Relaxation of interacting many-body systems under purely dissipative quantum dynamics

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We study the relaxation dynamics of quantum many-body systems that undergo purely dissipative dynamics through non-classical jump operators that can establish quantum coherence. We contrast the relaxation dynamics of this class of systems with those evolving via classical rate equations towards a stationary state with the same values of diagonal or “classical” observables as in the quantum system but where coherences are absent. We focus in particular on spin systems whose dynamics becomes correlated and complex due to dynamical constraints, inspired by kinetically constrained models (KCMs) of classical glasses. We show that in the quantum case the relaxation of the coherences can be orders of magnitude slower than that of diagonal observables. Finally, we show that the relaxation of strongly interacting Rydberg atoms under electromagnetically induced transparency (EIT) conditions can indeed, in an appropriate limit, be described by such a purely dissipative dynamics with non-classical jump operators. We establish a connection between the Rydberg system and the discussed KCMs and investigate the limitations of using a classical rate equation model to capture the non-equilibrium behaviour of this system.

Introduction. The study of many-body quantum systems that undergo purely dissipative dynamics but at the same time feature quantum coherences and superpositions is an emerging theme that has attracted much interest in the past few years [1–5]. The catalyst was the realization that an appropriately engineered dissipative dynamics with non-classical jump operators represents a route towards the preparation of specific many-body states and non-equilibrium quantum phases with potential to be a resource for quantum computation [6].

In this paper, we aim to study the role that quantum coherences and superpositions play in the relaxation of these dissipative many-body quantum systems. In particular, we will study systems with purely dissipative and Markovian dynamics generated by a Lindblad master equation formed from non-classical jump operators whose action can bring the system into quantum superpositions. The stationary state of these systems will be given by a pure state annihilated by all jump operators, an absorbing or “dark” state which in general will display quantum coherences. Moreover, these systems will be constructed such that in the stationary state the diagonal of the density matrix coincides with the equilibrium probability distribution of a completely classical rate equation. Our aim will be to compare the relaxation given by this dissipative, yet quantum, dynamics with the one obtained via classical rate equations.

In order to tackle this task we will focus on a class of quantum spin systems inspired by kinetically constrained models of glasses (KCMs) [7,8]. These are classical stochastic models with trivial static equilibrium properties but complex collective dynamics due to the imposed kinetic constraints. Here we construct the purely dissipative quantum counterparts of these systems by ensuring that the values of diagonal observables (e.g. the density of excitations) in the stationary state coincide with the classical ones. These models provide a good test bed to study the differences and similarities in the relaxation between the classical and purely dissipative quantum dynamics, in particular as all the complexity is found in the dynamics and not on their stationary state [9]. A significant finding is that even when the relaxation timescales of the diagonal observables are similar, the coherences in the quantum system can take orders of magnitude longer to relax.

Finally, as an application of the models we introduce here, we consider an ensemble of highly excited (Rydberg) atoms [10] under electromagnetically induced transparency (EIT) conditions, a physical system which is currently of substantial theoretical and experimental interest [11,12]. We show here that, in an appropriate limit, the relaxation of this system is indeed described by a purely dissipative quantum dynamics. We then analyze the adequacy and limitations of the use of an effective classical description which is employed in many current studies of this many-body system [17–22] to explore its non-equilibrium behaviour.

General setup. Consider a classical stochastic system described by the master equation \( \partial_t |P(t)\rangle = \mathbb{W} |P(t)\rangle \), where the vector \( |P(t)\rangle \equiv \sum_c P(C; t) |C\rangle \) represents the probability distribution at time \( t \), and \( \{ |C\rangle \} \) is an orthonormal configuration basis. For a continuous time Markov chain the classical master operator reads,

\[
\mathbb{W} \equiv \sum_{C \neq C'} W_{C \rightarrow C'} |C'\rangle \langle C| - \sum_C R_C |C\rangle \langle C|, \tag{1}
\]

where \( W_{C \rightarrow C'} \) is the transition rate from \( C \) to \( C' \), and \( R_C \equiv \sum_{C'} W_{C \rightarrow C'} \) the escape rate from \( C \). Let us assume that the dynamics obeys detailed balance with respect to the equilibrium probability \( p_{eq}(C) \), i.e., the transition rates satisfy, \( p_{eq}(C) W_{C \rightarrow C'} = p_{eq}(C') W_{C' \rightarrow C} \). This condition allows to transform the stochastic operator \( \mathbb{W} \) into a Hermitian one \( \mathbb{H} \) through a similarity transformation, \( \mathbb{H} \equiv -\mathbb{P}^{-1} \mathbb{W} \mathbb{P} \) with \( \mathbb{P} \equiv \sum_C \sqrt{p_{eq}(C)} |C\rangle \langle C| \).

