Entanglement manipulation under non-entangling operations

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Abstract. We demonstrate that entanglement shared by two or more parties can be asymptotically reversibly interconverted when one considers the set of operations which asymptotically cannot generate entanglement. In this scenario we find that the entanglement of every quantum state is uniquely characterized by a single quantity: the regularized relative entropy of entanglement. The main technical tool is a generalization of quantum Stein’s Lemma, which gives optimal discrimination rates in quantum hypothesis testing, to the case in which the alternative hypothesis might vary over sets of correlated states. We analyze the connection of our approach to recent rigorous formulations of the second law of thermodynamics.

1. Introduction
A basic feature of many physical settings is the existence of constraints on physical operations and processes that are available. These restrictions generally imply the existence of resources that can be consumed to overcome the constraints. Examples include an auxiliary heat bath in order to decrease the entropy of an isolated thermodynamical system or prior secret correlations for the establishment of secret key between two parties who can only operate locally and communicate by a public channel. In quantum information theory one often considers the scenario in which two or more distant parties want to exchange quantum information, but are restricted to act locally on their quantum systems and communicate classical bits. We find in this context that a resource of intrinsic quantum character, entanglement, allows the parties to completely overcome the limitations caused by the locality requirement on the quantum operations available.

In this respect resource theories are considered to determine when a physical system, or a state thereof, contains a given resource, to characterize the possible conversions from an state to another when one has access only to a restricted class of operations which cannot create the resource for free, and to quantify the amount of such a resource contained in a given system.

One may carry out these investigations at the level of individual systems, which is motivated by experimental considerations. It is natural however to expect that a simplified theory will emerge when instead one looks at the bulk properties of a large number of systems. The most successful example of such a theory is arguably thermodynamics. This theory was initially envisioned to describe the physics of large systems in equilibrium, determining their properties.
by a very simple set of rules of universal character. This was reflected in the formulation of the defining axiom of thermodynamics, the second law, in terms of quasi-static processes and heat exchange. However, the apparently universal applicability of thermodynamics suggested a deeper mathematical and structural foundation. Indeed, there is a long history of examinations of the foundations underlying the second law, starting with Carathéodory work in the beginning of last century [1]. Of particular interest in the present context is the work of Giles [2] and notably Lieb and Yngvason [3] stating that there exists a total ordering of equilibrium thermodynamical states that determines which state transformations are possible by means of an adiabatic process. In this sense, thermodynamics can be seen as a resource theory of order, dictating which transformations are possible between systems with different amounts of (dis)order by operations which cannot order out systems (adiabatic processes).

A remarkable aspect of these foundational works [2, 3] is that from simple, abstract, axioms they were able to show the existence of an entropy function $S$ fully determining the achievable transformations by adiabatic processes: given two equilibrium states $A$ and $B$, $A$ can be converted by an adiabatic process into $B$ if, and only if, $S(A) \leq S(B)$. As pointed out by Lieb and Yngvason [4], it is a strength of this abstract approach that it allows to uncover a thermodynamical structure in settings that may at first appear unrelated.

Early studies in quantum information indicate that entanglement theory could be one such possible setting. Possible connections between entanglement theory and thermodynamics were noted earlier on when it was found that for bipartite pure states a very similar situation to the second law holds in the asymptotic limit of an arbitrarily large number of identical copies of the state. Given two bipartite pure states $|\psi_{AB}\rangle$ and $|\phi_{AB}\rangle$, the former can be converted into the latter by LOCC if, and only if, $E(|\psi_{AB}\rangle) \geq E(|\phi_{AB}\rangle)$, where $E$ is the entropy of entanglement [5].

However, for mixed entangled states there are bound entangled states that require a non-zero rate of pure state entanglement for their creation by LOCC, but from which no pure state entanglement can be extracted at all [6]. As a consequence no unique measure of entanglement exists in the general case and no unambiguous and rigorous direct connection to thermodynamics appeared possible, despite various interesting attempts.

In this paper we identify a class of quantum operations which can be considered as a counterpart in entanglement theory of adiabatic processes. This, together with a new result on quantum hypothesis testing, allow us to establish a Theorem completely analogous to the Lieb and Yngvason formulation of the second law of thermodynamics for entanglement manipulation. Similar considerations may also be applied to resource theories in general, including e.g. theories quantifying the non-classicality, the non-Gaussian character, and the non-locality of quantum states. Indeed, the main conceptual message of this paper is an unified approach to deal with resources manipulation, by abstracting from theories where a resource appears due to the existence of some constraint on the type of operations available, to theories where the class of operations is derived from the resource considered, and chosen to be such that the latter cannot be freely generated. As it will be shown, such a shift of focus leads to a much simpler and elegant theory, which at the same time still gives relevant information about the original setting.

