5) and (1.6) depends on the degree of continuity of \( X \) w.r.t a structure \( (\{X^n\}_{n \geq 1}, \mathcal{F}) \) (see (3.13)) combined with large deviation principles related to a family of hitting times \( (\mathcal{T}_n^k)_{n \geq 1} \). The rate \( (r_k) \) in (1.5) and (1.6) depends on an equiconvergence property (Proposition 7.1) for Brownian martingales w.r.t the structure \( \mathcal{F} \) and it is independent of the controlled state. We stress the error estimates in (1.5), (1.6) and the number \( e(k,T) \) in (1.4) growth linearly w.r.t the driving Brownian motion dimension. The dependence on the state dimension appears only through the modulus of continuity of the coefficients of the state. See Remarks 4.2 and 7.1. Error estimates associated with approximations of (1.4) are left to a future project. See however [4] for the particular case of optimal stopping.

In contrast to the framework of one fixed probability measure in [39], in the present context it is essential to aggregate a given structure into a single deterministic finite sequence of maps due to a possible appearance of mutually singular measures induced by \( X^{k,\phi} \) as \( \phi \) varies over the set of controls \( U_0^{k,\epsilon}(k,T) \). Moreover, we mention it is possible to prove a pathwise differential-type representation result for (1.2) by extending the framework of [39] to controlling probabilities via measurable selection arguments combined with Proposition 7.1. Since it is not our purpose to focus on the representation of (1.2), we leave this construction to a future work.

It is important to emphasize that our methodology applies to any stochastic control problem driven by controlled Wiener functionals as long as Assumptions (A1) and (B1) are in force. However, our methodology requires effort on the part of the “user” in order to specify the “good” structure that is suitable for analyzing a given problem at hand. Theorems 7.1 and 7.2 provide explicit rates of convergence for the controlled states given by (1.7), (1.8) and (1.9).

1.2. Examples. As a test of the relevance of the theory, we show that our methodology can be applied to controlled Wiener functionals of the form

\[ X^u = F(B, (\alpha, \sigma)(X^u, u)), \]

where the non-anticipative functionals \((\alpha, \sigma)\) admit Lipschitz regularity w.r.t. control variables which in turn implies \( L^2\)-Lipschitz regularity of \( u \mapsto X^u \) w.r.t control processes (Assumption (B1)). Typical examples which fit into the theory developed in this paper are controlled SDEs with rather distinct types of path-dependence:

\[ dX^u(t) = \alpha(t, X^u, u(t))dt + \sigma(t, X^u, u(t))dB(t), \tag{1.7} \]

\[ dX^u(t) = \alpha(t, X^u, u(t))dt + \sigma dB_H(t), \tag{1.8} \]

and

\[
\begin{align*}
dX^u(t) &= X^u(t)\mu(u(t))dt + X^u(t)\vartheta(Z(t), u(t))dB(t) \\
dZ(t) &= \nu dB_H(t) - \beta(Z(t) - m)dt,
\end{align*} \tag{1.9}
\]
where $B_H$ is a fractional Brownian motion (henceforth abbreviated by FBM) with exponent $0 < H < 1$. In case of (1.7), the driving state is a CNM-SDE-BM and the lack of Markov property is due to non-anticipative functionals $\alpha$ and $\sigma$ which may depend on the whole path of $X^u$. In this case, the controlled state $X^u$ satisfies a pseudo-Markov property in the sense of [10]. The theory developed in this article applies to case (1.7) without requiring ellipticity conditions on the diffusion component $\sigma$. Case (1.8) illustrates a fully non-Markovian case: the controlled state $X^u$ is driven by a path-dependent drift and by a very singular transformation of the Brownian motion into a non-Markovian and non-semimartingale noise. In particular, there is no probability measure on the path space such that the controlled state in (1.8) is a semimartingale. Case (1.9) has been devoted to a considerable attention in recent years in the context of rough stochastic volatility models when $0 < H < \frac{1}{2}$. In this direction, see e.g. [1] and other references therein.

As far as (1.8), the current literature on the control theory for rough driving force relies on the characterization of value functions and existence theorems for optimal controls. The works [5, 23, 8] give a characterization of optimal controls via Pontryagin-type maximum principles based on BSDEs (with implicit or explicit FBM) at the expense of Malliavin differentiability of controls with exception of [8]. We mention that for non path-dependent quadratic costs, exact optimal controls for linear state controlled processes driven by FBM are obtained by [20] via solutions of BSDEs driven by FBM and Brownian motion. Infinite-dimensional lifts of linear quadratic control problems driven by non-Markovian stochastic Volterra-type equations (including FBM with $H \in (0, \frac{1}{2})$) have been studied by [27, 28]. By using rough path techniques, [14] studies a class of drift controlled differential equations driven by rough paths. They characterize the value function as an HJB-type equation and establish a form of the Pontryagin maximum principle. We also drive attention to [37] which studies the existence of optimal controls for controlled Young differential equations. Unfortunately, none of these works (except the linear quadratic ones via Ricatti-type equations) provide a concrete numerical solution of the control problem.

Recently, [1] presents discrete-type approximations in the sense of Hairer’s theory of regularity structures for (1.9) without the presence of controls, i.e., the classical linear expectation case. At this point, it is important to stress our approximation does not need any type of renormalization procedure because our structure is imbedded into the Brownian motion world which allows us to get rid of divergent Stratonovich’s correction terms. See Sections 6.3 and 10 for details.

To our best knowledge, despite the recent efforts on representation theorems for value processes (12, 13, 17, 21, 11) driven by CNM-SDEs-BM of the form (1.7) and numerical schemes for path-dependent PDEs (51, 52, 48), the explicit construction of optimal controls (either exact or near) with a precise rate of convergence at this level of generality is novel. In particular, we do not assume a priori regularity assumptions on the value process in the sense of functional Itô calculus, none elliptic condition on the controlled system is required and our methodology goes far beyond Markovian / semimartingale states. In contrast to previous frameworks, our methodology enables the numerical resolution of nonlinear stochastic control systems driven by FBM with $H \neq \frac{1}{2}$.

The remainder of this article is organized as follows. Section 2 presents some notations and summarizes the standing assumptions of this article. Section 3 presents the basic discretization of the Brownian structure. Section 4 presents the pathwise dynamic programming equation and the obtention of near-optimal controls for a given approximation level. Section 5 presents the abstract convergence results. Section 6 presents the controlled states (1.7), (1.8) and (1.9) as strongly controlled Wiener functionals. Section 2 presents the main results of this paper. The Appendix sections 8, 9 and 10 are devoted to the proofs of some technical results. Section 11 presents a simple numerical example.

1.3. Notation. The paper is quite heavy with notation, so here is a partial list for ease of reference: $U_{N,M}^k$ ($N,M$ stopping times), $U^k_m$, $\tilde{U}_m^k$, $U_{m}^{k,n}$: Set of admissible controls; equations (2.1), (3.6), (3.8) and (6.4), respectively. $e(k,T)$: Equation (3.10).
\( \mathcal{A}_k^N \): Equation (5.4).
\( u \otimes_N v \) (N stopping time), \( u^k \otimes_n u^k \) (n positive integer): Concatenations; equations (2.2) and (3.9).
\( \xi_X(u), \xi_X(u^k) \): The payoff functional \( \xi \) applied to controlled processes \( X \) and \( X^k \), respectively; equations (2.4) and (4.2).
\( \nu^k \): The probability kernel given by equation (3.5). See also (9.4).
\( Y_t \): Equation (6.1).
\( \mathbb{E}_k \): Equation (6.6).
\( \mathbb{S}_k \): Equations (3.3) and (4.5).
\( V(t, u), V^k(T^k, u^k) \): Equations (2.5) and (4.3).
\( \nu^k, t_k, t_k^k, t_k^\bar{u} \): Equations (4.18), (6.8) and (6.26), respectively.

2. Controlled stochastic processes

Throughout this article, we are going to fix a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a \( d \)-dimensional Brownian motion \( B = \{B^1, \ldots, B^d\} \) where \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \) is the usual \( \mathbb{F} \)-augmentation of the filtration generated by \( B \) under a fixed probability measure \( \mathbb{P} \). For a pair of finite \( \mathbb{F} \)-stopping times \((M, N)\), we denote

\[ ||M, N|| := \{(\omega, t); M(\omega) < t \leq N(\omega)\} \]

and \( ||M, +\infty|| := \{(\omega, t); M(\omega) < t < +\infty\} \). The action space is the compact

\[ A := \{(x_1, \ldots, x_r) \in \mathbb{R}^r; \max_{1 \leq i \leq r} |x_i| \leq \bar{a}\}, \]

for a given fixed \( 0 < \bar{a} < +\infty \). In order to set up the basic structure of our control problem, we first need to define the class of admissible control processes: for each pair \((M, N)\) of a.s finite \( \mathbb{F} \)-stopping times such that \( M < N \) a.s, we denote \( \mathbb{U}^N_M \)

\[ \mathbb{U}^N_M := \{\text{the set of all } \mathbb{F} - \text{predictable processes } u : ||M, N|| \to A; u(M+) \text{ exists}\}. \]

For such family of processes, we observe they satisfy the following properties:

- **Restriction:** \( u \in \mathbb{U}^N_M \Rightarrow u ||M, P|| \in \mathbb{U}^P_M \) for \( M < P \leq N \) a.s.
- **Concatenation:** If \( u \in \mathbb{U}^N_M \) and \( v \in \mathbb{U}^P_M \) for \( M < N < P \) a.s, then \( (u \otimes_N v)(\cdot) \in \mathbb{U}^P_M \), where

\[
(u \otimes_N v)(r) = \begin{cases} 
u(r) & \text{if } M < r \leq N \\ v(r) & \text{if } N < r \leq P. \end{cases}
\]

- **Finite Mixing:** For every \( u, v \in \mathbb{U}^N_M \) and \( G \in \mathcal{F}_M \), we have

\[ u1_G + v1_G \in \mathbb{U}^N_M. \]

Let \( \mathbb{L}^p(\mathbb{P} \times \text{Leb}) \) be the Banach space of all \( \mathbb{F} \)-adapted finite-dimensional processes \( Y \) such that

\[ \mathbb{E} \int_0^T \|Y(t)\|^p dt < \infty, \]

for \( 1 \leq p < \infty \), where \( \| \cdot \| \) is an Euclidean norm and \( 0 < T < +\infty \) is a fixed terminal time. Let us denote

\[ \|f\| \infty := \sup_{0 \leq t \leq T} \|f(t)\|. \]

We also define \( \mathbb{B}^p(\mathbb{F}) \) as the space of all \( \mathbb{F} \)-adapted càdlàg processes \( Y \) such that

\[ \text{ whenever necessary, we can always extend a given } u \in \mathbb{U}^N_M \text{ by setting } u = 0 \text{ on the complement of a stochastic set } ||M, N||. \]
\[ \|Y\|_{p} := \mathbb{E}\|Y\|_{\infty} < \infty, \]

for \( p \geq 1 \).

**Definition 2.1.** A continuous controlled Wiener functional is a map \( X : U_{0}^{T} \rightarrow L_{\mathcal{F}}^{p}(\mathbb{P} \times \text{Leb}) \) for some \( p \geq 1 \), such that for each \( t \in [0, T] \) and \( u \in U_{0}^{T} \), \( \{X(s, u); 0 \leq s \leq t\} \) depends on the control \( u \) only on \((0, t]\) and \( X(\cdot, u) \) has continuous paths for each \( u \in U_{0}^{T} \).

In the sequel, we denote \( D_{n,T} := \{h : [0, T] \rightarrow \mathbb{R}^{n} \text{ with càdlàg paths}\} \) and we equip this linear space with the uniform convergence on \([0, T]\). We now present the two standing assumptions of this article.

**Assumption (A1):** The payoff \( \xi : D_{n,T} \rightarrow \mathbb{R} \) satisfies the following regularity assumption: There exists \( \gamma \in (0, 1] \) and a constant \( C > 0 \) such that

\[ |\xi(f) - \xi(g)| \leq C\|f - g\|_{\infty}, \]

for every \( f, g \in D_{n,T} \).

**Assumption (B1):** There exists a constant \( C \) such that

\[ \|X(\cdot, u) - X(\cdot, \eta)\|_{B_{p}(\mathcal{F})}^{2} \leq C\mathbb{E}\int_{0}^{T}\|u(s) - \eta(s)\|^{2}ds, \]

for every \( u, \eta \in U_{0}^{T} \).

**Remark 2.1.** Path-dependent controlled SDEs with Lipschitz coefficients are typical examples of controlled Wiener functionals satisfying Assumption (B1). See Remark 6.7.

**Remark 2.2.** Even though we are only interested in controlled Wiener functionals with continuous paths, we are forced to assume the payoff functional is defined on the space of càdlàg paths due to a discretization procedure. However, this is not a strong assumption since most of the functionals of interest admits extensions from \( C_{n,T} \) to \( D_{n,T} \) preserving Assumption (A1).

From now on, we are going to fix a controlled Wiener functional \( X : U_{0}^{T} \rightarrow \mathcal{B}^{2}(\mathcal{F}) \). For a given functional \( \xi : D_{n,T} \rightarrow \mathbb{R} \), we denote

\[ \xi_{X}(u) := \xi(X(\cdot, u)); \quad u \in U_{0}^{T} \]

and

\[ V(t, u) := \text{ess sup}_{v \in U_{0}^{T}} \mathbb{E}[\xi_{X}(u \otimes_{t} v)|\mathcal{F}_{t}]; \quad 0 \leq t < T, \quad u \in U_{0}^{T}, \]

where \( V(T, u) := \xi_{X}(u) \) a.s. We stress the process \( V(\cdot, u) \) has to be viewed backwards for each control \( u \in U_{0}^{T} \). Throughout this paper, in order to keep notation simple, we omit the dependence of the value process in (2.5) on the controlled Wiener functional \( X \) and the payoff \( \xi \), so we write \( V \) meaning as a map \( V : U_{0}^{T} \rightarrow L_{\mathbb{P}}^{1}(\mathbb{P} \times \text{Leb}) \).

Since we are not assuming that \( \mathcal{F}_{0} \) is the trivial \( \sigma \)-algebra, we cannot say that \( V(0) \) is deterministic. However, the finite-mixing property on the class of admissible controls implies that \( \{\mathbb{E}[\xi_{X}(u \otimes_{t} \theta)|\mathcal{F}_{t}]; \theta \in U_{0}^{T}\} \) has the lattice property (see e.g Def 1.1.2 [30]) for every \( t \in [0, T] \) and \( u \in U_{0}^{T} \). In this case, it is known

\[ \mathbb{E}[V(0)] = \text{sup}_{v \in U_{0}^{T}} \mathbb{E}[\xi(X(\cdot, v))]. \]
Remark 2.3. One can easily check that under Assumptions (A1-B1), for any \( u \in U_0^T \), \( \{V(s, u); 0 \leq s \leq t\} \) depends only on the control \( u \) restricted to the interval \([0, t]\). Moreover, \( u \mapsto V(\cdot, u) \) is a continuous controlled Wiener functional in the sense of Definition 2.1 and \( V \) is an \( U_0^T \)-supermartingale, in the sense that \( V(\cdot, u) \) is an \( F \)-supermartingale for each \( u \in U_0^T \).

Definition 2.2. We say that \( u \in U_0^T \) is an \( \epsilon \)-optimal control if

\[
E[\xi_X(u)] \geq \sup_{\eta \in U_0^T} E[\xi_X(\eta)] - \epsilon.
\]

In case, \( \epsilon = 0 \), we say that \( u \) realizing (2.6) is an exact optimal control.

3. Discrete-type skeleton for the Brownian motion and controlled imbedded discrete structures

In this section, we introduce what we call a controlled imbedded discrete structure which is a natural extension of the approximation models presented in Leão, Ohashi and Simas [39]. Our philosophy is to view a controlled Wiener functional as a family of simplified models one has to build in order to extract the relevant information for the obtention of a concrete description of value processes and the construction of their associated (near or exact) optimal controls.

3.1. The underlying discrete skeleton. The discretization procedure will be based on a class of pure jump processes driven by suitable waiting times which describe the local behavior of the Brownian motion. We briefly recall the basic properties of this skeleton. For more details, we refer the reader to the work [39]. We start by constructing a sequence \( T := \{T_n^k; n \geq 0\} \) of hitting times which will be the basis for our discretization scheme. We set \( T_0^k := 0 \) and

\[
T_n^k := \inf\{T_{n-1}^k < t < \infty; \|B(t) - B(T_{n-1}^k)\| = \epsilon_k\}, \quad n \geq 1,
\]

where \( \sum_{k \geq 1} \epsilon_k < \infty \) and \( \| \cdot \| \) corresponds to the maximum norm on \( \mathbb{R}^d \). This implies

\[
\Delta T_n^k := T_n^k - T_{n-1}^k = \min_{j \in \{1, 2, \ldots, d\}} \{\Delta_n^{k,j}\} \text{ a.s.}
\]

where

\[
\Delta_n^{k,j} := \inf\{0 < t < \infty; |B^j(t + T_{n-1}^k) - B^j(T_{n-1}^k)| = \epsilon_k\}, \quad n \geq 1.
\]

Then, we define \( A^k := (A^{k,1}, \ldots, A^{k,d}) \) by

\[
A^{k,j}(t) := \sum_{n=1}^{\infty} \left( B^j(T_n^k) - B^j(T_{n-1}^k) \right) 1_{\{t \leq T_n^k\}}; \ t \geq 0, \ j = 1, \ldots, d,
\]

for integers \( k \geq 1 \).

The multi-dimensional filtration generated by \( A^k \) is naturally characterized as follows. Let \( \tilde{\mathcal{F}}^k := \{\tilde{\mathcal{F}}^k_t; 0 \leq t < \infty\} \) be the filtration generated by \( A^k \). We observe

\[
\tilde{\mathcal{F}}^k_t \cap \{T_n^k \leq t < T_{n+1}^k\} = \tilde{\mathcal{F}}^{k^*}_t \cap \{T_n^k \leq t < T_{n+1}^k\}; \ t \geq 0,
\]

where \( \tilde{\mathcal{F}}^{k^*} := \sigma(A^k(s \wedge T_n^k); s \geq 0) \) for each \( n \geq 0 \). Let \( \mathcal{F}^k_{\infty} \) be the completion of \( \sigma(A^k(s); s \geq 0) \) and let \( \mathcal{N}_k \) be the \( \sigma \)-algebra generated by all \( \mathbb{P} \)-null sets in \( \mathcal{F}^k_{\infty} \). We denote \( \mathbb{F}^k = (\mathcal{F}^k_t)_{t \geq 0} \), where \( \mathcal{F}^k_t \) is the usual \( \mathbb{P} \)-augmentation (based on \( \mathcal{N}_k \)) satisfying the usual conditions.

Definition 3.1. The structure \( \mathcal{D} = \{\mathcal{T}, A^k; k \geq 1\} \) is called a discrete-type skeleton for the Brownian motion.

By the strong Markov property, we observe that

1. The jumps \( \Delta A^{k,j}(T_n^k) = A^{k,j}(T_n^k) - A^{k,j}(T_{n-1}^k) \); \( n = 1, 2, \ldots \) are independent and identically distributed (iid).
(2) The waiting times $\Delta T^k_n; n = 1, 2, \ldots$ are iid random variables in $\mathbb{R}_+$.  
(3) The families $\{\Delta A^{k,i}(T^k_n); n = 1, 2, \ldots\}$ and $\{\Delta T^k_n; n = 1, 2, \ldots\}$ are independent.

Moreover, it is immediate that $A^{k,j}$ is a square-integrable $F^k$-martingale for each $j = 1, \ldots, d$.

**Remark 3.1.** The skeleton given in Definition 3.1 slightly differs from Def 2.1 in [39] and [40] if $d > 1$. In order to deal with controlled Wiener functionals, we adopt this modification. See also Remark 9.2. Since $(A^{1,k}, \ldots, A^{k,d})$ is a $d$-dimensional stepwise constant martingale process satisfying properties (1), (2) and (3) above, then the basic underlying differential structure presented in [39] is preserved. Since this article does not focus on representation results for controlled Wiener functionals, the limiting differential structure will not be investigated and we leave this construction to a future work.

### 3.2. Pathwise dynamics of the skeleton.

For a given choice of discrete-type skeleton $\mathcal{G}$, we will construct controlled functionals written on this structure. Before we proceed, it is important to point out that there exists a pathwise description of the dynamics generated by $\mathcal{G}$. Let us define

$$\mathbb{I}^k_n := \{(i^k_1, \ldots, i^k_d); i^k_j \in \{-1,0,1\} \; \forall j \in \{1, \ldots, d\} \; \text{and} \; \sum_{j=1}^d |i^k_j| = 1\}$$

and

$$\mathbb{I}_k := \{\epsilon_k \left(\mathbb{I}^k_1, z^k_1; \mathbb{I}^k_2, \ldots, z^k_d; \mathbb{I}^k_3, \mathbb{I}^k_4, \mathbb{I}^k_5, \mathbb{I}^k_6, \mathbb{I}^k_7, \mathbb{I}^k_8\right); (i^k_1, \ldots, i^k_d) \in \mathbb{I}^k_n, (z^k_1, \ldots, z^k_d) \in (-1,1)^d\}.$$  

The $n$-fold Cartesian product of $\mathbb{S}_k := (0, +\infty) \times \mathbb{I}_k$ is denoted by $\mathbb{S}^n_k$ and a generic element of $\mathbb{S}^n_k$ will be denoted by

$$b^k_n := (s^k, t^k, \ldots, s^k, t^k) \in \mathbb{S}^n_k,$$

where $(s^k, t^k) \in \mathbb{S}_k$ for $1 \leq r \leq n$. Let us define $\eta^k_n := (\eta^k_{1,j}, \ldots, \eta^k_{n,j})$, where

$$\eta^k_{n,j} := \Delta A^{k,j}(T^k_n) = B^j(T^k_n) - B^j(T^k_{n-1}),$$

for $1 \leq j \leq d; n,k \geq 1$. Let us define

$$A^k_n := \left(\Delta T^k_1, \eta^k_1, \ldots, \Delta T^k_n, \eta^k_n\right) \in \mathbb{S}^n_k \; \text{a.s.}$$

One should notice that

$$\mathcal{F}^k_{T^k_n} = (A^k_n)^{-1}(\mathcal{B}(\mathbb{S}^n_k)),$$

where $\mathcal{B}(\mathbb{S}^n_k)$ is the Borel $\sigma$-algebra generated by $\mathbb{S}^n_k; n \geq 1$.

By applying properties (1), (2) and (3), the law of the system will evolve according to the following probability measure defined by

$$\mathbb{P}^k_r(E_1 \times \cdots \times E_r) := P\{A^k_n \in \bigcap_{i=1}^r E_i\} = \prod_{i=1}^r \nu^k(E_i),$$

for $k,r \geq 1$, where $\nu^k$ is the probability measure defined by

$$\nu^k(E) := \mathbb{P}(\Delta T^k_n, \eta^k_n) \in E; E \in \mathcal{B}(\mathbb{S}_k).$$

By the very definition, $\nu^k(\cdot) = \mathbb{P}(\Delta T^k_{n+1}, \eta^k_{n+1} = \nu^k_n) \in |A^k_n = b^k_n|$ for every $b^k_n \in \mathbb{S}^n_k; n,k \geq 1$.

Let us start to introduce a subclass $U^k_{T^k_{m,n}} \subset U^k_{T^k_{m,n}}; 0 \leq m < n < \infty$. For $m < n$, let $U^k_{T^k_{m,n}}$ be the set of $F^k$-predictable processes of the form
\[ v^k(t) = \sum_{j=m+1}^{n} v^k_{j-1} \mathbb{1}_{\{T^k_{j-1} \leq t < T^k_j\}}; \quad T^k_m \leq t \leq T^k_n, \]

where for each \( j = m+1, \ldots, n \), \( v^k_{j-1} \) is an \( \mathbb{A} \)-valued \( \mathcal{F}^{k}_{T^k_{j-1}} \)-measurable random variable. To keep notation simple, we use the shorthand notations

\[ (3.10) \]

\[ e(k, t) := \left[ \frac{\varepsilon^2 t}{\chi_d} \right]; 0 \leq t \leq T, \]

where \([x]\) is the smallest integer greater or equal to \( x \geq 0 \) and

\[ (3.11) \]

\[ \chi_d := \mathbb{E} \min\{\tau^1, \ldots, \tau^d\}, \]

where \((\tau^j)_{j=1}^d\) is an iid sequence of random variables with distribution \(\inf\{t > 0; |W(t)| = 1\}\) for a real-valued standard Brownian motion \(W\). From Lemma 5.3, for each \( t \in [0, T] \), we know that

\[ T^k_{c(k,t)} \rightarrow t \text{ a.s and in } L^p(\mathbb{P}), \]

as \( k \rightarrow +\infty \), for each \( t \geq 0 \) and \( p \geq 1 \). Let \( O_T(\mathbb{F}^k) \) be the set of all stepwise constant \( \mathbb{F}^k \)-optional processes of the form

\[ Z^k(t) = \sum_{n=0}^{\infty} Z^k(T^k_n) \mathbb{1}_{\{T^k_n \leq t \land T^k_{c(k,t)} < T_n^k\}}; 0 \leq t \leq T, \]

where \( Z^k(T^k_n) \) is \( \mathcal{F}^{k}_{T^k_n} \)-measurable for every \( n \geq 0 \) and it has integrable quadratic variation \( \mathbb{E}[Z^k, Z^k](T) < \infty \), for every \( k \geq 1 \).
Let us now present two concepts which will play a key role in this work.

**Definition 3.2.** A **controlled imbedded discrete structure** $Y = \{(Y^k)_{k \geq 1}, D\}$ consists of the following objects: a discrete-type skeleton $D$ and a map $u^k : Y(\cdot, u^k) : X_{0}^{k, e(k, T)} \to O_T(\mathbb{R}^k)$ such that

\begin{equation}
Y^k(T_{n+1}, u^k) \text{ depends on the control only at } (u^k_0, \ldots, u^k_n),
\end{equation}

for each integer $n \in \{0, \ldots, e(k, T) - 1\}$.

**Definition 3.3.** A **strongly controlled Wiener functional** is a pair $(X, X)$, where $X$ is a controlled Wiener functional and $X = \{(X^k)_{k \geq 1}, D\}$ is a controlled imbedded discrete structure such that

\begin{equation}
\lim_{k \to +\infty} \sup_{\phi \in U_{k, e(k, T)}} \mathbb{E} \|X^k(\phi) - X(\phi)\|_{\gamma}^\gamma = 0,
\end{equation}

for $0 < \gamma \leq 1$.

The concepts of controlled imbedded discrete structures and strongly controlled Wiener functionals are nonlinear versions of the structures analyzed in [39]. The typical example we have in mind of a strongly controlled Wiener functional is a controlled state which drives a stochastic control problem. It is expected that functionals of strongly controlled Wiener functionals are continuous in some sense w.r.t imbedded discrete structures.

**4. The controlled imbedded discrete structure for the value process**

In this section, we are going to describe the canonical controlled imbedded discrete structure associated with an arbitrary value process

\begin{equation}
V(t, u) = \text{ess sup}_v \mathbb{E}[\xi_X(u \otimes_t v)|\mathcal{F}_t]; u \in U^T_0, 0 \leq t \leq T,
\end{equation}

where the payoff $\xi$ satisfies Assumption A1 and $X$ is an arbitrary controlled Wiener functional satisfying Assumption B1. Throughout this section, we are going to fix a controlled imbedded discrete structure

\begin{equation}
u^k \mapsto X^k(\cdot, u^k)
\end{equation}

and we set

\begin{equation}
\xi_X^k(u^k) := \xi(X^k(\cdot, u^k)),
\end{equation}

for $u^k \in U_{k, e(k, T)}$. We then define

\begin{equation}
V^k(T_n, u^k) := \text{ess sup}_{\phi^k \in U_{k, e(k, T)}} \mathbb{E}\left[\xi_X^k(u^k \otimes_n \phi^k)|\mathcal{F}^k_{T_n}\right]; n = 1, \ldots, e(k, T) - 1,
\end{equation}

with boundary conditions

\begin{equation}
V^k(0) := V^k(0, u^k) := \sup_{\phi^k \in U_{k, e(k, T)}} \mathbb{E}[\xi_X^k(\phi^k)], \quad V^k(T_{e(k, T)}, u^k) := \xi_X^k(u^k).
\end{equation}

This naturally defines the map $V^k : U_{k, e(k, T)} \to O_T(\mathbb{R}^k)$ with jumps $V^k(T_n, u^k); n = 1, \ldots, e(k, T)$ for $u^k \in U_{k, e(k, T)}$. One should notice that $V^k(T_n, u^k)$ only depends on $u^{k,n-1} := (u^k_0, \ldots, u^k_{n-1})$ so it is natural to write
Brownian filtration is replaced by the discrete-time filtration
with the convention that $u^{k, -1} = 0$. By construction, $V^k$ satisfies \eqref{3.12} in Definition 3.2.

Similar to the value process $V$, we can write a dynamic programming principle for $V^k$ where the Brownian filtration is replaced by the discrete-time filtration $\mathcal{F}^{k}_{i,n}; n = e(k,T) - 1, \ldots, 0$.