In
particularly, the ground state \(|g.s.\) of \(H\) (with eigenvalue 0) is directly related to the equilibrium probability as \(|g.s.\rangle \equiv \sum_C \sqrt{p_{eq}(C)}|C\rangle\).

Our aim is now to define a quantum model undergoing \textit{purely dissipative quantum dynamics} generated by a quantum master equation of the Lindblad form \[23\]

\[\partial_t \rho = \sum_{\mu} J_\mu \rho J_\mu^\dagger - \frac{1}{2} \left\{ J_\mu^\dagger J_\mu, \rho \right\}, \tag{2}\]

such that the dynamics converges to a stationary state \(\rho_{s.s.}\), where the expectation value of any classical operator \(\hat{O}\) (diagonal in the classical basis of configurations) is the same as in the classical equilibrium distribution \(\langle \hat{O} \rangle = \text{Tr}[\rho_{s.s.}\hat{O}]=\sum_C p_{eq}(C)\hat{O}\langle C\rangle\). This can be achieved by defining the following operators associated to the classical transitions between any pair of configurations \(\mu = (C, C')\),

\[J_\mu \equiv |\psi\rangle \langle \sqrt{W_{C\to C'}|C}\rangle - \sqrt{W_{C'\to C}|C'|}\langle C'|\rangle, \tag{3}\]

where \(|\psi\rangle\) can in principle be any normalised state. One can show that the Hermitian form of the master operator can be written as \(H = \sum_{\mu} J_\mu^\dagger J_\mu/2\). Moreover, \(J_\mu |g.s.\rangle = 0\) for all \(\mu\), that is, the state \(|g.s.\rangle\) is a \textit{dark state} for all operators \(J_\mu\). Thus, by considering the operators \(J_\mu\) of Eq. \(3\) as quantum jump operators in Eq. \(2\) the quantum dynamics converges to a pure stationary state \(\rho_{s.s.} = |g.s.\rangle \langle g.s.|\), where indeed the expectation values of classical operators correspond to the classical equilibrium ones. The key question is what difference, if any, there is between the quantum dynamics defined by Eqs. \(2\) \(3\) and the classical one generated by \(4\).

\textbf{Quantum vs. classical dissipative dynamics in kinetically constrained models.} To gain a general understanding of this question we consider a particular set of quantum spin models based on classical KCMs \[9\]. The models we consider are defined in terms of \(N\) binary variables on the sites of a lattice such that \(|0_k\rangle\) or \(|1_k\rangle\) corresponds to the \(k\)-th spin in the down or up state, respectively. The classical models can also be written in terms of a master equation like \(2\) but with a pair of jump operators for each site \(k = 1 \ldots N\) that act on a diagonal density matrix and connect a classical configuration with another one that differs by the flip of the \(k\)-th spin, \(J_k^+ = \sqrt{\lambda\kappa} f_k \sigma_k^+\) and \(J_k^- = \sqrt{\lambda(1-\kappa)} f_k \sigma_k^-\). Here, \(\lambda\) is the bare rate of jumps, \(\kappa \in [0, 1]\) is the average site occupation in equilibrium and \(\sigma_k^\pm\) are the spin-1/2 ladder operators. The operator \(f_k\) is a \textit{kinetic constraint} on site \(k\), that is, a function of diagonal operators \(n_j = \sigma_j^+ \sigma_j^-\) \(\forall j \neq k\), which conditions the dynamics of site \(k\) depending on the state of its neighbours.

