Geometrical Aspects of Lie Groups Representations and Their Optical Applications

J. Guerrero\(^1,2\), V. I. Man’ko\(^3\), G. Marmo\(^2\) and A. Simoni\(^2\)

Abstract

In this paper we present a new procedure to obtain unitary and irreducible representations of Lie groups starting from the cotangent bundle of the group (the cotangent group). We discuss some applications of the construction in quantum-optics problems.

1 Introduction

Any quantization procedure \[^{[1]}\], i.e., a way to construct a quantum system starting from a classical one, provides us with the construction of unitary representations of Lie groups, often arising in the picture as symmetry groups. Various symmetry groups (or dynamical symmetry groups \([^{[2]}\)) describe the properties of different quantum systems and quantum optical devices. The description of the energy spectrum of a quantum system like a trapped ion, or the evolution of the two-level atom in a cavity can be given either in terms of solutions of stationary and nonstationary Schrödinger equations or using matrix elements of the appropriate irreducible representations of the dynamical symmetry groups. The Heisenberg–Weyl group ISp(4,R) and noncompact group SU(1,1) as well as other groups are well-known examples (see \([^{[7]}\]) which are used to describe properties of optical and quantum optical phenomena. The application of group-theoretical methods in quantum mechanics and quantum optics is based on using the construction of matrix elements of irreducible representation of the various Lie groups and tools like Casimir operators whose eigenvalues determine the irreducible representations. In view of such an importance of the Lie group representations in physical applications, it is worth understanding the aspects of the irreducible representation constructions as deeply as possible. An elegant approach to the physical problems as well as to the problems of the dynamical symmetry groups is the geometrical approach. This approach permits to give a unified picture of various phenomena in quantum mechanics and quantum optics.

\(^1\)Departamento de Matemática Aplicada, Facultad de Informática Campus de Espinardo, 30100 Murcia, Spain

\(^2\)Dipartimento di Scienze Fisiche, Università "Federico II" di Napoli "Federico II" and Istituto Nazionale di Fisica Nucleare, sezione di Napoli, Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Napoli, Italy

\(^3\)P.N. Lebedev Physical Institute, Moscow
expressed in terms of the Lie group representation theory. The relation of the classical domain to the quantum one can be associated in this geometrical approach in terms of the procedure called “geometrical quantization.” One of the goals of this paper is a discussion of some geometrical aspects of the Lie group representation theory.

The notion of elementary system introduced by Wigner [11] requires the representation to be irreducible. Therefore, quantization procedures are often required to provide unitary irreducible representations of Lie groups. The so-called method of orbits, elaborated by Kirillov [12] and Kostant [13] on the mathematical side and by Souriau [14] on the physical side, was an attempt to geometrize and extend the quantization procedure used in physics starting with classical systems defined on phase-space (cotangent bundles of configuration spaces). The irreducibility problem, tackled by introducing polarizations, has been one of the major problems in this geometric quantization program.

As, to use the words of A. A. Kirillov [12], “the possibilities of geometric quantization are still far from having been exhausted”, it seems to us useful to consider what is the connection between the orbit method, the construction of representations on homogeneous spaces and the role and implementation of polarizing conditions to select irreducible representations.

Because the orbit method uses orbits of the co-adjoint action of a Lie group, say $G$, on the dual of its Lie algebra, say $G^*$, and homogenous spaces are usually realized as quotients $G/K$, with $K$ a closed subgroup of $G$, it is useful to consider these spaces as arising from $T^*G$ which is the “phase-space” on the configuration space $G$, using therefore a notion closer to the physicists view of quantization for classical systems.

Our paper is organized in the following way. We first consider geometrical entities on the cotangent bundle $T^*G$ of any Lie group $G$. Here we show the important role that Poisson subalgebras play in going from $T^*G$ to $G^*$. More generally we show that Poisson subalgebras are connected with quotient spaces that make the quotienting map a Poisson map. The left and right action of $G$ on itself, when extended to $T^*G$, provide a link with the problem of irreducibility of representations. The possibility of constructing the “quantization map” in terms of canonical variables, with one eye on the irreducibility problem, is tackled via a generalized Jordan-Schwinger map.

The geometrical picture emerging from these considerations will show the limitations of the construction with respect to the problem of locality versus globality, which is connected with the problem of self-adjointness of the constructed operators [13]. Here we should stress that usually problems of (common) domains of the operators constructed by quantization are not addressed.

The irreducible representations of classical Lie groups were constructed (see, e.g. [16]) in a purely mathematical context. On the other hand in physical applications it is desirable to consider the construction of irreducible representations as close as possible to the experience of theoretical physics. One of the aspects of this experience is the use of classical Mechanics methods, with the notion of phase space, Poisson brackets, and the notion of canonical quantization of the classical systems for which the classical positions and momenta become the operators acting in a Hilbert space of states of the quantum system. An attempt to find the link of the “physical picture” of the irreducible representations
construction with known mathematical formulae for the irreducible representations of Lie
groups was done in [17]. The considerations in this work was based on the properties
of the regular representation of the Lie group treated as classical motion in the “phase
space” of the group, but this consideration did not use the geometrical point of view.
Here we shall try to elucidate a geometrical picture of the group phase space construction
and the quantization picture for the known irreducible representations of Lie groups.

Important known example of the link of the group representation construction with the
physical system which is the two-mode quantum oscillator is the Jordan-Schwinger map
[18, 19]. This map was used to express the generators of the irreducible representations
of the compact group SU(2) in terms of the creation and annihilation operators of the
two-mode oscillator.

Geometrical properties of the Jordan-Schwinger map and its extension to quantum
groups were discussed in [20]. Using the experience of the Jordan-Schwinger map con-
struction we would like to generalize and apply it for considering generic Lie groups
representations. Thus the aim of this paper is to give a geometrical interpretation for
the Lie group “phase space” discussed in [17] and the irreducible representations. We
want to treat the group phase space as the known geometrical object called the cotangent
bundle $T^*G$. Also we suggest a conjecture that all the irreducible representations of the
group $G$ can be associated with the orbits of the group in the group phase space, repre-
senting the cotangent bundle $T^*G$ as the union of the orbits obtained due to right and
left action of the group on the group phase space. It is important that the generators of the
right and left regular representation can be considered for generic orbits as linear in
momenta functions. Of course, the geometrical construction of group representations is a
very well known procedure. But usually the construction is based on considering a fixed
orbit [13, 14], or on considering the group [21]. The novelty of the present geometrical
interpretation of the irreducible representations is in studying the extended object $T^*G$ as
the phase space of a dynamical system analog containing information on the irreducible
representations of the Lie group. It should be noted that the construction of [17] was used
[22] to consider new types of integrable systems.

2 Lie groups and their cotangent bundles

In this section we construct bundles over a Lie group $G$ which are themselves Lie groups.
For simplicity of notation we are assuming our Lie groups to be realized in terms of
matrices, however this assumption does not play any role in our considerations.

Given any Lie group we define on it left and right invariant 1-forms by introducing a
basis for the Lie algebra $\mathcal{G}$ of $G$, say $(\tau^1, \tau^2, \ldots, \tau^n)$, $n = \dim G$, and setting:
\[
g^{-1}dg = \theta^L_j \tau^j, \quad dg^{-1} = \theta^R_j \tau^j, \tag{1}
\]
where $g \in G$ and $\theta^L$ and $\theta^R$ are the left invariant and right invariant 1-forms, respectively.

Because $g^{-1}dg = g^{-1}(dg\ g^{-1})g$, we find $\theta^L_j \tau^j = g^{-1} \tau^k g \theta^R_k$, and by writing the adjoint
action as
\[
Ad(g)\_j^k \tau^j = g^{-1} \tau^k g, \tag{2}
\]
we find $\theta_j^L = Ad(g)^k_j \theta_j^R$, i.e. left and right invariant 1-forms are connected by the adjoint transformation. We also have $Ad(g^{-1})^k_j \theta_j^L = \theta_j^R$.

By using these 1-forms we construct the left invariant and right invariant volume elements, $\Omega^L = \theta_1^L \wedge \theta_2^L \wedge \ldots \wedge \theta_n^L$ and $\Omega^R = \theta_1^R \wedge \theta_2^R \wedge \ldots \wedge \theta_n^R$, respectively. When $\Omega^L = \Omega^R$ the Lie group $G$ is said to be unimodular.

By using the Riesz representation theorem we can associate a measure with either one of the two volumes. In any case we set

$$<\Psi|\Phi>^L = \int_G \Psi^* \Phi \Omega^L,$$

and

$$<\Psi|\Phi>^R = \int_G \Psi^* \Phi \Omega^R.$$

The left action of $G$ on itself preserves $\Omega^L$ and defines the left regular representation of $G$, and similarly for the right representation.

By using infinitesimal generators of the right and left action, $X^L$ and $X^R$, respectively, we find:

$$\theta_j^L((X^L)^k) = \delta^k_j, \quad \theta_j^R((X^R)^k) = \delta^k_j,$$

i.e. right generators $X^L$ are left-invariant and left generators $X^R$ are right-invariant. Clearly $\theta_j^L((X^R)^k) = Ad(g)^l_j \delta^k_l = Ad(g)^k_j$.

For any group $G$, we can construct the tangent bundle $TG$ and the cotangent bundle $T^*G$, these two bundles carry a Lie group structure associated with the adjoint representation and the co-adjoint representation respectively. On the one hand, if the group $G$ carries a natural metric, it is possible to define a non degenerate Lagrangian function on $TG$ and consider associated geodetical motions along with symmetries and constants of the motion. It is possible to define a Lagrangian symplectic structure and Lagrangian Poisson brackets, i.e. these structures are tied to a specific Lagrangian dynamics. On $T^*G$, on the other hand, it is always possible to define a symplectic structure along with canonical actions of $G$, left and right actions, with “Hamiltonian generating functions” closing on the Lie algebra of $G$ in terms of Poisson brackets [23].

The existence of a left invariant or right invariant volume element on $G$ allows to define a measure on $G$, the Haar measure, which can be used to define a Hilbert space of square integrable functions on $G$. On this Hilbert space $G$ has a unitary representation along with a decomposition into irreducible components. It is possible to go from one Hilbert space to the other by using the correspondence between left action and right action. The canonical action of $G$ on $T^*G$, allows to consider these Hilbert spaces realized in terms of functions on $T^*G$ bringing in the possibility of using symplectic and Poisson geometry in connection with the decomposition of unitary representation into irreducible ones.

This interplay is the hard core of Geometric Quantization and any other proposed quantization where geometry plays a relevant role.

In addition to previous considerations we also notice that $T^*G$, having “canonical” coordinates, may be thought of as a non-abelian generalization of $T^*R^n$ along with its
canonical structure of Heisenberg group, therefore we may try to generalize the Jordan-Schwinger map to the cotangent bundle of any Lie group.

By using left generators or right generators we can decompose $T^*G$ into $G \times_R G$ and $G \times_L G$, respectively, and similarly decompose $T^*G$ into $G \times_L G^*$ or $G \times_R G^*$ by using 1-forms.

Given any function $f : T^*G \to G^*$ which is a submersion onto, we can define two 1-forms on $T^*G$, $\theta_f^L = f(g^{-1}dg)$ and $\theta_f^R = f(dg g^{-1})$. These 1-forms are the potential for symplectic (non-degenerate) 2-forms $d\theta_f^L$ and $d\theta_f^R$. When $f$ is the identity map with respect to the splitting $T^*G \approx G \times_L G^*$ or $T^*G \approx G \times_R G^*$, we find the canonical 1-form $\Theta_0$ and we have

$$\Theta_0 = \mathcal{P}^L(g^{-1}dg) = \mathcal{P}^R(dg g^{-1}),$$

where $\mathcal{P}^L, \mathcal{P}^R : T^*G \to G^*$ are the left (right) invariant momenta which are the generating functions of the canonical right (left) actions of $G$ on $T^*G$.

