The Implications of the Microwave Background Anisotropies for Laser-Interferometer-Tested Gravitational Waves

L. P. Grishchuk *

Department of Physics and Astronomy
University of Wales, Cardiff CF2 3YB, UK

and

Moscow State University, 119899 Moscow V234, Russia

Abstract

The observed microwave background anisotropies in combination with the theory of quantum mechanically generated cosmological perturbations predict a well measurable amount of relic gravitational waves in the frequency intervals tested by LISA and ground based laser interferometers.

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At the first glance, the ground-based and space laser interferometers for gravity wave observations, as well as the Weber bar technique, do not have much in common with the ongoing and planned radio-astronomical measurements of the microwave background anisotropies. The solid-state detectors are sensitive to gravitational waves in the $10^3$ Hz frequency range, the laser interferometers are sensitive, correspondingly, to the frequencies $(10^{-1} - 10^3)$ Hz and $(10^{-4} - 10^{-1})$ Hz, whereas the microwave background anisotropies directly reflect only the variations in the cosmic temperature and, if they are caused by gravitational waves, can only provide us with information about extremely low-frequency gravitational waves - $(10^{-16} - 10^{-18})$ Hz and lower. However, the basic physics which enables us to see gravitational waves with the help of laser interferometers or microwave background anisotropies is exactly the same: alterations in frequency and phase of an electromagnetic signal propagating in the field of gravitational waves. Most importantly, the relic stochastic background of gravitational waves, to be discussed below, extends from very high frequencies of the order of $10^8$ Hz to extremely low frequencies of the order of $10^{-18}$ Hz and lower. It can be observed by all these techniques, and the predictions about the expected gravitational wave amplitudes in various frequency intervals are connected to each other by the theory.

Although the measured large-angular-scale anisotropies in the cosmic microwave background radiation (CMBR) could be expected on general grounds, their actual existence raises certain theoretical problems. The observed Universe is far from being homogeneous and isotropic, but becomes more and more so when one expands the study to larger and larger scales.
The anisotropies signify the presence in the Universe of cosmological perturbations with very small amplitudes (the dimensionless deviations are of the order of $10^{-5}$) but with extremely long wavelengths, of the order of and longer than the present-day Hubble radius $l_H$, $l_H \approx 2 \times 10^{28}$ cm. Regardless of nature of the perturbations responsible for the observed large-angular-scale anisotropy, that is, regardless of whether they are mostly density perturbations, or rotational perturbations, or gravitational waves, or all mixed together, there exists a puzzling question of their origin. The first wonder is whether they are remnants of the originally inhomogeneous and anisotropic Universe or, alternatively, were generated by some mechanism in the originally homogeneous and isotropic Universe. Since the perturbations of our interest are weak, we can use the linearized Einstein equations for the description of their evolution.

It is difficult to maintain that these perturbations are simply survived. The photons of the CMBR have become free and started their journey to us sometime at the beginning of the matter-dominated era. In the preceding radiation-dominated era, the general solution for a Fourier component of the metric perturbations is

$$h_n(\eta, x) = A \sin(n\eta + \chi) \frac{1}{a(\eta)} e^{i n \cdot x}$$  \hspace{1cm} (1)

where $A$ and $\chi$ are arbitrary constants, $a(\eta)$ is the scale factor of a FLRW universe

$$ds^2 = a^2(\eta) (d\eta^2 - dx^2 - dy^2 - dz^2)$$  \hspace{1cm} (2)

and $a(\eta) \sim \eta$ in the radiation-dominated era. Let us take the amplitude $A$ at the level $A \approx 10^{-5}$, in rough agreement with observations, and return back in
depth of the radiation-dominated era by sending $\eta$ to zero. The scale factor $a(\eta)$ diminishes by at least the factor $10^8$ by the time of reaching the era of the primordial nucleosynthesis. The dangerous term $A \cos n\eta \sin \chi e^{inx}/a(\eta)$ must be cancelled out with a great precision, if we do not want the $h_\eta(\eta, x)$ to become of the order of 1 and destroy in this way the homogeneity and isotropy of that era. That is, the phase $\chi$ must be 0 or $\pi$ with a precision of $10^{-3}$, or even much higher if we want to proceed further back in time. The neighboring solutions must all have the same (or differing by $\pi$) phases, that is, the distribution of the phases must be very narrow (highly “squeezed”); and the waves must be standing, and not traveling. (We will see below that all that is automatically guaranteed if the perturbations are generated quantum mechanically.) The solution (1) is strictly valid for gravitational waves, but the same argument is applicable for density perturbations as well.

