The $\Phi^4_3$ measure via Girsanov’s theorem

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Abstract

We construct the $\Phi^4_3$ measure on a periodic three dimensional box as an absolutely continuous perturbation of a random translation of the Gaussian free field. The shifted measure is constructed via Girsanov’s theorem and the relevant filtration is the one generated by a scale parameter. As a byproduct we give a self-contained proof that the $\Phi^4_3$ measure is singular wrt. the Gaussian free field.

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1 Introduction

The $\Phi^4_3$ measure on the three dimensional torus $\Lambda = T^3 = (R/2\pi Z)^3$ is the probability measure $\nu$ on distributions $\mathcal{D}^\prime(\Lambda)$ corresponding to the formal functional integral

$$
\nu(d\varphi) = \frac{1}{Z} \exp \left[ -\lambda \int_\Lambda (\varphi^4 - \infty \varphi^2) dx \right] \mu(d\varphi) \quad (1.1)
$$

where $\mu$ is the law of the Gaussian free field with covariance $(1 - \Delta)^{-1}$ on $\Lambda$, $Z$ a normalization constant and $\lambda$ the coupling constant. The $\infty$ appearing in this expression reminds us that many things are wrong with this recipe. The key difficulty can be traced to the fact that the measure we are looking for it is not absolutely continuous wrt. the reference measure $\mu$. This fact seems part of the folklore even if we could not find a rigorous proof for it in the available literature apart from a work of Albeverio and Liang [1], which however refers to the Euclidean fields at time zero, and the work of Feldman and Osterwalder [11] in infinite volume. The singularity of the $\Phi^4_3$ measure is indeed a major technical difficulty in a rigorous study of (1.1). Obtaining a complete

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construction of this formal object (both in finite and infinite volume) has been one of the main achievements of the constructive quantum field theory program \cite{12, 10, 26, 11, 22, 5, 8}.

In recent years the rigorous study of the $\Phi^4_3$ model has been pursued from the point of view of stochastic quantization. In the original formulation of Parisi–Wu \cite{25}, stochastic quantization is a way to introduce additional degrees of freedom (in particular a dependence on a fictitious time) in order to obtain an equation whose solutions describe a measure of interest, in this case the $\Phi^4_3$ measure on $\Lambda$ as in (1.1) or its counterpart in the full space. Rigorous analysis of stochastic quantization for simpler models like $\Phi^2_2$ (the two-dimensional analog of eq. (1.1)) started with the work \cite{20}. It has been only with the fundamental work of Hairer on regularity structures \cite{19} that the three dimensional model could be successfully attacked, see also \cite{9, 21}. This new perspective on this and related problems led to a series of new results on the global space-time control of the stochastic dynamics \cite{24, 15, 2, 23} and to a novel proof of the construction of non-Gaussian Euclidean quantum field theories in three dimensions \cite{14}.

A conceptual advantage of stochastic quantization is that it is a method which is insensitive to questions of absolute continuity wrt. to a reference measure. This, on the other hand, is the main difficulty of the Gibbisan point of view as expressed in eq. (1.1). In order to explore further the tradeoffs of different approaches we have recently developed a variational method \cite{4} for the construction and description of $\Phi^4_3$. We were able to provide an explicit formula for the Laplace transform of $\Phi^4_3$ in terms of a stochastic control problem in which the controlled process represents the scale-by-scale evolution of the interacting random field.

The present paper is the occasion to explore further this point of view by constructing a novel measure via a random translation of the Gaussian free field and by proving that the $\Phi^4_3$ measure can be obtained as an absolutely continuous perturbation thereof. Without entering into technical details right now, let us give the broad outline of this construction. We consider a Brownian martingale $(W_t)_{t \geq 0}$ with values in $\mathcal{S}'(\Lambda)$ and such that $W_t$ is a regularization of the Gaussian free field $\mu$ at (Fourier) scale $t$. Let us denote $P$ its law and $E$ the corresponding expectation. In particular, $W_t \to W_\infty$ in law as $t \to \infty$ and $W_\infty$ has law $\mu$. We can identify the $\Phi^4_3$ measure $\nu$ as the weak limit $\nu^T \to \nu$ as $T \to \infty$ of the family of probability measures $(\nu^T)_{T \geq 0}$ on $\mathcal{S}'(\Lambda)$ defined as

$$\nu^T(\cdot) = P^T(W_T \in \cdot),$$

where $P_T^T$ is the measure on paths $(W_t)_{t \geq 0}$ with density

$$\frac{dP_T^T}{dP} = \frac{1}{Z_T} e^{-V_T(W_T)},$$

and

$$V_T(\varphi) := \lambda \int_\Lambda (\varphi(x))^4 - a_T \varphi(x)^2 + b_T \varphi(x)dx,$$

is a quartic polynomial in the field $\varphi$ with $(a_T, b_T)_T$ a family of (suitably diverging) renormalization constants. The scale parameter $t \in \mathbb{R}_+$ allows to introduce a filtration and a measure $Q^T$ defined as the Girsanov transformation

$$\frac{dQ^T}{dP} |_{F_T} = \exp \left( L_T^0 - \frac{1}{2} \langle L^0 \rangle_T \right), \quad L_T^0 = \int_0^T <v_s, dW_s>_{L^2(\Lambda)}$$

where $(<L^0>_t)_{t \geq 0}$ is the quadratic variation of the (scalar) local martingale $(L^0_t)_{t \geq 0}$ and $(v_t)_{t \geq 0}$ is a progressively measurable process with values in $L^2(\Lambda)$. Let

$$D_T := \frac{1}{Z_T} e^{-V_T(W_T)} \left( \frac{dQ^T}{dP} \right)^{-1}.$$
be the density of $\mathbb{P}^T$ wrt. $Q^v$. We will show that it is possible to choose $v$ in such a way that the family $(D_T)_{T \geq 0}$ is uniformly integrable under $Q^v$ and that $D_T \to D_\infty$ weakly in $L^1(Q^v)$. With particular choice of $v$ we call $Q^v$ the drift measure: it is the central object of this paper. By Girsanov’s theorem the canonical process $(W_t)_{t \geq 0}$ satisfies the equation

$$dW_t = v_t dt + d\tilde{W}_t, \quad t \geq 0,$$

where $(\tilde{W}_t)_{t \geq 0}$ is a Gaussian martingale under $Q^v$ (and has law equal to that of $(W_t)_{t \geq 0}$ under $\mathbb{P}$, i.e. it is a regularized Gaussian free field). We will show also that the drift $v_t$ can be written as a (polynomial) function of $(\tilde{W}_s)_{s \in [0,t]}$, that is $v_t = \tilde{V}_t((\tilde{W}_s)_{s \in [0,t]})$. Therefore we have an explicit description of the process $(W_t)_{t \geq 0}$ under the drift measure $Q^v$ as the unique solution of the path-dependent SDE

$$dW_t = \tilde{V}_t((\tilde{W}_s)_{s \in [0,t]})dt + d\tilde{W}_t, \quad t \geq 0. \quad (1.3)$$

Let us note that this formula expresses the “interacting” random field $(W_t)$, as a function of the “free” field $(\tilde{W}_t)$. It is a formula which shares very similar technical merits with the stochastic quantization approach.

The drift measure $Q^v$ is half way between the variational description in [4] and the (formal) Gibbsian description of eq. (1.1). It constitutes a measure which is relatively explicit, easy to construct and analyze and which can be used as reference measure for $\Phi_3^1$, very much like the Gaussian free field can be used as reference measure for the stochastic quantization approach.

As another application of (1.4) we provide also a self-contained proof of the singularity of $\Phi_3^1$ measure $\nu$ wrt. the Gaussian free field $\mu$. We have already remarked that the singularity of $\Phi_3^1$ seems to belongs to the folklore and we were not able to trace any written proof of that. However, M. Hairer, during a conference at Imperial College in 2019, showed us an unpublished proof of his of singularity using the stochastic quantization equation. Our proof and his are very similar and we do not claim any essential novelty in this respect. Albeit the proof is quite straightforward we wrote down
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all the details in order to provide a reference for this fact. The basic idea is to consider the observable $\int_A [W^2]_t$ where the brackets denotes Wick products and prove that it diverge with different speed as $T \to \infty$ under the measure $P$ and $Q^\omega$ because in the first case the process $(W_t)$ is a Brownian martingale and therefore by the properties of Wick products also the process $(\int_A [W^2]_t)_t$ is a martingale with variance growing like $T$. Under the measure $Q^\omega$ however the presence of the drift $(V_t)_{t \geq 0}$ produces a deterministic contribution whose size is also $T$ and which dominates the fluctuations of the observable. Therefore the singularity of $\Phi^4_3$ can be directly linked with the pathwise properties of the scale-by-scale process $(W_t)_{t \geq 0}$ in the ultraviolet region and our proof of singularity shows also that the drift measure $Q^\omega$ is singular wrt. $P$. Intuitively, the drift $(V_t)_{t \geq 0}$ in the SDE (1.3) is not regular enough (as $t \to \infty$) to be along Cameron–Martin directions for the law $P$ of the process $(W_t)_{t \geq 0}$ and therefore the Girsanov transform (1.2) gives a singular measure when extended all the way to $T = +\infty$.

Let us stress that the main contribution of the present paper remains that of describing the drift measure as a novel object in the context of $\Phi^4_3$ and similar measures and pursuing the study of Euclidean quantum fields from the point of view of stochastic analysis.

Notations. Let us fix some notations and objects.

- For $a \in \mathbb{R}^d$ we let $\langle a \rangle := (1 + |a|^2)^{1/2}$, $B(x,r) \subseteq \mathbb{R}$ denotes the open ball of center $x \in \mathbb{R}$ and radius $r > 0$. We write $A \lesssim B$ for $A \leq C B$ for some constant $C$ and $A \asymp B$ for $A \lesssim B$ and $B \lesssim A$.

- The constant $\kappa > 0$ represents a small positive number which can be different from line to line.

- Denote with $\mathcal{S}(\Lambda)$ the space of Schwartz functions on $\Lambda$ and with $\mathcal{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ stands for the space Fourier transform of $f$ and we will write $g(D)$ to denote the Fourier multiplier operator with symbol $g : \mathbb{R}^d \to \mathbb{R}$, i.e. $\hat{f} = g(D)f$.

- $B^\alpha_{p,q} = B^\alpha_{p,q}(\Lambda)$ denotes the Besov spaces of regularity $\alpha$ and integrability indices $p, q$ as usual. $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\Lambda)$ is the H"older–Besov space $B^\alpha_{\infty,\infty}$, $W^{\alpha,p} = W^{\alpha,p}(\Lambda)$ denote the standard fractional Sobolev spaces defined by the norm $\|f\|_{W^{\alpha,p}} := \|\langle D \rangle^\alpha f\|_{L^p}$ and $H^\alpha = W^{\alpha,2}$. The symbols $\prec, \succ, \sim$ denotes spatial paraproducts wrt. a standard Littlewood–Paley decomposition. The reader is referred to Appendix A for an overview of the various functional spaces and paraproducts.

2 The setting

The setting of this paper is the same of that in our variational study [4]. In this section we will briefly recall it and also state some results from that paper which will be needed below. They concern the Boué–Dupuis formula and certain estimates which are relevant to the analysis of absolute continuity.

Let $\Omega := C(\mathbb{R}_+; \mathcal{C}^{-3/2}_{-\infty}(\Lambda))$ and $\mathcal{F}$ be the Borel $\sigma$–algebra of $\Omega$. On $(\Omega, \mathcal{F})$ consider the probability measure $P$ which makes the canonical process $(X_t)_{t \geq 0}$ a cylindrical Brownian motion on $L^2(\Lambda)$ and let $(\mathcal{F}_t)_{t \geq 0}$ the associated filtration completed with respect sets of $P$–measure 0. In the following $E$ without any qualifiers will denote expectations wrt. $P$ and $E_Q$ will denote expectations wrt. some other measure $Q$.

On the probability space $(\Omega, \mathcal{F}, P)$ there exists a collection $(B^n_t)_{n \in \mathbb{Z}^3}$ of complex (2-dimensional) Brownian motions, such that $B^n_t = B^{-n}_t$, $B^n_t$, $B^m_t$ independent for $m \neq \pm n$ and $X_t = \sum_{n \in \mathbb{Z}^3} e^{i(n \cdot \cdot \cdot)} B^n_t$, for example in $\mathcal{S}'(\Lambda)$.
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Fix some decreasing $\rho \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$ such that $\rho = 1$ on $B(0, 9/10)$ and $\text{supp}\, \rho \subset B(0, 1)$. For $x \in \mathbb{R}^3$ let $\rho(x) := \rho((x)/t)$ and

$$\sigma_t(x) := \left[ \frac{d}{dt}(\rho_t^2(x)) \right]^{1/2} = (-2((x)/t)\rho((x)/t)\rho'(x)/t)^{1/2}/t^{1/2}. $$

Denote $J_s = \sigma_s(D)(D)^{-1}$ and consider the process $(W_t)_{t \geq 0}$ defined by

$$W_t := \int_0^t J_s \, dX_s = \sum_{n \in \mathbb{Z}^3} e^{in x} \int_0^t \frac{\sigma_s(n)}{\langle n \rangle^2} \, dB_s^n, \quad t \geq 0. $$

(2.1)

It is a centered Gaussian process with covariance

$$E[(W_t, \varphi)(W_s, \psi)] = \sum_{n \in \mathbb{Z}^3} \frac{\rho_{\min(s,t)}^2(n)}{\langle n \rangle^2} \varphi(n) \overline{\psi(n)},$$

for any $\varphi, \psi \in \mathcal{S}(\Lambda)$ and $t, s \geq 0$, by Fubini theorem and Itô isometry. By dominated convergence $\lim_{t \to \infty} E[(W_t, \varphi)(W_t, \psi)] = \sum_{n \in \mathbb{Z}^3} (n)^{-2} \varphi(n) \overline{\psi(n)}$ for any $\varphi, \psi \in L^2(\Lambda)$. For any finite “time” $T$ the random field $W_T$ on $\Lambda$ has a bounded spectral support and the stopped process $W^T_t = W_t \wedge T$ for any fixed $T > 0$, is in $C(\mathbb{R}_+ \times C^{\infty}(\Lambda))$. Furthermore $(W^T_t)_t$ only depends on a finite subset of the Brownian motions $(B^n_{\mathbb{R}})_{n \in \mathbb{Z}^3}$.

