\textbf{\(\mathcal{W}\)-Extended Logarithmic Minimal Models}

Jørgen Rasmussen

\textit{Department of Mathematics and Statistics, University of Melbourne}
\textit{Parkville, Victoria 3010, Australia}

j.rasmussen@ms.unimelb.edu.au

\textbf{Abstract}

We consider the continuum scaling limit of the infinite series of Yang-Baxter integrable logarithmic minimal models \(\mathcal{LM}(p, p')\) as ‘rational’ logarithmic conformal field theories with extended \(\mathcal{W}\) symmetry. The representation content is found to consist of 6\(pp' - 2p - 2p'\) \(\mathcal{W}\)-indecomposable representations of which 2\(p + 2p' - 2\) are of rank 1, 4\(pp' - 2p - 2p'\) are of rank 2, while the remaining 2\((p - 1)(p' - 1)\) are of rank 3. We identify these representations with suitable limits of Yang-Baxter integrable boundary conditions on the lattice. The \(\mathcal{W}\)-indecomposable rank-1 representations are all \(\mathcal{W}\)-irreducible while we present a conjecture for the embedding patterns of the \(\mathcal{W}\)-indecomposable rank-2 and -3 representations. The associated \(\mathcal{W}\)-extended characters are all given explicitly and decompose as finite non-negative sums of \(\mathcal{W}\)-irreducible characters. The latter correspond to \(\mathcal{W}\)-irreducible subfactors and we find that there are 2\(pp' + (p - 1)(p' - 1)/2\) of them. We present fermionic character expressions for some of the \(\mathcal{W}\)-indecomposable representations.

\section{Introduction}

We consider the infinite series of Yang-Baxter integrable logarithmic minimal models \(\mathcal{LM}(p, p')\) \cite{1}. These are examples of two-dimensional lattice systems whose continuum scaling limits \cite{2} give rise to conformal field theories (CFTs). Our lattice approach to studying these CFTs is predicated on the supposition that, in the continuum scaling limit, a transfer matrix with prescribed boundary conditions gives rise to a representation of the Virasoro algebra. Different boundary conditions naturally lead to different representations. We further assume that, if in addition, the boundary conditions respect the symmetry of a larger conformal algebra \(\mathcal{W}\) \cite{3, 4}, then the continuum scaling limit of the transfer matrix will yield a representation of the extended algebra \(\mathcal{W}\).

A central question of much current interest \cite{5, 6, 7, 8} is whether an extended symmetry algebra \(\mathcal{W}\) exists for logarithmic CFTs \cite{9, 10, 11, 12} like the logarithmic minimal models. Such a symmetry should allow the countably infinite number of Virasoro representations to be reorganized into a finite number of extended \(\mathcal{W}\)-representations which close under fusion. In the case of the logarithmic minimal models \(\mathcal{LM}(1, p')\), the existence of such an extended \(\mathcal{W}\)-symmetry and the associated fusion rules are by now well established \cite{6, 13, 14, 15, 16, 17}. By stark contrast, although there are strong indications \cite{18, 19} that there exists a \(\mathcal{W}_{p,p'}\) symmetry algebra for general augmented minimal models, very little is known about the \(\mathcal{W}\)-extended fusion rules for the \(\mathcal{LM}(p, p')\) models with \(p \geq 2\). This situation was partly
resolved in our recent paper [20]. There we used a lattice approach on a strip, generalizing the approach of [17], to obtain fusion rules of critical percolation $\mathcal{LM}(2, 3)$ in the extended picture.

In [17], it was shown that symplectic fermions [21, 22] is just critical dense polymers $\mathcal{LM}(1, 2)$ [23, 24, 25, 26, 27, 28] viewed in the extended picture. Likewise in the case of critical percolation [20], the extended picture is described by the same lattice model as the Virasoro picture [1]. It is nevertheless useful to distinguish between the two pictures by denoting the extended picture by $\mathcal{WLM}(2, 3)$ and reserve the notation $\mathcal{LM}(2, 3)$ for critical percolation in the non-extended Virasoro picture. A similar distinction applies to the entire infinite series of logarithmic minimal models. Their extended pictures are thus denoted by $\mathcal{WLM}(p, p')$ and are the topic of the present work. The $\mathcal{W}$-extended fusion rules we obtain for these models are based on the fundamental fusion algebra in the Virasoro picture [29, 30] which is a subset of the full fusion algebra. The latter remains to be determined and may eventually yield a larger $\mathcal{W}$-extended fusion algebra than the one presented here.

The layout of this paper is as follows. In Section 2, we review the logarithmic minimal model $\mathcal{LM}(p, p')$ and its fusion algebra [1, 30]. In Section 3, we first summarize the representation content of $\mathcal{WLM}(p, p')$ consisting of $6pp' - 2p - 2p'$ $\mathcal{W}$-indecomposable representations of which $2p + 2p' - 2$ are of rank 1, $4pp' - 2p - 2p'$ are of rank 2, while the remaining $2(p - 1)(p' - 1)$ are of rank 3. The $\mathcal{W}$-indecomposable rank-1 representations are all $\mathcal{W}$-irreducible while we present a conjecture for the embedding patterns of the $\mathcal{W}$-indecomposable rank-2 and -3 representations. The associated $\mathcal{W}$-extended characters decompose as finite non-negative sums of $\mathcal{W}$-irreducible characters. The latter correspond to $\mathcal{W}$-irreducible subfactors and we find that there are $2pp' + (p - 1)(p' - 1)/2$ of them. These are all identified. To distinguish between inequivalent $\mathcal{W}$-indecomposable representations of identical characters, we introduce ‘refined’ characters carrying information also about the Jordan-cell content of a representation. We then present the fusion rules demonstrating closure of the associative and commutative fusion algebra of the $\mathcal{W}$-indecomposable representations. We conclude this section with a discussion of the $\mathcal{W}$-projective representations and find that there are $2pp'$ of them and that they generate a closed fusion subalgebra. In Section 4, we identify the $\mathcal{W}$-extended representations with suitable limits of Yang-Baxter integrable boundary conditions on the lattice and give details of their construction and properties. In particular, we explain how fusion is implemented on the lattice and analyze the ensuing closed fusion algebra. For $p > 1$, this fusion algebra does not contain an identity. We conclude with a short discussion in Section 5.

Notation

For $n, m \in \mathbb{Z}$,
\[
\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m]
\] (1.1)
denotes the set of integers from $n$ to $m$, both included. Certain properties of integers modulo 2 are written
\[
\epsilon(n) = \frac{1 - (-1)^n}{2}, \quad n \cdot m = \frac{3 - (-1)^{n+m}}{2}, \quad n, m \in \mathbb{Z}
\] (1.2)
where the dot product is seen to be associative. An $n$-fold fusion of the Virasoro representation $A$ with itself is abbreviated
\[
A^\otimes n = \underbrace{A \otimes A \otimes \ldots \otimes A}_n
\] (1.3)
By a direct sum of representations $A_n$ with unspecified lower summation bound, we mean a direct sum in steps of 2 whose lower bound is given by the parity $\epsilon(N)$ of the upper bound, that is,
\[
\bigoplus_n^{N} A_n = \bigoplus_{n=\epsilon(N), \text{by } 2}^{N} A_n, \quad N \in \mathbb{Z}
\] (1.4)
This direct sum vanishes for negative $N$. 

2
2 Logarithmic minimal model $\mathcal{LM}(p, p')$

A logarithmic minimal model $\mathcal{LM}(p, p')$ is defined $[1]$ for every coprime pair of positive integers $p < p'$. The model $\mathcal{LM}(p, p')$ has central charge

$$c = 1 - 6\frac{(p' - p)^2}{pp'}$$

(2.1)

and conformal weights

$$\Delta_{r,s} = \frac{(rp' - sp)^2 - (p' - p)^2}{4pp'}, \quad r, s \in \mathbb{N}$$

(2.2)

The fundamental fusion algebra $\langle (2,1), (1,2) \rangle_{p,p'}$ $[29, 30]$ of the logarithmic minimal model $\mathcal{LM}(p, p')$ is generated by the two fundamental Kac representations $(2,1)$ and $(1,2)$ and contains a countably infinite number of inequivalent, indecomposable representations of rank 1, 2 or 3. For $r, s \in \mathbb{N}$, the character of the Kac representation $(r, s)$ is

$$\chi_{r,s}(q) = \frac{q^{\frac{1}{2}rp - \Delta_{r,s}}}{\eta(q)} (1 - q^r) = \frac{1}{\eta(q)} (q^{(rp' - sp)^2/4pp'} - q^{(rp' + sp)^2/4pp'})$$

(2.3)

where the Dedekind eta function is given by

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

(2.4)

Such a representation is of rank 1 and is irreducible if $r \in \mathbb{Z}_{1,p}$ and $s \in p'\mathbb{N}$ or if $r \in p\mathbb{N}$ and $s \in \mathbb{Z}_{1,p'}$. It is a reducible yet indecomposable representation if $r \in \mathbb{Z}_{1,p-1}$ and $s \in \mathbb{Z}_{1,p'-1}$, while it is a fully reducible representation if $r \in p\mathbb{N}$ and $s \in p'\mathbb{N}$ where

$$(kp, k'p') = (k'p, kp) = \bigoplus_{j = |k - k'| + 1, \text{by} 2} (jp, p') = \bigoplus_{j = |k - k'| + 1, \text{by} 2} (p, jp')$$

(2.5)

These are the only Kac representations appearing in the fundamental fusion algebra. The characters of the reducible yet indecomposable Kac representations just mentioned can be written as sums of two irreducible Virasoro characters

$$\chi_{r,s}(q) = \text{ch}_{r,s}(q) + \text{ch}_{2p - r,s}(q) = \text{ch}_{r,s}(q) + \text{ch}_{r,2p' - s}(q), \quad r \in \mathbb{Z}_{1,p-1}, \quad s \in \mathbb{Z}_{1,p'-1}$$

(2.6)

In general and with $a \in \mathbb{Z}_{1,p-1}$, $b \in \mathbb{Z}_{1,p'-1}$ and $k \in \mathbb{N} - 1$, the irreducible Virasoro characters read $[31]$

$$\text{ch}_{a+kp,b}(q) = \frac{1}{\eta(q)} (q^{(kp+a)^2p'/4p} - q^{((k+2)p-a)^2p'/4p})$$

$$\text{ch}_{a+(k+1)p,p'}(q) = \frac{1}{\eta(q)} (q^{((k+1)p+b)^2p'/4p} - q^{((k+1)p+b)^2p'/4p'})$$

$$\text{ch}_{(k+1)p,b}(q) = \frac{1}{\eta(q)} (q^{kp+b)^2p'/4p} - q^{((k+1)p+b)^2p'/4p'})$$

$$\text{ch}_{(k+1)p,p'}(q) = \frac{1}{\eta(q)} (q^{k^2p'/4} - q^{(k+2)^2p'/4})$$

(2.7)

where $K_{n,\nu;k}(q)$ is defined as

$$K_{n,\nu;k}(q) = \frac{1}{\eta(q)} \sum_{j \in \mathbb{Z} \setminus \mathbb{Z}_{1,k}} q^{(\nu - jn)^2/2n}$$

(2.8)
For \( r \in \mathbb{Z}_{1,p}, s \in \mathbb{Z}_{1,p'}, a \in \mathbb{Z}_{1,p-1}, b \in \mathbb{Z}_{1,p'-1} \) and \( k \in \mathbb{N} \), the representations denoted by \( R_{kp,s}^{a,0} \) and \( R_{r,kp'}^{0,b} \) are indecomposable representations of rank 2, while \( R_{kp,p'}^{a,b} \equiv R_{p,kp'}^{a,b} \) is an indecomposaible representation of rank 3. Their characters read

\[
\chi[R_{kp,s}^{a,0}](q) = (1 - \delta_{k,1} \delta_{s,0} ch_{kp-a,s}(q) + 2 ch_{kp+a,s}(q) + ch_{(k+2)p-a,s}(q) \\
\chi[R_{r,kp'}^{0,b}](q) = (1 - \delta_{r,1} \delta_{p',0} ch_{r,kp'-b}(q) + 2 ch_{r,kp'+b}(q) + ch_{r,(k+2)p'-b}(q) \\
\chi[R_{kp,p'}^{a,b}](q) = (1 - \delta_{k,1} ch_{(k-1)p-a,b}(q) + 2 ch_{(k-1)p+a,b}(q) + 2(1 - \delta_{k,1} ch_{kp-a,p'-b}(q)
+ 4 ch_{kp+a,p'-b}(q) + 2 \delta_{k,1} ch_{(k+1)p-a,b}(q) + 2 ch_{(k+1)p+a,b}(q)
+ 2 ch_{(k+2)p-a,p'-b}(q) + ch_{(k+3)p-a,b}(q))
\]

Indecomposable representations of rank 3 appear for \( p > 1 \) only. For \( \alpha \in \mathbb{Z}_{0,p-1}, \beta \in \mathbb{Z}_{0,p'-1} \) and \( k, k' \in \mathbb{N} \), a decomposition similar to (2.5) also applies to the higher-rank decomposable representations \( R_{kp,kp'}^{\alpha,\beta} \) as we have

\[
R_{kp,kp'}^{\alpha,\beta} = R_{k'p,kp'}^{\alpha,\beta} = \bigoplus_{j=|k-k'|+1, by 2}^{k+k'-1} R_{jp,p'}^{\alpha,\beta} = \bigoplus_{j=|k-k'|+1, by 2}^{k+k'-1} R_{jp,p'}^{\alpha,\beta}
\]

Here we have introduced the convenient notation

\[
R_{r,s}^{0,0} \equiv (r, s), \quad r, s \in \mathbb{N}
\]

In the following, we will also use

\[
R_{0,s}^{\alpha,\beta} \equiv R_{r,0}^{\alpha,\beta} = 0, \quad \alpha \in \mathbb{Z}_{0,p-1}, \quad \beta \in \mathbb{Z}_{0,p'-1}, \quad r, s \in \mathbb{N}
\]

Fusion in the fundamental fusion algebra \( \langle (2,1),(1,2) \rangle_{p,p'} \) decomposes into ‘horizontal’ and ‘vertical’ components. With \( \alpha \in \mathbb{Z}_{0,p-1}, \beta \in \mathbb{Z}_{0,p'-1} \) and \( k \in \mathbb{N} \), we thus have

\[
R_{p,kp'}^{\alpha,\beta} = R_{p,1}^{\alpha,0} \otimes R_{1,kp'}^{\alpha,\beta} = R_{kp,1}^{\alpha,0} \otimes R_{1,p'}^{\alpha,\beta}
\]

The Kac representation (1,1) is the identity of the fundamental fusion algebra. For \( p > 1 \), this is a reducible yet indecomposable representation, while for \( p = 1 \), it is an irreducible representation.