It is still possible that the perturbations of our interest are classical, deterministic remnants of a strongly inhomogeneous anisotropic universe of a very distant past. It is also possible that, for some miraculous reason, the phases have been chosen rightly with enormous precision, so that the perturbations are classical, deterministic remnants of a universe which was almost homogeneous and isotropic from the very beginning. To the present author, these possibilities do not seem to be likely, even if they can be shown to be consistent with all available data. We need to turn to possibilities of generating the perturbations in an originally FLRW universe.

In principle, there are several options to do that. For instance, one can try to exploit the fact that the number of unknown functions of time participating in the perturbed Einstein equations is always greater than the number
of equations. These functions describe nonadiabatic pressure, entropy perturbations, anisotropic stresses, etc.. By manipulating with these functions and making additional assumptions, such as that these functions represent “cosmological defects” or “causal seeds”, one can produce the required perturbations, but essentially “by hand”. It seems to the writer that we should first try to build the theory on a minimal number of hypotheses.

It is known that the Einstein equations plus basic principles of quantum field theory allow (in fact, demand) the quantum-mechanical generation of cosmological perturbations from the vacuum state, as a result of parametric (superadiabatic) interaction of the quantized perturbations with strong variable gravitational field of the very early Universe. This process is possible for gravitational waves [3], for density perturbations [7, 8], and for rotational perturbations [4]. However, in the last two cases we need to assume that the primeval matter was capable of supporting the oscillations and that they were properly coupled (similar to gravitational waves) to the “pumping” gravitational field. If, as is assumed in the inflationary hypothesis, the very early Universe was governed by a scalar field, and if the field was minimally coupled to gravity, the quantum mechanical generation of density perturbations, in addition to the inevitable generation of gravitational waves, was possible. Below, we will follow the line of the quantum-mechanical generation of cosmological perturbations.

In the presence of perturbations, the metric tensor and the energy-momentum tensor can be written in the form

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij})dx^idx^j], \quad (3)$$
\[ T^\mu_\nu = T^\mu_{\nu}^{(0)} + T^\mu_{\nu}^{(1)}. \]

In cosmology, one usually considers the \( T^\mu_{\nu} \) for simple models of matter, such as perfect fluids or scalar fields. The perturbations \( h_{ij} \) and \( T^\mu_{\nu}^{(1)} \) are linked together by a set of the linearized Einstein equations. It is convenient to expand the perturbations over spatial harmonics \( e^{in\cdot x}, e^{-in\cdot x} \), where \( n = (n^1, n^2, n^3) \) is arbitrary wavevector and the wavenumber \( n \) is

\[ n = [n^1]^2 + [n^2]^2 + [n^3]^2]^{1/2}. \]

Demonstrating certain mathematical skill and understanding of the physical side of the problem, one can show that, for the simple models of matter, the perturbed Einstein equations for each \( n \)-mode and for each of the three types of cosmological perturbations (density perturbations, rotational perturbations, gravitational waves) can be reduced to a single differential equation of second order. It is only this one equation that defines the dynamical content of the problem and needs to be solved. All the components \( h_{ij}, T^\mu_{\nu}^{(1)} \) can then be found from its solutions by algebraic and differentiation/integration operations. This equation has the form of the equation for an oscillator with a variable frequency. The frequency varies due to the presence of the time-dependent scale factor \( a(\eta) \) which plays the role of the gravitational pump field. The very form of this equation explains why the cosmological perturbations can be parametrically amplified (if the initial classical amplitude was not zero) or quantum-mechanically generated from the zero-point quantum oscillations. In case of gravitational waves, this equation is
\[
\mu'' + [n^2 - \frac{a''}{a}] \mu_n = 0 \tag{4}
\]

and \( h_n(\eta, x) \sim \frac{1}{a} \mu_n e^{in.x} \).

