MOTIVES OF SMOOTH FAMILIES AND CYCLES ON THREEFOLDS

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Abstract. Let $X \to S$ be a smooth projective family of surfaces over a smooth curve $S$ whose generic fiber $X_\eta$ is a surface with $H^2_{\text{et}}(X_\eta, \mathbb{Q}(1))$ spanned by divisors on $X_\eta$ and $H^1_{\text{et}}(X_\eta, \mathbb{Q}) = 0$. We prove that, if the motive of $X/S$ is finite dimensional, the Chow group $\text{CH}^2(X)$ is generated by a multisection and vertical cycles, i.e. one-dimensional cycles lying in fibers of the above map. If $S = \mathbb{P}^1$, then $\text{CH}^2(X) = \mathbb{Q} \oplus \mathbb{Q}^\oplus n$, where $n \leq b_2$ and $b_2$ is the second Betti number of the generic fiber. Generators in $\text{CH}^2(X)$ can be concretely expressed in terms of spreads of algebraic generators of $H^2_{\text{et}}(X_\eta)$. We also show where such families are naturally arising from, and point out the connection of the result with Bloch’s conjecture.

1. Introduction

Let $X$ be a smooth projective variety defined over a field. An important problem in algebraic geometry is to compute the Chow group $\text{CH}^j(X)$ of codimension $j$ algebraic cycles modulo rational equivalence relation on $X$. In codimension one the problem is solved in the following sense: algebraically trivial divisors are parametrized by the Picard variety of $X$. But the next step, when $j = 2$, leads to hard problems. Even if $X$ is a surface, zero-dimensional cycles on $X$ are not understood by know. The conjecture due to S.Bloch asserts that, if $X$ is a complex surface with no non-trivial globally holomorphic 2-forms, the kernel of the Albanese mapping is trivial, [15]. If $X$ is of special type, i.e. the Kodaira dimension of $X$ is less than two, the conjecture was done in [4]. When $X$ is of general type, the problem is opened, except for a few cases (see [3], [12] and [19]).

In the last years it was discovered that Bloch’s conjecture is closely connected with the tensor structures in motivic categories. Namely, in [11] we have shown that the Albanese kernel is trivial for a given complex surface $X$ with $p_g = 0$ if and only if its motive $M(X)$ is finite dimensional in the sense of S.Kimura, [16]. The use of Kimura’s theory arises, actually, from simple facts in representation theory of symmetric groups applied in the setting of tensor categories. A purely algebraic version of that theory was independently developed by P.O’Sullivan, [2]. The connection with geometry is provided by
the following nilpotency theorem due to S.Kimura, [13, 7.5]: any numerically trivial endomorphism of a finite dimensional Chow motive is nilpotent.

The goal of the present paper is to show that motivic finite dimensionality can be also useful in the study of codimension two cycles on threefolds. In fact, we will try to extend the arguments from [11] to one-parameter families of smooth projective surfaces whose generic fiber has an algebraic second Weil cohomology group. For that we will use a more general version of Kimura’s theorem proved by Andre and Kahn, [2, 9.1.14]: any numerically trivial endomorphism of a finite dimensional object in a nice tensor category is nilpotent.

To state the result we need to fix some notation. All Chow groups will be with coefficients in \( \mathbb{Q} \). Let \( k \) be a field of characteristic zero, and let\[ \gamma : X \to S \]
be a smooth projective family of surfaces over a smooth connected quasi-projective curve \( S \) over \( k \). Define the Chow group\[ CH^2(X)_0 = \ker(\gamma_* : CH^2(X) \to CH_1(S) = \mathbb{Q}) \]
of 1-cycles of degree zero with respect to the map \( \gamma \), so that\[ CH^2(X) = CH^2(X)_0 \oplus \mathbb{Q} \].

A one-dimensional cycle class on \( X \) is called vertical, if it can be represented by a linear combination of curves lying in closed fibers of the map \( \gamma \). Let \( H^*(-) \) be a Weil cohomology theory over the function field \( k(S) \), say \( l \)-adic \'{e}tale cohomology groups. In particular, if \( X_\eta \) is the generic fiber of the map \( \gamma \), then\[ H^*(X_\eta) = H_{et}^*(X_\bar{\eta}, \mathbb{Q}_l) \],
where \( \bar{\eta} \) is the spectrum of an algebraic closure of \( k(S) \). Let\[ H^2_{tr}(X_\eta) \]
be the transcendental part of \( H^2(X_\eta) \), i.e. the second direct summand of \( H^2(X_\eta) \) after splitting of classes of divisors on \( X_\eta \). If \( H^2_{tr}(X_\eta) = 0 \), it follows that \( H^2(X_\eta) \) is generated by divisors \( D_1, \ldots, D_{b_2} \) on \( X_\eta \), where \( b_2 \) is the second Betti number of the generic fiber. For each \( i \) let\[ W_i \]
be a spread of \( D_i \) over \( S \) (see [8, 4.2] or Section 2.3 below for discussion of spreads). Our main result is then as follows:

**Theorem 1.** Let \( \gamma : X \to S \) be a family as above, and assume that\[ H^1(X_\eta) = H^2_{et}(X_\eta) = 0 \].

Then, if the relative motive \( M(X/S) \) is finite dimensional, any cycle class in \( CH^2(X)_0 \) is vertical, and it can be represented by a linear combination of intersections of the spreads \( W_i \) with closed fibers of the map \( \gamma \). If, moreover, \( S = \mathbb{P}^1 \), then\[ CH^2(X)_0 = \mathbb{Q}^{\oplus n} \].
with
\[ n \leq b_2 , \]
and any degree zero one-dimensional algebraic cycle on \( X \) is rationally equivalent to a linear combination of cycles
\[ W_1 \cdot F, \ldots, W_{b_2} \cdot F , \]
where \( F \) is a fixed closed fiber of \( \gamma \).

Geometrically this result can be illustrated as follows. Assume \( S = \mathbb{P}^1 \) and suppose we are given, for example, two curves \( C_1 \) and \( C_2 \) on \( X \) projecting onto \( \mathbb{P}^1 \) with the same degree. Then the difference \( C_1 - C_2 \) is rationally equivalent to a linear combination of one-cycles \( W_1 \cdot F, \ldots, W_{b_2} \cdot F \) on \( X \).

Bloch’s conjecture may be considered as a problem both on zero-dimensional cycles, and on codimension two cycles. If we look on it from the second viewpoint, a three-dimensional counterpart of Bloch’s conjecture, in the sense of a smooth one-parameter family of projective surfaces \( X \to S \), may be stated as follows: if \( H^2_{tr}(X_\eta) = 0 \) and \( H^1(X_\eta) = 0 \), then \( CH^2(X)_0 \) is generated by one-dimensional cycles lying in fibers of the map \( X \to S \). If we “collapse” the curve \( S \) into a point, the vertical generators disappear, so that we obtain the usual 2-dimensional situation. Thus, Theorem 1 partially supports the three-dimensional conjecture in those cases where we can prove finite dimensionality of \( M(X/S) \). But we cannot avoid the restriction \( H^1(X_\eta) = 0 \) yet, because we do not know any appropriate construction of the relative Murre decomposition for a smooth projective family of surfaces with non-trivial irregularity. On the contrary, if the first cogomology of the generic fiber vanishes, the relative Murre decomposition for the whole family can be easily constructed adding some vertical correcting terms to standard projectors. This method allows also to generalize Kimura’s theorem, \([16, 4.2]\), on motives of smooth projective curves over a field (Proposition 5).

The paper is organized as follows. In Section 2 we recall some well known facts about Chow motives, finite dimensional objects and spreads of algebraic cycles. Section 3 is devoted to splittings of the unit and a tensor power of the Lefschetz motive from the Chow motive of a family of relative dimension \( e \). The central Section 4 contains the proof of Theorem 1. In Section 5 we generalize Kimura’s theorem on motives of curves. In the last Section 6 we show how to construct families satisfying the assumptions of Theorem 1 by means of spreadings out of surfaces defined over \( \mathbb{C} \).

