On particle oscillations

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Abstract
It has been firmly established that neutrinos change their flavour during propagation. This feature is attributed to the feature that each flavour eigenstate is a superposition of three mass eigenstates, which propagate with different frequencies. This picture, although widely accepted, is wrong in the simplest approach and requires quite sophisticated treatment based on the wave-packet description within quantum field theory. In this communication we present a novel, much simpler explanation and show that oscillations among massive particles can be obtained in a natural way. We use the framework of quantum mechanics with time being a physical observable, not just a parameter.

Keywords: quantum mechanics, time evolution, particle oscillations, neutrino oscillations

(Some figures may appear in colour only in the online journal)

1. Introduction

All elementary particles are subject to mixing within their respective groups, i.e., quarks, neutral leptons (neutrinos), charged leptons, as well as gauge bosons. This peculiar feature of gauge theories underlying the Standard Model comes from the requirement that the quantum numbers should match those observed in nature. In other words, in order to arrive at a picture consistent with the experiment, one has to ‘rotate’ sectors of the Standard Model, with the rotation parameters fitted from the experimental data. As a consequence of mixing, particles should oscillate between their possible states, as is observed for neutrinos and some mesons.

However, from the theoretical point of view, not everything is clear in this picture. Take neutrino oscillations as an example. The widely accepted explanation is based on the assumption that neutrinos which are paired with the charged leptons ($e$, $\mu$, $\tau$) are not the same as the propagating neutrinos. From the so-called interaction basis, the interaction eigenstates $\nu_\alpha$, $\alpha = e$, $\mu$, $\tau$ have to be rotated by a unitary matrix to the physical basis, in which the states $\nu_i$, $i = 1$, 2, 3, are those which propagate. The latter are physical particles with well-defined masses, while the former are ill-defined, and therefore virtual particles.

The immediate question which arises is that of why we have to work in two bases—one for describing interactions (with unrealistic particles with no definite mass) and another for describing propagation (of physical particles which are not observed in nature as stand-alone objects). This counter-intuitive picture causes even more trouble when one wants to formulate a consistent description of, say, neutrino oscillations. Even assuming that in the process of emission three different physical particles are produced, each with well-defined mass, momentum, and energy, it is difficult to justify how these particles can arrive at the detection point as a single, detectable object. To resolve this problem, different authors have argued that these particles should share a common momentum or a common energy. Curiously, these two approaches lead to the same final expression for the phase of the oscillations. The same expression can also be reached when assuming nothing but neutrinos being ultra-relativistic [1]. The most correct derivation of the neutrino oscillations phase involves a full wave-packet treatment within quantum field theory [2].
In this communication we propose a different mechanism which leads to particle oscillations. Without referring to two different classes of states and working with the physical particles only, we show that under certain assumptions transitions between mass eigenstates can be observed. Our framework is the quantum mechanics in which time is no longer a parameter but now one of the space–time variables.

2. The model

Recent experimental progress in the field of quantum mechanics suggests that the ordinary formulation is not enough for properly describing what is being observed. In the so-called delayed-choice experiments [3–5] the cause and consequence seem to be inverted in time, implying that either causality is violated or our understanding of quantum phenomena should be altered. Also the newest experiments involving entangled systems [6] led to the conclusion that within the traditional framework of quantum mechanics and special relativity, superluminal communication between different parts of the system is observed unless we change some basic principles in the formalism. Only recently, an entangled system of two photons that never coexisted in time has been created [7]. Another example is the observation of interference fringes [8] which are in agreement with the hypothesis that the wavefunctions interfere in time, not in spatial variables.

With motivation coming from this line of research, a new quantum theory seems desirable, and one such model has been proposed in [9]. One of its main features is the inclusion of time as an observable, such that it is possible to consistently construct a time operator [10]. Consequently, no time evolution of the wavefunction, which is space as well as time dependent, is needed. Each measurement is represented by a projection of the wavefunction on the states of a properly constructed ‘detector’, according to the Dirac projection postulate. This model successfully described such phenomena as arrival time [11], delayed choice in quantum mechanics [9] and interference in time [12, 13].