Let us denote by $X^L$ ($X^R$) the infinitesimal canonical generators of the right (left) action of $G$ on $T^*G$, having generating functions the momenta $\mathcal{P}^L$ ($\mathcal{P}^R$).

From $(\mathcal{P}^L)^j \theta^L_j = (\mathcal{P}^R)^j \theta^R_j = \Theta_0$, we find $(\mathcal{P}^L)^j \text{Ad}(g)^k_j \theta^R_k = (\mathcal{P}^R)^k_j \theta^R_k$, i.e. $(\mathcal{P}^L)^j \text{Ad}(g)^k_j = (\mathcal{P}^R)^k_j$, and also $(X^L)^j \text{Ad}(g)^k_j = (X^R)^k_j$.

The symplectic volume, $\Omega_0 = (d\Theta_0)^n = d\Theta_0 \wedge \ldots \wedge d\Theta_0$, can be decomposed into

$$d(\mathcal{P}^L)^1 \wedge \ldots \wedge d(\mathcal{P}^L)^n \wedge \Omega^L,$$

or in

$$d(\mathcal{P}^R)^1 \wedge \ldots \wedge d(\mathcal{P}^R)^n \wedge \Omega^R,$$

respectively.

As for Poisson brackets, which in matrix notation can be written as $g^{-1}\{\mathcal{P}^L, g\} = I$, if we use Gaussian coordinates for the Lie group $G$, say $(\xi_1, \xi_2, \ldots, \xi_n)$ are the parameters of the 1-parameter subgroups associated with $((X^L)^1, (X^L)^2, \ldots, (X^L)^n)$, we find $\{(\mathcal{P}^L)^j, \xi_k\} = \delta^j_k$. Of course Gaussian coordinates only exist in a neighborhood of the identity (or, by translations, in the neighborhood of a point), therefore they do not capture the global aspects of our treatment.

By using the action (left and right) $G \times T^*G \to T^*G$, we find orbits of $G$ diffeomorphic to $G$ and giving a quotient space $T^*G/G$ diffeomorphic to $G^*$. As a matter of fact, if the lifted action of $G$ to $T^*G$ is defined via $\theta^L, \theta^R$ we find as projection map $f : T^*G \to T^*G/G$.

We stress that no matter which lift from $G$ to $T^*G$ is being used, the generating functions of the lifted action of $G$ are always linear in the corresponding momenta in $G^*$.

The projection map $f : T^*G \to T^*G/G \equiv G^*$ is a Poisson map with respect to the natural Poisson bracket available on $G^*$ by using the identification $G^* \equiv \text{Lin}(G, R)$ or $\mathcal{G} = (G^*)^*$.

Casimir functions on $T^*G$ are simply the inverse image of functions on $G^*$ which are central elements with respect to the natural Poisson bracket on $G^*$. They can also be defined as those functions on $T^*G$ which commute with $\mathcal{P}^L$ or $\mathcal{P}^R$ (then, as a result of previous analysis, they commute with both). We also have $\{\mathcal{P}^L, \mathcal{P}^R\} = 0$, along with $[X^L, X^R] = 0$ and $[X^L, \tilde{X}^R] = 0$. 

5
3 Digression: Manifolds and algebras of functions

As it is well known, manifolds can also be described by commutative rings of functions defined on them, i.e. \( M = \text{Hom}_A(\mathcal{F}, R) \), points of \( M \) are identified with algebra homomorphisms from the algebra \( \mathcal{F} \) to the algebra \( R \). When \( M \) is a Lie group \( G \), \( \mathcal{F}(G) \) also possesses a Hopf algebra, capturing the group multiplication property on the manifold \( G \).

If there is a smooth projection map, i.e. a submersion onto \( \phi : M \to N \), the pull-back \( \phi^*(\mathcal{F}(N)) \) defines a subalgebra of \( \mathcal{F}(M) \). Therefore subalgebras of the commutative algebra \( \mathcal{F}(M) \) constitute a generalization of projection maps onto “quotient manifolds”.

When \( M \) carries a Poisson structure, we can define, in addition, Poisson subalgebras. They are subalgebras which are also closed under Poisson brackets.

When \( N \) carries in addition a Poisson bracket, say \( \{f, h\}_N \) for \( f, g \in \mathcal{F}(N) \), we say that \( \phi \) is a Poisson map with respect to the Poisson bracket on \( M \), if we have:

\[
\{\phi^* f, \phi^* g\}_M = \phi^*(\{f, g\}_N),
\]

for any \( f, g \in \mathcal{F}(N) \). Therefore \( \phi^*(\mathcal{F}(N)) \) is a Poisson subalgebra. Thus Poisson subalgebras are a generalization of Poisson projection maps. We recall that Poisson subalgebras were considered by Lie under the name of “Functionen gruppen” [24].

The reason why subalgebras are generalizations of projection maps has to do with the fact that not all subalgebras arise from pull-backs via projection maps. To identify them we would need some regularity assumptions on the subalgebras involved.

In this duality between \( M \) and \( \mathcal{F}(M) \), vector fields on \( M \) arise as derivations on \( \mathcal{F}(M) \) and differential forms as \( \mathcal{F}(M) \)-multilinear antisymmetric maps from derivations to \( \mathcal{F}(M) \). Similarly, diffeomorphisms on \( M \) arise as automorphisms of the algebra structure on \( \mathcal{F}(M) \).

Given a subalgebra, say \( \mathcal{F}_N, \) in \( \mathcal{F}(M) \), the set of all derivations of \( \mathcal{F}(M) \) which annihilate \( \mathcal{F}_N \) are a Lie subalgebra of vector fields, say \( \mathcal{D}^0(N) \). The regularity assumption on \( \mathcal{F}_N \) can be stated by saying that maximal integral sub-manifolds of \( \mathcal{D}^0(N) \) define a regular foliation.

4 The Poisson algebra structure on \( \mathcal{G}^* \)

We recall that on the dual space \( \mathcal{G}^* \) of a Lie algebra \( \mathcal{G} \) we have a canonical Poisson bracket defined by

\[
\{f, h\}(x) = x[\text{df}(x), \text{dh}(x)],
\]

where \( \text{df} \in \mathcal{G} = (\mathcal{G}^*)^* \) and the commutator is taken in \( \mathcal{G} \), while the natural pairing between \( \mathcal{G} \) and \( \mathcal{G}^* \) is denoted by \( < \cdot | \cdot > \).

Casimir functions are defined to be those functions on \( \mathcal{G}^* \) which Poisson commute with any other function, i.e. the center of the Lie algebra defined by the Poisson structure on \( \mathcal{F}(\mathcal{G}^*) \).

The group \( G \) acts on \( \mathcal{G}^* \) via the co-adjoint action and defines the co-adjoint orbits. Each orbit is a symplectic manifold with symplectic structure given by inverting the
previous Poisson bracket on each orbit. The stability group of a point \( x \in G^* \) is a closed subgroup of \( G \) and will be denoted by \( G_x \), the corresponding Lie algebra will be denoted by \( \mathfrak{g}_x \).

For a generic \( x \in G^* \), \( G_x \) is abelian, and generic orbits are defined as level sets of the Casimir functions. Nongeneric orbits are level sets of \( Ad^* \)-invariant relations. They are invariant sub-manifolds defined by level sets of functions which are \( Ad^* \)-invariant only for specific values \( [25] \).

Any element \( x \in G^* \) defines a left invariant 1-form \( \theta_L^x \) and a right invariant 1-form \( \theta_R^x \) on the group \( G \). With these 1-forms we associate vector fields on \( G \) by considering \( \ker d\theta_L^x \) and \( \ker d\theta_R^x \). These vector fields close on the Lie algebra \( \mathfrak{g}_x \), in terms of right or left infinitesimal generators. We get two quotient spaces \( G / G_x^L \) and \( G / G_x^R \), respectively. These two spaces carry symplectic structures, namely the projections of \( d\theta_L^x \) and \( d\theta_R^x \), respectively. These quotient spaces are symplectomorphic with the symplectic orbit in \( G^* \) passing through \( x \in G^* \).

It is this correspondence which allows us to compare the orbit method with the construction of unitary representations via functions on homogeneous spaces. We shall discuss it again by using the reduction procedure on \( T^*G \).

5 Poisson subalgebras on \( T^*G \) and associated quotient spaces

The left invariant and right invariant splitting of \( T^*G \), namely \( G \times_L G^* \) and \( G \times_R G^* \), define a left invariant momentum map projection \( P^L : T^*G \to G^* \) and a right invariant momentum map projection \( P^R : T^*G \to G^* \). These maps are Poisson maps with respect to the standard Poisson bracket on \( T^*G \). Therefore \( \mathcal{F}^L(G^*) \equiv (P^L)^*(\mathcal{F}(G)) \) and \( \mathcal{F}^R(G^*) \equiv (P^R)^*(\mathcal{F}(G)) \) are Poisson subalgebras of \( \mathcal{F}(T^*G) \) with the additional property

\[
\{ \mathcal{F}^L(G^*), \mathcal{F}^R(G^*) \} = 0, \tag{11}
\]

i.e. left and right momenta define mutually commuting Poisson subalgebras in \( \mathcal{F}(T^*G) \). By considering these two projections we have the following diagram:

\[
\begin{array}{ccc}
G^L & \downarrow & P^R \\
G^R & \rightarrow & T^*G \rightarrow G^* \\
 & \downarrow & P^L \\
 & G^* & \\
\end{array}
\tag{12}
\]

where \( G^L \) and \( G^R \) stay to represent a typical fiber of the projections \( P^L \) and \( P^R \), respectively. It should be noticed that from the point of view of groups we have a split sequence of groups

\[
0 \to G^* \to T^*G \to G^L \to 1, \tag{13}
\]
neighborhoods of singular symplectic orbits in $G$ construction of previous decomposition. Even if we restrict to generic co-adjoint orbits, should make clear why our previously mentioned local decomposition cannot be global, arises in a quotient space and is not a sub-manifold of the original leaf. These remarks is a symplectic manifold, product of symplectic orbits. This makes clear that this product vector fields associated with Casimir functions. The quotient with respect to this kernel find a presymplectic structure. The kernel of this 2-form is generated by the hamiltonian.