**Lemma 4.1.** Let $0 \leq n \leq e(k,T) - 1$. For each $\phi^k$ and $\psi^k$ in $U^{k,e(k,T)}$, there exists $\theta^k \in U^{k,e(k,T)}$ such that

$$
E \left[ \xi_{X^k} (\pi^k \otimes \theta^k) | \mathcal{F}^{k}_{i,n} \right] = E \left[ \xi_{X^k} (\pi^k \otimes \psi^k) | \mathcal{F}^{k}_{i,n} \right] \quad \text{for every } \pi^k \in U^{k,n}.
$$

Therefore, for each $\pi^k \in U^{k,n}$

$$
E \left[ \text{ess sup}_{\pi^k \in U^{n+1}} E \left[ \xi_{X^k} (\pi^k \otimes \pi^k) | \mathcal{F}^{k}_{i,n} \right] | \mathcal{F}^{k}_{i,n} \right] = E \left[ \xi_{X^k} (\pi^k \otimes \pi^k) | \mathcal{F}^{k}_{i,n} \right] \quad \text{for } 0 \leq j \leq n \quad \text{and } 0 \leq n \leq e(k,T) - 1.
$$

**Proof.** For $\pi^k \in U^{k,n}$, let $G = \left\{ E \left[ \xi_{X^k} (\pi^k \otimes \theta^k) | \mathcal{F}^{k}_{i,n} \right] > \xi_{X^k} (\pi^k \otimes \psi^k) | \mathcal{F}^{k}_{i,n} \right\}$. Choose $\theta^k = \phi^k \mathbb{I}_G + \psi^k \mathbb{I}_{G^c}$ and apply the finite mixing property to exchange the esssup into the conditional expectation (see e.g Prop 1.1.4 in [36] to conclude the proof. □

**Proposition 4.1.** For each $u^k \in U^{k,e(k,T)}$, the discrete-time value process $V^k(. , u^k)$ satisfies

$$
V^k(T^k_n, u^k) = \text{ess sup}_{\theta^k \in U^{k,n+1}} E \left[ V^k \left( T^k_{n+1}, u^{k,n+1} \otimes \theta^k_n \right) | \mathcal{F}^{k}_{i,n} \right]; \quad 0 \leq n \leq e(k,T) - 1
$$

On the other hand, if a class of processes $\{Z^k(T^k_n, u^k); u^k \in U^{k,e(k,T)}; 0 \leq n \leq e(k,T)\}$ satisfies the dynamic programming equation \eqref{4.1} for every $u^k \in U^{k,e(k,T)}$, then $Z^k(T^k_n, u^k)$ coincides with $V^k(T^k_n, u^k)$ a.s for every $0 \leq n \leq e(k,T)$ and for every $u^k \in U^{k,e(k,T)}$.

**Proof.** Fix $u^k \in U^{k,e(k,T)}$. By using Lemma 4.1 and the identity

$$
\text{ess sup}_{\phi^k \in U^{k,e(k,T)}} E \left[ \xi_{X^k} (u^k \otimes \phi^k) | \mathcal{F}^{k}_{i,n} \right] = \text{ess sup}_{\theta^k \in U^{k,n+1}} \text{ess sup}_{\phi^k \in U^{k,e(k,T)}} E \left[ \xi_{X^k} (u^k \otimes (\theta^k \otimes \phi^k)) | \mathcal{F}^{k}_{i,n} \right]
$$

a.s for each $0 \leq n \leq e(k,T) - 1$, the proof is straightforward, so we omit the details. □

### 4.1. Construction of optimal controls: Measurable selection and $\epsilon$-controls

Let us now start the pathwise description. The whole dynamics will take place in the history space $\mathbb{H}^{n} := H \times S_1 \times \cdots \times S_n$.

We denote $\mathbb{H}^{k,n}$ and $\mathbb{I}_k$ as the $n$-fold Cartesian product of $\mathbb{H}^k$ and $\mathbb{I}_k$, respectively. The elements of $\mathbb{H}^{k,n}$ will be denoted by

$$
\phi^k_n := \left( (a^k_0, s^k_1, \tilde{s}^k_1), \ldots, (a^k_{n-1}, s^k_n, \tilde{s}^k_n) \right),
$$
where $(a^k_0, \ldots, a^k_{n-1}) \in H^n$, $(s^k_1, \ldots, s^k_n) \in (0, +\infty)^n$ and $(\tilde{s}^k_1, \ldots, \tilde{s}^k_n) \in \mathbb{I}_n$.

For a given admissible control $u^k \in U^{k,e(k,T)}$, we define $\Xi^k_{j} : \Omega \rightarrow \mathbb{H}^{k,j}$ as follows

$$
\Xi^k_{j} := \left( (u^k_0, \Delta T^k_1, \eta^k_1), \ldots, (u^k_{j-1}, \Delta T^k_j, \eta^k_j) \right),
$$
for $1 \leq j \leq e(k,T)$. We identify $\Xi_{j}^{k,u^{k}}$ as a constant (in the action space $A$) which does not necessarily depend on the control $u^{k}$. By definition, we obtain that $\Xi_{j}^{k,u^{k}}$ is an $\mathcal{F}^{k}_{T}$-measurable, for every $u^{k} \in U_{0}^{k,e(k,T)}$ and $j = 1,\ldots,m$.

In the sequel, it will be important to deal with universally measurable sets. For readers who are not familiar with this class of sets, we refer to e.g. [3]. If $(R, \beta(R))$ is a Borel space, let $P(R)$ be the space of all probability measures defined on the Borel $\sigma$-algebra $\mathcal{B}(R)$ generated by $R$. We denote

$$\mathcal{E}(R) := \bigcap_{p \in P(R)} \mathcal{B}(R, p),$$

where $\mathcal{B}(R, p)$ is the $p$-completion of $\mathcal{B}(R)$ w.r.t $p \in P(R)$.

**Lemma 4.2.** Let $G_{j}^{k} : \mathbb{H}^{k,j} \to E$ be a universally measurable function, where $(E, \mathcal{B}(E))$ is a Borel space. Then, the composition $G_{j}^{k} \circ \Xi_{j}^{k,u^{k}}$ is $\mathcal{F}_{T}^{k}$-measurable, for every $u^{k} \in U_{0}^{k,e(k,T)}$ and $j = 1,\ldots,m$.

**Proof.** We need to show that given $D \in \mathcal{B}(E)$, the inverse image $(\Xi_{j}^{k,u^{k}})^{-1}[(G_{j}^{k})^{-1}(D)] \in \mathcal{F}_{T}^{k}$. Since $(G_{j}^{k})^{-1}(D)$ is a universally measurable set, it is sufficient to check that $(\Xi_{j}^{k,u^{k}})^{-1}(H) \in \mathcal{F}_{T}^{k}$ for every universally measurable set $H \in \mathbb{H}^{k,j}$. Let $\mu$ be a probability measure on $\mathbb{H}^{k,j}$ given by

$$\mu(C) = \mathbb{P} \left( (\Xi_{j}^{k,u^{k}})^{-1}(C) \right), \quad C \in \mathcal{B}(\mathbb{H}^{k,j}).$$

For a given universally measurable set $H$ in $\mathbb{H}^{k,j}$, we can select (see e.g Lemma 7.26 in [3]) $\tilde{H} \in \mathcal{B}(\mathbb{H}^{k,j})$ such that

$$\mathbb{P} \left( (\Xi_{j}^{k,u^{k}})^{-1}(\tilde{H}) \right) = \mu \left( \tilde{H} \right) = 0.$$

The set $(\Xi_{j}^{k,u^{k}})^{-1}(\tilde{H}) \in \mathcal{F}_{T}^{k}$, so there exists $W \in \tilde{\mathcal{F}}_{T}^{k}$ such that $\mathbb{P} \left( W \cap (\Xi_{j}^{k,u^{k}})^{-1}(\tilde{H}) \right) = 0$. Then, $\mathbb{P} \left( W \cap (\Xi_{j}^{k,u^{k}})^{-1}(H) \right) = 0$ and hence, $(\Xi_{j}^{k,u^{k}})^{-1}(H) \in \mathcal{F}_{T}^{k}$. This concludes the proof.

Let us now present a selection measurable theorem which will allow us to aggregate the map $u^{k} \mapsto V^{k}(\cdot, u^{k})$ into a single list of upper semi-analytic functions $F_{j}^{k} : \mathbb{H}^{k,j} \to \mathbb{R}$; $j = 0,\ldots,e(k,T)$. More importantly, we will construct an $\epsilon$-optimal control at the level of the optimization problem

$$\sup_{\phi^{k} \in U_{0}^{k,e(k,T)}} \mathbb{E} \left[ \xi X_{\omega}(\phi^{k}) \right]; \quad k \geq 1.$$  

(4.7)

At first, we observe that for a given control $u^{k} \in U_{0}^{k,e(k,T)}$ and a given $x \in \mathbb{R}^{n}$, we can easily construct a Borel function $\gamma_{e(k,T)}^{k} : \mathbb{H}^{k,e(k,T)} \to \mathbb{D}^{n}_{T}$ such that $\gamma_{e(k,T)}^{k}(0) = x = X^{k}(0, u^{k})$ and

$$\gamma_{e(k,T)}^{k}( \Xi_{e(k,T)}^{k,u^{k}}(\omega)) \left( t \right) = X^{k}(t, \omega, u^{k}(\omega),$$

(4.8)

for a.a $\omega$ and for every $t \in [0,T]$. For concrete examples of these constructions, we refer to Section 6 (see [6.7] and [6.29]).

Let us now present the selection measurable theorem which will play a key role in our methodology. For this purpose, we will make a backward argument. To keep notation simple, in the sequel we set $m = e(k,T)$. Recall that a structure of the form (4.1) is fixed and it is equipped with a Borel function $\gamma_{m}^{k} : \mathbb{H}^{k,m} \to \mathbb{D}^{n}_{T}$ realizing (4.8) with a given initial condition $x \in \mathbb{R}^{n}$. For such structure, we write $V^{k}$ as the associated value process given by 4.3.
We start with the map $\psi_k^m : \mathbb{H}^{k,m} \to \mathbb{R}$ defined by

$$\psi_k^m(o_k^m) := \xi(\gamma_m(o_k^m)) ; o_m^k \in \mathbb{H}^{k,m}.$$  

By construction, $\psi_k^m$ is a Borel function.

**Lemma 4.3.** Assume that $\xi X_k(u^k) \in L^1(\mathcal{P})$ for $u^k \in U_0^{k,m}$. Then,

$$E\left[\xi X_k(u^k) | F_k^{T_m-1}\right] = \int_{\mathbb{S}_k} \psi_k^m \left(\left(\left(u^k_0, \Delta T^k_1, \eta^k_1\right), \ldots, \left(u^k_{m-1}, s^k_{m-1}, \tilde{r}^k_{m-1}\right)\right)\right) \nu^k(ds_m^k d\tilde{r}^k_{m-1}) \ a.s$$

**Proof.** At first, we observe

(4.10) $$E\left[\xi X_k(u^k) | F_k^{T_m-1}\right] = E\left[\xi X_k(u^k) | \tilde{F}_k^{T_m-1}\right] \ a.s$$

for every $u^k = (u^k_0, \ldots, u^k_{m-1}) \in U_0^{k,m}$. Elementary computation yields

$$E\left[\xi X_k(u^k) | \tilde{F}_k^{T_m-1}\right] = \int_{\mathbb{S}_k} \psi_k^m \left(\left(\left(u^k_0, \Delta T^k_1, \eta^k_1\right), \ldots, \left(u^k_{m-1}, s^k_{m-1}, \tilde{r}^k_{m-1}\right)\right)\right) \nu^k(ds_m^k d\tilde{r}^k_{m-1}) \ a.s$$

for every control $u^k = (u^k_0, \ldots, u^k_{m-1}) \in \tilde{U}_0^{k,m}$. Now, we recall for a given $u^k = (u^k_0, \ldots, u^k_{m-1}) \in U_0^{k,m}$, there exists $\tilde{u}^k = (v^k_0, \ldots, v^k_{m-1}) \in \tilde{U}_0^{k,m}$ which fulfills the following: for each $i$, there exists a set $\tilde{C}_i^k \in \tilde{F}_i^{k,m}$ of full probability such that $u^k_i(\omega) = v^k_i(\omega) \in \tilde{C}_i^k$ for $i = 0, \ldots, m-1$. Let us denote

$$\tilde{G}^k = \cap_{i=0}^{m-1} \tilde{C}_i^k \in \tilde{F}_m^{k,m}.$$  

Then, $u^k = v^k$ in $\tilde{G}^k$ and hence

$$E\left[\xi X_k(u^k) | \tilde{F}_m^{k,m}\right] = E\left[\xi X_k(u^k) \mathbb{I}_{(\tilde{G}^*)} | \tilde{F}_m^{k,m}\right] = E\left[\xi X_k(v^k) \mathbb{I}_{(\tilde{G}^*)} | \tilde{F}_m^{k,m}\right] = E\left[\xi X_k(u^k) \mathbb{I}_{(\tilde{G}^*)} | \tilde{F}_m^{k,m}\right]$$

Identities (4.10) and (4.11) allow us to conclude the proof. \hfill $\square$

**Lemma 4.4.** The map

$$o_{m-1}^k, a_{m-1}^k \mapsto \int_{\mathbb{S}_k} \psi_k^m(o_{m-1}^k, a_{m-1}^k, s_{m-1}^k, \tilde{r}_{m-1}^k) \nu^k(ds_m^k d\tilde{r}_{m-1}^k)$$

is a Borel function from $\mathbb{H}^{k,m-1} \times \mathbb{A}$ to $\mathbb{R}$.

**Proof.** We just need to imitate the proof of Prop. 7.29 in [x]. \hfill $\square$

**Lemma 4.5.** Let $\psi_{m-1}^k : \mathbb{H}^{k,m-1} \to \mathbb{R}$ be the function defined by

$$\psi_{m-1}^k(o_{m-1}^k) := \sup_{a_{m-1}^k \in \mathbb{A}} \int_{\mathbb{S}_k} \psi_k^m(o_{m-1}^k, a_{m-1}^k, s_{m-1}^k, \tilde{r}_{m-1}^k) \nu^k(ds_m^k d\tilde{r}_{m-1}^k),$$

for $o_{m-1}^k \in \mathbb{H}^{k,m-1}$. Then, $\psi_{m-1}^k$ is upper semianalytic and for every $\epsilon > 0$, there exists an analytically measurable function $C_{k,m-1}^\epsilon : \mathbb{H}^{k,m-1} \to \mathbb{A}$ which realizes

$$\psi_{m-1}^k(o_{m-1}^k) \leq \int_{\mathbb{S}_k} \psi_k^m(o_{m-1}^k, C_{k,m-1}^\epsilon(o_{m-1}^k), s_{m-1}^k, \tilde{r}_{m-1}^k) \nu^k(ds_m^k d\tilde{r}_{m-1}^k) + \epsilon,$$

for every $o_{m-1}^k \in \{\psi_{m-1}^k < +\infty\}$. \hfill $\square$
Proof. The fact that $\mathcal{V}_{m-1}$ is upper semianalytic follows from Prop 7.47 in [3] and Lemma [4.4] which say the map given by

$$f(\mathbf{o}_{m-1}, a_{m-1}^k) = \int_{\mathbb{S}_k} \mathbb{V}_{m-1}^k(\mathbf{o}_{m-1}, a_{m-1}^k, s_{m-1}^k, \mathbf{i}_{m-1}^k) \nu^k(d\mathbf{s}_{m-1} d\mathbf{i}_{m-1})$$

is a Borel function (hence upper semianalytic). Moreover, by construction, $\mathbb{H}^{k,m-1} \times \mathbb{A}$ is a Borel set. Let

$$\mathcal{V}_{m-1}(\mathbf{o}_{m-1}^k) = \sup_{a_{m-1}^k \in \mathbb{A}} f(\mathbf{o}_{m-1}^k, a_{m-1}^k); \mathbf{o}_{m-1}^k \in \mathbb{H}^{k,m-1}.$$ 

Prop 7.50 in [3] yields the existence of an analytically measurable function $C_{k,m-1}^\epsilon : \mathbb{H}^{k,m-1} \rightarrow \mathbb{A}$ such that

$$f(\mathbf{o}_{m-1}^k, C_{k,m-1}^\epsilon(\mathbf{o}_{m-1}^k)) \geq \mathcal{V}_{m-1}(\mathbf{o}_{m-1}^k) - \epsilon,$$

for every $\mathbf{o}_{m-1}^k \in \{\mathcal{V}_{m-1} < +\infty\}$. □

Lemma 4.6. If $\mathbb{H}_{m-1}^{k,m} = \{\mathcal{V}_{m-1} < +\infty\}$, then for every $\epsilon > 0$ and $u_k \in U_{0,m}^k$, there exists a control $\phi_{m-1}^{k,\epsilon} \in U_{m-1}^k$ such that

$$\mathcal{V}_{m-1}(\mathbb{Z}_{m-1}^{k,u_k}) = V_k(T_{m-1}^k, u_k) \leq E[V_k(T_{m-1}^k, u_k \otimes_{m-1} \phi_{m-1}^{k,\epsilon}) | \mathcal{F}_{T_{m-1}^k}^{k}] + \epsilon \text{ a.s.}$$

Proof. For every $\theta_{m-1}^k \in U_{m-1}^k$, it follows from Equation (4.9) that $E[\xi_k(u_k \otimes_{m-1} \theta_{m-1}^k) | \mathcal{F}_{T_{m-1}^k}^{k}]$ equals (a.s) to

$$\int_{\mathbb{S}_k} \mathbb{V}_{m-1}^k \left[ \left(u_1^k, \Delta T_1^k, \eta_1^k\right), \ldots, \left(u_{m-2}^k, \Delta T_{m-1}^k, \eta_{m-1}^k\right), \left(\theta_{m-1}^k, s_{m-1}^k, \mathbf{i}_{m-1}^k\right) \right] \nu^k(d\mathbf{s}_{m-1} d\mathbf{i}_{m-1}).$$

As a consequence, we obtain that

$$\mathcal{V}_{m-1}(\mathbb{Z}_{m-1}^{k,u_k}) \geq \mathbb{V}_{m-1} \left[ V_k(T_{m-1}^k, u_k \otimes_{m-1} \theta_{m-1}^k) \right] \text{ a.s.},$$

for every $\theta_{m-1}^k \in U_{m-1}^k$. By composing $\mathcal{V}_{m-1}$ with $\mathbb{Z}_{m-1}^{k,u_k}$ in (4.12), we obtain that $\mathcal{V}_{m-1}(\mathbb{Z}_{m-1}^{k,u_k})$ is an $\mathcal{F}_{T_{m-1}^k}^{k}$-measurable function (see, Lemma 4.2). Then, the definition of ess sup and Equation (4.14) yield that

$$\mathcal{V}_{m-1}(\mathbb{Z}_{m-1}^{k,u_k}) \geq \left[ V_k(T_{m-1}^k, u_k \otimes_{m-1} \theta_{m-1}^k) \right] \text{ a.s.},$$

for $\epsilon > 0$, let $C_{k,m-1}^\epsilon : \mathbb{H}^{k,m-1} \rightarrow \mathbb{A}$ be the analytically measurable function which realizes (4.12). We take $\phi_{m-1}^{k,\epsilon} := C_{k,m-1}^\epsilon(\mathbb{Z}_{m-1}^{k,u_k})$ as the composition of an analytically measurable function with an $\mathcal{F}_{T_{m-1}^k}$-measurable one. In this case, by Lemma 4.2 we know that $C_{k,m-1}^\epsilon \circ \mathbb{Z}_{m-1}^{k,u_k}$ is an $\mathcal{F}_{T_{m-1}^k}$-measurable function. This shows that $\phi_{m-1}^{k,\epsilon}$ is an admissible control. It follows from (4.12) that

$$\mathcal{V}_{m-1}(\mathbb{Z}_{m-1}^{k,u_k}) \leq \int_{\mathbb{S}_k} \mathbb{V}_{m-1}^k \left[ \left(u_1^k, \Delta T_1^k, \eta_1^k\right), \ldots, \left(u_{m-2}^k, \Delta T_{m-1}^k, \eta_{m-1}^k\right), \left(\phi_{m-1}^{k,\epsilon}, s_{m-1}^k, \mathbf{i}_{m-1}^k\right) \right] \nu^k(d\mathbf{s}_{m-1} d\mathbf{i}_{m-1}) + \epsilon$$

$$= \mathbb{E}[\xi_k(u_k \otimes_{m-1} \phi_{m-1}^{k,\epsilon}) | \mathcal{F}_{T_{m-1}^k}^{k}] + \epsilon = \mathbb{E}[V_k(T_{m-1}^k, u_k \otimes_{m-1} \phi_{m-1}^{k,\epsilon}) | \mathcal{F}_{T_{m-1}^k}^{k}] + \epsilon$$
Moreover, as a consequence of Equation (4.16), we obtain that
\begin{equation}
V^k(T_m^k, u_m^k) = V^k(T_m^k, u_m^k) \text{ a.s., } u_m^k \in U_{T_m^k}^k.
\end{equation}

Hence, it follows from Equations (4.15) and (4.17) that
\begin{equation}
V_{m-1}^k(\Xi_{m-1}^k, u_m^k) \leq V^k(T_m^k, u_m^k) \text{ a.s., } u_m^k \in U_{T_m^k}^k.
\end{equation}

Moreover, as a consequence of Equation (4.16), we obtain that
\begin{equation}
V^k(T_m^k, u_m^k) = V_{m-1}^k(\Xi_{m-1}^k, u_m^k) \leq E[V^k(T_m^k, u_m^k \otimes_{m-1} \phi_{m-1}^k), F_{T_m^k}^k] + \epsilon \text{ a.s.}
\end{equation}

□

We are now able to iterate the argument as follows. From a backward argument, we can define the sequence of functions $\mathcal{V}_j^k: \mathbb{H}^{k,\ell} \rightarrow \mathbb{R}$
\begin{equation}
\mathcal{V}_j^k(\mathbf{o}_j^k) := \sup_{a_{j+1}^k \in A} \int_{S_k} \mathcal{V}_{j+1}^k(\mathbf{o}_{j+1}^k, a_{j+1}^k, s_{j+1}^k, \tilde{r}_{j+1}^k) \nu^k(ds_{j+1}^k d\tilde{r}_{j+1}^k),
\end{equation}
for $\mathbf{o}_j^k \in \mathbb{H}^{k,\ell}$ and $\ell = m - 1, \ldots, 1$.

**Lemma 4.7.** For each $j = m - 1, \ldots, 1$, the map
\begin{equation}
(\mathbf{o}_j^k, a_j^k) \mapsto \int_{S_k} \mathcal{V}_{j+1}^k(\mathbf{o}_{j+1}^k, a_{j+1}^k, s_{j+1}^k, \tilde{r}_{j+1}^k) \nu^k(ds_{j+1}^k d\tilde{r}_{j+1}^k)
\end{equation}
is upper semianalytic from $\mathbb{H}^{k,j} \times A$ to $\mathbb{R}$.

**Proof.** The same argument used in the proof of Lemma 4.4 applies here. We omit the details. □

**Proposition 4.2.** The function $\mathcal{V}_j^k: \mathbb{H}^{k,j} \rightarrow \mathbb{R}$ is upper semianalytic for each $j = m - 1, \ldots, 1$. Moreover, for every $\epsilon > 0$, there exists an analytically measurable function $C_{k,j}^\epsilon: \mathbb{H}^{k,j} \rightarrow A$ such that
\begin{equation}
\mathcal{V}_j^k(\mathbf{o}_j^k) \leq \int_{S_k} \mathcal{V}_{j+1}^k(\mathbf{o}_{j+1}^k, C_{k,j}^\epsilon(\mathbf{o}_j^k), s_{j+1}^k, \tilde{r}_{j+1}^k) \nu^k(ds_{j+1}^k d\tilde{r}_{j+1}^k) + \epsilon,
\end{equation}
for every $\mathbf{o}_j^k \in \{\mathcal{V}_j^k < +\infty\}$, where $j = m - 1, \ldots, 1$.

**Proof.** We just repeat the argument of the proof of Lemma 4.5 jointly with Lemma 4.7 □

We are now able to define the value function at step $j = 0$ as follows
\begin{equation}
\mathcal{V}_0^k := \sup_{a_0^k \in A} \int_{S_k} \mathcal{V}_1^k(a_0^k, s_1^k, \tilde{r}_1^k) \nu^k(ds_1^k d\tilde{r}_1^k).
\end{equation}

Therefore, if $\mathcal{V}_0^k < +\infty$ then for $\epsilon > 0$, there exists $C_{k,0}^\epsilon \in A$ which realizes
\begin{equation}
\mathcal{V}_0^k < \int_{S_k} \mathcal{V}_1^k(C_{k,0}^\epsilon, s_1^k, \tilde{r}_1^k) \nu^k(ds_1^k d\tilde{r}_1^k) + \epsilon.
\end{equation}
Proposition 4.3. For each \( j = m - 1, \ldots, 0 \) and a control \( u^k \in U_0^{k,m} \), we have

\[
V^k_j(z_j^k u^k) = V^k(T_j^k, u^k) \quad \text{a.s.}
\]

Let \( C_{k,j}^\epsilon : \mathbb{H}^{k,j} \rightarrow \mathbb{R} \) be the functions given in Proposition 4.2. Observe that \( C_{k,j}^\epsilon \) can be computed from (4.18). If \( \mathbb{H}^{k,j} = \{ V_j^k < +\infty \} \), for \( j = m - 1, \ldots, 0 \), then for every \( \epsilon > 0 \), there exists a control \( u_j^{\epsilon,k} \) defined by

\[
u_j^{\epsilon,k} := C_{k,j}^\epsilon(z_j^k u^k): j = m - 1, \ldots, 0
\]

which realizes

\[
V^k(T_j^k, u^k) \leq E[V^k(T_{j+1}^k, u^k \otimes_j u_j^{\epsilon,k})| T_j^k] + \epsilon \quad \text{a.s.,}
\]

for every \( j = m - 1, \ldots, 0 \).

**Proof.** The statements for \( j = m - 1 \) hold true due to (4.13) in Lemma 4.6. Now, by using Proposition 4.2 and a backward induction argument, we conclude the proof. \( \square \)

The above construction suggests the following definition.

**Definition 4.1.** A pair \( (\xi, (X^k)_{k \geq 1}) \) is admissible w.r.t the control problem (4.7) if \( \mathbb{H}^{k,j} = \{ V_j^k < +\infty \} \) for every \( j = e(k, T) - 1, \ldots, 0 \) and \( k \geq 1 \).

Let us now summarize the essential results about the value functional. For any \((r,n)\) such that \( 1 \leq r \leq n \) and \( \alpha^k_n = ((a^k_0, s^k_1, \tilde{i}^k_1), \ldots, (a^k_{n-1}, s^k_n, \tilde{i}^k_n)) \), we denote

\[
\pi_r(\alpha^k_n) := \left((a^k_0, s^k_1, \tilde{i}^k_1), \ldots, (a^k_{r-1}, s^k_r, \tilde{i}^k_r)\right)
\]

as the projection of \( \alpha^k_n \in \mathbb{H}^{k,n} \) onto the first \( r \) coordinates. If \( F^k_{n} : \mathbb{H}^{k,r} \rightarrow \mathbb{R} ; \ell = 0, \ldots, e(k, T) \) is a list of universally measurable functions (\( F^k_{n} \) is a constant), we then define

\[
\mathcal{F}^k(\alpha^k_n, a^k_n) := \int_{\mathbb{S}_k} \left[F^k_{n+1}(\alpha^k_n, a^k_n, s^k_{n+1}, \tilde{i}^k_{n+1}) - F^k_{n}(\alpha^k_n)\right] ds_{n+1} d\tilde{i}^k_{n+1},
\]

for \( \alpha^k_n \in \mathbb{H}^{k,n} \).

**Corollary 4.1.** Let \( (\xi, (X^k)_{k \geq 1}) \) be an admissible pair w.r.t the control problem (4.7). For a given Borel function \( \gamma^k_m : \mathbb{H}^{k,n} \rightarrow \mathbb{R} \) satisfying (4.8) where \( m = e(k, T) \), the value function \( (\mathcal{V}^k_{n})_{n=0} \) associated with \( V^k \) is the unique solution of

\[
\sup_{\alpha^k_m \in A} \mathcal{F}^k(\pi_n(\alpha^k_m), a^k_n) = 0; \quad n = m - 1, \ldots, 0
\]

\[
\mathcal{V}^k_{m}(\alpha^k_m) = \xi(\gamma^k_m(\alpha^k_m)); \quad \alpha^k_m \in \mathbb{H}^{k,m}.
\]

We are now able to construct an \( \epsilon \)-optimal control in this discrete level.

