Certain properties of bounded variation of sequences of fuzzy numbers by using generalized weighted mean

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ABSTRACT

The class $bv_{F}(u, v)$ of bounded variation sequences of fuzzy numbers, introduced by Ojha and Srivastava (2015), has been investigated further using generalized weighted mean matrix $G(u, v)$. On imposing certain restrictions on the matrix $G(u, v)$, we have established its relation with different class of sequences of fuzzy numbers such as statistically null difference sequences and Cesáro sequences. We have examined the concepts of equivalent fuzzy number and symmetric fuzzy number over the quasi-linear space $bv_{F}(u, v)$.

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1. Introduction

The concept of fuzzy set and fuzzy set operations was first introduced by Zadeh (1965). Subsequently, several mathematicians have studied extensively different aspects of fuzzy number theory, fuzzy analysis, fuzzy topology, fuzzy measure, fuzzy decision making, etc.

In recent years, there has been an increasing interest in various mathematical aspects of operations defined on fuzzy sets. Based on these, many authors have introduced different new classes of sequences of fuzzy numbers which can be enumerated as below.

1. The set of all convergent sequences, $c_{F}$ is introduced by Matloka (1986) and convergent difference sequences, $c_{F}^{p}(\Delta)$ by Savas (2000).

2. Nanda (1989) defined the set of all $p$-summable sequences, $\ell_{F}^{p}(1 \leq p < \infty)$ and bounded sequences, $\ell_{F}^{\infty}$ which are further investigated by Talo and Basar (2009).

3. $p$-summable and bounded difference sequence, $\ell_{F}^{p}(\Delta)$ for $1 \leq p < \infty$ and $p = \infty$, are introduced by Mursaleen and Basarir (2003).

4. Nuray and Savas (1995) has introduced the concept of statistically convergent sequences, $S_{F}^{e}$ and statistically null sequences, $S_{F}^{0}$ for fuzzy numbers, whereas statistically null difference sequences, $S_{F}^{0}(\Delta)$ is defined by Bilgin (2003).

5. The set of all sequences of bounded variation, $bv_{F}$ is examined by several authors such as Talo and Basar (2008), Tripathy and Dutta (2010, 2013).

In some other developments, Ojha and Srivastava (2015) have introduced the notion of bounded variation over the class $\lambda_{F}(u, v)$ by defining a new class $bv_{F}(u, v)$ of sequences of fuzzy numbers using generalized weighted mean matrix $G(u, v)$ and studied various topological properties of this class.

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The aim of this paper is to investigate various relations of this class with other classes of sequences of fuzzy numbers, e.g., class of statistically null difference sequences, class of Cesáro sequences. The concepts like symmetric fuzzy number and equivalent fuzzy number are also discussed.

2. Definitions and preliminaries

**Definition 1:** A fuzzy real number \( X : \mathbb{R} \to [0,1] \) is a fuzzy set which is normal, fuzzy convex, upper semi-continuous and the support \( X^0 = \{ t \in \mathbb{R} : X(t) > 0 \} \) is compact.

Clearly, \( \mathbb{R} \) is embedded in \( L(\mathbb{R}) \), the set of all fuzzy numbers, in the following way:

For each \( r \in \mathbb{R} \), \( r \in L(\mathbb{R}) \) is defined as follows:

\[
\overline{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r. \end{cases}
\]

For \( 0 < \alpha \leq 1 \), \( \alpha \)-cut of a fuzzy number \( X \) is defined by \( X(\alpha) = \{ t \in \mathbb{R} : X(t) \geq \alpha \} \). It is easy to check that the set \( X(\alpha) \) is a closed, bounded, and non-empty interval for each \( \alpha \in [0,1] \).

For any two fuzzy numbers \( X, Y \), Matloka (1986) has shown that \( L(\mathbb{R}) \) is a complete metric space under the metric

\[
d(X, Y) = \sup_{0 \leq \alpha \leq 1} \max \left\{ |X(\alpha) - Y(\alpha)|, |\overline{X}(\alpha) - \overline{Y}(\alpha)| \right\}
\]

where \( X(\alpha) \) and \( \overline{X}(\alpha) \) are the lower and upper bounds of the \( \alpha \)-cut.