We will briefly summarize the main points of the quantum-mechanical generation of cosmological perturbations (for more details, see a recent paper [10] and references therein). The quantum-mechanical operator for \( h_{ij}(\eta, x) \) can be written as

\[
h_{ij}(\eta, x) = \frac{C}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 n \sum_{s=1}^{2} \hat{p}_{ij}(n) \frac{1}{\sqrt{2n}} \left[ \hat{c}_s(n, \eta) e^{in.x} + \hat{c}_s^\dagger(n, \eta) e^{-in.x} \right]. \tag{5}
\]

Each of the three types of perturbations has two polarisation states \((s = 1, 2)\) described by two polarisation tensors \( \hat{p}_{ij} \). The normalization constant \( C \) is, up to a numerical factor slightly different for each type of the perturbations, the Planck length \( l_{Pl} \), \( l_{Pl} = (G \hbar/c^3)^{1/2} \). For gravitational waves, \( C = \sqrt{16\pi l_{Pl}} \). The time dependent creation and annihilation operators \( \hat{c}_n(\eta), \hat{c}_n^\dagger(\eta) \) are governed by the Heisenberg equations of motion. The parametric nature of the interaction Hamiltonian allows one to apply the Bogoliubov transformation and to express the \( \hat{c}_n^\dagger(\eta), \hat{c}_n(\eta) \) in terms of their initial values \( \hat{c}_n^\dagger(0), \hat{c}_n(0) \). The operators \( \hat{c}_n^\dagger(0), \hat{c}_n(0) \) define the initial vacuum state \( |0\rangle \) for each \( n \) and \( s \); \( \hat{c}_n^\dagger|0\rangle = 0 \). The field operator (5) acquires the form

\[
h_{ij}(\eta, x) = \frac{C}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 n \sum_{s=1}^{2} \hat{p}_{ij}(n) \frac{1}{\sqrt{2n}} \left[ \hat{h}_n(\eta) e^{in.x} \hat{c}_n(0) + \hat{h}_n^*(\eta) e^{-in.x} \hat{c}_n^\dagger(0) \right]
\]

\[
\tag{6}
\]
where \( \hat{h}_n(\eta) \) are essentially the solutions to the single second-order differential equation (for each type of perturbations) mentioned above.

In the Schrodinger picture, the initial vacuum state \( |0_n\rangle|0_{-n}\rangle \), for every pair \( n, -n \) of modes, evolves into a two-mode squeezed vacuum quantum state. The modes affected by the amplification process will have a large mean value of the number operator \( N \) (large occupation number) and a large variance of \( N \). The conjugate variance of phase will be highly squeezed near the values \( \chi = 0, \pi \) (in the representation (1)). Classically, the generated perturbations can be treated as a stochastic collection of standing waves.

The scale factor \( a(\eta) \) of the very early Universe (well before the era of primordial nucleosynthesis) is not known. It depends on the unknown equation of state of the extremely dense matter (we are not quite sure even about the equation of state in cores of neutron stars). However, it is likely that the evolution was significantly different from the law of expansion of a radiation-dominated universe. If so, gravitational waves must have been generated, and density perturbations, as well as rotational perturbations, could be generated too.

Certain properties of the generated perturbations are universal, independent of a concrete form of \( a(\eta) \) in the very early Universe, which we still keep quite arbitrary. These properties are determined by the fact that the perturbations are placed in the squeezed vacuum quantum states. For instance, the expected (mean) quantum mechanical value of \( h_{ij}(\eta, \mathbf{x}) \) is zero in every spatial point and at every moment of time: \( \langle 0|h_{ij}(\eta, \mathbf{x})|0 \rangle = 0 \). However, the variance is not zero and does depend on time:
\[ \langle 0 | h_{ij}(\eta, \mathbf{x}) h^{ij}(\eta, \mathbf{x}) | 0 \rangle = \frac{C^2}{2\pi^2} \int_0^\infty n \sum_{s=1}^2 |h_n(\eta)|^2 dn. \] (7)

Eq. (7) defines the power spectrum

\[ P(n) = \frac{C^2}{2\pi^2} n \sum_{s=1}^2 |h_n(\eta)|^2. \] (8)

For numerical estimates, it is convenient to use the characteristic amplitude \( h(n) \) defining this amplitude as the standard deviation (square root of variance) per logarithmic frequency interval:

\[ h(n) = \left[ \frac{C^2}{2\pi^2} n^2 \sum_{s=1}^2 |h_n(\eta)|^2 \right]^{1/2}. \] (9)