2. Definitions and Main Results

A $m$-partite quantum state $\rho \in D(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m)$ contains only classical correlations, and is called separable [7], if there exist local density operators $\rho_j^k$ acting on $\mathcal{H}_k$ and a probability distribution $\{p_j\}$ such that

$$\rho = \sum_j p_j \rho_j^1 \otimes \ldots \otimes \rho_j^m.$$  

(1)
Operationally, the set of separable states $S$ is formed by all states that may be created from a pure product state $|0\rangle \otimes \ldots \otimes |0\rangle$, i.e. an uncorrelated state, by means of local operations and classical communication (LOCC). If $\rho$ cannot be written as in Eq. (1) we say it is entangled and its generation requires, in addition to LOCC, an exchange of quantum particles (quantum communication) or a supply of pre-existing pure entangled states that are consumed in the process. A central state in this context is the two qubit maximally entangled state,

$$\phi^+ := \frac{1}{2} \sum_{i,j=1}^2 |i, i\rangle \langle j, j|,$$

which defines the unit of entanglement [8]. The observation that LOCC alone does not create entanglement has been taken further to formulate as the basic law of entanglement manipulation that entanglement cannot be increased by LOCC. A principle that is similar in spirit to the second law of thermodynamics.

Our results are concerned with the asymptotic limit of a large number of identical copies of quantum states. Here it is also natural to consider transformations between states that may be approximate for any finite number of systems and are only required to become exact in the asymptotic limit. To describe this limiting process rigorously the well established trace distance $D(\rho, \sigma) = \sum_{i,j=1}^2 |\rho_{ij} - \sigma_{ij}|$ between quantum states is used.

We say that a state $\rho$ can be asymptotically converted into another state $\sigma$ by operations of a given class if there is a sequence of quantum operations $\{\Psi_n\}$ acting on $n$ copies of the first state such that $\lim_{n \to \infty} D(\Psi_n(\rho \otimes^n), \sigma \otimes^n) = 0$.

To formulate the main result we need two measures that will be used to quantify entanglement. It will be satisfying that these two seemingly distinct approaches will turn out to be equivalent in the asymptotic limit, as a consequence of the technical proof of our main theorem. We consider the relative entropy of entanglement [9, 10]

$$E_R(\rho) = \min_{\sigma \in S} S(\rho | | \sigma),$$

where $S(\rho | | \sigma) = tr \{ \rho \log \rho - \rho \log \sigma \}$ is the quantum relative entropy. and $S$ is the set of separable states. Furthermore, we consider the global robustness of entanglement [11, 12] which is defined as

$$R_G(\rho) = \min_{s \in \mathbb{R}} \left( s : \exists \sigma \ s.t. \ \frac{\rho + s\sigma}{1 + s} \in S \right).$$

Both are meaningful entanglement quantifiers and a more detailed account of their properties can be found in Refs. [8, 13]. They will play a crucial role in the proof of our main theorem. We will also use the log global robustness, given by $LR_G(\rho) := \log(1 + R_G(\rho))$.

As we are concerned with the entanglement properties in the asymptotic limit we should not consider entanglement measures at the single copy level, but rather their asymptotic, or regularized, counterpart. In the case of $E_R$, the relevant quantity to consider is actually the regularized relative entropy of entanglement, given by

$$E_R^\infty(\rho) = \lim_{n \to \infty} E_R(\rho \otimes^n).$$

This will turn out to be the central quantity in this work as it will emerge as the unique entanglement quantifier.

The correct choice of the set of operations employed is crucial for establishing reversibility in entanglement manipulation. To motivate this choice it is instructive to note that in the context of the second law it follows both from the approach of Giles [2] as well as Lieb and Yngvason [3]
that the class formed by all adiabatic processes is the largest class of operations which cannot decrease the entropy of an isolated equilibrium thermodynamical system.

Following such an inside we now identify the largest set of quantum operations that obeys the basic law of the non-increase of entanglement. Then one might expect to achieve reversibility in entanglement manipulation and thus a full analogy to the second law of thermodynamics. While operationally well motivated, the set of LOCC operations is not such a class.