Observe that $J_t$ satisfies the following bound

$$\|J_t f\|_{B_{p,p}^{s+1} \to} \lesssim \langle t \rangle^{-\alpha - 1/2} \|f\|_{B_{p,p}^s},$$

for any function $f \in B_{p,p}^s$ or $f \in W^{s,p}$ with $p \in [1, \infty]$ and $s \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$.

We will denote by $[\|W_t^s\|]$, $n = 1, 2, 3$, the $n$-th Wick-power of the Gaussian random variable $W_t$ (under $P$) and introduce the convenient notations $W_t^2 := 12\|W_t^2\|$, $W_t^3 := 4\|W_t^3\|$. Furthermore we will write $\|((D)^{-1/2}W_t^n)\|, n \in \mathbb{N}$ for the $n$-th Wick-power of $(D)^{-1/2}W_t$. It exists for any $0 < t < \infty$ and any $n \geq 1$ since it is easy to see that $(D)^{-1/2}W_t$ has a covariance with a diagonal behavior which can be controlled by $\log(t)$. These Wick powers converge as $T \to \infty$ in spaces of distributions with regularities given in the following table:

| $W$ | $W^2$ | $W^3$ | $[\|((D)^{-1/2}W_t^n)\|]$ |
|---|---|---|---|
| $C^{\infty}$ | $C^{\infty}$ | $C^{\infty}$ | $C^{\infty}$ |

We denote by $\mathcal{H}_a$ the space of $(\mathcal{F}_t)_{t \geq 0}$ progressively measurable processes which are $\text{P}$-almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element $v$ of $\mathcal{H}_a$ is a drift. Below we will also need drifts belonging to $H^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$ where $H^\alpha(\Lambda)$ is the Sobolev space of regularity $\alpha \in \mathbb{R}$ and we will denote the corresponding space with $\mathcal{H}_a^\alpha$. For any $v \in \mathcal{H}_a$ define the measure $Q^v$ on $\Omega$ by

$$\frac{dQ^v}{dp} = \exp \left[ \int_0^\infty v_s \, dX_s - \frac{1}{2} \int_0^\infty \|v_s\|^2_{L^2} \, ds \right].$$

Denote with $\mathcal{H}_c \subset \mathcal{H}_a$ the set of drifts $v \in \mathcal{H}_a$ for which $Q^v(\Omega) = 1$, and set $W^v := W - I(v)$, where

$$\int_t^s \frac{dQ^v}{dp} = \exp \left[ \int_0^\infty v_s \, dX_s - \frac{1}{2} \int_0^\infty \|v_s\|^2_{L^2} \, ds \right].$$

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We will need also the following objects. For all \( t \geq 0 \) let \( \theta_t : \mathbb{R}^3 \to [0, 1] \) be a smooth function such that
\[
\theta_t(\xi) \sigma_s(\xi) = 0 \quad \text{for } s \geq t, \\
\theta_t(\xi) = 1 \quad \text{for } |\xi| \leq t/2 \quad \text{provided that } t \geq T_0
\] (2.2)
for some \( T_0 > 0 \). For example one can fix smooth functions \( \tilde{\theta}, \eta : \mathbb{R}^3 \to \mathbb{R}_+ \) such that
\[
\tilde{\theta}(\xi) = 1 \quad \text{if } |\xi| \leq 1/2 \quad \text{and } \tilde{\theta}(\xi) = 0 \quad \text{if } |\xi| \geq 2/3, \\
\eta(\xi) = 1 \quad \text{if } |\xi| \leq 1 \quad \text{and } \eta(\xi) = 0 \quad \text{if } |\xi| \geq 2.
\]
Then let \( \theta_t(\xi) := (1 - \eta(\xi))\tilde{\theta}(\xi) + \zeta(t)\eta(\xi)\tilde{\theta}(\xi) \) where \( \zeta(t) : \mathbb{R}_+ \to \mathbb{R} \) is a smooth function such that \( \zeta(t) = 0 \) for \( t \leq 10 \) and \( \zeta(t) = 1 \) for \( t \geq 3 \). Then eq. (2.2) holds with \( T_0 = 3 \). Let
\[
f^\phi := \theta(D)f
\] (2.3)
for any \( f \in \mathcal{S}'(\Lambda) \).

Our aim here to study the measures \( \mu_T \) defined on \( \mathcal{C}^{-1/2-\kappa} \) as
\[
\frac{d\mu_T}{d\mathbb{P}} = e^{-V_T(W_T)},
\]
with
\[
V_T(\varphi) := \lambda \int_\Lambda (\varphi^4 - a_T \varphi^2 + b_T)dx, \quad \varphi \in C^\infty(\Lambda),
\] (2.4)
and suitable \( a_T, b_T \to \infty \). For convenience the measure \( \mu_T \) is not normalized and, wrt. to the notations in the introduction we have
\[
\frac{d\mathbb{P}^T}{d\mu^T} = \frac{1}{\mu^T(\Omega)}.
\]
Recall the following results of [4].

**Theorem 2.1.** For any \( a_T, b_T \in \mathbb{R} \), and \( f : \mathcal{C}^{-1/2-\kappa}(\Lambda) \to \mathbb{R} \) with linear growth let
\[
V_T^f(\varphi) := f(\varphi) + V_T(\varphi),
\]
where \( V_T \) is given by (2.4). Then the variational formula
\[
-\log \int_{\mathcal{S}'(\Lambda)} e^{-V_T^f(\varphi)}\mu_T(d\varphi) = -\log \mathbb{E}[e^{-V_T^f(W_T)}] = \inf_{u \in \mathcal{H}_u} \mathbb{E}\left[ V_T^f(W_T + I_T(u)) + \frac{1}{2} \int_0^T \|u_s\|_{L^2(\Lambda)}^2 dt \right]
\] (2.5)
holds for any finite \( T \).

This is a consequence of the more general Boué–Dupuis formula.

**Theorem 2.2 (BD formula).** Assume \( F : C([0, T], C^\infty(\Lambda)) \to \mathbb{R} \), be Borel measurable and such that there exist \( p, q \in (1, \infty) \), with \( 1/p + 1/q = 1 \), \( \mathbb{E}[|F(W)|^p] < \infty \) and \( \mathbb{E}[|e^{-F(W)}|^q] < \infty \) (where we can regard \( W \) as an element of \( C([0, T], C^\infty(\Lambda)) \) by restricting to \([0, T])\). Then
\[
-\log \mathbb{E}[e^{-F(W)}] = \inf_{u \in \mathcal{H}_u} \mathbb{E}\left[ F(W + I(u)) + \frac{1}{2} \int_0^T \|u_s\|_{L^2(\Lambda)}^2 dt \right].
\] (2.6)

We will use several times below eq. (2.6) in order to control exponential integrability of various functionals. By a suitable choice of renormalization and a change of variables in the control problem (2.5) we were able in [4] to control the functional in Theorem 2.1 uniformly up to infinity.
Theorem 2.3. There exist a sequence \((a_T, b_T)_T\) with \(a_T, b_T \to \infty\) as \(T \to \infty\), such that

\[
\mathbb{E} \left[ V^I_T(W_T + I_T(u)) + \frac{1}{2} \int_0^T \|u_t\|_{L^2(\Lambda)}^2 \, dt \right] = \mathbb{E} \left[ \Psi^I_T(W, I(u)) + \lambda \int (I_T(u))^4 + \frac{1}{2} \|I_T^2(u)\|_{H^2}^2 \right]
\]

where (recall that \(I^I_T(u) = \theta(D)I_T(u)\) by (2.3))

\[
I^I_T(u) := u_t + \lambda I_{t \leq T} W^{(3)} + \lambda I_{t \leq T} J_t(W^2_\perp \succ I^I_T(u)) \tag{2.7}
\]

and the functionals \(\Psi^I_T : C([0, T], C^\infty(\Lambda)) \times C([0, T], C^\infty(\Lambda)) \to \mathbb{R}\) satisfy the following bound

\[
|\Psi^I_T(W, I(u))| \leq Q_T(W) + \frac{1}{4} \|I_T(u)\|_{L^4}^4 + \|I_T^2(u)\|_{H^2}^2
\]

where \(Q_T(W)\) is a function of \(W\) independent of \(u\) and such that \(\sup_T \mathbb{E}[Q_T(W)] < \infty\).

As a consequence we obtain the following corollary (cfr. Corollary 1 and Lemma 6 in [4])

**Corollary 2.4.** For \(f : \mathcal{C}^{-1/2-\kappa}(\Lambda) \to \mathbb{R}\) with linear growth the bound

\[-C \leq \mathbb{E}_{\mu_T}[e^f] \leq C,
\]

holds, with a constant \(C\) independent of \(T\). In particular \(\mu_T\) is tight on \(\mathcal{C}^{-1/2-\kappa}\).

**3 Construction of the drift measure**

We start now to implement the strategy discussed in the introduction: identify a translated measure sufficiently similar to \(\Phi^1_T\). Intuitively the \(\Phi^1_T\) measure should give rise to a canonical process which is a shift of the Gaussian free field with a drift of the form given by eq. (2.7). Indeed this drift \(u\) should be the optimal drift in the variational formula. A small twist is given by the fact that the relevant Gaussian free field entering these considerations is not the process \(W = W(X)\) but that obtained from the shifted canonical process \(X^T_t = X_t - \int_0^t u_s \, ds\) which we denote by

\[W^u := W(X^u) = W - I(u).\]

Moreover, to prevent explosion at finite time, we have to modify the drift in large scales and add a coercive term. This will also allow later to prove some useful estimates. As a consequence, we define the functional

\[\Xi_s(W, u := -\lambda J_s W^3 + \lambda I_{s \geq T} J_s(W^2 \succ I^I_s(u)) + J_s(D)^{-1/2} (\|D^{-1/2} W_s\|^n) \tag{3.1}\]

where \(T > 0, n \in \mathbb{N}\) are constants which will be fixed later on and where we understand all the Wick renormalizations to be given functions of \(W\), i.e. polynomials in \(W\) where the constants are determined according to the law of \(W\) under \(P\). We look now for the solution \(u\) of the equation

\[u = \Xi(W^u, u) = \Xi(W - I(u), u).\] (3.2)

Expanding the Wick polynomials appearing in \(\Xi(W - I(u), u)\) we obtain the equation

\[
u_s = \Xi(W - I(u), u) = -\lambda J_s[W^3 + W^2 I_s(u) + 12W_s(I_s(u))^2 - 4(I_s(u))^3] - \lambda I_{s \geq T} J_s([W^2 - 24W_s I_s(u) + 12(I_s(u))^2] \succ I^I_s(u)) + \sum_{i=0}^n \binom{n}{i} J_s(D)^{-1/2} (\|D^{-1/2} W_s\|^n) \left[ -\lambda (D^{-1/2} I_s(u))^{n-i} \right] \tag{3.3}\]

\[\Xi \in \mathbb{E}^{P} (\text{The \Phi^1_T measure via Girsanov’s theorem})
\]
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for all $s \geq 0$. This is an integral equation for $t \mapsto u_t$ with smooth coefficients depending smoothly on $W$ and can be solved via standard methods. Since the coefficients are of polynomial growth the solution could explode in finite time. Note that for any finite time the process $(u_s)_{s \geq 0}$ has bounded spectral support. As a consequence we can solve the equation in $L^2$ and as long as $\int_0^t ||u_s||^2_{L^2} ds$ is finite we can see from the equation that $\sup_{s \leq t} ||u_s||^2_{L^2}$ is finite. By the existence of local solutions we have that, for all $N \geq 0$, the stopping time

$$
\tau_N := \inf\{ t \geq 0 | \int_0^t ||u_s||^2_{L^2} ds \geq N \},
$$
is strictly positive $\mathbb{P}$-almost surely and $u$ exists up to the (explosion) time $T_{\exp} := \sup_{N \in \mathbb{N}} \tau_N$. The following lemma will help to show that $\mathbb{P}$-almost surely $T_{\exp} = +\infty$ and will also be very useful below.

