GENERALISED FRIEZES AND A MODIFIED CALDERO-CHAPOTON MAP DEPENDING ON A RIGID OBJECT, II

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Abstract. It is an important aspect of cluster theory that cluster categories are “categorifications” of cluster algebras. This is expressed formally by the (original) Caldero-Chapoton map $X$ which sends certain objects of cluster categories to elements of cluster algebras.

Let $\tau c \rightarrow b \rightarrow c$ be an Auslander-Reiten triangle. The map $X$ has the salient property that $X(\tau c)X(c) - X(b) = 1$. This is part of the definition of a so-called frieze, see [1].

The construction of $X$ depends on a cluster tilting object. In a previous paper [14], we introduced a modified Caldero-Chapoton map $\rho$ depending on a rigid object; these are more general than cluster tilting objects. The map $\rho$ sends objects of sufficiently nice triangulated categories to integers and has the key property that $\rho(\tau c)\rho(c) - \rho(b)$ is 0 or 1. This is part of the definition of what we call a generalised frieze.

Here we develop the theory further by constructing a modified Caldero-Chapoton map, still depending on a rigid object, which sends objects of sufficiently nice triangulated categories to elements of a commutative ring $A$. We derive conditions under which the map is a generalised frieze, and show how the conditions can be satisfied if $A$ is a Laurent polynomial ring over the integers.

The new map is a proper generalisation of the maps $X$ and $\rho$.

0. Introduction

The (original) Caldero-Chapoton map $X$ is an important object in cluster theory. The arguments of $X$ are certain objects of a cluster category, and the values are the corresponding elements of a cluster algebra. The map $X$ expresses that the cluster category is a categorification of the cluster algebra, see [7], [9], [10], [13], [17]. For example, Figure 1 shows the Auslander-Reiten (AR) quiver of $C(A_5)$, the cluster category of Dynkin type $A_5$, with a useful “coordinate system”. Figure 2 shows the AR quiver again, with the values of $X$ on the indecomposable objects of $C(A_5)$. The values are Laurent polynomials over $\mathbb{Z}$; indeed, the cluster algebra consists of such Laurent polynomials.

It is a salient property of $X$ that it is a frieze in the sense of [1], that is, if $\tau c \rightarrow b \rightarrow c$ is an AR triangle then

$$X(\tau c)X(c) - X(b) = 1,$$

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see [12, theorem] and [7, prop. 3.10]. In the case of \( C \) we have

The Auslander-Reiten quiver of the cluster category \( C(A_5) \). The dotted lines should be identified with opposite orientations. The red vertices show the direct summands of a rigid object \( R \), and the red and blue vertices show the direct summands of a cluster tilting object \( T \).

Figure 2. The Auslander-Reiten quiver of \( C(A_5) \) with values of the original Caldero-Chapoton map \( X \). The map depends on the cluster tilting object \( T \) shown by red and blue vertices in Figure 1.

see [12, theorem] and [7, prop. 3.10]. In the case of \( C(A_5) \), this means that for each “diamond” in the AR quiver, of the form

\[
\begin{array}{c}
\tau c \\
\downarrow \\
\uparrow \\
b_2 \\
\end{array} \quad \begin{array}{c} b_1 \\
\downarrow \\
\uparrow \\
c, \\
\end{array}
\]

we have

\[
X(\tau c)X(c) - X(b_1)X(b_2) = 1.
\]
The definition of $X$ depends on a cluster tilting object $T$. For instance, the $X$ shown in Figure 2 depends on the $T$ which has the indecomposable summands shown by red and blue vertices in Figure 1.

This paper is about a modified Caldero-Chapoton map $\rho$ which is more general than $X$ in two respects: it depends on a rigid object $R$ and has values in a general commutative ring $A$. An object $R$ is rigid if $\text{Hom}(R, \Sigma R) = 0$. This is much weaker than being cluster tilting: recall that $T$ is cluster tilting if $\text{Hom}(T, \Sigma t) = 0 \iff t \in \text{add } T \iff \text{Hom}(t, \Sigma T) = 0$.

Our first main result gives conditions under which $\rho$ is a generalised frieze, in the sense that if $\tau c \to b \to c$ is an AR triangle then

$$\rho(\tau c)\rho(c) - \rho(b_1)\rho(b_2) = 1.$$ 

Our second main result is that the conditions can be satisfied if $A$ is chosen to be a Laurent polynomial ring over the integers.

Generalised friezes with values in the integers were introduced by combinatorial means in [3], and it was shown in [14] that they can be recovered from a modified Caldero-Chapoton map. The theory of [14] and the original Caldero-Chapoton map are both special cases of the theory developed here.

For example, consider $C(A_5)$ again and let $R$ be the rigid object which has the indecomposable summands shown by red vertices in Figure 1. Our results imply that we can choose $A = \mathbb{Z}[u^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$, and Figure 3 shows the AR quiver of $C(A_5)$ with the values of $\rho$ on the indecomposable objects. In this case, the generalised frieze property means that for each “diamond” in the AR quiver, of the form \((0,1)\), we have

$$\rho(\tau c)\rho(c) - \rho(b_1)\rho(b_2) \in \{0,1\}.$$ 

The solid grey diamonds in Figure 3 indicate where the displayed expression is equal to 1.

Let us explain how $\rho$ is defined. Let $k$ be an algebraically closed field, $C$ an essentially small Hom-finite $k$-linear triangulated category. Assume that $C$ has split idempotents and has a Serre functor. Note that these are the only assumptions on $C$ which is hence permitted to be a good...
deal more general than a cluster category. Let $\Sigma$ denote the suspension functor of $C$ and write $C(-, -)$ instead of $\text{Hom}_C(-, -)$. Let $R$ be a rigid object of $C$, assumed to be basic for reasons of simplicity, with endomorphism algebra $E = \text{End}_C(R)$. There is a functor

$$C \xrightarrow{\text{G}} \text{mod } E,$$

$$c \mapsto C(R, \Sigma c).$$

Let $A$ be a commutative ring and let $\alpha : \text{obj } C \to A$, $\beta : K_0(\text{mod } E) \to A$ be two maps, where $\text{obj } C$ is the set of objects of $C$ and $K_0$ denotes the Grothendieck group of an abelian category.