It follows from Eq. (8) that for perturbations with wavelengths shorter than the Hubble radius, the power spectrum \( P(n) \) is not a smooth but an oscillating function of the frequency (wave number) \( n \). Specifically for short gravitational waves, the generated stochastic background of the waves is not a stationary but a nonstationary noise, in the sense that the temporal correlation function of the field should depend on individual moments of time, and not only on the time difference. The large variance of \( N \) and hence the large variance of the amplitude of the perturbations will lead to large variations in the angular correlation function for the microwave background anisotropies, and so on. However, in this presentation, we will not go into the details of statistics and modulated spectra. We will be happy to show that we can get a right numerical level of the expected signal.

The amplitudes and spectra of the generated perturbations depend on the strength and variability of the pump field, which in our case is completely
determined by the scale factor $a(\eta)$. Having the freedom of playing with the
unknown part of $a(\eta)$ describing the very early Universe, one can derive, with
the degree of completeness allowed by quantum theory, all the characteris-
tics of the perturbations in the present Universe. For instance, some of the
gravity-wave spectra derived in this manner are shown in Fig. 1 (adopted
from [4] and updated). The original graph used, in the radio-astronomical
fashion, the spectral flux density $F_\nu$ (ergs / sec cm$^2$ Hz ster) as its vertical
axis. In Fig. 1, we have used the vertical axis in terms of the dimensionless
characteristic amplitude $h(\nu)$, Eq. (9), where the dimensionless frequency $n$
has been translated into the present day frequency $\nu$ measured in Hz. The
spectral cosmological $\Omega$-parameter due to the contribution of gravitational
waves, $\Omega_g(\nu)$, is defined as [4]

$$\Omega_g(\nu) = \frac{\epsilon_g(\nu)}{\epsilon_{cr}} = h^2(\nu) \left( \frac{\nu}{\nu_H} \right)^2$$  \hspace{1cm} (10)

where $\epsilon_{cr}$ is the critical cosmological energy density and $\nu_H \approx 10^{-18}$ Hz is the
Hubble frequency. Because the graph is the same, the vertical axis is now
not universally homogeneous.

In Fig. 1 one can also see the observational upper limits for stochastic
gravitational waves, marked by arrows, that were valid at that time, 10
years ago, as well as the expected sensitivities of then proposed new gravity-
wave detectors. Some of the sensitivity curves, in particular for the space
interferometer now called LISA, should be significantly modified, but we leave
them in the old shape because the modifications will not be very important
for our further discussion.
Figure 1: Possible spectra of relic gravitational waves. The solid line is the prediction based on the measured microwave background anisotropies.
Let us start from the two spectra with the maxima near $10^{-8}$ Hz and $10^3$ Hz. These spectra were derived from bizarre cosmological models designed specially to produce the maximum possible amount of gravitational waves in the frequency intervals where the millisecond pulsar (MSP) and bar-detector techniques, then most favorite, were operating. One can also make equally bizarre assumptions about localized sources (such as colliding bubbles, phase transitions, string loops, strings with attached monopoles at the ends, etc.) and produce maxima virtually in any frequency interval, for the benefit of every individual experimental group. Generally speaking, we should keep eyes open to all these possibilities, they all are not forbidden. A different question is whether we will be surprised if the predicted signal is not detected and what we will learn from that fact.

As was explained above, the quantum-mechanical (parametric) generating mechanism relies only on the validity of general relativity and basic principles of quantum field theory. The law of expansion of the very early Universe is not known but this is what we will learn, or at least will place restrictions on, by detecting or not detecting the predicted signal. For instance, the once popular cosmological model governed by matter with the stiff equation of state $p = \epsilon$ can already be ruled out, because the amount of the created high-frequency gravitons would be too big and would be inconsistent with available cosmological data.

A large class of expanding cosmological models is described by the scale factor

$$a(\eta) = l_0|\eta|^{1+\beta}$$

(11)
where \( l_0 \) and \( \beta \) are arbitrary constants. The \( \eta \) time grows from \(-\infty\) and \( \beta < -1 \) at the initial stage of expansion. The constant \( l_0 \) has the dimensionality of length and is effectively responsible for the Hubble-radius (time-dependent, unless \( \beta = -2 \)) of the very early Universe. With this scale factor, the Einstein equations require the equation of state to be in the form

\[
p = \frac{1 - \beta}{3(1 + \beta)} \epsilon. \tag{12}
\]

For \( \beta = -2 \) one has \( p = -\epsilon \), see Eq. (12), the case called inflation.

