Choldowsky type generalization of the $q$–Favard-Szász operators

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Abstract. In the present paper, we introduce a Choldowsky type generalization of the $q$–Favard-Szász operators and obtain weighted statistical approximation properties of these operators. We also establish the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Further, we study the local approximation properties of these operators.

Keywords: Choldowsky type generalization of the $q$–Favard-Szász operators, modulus of continuity, local approximation, Peetre’s K-functional, statistical convergence.

MSC: 41A25, 41A36

1. Introduction

The approximation of function by using the linear positive operators introduced via $q$–calculus is currently be come active research. After the paper of Phillips [6] who generalized the classical Bernstein polynomials base on $q$–integer, many generalization of well-known positive linear operators, base on $q$–integer were introduced and studied several authors. We refer to the following studies related to the approximation properties of the $q$–analogue polynomials. For example in [12, 13] $q$–analogue of Kantorovich-Bernstein operators; in [14] $q$–Baskakov-Kantorovich operators; [15] $q$–Szász-Mirakjan operators; in [10] modified $q$–Stancu-Beta operators were introduced and their statistical properties were investigated. For further information related to the statistical approximation of the operators, the followings are remarkable among others [18–21].

In this work, we generalized a Choldowsky type Favard-Szász operators base on $q$–integer and we study the weighted statistical approximation properties of the Choldowsky type $q$–Favard-Szász operators via Korovkin type approximation theorem. Further we compute the rate of statistical convergence by using modulus of continuity. Furthermore we also obtain some local approximation results of these new operators.

First of all, we recall some definitions and notations regarding the concept of $q$–calculus. For any none-negative integer $r$, the $q$– integer of the number $r$ is defied by

$$[r]_q := \begin{cases} \frac{1-q^r}{1-q}, & \text{if } q \neq 1 \\ 1, & \text{if } q = 1 \end{cases}.$$ 

The $q$–factorial $[n]_q!$ and $q$–binomial coefficients are defined as

$$[n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

and

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, 0 \leq k \leq n.$$ 

The $q$–derivative $D_q f$ of a function $f$ is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0.$$ 

Also, if there exists $\frac{df}{dx}(0)$, then $(D_q f)(0) = \frac{df}{dx}(0)$. The following $q$–derivatives of the product of the functions $f(x)$ and $g(x)$ are equivalent:

$$D_q (f(x)g(x)) = f(qx) D_q g(x) + g(x) D_q f(x)$$

and

$$D_q (f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x).$$

The $q$–analogue of the exponential function are given by

$$e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q !}.$$ 

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and

\[ E^q_x = \sum_{n=0}^{\infty} q \frac{n(n-1)}{[n]_q!} x^n. \]

The exponential functions have the following properties:

\[ D_q(e^{\alpha x}) = a e^{\alpha q x}, \quad D_q(E^{\alpha q x}) = a E^{\alpha q x}, \quad e^x E^{-x} = E^x e^{-x} = 1. \]

Further results can be found in [8].

2. CONSTRUCTION OF THE OPERATORS

In [9], Jakimovski and Leviatan introduced a Favard Szász type operator, by using Appell polynomials \( p_k(x) \geq 0 \) defined by

\[ g(u) e^{-ux} = \sum_{k=0}^{\infty} p_k(x) u^k, \]

where \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) is an analytic function in the disc \( |z| < R, R > 1 \) and \( g(1) \neq 0 \),

\[ P_{n,t}(f; x) = e^{-nx} g(1) \sum_{k=0}^{\infty} p_k(nx) f \left( \frac{k}{n} \right) \]

and they investigated some approximation properties of these operators.

Atakut at all. [7] defined a Choldowsky type of Favard-Szász operators as follows:

\[ P^*(f; x) = e^{-nx} g(1) \sum_{k=0}^{\infty} p_k \left( \frac{nx}{b_n} \right) f \left( \frac{k}{b_n} \right) \]

with \( b_n \) a positive increasing sequence with the properties \( \lim_{n \to \infty} b_n = \infty \) and \( \lim_{n \to \infty} \frac{b_n}{n} = 0 \). They also studied some approximation properties of the operators.