As we are concerned with asymptotic entanglement manipulation, it is physically natural and also convenient for mathematical reasons to define the set of asymptotically non-entangling operations, composed of sequences of operations that for a finite number of copies \( n \) may generate a small amount \( \epsilon_n \) of entanglement which vanishes asymptotically, \( \lim_{n \to \infty} \epsilon_n = 0 \). It is important to note that we do not simply require that the entanglement per copy vanishes but actually the total amount of entanglement. The next two definitions determine precisely the class of maps employed.

**Definition 2.1** Let \( \Omega : D(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m}) \to D(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m}) \) be a quantum operation. We say that \( \Omega \) is an \( \epsilon \)-non-entangling (or \( \epsilon \)-separability-preserving) map if for every separable state \( \sigma \in D(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m}) \),

\[
R_G(\Omega(\sigma)) \leq \epsilon.
\]

We denote the set of \( \epsilon \)-non-entangling maps by \( \text{SEPP}(\epsilon) \).

With this definition an asymptotically non-entangling operation is given by a sequence of CP trace preserving maps \( \{ \Lambda_n \}_{n \in \mathbb{N}} \), \( \Lambda_n : D(\mathcal{H}_{in}^{\otimes n}) \to D(\mathcal{H}_{out}^{\otimes n}) \), such that each \( \Lambda_n \) is \( \epsilon_n \)-non-entangling and \( \lim_{n \to \infty} \epsilon_n = 0 \). The use of the global robustness as a measure of the amount of entanglement generated is not arbitrary. The reason for this choice is explained in Ref. [13].

Having defined the class of maps we are going to use to manipulate entanglement, we can define the cost and distillation functions.

**Definition 2.2** We define the entanglement cost under asymptotically non-entangling maps of a state \( \rho \in D(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m}) \) as

\[
E^{ane}_C(\rho) := \inf_{\{k_n, \epsilon_n\} : \Lambda \in \text{SEPP}(\epsilon_n)} \left\{ \limsup_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \left( \min_{\Lambda \in \text{SEPP}(\epsilon_n)} \| \rho^{\otimes n} - \Lambda(\phi_2^{\otimes k_n}) \|_1 \right) = 0, \lim_{n \to \infty} \epsilon_n = 0 \right\},
\]

where the infimum is taken over all sequences of integers \( \{k_n\} \) and real numbers \( \{\epsilon_n\} \). In the formula above \( \phi_2^{\otimes k_n} \) stands for \( k_n \) copies of a two-dimensional maximally entangled state shared by the first two parties and the maps \( \Lambda_n : D((\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k_n}) \to D((\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m})^{\otimes n}) \) are \( \epsilon_n \)-non-entangling operations.

**Definition 2.3** We define the distillable entanglement under asymptotically non-entangling maps of a state \( \rho \in D(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m}) \) as

\[
E^{ane}_D(\rho) := \sup_{\{k_n, \epsilon_n\} : \Lambda \in \text{SEPP}(\epsilon_n)} \left\{ \liminf_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \left( \min_{\Lambda \in \text{SEPP}(\epsilon_n)} \| \Lambda(\rho^{\otimes n}) - \phi_2^{\otimes k_n} \|_1 \right) = 0, \lim_{n \to \infty} \epsilon_n = 0 \right\},
\]

where the infimum is taken over all sequences of integers \( \{k_n\} \) and real numbers \( \{\epsilon_n\} \). In the formula above \( \phi_2^{\otimes k_n} \) stands for \( k_n \) copies of a two-dimensional maximally entangled state shared by the first two parties and the maps \( \Lambda_n : D((\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k_n}) \to D((\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_m})^{\otimes n}) \) are \( \epsilon_n \)-non-entangling operations.

Note that when we do not specify the state of the other parties we mean that the non-local parts of their state is trivial (one-dimensional). Note furthermore that the fact that initially only two parties share entanglement is not a problem as the class of operations we employ include the swap operation. We are now in position to state the main result of this chapter.
Theorem 2.4 For every multipartite state $\rho \in D(C^{d_1} \otimes ... \otimes C^{d_m})$,

$$E_{C}^{\text{ane}}(\rho) = E_{D}^{\text{ane}}(\rho) = E_{R}^{\infty}(\rho).$$  \hfill (7)

We thus find that under asymptotically non-entangling operations entanglement can be interconverted reversibly. The situation is analogous to the Giles-Lieb-Yngvason formulation of the second law of thermodynamics. Indeed, Theorem 2.4 readily implies