Solving the gravity-wave equation (4) one can show that today’s values of the characteristic amplitude \( h(\nu) \) should be as follows (ignoring the modulation of the spectrum which takes place for \( \nu \gg \nu_H \)):

For \( \nu \leq \nu_H \),

\[
h(\nu) \approx \frac{l_{pl}}{l_0} \left( \frac{\nu}{\nu_H} \right)^{\beta + 2}. \tag{13}
\]

For \( \nu_H \leq \nu \leq \nu_m \), where \( \nu_m \) is determined by the time of transition from the radiation-dominated era to the matter-dominated era, \( \nu_m \approx 10^{-16} \) Hz,

\[
h(\nu) \approx \frac{l_{pl}}{l_0} \left( \frac{\nu}{\nu_H} \right)^{\beta}. \tag{14}
\]

For \( \nu_m \leq \nu < \nu_c \), where \( \nu_c \) labels the highest frequency waves marginally affected by the amplification process and above which the spectrum sharply falls down, \( \nu_c \approx 10^8 \) Hz in currently discussed models,

\[
h(\nu) \approx \frac{l_{pl}}{l_0} \left( \frac{\nu_m}{\nu_H} \right)^{\beta} \left( \frac{\nu}{\nu_m} \right)^{\beta + 1}. \tag{15}
\]

The scale factors (11) which are power-law dependent on \( \eta \) time generate spectra which are power-law dependent on frequency \( \nu \).
As we see, the numerical level of the predicted amplitudes depends on the fundamental constants $G, c, \hbar$ combined in $l_P$ and a couple of unknown cosmological parameters, such as $l_0$ and $\beta$. Of course, it is not for the first time that something observable is beautifully expressed in terms of the fundamental constants and a couple of parameters only. For instance, the maximal masses and radii of white dwarfs and neutron stars depend essentially only on $G, c, \hbar$, and the masses of an electron $m_e$ and a baryon $m_B$ (see, for example, [12]). In principle, if did not not know $m_e$ and $m_B$, we could derive them from accurate astronomical measurements! It seems to the author that in the case of relic gravitons we are dealing with the problem of a similar simplicity and deepness.

Let us continue our review of Fig. 1. A spectrum, part of which is shown as a horizontal line at the level $h(\nu) = 10^{-4}$, was derived from the $\beta = -2$ model and placed at the highest level allowed by then existing limits on the microwave background anisotropy $\Delta T/T$. It is known [13] that the long-wavelength cosmological perturbations produce the large-angular-scale anisotropy $\Delta T/T$ of the order of $\Delta T/T \approx h$. The upper limit of that time $\Delta T/T < 10^{-4}$ required $h < 10^{-4}$. The horizontal position of this part of the spectrum (use Eq. (13) for $\beta = -2$) explains why this spectrum is called “flat” (or Harrison-Zeldovich) spectrum. All waves with present frequencies $\nu > \nu_H$ were ordered long ago in the “flat” spectrum, but adiabatically decreased their amplitudes by now. In particular, if $\beta = -2$ and $\nu > \nu_m$ one has $h(\nu) \sim \nu^{-1}$ [14]. The full present-day spectrum is a continuation of the horizontal line to higher frequencies, as shown in Fig. 1. As was already emphasized above, we ignore oscillations in the power spectrum $P(n)$ of the $h$-
field itself. As for the power spectrum of the energy density $\epsilon_g$, it is expected to be smooth, because the $\sin^2(n\eta + \chi)$ oscillations in $h^2$ combine with the $\cos^2(n\eta + \chi)$ oscillations in $(h')^2$ to produce a smooth function for the sum. The $\beta = -2$ model and the $\Delta T/T$ limits of that time did not allow $\Omega_g$ in the frequency bands $\Delta \nu \approx \nu$ to be larger than $10^{-12}$ for all ground-based and space techniques. (The old graph for the space interferometer sensitivity does still appear promising for detecting such a signal but it is now known to be overly optimistic [1].)

