Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge

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It is well known that (1 + 1)-D bosonic symmetry-protected topological (SPT) phases with symmetry group $G$ can be identified by the projective representation of the symmetry at the edge. Here, we generalize this result to higher dimensions. We assume that the representation of the symmetry on the spatial edge of a $(d + 1)$-D SPT is local but not necessarily on-site, such that there is an obstruction to its implementation on a region with boundary. We show that such obstructions are classified by the cohomology group $H^{d+1}(G,U(1))$, in agreement with the classification of bosonic SPT phases proposed in [Chen et al, Science 338, 1604 (2012)]. Our analysis allows for a straightforward calculation of the element of $H^{d+1}(G,U(1))$ corresponding to physically meaningful models such as non-linear sigma models with a theta term in the action. SPT phases outside the classification of Chen et al are those in which the symmetry cannot be represented locally on the edge. With some modifications, our framework can also be applied to fermionic systems in (2+1)-D.
The classification of phases of matter in quantum systems at zero temperature has proven to be much richer than in classical statistical mechanical systems. For many such phases, the feature which distinguishes them from other phases is quantum mechanical and not related to spontaneous breaking of a symmetry. One such family of quantum phases which has been much studied in recent years is the symmetry-protected topological (SPT) phase. A system with a symmetry is considered to lie in an SPT phase if (a) the symmetry is not spontaneously broken; and (b) the system can be connected to one whose ground state is a trivial product state without a phase transition, but only if we allow the symmetry to be broken explicitly. In some sense, SPT phases are “trivial” in the bulk, but boundaries between different SPT phases are non-trivial and must either be gapless, break the symmetry (explicitly or spontaneously), or be topologically ordered.

The central problem in the study of SPT phases is classifying the different phases that can occur for a given symmetry. In bosonic systems with an internal symmetry group $G$, an early result was that in (1+1)-D systems, the possible SPT phases are classified by the second cohomology group $H^2(G, \text{U}(1))$. This result has a natural interpretation in terms of the symmetry transformation properties of an edge between the SPT and vacuum (or equivalently, of the entanglement spectrum). Such an edge will in general transform projectively under the symmetry. The second cohomology group arises naturally from a consideration of these projective representations.

It has been argued that, more generally, the SPT phases in $d$ spatial dimensions are classified by the cohomology group $H^{d+1}(G, \text{U}(1))$. This result was based on an explicit construction of field theories in discrete space-time which are believed to be representative of each SPT phase. However, making a definitive identification between these lattice field theories and other, more physically motivated, realizations of the corresponding SPT phases has proved difficult. In this paper, therefore, we propose to recast the cohomological classification in a different, hopefully more intuitive viewpoint, inspired by the original (1+1)-D treatment. The central idea is that, just as in the (1+1)-D case, the symmetry transformation on the edge of a $(d+1)$-D system will be, in some sense, anomalous. Specifically, if we have a system defined on a $d$-dimensional spatial manifold $M_{\text{bulk}}$ with a boundary, the edge symmetry acts on the boundary $\partial M_{\text{bulk}}$, which itself has no boundary [$\partial(\partial M_{\text{bulk}}) = 0$]. Therefore, there might be an obstruction to implementing the edge symmetry in a consistent way on a $(d-1)$-dimensional manifold $M$ with boundary $\partial M \neq 0$. We will argue that this obstruction is indeed classified by the cohomology group $H^{d+1}(G, \text{U}(1))$. [For (2+1)-D systems, our approach is related to, though more general than, that of Ref. 36, which was based on a tensor-network representation for the edge symmetry.] In fact, in (2+1)-D our approach also leads to a classification of SPT phases in interacting fermion systems, as we will show.

The remainder of this paper is organized as follows. In Section II we give the general demonstration that the obstruction is classified by $H^{d+1}(G, \text{U}(1))$. For (2+1)-D SPT’s, this argument can be given in full generality (assuming only that the symmetry acts locally on the edge), but in higher dimensions we will need to make additional assumptions about the form of the symmetry. In Section III we discuss by way of illustration a simple example of an anomalous symmetry that appears on the edge of a (2+1)-D SPT. In Section IV we use the ideas of this paper to prove that (2+1)-D SPT phases characterized by different elements of $H^3(G, \text{U}(1))$ are necessarily separated by a phase transition unless the symmetry is broken explicitly. In Section V we show how to use our approach to derive the element of the cohomology group corresponding to non-linear sigma models containing a topological
term. In Section V, we make explicit the connection between our work and the original classification of Ref. 7. In Section VI, we explain why, in the presence of anti-unitary symmetries, there exist bosonic SPT phases not captured by our arguments. In Section VII, we show how our ideas can be applied also to fermionic systems in (2+1)-D.