Finally, for later reference, we list the horizontal fusions

\[
(r, 1) \otimes (r', 1) = \bigoplus_{j=|r-r'|+1, by 2}^{p-p-r-r'|1} (j, 1) \oplus \bigoplus_{\alpha}^{r+r'-p-1} R_{p,1}^{\alpha,0}, \quad r, r' \in \mathbb{Z}_{1,p}
\]

and

\[
(kp, 1) \otimes (k'p, 1) = \bigoplus_{j=|k-k'|+1, by 2}^{k+k'-1} \{ p^{-1} \bigoplus_{\alpha}^{p-1} R_{jp,1}^{\alpha,0} \}
\]

\[
R_{kp,1}^{\alpha,0} \otimes (k'p, 1) = \bigoplus_{j=|k-k'|+2}^{k+k'} \delta_{j,\{k,k'\}} \bigoplus_{\alpha}^{a-1} R_{jp,1}^{\alpha,0} \bigoplus \bigoplus_{j=|k-k'|+1, by 2}^{k+k'-1} \{ p-a^{-1} \bigoplus_{\alpha}^{p-a-1} 2 R_{jp,1}^{\alpha,0} \}
\]

recalling the meaning of a direct sum with unspecified lower bound [1.3], and the horizontal triple fusions

\[
(p, 1)^{\otimes 3} = \bigoplus_{\alpha=0, by 2}^{p-2} (p-\alpha) \left\{ R_{p,1}^{\alpha,0} \oplus \frac{1}{2} R_{2p,1}^{\alpha,0} \right\}, \quad p \text{ even}
\]

\[
(p, 1)^{\otimes 3} = \bigoplus_{\alpha=0, by 2}^{p-1} (p-\alpha) R_{p,1}^{\alpha,0} \oplus \bigoplus_{\alpha=1, by 2}^{p-2} \frac{1}{2}(p-\alpha) R_{2p,1}^{\alpha,0}, \quad p \text{ odd}
\]
The Kronecker delta function combination appearing in (2.15) is defined by
\[ \delta^{(2)}_{j,\{k,k'\}} = 2 - \delta_{j,|k-k'|} - \delta_{j,k+k'} \quad (2.17) \]

Despite the factors of \(1/N\) in the decompositions (2.16), the multiplicities are all integer. Similar expressions for vertical fusions naturally apply. We refer to [30] for further details on the fundamental fusion algebra of \(\mathcal{LM}(p,p')\).

3 \(\mathcal{W}\)-extended logarithmic minimal model \(\mathcal{WL\mathcal{M}(p,p')}\)

In this section, we summarize our findings in the extended picture \(\mathcal{WL\mathcal{M}(p,p')}\) for the representation content and the associated embedding patterns, (refined) characters and closed fusion algebra. Unless otherwise specified, we let
\[ \kappa, \kappa' \in \mathbb{Z}_{1,2}, \quad r \in \mathbb{Z}_{1,p}, \quad s \in \mathbb{Z}_{1,p'}, \quad a, a' \in \mathbb{Z}_{1,p-1}, \quad b, b' \in \mathbb{Z}_{1,p'-1}, \quad \alpha \in \mathbb{Z}_{0,p-1}, \quad \beta \in \mathbb{Z}_{0,p'-1} \quad (3.1) \]
and \(k,k',n \in \mathbb{N}\).

3.1 Representation content

We have the \(2(p+p'-1)\) \(\mathcal{W}\)-indecomposable rank-1 representations
\[ \{(\kappa p, s)_{\mathcal{W}}, (r, \kappa' p')_{\mathcal{W}}\} \quad \text{subject to} \quad (p, \kappa' p')_{\mathcal{W}} \equiv (\kappa p, p')_{\mathcal{W}} \quad (3.2) \]
where \((p, p')_{\mathcal{W}}\) is listed twice, the \(2((p-1)p' + p(p'-1))\) \(\mathcal{W}\)-indecomposable rank-2 representations
\[ \{(\mathcal{R}^{a,0,\kappa p, s}_{\mathcal{W}}, \mathcal{R}^{0,b}_{\mathcal{W}})_{\mathcal{W}}\}\quad (3.3) \]
and the \(2(p-1)(p'-1)\) \(\mathcal{W}\)-indecomposable rank-3 representations
\[ \{(\mathcal{R}^{a, b}_{\mathcal{W}})_{\mathcal{W}}\} \quad \text{subject to} \quad (\mathcal{R}^{a, b}_{\mathcal{W}})_{\mathcal{W}} \equiv (\mathcal{R}^{a, b}_{\mathcal{W}})_{\mathcal{W}} \quad \text{and} \quad (\mathcal{R}^{a, b}_{\mathcal{W}})_{\mathcal{W}} \equiv (\mathcal{R}^{a, b}_{\mathcal{W}})_{\mathcal{W}} \quad (3.4) \]
Here we are asserting that these \(\mathcal{W}\)-representations are indeed \(\mathcal{W}\)-indecomposable. We furthermore believe that the \(\mathcal{W}\)-indecomposable representations (3.2) are \(\mathcal{W}\)-irreducible. Compactly, the various \(\mathcal{W}\)-indecomposable representations satisfy
\[ (\mathcal{R}^{a, \beta}_{(\kappa, \kappa')_{p, p'}})_{\mathcal{W}} \equiv (\mathcal{R}^{a, \beta}_{(\kappa p, \kappa' p')_{\mathcal{W}}})_{\mathcal{W}} \equiv (\mathcal{R}^{a, \beta}_{(\kappa, \kappa')_{p, p'}})_{\mathcal{W}} \quad (3.5) \]
where we have extended our notation to include \((\mathcal{R}^{a, \beta}_{(\kappa, \kappa')_{p, p'}})_{\mathcal{W}}\) for all \(\kappa, \kappa' \in \mathbb{Z}_{1,2}\). The numbers of rank-1 -2 and -3 \(\mathcal{W}\)-indecomposable representations are thus
\[ N_1(p, p') = 2(p+p'-1), \quad N_2(p, p') = 2(2pp' - p - p'), \quad N_3(p, p') = 2(p-1)(p'-1) \quad (3.6) \]
respectively. The total number of \(\mathcal{W}\)-indecomposable representations is therefore given by
\[ N_{\text{ind}}(p, p') = 6pp' - 2(p + p') \quad (3.7) \]
In the case of \(\mathcal{LM}(1,p')\), we recover the well-known numbers
\[ N_1(1,p') = 2p', \quad N_2(1,p') = 2(p'-1), \quad N_3(1,p') = 0, \quad N_{\text{ind}}(1,p') = 4p'-2 \quad (3.8) \]
The \(\mathcal{W}\)-extended logarithmic minimal models \(\mathcal{WL\mathcal{M}(1,p')}\) are discussed in \([6, 13, 14, 15, 16, 17]\) while \(\mathcal{W}\)-extended critical percolation \(\mathcal{WL\mathcal{M}(2,3)}\) is discussed in \([20]\). In the latter case, the numbers are
\[ N_1(2,3) = 8, \quad N_2(2,3) = 14, \quad N_3(2,3) = 4, \quad N_{\text{ind}}(2,3) = 26 \quad (3.9) \]

In terms of Virasoro-indecomposable representations, the \( \mathcal{W} \)-indecomposable rank-1 representations decompose as

\[
(\kappa p, s)_{\mathcal{W}} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)((2k - 2 + \kappa)p, s)
\]

\[
(r, \kappa p')_{\mathcal{W}} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)(r, (2k - 2 + \kappa)p')
\] (3.10)

where the two expressions for \( (p, p')_{\mathcal{W}} \) agree and where the identity \( (p, 2p')_{\mathcal{W}} \equiv (2p, p')_{\mathcal{W}} \) is verified explicitly. Similarly, the \( \mathcal{W} \)-indecomposable rank-2 representations decompose as

\[
(\mathcal{R}_{\kappa p,s}^{a,0})_{\mathcal{W}} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,s}^{a,0}, \quad (\mathcal{R}_{r,\kappa p'}^{0,b})_{\mathcal{W}} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p'}^{0,b}
\] (3.11)

while the \( \mathcal{W} \)-indecomposable rank-3 representations decompose as

\[
(\mathcal{R}_{\kappa p,p'}^{a,b})_{\mathcal{W}} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,p'}^{a,b} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,p'}^{a,b}
\] (3.12)

### 3.2 \( \mathcal{W} \)-extended characters

The characters of the \( \mathcal{W} \)-indecomposable rank-1 representations read

\[
\hat{\chi}_{\kappa p,s}(q) = \sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\chi_{(2k-2+\kappa)p,s}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} (2k - 2 + \kappa)q^{(2k-2+\kappa)p'-s}\eta_{p'/4p'}
\]

\[
\hat{\chi}_{r,\kappa p'}(q) = \sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\chi_{r,(2k-2+\kappa)p'}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} (2k - 2 + \kappa)q^{(2k-2+\kappa)p'-r}\eta_{p'/4p}
\] (3.13)

where it is recalled that \( (2p, p')_{\mathcal{W}} \equiv (p, 2p')_{\mathcal{W}} \). The characters of the \( \mathcal{W} \)-indecomposable rank-2 representations read

\[
\chi[(\mathcal{R}_{\kappa p,s}^{a,0})_{\mathcal{W}}](q) = \delta_{\kappa,1}(1 - \delta_{s,p'})\chi_{p,-a,s}(q) + 2\sum_{k \in \mathbb{N}} (2k + 1 - \kappa)\chi_{(2k+2-\kappa)p-a,s}(q)
\]

\[+ 2\sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\chi_{(2k-2+\kappa)p+a,s}(q)
\]

\[
\chi[(\mathcal{R}_{r,\kappa p'}^{0,b})_{\mathcal{W}}](q) = \delta_{\kappa,1}(1 - \delta_{r,p'})\chi_{r,-b}(q) + 2\sum_{k \in \mathbb{N}} (2k + 1 - \kappa)\chi_{r,(2k+2-\kappa)p'-b}(q)
\]

\[+ 2\sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\chi_{r,(2k-2+\kappa)p'+b}(q)
\] (3.14)

that is,

\[
\chi[(\mathcal{R}_{\kappa p,b}^{a,0})_{\mathcal{W}}](q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} (2k - 2 + \kappa)q^{(ap'-bp+(2k-2+\kappa)p)p'}/4p' - q^{(ap'+bp+(2k-2+\kappa)p)p'}/4p'
\]

\[
\chi[(\mathcal{R}_{\kappa p,p'}^{a,0})_{\mathcal{W}}](q) = \frac{2}{\eta(q)} \sum_{k \in \mathbb{Z}} q^{(a+(2k-1+\kappa)p)^2p'/4p}
\]

\[
\chi[(\mathcal{R}_{a,\kappa p'}^{0,b})_{\mathcal{W}}](q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} (2k - 2 + \kappa)q^{(-ap'+bp+(2k-2+\kappa)p)p'}/4p' - q^{(ap'+bp+(2k-2+\kappa)p)p'}/4p'
\]

\[
\chi[(\mathcal{R}_{p,\kappa p'}^{0,b})_{\mathcal{W}}](q) = \frac{2}{\eta(q)} \sum_{k \in \mathbb{Z}} q^{(b+(2k-1+\kappa)p')^2p'/4p'}
\] (3.15)
We note the character identities
\[
\chi[(\mathcal{R}_{p,p'}^{a,b})_W](q) = \chi[(\mathcal{R}_{2p,2p'}^{a,b})_W](q) \quad \chi[(\mathcal{R}_{p,p'}^{0,0})_W](q) = \chi[(\mathcal{R}_{p,2p'}^{0,0})_W](q) \tag{3.16}
\]
and the character relations
\[
\chi[(\mathcal{R}_{p,p'}^{a,b})_W](q) = \text{ch}_{p-a,b}(q) + \chi[(\mathcal{R}_{2p,2p'}^{a,b})_W](q)
\chi[(\mathcal{R}_{a,p'}^{0,b})_W](q) = \text{ch}_{a,p'-b}(q) + \chi[(\mathcal{R}_{a,2p'}^{0,b})_W](q) \tag{3.17}
\]
and
\[
\chi[(\mathcal{R}_{p,p'}^{a,b})_W](q) + \chi[(\mathcal{R}_{2p,p'}^{a,b})_W](q) = \chi[(\mathcal{R}_{a,p}^{0,b})_W](q) + \chi[(\mathcal{R}_{p-a,kp}^{0,b})_W](q) \tag{3.18}
\]
The characters of the $\mathcal{W}$-indecomposable rank-3 representations read
\[
\begin{align*}
\chi[(\mathcal{R}_{kp,p'}^{a,b})_W](q) &= 2\delta_{k,1}\text{ch}_{a,b}(q) + 2\delta_{k,2}\text{ch}_{p-a,b}(q) \\
&+ 4 \sum_{k \in \mathbb{N}} (2k - 2 + \kappa)(\text{ch}(2k-2+\kappa)p+a,p'-b(q) + \text{ch}(2k-2+\kappa)p+p-a,b(q)) \\
&+ 4 \sum_{k \in \mathbb{N}} (2k + 1 - \kappa)(\text{ch}(2k+1-\kappa)p+a,b(q) + \text{ch}(2k+1-\kappa)p'+b(q)) \\
&= \frac{2}{\eta(q)} \sum_{k \in \mathbb{Z}} \left(q^{(ap'-bp+(2k+1-\kappa)pp')/4pp'} + q^{(ap'+bp+(2k+1-\kappa)pp')/4pp'}\right) \tag{3.19}
\end{align*}
\]
and satisfy
\[
\chi[(\mathcal{R}_{(3-k)p,p'}^{a,b})_W](q) = \chi[(\mathcal{R}_{kp,p'}^{a,b})_W](q) = \chi[(\mathcal{R}_{kp,p'}^{a,b})_W](q) \tag{3.20}
\]
As we will discuss below, the rank-2 and -3 representations listed in (3.3) and (3.4) all have distinct Jordan-cell and general embedding structures, despite the character identities (3.10) and (3.20).

It is pointed out that some of the character expressions above are fermionic, namely $\chi[(\mathcal{R}_{kp,p'}^{a,b})_W](q)$ and $\chi[(\mathcal{R}_{kp,p'}^{a,b})_W](q)$ in (3.13) and (3.19). Although of great interest, it is beyond the scope of the present paper to work out fermionic character expressions for the other $\mathcal{W}$-indecomposable representations.

### 3.2.1 Irreducible subfactors

It is recalled that Virasoro-irreducible characters are denoted by $\text{ch}_{\rho,s}(q)$ where $\rho, s \in \mathbb{N}$ as we reserve the notation $\chi_{\rho,s}(q)$ for the characters of the (in general reducible) Kac representations $(\rho, s)$. Only if the Kac representation happens to be Virasoro-irreducible, cf. the discussion following (2.2), do we use both notations. In the $\mathcal{W}$-extended picture, on the other hand, we will denote the character of a $\mathcal{W}$-irreducible representation of conformal weight $\Delta_{\rho,s}$ simply by $\hat{\chi}_{\rho,s}(q)$.