Now, let us define Choldowsky type generalization of the \( q \)-Favard-Szász operators as follows:

\[ (2.1) \]

\[ P^*_q(f; q; x) = \frac{E^{\left[ \frac{nx}{b_n} \right]}_q}{A(1)} \sum_{k=0}^{\infty} P_k \left( q; \frac{nx}{b_n} \right) \frac{[k]_q!}{[n]_q!} f \left( \frac{k}{b_n} \right), \]

where \( \{ P_k(q; ) \} \geq 0 \) is a \( q \)-Appell polynomial set which is generated by

\[ (2.2) \]

\[ A(u) e_q = \sum_{k=0}^{\infty} \frac{P_k \left( q; \frac{nx}{b_n} \right) u^k}{[k]_q!} \]

and \( A(u) \) is defined by

\[ A(u) = \sum_{k=0}^{\infty} a_k u^k. \]

3. WEIGHTED STATISTICAL APPROXIMATION PROPERTIES

In this section, we give Korovkin type weighted statistical approximation properties of the operators \( P^*_q(f; q; x) \). Before proceeding further, let us give basic definition and notation on the concept of the statistical convergence which was introduced by Fast [10]. Let \( K \) be a subset of \( \mathbb{N} \), the set of natural numbers. Then, \( K_n = \{ k \leq n : k \in K \} \). The natural density of \( K \) is defined by \( \delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n| \) provided that the limit exists, where \( |K_n| \) denotes the cardinality of the set \( K_n \). A sequence \( x = (x_k) \) is called statistically convergent to the number \( \ell \in \mathbb{R} \), denoted by \( st \lim x = \ell \). For each \( \epsilon > 0 \), the set \( K_{\epsilon} = \{ k \in \mathbb{N} : |x_k - \ell| \geq \epsilon \} \) has a natural density zero, that is

\[ \lim_{n \to \infty} \frac{1}{n} |\{ k \leq n : |x_k - \ell| \geq \epsilon \}| = 0. \]

It is well know that every statistically convergence sequence is ordinary convergent, but the converse is not true. The concept of statistical convergence was firstly used in approximation theory by Gadgiev and Orhan [11]. They proved the Bolman–Korovkin type approximation theorem for statistical convergence.

A real function \( \rho_0 \) is called a weighted function if it is continuous on \( \mathbb{R} \) and \( \lim_{|x| \to 0} \rho_0(x) = \infty, \rho_0(x) \geq 1 \) for all \( x \in \mathbb{R} \), where \( \rho_0(x) = 1 + x^2 \). Let \( B_{\rho_0}(0, \infty) \) be the set of all functions \( f \) defined on \([0, \infty)\) satisfying the condition \( |f(x)| \leq M_f (1 + x^2) \), where \( M_f \) is a constant depending on \( f \). By \( C_{\rho_0}(0, \infty) \), we denote the subspace...
of all continuous function belonging to \(B_{p_0}[0,\infty)\). Also, \(C^*_{p_0}[0,\infty)\) be subspace of all continuous functions \(f \in C_{p_0}[0,\infty)\), for which \(\lim_{x \to \infty} \frac{f(x)}{1+x^2}\) is finite. The norm on \(C^*_{p_0}[0,\infty)\) is defined as follows:

\[
\|f\|_p = \sup_{x \in [0,\infty)} \frac{|f(x)|}{1+x^2}.
\]

Now, we may begin the following lemma which is needed proving our main result.

**Lemma 3.1.** For \(n \in \mathbb{N}\), \(x \in [0,\infty)\) and \(0 < q < 1\), we have

1. \(P_n^*(e_0; q; x) = 1\),
2. \(P_n^*(e_1; q; x) = x + \frac{D_q(A(1)) E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} b_n [n]_q\),
3. \(P_n^*(e_2; q; x) = x^2 + \frac{E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} [qD_q(A(q)) + D_q(A(1))] x b_n + \frac{E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} \frac{D_q^2(A(1))}{[n]_q^2} b_n^2\),

where \(e_i(x) = x^i, i = 0, 1, 2\).

