Hamiltonian thermodynamics of 2D vacuum dilatonic black holes

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Abstract

We consider the Hamiltonian dynamics and thermodynamics of the two-dimensional vacuum dilatonic black hole in the presence of a timelike boundary with a fixed value of the dilaton field. A canonical transformation, previously developed by Varadarajan and Lau, allows a reduction of the classical dynamics into an unconstrained Hamiltonian system with one canonical pair of degrees of freedom. The reduced theory is quantized, and a partition function of a canonical ensemble is obtained as the trace of the analytically continued time evolution operator. The partition function exists for any values of the dilaton field and the temperature at the boundary, and the heat

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capacity is always positive. For temperatures higher than \( \beta_c^{-1} = \hbar \lambda / (2\pi) \), the partition function is dominated by a classical black hole solution, and the dominant contribution to the entropy is the two-dimensional Bekenstein-Hawking entropy. For temperatures lower than \( \beta_c^{-1} \), the partition function remains well-behaved and the heat capacity is positive in the asymptotically flat space limit, in contrast to the corresponding limit in four-dimensional spherically symmetric Einstein gravity; however, in this limit, the partition function is not dominated by a classical black hole solution.

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I. INTRODUCTION

The observation that two-dimensional (2D) dilaton gravity theories admit classical and semiclassical black hole solutions \[1,2\] has inspired an intense interest in these theories as an arena for studying quantum black holes. At the very least, these theories provide a simplified arena for the study of the final stages of black hole evaporation, in the hope that the two-dimensional results would in some broad sense reflect on four-dimensional (4D) quantum gravity. More ambitiously, one may entertain the hope that two-dimensional dilaton gravity could in fact be closer to the ‘true’ theory of gravity in our universe than four-dimensional Einstein gravity. For a review, see Ref. \[3\].

The purpose of the present paper is to investigate the equilibrium thermodynamics of the two-dimensional vacuum dilatonic black hole \[1,2\] (“Witten’s black hole”) in the canonical ensemble. Within four-dimensional Einstein gravity, it is well known that the canonical ensemble for the radiating Schwarzschild black hole does not exist in asymptotically flat space \[4,5\], but the situation can be improved by postulating the black hole to be placed in a spherical, mechanically rigid ‘box’ on which the local temperature is then held fixed \[6–8\]; see Ref. \[9\] for a review. This motivates us to consider the canonical ensemble for the dilaton black hole under analogous boundary conditions, with the dilaton field providing the analogue of the box radius. As a limiting case, we shall also be able to address the canonical ensemble in asymptotically flat space.

For four-dimensional spherically symmetric Einstein gravity with a finite boundary, the evaluation of the thermodynamical partition function in the canonical ensemble has been addressed through a combination of Hamiltonian and path integral techniques \[6–8,10–14\]. Also, in two-dimensional dilaton gravity, a similar calculation of the thermodynamical partition function using the path integral approach exists \[15\]. In this paper we shall adapt to the dilaton gravity theory the Hamiltonian method of Ref. \[14\] (henceforth referred to as LW).

In this method, one first constructs a classical Lorentzian Hamiltonian theory of geometries such that, on the classical solutions, one end of the spacelike surfaces is on a timelike boundary in an exterior region of the black hole spacetime, and the other end is at the horizon bifurcation surface. One then canonically quantizes this theory, and obtains the thermodynamical partition function by suitably continuing the Schrödinger picture time evolution operator to imaginary time and taking the trace. A crucial input is how to do the analytic continuation at the bifurcation surface; in LW it was found that a choice motivated by smoothness of Euclidean black hole geometries yields a partition function that is in agreement with that obtained via path integral methods.

In order that the method can be implemented, one must be able to canonically quantize the Lorentzian theory in some practical fashion. In LW this was achieved for spherically symmetric four-dimensional Einstein gravity by using canonical variables that were first introduced by Kuchař under asymptotically flat, Kruskal-like boundary conditions \[16\]. In these variables the constraints become exceedingly simple, and the classical Hamiltonian theory could be explicitly reduced into an unconstrained Hamiltonian theory with just one canonical pair of degrees of freedom. For the two-dimensional dilaton gravity theory, an analogue of Kuchař’s variables was recently found by Varadarajan \[17\] under Kruskal-like
boundary conditions, and by Lau [18] under boundary conditions analogous to those in LW. We shall see that using these variables, it will be possible to construct a quantum theory and a thermodynamical partition function for the dilatonic theory in close analogy with LW.

As in spherically symmetric Einstein gravity in four dimensions, the partition function in the dilatonic theory will turn out to exist for all values of the dilaton and the temperature at the boundary. The heat capacity is always positive, implying thermodynamical stability of the canonical ensemble. For temperatures higher than the critical value $\beta_c^{-1} = \hbar \lambda / (2\pi)$, the partition function is dominated by a classical black hole solution, and the dominant contribution to the entropy is simply the two-dimensional Bekenstein-Hawking entropy $S_{BH}$,

$$S_{BH} = \beta_c M = \frac{2\pi M}{\hbar \lambda},$$

(1.1)

where $M$ is the ADM mass [1,12,13] of the hole. For temperatures lower than $\beta_c^{-1}$, on the other hand, the partition function is not dominated by a classical black hole solution. These properties are easily understood physically in terms of the gravitational blueshift effect and the fact that $\beta_c^{-1}$ is the Hawking temperature at infinity for the dilatonic black hole with any value of the mass [4]. The main difference from four-dimensional spherically symmetric Einstein gravity is that in the four-dimensional case, the condition that the entropy be dominated by the Bekenstein-Hawking entropy of a classical black hole solution is that the product of the temperature and the boundary curvature radius be larger than a critical numerical value [6,7]. Also, a four-dimensional black hole solution that dominates the partition function is necessarily so massive that the box is contained within the $3M$ radius [6], and for the partition function of Refs. [7,8] even inside the $\frac{3}{2}M$ radius; in contrast, for a two-dimensional black hole solution that dominates the partition function, the box can be arbitrarily large compared with the length scale set by the mass.