Here we assert that, in addition to the $2(p + p' - 1)$ $\mathcal{W}$-irreducible rank-1 representations listed in (3.2), there are $\frac{5}{2}(p-1)(p'-1)$ $\mathcal{W}$-irreducible rank-1 representations appearing as subfactors of the $\mathcal{W}$-indecomposable rank-2 and -3 representations. This brings the total number of $\mathcal{W}$-irreducible characters to
\[
N_{\text{irr}}(p,p') = N_1 + \frac{5}{2}(p-1)(p'-1) = 2pp' + \frac{1}{2}(p-1)(p'-1) \tag{3.21}
\]
$\frac{5}{2}(p-1)(p'-1)$ of these new $\mathcal{W}$-irreducible representations simply correspond to Virasoro-irreducible representations and have characters given by
\[
\hat{\chi}_{a,b}(q) = \hat{\chi}_{p-a,p'-b}(q) = \text{ch}_{a,b}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} \left(q^{(ap'-bp+2kpp')/4pp'} - q^{(ap'+bp+2kpp')/4pp'}\right) \tag{3.22}
\]
The remaining \(2(p-1)(p^\prime-1)\) new \(W\)-irreducible representations have characters

\[
\hat{\chi}_{\kappa p + a, b}(q) = \sum_{k \in \mathbb{N}} (2k - 2 + \kappa) \text{ch}_{(2k-2+\kappa)p+a,b}(q)
\]

\[
= \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} k(k - 1 + \kappa) \left( q^{(ap'-bp+(2k-2+\kappa)p')/4p'} - q^{(ap'+bp+(2k-2+\kappa)p')/4p'} \right)
\]

satisfying

\[
\hat{\chi}_{\kappa p + a, b'}(q) = \hat{\chi}_{p-a, \kappa p' + b}(q)
\]

We can now express the characters of the higher-rank \(W\)-indecomposable representations in terms of \(W\)-irreducible characters, and we find that the rank-2 and -3 characters enjoy the decompositions

\[
\chi[(\mathcal{R}_{\kappa p, s})_W](q) = \delta_{\kappa,1} (1 - \delta_{s,p'}) \hat{\chi}_{p-a, s}(q) + 2 \hat{\chi}_{(4-k)p-a, s}(q) + 2 \hat{\chi}_{\kappa p + a, b}(q)
\]

\[
\chi[(\mathcal{R}_{r, \kappa p})_W](q) = \delta_{\kappa,1} (1 - \delta_{r,p}) \hat{\chi}_{r, p'} - b(q) + 2 \hat{\chi}_{r, (4-k)p' - b}(q) + 2 \hat{\chi}_{r, \kappa p' + b}(q)
\]

and

\[
\chi[(\mathcal{R}_{r', \kappa p'})_W](q) = 2\delta_{\kappa,1} \hat{\chi}_{a, b}(q) + 2\delta_{\kappa,2} \hat{\chi}_{p-a, b}(q) + 4 \hat{\chi}_{\kappa p + a, p' - b}(q) + 4 \hat{\chi}_{\kappa p + p-a, b}(q) + 4 \hat{\chi}_{a, (3-k)p' + a, b}(q) + 4 \hat{\chi}_{a, (3-k)p' + a, b}(q)
\]

\subsection{Theta forms}

The characters of the \(N_{irr}(p, p')\) \(W\)-irreducible representations agree with those of [19]. In particular, they admit the expressions given there in terms of theta functions

\[
\theta_{\ell,k}(q, z) = \sum_{j \in \mathbb{Z} + \frac{k}{2\pi}} q^{kj^2} z^{kj}, \quad |q| < 1, \quad z \in \mathbb{C}, \quad k \in \mathbb{N}, \quad \ell \in \mathbb{Z}
\]

and so-called theta-constants

\[
\theta_{\ell,k}(q) = \theta_{\ell,k}(q, 1), \quad \theta_{\ell,k}^{(m)}(q) = \left. \left( z \frac{\partial}{\partial z} \right)^m \theta_{\ell,k}(q, z) \right|_{z=1}, \quad m \in \mathbb{N}
\]

Introducing the abbreviations

\[
\theta_{\ell}(q) = \theta_{\ell,pp'}(q), \quad \theta_{\ell,pp'}^{(1)}(q), \quad \theta_{\ell,pp'}^{(2)}(q)
\]

the theta forms are

\[
\hat{\chi}_{a, b}(q) = \frac{1}{\eta(q)} (\theta_{sp-rp'}(q) - \theta_{sp+rp'}(q))
\]

\[
\hat{\chi}_{r, s}^{+}(q) = \frac{1}{(pp')^2 \eta(q)} \left( \theta_{sp+rp'}^{''}(q) - \theta_{sp-rp'}^{''}(q) - (sp+rp') \theta_{sp+rp'}^{'}(q) + (sp-rp') \theta_{sp-rp'}^{'}(q) \right)
\]

\[
\hat{\chi}_{r, s}^{-}(q) = \frac{1}{(pp')^2 \eta(q)} \left( \theta_{sp'-sp-rp'}^{''}(q) - \theta_{pp'+sp-rp'}^{''}(q) + (sp + rp') \theta_{pp'-sp-rp'}^{'}(q) + (sp - rp') \theta_{pp'-sp-rp'}^{'}(q) - \frac{(sp - rp')^2 - (pp')^2}{4} \theta_{pp'-sp-rp'}^{''}(q) \right)
\]
where the Dedekind eta function is defined in $([2.4])$. Uniqueness of the theta forms $([3.30])$ is obtained by imposing $aq' + bp \leq pp'$. Besides the identification of the $\mathcal{W}$-irreducible characters $\hat{x}_{a,b}(q)$ in $([3.22])$ with the theta forms of the same names in $([3.30])$, the precise relations between the theta forms and our $\mathcal{W}$-irreducible characters are

$$
\hat{x}_{r,s}^+(q) = \hat{x}_{r,2p'-s}(q) = \hat{x}_{2p-r,s}(q), \quad \hat{x}_{r,s}^-(q) = \hat{x}_{r,3p'-s}(q) = \hat{x}_{3p-r,s}(q)
$$

(3.33)

### 3.3 Embedding patterns

We conjecture that every $\mathcal{W}$-indecomposable rank-2 representation has an embedding pattern of one of the types

$$
\mathcal{E}(\Delta_h, \Delta_v) : (\Delta_h)_W \rightarrow (\Delta_h)_W, \quad (\Delta_v)_W \rightarrow (\Delta_v)_W
$$

and

$$
\mathcal{E}(\Delta_h, \Delta_v; \Delta_v) : (\Delta_h)_W \rightarrow (\Delta_h)_W, \quad (\Delta_v)_W \rightarrow (\Delta_v)_W
$$

(3.34)

where the horizontal arrows indicate the non-diagonal action of the Virasoro mode $L_0$. Specifically, we conjecture that the $\mathcal{W}$-indecomposable rank-2 representations $([3.33])$ enjoy the embedding patterns

$$
(\mathcal{R}^{a,0}_{p,b})_W \sim \mathcal{E}(\Delta_{p+a,b}, \Delta_{3p-a,b}; \Delta_{p-a,b}), \quad (\mathcal{R}^{0,b}_{a,p'})_W \sim \mathcal{E}(\Delta_{a,p'+b}, \Delta_{a,3p'-b}; \Delta_{a,p'-b})
$$

$$
(\mathcal{R}^{a,0}_{p,p'})_W \sim \mathcal{E}(\Delta_{p+a,p'}, \Delta_{3p-a,p'}), \quad (\mathcal{R}^{0,b}_{p,p'})_W \sim \mathcal{E}(\Delta_{p,p'+b}, \Delta_{p,3p'-b})
$$

$$
(\mathcal{R}^{a,0}_{2p,s})_W \sim \mathcal{E}(\Delta_{2p+a,s}, \Delta_{2p-a,s}), \quad (\mathcal{R}^{0,b}_{r,2p'})_W \sim \mathcal{E}(\Delta_{r,2p'+b}, \Delta_{r,2p'-b})
$$

(3.35)

These embedding patterns demonstrate the inequivalence of the various rank-2 representations despite the character identities $([3.16])$.

We also conjecture that the $\mathcal{W}$-indecomposable rank-3 representations $([3.34])$ have embedding structures described by the patterns in $([3.33])$. Specifically, we conjecture that

$$
(\mathcal{R}^{a,b}_{\kappa p,p'})_W \sim \mathcal{E}((\mathcal{R}^{a,0}_{\kappa p,p'-b})_W, (\mathcal{R}^{a,0}_{(3-\kappa)p,b})_W) \sim \mathcal{E}((\mathcal{R}^{0,b}_{p-a,\kappa p'})_W, (\mathcal{R}^{0,b}_{a,(3-\kappa)p'})_W)
$$

(3.36)

where the $\mathcal{W}$-irreducible representations $(\Delta_h)_W$ and $(\Delta_v)_W$ have been replaced by $\mathcal{W}$-indecomposable rank-2 representations. It is noted that each of the $2(p - 1)(p' - 1)$ rank-3 representations is thus proposed to be viewable in two different ways. This corresponds to viewing it as an indecomposable ‘vertical’ combination of ‘horizontal’ rank-2 representations $(\mathcal{R}^{a,0})_W$ or as an indecomposable ‘horizontal’ combination of ‘vertical’ rank-2 representations $(\mathcal{R}^{0,b})_W$. As in the case of rank-2 representations, the conjectured embedding patterns $([3.36])$ demonstrate inequivalence of the various rank-3 representations despite the character identities $([3.20])$.

### 3.4 Jordan-cell structures and refined characters

To encode the Jordan-cell structure of a rank-$n$ representation of character $\chi(q)$, we introduce its ‘refined’ character

$$
\chi_{\text{ref}}(q) = \sum_{j=1}^n \mathcal{J}_j \chi^j_{\text{ref}}(q), \quad \chi(q) = \text{Tr} \chi_{\text{ref}}(q) = \sum_{j=1}^n j \chi^j_{\text{ref}}(q)
$$

(3.37)
where the \((j \times j)\)-dimensional canonical Jordan cell \(J_j\) is defined by

\[
J_j = \begin{pmatrix}
1 & 1 & 1 \\
& & \\
& & \\
& & \\
1 & 1 & 1
\end{pmatrix}, \quad (J_j)_{i'j'} = \delta_{i',j} + \delta_{i',i+1}
\]

and has trace \(\text{Tr}J_j = j\). Now, in the rank-\(n\) representation under consideration, the number of rank-\(j\) Jordan cells at a given level \(\ell\) is simply given by the multiplicity of \(q^\ell\) in \(\chi^2_{\text{ref}}(q)\). The mere number of Jordan cells at a given level \(\ell\) is therefore given by the multiplicity of \(q^\ell\) in \(\chi^\text{tot}_{\text{ref}}(q)\) where

\[
\chi^\text{tot}_{\text{ref}}(q) = \sum_{j=1}^{n} \chi^2_{\text{ref}}(q)
\]

For simplicity, we will omit the trivial matrix notation for \(j = 1\) and set \(J_1 = 1\).

We can also use this refined character notation when considering a decomposition of a character in terms of irreducible characters, for example. By

\[
2 \text{ch}_{r,s}(q) + J_2 (\text{ch}_{r',s'}(q) + \text{ch}_{r'',s''}(q))
\]

we thus mean a sum of 6 irreducible Virasoro characters where a Jordan cell of rank 2 is formed between every pair of matching states in the 2 modules labelled by \(r', s'\) and between every pair of matching states in the 2 modules labelled by \(r'', s''\) while no state in the modules labelled by \(r, s\) is part of a non-trivial Jordan cell. The refined characters of the \(\mathcal{W}\)-indecomposable rank-2 representations then read

\[
\chi^\text{ref}[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{s,p'})\text{ch}_{p-a,s}(q) + 2 \sum_{k \in \mathbb{N}} (2k + 1 - \kappa)\text{ch}_{(2k+2-\kappa)p-a,s}(q)
\]

\[
+ J_2 \sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\text{ch}_{(2k-2+\kappa)p+a,s}(q)
\]

\[
\chi^\text{ref}[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{r,p})\text{ch}_{r,p'-b}(q) + 2 \sum_{k \in \mathbb{N}} (2k + 1 - \kappa)\text{ch}_{r,(2k+2-\kappa)p'-b}(q)
\]

\[
+ J_2 \sum_{k \in \mathbb{N}} (2k - 2 + \kappa)\text{ch}_{r,(2k-2+\kappa)p'-b}(q)
\]

which can be re-expressed in terms of \(\mathcal{W}\)-irreducible characters as

\[
\chi^\text{ref}[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{s,p'})\hat{x}_{p-a,s}(q) + 2\hat{x}_{(4-\kappa)p-a,s}(q) + J_2 \hat{x}_{\kappa p+a,s}(q)
\]

\[
\chi^\text{ref}[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{r,p})\hat{x}_{r,p'-b}(q) + 2\hat{x}_{r,(4-\kappa)p'-b}(q) + J_2 \hat{x}_{r,\kappa p'+b}(q)
\]

It follows that the refined character components are given by

\[
\chi^\text{ref}^1[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{s,p'})\hat{x}_{p-a,s}(q) + 2\hat{x}_{(4-\kappa)p-a,s}(q)
\]

\[
\chi^\text{ref}^2[\mathcal{W}](q) = \hat{x}_{\kappa p+a,s}(q)
\]

and

\[
\chi^\text{ref}^1[\mathcal{W}](q) = \delta_{\kappa,1}(1 - \delta_{r,p})\hat{x}_{r,p'-b}(q) + 2\hat{x}_{r,(4-\kappa)p'-b}(q)
\]

\[
\chi^\text{ref}^2[\mathcal{W}](q) = \hat{x}_{r,\kappa p'+b}(q)
\]
We note that the refined character expressions contain enough information to distinguish between the different rank-2 representations. That is, the distinctions can be made by solely emphasizing the Jordan-cell structures without further reference to the complete embedding patterns. We also note that the refined character components are related to each other

\[
\chi_{\text{ref}}^1\left[(\mathcal{R}_{\kappa p,b}^{a,0})_W\right](q) = \chi_{\text{ref}}^1\left[(\mathcal{R}_{a,\kappa p'}^{0,b})_W\right](q), \quad \chi_{\text{ref}}^2\left[(\mathcal{R}_{\kappa p,b}^{a,0})_W\right](q) = \chi_{\text{ref}}^2\left[(\mathcal{R}_{p-a,\kappa p'}^{0,b})_W\right](q)
\]

from which it follows, in particular, that the refined characters themselves satisfy the relations

\[
\chi_{\text{ref}}\left[(\mathcal{R}_{\kappa p,b}^{a,0})_W\right](q) + \chi_{\text{ref}}\left[(\mathcal{R}_{p-a,\kappa p'}^{0,b})_W\right](q) = \chi_{\text{ref}}\left[(\mathcal{R}_{a,\kappa p'}^{0,b})_W\right](q) + \chi_{\text{ref}}\left[(\mathcal{R}_{\kappa p,b}^{0,p'})_W\right](q)
\]

and hence

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p'-1} \chi_{\text{ref}}\left[\left(\mathcal{R}_{\kappa p,b}^{a,0}\right)_W\right](q) = \sum_{a=1}^{p-1} \sum_{b=1}^{p'-1} \chi_{\text{ref}}\left[\left(\mathcal{R}_{a,\kappa p'}^{0,b}\right)_W\right](q)
\]

Refinements of the rank-3 characters similar to the refined characters (3.42) follow from the conjectured embedding patterns (3.36). Converting the two rank-2 Jordan cells linked by a horizontal arrow in (3.36) into a rank-3 and a rank-1 Jordan cell [32, 30, 20], we arrive at the refined characters

\[
\chi_{\text{ref}}\left[\left(\mathcal{R}_{\kappa p,b}^{a,b}\right)_W\right](q) = \delta_{\kappa,1}^{1} \mathcal{J}_{2}^{a,b}(q) + 2\delta_{\kappa,2}^{1} \hat{x}_{p-a,b}(q) + \{\mathcal{J}_{3}^{1} + 1\} \hat{x}_{\kappa p+a,p'-b}(q) + 4\hat{x}_{\kappa p+p-a,b}(q) + 2\mathcal{J}_{2}^{1} \hat{x}_{(3-\kappa)p+a,b}(q) + 2\mathcal{J}_{2}^{1} \hat{x}_{a,(3-\kappa)p'+b}(q)
\]

\[
= \sum_{j=1}^{3} \mathcal{J}_{j}^{1} \chi_{\text{ref}}^{j}\left[\left(\mathcal{R}_{\kappa p,b}^{a,b}\right)_W\right](q)
\]

(3.48)

where the refined character components are given by

\[
\chi_{\text{ref}}^{1}\left[\left(\mathcal{R}_{\kappa p,b}^{a,b}\right)_W\right](q) = 2\delta_{\kappa,2}^{1} \hat{x}_{p-a,b}(q) + \hat{x}_{\kappa p+a,p'-b}(q) + 4\hat{x}_{\kappa p+p-a,b}(q)
\]

\[
\chi_{\text{ref}}^{2}\left[\left(\mathcal{R}_{\kappa p,b}^{a,b}\right)_W\right](q) = \delta_{\kappa,1}^{1} \hat{x}_{a,b}(q) + 2\hat{x}_{(3-\kappa)p+a,b}(q) + 2\hat{x}_{a,(3-\kappa)p'+b}(q)
\]

\[
\chi_{\text{ref}}^{3}\left[\left(\mathcal{R}_{\kappa p,b}^{a,b}\right)_W\right](q) = \hat{x}_{\kappa p+a,p'-b}(q)
\]

(3.49)

In order to demonstrate inequivalence of the $\mathcal{W}$-indecomposable rank-3 representations, it suffices to focus on the presence of rank-3 Jordan cells. This follows from the fact that the $\mathcal{W}$-irreducible subfactors with characters $\hat{x}_{\kappa p+a,p'-b}(q)$ are distinct for every distinct choice of $\kappa, a, b$ (of which there are $2(p-1)(p'-1)$ possibilities).

