Low-Temperature Specific Heat of an Extreme Type-II Superconductor at High Magnetic Fields

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We present a detailed study of the quasiparticle contribution to the low-temperature specific heat of an extreme type-II superconductor at high magnetic fields. Within a T-matrix approximation for the self-energies in the mixed state of a homogeneous superconductor, the electronic specific heat is a linear function of temperature with a linear-T coefficient $\gamma_s(T)$ being a nonlinear function of magnetic field $H$. In the range of magnetic fields $H > 0.15 - 0.2 H_{c2}$ where our theory is applicable, the calculated $\gamma_s(H)$ closely resembles the experimental data for the borocarbide superconductor YNi$_2$B$_2$C.

PACS numbers: 74.25Bt, 74.60Ec, 74.70Dd

In the last two decades and particularly since the discovery of high temperature superconductors (HTS) almost all of the superconducting systems that hold the greatest promise for practical application are of the extreme-type-II variety. These materials are characterized by high transition temperatures ($T_c$), high upper critical fields ($H_{c2}$), and can be defined as materials in which the semiclassical $H_{c2}(0)$ in units of Tesla becomes comparable to, or even larger than, $T_c$ in units of Kelvin. In such systems, at low temperature and high magnetic fields near and around the semiclassical $H_{c2}(0)$, Landau level quantization of electronic energies within the superconducting state is well defined and has to be included in the description of the superconducting instability. Such a regime in which the Landau level structure is well resolved, i.e. cyclotron energy $\hbar \omega_c \geq \Delta(T, H), T, \Gamma$ (where $\Delta(T, H)$ is the BCS gap, and $\Gamma$ is the scattering rate due to disorder), represents a large portion of the $H - T$ phase diagram of an intrinsically extreme type-II superconductor. This region can extend down to magnetic fields as low as $\sim (0.2 - 0.5)H_{c2}(0)$ and temperatures as high as $\sim 0.3T_{c0}$. In contrast, the size of the similar region in an ordinary type-II superconductor (such as Nb) is expected to be negligible and confined to an immediate vicinity of $H_{c2}(0)$. Inside this high-field, low-temperature regime the superconducting state fundamentally differs from the familiar low-field mixed phase of the Abrikosov-Gorkov theory, primarily by the appearance of gapless quasiparticle excitations at the Fermi surface. These gapless excitations reflect a coherent quasiparticle propagation over many unit cells of a closely packed vortex lattice with fully overlapping vortex cores.

The presence of such low-lying quasiparticle excitations makes an s-wave, “conventional” superconductor in a high magnetic field somewhat similar to an anisotropic, “unconventional” superconductor with nodes in the gap. In the low-temperature and high-field regime, however, the nodes in the gap reflect the center-of-mass motion of the Cooper pairs in the magnetic field, in contrast to d-wave superconducting cuprates where such nodes are due to the relative orbital motion. This gapless behavior in three-dimensional systems is not restricted to fields very close to $H_{c2}$ but rather persists to a surprisingly low magnetic fields $H^* \sim (0.2 - 0.5)H_{c2}$. Below $H^*$ gaps start opening up in the quasiparticle spectrum and the system eventually reaches the low-field regime of localized states in the cores of isolated, well-separated vortex cores. Recently, an extensive numerical calculation of quasiparticle excitations in the mixed state for both s-wave and d-wave superconductors was performed and it was found that for fields $H > 0.5H_{c2}$ no qualitative difference in behavior can be seen between s- and d-wave cases. They are both characterized by coherent low-lying Landau level-like quasiparticles excitations. However, a marked difference appears at lower fields $H \ll 0.5H_{c2}$, where an s-wave superconductor is clearly in the regime of localized, bound vortex core states while a d-wave system still exhibits the extended nature of low-lying quasiparticle excitations as predicted by Franz and Tešanović. Furthermore, it was shown that the high-field gapless character of the excitation spectrum is not destroyed by a moderate level of nonmagnetic impurities present in either dirty homogeneous superconductor or dirty inhomogeneous superconducting systems.

