On Discrete U-duality in M-theory

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Abstract

We give a complete set of generators for the discrete exceptional U-duality groups of toroidal compactified type II theory and M-theory in $d \geq 3$. For this, we use the DSZ quantization in $d = 4$ as originally proposed by Hull and Townsend, and determine the discrete group inducing integer shifts on the charge lattice. It is generated by fundamental unipotents, which are constructed by exponentiating the Chevalley generators of the corresponding Lie algebra. We then extend a method suggested by the above authors and used by Sen for the heterotic string to get the discrete U-duality group in $d = 3$, thereby obtaining a quantized symmetry in $d = 3$ from a $d = 4$ quantization condition. This is studied first in a toy model, corresponding to $d = 5$ simple supergravity, and then applied to M-theory. It turns out that, in the toy model, the resulting U-duality group in $d = 3$ is strictly smaller than the one generated by the fundamental unipotents corresponding to all Chevalley generators. However, for M-theory, both groups agree. We illustrate the compactification to $d = 3$ by an embedding of $d = 4$ particle multiplets into the $d = 3$ theory.

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1 Introduction and Motivation

The study of nonperturbative duality symmetries in the last years has dramatically changed our understanding of string theory. The five perturbative string theories in ten dimensions are now understood as different limits of a unified eleven dimensional theory, called M-theory, with a web of duality symmetries connecting them [1, 2]. Since these symmetries act on the string coupling constant, they are difficult to test. Nonetheless, several quantities protected by supersymmetry admit such tests, such as BPS masses and spectra as well as BPS saturated amplitudes. Since the final form of a quantized M-theory is not yet known, most tests use the low-energy effective limit, which is eleven dimensional supergravity. Global quantum symmetries in this limit should extend to a fully quantized theory.

One of these nonperturbative dualities is the U-duality of toroidally compactified type II string theory, introduced in [3]. It was conjectured that this symmetry is generated by the perturbative target space duality (for a review see [4]) and the strong-weak coupling duality [5] of the type II string [6] that do not commute, a phenomenon first discovered in [8]. For compactifications on the torus $T^d$ of type II string theory, the target space duality group corresponds to $SO(d, d; \mathbb{Z})$, while the S-duality of the type IIB theory acts as an $SL(2, \mathbb{Z})$ group. The proposed U-duality group is then generated by

$$
G(\mathbb{Z}) = SL(2, \mathbb{Z}) \rtimes SO(d, d; \mathbb{Z}),
$$

where $\rtimes$ refers to the non-commuting action of both groups. The type IIB S-duality was interpreted as the modular group of the torus in the tenth and eleventh direction of M-theory [6, 1, 2]. For compactifications of M-theory on $T^{d+1}$, the modular group of the $d+1$-torus is $SL(d+1, \mathbb{Z})$, containing the above $SL(2, \mathbb{Z})$ as a subgroup. Therefore, rather than (1), the definition

$$
G(\mathbb{Z}) = SL(d+1, \mathbb{Z}) \rtimes SO(d, d; \mathbb{Z})
$$

may be adopted.

The discussion of U-duality has led to identify nonperturbative states in string theory with higher dimensional branes wrapping on the internal torus [3], as well as to include D-branes into the theory (see [10] for a review), improving dramatically our understanding of nonperturbative phenomena in string theory. Solitons of string theory corresponding to black hole solutions with regular horizon have made a microscopic interpretation of
black hole entropy possible (see [11] and [12] for a review). The entropy is known to be
given by invariants of the U-duality group.

In this context, the search for most general BPS black hole solutions has been impor-
tant ([13], see [14], [15] for recent reviews).

U-duality, rephrased in an algebraic language [16], has also been used in the infinite
momentum frame of Matrix theory, predicting BPS states for compactifications not yet
accessible to Matrix theory and new “mysterious” states especially for compactifications
to three dimensions. U-duality extends to a generalized electric-magnetic duality of the
Super-YM theory. Upon inclusion of the lightlike momentum on the M-theory circle, the
group was even proposed to be extended by rank, see [17] for a review and references
therein.

In [17], a recent review was given on U-duality in diverse dimensions. The definition
(2) was used for an algebraic definition of the U-duality group and was applied to study
BPS spectra in diverse dimensions, to generate U-duality invariant mass formulae and to
study $R^4$ corrections to type II theory as well as the implications on Matrix theory.

In this paper, we give generators for the discrete U-duality in four dimensions, thereby
determining higher dimensional U-dualities as well, and apply these for a definition of
U-duality in $d = 3$. Rather than using the string inspired definition [2] inferred into the
low energy effective action, we will recall the original conjecture of [3], using the hidden
symmetries of low energy supergravity [18]. The hidden symmetries have recently been
reinvestigated in [19, 20], where the symmetry groups $E_{d(+d)}$ were derived from $d = 11$
diffeomorphisms in internal space, and the change of the symmetry group with respect
to undualization of fields was investigated.

The low-energy effective field theory of type II theory on $T^6$ is $d = 4$ $N = 8$ maximal
supergravity and has $E_{7(+7)}$ global symmetry [21] acting in the fundamental 56 repre-
sentation. By an embedding into $E_{8(+8)}$, we will recover this representation in an easily
“manageable” way and give its Chevalley generators in this representation. This will
allow to address the discrete group explicitly.

The Dirac-Schwinger-Zwanziger quantization condition in four dimensions [23] dis-
cretizes electric and magnetic charges of the $U(1)$ gauge fields of the theory. The global
$E_{7(+7)}$ symmetry is therefore broken to a discrete $E_{7(+7)}(\mathbb{Z})$ symmetry, inducing integer
shifts on the charge lattice. Since the most general duality group for our field configura-
tion is known to be $Sp(2k, \mathbb{R})$ [23] and the discrete group $E_{7(+7)}(\mathbb{Z})$ was not determined
directly in [3], it was conjectured that the group
\[ E_{7(+7)}(\mathbb{Z}) = E_{7(+7)} \cap Sp(56, \mathbb{Z}) \]  

is a symmetry of the full type II string theory.

We will give a complete set of generators for the discrete \( E_{7(+7)}(\mathbb{Z}) \) group by demanding integer shifts on the lattice. We will use these to identify the known string dualities in our notation, and show that the definitions (4) and (6) agree with the one we have found, and comment on the algebraic generators reviewed in [17].

Duality groups of the type II string theory in dimensions larger than four are found by the intersections of the classical duality group with \( E_{7(+7)}(\mathbb{Z}) \). For \( d = 3 \), the situation is more complicated.

The classical continuous duality symmetry in three dimensions is \( E_{8(+8)} \), explicitly used for construction in [24]. But the meaning of a duality group is not clear since only scalars remain in the theory and electric charge seems ill defined. A DSZ quantization condition for \( d = 3 \) seems unclear, and no analogue to \( Sp(2k, \mathbb{Z}) \) and the construction (3) is known.

We will give an explicit construction of the U-duality group in three dimensions by extending a conjecture made by Hull and Townsend [3], parallel to a method applied to the heterotic string by Sen [8, 9]. This will enable us to describe the U-duality group in three dimensions explicitly and give a set of generators for this group as well. The procedure is illustrated in figure 4. By compactifying M-theory on the torus, we can choose eight different ways how to compactify first to four dimensions. This results in eight \( E_{7(+7)}(\mathbb{Z}) \) acting differently on M-theory fields. By reducing the theory further to three dimensions, these groups are merged together to form the three dimensional duality group.

We will illustrate duality in \( d = 3 \) by giving the embedding of M-theory particle solutions in \( d = 4 \), parallel to [8].

Before turning to the \( d = 11 \) case, we would like to introduce the main concepts first in a toy model for simplicity. For this, we will use five dimensional simple supergravity, which upon reduction to three dimensions exhibits a \( G_{2(+2)} \) global symmetry [23, 24]. It is known that this theory closely resembles \( d = 11 \) supergravity, the conjectured low energy limit of M-theory, in many respects, but no string compactification described by this no-moduli supergravity at low energies is known [28]. The reduction procedure for this theory is illustrated in figure 3.

After introducing the main concepts in this simple model, the \( d = 11 \) case will be
constructed strictly along the same lines. Quite surprisingly, we will see that these models differ in a major point. We will discuss this difference again in the last section, where the direct embedding of the toy model into M-theory will be given.

Figure 1: Construction of three dimensional U-duality

Figure 2: Construction of three dimensional U-duality in the $G_{2(+2)}$ toy model
2 The $G_{2(+2)}$ Toy Model

We start with $d = 5$ simple supergravity \textsuperscript{26, 27}. This theory is studied as a toy model for the low energy effective action of type II string theory and M-theory and introduce the main technical concepts. We will find an embedding into M-theory in the last section.

The bosonic part of the Lagrangian of simple $d = 5$ supergravity is given by

$$\mathcal{L} = -E^{(5)}(R^{(5)} + \frac{1}{4} F_{MN} F^{MN}) - \frac{1}{12 \sqrt{3}} \epsilon^{MNPQR} F_{MN} F_{PQ} A_R,$$

where $F_{MN} = 2 \partial [M A_N]$. Indices $M, N$ run from 0..4. We take the signature $(+---)$. To get a $d = 3$ U-duality group, we are interested in compactifications of this theory to three dimensions. The vielbein can be written as

$$E^{(5)A}_M = \begin{bmatrix} e^{-1}E^{(3)\alpha}_\mu & B_i^I e^a_i \\ 0 & e^a_i \end{bmatrix}$$

where the indices $i, a$ run from 1...2. $e_i^a$ is called the internal vielbein and is assumed to be of triangular shape. $e$ is its determinant. The vector field decomposes into $A_M = (A_\mu, A_{2+i})$, $i = 1...2$, leaving a vector field and two scalars in three dimensions.

In order to determine the $d = 4$ U-duality analogue in this theory and to see how it is embedded into $d = 3$, we will need to reduce in a stepwise fashion.

For this, we will interpret the coordinates $\{0, 1, 2, 3\}$ as coordinates of the five dimensional theory with compact dimension 4, corresponding to the left side of figure 2. We will study this effectively four dimensional theory and identify the U-duality group on the classical and quantum level.

We will then reduce the theory further to three dimensions with coordinates $\{0, 1, 2\}$ and interpret the three dimensional theory as four dimensional theory with compact dimension 3. We will show that this reduction leads to a $G_{2(\ast 2)}/SO(4)$ nonlinear coset sigma model. We will identify the four dimensional U-duality in this theory and show how it acts.

We will then study the other compactification procedure indicated on the right side of figure 2. Now the coordinates $\{0, 1, 2, 4\}$ will be interpreted as four dimensional coordinates, leading to a different U-duality group. We will then show how these two groups merge together to form the three dimensional U-duality.
2.1 The \( d = 4 \) Theory

Reducing the above Lagrangian to four dimensions yields \[26\]

\[
\mathcal{L} = -E^{(4)} \mathcal{R}^{(4)} + \frac{3}{2} E^{(4)} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} E^{(4)} \rho^{-2} \partial_\mu A_4 \partial^\mu A_4 \\
- \frac{1}{4} E^{(4)} \rho^3 B_{\mu \nu} B^{\mu \nu} - \frac{1}{4} E^{(4)} \rho F_{\mu \nu}^{(4)} F^{\mu \nu} \\
- \frac{1}{4 \sqrt{3}} \epsilon^{\mu \nu \rho \sigma} \left( A_4 F_{\mu \nu}^{(4)} F^{(4)} - A_4^2 A_{\mu}^{(4)} B_{\rho \sigma} + \frac{1}{3} A_4^3 B_{\mu \nu} B_{\rho \sigma} \right)
\]

(4)

where the fünfbtein and the vector field are parameterized as

\[
E^{(5)}_M = \begin{bmatrix} \rho^{-\frac{1}{2}} E^{(4)}_\bar{\alpha} & \rho B_{\bar{\alpha}} \\ 0 & \rho \end{bmatrix}, \quad A_M = [A_\bar{\mu}, A_4].
\]

(5)

\( \bar{\mu}, \bar{\nu}, \ldots \) are curved and \( \bar{\alpha}, \bar{\beta}, \ldots \) flat four dimensional indices. They run from 0\ldots3. We have defined \( F_{\mu \nu}^{(4)} = F_{\mu \nu}^{(4)} + B_{\mu \nu} A_4 \). \( F_{\mu \nu}^{(4)} = 2\partial_{[\mu} A_{\nu]} \) is the field strength of the Kaluza-Klein invariant vector field \( A_\mu' = A_\mu - B_\mu A_4 \), and \( B_{\mu \nu} = 2\partial_{[\mu} B_{\nu]} \). \( A_\mu' \) is dualized in the standard way by adding a Lagrange multiplier

\[
\mathcal{L}_{\text{Lag.mut.}} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} A_{\sigma} \partial_\rho F_{\mu \nu}^{(4)}.
\]

Defining the new field strength \( \hat{A}_{\mu \nu} = 2\partial_{[\mu} \hat{A}_{\nu]} \), we introduce the vector notation

\[
\mathcal{G}_{\mu \nu} = \begin{bmatrix} \hat{A}_{\mu \nu} \\ B_{\mu \nu} \end{bmatrix}, \quad \mathcal{H}_{\mu \nu} = \begin{bmatrix} H_{\mu \nu}^A \\ H_{\mu \nu}^B \end{bmatrix}
\]

where

\[
H_{\mu \nu}^A = -\frac{2}{E^{(4)}} \star \left( \frac{\delta \mathcal{L}}{\delta \hat{A}^{\mu \nu}} \right), \quad H_{\mu \nu}^B = -\frac{2}{E^{(4)}} \star \left( \frac{\delta \mathcal{L}}{\delta B^{\mu \nu}} \right).
\]

\( \star \equiv \frac{1}{2} E^{(4)-1} \epsilon^{\mu \nu \rho \sigma} \) denotes the space-time dual in four dimensions.

The scalar fields of the theory are \( A_4 \) and \( \rho \). For this sector, we introduce a field \( \mathcal{V}^{(4)} \in SL(2, \mathbb{R})/SO(2) \) in the 4 irreducible representation. Using the Iwasawa decomposition, \( \mathcal{V}^{(4)} \) is defined as

\[
\mathcal{V}^{(4)} = P^{-1} \exp \left( -\frac{1}{2} \ln \rho H \right) \exp \left( -\frac{1}{\sqrt{3}} A_4 E \right) P
\]

where the Chevalley generators \( H, E, F \) of \( SL(2, \mathbb{R}) \) have been chosen to be
\[ H = \begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}, F = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix} \]

and

\[ P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (P^{-1})^T. \]

The explicit form of these generators will become important in the next section.

From \( V^{(4)} \) a field \( P^{(4)}_\mu \) may be defined by

\[ \partial_\mu V^{(4)} V^{(4)-1} = Q^{(4)}_\mu + P^{(4)}_\mu, \quad Q^{(4)}_\mu \in so(2), \quad P^{(4)}_\mu \in sl(2) - so(2). \]

Using the above definitions, we note that the vector fields are related by

\[ \mathcal{F}_{\bar{\mu}\bar{\nu}} \equiv \begin{pmatrix} G_{\bar{\mu}\bar{\nu}} \\ \mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix} = \Omega V^{(4)T} V^{(4)} \begin{pmatrix} *G_{\bar{\mu}\bar{\nu}} \\ *\mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix}. \]

with the symplectic invariant

\[ \Omega = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \]

called twisted self duality in [19], present in supergravities in all even dimensions.