Due to the various character relations satisfied by the $\mathcal{W}$-indecomposable rank-2 representations, it follows from the embedding patterns (3.36) of the $\mathcal{W}$-indecomposable rank-3 representations that there are many relations for the (refined) rank-3 characters as well. As they do not seem to shed new light on the structure of the rank-3 representations, they will not be discussed any further here.

### 3.5 $\mathcal{W}$-extended fusion algebra

We denote the fusion product in the $\mathcal{W}$-extended picture by $\otimes$ and reserve the symbol $\circ$ for the fusion product in the Virasoro picture. The fusion rules underlying the fusion algebra in the $\mathcal{W}$-extended picture $\mathcal{WLM}(p,p')$

\[
\langle (\kappa p, s)_W, (r, \kappa' p')_W, (\mathcal{R}_{\kappa p,s}^{a,0})_W, (\mathcal{R}_{r,\kappa' p'}^{0,b})_W, (\mathcal{R}_{\kappa p,\kappa' p'}^{a,b})_W \rangle_{p,p'}
\]

(3.50)
are summarized in the following. Here it is recalled that the various $W$-indecomposable representations are subject to (3.45). The fusion of two $W$-indecomposable rank-1 representations is given by

\[
(k,p,s)_W \otimes (\kappa',s')_W = \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \} + \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \}
\]

(3.51)

The fusion of a $W$-indecomposable rank-1 representation with a $W$-indecomposable rank-2 representation is given by

\[
(k,p,s)_W \otimes (R_{r,\kappa'}^{a,b})_W = \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \} + \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \}
\]

(3.52)

The fusion of a $W$-indecomposable rank-2 representation with a $W$-indecomposable rank-3 representation is given by

\[
(k,p,s)_W \otimes (R_{r,\kappa'}^{a,b})_W = \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \} + \bigoplus_{\alpha} \bigoplus_{j=|s-s'|+1, \text{by 2}} \{ 2(R_{(\kappa,\kappa')}_{p,j})_W \oplus 2(R_{(2,\kappa')}_{(2,\kappa')}_{p,j})_W \}
\]

(3.53)
The fusion of two $W$-indecomposable rank-2 representations is given by

\[
\begin{align*}
(\mathcal{R}_{κ\mu,κ'}_{κp,κ'}^{α,β})_{W} ⊗ (\mathcal{R}_{κ\mu,κ'}_{κp,κ'}^{a',b'})_{W} &= \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \bigoplus \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \\
&= p'−|p′−s−s'|−1 \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \bigoplus \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \\
&= p−|p′−s−s'|−1 \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \bigoplus \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \\
&= s+s′−p′−1 \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \bigoplus \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W} \right\} \oplus \bigoplus_{\alpha} 2(\mathcal{R}_{κ\mu,κ'}^{α,β})_{W}
\end{align*}
\]
The fusion of a $W$-indecomposable rank-2 representation with a $W$-indecomposable rank-3 representation is given by

\[
\begin{align*}
(R_{\kappa,p,s}^{a,0})_W \otimes (R_{p',\kappa',p'}^{a',b'})_W &= p' - |p' - s - b'| - 1 \quad p - |a - a'| - 1 \quad |p - a - a'| - 1 \\
&\text{\quad} + \begin{cases}
\beta = |b' - s| + 1, \text{ by } 2 \quad s - b' - 1 \\
\beta = |b' - s| - 1, \text{ by } 2 \quad s + b'-1 \\
\end{cases}
\begin{align*}
&\begin{align*}
&\begin{cases}
\alpha,\beta \\
\end{cases} \\
&2(R_{\kappa,p',\kappa'}^{a,\beta})_W + 2(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\alpha,\beta \\
&4(R_{\kappa,p',\kappa'}^{a,\beta})_W + 4(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\alpha,\beta \\
&4(R_{\kappa,p',\kappa'}^{a,\beta})_W + 4(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\end{align*}
\end{align*}
\end{align*}
\]

\[
\begin{align*}
(R_{r',\kappa,p'}^{0,b})_W \otimes (R_{p',\kappa',p'}^{a',b'})_W &= p - |p - r - a'| - 1 \quad p' - |b - b'| - 1 \quad |p' - b - b'| - 1 \\
&\text{\quad} + \begin{cases}
\alpha,\beta \\
\end{cases} \\
&2(R_{\kappa,p',\kappa'}^{a,\beta})_W + 2(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\alpha,\beta \\
&4(R_{\kappa,p',\kappa'}^{a,\beta})_W + 4(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\alpha,\beta \\
&4(R_{\kappa,p',\kappa'}^{a,\beta})_W + 4(R_{\kappa,p',\kappa'}^{a',\beta})_W \\
&\end{align*}
\]

(3.55)
Finally, the fusion of two $\mathcal{W}$-indecomposable rank-3 representations is given by

$$
(R_{a,b}^{\alpha,b})_{\mathcal{W}} \otimes (R_{p,kp'}^{\alpha',b'})_{\mathcal{W}} = \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 4(R_{kp,kp'}^{\alpha\beta})_{\mathcal{W}} \right\} \bigoplus_{\alpha} \left\{ \bigoplus_{\beta} 4(R_{kp,kp'}^{\alpha\beta})_{\mathcal{W}} \right\}
$$

This fusion algebra is both associative and commutative, while there is no identity for $p > 1$. For $p = 1$, the $\mathcal{W}$-irreducible representation $(1,1)_{\mathcal{W}}$ is the identity.

### 3.6 $\mathcal{W}$-projective representations and their fusion algebra

Here it suffices to characterize a $\mathcal{W}$-projective representation as a $\mathcal{W}$-indecomposable representation which does not appear as a subfactor of any $\mathcal{W}$-indecomposable representation different from itself. It follows that there are 2 $\mathcal{W}$-projective representations of rank 1

$$
(kp,p')_{\mathcal{W}} \equiv (p,kp')_{\mathcal{W}} \quad (3.57)
$$

2$p + 2p' - 4$ $\mathcal{W}$-projective representations of rank 2

$$
(R_{a,0}^{\alpha,0})_{\mathcal{W}}, \quad (R_{p,kp}^{0,b})_{\mathcal{W}} \quad (3.58)
$$

and 2$(p - 1)(p' - 1)$ $\mathcal{W}$-projective representations of rank 3

$$
(R_{a,b}^{a,b})_{\mathcal{W}} \equiv (R_{p,p'}^{a,b})_{\mathcal{W}} \quad (3.59)
$$

This gives a total of $N_{\text{proj}}(p,p')$ $\mathcal{W}$-projective representations where

$$
N_{\text{proj}}(p,p') = 2pp' \quad (3.60)
$$

It also follows that every $\mathcal{W}$-indecomposable rank-3 representation is a $\mathcal{W}$-projective representation, while the 2 $\mathcal{W}$-indecomposable rank-1 representations $(3.57)$ are both $\mathcal{W}$-irreducible and $\mathcal{W}$-projective.
We will refer to a character of a \( \mathcal{W} \)-projective representation as a \( \mathcal{W} \)-projective character. Due to the character identities (3.16), the number of linearly independent \( \mathcal{W} \)-projective characters is smaller than \( N_{\text{proj}}(p, p') \) and is given by

\[
\frac{1}{2}(p + 1)(p' + 1)
\]

(3.61)

This agrees with the counting of \( \mathcal{W} \)-projective characters in [19]. It is noted that the fermionic character expressions appearing in (3.15) and (3.19) exactly correspond to the \( \mathcal{W} \)-projective representations of rank 2 in (3.58) and rank 3 in (3.59), respectively.

We find that the \( \mathcal{W} \)-indecomposable rank-1 representations

\[(\kappa p, b)_{\mathcal{W}}, \quad (a, \kappa p')_{\mathcal{W}}\]

(3.62)

appear as subfactors of the \( \mathcal{W} \)-projective rank-2 representations (3.58), while the \( \mathcal{W} \)-indecomposable rank-2 representations

\[(\mathcal{R}_{\kappa p, b})_{\mathcal{W}}, \quad (\mathcal{R}_{a, \kappa p'})_{\mathcal{W}}\]

(3.63)

appear as subfactors of the \( \mathcal{W} \)-projective rank-3 representations (3.59). This exhausts the set of \( \mathcal{W} \)-indecomposable representations appearing in the fusion algebra (3.50). The additional \( \mathcal{W} \)-irreducible representations introduced in Section 3.2.1

\[(a, b)_{\mathcal{W}}, \quad (2p - a, b)_{\mathcal{W}} \equiv (a, 2p' - b)_{\mathcal{W}}, \quad (3p - a, b)_{\mathcal{W}} \equiv (a, 3p' - b)_{\mathcal{W}}\]

(3.64)

all appear as subfactors of the \( \mathcal{W} \)-projective rank-3 representations (3.59).

As a simple inspection of the fusion rules in Section 3.5 reveals, the \( 2pp' \) \( \mathcal{W} \)-projective representations generate a closed fusion subalgebra of (3.50), naturally denoted by

\[\langle (\kappa p, p'), (\mathcal{R}_{\kappa p, p'}^{a, b}), (\mathcal{R}_{a, \kappa p'}^{a, b}), (\mathcal{R}_{\kappa p, p'}^{a, b}) \rangle_{p, p'}\]

(3.65)

We will comment on this fusion subalgebra in Section 5.

4 Lattice realization of \( \mathcal{WLM}(p, p') \)

In [17], we used the infinite series of logarithmic minimal lattice models \( \mathcal{LM}(1, p') \) to obtain \( \mathcal{W} \)-extended fusion rules applicable in the extended pictures \( \mathcal{WLM}(1, p') \). A crucial ingredient was the construction of a \( \mathcal{W} \)-invariant identity representation \( (1, 1)_{\mathcal{W}} \) defined as the infinite limit of a triple fusion of Virasoro-irreducible Kac representations in \( \mathcal{LM}(1, p') \). On the other hand, as indicated above and further discussed in Section 5 there is no obvious natural candidate for an identity in the lattice realization of \( \mathcal{WLM}(p, p') \) for \( p > 1 \). As in the case of the \( \mathcal{W} \)-extended picture of critical percolation \( \mathcal{WLM}(2, 3) \) [20], it nevertheless turns out fruitful to adopt the use of infinite limits of triple fusions of Virasoro-irreducible Kac representations. This also allows us to identify the various \( \mathcal{W} \)-representations with suitable limits of Yang-Baxter integrable boundary conditions on the lattice. Firmly based on the lattice-realization of the fundamental fusion algebra of \( \mathcal{LM}(p, p') \), our fusion prescription for \( \mathcal{WLM}(p, p') \) yields a commutative and associative fusion algebra. The analysis of this fusion algebra is greatly simplified by separating into ‘horizontal and vertical components’, much akin to the situation for the Virasoro picture of \( \mathcal{LM}(p, p') \) in [24, 30]. With \( p \) and \( p' \) unspecified, the two components have equivalent properties whose details only depend on the parities of \( p \) and \( p' \). We may therefore focus on the horizontal component and consider it for both possible parities knowing that the results can be translated straightforwardly to the vertical component. Once the two components are understood, we will describe how to merge them in order to construct the complete fusion algebra. Since the two characterizing parameters \( p \) and \( p' \) are coprime, we have two distinctively different situations, namely \( p \) and \( p' \) both odd or \( p \) and \( p' \) of different parities.
4.1 Horizontal component

Working in the fundamental fusion algebra of the logarithmic minimal model \( \mathcal{LM}(p, p') \), as opposed to the lesser-understood but larger full fusion algebra [29, 30], the only horizontal Kac representations at our disposal are \( \{(a, 1); a \in \mathbb{Z}_{1,p-1}\} \) and \( \{(kp, 1); k \in \mathbb{N}\} \). It is noted that the second and infinite set consists of Virasoro-irreducible representations only. There are many possible triple fusions to consider. We find it useful to introduce \( (E, 1)_W \) as the limit
\[
(E, 1)_W := \lim_{n \to \infty} (2np, 1) \otimes 3
\]
and \( (O, 1)_W \) as
\[
(O, 1)_W := \frac{1}{2p} (2p, 1) \otimes (E, 1)_W
\]
Here the notation \( E \) refers to “even” while \( O \) refers to “embedding” in (3.34). This dual role should not cause confusion. For \( p \) odd, the expression (4.2) coincides with the alternative limit
\[
(O, 1)_W = \lim_{n \to \infty} ((2n - 1)p, 1) \otimes 3
\]
whereas for \( p \) even, the two limits themselves coincide. These limits of Virasoro fusions decompose in terms of Virasoro-indecomposable representations. However, the technical part of the further analysis depends on the parity of \( p \) so we initially consider the two parities separately. This distinction is not conceptually relevant, and as we will see, the main results indeed have a very simple dependence on the parity.

4.1.1 \( p \) even

In this subsection 4.1.1, we let \( p \) be even and find
\[
(E, 1)_W = \bigoplus_{\alpha} (p - \alpha) \left\{ \bigoplus_{k \in \mathbb{N}} k \mathcal{R}_{kp,1}^{\alpha,0} \right\}
\]
\[
= \bigoplus_{\alpha} (p - \alpha) \left\{ \bigoplus_{k \in \mathbb{N}} (2k - 1) \mathcal{R}_{(2k-1)p,1}^{\alpha,0} \oplus \bigoplus_{k \in \mathbb{N}} 2k \mathcal{R}_{2kp,1}^{\alpha,0} \right\}
\]
where we recall the convenient notation (2.11), and
\[
(O, 1)_W = \bigoplus_{\alpha} (p - \alpha) \left\{ \bigoplus_{k \in \mathbb{N}} k \mathcal{R}_{kp,1}^{\alpha,0} \right\}
\]
\[
= \bigoplus_{\alpha} (p - \alpha) \left\{ \bigoplus_{k \in \mathbb{N}} (2k - 1) \mathcal{R}_{(2k-1)p,1}^{\alpha,0} \oplus \bigoplus_{k \in \mathbb{N}} 2k \mathcal{R}_{2kp,1}^{\alpha,0} \right\}
\]
Since \( p \) is even and therefore greater than 1, both of these decompositions in terms of Virasoro-indecomposable representations are non-trivial. Our next task is to disentangle these results and write them in terms of the \( W \)-indecomposable representations [3.2] and [3.3]. Following [20], we observe that the part of \( (E, 1)_W \) with \( \alpha = 0 \) corresponds to a direct sum of Virasoro-irreducible representations not taking part in any indecomposable combination. By selection of link states in the lattice description,

\[1\]Strictly speaking, the first and finite set for \( p = 2 \) is not generated by repeated fusions of the fundamental Kac representation (2, 1) even though it is present in the fundamental fusion algebra generated by repeated fusions of both of the fundamental representations (2, 1) and (1, 2).
we can thus project onto the set \( \{ \mathcal{R}_{0,0}^{(2k-1)p,1} \equiv ((2k-1)p,1); \ k \in \mathbb{N} \} \) or \( \{ \mathcal{R}_{2kp,1}^{0,0} \equiv (2kp,1); \ k \in \mathbb{N} \} \) separately, thereby allowing us to single out the two infinite direct sums
\[
(\kappa p, 1)_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)((2k - 2 + \kappa)p, 1) \tag{4.6}
\]
where \( \kappa \in \mathbb{Z}_{1,2} \). Asserting that these expressions indeed correspond to \( \mathcal{W} \)-indecomposable representations, we have thus identified the latter with limits of Yang-Baxter integrable boundary conditions on the lattice accompanied by specific selections of link states. Since the participating Virasoro representations all are of rank 1, the \( \mathcal{W} \)-indecomposable representation \((\kappa p, 1)_W\) itself is of rank 1.

