Integrable Multicomponent Perfect Fluid Multidimensional Cosmology II: Scalar Fields.

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Abstract
We consider anisotropic cosmological models with an universe of dimension 4 or more, factorized into \( n \geq 2 \) Ricci-flat spaces, containing an \( m \)-component perfect fluid of \( m \) non-interacting homogeneous minimally coupled scalar fields under special conditions. We describe the dynamics of the universe: It has a Kasner-like behaviour near the singularity and isotropizes during the expansion to infinity.

Some of the considered models are integrable, and classical as well as quantum solutions are found. Some solutions produce inflation from "nothing". There exist classical asymptotically anti-de Sitter wormholes, and quantum wormholes with discrete spectrum.

1. INTRODUCTION
It is well known that the isotropic cosmological model at present time gives a good description of the observables part of the universe. On the other hand, this very fact of our

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universe’s isotropy and homogeneity is puzzling [1]. Even in the papers which are devoted to the problem of inflation, they start mainly with the metric of the isotropic Friedmann universe [2]. However, it is possible that at early stages of its evolution the universe exhibits an anisotropic behaviour [3]. As it was shown in [4, 5], anisotropic cosmological models describe the most general approach to the cosmological singularity (the initial singularity at some instant $t_0$). Among anisotropic homogeneous models the Kasner solution [6] represents one of the most simple vacuum solutions of the Einstein equations. The Kasner solution is defined on a manifold

$$M = \mathbb{R} \times M_1 \times M_2 \times M_3,$$

(1)

where the differentiable manifold $M_i$ ($i = 1, 2, 3$) is either $\mathbb{R}$ or $S^1$.

Another very puzzling problem is the fact that the space-time of our universe is 4-dimensional. Fashionable theories of unified physical interactions (supergravity or superstrings [7, 8, 9]) use the Kaluza-Klein idea [10, 11] of hidden (or extra) dimensions, according to which our universe at small (Planckian) scales has a dimension more than four. If the extra dimensions are more than a mathematical construct, we should explain what dynamical processes lead from a stage with all dimensions developing with the same scale to the actual stage of the universe, where we have only four external dimensions and all internal spaces have to be compactified and contracted to unobservable scales.

Exploiting these two remarkable ideas (anisotropy and multidimensionality), it is natural to generalize the manifold (1) as follows

$$M = \mathbb{R} \times M_1 \times \ldots \times M_n,$$

(2)

where $M_i$ ($i = 1, \ldots, n$) is a $d_i$-dimensional space of constant curvature (or, more generally, an Einstein space). If $n = 3$ and $d_1 = d_2 = d_3 = 1$ or $n = 2$ and $d_1 = 2$, $d_2 = 1$ then this manifold describes an usual anisotropic homogeneous 4-dimensional space-time. For $n \geq 2$ and a total dimension $D = 1 + \sum_{i=1}^n d_i > 4$ we have an anisotropic multidimensional space-time where one of the spaces $M_i$ (say $M_1$) describes our 3-dimensional external space.

Multidimensional cosmological models of the type (2) (with arbitrary $n$) were investigated intensively in the recent decade (according to our knowledge, starting from the paper [12] investigating the stability of the internal spaces).

Quantization of a multidimensional model with a space-time (2) was first performed in [13]. Beside vacuum models, there were also cosmological models considered which contain different types of matter, and exact solutions of the Einstein equations, and of the Wheeler-De Witt equations in the quantum case, were obtained (see [14, 15] and the extended list of references there). Exact solutions are of special interest because they can be used for a detailed study of evolution of the universe (for example in the approach to the singularity), of the compactification of the internal spaces, and of the behaviour of matter fields.

In the present paper we consider an anisotropic homogeneous universe of type (2), where all $M_i$ are Ricci-flat. If $n = 3$ and $d_1 = d_2 = d_3 = 1$ it describes the usual 4-dimensional Bianchi type I model. We investigate this space-time in the presence of $m$ non-interacting minimally coupled scalar fields. Scalar fields are now popular in cosmology, because in most inflationary models the presence of a scalar field provides homogeneity, isotropy, and almost spatial flatness of the universe [2]. It was shown in the paper [16] that for a special form of the scalar field potentials these scalar fields are equivalent to a $m$-component perfect fluid. We exploit this equivalence in [16] to investigate a two-component model (a model with 2 scalar fields). Now we shall integrate this model in the presence of 3 scalar fields where one of them is equivalent to an ultra-stiff perfect fluid, the second one corresponds to dust, and the third one is equivalent to vacuum. The main features of the solutions are the following: If the parameters of the model permit the universe to run from the singularity to infinity, then the
universe has a Kasner-like behaviour near the singularity, with isotropization when it goes to infinity. In the 3-component integrable case, the universe has de Sitter-like behaviour in the infinite volume limit. Superficially, it seems this model is not a good candidate for a realistic multidimensional cosmology, because of the isotropization of all directions at late times. But we shall show that there are particular solutions, which describe a birth of the universe from "nothing". The parameters of the model in this case can be chosen in such a way that a scale factor of the external space undergoes inflation, while the other scale factors remain compactified near Planck length. However this model is really only good, if in addition we provide a graceful exit mechanism \cite{17}. For some of the parameters the infinite volume limit takes place in the Euclidean region which has asymptotically anti-de Sitter wormhole geometry. This is in fact possible because in the infinite volume limit the curvature of the spherical spatial sections of the anti-de Sitter geometry decays to zero, whence these spatial sections become asymptotically Ricci flat like the spacial sections of our solution.

Let us also clarify here that, below we refer only to the local properties of (anti-)de Sitter space. The relationships between different charts of de Sitter space, with different choices of spacial sections, are examined e.g. in \cite{18,19}, while \cite{20} recently provided a classification of different multidimensional representations of spaces of constant curvature in arbitrary dimension. For a pure $d+1$-dimensional geometry, without additional fields, different choices of time, i.e. different slicings into $d$-dimensional spatial hypersurfaces, should be equivalent due to general covariance. Note however that, in \cite{23} and here, the geometry hosts in general several additional time-dependent but spatially homogenous matter fields, and the spatial homogeneity of any such field is generally not preserved under a change of the slicing.

Another interesting Euclidean solution represents an instanton which describes tunnelling between a Kasner-like universe (a baby universe) and an asymptotically de Sitter universe. Sewing a number of these instantons may provide the Coleman mechanism \cite{21} for the vanishing cosmological constant.

Note that our asymptotically de Sitter solutions are different from the generalized de Sitter solutions considered in \cite{22}. There only one factor space (the external one) was Ricci flat, and the curvatures of the other (internal) factor spaces were fine tuned with the cosmological constant.

The previous paper \cite{23} has already considered multidimensional cosmological models in the presence of a $m$-component perfect fluid. In the case with one non-Ricci-flat space, say $M_1$, for $n = 2$ and $d_1 = 2$, $d_2 = 1$, this model describes a usual 4-dimensional Kantowski-Sachs universe (if $M_1$ has positive constant curvature) or a Bianchi III universe (if $M_1$ has negative constant curvature). We also found a 3-component integrable model, where the universe has a Kasner-like behaviour near the singularity as in the present paper, but there is no isotropization at all. All scale factors corresponding to Ricci-flat factor spaces $M_i$ are frozen in the infinite volume limit, but the negative curvature space $M_1$ grows in time. From this point of view, the model does not describe usual 4-dimensional space-time, because of the missing isotropization, but it may be a good candidate for a multidimensional cosmology, if all frozen internal scale factors are near Planck scale. For a positive curvature space $M_1$, the infinite volume limit takes place in the Euclidean region, which there, in contrast to the present paper, has wormhole geometry only w.r.t. the space $M_1$, and the wormhole is asymptotically flat.

In the present paper we consider homogeneous minimally coupled scalar fields as a matter source. Usually, only real scalar fields are taken. Here we admit also purely imaginary scalar fields. Such scalar fields imply a negative sign at the kinetic term in the Lagrangian. Such scalar fields may arise after conformal transformation of real scalar fields with arbitrary coupling to gravity \cite{24,25,26,27}. They appear also in the Brans-Dicke theories after the dimensional reduction from higher dimensional theories \cite{9,28,29}. Also the $C$-field of Hoyle...
and Narlikar has a negative sign in front of the kinetic term [31]. The authors of [31, 32] emphasize the need for scalar fields with negative kinetic terms in multidimensional theories in order to fit the observable data (see also a discussion of this topic in [33]). As we will show here, in the particular case of constant $\phi$, the imaginary scalar field is equivalent to a negative cosmological constant which results in an anti-de Sitter universe. In what follows we do not exclude the possible existence of imaginary scalar fields, whence in our paper we consider real as well as imaginary scalar fields.

This paper is organized as follows. In section 2 we describe our model and get an effective perfect fluid Lagrangian, exploiting the equivalence between an $m$-component perfect fluid and $m$ non-interacting scalar fields with a special class of potentials. In section 3, we investigate the general dynamics of the universe and its asymptotic behaviour. In section 4, classical solutions for the integrable 3-component models are obtained. Classical wormhole solutions are obtained in section 5 where it is also shown that they are asymptotically anti-de Sitter wormholes. Section 6 is devoted to the reconstruction of the scalar field potentials. Solutions to the quantized models are presented in section 7. In section 8 we summarize our results.

2. THE MODEL

We consider a cosmological model on a multidimensional manifold (2) with a metric

$$g = g_{MN} dx^M \otimes dx^N = -e^{2\gamma(\tau)} d\tau \otimes d\tau + \sum_{i=1}^{n} e^{2\beta_i(\tau)} g^{(i)},$$

(3)

where, for $i = 1, \ldots, n$, $g^{(i)} = g_{m_i n_i} dx^{m_i} \otimes dx^{n_i}$, $m_i, n_i = 1, \ldots, d_i$, is the metric form of the Ricci-flat factor space $M_i$ dimension $d_i$.

$$R_{m_i n_i} \left[ g^{(i)} \right] = 0, \quad i = 1, \ldots, n.$$ 

(4)

The action of the model is taken in the form

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{|g|} R[g] + S_\varphi + S_{GH},$$

(5)

where $S_{GH}$ is the standard Gibbons-Hawking boundary term, $\kappa^2$ is the gravitational coupling constant in dimension $D = \sum_{i=1}^{n} d_i + 1$, and $S_\varphi = \sum_{a=1}^{m} S_\varphi^{(a)}$ is the action of $m$ non-interacting minimally coupled homogeneous scalar fields

$$S_\varphi^{(a)} = -\int d^D x \sqrt{|g|} \left[ g^{MN} \partial_M \varphi^{(a)} \partial_N \varphi^{(a)} + U^{(a)}(\varphi^{(a)}) \right].$$

(6)