In this communication we outline the description of the oscillations of mass eigenstates, which ultimately can be used to describe neutrino oscillations.

3. Particle oscillations

Keeping in mind neutrinos as our primary example, we want to show that after emitting a particle of a certain mass, another particle of a close lying mass can be observed. This can all happen under the assumption that these particles share most if not all other properties, i.e., spin, electric charge etc.

We divide the process of describing particle oscillations into three stages:

1. the emission (creation) of the particle denoted by \( \tau_1 \),
2. propagation of the particle; for the sake of simplicity we assume free propagation and denote this stage by \( \tau_2 \),
3. detection of the particle (\( \tau_3 \)).

According to the general rules given above, one has to construct projection operators describing each stage, and project the initial wavefunction of the emitted particle subsequently using the appropriate operators. The final outcome will represent the probability density for detecting the particle. Denoting by \( \psi \) the initial wavefunction, the density matrix after the first stage is given by

\[
\rho_1(\tau_1) = |\psi\rangle \langle \psi|.
\]  

At the second stage the state changes into \( \rho_2 \) given by

\[
\rho_2(\tau_2) = \frac{\mathbb{E}(\tau_2)\rho_1(\tau_1)\mathbb{E}(\tau_2)}{\text{Tr}\left[\mathbb{E}(\tau_2)\rho_1(\tau_1)\mathbb{E}(\tau_2)\right]},
\]  

where \( \text{Tr} \) denotes the trace, which provides proper normalization of the expression, \( \mathbb{E}(\tau_2) \) is the projection operator which here describes the propagation, and \( \rho(\tau_2) \) is given by equation (1). The detection process introduces yet another operator \( \mathbb{E}(\tau_3) \) which defines the detector and acts according to

\[
\rho_3(\tau_3) = \frac{\mathbb{E}(\tau_3)\rho_2(\tau_2)\mathbb{E}(\tau_3)}{\text{Tr}\left[\mathbb{E}(\tau_3)\rho_2(\tau_2)\mathbb{E}(\tau_3)\right]}.
\]  

Now, the probability of the process \( \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \) is given by

\[
\text{Prob}(\tau_1 \rightarrow \tau_2 \rightarrow \tau_3) = \text{Tr}\left[\mathbb{E}(\tau_3)\mathbb{E}(\tau_2)\rho_1(\tau_1)\mathbb{E}(\tau_2)\mathbb{E}(\tau_3)\right].
\]  

To be more specific, let us assume three members of a family of particles having very similar masses. In the case of neutrinos, the mass differences are of the order of 10 meV or less which indicates really close lying states. Assume further that one of these particles is being created in a reaction. In the usual approach one neglects the time that this reaction takes. This implies further that the particle produced appears immediately in zero time, with sharply defined mass. On the other hand, if one takes into account that the reaction time is non-zero and finite, this introduces a kind of uncertainty in the time for the particle to be produced, which results in a broadening of its energy profile. Thus, the particle created is no longer sharply peaked in mass, but possesses also an
uncertainty in this parameter. One may also justify the broadening of mass of a particle in a more formal way. Namely, in our model the mass operator does not commute with the time operator. Therefore a kind of uncertainty relation between mass and time can formally be given, which prevents a sharply peaked mass distribution from appearing in any finite time.

The broadened mass profile overlaps with the neighbouring mass states, effectively turning into a linear combination of states with different masses, with the “mixing parameters” given by the values of the function describing the profile (see figure 1 for a graphical representation).

Let us work, without loss of generality, in two dimensions: one time and one spatial variable (t,x). It follows that the wavefunction of the emitted particle may be written in the form
\[
\psi(\tau; t, x) = \int d^2k \ a_M(k_0, k_1) \ \eta^*_\ell(t, x),
\]
where \( \ell \) denotes the first stage of the process, \( \eta^*_\ell(t, x) = \eta^*_{\ell_0}(t) \ \eta^*_{\ell_1}(x) = \exp(-i k_0 t) \ \exp(-i k_1 x)/2\pi, \) and the function \( a_M(k_0, k_1) \) of the 2-momentum describes the shape of the particle in the momentum space. We notice for future reference that \( k_0^2 - k_1^2 = m^2, \) so it is possible to change the integration variables from \( (k_0, k_1) \) to \( (m^2, k_1) \).

