Circles-in-the-sky searches and observable cosmic topology in a flat Universe

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In a Universe with a detectable nontrivial spatial topology the last scattering surface contains pairs of matching circles with the same distribution of temperature fluctuations — the so-called circles-in-the-sky. Searches for nearly antipodal circles-in-the-sky in maps of cosmic microwave background radiation have so far been unsuccessful. This negative outcome along with recent theoretical results concerning the detectability of nearly flat compact topologies is sufficient to exclude a detectable nontrivial topology for most observers in very nearly flat positively and negatively curved Universes, whose total matter-energy density satisfies $0 < \Omega_{\text{tot}} - 1 \lesssim 10^{-5}$. Here we investigate the consequences of these searches for observable nontrivial topologies if the Universe turns out to be exactly flat ($\Omega_{\text{tot}} = 1$). We demonstrate that in this case the conclusions deduced from such searches can be radically different. We show that, although there is no characteristic topological scale in the flat manifolds, for all multiply-connected orientable flat manifolds it is possible to directly study the action of the holonomies in order to obtain a general upper bound on the angle that characterizes the deviation from antipodicity of pairs of matching circles associated with the shortest closed geodesic. This bound is valid for all observers and all possible values of the compactification length parameters. We also show that in a flat Universe there are observers for whom the circles-in-the-sky searches already undertaken are insufficient to exclude the possibility of a detectable nontrivial spatial topology. It is remarkable how such small variations in the spatial curvature of the Universe, which are effectively indistinguishable geometrically, can have such a drastic effect on the detectability of cosmic topology. Another important outcome of our results is that they offer a framework with which to make statistical inferences from future circles-in-the-sky searches on whether the Universe is exactly flat.

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I. INTRODUCTION

Some fundamental open questions concerning the nature of our Universe are whether the Universe is spatially finite and what its shape and size are (see, e.g., the reviews [1]). An important point regarding these questions is that the spatial geometry constrains but does not determine the topology of the spatial sections $M$. As a result, general relativity, as well as any local metrical theory of gravitation, cannot determine the topology of the Universe, which can in principle be found through observations.1 A promising observational approach in the search for possible evidence of a nontrivial cosmic topology is based on searches for specific pattern repetitions in cosmic microwave background (CMB) temperature fluctuations — the so-called circles-in-the-sky [2–5] (see also the related Refs. [6]).

The CMB data have become available through many experiments including the ongoing Wilkinson Microwave Anisotropy Probe (WMAP) [7]. The forthcoming data from CMB mission Planck [8], which will be available in the near future, will combine all-sky high angular resolution and sensitivity with a wide frequency coverage, and will certainly be a powerful data set to be used in the search for a possible nontrivial topology of the Universe. These accumulation of high precision CMB data has at the same time provided strong support for the inflationary scenario, and the near flatness of the Universe.

A fundamental assumption in standard relativistic cosmological modelling is that the spacetime on large scales is well described by a 4-manifold $M = \mathbb{R} \times M$ endowed with the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -c^2dt^2 + a^2(t) \left[ d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where $t$ is the cosmic time, $a(t)$ is the scale factor and $f(\chi) = (\chi, \sin \chi, \sinh \chi)$ depending on the sign of the constant spatial curvature $k = (0, 1, -1)$. Furthermore, in the standard approach to modelling the Universe the spatial sections $M$ are often assumed to be the simply-connected 3-manifolds: Euclidean $\mathbb{E}^3$, spherical $S^3$, or hyperbolic space $\mathbb{H}^3$. These choices are, however, not

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1 In this work, in line with the usage in the literature, by topology of the Universe we mean the topology of its three dimensional spatial sections $M$.
unique, and depending on the sign of the spatial curvature, the 3-space \(M\) can be one of possible topologically

- distinct 3–manifolds, which are quotient spaces of the corresponding simply-connected constant curvature covering manifolds \(E^3\), \(S^3\) and \(H^3\), by the group of isometries (the so-called holonomy group) \(\Gamma\) that define the set of closed geodesics in each case. The presence of closed geodesics in these quotient manifolds leads to the existence of multiple images or pattern repetitions of radiating sources. By observing these images or pattern repetitions one can in principle directly obtain the elements of the holonomy group and hence deduce the topology of the Universe.