For the metric (3) the action (5) reads

$$S = \mu \int d\tau L_s,$$

(7)

with the Lagrangian

$$L_s = \frac{1}{2} e^{-\gamma + \gamma_0} \left( G_{ij} \dot{\varphi}^i \dot{\varphi}^j + \kappa^2 \sum_{a=1}^{m} (\varphi^{(a)})^2 \right) - \kappa^2 e^{\gamma + \gamma_0} \sum_{a=1}^{m} U^{(a)}(\varphi^{(a)}).$$

(8)
Here $\gamma_0 = \sum_{i=1}^{n} d_i \beta^i$ and $\mu = \prod_{i=1}^{n} V_i / \kappa^2$ where $V_i$ is the volume of the finite Ricci-flat spaces $(M_i, g^{(i)})$. The components of the minisuperspace metric read

$$G_{ij} = d_i \delta_{ij} - d_i d_j.$$  \hspace{1cm} (9)

As in [16] we subject the scalar fields to the perfect fluid energy-momentum constraints

$$P^{(a)} = \left( \alpha^{(a)} - 1 \right) \rho^{(a)},$$  \hspace{1cm} (10)

with constants $\alpha^{(a)}$, $a = 1, \ldots, m$, and the energy densities

$$\rho^{(a)} \equiv -T^{(a)0}_0 = \frac{1}{2} e^{-2\gamma(\dot{\varphi}^{(a)})^2} + U^{(a)}(\varphi^{(a)})$$  \hspace{1cm} (11)

and momenta

$$P^{(a)} = T^{(a)M}_M = e^{-2\gamma(\dot{\varphi}^{(a)})^2} - U^{(a)}(\varphi^{(a)}), \quad M = 1, \ldots, D - 1,$$  \hspace{1cm} (12)

according to the Lagrangian (8). In [16] it was proved that for cosmological models with a metric (3), the presence of $m$ non-interacting scalar fields satisfying the relations (10) is equivalent to the presence of an $m$-component perfect fluid with a Lagrangian

$$L_{\rho} = \frac{1}{2} e^{-\gamma+\gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j - \kappa^2 e^{\gamma+\gamma_0} \sum_{a=1}^{m} \rho^{(a)},$$  \hspace{1cm} (13)

and energy densities of the form

$$\rho^{(a)} = A^{(a)} V - \alpha^{(a)}, \quad a = 1, \ldots, m,$$  \hspace{1cm} (14)

with constants $A^{(a)}$ and a spatial volume scale

$$V = e^{\gamma_0} = \prod_{i=1}^{n} a^{d_i}$$  \hspace{1cm} (15)

defined by the scale factors $a_i = e^{\beta^i}$, $i = 1, \ldots, n$. Note that the total spatial volume is $V_{tot} = \mu \cdot V$. The energy density $\rho^{(a)}$ is then connected with the pressure $P^{(a)}$ via (10), and equations (11) and (12) imply $\alpha^{(a)} \rho^{(a)} = e^{-2\gamma(\dot{\varphi}^{(a)})^2}$. So, for real scalar fields and positive $\alpha^{(a)}$, the energy density of the perfect fluid is positive. But, keeping in mind the possibility of imaginary scalar fields (see Introduction), for the general model we shall also consider the case $\rho^{(a)} < 0$. Then, the constants $A^{(a)}$ may have any sign.

Assuming the speed of sound in each component of the perfect fluid to be less than the speed of light,

$$-|\rho^{(a)}| \leq P^{(a)} \leq |\rho^{(a)}|, \quad a = 1, \ldots, m.$$  \hspace{1cm} (16)

With (10) this implies the inequalities

$$0 \leq \alpha^{(a)} \leq 2, \quad a = 1, \ldots, m.$$  \hspace{1cm} (17)

Note that, with $\rho = \sum_{a=1}^{m} \rho^{(a)}$ and $P = \sum_{a=1}^{m} P^{(a)}$, the energy dominance condition requires only $-|\rho| \leq P \leq |\rho|$, rather than (10). In this paper however, although it might be possible to generalize results for arbitrary $\alpha^{(a)}$, for simplicity we keep the assumption (16) in order to make use of the inequalities (17).
Exploiting the mentioned equivalence between scalar fields and perfect fluid, we investigate the dynamics of the universe via the Euler-Lagrange equations of \[(13)\], and reconstruct the scalar field potentials \(U^{(a)}(\varphi^{(a)})\) satisfying the perfect fluid constraint \[(14)\].

3. GENERAL DYNAMICS OF THE UNIVERSE

In the harmonic time gauge \(\gamma = \gamma_0 = \sum_{i=1}^{n} d_i \beta^i\) (see e.g. \[13, 25\]), the Lagrangian \[(13)\] with energy densities \[(14)\] just reads
\[
L_\rho = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - \kappa^2 e^{2\gamma_0} \sum_{a=1}^{m} \rho^{(a)}.
\]
(18)

Then the corresponding scalar (zero energy) constraint can be imposed as
\[
\frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j + \kappa^2 e^{2\gamma_0} \sum_{a=1}^{m} \rho^{(a)} = 0.
\]
(19)

The minisuperspace metric may be diagonalized (see also \[13\]) to
\[
G = \eta_{kl} dz^k \otimes dz^l = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i,
\]
(20)

where
\[
z^0 = q^{-1} \sum_{j=1}^{n} d_j \beta^j = q^{-1} \ln V,
\]
(21)
\[
z^i = \left[ d_i / \Sigma_i \Sigma_{i+1} \right]^{1/2} \sum_{j=i+1}^{n} d_j \left( \beta^j - \beta^i \right),
\]
(22)

\(i = 1, \ldots, n-1\), and
\[
q := \left[ (D - 1)/(D - 2) \right]^{1/2}, \quad \Sigma_k := \sum_{i=k}^{n} d_i.
\]
(23)

With the aid of these transformations the Lagrangian \[(18)\] and the scalar constraint \[(19)\] can be rewritten as
\[
L_\rho = \frac{1}{2} \eta_{kl} z^k z^l - \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z^0),
\]
(24)
\[
\frac{1}{2} \eta_{kl} z^k z^l + \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z^0) = 0
\]
(25)

respectively. Here, \(k^{(a)} := 2 - \alpha^{(a)}\) \((a = 1, \ldots, m)\), whence the inequalities \[(17)\] for \(\alpha^{(a)}\) hold also for \(k^{(a)}\),
\[
0 \leq k^{(a)} \leq 2.
\]
(26)

The equations of motion for \(z^i, i = 1, \ldots, n-1\), simply read
\[
z^{ii} = 0,
\]
(27)
and readily yield

\[ z^i = p^i \tau + q^i, \quad (28) \]

where \( \tau \) is the harmonic time, \( p^i \) and \( q^i \) are constants. Clearly the geometry is real if \( p^i \) and \( q^i \) are real. The dynamics of \( z^0 \) is then given by the scalar constraint \((25)\), which may now be written as

\[ -\frac{1}{2}(\dot{z}^0)^2 + \varepsilon + \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)}qz^0) = 0, \quad (29) \]

for a real geometry with

\[ \varepsilon := \frac{1}{2} \sum_{i=1}^{n-1} (p^i)^2 \geq 0. \quad (30) \]

The coordinate transformations \((21)\) and \((22)\) can be written as

\[ z^k = \sum_{i=1}^{n} t^k_i \beta^i, \quad k = 0, \ldots, n - 1, \quad (31) \]

whence the inverse is given by

\[ \beta^i = \sum_{k=0}^{n-1} \bar{t}^k k z^k, \quad i = 1, \ldots, n. \quad (32) \]

For \( i = 1, \ldots, n \), with \( t^0_i = d_i/q \) and \( \bar{t}^0_0 = [q(D - 2)]^{-1} \) we obtain the scale factors

\[ a_i = A_i V^{1/(D-1)} e^{\alpha^i \tau}, \quad (33) \]

where

\[ A_i := e^{\gamma^i}, \quad \gamma^i := \sum_{l=1}^{n} \bar{t}^i q^l, \quad \alpha^i := \sum_{l=1}^{n-1} \bar{t}^i p^l. \quad (34) \]

The parameters \( \alpha^i \) satisfy the relations

\[ \sum_{i=1}^{n} d_i \alpha^i = 0, \quad (35) \]

\[ \sum_{i=1}^{n} d_i (\alpha^i)^2 = \sum_{l=1}^{n-1} (p^l)^2 = 2\varepsilon, \quad (36) \]

and, analogously the parameters \( \gamma^i \) fulfil

\[ \sum_{i=1}^{n} d_i \gamma^i = 0, \quad (37) \]

\[ \sum_{i=1}^{n} d_i (\gamma^i)^2 = \sum_{l=1}^{n-1} (q^l)^2. \quad (38) \]

From the definition \((34)\) and the relation \((37)\) it follows that

\[ \prod_{i=1}^{n} A^d_i = 1. \quad (39) \]
Note that using the constraints (35) and (39) the equation (33) yields again (15), i.e. \( \prod_{i=1}^{n} a_i = V \). Recall that \( \tau \) in (33) is the harmonic time. The synchronous and harmonic times are related by
\[
t = \pm \int e^{\gamma \theta} d\tau + t_0 = \pm \int V d\tau + t_0. \tag{40}
\]
The expression (33) shows that the general model does not belong to a class with static internal spaces (see e.g. \[34\]), but just for \( \varepsilon = 0 \) (i.e. \( \alpha_i = 0, \ i = 1, \ldots, n \)), there is a solution
\[
a_i = A_i V^{1/(D-1)}, \quad i = 1, \ldots, n, \tag{41}
\]
which is isotropic.