Goetschel and Voxman (1986) has introduced the following theorem for fuzzy numbers and their respective \( \alpha \)-cuts.

**Theorem 1 (Representation Theorem):** Let \( X(\alpha) = [X^L(\alpha), X^U(\alpha)] \) for \( X \in L(\mathbb{R}) \) and for each \( \alpha \in [0,1] \). Then, the following statements hold:

1. \( X^L(\alpha) \) is a bounded and non-decreasing left continuous function on \( (0,1] \).
2. \( X^U(\alpha) \) is a bounded and non-increasing left continuous function on \( (0,1) \).
3. The functions \( X^L(\alpha) \) and \( X^U(\alpha) \) are right continuous at the point \( \alpha = 0 \).
4. \( X(1) \leq \overline{X}(1) \).

Conversely, if a pair of real functions \( P \) and \( Q \) satisfies the above conditions (1)–(4), then there exists a unique \( X \in L(\mathbb{R}) \) such that \( X(\alpha) = \{ P(\alpha), Q(\alpha) \} \) for each \( \alpha \in [0,1] \). The fuzzy number \( X \) corresponding to the pair of functions \( P \) and \( Q \) is defined by \( X : \mathbb{R} \to [0,1] \), \( X(t) = \sup\{ \alpha : P(\alpha) \leq t \leq Q(\alpha) \} \).

Thus, for every fuzzy number \( X \in L(\mathbb{R}) \), we can have \( d(X, \overline{0}) = \max\{|X^L|, |\overline{X}^0|\} \). Let \( w^F \) be the set of all sequences of fuzzy numbers.

The concept of generalized weighted mean is given as follows.

Let \( U \) be the set of all real sequences \( u = (u_n) \) such that \( u_n \neq 0 \) for all \( n \in \mathbb{N} \). Then, consider the matrix \( G(u, v) = (g_{nk}) \), where
\[(g_{nk}) = \begin{cases} u_n v_k, & 1 \leq k \leq n \\ 0, & k > n, \end{cases}\]

i.e. \(G(u, v)\) is of the form
\[
\begin{pmatrix}
u_1 & 0 & 0 & \cdots & 0 \\
u_2 v_1 & u_2 v_2 & 0 & \cdots & 0 \\
u_3 v_1 & u_3 v_2 & u_3 v_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_n v_1 & u_n v_2 & u_n v_3 & \cdots & u_n v_n
\end{pmatrix}
\]

for all \(k, n \in \mathbb{N}\), where \(u_n\) depends only on \(n\) and \(v_k\) depends only on \(k\). The matrix \(G(u, v)\), defined above, is called as generalized weighted mean matrix or factorable matrix. With this matrix, Polat, Karakaya, and Simsek (2011) have defined some new sequence spaces of real or complex numbers and established various properties of it.

Motivated by their work, Ojha and Srivastava (2014) defined a new class of sequences of fuzzy numbers with the help of the generalized mean matrix \(G(u, v)\) as follows:
\[
\lambda^F(u, v) = \left\{ X = (X_k) \in w^F : (y_k) \in \lambda \right\}
\]
where the scalar \(y_n = G(u, v)\)
\[
\begin{pmatrix}
\delta(X_1, \tilde{0}) \\
\delta(X_2, \tilde{0}) \\
\delta(X_3, \tilde{0}) \\
\vdots \\
\delta(X_n, \tilde{0})
\end{pmatrix}
\]

In other words
\[
\lambda^F(u, v) = \left\{ X = (X_k) \in w^F : \left( \sum_{i=1}^{k} u_i v_i \delta(X_i, \tilde{0}) \right) \in \lambda \right\}
\]
for \(\lambda = c, c_0, \ell_\infty, \ell_p (1 \leq p < \infty)\).