The rules for the free propagation of the particle are governed by the structure of the vacuum. From this point of view the vacuum cannot distinguish the broadened state \( \psi(\tau) \) from three separate mass states propagating together. We therefore construct the projection operator describing the propagation as
\[
\mathcal{H}(\tau_2)\psi(\tau_1; t, x) = \int d^2k \ \left\{ \eta^*_{\ell_1}(t, x) \right\} \times \eta^*_\ell(t, x),
\]
where \( \Delta_\ell \) is the set of 2-momenta that can be transmitted through the vacuum during propagation. This, after a variable change \( (k_0, k_1) \rightarrow (m^2, k_1) \), using the mass-shell relation, turns into a disjoint set of narrow peaks around \( m_1, m_2, \) and \( m_3 \). We therefore assume that the structure of the vacuum permits propagation of some chosen set of masses, which defines our Standard Model. Evaluating equation (6) using (5) one gets
\[
\mathcal{H}(\tau_2)\psi(\tau_1; t, x) = \int d^2k \ a_M(k_0, k_1) \eta^*_\ell(t, x).
\]

Finally, let us denote the wavefunction of the detector by \( \phi(\tau; s, X) \). The detector is two dimensional (one time and one spatial dimension) and located at the space–time point \( (s, X) \). Notice, that by construction the measurement is time dependent. The projection operator representing the detection is now
\[
\mathcal{E}(\tau_1) = |\psi(\tau_1; s, X)\rangle \langle \psi(\tau_1; s, X)|.
\]

If we want to distinguish the detected particles by their masses, the detector function \( \phi \) should describe states with definite mass, or should be peaked around some mass in the mass–momentum space. One such example is provided by the infinite potential well described in the next section.

3.1. An example: the infinite potential well
Let us represent the detector as eigenfunctions of the Klein–Gordon equation in a two-dimensional infinite potential well \( \phi_{m(x)}^{(\infty)}(\tau_1; t, x) \). Denote its dimensions and localization by \( L_0 \times L_1 \) with the central point \( (s + L_0/2, X + L_1/2) \), i.e., it is a rectangle with the closer corner given by the space–time point \( (s,X) \), extending by \( L_0 \) in the time direction and by \( L_1 \) in the spatial direction. The detector has to be tuned to detect a certain mass \( m_\ell \) given by
\[
(m_\ell)^2 = \frac{n_0^2}{L_0^2} - \frac{n_1^2}{L_1^2},
\]
where \( n_0, n_1 \) signify different modes of the wavefunction within the well. The probability of detection is in this case given by
\[
\text{Prob}(\tau_1 \rightarrow \tau_2; s, X) = \sum_{n_0, n_1} \left| \int d^2k \ \langle \eta^*_{n_1}(t, x) \rangle \right|^2.
\]
Working this example out explicitly, the full formula reads
\[
\text{Prob}(\tau_1 \rightarrow \tau_2; s, X) = \mathcal{N} \sum_{n_0, n_1} (n_0 n_1)^2 \left| \int d^2k \ a_M(m^2, k) \exp\left\{-i \left[ \sqrt{m^2 + k^2} L_0 (s + 1) + k (X + L_1) \right] \right\} \times \sin\left(\sqrt{m^2 + k^2} L_0/2 - n_0 \pi/2\right) \sin\left(k L_1/2 - n_1 \pi/2\right) \right|^2,
\]
where we have changed the variables to integrate over masses squared. Here \( \mathcal{N} \) is an overall normalization factor, and we recall that in normal units all masses should be read as \( m/(\hbar c) \).

