Gorenstein flat dimension with group ring coefficients

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Abstract

For any group \(G\) and any commutative ring \(R\), the Gorenstein homological dimension \(\text{Ghd}_RG\), defined as the Gorenstein flat dimension of the trivial \(RG\)-module \(R\), is characterized. We prove that \(\text{Ghd}_RG < \infty\) if and only if Gorenstein flat dimension of any \(RG\)-module is finite, whenever the supremum of flat dimension of injective modules \(\text{sfli} R\) over the coefficient ring \(R\) is finite. As applications, properties of \(\text{Ghd}_RG\) on subgroup, quotient group, extension of groups as well as Weyl group are investigated, and moreover, we compare the relations between some invariants such as \(\text{sfli} RG\), \(\text{silf} RG\), \(\text{spli} RG\), \(\text{silp} RG\), and Gorenstein projective, Gorenstein flat and PGF dimensions of modules over group rings \(RG\).

1. Introduction

Let \(G\) be any group. Recall that the cohomological dimension \(\text{cd}_ZG\) and homological dimension \(\text{hd}_ZG\) are defined as the projective and flat dimension of the \(ZG\)-module \(Z\) respectively, where \(G\) acts trivially. Studying groups through these dimensions arose from both topological and algebraic sources, and has a long history in group theory.

Let \(R\) be a commutative ring of coefficients. For any group \(G\), the Gorenstein cohomological dimension \(\text{Gcd}_RG\) \[3, 11\], and the Gorenstein homological dimension \(\text{Ghd}_RG\) \[1, 19\], are defined as the Gorenstein projective and Gorenstein flat dimension of the trivial \(RG\)-module \(R\), respectively. The notions of Gorenstein projective, injective and flat modules were introduced by Enochs and co-authors \[12, 13\], which have the origin dating back to the study of \(G\)-dimension by Auslander and Bridger \[2\] in 1960s, and now form the basis of a version of relative homological algebra known as Gorenstein homological algebra. The Gorenstein projective (resp. Gorenstein flat) dimension generalizes the projective (resp. flat) dimension, in the sense that whenever the latter is finite, then they coincide.

For any group \(G\), there is an interesting result \[3, \text{Theorem 2.7}\] that \(\text{Gcd}_ZG < \infty\) if and only if there is a \(Z\)-split \(ZG\)-exact sequence \(0 \rightarrow Z \rightarrow A\), where \(A\) is a \(Z\)-projective (\(Z\)-free) \(ZG\)-module such that the projective dimension \(\text{pd}_ZA\) of \(A\) is finite; in particular, \(\text{pd}_ZA = \text{Gcd}_ZG\).

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was generalized by Emmanouil and Talelli [11, Theorem 1.7] from the coefficient ring of integers $\mathbb{Z}$ to any commutative ring with finite global dimension. In fact, we can relax the coefficient ring further to be of finite Gorenstein global dimension [22, Theorem 2.4]. Moreover, $\text{Gcd}_Z G < \infty$ if and only if every $ZG$-module has finite Gorenstein projective dimension, and it is conjectured in [3] that $\text{Gcd}_Z G < \infty$ if and only if the group $G$ admits a finite dimensional classifying space $EG$ for proper actions; in this case, there is a finite dimensional contractible $G$-CW-complex with finite stabilizers.

First, we intend to characterize the finiteness of $\text{Ghd}_R G$, and its relations with Gorenstein flat dimension of modules over the group ring $RG$. However, the arguments are not simply analogous to those for Gorenstein cohomological dimension $\text{Gcd}_R G$. There are some evidences for this phenomenon. Unlike the well-known fact that every projective module is flat, the relation between Gorenstein projective and Gorenstein flat modules is not fully understood, and it is still an open problem whether Gorenstein projective modules are Gorenstein flat. There is a basic and important property that the class of Gorenstein projective modules is closed under extensions; see for example [16, Theorem 2.5]. However, the analogous result for Gorenstein flat modules is not easy to prove. In [16, Theorem 3.7], an additional assumption that the base ring is coherent is needed. Until recently, the assertion was proved affirmatively over any ring by Šaroch and Šťovíček (see [23, Corollary 4.12]), where a notion of PGF-modules was invented.

In fact, projectivity connects closely to the splitness, however, flatness is a bit inseparable from purity. We succeed in getting the following, which is formally analogous to the aforementioned results in [3, 11, 22]. Let $R$ be a commutative ring such that the supremum of flat dimension (length) of injective $R$-modules $\text{sfli} R$ is finite. Then $\text{Ghd}_R G < \infty$ if and only if there exists an $R$-pure $RG$-exact sequence $0 \to R \to A$, where $A$ is an $R$-flat $RG$-module of finite flat dimension, if and only if every $RG$-module has finite Gorenstein flat dimension; see Theorem 3.9. In this case, we have an equality $\text{Ghd}_R G = \text{fd}_{RG} A$, and the following inequalities

$$\text{sfli} R \leq \text{sfli} RG = \text{G.wgldim} RG \leq \text{Ghd}_R G + \text{sfli} R,$$

where $\text{G.wgldim} RG$ denotes the Gorenstein weak global dimension of the group ring $RG$.

It is worth to remark that the assumption $\text{sfli} R < \infty$ on the coefficient ring $R$ is not too restrictive. It follows from [9, Theorem 5.3] that $\text{sfli} R < \infty$ if and only if every $R$-module has finite Gorenstein flat dimension. In this case, $R$ is called a ring with finite Gorenstein weak global dimension. In fact, we frequently concern modules with finite, rather than infinite, Gorenstein flat dimension. Rings with finite weak global dimension are strictly contained in those of finite Gorenstein weak global dimension. The most interesting examples of coefficient rings in dealing with applications of group rings in geometry and representation theory, such as the ring of integers $\mathbb{Z}$, the field of rationals $\mathbb{Q}$ and finite fields, are all of finite “$\text{sfli}$”.

For further applications, we strengthen the role of Gorenstein homological dimension of groups by proving the following properties that are standard for ordinary group cohomology. As shown
in Section 4, the arguments heavily rely on the above $R$-pure monomorphism $R \to A$. More precisely, for any commutative ring $R$ with $\text{sfli}_R < \infty$ and any group $G$, we show that

1. $\text{Ghd}_R H \leq \text{Ghd}_R G$ for any subgroup $H$ of $G$ (see Proposition 4.1).
2. If $1 \to H \to G \to L \to 1$ is an extension of groups, then $\text{Ghd}_R G \leq \text{Ghd}_R H + \text{Ghd}_R L$ (see Proposition 4.3).
3. $\text{Ghd}_R G = \text{Ghd}_R (G/H)$ for any finite normal subgroup $H$ of $G$ (see Proposition 4.4).
4. If $H$ is a finite subgroup of $G$, and $N_G(H)$ its normalizer in $G$, then $\text{Ghd}_R W \leq \text{Ghd}_R G$ for the Weyl group $W = N_G(H)/H$ (see Corollary 4.5).

Moreover, we compare Gorenstein homological dimension of groups with the generalized homological dimension of groups introduced by Ikenaga [17, III, Definition]; see details in Proposition 4.6 and 4.7. In particular, we obtain in Corollary 4.9 that $\text{Ghd}_2 G < \infty$ if and only if $\text{hd}_2 G < \infty$, and furthermore, $\text{Ghd}_2 G = \text{hd}_2 G$ in this case. Consequently, we get that $\text{hd}_2 G = 0$ if and only if $G$ is a finite group; see Corollary 4.10. This generalizes a result due to Ikenaga [17, Proposition 7], where only the sufficient condition was proved.

In Section 5, we are devoted to study the relation between Gorenstein flat, Gorenstein projective and PGF dimensions of modules over group rings, as well as some invariants such as $\text{spl}_R G$, $\text{spl}_R G$ and $\text{sfli}_R G$, which are proved to be closely related to the theory of Gorenstein projective, flat and PGF dimensions; see for example [6, 7, 9].