Currently, the most promising method for searching for an observable non-trivial topology is based on the existence of pattern repetitions in the CMB anisotropies on the last scattering surface (LSS). In a Universe with a detectable non-trivial topology \([9]\) the LSS intersects some of its images along the so-called circles-in-the-sky, which are pairs of matching circles with the same distribution of temperature fluctuations, identified by \(\Gamma\), i.e. with the same distribution of temperature fluctuations (up to a phase). Recent searches restricted to antipodal (back to back) or nearly antipodal circles have been undertaken without success \([3, 4]\). Parallel to this it was proven in Ref. \([10]\) that in the case of very nearly, but not exactly, flat manifolds (i.e., those compatible with typical inflationary models), a generic compact manifold is ‘locally’ well approximated by either a slab space \((R^2 \times S^1)\) or chimney space \((R \times T^2)\) manifold, irrespective of its global topology. This in turn allows an upper bound to be placed on the angle corresponding to the deviation from antipodicity in the inflationary limit, which turns out to be less than 10 degrees for the majority of observers \([11]\). This result, coupled with the aforementioned searches, which covered deviations from antipodicity of up to 10 degrees, would in principle be sufficient to exclude a detectable manifold with non-trivial topology for the overwhelming majority of observers in a very nearly flat Universe.\(^2\)

An important remaining question is how the above results would be modified if the Universe turns out to be exactly flat. Apart from being allowed by the observations, this limiting case \((\Omega_{tot} = 1)\) is also significant, since it is compatible with some inflationary scenarios. Indeed, it has been argued that flat compact manifold should be regarded as typical for generic inflationary scenarios (either along with hyperbolic compact manifolds \([12]\) or exclusively \([11]\)).

Here, to answer this question we investigate what constraints the above mentioned existing searches would impose on the detectability of topology in a flat Universe in which the spatial section \(M\) is an Euclidean space endowed with any orientable nontrivial topology.

In this paper, by considering all possible orientable multiply connected flat 3–manifolds, we show that the deviation from antipodicity of the pair of circles-in-the-sky can be, for some observers, larger than those expected in the case of nearly flat Universes. This therefore implies that, for all globally inhomogeneous flat manifolds with a nontrivial topology, there remains a substantial fraction of observers for which the searches for the circles-in-the-sky so far undertaken are not enough to exclude the possibility of a detectable non-trivial flat topology.

The structure of the paper is as follows. In Section \(\text{II}\) we present a detailed study of the circles-in-the-sky in all multiply connected flat orientable 3–manifolds with a detectable topology, and derive bounds on deviations from antipodicity of the circles in each of these manifolds. In Section \(\text{III}\) we conclude with a brief discussion of the significance of our results for possibility of detectable nontrivial topology for the Universe in view of the present and future CMB observations.

II. CIRCLES-IN-THE-SKY IN ORIENTABLE FLAT 3-MANIFOLDS

We wish to study the observable signatures of nontrivial topologies of flat 3–dimensional orientable manifolds. We begin by recalling that these flat quotient manifolds are not rigid, in the sense that topologically equivalent flat quotient manifolds, defined by a given holonomy group \(\Gamma\), can have different sizes. Thus, there is no invariant characteristic topological scale with which one can establish a detectability criterion (as given in Ref. \([9]\), for example). Nevertheless, we know that the action of each pair of elements \((\gamma, \gamma^{-1})\) of the group \(\Gamma\) may generate one pair of matching circles in the CMB maps when the LSS intersects its images under the action of \(\gamma\) and \(\gamma^{-1}\). Using this basic fact, it is possible to directly study the holonomies in each orientable flat manifold with nontrivial topology in order to obtain the maximum deviation from antipodicity of the pairs of matching circles associated with the shortest closed geodesic which contains the observer’s position in that manifold.\(^3\) This specific pair of circles is important because if any other pair of matching circles is detectable, then the pair associated with the shortest closed geodesic will necessarily also be detectable.\(^4\) Conversely, if this pair from the closest images of the LSS is not detectable, then we can be sure that no other pair of matching circles will be detectable either.