Polarized co-adjoint orbits: The generalized Jordan-Schwinger map

The appearance of co-adjoint orbits of $G$ in $G^*$ in the decomposition of $T^*G$ or in the homogeneous spaces $G/G_x$, shows that to use the standard canonical quantization procedure would be useful to find canonical (Darboux) coordinates for each co-adjoint orbit in $G^*$, in this way we could use the standard $p \to -i\hbar \frac{\partial}{\partial p_x}$ and $x \to \hat{x}$ correspondence for the quantization. We can rephrase this procedure by looking for a Poisson map $S : G^* \to T^*Q$ which would allow to realize the algebra of $G$ in terms of Poisson brackets on $T^*Q$, with

\begin{align*}
G^* \text{ is considered an abelian (vector) group.}

We can use now the Marsden-Weinstein reduction procedure in a revised form. We consider a point $m \in T^*G$ along with orbits $G^L\cdot m$ and $G^R\cdot m$. By restricting $\omega_0$ (the canonical symplectic structure on $T^*G$) to this orbit we find $\omega_m^L = \omega_0|_{G^L\cdot m}$ and $\omega_m^R = \omega_0|_{G^R\cdot m}$, it is not difficult to show that $\omega_m^L = d\theta^L_{P^L(m)}$ and $\omega_m^R = d\theta^R_{P^R(m)}$, where $P^L,R(m)$ stay for the corresponding point $x$ in $G^*$, according to the appropriate projection. The kernel of $\omega_m^{L,R}$ are generated by vector fields associated with Casimir functions (or with invariant relations, as the case may be) restricted to $G^{L,R}\cdot m$, respectively. When projected onto $G$ they are represented by vector fields on $G$ corresponding to $G_x$, $x = P^L,R(m)$. With $G^{L,R}\cdot m$ we associate two manifolds, $(G^L\cdot m) \cup (G^R\cdot m)$ and $(G^L\cdot m) \cap (G^R\cdot m)$. The union of the two orbits is simply the level set through $m$ of Casimir functions from $G^*$ to $T^*G$. The intersection is the orbit (possibly a union of orbits) of $G^L_x$ or $G^R_x$ with $x = P^L,R(m)$. Therefore a neighborhood of a generic $m$ in $T^*G$ decomposes into the product of two copies of symplectic co-adjoint orbits passing through $x$ (for a generic $x$) and a symplectic manifold diffeomorphic with $T^*G_x$ (this is not a symplectomorphism with the standard symplectic structure on $T^*G$). This decomposition does not hold for neighborhoods of singular symplectic orbits in $G^*$.

To have a feeling of the problems arising at a global level, we mention a different construction of previous decomposition.

On $G^*$ we restrict ourselves to the open sub-manifold of all generic orbits. The pull-back of this sub-manifold to $T^*G$ defines an open symplectic sub-manifold on it. Level sets of Casimir functions define a regular foliation. On these leaves we restrict $\omega_0$ and find a presymplectic structure. The kernel of this 2-form is generated by the hamiltonian vector fields associated with Casimir functions. The quotient with respect to this kernel is a symplectic manifold, product of symplectic orbits. This makes clear that this product arises in a quotient space and is not a sub-manifold of the original leaf. These remarks should make clear why our previously mentioned local decomposition cannot be global, even if we restrict to generic co-adjoint orbits.

For instance, in the case of $T^*SU(2)$ the product would give $S^2 \times S^2 \times T^*S^1$, which clearly is not diffeomorphic to $S^3 \times R^3 \equiv T^*SU(2)$. The open sub-manifold of generic co-adjoint orbits would give rise to the open sub-manifold $S^3 \times S^2 \times R$, again not diffeomorphic to the mentioned decomposition.
the Poisson bracket on $T^*Q$ not being necessarily the standard one, i.e. we can afford using commuting positions (on Q) and non-commuting momenta (like the commutations relations of gauge invariant momenta in the presence of magnetic fields, see [26]). In this formulation it is clear that co-adjoint orbits should be noncompact for them to be diffeomorphic with cotangent bundles (i.e. they should satisfy Pukanzsky condition, see [26]), moreover to realize the Lie algebra $G$ in terms of Poisson brackets on $T^*Q$, this should allow for an action (at least local) of $G$ so that $S$ becomes a $G$-invariant map. This Poisson map is a classical version of the Jordan-Schwinger map, where Lie algebras are realized in terms of creation and annihilation operators, constituting a complexification of $(x, -i \hbar \partial_x)$.

We shall give more details of this construction, in order to apply it to particular examples.

Let $O_x$ be the co-adjoint orbit passing through the point $x \in G^*$. Let $\omega$ be the symplectic form on $O_x$, obtained by inverting the Poisson brackets of $G^*$ on $O_x$. If \{p^1, p^2, \ldots, p^n\} is a basis\footnote{We shall use the same notation $p^i$ for the momentum coordinates in $T^*G$ and the coordinates in $G^*$.} of $G^*$ (dual to some selected basis in $G$), we can write $\omega$ as:

$$\omega = w_{ij}(p) dp^i \wedge dp^j. \quad (14)$$

Let $S : G^* \to T^*Q$ be a Poisson map which allows to write $\omega$ in terms of Darboux coordinates $q^i = q^i(p), \pi_i = \pi_i(p), \ i = 1, \ldots, k, 2k$ being the dimension of the co-adjoint orbit $O_x, \omega = \sum_{i=1}^k d\pi_i \wedge dq^i$. Then we define the following functions on $T^*G$:

$$X^i \equiv (P^L)^* \cdot S^*(q^i)$$
$$P_{X^i} \equiv (P^L)^* \cdot S^*(\pi_i) \quad (15)$$

$$Y^i \equiv (P^R)^* \cdot S^*(q^i)$$
$$P_{Y^i} \equiv (P^R)^* \cdot S^*(\pi_i), \quad (16)$$

for $i = 1, 2, \ldots, k$, where $P^{L,R} : T^*G \to G^*$ are the generating functions of the right (left) action of the group $G$ on $T^*G$ (momentum maps). Then, by definition, $X^i, P_{X^i}$ (resp. $Y^i, P_{Y^i}$) are invariant under the left (resp. right) action of the group, and since $P^{L,R}$ are Poisson maps, they verify:

$$\{P_{X^i}, X^j\} = \delta_i^j$$
$$\{X^i, X^j\} = 0$$
$$\{P_{X^i}, P_{X^j}\} = 0$$

$$\{P_{Y^i}, Y^j\} = \delta_i^j$$
$$\{Y^i, Y^j\} = 0$$
$$\{P_{Y^i}, P_{Y^j}\} = 0. \quad (17)$$

\footnote{We shall use the same notation $p^i$ for the momentum coordinates in $T^*G$ and the coordinates in $G^*$.}
As discussed above, we can generalize this situation to the case in which the commutator between the \( \pi_i \) among themselves do not commute, to allow, for instance, for the description of gauge invariant momenta in the presence of magnetic fields. In this case the commutator between the \( P_{Xi} \) (and the commutator between the \( P_{Yi} \)) will be different from zero.

Also, due to the fact that the left and the right action commute, both set of coordinates, \((X^i, P_{Xi})\) and \((Y^i, P_{Yi})\), commute (in other words, since \((X^i, P_{Xi})\) are functions of the left invariant momenta \((P^L)^i\) and \((Y^i, P_{Yi})\) are functions of the right invariant momenta \((P^R)^i\), they commute).

Note that we can use two different sets \((q^i, \pi_i)\) and \((q'^i, \pi'_i)\) for defining the coordinates \((X^i, P_{Xi})\) and \((Y^i, P_{Yi})\), respectively, and these would be related to the ones given before by a canonical transformation in \(T^*G\), since the two pairs of Darboux coordinates are related through a canonical transformation in \(T^*Q\).

Casimirs are functions on \(G^*\) which are central with respect to the natural Poisson bracket in \(G^*\) (which is isomorphic to the Lie algebra bracket of \(G\)). For every Lie Poisson algebra there are a number of independent Casimirs (generally polynomials in \(G^*\)) that is equal to the rank \(r\) of the algebra. Taking the pullback with \(P^L,R\) of the Casimirs we obtain \(r\) functions on \(T^*G\) which are right and left invariant (and therefore central with respect to the natural Poisson bracket in \(T^*G\)). We define \(P_{Zi}, i = 1, \ldots, r\), as \(r\) independent functions of the Casimirs in such a way that, inverting the equations (15,16), \((P^L)^i\) could be written as linear functions in the momenta \(P_{Xj}\), and \((P^R)^i\) could be linear in the momenta \(P_{Yj}\). Clearly, this will not be always possible since it would imply that all co-adjoint orbits for any group are polarizable (in the sense of Geometric Quantization, see [27]), as we shall see in the next section. But in the cases in which this construction is possible, it simplifies very much the construction of unitary irreducible representations (see next section).

In some cases, we shall get only linearity in the momenta \(P_{Xj}\) (resp. \(P_{Yj}\)) but not in the Casimirs \(P_{Zj}\). This problem is harmless for our purposes, since it is still possible to define uniquely the quantization (irreducible representations).

To complete the discussion, we only need to compute \(r\) functions \(Z^i\) on \(T^*G\) which are canonically conjugated to the \(P_{Zi}\) and such that \((g^i, p^i) \rightarrow (X^i, Y^j, Z^k, P_{Xi}, P_{Yj}, P_{Zk})\) be a canonical transformation in \(T^*G\). These functions are neither left nor right invariant, instead they transform according to an additive function on the group satisfying the properties of 1-cocycle on the group (see [28]).

7 Irreducible representations

Once we have found the canonical transformation which “separates” the dependence of the right and left momentum maps, it is very easy to obtain irreducible representations of \(G\) using the techniques of Geometric Quantization in its simpler version. First of all, note that from the construction realized above, we can deduce few simple facts which will be useful in the following:
• The variables $P_{Z^i}$, being functions of the Casimirs in $T^*G$, are invariant under the left and right action of $G$.

• The variables $Z^i$, conjugated to $P_{Z^i}$, are cyclic in all the generators of either the right or the left action. This implies that the vector fields $\frac{\partial}{\partial Z^j}$ commute with the generators of the right and left action of $G$. Therefore, they must be constant on each irreducible representation.

Our aim is to construct irreducible representations of $G$ using the previous considerations in $T^*G$. We can proceed in two different ways, one starting directly from a complex line bundle on the phase-space $T^*G$ (the connection 1-form given by $\Theta_0$, the curvature being the symplectic 2-form $d\Theta_0$ and the invariant measure given by $\Omega_0 = (d\Theta_0)^n$) and impose the appropriate polarization (and constraints) conditions to obtain, finally, the desired representations of $G$. The second possibility is to start with a certain subspace of $T^*G$, and introduce on it a contact structure derived from the symplectic structure in $T^*G$, and an invariant measure derived from the natural volume in $T^*G$. We shall concentrate on the second possibility since the first one requires more elaborated procedures, involving (second class) constraints.

Consider the left action of $G$ on $T^*G$. To this action, as usual, we can associate a momentum map $P^R : T^*G \rightarrow \mathcal{G}^*$. If we consider a point $p$ in $\mathcal{G}^*$, we can define a hypersurface $\Sigma^R \subset T^*G$ as $\Sigma^R \equiv (P^R)^{-1}(p)$, i.e. the level set of $P^R$ associated with $p \in \mathcal{G}^*$.

On $\Sigma^R$ we have that $dP^R = 0$, and from this fact and the independence of the components of $P^R$ in $\mathcal{G}^*$ we deduce that $d\Theta_0 = P_X^i dX^i$ when restricted to $\Sigma^R$ (remember that, in $T^*G$, $\Theta_0$ could be written as $\Theta_0 = P_X^i dX^i + P_Y^i dY^i + P_{Z^j} dZ^j$).

Therefore, we can consider $\Sigma^R$ a contact manifold with presymplectic form $d\Theta_0|_{\Sigma^R}$. The kernel of this presymplectic 2-form is generated by the vector fields $\frac{\partial}{\partial Z^j}$, and if we quotient with the distribution generated by them we obtain a symplectic manifold $S$ to which we can apply the techniques of Geometric Quantization. The symplectic 2-form is $\omega = d\Theta_0|_S = P_X^i \wedge dX^i$ and there exist an invariant volume (under the left action of the group) on this symplectic manifold, which is clearly given by $\Omega \equiv \omega^k = P_X^1 \wedge dX^1 \wedge \ldots \wedge P_X^k \wedge dX^k$, where $2k$ is the dimension of the symplectic manifold $S$. This symplectic manifold is symplectomorphic to the co-adjoint orbit $O_p$ passing through the point $p \in \mathcal{G}^*$ under the co-adjoint action of $G$.