Corollary 2.5 For two multipartite states $\rho \in D(C^{d_1} \otimes ... \otimes C^{d_m})$ and $\sigma \in D(C^{d'_1} \otimes ... \otimes C^{d'_m})$, there is a sequence of quantum operations $\Lambda_n$ such that

$$\Lambda_n \in SEPP(\epsilon_n), \quad \lim_{n \to \infty} \epsilon_n = 0,$$  \hfill (8)

and

$$\lim_{n \to \infty} ||\Lambda_n(\rho^{\otimes n}) - \sigma^{\otimes n - o(n)}||_1 = 0$$  \hfill (9)

if, and only if,

$$E_{R}^{\infty}(\rho) \geq E_{R}^{\infty}(\sigma).$$  \hfill (10)

In addition we have also identified the regularized relative entropy of entanglement as the natural counterpart in entanglement theory to the entropy function in thermodynamics. As will be shown in section 3, the regularized relative entropy measures how distinguishable an entangled state is from an arbitrary sequence of separable states. Therefore, we see that under asymptotically non-entangling operations, the amount of entanglement of any multipartite state is completely determined by how distinguishable the latter is from a state that only contains classical correlations. Furthermore, in section 3 we will also see that the amount of entanglement is equivalently uniquely defined in terms of the robustness of quantum correlations to noise in the form of mixing.

In the following sections we outline the structure of the proof of Theorem 2.4. The proof is rather lengthy and hence it is out of the scope of the present paper to give it in full. The interested reader is referred to Refs. [13, 14, 15] for a full rigorous proof of the results of this paper.

3. A Generalization of Quantum Stein’s Lemma

We now turn to the main technical tool for establishing our main result. This is not directly related to entanglement theory, but actually to quantum hypothesis testing. Hypothesis testing refers to a general set of tools in statistics and probability theory for making decisions based on experimental data from random variables. In a typical scenario, an experimentalist is faced with two possible hypothesis and must decide based on experimental observation which of them was actually realized.

Suppose we are faced with a source that emits several i.i.d. copies of one of two quantum states $\rho$ and $\sigma$, and we should decide which of them is being produced. Since the quantum setting also encompasses the classical, we will focus on the former, noting the differences between the two when necessary.

In order to learn the identity of the state the observer measures a two outcome POVM $\{M_n, I - M_n\}$ given $n$ realizations of the unknown state. If he obtains the outcome associated to $M_n$ ($I - M_n$) then he concludes that the state was $\rho$ ($\sigma$). The state $\rho$ is seen as the null hypothesis, while $\sigma$ is the alternative hypothesis. There are two types of errors here:

\footnote{In Eq. (9) $o(n)$ stands for a sublinear term in $n$.}
• Type I: The observer finds that the state was $\sigma$, when in reality it was $\rho$. This happens with probability $\alpha_n(M_n) := \text{tr}(\rho^\otimes n (I - M_n))$.

• Type II: The observer finds that the state was $\rho$, when it actually was $\sigma$. This happens with probability $\beta_n(M_n) := \text{tr}(\sigma^\otimes n M_n)$.

There are several distinct settings that might be considered, depending on the importance we attribute to the two types of errors [16].

In one such setting, the probability of type II error should be minimized to the extreme, while only requiring that the probability of type I error is bounded by a small parameter $\epsilon$. The relevant error quantity in this case can be written as

$$\beta_n(\epsilon) := \min_{0 \leq M_n \leq I} \{ \beta_n(M_n) : \alpha_n(M_n) \leq \epsilon \}.$$  

Quantum Stein’s Lemma [17, 18] tell us that for every $0 \leq \epsilon \leq 1$,

$$\lim_{n \to \infty} -\frac{\log(\beta_n(\epsilon))}{n} = S(\rho||\sigma). \quad (11)$$

This fundamental result gives a rigorous operational interpretation for the relative entropy and was proven in the quantum case by Hiai and Petz [17] and Ogawa and Nagaoka [18]. Different proofs have since be given in Refs. [19, 20, 16]. The relative entropy is also the asymptotic optimal exponent for the decay of $\beta_n$ when we require that $\alpha_n \to 0$ [20].