I. THE GENERAL FORMALISM

Consider a system in a bosonic SPT phase. By definition, this means it is gapped and non-degenerate in the bulk, and (disregarding symmetry considerations) can be continuously connected to a product state without a phase transition. However, in a system with boundary, we can define an effective low-energy theory for the boundary, which may be gapless notwithstanding the gap in the bulk. A key property of SPT phases is that the boundary theory of an SPT phase in \(d\) spatial dimensions can always be realized at the microscopic level in a strictly \((d-1)\)-dimensional system (see Appendix A for a careful proof of this well-known fact.) This is in contrast to, for example, integer quantum Hall states in which the boundary is chiral and cannot be realized as a stand-alone system. For SPT phases, the anomalous nature of the edge arises not from the boundary theory itself but from the way it is acted upon by the symmetry.

We assume that the symmetry in the bulk is unitary and on-site, that is, for a lattice system with \(N\) sites, the symmetry group \(G\) is is represented as a unitary tensor-product \(U(g) = \prod_{i=1}^{N} u_i(g)\) of operators acting on each site. (We may need to group several sites together into a single effective site in order to satisfy this condition.) We now consider the low-energy Hilbert space of states with energies below some cutoff that is less than the bulk gap; these states are edge excitations. Projecting the unitary representation of the symmetry group onto this low-energy Hilbert space, we obtain a unitary representation, acting only on the boundary degrees of freedom, that may not be on-site. On the contrary, it appears to be a characteristic of non-trivial SPT phases that the symmetry is realized on the boundary in a fundamentally non-on-site way. Nevertheless, the key assumption that we make in this paper is that the boundary symmetry, albeit not on-site, is nevertheless still \(local\) in the sense of Ref. 2 (e.g. it can be represented as a finite-depth quantum circuit.) This seems a natural assumption, but we expect it to be violated by SPT phases not captured by the cohomological classification (see Section VI for further discussion).

For a non-on-site symmetry, there is the possibility that there is an obstruction to implementing the symmetry on a manifold with boundary in a consistent way. We intend to show that, by classifying these obstructions, one recovers the cohomological classification of SPT phases. A simple example of this idea is the well-known connection between (1+1)-D SPT’s and the projective symmetry transformation of the edg. Nevertheless, the key assumption that we make in this paper is that the boundary symmetry, albeit not on-site, is nevertheless still \(local\) in the sense of Ref. 2 (e.g. it can be represented as a finite-depth quantum circuit.) This seems a natural assumption, but we expect it to be violated by SPT phases not captured by the cohomological classification (see Section VI for further discussion).

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A. (1+1)-D SPT’s

The boundary of a one-dimensional system simply comprises a pair of points \(a\) and \(b\). Let \(U(g)\) be the representation of the symmetry group \(G\) on this boundary. Assuming that we chose the system size such that the end-points \(a\) and \(b\) are well-separated (i.e. by a distance large compared to all intrinsic length scales), locality of \(U(g)\) simply implies that it must act on \(a\) and \(b\) separately; that is, it must be a tensor product \(U(g) = U_a(g) \otimes U_b(g)\). We can think of \(U_a(g)\) as the \(restriction\) of \(U(g)\) to the point \(a\). Importantly, however, this
restriction is uniquely defined only modulo phase factors. Indeed, $U(g)$ is left invariant under $U_a(g) \rightarrow \beta(g)U_a(g), U_b(g) \rightarrow \beta(g)^{-1}U_b(g)$ for any $U(1)$-valued function $\beta(g)$. Thus, while $U(g)$ is always a representation of the symmetry group $G$, that is $U(g_1)U(g_2) = U(g_1g_2)$, the non-uniqueness of the restriction procedure implies that $U_a(g)$ need only be a projective representation of $G$, which is to say that $U_a(g_1)U_a(g_2) = \omega(g_1, g_2)U_a(g_1g_2)$ for some $U(1)$-valued function $\omega(g_1, g_2)$. The function $\omega$ describes the obstruction to consistently (i.e. non-projectively) implementing the symmetry on the point $a$.