For any commutative Gorenstein ring $R$ and any group $G$, Gedrich and Gruenberg have proved in [14, Theorem 2.4] that $\text{spl}_R G < \infty$ implies $\text{sfli}_R G < \infty$. For the finiteness of $\text{sfli}_R G$ and $\text{sfli}_R G$, we show in Proposition 5.2 that $\text{sfli}_R G < \infty$ if and only if both $\text{sfli}_R G < \infty$ and $\text{sfli}_R G < \infty$, where the notation of the invariant $\text{sfli}_R G$, due to Raynaud and Gruson [20, Section II.3.3], is introduced in [10]. We show that if $\text{spl}_R G < \infty$, then $\text{Ghd}_R G < \infty$ and every Gorenstein projective $RG$-module is Gorenstein flat; moreover, for any Gorenstein flat $RG$-module $M$, $\text{PGF-dim}_{RG} M = \text{Gpd}_{RG} M < \infty$; see Corollary 5.4. In particular, if $G$ is a group such that $RG$ is a coherent ring, then $\text{sfli}_R G < \infty$ if and only if $\text{sfli}_R G < \infty$ in this case $\text{sfli}_R G = \text{sfli}_R G$; see Proposition 5.6. Remark that Hirschhorn characterized in [15] a class of groups $G$ such that $ZG$ is a coherent ring, where $G \simeq \pi_1 X$ for some CW-complex $X$ with finite skeletons. Finally, we show in Proposition 5.9 that there is an inequality $G.\text{wgldim} RG \leq \text{PGF-gldim} RG = G.\text{gldim} RG$ for any commutative ring $R$ and any group $G$.

2. Preliminaries

In this section, we recall some notions and facts which will be needed in the following.

Gorenstein flat dimension of modules. Let $\Lambda$ be an associative ring with identity. Recall that $F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ is called a totally acyclic complex of flat modules, provided it is acyclic with each $F_i$ being a flat left $\Lambda$-module, and for any injective right $\Lambda$-module $I$, the
complex remains acyclic after applying $I \otimes_{\Lambda} -$. A left $\Lambda$-module $M$ is called Gorenstein flat \([13]\), if there exists a totally acyclic complex of flat modules $F$, such that $M \cong \Ker(F_0 \to F_{-1})$.

Let $M$ be any left $\Lambda$-module. The Gorenstein flat dimension of $M$, denoted by $\text{Gfd}_\Lambda M$, is defined by declaring that $\text{Gfd}_\Lambda M \leq n$ if and only if $M$ has a Gorenstein flat resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ of length $n$, where each $P_i$ is a Gorenstein flat left $\Lambda$-module. The Gorenstein weak global dimension \([4]\) of $\Lambda$, denoted by $\text{G.wgldim}\Lambda$, is defined as the supremum of Gorenstein flat dimensions of all left $\Lambda$-modules.

It was proved in \([16, \text{Theorem 3.7}]\) that for a right coherent ring, the class of Gorenstein flat left modules is closed under extensions and direct summands. This result was extended and generalized to any associative ring by \([23, \text{Corollary 4.12}]\). This basic property is crucial for studying homology of Gorenstein flat modules, however, it is not easy to prove. Thanks to \([23, \text{Corollary 4.12}]\), now we can remove the assumption of coherent rings in many situations when dealing with Gorenstein flat modules and Gorenstein flat dimension of modules, see for example \([16, \text{Theorem 3.14}]\).

**Modules over group rings.** Let $G$ be a group, and $R$ be a commutative ring. A module over the group ring $RG$ is simply an $R$-module $M$ together with an action of $G$ on $M$. In particular, $R$ is an $RG$-module with trivial $G$-action, that is $gr = r$ for any $g \in G$ and any $r \in R$.

For an $RG$-module $M$, the module of invariants of $M$ is defined as the largest submodule of $M$ on which $G$ acts trivially, that is, $M^G := \{m \in M | gm = m \text{ for all } g \in G\}$. Analogously, the module of coinvariants of $M$, denoted by $M_G$, is defined to be the largest quotient of $M$ on which $G$ acts trivially.

Since $R$ is assumed to be commutative, for any $RG$-module, one can avoid considering both left and right modules by using the anti-automorphism $g \mapsto g^{-1}$ of $G$. Thus, by setting $gm = mg^{-1}$ for any $g \in G$ and $m \in M$, we can regard any left $RG$-module $M$ as a right $RG$-module. In this way, for any two left $RG$-modules $M$ and $N$, the tensor product $M \otimes_{RG} N$ makes sense by introducing the relations $gm \otimes n = mg^{-1} \otimes n = m \otimes g^{-1}n$.

For any $RG$-modules $M$ and $N$, $M \otimes_R N$ is an $RG$-module where $G$ acts diagonally, that is, $g(m \otimes n) = gm \otimes gn$. By replacing $n$ in the above relations by $gn$, we get that $gm \otimes gn = m \otimes n$. Hence, $M \otimes_{RG} N = (M \otimes_R N)_G$, where $G$ acts diagonally on $M \otimes_R N$. Moreover, there is an isomorphism $M \otimes_{RG} N \cong N \otimes_{RG} M$, that is, the bifunctor $- \otimes_{RG} -$ is commutative.

Let $H$ be any subgroup of $G$. There exist simultaneously an induction functor $\text{Ind}_H^G = RG \otimes_{RH} -$ and a coinduction functor $\text{Coind}_H^G = \Hom_{RH}(RG, -)$ from the category of $RH$-modules $\text{Mod}(RH)$ to the category of $RG$-modules $\text{Mod}(RG)$. We denote the restriction functor from $\text{Mod}(RG)$ to $\text{Mod}(RH)$ by $\text{Res}_H^G$. It is clear that $(\text{Ind}_H^G, \text{Res}_H^G)$ and $(\text{Res}_H^G, \text{Coind}_H^G)$ are adjoint pairs of functors.

**Gorenstein homological dimension of groups.** Recall that for any group $G$, the Gorenstein homological dimension of $G$ is defined to be the Gorenstein flat dimension of the trivial
$\mathbb{Z}G$-module $\mathbb{Z}$; see [1, Definition 4.5]. Analogously, we may define the Gorenstein homological dimension of $G$ over any commutative ring $R$, denoted by $\text{Ghd}_R G$, to be the Gorenstein flat dimension of the trivial $RG$-module $R$; see [19, Definition 2.5]. It follows from [19, Corollary 2.8] that if the coefficient ring $R$ is $\mathbb{Z}$-torsion-free, then $\text{Ghd}_R G$ is a refinement of $\text{Ghd}_\mathbb{Z} G$, that is, $\text{Ghd}_R G \leq \text{Ghd}_\mathbb{Z} G$.

**Invariants $\text{silp}$, $\text{spli}$, $\text{silf}$ and $\text{sfli}$.** The Gorenstein flat dimension is closely related to some homological invariants. Recall that in connection with the existence of complete cohomological functors in the category of left $\Lambda$-modules, Gedrich and Gruenberg have defined in [14] the invariant $\text{silp}\Lambda$ as the supremum of the injective length (dimension) of projective left $\Lambda$-modules, and the invariant $\text{spli}\Lambda$ as the supremum of the projective length (dimension) of injective left $\Lambda$-modules. Analogously, we use $\text{silf}\Lambda$ to denote the supremum of the injective length (dimension) of flat left $\Lambda$-modules, and use $\text{sfli}\Lambda$ to denote the supremum of the flat length (dimension) of injective left $\Lambda$-modules.

