ABSTRACT

Multiplicative noise removal has been a focus of research in recent years. Aiming at solving the problem that the total variation regularization method can remove noise well but sometimes produce stair-case effect, this paper proposes an efficient variational model and gives the iterative algorithm to remove Gamma multiplicative noise. This paper also shows that the iterative sequence converges to the optimal solution of the model. Through simulation experiments the proposed model has proved highly effective. That is, this model can preserve the edges of the image well and significantly reduce the stair-case effect in the smooth regions while removing Gamma multiplicative noise effectively.

Keywords: Variational model; multiplicative noise; noise removal; stair-case effect.
1. INTRODUCTION

It is well known that images deteriorate during formation, transmission, recording, and processing. Image denoising is of momentous significance in coherent imaging systems and various image processing applications [1]. Therefore, the image noise removal problems have attracted much attention in recent years [2, 3].

Image noise can be roughly divided into additive noise and multiplicative noise. Models for removing additive noise have been studied extensively. Since additive noises are dominant for high frequencies, their effects can be removed using some kinds of low-pass filters [4]. Other kinds of methods, such as variational methods, have also been proposed. In [5], for example, Rudin, Osher, and Fatemi proposed a total variation regularization model (ROF model), which has been widely accepted as a reliable tool for removing additive noise. Over the last few years, various research efforts have been devoted to studying, solving, and extending the ROF model [6-9]. Multiplicative noises are commonly found in many real-world image processing applications, such as synthetic aperture radar, medical ultrasound images, and particle tomography [10, 11]. Therefore, how to effectively remove multiplicative noises becomes a significant study in recent years [12-15].

In general, the recorded image \( g \), defined in \( D \subset \mathbb{R}^2 \), is the multiplication of an original image \( u \) and a noise \( v \) [16]:

\[
g = uv. \tag{1.1}
\]

In 2003, Rudin, Lions, and Osher proposed the following denoising model in [17]:

\[
\min_u J(u) = \min_u \left\{ \int_D |D_u| \, dx \, dy + \lambda_1 \int_D \left( \frac{g}{u} + \log u \right) \, dx \, dy \right\}, \tag{1.2}
\]

This model is called RLO model, it was first used to remove Gaussian noise, but the experimental results show that the texture detail of the restored image has been destroyed. When the multiplicative noise is out of the Gauss distribution, the above RLO model is no longer available.

In 2008, Aubert and Aujol [13] used the maximum a posteriori (MAP) regularization approach and derived a functional whose minimizer corresponds to the denoised image to be recovered. Their model is called AA model which can be described as follows:

\[
\min_u J(u) := \min_u \left\{ \int_D |D_u| \, dx \, dy + \lambda \int_D \left( \frac{g}{u} + \log u \right) \, dx \, dy \right\}, \tag{1.3}
\]

where \( \lambda \) is the weighted parameter. Although the functional is not convex, the related theory, such as the existence of a minimizer and the associated evolution problem and so forth, have been still set up, and the capability of their model on some numerical examples was shown. Since the computed solutions by some optimization methods are not necessary to be a global optimal solution for (1.3), the quality of the restored image solution may not be good.

In 2009, Huang, Ng and Wen [14] added a fitting term \( \int_D (z - w)^2 \, dx \, dy \) based on AA model, used a logarithmic transformation \( z = \log u \) in the fidelity term of AA model and proposed the following model (Named HNW model):

\[
\min_{z,w} J(z, w) := \min_{z,w} \left\{ \int_D |D_w| \, dx \, dy + \lambda_1 \int_D (z - w)^2 \, dx \, dy + \lambda_2 \int_D (z + ge^{-z}) \, dx \, dy \right\}. \tag{1.4}
\]

HNW model got over the ill-posedness of the restoration problem tactfully, and some experimental results also showed the quality of the images restored was better than that of the previous main models. However, its denoising effect is not ideal enough, and the restored image used by the model results in stair-case effect in smooth regions of the image.

In 2012, reference [18] proposed the following model (called HZJ model)

\[
\min_u J(u) := \min_u \left\{ \int_\Omega |D_u| \log(1 + |D_u|) \, dx \, dy + \lambda_1 \int_\Omega ge^{-u} \, dx \, dy + \lambda_2 \int_\Omega (ge^{-u} - 1)^2 \, dx \, dy \right\}. \tag{1.5}
\]

The model (1.5) has improved distinctly the denoising effect and preserved the details of images. Inspired by [14] and [18], Hu and Lou proposed a new total variational model as follows (called HL model) [19]:
The numerical experiments showed that the texture details in the images restored by HL model were kept and the ‘stair-case effect’ was suppressed at some level.