The quantization of the symplectic manifold $S$ would proceed as follows (see [27] for instance). Consider the complex hermitian line bundle $L$ on $S$ with connection $\nabla$ and curvature $\omega = dP_X^i \wedge dX^i$. The invariant (under the left action) volume is given by $\Omega$ and let us choose $\theta = P_X^i dX^i$ as the potential 1-form. The Hilbert space $\mathcal{H}$ is given by (the completion of) the space of smooth sections on $L$ with scalar product given by the scalar product on the vector bundle $L$.

\[2\] For concreteness, we shall consider one of the two possible actions of the group, and construct our procedure in terms of this action. Obviously, there exist the analogous constructions which lead to completely equivalent results.

\[3\] For the moment we shall restrict ourselves to regular points in $\mathcal{G}^*$.  

11
volume $\Omega$. The “quantum operator” $\hat{f}$ associated with the “classical” function $f$ on $S$ is defined in the usual way, in terms of the lift of the vector field $V_f$ associated with $f$:

$$\hat{f} = -i \nabla_{V_f} + f I,$$

where $V_f \equiv \{ f, \cdot \} = \frac{\partial f}{\partial P_{X_i}} \frac{\partial}{\partial X^i} - \frac{\partial f}{\partial X^i} \frac{\partial}{\partial P_{X_i}}$ and $I$ is the identity operator. If $S$ is a cotangent bundle, the connection is written like $\nabla_{V} \Psi = V \Psi - i \theta(V) \Psi$, for any vector field $V$ in $S$.

The left-invariant momenta $P^L_i$, which can be written in terms of the left-invariant variables $X^j$, $P_{X^j}$ and the Casimirs $P^Z_k$, restrict to the hypersurface $\Sigma^R$, and also to the symplectic manifold $S$, i.e. they continue to be the generating functions of the (right) action of $G$ on $\Sigma^R$ and $S$ since the Casimirs $P^Z_k$ are constant on $\Sigma^R$ and $S$. The quantum operators $\hat{P^L_i}$, obtained as the lift of the associated vector fields on $S$, provide us with a representation of the group $G$ acting on $\mathcal{H}$. This representation is unitary with the scalar product given by the invariant measure $\Omega$. As it is well-known, this representation is not irreducible and some invariant subspace $\mathcal{H}' \subset \mathcal{H}$ should be selected. This task is achieved by means of the polarization conditions, expressed in terms of operators acting on $\mathcal{H}$ obtained as the “horizontal lift” of vector fields associated with certain functions on $S$, which is given simply by $\hat{f} = \nabla_{V_f}$.

The polarization conditions are imposed through a Poisson subalgebra $\mathcal{P}$ of the Poisson algebra of functions on $S$. This Poisson subalgebra $\mathcal{P}$ defines a Hilbert subspace $\mathcal{H}' \subset \mathcal{H}$ on which the “horizontal lift” of all vector fields associated with functions in $\mathcal{P}$ vanish (in other words, $\mathcal{H}'$ is the subspace of sections which are “parallel” to the integral curves of all vector fields associated to functions in $\mathcal{P}$). Clearly, for this condition to be compatible with the representation of $G$, i.e. for $\mathcal{H}'$ to be an invariant subspace under the action of $G$, it is necessary that the condition $\{ P^L_i, \mathcal{P} \} \subset \mathcal{P}$ be satisfied. In this case we shall say that the polarization subalgebra $\mathcal{P}$ is admissible. This condition imposes strong restrictions on the choices (and even the existence) of polarization subalgebras $\mathcal{P}$ for each given group $G$.

Also, if the left invariant momenta are polynomials of degree up to one in the momenta $P_{X^j}$, we find easily an admissible polarization subalgebra, since the abelian subalgebra $\mathcal{P}^X$ of functions of $X^j$ is always preserved by the momentum maps $P^L_j$. Therefore, imposing the restrictions:

$$\hat{X}^i \Psi = \nabla_{V_{X^i}} \Psi = -\frac{\partial}{\partial P_{X^i}} \Psi = 0,$$

we select a subspace $\mathcal{H}^X$ of wave functions in $\mathcal{H}$ that depend only on the coordinates (coordinate representation) and such that it is invariant under the action of $G$. This subspace will provide, in most of the cases, an irreducible representation of $G$ with measure (not necessarily invariant) $dX^1 \wedge \ldots \wedge dX^k$. To guarantee the irreducibility, we should also take into account the possible existence of higher-order differential operators (see [29, 30]) or even discrete operators commuting with the representation, and which escapes to our

\footnote{This will always be our case by construction, since we are assuming that we can find (global) Darboux coordinates in the co-adjoint orbit diffeomorphic to $S$.}
local treatment. In the case in which the measure is not invariant, it will be quasi-invariant and therefore the representation can be unitarized by means of the Radon-Nikodym derivative (see [31]).

Depending on the form of the left-invariant momenta, there can exist more admissible polarization subalgebras leading to equivalent or inequivalent representations, this may change from case to case. If they are polynomials of degree up to one in the coordinates $X^i$, then the abelian subalgebra $P^{PX}$ of functions of $P_X$ defines an admissible polarization subalgebra. Therefore, imposing the restrictions:

$$\tilde{P}_{X^i} \Psi = \left( \frac{\partial}{\partial X^i} - iP_{X^i} \right) \Psi = 0,$$  \hspace{1cm} (20)

we select a subspace $\mathcal{H}^{PX}$ of wave functions depending, up to a phase factor, on the momenta (momentum representation), which is also invariant under the action of $G$.

It is worth to note that (in the case in which both polarizations are admissible) they are related by the “adjoint” action of a certain function in $S$. This is nothing other than (the generator of) the Fourier transform, associated with the function $\mathcal{F} \equiv \frac{1}{2} \sum_{i=1}^{k} (X^{i2} + P_{X^i}^2)$, verifying $\{ \mathcal{F}, P^X \} = P^{PX}$ and $\{ \mathcal{F}, P^{PX} \} = P^X$. The coordinate and momentum representations are unitarily equivalent, since the Fourier transform is a unitary operator. However, the Fourier transform is an integral operator and cannot be obtained within the framework of Geometric Quantization (one has to resort to the Meaplectic representation, see for instance [32]). The generator of the Fourier transform can be obtained as a second order differential operator using the technique of higher-order polarizations [29, 30] (see below).

It is interesting to note that, even in the case in which only one of the polarization subalgebras $P^X$ or $P^{PX}$ is admissible, this does not mean that the other representation does not exist. In fact, one can apply the Fourier transformation and obtain an equivalent representation. The difference arises in the fact that in this representation the operators $\tilde{P}^L_{X^i}$ will be higher-order differential operators, which cannot be obtained within the framework of Geometric Quantization. Using other techniques, like the technique of higher-order polarizations (see [33, 24, 30], for instance), one can obtain them, even in the anomalous case in which none of the polarizations exist, as it happens for the Schrödinger group.

Essentially, the technique of higher-order polarizations consists in allowing to enter the polarization subalgebra higher-order differential operators, which are available in the enveloping algebra of the Lie algebra. More precisely, a higher-order polarization is a maximal subalgebra of the (left or right) universal enveloping algebra with no intersection with the identity operator on sections. With this definition, a higher-order polarization contains the maximum number of conditions compatible with the hermitian line bundle structure of $\mathcal{L}$ and with the action of the quantum operators.

The use of higher-order polarizations allows, for instance, to obtain the Schrödinger equation in configuration space, which is a second order differential equation, or to find a polarization for the anomalous case of the Schrödinger group, which does not admit any
usual, first order, polarization. However, in this work we shall restrict to usual, first order polarizations.

Finally, there exist polarization conditions which are not related to polarization subalgebras, i.e. they are not the “horizontal lift” of any vector field in $\mathcal{S}$, but define a subspace $\mathcal{H}' \subset \mathcal{H}$ that is left invariant by the action of $G$. We shall see an example of such polarization condition for the case of the 2-dimensional group.

8 Examples

In this section we shall apply the results of previous sections to some particular examples, the simplest case of the non-trivial two dimensional Lie group, and all three dimensional Lie groups.

For all these groups, generic co-adjoint orbits have dimension 2, for which Darboux coordinates $p, q$ can be found and their inverse images by the (left and right invariant) momentum maps provide us with the coordinates (left and right invariant, respectively) $X, P_X$ and $Y, P_Y$. Choosing an arbitrary point $p$ in each co-adjoint orbit, we construct the hypersurface $\Sigma^R$ as explained in the previous section, which is parametrized by $X, P_X$ and $Z$ (in the case of the 2-dimensional group the variable $Z$ is not present since there is no Casimir). The presymplectic 2-form $d\Theta_0|_{\Sigma^R}$ is $dP_X \wedge dX$, which has $\frac{\partial}{\partial Z}$ as kernel (except for the 2-dimensional group, for which it is directly symplectic). Taking quotient by the distribution generated by this kernel we obtain a symplectic manifold $\mathcal{S}$ with symplectic 2-form $\omega = d\Theta_0|_{\mathcal{S}} = dP_X \wedge dX$. Although apparently the construction is identical for all the groups (this is because we are working with Darboux coordinates), $\mathcal{S}$ differs for each group, since it is a homogeneous manifold with respect to its corresponding group, and the range of the parameters $X$ and $P_X$ can also be different for each group.

The construction of the hermitian line bundle $\mathcal{L}$ is also similar in all cases, with the connection $\nabla$ given by the potential 1-form $\theta = P_X dX$ and curvature $\omega = dP_X \wedge dX$. The Hilbert space $\mathcal{H}$ of smooth sections on $\mathcal{L}$ completed with respect to the scalar product given by the invariant measure $\Omega = \omega$ is also constructed along the guidelines of previous section. The “vertical lift” of a vector field on $\mathcal{S}$ to a (quantum) operator on $\mathcal{H}$ is constructed as explained previously, and the same for the construction of polarization conditions by means of the “horizontal lift” of vector fields. The differences for each groups lies in the explicit form of the operators $\hat{\mathcal{P}}^L$ associated with the left-invariant momentum maps of the action of $G$ on and in the existence or not of certain polarization subalgebras $\mathcal{P}$.

8.1 The 2-dim group

Consider the group of matrices

$$
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix},
$$

(21)
with \( x \in R - \{0\} \) and \( y \in R \). This group can be identified with the group of scale transformations and translations on the real line:

\[
u' = xu + y
\]

It is not connected unless \( x \) is restricted to be positive, \( G^+ \) will denote the component of \( G \) connected to the identity. This group is solvable and therefore unitary representations are either one dimensional or infinite dimensional.

It is also diffeomorphic to \( SB(2, R) \), the diffeomorphism being

\[
\begin{pmatrix}
x y \\
0 \frac{1}{x}
\end{pmatrix} \iff \begin{pmatrix}
x^3 xy \\
0 1
\end{pmatrix}
\]

\( \odot \) From matrix multiplication we can obtain easily the group law:

\[
x'' = x'x \\
y'' = y' + x'y,
\]

and from this right and left invariant vector fields are easily computed:

\[
\begin{align*}
X^L_x &= x \frac{\partial}{\partial x} & X^L_y &= \frac{\partial}{\partial y} \\
X^R_x &= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} & X^R_y &= -\frac{\partial}{\partial y}.
\end{align*}
\]

The Lie algebra for this group is given by:

\[
[X^L_x, X^L_y] = X^L_y.
\]

This Lie algebra, being two dimensional and non-trivial, admits no Casimirs. The left and right-invariant momentum maps \( \mathcal{P}^{L,R} : T^* G^+ \to G^* \) of the action of \( G^+ \) on \( T^* G^+ \) are computed, as usual, by \( \mathcal{P}^{L,R}_i \equiv i_{\mathfrak{g}^{L,R}} \Theta_0 \), where \( \Theta_0 \) is the canonical 1-form on \( T^* G^+ \), \( \Theta_0 = p_x dx + p_y dy \). We obtain:

\[
\begin{align*}
\mathcal{P}^L_x &= xp_x & \mathcal{P}^R_x &= -xp_x - yp_y \\
\mathcal{P}^L_y &= xp_y & \mathcal{P}^R_y &= -p_y
\end{align*}
\]

To find the canonical variables \((X, Y, P_X, P_Y)\) in \( T^* G^+ \), we proceed as explained before, finding Darboux coordinates on co-adjoint orbits. In this case, co-adjoint orbits by the action of \( G^+ \) are given by two half-planes \( A^+ = \{p_y > 0\} \) and \( A^- = \{p_y < 0\} \) and the zero-dimensional orbits \( A^0_{px} = \{p_x, p_y = 0\} \). If the non-connected group \( G \) is considered, \( A^+ \cup A^- \) is a single two-dimensional disconnected orbit.

