On the Canonical Structure of the De Donder-Weyl Covariant Hamiltonian Formulation of Field Theory

I. Graded Poisson brackets and equations of motion

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Abstract
As opposed to the conventional field-theoretical Hamiltonian formalism, which requires the space+time decomposition and leads to the picture of a field as a mechanical system with infinitely many degrees of freedom, the De Donder-Weyl (DW) Hamiltonian canonical formulation of field theory (which is known for about 60 years) keeps the space-time symmetry explicit, works in the finite dimensional analogue of the phase space and leads to the Hamiltonian and Hamilton-Jacobi formulations of field equations in terms of partial derivative equations. No field quantization procedure based on this “finite dimensional” covariant canonical formalism is known. As a first step in this direction we consider the appropriate generalization of the Poisson bracket concept to the DW Hamiltonian formalism and the expression of the DW Hamiltonian form of field equations in terms of these generalized Poisson brackets. Starting from the Poincaré-Cartan form of the multidimensional variational calculus we argue that the analogue of the Poisson brackets is defined on forms of different degrees and is related to the Schouten-Nijenhuis bracket of the corresponding multivector fields. The forms generalize the dynamical variables (functions) of mechanics and the multivector fields generalize the Hamiltonian vector fields associated with dynamical variables. The corresponding map between forms and multivectors is determined by the "polysymplectic" \((n+1)\)-form (given by the Poincaré-Cartan form) which we consider as the analogue of the symplectic form in the DW Hamiltonian formalism for fields. The space of "Hamiltonian forms" equipped with the exterior product and our Poisson bracket is shown to constitute the Gerstenhaber graded algebra. We also demonstrate that the Poisson bracket of any form with the \(n\)-form \(H \omega\), where \(H\) is the DW Hamiltonian function, generates its exterior differential and this enables us to write the DW Hamiltonian field equations in the bracket form. Finally, we present few simple examples illustrating how the formalism works in some field-theoretical models, and also briefly discuss the relation to the conventional Hamiltonian description of fields.

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1 Introduction

There are known two different approaches to the Hamiltonian formulation of field theory: the first is built on the infinite dimensional "instantaneous" phase space and implies certain space+time decomposition while the second is formulated on the finite dimensional analogue of the phase space and is manifestly space-time covariant. Both are based on certain extensions of the structures of classical analytical mechanics and one dimensional variational calculus. The first approach singles out the time dimension and treats a field as a mechanical system with continually infinite number of degrees of freedom. The generalized coordinates are the values of fields $y^a$ at each point of the space at a given instant of time $y^a(x)$, and the generalized canonical momenta are defined from the Lagrangian density $L$ to be $p_a(x) = \partial L / \partial (\partial_t y^a(x))$ as in mechanics. This is, of course, a well known conventional treatment used for example when canonically quantizing the fields. Recent discussion of the covariant version of this approach may be found for example in [1].

The second approach, that we are concerned with in this paper, originates from the approaches to the multidimensional variational problems due to De Donder [2], Carathéodory [3], Weyl [4] and some others (see for example [3]) and [5] for a review). It is entirely space-time covariant because a field is treated as a sort of generalized Hamiltonian dynamical system with many "times". This means that both space and time enter the formalism on an equal footing as variables over which a field "evolution" proceeds. By "evolution" one means here not merely a time evolution from the given Cauchy data, as usual, but any space-time development or variation of a field. In this approach[6] the generalized coordinates are the field variables $y^a$ to which a set of canonically conjugate momenta $p^i_a := \partial L / \partial (\partial_i y^a)$ is associated. The De Donder-Weyl Hamiltonian function is defined as $H_{DW} := p^i_a \partial_i y^a - L$ provided the

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1 In fact, we consider here only the simplest particular representative of a whole variety of finite dimensional canonical theories for fields which differ by an extra "Lepagean" term added to the canonical Hamilton-Poincare-Cartan form (see eq. (2)); an excellent survey may be found in [1], see also [3, 4]. This particular case is sometimes called the De Donder-Weyl (DW) canonical theory and we also will use this term.
corresponding generalized Legendre transform is regular. Note that unlike the first approach, the Hamiltonian function is scalar, but its direct physical interpretation, if there is any, is not evident. The generalization of the extended phase space of mechanics in this approach is a finite dimensional phase space of the variables \( (y^a, p_a, x^i) \) which replaces the infinite dimensional phase space of the instantaneous approach. As a consequence, the Euler-Lagrange field equations may be written in the corresponding Hamiltonian form in an entirely covariant way (see eqs. (7) below) and in terms of partial derivative equations. The corresponding Hamilton-Jacobi theory (see for example [5, 6]) is also formulated in terms of the covariant partial differential equation as opposed to the first approach leading to the functional derivative equation. The connection between the instantaneous and the covariant finite dimensional formulations was studied recently in detail by Gotay [9] (see also the book [10]).

Despite all of the attractive features of the second treatment which look especially relevant in the context of general relativity and string theory, there is surprisingly small number of its applications to relativistic field theories [11, 12, 13], gauge fields [14, 15], classical bosonic string [16, 17, 18, 19, 20, 21] and general relativity [22, 23] in the literature (see also [10]). In particular, it remains unclear till now how to develop a field quantization starting from this finite dimensional Hamiltonian treatment on the classical level and whether it is possible or has a sense at all. Indeed, is it really necessary to split at first the space-time in order to obtain the Hamiltonian formulation, and then to quantize a field according to standart prescriptions of quantum theory and to prove the procedure to be consistent with the relativistic symmetries, or it is possible instead to develop a field quantization based on the finite dimensional covariant Hamiltonian framework and then obtain the space-time splitted results, as it is operationally required, from the manifestly covariant quantum field theory? Another related question is whether there exists a quasiclassical transition from some formulation of a quantum field theory to the Hamilton-Jacobi equations corresponding to the finite dimensional canonical formulations of
classical fields.

The problem of field quantization based on the finite dimensional Hamiltonian formalism, which is the main motivation of our study, was shortly discussed in thirties by Born \cite{24} and Weyl \cite{25}. In early seventies a considerable progress was made in understanding the differential geometric structures of the De Donder-Weyl canonical formalism \cite{26,27,30,31} (see also Dedecker \cite{28}, who studied more general canonical theories and the recent paper by Gotay \cite{7} for a subsequent development), however the attempts \cite{29,26,30,31} to approach from this viewpoint a quantum field theory did not lead to any new formulation but have established some links with the conventional one which is based on the instantaneous Hamiltonian formalism. More recently the attempt to construct a quantum field theoretical formalism based entirely on the finite dimensional DW canonical theory was reported by Günther in \cite{33} who used his own \cite{32} geometrical version of the DW canonical theory, the ”polysymplectic Hamiltonian formalism”. Unfortunately, the ideas of his brief report \cite{33} were not developed to the extent which would allow us to compare the outcome with something known from the conventional quantum field theory.

The main obstacle in the direction of a ”finite dimensional field quantization” seems to be the lack of an appropriate generalization or analogue of the Poisson brackets in the classical canonical theories under discussion. Within the DW Hamiltonian theory, the brackets of the \((n-1)\)-forms corresponding to observables in field theory were proposed in \cite{27,29,30,31}, but the related construction proved to be too restrictive to reproduce the algebra of observables in the theories of sufficiently general type and were not appropriate for representing the canonical Hamiltonian field equations in the bracket form. Another approaches due to Good \cite{34}, Edelen \cite{35} and Günther \cite{32} enable one to write the canonical equations in the bracket form, however, the group theoretical properties of their brackets are not evident.

The purpose of the present study is to develop those elements of
the finite dimensional canonical formalism for fields which are essential for the canonical quantization. Some of the questions (chosen more or less randomly) which arise as soon as we are trying to quantize a field theory (let’s say in the Schrödinger picture) proceeding from a finite dimensional canonical framework are the following: (i) which variables may be considered as the canonically conjugate ones and in which sense (notice, that the number of generalized coordinates $y^a$ and generalized momenta $p_a^i$ is different!), (ii) what are the Poisson brackets corresponding to the finite dimensional canonical formalism, (iii) if these brackets exist, do they generate the equations of motion, like in mechanics, if one of the arguments is the DW Hamiltonian function, and lastly (iv) which operation replaces in the bracket representation of the DW canonical equations the total time derivative and describes ”evolution” in the sense explained above?

To approach these and other related questions, we suggest to proceed from the most fundamental object of any canonical theory, the Hamilton-Poincaré-Cartan (HPC) $n$-form ($n =$space-time dimension), and try to develop the subsequent formalism by searching for the proper generalizations of the corresponding elements of the canonical formalism of mechanics (see e.g. [36, 37]) to the finite-dimensional canonical formulation of field theory.

