Gravitational constraint combinations generate a Lie algebra

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Abstract. We find a first-order partial differential equation whose solutions are all ultralocal scalar combinations of gravitational constraints with Abelian Poisson brackets between themselves. This is a generalization of the Kuchař idea of finding alternative constraints for canonical gravity. The new scalars may be used in place of the Hamiltonian constraint of general relativity and, together with the usual momentum constraints, replace the Dirac algebra for pure gravity with a true Lie algebra: the semidirect product of the Abelian algebra of the new constraint combinations with the algebra of spatial diffeomorphisms.

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1. Introduction

In the Hamiltonian formulation of general relativity one assumes that the spacetime manifold \( \mathcal{M} \), with metric \( \gamma_{\alpha\beta}(X) \), can be foliated as \( \Sigma \times \mathbb{R} \) where \( \Sigma \) is a three-dimensional manifold and the real line provides a global time coordinate‡. The physical information of the four-dimensional theory is then contained in the intrinsic metric \( g_{ij}(x) \) on \( \Sigma \)—the pullback of \( \gamma_{\alpha\beta}(X) \) on \( \Sigma \)—and its conjugate momentum \( p^{ij}(x) \).

On the foliated manifold, the Einstein equations of general relativity become ten equations that the canonical data \( (g_{ij}(x), p^{ij}(x)) \) must satisfy. Six of these are first-order dynamical equations and describe how the spatial metric and its conjugate momentum change with time; the remaining four are constraints that \( (g_{ij}(x), p^{ij}(x)) \) must obey at any point \( x \) on \( \Sigma \) and reflect the symmetries of the four-dimensional theory. The momentum constraints \( \mathcal{H}_i(x) \) generate spatial diffeomorphisms on a slice \( \Sigma \), and the Hamiltonian constraint \( \mathcal{H}_\perp(x) \) propagates \( \Sigma \) in the direction normal to the hypersurface. As functions of \( g_{ij}(x), p^{ij}(x) \) and the determinant of the spatial metric, \( g(x) \), the gravitational constraints are [1, 2]

\[
\mathcal{H}_\perp(x; g, p) = G_{ijkl}(x; g)p^{ij}(x)p^{kl}(x) - g^{1/2}(x) R(x; g)
\]  

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‡ The notation used in this paper is: events on \( \mathcal{M} \) are denoted by \( X^\alpha \), while spatial points on \( \Sigma \) are labelled by \( x^i \). Indices \( \alpha, \beta, \ldots \) are spacetime indices running from 0 to 3; \( i, j, \ldots \) are indices on \( \Sigma \) that run from 1 to 3. A standard notation is used to distinguish functions from functionals, namely, round brackets, ( ), indicate that the object is a function of the arguments in the brackets, square brackets, [ ], denote a functional and an object with a mixed bracket, ( ; [ ], is a function of the arguments that appear on the left and a functional of those on the right. Finally, the covariant derivative on \( (\Sigma, g) \) is written as \( D_i \).
where \( G_{ijkl}(x; g) = \frac{1}{2} g^{-1/2}(x)(g_{ik}g_{jl} + g_{ij}g_{lk} - g_{ij}g_{kl}) \)
\[
\mathcal{H}_i(x; g, p) = -2D_j p^j_i(x).
\]

The constraint system,
\[
\mathcal{H}_\perp(x) = 0 = \mathcal{H}_i(x),
\]
satisfies the Dirac algebra
\[
\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\} = g^{ij}(x)\mathcal{H}_i(x)\delta_j(x, x') - (x \leftrightarrow x')
\]
\[
\{\mathcal{H}_\perp(x), \mathcal{H}_i(x')\} = \mathcal{H}_\perp(x)\delta_i(x, x') + \mathcal{H}_{\perp}(x)\delta_i(x, x')
\]
\[
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_j(x)\delta_i(x, x') - (ix \leftrightarrow jx').
\]