The modified Caldero-Chapoton map is the map $\rho : \text{obj } C \to A$ defined as follows.

$$\rho(c) = \alpha(c) \sum_e \chi(\text{Gr}_e(Gc)) \beta(e)$$

Here $c \in C$ is an object, the sum is over $e \in K_0(\text{mod } E)$, and $\chi$ denotes the Euler characteristic defined by étale cohomology with proper support. By $\text{Gr}_e$ is denoted the Grassmannian of submodules $M \subseteq Gc$ with $K_0$-class $[M] = e$.

The original Caldero-Chapoton map is the special case where $R$ is a cluster tilting object and $\alpha$ and $\beta$ are two particular maps; see Remark 2.9. The modified Caldero-Chapoton map of 14 is the special case where $A = \mathbb{Z}$ and $\alpha$ and $\beta$ are identically equal to 1. In general, $\rho$ is only likely to be interesting if the maps $\alpha$ and $\beta$ are chosen carefully, and we formalise this by saying that $\alpha$ and $\beta$ are frieze-like for the AR triangle $\Delta = \tau c \to b \to c$ if they satisfy the technical conditions in Definition 1.4. Observe that the conditions are trivially satisfied if $\alpha$ and $\beta$ are identically equal to 1. Our first main result is the following.

**Theorem A.** If $\alpha$ and $\beta$ are frieze-like for each AR triangle in $C$, then the modified Caldero-Chapoton map $\rho : \text{obj } C \to A$ is a generalised frieze in the sense of [14, def. 3.4]. That is,

(i) $\rho(c_1 \oplus c_2) = \rho(c_1)\rho(c_2)$,

(ii) if $\Delta = \tau c \to b \to c$ is an AR triangle then $\rho(\tau c)\rho(c) - \rho(b) \in \{0, 1\}$.

This follows from Proposition 1.3 and Theorem 1.5 which even show

$$\rho(\tau c)\rho(c) - \rho(b) = \begin{cases} 
0 & \text{if } G(\Delta) \text{ is a split short exact sequence}, \\
1 & \text{if } G(\Delta) \text{ is not a split short exact sequence}.
\end{cases}$$

Note that $G(\Delta)$ is never split exact when $R$ is a cluster tilting object, that is, in the case of the original Caldero-Chapoton map.

Our second main result is that one can find frieze-like maps $\alpha$ and $\beta$, and hence generalised friezes, with values in Laurent polynomials.

**Theorem B.** Assume that $C$ is 2-Calabi-Yau and that the basic rigid object $R$ has $r$ indecomposable summands and is a direct summand of a cluster tilting object.

Then there are maps $\alpha$ and $\beta$ with values in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$, using all the variables $x_1, \ldots, x_r$, which are frieze-like for each AR triangle in $C$.

Hence there is a modified Caldero-Chapoton map $\rho : \text{obj } C \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$, using all the variables $x_1, \ldots, x_r$, which is a generalised frieze.
This is established in Definition 2.8, Theorem 2.11, and Remark 2.12. We leave it vague for now what it means to “use all the variables $x_1, \ldots, x_r$”, but see Remark 2.12. In fact, it is sometimes possible to get values in Laurent polynomials in more than $r$ variables. For example, the basic rigid object $R$ defined by the red vertices in Figure 1 has $r = 2$ indecomposable summands, but the corresponding $\rho$ has values in $\mathbb{Z}[u^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$ as shown in Figure 3.

It is natural to ask if the $\mathbb{Z}$-algebra generated by the values of $\rho$ is an interesting object. In particular, one can ask how it is related to cluster algebras and how it is affected by mutation of rigid objects as defined in [16, sec. 2]; see the questions in Section 4.

The paper is organised as follows: Section 1 defines what it means for $\alpha$ and $\beta$ to be frieze-like and proves Theorem A. Section 2 proves Theorem B. Section 3 shows how to obtain the example in Figure 3. Section 4 poses some questions.

1. The frieze-like condition on the maps $\alpha$ and $\beta$ implies that $\rho$ is a generalised frieze

This section proves Theorem A in the introduction. It is a consequence of Proposition 1.3 and Theorem 1.5.

We start by setting up items to be used in the rest of the paper. Instead of a basic rigid object $R$ we will work with a rigid subcategory $\mathcal{R}$. This is more general since we can set $\mathcal{R} = \text{add } R$ when $R$ is given, but not every $\mathcal{R}$ has this form, see [15, sec. 6]. The higher generality means that the definitions of $G$ and $\beta$ are different from those in the introduction.

**Setup 1.1.** Let $k$ be an algebraically closed field and let $\mathcal{C}$ be an essentially small $k$-linear Hom-finite triangulated category with split idempotents. Hence $\mathcal{C}$ is a Krull-Schmidt category. Assume that $\mathcal{C}$ has a Serre functor. Hence it has AR triangles by [18, thm. A], and the Serre functor is $\Sigma \circ \tau$ where $\Sigma$ is the suspension functor and $\tau$ is the AR translation.

Let $\mathcal{R}$ be a full subcategory of $\mathcal{C}$ which is closed under direct sums and summands, is functorially finite, and rigid, that is, $\mathcal{C}(\mathcal{R}, \Sigma \mathcal{R}) = 0$.