For temperatures lower than $\beta_c^{-1}$, taking the boundary to infinity yields a well-defined partition function, which can be identified with the partition function associated with asymptotically flat boundary conditions at infinity. This is in a striking contrast with four-dimensional spherically symmetric Einstein theory, where the partition function diverges in the asymptotically flat space limit [3,7]. In the two-dimensional case, the heat capacity turns out to be again positive, but it diverges as the temperature approaches the critical value $\beta_c^{-1}$. Again, this behavior is easily understandable in view of the classical black hole solutions. As the Hawking temperature at infinity is independent of the ADM mass, the hole can absorb or emit energy without changing its temperature: the heat capacity can thus be regarded as infinite.

The rest of the paper is as follows. In Section II we briefly recall the two-dimensional vacuum dilaton gravity theory and Witten’s black hole solution, establishing our notation which is motivated by the reduction from four-dimensional dilaton gravity [20,21]. In Section III we present, in these variables, a canonical transformation which is equivalent to that given by Lau [18], and differs from that given by Varadarajan [17] in essence only in the boundary conditions. We also reduce the theory to a single true pair of canonical variables.

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1Ref. [18] appeared while the present paper was in preparation.
The partition function is constructed and the thermodynamical properties are discussed in Section IV, and the limit of asymptotically flat space is explored in Section V. Section VI offers brief concluding remarks.

We shall work throughout in units in which $c = G = 1$. The action will then be dimensionless, which means that also the two-dimensional Planck’s constant $\hbar$ is dimensionless.

II. 2D DILATON GRAVITY

A. Dynamics

The CGHS action for a two-dimensional theory of gravity coupled to a dilaton field $\phi$ is

$$S = \frac{1}{2} \int dt \int dr \sqrt{-g} e^{-2\phi} [R^{(2)} + 4(\nabla \phi)^2 + 4\lambda^2],$$

where $R^{(2)}$ is the two-dimensional Ricci scalar and $\lambda^2$ is a cosmological constant term. We shall take $\lambda > 0$. The two-dimensional metric can be written in the ADM form

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2,$$

where the lapse-function $N$, the shift vector $N^r$, and the spatial one-metric component $\Lambda^2$ are functions of $t$ and $r$. We shall assume $\Lambda$ and $N$ to be positive, which guarantees that the metric is nondegenerate with signature $(-+)$. Viewing the two-dimensional CGHS theory as a reduced four-dimensional dilatonic gravity theory, we define the positive-definite field

$$R = e^{-2\phi}/(2\lambda),$$

which can be interpreted as the radial coordinate for the four-dimensional classical solution. Up to boundary terms, the action (2.1) takes then the ADM form

$$S = \int dt \int dr \left\{ \frac{-2\lambda}{RN} \left[ R \left( -\dot{A} + (N^r A)' \right) (\dot{R} + R'N^r) + \frac{1}{2} \Lambda (\dot{R} + R'N^r)^2 \right] + 2\lambda R'\Lambda^{-1}N^r + \lambda\Lambda^{-1}NR^{-1}(R')^2 + 4\lambda^3 NRA \right\},$$

where $\dot{} = \frac{\partial}{\partial t}$ and $' = \frac{\partial}{\partial r}$.

For concreteness, we take both $r$ and $t$ to have the dimension of length, which implies that $\Lambda$, $N$, and $N^r$ are functions of $t$ and $r$. The constant $\lambda$ has the dimension of inverse length, $R$ has the dimension of length, and the action is dimensionless. We shall follow the convention of Kuchař in denoting by Latin letters those canonical coordinates that are spatial scalars (e.g. $R$), and by Greek letters those that are spatial densities (e.g. $\Lambda$).

The momenta conjugate to $\Lambda$ and $R$ are respectively

$$\Pi_\Lambda = -2\lambda N^{-1}(\dot{R} - R'N^r), \quad \Pi_R = -2\lambda N^{-1} \left[ \Lambda R^{-1}(\dot{R} - R'N^r) + (\dot{\Lambda} - (N^r \Lambda)') \right].$$

The dimension of $\Pi_\Lambda$ is inverse length, and that of $\Pi_R$ is inverse length squared. Note that the momentum with a Greek subscript is a spatial scalar and that with a Latin subscript
is a spatial density. A Legendre transformation leads to the canonical bulk action (i.e., the action up to boundary terms):

\[ S_{\Sigma}[R, \Lambda, \Pi_R, \Pi_\Lambda; N, N^r] = \int dt \int dr \left( \dot{R} \dot{\Pi}_R + \dot{\Lambda} \dot{\Pi}_\Lambda - NH - N^r H_r \right), \tag{2.5} \]

where the super-Hamiltonian \( H \) and the supermomentum \( H_r \) are given respectively by

\[
\begin{align*}
H &= -(2\lambda)^{-1} \Pi_R \Pi_\Lambda + (4\lambda)^{-1} R^{-1} \Lambda \Pi_\Lambda^2 \\
&\quad + 2\lambda \Lambda^{-1} R'' - 2\lambda \Lambda^{-2} R' \Lambda' - \lambda \Lambda^{-1} R^{-1} R'' - 4\lambda^3 \Lambda R, \tag{2.6a} \\
H_r &= \Pi_R R' - \Lambda \Pi_\Lambda'. \tag{2.6b}
\end{align*}
\]

We shall discuss the boundary conditions and boundary terms in Section III.

**B. Witten’s black hole**

The general solution [2,21] to the equations of motion derived from the action (2.3) is

\[ ds^2 = -F(R)dt^2 + \frac{dR^2}{4\lambda^2 R^2 F(R)}, \tag{2.7} \]

where

\[ F(R) = 1 - \frac{M}{2\lambda^2 R}. \tag{2.8} \]

Here \( M \) is a parameter that can be interpreted as the ADM mass [1,17,19]. For positive \( M \), Eq. (2.7) describes a two-dimensional black hole geometry [2], with a horizon at \( R = M/(2\lambda^2) \) and a spacelike singularity at \( R = 0 \). The \((T,R)\) coordinates are analogous to the curvature coordinates ("Schwarzschild coordinates") of the four-dimensional Schwarzschild metric [16], and they cover at a time only one quadrant of the full spacetime. In the global, Kruskal-like coordinates \((x^+, x^-)\) defined via [21]

\[
\begin{align*}
\lambda^2 x^+ x^- &= \frac{M}{\lambda} - 2\lambda R, \tag{2.9a} \\
\ln |x^+/x^-| &= 2\lambda T, \tag{2.9b}
\end{align*}
\]

the metric takes the more familiar form

\[ ds^2 = \frac{-dx^+ dx^-}{-\lambda^2 x^+ x^- + M/\lambda}. \tag{2.10} \]