The Dirac algebra is often thought of as the algebra of spacetime diffeomorphisms, \( \text{LDiff} \, M \), ‘projected’ onto the foliation \( \Sigma \times \mathbb{R} \). Equation (6) is the algebra of spatial diffeomorphisms, generated by \( \mathcal{H}_i(x) \). The Poisson bracket (5) shows that \( \mathcal{H}_\perp(x) \) transforms as a scalar density of weight 1 under spatial diffeomorphisms. The first equation, (4), however, means that two infinitesimal normal deformations on \( \Sigma \), performed in arbitrary order, end on the same final hypersurface but not on the same point on that hypersurface [3]. The fact that the right-hand side of this Poisson bracket involves the metric explicitly is a source of problems in any attempt to use the Dirac algebra in a canonical quantization of gravity. The Dirac algebra is closed, in the sense that the Poisson brackets between constraints are constraints themselves, however, the structure coefficients in equation (4) depend on the canonical variables. In a quantum version of the theory where, presumably, quantum counterparts of the classical gravitational data and the constraints will appear, the Poisson bracket algebra will go over to a commutator algebra which is not a Lie algebra.

Recently, Brown and Kuchař [5] obtained some surprising results which provide a promising proposal on the problem of the algebra of canonical gravity. They studied a spacetime filled with incoherent dust; in such a system the coupling of the metric to matter introduces into spacetime a privileged dynamical frame and time foliation. The system has a very interesting feature: the scalar constraint for the combined system, which in [5] was denoted by \( \mathcal{H}_\uparrow(x) \), can be split into two separate parts, one for matter and one for gravity, and cast in the form
\[
\mathcal{H}_\uparrow(x) := P(x) + h(x; g_{ij}, p^{ij}) = 0,
\]
where \( P(x) \) is the momentum conjugate to the ‘dust time’ variable \( T(x) \), and \( h(x) \) is the following scalar combination of the gravitational constraints:
\[
h(x)^2 = G(x) := \mathcal{H}_{\perp}^2(x) - g^{ij}\mathcal{H}_i(x)\mathcal{H}_j(x).
\]

The truly remarkable result of [5] was that \( G(x) \) and \( h(x) \), scalar functions of gravitational variables only, have strongly vanishing Poisson brackets among themselves. One then has a possible alternative set of constraints for pure gravity,
\[
G(x) = 0 = \mathcal{H}_i(x),
\]
which generate the true Lie algebra,
\[
\{G(x), G(x')\} = 0,
\]
\[
\{G(x), \mathcal{H}_i(x')\} = g_{,ij}(x)\delta_j(x, x') + 2G(x)\delta_i(x, x'),
\]
\[
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_j(x)\delta_i(x, x') - (ix \leftrightarrow jx').
\]

This algebra is the semidirect product of the Abelian algebra generated by \( G(x) \) (equation (10)), and the algebra of spatial diffeomorphisms \( \text{LDiff} \Sigma \) generated by \( \mathcal{H}_i(x) \).
Gravitational constraint combinations generate a Lie algebra

Previous investigations on the use of matter time and reference fluids in canonical gravity [6, 7] led to the ‘phenomenological’ approach to the problem of time in canonical gravity of [5]. The significance of the new constraints (8), however, and the role they could play in pure gravity was left to be investigated at some later stage. That dust was not, in itself, essential in the process was made clear when Kuchař and Romano [8] found a simpler way of producing results similar to those of [5]. They coupled gravity to a massless scalar field, and the scalar density they obtained is the following combination of gravitational constraints:

\[ \Lambda^\pm(x) = g^{1/2}(x)(-H_\perp(x) \pm \sqrt{G(x)}). \]  

(13)

As in the case of \( G(x) \), the weight-2 scalar densities \( \Lambda^\pm(x) \) have strongly vanishing Poisson brackets with themselves. They can be used to rewrite the constraint system for pure gravity as

\[ \Lambda^\pm(x) = 0 = H(x) \]  

(14)

and, again, if \( \Lambda^\pm(x) \) can be taken to replace \( H_\perp(x) \) a true Lie algebra for vacuum gravity has been found.

Although coupling to matter was an essential step for [5, 8] to find the explicit form of the new scalar constraint, the resulting combinations, \( G(x) \) and \( \Lambda(x) \), involve gravitational variables only and do not depend on the matter variables to which the gravitational field was originally coupled. The question that naturally arises therefore is whether a transformation of the scalar constraint to a form that generates a true Lie algebra can be made within the context of pure gravity only.

In this paper we take the first step towards answering this question by finding the full set of scalar densities, combinations of the original pure gravity constraints, that have the crucial property of generating a Lie algebra (with vanishing Poisson brackets between themselves). We find that the calculation can be carried out by working directly with the Dirac algebra, without any reference to matter couplings. Thus, we are not restricted by any need to include matter fields and there is greater freedom in using the new algebra in pure gravity. A complementary view of these combinations of pure gravity constraints is provided in a paper by Kouletsis [9], to appear shortly. In this work the constraint combinations are shown to arise naturally in a system of pure gravity with non-derivative coupling to an action of a single scalar field and two arbitrary functions of a Lagrange multiplier.