In order to obtain an efficient denoising method for Gamma noise, it is a natural choice to build a novel model by introducing an appropriate coordinating term and regularization term. In this paper we propose a novel denoising model which not only ensures that our model removes Gamma noise very well, but also it can effectively protect the edge details and texture features of the image. Our experimental results show that the quality of images restored by the proposed method is better than that by several popular models.

The rest of this paper is organized as follows. In next section, we introduce our model briefly. In Section 3, the minimizing algorithm and the convergence of the proposed method are given. In Section 4, we show experimental results to demonstrate the performance of our proposed model. Finally, concluding remarks are given in section 5.

2. THE PROPOSED MODEL

What is usually referred to as multiplicative noise removal is of course nothing but the estimation of the reflectance of the underlying scene in imaging systems. This is an inverse problem calling for regularization, which usually consists in assuming that the underlying reflectance image is piecewise smooth. This assumption has been formalized, in a Bayesian estimation framework and variational approaches. Both the variational and the Bayesian MAP formulations to image denoising (under multiplicative, Gaussian, or other noise models) lead to optimization problems with two terms: a data fidelity term and a regularizer.

\[
\min_{z,w} J(z,w) := \min_{z,w} \left\{ \int_D \left| Dw \right| \log(e^+ | Dw |) dx dy + \lambda_1 \int_D \left( z - w \right)^2 dx dy + \lambda_2 \int_D \left( z + ge^{-z} \right) dx dy \right\}. \tag{1.6}
\]

For the noise model (1.1), invoking the Bayesian rules and the MAP criterion, and using logarithm transformation, the fidelity term \( \int (z + ge^{-z}) \) for removing Gamma noise can be obtained [20].

Recent years, there are some method for removing multiplicative noise [17-19,21-24]. In this paper, we consider the following multiplicative noise model which provides an accurate description of many special imaging systems [10]

\[
g = u + \sqrt{uv}. \tag{2.1}
\]

That is to say, \( v = \left( \frac{g}{u} - u \right) \). We have known that the fidelity term of AA is \( \int_D \left( \frac{g}{u} + \log u \right) dx dy \), and many existing models used a logarithmic transformation \( z = \log u \) in AA model to get a new fidelity term \( \int_D \left( z + ge^{-z} \right) dx dy \). In this paper, we can get \( \int_D \left( \frac{g - u}{u} + \log u \right) dx dy \) by using the new \( v \) to replace the original \( \frac{g}{u} \). Now, we also use logarithmic transformation \( z = \log u \), and then get a new the fidelity term \( \int_D \left( z + ge^{-z} - 2g + e^z \right) \) for removing Gamma noise.

Inspired by [19], this paper uses the new hybrid measure \( \int \left| Dw \right| \ln(e^+ | Dw |) \) as energy functional regularization term, and invoke the coordination term \( \int (z-w)^2 \) to propose the new hybrid variational model for removing Gamma multiplicative noise as follows:

\[
\min_{z,w} J(z,w) := \min_{z,w} \left\{ \int_D \left( z + ge^{-z} + e^z - 2g \right) dx dy + \lambda_1 \int_D \left( z-w \right)^2 dx dy + \lambda_2 \int_D | Dw | \ln(e^+ | Dw |) dx dy \right\}. \tag{2.2}
\]

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The first term of model (2.2) is called the loyalty term which ensures recovering image $u$ to retain the main features from the virtual image log $g$.

The second term is the coordination term which measures the influence between the fitting term and the regularization term and plays a role in undertaking and tradeoffs. The final term is the regularization term which ensures the smooth of the denoising image $w$, and prevents stair-case effects.

Before removing image noise by using our model, we first do a convolution operation for the original degraded image so as to avoid mistaking some noise gradient for the edge gradient.