Further the quasi-linear space \(bv^F(u, v)\), bounded variation of sequences of fuzzy numbers using generalized weighted mean, is introduced by Ojha and Srivastava (2015) as follows:
\[
bv^F(u, v) = \left\{ X = (X_k) \in w^F : \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_i v_i \delta(\Delta X_i, \tilde{0}) \right| < \infty \right\}
\]
where \(\Delta X_i = X_i - X_{i+1}\) and established many topological properties such as completeness, solidness, and symmetry for it. Here, in this paper, we investigate some inclusion relations of the class \(bv^F(u, v)\) with other known sets of sequences of fuzzy numbers, defined earlier, by imposing conditions on \(u, v\) or on the matrix \(G(u, v)\).

Note 1: The notation \(\lambda^F\) is used to differentiate the fuzzy sense from the usual one \(\lambda\).

Note 2: Clearly, the \(k\)th row sum of \(G(u, v)\) is \(\sum_{i=1}^{k} u_i v_i\).

3. Main results

Theorem 2:

1. The set \(\ell^F_p \subset bv^F(u, v)\) if \(v = (v_i) \in \ell_q\) and \(u = (u_i) \in \ell_1\) for \(1 < p < \infty\), where \(1/p + 1/q = 1\).
(2) For $p = 1$ and $p = \infty$, $\ell_p^F \subset bv^F(u, v)$ if the row sum of $G(u, v)$ is in $\ell_1$.

The inclusions are strict in both cases.

Proof:

(1) Let $(X_k) \in \ell_p^F$ and $(v_i) \in \ell_q$. Then, $\exists M_1, M_2 > 0$ such that $\sum_{k=1}^{\infty} d(X_k, \bar{0})^p = M_1$ and $\sum_{i=1}^{\infty} |v_i|^q = M_2$. Now,

$$
\left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \leq \left| \sum_{i=1}^{k} u_k v_i d(X_i, \bar{0}) \right| + \left| \sum_{i=1}^{k} u_k v_i d(X_{i+1}, \bar{0}) \right|
$$

$$
\leq |u_k| \sum_{i=1}^{k} |v_i| d(X_i, \bar{0}) + |u_k| \sum_{i=1}^{k} |v_i| d(X_{i+1}, \bar{0})
$$

$$
\leq |u_k| \left[ \left( \sum_{i=1}^{k} |v_i|^q \right)^{1/q} \left( \sum_{i=1}^{k} d(X_i, \bar{0})^p \right)^{1/p} \right]
$$

$$
+ \left( \sum_{i=1}^{k} |v_i|^q \right)^{1/q} \left( \sum_{i=1}^{k} d(X_{i+1}, \bar{0})^p \right)^{1/p}
$$

(Using Hölder’s inequality)

$$
\leq |u_k| \left( \sum_{i=1}^{k} |v_i|^q \right)^{1/q} 2M_1^{1/p}
$$

$$
\leq 2M_1^{1/p} M_2^{1/q} |u_k|
$$

Taking sum over $k = 1$ to $\infty$ and assuming $(u_i) \in \ell_1$, we have $(X_k) \in bv^F(u, v)$. This proves $\ell_p^F \subset bv^F(u, v)$ for $1 < p < \infty$.

To prove that the inclusion is strict, let us consider the following example.

Example 1: Let $u_k = \frac{1}{k^{3/2}}$ and $v_k = \frac{1}{k}$ for all $k \geq 1$. Choose $p = q = 2$. Then $(u_k) \in \ell_1$ and $(v_i) \in \ell_2$. Let

$$
X_i = \bar{i} \quad \text{for all } i \geq 1
$$

Then, $d(X_i, \bar{0}) = i$ and so $(X_i) \notin \ell_p^F$.

But $d(\Delta X_i, \bar{0}) = 1$ for all $i \geq 1$ and so

$$
\sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) = \frac{1}{k^{3/2}} \sum_{i=1}^{k} \frac{1}{i} = \frac{1}{k^{3/2}} \left[ 1 + \frac{1}{2} + \cdots + \frac{1}{k} \right]
$$

So $(X_k) \in bv^F(u, v)$.

(2) The case $p = 1$ and $p = \infty$ follows almost on the same lines. So, we omit it.

\[ \square \]

Note 3: The conditions on $u, v$ are necessary for the inclusion relation. One cannot drop any one of the conditions. To prove this, consider the following examples. For simplicity, take $p = q = 2$.