The strongest evidence for Landau level quantization within the superconducting state comes from the experimental observation of de Haas-van Alphen (dHvA) oscillations in various A-15 and borocarbide superconductors. The persistence of the dHvA signal deep within the mixed state of these three-dimensional extreme type-II systems can be attributed to the presence of a small portion of the Fermi surface containing gapless quasiparticle excitations, surrounded by regions where the gap is large. At the same time, careful measurements of thermal properties (i.e. thermal transport and/or specific heat) at low temperatures and high magnetic fields are also expected to reveal the novel
gapless behavior in these systems. The presence of extended gapless quasiparticle states at low temperatures should lead to qualitatively different thermal behavior than those found in an s-wave superconductor at low fields, where the number of quasiparticles excited above the gap is exponentially small and the only contribution might come from the bound states localized in the vortex cores. Recently, we studied the quasiparticle contribution to the thermal conductivities $\kappa_{ij}(\Omega, T)$ of an extreme type-II superconductor placed in magnetic field $H$ such that $H_{c1} \ll H \lesssim H_{c2}$. We examined the transport coefficients $\kappa_{ij}/T$ in the limit of $\Omega \to 0$ and $T \to 0$ and found that there was considerable enhancement of thermal transport in the mixed state of an s-wave superconductor due to the creation of gapless excitations in the magnetic field. This is in contrast to the zero field thermal transport which is exponentially small for an s-wave superconductor with no nodes in the gap. The agreement of our theoretical curves with the experimental data for the borocarbide superconductor LuNi$_2$B$_2$C and A-15 superconductor V$_3$Si by Boaknin et al. is very good over a wide range of fields used in the experiments.

The low-temperature electronic specific heat $C(T, H)$ is yet another probe of the quasiparticle excitations in the mixed state of a superconductor. In a fully gapped s-wave superconductor at low magnetic fields there is an exponentially small contribution to $C(T, H)$ at low temperatures and the only significant contribution to $C(T, H)$ comes from the quasiparticles localized near the vortex core. Assuming that the vortex core can be approximated as a “normal” metal embedded in a superconducting medium, this contribution is then proportional to the quasiparticle density of states which is finite and approximately equal to its normal state value. From here it follows that $C(T, H)$ varies linearly with $T$ and $\gamma_s(H) \equiv C(T, H)/T$ is proportional to $H$ as $T \to 0$. On the other hand, it was predicted that in unconventional d-wave superconductors the density of states and therefore the linear-T specific heat coefficient varies as $\sqrt{H}$ at low fields $H \gtrsim H_{c1}$. This field dependence is a consequence of the delocalized quasiparticles that can move along the nodal directions of the order parameter. Experiments on HTS materials YBa$_2$Cu$_3$O$_{7-\delta}$ (Ref. 20) and La$_{2-x}$Sr$_x$CuO$_4$ (Ref. 21) have confirmed this theoretical prediction. The consensus has been reached that nonlinear field dependence of $\gamma_s(H)$ in HTS systems is one of the signatures of an order parameter with $d_\lambda$ symmetry. However, a number of experimental studies do not conform to this interpretation: a nonlinear $H$-dependence of $\gamma_s(H)$ in almost the entire regime of the mixed state is observed in s-wave superconductors, such as A-15’s V$_3$Si (Ref. 22) and NbSe$_2$ (Refs. 23 and 24) as well as in borocarbides superconductors LuNi$_2$B$_2$C (Ref. 25) and YNi$_2$B$_2$C (Refs. 24 and 26). It is well established that A-15’s are fully gapped superconductors at zero field while there might be a significant anisotropy of the borocarbide’s s-wave order parameter at very low fields $H \sim H_{c1}$. Sonier et al. accounted quantitatively for this nonlinear behavior of $\gamma_s(H)$ in NbSe$_2$ by the expansion of the vortex cores at low fields. On balance, it seems that the specific heat behavior of s-wave superconductors in the whole regime of the mixed state is not fully understood and merits further attention.

