ON THE STABILITY OF TIME-DOMAIN INTEGRAL EQUATIONS FOR ACOUSTIC WAVE PROPAGATION

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In honor of Peter Lax on the occasion of his 90th birthday.

Abstract. We give a principled approach for the selection of a boundary integral, retarded potential representation for the solution of scattering problems for the wave equation in an exterior domain.

1. Introduction. Let \( D \) be a bounded domain with smooth boundary, \( \Gamma \), and set \( \Omega = \overline{D}^c \). The determination of the outgoing part of the solution to the classical acoustic scattering problem for a soft scatterer requires solving the mixed Cauchy problem for the wave equation:

\[
\begin{align*}
\partial_t^2 u &= c^2 \Delta u \text{ in } \Omega \times [0, \infty), \\
\text{with } u(0, x) &= \partial_t u(0, x) = 0 \text{ and } \\
u(x, t) &= g(x, t) \text{ for } (x, t) \in \partial \Omega \times [0, \infty). 
\end{align*}
\]

Various other types of scatterers are modeled by using boundary conditions of the form \( \beta(x)u(x, t) + \partial_n u(x, t) = g(x, t) \), for \((x, t) \in \partial \Omega \times [0, \infty)\).

Many numerical approaches to solving this problem begin by expressing the solution in terms of the retarded potentials derived from the fundamental solution to

2010 Mathematics Subject Classification. Primary: 65M80, 31B10; Secondary: 35P25, 35L20.

Key words and phrases. Wave equation, boundary integral equations, time dependent, scattering poles, exponential decays.

The first author is partially supported by NSF grant DMS12-05851 and ARO grant W911NF-12-1-0552; the second author is partially supported by the U.S. Department of Energy under contract DEFG0288ER25053 and by the Office of the Assistant Secretary of Defense for Research and Engineering and AFOSR under NSSEFF Program Award FA9550-10-1-0180; the third author is partially supported by grants ARO W911NF-09-1-0344 and the NSF DMS-1418871.

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the wave equation. By analogy with the time harmonic case, we define the single and double layer potentials by the formulae

$$S\mu(x,t) = \int_{\Gamma} \frac{\mu(y,t - |x - y|)}{4\pi|x - y|} dS_y$$

$$D\mu(x,t) = \int_{\Gamma} n_y \cdot (x - y) \left[ \frac{\mu(y,t - |x - y|)}{|x - y|^2} + \frac{\partial \mu(y,t - |x - y|)}{|x - y|} \right] dS_y,$$

(2)

for $x \in \Gamma^c$. Here, $\partial_s \mu(y,s)$.

Suppose now that we represent the solution in the form

$$u(x,t) = D\mu(x,t) + 4\pi \int_{\Gamma} \left[ \frac{\partial \mu(y,t - |x - y|)}{|x - y|^2} + \frac{\partial \mu(y,t - |x - y|)}{|x - y|} \right] dS_y, \quad x \in \Gamma^c. \quad \text{(3)}$$

To enforce the boundary condition on $\Gamma \times [0,\infty)$, we take the limit of this representation from the exterior of $D$, yielding the integral equation

$$\left[ \frac{\mu(x,t)}{2} + N\mu(x,t) + K(a(y)\mu + b(y)\mu)(x,t) \right] = g(x,t), \quad \text{(4)}$$

for $(x,t) \in \Gamma \times [0,\infty)$. Here $N$ is the weakly singular principal part of the double layer restricted to $\Gamma \times \Gamma$, and $K$ is the single layer on $\Gamma \times \Gamma$.

There is a substantial literature on using time-domain integral equations for the solution of scattering problems, which we do not seek to review here. (See, for example, [3, 4, 11, 6, 5, 10] and the references therein.) We simply note that, while prior work has considered the use of the single layer alone, the double layer alone, as well as linear combinations of the two, there is relatively little discussion of a principled analytic approach for selecting one representation over another.

In many physical circumstances the solution to the wave equation in the exterior domain is known to decay exponentially in time, at least at fixed spatial locations. If the disturbance $g(y,t)$ has compact support in time and $\Gamma$ is a non-trapping obstacle, then classical results from the scattering theory of Lax, Morawetz, and Phillips [8] show that the solution, $u(x,t)$ to (1) will eventually exhibit local exponential decay. Non-trapping is a hypothesis about the behavior of reflected light rays in the exterior of $D$ that is always satisfied if $D$ is convex or star-shaped.

Nonetheless, the source density $\mu(x,t)$ obtained by many numerical methods based on boundary integral equations does not itself decay exponentially. For large times, the accuracy of the map from $\mu$ to $u(x,t)$ must therefore rely upon catastrophic cancellations. Our focus in this article is to find combinations of the single and double layer potentials that are guaranteed to produce exponentially decaying sources when the solution itself is known to exhibit such decay.

Ideally, the source $\mu(y,t)$ should display the same rate of exponential decay in time as the solution itself. If this is not the case, then the mapping from $\mu \mapsto u(x,t)$ in (3) must, after a certain time, involve some catastrophic cancellation, and hence can be expected to lose relative accuracy as time progresses. This goal can be achieved exactly for round spheres. For more general non-trapping obstacles, we show (modulo a technical hypothesis) that certain choices of $(a,b)$ lead to sources that decay exponentially, though perhaps not at the same rate as $u$ itself. As is shown in Section 5, in the smooth, strictly convex case, microlocal analysis indicates that the choice $a = 1$ should be be optimal.