In the dual space of the Lie algebra there is a natural Poisson bracket

\[
\Lambda = p_y \frac{\partial}{\partial p_x} \wedge \frac{\partial}{\partial p_y},
\]
which is non degenerate on $A^+$ and $A^-$. On $A^+$ and $A^-$, $\Lambda$ has an inverse

$$\omega = (\Lambda)^{-1} = \frac{1}{p_y} dp_x \wedge dp_y$$

A set of Darboux coordinates for $\omega$ is given by the map $S : \mathcal{G}^* \to T^*R$:

$$S(p_x, p_y) = (p_y, \frac{p_x}{p_y}) \equiv (\pi, q) \quad (25)$$

Making use of the left and right-invariant momentum maps, we obtain:

$$X = (\mathcal{P}^{L})^* S^*(q) = \frac{\mathcal{P}^L_x}{\mathcal{P}^L_y} = \frac{p_x}{p_y}$$

$$P_X = (\mathcal{P}^{L})^* S^*(\pi) = \mathcal{P}^L_y = xp_y \quad (26)$$

$$Y = (\mathcal{P}^{R})^* S^*(q) = \frac{\mathcal{P}^R_x}{\mathcal{P}^R_y} = \frac{x p_x}{p_y} + y$$

$$P_Y = (\mathcal{P}^{R})^* S^*(\pi) = \mathcal{P}^R_y = -p_y \quad (27)$$

Inverting these relations we get the expressions of $\mathcal{P}^{L,R}_i$ in terms of $X, P_X$ (resp. $Y, P_Y$):

$$\mathcal{P}^L_x = XP_X$$

$$\mathcal{P}^L_y = P_X \quad (28)$$

$$\mathcal{P}^R_x = YP_Y$$

$$\mathcal{P}^R_y = P_Y.$$  

Now we proceed with the computation of the irreducible representations. We shall consider only the 2-dimensional orbits in $\mathcal{G}^*$, $A^+$ and $A^-$, which are generic. The non-generic ones are only, in this case, the zero-dimensional orbits. These orbits are associated with one-dimensional representations of $G^+$, i.e. characters of $G^+$, and can be computed easily with other procedures. Also, the representations associated to $A^+$ and $A^-$ are equivalent, since the discrete operator of $G$ which sends one orbit to the other is unitary.

Thus, let us consider a point $p \in A^+$, for instance $p = (0, p_y^0 > 0)$. The hypersurface $\Sigma^R$ associated with this point is given by $\Sigma^R \equiv \mathcal{P}^{R-1}(p) = \{(x, y, p_x, p_y) \in T^*G^+, \text{ such that } xp_x + yp_y = 0, \ p_y = p_y^0\}$. The variables $Y$ and $P_Y$ are constant on $\Sigma^R$, which is parameterized by $X = -y/x$ and $P_X = xp_y^0 > 0$.

In this particular case the presymplectic 2-form $d\Theta_0|_{\Sigma^R}$ has no kernel and therefore the symplectic manifold $S$ coincides with $\Sigma^R$ and $\omega$ with $d\Theta_0|_{\Sigma^R}$.

The vertical lift of the vector fields associated with $\mathcal{P}^L_x$ and $\mathcal{P}^L_y$ are:

$$\hat{\mathcal{P}}^L_x = -iX \frac{\partial}{\partial X} + iP_X \frac{\partial}{\partial P_X}, \quad \hat{\mathcal{P}}^L_y = -i \frac{\partial}{\partial X}. \quad (30)$$
Since the left invariant momenta $P^L_x, P^L_y$ are both polynomials of degree up to one in $P_X$ and $X$, the polarization subalgebras $P^X$ and $P^P_X$ are both admissible, and lead to equivalent representations related by the Fourier transform. Using $P^X$, for instance, the restriction that we have to impose is given by $\tilde{X} \Psi = -\frac{\partial}{\partial P_X} \Psi = 0$, therefore obtaining a Hilbert subspace $\mathcal{H}^X$ of sections depending only on $X$ on which the operators $\hat{P}^L_x$ and $\hat{P}^L_y$ have the form:

$$
\hat{P}^L_x \Psi(X) = -iX \frac{\partial}{\partial X} \Psi(X)
$$

$$
\hat{P}^L_y \Psi(X) = -i \frac{\partial}{\partial X} \Psi(X).
$$

This clearly constitute an irreducible representation of the 2-dimensional group. It is not unitary since the measure $dX$ on $\mathcal{H}^X$ is not invariant, but quasi-invariant. Introducing the Radon-Nykodim derivative we obtain a unitary representation in which $\hat{P}^L_x \Psi(X) = -i(X \frac{\partial}{\partial X} + \frac{i}{2}) \Psi(X)$, as it should be.

There exist another polarization which does not come from any polarizing subalgebra $P$ in $\mathcal{S}$. This is given by the condition:

$$
\frac{\partial}{\partial X} \Psi = 0,
$$

which leads to sections depending only on $P_X$ (do not confuse with the ones given by the polarization $P^P_X$, which carry an additional phase of the form $e^{iXP_X}$). This defines an invariant Hilbert space $\mathcal{H}^{P_X'}$ on which the operators act as:

$$
\hat{P}^L_x \Psi(P_X) = iP_X \frac{\partial}{\partial P_X} \Psi(P_X)
$$

$$
\hat{P}^L_y \Psi(P_X) = 0.
$$

This representation, with measure $dP_X$ and corrected with the Radon-Nikodym derivative\[^{[5]}\] is clearly non equivalent to the previous one, since one is faithful and the other one is not (the operator $\hat{P}^L_y$ is represented trivially).

### 8.2 The Heisenberg-Weyl group

In quantum optics, the Heisenberg–Weyl group plays a very important role. The quadrature components of photon are the generator of Lie algebra of the Heisenberg–Weyl group. The photon creation and annihilation operators are linear combinations of the Lie algebra generators.

The Heisenberg-Weyl group in one dimension is a non-trivial central extension of $R^2$ by $R$ or $U(1)$. Its elements can be written in matrix form:

$$
g = \begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}.
$$

\[^{[5]}\]In this particular case it is already unitary with respect to the invariant measure $dP_X/P_X$, since the translations are represented trivially.
The group law $g'' = g' \ast g$ for this group is given by:

$$
x'' = x' + x
\quad
y'' = y' + y
\quad
z'' = z' + z + x'y.
$$

(35)

Left and right-invariant vector fields are given by:

$$
X^L_x = \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}
\quad
X^L_y = \frac{\partial}{\partial y}
\quad
X^L_z = \frac{\partial}{\partial z}
\quad
X^R_x = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}
\quad
X^R_y = -\frac{\partial}{\partial y}
\quad
X^R_z = -\frac{\partial}{\partial z}.
$$

(36)

The commutation relations are:

$$
[X^L_x, X^L_y] = X^L_z
$$

(37)

This Lie algebra admits a Casimir, which is given by the central element of the Lie algebra $X_z^L = -X_z^R = \frac{\partial}{\partial z}$.

We compute the left and right-invariant momentum maps $P^{L,R}_i : T^*G \to G^*$ as usual, $P_i^{L,R} \equiv i_{X_i^{L,R}} \theta_0$, where $\theta_0$ is the canonical 1-form on $T^*G$, $\theta_0 = p_\theta d\theta + p_x dx + p_y dy$. We obtain:

$$
P^L_x = p_x
\quad
P^L_y = p_y + xp_z
\quad
P^L_z = p_z
\quad
P^R_x = -p_x - yp_z
\quad
P^R_y = -p_y
\quad
P^R_\theta = -p_z.
$$

(38)

The Lie algebra, in terms of Poisson brackets, generated by $P_i^{L,R}$ is isomorphic to that of $G$ in (37).

The classical Casimir is given by $p_z$. To find the canonical variables $(X, Y, Z, P_X, P_Y, P_Z)$ in $T^*G$ we proceed as explained before, computing Darboux coordinates on co-adjoints orbits. For the H-W group, the co-adjoints orbits are of two classes: two-dimensional orbits of the form $O_{p_z} = \{(p_x, p_y, p_z), p_x, p_y \in R\}$, with $p_z = p_z^0 \neq 0$, and zero-dimensional orbits of the form $O_{p_x, p_y} = \{(p_x, p_y, 0)\}$. Obviously, we are interested in the two-dimensional orbits $O_{p_z}$.

The symplectic form $\omega$ for $O_{p_z}$, obtained by inverting the Poisson brackets on the orbit, has the form:

$$
\omega = \frac{1}{p_z^0} dp_x \wedge dp_y,
$$

(39)

(remember that $\{p_x, p_y\} = p_z$ and $p_z$ is central). A set of Darboux coordinates is given, for example, by the map $S : G^* \to T^*R$ with $S(p_x, p_y, p_z) = (\frac{2p_y}{p_z}, p_z) = (q, \pi)$ or by the map $S' : G^* \to T^*R$ with $S'(p_x, p_y, p_z^0) = (\frac{2p_y}{p_z^0}, -p_y) = (q', \pi')$. Therefore, making use of the left and right momentum maps, we obtain:

$$
X = (P^L)^* S^*(q) = \frac{p_y}{p_z^0} = x + \frac{p_y}{p_z^0}
$$

18
\[ P_X = (\mathcal{P}^L)^*S^*(\pi) = P_x^L = p_x \quad (40) \]

\[ Y = (\mathcal{P}^R)^*S^*(q') = \frac{P_x^R}{P_z^R} = y + \frac{p_x}{p_z^0} \]

\[ P_Y = (\mathcal{P}^R)^*S^*(\pi') = P_y^R = p_y. \]

Inverting these relations, we can obtain the expressions of \( \mathcal{P}_i^{L,R} \) in terms of \( X, P_X, P_Z \) (resp. \( Y, P_Y, P_Z \)):

\[
\begin{align*}
\mathcal{P}_x^L &= P_X \\
\mathcal{P}_y^L &= XP_Z \\
\mathcal{P}_z^L &= P_Z \\
\mathcal{P}_x^R &= YP_Z \\
\mathcal{P}_y^R &= P_Y \\
\mathcal{P}_z^R &= P_Z,
\end{align*}
\]

(41)

where the Casimir \( P_Z \) is easily computed to be \( p_z \). To compute the cyclic variable \( Z \), we make use of the fact that

\[ \Theta_0 = p_x dx + p_y dy + p_z dz = P_X dX + P_Y dY + P_Z dZ, \]

from which we obtain directly that

\[ Z = z - \frac{p_x p_y}{p_z^0}. \]

For the computation of the irreducible representations, let us consider, as explained in Sec. 4, a point \( p \) belonging to a generic orbit in \( G^* \). In this case the generic orbits are the two dimensional orbits \( O_{p^0} \), with \( p_z^0 \neq 0 \). We can choose, for simplicity, \( p = (0, 0, p_z^0) \). This defines a hypersurface of the form \( \Sigma^R = \mathcal{P}^{L-1}(p) = \{(x, y, z, p_x, p_y, p_z) \in T^*G \text{ such that } p_x + yp_z = 0, p_y = 0, p_z = p_z^0\} \) on \( T^*G \), which is parameterized, in this case, by the coordinates \( X = x, P_X = p_x \) and \( Z = z \).