2. Constraint combinations are generated by a differential equation

From the work of [5, 8], it seems clear that there might be more combinations of gravitational constraints obeying the algebra (10)–(12). We want to find them all, and furthermore, as explained in the introduction, we wish to investigate the new algebra without any reference to a matter coupling. We therefore choose to work directly with the Dirac algebra of pure gravity.

A particularly interesting feature of the constraints \( G(x) \) and \( \Lambda^\pm(x) \) is that they are scalar densities of weight 2, unlike the usual Hamiltonian constraint \( H_\perp(x) \) which is of weight-1 scalar density. An important question therefore is whether a weight of 2 is in some way a natural choice for a scalar constraint for pure gravity that has Abelian Poisson brackets with itself.
For generality then, we assume that the candidate new constraint we are looking for is an ultralocal scalar density \( K(x) \) of arbitrary weight \( \omega \). Such a \( K(x) \) can only be a function of the scalars \( H_\perp, (g^{ij} H_i H_j) \), and the determinant of the spatial metric \( g \) (there are no derivatives of the gravitational constraints). Together with the usual momentum constraint, \( K(x) \) is assumed to satisfy the Lie algebra

\[
\{ K(x), K(x') \} = 0, \tag{15}
\]

\[
\{ K(x), H_i(x') \} = K_{,i}(x) \delta(x, x') + \omega K(x) \delta_{ij}(x, x'), \tag{16}
\]

\[
\{ H_i(x), H_j(x') \} = H_j(x) \delta_{ij}(x, x') - (ix \leftrightarrow jx'). \tag{17}
\]

The bracket (16), consistent with our original assumptions, means simply that \( K(x) \) transforms as a scalar of weight \( \omega \) under \( \text{Diff} \). The dependence of \( K(x) \) on \( H_\perp(x) \) and \( H_i(x) \) lies in the first bracket (15), which we shall now evaluate.

The calculations that need to be done are considerably simplified if we define \( h \) and \( f \) as the two obvious scalar densities of weight 0 that can be formed from the gravitational constraints \( H_\perp \) and \( H_i \):

\[
h := g^{-1/2} H_\perp \quad \text{and} \quad f := (g^{-1})^{ij} H_i H_j. \tag{18}
\]

Using \( h \) and \( f \), one can define another weight-0 scalar density \( K(x) \), related to the weight-\( \omega \) density \( K(x) \) by

\[
K(x; h, f, g) = g^{\omega/2}(x) K(x; h, f). \tag{19}
\]

It is easy to construct a \( K(x) \) by premultiplying \( K(x) \) by the \( g^{\omega/2}(x) \) factor, so for simplicity, our discussion is centred around \( K(x) \). One only needs to remember that \( K(x) \) itself does not satisfy the Abelian Poisson bracket (15). Its dependence on \( h, f \), however, is the same as that of \( K(x) \).

Imposing the central requirement that \( K \) satisfies the Poisson bracket (15), we can now find, by simply working out the Poisson brackets, which (weight-0 densities) that \( K \) corresponds to the (weight-\( \omega \) densities) \( K \) that generate a true Lie algebra. Our calculation is a generalized analogue of the calculation of Kuchař and Romano, so we concentrate on the steps needed to adapt their calculation for arbitrary-weight densities.

By the Leibniz rule, equation (15) gives

\[
\{ K, K' \} = g^{\omega/2} g^{\omega/2} \{ K, K' \} + (g^{\omega/2} K \{ g^{\omega/2}, K' \} - (x \leftrightarrow x')), \tag{20}
\]

where, to avoid a clumping of symbols, primes have been used on quantities that have \( x' \) as their spatial argument. The Poisson brackets above can be expanded to

\[
\{ K, K' \} = \left( \frac{\partial K}{\partial h} \frac{\partial K'}{\partial h'} \{ h, h' \} + \frac{\partial K}{\partial f} \frac{\partial K'}{\partial f'} \{ f, f' \} + \left( \frac{\partial K}{\partial h} \frac{\partial K'}{\partial f'} \right) \{ h, f' \} - (x \leftrightarrow x') \right). \tag{21}
\]

