Abstract. Let $G = (V, E)$ be a connected graph. A set $U \subseteq V$ is convex if $G[U]$ is connected and all vertices of $V \setminus U$ have at most one neighbor in $U$. Let $\sigma(W)$ denote the unique smallest convex set that contains $W \subseteq V$. Two players play the following game. Consider a convex set $U$ and call it the 'playground.' Initially, $U = \emptyset$. When $U = V$, the player to move loses the game. Otherwise, that player chooses a vertex $x \in V \setminus U$ which is at distance at most two from $U$. The effect of the move is that the playground $U$ changes into $\sigma(U \cup \{x\})$ and the opponent is presented with this new playground.

A graph is chordal bipartite if it is bipartite and has no induced cycle of length more than four. In this paper we show that, when $G$ is chordal bipartite, there is a polynomial-time algorithm that computes the Grundy number of the $P_3$-game played on $G$. This implies that there is an efficient algorithm to decide whether the first player has a winning strategy.

1 Introduction

The $P_3$-convexity in graphs was introduced in [1].

Definition 1. Let $G$ be a connected graph. A set $U$ is convex if

\[ G[U] \text{ is connected and } \forall x \in V \setminus U \mid N(x) \cap U \mid \leq 1. \]

Notice that the intersection of any two convex sets is convex. Since $\emptyset$ and $V(G)$ are also convex, it follows that this convexity is an alignment. Let $\mathcal{L} = \mathcal{L}(G)$ denote the collection of convex sets in $G$.

The convex closure of a set $W \subseteq V$ is defined as the smallest convex set that contains $W$, that is,

\[ \sigma(W) = \bigcap \{ U \mid W \subseteq U \text{ and } U \in \mathcal{L} \}. \]
We introduced the P₃-game in [5]. It is played as follows. Let G be a connected graph. When it is his turn, each of the two players is presented with a playground. The playground is a convex set U in G. Initially, U = ∅. When it is a player’s turn to move, he loses the game if U = V. Otherwise, he selects a vertex \( x \in V \setminus U \) at distance at most two from U. The effect of the move is that the playground changes into \( \sigma(U \cup \{x\}) \), and it is now the opponent’s turn to move.

In this paper we show that, when G is chordal bipartite, there exists an efficient algorithm that decides whether the player who is first to move has a winning strategy.

2 Preliminaries

2.1 Chordal bipartite graphs

Golumbic and Goss launched the studies on chordal bipartite graphs. They defined the class of graphs as follows.

**Definition 2.** A graph is chordal bipartite if it is bipartite and has no induced cycles of length more than 4.

Several characterizations of this class of graphs are available. We refer to [6], which contains a short survey. In this paper also appears the following lemma [6, Lemma 2 and Corollary 1].

**Lemma 1.** Let \( G = (A, B, E) \) be chordal bipartite, let \( S \) be a minimal separator in \( G \), and let \( C \) be a component that is close to \( S \), that is, \( C \) is a component of \( G - S \) and \( N(C) = S \). Then

(I) \( G[S] \) is complete bipartite (possibly an independent set), if \( S \cap A \neq \emptyset \), then there exists a vertex \( x \in C \) satisfying

\[
N(x) \cap S = S \cap A,
\]

(II) if \( S \cap A \neq \emptyset \) and \( S \cap B \neq \emptyset \), there exist two adjacent vertices \( x \) and \( y \) satisfying

\[
N(x) \cap S = S \cap A \quad \text{and} \quad N(y) \cap S = S \cap B.
\]

This implies that each color class of a minimal separator is the common neighborhood of two nonadjacent vertices. If the separator has vertices in both color classes, then there exists a 2K₂ such that each color class of S is the common neighborhood of one of the two nonadjacent pairs, that are in the same color class of G, and that are in the 2K₂. Golumbic and Goss show that this condition on the separator with vertices in both color classes, characterizes chordal bipartite graphs; namely, a graph is chordal bipartite if and only if every minimal edge-separator (separating two edges of G) is complete bipartite.
2.2 Game graphs and Grundy numbers

The game graph, for the $P_3$-game defined above on a chordal bipartite graph $G$, is a directed acyclic graph $P$, whose vertices are the playgrounds. There is an arc from a playground $A$ to a playground $B$ if $B$ can be reached from $A$ within one move.

The game graph $P$ is labeled as follows. The unique sink-node, $V$, is labeled 0. Recursively, let $A$ be an unlabeled playground for which all outgoing neighbors are labeled. Then the label of $A$ is the $\text{mex}$-value of its successors.