The vertical lift of the vector fields associated with the left invariant momenta are:

\[
\begin{align*}
\hat{\mathcal{P}}_x^L &= -i \frac{\partial}{\partial X} \\
\hat{\mathcal{P}}_y^L &= iP_Z \frac{\partial}{\partial P_X} + XP_Z I \\
\hat{\mathcal{P}}_z^L &= P_Z I.
\end{align*}
\]

(43)

Again, the left invariant momenta are all polynomials of degree up to one in the variables \( X \) and \( P_X \), implying that both polarization subalgebras \( \mathcal{P}^X \) and \( \mathcal{P}^{P_X} \) are admissible and lead to equivalent representations, related by the Fourier Transform. Using, for instance, \( \mathcal{P}^X \) we obtain the same Hilbert subspace \( \mathcal{H}^X \) of sections depending only on the
variable $X$ as in the case of the 2-dimensional group. The action of the operators in (43) on this Hilbert space is given by:

\[
\hat{P}_x \Psi(X) = -i \frac{\partial}{\partial X} \Psi(X), \\
\hat{P}_y \Psi(X) = X P_Z \Psi(X), \\
\hat{P}_z \Psi(X) = P_Z \Psi(X).
\]

This constitutes a unitary and irreducible representation of the H-W group, and coincides with the standard Schrödinger representation with the identification $P_Z \equiv \hbar$. Thus the eigenvalue of the Casimir operator is identified with the Planck’s constant.

Unlike the case of the 2-dimensional group, $\frac{\partial}{\partial X}$ does not define a polarization condition, since the subspace of sections depending only on $P_X$ (without any additional phase as the one that results from the polarization $P^{P_X}$) is not invariant under the H-W group. Therefore these (one for each value of $P_Z$) are the only unitary irreducible representations of the Heisenberg-Weyl group, apart from the one-dimensional ones.

### 8.3 The Euclidean group $E(2)$

The Euclidean group in two dimensions is the semidirect action of $U(1)$ on $\mathbb{R}^2$, i.e. it is constituted by translations and rotations in the plane. Its elements can be written in a matrix form as:

\[
g = \begin{pmatrix} \cos \theta & \sin \theta & x \\ -\sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix}.
\]

The group law $g'' = g' \ast g$ for this group is given by:

\[
\theta'' = \theta' + \theta, \\
x'' = x' + \cos \theta' x + \sin \theta' y, \\
y'' = y' - \sin \theta' x + \cos \theta' y.
\]

Left and right-invariant vector fields are given by:

\[
X^L_\theta = \frac{\partial}{\partial \theta}, \\
X^L_x = \cos \theta \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y}, \\
X^L_y = \cos \theta \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial x}, \\
X^R_\theta = -\frac{\partial}{\partial \theta} + y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \\
X^R_x = -\frac{\partial}{\partial x}, \\
X^R_y = -\frac{\partial}{\partial y}.
\]

The commutation relations are:

\[
[X^L_\theta, X^L_x] = X^L_y, \\
[X^L_\theta, X^L_y] = -X^L_x.
\]

This Lie algebra admits a Casimir, which is given by the second order operator

\[
\hat{C}_2 = (X^L_x R)^2 + (X^L_y R)^2.
\]
We compute the left and right-invariant momentum maps $P^L, P^R : T^*G \to G^*$ as usual,
$P^L_i, P^R_i \equiv \iota_{X^L, R}^\# \Theta_0$, where $\Theta_0$ is the canonical 1-form on $T^*G$, $\Theta_0 = p_\theta d\theta + p_x dx + p_y dy$. We obtain:

\begin{align*}
P^L_\theta &= p_\theta \\
P^L_x &= \cos \theta p_x - \sin \theta p_y \\
P^L_y &= \cos \theta p_y + \sin \theta p_x \\
\end{align*}

The classical Casimir is given by $C_2 = (P^L_x)^2 + (P^L_y)^2 = p_x^2 + p_y^2$.

To find the canonical variables $(X, Y, Z, P_X, P_Y, P_Z)$ in $T^*G$ we proceed as explained before, computing Darboux coordinates on co-adjoints orbits. For the case of $E(2)$, co-adjoint orbits are easily computed. The two-dimensional co-adjoints orbits correspond to the cylinders given by the equation $C_2 = p_x^2 + p_y^2 = R^2$, for $R > 0$, and the zero-dimensional orbits are given by $O_{p_\theta} = \{(p_\theta, 0, 0)\}$. We are interested in the two-dimensional orbits, for which the symplectic form is (computed again by inverting the Poisson brackets on the co-adjoint orbit):

$$\omega = \frac{1}{R^2} dp_\theta \wedge (p_y dp_x - p_x dp_y).$$

A set of Darboux coordinates for $\omega$ is given by the map $S : G^* \to T^*S^1$:

$$S(p_\theta, p_x, p_y) = (p_\theta, \tan^{-1}\frac{p_x}{p_y}) = (\pi, q).$$

Once we have found Darboux coordinates on the co-adjoint orbits, we simple define $X, P_X$ (resp. $Y, P_Y$) as the pullback by $P^L$ (resp. $P^R$) of $S^*(q)$ and $S^*(\pi)$, respectively:

\begin{align*}
X &= (P^L)^* S^*(q) = \tan^{-1}\frac{P^L_x}{P^L_y} = \tan^{-1}\frac{\cos \theta p_x - \sin \theta p_y}{\cos \theta p_y + \sin \theta p_x} \\
P_X &= (P^L)^* S^*(\pi) = P^L_\theta = p_\theta \\
Y &= (P^R)^* S^*(q) = \tan^{-1}\frac{P^R_x}{P^R_y} = \tan^{-1}\frac{p_x}{p_y} \\
P_Y &= (P^R)^* S^*(\pi) = P^R_\theta = -p_\theta + y p_x - x p_y.
\end{align*}

Inverting these relations we obtain the expressions of $P^L_i$ in terms of $X, P_X$ and $P_Z$:

\begin{align*}
P^L_\theta &= P_X \\
P^L_x &= \sin X \ P_Z \\
P^L_y &= \cos X \ P_Z,
\end{align*}

with analogous expressions for $P^R_i$ in terms of $Y, P_Y$ and $P_Z$, where $P_Z$ is the function of the Casimir $P_Z = R = \sqrt{C_2}$. The only thing we have to compute is the cyclic coordinate $Z$ canonically conjugated to $P_Z$. As in the previous example, using the fact that the
canonical 1-form $\theta_0$ in $T^*G$ can be written as $P_X dX + P_Y dY + P_Z dZ$, we derive the expression for $Z$:

$$Z = \frac{xp_x + yp_y}{\sqrt{p_x^2 + p_y^2}}. \quad (55)$$

Following Sec. 7, let us compute the irreducible representations. Choose a point $p$ in each 2-dimensional co-adjoint orbit, characterized by radius $R$, for instance $p = (0, R, 0)$. The hypersurface $\Sigma^R$ is given by $\Sigma = \mathcal{P}^{L-1}(p) = \{(x, y, \theta, p_x, p_y, p_\theta) \in T^*G$ such that $p_\theta - yp_x - xp_y = 0, p_x = R, p_z = 0\}$. $\Sigma^R$ is then parametrized by the coordinates $X = \frac{\pi}{2} - \theta$, $P_X = yR$ and $Z = x$.

The vertical lift of the momentum maps are given by:

$$\hat{\mathcal{P}}^L_{\theta} = -i \frac{\partial}{\partial X}$$

$$\hat{\mathcal{P}}^L_x = i \cos XP_Z \frac{\partial}{\partial P_X} + \sin XP_Z I \quad (56)$$

$$\hat{\mathcal{P}}^L_y = -i \sin XP_Z \frac{\partial}{\partial P_X} + \cos XP_Z I.$$

In this case, the momentum maps are all up to first order in the momentum $P_X$, therefore the polarization subalgebra $\mathcal{P}^X$ is admissible, but are not polynomial in $X$, implying that the polarization subalgebra $\mathcal{P}^{P_X}$ is not admissible. Using $\mathcal{P}^X$, we obtain the representation in coordinate space, with sections depending only on $X$, and operators with the form:

$$\hat{\mathcal{P}}^L_{\theta} \Psi(X) = -i \frac{\partial}{\partial X} \Psi(X)$$

$$\hat{\mathcal{P}}^L_x \Psi(X) = P_Z \sin X \Psi(X) \quad (57)$$

$$\hat{\mathcal{P}}^L_y \Psi(X) = P_Z \cos X \Psi(X).$$

This representation is irreducible and unitary with respect to the measure $dX$. Up to equivalence, it is the only representation for each value of $P_Z = R$ of the group $E(2)$ (that is, there are no other polarization conditions leading to inequivalent representations such as happened in the 2-dimensional group with $\frac{\partial}{\partial X}$).

8.4 The $SB(2, C)$ group

The $SB(2, C)$ group is given by the following matrix representation:

$$\begin{pmatrix} x & y + iz \\ 0 & x^{-1} \end{pmatrix}, \quad (58)$$

with $x \in R - \{0\}$, and $y, z \in R$. As in the case of $SB(2, R)$, it is a non-connected group, the connected component of the identity $G^+$ is given by $x \in R^+$.  

22
The group law is easily obtained from matrix multiplication, being:

\[ x'' = xx' \]
\[ y'' = \frac{yy'}{x} + x'y \]
\[ z'' = \frac{zz'}{x} + x'z \]  \hfill (59)

Left and right invariant vector fields are:

\[ X^L_x = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \]
\[ X^L_y = x \frac{\partial}{\partial y} \]
\[ X^L_z = x \frac{\partial}{\partial z} \]
\[ X^R_x = -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \]
\[ X^R_y = -x^{-1} \frac{\partial}{\partial y} \]
\[ X^R_z = -x^{-1} \frac{\partial}{\partial z} \]  \hfill (60)

The commutations relations are given by:

\[ [X^L_x, X^L_y] = 2X^L_y \]
\[ [X^L_x, X^L_z] = 2X^L_z \]
\[ [X^L_y, X^L_z] = 0 \]  \hfill (61)

This algebra admits a non polynomial (and non smooth) Casimir, given by:

\[ \hat{C} = \frac{X^{L,R}_y X^{L,R}_z}{X^{L,R}_z} \]  \hfill (62)

The momentum maps for the left and right action of \( G \) on \( T^*G \) are:

\[ P^L_x = xp_x - yp_y - zp_z \]
\[ P^L_y = xp_y \]
\[ P^L_z = xp_z \]
\[ P^R_x = -xp_x - yp_y - zp_z \]
\[ P^R_y = -x^{-1}p_y \]
\[ P^R_z = -x^{-1}p_z \]  \hfill (63)

The classical Casimir is given by \( C = \frac{P^{L,R}_y}{P^{L,R}_z} = \frac{p_y}{p_z} \).