Note that if $\text{sfli}\Lambda$ is finite, then any acyclic complex of flat left $\Lambda$-modules is totally acyclic. It follows from [9, Theorem 5.3] and [6, Corollary 1.5] that $G.\text{wgldim}\Lambda = G.\text{wgldim}\Lambda^\text{op}$ is finite if and only if $\text{sfli}\Lambda = \text{sfli}\Lambda^\text{op}$ is finite; moreover, in this case one has $G.\text{wgldim}\Lambda = \text{sfli}\Lambda < \infty$, and $\Lambda$ is called a ring with finite Gorenstein weak global dimension.

### 3. Finiteness of Gorenstein Homological Dimension of Groups

Throughout the paper, $R$ is assumed to be a commutative ring, $G$ is a group. In this section, we intend to characterize the finiteness of $\text{Ghd}_R G$, which is equivalent to that every $RG$-module has finite Gorenstein flat dimension; see Theorem 3.9.

**Lemma 3.1.** If $\text{sfli} R < \infty$, then any Gorenstein flat $RG$-module is also a Gorenstein flat $R$-module.

**Proof.** Let $M$ be a Gorenstein flat $RG$-module. Then there is a totally acyclic complex of flat $RG$-modules $\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $M \cong \text{Ker}(F_0 \to F_{-1})$. Since any flat $RG$-module is also $R$-flat, by restricting this totally acyclic complex, we get an acyclic complex of flat $R$-modules. Noting that $\text{sfli} R < \infty$, every acyclic complex of flat $R$-modules is totally acyclic. Hence, $M$ is also Gorenstein flat as an $R$-module. \qed

**Lemma 3.2.** Let $H$ be any subgroup of $G$. For any Gorenstein flat $RH$-module $M$, the $RG$-module $\text{Ind}_H^G M$ is also Gorenstein flat.

**Proof.** Let $M$ be a Gorenstein flat $RH$-module. There is a totally acyclic complex of flat $RH$-modules $F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $M = \text{Ker}(F_0 \to F_{-1})$. We imply that $\text{Ind}_H^G F = RG \otimes_{RH} F$ is an acyclic complex of flat $RG$-modules such that $\text{Ind}_H^G M = \text{Ker}(\text{Ind}_H^G F_0 \to \text{Ind}_H^G F_{-1})$. For any injective $RG$-module $E$, it is restricted to be an injective $RH$-module. Then,
we infer from the isomorphism
\[ E \otimes_{RG} \text{Ind}_H^G F = E \otimes_{RG} RG \otimes_{RH} F \cong E \otimes_{RH} F \]
that \( E \otimes_{RG} \text{Ind}_H^G F \) is acyclic. Hence, \( \text{Ind}_H^G F \) is a totally acyclic complex of flat \( RG \)-modules, and this implies that \( \text{Ind}_H^G M \) is a Gorenstein flat \( RG \)-module. \( \square \)

**Lemma 3.3.** Assume \( \text{sfli} R < \infty \). For any \( RG \)-module \( M \), if \( M \) is Gorenstein flat as an \( R \)-module, then for any flat \( RG \)-module \( F \), the induced \( RG \)-module \( F \otimes_R M \) is also Gorenstein flat.

**Proof.** Let \( H = \{1\} \) be the subgroup formed by the identity element of \( G \). Then, the induction functor \( \text{Ind}_H^G = RG \otimes_R - \) sends every \( R \)-module to be an \( RG \)-module. Let \( M \) be an \( RG \)-module, and assume that \( M \) is Gorenstein flat as an \( R \)-module. Since any flat \( RG \)-module \( F \) is also restricted to be a flat \( R \)-module and \( \text{sfli} R \) is assumed to be finite, it follows that \( F \otimes_R M \) is a Gorenstein flat \( R \)-module. Then, by Lemma 3.2 we infer that the induced \( RG \)-module \( \text{Ind}_H^G(F \otimes_R M) \) is Gorenstein flat.

There is an isomorphism of \( RG \)-modules \( F \otimes_R \text{Ind}_H^G M \cong \text{Ind}_H^G(F \otimes_R M) \), where \( G \) acts diagonally on the left tensor product; see for example [5, Section III 5]. Note that the diagonal \( RG \)-module structure of \( F \otimes_R \text{Ind}_H^G M \) coincides with the one induced by \( RG F \otimes_R - \). As \( R \)-modules, \( M \) is a direct summand of \( \text{Ind}_H^G M \). Then, the induced \( RG \)-module \( F \otimes_R M \) is a direct summand of the induced \( RG \)-module \( F \otimes_R \text{Ind}_H^G M \). Hence, \( F \otimes_R M \) is a Gorenstein flat \( RG \)-module. \( \square \)

**Proposition 3.4.** Let \( M \) be an \( RG \)-module with \( \text{Gfd}_{RG} M < \infty \). Assume \( \text{sfli} R < \infty \). If the flat dimension \( \text{fd}_R M \) is finite, then there exists an \( R \)-pure \( RG \)-exact sequence \( 0 \to M \to N \to L \to 0 \), for which \( L \) is an \( R \)-flat \( RG \)-module and \( \text{Gfd}_{RG} M = \text{fd}_{RG} N \).

