Weak lensing and the Dyer–Roeder approximation

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Abstract

The distance–redshift relation plays an important role in cosmology. In the standard approach to cosmology, it is assumed that this relation is the same as in a homogeneous universe. As the real Universe is not homogeneous, there are several methods used to calculate the correction. The weak-lensing approximation and the Dyer–Roeder relation are among them. This paper establishes a link between these two approximations. It is shown that if the Universe is homogeneous with only small density fluctuations along the line of sight that vanish after averaging, then the distance correction is negligible. It is also shown that a vanishing three-dimensional average of density fluctuations does not imply that the mean of density fluctuations along the line of sight is zero. In this case, even within the linear approximation, the distance correction is not negligible. A modified version of the Dyer–Roeder relation is presented and it is shown that this modified relation is consistent with the correction obtained within the weak-lensing approximation. The correction to the distance for a source at \( z \sim 2 \) is of the order of a few per cent. Thus, with the increasing precision of cosmological observations, an accurate estimation of the distance is essential. Otherwise errors due to miscalculation of the distance can become a major source of systematics.

Key words: gravitational lensing: weak – cosmology: theory – large-scale structure of Universe.

1 Introduction

The distance–redshift relation plays an important role in cosmology. In fact, almost all cosmological observations depend, either explicitly or implicitly, on this relation. However, in the general case, to calculate the distance knowledge of the matter distribution (as well as its evolution) between the observer and the source is needed. Therefore, approximate relations are of great use to astronomers. The simplest approximation assumes homogeneity of the Universe and is based on the Friedmann model. However, as the real Universe is not homogeneous, more elaborate relations are needed. Zel’dovich (1964) proposed an approximation that takes into account that light propagates through emptier rather than denser regions of the Universe. This approximation is now known as the Dyer–Roeder approximation (Dyer & Roeder 1972, 1973). It also assumes homogeneity but allows for a different density from that in the background model. The density difference is modelled by a constant, the so-called smoothness parameter \( \alpha \). As the Dyer–Roeder equation is a differential equation, an approximate analytic solution was presented and discussed by Demiański et al. (2003). Generalization to \( \alpha(z) \) was first suggested by Linder (1988) and the effect of the change of the expansion rate was discussed by Mattsson (2010).

The Dyer–Roeder relation was tested against cosmological observations by Santos & Lima (2006), Santos, Cunha & Lima (2008) and Yu et al. (2010).

However, it has been argued that the Dyer–Roeder relation may not properly describe the effect of matter clustering (Räsänen 2009; Ellis 2009) and therefore may not be an appropriate approximation for the distance–redshift relation in the real Universe. Also, if the smoothness parameter changes with redshift then the distance formula depends on several free parameters, the interpretation of which can be ambiguous. Therefore, although cosmologists are aware that not taking into account inhomogeneities introduces additional systematics, the Dyer–Roeder relation is not widely used. Instead, several alternatives based on either the linear perturbative scheme (see Bonvin, Durrer & Gasparini 2006 and references therein) or non-linear models (see Bolejko et al. 2009 and references therein) have been developed.

This paper explores the connection between the Dyer–Roeder approximation and the weak-lensing approximation. It shows that the Dyer–Roeder relation can be modified so that it is consistent with the lensing approximation. The modified version contains only one free parameter of clear interpretation, i.e. the mean of density fluctuations along the line of sight \( \langle \delta \rangle_{1D} \).

The structure of this paper is as follows: Sections 2 and 3 present the Dyer–Roeder and weak-lensing approximations respectively, Section 4 presents the comparison of these two methods and Section 5 discusses the results.
2 THE DYER–ROEDER APPROXIMATION