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2. Preliminary results

2.1. Chow motives over a base. Let $S$ be a smooth connected quasi-projective variety over a field $k$, and let $\mathcal{SP}(S)$ be the category of all smooth and projective schemes over $S$. Assume we are given two objects of that category, and let $X = \bigcup_j X_j$ be the connected components of $X$. For any non-negative $m$ let

$$\text{Corr}_m^S(X, Y) = \bigoplus_j \text{CH}^{e_j+m}(X_j \times_S Y)$$

be the group of relative correspondences of degree $m$ from $X$ to $Y$ over $S$, where $e_j$ is the relative dimension of $X_j$ over $S$. For example, given a morphism $f : X \to Y$ in $\mathcal{SP}(S)$, the transpose $\Gamma^t_f$ of its graph $\Gamma_f$ is in $\text{Corr}_0^S(X, Y)$. For any two correspondences $f : X \to Y$ and $g : Y \to Z$ their composition $g \circ f$ is defined, as usual, by the formula

$$g \circ f = p_{13}^*(p_{12}^*(f) \cdot p_{23}^*(g)),$$

where the central dot denotes the intersection of cycle classes in the sense of [6].

The category $\text{CHM}(S)$ of Chow-motives over $S$ with coefficients in $\mathbb{Q}$ can be defined then as a pseudoabelian envelope of the category of correspondences with certain “Tate twists” indexed by integers (see [14]). For any smooth projective $X$ over $S$ its motive $M(X/S)$ is defined by the relative diagonal $\Delta_{X/S}$, and for any morphism $f : X \to Y$ in $\mathcal{SP}(S)$ the correspondence $\Gamma_f$ defines a morphism $M(f) : M(Y/S) \to M(X/S)$. The category $\text{CHM}(S)$ is rigid with a tensor product satisfying the formula

$$M(X/S) \otimes M(Y/S) = M(X \times_S Y),$$

so that the functor

$$M : \mathcal{SP}(S) \to \text{CHM}(S)$$

is tensor. The scheme $S/S$ indexed by 0 gives the unite $1_S$ in $\text{CHM}(S)$, and when it is indexed by $-1$, it gives the Lefschetz motive $\mathbb{L}_S$. If $E$ is a multi-section of degree $w > 0$ of $X/S$, we set

$$\pi_0 = \frac{1}{w} [E \times_S X] \quad \text{and} \quad \pi_{2e} = \frac{1}{w} [X \times_S E],$$

where $e$ is the relative dimension of $X/S$. Then one has the standard isomorphisms $1_S \cong (X, \pi_0, 0)$ and $\mathbb{L}_S^{\otimes e} \cong (X, \pi_{2e}, 0)$. Finally, if $f : T \to S$ is a morphism of base schemes over $k$, then $f$ gives a base change tensor functor $f^* : \text{CHM}(S) \to \text{CHM}(T)$. All the details about Chow motives over a smooth base can be found, for instance, in [7].

2.2. Finite dimensional objects. Below we will use basic facts from the theory of finite dimensional motives, or, more generally, finite dimensional objects, see [16] or [2]. Roughly speaking, as soon as we have a tensor pseudoabelian category $\mathcal{C}$ with coefficients in a field of characteristic zero, one may speak about wedge and symmetric powers of any object in $\mathcal{C}$. Then we say that $X \in \text{Ob}(\mathcal{C})$ is finite dimensional, if it can be decomposed into a direct sum, $X = Y \oplus Z$, such that $\wedge^m Y = 0$ and $\text{Sym}^n Z = 0$ for some non-negative integers $m$ and $n$. The property to be finite dimensional is closed under direct
soms and tensor products, etc. A morphism \( f : X \to Y \) in \( \mathcal{C} \) is said to be numerically trivial if for any morphism \( g : Y \to X \) the trace of the composition \( g \circ f \) is equal to zero, [1, 2.3]. The important role in the below arguments is played by the following result:

**Proposition 2.** Let \( \mathcal{C} \) be a tensor pseudoabelian category with coefficients in \( \mathbb{Q} \). Let \( X \) be a finite dimensional object in \( \mathcal{C} \) and let \( f \) be a numerically trivial endomorphism of \( X \). Then \( f \) is nilpotent in the ring of endomorphisms of \( X \).

**Proof.** See [16, 7.5] for Chow motives and [2, 9.1.14] in the abstract setting. □

**Lemma 3.** Let \( \Xi : \mathcal{C}_1 \to \mathcal{C}_2 \) be a tensor functor between two rigid tensor pseudoabelian categories, both with coefficients in \( \mathbb{Q} \). Assume \( \Xi \) induces an injection \( \text{End}(1_{\mathcal{C}_1}) \hookrightarrow \text{End}(1_{\mathcal{C}_2}) \). Then, if \( X \) is a finite dimensional object in \( \mathcal{C}_1 \) and \( \Xi(X) = 0 \), it follows that \( X = 0 \) as well.

**Proof.** Let \( g \) be an endomorphism of \( X \). Then \( F(\text{tr}(g \circ 1_X)) = \text{tr}(F(g \circ 1_X)) \), see [5], page 116. Since \( F(X) = 0 \), we have \( F(g \circ 1_X) = 0 \). Then \( \text{tr}(g \circ 1_X) = 0 \) because \( F \) is an injection on the rings of endomorphisms of units. Hence, the identity morphism \( 1_X \) is numerically trivial. Since \( X \) is finite dimensional, it is trivial by Proposition 2. □

**Lemma 4.** Let \( f : T \to S \) be a finite étale cover of the base \( S \), and let \( f^* : \text{CHM}(S) \to \text{CHM}(T) \) be the pull-back on the corresponding categories of Chow motives. For any object \( M \in \text{CHM}(S) \), the Chow motive \( M \) is finite dimensional if and only if the motive \( f^*M \) is finite dimensional.

**Proof.** The detailed proof is given in [9]. □

2.3. **Spreads.** The notion of a spread is described, for instance, in [8, 4.2]. Here we recall and adapt it for our goals. Let \( \gamma : X \to S \) be a projective family over an irreducible projective base \( S \). Let \( V \) be a closed irreducible subvariety in the generic fiber \( X_\eta \) defined over the field \( k(S) \). Since the family is projective, the generic fiber can be embedded into a projective space \( \mathbb{P}^n_\eta \) as a Zariski closed subset. Then \( V \) can also be considered as a closed subset in the same \( \mathbb{P}^n_\eta \). Let

\[
F_i(x) = F_i(x_0 : \cdots : x_n) = 0
\]

be a finite collection of homogeneous equations with coefficients in \( k(S) \) defining \( V \) in \( \mathbb{P}^n_\eta \). At the same time, the variety \( S \) being projective over \( k \) can be embedded into a projective space \( \mathbb{P}^m_k \). Let \( U \) be the intersection of \( S \) with an affine chart in \( \mathbb{P}^m_k \). Being a closed irreducible subvariety in \( \mathbb{A}^m_k \), it can be defined by equations

\[
G_j(y) = G_j(y_1, \ldots, y_m) = 0
\]

with coefficients in \( k \). Any coefficient of \( F_i \) can be represented then as a fraction \( \frac{f}{g} \), where \( f \) and \( g \) are elements in the coordinate ring \( k[U] \) of the affine variety \( U \). Multiplying all the coefficients of the polynomials \( F_i \) on the product of
their denominators, we can assume that they have coefficients in $k[U]$. Let $Y = \gamma^{-1}(U)$. Any element in $k[U]$ can be represented by a polynomial in variables $y_1, \ldots, y_m$ with coefficients in $k$. Substituting polynomials instead of coefficients in $F_i(x)$ we obtain the equations

$$F_i(x, y) = F_i(x_0 : \cdots : x_n; y_1, \ldots, y_m) = 0.$$

Then define a quasi-projective variety

$$sV_U$$

in $\mathbb{P}^n_k \times \mathbb{A}^m_k$ by the system of equations

$$\begin{cases}
F_i(x; y) = 0 \\
G_j(y) = 0
\end{cases}$$

Certainly, it is a closed subvariety in $Y$, and, as such, it is a quasi-projective subvariety in $X$. This $sV$ may be called a spread of $V$ over $U$ with respect to the map $\gamma$. If we choose another affine chart $W$ in $S$, then we will construct a spread $sV_W$ over $W$. If we fix polarizations of varieties and system of defining equations, the local spreads $sV_U$, $sV_W$, etc are coherent, so that we can glue them into the global spread

$$sV$$

of $V$ w.r.t. $\gamma$. However, the construction is not defined uniquely because of a few ambiguities - in the choice of polarizations of the varieties $X$ and $S$, and the equations $F_i = 0$ and $G_j = 0$. If $X \to S$ is a family over a quasi-projective base $S$, then we can consider any closure $\bar{S}$, build a spread of $V$ with respect to the map $X \to \bar{S}$ and restrict it on the preimage of the intersection $U \cap S$ in $\bar{S}$.