The category of $k$-vector spaces is denoted $\text{Mod } k$ and the category of $k$-linear functors $\mathcal{R} \rightarrow \text{Mod } k$ is denoted $\text{Mod } \mathcal{R}$. It is a $k$-linear abelian category and its full subcategory of objects of finite length is denoted $\text{fl } \mathcal{R}$.

Let $A$ be a commutative ring and let

$$\alpha : \text{obj } \mathcal{C} \rightarrow A, \quad \beta : K_0(\text{fl } \mathcal{R}) \rightarrow A$$

be maps which are “exponential” in the sense that

$$\alpha(0) = 1, \quad \alpha(c \oplus d) = \alpha(c) \alpha(d),$$

$$\beta(0) = 1, \quad \beta(e + f) = \beta(e) \beta(f).$$

**1.2 (The modified Caldero-Chapoton map).** There is a functor

$$\mathcal{C} \xrightarrow{\mathcal{C}} \text{Mod } \mathcal{R},$$

$$c \mapsto \mathcal{C}(-, \Sigma c)|_{\mathcal{R}}.$$ 

The modified Caldero-Chapoton map is defined by the following formula.

$$\rho(c) = \alpha(c) \sum_e \chi(\text{Gr}_e(Gc)) \beta(e)$$

(1.1)
The sum is over \( e \in K_0(\text{fl}\, R) \), and \( \text{Gr}_e(Gc) \) is the Grassmannian of submodules \( M \subseteq Gc \) where \( M \) has finite length in \( \text{Mod}\, R \) and class \([M] = e\) in \( K_0(\text{fl}\, R)\). The notation is otherwise as explained in the introduction.

The formula may not make sense for each \( c \in C \), but it does make sense if \( Gc \) has finite length in \( \text{Mod}\, R \) since then \( \text{Gr}_e(Gc) \) is finite-dimensional and non-empty for only finitely many values of \( e \); see \([13, 1.6\text{ and } 1.8]\). When the formula makes sense, it defines an element \( \rho(c) \in A \). Note that

\[
\rho(0) = 1.
\]

**Proposition 1.3.** Let \( a, c \in C \) be objects such that \( Ga, Gc \) have finite length in \( \text{Mod}\, R \). Then \( G(a \oplus c) \) has finite length in \( \text{Mod}\, R \) and

\[
\rho(a \oplus c) = \rho(a)\rho(c).
\]

**Proof.** The statement about the length of \( G(a \oplus c) \) is clear.

It is immediate that

\[
\rho(a)\rho(c) = \alpha(a \oplus c) \sum_g \left( \sum_{e+f = g} \chi(\text{Gr}_e(Ga) \times \text{Gr}_f(Gc)) \right) \beta(g) = (*). \tag{1.2}
\]

In \([14\text{ sec. } 2]\) we considered a pair of morphisms \( a \to b \to c \) in \( C \) and introduced auxiliary spaces \( X_{e,f} \). If we set \( a \to b \to c \) equal to the canonical morphisms \( a \to a \oplus c \to c \), then \([14\text{ lem. } 2.4(i+v)]\) and \([14\text{ rmk. } 2.5]\) mean that we can compute as follows.

\[
(*) = \alpha(a \oplus c) \sum_g \left( \sum_{e+f = g} \chi(X_{e,f}) \right) \beta(g) = \alpha(a \oplus c) \sum_g \chi(\text{Gr}_g(G(a \oplus c))) \beta(g) = \rho(a \oplus c)
\]

\( \square \)

**Definition 1.4** (Frieze-like \( \alpha \) and \( \beta \)). Let

\[
\Delta = \tau c \to b \to c
\]

be an AR triangle in \( C \) and assume that \( Gc \) and \( G(\tau c) \) have finite length in \( \text{Mod}\, R \). We say that \( \alpha \) and \( \beta \) are frieze-like for \( \Delta \) if the following hold.

(i) If \( c \not\in R \cup \Sigma^{-1}R \) and \( G(\Delta) \) is a split short exact sequence, then

\[
\alpha(b) = \alpha(c \oplus \tau c).
\]

(ii) If \( c \not\in R \cup \Sigma^{-1}R \) and \( G(\Delta) \) is a non-split short exact sequence, or if \( c = \Sigma^{-1}r \in \Sigma^{-1}R \), then

\[
\alpha(b) = \alpha(c \oplus \tau c), \quad \alpha(c \oplus \tau c)\beta([Gc]) = 1.
\]

(iii) If \( c = r \in R \), then

\[
\alpha(c \oplus \tau c) = 1, \quad \alpha(b) = \beta([Sr])
\]

where \( Sr \in \text{Mod}\, R \) is the simple object supported at \( r \); see \([2\text{ prop. } 2.2]\).

**Theorem 1.5.** Let

\[
\Delta = \tau c \to b \to c
\]

be an AR triangle in \( C \). Assume that \( Gc \) and \( G(\tau c) \) have finite length in \( \text{Mod}\, R \) and that \( \alpha \) and \( \beta \) are frieze-like for \( \Delta \). Then \( Gb \) has finite length in \( \text{Mod}\, R \) and

\[
\rho(\tau c)\rho(c) - \rho(b) = \begin{cases} 
0 & \text{if } G(\Delta) \text{ is a split short exact sequence,} \\
1 & \text{if } G(\Delta) \text{ is not a split short exact sequence.}
\end{cases}
\]
Proof. It is clear from the definition of $G$ that it is a homological functor, so $G(\Delta)$ is an exact sequence (albeit not necessarily short exact). The statement about the length of $Gb$ follows.

We split into cases. First some preparation: setting $a = \tau c$ in Equation (1.2) gives

$$\rho(\tau c)\rho(c) = \alpha(c \oplus \tau c) \sum_g \left( \sum_{e+f=g} \chi \left( \text{Gr}_e(G(\tau c)) \times \text{Gr}_f(Gc) \right) \right) \beta(g) = (*) \quad (1.3)$$

Moreover, we use the morphisms $a \to b \to c$ and auxiliary spaces $X_{e,f}$ from [14, sec. 2] again, this time setting $a \to b \to c$ equal to the AR triangle $\tau c \to b \to c$. We can then use the results of [14, sec. 2].