In order to find the dynamical behaviour of the universe we should now solve the constraint (29), i.e. the mechanical energy conservation equation
\[
\varepsilon = T + U \tag{42}
\]
with kinetic energy \( T := \frac{1}{2} (\dot{z}^0)^2 \) and potential \( U := -\kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z^0) \). Depending on the parameters \( A^{(a)} \) and their signs, the potential \( U \) may exhibit a rich structure with several extrema, and a classical Lorentzian trajectory is bound by possible turning points at \( \varepsilon = U \). Since the general dynamics is very complex, we investigate the asymptotic behaviour of our model universe in the limit of large spatial geometries \( V \to \infty \) and near the singularity \( V \to 0 \). Without restriction we suppose now
\[
0 \leq k^{(1)} < \ldots < k^{(m)} \leq 2. \tag{43}
\]

1. Limit \( V \to \infty \): In the limit \( V \to \infty \) (i.e. \( z_0 \to \infty \)) the term \( -\kappa^2 A^{(m)} \exp(k^{(m)} q z^0) \) dominates the potential \( U \), whence, for \( k^{(m)} \neq 0 \), there are two cases to be distinguished:

   i) \( A^{(m)} > 0 \): Here, for \( V \to \infty \), the term \( \varepsilon \) may be neglected. So, the constraint equation (42) has the asymptotic solution
\[
e^{qz_0} = V \approx (2\kappa^2 A^{(m)})^{-1/k^{(m)}} |q|^{-2/k^{(m)}}, \tag{44}
\]
with \( 2q := k^{(m)} q \), where (without restriction) we have chosen initial conditions such that \( V \to \infty \) at \( \tau \to 0 \), when according to (33) the system is subject to an isotropization,
\[
a_i \sim V^{1/(D-1)}, \quad V \to \infty, \quad i = 1, \ldots, n. \tag{45}
\]

In this limit the harmonic and synchronous times are connected by
\[
|\tau| \sim |t|^{k^{(m)}/(k^{(m)} - 2)}, \quad k^{(m)} \neq 2, \tag{46}
|\tau| \sim \exp(-\sqrt{2\kappa^2 A^{(m)} q|t|}), \quad k^{(m)} = 2. \tag{47}
\]

So the synchronous time evolution of the spatial volume is (asymptotically for \( t \to \infty \)) given by
\[
V \sim |t|^{2/\alpha^{(m)}}, \quad k^{(m)} \neq 2, \tag{48}
V \sim \exp(\sqrt{2\kappa^2 A^{(m)} q|t|}), \quad k^{(m)} = 2. \tag{49}
\]
with scale factors (according to isotropization)

\[ a_i \sim |t|^{2/\alpha(m)(D-1)} , \quad k^{(m)} \neq 2 , \quad (50) \]

\[ a_i \sim \exp \left( \frac{\sqrt{2\kappa^2 A(m)} q}{D-1} |t| \right) , \quad k^{(m)} = 2 . \quad (51) \]

Taking a usual anisotropic space-time model \((D = 4, n = 3, d_1 = d_2 = d_3 = 1)\) then for large (synchronous) times the formulas \((50)\) and \((51)\) yield scale factors \(a_i \sim |t|^{2/3}\) for \(k^{(m)} = 1\) (dust) and \(a_i \sim \exp \left( \frac{\sqrt{2\kappa^2 A(m)} q}{3|t|} \right)\) for \(k^{(m)} = 2\) (vacuum). Asymptotically, power-law inflation (with power \(p > 1\)) takes place for \(0 < \alpha^{(m)} < 2/(D-1)\), and \(\alpha^{(m)} = 2/(D-1)\) yields a generalized Milne universe.

ii) \(A^{(m)} < 0\): Here, the Lorentzian region has a boundary at the turning point \(V_{\text{max}}\) of the volume scale, which in the large energy limit \(\varepsilon \to \infty\) is asymptotically given as

\[ V_{\text{max}} \approx \left[ \frac{\varepsilon}{\kappa^2 |A^{(m)}|} \right]^{1/k^{(m)}} . \quad (52) \]

The region with \(V > V_{\text{max}}\) is the Euclidean sector. For \(V >> V_{\text{max}}\), we obtain the asymptotically isotropic solution

\[ a_i \sim V^{1/(D-1)} \approx \left[ \frac{\varepsilon}{2\kappa^2 |A^{(m)}| q/|t|} \right]^{-2/k^{(m)}(D-1)} . \quad (53) \]

In the Euclidean region, we obtain a classical wormhole w.r.t. each factor space. With constants of integration (in \((28)\)) \(p_i = 0\) (i.e. \(\alpha^i = 0\), \(i = 1, \ldots, n\)), the wormhole takes its most simple and symmetric form. Then the throats are given by

\[ a_{(th)}^{(i)} \approx A_i \left[ \varepsilon/\kappa^2 |A^{(m)}| q/|t| \right]^{1/k^{(m)}(D-1)} . \quad (54) \]

In the case \(k^{(m)} = 2\) we obtain asymptotically (for \(t \to \infty\)) anti-de Sitter wormholes with synchronous time scale factors

\[ a_i \sim \exp \left( \frac{\sqrt{2\kappa^2 A^{(m)}} q}{D-1} |t| \right) , \quad i = 1, \ldots, n . \quad (55) \]

2. Limit \(V \to 0\): For \(k^{(1)} \neq 0\), in the small volume limit \(V \to 0\), i.e. \(z^0 \to -\infty\), the potential vanishes \(U \to 0\). So, for \(\varepsilon > 0\), we obtain (asymptotically for \(t \to 0\)) a (multidimensional) Kasner universe \([35, 36]\), with scale factors

\[ a_i \sim |t|^\bar{\alpha}^i , \quad i = 1, \ldots, n . \quad (56) \]

with parameters \(\bar{\alpha}^i\) satisfying

\[ \sum_{i=1}^{n} d_i \bar{\alpha}^i = 1 , \quad \sum_{i=1}^{n} d_i (\bar{\alpha}^i)^2 = 1 . \quad (57) \]

If \(k^{(1)} = 0\), then \(U \to -\kappa^2 A^{(1)}\) for \(z^0 \to -\infty\). Here, for \(E := \varepsilon + \kappa^2 A^{(1)} > 0\), we obtain (asymptotically for \(t \to 0\)) a generalized Kasner universe \([35]\), i.e. scale factors \((56)\) with parameters \(\bar{\alpha}^i\) satisfying

\[ \sum_{i=1}^{n} d_i \bar{\alpha}^i = 1 , \quad \sum_{i=1}^{n} d_i (\bar{\alpha}^i)^2 = 1 - \bar{\alpha}^2 . \quad (58) \]

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with the parameter $\bar{\alpha} \to 0$ for $A^{(1)} \to 0$.

In the exceptional case $E = \varepsilon + \kappa^2 A^{(1)} = 0$ the term of the matter component $a = 2$ dominates the constraint (29), whence we obtain (compare also (16))

\begin{align}
V & \sim t^{2/\alpha^{(2)}} , \\
a_i & \sim t^{2/((D-1)\alpha^{(2)})} \exp \left\{ \alpha^{(2)} f(t^{2/\alpha^{(2)}}) \right\} , \quad i = 1, \ldots, n ,
\end{align}

where $f(x) := \left( \frac{2-x}{x} \right)^{\frac{2-x}{x}} \left[ \frac{2}{(2-x)^{1/2}} \right]^{\frac{1}{2}}$. In another exceptional case where $\varepsilon = 0$ (i.e. $\alpha^i = 0, i = 1, \ldots, n$) the universe is isotropic everywhere, i.e. $a_i \sim V^{1/(D-1)}, i = 1, \ldots, n$. If, for example, $k^{(1)} = 0$ (and $A^{(1)} = 0$) we obtain from (59) or (60)

\begin{equation}
a_i \sim t^{2/((D-1)\alpha^{(2)})} .
\end{equation}

4. INTEGRABLE 3-COMPONENT MODEL. CLASSICAL SOLUTIONS

In this chapter, we consider the integrable case of a three-component perfect fluid ($m = 3$) where one of them ($a = 1$) is ultra-stiff matter ($k^{(1)} = 0, \alpha^{(1)} = 2$), the second one ($a = 2$) is dust ($k^{(2)} = 1, \alpha^{(2)} = 1$), and the third one ($a = 3$) is vacuum ($k^{(3)} = 2, \alpha^{(3)} = 0$). The case $k^{(1)} = 0, k^{(3)} = 2k^{(2)}$ with $0 < k^{(2)} \leq 2$ is also included if one substitutes $q$ by $\bar{q} = k^{(2)}q$.

The constraint equation (29) reads in this case

\begin{equation}
- \frac{1}{2} \left( \frac{\dot{z}^0}{e^{\kappa^2 z^0}} \right)^2 + \varepsilon + \kappa^2 A^{(1)} + \kappa^2 A^{(2)} e^{qz^0} + \kappa^2 A^{(3)} e^{2qz^0} = 0
\end{equation}

and can be rewritten like

\begin{equation}
E = \frac{1}{2} \left( \frac{\dot{z}^0}{e^{\kappa^2 z^0}} \right)^2 + U(z^0) ,
\end{equation}

where

\begin{equation}
E := \varepsilon + \kappa^2 A^{(1)}
\end{equation}

and the potential $U(z^0)$ is

\begin{equation}
U(z^0) := -Be^{qz^0} - Ce^{2qz^0}
\end{equation}

with the definitions $B := \kappa^2 A^{(2)}$ and $C := \kappa^2 A^{(3)}$.

As mentioned in the introduction, for a complete description of the model the parameters $E$, $B$, and $C$ are considered to take positive and negative values. Then, we have four qualitatively different shapes of the potential (63) (see Fig. 1 and Fig. 2). For each of them, we shall solve the constraint equation separately. Eq. (63) integrates to

\begin{equation}
\int \frac{dV}{V\sqrt{E + BV + CV^2}} = \pm \sqrt{2q} (\tau - \tau_0) ,
\end{equation}

where $\tau$ is the harmonic time coordinate, and $\tau_0$ is a constant of integration.
i) $B > 0$, $C > 0$ (see Fig. 1): The solutions of equations (66) are

$$V = \frac{1}{B} \mathbb{E}^2 \left[ \frac{f^2 - 2Cq}{qB} \right], \quad E = 0, \quad (67)$$

$$V = \frac{4Ef}{(B - f)^2 - 4EC}, \quad E > 0, \quad (68)$$

$$V = \frac{2|E|}{B + \sqrt{\Delta}|f|}, \quad E < 0, \quad (69)$$

where $\Delta := 4EC - B^2$ ($|\Delta| = B^2 + 4|E|C$ for $E < 0$) and

$$f = \tau - \tau_0, \quad E = 0, \quad \frac{\sqrt{2C}}{qB} \leq |\tau - \tau_0| < +\infty, \quad (70)$$

$$f = \exp \left( \sqrt{2E}q(\tau - \tau_0) \right), \quad E > 0, \quad \ln \left( B + 2\sqrt{EC} \right) \leq \ln f < +\infty, \quad (71)$$

$$f = \sin \left( \sqrt{2}|E|q(\tau - \tau_0) \right), \quad E < 0, \quad -\arcsin \left( \frac{B}{\sqrt{\Delta}} \right) \leq \arcsin f \leq \frac{\pi}{2}. \quad (72)$$

The synchronous and harmonic time coordinate are related via

$$\tau - \tau_0 = \frac{\sqrt{2C}}{qB} \coth \left( \sqrt{\frac{C}{2}}qt \right), \quad E = 0, \quad (73)$$

$$\exp \left( \sqrt{2E}q(\tau - \tau_0) \right) = B + \sqrt{4EC} \coth \left( \sqrt{\frac{C}{2}}qt \right), \quad E > 0, \quad (74)$$

$$\tan \left( \sqrt{|E|/2q(\tau - \tau_0)} \right) = \frac{\sqrt{\Delta}}{B} \left[ \frac{4|E|C}{|\Delta|} \coth \left( \sqrt{\frac{C}{2}}qt \right) - 1 \right], \quad E < 0. \quad (75)$$