**Proof.** Let \( \text{Gfd}_{RG} M = n \). It follows from [16, Theorem 3.23] that there exists an exact sequence \( 0 \to K \to X \to M \to 0 \), where \( X \) is a Gorenstein flat \( RG \)-module, and \( \text{fd}_{RG} K = n - 1 \). For \( X \), there is an exact sequence of \( RG \)-modules \( 0 \to X \to F \to L \to 0 \), where \( F \) is flat and \( L \) is Gorenstein flat. We consider the following pushout of \( X \to M \) and \( X \to F \):

```
0 \rightarrow 0 \\
| \downarrow \\
0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0 \\
| \downarrow \\
| \downarrow \\
0 \rightarrow L \rightarrow L \\
| \downarrow \\
0 \rightarrow 0 \\
```

```
From the middle row we infer that $fd_{RG} N = fd_{RG} K + 1 = n$, which also implies the finiteness of $fd_R N$. In view of the assumption $fd_R M < \infty$, we infer from the right column that $fd_R L < \infty$. For the Gorenstein flat $RG$-module $L$, it follows from Lemma 3.1 that $L$ is also Gorenstein flat as an $R$-module. Furthermore, we imply that $L$ is $R$-flat since flat dimension of any Gorenstein flat module is either zero or infinity; see [12, Corollary 10.3.4]. Hence, the exact sequence of $RG$-modules $0 \to M \to N \to L \to 0$ is $R$-pure exact. This completes the proof.

Corollary 3.5. Assume $sfl_i R < \infty$. If $Ghd_R G$ is finite, then there exists an $R$-pure $RG$-exact sequence $0 \to R \to A$, where $A$ is an $R$-flat $RG$-module such that $Ghd_R G = fd_{RG} A$.

Lemma 3.6. Let $\iota : R \to A$ be an $R$-pure monomorphism of $RG$-modules, where $A$ is $R$-flat. For any $RG$-module $N$, if $A \otimes_R N$ is a Gorenstein flat $RG$-module, then so is $N$.

Proof. Since $A \otimes_R N$ is a Gorenstein flat $RG$-module, there is an exact sequence of $RG$-modules $0 \to A \otimes_R N \overset{\alpha}{\to} F_0 \to L \to 0$, where $F_0$ is flat and $L$ is Gorenstein flat. Let $\beta = \alpha(\iota \otimes \text{Id}_N) : N \to F_0$, and consider the following commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & N & \overset{\iota \otimes \text{Id}_N}{\to} & A \otimes_R N & \overset{\alpha}{\to} & B \otimes_R N & \to 0 \\
0 & \to & N & \overset{\beta}{\to} & F_0 & \to & N' & \to 0 \\
& & L & \to & L & & \\
& & 0 & \to & 0 & & \\
\end{array}
$$

where $B = \text{Coker} \iota$, and $N' = \text{Coker} \beta$.

Let $E$ be any injective $RG$-module. It is clear that $\text{Id}_E \otimes \alpha : E \otimes_{RG} A \otimes_R N \to E \otimes_{RG} F_0$ is monic. Since $\iota : R \to A$ is an $R$-pure monomorphism, we infer that $\text{Id}_E \otimes \iota \otimes \text{Id}_N : E \otimes_{RG} R \otimes_R N = E \otimes_{RG} N \to E \otimes_{RG} A \otimes_R N$ is monic. Hence, $\text{Id}_E \otimes \beta : E \otimes_{RG} N \to E \otimes_{RG} F_0$ is a monomorphism, and moreover, we infer from the middle row of the above diagram that $\text{Tor}^1_{RG}(E, N') = 0$.

By applying $A \otimes_R -$ to the middle row, we get an exact sequence of induced $RG$-modules

$$0 \to A \otimes_R N \to A \otimes_R F_0 \to A \otimes_R N' \to 0,$$

where $A \otimes_R F_0$ is flat and $A \otimes_R N$ is assumed to be Gorenstein flat. Since $A$ is a flat $R$-module, we infer from $\text{Tor}^1_{RG}(E, N') = 0$ that $\text{Tor}^1_{RG}(E, A \otimes_R N') = 0$. Note that the class of all Gorenstein flat modules is always closed under extensions, we then get rid of the coherent assumption on rings in [16, Proposition 3.8], and obtain that $A \otimes_R N'$ is also a Gorenstein flat $RG$-module.
Proceed in this manner, we obtain an acyclic complex

\[
\begin{array}{c}
F_{\leq 0} = \\
0 \rightarrow N \rightarrow F_0 \rightarrow F_{-1} \rightarrow F_{-2} \rightarrow \cdots
\end{array}
\]

with each \( F_i \) being a flat \( RG \)-module, which remains acyclic after applying \( E \otimes_{RG} - \) for any injective \( RG \)-module \( E \). Now consider a flat resolution of \( N \):

\[
F_{>0} = \cdots \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow N \rightarrow 0.
\]

We will show that the complex \( E \otimes_{RG} F_{>0} \) is acyclic for any injective \( RG \)-module \( E \). Then, by pasting \( F_{\leq 0} \) and \( F_{>0} \) we will get a totally acyclic complex of flat \( RG \)-modules such that \( N \cong \ker(F_0 \rightarrow F_{-1}) \). This yields that \( N \) is a Gorenstein flat \( RG \)-module.

We now return to prove \( E \otimes_{RG} F_{>0} \) is acyclic. From the \( R \)-pure exact sequence of \( RG \)-modules \( 0 \rightarrow R \rightarrow A \rightarrow B \rightarrow 0 \), we induce an exact sequence of \( RG \)-modules

\[
0 \rightarrow E \rightarrow A \otimes_R E \rightarrow B \otimes_R E \rightarrow 0.
\]

Since the \( RG \)-module \( E \) is injective, the above sequence is split. Therefore, the complex \( E \otimes_{RG} F_{>0} \) is a direct summand of \( (A \otimes_R E) \otimes_{RG} F_{>0} \). Since \( A \) is \( R \)-flat, it follows that \( A \otimes_R F_{>0} \) is acyclic with each \( A \otimes_R F_i \) being a flat \( RG \)-module. Invoking the assumption that \( A \otimes_R N \) is a Gorenstein flat \( RG \)-module, we get \( \text{Tor}^i_{RG}(A \otimes_R N, E) = 0 \) for each \( i \geq 1 \), and this implies that the complex \( (A \otimes_R F_{>0}) \otimes_{RG} E \) is acyclic. Thus, the isomorphisms

\[
(A \otimes_R E) \otimes_{RG} F_{>0} \cong A \otimes_R (E \otimes_{RG} F_{>0})
\]

\[
\cong A \otimes_R (F_{>0} \otimes_{RG} E)
\]

\[
\cong (A \otimes_R F_{>0}) \otimes_{RG} E
\]

imply that \( (A \otimes_R E) \otimes_{RG} F_{>0} \) is acyclic. Hence, its direct summand \( E \otimes_{RG} F_{>0} \) is acyclic, as expected. This completes the proof. \( \square \)

**Proposition 3.7.** Assume \( sfli R < \infty \). Let \( M \) be any \( RG \)-module which is Gorenstein flat as an \( R \)-module. If there exists an \( R \)-pure monomorphism of \( RG \)-modules \( i : R \rightarrow A \), where \( A \) is \( R \)-flat with \( \text{fd}_{RG} A < \infty \), then \( \text{Gfd}_{RG} M \leq \text{fd}_{RG} A \).

**Proof.** Let \( \text{fd}_{RG} A = n \). There exists an exact sequence \( 0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \) for which each \( F_i \) is a flat \( RG \)-module. Since \( A \) is a flat \( R \)-module and each \( F_i \) restricts to be a flat \( R \)-module, the sequence is \( R \)-pure exact.

By applying \( - \otimes_R M \) to the above \( R \)-pure exact sequence, we get an exact sequence of \( RG \)-modules

\[
0 \rightarrow F_n \otimes_R M \rightarrow \cdots \rightarrow F_1 \otimes_R M \rightarrow F_0 \otimes_R M \rightarrow A \otimes_R M \rightarrow 0.
\]

By Lemma 3.3 \( F_i \otimes_R M \) are Gorenstein flat \( RG \)-modules, and then \( \text{Gfd}_{RG}(A \otimes_R M) \leq n \).

Now we consider a flat resolution \( \cdots \rightarrow F'_n \rightarrow \cdots \rightarrow F'_1 \rightarrow F'_0 \rightarrow M \rightarrow 0 \) of the \( RG \)-module \( M \), and let \( K = \ker(F'_{n-1} \rightarrow F'_{n-2}) \). By applying \( A \otimes_R - \), we get an exact sequence