**Proposition 4.4.** Let \( (\xi, (X^k)_{k \geq 1}) \) be an admissible pair w.r.t the control problem (4.7). Then, \( \phi^{*\epsilon,k} = (\hat{\phi}^{*,k}(\xi), \phi_1^{*,k}(\xi), \ldots, \phi_{m-1}(\xi)) \) constructed via (4.21) is \( \epsilon \)-optimal, where \( \eta_k(\epsilon) = \frac{\epsilon}{e(k, T)} \). In other words, for every \( \epsilon > 0 \) and \( k \geq 1 \), \( \phi^{*,k,\epsilon} \in U_0^{k,m} \) realizes

\[
\sup_{u^k \in U_0^{k,m}} E[X^{\epsilon,k}(u^k)] \leq E[X^{\epsilon,k}(\hat{\phi}^{*,k,\epsilon})] + \epsilon.
\]
Proof. Fix \( \epsilon > 0 \) and let \( \eta_k(\epsilon) = \frac{\epsilon}{m} \), where we recall \( m = e(k, T) \). The candidate for an \( \epsilon \)-optimal control is

\[
\phi^*,k,\epsilon = (\phi_0^{k,\eta_k(\epsilon)}, \phi_1^{k,\eta_k(\epsilon)}, \ldots, \phi_{m-1}^{k,\eta_k(\epsilon)}),
\]

where \( \phi_i^{k,m(\epsilon)}; i = m-1,\ldots,0 \) are constructed via (4.21). Let us check it is indeed \( \epsilon \)-optimal. From (4.22), we know that

\[
\sup_{u^k \in U_0^{k,e}(k,T)} \mathbb{E}[\xi_{X^k}(u^k)] \leq \mathbb{E}[V^k(T_1^k, \phi_0^{k,\eta_k(\epsilon)} \otimes 1 \phi_1^{k,\eta_k(\epsilon)} \otimes \ldots \otimes \phi_{m-1}^{k,\eta_k(\epsilon)})] + \eta_k(\epsilon) a.s.
\]

Inequalities (4.24) and (4.25) yield

\[
\sup_{u^k \in U_0^{k,e}(k,T)} \mathbb{E}[\xi_{X^k}(u^k)] \leq \mathbb{E}[V^k(T_1^k, \phi_0^{k,\eta_k(\epsilon)} \otimes 1 \phi_1^{k,\eta_k(\epsilon)} \otimes \ldots \otimes \phi_{m-1}^{k,\eta_k(\epsilon)})] + 2\eta_k(\epsilon),
\]

where (4.22) implies \( V^k(T_{j+1}^k, \phi_0^{k,\eta_k(\epsilon)} \otimes 1 \phi_1^{k,\eta_k(\epsilon)} \otimes \ldots \otimes \phi_{j-1}^{k,\eta_k(\epsilon)}(\omega) \leq \frac{1}{2}\eta_k(\epsilon) a.s.
\]

for \( j = 1,\ldots,m-1 \). By iterating the argument starting from (4.26) and using (4.27), we conclude

\[
\eta_k(\epsilon) \leq \frac{1}{2}\eta_k(\epsilon) a.s.
\]

Remark 4.1. If the map \( \mathbb{V}^k_m : \mathbb{R}^{k,m} \to \mathbb{R} \) is upper semicontinuous for \( m = e(k, T) \), then we can apply the Borel measurable selection theorem (see, Prop 7.33 in [3]) to conclude that the function \( \mathbb{V}^k_j : \mathbb{R}^{k,j} \to \mathbb{R} \) is also upper semicontinuous for each \( j = m-1,\ldots,1 \). Moreover, there exists a Borel measurable function \( C^*_{k,j} : \mathbb{R}^{k,j} \to \mathbb{R} \) such that

\[
\mathbb{V}^k_j(o_j^k) = \int_{\mathbb{S}_k} \mathbb{V}^k_{j+1}(o_j^k, C^*_{k,j}(o_j^k), s_j^k, l_j^k, d_j^k) \nu^k(ds_j^k, dl_j^k, dk^k),
\]

for every \( o_j^k \in \{\mathbb{V}^k_j < +\infty\} \), where \( j = m-1,\ldots,1 \). If \( \mathbb{V}^k_j \in \{\mathbb{V}^k_j < +\infty\}; j = m-1,\ldots,0 \) then there exists a control \( u_{j,k}^* \) defined by

\[
u_j^{k,*} := C^*_{k,j}(o_j^k, u_j^k); j = m-1,\ldots,0,
\]

for \( u_j^k \in U_0^{k,m} \), which realizes

\[
V^k(T_j^k, u^k) = \mathbb{E}[V^k(T_{j+1}^k, u^k \otimes \ldots \otimes u_j^{k,*} \otimes \ldots \otimes u_{m-1}^{k,*})] a.s.
\]

for every \( j = m-1,\ldots,0 \).

Remark 4.2. One can easily check that (see e.g. the proof of Lemma 3 in [3]) for every \( p > 0 \), there exists a constant \( C_p \) such that \( f(t) \leq C_p t^{p-\frac{1}{2}}; t \geq 0 \), where \( f \) is the density of \( \tau^1 \) in [3,11]. By choosing \( p = \frac{3}{2} \), we get assumption (1) in [22] and, by Th 5 in [22], this is sufficient to get a lower bound

\[
\frac{1}{2C_p^2} \frac{1}{d} \leq \chi_d.
\]

Therefore, for given \( k \geq 1 \) and \( T \), the number of periods \( e(k, T) \) grows no faster than the dimension of the driving Brownian motion.
5. Convergence of optimal values and \( \varepsilon \)-optimal controls

Throughout this section, we assume that \((X, \mathcal{X})\) is a strongly controlled Wiener functional. In this section, we aim to prove the controlled imbedded discrete structure \(\mathcal{V} = ((V^k)_{k \geq 1}, \mathcal{P})\) given by (4.3) and (4.20) yields the solution of the control problem (1.1) for \(k \geq 1\) large enough. In the sequel, we present a density result which will play a key role in this article: we want to approximate any control \(u \in U_0^T\) by means of controls in the sets \(U_0^{k,\varepsilon(k,T)}\).

**Lemma 5.1.** If \(S\) is an \(\mathbb{F}\) - stopping time, then there exists a sequence of positive random variables \((S_k)_{k \geq 1}\) such that \(S_k\) is an \(\mathbb{F}^k\) - stopping time for each \(k \geq 1\) and \(\lim_{k \to \infty} \mathbb{P}(S_k = S) = 1\). Moreover, for any \(G \in \mathcal{F}_S\) there exists a sequence of sets \((G_k)_{k \geq 1}\) such that \(G_k \in \mathcal{F}_{S_k}\), \(G_k \subset G \cap \{S < \infty\}\) for every \(k \geq 1\), and

\[
\lim_{k \to \infty} \mathbb{P}(G \cap \{S < \infty\} - G^k) = 0.
\]

Therefore, if \(Z\) is an \(\mathcal{F}_S\)-bounded random variable, then there exists a sequence \(\{Z_k; k \geq 1\}\) such that \(Z_k\) is \(\mathcal{F}_{S_k}\)-measurable for each \(k \geq 1\) and \(\lim_{k \to \infty} Z_k = Z\) in probability.

**Proof.** The proof is identical to Lemma 3.3 in [38], so we omit the details. \(\square\)

**Lemma 5.2.** The subset \(\cup_{k \geq 1} U_0^{k,\varepsilon(k,T)}\) is dense in \(U_0^T\) w.r.t the \(L_2^a(\mathbb{P} \times \text{Leb})\)-strong topology.

**Proof.** It is well known (see e.g [46] Th 2, pp. 156) that

\[
\mathbb{L}_b = \{u; u \text{ is left-continuous } \mathbb{F} \text{- adapted and } \sup_{0 \leq t \leq T} \|u(t)\|_{\mathbb{R}^c} \leq \bar{a} \text{ a.s}\}
\]

is a dense subset of \(U_0^T\) w.r.t \(L_2^a(\mathbb{P} \times \text{Leb})\)-topology. A process \(Z\) is said to be a simple \(\mathbb{F}\)-predictable process if it has the following representation

\[
Z = Z(0)\mathbb{I}_{[0]} + \sum_{i=1}^q Z_i\mathbb{I}_{[[J_{i-1}, J_i]]}; q \in \mathbb{N},
\]

where \(0 = J_0 < J_1 < \cdots < J_q < \infty\) is a finite sequence of \(\mathbb{F}\)-stopping times and \(Z_i\) is a bounded \(\mathcal{F}_{J_{i-1}}\)-random variable such that \(\|Z_i\|_{L^\infty} \leq \bar{a}\) for every \(i = 1, \ldots, q\). It is also well-known (see e.g [46] Th 10 pp. 57) that the set \(\mathcal{S}\) of simple \(\mathbb{F}\)-predictable processes is a dense subset of \(\mathbb{L}_b\) w.r.t \(L_2^a(\mathbb{P} \times \text{Leb})\)-topology. Therefore, in order to prove our claim, it is sufficient to check that \(\cup_{k \geq 1} U_0^{k,\varepsilon(k,T)}\) is a dense subset of \(\mathcal{S}\) w.r.t the \(L_2^a(\mathbb{P} \times \text{Leb})\)-topology. Clearly, it is sufficient to check that for a given real-valued process \(Y = Z\mathbb{I}_{[[J, L]]} \in \mathcal{S}\), there exists a sequence \(\{u^k \in U_0^{k,\varepsilon(k,T)}; k \geq 1\}\) such that

\[
\lim_{k \to +\infty} \|Y - u^k\|_{L_2^a} = 0.
\]

By Lemma 5.1 there exists a sequence \(\{S^k; k \geq 1\}\) such that \(S^k\) is an \(\mathbb{F}\)-stopping time and \(\lim_{k \to +\infty} S^k = J, \lim_{k \to +\infty} S^k = L\) in probability. Moreover, there exists a sequence \(\{Z^k; k \geq 1\}\) such that \(Z^k\) is a bounded \(\mathcal{F}_{S^k}\)-random variable with \(\|Z^k\|_{L^\infty} \leq \bar{a}\) and \(\lim_{k \to +\infty} Z^k = Z\) in probability. Then, we propose the following approximating control

\[
u^k = \sum_{i=1}^{\varepsilon(k,T)} \left( Z^k \mathbb{I}_{[S_{i1} \leq T_{i1} \leq S_{i2}]} \right) \mathbb{I}_{[T_{i1} \leq T_{i1}, T_{i2}]}.
\]

Since \(Z^k \mathbb{I}_{[S_{11} \leq T_{11} \leq S_{22}]} = Z^k \mathbb{I}_{[S_{11} \leq T_{11} \leq S_{22}]} \mathbb{I}_{[T_{11} \leq T_{11}, T_{22}]}\) is an \(\mathcal{F}_{T_{11}}\)-measurable random variable (see e.g Corollary 3.5.2 in [23]) and bounded by \(\bar{a}\), then \(u^k \in U_0^{k,\varepsilon(k,T)}\) for every \(k \geq 1\). If we set \(W^k = Z^k \mathbb{I}_{[S_{11} \leq S_{22}]}\), then
\[ \|W^k - Y\|_{L^2_S} \leq \|Z^k I_{[s_1^k, s_2^k]} - Z I_{[s_1^k, s_2^k]}\|_{L^2_S} + \|Z I_{[s_1^k, s_2^k]} - Z I_{[J, L]}\|_{L^2_S} \]

(5.4)

\[ \leq T(E|Z^k - Z|^2) + \tilde{\alpha} \|I_{[s_1^k, s_2^k]} - I_{[J, L]}\|_{L^2_S} \rightarrow 0, \]

as \( k \to +\infty \). Now, we observe

\[ \mathbb{E} \int_0^T (W^k(s) - u^k(s))^2 \, ds = \mathbb{E} \int_0^T \left( Z^k I_{[s_1^k, s_2^k]}(s) - \sum_{i=1}^{\epsilon(k,T)} \left( Z^k I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, T_i^k\}}(s) \right)^2 \, ds \]

(5.5)

\[ \leq \tilde{\alpha}^2 \mathbb{E} \int_0^T \left( I_{[s_1^k, s_2^k]}(s) - \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, T_i^k\}}(s) \right)^2 \, ds \]

\[ + \tilde{\alpha}^2 \mathbb{E} \int_0^T \left( I_{[s_1^k, s_2^k]}(s) - \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, T_i^k\}}(s) \right)^2 \, ds \]

\[ =: I_1^k + I_2^k + I_3^k. \]

Observe we can write

\[ \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, T_i^k\}}(s) = I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) - I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) + I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s), \]

on \( \{S_2^k \leq T_{e(k,T)}^k\} \), where \( T_{e(k,T)}^k := \min\{T_i^k, S_j^k \leq T_{e(k,T)}^k\} \) for \( j = 1, 2 \). By using Young inequality and \( (5.6) \), we get

\[ I_1^k \leq 3 \tilde{\alpha}^2 \mathbb{E} \max_{1 \leq i \leq \epsilon(k,T)} \|\Delta T_i^k\| I_{\{T_{e(k,T)}^k \geq S_2^k\}}. \]

We observe \( \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, s_1^k \leq s_2^k\}}(s) = 0 \) on \( \{T_{e(k,T)}^k < S_1^k \leq S_2^k\} \) and hence

\[ \left| I_{\{s_1^k \leq s_2^k\}}(s) - \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, s_1^k \leq s_2^k\}}(s) \right| = I_{\{s_1^k \leq s_2^k\}}(s) \]

on \( \{T_{e(k,T)}^k < S_1^k \leq S_2^k\} \). Therefore,

\[ I_2^k \leq \tilde{\alpha}^2 \mathbb{E}|T - T_{e(k,T)}^k| I_{\{T_{e(k,T)}^k < S_1^k \leq S_2^k\}}. \]

Observe we can write

\[ \sum_{i=1}^{\epsilon(k,T)} \left( I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) \right) I_{\{T_{i-1}^k, s_1^k \leq s_2^k\}}(s) = I_{\{s_1^k \leq s_2^k\}}(s) - I_{\{s_1^k \leq T_{i-1}^k \leq s_2^k\}}(s) - I_{\{T_{e(k,T)}^k < s_2^k\}}(s), \]

on \( \{S_1^k \leq T_{e(k,T)}^k < S_2^k\} \). Identity \( (5.9) \) yields
Proposition 5.2. \( I^k_3 \leq 3\hat{a}^2 \left( \mathbb{E} \max_{1 \leq t \leq (k,T)} |\Delta T^k_4| \mathbb{1}_{\{S^k_1 \leq T^k_{e(k,T)}, \leq S^k_2\}} + \mathbb{E} |T^k_{e(k,T)} - T| \mathbb{1}_{\{S^k_1 \leq T^k_{e(k,T)}, \leq S^k_2\}} \right) \).

By applying Lemmas 8.2-8.3 and using (5.5), (5.7), (5.8), (5.10) and (5.4), we conclude (5.3) holds true. This concludes the proof. \( \square \)

Proposition 5.1. Under Assumptions A1 and B1, we have

\[
V(0) = \sup_{\psi \in U_{k}^{k,k+1}(k,T)} \mathbb{E} \left[ \xi_X(\psi) \right] = \lim_{k \to +\infty} \sup_{\psi \in U_{0}^{k,k+1}(k,T)} \mathbb{E} \left[ \xi_X(\psi) \right].
\]

Proof. By Lemma 5.2, we know that \( U_{k \geq 1} U_{0}^{k,k+1}(k,T) \) is dense in the \( L^2(\mathbb{P} \times \text{Leb}) \)-strong topology. In the sequel, we fix \( \epsilon > 0 \). For every \( \phi \in U_{0}^{k,T} \), there exists \( \psi \in U_{k \geq 1} U_{0}^{k,k+1}(k,T) \) such that

\[
\mathbb{E} \int_{0}^{T} |\phi(s) - \psi(s)|^2 ds < \epsilon.
\]

By applying Assumptions A1-B1 and Hölder’s inequality for \( p = \frac{2}{\gamma} > 1 \), we obtain

\[
\mathbb{E} \left| \xi_X(\phi) - \xi_X(\psi) \right| \leq C \left( \mathbb{E} \int_{0}^{T} |\psi(s) - \phi(s)|^2 ds \right)^{\gamma/2} \leq C\epsilon^{\frac{\gamma}{2}},
\]

for a constant \( C \) which depends on \( \xi \) and Assumption B1. As a consequence, we obtain that

\[
\mathbb{E} \left[ \xi_X(\phi) \right] \leq \mathbb{E} \left[ \xi_X(\psi) \right] + C\epsilon^{\frac{\gamma}{2}}.
\]

Then, we conclude that

\[
\mathbb{E} \left[ \xi_X(\phi) \right] \leq \sup_{\psi \in U_{k \geq 1} U_{0}^{k,k+1}(k,T)} \mathbb{E} \left[ \xi_X(\psi) \right] + C\epsilon^{\frac{\gamma}{2}}, \text{ for every } \phi \in U_{0}^{k,T}.
\]

The fact that \( \epsilon > 0 \) is arbitrary yields

\[
\mathbb{E} \left[ \xi_X(\phi) \right] \leq \sup_{\psi \in U_{k \geq 1} U_{0}^{k,k+1}(k,T)} \mathbb{E} \left[ \xi_X(\psi) \right], \text{ for every } \phi \in U_{0}^{k,T}.
\]

This shows that \( V(0) = \sup_{\psi \in U_{k \geq 1} U_{0}^{k,k+1}(k,T)} \mathbb{E} \left[ \xi_X(\psi) \right] \). By using the fact \( U_{k \geq 1} U_{0}^{k,k+1}(k,T) \subset U_{0}^{k+1,k+1+1}(k+1,T) \) for every \( k \geq 1 \), the same argument allows us to conclude \( V(0) = \lim_{k \to +\infty} \mathbb{E} \left[ \xi_X(\psi) \right] \).

We are now able to state two important consequences.

Proposition 5.2. Let \( V(t,u) = \text{ess sup}_{\theta \in U_{0}^{U}(t,T)} \mathbb{E} \left[ \xi_X(u \otimes \theta) | \mathcal{F}_t \right] ; 0 \leq t \leq T \) be the value process associated with a payoff \( \xi \) and a strongly controlled Wiener functional \( (X,X) \) satisfying Assumption A1 and B1, respectively. Let \( \mathcal{V} = \{(V^k)_{k \geq 1}, \mathcal{S}\} \) be the value process \( \{(X^k)_{k \geq 1}, \mathcal{S}\} \) associated with \( X = ((X^k)_{k \geq 1}, \mathcal{S}) \).

Then,

\[
\lim_{k \to +\infty} |V^k(0) - V(0)| = 0.
\]

Proof. By the property (3.13) and Proposition 5.1, there exist two sequences of positive numbers \( (h_k,r_k) \) such that \( \lim_{k \to +\infty} (h_k + r_k) = 0 \),

\[
\sup_{\phi \in U_{0}^{k,k+1}(k,T)} \mathbb{E} \left\| X^k(\phi) - X(\phi) \right\|_{\infty} = O(h_k),
\]
for $0 < \gamma \leq 1$ and
\[
(V(0) - \sup_{\phi \in U^p_{0,T}(k,T)} E[\xi_X(\phi)]) = O(r_k),
\]
for $k \geq 1$. Assumption A1 and (3.13) yield
\[
|V^k(0) - \sup_{\phi \in U^p_{0,T}(k,T)} E[\xi_X(\phi)]| \leq \|\xi\| \sup_{\phi \in U^p_{0,T}(k,T)} E\|X^k(\phi) - X(\phi)\|_{\infty} = O(h_k).
\]
Then, (5.12) jointly with triangle inequality yield (5.11).

**Remark 5.1.** By Assumptions A1-B1,
\[
V(0) = \sup_{\phi \in S} E[\xi_X(\phi)] = \sup_{\phi \in S \circ} E[\xi_X(\phi)],
\]
where $\mathbb{L}_0$ and $S$ are given by (5.1) and (5.2), respectively. In general,
\[
V(0) = \sup_{\phi \in \mathcal{R}} E[\xi_X(\phi)],
\]
for any dense subset $\mathcal{R}$ of $U^p_{0,T}$ w.r.t $L^2_0(\mathbb{P} \times \text{Leb})$-topology.

An important consequence of Proposition 5.2 is the next result which states that optimal controls (either exacts or near) computed for the approximating problem (4.7) are near optimal controls for the control problem sup$_{u \in U^p_{0,T}} E[\xi_X(u)]$.

**Proposition 5.3.** Assume the pair $(\xi, X)$ satisfies Assumptions A1-B1 and $(X, \mathcal{X})$ is a strongly controlled Wiener functional. For a given $\epsilon > 0$ and $k \geq 1$, let $\phi^{*,k,\epsilon}$ be an $\epsilon$-optimal control associated with the control problem (4.7) written on $X$ (e.g the near optimal control constructed in Propositions 4.3 and 4.4). Then, $\phi^{*,k,\epsilon} \in U^p_{0,T}$ is a near optimal control for the Brownian motion driving stochastic control problem, i.e.,
\[
E[\xi_X(\phi^{*,k,\epsilon})] \geq \sup_{u \in U^p_{0,T}} E[\xi_X(u)] - \epsilon,
\]
for every $k$ sufficiently large.

**Proof.** Let us fix $\epsilon > 0$ and a positive integer $k \geq 1$. By definition of $\phi^{*,k,\epsilon}$, we have
\[
E[\xi_X(\phi^{*,k,\epsilon})] \geq \sup_{\theta \in U^p_{0,T}(k,T)} E[\xi_X(\theta)] - \frac{\epsilon}{3}, \quad k \geq 1.
\]
Proposition 5.2 yields
\[
|E[\xi_X(\theta)] - E[\xi_X(\phi^{*,k,\epsilon})]| < \frac{\epsilon}{3},
\]
for every $k$ sufficiently large. By using assumptions (3.13) and (A1), we also know there exists a positive constant $C$ such that
\[
E[\xi_X(\phi^{*,k,\epsilon})] - E[\xi_X(\phi^{*,k,\epsilon})] \leq C(E\|X^k(\phi^{*,k,\epsilon}) - X(\phi^{*,k,\epsilon})\|_{\infty}^p \frac{1}{3} < \frac{\epsilon}{3},
\]
for every $k$ sufficiently large and $\alpha = p/\gamma$. Summing up inequalities (5.14), (5.15) and (5.16), we conclude the proof.

In Section 7 we analyze in detail the impact of the choice of $\epsilon_k$ in our approximation scheme. We illustrate these results in three rather distinct types of non-Markovian controlled states as described in the next section.
6. Controlled imbedded discrete structures for non-Markovian states

We now start to investigate how the abstract results obtained in previous sections can be applied to the examples mentioned in the Introduction. The goal here is to construct imbedded discrete structures $X$ associated with controlled states $X$ of the form (1.7), (1.8), and (1.9) in such way that $(X, X)$ is a strongly controlled Wiener functional.

6.1. Path-dependent controlled SDEs. In the sequel, we make use of the following notation

$$
\omega_t := \omega(t \wedge \cdot); \omega \in D_{n,T}.
$$

This notation is naturally extended to processes. We say that $F$ is a non-anticipative functional if it is a Borel mapping and

$$
F(t, \omega) = F(t, \omega_t); (t, \omega) \in [0, T] \times D_{n,T}.
$$

The underlying state process is the following $n$-dimensional controlled SDE

$$
dX^\omega(t) = \alpha(t, X^\omega_t, u(t))dt + \sigma(t, X^\omega_t, u(t))dB(t); 0 \leq t \leq T,
$$

with a given initial condition $X^\omega(0) = x \in \mathbb{R}^n$. We define $\Lambda_T := \{ (t, \omega_t); t \in [0, T]; \omega \in D_{n,T} \}$ and we endow this set with the metric

$$
\| \omega_t - \omega'_t \|_\infty + | t - t' |^{1/2}.
$$

Then, $(\Lambda_T, d_{1/2})$ is a complete metric space equipped with the Borel $\sigma$-algebra. The coefficients of the SDE will satisfy the following regularity conditions:

**Assumption (C1):** The non-anticipative mappings $\alpha : \Lambda_T \times \mathbb{A} \to \mathbb{R}^n$ and $\sigma : \Lambda_T \times \mathbb{A} \to \mathbb{R}^{n \times d}$ are Lipschitz continuous, i.e., there exists a pair of constants $K_{1,\text{Lip}} = (K_{1,\text{Lip}}, K_{2,\text{Lip}})$ such that

$$
\| \alpha(t, \omega, a) - \alpha(t', \omega', b) \| + \| \sigma(t, \omega, a) - \sigma(t', \omega', b) \|
\leq K_{1,\text{Lip}}d_{1/2}((t, \omega); (t', \omega')) + K_{2,\text{Lip}}\| a - b \|,
$$

for every $t, t' \in [0, T]$ and $\omega, \omega' \in D_{n,T}$ and $a, b \in \mathbb{A}$. One can easily check by routine arguments that the SDE (6.2) admits a strong solution and

$$
\sup_{u \in U_T^T} \mathbb{E}\|X^u\|_{\infty}^{2p} \leq C(1 + \|x_0\|^{2p}) \exp(CT),
$$

where $X(0) = x_0$, $C$ is a constant depending on $T > 0, p \geq 1$, $K_{\text{Lip}}$ and the compact set $\mathbb{A}$.

**Remark 6.1.** Due to Assumption C1, one can apply Jensen and Burkholder-Davis-Gundy’s inequalities to arrive at the following estimate: there exists a constant $C = C(d, T, K_{1,\text{Lip}}, K_{2,\text{Lip}})$ such that

$$
\mathbb{E}\|X^u_1 - X^u_2\|_{\infty}^2 \leq C \int_0^t \mathbb{E}\|X^u_1 - X^u_2\|_{\infty}^2 dt + C \int_0^t \mathbb{E}\|u^1(s) - u^2(s)\|^2 ds,
$$

for $0 \leq t \leq T$ and $u_1, u_2 \in U_0^T$. Then, by applying Grönwall’s inequality, we observe the controlled SDE (6.2) satisfies Assumption B1.
Construction of the controlled imbedded discrete structure for (6.2). Let us now construct a controlled imbedded structure \((X^k)_{k \geq 1}, \mathcal{D})\) associated with (6.2). In the sequel, in order to alleviate notation, we are going to write controlled processes as

\[
X^{k, \phi}, \quad X^{\phi}
\]

rather than \(X^k(\cdot, \phi)\) and \(X(\cdot, \phi)\), respectively, as in previous sections. In this section, we will establish a slightly stronger property than (3.13), namely we construct \(X = ((X^k)_{k \geq 1}, \mathcal{D})\) jointly with a sequence \(h_k \downarrow 0\) such that

\[
\tag{6.3}
\sup_{\phi \in \mathcal{D}} \mathbb{E} \|X^{k, \phi} - X^{\phi}\|_{\infty} = O(h_k),
\]

for \(0 < \gamma \leq 1\) and \(k \geq 1\). Here, for integers \(0 \leq n < m\), \(U^{k,m}_n\) \((0 \leq n < m)\) is the set of all \(\mathbb{F}\)-predictable processes of the form

\[
\phi(t) = \sum_{j=n+1}^{m} v_j^{k-1} \mathbf{1}_{(T_j^k, t \leq T_j^k)},
\]

where for each \(j = n+1, \ldots, m\), \(v_j^{k-1}\) is an \(\mathcal{A}\)-measurable \(\mathcal{F}_{T_{j-1}^k}\)-measurable random variable. The sequence \((h_k)_{k \geq 1}\) will be precisely computed as a function of \(\mathcal{D}\). Of course, for each \(m \geq 1\), \(U_n^{k,m} \subset U^{k,m}_n\) for every \(n \in \{0, \ldots, m-1\}\). We denote \(U^{k,\infty}_n\) similar to (3.7) but the jumps of the controls are \(\mathcal{F}_{T_{n-1}^k}\)-measurable for \(n \geq 1\).

Let us fix a control \(\phi = (v_0^k, \ldots, v_m^k) \in U^{k,\infty}_0\). At first, we construct an Euler-Maruyama-type scheme based on the random partition \((T_n^k)_{n \geq 0}\) as follows. Let us define \(X^{i,k,\phi}(0) := X_0^{i,k,\phi} := X^{i,k,\phi} := x\) is the constant function over \([0, T]\). Let us define \(X^k(T_q^k) := (X^{1,k,\phi}(T_q^k), \ldots, X^{n,k,\phi}(T_q^k))\) as follows

\[
X^{1,k,\phi}(T_q^k) := X^{i,k,\phi}(T_{q-1}^k) + \alpha^i(T_q^k, X^{i,k,\phi}(T_{q-1}^k), v_q^k) \Delta T_q^k
\]

\[
+ \sum_{j=1}^{d} \sigma^{ij}(T_q^k, X^{i,k,\phi}(T_{q-1}^k), v_q^k) \Delta A^{k,j}(T_q^k),
\]

for \(q \geq 1\) and \(1 \leq i \leq n\), where \(X^{i,k,\phi}(t) = \sum_{m=0}^{\infty} X^{i,k,\phi}(T_m^k) \mathbf{1}_{(T_m^k, t \leq T_{m+1}^k)}; t \geq 0\). The controlled structure is naturally defined by

\[
X^{k,\phi}(t) := X^{k,\phi}(t \wedge T^k_{c(k,T)}),
\]

for \(t \in [0, T]\) and \(\phi \in U^{e(k,T)}_0\).