Putting all together, the Lagrangian may be rewritten in the form [26]

\[ \mathcal{L} = -E^{(4)} R^{(4)} + E^{(4)} \left( \frac{1}{4} G_{\mu\nu}^T * H^{\mu\nu} + \frac{3}{10} \text{Tr}(P^{(4)}_\mu P^{(4)\mu}) \right). \]

### 2.2 Duality in \( d = 4 \)

Following [3], we will define the analogue of U-duality. The classical theory has a continuous \( SL(2, \mathbb{R}) \) duality symmetry that interchanges Bianchi identities and equations of motion. Its action is

\[ \mathcal{F}_{\bar{\mu}\bar{\nu}} \rightarrow \Lambda^{-1} \mathcal{F}_{\bar{\mu}\bar{\nu}}, \quad V^{(4)} \rightarrow V^{(4)} \Lambda, \quad \Lambda \in SL(2, \mathbb{R}). \]

The transformation of \( V^{(4)} \) is accompanied by a compensating local \( SO(2) \) transformation

\[ V^{(4)} \rightarrow h(x)V^{(4)} \Lambda, \quad h(x) \in SO(2) \] to restore the parameterization of the coset space.
If we define charges

\[ Z = \begin{bmatrix} p \\ q \end{bmatrix}, \quad p = \frac{1}{2\pi} \oint G, \quad q = \oint H \]

and

\[ p = \begin{bmatrix} p^A \\ p^B \end{bmatrix}, \quad q = \begin{bmatrix} q^A \\ q^B \end{bmatrix}, \]

the \( p \) charges are interpreted as magnetic, the \( q \) charges as Noether electric charges.

We will introduce a charged “elementary” soliton multiplet in the \( d = 11 \) section and study it. Due to its simplicity, the soliton solution presented there is actually a soliton in our toy-model as well. In M-theory, these solitons have been identified with fundamental and solitonic string states and D-brane states that fill the U-duality multiplets [3].

The charge vector \( Z \) transforms as

\[ Z \rightarrow \Lambda^{-1} Z, \ \Lambda \in SL(2, \mathbb{R}) \]

under the classical duality symmetry.

Upon quantization, the charges have to obey the Dirac-Schwinger-Zwanziger charge quantization condition. If all electric and magnetic charges exist, they are restricted to live on an integer lattice, and the \( SL(2, \mathbb{R}) \) symmetry is broken to a discrete subgroup inducing integer shifts, which is \( SL(2, \mathbb{Z}) \). Note that the situation is complicated due to the fact that the two gauge fields \( \tilde{A}_{\mu\bar{\nu}} \) and \( B_{\mu\bar{\nu}} \) are not treated on the same footing by the \( SL(2, \mathbb{R}) \) symmetry, but \( B_{\mu\bar{\nu}} \) carries spin 3/2 and \( \tilde{A}_{\mu\bar{\nu}} \) spin -1/2 with respect to \( SL(2, \mathbb{R}) \), that is, they correspond to weights of different length.

To see how \( SL(2, \mathbb{Z}) \) acts, we will not try to uncover the intersection of \( SL(2, \mathbb{R}) \) with \( Sp(4, \mathbb{Z}) \), but follow a more direct path. For this, we change the basis of the \( SL(2, \mathbb{R}) \) representation space. We define

\[ \tilde{\mathcal{F}} = U \mathcal{F}, \ \tilde{\mathcal{V}}^{(4)} = U \mathcal{V}^{(4)} U^{-1}, \text{etc.} \]

with

\[ U = \begin{bmatrix} 1 & 1 \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}. \]

Then the Lie algebra of \( SL(2, \mathbb{R}) \) is transformed into
\[
\tilde{E} = \begin{bmatrix}
0 & 3 \\
0 & 2 \\
0 & 1
\end{bmatrix}, \quad \tilde{F} = \begin{bmatrix}
0 & 0 & 1 \\
2 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}, \quad \tilde{H} = H.
\]

The DSZ quantization condition now reads
\[
\tilde{Z}^T U^{-1} \Omega U^{-1} \tilde{Z}' = \tilde{p}^A \tilde{q}'_A - \tilde{p}'^A \tilde{q}_A + \frac{1}{3} (\tilde{p}^B \tilde{q}'_B - \tilde{p}'^B \tilde{q}_B) = n, \quad n \in \mathbb{Z}.
\]

The maximal subgroup of \(SL(2, \mathbb{R})\) preserving this discretization is \(SL(2, \mathbb{Z})\), generated by the modular group generators \(P^{-1}SP\) and \(P^{-1}TP\), where
\[
S = \exp(-\tilde{F}) \exp(\tilde{E}) \exp(-\tilde{F}) = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}, \quad T = \exp(\tilde{E}) = \begin{bmatrix}
1 & 3 & 3 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

This symmetry can be interpreted as the analogue of \(d = 4\) U-duality in our toy model.

If the \(\{\tilde{p}^A, \tilde{q}_A\} = \{p^A, q_A\}\) are chosen to be integer, the \(SL(2, \mathbb{Z})\) symmetry and DSZ condition yield that \(\{\tilde{p}^B, \tilde{q}_B\}\) are in \(3\mathbb{Z}\).

Note that on the scalar \(z \equiv -1/\sqrt{3}A_1 + i\rho\) this definition of \(SL(2, \mathbb{Z})\) induces the familiar modular transformations \(z \rightarrow z + 1\) and \(z \rightarrow -1/z\) under \(T\) and \(S\), respectively\(^1\).

We will now assume as in \([8]\) that this quantum symmetry is not broken when we further compactify to three dimensions.

To recover \(d = 4\) solitons in the \(d = 3\) theory, one may take an array of solitonic solutions aligned along a specific direction \([8, 3, 11, 17]\). Compactification of this direction then corresponds to identify this array periodically and taking the period to be small. We will see in the M-theory section that this leads to vortex solutions in three dimensions. We will recover the \(d = 4\) U-duality in the reduced theory acting exactly on these solutions and their \(d = 4\) charges. The equivalence of such periodic array solutions and fundamental string states under duality for the \(d = 3\) heterotic string was studied closely in \([8]\).

\section*{2.3 The \(d = 3\) Theory}

We will now reduce the theory further to three dimension \([9]\). For this, we assume that the direction 3 is compact and the fields do not depend on this coordinate, which corresponds

\(^1\)Looking at the Lagrangian, it is obvious that, in the asymptotic limit, \(z\) is exactly the \(\tau\) parameter of electromagnetism plus theta term for the field \(F''_{\mu\nu}\), when \(B_{\mu\nu} \equiv 0\). However, the above \(Sl(2, \mathbb{Z})\) will always mix all four types of charges and not preserve such a truncation.

\(^2\)The reduction from \(d = 4\) to \(d = 3\) in the context of black holes in Kaluza Klein theory was studied in \([3]\).
to keeping the zeroth Fourier component with respect to the compact direction.

We will “maximally dualize” all fields such that only scalars remain in the theory. Quite analogously to the reduction step before, we choose the vierbein to be

\[ E^{(4)\alpha}_\mu = \begin{bmatrix} e^{\phi/2} E^{(3)\alpha}_\mu & e^{-\phi/2} \hat{B}_\mu \\ 0 & e^{-\phi/2} \end{bmatrix} , \]

where \( \mu, \alpha \) now run from 0...2 and define the \( U(1) \) vector field strengths \( G'_{\mu\nu} = 2 \partial_{[\mu} G'_{\nu]} \) by taking \( G'_\mu = G_\mu - \hat{B}_\mu G_3 \).

The vector fields are then dualized by adding the Lagrange multiplier

\[ \mathcal{L}_{\text{Lag.muli.}} = \frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\rho G'_{\mu\nu} \]

and integrating out \( G'_{\mu\nu} \). This yields

\[ \partial_\mu \bar{\eta} = H_{\mu 3} \quad (10) \]

which together with the definition

\[ \partial_\mu \eta = G_{\mu 3} \]

and

\[ \mathcal{V} = \begin{bmatrix} \eta \\ \bar{\eta} \end{bmatrix} \]

will be of crucial importance in the following.

It remains to dualize the Kaluza-Klein field strength \( \hat{B}_{\mu\nu} \) by adding

\[ \mathcal{L}_{\text{Lag.muli.}} = \frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\rho \hat{B}_{\mu\nu} \]

which by integrating out \( \hat{B}_{\mu\nu} \) yields

\[ \partial_\mu f = -\frac{1}{2} E^{(3)} \epsilon^{\mu\nu\rho} e^{-2\phi} \hat{B}_{\nu\rho} - \frac{1}{2} \mathcal{V}^t \Omega \partial_\mu \mathcal{V}. \]

The Lagrangian becomes

\[ \mathcal{L} = -E^{(3)} R^{(3)} + \frac{1}{2} E^{(3)} \partial_\mu \phi \partial^\mu \phi + \frac{3}{10} E^{(3)} \text{Tr}(P_\mu (P^{(4)\mu} P^{(4)\mu}) ) + E^{(3)} e^{2\phi} (\partial_\mu f + \mathcal{V}^t \Omega \partial_\mu \mathcal{V})(\partial^\mu f + \mathcal{V}^t \Omega \partial^\mu \mathcal{V}) + 2 E^{(3)} e^\phi \partial_\mu \mathcal{V}^t \mathcal{V}(\mathcal{V}^{(4)\mu}) \partial^\mu \mathcal{V}. \quad (11) \]
2.4 The $G_{2(+2)}$ Coset in $d = 3$

We will now show that the scalars of this theory fit into a $G_{2(+2)}/SO(4)$ coset, where $G_{2(+2)}$ is the normal real form \[29\] of $G_2$. The Lagrangian gets

$$\mathcal{L} = -E^{(3)}R^{(3)} + \frac{1}{16}E^{(3)}\text{Tr}(P^{(3)}_\mu P^{(3)}\mu)$$

(12)

where the scalars are contained in a field $\mathcal{V}^{(3)} \in G_{2(+2)}/SO(4)$ and we define

$$\partial_\mu \mathcal{V}^{(3)}\mathcal{V}^{(3)-1} = Q^{(3)}_\mu + P^{(3)}_\mu, \quad Q^{(3)}_\mu \in \mathfrak{so}(4), \quad P^{(3)}_\mu \in \mathfrak{g}_{2(+2)} - \mathfrak{so}(4).$$

For the algebra $\mathfrak{g}_{2(+2)}$ of $G_{2(+2)}$, we will use Freudenthal’s realization of exceptional Lie algebras explained in appendix A. This will allow us to stay as closely as possible to the $d = 11$ case.

For the positive roots, the corresponding generators are $E^{i}_j, 1 \leq i \leq j \leq 3$, $E^{i}, 1 \leq i \leq 2$ and $E^{*}_3$, for the Cartan subalgebra we use generators $h_i, i = 1, 2$. Their definitions and commutators are given in the appendix. The generator $E^{1}_2$ corresponds to the long simple positive root, the generator $E^{2}$ to the short one.

We will now use the fact that $G_{2(+2)}$ has a maximal subgroup $SL(2) \times SL(2)$, where the two $SL(2)$ groups are generated by the short simple and the lowest root. In the Iwasawa decomposition, the $SL(2)$ generated by the short simple root will be associated with $\mathcal{V}^{(4)}$, while the $SL(2)$ generated by the lowest root will carry the 3d dilaton and dualized Kaluza-Klein gauge field.

![Figure 3: Decomposition of $G_{2(+2)} \supset SL(2) \times SL(2)$. The root surrounded by a circle is the lowest root added to the Dynkin diagram.](image)

To see this, we define the vector

$$S^t = (\frac{1}{\sqrt{3}}E^*_3, -E^{1}_2, \frac{1}{\sqrt{3}}E^{1}, E^{2}_3).$$

From the $\mathfrak{g}_{2(+2)}$ commutation relations given in the appendix one may then easily deduce that

$$[E^2, S_i] = (P^{-1}EP)_{ij}^t \ S_j, \quad [E^*_2, S_i] = (P^{-1}FP)_{ij}^t \ S_j,$$
with

\[ [\mathcal{S}_i, \mathcal{S}_j] = \Omega_{ij} E_3^1 \]

where the \( E, F \) are the \( \mathfrak{sl}(2, \mathbb{R}) \) generators given in (6) and \( P \) is given in (7).

Using these relations, it is straightforward to verify that the Lagrangian (11) is identical to (12) if

\[
\mathcal{V}^{(3)} = \exp \left( -\frac{1}{2} \ln \rho (h_2 - h_1) \right) \exp \left( -\frac{1}{\sqrt{3}} A_4 E^2 \right) \exp \left( \frac{1}{2} \phi (h_1 + h_2) \right) \exp (\mathcal{V}_i \mathcal{S}_i) \exp \left( f E_3^1 \right) = \mathcal{V}^{(4)} \exp \left( \frac{1}{2} \phi (h_1 + h_2) \right) \exp (\mathcal{V}_i \mathcal{S}_i) \exp \left( f E_3^1 \right). \tag{13} \]

The prime refers to the fact that \( \mathcal{V}^{(4)} \) is not in the spin 3/2 representation introduced in (3) of \( SL(2, \mathbb{R}) \) any more. \( G_{2(+2)} \) decomposes as

\[
G_{2(+2)} \supset SL(2, \mathbb{R}) \times SL(2, \mathbb{R})
\]

\[
14 \supset (2, 4) + (1, 3) + (3, 1) \tag{14}
\]

In the \( 14 \) representation, \( \mathcal{V}^{(4)} \), corresponding to the latter \( SL(2, \mathbb{R}) \), therefore has block structure

\[
\mathcal{V}^{(4)} = \begin{bmatrix}
4 & 4 & 3 & 1 & 1 & 1
\end{bmatrix},
\]

containing the familiar 4 block of \( d = 4 \).

\(^3\)Note that no representation of \( G_{2(+2)} \) needs to be specified for this. The calculation uses the algebra, the representation influences only the prefactor of the scalar term in the Lagrangian which could be chosen according to which representation is used. But due to the decomposition (14), we have used the adjoint of \( G_{2(+2)} \).
2.5 Identifying $d = 4$ U-Duality in the $d = 3$ Theory

Defining an element

$$\Lambda \in SL(2, \mathbb{R}), \quad \Lambda = e^X, \quad X = a E^2 + b E^2_+ + c (h_2 - h_1), \quad a, b, c \in \mathbb{R}$$

we may look at

$$\mathcal{Y}^{(3)} \rightarrow \mathcal{Y}^{(3)} \Lambda^{-1} = \mathcal{Y}^{(4)} \Lambda^{-1} \exp \left( \frac{1}{2} \phi (h_1 + h_2) \right) \Lambda \exp (\mathcal{Y}_i S_i) \Lambda^{-1} \exp \left( f E^1_3 \right)$$

$$= \mathcal{Y}^{(4)} \Lambda^{-1} \exp \left( \frac{1}{2} \phi (h_1 + h_2) \right) \exp \left( \mathcal{Y}_i [\exp D(X)] S_i \right) \exp \left( f E^1_3 \right)$$

$$= \mathcal{Y}^{(4)} \Lambda^{-1} \exp \left( \frac{1}{2} \phi (h_1 + h_2) \right) \exp (\mathcal{Y} [\Lambda] S_i) \exp \left( f E^1_3 \right)$$

(15)

therefore

$$\mathcal{Y}^{(4)} \rightarrow \mathcal{Y}^{(4)} \Lambda^{-1} \text{ and } \mathcal{Y} \rightarrow D(\Lambda) \mathcal{Y}$$

where $D(.)$ is the spin 3/2 representation introduced in (6). Since the vector $\mathcal{Y}$ carries the $d = 4$ charges, this is exactly the transformation behavior (8) and resembles the four dimensional U-duality in the reduced model. We will illustrate this in the $d = 11$ case by studying an “elementary” soliton multiplet.

In the reduced model, the discrete group is now generated by

$$S^2 = \exp(-E^*_2) \exp(E^2) \exp(-E^*_2), \quad T^2 = \exp(E^2).$$

2.6 Connection to $d = 5$ Fields

We will now take a step back and look at the reduction we performed so far. The choice we took for the $d = 5$ vielbein was

$$E^{(5)A}_M = \begin{bmatrix}
    e^{-1} E^{(3)\alpha}_\mu & B^i_i & e_i^a \\
    0 & e^a_i \\
    e^a_i & e^a_i & e^{-\phi/2} \rho^{-1/2} E^{(3)\alpha}_\mu & e^{-\phi/2} \rho^{-1/2} \hat{B}_\mu & \rho B_\mu
\end{bmatrix}.$$  

(16)

In order to compare the different compactification in figure 2 and join the $d = 4$ U-duality groups together to a $d = 3$ U-duality group, we reexpress the coset matrix (13).