Our analysis relies upon taking the Fourier transform in the time variable and using recent estimates for the kernel of the CFIE, $D_k - i(ku + ib)S_k$, done by Chandler-Wilde and Monk, and Baskin, Spence, and Wunsch, see [1, 2]. The results
of [1] and our own also rely on several modern improvements to the classical results in scattering theory alluded to above. This analysis occupies Sections 2–3. In this case the problem splits into the analysis of a single 1-dimensional integral equation for each spherical harmonic subspace. We conclude this section with some numerical examples illustrating these results. In particular the form of the equations directly identifies \( a = 1 \) as a distinguished choice and also indicates that choosing \( b > 0 \) eliminates a pole at \( k = 0 \). We illustrate the dramatic differences in the long time behavior of the density for different choices of \( a \) and \( b \). These range from oscillation or linear growth if the single layer (\( a = b = 0 \)) is used to the desired exponential decay for \( a = 1 \) and \( b > 0 \). Asymptotic analysis for high frequency, presented in Section 5, indicates that the choice \( a = 1 \) is natural for all convex obstacles. Finally in Section 5 we consider the high frequency behavior of the CFIE, \( D_k - i(ka + ib)S_k \), when \( \Gamma \) is the boundary of a smooth convex domain. In this context we show that choosing \( a = 1 \) provides optimal cancellation for the leading part of the “delay” term as \( k \to \infty \).

2. Time-harmonic analysis. Recall that

\[
g_k(x) = \frac{e^{ik|x|}}{4\pi |x|},
\]

is the outgoing fundamental solution for \((\Delta + k^2)\), with the corresponding single and double layer potentials defined by:

\[
S_k f(x) = \int_{\partial D} g_k(x-y)f(y) dS_y, \quad D_k f(x) = \int_{\partial D} \partial_n g_k(x-y)f(y) dS_y.
\]

(6)

With respect to the complex linear pairing, \( \langle u, v \rangle = \int_{\partial D} uv dS \), the single layer \( S_k \) is self-dual and

\[
D'_k f(x) = \int_{\partial D} \partial_n g_k(x-y)f(y) dS_y.
\]

(7)

If we take the Fourier transform in time:

\[
\hat{\mu}(x, k) = \int_0^\infty \mu(x, t)e^{itk} dt,
\]

(normalized so that it has an analytic extension to \( \text{Im} k > 0 \)), then equation (4) becomes

\[
\left[ \frac{\hat{\mu}(x, k)}{2} + D_k \hat{\mu}(x, k) - iS_k[a(y)k\hat{\mu} + ib(y)\hat{\mu}](x, k) \right] = \hat{g}(x, k).
\]

(9)

As before the limit is taken from the exterior of \( D \). Here and in the sequel we assume that \( g(x, t) = 0 \) for \( t \leq 0 \); by causality \( \mu(x, t) \) can also be taken to vanish for \( t \leq 0 \).

The combined field operators, acting on functions on \( \Gamma \), given by

\[
A(a, b; k) = \frac{I}{2} + D_k - iS_k[a(y)k\hat{\mu} + ib(y)\hat{\mu}],
\]

(10)

are well known to be Fredholm operators of second kind, provided that \( \Gamma \) is at least \( C^1 \). For functions \( a(y) > 0 \) and \( b(y) > 0 \) for all \( y \in \Gamma \), it can be shown, by a simple integration by parts argument, that \( A(a, b; k) \) is invertible for \( k \) in the closed upper
half plane. To prove this we use that the dual of \( A(a, b; k) \) with respect to the complex linear pairing is

\[
A'(a, b; k) = \frac{I}{2} + D_k' - i(a(y)k + ib(y))S_k.
\]  

Suppose that for some \( \text{Im } k \geq 0 \), there is a non-trivial solution to \( A(a, b; k)f = 0 \). By the Fredholm theory this means that there is also a non-trivial solution \( h \) to \( A'(a, b; k)h = 0 \). Let

\[
u(x) = S_kh(x).
\]

We have occasion to consider the function \( u(x) \) in both the interior of \( D \) and its exterior. To make this distinction clear, we refer to the corresponding restrictions of \( u \) by \( u_- \) and \( u_+ \), respectively.

Observe now that \( u_- \neq 0 \). If it were, then \( u_+ \mid_{\partial D} = 0 \) as well. In this case \( u_+ \) would be a solution to \( (\Delta + k^2)u_+ = 0 \) with vanishing Dirichlet data, satisfying the Sommerfeld radiation condition and it would therefore also be zero. The standard jump condition,

\[
\partial_n u_+ - \partial_n u_- = h,
\]

would then imply that \( h \equiv 0 \). Thus, when \( \text{Im } k \geq 0 \), this gives a non-trivial solution in \( D \) to

\[
(\Delta + k^2)u_- = 0 \quad \text{with} \quad \partial_n u_-(x) - i(ka(x) + ib(x))u_-(x) = 0.
\]

The boundary condition is equivalent to the fact that \( A'(a, b; k)h = 0 \).