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\(^2\) Throughout this paper we assume that these searches would reliably detect any pairs of matching circles present in the CMB data as long as their parameters fall within the scope of the search (see also Ref. \([12]\)).

\(^3\) The ‘end-points’ of these geodesics, i.e. the observer’s position \(p\) and its nearest image \(p\), are the centers of the closest neighboring images of the sphere of last scattering (LSS).

\(^4\) Of course, we are assuming in this article that all circles that arise from the intersection of the LSS with any of its topological copies are in practice statistically extractable from CMB maps.
We recall that in addition to the simply connected flat Euclidean space $\mathbb{E}^3$, there are 17 multiply-connected 3-dimensional flat manifolds which are quotient spaces of the form $\mathbb{E}^3/\Gamma_i$, where $\mathbb{E}^3$ is the covering space and $\Gamma_i$ is the discrete and fixed point free group of holonomies. Only nine out of these quotient manifolds are orientable. These consist of the six compact manifolds, namely $E_1$ (3-torus), $E_2$ (half turn space), $E_3$ (one quarter turn space), $E_4$ (one third turn space), $E_5$ (one sixth turn space), $E_6$ (Hantzsche-Wendt space), plus three non-compact ones: the chimney space $E_{11}$, the chimney space with half turn $E_{12}$ and the slab space $E_{16}$ (for details on the names and a description of the fundamental domain of these manifolds see, e.g., Refs. [15–17]).

The orientable manifolds $E_1$, $E_{11}$ and $E_{16}$ are globally homogeneous and hence would only produce antipodal pairs of circles-in-the-sky. The question then is what type of circles-in-the-sky would the other remaining 6 manifolds produce, and specifically what would be the deviations from antipodcity for the pairs of matching circles in each of these manifolds, and in particular for the circles associated with the shortest closed geodesics.

Any holonomy $\gamma$ of an orientable Euclidean 3-space can always be expressed as a so-called screw motion (in the covering space), consisting of a combination of a rotation $R(\alpha, \hat{u})$ by an angle $\alpha$ around an axis of rotation $\hat{u}$, plus a translation along the vector $L = \ell \hat{u}$. The action$^5$ of $\gamma$ on any point $p$ in the covering manifold is then given by $p \to R(p) + L$. When there is no rotation part in the screw motion, i.e., when $\alpha = 0$, the holonomy is a translation, and its action is exactly the same at every point; in particular the length of the closed geodesic associated with $\gamma$, $\ell_\gamma = |\gamma p - p| = L$, is the same everywhere. For a general screw motion, with $\alpha \neq 0$, $\ell_\gamma$ depends on $p$, and in particular on the distance between $p$ and the axis of rotation.

The two matching circles associated with the holonomy $\gamma = (R(\alpha, \hat{v}), L)$ are produced by the intersections of the sphere of last scattering with its images under the isometries $\gamma$ and $\gamma^{-1}$. The deviation from antipodcity, $\theta$, for any holonomy $\gamma$, is given by

$$\cos \theta = -\frac{(\gamma p - p) \cdot (\gamma^{-1} p - p)}{|\gamma p - p||\gamma^{-1} p - p|},$$

where a dot denotes the usual scalar product in $\mathbb{E}^3$.

Let us first calculate $\theta$ for the manifolds $E_i$, $i = 1, \ldots, 5$, which are quotient 3-manifolds that have as generators of the holonomy group $\Gamma$ two translations plus a screw motion. In these manifolds, the axis of rotation and the direction of the translation of the screw motion are parallel. This common direction is perpendicular to the directions of the translations associated with the other two remaining generators. The rotation angle in these manifolds takes the form $\alpha = 2\pi/n$, where the so-called screw motion parameter $n$ is 1, 2, 4, 3 and 6 respectively. Of course, any translation will correspond to the case $n = 1$.

In order to obtain the value of $\theta_{\text{max}}$, the maximum deviation from antipodcity between the pair of matching circles associated with the shortest closed geodesic which contains the observer’s position, we should in principle compare the lengths of the closed geodesics generated by all elements of $\Gamma$ at every point in the manifold. We note, however, that as the scale of the translational generators is not fixed, the length of the smallest geodesic corresponding to pure translations can be arbitrary.