Let us now turn to describe the setting of quantum hypothesis setting that we are interest in. Given a set of states $\mathcal{M} \subseteq \mathcal{D}(\mathcal{H})$, we define

$$E_\mathcal{M}(\rho) := \inf_{\sigma \in \mathcal{M}} S(\rho||\sigma), \quad (12)$$

and

$$LR_\mathcal{M}(\rho) := \inf_{\sigma \in \mathcal{M}} S_{\max}(\rho||\sigma), \quad (13)$$

where

$$S_{\max}(\rho||\sigma) := \inf\{s : \rho \leq 2^s \sigma\}. \quad (14)$$

is the maximum relative entropy introduced by Datta [21]. Note that if we take $\mathcal{M}$ to be the set of separable states, then $E_\mathcal{M}$ and $LR_\mathcal{M}$ reduce to the relative entropy of entanglement and the logarithm global robustness of entanglement. This connection is the reason for the measures nomenclature. We will also need the smooth version of $LR_\mathcal{M}$, defined as

$$LR_{\mathcal{M}}^\epsilon(\rho) := \min_{\tilde{\rho} \in B_\epsilon(\rho)} LR_\mathcal{M}(\tilde{\rho}), \quad (15)$$

where $B_\epsilon(\rho) := \{\tilde{\rho} \in \mathcal{D}(\mathcal{H}) : \|\rho - \tilde{\rho}\|_1 \leq \epsilon\}$.

Let us specify the set of states over which the alternative hypothesis can vary. We will consider any family of sets $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$, with $\mathcal{M}_n \in \mathcal{D}(\mathcal{H}^\otimes n)$, satisfying the following properties

(i) Each $\mathcal{M}_n$ is convex and closed.

(ii) Each $\mathcal{M}_n$ contains the maximally mixed state $I^\otimes n / \text{dim}(\mathcal{H})^n$.

(iii) If $\rho \in \mathcal{M}_{n+1}$, then $tr_{n+1}(\rho) \in \mathcal{M}_n$.

(iv) If $\rho \in \mathcal{M}_n$ and $\sigma \in \mathcal{M}_m$, then $\rho \otimes \sigma \in \mathcal{M}_{n+m}$.

(v) If $\rho \in \mathcal{M}_n$, then $P_n \rho P_n \in \mathcal{M}_n$, for every $\pi \in S^2_n$.

2 $P_\pi$ is the standard representation in $\mathcal{H}^\otimes n$ of an element $\pi$ of the symmetric group $S_n$. 


We define the regularized version of the quantity given by Eq. (12) as

$$E^\infty_M(\rho) := \lim_{n \to \infty} \frac{1}{n} E_{\mathcal{M}_n}(\rho^\otimes n).$$

(16)

We now turn to the main result of this section. Suppose we have one of the following two hypothesis:

(i) For every $n \in \mathbb{N}$ we have $\rho^\otimes n$.

(ii) For every $n \in \mathbb{N}$ we have an unknown state $\omega_n \in \mathcal{M}_n$, where $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ is a family of sets satisfying properties i-v.

The next Theorem gives the optimal rate limit for the type II error when one requires that type I error vanishes asymptotically.

**Theorem 3.1** Given a family of sets $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ satisfying properties i-v and a state $\rho \in \mathcal{D}(\mathcal{H})$, for every $\epsilon > 0$ there exists a sequence of POVMs $\{A_n, \mathbb{I} - A_n\}$ such that

$$\lim_{n \to \infty} \text{tr}((\mathbb{I} - A_n)\rho^\otimes n) = 0$$

and for all sequences of states $\{\omega_n \in \mathcal{M}_n\}_{n \in \mathbb{N}},$

$$-\log \frac{\text{tr}(A_n\omega_n)}{n} + \epsilon \geq E^\infty_M(\rho) = \lim_{\epsilon' \to 0} \limsup_{n \to \infty} \frac{1}{n} LR'_{\mathcal{M}_n}(\rho^\otimes n).$$

Conversely, if there is a $\epsilon > 0$ and sequence of POVMs $\{A_n, \mathbb{I} - A_n\}$ satisfying

$$-\log \frac{\text{tr}(A_n\omega_n)}{n} - \epsilon \geq E^\infty_M(\rho)$$

for all sequences $\{\omega_n \in \mathcal{M}_n\}_{n \in \mathbb{N}},$ then

$$\lim_{n \to \infty} \text{tr}((\mathbb{I} - A_n)\rho^\otimes n) = 1.$$

The proof of Theorem 3.1 can be found in Refs. [13, 15]. On general lines, the main idea of the proof is to employ Renner’s exponential de Finetti Theorem [22] to reduce the correlated instance at hand to an almost i.i.d. setting, which can then be handled by standard techniques.