Using the boundary condition, and integration by parts we obtain:

\[
0 = \int_D (\Delta u_- + k^2u_-)\bar{u}_- \, dx = \int_D [k^2|u_-|^2 - |\nabla u_-|^2] \, dx + \int_{\partial D} \partial_n u_- \bar{u}_- \\
= \int_D [k^2|u_-|^2 - |\nabla u_-|^2] \, dx + i\int_{\partial D} (ka(x) + ib(x))|u_-|^2 \, dS.
\]

Let \( k = k_1 + ik_2 \), and take the real and imaginary parts to see that this implies

\[
k_1 \left[ 2k_2 \int_D |u_-|^2 \, dx + \int_{\partial D} a(x)|u_-|^2 \, dS \right] = 0 \quad \text{and} \quad \int_{\partial D} (k_2^2 - k_2^2)|u_-|^2 \, dx - \int_{\partial D} (k_2a(x) + b(x))|u_-|^2 \, dS = 0.
\]

The first relation shows that if \( k_1 \neq 0 \) and \( k_2 > 0 \), then \( u_- = 0 \) as well. If \( k_1 = 0 \) and \( k_2 \geq 0 \), then the second relation shows that \( u_- = 0 \). If \( k_2 = 0 \), but \( k_1 \neq 0 \), then the first relation implies that \( u_- \mid_{\partial D} = 0 \); the boundary condition in (14) then implies that \( \partial_n u_- = 0 \) as well. It then follows from Green’s formula that \( u_- = 0 \). This proves the following basic theorem:

**Theorem 2.1.** If \( a(x) \) and \( b(x) \) are positive functions defined on \( \partial D \), a compact \( C^1 \)-surface, then the operators \( A(a, b; k) \) and \( A'(a, b; k) \) are invertible for \( k \) with \( \text{Im } k \geq 0 \).

**Remark 1.** If \( b = 0 \), then \( A(a, b; 0) \) has a non-trivial nullspace. Such a nullspace would generically destroy any possibility for exponential decay in the source function \( \mu(x, t) \). In Kress’ early paper [7] on the combined field operator, he shows that the optimal result (for the disk) when \( k \) is close to zero results from taking \( b =
−\frac{i}{2} + O(k^2) in our notation. This is somewhat at odds with our choice to take \( b \) real and positive.

Careful examination shows that Kress’ choice only works if \( k_1 \geq 0 \) : In the set \( k_2 < |k_1| \) the real part of the quadratic form in (15) is indefinite (regardless of the values that \( a \) and \( b \) take) and so to obtain the desired result we need to employ the imaginary part of the quadratic form. If \( b = b_1 + ib_2 \), then this form would become

\[
 k_1 \left[ 2k_2 \int_D |u_-|^2 \, dx + a \int_{\partial D} |u_-|^2 \, dS \right] - b_2 \int_{\partial D} |u_-|^2 \, dS. \tag{17}
\]

In order for the term in the brackets to be definite where \( b_2 > 0 \), we need to take \( a > 0 \). In order for this expression to be definite in both components of the set \( 0 < k_2 < |k_1| \) it is clearly necessary to take \( b_2 = 0 \). Of course one could use functions of \( k \) more complicated than \( ak + ib \) as the coefficient of \( S_k \), but this would considerably complicate the relationship between the solutions in the frequency and the time domains.

3. Scattering theory. Let \((a(x), b(x))\) be positive functions defined on \( \partial D \). In the recent paper of Baskin, Spence, and Wunsch; see Theorem 1.10 of [1], it is shown that, in 2 or 3 dimensions, there is a positive number \( \beta_1 \) so that the operator \((\Delta + k^2)u\) acting on data in \( D \) that satisfies the boundary condition \((\partial_n u - i(a(x)k + ib(x))u = 0\), is invertible for Im \( k > -\beta_1 \). This result does not assume that \( \partial D \) is non-trapping. Using this result, along with modern refinements of theorems of Lax and Phillips and Lax, Morawetz and Phillips [8, 9], we prove the following result:

**Theorem 3.1.** Let \( D \) be a non-trapping region in \( \mathbb{R}^3 \). If \( a \) and \( b \) are both positive, then there is a positive number \( \alpha \) so that \( A(a,b;k) \) is invertible in Im \( k > -\alpha \).

**Proof.** The facts we use, in addition to the result of [1] are

1. If \( D \) is a non-trapping region, then \( B \), the generator of \( Z(t) \), the compressed wave-semigroup of Lax and Phillips, has its spectrum, \( \sigma(B) \), in a lower half plane of the form Im \( k \leq -\beta_0 < 0 \).

2. A number \( k \) belongs to \( \sigma(B) \) if and only if there is an “eventually outgoing” solution \( v_+ \) to \((\Delta + k^2)v_+ = 0 \) with \( v_+ \big|_{\partial D} = 0 \). See Theorem V-4.1 in [9].

By definition (see [9] pp. 126-7) a solution is eventually outgoing (even for Im \( k < 0 \)) if and only if it can be represented in the form

\[
v_+(x) = \int_{\partial D} [\partial_{\nu_y} v_+(y) g_k(x - y) - v_+(y) \partial_{\nu_y} g_k(x - y)] \, dS_y, \tag{18}
\]

with \( \nu_y = -n_y \), the outer normal to \( D^c \). See Theorem IV-4.3 in [9]. Here \( g_k(x) \) is the “outgoing” fundamental solution for \( \Delta + k^2 \) in \( \mathbb{R}^{2n+1} \).

We argue as before: if for some \( k \), with Im \( k < 0 \), the operator \( A(a,b;k) \) has a non-trivial null-space, then so does \( \tilde{A}'(a,b;k) \). Let \( \varphi \neq 0 \) be in this latter nullspace and set

\[
u = \int_{\partial D} \varphi g_k(x - y) \, dS_y. \tag{19}
\]

Unlike the case where Im \( k \geq 0 \), we do not know, a priori, that \( u_- \neq 0 \). Indeed, there are just the two cases to consider: \( u_- = 0 \) or not. If \( u_- \neq 0 \), then \( u_- \) is a non-trivial solution to the boundary value problem considered in [1], and therefore Im \( k \leq -\beta_1 \).
The other possibility is that \( u_− = 0 \). This implies that \( u_+ \mid_{\partial D} = 0 \) as well, since \( u \) is continuous across \( \partial D \). However \( u_+ \) need not be zero, since it does not satisfy the Sommerfeld radiation condition, and is, in fact, exponentially growing at infinity. Indeed, if \( \varphi \neq 0 \), and \( u_− = 0 \), then the jump condition, (13), for \( \partial_n u_\pm \) shows that \( u_+ \) cannot be zero. Because \( u_+ \mid_{\partial D} = 0 \), and \( \partial_n u_+ = -\varphi \), it follows that \( u_+ \) satisfies (18) and therefore that \( u_+ \) is an eventually outgoing solution in the sense of Lax and Phillips. Hence \( k \) must belong to the spectrum of \( B \), and therefore \( \text{Im} \, k \leq -\beta_0 \). This shows that if \( A(a, b; k) \) is non-invertible, then \( \text{Im} \, k \leq \max \{ -\beta_1, -\beta_0 \} = -\alpha \).