Having identified \((\kappa p, 1)_W\) for each \( \kappa \in \mathbb{Z}_{1,2} \), we now define the \( \mathcal{W} \)-indecomposable rank-2 representation
\[
(\mathcal{R}_{\kappa p,1}^{1,0})_W := (2, 1) \otimes (\kappa p, 1)_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,1}^{1,0} \tag{4.7}
\]
This would complete the disentanglement of (4.1) for \( p = 2 \). For \( p > 2 \), we continue by decomposing the more general fusion
\[
(r, 1) \otimes (\kappa p, 1)_W = \bigoplus_{r-1} \bigg\{ \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,1}^{\alpha,0} \bigg\} \tag{4.8}
\]
where we recall the direct-sum convention (1.4). For \( r = 3 \), the sum over \( \alpha \) involves two terms of which the one for \( \alpha = 0 \) is recognized as \((\kappa p, 1)_W\). The other term is subsequently identified with
\[
(\mathcal{R}_{\kappa p,1}^{2,0})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,1}^{2,0} \tag{4.9}
\]
For \( r = 4 \), the sum over \( \alpha \) also involves two terms of which the one for \( \alpha = 1 \) is recognized as \((\mathcal{R}_{\kappa p,1}^{1,0})_W\). The other term is subsequently identified with
\[
(\mathcal{R}_{\kappa p,1}^{3,0})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,1}^{3,0} \tag{4.10}
\]
For \( r = 5 \), the sum over \( \alpha \) involves three terms of which the ones for \( \alpha = 0 \) and \( \alpha = 2 \) are recognized as \((\kappa p, 1)_W\) and \((\mathcal{R}_{\kappa p,1}^{2,0})_W\), respectively. We can subsequently identify the remaining term with \((\mathcal{R}_{\kappa p,1}^{4,0})_W\). It is now clear how a bootstrapping procedure, as \( r \) increases to its greatest possible value \( p \), allows us to identify
\[
(\mathcal{R}_{\kappa p,1}^{a,0})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}_{(2k-2+\kappa)p,1}^{a,0} \tag{4.11}
\]
for all \( a \in \mathbb{Z}_{1,p-1} \). We assert that these representations are \( \mathcal{W} \)-indecomposable and note that they all are of rank 2. In conclusion, we have found that the representations \((\mathcal{E}, 1)_W\) and \((\mathcal{O}, 1)_W\) are \( \mathcal{W} \)-indecomposable as they can be written as the following direct sums of \( \mathcal{W} \)-indecomposable representations
\[
(\mathcal{E}, 1)_W = \bigoplus_{\alpha} (p - \alpha)\left\{ (\mathcal{R}_{p,1}^{\alpha,0})_W \oplus (\mathcal{R}_{2p,1}^{\alpha,0})_W \right\} \tag{4.12}
\]

Of interest in their own right, but also of importance for the evaluation of fusion products below, we find that the \( \mathcal{W} \)-indecomposable representations (4.6) and (4.11) have the ‘stability properties’
\[
((2n - 2 + \kappa)p, 1) \otimes (\kappa'p, 1)_W = (2n - 2 + \kappa)\bigg\{ \bigoplus_{\alpha} (\mathcal{R}_{(2n-2+\kappa)p,1}^{\alpha,0})_W \bigg\} \tag{4.13}
\]
\[
((2n - 2 + \kappa)p, 1) \otimes (\mathcal{R}_{\kappa'p,1}^{a,0})_W = 2(2n - 2 + \kappa)\bigg\{ \bigoplus_{\alpha} (\mathcal{R}_{(2n-2+\kappa)p,1}^{\alpha,0})_W \bigg\} \tag{4.13}
\]
and

\[(r, 1) \otimes (\kappa p, 1)_W = \bigoplus_{\alpha} (R^\alpha_{\kappa p, 1})_W \]

\[(r, 1) \otimes (R^0_{\kappa p, 1})_W = \bigoplus_{\alpha=a-\alpha+1, by 2} (R^\alpha_{\kappa p, 1})_W \bigoplus 2(R^0_{\kappa p, 1})_W \bigoplus 2(R^0_{\kappa p, 1})_W \quad (4.14)\]

We note that the two expressions for \((p, 1) \otimes (\kappa p, 1)_W\) and likewise for \((p, 1) \otimes (R^0_{\kappa p, 1})_W\) appearing in (4.13) and (4.14), respectively, agree. It also follows that the \(W\)-representations \((E, 1)_W\) and \((O, 1)_W\) are related by the remarkably simple stability properties

\[(2np, 1) \otimes (E, 1)_W = 2np(O, 1)_W, \quad (2np, 1) \otimes (O, 1)_W = 2np(E, 1)_W \quad (4.15)\]

and

\[(r, 1) \otimes (E, 1)_W = \begin{cases} r(O, 1)_W, & r \text{ even} \\ r(E, 1)_W, & r \text{ odd} \end{cases} \quad (r, 1) \otimes (O, 1)_W = \begin{cases} r(E, 1)_W, & r \text{ even} \\ r(O, 1)_W, & r \text{ odd} \end{cases} \quad (4.16)\]

As we will see in the following, there are many more such properties, but this list suffices for now.

From the lattice, we define the \(W\)-extended fusion product \(\hat{\otimes}\) by

\[(E, 1)_W \hat{\otimes} (A)_W := \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 (2np, 1)_{\otimes^3} \otimes (A)_W \quad (4.17)\]

First, we consider the two cases \((A)_W = (\kappa p, 1)_W\) where \(\kappa \in \mathbb{Z}_{1,2}\) and find

\[(E, 1)_W \hat{\otimes} (\kappa p, 1)_W = \bigoplus_{\alpha} (p - \alpha) \left\{ (R^\alpha_{p, 1})_W \otimes (R^0_{2p, 1})_W \right\} \otimes (\kappa p, 1)_W \]

\[= \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 (2np, 1)_{\otimes^3} \otimes (\kappa p, 1)_W \]

\[= \lim_{n \to \infty} \left( \frac{1}{2n} \right)^2 (2np, 1)_{\otimes^2} \otimes \bigoplus_{\alpha} (R^\alpha_{(2\kappa p, 1)})_W \]

\[= \lim_{n \to \infty} \left( \frac{1}{2n} \right) (2np, 1) \otimes \bigoplus_{\alpha} (p - \alpha) \left\{ (R^\alpha_{p, 1})_W \otimes (R^0_{2p, 1})_W \right\} \]

\[= \bigoplus_{\alpha} p(p - \alpha) \left\{ (R^\alpha_{p, 1})_W \otimes (R^0_{2p, 1})_W \right\} \]

\[= p(O, 1)_W \quad (4.18)\]

which is seen to be independent of \(\kappa\). We are still faced with the task of disentangling these results since the identification of the individual fusions such as \((p, 1)_W \hat{\otimes} (p, 1)_W\) is ambiguous at this point. To this end, we use (2.15) to deduce that the decomposition of the fusion \((\kappa p, 1)_W \otimes (\kappa' p, 1)_W\) only involves representations of the form \((R^\alpha_{(\kappa, \kappa')p, 1})_W\) with \(\alpha\) odd and that these \(W\)-indecomposable rank-2 representations only appear there in multiples of the combination \(\bigoplus_{\alpha} (R^\alpha_{(\kappa, \kappa')p, 1})_W\). It also follows from (2.15) that, in the fusion \((E, 1)_W \hat{\otimes} (p, 1)_W\), the \(W\)-indecomposable rank-2 representation \((R^p_{p, 1})_W\) is only produced by \((p, 1)_W \hat{\otimes} (p, 1)_W\). Since \((p, 1)_W\) appears with multiplicity \(p\) in the decomposition of \((E, 1)_W\), and \((R^p_{p, 1})_W\) appears with multiplicity \(p\) in the fusion \((E, 1)_W \hat{\otimes} (p, 1)_W\), it follows that \((R^p_{p, 1})_W\) is produced with multiplicity 1 in the fusion \((p, 1)_W \hat{\otimes} (p, 1)_W\).

We thus conclude that

\[(p, 1)_W \hat{\otimes} (p, 1)_W = \bigoplus_{\alpha} (R^\alpha_{p, 1})_W \quad (4.19)\]
We likewise find that

$$(p, 1)_W \otimes (2p, 1)_W = \bigoplus_{\alpha} (R^{\alpha,0}_{2p,1})_W, \quad (2p, 1)_W \otimes (2p, 1)_W = \bigoplus_{\alpha} (R^{\alpha,0}_{p,1})_W$$

(4.20)

In order to evaluate fusions involving the $W$-indecomposable rank-2 representations $(R^{\alpha,0}_{\kappa p,1})_W$, we note that (4.14) implies

$$(a + 1, 1) \otimes (\kappa p, 1)_W = (R^{\alpha,0}_{\kappa p,1})_W \oplus ((a - 1, 1) \otimes (\kappa p, 1)_W)$$

(4.21)

It follows that

$$(R^{\alpha,0}_{\kappa p,1})_W \otimes (A)_W = (a + 1, 1) \otimes ((\kappa p, 1)_W \otimes (A)_W) \oplus (a - 1, 1) \otimes ((\kappa p, 1)_W \otimes (A)_W)$$

(4.22)

where $\otimes$ denotes direct subtraction) which for $(A)_W = (\kappa' p, 1)_W$ yields

$$(R^{\alpha,0}_{\kappa p,1})_W \otimes (\kappa' p, 1)_W = \bigoplus_{\alpha} 2(R^{\alpha,0}_{(\kappa - \kappa')p,1})_W \oplus \bigoplus_{\alpha} 2(R^{\alpha,0}_{(2 - \kappa - \kappa')p,1})_W$$

(4.23)

By re-cycling these results with $(A)_W = (R^{\alpha',0}_{\kappa' p,1})_W$, we finally obtain

$$(R^{\alpha,0}_{\kappa p,1})_W \otimes (R^{\alpha',0}_{\kappa' p,1})_W = \bigoplus_{\alpha} 2(R^{\alpha,0}_{(\kappa - \kappa')p,1})_W \oplus \bigoplus_{\alpha} 2(R^{\alpha,0}_{(2 - \kappa - \kappa')p,1})_W$$

(4.24)

### 4.1.2 $p$ odd

In this subsection 4.1.2, we let $p$ be odd and find

$$(\mathcal{E}, 1)_W = \bigoplus_{\alpha} (p - \alpha)\left\{ \bigoplus_{k \in \mathbb{N}} 2k R^{\alpha,0}_{2kp,1} \right\} \bigoplus \bigoplus_{\alpha} (p - \alpha)\left\{ \bigoplus_{k \in \mathbb{N}} (2k - 1) R^{\alpha,0}_{(2k - 1)p,1} \right\}$$

(4.25)

and

$$(\mathcal{O}, 1)_W = \bigoplus_{\alpha} (p - \alpha)\left\{ \bigoplus_{k \in \mathbb{N}} (2k - 1) R^{\alpha,0}_{(2k - 1)p,1} \right\} \bigoplus \bigoplus_{\alpha} (p - \alpha)\left\{ \bigoplus_{k \in \mathbb{N}} 2k R^{\alpha,0}_{2kp,1} \right\}$$

(4.26)

Decompositions of these results in terms of $W$-indecomposable representations are obtained by mimicking the disentangling procedure employed above for $p$ even. That is, by appropriately selecting the link states in the lattice description, we first isolate the terms corresponding to $\alpha = 0$ in (4.25) and (4.26)

$$(2p, 1)_W = \bigoplus_{k \in \mathbb{N}} 2k(2kp, 1), \quad (p, 1)_W = \bigoplus_{k \in \mathbb{N}} (2k - 1)(2k - 1)p, 1$$

(4.27)

Having identified these $W$-indecomposable rank-1 representations, we apply the bootstrapping procedure where the $W$-indecomposable rank-2 representations

$$(R^{\alpha,0}_{\kappa p,1})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa) R^{\alpha,0}_{(2k - 2 + \kappa)p,1}$$

(4.28)
are identified one by one as \( a \) increases from 1 to \( p - 1 \) in
\[
(a + 1, 1) \otimes (\kappa p, 1)_W = \bigoplus_{\alpha} \left\{ \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)\mathcal{R}^{\alpha,0}_{(2k-2+\kappa)p,1} \right\} \tag{4.29}
\]

Asserting that the representations \((4.27)\) and \((4.28)\) indeed are \( W \)-indecomposable, we see that the decompositions of \((\mathcal{E}, 1)_W\) and \((\mathcal{O}, 1)_W\) in terms of \( W \)-indecomposable representations read
\[
(\mathcal{E}, 1)_W = \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{p,1})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{2p,1})_W
\]
\[
(\mathcal{O}, 1)_W = \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{p,1})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{2p,1})_W \tag{4.30}
\]

We note that the horizontal \( W \)-indecomposable representations have stability properties which our notation \( (4.4) \) allows us to write in the exact same way as for \( p \) even, namely \((4.13)\) and \((4.14)\). Likewise, the representations \((\mathcal{E}, 1)_W\) and \((\mathcal{O}, 1)_W\) have the same simple stability properties \((4.15)\) and \((4.16)\) as for \( p \) even. Fusion is naturally defined as for \( p \) even \((4.17)\). Thus following the derivation of the fusion rules for \( p \) even, but now based on the stability properties just listed, we obtain the fusion rules for \( p \) odd. They are given in the summary below.