In contrast, in the quantum model there is a \textit{single jump operator per site}, which establishes superpositions between \(|0_k\rangle\) and \(|1_k\rangle\):

\[J_k \equiv \sqrt{\lambda} U_k\ |B_k\rangle \langle B_k| f_k, \tag{4}\]

where \(|B_k\rangle = \sqrt{\kappa} |0_k\rangle - \sqrt{1-\kappa} |1_k\rangle\) and \(U_k\) is a free unitary that, together with \(f_k\), controls the relaxation of the system. This choice of jump operators ensures that the stationary state of the dynamics generated by \(2\) is, regardless of \(U_k\) and \(f_k\), a pure state of the direct product form, \(\rho_{s.s.} = \bigotimes_k |S_k\rangle \langle S_k|\), where \(|S_k\rangle = \sqrt{\kappa} |0_k\rangle + \sqrt{1-\kappa} |1_k\rangle\), so that \(\langle S_k| B_k \rangle = 0\). While the quantum dynamics can differ from the classical one, in the stationary state the expectation values of all diagonal operators in the \(|0, 1\rangle\) basis (such as the density of excitations or the density-density correlations) determined in the quantum problem by \(\rho_{s.s.}\) \textit{coincide} with those at equilibrium in the classical one.

As a first elementary comparison we can consider an unconstrained \((f = 1)\) single spin. Its classical dynamics is determined by Eq. \(2\) with the two jump operators \(J_1 = \sqrt{\lambda(1-\kappa)} \sigma^-\) and \(J_2 = \sqrt{\lambda\kappa} \sigma^+\). In this case, the relaxation of the density of excitations \((n(t))\) is exponential with a single timescale given by \(\tau_{eq} = \lambda^{-1}\). The quantum model has a single jump operator of the form \(4\), and we choose \(U = \exp (i\theta \sigma_y)\) with angle \(0 \leq \theta \leq \pi\). In this case there are two relaxation timescales for the density of excitations, \(\tau_{eq} = 2\lambda^{-1}\) and \(\tau_q^+ = (\sin \theta)^{-2}\lambda^{-1}\). The emergence of the second \(\theta\)-dependent timescale, which is due to \(U\) and hence absent in the classical case, can be understood by unravelling the dynamics in terms of quantum jump trajectories with respect to the jump operators \(J\) \[24\] \[25\]. Here, each trajectory is obtained by a “no-jump” non-Hermitian time-evolution of the system via \(H_{\text{eff}} = -iJ^J/2\) interspersed with quantum jumps. While the evolution between jumps is \(U\)-independent and thus \(\theta\)-independent, each quantum jump may take the system closer or further away from the stationary state \(|S\rangle\) depending on the angle \(\theta\). In particular, here when \(\theta = \pi/2\) the time \(\tau_q^+\) coincides with \(\tau_{eq}^+\) \[26\].

We now turn to the actual constrained many-body models \[27\]. We will consider two models determined by the kinetic constraints: \(f_k^{\text{East}} = n_{k+1}\), which allows transitions at site \(k\) only if site \(k + 1\) has a projection on the spin state \(|1\rangle\) (in analogy with the classical East model \[8\] \[9\] \[28\]), and \(f_k^{\text{FA}} = n_{k+1} + n_{k-1} - n_{k+1}n_{k-1}\), which allows transitions at site \(k\) if and only if at least one of the sites at \(k \pm 1\) has a projection on \(|1\rangle\) (in analogy with the classical Fredrickson-Andersen (FA) model \[7\] \[9\]).

We first focus on the East model. When \(\kappa\) is small the system relaxation encounters a conflict between the probability cost of flipping spins up, and the need for excited sites to facilitate neighbouring ones through the constraint \(f_k^{\text{East}}\). In the classical model this gives rise to hierarchical dynamics \[9\] \[28\], which manifests for example in metastable plateaus in the relaxation of the density, see Fig. \[1\]. For the quantum counterpart, Eqs. \(2\) \(3\), the dynamics depends on the angle \(\theta\) that defines the unitaries \(U_k = \exp (i\theta \sigma_y^k)\), which for simplicity we consider to be local rotations on the \(k\)-th site. We use quantum
FIG. 1. Relaxation of both classical and quantum dissipative East models for $\kappa/(1-\kappa) = 10^{-2}$ (size is $N = 10$ spins). The upper panel shows the average excitation density, $\langle n(t) \rangle$. The lower panel shows the evolution of the coherences, represented by $\langle \sigma(t) \rangle$, in the quantum case. Inset: time-evolution of $\langle n_k \rangle$ for $k = 1 \ldots N$ in a single trajectory for $\theta = \pi/2$.

jump Monte Carlo simulations to study the time evolution of the system numerically starting from a state of maximum density, see Fig. 1.