The converse is also true.

**Theorem 3.2.** If \( k \) is a scattering pole for the Dirichlet Laplacian, then \( A(a, b; k) \) has a non-trivial null-space.

**Proof.** The theorem of Lax and Phillips states that \( k \) is a scattering pole if and only if there is an outgoing solution to the boundary value problem

\[
(\Delta + k^2)u = 0 \text{ in } D^c \text{ and } u \mid_{\partial D} = 0; \tag{20}
\]

the outgoing condition in this case is equivalent to

\[
u(x) = -\int_{\partial D} g_{k}(x - y)\partial_n u(y) dS_y \text{ for } x \in D^c. \tag{21}\]

If we let \( \varphi = \partial_n u \), then \( S_k \varphi \) vanishes on \( \partial D \) and

\[
\frac{\varphi}{2} - D_k' \varphi = \varphi, \tag{22}\]

from which it is immediate that

\[
A'(a, b; k) \varphi = 0, \tag{23}\]

which proves the claim. Thus every scattering resonance occurs among the set \( \{ k : \ker A(a, b; k) \neq 0 \} \), for any choice of positive \( a \) and \( b \).

**Remark 2.** These arguments apply generally to identify the frequencies for which the null-space of any combination of single and double layer potentials is non-trivial. The only cases not explicitly covered are those of the single and double layers alone. One easily establishes that \( S_k \) fails to be invertible for \( k^2 \) in the Dirichlet spectrum of \( D \) union with the scattering poles of the Dirichlet operator in \( \Omega \). The (exterior) double layer \( \text{Id}/2 + D_k \) fails to be invertible for \( k^2 \) in the Neumann spectrum of \( D \) union with the scattering poles of the Dirichlet operator in \( \Omega \).

In [1], Baskin, Spence, and Wunsch show that the resolvent kernel for the interior impedance problem satisfies the norm estimate

\[
\|R_{a,b}(k^2)\| \leq \frac{C}{1 + |k|} \text{ for } k \text{ in a strip around the real axis}. \tag{24}\]

In light of this it seems quite likely that there a constant \( m \) so that \( \|A(a, b; k)^{-1}\|_{L^2} \) satisfies a bound
of the form \( \| A(a, b; k)^{-1} \|_{L^2} \leq M(1 + |k|)^m \) in any smaller strip. We will return to this question in a later publication.

Suppose that \( g(x, t) \) is data for the wave equation on the boundary of \( D \) with support for \( t \in [0, T] \), and let

\[
\hat{g}(x, k) = \int_0^\infty g(x, t) e^{ikt} dt. \tag{25}
\]

The source term on the boundary as a function of \( t \) will be given by the inverse transform

\[
\mu(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikt} [A(a, b; k)]^{-1} \hat{g}(x, k) dk. \tag{26}
\]

If this is the case, then the corresponding time-domain representation of a solution to the wave-equation is

\[
u(x, t) = D\mu + S(a\partial_t + b)\mu, \tag{27}
\]

with \( S \) and \( D \) the retarded potential single and double layers for the wave equation.

Assume that for any \( \epsilon > 0 \) and \( \text{Im} \, k > -(\alpha - \epsilon) \), we have the estimate

\[
\| [A(a, b; k)]^{-1} \|_{L^2} \leq C_{\epsilon}(1 + |k|)^m.
\]

For \( g \) with sufficient smoothness in \( t \) we can deform the contour to conclude that, for \( \sigma < \alpha \), we have the representation:

\[
\mu(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i(k-i\sigma)} [A(a, b; k-i\sigma)]^{-1} \hat{g}(x, k-i\sigma) dk, \tag{28}
\]

and therefore

\[
\| \mu(\cdot, t) \|_{L^2} \leq C e^{-\sigma t} \| g \|_m, \tag{29}
\]

for a conveniently defined norm, \( \| \cdot \|_m \). This is essentially what we wanted to prove.

Of course this may not give the “optimal” rate of decay. It should be noted that the rate of decay \( \alpha \) is directly related to two spectral invariants, one being the genuine scattering resonances, and the other the spectrum of the boundary value problem for \( \Delta \) on \( L^2(D) \) with impedance boundary condition

\[
\partial_n u - i(ak + ib)u = 0 \quad \text{on} \quad \partial D. \tag{30}
\]

At least near the negative imaginary axis, we can push this down into the lower half plane so that the first eigenvalue encountered is a genuine scattering resonance. This follows from a careful examination of the integration by parts argument given above.