The pathwise version of (6.5) is given by the following objects: for each \(b_q^k = (s_1^k, t_1^k, \ldots, s_q^k, t_q^k)\), we define

\[
l_q^k(b_q^k) := \sum_{\ell=1}^{q} s_{\ell}^k.
\]

We set \(h_0^k := x\) and \(\gamma_0^k = x\) is the constant function over \([0, T]\). For a given

\[
a_q^k := (a_0^k, s_1^k, t_1^k), \ldots, (a_{q-1}^k, s_q^k, t_q^k) \in \mathbb{H}^{k,\phi},
\]

we set \(h_q^k(a_q^k) := (h_q^k(a_q^k), \ldots, h_q^k(a_q^k))\), where
\[ h_t^{i,k}(o_q^k) := h_{q-1}^{i,k}(o_{q-1}^k) + \alpha^i(t_{q-1}^k, \bar{\gamma}_q^k(o_{q-1}^k), o_{q-1}^k)x^k \]
\[ + \sum_{j=1}^d \sigma^{ij}(t_{q-1}^k, \bar{\gamma}_q^k(o_{q-1}^k), o_{q-1}^k)\epsilon_k(t_{q,j}^k \mathbb{1}_{\{|v^k_{q,j}|=1\}} + z_{q,j}^k \mathbb{1}_{\{|v^k_{q,j}| \neq 1\}}), \]

for \(1 \leq i \leq n\), where

\[ \bar{\gamma}_q^k(o_{q-1}^k)(t) := \sum_{\ell=0}^{q-2} h_t^k(\pi_t(o_{q-1}^k))\mathbb{1}_{\{t_{\ell}^k \leq t < t_{\ell+1}^k\}} + h_t^k(o_{q-1}^k)\mathbb{1}_{\{t_{q-1}^k \leq t\}}, \]

for \(0 \leq t \leq T\). In the above expression, \(\pi_t(o_{q-1}^k)\) is the projection of the variable \(o_{q-1}^k\) onto the first \(t\) variables. We then define

\[ \gamma^k(o_{\infty}^k)(t) := \sum_{n=0}^{\infty} h_n^k(o_{n}^k)\mathbb{1}_{\{t_n^k \leq t < t_{n+1}^k\}}, \]

for \(o_{\infty}^k \in \mathbb{R}^{k,\infty}\) and we set

\[ (6.7) \quad \gamma_{\epsilon(k,T)}(o_{\epsilon(k,T)}^k)(t) := \gamma^k(o_{\infty}^k)(t \wedge t_{\epsilon(k,T)}^k); 0 \leq t \leq T. \]

By construction, \(\gamma_{\epsilon(k,T)}\) realizes (4.8) for the controlled structure (6.6).

\[ \bullet \text{ Checking that } (X, \mathcal{X}) \text{ with } \mathcal{X} = (\langle X^k \rangle_{k \geq 1}, \mathcal{D}) \text{ is a strongly controlled Wiener functional.} \]

Let us now check that \((X, \mathcal{X})\) satisfies (6.3). For a given control \(\phi = (v_0^k, \ldots, v_{n-1}^k, \ldots) \in \mathcal{U}^{k,\infty}_{0}\), we set

\[ \Sigma_{t,\phi}^i(t) := 0\mathbb{1}_{\{t=0\}} + \sum_{n=1}^{\infty} \sigma^{ij}(T_{n-1}^k, \bar{\gamma}_n^k, v_{n-1}^k)\mathbb{1}_{\{T_{n-1}^k \leq t < T_n^k\}}, \]

for \(0 \leq t \leq T, 1 \leq i \leq n, 1 \leq j \leq d\). In the sequel, it is convenient to introduce the following notation: for each \(t \geq 0\), we set

\[ (6.8) \quad \bar{t}_k := \sum_{n=1}^{\infty} T_{n}^k \mathbb{1}_{\{T_{n}^k \leq t < T_{n+1}^k\}}, \]

We define

\[ (6.9) \quad \widehat{X}^{i,k,\phi}(t) := x_0^i + \int_0^t \alpha^i(\delta_t, X^{k,\phi}_t, \phi(s))\mathbb{1}_{\{t \wedge \bar{t}_k \leq T_n^k\}}\mathbb{1}_{\{T_{n-1}^k \leq t < T_n^k\}} ds + \sum_{j=1}^d \int_0^t \Sigma_{t,\phi}^j(s)\mathbb{1}_{\{t \wedge \bar{t}_k \leq T_n^k\}} ds, \]

for \(0 \leq t \leq T, 1 \leq i \leq n\). The differential \(dA^{k,j}\) in (6.9) is interpreted in the Lebesgue-Stieljes sense. One should notice that

\[ X^{k,\phi}(t) = X^{k,\phi}(\bar{t}_k) = \widehat{X}^{i,k,\phi}(t), \quad \Sigma_t^k = \Sigma_t^{k,\phi}, \]

for every \(t \in [0,T]\). For a given \(1 \leq i \leq n\), the idea is to analyse

\[ E\|X^{i,k,\phi} - X^{i,\phi}\|^2 \leq 2E\|X^{i,k,\phi} - \widehat{X}^{i,k,\phi}\|^2 + 2E\|\widehat{X}^{i,k,\phi} - X^{i,\phi}\|^2. \]

Let us now present a couple of lemmas towards the final estimate. In the remainder of this section, \(C\) is a constant which may differ from line to line in the proofs of the Lemmas below. The following result is very simple but very useful to our argument.
Lemma 6.1. For every $t \geq 0$, $1 \leq i \leq n, 1 \leq j \leq d$ and $\phi \in \mathcal{U}_0^k$, we have

$$
\int_0^t \Sigma_{i,j,k,\phi}(s)dB_j(s) = \int_0^t \Sigma_{i,j,k,\phi}(s)dA_{k,j}(s) + \sigma^{ij}(\bar{t}_k, \bar{X}_{i,k,\phi}, \phi(\bar{t}_k))(B_j(t) - B_j(\bar{t}_k)) \text{ a.s.}
$$

where $dB_j$ is the Itô integral and $dA_{k,j}$ is the Lebesgue Stieltjes integral.

Lemma 6.2.

$$
\sup_{k \geq 1} \sup_{\phi \in \mathcal{U}_0^k} \mathbb{E}\|X_{T,\phi}^k\|_\infty < \infty, \quad \forall p > 1.
$$

Proof. We fix $1 \leq i \leq n$. At first, we notice that

$$
\int_0^T \mathbb{E}\|X_{T,\phi}^k\|_\infty^p ds < \infty,
$$

for every $k \geq 1$, $p > 1$ and $\phi \in \mathcal{U}_0^k$. From Assumption C1, there exists a constant $C$ such that

$$
|\alpha^i(t, \omega, a)| + |\sigma^{ij}(t, \omega, a)| \leq C(1 + \|\omega_i\|_\infty),
$$

for every $(t, \omega, a) \in [0, T] \times \mathcal{D}_n.T \times \mathcal{A}$ and $1 \leq i \leq n, 1 \leq j \leq d$, where $C$ only depends on $T, \alpha(0, 0, 0), \sigma(0, 0, 0)$ and the compact set $\mathcal{A}$. Identity (6.42) and Lemma 6.1 yield

$$
\|X_{t,\phi}^k\|_\infty \leq \|X_{0,\phi}^k\|_\infty + \int_0^t \mathbb{E}\|X_{s,\phi}^k\|_\infty^p ds + \sum_{j=1}^d \int_0^t \Sigma_{i,j,k,\phi}(s)ds + \sum_{j=1}^d \int_0^t \Sigma_{i,j,k,\phi}(s)dB_j(s).
$$

Now, (6.10) and representation (6.11) allow us to make use of routine arguments based on Burkholder-Davis-Gundy and Jensen inequalities to apply Grönwall's inequality on the function $s \mapsto \mathbb{E}\|X_{s,\phi}^k\|_\infty^p$. This concludes the proof.

Lemma 6.3. For every $0 < \beta < 1$, there exists a constant $C$ which only depends on $\alpha, \sigma, \beta$ such that

$$
\mathbb{E}\|\bar{X}_{T,\phi}^k - X_{T,\phi}^k\|_\infty^2 \leq C \left\{ \epsilon_k^2 \left\| \int_0^T \alpha^i(\bar{s}_k, \bar{X}_{i,k,\phi}^k, \phi(s)) - \alpha^i(\bar{s}_k, \bar{X}_{i,\phi}^k, \phi(s)) \right\|_\infty^2 \right\},
$$

for every $\phi \in \mathcal{U}_0^k$ and $k \geq 1$.

Proof. Let us fix $1 \leq i \leq n$ and $\phi \in \mathcal{U}_0^k$. Lemma 6.1 allows us to write

$$
\bar{X}_{i,k,\phi}(t) - X_{i,\phi}(t) = \int_0^t \left[ \alpha^i(\bar{s}_k, \bar{X}_{i,k,\phi}^k, \phi(s)) - \alpha^i(s, X_{i,\phi}^k, \phi(s)) \right] ds
$$

$$
+ \sum_{j=1}^d \int_0^t \left[ \Sigma_{i,j,k,\phi}(s) - \sigma^{ij}(s, \bar{X}_{i,\phi}^k, \phi(s)) \right] dB_j(s)
$$

$$
- \sum_{j=1}^d \sigma^{ij}(\bar{t}_k, \bar{X}_{i,k,\phi}^k, \phi(\bar{t}_k))(B_j(t) - B_j(\bar{t}_k))
$$

$$
= I_{1,i}(t) + I_{2,i}(t) + I_{3,i}(t).
$$

Assumption C1 and Lemma 6.2 yield

$$
\mathbb{E}\|I_{1,i}(t)\|_\infty^2 \leq C \left\{ \epsilon_k^2 \left\| \int_0^T \alpha^i(\bar{s}_k, \bar{X}_{i,k,\phi}^k, \phi(s)) - \alpha^i(s, X_{i,\phi}^k, \phi(s)) \right\|_\infty^2 \right\},
$$

where
for $0 < \beta < 1$. The estimate \(6.10\), Lemma \(6.2\) and the fact that \(|B^j(t) - B^j(\bar{t}_k)| \leq \epsilon_k \ a.s\) for every \(t \geq 0\) yield

\[
(6.14) \quad \mathbb{E}\|I^{k,\phi}_t\|_\infty^2 \leq C\epsilon_k^2.
\]

By using Burkholder-Davis-Gundy's inequality, we have

\[
(6.15) \quad \mathbb{E}\|I^{k,\phi}_t\|_\infty^2 \leq C \sum_{j=1}^d \mathbb{E}\int_0^T |\sum_{i=j}^{d-1} |\sigma^{i,j}(s, X^\phi_s, \phi(s))| |^2 \, ds.
\]

Assumption C1 yields

\[
(6.16) \quad \left| \sum_{i=j}^{d-1} \sigma^{i,j}(s, X^\phi_s, \phi(s)) \right| \leq C \sup_{0 \leq t \leq T} \|X^\phi_t - X^\phi_s\|.
\]

We observe

\[
(6.17) \quad \|X^\phi_t - X^\phi_s\| \leq \|X^\phi_t - X^\phi_s\|_\infty.
\]

Summing up \(6.12\), \(6.13\), \(6.14\), \(6.15\), \(6.16\) and \(6.17\), we can conclude the proof.

\[\square\]

**Lemma 6.4.** For every $0 < \beta < 1$ and $p > 1$, there exists a constant $C$ which only depends on $T, p, \alpha$ and $\beta$ such that

\[
\mathbb{E}\|X^\phi_{t_T} - \bar{X}^\phi_{t_T}\|_\infty^2 \leq C\epsilon_k^2 \left(\frac{\epsilon_k^2 T}{\lambda_d}\right)^{\frac{1-\beta}{p}},
\]

for every $\phi \in \bar{U}_{0,T}^{k,\infty}$ and $k \geq 1$.

**Proof.** Fix $\phi \in \bar{U}_{0,T}^{k,\infty}$. By the very definition, for a given $i \in \{1, \ldots, n\}$, we have

\[
|\chi_{\gamma,\phi}(t) - X^\phi(t)| \leq \int_{\bar{t}_k}^t |\alpha^i(s, X^\phi_s, \phi(s))| \, ds,
\]

because $t - \bar{t}_k > 0$ a.s for every $t > 0$ and $\sum_{j=1}^d \int_{\bar{t}_k}^{t} \sum_{i=j}^{d-1} |\sigma^{i,j}(s)| \, ds \leq \int_{0}^{t} \sum_{j=1}^d \sum_{i=j}^{d-1} |\sigma^{i,j}(s)| dA_{ij}(s)$ a.s for every $t \geq 0$. Now, we use \(6.10\), Lemma \(6.2\) and Jensen and Hölder inequalities to get the existence of a constant $C$ which only depends on $\alpha, p$ and $T$ such that

\[
\mathbb{E}\|X^\phi_{t_T} - \bar{X}^\phi_{t_T}\|_\infty^2 \leq C \left(\mathbb{E}\|\Delta T^\phi_n\|_I \right. \left. \{T < t_T\} \right)^{\frac{p}{2}},
\]

for each $p > 1$. Apply Lemma \(8.2\) to conclude the proof.

\[\square\]

**Corollary 6.1.** For given $\alpha, \sigma, T$, $\beta \in (0, 1)$ and $\bar{a}$, there exists a constant $C > 0$ which depends on these parameters such that

\[
(6.18) \quad \mathbb{E}\|X^\phi_{t_T} - \bar{X}^\phi_{t_T}\|_\infty^2 \leq C\epsilon_k^{2\beta},
\]

for every $\phi \in \bar{U}_{0,T}^{k,\infty}$ and $k \geq 1$.

**Proof.** Apply Lemmas \(6.3\) and \(6.4\) and use Grönwall’s inequality to arrive at

\[
\mathbb{E}\|X^\phi_{t_T} - \bar{X}^\phi_{t_T}\|_\infty^2 \leq C\left\{\epsilon_k^2 \left(\frac{\epsilon_k^2 T}{\lambda_d}\right)^{\frac{1-\beta}{2}} + \epsilon_k^2 \right\},
\]
for every $\phi \in U_{0}^{k,\epsilon(t)}$, $\epsilon > 0$, $t > 0$ and $k, T > 0$.

By using Lemma 6.2, Hölder’s inequality for $p > 1$ and Lemma 8.3, we get

\[ \max_{p \geq 1} \left( \int_{T_{c}(k, T)}^{T} |T_{p}^{k} - T_{c}(k, T)| \right)^{2} \leq C|T - T_{c}(k, T)| \left( 1 + \|X_{T}^{k, \phi} \|_{\infty} \right)^{2}. \]
We observe

\[ \sum_{1 \leq i < j \leq d} \sigma_{ij}^k, \phi(s) \, dB^i(s) \]

for every \( k \geq 1 \), where the constant \( C \) in (6.21) does not depend on \( \phi \). Let us now treat the stochastic integral. For each \( \eta > 0 \) and \( k \geq 1 \), let us denote

\[ E_{k, \eta} \triangleq \{ T_k^{e(k,T)} < T_{N_k^k} \} \]

and

\[ E_{k, \eta}^2 \triangleq \{ T_k^{e(k,T)} < T_{N_k^k} \} \]

We observe

\[ J_k^1(\eta, \phi) = \max_{p \geq 1} \left[ \sum_{e(k,T) < n-1 \leq p \leq N_k^k} \sigma_{ij}^k \left( T_{n-1}^k, \eta_{n-1}^k, \epsilon_{n-1}^k \right) \Delta A_{ij}^k \left( T_n^k \right) \right] \]

for \( \ell = 1, 2 \). By the additivity of the stochastic integral, we have

\[ J_k^\ell(\eta, \phi) \leq 2 \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{ij}^k(s) dB^i(s) \right| \]

By applying Burkholder-Davis-Gundy and Hölder inequalities to (6.22) and using (6.10) and Lemma 6.2 we get

\[ \mathbb{E} J_k^1(\eta, \phi) \leq C \left( \mathbb{P} \{ \eta + T_k^{e(k,T)} < T_{N_k^k} \} \right)^{1/2}, \]

for any \( \zeta > 1, \eta > 0 \) and a control \( \phi \in \mathcal{U}_0^{e(k,T)} \). Here, the constant \( C \) does not depend on controls, \( \eta > 0 \) and \( k \geq 1 \). By applying Lemma 8.3 we get

\[ \sup_{\phi \in \mathcal{U}_0^{e(k,T)}} \mathbb{E} J_k^1(\eta, \phi) \leq C \left( \mathbb{P} \{ \eta + T_k^{e(k,T)} < T \} \right)^{1/2} \]

(6.23)

for every \( k \geq 1 \) and \( 0 < \eta < T \). In order to estimate \( J_k^2(\eta, \phi) \), we denote

\[ m_{ij,k}(h, T, \phi) = \sup_{t, s \in [0, T], |t-s| \leq h} \left| \int_t^s \sigma_{ij}^k(s) dB^i(s) \right|, \]

for \( h > 0 \). We notice

\[ J_k^2(\eta, \phi) \leq \max_{p \geq 1} \left\{ m_{ij,k}(\eta, T, \phi), \ldots, m_{ij,k}(\eta, T, \phi) \right\} \mathbb{E} J_{k, \eta}^2 \cap \{ T_k^{e(k,T)} < T_{N_k^k} \} \]

for every \( k \geq 1 \), where the constant \( C \) in (6.21) does not depend on \( \phi \). Let us now treat the stochastic integral. For each \( \eta > 0 \) and \( k \geq 1 \), let us denote

\[ E_{k, \eta} \triangleq \{ T_k^{e(k,T)} < T_{N_k^k} \} \]

and

\[ E_{k, \eta}^2 \triangleq \{ T_k^{e(k,T)} < T_{N_k^k} \} \]

We observe

\[ J_k^1(\eta, \phi) = \max_{p \geq 1} \left[ \sum_{e(k,T) < n-1 \leq p \leq N_k^k} \sigma_{ij}^k \left( T_{n-1}^k, \eta_{n-1}^k, \epsilon_{n-1}^k \right) \Delta A_{ij}^k \left( T_n^k \right) \right] \]

for \( \ell = 1, 2 \). By the additivity of the stochastic integral, we have

\[ J_k^\ell(\eta, \phi) \leq 2 \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{ij}^k(s) dB^i(s) \right| \]

By applying Burkholder-Davis-Gundy and Hölder inequalities to (6.22) and using (6.10) and Lemma 6.2 we get

\[ \mathbb{E} J_k^1(\eta, \phi) \leq C \left( \mathbb{P} \{ \eta + T_k^{e(k,T)} < T_{N_k^k} \} \right)^{1/2}, \]

for any \( \zeta > 1, \eta > 0 \) and a control \( \phi \in \mathcal{U}_0^{e(k,T)} \). Here, the constant \( C \) does not depend on controls, \( \eta > 0 \) and \( k \geq 1 \). By applying Lemma 8.3 we get

\[ \sup_{\phi \in \mathcal{U}_0^{e(k,T)}} \mathbb{E} J_k^1(\eta, \phi) \leq C \left( \mathbb{P} \{ \eta + T_k^{e(k,T)} < T \} \right)^{1/2} \]

(6.23)

for every \( k \geq 1 \) and \( 0 < \eta < T \). In order to estimate \( J_k^2(\eta, \phi) \), we denote

\[ m_{ij,k}(h, T, \phi) = \sup_{t, s \in [0, T], |t-s| \leq h} \left| \int_t^s \sigma_{ij}^k(s) dB^i(s) \right|, \]

for \( h > 0 \). We notice

\[ J_k^2(\eta, \phi) \leq \max_{p \geq 1} \left\{ m_{ij,k}(\eta, T, \phi), \ldots, m_{ij,k}(\eta, T, \phi) \right\} \mathbb{E} J_{k, \eta}^2 \cap \{ T_k^{e(k,T)} < T_{N_k^k} \} \]
\[ \leq m^2_{ij,k}(\eta, T, \phi) \text{ a.s.} \]

for every \( k \geq 1 \). By applying Th 1 in [20] jointly with (6.10) and Lemma 6.2, there exists a constant \( C \) which does not depend on controls such that

\[ \mathbb{E}J_k^2(\eta, \phi) \leq C\eta \ln \left( \frac{2T}{\eta} \right), \]

for every \( k \geq 1 \) and \( 0 < \eta < T \). Then,

\[ (6.24) \sup_{\phi \in U_{k,\eta}(k, T)} \mathbb{E}J_k^2(\eta, \phi) \leq C\eta \ln \left( \frac{2T}{\eta} \right). \]

In (6.23) and (6.24), \( \eta \in (0, T) \) can be chosen arbitrarily and independently from \( k \geq 1 \). Summing up (6.18), (6.19), (6.21), (6.23) and (6.24), we conclude the proof.

6.2. Path-dependent controlled SDEs driven by FBM. In this section, we investigate controlled Wiener functionals of the form

\[ dX^u(t) = \alpha(t, X_t, u(t))dt + \sigma dB_H(t), \]

where \( X(0) = x_0 \in \mathbb{R} \), \( \sigma \) is a constant, \( \alpha \) is a non-anticipative functional satisfying Assumption (C1) and \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). For simplicity of presentation, the payoff functional \( \xi \) will be Lipschitz and we set the state dimension equals to one. The path-dependence feature is much more sophisticated than previous example because the lack of Markov property comes from distorting the driving Brownian motion by a singular kernel and also from a non-anticipative drift \( \alpha \) which depends on the whole solution path.

Under Assumption C1, by a standard fixed point argument, one can show there exists a unique strong solution for (6.25) for each \( u \in U^T_T \). Moreover, the following simple remark holds true.

Remark 6.2. The controlled SDE (6.25) satisfies Assumption B1 due to modulus of continuity of \( \alpha \) given by Assumption C1.

Clearly, the first step is to construct an imbedded discrete structure for \( B_H \). If \( \frac{1}{2} < H < 1 \), then there is a pathwise representation

\[ B_H(t) = \int_0^t \rho_H(t, s)B(s)ds; 0 \leq t \leq T, \]

where

\[ \rho_H(t, s) := d_H \left[ (H - \frac{1}{2})s^{-2H+\frac{1}{2}} \int_s^t \left( u^{-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du - s^{-\frac{1}{2}}t^{H+\frac{3}{2}}(t-s)^{H-\frac{3}{2}} \right) \right], \]

for a constant \( d_H \). For further details, see e.g. [40]. We define

\[ Y^k_H(t) := \int_0^t \rho_H(t, s)A^k(s)ds; 0 \leq t \leq T. \]

To get a piecewise constant process, we set

\[ B^k_H(t) := \sum_{n=0}^{\infty} Y^k_H(T^k_n)1_{\{T^k_n \leq t < T^k_{n+1}\}}; 0 \leq t \leq T. \]

The case \( 0 < H < \frac{1}{2} \) is much more delicate. In the philosophy of [39], it is important to work with a pathwise representation of FBM which will be denoted by \( B_H \) for \( H \in (0, \frac{1}{2}) \). Let \( C_0^H \) be the space of Hölder continuous real-valued functions on [0, T] and starting at zero equipped with the usual norm. By Lemma 10.1 there exists a bounded linear operator \( \mathcal{A}_H : C_0^H \rightarrow C_{1,T} \) for \( \frac{1}{2} - H < \lambda < \frac{1}{2} \) such that
In the sequel, we recall (see (6.8))
\[ \ell_k = \max\{T_n^k; T_n^k \leq t\}, \]
and we define
\[ \hat{t}_k^+ := \min\{T_n^k; \hat{t}_k < T_n^k\} \wedge T \quad \text{and} \quad \hat{t}_k^- := \max\{T_n^k; T_n^k < \hat{t}_k\} \vee 0, \]
where we set \( \max 0 = -\infty \). The discrete structure for FBM with \( 0 < H < \frac{1}{2} \) is given by
\[ B_H^k(t) := \int_0^{\hat{t}_k^-} \partial_s K_{H,1}^k(\hat{t}_k, s) [A^k(\hat{t}_k) - A^k(s_k^+)] ds - \int_0^{\hat{t}_k} \partial_s K_{H,2}^k(\hat{t}_k, s) A^k(s) ds \]
where \( (\partial_s K_{H,1}, \partial_s K_{H,2}) \) is defined by [10.3]. We observe
\[ B_H^k(t) = \sum_{n=0}^{\infty} B_H^k(T_n^k) \mathbb{1}_{\{T_n^k \leq t \leq T_{n+1}^k\}}; 0 \leq t \leq T. \]

The controlled structure \( (X_t^k)_{t \geq 0} \) is given as follows: let us fix a control \( \phi \in U_0^{k,c(k,T)} \) with jumps given by \( v_{n-1}^k \); \( 1 \leq n \leq e(k,T) \). Let us define
\[ X^k,\phi(T_m) := X^k,\phi(T_{m-1}) + \alpha(T_{m-1}, X^k,\phi(T_{m-1}), v_n^k) \Delta T_m^k + \sigma \Delta B_H^k(T_m^k), \]
where \( X^k,\phi(t) = \sum_{m=0}^{\infty} X^k,\phi(T_m^k) \mathbb{1}_{\{T_m^k \leq t \leq T_{m+1}^k\}}; t \geq 0 \). Then, we set \( \mathcal{X} = ((X_t^k)_{t \geq 0}, \mathcal{F}) \) given by
\[ X^k,\phi(t) := \sum_{t=0}^{\infty} X^k,\phi(T_t^k) \mathbb{1}_{\{T_t^k \leq t \leq T_{t+1}^k\}}; 0 \leq t \leq T. \]
The pathwise version of (6.27) is given by the following objects: for each sequence \( b^k_\infty \in \mathbb{S}_k^\infty \), we denote
\[ w^k(s) = \sum_{n=0}^{\infty} \epsilon_k \tilde{w}^{k,n}_s \mathbb{1}_{\{t_n^k \leq s < t_{n+1}^k\}}, \]
for \( s \in [0, T] \). Moreover, with a slight abuse of notation, for each \( s \in [0, T] \), we set \( \tilde{s}_k = \max\{t_n^k; t_n^k \leq s\} \) and \( \hat{s}_k = \min\{t_n^k; \hat{t}_k < t_n^k\} \). With these objects at hand, we define
\[ w_H^k(t_n^k) := \int_0^{t_n^k} \partial_s K_{H,1}(t^k_n, s) [w^k(t_n^k) - w^k(\hat{s}_k^+)] ds - \int_0^{t_n^k} \partial_s K_{H,2}(t_n^k, s) w^k(s) ds, \]
for \( 0 < H < \frac{1}{2} \) and
\[ w_H^k(t_n^k) := \int_0^{t_n^k} \rho_H(t_n^k, s) w^k(s) ds, \]
for \( \frac{1}{2} < H < 1 \). We then set \( h_n^k := x \) and \( \tilde{\gamma}_0^k = x \) is the constant function over \([0, T]\). For a given information set
\[ o_0^k = \left( (a_0^k, s_0^k, \tilde{q}_0^k), \ldots, (a_{q-1}^k, s_{q-1}^k, \tilde{q}_{q-1}^k) \right) \in \mathbb{H}^{k,q}, \]
we define inductively
\[ h_q^k(o_q^k) := h_{q-1}^k(o_{q-1}^k) + \alpha(t_{q-1}^k, s_{q-1}^k\tilde{q}_{q-1}^k, o_{q-1}^k, a_{q-1}^k) s_q^k \]
Proof. Fix $\phi \in \mathbb{H}_k^1$. There exists a constant $C$ such that

$$\gamma_{\phi}^k(t) := \sum_{\ell=0}^{q-2} h^k_\ell (\sigma_{\phi}^k_{\ell}) I_{\{t^k_\ell < t < t^k_{\ell+1}\}} + h^k_{q-1} (\sigma_{\phi}^{k,1}) I_{\{t^k_{q-1} \leq t\}},$$

for $0 \leq t \leq T$, where $\sigma_{\phi}$ is the projection onto $\mathbb{H}_k^1$. We then define

$$\gamma_{\phi}^k(t) := \sum_{n=0}^{\infty} h^k_n (\sigma_{\phi}^k) I_{\{t^k_n < t < t^k_{n+1}\}},$$

for $\sigma_{\phi}^k \in \mathbb{H}_k^\infty$ and we set

$$\gamma_{\phi}^k(t) := \gamma_{\phi}^k(t) (t^k_{\ell}) := \gamma_{\phi}^k(t \wedge t^k_{\ell}); 0 \leq t \leq T.$$

By construction, $\gamma_{\phi}^k(t)$ realizes $(\Delta_k, \lambda_k)$ for the controlled structure $(6.27)$. Proposition 6.2. Assume $\alpha$ satisfies Assumption C1. We fix $\frac{1}{2} < H < 1$ and $H - \frac{1}{2} < \lambda < \frac{1}{2}$. Then, there exists a constant $C$ which depends on $\alpha$, $T$, and $\lambda$ such that

$$\sup_{\phi \in \mathbb{H}_0^1} \mathbb{E} \left\| X^{k,\phi} - X^\phi \right\|_\infty \leq C \epsilon_k^{1-2\lambda}; k \geq 1.$$