The structure of the paper is the following. At first we recall in Sect. 2 how the DW Hamiltonian field equations comes directly from the canonical HPC form. This consideration indicates that in field theory the suitable generalization of the notion of the canonical Hamiltonian vector field is the multivector field of degree $n$, and also suggests the analogue of the symplectic form to be certain $(n + 1)$-form (see eq. (10)), called polysymplectic, which is obtained from the HPC form. Then, in Sect. 3 we suggest the generalization of the principle of preservation of the symplectic structure, which is a cornerstone of the classical canonical transformation theory, to field theory. This generalization involves the extension of the notion of Lie derivative giving sense to the Lie derivative with respect to a multivector field. The symmetry
postulate that the generalized Lie derivative of the polysymplectic form with respect to the Hamiltonian multivector field vanishes, turns out to be consistent with the canonical DW equations. Moreover, using the generalized Lie derivative allows us to define the bracket operation of both the generalized Hamiltonian fields which are *vertical* (in the sense defined in Sect. 3) multivectors of various degrees \( p = 1, \ldots, n \) and the generalised Hamiltonian functions which are *horizontal* forms of degrees \( q = 0, \ldots, n - 1 \). The first bracket turns out to be the Schouten-Nijenhuis (SN) bracket of multivectors and the second one, which is the analogue of the Poisson bracket acting on Hamiltonian forms, is related to the SN bracket and to the polysymplectic form in just the same way as the Lie bracket of Hamiltonian vector fields is related to the Poisson bracket of Hamiltonian functions in mechanics. We also show that the space of Hamiltonian forms equipped with the exterior product and the Poisson bracket of forms becomes essentially the *Gerstenhaber algebra* [39]. The generalized Poisson bracket of Hamiltonian forms (see Sect. 3 for an explanation of this term) is used in Sect. 4 for representing the DW Hamiltonian field equations of \((n - 1)\)-forms in the bracket form. As a by-product we also discuss the proper generalization of the notions of integral of motion and canonically conjugate variables to the DW canonical formalism. Then, in Sect.5 we enlarge the set of Hamiltonian forms by adding the \( n \)-forms. It enables us to express the equations of motion of Hamiltonian forms of any degree in the bracket form. However, this enlargement implies also an extension of the space of Hamiltonian multivector fields and the algebraic closure of this enlarged space is argued to involve both the (vertical) vector-valued one-forms corresponding to \( n \)-forms and the multivector-valued forms of higher degrees. Thus the algebraic closure of the space of generalized Hamiltonian ”vector” fields leads to the problem of embedding of both the Schouten-Nijenhuis and the Frölicher-Nijenhuis graded Lie algebras in some larger algebraic structure which includes the multivector-valued forms of all possible degrees. Solving this problem remains beyond the scope of our paper. Finally, some simple applications of our Poisson brackets to interacting scalar fields, electrodynamics and the Nambu-Goto string are presented in Sect. 6, and a general discussion including
some remarks on the connections with the conventional Hamiltonian formulation may be found in the concluding Sect. 7.

2 Poincaré-Cartan form and the De Donder-Weyl Hamiltonian field equations

Given a first order multidimensional variational problem

$$\delta \int L(y^a, \partial_i y^a, x^i) \tilde{\text{vol}} = 0,$$

where \((y^a), 1 \leq a \leq m\) are field variables, \((x^i), 1 \leq i \leq n\) are space-time variables, and \(\tilde{\text{vol}} := dx^i \wedge ... \wedge dx^n\) it is known that the Hamilton-Poincaré-Cartan (HPC) fundamental \(n\)-form is defined within the De-Donder-Weyl (DW) approach to multidimensional variational problems as (see for example [6, 10] and [27])

$$\Theta_{DW} = p^i_a \wedge dy^a \wedge \partial_i \tilde{\text{vol}} - H_{DW} \tilde{\text{vol}}$$

so that its exterior differential is given by

$$\Omega_{DW} = dp^i_a \wedge dy^a \wedge \partial_i \tilde{\text{vol}} - dH_{DW} \wedge \tilde{\text{vol}}.$$  

Here \(p^i_a := \partial L/\partial(\partial_i y^a)\) are the DW canonical momenta and

$$H_{DW}(y^a, p^i_a, x^i) := p^i_a \partial_i y^a - L$$

is the DW Hamiltonian function. The symbol \(\lrcorner\) denotes the interior product of a (multi)vector on the left and a form on the right. In the following we will omit the subscript DW, but the quantity \(H\) which we call the (DW) Hamiltonian function should not be confused with usual Hamiltonian which is related to energy.

The form \(\Omega_{DW}\) contains in a sense all the information about field dynamics. In particular, one can derive the appropriate Hamiltonian form of field equations directly from \(\Omega_{DW}\). Indeed, the solutions of the variational problem (1) may be considered as n-dimensional distributions

\[\text{In order to simplify formulae we imply in the following that the coordinates on the x-space are chosen such that the metric determinant } |g| = 1.\]
in the extended DW phase space with the coordinates

\[ Z^M := (y^a, p_i^b, x^i). \]

These distributions one can describe by the n-multivector (or n-vector, in short) field \( \bar{X} \):

\[ \bar{X} := X^{M_1 \ldots M_n}(Z) \partial_{M_1} \wedge \ldots \wedge \partial_{M_n} \]

representing their tangent n-planes. Then the condition on \( \bar{X} \) to give the classical extremals is that the form \( \Omega_{DW} \) should vanish on \( \bar{X} \) (cf. e.g. [6, 8, 10, 11, 27, 29]), i.e.

\[ \bar{X} \cdot \Omega_{DW} = 0. \]

The n-vector field \( \bar{X} \) naturally generalizes the velocity field of the canonical Hamiltonian flow in classical mechanics corresponding to \( n = 1 \) to field theory, which corresponds to \( n > 1 \). Eq. (5) gives the components of the n-vector annihilating the \((n+1)\)-form \( \Omega_{DW} \) and together with the following natural parametrization of the components of \( \bar{X} \):

\[ \bar{X}^{M_1 \ldots M_n} = \frac{\partial (Z^{M_1}, \ldots, Z^{M_n})}{\partial (x^1, \ldots, x^n)}, \]

leads to the set of equations\(^3\)

\[ \partial_i p_a^i = -\partial_a H := -\frac{\partial H}{\partial y^a}, \]

\[ \partial_i y^a = \partial_a^i H := \frac{\partial H}{\partial p_a^i}, \]

which we will refer to as the (DW) Hamiltonian field equations. They are equivalent to the Euler-Lagrange equations one gets from the variational problem (1) and are the simplest field theoretic generalization of the canonical Hamilton’s equations of motion.

\(^3\)In fact, the components \( X^{a_i i_1 \ldots i_{n-2}} \) of the n-vector \( \bar{X} \) yield also the third equation which may be shown to be a consequence of eqs. (7,8). Thus the information about the classical dynamics of field is essentially encoded in the "vertical", as we call them below, components \( X^{a_i i_1 \ldots i_{n-1}} \) and \( X^{a_i i_1 \ldots i_{n-1}} \). This is the observation which motivates our construction in Sect.3.
3 Polysymplectic form, Hamiltonian multivector fields and forms and the generalization of the Poisson bracket.

Our task in this section is to find the appropriate generalization of the basic structures of classical Hamiltonian mechanics, such as the Hamiltonian vector fields and functions, the symplectic structure, Poisson brackets etc., to the DW Hamiltonian formulation of field theory.