Example 2: Let $(u_i) \in \ell_1$ but $(v_i) \notin \ell_q$, then we will show that $\ell_p^F \not\subset bv^F(u, v)$. 

\[ \square \]
Take $u_i = \frac{1}{t^2}$ and $v_i = i$ for all $i \in \mathbb{N}$. Now define the sequence $(X_i)$ as follows:

$$X_i(t) = \begin{cases} 1 + it, & -\frac{1}{i} \leq t \leq 0 \\ 1 - it, & 0 \leq t \leq \frac{1}{i} \\ 0, & \text{otherwise} \end{cases}$$

when $i$ is odd

otherwise.

Then, $d(X_i, \bar{0}) = \frac{1}{t}$ for all odd $i$ and 0 elsewhere. Therefore, $(X_i) \in \ell^F_2$. We have to show that $(X_i) \notin bv^F(u, v)$.

For the above $(X_i)$, we have

$$d(\Delta X_i, \bar{0}) = \begin{cases} i^{-1}, & \text{when $i$ is odd} \\ (i + 1)^{-1}, & \text{otherwise} \end{cases}$$

Thus

$$\sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) = \frac{1}{k^2} \left[ \sum_{i=1, i=\text{odd}}^{k} i \frac{1}{i} + \sum_{i=1, i=\text{even}}^{k} i \frac{1}{i + 1} \right]$$

$$\geq \frac{1}{k^2} \frac{[k + 1]}{2} \geq \frac{1}{2k}.$$

Therefore, $(X_i) \notin bv^F(u, v)$ and thus $\ell^F_2 \not\subset bv^F(u, v)$.

**Example 3:** Now let $(u_i) \notin \ell_1$ but $(v_i) \in \ell_q$. To show that $\ell^F_p \not\subset bv^F(u, v)$.

Choose $u_i = i$ and $v_i = \frac{1}{i}$. Let us take $(X_i)$ as follows:

$$X_i = \begin{cases} \left(\frac{1}{i}\right), & \text{whenever $i = n^2$ for some integer $n$} \\ \bar{0}, & \text{otherwise} \end{cases}$$

Then, $d(X_i, \bar{0}) = \frac{1}{i}$ whenever $i$ is a square integer and 0 otherwise and so $(X_i) \in \ell^F_2$. Now

$$d(\Delta X_i, \bar{0}) = \begin{cases} \frac{1}{i^2}, & \text{whenver $i = n^2$ or $i = n^2 - 1$ for some integer $n$} \\ 0, & \text{otherwise} \end{cases}$$

Thus

$$\sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) = k \sum_{i=n^2, n^2 - 1}^{k} \frac{1}{i^2}$$

$$= \begin{cases} k \left[ 1 + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k^2} \right], & k = n^2, n^2 - 1 \\ k \left[ 1 + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{p^2} \right], & \text{otherwise} \end{cases}$$

where $p$ is the nearest square number from $k$ and $p < k$. Taking sum over $k = 1$ to $\infty$, we can see $(X_i) \notin bv^F(u, v)$.

**Corollary 1:** $\ell^F_p(\Delta) \subset bv^F(u, v)$ by taking the same above conditions for $1 \leq p \leq \infty$.

**Theorem 3:** The limit of a sequence in $c^F(\Delta)$ is preserved under the metric in $bv^F(u, v)$ if the row sum of the matrix $G(u, v)$ are in $\ell_1$.
**Proof:** According to Savas (2000), $c^F(\Delta)$ is a complete metric space under the metric

\[ \rho(X, Y) = d(X_1, Y_1) + \sup_i d(\Delta X_i, \Delta Y_i) \text{ for } X, Y \in c^F(\Delta) \]

Ojha and Srivastava (2015) have established that $bv^F(u, v)$ is a complete metric space under the metric

\[ D(X, Y) = |u_1 v_1 d(X_1, Y_1)| + \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \Delta Y_i) \right| \text{ for } X, Y \in bv^F(u, v). \]