The angular diameter distance $D_{\Lambda}$ is given by the following relation (Sachs 1961):

$$\frac{d^2D_{\Lambda}}{dz^2} = -\left(\sigma^2 + \frac{1}{2} R_{\alpha\beta}k^\alpha k^\beta\right)D_{\Lambda},$$

where $\sigma$ is the shear of the light bundle, $k^\alpha$ is a vector tangent to the light ray, $R_{\alpha\beta}$ is the Ricci tensor and $R_{\alpha\beta}k^\alpha k^\beta = \kappa T_{\alpha\beta}k^\alpha k^\beta$ (where $T_{\alpha\beta}$ is the energy–momentum tensor). In comoving and synchronous coordinates, for pressureless matter, $T_{\alpha\beta}k^\alpha k^\beta = \kappa \rho k^\alpha k^\beta$. The Dyer–Roeder approach assumes homogeneity ($\sigma = 0 = \delta \rho$) but takes into account that light propagates through a vacuum. Therefore, the $\Omega_m$ that photons ‘feel’ is different from the true $\Omega_m$. This is modelled by a constant parameter $\sigma$ (of value between 0 and 1) that multiplies $\Omega_m$. In this case (1) reduces to

$$\frac{d^2D_{\Lambda}}{dz^2} + \left(\frac{1}{H(z)} \frac{dH}{dz} + \frac{2}{1 + z}\right) \frac{dD_{\Lambda}}{dz} + 3\sigma\Omega_mH^2_0(1 + z)D_{\Lambda} = 0,$$

where $H(z) = H_0\sqrt{\Omega_m(1 + z)^3 + \Omega_{\Lambda}(1 + z)^2 + \Omega_{\Lambda}}$. The initial conditions needed to solve (2) are $D_{\Lambda} = 0$ and $dD_{\Lambda}/dz = 1/H_0$.

Generalization to $\alpha(z)$ was explored by Linder (1988), who suggested several algebraic forms like $\alpha(z) = a_0 + a_1z$ or $\alpha(z) = a_\alpha + a_\sigma(1 + 3\gamma^2)^{\alpha_\sigma}$. Santos & Lima (2006) proposed $\alpha(z) = \beta_0(1 + z)^{\beta_0} / [1 + \beta_1(1 + z)^{\beta_1}]$. This paper studies the following form of $\alpha(z)$:

$$\alpha(z) = 1 + D(z)(\delta)_{1D},$$

where $\delta_{1D}$ is the mean of the present-day density fluctuations along the line of sight, and $D(z)$ describes its evolution. Below it is shown that if $D = (1 + z)^{-5/4}$, then the Dyer–Roeder equation gives results consistent with the results obtained under the assumption of the lensing approximation.

3 LINEAR PERTURBATIONS AND THE LENSING APPROXIMATION

Writing the distance as

$$D_{\Lambda}(z) = D_{\Lambda}(1 + \delta_0),$$

where $D_{\Lambda}$ is the distance in a homogeneous universe, one can derive a formula for $\delta_0$ using a linear perturbative scheme. The most general form was presented and discussed by Pyne & Birkshaw (2004), Bonvin et al. (2006), Hui & Greene (2006) and Enqvist, Mattsson & Rigopoulos (2009). Excluding the contribution from the motion of the observer and source, and taking the leading term, $\delta_{1D}$ reduces to

$$\delta_{1D} = -\int_0^\chi \frac{Xe - X}{X_e} \nabla^2\phi(\chi) d\chi,$$

where $\chi$ is the comoving coordinate $d = d z/H(z) + \phi$ is the gravitational potential, which can be related to density perturbations $\rho \delta$ via the Poisson equation $\nabla^2\phi = 4\pi G \rho / c^2 \rho \delta$. Equation (5) is equivalent to convergence in the lensing approximation and is known as the Born approximation. As seen, voids ($\delta < 0$) increase the distance while regions of $\delta > 0$ decrease it.

In order to solve (5), one needs to know $\nabla^2\phi$ along the line of sight. Using the Poisson equation, the gravitational potential is related to the density fluctuations. Thus when calculating the variance one can Fourier-transform $\delta$ and use the matter power spectrum instead (Munshi & Jain 2000, 2001). However, in this paper we are not interested in the variance or higher order moments, just in the distance itself. Therefore, to solve (5) actual density fluctuations along the line of sight are needed.

