Localisation of guided wave in stratified elastic reflector sandwiched between two elastic semi-spaces

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Abstract. The guided shear wave localisation is considered in the problem of reflection and refraction through bi–material stratified elastic reflector perfectly sandwiched between two elastic semi–spaces. Bi–material stratified reflector consists of finite number periodically arranged and perfectly bonded elastic sub–layers. A shear wave incident at the interface of layer from the first semi–space will give rise to a guided wave in reflector, a reflected shear wave in the same semi–space and a refracted shear wave in the second semi–space. It is shown that guided wave amplitude is localized at the neighbourhood of the layer interface adjacent to the incidence elastic semi-space and monotonously attenuates with increasing of cell number, if the frequencies of incident wave are in the stopband ranges.

1. Introduction
Recently much attention has been given to the propagation of elastic waves which occurs in elastic periodic structures consisting of an arrangement of coupled substructures with highly contrasting mechanical properties (elastic stiffness, mass density). The most notable feature of phonon crystals is the existence of stopbands of frequencies in which elastic waves are unable to propagate. The reflection and refraction of electromagnetic waves through periodically stratified medium were considered in [1], where the analytical expression of the reflectivity of a finite multilayer two phase dielectric reflector was presented. In the framework of matrix analysis the implication of the band structure of an infinite periodic structure for reflection by a finite structure are demonstrated in [2] for electromagnetic waves in stratified dielectric media. Numerous problems of wave propagation in elastic multilayered medium were considered in [3]. Reflection and refraction of shear waves from micro-polar elastic layer is considered in [4].

2. Formulation of the problem
We consider the wave localization effect in the problem of reflection and transmission of SH shear wave through stratified reflector of finite width. Let two different elastic homogeneous semi spaces be in perfect contact with the stratified elastic layer constituted by a finite number of repeated two sub–layers of different elastic materials A and B. Each of these sub–layers of widths $d_1$, $d_2$, is labeled by the index $(s) = 1$, $(s) = 2$ within the unit cell labeled by the index $n \ (n = 1, 2, 3, \ldots, N)$. The sub–layers are assumed to be perfectly bonded to the adjoining sub–layers. The layer extends from the top surface $x = 0$ to the bottom surface $x = Nd$, and $d = d_1 + d_2$, $N$ is the number of elementary units. A shear wave incident at the interface of
Figure 1. Stratified layer consisting from sub-layers sandwiched between two elastic semi spaces

layer from the top semi-space will give rise to a reflected shear wave in the same semi-space and a reflected (transmitted) shear wave in the lower semi-space, as described in figure 1.

First we wish to determine a correlation link between elastic field values (displacements and shear stresses) at the top $x = 0$ and the bottom $x = N d$ interfaces of the reflector.

For the SH wave the elastic displacements and stresses obey anti-plane equations of motion and Hooke’s law. Choosing the anti-plane deformation in the $z$-direction one has

$$
\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = \rho \partial_t u_z, \quad \sigma_{xz} = \mu \partial_x u_z, \quad \sigma_{yz} = \mu \partial_y u_z,
$$

(1)

where $u_z(x, y, t)$ is the displacement in $z$-direction.

Considering a steady SH wave propagating we have

$$
\begin{align*}
    u_z(x, y, t) &= u(x) \exp[i(k y - \omega t)], \\
n_{s} &= \sqrt{\omega^2 / c_s^2 - k^2}, \quad c_s^2 = \mu_s / \rho_s, \quad \mu_s, \rho_s \text{ are the shear modulus and density of the sub-layer.}
\end{align*}
$$

(2)

Solutions for functions $u_n^{(s)}(x)$ within $A, B$ materials of the sub-layer can be found as

$$
    u_n^{(s)}(x) = \alpha_n^{(s)} \exp(iq_s x) + \alpha_{n+1}^{(s)} \exp(-iq_s x).
$$

(3)

Where $q_s = \sqrt{\omega^2 / c_s^2 - k^2}$, $c_s^2 = \mu_s / \rho_s$, $\mu_s, \rho_s$ are the shear modulus and density of the sub-layer.