Let \( a \) and \( b \) be positive constants. We assume that \( (\Delta u + k^2 u) = 0 \), and

\[
\partial_n u - i(ak + ib)u \big|_{\partial D} = 0. \tag{31}
\]

If we integrate by parts we see that, with \( k = k_1 + ik_2 \):

\[
0 = \int_D ((\Delta u + k^2 u) \bar{u} dx
\]

\[
= - \int_D |\nabla u|^2 + (k_2^2 - k_1^2)|u|^2 dx - (ak_2 + b) \int_{\partial D} |u|^2 dS + \tag{32}
\]
\[ ik_1 \left( 2k_2 \int_D |u|^2 \, dx + a \int_{\partial D} |u|^2 \, dS \right). \]

If \( k_1 \neq 0 \), then we can substitute from the imaginary part the fact that
\[ \int_{\partial D} |u|^2 \, dS = -\frac{2k_2}{a} \int_D |u|^2 \, dx \] into the real part to see that
\[ 0 = \int_D \nabla |u|^2 \, dx - \left[ a\left( k_1^2 + k_2^2 \right) + 2bk_2 \right] \int_D |u|^2 \, dx. \]

Evidently the quantity in the brackets must be positive, which implies that
\[ k_1^2 + \left( k_2 + \frac{b}{a} \right)^2 > \left( \frac{b}{a} \right)^2. \]

Thus the impedance boundary value problem has a pole free region lying in the disk minus the imaginary axis
\[ k_1^2 + \left( k_2 + \frac{b}{a} \right)^2 \leq \left( \frac{b}{a} \right)^2 \{ k_1 = 0 \}. \]

Along the imaginary axis it’s clear that \( k_2 < -\frac{b}{a} \) is needed for (32) to hold. A somewhat better result can be obtained by using the fact that there is an estimate of the form
\[ \int_{\partial D} |u|^2 \, dS \leq C_1 \left[ \int_D |\nabla u|^2 + |u|^2 \, dx \right], \]

which implies that, if \( k_1 = 0 \), then
\[ k_2 \leq \max\{ \tau : 1 + (a\tau + b)C_1 \leq 0 \text{ and } \tau^2 + C_1(a\tau + b) \leq 0 \}. \]

Taking \( b/a \) large, we can arrange to have the pole-free region encompass as much of the lower half plane as we want.

4. The round sphere. In this section we consider the case that \( \Gamma \) is equal to the unit sphere in \( \mathbb{R}^3 \). We take \( a \) to be a constant \( \alpha \) and \( b \) a constant \( \beta \). Since everything commutes with the action of the rotation group, we can analyze this problem one spherical harmonic subspace at a time. Each choice of \( (\alpha, \beta) \) leads to a single 1-dimensional integral equation for each spherical harmonic subspace; see (49). While the detailed spectral theories of these operators are by no means obvious, the roles played by \( \alpha \) and \( \beta \) are much clearer here than in the general case. In particular, we see that \( \alpha = 1 \) is a very good choice; we show in Section 5 that this remains true for a general convex body. We close this section by examining the consequences of different choices for these parameters in numerical experiments.

4.1. Integral equations on spherical harmonic subspaces. We need to analyze the action of the operator:
\[ G^{\alpha,\beta}(x,t) = \frac{\mu(x,t)}{2} + \frac{1}{4\pi} \int_{\Gamma} \frac{n_y \cdot (x-y)}{|x-y|^2} \left( \frac{\mu(y,\tau)}{|x-y|^2} + \frac{\partial_{\nu} \mu(y,\tau)}{|x-y|} \right) dS_y + \frac{1}{4\pi} \int_{\Gamma} \alpha \partial_{\nu} \mu(y,\tau) + \beta \mu(y,\tau) \frac{|x-y|}{|x-y|} dS_y, \]  

(39)

where \( \tau = t - |x-y| \), on data of the form \( \mu(y,t) = Y^m_n(y) f(t) \). Notationally it’s easier to study the general class of operators of the form:

\[ K(Y^m_n f)(x,t) = \frac{1}{4\pi} \int_{\Gamma} Y^m_n(y) f(t - |x-y|) k(x \cdot y) dS_y. \]  

(40)

Up to taking time derivatives of \( f \), all the operators we consider are of this form.

The key observation is that if we let \( x^\perp \) be a unit vector orthogonal to \( x \), then we can define a coordinate system \((\theta, \phi)\) on the unit sphere by setting

\[ y = \cos \theta x + \sin \theta R_\phi x^\perp. \]  

(41)

Here, \( R_\phi \) is a rotation through angle \( \phi \) about \( x \). Substituting, we see that

\[ K(Y^m_n f)(x,t) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi Y^m_n(\cos \theta x + \sin \theta R_\phi x^\perp) \]

\[ f(t - \sqrt{2(1 - \cos \theta)}) k(\cos \theta) \sin \theta d\phi d\theta. \]  

(42)

A simple calculation shows that

\[ \frac{1}{2\pi} \int_0^{2\pi} Y^m_n(\cos \theta x + \sin \theta R_\phi x^\perp) d\phi = Y^m_n(x) P_n(\cos \theta). \]  

(43)

Here \( P_n(z) \) is the degree \( n \) Legendre polynomials normalized so that \( P_n(\pm1) = (\pm1)^n \). Therefore we have that

\[ K(Y^m_n f)(x,t) = \frac{Y^m_n(x)}{2} \int_0^\pi P_n(\cos \theta) f(t - \sqrt{2(1 - \cos \theta)}) k(\cos \theta) \sin \theta d\theta. \]  

(44)

Let \( s = \sqrt{2(1 - \cos \theta)} \), and \( \tilde{k}(s) = \sqrt{2(1 - s^2)} k(s) \), to obtain:

\[ K(Y^m_n f)(x,t) = \frac{Y^m_n(x)}{2} \int_0^2 P_n(1 - s^2/2) f(t-s) \tilde{k}(1 - s^2/2) ds. \]  

(45)