Notice first that, as it follows from (5), the Hamiltonian field equations (7),(8) can be derived also from the condition

\[ \overset{v}{X} \cdot \Omega = d^v H, \quad (9) \]

where the superscript \( v \) shows that we take a vertical' part of the quantity. We call *vertical* the variables \( z^V = (y^a, p_i^a) \) and the corresponding subspace of the extended DW phase space and *horizontal* the space-time variables \( (x^i) \); the index \( V \) corresponds to vertical variables. Further, the \( p \)-multivector is called vertical if it has one vertical and \((p-1)\) horizontal indices, i.e.

\[ \overset{p}{X} \overset{v}{:=} \overset{p}{X}^{V i_1...i_{p-1}}(Z) \partial^V \wedge \partial_{i_1} \wedge ... \wedge \partial_{i_{p-1}}, \]

and the vertical exterior differential \( d^v \) of any form \( \omega \) is defined as

\[ d^v \omega := dz^V \wedge \partial_V \omega, \]

so that, in particular,

\[ d^v H = \partial_a H dy^a + \partial^a_i H dp^i_a. \]

In the following we will also use the notion of *horizontal* \( p \)-forms which are defined to have a form

\[ \overset{p}{F} := F_{i_1...i_p}(Z) dx^{i_1} \wedge ... \wedge dx^{i_p}. \]

Finally, the form \( \Omega^v \) in eq.(9) is defined as

\[ \Omega^v := dp^i_a \wedge dy^a \wedge \partial_i \cdot \overset{\sim}{\text{vol.}} \quad (10) \]

so that it is given by the vertical exterior differential of the vertical part of the HPC form:

\[ \Omega^v := d^v \Theta^v, \]
\[ \Theta^v := p_i^a dy^a \land \partial_i \tilde{\text{vol}}. \]

In the following, the closed \((n+1)\)-form \(\Omega^v\) will be denoted as \(\Omega\), and we shall call it the \textit{polysymplectic} form adopting the term introduced in a similar context earlier \([32]\). We will also omit all the superscripts \(v\) of the multivectors, since all of them appearing in the following will be taken to be vertical, unless the opposite will explicitly be stated. Note also that the polysymplectic form is related to the HPC form in exactly the same way as the symplectic form in mechanics is related to the HPC form of the 1-dimensional variational problem.

Let us recall now (see for details \([36, 37]\)) that the structures of classical Hamiltonian mechanics are contained essentially in a single statement that Lie derivative of a symplectic form \(\omega\) with respect to the vertical vector fields \(X\) generating the infinitesimal canonical transformations vanishes: \(\mathcal{L}_X \omega = 0\). Since \(\omega\) is closed, it implies locally that \(X_f \omega = df\) for some function \(f\) of the phase space variables. If this equality holds globally, the vector field \(X_f\) is said to be (globally) Hamiltonian vector field associated with the Hamiltonian function \(f\). When \(f\) is taken to be the canonical Hamilton’s function \(H\), the equations of the integral curves of \(X_H\) (the canonical Hamiltonian vector field) reproduce Hamilton’s canonical equations of motion.

From the previous considerations one could already notice that in field theory the \(n\)-vector field \(\tilde{\mathcal{X}}\) associated with the DW Hamilton’s function as in eq.(9) is similar to the canonical Hamiltonian vector field in mechanics and the \((n+1)\)-form \(\Omega\) is similar to the symplectic 2-form. To pursue this parallel further, let us introduce the generalized Lie derivative \(-\mathcal{L}_{\tilde{\mathcal{X}}}^{\tilde{p}}\) – with respect to a multivector field of degree \(p\) and postulate, as a fundamental symmetry principle, that

\[ \mathcal{L}_{\tilde{\mathcal{X}}}^{\tilde{p}} \Omega = 0. \]  

(11)

We define the generalized Lie derivative of any form \(\omega\) with respect to the multivector field \(\tilde{\mathcal{X}}\) of degree \(p\) (not necessarily vertical) by the formula
\[ \mathcal{L}_X^p \omega := \mathcal{P}_X \mathcal{D} \omega - (-1)^p d(\mathcal{P}_X \mathcal{D} \omega) \] (12)

which is the simplest generalization of the Cartan formula relating the Lie derivative of a form along the vector field to the exterior derivative and the inner product with the vector; this relation is recovered when \( p = 1 \). Note however that, unlike the \( p = 1 \) case, the operation \( \mathcal{L}_X^p \) does not preserve the degree of a form it acts on: it maps \( q \)-forms to \((q + 1 - p)\)-forms. If \( \mathcal{P}_X \) is vertical, then the definition of \( \mathcal{L}_X^p \) is modified by replacing the exterior differentials in eq.(12) by the vertical \( (d^v) \) ones.

Since \( \Omega \) is closed with respect to the vertical exterior differential, from the symmetry postulate, eq.(11), and the definition of the generalized Lie derivative, eq.(12), it follows

\[ d^v (\mathcal{P}^n \mathcal{D} \Omega) = 0, \] (13)

so that locally one can write

\[ \mathcal{P}^n \mathcal{D} \Omega = d^v 0^F \] (14)

for some 0-form \( 0^F \) depending on the phase space variables \( Z^M \). By analogy with mechanics, if such a form exists globally, we call \( \mathcal{P}^n \mathcal{D} \) the (globally) Hamiltonian \( n \)-vector field (associated with the Hamiltonian 0-form \( 0^F \)) while the \( \mathcal{P}^n \mathcal{D} \) satisfying eq. (13) is called locally Hamiltonian. We see from eqs. (9) and (13) that our postulate, eq. (11), together with the definition of the generalized Lie derivative in eq. (12) leads to the correct DW Hamiltonian field equations if the the 0-form \( 0^F \) at the r.h.s. of eq. (13) is taken to be the DW Hamiltonian \( H \).

Given two locally Hamiltonian \( n \)-vector fields it is natural to define their bracket as

\[ [\mathcal{P}^n \mathcal{D}_1, \mathcal{P}^n \mathcal{D}_2] \mathcal{D} \Omega := \mathcal{L}^n_{\mathcal{P}^n \mathcal{D}_1} (\mathcal{P}^n \mathcal{D}_2 \mathcal{D} \Omega), \] (15)

which is obviously in accordance with the invariance property we have postulated in eq. (11). From the definition in eq. (15) it follows

\[ d^v ([\mathcal{P}^n \mathcal{D}_1, \mathcal{P}^n \mathcal{D}_2] \mathcal{D} \Omega) = 0, \] (16)
so that $[\vec{X}_1, \vec{X}_2]$ is also locally Hamiltonian. However, this bracket does not map $n$-vector fields to $n$-vector ones; instead it mixes the multivectors of different degrees. Moreover, as the counting of degrees in eq. (15) gives $\text{deg}(\vec{X}_1, \vec{X}_2) = 2n - 1$, the bracket vanishes identically if $2n - 1 > n$, i.e. for $n > 1$. These observations indicate that the multivectors and, therefore, the forms of various degrees should come into play.

Thus, given the polysymplectic $(n + 1)$-form $\Omega$, we shall define the set of \textit{locally Hamiltonian} (LH) multivector fields as the set of \textit{vertical} \(p\)-vector fields $\vec{X}$, $1 \leq p \leq n$, for which

$$\mathcal{L}_{\vec{X}} \Omega = 0.$$ (17)

Then the $p$-vector fields are defined to be \textit{Hamiltonian} if there exist \textit{horizontal} $q$-forms $\tilde{F} := F_{i_1 \ldots i_q}(Z) dx^{i_1} \wedge \ldots \wedge dx^{i_q}, 0 \leq q \leq (n - 1)$, such that

$$\vec{X}_F \mathcal{J} \Omega = d^q \tilde{F},$$ (18)

where $p = n - q$. In the following we call the forms $\tilde{F}$ the \textit{Hamiltonian forms} and the multivector fields $\vec{X}_F$ the \textit{Hamiltonian multivector fields} generated by (or associated with) the forms $\tilde{F}$. The set of Hamiltonian forms extends to field theory the notion of Hamiltonian functions or dynamical variables in mechanics. The inclusion of forms of various degrees is motivated by the fact that the dynamical variables of interest in field theory in $n$ dimensions can be the forms of any degree $p \leq n$ (the $n$-forms are incorporated in Sect.5). It should be noted that, in contrast with mechanics, eq. (18) imposes rather strong restriction on the functional dependence of the components of Hamiltonian forms on the DW momenta (see eq. (29) below for the case of $(n - 1)$-forms).

The bracket of two locally Hamiltonian fields may be defined now similarly to eq. (15):

$$[\vec{X}_1, \vec{X}_2] \mathcal{J} \Omega := \mathcal{L}_{\vec{X}_1} (\vec{X}_2 \mathcal{J} \Omega),$$ (19)

and it is easy to show that it maps the LH fields to LH ones. This bracket (i) generalizes the Lie bracket of vector fields, (ii) its degree is
easily found to be

\[ \text{deg}([\mathcal{P}_1, \mathcal{X}_2]) = p + q - 1, \quad (20) \]

(iii) it can be both odd and even

\[ [\mathcal{P}_1, \mathcal{Q}_2] = -(-1)^{(p-1)(q-1)} [\mathcal{Q}_2, \mathcal{P}_1] \]

and, finally, (iv) it fulfils the graded Jacobi identities

\[
\begin{align*}
(-1)^{g_1 g_3} [\mathcal{P}_1, [\mathcal{Q}_2, \mathcal{R}_3]] & + \\
(-1)^{g_2 g_3} [\mathcal{Q}_2, [\mathcal{P}_1, \mathcal{R}_3]] & + (-1)^{g_2 g_3} [\mathcal{R}_3, [\mathcal{P}_1, \mathcal{Q}_2]] = 0,
\end{align*}
\]

where \( g_1 = p - 1, \ g_2 = q - 1 \) and \( g_3 = r - 1 \). Therefore, the bracket defined in eq.(19) may be identified with the Schouten-Nijenhuis (SN) bracket of multivector fields [38] and the set of \( LH \) multivector fields equipped with the SN bracket constitutes a \( \mathbb{Z} \)-graded Lie algebra.

