Research Article

Finite 1-Regular Cayley Graphs of Valency 5

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Let \( \Gamma = \text{Cay}(G, S) \) and \( G \leq X \leq \text{Aut}\Gamma \). We say \( \Gamma \) is \((X, 1)\)-regular Cayley graph if \( X \) acts regularly on its arcs. \( \Gamma \) is said to be core-free if \( G \) is core-free in some \( X \leq \text{Aut(Cay}(G, S)) \). In this paper, we prove that if an \((X, 1)\)-regular Cayley graph of valency 5 is not normal or binoimal, then it is the normal cover of one of two core-free ones up to isomorphism. In particular, there are no core-free 1-regular Cayley graphs of valency 5.

1. Introduction

We assume that all graphs in this paper are finite, simple, and undirected.

Let \( \Gamma \) be a graph. Denote the vertex set, arc set, and full automorphism group of \( \Gamma \) by \( V\Gamma , A\Gamma \), and \( \text{Aut}\Gamma \), respectively. A graph \( \Gamma \) is called \( X\)-vertex-transitive or \( X\)-arc-transitive if \( X \) acts transitively on \( V\Gamma \) or \( A\Gamma \), where \( X \leq \text{Aut}\Gamma \). \( \Gamma \) is simply called vertex-transitive, arc-transitive for the case where \( X = \text{Aut}\Gamma \). In particular, \( \Gamma \) is called \((X, 1)\)-regular if \( X \leq \text{Aut}\Gamma \) acts regularly on its arcs and then 1-regular when \( X = \text{Aut}\Gamma \).

Let \( G \) be a finite group with identity element 1. For a subset \( S \) of \( G \) with \( 1 \notin S = S^{-1} := \{x^{-1} \mid x \in S\} \), the Cayley graph \( \text{Cay}(G, S) \) of \( G \) (with respect to \( S \)) is defined as the graph with vertex set \( G \) such that \( x, y \in G \) are adjacent if and only if \( yx^{-1} \in S \). It is easy to see that a Cayley graph \( \text{Cay}(G, S) \) has valency \( |S| \), and it is connected if and only if \( S = G \).

Li proved in [1] that there are only finite number of core-free \( s \)-transitive Cayley graphs of valency \( k \) for \( s \in \{2, 3, 4, 5, 7\} \) and \( k \geq 3 \) and that, with the exceptions \( s = 2 \) and \( (s, k) = (3, 7) \), every \( s \)-transitive Cayley graph is a normal cover of a core-free one. It was proved in [2] that there are 15 core-free \( s \)-transitive cubic Cayley graphs up to isomorphism, and there are no core-free 1-regular cubic Cayley graphs. A natural problem arises. Characterize \( 1 \)-transitive Cayley graphs, in particular, which graphs are 1-regular? Until now, the result about 1-regular graphs mainly focused constructing examples. For example, Frucht gave the first example of cubic 1-regular graph in [3]. After then, Conder and Praeger constructed two infinite families of cubic 1-regular graphs in [4], Marušič [5] and Malnič et al. [6] constructed two infinite families of tetravalent 1-regular graphs. Classifying such graphs has aroused great interest. Motivated by above results and problem, we consider 1-regular Cayley graphs of valency 5 in this paper.

A graph \( \Gamma \) can be viewed as a Cayley graph of a group \( G \) if and only if \( \text{Aut}\Gamma \) contains a subgroup that is isomorphic to \( G \) and acts regularly on the vertex set. For convenience, we denote this regular subgroup still by \( G \). If \( X \leq \text{Aut}\Gamma \) contains a normal subgroup that is regular and isomorphic to \( G \), then \( \Gamma \) is called an \( X\)-normal Cayley graph of \( G \); if \( G \) is not normal in \( X \) but has a subgroup which is normal in \( X \) and semiregular on \( V\Gamma \) with exactly two orbits, then \( \Gamma \) is called an \( X\)-bi-normal Cayley graph; furthermore if \( X = \text{Aut}\Gamma \), \( \Gamma \) is called normal or bi-normal. Some characterization of normal and bi-normal Cayley graphs has given in [1, 2].