We define
\[ \varphi = \eta^{\dagger} + \frac{1}{\sqrt{3}}A_3A_4 - \frac{1}{\sqrt{3}}B_3A_4^2, \]
\[ \Psi_1 = f + \frac{1}{2}B_3\Psi_2 - \frac{1}{4}B_3A_4\varphi + \frac{1}{6\sqrt{3}}A_3A_4(A_3 - B_3A_4), \]
\[ \Psi_2 = \bar{\eta}_B - \frac{1}{2}A_4\varphi - \frac{1}{3\sqrt{3}}A_3^2(A_3 - B_3A_4). \]

Note that the new field are polynomial in the old fields. One may now use (16) and (42) to get

\[ V(3) = \exp \left( -\ln(e_1^i h_1) - \ln(e_1^j e_2^j) h_2 \right) \exp \left( -e_1^i e_2^j E_1^j \right) \]
\[ \exp \left( \Psi_i E_i^j \right) \]
\[ \exp \left( \frac{1}{\sqrt{3}}(-A_{2+j} E_i^j + \varphi E_3^j) \right) \]
\[ = \exp \left( \frac{1}{2}((\phi + \ln \rho) h_1 + (\phi - \ln \rho) h_2) \right) \exp \left( -B_3 E_1^j \right) \exp \left( \Psi_1 E_3^l + \Psi_2 E_3^j \right) \]
\[ \exp \left( \frac{1}{\sqrt{3}}(-A_3 E_1^i - A_4 E^2 + \varphi E_3^j) \right) \]

where dotted indices are flat internal indices. Explicitly, \( \varphi \) and \( \Psi_i \) obey

\[ \partial_{\mu}\varphi = -e^2 E^{(3)}\epsilon_{\mu\nu\rho}(\partial^{[\nu} A^\rho] + B^{[i} \partial^{j]} A_{(2+i)}) + \frac{1}{\sqrt{3}}\epsilon^{ij} A_{2+i} \partial_\mu A_{2+j}, \]
\[ \partial_{\mu}\Psi_i = -\frac{1}{2}e^2 E^{(3)}\epsilon_{\mu\nu\rho} B^{\nu\rho} - \frac{1}{2}(\varphi \partial_\mu A_{2+i} - \partial_\mu \varphi A_{2+i}) - \frac{1}{3\sqrt{3}}\epsilon^{jk} A_{2+i} A_{2+j} \partial_\mu A_{2+k}. \]

This exactly reproduces the result of [26] obtained by direct reduction to \( d = 3 \).

### 2.7 Different Orders of Compactification

We will now exploit this connection to \( d = 5 \) fields. We chose to reduce in the order of the left side of figure 2 and arrived at a theory with scalar coset matrix (17) called \( \mathcal{V}^{(3)}_{#1} \) in the following. The \( d = 4 \) U-duality is generated by

\[ S^2 = \exp(-E_2^*) \exp(E^2) \exp(-E_2^*), \quad T^2 = \exp(E^2). \]
We will now turn to the right side of figure 2. We consider first the $d = 5$ vielbein. We need a convenient parameterization for the second reduction. For this, we perform a local Lorentz transformation

$$\Lambda_{B}^{A} = \left[ \begin{array}{ccc} \delta_{\beta}^{\alpha} & 0 & \cos \theta \\ 0 & \sin \theta & -\sin \theta \\ 0 & \cos \theta & \cos \theta \end{array} \right]$$

such that

$$\left[ \begin{array}{c} \rho' \\ \rho' B_3' \\ \rho' e^{-\phi/2} \end{array} \right] = \left[ \begin{array}{cc} \rho e^{-\phi/2} \\ 0 \\ \rho \end{array} \right] \left[ \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right]$$

which yields $\tan \theta = \rho^{3/2} e^{\phi/2} B_3$.

Using this parameterization, we can perform the reduction strictly parallel to the one before, only a sign change in the Chern-Simons term has to be taken into account. The result is

$$V_{#2}^{(3)} = \exp \left( \frac{1}{2} \left( (\phi' + \ln \rho') h_1 + (\phi' - \ln \rho') h_2 \right) \right) \exp \left( -B_3' E_2 \right) \exp \left( \Psi_2 E_1 + \Psi_1 E_3 \right) \exp \left( \frac{1}{\sqrt{3}} \left( A_4 E_1 + A_3 E_2 - \varphi E_3^{*} \right) \right).$$

The $d = 4$ discrete U-duality is again generated by

$$S^2 = \exp(-E_2^{*}) \exp(E_2) \exp(-E_2^{*}), \quad T^2 = \exp(E_2).$$

2.8 Joining $d = 4$ U-dualities in $d = 3$

In order to join the two compactifications and U-dualities, we recognize that the two scalar matrices are related by

$$V_{#2}^{(3)} = (PS_{2})^{-1} \exp \left( \theta(E_1^{1} - E_2^{1}) \right) V_{#1}^{(3)} PS_{2}^{1}$$

(18)

$\theta$ is given by $\tan \theta = \rho^{3/2} e^{\phi/2} B_3$ as above, the factor $\exp(\theta(E_2^{1} - E_1^{1})) \in SO(4)$ is nothing but the local Lorentz transformation we performed. $S_{2}^{1}$ is given by

$$S_{2}^{1} = \exp(-E_1^{1}) \exp(E_2^{1}) \exp(-E_2^{1})$$

and corresponds to the T-duality transformation in figure 2. It represents the needed Weyl reflection in the root space of $g_{2(+2)}$. 

15
Somewhat unexpected is the appearance of $P$ in $\mathfrak{R}$. $P$ is a “parity” transformation given by

$$P = (-1)^{(h_1 + h_2)} = (S^1_3)^2 = (S^2)^2$$

and is an element of $d = 4$ U-duality corresponding to a charge conjugation of the $d = 4$ charges.

Turning to the U-duality transformations, consider a transformation $U$ on $\mathcal{V}^{(3)}_{#1}$. We have

$$\mathcal{V}^{(3)}_{#2} = (PS^1_2)^{-1} \exp \left( \theta(E^1_2 - E^2_1) \right) \mathcal{V}^{(3)}_{#1} PS^1_2$$

$$= (PS^1_2)^{-1} \exp \left( \theta(E^1_2 - E^2_1) \right) \mathcal{V}^{(3)}_{#1} PS^1_2 (PS^1_2)^{-1} U (PS^1_2)$$

$$= \mathcal{V}^{(3)}_{#2} (PS^1_2)^{-1} U (PS^1_2)$$

$$= \mathcal{V}^{(3)}_{#2} \tilde{U}$$

$\tilde{U}$ is a matrix generated by

$$S^1 = \exp(-E^*_1) \exp(E^1) \exp(-E^*_1), \quad T^1 = \exp(E^1).$$

The joint U-duality group $U(\mathbb{Z})$ in three dimensions is therefore generated by

$$S^1, S^2, T^1, T^2.$$
the 4 of \( SL(2, \mathbb{R}) \) do not belong to this class. But, as stated in [46], the notion of the group \( G_{2(+2)}(\mathbb{Z}) \) is independent of its representation.

The disagreement of \( U(\mathbb{Z}) \) with \( G_{2(+2)}(\mathbb{Z}) \) might be surprising at first. We will now turn to the \( d = 11 \) case to see if the same is true in this model.

3 The \( d = 11 \) Case

M-theory is supposed to have eleven dimensional supergravity as low energy limit. The bosonic part of the Lagrangian is given by

\[
\mathcal{L} = \frac{1}{4} E^{(11)}(R^{(11)} + \frac{1}{12} F_{MNPQ} F^{MNPQ}) + \frac{2}{12} \epsilon^{MNPQRSTU VWX} F_{MNPQ} F_{RSTU} A_{VWX},
\]

where \( F_{MNPQ} = 4 \partial_{[M A_{NPQ}]}. \) Indices \( M, N, \ldots \) now run from 0...10, the metric has signature \((+, -, \ldots, -)\).

Quite analogously to the last section, if the theory is compactified to three dimensions, the vielbein can taken to be of the form

\[
E^{(11)A}_{M} = \begin{bmatrix}
E^{-1}_{\mu} & \begin{array}{c}
B^{(3)i} e_{i}^{a} \\
0
\end{array} \\
0 & e_{i}^{a}
\end{bmatrix}.
\]

Again, we define the internal vielbein \( e_{i}^{a} \) with indices \( i, a \) running in 1...8. \( e_{i}^{a} \) is assumed to be of triangular shape in the following.

In order to identify the \( d = 4 \) U-duality in \( d = 3 \), we will proceed exactly as in the toy model in a stepwise fashion.

We will first study M-theory compactified to \( d = 4 \) and U-duality in \( d = 4 \).

3.1 The \( d = 4 \) Theory

The reduction to \( d = 4 \) was carried out in detail e.g. in [21] (see also [19] for a more recent treatment). The reduced Lagrangian reads

\[
\mathcal{L} = \frac{1}{4} E^{(4)} R^{(4)} + \frac{1}{32} E^{(4)} \partial_{\mu} \ln \Delta \partial^{\mu} \ln \Delta - \frac{1}{16} E^{(4)} \partial_{\mu} g_{\bar{m}\bar{n}} \partial^{\mu} g^{\bar{m}\bar{n}} \\
- \frac{1}{12} E^{(4)} \partial_{\mu} A^{(i+2)(j+2)(k+2)} \partial^{\mu} A^{(i+2)(j+2)(k+2)} \\
+ \frac{1}{16} E^{(4)} \sqrt{\Delta} D_{\mu}^{(4)} D_{i}^{(4)} \bar{D}_{\mu}^{(4)} \bar{D}_{i}^{(4)} - \frac{1}{12} E^{(4)} \Delta F_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\tau}}^{(4)} F^{(4)} \bar{D}_{\mu}^{(4)} \bar{D}_{i}^{(4)} - \frac{1}{8} E^{(4)} \sqrt{\Delta} F_{\mu\nu\bar{\rho}\bar{\tau}}^{(4)} F^{(4)} \bar{D}_{i}^{(4)} \bar{D}_{j}^{(4)} 
\]
\[- \frac{2}{123} \epsilon_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \epsilon^{\bar{\jmath} \bar{k} \bar{m} \bar{n}} \left( 4 F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} \partial_{\bar{\rho}} A_{(j+2)(k+2)(l+2)} A_{(m+2)(n+2)(\bar{o}+2)} \right.
\left. - 9 F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} F_{\bar{\rho} \bar{\sigma} \bar{k} \bar{l}}^{(4)} A_{(m+2)(n+2)(\bar{o}+2)} \right.
\left. + 9 F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} B_{\bar{\rho} \bar{\sigma}}^{(4)p} A_{(p+2)(k+2)(l+2)} A_{(m+2)(n+2)(\bar{o}+2)} \right.
\left. - 3 B_{\bar{\mu}\bar{\nu}}^{(4)p} B_{\bar{\rho} \bar{\sigma}}^{(4)\bar{q}} A_{(p+2)(\bar{j}+2)(\bar{j}+2)} A_{(\bar{q}+2)(k+2)(l+2)} A_{(m+2)(n+2)(\bar{o}+2)} \right) \right]

(20)

where the elfbein is parameterized as

\[
E_{(11)\bar{M}}^{(1)} = \begin{bmatrix} \Delta^{-\frac{1}{4}} E^{(4)\bar{\alpha}}_{\bar{\mu}} & B_{\bar{\mu}}^{(4)\bar{\nu}} \rho_{\bar{\nu}}^{\bar{a}} \\ 0 & \rho_{\bar{\nu}}^{\bar{a}} \end{bmatrix}
\]

(21)

with the internal triangular vielbein \( \rho_{\bar{\nu}}^{\bar{a}} \). \( \bar{\nu}, \bar{a} \) are curved and flat internal indices respectively. They have been chosen to run in 2...8 for later convenience. The internal metric \( g_{\bar{m}\bar{n}} \) is defined as usual to be \( g_{\bar{m}\bar{n}} = \rho_{\bar{m}} \rho_{\bar{n}}^{\bar{a}} \) and has signature \(-\ldots-\). Its determinant is \( \sqrt{\Delta} = \det \rho_{\bar{m}}^{\bar{a}} \).

\( \bar{\mu}, \bar{\nu}, \ldots \) are curved and \( \bar{\alpha}, \bar{\beta}, \ldots \) flat four dimensional indices. They run from 0...3. For the fields \( F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} \) and \( F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} \), the following definitions were used in order to ensure the suitable transformation properties with respect to internal diffeomorphisms:

\[
F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} = F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} + B_{\bar{\mu}\bar{\nu}}^{(4)\bar{k}} A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}
\]

\[
F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} = 2 \partial_{\bar{\mu}} A_{\bar{i}}^{(4)\bar{a}} A_{(\bar{i}+2)(\bar{a}+2)}
\]

\[
A_{\bar{\mu}}^{(4)\bar{a}}(\bar{i}+2)(\bar{j}+2) = A_{\bar{\mu}}(\bar{i}+2)(\bar{j}+2) - B_{\bar{\mu}}^{(4)\bar{a}} A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}
\]

\[
B_{\bar{\mu}\bar{\nu}}^{(4)\bar{a}} = 2 \partial_{\bar{\mu}} B_{\bar{\nu}}^{(4)\bar{a}}
\]

\[
F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} = F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} + 3 B_{\bar{\mu}\bar{\nu}}^{(4)\bar{a}} A_{\bar{a}}(\bar{i}+2)(\bar{k}+2)
\]

\[
F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} = 3 \partial_{\bar{\nu}} A_{\bar{i}}^{(4)\bar{a}}(\bar{i}+2)
\]

\[
A_{\bar{\mu}}^{(4)\bar{a}}(\bar{i}+2) = A_{\bar{\mu}}(\bar{i}+2) - 2 B_{\bar{\mu}}^{(4)\bar{a}} A_{\bar{a}}(\bar{i}+2)(\bar{j}+2) - B_{\bar{\mu}}^{(4)\bar{j}} B_{\bar{\nu}}^{(4)\bar{k}} A_{(\bar{a}+2)(\bar{k}+2)(\bar{i}+2)}
\]

The fields \( F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} \) and \( F_{\bar{\mu}\bar{\nu} \bar{ij}}^{(4)} \) are then dualized by adding

\[
\mathcal{L}_{\text{Lag.mult.}} = \frac{1}{12} \epsilon_{\bar{\nu}\bar{\rho}\bar{a}}^{(4)} \partial_{\bar{\mu}} F_{\bar{\rho} \bar{\nu} \bar{a}}^{(4)} + \frac{1}{4} \epsilon_{\bar{\nu}\bar{a}}^{(4)} \partial_{\bar{\mu}} F_{\bar{\nu} \bar{a}}^{(4)}
\]

In order to simplify the Lagrangian and make the hidden \( E_{7(+7)} \) symmetry manifest, the new field strength \( \tilde{A}_{\bar{\mu}\bar{\nu}}^{(4)} = 2 \partial_{\bar{\mu}} A_{\bar{\nu}}^{(4)} \) is introduced, as well as the vector notation

\[
\mathcal{G}_{\bar{\mu}\bar{\nu}} = \begin{bmatrix} \tilde{A}^{(4)\bar{\mu}}_{\bar{\nu}} \\ A_{\bar{\mu}\bar{\nu}} \end{bmatrix}, \quad \mathcal{H}_{\bar{\mu}\bar{\nu}} = \begin{bmatrix} H^{(4)\bar{\mu}}_{\bar{\nu}} \\ H_{\bar{\mu}\bar{\nu}} \end{bmatrix}
\]
where $A_{\bar{i}\bar{j}} = -\frac{1}{2} B_{\mu\nu}$. The dual fields $\mathcal{H}_{\mu\nu}$ obey

$$H_{\bar{\mu}\bar{\nu} \bar{i} \bar{j}} = -\frac{4}{E^{(4)}} \left( \frac{\delta \mathcal{L}}{\delta A_{\mu\nu \bar{i} \bar{j}}} \right), \quad H_{\bar{i} \bar{j} \bar{k} \bar{l}} = -\frac{4}{E^{(4)}} \left( \frac{\delta \mathcal{L}}{\delta A_{\mu\nu \bar{i} \bar{j}}} \right).$$

The 70 scalars of the theory, $A_{(i+2)(j+2)(k+2)}$, $\rho_{\bar{i}}$ and $\varphi^{(4) i}$, are joint together in a field $\mathcal{V}^{(4)} \in E_{7(+7)}/SU(8)$, a representation matrix in the fundamental 56 representation [21].