By linearity, one can define now a spread of any given algebraic cycle $Z = \sum_i n_i Z_i$ on the generic fiber $X_\eta$ provided all the components $Z_i$ are rational over $k(S)$:

$$sZ = \sum_i sZ_i.$$

Due to the above ambiguities, one algebraic cycle $Z$ can have two spreads $sZ'$ and $sZ''$. However, the difference $sZ' - sZ''$ is an algebraic cycle whose projection on $S$ w.r.t. the map $\gamma$ is a proper subvariety in $S$, see [8]. For example, if $X \to S$ is a family of surfaces over a curve, then the ambiguity in spreads of a divisor on $X_\eta$ is a vertical two-dimensional cycle, i.e. it is generated by fibers of the map $\gamma$.

3. Splitting $\mathbb{1}$ and $\mathbb{L}^{\otimes e}$

Let $S$ be a smooth connected quasi-projective variety over a field $k$, let

$$\gamma : X \longrightarrow S$$

be a smooth projective family over $S$, let $d = \dim(S)$ and let $e = \dim(X/S)$ be the relative dimension of $X$ over $S$. For any point $s \in S$ let $X_s$ be the scheme-theoretical fiber of the morphism $\gamma$. Then $X_s$ is a smooth projective variety
over the residue field $k(s)$. If $\eta$ is the generic point of $S$, then $k(\eta) = k(S)$ and $X_\eta = X \times_S \eta$ is the generic fiber of the family $\gamma$.

Assume $X_\eta$ has a rational point over $\eta$. Then, locally on the base, the structural morphism $\gamma$ has a section, i.e. there exists a Zariski open subset $U \subset S$ with $Y = \gamma^{-1}(U)$, and a morphism

$$\sigma : U \longrightarrow Y,$$

such that $\gamma \circ \sigma = 1_U$. Let $\bar{E}$ be the Zariski closure of $E = \sigma(U)$ in $X$. Since the structure map $\gamma$ is proper, $\gamma(\bar{E})$ is a Zariski closed subset in $S$ containing $U$. Then $\gamma(\bar{E}) = S$ because the closure of $U$ is the whole $S$. Since $U$ is an open subset in the irreducible variety $S$, $U$ is irreducible itself. Therefore, $\bar{E}$ is also irreducible. Moreover, $\dim(\bar{E}) = \dim(E) = d$. Then $\bar{E}$ is a one-dimensional cycle of degree one over $S$, and we consider two relative projectors

$$\bar{\pi}_0 = [\bar{E} \times_S X] \quad \text{and} \quad \bar{\pi}_{2e} = [X \times_S \bar{E}].$$

If $[\bar{E}] \cdot [\bar{E}] = 0$, then $\bar{\pi}_0$ and $\bar{\pi}_2$ are pair-wise orthogonal. Assume that

$$[\bar{E}] \cdot [\bar{E}] \neq 0.$$

In that case $\bar{\pi}_0$ and $\bar{\pi}_2$ are not orthogonal, so that we cannot use them in order to split the unit and the Lefschetz motives from $M(X/S)$ simultaneously. Let

$$\zeta = \gamma_*(\bar{E} \cdot \bar{E}).$$

For any natural $n$ let

$$\gamma_n : X \times_S \cdots \times_S X \longrightarrow S,$$

where the fibered product is taken $n$-times. The pull-back

$$\theta = \gamma_2^*(\zeta)$$

is a cycle class in $CH_{d+1}(X \times_S X)$. It can be considered as a vertical correcting term for the projector $\bar{\pi}_0$ in the following sense. Set

$$\tau_0 = \bar{\pi}_0 - \theta.$$

As we will show right now, $\tau_0$ and $\bar{\pi}_{2e}$ are now pairwise orthogonal, so that we can use them for simultaneous splitting of $1$ and $\mathbb{L}^{\otimes e}$.

For any $i, j \in \{1, 2, 3\}$ let

$$p_{ij} : X \times_S \cdots \times_S X \longrightarrow X \times_S X$$

and

$$p_i : X \times_S \cdots \times_S X \longrightarrow X.$$
be the projections corresponding to their indexes. From the commutative diagram

\[
\begin{array}{ccc}
X \times_S X \times_S X & \xrightarrow{p_{ij}} & X \times_S X \\
\gamma_3 & \downarrow & \\
S & \xrightarrow{\gamma_2} & \end{array}
\]

one has

\[
p_{ij}^*(\theta) = p_{ij}^* \gamma_2^*(\zeta) = \gamma_3^*(\zeta).
\]

Therefore,

\[
\theta \circ \theta = p_{13*}(p_{12}^*(\theta) \cdot p_{23}^*(\theta)) = p_{13*} \gamma_3^*(\zeta \cdot \zeta) = p_{13*} \gamma_2^*(\zeta \cdot \zeta) = \gamma_2^*(\zeta \cdot \zeta) \cdot p_{13*}([X \times_S X \times_S X]) = \gamma_2^*(\zeta \cdot \zeta) \cdot 0 = 0.
\]

The commutative diagram

\[
\begin{array}{ccc}
X \times_S X \times_S X & \xrightarrow{p_{23}} & X \times_S X \\
\gamma_3 & \downarrow & \\
S & \xrightarrow{\gamma_2} & \end{array}
\]

gives

\[
p_{23}^*(\pi_0) = p_{23}^* p_{1*}([\bar{E}]) = p_2^*([E]).
\]

Then we compute:

\[
\pi_0 \circ \theta = p_{13*}(p_{12}^*(\theta) \cdot p_{23}^*(\pi_0)) = p_{13*} \gamma_3^*(\zeta \cdot p_2^*([E])) = p_{13*} \gamma_2^*(\zeta \cdot p_2^*([E])) = p_{13*} \gamma_1^*(\zeta \cdot [E]) = \gamma_2^*(\zeta \cdot \gamma_1^*([E])) = \gamma_2^*(\zeta \cdot [S]) = \gamma_2^*(\zeta) = \theta.
\]

The transposition of the cycle class $\theta$ is $\theta$, and the transposition of the cycle class $\pi_0$ is $\pi_{2e}$. Hence the last equality yields:

\[
\theta \circ \pi_{2e} = \theta.
\]
Since $\pi_{2e} = [X \times_S E]$, we have:

\[
\pi_{2e} \circ \theta = p_{13*}(p_{12}^*(\theta) \cdot p_{23}^*(\pi_{2e})) = p_{13*}(\gamma_2^*(\zeta) \cdot p_3^*([E])) = p_{13*}(p_3^*(\gamma_1^*(\zeta) \cdot [E]) = 0
\]

because $p_{13*}p_3^* = 0$ in general. Transposing the cycles we get:

\[
\theta \circ \pi_0 = 0.
\]