For a Cayley graph \( \Gamma = \text{Cay}(G, S) \), \( \Gamma \) is said to be core-free (with respect to \( G \)) if \( G \) is core-free in some \( X \leq \text{Aut}\Gamma \); that is, \( \text{Core}_X(G) = \bigcap_{x \in X} G_x = 1 \).

The main result of this paper is the following assertion.

Theorem 1. Let \( \Gamma = \text{Cay}(G, S) \) be an \((X, 1)\)-regular Cayley graph of valency 5, where \( G \leq X \leq \text{Aut}\Gamma \). Let \( n(G) \) be the number of nonisomorphic core-free \((X, 1)\)-regular Cayley
Table 1

| Number | 𝑋 | 𝐺 | 𝑛(𝐺) | 1-regular | Remark |
|--------|----|----|------|-----------|--------|
| 1      | 𝐴₅ | 𝐴₄ | 1    | No        | Icosahedron |
| 2      | 𝑆₃ | 𝑆₃ | 1    | No        |         |

The graph of valency 5 with the regular subgroup equal to 𝐺. Then either

(i) 𝐇 is an 𝑋-normal or 𝑋-bi-normal Cayley graph or

(ii) 𝐇 is a nontrivial normal cover of one line of Table 1.

In particular, there are no core-free 1-regular Cayley graphs of valency 5.

By Theorem 1, we can get the following remark immediately.

Remark 2. Let 𝛾 = Cay(𝐺, 𝑆) be an 1-regular Cayley graph of valency 5. Then 𝛾 is normal or bi-normal.

2. Examples

In this section we give some examples of graphs appearing in Theorem 1.

Example 3. Let 𝑀 = ⟨𝑎⟩ ≅ ℤ₁₁ be a cyclic group. Assume that 𝜏 ∈ Aut(𝑀) is of order 10 and 𝑋 = 𝑀 : ⟨𝜏⟩ ≅ ℤ₁₁ : ℤ₁₀. Let

\[ 𝐺 = 𝑀 : ⟨τ^5⟩ ≅ D_{22}. \]  

Suppose that

\[ 𝑆 = ℤ^5 = \{g, g^τ, g^{-1}, g^{-2}, g^{-3}\}. \]  

where \( g \in G \) is an involution such that \( g \neq τ^5 \). Let \( Γ = Cay(G, S) \) be the Cayley graph of the dihedral group \( G \) with respect to \( S \). Then \( Γ \) is a connected \((X,1)\)-regular Cayley graph of valency 5. In particular, \( Γ \) is \( X \)-normal.

Proof. 

\[ G = ⟨a⟩ : ⟨b⟩ = \{1, a, a^2, \ldots, a^{10}, b, ab, a^2b, \ldots, a^{10}b⟩ \cong D_{22}, \]  

where \( b = τ^5 \).

Noting \( o(a) = 11 \), we may assume that \( a^7 = a^5 \). Since the involution \( g \in G \) is not equal to \( b \), we may let \( g = a^ib \) for some \( 1 \leq i < 11 \) such that \( (g, i) = 1 \). Then \( g^7 = (a^ib)^7 = a^{8i}b = a^{4i}b \), and so \( g^2g^{-1} = a^{3i} \in ⟨g^3⟩ = ⟨S⟩ \). Thus the element \( g^2g^{-1} \) is of order 11 as \( (3i, 11) = 1 \). So \( ⟨S⟩ = ⟨g^7⟩ = G \); that is, \( Γ = Cay(G, S) \) is connected.

Obviously, \( G \trianglelefteq X \trianglelefteq Aut(Γ) \) and \( X = ⟨τ⟩ \). However, \( |X| = 55 = |Aut(Γ)| \); then \( Γ \) is an \((X,1)\)-regular normal Cayley graph of \( G \) of valency 5.

Example 4. Let \( G = ⟨a, b \mid a^5 = b^2 = 1, a^b = a^{-1}⟩ \cong D_{10} \). Set \( S = \{b, ab, a^2b, a^3b, a^4b\} \) and \( Γ = Cay(G, S) \). Then \( Γ \cong K_5 \) and \( Aut(Γ) = S_5 \). Let \( X = (Z_5 \times Z_5) : Z_2 \cong D_{10} \times Z_5 \) such that \( G \trianglelefteq X \trianglelefteq Aut(Γ) \). It follows that \( Core_X(Γ) \cong Z_5 \). Then \( X_{Aut(Γ)} = Z_5 \) for \( α \in VT \), and furthermore \( Γ \) is \((X,1)\)-regular. Obviously \( G \) is not normal in \( X \). However, \( Core_X(Γ) \trianglelefteq X \) is semiregular and has exactly two orbits on \( VT \); then \( Γ \) is an \((X,1)\)-regular Cayley graph of valency 5. In particular, \( Γ \) is \( X \)-bi-normal.