The situation has considerably changed in the recent years.

First, the large-angular-scale anisotropy has been actually detected [2], so we are now dealing with the detected signal $\Delta T/T \approx 10^{-5}$, and not with the upper limit $\Delta T/T < 10^{-4}$. A very important question is whether we can attribute, say, a half of the detected signal to gravitational waves, assuming of course that the $\Delta T/T$ is caused by cosmological perturbations of quantum-mechanical origin, as we argued in the very beginning of this paper. Without being able presently to answer this question observationally, we should rely on the theory. The theory definitely says yes. A possible contribution of quantum mechanically generated density perturbations can be of the same order of magnitude as (in fact, somewhat smaller than) the gravity-wave contribution, but cannot be much higher [3]. Specifically for models with the scale factors (11) governed by a scalar field, the characteristic amplitude of the long-wavelengths metric perturbations $h(\nu)$ associated with the density perturbations and responsible for $\Delta T/T$ is described by exactly the same formula as formula (13) for gravitational waves. This is not surprising since the basic dynamical equation for the scalar field density perturbations

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\[ \mu_n'' + [n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}}]\mu_n = 0 \]

(where \( \gamma \equiv 1 + (a/a')' \) and the scalar field potential is arbitrary) is not only similar, but is exactly the same as Eq. (4) for gravitational waves, when the scale factor \( a(\eta) \) at the initial stage of expansion is taken as a power-law function (11) (in which case, \( \gamma = \text{const.} \) and \( \gamma \) drops out of the equation). The numerical coefficients in formula (13) are somewhat in favor of gravitational waves, but it is not numerical coefficients of order 1 that we are now discussing. What is important for us is that the observed \( \Delta T/T \) can now be taken as an experimental point for relic gravitational waves. If we have still used the \( \beta = -2 \) model, the entire inflationary graph of Fig. 1 should have been shifted down by 1 order of magnitude in terms of \( h(\nu) \), predicting a hopelessly small \( \Omega_g \approx 10^{-14} \) for frequencies of our interest, where direct measurements of the gravity-wave background are possible.

Second, the processing of the COBE data has allowed us to obtain some information [13, 16] about the power-law spectral index of primordial perturbations. Usually, the COBE data are processed under the assumption that the anisotropies are caused by density perturbations. Fortunately, it does not matter what one thinks about the perturbations while processing the data: the effects of gravitational waves and density perturbations on the large-angular-scale anisotropies are about the same when amplitudes and spectral indexes \( (\beta + 2 \) in our notation) of the corresponding metric perturbations are the same, see Eq. (13).

Density perturbations with the Harrison-Zeldovich spectrum are often
described by the spectral index $n = 1$ (this $n$ should not be confused with the dimensionless wave number $n$ used in this paper). The relationship between this spectral index $n$ and the parameter $\beta$ is

$$n \equiv 2\beta + 5. \quad (16)$$

Information about the primordial spectral index extracted from the large-angular-scale anisotropies is information about gravitational waves (when they dominate or provide, at least, a half of the signal) even if one thinks that the processed anisotropies are entirely produced by density perturbations.

The authors of [15] concluded: “...we find a power-law spectral index of $n = 1.2 \pm 0.3 \ldots$”, “The power spectrum of the COBE DMR data is consistent with [a Peebles-Harrison-Zeldovich $n = 1$ universe]”. The authors of [16], who processed the same set of data but in a different manner, concluded: “The spectral parameter of the power spectrum of primordial perturbations $n = 1.84 \pm 0.29$ [is] estimated”, “The power spectrum estimation results are inconsistent with the Harrison-Zeldovich $n = 1$ model with the confidence 99 %”.

It is difficult for us to judge whether the $n = 1$ model is ruled out at the confidence level 99 %, according to [16], or at the confidence level 60 % or so, according to [15]. However, in these two results, we see the indication that the spectral index $n$ is indeed larger than 1. As a compromise, we will first take the value $n = 1.4$ and derive its consequences for gravitational waves.