### 4.1.3 General \( p \)

In summary, valid for both parities of \( p \), we have determined the horizontal fusion rules
\[
(\kappa p, 1)_W \hat{\otimes} (\kappa' p, 1)_W = \bigoplus_{\alpha} (\mathcal{R}^{\alpha,0}_{(\kappa \kappa')p,1})_W
\]
\[
(\kappa p, 1)_W \hat{\otimes} (\mathcal{R}^{\alpha,0}_{\kappa p,1})_W = \bigoplus_{\alpha} 2(\mathcal{R}^{\alpha,0}_{(\kappa \kappa')p,1})_W \oplus \bigoplus_{\alpha} 2(\mathcal{R}^{\alpha,0}_{(2 \kappa \kappa')p,1})_W
\]
\[
(\mathcal{R}^{\alpha,0}_{\kappa p,1})_W \hat{\otimes} (\mathcal{R}^{\alpha',0}_{\kappa' p,1})_W = \bigoplus_{\alpha} 2(\mathcal{R}^{\alpha,0}_{(\kappa \kappa')p,1})_W \oplus \bigoplus_{\alpha} 2(\mathcal{R}^{\alpha,0}_{(2 \kappa \kappa')p,1})_W \oplus \bigoplus_{\alpha} 2(\mathcal{R}^{\alpha,0}_{(2 \kappa \kappa')p,1})_W \tag{4.31}
\]
governing the \(2p\)-dimensional closed fusion (sub)algebra
\[
\langle (\kappa p, 1)_W, (\mathcal{R}^{\alpha,0}_{\kappa p,1})_W \rangle_{p,p'} \tag{4.32}
\]

We note that the two representations \((\mathcal{E}, 1)_W\) and \((\mathcal{O}, 1)_W\) form a two-dimensional subalgebra of this fusion algebra
\[
(\mathcal{E}, 1)_W \hat{\otimes} (\mathcal{E}, 1)_W = (\mathcal{O}, 1)_W \hat{\otimes} (\mathcal{O}, 1)_W = p^3(\mathcal{O}, 1)_W, \quad (\mathcal{E}, 1)_W \hat{\otimes} (\mathcal{O}, 1)_W = p^3(\mathcal{E}, 1)_W \tag{4.33}
\]

### 4.2 Vertical component

The vertical component is obtained from the horizontal component simply by interchanging the first and second indices and replacing \( p \) by \( p' \) as in
\[
\{ (\mathcal{R}^{\alpha,0}_{\kappa p,1})_W; \, \alpha \in \mathbb{Z}_{0,p-1} \} \quad \longrightarrow \quad \{ (\mathcal{R}^{\alpha,0}_{\kappa p',1})_W; \, \beta \in \mathbb{Z}_{0,p'-1} \} \tag{4.34}
\]
where the vertical $W$-representations decompose in terms of Virasoro-indecomposable representations as

\[
(1, \kappa p')_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)(1, (2k - 2 + \kappa)p')
\]

\[
(R^{0,b}_{1,\kappa p'})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)R^{0,b}_{1,(2k-2+\kappa)p'} \tag{4.35}
\]

The $2p'$-dimensional vertical fusion (sub)algebra

\[
\langle (1, \kappa p')_W, (R^{0,b}_{1,\kappa p'})_W \rangle_{p,p'} \tag{4.36}
\]

is thus governed by the fusion rules

\[
(1, \kappa p')_W \otimes (1, \kappa' p')_W = \bigoplus_{\beta}^{p'-1} (1, (\kappa - \kappa')p')_W
\]

\[
(1, \kappa p')_W \otimes (R^{0,b}_{1,\kappa' p'})_W = \bigoplus_{\beta}^{p'-b'-1} 2(R^{0,b}_{1,\kappa p'})_W \oplus 2(R^{0,b}_{1,2(\kappa - \kappa')p'})_W \tag{4.37}
\]

\[
(R^{0,b}_{1,\kappa p'})_W \otimes (R^{0,b}_{1,\kappa' p'})_W = \bigoplus_{\beta}^{p'-|b-b'|-1} 2(R^{0,b}_{1,\kappa p'})_W \pm \bigoplus_{\beta}^{|b-b'|-1} 2(R^{0,b}_{1,2(\kappa - \kappa')p'})_W
\]

Of course, these results can be obtained 'directly' from the lattice by introducing the vertical representation $(1, \mathcal{E})_W$ as the limit

\[
(1, \mathcal{E})_W := \lim_{n \to \infty} (1, 2np') \otimes^3 \tag{4.38}
\]

and its companion $(1, \mathcal{O})_W$ by

\[
(1, \mathcal{O})_W := \frac{1}{2p'}(1, 2p') \otimes (1, \mathcal{E})_W \tag{4.39}
\]

They have the stability properties

\[
(1, 2np') \otimes (1, \mathcal{E})_W = 2np'(1, \mathcal{O})_W, \quad (1, 2np') \otimes (1, \mathcal{O})_W = 2np'(1, \mathcal{E})_W \tag{4.40}
\]

and

\[
(1, \mathcal{E})_W \otimes (1, s) = \begin{cases} s(1, \mathcal{O})_W, & s \text{ even} \\ s(1, \mathcal{E})_W, & s \text{ odd} \end{cases}, \quad (1, \mathcal{O})_W \otimes (1, s) = \begin{cases} s(1, \mathcal{E})_W, & s \text{ even} \\ s(1, \mathcal{O})_W, & s \text{ odd} \end{cases} \tag{4.41}
\]

while the $W$-indecomposable representations have the stability properties

\[
(1, (2n - 2 + \kappa)p') \otimes (1, \kappa' p')_W = (2n - 2 + \kappa)\bigoplus_{\beta}^{p'-1} (R^{0,b}_{1,(\kappa - \kappa')p'})_W
\]

\[
(1, (2n - 2 + \kappa)p') \otimes (R^{0,b}_{1,\kappa' p'})_W = 2(2n - 2 + \kappa)\bigoplus_{\beta}^{b-1} (R^{0,b}_{1,2(\kappa - \kappa')p'})_W \oplus \bigoplus_{\beta}^{p'-b-1} (R^{0,b}_{1,2(\kappa - \kappa')p'})_W \tag{4.42}
\]
and
\[(1, s) \otimes (1, \kappa p')_W = \bigoplus_{\beta} (\mathcal{R}_{1, \kappa p'}^{0, \beta})_W \]
\[(1, s) \otimes (\mathcal{R}_{1, \kappa p'}^{0, b})_W = \bigoplus_{\beta \equiv b-s+1, \text{by } 2} (\mathcal{R}_{1, \kappa p'}^{0, \beta})_W \oplus 2(\mathcal{R}_{1, \kappa p'}^{0, \beta})_W \oplus 2(\mathcal{R}_{1, (2\cdot n)p'}^{0, \beta})_W \] (4.43)

In accordance with the definition (4.17), the individual vertical fusions then follow from appropriately disentangling the result of evaluating
\[(A)_W \otimes (1, \mathcal{E})_W = \lim_{n \to \infty} \left(\frac{1}{2n}\right)^3 (A)_W \otimes (1, 2np')^{\otimes 3} \] (4.44)

For \(p'\) even, the decompositions of \((1, \mathcal{E})_W\) and \((1, \mathcal{O})_W\) in terms of \(\mathcal{W}\)-indecomposable representations read
\[(1, \mathcal{E})_W = \bigoplus_{\beta} (p' - \beta)\left\{ (\mathcal{R}_{1, p'}^{0, \beta})_W \oplus (\mathcal{R}_{1, 2p'}^{0, \beta})_W \right\} \]
\[(1, \mathcal{O})_W = \bigoplus_{\beta} (p' - \beta)\left\{ (\mathcal{R}_{1, p'}^{0, \beta})_W \oplus (\mathcal{R}_{1, 2p'}^{0, \beta})_W \right\} \] (4.45)

while for \(p'\) odd they read
\[(1, \mathcal{E})_W = \bigoplus_{\beta} (p' - \beta)(\mathcal{R}_{1, p'}^{0, \beta})_W \oplus \bigoplus_{\beta} (p' - \beta)(\mathcal{R}_{1, 2p'}^{0, \beta})_W \]
\[(1, \mathcal{O})_W = \bigoplus_{\beta} (p' - \beta)(\mathcal{R}_{1, p'}^{0, \beta})_W \oplus \bigoplus_{\beta} (p' - \beta)(\mathcal{R}_{1, 2p'}^{0, \beta})_W \] (4.46)

The two-dimensional fusion subalgebra generated by \((1, \mathcal{E})_W\) and \((1, \mathcal{O})_W\) is governed by the fusion rules
\[(1, \mathcal{E})_W \otimes (1, \mathcal{E})_W = (1, \mathcal{O})_W \otimes (1, \mathcal{O})_W = (p')^3(1, \mathcal{E})_W, \quad (1, \mathcal{E})_W \otimes (1, \mathcal{O})_W = (p')^3(1, \mathcal{E})_W \] (4.47)

### 4.3 Horizontal and vertical components combined

Here we describe the merge of the horizontal and vertical components by completing the set of \(N_{\text{ind}}(p, p')\) \(\mathcal{W}\)-representations announced in (3.7). The derivation of the ensuing fusion algebra (3.50) is discussed in Section 4[3].

#### 4.3.1 Representation content

New representations are constructed by fusing the horizontal representations above by the simple vertical (Virasoro-indecomposable) Kac representations \((1, s)\). For \(s \in \mathbb{Z}_{2p'}\), we thus define the \(\mathcal{W}\)-indecomposable rank-1 representations
\[(\kappa p, s)_W := (\kappa p, 1)_W \otimes (1, s) = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)((2k - 2 + \kappa)p, s) \] (4.48)
For convenience of notation, we also introduce
and the $\mathcal{W}$-indecomposable rank-2 representations
\[
(\mathcal{R}_{kp,s}^{α,0})_W := (\mathcal{R}_{kp,1}^{α,0})_W \otimes (1, s) = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa) \mathcal{R}_{(2k-2+\kappa)p,s}^{α,0} \tag{4.49}
\]
For $s = 1$, these identities are valid but do not constitute definitions. We also define $\mathcal{W}$-indecomposable rank-3 representations by fusing the $\mathcal{W}$-indecomposable rank-2 representations with vertical Virasoro-indecomposable representations of rank 2
\[
(\mathcal{R}_{kp,p'}^{α,b})_W := (\mathcal{R}_{kp,1}^{α,0})_W \otimes \mathcal{R}_{1,p'}^{0,b} = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa) \mathcal{R}_{(2k-2+\kappa)p,p'}^{α,b} \tag{4.50}
\]
We could just as well have fused the vertical $\mathcal{W}$-representations by horizontal Virasoro-indecomposable representations. For $r \in \mathbb{Z}_{2, p'}$, this yields the $\mathcal{W}$-indecomposable rank-1 representations
\[
(r, kp')_W := (r, 1) \otimes (1, kp')_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa)(r, (2k - 2 + \kappa)p') \tag{4.51}
\]
the $\mathcal{W}$-indecomposable rank-2 representations
\[
(\mathcal{R}_{r,kp'}^{0,b})_W := (r, 1) \otimes (\mathcal{R}_{1,kp'}^{0,b})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa) \mathcal{R}_{r,(2k-2+\kappa)p'}^{0,b} \tag{4.52}
\]
and the $\mathcal{W}$-indecomposable rank-3 representations
\[
(\mathcal{R}_{p,kp'}^{α,b})_W := \mathcal{R}_{p,1}^{α,0} \otimes (\mathcal{R}_{kp'}^{0,b})_W = \bigoplus_{k \in \mathbb{N}} (2k - 2 + \kappa) \mathcal{R}_{p,(2k-2+\kappa)p'}^{α,b} \tag{4.53}
\]
The identities $\mathcal{R}_{kp,k'p'}^{α,β} = \mathcal{R}_{k'p,kp'}^{α,β}$ between Virasoro representations imply the $\mathcal{W}$-representation identities
\[
(2p, p')_W = (p, 2p')_W, \quad (\mathcal{R}_{2p,p'}^{α,b})_W = (\mathcal{R}_{p,2p'}^{α,b})_W \tag{4.54}
\]
For convenience of notation, we also introduce
\[
(\mathcal{R}_{2p,2p'}^{α,β})_W := \frac{1}{2} \mathcal{R}_{2p,1}^{α,0} \otimes (\mathcal{R}_{1,2p'}^{0,β})_W = \bigoplus_{k \in \mathbb{N}} k \mathcal{R}_{2p,2p'}^{α,β} = \bigoplus_{k \in \mathbb{N}} (2k - 1) \mathcal{R}_{p,(2k-1)p'}^{α,β} = (\mathcal{R}_{p,p'}^{α,β})_W \tag{4.55}
\]
Compactly, our notation allows us to write
\[
(\mathcal{R}_{(κ,κ')p,p'})_W \equiv (\mathcal{R}_{kp,k'p'}^{α,β})_W \equiv (\mathcal{R}_{p,(κ-κ')p'})_W \tag{4.56}
\]
and
\[
\frac{1}{κ'}(\mathcal{R}_{kp,1}^{α,0})_W \otimes \mathcal{R}_{1,k'p'}^{α,β} = (\mathcal{R}_{kp,k'p'}^{α,β})_W = \frac{1}{κ} \mathcal{R}_{kp,1}^{α,0} \otimes (\mathcal{R}_{1,k'p'}^{0,β})_W \tag{4.57}
\]
Having ventured into the bulk part of the Kac table, we note the stability properties
\[
(\mathcal{R}_{kp,1}^{α,0})_W \otimes (1, (2n - 2 + κ)p') \equiv (2n - 2 + κ')(\mathcal{R}_{(κ,κ')p,p'})_W \\
((2n - 2 + κ)p, 1) \otimes (\mathcal{R}_{1,k'p'}^{0,β})_W \equiv (2n - 2 + κ)(\mathcal{R}_{p,(κ-κ')p'})_W \tag{4.58}
\]
Combining all the $\mathcal{W}$-indecomposable representations discussed so far in this Section we arrive at the classification \(\text{[3.2]}, (3.3)\) and \(\text{[3.4]}\).
4.3.2 Some linear relations

Here we list some intriguing linear relations involving the horizontal representations \((\mathcal{E}, 1)_W\) and \((\mathcal{O}, 1)_W\) and the vertical representations \((1, \mathcal{E})_W\) and \((1, \mathcal{O})_W\). These linear relations will resurface when discussing fusion subalgebras in Section 4.5. For \(p\) even and \(p'\) odd, we find that

\[
\bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}_{1,p'}^{0,\beta} = \bigoplus_{\alpha}^{p-2} (p - \alpha)\mathcal{R}_{p,1}^{\alpha,0} \otimes \left\{ (1, \mathcal{E})_W \oplus (1, \mathcal{O})_W \right\}
\]

where it is noted that the summations over \(\alpha\) are in steps of 1 while the summations over \(\beta\) are in steps of 1.

For \(p'\) even and \(p\) odd, we have

\[
\bigoplus_{\beta=0}^{p-1} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}_{1,p'}^{0,\beta} = \bigoplus_{\alpha}^{p-1} (p - \alpha)\mathcal{R}_{p,1}^{\alpha,0} \otimes \left\{ (1, \mathcal{E})_W \oplus (1, \mathcal{O})_W \right\}
\]

which will resurface when discussing fusion subalgebras.

For \(p\) odd and \(p'\) even, we similarly have

\[
\bigoplus_{\beta=0}^{p'-2} (p' - \beta)\left\{ (\mathcal{E}, 1)_W \oplus (\mathcal{O}, 1)_W \right\} \otimes \mathcal{R}_{1,p'}^{0,\beta} = \bigoplus_{\alpha=0}^{p-1} (p - \alpha)\mathcal{R}_{p,1}^{\alpha,0} \otimes (1, \mathcal{E})_W
\]

where it is noted that the summations over \(\alpha\) are in steps of 2 while the summations over \(\beta\) are in steps of 1.