The fundamental 56 representation is given in the following way: The representation space is spanned by two antisymmetric tensors $x^{ij}, y^{ij}$, where indices $i, j$ run in $2 \ldots 9$. On a vector $(x^{ij}|y^{ij})^t$, the algebra $\mathfrak{e}_7^{(+7)}$ acts by the real matrices

$$\Lambda = \begin{pmatrix} 2\Lambda_{[i}^{\emptyset} \delta_{j]}^j \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{ijkl} \\ \Sigma^{ijkl} \end{pmatrix},$$

where $\Lambda_{[i}^{\emptyset} \delta_{j]}^j = -\Lambda_{[i}^{\emptyset} \delta_{j]}^j$, $\Lambda_{[i}^{\emptyset} = 0$ obviously represents a $\mathfrak{sl}(8)$ subalgebra, and $\Sigma_{ijkl}$ is totally antisymmetric. We have $\Sigma^{ijkl} = \frac{1}{24} \epsilon_{ijklmnopq} \Sigma_{mnopq}$.

The representation (22) is symplectic, it preserves the symplectic form

$$\Omega = \begin{pmatrix} 1_{ijkl} & -1_{ijkl} \end{pmatrix}.$$

We will now recover this representation of $\mathfrak{e}_7^{(+7)}$ embedded into the realization of $\mathfrak{e}_{8(+8)}$ given in appendix A, using Freudenthal’s realization of exceptional Lie algebras [30].

![Diagram](image)

Figure 4: Decomposition of $E_{8(+8)} \supset SL(2) \times E_{7(+7)}$ and $E_{7(+7)} \supset SL(8)$
From the decomposition illustrated in figure 4 we learn that the \( e_{7(+7)} \) subalgebra of \( e_{8(+8)} \) may be generated by

\[
h_{1\bar{9}}, \ E^{1\bar{9}}, \ E^{i}_{1\bar{9}}, \ E^{ij}_{i\bar{9}}, \ E^{*}_{i\bar{9}},
\]

(23)

Let us define

\[
\Lambda e_{8(+8)} = \sum_{\bar{i} = 2}^{8} \left( \Lambda_{\bar{i}} h_{1\bar{9}} + \Lambda_{\bar{i}}^{9} E^{1\bar{9}} + \Lambda_{\bar{i}}^{9} E^{*}_{1\bar{9}} \right) + \sum_{\bar{i}, \bar{j} = 2}^{8} \Lambda_{\bar{i}}^{\bar{j}} E^{\bar{i}\bar{j}}.
\]

(24)

For the representation space basis, we define the vectors

\[
S^{t} = (-E^{*}_{ij\bar{9}}, +E_{1\bar{9}}, -E^{1\bar{9}}, -E^{i\bar{9}}),
\]

\[
X^{t} = \left( x^{i\bar{j}}_{\bar{9}} | x^{i\bar{9}}_{\bar{9}} | y^{i\bar{j}}_{\bar{9}} | y^{i\bar{9}}_{\bar{9}} \right).
\]

(25)

Using the relations among the generators (50), one may verify that

\[
[\Lambda e_{8(+8)}, \ X \cdot S] = \mathcal{X}' \cdot S, \quad [\Sigma e_{8(+8)}, \ X \cdot S] = \mathcal{X}'' \cdot S
\]

(26)

with

\[
\mathcal{X}' = \Lambda \cdot \mathcal{X}, \quad \mathcal{X}'' = \Sigma \cdot \mathcal{X}
\]

reproducing exactly the action of (22). We can therefore use the adjoint action on \( S \) to define the \( 56 \) representation \( \rho_{56} \) of \( e_{7(+7)} \) as subalgebra of \( e_{8(+8)} \).

Later we use \( i, j, \ldots = 1, 2, \ldots, 8 \) as the indices for the eight dimensional torus in the dimensional reduction to \( d = 3 \). Note that the \( SL(8, \mathbb{R}) \) subgroup generated by \( \Lambda \)'s is not the modular group of this torus.

The scalar field \( V^{(4)} \) is given explicitly by

\[
V^{(4)} = \exp \left( -\sum_{\bar{m} = 2}^{8} \ln \left( \rho_{\bar{m}}^{\bar{m}} \right) h_{1\bar{m}9} + \frac{1}{8} \ln \Delta h_{1\bar{m}9} \right) \prod_{\bar{p} = 0}^{5} \exp \left( -\sum_{q, \bar{r} = 8-\bar{p}}^{8} \rho_{7-\bar{p}}^{\bar{q}} \rho_{7-\bar{p}}^{\bar{r}} \right) E_{\bar{7}\bar{p}}^{\bar{r}} \exp \left( -\sum_{i = 2}^{8} \varphi^{(4)}_{i} E^{*}_{i\bar{9}} \right)
\]

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\[ \exp \left( \frac{2}{3!} \sum_{i,j,k=2}^{8} A_{(2+i)(2+j)(2+k)} \bar{E}_{ijk}^{\hat{i}\hat{j}\hat{k}} \right) \] (27)

in the above \( \rho_{56} \) representation, where \( \rho_i^{(n)\bar{a}} \) is the submatrix of \( \rho_i \bar{a} \) with columns and rows \((n+1)\ldots8\). This result can be seen to correspond to the one in \([19]\).

From \( \mathcal{V}^{(4)} \), again a field \( P_{\mu}^{(4)} \) may be defined by

\[ \partial_\mu \mathcal{V}^{(4)} \mathcal{V}^{(4)-1} = Q_\mu^{(4)} + P_\mu^{(4)}, \quad Q_\mu^{(4)} \in \mathfrak{su}(8), \quad P_\mu^{(4)} \in \mathfrak{e}_7(+7) - \mathfrak{su}(8). \]

The vector fields are related by the twisted self-duality relation

\[ \mathcal{F}_{\hat{\mu}\hat{\nu}} \equiv \begin{bmatrix} \mathcal{G}_{\hat{\mu}\hat{\nu}} \\ \mathcal{H}_{\hat{\mu}\hat{\nu}} \end{bmatrix} = \mathcal{V}^{(4)T} \mathcal{V}^{(4)} \begin{bmatrix} *\mathcal{G}_{\hat{\mu}\hat{\nu}} \\ *\mathcal{H}_{\hat{\mu}\hat{\nu}} \end{bmatrix}. \]

The Lagrangian finally takes the form

\[ \mathcal{L} = -\frac{1}{4} E^{(4)} R^{(4)} + E^{(4)} \left( \frac{1}{8} \mathcal{G}_{\hat{\mu}\hat{\nu}}^T \star \mathcal{H}^{\hat{\mu}\hat{\nu}} + \frac{1}{48} \mathrm{Tr}(P_\mu^{(4)} P^{(4)\mu}) \right). \] (28)

The scalar part of the action is invariant under \( \mathcal{V}^{(4)} \rightarrow h(x) \mathcal{V}^{(4)} \Lambda, \Lambda \in E_{7(+7)}, h(x) \in SU(8) \), where the local \( SU(8) \) is used to restore the parameterization of coset space.

The combined vector field \( \mathcal{F} \) transforms as a vector with respect to \( E_{7(+7)} \), but not the entire \( E_{7(+7)} \) is a symmetry of the action. Writing the vector field part of the action as

\[ \frac{1}{16} E^{(4)} \left( \mathcal{F}_{\mu\nu}^T L \star \mathcal{F}^{\mu\nu} \right) \]

with

\[ L = \begin{bmatrix} 1 \\ \hat{1}_{ijkl} \end{bmatrix} \]

we see that a symmetry of the Lagrangian has to preserve \( L \). It follows that only \( \Lambda_i \hat{k} \) in \( \mathcal{F}_{\mu\nu} \) can be nonzero, the action is invariant under the \( SL(8) \) subgroup in figure \( 7 \). We will treat the symmetries and subgroups more closely in the following section.

### 3.2 Duality in \( d = 4 \)

The equations of motion of the theory show the full \( E_{7(+7)} \) invariance. The classical duality symmetry is given by

\[ \mathcal{F}_{\mu\nu} \rightarrow \Lambda^{-1} \mathcal{F}_{\mu\nu}, \quad \mathcal{V}^{(4)} \rightarrow \mathcal{V}^{(4)} \Lambda, \quad \Lambda \in E_{7(+7)}. \]
Again, the transformation of $V^{(4)}$ is accompanied by a compensating local $SU(8)$ transformation $V^{(4)} \rightarrow h(x)V^{(4)}\Lambda$, $h(x) \in SU(8)$ to restore the parameterization of the coset space.

We again define charges

$$Z = \begin{pmatrix} p \\ q \end{pmatrix}, \quad p = \frac{1}{2\pi} \oint_{\Sigma} G, \quad q = \oint_{\Sigma} H.$$

It has been argued in [3] that all magnetic and electric charges exist. It is actually clear from the basis (25) and the fact that the 56 representation of $E_{7(+7)}$ is minimal that, if a solution with one nonzero charge exists, solutions with a single charge carried by all other gauge fields may be obtained by Weyl reflections that are group elements of $E_{7(+7)}$, to be introduced below.

The DSZ condition

$$Z^t \Omega Z' = n, \quad n \in \mathbb{Z}$$

breaks $E_{7(+7)}$ to $E_{7(+7)}(\mathbb{Z})$, demanding integer shifts on the lattice defined by the basis (25). This group has been proposed to be a unified duality symmetry of type II string theory in [3], called U-duality for short, unifying strong-weak coupling dualities and target space dualities and putting all 70 moduli of the theory, including the string coupling constant, on the same footing. We will analyse the subgroups corresponding to T- and S-duality in our notation after introducing generators for the discrete group.

What are generators of this $E_{7(+7)}(\mathbb{Z})$? It may be checked that the basis $S$ in (23) forms an admissible lattice (see appendix B) of $e_{7(+7)}$ in the representation $\rho_{56}$ defined by (24), (25), (26). The 56 representation of $e_{7(+7)}$ is furthermore the unique minimal representation [47]. From the proof in appendix B it follows that the subgroup inducing integer shifts on the lattice with base $S$, called $E_{7(+7)}(\mathbb{Z})$ in the following, is generated by “fundamental unipotents”, that is,

$$T_{\bar{i} \bar{j}} = \exp(E_{\bar{i} \bar{j}}), \quad \bar{i} < \bar{j}; \quad T_{\bar{i} \bar{j}} = \exp(E_{\bar{i} \bar{j}}), \quad \bar{i} > \bar{j},$$

$$T_{1 \bar{i} \bar{j}} = \exp(E_{1 \bar{i} \bar{j}}); \quad T_{1 \bar{i} \bar{j}} = \exp(E_{1 \bar{i} \bar{j}}),$$

$$T_{\bar{i} \bar{j} \bar{k}} = \exp(E_{\bar{i} \bar{j} \bar{k}}); \quad T_{\bar{i} \bar{j} \bar{k}} = \exp(E_{\bar{i} \bar{j} \bar{k}})$$

or alternatively

\begin{align*}
T_{\bar{i} \bar{j}} &= \exp(E_{\bar{i} \bar{j}}), \quad \bar{i} < \bar{j}; \\
T_{\bar{i} \bar{j}} &= \exp(E_{\bar{i} \bar{j}}), \quad \bar{i} > \bar{j}, \\
T_{1 \bar{i} \bar{j}} &= \exp(E_{1 \bar{i} \bar{j}}); \\
T_{1 \bar{i} \bar{j}} &= \exp(E_{1 \bar{i} \bar{j}}), \\
T_{\bar{i} \bar{j} \bar{k}} &= \exp(E_{\bar{i} \bar{j} \bar{k}}); \\
T_{\bar{i} \bar{j} \bar{k}} &= \exp(E_{\bar{i} \bar{j} \bar{k}})
\end{align*}
\[ T_{\tilde{i}\tilde{j}} = \exp(E_{\tilde{i}\tilde{j}}), \quad S_{\tilde{i}\tilde{j}} = \exp(-E_{\tilde{j}\tilde{i}}) \exp(E_{\tilde{i}\tilde{j}}) \exp(-E_{\tilde{j}\tilde{i}}) \quad \tilde{i} < \tilde{j}, \]
\[ T_{\tilde{i}\tilde{1}\tilde{9}} = \exp(E_{\tilde{i}\tilde{1}\tilde{9}}), \quad S_{\tilde{i}\tilde{1}\tilde{9}} = \exp(-E_{\tilde{1}\tilde{9}\tilde{i}}) \exp(E_{\tilde{i}\tilde{1}\tilde{9}}) \exp(-E_{\tilde{1}\tilde{9}\tilde{i}}), \]
\[ T_{\tilde{i}\tilde{j}\tilde{k}} = \exp(E_{\tilde{i}\tilde{j}\tilde{k}}), \quad S_{\tilde{i}\tilde{j}\tilde{k}} = \exp(-E_{\tilde{k}\tilde{i}\tilde{j}}) \exp(E_{\tilde{i}\tilde{j}\tilde{k}}) \exp(-E_{\tilde{k}\tilde{i}\tilde{j}}) \]
\[ \text{in the representation } \rho_{56}, \text{ where the } S \text{ generators are known to carry a representation of the Weyl group modulo } \mathbb{Z}_2 \text{ (see e.g. [33]).} \]

This yields a complete set of discrete generators of the $E_{7(+7)}(\mathbb{Z})$ U-duality. U-duality groups in higher dimensions are found by their embeddings. Since the notion of the above generators is representation independent, the $E_{6(+6)}(\mathbb{Z})$ etc. U-duality generators follow directly from truncating the Dynkin diagram. Furthermore, the corresponding representations in these truncations are always minimal, and the corresponding discrete group generator representations can be read off explicitly from the basis (23). Actually, in this sense, all U-dualities follow from the adjoint representation of $E_{8(+8)}$.

The equations (24), (25), (26) together with (27) and the definition of $F$ enable us to give U-duality transformations explicitly in $d = 4$, but give a direct contact to the algebraic notations of [16] as well. We will illustrate this below.

It is instructive to identify T- and S-duality symmetries in the theory along the lines of [34, 8].

The known superstring theories in ten dimensions have a common low-energy sector whose spectrum is the same as the NS-NS sector of type II theories. The corresponding low-energy fields will be called NS-NS fields.

In $d = 4$, the fundamental string can carry electric charge with respect to the $U(1)$ fields in the NS-NS sector [35]. T-duality is identified with the subgroup of $E_{7(+7)}$ that stabilizes the NS-NS charge lattice.

The NS-NS sector in $d = 10$ consists of the metric, the dilaton and an antisymmetric two-form. Looking at (19) and considering the direction 10 as eleventh spatial dimension, the NS-NS fields may be identified with

\[ E^{(4)\tilde{a}}_{\mu}, \rho^8_{\tilde{a}}, \rho_{\tilde{b}}, B^{(4)i}_{\mu}, A_{\mu\tilde{8}}, A_{\tilde{a}\tilde{8}}, A_{i\tilde{j}\tilde{8}} \]

where indices with tilde run from 2 \ldots 7.

The $d = 4$ $U(1)$ field strengths in the NS-NS sector are
\[ B^{(4)}_{\mu \nu} \tilde{i} = -2 \tilde{A}^{i9}_{\mu \nu}, \quad F'_{\mu \nu} \tilde{i}8 = -H_{\mu \nu} \tilde{i}8, \quad \tilde{i} \in \{2, \ldots, 7\}. \]

where we assume that the RR fields are set to zero.