Using these relations, we obtain the expressions for the volume of the universe in synchronous time:

$$V = \frac{B}{C} \sinh^2 \left( \sqrt{\frac{C}{2}}qt \right), \quad E = 0, \quad |t| < \infty, \quad (76)$$

$$V = \frac{1}{C} \left[ B + \sqrt{4EC} \coth \left( \sqrt{\frac{C}{2}}qt \right) \right] \sinh^2 \left( \sqrt{\frac{C}{2}}qt \right), \quad E > 0, \quad 0 \leq t < +\infty, \quad (77)$$

$$V = \frac{2|E|(1 + \tan^2(y/2))}{B(1 + \tan^2(y/2)) + 2\sqrt{|\Delta|} \tan(y/2)}, \quad E < 0, \quad (78)$$

where $\tan(y/2) = \tan \left( \frac{\sqrt{|E|/2q(\tau - \tau_0)}}{2} \right)$ is given by (74). Expression (78) can be written in a more convenient way if the parameter $\tau_0$ is chosen such that equation (69) is symmetric with respect to the turning point $V_0 = \left( -B + \sqrt{|\Delta|} \right)/2C$, namely

$$V = \frac{2|E|}{B + \sqrt{|\Delta|} \cos \left( \sqrt{2|E|q}\tau \right)}, \quad |\tau| < \frac{1}{\sqrt{2|E|q}} \left[ \pi \arcsin \left( \frac{B}{\sqrt{|\Delta|}} \right) \right]. \quad (79)$$
In this case,
\[ \tan \left( \sqrt{|E|/2q^2} \right) = \frac{\sqrt{4|E|C}}{\sqrt{|\Delta|} - B} \tanh \left( \sqrt{\frac{C}{2}} q \right) \]  
and for the volume results
\[ V = \frac{1}{2C} \left[ \sqrt{|\Delta|} - B + \left( \sqrt{|\Delta|} + B \right) \tanh^2 \left( \sqrt{\frac{C}{2}} q \right) \right] \cosh^2 \left( \sqrt{\frac{C}{2}} q \right), \quad |t| < \infty. \]  
The region $V < V_0$ is the Euclidean sector and we obtain the instanton by analytic continuation $t \to -it$ in formula (81):
\[ V = \frac{1}{2C} \left[ \sqrt{|\Delta|} - B - \left( \sqrt{|\Delta|} + B \right) \tan^2 \left( \sqrt{\frac{C}{2}} q \right) \right] \cos^2 \left( \sqrt{\frac{C}{2}} q \right) \]  
with $|t| \leq \frac{2}{\sqrt{2Cq}} \arctan \frac{\sqrt{|\Delta| - B}}{\sqrt{|\Delta| + B}}$.

On the quantum level, this instanton is responsible for the birth of the universe from “nothing”.

ii) $B < 0, C > 0$ (see Fig. 2): In this case, the maximal value of the potential $U(z^0)$ is $U_m = B^2/4C$ at $z_m^0 = \frac{1}{q} \ln \left( |B|/2C \right)$ and for $0 < E < V_m$ we have two turning points, namely:
\[ V^{(1,2)}_0 = \left( |B| \pm \sqrt{|\Delta|} \right) / 2C, \]  
where $|\Delta| = B^2 - 4EC$. Classical motion takes place either for $0 \leq V \leq V^{(1)}_0$ or for $V^{(2)}_0 \leq V < +\infty$.

If $E \leq 0$, we have one turning point only, namely
\[ V_0 = |B|/C, \quad E = 0, \]  
\[ V_0 = \left( |B| + \sqrt{|\Delta|} \right) / 2C, \quad E < 0, \]  
where $|\Delta| = B^2 + 4|E|C$ and classical motion takes place for $V \geq V_0$.

The solutions of the equation (66) read
\[ V = \frac{1}{|B|} \frac{\sqrt{\frac{2}{B'} - f^2}}{2}, \quad E = 0, \]  
\[ V = \frac{4Ef}{(|B| + f)^2 - 4EC \cdot 0 < E < U_m, \quad 0 \leq V \leq V^{(1)}_0, \]  
\[ V = \frac{4Ef}{4EC - (|B| - f)^2}, \quad 0 < E < U_m, \quad V^{(2)}_0 \leq V < +\infty, \]  
\[ V = \frac{4Ef}{(|B| + f)^2 - 4EC}, \quad E > U_m, \]  
\[ V = \frac{2|E|}{\sqrt{|\Delta|} |f - |B|}, \quad E < 0, \]  

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where

\[ f = \tau - \tau_0, \quad E = 0, \quad |\tau - \tau_0| < \sqrt{2C/q|B|}, \]  
\[ f = \exp \left( \sqrt{2E}q(\tau - \tau_0) \right), \quad 0 < E < U_m, \quad V \leq V_0^{(1)}, \]  
\[ f = \exp \left( \sqrt{2E}q(\tau - \tau_0) \right), \quad 0 < E < U_m, \quad V \geq V_0^{(2)}, \]  
\[ f = \exp \left( \sqrt{2E}q(\tau - \tau_0) \right), \quad E > U_m, \]  
\[ f = \sin \left( \sqrt{2|E|}q(\tau - \tau_0) \right), \quad E < 0, \]  
\[ \arcsin \frac{|B|}{\sqrt{|\Delta|}} \leq \arcsin f \leq \frac{\pi}{2}. \]

The harmonic and synchronous time coordinates are related via

\[ \tau = \frac{2C}{|B|q} \tanh \left( \frac{C}{2}qt \right), \quad E = 0, \]  
\[ f = \sqrt{4EC} \cot \left( \frac{C}{2}qt \right) - |B|, \quad 0 < E < B^2/4C, \quad V < V_0^{(1)}, \]  
\[ f = -\sqrt{4EC} \tanh \left( \frac{C}{2}qt \right) + |B|, \quad 0 < E < B^2/4C, \quad V > V_0^{(2)}, \]  
\[ f = \sqrt{4EC} \coth \left( \frac{C}{2}qt \right) - |B|, \quad E > B^2/4C, \]  
\[ \tan \left( \frac{|E|}{2q\tau} \right) = \frac{\sqrt{4|E|C}}{\sqrt{|\Delta|} + |B|} \tanh \left( \frac{C}{2}qt \right), \quad E < 0, \]

where in (96) and (100) the constant \( \tau_0 \) is chosen such that the expressions are symmetric with respect to the turning point at the minimum. Then, the volume of the universe is

\[ V = \frac{|B|}{C} \cosh^2 \left( \frac{C}{2}qt \right), \quad E = 0, \quad |t| < +\infty, \]  
\[ V = \frac{1}{C} \left[ \sqrt{4EC} \coth \left( \frac{C}{2}qt \right) - |B| \right] \sinh^2 \left( \frac{C}{2}qt \right), \]  
\[ 0 < E < B^2/4C, \quad V < V_0^{(1)}, \quad 0 \leq t \leq \frac{2}{\sqrt{2Cq}} \arccosh \frac{|B| + \sqrt{|\Delta|}}{\sqrt{4EC}}, \]  
\[ V = \frac{1}{2C} \left[ |B| + \sqrt{|\Delta|} - \left( |B| - \sqrt{|\Delta|} \right) \tanh \left( \frac{C}{2}qt \right) \right] \cosh^2 \left( \frac{C}{2}qt \right), \]
\[ 0 < E < B^2/4C, \quad V > V_0^{(2)}, \quad |t| < +\infty \]

\[ V = \frac{1}{C} \left[ \sqrt{4EC} \coth \left( \frac{C}{2}qt \right) - |B| \right] \sinh^2 \left( \frac{C}{2}qt \right), \quad (104) \]

\[ E > B^2/4C, \quad 0 \leq t < +\infty \]

\[ V = \frac{1}{2C} \left[ \sqrt{\Delta} + |B| + \left( |B| - \sqrt{\Delta} \right) \tan^2 \left( \frac{C}{2}qt \right) \right] \cosh^2 \left( \frac{C}{2}qt \right), \quad (105) \]

\[ E < 0, \quad |t| < +\infty. \]

Eqs. (101), (103), and (105) are written in a symmetric way with respect to the turning point at \( t = 0 \). The instanton solutions can be obtained by analytic continuation of these symmetric expressions and result in:

\[ V = \frac{|B|}{C} \cos^2 \left( \sqrt{\frac{C}{2}qt} \right), \quad E = 0, \quad |t| \leq \pi/\sqrt{2Cq}, \quad (106) \]

\[ V = \frac{1}{2C} \left[ |B| + \sqrt{|\Delta|} + \left( |B| - \sqrt{|\Delta|} \right) \tan^2 \left( \sqrt{\frac{C}{2}qt} \right) \right] \cos^2 \left( \sqrt{\frac{C}{2}qt} \right), \quad (107) \]

\[ 0 < E < B^2/4C \quad (|\Delta| = B^2 - 4EC), \quad |t| \leq \pi/\sqrt{2Cq}, \]

\[ V = \frac{1}{2C} \left[ |B| + \sqrt{|\Delta|} + \left( |B| - \sqrt{|\Delta|} \right) \tan^2 \left( \sqrt{\frac{C}{2}qt} \right) \right] \cos^2 \left( \sqrt{\frac{C}{2}qt} \right), \quad (108) \]

\[ E < 0 \quad (|\Delta| = B^2 + 4|E|C), \quad |t| \leq \frac{2}{\sqrt{2Cq}} \arctan \sqrt{\frac{\sqrt{\Delta} + |B|}{\sqrt{\Delta} - |B|}} \]

In equation (107), the instanton is symmetric with respect to the turning point \( V_0^{(2)} \). For the same instanton but now symmetric with respect to the turning point at \( V_0^{(1)} \), we have:

\[ V = \frac{1}{2C} \left[ |B| - \sqrt{|\Delta|} + \left( |B| + \sqrt{|\Delta|} \right) \tan^2 \left( \sqrt{\frac{C}{2}qt} \right) \right] \cos^2 \left( \sqrt{\frac{C}{2}qt} \right), \quad (109) \]

\[ 0 < E < B^2/4C, \quad |t| \leq \pi/\sqrt{2Cq}. \]

All the instantons (104) to (109) are responsible on the quantum level for the birth of the universe from “nothing”. The instanton (107), (108) is of special interest. Its qualitative shape is seen in Fig. 3 where \( V_{\min} = V_0^{(1)} \) and \( V_{\max} = V_0^{(2)} \). The instanton describes tunnelling between a multidimensional Kasner-like universe (a baby universe) and a multidimensional de Sitter universe because, as was mentioned in chapter 3 and as we shall demonstrate more precisely latter, the limit \( V \to 0 \) corresponds to a Kasner-like universe and in the limit \( V \to \infty \) we obtain an (isotropic) de Sitter universe (in \[ 37 \] to \[ 40 \] analogous types of an instanton describing tunnelling between a Friedmann universe and a de Sitter universe were obtained for a different model). The instanton may also represent the birth (demise) of a de Sitter universe (see \[ 107 \]) and a baby universe (see \[ 109 \]) from (into) “nothing”. As was demonstrated in \[ 37 \] to \[ 38 \], the instanton may be extended beyond \( V = V_{\min,max} \) gluing together a number of Euclidean manifolds (see Fig. 4). Such gluing may provide the Coleman mechanism \[ 21 \] that establishes the vanishing of the cosmological constant.
The case with $E = B^2/4C$ is degenerated. Here, two turning points coincide with each other: $V_0^{(1)} = V_0^{(2)} = V_0 = |B|/2C$. In this case, we obtain with the synchronous time coordinate $t$ for the two volumes the expressions

$$V = \frac{2E}{|B|} \left[ 1 - \exp \left( \frac{|B|}{\sqrt{2E}} t \right) \right], \quad 0 \leq t < +\infty,$$

(110)

which describes an infinitely long lasting rolling down from the unstable equilibrium position $V_0$ to the singularity $V = 0$ and

$$V = \frac{2E}{|B|} \left[ 1 + \exp \left( \frac{|B|}{\sqrt{2E}} t \right) \right], \quad |t| < +\infty$$

(111)

describing the infinitely long lasting rolling down with $V \to \infty$.

