LINEAR EMBEDDINGS OF $K_{9}$ ARE TRIPLE LINKED

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Abstract. We use the theory of oriented matroids to show that any linear embedding of $K_{9}$, the complete graph on nine vertices, contains a non-split link with three components.

1. Introduction

In two seminal works in the early 1980’s, Sachs [10] and Conway and Gordon [2] showed that $K_{6}$, the complete graph on six vertices, is intrinsically linked, i.e., every embedding of it in $\mathbb{R}^{3}$ (or $S^{3}$) contains two disjoint cycles that form a nontrivial link. In [2], Conway and Gordon also showed that $K_{7}$ is intrinsically knotted, i.e., every embedding of it in $\mathbb{R}^{3}$ contains a nontrivial knot. Since then, these results have been extended to other graphs, to more “complex” types of linking and knotting, and to linear embeddings (or straight-edge embeddings), i.e., embeddings of graphs in $\mathbb{R}^{3}$ in which every edge of the graph is a straight line segment. For example, Negami [8] showed that given any knot or link, for all sufficiently large $n$, every linear embedding of $K_{n}$ contains that knot or link. Miyauchi [6] extended this result to complete bipartite graphs. And Ramirez Alfonsin [9] showed that every linear embedding of $K_{7}$ contains a trefoil.

Going back to “not-necessarily-linear embeddings,” a graph is called intrinsically $n$-linked (InL, for short) if every embedding of it in $\mathbb{R}^{3}$ contains a non-split $n$-component link, i.e., for every embedded 2-sphere $S^{2}$ disjoint from the link, all $n$ components of the link are contained in the same component of $\mathbb{R}^{3} \setminus S^{2}$. The smallest complete graph that could possibly be InL is $K_{9}$, since $K_{8}$ doesn’t have enough vertices to contain a 3-link (each of the three components requires at least 3 vertices). In [4], Flapan, Naimi, and Pommersheim showed that $K_{9}$ is not InL, but $K_{10}$ is; they illustrated an embedding of $K_{9}$ that contains no non-split 3-link. This embedding, however, was not a linear embedding. So it is natural to ask whether there also exists a linear embedding of $K_{9}$ that contains no non-split 3-link. In this paper we show that there is no such embedding.

Theorem 1.1. Every linear embedding of $K_{9}$ in $\mathbb{R}^{3}$ contains a non-split link with three components.

It is not difficult to show that every finite graph has only finitely many linear embeddings up to isotopy. So, in theory, one could check every linear embedding of $K_{9}$ to see whether or not there is one with no non-split 3-links. Finding all linear embeddings of $K_{9}$ up to isotopy, however, is a different story. We were inspired by the paper of Ramirez Alfonsin [9] to use Oriented Matroid theory to approach this problem. To every
linear embedding of a graph one can in a natural way associate an oriented matroid. The oriented matroid determines up to "linear embedding isotopy" the embedding of the graph, in particular, it determines which pairs of disjoint cycles in the embedded graph are linked. Linear embeddings of $K_9$, in particular, give non-degenerate uniform oriented matroids of rank 4 on 9 elements. And the set of all such oriented matroids has already been found by the use of computers \[3\]. This provided us with a great tool for working on the $K_9$ problem. It turns out that there are over two billion such oriented matroids. We wrote a Mathematica computer program, which we ran in parallel on 31 computers for 36 hours on average, to go through these oriented matroids. The program, with full documentation, is available for download at \[7\].

2. Acknowledgements

We thank Lew Ludwig for suggesting this problem to us; Lukas Finschi for his help on retrieving the oriented matroid database and for maintaining an awesome website on oriented matroids; and Sonoko Moriyama for sharing with us the oriented matroid database we needed. We are also grateful to the faculty of the mathematics department at Occidental College for offering to run our program on their computers.

3. Background on Oriented Matroids

Before giving a precise and detailed account of how oriented matroids are associated with linear embeddings of graphs, we first give an informal explanation for the case of $K_9$. Every linear embedding of a graph in $\mathbb{R}^3$ is determined solely by where its vertices are embedded. In a linearly embedded complete graph, no four vertices are coplanar, since otherwise at least two edges would intersect each other in their interiors. Suppose we are given a linear embedding $\Gamma$ of $K_9$. We label the vertices $1, 2, \ldots, 9$. Given any four vertices $a_1, a_2, a_3, a_4$ with $a_i < a_{i+1}$, we can form three vectors, $v_i = a_4 - a_i$, $i = 1, 2, 3$. Since the four points $a_1, a_2, a_3, a_4$ are not coplanar, $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$ and the determinant of the matrix $M = [v_1|v_2|v_3]$ is nonzero. We assign a $+$ or $-$ sign to $\{a_1, a_2, a_3, a_4\}$ according to whether $\det(M)$ is positive or negative. Doing this for every set of four vertices of $\Gamma$ amounts to associating an oriented matroid to $\Gamma$: a uniform oriented matroid of rank $r$ on $n$ elements is an assignment of a $+$ or $-$ sign to every $r$-subset of the $n$ elements (called bases), subject to certain conditions (called "chirotope conditions"). It turns out that this set of $+$ and $-$ signs captures enough information for determining which pairs of triangles in $\Gamma$ are linked. For three non-collinear points, we will refer to both the union of the three edges they determine and to their convex hull as a triangle.