To represent \( G^{\alpha,\beta} \) acting on data of this type we need to determine what \( \tilde{k} \) is for each of the terms in (39). The kernels are

\[ \frac{n_y \cdot (x-y)}{|x-y|^3} = \frac{x \cdot y - 1}{|2(1 - x \cdot y)|^2} \leftrightarrow \tilde{k}(s) = -\frac{1}{2} \]

\[ \frac{n_y \cdot (x-y)}{|x-y|^2} = \frac{x \cdot y - 1}{|2(1 - x \cdot y)|} \leftrightarrow \tilde{k}(s) = -\frac{\sqrt{2(1 - s)}}{2} \]

\[ \frac{1}{|x-y|} = \frac{1}{\sqrt{2(1 - x \cdot y)}} \leftrightarrow \tilde{k}(s) = 1 \]  

(46)
Acting on $Y_n^m f$ we get

$$G^{\alpha,\beta}(Y_n^m f)(x,t) =$$

$$\frac{Y_n^m(x)}{2} \left[ f + \frac{1}{2} \int_0^2 P_n(1-s^2/2) \left[ f(t-s)(-1) + \partial_s f(t-s)(s) - 2\alpha \partial_s f(t-s) + 2\beta f(t-s) \right] ds \right].$$

(47)

Note that we have replaced $\partial_t$ with $-\partial_s$. Integrating by parts with respect to $s$ gives

$$G^{\alpha,\beta}(Y_n^m f)(x,t) = Y_n^m(x) \left[ \frac{(1+\alpha)}{2} f(t) + (-1)^n \frac{(1-\alpha)}{2} f(t-2) - \frac{1}{4} \int_0^2 \left[ (2-2\beta) P_n(1-s^2/2) - s(s-2\alpha) P_n'(1-s^2/2) \right] f(t-s) ds \right].$$

(48)

Denote the 1-d operator in the brackets by $G_n^{\alpha,\beta} f$. If we let $\alpha = \beta = 1$, we get a very simple integral equation of the second kind to solve for $f$:

$$G_n^{1,1} f_n^m(t) = f_n^m(t) + \frac{1}{4} \int_0^2 s(s-2) P_n'(1-s^2/2) f_n^m(t-s) ds = g_n^m(t).$$

(49)

If $n = 0$, then this is simply $f_0^m(t) = g_0^m(t)$.

### 4.2. The roles of $\alpha$ and $\beta$

The equation

$$G_0^{\alpha,1} f(t) = \frac{(1+\alpha)}{2} f(t) + \frac{(1-\alpha)}{2} f(t-2) = g(t),$$

(50)

is quite informative as regards the rate of decay and regularity of the solution. Assuming that $0 < \alpha < 1$, we simplify this equation to obtain

$$f(t) + \frac{1-\alpha}{1+\alpha} f(t-2) = \frac{2}{1+\alpha} g(t).$$

(51)

If $g(t) = 0$ for $t < 0$, then we can let $f(t) = 0$, for $t < 0$ as well. The solution to this equation is formally given by the infinite series

$$f(t) = \frac{2}{1+\alpha} \sum_{j=0}^{\infty} (-\lambda)^j g(t-2j), \text{ with } \lambda = \frac{1-\alpha}{1+\alpha}.$$  

(52)

For data supported in $[0, \infty)$ this sum is finite for any $t$. In particular:

$$f(t) = \frac{2}{1+\alpha} g(t) \text{ for } t \in (-\infty, 2).$$

(53)

If $g$ is supported in $[0, 2M]$ then the sum can be made more explicit:

$$f(t) = \frac{2}{1+\alpha} \sum_{j=\lfloor \frac{t}{2} \rfloor}^{\lfloor \frac{t}{2} \rfloor + M} (-\lambda)^j g(t-2j).$$

(54)

Eventually $f$ satisfies an estimate of the form

$$|f(t)| \leq C_{M,\lambda} \left( \frac{1-\alpha}{1+\alpha} \right)^{\frac{t}{2}} ||g||_{L^\infty}.$$  

(55)
This shows how the choice of $\alpha$ affects the decay of the source term on the boundary. The $n = 0, \beta = 0$ case can be solved by using Laplace transform:

$$Lg(\tau) = \gamma_0^\alpha(\tau) Lf(\tau),$$

where

$$\gamma_0^\alpha(\tau) = \frac{1}{2} \left[ 1 + e^{-2s} + \left( \alpha - \frac{1}{s} \right) (1 - e^{-2s}) \right].$$

Note, however, that no matter what value $\alpha$ takes, this multiplier has a root at $s = 0$. This explains why we need to take $\beta > 0$ in order to get exponential decay.

4.3. **Numerical illustrations.** To provide concrete examples of the implications of our analysis, we have carried out representative numerical simulations in the case $n = 0$. We consider two choices for the Dirichlet data:

$$g(t) = 8 \sin (50t) \cdot e^{-40(t-1)^2} \quad \text{Oscillatory},$$
$$g(t) = 8e^{-40(t-1)^2} \quad \text{Non-oscillatory}.$$  

We also consider three choices for the parameters in the integral equation:

- $a = \alpha = 0, \quad b = \beta = 0.$
- $a = \alpha = 1, \quad b = \beta = 0.$
- $a = \alpha = 1, \quad b = \beta = \frac{1}{2}.$

(Note that the optimal choice for the sphere, $a = b = 1$, leads to a trivial equation when $n = 0$ so we avoid it.)