**Proof.** By (2.22) and definition of \(q\) derivative, we obtain that

\[
\sum_{k=0}^{\infty} p_k \left( q; \frac{[n]_q}{b_n} x \right) \frac{[k]_q!}{[k]_q} = A(1) e_q^{[n]_q} x,
\]

and

\[
\sum_{k=0}^{\infty} \frac{p_k \left( q; \frac{[n]_q}{b_n} x \right)}{[k]_q!} [k]_q = A(1) \frac{[n]_q}{b_n} x e_q^{[n]_q} + e_q^{[n]_q} D_q A(1)
\]

By using the relations (3.4)-(3.6), from (2.1), we obtain the results

\[
P_n^*(e_0; q; x) = \frac{E_q^{[n]_q}}{A(1)} \sum_{k=0}^{\infty} p_k \left( q; \frac{[n]_q}{b_n} x \right) \frac{[k]_q!}{[n]_q} = \frac{E_q^{[n]_q}}{A(1)} A(1) e_q^{[n]_q} x = 1,
\]

\[
P_n^*(e_1; q; x) = \frac{E_q^{[n]_q}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left( q; \frac{[n]_q}{b_n} x \right)}{[k]_q!} \left( \frac{[k]_q}{[n]_q} b_n \right)
= \frac{E_q^{[n]_q} b_n}{A(1) [n]_q} \sum_{k=0}^{\infty} \frac{p_k \left( q; \frac{[n]_q}{b_n} t \right)}{[k]_q!} [k]_q
= \frac{E_q^{[n]_q} b_n}{A(1) [n]_q} \left( A(1) \frac{[n]_q}{b_n} x e_q^{[n]_q} + e_q^{[n]_q} D_q A(1) \right)
= x + \frac{E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} \frac{D_q(A(q)) + D_q(A(1))}{[n]_q} b_n
\]

and

\[
P_n^*(e_2; q; x) = \frac{E_q^{[n]_q}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left( q; \frac{[n]_q}{b_n} x \right) [k]_q^2}{[n]_q^2} b_n^2
= \frac{E_q^{[n]_q} b_n^2}{A(1) [n]_q^2} \left\{ D_q^2(A(1)) e_q^{[n]_q} + e_q^{[n]_q} [n]_q x [qD_q(A(q)) + D_q(A(1)) + A(1) \frac{[n]_q^2}{b_n^2} x^2 e_q^{[n]_q}] \right\}
= x^2 + \frac{E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} \frac{[qD_q(A(q)) + D_q(A(1))] x b_n}{[n]_q} + \frac{E_q^{[n]_q} x e_q^{[n]_q}}{A(1)} \frac{D_q^2(A(1))}{[n]_q^2} b_n^2.
\]
Hence, the proof is completed.

\[ \text{Theorem 3.2. Assume that } q := (q_n), 0 < q_n < 1 \text{ be a sequence satisfying the following conditions:} \]

\[ (3.7) \quad st - \lim_n \frac{b_n}{|n|_q} = 0, \quad st - \lim_n q_n = b. \]

Let \( f \in C[0, \infty) \cap E, \) where \( E = \{ f \in C[0, \infty) : |f(x)| \leq \alpha e^{\beta x} \text{ for some } \alpha \in \mathbb{R}, \beta \in \mathbb{R} \}. \) Then we have the following:

\[ st - \lim_n \| P_n^* (f, q_n) - f \|_{\rho_0} = 0. \]

\[ Proof. \text{ It is enough to prove that} \]

\[ st - \lim_n \| P_n^* (e_v, q_n; \cdot) - e_v \|_{\rho_0} = 0 \]

where \( v = 0, 1, 2. \)

From the equation (3.1), it is easy to obtain that

\[ st - \lim_n \| P_n^* (e_0, q_n; \cdot) - e_0 \|_{\rho_0} = 0. \]

By (3.2) and combining with \( E_{\delta} < \frac{q_n}{b_n} < \frac{1}{\epsilon} \), we get

\[ \| P_n^* (e_1, q_n; \cdot) - e_1 \|_{\rho_0} \leq \frac{D_q (A(1))}{A(1)} \left[ \frac{b_n}{|n|_{q_n}} \right] \| e_0 \|_{\rho_0}. \]