At last,

\[
\bar{\pi}_0 \circ \bar{\pi}_{2e} = p_{13*}(p_{12}^*(\bar{\pi}_{2e}) \cdot p_{23}^*(\bar{\pi}_0)) = p_{13*}(\gamma_2^*(\bar{\pi}_{2e}) \cdot p_3^*([E])) = \gamma_2^*(\bar{\pi}_{2e} \cdot [E]) = \gamma_2^*(\zeta)
\]

and

\[
\bar{\pi}_{2e} \circ \bar{\pi}_0 = p_{13*}(p_{12}^*(\bar{\pi}_0) \cdot p_{23}^*(\bar{\pi}_{2e})) = p_{13*}(\gamma_2^*(\bar{\pi}_0) \cdot p_3^*([E])) = p_{13*}(\gamma_2^*(\bar{\pi}_0) \cdot [E]) = 0.
\]

Now, using the obtained equalities, one can directly compute:

\[
\tau_0 \circ \tau_0 = (\bar{\pi}_0 - \theta) \circ (\bar{\pi}_0 - \theta)
\]

\[
= \bar{\pi}_0 \circ \bar{\pi}_0 - \bar{\pi}_0 \circ \theta - \theta \circ \bar{\pi}_0 + \theta \circ \theta
\]

\[
= \bar{\pi}_0 - \theta - 0 + 0
\]

\[
= \tau_0,
\]

\[
\tau_0 \circ \bar{\pi}_{2e} = (\bar{\pi}_0 - \theta) \circ \bar{\pi}_{2e}
\]

\[
= \bar{\pi}_0 \circ \bar{\pi}_{2e} - \theta \circ \bar{\pi}_{2e}
\]

\[
= \theta - \theta
\]

\[
= 0
\]

and

\[
\bar{\pi}_{2e} \circ \tau_0 = \bar{\pi}_{2e} \circ (\bar{\pi}_0 - \theta)
\]

\[
= \bar{\pi}_{2e} \circ \bar{\pi}_0 - \bar{\pi}_{2e} \circ \theta
\]

\[
= 0 - 0
\]

\[
= 0.
\]

So, we see that $\tau_0$ and $\bar{\pi}_{2e}$ are pair-wise orthogonal idempotents in the associative ring $\text{Corr}^0_S(X, X)$. Let

\[
\tilde{\pi} = \Delta_{X/S} - \tau_0 - \bar{\pi}_2.
\]

Since $\tau_0$ and $\bar{\pi}_2$ are pair-wise orthogonal, it follows that all the projectors $\tau_0$, $\tilde{\pi}$ and $\bar{\pi}_{2e}$ are pair-wise orthogonal. Let

\[
A = (X/S, \tau_0, 0),
\]

and

\[
\tilde{M}(X/S) = (X/S, \tilde{\pi}, 0)
\]
be two motives in $\text{CHM}(S)$. The projector $\tilde{\pi}_{2e}$ defines, of course, the $e$-th tensor power of relative Lefschetz motive $\mathbb{L}_S$. Then one has:

$$M(X/S) = A \oplus \tilde{M}(X/S) \oplus L_S^{\otimes e}.$$ 

Certainly, we can split the motive $M(X/S)$ in another way. Set

$$\tau_{2e} = \pi_{2e} - \theta,$$

$$\hat{\pi} = \Delta_{X/S} - \pi_0 - \tau_{2e}.$$

By analogous arguments, we have that $\pi_0$, $\hat{\pi}$ and $\tau_{2e}$ are pair-wise orthogonal idempotents, so that we can consider the motives

$$B = (X/S, \tau_{2e}, 0)$$

and

$$\tilde{M}(X/S) = (X/S, \hat{\pi}, 0)$$

as submotives in $M(X/S)$. Then $M(X/S)$ can be decomposed as

$$M(X) = 1_S \oplus \tilde{M}(X/S) \oplus B.$$

Finally, it is easy to show that $A$ is isomorphic to $1_S$ and $B$ is isomorphic to the $e$-th tensor power Lefschetz motive $\mathbb{L}$. Let us prove this assertion, for example, the case of the motive $B$. Since $\mathbb{L}^{\otimes e} = (S/S, \Delta_{S/S}, -e)$, we consider two correspondences

$$a := [X] \in Corr_{\tilde{S}}^e(X, S) = CH^0(X)$$

and

$$b := [\tilde{E}] - \sum_{i=1}^m n_i[V_i] \in Corr_{\tilde{S}}^e(S, X) = CH^e(X).$$

Then it is easy to compute:

$$b \circ a = p_{13*}(p_{12}^*(a) \cdot p_{23}^*(b)) = [X \times_S \tilde{E}] - \sum_{i=1}^m n_i[V_i \times_S V_i] = \tau_2$$

and

$$a \circ b = p_{13*}(p_{12}^*(b) \cdot p_{23}^*(a)) = \gamma_1*(b) = \Delta_{S/S}.$$

As a result, one has the following global decomposition

$$M(X/S) = 1_S \oplus M^\dagger(X/S) \oplus L_S^{\otimes e},$$

where $M^\dagger(X/S)$ is either $\tilde{M}$ or $\hat{M}$.

Note that in two cases the above construction gives the complete relative Murre decomposition of the motive $M(X/S)$. For example, this is so if $X/S$ is a smooth projective family of curves over $S$, or if $X/S$ is a family of surfaces over $S$, with $H^1(X_\eta) = H^2_{et}(X_\eta) = 0$. Let now $\gamma : X \to S$ be a smooth projective family of surfaces over a smooth quasi-projective curve $S$, and let

$$\Delta_\eta = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$$

be the Chow-Künneth decomposition of the diagonal for the surface $X_\eta$, see [17]. Recall that all the summands are pairwise orthogonal projectors on $X_\eta,$
and the cycle class homomorphism maps $\pi_i$ into $(i, 4 - i)$-component of the diagonal in the K"unneth decomposition. For each index $i$ let

$$s_\pi_i$$

be a spread of the projector $\pi_i$. The natural question is now as follows: is it possible to add vertical correcting terms to the spreads $s_\pi_i$ getting a nice decomposition of the diagonal $\Delta_{X/S}$. Here the word “nice” means that we wish to get a sum of pairwise orthogonal relative projectors, which could be considered as a relative Chow-K"unneth decomposition under any reasonable cohomology of $X/S$.

4. The proof of Theorem 1

Now we are ready to prove Theorem 1. Let $\gamma : X \to S$ be a smooth projective one-parameter family of surfaces over a smooth irreducible curve $S$ with $H^1(X_\eta) = H^2_{\text{tr}}(X_\eta) = 0$. We first consider the case when the generic fiber $X_\eta$ has a rational point over $k(S)$. In that case we have the projector $\tilde{\pi}$ constructed in the previous section. Let us now denote this projector as

$$\tilde{\pi}_2,$$

since it is a relative second Murre projector for the whole family $\gamma$. Certainly, this is not a unique choice of Murre’s projector, because we can also take $\hat{\pi}$, which can be denoted as

$$\hat{\pi}_2.$$

Let then $M^2(X/S)$ be the motive defined by either $\tilde{\pi}_2$ or $\hat{\pi}_2$. As we know from the previous section, one has the decomposition

$$M(X/S) = \mathbb{1}_S \oplus M^2(X/S) \oplus L_S^{\otimes 2}.$$

Assume now $M(X/S)$ is finite dimensional. It follows that $M^2(X/S)$ is finite dimensional as well. But, actually, $M^2$ is evenly finite dimensional of dimension $b_2$, where $b_2$ is the second Betti number for the generic fiber. Indeed, apriori one has

$$M^2(X/S) = K \oplus L,$$

where $\wedge^n K = 0$ and $\text{Sym}^n L = 0$. Consider the base change functor

$$\Xi : \text{CHM}(S) \to \text{CHM}(\eta).$$

The functor $\Xi$ is tensor, so that it respects finite dimensionality. In particular, the motive of the generic fiber

$$M(X_\eta) = \Xi(M(X/S))$$

is finite-dimensional because the relative motive $M(X/S)$ is so. In addition, $\Xi$ induces an isomorphism between $\text{End}(\mathbb{1}_S) = \mathbb{Q}$ and $\text{End}(\mathbb{1}_\eta) = \mathbb{Q}$. Since

$$\Xi(M^2(X/S)) = M^2(X_\eta)$$

it follows that

$$\Xi(L) = 0$$
because $M^2(X_\eta)$ is evenly finite dimensional. Then

$$L = 0$$

by Lemma [3]. Thus, $M^2(X/S)$ is evenly finite dimensional. Using the same arguments one can also show that it can be annihilated by $\wedge^{b_2+1}$.