Fix $0 < H < \frac{1}{2}$, $0 < \epsilon < H$ and $H, \lambda, \epsilon$ such that $\delta < (0,1)$, $\lambda < H + \frac{2k-2}{2} + \frac{2k-2}{2}$. Then, there exists a constant $C$ which depends on $H, \delta, \lambda, \epsilon$ such that

$$\sup_{\phi \in \mathbb{H}_0^1} \mathbb{E} \left\| X^{k,\phi} - X^\phi \right\|_\infty \leq C (\epsilon_k^{H-2\epsilon} + \epsilon_k^{1-2\lambda} + \epsilon_k^{2(H-1-\lambda)} + \epsilon_k^{2(H-1-\lambda)+2(\delta-1)}); k \geq 1.$$

In particular, $(X, \mathcal{X})$ is a strongly controlled Wiener functional. Hence, Propositions 5.2 and 5.3 apply to the control problem driven by $(6.27)$ for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

Proof. At first we treat $\frac{1}{2} < H < 1$. In the sequel, $C$ is a constant which may differ from line to line. Fix $\phi \in \mathbb{H}_0^1$ and a number $\beta \in (0,1)$. By repeating exactly the same steps (with the obvious modification by replacing $A^k$ by $B^k$) as in the proofs of Lemmas 6.2 and 6.3 and 6.4, we can find a constant $C$ (depending on $\alpha, T, H, \beta \in (0,1)$ and $\lambda \in (H - \frac{1}{2}, \frac{1}{2})$) such that

$$\mathbb{E} \left\| X^{k,\phi} - X^\phi \right\|_\infty \leq C \left( \epsilon_k^{1-2\beta} + \sup_{0 \leq t \leq T} |B^k_H(t) - B_H(t)| \right)$$

(6.30)

$$\leq C \left( \epsilon_k^{1-2\beta} + \epsilon_k^{1-2\lambda} \right) \leq C \left( \epsilon_k^{\beta} + \epsilon_k^{1-2\lambda} \right) \leq C \epsilon_k^{1-2\lambda},$$

where the second inequality in (6.30) is due to Prop. 5.2. In order to estimate $\|X^{k,\phi} - X^\phi\|_\infty$, we observe

$$\|X^{k,\phi} - X^\phi\|_\infty \leq \|X^{k,\phi} - X^\phi\|_\infty + \sup_{0 \leq t \leq T} |X^\phi(t \wedge T^k_{\phi} - X^\phi(t)|,$$

where
From (6.33) and (8.4), there exists a constant $C$
\begin{equation}
\sup_{0 \leq s \leq t \leq T} |X^\phi(t \wedge T^k_{e(k,T)}\wedge T^k_{e(k,T)}) - X^\phi(t)| \leq C(1 + \|X^\phi\|_\infty) |T - T^k_{e(k,T)}| + \sigma^2 \|B_H(t \wedge T^k_{e(k,T)}) - B_H\|_\infty
\end{equation}
(6.31)
\[
= I^k_T + I^k_T.
\]
One can easily check $\sup_{u \in U_3} \mathbb{E}\|X^u\|\_p^2 \leq C(1 + |x_0|^p \exp(CT))$ for every $p \geq 1$ where $C$ is a constant which depends on $\bar{a}$ and $B_H$. Moreover, we shall invoke Lemma 8.3 (see (8.3)) to arrive at
\begin{equation}
\mathbb{E}I^k_T \leq C\epsilon^2
\end{equation}
(6.32)
for every $k \geq 1$. Now, from Garsia-Rodemich-Rumsey’s inequality (see e.g Lemma 7.4 in [3]), for all $\epsilon \in (0,H)$ there exists a nonnegative random variable $G_{\epsilon,T}$ with $\mathbb{E}|G_{\epsilon,T}|^p < +\infty$ for all $p \geq 1$ such that
\[
|B_H(t) - B_H(s)| \leq G_{\epsilon,T}|t - s|^{H-\epsilon} \ \text{a.s.},
\]
for all $s,t \in [0,T]$. Therefore,
\begin{equation}
\sup_{0 \leq s \leq t \leq T} |B_H(t \wedge T^k_{e(k,T)}) - B_H(t)| \leq |T - T^k_{e(k,T)}|^{H-\epsilon} G_{\epsilon,T} a.s.
\end{equation}
(6.33)
From (6.33) and (8.4), there exists a constant $C$ which depends on $H$ such that
\begin{equation}
\mathbb{E}I^k_T \leq C\epsilon^2
\end{equation}
(6.34)
for every $k \geq 1$. Summing up (6.30), (6.31), (6.32) and (6.34) and noticing we shall take $(H - \epsilon) + \lambda > \frac{1}{2}$, we then conclude the proof for $H \in \left(\frac{1}{2},1\right)$.

We now fix $H \in (0,1)$, $0 < \epsilon < H$, $\beta = 2(H - \epsilon)$, $\delta \in (0,1)$, $\lambda \in \left(\frac{1}{2} - H + \frac{2\delta-2}{2}, \frac{1}{2} + \frac{2\delta-2}{2}\right)$. By applying Theorem 10.2 and repeating exactly the same steps (with the obvious modification by replacing $A^k$ by $B^k_H$) as in the proofs of Lemmas 6.2, 6.3 and 6.4 we can find a constant $C$ (depending on $\alpha, T, H, \delta, \lambda$) such that
\[
\mathbb{E}\|X^k - X^\phi\|_\infty \leq C \left\{ \epsilon_k \left[ \epsilon_k^{-2(T)} \right]^{\frac{1-\beta}{2}} + \mathbb{E}\|B_H^k - B_H\|_\infty \right\}
\]
\[
\leq C \left\{ \epsilon_k \left[ \epsilon_k^{-2(T)} \right]^{\frac{1-\beta}{2}} + \epsilon_k^{1-2\lambda) + 2(\delta-1) + \epsilon_k^{2(H-\frac{1}{2} + \lambda) + 2(\delta-1)} + \epsilon_k^{2(H-\epsilon)} \right\}
\]
\[
\leq C \left\{ \epsilon_k^{\beta} + \epsilon_k^{(1-2\lambda) + 2(\delta-1)} + \epsilon_k^{2(H-\frac{1}{2} + \lambda) + 2(\delta-1)} + \epsilon_k^{2(H-\epsilon)} \right\}
\]
\[
= C \left( \epsilon_k^{2(H-\epsilon)} + \epsilon_k^{(1-2\lambda) + 2(\delta-1)} + \epsilon_k^{2(H-\frac{1}{2} + \lambda) + 2(\delta-1)} \right).
\]
The bound for $\mathbb{E}\|X^k - X^\phi\|_\infty$ follows the same lines of (6.31), (6.32), (6.33) and (6.34). We stress (6.34) holds as well for $H \in (0, \frac{1}{2})$. We then conclude the proof.

6.3. Controlled rough stochastic volatility. Let us now investigate the third type of non-Markovian controlled state:
\begin{equation}
\begin{cases}
\frac{dX^u(t) = X^u(t)\mu(u(t))dt + X^u(t)\theta(Z(t), u(t))dB^1(t)}{dZ(t) = \nu dW_H(t) - \beta(Z(t) - m)dt} ,
\end{cases}
\end{equation}
(6.35)
where $m = 0$ (for simplicity), $\beta, \nu > 0$ and $\theta, \mu$ satisfy the following assumption:
**Assumption (D1):** $\vartheta$ is bounded and there exists a constant $C$ such that

$$|\mu(a) - \mu(b)| \leq C|a - b|, \quad |\vartheta(x, a) - \vartheta(y, b)| \leq C\{|x - y|^\theta + |a - b|\}$$

for every $x, y \in \mathbb{R}$, $a, b \in \mathbb{A}$, where $0 < \theta \leq 1$.

The noise $W_H$ in (6.35) is a FBM with exponent $H \in (0, \frac{1}{2})$. We assume $W_H$ is a functional of the Brownian motion

$$W := \rho B^1 + \bar{\rho} B^2,$$

where $\bar{\rho} := \sqrt{1 - \rho^2}$ for $-1 < \rho < 1$. By Theorem 10.1, we can write

$$W_H = \rho (\mathcal{A}_H B^1) + \bar{\rho} (\mathcal{A}_H B^2).$$

By Itô’s formula,

$$X^u(t) = X^u(0) \mathcal{E} \left( \int_0^t \vartheta(Z(s), u(s))dB^1(s) \right) \exp \left( \int_0^t \mu(u(s))ds \right),$$

where $\mathcal{E}$ is the Doléans-Dade exponential. For further details, see e.g. [1]. It is well-known (see e.g. Prop A1 [9])

$$Z(t) = e^{-\beta t} z_0 + \nu W_H(t) - \beta \nu e^{-\beta t} \int_0^t W_H(u)e^{\beta u}du; 0 \leq t \leq T.$$

Representation (6.36), Th 1 in [23] and the boundedness assumption on $\vartheta$ yields sup$_{u \in U^T} \mathbb{E}\|X^u\|_\infty < \infty$ for every $p \geq 1$. Moreover, a similar computation as explained in Remark 6.1 gives Assumption (B1) for the controlled state (6.35). An imbedded discrete structure for the volatility process will be described by

$$Z^k(t) := e^{-\beta t} z_0 + \nu W^k_H(t) - \beta \nu e^{-\beta t} \int_0^{\hat{t}_k} W^k_H(u)e^{\beta u}du,$$

for $0 \leq t \leq T$, where, for each $i = 1, 2$, we define

$$W^{k,i}_H(t) := \int_0^{\hat{t}_k} \partial_s K_{H,i}(\bar{f}_k, s)[A^{k,i}(\bar{f}_k) - A^{k,i}(s^+)]ds - \int_0^{\hat{t}_k} \partial_s K_{H,2}(\bar{f}_k, s)A^{k,i}(s)ds,$$

and

$$W^k_H(t) := \rho W^{k,1}_H(t) + \bar{\rho} W^{k,2}_H(t); 0 \leq t \leq T.$$

Clearly, there exists a constant $C$ which only depends on $H$ such that

$$\|Z - Z^k\|_\infty \leq C \left\{ |\beta\nu| \left( \max_{n \geq 1} \Delta T^k_n \mathbb{1}_{(T^k_n \leq T)} \right) + \nu \|W^k_H - W_H\|_\infty + |2\beta^2\nu| \left( \max_{n \geq 1} \Delta T^k_n \mathbb{1}_{(T^k_n \leq T)} \right) \|W^k_H\|_\infty e^{\beta T} \right\} \text{ a.s.}$$

Then, Lemma 8.2 and Theorem 10.2 applied to (6.39) allow us to state the following result.
Corollary 6.2. Fix $0 < H < \frac{1}{2}$. For a given $p > 1$, $0 < \varepsilon < H$ and a pair $(\delta, \lambda)$ such that $\delta \in (0, 1)$ and $\lambda \in \left(\frac{1}{2} - H + \frac{2s-2}{2p}, \frac{1}{2} + \frac{2s-2}{2p}\right)$, there exists a constant $C$ which depends on $\beta, \nu, p, \delta, H, T, \lambda, \varepsilon, \rho, \chi_2$ such that

$$E\|Z^k - Z\|_\infty^p \leq Cz_k(\delta, p, \lambda, H, \varepsilon); \quad k \geq 1,$$

where $z_k(\delta, p, \lambda, H, \varepsilon) := \epsilon_k^{2(\delta-1+p)} + \epsilon_k^{p(1-2\lambda)+2(\delta-1)} + \epsilon_k^{2p(H-\frac{1}{2}+\lambda)+2(\delta-1)} + \epsilon_k^{p(H-\varepsilon)}$ for $k \geq 1$.

Let us fix a control $\phi = (v_0^k, \ldots, v_n^k, \ldots) \in U_0^{T,\infty}$. Starting from $X^k(0) = x$, we define

$$X^k,\phi(T^k_q) := X^k,\phi(T^k_{q-1}) + X^k,\phi(T^k_{q-1})\mu(Z^k(T^k_{q-1}), v^k_{q-1})\Delta T^k_q$$

(6.40)

and

$$\bar{X}^k,\phi(T^k_{0-1})\mu(Z^k(T^k_{q-1}), v^k_{q-1})\Delta A^{k,1}(T^k_q),$$

for $q \geq 1$. We then define $X^k,\phi$ similar to (6.6). This provides a controlled imbedded discrete structure $X = ((X^k)_k \geq 1, \mathcal{D})$. A pathwise representation of $X^k,\phi$ is clear by looking at the previous examples and identities (6.36), (6.38), (6.28) and (6.40). We left the details of this representation to the reader.

Let us define

$$\tilde{\mu}(t, f^1, f^2, f^3) := f^1(t)\mu(f^3(t)), \quad \tilde{\vartheta}(t, f^1, f^2, f^3) := f^1(t)\vartheta(f^2(t), f^3(t)),$$

for $f^1, f^2, f^3 \in D_{1,T}$ and $t \in [0, T]$. We set

$$\Sigma^{k,\phi}(t) := 0\mathbb{1}_{\{t=0\}} + \sum_{n=1}^{\infty} \tilde{\vartheta}(T^k_{n-1}, X^k,\phi, Z^k, \phi)\mathbb{1}_{\{T^k_{n-1} < t \leq T^k_n\}},$$

for $0 \leq t \leq T$. We define

$$\hat{X}^{k,\phi}(t) := x_0 + \int_0^t \tilde{\mu}(s, \hat{X}^{k,\phi}(s))ds + \int_0^t \Sigma^{k,\phi}(s)dA^{k,1}(s),$$

(6.41)

for $0 \leq t \leq T$. The differential $dA^{k,1}$ in (6.41) is interpreted in the Lebesgue-Stieljtes sense. One should notice that

$$X^{k,\phi}(t) = \hat{X}^{k,\phi}(\bar{t}_k) = X^{k,\phi}(\bar{t}_k), \quad X^{k,\phi} = X^{k,\phi},$$

for every $t \in [0, T]$. The idea is to analyse

$$E\|X^{k,\phi} - X^{\phi}\|_\infty^2 \leq 2E\|X^{k,\phi} - \hat{X}^{k,\phi}\|_\infty^2 + 2E\|\hat{X}^{k,\phi} - X^{\phi}\|_\infty^2.$$

By using Assumption (D1), one can follow the same arguments as described in Lemmas 6.1, 6.2, 6.3, 6.4 and Corollary 6.1 to arrive at the following estimates:

$$\sup_{\phi \in \mathbb{T}_{0}^{T,\infty}(k, T)} E\|X^{k,\phi} - X^{\phi}\|_\infty^2 \leq C\left\{\epsilon_k^2 + z_k(\delta, p, \lambda, H, \varepsilon) + E \int_0^T \|X^{k,\phi} - X^{\phi}\|_\infty^2 ds\right\},$$

(6.43)

for $p > 2$ and $(\delta, \lambda, H, \varepsilon)$ satisfying the compatibility condition described in Corollary 6.2. Moreover,

$$\sup_{\phi \in \mathbb{T}_{0}^{T,\infty}(k, T)} E\|X^{k,\phi} - X^{\phi}\|_\infty^2 \leq C\left\{\epsilon_k^2 + z_k(\delta, p, \lambda, H, \varepsilon) + \epsilon_k^{2\beta}\right\},$$

(6.44)

for a constant $C$ which depends on $\mu, \bar{a}, T, \chi_2$ and $\beta$. Summing up the estimates (6.43) and (6.44) and applying Grönwall’s inequality, we get

$$E\|X^{k,\phi} - X^{\phi}\|_\infty^2 \leq C\left\{\epsilon_k^2 + z_k(\delta, p, \lambda, H, \varepsilon) + \epsilon_k^{2\beta}\right\},$$
for every $k \geq 1$. Following the same argument as described in Proposition 6.1, we arrive at the following result. Let $\tilde{I}_k$ be the Cramer-Legendre transform of $\min\{\tau^1, \tau^2\}$, for an iid pair $(\tau^j)_{j=1}^{2}$ of random variables with distribution $\inf\{t > 0; |Y(t)| = 1\}$ for a real-valued standard Brownian motion $Y$.

**Proposition 6.3.** Fix $0 < H < \frac{1}{2}$. For a given $\beta \in (0, 1)$, $\zeta > 1$, $p > 2$, $0 < \varepsilon < H$ and a pair $(\delta, \lambda)$ as described in Corollary 6.2, there exists a constant $C = C(\mu, \partial, T, \beta, \zeta, \chi_2, p, H, \varepsilon, \bar{a})$ such that

$$\sup_{\phi \in \mathcal{F}^{k,\phi}_0(\omega, T)} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty^2 \leq C \left\{ \frac{2^2}{k} + z_k(\delta, p, \lambda, H, \varepsilon) + \exp \left[ - \frac{c(k, T)}{\zeta} \tilde{I}_k \left( \chi_2 \left( 1 - \frac{\eta}{T} \right) \right) \right] + \eta \ln \left( \frac{2T}{\eta} \right) \right\},$$

for every $k \geq 1$ and $\eta \in (0, T)$ which can be chosen arbitrarily and independently from $k$. In particular, $(X, \mathcal{X})$ is a strongly controlled Wiener functional. Hence, Propositions 5.2 and 5.3 apply to the control problem driven by (6.35).

7. MAIN RESULTS

In this section, we provide a detailed analysis on the convergence rate of our scheme. The argument is divided into two parts. At first, we will prove it is possible to enlarge the set $U_{0}^{k,\epsilon(k, T)}$ to $\overline{U}_{0}^{k,\epsilon(k, T)}$ without affecting $V^k(0)$. Second, we will establish an equiconvergence result for the family of Brownian martingales parameterized by $U_{0}^{k}$. Finally, we invoke the property of the strongly controlled Wiener functionals as demonstrated in Section 5 for a variety of examples.

Since $T_{k}^n < \infty \ a.s$ for each $n \geq 1$, it is known (see e.g. Corollary 3.22 in [25]) that $(\hat{\Phi}_k)^{-1}(\mathcal{O}) = \mathcal{F}_{T_{k}^{n}}$, where $\mathcal{O}$ is the optional $\sigma$-algebra on $\Omega \times \mathbb{R}_+$ and $\Phi_k(\omega) := \Phi_k^m(\omega, T_{k}^n(\omega)) := (\omega, T_{k}^n(\omega)); \omega \in \Omega^*, j \geq 1$, where $\mathbb{P}(\Omega^*) = 1$. To keep notation simple, we choose a version of $\Phi_k^m$ defined everywhere and with a slight abuse of notation we write it as $\Phi_k$. Based on this fact, for each $\mathcal{A}$-valued and $\mathcal{F}_{T_{k}^{n}}$-measurable variable $v_\ell$; $\ell = e(k, T) - 1, \ldots, n$, there exists a list of Borel functions $\varphi_k^k : \Omega \times \mathbb{R}_+ \to \mathcal{A}$ which realizes

$$\varphi_k^k(v_\ell) = \psi_k^k(\Phi_k^m) \ a.s.,$$

for $\ell = e(k, T) - 1, \ldots, n$. In the sequel, in order to keep notation simple, we set $m = e(k, T)$. For a given $u^k \in U_{0}^{k,n}$ and a list of Borel functions $(\varphi_k^k)_{\ell=n}^{e(k, T)}$ realizing (7.1), we then define the map $\Xi_k^{u^k \otimes_\mathcal{A} \varphi_k^k} : \Omega \times \mathbb{R}_+^{m-n} \times \mathbb{S}^k_n \to \mathbb{H}^{k,n} \times \mathbb{H}^{k,m-n}$ by

$$\Xi_k^{u^k \otimes_\mathcal{A} \varphi_k^k}(\omega, x_n, \ldots, x_{m-1}, b_m^k),$$

where $\varphi_k^k$ is a pathwise representation for a structure $\mathcal{X} = ((X^k)_{k \geq 1}, \mathcal{G})$. Let us define

$$J_{m,n}^k := (\text{Id}, T_{k,n-1}, \ldots, T_{k,0}, \mathcal{A}_n^k),$$

where $\text{Id} : \Omega \to \Omega$ is the identity map. For each $\bar{\varphi}_k = (v_{n}^{k}, \ldots, v_{m-1}^{k}) \in \overline{U}_{0}^{k,m}$ represented by Borel functions $\varphi_n^k, \ldots, \varphi_{m-1}^k$ realizing (7.1) and $u^k \in U_{0}^{k,n}$, we can represent

$$\xi_{X^k}(u^k \otimes_\mathcal{A} \bar{\varphi}_k) = \xi \circ \gamma_{m}^k(\Xi_{m}^{u^k \otimes_\mathcal{A} \varphi_k^k}(J_{m,n}^k)); n = m - 1, \ldots, 0$$

where $\gamma_{m}^k$ is a pathwise representation for a structure $\mathcal{X} = ((X^k)_{k \geq 1}, \mathcal{G})$.

Let $H_{n,m}^k : \mathcal{B}(\Omega \times \mathbb{R}_+^{m-n} + 1) \times \mathbb{S}_n^k \to [0, 1]$ be the disintegration of $\mathcal{P} \circ \rho_{n,m}^k \text{ w.r.t } \mathcal{P}_{m}^k$ and let $\nu_{n,m}^k$ be the disintegration of $\mathcal{P}_{m}^k \text{ w.r.t } \mathcal{P}_{m}^k$ for $n = m - 1, \ldots, 1$. In the sequel, $\rho_{n,m}^k$ is the projection of $b_n^k$ onto the last $(r - n)$ components. The proof of the following result is elementary, so we omit the details.
Lemma 7.1. Let $u^k \in U_0^{k,n}$ and let $\bar{\varphi}^k = (\bar{v}_n^k, \ldots, \bar{v}_{m-1}^k) \in \overline{U}_n^{k,m}$ be a control associated with Borel functions $(\varphi_j^k)_{j=n-1}^{m-1}$ realizing \eqref{7.1} for $n = m-1, \ldots, 0$ and $m = e(k,T)$. If $\xi \in L^1(P)$, then $\xi \circ \gamma_m u^k \in L^1(P)$, then we can represent
\[
\mathbb{E}[\xi \in L^1(P) \mid F_{T_n}^k] = \int_{\mathbb{R}^k_{m-n}} \int_{\Omega \times \mathbb{R}^k_{m-n} \times \mathbb{R}^k_{m-n}} \xi \circ \gamma_m u^k \in L^1(P) \mid F_{T_n}^k]
\]
where $\tilde{u}^k \in \tilde{U}_0^{k,n}$ and $\bar{\varphi}^k \in \overline{U}_n^{k,m}$ are versions of $u^k$ and $\varphi^k$, respectively.

The following result shows the set of controls $U_0^{k,e(k,T)}$ and $U_0^{k,e(k,T)}$ are equivalent in a suitable sense.

**Lemma 7.2.** For $m = e(k,T)$ and $k \geq 1$, we have

\[
\sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \int_{\mathbb{R}^k_{m-n}} \xi \circ \gamma_m u^k \in L^1(P) \mid F_{T_n}^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k].
\]

**Proof.** Following the same argument employed in the proof of Lemma 4.3 we have

\[
\sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \int_{\mathbb{R}^k_{m-n}} \xi \circ \gamma_m u^k \in L^1(P) \mid F_{T_n}^k]
\]

We now want to check $\sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k]$ equals to the expression \eqref{7.2}. For a given control $u^k \in \tilde{U}_0^{k,m}$, let $g^k$ be a list of Borel functions such that

\[
g^k(A_j^k) = u^k_j; j = 0, \ldots, m-1.
\]

Let us define $\Xi_k^j : S_k \rightarrow \mathbb{H}^{k,j}$

\[
\Xi_k^j : \begin{cases}
(\bar{v}_j^k) & j = 0, \ldots, m-1, \\
(\bar{v}_{j-1}^k) & j = m.
\end{cases}
\]

where $1 \leq j \leq m$. Let $\bar{\varphi}^k = (\bar{v}_n^k, \ldots, \bar{v}_{m-1}^k) \in \overline{U}_n^{k,m}$ be a control associated with Borel functions $(\varphi_j^k)_{j=n-1}^{m-1}$ realizing \eqref{7.1} for $n = m-1, \ldots, 0$. Moreover, we define the map $\Xi_k^j : S_k \rightarrow \mathbb{H}^{k,j}$ given by

\[
(\Xi_k^j)_{\bar{v}_n^k} = (\Xi_k^j)_{\bar{v}_{m-1}^k} = \begin{cases}
(\bar{v}_n^k) & n = m-1, \\
(\bar{v}_{m-1}^k) & n = m-1.
\end{cases}
\]

for $\omega \in \Omega$, $x_n \in \mathbb{R}^{m-n}$ and $b_m \in S_k^m$. We define $\nabla^k(T_n^k, u^k) := \xi \in L^1(P)$ and

\[
\nabla^k(T_n^k, u^k) := \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k].
\]

with initial condition $\nabla^k(0) := \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k]$. By Lemma 7.2

\[
\nabla^k(T_n^k, u^k) := \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k] = \sup_{\varphi^k \in U_0^{k,m}} \mathbb{E}[\xi \circ \varphi^k].
\]
where \( \text{ess sup} \) is computed over the set of all Borel functions \( \varphi_k : \Omega \times \mathbb{R}_+ \to \mathbb{A}; \ell = m - 1, \ldots, n \) for \( n = m - 1, \ldots, 0 \). Now, we define \( \mathcal{V}_n^k : \mathbb{H}^{k,n} \to \mathbb{R} \) by

\[
\mathcal{V}_n^k(\varphi_n^k(A_n^k)) = \mathcal{V}_n^k(T_n, u^k)
\]

\( (7.4) \)

for \( n = m - 1, \ldots, 0 \), where \( (7.4) \) is due to \( (7.3) \). This concludes the proof of \( (7.2) \). \( \Box \)

### 7.1. Equiconvergence of martingales

Recall that \( \mathcal{S} \) is the set of all stepwise constant processes of the form

\[
Z = Z(0)\mathbb{1}_{[0]} + \sum_{i=1}^{q} Z_i \mathbb{1}_{[J_{i-1}, J_i]}; q \in \mathbb{N},
\]

where \( 0 = J_0 < J_1 < \cdots < J_q < \infty \) is a finite family of \( \mathcal{F} \)-stopping times and \( Z_i \) is an \( \mathcal{A} \)-valued \( \mathcal{F}_{J_{i-1}} \)-random element for \( i = 1, \ldots, q \). By Assumptions (A1-B1) (see Remark 5.1), we have

\[
V(0) = \sup_{\varphi \in \mathcal{S}} \mathbb{E}[\xi x(\varphi)] = \sup_{\varphi \in \bigcup_{i \geq 1} \mathcal{U}_0^{i,\varepsilon}(i, T)} \mathbb{E}[\xi x(\varphi)].
\]

Moreover, from Lemma 7.2, we know that

\[
V^k(0) = \sup_{\varphi^k \in \mathcal{U}_0^{k,\varepsilon}(k, T)} \mathbb{E}[\xi x^k(\varphi^k)] = \sup_{\varphi^k \in \mathcal{U}_0^{k,\varepsilon}(k, T)} \mathbb{E}[\xi x^k(\varphi^k)].
\]

Now, for each \( \varphi \in \mathcal{S} \) with representation \( (7.5) \), we define

\[
\mathbf{S}_t^k(\varphi) := \sum_{i=1}^{q} Z_i \mathbb{1}_{\{T_t^{k,+} \leq t \wedge T^{k}(i, T) < T_{J_{i}}^{k,+}\}},
\]

for \( t \in [0, T] \), where \( T_{J_{i}}^{k,+} := \min\{T_n^k; J_t \leq T_n^k\} \) is a finite \( \mathcal{F} \)-stopping time such that \( J_t \leq T_{J_{i}}^{k,+} \) a.s for \( \ell = 0, \ldots, q \). Then, \( \mathbf{S}_t^k(\varphi) \) is an \( \mathcal{F} \)-adapted càdlàg process which jumps only at the hitting times \( \{T_{J_{i}}^{k}; n \geq 1\} \). Let \( \mathcal{D}_r T \) be the space of càdlàg functions from \( [0, T] \) to \( \mathbb{R}^d \) equipped with the Skorohod metric

\[
d(f, g) := \inf \{\|\lambda - \text{Id}\|_{\infty} \vee \|f - g \circ \lambda\|_{\infty}; \lambda \in \Lambda\},
\]

where \( \Lambda \) is the set of all strictly increasing continuous functions from \( [0, T] \) onto \([0, T]\) and \( \text{Id} \) is the identity function.
**Lemma 7.3.**

\[
\left( \int_0^T \| f(t) - g(s) \|^2 ds \right)^{1/2} \leq (5 \| f \|_\infty \vee 5) (d(f, g) + d^2(f, g) + d^1(f, g)),
\]

for every \( f, g \in D_{r, T} \).

**Proof.** In the sequel, without any loss of generality, we set \( r = 1 \). Take \( f, g \in D_{1, T} \) with \( d(f, g) < \epsilon \).