**Lemma 3.1.** Let

$$
\text{Aux}_s(W, w) := \sum_{i=0}^{n} \binom{n}{i} J_s((D)^{-1/2}(\{((D)^{-1/2}W_s)^i\}((D)^{-1/2}I_s(w))^{n-i}).
$$

then we have

$$
\begin{align*}
\mathbb{E} \int_0^t ||w_s||^2_{L^2} ds + \sup_{s \leq t} \mathbb{E} ||I_s(w)||_{W^{-1/2, n+1}}^{n+1} \\
\leq 1 + \int_0^t (2\mathbb{E} ||w_s + g_s||^2_{L^2} + 4\mathbb{E} ||g_s - \text{Aux}_s(W, w)||^2_{L^2}) ds,
\end{align*}
$$

uniformly in $t \geq 0$, for any pair of adapted processes $w, g \in L^2(\mathbb{P}, \mathcal{H})$ such that

$$
\mathbb{E} \int_0^t ||g_s - \text{Aux}_s(W, w)||^2_{L^2} ds < \infty.
$$

**Proof.** Take $t_n = \inf\{ t > 0 : \int_0^t ||w_s||^2_{L^2} ds > N \}$. By Itô’s formula we have

$$
\int_0^{\tau_{N+1}} \int_\Lambda \text{Aux}_s(W, w) w_s \, ds = \overline{\text{Aux}}_{t, \Lambda, N}(W, w) + \text{martingale}
$$

where

$$
\overline{\text{Aux}}_t(W, w) := \sum_{i=0}^{n} \frac{1}{n+1-i} \binom{n}{i} \int_\Lambda \{((D)^{-1/2}W_t)^i\}((D)^{-1/2}I_t(w))^{n+1-i}.
$$

Integrating over the probability space and using Cauchy–Schwarz inequality, we obtain

$$
\frac{d}{dt} \mathbb{E} \left( \int_0^{\tau_{N+1}} ||w_s||^2_{L^2} ds + 4\overline{\text{Aux}}_{t, \Lambda, N}(W, w) \right)
= \mathbb{E} \left[ \int_\Lambda I_{\{t \leq \tau_N\}} (w_t^2 + 4 \text{Aux}_t(W, w) w_t) \right]
\leq \mathbb{E} \left[ \int_\Lambda I_{\{t \leq \tau_N\}} \left( 2||w_t + g_t||^2_{L^2} + 4 \text{Aux}_t(W, w) - g_t \right) w_t - ||w_t||^2_{L^2} \right]
\leq 2\mathbb{E} \left[ \int_\Lambda I_{\{t \leq \tau_N\}} ||w_t + g_t||^2_{L^2} + 4\text{Aux}_t(W, w) - \text{Aux}_t(W, w) ||^2_{L^2} \right].
$$

where $g_t$ is an arbitrary function. By Lemma 3.7 below, we have constants $c, C$ and a random variable $Q_T(W)$ such that

$$
\sup_{t \in \mathbb{R}} \sup_{N \in \mathbb{N}} \mathbb{E} [Q_{t, \Lambda, N}(W)] < \infty,
$$

and

$$
c \int_0^t ||w_s||^2_{L^2} ds + c ||I_t(w)||_{W^{-1/2, n+1}}^{n+1} - Q_t(W) \leq \int_0^t ||w_s||^2_{L^2} ds + \overline{\text{Aux}}_t(W, w)
$$

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where we introduced the notations $$W$$ which implies $$T$$ As a consequence, we deduce

$$N$$

Then, for any $$n$$ provided $$n$$ is chosen sufficiently large. Using Gronwall’s inequality this gives

And we can conclude by sending $$N \to \infty$$ and using monotone convergence. □

In particular, taking $$w = -I_{t \leq \tau_N} u$$ and $$g = -w$$, we have

$$E \int_0^t \|I_{s \leq \tau_N} u_s\|^2_{L^2} ds + \sup_{s \leq t} E\|I_s(I_{s \leq \tau_N} u)\|_{W^{-1/2,n+1}}^{n+1} \leq 1 + \int_0^t E(I_{s \leq \tau_N} \|u_s - Aux_s(W,-u)\|^2_{L^2}) ds,$$

for all $$t \leq T$$, where, using (3.3),

$$u_s - Aux_s(W,-u) = -\lambda J_s[W_s^3 - W_s^2 I_s(u) + 12 W_s(I_s(u))^2 - 4(I_s(u))^3] - J_s[\langle (W_s^2 - 4W_s I_s(u) + 12(I_s(u))^2) \rangle + \langle T_s(u) \rangle].$$ (3.5)

Then, for any $$s \leq T$$ we have

$$E(I_{s \leq \tau_N} \|u_s - Aux_s(W,-u)\|^2_{L^2}) \leq C_T + E\|I_s(I_{s \leq \tau_N} u)\|_{W^{-1/2,n+1}}^{n+1},$$

provided $$n$$ is chosen sufficiently large. Using Gronwall’s inequality this gives

$$E \int_0^T \|I_{s \leq \tau_N} u_s\|^2_{L^2} ds \leq C_T,$$

and we can let $$N \to \infty$$ to obtain

$$E \int_0^T \|u_s\|^2_{L^2} ds \leq C_T,$$

which implies $$T_{exp} = +\infty$$. In addition and by construction, the process $$u_t^n := I_{t \leq \tau_N} u_t$$ satisfies Novikov’s condition, so it is in $$H_s$$ and Girsanov’s transformation allows us to define the probability measure $$Q^{u^n}$$ on $$C(\mathbb{R}_+, \mathcal{C}^{-1/2-\kappa}(A))$$ given by

$$dQ^{u^n} := \exp \left[ \int_0^\infty u^n_s dX_s - \frac{1}{2} \int_0^\infty \|u^n_s\|^2_{L^2(A)} ds \right] dP,$$

under which $$X_t^n = X_t - \int_0^t u^n_s ds$$ is a cylindrical Brownian motion. Moreover, under $$Q^{u^n}$$ the process $$(W_t^n := \int_0^t J_s dX_s^n)_{t \geq 0}$$ has the same law as $$(W_t)_{t \geq 0}$$ under $$P$$. We observe also that $$W^n_s = W^n_u$$ for $$0 \leq s \leq \tau_N$$ and that $$u$$ satisfies the equation

$$u_s = -\lambda J_s W^n_s, -\lambda I_{s \geq T} J_s(W^n_s) + J_s(D)^{-1/2}([[(1-(D)^{-1/2}W^n_s)])],$$ (3.6)

where we introduced the notations $$W^n_s := 4[[W^n_s]]$$ and $$W^n_u := 12[[W^n_u]]$$. Note that here the Wick powers are still taken to be given functions of $$W$$, i.e. we are still taking
The \( \Phi^3 \) measure via Girsanov’s theorem

the Wick ordering with respect to the law of \( W \) under \( P \) (or, equivalently, the law of \( W^u \) under \( Q^u \)).

If we think of the terms containing \( W^u \) as given (that is, we ignore their dependence on \( u \)), eq. (3.6) is a linear integral equation in \( u \) which can be estimated via Gronwall-type arguments. In order to do so, let us denote by \( U : H \mapsto \tilde{u} \) the solution map of the equation

\[
\tilde{u} = \Xi(H, \tilde{u}).
\]  

(3.7)

This last equation is linear and therefore has nice global solutions (let’s say in \( C(\mathbb{R}_+, L^2) \)) and by uniqueness and eq. (3.6) we have \( u_t = U_t(W^u) \) for \( t \in [0, T_{\exp}) \). From this perspective the residual dependence on \( u \) will not play any role since under the shifted measure the law of the process \( W^u \) does not depend on \( u \). By standard paraproduct estimates (see Appendix A) we have

\[
\| I_t(u) \|_{L^\infty} \lesssim \tilde{H}_t + \int_0^t \mathbf{1}_{\{s \geq T \}} \| J_s^2(W^u) \|_{L^\infty} \, ds
\]

\[
\lesssim \tilde{H}_t + \tilde{T}^{-\kappa} \int_0^t \langle s \rangle^{-3/2} \| W^u \|_{L^{q-1-\kappa}} \| I_s^0(u) \|_{L^\infty} \, ds,
\]

where we have crucially exploited the presence of the cutoff \( \mathbf{1}_{\{s \geq T \}} \) to introduce the small factor \( \tilde{T}^{-\kappa} \) and we have employed the notation

\[
\tilde{H}_t := \int_0^t \| J_s^2(W^u) \|_{L^\infty} + \| J_s(D)^{-1/2}(\langle (D)^{-1/2} W^u \rangle^n) \|_{L^\infty} \, ds
\]

\[
\lesssim \int_0^t \frac{1}{\langle s \rangle^{1/2+\kappa}} \| J_s W^u \|_{L^{q-1-\kappa}} + \int_0^t \frac{1}{\langle s \rangle^{1/2}} \| \langle (D)^{-1/2} W^u \rangle^n \|_{H^{-1/2}} \, ds.
\]

By Gronwall’s lemma,

\[
\sup_{t \leq \tau_N} \| I_t(u) \|_{L^\infty} \lesssim \tilde{H}_{\tau_N} \exp \left[ C \tilde{T}^{-\kappa} \int_0^\tau_N \| W^u \|_{L^{q-1-\kappa}} \, ds \right] .
\]  

(3.8)

Under \( Q^u \), the terms in \( \tilde{H}_{\tau_N} \) are in all the \( L^p \) spaces by hypercontractivity and moreover for any \( p > 1 \) one can choose \( \tilde{T} \) large enough so that also the exponential term is in \( L^p \). Using eq. (3.6) it is then not difficult to show that the expectation \( E_{Q^N} \| u^{N_2} \|_{H^{-1/2-\kappa}}^p < \infty \) is finite for any \( p > 1 \) (again provided we take \( \tilde{T} \) large enough depending on \( p \)) as long as \( N_1 > N_2 \). By the spectral properties of \( J \) and the equation for \( u \), the process \( t \mapsto \mathbf{1}_{\{t < T \}} u_t \) is spectrally supported in a ball of radius \( T \), so we get in particular that

\[
E_{Q^N} \left[ \int_0^{\tau_{N_2}} \| u_s \|_{L^2}^2 \, ds \right] \lesssim T^{1+\kappa},
\]

uniformly for any choice of \( N_1 > N_2 \).

**Lemma 3.2.** The family \( \{Q^u_N\}_N \) weakly converges to a limit \( Q^u \) on \( C(\mathbb{R}_+, \mathcal{F}^{-3/2-\kappa}) \). Under \( Q^u \) it holds \( T_{\exp} = \infty \) almost surely and \( \text{Law}_{Q^u}(X^u) = \text{Law}_P(X) \). Moreover for any finite \( T \)

\[
\frac{dQ^u}{dP} \big|_{\mathcal{F}_T} = \exp \left[ \int_0^T u_s dX_s - \frac{1}{2} \int_0^T \| u_s \|_{L^2}^2 \, ds \right].
\]

**Proof.** Consider the filtration \( (\mathcal{F}_N = \mathcal{F}^{T_N}_N) \) and observe that \( \left( Q^N \big|_{\mathcal{F}_T} \right)_N \) is a consistent family of inner regular probability distributions and therefore there exists a unique

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This establishes that Law as claimed.

Lemma 3.3. For any $p > 1$ there exists a suitable choice of $\tilde{T}$ such that $\mathbb{E}_{\Phi_3} \left[ \sup_{t \geq 0} \|I_t(u)\|_{L^p}^p \right] < \infty$.

Proof. This follows from the bound (3.8), after choosing $\tilde{T}$ large enough, where we recall that $\tilde{T}$ has been introduced in the definition of the drift through eq. (3.1). \qed
The $\Phi_3^*$ measure via Girsanov’s theorem

3.1 Proof of absolute continuity

In this section we prove that the measure $\mu_T$ is absolutely continuous with respect to the measure $Q^u$ which we constructed in Lemma 3.2. First recall that the measures $\mu_T$ defined on $\Omega$ as

$$\frac{d\mu_T}{d\mathbb{P}} = e^{-V_T(W_T)}$$

can be described, using Lemma 3.2, as a perturbation of $Q^u$ with density $D_T$ given by

$$D_T := \left. \frac{d\mu_T}{dQ^u} \right|_{\mathcal{F}_T} = \frac{d\mu_T}{d\mathbb{P}} \left|_{\mathcal{F}_T} \right. \mathbb{P} = e^{-V_T(W_T) - \int_0^T \theta_t \, dt},$$

at least on $\mathcal{F}_T$.

**Lemma 3.4.** There exists a $p > 1$, such that for any $K > 0$,

$$\sup_T E_{Q^u} \left[ |D_T|^p \mathbb{1}_{\{W_\infty \geq 1/2 \geq K\}} \right] < \infty. \tag{3.9}$$

In particular, the family $(D_T)_T$ is uniformly integrable under $Q^u$.

**Proof.** The proof eq. (3.9) is given in Section 3.2 below. For the uniform integrability fix $\epsilon > 0$. Our aim is to show that there exists $\delta > 0$ such that $Q^u(A) < \delta$ implies $\int_A D_T \, dQ^u < \epsilon$. From Corollary 2.4, for any $\epsilon > 0$ there exists a $K > 0$ such that

$$\epsilon/2 > \mu_T\left(\{\|W_\infty\|_{\mathcal{F}} \geq K\}\right) = \int_{\{\|W_\infty\|_{\mathcal{F}} \geq K\}} D_T \, dQ^u.$$

Then for any $A \in \mathcal{F}$ such that $Q^u(A)^{(p-1)/p} < \epsilon/ \left(2 \sup_T E_{Q^u} \left[ |D_T|^p \mathbb{1}_{\{W_\infty \geq 1/2 \geq K\}} \right]\right)$

$$\int_A D_T \, dQ^u = \int_A \mathbb{1}_{\{\|W_\infty\|_{\mathcal{F}} \geq K\}} D_T \, dQ^u + \int_{A^c} \mathbb{1}_{\{\|W_\infty\|_{\mathcal{F}} < K\}} D_T \, dQ^u \leq \epsilon/2 + \sup_T E_{Q^u} \left[ |D_T|^p \mathbb{1}_{\{W_\infty \geq 1/2 \geq K\}} \right] Q^u(A)^{(p-1)/p} \leq \epsilon$$

**Corollary 3.5.** The family of measures $(\mu_T)_{T \geq 0}$ is sequentially compact w.r.t. strong convergence on $(\Omega, \mathcal{F})$. Furthermore any accumulation point is absolutely continuous with respect to $Q^u$.