Let $b_2 = \dim H^2(X_\eta)$ and let $D_1, \ldots, D_{b_2}$ be divisors on $X_\eta$ generating $H^2(X_\eta)$. Since the motive $M(X_\eta)$ is finite dimensional, and $X_\eta$ is a surface with $H^1(X_\eta) = H^2_0(X_\eta) = 0$, the second piece $M^2(X_\eta)$ in the Murre decomposition of $M(X_\eta)$ can be computed as follows:

$$M^2(X_\eta) = \mathbb{L}^\oplus b_2,$$

see [10, Theorem 2.14]. Actually, these $b_2$ copies of $\mathbb{L}_\eta$ are arising from the collection of divisor classes $[D_1], \ldots, [D_{b_2}]$ on $X_\eta$, and their Poincaré dual $[D'_1], \ldots, [D'_{b_2}]$.

loc.cit. In other words, if $\nu_2 = \Xi(\tilde{\pi}_2) = (\tilde{\pi}_2)_\eta$ is a projector determining the middle motive $M^2(X_\eta)$, the difference

$$\xi_\eta = \nu_2 - \sum_{i=1}^{b_2} [D_i \times_\eta D'_i]$$

is homologically trivial. Then

$$\xi^n_\eta = 0$$

in the associative ring $\text{End}(M^2_\eta)$ for some $n$ by Kimura’s nilpotency theorem.

Now let

$$W_i = sD_i$$

and

$$W'_i = sD'_i$$

be spreads of the above divisors $D_i$ and $D'_i$ over $S$. They are defined not uniquely, of course. The cycles $W_i \times_S W'_i$ are in $\text{Corr}_S^0(X \times_S X)$, and we set

$$\xi = \tilde{\pi}_2 - \sum_{i=1}^{b_2} [W_i \times_S W'_i].$$

Let $\omega$ be any endomorphism of the motive $M^2(X/S)$.

$$\Xi(\text{tr}(\omega \circ \xi)) = \text{tr}(\Xi(\omega \circ \xi)) = \text{tr}(\omega_\eta \circ \xi_\eta) = 0$$

because $\xi_\eta$ is homologically trivial (here we use the formula on page 116 in [5] again). Since the functor $\Xi$ induces an isomorphism $\text{End}(\mathbb{1}_S) \cong \text{End}(\mathbb{1}_\eta)$,

$$\text{tr}(\omega \circ \xi) = 0$$

for any $\omega$, i.e. $\xi$ is numerically trivial. Therefore,

$$\xi^n = 0$$
in \( \text{End}(M^2(X/S)) \) by Proposition \( \mathbb{P} \).

For any cycle class \( z \in CH^i(X) \) and for any correspondence \( c \in \text{Corr}_S^j(X, X) \) let, as usual,

\[
c_*(z) = p_{2*}(p_1^*(z) \cdot c) \in CH^{i+j}(X)
\]

be the action of the correspondence \( c \) on \( z \). In particular, if \( z \in CH^2(X) \), one has a decomposition

\[
z = \Delta_{X/S}^*(z) = (\bar{\pi}_0)_*^*(z) + (\pi_2)_*^*(z) + (\tau_4)_*^*(z)
\]

in \( CH^2(X) \). Let us compute each term separately:

\[
(\bar{\pi}_0)_*^*(z) = p_{2*}(p_1^*(z) \cdot \bar{\pi}_0)
= p_{2*}(p_1^*(z) \cdot p_1^*([\bar{E}]))
= p_{2*}p_1^*(z \cdot [\bar{E}])
= \gamma^*\gamma^*_{\ast}(z \cdot [\bar{E}]),
\]

\[
(\bar{\pi}_4)_*^*(z) = p_{2*}(p_1^*(z) \cdot \bar{\pi}_4)
= p_{2*}((z \times_S [X]) \cdot ([X] \times_S [\bar{E}]))
= p_{2*}(z \times_S [\bar{E}])
= \gamma_\ast(z) \times_S [\bar{E}]
\]

and

\[
\theta_\ast(z) = p_{2*}(p_1^*(z) \cdot \theta)
= p_{2*}(p_1^*(z) \cdot \gamma_\ast^* \zeta)
= p_{2*}(p_1^*(z) \cdot p_1^*\gamma_\ast^* \zeta)
= p_{2*}p_1^*(z \cdot \gamma_\ast_{\ast} \zeta)
= \gamma^*\gamma^*_{\ast}(z \cdot \gamma^* \zeta)
= \gamma^*\gamma^*_{\ast}(z \cdot \zeta).
\]

Now assume that \( z \in CH^2(X)_0 \), i.e. \( \gamma_\ast(z) = 0 \). In that case:

\[
(\bar{\pi}_4)_*^*(z) = 0
\]

and

\[
\theta_\ast(z) = 0.
\]

In addition, \( z \cdot [\bar{E}] = 0 \) because both cycle classes have codimension two in a smooth threefold, so that \( (\bar{\pi}_0)_*^*(z) = 0 \) as well. Then,

\[
z = (\pi_2)_*^*(z).
\]

On the other hand, we know that

\[
\bar{\pi}_2 = \sum_{i=1}^{b_2} [W_i \times_S W'_i] + \xi,
\]

whence we get:

\[
(\bar{\pi}_2)_*^*(z) = \sum_{i=1}^{b_2} ([W_i \times_S W'_i]_*^*(z)) + \xi_\ast(z).
\]
Let
\[ v_1 = -\sum_{i=1}^{b_2} ([W_i \times_S W'_i]_s(z)) \]
and write \( z \) as
\[ z = \left[ \sum_j n_j Z_j \right], \]
where the cycles \( Z_j \) are irreducible curves on \( X \). For any \( i \) and \( j \) one has:
\[ [W_i \times_S W'_i]_s[Z_j] = p_2_*([Z_j \times_S X] \cdot [W_i \times_S W'_i]). \]
Since \( Z_j \) is of codimension two and \( W_i \) is of codimension one in the threefold \( X \), it follows that a most intersection of \( Z_j \times_S X \) with \( W_i \times_S W'_i \) can be represented by a sum of algebraic cycles of type
\[ p \times_{k(q)} (W'_i \cdot X_q), \]
where \( q \) is a closed point on \( S \), \( X_q \) is a cycle-theoretic fiber of the morphism \( \gamma \) at \( q \), and \( p \) is a zero-dimensional point in the fiber \( X_q \). Hence, \( [W_i \times_S W'_i]_s[Z_j] \) is a linear combination of vertical cycles of type
\[ W'_i \cdot X_q. \]
But then so is \( v_1 \). In particular, \( v_1 \) is a vertical cycle class and
\[ \xi_* (z) = z + v_1. \]
Applying \( \xi \) once again we see that
\[ \xi_*^2(z) = \xi_* (z + v_1) = z + v_1 + \xi_* (v_1). \]
Now we need to show that \( \xi_* (v_1) \) can be also represented by a linear combination of cycles \( W'_i \cdot X_q \). By construction,
\[ \xi = \Delta_{X/S} - \tilde{\pi}_0 + \theta - \tilde{\pi}_2 - \sum_i [W_i \times_S W'_i]. \]
The one-dimensional cycle class \( (\tilde{\pi}_0)_*[W'_i \cdot X_q] \) is represented by the pushforward of the algebraic cycle
\[ (\bar{E} \cdot W'_i \cdot X_q) \times_{k(q)} X_q \]
on \( X \times_S X \) with respect to the projection \( p_2 : X \times_S X \to X \). But the intersection
\[ \bar{E} \cdot W'_i \cdot X_q \]
is zero because of codimensional reasons, whence
\[ (\tilde{\pi}_0)_*[W'_i \cdot X_q] = 0. \]
Also it is easy to show that
\[ (\tilde{\pi}_2)_*[W'_i \cdot X_q] = 0 \]
and
\[ \theta_*[W'_i \cdot X_q] = 0. \]
The diagonal does not change anything. Finally, since we have seen already that for any codimension two subvariety $Z$ in $X$ the cycle class

$$[W_i \times_S W'_i]_* [Z]$$

is a linear combination of vertical cycles of type $W'_i \cdot X_q$, it is so for $W'_i \cdot X_q$ itself. Thus, the cycle class $\xi(v_1)$ can be represented by a linear combination of cycles $W'_i \cdot X_q$. In particular, $\xi(v_1)$ is vertical again.