**Definition 3.** The $\text{mex}$-value of a set of nonnegative integers is the smallest nonnegative integer that is not in the set.

**Definition 4.** The Grundy value of $G$ is the label of $\emptyset$ in the game graph $P$.

We denote the Grundy value of the graph $G$ by $g(G)$.

The player who is to move first, wins the game if and only if the Grundy value is not 0. Thus, the game graph $P$ provides an (exponential) algorithm to decide the $P_3$-game on a graph $G$.

The Sprague-Grundy theorem deals with products of games. Let $\mathcal{G}$ be a collection of games. The product game of $\mathcal{G}$ is the game in which each player makes a (legal) move in one of the games of $\mathcal{G}$. The Sprague-Grundy theorem is the following.

**Theorem 1.** Let $\mathcal{G}$ be a collection of impartial 2-person games. Then the Grundy value of the product game is the nim-sum of the Grundy values of the games in $\mathcal{G}$.

3 The $P_3$-game on biconnected chordal bipartite graphs

Centeno, et al., showed that, in a biconnected chordal graph $G$, if $x$ and $y$ are two vertices at distance at most two, then $\sigma(\{x,y\}) = V(G)$. The following lemma shows that a similar statement holds in biconnected, chordal bipartite graphs.

**Lemma 2.** Let $G = (A, B, E)$ be a biconnected and chordal bipartite graph. Let $U$ be a convex set that contains two nonadjacent vertices $x$ and $y$ that are in a $C_4$ of $G$. Then $\sigma(\{x,y\}) = V(G)$.

**Proof.** Let $S$ be a minimal $x,y$-separator, and let $C_x$ and $C_y$ be the components that contain $x$ and $y$. Then $S$ contains the common neighbors of $x$ and $y$.

By Lemma 1 there exist vertices in $C_x$ and $C_y$ that are adjacent to all vertices of $S$ in one color class. It follows that there exist vertices in $C_x$ and in $C_y$ that have two neighbors in $U$, and so, they are also in $U$. In turn, this implies that $S \subseteq U$. 
Let $\Omega \subseteq C_x$ be the set of vertices that are adjacent to every vertex of one of the two color classes of $S$. Let $X_1, \ldots, X_t$ be the components of $C_x \setminus \Omega$. Then $S_i = N(X_i)$ is a minimal separator, and it is contained in $U$. By Lemma 1 and by induction, it now follows that $X_i \subseteq U$. This proves that $C_x \subseteq U$ and, similarly, $C_y \subseteq U$.

Let $D$ be a component of $G - S$, other than $C_x$ or $C_y$. Then $N(D)$ is a minimal separator. Since $G$ is biconnected, $N(D) \subseteq S$ and $|N(D) \cap S| \geq 2$.

By the same argument as above, $D \subseteq U$.

This proves the lemma. \hfill $\Box$

**Example 1.** Consider a ladder $L$. Notice that every convex set is either

(a) $\emptyset$, or $V(L)$, or a single vertex, or
(b) a rung of $L$, or
(c) a connected subpath of a stile (stringer) of $L$.

**Theorem 2.** Assume $G$ is a biconnected, chordal bipartite graph with at least two vertices. Then the second player to move has a winning strategy.

**Proof.** If $G$ is an edge, the second player wins the $P_3$-convex game. Assume $G$ has more than 2 vertices. Assume the first move labels a vertex $s$. Notice that $s$ is in a $C_4$, otherwise $G$ would have a cutvertex. The player to move, chooses a vertex $s'$ which is not adjacent to $s$ and which is in a $C_4$ together with $s$. Then, by Lemma 2 the second move changes the playground into $V(G)$, which ends the game. \hfill $\Box$

**Remark 1.** Similarly, when $s$ is a pendant vertex and $G - x$ is a biconnected and chordal bipartite graph with at least two vertices, then the player who is first to move has a winning strategy. When $G$ is $P_3$, then the winning move is to play the midpoint. Otherwise, when $G$ has at least 4 vertices, the player who moves first labels the pendant vertex $s$. His opponent either adds a vertex or an edge of the biconnected component to the playground and, since the vertex or edge is in a $C_4$, the player who made the first move can then end the game.

## 4 Splitters

Let $G$ be a connected chordal bipartite graph. A generalization of the $P_3$-game is, where the initial playground is some (arbitrary) convex set $U$, instead of $\emptyset$. We denote the Grundy number of this game by $g^*(U)$, or by $g^*(G, U)$, when the graph $G$ is not clear from the context. Then we have, for the Grundy value $g(G)$ of $G$,

$$g(G) = g^*(\emptyset) = g^*(G, \emptyset).$$
**Definition 5.** A splitter is a playground that contains a minimal separator of $G$.