Given the canonical data \((R, \Lambda, \Pi_R, \Pi_\Lambda)\) on one hypersurface in the solution (2.7), one can read from this data the value of the mass parameter \( M \), and also the location of the surface up to translations in the Killing time \( T \). Adapting Kuchař’s analysis of the Schwarzschild black hole [16] as in Refs. [17,18], one finds

\[ F = (2\lambda)^{-2} \left[ \left( \frac{R'}{\lambda R} \right)^2 - \left( \frac{\Pi_\Lambda}{2\lambda R} \right)^2 \right] \tag{2.11} \]
and

\[-T' = (2\lambda)^{-2}R^{-1}F^{-1}\Lambda\Pi_{\Lambda}.
\]  

(2.12)

\(M\) is then recovered from (2.8) using (2.11).

It is now possible to follow Kuchař \[16\] and promote the on-shell expressions for \(M\) and \(T'\) into a canonical transformation, provided the boundary conditions can be handled in a satisfactory manner. For Kruskal-like boundary conditions this was achieved in Ref. \[17\], and for thermodynamically motivated boundary conditions analogous to those of LW in Ref. \[18\]. In the following section we shall review this analysis under the thermodynamical boundary conditions, in terms of the variables introduced in subsection \[II A\], and we shall explicitly reduce the theory into its unconstrained Hamiltonian form.

III. GEOMETRODYNAMICS OF WITTEN’S BLACK HOLE IN A BOX

A. Hamiltonian formulation

Our first task is to specify a set of boundary conditions analogous to those of LW, and to add to the Hamiltonian bulk action (2.5) appropriate boundary terms. As the spatial proper distance will under our boundary conditions be finite, we follow Refs. \[14,18\] and take \(r\) to have the range \([0, 1]\).

Consider first the left end of the spacelike surfaces. At the limit \(r \to 0\), we adopt the fall-off conditions

\[
\Lambda(t, r) = \Lambda_0(t) + O(r^2), \\
R(t, r) = R_0(t) + R_2(t)r^2 + O(r^4), \\
\Pi_{\Lambda}(t, r) = O(r^3), \\
\Pi_R(t, r) = O(r), \\
N(t, r) = N_1(t)r + O(r^3), \\
N^r(t, r) = N_1^r(t)r + O(r^3),
\]

(3.1)

where \(\Lambda_0\) and \(R_0\) are positive, and \(N_1 \geq 0\). The consistency of these conditions with the equations of motion can be shown as in LW \[18\]. When the equations of motion hold, the conditions enforce \(r = 0\) to be at the horizon bifurcation point of a black hole solution.

Depending on what boundary conditions one chooses to impose on the canonical data, the bulk action in (2.5) needs to be supplemented with boundary terms such that the variation of the total action under the chosen boundary conditions leaves only a bulk term that gives the equations of motion. Consider now the total action

\[S = S_{\Sigma} + S_{\partial\Sigma},\]

where the bulk action \(S_{\Sigma}\) was given in (2.5) and the boundary action \(S_{\partial\Sigma}\) is given by

\[
S_{\partial\Sigma}[R, \Lambda, \Pi_R, \Pi_{\Lambda}; N, N^r] = 2\lambda \int dt [RN'\Lambda^{-1}]_{r=0} \\
+ \int dt \left[2\lambda N'R'\Lambda^{-1} - N^r\Lambda\Pi_{\Lambda} - \lambda\dot{R} \ln \left(\frac{N + \Lambda N^r}{N - \Lambda N^r}\right)\right]_{r=1}.
\]

(3.2)
The variation of the total action contains a bulk term that yields the equations of motion, as well as several boundary terms. The boundary terms on the initial and final surfaces have the usual form \( \pm \int_0^1 dr (\Pi_A \delta \Lambda + \Pi_R \delta R) \), and they vanish provided one fixes the one-metric and the dilaton field on the initial and final surfaces. The boundary terms at \( r = 0 \) take the form

\[
2\lambda \int dt [R \delta (N' \Lambda^{-1})]_{r=0} = 2\lambda \int dt R_0 \delta (N_1 \Lambda_0^{-1}),
\]

which vanishes if we set \( \delta (N_1 \Lambda_0^{-1}) = 0 \). As in the four-dimensional case of LW, fixing the quantity \( N_1 \Lambda_0^{-1} \) means fixing in the classical solution the rate at which the unit normal to the constant \( t \) surface is boosted at the coordinate singularity at the bifurcation point. The boundary term from \( r = 1 \) is cumbersome, but it can be verified to vanish for the classical solutions provided one fixes the timelike one-metric component \( g_{tt} = -N^2 + (\Lambda N^r)^2 \) and \( R \).

Thus, the total action \( S_\Sigma + S_{\partial \Sigma} \) is appropriate for a variational principle that fixes the one-metric and the dilaton field on the initial and final surfaces and also on the timelike boundary \( r = 1 \), and in addition the quantity \( N_1 \Lambda_0^{-1} \) at \( r = 0 \). As will be seen in Section [V], these boundary conditions are tailored in view of the thermodynamics of Witten’s black hole in a box. Fixing the one-metric on the timelike boundary will translate into fixing the temperature at the box that encloses the black hole, and fixing the value of the dilaton field at the timelike boundary will specify the “radius” of this box: these conditions will lead into the thermodynamical canonical ensemble. Fixing \( N_1 \Lambda_0^{-1} \) at the bifurcation point will turn out to yield the black hole entropy, in a way that can be related to the regularity of the Euclidean black hole solutions.