Applying $\xi$ once again we see that

$$\xi^2(z) = z + v_2,$$

where

$$v_2 = v_1 + \xi(v_1)$$

is a vertical class. And so forth. After $n$ steps we will get:

$$\xi^n(z) = z + v_n,$$

where $v_n$ is represented by a linear combination of the cycles $W'_i \cdot X_q$, and, as such, it is a vertical class cycle again. But we know that $\xi$ is a nilpotent correspondence in $Corr^0_S(X, X)$, so that

$$\xi^n(z) = 0$$

for big enough $n$. It follows that

$$z = -v_n$$

is a vertical cycle class.

If the generic fiber $X_\eta$ has no rational points over $k(S)$, we consider a finite normal extension $L$ of $k(S)$, such that $X_\eta(L) \neq \emptyset$. Let $b : S' \rightarrow S$ be a finite étale cover of the curve $S$ with $k(S') = L$, and let

$$\begin{array}{ccc}
X' & \xrightarrow{\gamma'} & S' \\
\downarrow a & & \downarrow b \\
X & \xrightarrow{\gamma} & S
\end{array}$$

be a Cartesian square. Since $b$ is flat and $\gamma$ is proper, one has $b^* \gamma_* = \gamma'_* a^*$ (see [6, 1.7]). Then we have the following commutative diagram

$$\begin{array}{c}
CH^2(X'_0) \xrightarrow{a_*} CH^2(X') \xrightarrow{\gamma'_*} CH^0(S') \\
\downarrow a^* \quad \downarrow a^* \quad \downarrow b^* \\
CH^2(X_0) \xrightarrow{a_*} CH^2(X) \xrightarrow{\gamma_*} CH^0(S)
\end{array}$$

By Lemma 4, the motive $M(X'/S')$ is finite dimensional because $M(X/S)$ is so. Since the generic fiber of the family $\gamma'$ has a rational point over $k(S')$,
we can apply the above arguments to show that any cycle class in $CH^2(X')_0$ is vertical. The morphism $a : X' \to X$ is a finite étale cover, whence the composition $a_* a^*$, considered on the group $CH^2(X)_0$, is just the multiplication by $\deg(X'/X)$. Then it is easy to see that any cycle class in $CH^2(X)_0$ is vertical as well. So, the first part of Theorem 1 is done.

As to the second part, assume that $S = \mathbb{P}^1$, whence $\text{Pic}^0(S) = 0$, i.e. any two closed points $q$ and $q'$ are proportional modulo rational equivalence on $\mathbb{P}^1$. It follows that the corresponding fibers $X_q$ and $X_{q'}$ are proportional as divisors in $X$. Assume $X_\eta(k(S)) \neq 0$. Then all the above vertical cycles are linear combination of cycles $W_i q$ for some fixed closed point $q$ on $\mathbb{P}^1$. Therefore, $CH^2(X)_0 = \mathbb{Q}^n$ and $n \leq b_2$, where $b_2$ is the second Betti number of the generic fiber.

If $X_\eta$ has no $k(\mathbb{P}^1)$-rational points, then we consider a finite étale covering $S' \to S = \mathbb{P}^1$, such that, if $\eta' = \text{Spec}(k(S'))$, the generic fiber $X_{\eta'}$ has a rational point over $k(S')$. Note that any finite étale covering can be dominated by Galois one. Therefore we may even assume that $S'/S$ is Galois. Then $X'/X$ is Galois as well.

Since $k(S')$ is a finite extension of $k(S)$, we can take an algebraic closure $\overline{k(S)}$ of $k(S)$ containing $k(S')$, whence $\overline{\eta} = \overline{\eta'}$. Fix an embedding of $k(S')$ into the field $\mathbb{C}$. Without loss of generality, we may also assume that the algebraic closure $\overline{k(S)} = \overline{k(S')} = \overline{k(S')}$. Then we consider also the surface $X_\eta \otimes \mathbb{C}$ over $\mathbb{C}$ and the corresponding commutative diagram:

\[
\begin{array}{ccc}
CH^1(X_\eta \otimes \mathbb{C}) & \xrightarrow{cl_C} & H^2(X_\eta \otimes \mathbb{C}) \\
| & & |
\downarrow cl' & \downarrow cl & \downarrow cl'
\end{array}
\]

Since $cl$ is a surjection, $cl'$ is a surjection. But then $cl_C$ is a surjection as well. Since $H^1(X_\eta) = H^1(X_{\eta'}) = 0$, it follows that the Albanese variety is trivial for $X_\eta$. Then it is trivial also for the complex surface $X_\eta \otimes \mathbb{C}$. Thus, we see that $X_\eta \otimes \mathbb{C}$ is a complex surface with $p_g = q = 0$. In that case, using the $(1,1)$-Lefschetz theorem and the exponential sequence, one can see that $cl_C$ is an isomorphism. Then the cycle class maps $cl$ and $cl'$ are injections. Since
they are also surjective, we claim that cl and cl' are isomorphisms. Then the pull-back $a^*$ is an isomorphism as well.

In other words, we may use the divisors $a^*D_1, \ldots, a^*D_{b_2}$ as an algebraic basis for $H^2(X'_{\eta'})$, where $D_i'$s are the above algebraic basis for $H^2(X_{\eta})$. Consider the spreads $W_i$ of $D_i$. It is easy also to see that $a^*W_i$ is a spread of $a^*D_i$ for all $i$.

Let $z$ be any algebraic cycle in $CH^2(X_0)$. By the proven part of Theorem 1 we have that $a^*(z)$ can be represented by a linear combination
\[ \sum_i m_i(a^*W_i \cdot X'_{p_i}) \]
of cycles $a^*W_i \cdot X'_{p_i}$ where $p_i$ is a closed point on $S'$ and $X'_{p_i}$ is a cycle-theoretic preimage of $p_i$. By the projection formula:
\[ a^*a^*(z) = \sum_i m_i a^*(a^*W_i \cdot X'_{p_i}) = \sum_i m_i a^*(W_i \cdot a^*(X'_{p_i})) = \sum_i m_i(W_i \cdot a^*(X'_{p_i})). \]
Since $a^*$ is injective, it follows that
\[ z = \frac{1}{d} \cdot a^*a^*(z) = \frac{1}{d} \cdot \sum_i m_i(W_i \cdot a^*(X'_{p_i})). \]
But $a^*(X'_{p_i})$ is the cycle-theoretic fiber $X_{q_i}$ of the map $\gamma$ over the image $q_i$ of the point $p_i$ under the map $S' \to S$ by [6, 1.7].

So, we see that, again, any cycle class in $CH^2(X_0)$ can be represented by a linear combination of cycles of type $W_i \cdot X_q$ where $X_q$ is a cycle-theoretic fiber over a closed point $q \in S$. Since $S = \mathbb{P}^1$, any two fibers $X_q$ and $X_{q'}$ are proportional divisors on $X$, whence we can use only one fiber. This finishes the proof of Theorem 1.

