Renormalizability of Nonrenormalizable Field Theories

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We give a simple and elegant proof of the Equivalence Theorem, stating that two field theories related by nonlinear field transformations have the same S matrix. We are thus able to identify a subclass of nonrenormalizable field theories which are actually physically equivalent to renormalizable ones. Our strategy is to show by means of the BRS formalism that the “nonrenormalizable” part of such fake nonrenormalizable theories, is a kind of gauge fixing, being confined in the cohomologically trivial sector of the theory.

Recently there has been a renewed interest, triggered by the work of Gomis and Weinberg [1] on the apparently nonrenormalizable theories. The main point was analyzed by Bergère and Lam [2] who showed that two quantum field theories related by a nonlinear field transformation of the kind

\[ \phi = \hat{\phi} + \alpha \hat{\phi}^2 g(\hat{\phi}; \alpha) \]  

have the same S-matrix. This statement is known in the literature as the “equivalence theorem” since more than fifty years [3], and we propose here an alternative very simple proof, which is easily adaptable to all situations, since it relies neither on the use of the equations of motion, nor on any particular renormalization scheme.

The strategy is to approach the problem with the technology of nilpotent operators, as it is applied in gauge field theories, and hence to interpret the effect on the action generated by the nonlinear part of the transformation (1) as a “gauge fixing” term. If this is possible, we have immediately at our disposal the standard results of gauge field theories which insure that “physics” is independent from the gauge choice.

In order to emphasize the relevant features we shall treat only the simplest case, i.e. a scalar field theory with quartic interaction in four dimensions; the method can be straightforwardly extended to the physically interesting cases.

We begin with the renormalizable classical action

\[ \Gamma_R^{(0)}[\phi] = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right) \]  

and the related path integral representation for the vertex functional

\[ \Gamma_R = \int D(\phi) \exp (-\Gamma_R^{(0)}[\phi]) \]  

Now we perform the nonlinear field redefinition (1), where the nonlinear part of the field transformation is identified by the parameter \( \alpha \), and \( g(\hat{\phi}; \alpha) \) is an analytic function of both \( \phi \) and \( \alpha \). The introduction of the \( \alpha \)-parameter, although it could appear as a computational artifact, is indeed natural since to preserve the dimensionally homogeneous character of (1), \( \alpha \) has the dimension of \( \frac{1}{m^d} \). By applying (1) to (3), we obtain a new classical action, but we have also to take into account the Jacobian of the field transformation, which is conveniently exponentiated by means of anticommuting variables \( c(x) \) and \( \bar{c}(x) \). Thus we find a new classical action, non renormalizable by power counting

\[ \Gamma_{NR}^{(0)}[\hat{\phi}, \bar{c}; \alpha] = \Gamma_R^{(0)}[\hat{\phi}] + \alpha \Gamma_R^{(0)}[\hat{\phi}] + \int d^4x \bar{c} \left( 1 + 2 \alpha \hat{\phi} g(\hat{\phi}; \alpha) + \alpha \hat{\phi}^2 g'(\hat{\phi}; \alpha) \right) c \]  

where \( \alpha \Gamma_R^{(0)}[\hat{\phi}; \alpha] \) is obtained from

\[ \alpha \Gamma_R^{(0)}[\hat{\phi}; \alpha] = \Gamma_R^{(0)}[\hat{\phi} + \alpha \hat{\phi}^2 g(\hat{\phi}; \alpha)] - \Gamma_R^{(0)}[\hat{\phi}] \]
The corresponding path integral formulation for the proper functional now reads

$$\Gamma_{NR} = \int D(\hat{\phi}) D(c) D(\bar{c}) \exp \left(-\Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha]\right)$$

(6)

It is precisely the part of $\Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha]$ not coinciding with $\Gamma_R^{(0)}[\hat{\phi}]$, which we would like to identify as a “gauge fixing term” with “gauge parameter” $\alpha$.

With this in mind we introduce two ghosts $\beta$ and $b(x)$ of which the first is global, while the second is local, and the BRS transformations

$$s\hat{\phi}(x) = sc(x) = s\bar{c}(x) = s\alpha = 0 \ ,$$

$$sb(x) = \bar{c}(x) \ ,$$

$$s\beta = \alpha \quad (7)$$

Correspondingly, the classical action (4) reads

$$\Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] = \Gamma_{NR}^{(0)}[\hat{\phi}] + s Y[\hat{\phi}, c, \bar{c}; \alpha]$$

(8)

where

$$Y[\hat{\phi}, c, \bar{c}; \alpha] = \beta \Gamma^{(1)} + \int d^4x \ b(1 + 2\alpha \hat{\phi}g(\hat{\phi}; \alpha) + \alpha \hat{\phi}^2 g'(\hat{\phi}; \alpha))c$$

(9)

The classical action $\Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha]$ satisfies the linear Slavnov–Taylor identity

$$S\Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] = \left(\int d^4x \ \bar{c}(x) \frac{\delta}{\delta b(x)} + \alpha \frac{\partial}{\partial \beta}\right) \Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] = 0$$

(10)

Moreover, the action is uncharged with respect to the Faddeev-Popov assignments written in the Table

| $\Phi$ | $\Pi$ | $c$ | $\bar{c}$ | $b$ | $\alpha$ | $\beta$ |
|-------|------|-----|--------|-----|---------|------|
| $01$  | $1$  | $-1$ | $-2$   | $0$ | $-1$    |

Table: Faddeev–Popov charges.