3. THE ITERATIVE ALGORITHM

Inspired by the idea of the literature [14,19], we use an adaptive alternating iterative algorithm to solve the model (2.2). It is divided into two optimization problems. Starting from the initial data $w^{(0)}$, we solve the following variational problems

$$
\begin{align*}
z^{(m)} &= \arg \min_z \left\{ \int_D (z + g^2 e^{z} - 2g + e^z) + \lambda \int_D (z - w^{(m-1)})^2 \right\} \\
w^{(m)} &= \arg \min_w \left\{ \lambda \int_D (z^{(m)} - w)^2 + \lambda \int_D |Dw| \ln(e + |Dw|) \right\}
\end{align*}
$$

(3.1) (3.2)

Cycling alternately, we can get the following iterative sequence

$w^{(0)}, z^{(1)}, w^{(1)}, z^{(2)}, w^{(2)}, ..., z^{(m)}, w^{(m)}, ....$

Step 1. To solve the variational problem (3.1), we need to solve its discretization:

$$
z^{(m)} = \arg \min_z \left\{ \sum_{i,j} \left( z(i,j) + g(i,j) e^{-z(i,j)} - 2g(i,j) + e^{z(i,j)} \right) + \lambda \sum_{i,j} \left( z(i,j) - w^{(m-1)}(i,j) \right)^2 \right\}.
$$

(3.3)

Letting $f(z(i,j)) = \left( z(i,j) + g(i,j) e^{-z(i,j)} - 2g(i,j) + e^{z(i,j)} \right) + \lambda \left( z(i,j) - w^{(m-1)}(i,j) \right)^2$. Since $f$ is continuous and derivable in the specified range, this problem (3.3) is equivalent to solving the regular system with $n^2$ equations:

$$
f'(z(i,j)) = 1 - g(i,j) e^{-z(i,j)} + e^{z(i,j)} + 2\lambda (z(i,j) - w^{(m-1)}(i,j)) = 0, \quad i,j = 1,2,...n
$$

(3.4)

Noticing that $f''(z(i,j)) = g(i,j) e^{-z(i,j)} + e^{z(i,j)} + 2\lambda > 0$, $i,j = 1,2,...n$, the function $f$ is convex strictly with respect to $z$, so we can find has a unique solution of the system (3.4) by using Newton iterative method

$$
z^{(m)}(i,j) = z^{(m-1)}(i,j) - f'(z^{(m-1)}(i,j)) / f''(z^{(m-1)}(i,j)).
$$

(3.5)

This is the solution of the problem (3.1).

Step 2. To find the solution of the variational problem (3.2). Letting

$$
H(x,y,w,w_x,w_y) = \lambda \int_D (z - w)^2 + \lambda_2 \int_D |Dw| \ln(e + |Dw|),
$$

we get the corresponding Euler-Lagrange equation of the variational problem (3.2) as follows

$$
\lambda_2 \frac{\text{div}(Dw + (e + |Dw|) \ln(e + |Dw|))}{(e + |Dw|)|Dw|} Dw + 2\lambda (z - w) = 0.
$$

(3.6)

Let $\psi(x) = (x + (e + x) \ln(e + x)) / (e + x)$, (3.6) is simplified as
\[ \lambda_2 \text{div}(\psi(\vert Dw \vert) Dw) + 2\lambda_1 (z - w) = 0 , \]  

(3.7)

In this paper, \(Dw(i, j)\) is the gradient at the location \( \left( i, j \right) \), \(Dw(i, j) = (w_i(i, j), w_j(i, j))\) and

\[ |Dw(i, j)| = \sqrt{w_i(i, j)^2 + w_j(i, j)^2}, \quad i, j = 1, 2, \ldots, n. \]

where

\[ w_i(i, j) = \begin{cases} w(i + 1, j) - w(i, j), & i < n, \\ 0, & i = n. \end{cases} \]

and

\[ w_j(i, j) = \begin{cases} w(i, j + 1) - w(i, j), & j < n, \\ 0, & j = n. \end{cases} \]

Invoking the numerical method in [19], we can obtain the numerical solution of (3.7) as follows

\[ w_\perp t = \lambda_2 \sum_{p,k} \left[ \psi(\vert Dw(p) \vert) (w(p) - w(i, j)) \right] + 2\lambda_1 (w - z^{(m)}) \]  

(3.8)

and iterative formula

\[ w^{(m)} = w^{(m-1)} + dt \times w_\perp t . \]  

(3.9)