**Proof.** We choose a sub-sequence (not relabeled) such that $D_T \to D_\infty$ weakly in $L^1(Q^u)$, for some $D_\infty \in L^1(Q^u)$. It always exists by uniform integrability. We now claim that for any $A \in \mathcal{F}$

$$\lim_{T \to \infty} \mu_T(A) = \int_A D_\infty \, dQ^u.$$

It is enough to check this for $A \in \mathcal{F}_S$ for any $S \in \mathbb{R}_+$ since these generate $\mathcal{F}$. But there we have for $T \geq S$,

$$\mu_T(A) = \int_A D_T \, dQ^u \to \int_A D_\infty \, dQ^u$$

by weak $L^1$ convergence. \qed

Recall that the $\Phi_3^*$ measure can be defined as a weak limit of the measures $\tilde{\mu}_T$ on $\mathcal{F}^{1/2}$ given by

$$\int f(\varphi) \tilde{\mu}_T(d\varphi) = \int f(\varphi) e^{-V_T(\varphi)} \theta_T(d\varphi) = E_P[f(W_T)e^{-V_T(W_T)}]$$
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where $\theta_T$ is the Gaussian measure with covariance $\rho^2_T(D)(D)^{-2}$. From this, together with the above considerations, we see that any accumulation point $\tilde{\mu}_\infty$ of $\tilde{\mu}_T$ satisfies

$$\tilde{\mu}_\infty(A) = \mathbb{E}_{Q^w}[\mathbb{I}_A(W_\infty)D_\infty],$$

for some $D_\infty \in L^1(Q^w)$.

### 3.2 $L^p$ bounds

Now we will prove local $L^p$-bounds on the density $D_T$. In the sequel we will denote $\tilde{W} = W^u$, with $u$ satisfying \((3.3)\), namely $u = U(\tilde{W})$. Before we proceed let us study how the functional $U(\tilde{W})$ behaves under shifts of $\tilde{W}$, since later we will want to apply the Boué–Dupuis formula and this kind of behavior will be crucial. Let $w \in L^2([0, \infty) \times \Lambda)$ and denote

$$u^w := U(\tilde{W} + I(w)) \quad \text{and} \quad h^w := U(\tilde{W} + I(w)) + w = u^w + w.$$  

The process $h^w$ satisfies

$$h^w - w = u^w = \Xi(\tilde{W} + I(w), u^w).$$

More explicitly, for all $s \geq 0$ we have

$$h^w_s - w_s = -4\lambda J_s[\tilde{W}^3_s] - 12\lambda J_s[\tilde{W}_s^2]I_s(w) - 12\lambda J_s[\tilde{W}_s^2I_s(w)] - 4\lambda J_s[\tilde{W}_s^2I_s(w)]^3 - 12\lambda J_s[\tilde{W}_s^2I_s(w)] - 24\lambda I_4(\theta_T)(J_s(\tilde{W}_sI_s(w) \sim I_s^4(u^w))) - 12\lambda J_s[\tilde{W}_s^2I_s(w)] - 24\lambda I_4(\theta_T)(J_s(\tilde{W}_sI_s(w) \sim I_s^4(u^w))) - 12\lambda J_s[\tilde{W}_s^2I_s(w)] - 24\lambda I_4(\theta_T)(J_s(\tilde{W}_sI_s(w) \sim I_s^4(u^w))) + \sum_{i=0}^{\infty} \binom{n}{i} J_s((D^{-1/2}\tilde{W}_s)^i)((D^{-1/2}I_s(w))^{n-i}).$$

Decomposing

$$[\tilde{W}_s^2]J_s(w) = [\tilde{W}_s^2]I_s(w) + [\tilde{W}_s^2] - (1 - \theta_s)I_s(w) + [\tilde{W}_s^2I_s(w)] - I_s(w) \sim I_s^4(u^w) - I_s(w) \sim I_s^4(u^w) - I_s(w) \sim I_s^4(u^w) + \sum_{i=0}^{\infty} \binom{n}{i} J_s((D^{-1/2}\tilde{W}_s)^i)((D^{-1/2}I_s(w))^{n-i}).$$

The first two terms in (3.11) will be used for renormalization while the remainder $r_s^w$ contains terms of higher regularity which will have to be estimated in the sequel.

**Proof.** Proof of eq. (3.9) Observe that

$$|\{ \|W_\infty\|_{\theta_{1/2-\epsilon}} \leq K \}| \leq K_n \exp \left( -\|W_\infty\|_{\theta_{1/2-\epsilon}}^n \right) = \exp \left( -\|W_\infty + I_\infty(U(W))\|_{\theta_{1/2-\epsilon}}^n \right)$$

and

$$|D_T|^n = e^{-p[V_T(\tilde{W}_T + I(U(\tilde{W}))) + \int_0^T U_t(\tilde{W})d\tilde{X} + \frac{1}{2} \int_0^T \|U_t(\tilde{W})\|_2^2 dt]}.$$

Combining these two facts we have

$$\mathbb{E}_{Q^w}[|D_T|^n | \{ \|W_\infty\|_{\theta_{1/2-\epsilon}} \leq K \}] \leq \mathbb{E}_{Q^w} \left[ \exp \left( -p \left( V_T(\tilde{W}_T + I_T(U(\tilde{W}))) + \int_0^T U_t(\tilde{W})d\tilde{X} + \frac{1}{2} \int_0^T \|U_t(\tilde{W})\|_2^2 dt \right) - \|W_\infty + I_\infty(U(W))\|_{\theta_{1/2-\epsilon}}^n \right] \right]$$

$$= \mathbb{E} \left[ \exp \left( -p \left( V_T(W_T + I_T(U(W))) + \int_0^T U_t(W)dX_t + \frac{1}{2} \int_0^T \|U_t(W)\|_2^2 dt \right) - \|W_\infty + I_\infty(U(W))\|_{\theta_{1/2-\epsilon}}^n \right] \right].$$
The Boué–Dupuis formula (2.6) provides the variational bound

\[- \log \mathbb{E}_{Q'_{w}} \left[ \left| D_{T} \right|^{p} \mathbb{I}_{\{ \| W \|_{\varphi_{-1/2,-\kappa}} \leq K \}} \right] \geq \inf_{w \in \mathbb{H}} \mathbb{E} \left[ p \left( V_{T}(W_{T} + I_{T}(h^{w})) + \frac{1}{2} \int_{0}^{T} \| h^{w} \|_{L_{2}^{1}}^{2} dt \right) \right. \]
\[+ \left. \frac{1}{2^{p}} \int_{0}^{T} \| w \|_{L_{2}^{1}}^{2} dt + \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} + \frac{1}{2} \int_{T}^{\infty} \| w \|_{L_{2}^{1}}^{2} dt \right] \]

where we have set \( h^{w} = w + U(W + I(w)) \) as above. Recall now that from Theorem 2.3 there exists a constant \( C \), independent of \( T \), such that for each \( h^{w} \),

\[ \mathbb{E} \left[ p \left( V_{T}(W_{T} + I_{T}(h^{w})) + \frac{1}{2} \int_{0}^{T} \| h^{w} \|_{L_{2}^{1}}^{2} dt \right) \right] \geq - C + \frac{1}{4} \mathbb{E} \left[ \lambda \| I_{T}(h^{w}) \|_{L_{2}^{1}}^{4} + \int_{0}^{T} \| I^{T}(h^{w}) \|_{L_{2}^{1}}^{2} dt \right] \]

where

\[ I^{T}(h^{w}) = h_{t}^{w} + \lambda \mathbb{I}_{t \leq T}(W_{t}^{(3)}) + \lambda \mathbb{I}_{t \leq T}(W_{t}^{(4)}) \]

Using eq. (3.11) we compute

\[ I_{t \leq T}^{(3)}(h^{w}) = \mathbb{I}_{t \leq T}^{(3)}(W_{t}^{(3)}) + \mathbb{I}_{t \leq T}^{(3)}(W_{t}^{(4)}) \]

At this point we need a lower bound for

\[ \mathbb{E} \left[ \frac{1}{4} \left( \lambda \| I_{T}(h^{w}) \|_{L_{2}^{1}}^{4} + \int_{0}^{T} \| r_{t}^{w} + w_{t} \|_{L_{2}^{1}}^{2} dt \right) + \frac{1}{2^{p}} \int_{0}^{T} \| w \|_{L_{2}^{1}}^{2} dt \right. \]
\[+ \left. \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} + \frac{1}{2} \int_{T}^{\infty} \| w \|_{L_{2}^{1}}^{2} dt \right] - C. \]

Given that we need to take \( p > 1 \), this expression presents a difficulty in the fact that the term \( \int_{0}^{T} \| w \|_{L_{2}^{1}}^{2} dt \) appears with a negative coefficient. Note that this term cannot easily be controlled via \( \int_{0}^{T} \| r_{t}^{w} + w_{t} \|_{L_{2}^{1}}^{2} dt \) since the contribution \( r_{t}^{w} \), see eq. (3.12), contains factors which are homogeneous in \( w \) of order up to 3. This is the reason we had to localize the estimate, introduce the "good" term \( \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \), and introduce the term \( J_{s}(D)^{-1/2} [(D)^{-1/2} W_{s}^{(w)}] \) in (3.1) which will help us to control the growth of \( r_{t}^{w} \). Indeed in Lemma 3.6 below, a Gronwall argument will allow us to show that \( \int_{0}^{T} \| w \|_{L_{2}^{1}}^{2} dt \) can be bounded by a combination of the other "good" terms as

\[ \mathbb{E} \left[ \int_{0}^{T} \| w \|_{L_{2}^{1}}^{2} dt \right] \leq \mathbb{E} \left[ \| P_{T}^{2}(h) \|_{L_{2}^{1}}^{4} + \| P_{T}^{2}(h) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} + \int_{0}^{T} \| w_{t} + r_{t}^{w} \|_{L_{2}^{1}}^{2} dt + 1 \right]. \]

This implies that for \( 1 < p \ll 2 \),

\[- \log \mathbb{E}_{Q'_{w}} \left[ \left| D_{T} \right|^{p} \mathbb{I}_{\{ \| W \|_{\varphi_{-1/2,-\kappa}} \leq K \}} \right] \geq \inf_{w \in \mathbb{H}} \mathbb{E} \left[ \frac{1}{4} \left( \lambda \| I_{T}(h^{w}) \|_{L_{2}^{1}}^{4} + \int_{0}^{T} \| I^{T}(h^{w}) \|_{L_{2}^{1}}^{2} dt \right) \right. \]
\[+ \left. (1 - p)C \left( \| P_{T}^{2}(h^{w}) \|_{L_{2}^{1}}^{4} + \| P_{T}^{2}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} + \int_{0}^{T} \| I^{T}(h^{w}) \|_{L_{2}^{1}}^{2} dt \right) + \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \right] - C \]
\[ \geq - C \]

which gives the claim. Note that here we used the bound

\[ \mathbb{E} \| I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \leq \mathbb{E} \| W_{\infty} \|_{\varphi_{-1/2,-\kappa}}^{\kappa} + \mathbb{E} \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \]
\[ \leq C + \mathbb{E} \| W_{\infty} + I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \]

as well as the fact that \( \| I_{t}^{(3)}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \leq \| I_{\infty}(h^{w}) \|_{\varphi_{-1/2,-\kappa}}^{\kappa} \) to conclude. \[ \square \]
The following lemmas complete the proof.