According to (1) one can define $\sigma^{(s)}_{xz}$ as

$$
\begin{align*}
    \sigma^{(s)}_{xz} &= \tau^{(s)}_{n+1}(x) \exp[i(k y - \omega t)], \\
    \tau^{(s)}_{n}(x) &= i\mu_s q_s [\alpha_{n+1}^{(s)} \exp(iq_s x) - \alpha_n^{(s)} \exp(-iq_s x)].
\end{align*}
$$

(4)

(5)

At the interfaces of two materials the transmission conditions of the stress and displacement continuities can be imposed, that is

$$
\begin{align*}
    u_n^{(1)}(x) &= u_n^{(2)}(x), & \tau_n^{(1)}(x) &= \tau_n^{(2)}(x), & x &= (n - 1)d + d_1, \\
    u_n^{(2)}(x) &= u_{n+1}^{(1)}(x), & \tau_n^{(2)}(x) &= \tau_{n+1}^{(1)}(x), & x &= nd, & n &= 1, 2, \ldots, N - 1.
\end{align*}
$$

(6)
Since the interface continuity conditions are imposed on functions \( u_n^{(s)}(x) \), \( \tau_n^{(s)}(x) \) it is convenient to introduce the following column vectors

\[
\hat{U}_n^{(s)}(x) = \begin{pmatrix} u_n^{(s)} \\ \tau_n^{(s)} \end{pmatrix}, \quad \hat{A}_n^{(s)} = \begin{pmatrix} \alpha_n^{(s)} \\ \alpha_n^{(sr)} \end{pmatrix}.
\]

In matrix form the solutions (4), (5) can be cast as

\[
\hat{U}_n^{(s)}(x) = \hat{F}_n^{(s)}(x)\hat{A}_n^{(s)},
\]

where

\[
\hat{F}_n^{(s)}(x) = \begin{pmatrix} \exp(iq_x x) & \exp(-iq_x x) \\ i\mu_s q_x \exp(iq_x x) & -i\mu_s q_x \exp(-iq_x x) \end{pmatrix}.
\]

Let note that the transmission conditions reported in (6) lead to the conditions of continuities of the field vectors \( \hat{U}_n^{(s)}(x) \) at separation interfaces of the sub-layers

\[
U_n^{(1)}(n-1)d + d_1 = U_n^{(2)}((n-1)d + d_1), \quad U_n^{(2)}(nd) = U_n^{(1)}(nd).
\]

3. Propagator matrix approach

With the view of linking the field values of the vectors \( \hat{U}_1^{(1)}(x) \), \( \hat{U}_2^{(2)}(x) \), between top \( x = 0 \) and bottom \( x =Nd \) surfaces of the layer, a propagator matrix method will be used [2, 5, 6].

The method considers two neighboring points \( x_{1n}^{(s)} \), \( x_{2n}^{(s)} \) within each material in domains of the sub-layers \( A, B \) of the \( n \)th cell. For values of field vectors \( \hat{U}_n^{(s)}(x) \) in these points the following conditions hold valid

\[
\hat{U}_n^{(s)}(x_{1n}^{(s)}) = \hat{F}_n^{(s)}(x_{1n}^{(s)})\hat{A}_n^{(s)}, \quad \hat{U}_n^{(s)}(x_{2n}^{(s)}) = \hat{F}_n^{(s)}(x_{2n}^{(s)})\hat{A}_n^{(s)}.
\]

Eliminating vectors \( \hat{A}_n^{(s)} \) from (10) the relation linking \( \hat{U}_n^{(s)} \) vector field values within each material can be found. This is:

\[
\hat{U}_n^{(s)}(x_{2n}^{(s)}) = \hat{T}_n^{(s)}(x_{1n}^{(s)}, x_{2n}^{(s)})\hat{U}_n^{(s)}(x_{1n}^{(s)}).
\]

Herein \( \hat{T}_n^{(s)}(x_{1n}^{(s)}, x_{2n}^{(s)}) = \hat{F}_n^{(s)}(x_{2n}^{(s)})[\hat{F}_n^{(s)}(x_{1n}^{(s)})]^{-1} \) is the transfer matrix in each sub-layer:

\[
\hat{T}_n^{(s)}(x_{1n}^{(s)}, x_{2n}^{(s)}) = \begin{pmatrix} \cos[q_s(x_{2n}^{(s)} - x_{1n}^{(s)})] & (\mu_s q_s)^{-1}\sin[q_s(x_{2n}^{(s)} - x_{1n}^{(s)})] \\ -\mu_s q_s \sin[q_s(x_{2n}^{(s)} - x_{1n}^{(s)})] & \cos[q_s(x_{2n}^{(s)} - x_{1n}^{(s)})] \end{pmatrix}.
\]

Let now consider the \( n \)th cell of the structure. Using the continuity conditions of field vectors \( \hat{U}_n^{(s)}(x) \) at interfaces \( d_0 = (n - 1)d + d_1 \) one obtains

\[
\hat{U}_n^{(1)}(d_0) = \hat{U}_n^{(2)}(d_0),
\]

while (11) leads to the matrix equations

\[
\hat{U}_n^{(2)}(nd) = \hat{M}\hat{U}_n^{(1)}((n - 1)d),
\]

where

\[
\hat{M} = \hat{T}_n^{(2)}(d_0, nd)\hat{T}_n^{(1)}((n - 1)d, d_0).
\]
Herein $\hat{M}$ is the unimodal propagator matrix for shear wave field, which links the field vectors at the top and bottom of the $n$th cell.