Two remarks are in order. Firstly, although we have here found it convenient to introduce the boundary conditions and boundary terms intrinsically within the Hamiltonian theory, it would of course be possible to translate the conditions into the Lagrangian theory and introduce corresponding boundary terms to be added to the (1+1) split Lagrangian action (2.3) or to the covariant CGHS action (2.1). As discussed in Ref. [18], the boundary terms to be added to the CGHS action would consist of the contributions \( \pm \frac{1}{2} \int dx^a \sqrt{\mp (1) g} e^{-2\phi} K \) from the spacelike and timelike boundaries, where \( (1) g \) is the one-metric component, \( K \) is the extrinsic curvature, and \( x^a \) is respectively \( r \) or \( t \), as well as additional contributions from the bifurcation point and from the corners where the timelike boundary meets the spacelike boundaries. We shall, however, not need the explicit form of the Lagrangian actions here.

Secondly, given our boundary conditions, the choice of the boundary action is not unique. For example, it would be possible to replace \( S_{\partial \Sigma} (3.2) \) by any expression that is equivalent when the classical equations of motion hold: as the bulk term in the variation enforces the classical equations of motion for \( 0 < r < 1 \), continuity of the variables implies that such a replacement does not change the critical points of the total action. Also, as discussed in LW, it would be possible to leave \( N_1 \Lambda_0^{-1} \) free at \( r = 0 \) if the first term in (3.2) were replaced by \( 2\lambda \int dt \tilde{N}_0 R_0 \), where \( \tilde{N}_0(t) \) is a new quantity that is fixed in the variational principle: the stationarity of the new action gives the equation of motion \( N_1 \Lambda_0^{-1} = \tilde{N}_0 \), and the boundary data for the classical solutions remains exactly the same as before. Modifications of this kind would not affect the reduced Hamiltonian theory that we shall arrive at in Subsection [III C], or the thermodynamical analysis of Section [V]. For concreteness, we shall adhere to the boundary term \( S_{\partial \Sigma} (3.2) \).
B. Canonical transformation

We pass from the old canonical variables \((R, \Lambda, \Pi_R, \Pi_\Lambda)\) to the new canonical set \((M, R, P_M, P_R)\) via the equations

\[
M = (8\lambda^2)^{-1} R^{-1} \Pi_\Lambda^2 - \frac{1}{2} \Lambda^{-2} R^{-1} R'^2 + 2\lambda^2 R, \tag{3.4a}
\]

\[
P_M = (2\lambda)^{-2} R^{-1} F^{-1} \Lambda \Pi_\Lambda, \tag{3.4b}
\]

\[
P_R = \Pi_R - \frac{1}{2} R^{-1} \Lambda \Pi_\Lambda - \frac{1}{2} R^{-1} F^{-1} \Lambda \Pi_\Lambda
- (2\lambda R)^{-2} F^{-1} \Lambda^{-2} [R'(\Lambda \Pi_\Lambda)' - R''(\Lambda \Pi_\Lambda)], \tag{3.4c}
\]

where \(F\) is understood to be defined by (2.11). On a classical solution, the variable \(M\) is a constant whose value is the ADM mass parameter in the metric (2.7). Note that the variable \(R\) appears both in the old and new canonical sets, but the transformation changes its conjugate momentum.

To prove that the transformation is canonical, one starts from the identity

\[
\Pi_\Lambda \delta \Lambda + \Pi_R \delta R - P_M \delta M - P_R \delta R = \delta \left( \Lambda \Pi_\Lambda + \lambda R' \ln \left| \frac{2\lambda R' - \Lambda \Pi_\Lambda}{2\lambda R' + \Lambda \Pi_\Lambda} \right| \right)
- \left( \lambda \delta R \ln \left| \frac{2\lambda R' - \Lambda \Pi_\Lambda}{2\lambda R' + \Lambda \Pi_\Lambda} \right| \right)', \tag{3.5}
\]

and integrates both sides with respect to \(r\) from \(r = 0\) to \(r = 1\). The contribution from the second term on the right hand side vanishes, and we obtain

\[
\int_0^1 dr (\Pi_\Lambda \delta \Lambda + \Pi_R \delta R) - \int_0^1 dr (P_M \delta M + P_R \delta R) = \delta \omega, \tag{3.6}
\]

where

\[
\omega[\Lambda, \Pi_\Lambda, R] = \int_0^1 dr \left( \Lambda \Pi_\Lambda + \lambda R' \ln \left| \frac{2\lambda R' - \Lambda \Pi_\Lambda}{2\lambda R' + \Lambda \Pi_\Lambda} \right| \right). \tag{3.7}
\]

The functional (3.7) is well-defined, and the difference of the old and new Liouville forms is thus an exact form. This shows that the transformation is canonical.

The constraint terms \(NH + N^r H_r\) in the old surface action (2.3) take the form \(N^M M' + N^R P_R\), where

\[
N^M = (2\lambda)^{-2} N^r R^{-1} F^{-1} \Lambda \Pi_\Lambda - (2\lambda R)^{-1} NF^{-1} \Lambda^{-1} R', \tag{3.8a}
\]

\[
N^R = R' N^r - (2\lambda)^{-1} \Pi_\Lambda N. \tag{3.8b}
\]

We can thus write the new surface action as

\[
S_\Sigma[M, R, P_M, P_R; N^M, N^R] = \int dt \int_0^1 dr (P_M \dot{M} + P_R \dot{R} - N^M M' - N^R P_R), \tag{3.9}
\]

where the quantities to be varied independently are \(M, R, P_M, P_R, N^M, \) and \(N^R\).
We take the total action to be
\[ S[M, R, P_M, P_R; N^M, N^R] = S_\Sigma[M, R, P_M, P_R; N^M, N^R] + S_\partial \Sigma[M, R, P_M, P_R; N^M, N^R], \]
where
\[ S_\partial \Sigma[M, R, P_M, P_R; N^M, N^R] = -\int dt[M N^M]_{r=0} + \int dt \left[ 4\lambda^2 R \sqrt{F Q^2 + (2\lambda R)^{-2} R^2} \right. \\
+ \lambda \dot{R} \ln \left( \frac{\sqrt{F Q^2 + (2\lambda R)^{-2} R^2} - (2\lambda R)^{-1} \dot{R}}{\sqrt{F Q^2 + (2\lambda R)^{-2} R^2} + (2\lambda R)^{-1} \dot{R}} \right) \left. \right]_{r=1}. \]

Here \( F \) is defined by (2.8) as before, and \( Q^2 \) is defined by
\[ Q^2 = -g_{tt} = F(N^M)^2 - (2\lambda R)^{-2} F^{-1}(N^R)^2. \]

\( Q^2 \) need not have a definite sign for all values of \( r \), but at \( r = 1 \) it is positive by virtue of our boundary conditions.