With (25) this corresponds to the representation space basis

\[ (E^{118}, \quad -E^{1}_{i}). \quad (31) \]

Using (50) one may verify that the subgroup stabilizing this basis is the obvious \( O(6,6) \) subgroup generated as indicated in figure 5.

Figure 5: Decomposition of \( E_{7(+7)} \supset O(6,6) \times SL(2) \)

With the above representation basis one may verify that it acts in the fundamental representation and preserves the metric

\[ \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \quad (32) \]

where \( 1 \) is the six dimensional unit matrix. The dual fields

\[ -\frac{1}{2} H_{\mu \nu} \tilde{i}9, \quad -\tilde{A}^{i8}_{\mu \nu}, \quad \tilde{i} \in \{2, \ldots, 7\}. \]

correspond to the basis

\[ (E^{*}_{i89}, \quad E^{i}_{9}) \]

and transform in the transpose inverse representation of the above fundamental representation of \( O(6,6) \) as expected. We explicitly have the decomposition

\[ E_{7(+7)} \supset O(6,6) \times SL(2, \mathbb{R}) \]

\[ 56 \supset (12, 2) + (32, 1) \]
mentioned in [3]. The $U(1)$ field strength in the R-R sector transform in the 32 spinor representation of $O(6,6)$, which means that T-duality mixes electric and magnetic fields in the R-R sector.

The action of $O(6,6)$ on the Cartan subalgebra part of (27) can be seen by looking at the submatrix of (27) corresponding to the basis (31). This yields a matrix with the $\rho_m \Delta^{-\frac{1}{8}}$ on the diagonal. The $\rho_m$ correspond to the radii of the compactification torus transforming under $O(6,6)$, while $\Delta^{1/2}$ is the volume of the 7-torus including the eleventh direction. The appearance of the factor $\Delta^{-1/8}$ indicates that $\rho_8$, corresponding to the radius of the eleventh compact direction, transforms as well under T-duality and is not decoupled, as was pointed out e.g. in [17]. This transformation therefore mixes strong weak coupling duality of the type IIB string [6, 7], that corresponds to modular transformations involving $\rho_8$, with a transformation of the moduli corresponding to the compactification torus of type II string theory.

Note that the $O(6,6)$ in our formulation is not a symmetry of the action, but only of the equations of motion.

We now turn to the commuting $SL(2,\mathbb{R})$ factor in figure 5, which is a symmetry of the action (28). It was suggested in [3] to interpret this factor like in the heterotic case [34] as a $d=4$ S-duality, not to be confused with the $d=10$ S-duality of the type IIB string.

Interpreting the above $SL(2,\mathbb{R})$ factor as S-duality, a $\mathbb{Z}_2$ symmetry exchanging electric and magnetic sector is expected to be present. The natural candidate is $S^{189}$. Using (50), it is interesting to note that $S^{189}$ transforms the magnetic into the electric sector and vice versa, but takes the Kaluza-Klein sector into the 3-form field sector (and vice versa) as well. In the basis we have chosen, $S^{189}$ has to be accompanied by an $O(6,6)$ transformation in order to transform the magnetic to the electric field strength of a specific vector field. This corresponds to the above preserved metric (32), which is equal to

$$\prod_{i<j} S_{ij} \prod_{i<j} S_{8i8j}.$$  

A suitable choice of basis can prevent us from needing this additional $O(6,6)$ transformation.

We will now compare our notion of $E_{7(+7)}(\mathbb{Z})$ in (29), (30) with the definition (2) used in [17].

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This is analogous to Sen’s manifestly $SL(2,\mathbb{R})$ invariant action from the dual $N = 1$ $d = 10$ supergravity theory. The connection is obvious when all R-R fields are set to zero.
In fact, since $E_7$ is simply laced, the generators in (29) corresponding to simple roots generate all other generators and therefore the whole $E_{7(+7)}(\mathbb{Z})$. On the other hand, the same is true for the simply laced groups $O(6,6,\mathbb{Z})$ and $SL(7,\mathbb{Z})$ as subgroups of $E_{7(+7)}(\mathbb{Z})$. Both groups correspond to subdiagrams of the Dynkin diagram of $E_7$ and act in minimal representations. The subgroup $O(6,6)$ is indicated in figure [5], the subgroup $SL(7)$ simply corresponds to erasing the root $e_6$ for the $E_7$ Dynkin diagram. Joining their generators together, we get the whole set of generators of $E_{7(+7)}(\mathbb{Z})$. The two definitions are therefore equivalent.

In the algebraic approach reviewed in [17], Weyl and Borel generators were used to define the discrete group. These actually correspond to the $S$ and $T$ generators in (30). For the Weyl group, the identification
\[
S^{ij} = \hat{S}_{ij}, \quad S^{ijk} = \hat{T}_{ijk}
\]
holds, where hatted indices are the Weyl generators of [17]. The $S^i_j$ correspond to the exchange of two radii, while the $S^{ijk}$ correspond to a simultaneous inversion of three radii. Only the $S^{ij8}$ are elements of the T-duality group $O(6,6)$, corresponding, as pointed out above, to a simultaneous inversion of two radii and the radius corresponding to the eleventh dimension connected to type IIB S-duality. It should be kept in mind that the Weyl group representation carried by $S$ is a representation only modulo $\mathbb{Z}_2$ shifts on the Chevalley generators.

The Borel generators are identified correspondingly, noting for the T-duality group that the $d=10$ antisymmetric tensor field corresponds to $A^{i\overline{j}8}$.

### 3.3 Elementary Solitons

In [3], “elementary” solitonic BPS solutions have been identified that each carry a single type of charge and therefore fill out an $E_{7(+7)}$ multiplet. This also accounts for the perturbative electrically charged string states that have their solitonic counterparts in these multiplets. Note that these solitons in the NS-NS sector are exact classical solutions to type II string theory [3], not only of the low energy sector, and therefore the whole multiplet consists of exact solutions.

Other solutions corresponding to BPS solitons that break 1/4 and 1/8 supersymmetry have been identified with multi-particle bound states of the “elementary” solitons charged with respect to different gauge fields [37]. Generically these correspond to intersecting p-brane configurations in higher dimensions [38]. The most general 1/8 BPS soliton
is known to depend on five parameters, four charges and one relative phase ([13] and references therein).

As already pointed out, the fact that the 56 representation is minimal guarantees that each gauge field can be reached by a Weyl reflection from one nonzero gauge field. We will sketch the well known $a = \sqrt{3}$ black hole solution [36] with $A_{MNP} = 0$ in (19), which corresponds to the above solitons charged under one Kaluza-Klein gauge field, as an example. For the ten dimensional origin of this multiplet see e.g. [3] and [17].

It is governed by one harmonic function $H$, $\partial_i \partial_i H = 0$, where we have chosen Cartesian coordinates. The metric is of the form

$$ds^2 = H^{-1/2}dt^2 - H^{1/2}dx_i^2$$

which directly resembles its origin as pp-wave of M-theory traveling in the compact direction that corresponds to the KK gauge field it is charged under. Choosing a specific $\bar{k} \in \{1...7\}$, the electric solution reads in our notation

$$B_{ti}^{(4)\bar{k}} = H^{-2} \partial_i H, \quad g_{\bar{k}\bar{k}} = -H,$$

all other fields zero. The harmonic form is chosen to be

$$H = 1 + \frac{Q}{|\vec{r}|}$$

for a single BPS soliton, $Q$ corresponds to the electric charge.

For the corresponding multi black hole solution, the harmonic form reads

$$H = 1 + \sum_n \frac{Q_n}{|\vec{r} - \vec{r}_n|}$$

reflecting the fact that the Bogomol’nyi bound is saturated and there are no forces between the constituents, allowing a superposition despite the nonlinearity of the Einstein equations.

The magnetic dual solution, corresponding to the wrapped M-theory KK6-brane, is given by

$$B_{ij}^{(4)k} = \epsilon_{ijk} \partial_k H, \quad g_{\bar{k}\bar{k}} = -H^{-1},$$

with the harmonic form

$$H = 1 + \frac{P}{|\vec{r}|}$$
for the single soliton with magnetic charge $P$, and

$$H = 1 + \sum_n \frac{P_n}{|\vec{r} - \vec{r}_n|}$$

for the multi magnetically charged black hole.

Solutions charged with respect to the other gauge fields of the theory may now be obtained by the action of $E_{7(+7)}$ resp. $E_{7(+7)}(\mathbb{Z})$. They correspond to wrapped membrane and fivebrane solutions of M-theory. While the $S$ generators take single charge solutions to single charge solutions, the $T$ generators will yield solitons with multiple charge.

Note that the solitons may carry electric or magnetic charges, but there are no dyonic states in this multiplet. Looking at (25), one sees that no corresponding $T$ generators exist. This is not surprising, since the dyonic solutions are known to have a different conformal structure, while U-duality does not act on the space-time metric.

We will now turn to the compactification to three dimensions. We will assume as in § that the $E_{7(+7)}(\mathbb{Z})$ symmetry is not broken.

### 3.4 The $d = 3$ Theory

The reduction to $d = 3$ is strictly parallel to the toy model. We will give the reduction here explicitly by using the embedding of $E_{7(+7)}$ in $E_{8(+8)}$ discussed above\footnote{The $E_{8(+8)}$ symmetry in $d = 3$ by reduction from $d = 4$ was addressed in \cite{40}.}

Dropping the dependence on the compact coordinate, we write the vierbein as

$$E^{(4)\alpha}_{\mu} = \begin{bmatrix} e^{\phi/2}E^{(3)\alpha}_{\mu} & e^{-\phi/2}\hat{B}_{\mu} \\ 0 & e^{-\phi/2} \end{bmatrix},$$

where $\mu, \alpha$ now run from $0\ldots2$, and define the Kaluza-Klein invariant vector field strengths $G'_{\mu\nu} = 2\partial_{[\mu}G'_{\nu]}$ with $G'_{\mu} = G_{\mu} - \hat{B}_{\mu}G_3$. The vector fields are dualized by adding the Lagrange multipliers

$$L_{\text{Lag.mult.}} = \frac{1}{4}\epsilon^{\mu\nu\rho}\tilde{\eta}\partial_{[\mu}G'_{\nu]} + \frac{1}{8}\epsilon^{\mu\nu\rho}f\partial_{[\mu}\hat{B}_{\nu]}.$$

With

$$\partial_{\mu}\eta = G_{\mu3}, \quad \partial_{\mu}\tilde{\eta} = H_{\mu3}, \quad Y = \begin{bmatrix} \eta \\ \tilde{\eta} \end{bmatrix},$$

$$\partial_{\mu}f = -\frac{1}{2}E^{(3)}\epsilon^{\mu\nu\rho}e^{-2\phi}\hat{B}_{\nu\rho} - \mathcal{Y}^\mu\mathcal{O}_{\mu}Y$$

\cite{8}.
the Lagrangian gets

\[ \mathcal{L} = -\frac{1}{4} E^{(3)} R^{(3)} + \frac{1}{8} E^{(3)} \partial_\mu \phi \partial^\mu \phi + \frac{1}{48} E^{(3)} \text{Tr}(P^{(4)}_\mu P^{(4)\mu}) + \frac{1}{8} E^{(3)} e^{2\phi} (\partial_\mu f + Y^t \Omega \partial_\mu Y)(\partial^\mu f + Y^t \Omega \partial^\mu Y) + \frac{1}{4} E^{(3)} e^\phi \partial_\mu Y^t \Omega (4)^t (4) \partial^\mu Y. \]  

(33)

3.5 Elementary solitons in \( d = 3 \)

We will now study how the \( d = 4 \) solitons appear in the compactified theory. For this, we study multi-BPS solitons where copies of a single BPS soliton of the four dimensional theory are put along the 3-direction with distance \( 2\pi R \) among two of them\(^6\).

This corresponds to the harmonic form

\( H = 1 + \sum_n \frac{Z}{\sqrt{x_1^2 + x_2^2 + (x_3 + 2\pi R n)^2}} \)

where \( Z = Q, P \) for the electric resp. magnetic case.

The above sum is logarithmically divergent, but the divergence can be regularized by adjusting an additive constant. We can e.g. add a regulator of the form\(^7\)

\( -\frac{Z}{2\pi R n} \).

Let us study the derivatives of \( H \) with respect to the cartesian coordinates. It has been shown in \(^8\) that the dependence of \( H \) on \( x_3 \) falls off exponentially if the compactification radius is small. To study the derivatives with respect to \( x_1, x_2 \), the summation is approximated by an integration\(^9\), which finally yields

\[ H \propto \frac{Z}{2\pi R} \ln \rho + C \]

where we introduced polar coordinates \( \rho = \sqrt{x_1^2 + x_2^2} \) and \( \theta \).\(^6\) The result depends logarithmically on \( \rho \) and therefore corresponds to the naïve solution of the \( d = 3 \) reduced field equations. The \( d = 3 \) fields read

\[ ds^2 = H^{-1} dt^2 - dx_i^2, \quad e^\phi = H^{-1/2}, \]

\(^6\)Note that such solutions exist for the Schwarzschild case as well\(^11\).

\(^7\)We would like to thank Hermann Nicolai for this comment.

\(^8\)See also\(^42\).
plus

\[ g_{kk} = H, \quad \bar{\eta}^{k9} = \frac{Q}{4\pi R} \theta \]
in the electric and

\[ g_{kk} = H^{-1}, \quad \eta^{k9} = \frac{P}{4\pi R} \theta \]
in the magnetic case. This corresponds to the vortex solutions studied in \[8\]. Note that this solution is not asymptotically flat, therefore an interpretation as soliton seems difficult \[17\].

Under \( \theta \to \theta + 2\pi \), we encounter the discrete \( d = 4 \) charges by the shifts \( \bar{\eta}^{k9} \to \bar{\eta}^{k9} + Q/2R \) and \( \eta^{k9} \to \eta^{k9} + P/2R \). Full rotations therefore translate to discrete \( E_7(+7) \) transformations parallel to \[8\].

Note that the electric and magnetic solutions are embedded in a perfectly equal way. We actually expect the spectrum to significantly unify in \( d = 3 \), since only particle and string-like configurations survive.

This concludes our example of \( d = 4 \) solitons in \( d = 3 \). The whole spectrum in \( d = 3 \) will be spanned by \( E_{8(+8)} \) resp. a discrete subgroup \( E_{8(+8)}(\mathbb{Z}) \), which we will study now.

3.6 The \( E_{8(+8)} \) Coset in \( d = 3 \)

We will recover the \( E_{8(+8)}/SO(16) \) coset in this section. The decomposition of \( E_{8(+8)} \supset SL(2, \mathbb{R}) \times E_7(+7) \) in figure \[4\] was already described in section \[3.4\]. Again, the \( SL(2, \mathbb{R}) \) factor will carry the \( d = 3 \) dilaton and dualized Kaluza-Klein gauge field.