**Lemma 3.6.** For $n \in \mathbb{N}$ odd and large enough

\[
E \int_0^T \|w_s\|_{L^2}^2 \, ds \lesssim E \int_0^T \|w_s + r_w\|_{L^2}^2 \, ds + E\|P_T(h_w)\|_{C^{-1/2}}^{n+1} + \|P_T(h_w)\|^4_{L^4} + 1.
\]

**Proof.** Let us recall the notation

\[
\text{Aux}_s(W, w) := \sum_{i=0}^{n} \binom{n}{i} J_i(D)^{-1/2}(I((D)^{-1/2}W_s)^i)((D)^{-1/2}I_s(w))^{n-i}.
\]

Write $r_w = r_w + \text{Aux}_s(W, w)$ and observe that by Lemma 3.1 we with $g = r_w$ we have

\[
E \int_0^T \|w_s\|_{L^2}^2 \, ds + \sup_{s \in [0, 1]} E\|I_s(w)\|_{W^{-1/2, n+1}}^{n+1} \lesssim 1 + \int_0^T (2E\|w_s + r_w\|_{L^2}^2 + 4E\|r_w\|_{L^2}^2) \, ds,
\]

Now by Lemma 3.8 below

\[
(t_1 + \epsilon)\|\tilde{r}_T\|_{L^2}^2 \lesssim \int_0^T \|w_s\|_{L^2}^2 \, ds + \|I_t(w)\|_{W^{-1/2, n+1}}^{n+1} + \|P_T(h_w)\|_{C^{-1/2}}^{n+1} + \|P_T(h_w)\|^4_{L^4} + Q_t(W)
\]

for a random variable $Q_t(W)$ such that for any stopping time $\tau$

\[
\sup_T E[|Q_{\tau\wedge T}(W)|] < \infty,
\]

and

\[
-Q_T(W) + c \int_0^T \|w_s\|_{L^2}^2 \, ds + c\|I_T(w)\|_{W^{-1/2, n+1}}^{n+1} \leq \int_0^T \|w_s\|_{L^2}^2 \, ds + \text{Aux}_T(W, w)
\]

\[
\leq C\|I_T(w)\|_{W^{-1/2, n+1}}^{n+1} + c \int_0^T \|w_s\|_{L^2}^2 \, ds + Q_T(W)
\]

**Proof.** We recall that (see eq. (3.4))

\[
\text{Aux}_T(W, w) = \sum_{i=0}^{n} \frac{1}{n + 1 - i} \binom{n}{i} \int \left\langle \left((D)^{-1/2}W_s\right)^i\right\rangle ((D)^{-1/2}I_T(w))^{n+1-i}
\]

\[
= \sum_{i=1}^{n} \frac{1}{n + 1 - i} \binom{n}{i} \int \left\langle \left((D)^{-1/2}W_s\right)^i\right\rangle ((D)^{-1/2}I_T(w))^{n+1-i}
\]

\[
+ \frac{1}{n + 1} \|I_T(w)\|_{W^{-1/2, n+1}}^{n+1} + \|I_T(w)\|_{L^2}^2 \lesssim \int_0^T \|w_s\|_{L^2}^2 \, ds.
\]

By interpolation we can estimate, for $i \geq 1$

\[
\|((D)^{-1/2}I_T(w))^{n+1-i}\|_{B^i_{r, 1}} \lesssim \|((D)^{-1/2}I_T(w))\|_{B^{n+1}_r} + C
\]

\[
\lesssim \|I_T(w)\|_{W^{-1/2, n+1}}^{n+1-i} \|I_T(w)\|_{H^1}^{1+i} + C
\]

(\text{let } \epsilon = \frac{1}{n(n-1)})

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Choosing $q = n/(n - (1/2 - \kappa)) > 1$, we have

$$\left(\|I_T(w)\|_{W_{n-2}^{1/2, n+1}}^{n/(n-1)}\right)^q = \|I_T(w)\|_{W_{n-1}^{1/2, n+1}}^{n/(n-1)} \cdot \|I_T(w)\|_{W_{n-1}^{1/2, n+1}}^{n/(n-1)}. $$

Now for $n$ large enough $\frac{n}{n-1} \leq \frac{2}{n+1}$ and using Young’s inequality we can estimate

$$\|I_T(w)\|_{W_{n-2}^{1/2, n+1}}^{n/(n-1)} \lesssim \|I_T(w)\|_{W_{n-1}^{1/2, n+1}}^{n/(n-1)} \left(\|I_T(w)\|_{H^1}^{n/(n-1)} + 1\right)
\lesssim \|I_T(w)\|_{W_{n-1}^{1/2, n+1}}^{n+1} + \|I_T(w)\|_{H^1}^{n/(n-1)} + 1$$

Lemma 3.8. Let

$$\tilde{r}_s^w = -12\lambda J_1(W_s^2) \times (1 - \theta_s) I_s(w) + 12 \lambda J_1(W_s^2) \times I_s(w)
- 12 \lambda J_1(W_s^2) \times I_s(w)^2 - 4 \lambda J_1(W_s^2) \times 24 \lambda J_1(W_s^2) \times I_s(w) \times \theta_s I_s(w)^2
- 12 \lambda J_1(W_s^2) \times \theta_s I_s(w)^2 + \lambda I_{s < T} J_s(W_s^2) \times I_s(w)^2.$$

Setting $h^w = u + w$, there exists a random variable $Q_t(W)$ such that $\sup_t \mathbb{E}[|Q_t(W)|] < \infty$ and

$$(t)^{1+\kappa} \|r_t^w\|^2 \lesssim \int_0^t \|w_s\|_{L^2}^2 ds + \|I_t(w)\|_{W_{n-2}^{1/2, n+1}}^{n+1} + \|I_t(h^w)\|_{W_{n-2}^{1/2, n+1}}^{n+1} + \|I_t(h^w)\|_{L^2}^2 + Q_t(W).$$

Proof. Note that

$$\|I_{s < T} J_s(W_s^2) \times I_s(u^w)\|_{L^2}^2 \lesssim \int_0^T \|W_s^2\|_{W_{n-1}^{1/2, n+1}}^{n+1} + \|I_s(u^w)\|_{L^2}^4.$$

Moreover $h^w = u^w + w$ implies

$$\|I_t(h^w)\|_{W_{n-2}^{1/2, n+1}}^{n+1} \lesssim \|I_t(u^w)\|_{W_{n-2}^{1/2, n+1}}^{n+1} + \|I_t(h^w)\|_{W_{n-2}^{1/2, n+1}}^{n+1},$$

and $\|I_t(h^w)\|_{L^2}^4 \lesssim \|I_t(h^w)\|_{L^2}^4 + \|I_t(u^w)\|_{L^2}^4$. From Lemma 5.2 we get

$$\|I_t(u^w)\|_{L^2}^4 \lesssim C + \int_0^t \|w_s\|_{L^2}^2 ds + \|I_t(w)\|_{W_{n-2}^{1/2, n+1}}^{n+1}.$$ 

The estimation for the other terms is easy but technical and postponed until Section 5.

4 Singularity of $\Phi^4_3$ w.r.t. the free field

The goal of this section is to prove that the $\Phi^4_3$ measure is singular with respect to the Gaussian free field. For this we have to find a set $S \subseteq \mathcal{C}_{-1/2-\kappa}(\Lambda)$ such that $P(W_\infty \in S) = 1$ and $Q^4(W_\infty \in S) = 0$. Together with (3.10), this will imply singularity. We claim that setting

$$S := \left\{ f \in \mathcal{C}_{-1/2-\kappa}(\Lambda) : \frac{1}{T_n^{1+\kappa}} \int_\Lambda \mathbb{E}[|\theta_f|^4] \to 0 \right\}$$

for some suitable sequence $(T_n)_n$ such that $T_n \to \infty$, does the job. Here

$$\mathbb{E}[|\theta_f|^4] = \mathbb{E}[|\theta_f|^4 - 6\mathbb{E}[|\theta_f W_\infty(0)|^2] |\theta_f|^2 + 3\mathbb{E}[|\theta_f W_\infty(0)|^2]^2]$$

denotes the Wick ordering with respect to the Gaussian free field. For later use we define

$$W_t^{\theta_f, 3} = 4(\theta_t W_t)^3 - 12\mathbb{E}[|\theta_t W_t(0)|^2] (\theta_t W_t)$$

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and

$$W_t^{\theta_T^2} = 12([\theta_T W_t]^2 - E([\theta_T W_t(0)]^2)).$$

Before we proceed with the proof let us briefly motivate the choice of the event $S$ and give a sketch of the proof below. By Ito’s formula one can show that

$$\int_{\Lambda} [\theta_T W_{\infty}^4] = \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t$$

(4.1)

where we recall that under $\mathbb{P}$ $X_t$ is a cylindrical Wiener process. From this formula using the properties of $W_t^{\theta_T}$ and Ito isometry we will deduce that

$$E \left[ \left( \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t \right)^2 \right] = T,$$

(4.2)

and extracting a subsequence we get $\mathbb{P}(W_{\infty} \in S) = 1$. On the other hand with $R$ denoting a regular remainder we have

$$\int_{\Lambda} [\theta_T W_{\infty}^4] = \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t - \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} J_t W_t^{\theta_T} dX_t - \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t + R,$$

(4.3)

and under $Q^{\alpha}$ the process $X^{\alpha}_t$ is again a cylindrical Wiener process. Therefore, as in (4.2) we have a martingale whose variance is estimated as

$$E_{Q^{\alpha}} \left[ \left( \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t \right)^2 \right] \approx T.$$

However now the additional drift term in (4.3) grows faster than $T^{1/2+\delta}$ since it behaves as the positive term

$$\int_0^\infty \int_{\Lambda} (\theta_T J_t W_t^{\theta_T})^2 dX_t,$$

whose average can be estimated exactly as in (4.3), that is proportional to $T$. To make the argument rigorous we need only to show that all the neglected terms cannot compensate for this divergence, this will be done by estimating their average size and then using Borel–Cantelli.

Let us start by proving that $\mathbb{P}(W_{\infty} \in S) = 1$ for some sequence $(T_n)$.\n
**Lemma 4.1.** For any $\delta > 0$

$$\lim_{T \to \infty} E \left[ \left( \frac{1}{T^{(1+\delta)/2}} \int_{\Lambda} [(\theta_T W_{\infty})^4] \right)^2 \right] = 0.$$

**Proof.** Wick products corresponds to iterated Ito integrals. Introducing the notation

$$d\omega_t^{\theta_T} = \theta_T J_t dX_t,$$

we can verify by Ito formula that

$$\int_{\Lambda} [\theta_T W_{\infty}^4] = \int_0^\infty \int_{\Lambda} W_t^{\theta_T} d\omega_t^{\theta_T} = \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t.$$

Since $\theta_T J_t = 0$ for $t \geq T$, Ito isometry gives

$$E \left[ \left( \int_0^\infty \int_{\Lambda} \theta_T J_t W_t^{\theta_T} dX_t \right)^2 \right] = E \int_0^T \left( \int_{\Lambda} (\theta_T J_t W_t^{\theta_T})^2 dX_t \right).$$
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Then, again by Ito formula the expectation on the r.h.s. can be estimated as

$$\mathbb{E}\left[ \int_{\Lambda} (W^\theta_{t^r,3})^2 \right] = 4\mathbb{E}\left[ \sum_{k_1,k_2,k_3} \int_0^{t^r} \int_0^{t^r} \int_0^{t^r} d\omega^\theta_{s_1}(k_1) d\omega^\theta_{s_2}(k_2) d\omega^\theta_{s_3}(k_3) \right]^2$$

$$= 24\mathbb{E}\left[ \sum_{k_1,k_2,k_3} \int_0^{t^r} \int_0^{t^r} \int_0^{t^r} \frac{\theta^2_T(k_1) \sigma^2_{s_1}(k_1) \theta^2_T(k_2) \sigma^2_{s_2}(k_2) \theta^2_T(k_3) \sigma^2_{s_3}(k_3)}{\langle k_1 \rangle^2 \langle k_2 \rangle^2 \langle k_3 \rangle^2} \, ds_1 \, ds_2 \, ds_3 \right]$$

$$\leq 24\mathbb{E}\left[ \sum_{k_1,k_2,k_3} \int_0^{t^r} \int_0^{t^r} \int_0^{t^r} \frac{\sigma^2_{s_1}(k_1) \sigma^2_{s_2}(k_2) \sigma^2_{s_3}(k_3)}{\langle k_1 \rangle^2 \langle k_2 \rangle^2 \langle k_3 \rangle^2} \, ds_1 \, ds_2 \, ds_3 \right]$$

$$\lesssim t^3$$

Now recall that $\|J_t f\|_{L^2(\Lambda)} \lesssim (t)^{-3/2}\|f\|_{L^2(\Lambda)}$ to conclude:

$$\mathbb{E}\left[ \int_0^T \int_{\Lambda} (\theta_T J_t W^\theta_{t^r,3})^2 \, dt \right] \leq \frac{1}{T^{1+\delta}} \mathbb{E}[\|J_t \theta_T W^\theta_{t^r,3}\|^2_{L^2(\Lambda)}] dt \to 0. \Box$$

The lemma implies that $T^{-(1+\delta)/2} \int_{\Lambda} (\theta_T W^-)^t \to 0$ in $L^2(\mathbb{P})$. So there exists a sequence $(T_n)_n$ such that $T_n^{-(1+\delta)/2} \int_{\Lambda} (\theta_{T_n} W^-)^t \to 0$ almost surely.

The next step of the proof is to check that $Q^a(W_\infty \in S) = 0$. More concretely we will show that for a sub-sequence of $(T_n)_n$ (not relabeled)

$$\frac{1}{T_n^{1+\delta}} \int_{\Lambda} (\theta_{T_n} W^-)^t \to -\infty,$$

$Q^a$ almost surely. Observe that

$$\int_{\Lambda} (\theta_T W^-)^t = \int_0^\infty \int_{\Lambda} \theta_T J_t W^\theta_{t^r,3} dX_t$$

$$= \int_0^\infty \int_{\Lambda} \theta_T J_t W^\theta_{t^r,3} dX_t^u + \int_0^\infty \int_{\Lambda} \theta_T J_t W^\theta_{t^r,3} u d\mathbb{P}\, dt$$

$$= \int_0^\infty \int_{\Lambda} \theta_T J_t W^\theta_{t^r,3} dX_t^u - \lambda \int_0^\infty \int_{\Lambda} (\theta_T J_t W^\theta_{t^r,3}) J_t (W_t^u) \, dt$$

$$- \lambda \int_0^\infty \int_{\Lambda} (\theta_T J_t W^\theta_{t^r,3}) J_t (D)^{-1/2} \|J_t (D)^{-1/2} W^u_t\|^2 dt.$$

We expect the term

$$\int_0^\infty \int_{\Lambda} (\theta_T J_t W^\theta_{t^r,3}) J_t W_t^u \, dt$$

to go to infinity faster than $T^{1-\delta}$, $Q^a$-almost surely. To actually prove it, we start by a computation in average.