A different but equivalent way to assign an oriented matroid to the embedding $\Gamma$ is via circuits instead of bases. For each 5-subset of the 9 vertices of $\Gamma$, either one of the vertices will be inside the tetrahedron determined by the remaining four vertices, or two of the vertices will determine an edge that "piecess" the triangle determined by the remaining three vertices, as in Figure $\[1\]$. Accordingly, each 5-subset of the 9 vertices is given a 4-1 or a 3-2 partition. This assignment of a 4-1 or a 3-2 partition to every 5-subset is sufficient for describing an oriented matroid of rank 4 on 9 elements: a uniform oriented matroid of rank $r$ on $n$ elements can be defined by assigning a partition to each of the $(r+1)$-subsets (called circuits) of its elements, subject to certain conditions (called
“circuit axioms”). There is a procedure for obtaining the signs of the bases from the partitioned circuits, and vice versa. Knowing exactly which edges pierce which triangles is sufficient for determining whether any two disjoint triangles are linked or not. Thus, knowing the circuit partitions of the oriented matroid associated to \( \Gamma \) is all we need to determine the links in it. With this informal description of oriented matroids and their connection to linear embeddings and linking, we now move on to a more formal and detailed presentation.

We first give the definition via circuits. A signed set \( X \) is an ordered pair of disjoint sets, written as \((X^+, X^-)\). The opposite of a signed set \( X \) is the signed set \(-X = (X^-, X^+)\). Thus, \((-X)^+ = X^-\), and \((-X)^- = X^+\). The underlying set of \( X \) is defined as \( X = X^+ \cup X^-\).

An oriented matroid \( M \) on a finite set \( E \) is defined by a collection \( C \) of signed subsets of \( E \), called circuits of \( M \), satisfying the following axioms:

1. for all \( C_1 \in C \), \( C_1 \neq \emptyset \) and \(-C_1 \in C\); (symmetry)
2. for all \( C_1, C_2 \in C \), if \( C_1 \subseteq C_2 \), then \( C_1 = C_2 \) or \( C_1 = -C_2\); (incomparability)
3. for all \( C_1, C_2 \in C \) with \( C_1 \neq C_2 \), if \( e \in C_1^+ \cap C_2^- \), then there exists \( C_3 \in C \) such that \( C_3^+ \subseteq (C_1^+ \cup C_2^-) \setminus \{e\} \) and \( C_3^- \subseteq (C_1^- \cup C_2^+) \setminus \{e\} \). (weak elimination)

There is a natural way to obtain a new oriented matroid from a given one by an operation called reorientation: For a subset \( A \subseteq E \) consider a new collection of signed sets, \( -_A C = \{-_A C \mid C \in C\} \), where \( -_A C \) is the signed set with \( (_A C)^+ = (C^+ \setminus A) \cup (C^- \cap A) \) and \( (_A C)^- = (C^- \setminus A) \cup (C^+ \cap A) \). The collection \( -_A C \) represents the set of circuits for an oriented matroid, denoted by \( -_A M \), which is said to be obtained from \( M \) by reorientation on the set \( A \). This operation defines an equivalence relation on the set of oriented matroids of fixed rank on a ground set \( E \). Two oriented matroids on \( E \), \( M \) and \( M' \), belong to the same orientation class if there exists a set \( A \subseteq E \) such that \( M' = -_A M \).

An important class of oriented matroids is the class of affine oriented matroids. To define this class, let \( E \) denote a finite set of points spanning \( \mathbb{R}^r \) and let \( C \) be the collection of non-empty signed subsets of \( E \) such that for all \( C \in C \)

1. no proper subset of \( C \) is in \( C \), and
2. for all \( x \in C \), there exists \( \alpha(x) \in \mathbb{R} \) such that

\[
\sum_{x \in C} \alpha(x) \cdot x = 0, \quad \sum_{x \in C} \alpha(x) = 0
\]

We partition each \( C \in C \) by \( C^+ = \{x \in C \mid \alpha(x) > 0\} \), and \( C^- = \{x \in C \mid \alpha(x) < 0\} \).