To make contact with the initial problem, and to identify the physical subspace of our example, we restrict the space to that of analytic functions of the $\alpha$-parameter. Within this subspace we can analyze the cohomology of the BRS operator $\hat{\mathcal{L}}[\beta]$ and easily find that it contains only $\alpha$-independent local functionals of $\hat{\phi}(x)$ and $c(x)$, since $\{b(x); \bar{c}(x)\}$ and $\{\beta; \alpha\}$ appear in $\mathcal{L}$ as BRS-doublets $\hat{\mathcal{L}}$.

Thus we have the parametric equation

$$\alpha \frac{\partial}{\partial \alpha} \Gamma = s \left(\int d^4x \ \hat{X}\right) \cdot \Gamma$$

(11)

for a suitable local functional $\hat{X}$. Notice that we have to employ the $\alpha \frac{\partial}{\partial \alpha}$ operator which leaves the cohomology invariant and not simply the $\frac{\partial}{\partial \alpha}$ operator which mixes the cohomological separation of the target space. Indeed, from the expression (11) we have that

$$\frac{\partial}{\partial \alpha} \mathcal{S} \mathcal{F} - \mathcal{S} \frac{\partial}{\partial \alpha} \mathcal{F} = \frac{\partial}{\partial \beta} \mathcal{F}$$

(12)

where $\mathcal{F}$ is a generic functional. Hence
\[ \frac{\partial}{\partial \alpha} \Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] = \frac{\partial}{\partial \alpha} S Y[\hat{\phi}, c, \bar{c}; \alpha] = \left( S \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \right) Y[\hat{\phi}, c, \bar{c}; \alpha] = S \frac{\partial}{\partial \alpha} Y[\hat{\phi}, c, \bar{c}; \alpha] + \Gamma^{(1)} \] (13)

So \( \frac{\partial}{\partial \alpha} \Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] \) is not cohomologically trivial, but \( \alpha \frac{\partial}{\partial \alpha} \Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] \) on the contrary does, since

\[ \alpha \frac{\partial}{\partial \alpha} \Gamma_{NR}^{(0)}[\hat{\phi}, c, \bar{c}; \alpha] = S \left( \alpha \frac{\partial}{\partial \alpha} Y[\hat{\phi}, c, \bar{c}; \alpha] + \beta \Gamma^{(1)} \right) \] (14)

Equation (11) is the statement that only the \( \alpha \)-independent Green functions are “physical” and these are built with the vertices and the propagator obtained by \( \Gamma_{NR}^{(0)}[\hat{\phi}] \). One final remark may be in order; to implement the stability of the theory, we may impose on \( \Gamma \) the further conditions

\[ \frac{\partial \Gamma}{\partial \beta} = \frac{\delta \Gamma}{\delta b(x)} = 0 \] (15)

which are trivially true at the classical level.

Finally, we would like to provide a simple method to decide whether or not a theory specified by a classical action which appears to be nonrenormalizable by power counting can be obtained by a power counting renormalizable action through a nonlinear field redefinition. To be definite, consider the example treated in this letter, i.e. a scalar field \( \phi(x) \) with a certain classical action \( \Gamma_{NR}(\phi) \). First collect in \( \Gamma_{R}(\phi) \) all terms which are power counting renormalizable. The remaining contributions, being non power counting renormalizable, contain at least a power of a parameter \( \alpha \) with the dimension of an inverse of a mass; therefore we can write

\[ \Gamma_{NR}(\phi) = \Gamma_{R}(\phi) + \alpha \Gamma^{*}(\phi) \] (16)

If \( \Gamma_{NR}(\phi) \) can be obtained from \( \Gamma_{R}(\phi) \) by a nonlinear field transformation

\[ \phi \rightarrow \phi + \alpha X(\phi; \alpha) \] (17)

we have to solve for \( X(\phi; \alpha) \) the equation

\[ \Gamma_{R}(\phi + \alpha X(\phi; \alpha)) - \Gamma_{R}(\phi) = \alpha \Gamma^{*}(\phi) \] (18)

A very simple criterion is to analyze it in a descending way, beginning from the highest order monomial in \( \Gamma^{*}(\phi) \) and \( \Gamma_{R}(\phi) \) and remembering that \( X(\phi; \alpha) \) is at least quadratic in \( \phi(x) \). For example, an action \( \Gamma_{NR}(\phi) \) which contains \( \phi^2, \phi^4 \) and \( \phi^6 \) terms only is truly non power counting renormalizable since, through a bilinear transformation containing only one parameter we are bound to obtain a \( \phi^8 \) contribution, too.

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