Case (i): $c \not\in R \cup \Sigma^{-1}R$ and $G(\Delta)$ is a split short exact sequence.

We start from Equation (1.3) and compute as follows:

$$(*) \overset{(a)}{=} \alpha(b) \sum_g \left( \sum_{e+f=g} \chi(X_{e,f}) \right) \beta(g) \overset{(b)}{=} \alpha(b) \sum_{g} \chi \left( \text{Gr}_g(Gb) \right) \beta(g) = \rho(b),$$

where (a) is by Definition [14(i)] and [14, lem. 2.4(i+v)], and (b) is by [14, rmk. 2.5].

Case (ii): $c \not\in R \cup \Sigma^{-1}R$ and $G(\Delta)$ is a non-split short exact sequence.

We start from Equation (1.3) and compute as follows:

$$(*) = \alpha(c \oplus \tau c) \left\{ \chi \left( \text{Gr}_0(G(\tau c)) \times \text{Gr}_{[Gc]}(Gc) \right) \beta \left( [Gc] \right) \right. + \sum_{g} \left( \sum_{e+f=g} \chi \left( \text{Gr}_e(G(\tau c)) \times \text{Gr}_f(Gc) \right) \right) \beta(g) \right\} \overset{(c)}{=} \alpha(c \oplus \tau c) \left\{ \beta \left( [Gc] \right) + \sum_{g} \left( \sum_{e+f=g \neq 0, [Gc]} \chi(X_{e,f}) \right) \beta(g) \right\} \overset{(d)}{=} 1 + \alpha(b) \sum_g \left( \sum_{e+f=g \neq 0, [Gc]} \chi(X_{e,f}) \right) \beta(g) \overset{(e)}{=} 1 + \alpha(b) \sum_g \chi(X_{e,f}) \beta(g) \overset{(f)}{=} 1 + \alpha(b) \sum_g \chi(G_G(Gb)) \beta(g) \overset{(e)}{=} 1 + \rho(b),$$

where (c) follows from [14, lem. 2.4(ii)+(iv)+(v)], (d) is by Definition [14(ii)], (e) is by [14, lem. 2.4(iii)], and (f) is by [14, rmk. 2.5].

Case (iii): $c = \Sigma^{-1}r \in \Sigma^{-1}R$. Then $G(\Delta)$ is not a split short exact sequence, but

$$G(\Delta) = G(\tau c \to b \to c) = 0 \to \text{rad} \ P_r \to P_r \quad (1.4)$$

by [14, lem. 1.12(i)], where $P_r = C(-, r) |_R$ is the indecomposable projective object of $\text{ModR}$ associated with $r$, see [14, 1.5]. In particular, we have $G(\tau c) = 0$ whence $\rho(\tau c) = \alpha(\tau c)$, and...
this gives the first equality in the following computation.
\[
\rho(\tau c) \rho(c) = \alpha(\tau c) \alpha(c) \sum_f \chi(\text{Gr}_f(Gc)) \beta(f)
\]
\[
= \alpha(c \oplus \tau c) \sum_f \chi(\text{Gr}_f(Gc)) \beta(f)
\]
\[
= \alpha(c \oplus \tau c) \left\{ \chi(\text{Gr}_{[Gc]}(Gc)) \beta([Gc]) + \sum_{f \neq [Gc]} \chi(\text{Gr}_f(Gc)) \beta(f) \right\}
\]
\[
\overset{(g)}{=} \alpha(c \oplus \tau c) \left\{ \beta([Gc]) + \sum_{f \neq [Gc]} \chi(\text{Gr}_f(Gc)) \beta(f) \right\}
\]
\[
\overset{(h)}{=} \alpha(c \oplus \tau c) \left\{ \beta([Gc]) + \sum_f \chi(\text{Gr}_f(Gb)) \beta(f) \right\}
\]
\[
\overset{(i)}{=} 1 + \alpha(b) \sum_f \chi(\text{Gr}_f(Gb)) \beta(f)
\]
\[
= 1 + \rho(b)
\]
To see (g), note that for \(M' \subseteq Gc\) we have
\[
[M'] = [Gc] \iff M' = Gc
\]
by \([14, \text{eq. (1.2)}]\). Hence \(\text{Gr}_{[Gc]}(Gc)\) has only a single point whence \(\chi(\text{Gr}_{[Gc]}(Gc)) = 1\). To see (h), note that Equation (1.4) says that \(Gc\) is an indecomposable projective object with radical \(Gb\). So the proper submodules of \(Gc\) are precisely all the submodules of \(Gb\), whence Equation (1.5) implies
\[
\text{Gr}_f(Gb) = \begin{cases} 
\text{Gr}_f(Gc) & \text{for } f \neq [Gc], \\
\emptyset & \text{for } f = [Gc]
\end{cases}
\]
and (h) follows. Finally, (j) holds by Definition 1.4(ii).

Case (iv): \(c = r \in R\). Then \(G(\Delta)\) is not a split short exact sequence, but we have
\[
G(\Delta) = G(\tau c \to b \to c) = I_r \to \text{corad } I_r \to 0
\]
by \([14, \text{lem. 1.12(ii)}]\), where \(I_r = \text{C}(\text{C}(r, \Sigma \tau r))\) is the indecomposable injective object of \(\text{Mod } R\) associated with \(r\), see \([14, 1.10]\), and \(\text{corad}\) denotes the quotient by the socle. Now proceed dually to Case (iii), replacing Definition 1.4(ii) by Definition 1.4(iii).

\[\square\]

2. A construction of frieze-like maps \(\alpha\) and \(\beta\) with values in Laurent polynomials

This section proves Theorem B in the introduction. It is a consequence of Definition 2.8, Theorem 2.11, and Remark 2.12.