Since multiplication of the $U_a$’s must be associative, one can derive a consistency condition on $\omega$ by evaluating $U_a(g_1)U_a(g_2)U_a(g_3)$ in two different ways, namely

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_2, g_3)\omega(g_1, g_2g_3).$$

(1)

A function $\omega$ satisfying Eq. (1) is known as a 2-cocycle. Furthermore, due to the fact that $U_a(g)$ is only defined up to a $g$-dependent phase factor $\beta(g)$, it follows that we have an equivalence relation on 2-cocycles:

$$\omega(g_1, g_2) \sim \omega(g_1, g_2)\beta(g_1)\beta(g_2)\beta(g_1g_2)^{-1}.$$  

(2)

The group of 2-cocycles quotiented by the above equivalence relation is, by definition, the second cohomology group $H^2(G, U(1))$. One can then show that two models are in the same SPT phase if and only if they correspond to the same element of $H^2(G, U(1))$. Therefore, SPT phases in (1+1)-D are classified by $H^2(G, U(1))$.

B. (2+1)-D SPT’s

When presented as it was above, the (1+1)-D case suggests an obvious generalization to higher dimensions: we consider the symmetry $U(g)$ acting on the boundary $C$, then restrict it to a subregion $M$, which in general will be a manifold with boundary ($C$ itself has no boundary as it the boundary of a higher-dimensional manifold), to see if the symmetry is implemented consistently or not.
First, we need to give a more general definition of what it means to restrict a local unitary $U$ acting on a spatial manifold $C$ to a sub-manifold $M$, which for the case discussed above was obvious due to the tensor-product structure. Specifically, we say that a local unitary $U_M$ acting on the region $M$ is the restriction of $U$ to the region $M$ if it acts the same as $U$ in the interior of $M$, well away from the boundary $\partial M$. We observe two properties about this restriction:

(a) It always exists for any local unitary. This can easily be seen from, for example, the quantum circuit description.

(b) It is defined modulo local unitaries acting in the vicinity of the boundary $\partial M$.

The second property is the higher-dimensional generalization of the restriction being defined only up to phase factors. Thus, in general, if $U(g)$ is a representation of the symmetry group $G$, then $U_M(g)$ need only satisfy

$$U_M(g_1)U_M(g_2) = \Omega(g_1, g_2)U_M(g_1 g_2)$$

(3)

where $\Omega(g_1, g_2)$ is a local unitary acting in the vicinity of $\partial M$ which represents the obstruction to a consistent representation on $M$ due to the fact that it is a manifold with
boundary. Thus, we have reduced the problem of classifying local unitary representations $U(g)$ on a $d$-dimensional manifold to that of classifying local unitary obstructions $\Omega(g_1, g_2)$ on a $(d - 1)$-dimensional manifold. The idea now is to perform more such reductions, each time reducing by 1 the dimensionality of the manifold acted upon, until we get down to the simplest case of 0 dimensions (i.e. points).

For (2+1)-D SPT’s, this reduction can be completed as follows. In this case the boundary has only one spatial dimension, and so $\Omega(g_1, g_2)$ as constructed above already acts on just a pair of points $a$ and $b$. We observe that Eq. (3), together with the associativity of the operators $U_M(g)$, implies that $\Omega$ must satisfy

$$\Omega(g_1, g_2)\Omega(g_1g_2, g_3) = U_M(g_1)\Omega(g_2, g_3)\Omega(g_1, g_2g_3),$$

which is a non-abelian analogue of Eq. (1), and where we have introduced the conjugation notation $x^y = xyx^{-1}$. Now we perform a second restriction, from $\partial M = \{a, b\}$ to the single point $a$. The restriction $\Omega \to \Omega_a$ is defined only up to phase factors, and so we conclude that $\Omega_a$ satisfies Eq. (4) only up to phase factors:

$$\Omega_a(g_1, g_2)\Omega_a(g_1g_2, g_3) = \omega(g_1, g_2, g_3) U_M(g_1)\Omega_a(g_2, g_3)\Omega_a(g_1, g_2g_3),$$

where $\omega(g_1, g_2, g_3) \in U(1)$. We show in Appendix B that $\omega$ must satisfy the 3-cocycle condition

$$\omega(g_1, g_2, g_3)\omega(g_1g_2, g_3, g_4)^{-1}\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3g_4)^{-1}\omega(g_2, g_3, g_4) = 1.$$  