The region \( \Delta_{m(x)} \) consists of three narrow peaks around the masses \( m_1, m_2, \) and \( m_3 \). One may therefore simplify the

\[\text{We use natural units: } \hbar = c = 1.\]
integral over $m^2$ and substitute it by a sum

$$\int_{\mathcal{A}[m^2]} d(m^2) F(m^2, k) \rightarrow \sum_{j=1,2,3} (4m_j\delta_j) F(m_j^2, k)$$

where $\delta_j$ is the (common, for simplicity) width of the peaks around the masses $m_j$, characteristic for the second stage $\tau_j$. This shows clearly that in the final formula (after the modulus squared is applied), interference terms involving different masses $m_j$ will appear, leading to possible oscillations. We have shown therefore that the emission of one mass results in a non-zero probability of detection of another mass. This probability is some function of the localization of our model in a non-zero probability of detection of another mass. We propose in this example the following explicit form of the profile function:

$$a_{n_0^2}(m^2, k) = \cos \left( \left( m^2 - m_0^2 \right) \frac{\pi}{\delta} \right) \text{rect}_w(m^2 - m_0^2)$$

$$\times \text{rect}_w(k),$$

where

$$a_{n_0^2}(m^2, k) = \cos \left( (k_0^2 - k^2 - m_0^2) \frac{\pi}{\delta} \right)$$

$$\times \text{rect}_w(k_0^2 - k^2 - m_0^2) \text{rect}_w(k).$$

Figure 2. Plot of the profile function $a_{n_0^2}(m^2, k)$, equation (12), for $m_0^2 = 5$, $\delta = 2$, $W=2$.

Here the box function $\text{rect}_w(x)$ is a rectangle of width $w$ and height 1, centred around 0 in the $x$ variable. The distribution (12) gives a cosine-shaped smearing of width $\delta$ in the masses squared around $m_0^2$, and a flat smearing in spatial momenta of width $W$. This choice is physically reasonable. The function $a_{n_0^2}(m^2, k)$ is depicted in figure 2 for a certain choice of the width parameters.

We further assume that the process of detection is in fact quite similar to the process of emission. For the case of neutrinos, we expect them to be created and detected in a weak process, so the interaction vertices in the Feynman diagrams are similar (e.g., a beta and an inverse-beta decay). Therefore we define the initial wavefunction and the detector wavefunction in similar ways, i.e.,

$$\psi(t, x) = \int_{\mathcal{A}[m^2]} d^2 k \ a_{n_0^2}(k) \ \eta^*_{n_0}(t, x),$$

$$\phi(t, x) = \int_{\mathcal{A}[m^2]} d^2 k \ a_{n_0^2}(k) \ \eta^*_{n_0}(t, x),$$

$m_n$ and $m_p$ being the emitted and detected mass, respectively. Now, the detector needs to be shifted from the origin to the point $(s, X)$, resulting in

$$\phi(t, x) \rightarrow \phi(t - s, x - X) = e^{i k_0} e^{i k X} \phi(t, x).$$

Finally, we arrive at the following formula for the probability of detection:

$$\text{Prob}(s, X) = N \int_{\mathcal{A}[m^2]} d^2 k \int_{\Delta[m^2]} d(m^2)$$

$$\left| e^{i k_0} e^{i k X} a_{n_0^2}(m^2, k_1) a_{n_0^2}(m^2, k_1) \right|^2,$$

where $N$ is an overall normalization factor and the functions $a$ are given by equation (12). The integration range $\Delta[m^2]$ consists of three narrow peaks around the masses $m_{1,2,3}$. Assuming the width of the peaks $\delta_n$ to be small, one may approximate this integration by taking the value of the integrand at the central points times the width of the peaks, which

Figure 3. Density plot of the probability $\text{Prob}(s, X)$ for $m_n = m$, and $m_p = m$. A clear maximum is visible for $s=X$. 4. The truncated cosine distribution

To better control which mass is emitted, let us write down the initial profile $a_n(k_w, k_0)$ as $a_{n_0^2}(m^2, k_0)$, with $m_0$ being the central value of the emitted mass. We propose in this example the following explicit form of the profile function:

$$a_{n_0^2}(m^2, k) = \cos \left( \left( m^2 - m_0^2 \right) \frac{\pi}{\delta} \right) \text{rect}_w(m^2 - m_0^2)$$

$$\times \text{rect}_w(k_0),$$

which in the momentum space is given by

$$a_{n_0^2}(k_w, k_0) = \cos \left( (k_0^2 - k^2 - m_0^2) \frac{\pi}{\delta} \right)$$