The present-day fluctuations are non-linear and therefore no longer Gaussian. Only when density fluctuations are in the linear regime can their probability distribution function (PDF) be approximated by the Gaussian PDF. Once density fluctuations are in the non-linear regime, they are no longer Gaussian (there is no symmetry as $-1 < \delta < \infty$). However, it has been shown that in the non-linear regime density fluctuations can be approximated by the one-point log-normal PDF (Kayo, Taruya & Suto 2001; Lahav & Suto 2004):

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma^2_{nl}}} \exp\left[-\frac{(\ln(1 + \delta) + \sigma_{nl}^2/2)^2}{2\sigma_{nl}^2}\right] \frac{1}{1 + \delta},$$

where

$$\sigma_{nl}^2 = \ln[1 + \sigma_R^2]$$

and

$$\sigma_R^2 = \int_0^\infty \mathcal{P}(k) W^2(k) k^2 dk,$$

where $\mathcal{P}(k)$ is the matter power spectrum.

Two methods for generating density fluctuations along the line of sight are considered. Both have a log-normal PDF of $\delta$ (3D). The first method ensures that $\langle \delta \rangle_{1D} = 0$ (the mean of the present-day fluctuations along the line of sight). The second method, on the other hand, allows for $\langle \delta \rangle_{1D} \neq 0$.

4 RESULTS

4.1 Vanishing mean of the density fluctuations along the line of sight

First let us focus on generating the density fluctuations using the log-normal PDF directly. Using (6), $\delta$ smoothed in a sphere of radius $R$ can be generated. Density fluctuations in this model are schematically presented in Fig. 1(a) – when the light ray exits one sphere it enters another one of different $R$ and $\delta$ (for details see Appendix A).

The distance correction $\delta_D$ calculated from (5) is presented in Fig. 2. One point about Fig. 2 needs to be emphasized. As follows from (5), the distance correction depends on the position of the source, for example $\delta_D(z_{*} = 0.5)$ for a source at $z_{*} = 2$ is different from $\delta_D(z_{*} = 0.5)$ for a source at $z_{*} = 0.5$ even if density fluctuations along the line of sight are the same. Thus, the distance correction presented in Fig. 2 is the distance correction for the source at a given $z$, i.e. $\delta_D(z_{*} = z_{*})$. Fig. 2 presents the mean and variance: at each $z$ (a discrete number of $z$ with an interval $\Delta z = 0.01$ was considered) based on 100 000 runs (each with a different distribution of density fluctuations along the line of sight); the mean and variance were calculated. The mean and variance are represented by solid lines. For comparison, the Dyer–Roeder approximation with $\alpha = 0.99$ is also presented (when $\alpha = 1$ the Dyer–Roeder approximation reduces to the standard Friedmann relation for the distance). As seen, $\delta_D$ is of negligible amplitude $\sim 10^{-3}$. Thus for this configuration, and under the assumptions considered, there is no need to take inhomogeneities into account. Also, another important fact is that $\delta_D$ in the weak-lensing approximation is negative, while the Dyer–Roeder approximation implies positive $\delta_D$. This is a consequence of positive density fluctuations along the line of sight. For example, for a source at $z_{*} = 3$ (based on 100 000 runs) the mean of the present-day density fluctuations along the line of sight defined as

$$\langle \delta \rangle_{1D}(r) = \frac{1}{r} \int_0^r d\tilde{r} \delta(t_0, \tilde{r})$$

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Figure 1. Density inhomogeneities along the line of sight. (a) Density fluctuations generated from the log-normal PDF (6): when the light ray exits one structure, the next one is generated. (b) Compensated density fluctuations: when the light ray exits a structure, another one is generated together with an initial angle at which the ray enters the structure. When $\gamma$ is large then the light enters and exits the structure at large angles. In this case structures can overlap. (c) Light propagation in the real Universe.

Figure 2. Distance correction $\delta_D$ within the weak-lensing approximation for a model with a negligible mean of density fluctuations along the line of sight (Section 4.1). The dash–dotted line presents $\delta_D$ obtained using the Dyer–Roeder relation with the smoothness parameter $\alpha = 0.99$ ($\alpha = 1$ is equivalent to the standard Friedmann relation).

is $\langle \delta \rangle_{3D} = 6 \times 10^{-3}$ and the standard deviation is $2 \times 10^{-2}$, which implies $\delta_D = -5 \times 10^{-4} \pm 3 \times 10^{-3}$. Thus, as pointed out by Räsänen (2009), in order to describe properly the effect of clustering, the smoothness parameter should be larger than 1 and not smaller than 1 as in the Dyer–Roeder approximation. However, in this case the difference as well as the total correction are negligible. Still, even if the mean is negligible, the variance can provide additional information – see Linder (2008) for a discussion.