0 \to A \otimes_R K \to A \otimes_R F'_{n-1} \to \cdots \to A \otimes_R F'_0 \to A \otimes_R M \to 0$, where $A \otimes_R F'_i$ are flat $RG$-modules. We infer from Gfd$_{RG}(A \otimes_R M) \leq n$ that $A \otimes_R K$ is a Gorenstein flat module. Hence, it follows from Lemma 3.6 that $K$ is a Gorenstein flat $RG$-module, and then the desired inequality Gfd$_{RG}M \leq n$ holds.

\[\square\]

**Corollary 3.8.** Assume sfil$_R < \infty$. If there exists an $R$-pure monomorphism of $RG$-modules $\iota : R \to A$, where $A$ is $R$-flat with fd$_{RG}A < \infty$, then for any $RG$-module $M$, we have

\[Gfd_{RG}M \leq fd_{RG}A + Gfd_{RG}M \leq fd_{RG}A + sfilR.\]

**Proof.** Let fd$_{RG}A = n$. Since sfil$_R < \infty$, we assume that for an $RG$-module $M$, Gfd$_{RG}M = m$ is finite. In this case, we may prove the result by induction on $m$. If $m = 0$, that is, $M$ is a Gorenstein flat $R$-module, then the assertion is immediate from Proposition 3.7.

We now assume $m > 0$, and consider a short exact sequence $0 \to K \to F \to M \to 0$ of $RG$-modules, where $F$ is flat. Since $F$ is restricted to be a flat $R$-module, we have Gfd$_{RG}K = m - 1$. Invoking the induction hypothesis, we may conclude that Gfd$_{RG}K \leq n + (m - 1)$, and hence Gfd$_{RG}M \leq n + m$. This proves the first inequality, and the second inequality is clear. \[\square\]

Now, we are in a position to state the main result of this section.

**Theorem 3.9.** Let $G$ be a group, $R$ a commutative ring with sfil$_R < \infty$. The following are equivalent:

1. Ghd$_{RG}G$ is finite.
2. There exists an $R$-pure $RG$-exact sequence $0 \to R \to A$, where $A$ is an $R$-flat $RG$-module of finite flat dimension.
3. Any $RG$-module has finite Gorenstein flat dimension.
4. sfil$_{RG}$ is finite.

In this case, we have an equality Ghd$_{RG}G = fd_{RG}A$ and the following inequalities

\[sfilR \leq sfil_{RG} = G.wgldim{RG} \leq Ghd_{RG}G + sfilR.\]

**Proof.** The implication (1)$\implies$(2) follows by Corollary 3.5, and the implication (2)$\implies$(3) follows from Corollary 3.8. The implication (3)$\implies$(1) is trivial, and (3)$\iff$(4) follows from [9, Theorem 5.3].

The inequality sfil$_{RG} = G.wgldim{RG} \leq Ghd_{RG}G + G.wgldimR = Ghd_{RG}G + sfilR$ follows from Corollary 3.5, 3.8 and [9, Theorem 5.3]. For the first inequality sfil$_R \leq sfil_{RG}$, it suffices to assume that sfil$_{RG} = n$ is finite. We consider any injective $R$-module $I$. Let $H = \{1\}$ be the subgroup of $G$. Note that Coind$_{H}^{G}I$ is an injective $RG$-module. Then fd$_{RG}Coind_{H}^{G}I \leq n$, which induces fd$_{R}Coind_{H}^{G}I \leq n$. Moreover, as $R$-modules $I$ is a direct summand of Coind$_{H}^{G}I$, and hence fd$_{R}I \leq n$. This yields that sfil$_R \leq n$. \[\square\]
4. Properties of Gorenstein homology of groups

Using the above characterization on the finiteness of \( \text{Ghd}_R G \), we obtain the following properties of the Gorenstein homological dimensions of groups.

**Proposition 4.1.** Let \( H \) be any subgroup of \( G \). If \( \text{sfl}_I R < \infty \), then \( \text{Ghd}_R H \leq \text{Ghd}_R G \).

**Proof.** There is nothing to prove if \( \text{Ghd}_R G = \infty \) and hence we may assume that \( \text{Ghd}_R G = n \) is finite. It follows from Corollary [3.5] that there exists an \( R \)-pure \( RG \)-exact sequence \( 0 \to R \to A \) for which \( A \) is \( R \)-flat such that \( \text{fd}_{RG} A = n \). Note that every flat \( RG \)-module is restricted to be a flat \( RH \)-module, and \( 0 \to R \to A \) is also an exact sequence of \( RH \)-modules. We infer that \( \text{fd}_{RH} A \leq \text{fd}_{RG} A \), and moreover, for the trivial \( RH \)-module \( R \), by Proposition [3.7] we have \( \text{Ghd}_R H = \text{Gfd}_{RH} R \leq \text{fd}_{RH} R \leq n \). This completes the proof. \( \Box \)

The following generalizes [1, Proposition 4.11]: let \( H \) be a subgroup of \( G \) of finite index, then \( \text{Ghd}_Z H \leq \text{Ghd}_Z G \). We remove the assumption of finite index therein.

**Corollary 4.2.** Let \( H \) be any subgroup of \( G \). Then \( \text{Ghd}_Z H \leq \text{Ghd}_Z G \), and \( \text{Ghd}_Q H \leq \text{Ghd}_Q G \).

**Proposition 4.3.** Let \( 1 \to H \to G \to L \to 1 \) be an extension of groups. If \( \text{sfl}_I R < \infty \), then

\[
\text{Ghd}_R G \leq \text{Ghd}_R H + \text{Ghd}_R L.
\]

**Proof.** It suffices to assume that both \( \text{Ghd}_R H = m \) and \( \text{Ghd}_R L = n \) are finite. Then, it follows from Corollary [3.5] that there exists an \( R \)-pure \( RL \)-exact sequence \( 0 \to R \to A \) for which \( A \) is \( R \)-flat such that \( \text{fd}_{RL} A = \text{Ghd}_R L = n \).

For the quotient group \( L = G/H \), we may consider every \( RL \)-module as an \( RG \)-module through the natural morphism of group rings \( RG \to RL \). We claim that for any flat \( RL \)-module \( F \), one has \( \text{Gfd}_{RG} F \leq \text{Ghd}_R H \). Note that \( RL = R[G/H] = \text{Ind}^G_H R \); see for example [3, Proposition III 5.6]. Since \( \text{Ghd}_R H = \text{Gfd}_{RH} R = m \), there is a Gorenstein flat \( RH \)-resolution

\[
0 \to M_m \to \cdots \to M_1 \to M_0 \to R \to 0
\]

of length \( m \). By applying the induction functor, we get an exact sequence of \( RG \)-modules

\[
0 \to \text{Ind}^G_H M_m \to \cdots \to \text{Ind}^G_H M_1 \to \text{Ind}^G_H M_0 \to \text{Ind}^G_H R = RL \to 0.
\]

By Lemma [3.2] we infer that all \( \text{Ind}^G_H M_i \) are Gorenstein flat \( RG \)-modules. Then, we get that \( \text{Gfd}_{RG} RL \leq m \), and moreover, by [16, Proposition 3.13] we induce that \( \text{Gfd}_{RG} P \leq m \) for any free \( RL \)-module \( P \). For any flat \( RL \)-module \( F \), it follows from Lazard-Govorov Theorem that \( F = \lim P_i \) is a direct limit of finitely generated free \( RL \)-modules \( P_i \). Furthermore, since the subcategory of Gorenstein flat modules is closed under direct limits (see [23, Corollary 4.12] and [24, Lemma 3.1]), we prove the claim that \( \text{Gfd}_{RG} F = \text{Gfd}_{RG} (\lim P_i) \leq m = \text{Ghd}_R H \).

Recall that \( \text{fd}_{RL} A = \text{Ghd}_R L = n \). We consider an exact sequence of \( RL \)-modules

\[
0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0
\]

where
for which $F_i$ are flat $RL$-modules. It follows from that above argument that $\text{Gfd}_{RG}F_i \leq m$ for $0 \leq i \leq n$. By a standard argument, we infer from the above exact sequence that $\text{Gfd}_{RG}A \leq m + n$.

Since $A$ is an $R$-flat module and $\text{Gfd}_{RG}A \leq m + n$ is finite, it follows from Proposition 3.4 that there is an $R$-pure $RG$-exact sequence $0 \to A \xrightarrow{\beta} B$, where $B$ is an $R$-flat $RG$-module such that $\text{fd}_{RG}B = \text{Gfd}_{RG}A \leq m + n$. Let $\iota = \beta\alpha$. Then, we obtain an $R$-pure $RG$-exact sequence $0 \to R \xrightarrow{\iota} B$, and consequently, it follows from Proposition 3.7 that