Setup 2.1. We continue to work under Setup 1.1 and henceforth add the assumption that \(C\) is a 2-Calabi-Yau category with a cluster tilting subcategory \(T\) which belongs to a cluster structure in the sense of \([4, \text{sec. II.1}]\), and which satisfies \(R \subseteq T\).

Note that the AR translation of \(C\) is
\[
\tau = \Sigma
\]
and that the Serre functor is \(\Sigma^2\).
Remark 2.2. When $R$ is a rigid subcategory of $C$, it is often possible to find a cluster tilting subcategory $T$ with $R \subseteq T$. Not always, however: if $C$ is a cluster tube, then such a $T$ cannot be found since cluster tubes have no cluster tilting subcategories, see [6, cor. 2.7].

2.3 (Mutation and exchange triangles). Let $\text{ind } T$ denote the set of (isomorphism classes of) indecomposable objects of $T$. Each $t \in \text{ind } T$ has a mutation $t^*$ which is the unique indecomposable object in $C$ such that $T$ remains a cluster tilting subcategory if $t$ is replaced by $t^*$. There are distinguished triangles

$$t^* \to a \to t, \quad t \to a' \to t^*$$

with $a, a' \in \text{add } \left( \text{ind } T \setminus \text{ind } R \right)$, known as exchange triangles, see [4, sec. II.1].

Definition 2.4 (The subgroup $N$). The split Grothendieck group of an additive category is denoted by $K_0^{\text{split}}$. It has a relation $[a + b] = [a] + [b]$ for each pair of objects $a, b$, where $[a]$ is the $K_0^{\text{split}}$-class of $a$. Define a subgroup of $K_0^{\text{split}}(T)$ as follows.

$$N = \left\{ [a] - [a'] \mid s^* \to a \to s, \ s \to a' \to s^* \text{ are exchange triangles with } s \in \text{ind } T \setminus \text{ind } R \right\}$$

Let $Q : K_0^{\text{split}}(T) \to K_0^{\text{split}}(T)/N$ denote the canonical surjection.

2.5 (Simple objects, K-theory, and the homomorphism $\overline{\theta}$). The inclusion functor $i : R \to T$ induces an exact functor

$$i^* : \text{Mod } T \to \text{Mod } R, \quad i^*(M) = M \circ i.$$

Each indecomposable object $t \in \text{ind } T$ gives rise to a simple object $S_t \in \text{Mod } T$, and each $r \in \text{ind } R$ gives rise to a simple object $S_r \in \text{Mod } R$, see [2, prop. 2.3(b)]. It is not hard to show

$$i^* S_t = \begin{cases} S_t & \text{if } t \in \text{ind } R, \\ 0 & \text{if } t \in \text{ind } T \setminus \text{ind } R. \end{cases}$$

Since $i^*$ is exact and sends simple objects to simple objects or 0, it preserves finite length so restricts to an exact functor

$$i^* : \text{fl } T \to \text{fl } R.$$

Let

$$\kappa : K_0(\text{fl } T) \to K_0(\text{fl } R)$$

be the induced homomorphism. The source is a free group on the classes $[S_t]$ for $t \in \text{ind } T$ and the target is a free group on the classes $[S_r]$ for $r \in \text{ind } R$. The homomorphism $\kappa$ is surjective and given by

$$\kappa([S_t]) = \begin{cases} [S_t] & \text{if } t \in \text{ind } R, \\ 0 & \text{if } t \in \text{ind } T \setminus \text{ind } R. \end{cases}$$

There is a functor

$$C^\to : \text{Mod } T, \quad c \mapsto C(-, \Sigma c)|_T,$$

and $i^* C^\to = G$ where $G$ is the functor from Subsection 1.2.

We define a homomorphism as follows,

$$\overline{\theta} : K_0(\text{fl } T) \to K_0^{\text{split}}(T), \quad \overline{\theta}([S_t]) = [a] - [a'],$$

(2.4)
where $a, a'$ come from the exchange triangles \((2.1), \text{ see \[15, 1.5(ii)\]}\).

2.6 (The homomorphism $\theta$). It is clear from Equations \((2.2), (2.3), \text{ and \(2.4\)}\) that there is a unique homomorphism $\theta$ which makes the following square commutative.

\[
\begin{array}{ccc}
K_0(\text{fl } T) & \overset{\theta}{\longrightarrow} & K_0^{\text{split}}(T) \\
\kappa \downarrow & & \downarrow Q \\
K_0(\text{fl } R) & \overset{\theta}{\longrightarrow} & K_0^{\text{split}}(T)/N
\end{array}
\] (2.5)

2.7 (Index and coindex). For $c \in C$ there is a distinguished triangle $t_1 \to t_0 \to c$ with $t_0, t_1 \in T$ by \[14\] sec. 1, and the index $\text{ind } c = [t_0] - [t_1]$ is a well-defined element of $K_0^{\text{split}}(T)$. Similarly there is a distinguished triangle $c \to \Sigma^2 t^0 \to \Sigma^2 t^1$ with $t^0, t^1 \in T$, and the coindex $\text{coind } c = [t^0] - [t^1]$ is a well-defined element of $K_0^{\text{split}}(T)$.

**Definition 2.8** (The maps $\alpha$ and $\beta$). Recall that $A$ is a commutative ring. Let $\varepsilon : K_0^{\text{split}}(T)/N \to A$ be a map which is “exponential” in the sense that

$$
\varepsilon(0) = 1, \quad \varepsilon(e + f) = \varepsilon(e)\varepsilon(f).
$$

Define $\alpha : \text{obj } C \to A, \beta : K_0(\text{fl } R) \to A$ by

$$
\alpha(c) = \varepsilon(Q(\text{ind } c)), \quad \beta(e) = \varepsilon(\theta(e)).
$$

(2.7)

It is easy to see that $\alpha$ and $\beta$ satisfy the conditions in Setup \[1.1\] and Equation \((1.1)\) now defines a modified Caldero-Chapoton map $\rho$ with values in $A$.