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4.2 Non-zero mean of the density fluctuations along the line of sight

In the previous section, random matter fluctuations along the line of sight were considered. In such a case, $\langle \delta \rangle_{3D} = 0$ implies $\langle \delta \rangle_{1D} = 0$ and the distance correction is negligible. However, present-day matter fluctuations in the Universe are not purely random but are organized – matter in the Universe forms the cosmic web. In this case $\langle \delta \rangle_{3D} = 0$ does not necessarily imply $\langle \delta \rangle_{1D} = 0$. As seen in Fig. 1(c), the cosmic web contains large voids with fairly compact filaments. Therefore, photons spend more time in voids than in overdense regions. In order to model this phenomenon let us consider a universe that consists of voids surrounded by filaments, each structure compensated. Thus, by construction a three-dimensional (3D) average of density fluctuations is zero (if averaged over sufficiently large scales). Moreover, if the parameters of the system are adjusted (see Appendix B for details) then the PDF of density fluctuations can be almost log-normal. This feature is presented in Fig. 3, which shows the PDF of density fluctuations smoothed within a sphere of radius 4 Mpc. As seen, the PDF is similar to a log-normal PDF, apart from for high $\delta$ where the PDF is of lower amplitude – but see Jain, Seljak & White (2000) where the PDF of density fluctuations smoothed on scales of $3h^{-1}$ Mpc also has a lower amplitude for high $\delta$ than the log-normal distribution.

When the light ray exits one compensated structure before it enters another one, not only are the parameters of the next structure generated, but also an angle $\gamma$ at which the light ray enters another structure. Therefore, four different methods are going to be considered.

(i) $\gamma = 0$. The light ray always enters another structure at $\gamma = 0$, thus passing through the centre of the structure.

(ii) $\gamma$ is randomly generated from a uniform distribution between 0 and 0.5$\pi$. However, as shown in Fig. 1(b), in this case structures can overlap.

(iii) To reduce overlapping, $\gamma$ is generated from the Gaussian distribution with mean 0 and $\sigma = 0.25\pi$ (if $|\gamma| > 0.5\pi$ then $|\gamma| \rightarrow \pi - |\gamma|$).

(iv) $\gamma$ is randomly generated from a uniform distribution between 0 and 0.25$\pi$. 

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The mean density fluctuations along the line of sight for these cases are presented in Fig. 4. The distance correction $\delta_D$ (the mean and variance) is presented in Fig. 5. As seen, if $\gamma$ is randomly generated from a uniform distribution between 0 and $0.5\pi$ then the mean of density fluctuations along the line of sight is small and $\delta_D$ is almost negligible (cf. Brouzakis, Tetradis & Tzavara 2008; Vanderveld, Flanagan & Wasserman 2008).

However, as pointed out above, if $\gamma$ changes between 0 and $0.5\pi$ then structures can overlap (see Fig 1b). To reduce the overlapping, $\gamma$ needs to be chosen from a smaller range of angles. In this case $\delta_D$ increases. This is because the light rays more likely propagate through voids than through filaments.

In Fig. 5, the Dyer–Roeder relation is also presented. The dash–dotted line presents $\delta_D$ obtained from the original Dyer–Roeder formula, while the dashed and dotted lines present the evolving smoothness parameter (3). Intuitively, if $\langle \delta \rangle_{1D}$ were just an ordinary density perturbation, in the linear regime it should evolve according to (A1). Thus, the dotted line presents a model with $\alpha(z) = 1 + \langle \delta \rangle_{1D}$, where $\delta(z)$ is given by (A1) with an initial condition $\delta(z = 0) = \langle \delta \rangle_{1D}$.