Now if a pure translation, such as one of the translational generators with axis perpendicular to the screw motion, happens to produce the shortest geodesic, then its resulting circles-in-the-sky are back to back (zero deviation from antipodcity). If, on the other hand, pure translations do not produce the shortest geodesic, then we need to look at the remaining holonomies which consist of the screw motion generator $\gamma_{SM}$, and its higher powers, $\gamma_{SM}^m$, for integer $m$. Since we want to obtain an upper bound on $\theta$ applicable to all choices of topological length parameter, then clearly the configurations where this angle assumes its maximum value are those in which the length of the closed geodesics associated with the translational generators are long enough so that the screw motion generator and its powers always generate the shortest geodesics. To obtain our upper bound we thus need to only consider this latter case.

Consider an arbitrary observer at a point $p$, whose distance from the axis of rotation is $r$. Without loss of generality, we can choose coordinates $(x, y, z)$ such that the rotation axis lies along the direction of $\hat{x}$, and the coordinates of observer’s position are given by $p = (0, 0, r)$. In this coordinate system one has $\gamma_{SM} p = (L, r \sin \alpha, r \cos \alpha)$ and $\gamma_{SM}^m p = (-L, -r \sin \alpha, r \cos \alpha)$. Then the lengths of the closed geodesics associated with screw motion holonomies $\gamma_{SM}^m$ are given by

$$\ell_{\gamma_{SM}^m} = |\gamma_{SM}^m p - p| = \sqrt{(mL)^2 + 2 r^2 (1 - \cos m\alpha)}.$$  

Now to decide which of these holonomies produces the smallest geodesic for each point $p$ in each of the considered five manifolds $E_i$, we recall that the screw motion rotation angles are $\alpha = 2\pi/n$ with $n = 1, 2, 4, 3, 6$. Note also that we need not concern ourselves with the cases in which the translational generators produce the shortest geodesic, since the maximum deviation from antipodcity for the circles associated with the shortest geodesic will occur when the (essentially arbitrary) parameters that define their lengths are large enough so that the screw motion generates the shortest geodesic.

The question therefore now becomes: For which positive integer $m$ is $\ell_{\gamma_{SM}^m}$ minimum? From Eq. (3) it is clear that if $m = sn + m'$, where $s$ is a positive integer and $1 \leq m' \leq n$, then $\ell_{\gamma_{SM}^{sn}} < \ell_{\gamma_{SM}^{m'}}$. We can thus without

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$^5$ The choice of axes to describe a screw motion is not unique, but one can always find a rotation axis parallel to the direction of translation.
loss of generality assume that \(1 \leq m \leq n\).

To proceed, we need to look inside the square root at the right hand side of Eq. (3). It is the sum of two positive terms. The first, \((mL)^2\), obviously increases with \(m\) and is minimum for \(m = 1\). The second term, \(r^2 \left(1 - \cos 2\pi m/n \right)\), assumes its minimum value when \(m = n\) and its next-to-minimum value when \(m = 1\) (and \(m = n - 1\)). Therefore, in all cases \(\ell_{\gamma_{SM}}\) is minimized for either \(m = 1\) or \(m = n\), and the length of the smallest geodesic is either \(\ell_{\gamma_{SM}}\) or \(\ell_{\gamma_{SM}}'\). It is important to note that the latter length is the same for all points of the manifold, since \(\gamma_{SM}\) is a translation.

Now that we have narrowed down the holonomies we need to consider to only two, \(\ell_{\gamma_{SM}}\) and \(\ell_{\gamma_{SM}}'\), we only need to determine for which points (as determined by the value of \(r\)) each of the two holonomies under consideration generates the shortest geodesic. To that end, we note that at the axis of rotation the length associated with \(\gamma_{SM}\) is \(\ell_{\gamma_{SM}} = L\) (which is smaller than \(\ell_{\gamma_{SM}}' = nL\) for \(n > 1\)), and this length increases monotonically as the distance \(r\) from the axis of the screw motion is increased, as can be seen from Eq. (4). On the other hand, \(\ell_{\gamma_{SM}}'\) will of course remain fixed at \(nL\). Therefore, there is a sufficiently large \(r\) such that the equation

\[
\ell_{\gamma_{SM}} = \ell_{\gamma_{SM}}',
\]

holds. This limiting case corresponds to the maximum value \(r_{\max}\) of the distance \(r\) for which the shortest geodesic is generated by the screw motion. Furthermore, by using the expression for the length given by Eq. (3) on both sides of Eq. (4) (for \(m = 1\) and \(m = n\)), we obtain

\[
n^2 = 1 + 2 \frac{L}{L^2} \left(1 - \cos \frac{2\pi}{n} \right). \tag{5}
\]