\[
\{ g^{\omega/2}, K' \} = \left( \frac{\partial K'}{\partial h'} \{ g^{\omega/2}, h' \} + \frac{\partial K'}{\partial f'} \{ g^{\omega/2}, f' \} \right). \tag{22}
\]

and, therefore, one needs to calculate Poisson brackets involving \( h, f \) and \( g^{\omega/2} \). To handle these brackets, the identity

\[
g^{\omega/2} \{ g^{\omega/2}, f' \} = \omega g^{(\omega-1)/2} \{ g^{1/2}, f' \}. \tag{23}
\]

is used, while the bracket between \( g^{\omega/2} \) and \( h \) is proportional to \( \delta(x, x') \) and therefore its contribution cancels with the \((x \leftrightarrow x')\) terms that appear in equation (20). For the brackets
of the weight-0 gravitational variables $h$ and $f$, it is straightforward, using the Jacobi identity and the standard Dirac relations (4)–(6), to verify that they satisfy the algebra

\[ \{ h, h' \} = g^{-1/2}g^{-1/2}[\mathcal{H}_\perp, \mathcal{H}'_\perp], \]
\[ \{ f, f' \} = -4g^{-1/2}g^{-1/2}f(\mathcal{H}_\perp, \mathcal{H}'_\perp), \]
\[ \{ h, f' \} = \propto \delta(x, x'), \]

where $\propto \delta(x, x')$ stands for terms proportional to $\delta(x, x')$ which, as always, cancel with the $(x \leftrightarrow x')$ terms. One now has all the necessary information to extend the procedure of Kuchař and Romano [8] for $K(x)$.

The requirements on the scalar density $K(x)$ are minimal. Thus one would expect that the rather lengthy calculation of the Poisson brackets between the $K$’s would lead to a fairly cumbersome result. It is surprising to find that it is instead quite simple:

\[ \{ K, K' \} = g^{\omega-1}\left[ \left( \frac{\partial K}{\partial h} \right)^2 - 4f \left( \frac{\partial K}{\partial f} \right)^2 + 2\omega K \left( \frac{\partial K}{\partial f} \right) \right]\{ \mathcal{H}_\perp, \mathcal{H}'_\perp \}. \]

All functions $K$ that turn the quantity inside the square brackets in (27) into zero correspond, via equation (19), to candidate new constraints $\mathcal{K}$. The family of these scalar functions are therefore the solutions of the nonlinear, first-order partial differential equation for $K$

\[ \frac{\omega}{2} K \left( \frac{\partial K}{\partial f} \right)^2 = f \left( \frac{\partial K}{\partial f} \right)^2 - \frac{1}{4} \left( \frac{\partial K}{\partial h} \right)^2. \]

This unexpectedly compact equation is the key result because any of its solutions can be used directly to give a Lie algebra for pure gravity. Also, if by some other method one has already obtained a combination of gravitational constraints which is suspected to satisfy a Lie algebra, it is enough (and far easier) to instead check that it satisfies the differential equation. For example, it can be easily verified that the differential equation (28) is satisfied by the scalars that were discovered in [5, 8] (in their respective weight-0 form),

\[ K \equiv g^{-1} G = h^2 - f \quad \text{and} \quad K \equiv g^{-1} \Lambda_\pm = -h \pm \sqrt{h^2 - f}, \]

for weight $\omega = 2$. It is also satisfied by the weight $\omega = 1$ scalar $\sqrt{h^2 - f}$ that appeared in [5].

3. The solution of the differential equation

The general solution to the first-order nonlinear partial differential equation (28) of the two independent variables $h$ and $f$, for $\omega \neq 0$, is expected to be an expression involving $\mathcal{H}_\perp$, $\mathcal{H}_i$ and an arbitrary function of one parameter.