Let $X^n = (X^n_i)$ be a sequence in $c^F(\Delta)$ converging to $X = (X_i)$. Then, from the above metric $\rho$, we have

\[ \rho(X^n, X) \to 0 \text{ as } n \to \infty \]

i.e. $d(X^n_i, X_i) + \sup_i d(\Delta X^n_i, \Delta X_i) \to 0 \text{ as } n \to \infty$

So $d(X^n_i, X_i) \to 0$ and $\sup_i d(\Delta X^n_i, \Delta X_i) \to 0 \text{ as } n \to \infty$ \hspace{1cm} (1)

Since $c^F(\Delta) \subset l^F_{\infty}(\Delta)$ and the row sum of the matrix $G(u, v)$ are in $\ell_1$, so $(X^n_i), (X_i) \in bv^F(u, v)$ by Corollary 1. Let \( \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i \right| = M. \)

Since $(u_i), (v_i) \in U$, hence from (1), it follows $u_1 v_1 d(X^n_i, X_i) \to 0 \text{ as } n \to \infty$ i.e. for some $\varepsilon > 0$, there exist an integer $n_0$ such that

\[ u_1 v_1 d(X^n_i, X_i) < \frac{\varepsilon}{2} \text{ for all } n \geq n_0. \]

Also, since $\sup_i d(\Delta X^n_i, \Delta X_i) \to 0 \text{ as } n \to \infty$ from (1), so there exists an integer $n_1 \in \mathbb{N}$ such that

\[ \sup_i d(\Delta X^n_i, \Delta X_i) < \frac{\varepsilon}{2M} \text{ for all } n \geq n_1. \]

So for all $n \geq \max\{n_0, n_1\}$, we have

\[ D(X^n, X) = |u_1 v_1 d(X^n_1, X_1)| + \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i d(\Delta X^n_i, \Delta X_i) \right| \]

\[ \leq \frac{\varepsilon}{2} + \sup_i d(\Delta X^n_i, \Delta X_i) \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i \right| \]

\[ \leq \frac{\varepsilon}{2} + M \frac{\varepsilon}{2M} = \varepsilon. \]

So, we conclude $D(X^n, X) \to 0 \text{ as } n \to \infty$, i.e. $X^n \to X$ as $n \to \infty$ under the metric $D$ in $bv^F(u, v)$. This completes the proof. \( \square \)
Theorem 4: If \( \inf_{k, i} u_k v_i > 0 \), then \( bv^F(u, v) \subset S_0^F(\Delta) \).

Proof: Bilgin (2003) has shown that a sequence of fuzzy numbers \( (X_k) \) will belong to \( S_0^F(\Delta) \) if

\[
\frac{1}{n} \left| \left\{ k \leq n : d(\Delta X_i, \bar{0}) \geq \varepsilon \right\} \right| \to 0 \quad \text{as} \quad n \to \infty
\]

where \( |\cdot| \) means the cardinality of the enclosed set.

Let \( (X_k) \in bv^F(u, v) \), so \( \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| = M \) (say). Now for a given \( \varepsilon > 0 \) and \( n \in \mathbb{N} \),

\[
M \geq \sum_{k=1}^{n} \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \left| \left\{ k \leq n : \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \right\} \right|. \quad (2)
\]

Let \( \inf_{k, i} u_k v_i = m \). Then, \( m > 0 \). So, we have

\[
\left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq m \sum_{i=1}^{k} d(\Delta X_i, \bar{0})
\]

and so \( \left\{ k \leq n : m \sum_{i=1}^{k} d(\Delta X_i, \bar{0}) \geq \varepsilon \right\} \subseteq \left\{ k \leq n : \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \right\} \)

i.e. \( \left\{ k \leq n : \sum_{i=1}^{k} d(\Delta X_i, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \subseteq \left\{ k \leq n : \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \right\} \). \quad (3)

Also, \( \sum_{i=1}^{k} d(\Delta X_i, \bar{0}) \geq d(\Delta X_k, \bar{0}) \) for each \( k \). Thus, we get

\[
\left\{ k \leq n : d(\Delta X_k, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \subseteq \left\{ k \leq n : \sum_{i=1}^{k} d(\Delta X_i, \bar{0}) \geq \frac{\varepsilon}{m} \right\}. \quad (4)
\]