Remark 2.9. The definition of $\alpha$ and $\beta$ is motivated by the original Caldero-Chapoton map which is recovered as follows when $R = T$: in this case, $N = 0$ and $Q$ is the identity while $\theta = \theta$, so Equations \((2.7)\) read

$$
\alpha(c) = \varepsilon(\text{ind } c), \quad \beta(e) = \varepsilon\theta(e).
$$

The group $K_0^{\text{split}}(T)$ is free on the classes $[t]$ for $t \in \text{ind } T$. Let $A$ be the Laurent polynomial ring on generators $x_t$ for $t \in \text{ind } T$ and set $\varepsilon([t]) = x_t$ for $t \in \text{ind } T$. Then the map $\rho$ from Equation \((1.1)\) is the original Caldero-Chapoton map, see \[15, 1.8\].

**Lemma 2.10.** The map $\theta$ from Equation \((2.4)\) satisfies the following.

(i) If $Gc$ has finite length in $\text{Mod } T$ then $\theta([Gc]) = -(\text{ind } c + \text{ind } \Sigma c)$.

(ii) Let $\Sigma c \overset{\phi}{\to} b \to c$ be an AR triangle in $C$. If $G(\Sigma c)$ and $Gc$ have finite length in $\text{Mod } T$ then \(\text{ind } b = \left\{ \begin{array}{ll}
-\theta([Gc]) & \text{if } c \not\in T, \\
\theta([Sb]) & \text{if } c = t \in T.
\end{array} \right.\)
Proof. First note that [17, lem. 2.1(2) and prop. 2.2] apply to the present setup by [15, 1.3].

(i) Combine [15, 1.5] with [17, lem. 2.1(2)].

(ii) Observe that
\[ \text{ind } b = \overline{\theta}([\text{Ker } G\varphi] - [Gc]). \] (2.8)
This can be seen by combining part (i) of the lemma with [17, prop. 2.2].

The case \( c \not\in T \): then \( C(t, b) \to C(t, c) \) is surjective because \( \Sigma c \xrightarrow{\varphi} b \to c \) is an AR triangle. The long exact sequence \( C(t, b) \to C(t, c) \to C(t, \Sigma^2 c) \xrightarrow{G} C(t, \Sigma b) \) shows that \( (\Sigma \varphi)_* : C(t, \Sigma^2 c) \to C(t, \Sigma b) \) is injective. This implies that \( G\varphi \) is injective whence Equation (2.8) gives
\[ \text{ind } b = -\overline{\theta}([Gc]) \] as desired.

The case \( c = t \in T \): then \( G(\Sigma c) = \mathcal{T}_t \xrightarrow{G_t} \text{corad } \mathcal{T}_t \to 0 \) by [14, lem. 1.12(ii)]. Here \( \mathcal{T}_t = C(-, \Sigma^2 t)|_T \) is the indecomposable injective object of \( \text{Mod } T \) associated with \( t \), and corad denotes the quotient by the socle. Hence \( \text{Ker } G\varphi = \mathcal{T}_t \) and \( Gc = 0 \), whence Equation (2.8) reads \[ \text{ind } b = \overline{\theta}([Gc]) \] as desired. \( \square \)

**Theorem 2.11.** Let \( \Delta = \Sigma c \to b \to c \) be an AR triangle in \( C \) such that \( Gc \) and \( G(\Sigma c) \) have finite length in \( \text{Mod } T \).

Then \( Gc \) and \( G(\Sigma c) \) have finite length in \( \text{Mod } R \), and the maps \( \alpha \) and \( \beta \) from Definition 2.8 are frieze-like for \( \Delta \).

Proof. The statement on lengths holds because \( G = i^*G \) and \( i^* \) preserves finite length, see Subsection 2.5. We must now check the conditions of Definition 1.4.

First, we show that
\[ \alpha(c \oplus \Sigma c)\beta([Gc]) = 1, \] (2.9)
in particular establishing the second equation in Definition 1.4(ii). Equation (2.7) gives
\[ \alpha(c \oplus \Sigma c) = \varepsilon Q(\text{ind } c \oplus \Sigma c) = \varepsilon Q(\text{ind } c + \text{ind } \Sigma c). \] (2.10)

On the other hand, combining Equation (2.7), the fact that \( [Gc] = [i^*Gc] = \kappa[Gc] \), the commutative square (2.5), and Lemma 2.10(i) gives
\[ \beta([Gc]) = \varepsilon \theta([Gc]) = \varepsilon \theta \kappa([Gc]) = \varepsilon Q \theta([Gc]) = \varepsilon Q \left( (\text{ind } c + \text{ind } \Sigma c) \right). \]

Multiplying the last two equations proves Equation (2.9).

Secondly, if \( c = t \in T \) then it is direct from the definition of index and coindex that
\[ \text{ind } c = [t] \land \text{ind } \Sigma c = -[t]. \]

Inserting into Equation (2.10) gives
\[ c \in T \Rightarrow \alpha(c \oplus \Sigma c) = 1, \] (2.11)
in particular establishing the first equation in Definition 1.4(iii).

Thirdly, suppose \( c \not\in R \). We will show
\[ \alpha(b) = \alpha(c \oplus \Sigma c), \] (2.12)
establishing Definition 1.4(i) as well as the first equation in Definition 1.4(ii).
The case $c = t \in T$: Note that $c \not\in R$ implies $\overline{\theta}([S_r]) \in \mathbb{N}$ by Equations (2.4) and (2.2). Using Equation (2.7) and Lemma 2.10(ii) therefore gives
\[ \alpha(b) = \varepsilon Q(\text{ind } b) = \varepsilon Q\overline{\theta}([S_r]) = \varepsilon(0) = 1. \]
Combining with Equation (2.11) shows Equation (2.12).