The spectral index $n = 1.4$ translates into $\beta = -1.8$, see Eq. (16). The $\nu < \nu_H$ part of the spectrum is not any longer “flat” but gives more power to higher frequencies, $h(\nu) \sim \nu^{0.2}$, see Eq. (13). The value $h(\nu_H) = 10^{-5}$
is fixed by observations. The number $\beta = -1.8$ should also be used in Eq. (14) and Eq. (15). This gives $h(\nu_m) = 10^{-8.6}$, a slightly smaller amplitude than $h(\nu_m) = 10^{-8}$ for the $\beta = -2$ model. But for the higher frequencies the results are very encouraging. At the LISA-tested frequency $\nu = 10^{-3}$ Hz we get $h = 10^{-19}$ and $\Omega_g = 10^{-8}$. At the LIGO/VIRGO-tested frequency $\nu = 10^2$ Hz we get $h = 10^{-23}$ and $\Omega_g = 10^{-6}$. To plot the full spectrum, one can essentially shift down the entire graph of the $\beta = -2$ model and slightly rotate the graph around the fixed observational point $h = 10^{-5}$ at $\nu = \nu_H$. This produces a solid line shown in Fig. 1. This line is no longer a result of a simple model-building, it is now supported, at least partially, by observations. The fact that the graph consists of straight line pieces meeting at corners is accounted for by the nature of our approximation: strictly power-law scale factors joined at the transition points between the initial, radiation-dominated, and matter-dominated eras. At a more accurate graph, the corners will be rounded and lines will be slightly bent.

We have to admit that it is not so easy to give a “microphysical” explanation to the derived parameter $\beta = -1.8$. This value of $\beta$ corresponds to the equation of state $p = -1.2 \epsilon$, see Eq. (12). With the equation of state of this kind, energy density of the matter, as well as curvature of the space-time, increase in course of expansion. However, the value $\beta = -1.8$ is marginally consistent with the assumption that the Planck densities are not encountered in course of the evolution [17]. The scalar fields, often considered in the context of the inflationary hypothesis, are only capable of providing an equation of state with $\beta \leq -2$. Possibly, a solution to the “microphysical” side of the problem can be found along the lines of the “superstring
motivated” cosmologies [18].

The direct detection of the useful noise (a stochastic gravity-wave signal) by noisy detectors can be achieved with the help of a standard technique of cross-correlating the outputs of two or more detectors [19, 20, 21]. The LISA is not planned to have two independent detectors [1]. However, the predicted signal is so high, $\Omega_g = 10^{-8}$ at $\nu = 10^{-3}$ Hz and $\Omega_g = 5 \times 10^{-8}$ at $\nu = 10^{-1}$ Hz, that one can probably recognize the signal by comparing the observational data with the calculated sensitivity of the instrument. Fortunately, at frequencies around $10^{-2}$ Hz and higher, the contaminating gravity-wave noise from compact binaries is expected to be below the projected LISA sensitivity. It is important to note that the conservative value $n = 1.2$ ($\beta = -1.9$) of the spectral index does still lead to a quite well measurable signal: $h = 10^{-20.5}$, $\Omega_g = 10^{-11}$ at $\nu = 10^{-3}$ Hz and $h = 10^{-25}$, $\Omega_g = 10^{-10}$ at $\nu = 10^{2}$ Hz.

The first evaluation [19] of a possibility to detect relic gravitons by a cross-correlating technique was based on the assumptions that the flux density behaves as $F_\nu \sim \nu^{-1}$ (that is, $h(\nu) \sim \nu^{-1}$, like in the $\beta = -2$ model considered above), that the expected amplitude is at the level (in current notation) of $\Omega_g = 10^{-4}$, and that the electromagnetic detectors operating in the high-frequency band $\nu = 10^{7}$ Hz are being used. At that time, these assumptions were of kind of a stretched imagination. Presently, this possibility may turn out to be more realistic in view of the fact that $\Omega_g(\nu)$ is growing toward the higher frequencies, see Eq. (15) and Fig. 1. However, the prospect for the high-frequency techniques, such as bars and electromagnetic detectors, may be not as good as it looks on the graph. The high-frequency parts of spectra generated in simple models of the very early Universe have
tendency to deviate down from the straight line corresponding to a strict power-law dependence (11). Generally speaking, the farther we extrapolate the spectrum from the experimental point at $\nu = \nu_H$, the less confident we are. In this respect, the LISA has an additional advantage of operating at relatively low frequencies.

So, what is my conclusion? Urgently fly LISA!

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