The variation of (3.10) contains a bulk term proportional to the equations of motion, and several boundary terms. The initial and final surfaces contribute \( \pm \int_0^1 dr(P_M \delta M + P_R \delta R) \), which vanish if we fix \( M \) and \( R \) on these surfaces. The boundary term from \( r = 0 \) is
\[ -\int dt[M \delta N^M]_{r=0} = -\int dt R_0 \delta N_0^M. \]
which vanishes provided we fix \( N_0^M = \lim_{r \to 0} N^M \). As the fall-off conditions (3.3) imply
\[ \Lambda_0^{-1} N_1 = -\lambda N_0^M, \]
fixing \( N_0^M \) has the interpretation mentioned in the lines following (3.3) in terms of the unit normal to the constant \( t \) surfaces at \( r \to 0 \). Finally, the boundary terms from \( r = 1 \) are cumbersome, but it can be verified as in LW that they vanish on the classical solutions provided one fixes \( R \) and \( Q^2 \).

Our final action (3.10) agrees with that obtained by Lau [18]: it can be recovered from Eqs. (5.8) and (4.36) in Ref. [18] by setting \( \alpha = 2, y = 2, \Psi = -(1/2) \ln(2\lambda R), \) and \( \bar{N} = Q \).

C. Hamiltonian reduction

Reduction of the action (3.10) by solving the constraints proceeds as in the four-dimensional case of LW. The constraint \( P_R = 0 \) implies that \( R \) and \( P_R \) drop out altogether. The constraint \( M' = 0 \) implies
\[ M(t, r) = m(t). \]
Substituting (3.15) into (3.10) gives the action
\[ S[m, p; N^M_0; R_B, Q_B] = \int dt (p\dot{m} - h), \]  

(3.16)

where

\[ p = \int_0^1 dr P_M. \]  

(3.17)

The reduced Hamiltonian \( h \) is given by

\[ h = h_H + h_B \]  

(3.18)

where

\[ h_H = N^M_0 m, \]  

(3.19a)

\[ h_B = -4\lambda^2 R_B \sqrt{F_BQ^2_B + (2\lambda R_B)^{-2} \dot{R}_B^2} \]

\[ -\lambda \dot{R}_B \ln \left( \frac{\sqrt{F_BQ^2_B + (2\lambda R_B)^{-2} \dot{R}_B^2}}{\sqrt{F_BQ^2_B + (2\lambda R_B)^{-2} \dot{R}_B^2}} \right). \]  

(3.19b)

Here \( R_B \) and \( Q^2_B \) stand for the values of \( R \) and \( Q^2 \) at \( r = 1 \), and \( F_B = 1 - m(2\lambda^2 R_B)^{-1} \). \( R_B, Q^2_B, \) and \( N^M_0 \) are considered to be prescribed functions of \( t \), satisfying \( R_B > 0, Q^2_B > 0, \) and \( N^M_0 \leq 0 \).

The variational principle associated with the action \( 3.16 \) fixes the initial and final values of \( m \). The equation of motion for \( m \) reads \( \ddot{m} = 0 \), which reflects the fact that on a classical solution \( m \) is equal to the mass of the black hole. The equation of motion for \( p \) can be related via \( 2.12, 3.4b, \) and \( 3.17 \) to the difference of the evolution rates \( \dot{T} \) of the Killing time \( T \) at the two ends of the constant \( t \) surfaces.

IV. HAMILTONIAN THERMODYNAMICS

A. Quantum theory

We now take the value of the dilaton field at the boundary to be time-independent, \( \dot{R}_B = 0 \). The action becomes

\[ S[m, p; N^M_0, Q_B; B] = \int dt (p\dot{m} - h), \]  

(4.1)

where the Hamiltonian \( h \) is

\[ h = \left( 1 - \sqrt{1 - m(2\lambda^2 B^{-1})} \right) 4\lambda^2 BQ_B + N^M_0 m. \]  

(4.2)

Here \( B \) denotes the positive time-independent value of \( R_B \), and \( Q_B > 0 \) and \( N^M_0 \leq 0 \) are prescribed functions of \( t \) as before. To obtain \( 4.2 \), we have added to \( 3.18 \) the term \( 4\lambda^2 BQ_B \), which renormalizes the value of the Hamiltonian so that \( h(m = 0) = 0 \). As the added term is independent of the canonical variables, this renormalization does not affect
the equations of motion; it is analogous to the addition of the $K_0$ term in four-dimensional Einstein gravity. The canonical momentum $p$ takes all real values, but the range of the canonical coordinate $m$ is $0 < m < 2\lambda^2 B$.

In the quantum theory, we take the Hilbert space to be $H = L^2([0, 2\lambda^2 B]; \mu)$, that is, the space of square integrable functions of $m$ with respect to the inner product

$$\langle \psi, \chi \rangle = \int_0^{2\lambda^2 B} \mu(m) \overline{\psi(m)} \chi(m),$$

where $\mu(m; B)$ is a smooth positive weight function. More specific assumptions about $\mu$ will be made in subsection IV C.

We take the Hamiltonian operator $\hat{h}$ to act by pointwise multiplication by the function $h(m)$ (4.2): $$(\hat{h} \psi)(m) = h(m) \psi(m).$$

The unitary time evolution operator is

$$\hat{K}(t_2; t_1) = \exp \left[ -i \hbar^{-1} \int_{t_1}^{t_2} dt' \hat{h}(t') \right],$$

and it acts in $H$ by pointwise multiplication by the function

$$K(m; T_B; \Theta_H) = \exp \left[ -i \hbar^{-1} \left( 1 - \sqrt{1 - m(2\lambda^2 B)^{-1}} \right) 4\lambda^2 BT_B + i\hbar^{-1} \lambda^{-1} m \Theta_H \right],$$

where

$$T_B = \int_{t_1}^{t_2} dt Q_B,$$

$$\Theta_H = -\lambda \int_{t_1}^{t_2} dt N^M_0.$$

This means that $T_B$ and $\Theta_H$ are two independent evolution parameters. $T_B$ is the proper time elapsed at the timelike boundary, and $\Theta_H$ is the boost parameter elapsed at the bifurcation point. $\hat{K}(t_2; t_1)$ depends on $t_1$ and $t_2$ only through $T_B$ and $\Theta_H$, and we can write it as $\hat{K}(T_B, \Theta_H)$.