The case $c \not\in T$: combining the two parts of Lemma 2.10 shows $\text{ind } b = \text{ind } c + \text{ind } \Sigma c$. Applying $\varepsilon Q$ shows $\alpha(b) = \alpha(c)\alpha(\Sigma c)$ which is equivalent to Equation (2.12).

Finally, suppose $c = r \in R$. We show that
\[ \alpha(b) = \beta([S_r]), \]
establishing the second equation in Definition 1.4(iii). Lemma 2.10(ii) says
\[ \text{ind } b = \overline{\theta}([S_r]). \]
Applying $\varepsilon Q$ gives the first of the following equalities.
\[ \alpha(b) = \varepsilon Q\overline{\theta}([S_r]) = \varepsilon\theta([S_r]) = \theta([S_r]) = \beta([S_r]) \]
The other equalities are by the commutative diagram (2.5) and Equations (2.3) and (2.7). □

Remark 2.12. The maps $\alpha$ and $\beta$ from Definition 2.8 and hence the modified Caldero-Chapoton map $\rho$ from Equation (1.11), depend on the map $\varepsilon : K^\text{split}_0(T)/N \to A$. The possible choices of $\varepsilon$ are determined by the structure of $K^\text{split}_0(T)/N$ which we do not know in general.

However, let us suppose that $\text{ind } R$ and $\text{ind } T$ are finite, with $r$, respectively $r + s$, objects. This is the situation from Theorem B in the introduction if we set $R$, respectively $T$, equal to the direct sum of the indecomposable objects in $\text{ind } R$, respectively $\text{ind } T$. Then we can set $A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ and use all the variables $x_1, \ldots, x_r$ thereby proving Theorem B.

Namely, $K^\text{split}_0(T)$ is a free abelian group on $r + s$ generators, one per object in $\text{ind } T$, and the subgroup $N$ has $s$ generators, one per object in $\text{ind } T \setminus \text{ind } R$. So $K^\text{split}_0(T)/N$ has a quotient group $F$ which is free abelian of rank $(r+s) - s = r$. The desired map $\varepsilon : K^\text{split}_0(T)/N \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ can be obtained by sending each generator of $F$ to a generator of $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$.

Note that $K^\text{split}_0(T)$ may have a quotient group which is free abelian of rank $n > r$, see the example in Section 3. In this case, the above method means that we can even set $A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and use all the variables $x_1, \ldots, x_n$.

3. Example: A modified Caldero-Chapoton map on the cluster category of Dynkin type $A_5$

This section shows how to obtain the example in Figure 3 in the introduction.

Setup 3.1. Let the category $C$ of Setups 2.1 and 2.1 be $C(A_5)$, the cluster category of Dynkin type $A_5$. There is a bijection between $\text{ind } C$ and the diagonals of a 8-gon, see [8] sec. 2 and 5]. We let $\text{ind } R$, respectively $\text{ind } T$, be given by the red diagonals, respectively all the red and blue diagonals, in Figure 4. These data satisfy our assumptions, see [5] sec. 1].

3.2 (Some properties of $C$). We denote diagonals and their corresponding indecomposable objects by pairs of vertices, so $\{2,7\}$ is both a red diagonal in Figure 4 and an object of $\text{ind } R$. The AR quiver of $C$ and the objects of $\text{ind } R$, respectively $\text{ind } T$, are shown in Figure 4 in the introduction.

At the level of objects, the suspension functor $\Sigma$ is given by $\Sigma\{i, j\} = \{i - 1, j - 1\}$. Note that vertex numbers are taken modulo 8.
Figure 4. The diagonals of the 8-gon correspond to the indecomposable objects of $C(A_5)$. The red diagonals define $\text{ind } R$ and all the red and blue diagonals define $\text{ind } T$.

If $x, y \in \text{ind } C$ then

$$C(x, \Sigma y) = \begin{cases} k & \text{if the diagonals corresponding to } x \text{ and } y \text{ cross}, \\ 0 & \text{if not}. \end{cases} \quad (3.1)$$

If $i, k, j, \ell$ are four vertices in anticlockwise order on the polygon, then $\{i, j\}$ and $\{k, \ell\}$ are crossing diagonals, and there are the following non-split distinguished triangles,

$$\{i, j\} \to \{i, k\} \oplus \{j, \ell\} \to \{k, \ell\}, \quad \{k, \ell\} \to \{i, k\} \oplus \{j, \ell\} \to \{i, j\}, \quad (3.2)$$

where a pair of neighbouring vertices must be interpreted as 0.

3.3 (K-theory). The category $T$ has the following indecomposable objects.

$$\{1, 7\}, \{2, 4\}, \{2, 5\}, \{2, 7\}, \{5, 7\}$$

Their $K_0^{\!\text{split}}$-classes are free generators of $K_0^{\!\text{split}}(T)$. To save parentheses, the classes are denoted $[1, 7]$ etc. The objects in $\text{ind } T \setminus \text{ind } R$ are

$$\{1, 7\}, \{2, 4\}, \{5, 7\},$$

and Equation (3.2) means that they sit in the following exchange triangles.

$$\begin{array}{cccccc}
\{2, 8\} & \to & 0 & \to & \{1, 7\} & \to & \{2, 8\} \\
\{3, 5\} & \to & 0 & \to & \{2, 4\} & \to & \{3, 5\} \\
\{2, 6\} & \to & \{2, 7\} & \to & \{5, 7\} & \to & \{2, 6\}
\end{array}$$