Then, \( \| g \circ \lambda - f \|_\infty < \epsilon \) and \( \| \lambda - \text{Id} \|_\infty < \epsilon \) for some \( \lambda \in \Lambda \). Moreover,

\[
\| g \|_\infty = \| g \circ \lambda \|_\infty = \| g \circ \lambda - f + f \|_\infty
\]

(7.7)

\[
\leq \| f \|_\infty + \epsilon < \infty.
\]

Let us consider

\[
g^+(t) := \sup_{0 \leq s \leq t} g(s), \quad g^-(t) := g^+(t) - g(t); 0 \leq t \leq T.
\]

Then, \((g^-, g^+)\) is a pair of non-decreasing functions in \( D_{1, T} \). Then, there exist Lebesgue-Stieltjes measures \( \mu^\pm \) which realize

\[
\mu^\pm(0, t] = g^\pm(t) - g(0); 0 < t \leq T.
\]

Moreover,

(7.8)

\[
\| \mu^+ \|_{TV} \leq 2\| g \|_\infty \quad \text{and} \quad \| \mu^- \|_{TV} \leq 2\| g \|_\infty,
\]

where \( \cdot \|_{TV} \) denotes the total variation norm of a signed measure. Let us consider \( \mu := \mu^0 + \mu^+ - \mu^- \), where \( \mu^0 := g(0)\delta_0 \) and \( \delta_0 \) is the Dirac measure concentrated at zero. Clearly

\[
\mu(0, t] = g(t); 0 < t \leq T,
\]

where \( \mu(0] = g(0) \) and (7.8) yields

(7.9)

\[
\| \mu \|_{TV} \leq 5\| g \|_\infty.
\]

Let us consider

\[
Q(u, v, t) = \left( \mathbb{1}_{[0, \lambda(t)]}(u) - \mathbb{1}_{[0, t]}(u) \right) \left( \mathbb{1}_{[0, \lambda(t)]}(v) - \mathbb{1}_{[0, t]}(v) \right).
\]

By Fubini’s theorem, we can write

\[
\int_0^T |g(\lambda(t)) - g(t)|^2 dt = \int_{[0, T]^3} Q(u, v, t)\mu(du)\mu(dv)dt.
\]

Now, we observe

\[
|Q(u, v, t)| \leq |\mathbb{1}_{[0, \lambda(t)]}(u) - \mathbb{1}_{[0, t]}(u)|
\]

(7.10)

\[
\leq \mathbb{1}_{\{\lambda(t) < u \leq 1\}} + \mathbb{1}_{\{t < u \leq \lambda(t)\}} = \mathbb{1}_{\{\lambda^{-1}(u) < t \leq u\}} + \mathbb{1}_{\{u \leq \lambda^{-1}(u)\}},
\]

for every \( u, v, t \in [0, T] \). The equality in the right-hand side of (7.10) holds true because \( \lambda^{-1} \) is strictly increasing which allows us to state

\[
t < u \leq \lambda(t) \iff \lambda^{-1}(u) \leq t < u
\]
and
\[ \lambda(t) < u \leq t \iff u \leq t < \lambda^{-1}(u). \]
Then, (7.10) yields
\[
\int_0^T |Q(u, v, t)|dt \leq |u - \lambda^{-1}(u)|
\]
\[
\leq \sup_{0 \leq u \leq T} |u - \lambda^{-1}(u)|
\]
(7.11)
\[
= \sup_{0 \leq x \leq T} |x - \lambda(x)| < \epsilon,
\]
for every \( u, v \in [0, T] \). Hence, (7.7), (7.9), (7.11) yield
\[
\int_0^T |g(\lambda(t)) - g(t)|^2 dt = \int_{[0, T]} Q(u, v, t)\mu(du)\mu(dv)dt
\]
(7.12)
\[
\leq \epsilon\mu\|T\|_V \leq \epsilon(5\|g\|_\infty)^2 \leq 25\epsilon(\|f\|_\infty + \epsilon)^2.
\]
Of course, \( \|g \circ \lambda - f\|_{L^2([0, T])} \leq \|g \circ \lambda - f\|_\infty < \epsilon \) and hence (7.12) yields
\[
\|g - f\|_{L^2([0, T])} \leq \|g \circ \lambda - f\|_{L^2([0, T])} + \|g \circ \lambda - g\|_{L^2([0, T])}
\]
\[
< \epsilon + 5\sqrt{\epsilon}(\|f\|_\infty + \epsilon) \leq (5\|f\|_\infty \lor 5)(\epsilon + \epsilon^2 + \epsilon^2).
\]
Since \( \epsilon > d(f, g) \) is arbitrary, we conclude the proof.

For a given \( \phi \in \mathcal{S} \) of the form (7.5), let \( \phi_+(t) = \sum_{i=1}^r Z_{\ell-i} 1_{\{J_{\ell-i} \leq t < J_{\ell}\}}. \)

**Lemma 7.4.**
\[
d(S^k(\phi), \phi_+) \leq \max_{n \geq 1} \Delta T^k_n 1_{\{T^k_n \leq T\}} \text{ a.s.}
\]
for every \( \phi \in \mathcal{S} \).

**Proof.** Let \( J_n^* \leq T < J_{n+1}^* \). Let \( T^k_{J_{\ell}} := \max\{T^k_n; J_{\ell} \geq T^k_n\} \) for \( \ell = 0, \ldots, q \).

Let us define
\[
\lambda^T_n(t) := \begin{cases} J_{\ell} + \frac{T^k_{J_{\ell+1}} - T^k_{J_{\ell}}}{T^k_{J_{\ell+1}} - T^k_{J_{\ell}}} (t - T^k_{J_{\ell}}); & \text{if } T^k_{J_{\ell}} \leq t < T^k_{J_{\ell+1}}; \ell \leq n^* - 1, \\
J_{n^*} + t - T^k_{J_{n^*}}; & \text{if } T^k_{J_{n^*}} \leq t \leq T.
\end{cases}
\]
Then, clearly \( \lambda^T_n \in \Lambda \text{ a.s.} \) We observe
\[
\|\lambda^T_n - I\|_\infty \leq \max_{0 \leq \ell \leq n^*} |J_{\ell} - T^k_{J_{\ell}}| \leq \max_{0 \leq \ell \leq n^*} |T^k_{J_{\ell}} - T^k_{J_{\ell+1}}| \leq \max_{n \geq 1} \Delta T^k_n 1_{\{T^k_n \leq T\}} \text{ a.s}
\]
and
\[
\|S^k(\phi) - \phi_+ \circ \lambda^T_n\|_\infty = \max_{0 \leq \ell \leq n^*} \|Z_{\ell} - Z_{\ell}\| = 0 \text{ a.s.}
\]
This shows that \( d(S^k(\phi), \phi_+) \leq \max_{n \geq 1} \Delta T^k_n 1_{\{T^k_n \leq T\}} \text{ a.s.} \).

\( \square \)
For every \( k \)

\[
f = \sum_{i=1}^{q} z_{t-1} I_{\{a_{t-1} \leq t < a_t\}},
\]

we set \( f_- = f(0) I_{t=0} + \sum_{t=1}^{q} z_{t-1} I_{\{a_{t-1} < t \leq a_t\}} \). Lemma 7.4 allows us to state the following result.

**Proposition 7.1.** For a given \( \beta \in (0, 1) \), there exists a constant \( C \) which depends only on \( \alpha \) and \( \beta \) such that

\[
(7.13) \quad \sup_{\phi \in S} E \int_0^T \| S^k_{-t} (\phi) - \phi(t) \|^2 dt \leq C \varepsilon_k^{2\beta},
\]

for every \( k \geq 1 \). Moreover, \( S^k_{-t} (\phi) \in \bar{U}^k_{e(k,T)} \) for every \( \phi \in S \).

**Proof.** Fix \( \beta \in (0, 1) \). By applying Lemmas 7.3, 7.4 and 8.2 there exists a constant \( C_1 \) which depends on \( \beta \) and \( \alpha \) such that

\[
\sup_{\phi \in S} E \int_0^T \| S^k_{-t} (\phi) - \phi(t) \|^2 dt \leq C_1 \left( \frac{\varepsilon_k^{-2T}}{\chi d} \right)^{(1-\beta)},
\]

for every \( k \geq 1 \). There exists a constant \( C_2 \) depending on \( \beta \) such that

\[
\left( \frac{\varepsilon_k^{-2T}}{\chi d} \right)^{(1-\beta)} \leq C_2 \left( 1 + \varepsilon_k^{2(\beta-1)} \right),
\]

for every \( k \geq 1 \). Hence, there exists a constant \( C_3 \) which depends on \( \beta \) and \( \alpha \) such that

\[
\sup_{\phi \in S} E \int_0^T \| S^k_{-t} (\phi) - \phi(t) \|^2 dt \leq C_3 \varepsilon_k^{2\beta},
\]

for every \( k \geq 1 \). This shows (7.13). In order to check the second assertion, we observe if we denote

\[
\delta^k S^k_{t} (\phi) = \sum_{\ell=1}^{\infty} S^k_{T^k_{n}(\phi)} I_{\{ T^k_{n}(\phi) \leq t < T^k_{n+1}(\phi) \}}
\]

for \( 0 \leq t \leq T \), then

\[
\delta^k S^k_{t} (\phi) = S^k_{T^k_{J_{t-1}}(\phi)} = S^k_{T^k_{J_{t+1}}(\phi)}
\]

whenever \( T^k_{J_{t+1}} \leq T^k_{J_{t-1}} \leq t < T^k_{J_{n+1}} < T^k_{J_{n+1}} \). This shows that \( \delta^k S^k_{t} (\phi) \) is actually equal to \( S^k_{t} (\phi) \) over \([0, T \wedge T^k_{e(k,T)}]\). Hence, \( S^k_{-t} (\phi) \in \bar{U}^k_{e(k,T)} \) for every \( k \geq 1 \) and \( \phi \in S \). \(\square\)

We are now able to state the convergence rate of our scheme. The idea is the following. In one hand, Assumption A1 and Lemma 7.2 yield

\[
(7.14) \quad \left| \chi^k (0) - \sup_{\phi \in \bar{U}^k_{e(k,T)}} E [\chi X (\phi)] \right| \leq \| \xi \| \sup_{\phi \in \bar{U}^k_{e(k,T)}} E \| X^k (\phi) - X (\phi) \|_{\infty} = O(h_k),
\]

for every \( k \geq 1 \). In typical examples, \( (h_k) \) can be explicitly computed as demonstrated in Section 6.

On the other hand, for a given \( \beta \in (0, 1) \), Assumption A1-B1, Proposition 7.1 and Hölder’s inequality yield

\[
\left| E [\chi X (\phi)] - E [\chi (S^k_{-t} (\phi))] \right| \leq \| \xi \| E \| X (S^k_{t} (\phi)) - X (\phi) \|_{\infty}
\]
\[ \leq C \left( \epsilon_k^2 \left[ \frac{\epsilon_k^{2T}}{\chi_d^T} \right]^{(1-\beta)} \right)^{\frac{2}{\beta}} \]

\[ \leq C \epsilon_k^{2\gamma}, \]

for every \( k \geq 1 \), where \( C \) is a constant which depends on \( \beta, T, \xi \) and the constant appearing in Assumption B1, i.e., (2.3). Then,

\[ \mathbb{E}[\xi_X(\phi)] \leq C \epsilon_k^{2\gamma} + \mathbb{E}[\xi_X(S^k(\phi))], \]

for every \( \phi \in \mathcal{S} \) and \( k \geq 1 \). Hence,

\[ \sup_{\phi \in \mathcal{S}} \mathbb{E}[\xi_X(\phi)] \leq C \epsilon_k^{2\gamma} + \sup_{\psi \in \mathcal{U}^{k,\epsilon}(k,T)} \mathbb{E}[\xi_X(\psi)]; k \geq 1, \]

and, more importantly, (7.6) yields

\[ (7.15) \quad \left| V(0) - \sup_{\psi \in \mathcal{U}^{k,\epsilon}(k,T)} \mathbb{E}[\xi_X(\psi)] \right| \leq C \epsilon_k^{2\gamma}, \]

for every \( k \geq 1 \). Therefore, from (7.14) and (7.15), we have

\[ |V^k(0) - V(0)| = O(h_k + \epsilon_k^{2\gamma}), \]

for \( k \geq 1 \). We then arrive at the following results.

**Theorem 7.1.** Let \( V(t, u) = \text{ess} \sup_{\theta \in \mathcal{U}^T} \mathbb{E}[\xi_X(u \circ \theta)|\mathcal{F}_t]; 0 \leq t \leq T \) be the value process associated with a payoff \( \xi \) satisfying Assumption A1 with \( \gamma \)-Hölder regularity and a strongly controlled Wiener functional \((X, \mathcal{X})\) satisfying Assumption B1. Let \(( (V^k)_{k \geq 1}, \mathcal{D}) \) be the value process (4.3) associated with a controlled imbedded discrete structure \( \mathcal{X} = (X^k)_{k \geq 1}, \mathcal{D} \). If

\[ (7.16) \quad \sup_{\phi \in \mathcal{U}^{k,\epsilon}(k,T)} \mathbb{E}[\|X^k(\phi) - X(\phi)\|_\infty] = O(h_k); k \geq 1, \]

then, for a given \( \beta \in (0,1) \), we have

\[ (7.17) \quad |V^k(0) - V(0)| = O(h_k + \epsilon_k^{2\gamma}); k \geq 1. \]

For a given error bound \( \epsilon \geq 0 \), we can provide the rate of convergence of the payoff associated with the optimal control constructed via (4.21) and Proposition 4.2.

**Theorem 7.2.** Assume the pair \((\xi, X)\) satisfies Assumptions (A1-B1) and \((X, \mathcal{X})\) is a strongly controlled Wiener functional associated with an imbedded discrete structure \( \mathcal{X} = (X^k)_{k \geq 1}, \mathcal{D} \) satisfying (7.16). For a given error bound \( \epsilon \geq 0 \) and \( k \geq 1 \), let \( \phi^{*,\epsilon,k} \) be an \( \epsilon \)-optimal control associated with the control problem (4.7) driven by \( \mathcal{X} \) and which can be constructed via (4.21) and Proposition 4.2. Then, \( \phi^{*,\epsilon,k} \in U^T_0 \) is a near optimal control for the stochastic control problem (2.6). More precisely, if \( (h_k)_{k \geq 1} \) is the order of convergence of \( V^k \) in (7.16), then for \( \beta \in (0,1) \) and \( 0 < \gamma \leq 1 \), we have

\[ (7.18) \quad \left| \sup_{u \in \mathcal{U}^T_0} \mathbb{E}[\xi_X(u)] - \mathbb{E}[\xi_X(\phi^{*,\epsilon,k})] \right| = O(h_k + \epsilon_k^{2\gamma} + \epsilon); k \geq 1. \]

**7.2 Applications.** We now apply Theorems 7.1 and 7.2 to the examples of Section 6. Proposition 6.1 and 6.2 present the rates associated with \( h_k \) in (7.17) and (7.18) as follows: for the control problem based on the path-dependent SDE (6.2), we have

\[ h_k = \left\{ \epsilon_k^\beta + \exp \left[ \frac{\epsilon(k,T)^2}{\zeta} \left( \chi_d \left( 1 - \frac{\eta}{T} \right) \right) \right] + \eta \ln \left( \frac{2T}{\eta} \right) \right\}^{\frac{2}{\beta}}; \quad k \geq 1, \]
Lemma 8.2. For every $\beta$ such that $\eta > 0$ can be chosen arbitrarily and independently from $k \geq 1$ and $\beta, \zeta$ and $\gamma$ are suitable constants as described in Proposition 6.1. For the control problem based on the path-dependent SDE (6.25) driven by FBM with $\frac{1}{2} < H < 1$, we have

$$h_k = \epsilon_k^{1-2\lambda}; \quad k \geq 1,$$

for $H - \frac{1}{2} < \lambda < \frac{1}{2}$. For the control problem based on the path-dependent SDE (6.25) driven by FBM with $0 < H < \frac{1}{2}$, we have

$$h_k = \epsilon_k^{2(H-\epsilon)} + \epsilon_k^{(1-2\lambda)+2(\delta-1)} + \epsilon_k^{2(H-\frac{1}{2}+\lambda)+2(\delta-1)}; \quad k \geq 1,$$

for suitable constants $\epsilon, \lambda, \delta$ as described in Proposition 6.2. For the control problem based on the rough stochastic volatility model (6.35) driven by FBM with $H \in (0, \frac{1}{2})$, we have

$$h_k = \left\{ \epsilon_k^{2\beta} + z_k(\delta, p, \lambda, H, \epsilon) + \exp \left[ - \frac{e(k, T) \widetilde{\chi}_2(1 - \eta \frac{T}{T})}{\zeta} \right] + \eta \ln \left( \frac{2T}{\eta} \right) \right\}^{\frac{1}{2}}; \quad k \geq 1,$$

where $\eta > 0$ can be chosen arbitrarily and independently from $k \geq 1$ and $\beta, \zeta, \gamma, p, \epsilon, \lambda, \delta$ are suitable constants as described in Proposition 6.3.

Remark 7.1. We stress our error bounds depend on the driving Brownian motion and state dimensions via $\chi_d$ and $(K_1, L_{\beta}, K_2, L_{\beta})$ in Assumption (C1), respectively. In particular, they growth linearly w.r.t driving Brownian motion dimension in the form $\frac{1}{\chi_d}$. In this direction, see Remark 4.2.

Remark 7.2. If the driving Brownian motion dimension equals to one, then there exists an analytical expression for $\tilde{I}_*$ (see e.g [7]) given by

$$\tilde{I}_*(x) = \sup_{\lambda > 0} \left[ \lambda x - \ln \left( \frac{1}{\cosh(\sqrt{2}|\lambda|)} \right) \right]; \quad x < 1.$$

In higher dimensions, we do not know closed form expressions of the Cramer-Legendre transform $\tilde{I}_*$.

8. Appendix A. Random mesh and large deviations

At first, we recall the following elementary inequality (see the Supplementary material of [39]).

Lemma 8.1. Let $Z_1, \ldots, Z_n$ be an i.i.d. sequence of absolutely continuous positive random variables. Then, for every $\beta \in (0, 1)$ and $r \geq 1$, we have

$$\mathbb{E} \left[ \sum_{i=1}^{n} Z_i^r \right]^\beta \leq \left( \mathbb{E} \left[ Z_1^{\beta(1-\beta)} \right] \right)^{(1-\beta)n}1^{1-\beta}.$$

Lemma 8.2. For every $q \geq 1$ and $\beta \in (0, 1)$, there exists a constant $C$ which depends on $q \geq 1$ and $\beta$ such that

$$\mathbb{E} \left[ \sup_{n \geq 1} \Delta T_{n, k}^{\beta, q} \mathbb{1}_{[T_n \leq T]} \right] \leq C \left( \epsilon_k^{2\beta} \frac{T^2}{\chi_d} \right)^{(1-\beta)},$$

for every $k \geq 1$.

Proof. The proof is almost identical to Lemma 2.2 in [39]. For sake of completeness, we give the details. We start by noticing (see (3.2)) that $\Delta T_{n, k}^{\beta, q} = \min_{j \in \{1, 2, \ldots, d\}} \{ \Delta T_{n, j}^{\beta, q} \}$ a.s where $\Delta T_{n, j}^{\beta, q} \equiv \epsilon_k^{2\beta} \tau_{n, j}$ for an iid sequence of random variables $(\tau_{n, j})_{j \geq 1}$ with distribution equals to $\inf\{ t > 0; |W^j(t)| = 1 \}$ for a sequence of independent real-valued Brownian motions $(W^j)_{j=1}^d$. Let $N^k(t) = \max\{ n; T_n \leq t \}; 0 \leq t \leq T$. We observe there exists $\lambda > 0$ such that

$$\varphi(\lambda) := \mathbb{E} \exp(\lambda \min\{ \tau_1^{(1)}, \ldots, \tau_1^{(d)} \}) \leq \varphi(\lambda) := \mathbb{E} \exp(\lambda \tau_1^{(1)}) = \frac{1}{\cosh(\sqrt{2}\lambda)} < \infty.$$
Let $\tilde{\psi}(\lambda) := \ln \tilde{\varphi}(\lambda)$ defined on $\{ \lambda \in \mathbb{R}; \tilde{\varphi}(\lambda) < \infty \}$ and the Cramer transform is defined by $\tilde{I}_s(a) := \sup_{\lambda < 0} |\lambda a - \tilde{\psi}(\lambda)|; a < \chi_d$. Similarly, we set $\psi(\lambda) := \ln \varphi(\lambda)$ defined on $\{ \lambda \in \mathbb{R}; \varphi(\lambda) < \infty \}$ and the Cramer transform is defined by $I_s(a) := \sup_{\lambda < 0} |\lambda a - \psi(\lambda)|; a < 1$. Since $\tilde{\psi}(\lambda) \geq \psi(\lambda)$ for $\lambda < 0$, then $\tilde{\psi}(\lambda) \geq \psi(\lambda); \lambda < 0$ and hence

$$\lambda a - \tilde{\psi}(\lambda) \leq \lambda a - \psi(\lambda),$$

for every $\lambda < 0$ and $a < \chi_d \leq 1$. Then, we get $\tilde{I}_s(a) \leq I_s(a)$ for every $a < \chi_d \leq 1$. Now, we split

$$\nabla_{n=1}^{N_k(T)} \Delta T_n \begin{cases} \nabla_{n=1}^{N_k(T)} \Delta T_n \mathbb{I}_{\{N_k(T) < 2e(k,T)\}} \\
abla_{n=1}^{N_k(T)} \Delta T_n \mathbb{I}_{\{N_k(T) \geq 2e(k,T)\}}. \end{cases}$$

Hence,

$$\mathbb{E} \left| \nabla_{n=1}^{N_k(T)} \Delta T_n \right|^q \leq 2C\mathbb{E} \left| \nabla_{n=1}^{2e(k,T)} \Delta T_n \right|^q + C T^q \mathbb{P}\{N_k(T) \geq 2e(k,T)\} =: I_1^k + I_2^k,$$

where we recall $e(k,T) = \left[ \frac{\chi_d^2 T}{\chi_d} \right]$. Clearly,

$$\mathbb{P}\{N_k(T) \geq 2e(k,T)\} = \mathbb{P}\{T_{2e(k,T)}^k \leq T\}.$$

We can write

$$T_n^k = \epsilon^2 \sum_{\ell=1}^n \alpha_\ell,$$

where $\alpha_\ell \overset{d}{=} \min\{\tau_1^d, \ldots, \tau_d^d\}$. Then, $(\alpha_\ell)_{\ell=1}^\infty$ is an iid sequence with mean $\chi_d$ given by (3.11). In particular,

$$\mathbb{P}\{T_{2e(k,T)}^k \leq T\} = \mathbb{P}\left\{ \frac{2e(k,T)}{2e(k,T)} \sum_{\ell=1}^{2e(k,T)} \alpha_\ell \leq T \right\} = \mathbb{P}\left\{ \frac{1}{2e(k,T)} \sum_{\ell=1}^{2e(k,T)} \alpha_\ell \leq \frac{T}{2e(k,T)\epsilon_k^2} \right\} \leq \mathbb{P}\left\{ \frac{1}{2e(k,T)} \sum_{\ell=1}^{2e(k,T)} \alpha_\ell \leq \frac{\chi_d}{2} \right\} \leq \exp \left( -2e(k,T)\tilde{I}_s \left( \frac{\chi_d}{2} \right) \right).$$

Let us fix $\beta \in (0,1)$. Take $n = 2e(k,T)$ in Lemma 8.1 and notice that

$$\left( \mathbb{E}[|\Delta T_n^k|^{q/(1-\beta)}] \right)^{1-\beta} = \epsilon_k^{2q} \left( \mathbb{E}[\min\{\tau_1^d, \ldots, \tau_d^d\}^{q/(1-\beta)}] \right)^{1-\beta} :\! = : C k^{2q},$$

where $C$ is a constant depending on $\beta$ and $q$. Therefore, by applying Lemma 8.1 we have

$$I_1^k \leq 2^{-\beta} C k^{2q} e(k,T)^{(1-\beta)}; k \geq 1.$$

This concludes the proof.

Let $\tilde{I}_s(a) := \sup_{\lambda < 0} |\lambda a - \tilde{\psi}(\lambda)|; a < \chi_d$ be the Cramer-Legendre transform of $\min\{\tau_1^d, \ldots, \tau_d^d\}$, where $\tilde{\psi}$ is given by (8.1).
Moreover, classical large deviation principle yields
\[ E \]  
(8.4)  
and
(8.3)  
\[ P \{ T_{e(k,t)}^k < t - \delta \} \leq \exp \left[ - e(k,t) \bar{I}_s \left( \chi_d \left( 1 - \frac{\delta}{t} \right) \right) \right]; k \geq 1. \]

Moreover, there exists a constant \( C \) which depends on \( p \geq 1 \) and \( \chi_d \) such that
\[ E |t - T_{e(k,t)}^k|^p \leq C e_k^{2p}; k \geq 1. \]

**Proof.** Fix \( t \in (0, T] \). In the sequel, we use the notation employed in the proof of Lemma 8.2. Assertion (8.2) is a consequence of the strong law of large numbers and the fact that \( \lim_{k \to +\infty} e(k,t) e_k^2 \chi_d = t \). Moreover, classical large deviation principle yields
\[ P \{ T_{e(k,t)}^k \leq t - \delta \} = P \left\{ \sum_{\ell=1}^{e(k,t)} \alpha_{\ell t} \leq t - \delta \right\} \]
\[ = P \left\{ \frac{1}{e(k,t)} \sum_{\ell=1}^{e(k,t)} \alpha_{\ell t} \leq \frac{t - \delta}{e(k,t) e_k^2} \right\} \]
\[ \leq P \left\{ \frac{1}{e(k,t)} \sum_{\ell=1}^{e(k,t)} \alpha_{\ell t} \leq \chi_d \left( 1 - \frac{\delta}{t} \right) \right\} \]
\[ \leq \exp \left( - e(k,t) \bar{I}_s \left( \chi_d \left( 1 - \frac{\delta}{t} \right) \right) \right). \]

This shows (8.3). Now, we write \( E |T_{e(k,t)}^k - t|^p = E |e_k^{-2} (T_{e(k,t)}^k - t)|^p = e_k^{2p} E |e_k^{-2} (T_{e(k,t)}^k - t)|^p \) and we claim that \( \sup_{k \geq 1} E |e_k^{-2} (T_{e(k,t)}^k - t)|^p < \infty \). We want to check
\[ \sup_{k \geq 1} \int_{T_{e(k,t)}^k > t} \left[ e_k^{-2} (T_{e(k,t)}^k - t) \right]^p dP + \sup_{k \geq 1} \int_{T_{e(k,t)}^k \leq t} \left[ e_k^{-2} (t - T_{e(k,t)}^k) \right]^p dP < \infty. \]

Since \( \left[ e_k^{-2} (T_{e(k,t)}^k - t) \right]^p I_{(T_{e(k,t)}^k > t)} \) is a positive random variable, we have
\[ \int_{T_{e(k,t)}^k > t} \left[ e_k^{-2} (T_{e(k,t)}^k - t) \right]^p dP = \int_0^\infty P \left\{ \left[ e_k^{-2} (T_{e(k,t)}^k - t) \right]^p I_{(T_{e(k,t)}^k > t)} > s \right\} ds \]
\[ = \int_0^\infty P \{ T_{e(k,t)}^k - t > e_k^2 s^{1/p}, T_{e(k,t)}^k > t \} ds \]
\[ \leq \int_0^\infty P \{ T_{e(k,t)}^k > t + e_k^2 s^{1/p} \} ds \]
\[ = \int_0^{+\infty} P \left\{ \frac{1}{e(k,t)} \sum_{j=1}^{e(k,t)} \alpha_{j t} > \frac{t + e_k^2 s^{1/p}}{e_k^2 e(k,t)} \right\} ds \]
\[ \leq \int_0^{+\infty} P \left\{ \frac{1}{e(k,t)} \sum_{j=1}^{e(k,t)} \alpha_{j t} > \chi_d \frac{t + e_k^2 s^{1/p}}{t + e_k^2} \right\} ds \]
\[ \leq \int_1^\infty \exp \left( -e(k,t)\tilde{I}_s \left( \frac{t + e_k^2 s^{1/p}}{t + e_k^2 |x_d|} \right) \right) ds + 1, \]

where \( s \mapsto \tilde{I}_s \left( \frac{t + e_k^2 s^{1/p}}{t + e_k^2 |x_d|} \right) \) is positive strictly increasing on \([1, \infty)\). Recall that \( \tilde{I}_s \) is increasing on \([|x_d|, \infty)\) and \( \tilde{I}_s(x_d) = 0 \). The analysis for the second term in (8.5) is similar. This shows (8.4). \( \square \)

9. Appendix B. Transition Probabilities

In this section, we provide a closed form expression for the transition probabilities (3.5) given by

\[ \nu^k(E) = \mathbb{P}\left\{ (\Delta T^k_n, \eta^k_n) \in E \right\} \]

\[ = \mathbb{P}\{(\Delta T^k_{n+1}, \eta^k_{n+1}) \in E | A^k_n = b^k_n \}, \]

For \( E \in \mathcal{B}(\mathbb{S}_k) \) and \( b^k_n \in \mathbb{S}^n_k; n, k \geq 1 \). Even though \( \nu^k \) does not depend on \( b^k_n \), we adopt the strategy of conditioning to facilitate computation in the derivation of the formula. In the sequel, \( f_r \) is the density function of \( \tau = \inf \{ t > 0; |Y(t)| = 1 \} \) for a standard Brownian motion \( Y \). Formula for \( d = 1 \) is a simple consequence of the strong Markov property of the Brownian motion. For simplicity, we present the formula for \( d = 2 \). The argument for \( d \geq 2 \) is similar. In the sequel, we consider generic Borel sets of the form \( \{(a,b) \times \{ \pm \epsilon_k \} \times (y, \bar{y})\} \) and \( \{(a,b) \times (y, \bar{y}) \times \{ \pm \epsilon_k \}\} \) for arbitrary open sets \( (a,b) \) and \( (y, \bar{y}) \subset (-\epsilon_k, \epsilon_k) \). Clearly, this class of sets generates the Borel sigma algebra of \( \mathbb{S}_k \).