Using (50) in appendix A, one may verify that the Lagrangian (33) takes the form

\[ \mathcal{L} = -\frac{1}{4} E^{(3)} R^{(3)} + \frac{1}{240} E^{(3)} \text{Tr}(P^{(3)} P^{(3)\mu}) \]

with

\[ \partial_{\mu} \mathcal{V}^{(3)} \mathcal{V}^{(3)-1} = Q^{(3)}_{\mu} + P^{(3)}_{\mu}, \quad Q^{(3)}_{\mu} \in \mathfrak{e}_{8(+8)} \), \quad P^{(3)}_{\mu} \in \mathfrak{e}_{8(+8)} - \mathfrak{so}(16) \]

where

\[ \mathcal{V}^{(3)} = \mathcal{V}^{(4)} \exp \left( \frac{1}{2} \phi \sum_{i=1}^{8} h_i \right) \exp \left( \mathcal{Y} \cdot S \right) \exp \left( f \ E^{(1)}_9 \right). \quad (34) \]

\( \mathcal{V}^{(4)} \) is identical to (27), but is now in the \textbf{248} adjoint representation of \( E_{8(+8)} \).
The Lagrangian admits local $SO(16)$ and global $E_{8(+8)}$ symmetry. Quite similar to the toy model, $E_{8(+8)}$ decomposes as

$$E_{8(+8)} \supset SL(2,\mathbb{R}) \times E_{7(+7)}$$

$$248 \supset (2,56) + (1,133) + (3,1).$$

(35)

$\mathcal{V}^{(4)}$ therefore has block structure

$$\mathcal{V}^{(4)} = \begin{bmatrix}
56 \\
56 \\
133 \\
1 \\
\end{bmatrix} .$$

### 3.7 Identifying $d = 4$ U-Duality in the $d = 3$ Theory

To recover $d = 4$ duality, consider an element

$$\Lambda \in E_{7(+7)}, \quad \Lambda = e^X, \quad X \in \mathfrak{e}_{7(+7)}$$

where $\mathfrak{e}_{7(+7)}$ is generated by the set (23). Parallel to (15) one gets

$$\mathcal{V}^{(4)} \rightarrow \mathcal{V}^{(4)} \Lambda^{-1} \text{ and } \mathcal{Y} \rightarrow \rho_{56}(\Lambda) \mathcal{Y}$$

and therefore recovers the $d = 4$ U-duality.

As we have seen, the discrete $d = 4$ duality in $d = 3$ corresponds to traveling around vortex solutions on a full circle. The generators of the discrete group are identical to (29) resp. (30) in the $248$ adjoint representation of $E_{8(+8)}$.

### 3.8 Connection to $d = 11$ fields

In order to compare different orders of compactification as indicated in figure [1], we put the coset matrix $\mathcal{V}^{(3)}$ in a convenient form. Remembering the $d = 11$ vielbein,

$$E^{(11)A}_M = \begin{bmatrix}
e^{-1}E^{(3)\alpha}_\mu B^{(3)i}_\mu e^a \hat{e}_i^a \\
0 \\
e^{\phi/2} \Delta^{-\frac{i}{4}} E^{(3)\alpha}_\mu e^{\alpha} \\
0 \\
\end{bmatrix} = \begin{bmatrix}
e^{-\phi/2} \Delta^{-\frac{i}{4}} E^{(3)\alpha}_\mu e^{\alpha} \\
e^{-\phi/2} \Delta^{-\frac{i}{4}} \dot{B}_\mu \\
\Delta^{-\frac{i}{4}} e^{-\phi/2} \dot{B}_\mu B^{(4)i}_\mu \rho_i^a \\
0 \\
\end{bmatrix} ,$$

(36)
we define

\[ \varphi^{ij} = \varphi^{(4)i}, \]
\[ \varphi^{ij} = 2\eta^{ij} + 4\varphi^{(4)ij}\eta^{09} + \frac{1}{6}\epsilon^{ijklmnpq}A_{(l+2)(m+2)(n+2)}\eta_{pq}. \]
\[ \Psi_1 = f - \eta^{09}(-2\eta_{i9} + 2\eta_{ij}\varphi^j - 2\eta^{jk}A_{(j+2)(k+2)}) \]
\[ - \frac{1}{36}\epsilon^{ijklmnop}r(\eta_{ij} - 2\eta^{m9}A_{(i+2)(j+2)(m+2)}A_{(j+2)(q+2)(r+2)}\eta_{kl}). \]
\[ \Psi_i = -\eta_{i9} + \eta_{ij}\varphi^j - A_{(i+2)(j+2)(k+2)}\eta^{jk} - \frac{1}{36}\epsilon^{ijklmnop}A_{(i+2)(j+2)(k+2)}A_{(j+2)(m+2)(n+2)}\eta_{pq}. \]

Using (36) it may be checked that (34) is identical to

\[ \nu^{(3)} = \exp \left( \sum_{m=1}^{8} \ln \left( -\prod_{n=1}^{m} e^n \right) h_m \right) \prod_{p=0}^{6} \exp \left( -\sum_{q,r=8-p}^{8} e_{7-p} \left( e_{(7-p)} - 1 \right)^q \right) E_{7-p}^r \]
\[ \exp \left( \sum_{i=1}^{8} \Psi_i E^i \right) \]
\[ \exp \left( \frac{2}{3!} \sum_{i,j,k=1}^{8} A_{(2+i)(2+j)(2+k)} E^{ijk} - \frac{1}{2!} \sum_{i,j=1}^{8} \varphi^{ij} E^{i9} \right) \]

where summations have been spelled out. The \( \varphi^{ij}, \Psi_i \) obey

\[ \partial_\mu \varphi^{ij} = -2e^2 E^{(3)} \epsilon_{\mu\rho\nu} (\partial^\nu A^{\rho(i+2)(j+2)} + B^{(3)k} \partial^\nu A^{(k+2)(i+2)(j+2)} + \frac{1}{18} \epsilon^{ijklmnop} \partial_\mu A_{(2+k)(2+l)(2+m)} A_{(2+n)(2+p)(2+q)}) \]
\[ \partial_\mu \Psi_i = -\frac{1}{2} e^2 E^{(3)} \epsilon_{\mu\rho\nu} B^{\nu p} - \frac{1}{2} (\varphi^{kl} \partial_\mu A_{(2+k)(2+l)(2+i)} - \varphi^{kl} A_{(2+k)(2+l)(2+i)}) - \frac{1}{54} \epsilon^{ijklmnop} A_{(2+i)(2+j)(2+k)} \partial_\mu A_{(2+i)(2+j)(2+k)} A_{(2+i)(2+j)(2+k)} A_{(2+i)(2+j)(2+k)} A_{(2+i)(2+j)(2+k)} A_{(2+i)(2+j)(2+k)} A_{(2+i)(2+j)(2+k)} \]

where \( i, j, k, \ldots = 1 \ldots 8 \). This is exactly the result of (32) found by direct reduction to \( d = 3 \).

### 3.9 Different Orders of Compactification

To study different orders of compactification, consider the internal vielbein \( e_i^a \). Identifying the coordinate \( 2 + n \) as fourth coordinate in four dimensions, as indicated in figure [I], corresponds to an exchange of the first and \( n \)th row of \( e_i^a \). Ordering the columns as
\{n, 2, 3, \ldots, n-1, 1, n+1, \ldots, 8\}, where the 1 appears at the nth position, we need the new vielbein \( \tilde{e}_i^a \) to be triangular. For this, one may always find an \( SO(8) \) transformation such that the einbein transforms like

\[
\begin{bmatrix}
  e^{\phi/2} \Delta^{-\frac{1}{4}} E^{(3)\alpha}_\mu & e^{-\phi/2} \Delta^{-\frac{1}{4}} B_\mu & B^{(4)i}_\mu \bar{\rho}_i^a \\
  \Delta^{-\frac{1}{2}} e^{-\phi/2} B_3^{(4)i} \bar{\rho}_i^2 & B_3^{(4)i} \bar{\rho}_i^3 & \cdots & B_3^{(4)i} \bar{\rho}_i^8 \\
  0 & \rho_2^2 & \rho_3^3 & \cdots & \rho_2^8 \\
  0 & 0 & \rho_3^3 & \cdots & \rho_3^8 \\
  0 & \cdots & 0 & \rho_n^i & \cdots & \rho_n^8 \\
  0 & 0 & \cdots & 0 & \rho_8^8
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
  e^{\tilde{\phi}/2} \Delta^{-\frac{1}{4}} E^{(3)\alpha}_\mu & e^{-\tilde{\phi}/2} \Delta^{-\frac{1}{4}} B^i_\mu & \tilde{B}^{(4)i}_\mu \bar{\rho}_i^a \\
  0 & \cdots & 0 & \tilde{\rho}_1^i & \cdots & \tilde{\rho}_1^8 \\
  0 & \tilde{\rho}_2^2 & \cdots & \tilde{\rho}_2^8 \\
  0 & 0 & \cdots & \tilde{\rho}_3^3 & \cdots & \tilde{\rho}_3^8 \\
  \tilde{\Delta}^{-\frac{1}{2}} e^{-\tilde{\phi}/2} \tilde{B}^{(4)i}_{2+n} \rho_i^2 & \tilde{B}^{(4)i}_{2+n} \rho_i^3 & \cdots & \tilde{B}^{(4)i}_{2+n} \rho_i^8 \\
  0 & 0 & \cdots & 0 & \tilde{\rho}_8^8
\end{bmatrix}
\]

(37)

Taking the sign change in the Chern-Simons term into account, we may then write the scalar coset matrix for the compactification obtained by taking \( \{0, 1, 2, (2+n)\} \) as four dimensional coordinates as

\[
\mathcal{V}^{(3)}_{\#n} = \mathcal{V}^{(3)}_{\#1}(\tilde{e}_i^a, \tilde{\Psi}_i, \tilde{A}_{(i+2),(j+2),(k+2)}, \tilde{\varphi}^{ij})
\]

with

\[
\tilde{A}_{(i+2),(j+2),(k+2)} = -A_{(i+2),(j+2),(k+2)}, \quad i, j, k \neq 1, n,
\]

\[
\tilde{A}_{3,(j+2),(k+2)} = -A_{(n+2),(j+2),(k+2)}, \quad j, k \neq 1, n,
\]

\[
\tilde{A}_{(n+2),(j+2),(k+2)} = -A_{3,(j+2),(k+2)}, \quad j, k \neq 1, n,
\]

\[
\tilde{A}_{3,(n+2),(k+2)} = -A_{(n+2),(3,(k+2)), \quad k \neq 1, n}
\]

and

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\[ \tilde{\varphi}^{ij} = -\varphi^{ij}, \ i, j \neq 1, n, \]
\[ \tilde{\varphi}^{1j} = -\varphi^{1j}, \ j \neq 1, n, \]
\[ \tilde{\varphi}^{nj} = -\varphi^{1j}, \ j \neq 1, n, \]
\[ \tilde{\varphi}^{1n} = -\varphi^{1n}, \]

finally

\[ \tilde{\Psi}_i = \Psi_i, \ i \neq 1, n, \]
\[ \tilde{\Psi}_1 = \Psi_n, \]
\[ \tilde{\Psi}_n = \Psi_1, \]

while the U-duality group in this compactification is generated by \((23)\) and the discrete group by \((29)\) resp. \((30)\) for all \(\mathcal{V}^{(3)}_{\#n}\).

### 3.10 Joining U-dualities in \(d = 3\)

The above compactifications are related by

\[ \mathcal{V}^{(3)}_{\#n} = \left( P_n S^1_n \right)^{-1} \ h_n \ \mathcal{V}^{(3)}_{\#1} \ P_n S^1_n \]

where \(h_n\) is the natural lift to \(E_8(+8)\) of the local transformation \((37)\), \(S^1_n\) generates the Weyl reflection exchanging \(e_1\) with \(e_n\) and \(P_n\) is again a “parity” transformation. One has

\[ P_n = (-1)^{h_{678} + h_6 + \sum_{m=n}^{7} h_m} = (S^{678})^2 (S^6)^2 (S^8)^2 \quad 2 \leq n \leq 7, \]
\[ P_8 = (-1)^{h_{678} + h_6} = (S^{678})^2 (S^6)^2. \]

\(P_n\) is an \(E_7(+7)(\mathbb{Z})\) transformation corresponding to a charge conjugation in \(d = 4\), but leaves the fields \(A^{(ij)}_{\mu\nu}\), \(B^{(4ij)}_{\mu\nu}\), \(\tilde{j} \neq n\) unchanged. Note that for \(n = 8\) they exactly correspond to the NS-NS fields in section 3.2.

The \(d = 3\) U-duality is therefore given by joining all

\[ \Lambda_n = P_n S^1_n \Lambda (P_n S^1_n)^{-1} \]
where \( \Lambda \) is an element of \( E_{7(+7)} \) spanned by (23), and in the discrete case by (29) resp. (30).

Note that the intersection of two different U-dualities is exactly \( E_{6(+6)} \) as expected! Joining all \( \Lambda_n \) gives the whole of \( E_{8(+8)} \), and the \( d = 3 \) discrete U-duality is generated by the set of generators obtained by exponentiating the Chevalley generators for all roots. This coincides with \( E_{8(+8)}(\mathbb{Z}) \) as defined in appendix B.

We have therefore given a complete set of generators for the U-duality group in three dimensions.

3.11 \( G_{2(+2)} \) in \( E_{8(+8)} \)

Before concluding, we would like to turn back to our toy model.

\( g_{2(+2)} \) may be found in the algebra of \( e_{8(+8)} \) by considering the direct embedding \( \mathfrak{sl}(9) \supset \mathfrak{sl}(3) \). We may choose for example

\[
E_{i12} + E_{i34} + E_{i56}^* + E_{i12}^* + E_{i34}^* + E_{i56}^* + E_j^i (i, j = 7, 8, 9).
\]

It may be checked by using (50) that this choice generates \( g_{2(+2)} \). From the point of view of the physical theory, this truncation is suggested by identifying \( \{0,1,2,3,4\} \) as \( d = 5 \) coordinates

\[
ds^{(11)} = ds^{(5)} + ds^{(E6)}
\]

\[
A^{(11)} = -\frac{1}{\sqrt{3}} A^{(5)} \wedge J, \quad J = \frac{1}{2} (dx^5 \wedge dx^6 + dx^7 \wedge dx^8 + dx^9 \wedge dx^{10})
\]

where \( E6 \) is the flat six dimensional Euclidean space and \( J \) is its Kähler form, as proposed in [38], and \( A^{(11)}, A^{(5)} \) is the eleven dimensional three form and the five dimensional one-form potential.

Note that reexpressing the \( d = 5 \) one form by its M-theory pendant exactly cancels the factor \( \frac{1}{3} \) in (3). The factor 3 is actually symptomatic for the truncation \( E_{8(+8)} \) to \( G_{2(+2)} \), e.g the length squared of the long roots is 3 times larger than the one of the short roots.

In the toy model, we found no agreement of the discrete U-duality group with the definition of appendix B, while there was agreement for M-theory. However, this may be seen as a consequence of the rather complicated embedding of \( g_{2(+2)} \) into \( e_{8(+8)} \), involving generator sums at the level of the algebra, and leading to a non-simply laced algebra.
4 Conclusions and Outlook

In this paper, we have studied the discrete U-duality groups in the context of low energy supergravity as proposed by Hull and Townsend \footnote{3}. We have proposed a set of generators for $E_{7(+7)}$, corresponding to the $d=4$ U-duality group, and presented a proof. Higher dimensional U-dualities are found by direct embedding.

In studying this group, we have made the 56 representation of $E_{7(+7)}$ explicit by an embedding into $E_{8(+8)}$. The way the representation was given might make it “manageable” in order to investigate U-duality transformations connecting BPS solutions as mentioned in \footnote{13}.

In comparison to the other definitions of the U-duality group, we have seen that they agree with our definition and that the set of generators used in the algebraic approach to U-duality can be identified with ours.

We have extended a proposal by Hull and Townsend along the lines of Sen to determine the $d=3$ U-duality group and found that this indeed yields the full $E_{8(+8)}(\mathbb{Z})$ in the definition of appendix B and \footnote{46}. It is interesting to note that the transition from one compactification to the other involves a “parity” transformation, corresponding to a charge conjugation in $d=4$. The fact that we obtained a proposed $d=3$ quantized symmetry from a $d=4$ quantization condition is interesting in this context and might yield new insights into the quantization in $d=3$.

We have studied this procedure also in a truncated model corresponding to simple $d=5$ supergravity, and seen that the $d=3$ U-duality group in this case fails to agree with the definition of appendix B and \footnote{46}, that is, the U-duality in $d=3$ is strictly smaller. We conclude that truncations of M-theory need to be studied very carefully, especially in a quantized context.