$$\text{Ghd}_{RG} = \text{Gfd}_{RG} \leq \text{fd}_{RG}B \leq m + n.$$ 

This completes the proof. \hfill \Box

**Proposition 4.4.** Let $H$ be a finite normal subgroup of $G$, and $R$ be a commutative ring such that $\text{sfli} R < \infty$. Then $\text{Ghd}_{RG} = \text{Ghd}_R(G/H)$.

**Proof.** By Proposition 4.3, we have $\text{Ghd}_{RG} \leq \text{Ghd}_R + \text{Ghd}_R(G/H)$. By [19, Theorem 3.1] the subgroup $H$ is finite if and only if $\text{Ghd}_R = 0$, which yields $\text{Ghd}_{RG} \leq \text{Ghd}_R(G/H)$.

Since the inequality $\text{Ghd}_R(G/H) \leq \text{Ghd}_R(G)$ is obvious if $\text{Ghd}_R = \infty$, it suffices to assume $\text{Ghd}_R = n$ is finite. In this case, it follows immediately from Corollary 3.5 that there exists an $R$-pure $RG$-exact sequence $0 \to R \xrightarrow{\iota} A$ for which $A$ is an $R$-flat such that $\text{fd}_{RG}A = n$. Since the group $G$ acts trivially on $R$, it implies that $\text{lim} \iota \subseteq A^G \subseteq A^H$. We may therefore consider the $R[G/H]$-module $A^H$, and the $R$-pure $R[G/H]$-monomorphism $R \to A^H$.

In the following, we will prove that $A^H$ is $R$-flat and $\text{fd}_{R[G/H]} A^H \leq n$. Then, the desired inequality $\text{Ghd}_R(G/H) = \text{Gfd}_{R[G/H]} R \leq n$ will hold by Proposition 3.7.

Since $A$ is $R$-flat with $\text{fd}_{RG}A < \infty$ and $H$ is a finite group, we infer that $A$ is a flat $RH$-module, and then $A = \lim P_i$ for some finitely generated free $RH$-module. Since $(RH)^H \cong R$, we imply that $P^H_i$ are finitely generated free $R$-modules. Note that for any $RG$-module $M$, $M^H \cong \text{Hom}_{RH}(R, M)$ as $R[G/H]$-modules; see for example [5, pp.56]. Since $H$ is a finite group, $R$ is a finitely presented $RH$-module. Hence, we induce from the isomorphisms

$$A^H \cong \text{Hom}_{RH}(R, \lim P_i) \cong \lim \text{Hom}_{RH}(R, P_i) \cong \lim P^H_i$$

that $A^H$ is a flat $R$-module.

For the finite group $H$, $R \to RH$ is a Frobenius extension of rings, and then $\text{RHom}(RH, R) \cong RH$ in the derived category $\text{D}(RH)$. Hence, we have

$$\text{RHom}_{RH}(R, RH) \cong \text{RHom}_{RH}(R, \text{RHom}_R(RH, R))$$

$$\cong \text{RHom}_R(R \otimes^{L}_{RH} RH, R) \cong \text{RHom}_R(R, R) = R,$$

which yield that $\text{Ext}^1_{RH}(R, RH) = 0$. This result can also be observed if one notices that $R$ is a Gorenstein projective $RH$-module; see [21, Theorem 2.2]. Moreover, we infer that $\text{Ext}^1_{RH}(R, F) = 0$ for any flat $RH$-module $F$. 

11
Since \( \text{fd}_{RG}A = n \), there is an exact sequence \( 0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \) for which \( F_i \) are flat \( RG \)-modules. Analogous to the above argument, we get that \( F_i^H \cong \text{Hom}_{RH}(R, F) \) are flat modules over \( R[G/H] \). By restriction, \( F_i \) are flat \( RH \)-modules, and then \( \text{Ext}^1_{RH}(R, F_i) = 0 \). By applying \( \text{Hom}_{RH}(R, -) \) to the above sequence, we can obtain an exact sequence of \( R[G/H] \)-modules

\[
0 \longrightarrow F_n^H \longrightarrow \cdots \longrightarrow F_1^H \longrightarrow F_0^H \longrightarrow A^H \longrightarrow 0.
\]

Hence, \( \text{fd}_{R[G/H]}A^H \leq n \), as expected. This completes the proof.

The Weyl groups of the subgroups of a given group \( G \) are useful tools in the study of actions of \( G \) on topological spaces, as these Weyl groups act naturally on the fixed point spaces of the actions. Let \( H \) be a subgroup of \( G \). We denote by \( N_G(H) \) the normalizer of \( H \) in \( G \).

Corollary 4.5. Let \( H \) be a finite subgroup of \( G \), and \( R \) be a commutative ring such that \( \text{sfli}R < \infty \). Then for the Weyl group \( W = N_G(H)/H \), one has \( \text{Ghd}_R W \leq \text{Ghd}_R G \).

Proof. In view of Proposition 4.4, we have \( \text{Ghd}_R N_G(H) \leq \text{Ghd}_R G \). It follows from Proposition 4.4 that \( \text{Ghd}_R W = \text{Ghd}_R N_G(H) \). Then, we get the inequality \( \text{Ghd}_R W \leq \text{Ghd}_R G \).

For a group \( G \), recall that Ikenaga introduced the generalized homological dimension of \( G \) over the ring of integers \( \mathbb{Z} \); see [17, III, Definition]. Analogously, we may define the generalized homological dimension of \( G \) over any coefficient ring \( R \) as follows:

\[
\text{hd}_RG = \sup\{i \in \mathbb{N} \mid \text{Tor}_i^{RG}(M, I) \neq 0, \exists M \text{ R-flat}, \exists I \text{ RG-injective}\}.
\]

We conclude this section by comparing the Gorenstein homological dimension and generalized homological dimension of groups.

Proposition 4.6. Let \( R \) be a commutative ring of coefficients. If \( \text{Ghd}_RG < \infty \), then \( \text{Ghd}_RG \leq \text{hd}_RG \). Moreover, if \( \text{sfli}R < \infty \), then \( \text{Ghd}_RG = \text{hd}_RG \).

Proof. Assume \( \text{Ghd}_RG = \text{Gfd}_{RG}R \) is finite. It follows from [16, Theorem 3.14] that

\[
\text{Ghd}_RG = \sup\{i \in \mathbb{N} \mid \text{Tor}_i^{RG}(M, I) \neq 0, \exists M \text{ R-flat}, \exists I \text{ RG-injective}\}.
\]

Then, we infer from the definition of \( \text{hd}_RG \) that \( \text{Ghd}_RG \leq \text{hd}_RG \).

Moreover, if we assume \( \text{sfli}R < \infty \), then every \( RG \)-module has finite Gorenstein flat dimension; see Theorem 3.9. For any \( RG \)-module \( M \), by [19, Proposition 3.2] we have

\[
\text{Gfd}_{RG}M \leq \text{Ghd}_RG + \text{fd}_RM,
\]

and so \( \text{hd}_RG = \sup\{\text{Gfd}_{RG}M \mid M \text{ R-flat}\} \leq \text{Ghd}_RG \). Hence, the equality \( \text{Ghd}_RG = \text{hd}_RG \) holds.

In the following, we denote by \( \text{wgldim}R \) the weak global dimension of \( R \).
Proposition 4.7. For any group $G$ and any commutative ring $R$, there are inequalities 
\[ \hd R G \leq \sfli R G \leq \hd R G + \text{wgl.dim}R. \]

Proof. If we assume $\sfli R G < \infty$, it follows from [9, Theorem 5.3] that $\sfli R G = G.\text{wgl.dim}R G$, and then the first inequality is clear. In order to prove the second inequality, it suffices to assume that both $\hd R G = m$ and $\text{wgl.dim}R = n$ are finite. In this case, we may prove the result by induction on $n$.

If $n = 0$, then every $R$-module is flat, and moreover, $\hd R G = G.\text{wgl.dim}R G$ holds immediately from the definitions. Note that $\hd R G$ is assumed to be finite, and then we have $\sfli R G = \hd R G$.

Now assume $n > 0$. For any $RG$-module $M$, we consider the exact sequence of $RG$-modules 
\[ 0 \to K \to P \to M \to 0 \]
for which $P$ is projective. Since $P$ is restricted to be a projective $R$-module, as an $R$-module we have $\text{fd} R K \leq n - 1$. For any injective $RG$-module $I$ and any $i > 0$, there are isomorphisms $\text{Tor}^{RG}_i(M, I) \cong \text{Tor}^{RG}_{i-1}(K, I)$. Invoking the induction hypothesis, we infer that $\text{Tor}^{RG}_{i-1}(K, I) = 0$ for any $i - 1 > m + (n - 1)$, and then $\text{Tor}^{RG}_i(M, I) = 0$ for any $i > m + n$. Hence, we have $\text{fd} R I \leq m + n$, and consequently, $\sfli R G \leq m + n$. □