The oriented matroid defined this way by the set of circuits \( C \) is called an affine oriented matroid of rank \( r + 1 \) on \( E \). Geometrically, every circuit \( C \) of this matroid is an inclusion minimal signed \((r + 2)\)-set of points in \( E \) such that the convex hull of the points in \( C^+ \) intersects the convex hull of the points in \( C^- \). An oriented matroid is said to be acyclic if for each of its circuits \( C \), \( C^- \neq \emptyset \) and \( C^+ \neq \emptyset \); otherwise it is said to be cyclic.

Thus, affine oriented matroids are acyclic: not all \( \alpha(x) \) are positive and not all \( \alpha(x) \) are negative, since \( \sum_{x \in C} \alpha(x) = 0 \).

By considering a set of points \( E \) spanning \( \mathbb{R}^3 \) such that no four points are coplanar (as are the vertices of a straight-edge embedding of a graph) we obtain an affine oriented matroid of rank 4. The non-coplanarity implies that every subset of \( E \) with five elements
is the underlying set of a circuit. The relative positions of any five points fits into one of two cases:

1. The convex hull of two of the points intersects the convex hull of the other three points (we say the circuit determined by the five points has a (3-2)-partition), or
2. One of the points is in the convex hull of the other four (we say the circuit determined by the five points has a (4-1)-partition).

These two cases are depicted in Figure 1, where, for a subset $A \subset \mathbb{R}^3$, we denote the convex hull of $A$ by $A_{CH}$.

![Figure 1](image)

An oriented matroid $M$ of rank $r$ with ground set $E$ is said to be uniform if every $(r + 1)$-subset of $E$ is the underlying set of some circuit in $M$. We denote by $\text{OM}(9, 4)$ the set of all uniform oriented matroids of rank 4 on a set with 9 elements.

Oriented matroids can also be defined via basis orientations [5]. For an integer $r \geq 1$ and a finite set $E$, an alternating map $\chi : E^r \to \{-1, 0, 1\}$ with certain properties called “chirotope properties” ([1], p.124) gives a basis orientation for an oriented matroid of rank $r$ on $E$. The map $\chi$ specifies which $r$-subsets of $E$ are bases by assigning a nonzero value to them, and assigns a sign to each subset which is a basis.

A chirotope $\chi : E^r \to \{-1, 1\}$ defines a uniform oriented matroid (all $r$-subsets of $E$ are bases). An order on the elements of $E$ induces the lexicographic order on the set of $r$-subsets of $E$. A chirotope defining a uniform oriented matroid can be identified with a string of + and - signs, of length $(|E|)$. The $k$th sign in the string represents the sign of the $k$th basis in lexicographic order. The string obtained by replacing every + with − and vice versa is defined to represent the same oriented matroid.

Given any uniform oriented matroid $M$ of rank $r$ defined by a chirotope $\chi$, we can obtain a collection $C$ of circuits that define the same oriented matroid $M$, as follows. Let $\{x_1, x_2, \ldots, x_{r+1}\}$ be any ordered $(r + 1)$-subset of the ground set $E$ of $M$. We define the sign of each $x_i$ by

$$\text{sgn}(x_i) = (-1)^{i-1}\chi(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+1}\})$$

We then let $C = (C^+, C^-)$, where $C^+ = \{x_i | \text{sgn}(x_i) > 0\}$ and $C^- = \{x_i | \text{sgn}(x_i) < 0\}$. Then, the collection $C$ of all circuits $C$ defined this way will satisfy the circuit axioms and give the same oriented matroid as given by $\chi$. 

![Figure 1](image)
4. Proof of the Main Theorem

Our proof of Theorem 1.1 relies on the program which we describe at the end of this section. The program relies on the following lemmas.

Lemma 4.1. Let \( a, b, c, d, e, \) and \( f \) be six distinct points in \( \mathbb{R}^3 \) such that no four are coplanar. Let \( T \) be the triangle determined by \( \{a, b, c\} \) (i.e., the boundary of \( \{a, b, c\}_{CH} \)) and let \( T' \) be the triangle determined by \( \{d, e, f\} \). The following are equivalent:

- (1) The triangles \( T \) and \( T' \) are linked.
- (2) Exactly one edge of \( T \) intersects \( \{d, e, f\}_{CH} \).
- (3) Exactly one edge of \( T' \) intersects \( \{a, b, c\}_{CH} \).