The purpose of this work is to examine in detail the quasiparticle contribution to the low-temperature specific heat in the mixed state of a three-dimensional s-wave extreme type-II superconductor starting from the high field limit. It was already suggested in Ref. 5 that in a pure superconducting system close to $H_{c2}$, the specific heat $C(H, T)$ should be an algebraic function of temperature with the power dependent on the dimensionality of the system. This behavior was attributed to the strong dispersion around the gapless points in the quasiparticle excitation spectrum. In order to obtain analytical results, the authors of Ref. 5 assumed a relatively small number $n_c$ of Landau levels occupied by the electrons participating in the superconducting pairing ($n_c = E_F/h\omega_c$, where $E_F$ is the Fermi energy and $\omega_c = eH/m^*$ is the cyclotron frequency). However, this assumption is not expected to be quantitatively valid in the typical range of fields used in experiments. On the contrary, the number of occupied Landau levels $n_c$ in the mixed state $H_{c1} \ll H \lesssim H_{c2}$ is often quite large, typically $n_c \sim 30 - 270$ for borocarbide and $n_c \sim 250 - 4500$ for A-15 superconductors. The intention of the present work is to numerically evaluate the quasiparticle specific heat in the mixed state starting from the high-field limit of the Landau level pairing scheme, but under more realistic assumptions for the microscopic properties of materials studied so that a direct comparison can be made with available experimental data.

We begin by considering the density of states $N(\omega, H)$ in a dirty but homogeneous superconductor in the presence of nonmagnetic impurities in high magnetic field. In such a superconductor, the coherence length is much longer than the effective range of the impurity potential, so that under these conditions the order parameter in the mixed state $\Delta(r)$ is not substantially affected and still forms a perfect vortex lattice. We follow Green’s function perturbative approach to impurity effects in high magnetic fields developed in our previous work. Normal and anomalous Green’s functions for a clean superconductor are expanded in terms of the complete set of eigenfunctions in a magnetic sublattice representation (MSR). In the Landau gauge $A = H(-y, 0, 0)$, the eigenfunction $\phi_{k_z, q, n}(r)$ belonging to $n$th Landau level can be written as

$$
\phi_{k_z, q, n}(r) = \sqrt{\frac{b_y}{2\pi n!\sqrt{\pi l^3}}} \exp\left(ik_z\zeta\right) \times \sum_k \exp\left(i\frac{\pi b_x}{2a} k^2 - i k q_y b_y\right) \times \exp\left[i(q_x + \frac{\pi k^2}{a}) \frac{1}{2}(y/l + q_x l + \frac{\pi k^2}{a})^2\right]
$$
where $\zeta$ is the spatial coordinate and $k_z$ is the momentum along the field direction, $a = (a, 0)$ and $b = (b_x, b_y)$ are the unit vectors of the triangular vortex lattice, $l = \sqrt{\hbar/cH}$ is the magnetic length, and $V$ is the volume of the system. $H_n(x)$ is the Hermite polynomial of order $n$. Quasimomentum $\mathbf{q}$, perpendicular to the direction of the magnetic field, is restricted to the first magnetic brillouin zone (MBZ) spanned by vectors $\mathbf{Q}_1 = (b_y/l^2, -b_z/l^2)$ and $\mathbf{Q}_2 = (0, 2a/l^2)$. In this representation, the "Fourier transforms" of superconducting Green's functions in this quasimomentum space expressed in the Nambu formalism can be written as

$$\hat{G}_n(k_z, \mathbf{q}, i\omega) = \frac{1}{(i\omega)^2 - E_n(k_z, \mathbf{q})} \times \left( i\omega + \epsilon_n(k_z) - \Delta_{nn}(\mathbf{q}) - i\epsilon_n(k_z) \right)$$

where

$$E_{n,p}(k_z, \mathbf{q}) = \hbar \omega_c \pm \sqrt{\epsilon_n^2(k_z) + |\Delta_{n+p,n-p}(\mathbf{q})|^2}$$

$$\epsilon_n(k_z) = \hbar^2 k_z^2 / 2m + \hbar \omega_c (n + 1/2) - \mu$$

is the quasiparticle excitation spectrum of the superconductor in high magnetic field near the points $k_z = \pm k_{Fn} = (2\sqrt{m(\mu - \hbar \omega_c (n + 1/2))/\hbar})$. This spectrum is calculated within the diagonal approximation, where only the electrons belonging to mutually degenerate Landau Levels at the Fermi surface are involved in the superconducting pairing. Contributions to the pairing from the Landau levels separated by $\hbar \omega_c$ or more are included in the renormalization of the effective BCS coupling constant $[g \rightarrow g(H, T)]^{30}$.