**B. Partition function**

We wish to obtain the partition function by analytically continuing $\hat{K}(T_B, \Theta_H)$ to imaginary time and taking the trace. Since $T_B$ is the Lorentzian proper time elapsed at the timelike boundary, we set it equal to $-i\hbar \beta$, and we interpret $\beta$ as the inverse temperature at the boundary. The continuation of $\Theta_H$ is motivated by consistency with the Euclidean path integral approach (i.e., requiring the absence of a conical singularity in the Euclidean solution) as in LW, leading us to set $\Theta_H = -2\pi i$. In this way we obtain for the partition function the formal expression

$$Z(\beta) = \text{Tr} \left[ \hat{K}(-i\hbar \beta, -2\pi i) \right]$$

$$= \int_0^{2\lambda^2 B} \mu(m) \langle m | \hat{K}(-i\hbar \beta, -2\pi i) | m \rangle$$

$$= \int_0^{2\lambda^2 B} \mu(m) K(m; -i\hbar \beta; -2\pi i) \langle m | m \rangle,$$
where the last expression is divergent because of the inner product \( \langle m|m \rangle \). The origin of this divergence can be traced to the absence of a kinetic term in the Hamiltonian operator \( \hat{h} \) appearing in the expression for the time evolution operator \( \hat{K} \) given in (4.4). To renormalize this divergent trace, we define the following renormalized partition function in terms of a small kinetic term \( -\alpha(\mu^{-1}d/dm)^2 \):

\[
Z_{\text{ren}}(\beta) = \lim_{\alpha \to 0^+} \frac{\text{Tr} \left[ \exp \left( -\frac{1}{2} \alpha \hat{A} \right) \hat{K} \exp \left( -\frac{1}{2} \alpha \hat{A} \right) \right]}{\text{Tr} \left[ \exp \left( -\alpha \hat{A} \right) \right]},
\]

(4.8)

where \( \hat{A} \) is the positive self-adjoint operator \( -(\mu^{-1}d/dm)^2 \) associated with the boundary condition that \( \mu^{-1}d/dm \) acting on its eigenfunctions vanishes at the boundaries \( m = 0 \) and \( m = 2\lambda^2B \) (see Appendix A in LW for more details). Taking the limit \( \alpha \to 0^+ \) in (4.8) gives

\[
Z_{\text{ren}}(\beta) = \left( \int_0^{2\lambda^2B} \mu dm \right)^{-1} \times \int_0^{2\lambda^2B} \mu dm \exp \left[ -\left( 1 - \sqrt{1 - m(2\lambda^2B)^{-1}} \right) 4\lambda^2B\beta + 2\pi\hbar^{-1}\lambda^{-1}m \right].
\]

(4.9)

In terms of the dimensionless variable \( x = (2\lambda^2B)^{-1}m \), (4.9) can be written in the notation of Ref. [7] as

\[
Z_{\text{ren}}(\beta) = \left( \int_0^1 \mu dx \right)^{-1} \int_0^1 \mu dx \exp \left[ -I_*(x)/\hbar \right],
\]

(4.10)

where the effective action \( I_*(x) \) is

\[
I_*(x) = 4\pi\lambda B \left( 2 \left( 1 - \sqrt{1 - x} \right)^{\beta/\beta_c} - x \right),
\]

(4.11)

and the critical inverse temperature \( \beta_c \) is given by

\[
\beta_c = \frac{2\pi}{\lambda\hbar}.
\]

(4.12)

C. Thermodynamics

A first observation from the partition function (4.10) is that the (constant volume) heat capacity, \( C = \beta^2(\partial^2(\ln Z_{\text{ren}})/\partial \beta^2) \), is always positive\(^2\). The canonical ensemble with our boundary conditions is thus thermodynamically stable. This is analogous to the stability

\[^2\text{In general, suppose that a partition function can be written as } Z(\beta) = \int dx \nu(x)e^{-\beta x}, \text{ where } \nu(x) \geq 0. \text{ As } Z(\partial^2Z/\partial \beta^2) - (\partial Z/\partial \beta)^2 = \frac{1}{2} \int dx dy (x - y)^2 \nu(x)\nu(y)e^{-\beta(x+y)}, \text{ the heat capacity, } C = \beta^2(\partial^2(\ln Z)/\partial \beta^2), \text{ is then necessarily positive.}\]
of the fixed volume canonical ensemble for spherically symmetric Einstein gravity in four dimensions \[6,7\].

We shall now assume that \(\mu\) varies slowly compared with the exponential factor in (4.10). This will enable us to evaluate the integral by the saddle point approximation.

The behavior of the partition function exhibits two qualitatively different regions. Let us first suppose \(\beta < \beta_c\). In this case \(I_*(x)\) has a global minimum at \(x_0 = 1 - (\beta/\beta_c)^2\), and \(I_*(x_0) = -4\pi\lambda B[1 - (\beta/\beta_c)^2] < 0\). The saddle point approximation yields

\[
Z_{\text{ren}}(\beta) \sim \exp \left[ -\frac{I_*(x_0)}{\hbar} \right].
\] (4.13)

The dominant contribution therefore comes from a classical Euclidean black hole solution with the mass \(m_0 = 2\lambda^2 B x_0\), whose Euclidean action is \(I_*(x_0)\). This is what one would have expected already from the Lorentzian viewpoint: for the Lorentzian black hole solution, the Hawking temperature at infinity is \(\beta_c^{-1}\) for any value of the mass \[4\], and \((2.7)\) shows that the local Hawking temperature at a finite distance is obtained by multiplication with the blueshift factor \(F^{-1/2}\). Note that for the dominating classical solutions, \(B\) can be arbitrarily large compared with the length scale \(\lambda^{-2} m_0\) that is set by the mass. This feature is qualitatively different from four-dimensional spherically symmetric Einstein theory, where the mass of a dominant saddle point solution is always so large that the box lies within the closed photon orbit \[6,7\]. From (4.13) we recover for the energy expectation value \(\langle E \rangle\) the expression

\[
\langle E \rangle = -\frac{\partial (\ln Z_{\text{ren}})}{\partial \beta} \simeq 4\lambda^2 B \left[1 - (\beta/\beta_c)\right].
\] (4.14)

Expressing \(\beta\) in terms of \(m_0\) and inverting (4.14) yields

\[
m_0 \approx \langle E \rangle - \frac{(\langle E \rangle)^2}{8\lambda^2 B}.
\] (4.15)

Eq. (4.15) gives an interpretation to the ADM mass as the sum of the thermal energy and the gravitational self-energy associated with the thermal energy. An analogous formula holds for the four-dimensional Schwarzschild hole \[3\].