**Lemma 4.2.** It holds

$$\mathbb{E}\left[ \int_0^\infty \int_{\Lambda} (\theta_T J_t W^\theta_{t^r,3}) J_t W^3_t \, dt \right] \simeq T.$$

**Proof.** Recall that $dW^\theta_{t^r} = \theta_T J_t dX_t$. With a slight abuse of notation we can write

$$\int_0^\infty \int_{\Lambda} \frac{\theta^2_T(k) \langle k \rangle^2}{\langle k \rangle^3} \left( \sum_{k_1+k_2+k_3=k} f^l_0 f^s_0 f^r_0 d\omega^\theta_{s_1}(k_1) d\omega^\theta_{s_2}(k_2) d\omega^\theta_{s_3}(k_3) \right) dt$$

$$\times \int_0^l \int_0^s \int_0^r d\omega^\theta_{s_1}(k_1) d\omega^\theta_{s_2}(k_2) d\omega^\theta_{s_3}(k_3)$$

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and by Ito isometry
\[
E \left[ \sum_{k_1, k_2, k_3} \int_0^t \int_0^{s_1} \int_0^{s_2} \theta^\omega_{1/s_1}(k_1) \theta^\omega_{1/s_2}(k_2) d\omega_{s_3}^\omega(k_3) \right] = 6 \sum_{k_1, k_2, k_3} \int_0^t \int_0^{s_1} \int_0^{s_2} \theta^\omega_{1/s_1}(k_1) \theta^\omega_{1/s_2}(k_2) \frac{\theta^\omega_{1/s_3}(k_3)}{(k_3)^2} ds_1 ds_2 ds_3
\]

For $T$ large enough and since $\sigma^2$ and $\theta$ are positive, we have denoting $\tilde{\sigma}_s = \theta^\omega_{1/s}$
\[
\int_0^{T/2} \sum_k \frac{\tilde{\sigma}_s^2(k)}{(k)^2} \sum_{k_1, k_2, k_3} \int_0^{s_1} \int_0^{s_2} \frac{\tilde{\sigma}_s^2(k_1)}{(k_1)^2} \frac{\tilde{\sigma}_s^2(k_2)}{(k_2)^2} \frac{\tilde{\sigma}_s^2(k_3)}{(k_3)^2} ds_1 ds_2 ds_3 dt
\]

Introduce the notation $Z^\omega_n = \{ n \in \mathbb{Z}^3 : n = (n_1, n_2, n_3) \text{ with } n_i \geq 0 \}$. Furthermore we denote $\bar{\rho}_T(k) = \rho_{T/2}(k) - \rho_{T/s}(k)$ After restricting the sum to $(Z^\omega_n)^3$ we get the bound
\[
\geq \int_{T/8}^{T/2} \sum_k \frac{\sigma^2(k)}{(k)^2} \sum_{k_1, k_2, k_3} \int_0^{s_1} \int_0^{s_2} \frac{\sigma^2(k_1)}{(k_1)^2} \frac{\sigma^2(k_2)}{(k_2)^2} \frac{\sigma^2(k_3)}{(k_3)^2} ds_1 ds_2 ds_3 dt
\]

Now, for large enough $T$ if $k_1 + k_2 + k_3 = k$ and $\langle k \rangle \leq T/8$ then $\langle k \rangle \leq T/2 \times 0.9$. Furthermore if $T$ large enough and $k_1, k_2, k_3 \in \mathbb{Z}^3_+$ and $k_1 + k_2 + k_3 = k$, while $\langle k \rangle \geq (3T/32) \times 0.9$ (recall that if $\langle k \rangle < (3T/32) \times 0.9$ and $s > 3T/32$ then $\sigma_s(k_1) = 0$) we have $\langle k \rangle \geq T/8$. So for any $k$ for which the integral is nonzero we have $\bar{\rho}_{T/2}(k) - \rho_{T/s}(k) = 1$ (recall that $\rho = 1$ on $B(0,9/10)$ and $\rho = 0$ outside of $B(0,1)$). This implies
\[
\frac{1}{T^2} \sum_{k \in \mathbb{Z}^3_+} \bar{\rho}_T(k) \sum_{k_1, k_2, k_3} \int_0^{s_1} \int_0^{s_2} \frac{\sigma^2(k_1)}{(k_1)^2} \frac{\sigma^2(k_2)}{(k_2)^2} \frac{\sigma^2(k_3)}{(k_3)^2} ds_1 ds_2 ds_3
\]

\[
= \frac{1}{T^2} \sum_{k_1, k_2, k_3} \int_0^{s_1} \int_0^{s_2} \frac{\sigma^2(k_1)}{(k_1)^2} \frac{\sigma^2(k_2)}{(k_2)^2} \frac{\sigma^2(k_3)}{(k_3)^2} ds_1 ds_2 ds_3
\]

\[
\geq T
\]

The upper bound $\lesssim T$ it is easier to obtain, essentially as in Lemma 4.1.

Next we upgrade this average bound to almost sure divergence of the random variable at least as $T^{1-\delta}$ for some $\delta$ small.

**Lemma 4.3.** There exists a $\delta_0 > 0$ such that for any $\delta_0 \geq \delta > 0$, there exists a sequence $(T_n)_n$ such that $P$ – almost surely
\[
\frac{1}{T_n^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_{T_n} J_t W_t^{\theta T_n,3}) J_t W_t^3 dt \to \infty.
\]

**Proof.** Define
\[
G_T := \frac{1}{T} \int_0^\infty \int_\Lambda (\theta_{T} J_t W_t^{\theta T,3}) J_t W_t^3 dt + \sup_{t<\infty} ||W_t||_{H_{\omega_{1/2-\delta}}}^3.
\]
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We will show that $e^{-G_{T_n}} \to 0$ in $L^1(\mathbb{P})$, which implies that there exists a sub-sequence $(T_{n_k})$ such that $e^{-G_{T_{n_k}}} \to 0$ almost surely. From this our statement follows. By the Boué-Dupuis formula

$$- \log \mathbb{E}[e^{-G_T}] = \inf_{v \in \mathcal{U}_n} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_{\Lambda} (\theta_T J_i[(\theta_T(W_t + I_t(v))^3)]J_i[(W_t + I_t(v))^3] dt +
+ \sup_{t < \infty} \|W_t + I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 dt \right]$$

$$= \inf_{v \in \mathcal{U}_n} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_{\Lambda} (\theta_T J_i W_t^{3}) J_i W_t^3 dt +
+ \frac{1}{T^{1-\delta}} \sum_{(i,j) \in \{0,1,2,3\}^3 \setminus \{0,0,0\}} \int_0^T \int_{\Lambda} A^j_i B^j_i dt
+ \sup_{t < \infty} \|W_t + I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 dt \right]$$

$$\geq \inf_{v \in \mathcal{U}_n} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_{\Lambda} (\theta_T J_i W_t^{3-\delta}) J_i W_t^3 dt +
+ \frac{1}{T^{1-\delta}} \sum_{(i,j) \in \{0,1,2,3\}^3 \setminus \{0,0,0\}} \int_0^T \int_{\Lambda} A^j_i B^j_i dt
+ \frac{1}{2} \sup_{t < \infty} \|I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} - C \sup_{t < \infty} \|W_t\|_{K_{\theta^{-1/2-\epsilon}}} + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 dt \right]$$

where where have used that $\theta_T J_i = 0$ for $t \geq T$ and introduced the notations, for $0 \leq i \leq 3$,

$$A^j_i := 4 \binom{3}{i} J_i \theta_T ([|\theta_T W_t|^3] |\theta_T I_t(v)|^i),$$

and

$$B^j_i := 4 \binom{3}{i} J_i ([|W_t^{3-\delta}|] (I_t(v))^i).$$

Our aim now to prove that the last three terms are bounded below uniformly as $T \to \infty$ (while we already know that the first one diverges). For $i \in \{1, 2, 3\}$

$$\|A^j_i\|^2_{L^2} + \|B^j_i\|^2_{L^2} \lesssim \langle t \rangle^{-1+\delta} \left( \|I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} + \|I_t(u)\|_{L^1} + Q_t(W) \right),$$

by Lemmas 5.4 and 5.6. Here $Q_t(W)$ is a random variable only depending on $W$ such that $\sup_t \mathbb{E}[|Q_t(W)|^p] < \infty$ for any $p < \infty$. Then

$$\leq \frac{1}{T^{1-\delta}} \sum_{(i,j) \in \{0,1,2,3\}^3 \setminus \{0,0,0\}} \int_0^T \int_{\Lambda} |A^j_i B^j_i| dt
+ \frac{1}{T^{1-\delta}} \sum_{i \in \{1,2,3\}} \int_0^T \|A^j_i\|^2_{L^2} + \|B^j_i\|^2_{L^2} dt
+ \frac{1}{T^{1-\delta}} \sum_{i \in \{1,2,3\}} \int_0^T \|A^j_i\|_{L^2}^2 \|B^j_i\|_{L^2}^2 dt + \frac{1}{T^{1-\delta}} \sum_{i \in \{1,2,3\}} \int_0^T \|A^j_i\|^2_{L^2} \|B^j_i\|^2_{L^2} dt.$$

Now for the first term we obtain

$$\mathbb{E} \left[ \frac{1}{T^{1-\delta}} \sum_{(i,j) \in \{0,1,2,3\}^3 \setminus \{0,0,0\}} \int_0^T \|A^j_i\|^2_{L^2} + \|B^j_i\|^2_{L^2} dt \right]
= \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \sum_{(i,j) \in \{0,1,2,3\}^3 \setminus \{0,0,0\}} \int_0^T \langle t \rangle^{-1+\delta} \left( \|I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} + \|I_t(u)\|_{L^1} + Q_t(W) \right) dt \right]
= \frac{C}{T^{1-\delta}} \mathbb{E} \left[ \sup_t \left( \|I_t(v)\|_{K_{\theta^{-1/2-\epsilon}}} + \|I_t(u)\|_{L^1} + Q_t(W) \right) \right] + \frac{C}{T^{1-\delta}}.$$
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For the second term we use that $\|A^0_t\|_{L^2} \leq Q_t(W)$ so
\[
\frac{1}{T^{1+\delta}} E \left[ \int_0^T \|A^0_t\|_{L^2}^2 dt \right] \
\leq \frac{1}{T^{1+\delta}} E \left[ \int_0^T (t)^{-1/2} \|A^0_t\|_{L^2}^2 dt + \int_0^T (t)^{1/2} \|B^0_t\|_{L^2}^2 dt \right] \
\leq \frac{1}{T^{1+\delta}} E \left[ \int_0^T (t)^{-1/2} \|A^0_t\|_{L^2}^2 dt \right] \
+ \frac{1}{T^{1+\delta}} E \left[ \int_0^T (t)^{-1/2} \delta \left( \|I_t(v)\|^K_{L^2} + \|I_t(v)\|^2_{H^1} + Q_t(W) \right) dt \right] \
\leq \frac{C}{T^{1/2-\delta}} E \left[ \sup_t \left( \|I_t(v)\|^K_{L^2} + \|I_t(v)\|^2_{H^1} \right) \right] + \frac{C}{T^{1/2-\delta}}
\]
Since $\sup_t \left( \|I_t(v)\|^K_{H^1} \right) \leq \int_0^\infty \|v_t\|^2_{L^2} dt$ in total we obtain for $T$ large enough. The third term is estimated analogously. Then
\[
- \log E[e^{-GT}] \
\geq \inf_{\varphi \in H_n} E \left[ \frac{1}{T^{1+\delta}} \int_0^\infty \left( \varphi_T - J_t W_t^{\delta_T} \right) J_t W_t^{3} dt + \left( \frac{1}{2} - \frac{C}{T^{1/2-\delta}} \right) \sup_{t<\infty} \|I_t(v)\|^K_{L^2} \right] - \frac{C}{T^{1/2-\delta}} \
\geq E \left[ \frac{1}{T^{1+\delta}} \int_0^\infty \left( \varphi_T - J_t W_t^{\delta_T} \right) J_t W_t^{3} dt \right] - C \rightarrow \infty
\]
as claimed.

Next we obtain an estimate which will help with the proof of the main theorem.

**Lemma 4.4.** We have
\[
\sup_T E_{Q^u} \left[ \int_0^\infty \int_\Lambda \frac{1}{1+\tau^3} (\varphi_T - J_t W_t^{\delta_T})^2 dt \right] < \infty.
\]
Furthermore, there exists a (deterministic) sub-sequence $(T_n)_n$ such that
\[
\frac{1}{T_n^{1/2+\delta}} \left| \int_0^\infty \int_\Lambda \varphi_T J_t W_t^{\delta_T} \cdot dX_t^u \right| \rightarrow 0
\]
$Q^u$ almost surely.

**Proof.** Recall that under $Q^u$ we have $W_t = W_t^u + I_t(u)$ where $u$ is defined above by (3.3) and $\text{Law}_{Q^u}(W^u) = \text{Law}_{Q}(W)$. With this in mind we compute
\[
\int_0^T \int_\Lambda \frac{1}{1+\tau^3} (\varphi_T - J_t W_t^{\delta_T})^2 dt = \sum_{i,j \leq 3} \int_0^T \int_\Lambda \frac{1}{1+\tau^3} A_i^j A_i^j dt,
\]
where, as above,
\[
A_i^j = 4 \binom{3}{i} J_i \theta_T (L^u)^{\delta_T} (\varphi_T - J_t W_t^{\delta_T}) dt.
\]
By Lemmas 5.4 and 5.6 we have that $E_{Q^u} \left[ \|A_i^j\|^2_{L^2} \right] \leq C$ so the Cauchy-Schwartz inequality gives the result. 