Now, taking \( \mathcal{P}_1 \) and \( \mathcal{Q}_2 \) to be Hamiltonian fields, one gets from eqs. (18) and (19):

\[
[\mathcal{P}_1, \mathcal{Q}_2] \downarrow \Omega = \mathcal{L}_{\mathcal{P}_1} d^r \mathcal{F}_2
\]

\[ = (-1)^{p+1} d^r (\mathcal{P}_1 \downarrow d^r \mathcal{F}_2) \]

\[ = -d^r \{ \mathcal{F}_1, \mathcal{F}_2 \}, \]

where \( r = n - p \) and \( s = n - q \). The last equality in eq.(21) defines the analogue of the Poisson bracket acting on Hamiltonian forms of various degrees. As it is seen from the definition, it is related to the Schouten-Nijenhuis bracket of multivector Hamiltonian fields and the polysymplectic \((n + 1)\)-form \( \Omega \) in just the same way as the usual Poisson bracket is related to the Lie bracket of Hamiltonian vector fields and the symplectic form. In eq.(21) we have actually shown that the S-N bracket of two Hamiltonian multivector fields is Hamiltonian (see the second equality), and then postulated in the third equality that the Hamiltonian form corresponding to the S-N bracket of two Hamiltonian fields defines the Poisson bracket of the Hamiltonian forms they are associated with according to the map given by eq.(18).
The degree counting in eq. (21) gives
\[
\text{deg}\{ F^r_1, F^s_2 \} = r + s - n + 1 \quad (22)
\]
and with the help of eq.(20) one also finds
\[
\{ F^r_1, F^s_2 \} = -(-1)^\sigma \{ F^s_2, F^r_1 \}, \quad (23)
\]
where \( \sigma = (n - r - 1)(n - s - 1) \). From the definition in eq. (21) the following useful formulae for the Poisson bracket can also be obtained:
\[
\{ F^r_1, F^s_2 \} = (-1)^{(n-r)} X_1 \, \hat{\wedge} \, d^\flat F^s_2 = (-1)^{(n-r)} X_1 \, \hat{\wedge} \, X_2 \, \hat{\wedge} \, \Omega. \quad (24)
\]
These equations resemble familiar definitions of the Poisson bracket in mechanics, but they are merely a consequences, as in mechanics, of the fundamental definition, eq. (21), which is directly related to the basic symmetry principle, eq. (11). Note, that in spite of the fact that the definition in eq. (21) determines only a vertical exterior derivative of the Poisson bracket there is no arbitrariness "modulo exact form" in the definition of the Poisson bracket itself (cf. \[27, 29, 30\]) since the latter is required to map the horizontal forms to horizontal ones while the \( d^\flat\)-exact addition would necessarily be vertical.

It should be noted that, unlike mechanics, in field theory we have a nontrivial set of primitive Hamiltonian multivector fields \( \hat{X}_0 \) satisfying
\[
\hat{X}_0 \, \hat{\wedge} \, \Omega = 0,
\]
\( p = 1, ..., n \). Evidently they constitute the subalgebra \( \mathcal{X}_0 \) of the algebra \( \mathcal{X} \) of Hamiltonian multivector fields w.r.t. SN brackets and the map in eq.(18) actually maps the Hamiltonian forms \( \hat{q}^p \) to the equivalence classes of Hamiltonian multivector fields \( \hat{X}^p \) with respect to the addition of the primitive Hamiltonian \( p \)-vector fields: \( \hat{X}^p = [\hat{X} + \hat{X}_0] \). Therefore, the quotient algebra \( \mathcal{X}/\mathcal{X}_0 \) is more adequate field theoretical analogue of the Lie algebra of Hamiltonian vector fields in mechanics than the original algebra \( \mathcal{X} \) (cf. also the related discussion in \[31\]).
By a straightforward calculation one can obtain the following properties of the Poisson brackets: (i) the graded analogue of the Leibniz rule

\[ \{ \tilde{F}, \{ \tilde{F}, \tilde{F} \} \} = \{ \tilde{F}, \tilde{F} \} \wedge \tilde{F} - (-1)^{g_1(g_3 - 1)} \tilde{F} \wedge \{ \tilde{F}, \tilde{F} \} \]  

which means that the Poisson bracket with a $p$-form acts as a graded derivation of degree $(n - p - 1)$ (see also eq. (22)), and also (ii) the graded Jacobi identities:

\[ (-1)^{g_1 g_3} \{ \tilde{F}, \{ \tilde{F}, \tilde{F} \} \} + (-1)^{g_2 g_3} \{ \tilde{F}, \{ \tilde{F}, \tilde{F} \} \} = 0, \]

where $g_1 = n - p - 1$, $g_2 = n - q - 1$ and $g_3 = n - r - 1$. Thus, the space of Hamiltonian forms equipped with the Poisson bracket operation as defined above constitutes a $\mathbb{Z}$-graded Lie algebra. Moreover, since the exterior algebra of forms (i.e. the algebra w.r.t. the $\wedge$-product) is itself a $\mathbb{Z}$-graded supercommutative associative algebra, one can conclude that the Hamiltonian forms constitute the so-called Gerstenhaber algebra [39] (see also [45]) with respect to our Poisson brackets and the exterior product of forms.

4 Equations of motion of Hamiltonian $(n-1)$-forms and the DW canonical equations in the bracket form.

In this section we consider the operation on the Hamiltonian forms which is generated by the Poisson bracket with the DW Hamiltonian function and show how the equations of motion of $(n - 1)$-forms can be written with the help of this bracket. Then we shall shortly discuss the field theoretic analogues of the integrals of motion and the suitable choice of the canonically conjugate variables.

From the degree counting in eq. (22) we see that only the $(n - 1)$-forms have nonvanishing brackets with $H$ and these are the 0-forms. Let us calculate the bracket of the general Hamiltonian $(n-1)$-form

\[ F := F^i \partial_i \int \text{vol} \]
with the DW Hamiltonian function \( H \):

\[
\{ F , H \} = - \frac{1}{X_F} \int d^v H. \tag{26}
\]

The components of the vector field \( X_F := X^a \partial_a + X^i \partial_i \) associated with \( F \) are to be calculated from the equation

\[
\frac{1}{X_F} \int \Omega = d^v F \tag{27}
\]

which reads in components

\[
(-X^a dp^i_a + X^i a dy^a) \wedge \partial_i \tilde{vol} = (\partial_a F^i dy^a + \partial_j F^a dp^a_j) \wedge \partial_i \tilde{vol} \tag{28}
\]

and yields

\[
X^a \delta^i_j = - \partial_j^a F^i, \tag{29}
\]

\[
X^i_a = \partial_a F^i. \tag{30}
\]

Hence, in contrast with mechanics, no arbitrary \((n - 1)\)-forms can be Hamiltonian (i.e. to ensure the consistency of both sides of eq. (27) and to give rise to some Hamiltonian vector field), but only those which satisfy the condition (29) which restricts the dependence of the components of \( F \) on the DW canonical momenta \( p^i_a \). For such \((n - 1)\)-forms one has:

\[
\{H, F\} = \partial_a F^i \partial^a_i H + X^a \partial_a H. \tag{31}
\]

Now, the total (i.e. taken on extremals) exterior differential \( d \) of \( F \)

\[
dF := (\partial_a F^j \partial_i y^a + \partial^k_i F^j \partial_k p^i_a + \partial_i F^j)dx^j \wedge \partial_j \tilde{vol}
\]

on account of the condition (29) takes the form

\[
dF = (\partial_a F^i \partial_i y^a - X^a \partial_i p^i_a + \partial_i F^i)\tilde{vol}.
\]

Thus, with the help of the DW Hamiltonian field equations, eqs. (7,8), for an arbitrary Hamiltonian \((n - 1)\)-form \( F \) one obtains:

\[
dF = \{ H , F \} \tilde{vol} + d^{\text{hor}} F. \tag{32}
\]

The last term \( d^{\text{hor}} F = (\partial_i F^i)\tilde{vol} \) in eq.(32) appears for forms having explicit dependence on the space-time variables. The inverse Hodge


\( \star^{-1} d F = \{ H, F \} + \partial_i F^i \)  \hspace{1cm} (33)

shows that the Poisson bracket with the DW Hamiltonian function is related to the Hodge dual of the exterior differential in essentially the same way as the time derivative is related to the Poisson bracket with Hamilton’s canonical function in mechanics.