For quasiparticles near the Fermi surface ($k_z \sim k_{Fn}$) it is enough to consider only the $E_{n,p=0}$ bands. The gap function, $\Delta_{nn}(\mathbf{q})$, in the MSR can be written as

$$\Delta_{nm}(\mathbf{q}) = \frac{\Delta}{\sqrt{2}} \frac{(-1)^m}{2^{m+1} \sqrt{m!}} \exp(i \frac{b_y}{a} k_z^2) \times \exp(2ik_y b_y - (q_x + \frac{\pi k_z}{a})^2 l^2) H_n + m(\sqrt{2} (q_x + \frac{\pi k_z}{a}) l)$$

The function $\Delta_{nm}(\mathbf{q})$ turns to zero on the Fermi surface at the set of points in the MBZ with a strong linear dispersion in $q$. The excitations from other, $p \neq 0$ in Eq. (3), bands are gapped by at least the cyclotron energy $\hbar \omega_c$ and their contribution to a superconductor's thermodynamics can be neglected at low temperatures ($T \ll \Delta(T, H) < \hbar \omega_c$). Once the off-diagonal contribution is included in the superconducting pairing, the excitation spectrum cannot be written in the simple form, Eq. (3), and a closed analytic expression for the superconducting Green's function cannot be found. A detailed study of the effects of off-diagonal terms on the superconducting state has been pioneered by Norman, MacDonald and collaborators (see Ref. 30 and references therein). Still, when these off-diagonal terms are treated analytically within the perturbation theory of Ref. 6, the qualitative behavior of the quasiparticle excitations at the Fermi surface, as characterized by nodes in the MBZ, remains the same. This statement is correct in all orders of perturbation theory and therefore is exact as long as the perturbative expansion itself is well defined, i.e. as long as the magnetic field is larger than some critical field $H^*(T)$. The critical field $H^*$ at $T \sim 0$ can be estimated from the dHvA experiments to be $\sim 0.5H_{c2}$ for A-15 and $\sim 0.2H_{c2}$ for borocarbide superconductors.

Recent measurements of thermal transport in borocarbides in the mixed state suggest a strong anisotropy of the s-wave order parameter, so that the value for $H^*$ in these systems can be even lower than the estimate obtained from dHvA measurements. Once the magnetic field is lowered all the way to $H^*$, the contribution of the off-diagonal pairing terms becomes essential, and gaps start opening up at the Fermi surface signaling the crossover to the low-field regime of quasiparticle states localized in the cores of widely separated vortices.

In the presence of disorder the bare Green's function in Eq. (2) is dressed via scattering through the diagonal (normal) self-energy $\Sigma^N(\omega)$ and off-diagonal (anomalous) self-energy $\Sigma^A_{nn}(\mathbf{q}, \omega)$.

A dressed Green's function is obtained by replacing $\omega$ with $i\omega$ and $\Delta_{nn}(\mathbf{q})$ with $\Delta_{nn}(\mathbf{q})$ in Eq. (2) where

$$i\omega \equiv \omega - \Sigma^N(\omega)$$

$$\hat{\Delta}_{nn}(\mathbf{q}) \equiv \Delta_{nn}(\mathbf{q}) + \Sigma^A_{nn}(\mathbf{q}, \omega).$$

We follow a $T$-matrix approximation originally developed for heavy fermion superconductors and adapted by us to treat self-consistently impurity scattering at high magnetic field.

Within this approximation both weak-scattering and strong scattering limits can be treated on equal footing. However, we anticipate that the experimentally determined disorder parameters will put our calculation into the weak scattering limit of this theory with a dilute concentration of impurities. Within a $T$-matrix approximation the self-energies $\Sigma^N(\omega)$ and $\Sigma^A_{nn}(\mathbf{q}, \omega)$ of the superconducting system are closely related to the diagonal (with respect to the magnetic translation group basis) $T$-matrix elements in a single-site approximation as

$$\Sigma^N(\omega) = n_t < T_{nn}^{11}(k_z, \mathbf{q}, \omega)>_R$$

$$\Sigma^A_{nn}(\mathbf{q}, \omega) = -n_t < T_{nn}^{12}(k_z, \mathbf{q}, \omega)>_R$$

where $\langle ... >_R$ denotes the average over the impurity positions and $n_t$ is the impurity concentration. $T_{nn}^{1\alpha}(k_z, \mathbf{q}, \omega)$ are the coefficients in the $T$-matrix expansion over the complete set of MSR eigenstates. The $2 \times 2$ matrix $\hat{T}(\mathbf{r}, \mathbf{r}'; i\omega)$ obeys the Lippmann-Schwinger equations

$$\hat{T}(\mathbf{r}, \mathbf{r}'; i\omega) = U(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \hat{\sigma}_z + \int d\mathbf{r}_1 U(\mathbf{r})$$

$$\times \hat{\sigma}_z \hat{G}(\mathbf{r}, \mathbf{r}_1; i\omega) \hat{T}(\mathbf{r}_1, \mathbf{r}'; i\omega)$$