Where

\[ K = \{(i - 1, j), (i + 1, j), (i, j - 1), (i, j + 1)\} , \]

\[ |Dw(p)| = \sqrt{\epsilon + w_i^2(p) + w_j^2(p)} , \quad \epsilon \quad \text{is an arbitrary small positive constant,} \]

\[ dt \quad \text{is a constant, we take} \]  

\[ dt = 0.12 \quad \text{in this paper.} \]

Step 3. Repeat the above process until a stopping criterion is satisfied. The cyclic condition is as follows

\[ \left\| w^{(m+1)} - w^{(m)} \right\| / \left\| w^{(m)} \right\| \leq 10^{-4} . \]

Slightly modifying the corresponding procedures in e.g. [19], we can prove the following theorem easily.

**Theorem 3.1** For any initial data \( w^{(0)} \), iterative sequence \( \{w^{(m)}\} \) converges to the optimal solution of the problem (2.2).

Proof . This proof please refers to Appendix.

4. EXPERIMENT RESULTS AND ANALYSIS

In this section, numerical results are presented to demonstrate the performance of our proposed model. The results are compared with several popular models such as AA model, HNW model and HL model. We used three images with the size 256 × 256, Lena, Barbara and Butterfly, in our experiments; see Fig. 1-7. Image Lena is a good test image because it has a nice mixture of detail, flat regions, shading are, and texture. Both Barbara and Butterfly also consist of complex components in different scales, with different patterns and under inhomogeneous illuminations. The three images are suited for our experiments.

In the tests, each pixel of an original image is degraded by a Gamma noise with mean one, and the noise level is controlled by the value of \( L \) in the experiments. The experiments are performed in MATLAB under the same software and hardware conditions.

The original image Lena and Barbara are shown in Fig. 1. The Lena image in Fig. 1 (a) is distorted by a Gamma noise with \( L = 20 \) and \( L = 10 \), respectively. The noisy images are shown in Fig. 2 (a) and Fig. 3 (a), respectively. The Barbara image in Fig. 1 (b) is distorted by a Gamma noise with \( L = 5 \) and \( L = 10 \) respectively.
The noisy images are shown in Fig. 4 (a) and Fig. 5 (a), respectively. As $L$ is smaller, the pictures are more noisy.

![Lena](image1.png) ![Barbara](image2.png)

**Fig. 1. The original images**

From the denoising contrast experiments shown in Figs. 2-5, we see that the denoising results obtained by the proposed model and the $HL$ model are visually much better than those by the others. Obviously, the restoration result by the proposed model is the best in the four models.

In Fig. 2, we show how the four models behave with a small noisy image. Notice that in this case all the models perform well. Besides, the proposed model is superior to the other three models for preserving the textures and fine details of the images. In Figs. 3-5, we see that the proposed model gets a very good visual effect and works well in the case of heavy noise.

In addition to visual comparison, we use the peak signal to noise ratio (PSNR) and relative error rate (ReErr) of the images to assess the quality of the restored images [14]. Greater PSNR or the smaller ReErr is, the denoising effect is better. From Table 1, we can see that the PSNR obtained by our model is the biggest in those by all four models for the same noisy image. Similarly the ReErr obtained by the proposed model is smallest.

Both denoising visual effect and objective index have showed that the proposed model behaves very well. The model can remove Gamma noise effectively, but also it can protect the edge details and texture features of the image, prevent from producing stair-case phenomenon at some level.

![Lena](image3.png) ![Barbara](image4.png)

**Fig. 2. The noisy image Lena and denoising contrast experiment 1**
Fig. 3. The noisy image Lena and denoising contrast experiment 2

Fig. 4. The noisy image Barbara and denoising contrast experiment 3
Finally, we also display the restoration of the color Lena image and the Butterfly image. From Figs. 6-7, it is clear that the restoration results by the proposed method are visually better than those by HL model. Especially, our model is superior in avoiding or reducing the stair-case effect.