Our numerical method is based on the standard Adams predictor-corrector idea. Recall that for this special case the equation to be solved is

$$\mu(t) + \left( \frac{1 - \alpha}{1 + \alpha} \right) \mu(t - 2) - \left( \frac{1 - \beta}{1 + \alpha} \right) \int_{t-2}^{t} \mu(s) \, ds = \frac{2}{1 + \alpha} g(t).$$

Introducing a time step $\Delta t$ we rewrite in correction form:

$$\mu(t + \Delta t) = \mu(t) - \left( \frac{1 - \alpha}{1 + \alpha} \right) \left( \mu(t + \Delta t - 2) - \mu(t - 2) \right)$$
$$+ \left( \frac{1 - \beta}{1 + \alpha} \right) \int_{t}^{t+\Delta t} \mu(s) \, ds - \left( \frac{1 - \beta}{1 + \alpha} \right) \int_{t-2}^{t+\Delta t - 2} \mu(s) \, ds$$
$$+ \frac{2}{1 + \alpha} \left( g(t + \Delta t) - g(t) \right).$$

In the examples which follow we interpolate past solution data on an interval $(j\Delta t, (j+1)\Delta t)$ by the Lagrange polynomial of degree 5 using the approximate solution at $t = k\Delta t, k = j-3, \ldots, j+2$. Precisely the interpolant is used to calculate $\mu(t - 2), \mu(t + \Delta t - 2), \int_{t-2}^{t+\Delta t - 2} \mu(s) \, ds$ appearing on the right hand side of (59). The future integral uses the 6th order Adams-Bashforth-Moulton predictor-corrector method. That is, we compute a predicted value $\mu^{(p)}(t + \Delta t)$ using the degree 5 Lagrange interpolant of $\mu(t - k\Delta t), k = 0, \ldots, 5$ to approximate $\int_{t}^{t+\Delta t} \mu(s) \, ds$. We then compute $\mu(t + \Delta t)$ using the interpolant of $\mu^{(p)}(t + \Delta t)$ and $\mu(t - k\Delta t), k = 0, \ldots, 4$ inside the integral. We have verified that convergence at 6th order is generally obtained (in some experiments the observed rate was reduced to 5 when $\alpha = \beta = 0$) and we have also cross-checked the results with those computed using 2nd and 4th order solvers constructed the same way. The solutions displayed in the graphs were calculated with a time step $\Delta t = 97/6400$ for the non-oscillatory
data and $\Delta t = 97/12800$ for the oscillatory data. Comparisons with a coarsened computation suggest that the solutions are accurate to at least 7 digits in the non-oscillatory case and 4-6 digits when the data was oscillatory. Here the maximum errors in the oscillatory case are two orders of magnitude larger when $\alpha = \beta = 0$; looking at the solutions below we attribute the difference to dispersion effects.

Solutions of the integral equation for the choice $\alpha = \beta = 0$ are displayed in Figure 1. We see that in the case where the data is oscillatory $\mu$ does not decay but oscillates at high frequency. When the data is non-oscillatory we observe linear growth. In each case we note that the solution of the wave equation itself decays exponentially in time, so, as our analysis shows, the long time behavior of the density $\mu(t)$ is quite different than that of the function it represents.

![Figure 1](image1.png)

Figure 1: Density $\mu(t)$ at mode 0 with oscillatory data (left) and non-oscillatory data (right) for $\alpha = \beta = 0$.

Solutions of the integral equation for the choice $\alpha = 1$, $\beta = 0$ are displayed in Figure 2. We see that in the case where the data is oscillatory $\mu$ apparently decays in time. However a closer look indicates that it in fact approaches a very small steady state. When the data is non-oscillatory $\mu$ clearly approaches an $O(1)$ steady state, again in contrast with the solution of the wave equation itself. This behavior corresponds to the pole at the origin discussed above.

![Figure 2](image2.png)

Figure 2: Density $\mu(t)$ at mode 0 with oscillatory data (left) and non-oscillatory (right) for $\alpha = 1$, $\beta = 0$.

Lastly we consider the solutions which arise when $\alpha = 1$, $\beta = \frac{1}{2}$; these are displayed in Figure 3. Here in each case we have exponential decay of $\mu$ in time.
5. **High frequency asymptotics for convex bodies.** In the previous section we saw that, at least for round spheres, a principal determinant of the decay rate for the solution of the integral equations on the boundary, \( G^a,b f = g \), is the delay term \( \frac{1}{2} [(1+\alpha)f(t) + (-1)^n (1-\alpha) f(t-2t)] \). Choosing \( \alpha = 1 \) removes the sharp contribution from the antipodal point, and gives the optimal rate of decay available for this case. In this section we study the high frequency behavior of \( D_k - i(ka + ib)S_k \) for a smooth surface \( \Gamma \). In general one cannot expect to have the sort of exact cancellation attained for a round sphere, however we demonstrate the rather remarkable fact that for convex bodies, the leading order part of the delay term is optimally canceled by taking \( a = 1 \). This analysis does not elucidate the choice of \( b > 0 \), which is unsurprising as it is related to insuring invertibility of \( A(a,b;k) \) at zero frequency.

We begin by recalling the formulæ for the relevant kernels:

\[
S_k f(x) = \int_{\Gamma} e^{ik|x-y|} f(y) dS_y / \left(4\pi |x-y|^3\right),
\]

\[
D_k f(x) = \int_{\Gamma} e^{ik|x-y|} \langle (x-y), n_y \rangle \left[ \frac{1}{|x-y|} - ik \right] f(y) dS_y.
\] (60)

As is well known, these are weakly singular integral operators on a \( C^1 \) surface in \( \mathbb{R}^3 \).