Equation (33) contains, as a special case, the entire set of the DW Hamiltonian field equations, eqs.(7,8). For, on account of \( d \text{hor} y^a = 0 \) and \( d \text{hor} p_a^i = 0 \), by substituting \( p_a := p_a^i \partial_i \tilde{\text{vol}} \) and then \( y_a^i := y^a \partial_i \tilde{\text{vol}} \) for the \((n-1)\)-form \( F \), from eqs. (30) and (32) we obtain

\[ \star^{-1} dp_a = \{ H, p_a \} = -\partial_a H \]

and

\[ \star^{-1} dy_a^i = \{ H, y_a^i \} = \partial_i H. \]

Note that the dual of the total exterior derivative \( \star^{-1} d \) in eq. (33) is in fact nothing else than the generalized Lie derivative with respect to the total n-vector field \( \tilde{X}^{\text{tot}} = \tilde{X}^v + \tilde{X}^{\text{hor}} \) annihilating \( \Omega_{DW} \) (see eqs.(4)-(6) above). The term including the Poisson bracket is due to the Lie derivative w.r.t. the vertical part \( \tilde{X}^v \) of this n-vector field while the last term in eq. (28) is due to the Lie derivative w.r.t. its horizontal part \( \tilde{X}^{\text{hor}} \); thus, \( \star^{-1} d = (-1)^n \mathcal{L}_{\tilde{X}^{\text{tot}}} \) when acting on the Hamiltonian \((n-1)\)-forms. In the subsequent section we will discuss how this bracket representation of the equations of motion may be generalized to the Hamiltonian forms of degrees \( p \leq (n-1) \).

Equations of motion in the bracket form suggest a natural generalization of the classical notion of an integral of motion to field theory. Let \( \mathcal{J} \) be the Hamiltonian \((n-1)\)-form which does not depend explicitly on space-time coordinates and has vanishing Poisson bracket with the DW Hamiltonian function. Then from eq. (32) the conservation law follows:

\[ d \mathcal{J} = 0. \]

\( \star^{-1} \star = \sigma \), where \( \sigma = +1 \) for Euclidean and \( \sigma = -1 \) for Minkowski signature of the metric; \( \star^{-1} \star := 1 \), therefore on \( n \)-forms \( \star^{-1} = \sigma \star \) or, in general, \( \star^{-1} = \sigma (-1)^p (n-p) \star \) on \( p \)-forms.

\footnote{Recall that \( \star \text{vol} = \sigma \), where \( \sigma = +1 \) for Euclidean and \( \sigma = -1 \) for Minkowski signature of the metric; \( \star^{-1} \star := 1 \), therefore on \( n \)-forms \( \star^{-1} = \sigma \star \) or, in general, \( \star^{-1} = \sigma (-1)^p (n-p) \star \) on \( p \)-forms.
Thus, the field theoretical analogues of integrals of motion in the present formulation are the \((n-1)\)-forms corresponding to conserved currents. Like the conserved quantities in mechanics they are characterized by the condition

\[
\{ J, H \} = 0. \tag{34}
\]

Taking \(J_1\) and \(J_2\) to be the \((n-1)\)-forms satisfying eq.(34) and using the Jacobi identities, eq.(27), one gets

\[
\{ H, \{ J_1, J_2 \} \} = 0.
\]

Therefore, the Poisson bracket of two conserved currents fulfilling eq.(34) is again a conserved current of the same kind. Latter statement extends to field theory the Poisson theorem\cite{37} that the Poisson bracket of two integrals of motion of a Hamiltonian flow is again an integral of motion. One can also conclude that the set of conserved \((n-1)\)-form currents having vanishing Poisson bracket with the DW Hamiltonian function is closed with respect to the Poisson bracket and thus forms a Lie algebra being a subalgebra of the graded algebra of all Hamiltonian forms.

Furthermore, eq.(34) means that the Lie derivative of \(H\) w.r.t. a vertical vector field \(X\) associated with \(J\) vanishes, i.e. \(H\) is invariant w.r.t. a symmetry generated by \(X\) and \(J\) is a conserved current corresponding to this symmetry of the DW Hamiltonian. We arrive thus at a sort of a field theoretical extension of the Hamiltonian Noether theorem (cf. for example \cite[§40 or §15.1]{37b}). Note that this extension concerns only the symmetries generated by the vertical vector fields.

The way in which the canonical Hamiltonian field equations are represented in terms of the Poisson brackets sheds light on the question as to which variables may be considered in the present formalism as the canonically conjugate ones. As we know from mechanics, canonically conjugate variables (i) have ”simple” mutual Poisson brackets (leading to the Heisenberg algebra structure) and (ii) their products have the dimension of action. It is easy to see that in our approach the pair of variables

\[
(y^a, p_a := p_i^a \delta_i \overline{\text{vol}}), \tag{35}
\]
one of which is 0-form and another is \((n - 1)\)-form, may be considered as a pair of canonically conjugate variables. The Poisson brackets of these variables

\[
\{ y^a, p_b \} = -\delta_b^a; \quad \{ y^a, y^b \} = 0; \quad \{ p_a, p_b \} = 0
\]  

(36)

turn out to be the same as those of coordinates and canonically conjugate momenta in mechanics. Indeed, from eqs. (23),(24) one obtains

\[
\{ y^a, p_b \} = -\{ p_b, y^a \} = \frac{1}{X(p_b)} \int dy^a = f - \delta_b^a;
\]

where one has used in the last equality that the vector field \(\frac{1}{X(p_b)}\) associated with the \((n - 1)\)-form \(p_b\) is given by

\[
\frac{1}{X(p_b)} \int \Omega = dp_b,
\]

so that

\[
\frac{1}{X(p_b)} = -\partial_b
\]

(cf. also eqs. (29),(30)).

It should be noted, however, that in principle this choice is not unique. For example, we could also choose the pair \((y^a \partial_i \tilde{\text{vol}}, p^i_b)\): the (nonvanishing) Poisson bracket in this case is also remarkably simple, namely

\[
\{ y^a \partial_i \tilde{\text{vol}}, p^i_b \} = -\delta_i^j \delta_b^a.
\]  

(37)

Such a freedom is due to the "canonical supersymmetry", eq.(17), mixing the forms of different degrees. It might be especially useful in field theories in which the field variables themselves are forms, like a 1-form potential \(A_\nu dx^\nu\) in electrodynamics or a 2-form potential in the Kalb-Ramond field theory: to the \(p\)-form field variable the \((n - p - 1)\)-form conjugated momentum may be associated, and their mutual Poisson bracket may be shown (cf. Sect. 6.2) to be equal to one (up to a sign). Note that the extension of our construction in the following section, which involves the \(n\)-forms as Hamiltonian forms, provides in principle still more freedom in specifying the canonically conjugate variables.
5 Equations of motion of Hamiltonian forms of an arbitrary degree

In Sect. 3 we have found that the proper field theoretical generalization of the Hamiltonian functions of mechanics are the horizontal forms of various degrees from 0 to \((n - 1)\), on which the analogue of the Poisson bracket operation involving the forms of various degrees was defined. However, in Sect. 4 only the equations of motion of \((n - 1)\)-forms were formulated in terms of these generalised Poisson brackets. It seems natural to ask whether this circumstance is due to some privileged position of \((n - 1)\)-forms in the formalism (which might indeed be the case since some of them yield classical observables after integrating over the spacelike surface orthogonal to the time-direction of an observer) or there exists a possibility to write in bracket form the equations of motion of Hamiltonian forms of any degree. In this Section we argue that the second alternative may be realized indeed by means of a slight generalization of the construction of Sect. 3 leading to further extension of the notion of Hamiltonian forms and the associated Hamiltonian fields.