3. The Proof of Main Results

In this section, we will prove our main results. We first present some properties about normal Cayley graphs.

For a Cayley graph \( Γ = Cay(G, S) \), we have a subgroup of \( Aut(G) \):

\[ Aut(G, S) = \{σ ∈ Aut(G) \mid S^σ = S\}. \]  

Clearly it is a subgroup of the stabilizer in \( Aut(Γ) \) of the vertex corresponding to the identity 1 of \( G \). Since \( Γ \) is connected, \( Aut(G, S) \) acts faithfully on \( S \). By Godsil [7, Lemma 2.1], the normalizer \( N_{Aut(Γ)}(G) = G : Aut(S) \); So \( Γ = Cay(G, S) \) is a normal Cayley graph if and only if \( Aut(G, S) = \{Aut(Γ)\} \).

Let \( Γ = Cay(G, S) \) be an \((X,1)\)-regular Cayley graph of valency 5 such that \( G \trianglelefteq X \trianglelefteq Aut(Γ) \). Then \( S \) contains at least one involution. Let \( K = Core_X(Γ) \), which is the core of \( G \) in \( X \).

Lemma 5. Assume that \( K = 1 \). Then \( (X, G) = (A_5, A_4) \) or \((S_3, S_4)\).

Proof. Let \( H \) be the stabilizer in \( X \) of the vertex corresponding to the identity 1 of \( G \). Then \( H ≅ Z_2, H \cap G = 1, \) and \( X = GH \). Let \( [X : G] \) be the set of right cosets of \( G \) in \( X \). Consider the action of \( X \) on \([X : G]\) by the right multiplication. Then we get that \( X \) is a primitive permutation group of degree 5 and \( G \) is a stabilizer of \( X \). Since \( Γ \) has valency 5, \( |G| = |VT| ≥ 6, \) and \( |X| = |G||H| ≥ 30 \). Then we can show \( X ≅ A_5 \) or \( S_5 \), and then \( G = A_4 \) or \( S_4 \), respectively.

Lemma 6. Suppose that \( G = A_4 \) and \( X = A_5 \). Then \( Γ \) is the icosahedron graph. Moreover, \( Aut(Γ) = A_5 \times Z_2 \) and \( Γ \) is not 1-regular.

Proof. Note that \( X = GH \), where \( X ≅ A_5, G ≅ A_4, \) and \( H ≅ Z_5 \). Since \( X \) has no nontrivial normal subgroup, \( Γ \) is not bipartite. So \( Γ \) is the icosahedron graph. Further by Magma [8], \( Aut(Γ) = A_5 \times Z_2 \), so \( Γ \) is not 1-regular.

Lemma 7. Suppose that \( G = S_3 \) and \( X = S_4 \). Then the graph \( Γ \) is not 1-regular and there is only one isomorphism class of these graphs.

Proof. Note that \( G = S_4, X = S_5, \) and \( X = GH \). Let \( H = ⟨σ⟩, \) where \( σ = (1 2 3 4 5) \). By considering the right multiplication action of \( X \) on the right cosets of \( G \) in \( X, G \) can be viewed as a stabilizer of \( X \) acting on \( [1, 2, 3, 4, 5] \). Without lost generality, we may assume that \( 1 \) is fixed by \( G \). Take an involution \( τ \in S \). Then, by \([2], \tau \in S_5 \setminus N_{S_5}(H) \) and we can identify \( S \) with \( HT \cap \Gamma \). Note that \( τ \in G \trianglelefteq S_4 \).
and \( N_5(H) = H : \text{Aut}(H) = \langle (1 2 3 4 5) : (2 3 5 4) \rangle \equiv Z_5 : Z_3 \); then \( \tau \) is one of the following: \( (2 5), (3 5), (2 3), (3 4), (4 5), (2 4), (2 3)(4 5), \) and \( (2 4)(3 5) \). Note \( H = \langle (1 2 3 4 5) \rangle \). Assume that \( \tau = (2 5) \); by calculation, we have \( (2 5) \Rightarrow h_1, \tau \cdot (1 2 3 4 5) = (1 2)(3 4 5) \Rightarrow h_1 \), \( \tau \cdot (1 3 5 2 4) = (1 3 5 4) \Rightarrow h_3, \tau \cdot (1 4 2 5 3) = (1 4 2 3) \Rightarrow h_4, \) and \( \tau \cdot (1 5 4 3 2) = (1 5)(2 4 3) \Rightarrow h_2 \). Then \( H(2 5)H = \{ H h \}_{h \in H} \). Assume that \( H(2 5)H = \{ H h \}_{h \in H} \).