A change of variables can turn (28) into a more manageable equation. If we set

\[ C = \ln K \]

(we assume that $C$ in the above definition is well defined) and then divide the differential equation (28) through by $K^2$, we can rearrange and rewrite it as a partial differential equation for $C$:

\[ f \left( \frac{\partial C}{\partial f} \right)^2 - \frac{\omega}{2} \frac{\partial C}{\partial f} - \frac{1}{4} \left( \frac{\partial C}{\partial h} \right)^2 = 0. \]
The advantage of this rearrangement is that this equation does not involve \( C \) explicitly. Because the variables \( h, f \) are independent, the differential of the function \( C(h, f) \) is given by the expression

\[
dC(h, f) = \left( \frac{\partial C}{\partial f} \right) df + \left( \frac{\partial C}{\partial h} \right) dh.\tag{32}
\]

In equation (31) there are now no terms involving both variables \( h \) and \( f \), and thus it can be ‘split’ into two parts, for \( \partial C/\partial f \) and \( \partial C/\partial h \); namely, it holds [11] that any solution to the coupled set

\[
f \left( \frac{\partial C}{\partial f} \right)^2 - \omega \frac{\partial C}{2 \partial f} = \frac{\alpha^2}{4} \left( \frac{\partial C}{\partial h} \right)^2 = \alpha^2\tag{33}
\]

for some constant \( \alpha \), will also satisfy (31). These two, single-variable equations can be readily solved as

\[
\frac{\partial C}{\partial f} = \frac{1}{4f} \left( \omega \pm \sqrt{\omega^2 + 16\alpha^2f} \right) \quad \text{and} \quad \frac{\partial C}{\partial h} = \pm 2\alpha\tag{34}
\]

(note that the \( \pm \) signs appearing in the above pair of equations are independent) and therefore the differential of \( C(h, f) \) in equation (32) can be integrated to

\[
C(h, f, \alpha, \beta) = \frac{\omega}{4} \ln f \pm \frac{1}{4} \left[ 2\sqrt{\omega^2 + 16\alpha^2f} + \omega \ln \frac{\sqrt{\omega^2 + 16\alpha^2f} - \omega}{\sqrt{\omega^2 + 16\alpha^2f} + \omega} \right] \pm 2\alpha h + \beta \tag{35}
\]

(where, again, the two possible choices of \( \pm \) sign are independent). This is a complete integral of the differential equation for \( C(h, f) \); namely, a solution of the equation involving two arbitrary independent constants \( \alpha, \beta \) which come from the two integrations in equation (32). The complete integral then is a two-parameter family of surfaces, since a different surface is obtained for each choice of \( \alpha \) and \( \beta \).

The general solution should also describe this family of surfaces. The difference is that, instead of depending on two arbitrary parameters, the general solution should contain an arbitrary function of one parameter. That is, we suppose that

\[
\beta = \phi(\alpha)\tag{36}
\]

so that \( C(h, f, \alpha, \phi(\alpha)) \) describes a one-parameter family of solutions. One can think of \( \phi(\alpha) \) as the two-variable analogue of the integration constant of the familiar one-variable partial differential equation.

One can go further by bringing in a basic theorem in the theory of first-order partial differential equations [11], that the envelope of any family of solutions of a first-order equation, depending on some parameter, is again a solution. The envelope of the one-parameter family of solutions \( C(h, f, \alpha, \phi(\alpha)) \) is essentially a surface tangent to the family of surfaces that are the solutions. It should then be given by \( C(h, f, \alpha, \phi(\alpha)) \), together with its differential with respect to \( \alpha \), namely,

\[
\frac{\partial C}{\partial \alpha} = \pm \sqrt{\frac{\omega^2 + 16\alpha^2f}{2\alpha}} \pm 2h + \phi'(\alpha) = 0,\tag{37}
\]

where a prime denotes differentiation with respect to \( \alpha \). One now only has to eliminate \( \alpha \) from the above equation and \( C(h, f, \alpha, \phi(\alpha)) \) to have a single expression for the envelope. This will also be the general solution involving the arbitrary function \( \phi(\alpha) \) [11]. If we
Gravitational constraint combinations generate a Lie algebra

return to the original equation for \( K \) by exponentiating \( C \) in equation (35), we can write

\[
K_{\pm}(h, f, \phi(\alpha)) = \left[ (h \pm \frac{1}{2}\phi'(\alpha)) \pm \sqrt{(h \pm \frac{1}{2}\phi'(\alpha))^2 - f} \right]^{\omega/2}
\]

\[
\times \exp \left( \phi(\alpha) \pm \frac{\omega}{2} \frac{\frac{1}{2}\phi'(\alpha)}{\sqrt{(h \pm \frac{1}{2}\phi'(\alpha))^2 - f}} \right)
\]

(where the \( \pm \) signs in front of \( \phi(\alpha) \) are independent of those in front of the square roots) together with equation (37). Note that, by a trivial rewriting of the exponential factor, the whole solution can be raised to the power of \((\omega/2)\), and hence the constraint combination \( K(x) \), is simplest but non-trivial when \( \omega = 2 \). This is consistent with the fact that the \('physical\'\) matter couplings used by Kuchař \textit{et al}, also produced weight-2 densities.