The problem we meet trying to extend the equation of motion of \((n - 1)\)-forms to Hamiltonian forms of arbitrary degree \(p < (n - 1)\) is essentially that \(\{\tilde{F}^p, H\}\) vanishes identically when \(p < (n - 1)\). The possible way out of that is suggested by the observation that for all \(p\) the bracket \(\{\tilde{F}^p, \widetilde{Hvol}\}\) would not vanish, as the formal degree counting based on eq. (20) indicates, if one could extend our hierarchy of equations relating the Hamiltonian forms and the Hamiltonian multivector fields, eq. (18), so as to supplement the set of Hamiltonian forms with the horizontal forms of degree \(n\), as the form \(\widetilde{Hvol}\) is. This is possible indeed, if the object \(\tilde{X}^v\) which one associates with the horizontal \(n\)-form \(\tilde{F}\) by means of the map

\[
\tilde{X}^v \cup \Lambda^v = d^v \tilde{F}^n
\]

is thought to be the vertical-vector-valued horizontal one-form

\[
\tilde{X}^v := X^v_k dx^k \otimes \partial_V,
\]

and the inner product \(\cup\) to be the Frölicher-Nijenhuis (FN) inner prod-
uct of a vector-valued form and a form \[40, 41\]:

\[
\tilde{X} \bigoplus \Omega := X^V_k dx^k \wedge (\partial V \bigoplus \Omega). \tag{40}
\]

Here we use the usual symbol of the inner product of vectors and forms and imply that the tilde over the argument at the l.h.s. indicates that it is a vector-valued form so that \[\bigoplus\] in this case denotes the FN inner product of a vector-valued form and a form.

By extending formulae (24) one can define now the left Poisson bracket of the p-form \(\vec{F}\) with the n-form \(\vec{F}'\) as follows:

\[
\{ \vec{F}_1, \vec{F}_2 \} = \tilde{X}^{v}_{F_1} \bigoplus d^v \vec{F}_2. \tag{41}
\]

This expression may be substantiated by the considerations similar to those which led from eq. (17) to eq. (24), provided one supplements the hierarchy of symmetries in eq. (17) with the additional assumption

\[
\mathcal{L}_{\tilde{X}^v} \Omega = 0 \tag{42}
\]

formally corresponding to \(p = 0\) and defines the generalized Lie derivative of an arbitrary form \(\omega\) with respect to the vertical-vector-valued form \(\tilde{X}^v\) as

\[
\mathcal{L}_{\tilde{X}^v} \omega := \tilde{X}^v \bigoplus d^v \omega - d^v (\tilde{X}^v \bigoplus \omega). \tag{43}
\]

Note that \(\mathcal{L}_{\tilde{X}^v}\) maps p-forms to \((p+1)\)-forms.

Taking \(\vec{F} = H_{\text{vol}}\), the components of the associated vector-valued form \(\tilde{X}_H\) may be found from eq. (38) to be

\[
\tilde{X}^a = \partial^a H, \quad \tilde{X}^i_{\delta i} = -\partial_i H. \tag{44}
\]

We see from eq. (44) that \(\tilde{X}_H\) is also suitable generalization, as \(\tilde{X}\) is, of the canonical Hamiltonian vector field in mechanics because using the natural parametrization of \(\tilde{X}_H\)

\[
\tilde{X}^V_k = \frac{\partial z^V}{\partial x^k} \tag{45}
\]

leads to the DW Hamiltonian field equations again. It is obvious that the horizontal counterpart of the vertical-vector-valued form \(\tilde{X}^v_H\) associated with \(H_{\text{vol}}\) is

\[
\tilde{X}^{\text{hor}} = \delta^i_k dx^k \otimes \partial_i. \tag{46}
\]
Now, the total exterior differential of the $p$-form has the form
\[ d\tilde{F}^p = dx^k \wedge \partial_k z^V \partial_V \tilde{F}^p + d^{hor} \tilde{F}^p, \]
and from eq. (41) it follows
\[ \{ H\tilde{vol}, \tilde{F}^p \} = \tilde{X}_V^k dx^k \wedge \partial_V \tilde{F}^p. \] (46)
Thus the DW canonical equations encoded in eqs. (44),(45) imply the following equation of motion of an arbitrary $p$-form
\[ d\tilde{F}^p = \{ H\tilde{vol}, \tilde{F}^p \} + d^{hor} \tilde{F}^p. \] (47)
The l.h.s. of eq. (47) generalizes the total time derivative in the equations of motion of a dynamical variable in mechanics, whereas its last term generalizes the partial time derivative. It is clear from the definition, eq. (43), that $d = \mathcal{L}_{\tilde{X}_t}^t$, where $\tilde{X}_t^t = \tilde{X}_t^v + \tilde{X}_t^{hor}$.

It should be noted that enlargening the set of the Hamiltonian multivector fields of Sect. 3 by the vector-valued one-forms associated with the Hamiltonian $n$-forms implies certain extension of the algebra of Hamiltonian (and LH) fields w.r.t. the SN bracket. Let us closer look at this extension. First, one defines the bracket of the vector-valued-one-forms and multivectors:
\[ [\tilde{X}, \tilde{X}]_\perp \Omega := \mathcal{L}_{\tilde{X}}(\tilde{X}_\perp \Omega). \] (48)
Therefore,
\[ [\tilde{X}_1, \tilde{X}_2]_\perp \Omega \in \Lambda^1_q, \] (49)
where $\Lambda^p_q$ denotes the space of vertical-$p$-vector-valued horizontal $q$-forms. Second, the bracket of the vector-valued forms would be natural to define by the equality
\[ [\tilde{X}_1, \tilde{X}_2]_\perp \Omega := \mathcal{L}_{\tilde{X}_1}(\tilde{X}_2_\perp \Omega), \] (50)
however, its r.h.s. is identically zero, as it follows from the formal degree counting. Hence, it does not actually define the bracket, but only indicates that
\[ [\tilde{X}_1, \tilde{X}_2] \in \Lambda^1_2. \] (51)
This is exactly the property which the Frölicher-Nijenhuis bracket of two vector-valued forms has \[^{40, 41}\]. Thus, it is natural to identify the bracket of two vector-valued forms with the FN bracket. As a result, the closure of the algebra will involve the vector-valued $p$-forms of all degrees $p \leq n$ because

$$[\tilde{X}_1, \tilde{X}_2]_{FN} \in \Lambda^1_{p+q}$$

if $\tilde{X}_1 \in \Lambda^1_p$ and $\tilde{X}_2 \in \Lambda^1_q$. Notice, that an appearance the of vector-valued forms of higher degrees does not lead to any extension of the algebra of Hamiltonian forms: for $\tilde{X} \in \Lambda^1_{p>1}$

$$\tilde{X} \int \Omega = 0, \quad (53)$$

where $\int$ denotes the FN inner product of a vector-valued $q$-form $\tilde{X}$ and a form $\omega$, defined as follows:

$$\tilde{X} \int \omega := X^V_{k_1...k_q} dx^{k_1} \wedge ... \wedge dx^{k_q} \wedge (\partial_V \int \omega). \quad (54)$$

It follows from eq. (53) that the vector-valued forms of the degree higher than one extend only the subalgebra of primitive Hamiltonian fields (see Sect.3).

Let us return now to the bracket in eq.(49): the bracket of a vector-valued form and a $p$-vector gives a $p$-vector-valued form. According to our scheme one should associate these objects with the ”Hamiltonian” (here in vague sense) forms via the polysymplectic form, and then to define somehow the corresponding brackets. However, the first task meets the problem of unique definition of the inner product of $\tilde{X} \in \Lambda^1_p$ with forms while the second one leads to the related problem of an appropirate definition of the ”Lie derivative” w.r.t. these $\tilde{X}$-s. Moreover, since ”most probably” the mutual brackets (yet to be properly defined) of those $\tilde{X}$-s will yield the elements from all the spaces $\Lambda^1_p$, we actually have to solve the same problems for arbitrary vertical-multivector-valued forms. Thus, the problem is essentially to construct a graded algebra of multivector-valued forms equipped with some appropriate bracket operation generalizing both the Schouten-Nijenhuis and the Frölicher-Nijenhuis bracket. This is in fact the ”well-known”
mathematical problem. However, recently A.M. Vinogradov has published his "unification theorem" [42] which states that SN and FN algebras may be imbedded in certain \( \mathbb{Z} \)-graded quotient algebra of the algebra of super-differential operators on the exterior algebra of forms. Although this result sounds highly relevant, the solution of the problem outlined above, which would be satisfactory for our purpose, as yet is not obtained by the author.