Theorem 3.1 gives an operational interpretation to the regularized relative entropy of entanglement. Taking $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ to be the sets of separable states over $\mathcal{H}^\otimes n$, it is a simple exercise to check that they satisfy conditions i-v. Therefore, we conclude that $E^\infty_M(\rho)$ gives the rate limit of the type II error when we try to decide if we have several realizations of $\rho$ or a sequence of arbitrary separable states. This rigorously justify the use of the regularized relative entropy of entanglement as a measure of distinguishability of quantum correlations from classical correlations. Another noteworthy point of Theorem 3.1 is the formula

$$\lim_{n \to \infty} \frac{1}{n} E_{\mathcal{M}_n}(\rho^\otimes n) = \lim_{\epsilon' \to 0} \limsup_{n \to \infty} \frac{1}{n} LR'_{\mathcal{M}_n}(\rho^\otimes n).$$

(17)

Taking once more $\{\mathcal{M}_n\}$ as the sets of separable states over $\mathcal{H}^\otimes n$, this Equation shows that the regularized relative entropy of entanglement is a smooth asymptotic version of the log global robustness of entanglement. We hence have a connection between the robustness of quantum correlations under mixing and their distinguishability to classical correlations. For later use we denote the quantity appearing in Eq. (17) by $LG$ when $\{\mathcal{M}_n\}$ are taken to be the sets of separable states.
4. Proof of Theorem 2.4

4.1. The Entanglement Cost under Asymptotically non-Entangling Maps

Proposition 4.1 For every multipartite state \( \rho \in D(\mathbb{C}^{d_1} \otimes ... \otimes \mathbb{C}^{d_2}) \),

\[
E_{\text{C}}^\text{ane}(\rho) \leq E_{\text{R}}^\infty(\rho).
\]  

Proof To prove the Proposition we consider a specific sequence of operations achieving the rate \( LG(\rho) = E_{\text{R}}^\infty(\rho) \). We consider maps of the form:

\[
\Lambda_n(A) = \text{tr}(A\Phi(K_n))\rho_n + \text{tr}(A(\mathbb{I} - \Phi(K_n)))\pi_n,
\]

where (i) \( \{\rho_n\} \) is an optimal sequence of approximations for \( \rho \otimes n \) achieving the infimum in \( LG(\rho) \), (ii) \( \log(K_n) = \lfloor \log(1 + RG(\rho_n)) \rfloor \), and (iii) \( \pi_n \) is a state such that \( (\rho_n + (K_n - 1)\pi_n)/K_n \) is separable which always exists as \( K_n \geq 2^{\log(1 + RG(\rho_n))} = RG(\rho_n) + 1 \). As \( \pi_n \) and \( \rho_n \) are states, each \( \Lambda_n \) is completely positive and trace-preserving.

We now show that each \( \Lambda_n \) is a \( 1/(K_n - 1) \)-non-entangling map. From (ii) and (iii) we find

\[
\frac{\pi_n + (K_n - 1)^{-1}\rho_n}{1 + (K_n - 1)^{-1}} \in \mathcal{S},
\]

and, thus,

\[
RG(\pi_n) \leq \frac{1}{K_n - 1}.
\]

A simple calculation using the \( UU^* \)-symmetry of the maximally entangled state, together with the convexity of \( RG \) indeed show that \( \Lambda_n \) is a \( 1/(K_n - 1) \)-separability-preserving map.

A Corollary of Theorem 3.1 shows that \( LG(\rho) = E_{\text{R}}^\infty(\rho) > 0 \) for every entangled state (see Refs. [13, 15]). Therefore

\[
\lim_{n \to \infty} (K_n - 1)^{-1} \leq \lim_{n \to \infty} RG(\rho_n)^{-1} = 0.
\]

Moreover, as

\[
\lim_{n \to \infty} ||\rho_n \otimes n - \Lambda_n(\Phi(K_n))||_1 = \lim_{n \to \infty} ||\rho_n \otimes n - \rho_n||_1 = 0,
\]

it follows that \( \{\Lambda_n\} \) is an admissible sequence of maps for \( E_{\text{C}}^\text{ane}(\rho) \) and, thus,

\[
E_{\text{C}}^\text{ane}(\rho) \leq \lim sup_{n \to \infty} \frac{1}{n} \log(K_n)
= \lim sup_{n \to \infty} \frac{1}{n} \log(1 + RG(\rho_n))
= LG(\rho).
\]  