The procedure we have presented to determine low dimensional discrete U-duality groups can be extended to $d=2$ and even $d=1$. In these theories, a classical Kač-Moody symmetry algebra known as $e_9$ and Yangian algebra $Y(e_8)$ upon quantization (see \footnote{48} for a recent account) is present. By a lightlike reduction, which makes an intriguing connection to the infinite momentum frame of Matrix theory, a hyperbolic Kač-Moody algebra denoted by $e_{10}$ is expected (\footnote{32}, see \footnote{49} for an introduction to hyperbolic Kač-Moody algebras). In this context, it is interesting to note that the notion of discrete groups over $\mathbb{Z}$ has been connected to homeomorphisms of the group ring over $\mathbb{Z}$ to $\mathbb{Z}$ \footnote{11, \footnote{45}, \footnote{47}}, thereby making a connection to Hopf algebra structures. It should be interesting to see how the notion of discrete duality groups extends for $d=2$ and $d=1$. 

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Work in this direction is in progress.

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A Exceptional Lie Algebras

In this appendix, we will present the representation of exceptional Lie algebras parallel to Freudenthal’s paper [30]. We will start with $g_{2(\pm 2)}$, and then turn to $e_{8(\pm 8)}$.

A.1 Realization of $g_2$

The Lie algebra $g_2$ is known to allow a $\mathbb{Z}_3$ grading and decomposes into the adjoint and two fundamental representations of $\mathfrak{sl}(3)$:

$$14 = 8 \oplus 3 \oplus 3.$$ (39)

Thus any element of $g_2$ can be specified by a traceless matrix $X_{\nu \nu'}^{\nu''}$, $1 \leq \nu, \nu' \leq 3$ and two vectors $v_{\nu'}$ and $v^{*\nu'}$, $1 \leq \nu' \leq 3$ which transform as $3$ and $\overline{3}$ under $\mathfrak{sl}(3)$. The Lie bracket of the two elements $[X_{(1)}^{(1)}, v_{(1)}^{(1)}], [X_{(2)}^{(2)}, v_{(2)}^{(2)}]$ gives rise to a new element of $g_2$ specified by $[X_{(3)}^{(3)}, v_{(3)}^{(3)}, v_{(3)}^{*}]$ with

$$X_{(3)\nu\nu'}^{\nu''} = X_{(1)\nu\nu'}^{\nu''} X_{(2)\nu'\nu''}^{\nu''} - X_{(2)\nu\nu'}^{\nu''} X_{(1)\nu'\nu''}^{\nu''}$$

$$-3 \left( v_{(1)\nu\nu'}^{\nu''} v_{(2)\nu'\nu''}^{\nu''} - v_{(2)\nu\nu'}^{\nu''} v_{(1)\nu'\nu''}^{\nu''} \right) - \frac{1}{3} \left( v_{(1)\nu\nu'}^{\nu''} v_{(2)\nu'\nu''}^{\nu''} - v_{(2)\nu\nu'}^{\nu''} v_{(1)\nu'\nu''}^{\nu''} \right)$$

$$v_{(3)\nu'} = X_{(1)\nu'}^{\nu'} v_{(2)\nu'}^{(2)} - X_{(2)\nu'}^{\nu'} v_{(1)\nu'}^{(1)}$$

$$-2 \epsilon_{\nu'\nu'\nu''} v_{(1)\nu'}^{\nu''} v_{(2)\nu'}^{\nu''}$$

$$v_{(3)}^{*\nu'} = -(X_{(1)\nu'}^{\nu'} v_{(2)\nu'}^{(2)} - X_{(2)\nu'}^{\nu'} v_{(1)\nu'}^{(1)})$$

$$+2 \epsilon_{\nu'\nu'\nu''} v_{(1)\nu'}^{\nu''} v_{(2)\nu'}^{\nu''},$$ (40)

where $\epsilon_{\nu'\nu'\nu''} = -\epsilon_{\nu'\nu'\nu''}$. If all $X_{\nu}^{\nu'}$, $v_{\nu'}$ and $v^{*\nu'}$ are restricted to real numbers, the relations (40) define the real form $g_{2(+2)}$.

The indices are raised and lowered by the metric diag$[-1, -1, -1]$. 

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The roots of \( \mathfrak{g}_2 \) are shown in Figure 3. Regarded as weights of \( \mathfrak{sl}(3) \), they are naturally embedded in a hyperplane in \( \mathbb{R}^3 \):

\[
\mathbf{e}_{ij} \equiv \mathbf{e}_i - \mathbf{e}_j \quad (1 \leq i \neq j \leq 3),
\]

\[
\pm \mathbf{e}_i \equiv \pm (\mathbf{e}_i - \frac{1}{3} \sum_{l=1}^{3} \mathbf{e}_l) \quad (1 \leq i \leq 3),
\]

where \( \{ \mathbf{e}_i \mid 1 \leq i \leq 3 \} \) is a set of orthonormal vectors in \( \mathbb{R}^3 \). The hyperplane is normal to \( \sum_{i=1}^{3} \mathbf{e}_i \).

Let us write \( E_{\mathbf{e}_{ij}} = E_{ij} \), \( E_{\mathbf{e}_i} = E_i \) and \( E_{-\mathbf{e}_i} = E_{i}^* \) corresponding to each non-zero root, respectively. We also take \( \{ h_i \equiv [E_{i+1}^i, E_{i+1}^{i+1}] \mid i = 1, 2 \} \) as the basis of the Cartan subalgebra. Their commutators are:

\[
[ h_i , h_j ] = 0,
\]

\[
[ h_i , E_{k}^j ] = \delta_i^j E_{k}^i - \delta_{i+1}^j E_{k}^{i+1} - \delta_i^j E_{i}^{i+1} + \delta_{i+1}^j E_{i+1}^i,
\]

\[
[ h_i , E_j ] = \delta_i^j E_i - \delta_{i+1}^j E_{i+1}^i,
\]

\[
[ h_i , E_{i}^* ] = - (\delta_{i+1}^j E_{i+1}^{i} - \delta_{i+1}^j E_{i+1}^{i+1}),
\]

\[
[ E_{i}^j , E_{k}^l ] = \delta_{i}^k E_{i}^l - \delta_{i}^l E_{i}^k,
\]

\[
[ E_{i}^j , E_{k} ] = \delta_{j}^k E_{i}^l,
\]

\[
[ E_{i}^j , E_{k}^* ] = - \delta_{j}^k E_{j}^*,
\]

Figure 6: Roots of \( \mathfrak{g}_{2(+2)} \)
\[ [E^i, E^j] = -2 \sum_k \epsilon^{ijk} E^*_k, \]
\[ [E^*_i, E^*_j] = -2 \sum_k \epsilon^{ijk} E^k, \]
\[ [E^i, E^*_j] = 3E^i_j \text{ if } i \neq j; \]
\[ [E^1, E^*_i] = 2h_1 + h_2, \]
\[ [E^2, E^*_i] = -h_1 + h_2, \]
\[ [E^3, E^*_3] = -h_1 - 2h_2. \] (42)

Their expressions in terms of the triple are
\[
h_i = [\delta^i_i \delta^j_j - \delta^i_{i+1} \delta^j_{j+1}, 0, 0],
\]
\[
E^i_j = [\delta^i_i \delta^j_j, 0, 0],
\]
\[
E^i = [0, \delta^i_i, 0],
\]
\[
E^*_i = [0, 0, -\delta^i_i]. \] (43)

### A.2 Realization of $E_8$

The adjoint representation of $E_8$ is decomposed into a sum of irreducible representations of the subalgebra $SL(9)$ as
\[
248 = 80 \oplus 84 \oplus 84, \] (44)
where $80$ is the defining representation, while $84$ ($84$) is $(3,0)$ ($(0,3)$) antisymmetric tensor representation. One may specify any element of $E_8$ by a triple:
\[
[X^j_{ij'}, v^i_{ij'}, v^{*ij'k'}] \text{ (} X \text{ is traceless; } v, v^{*} \text{ are totally antisymmetric),} \] (45)

where $i', j', \ldots = 1, \ldots, 9$ are the indices of the nine dimensional vector space on which $X$, $v$ and $v^{*}$ act. The Lie bracket of the two elements $[X(1), v(1), v^{*}(1)]$ and $[X(2), v(2), v^{*}(2)]$ is again a triple $[X(3), v(3), v^{*}(3)]$ with $\mathfrak{f}$ (Again, the repeated indices are summed over.)

\[
X_{(3)i'j'} = X_{(1)i'j'} X_{(2)j'} - X_{(2)i'j'} X_{(1)i'},
\]
\[
- \frac{1}{2!} \left( v_{(1)i'j'} v_{(2)j'} v^{*j'i'q'} - v_{(2)i'j'} v_{(1)j'} v^{*j'i'q'} \right) \delta_{i'}^j,
\]
\[
v_{(3)i'j'k'} = 3(X_{(1)i'k'} v_{(2)j'k'}) i' - X_{(2)i'k'} v_{(1)i'j'k'} i'. \]
\[
v^{s'}j'k' = \frac{1}{3!} \epsilon^{s'j'k'}m'n'p'q'r' v^{s'm'n'}_{(1)} v^{p'q'r'}_{(2)} = \frac{1}{3!} \epsilon^{s'j'k'}m'n'p'q'r' v^{s'm'n'}_{(1)} v^{p'q'r'}_{(2)} \\
-3 (X^{(1)}_{(1)} v^{s'j'k'}_{(1)} - X^{(2)}_{(2)} v^{s'j'k'}_{(2)}) + \frac{1}{3!} \epsilon^{s'j'k'}m'n'p'q'r' v^{s'm'n'}_{(1)} v^{p'q'r'}_{(2)},
\]

where \( \epsilon^{s'j'k'}m'n'p'q'r' = -\epsilon^{s'j'k'}m'n'p'q'r' \) \(^{10}\). If all the tensors \( X^{(1)}, v^{s'j'k'} \) and \( v^{s'm'n'} \) are restricted to real numbers, then the relations (46) define the real form \( E_{8(8)} \).

Let us think of the eight dimensional root space of \( E_8 \) as a hyperplane lying in \( \mathbf{R}^9 \). Our non-zero roots are

\[
e_{ij} \equiv e_i - e_j \quad (1 \leq i \neq j \leq 9),
\]

\[
\pm e_{ijk} \equiv \pm (e_i + e_j + e_k - \frac{1}{3} \sum_{l=1}^{9} e_l) \quad (1 \leq i < j < k \leq 9),
\]

where \( \{e_i \mid 1 \leq i \leq 9\} \) is a set of orthonormal vectors in \( \mathbf{R}^9 \). The hyperplane is normal to \( \sum_{i=1}^{9} e_i \).

We write the corresponding generators as

\[
E_{e_{ij}} = E^i_j \quad (\text{total 72}),
\]

\[
E_{e_{ijk}} = E^{ijk} \quad (\text{total 84}),
\]

\[
E_{-e_{ijk}} = E^{*}_{ijk} \quad (\text{total 84})
\]

and take the commutators:

\[
h_i \equiv [E^i_{i+1}, E^{i+1}_{i}] \quad (\text{total 8})
\]

as the basis of the Cartan subalgebra \( \{h_i \mid i = 1, \ldots, 8\} \). These 72+84+84+8 = 248 generators form a basis of \( E_8 \) and satisfy the relations (Note that the repeated indices are not summed over unless stated explicitly.)

\[
[ h_i, h_j ] = 0,
\]

\[
[ h_i, E^j_k ] = \delta^j_k E^i_k - \delta^j_{i+1} E^{i+1}_k - \delta^j_k E^i_{i+1} + \delta^j_{k+1} E^{i+1}_{i+1},
\]

\[
[ h_i, E^{ijkl} ] = 3(\delta^k_i E^{ijkl} - \delta^j_i E^{iklj} - \delta^j_i E^{ilkj} + \delta^j_i E^{ikjl}),
\]

\[
[ h_i, E^{*}_{ijk} ] = -3(\delta^j_i E^{*}_{ijk} - \delta^j_{i+1} E^{*}_{ijk} + \delta^j_{i+1} E^{*}_{i+1j}),
\]

\[
[ E^i_j, E^k_l ] = \delta^k_j E^i_l - \delta^j_l E^k_j,
\]

\[
[ E^i_j, E^{klm} ] = 3 \delta^j_l E^{klm},
\]

\(^{10}\) The indices are raised and lowered by the metric diag\([-1, \ldots, -1]\).
\[
\begin{align*}
\lbrack E^i_{\ j}, E^*_m \rbrack &= -3\delta^i_m E^*_{mj}, \\
\lbrack E^{ijk}, E^{lmn} \rbrack &= -\frac{1}{3!}\sum_{p,q,r} \epsilon^{ijkmnpr} E^*_{pq}, \\
\lbrack E^{*}_{ijk}, E^{*}_{lmn} \rbrack &= -\frac{1}{3!}\sum_{p,q,r} \epsilon_{ijklmnpqr} E^*_{pqr}, \\
\lbrack E^{ijk}, E^{*}_{lmn} \rbrack &= 6\delta^i_m \delta^k_n E^j_l \quad \text{if } i \neq l, m, n, \\
\lbrack E^{*}_{ijk}, E^{*}_{ijk} \rbrack &= -\frac{1}{3!}\sum_{l=1}^9 l h_l + \sum_{l=i}^9 h_l + \sum_{l=j}^9 h_l + \sum_{l=k}^9 h_l \\
&= h_{ijk}. \quad (50)
\end{align*}
\]

We also extended the definition of \( E^{ijk} \) as
\[
E^{ijk} = E^{jki} = E^{kij} = -E^{ikj} = -E^{jik} = -E^{kji}, \quad (51)
\]
and likewise for \( E^{*}_{ijk} \). Their expressions in terms of the triple are
\[
\begin{align*}
h_i &= [\delta^i_i \delta^j_j - \delta^i_{i+1} \delta^j_{j+1}, 0, 0], \\
E^i_{\ j} &= [\delta^i_i \delta^j_j, 0, 0], \\
E^{ijk} &= [0, 3! \delta^i_i \delta^j_j \delta^k_k, 0], \\
E^{*}_{ijk} &= [0, 0, -3! \delta^i_i \delta^j_j \delta^k_k]. \quad (52)
\end{align*}
\]

Also
\[
h_{ijk} = [\delta^i_i \delta^j_j + \delta^i_i \delta^j_j + \delta^k_k \delta^j_j - \frac{1}{3} \delta^j_j, 0, 0]. \quad (53)
\]

Thus \( h_i, h_{ijk} \) may be thought of as \( E^i_i - E^{i+1}_{i+1}, E^i_i + E^k_k - \frac{1}{3} \sum_{l=1}^9 E^l_l \), respectively.

Finally, we give some useful trace formulas:
\[
\begin{align*}
\text{Tr}_{248} E^i_{\ j} E^k_{\ l} &= 60 \delta^i_i \delta^k_k, \\
\text{Tr}_{248} E^{ijk} E^{*}_{lmn} &= 60 \cdot 3! \delta^i_i \delta^j_j \delta^k_k. \quad (54)
\end{align*}
\]

\( i, j, k = 1, 2, \ldots, 9. \)

\section*{B Discrete Subgroups of Lie Groups}

In this appendix we study the properties of the discrete subgroup \( G(\mathbb{Z}) \) used in this paper.

We give a definition of \( G(\mathbb{Z}) \) and propose a set of generators. We use properties of a special
class of representations relevant to dualities, together with the Birkhoff decomposition of Lie groups [43] to prove that all elements of $G(\mathbb{Z})$ can be generated from this set.

Let $G$ be a complex simple Lie group, and $\mathfrak{g}$ be the Lie algebra of $G$. Let $h$ be a Cartan subalgebra of $\mathfrak{g}$, and $H$ be its Lie group. Let $\Phi$ be the set of roots of $G$ relative to $H$, $\Delta$ a fixed set of simple roots in $\Phi$ and $\Phi^+(\Phi^-)$ the set of positive(negative) roots with respect to $\Delta$.