$$\times \text{rect}_w(k_0^2 - k^2 - m_0^2) \text{rect}_w(k).$$

Here the box function $\text{rect}_w(x)$ is a rectangle of width $w$ and height 1, centred around 0 in the $x$ variable. The distribution (12) gives a cosine-shaped smearing of width $\delta$ in the masses squared around $m_0^2$, and a flat smearing in spatial momenta of width $W$. This choice is physically reasonable. The function
yields
\[ \text{Prob}(s, X) = N \left[ \int_{\mathbb{R}} dk_i \sum_{j=1}^{3} (4m_\delta) \frac{e^{i \left( m_i^2 + m_j^2 \right) X}}{m_i^2 + k_i^2} a_{m_j^2} \left( m_j^2, k_i \right) a_{m_i^2} \left( m_i^2, k_i \right) \right]. \]  

(18)

Notice that both \( m_0 \) and \( m_D \) are among the masses \( m_j \).

First of all let us check the overall behaviour of the formula (18). We present a 3D plot of the values of Prob as a function of the localization \((s, X)\) of the detector in figure 3. This is a neutrino-inspired example with the masses given by (14):

\[
\begin{align*}
  m_1 &= 0.1 \text{ eV}, \\
  m_2 &= m_1 + \Delta m_{12}^2, \\
  m_3 &= m_1 + \Delta m_{13}^2, \\
  \Delta m_{12}^2 &= 7.6 \times 10^{-5} \text{ eV}^2, \\
  \Delta m_{13}^2 &= 2.5 \times 10^{-3} \text{ eV}^2.
\end{align*}
\]

The smearing parameters were chosen as
\[
\delta_0 = \delta_\delta = 1.1 \times \Delta m_{12}^2, \quad \delta_1 = 0.001 \times \Delta m_{13}^2.
\]

We first notice that there is a strong maximum of the detecting probability along the \( s = X \) line. This indicates that the particles propagate with some maximum speed which corresponds approximately to the speed of light (in our units \( c = 1 \)). So the choice of small masses implies the particles being ultra-relativistic. Another feature is that the probability is exactly zero until the fastest particles reach the point \( s = X \) (lower light triangle), but remains non-zero for later times, as slower particles (or tails of the wavefunctions of particles that have already passed this point) may still be detected. This presents a physically consistent picture of particle propagation. Notice that in figure 3 the case of \( m_0 \neq m_D \) is demonstrated.

A detailed analysis is shown in figure 4, in which we present the shapes of the detection probability functions for time \( s = 20 \). We include the probabilities of detection of \( m_1 \) or \( m_3 \), when \( m_1 \) or \( m_3 \) is emitted. The curve describing \( m_2 \) will be lying between these two. The probability is not normalized, so the units on the vertical axes are arbitrary. One clearly sees that the probability shape is strongly peaked around \( x = 20 \), which corresponds to the choice of \( s \), and vanishes to zero for greater distances. Also, the maximum probability is obtained for the emitted mass, but some admixture of the other mass is also observed. This will lead to a drop in the observed flux of the particles, which is widely regarded as proof of the particle undergoing oscillations. In fact, with our choice of the profile function \( a_{m^2} \) being proportional to the cosine, some oscillations of the probability of detection are also visible. These oscillations are more clearly visible after the inclusion of the proper normalization \( N \), but we will discuss this topic in more detail in an upcoming paper.

5. Conclusions and outlook

We have shown that particle oscillations may be a pure quantum mechanical phenomenon and do not require invoking unphysical interaction eigenstates. In our model the effect comes from the uncertainty principle, after taking into account the fact that no physical process can happen in zero time. The broadening of the mass spectrum of the emitted particle implies overlaps with the neighbouring mass states, which effectively adds them as admixtures to the propagating state. In this model it is natural that heavier particles (charged leptons for example) will not mix due to the huge mass differences between them. The mixing of light quarks and almost no mixing between charged leptons is in excellent agreement with observations.

More work is needed to check to what extent the effect depends on the detailed choice of the \( a_{m^2} \) profile. We suspect that each distribution (Gaussian, inverted parabola etc) should lead to similar results, but no formal proof of this statement has been constructed. It is also interesting to investigate this problem for different constructions of the detector.

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