The explicit expressions of the unimodal propagator matrix $\hat{M}$ elements can be derived as

$$
m_{11} = \cos(d_1q_1) \cos(d_2q_2) - \frac{q_1\mu_1}{q_2\mu_2} \sin(d_1q_1) \sin(d_2q_2),
$$

$$
m_{12} = \frac{\cos(d_1q_1) \sin(d_2q_2)}{q_2\mu_2} \frac{q_1\mu_2}{q_1\mu_1} \sin(d_1q_1) \cos(d_2q_2),
$$

$$
m_{21} = -q_2\mu_2 \cos(d_1q_1) \sin(d_2q_2) - q_1\mu_1 \cos(d_2q_2) \sin(d_1q_1),
$$

$$
m_{22} = \cos(d_1q_1) \cos(d_2q_2) - \frac{q_2\mu_2}{q_1\mu_1} \sin(d_1q_1) \sin(d_2q_2).
$$

(16)

Let note that elements of matrix $\hat{M}$ do not depend on cell number $n$. Repeating this procedure the $n$th times the propagator unimodal matrix $\hat{M}^n$ can be found. The matrix $\hat{M}^n$ links the field vectors at $x = 0$ and $x = Nd$ surfaces of the layer.

$$
\hat{M}^n \vec{U}^{(1)}(0) = \vec{U}^{(2)}(nd), \quad n = 1, 2, \ldots, N.
$$

(17)

According to Sylvester’s matrix polynomial theorem [7] for $2 \times 2$ matrices the elements of the $n$th power of a unimodal matrix $\hat{M}^n$ can be cast as

$$
\hat{M}^n = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}
$$

(18)

and can be simplified using the following matrix identity

$$
M_{11} = m_{11}S_{n-1}(\eta) - S_{n-2}(\eta), \quad M_{12} = m_{12}S_{n-1}(\eta),
$$

$$
M_{21} = m_{21}S_{n-1}(\eta), \quad M_{22} = m_{22}S_{n-1}(\eta) - S_{n-2}(\eta),
$$

(19)

where $S_n(\eta)$ are the Chebyshev polynomials of second kind, namely

$$
S_n(\eta) = \frac{\sin[(n + 1)\phi]}{\sin \phi}, \quad \cos \phi = \eta,
$$

$$
\eta = \frac{1}{2} \text{tr} \hat{M} = \frac{1}{2} (m_{11} + m_{22}).
$$

(20)

The first Chebyshev polynomials are $S_0(\eta) = 1$, $S_1(\eta) = 2\eta$, $S_2(\eta) = 4\eta^2 - 1$. Subsequent polynomials may be obtained from the recurrence relation of Chebyshev polynomials [7]:

$$
S_m(\eta) = 2\eta S_{m-1}(\eta) - S_{m-2}(\eta).
$$

(21)

The matrix trace $\text{tr} \hat{M}$, namely the condition $|\text{tr}(M)| > 2$, defines the stopband of frequencies ranges of eigenfrequencies where waves cannot propagate in the infinite periodic medium consisting of periodically repeated sub-layers of materials $A$ and $B$ [8–10]. The stopband edges are given by $|\text{tr} \hat{M}| = 2$.

By drawing an analogy with an infinite periodic medium we define by stopbands the ranges of eigenfrequencies where $|\eta| \geq 1$ and by passbands the ranges where $|\eta| < 1$. 


4. Solution of transmission/reflection problem

Considering a steady shear wave propagation in semi-spaces the solutions in the upper and the lower semi-spaces can be written as

\[ v_{01}(x, y, t) = v_1(x) A_t \exp(i q_{01} x) + A_r \exp(-i q_{01} x) \exp[i(y - \omega t)], \]
\[ v_{02}(x, y, t) = A_t v_2(x) \exp[i(y - \omega t)], \]
\[ v_1(x) = A_t \exp(i q_{01} x) + A_r \exp(-i q_{01} x), \]
\[ v_2(x) = \exp(i q_{02} x), \]
\[ \tau_{01}(x) = i \mu_0 q_{01} [A_t \exp(i q_{01} x) - A_r \exp(-i q_{01} x)], \]
\[ \tau_{02}(x) = A_t i \mu_0 q_{02} \exp(i q_{02} x), \]