Following Mattsson (2010), let us consider the Dyer–Roeder relation with a perturbed expansion rate. From the continuity equation $\dot{\rho} + 3H\rho = 0$, the perturbation in the expansion rate is
\[ \Delta H = -\dot{\delta}/3, \] where \( \dot{\delta} \) follows from (A1) with an initial condition \( \delta(z = 0) = (\delta)_{1D} \). We use this relation to calculate the perturbed \( H(z) \) and \( dH/dz \), and then insert this into (2) and, using \( a(z) = 1 + \delta(z) \) where \( \delta(z) \) is given by (A1), we obtain the distance correction that is presented using the dashed lines.

As seen, neither a constant \( \alpha \), nor the evolving one with \( \delta(z) \) given by (A1), nor a perturbed expansion rate are consistent with the weak-lensing approximation (expect for the case where \( \delta_0 \) is almost negligible). However, if \( a(z) = 1 + (\delta)_{1D}/(1 + z)^\gamma \), where \( \gamma \approx 5/4 \), then the Dyer–Roeder approximation produces results comparable with the weak-lensing approximation. As seen, the evolution of \( \alpha \) is not as intuitively expected, i.e. it does not directly follow from (A1). Here an empirical approach was employed, and it was found that if \( \gamma \approx 5/4 \) then the Dyer–Roeder relation modified in this way leads to agreement with the lensing approximation.

An important result of the above analysis is that a non-zero mean of density fluctuations along the line of sight can modify the distance by a few per cent. As cosmological observations are now reaching a precision of a few per cent, a proper handling of the distance is essential. Otherwise the errors in the estimation of the distance can become a major source of systematics. For example, a proper handling of distance will be of great importance in the future analysis of measurements of the baryon acoustic oscillations (Bolejko 2011).

5 CONCLUSIONS

The distance–redshift relation plays a central role in cosmology. If the homogeneous Friedmann model correctly describes the evolution of the Universe on large scales and, in addition, if density fluctuations along the line of sight vanish after averaging, then within the linear approximation the distance correction is negligible – it is sufficient to apply the Friedmann relation. However, if the mean of density fluctuations along the line of sight is not zero then, even within the linear approximation, the distance can change by several per cent.

In Section 4.2 it was shown that a vanishing 3D average of density fluctuations does not imply that the mean of density fluctuations along the line of sight is zero. It is argued that in the real Universe this may be the case. In the real Universe, voids occupy large regions while overdensities are more compact. Moreover, if light propagates for a long time through filaments then it is more likely to be absorbed or scattered. Thus, if a remote galaxy is observed then most likely its photons propagated through emptier rather than denser regions. In this case the weak-lensing approximation produces results that are similar to results obtained from the modified version of the Dyer–Roeder equation. The modified relation is

\[
\frac{d^2 D_A}{dz^2} + \left( \frac{1}{H} \frac{dH}{dz} + \frac{2}{1 + z} \right) \frac{dD_A}{dz} + \frac{3}{2} \frac{H_0^2}{H^2} \Omega_m(1 + z) \left[ 1 + \left( \frac{\delta}{1 + z} \right)_{1D} \right] D_A = 0, \tag{9}
\]

with initial conditions \( D_0 = 0 \) and \( dD_A/dz = 1/H_0 \). Thus, apart from the background cosmological parameters, the only free parameter left is the mean of density fluctuations along the line of sight \( (\delta)_{1D} \). Since (9) is just an ordinary differential equation, it is as easy to implement and solve numerically as the standard relation for distance in the Friedmann model. The mean of density fluctuations along the line of sight \( (\delta)_{1D} \) could be deduced from either galaxy redshift surveys or N-body simulations.

It should be noted that the analysis presented here was based on the linear approximation (the lensing approximation), i.e. higher order corrections to the distance were not taken into account. It is likely that quadratic corrections may lead to non-negligible \( \delta_0 \) even when \( (\delta)_{1D} \approx 0 \). In this case the modified version of the Dyer–Roeder equation presented above would not be consistent with the actual non-linear distance-redshift relation, as when \( (\delta)_{1D} = 0 \) it reduces to the Friedmann relation.