Co-adjoint orbits are given, as usual, by the co-adjoint action of the group on \( G^* \), parametrized by \((p_x, p_y, p_z)\). In this case there are orbits given by constant values of the Casimir function on \( G^* \), and orbits given by invariant relations (functions \( f_i \) on \( G^* \) which are not invariant under the co-adjoint action but such that the equations \( f_i = 0 \) are preserved) \[23\]. The invariant relations are \( p_y = 0, p_z = 0 \), determining the points \((p_x, 0, 0)\), for \( p_x \in \mathbb{R} \) as zero dimensional co-adjoint orbits.

The co-adjoint orbits given by the constant values of the Casimir function are given by \( p_y/p_z = c \), which, for convenience we denote as \( c = \tan \theta \). These orbits are planes distributed as a "book", i.e. all of their closures "intersect" along the line \( p_y = p_z = 0 \). This line, however, does not lies in these co-adjoint orbits, since each point of the line is a co-adjoint orbit by itself. It is convenient to parametrize these planes in the form \((p_x, r \sin \theta, r \cos \theta)\), with \( r \neq 0 \), where we can interpret \( \theta \) as the angle between the plane and the \( p_y \)-axis. As in the case of the 2-dim group (isomorphic to \( SB(2, \mathbb{R}) \)), these orbits
are disconnected \((r > 0 \text{ and } r < 0)\) if we consider the action of \(G\) or each one decomposes in two connected co-adjoint orbits if we consider \(G^+\).

The Poisson structure can be inverted in each 2-dimensional co-adjoint orbit, providing a symplectic structure of the form:

\[ w_\theta = \frac{1}{2r} dp_x \wedge dr. \]  

Clearly, canonical coordinates are given by the Darboux map \(S(p_x, p_y, p_z) = (\frac{p_z}{2}, r) = (\pi, q)\). This allows to define canonical coordinates in \(T^*_G\) by means of the pull-back by left and right invariant momenta:

\[
\begin{align*}
X &= (\mathcal{P}^L)^* S^*(q) = \frac{P_y}{\sin \theta} = x \sqrt{p^2_y + p^2_z} \\
P_X &= (\mathcal{P}^L)^* S^*(\pi) = \frac{P_x}{2P_y} \sin \theta = \frac{x p_x - y p_y - z p_z}{2x \sqrt{p^2_y + p^2_z}} \\
Y &= (\mathcal{P}^R)^* S''(q) = \frac{P_z}{\sin \theta} = \frac{\sqrt{p^2_y + p^2_z}}{x} \\
P_Y &= (\mathcal{P}^R)^* S''(\pi) = \frac{P_x}{2P_y} \sin \theta = \frac{x(p_x + y p_y + z p_z)}{2 \sqrt{p^2_y + p^2_z}},
\end{align*}
\]

where we have replaced \(\tan \theta\) with \(p_y/p_z\), and used the fact that it remains unchanged under the left and right pull-backs. Inverting these relations we obtain:

\[
\begin{align*}
\mathcal{P}^L_x &= 2X P_X \\
\mathcal{P}^L_y &= \frac{P_z}{\sqrt{1 + P^2_z}} X = \sin \theta X \\
\mathcal{P}^L_z &= \frac{1}{\sqrt{1 + P^2_z}} X = \cos \theta X,
\end{align*}
\]

where \(P_z \equiv \tan \theta\), and identical expressions for the \(\mathcal{P}^R\)'s in terms of \(P_Y\), \(Y\) and \(P_Z\).

To complete the canonical transformation from \((x, y, z, p_x, p_y, p_z)\) to \((X, Y, Z, P_X, P_Y, P_Z)\), we need the expression of the cyclic variable \(Z\). This is easily obtained to be:

\[
Z = \frac{y p_z - z p_y}{1 + P^2_z} = \frac{p^2_z(y p_z - z p_y)}{p^2_y + p^2_z}.
\]

Let us compute the irreducible representations using the procedure explained in Sec. 4. For each 2-dimensional co-adjoint orbit, characterized by the Casimir \(c = \tan \theta\), we choose a point \(p = (0, \sin \theta, \cos \theta)\). The hypersurface \(\Sigma^R\) associated with it is given by \(\Sigma^R = \mathcal{P}^L^{-1}(p) = \{(x, y, z, p_x, p_y, p_z) \in T^*_G \text{ such that } x p_x + y p_y + z p_z = 0, p_y/x = \sin \theta, p_y/x = \).

24
The coordinates which parametrize $\Sigma^R$ are $X = x^2$, $P_X = -(y \sin \theta + z \cos \theta)/x$ and $Z = x \cos^2 \theta (y \cos \theta - z \sin \theta)$.

The vertical lift of the momentum maps are given by:

\[
\hat{P}^L_x = -2i X \frac{\partial}{\partial X} + 2i P_X \frac{\partial}{\partial P_X} \\
\hat{P}^L_y = i \sin \theta \frac{\partial}{\partial P_X} + \sin \theta X I \\
\hat{P}^L_z = i \cos \theta \frac{\partial}{\partial P_X} + \cos \theta X I.
\]

Since the momentum maps are polynomials up to first order in the momentum $P_X$ and the coordinate $X$, both polarization subalgebras $P^X$ and $P^{P_X}$ are admissible, leading to equivalent representations related by the Fourier transform. Imposing, for instance, $P^X$, we obtain a Hilbert space of sections depending only of $X$, with the action of the operators given by:

\[
\hat{P}^L_x \Psi(X) = -i2X \frac{\partial}{\partial X} \Psi(X) \\
\hat{P}^L_y \Psi(X) = X \sin \theta \Psi(X) \\
\hat{P}^L_z \Psi(X) = X \cos \theta \Psi(X).
\]

This irreducible representation is very similar to the one obtained for the 2-dimensional group, and as happened there, it is not unitary with respect to the measure $dX$, and needs to be corrected with the Radon-Nikodym derivative, which changes the first operator to its correct expression $\hat{P}^L_x \Psi(X) = -i2(X \frac{\partial}{\partial X} + \frac{1}{2}) \Psi(X)$. The representations for different values of $\tan \theta$ are, in this case, equivalent, since there are unitary operator relating all of them. These operators are rotations in the angle $\theta$, and are associated with the rotations in $G^*$ which relate all 2-dimensional co-adjoint orbits. These operators, as happens with the Fourier transform, are not inner operators.

As it happens with the 2-dimensional group, there exist another polarization condition which does not come from any polarization subalgebra. It is given by the operator $\frac{\partial}{\partial P_X} - iXI$, which leads to a Hilbert subspace preserved by the group $SB(2, C)$, where the sections have the form $\Psi = e^{iX P_X} \Phi(X)$. The action of the operators on this sections are given by:

\[
\hat{P}^L_x (e^{iX P_X} \Phi(X)) = e^{iX P_X} (-i2X \frac{\partial}{\partial X}) \Phi(X) \\
\hat{P}^L_y (e^{iX P_X} \Phi(X)) = 0 \\
\hat{P}^L_z (e^{iX P_X} \Phi(X)) = 0.
\]

This representation is irreducible and unitary with respect to the measure $dX/X$ (or with the measure $dX$ but corrected with the Radon-Nikodym derivative), is the same for all 2-dimensional co-adjoint orbits, and it is not equivalent to the previous one for the same reason as in the case of the 2-dimensional group.
8.5 The \( SL(2, \mathbb{R}) \) group

The \( SL(2, \mathbb{R}) \) group is the group of real \( 2 \times 2 \) matrices of determinant one:

\[
G = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ such that } \alpha \delta - \beta \gamma = 1 \right\}.
\] (71)

We will use a Gauss decomposition for \( SL(2, \mathbb{R}) \) of the form:

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\] (72)

The group law \( g'' = g' \ast g \) for this group in terms of the variables \( x, y, z \) is given by:

\[
\begin{align*}
x'' &= x + e^{-2z} \frac{x'}{1 + x'y} \\
y'' &= y' + e^{-2z'} \frac{y}{1 + x'y} \\
z'' &= z' + z + \log(1 + x'y).
\end{align*}
\] (73)

Left and right-invariant vector fields are given by:

\[
\begin{align*}
X^L_x &= \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial z} \\
X^L_y &= e^{-2z} \frac{\partial}{\partial y} - x^2 \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \\
X^L_z &= \frac{\partial}{\partial z} \\
X^R_x &= -e^{-2z} \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \\
X^R_y &= -\frac{\partial}{\partial y} \\
X^R_z &= -\frac{\partial}{\partial z} + 2y \frac{\partial}{\partial y}.
\end{align*}
\] (74)

The commutation relations are:

\[
\begin{align*}
[X^L_x, X^L_y] &= X^L_z \\
[X^L_x, X^L_z] &= -2X^L_x \\
[X^L_y, X^L_z] &= 2X^L_y.
\end{align*}
\] (75)

This Lie algebra admits a Casimir, which is given by the second order operator

\[
\hat{C}_2 = (X^L_x)^2 + 2(X^L_x X^L_y + X^L_y X^L_x).
\] (76)

We compute the left and right-invariant momentum maps \( \mathcal{P}^{L,R} : T^*G \to \mathcal{G}^* \) as usual, \( \mathcal{P}^{L,R}_i \equiv i_{X^L,R_i} \Theta_0 \), where \( \Theta_0 \) is the canonical 1-form on \( T^*G \), \( \Theta_0 = p_x dx + p_y dy + p_z dz \). We obtain:

\[
\begin{align*}
\mathcal{P}^L_x &= p_x \\
\mathcal{P}^L_y &= e^{-2z} p_y - x^2 p_x + xp_z \\
\mathcal{P}^L_z &= p_z - 2xp_x \\
\mathcal{P}^R_x &= -e^{-2z} p_x + y^2 p_y - yp_z \\
\mathcal{P}^R_y &= -p_y \\
\mathcal{P}^R_z &= -p_z + 2yp_y.
\end{align*}
\] (77)

The classical Casimir is given by \( C_2 = (\mathcal{P}^{L,R}_z)^2 + 4\mathcal{P}^{L,R}_x \mathcal{P}^{L,R}_y = p_z^2 + 4e^{-2z} p_x p_y \).
To find the canonical variables \((X, Y, Z, P_X, P_Y, P_Z)\) in \(T^*G\) we proceed as explained before, computing Darboux coordinates on co-adjoint orbits. For the case of \(SL(2, R)\), co-adjoint orbits are characterized by the positive, null or negative values of the classical Casimir \(C_2\). Each positive value of the Casimir corresponds to a single two-dimensional orbit (a one-sheet hyperboloid), each negative value of the Casimir corresponds to two two-dimensional orbit (the two-sheet hyperboloid), meanwhile the zero value of the Casimir is associated with three orbits, the origin (zero-dimensional) and the upper and lower sheets of the cone.

As before, we shall restrict to the two-dimensional orbits (the only zero-dimensional orbit, the origin in \(G^*\), is associated with the trivial representation, the only one-dimensional representation of \(SL(2, R)\)). We shall consider first the cases of the 1-sheet hyperboloid (with positive Casimir) and the two cones. The 2-sheet hyperboloids will be considered later.

Define, for \(C_2 \geq 0\), \(R \equiv \sqrt{C_2}\), then the 2-form \(\omega\) for the 1-sheet hyperboloid and cone orbits, obtained by inverting the Poisson brackets on the orbits, can be written as

\[
\omega = \frac{1}{2p_x} dp_z \wedge dp_x ,
\]

and Darboux coordinates are given by the map \(S : G^* \to T^*R:\)

\[
S(p_x, p_y, p_z) = (p_x, \frac{R - p_z}{2p_x}) = (\pi, q).
\]

It is convenient to introduce a second set of Darboux coordinates given by the map \(S' : G^* \to T^*R:\)

\[
S'(p_x, p_y, p_z) = \left(\frac{R - p_z}{2p_y}, -p_y\right) = (\pi', q').
\]

These maps parametrize the 1-sheet hyperboloid when \(R > 0\). When \(R = 0\) we have three different orbits, according to the values \(\pi = 0\) (zero-dimensional orbit, the origin in \(G^*\)) and the two cones given by \(\pi < 0\) and \(\pi > 0\).