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**Theorem 4.5.** There exists a sequence $(T_n)_n$ such that, $Q^n$-almost surely,

$$
\frac{1}{T_n} \int \lambda [(\theta_{T_n} W)^3] \to -\infty.
$$

**Proof.** We have

$$
\int \lambda [(\theta_{T} W)^3] = \int_0^\infty \int \lambda \theta_{T} J_t W_{t,3}^\theta dX_t.
$$

Since $dX_t = dX_t^u + \omega_t dt$ we have also

$$
\frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta dX_t = \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta dX_t^u + \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta \omega_t dt = \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta dX_t^u - \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta J_t (W_t^u)^2 dI_t(u) dt + \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta J_t \langle D \rangle^{-1/2} [(D)^{-1/2} W_t^n]^n] dt.
$$

The first term goes to 0 $Q^n$-almost surely by Lemma 4.4. To analyze the third term we estimate

$$
\begin{aligned}
\frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta J_t (W_t^u)^2 dI_t(u) dt &\leq \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta J_t \langle D \rangle^{-1/2} L \|W_t^n\|^2_{L^2} \leq \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta \|\|J_t \|L^2\| \leq \frac{1}{T_n} \int_0^T \int \lambda \theta_{T} J_t W_{t,3}^\theta \|\|J_t \|L^2\||I_t(u)| \|L^2\| dt \\
&\leq T^{-1/2-\delta} \left( \int_T \|\theta_{T} J_t W_{t,3}^\theta \|^2_{L^2} dt \right)^{1/2} \\
&\times T^{-1/2+\delta} \left( \int_T t^{-1-\delta} (\|W_t^n\|^2_{L^2} \|I_t(u)\|^2_{L^2})^{1/2} \right)^{1/2}.
\end{aligned}
$$

By the computation from Lemma 4.4 we have then

$$
E_{Q^n} \left[ T^{-1/2-\delta} \left( \int_T \|\theta_{T} J_t W_{t,3}^\theta \|^2_{L^2} dt \right)^{1/2} \right] \to 0,
$$

and $\sup_t E_{Q^n} [(\|W_t^n\|^2_{L^2} \|I_t(u)\|^2_{L^2}) < \infty$. Therefore (4.4) converges to 0 in $L^1(Q^n)$.

For the fourth term we proceed in the same way:

$$
\begin{aligned}
&\leq \int_0^T \|\theta_{T} J_t W_{t,3}^\theta \|L^2 \|I_t \|D^{-1/2} \|W_t^n\|^n \|L^2\| dt \\
&\leq \int_0^T \|\theta_{T} J_t W_{t,3}^\theta \|L^2 \|I_t \|D^{-1/2} \|W_t^n\|^n \|L^2\| dt \\
&\leq \int_0^T \|\theta_{T} J_t W_{t,3}^\theta \|L^2 \|I_t \|D^{-1/2} \|W_t^n\|^n \|L^2\| dt \\
&\leq \int_0^T (\|\theta_{T} J_t W_{t,3}^\theta \|L^2)^2 t^{-1/2} \|\|D\|^{-1/2} W_t^n\|^n \|L^2\| dt \\
&\leq \left( \int_0^T t^{-1/2} (\|\theta_{T} J_t W_{t,3}^\theta \|L^2)^2 dt \right)^{1/2} \left( \int_0^T t^{-1/2} (\|\|D\|^{-1/2} W_t^n\|^n \|L^2\| dt \right)^{1/2}
\end{aligned}
$$

which is bounded in expectation uniformly in $T$, so the fourth term goes to 0 in $L^1(Q^n)$ as well. It remains to analyze the second term. Again introducing the notation

$$
A_t^i = 4 \left( \frac{3}{3} \right) J_t \theta_t [(\theta_t W_{t,3}^\theta)^{3-1} [(\theta_t W_{t,3}^\theta)^3, \quad W_{t,3}^\theta u, \quad 4] [(\theta_t W_{t,3}^\theta)^3],
$$

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we have
\[
\frac{1}{T^{1-\delta}} \int_0^T \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho dt = \frac{1}{T^{1-\delta}} \int_0^T \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho dt + \sum_{1 \leq i \leq 3} \frac{1}{T^{1-\delta}} \int_0^T A_i^3 J_t W_{t,3}^\rho dt.
\]

Now observe that
\[
\frac{1}{T^{1-\delta}} \int_0^T \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho Q_u \sim P \frac{1}{T^{1-\delta}} \int_0^T \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho dt,
\]
so the $\limsup$ of this is $\infty$ almost surely. To estimate the sum we again observe that for $i \geq 3$ $E\nu[|A_i^3|_{L^2}] \lesssim (t)^{-1+\delta}$ and by Young’s inequality
\[
\int_0^T \int_\Lambda A_i^3 J_t W_{t,3}^\rho dt \leq \int_0^T \int_\Lambda |A_i^3|_{L^2} |J_t W_{t,3}^\rho|_{L^2} dt \\
\leq \int_0^T \int_\Lambda (t)^{1/3} |A_i^3|_{L^2} (t)^{-1/3} |J_t W_{t,3}^\rho|_{L^2} dt \\
\leq \int_0^T \int_\Lambda (t)^{2/3} |A_i^3|^2_{L^2} + \int_0^T \int_\Lambda (t)^{-2/3} |J_t W_{t,3}^\rho|^2_{L^2} dt.
\]

Taking expectation we obtain
\[
\frac{1}{T^{1-\delta}} E \left[ \int_0^T \int_\Lambda A_i^3 J_t W_{t,3}^\rho dt \right] \leq \frac{1}{T^{1-\delta}} E \left[ \int_0^T \int_\Lambda (t)^{2/3} |A_i^3|^2_{L^2} dt \right] + \frac{1}{T^{1-\delta}} E \left[ \int_0^T \int_\Lambda (t)^{-2/3} |J_t W_{t,3}^\rho|^2_{L^2} dt \right] \\
\leq \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda (t)^{-1/3} dt + \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda (t)^{-2/3} dt \to 0.
\]

We have deduced that
\[
\frac{1}{T^{1-\delta}} \int_0^T \|\theta_t W_{t,3}\|^4 = \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho dt + R_T,
\]
where $R_T \to 0$ in $L^1(Q^u)$. We can conclude by selecting a sub-sequence $(T_n)_n$ such that
\[
\frac{1}{T_n^{1-\delta}} \int_0^{T_n} \int_\Lambda \theta_t J_t W_{t,3}^\rho J_t W_{t,3}^\rho dt \to \infty
\]
$Q^u$-almost surely and $R_{T_n} \to 0$, $Q^u$-almost surely. \hfill $\square$

5 Some analytic estimates

We collect in this final section various technical estimates needed to complete the proof of Lemma 3.8.

**Proposition 5.1.** Let $1 < p < \infty$ and $p_1, p_2, p_1', p_2' > 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p}$. Then for any $s, \alpha \geq 0$
\[
\|D^s f\|_{L^p} \lesssim \|D^{s+\alpha} f\|_{L^{p_2}} \|D^{-\alpha} g\|_{L^{p_1}} + \|D^{s+\alpha} g\|_{L^{p_1'}} \|D^{-\alpha} f\|_{L^{p_2'}}.
\]

*Proof.* See [18]. \hfill $\square$

**Lemma 5.2.** There exists $\varepsilon > 0$, $n \in \mathbb{N}$ such that for any $\delta > 0$ there exists $C_\delta < \infty$ for which the following inequality holds for any $\phi \in H^1(\Lambda)$
\[
\|\phi\|_{L^4}^{4+\varepsilon} \leq C\|\phi\|_{W^{-1/2, n+1}}^{n+1} + \delta\|\phi\|_{H^1}^2 + C_\delta.
\]
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**Proof.**

\[
\int \phi^t \mathrm{d}x \leq \| \langle D \rangle^{-1/2} \phi \|_{L^2} \| \langle D \rangle^{1/2} \phi \|_{L^{2\gamma}}
\]
\[
\leq \| \langle D \rangle^{-1/2} \phi \|_{L^2} \| \langle D \rangle^{1/2} \phi \|_{L^{2\gamma}}^2
\]
\[
\leq \| \langle D \rangle^{-1/2} \phi \|_{L^2} \| \phi \|_{H^1}^2 \| \phi \|_{L^4}^{5/2}
\]

So

\[
\| \langle D \rangle^{-1/2} \phi \|_{L^2}^{21/20} \leq \| \langle D \rangle^{-1/2} \phi \|_{L^2}^{21/20} \| \phi \|_{H^1}^{21/40} \| \phi \|_{L^4}^{104/40}
\]

and applying Young’s inequality with the exponents \((32, 32/9, 32/22)\), we obtain

\[
\| \langle D \rangle^{-1/2} \phi \|_{L^2} \| \phi \|_{H^1} \| \phi \|_{L^4} \leq C_4 \| \langle D \rangle^{-1/2} \phi \|_{L^2}^{168/5} + \delta \| \phi \|_{H^1}^{16/9} + \delta \| \phi \|_{L^4}^{208/55}
\]

and subtracting \(\delta(\| \phi \|_{L^2})^{21/20}\) on both sides of the inequality gives the result. \(\square\)

**Lemma 5.3.** The following estimates hold with \(\varepsilon > 0\) small enough

\[
\| J_t([W^2_s]) \cdot (1 - \theta_t)I_t(w) \|_{L^2}^2 \leq \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \right)
\]

\[
\| J_t([W^2_s] \circ I_t(w)) \|_{L^2}^2 \leq \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \right)
\]

\[
\| J_t([W^2_s] \prec I_t(w)) \|_{L^2}^2 \leq \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \right)
\]

**Proof.** We observe that since \([W^2_s]\) is spectrally supported in a ball of radius \(\sim t\)

\[
\| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}} \lesssim \langle t \rangle^{2\varepsilon} \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}
\]

For the first estimate we know that \((1 - \theta_t)I_t(w)\) is supported in an annulus of radius \(\sim t\), so

\[
\| (1 - \theta_t)I_t(w) \|_{L^2} \lesssim \langle t \rangle^{-1+\varepsilon} \| I_t(w) \|_{H^{1-\varepsilon}} \quad \text{and furthermore by interpolation} \quad \| I_t(w) \|_{H^{1-\varepsilon}} \lesssim \| I_t(w) \|_{H^{-1/2}}^2 \| I_t(w) \|_{L^2}^2 \lesssim \| I_t(w) \|_{H^{-1/2}} \| I_t(w) \|_{L^2}^2.
\]

By definition \((t)^{1/2} J_t\) is a uniformly bounded Fourier multiplier regularizing by 1, and putting everything together, by paraproduct estimates

\[
\| J_t([W^2_s]) \cdot (1 - \theta_t)I_t(w) \|_{L^2}^2 \lesssim \langle t \rangle^{-1} \langle t \rangle^{2\varepsilon} \langle t \rangle^{-2+2\varepsilon} \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \| I_t(w) \|_{H^{1-\varepsilon}}^2
\]

\[
\langle \varepsilon = 2/7 \rangle \lesssim \langle t \rangle^{-1} \langle t \rangle^{2\varepsilon} \langle t \rangle^{-2+2\varepsilon} \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \| I_t(w) \|_{H^{1-\varepsilon}}^2
\]

\[
\lesssim \langle t \rangle^{-3/2} \langle \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \rangle
\]

For the second term in addition observe that the function \((t)^{1/2} J_t\) is spectrally supported in an annulus of radius \(\sim t\), and regularizes by 1 so again by estimates for the resonant product

\[
\| J_t([W^2_s] \circ I_t(w)) \|_{L^2}^2 \lesssim \langle t \rangle^{-3} \| [W^2_s] \|_{\mathcal{E}^{-1+\varepsilon}}^2 \| I_t(w) \|_{H^{1-\varepsilon}}^2
\]

\[
\lesssim \langle t \rangle^{-3} \langle \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1+\varepsilon}}^2 \rangle
\]

For the third estimate again applying paraproduct estimates and the properties of \(J\),

\[
\| J_t([W^2_s] \prec I_t(w)) \|_{L^2}^2 \lesssim \langle t \rangle^{-3} \langle \int_0^t \| w_s \|_{L^2}^2 \mathrm{d}s + \| I_t(w) \|_{W^{-1/2, n+1}}^n + \| [W^2_s] \|_{\mathcal{E}^{-1-\varepsilon}}^2 \rangle
\]

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Now, the claim follows from interpolation and Young’s inequality
\[
\|W_t\|_{L^{2^{-\epsilon}}}^2 \leq \|I_t(w)\|_{H^{1-\epsilon}}^2 \\
(\epsilon = 2/7)
\]

Applying Young’s inequality with exponents \(2, 2\)
\[
\|W_t\|_{L^{2^{-\epsilon}}}^2 \leq \|I_t(w)\|_{H^{1-\epsilon}}^2 + \|I_t(w)\|_{H^1}^2 + \|I_t(w)\|_{L^4}^4 \\
\leq \left(\int_0^t \|w_t\|_{L^2} dt + ||I_t(w)\|_{W^{-1/2,n+1}}^2 + \|W_t\|_{L^{2^{-\epsilon}}}^2\right).
\]

\[\]  