Finally, for both odd, we have

\[
\left\{ \bigoplus_{\beta=0}^{p'-2} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}_{1,p'}^{0,\beta} \right\} \oplus \left\{ \bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}_{1,p'}^{0,\beta} \right\} = \left\{ \bigoplus_{\alpha=0}^{p-2} (p - \alpha)\mathcal{R}_{p,1}^{\alpha,0} \otimes (1, \mathcal{E})_W \right\} \oplus \left\{ \bigoplus_{\alpha=0}^{p-1} (p - \alpha)\mathcal{R}_{p,1}^{\alpha,0} \otimes (1, \mathcal{O})_W \right\}
\]

4.4 \(\mathcal{W}\)-extended fusion

There are two obvious approaches to the examination of fusions between horizontal and vertical representations, namely (4.17) and (4.44). Self-consistency of our fusion prescription requires that the evaluation of a given fusion product based on (4.17) must yield the same result as the evaluation of the same fusion product based on (4.44), when both methods are applicable. Since the parameters \(p\) and \(p'\) are coprime, at least one of them must be odd. Without loss of generality, we assume that \(p\) is odd and initially use (4.17). Had we instead assumed that \(p'\) is odd, we would initially use (4.44). We will subsequently address the question of self-consistency. We thus consider

\[
(\mathcal{E}, 1)_W \hat{\otimes} (\mathcal{R}_{1,n',p'})_W = \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 (2np, 1) \otimes (\mathcal{R}_{1,n',p'})_W
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2n} \right)^2 (2np, 1) \otimes (\mathcal{R}_{1,2n',p'})_W
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2n} \right)^2 (p, 1) \otimes (2np, 1) \otimes (\mathcal{R}_{1,2n',p'})_W
\]

\[
= (p, 1) \otimes (\mathcal{R}_{1,2n',p'})_W \quad (4.62)
\]
The further analysis of this depends on the parity of \( p \) \((2.16)\), but having assumed that \( p \) is odd, we use the decomposition of \( (\mathcal{E}, 1)_W \) in \((4.30)\) to obtain

\[
\left\{ \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{2p,1})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{p,1})_W \right\} \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,\beta}_{2p,k'})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,\beta}_{p,k'})_W
\]

\((4.63)\)

We are now faced with yet another disentangling task in order to identify the individual fusion products. Since the result of fusing \((k(A)_W)\) with \((\mathcal{R}^{0,\beta}_{1,k'})_W\) must be divisible by \( k \), we find, as \( \alpha \) increases from 0 to \( p - 1 \), that

\[
(\mathcal{R}^{\alpha,0}_{2p,1})_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = (\mathcal{R}^{\alpha,\beta}_{2p,k'})_W, \quad \alpha \text{ even}
\]

\((4.64)\)

\[
(\mathcal{R}^{\alpha,0}_{p,1})_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = (\mathcal{R}^{\alpha,\beta}_{p,k'})_W, \quad \alpha \text{ odd}
\]

Since \( p \) is odd, we also have

\[
(\mathcal{O}, 1)_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 ((2n - 1)p, 1)^{\otimes 3} \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W
\]

\((4.65)\)

that is,

\[
\left\{ \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{p,1})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,0}_{2p,1})_W \right\} \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,\beta}_{p,k'})_W \oplus \bigoplus_{\alpha} (p - \alpha)(\mathcal{R}^{\alpha,\beta}_{2p,k'})_W
\]

\((4.66)\)

from which it follows that

\[
(\mathcal{R}^{\alpha,0}_{p,1})_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = (\mathcal{R}^{\alpha,\beta}_{p,k'})_W, \quad \alpha \text{ even}
\]

\((4.67)\)

\[
(\mathcal{R}^{\alpha,0}_{2p,1})_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = (\mathcal{R}^{\alpha,\beta}_{2p,k'})_W, \quad \alpha \text{ odd}
\]

\((4.68)\)

Combining these results for general \( \alpha \in \mathbb{Z}_{0,p-1} \), we see that

\[
(\mathcal{R}^{\alpha,0}_{\kappa p,1})_W \otimes (\mathcal{R}^{0,\beta}_{1,k'})_W = (\mathcal{R}^{\alpha,\beta}_{\kappa p,k'})_W
\]

\((4.69)\)

Returning to the question of self-consistency, it is obvious that one arrives at the same result \((4.68)\) using \((4.44)\) if both \( p \) and \( p' \) are odd. For \( p \) odd and \( p' \) even, self-consistency requires that

\[
(\mathcal{R}^{\alpha,0}_{p,1})_W \otimes \left( \bigoplus_{\beta} (p' - \beta) \left\{ (\mathcal{R}^{0,\beta}_{1,p'})_W \oplus (\mathcal{R}^{0,\beta}_{1,2p'})_W \right\} \right)
\]

\[= (\mathcal{R}^{\alpha,0}_{p,1})_W \otimes (1, \mathcal{E})_W = \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 (\mathcal{R}^{\alpha,0}_{p,1})_W \otimes (1, 2np')^{\otimes 3}
\]

\[= (\mathcal{R}^{\alpha,0}_{p,1})_W \otimes \left( \bigoplus_{\beta} (p' - \beta) \left\{ (\mathcal{R}^{0,\beta}_{1,p'})_W \oplus \frac{1}{2} (\mathcal{R}^{0,\beta}_{1,2p'})_W \right\} \right)
\]

\[= \bigoplus_{\beta} (p' - \beta) \left\{ (\mathcal{R}^{\alpha,\beta}_{p,p'})_W \oplus (\mathcal{R}^{\alpha,\beta}_{p,2p'})_W \right\}
\]

\((4.69)\)
when the left-hand side is evaluated using (4.68). This is easily verified. Our ‘symmetric’ notation finally ensures that our fusion prescription is self-consistent also in the case where \( p \) is even and \( p' \) is odd.

Together with the definitions of \( \mathcal{W} \)-indecomposable representations as simple fusions of \( \mathcal{W} \)- and Virasoro-indecomposable representations in Section 4.3.1, the remarkably simple \( \mathcal{W} \)-extended fusion products (4.68) demonstrate that the evaluation of the fusion algebra of \( \mathcal{W}\mathcal{L}\mathcal{M}(p,p') \) separates into horizontal and vertical parts. Furthermore, associativity and commutativity of the fusion algebra of \( \mathcal{W}\mathcal{L}\mathcal{M}(p,p') \) are inherited from the associative and commutative fundamental fusion algebra of \( \mathcal{L}\mathcal{M}(p,p') \) in the Virasoro picture.

It is now straightforward to complete the derivation of the fusion algebra of \( \mathcal{W}\mathcal{L}\mathcal{M}(p,p') \) as summarized in Section 3.3. To illustrate this, we first consider

\[
(r, \kappa p')_W \hat{\otimes} (r', \kappa' p')_W = \left\{ (r, 1) \otimes (r', 1) \right\} \hat{\otimes} \left\{ (1, \kappa p')_W \hat{\otimes} (1, \kappa' p')_W \right\} = \left\{ \bigoplus_{j=|r'-r|+1, \text{by } 2}^{p-|p-r-r'|-1} \bigoplus_{\alpha}^{j} \mathcal{R}^{\alpha,0}_{p,1} \right\} \hat{\otimes} \left\{ \bigoplus_{\beta}^{p'-1} (\mathcal{R}^{0,\beta}_{1,(\kappa,\kappa')p'})_W \right\}
\]

\[
= \left\{ \bigoplus_{j=|r'-r|+1, \text{by } 2}^{p-|p-r-r'|-1} \bigoplus_{\alpha}^{j} \mathcal{R}^{\alpha,0}_{p,1} \right\} \hat{\otimes} \left\{ \bigoplus_{\beta}^{p'-1} (\mathcal{R}^{0,\beta}_{1,(\kappa,\kappa')p'})_W \right\}
\]

in accordance with (3.51). In the second and final example, we consider

\[
(\mathcal{R}^{\alpha,0}_{p,\kappa p})_W \hat{\otimes} (\mathcal{R}^{\alpha',0}_{p,\kappa' p'})_W = \left\{ \bigoplus_{\alpha}^{p-|p-a-a'|-1} 2(\mathcal{R}^{\alpha,0}_{p,\kappa p})_W \hat{\otimes} \bigoplus_{\alpha}^{|p-a-a'|-1} 2(\mathcal{R}^{\alpha,0}_{p,\kappa p})_W \right\} \hat{\otimes} \left\{ \bigoplus_{\beta}^{p'-|p'-b-b'|-1} \bigoplus_{\beta}^{s-b'-1} (\mathcal{R}^{0,\beta}_{1,(\kappa,\kappa')p'})_W \hat{\otimes} \bigoplus_{\beta}^{b'+s-p'-1} 2(\mathcal{R}^{0,\beta}_{1,(\kappa,\kappa')p'})_W \right\}
\]

By recombining the two components using (4.68), we immediately recognize the first fusion rule in (3.55).

### 4.5 Fusion subalgebras without disentanglement

There are many fusion subalgebras of the \( \mathcal{W} \)-extended fusion algebra (3.50). We have already encountered some of them, namely the projective fusion algebra discussed in Section 4.6 as well as the horizontal and vertical fusion algebras discussed in Section 4.1.3 and Section 4.2 respectively. For \( p > 1 \), there is also a six-dimensional fusion subalgebra

\[
\langle (\mathcal{E}, 1)_W, (\mathcal{O}, 1)_W, (1, \mathcal{E})_W, (1, \mathcal{O})_W, (\mathcal{A})_W, (\mathcal{B})_W \rangle_{p,p'}
\]

where \( (\mathcal{A})_W \) and \( (\mathcal{B})_W \) depend on the parities of \( p \) and \( p' \). To describe this fusion subalgebra, we introduce the abbreviations

\[
(\mathcal{E}, \mathcal{E})_W := (\mathcal{E}, 1)_W \hat{\otimes} (1, \mathcal{E})_W, \quad (\mathcal{E}, \mathcal{O})_W := (\mathcal{E}, 1)_W \hat{\otimes} (1, \mathcal{O})_W
\]

\[
(\mathcal{O}, \mathcal{E})_W := (\mathcal{O}, 1)_W \hat{\otimes} (1, \mathcal{E})_W, \quad (\mathcal{O}, \mathcal{O})_W := (\mathcal{O}, 1)_W \hat{\otimes} (1, \mathcal{O})_W
\]
For $p$ even and $p'$ odd, we have

$$ (A)_W = (\mathcal{E}, \mathcal{E})_W = (\mathcal{E}, \mathcal{O})_W = \bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}^{0,\beta}_{1,p'} $$

$$ = \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-1} (p' - \beta) \left( (\mathcal{R}_{p,p'})^0_W \oplus (\mathcal{R}_{2p,p'})^0_W \right) \right\} $$

$$ (B)_W = (\mathcal{O}, \mathcal{E})_W = (\mathcal{O}, \mathcal{O})_W = \bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{O}, 1)_W \otimes \mathcal{R}^{0,\beta}_{1,p'} $$

$$ = \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-1} (p' - \beta) \left( (\mathcal{R}_{p,p'})^0_W \oplus (\mathcal{R}_{2p,p'})^0_W \right) \right\} $$

For $p$ odd and $p'$ even, we have

$$ (A)_W = (\mathcal{E}, \mathcal{E})_W = (\mathcal{O}, \mathcal{E})_W = \bigoplus_{\beta=0}^{p'-2} (p' - \beta)(\mathcal{E}, 1)_W \oplus (\mathcal{O}, 1)_W \bigotimes \mathcal{R}^{0,\beta}_{1,p'} $$

$$ = \bigoplus_{\beta}(p' - \beta) \left\{ \bigoplus_{\alpha=0}^{p'-1} (p - \alpha) \left( (\mathcal{R}_{p,p'})^0_W \oplus (\mathcal{R}_{2p,p'})^0_W \right) \right\} $$

$$ (B)_W = (\mathcal{O}, \mathcal{O})_W = \bigoplus_{\beta}(p' - \beta) \left\{ \bigoplus_{\alpha=0}^{p'-1} (p - \alpha) \left( (\mathcal{R}_{p,p'})^0_W \oplus (\mathcal{R}_{2p,p'})^0_W \right) \right\} $$

Finally, for $p$ and $p'$ both odd, we have

$$ (A)_W = (\mathcal{E}, \mathcal{E})_W = (\mathcal{O}, \mathcal{O})_W = \bigoplus_{\beta=0}^{p'-2} (p' - \beta)(\mathcal{E}, 1)_W \otimes \mathcal{R}^{0,\beta}_{1,p'} $$

$$ = \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{R}_{p,p'})^0_W \right\} \oplus \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-2} (p' - \beta)(\mathcal{R}_{p,p'})^0_W \right\} $$

$$ (B)_W = (\mathcal{O}, \mathcal{E})_W = (\mathcal{O}, \mathcal{O})_W = \bigoplus_{\beta}(p' - \beta)(\mathcal{O}, 1)_W \otimes \mathcal{R}^{0,\beta}_{1,p'} $$

$$ = \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-1} (p' - \beta)(\mathcal{R}_{p,p'})^0_W \right\} \oplus \bigoplus_{\alpha}(p - \alpha) \left\{ \bigoplus_{\beta=0}^{p'-2} (p' - \beta)(\mathcal{R}_{p,p'})^0_W \right\} $$

We note that the linear relations discussed in Section 4.3.2 simply correspond to the identities
Figure 1: Cayley table of the six-dimensional $E, O$ fusion subalgebra for $p$ even and $p'$ odd.

| $\cdot$ | $(E, 1)_W$ | $(O, 1)_W$ | $(1, E)_W$ | $(1, O)_W$ | $(E, E)_W$ | $(O, O)_W$ |
|---------|-------------|-------------|------------|------------|------------|------------|
| $(E, 1)_W$ | $p^3(O, 1)_W$ | $p^3(E, 1)_W$ | $(E, E)_W$ | $(E, E)_W$ | $p^3(O, O)_W$ | $p^3(E, E)_W$ |
| $(O, 1)_W$ | $p^3(E, 1)_W$ | $p^3(O, 1)_W$ | $(O, O)_W$ | $(O, O)_W$ | $p^3(E, E)_W$ | $p^3(O, O)_W$ |
| $(1, E)_W$ | $(E, E)_W$ | $(O, O)_W$ | $p^3(1, O)_W$ | $p^3(1, E)_W$ | $p^3(O, O)_W$ | $p^3(E, E)_W$ |
| $(1, O)_W$ | $(E, E)_W$ | $(O, O)_W$ | $p^3(1, E)_W$ | $p^3(1, O)_W$ | $p^3(E, E)_W$ | $p^3(O, O)_W$ |
| $(E, E)_W$ | $p^3(O, O)_W$ | $p^3(E, E)_W$ | $p^3(E, E)_W$ | $p^3(E, E)_W$ | $(pp')^3(O, O)_W$ | $(pp')^3(E, E)_W$ |
| $(O, O)_W$ | $p^3(E, E)_W$ | $p^3(O, O)_W$ | $p^3(O, O)_W$ | $p^3(O, O)_W$ | $(pp')^3(E, E)_W$ | $(pp')^3(O, O)_W$ |

Figure 2: Cayley table of the six-dimensional $E, O$ fusion subalgebra for $p$ odd and $p'$ even.

| $\cdot$ | $(E, 1)_W$ | $(O, 1)_W$ | $(1, E)_W$ | $(1, O)_W$ | $(E, E)_W$ | $(O, O)_W$ |
|---------|-------------|-------------|------------|------------|------------|------------|
| $(E, 1)_W$ | $p^3(O, 1)_W$ | $p^3(E, 1)_W$ | $(E, E)_W$ | $(E, E)_W$ | $p^3(O, E)_W$ | $p^3(E, E)_W$ |
| $(O, 1)_W$ | $p^3(E, 1)_W$ | $p^3(O, 1)_W$ | $(O, E)_W$ | $(O, E)_W$ | $p^3(E, E)_W$ | $p^3(O, E)_W$ |
| $(1, E)_W$ | $(E, E)_W$ | $(O, E)_W$ | $p^3(1, O)_W$ | $p^3(1, E)_W$ | $p^3(O, E)_W$ | $p^3(E, E)_W$ |
| $(1, O)_W$ | $(O, E)_W$ | $(E, E)_W$ | $p^3(1, E)_W$ | $p^3(1, O)_W$ | $p^3(E, E)_W$ | $p^3(O, E)_W$ |
| $(E, E)_W$ | $p^3(O, E)_W$ | $p^3(E, E)_W$ | $p^3(O, E)_W$ | $p^3(O, E)_W$ | $(pp')^3(E, E)_W$ | $(pp')^3(O, E)_W$ |
| $(O, E)_W$ | $p^3(E, E)_W$ | $p^3(O, E)_W$ | $p^3(O, E)_W$ | $p^3(O, E)_W$ | $(pp')^3(O, E)_W$ | $(pp')^3(E, E)_W$ |

Figure 3: Cayley table of the six-dimensional $E, O$ fusion subalgebra for $p$ and $p'$ both odd.
The decompositions of these vertical representations given in (4.45) for \( p \) even and in the Cayley table in Figure 3 for \( p \) odd are unaffected by setting \( p = 1 \). Thus, there is a four-dimensional fusion subalgebra

\[
\langle (1,1)_W, (2,1)_W, (1,\mathcal{E})_W, (1,\mathcal{O})_W \rangle_{1,p'}
\]

of the \( \mathcal{W} \)-extended fusion subalgebra of \( \mathcal{WL}\mathcal{M}(1,p') \), where

\[
(\mathcal{E},1)_W = (2,1)_W, \quad (\mathcal{O},1)_W = (1,1)_W
\]

The fusion rules governing (4.78) are given in the Cayley table in Figure 4 for \( p' \) even and in the Cayley table in Figure 5 for \( p' \) odd.