Combining (3) and (4), we get

\[
\left\{ k \leq n : d(\Delta X_k, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \subseteq \left\{ k \leq n : \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \right\}
\]

\[
\Rightarrow \left| \left\{ k \leq n : d(\Delta X_k, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \right| \leq \left| \left\{ k \leq n : \left| \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \right| \geq \varepsilon \right\} \right|.
\]

Thus, multiplying the above by \( \varepsilon \) and with the help of (2), we get

\[
\varepsilon \left| \left\{ k \leq n : d(\Delta X_k, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \right| \leq M
\]

\[
\varepsilon \cdot \frac{1}{n} \left| \left\{ k \leq n : d(\Delta X_k, \bar{0}) \geq \frac{\varepsilon}{m} \right\} \right| \leq \frac{M}{n} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( \varepsilon > 0 \) is arbitrary, so we can conclude that \( (\Delta X_n) \) is statistically convergent to \( \bar{0} \) as \( n \to \infty \). This completes the proof. \( \Box \)
Theorem 5: Let \((X_k)\) be a sequence of fuzzy numbers which is Cesáro summable to some \(L \in L(R)\), then \((X_k) \in \text{bv}^F(u, v)\) if the row sum of the matrix \(G(u', v)\) is in \(l_1\) where \(u' = (u'_k) = (2k + 1)u_k\).

Proof: Since \((X_k)\) is Cesáro summable to \(L\) i.e. \(\frac{1}{n} \sum_{i=1}^{n} d(X_i, L) \to 0\) as \(n \to \infty\), so for some integer \(k_0\) such that

\[
\frac{1}{k} \sum_{i=1}^{k} d(X_i, L) \leq \varepsilon \quad \forall k \geq k_0.
\]

So, \(\frac{1}{k} \sum_{i=1}^{k} d(X_i, \bar{0}) \leq \frac{1}{k} \sum_{i=1}^{k} [d(X_i, L) + d(L, \bar{0})]
\]

\[
\leq \varepsilon + d(L, \bar{0}) \text{ whenever } k \geq k_0
\]

i.e. \(\sum_{i=1}^{k} d(X_i, \bar{0}) \leq k \varepsilon + d(L, \bar{0})\) whenever \(k \geq k_0\).

Now for all \(k \geq k_0\)

\[
\left| \sum_{i=1}^{k} u_kv_i d(\Delta X_i, \bar{0}) \right| 
\leq \left| \sum_{i=1}^{k} u_kv_i d(X_i, \bar{0}) \right| + \left| \sum_{i=1}^{k} u_kv_i d(X_{i+1}, \bar{0}) \right|
\]

\[
\leq \sum_{i=1}^{k} u_k \left| \sum_{i=1}^{k} d(X_i, \bar{0}) \right| + \sum_{i=1}^{k} u_k \left| \sum_{i=2}^{k} d(X_i, \bar{0}) \right|
\]

\[
\leq |u_k| \left| \sum_{i=1}^{k} v_i (\varepsilon + d(L, \bar{0})) \right| + |u_k| \left| \sum_{i=1}^{k} v_i (k + 1)(\varepsilon + d(L, \bar{0})) \right|
\]

\[
\leq (\varepsilon + d(L, \bar{0})) |u_k| \left| \sum_{i=1}^{k} v_i (2k + 1) \right|
\]

\[
= (\varepsilon + d(L, \bar{0})) |(2k + 1)u_k| \left| \sum_{i=1}^{k} v_i \right|
\]

\[
= (\varepsilon + d(L, \bar{0})) |(2k + 1)u_k \sum_{i=1}^{k} v_i|
\]

Now considering, \(u'_k = (2k + 1)u_k\), we get our required result. \(\square\)

4. Equivalent fuzzy numbers on \(\text{bv}^F(u, v)\)

Now, we list some basic definitions of fuzzy numbers as discussed in Qiu and Weiquan (2013) in order to get some relation for the class \(\text{bv}^F(u, v)\).
Definition 2:

1. A fuzzy number \( S \in L(R) \) is said to be symmetric if \( S(t) = S(-t) \) for all \( t \in \mathbb{R} \), i.e. \( S = -S \).
2. For \( X, Y \in L(R) \), the fuzzy number \( X \) is said to be equivalent to \( Y \) or \( X \sim Y \) if and only if there exists two symmetric fuzzy numbers \( S_1, S_2 \) such that

\[
X + S_1 = Y + S_2
\]

3. The midpoint function \( X_M : [0, 1] \rightarrow \mathbb{R} \) of a fuzzy number \( X \) is defined by assigning the mid point of each \( \alpha \) - level sets to \( X_M(\alpha) \) for all \( \alpha \in [0, 1] \) i.e.

\[
X_M(\alpha) = \frac{X(\alpha) + X(\alpha)}{2}
\]

It is interesting to note that the midpoint function \( X_M \) is actually a gradual number.

With the above definitions, Qiu et al. (2014) have proved the following theorem.

**Theorem 6:** For any \( X, Y \in L(R) \), \( X \sim Y \Leftrightarrow X_M = Y_M \).

On the basis of these above concepts, now we give some interesting result on the class \( bv^F(u, v) \).

**Theorem 7:** Let \( (X_i), (Y_i) \) be two sequences of fuzzy numbers such that \( X_i \sim Y_i \), i.e. \( X_i + S_i = Y_i + S_i' \), where \( S_i, S_i' \) are symmetric fuzzy numbers. If \( S_i, S_i' \in \ell_1^F \) and the row sum of \( G(u, v) \) are in \( \ell_1 \), then \( (X_i) \in bv^F(u, v) \Leftrightarrow (Y_i) \in bv^F(u, v) \).

**Proof:** It is enough to prove it for one-sided implication. Let \( (X_i) \in bv^F(u, v) \). Since \( (S_i) \in \ell_1^F \), so \( (S_i) \in bv^F(u, v) \) if the row sum of \( G(u, v) \) are in \( \ell_1 \).

\[
\Rightarrow (X_i + S_i) \in bv^F(u, v) \text{ as } bv^F(u, v) \text{ is closed under addition.}
\]

\[
\Rightarrow (Y_i + S_i') \in bv^F(u, v) \text{ since } X_i + S_i = Y_i + S_i'.
\]

Now, we only have to show that \( (Y_i) \in bv^F(u, v) \).

\[
\left| \sum_{i=1}^{k} u_k v_i d(\Delta Y_i, \bar{0}) \right| \leq \left| \sum_{i=1}^{k} u_k v_i d(\Delta Y_i, \Delta (Y_i + S_i')) \right| + \left| \sum_{i=1}^{k} u_k v_i d(\Delta (Y_i + S_i'), \bar{0}) \right|
\]

\[
\leq \left| \sum_{i=1}^{k} u_k v_i d(\bar{0}, \Delta S_i') \right| + \left| \sum_{i=1}^{k} u_k v_i d(\Delta (Y_i + S_i'), \bar{0}) \right|
\]

since \( d(X + Z, Y + Z) = d(X, Y) \) where \( X, Y, Z \in L(R) \). Also as \( (S_i') \in \ell_1^F \) and so \( (S_i') \in bv^F(u, v) \) by given condition. Therefore from the above inequality we get

\[
\sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} u_k v_i d(\Delta Y_i, \bar{0}) \right| < \infty
\]

i.e. \( (Y_i) \in bv^F(u, v) \)

The converse implication follows the same lines. This proves the theorem. \( \square \)
Theorem 8: Let \((X_{M,i})\) be the sequence of midpoint function of the sequence of fuzzy numbers \((X_i)\) for each \(i\). Then \((X_i) \in bv^F(u, v) \implies (\hat{X}_{M,i}) \in bv^F(u, v)\) where \(\hat{X}_M\) is the fuzzy embedding of \(X_M\) in the sense that \(X_M(\alpha) = \hat{X}_M^\alpha\) for each \(\alpha \in [0, 1]\).