On the other hand, to obtain \(\theta_{\max}\) we first substitute the explicit expressions for \(p, \gamma_{SM}p\) and \(\gamma_{SM}^{-1}p\) in the expression (2) in order to relate the deviation from antipodicity \(\theta\) to the screw motion parameters \((\alpha = 2\pi/n, L)\) and the position of the observer (see also Ref. [1]):

\[
\cos \theta = 1 - \frac{2r^2(1 - \cos \alpha)^2}{L^2 + 2r^2(1 - \cos \alpha)}. \tag{6}
\]

This shows that the further away the observer is from the axis of rotation of the screw motion, the greater is the deviation from antipodicity. As a result, the maximum value of \(r\) for which the shortest geodesic is generated by \(\gamma_{SM}\) instead of the translation \(\gamma_{SM}^n\) corresponds to the maximum value of \(\theta\), \(\theta_{\max}\).

We can now obtain \(\theta_{\max}\) for the manifolds \(E_i, i = 1, \ldots, 5\) and \(E_{12}\) by substituting the value of \(r_{\max}\) given by Eq. (4) into Eq. (6) to obtain

\[
\cos(\theta_{\max}) = 1 - \frac{n^2 - 1}{n^2} \left[1 - \cos \left(\frac{2\pi}{n}\right)\right], \tag{7}
\]

which readily allows the calculation of the values of \(\theta_{\max}\) for each of these manifolds.

We note that the results concerning the manifolds \(E_i, i = 1, \ldots, 5\) also apply to the chimney with half turn \(E_{12}\), as this manifold can be considered as a limiting case of \(E_2\) (half turn space) when the length parameter of one of the translational generators goes to infinity. Furthermore, the chimney and slab spaces, \(E_{11}\) and \(E_{16}\), are clearly the limiting cases of the three-torus \(E_1\) in which one or two of the length parameters go to infinity.

| Symbol | Manifold                           | \(n\) | \(\theta_{\max}\) |
|--------|------------------------------------|-------|-------------------|
| \(E_1\) | three-torus                         | 1,1,1 | 0°                |
| \(E_2\) | half turn space                     | 1,1,2 | 120°              |
| \(E_3\) | quarter turn space                  | 1,1,4 | 86°               |
| \(E_4\) | third turn space                    | 1,1,3 | 109°              |
| \(E_5\) | sixth turn space                    | 1,1,6 | 59°               |
| \(E_6\) | Hantzsche-Wendt space               | 2,2,2 | 120°              |
| \(E_{11}\) | chimney space                       | 1,1   | 0°                |
| \(E_{12}\) | chimney space with half turn        | 1,2   | 120°              |
| \(E_{16}\) | slab space                          | 1     | 0°                |

TABLE I: Multiply-connected flat orientable manifolds and the maximum deviation from antipodicity of the circles-in-the-sky for each manifold. The screw motion twist parameters \(n\) for all the generators are also indicated.

In Table I we collect the values of \(\theta_{\max}\), the maximum possible deviation from antipodicity of the circle pairs associated with the shortest closed geodesic (i.e., the most readily detectable), for all orientable flat three-manifolds with a nontrivial topology. We note that the calculation of \(\theta_{\max}\) for \(E_6\) (Hantzsche-Wendt space) is more involved and is presented in detail in the Appendix. This Table shows that although the flat manifolds are not rigid, a general maximum, applicable to all observers and choices of length parameters, can be obtained for the values of the deviation from antipodicity that needs to be included in a comprehensive search for a detectable cosmic topology by the circles in the sky method. The relatively large numerical values in the table make it apparent that the circles searches so far undertaken [3, 4] (see also Ref. [3]) do not rule out the possibility of a detectable nontrivial flat topology.