4.2. The Distillable Entanglement under non-Entangling Operations

Before we turn to the proof of the main Proposition of this section, we state and prove an auxiliary Lemma which will be used later on. It can be considered the analogue for non-entangling maps of Theorem 3.3 of Ref. [24], which deals with PPT maps. Its proof, which is a simple exercise in duality of convex optimization problems, is given in [14, 13].
Lemma 4.2 For every multipartite state $\rho \in \mathcal{D}(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n})$ the singlet-fraction under non-entangling maps,

$$F_{\text{sep}}(\rho; K) := \max_{\Lambda \in \mathcal{SEPP}} \text{tr}(\Phi(K)\Lambda(\rho)),$$

where $\Phi(K)$ is a $K$-dimensional maximally entangled state shared by the first two parties, satisfies

$$F_{\text{sep}}(\rho; K) = \min_{\sigma \in \cone(S)} \text{tr}(\rho - \sigma)_+ + \frac{1}{K} \text{tr}(\sigma).$$

(24)

It turns out that for the distillation part we do not need to allow any generation of entanglement from the maps. In analogy to Definition 2.3, we can define the distillable entanglement under non-entangling maps as

$$E_{\text{D}}^{\text{ne}}(\rho) := \sup_{\{k_n\}} \liminf_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \left( \min_{\Lambda \in \mathcal{SEPP}} \|\Lambda(\rho^{\otimes n}) - \phi_2^{k_n}\|_1 \right) = 0 \right\}.$$

(26)

Proposition 4.3 For every multipartite entangled state $\rho \in \mathcal{D}(\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n})$,

$$E_{\text{D}}^{\text{ne}}(\rho) \geq E_{\text{D}}^{\text{ne}}(\rho) = E_{\text{R}}^{\infty}(\rho).$$

(27)

Proof On one hand, from Theorem 3.1 we have

$$\lim_{n \to \infty} \min_{\sigma_n \in \mathcal{S}} \text{tr}(\rho^{\otimes n} - 2^{ny}\sigma_n)_+ \begin{cases} 0, & y > E_{\text{R}}^{\infty}(\rho) \\ 1, & y < E_{\text{R}}^{\infty}(\rho). \end{cases}$$

(28)

On the other, from Lemma 4.2 we find

$$F_{\text{sep}}(\rho^{\otimes n}, 2^{ny}) := \min_{\sigma_n \in \mathcal{S}, b \in \mathbb{R}} \text{tr}(\rho^{\otimes n} - 2^{nb}\sigma)_+ + 2^{-(y-b)n}.$$

(29)

Let us consider the asymptotic behavior of $F_{\text{sep}}(\rho^{\otimes n}, 2^{ny})$. Take $y = E_{\text{R}}^{\infty}(\rho) + \epsilon$, for any $\epsilon > 0$. Then we can choose, for each $n$, $b = 2^{n(E_{\text{R}}^{\infty}(\rho) + \frac{\epsilon}{4})}$, giving

$$F_{\text{sep}}(\rho^{\otimes n}, 2^{ny}) \leq \min_{\sigma_n \in \mathcal{S}} \text{tr}(\rho^{\otimes n} - 2^{n(E_{\text{R}}^{\infty}(\rho) + \frac{\epsilon}{4})}\sigma)_+ + 2^{-\frac{n\epsilon}{4}}.$$

We then see from Eq. (28) that $\lim_{n \to \infty} F_{\text{sep}}(\rho^{\otimes n}, 2^{ny}) = 0$, from which follows that $E_{\text{D}}^{\text{ne}}(\rho) \leq E_{\text{R}}^{\infty}(\rho)$. As $\epsilon$ is arbitrary, we find $E_{\text{D}}^{\text{ne}}(\rho) \leq E_{\text{R}}^{\infty}(\rho)$.

Conversely, let us take $y = E_{\text{R}}^{\infty}(\rho) - \epsilon$, for any $\epsilon > 0$. The optimal $b$ for each $n$ has to satisfy $b_n \leq 2^{m\epsilon}$, otherwise $F_{\text{sep}}(\rho^{\otimes n}, 2^{ny})$ would be larger than one, which is not true. Therefore,

$$F_{\text{sep}}(\rho^{\otimes n}, 2^{ny}) \geq \min_{\sigma_n \in \mathcal{S}} \text{tr}(\rho^{\otimes n} - 2^{n(E_{\text{R}}^{\infty}(\rho) - \epsilon)}\sigma)_+,$$

which goes to one again by Eq. (28). This then shows that $E_{\text{D}}^{\text{ne}}(\rho) \geq E_{\text{R}}^{\infty}(\rho) - \epsilon$. Again, as $\epsilon$ is arbitrary, we find $E_{\text{D}}^{\text{ne}}(\rho) \geq E_{\text{R}}^{\infty}(\rho)$. 