Accordingly, the subgroup $N$ of Definition 2.4 is

$$N = \langle -[2, 7], -[2, 5], [2, 7] - [2, 5] \rangle = \langle [2, 5], [2, 7] \rangle,$$

and $K_0^{\!\text{split}}(T)/N$ is the free abelian group generated by

$$[1, 7] + N, \ [2, 4] + N, \ [5, 7] + N.$$

The category $\text{fl } R$ has the simple objects

$$S_{\{2, 5\}}, \ S_{\{2, 7\}}.$$
whose $K_0$-classes are free generators of $K_0(\text{fl } R)$, and $\text{fl } T$ has the simple objects

\[ \mathcal{S}_{\{1,7\}}, \mathcal{S}_{\{2,4\}}, \mathcal{S}_{\{2,5\}}, \mathcal{S}_{\{2,7\}}, \mathcal{S}_{\{5,7\}} \]

whose $K_0$-classes are free generators of $K_0(\text{fl } T)$. To save parentheses, the simple objects will be denoted $S_{2,5}$, respectively $\mathcal{S}_{1,7}$, etc.

**Definition 3.4.** Let the map

\[ \varepsilon : K_0^{\text{nil}}(T)/N \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}, z^{\pm 1}] \]

be given by

\[ \varepsilon([1, 7] + N) = u, \quad \varepsilon([2, 4] + N) = v, \quad \varepsilon([5, 7] + N) = z. \]  \hspace{1cm} (3.3)

Equation (2.7) defines the maps $\alpha$ and $\beta$, and the modified Caldero-Chapoton map $\rho$ is defined by Equation (2.1).

**Example 3.5.** Let us compute $\rho(\{4, 6\})$.

Equation (2.7) gives

\[ \alpha(\{4, 6\}) = \varepsilon Q(\text{ind}\{4, 6\}) = (*). \]

Now $\{4, 6\} = \Sigma\{5, 7\}$ so $\text{ind}\{4, 6\} = \text{ind} \Sigma\{5, 7\} = -[5, 7]$, where the last equality is direct from the definition of index because $\{5, 7\} \in T$, see Subsection 2.7. Hence Equation (3.3) gives

\[ (*) = \varepsilon Q(-[5, 7]) = \varepsilon(-[5, 7]+N) = z^{-1}. \]

We have $G(\{4, 6\}) = C(-, \Sigma\{4, 6\})|_R$. Moreover, $R = \text{add} \{\{2, 5\}, \{2, 7\}\}$, and it is direct from Equation (3.1) that $G(\{4, 6\})$ is supported only at $\{2, 5\}$ where it has the value $k$. That is,

\[ G(\{4, 6\}) = S_{2,5}. \]

It follows that the only non-empty Grassmannians appearing in Equation (1.1) when computing $\rho(\{4, 6\})$ are $\text{Gr}_0(G(\{4, 6\}))$ and $\text{Gr}_{[S_{2,5}]}(G(\{4, 6\}))$, and it is clear that each is a point so has Euler characteristic 1.

Finally, Equations (2.3) and (2.7) and diagram (2.5) give

\[ \beta([S_{2,5}]) = \varepsilon \theta([S_{2,5}]) = \varepsilon \theta K([\mathcal{S}_{2,5}]) = \varepsilon Q \bar{\theta}([\mathcal{S}_{2,5}]) = (**). \]

Equation (3.2) gives exchange triangles

\[ \{4, 7\} \rightarrow \{2, 4\} \oplus \{5, 7\} \rightarrow \{2, 5\} \quad \{2, 5\} \rightarrow \{2, 7\} \rightarrow \{4, 7\} \]

and Equation (2.4) gives $\bar{\theta}(\mathcal{S}_{2,5}) = [2, 4] + [5, 7] - [2, 7]$ whence Equation (3.3) gives

\[ (** \rightarrow) = \varepsilon Q([2, 4] + [5, 7] - [2, 7]) = vz. \]

Hence Equation (1.1) says

\[ \rho(\{4, 6\}) = \alpha(\{4, 6\}) \sum_e \chi(\text{Gr}_e(G\{4, 6\})) \beta(e) \]

\[ = z^{-1} \left( \chi(\text{Gr}_0(S_{2,5})) \beta(0) + \chi(\text{Gr}_{[S_{2,5}]}(S_{2,5})) \beta([S_{2,5}]) \right) \]

\[ = z^{-1} \left( 1 + vz \right) \]

\[ = \frac{1 + vz}{z}. \]

Similar computations for the other indecomposable objects finally produce the generalised frieze in Figure 3 in the introduction.
4. Questions

We end the paper with some questions.

(i) The group $K_0^{\text{split}}(T)$ is free abelian on $\text{ind } T$, and the subgroup $N$ of Definition 2.4 is generated by all expressions $[a] - [a']$ where $s^* \to a \to s$ and $s \to a' \to s^*$ are exchange triangles with $s \in \text{ind } T \setminus \text{ind } R$.

What is the rank $n$ of the quotient $K_0^{\text{split}}(T)/N$? Note that when $n$ is finite, it is the largest integer such that the method of Remark 2.12 results in a modified Caldero-Chapoton map $\rho: \text{obj } C \to \mathbb{Z}[x^{\pm 1}, \ldots, x_n^{\pm 1}]$ using all the variables $x_1, \ldots, x_n$.

(ii) Consider the $\mathbb{Z}$-subalgebra of $\mathbb{Z}[x^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by the values of the modified Caldero-Chapoton map $\rho$.

What is its relation to the cluster algebra?

(iii) Let $T$ be a cluster tilting object and use it to define a Caldero-Chapoton map $X$. If $T$ is subjected to cluster mutation, then the values of $X$ change in a well-understood way, see [17, proof of cor. 5.4].

There is a notion of mutation of rigid objects due to [16, sec. 2]. What happens to the values of the modified Caldero-Chapoton map under such mutation?

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