Another important outcome of the results of Table I is that they offer a theoretical framework to draw conclusions from future circles-in-the-sky searches regarding the presence or absence of spatial curvature in the Universe. Given that the deviation from antipodicity for a very nearly flat (\(\theta < |\Omega_{\text{tot}}| - 1 \lesssim 10^{-5}\)) multiply connected Universe is less than 10° for the majority of observers [1], the detection of a pair of circles-in-the-sky with a higher value of \(\theta\) (e.g., \(\theta \approx 60°\)) would imply that the Universe is very likely to be flat (with \(\Omega_{\text{tot}} = 1\)), in the sense that in nearly flat manifolds such high values for \(\theta\) could only occur for a vanishingly small subset of observers. Note that no such definite determination can conceivably be made by purely geometrical methods.

The use of the patterns of image repetitions in cosmic topology to constrain spatial curvature has been discussed at some length in the literature (see, e.g., Refs. [1, [3].
and [9]). What we propose above is a direct observational test that may be able to determine the curvature sign based on a partial detection of the cosmic topology. Indeed, it should not be surprising that flat and very nearly flat manifolds present such radically different pictures regarding the detectable circles-in-the-sky patterns. Although such manifolds may be geometrically indistinguishable (geometry being a local property), they remain quite distinct topologically, since the possible holonomy classes for zero, positive and negative spatial curvature are completely different from each other (this is in turn a reflection of the differences in the global properties of the covering spaces these groups tesselate). So, even locally, the detectable topology contains information about the global shape of the Universe.

III. FINAL REMARKS

The existence of correlated pairs of circles in the CMB anisotropy maps, the so-called circles-in-the-sky, is a generic prediction of a detectable non-trivial cosmic topology, regardless of the background geometry. Detecting such circles, and measuring their position, angular radii and relative phase, would allow us to characterize, and possibly determine, the topology of the spatial section of the Universe.

Searches for circles-in-the-sky whose centers are separated by more than 170° (nearly antipodal circles) have been performed with negative results. Recent theoretical results together with this negative outcome would be sufficient to exclude a detectable non-trivial topology for most observers in very nearly flat Universe.

In this work, we have studied what happens if the Universe turns out to be exactly flat ($\Omega_{\text{tot}} = 1$), rather than nearly flat. Despite the fact that flat manifolds have no fixed topological scales, we have been able, by studying the action of holonomies, to derive upper bounds on the deviations from antipodicity of the pairs of matching circles-in-the-sky associated with the shortest closed geodesics in all orientable manifolds with a nontrivial flat topology. The key point to bear in mind is that, if the circle pair associated with the shortest closed geodesic is not detectable, then no other circle pair will be detectable; but conversely, the failure to detect any other (possibly back-to-back, or nearly so) pair of matching circles does not guarantee that the pair of circles corresponding to shortest geodesic is not detectable either. Thus, it can only be said that a given search is comprehensive if it is designed to be able to detect the shortest geodesic’s circle pair for all observers. Our bounds can be regarded as defining what constitutes a comprehensive search, in the idealized case in which detection is certain if a detectable non-trivial topology exists.

The derived bounds show that the searches already undertaken are not sufficient to exclude the possibility of a nontrivial flat topology for the Universe. Our results also demonstrate that a slight variation in the value of the spatial curvature by making it exactly flat, can have striking consequences for the analysis of the detectability of cosmic topology.

Finally, the theoretical results presented here would allow us to make inferences regarding the spatial curvature sign of the universe, in the case of the detection of some correlated pairs of circles in upcoming circles-in-the-sky searches.

Appendix

The the calculation of $\theta_{\text{max}}$ for $E_6$ (Hantzche-Wendt space) manifold is somewhat more involved, since the generators of its holonomy group are three half-turn screw motions (the choice of generators is not unique, but always include at least three screw motions), and one needs to be careful in considering all the possible combinations of the generators to find which holonomies generate the shortest closed geodesic at each point.