![Fig. 5. The noisy image Barbara and denoising contrast experiment 4](image1)

![Fig. 6. The original color Lena image and denoising contrast experiment 5](image2)
Table 1. The PSNRs and ReErrs of the restored images

| Image          | AA model | HNW model | HL model | Proposed model |
|----------------|----------|-----------|----------|----------------|
| “Lena” in Fig.2 (L=20) | PSNR=61.55 | PSNR=61.77 | PSNR=63.51 | PSNR=65.60 |
| | ReErr=0.091 | ReErr=0.087 | ReErr=0.084 | ReErr=0.074 |
| “Lena” in Fig.3 (L=10) | PSNR=55.78 | PSNR=57.58 | PSNR=57.90 | PSNR=61.51 |
| | ReErr=0.129 | ReErr=0.115 | ReErr=0.113 | ReErr=0.092 |
| “Barbara” in Fig.4 (L=10) | PSNR=48.31 | PSNR=48.65 | PSNR=49.36 | PSNR=49.42 |
| | ReErr=0.207 | ReErr=0.203 | ReErr=0.182 | ReErr=0.197 |
| “Barbara” in Fig.5 (L=5) | PSNR=45.11 | PSNR=46.58 | PSNR=47.77 | PSNR=48.86 |
| | ReErr=0.257 | ReErr=0.201 | ReErr=0.200 | ReErr=0.197 |

Fig. 7. The original color Butterfly image and denoising contrast experiment 6.

5. CONCLUSION

In this paper, we propose an efficient total variation model for Gamma multiplicative noise removal. Strictly convex objective function not only ensures that the new model has a unique solution, but also improves the quality of the restored image. Our model can reduce stair-case effect significantly while removing Gamma multiplicative noise quite well. Our experimental results also show that the method is superior to some of existing total variation methods.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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APPENDIX

Proof of Theorem 3.1

According to (3.1) and (3.2), there
\[ m^{(m)} = T(S(w^{(m-1)})) = R(w^{(m-1)}). \]
To prove this theorem, we give the following four lemmas, and the appropriate proof.

**Lemma 1** The operator \( S \) is non-expansive; \( T \) is \( \frac{1}{2} \)-averaged non-expansive; \( R \) is non-expansive. (If for any \( x, y \in \mathbb{R}^n \), we have \( \|Tx - Ty\| \leq \langle Tx - Ty, x - y \rangle \), the operator \( T \) is called non-expansive; if there exists \( \alpha \in (0,1) \), such that \( T_\alpha = (1-\alpha)I + \alpha T \), then \( T_\alpha \) is called \( \alpha \)-averaged non-expansive)

**Proof.** We know that \( S \) is a non-expansive operator from (3.1) formula and literature [19].
Now we define
\[ S(w^{(m-1)}) = \arg \min_w \left( \frac{1}{2} \|z - w^{(m-1)}\|^2 + \frac{1}{2\lambda_1} (z + g^2 e^2 - 2g + e^2) \right) \]
and \( \phi = \frac{1}{2\lambda_1} (z + g^2 e^2 - 2g + e^2) \). According to the definition of the neighboring operator [19], we know that \( S \) is a neighboring operator as respect to \( \phi \), strictly convex function on \( z \) and under semi-continuous. We can get the consequence from Proposition 2.4 in the literature [8] in which shows
\[ \|S(w) - S(w)\|^2 \leq \langle S(w) - S(w), w - w \rangle. \]
It proves that \( S \) is non-expansive.
Now, we define \( M = 2T - I \), then
\[
\|M(z_1) - M(z_2)\|^2 \\
= \left\| (2T - I)(z_1) - (2T - I)(z_2) \right\|^2 \\
= \left\| 2(T(z_1) - T(z_2)) - (z_1 - z_2) \right\|^2 \\
= 4\left\| T(z_1) - T(z_2) \right\|^2 + \|z_1 - z_2\|^2 - 4\langle T(z_1) - T(z_2), z_1 - z_2 \rangle. \tag{4.1}
\]
With the similar way to prove \( S \), we can get the following function
\[ T(z^{(m)}) = \arg \min_w \left[ \frac{1}{2} \|z^{(m)} - w\|^2 + \frac{\lambda_2}{2\lambda_1} \int_{\Omega} Dw \log(e + |Dw|) \right], \]
Where \( \varphi \) is given by
\[ \varphi = \frac{\lambda_2}{2\lambda_1} \int_{\Omega} Dw \log(e + |Dw|). \]
On the other hand, \( \varphi \) is a convex function and under semi-continuous. So \( T \) satisfies
\[ \|T(z_1) - T(z_2)\|^2 \leq \langle T(z_1) - T(z_2), z_1 - z_2 \rangle. \tag{4.2} \]
Combined with (4.1) and (4.2), we obtain
\[ \|M(z_j) - M(z_j) \|_2^2 \leq \|z_j - z_j \|_2^2. \]