**Proof:** For any fuzzy number \(X\), we know \(X_M(\alpha) = X(\alpha) = \frac{X^0 + X^\alpha}{2}\) and \(X_M(\alpha) = \hat{X}_M^\alpha\) i.e. \(\alpha\)-cut of \(\hat{X}_M\). Also, from the definition of fuzzy number, it is clear that \(d(X, \bar{0}) = \max\{|X^0|, |X^\alpha|\}\). So

\[
d(\hat{X}_M, \bar{0}) = \left| \frac{X^0 + X^\alpha}{2} \right|
\]

As the upper and lower cut are same for \(\hat{X}_M\)

\[
d(\hat{X}_M, \bar{0}) \leq \max\{|X^0|, |X^\alpha|\} = d(X, \bar{0}) \tag{5}
\]

Now for a sequence of midpoint fuzzy numbers \((\hat{X}_{M,i})\) of \((X_i)\), we can have

\[
\Delta \hat{X}_{M,i} = \hat{X}_{M,i} - \hat{X}_{M,i+1} \quad (\Delta \hat{X}_{M,i})^{(\alpha)} = (\hat{X}_{M,i})^{(\alpha)} - (\hat{X}_{M,i+1})^{(\alpha)}
\]

\[
= \frac{X_i^{(\alpha)} + X_i^{(\alpha)}}{2} - \frac{X_{i+1}^{(\alpha)} + X_{i+1}^{(\alpha)}}{2}
\]

\[
= \frac{X_i^{(\alpha)} - X_{i+1}^{(\alpha)}}{2} + \frac{X_{i+1}^{(\alpha)} - X_i^{(\alpha)}}{2}
\]

\[
= \frac{(\Delta X_i)^{(\alpha)}}{2} + \frac{(\Delta X_{i+1})^{(\alpha)}}{2}
= [(\Delta \hat{X}_i)_M]^\alpha
\]

and so

\[
d(\Delta \hat{X}_{M,i}, \bar{0}) = d((\Delta \hat{X}_i)_M, \bar{0})
\]

\[
\sum_{i=1}^{k} u_k v_i d(\Delta \hat{X}_{M,i}, \bar{0}) \leq \sum_{i=1}^{k} u_k v_i d((\Delta \hat{X}_i)_M, \bar{0}) \leq \sum_{i=1}^{k} u_k v_i d(\Delta X_i, \bar{0}) \quad \text{Follows from (6)}
\]

From the above inequality it is clear that if \((X_i) \in bv^F(u, v)\), then \((\hat{X}_{M,i}) \in bv^F(u, v)\).

**Remark 1:** The converse of the above is not true in general, i.e. \((\hat{X}_{M,i}) \in bv^F(u, v)\) for some \((X_i)\) does not imply \((X_i) \in bv^F(u, v)\). To show this, let us take \((X_k)\) as follows:

\[
X_k(t) = \begin{cases} 
\frac{t+k}{k}, & -k \leq t < 0 \\
\frac{k-t}{k}, & 0 \leq t \leq k \\
0, & \text{otherwise}
\end{cases}
\]

Then, we have \(\hat{X}_{M,k} = \bar{0}\) for all \(k\) (due to symmetry). So, for any choice of \((u, v)\), \((X_{M,i}) \in bv^F(u, v)\).
Now
\[
\Delta X_k(t) = \begin{cases} 
\frac{t+2k+1}{2k+1}, & -2k - 1 \leq t \leq 0 \\
\frac{2k+1-t}{2k+1}, & 0 \leq t \leq 2k + 1 \\
0, & \text{otherwise}
\end{cases}
\]
and therefore \(d(\Delta X_k, \tilde{0}) = 2k + 1\) which does not belong to \(bv^F(u, v)\) for some suitable choice of \((u, v)\).

It is well known that equivalent fuzzy numbers have same midpoint. But due to the fact that sequence of midpoint fuzzy numbers belong to \(bv^F(u, v)\) does not confirm \((X_i) \in bv^F(u, v)\), so, the above two theorems can not be equivalent.

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No potential conflict of interest was reported by the authors.

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