**An application of Theorem 1** Note that, joint with the localization sequence for Chow groups, Theorem 1 allows also to compute the second Chow group in a more general situation. Indeed, let $X$ be a smooth quasi-projective threefold defined over a field $k$, $char(k) = 0$, and assume we are given with a regular projective flat and dominant map $\gamma : X \to S$ onto a smooth connected quasi-projective curve $S$. We can take, for example, a Lefschetz pencil
\[ \gamma : X' \to \mathbb{P}^1 \]
under some embedding $X \subset \mathbb{P}^n$. To compute $CH^2(X')$ is the same as to compute $CH^2(X)$ because $X'$ is just a blow up of $X$. Now the morphism $\gamma$ is not necessarily smooth. Suppose, moreover, that $H^1(X_{\eta}) = H^2_{tr}(X_{\eta}) = 0$ and that the motive of the generic fiber $M(X_{\eta})$ is finite dimensional. The category of Chow motives over the generic fiber can be considered as a colimit of categories of Chow motives over Zariski open subsets $U$ in the base curve $S$:
\[ CHM(\eta) = \text{colim}_{U \subset S} \text{CHM}(U), \]
where the canonical morphisms $\text{CHM}(U) \to \text{CHM}(\eta)$ are pull-backs with respect to the morphisms $\eta \to U$. Since the motive $M(X_\eta)$ is finite dimensional, there exists a Zariski subset $U$, such that, if
$$Y = \gamma^{-1}(U),$$
the restriction
$$\gamma|_Y : Y \to U$$
is a smooth projective family and the motive $M(Y/U)$ is finite dimensional. In other words, we “spread out” the finite dimensionality of the motive $M(X_\eta)$ over a Zariski open subset in $S$. By Theorem \[\text{H}\] the Chow group $\text{CH}^2(Y)$ is generated by a multisection and vertical cycles. Let
$$Z = X' - Y,$$
and let
$$i : Y \hookrightarrow X' \quad \text{and} \quad j : Z \hookrightarrow X'$$
be, respectively, the open and closed embeddings. Using the exact localization sequence
$$\text{CH}^1(Z) \xrightarrow{j^*} \text{CH}^2(X') \xrightarrow{i^*} \text{CH}^2(Y) \to 0$$
it is easy now to show that any codimension two cycle class $z$ on $X'$ is represented by a linear combination of a multisection, vertical cycles of type $W_i \cdot F$, where $F$ is a smooth fiber over a closed point on $S$, and vertical one-dimensional cycles lying in singular fibers of the map $\gamma$. In particular, if $S = \mathbb{P}^1$ and the singular fibers have finitely generated groups of divisors, then $\text{CH}^2(X')$ is finitely generated.

5. Families of curves.

The relative middle projector for a smooth projective curve over a quasi-projective base can be obtained by splitting the relative unit and Lefschetz motives simultaneously using vertical correcting terms $\theta$, see Section \[\text{I}\]. In addition, these $\theta$ gives the equality $\bar{\pi}_0 \cdot \bar{\pi}_1 = 0$. These two things allow to generalize Kimura’s theorem, [16, 4.4], to a smooth projective curve over an arbitrary smooth quasi-projective base:

**Proposition 5.** Let $S$ be a smooth quasi-projective variety over a field, let $X \to S$ be a smooth projective family of curves over $S$, and let $g$ be the genus of its generic fiber. Then the motive $M(X/S)$ is finite dimensional. To be more precise, it splits in $\text{CHM}(S)$ as usual: $M(X/S) = 1_S \oplus M^1(X/S) \oplus \mathbb{L}_S,$ and
$$\text{Sym}^{2g+1} M^1(X/S) = 0.$$

**Proof.** Let $X/S$ be a smooth projective family of relative dimension one. Assume first that the structure morphism $\gamma : X \to S$ has a section $\sigma : S \to X$. For any natural $n$ let
$$h : \text{Sym}^{n-1}(X/S) \times_S X \to \text{Sym}^n(X/S)$$
be the evident map\(^1\), and let
\[ i : \text{Sym}^{n-1}(X/S) \longrightarrow \text{Sym}^{n-1}(X/S) \times S X \]
be the embedding induced by the section \( \sigma \). Then let \( M_n \) be the codimension one subvariety in \( \text{Sym}^n(X/S) \), which is the image of the composition \( h \circ i \). Let also
\[ M_n \]
be the invertible sheaf corresponding to the divisor \( M_n \) respectively.

**Lemma 6.** Under the above assumptions, there exists a locally free sheaf \( \mathcal{E} \) on the relative Jacobian
\[ J = \text{Pic}_X^0 \]
of \( X/S \), and an isomorphism
\[ r_n : \text{Sym}^n(X/S) \longrightarrow \mathbb{P}(\mathcal{E}) , \]
such that the diagram
\[ \text{Sym}^n(X/S) \longrightarrow \mathbb{P}(\mathcal{E}) \]
\[ \downarrow \]
\[ \text{Sym}^n(X/S) \times S X \]
\[ \downarrow \]
\[ J \]
commutes and
\[ r_n^* \mathcal{O}(1) = M_n , \]
where all three maps in the above commutative diagram are over \( S \).

**Proof.** Follow [18] in the relative setting. \( \square \)

If the generic fiber \( X_\eta \) has no \( k(S) \)-rational points, then, as in the proof of Theorem [1] we take a finite normal extension \( L \) of the field \( k(S) \) with \( X_\eta(L) \neq \emptyset \), and apply Lemma [3]. In other words, without loss of generality, we may assume from the beginning that \( X_\eta \) has a point rational over \( k(S) \). Then, locally on the base, Proposition [5] is a straightforward generalization of results from [16]. We first recall the local situation and then prove the theorem globally.

Since \( X_\eta(k(S)) \neq \emptyset \), there exists a Zariski open subset \( U \in S \), such that \( \gamma : Y \rightarrow U \) has a section \( \sigma \), where \( Y = \gamma^{-1}(U) \) (see Section [3]). We can make \( U \) smaller, so that the self-intersection \( [E] \cdot [E] \) is zero in the Chow group \( CH_d(Y \times_U Y) \), where \( E = \sigma(S) \). In that case the correspondences \( \pi_0 = [E \times_U Y] \) and \( \pi_2 = [Y \times_U E] \) are pair-wise orthogonal, so that we can directly define the middle projector for \( Y/U \) by the formula
\[ \pi_1 = \Delta_{Y/U} - \pi_0 - \pi_2 . \]

\(^1\) all symmetric products here and below are over the base \( S \)
Then we get the decomposition
\[
M(Y/U) = \mathbb{1}_U \oplus M^1(Y/U) \oplus \mathbb{1}_U,
\]
where
\[
M^1(Y/U) = (Y/U, \pi_1, 0).
\]
The morphism \( Y \to U \) is smooth projective with a section. By Lemma 6 there exists a locally free sheaf \( \mathcal{E} \) on the Jacobian \( J \) of \( Y/U \), such that \( \text{Sym}^n(Y/U) \) is isomorphic to \( \mathbb{P}(\mathcal{E}) \) provided \( n > 2g - 2 \). And, moreover, the pull-back of the corresponding bundle \( \mathcal{O}(1) \) to \( \text{Sym}^n(Y/U) \) coincides with the class of the relative divisor \( \mathcal{M}_n \). The equality \( [E] \cdot [E] = 0 \) gives
\[
\pi_0 \cdot \pi_0 = 0,
\]
whence
\[
\pi_1 \cdot \pi_0 = 0.
\]
Therefore, one can directly generalize Kimura’s arguments from [16, §4] to show that
\[
\text{Sym}^{2g+1}M^1(Y/U) = 0.
\]