To obtain solutions in the two remaining cases $B > 0, \quad C < 0$ (see Fig. 2) and $B < 0, \quad C < 0$ (see Fig. 1), it is not necessary to solve equation (60) again. We can instead take the solutions found already in subsections i) and ii). It is clear that the Euclidean solutions obtained there are Lorentzian ones here and vice versa Lorentzian solutions instead take the solutions found already in subsections i) and ii). What we have to do is the evident substitutions $B \to |B|, \quad C \to |C|$, and $E \to -E$ where it is necessary. For example:

iii) $B > 0, \quad C < 0$: From (106), (108), and (109), we obtain respectively

$$V = \frac{B}{|C|} \cos^2 \left( \sqrt{|C|}/2qt \right), \quad E = 0, \quad |t| \leq \pi/q \sqrt{2|C|},$$

(112)

$$V = \frac{1}{2|C|} \left[ \sqrt{|\Delta|} + B - \left( \sqrt{\Delta} - B \right) \tan^2 \left( \sqrt{|C|}/2qt \right) \right] \cos^2 \left( \sqrt{|C|}/2qt \right),$$

(113)

$$E > 0, \quad |t| \leq \frac{2}{2|C| q} \arctan \left[ \left( \sqrt{\Delta} + B \right) / \left( \sqrt{\Delta} - B \right) \right]^{1/2},$$

$$V = \frac{1}{2|C|} \left[ B - \sqrt{|\Delta|} + \left( B + \sqrt{\Delta} \right) \tan^2 \left( \sqrt{|C|}/2qt \right) \right] \cos^2 \left( \sqrt{|C|}/2qt \right),$$

(114)

$$-B^2/4|C| < E < 0, \quad |t| < \pi/\sqrt{2|C| q}.$$  

Solution (113) is symmetric with respect to the classical turning point. To investigate the limit $|t| \to 0$ it is better to give another representation of the same solution, namely

$$V = \frac{1}{|C|} \left[ B + \sqrt{4E|C|} \cot \left( \sqrt{|C|}/2qt \right) \right] \sin^2 \left( \sqrt{|C|}/2qt \right),$$

(115)

$$E > 0, \quad 0 \leq t \leq \frac{2}{\sqrt{2|C| q}} \arctan \frac{\sqrt{\Delta} - B}{\sqrt{4E|C|}}.$$  

In this case, the harmonic time coordinate and the synchronous one are related via

$$\exp \left( \sqrt{2E} q (\tau - \tau_0) \right) = B + \sqrt{4E|C|} \cot \left( \sqrt{|C|}/2qt \right).$$

(116)

Equation (114) is symmetric with respect to the turning point $V_0^{(1)} = (B - \sqrt{\Delta}) / 2|C|$. Its analytic continuation gives a “parent instanton” (see (102)) with

$$V = \frac{1}{|C|} \left[ \sqrt{4EC} \coth \left( \sqrt{|C|}/2qt \right) - B \right] \sinh^2 \left( \sqrt{|C|}/2qt \right),$$

(117)

$$-B^2/4|C| < E < 0, \quad 0 \leq t \leq \frac{2}{\sqrt{2|C| q}} \arcoth \frac{B + \sqrt{\Delta}}{\sqrt{4EC}},$$
which is responsible for the birth of a baby universe from “nothing”. The Lorentzian solution (114) symmetrically written with respect to the turning point $V^{(2)}_0 = \left(B + \sqrt{|\Delta|}\right)/2|C|$ reads

$$V = \frac{1}{2|C|} \left[ B + \sqrt{|\Delta|} + \left( B - \sqrt{|\Delta|} \right) \tan^2 \left( \sqrt{|C|}/2qt \right) \right] \cos^2 \left( \sqrt{|C|}/2qt \right),$$  \hspace{1cm} (118)

$$-B^2/4|C| < E < 0, \quad |t| \leq \pi/\sqrt{2|C|}.$$

iv) $B < 0, C < 0$: Here, a Lorentzian region exists for $E > 0$ only. From equation (66), we obtain

$$V = \frac{1}{|C|} \left[ 4E|C| \cot \left( \sqrt{|C|}/2qt \right) - |B| \right] \sin^2 \left( \sqrt{|C|}/2qt \right),$$  \hspace{1cm} (119)

$$E > 0, \quad 0 \leq t \leq \frac{2}{\sqrt{2|C|} \arg \left( \sqrt{|\Delta|} + |B| \right)},$$

and the equation relating the harmonic time coordinate and the synchronous one reads

$$\exp \left( \sqrt{2E}q(\tau - \tau_0) \right) = \sqrt{4E|C| \cot \left( \sqrt{|C|}/2qt \right) - |B|}.$$  \hspace{1cm} (120)

These equations are useful for the investigation of the small time limit $|t| \to 0$.

To obtain an instanton solution (wormhole) it is necessary to rewrite equation (119) symmetrically with respect to the classical turning point $V_0 = (-|B| + \sqrt{|\Delta|})/2|C|$. We can reformulate equation (119) or use directly equation (82). The result is

$$V = \frac{1}{2|C|} \left[ \sqrt{|\Delta|} - |B| - \left( \sqrt{|\Delta|} + |B| \right) \tan^2 \left( \sqrt{|C|}/2qt \right) \right] \cos^2 \left( \sqrt{|C|}/2qt \right),$$  \hspace{1cm} (121)

$$E > 0, \quad |t| \leq \frac{2}{\sqrt{2|C|} \arg \left( \sqrt{|\Delta|} - |B| \right)/\sqrt{|\Delta|} + |B|}.$$

We shall investigate now the small time limit $|t| \to 0$ for Lorentzian solutions. As we shall see, it corresponds to the vanishing volume limit $V \to 0$ and takes place for $E \geq 0$ if $B, C > 0$ or $B > 0, C < 0$ and for $E > 0$ if $B, C < 0$ or $B < 0, C > 0$ (see Fig. 1 and Fig. 2). First, we consider the case of positive energies $E > 0$. As follows from (74), (97), (116), and (120)

$$\exp \left( \sqrt{2E}qt \right) \sim t^{-1}, \quad t \to 0$$  \hspace{1cm} (122)

and with the help of equations (33), (35), and (36) we obtain for the scale factors in this limit the expressions

$$a_i \approx \bar{A}_i t^{\bar{\alpha}_i}, \quad t \to 0,$$  \hspace{1cm} (123)

where

$$\bar{\alpha}_i = \frac{1}{D-1} - \frac{1}{\sqrt{2E}q} \alpha^i$$  \hspace{1cm} (124)

and the parameters satisfy the conditions

$$\sum_{i=1}^{n} d_i \bar{\alpha}_i = 1,$$  \hspace{1cm} (125)
\[
\sum_{i=1}^{n} d_i (\bar{\alpha}_i)^2 = 1 - \frac{1}{q^2} \frac{\kappa^2 A^{(1)}(1)}{\varepsilon + \kappa^2 A^{(1)}} \to 1 \quad \text{for} \quad A^{(1)} \to 0 \quad (126)
\]

in accordance with the Eqs. (56) to (58). For the volume of the universe, we obtain in this limit

\[ V \sim t, \quad t \to 0. \quad (127) \]

Thus, for positive energy, \( E > 0 \), and small synchronous times the universe behaves like the Kasner universe.

Now, we consider the exceptional case \( E = 0 \). It follows from (73) that

\[ \tau \approx \frac{2}{q^2 B t}, \quad t \to 0 \quad (128) \]

and for the volume, we obtain from (76)

\[ V \sim t^2, \quad t \to 0. \quad (129) \]

With the help of Eq. (33), we conclude that the approximation of the scale factors is given by

\[ a_i \approx \bar{A}_i t^{2/(D-1)} \exp \left( \frac{2\alpha_i}{q^2 B t} \right), \quad t \to 0 \quad (130) \]

in accordance with expression (60) for \( \alpha^{(2)} = 1 \).

Thus, the scale factors behave either anisotropically and exponentially like

\[ a_i \sim \exp \left( \frac{2\alpha_i}{q^2 B t} \right), \quad t \to 0 \quad (131) \]

if \( \alpha_i \neq 0 \) \((\varepsilon > 0, A^{(1)} < 0)\) or they have power law behaviour like

\[ a_i \sim t^{2/(D-1)}, \quad t \to 0 \quad (132) \]

if \( \alpha_i = 0 \) \((\varepsilon = 0, A^{(1)} = 0)\) (see Eq. (61) for \( \alpha^{(2)} = 1 \)). In the latter case, the free minimally coupled scalar field is absent \((A^{(1)} = 0)\).

Similar investigations can be done for the equation (112) shifted in time such that \( V \approx t^2, \quad t \to 0 \).

If \( E < 0 \), the universe has in the Lorentzian region a classical turning point at the minimal volume \( V_{min} \) and reaches never \( V = 0 \) (see Fig. 1 and Fig. 2).

Now, let us consider the infinite volume limit \( V \to 0 \) which, as we shall see, coincides with the limit \( t \to +\infty \). As follows from Fig. 1 and Fig. 2, this is possible if \( B, C > 0 \) or \( B < 0, C > 0 \). With Eqs. (73) to (75) and (96), (98) to (100) one can demonstrate that \( \tau \) becomes asymptotically constant for \( t \to +\infty \) and the constant can be put equal to zero (with a proper choice of the integration constant \( \tau_0 \)). From (33) follows that isotropization takes place in this limit, namely

\[ a_i \sim V^{1/(D-1)}, \quad t \to +\infty \quad (133) \]

and from Eqs. (76) to (78) and (101), (103) to (105) we get

\[ V \approx \exp \left( \sqrt{2Cq} t \right), \quad t \to +\infty \quad (134) \]

in accordance with (49).
Thus, if $C > 0$ we obtain in the limit $t \to +\infty$ an (isotropic) de Sitter universe. If $C < 0$, the universe has a classical turning point at maximal volume $V_{\text{max}}$ and the volume can not go to infinity.