Since we are interested in the high frequency asymptotics, the contribution of the diagonal singularity is of less interest to us than the that of the other critical points of the phase function

\[
\phi_x(y) = |x-y|.
\] (61)

For completeness we state the contribution from the diagonal:

\[
S_k^{\text{diag}} f(x) = -\frac{f(x)}{2ik} + O(k^{-2})
\]

\[
D_k^{\text{diag}} f(x) = \frac{H(x)f(x)}{ik} + O(k^{-2}),
\] (62)

where \( H(x) = \frac{1}{2}(\kappa_1(x) + \kappa_2(x)) \) is the mean curvature of \( \Gamma \) at \( x \) with respect to the outer normal vector \( n_x \). Observe that the diagonal asymptotics of the combined
field operator are given by:

\[ A^{\text{diag}}(a, b; k) f(x) = \frac{1 + a}{2} f(x) + (2H(x) - b) \frac{f(x)}{2ik} + O(k^{-2}) \]  

(63)

Now we turn to the other critical points of \( \phi_x \):

\[ C_x = \{ y \in \Gamma : \nabla_y \phi_x(y) = \pm n_y \} \setminus \{ x \} \]
\[ = \{ y \in \Gamma : x = y \mp \phi_x(y)n_y \} \setminus \{ x \}. \]  

(64)

Even for a strictly convex body, these critical points can be degenerate. For example, if \( \Gamma \) contains a degenerate critical point of \( \phi_x \) then there are also points \( x \) with different signatures. If this happens then there are also points \( x \) so that \( C_x \) contains degenerate critical points.

To apply the stationary phase formula (at least to leading order) we need only compute the Hessian of the phase function at \( 0 \); it is

\[ S_k^2 f(x) \sim \int \frac{e^{i k (|z|^2 + 2d h(z) + h(z)^2 + d^2)^{\frac{3}{2}}}}{4 \pi (|z|^2 + 2d h(z) + h(z)^2 + d^2)^{\frac{3}{2}}} f(z) \psi(z) dz. \]  

(66)

To apply the stationary phase formula (at least to leading order) we need only compute the Hessian of the phase function at \( 0 \); it is

\[ H_{xz}(0) = \frac{1}{d} \begin{pmatrix} 1 & \mp dh_{11}(0) & \mp dh_{12}(0) \\ \mp dh_{21}(0) & 1 & \mp dh_{22}(0) \end{pmatrix}. \]  

(67)

According to stationary phase we see that

\[ S_k^2 f(x) \sim \frac{e^{ik \text{sgn} H_{xz}(0)}}{2k \sqrt{\det H_{xz}(0)}} \frac{e^{ikd} f(x)}{d} + O(k^{-2}). \]  

(68)

Recall that \( \text{sgn} H_{xz}(0) \) is the signature of the Hessian, which is the number of positive eigenvalues minus the number of negative eigenvalues. For a maximum \( \text{sgn} H_{xz}(0) = -2 \). Even for strictly convex bodies there can be several critical points with different signatures. If this happens then there are also points \( x \) so that \( C_x \) contains degenerate critical points.

The calculation for the double layer is similar:

\[ D_k^2 f(x) \sim \int \frac{e^{i k (|z|^2 + 2d h(z) + h(z)^2 + d^2)^{\frac{3}{2}}}}{4 \pi (|z|^2 + 2d h(z) + h(z)^2 + d^2)^{\frac{3}{2}}} (h(z) - \nabla h(z) \cdot z \pm d) f(z) \psi(z) dz \times \]

\[ \left[ \left( \frac{1}{(|z|^2 + 2d h(z) + h(z)^2 + d^2)^{\frac{3}{2}}} - ik \right) dz. \right] \]  

(69)
Once again, applying the stationary phase formula we see that

$$D^k_{\hat{x}}f(x) \sim \frac{e^{\pi \mathrm{sign} \, \mathcal{H}_{\hat{x}}(0)}}{2\sqrt{\det \mathcal{H}_{\hat{x}}(0)}} \frac{\mp ie^{ikx} f(\hat{x})}{d} + O(k^{-1}).$$

(70)

As noted above, if $\Gamma$ is convex, then $x = (0,0,-d)$, and therefore, we see that

$$D^k_{\hat{x}}f(x) - ikS^k_{\hat{x}}f(x) = O(k^{-1}),$$

(71)

for all $x$ and non-degenerate critical points $\hat{x} \in \mathcal{C}_e$. Unless each $\mathcal{C}_e$ consists of a single maximum, for every $x \in \Gamma$, then, for some points $x \in \Gamma$, $\phi_x$ is sure to have degenerate critical points. If these are isolated and of finite degeneracy, then this difference is still likely to be $o(1)$ as $k \to \infty$. Of course if $\Gamma$ is not convex, then it is perfectly possible that we may sometimes get the $-$ sign in the asymptotics for $D^k_{\hat{x}}f(x)$, equation (70), so that we obtain contributions of the form:

$$D^k_{\hat{x}}f(x) - ikS^k_{\hat{x}}f(x) = -i(1+a) \frac{e^{\pi \mathrm{sign} \, \mathcal{H}_{\hat{x}}(0)}}{4\sqrt{\det \mathcal{H}_{\hat{x}}(0)}} \frac{e^{ik|x-\hat{x}|} f(\hat{x})}{|x-\hat{x}|} + O(k^{-1}).$$

(72)

Since $a > 0$ is needed to keep poles out of the upper half plane, this may prevent any source for the wave equation, defined by solving (4) with an allowable choice of $(a,b)$, from decaying exponentially.

**Acknowledgments.** We would like to thank Euan Spence and Jared Wunsch for several helpful conversations, and the referee for suggestions that improved the clarity of our paper and for the reference to Vainberg’s book.

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Received April 2015; revised October 2015.

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