We finally stress that, for every \( \mathcal{W} \)-extended logarithmic minimal model \( \mathcal{WL}\mathcal{M}(p,p') \), a virtue of the six-dimensional (or four-dimensional for \( p = 1 \)) fusion subalgebra just described is that it does not rely on any disentangling procedure.

5 Discussion

There is an infinite series of Yang-Baxter integrable logarithmic minimal models \( \mathcal{LM}(p,p') \) \cite{1}. As in the rational case \cite{33}, the Yang-Baxter integrable boundary conditions give insight into the conformal boundary conditions \cite{34} in the continuum scaling limit as well as into the fusion of their associated Virasoro representations. This enabled us in \cite{1} to construct integrable boundary conditions labelled by \((r,s)\) and corresponding to so-called Kac representations with conformal weights in an infinitely extended Kac table. Moreover, from the lattice implementation of fusion, we obtained \cite{29,30} the closed (fundamental) fusion algebra generated by these Kac representations finding that indecomposable representations of ranks 1, 2 and 3 are generated by the fusion process. In the special case where
p = 1, only indecomposable representations of rank 1 or 2 arise. Although there is a countable infinity of representations for general $\mathcal{LM}(p, p')$, the ensuing fusion rules are quasi-rational in the sense of Nahm $[35]$, that is, the fusion of any two indecomposable representations decomposes into a finite sum of indecomposable representations. This is the relevant picture in the case where the conformal algebra is the Virasoro algebra. Of course, there is no claim, in the context of this logarithmic CFT, that the representations generated in this picture exhaust all of the representations associated with conformal boundary conditions. This is in stark contrast to the situation in rational CFTs where all representations decompose into direct sums of a finite number of irreducible representations.

In this paper, we have reconsidered the lattice description of the logarithmic minimal model $\mathcal{LM}(p, p')$ in the continuum scaling limit to expose its nature as a ‘rational’ logarithmic CFT with respect to a $\mathcal{W}$-extended conformal algebra. Under the extended symmetry, the infinity of Virasoro representations are reorganized into a finite number of $\mathcal{W}$-representations. Following the approach of $[17,20]$, we have constructed new solutions of the boundary Yang-Baxter equation which, in a particular limit, correspond to these representations. With respect to a suitably defined $\mathcal{W}$-fusion implemented on the lattice, we find that the representation content of the ensuing closed, associative and commutative fusion algebra is finite containing $6pp' - 2p - 2p'$ $\mathcal{W}$-indecomposable representations with $2p + 2p' - 2$ rank-1 representations, $4pp' - 2p - 2p'$ rank-2 representations and $2(p - 1)(p' - 1)$ rank-3 representations. The $\mathcal{W}$-indecomposable rank-1 representations are all $\mathcal{W}$-irreducible while we have presented a conjecture for the embedding patterns of the $\mathcal{W}$-indecomposable rank-2 and -3 representations. We have also identified their associated $\mathcal{W}$-extended characters which decompose as finite non-negative sums of $2pp' + (p - 1)(p' - 1)/2$ distinct $\mathcal{W}$-irreducible characters. For $2p' > 2p' - 4$ of the rank-2 and all of the rank-3 $\mathcal{W}$-indecomposable representations, we have presented fermionic character expressions. To distinguish between inequivalent $\mathcal{W}$-indecomposable representations of identical characters, we have introduced ‘refined’ characters carrying information also about the Jordan-cell content of a representation. Furthermore, we have found that $2pp'$ of the $\mathcal{W}$-indecomposable representations are in fact $\mathcal{W}$-projective representations and shown that they generate a closed fusion subalgebra. Finally, we interpret the closure of the $\mathcal{W}$-indecomposable representations among themselves under fusion as confirmation of the proposed extended symmetry.

The results presented in this paper apply to the entire infinite series $\mathcal{WL}\mathcal{M}(p, p')$. Some of these models are of great interest and have been studied before. In particular, symplectic fermions $\mathcal{WL}\mathcal{M}(1, 2)$ (which are critical dense polymers $\mathcal{LM}(1, 2)$ viewed in the $\mathcal{W}$-extended picture $[17]$) and more generally the infinite series $\mathcal{WL}\mathcal{M}(1, p')$ are discussed in $[6,13,14,15,16,17]$, while $\mathcal{W}$-extended critical percolation $\mathcal{WL}\mathcal{M}(2, 3)$ is discussed in $[20]$. One may verify explicitly that our general expressions for characters and fusion rules indeed reduce to the expressions given in those papers when fixing $p$ and $p'$ to their relevant values. Among the many other interesting models are the $\mathcal{W}$-extended logarithmic Yang-Lee model $\mathcal{WL}\mathcal{M}(2, 5)$ and the $\mathcal{W}$-extended logarithmic Ising model $\mathcal{WL}\mathcal{M}(3, 4)$. The
numbers of $\mathcal{W}$-indecomposable and $\mathcal{W}$-irreducible representations are rather large as they are given by

\begin{align*}
N_1(2,5) &= 12, \quad N_2(2,5) = 26, \quad N_3(2,5) = 8, \quad N_{\text{ind}}(2,5) = 46, \quad N_{\text{irr}}(2,5) = 22 \\
N_1(3,4) &= 12, \quad N_2(3,4) = 34, \quad N_3(3,4) = 12, \quad N_{\text{ind}}(3,4) = 58, \quad N_{\text{irr}}(3,4) = 27
\end{align*}

(5.1)

while the numbers of $\mathcal{W}$-projective representations are

\begin{align*}
N_{\text{proj}}(2,5) &= 20, \quad N_{\text{proj}}(3,4) = 24
\end{align*}

(5.2)

A somewhat surprising feature of our closed $\mathcal{W}$-extended fusion algebra of $\mathcal{WL}M(p,p')$ is that there appears to be no natural identity $\mathcal{I}_\mathcal{W}$ expressed in terms of the fundamental Virasoro fusion algebra and with respect to the fusion multiplication $\hat{\otimes}$. Since the Kac representation $(1,1)$ is the identity of the fundamental fusion algebra itself, it may be tempting to include it in the $\mathcal{W}$-extended spectrum and identify it with $\mathcal{I}_\mathcal{W}$. However, we have

\[ (\mathcal{E},1)_\mathcal{W} \hat{\otimes} \mathcal{I}_\mathcal{W} := \lim_{n \to \infty} \left( \frac{1}{2n} \right)^3 (2np,1)^{3} \otimes (1,1) = 0 \]

(5.3)

demonstrating that this simple extension fails. We find it natural, though, to expect that one can extend our fusion algebra of $\mathcal{WL}M(p,p')$ by working with the full Virasoro fusion algebra. We hope to discuss this and re-address the identity question elsewhere.

Comparing the sets of $\mathcal{W}$-irreducible and $\mathcal{W}$-indecomposable rank-1 representations with the results of [19], we find complete agreement. In a further comparison, the sets of $\mathcal{W}$-projective characters agree as well. We therefore find it natural to suspect that our construction is with respect to their extended conformal algebra $\mathcal{W}_{p,p'}$. Here we wish to point out that these $\mathcal{W}$-projective characters were found to constitute a representation of the modular group in [19] and that several modular invariants can be formed out of these. Combining this with our observation that the corresponding $\mathcal{W}$-projective representations generate a closed fusion algebra, yields an intriguing hint towards the classification of torus amplitudes in $\mathcal{WL}M(p,p')$. We also wish to emphasize that the works [19, 36] address fusion only by studying the Grothendieck ring of a related quantum group as an approximation to the fusion algebra of their $2pp'$ representations $K_{r,s}^{\pm}$. In this context, the Grothendieck ring may be regarded as the ‘fusion algebra’ of the corresponding set of $\mathcal{W}$-characters as opposed to the much richer fusion algebra of $\mathcal{W}$-representations. The latter is given explicitly in Section 3.5 above and is one of our main results.

Acknowledgments

This work is supported by the Australian Research Council (ARC). The author thanks Paul A. Pearce and Ilya Yu. Tipunin for useful discussions and comments.

References

[1] P.A. Pearce, J. Rasmussen, J.-B. Zuber, Logarithmic minimal models, J. Stat. Mech. (2006) P11017, arXiv:hep-th/0607232.

[2] J. Cardy, Conformal invariance, in C. Domb and J. L. Lebowitz, eds., Phase Transitions and Critical Phenomena vol. 11, Acad. Press (1987).

[3] A.B. Zamolodchikov, Infinite additional symmetries in two-dimensional conformal quantum field theory, Theor. Math. Phys. 65 (1985) 1205–1213.

[4] P. Bouwknegt, K. Schoutens, W-symmetry in conformal field theory, Phys. Rept. 223 (1993) 183–276, arXiv:hep-th/9210010.
[5] M. Flohr, *On modular invariant partition functions of conformal field theories with logarithmic operators*, Int. J. Mod. Phys. **A11** (1996) 4147–4172, [arXiv:hep-th/9509166](http://arxiv.org/abs/hep-th/9509166).

[6] M.R. Gaberdiel, H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. **B386** (1996) 131–137, [arXiv:hep-th/9606050](http://arxiv.org/abs/hep-th/9606050).

[7] M. Flohr, M.R. Gaberdiel, *Logarithmic torus amplitudes*, J. Phys. **A39** (2006) 1955–1968, [arXiv:hep-th/0509075](http://arxiv.org/abs/hep-th/0509075).

[8] M.R. Gaberdiel, I. Runkel, *The logarithmic triplet theory with boundary*, J. Phys. **A39** (2006) 14745–14780, [arXiv:hep-th/0608184](http://arxiv.org/abs/hep-th/0608184).

[9] V. Gurarie, *Logarithmic operators in conformal field theory*, Nucl. Phys. **B410** (1993) 535–549, [arXiv:hep-th/9303160](http://arxiv.org/abs/hep-th/9303160).

[10] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, Int. J. Mod. Phys. **A18** (2003) 4497–4592, [arXiv:hep-th/0111228](http://arxiv.org/abs/hep-th/0111228).

[11] M.R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory*, Int. J. Mod. Phys. **A18** (2003) 4593–4638, [arXiv:hep-th/0111260](http://arxiv.org/abs/hep-th/0111260).

[12] S. Kawai, *Logarithmic conformal field theory with boundary*, Int. J. Mod. Phys. **A18** (2003) 4655–4684, [arXiv:hep-th/0204169](http://arxiv.org/abs/hep-th/0204169).

[13] J. Fuchs, S. Hwang, A.M. Semikhatov, I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. **247** (2004) 713–742, [arXiv:hep-th/0306274](http://arxiv.org/abs/hep-th/0306274).

[14] B. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun. Math. Phys. **265** (2006) 47–93, [arXiv:hep-th/0504093](http://arxiv.org/abs/hep-th/0504093).

[15] M.R. Gaberdiel, I. Runkel, *From boundary to bulk in logarithmic CFT*, J. Phys. **A41** (2008) 075402, [arXiv:0707.0388](http://arxiv.org/abs/0707.0388) [hep-th].

[16] A.M. Gainutdinov, I. Yu. Tipunin, *Radford, Drinfeld and Cardy boundary states in (1,p) logarithmic conformal field models*, [arXiv:0711.3430](http://arxiv.org/abs/0711.3430) [hep-th].

[17] P.A. Pearce, J. Rasmussen, P. Ruelle, *Integrable boundary conditions and W-extended fusion in the logarithmic minimal models LM(1,p)*, [arXiv:0803.0785](http://arxiv.org/abs/0803.0785) [hep-th].

[18] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Kazhdan-Lusztig dual quantum group for logarithmic extensions of Virasoro minimal models*, J. Math. Phys. **48** (2007) 032303, [arXiv:math.QA/0606506](http://arxiv.org/abs/math.QA/0606506).

[19] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Logarithmic extensions of minimal models: characters and modular transformations*, Nucl. Phys. **B757** (2006) 303–343, [arXiv:hep-th/0606196](http://arxiv.org/abs/hep-th/0606196).

[20] J. Rasmussen, P.A. Pearce, *W-extended fusion algebra of critical percolation*, [arXiv:0804.4335](http://arxiv.org/abs/0804.4335) [hep-th].

[21] H.G. Kausch, *Curiosities at c = −2*, [arXiv:hep-th/9510149](http://arxiv.org/abs/hep-th/9510149).

[22] H.G. Kausch, *Symplectic fermions*, Nucl. Phys. **B583** (2000) 513–541, [arXiv:hep-th/0003029](http://arxiv.org/abs/hep-th/0003029).

[23] P.G. de Gennes, *Scaling Concepts in Polymer Physics*, Cornell University, Ithaca (1979).
[24] J. des Cloizeaux, G. Jannink, Polymers in Solution: Their Modelling and Structure, Clarendon Press (1990).

[25] H. Saleur, New exact exponents for the two-dimensional self-avoiding walks, J. Phys. A19 (1986) L807–L810.

[26] H. Saleur, Magnetic properties of the two-dimensional n = 0 vector model, Phys. Rev. B35 (1987) 3657–3660.

[27] B. Duplantier, Exact critical exponents for two-dimensional dense polymers, J. Phys. A19 (1986) L1009–L1014.

[28] P.A. Pearce, J. Rasmussen, Solvable critical dense polymers, J. Stat. Mech. P02015 (2007), arXiv:hep-th/0610273.

[29] J. Rasmussen, P.A. Pearce, Fusion algebra of critical percolation, J. Stat. Mech. P09002 (2007), arXiv:0706.2716 [hep-th].

[30] J. Rasmussen, P.A. Pearce, Fusion algebras of logarithmic minimal models, J. Phys. A40 (2007) 13711–13733, arXiv:0707.3189 [hep-th].

[31] P. Di Francesco, H. Saleur, J.-B. Zuber, Modular invariance in nonminimal two-dimensional conformal theories, Nucl. Phys. B285 (1987) 454-480.

[32] H. Eberle, M. Flohr, Virasoro representations and fusion for general augmented minimal models, J. Phys. A39 (2006) 15245–15286, arXiv:hep-th/0604097.

[33] R.E. Behrend, P.A. Pearce, Integrable and conformal boundary conditions for lattice A,D,E and unitary minimal sl(2) models, J. Stat. Phys. 102 (2001) 577–640, arXiv:hep-th/0006094.

[34] R.E. Behrend, P.A. Pearce, V.B. Petkova, J.-B. Zuber, Boundary conditions in rational conformal field theories, Nucl. Phys. B579 (2000) 707–773, arXiv:hep-th/9908036.

[35] W. Nahm, Quasi-rational fusion products, Int. J. Mod. Phys. B8 (1994) 3693–3702, arXiv:hep-th/9402039.

[36] A.M. Semikhatov, A note on the logarithmic (p,p') fusion, arXiv:0710.5157 [hep-th].