Once we have found Darboux coordinates on the co-adjoint orbits, we simply define \(X, P_X\) (resp. \(Y, P_Y\)) as the pullback by \(P^L\) (resp. \(P^R\)) of \(S^*(q)\) and \(S^*(\pi)\) (resp. \(S'^*(q')\) and \(S'^*(\pi')\)), respectively:

\[
\begin{align*}
X &= (P^L)^* \cdot S^*(q) = \frac{R - P^L_z}{2P^L_x} = \frac{R - p_z + 2xp_x}{2p_x} \\
Y &= (P^R)^* \cdot S'^*(q') = \frac{R - P^R_z}{2P^R_y} = \frac{R + p_z - 2yp_y}{2p_y}
\end{align*}
\]

\[
\begin{align*}
P_X &= (P^L)^* \cdot S^*(\pi) = P^L_x = p_x \\
P_Y &= (P^R)^* \cdot S'^*(\pi') = -P^R_y = p_y.
\end{align*}
\]
where \( R \), since it is a function of the Casimir \( C_2 \), it remains unchanged to the value \( R = \sqrt{p_x^2 + 4e^{-2z}p_x p_y} \). Inverting these relations we obtain the expressions of \( \mathcal{P}^L, \mathcal{P}^R \) in terms of \( X \) and \( P_X \) (resp. \( Y \) and \( P_Y \)):}

\[
\begin{align*}
\mathcal{P}_x^L &= P_X \\
\mathcal{P}_y^L &= XP_Z - X^2 P_X \\
\mathcal{P}_y^L &= P_Z - 2X P_X \\
\mathcal{P}_x^R &= Y^2 P_Y - Y P_Z \\
\mathcal{P}_y^R &= -P_Y \\
\mathcal{P}_z^R &= 2Y P_Y - P_Z,
\end{align*}
\]

(82)

where \( P_Z \) has been introduced, being the function of the Casimir \( P_Z = R = \sqrt{C_2} \). This expressions are clearly linear in the momenta, so they can be properly “quantized”. The only thing we have to compute is the cyclic coordinate \( Z \) canonically conjugated to \( P_Z \). As in the previous example, using the fact that the canonical 1-form \( \Theta_0 \) in \( T^*G \) can be written as \( P_X dX + P_Y dY + P_Z dZ \), we derive the expression for \( Z \):

\[
Z = \log \frac{p_x}{p_y} + \tanh^{-1} \frac{p_z}{P_Z}.
\]

(83)

Let us compute the irreducible representations associated with the 1-sheet hyperboloid and cone co-adjoint orbits. In each orbit, we choose \( p = (R/2, R/2, 0) \), and the hypersurface \( \Sigma^R \) associated with it is given by \( \Sigma = \mathcal{P}^{L-1}(p) = \{(x, y, z, p_x, p_y, p_z) \in T^*G \text{ such that } e^{-2z}p_x - y^2 p_y - yp_z = R/2, p_y = R/2, p_z - 2yp_y = 0 \} \). This hypersurface is parameterized by the coordinates \( X = x + \frac{e^{-2z}}{1+y}, P_X = \frac{R}{2} e^{2x}(1-y^2) \) and \( Z = 2z + \log (1-y^2) + \tanh^{-1} y \).

The vertical lift of the left invariant momentum maps are given by:

\[
\begin{align*}
\hat{\mathcal{P}}_x^L &= -i \frac{\partial}{\partial X} \\
\hat{\mathcal{P}}_y^L &= iX^2 \frac{\partial}{\partial X} - 2iXP_X \frac{\partial}{\partial P_X} + XP_Z I \\
\hat{\mathcal{P}}_z^L &= 2iX \frac{\partial}{\partial X} - 2iP_X \frac{\partial}{\partial P_X} + P_Z I.
\end{align*}
\]

(84)

Since the left-invariant momentum maps are at most of first order in the momentum \( P_X \), the polarization subalgebra \( \mathcal{P}^X \) is admissible. However, \( \mathcal{P}^{PX} \) is not admissible since they are of second order in \( X \). The Hilbert space \( \mathcal{H}^X \) is made of sections depending only on \( X \), and the operators (84) reduce to:

\[
\begin{align*}
\hat{\mathcal{P}}_x^L \Psi(X) &= -i \frac{\partial}{\partial X} \Psi(X) \\
\hat{\mathcal{P}}_y^L \Psi(X) &= (iX^2 \frac{\partial}{\partial X} + XP_Z) \Psi(X) \\
\hat{\mathcal{P}}_z^L \Psi(X) &= (2iX \frac{\partial}{\partial X} + P_Z) \Psi(X).
\end{align*}
\]

(85)

This representation is irreducible but not unitary with respect to the scalar product given by the measure \( dX \). It requires the addition of the Radon-Nikodym derivative,
which transforms the operators into:

\[
\hat{P}_x \Psi(X) = -i \frac{\partial}{\partial X} \Psi(X)
\]

\[
\hat{P}_y \Psi(X) = [iX^2 \frac{\partial}{\partial X} + X(i + P_Z)] \Psi(X)
\]

\[
\hat{P}_z \Psi(X) = [2iX \frac{\partial}{\partial X} + (i + P_Z)] \Psi(X)
\]

providing a unitary representation of \(SL(2, R)\). The value of the Casimir operator in each representation is \(\hat{C}_2 = 1 + P_Z^2\).

There are no other polarization conditions, therefore the only irreducible unitary representations for each value of \(P_Z^2 > 0\) are the ones presented here, which correspond to the continuous series of representations of \(SL(2, R)\).

9 Conclusions

We have shown that the irreducible representations of Lie groups can be constructed relating the classical configuration space (G group) and classical phase space \(T^*G\), in which the orbits of the group action on the points in the spaces are defined, to the Lie group. The quantization procedure, which uses canonical quantization replacing the classical momenta in the phase space by the standard momentum operators, provides the construction of the generators of the Lie group representation. In quantum mechanics and quantum optics, the presented geometrical picture of the Lie group representations clarifies the group-theoretical meaning of the quadrature components and their statistical properties associated with different basis vectors in the representation on Hilbert space. One should point out that in the mathematical context the Lie group representation theory has been constructed long time ago, but to apply this formalism in quantum mechanics and quantum optics, one needs a tutorial presentation of the Lie groups and their irreducible representations. The geometrical picture developed in this work, which treats the \(T^*G\) as the phase space associated to the G group and connects the irreducible representations with group “trajectories” in this phase space, provides a conventional tool to access the rigorous mathematics to physical intuition. The considered examples of Lie groups of low dimensions show the properties of the group representations in visible physical images like positions and momenta and their change in the process of evolution. The evolution itself is treated as a simple trajectory in the phase space of quadratures under the action of the evolution operator obtained in terms of very simple Hamiltonian which is linear in the position operator.

References

[1] P.A.M. Dirac, *The principles of quantum mechanics, 4th ed.*, Oxford:Pergamon (1958)
[2] A. Simoni, F. Zaccaria and B. Vitale, Nuovo Cim. 51A, 448 (1957)

[3] A. Barut and A. Bohm, Phys.Rev.10, 2331 (1964)

[4] E.C.G. Sudarshan, N. Mukunda and L. O’Raifeartaigh, Phys.Lett. 17, 32 (1965)

[5] M. Gell-Mann, J. Dothan and Y. Neeman, Phys.Rev.Lett.14,136 (1965)

[6] I.A. Malkin and V.I. Man’ko, JEPT Letters 2, 230 (1965)

[7] R. Simon and N. Mukunda, JOSA 17, 2440 (2000)

[8] J.R. Klauder and E.C.G. Sudarshan, Fundamentals of Quantum Optics, W.A. Benjamin, New York, 1968

[9] V.I. Man’ko and K.B. Wolf Symplectic and Euclidean groups of transformations in optics in Theory of the Interaction of Multilevel Systems with Quantized Fields, V.I. Man’ko and M.A. Markov, eds. P.N. Lebedev Phys-Inst. 209, 163 (1996)

[10] V.V. Dodonov and O.V. Man’ko, JOSA 17,2403 (2000)

[11] E.P. Wigner, Ann. of Math. 40, 149-204 (1939)

[12] A. A. Kirillov, Elements of the Theory of Representations, Springer-Verlag, Berlin (1975)

[13] B. Kostant, Quantization and Unitary Representations. Part I: Prequantization. In Lectures in Modern Analysis and Applications III. Lect. Notes Math. 1970, Springer Verlag (1970)

[14] J. M. Souriau, Structure des Systemes Dynamique, Dunod, Paris (1970)

[15] Chengjun Zhu and J.R. Klauder, Am. J. Phys,61,605-611,(1993)

[16] G.W. Mackey, Ann. of Math. 55, 101-139 (1952)

[17] A.N. Leznov, I.A. Malkin and V.I. Manko, Canonical Transformations and the Theory of Representations of Lie Groups, in “Problems in the General Theory of Relativity and Theory of Group Representations”, Proceedings of the P.N. Lebedev Physcis Institute, vol. 96, 25 (1978)

[18] P. Jordan, Z. Phys. 94, 531 (1935)

[19] J. Schwinger, in Quantum Theory of Angular Momentum, eds. L.C. Biedenharn and H. Van Dam, Academic (1965)

[20] V.I. Man’ko, G. Marmo, P. Vitale and F. Zaccaria, Int. J. Mod. Phys. A9, 5541-5561 (1994)

30
[21] V. Aldaya and J.A. Azcárraga, J. Math. Phys. 23, 1297 (1982)

[22] A.N. Leznov and M.V. Saveliev, Communs.Math.Phys. 74, 111 (1980)

[23] G. Marmo, E.J. Saletan, A. Simoni and B. Vitale, Dinamical Systems. A Differential Geometric Approach to Symmetry and reduction, J.Wiley, Chichester, 1985)

[24] S. Lie, Theorie der Transformationengruppen, Christ. Forth. Aar 1888, Nr. 13

[25] T. Levi-Civita and U. Amaldi, Lezioni di Meccanica Razionale, Zanichelli, Bologna, 1974 (reprinted version of 1949 edition).

[26] C. Duval, J. Elhadad and G.M. Tuynman, Pukanszky’s Condition and Symplectic Induction, preprint CPT-90/P.2398 (1990)

[27] N. Woodhouse, Geometric Quantization, Clarendon Press, Oxford (1980)

[28] I.A. Malkin and V.I. Man’ko, Dynamical symmetries and coherent states of quantum systems, Moskow, Nauca, 1979 [in Russian]

[29] V. Aldaya, J. Guerrero and G. Marmo, Int. J. Mod. Phys. A12, 3-11 (1997)

[30] V. Aldaya, J. Guerrero and G. Marmo, Quantization in a Lie Group: Higher-Order Polarizations, in Symmetries in Science X, Eds. B. Gruber and M. Ramek, Plenum Press, New York (1998) (hep-th/9710002)

[31] A. O. Barut and R. Raczka, Theory of Group Representations and Applications, World Scientific, Singapore, (1986)

[32] G. B. Folland Harmonic Analysis in Phase Space, Annals of Mathematics Studies 122, Princeton University Press, Princeton (1989)

[33] V. Aldaya, J. Navarro-Salas, J. Bisquert, and R. Loll, J. Math. Phys. 33, 3087 (1992)