6 Several simple applications

6.1 Interacting scalar fields

As a simplest example of how the formalism we have constructed works, let us consider the system of interacting real scalar fields \( \{ \phi^a \} \) described by the Lagrangian density

\[
L = \frac{1}{2} \partial_i \phi^a \partial^i \phi_a - V(\phi^a).
\]

(55)

The DW canonical momenta are

\[
p^i_a := \frac{\partial L}{\partial (\partial_i \phi^a)} = \partial^i \phi_a
\]

and for the DW Hamiltonian function we easily obtain

\[
H = \frac{1}{2} p^i_a p^a_i + V(\phi).
\]

(57)

In terms of the canonically conjugate (in the sense of Sect. 4) variables \( \phi^a \) and \( \pi_a := p^i_a \partial_i \widetilde{vol} \) which have the following nonvanishing mutual Poisson bracket

\[
\{ \phi^a, \pi_b \} = -\delta^a_b,
\]

(58)

we can also write

\[
H \widetilde{vol} = \frac{1}{2} \sigma(\star \pi^a) \wedge \pi_a + V(\phi) \widetilde{vol}.
\]

(59)

Finally, the canonical DW equations may be written in the bracket form

\[
\begin{align*}
\text{d} \pi_a &= \{ H \widetilde{vol}, \pi_a \} = -\partial_a H \widetilde{vol}, \\
\text{d} \phi^a &= \{ H \widetilde{vol}, \phi^a \} = \sigma \star \pi^a,
\end{align*}
\]

(60)
which is equivalent to the field equations following from the Lagrangian (55):
\[ \Box \phi_a = -\partial_a V. \] (61)

6.2 The electromagnetic field

Let us start from the conventional Lagrangian density
\[ L = -\frac{1}{4} F_{ij} F^{ij} - j_i A^i, \] (62)
where \( F_{ij} := \partial_i A_j - \partial_j A_i \). For the canonical DW momenta we get
\[ \pi_{ij} := \frac{\partial L}{\partial (\partial_i A^j)} = -F_{ij} \] (63)
whence the primary constraints
\[ \pi^{ij} + \pi^{ji} = 0 \] (64)
follow. Despite the DW Legendre transformation is singular, we can define the canonical DW Hamiltonian function as usual:
\[ H = -\frac{1}{4} \pi_{ij} \pi^{ij} - j_i A^i. \] (65)

However, due to the constraints using this Hamiltonian in the DW Hamiltonian field equations leads to the incorrect equation \( \partial_i A^j = \partial H/\partial \pi_{ij} = \pi_i^j \): only its antisymmetric part is right. Usually the problems of this sort are handled by substituting the constraints with some Lagrange multiplies to the canonical Hamiltonian function and then applying the well known Dirac’s procedure. Our formalism offers another possibility based on a freedom in choosing the canonically conjugate variables which we have already mentioned in Sect.4.

Namely, let us try to use as a canonical field variable the one-form \( \alpha = A_i dx^i \) instead of the set of its components \( \{A_i\} \). Then the canonically conjugate momentum may be found to be the \((n-2)\)-form
\[ \pi := -F^{ij} \partial_i \int \partial_j \int \widetilde{vol} = \pi^{ij} \partial_i \int \partial_j \int \widetilde{vol}. \]
To see this let us calculate the Poisson bracket of the 1-form \( \alpha \) and the \((n-2)\)-form \( \pi \). Remark first, that the form \( \pi \) is Hamiltonian form as opposite to its dual 2-form \( F_{ij} dx^i \wedge dx^j \) which we might naively try to associate with \( \alpha \) as its conjugate momentum; moreover, the bracket of
the latter two forms would vanish for \( n > 4 \), as a simple degree counting shows (see eq. (22)). Further, the components of the \((n - 1)\)-vector field \( X_\alpha \) associated with \( \alpha \) are defined by

\[
X_\alpha \oint \Omega = d\alpha = dA_i \wedge dx^i,
\]

where

\[
\Omega = -dA_i \wedge d\pi^j \wedge (\partial_j \overline{vol}).
\]

For the only nonvanishing component of \( X_\alpha \) we get

\[
X^{j_1 \ldots j_{n-2}}_{i} \varepsilon^{i_1 \ldots i_{n-2} j k} = g^i_k,
\]

where the first column of indices is a single index corresponding to the direction \( \partial^i_j := \frac{\partial}{\partial \pi^j} \) in the tangent space of the DW phase space. The Poisson bracket of \( \alpha \) and \( \pi \) is easily obtained from its definition

\[
\{ \alpha , \pi \} = (-1)^{n-1} X_\alpha \oint d\pi = -1.
\]

This property justifies our choice of the canonically conjugate momentum of the one-form potential \( \alpha \).

In terms of new canonical variables \( \alpha \) and \( \pi \) the DW Hamiltonian \( n \)-form is expressed as

\[
H_{\tilde{vol}} = -\frac{1}{4} \sigma \pi \wedge (\ast \pi) - \alpha \wedge j,
\]

where \( j := j^i \partial_i \overline{vol} \) is the current density \((n - 1)\)-form. Now, the Maxwell equations acquire the following Hamiltonian form in terms of new variables and the Poisson brackets:

\[
d\alpha = \{ H_{\tilde{vol}} , \alpha \} = \ast^{-1} \pi,
\]

\[
d\pi = \{ H_{\tilde{vol}} , \pi \} = j.
\]

Thus we have obtained a covariant Hamiltonian formulation of Maxwell’s electrodynamics without recourse to the formalism of the fields with constraints. The constraints, both gauge and initial data, which, of course, did not disappear nowhere can be taken into account after the covariant Hamiltonian formulation was constructed.
6.3 The Nambu-Goto string

The classical dynamics of a string sweeping in space-time the world-sheet \( x^a = x^a(\sigma, \tau) \) is determined by the Nambu-Goto Lagrangian

\[
L = -T \sqrt{(\dot{x} \cdot \dot{x'})^2 - x'^2 \dot{x}^2} = -T \sqrt{-\det \parallel \partial_\tau x^a \partial_\sigma x_a \parallel},
\]

(70)

where \( \dot{x} = \partial_\tau x^a, x'^a = \partial_\sigma x^a, T \) is a string rest tension and we have also used the following notation for the world-sheet parameters \( (\sigma, \tau) = (\tau^0, \tau^1) := (\tau^i); i = 0, 1. \)

Define the canonical DW momenta:

\[
p_0^a := \frac{\partial L}{\partial \dot{x}^a} = T^2 \frac{(\dot{x}' \cdot \dot{x'}) x'_a - x^2 \dot{x}_a}{L},
\]

\[
p_1^a := \frac{\partial L}{\partial x'^a} = T^2 \frac{(\dot{x}' \cdot \dot{x}) x_a - \dot{x}^2 x'_a}{L},
\]

(71)

From eqs. (71) the following identities follow

\[
p_0^a x^a = 0, \quad (p_0^0)^2 + T^2 x^2 = 0, \]

\[
p_1^a x^a = 0, \quad (p_1^0)^2 + T^2 \dot{x}^2 = 0,
\]

(72)

however, they do not have a meaning of the Hamiltonian constraints within the DW canonical formalism since they do not imply any relations between the generalized coordinates \( x^a \) and the generalized momenta \( p_i^a \). In fact, with the help of these identities the eqs. (71) may be easily solved (if \( L \neq 0 \)) yielding the expression of the generalized velocities \( (\dot{x}, \dot{x}') \) in terms of the DW momenta; it proves de facto that the DW Legendre transform for the Nambu-Goto Lagrangian is regular! In terms of the DW momenta the DW Hamiltonian function takes the form

\[
H = -\frac{1}{T} \sqrt{-\det \parallel p_i^a p^a_j \parallel}
\]

and can also be expressed in terms of the 1-form momentum variables

\[
\pi_a := p_i^a \varepsilon_{ij} d\tau^j
\]

canonically conjugate (in the sense of Sect. 4) to \( x^a \). It is easily checked that

\[
\det \parallel p_i^a p^a_j \parallel = \frac{1}{2} (\varepsilon_{ij} p_i^a p^a_j)(\varepsilon_{ij} p^a_i p^b_j)
\]
and
\[ \varepsilon_{ijp}p_i^j = \star(\pi_a \wedge \pi_b). \]
The string equations of motion in terms of the Poisson brackets can be written now as
\[
\begin{align*}
\mathbf{d}x^a &= \{ H \mathrm{vol}, x^a \} = \frac{\partial H}{\partial p^i_a} d\tau^i, \\
\mathbf{d}\pi_a &= \{ H \mathrm{vol}, \pi_a \} = 0.
\end{align*}
\]
(74)

As yet another application we show how the Poincaré algebra is reproduced with the help of our brackets. In the \( x^a \)-space the translations are generated by the vector fields \( X_a := \partial_a \) and the Lorentz rotations by the bivectors \( X_{ab} := x_a \partial_b - x_b \partial_a \). The corresponding conserved current densities are the one-forms:
\[
\pi_a \quad \text{and} \quad \mu_{ab} := x_a \pi_b - x_b \pi_a,
\]
(75)
and from the string equations of motion it follows
\[
\mathbf{d}\pi_a = 0 \quad \text{and} \quad \mathbf{d}\mu_{ab} = 0.
\]
(76)

Now, a straightforward calculation of the Poisson brackets of these 1-forms yields:
\[
\begin{align*}
\{ \pi_a, \pi_b \} &= 0, \\
\{ \mu_{ab}, \pi_c \} &= g_{ac} \pi_b - g_{bc} \pi_a, \\
\{ \mu_{ab}, \mu_{cd} \} &= C_{abcd}^{ef} \mu_{ef},
\end{align*}
\]
(77)
where \( g_{ab} \) is the \( x \)-space metric and
\[
C_{abcd}^{ef} = -g_{ce}^e g_{bd}^f + g_{ce}^e g_{ad}^f + g_{ac}^e g_{bd}^f + g_{bc}^e g_{ad}^f
\]
are the Lorentz group structure constants. Thus the internal Poincaré symmetry of a string is represented with the help of the Poisson brackets of 1-forms corresponding to the conserved currents related to this symmetry.