Corollary 4.8. Let $R$ be a commutative ring with finite weak global dimension. If $\hd R G$ is finite, then $\text{Ghd}_R G = \hd R G$.

Proof. Following the assumptions, we infer that $\sfli R G \leq \hd R G + \text{wgl.dim}R < \infty$. By Theorem 3.9, this induces $\text{Ghd}_R G < \infty$, and then the equality holds by Proposition 4.6 immediately. □

Corollary 4.9. $\text{Ghd}_Z G < \infty$ if and only if $\hd Z G < \infty$. In this case, $\text{Ghd}_Z G = \hd Z G$.

By [17, Proposition 7], if $G$ is a finite group then $\hd Z G = 0$. This result can be generalized as follows. In view of Corollary 4.9 and [1, Proposition 4.12], we infer that the converse also holds. Compare to [8, Theorem A] for a similar result with respect to $\text{cd}_Z G$, the generalized cohomological dimension of the group.

Corollary 4.10. Let $G$ be a group. Then $G$ is a finite group if and only if $\hd Z G = 0$.

5. Some relative homological dimensions and invariants

Let $\Lambda$ be any associative ring with identity. We consider an acyclic complex of projective $\Lambda$-modules
\[ P = \cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \]
and a $\Lambda$-module $M \cong \text{Ker}(P_0 \to P_{-1})$. Recall that the module $M$ is $\text{Gorenstein projective}$ [12] if the complex $P$ remains acyclic after applying $\text{Hom}_\Lambda(-, Q)$ for any projective module $Q$; $M$ is called a $\text{projectively coresolved Gorenstein flat module}$, or a $\text{PGF-module}$ for short [23], provided that the complex $P$ remains acyclic after applying $I \otimes_\Lambda -$ for any injective right $\Lambda$-module $I$. It is clear that PGF-modules are Gorenstein flat. As shown in [23, Theorem 4.4], PGF-modules are also Gorenstein projective.
For any module $M$, the *Gorenstein projective dimension*, denoted by $\text{Gpd}_\Lambda M$, is defined in the standard way by using resolutions by Gorenstein projective modules; see [12, 16]. The *Gorenstein global dimension* of $\Lambda$, denoted by $\text{G.gldim}\Lambda$, is defined as the supremum of Gorenstein projective dimension of all $\Lambda$-modules [4]. Analogously, the PGF-dimension $\text{PGF-dim}_\Lambda M$ of any $\Lambda$-module $M$, and the PGF-global dimension $\text{PGF-gldim}\Lambda$ are introduced in [7]. The Gorenstein projective dimension is a refinement of the PGF-dimension in the sense that $\text{Gpd}_\Lambda M \leq \text{PGF-dim}_\Lambda M$ for any module $M$, and $\text{Gpd}_\Lambda M = \text{PGF-dim}_\Lambda M$ if $\text{PGF-dim}_\Lambda M < \infty$; see [7, Corollary 3.7].

There is a widely accepted result (see [9, Theorem 4.1]) that $\text{G.gldim}\Lambda < \infty$ if and only if $\text{spli}\Lambda = \text{silp}\Lambda < \infty$; moreover, under these finiteness conditions, one has

$$\text{G.gldim}\Lambda = \text{spli}\Lambda = \text{silp}\Lambda.$$  

For PGF-global dimension there is an interesting result [7, Theorem 5.1] due to Dalezios and Emmanouil, which states that $\text{PGF-gldim}\Lambda < \infty$ if and only if $\text{spli}\Lambda = \text{silp}\Lambda < \infty$ and $\text{sfli}\Lambda = \text{sfli}\Lambda^{\text{op}} < \infty$; in this case one has

$$\text{PGF-gldim}\Lambda = \text{spli}\Lambda = \text{silp}\Lambda = \text{G.gldim}\Lambda.$$  

In the following, we will compare invariants $\text{sfli}$, $\text{siff}$, $\text{spli}$ and $\text{silp}$, which are closely related to Gorenstein projective, Gorenstein flat and PGF dimensions.

Let $\Lambda$ be a ring. Gedrich and Gruenberg [14, 1.6] noted that the finiteness of both $\text{spli}\Lambda$ and $\text{silp}\Lambda$ implies that $\text{spli}\Lambda = \text{silp}\Lambda$. Jensen has proved in [18, 5.9] that the equality $\text{spli}\Lambda = \text{silp}\Lambda$ holds if $\Lambda$ is a commutative noetherian ring. By [8, Theorem 4.4], for any group $G$, $\text{spli}RG = \text{silp}RG$ if $R$ is a *commutative Gorenstein ring* (noetherian ring with finite self-injective dimension).

Since projective modules are flat, for any ring $\Lambda$ it is immediate that $\text{silp}\Lambda \leq \text{siff}\Lambda$. In fact, Emmanouil and Talelli have proven in [10, Proposition 2.1] that the equality always holds.

**Lemma 5.1.** Let $G$ be any group, $R$ be a commutative ring. then $\text{silp}RG = \text{siff}RG$.

Let $R$ be a commutative Gorenstein ring. It follows from [14, Theorem 2.4] that $\text{spli}RG < \infty$ implies $\text{silp}RG < \infty$. The finiteness of $\text{sfli}RG$ and $\text{siff}RG$ will be compared as follows. We shall also consider the invariant $\text{splf}RG$, which is defined as the supremum of projective length (dimension) of flat $RG$-modules. The invariant is due to Raynaud and Gruson [20, Section II.3.3], while the notation “splf” is introduced in [10]. We will show that the comparison between $\text{sfli}RG$ and $\text{siff}RG$ is essentially about a problem on the relation between the projective and flat dimensions of modules. Let $\overline{P}(RG)$ and $\overline{F}(RG)$ denote the classes of $RG$-modules with finite projective dimension and finite flat dimension, respectively.

**Proposition 5.2.** Let $R$ be a commutative Gorenstein ring, $G$ be a group. The following are equivalent:

1. $\text{siff}RG < \infty$. 

### Proposition 5.2.** Let $R$ be a commutative Gorenstein ring, $G$ be a group. The following are equivalent:

1. $\text{siff}RG < \infty$. 

(2) $\text{sfli}RG < \infty$ and $\text{splf}RG < \infty$.

(3) $\text{sfli}RG < \infty$ and $\overline{P}(RG) = \overline{F}(RG)$.

Proof. $(1) \implies (2)$ Since the ring $R$ is commutative Gorenstein, combining with [8, Theorem 4.4] and Lemma 5.1 we have $\text{spli}RG = \text{silp}RG = \text{silf}RG$. Hence, we infer from $\text{silf}RG < \infty$ that $\text{sfli}RG \leq \text{spli}RG < \infty$. Moreover, the finiteness of both $\text{silf}RG$ and $\text{spli}RG$ yields that the projective dimension of any flat $RG$-module is finite, and hence $\text{splf}RG < \infty$.

$(2) \implies (1)$ Let $I$ be any injective $RG$-module. The statements of (2) imply that the projective dimension of $I$ is finite, hence $\text{spli}RG < \infty$. By [14, Theorem 2.4], $\text{silp}RG$ is also finite, and then $\text{spli}RG = \text{silp}RG$. By Lemma 5.1 it follows that $\text{silf}RG = \text{silp}RG < \infty$.

$(2) \iff (3)$ Since projective modules are flat, it is clear that $\overline{P}(RG) \subseteq \overline{F}(RG)$. Moreover, it is easy to see that the projective dimension of any flat $RG$-module is finite if and only if $\overline{F}(RG) \subseteq \overline{P}(RG)$. Hence, the assertion follows.