The entropy associated with the gravitational field of a two-dimensional black hole, \(S_{GF}\), is given to the leading order by

\[
S_{GF} = \left(1 - \beta \frac{\partial}{\partial \beta} \right) \ln (Z_{\text{ren}}(\beta)) \simeq \beta_c m_0.
\] (4.16)

This is precisely the two-dimensional Bekenstein-Hawking black hole entropy, \(S_{BH} = \beta_c m_0\) \[23\]. The (constant volume) heat capacity is

\[
C = \beta^2 \frac{\partial^2 (\ln Z_{\text{ren}})}{\partial \beta^2} \simeq \frac{4\lambda^2 B \beta^2}{\beta_c}.\] (4.17)

The higher order corrections to \(\langle E \rangle\) and \(S_{GF}\) depend on the choice of the weight function \(\mu\). As an example, let us consider the case where \(\mu\) is independent of \(m\). To the next-to-leading order, one then obtains
When the radius of the box is much larger than the horizon radius, we have \( \lambda B \gg \lambda^{-1}m_0 \) and \( \beta/\beta_c \approx 1 \). In addition, if \( \lambda^{-1}m_0 \gg \bar{\hbar} \), so that the semiclassical approximation is good, the next-to-leading order contributions to \( S_{GF} \) are dominated by the second term on the right hand side of (4.18). Eq. (4.18) appears thus to be in agreement with the quantum corrections from entanglement entropy discussed in Refs. [23,24]. Note that the expression given in (4.18) does not involve a renormalization cutoff parameter; however, a renormalization was involved in obtaining the partition function from the divergent expression (4.7).

It is of interest to compare our partition function to the partition function that is obtained from a Euclidean path integral via the Hamiltonian reduction method that Whiting and York introduced in the four-dimensional context [7,8]. Adapting the Whiting-York method to our case leads to a partition function that is obtained from (4.9) by replacing \((\int \mu d\mathbf{m})^{-1}\mu d\mathbf{m}\) by \(d(S_{BH}) = 2\pi \lambda^{-1}\bar{\hbar}^{-1}d\mathbf{m}\). We see that if \( \mu \) is chosen independent of \( \mathbf{m} \), our partition function (4.10) differs from the Whiting-York weighted partition function only by the overall factor \(4\pi\lambda B\bar{\hbar}^{-1}\). The two partition functions thus yield identical results for quantities that only involve logarithmic temperature derivatives of the partition function, such as the energy expectation value and the heat capacity. The quantum corrections to the entropy differ, however: for the Whiting-York weighting, (4.18) is replaced by

\[
S_{GF} \simeq S_{BH} - \frac{1}{2} \ln \left( \frac{\lambda B}{\bar{\hbar}} \right) + \ln(\beta/\beta_c) - 1 + \ln(4\pi). \tag{4.19}
\]

As \( \beta \) approaches \( \beta_c \) from below, \( \beta \to \beta_c^- \), one has \( x_0 \to 0 \), and \( \lambda^2 Bm_0^{-1} \) diverges. For fixed \( B \) this means that the mass of the saddle point black hole approaches zero, and the saddle point approximation is no longer expected to be good in this limit. However, if one takes the limits \( \beta \to \beta_c^- \) and \( \lambda B \to \infty \) simultaneously, in such a way that \( \lambda^{-1}m_0 \) is fixed and much larger than \( \bar{\hbar} \), the saddle point approximation remains valid. In this limit, the energy expectation value \( \langle E \rangle \) becomes just the ADM mass, and the saddle point describes a black hole of mass \( \mathbf{m}_0 \) in asymptotically flat space. Equivalently, taking \( \lambda B \to \infty \) while keeping \( \lambda^{-1}m_0 \) fixed implies, through the saddle point condition (for \( \beta < \beta_c \)), that \( \beta \to \beta_c^- \). So, the temperature at asymptotic infinity is \( \beta_c \). This is the solution usually referred to as the Witten black hole. Note that in this limit, the (constant volume) heat capacity given in Eq. (4.17) diverges. This is consistent with the observation that for the Witten black hole, the Hawking temperature at infinity is independent of the mass: the hole can change its energy without changing the temperature at infinity, and the heat capacity can thus be regarded as infinite.

We finally turn to the case \( \beta > \beta_c \). The global minimum of \( I_*(x) \) is now at \( x = 0 \), and \( I_*(x = 0) = 0 \). There are no saddle points, and the dominant contribution to (4.10) comes from the vicinity of \( x = 0 \). Again, this agrees with Lorentzian expectations: for a Lorentzian classical solution, the local Hawking temperature is always higher than the Hawking temperature \( \beta_c^{-1} \) at the infinity. As in four-dimensional spherically symmetric Einstein gravity [1,7], one may see this as evidence for a phase transition between a black hole sector and a topologically different “hot flat space” sector of the theory. A difference
is, however, that in the dilatonic theory the transition is sharply related to the existence of a saddle point. In the four-dimensional case of Refs. [7,8] the transition occurs while the effective action still has a local minimum, but this minimum no longer gives the dominant contribution to the partition function.

V. ASYMPTOTICALLY FLAT SPACE

In this section we consider briefly the situation where the timelike boundary is replaced by an asymptotically flat infinity.