7 Discussion

In this paper we have discussed a possible extension to the finite-dimensional De Donder-Weyl Hamiltonian formulation of field theory
of some of the structures of the classical Hamiltonian mechanics, in particular, those which are known (or commonly believed) to be important for canonical quantization. Although the symplectic form is explicitly needed only in the geometric quantization schemes, we started first from its appropriate field theoretical (within the DW framework) analogue since this seems to provide the only reliable basis for generalizations. Unlike the symplectic 2-form in mechanics \((n = 1)\) its field theoretical \((n\) dimensions) generalization in the present formalism, the polysymplectic \((n+1)\)-form which is determined from the Poincaré-Cartan canonical form, is not purely vertical. As a result, it determines the map between multivectors and forms of various degrees, which generalizes the map between vectors and functions given by the symplectic form in mechanics. However, our map is rather a map between the equivalence classes of Hamiltonian multivector fields modulo the addition of the primitive Hamiltonian fields, which are defined in Sect. 3, and the specific class of forms, called the Hamiltonian forms; the latter term in general implies certain limitation for the dependence of forms on the DW canonical momenta. The meaning of this limitation, which is obtained only as an analytic relation following from the consistency of the equation defining the components of the multivector associated with a given form, is not quite clear.

Introduction of the generalized Lie derivative of forms with respect to the multivector fields enables us to define the Poisson brackets of Hamiltonian forms which mix the forms of different degrees and are shown to be connected with the Schouten-Nijenhuis brackets of Hamiltonian multivector fields they are associated with. The brackets can be both odd or even depending on the degrees of their arguments and are proved to fulfil the graded derivation property and the graded Jacobi identity. Thus, the set of Hamiltonian multivector fields and the set of Hamiltonian forms possess a clear graded Lie algebra structure and, moreover, the Gerstenhaber algebra structure. One of the consequences of the "mixing property" of our brackets is that the natural pairs of canonically conjugate field and momentum variables consist typically of forms of different degrees whose components are field vari-
ables or DW momenta. We also show that the Poisson bracket with the DW Hamiltonian function generates the Hodge dual of the exterior differential on the space of Hamiltonian \((n-1)\)-forms, whereas the bracket with the \(n\)-form \(H\vDash = \star H\) generates the exterior differential of an arbitrary (Hamiltonian) form. This leads, in particular, to the representation of the De Donder-Weyl Hamiltonian field equations in the bracket form which is similar to that in mechanics where the time derivative of a dynamical variable is generated by the Poisson bracket with Hamilton’s function.

However, the \(n\)-forms, as the Hamiltonian forms, are associated with the vector-valued one-forms enlargening the space of Hamiltonian multivector fields. As we have argued, this implies a certain extension of the graded Lie algebra of Hamiltonian multivector fields. The algebraic closure of this extension involves the multivector-valued forms of all possible degrees and requires the appropriate definition of some bracket operation for these objects, which would extend the Lie, Schouten-Nijenhuis and Frölicher-Nijenhuis brackets. This problem is related to the problem of construction of the graded algebra of multivector-valued forms covering both SN and FN graded algebras. The techniques we have used in this paper did not allow us to construct this superalgebra; perhaps the recent unification theorem by A.M. Vinogradov [42] might be helpful in this connection. As a speculation, one could expect that taking into consideration of all the elements of this enlarged graded algebra can also lead to a certain extension of the algebra of Hamiltonian forms and possibly will allow us to weaken or avoid the restrictive analytical condition on the Hamiltonian forms. In other words, the question is whether one can associate the objects of more general nature, the multivector-valued forms, with the forms which are not Hamiltonian according to the definition of Sect.3. An extension of the class of Hamiltonian forms seems also to be desirable in view of the fact that the (horizontal) Hodge duals of the Hamiltonian forms which we seemingly need to consider together with the Hamiltonian forms, as the examples in Sect. 6 indicate, are not Hamiltonian in general [43].
Let us sketch the connections of the formalism presented here with the conventional instantaneous Hamiltonian formalism for fields (more detailed treatment will be presented elsewhere). Let us choose a space-like surface $\Sigma$ in the $x$-space (here we will assume it to be pseudo-Euclidean with the signature $++...+-$). The restrictions of the DW phase space variables to $\Sigma$ will be the functions of $x$-s. In particular, if $\Sigma$ is given by the equation $x^n = t$ ($n$ is the number of the time-like component of $\{x^i\} = \{x^1, ..., x^{n-1}, x^n\}$, not an index), we have $y^a|_\Sigma = y^a(x, t)$ and $p^n_a|_\Sigma = p^n_a(x, t)$, where $x$ denotes the space-like components of $\{x^i\}$.

Moreover, the restriction of forms to $\Sigma$ implies setting $dx^n = 0$, so that for $p^a := p^i_a \partial_i \tilde{vol}$ we have $p^a|_\Sigma = p^n_a(x, t) \partial_n \tilde{vol}$, where $\partial_n \tilde{vol}$ is obviously the $(n - 1)$-volume form on $\Sigma$, which we shall denote as $dx$. The functional symplectic 2-form $\omega$ on the phase space of the instantaneous formalism may be related now to the restriction of the polysymplectic form $\Omega$ to $\Sigma$ in the following way (cf. [9, 10]):

$$\omega = \int_\Sigma (\Omega^n|_\Sigma) = - \int_\Sigma dy^n(x) \wedge dp^n_a(x)dx.$$

Then, the equal-time Poisson bracket of $y^a(x)$ with the canonical conjugate momentum $p^a_n(x)$:

$$\{y^a(x), p^a_n(y)\}_{PB} = \delta^a_n \delta(x - y)$$

may be related to the Poisson bracket of the canonically conjugate variables $y^a$ and $p_a$ of the DW theory (see Sect. 4) as follows:

$$\int_{\Sigma_x} \int_{\Sigma_y} \{y^a(x), p^a_n(y)\}_{PB} f(x)f(y)dxdy = - \int_\Sigma \{y^a, p_a\}f(x)dxdy,$$

where $f(x)$ is a test function. In general, one can anticipate the following relationship between the generalized Poisson bracket of Hamiltonian forms and the equal-time Poisson bracket of their restrictions to the space-like surface $\Sigma$:

$$\int_{\Sigma_x} \int_{\Sigma_y} \phi_1(x) \wedge (\tilde{F}_1|_{\Sigma_x}(x), \tilde{F}_2|_{\Sigma_y}(y))_{PB} \wedge \phi_2(y) \sim \int_\Sigma \phi_1 \wedge \{ \tilde{F}_1, \tilde{F}_2 \} \wedge \phi_2,$$

where $\phi_1$ and $\phi_2$ denote the "test forms" of degree $(n - p - 1)$ and $(n-q-1)$ respectively and the standart Poisson bracket $\{ , \}_{PB}$ of forms is defined via the Poisson brackets of their components. This formula reproduces, in particular, the canonical equal-time Poisson brackets.
from the generalized Poisson brackets of the admissible pairs of canonically conjugate variables (see the end of Sect. 4). However, it fails to reproduce some of the Poisson brackets of interest in field theory from the generalized Poisson brackets of Hamiltonian forms, although all such examples known to the author are related to the quantities which are not Hamiltonian forms. This is another indication that our canonical scheme should be extended.

It is interesting to note in conclusion that the algebraic structures arised in our formalism are cognate with those appearing in the BRST-inspired approaches in field theory, in particular, in the antibracket formalism (see for example [44]). The latter, of course, are established within the functional framework which is conceptually different from the spirit of this paper. Nevertheless, deeper relationship rather than a superficial algebraic analogy may be expected in view of the connection discussed above between the usual Poisson brackets and those suggested in this paper; in this case one could hope to clarify the geometrical origin of the BRST formalism. It is worthy of noting in this connection that the Gerstenhaber algebra structure which we have found for graded Poisson bracket algebra of Hamiltonian forms has appeared recently also in the discussion of the BRST-algebraic structure of string theory [45].

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