For \( \epsilon > 0 \), define the following sets:

\[ \mathcal{M} := \{ k : \| P_n^* (e_1, q_k; \cdot) - e_1 \|_{\rho_0} \geq \epsilon \}, \]

\[ \mathcal{M}_1 := \{ k : \frac{D_q (A(1))}{A(1)} \frac{b_k}{|k|_{q_k}} \geq \epsilon \}. \]

such that \( \mathcal{M} \subseteq \mathcal{M}_1. \)

By (3.7), one can write the following

\[ \delta \{ k \leq n : \| P_n^* (e_1, q_k; \cdot) - e_1 \|_{\rho_0} \geq \epsilon \} \leq \delta \{ k \leq n : \frac{D_q (A(1))}{A(1)} \frac{b_k}{|k|_{q_k}} \geq \epsilon \}. \]

Hence, we get

\[ (3.8) \quad st - \lim_n \| P_n^* (e_1, q_n; \cdot) - e_1 \|_{\rho_0} = 0. \]

By (3.3), one can see that

\[ \| P_n^* (e_2, q_n; \cdot) - e_2 \|_{\rho_0} \leq \frac{q_n D_q (A(q)) + D_q (A(1))}{A(1)} \left[ \frac{b_n}{|n|_{q_n}} \right] \| e_1 \|_{\rho_0} \]

\[ + \frac{D_q (A(1)) + D_q^2 (A(1))}{A(1)} \left( \frac{b_k^2}{|k|_{q_k}} \right) \| e_0 \|_{\rho_0}. \]

Now, let \( \epsilon > 0 \) be given, we define the following sets:

\[ \mathcal{V} := \{ k : \| P_n^* (e_2, q_k; \cdot) - e_2 \|_{\rho_0} \geq \epsilon \}, \]

\[ \mathcal{V}_1 := \{ k : \frac{q_k D_q (A(q)) + D_q (A(1))}{A(1)} \frac{b_k}{|k|_{q_k}} \geq \epsilon \}, \]

\[ \mathcal{V}_2 := \{ k : \frac{D_q (A(1)) + D_q^2 (A(1))}{A(1)} \frac{b_k^2}{|k|_{q_k}} \geq \epsilon \}. \]

such that \( \mathcal{V} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2. \)

Thus, we obtain

\[ \delta \{ k \leq n : \| P_n^* (e_2, q_k; \cdot) - e_2 \|_{\rho_0} \geq \epsilon \} \leq \delta \left\{ k \leq n : \frac{q_k D_q (A(q)) + D_q (A(1))}{A(1)} \frac{b_k}{|k|_{q_k}} \right\} \]

\[ + \delta \left\{ k \leq n : \frac{D_q (A(1)) + D_q^2 (A(1))}{A(1)} \frac{b_k^2}{|k|_{q_k}} \geq \epsilon \right\}. \]

Hence, (3.7) and (3.9) imply that

\[ (3.10) \quad st - \lim_n \| P_n^* (e_2, q_n; \cdot) - e_2 \|_{\rho_0} = 0. \]
In this section, we give the rates of statistical convergence of the operators \( P_n^*(f; q; x) \) by means of modulus of continuity with the help of functions from Lipschitz class.

\( C_B[0, \infty) \) denotes all real valued continuous bounded functions. The modulus of continuity for the function \( f \in C_B[0, \infty) \) is defined as

\[
\omega(f, \delta)_{\rho_0} = \sup_{|x-y| \leq \delta, x, y \in [0, \infty)} \frac{|f(x) - f(y)|}{1 + x^2 + y^2},
\]

where \( \omega(f, \delta)_{\rho_0} \) for \( \delta > 0, \lambda \geq 0 \) satisfy the following conditions: for every \( f \in C_B[0, \infty) \),

\[
\lim_{\delta \to \infty} \omega(f, \delta)_{\rho_0} = 0
\]

and

\[
|f(x) - f(y)| \leq \omega(f, \delta)_{\rho_0} \left( \frac{|x - y|}{\delta} + 1 \right).
\]

Now, we prove the following theorem for the rate of pointwise convergence of the operators \( P_n^*(f; q; x) \) to the function \( f(x) \) by means of modulus of continuity.