Let us come back once more to the case $C > 0$ describing a universe arising from “nothing”. The volume is given by (81), (105), and (103) and the harmonic time coordinate and the synchronous one are related via (80), (100), and (98), respectively. We shall restrict ourselves to the case $E < 0$ for simplicity. In this case, we get the asymptotic expression

$$
\tau \approx \frac{1}{\sqrt{|E|/2q}} \arctan \frac{\sqrt{2|E|C}}{\sqrt{|\Delta|} - B} \equiv A,
$$

(135)

if $t \gg \left(\sqrt{\frac{|C|}{2q}}\right)^{-1}$ (it is sufficient to satisfy $\sqrt{\frac{|C|}{qt}} \geq 2$). Then, as follows from equation (33), the scale factors are given by

$$
a_i \approx A_i \exp(\alpha_i A) V^{1/(D-1)}.
$$

(136)

In [17] was shown that for $4 \lesssim \sqrt{|C|/2q} \ll D - 1$ the parameters of the model can be chosen such that, due to the exponential prefactor in (136), some of the factor spaces (with $\alpha^i > 0$) undergo inflation after birth from “nothing” while other factor spaces (with $\alpha^i < 0$) remain compactified near the Planck length $L_{Pl} \approx 10^{-33} \text{ cm}$. The (graceful exit) mechanism responsible for the transition from the inflationary stage to the Kasner-like stage, in which the scale factors of the external spaces $M_i$ exhibit power-law behaviour while the internal spaces remain frozen in near the Planck scale, deserves still more detailed investigations, similar those of [17]. (There the complementary case of multidimensional cosmological models with cosmological constant was considered.)

5. CLASSICAL WORMHOLES

In this chapter we consider in more detail a special type of instantons, called wormholes. These usually are Riemannian metrics, consisting of two large regions joined by a narrow throat (handle). Obviously, they appear if the classical Lorentzian solutions of the model have turning points at some maximum, Namely, according to Fig. 1 and 2, for models with $C < 0$ (the parameter $B$ may be positive as well as negative). Let us show this explicitly. We consider instantons which can be obtained by analytic continuation $t \to -it$ of the Lorentzian solutions (112), (113), (118) and (121) respectively.

$$
V = \frac{B}{|C|} \cosh^2 \left(\sqrt{|C|/2qt}\right), \quad E = 0, \quad |t| < \infty,
$$

(137)

$$
V = \frac{1}{2|C|} \left[ \sqrt{|\Delta|} + B + \left(\sqrt{|\Delta|} - B\right) \tanh^2 \left(\sqrt{|C|/2qt}\right) \cosh^2 \left(\sqrt{|C|/2qt}\right) \right], \quad E > 0, \quad |t| < \infty,
$$

(138)

$$
V = \frac{1}{2|C|} \left[ B + \sqrt{|\Delta|} - \left( B - \sqrt{|\Delta|} \right) \tanh^2 \left(\sqrt{|C|/2qt}\right) \cosh^2 \left(\sqrt{|C|/2qt}\right) \right], \quad -B^2/4|C| < E < 0, \quad |t| < \infty,
$$

(139)

$$
V = \frac{1}{2|C|} \left[ \sqrt{|\Delta|} - |B| + \left( \sqrt{|\Delta|} + |B| \right) \tanh^2 \left(\sqrt{|C|/2qt}\right) \cosh^2 \left(\sqrt{|C|/2qt}\right) \right], \quad E > 0, \quad |t| < \infty.
$$

(140)
As mentioned before, these equations correspond (with evident substitutions) to the Lorentzian equations (101), (105), (103) and (81), respectively. The harmonic and synchronous times are related respectively by

\[ \tau = \frac{\sqrt{2|C|}}{Bq} \tanh \left( \frac{|C|}{2qt} \right), \quad E = 0, \quad (141) \]

\[ \tan \left( \frac{E}{2q\tau} \right) = \frac{\sqrt{4E|C|}}{\sqrt{|\Delta|} + B} \tanh \left( \frac{|C|}{2qt} \right), \quad E > 0, \quad (142) \]

\[ \tanh \left( \frac{|E|}{2q\tau} \right) = \frac{\sqrt{|\Delta|} - B}{\sqrt{4EC}} \tanh \left( \frac{|C|}{2qt} \right), \quad -\frac{B^2}{4|C|} < E < 0, \quad (143) \]

\[ \tan \left( \frac{E}{2q\tau} \right) = \frac{\sqrt{4E|C|}}{\sqrt{|\Delta|} - |B|} \tanh \left( \frac{|C|}{2qt} \right), \quad E > 0. \quad (144) \]

(See (96), (100), (98) and (80) respectively. (98) looks like (143), if we choose the constant of integration \( \tau_0 \) such that \( f|_{\tau=0} = V_0^{(2)} \), whence \( f = \sqrt{|\Delta|} \exp \left( \sqrt{2E}q\tau \right) \), and use the relation \( f = |B| - \sqrt{4EC} \frac{\arctanh \left( \frac{|C|}{2qt} \right)}{\sqrt{4EC}} \), where a turning point appears for \( t = 0 \).)

It can easily be seen from (141) to (144) that the harmonic time \( \tau \) is finite for the full range \(-\infty < t < \infty\) and goes to constants when \( |t| \to +\infty \).

For the spatial volume of the universe we have the asymptotic behaviour

\[ V \sim \exp \left( \sqrt{2|C|} q |t| \right), \quad |t| \to \infty, \quad (145) \]

for all cases (137) to (140).

In the Euclidean region (23) holds unchanged, since the Wick rotation \( \tau \to -i\tau \) has to be accompanied by the transformation \( \alpha^j \to i\alpha^j \) (\( p^j \to ip^j \)). This means that the parameter \( \varepsilon \) in the constraint equation (29) remains unchanged (see (28)).

Thus, the Euclidean metric reads

\[ ds^2 = dt^2 + a_1(t)g(1) + \ldots + a_n(t)g(n), \quad (146) \]

where each scale factor \( a_i \) has its own turning point at "time" \( t_i \), when \( \frac{dq}{dv}a_i = 0 \). The metric has its most simple and symmetric form in the case \( \varepsilon = 0 \) (\( \alpha^i = 0, i = 1, \ldots, n \)), whence

\[ ds^2 = dt^2 + V V^{2\tau} \left( g(1) + \ldots + g(n) \right), \quad (147) \]

where \( V \) is given by equations (137) to (140), and the throat is located at \( t = 0 \). In the limit \( |t| \to \infty \), the metric (146) converges to

\[ ds^2 = dt^2 + \exp \left( \frac{2\sqrt{2|C|} q |t|}{D - 1} \right) \left( g(1) + \ldots + g(n) \right), \quad (148) \]

which describes an asymptotically anti-de Sitter Euclidean universe. Thus, the metric (146) describes asymptotically anti-de Sitter wormholes.

The structure of a universe for models with classical Euclidean wormholes is shown schematically in Fig. 5 and Fig. 6 for the symmetric case \( \alpha^i = 0 \) (\( i = 1, \ldots, n \)) with a metric (147). (Note that, (147) has a common scale factor for all factor spaces, while for the universe of Eq. (87) in [23], sketched by similar looking Fig. 3 resp. Fig. 4 there, internal spaces are static, but external space is not.) There are two qualitatively different pictures.
The first case (see Fig. 5) takes place for $E \geq 0 \ (B > 0, \ C < 0)$ and for $E > 0 \ (B, \ C < 0)$ and describes asymptotically an anti-de Sitter wormhole and a baby universe which can branch off from this wormhole. The second case (see Fig. 6) takes place for $-B^2/4|C| < E < 0 \ (B > 0, \ C < 0)$ and describes, besides wormhole and baby universe, an additional parent instanton which is responsible for the birth of the universe from nothing.

### 6. RECONSTRUCTION OF THE POTENTIALS

The effective perfect fluid Lagrangian [13] has its origin in the scalar field Lagrangian [8]. In this Chapter, we calculate a class of potentials $U^{(a)}(\varphi^{(a)})$ which ensure the equivalence of these Lagrangians. The procedure of potential reconstruction was proposed in [16] and is applied in the following. For the integrable 3-component model holds

$$\varphi^{(a)} = \pm \sqrt{\frac{\alpha^{(a)}}{q}} \int_0^{V_a} \frac{dV}{\rho^{(a)}(V)} + \varphi^{(a)}_0, \quad a = 1, 2, 3, \quad (149)$$

where the energy density $\rho^{(a)}$ is given by (14) and $\alpha^{(1)} = 2, \ \alpha^{(2)} = 1, \ \alpha^{(3)} = 0$. $\varphi^{(a)}_0$ is a constant of integration. We should stress that equation (149) was obtained for Lorentzian regions. As a result, we get the scalar fields $\varphi^{(a)}$ as a function of the spatial volume $V$. Inverting this expression, we find the spatial volume as a function of the scalar field $\rho^{(a)}(\varphi^{(a)})$. Consequently, a dependence of the energy density $\rho^{(a)}$ on the scalar field $\varphi^{(a)}$, $\rho^{(a)} = \rho^{(a)}(\varphi^{(a)})$. Then, using Eqs. (10) to (12), we find the potential $U^{(a)}(\varphi^{(a)})$ in the form

$$U^{(a)}(\varphi^{(a)}) = \frac{1}{2} (2 - \alpha^{(a)}) \rho^{(a)}(\varphi^{(a)}), \quad a = 1, 2, 3, \quad (150)$$

where

$$\rho^{(a)} = A^{(a)} \left[ V(\varphi^{(a)}) \right]^{-\alpha^{(a)}}. \quad (151)$$

The third component of the scalar field has $\alpha^{(3)} = 0$. Then, from (151) and (151) it follows that $\varphi^{(3)}$, $U^{(3)}$, and $\rho^{(3)}$ are constant. This scalar field component with the equation of state $P^{(3)} = -\rho^{(3)}$ is equivalent to the cosmological constant $\Lambda \equiv \kappa^2 U^{(3)} = \kappa^2 A^{(3)} = C$. For $\alpha^{(1)} = 2$, we have $U^{(1)} \equiv 0$ (free scalar field). In this case, the scalar field $\varphi^{(1)}$ is equivalent to a ultra-stiff perfect fluid ($P^{(1)} = -\rho^{(1)}$). Equation (149) reads in this case

$$\varphi^{(1)} - \varphi^{(1)}_0 = \pm \sqrt{\frac{A^{(1)}}{q}} \int_0^E \frac{dV}{\sqrt{E + BV + CV^2}}, \quad (152)$$

where $E$ and $B$ are defined by (14) and (15) respectively.