is trivial to see that \( H = \langle (1 2 3 4 5) \rangle \). Assume that \( h = (1 4 2 3) \); then \( \tau \cdot (1 2 3 4 5) = (1 2)(3 4 5) \Rightarrow h_1 \), \( \tau \cdot (1 3 5 2 4) = (1 3 5 4) \Rightarrow h_3, \tau \cdot (1 4 2 5 3) = (1 4 2 3) \Rightarrow h_4, \) and \( \tau \cdot (1 5 4 3 2) = (1 5)(2 4 3) \Rightarrow h_2 \). Then \( H(2 5)H = \{ H h \}_{h \in H} \).

is one of the following: \( (1 4 2 3), (1 2 5 3), (1 2 3 5), (1 2 3 5 4), (1 5 4 3 2) \).

To finish our proof, we need to introduce some definitions and properties. Assume that \( \Gamma \) is an \( X \)-regular Cayley graph with \( X \) being a subgroup of \( \text{Aut}(\Gamma) \). Let \( N \) be a normal subgroup of \( X \). Denote the set of \( N \)-orbits in \( V_\Gamma \) by \( V_\Gamma(N) \). The normal quotient \( \Gamma_N \) of \( \Gamma \) induced by \( N \) is defined as the graph with vertex set \( V_\Gamma \), and two vertices \( B, C \in V_\Gamma \) are adjacent if there exist \( u \in B \) and \( v \in C \) such that they are adjacent in \( \Gamma \). It is easy to show that \( X/N \) acts transitively on the vertex set of \( \Gamma_N \). Assume further that \( \Gamma \) is \( X \)-edge-transitive. Then \( X/N \) acts transitively on the edge set of \( \Gamma_N \), and the valency \( \text{val}(\Gamma) = \text{val}(\Gamma_N) \) for some positive integer \( m \). If \( m = 1 \), then \( \Gamma \) is called a normal cover of \( \Gamma_N \).

**Proof of Theorem 1.** Let \( \Gamma = \text{Cay}(G, S) \) be an \( (X, 1) \)-regular Cayley graph of valency 5, where \( G \leq X \leq \text{Aut}(\Gamma) \). Then it is trivial to see that \( \Gamma \) is connected. Let \( N = \text{Core}_X(G) \) be the core of \( G \) in \( X \). Assume that \( N \) is not trivial. Then either \( G = N \) or \( |G : N| \geq 2 \). The former implies \( G \leq X \); that is, \( \Gamma \) is an \( X \)-normal Cayley graph with respect to \( G \). For the case where \( |G : N| = 2 \), it is easy to verify \( \Gamma \) is an \( X \)-bi-normal Cayley graph. Suppose that \( |G : N| > 2 \); namely, \( N \) has at least three orbits on \( V_\Gamma \). Since \( \text{val}(\Gamma) = 5 \) is a prime and \( \Gamma \) is \( (X, 1) \)-regular, \( \Gamma \) is a cover of \( \Gamma_N \). Hence we have that \( \Gamma_N \) is a Cayley graph of \( G/N \) and \( \Gamma_N \) is core-free with respect to \( G/N \). Now suppose that \( N \) is trivial, then \( \Gamma \) is a core-free one. According to Lemmas 5, 6, and 7, there are two core-free \( (X, 1) \)-regular Cayley graphs of valency 5 (up to isomorphism) as in Table 1. As far as Theorem 1 holds.

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