At first, for each non-negative integer \( n, a < b \) and \( b^k_n \in \mathbb{S}^n_k \), we clearly have

\[ \mathbb{P}\left\{ \Delta T^k_{n+1} \in (a,b) | A^k_n = b^k_n, \eta^k_{n+1} \in \{ \pm \epsilon_k \} \times (y, \bar{y}) \right\} = \int_{\epsilon_k}^{\bar{\epsilon}_k} f_r(x) dx, \]

because \( \Delta T^k_{n+1} \) is independent of \( A^k_n \). Moreover,

\[ \mathbb{P}\left\{ \eta^k_{n+1} = \epsilon_k | A^k_n = b^k_n \right\} = \mathbb{P}\left\{ \eta^k_{n+1} = -\epsilon_k | A^k_n = b^k_n \right\} = \frac{1}{2d}, \]

\[ \mathbb{P}\left\{ \eta^k_{n+1} = \epsilon_k | A^k_n = b^k_n \right\} = \mathbb{P}\left\{ \eta^k_{n+1} = -\epsilon_k | A^k_n = b^k_n \right\} = \frac{1}{2d}. \]

We observe the regular conditional probabilities are well-defined so that

\[ \mathbb{P}\left\{ (\Delta T^k_{n+1}, \eta^k_{n+1}) \in ((a,b) \times \{ \pm \epsilon_k \} \times (y, \bar{y})) | A^k_n = b^k_n \right\} = \mathbb{P}\left\{ \Delta T^k_{n+1} \in (a,b) | A^k_n = b^k_n, \eta^k_{n+1} \in \{ \pm \epsilon_k \} \times (y, \bar{y}), A^k_n = b^k_n \right\} \]

\[ \times \mathbb{P}\left\{ \eta^k_{n+1} \in \{ \pm \epsilon_k \} \times (y, \bar{y}) | A^k_n = b^k_n \right\}, \]

\[ \mathbb{P}\left\{ \eta^k_{n+1} \in \{ \pm \epsilon_k \} \times (y, \bar{y}) | A^k_n = b^k_n \right\} = \mathbb{P}\left\{ \eta^k_{n+1} = \epsilon_k, \eta^k_{n+1} \in (y, \bar{y}) | A^k_n = b^k_n \right\} \]

\[ \times \mathbb{P}\left\{ \eta^k_{n+1} \in (y, \bar{y}) | \eta^k_{n+1} = \epsilon_k, A^k_n = b^k_n \right\}, \]

where the last identity is due (9.2). Therefore, (9.1) and (9.2) yield

\[ \mathbb{P}\left\{ (\Delta T^k_{n+1}, \eta^k_{n+1}) \in (a,b) \times \{ \pm \epsilon_k \} \times (y, \bar{y}), A^k_n = b^k_n \right\} = \frac{1}{4} \mathbb{P}\left\{ \Delta T^k_{n+1} \in (a,b) | \eta^k_{n+1} \in \{ \pm \epsilon_k \} \times (y, \bar{y}), A^k_n = b^k_n \right\} \]

\[ \times \mathbb{P}\left\{ \eta^k_{n+1} \in (y, \bar{y}) | \eta^k_{n+1} = \epsilon_k, A^k_n = b^k_n \right\}. \]
for a positive constant $c$. It is a well-known fact that the FBM can be represented w.r.t a Brownian motion $B$ which will play a key role in constructing an imbedded discrete structure (in the sense of [39]) for FBM and its functionals.

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We stress the choice of the hitting times

Remark 9.1.

(9.4)

In order to conclude the proof, we only need to compute $\mathbb{P}\left\{\eta_{n+1}^{2} \in (y, \bar{y}) \mid \eta_{n+1}^{1} = \pm \epsilon_k, A_n^k = b_n^k\right\}$ whose proof is elementary and it is left to the reader.

Lemma 9.1. For each non-negative integer $n, j \in \{1, 2\}$ and $b_n^k \in S_n^k$, we have

$$\mathbb{P}\left\{\eta_{n+1}^{2} \in (y, \bar{y}) \mid \eta_{n+1}^{1} = \pm \epsilon_k, A_n^k = b_n^k\right\} = \epsilon_k^{-2} \int_y^{\bar{y}} \int_0^\infty \frac{1}{\sqrt{2\pi t}} \frac{\exp\left(-\frac{x^2}{2t}\right)}{2\Phi\left(\frac{\epsilon_k}{\sqrt{t}}\right)} f_r(\epsilon_k^2 t) dt dx.$$

From Lemma 9.1 and identity (9.3), we are able to state the following formula. In the sequel, $\Phi$ is the cumulative distribution function of the standard Gaussian distribution.

Proposition 9.1. Assume the underlying Brownian motion is $d$-dimensional with $d = 2$. For each $a < b$, $\epsilon_k > 0$ and $-\epsilon_k < y < \bar{y} < \epsilon_k$, we have

$$\nu^k((a, b) \times \{\pm \epsilon_k\} \times (y, \bar{y})) = \epsilon_k^{-2} \int_y^{\bar{y}} \int_0^\infty \frac{1}{\sqrt{2\pi t}} \frac{\exp\left(-\frac{x^2}{2t}\right)}{2\Phi\left(\frac{\epsilon_k}{\sqrt{t}}\right)} f_r(\epsilon_k^2 t) dt dx$$

(9.4)

$$\times \frac{1}{2a} \int_{\epsilon_k^{-2} a}^{\epsilon_k^{-2} b} f_r(x) dx; k \geq 1.$$

Remark 9.1. We stress the choice of the hitting times $\{T_n^k; n \geq 1\}$ differ from [39]. In the present work, we adopt (3.1) rather than Th 2.1 in [4] for $j \in \{1, \ldots, d\}$. This allows us to reduce the complexity of the discretization scheme and, in contrast to Th 2.1 in [4], the transition probabilities are homogeneous in time.

10. Appendix C. A pathwise representation of FBM with $H \in (0, \frac{1}{2})$ and its imbedded discrete structure

In the sequel, we will derive a pathwise FBM representation for $0 < H < \frac{1}{2}$ which will play a key role in constructing an imbedded discrete structure (in the sense of [39]) for FBM and its functionals. It is a well-known fact that the FBM can be represented w.r.t a Brownian motion $B$ as follows

$$B_H(t) := \int_0^t K_H(t, s) dB(s); 0 \leq t \leq T,$$

where $K_H$ is a deterministic kernel described by

$$K_H(t, s) := c_H \left[ t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t - s)^{H - \frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2} - H} \int_s^t u^{H-\frac{1}{2}} (u - s)^{H - \frac{1}{2}} du \right]; 0 < s < t,$$

for a positive constant $c_H$.

Let us define
\[ K_{H,1}(t,s) := c_H t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H}, \]
\[ K_{H,2}(t,s) := c_H \left( \frac{1}{2} - H \right) s^{\frac{1}{2} - H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} du, \]
for \(0 < s < t\). By definition, \( K_H(t,s) = K_{H,1}(t,s) + K_{H,2}(t,s)\); \(0 < s < t\). By making change of variables \( v = \frac{u}{s} \), we can write

\[(10.1) \quad \int_s^t u^{\frac{1}{2}} (u-s)^{H-\frac{1}{2}} du = \theta_H(t,s) s^{2H-1},\]
where \( \theta_H(t,s) := \int_1^t u^{(H-\frac{1}{2})} (v-1)^{(H-\frac{1}{2})} dv \), for \(0 < s \leq t\). We observe

\[(10.2) \quad \int_1^{+\infty} v^{(H-\frac{1}{2})} (v-1)^{(H-\frac{1}{2})} dv < \infty,\]
for \(0 < H < \frac{1}{2}\). Therefore,

\[ K_{H,2}(t,s) = c_H \left( \frac{1}{2} - H \right) s^{H-\frac{1}{2}} \int_1^t v^{(H-\frac{1}{2})} (v-1)^{(H-\frac{1}{2})} dv \]
and

\[(10.3) \quad \partial_s K_{H,1}(t,s) = c_H \left( \frac{1}{2} - H \right) \left[ t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H} + t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H} \right],\]
\[ \partial_s K_{H,2}(t,s) = c_H \left( \frac{1}{2} - H \right) \left[ -t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H} + \left( H - \frac{1}{2} \right) s^{H-\frac{1}{2}} \right].\]

for \(0 < s < t\). Hence,

\[ \partial_s K_H(t,s) = c_H \left( \frac{1}{2} - H \right) \left[ t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H} + \left( H - \frac{1}{2} \right) s^{H-\frac{1}{2}} \theta_H(t,s) \right] \]
\[ = c_H \left( \frac{1}{2} - H \right) \left[ t^{H-\frac{1}{2}} s^{\frac{1}{2} - H} (t-s)^{\frac{1}{2} - H} + \left( H - \frac{1}{2} \right) s^{H-\frac{1}{2}} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} du \right].\]

In the sequel, if \( f : [0,T] \rightarrow \mathbb{R} \), we denote

\[ \|f\|_\lambda := \sup_{0 < r < s \leq T} \frac{|f(r) - f(s)|}{|r-s|^\lambda}, \]
for \(0 < \lambda \leq 1\). Let \( \mathcal{C}_0^\lambda \) be the space of all Hölder continuous functions \( f : [0,T] \rightarrow \mathbb{R} \) with \( f(0) = 0 \) equipped with the norm \( \| \cdot \|_\lambda \). For each \( f \in \mathcal{C}_0^\lambda \), we define

\[ (\mathcal{A}_H f)(t) := \int_0^t [f(t) - f(s)] \partial_s K_{H,1}(t,s) ds - \int_0^t \partial_s K_{H,2}(t,s) f(s) ds; 0 \leq t \leq T. \]

**Lemma 10.1.** If \( \frac{1}{2} - H < \lambda < \frac{1}{2} \), there exists a constant \( C \) which depends on \( H \) such that

\[ \sup_{0 \leq t \leq T} |(\mathcal{A}_H f)(t)| \leq CT^{H-\frac{1}{2} + \lambda} \| f \|_\lambda, \]

for every \( f \in \mathcal{C}_0^\lambda \).
Proof. In the sequel, $C$ is a constant which may differ from line to line and we fix $\frac{1}{2} - H < \lambda < \frac{1}{2}$.

We observe

$$|(A_H f)(t)| \leq \left( \int_0^t |\partial_t K_{H,1}(t, s)||f(t) - f(s)||ds + \int_0^t |\partial_s K_{H,2}(t, s)||f(s)||ds \right),$$

and

$$\int_0^t |\partial_t K_{H,1}(t, s)||f(t) - f(s)||ds \leq C t^{H - \frac{1}{2}} \|f\|_{L^\infty} \left[ \int_0^t s^{\frac{1}{2} - H}(t - s)^{H - \frac{1}{2}} ds \right]$$

$$+ \left[ \int_0^t s^{\frac{1}{2} - H}(t - s)^{H - \frac{1}{2}} ds \right]$$

$$\leq C t^{H - \frac{1}{2}} \|f\|_{L^\infty} 2t^\lambda$$

$$\leq C \|f\|_{L^\infty} t^{H + \lambda - \frac{1}{2}} \leq C \|f\|_{L^\infty} T^{H + \lambda - \frac{1}{2}}.$$

Similarly,

$$\int_0^t |\partial_s K_{H,2}(t, s)(t, s)||f(s)||ds \leq \left[ \int_0^t \left( \frac{H + \lambda - \frac{1}{2}}{H - \frac{1}{2}} \right) u^{\lambda - H - \frac{1}{2}}(u - s)^{H - \frac{1}{2}} ds \right]$$

$$\leq C \|f\|_{L^\infty} T^{H + \lambda - \frac{1}{2}}.$$

This concludes the proof.

In order to deal with singularities, for a given $\epsilon > 0$, we set

$$K'_H(t, s) := K_{H,1}(t, s) + K_{H,2}(t, s),$$

$$K_{H,1}(t, s) := c_H t^{H - \frac{1}{2}}(s + \epsilon)^{\frac{1}{2} - H}(t - s + \epsilon)^{H - \frac{1}{2}},$$

$$K_{H,2}(t, s) := c_H \left( \frac{1}{2} - H \right) (s + \epsilon)^{H - \frac{1}{2}} \int_1^{s + \epsilon} v^{H - \frac{1}{2}}(v - 1)^{H - \frac{1}{2}} dv.$$

We set $K_{H,2}(t, s) = 0$ if $s + \epsilon \geq t$. A direct computation shows that

$$\partial_s K_{H,1}(t, s) = c_H \left( \frac{1}{2} - H \right) \left[ t^{H - \frac{1}{2}}(s + \epsilon)^{H - \frac{1}{2}}(t - s + \epsilon)^{H - \frac{1}{2}} + t^{H - \frac{1}{2}}(s + \epsilon)^{H - \frac{1}{2}}(t - s + \epsilon)^{H - \frac{1}{2}} \right],$$

for $0 < s < t$. Moreover,

$$\partial_s K_{H,2}(t, s) = c_H \left( \frac{1}{2} - H \right) \left[ \left( H - \frac{1}{2} \right)(s + \epsilon)^{H - \frac{1}{2}} \int_1^{s + \epsilon} v^{H - \frac{1}{2}}(v - 1)^{H - \frac{1}{2}} dv \right.$$

$$- t^{H - \frac{1}{2}}(t - s - \epsilon)^{H - \frac{1}{2}}(s + \epsilon)^{- \frac{1}{2} - H} \right].$$
for $0 < s < t - \epsilon$ and $\partial_s K_{H,i}^r(t,s) = 0$ for $s > t - \epsilon$. We observe $s \mapsto \partial_s K_{H,i}^r(t,s)$ is integrable over $[0,t-\epsilon]$ for each $i = 1, 2$. Hence, $s \mapsto K_{H,i}^r(t,s)$ is absolutely continuous over $[0,t-\epsilon]$ for each $i = 1, 2$.

Lemma 10.2. For each $t \in (0,T]$, 
\[
\lim_{\epsilon \downarrow 0} K_{H}^r(t,\cdot)\mathbb{I}_{[0,t-\epsilon]} = K_H(t,\cdot) \text{ in } L^2([0,t]).
\]

Moreover,
\[
\lim_{\epsilon \downarrow 0} \left( B(t) - B(\cdot) \right) \partial_s K_{H,1}^r(t,\cdot)\mathbb{I}_{[0,t-\epsilon]} = \left( B(t) - B(\cdot) \right) \partial_s K_{H,1}^r(t,\cdot)
\]
and
\[
\lim_{\epsilon \downarrow 0} B(\cdot) \partial_s K_{H,2}^r(t,\cdot)\mathbb{I}_{[0,t-\epsilon]} = B(\cdot) \partial_s K_{H,2}^r(t,\cdot),
\]
in $L^1(\Omega \times [0,t])$ for each $0 < t \leq T$.

Proof. In the sequel, $C$ is a constant which may differ from line to line. Let us fix $t \in (0,T)$. We observe that
\[
\lim_{\epsilon \downarrow 0} \partial_s K_{H,1}^r(t,s)\mathbb{I}_{[0,t-\epsilon]}(s) = \partial_s K_{H,1}^r(t,s) \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \partial_s K_{H,2}^r(t,s)\mathbb{I}_{[0,t-\epsilon]}(s) = \partial_s K_{H,2}^r(t,s),
\]
for each $s \in (0,t)$. Moreover,
\[
\left| (B(t) - B(s)) \partial_s K_{H,1}^r(t,s) \right| \leq C\|B\|_x \left[ t^{H - \frac{1}{2}} s^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2} + \lambda} + t^{H - \frac{1}{2}} (T + 1)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2} + \lambda} \right]; 0 < s < t \leq T,
\]
almost surely for every $0 < \epsilon < 1$ for some $0 < \lambda < \frac{1}{2}$. By recalling
\[
\int_a^b (s-a)^{u-1}(b-s)^{v-1}ds = (b-a)^{u+v-1}Beta(u,v), \quad \text{Re}(u) > 0, \quad \text{Re}(v) > 0,
\]
and $\mathbb{E}\|B\|_x < \infty$ for every $p > 1$, we observe the right-hand side of (10.7) belongs to $L^1(\Omega \times [0,t])$ provided that $\frac{1}{2} - H < \lambda < \frac{1}{2}$. This shows (10.5).

We claim that $\{ B(\cdot) \partial_s K_{H,2}^r(t,\cdot)\mathbb{I}_{[0,t-\epsilon]}; 0 < \epsilon < 1 \}$ is uniformly integrable over $[0,t]$. Indeed, we observe
\[
\left| B(s) \partial_s K_{H,2}^r(t,s)\mathbb{I}_{[0,t-\epsilon]}(s) \right| \leq |\partial_s K_{H,2}^r(t,s)| |B(s)|
\]
\[
+ \|B\|_x t^{H - \frac{1}{2}} s^{-\frac{1}{2} - H + \lambda} \int_0^t (t - s - \epsilon)^{\frac{1}{2} - H} ds \mathbb{I}_{[0,t-\epsilon]}(s) \text{ a.s.}
\]
We claim there exists $p > 1$ such that
\[
\sup_{0 < \epsilon < 1} \int_0^{t-\epsilon} \frac{s^{-\frac{1}{2} - H + \lambda}}{(t-s-\epsilon)^{\frac{1}{2} - H} ds} < \infty.
\]
Indeed, take $p, \alpha, \beta > 1$ and $\frac{1}{2} - H < \lambda < \frac{1}{2}$ such that
\[
\lambda + \frac{1}{p\alpha} > H + \frac{1}{2}, \quad 1 < p\beta < \frac{1}{\frac{1}{2} - H}.
\]
By using Hölder’s inequality,
\[ \int_0^{t-\epsilon} \frac{s^{(-\frac{1}{2} - H + \lambda)p}}{(t-s-\epsilon)^{\frac{1}{2} - H + \lambda)}p} ds \leq \left( \int_0^t s^{(-\frac{1}{2} - H + \lambda)p_0} ds \right)^{\frac{1}{p}} \left( \int_0^t (t-s-\epsilon)^{(H-\frac{1}{2})p_0} ds \right)^{\frac{1}{p}} \]
\[ \leq CT^{(H-\frac{1}{2})(\beta-1)}(\epsilon^{\frac{1}{2} - H + \lambda(p+\alpha^{-1})) \forall \epsilon \in (0,1), \]
for a constant C which depends on \(\lambda, H, p, \alpha\) and \(\beta\). This shows (10.9). By using (10.4) and (10.9) into (10.8), we conclude (10.6). Finally, there exists a constant C (which only depends on \(H\)) such that

\[ (10.10) \quad |K_H(t,s)| \leq C|K_H(t,s)| + C\left\{\epsilon^{\frac{1}{2} - H + \frac{1}{2}} + (t-s)^{H-\frac{1}{2}}\right\}; 0 < s < t, \]
for every \(\epsilon > 0\), where the right-hand side of (10.10) belongs to \(L^2([0, t])\). This concludes the proof. \(\square\)

We are now able to prove a pathwise representation for the FBM with \(0 < H < \frac{1}{2}\).

**Theorem 10.1.** If \(B\) is a real-valued standard Brownian motion, then

\[ B_H = (A_H B) \]

is a FBM with exponent \(0 < H < \frac{1}{2}\).

**Proof.** Starting from \(B_H(t) = \int_0^t K_H(t,s)dB(s)\), the proof is an almost immediate consequence of Lemmas 10.1, 10.2 and a routine use of integration by parts formula. For sake of completeness, we give the details. We fix \(t \in (0, T]\) and a Brownian motion \(B\). For \(0 < \epsilon < 1\), \(|K_H'(t,t-\epsilon)| + |K_H'(t,0)| < \infty \) and integration by parts yields

\[ [K_H'(t,\cdot), B](t-\epsilon) = K_H'(t, t-\epsilon)B(t-\epsilon) - K_H'(t,0)B(0) - \int_0^{t-\epsilon} B(s)\partial_s K_H'(t,s)ds - \int_0^{t-\epsilon} K_H'(t,s)dB(s). \]

In other words,

\[ \int_0^{t-\epsilon} K_H'(t,s)dB(s) = K_H'(t,t-\epsilon)B(t-\epsilon) - \int_0^{t-\epsilon} B(s)\partial_s K_H'(t,s)ds \]
\[ = K'_{H,1}(t, t-\epsilon)B(t-\epsilon) + K'_{H,2}(t, t-\epsilon)B(t-\epsilon) + \int_0^{t-\epsilon} [B(t) - B(s)]\partial_s K_H'(t,s)ds \]
\[ - B(t) \int_0^{t-\epsilon} \partial_s K'_{H,1}(t,s)ds - \int_0^{t-\epsilon} B(s)\partial_s K'_{H,2}(t,s)ds \]
\[ = \int_0^{t-\epsilon} [B(t) - B(s)]\partial_s K'_{H,1}(t,s)ds - \int_0^{t-\epsilon} B(s)\partial_s K'_{H,2}(t,s)ds \]
\[ + |B(t-\epsilon) - B(t)|K'_{H,1}(t, t-\epsilon) + K'_{H,2}(t, t-\epsilon)B(t-\epsilon) + B(t)K'_{H,1}(t,0) \text{ a.s.} \]

We observe

\[ (10.12) \quad \lim_{\epsilon \downarrow 0} K'_{H,1}(t,0) = \lim_{\epsilon \downarrow 0} K'_{H,2}(t, t-\epsilon) = 0 \]

for \(t > 0\). Moreover, if \(\frac{1}{2} > \lambda > \frac{1}{2} - H\), then

\[ |[B(t-\epsilon) - B(t)]K'_{H,1}(t, t-\epsilon)| \leq c_Ht^{H-\frac{1}{2}}(t-\epsilon)^{H-\frac{1}{2}}(t-\epsilon)^{H-\frac{1}{2}}\|B\|\lambda e^\lambda \]
as \( \epsilon \downarrow 0 \). Lemmas \((10.1)\) and \((10.2)\) and \((10.11)\), \((10.12)\), \((10.13)\) allow us to conclude the proof. \( \square \)

Recall \((6.8)\) and \((6.20)\). By construction, \( t_k \leq \bar{t}_k^+ \) a.s for each \( t \geq 0 \). Let us define

\[
B^k_H(t) := \int_0^{\bar{t}_k} \partial_s K_{H,1}(t_k, s)[A^k(t_k) - A^k(s_k)]ds - \int_0^{\bar{t}_k} \partial_s K_{H,2}(t_k, s)A^k(s)ds; 0 \leq t \leq T.
\]

Clearly, \( B^k_H \) is a pure jump \( \mathbb{F}^k \)-adapted process of the form

\[
B^k_H(t) = \sum_{n=0}^{\infty} B^k_H(T^k_n) \mathbb{1}_{\{T^k_n \leq t < T^k_{n+1}\}}; 0 \leq t \leq T.
\]

Let us define

\[
\|A^k - B\|_{-,\lambda} := \sup_{0 \leq t < t_k \leq t \leq T} \frac{|A^k(s_k) - B(s)|}{(t_k - s)^{\lambda}},
\]

\[
\|A^k - B\|_{T^k_1,\lambda} := \sup_{T^k_1 \leq t \leq s \leq T} \frac{|A^k(s) - B(s)|}{s^{\lambda}}.
\]

**Lemma 10.3.** If \( \frac{1}{2} - H < \lambda < \frac{1}{2} \) and \( 0 < \epsilon < H \), then there exists a constant \( C \) which only depends on \( H \) such that

\[
\|B^k_H - B_H\|_{\infty} \leq C \|A^k - B\|_{-,\lambda} T^{H-\frac{1}{2}+\lambda} + C \|B\|_{\lambda}(\max_{n \geq 1} \Delta T^k_n)^{H-\frac{1}{2}+\lambda} \mathbb{1}_{\{T^k_n \leq T\}}
\]

\[
+ \ C \left( \|B\|_{\lambda} (T^k_1 \wedge T)^{\lambda+H-\frac{1}{2}} + \|A^k - B\|_{T^k_1,\lambda} T^{\lambda+H-\frac{1}{2}} \right)
\]

\[
+ \ \|B_H\|_{H-\epsilon} (\max_{n \geq 1} \Delta T^k_n \mathbb{1}_{\{T^k_n \leq T\}})^{H-\epsilon} \ a.s.,
\]

for every \( k \geq 1 \).

**Proof.** In the sequel, \( C \) is a constant which may differ from line to line and we fix \( \frac{1}{2} - H < \lambda < \frac{1}{2}, 0 < \epsilon < H \). First of all, we observe

\[
|B^k_H(t) - B_H(t)| \leq |B^k_H(t) - B_H(t_k)| + |B_H(t_k) - B_H(t)|
\]

\[
\leq \|B^k_H(t) - B_H(t_k)\| + \|B_H\|_{H-\epsilon} |t_k - t|^{H-\epsilon}
\]

\[
\leq |B^k_H(t) - B_H(t_k)| + \|B_H\|_{H-\epsilon} (\max_{n \geq 1} \Delta T^k_n \mathbb{1}_{\{T^k_n \leq T\}})^{H-\epsilon} \ a.s.
\]

Then,

\[
(10.14) \quad \|B^k_H - B_H\|_{\infty} \leq \sup_{0 \leq t \leq T} |B^k_H(t) - B_H(t_k)| + \|B_H\|_{H-\epsilon} (\max_{n \geq 1} \Delta T^k_n \mathbb{1}_{\{T^k_n \leq T\}})^{H-\epsilon} \ a.s.
\]

To keep notation simple, we denote
\( \varphi^k(t, s) := A^k(t) - A^k(s^+) - (B(t) - B(s)) \)
\( \varphi^k(s) := A^k(s) - B(s) \)
\( \|A^k - B\|_{T^+} := \sup_{t_k \leq s < t_k \leq t \leq T} \frac{|B(t_k) - A^k(t) - B(s) + A^k(s^+)|}{(t - s)^\lambda} = \frac{\sup_{t_k \leq s < t_k \leq t \leq T} |A^k(s^+) - B(s)|}{(t - s)^\lambda} \)
\( \|A^k - B\|_{T^-} := \sup_{0 \leq s \leq T^\lambda \wedge T} \frac{|A^k(s) - B(s)|}{s^\lambda} \).