Consider a nontrivial irreducible representation $\rho$ of $G$ and let $\Lambda^\rho$ be the set of weights of $\rho$. Let us call $\rho$ a minimal representation if all $\lambda \in \Lambda^\rho$ are transformed into each other by the Weyl group $W$ of $G$. It is easy to show that $\rho$ is a minimal representation if and only if $\langle \lambda, \alpha \rangle (= 2(\lambda, \alpha)/(\alpha, \alpha)) = 0$ or $\pm 1$ for all $\lambda \in \Lambda^\rho$, $\alpha \in \Phi$. Each coset of the root lattice $\Lambda_r$ in the weight lattice $\Lambda$ contains precisely one highest weight of a minimal representation. Thus there are $|\Lambda/\Lambda_r| - 1$ such representations. In fact they are all fundamental representations. (See [47], page 72.) It is enough to consider only this particular class of representations, since they include all the representations relevant to our discussion such as the $56$ of $E_7$, the $N$ of $SL(N)$ and the $2N$ of $SO(2N)$ \[11\].

Let $V$ be a representation space of a minimal representation $\rho$. A lattice $V_\mathbb{Z}$ in $V$ is defined to be the $\mathbb{Z}$-span of a basis of $V$. With a fixed Chevalley basis $\{e_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq N\}$ of $\mathfrak{g}$, a lattice $V_\mathbb{Z}$ is said to be admissible if $V_\mathbb{Z}$ is invariant under the action of the $\mathbb{Z}$-form $U(\mathfrak{g})_\mathbb{Z}$ \[17\] of the universal enveloping algebra $U(\mathfrak{g})$, that is, invariant under the actions of all $\rho(e_\alpha)^n/n!$, $n \in \mathbb{N}$. The discrete group $G(\mathbb{Z})$ of $G$ is defined as the subgroup of $G$ which consists of all $g \in G$ such that $\rho(g)$ stabilizes $V_\mathbb{Z}$ \[17\].

Remark. An admissible lattice fixes in which basis of $V$ the entries of the elements of $G(\mathbb{Z})$ are restricted to integer values. For example,

$$SL(2, \mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

stabilizes the admissible lattice

$$\mathbb{Z}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

In other generic basis of $V$, $SL(2, \mathbb{Z})$ is not represented as a group of matrices with integral entries. In $d = 4$ duality in general, each $U(1)$ gauge field corresponds to different weight space and one can always normalize the charge lattice so that it may coincide with an admissible lattice. Therefore this definition of $G(\mathbb{Z})$ and the concept of discrete duality groups agree.

\[11\] In Ref. [46] a larger class of representations which may include zero weights are considered and called basic representations. It is also claimed that $G(\mathbb{Z})$ does not depend on the representation.
One can construct the smallest admissible lattice $V_\mathbb{Z}$ by acting $U(g)\mathbb{Z}$ to a given highest weight vector of $V$.

We will now prove the following proposition:

**Proposition 46.** $G(\mathbb{Z})$ coincides with the group generated by

$$\{\exp e_\alpha | \alpha \in \Phi\}$$

denoted by $E(\mathbb{Z})$ in the following.

**Example.** As is well known, $SL(2,\mathbb{Z})$ is generated by

$$\exp e_\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \exp e_{-\alpha} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

or equivalently $T = \exp e_\alpha$ and $S = \exp e_\alpha \exp(-e_{-\alpha}) \exp e_\alpha$.

**Proof of Proposition.** Let $\alpha_1$ be the simple root dual to the highest weight $\lambda_1$ of $\rho$. Let $\Phi^{r+}$ be the set of positive roots orthogonal to $\lambda_1$: $\Phi^{r+} = \{\alpha \in \Phi^+ | (\lambda_1, \alpha) = 0\}$. We introduce and fix an order (denoted by $<$) among the elements of $\Phi^+$ in such a way that $\alpha < \beta$ if $\alpha \in \Phi^{r+}$ and $\beta \in \Phi^+ - \Phi^{r+}$. Similarly, for $-\alpha, -\beta \in \Phi^-$, we define $-\beta < -\alpha$ if $\alpha < \beta$.

We can take a basis $\{v_\lambda | \lambda \in \Lambda_\rho\}$ of $V_\mathbb{Z}$ as

$$v_{\lambda_1},
\begin{align*}
v_{\lambda_1 - \alpha} &= \rho(e_{-\alpha})v_{\lambda_1} \quad (\alpha \in \Phi^+ - \Phi^{r+}), \\
v_{\lambda_1 - \alpha - \beta} &= \rho(e_{-\beta})\rho(e_{-\alpha})v_{\lambda_1} \quad (\alpha, \beta \in \Phi^+ - \Phi^{r+}), \\
\ldots
\end{align*}$$

where the product of $\rho(e_{-\alpha})$’s are “normal-ordered” according to the order introduced above, that is, $\rho(e_{-\beta})$ comes to the left of $\rho(e_{-\alpha})$ if $-\beta < -\alpha$.

It is known that any $g \in G$ can be written in the form (the Birkhoff decomposition)

$$g = \prod_{\alpha \in \Phi^+} \exp(c_{-\alpha}e_{-\alpha}) \cdot w(g) \cdot \prod_{\alpha \in \Phi^+} \exp(c_\alpha e_\alpha)$$

(55)

with $w(g)$ being an element of the normalizer of $H$. The multiple products are “normal-ordered” similarly. Since $\rho$ is a minimal representation, $\rho(e_\alpha)^2 = 0$ for any root $\alpha$. Thus each factor $\rho(\exp c_\alpha e_\alpha)$ in $\rho(g)$ (55) can be replaced by $1 + c_\alpha \rho(e_\alpha)$.

Let us write

$$\rho(g)v_{\lambda_1} = \sum_{\lambda \in \Lambda_\rho} \rho(g)_{\lambda\lambda_1} v_\lambda, \quad \rho(g)_{\lambda\lambda_1} \in \mathbb{Z}.$$  

(56)
Let \( s_\alpha = \exp(-e_\alpha) \exp e_\alpha \exp(-e_\alpha) \in E_Z \), \( \alpha \in \Phi \). Then \( \rho(s_\alpha) \) sends \( v_\lambda \) to a vector proportional to \( v_{\sigma_\alpha(\lambda)} \), where \( \sigma_\alpha \) is the Weyl reflection with respect to \( \alpha \). Since any weight of \( \rho \) is transformed to the highest weight by the Weyl group, one can find some \( s \in E_Z \) (written as a product of \( s_\alpha \)'s) such that \( \rho(s)v_\lambda \propto v_{\lambda_1} \) for any weight \( \lambda \). Therefore, if \( \rho(g)_{\lambda_1 \lambda_1} = 0 \), one may still have \( \rho(sg)_{\lambda_1 \lambda_1} \neq 0 \) for some element \( s \) of \( E_Z \). Thus we assume that \( \rho(g)_{\lambda_1 \lambda_1} \neq 0 \) from the beginning, without loss of generality.

Since \( \rho(e_\alpha)v_{\lambda_1} = 0 \) for any positive root \( \alpha \), and since \( \rho(e_\alpha)_{\lambda_1 \lambda} \neq 0 \) only if \( \lambda = \lambda_1 \) for any negative root \( \alpha \) and any weight \( \lambda \), our assumption \( \rho(g)_{\lambda_1 \lambda_1} \neq 0 \) implies that \( w(g) \) in (56) stabilizes the highest weight \( \lambda_1 \), i.e. \( \rho(w(g))v_{\lambda_1} = cv_{\lambda_1} \) for some constant \( c \in \mathbb{C} \). Thus in the first column and row of the matrix \( \rho(g) \) we have

\[
\rho(g)_{\lambda_1 \lambda_1} = c \cdot \rho \left( \prod_{\alpha \in \Phi^+ - \Phi^{++}} (1 + c_{-\alpha} e_{-\alpha}) \right)_{\lambda_1 \lambda_1} \in \mathbb{Z},
\]

\[
\rho(g)_{\lambda_1 \lambda} = c \cdot \rho \left( \prod_{\alpha \in \Phi^+ - \Phi^{++}} (1 + c_\alpha e_\alpha) \right)_{\lambda_1 \lambda} \in \mathbb{Z}. \tag{57}
\]

In particular, if \( \lambda_1 - \lambda = \alpha \in \Phi^+ - \Phi^{++} \), we have

\[
\rho(g)_{\lambda_1 \lambda_1} = cc_{-\alpha} \in \mathbb{Z},
\]

\[
\rho(g)_{\lambda_1 \lambda} = cc_\alpha \in \mathbb{Z}, \tag{58}
\]

\[
\rho(g)_{\lambda_1 \lambda_1} = c \in \mathbb{Z}.
\]

Note, again, that the factors in the multiple products are ordered according to some fixed ordering of the roots. If \( \lambda \) is a weight but cannot be reached from the highest weight \( \lambda_1 \) by a single Weyl reflection, then \( \rho(g)_{\lambda_1 \lambda} \) \( (\rho(g)_{\lambda_1 \lambda_1}) \) is expressed as a polynomial of \( c_{-\alpha} \) \( (c_\alpha) \).

We have defined \( E_Z \) to be the group generated by \( \{ \exp e_\alpha | \alpha \in \Phi \} \). We will now show that if \( g \in G(Z) \), then \( g \) can be reduced, by multiplying some elements in \( E_Z \) to \( g \), to an element of a subgroup \( G'(Z) \) constructed from \( G' \subset G \) with rank \( G' < \text{rank } G \). Once if this is proved, then repeating these operations, one may reduce any \( g \) to the identity. By induction this will show that \( G(Z) = E_Z \).

Let us first consider the case when \( G \) is simply laced. In this case we fix a lexicographic order in \( \Phi^+ \) defined by the inner product in the weight space with some ordered basis \( \mu_i, i = 1, \ldots, \text{rank } G \). (The reason for why we need this will be explained soon.) For example, one may first define a partial order in \( \Phi^+ \) by the height, that is, the inner product \( \mu_1 = \delta \), half the sum of positive roots. For \( \alpha, \beta \in \Phi^+ \) with the same height,
one may next take some arbitrary \( \mu \) (linearly independent of \( \delta \)) and define \( \alpha < \beta \) if \( (\mu, \alpha) < (\mu, \beta) \). For those which have the same inner product with \( \mu \), one may consider the product with another independent \( \mu' \). In this way one may have a lexicographic order in \( \Phi^+ \).

Suppose first that \( \rho(g)_{\lambda \lambda_1} \neq 0, \lambda = \lambda_1 - \alpha \) for the minimal \( \alpha \in \Phi^+ - \Phi'^+ \) with respect to the above lexicographic order (that is, \( \alpha < \beta \) for any \( \beta \neq \alpha \) in \( \Phi^+ - \Phi'^+ \)). Let us put \( t_\alpha = \exp e_\alpha \in E_Z \), and \( s_\alpha = t_\alpha^{-1} t_{-\alpha} t_\alpha^{-1} \in E_Z^\prime \) as before. Then for \( n \in \mathbb{Z} \) we have

\[
\rho(t^n_{-\alpha}s_\alpha g)_{\lambda \lambda_1} = n \rho(g)_{\lambda_1 \lambda_1} - \rho(g)_{\lambda \lambda_1},
\]

\[
\rho(t^n_{-\alpha}s_\alpha g)_{\lambda_1 \lambda_1} = -\rho(g)_{\lambda \lambda_1}.
\]

There exists some integer \( n \) such that \( \rho(t^n_{-\alpha}s_\alpha g)_{\lambda_1} < \rho(g)_{\lambda \lambda_1} \). Therefore, by repeating this operation, one can reduce the \( (\lambda, \lambda_1) \) entry of \( \rho(g) \) to 0 (Euclidean algorithm).

We next assume that there exists some \( \beta \in \Phi^+ - \Phi'^+ \) such that \( \rho(g)_{\beta \lambda_1} = 0 \) for all \( \lambda = \lambda_1 - \alpha, \alpha < \beta \), and \( \rho(g)_{\lambda_1 - \beta, \lambda_1} \neq 0 \). Then applying a similar operation to \( \lambda_1 - \beta \), we have \( \rho(u_{-\alpha}g)_{\lambda_1 - \beta, \lambda_1} = 0 \) for some element \( u_\alpha \) of \( E_Z \).

Thus, by induction, we have \( \rho(u_{-\alpha}g)_{\lambda_1 - \alpha, \lambda_1} = 0 \) for all \( \alpha \in \Phi^+ - \Phi'^+ \) for some element \( u_\alpha \in E_Z \).

Let \( \alpha \in \Phi^+ - \Phi'^+ \), that is, a positive root that contains \( \alpha_1 \). Then \( \rho(e_\alpha) \) does not kill \( v_{\lambda_1} \): \( \rho(e_\alpha)v_{\lambda_1} \neq 0 \). For any minimal representation of a simply laced simple Lie group \( G \), such \( \alpha \) can contain only one \( \alpha_1 \). Therefore, any root in \( \Phi^+ - \Phi'^+ \) cannot be a sum of any other two such roots. Then the order in \( \Phi^+ - \Phi'^+ \) (induced by that in \( \Phi^+ \)) ensures that the inductive assumption \( \rho(g)_{\lambda \lambda_1} = 0 \) for all \( \lambda = \lambda_1 - \alpha, \alpha < \beta \) can never be violated by the operation for \( \lambda_1 - \beta \).

We may now write \( u_{-\alpha}g \) in the Birkhoff decomposition as

\[
u_{-\alpha}g = \prod_{\alpha \in \Phi'^+} \exp(c'_\alpha e_\alpha) \cdot w(g) \cdot \prod_{\alpha \in \Phi^+} \exp(c'_\alpha e_\alpha).
\]

Therefore, for any weight \( \lambda \neq \lambda_1 \), we have \( \rho(u_{-\alpha}g)_{\lambda \lambda_1} = 0 \) and \( \rho(u_{-\alpha}g)_{\lambda_1 \lambda_1} = \pm 1 \) for some \( u \in E_Z \). Actually we may take \( \rho(u_{-\alpha}g)_{\lambda_1 \lambda_1} = +1 \) since \( s_\alpha^2 = -1 \). Using (58), we find

\[
\rho(u_{-\alpha}g)_{\lambda \lambda_1} = 0,
\]

\[
\rho(u_{-\alpha}g)_{\lambda_1 \lambda} = c'_\alpha \in \mathbb{Z},
\]

\[
\rho(u_{-\alpha}g)_{\lambda_1 \lambda_1} = 1.
\]

\footnote{This lexicographic order can be different from the order introduced above equation (54), but they can be made consistent if \( \mu_1 \) is taken = \( \lambda_1 \). Thus we use the same symbol < here for simplicity.}
where \( \alpha \in \Phi^+ - \Phi'^{+} \). Thus \( u_+ = \prod_{\alpha \in \Phi^+ - \Phi'^+} \exp(-c'_\alpha e_\alpha) \) (in an arbitrary order) belongs to \( E_{\mathbb{Z}} \), and

\[
\begin{align*}
\rho(u-gu_+o_{\lambda_1}) &= 0, \\
\rho(u-gu_+o_{\lambda_1}) &= 0, \\
\rho(u-gu_+)^{o_{\lambda_1}} &= 1
\end{align*}
\]

for all the weights \( \lambda \neq \lambda_1 \). Therefore, \( u_+g_+ \) belongs to the subgroup \( G' \subset G \) of whose simple roots are \( \Delta - \alpha_1 \), and the minimal representation \( \rho \) of \( G \) is again decomposed into a direct sum of minimal representations of \( G' \). This is what we desired, and completes the proof for a simply laced Lie group \( G \).

Finally, if \( G \) is not simply laced, the minimal representation \( \rho \) is either the \( 2^N \) for \( SO(2N + 1) \) or the \( 2^N \) for \( Sp(2N) \). One may also apply the same argument to these cases except for the modification in the consistent order of operations; since some roots in \( \Phi^+ - \Phi'^{+} \) can be written as a sum of two roots in \( \Phi^+ - \Phi'^{+} \) if \( G \) is not simply laced, the lexicographic order fails to ensure the consistency of induction. Nevertheless, one may still find alternative consistent orders of operations explicitly, and prove the proposition in both cases. We omit the detail in this paper; a way to find this is to write the Hasse diagrams for these representations.

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