The most important conclusion of this paper is that even within the idealized case of a large-scale homogeneous universe with only Mpc-scale inhomogeneities, the distance can be different from that in a fully homogeneous universe. The difference can be of the order of a few per cent. Thus, with the increasing precision of cosmological observations, accurate estimation of the distance is essential. Otherwise errors due to miscalculation of the distance can become a major source of systematics.

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APPENDIX A: ALGORITHM FOR CALCULATING $\delta_D$ FOR THE MODEL DISCUSSED IN SECTION 4.1

The algorithm for calculating $\delta_D$ along the line of sight is as follows.

(i) The radius of a structure $R$ is randomly generated from a uniform distribution, from 0–3 Mpc.

(ii) Then $\sigma_R$ is calculated using (7). The primordial power spectrum was chosen in agreement with WMAP7 data (Komatsu et al. 2010): $A_{k_n}$ with $n_s = 0.969$ and the amplitude $A$ chosen so that $\sigma_8 = 0.803$. The transfer function was calculated according to Eisenstein & Hu (1998).

(iii) An initial value of a density fluctuation $\delta_0$ is generated from the log-normal distribution (6).

(iv) The evolution of $\delta$ (at a fixed point) is calculated using the linear approximation (Peebles 1980)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \frac{4\pi G}{c^2} \rho \delta. \tag{A1}$$

(v) Using the Poisson equation, $\nabla^2 \phi$ is calculated and inserted into (5), which is solved from $\chi_i$ to $\chi_e$.

The cosmological parameters are the same as those derived from the seven-year WMAP data: $H_0 = 71.4$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_m = 0.262, \Omega_\Lambda = 0.738$ (Komatsu et al. 2010).

APPENDIX B: ALGORITHM FOR CALCULATING $\delta_D$ FOR THE MODEL DISCUSSED IN SECTION 4.2

The algorithm for calculating $\delta_D$ is as follows.

(i) First the radius of a void $R_v$ is generated from the Gaussian distribution with mean 12 Mpc and standard deviation 2 Mpc.

(ii) Density within the void $\Omega_v$ is generated from the Gaussian distribution with mean $0.2 \Omega_m$ and $\sigma = 0.27 \Omega_m$. If the $\Omega_v$ generated in this way is lower than 0.01 $\Omega_m$ then the generation is repeated. If after six attempts it is still less than 0.01 $\Omega_m$ then $\Omega_v$ is generated for a uniform distribution between 0 and 0.01 $\Omega_m$. If $\Omega_v \geq 0.01 \Omega_m$ then $\Omega_v$ is generated one more time. If after six attempts $\Omega_v \geq 0.01 \Omega_m$ then its value is chosen for a uniform distribution from 0.85 $\Omega_m$ to $\Omega_m$.

(iii) Density of the surrounding shell $\Omega_s$ is generated from the Gaussian distribution with mean $1.75 \Omega_m$ and $\sigma = 0.7 \Omega_m$. If $\Omega_s \leq 0.01 \Omega_m$ then its value is generated again. If after six attempts $\Omega_s \leq 0.01 \Omega_m$ then its value is generated for a uniform distribution between $1.75 \Omega_m$ and $1.95 \Omega_m$.

(iv) The condition that the structure is compensated implies that the radius of the whole structure is

$$R = R_v \left( \frac{\Omega_v - \Omega_s}{\Omega_v - \Omega_m} \right)^{1/3}. \tag{B1}$$

(v) The angle at which the light ray enters the structure is generated using four different methods – for details see Section 4.2.

(vi) The evolution of $\delta$ (at a fixed point) is calculated using (A1).

(vii) Integral (5) is solved from $\chi_i$ to $\chi_f$ (where $\chi_i$ is the comoving coordinate of the entry point and $\chi_f$ the point where the light ray exits the structure).

(viii) Steps (i)–(vii) are repeated so that (5) is solved from $\chi = 0$ to $\chi_e$.

If, instead of the Gaussian, a uniform PDF is used in steps (i)–(iii) then the PDF of $\delta$ is not a log-normal PDF as presented in Fig. 3.

The cosmological parameters are the same as those derived from the seven-year WMAP data: $H_0 = 71.4$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_m = 0.262, \Omega_\Lambda = 0.738$ (Komatsu et al. 2010).

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