**Proof.** We prove (1) \( \iff \) (2). By symmetry we get (1) \( \iff \) (3).

(1) \( \Rightarrow \) (2). Assume the triangles \( T \) and \( T' \) are linked. Choose orientations for \( \{a, b, c\}_{CH} \) and \( \{d, e, f\}_{CH} \) and take the induced orientations on \( T \) and \( T' \), respectively. Since \( \text{lk}(T, T') \neq 0 \) represents the signed sum of the intersections of \( T \) with \( \{d, e, f\}_{CH} \), at least one of the edges of \( T \) intersects \( \{d, e, f\}_{CH} \). Assume exactly two edges of \( T \) intersect \( \{d, e, f\}_{CH} \). Since the two edges share a vertex, they intersect \( \{d, e, f\}_{CH} \) with different signs and hence \( \text{lk}(T, T') = 0 \), contradicting the hypothesis. Three edges of \( T \) cannot possibly intersect \( \{d, e, f\}_{CH} \), since this would imply that the two planes determined by \( \{a, b, c\}_{CH} \) and \( \{d, e, f\}_{CH} \) share three non-collinear points, and hence they coincide, contradicting the non-planarity hypothesis.

(2) \( \Rightarrow \) (1). Assume \( \{a, b\}_{CH} \) is the unique edge of \( T \) intersecting \( \{d, e, f\}_{CH} \). Then \( \text{lk}(T, T') = \pm 1 \), that is, \( T \) and \( T' \) are linked. \( \square \)

In light of the above lemma, the following definition gives a “natural extension” of the concept of linked triangles to oriented matroids.

Definition 4.2. Let \( \mathcal{M} \) be an oriented matroid of rank 4 on a set \( E \) with \( |E| \geq 6 \). To a pair of disjoint triangles \( T = \{a, b, c\} \subset E \) and \( T' = \{d, e, f\} \subset E \) we associate the sets \( S = \{(a, b, c), \{d, e\}, (a, b, c), \{d, f\}, (a, b, c), \{e, f\}\} \) and \( S' = \{(a, b), \{d, e, f\}, \{a, c\}, \{d, e, f\}, \{b, c\}\} \). Then \( T \) and \( T' \) are said to be linked if exactly one element of each of \( S \) and \( S' \) is a circuit of \( \mathcal{M} \).

Lemma 4.3. If \( C = (C^+, C^-) \) is a circuit of \( \mathcal{M} \) and \( A \subset E \) is such that \( A \cap C^- = C^+ \) or \( A \cap C^- = C^- \) then \( -A \mathcal{M} \) is cyclic.

**Proof.** If \( A \cap C^- = C^+ \) then \( (\neg A)^+ = \emptyset \). If \( A \cap C^- = C^- \) then \( (\neg A)^- = \emptyset \). \( \square \)

To obtain the orientation class of an oriented matroid \( \mathcal{M} \), it suffices to reorient only on sets \( A \subset E \) with \( |A| \leq \frac{1}{2}|E| \), since reorientation on \( E \setminus A \) gives the same matroid as reorientation on \( A \). There are \( \sum_{i=0}^{4} \binom{9}{i} = 256 \) such reorientation sets \( A \). Since linear spatial graphs correspond to affine oriented matroids, and affine oriented matroids are acyclic, our program only tests acyclic oriented matroids for non-split 3-links. We let \( \mathcal{A}_{\text{cyclic}} \) denote the collection of subsets \( A \subset E \) such that \( -A \mathcal{M} \) is cyclic. These are sets \( A \) as in Lemma 4.3.

4.1. Description of the Mathematica Program. Our program checks, as follows, that in every acyclic \( \mathcal{M} \in \text{OM}(9, 4) \) there are three disjoint triangles \( T, T', \) and \( T'' \) such that \( T \) links both \( T' \) and \( T'' \).
(1) Read in, from the given database of OM(9, 4) orientation classes, the oriented matroid \(M\) given for each orientation class; \(M\) is given in chirotope form (as a string of \(\binom{9}{4} = 126\) + and – signs).

(2) Compute the set of \(\binom{9}{5} = 126\) circuits for each \(M\), as described in Equation 3.1.

(3) Using Lemma 4.3, compute the set \(A_{\text{cyclic}}\).

(4) For each \(M\), and for each reorientation set \(A\) not in \(A_{\text{cyclic}}\), compute the circuits of the oriented matroid \(-AM\).

(5) For each triangle \(T\) in \(K_9\), find, from the (3-2)-partitioned circuits of \(-AM\), the set of all edges that pierce \(T\); use this to find the set of all triangles \(T'\) that link \(T\); and use this in turn to determine if \(T\) belongs to a non-split 3-link.

There are 9,276,595 orientation classes in the set OM(9, 4), and each orientation class consists of 256 oriented matroids (including cyclic and acyclic ones). Among these 2,374,808,320 oriented matroids, the program checked all the acyclic ones and found a non-split 3-link in each one.

Remark 4.4. The program gives us a slightly stronger result than Theorem 1.1 since it found a non-split link in all the acyclic oriented matroids, not just the affine ones.

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