**Theorem 4.1.** Let \( q := (q_n) \) be a sequence and \( x \in [0, \infty) \), then we have

\[
|P_n^*(f; q; x) - f(x)| \leq 2\omega(f, \sqrt{\delta_n})_{\rho_0},
\]

for all \( f \in C_B[0, \infty) \cap E \), where

\[
(4.2) \quad \delta_n = \frac{q_n D_q(A(q)) + D_q(A(1))}{A(1)} \frac{b_n}{|n| q_n} ||e_1||_{\rho_0} + \frac{D_q^2(A(1))}{A(1)} \frac{b_n^2}{|n|^2 q_n} ||e_0||_{\rho_0}
\]

**Proof.** By the linearity and positivity of the operators \( P_n^*(f; q; x) \), one can write

\[
|P_n^*(f; q; x) - f(x)| \leq \frac{E_q^{[n]x}}{A(1)} \sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \left| f \left( \frac{[k] q_n}{|n| q_n} b_n \right) - f(x) \right|
\]

\[
\leq \frac{E_q^{[n]x}}{A(1)} \sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \omega(f, \delta) \left\{ \frac{1}{\delta} \left| \frac{[k] q_n}{|n| q_n} b_n - x \right| + 1 \right\}
\]

If we apply Cauchy-Schwarz inequality for sums, we obtain

\[
\sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \left| \frac{[k] q_n}{|n| q_n} b_n - x \right|^2 \leq \left( \sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \left| \frac{[k] q_n}{|n| q_n} b_n - x \right| \right)^2.
\]

Using above inequality and by (3.1), we have

\[
|P_n^*(f; q; x) - f(x)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \left[ \frac{E_q^{[n]x}}{A(1)} \sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \left| \frac{[k] q_n}{|n| q_n} b_n - x \right|^2 \right]^{1/2} \right\}
\]

\[
= \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \left[ \frac{E_q^{[n]x}}{A(1)} \sum_{k=0}^{[n]x} p_k \left( q; \frac{|n| q_n}{b_n} \right) \left| \frac{[k] q_n}{|n| q_n} b_n - x \right|^2 \right]^{1/2} \right\}
\]

\[
= \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \left[ \frac{1}{E_q^{[n]x}} \frac{q_n D_q(A(q)) + D_q(A(1))}{A(1)} \frac{b_n}{|n| q_n} ||e_1||_{\rho_0} + \frac{D_q^2(A(1))}{A(1)} \frac{b_n^2}{|n|^2 q_n} ||e_0||_{\rho_0} \right] \right\}^{1/2}.
\]

From Lemma 3.1 and the fact \( E_q^{[n]x} e_q^{[n]x} < 1 \), we have

\[
(4.3) \quad P_n^*((s - e_1)^2; q_n; x) \leq \frac{q_n D_q(A(q)) + D_q(A(1))}{A(1)} \frac{b_n}{|n| q_n} ||e_1||_{\rho_0} + \frac{D_q^2(A(1))}{A(1)} \frac{b_n^2}{|n|^2 q_n} ||e_0||_{\rho_0}.
\]

By (4.3) and choose \( \delta = \sqrt{\delta_n} \), we get

\[
|P_n^*(f; q; x) - f(x)| \leq 2\omega(f, \sqrt{\delta_n})_{\rho_0}.
\]
This step concludes the proof. □

We know that a function \( f \in C[0, \infty) \) is in \( \text{Lip}_M(\alpha) \) on \( F \), \( 0 < \alpha \leq 1 \), \( F \) is a any bounded subset of the interval \([0, \infty)\) if it satisfies the conditions

\[
|f(y) - f(x)| \leq M|x - y|^\alpha, \quad y \in [0, \infty) \text{ and } x \in F,
\]

where \( M \) is a constant depending only \( \alpha \) and \( f \).