(7)
where $\hat{G}$-matrix elements are given by Eq. (2) and $U(r) = \sum U_0 \delta (r - R_i)$ represents a short-range impurity potential with the location of impurities $R_i$ taken to be randomly distributed everywhere in the sample. The scalar scattering amplitude $U_0$ is assumed to be isotropic. In high magnetic field we can assume that the scattering potential is weak compared to the separation between Landau levels, given by $\hbar \omega_c$. Under these circumstances the electrons scatter into the states belonging to the same Landau level so that the off-diagonal (with respect to Landau level index $n$) $T$-matrix elements in Eq. (6) can be neglected. Note that this approximation might be valid even under more general circumstances, when the effective scattering is larger than $\hbar \omega_c$, as is the case in the Quantum Hall Effect (QHE) problem. However, our case, unlike the QHE, also contains strong pairing potential and thus the interference between the two is generally a rather formidable problem. Following the formalism outlined in Ref. 11, Lippmann-Schwinger equations (7) are averaged over the impurity position and from there can be solved as

$$T_{nn}^1(i\omega) = \frac{(b_y/V \sqrt{\pi l}) \sum_{k_x,k,m} g_{11}^{(1)}(k_x,k,i\omega)}{1/U_0^2 - [(b_y/V \sqrt{\pi l}) \sum_{k_x,k,m} g_{11}^{(1)}(k_x,k,i\omega)]^2}$$

and

$$T_{nn}^{2\dagger}(q; i\omega) = -\left(\sqrt{2} b_y/V \sqrt{\pi l}\right) f_{nn}(q) \sum_{k_x,k,m} f_{mm}(k) g_{m}^{(2\dagger)}(k_x,k,i\omega) / U_0^2 - [(b_y/V \sqrt{\pi l}) \sum_{k_x,k,m} g_{11}^{(1)}(k_x,k,i\omega)]^2$$

where $g_{m}^{(2\dagger)}(k_x,k,i\omega)$ is a matrix element of a Nambu matrix (2), $f_{nn}(k) = \Delta n_n(k)/\Delta$ and $V$ is the volume of the system. $\sum_k$ goes over the entire MBZ while $\sum_n$ is over all the occupied Landau levels. Replacing $\omega$ with $\tilde{\omega}$ and $\Delta n_n(q)$ with $\tilde{\Delta} n_n(q)$ in (2) with the help of definitions (3) and (4), Eqs. (8) and (9) can be brought to the form

$$u = \frac{\omega}{\Delta} + \zeta \sum_{n=0}^{n_c} \left( m^*/4\pi^3 k_{F n} N(0) \right) \int d\mathbf{q}(1 - \sqrt{2}|f_{nn}(q)|^2) u/\sqrt{u^2 + |f_{nn}(q)|^2}$$

$$= \frac{\sum_{n=0}^{n_c} \left( m^*/4\pi^3 k_{F n} N(0) \right) \int d\mathbf{q}/\sqrt{u^2 + |f_{nn}(q)|^2}}{2} + \frac{\omega}{\Delta}$$

(10)

where $\zeta = \Gamma/\Delta$ and $u = \tilde{\omega}/\Delta$. $N(0)$ is the normal state density of states at the Fermi level. Disorder is characterized with two parameters: $\Gamma = n_i/N(0)\pi = (n_i/n) E_F$, which measures the concentration of impurities $n_i$ relative to the electron density $n$, and $c = 1/\pi N(0) U_0$, which measures the strength of the scattering potential. The weak-scattering limit in (10) is approached when $c^2$ is much larger than the second term in the denominator of (10), typically when $c \sim 1$. The strong (i.e. unitary) scattering limit is approached when $c = 0$. The normal state inverse scattering rate $\Gamma_0$ is found by taking $f_{nn}(q) = 0$ in (10) and letting $\omega \to 0$. This procedure yields

$$\Gamma_0 = \frac{\Gamma}{1 + c^2}$$

(11)

and establishes the connection between the experimentally determined disorder parameter $\Gamma_0$ and the parameters $\Gamma$ and $c$ in our theory. Equation (10) is an implicit equation from which $u = f(\omega/\Delta)$ has to be calculated numerically. In fact, once the analytic continuation to real frequencies $i\omega = \omega + i\delta$ is performed, this equation transforms in a nonlinear system of equations. Finally, once $u$ is known the superconducting density of states and other thermodynamic quantities can be calculated.