Therefore, for each choice of the function \( \phi(\alpha) \), one solves (37) algebraically to obtain \( \phi(\alpha) \) as a function of \( h \) and \( f \), and then substitutes the result in the general solution (38). For example, the Brown–Kuchař solution is reproduced when \( \phi(\alpha) = -\ln \alpha - 1 \) and \( \omega = 2 \) (and the minus signs in front of the square roots), so that \( \phi'(\alpha) = -1/\alpha \). Then, solving (37) gives

\[
\alpha = -\frac{h}{h^2 - f},
\]

(39)

which implies \( \phi'(\alpha) = h - f/h \), and which, when inserted in (38), produces \( K = \pm(h^2 - f) \).

Obtaining the Kuchař–Romano solution is also straightforward, it only requires setting \( \phi(\alpha) = 0 \). With the same method one can also generate new solutions, in fact an infinite number of them, for all possible choices of \( \phi(\alpha) \). New solutions, and also their relationship to scalar field actions, have been investigated by Kouletsis in [9]. Generally, the only simple ones are those that have been already found by Kuchař \textit{et al}.

The weight-\( \omega = 0 \) case has to be treated separately since then equation (28) becomes a homogeneous partial differential equation,

\[
f \left( \frac{\partial K^0}{\partial f} \right)^2 - \frac{1}{4} \left( \frac{\partial K^0}{\partial h} \right)^2 = 0,
\]

(42)

whose solution is of the form \( K^0 = \propto(h \pm \sqrt{f}) + \text{constant} \).

Summarizing, we have shown that for each choice of the function \( \phi(\alpha) \) in the pair of equations (37) and (38), a combination of the gravitational constraints \( h \) and \( f \) is generated. The solution, via equation (19), provides the Abelian weight-\( \omega \) scalar density \( K(x) \) which can be used in the construction of a true Lie algebra for pure gravity.

† For example, the function \( \phi(\alpha) = A \ln \alpha \), where \( A \) is an arbitrary constant, leads to the new solution [10]:

\[
K = \left( -\frac{h}{A} + R_1 + R_2 \right) \left( \frac{Ah + R_1}{2(h^2 - f)} \right)^A \exp \left( \frac{Ah - R_1}{R_2} \right),
\]

(40)

where

\[
R_1 = \sqrt{h^2 + (A^2 - 1)f} \quad \text{and} \quad R = \sqrt{\frac{A^2 + 1}{A^2}h^2 + \frac{A^2 - 1}{A^2}f - \frac{2}{A}hR_1}.
\]

(41)
4. Conclusions

In this paper we started from observations of the alternative Hamiltonian constraints found by Kuchař et al., which generate a true Lie algebra of gravity. We generalized to a much larger family of such scalar densities in the hope that knowing their general form would make their origin clearer. All previous work on these constraints has been based on introducing some simple form of matter into spacetime, either dust [5], a scalar field [8] or the deWitt action for a perfect fluid [12]. We found that it is possible, and in fact simpler, to arrive at these constraint combinations working solely with the algebra of constraints. This is also the least restrictive method and enabled us to generalize to ultralocal scalar densities of arbitrary weight. Here we saw that, although in principle scalar densities of any weight can be employed in the construction of a Lie algebra, the weight-2 ones are the most simple and natural choice. In the present work, we have not addressed the problem of the global equivalence of the scalar densities found to the usual Hamiltonian constraint of general relativity. We have also not discussed the conditions for the constraint combinations obtained to be well defined. For example, it has already been noticed by Brown and Kuchař that the Hamiltonian vector field associated with the new scalars vanishes on the constraint surface. The criteria determining the answer to these questions may depend on the context in which the new scalars will be applied.

There are certainly other interesting aspects of these gravitational constraint combinations. They are related to matter time in canonical gravity and to coordinate conditions [6]. Possibly, they can also come out of a canonical transformation of the geometric canonical data that separates the true degrees of freedom (in the spirit of [13]). The possibilities they open in the quantum theory are certainly worth investigating and we will discuss them in future work.

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