A consequence of (151) is

$$\varphi^{(1)} - \varphi^{(1)}_0 = \pm \sqrt{2A^{(1)}} \tau. \quad (153)$$

This result is expected for a free minimal coupled scalar field in the harmonic time gauge where $\varphi^{(1)} = 0$. After integration in (152),

$$\varphi^{(1)} - \varphi^{(1)}_0 = \pm i \frac{2\sqrt{|A^{(1)}|}}{q} \frac{\sqrt{BV + CV^2}}{BV}, \quad E = 0, \quad (154)$$

$$\varphi^{(1)} - \varphi^{(1)}_0 = \pm \sqrt{\frac{A^{(1)}}{q}} \ln \frac{2E + BV - 2\sqrt{ER}}{2V}, \quad E > 0, \quad (155)$$
\[ \phi^{(1)} - \phi_0^{(1)} = \pm i \frac{\sqrt{|A^{(1)}|}}{\sqrt{|E|} q} \arcsin \frac{BV - 2|E|}{V \sqrt{|\Delta|}}, \quad E < 0, \quad B^2 - 4EC > 0, \quad (156) \]

with \( R := E + BV + CV^2 \) and \(|\Delta| = B^2 - 4EC. \) For \( E \leq 0 \) (i.e. \( A^{(1)} < 0 \)) this scalar field is imaginary.

Let us now consider the second component with \( \alpha^{(2)} = 1. \) The scalar field \( \phi^{(2)} \) is equivalent to dust \( (P^{(2)} = 0). \) Equation (149) reads now

\[ \phi^{(2)} - \phi_0^{(2)} = \pm \frac{\sqrt{A^{(2)}}}{2q} \int \frac{dV}{\sqrt{V \sqrt{E + BV + CV^2}}}, \quad (157) \]

\( \phi^{(2)} \) is imaginary for \( A^{(2)} < 0, \) i.e. \( B < 0. \) The integral in (157) is an elliptic one and, in general, it is not possible to express it by elementary functions. But in the particular case \( E = 0, \) which expresses the asymptotic behaviour of the scalar field (157), we get

\[ \phi^{(2)} - \phi_0^{(2)} = \mp \frac{\sqrt{2}}{\kappa q} \arcoth \left( 1 + \frac{C}{B} V \right)^{1/2}, \quad B, C > 0, \quad (158) \]

\[ \phi^{(2)} - \phi_0^{(2)} = \mp \frac{\sqrt{2}}{\kappa q} \tanh \left( 1 - \frac{|C|}{B} V \right)^{1/2}, \quad B > 0, C < 0, \quad (159) \]

\[ \phi^{(2)} - \phi_0^{(2)} = \mp i \frac{\sqrt{2}}{\kappa q} \arctan \left( \frac{C}{|B|} V - 1 \right)^{1/2}, \quad B < 0, C > 0, \quad (160) \]

the volume of the universe

\[ V = \frac{B}{C} \sinh^{-2} \left[ \frac{\kappa q}{\sqrt{2}} \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B, C > 0, \quad (161) \]

\[ V = \frac{B}{|C|} \cosh^{-2} \left[ \frac{\kappa q}{\sqrt{2}} \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B > 0, C < 0, \quad (162) \]

\[ V = \frac{|B|}{C} \cos^{-2} \left[ \frac{\kappa q}{\sqrt{2}} i \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B < 0, C > 0, \quad (163) \]

and the potential of the scalar field

\[ U^{(2)}(\phi^{(2)}) = \frac{C}{2\kappa^2} \sinh^2 \left[ \frac{\kappa q}{\sqrt{2}} \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B, C > 0, \quad (164) \]

\[ U^{(2)}(\phi^{(2)}) = \frac{|C|}{2\kappa^2} \cosh^2 \left[ \frac{\kappa q}{\sqrt{2}} \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B > 0, C < 0, \quad (165) \]

\[ U^{(2)}(\phi^{(2)}) = \frac{C}{2\kappa^2} \cos^2 \left[ \frac{\kappa q}{\sqrt{2}} i \left( \phi^{(2)} - \phi_0^{(2)} \right) \right], \quad B < 0, C > 0. \quad (166) \]

It follows from these equations that for \( B, C > 0 \) and \( B, C < 0 \) the volume goes to infinity like

\[ V \sim \frac{1}{|\phi^{(2)}|^2} \to +\infty, \quad |\phi^{(2)}| \to 0. \quad (167) \]

The general expression (157) should have the same asymptotic behaviour in all the cases where the limit \( V \to +\infty \) is permitted, because we can drop in this limit the term \( E \) in the denominator of (157).

If \( E > 0, \) from (157) results

\[ \phi^{(2)} - \phi_0^{(2)} \approx \frac{2A^{(2)}/E}{q} \sqrt{V} \to 0, \quad V \to 0. \quad (168) \]
Let us now consider two particular cases of (157) for \( E \neq 0 \). The first case is that one when the classical trajectory has two turning points \( V_{0}^{(1,2)} \), i. e. when either \( B > 0, C < 0 \) or \( B < 0, C > 0 \) (for both the cases \( B^2 > 4EC \)). Then (see equation (3.131) in [11]),

\[
\varphi^{(2)} = \pm \frac{2}{\kappa q} \frac{\sqrt{B}}{|B| + \sqrt{|\Delta|}} F(\psi|m),
\]

where \( F(\psi|m) \) is the elliptic integral of the first kind [12] and

\[
\psi = \arcsin \sqrt{\frac{V_0^{(1)} - V}{V_0^{(2)} - V_0^{(1)}}}, \quad V_0^{(2)} > V \geq V_0^{(1)}, \quad B > 0, C < 0, \tag{170}
\]

\[
\psi = \arcsin \sqrt{\frac{V}{V_0^{(1)}}}, \quad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0, C > 0, \tag{171}
\]

\[
\psi = \arcsin \sqrt{\frac{V - V_0^{(2)}}{V - V_0^{(1)}}}, \quad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0, C > 0, \tag{172}
\]

\[
m = \sqrt{1 - \frac{V_0^{(1)}}{V_0^{(2)}}}, \quad V_0^{(2)} > V \geq V_0^{(1)}, \quad B > 0, C < 0, \tag{173}
\]

\[
m = \sqrt{\frac{V_0^{(1)}}{V_0^{(2)}}}, \quad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0, C > 0, \tag{174}
\]

\[
m = \sqrt{\frac{V_0^{(1)}}{V_0^{(2)}}}, \quad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0, C > 0. \tag{175}
\]

The turning points are

\[
V_0^{(1,2)} = \frac{|B| + \sqrt{|\Delta|}}{2|C|}. \tag{176}
\]

The minus sign is related to \( V_0^{(1)} \), the plus sign to \( V_0^{(2)} \), and \( |\Delta| = B^2 - 4EC \). The scalar field \( \varphi^{(2)} \) is imaginary for \( B < 0 \).

With the Jacobian elliptic functions [12], inverting (169), the volume of the universe is given by

\[
V = V_0^{(2)} - s_\text{n}^2(V_0^{(2)} - V_0^{(1)}), \quad V_0^{(2)} > V \geq V_0^{(1)}, \quad B > 0, C < 0, \tag{177}
\]

\[
V = s_\text{n}^2V_0^{(1)}, \quad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0, C > 0, \tag{178}
\]

\[
V = \frac{V_0^{(2)} - s_\text{n}^2V_0^{(1)}}{1 - s^2}, \quad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0, C > 0, \tag{179}
\]

where \( s_\text{n} \equiv s_\text{n}(W_{1}\varphi^{(2)}|m) = \sin \psi \) and \( W_{1}^{-1} = \pm \frac{2\sqrt{B}}{\kappa q \sqrt{|B| + |\Delta|}} \). The corresponding potential terms are then given as \( U^{(2)} = A^{(2)}/2V \) (see (150) and (151) for \( \alpha^{(2)} = 1 \)). According to the properties of the Jacobian elliptic functions [12] asymptotic estimates for (178) and (179) are \( V \approx (q^2E/2A^{(2)})(\varphi^{(2)})^2 \) for \( |\varphi^{(2)}| \to 0 \) (in accordance with (168)) and \( V \sim 1/|\varphi^{(2)}|^2 \) for \( |\varphi^{(2)}| \to 0 \) (in accordance with (167)).

Another particular case for \( C > 0 \) is that with \( E > B^2/4C \). Here (see (3.138 (7)) in [11]), we obtain

\[
\varphi^{(2)} = \pm \frac{\sqrt{B}/|2C|}{\kappa q} \frac{1}{(E/C)^{1/4}} F(\psi|m), \tag{180}
\]

where

\[
\psi = 2 \arctan \sqrt{\frac{V}{\sqrt{E/C}}}, \tag{181}
\]

and

\[
m = \sqrt{(2\sqrt{EC} - B)/4\sqrt{EC}}. \tag{182}
\]
Inverting equation (180), we obtain
\[ V = \frac{2 - \text{sn}^2}{\text{sn}^2} \sqrt{E/C} \pm \frac{\sqrt{2 - \text{sn}^2} \sqrt{E/C}}{\text{sn}^2} - \frac{E}{C} \]  
(183)
with \( \text{sn} \equiv \text{sn}(I_2 \varphi^{(2)}|m) = \sin \psi \) and \( I_2^{-1} = \pm \sqrt{\frac{B}{2C}/m} \). For the branch with the plus sign, 
\[ V \sim \frac{1}{|\varphi^{(2)}|} \rightarrow \infty \text{ for } |\varphi^{(2)}| \rightarrow 0 \text{ (in accordance with (167)) and for the branch with the minus sign, } V \approx \left( q^2 E/2A^{(2)} \right)^2 \rightarrow 0 \text{ for } |\varphi^{(2)}| \rightarrow 0 \text{ (in accordance with (168)).} \]

To find the scalar field potential, we have to substitute (183) into \( U^{(2)}(\varphi^{(2)}) = A^{(2)}/2V \).

7. SOLUTIONS TO THE QUANTIZED MODEL

At the quantum level, the constraint equation (25) is replaced by the Wheeler-DeWitt (WDW) equation. The WDW equation is covariant with respect to gauge as well as minisuperspace coordinate transformations [25]. In the harmonic time gauge [13, 25] it reads
\[ \left( \frac{1}{2} \frac{\partial^2}{\partial z_0^2} - \frac{1}{2} \sum_{i=1}^{n-1} \frac{\partial^2}{\partial z_i^2} + \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z_0) \right) \Psi = 0. \]  
(184)
Formally, this WDW equation has the same structure as that of [23]. However, on the semiclassical level the dynamics of the universe is quite different for the models in both the papers. Semiclassical equations were considered in [43].