The superconducting density of states in the presence of disorder is defined as

$$N(\omega, H) = -\frac{1}{2\pi V} \Im \sum_{n,k,z} T_{nn}^{\dagger} \hat{G}_n(k_x, q; i\omega)|_{i\omega=\omega+i\delta}$$

(12)

where $\hat{G}_n(k_x, q, i\omega)$ is a Nambu matrix (2) in which the replacement $\omega$ with $\tilde{\omega}$ and $\Delta n_n(q)$ with $\tilde{\Delta} n_n(q)$ has been implemented. Once the analytic continuation to real frequencies is performed in Eq. (10) and Eq. (12) and with the help of the definition $u = \tilde{\omega}/\Delta$, the density of states (12) can be obtained as

$$N(\omega, H) = \frac{1}{N(0)} \frac{\Im \sum_{n=0}^{n_c} m 4\pi^3 k_{F n} \int d\mathbf{q} u/\sqrt{|f_{nn}(q)|^2 - u^2}}{3m \sum_{n=0}^{n_c} m 4\pi^3 k_{F n}}$$

(13)

In deriving this equation, we assume that the impurity scattering does not change the $\mathbf{q}$-dependence of the quasiparticle excitation spectrum, i.e. $\Delta(q) = \Delta f_{nn}(q)$ is given by Eq. (4). This assumption is shown to be valid for the high-field quasiparticle excitations close to the gapless points at the Fermi surface while it is less reliable for excitations gapped by large $\Delta$. We are primarily interested in the behavior of the specific heat when $T \to 0$ which is governed by the excitations around nodes while the contribution of the gapped regions in the MBZ is exponentially small. Therefore, this approximation seems a modest sacrifice in the quantitative accuracy when faced with the overwhelming numerical difficulty in determin-
ing fully self-consistent $\tilde{\Delta}(q)$ in the presence of the disorder.

Once the superconducting density of states is obtained from Eq. (13) the quasiparticle contribution to the specific heat in the mixed state at low temperatures can be computed as:

$$C(T, H) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega N(\omega, H) \frac{\omega^2}{T^2} \cosh^{-2} \frac{\omega}{2T}$$  \hspace{1cm} (14)$$

and then $\gamma_s(H) \equiv C(H)/T$ when $T \to 0$ can be determined. The most challenging part of our density of state and specific heat calculations for realistic superconducting materials in the range of the fields used in experiments is solving Eq. (11) when the number of Landau levels involved in the superconducting pairing $n_c = E_F/h\omega_c$ is very large. For the A-15 superconductor $V_3Si$, we estimate $n_c \sim 241$ at $H_{c2} = 18.5$ Tesla and $n_c \sim 4470$ at $H = 1$ Tesla (we used an effective mass of $m^* = 1.7m_e$ and Fermi velocity $v_F = 2.8 \times 10^7$ cm/s from Ref. 34 in this estimate). For the borocarbide superconductor YNi$_2$B$_2$C, we get $n_c \sim 25$ at $H_{c2} = 8$ Tesla (from Ref. 24) and $n_c \sim 200$ at $H = 1$ Tesla, obtained using $m^* = 0.35m_e$, $v_F = 2.5 \times 10^7$ cm/s, a normal scattering rate $\Gamma_0 = 0.53$ meV from the dHvA experiment of Terashima et al. (these values reproduce the mean free path $l = 1500$ Å reported for a clean sample by Nohara et al.). Faced with the overwhelming computational difficulty of solving a nonlinear system of equations that Eq. (11) becomes for $n_c \gg 100$, we limit our study to the borocarbide superconductor YNi$_2$B$_2$C only.