Let $S$ be a minimal separator, and assume $S \subseteq U$, for some playground $U$, and let $C_1, \ldots, C_t$ be the components of $G - S$. Denote

$$\hat{C}_i = C_i \cup N(C_i), i \in \{1, \ldots, t\}.$$  

Then each player, when it is his move, plays a vertex in one of the components $C_i$, that is, he plays a move in one of the games $G_i = G[\hat{C}_i]$ with playground $U_i = U \cap V(G_i)$. By Theorem 1, this proves the following theorem.

**Theorem 3.** Let $G$ be a connected chordal bipartite graph. Let $U$ be a splitter of the $P_3$-game played on $G$. Then

$$g^*(G, U) = \text{nim-sum} \{ g^*(G_i, U_i) | i \in \{1, \ldots, t\} \}.$$  

**Remark 2.** Notice that the definition of $\hat{C}_i$ guarantees that any move made in the product game is a legal move in the $P_3$-game on $G$. Notice also the necessity of the condition that the separator $S$ is part of the current playground; otherwise, a move within $S$ would be a move in all games $G_i$, which is not allowed in the product game.

### 5 Deciding the $P_3$-game on chordal bipartite graphs

Let $G$ be a connected, chordal bipartite graph. Let $P$ be the game graph of the $P_3$-game played on $G$. Let $P'$ be the labeled digraph obtained from $P$ as follows. Let $U$ be a playground. Let $H(U)$ be the graph obtained from $G$ by removing those vertices $x \in V(G)$ that satisfy

$$\sigma(U \cup \{x\}) = V.$$  

**Definition 6.** The augmented game graph $P^*$ is the labeled digraph obtained from $P'$, by adding an arc from each sink in $P'$ to a new sink node $V$.

**The augmented game.** The augmented game graph $P^*$ represents the following ‘augmented’ game. When it is a player’s move, and when the playground is a convex set $U$, then he chooses a vertex from $H(U)$ which is at distance at most 2 from $U$. When he cannot make a move, he loses the game.

**Lemma 3.** Assume that $G$ is biconnected, and chordal bipartite, and assume that $G$ has at least 2 vertices. The Grundy value of $G$ satisfies

$$g(G) = g^*(G),$$  

where $g^*(G)$ is the Grundy value of the augmented game.
**Proof.** Each move in the $P_3$-game on $G$ is a move in $P$, 
\[ A \rightarrow B, \]
where $A$ and $B$ are convex sets. That is, either it is a move 
\[ A \rightarrow V, \]
or else it is a move in $P'$. 
It follows that for each convex set $U$ in $P$, the Grundy value is 
\[
g(U) = \begin{cases} 
\text{mex } \{ 0, g^*(U) \}, & \text{if } U \rightarrow V \text{ in } P, \\
\text{otherwise}, & 
\end{cases}
\] (1)
where $g^*(U)$ is the Grundy value of $U$ in the augmented game. By Lemma 2 on page 3, since $G$ is biconnected, when $U \neq V$ and $U \neq \emptyset$, there exists a vertex $x \in V \setminus U$ satisfying 
\[ \sigma(U \cup \{x\}) = V. \]
This implies that, unless $U = V$ or $U = \emptyset$, there is an arc $U \rightarrow V$, and so, 
\[ g^*(U) = \text{mex } \{ 0, g^*(U) \}. \]
Notice that $g(\emptyset) = g^*(\emptyset)$, since no vertex played as an initial move ends the game (since $G$ has at least two vertices). Finally, by definition, 
\[ g(V) = g^*(V) = 0. \]
This proves the lemma. \(\square\)

**Theorem 4.** There exists a polynomial-time algorithm to compute the Grundy value of the $P_3$-game on chordal bipartite graphs.

**Proof.** We may assume that $G$ is connected. Consider a playground $U$ that contains an induced $P_3$ in $G$, say $[x, y, z]$. We claim that $U$ is a splitter. We may assume that none of $x$, $y$ or $z$ is a cutvertex in $G$, otherwise we are done. Thus $[x, y, z]$ is contained in a biconnected component of $G$. We may assume also that $U \neq V$.