Proceeding as in Ref. [17], one arrives at a classically reduced action that can be obtained from Eqs. (4.1) and (4.2) by taking the limit $\lambda B \to \infty$. $Q_B$ has then the interpretation as the proper time elapsed at infinity. One can quantize as in subsection IV A, the only difference being that the range of $m$ is now $0 < m < \infty$. The analytic continuation of the time evolution operator is performed as in subsection IV B. The trace of the analytically continued time evolution operator is again divergent, and the infinite range of $m$ renders the renormalization technique used in subsection IV B not directly applicable; however, in the limit $m_B - 1 \to 0$, the effective action in (4.11) $I^* (x)$ approaches the expression $(\beta - \beta_c) \bar{h} m$, and one can argue that the final expression for the renormalized partition function obtained from (4.10) should be

$$Z_{\text{ren}} (\beta) = A \int_0^\infty \mu dm \exp \left[ - (\beta - \beta_c) m \right],$$

(5.1)

where the normalization factor $A$ depends on the details of the renormalization, and may depend on $\lambda$, but is independent of $\beta$.

Let us again assume that $\mu$ varies slowly compared with the exponential factor in (5.1). The expression (5.1) is then divergent for $\beta < \beta_c$, but for $\beta > \beta_c$ the integral converges and yields a well-defined partition function. This is in a striking contrast with four-dimensional spherically symmetric Einstein gravity, where the asymptotically flat space limit yields a divergent integral for all values of the temperature [7,14]. The reason for this difference is that the second term $\beta_c m$ in the exponent in (5.1) grows only linearly in $m$, whereas in four-dimensional spherically symmetric Einstein gravity the corresponding term grows quadratically in the mass variable. Note that in both cases this term can be interpreted as the Bekenstein-Hawking entropy [7].

We now assume $\beta > \beta_c$. The heat capacity is again positive, and the canonical ensemble is thus stable. However, the integral in (5.1) does not admit a saddle point approximation, and the partition function gets its dominant contribution from the vicinity of $m = 0$. This is analogous to what happened also in the finite boundary case for $\beta > \beta_c$, and reflects the fact that there are no classical black hole solutions with the Hawking temperature at infinity lower than $\beta_c^{-1}$.

For concreteness, let us set $\mu = m^p$, $p > -1$ in (5.1). The energy expectation value, the heat capacity, and the entropy are then given by

$$\langle E \rangle = \frac{(p + 1)}{(\beta - \beta_c)},$$

(5.2a)
\[ C = \frac{(p + 1)\beta^2}{(\beta - \beta_c)^2}, \quad (5.2b) \]

\[ S = (p + 1) \left[ \frac{\beta_c}{\beta - \beta_c} + \ln \left( \frac{\beta_c}{\beta - \beta_c} \right) \right] + \text{constant}. \quad (5.2c) \]

At the limit \( \beta \to \beta_c^+ \), both \( \langle E \rangle \), \( C \), and \( S \) diverge. A way to understand the divergence in \( \langle E \rangle \) physically is to recall that a classical two-dimensional black hole solution satisfies \( \beta = \beta_c \) with any value of the mass. For \( \beta \to \beta_c^+ \), arbitrarily high mass black holes would thus be expected to contribute to \( \langle E \rangle \) with roughly equal weights, resulting in a divergence. A similar interpretation accounts for the divergence in \( S \). The divergence in \( C \) can be interpreted as in subsection IV C, in terms of the fact that the Hawking temperature at infinity is independent of the mass.

VI. CONCLUDING REMARKS

In this paper we have investigated the equilibrium thermodynamics of the two-dimensional vacuum dilatonic black hole in the canonical ensemble. The classically reduced Hamiltonian theory of Ref. [18] was quantized, and the thermodynamical partition function was obtained as the trace of the analytically continued time evolution operator. When the system is confined in a finite box that is characterized by the boundary value of the dilaton field, the partition function is well-defined for all boundary temperatures, and the heat capacity is always positive. For temperatures higher than \( \beta_c^{-1} = \hbar \lambda/(2\pi) \), the partition function is dominated by a classical black hole solution, and the dominant contribution to the entropy is the two-dimensional Bekenstein-Hawking entropy, \( S_{BH} = \beta_c m_0 \), where \( m_0 \) is the mass of the hole. The situation is thus qualitatively very similar to that with four-dimensional Schwarzschild black holes [6,7,14]. The main difference is that in our two-dimensional case the condition for a saddle point to dominate the partition function only depends on the temperature, whereas in the Schwarzschild case the corresponding condition involves both the temperature and the boundary radius.

In the limit of asymptotically flat boundary conditions our partition function remains well-defined for temperatures lower than \( \beta_c^{-1} \). The heat capacity is again positive, but the partition function cannot be approximated by a classical black hole solution. When the temperature approaches \( \beta_c^{-1} \), the energy expectation value, the heat capacity, and the entropy all diverge, provided the measure in the partition function is sufficiently slowly varying in the mass variable; this divergence can be understood physically in terms of the fact that for a classical black hole, the Hawking temperature at infinity is independent of the black hole mass. This behavior is in a striking contrast with four-dimensional spherically symmetric Einstein gravity, where the asymptotically flat space canonical ensemble does not exist for any temperature [3,4]. The underlying reason for the difference is that for the dilatonic black hole the Bekenstein-Hawking entropy \( S_{BH} \) is linear in the mass, whereas for the four-dimensional Schwarzschild black hole \( S_{BH} \) is quadratic in the mass.

When the partition function of the finite boundary canonical ensemble is dominated by a classical black hole solution, the next-to-leading order corrections to the energy expectation value and to the entropy depend sensitively on the inner product that is adopted in the
Lorentzian Hamiltonian quantum theory. If the weight factor $\mu$ in the inner product is chosen to be independent of the mass variable, the next-to-leading order correction to the entropy appears to be compatible with the first quantum corrections from the entanglement entropy [23, 24]. The correction does not involve an explicit renormalization parameter; however, a renormalization was required when recovering the partition function from a formally divergent trace. These results seem to be in agreement with Frolov’s observation [25] that the 1-loop thermodynamical entropy is finite, while the entanglement (or statistical mechanics) 1-loop entropy diverges and requires a renormalization cutoff [24]. The study of two-dimensional black holes in the presence of fields other than the gravitational field may help to clarify this point [27].

It might be interesting to investigate along the lines of the present paper the thermodynamics of the one-loop corrected model of Ref. [28] with matter fields. Work in this direction is in progress.

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