where \( A_t, A_r, A_s \) stand for the amplitudes of incident, reflected, and refracted waves, respectively, \( q_{01} = \sqrt{\omega^2/c_{01}^2 - k^2}, \ c_s^2 = \mu_0/\rho_0, \ q_{02} = \sqrt{\omega^2/c_{02}^2 - k^2}, \ c_s^2 = \mu_2/\rho_2, \ \mu_0, \ \mu_2 \) are shear moduli, \( \rho_0, \ \rho_2 \) are bulk densities of semi-spaces.

At the interfaces \( x = 0, \ x = N d \) the transmission conditions of the stress and displacement continuities can be imposed, that is

\[ v_{01}(0) = u_1(0), \quad \tau_{01}(0) = \tau_0(0), \]
\[ v_{01}(N d) = u_2(N d), \quad \tau_{02}(N d) = \tau_0(2)(N d), \]

or

\[ \hat{U}_1^{(1)}(0) = \begin{pmatrix} v_1(0) \\ \tau_0(0) \end{pmatrix}, \quad \hat{U}_1^{(2)}(N d) = \begin{pmatrix} v_2(N d) \\ \tau_0(2)(N d) \end{pmatrix}. \]

Taking into account the transmission condition (24) and the link relation (17), the amplitudes \( A_r, A_t \) via \( A_t \) can be found by solving the following matrix equation

\[ \hat{M}^N \begin{pmatrix} A_t + A_r \\ i \mu_0 q_{01} (A_t - A_r) \end{pmatrix} = \begin{pmatrix} A_r \exp(i q_{02} N d) \\ i \mu_0 q_{02} A_t \exp(i q_{02} N d) \end{pmatrix}. \]

Solution of (25) can be found in the form

\[ A_r = A_t \frac{(q_{01} \mu_0 q_{02} \mu_0 M_{12} + M_{21}) + i(M_{22} q_{01} \mu_0 - q_{02} \mu_2 M_{11})}{(q_{01} \mu_0 q_{02} \mu_0 M_{12} - M_{21}) + i(M_{22} q_{01} \mu_0 + q_{02} \mu_2 M_{11})}, \]
\[ A_t = \frac{2 A_t q_{01} \mu_0 \exp(-i q_{02} N d)}{(M_{22} q_{01} \mu_0 + q_{02} \mu_2 M_{12}) - i(q_{01} \mu_0 q_{02} \mu_0 M_{12} - M_{21})}. \]

Energy flux conservation is then expressed via reflection and refraction amplitudes by the following algebraic identity

\[ q_{01} \mu_0 |A_r|^2 + q_{02} \mu_2 |A_t|^2 = q_{01} \mu_0 |A_t|^2. \]

Here

\[ |A_r|^2 = A_t^2 \left[ 1 - \frac{4 q_{10} q_{20} \mu_0 \mu_2}{M_{21}^2 + M_{11}^2 q_{01}^2 \mu_0^2 + 2 q_{01} q_{02} \mu_0 \mu_2 q_{00}^2 + q_{02}^2 \mu_0^2 (M_{22}^2 + M_{12}^2 q_{01}^2 \mu_0^2)} \right], \]
\[ |A_t|^2 = \frac{4 A_t^2 q_{01}^2 \mu_0^2}{M_{21}^2 + M_{11}^2 q_{01}^2 \mu_0^2 + 2 q_{01} q_{02} \mu_0 \mu_2 q_{00}^2 + q_{02}^2 \mu_0^2 (M_{22}^2 + M_{12}^2 q_{01}^2 \mu_0^2)}. \]

Taking into account Eq. (28) and determining from the matrix equation

\[ U_N^{(2)}(N d) = M^{N-m+1} U_m^{(1)}((m - 1) d), \]

\[ U_N^{(2)}(N d) = M^{N-m+1} U_m^{(1)}((m - 1) d), \]

\[ U_N^{(2)}(N d) = M^{N-m+1} U_m^{(1)}((m - 1) d), \]
the expression for displacement of the \(m\)th cell at \(x = (m - 1)d\) comes to the following form

\[
\tilde{u}_N = \left| \frac{u^{(2)}_N(Nd)}{u^{(1)}_m((m-1)d)} \right| = \frac{1}{\sqrt{[m_{22}S_{N-m}(\eta) - S_{N-m-1}(\eta)]^2 + [m_{12}q_02\mu_2S_{N-m}(\eta)]^2}}. \tag{30}
\]