Let $C_y$ be the component of $G - N[x]$ that contains $y$ and let 
\[ S = N(C_y). \]
Then $S \subseteq N(x)$, and so $S$ is an independent set (since $G$ is bipartite). By Lemma 1 on page 2, there exists a nonempty set of vertices $Y' \subseteq C_y$ which are adjacent to all vertices of $S$. We claim that $Y' \cap H(U) = \emptyset$, where $H(U)$ is the label of $U$ in the augmented game graph $P^*$. 

\[ g(U) = \begin{cases} 
\text{mex } \{ 0, g^*(U) \}, & \text{if } U \rightarrow V \text{ in } P, \\
\text{otherwise}, & 
\end{cases}
\] (1)
To see this, first notice that \( y' \notin U \), for \( y' \in Y' \). Otherwise, since \( U \) is contained in a biconnected component of \( G \), there exists a vertex \( q \in S \setminus \{z\} \), which is a common neighbor of \( x \) and \( y' \). This implies that \( \{x, z, y', q\} \) induces a \( C_4 \). By Lemma 2 on page 3, \( y' \in U \) implies that \( \sigma(U) = U = V \), which is a contradiction.

Every vertex of \( Y' \) represents a legal move, since they are adjacent to \( z \in U \). Since \( Y' \) contains legal moves \( y' \) for which

\[ \sigma(U \cup \{y'\}) = V, \]

we have, by definition of the augmented game graph, \( Y' \cap H(U) = \emptyset \).

In the augmented game, the vertices of \( Y' \) are removed from the graph (recall that these represent moves that point to \( V \)). Let \( G' = G - Y' \). Let \( \{D_i\} \) represent the set of components of \( G[C_u] - Y' \). First consider a component \( D_i \) that contains a vertex of \( U \). Let \( S_i = N(D_i) \). Then \( S_i \subseteq S \). As long as \( |S_i| > 0 \), by the argument above, we find new vertices in \( D_i \) that are not in \( H(U) \). Thus, after removal of all vertices that are not in \( H(U) \), we have that all components \( D_i \) that contain a vertex of \( U \), satisfy \( N(D_i) = \{z\} \). This proves that \( H(U) \) has a cutvertex, and so, in the augmented game graph, \( U \) is a splitter.

Notice that there are at most \( n^3 \) minimal splitters, since it is bounded from above by the number of induced \( P_3 \)'s in \( G \). Each connected subgraph of \( P^* \), without splitters has at most \( O(n^2) \) nodes, since otherwise it contains a splitter. Therefore, the number of nodes in the augmented game is \( O(n^3) \). It follows that the computation of the augmented game graph can be carried out in polynomial time. The Grundy values can be computed using Theorem 3 (ie, the nim-sum operator, and (1) (which relates the Grundy values in \( P \) and \( P^* \)), and the mex operator.

This proves the theorem. \( \square \)

6 Concluding remark

In this paper we introduced a new technique, dubbed ‘splitters,’ which is used for the computation of the Grundy numbers of certain games on graphs. The technique attempts to reduce the game graph, by using splitters, to an equivalent game graph which has polynomial size. In the case of the \( P_3 \)-game on chordal bipartite graphs, this turned out to be successful. At the moment we are investigating for which classes of graphs, and for which games, this technique is applicable. It would be nice to have a characterization of the classes of graphs, say with a polynomial number of separators, for which the \( P_3 \)-game is solvable in polynomial time.

5 Notice that the same argument applies to components of \( G - N[x] \) that have no vertices of \( U \), but are adjacent to \( z \). After removal of vertices that are not in \( H(U) \), these components split also off as a component of \( G - \{z\} \).
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References

1. Centeno, C., M. Dourado, L. Penso, D. Rautenbach and J. Szwarcfiter, Irreversible conversion of graphs, *Theoretical Computer Science* **412** (2011), pp. 3693–3700.
2. Edelman, P. and R. Jamison, The theory of convex geometries, *Geometriae Dedicata* **19** (1985), pp. 247–270.
3. Golumbic, M. and C. Goss, Perfect elimination and chordal bipartite graphs, *Journal of Graph Theory* **2** (1978), pp. 155–163.
4. Grundy, P., Mathematics and games, *Eureka* **2** (1939), pp. 6–8.
5. Hon, W., T. Kloks, F. Liu, H. Liu and T. Wang, $P_3$-games. Manuscript on arXiv: 1608.05169, Accepted for TAMC’16, 2016.
6. Kloks, T., Ching-Hao Liu and Sheung-Hung Poon, Feedback vertex set on chordal bipartite graphs. Manuscript on arXiv: 1104-3915, 2012.
7. Kloks, T. and Y. Wang, *Advances in graph algorithms*. Manuscript on ViXrA:1409.0165, 2014.
8. Pelayo, I., *Geodesic convexity in graphs*, Series SpringerBriefs in Mathematics, Springer-Verlag New York, 2013.