Figure 1 represents the quasiparticle density of states $N(\omega)$ in the mixed state of the borocarbide superconductor YNi$_2$B$_2$C computed from Eq. (13) and rescaled by the normal state density of state $N(0)$ as a function of the reduced energy $\omega/\Delta$. Disorder parameter $c$ in Eq. (10) is chosen to be $c = 0.65$ which, at the same time, determines a value for the second disorder parameter $\zeta$ to be $\zeta = 0.33$ if the experimentally determined normal state scattering rate of $\Gamma_0/\Delta$ from Ref. 3 is to be reproduced using Eq. (11). It can be seen in Fig. 1 that the density of states in the mixed state $N(\omega)$ at low energies diminishes as the magnetic field is lowered from $H \gtrsim H_{c2}$ to $H \sim 0.2H_{c2}$. This is a consequence of the depletion of gapless or near gapless quasiparticle excitations at the Fermi surface at lower magnetic fields.

The quasiparticle specific heat at low temperatures as computed from Eq. (14) is, to leading order, a linear function of temperature $T$ due to the creation of a finite density of states at the Fermi level in Fig. 1. In Fig. 2, we plot $C(H, T)/T$ as $T \to 0$ (i.e. the coefficient of a linear-T term in the quasiparticle specific heat) normalized by the Sommerfeld constant $\gamma_s(H)/\gamma_N$ for YNi$_2$B$_2$C as a function of the reduced magnetic field $H/H_{c2}$. Full circles represent the experimental data of Nohara et al. (Ref. 24). We plot two theoretical curves calculated from Eq. (14) with $c = 0.60$ (full line) and $c = 0.65$ (broken line). The second disorder parameter $\zeta$ is calculated from Eq. (11) where the experimentally determined value for the normal state scattering rate $\Gamma_0 = 0.53$ meV from Ref. 3 is used. The values of the other physical quantities needed in our theory, effective mass $m^* = 0.35$, BCS gap $\Delta = 2.3$ meV and upper critical field $H_{c2} = 8$ Tesla in $\Delta(H) = \Delta\sqrt{1 - H/H_{c2}}$, are taken from experiments.
of Nohara et al. (Ref. 24) and/or Terashima et al. (Ref. 3).

Fig. 2 demonstrates that the specific heat coefficient computed from the theory presented in this paper exhibits a nonlinear dependence on the magnetic field $H$, i.e., $\gamma_s(H) \sim H^{0.37}$ for $c = 0.60$ and $\gamma_s(H) \sim H^{0.46}$ for $c = 0.65$. This is in contrast to the linear $H$ dependence predicted for a fully gapped $s$-wave superconductor in the mixed state as $T \to 0$. We attribute this nonlinear behavior to the creation of coherent, gapless quasiparticle excitations at the Fermi surface in the mixed state of an extreme type-II superconductor at high magnetic field $H$ such that $H^* \lesssim H < H_{c2}$. The estimated critical field $H^*$ for pure YNi$_2$B$_2$C is $\sim (0.15 - 0.2)H_{c2}$. However, this estimate depends on the value of the $s$-wave gap function and can be much smaller than $0.15H_{c2}$ if the value of the minimum gap in the strongly anisotropic $s$-wave case is significantly different from the accepted BCS value. Furthermore, it seems that the unusual behavior of the specific heat coefficient $\gamma_s(H)$ is also a consequence of disorder present in the superconducting system. The finite amount of impurities in the system leads to the creation of a finite density of states at the Fermi level $N(0)$, in contrast to the perfectly clean superconductor where $N(\omega)$ exhibits an algebraic behavior at low energies.

In summary, we have computed the quasiparticle contribution to the specific heat $C(H,T)$ within the $T$-matrix approximation for the self-energies and found a nonlinear behavior of $\gamma_s(H) = C(H,T)/T$ when $T \to 0$. In the range of magnetic fields where our theory is applicable $H^* \leq H < H_{c2}$, the calculated $\gamma_s(H)$ closely resembles the experimental data for the borocarbide superconductor YNi$_2$B$_2$C.

This work is supported by an award from Research Corporation (A.L.C., J.J.T. and S.D) and by NSF grant No. DMR00-94981 (Z.T.).
In a clean s-wave superconductor \( x = H^*/H_{c2} \) can be estimated as a solution of the equation

\[
x^3 = \frac{2\Delta^4}{\pi E_F(\bar{\hbar} \omega_c^2)} (1-x)^2
\]

and yields \( H^* \approx (0.15 - 0.2)H_{c2} \) for the system in question.