**Theorem 4.2.** Let \( f \in C[0, \infty) \cap \text{Lip}_M(\alpha) \), \( 0 < \alpha \leq 1 \). Then, we get

\[
|P_n^*(f; q_n; x) - f(x)| \leq M \left( \delta_n^{\alpha/2} + d^\alpha(x, F) \right), \quad x \in [0, \infty),
\]

where \( M \) is a constant depending only \( \alpha \) and \( f \) and \( d(x, F) \) is the distance between \( x \) and \( F \) defined by \( d(x, F) = \inf\{|x - t| : t \in F\} \).

**Proof.** Let \( F \) be the closure of \( F \) in \([0, \infty)\). There exists at least point \( x_0 \in \overline{F} \) such that \( d(x, F) = |x - x_0| \).

By our assumption and monotonicity of \( P_n^*(f; q_n; x) \), one can write the following

\[
|P_n^*(f; q_n; x) - f(x)| \leq P_n^* \left( |f(t) - f(x)|; q_n; x \right) + P_n^* \left( |f(x) - f(x_0)|; q_n; x \right)
\]

\[
\leq M \left\{ P_n^* \left( |t - x|^\alpha; q_n; x \right) + |x - x_0|^\alpha \right\}.
\]

Next, applying the Hölder inequality with \( p = \frac{1}{\alpha} \), \( q = \frac{\alpha}{2 - \alpha} \) and we get

\[
|P_n^*(f; q_n; x) - f(x)| \leq M \left\{ P_n^* \left( |t - x|^2; q_n; x \right) |x - x_0|^\alpha \right\}.
\]

This step conclude the proof. □

Now, we obtain local direct estimate of the operators \( P_n^*(f; q_n; x) \) using the Lipschitz-type maximal function of order \( \alpha \) introduce Lenz [17] as

\[
\omega_\alpha(f, x) = \sup_{t \neq x, t \in [0, \infty)} \frac{|f(t) - f(x)|}{|t - x|^\alpha}, \quad x \in [0, \infty) \text{ and } \alpha \in (0, 1].
\]

**Theorem 4.3.** Let \( f \in \text{Lip}_M(\alpha) \), \( 0 < \alpha \leq 1 \), then we have

\[
|P_n^*(f; q_n; .) - f(.)| \leq \omega_\alpha(f, x) \delta_n^{\alpha/2},
\]

where \( \delta_n \) defined in \( (5.1) \).

**Proof.** From \((4.5)\), we obtain

\[
|P_n^*(f; q_n; x) - f(x)| \leq P_n^* \left( |f(t) - f(x)|; q_n; x \right)
\]

\[
\leq \omega_\alpha(f, x) P_n^* \left( |t - x|^\alpha; q_n; x \right).
\]

Again for \( p = \frac{1}{\alpha} \), \( q = \frac{\alpha}{2 - \alpha} \), applying the Hölder inequality, we get

\[
|P_n^*(f; q_n; x) - f(x)| \leq P_n^* \left( |f(t) - f(x)|; q_n; x \right)
\]

\[
\leq \omega_\alpha(f, x) \left\{ P_n^* \left( |e|; q_n; x \right) \right\} \alpha/2 = \omega_\alpha(f, x) \delta_n^{\alpha/2}.
\]

Hence, we get the desired result. □

### 5. Local Approximation

In this section, we state the local approximation theorem of the operators \( P_n^*(f; q_n; x) \). Let \( C_B[0, \infty) \) be the space of all real valued continuous bounded functions \( f \) on \([0, \infty)\) with the norm \( \| f \| = \sup\{ |f(x)| : x \in [0, \infty) \} \).

The K-functional of \( f \) is defined by

\[
K_2(f; \delta) = \inf_{g \in W^2} \{ \| f - g \| + \delta \| g'' \| \},
\]

where \( \delta > 0 \) and \( W^2 = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \} \). By Devore-Lorentz [5, p. 177], there exists an absolute constant \( C > 0 \) such that

\[
(5.1) \quad K_2(f; \delta) \leq C \omega_2 \left( f, \sqrt{\delta} \right)
\]

where

\[
\omega_2 \left( f, \sqrt{\delta} \right) = \sup_{0 < h \leq \delta, 0 < x < \infty} |f(x + 2h) - 2f(x + h) + f(x)|
\]
is the second order modulus of smoothness of \( f \). Moreover,

\[
\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{0 \leq x \leq 1} |f(x + h) - f(x)|
\]

denotes the modulus of continuity of \( f \).