At first, we observe \( \|A^k - B\|_{+\lambda} \leq \|B\| \lambda \) a.s and \( \|A^k - B\|_{T^- \lambda} \leq \|B\| \lambda \) a.s for every \( k \geq 1 \). Furthermore, we have

\[
(10.15) \quad \sup_{0 \leq t \leq T} |B_H^k(t) - B_H(t)| = \sup_{T^\lambda \leq \omega \leq T} \left| \int_0^{t_k} \partial_s K_{H,1}(\tilde{t}, s) \varphi^k(\tilde{t}, s) ds - \int_0^{t_k} \partial_s K_{H,2}(\tilde{t}, s) \varphi^k(s) ds \right|
\]
\[
\leq \sup_{T^\lambda \leq \omega \leq T} \int_0^{t_k} |\partial_s K_{H,1}(\tilde{t}, s)| |\varphi^k(\tilde{t}, s)| ds + \sup_{T^\lambda \leq \omega \leq T} \int_0^{t_k} |\partial_s K_{H,2}(\tilde{t}, s)| |\varphi^k(s)| ds a.s.
\]

We observe
\[
\int_0^{t_k} |\partial_s K_{H,1}(\tilde{t}, s)| |\varphi^k(\tilde{t}, s)| ds \leq \int_0^{t_k} \|A^k - B\|_{-\lambda} |\partial_s K_{H,1}(\tilde{t}, s)| (\tilde{t} - s)^\lambda ds + \int_0^{t_k} \|A^k - B\|_{+\lambda} |\partial_s K_{H,1}(\tilde{t}, s)| (\tilde{t} - s)^\lambda ds
\]
\[
\leq C \|A^k - B\|_{-\lambda} (\tilde{t} - s)^{H - \frac{1}{2}} \int_0^{t_k} s^{-\frac{1}{2} - H} (\tilde{t} - s)^{H - \frac{1}{2} + \lambda} ds + C \|A^k - B\|_{+\lambda} (\tilde{t} - s)^{H - \frac{1}{2}} \int_0^{t_k} s^{\frac{1}{2} - H} (\tilde{t} - s)^{H - \frac{1}{2} + \lambda} ds
\]
\[
+ C \|A^k - B\|_{-\lambda} (\tilde{t} - s)^{H - \frac{1}{2}} \int_0^{t_k} s^{-\frac{1}{2} - H} (\tilde{t} - s)^{H - \frac{1}{2} + \lambda} ds + C \|A^k - B\|_{+\lambda} (\tilde{t} - s)^{H - \frac{1}{2}} \int_0^{t_k} s^{\frac{1}{2} - H} (\tilde{t} - s)^{H - \frac{1}{2} + \lambda} ds
\]
\[
=: I_1^k(\tilde{t}, s) + I_2^k(\tilde{t}, s) + I_3^k(\tilde{t}, s) + I_4^k(\tilde{t}, s) a.s.
\]

The following estimates hold true a.s

\[
I_1^k(\tilde{t}, s) \leq C \|A^k - B\|_{-\lambda} (\tilde{t} - s)^{H - \frac{1}{2}} \int_0^{t_k} s^{-\frac{1}{2} - H} (\tilde{t} - s)^{H - \frac{1}{2} + \lambda} ds
\]
\[
\leq C \|A^k - B\|_{-\lambda} T^{H - \frac{1}{2} + \lambda},
\]
At first, we notice

\[ I_k^1(t_k, s) \leq C\|A^k - B\|_{-\lambda} (t_k) H^{-\frac{1}{2}} \int_0^{t_k} s^{-\frac{1}{2} - H} (t_k - s)^{H-\frac{3}{2} + \lambda} ds \leq C\|A^k - B\|_{-\lambda} T H^{-\frac{1}{2} + \lambda}, \]

and

\[
I_k^1(t_k, s) + I_k^2(t_k, s) \leq C\|A^k - B\|_{+\lambda} (t_k) H^{-\frac{1}{2}} \left( \int_0^{t_k} s^{-\frac{1}{2} - H} (t_k - s)^{H-\frac{3}{2} + \lambda} ds + \int_{\tilde{t}_k}^{t_k} (s - \tilde{t}_k)^{-\frac{1}{2} - H} (t_k - s)^{H-\frac{3}{2} + \lambda} ds \right)
\]

\[= C\|A^k - B\|_{+\lambda} (t_k) H^{-\frac{1}{2}} \left( (t_k) \left( \frac{1}{2} - H \right) (\Delta t_k)^{H-\frac{3}{2} + \lambda} + (\Delta t_k)^\lambda \right) \]

\[= C\|A^k - B\|_{+\lambda} (t_k) H^{-\frac{1}{2}} \left( (t_k) \left( \frac{1}{2} - H \right) (\Delta t_k)^{H-\frac{3}{2} + \lambda} + (\Delta t_k) \left( \frac{1}{2} - H \right) (\Delta t_k)^{H-\frac{3}{2} + \lambda} \right) \]

\[\leq C\|A^k - B\|_{+\lambda} (t_k) H^{-\frac{1}{2}} \frac{1}{2} 2(t_k) \left( \frac{1}{2} - H \right) (\Delta t_k)^{H-\frac{3}{2} + \lambda} \]

\[(10.18) \leq C\|B\|_{\lambda} (\max_{m \geq 1} \Delta T_n^k) H^{-\frac{1}{2} + \lambda} \mathbb{1}_{\{T_n^k \leq T\}}, \]

where \( \Delta t_k := t_k - t_k^- \). Summing up (10.16), (10.17) and (10.18), we arrive at the following estimate

\[(10.19) \sup_{T_1^k \leq t \leq T} \int_0^{t_k} |\partial_{\alpha} K_{H, 1}(\tilde{t}_k, s)| \varphi^k(\tilde{t}_k, s)| ds \leq C\|A^k - B\|_{-\lambda} T H^{-\frac{1}{2} + \lambda} + C\|B\|_{\lambda} (\max_{m \geq 1} \Delta T_n^k) H^{-\frac{1}{2} + \lambda} \mathbb{1}_{\{T_n^k \leq T\}}, \]

almost surely for every \( k \geq 1 \). Let us now estimate the second term in the right-hand side of (10.15). At first, we notice

\[ \sup_{T_1^k \leq t \leq T} \int_0^{t_k} |\partial_{\alpha} K_{H, 2}(\tilde{t}_k, s)| \varphi^k(\tilde{t}_k, s)| ds \leq C \sup_{T_1^k \leq t \leq T} (t_k) H^{-\frac{1}{2}} \int_0^{t_k} s^{-\frac{1}{2} - H}(t_k - s)^{H-\frac{3}{2}} |\varphi^k(s)| ds \]

\[+ C \sup_{T_1^k \leq t \leq T} \int_0^{t_k} s^{-\frac{1}{2} - H} \int_s^{t} u^{-\frac{1}{2}} (u - s)^{H-\frac{3}{2}} du |\varphi^k(s)| ds \]

\[=: J_{k, 1} + J_{k, 2} a.s., \]

where
The following result is an immediate consequence of Lemmas 2 and 3 in [6].

**Lemma 10.4.** If $\tau = \inf\{t > 0; |Y(t)| = 1\}$ for a one-dimensional Brownian motion $Y$, then for every $q > 0$,
In particular this shows (10.26). We observe (10.28) and (10.27) exist a constant 

\[ C > 0 \]

Proof. Let \( f \) be the density of \( \tau = \inf \{ t > 0; |Y(t)| = 1 \} \). We recall \( f \) has exponential tails for \( x \to +\infty \). By Lemma 3 in [6], for every \( m \geq 1 \), \( f(x) = o(x^m) \) as \( x \to 0 \). This shows [10.24].

**Lemma 10.5.** For every \( 0 < \lambda < \frac{1}{2} \), we have

\[
\sup_{0 \leq s < t_k \leq T} \frac{|A^k(s^+ - B(s))|}{(t_k - s)\lambda} \leq 2 \|B\|_\lambda \text{ a.s}
\]

and

\[
\sup_{0 < s \leq T} \frac{|B(s) - A^k(s)|}{s^\lambda} \leq \|B\|_\lambda \text{ a.s},
\]

for every \( k \geq 1 \). Moreover, for each \( \lambda, \delta, p \) satisfying \( 0 < \lambda < \frac{1}{2} + \frac{2\delta - 2}{2p} \), \( 0 < \delta < 1 \) and \( p \geq 1 \), there exists a constant \( C > 0 \) depends on \( \lambda \) and \( \delta \), such that

\[
E\|A^k - B\|_{-\lambda}^p \leq C e^{p(1-2\lambda)} \left[ \epsilon_k^{2-p} \right]^{1-\delta},
\]

\[
E \left( \sup_{t_k^- < s < t_k \leq T} \frac{|A^k(s^+ - B(s))|}{(t_k - s)\lambda} \right)^p \leq 2^p E\|B\|_\lambda^p,
\]

and

\[
E\|A^k - B\|_{T^\lambda}^p \leq C e^{p(1-2\lambda)},
\]

for every \( k \geq 1 \).

Proof. At first, we recall that \( (T_n^k)_{n \geq 1} \) is a sequence of totally inaccessible \( \mathbb{P}^k \)-stopping times so that \( \mathbb{P}\{s_k < s\} = 1 \) for each \( s > 0 \). Since \( \|B\|_\lambda < \infty \text{ a.s} \), then we can find a set of full probability such that

\[
\sup_{0 \leq s \leq T} \frac{|B(s) - A^k(s)|}{s^\lambda} \leq \sup_{0 \leq s \leq T} \frac{|B(s) - A^k(s)|}{(s - s_k)^\lambda} = \sup_{0 \leq s \leq T} \frac{|B(s) - B(s_k)|}{(s - s_k)^\lambda} \leq \|B\|_\lambda \text{ a.s},
\]

and

\[
\sup_{0 \leq s < t_k \leq T} \frac{|A^k(s^+ - B(s))|}{(t_k - s)\lambda} \leq \sup_{0 \leq s < t_k \leq T} \frac{|B(s) - A^k(s^+) - B(s)|}{(t_k - s)\lambda} + \sup_{t_k^- < s < t_k \leq T} \frac{|B(s) - A^k(s^+) - B(s)|}{(t_k - s)\lambda} \text{ a.s.}
\]

In particular this shows [10.26]. We observe

\[
\sup_{0 \leq s \leq t_k^- \leq T} \frac{|B(s) - A^k(s^+) - B(s)|}{(t_k - s)\lambda} \leq \sup_{0 \leq s \leq t_k^- \leq T} \frac{|B(s) - A^k(s^+) - B(s)|}{(s_k^+ - s)\lambda} \leq \|B\|_\lambda \text{ a.s.}
\]

Now, \( \{t_k^- < s < t_k\} \subset \{A^k(s^+) = B(t_k)\} \text{ a.s.} \). Then,
\begin{equation}
\sup_{t_k^- < s < t_k^+} \frac{|B(s) - A^k(s_k^+)|}{(t_k - s)\lambda} = \sup_{t_k^- < s < t_k^+} \frac{|B(t_k) - B(s)|}{(t_k - s)\lambda} \leq \|B\|_\lambda \text{ a.s.}
\end{equation}

From (10.30) and (10.31), we conclude (10.25). Notice if \(0 \leq t \leq t_k^-\), then \((t_k - s) \geq t_k - t_k^- = \Delta t_k\). One can easily check Lemmas [8.1] and [8.2] also hold for negative exponents. By applying Lemma 10.4 jointly with such estimates, we have

\begin{align}
\mathbb{E}\|A^k - B\|_{T_k^+, \lambda}^p &\leq \mathbb{E} \left( \sup_{0 \leq s \leq t_k^+} \frac{|B(s) - A^k(s_k^+)|}{(\Delta t_k)\lambda} \right)^p \\
&\leq \epsilon_k^p \mathbb{E} \left( \max_{n \geq 1} \frac{1}{\Delta T_n^k} \mathbb{I}_{(T_k^+ \leq T)} \right)^{\lambda p} \\
&\leq C \left( \epsilon_k^{p - 2}\lambda \epsilon_k^{-2T} \right)^{1 - \delta},
\end{align}

for a constant \(C\) which depends on \(\delta, p, \lambda\), where \(0 < \lambda < \frac{1}{2} + \frac{2\delta - 2}{2p}, 0 < \delta < 1\) and \(p \geq 1\). This shows (10.27). Now, we observe

\begin{equation}
\left( \sup_{T_k^+ < t \leq T} \frac{|A^k(s) - B(s)|}{s^\lambda} \right)^p \leq \epsilon_k^p (T_k^+)^{-\lambda p} \text{ a.s.,}
\end{equation}

for every \(k \geq 1\). By definition, \(T_k^+ = \inf \{t > 0; |B(t)| = \epsilon_k \} \equiv s_k^+\tau_k\), where \(\tau\) is given in Lemma 10.4. Then, (10.24) yields

\[ \mathbb{E}(T_k^+)^{-\lambda p} \leq C\epsilon_k^{-2\lambda p}, \]

for a constant \(C\) which depends on \(\lambda\). We then get

\[ \mathbb{E}\|A^k - B\|_{T_k^+, \lambda}^p \leq C\epsilon_k^{p(1 - 2\lambda)}, \]

which shows (10.29). Assertion (10.28) is a consequence of (10.25). This concludes the proof. \[\square\]

**Theorem 10.2.** Fix 0 < \(H < \frac{1}{2}\), \(p \geq 1\), \(0 < \varepsilon < H\) and a pair \((\delta, \lambda)\) such that \(\delta \in (0, 1), \lambda \in (\frac{1}{2} - H + \frac{2\delta - 2}{2p}, \frac{1}{2} + \frac{2\delta - 2}{2p})\). Then, there exists a constant \(C\) which depends on \(p, \delta, H, T, \lambda, \varepsilon\) such that

\[ \mathbb{E}\|B^k_H - B_H\|_{\infty}^p \leq C \left( \epsilon_k^{p(1 - 2\lambda) + 2(\delta - 1)} + \epsilon_k^{2p(H - \frac{1}{2} + \lambda) + 2(\delta - 1)} + \epsilon_k^{2p(H - \varepsilon)} \right), \]

for every \(k \geq 1\).

**Proof.** In the sequel, \(C\) is a constant which may differ from line to line. Let us fix 0 < \(H < \frac{1}{2}\). By Lemma 10.3 if \(\frac{1}{2} - H < \lambda < \frac{1}{2}\) and 0 < \(\varepsilon < H\), then there exists a constant \(C\) which depends on \(H, T\) and \(p \geq 1\) such that

\begin{align*}
\|B^k_H - B_H\|_{\infty}^p &\leq C\|A^k - B\|_{T_k^+, \lambda}^p + C\|B\|_{\infty}^p \left( \max_{m \geq 1} \Delta T_m^k \right)^{p(H - \frac{1}{2} + \lambda)} \mathbb{I}_{(T_k^+ \leq T)} \\
&\quad + C\|B\|_{T_k^+, \lambda}^{p(\lambda + H - \frac{1}{2})} + C\|A^k - B\|_{T_k^+, \lambda}^p \\
&\quad + \|B_H\|_{H - \varepsilon}^p \left( \max_{m \geq 1} \Delta T_m^k \mathbb{I}_{(T_k^+ \leq T)} \right)^{p(H - \varepsilon)} \text{ a.s.,}
\end{align*}
for every $k \geq 1$. Now, for each $\lambda, \delta, p$ satisfying $0 < \lambda < \frac{1}{2} + \frac{2p}{3p}, 0 < \delta < 1, p \geq 1$ and $0 < \varepsilon < H$, we make use of the Gaussian tails of the Brownian motion and FBM jointly with Lemmas 10.5 and 8.2 to get a constant $C > 0$ which depends on $H, T, p, \lambda, H - \varepsilon$ and $\delta$ such that
\[
\mathbb{E}\|B^k_H - B_H\|_{\infty} \leq C\epsilon_k^{p(1-2\lambda)} \left[ \epsilon_k^2 T \right]^{1-\delta} + C\epsilon_k^{2p(H-\frac{1}{2}+\lambda)} \left[ \epsilon_k^2 T \right]^{1-\delta} + C\epsilon_k^{2p(H-\varepsilon)}\]
for every $k \geq 1$. Finally, by noticing that $|x| \leq 1 + x$ for every $x \geq 0$, we conclude the proof.

11. Appendix D. Pseudocode and a numerical example

In this section, we explain how the methodology can be implemented in a concrete simple example. At first, we present a pseudo-code for a given stochastic control problem. For simplicity, we set the dimension of the Brownian motion equals to 2.

**Algorithm 1:** Pseudocode for a near optimal control specified in Propositions 4.2, 4.3 and (4.21).

**Data:** Level of discretization $\epsilon_k$, number of periods $m := \left\lceil \frac{\bar{a}^2 T}{\lambda^2} \right\rceil, \bar{a} > 0$ and a Bernoulli-(1/2) distribution $Q$ with support in $\{-1, 1\}$.

**Result:** Vector of optimal control $u^k$ given by (4.21).

1. **Initialization:**
2. for $\ell \leftarrow 1$ to $m$ do
3. \begin{align*}
&\text{Generate } \Delta T^k_{\ell - 1} \text{ and } T^k_{\ell - 1} \text{ according to the algorithm described in [6].} \\
&\text{Compute } \Delta T^k_{\ell} = \min\{\Delta T^k_{\ell - 1}, \Delta T^k_{\ell - 2}\}.
\end{align*}
4. if $\min\{\Delta T^k_{\ell - 1}, \Delta T^k_{\ell - 2}\} = \Delta T^k_{\ell - 1}$ then
5. \begin{align*}
&\text{generate } \Delta T^k_{\ell} \overset{d}{=} Q \epsilon_k \text{ and } \Delta A^{k,1}(T^k_{\ell}) \overset{d}{=} z, \text{ where } z \text{ is a truncated normal distribution with parameters } (0, T^k_{\ell - 1}, -\epsilon_k, \epsilon_k); \\
&\text{generate } \Delta A^{k,2}(T^k_{\ell}) \overset{d}{=} Q \epsilon_k \text{ and } \Delta A^{k,2}(T^k_{\ell}) \overset{d}{=} z, \text{ where } z \text{ is a truncated normal distribution with parameters } (0, T^k_{\ell - 1}, -\epsilon_k, \epsilon_k);
\end{align*}
6. \begin{align*}
&\text{Generate } \phi^k_{\ell} \text{ according to a uniform distribution on } [-\bar{a}, \bar{a}].
\end{align*}
7. Store the information $\mathcal{T} = \{T^k_{\ell}, \ell = 1 \ldots m\}$ and $\eta^k_{\ell} = (\Delta A^{k,1}(T^k_{\ell}), \Delta A^{k,2}(T^k_{\ell}))$ and $\phi^k_{\ell}$.
8. With the information set generated in previous step, calculate $\{X^k(T^k_{\ell}, \phi^k); \ell = 1, \ldots, m\}$ as a function of $\mathcal{T}, \eta^k_{\ell}$ and $\phi^k_{\ell}$. Store $\phi^k_{\ell} := (\Delta T^k_{\ell}, \eta^k_{\ell}, \phi^k_{\ell}, \cdots, \Delta T^k_{\ell}, \eta^k_{\ell}, \phi^k_{\ell})$ for $\ell = 1, \ldots, m$.
9. for $\ell \leftarrow m$ to 1 do
10. \begin{align*}
&\text{Starting with } \forall^k_m (\phi^k_0) := \xi(\{X^k(T^k_{\ell}, \phi^k); \ell = 1, \ldots, m\}), \text{ solve backwards (11.1)}\]
\end{align*}
11. Let us now present a simple example to illustrate the theory developed in this article. We choose the example of hedging in a two-dimensional Black-Scholes model. This is a classical problem in Finance which can be briefly described as follows. For a given $c \in \mathbb{R}$ and a Lipschitz function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we define $g_c : \mathbb{R}^3 \to \mathbb{R}$ by
\[
g_c(x, y, z) := (c + x - \varphi(y, z))^2; (x, y, z) \in \mathbb{R}^3.
\]
Let us consider
\[ dS^1(t) = S^1(t) (\mu_1 dt + \sigma_1 dB^1(t)) \]
\[ dS^2(t) = S^1(t) (\mu_2 dt + \sigma_2 dB^2(t)) \]
where, for simplicity, we assume \([B^1, B^2] = 0, \mu_1 = \mu_2 = 0, T = 1\) and the riskless rate equals zero.

The problem is
\[ \begin{align*}
\text{minimize} \quad & E[\varrho_c(X(T, \phi), S^1(T), S^2(T))] \\
\text{over all } & \phi \in U^T_0, \ c \in \mathbb{R},
\end{align*} \]
where
\[ X(t, \phi) = \sum_{j=1}^{2} \int_0^t \phi_j(r) dS^j(r); \phi \in U^T_0, 0 \leq t \leq T, \]
and the controls \(\phi(t) = (\phi_1(t), \phi_2(t))\) represent the absolute percentages of the securities \((S^1, S^2)\) which an investor holds at time \(t \in [0, T]\). In this example, we choose \(\varphi(y, z) := \max(y-z, 0)\) and \(a = 1\). It is well-known there exists a unique choice of \((c^*, \phi^*) \in \mathbb{R} \times U^T_0\) such that
\[ \inf_{(c, \phi) \in \mathbb{R} \times U^T_0} E[\varrho_c(X(T, \phi), S^1(T), S^2(T))] = E[\varrho_{c^*}(X(T, \phi^*), S^1(T), S^2(T))] = 0, \]
where by Margrabe's formula, we have
\[ c^* = S_0^1 \Phi(d_1) - S_0^2 \Phi(d_2), \]
where
\[ \sigma = \sqrt{\sigma_1^2 + \sigma_2^2}, \quad d_1 = \frac{\log \left( \frac{S^1(0)}{S^2(0)} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}, \]
and \(\Phi\) is the cumulative distribution function of the standard Gaussian variable. We recall \(c^*\) is the price of the option and \(\phi^*\) is the so-called delta hedging which can be computed by means of the classical PDE Black-Scholes as a function of \(\Phi\).

We set \(S^1(0) = 49, S^2(0) = 52, \sigma_1 = 0.2, \sigma_2 = 0.3\) and \(\epsilon_k = 2^{-k}\). At first, for a given \(c \in \mathbb{R}\), we apply the algorithm described above to get a Monte Carlo optimal control approximation \(\phi^{k,*}\) related to \((11.2)\). In this particular case, we can analytically solve \((11.1)\) and the estimated value \(c^{k,*}\) is computed according to
\[ c^{k,*} \in \arg \min_{c \in \mathbb{R}} E \left[ T c^k \left( X^k(T, \phi^{k,*}), S^{k,1}(T \wedge T^{k}_{c(k,T)}), S^{k,2}(T \wedge T^{k}_{c(k,T)}) \right) \right]. \]

In other words, \(E \left[ T c^k \left( X^k(T, \phi^{k,*}), S^{k,1}(T \wedge T^{k}_{c(k,T)}), S^{k,2}(T \wedge T^{k}_{c(k,T)}) \right) \right] = 0\). Here
\[ X^k(t, \phi^k) = X^k(t, T^{k}_{c(k,T)}), X^k(t, \phi^k) = \sum_{j=1}^{2} \int_0^{T_k} \phi_j^k(r) dS^j(r); \phi^k \in U^k_{c(k,T)}, \]
and \(S^{k,i}(t) = S^{k,i}(t \wedge T^{k}_{c(k,T)}), \) where \(S^k = (S^{k,1}, S^{k,2})\) follows \((1.5)\) (without the presence of controls) with the coefficients \(\alpha^i(t, f) = \mu_i(f(t), \sigma_i^2), \sigma^{ij}(t, f) = \sigma_i f(t) \delta_{i,j}\) for \(i, j = 1, 2\), where \(\delta_{i,j}\) is the delta Dirac function concentrated at \(i = j\). Table \(1\) presents a comparison between the true call option price \(c^*\) and the associated Monte Carlo price \(c^{k,*}\). Figure 1 presents the Monte Carlo experiments for \(c^{k,*}\) with \(k = 1, 2\) and \(3\). The number of Monte Carlo iterations in the experiment is \(3 \times 10^4\).
Table 1. Comparison between $c^*$ and $c^{k,*}$ for $\epsilon_k = 2^{-k}$

| $k$ | Result | Mean Square Error | True Value | Difference | % Error |
|-----|--------|------------------|------------|------------|---------|
| 1   | 5.9740 | 0.01689567       | 5.821608   | 0.152458   | 0.0261% |
| 2   | 5.8622 | 0.01158859       | 5.821608   | 0.04059157 | 0.0069% |
| 3   | 5.7871 | 0.00821813       | 5.821608   | 0.03441365 | 0.0059% |

Figure 1. Monte Carlo experiments for $c^{k,*}$ with $\epsilon_k = 2^{-k}$

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References

[1] Bayer, C., Friz, P. K., Gassiat, P., Martin, J. and Stemper, B. (2019). A regularity structure for rough volatility. *Math. Finance*, 1-51.

[2] Barles, G. and Souganidis, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4, 271-283.

[3] Bertsekas, D. P. and Shreve, S. *Stochastic optimal control: The discrete-time case*. Athena Scientific Belmont, Massachusetts, 1996.

[4] Bezerra, S. C., Ohashi, A. and Russo, F. and de Souza, F. (2019) Discrete-type approximations for non-Markovian optimal stopping problems: Part II. arXiv: 1707.05250. To appear in Methodol. Comput. Appl. Probab.

[5] Biagini, F., Hu, Y., Øksendal, B., and Sulem, A. (2002). A stochastic maximum principle for processes driven by fractional Brownian motion. *Stochastic Process Appl.*, 100, 1-2, 233-253.

[6] Burq, Z. A. and Jones, O. D. (2008). Simulation of brownian motion at first-passage times. *Math. Comput. Simul.*, 77, 1, 64-71.

[7] Borodin, A. N. and Salminen, P., *Handbook of Brownian Motion: Facts and Formulae*. Birkhauser, 2002.

[8] Buckdahn, R., and Shuai, J. (2014). Peng’s maximum principle for a stochastic control problem driven by a fractional and a standard Brownian motion. *Science China Mathematics*, 57, 10, 2025-2042.

[9] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003). Fractional Ornstein-Uhlenbeck processes. *Electron. J. Probab.*, 8, 3, 14 p.

[10] Davis, M. Martingale methods in stochastic control, in *Stochastic Control and Stochastic Differential Systems*, Parts and Functional Ito calculus (Lectures Notes of the Barcelona Summer School on Stochastic Analysis, Centro de Recerca de Matematica, July 2012), Springer: 2016.

[11] Diehl, J., Friz, P. K. and Gassiat, P. (2017). Stochastic control with rough paths. *Appl. Math. Optim.*, 75, 285315.

[12] Dolinsky, Y. (2012). Numerical schemes for G-expectations. *Math. Comput. Simul.*, 876.

[13] El Karoui, N. (1979). *Les Aspects Probabilistes du Contrôle Stochastique*, in *Ecole d’Eté de Probabilités de Saint-Flour IX*, Lecture Notes in Math., 876.

[14] Fahim, A., Touzi, N. and Warin, X. (2011). A probabilistic numerical method for fully nonlinear parabolic PDEs: Part I. *Ann. Probab.*, 44, 1, 109-133.

[15] Fernholz, R. L. (2002). *Stochastic Calculus and Financial Applications*. Springer-Verlag, Berlin 1979.

[16] Fischer, M. and Nappo, G. (2009). On the Moments of the Modulus of Continuity of Itô Processes. *Stoch. Analysis Appl.*, 27, 4, 1322-1364.

[17] Fuhrman, M. and Pham, H. (2015). Randomized and backward SDE representation for optimal control of non-Markovian SDEs. *Ann. Probab.*, 43, 2, 1245-1277.

[18] Gordon, Y., Litvak, A.E., Schütt, C. and Werner, E. (2006). On the Minimum of Several Random Variables. *P. Am. Math. Soc.*, 134, 12, 3665-3675.

[19] Grigelionis, B. and Mackevicius, V. (2003). The finiteness of moments of a stochastic exponential. *Stat. Probab. Letters*, 64, 243-248.

[20] Han, Y., Hu, Y. and Song, J. (2013). Maximum principle for general controlled systems driven by fractional Brownian motions. *Appl Math Optim.*, 7, 279-322.

[21] He, S.-w., Wang, J.-g., and Yan, J.-a. *Semimartingale Theory and Stochastic Calculus*, CRC Press, 1992.

[22] Hu, Y. and Zhou, X. (2005). Stochastic control for linear systems driven by fractional noises. *SIAM J Control Optim.*, 43, 2245-2277.

[23] Jaber, E. A., Enzo Miller, E. and Pham, H. (2019). Integral operator Riccati equations arising in stochastic Volterra control problems. *arXiv:1911.01993*.

[24] Jaber, E. A., Enzo Miller, E. and Pham, H. (2019). Linear-Quadratic control for a class of stochastic Volterra equations: solvability and approximation. *arXiv: 1911.01900*.

[25] Jacod, J., and Shkhorid, A.V. (1994). Jumping filtrations and martingales with finite variation. *Lecture Notes in Math.*, 1583, 21-35. Springer.
[30] Kharroubi, I. and Pham, H. (2014). Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE. *Ann. Probab.*, **43**, 4, 1823-1865.

[31] Kharroubi, I., Langrené, N. and Pham, H. (2014). A numerical algorithm for fully nonlinear HJB equations: An approach by control randomization. *Monte Carlo Methods and Applications*, **20**, 2.

[32] Kharroubi, I., Langrené, N. and Pham, H. (2015). Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps. *Ann. Appl. Probab.*, **25**, 4, 2301-2338.

[33] Krylov, N. V. (1999). Approximating value functions for controlled degenerate diffusion processes by using piecewise constant policies. *Electron. J. Probab.*, **4**, 1-19.

[34] Lamberton, D. Optimal stopping and American options, Daiwa Lecture Ser., Kyoto, 2008.

[35] Mazliak, L. and Nourdin, I. (2008). Optimal control for rough differential equations. *Stoch. Dyn.*, **8**, 3.

[36] Nutz, M. and van Handel, R. (2013). Constructing Sublinear Expectations on Path Space. *Stochastic Process. Appl.*, **123**, 8, 3095-3121.

[37] Qiu, J. (2018). Viscosity Solutions of Stochastic Hamilton–Jacobi–Bellman Equations. *SIAM J. Control Optim.*, **56**, 5, 3708-3730.

[38] Soner, M., Touzi, N. and Zhang, J. (2012). The wellposedness of second order backward SDEs. *Probab. Theory Related Fields*, **153**, 149-190.

[39] Tan, X. (2014). Discrete-time probabilistic approximation of path-dependent stochastic control problems. *Ann. Probab.*, **42**, 5, 1803-1834.

[40] Zhang, J and Zhuo, J. (2014). Monotone schemes for fully nonlinear parabolic path dependent PDEs, *Journal of Financial Engineering*, **1**.

[41] Zhou, X.Y. (1998). Stochastic near-optimal controls: Necessary and sufficient conditions for near optimality. *SIAM. J. Control Optim.*, **36**, 3, 929-947.

Departamento de Matemática Aplicada e Estatística. Universidade de São Paulo, 13560-970, São Carlos - SP, Brazil

E-mail address: leao@estatcamp.com.br

Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília - Distrito Federal, Brazil

E-mail address: amfohashi@gmail.com

Instituto de Matemática, Estatística e Computação Científica. Universidade de Campinas, 13083-859, Campinas - SP, Brazil

E-mail address: francysouz@gmail.com