More specifically, the action of the $E_6$ generators can be expressed as three screw motions, $\gamma_1$, $\gamma_2$ and $\gamma_3$, consisting of rotations of $\pi$ around respectively the $z$, $x$ and $x$-axes, followed by translations of, respectively, $a$ along the $z$-axis, $b$ along the $z$-axis plus $c$ along the $y$-axis and $-b$ along the $z$-axis plus $-c$ along the $y$-axis respectively. Thus, the actions of the generators on a point $p$ on the manifold is given by

$$\begin{align*}
\gamma_1 p &= R(\pi, \hat{z}) p + a\hat{z} \\
\gamma_2 p &= R(\pi, \hat{x}) p + b\hat{x} + c\hat{y} \\
\gamma_3 p &= R(\pi, \hat{x}) p + b\hat{x} - c\hat{y}.
\end{align*}$$

The key point to note is that, since $\gamma_1$, $\gamma_2$ and $\gamma_3$ are all half-turn screw motions around one of the coordinate axes, then any combination of these holonomies and their inverses will also be either a half-turn screw motion around one of the coordinate axis or a translation (a "full turn" screw motion). In the case of a half-turn screw motion, the translation axis will not in general be parallel to the axis of rotation. But it is well known (see e.g. Ref. [10]) that for a suitable change of axis any screw motion can be expressed as a rotation followed by a translation along the appropriately chosen axis. Moreover, two equivalent screw motions will have parallel axes and equal values for the rotation angle.

6 To see this, note that generally the combination of screw motions is itself a screw motion, with an $R$ matrix that is the product of the component $R$ matrices. Since in this case the $R$ matrices for the generators are diagonal, with the diagonal elements being either $1$ or $-1$ such that $\det(R) = 1$, then the diagonal elements and determinant for the resulting $R$ matrix will be simply the product of respectively the diagonal elements and determinants of the generators’ $R$ matrices. In other words, the resulting $R$ matrix will be diagonal, with elements being either $1$ or $-1$, and $\det(R) = 1$, i.e., $R$ will be either a rotation of $\pi$ around one of the coordinate axes or the identity matrix.
Now, let us assume that for an observer with position \( \mathbf{p} \), and for some specific values of the constants \( a, b \) and \( c \), the holonomy \( \gamma_S \) generates the shortest closed geodesic. From the brief discussion above, it is clear that \( \gamma_S \) is either a translation, in which case the deviation from antipodality for the most readily detectable pair of circles is 0, or \( \gamma_S \) is a half-turn screw motion. In the latter case \( \gamma_S \) consists of a rotation around some axis \( \hat{w} \) (parallel to one of the coordinate axes) plus a translation of \( L \) along \( \hat{w} \): \( \gamma_S \mathbf{p} = R(\pi, \hat{w}) \mathbf{p} + L \hat{w} \). The length of the associated closed geodesic is given by Eq. (3) (with \( m = 1 \) and \( \alpha = \pi \)). Furthermore, the holonomy \( \gamma_S \mathbf{p} = \mathbf{p} + 2L \hat{w} \) is a translation with length \( 2L \). Thus, by the same reasoning we applied to the other manifolds, the maximum deviation from antipodality for which \( \gamma_S \) generates a shorter closed geodesic than \( \gamma^2_2 \) occurs when \( \ell_{\gamma_0} = \ell_{\gamma_2} \). Performing the same calculation we used to estimate \( \theta_{\text{max}} \) for \( E_2 \), we obtain from Eq. (7) (with \( n = 2 \)) that \( \theta_{\text{max}} \leq 120^\circ \).

Finally, all that remains is to prove that there exists at least one combination of \( a, b, c \) and \( \mathbf{p} \) such that the deviation \( \theta \) from antipodity associated with the holonomy generating the shortest geodesic is exactly \( 120^\circ \). To show that this is indeed the case, note that for \( b > 2a \) and \( c > 2a \), either \( \gamma_1 \) or \( \gamma_2^2 \) will generate the shortest closed geodesic at any position, and that \( \mathbf{p} \) can be chosen so that \( \ell_{\gamma_1} = \ell_{\gamma_2^2} \), which again from Eq. (7) results in \( \theta = 120^\circ \). Since \( \theta_{\text{max}} \) is by definition the maximum value of \( \theta \) for all possible parameter and position combinations, this implies that \( \theta_{\text{max}} \geq 120^\circ \).

By combining both inequalities, we then obtain that \( \theta_{\text{max}} = 120^\circ \).

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