According to the definition of the non-expansive operator and the average non-expansive, operator \( M \) is non-expansive. However, \( T = (1 - 1/2)I + 1/2M \), so the operator \( R \) is a \( 1/2 \) -averaged non-expansive.

\[ \square \]

**Lemma 2** \( \sum_{m=1}^{\infty} \| w^{(m-1)} - w^{(m)} \|_2^2 \) is convergence.

**Proof.** Denote \( J_1(z, w) = \|z - w\|_2^2 \) and \( J_2(w) = \int_D |Dw| \ln(e + |Dw|) dx dy \). Hence, we have
\[
J(z^{(n)}, w^{(n)}) - J(z^{(n)}, w^{(n+1)}) \\
= \lambda_1 \left[ J_1(z^{(n)}, w^{(n)}) - J_1(z^{(n)}, w^{(n+1)}) \right] \\
+ \lambda_2 \left[ J_2(w^{(n)}) - J_2(w^{(n+1)}) \right]
\]

That is to say
\[ \frac{\partial J}{\partial w} = \lambda_1 \frac{\partial J_1}{\partial w} + \lambda_2 \frac{\partial J_2}{\partial w}. \]

We consider the Taylor series expansion of \( J_1(z^{(n)}, w^{(n)}) \) in the second variable,
\[
J_1(z^{(n)}, w^{(n)}) \\
= J_1(z^{(n)}, w^{(n+1)}) + (w^{(n+1)} - w^{(n)}) \frac{\partial J_1}{\partial w}(z^{(n)}, w^{(n+1)}) \\
+ \frac{1}{2} (w^{(n+1)} - w^{(n)}) \frac{\partial^2 J_1}{\partial w^2}(z^{(n)}, w^{(n+1)}) (w^{(n+1)} - w^{(n)})
\]

Here we note that \( J_1 \) is quadratic in \( w \) and \( x' \) denotes a transpose of \( x \). As \( J_2 \) is a convex function, we have
\[
J_2(w^{(n)}) \geq J_2(w^{(n+1)}) + (w^{(n)} - w^{(n+1)}) \frac{\partial J_2}{\partial w}(w^{(n+1)})
\]

By considering \( \frac{\partial^2 J_1}{\partial w^2} = I \) and substituting (4.4) and (4.5) into (4.3), we obtain
\[
J(z^{(n)}, w^{(n)}) - J(z^{(n)}, w^{(n+1)}) \\
\geq (w^{(n+1)} - w^{(n)}) \left[ \lambda_1 \frac{\partial J_1}{\partial w}(z^{(n)}, w^{(n+1)}) + \lambda_2 \frac{\partial J_2}{\partial w}(w^{(n+1)}) \right] \\
+ \lambda_2 \frac{1}{2} \|w^{(n+1)} - w^{(n)}\|_2^2
\]

As \( w^{(n+1)} \) is the minimizer of \( J(z^{(n)}, w) \), we obtain
\[ \frac{\partial J}{\partial w}(z^{(m)}, w^{(m+1)}) = \lambda \frac{\partial J}{\partial w}(z^{(m)}, w^{(m+1)}) + \frac{\partial J}{\partial w}(w^{(m+1)}) = 0 \]

It is easy to prove the following fact
\[ J(z^{(m)}, w^{(m)}) \geq J(z^{(m)}, w^{(m+1)}) \]
and we use it to show
\[ J(z^{(m)}, w^{(m)}) - J(z^{(m+1)}, w^{(m+1)}) \geq J(z^{(m)}, w^{(m)}) - J(z^{(m+1)}, w^{(m+1)}) \]
\[ \geq \frac{\lambda}{2} \|w^{(m+1)} - w^{(m)}\|^2 \]

So we can say that \( \sum_{m=1}^{\infty} \|w^{(m+1)} - w^{(m)}\|^2 \) is bounded and converges.