Now we want to prove the theorem globally on the base. We consider the projectors
\[
\overline{\pi}_0 = [\overline{E} \times_S X] \quad \text{and} \quad \overline{\pi}_2 = [X \times_S \overline{E}],
\]
introduce the vertical correcting term \( \theta \), and take the correspondences \( \tau_0 \) and \( \tau_2 \) from Section 3. Then we can define
\[
\overline{\pi}_1 = \Delta_{X/S} - \tau_0 - \pi_2
\]
in order to get the splitting with
\[
M^1(X/S) = (X/S, \overline{\pi}_1, 0).
\]
(see Section 3). We need to show that \( M^1(X/S) \) is finite dimensional. Let again \( g \) be the genus of the generic fiber of the structural morphism \( \gamma : X \to S \). For any natural number \( n \) let \( (X/S)^n_1 \) be the fibered product
\[
X \times_S \cdots \times_S \overline{E} \times_S \cdots \times_S X,
\]
where \( \overline{E} \) is located on the \( i \)-th place. Since \( X/S \) has a relative divisor of degree one over \( S \) (subvariety \( \overline{E} \)), we can again apply Kimura’s arguments to the family \( X/S \). Therefore, in order to prove that \( \text{Sym}^{2g+1}M^1(X/S) \) vanishes we have only to show that the intersection of the cycle
\[
[(X/S)^{2g+1}_1 \times_S (X/S)^{2g+1}]
\]
with
\[
\overline{\pi}_1^{(2g+1)}
\]
is equal to zero. But
\[
\overline{\pi}_0 \cdot \overline{\pi}_1 = \overline{\pi}_0 \cdot \Delta_{X/S} - \overline{\pi}_0 \cdot \tau_0 - \overline{\pi}_0 \cdot \overline{\pi}_2
\]
\[
= [\overline{E} \times_S \overline{E}] - \overline{\pi}_0 \cdot (\overline{\pi}_0 - \theta) - [\overline{E} \times_S \overline{E}]
\]
\[
= \overline{\pi}_0 \cdot \overline{\pi}_0 - \overline{\pi}_0 \cdot \theta.
\]
On the other hand,
\[
\bar{\pi}_0 \cdot \theta = p^*_1(\bar{E}) \cdot \gamma^*_1 \gamma_1^*([\bar{E}] \cdot [\bar{E}]) \\
= p^*_1(\bar{E}) \cdot p^*_1 \gamma^*_1 \gamma_1^*([\bar{E}] \cdot [\bar{E}]) \\
= p^*_1([\bar{E}] \cdot \gamma^*_1 \gamma_1^*([\bar{E}] \cdot [\bar{E}])) \\
= p^*_1([\bar{E}] \cdot \gamma^*_1 \gamma_1^*([\bar{E}] \cdot [\bar{E}])) \\
= p^*_1([\bar{E}]) \cdot p^*_1 p^*_2([\bar{E}] \cdot [\bar{E}])) \\
= p^*_1([\bar{E}]) \cdot p^*_1 p^*_2([\bar{E}]) \\
= \bar{\pi}_0 \cdot \bar{\pi}_0.
\]
Therefore,
\[
\bar{\pi}_0 \cdot \bar{\pi}_1 = \bar{\pi}_0 \cdot \bar{\pi}_0 - \bar{\pi}_0 \cdot \theta \\
= \bar{\pi}_0 \cdot \bar{\pi}_0 - \bar{\pi}_0 \cdot \bar{\pi}_0 \\
= 0.
\]
Then,
\[
[(X/S)^{2g+1} \times S (X/S)^{2g+1}] \cdot \tilde{\pi}_1^{(2g+1)} = 0
\]
because \(\bar{\pi}_0 \cdot \bar{\pi}_1 = 0\).

Certainly, we can split the motive \(M(X/S)\) in another way. Set
\[
\tilde{\pi}_1 = \Delta_{X/S} - \bar{\pi}_0 - \tau_2.
\]
and
\[
M^1(X/S) = (X/S, \tilde{\pi}_1, 0).
\]
In that case the proof is analogous.

Of course, Proposition 5 has the following standard corollaries. Let
\[
\text{CHM}(S)
\]
be the full tensor pseudoabelian subcategory in \(\text{CHM}(S)\) generated by motives of relative curves. Then, by Proposition 5 and using abstract properties of finite dimensional objects, we obtain that all objects in \(\text{CHM}(S)\) are finite dimensional. Consequently, if \(M \in \text{CHM}(S)\) and \(f : M \rightarrow M\) is a numerically trivial endomorphism of the motive \(M\), then \(f\) is nilpotent in the associative ring \(\text{End}(M)\) by Proposition 2. The last assertion can help to detect algebraic cycles in families made by relative curves.

6. Examples

Let us now indicate where threefolds satisfying the assumptions of the above theorem are arising from.

Let \(Y\) be a smooth projective complex surface with \(p_g = q = 0\). Actually, \(Y\) is defined over the minimal extension \(K_0\) of the prime field \(\mathbb{Q}\), such that all the coefficients of equations defining \(Y\) are in \(K_0\). Clearly, \(K_0\) is finitely generated over \(\mathbb{Q}\). Since \(q = 0\) for \(Y/\mathbb{C}\), the surface \(Y\) has trivial irregularity also over \(K_0\), whence \(H^1(Y) = 0\) for \(Y\) considered over \(K_0\). On the other hand, it might be possible that there are no a system of divisors on \(Y/K_0\) generating \(H^2(Y)\). Then let \(K\) be any finitely generated extension of \(K_0\), such that the variety \(Y/K\) satisfies the condition \(H^2_{K_0}(Y) = 0\) over \(K\).
The field $K$ can be viewed as the field of rational functions over an irreducible quasi-projective variety defined over $\mathbb{Q}$. If we take $K$ so that its transcendence degree over $\mathbb{Q}$ is positive, then we can also consider $K$ as the function field $k(S')$ of an irreducible quasi-projective curve $S'$ defined over some extension $k$ of $\mathbb{Q}$. This $k$ can have a non-trivial transcendental degree over $\mathbb{Q}$, of course. Now we spread out $Y/k(S')$ getting a family of surfaces $\gamma : X \to S$, where $S$ is a Zariski open subset in the curve $S'$. The generic fiber $X_\eta$ of the morphism $\gamma$ is then our surface $Y$ considered over $\eta = \text{Spec}(k(S))$, where $k(S) = K$.

Now assume Bloch’s conjecture holds for $Y/\mathbb{C}$. It should be noted that, by now, we know only four types of surfaces with $p_g = q = 0$ and known Bloch’s conjecture:

(i) Enriques surfaces, [4];
(ii) the classical Godeaux surface, [12];
(iii) some surfaces originated by modular groups, [3], and
(iv) finite quotients of intersections of four quadrics in $\mathbb{P}^6$, [19].

Since Bloch’s conjecture holds for $Y/\mathbb{C}$, the motive $M(Y/\mathbb{C})$ is finite dimensional, [11]. Finite dimensionality of a Chow motive is, actually, a rational triviality of its determining projector, and the last thing can be viewed as an existence of an algebraic cycle on a variety. Therefore, we can say that $M(Y)$ is finite dimensional over some finitely generated extension of the primary field $\mathbb{Q}$ over which a trivializing algebraic cycle is rational. Extending $K$ more, if necessary, we can assume that the motive $M(Y/k(S))$ is finite dimensional. But $M(Y/k(S))$ is nothing else than the motive $M(X_\eta)$. Spreading out finite dimensionality of the motive $M(X_\eta)$ over some Zariski open subset in $S$ we get a family whose relative motive is finite dimensional.

So, we see that any smooth projective surface $X/\mathbb{C}$ with $p_g = q = 0$ and finite dimensional $M(X)$ naturally gives rise to a family of surfaces satisfying the assumptions of Theorem 1 by means of the above spreading construction. In particular, we can now describe codimension 2 algebraic cycles on threefolds arising from spreadings of surfaces of the above four types (i)-(iv).

Another one way to introduce such families is to consider Lefschetz pencils of appropriate threefolds in a projective space.

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