There are several regimes in the quantum relaxation of the average density of excited sites. For short times we have effectively unconstrained dynamics, as the density of excitations is high and thus the constraints do not play a role. When $\theta = \pi/2$, both the "no jump" dynamics due to $H_{\text{eff}}$ and the action of the non-classical jump operators help the system to reach a configuration where the excitations are isolated (see inset to Fig. 1). Further relaxation needs excitations to effectively propagate to meet and coalesce with others. This slow propagation leads to the different plateaus, in analogy with the classical East model [9, 29]. In contrast, when $\theta \gtrsim 0$ every jump brings the system away from the stationary state, making another jump likely to occur. This different behaviour between $\theta = \pi/2$ and $\theta \gtrsim 0$ becomes clear in the distributions of waiting times between jumps, see Fig. 2.

The key aspect in the dynamics of the quantum version of the East model is in the relaxation of coherences. The lower panel of Fig. 1 shows the evolution of $\langle \sigma(t) \rangle = 1/N \sum_k \langle \sigma_k \rangle$. For any angle $\theta$ this observable takes approximately three orders of magnitude longer to relax than the density of excitations. The reason for this mismatch between the two timescales is that the strongly constrained propagation-coalescence process that allows for the relaxation of the density leaves the system in a very inhomogeneous configuration with a single excitation, where the density profile is roughly $\ldots \kappa \kappa 1 \kappa \kappa \ldots$ (see inset to Fig. 1). The stationary state $\rho_{\text{s,s.}}$, however, has uniform density, with larger coherences. Thus, since the quantum jumps tend to localise the density, the last isolated excitation must be distributed through the lattice by the "no-jump" dynamics, which due to the constraints takes a long time.

In other quantum models with less restrictive constraints diagonal and off-diagonal observables are able to relax simultaneously. This is the case, for example, in the quantum counterpart of the FA model. Figure 3 shows that indeed the coherence and the excitation density relax simultaneously. The reason is that due to the bidirectional nature of the constraints $f_{\text{FA,k}}$, isolated excitations can become delocalised through both jumps and no-jump evolution.

Strongly interacting Rydberg gas. Up until now we have studied a class of idealized models with the common factor of obeying purely dissipative dynamics where the non-classical form of the jump operators gives rise to complex relaxation dynamics. Now we consider a physical many-body system which is currently extensively studied both theoretically [17,22] and experimen-
yield $\sigma \propto V$. For short times, constraints are not important and we find in the classical case $\langle n_{\xi}(t) \rangle = x^2/(1+x^2) \left[ 1 - e^{-\lambda t} \right]$, whereas in the quantum case we obtain $\langle n_{\xi}(t) \rangle = x^2/(1+x^2) \left[ 1 + (1+x^2)/(1-x^2) e^{-t} - 2/(1-x^2)e^{-((1+x^2)/2)} \right] \approx x^2/(1+x^2) \left[ 1 - e^{-t} \right]$ in the limit of $x \gg 1$. This choice for $\lambda$ is used in all simulations shown in Fig. 4(b).

Away from these two limits the classical model reproduces neither the stationary expectation values of the diagonal observables nor the relaxation dynamics towards them, as it can be observed in Fig. 4(b) for $x = 1$. Moreover, let us note that while in this system the coherences relax again simultaneously with the diagonal observables, in the stationary state their value is, in general, not negligible (e.g. for $x = 1$ the stationary value of $\langle \sigma_x \rangle$ is close to $1/2$) except in the limit $x \gg 1$.

**Outlook.** We have introduced here a general framework for contrasting the purely dissipative dynamics of quantum many-body systems with those described by classical rate equations. Even within the framework of idealised spin models many questions remain open. For example, it will be interesting to consider classical models where the configurations are connected by more than one spin flip at a time and their relation to atomic many-body quantum systems with dissipation that acts on a correlated way on several atoms at the same time. This collective dissipation has been shown to give rise to interesting physical phenomena such as binding of strongly interacting atoms and fermionic pairing.

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