**Lemma 3** Model (2.2) corresponds to the objective function \( J(z, w) \) is compulsory.

**Proof.** Denote \( L = \begin{pmatrix} L_x & \cdot \\ \cdot & L_y \end{pmatrix} \). On one hand, \( L_x \) is a one-sided difference matrix on the horizontal direction, and \( L_y \) is on the vertical direction. On the other hand, \( L \) is not a full-rank matrix. The lower bound of the discrete TV is given by
\[
\int_{\Omega} |Dw| \ln(e + |Dw|) dx dy = \sum_{i,j,k,l} |Vw|_{i,j} \ln(e + |Vw|_{i,j})
\]
\[
= \sum_{i,j,k,l} \sqrt{\left(\frac{Vw}{e}\right)_{i,j}} + \left(\frac{Vw}{e}\right)_{i,j} \ln\left(e + \sqrt{\left(\frac{Vw}{e}\right)_{i,j}} + \left(\frac{Vw}{e}\right)_{i,j}\right), \quad (4.6)
\]

where \( \ln\left(e + |Vw|_{i,j}\right) \geq \ln(e) = 1 \), so

\[
(4.6) \geq \sum_{i,j,k,l} \sqrt{\left(\frac{Vw}{e}\right)_{i,j}} + \left(\frac{Vw}{e}\right)_{i,j} \geq \frac{1}{\sqrt{2}} \sum_{i,j,k,l} \left|\frac{Vw}{e}\right|_{i,j} + \left|\frac{Vw}{e}\right|_{i,j} = \frac{1}{\sqrt{2}} \|Lw\|
\]

The next proof refers to Lemma 3.8 in [19].

**Lemma 4** The operator \( R \) has at least one fixed point.

**Proof.** From Lemma 3, we can know that objective functional \( J \) is compulsory, so the set of fixed points of \( J \) is non-empty. Assume \( (z, w) \) is a minimizer of \( J(z, w) \), i.e.,
\[
\frac{\partial J}{\partial z}(z, w) = 0, \quad \frac{\partial J}{\partial w}(z, w) = 0,
\]
\[
w = T(z) = T(S(w)) = R(w).
\]

We can obtain that \( w \) is a fixed point of \( R \).

**Proof of Theorem 1.** Since objective functional (2.2) is strictly convex, and \( J \) is differentiable with respect to \( z \), so the fixed point set of \( J \) is as well as its minimum point set. According to Lemma 4, the operator \( R \) exists fixed points, while strictly convex of the functional \( J \) ensures that \( R \) is at most one fixed point, so the operator \( R \) has a unique fixed point denoted \( w_0 \). There, we can also get
\[
w^{(m)} = R^m(w_0).
\]
We have proved that $R$ is non-expansive in Lemma 1, whereby $R^n(w_0)$ and $\|R^n w - w_0\|_2$ are also non-expansive. From the following inequality

$$\|R^{n+1} w - w_0\|_2 = \|R(R^n w) - Rw_0\|_2 \leq \|R^n w - w_0\|_2, \quad m = 0, 1, \ldots$$

we can get the below

$$d(w_0) = \lim_{m \to \infty} \|R^n w - w_0\|_2.$$

Next work is to prove $R^n w - w_0 \to 0 (m \to \infty)$.

(Proof by contradiction) Now if $\{R^n w\}$ subsequence of $\{R^n w\}$ has limits $w'$, and $w' \neq w_0$.

Known by the asymptotically regular of $R$, $\lim_{n_i \to \infty} \{(I - R)(R^n w)\} = 0$, or $(I - R)(w') = 0$, i.e. $Rw' = w'$, so $w'$ is also a fixed point of $R$. Which, however, is contradiction with the truth that $R$ has a unique fixed point $w_0$. So the limit is only one given by $w_0$.

For further proof the Lemm3.1, we denote

$$w = \lim_{m \to \infty} R^n(w_0) = \lim_{m \to \infty} w^{(m)}.$$

Therefore

$$S(w) = \lim_{m \to \infty} S(w^{(m)}).$$

Using the non-expansive of $S$, we can obtain

$$\|z^{(m)} - S(w)\|_2 = \|S(w^{(m-1)}) - S(w)\|_2 \leq \|w^{(m-1)} - w\|_2.$$

So $\lim_{m \to \infty} w^{(m-1)} - w = 0$, $\lim_{m \to \infty} z^{(m)} = S(w)$, that is to say, $\{z^{(m)}\}$ converges to a unique fixed point, when $m \to \infty$.

This completes the proof.