We look for solutions of (184) by separation of the variables and try the ansatz
\[ \Psi(z) = \Phi(z^0) \exp(i \mathbf{p} \cdot \mathbf{z}), \]  
(185)
where \( \mathbf{p} := (p^1, \ldots, p^{n-1}) \) is a constant vector, \( \mathbf{z} := (z^1, \ldots, z^{n-1}) \), \( p_i = p^i \) and \( \mathbf{p} \cdot \mathbf{z} := \sum_{i=1}^{n-1} p_i z^i \). Substitution of (185) into (184) gives
\[ \left[ \frac{1}{2} \frac{d^2}{dz_0^2} + \frac{1}{2} \sum_{i=1}^{n-1} (p_i)^2 + \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z_0) \right] \Phi = 0. \]  
(186)
For the integrable 3-component model this equation reduces to
\[ \left[ -\frac{1}{2} \frac{d^2}{dz_0^2} + U(z^0) \right] \Phi = E \Phi \]  
(187)
in the notation of (64) and (65). Following [23], we rewrite this equation like
\[ x^2 \frac{d^2 \Phi}{dx^2} + x \frac{d \Phi}{dx} + \left( \tilde{E} + \tilde{B} x + \tilde{C} x^2 \right) \Phi = 0, \]  
(188)
where \( \tilde{E} = 2E/q^2, \tilde{B} = 2B/q^2, \tilde{C} = 2C/q^2, \) and \( x = \exp(q z_0) \) (\( x \) is identical to the spatial volume of the universe \( V \)). (188) is equivalent to the Whittaker equation
\[ \frac{d^2 y}{d\xi^2} + \left[ -\frac{1}{4} + \frac{\tilde{B}/T}{\xi} + \frac{\tilde{E} + 1/4}{\xi^2} \right] y = 0, \]  
(189)
where $T := \pm 2i\sqrt{C}$, $\xi := Tx$, and $\Phi =: x^{-1/2}y(\xi)$ and also equivalent to the Kummer equation
\[
\xi \frac{d^2 w}{d\xi^2} + (1 + 2\mu - \xi) \frac{dw}{d\xi} - \left[ \frac{1}{2} + \mu - \frac{\tilde{B}}{T} \right] w = 0,
\]  
(190)
where $\mu^2 := -\tilde{E}$ and $\Phi =: x^{-1/2} \exp \left( -\frac{i}{x} \xi \right) \xi^{1/2+\mu} w(\xi)$. In the first case, the solutions are the Wittaker functions $w_1 := M_{k,\mu}(\xi)$ and $w_2 := W_{k,\mu}(\xi)$ with $k := \tilde{B}/T$ and $\mu^2 := -\tilde{E}$. In the second case, the solutions are the Kummer functions $w_1 := M(a, b, \xi)$ and $w_2 := U(a, b, \xi)$ with $a := \frac{1}{2} + \mu - \tilde{B}/T$ and $b := 1 + 2\mu$.

The general solution of equation (184) for the 3-component model is
\[
\Psi(z) = \sum_{i=1,2} \int d^{n-1} p C_i(p) \exp (i p \cdot z) \Phi_E^{(i)}(\exp(qz^0)),
\]  
(191)
where $\Phi_E^{(1,2)} = \frac{1}{\sqrt{x}} y_{1,2}(\xi)$, or $\Phi_E^{(1,2)} = \frac{1}{\sqrt{x}} \exp \left( -\frac{1}{2} \xi \right) \xi^{1/2+\mu} w_{1,2}(\xi)$. It is convenient to set $T = +2i\sqrt{C}$ for $C > 0$ and $T = -2i\sqrt{C}$ for $C < 0$, and $\mu := +\sqrt{-E}$.

In [43], it was argued that the parameter $E$ can be interpreted as energy. So, the state with $E = 0$, vanishing momenta $p_i (i = 1, \ldots, n - 1)$, and $A^{(1)} = 0$ (absence of free scalar field excitations) is the ground state of the system. Thus, its wave function reads
\[
\Psi_0 = \Phi_0^{(i)} \left( \exp(qz^0) \right), \quad i = 1, 2.
\]  
(192)
The limit of large spatial geometries in (188) $z^0 \to +\infty$ (remember $x \equiv V = \exp(qz^0)$) is equivalent to $\tilde{B} \to 0$. In this limit, the Wittaker functions reduce to Bessel functions [42], namely
\[
M_{k,\mu}(\xi) \underset{k \to 0}{\longrightarrow} \sqrt{V} J_\mu \left( \sqrt{C} V \right),
\]  
(193)
\[
W_{k,\mu}(\xi) \underset{k \to 0}{\longrightarrow} \sqrt{V} H_{\mu}^{(2)} \left( \sqrt{C} V \right)
\]  
(194)
for $C > 0$ and
\[
M_{k,\mu}(\xi) \underset{k \to 0}{\longrightarrow} \sqrt{V} I_\mu \left( \sqrt{|C|} V \right),
\]  
(195)
\[
W_{k,\mu}(\xi) \underset{k \to 0}{\longrightarrow} \sqrt{V} K_\mu \left( \sqrt{|C|} V \right)
\]  
(196)
for $C < 0$.

Following the ideas of [43, 44], one can demonstrate that for $C > 0$ the wave function
\[
\Psi_0^{HH} = \Phi_0^{(1)} \underset{k \to 0}{\longrightarrow} J_0 \left( \frac{\sqrt{2C}}{q} V \right) \sim \cos \left( \frac{\sqrt{2C}}{q} V \right)
\]  
(197)
corresponds to the Hartle-Hawking boundary condition [45] and the wave function
\[
\Psi_0^{V} = \Phi_0^{(2)} \underset{k \to 0}{\longrightarrow} H_0^{(2)} \left( \frac{\sqrt{2C}}{q} V \right) \sim \exp \left( -i \frac{\sqrt{2C}}{q} V \right)
\]  
(198)
corresponds to the Vilenkin boundary condition [46].

In the case $C < 0$, we get
\[
\Psi_0^{HH} = \Phi_0^{(1)} \underset{k \to 0}{\longrightarrow} I_0 \left( \frac{\sqrt{2|C|}}{q} V \right) \sim \exp \left( \frac{\sqrt{2|C|}}{q} V \right)
\]  
(199)
The potential has for $B > 0$, $C < 0$ a well and for $E < 0$ the energy spectrum is discrete (see Fig. 2). In this case, the finite solutions of the wave equation (187) [47] are

$$\Phi_n = \exp \left( -\frac{1}{2} \xi \right) \xi^\mu M(-n, b, \xi).$$

(200)

The energy levels are given by

$$-E_n = \left[ \frac{B}{2\sqrt{|C|}} - \frac{q}{\sqrt{2}} \left( n + \frac{1}{2} \right) \right]^2.$$  

(201)

$n$ is a non negative integer and restricted to

$$n < \frac{B}{q\sqrt{2|C|}} - \frac{1}{2}.$$  

(202)

Thus, the discrete spectrum has a finite number of eigenvalues. If $\frac{B}{q\sqrt{2|C|}} < \frac{1}{2}$, there is no discrete spectrum. It was demonstrated in [23] that the wave functions (200) satisfy the quantum wormhole boundary conditions [15].

8. CONCLUSIONS

We considered the generalization of a homogeneous cosmological model of Bianchi type I to an anisotropic multidimensional one with $n \geq 2$ Ricci-flat spaces of arbitrary dimensions, in the presence of $m$ homogeneous non-interacting minimally coupled scalar fields. Under certain conditions these models are equivalent to multidimensional cosmological models in the presence of an $m$-component perfect fluid with equations of state $P^{(a)}(a) = (\alpha^{(a)} - 1) \rho^{(a)}$ with matter constants $\alpha^{(a)}$ for $a = 1, \ldots, m$. Using this equivalence, for $m = 3$, we find integrable models when one of the scalar fields is equivalent to an ultra-stiff perfect fluid component, the second one corresponds to dust, and the third one is equivalent to a vacuum component. Dynamics of the universe was investigated in general, as well as in a particular 3-component integrable case. For integrable models, there are four qualitatively different types of evolution of the universe, depending on the potential $U(z_0)$ (see Fig. 1 and Fig. 2), but in all four cases the universe has a Kasner-like behaviour near the cosmological singularity (see [23]). In the cases where the universe can expand to infinity, an isotropization takes place which results in an asymptotically de Sitter universe (see [33] and [34]).

In quantum cosmology, instantons, solutions of the classical Einstein equations in Euclidean space, play an important role, giving significant contributions to the path integral. They are connected with the changing geometry of the model. We found here three interesting types of instantons. The first one (see (107) and (109)) describes tunnelling between a Kasner-like universe and an asymptotically de Sitter universe. Sewing a number of these instantons (see Fig. 4) may provide the Coleman mechanism for the vanishing of the cosmological constant. Another type of instanton (see (32) and (108)) is responsible for the birth of the universe from "nothing". It was shown that corresponding Lorentzian solutions (81) and (105) can ensure inflation of the external space (see (136)) and compactification of the internal ones. This problem needs a more detailed investigation in a separate paper. The third type of instantons (see equations (137) to (140)) describes the Euclidean space which has an asymptotically anti-de Sitter wormhole geometry (see Fig. 5 and Fig. 6).
The scalar field potentials $U^{(a)}(\varphi^{(a)}) \ (a = 1, \ldots, m)$ can be reconstructed by the method described in \([16]\). We performed this procedure for integrable models, and exact forms of potentials were presented in section 6.

The equivalence between a scalar field and a perfect fluid component helps also to investigate the quantum behaviour of the universe. We obtained the Wheeler-de Witt equation from the effective perfect fluid Lagrangian. Exact solutions are found, some of which describe cosmological transitions with a signature change of the metric, e.g. universe nucleation as quantum tunnelling from an Euclidean region. Other solutions are given as quantum wormholes with discrete spectrum (see \([201]\)).

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FIGURE CAPTIONS

Fig. 1. The potential \( U_0(z^0) \) (solid line) and the energy levels \( E \) (dashed lines) in the cases \( B,C > 0 \) and \( B,C < 0 \). Lorentzian regions exist for \( E > U_0(z^0) \).

Fig. 2. The potential \( U_0(z^0) \) (solid line) and the energy levels \( E \) (dashed lines) in the cases \( B > 0, C < 0 \) and \( B < 0, C > 0 \). In the former case we get a potential well, and in the latter case we obtain a potential barrier. Lorentzian regions exist for \( E > U_0(z^0) \). Here, \( U_m = B^2/4C \) and \( z_m^0 = \frac{1}{q} \ln |B/2C| \).

Fig. 3. The qualitative shape of the instanton \( (107) \) (or \( (109) \)). The instanton describes tunnelling between a Kasner-like (baby) universe and a de Sitter universe.

Fig. 4. Examples of instantons constructed by sewing together several copies of the the instanton illustrated in Fig. 3. Such instantons may describe tunnelling between (a) Kasner and de Sitter universe, (b) two Kasner universes, (c) two de Sitter universes.

Fig. 5. An asymptotically anti-de Sitter wormhole is shown schematically for energies \( E \geq 0 \) \( (B > 0, C < 0) \) and \( E \geq 0 \) \( (B,C < 0) \) in the symmetrical case \( \alpha^i = 0 \) \( (i = 1,\ldots,n) \). (Note that due to suppressed internal degrees of freedom this looks like Fig. 3 in [23], although the solution and its context are quite different there.)

Fig. 6. The qualitative structure of the universe is is shown schematically for energies \( -U_m < E < 0 \) \( (B > 0, C < 0) \) in the symmetrical case \( \alpha^i = 0 \) \( (i = 1,\ldots,n) \). (Note that due to suppressed internal degrees of freedom this looks like Fig. 4 in [23], although the solution and its context are quite different there.)