Using Eq. (30) the relationship between displacements of adjacent cells \(n = m\) and \(n = m + 1\) is obtained as

\[
\tilde{u}_m = \left| \frac{u^{(1)}_{m+1}(md)}{u^{(1)}_m((m-1)d)} \right| = \frac{\sqrt{[m_{22}S_{N-m-1}(\eta) - S_{N-m-2}(\eta)]^2 + [m_{12}q_02\mu_2S_{N-m-1}(\eta)]^2}}{\sqrt{[m_{22}S_{N-m}(\eta) - S_{N-m-1}(\eta)]^2 + [m_{12}q_02\mu_2S_{N-m}(\eta)]^2}}, \tag{31}
\]

while the relationship between displacements at the top and bottom surfaces of the layer can be obtained as

\[
\tilde{u}_0 = \left| \frac{u^{(2)}_N(Nd)}{u^{(1)}_m(0)} \right| = \frac{1}{\sqrt{[m_{22}S_{N-1}(\eta) - S_{N-2}(\eta)]^2 + [m_{12}q_02\mu_2S_{N-2}(\eta)]^2}}. \tag{32}
\]

If the frequencies of incident wave are in the stopband range \(|\eta| > 1\) of stratified layer, then \(\phi = m\pi + i\theta\) and

\[
S_n(\eta) = \frac{\sinh[(n+1)\theta]}{\sinh \theta}, \quad \cosh \theta = |\eta|. \tag{33}
\]

For large \(N \gg 1\) we have that

\[
\begin{align*}
S_{N-1}(\eta) &= \frac{\sinh(N\theta)}{\sinh \theta} \approx \exp[(N-1)\theta], \\
S_{N-2}(\eta) &= \frac{\sinh((N-1)\theta)}{\sinh(N\theta)} = \cosh \theta - \coth(N\theta) \sinh \theta \approx \exp(-\theta). \tag{34}
\end{align*}
\]

5. Discussion of results

From this set of equations (30)–(34) it follows that if the frequencies of the incident wave are in stopbands range \(|\eta| > 1\) of stratified layer, the dimensionless displacement \(\tilde{u}_0\) exponentially attenuate to zero with increasing of layer units number \(N \to \infty\). Moreover since in this range \(m_{22}(\omega) \leq 0\), from Eq. (29) follows that \(\tilde{u}_m < 1\). This implies that, if the frequencies of incident wave are in the stopbands simultaneously the guided wave is localized at the neighborhood of the layer surface adjacent to the incidence elastic medium and, due to the condition \(m_{22}(\omega) \leq 0\) is monotonously attenuated with increasing of cell number.

From the wave localization at the neighborhood of the layer surface adjacent to the incidence elastic it follows that

\[
\left| \frac{A_t}{A_r} \right| \to 0, \quad \left| \frac{A_r}{A_t} \right| \to 1 \quad \text{as} \quad N \to \infty.
\]

If the frequencies of incident wave are in the passband range, the guided wave is periodically distributed (non-localized) along thickness of stratified layer, since in the passband range \(|\eta| < 1\) the Chebyshev polynomials of second kind \(S_n(\eta)\) are periodic functions; the layer elastic displacements have the same magnitude at adjacent to elastic medium surfaces.

Let note that the all above mentioned results are also valid when \(\mu_2 = 0\), which corresponds to the case where the bottom surface of the stratified reflector is a traction free one. In this case for reflection amplitude in the incident semi-space the following relation is valid

\[
A_r = -A_t \frac{M_{21} + iM_{22}q_{01}\mu_{01}}{M_{21} - iM_{22}q_{01}\mu_{01}}, \quad |A_r|^2 = |A_t|^2.
\]
Conclusions
The guided wave localization is established in the problem of shearwave reflection and refraction through stratified layer/reflector perfectly sandwiched between two elastic semi spaces. The stratified layer is constituted by a finite number of alternative repeated perfectly bonded elastic sub-layers. Based on propagator matrix method the analytic expressions for amplitudes and flux energies of incident and reflection waves are derived. It is shown that in the reflector the guided wave amplitude can be strictly localized at the neighborhood of the surface adjacent to incidence elastic medium and monotonously attenuated with the increasing of the reflector number of sub-layers to the surface adjacent to the other elastic medium. This localization occurs in the range of incident wave stopband ranges, only. In the passband range of the incident wave there no localization, since the guided wave is distributed periodically along the thickness of the stratified reflector and the elastic displacements have the same magnitude at adjacent to elastic semi-spaces interfaces.

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