From Propositions 4.1 and 4.3 it is clear that in order to prove Theorem it suffices to show that $E_{\text{C}}^{\text{ne}}(\rho) \geq E_{\text{D}}^{\text{ne}}(\rho)$ for every state $\rho$. This can shown using standard methods in entanglement theory and is left out of this paper (see Refs. [14, 13] for a proof). We also do not present the proof of Corrolary 2.5, which follows straightforwardly from Theorem 2.4. A full proof can be found in Refs. [14, 13].
5. Connection to the Axiomatic Formulation of the Second Law of Thermodynamics

In this section we comment on the similarities and differences of entanglement manipulation under asymptotic non-entangling operations and the axiomatic approach of Giles [2] and more particularly of Lieb and Yngvason [3] for the second law of thermodynamics.

Let us start by briefly recalling the axioms used by Lieb and Yngvason [3] in order to derive the second law. Their starting point is the definition of a system as a collection of points called state space and denoted by $\Gamma$. The individual points of a state space are the states of the system. The composition of two state spaces $\Gamma_1$ and $\Gamma_2$ is given by their Cartesian product. Furthermore, the scaled copies of a given system is defined as follows: if $t > 0$ is some fixed number, the state space $\Gamma(t)$ consists of points denoted by $tX$ with $X \in \Gamma$. Finally, a preorder $\prec$ on the state space satisfying the following axioms is assumed:

(i) $X \prec X$.
(ii) $X \prec Y$ and $Y \prec Z$ implies $X \prec Z$.
(iii) If $X \prec Y$, then $tX \prec tY$ for all $t > 0$.
(iv) $X \prec (tX, (1-t)X)$ and $(tX, (1-t)X) \prec X$.
(v) If, for some pair of states, $X$ and $Y$, $(X, \epsilon Z_0) \prec (Y, \epsilon Z_1)$ (30) holds for a sequence of $\epsilon$'s tending to zero and some states $Z_0$, $Z_1$.
(vi) $X \prec X'$ and $Y \prec Y'$ implies $(X, Y) \prec (X', Y')$.

Lieb and Yngvason then show that these axioms, together with the comparison hypothesis, which states that

Comparison Hypothesis: for any two states $X$ and $Y$ either $X \prec Y$ or $Y \prec X$,

are sufficient to prove the existence of a single valued entropy function completely determining the order induced by the relation $\prec$.

In the context of entanglement transformations, we interpret the relation $\rho \prec \sigma$ as the possibility of asymptotically transforming $\rho$ into $\sigma$ by asymptotically non-entangling maps. Then, the composite state $(\rho, \sigma)$ is nothing but the tensor product $\rho \otimes \sigma$. Moreover, $t\rho$ takes the form of $\rho^{\otimes t}$, which is a shortcut to express the fact that if $\rho^{\otimes t} \prec \sigma$, then asymptotically $t$ copies of $\rho$ can be transformed into one of $\sigma$. More concretely, we say that

$$
(X, \epsilon Z_0) \prec (Y, \epsilon Z_1)
$$

(30)

holds for a sequence of $\epsilon$'s tending to zero and some states $Z_0$, $Z_1$, then $X \prec Y$.

Property 5 is also true is proven in Refs. [13].

The Compensation Hypothesis, in turn, follows from Corollary 2.5: it expresses the total order induced by the regularized relative entropy of entanglement.
We cannot decide if the theory we are considering for entanglement satisfy axiom 6. This is fundamentally linked to the possibility of having entanglement catalysis [23] under asymptotically non-entangling transformations. As shown in Theorem 2.1 of Ref. [3], given that axioms 1-5 are true, axiom 6 is equivalent to

(i) \((X, Y) \prec (X', Y)\) implies \(X \prec X'\),

which is precisely the non-existence of catalysis. Interestingly, we can link such a possibility in the bipartite case to an important open problem in entanglement theory, the full additivity of the regularized relative entropy of entanglement.

**Lemma 5.1** The regularized relative entropy of entanglement is fully additive for bipartite states, i.e. for every two states \(\rho \in \mathcal{D}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})\) and \(\pi \in \mathcal{D}(\mathbb{C}^{d'_1} \otimes \mathbb{C}^{d'_2})\),

\[
E_R^\infty (\rho \otimes \pi) = E_R^\infty (\rho) + E_R^\infty (\pi),
\]

if, and only if, there is no catalysis for entanglement manipulation under asymptotically non-entangling maps.

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