Every projective module is flat, however, it is not at all clear from the definitions that Gorenstein projective modules are Gorenstein flat. Holm has shown in [16, Proposition 3.4] that every left Gorenstein projective module is Gorenstein flat if the base ring is right coherent and has finite left finitistic dimension. In general, the relation between Gorenstein projective and Gorenstein flat modules remains somehow mysterious.

We have the following observation.

**Proposition 5.3.** Let $G$ be a group and $R$ be a commutative ring. If $\text{Ghd}_RG < \infty$, then $\text{sfli}R < \infty$ if and only if $\text{sfli}RG < \infty$. Moreover, if both $\text{Ghd}_RG$ and $\text{sfli}R$ are finite, then every Gorenstein projective $RG$-module is a PGF-module, and furthermore is Gorenstein flat.

Proof. The “if” part follows from the inequality $\text{sfli}R \leq \text{sfli}RG$. The “only if” part follows from Theorem 3.9.

Note that the finiteness of both $\text{Ghd}_RG$ and $\text{sfli}R$ implies $\text{sfli}RG < \infty$. In this case, by induction on the flat dimension of any injective $RG$-module $I$, we infer that any acyclic complex of projective $RG$-modules remains acyclic after applying $I \otimes_{RG} -$. Hence, invoking the definitions it is immediate that every Gorenstein projective $RG$-module is a PGF-module, and moreover, is a Gorenstein flat module; see also [9, Remark 2.3 (ii)].

**Corollary 5.4.** Let $R$ be a commutative Gorenstein ring, $G$ be any group. If $\text{silf}RG (= \text{silp}RG)$ is finite, then the following hold:

1. $\text{Ghd}_RG < \infty$, and every Gorenstein projective $RG$-module is Gorenstein flat.
2. For any Gorenstein flat $RG$-module $M$, $\text{PGF-dim}_{RG}M = \text{Gpd}_{RG}M < \infty$.

Proof. By Proposition 5.2 the finiteness of $\text{silf}RG$ implies both $\text{sfli}R < \infty$ and $\text{splf}RG < \infty$. Then the assertion (1) follows by Theorem 3.9 and Proposition 5.3, and the assertion (2) is immediate from [7, Proposition 3.9] and [7, Corollary 3.7 (2)].
It follows from [12, Proposition 10.2.3] that any Gorenstein projective module of finite projective dimension is necessarily projective. For modules over group rings, the hypothesis on the finiteness of the projective dimension may be relaxed as shown below.

**Corollary 5.5.** Let $R$ be a commutative Gorenstein ring, and $G$ be a group. If $\text{silf}RG < \infty$, then any Gorenstein projective $RG$-module of finite flat dimension is necessarily projective.

Inspired by [1, Theorem 3.7], we have the following.

**Proposition 5.6.** Let $R$ be a commutative Gorenstein ring. Let $G$ be a group such that $RG$ is a coherent ring. Then $\text{silf}RG < \infty$ if and only if $\text{silf}RG < \infty$. In this case $\text{silf}RG = \text{silf}RG$.

**Proof.** It follows from Proposition 5.2 that for a commutative Gorenstein ring $R$ and any group $G$, the “if” part holds, that is, $\text{silf}RG < \infty$ implies $\text{silf}RG < \infty$. By Lemma 5.1, we have $\text{silf}RG = \text{silp}RG$. Moreover, we infer from [8, Proposition 4.2] that there is an inequality $\text{silf}RG \leq \text{silp}RG$. Hence, we have $\text{silf}RG \leq \text{silf}RG$.

For the “only if” part, we assume that $RG$ is a coherent ring, and $\text{silf}RG = n < \infty$. Let $F$ be any flat $RG$-module. Consider the exact sequence $0 \to F \to I_0 \to \cdots \to I_{n-1} \to L \to 0$ of $RG$-modules, for which $I_i$ ($0 \leq i \leq n-1$) are injective modules. Denote by $(-)^+$ the Pontryagin dual $\text{Hom}_Z(-, \mathbb{Q}/\mathbb{Z})$. Then we get an exact sequence

$$0 \to L^+ \to I_{n-1}^+ \to \cdots \to I_1^+ \to I_0^+ \to F^+ \to 0.$$ 

It is clear that $F^+$ is an injective $RG$-module. Since $RG$ is coherent, it follows that $I_i^+$ are flat $RG$-modules for $0 \leq i \leq n-1$. Hence, the assumption $\text{silf}RG = n$ implies that $L^+$ is a flat module, and consequently, $L$ is an injective module. Thus, $\text{silf}RG \leq n$. This completes the proof.

**Remark 5.7.** Hirschhorn characterized in [15] a class of groups $G$ such that $ZG$ is a coherent ring, and $G \simeq \pi_1X$ for some CW-complex $X$ with finite skeletons. The class of groups includes all finite groups, finitely generated abelian groups, finitely generated nilpotent groups, finitely generated free groups, and free products of any of these.

Gedrich and Gruenberg proved in [14, Theorem 2.4] that $\text{silp}RG \leq \text{spli}RG$ if the coefficient ring $R$ is commutative noetherian. This inequality was generalized recently by Dalezios and Emmanouil to any commutative coefficient ring $R$; see [7, Corollary 5.4]. In fact, if $R$ is noetherian, then $\text{silf}RG = \text{silp}RG \leq \text{spli}RG$ with the equality if the latter is finite. Moreover, we have the following.

**Proposition 5.8.** Let $R$ be a commutative ring, and $G$ be a group. Then $\text{spli}RG < \infty$ if and only if $\text{PGF-gldim}RG = G.\text{gldim}RG < \infty$; in this case, $\text{PGF-gldim}RG = \text{spli}RG$.

**Proof.** It suffices to prove the “only if” part. It follows from [7, Corollary 5.4] and [14, 1.6] that $\text{silp}RG \leq \text{spli}RG$ with the equality if $\text{spli}RG < \infty$. Invoking the finiteness of $\text{spli}RG$, we infer
that $\text{sfli}(RG)^{op} = \text{sfli}RG < \infty$. Then the assertion and the equality $\text{PGF-gldim}RG = \text{spli}RG$ follow immediately from [7, Theorem 5.1].

It is well known that flat dimension of any module is a refinement of its projective dimension, while it is not clear if the inequality $\text{Gfd}M \leq \text{Gpd}M$ holds “locally” for any module $M$. However, for group rings the inequality follows “globally” as shown below.

**Proposition 5.9.** Let $R$ be an commutative ring and $G$ a group. Then there is an inequality 
\[ \text{G.wgldim}RG \leq \text{PGF-gldim}RG = \text{G.gldim}RG. \]

**Proof.** The inequality $\text{G.wgldim}RG \leq \text{G.gldim}RG$ follows from [4, Corollary 1.2 (1)]. For completeness, we briefly include an argument in [9, Remark 5.4 (ii)], which is different from that of [4]. The inequality is obvious if $\text{G.gldim}RG = \infty$ and hence we may assume that $\text{G.gldim}RG = n < \infty$. Then it follows from [9, Theorem 4.1] that $\text{silp}RG = \text{spli}RG = n$. By the definition, we have $\text{sfli}RG \leq \text{spli}RG$. Therefore, it follows from [9, Theorem 5.3] that $\text{G.wgldim}RG = \text{sfli}RG \leq n$.

It remains to prove the equality $\text{PGF-gldim}RG = \text{G.gldim}RG$. It follows from [23, Theorem 4.4] that every PGF-module is Gorenstein projective, and then $\text{G.gldim}RG \leq \text{PGF-gldim}RG$. Conversely, in proving the inequality $\text{G.gldim}RG \geq \text{PGF-gldim}RG$, it suffices to assume that $\text{G.gldim}RG = n$ is finite. Then $\text{sfli}RG \leq \text{spli}RG = \text{silp}RG = n$. Hence, we infer from [7, Theorem 5.1] that $\text{PGF-gldim}RG < \infty$, and moreover $\text{G.gldim}RG = \text{PGF-gldim}RG$. □

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