Now, we give the direct local approximation theorem for the operators \( P_n^* (f; q; x) \).

**Theorem 5.1.** Let \( q \in (0, 1) \). We have

\[
|P_n^* (f; q; x) - f(x)| \leq K\omega_2(f, \varphi_n) + \omega \left( f, \frac{D_q(A(1)) b_n}{|n|q} \right)
\]

\( \forall x \in [0, \infty), f \in C_B[0, \infty), \) where \( K \) is a positive constant and

\[
\varphi_n = \frac{q_n D_q(A(q)) + D_q(A(1))}{A(1)} b_n + \frac{(1)D_q^2(A(1)) + [D_q(A(1))]^2}{A^2(1)} \frac{b_n^2}{|n|q^n}
\]

**Proof.** Let us define the following operators

\[
P_n^* (f; q; x) = P_n^*(f; q; x) - f \left( x + \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} \right) + f(x),
\]

\( x \in [0, \infty) \). The operators \( P_n^*(f; q; x) \) are linear. Thus, we have the following:

\[
P_n^* (s-x; q; x) = 0,
\]

(see Lemma 3.1). Let \( g \in W^2 \), from Taylor’s expansion

\[
g(s) = g(x) + g'(x)(s-x) + \int_x^s (s-u)g''(x)du,
\]

\( s \in [0, \infty) \) and \( 5.3 \) we obtain

\[
P_n^* (g; q; x) = g(x) + P_n^*(\int_x^s (s-u)g''(x)du).
\]

By \( 5.2 \), we have the following

\[
|P_n^* (g; q; x) - g(x)| \leq \left| P_n^* \left( \int_x^s (s-u)g''(u)du \right) \right|
\]

\[
+ \left| \int_x^s \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} \left( x + \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} - u \right) g''(u)du \right|
\]

\[
\leq P_n^* \left( \left| \int_x^s (s-u)g''(u)du \right|, x \right)
\]

\[
+ \left| \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} \right| \left| \int_x^s x + \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} - u \right| \left| g''(u) \right| du
\]

\[
\leq \left[ P_n^* (s-x)^2 + \left( \frac{D_q(A(1)) E_q^{\frac{|n|q}{|n|q+n}} e_q^{\frac{|n|q}{|n|q+n}} b_n}{A(1)} \right)^2 \right] \left| g'' \right|
\]

(5.4)

Therefore, from \( 5.4 \), we obtain

\[
|P_n^* (g; q; x) - g(x)| \leq \varphi_n.
\]

By \( 5.1, 5.2 \) and \( 5.4 \), we get

\[
|P_n^* (f; q; x)| \leq P_n^* (f; q; x) + 2\|f\|
\]

\[
\leq \|f\| P_n^* (1; q; x) + 2\|f\|
\]

\[
\leq 3\|f\|
\]

(5.6)
and by (5.2), (5.5) and (5.6)

\[ |P_n^*(f; q; x) - f(x)| \leq |\tilde{P}_n^*(f - g; q; x) - (f - g)(x)| + |\tilde{P}_n^*(g; q; x) - g(x)| \]

\[ + \left| f \left( x + \frac{D_q(A(1))}{A(1)} \frac{\frac{[n]_q}{q^n} e_{\alpha q}^{[n]_q} b_n}{[n]_q} \right) - f(x) \right| \]

\[ \leq 4\|f - g\| + \varphi_n\|g''\|. \]

From (5.1), one can see that

\[ |P_n^*(f; q; x) - f(x)| \leq K\omega_2(f, \varphi_n) + \omega \left( f, \frac{D_q(A(1))}{A(1)} \frac{b_n}{[n]_q} \right), \]

and this concludes the proof. □

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