**Lemma 5.4.** Let \(f = C([-\infty, \infty], \mathbb{R})\) and \(g = C([-\infty, \infty], H^1)\) such that \(f_t, g_t\) have spectral support in a ball of radius proportional to \(t\). There exists \(n \in \mathbb{N}\) such that the following estimates hold:
\[
\|J_t(f_t g_t^2)\|_{L^2}^2 \lesssim \langle t \rangle^{3/2} \|f_t\|_{L^{2^{-\epsilon}}}^2 \|g_t\|_{L^4}^4, \\
\|J_t(f_t g_t^2)\|_{L^2}^2 \lesssim \langle t \rangle^{-3/2} \|f_t\|_{W^{-1/2,n+1}} + \|g_t\|_{W^{-1/2,n+1}}.
\]

and
\[
\|J_t(g_t^2)\|_{L^2}^2 \lesssim \langle t \rangle^{-3/2} \|g_t\|_{H^1}^2 + \|g_t\|_{W^{-1/2,n+1}}.
\]

**Proof.** By the spectral properties of \(J_t\),
\[
\|J_t(f_t g_t^2)\|_{L^2}^2 \lesssim \langle t \rangle^{-3/2} \|f_t\|_{L^{2^{-\epsilon}}}^2 \|g_t\|_{L^4}^4 \lesssim \langle t \rangle^{-3/2} \|f_t\|_{L^{2^{-\epsilon}}}^2 \|g_t\|_{L^4}^4.
\]

Applying Young’s inequality with exponents \(\frac{n}{2}, \frac{n}{2}\) with \(\frac{n}{2} \leq 4 + \epsilon\) where \(\epsilon\) is chosen as in Lemma 5.2 we have
\[
\langle t \rangle^{-3/2} \|f_t\|_{L^{2^{-\epsilon}}}^2 \lesssim \langle t \rangle^{-3/2} \|f_t\|_{L^{2^{-\epsilon}}}^2 + \|g_t\|_{L^4}^4.
\]

Now the second estimate follows from choosing \(n\) large enough (depending on \(\delta\)) and using Besov embedding after taking \(f = g\).

**Lemma 5.5.** The following estimates hold
\[
\langle t \rangle \|J_t(W_s I_t(w) \ast I_t^s(u))\|_{L^2}^2 \lesssim \|I_t(w)\|_{L^{2^{-\epsilon}}}^2 + \|I_t(u)\|_{L^{2^{-\epsilon}}}^2 + \|W_s\|_{L^{2^{-\epsilon}}}^2, \\
\langle t \rangle \|J_t(W_s I_t(w) \ast I_t^s(u))\|_{L^2}^2 \lesssim \|I_t(w)\|_{L^{2^{-\epsilon}}}^2 + \|I_t(u)\|_{L^{2^{-\epsilon}}}^2.
\]

**Proof.** For the first estimate we again use the spectral properties of \(W, I, J\) and obtain by paraproduct estimate
\[
\|J_s(W_t I_t(w) \ast I_t^s(u))\|_{L^2}^2 \lesssim \langle t \rangle^{-3} \|W_t\|_{L^{2^{-\epsilon}}}^2 \|I_t(w)\|_{L^{2^{-\epsilon}}}^2 + \|I_t(w)\|_{L^{2^{-\epsilon}}}^2.
\]

and the claim follows by Young’s inequality. For the second
\[
\|J_s((I_t(w) \ast I_t^s(u))\|_{L^2}^2 \lesssim \langle t \rangle^{2-2\epsilon} \|I_t(w)\|_{L^{2^{-\epsilon}}}^2 \|I_t^s(u)\|_{L^{2^{-\epsilon}}}^2.
\]

and the claim follows again by Young’s inequality.

**Lemma 5.6.** Let \(f_t \in C([-\infty, \infty], R^{1/2})\) and \(g_t \in C([-\infty, \infty], H^1)\) such that \(f_t, g_t\) have spectral support in a ball of radius proportional to \(t\). Then the following estimates hold
\[
\|\|J_t(f_t g_t)\|_{L^2}^2 \lesssim \langle t \rangle^{-1+2\epsilon} \|f_t\|_{L^{2^{-\epsilon}}}^2 \|g_t\|_{L^2}^2 \\
\|\|J_t(f_t g_t)\|_{L^2}^2 \lesssim \langle t \rangle^{-1+2\epsilon} \|f_t\|_{L^{2^{-\epsilon}}}^2 + \|g_t\|_{H^1}^2.
\]
This proves the first estimate. For the second we continue
\[ \langle t \rangle^{-1+2\delta} \| f_t \|_{B_{\infty}^{1-\delta}}^2 \lesssim \langle t \rangle \| g_t \|_{L^2}^2 \lesssim \langle t \rangle^{-1+2\delta} \| f_t \|_{B_{\infty}^{1-\delta}}^2 \| g_t \|_{L^2}^2 \lesssim (\langle t \rangle^{-1+2\delta} \| f_t \|_{B_{\infty}^{1-\delta}}^2 + \| g_t \|_{L^1}^4 + \| g_t \|_{H^{-1}}^2). \]

**Lemma 5.7.** It holds
\[ \int_0^T \int_\Lambda (J_t(W_t^2 \ast I_t^\delta(w)))^2 \lesssim T^{3\delta} \left( \sup_t \| W_t^2 \|_{B_{\infty}^{1-\delta}} \right) \left( \sup_t \| I_t(w) \|_{L^2} \right), \]
and
\[ \int_0^T \int_\Lambda (J_t(W_t^2 \ast I_t^\delta(w)))^2 \lesssim T^{3\delta} \left( \sup_t \| I_t(w) \|_{H^{-1}}^4 + \int_0^T \| w_t \|_{L^2}^2 dt + \sup_t \| W_t^2 \|_{B_{\infty}^{1-\delta}} \right). \]

**Proof.** This follows in the same fashion as Lemma 5.6. \(\square\)

## A Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [3, 16].

First recall the definition of Littlewood–Paley blocks. Let \( \chi, \varphi \) be smooth radial functions \( \mathbb{R}^d \to \mathbb{R} \) such that

- \( \text{supp } \chi \subseteq B(0, R), \text{supp } \varphi \subseteq B(0, 2R) \setminus B(0, R); \)
- \( 0 \leq \chi, \varphi \leq 1, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \) for any \( \xi \in \mathbb{R}^d; \)
- \( \text{supp } \varphi(2^{-j}) \cap \text{supp } \varphi(2^{-i}) = \emptyset \) if \( |i-j| > 1. \)

Introduce the notations \( \varphi_{-1} = \chi, \varphi_j = \varphi(2^{-j} \cdot) \) for \( j \geq 0. \) For any \( f \in \mathcal{S}'(\Lambda) \) we define the operators \( \Delta_j f := \mathcal{F}_{\xi}^{-1} (\varphi_j(\xi) \hat{f}(\xi)) \), \( j \geq -1. \)

**Definition A.1.** Let \( s \in \mathbb{R}, p, q \in [1, \infty]. \) For a Schwarz distribution \( f \in \mathcal{S}'(\Lambda) \) define the norm
\[ \| f \|_{B_{p,q}^s} := \| (2^j s) \|_{\mathcal{S}'(\Lambda)} \). \]

Then the space \( B_{p,q}^s \) is the closure of Schwarz distributions under this norm. We denote \( \mathcal{D}' = B_{\infty,\infty}^0 \) the Besov–Hölder space and \( H^s = B_{2,2}^s \) the Sobolev spaces.

**Proposition A.2.** Let \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq q_1 \leq q_2 \leq \infty. \) Then \( B_{p_2,q_2}^{s-d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \) is continuously embedded in \( B_{p_1,q_1}^s. \)

**Proposition A.3.** For any \( s_1, s_2 \in \mathbb{R} \) such that \( s_1 < s_2, \) any \( p, q \in [1, \infty] \) the Besov space \( B_{p,q}^{s_2} \) is compactly embedded into \( B_{p,q}^{s_1}. \)

**Definition A.4.** Let \( f, g \in \mathcal{S}(\Lambda). \) We define the paraproducts
\[ f \triangleright g := \sum_{j \leq -1} \Delta_j f \Delta_j g, \quad \text{and} \quad f \triangleleft g := \sum_{j > i+1} \Delta_i f \Delta_j g = g \triangleright f. \]

Moreover we introduce the resonant product
\[ f \circ g := \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g. \]

Then \( fg = f \triangleright g + f \circ g + f \triangleright g. \)
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**Proposition A.5.** Let $\alpha < 0$, $\beta \in \mathbb{R}$. For $f, g \in \mathcal{S}(\Lambda)$ we have the estimates

\[
\begin{align*}
\|f \ast g\|_{H^{s-\delta}} & \lesssim \|f\|_{\mathcal{E}^s} \|g\|_{L^2}, \\
\|f \ast g\|_{H^{s-\alpha}} & \lesssim \|f\|_{\mathcal{E}^s} \|g\|_{H^\alpha}, \\
\|f \ast g\|_{H^{s-\omega}} & \lesssim \|f\|_{\mathcal{E}^s} \|g\|_{L^\infty}, \\
\|f \ast g\|_{H^{s-\theta}} & \lesssim \|f\|_{\mathcal{E}^s} \|g\|_{\mathcal{E}^\theta}.
\end{align*}
\]

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$. Then

\[
\|f \circ g\|_{H^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{E}^\alpha} \|g\|_{H^\theta}, \quad \|f \circ g\|_{W^{\alpha,\beta}} \lesssim \|f\|_{\mathcal{E}^\alpha} \|g\|_{W^{\alpha,\beta}}.
\]

By density the paraproduct and resonant product also extend to bilinear operators on the respective spaces.

**Proposition A.6.** Let $\alpha \in (0, 1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$. Then for $f, g, h \in \mathcal{S}$, and for any $\delta > 0$,

\[
\begin{align*}
\|(f \ast g) \circ h - g(f \circ h)\|_{H^{\alpha+\beta+\gamma-\delta}} & \lesssim \|f\|_{\mathcal{E}^\alpha} \|h\|_{\mathcal{E}^\beta} \|g\|_{H^\gamma}, \\
\|(f \ast g) \circ h - g(f \circ h)\|_{H^{\alpha+\beta+\gamma}} & \lesssim \|f\|_{\mathcal{E}^\alpha} \|h\|_{\mathcal{E}^\beta} \|g\|_{\mathcal{E}^\gamma}.
\end{align*}
\]

**Proposition A.7.** Assume $f \in \mathcal{G}^\alpha$, $g \in H^\beta$, $h \in H^\gamma$ and $\alpha + \beta + \gamma = 0$. Then

\[
\int_{\mathbb{T}^d} |(f \ast g)h - (f \circ g)h| \leq \|f\|_{\mathcal{E}^\alpha} \|g\|_{H^\beta} \|h\|_{H^\gamma}.
\]

**Remark A.8.** Proposition A.7 is not proven in the above references but is quite easy and the reader can fill out a proof.

**Definition A.9.** A smooth function $\eta : \mathbb{R}^d \to \mathbb{R}$ is said to be an $S^m$-multiplier if for every multi-index $\alpha$ there exists a constant $C_\alpha$ such that

\[
\left| \frac{\partial^\alpha}{\partial \xi^\alpha} f(\xi) \right| \lesssim \alpha (1 + |\xi|)^{m-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d. \tag{A.1}
\]

We say that a family $(\eta_t)_{t \geq 0}$ is a uniform $S^m$-multiplier if (A.1) is satisfied for every $\eta_t$ with $C_\alpha$ independent of $t \geq 0$.

**Proposition A.10.** Let $\eta$ be an $S^m$-multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}(\mathbb{T}^d)$, then

\[
\|\eta(D)f\|_{B^{-m}_{p,q}} \lesssim \|f\|_{B^s_{p,q}}.
\]

Furthermore the constant depends only on $s$, $p$, $q$, $d$ and the constants $C_\alpha$ in (A.1).

**Proposition A.11.** Assume $m \leq 0$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. Let $\eta$ be an $S^m$-multiplier, $f \in \mathcal{G}^\beta$, $g \in H^\alpha$. Then for any $\delta > 0$,

\[
\|\eta(D)(f \ast g) - (\eta(D)f \ast g)\|_{H^{\alpha+\beta-\delta}} \lesssim \|f\|_{\mathcal{E}^\beta} \|g\|_{H^\alpha}.
\]

Again the constant depends only on $\alpha, \beta, \delta$ and the constants in (A.1).

**Proposition A.12.** Let $\delta > 0$. We have for any $q_1, q_2 \in [1, \infty]$, $q_1 < q_2$

\[
\|f\|_{B^\delta_{q_2,q_2}} \lesssim \|f\|_{B^\delta_{q_1,q_1}} \lesssim \|f\|_{B^{\delta+\delta}_{q_1,\infty}}.
\]

Furthermore, if we denote by $W^{s,p}$, $s \in \mathbb{R}$, $p \in [1, \infty]$ the fractional Sobolev spaces defined by the norm $\|f\|_{W^{s,p}} := \|(D)^s f\|_{L^p}$, then, for any $q \in [1, \infty]$,

\[
\|f\|_{B^\delta_{p,q}} \lesssim \|f\|_{W^{\delta+\delta,p}} \lesssim \|f\|_{B^{\delta+\delta}_{p,\infty}}.
\]
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