Furthermore, as $\Omega_a(g, g')$ is only defined up to phase factors $\beta(g, g')$, we must identify

$$\omega(g_1, g_2, g_3) \sim \omega(g_1, g_2, g_3) \beta(g_1, g_2)\beta(g_1g_2, g_3)\beta(g_2, g_3)^{-1}\beta(g_1, g_2g_3)^{-1}. $$

We show in Appendix B that, up to equivalence, the choice of restriction $U(g) \to U_M(g)$ does not affect the 3-cocycle. The group of 3-cocycles quotiented by the equivalence relation Eq. (7) is, by definition, the third cohomology group $H^3(G, U(1))$. Hence, we recover the cohomological classification of (2+1)-D SPT’s.

C. Higher dimensions

In higher dimensions it is not clear whether we can still do the reduction procedure in complete generality as in the (2+1)-D case. Nevertheless, we can still perform the reduction if we make some simplifying assumptions about the action of the symmetry on the boundary. (The non-linear sigma models discussed in Section IV are a non-trivial example in which the symmetry on the edge takes the required form.) Specifically, we consider a symmetry group $G$ acting on a Hilbert space equipped with a set of basis states labeled by the variables $\alpha(x)$ associated with each spatial location in a closed $(d - 1)$-dimensional space $C_1$. We can take the spatial coordinate $x$ to be either discrete (i.e., a lattice) or continuous. The class of symmetry actions that we consider are those that can be written in the form

$$U(g) = N(g)S(g),$$

such that:
(a) $S(g)$ is the on-site part of the symmetry which can be written in the form

$$S(g) = \sum_\alpha |g\alpha\rangle\langle\alpha|,$$

where $\alpha \to g\alpha$ is some on-site action of the symmetry on the classical labels $\alpha$; and

(b) in the same basis, the non-on-site part $N(g)$ is diagonal, namely

$$N(g) = \sum_\alpha e^{i\mathcal{N}(g)[\alpha]|\alpha\rangle\langle\alpha|}$$

where $\mathcal{N}(g)$ are functionals of the configuration $\alpha$. We require these functionals to be sufficiently local that $N(g)$, and hence $U(g)$, are local unitaries.

The requirement that $U(g)$ be a representation, $U(g_1)U(g_2) = U(g_1g_2)$, can be written in terms of the functionals $\mathcal{N}(g)$ as

$$g_1\mathcal{N}(g_2) + \mathcal{N}(g_1) - \mathcal{N}(g_1g_2) = 0 \pmod{2\pi},$$

where we have defined the action of group elements on functionals in the obvious way: $(g\mathcal{F})[\alpha] = \mathcal{F}[g^{-1}\alpha]$. Henceforth we will take the $(\mod 2\pi)$ to be implied, or in other words we consider the functionals to take values in $\mathbb{R}/(2\pi\mathbb{Z})$.

Now as before, we can restrict $U(g)$ to a subregion $M_1$ with boundary, which (since $S(g)$ can be trivially restricted) amounts to restricting the functionals $\mathcal{N}(g)$. Then Eq. (11) need be satisfied by the restricted functionals $\mathcal{N}_M^{(1)}(g)$ only up to boundary terms,

$$g_1\mathcal{N}_M^{(1)}(g_2) + \mathcal{N}_M^{(1)}(g_1) - \mathcal{N}_M^{(1)}(g_1g_2) = \mathcal{N}^{(2)}(g_1, g_2)$$

where the $\mathcal{N}^{(2)}(g_1, g_2)$ are functionals which depend only on the value of $\alpha$ near the boundary $\partial M_1$ and describe the obstruction. This corresponds to Eq. (3).

In order to continue the reduction process, we find it useful to define the group coboundary operators $\delta_k$ which map functionals depending on $k$ group elements into functionals depending on $k + 1$ group elements, as follows:

$$(\delta_k\mathcal{N}^{(k)})(g_1, \ldots, g_{k+1}) = g_1\mathcal{N}^{(k)}(g_2, \ldots, g_n) + (-1)^{k+1}\mathcal{N}^{(k)}(g_1, \ldots, g_k)$$

$$+ \sum_{i=1}^k (-1)^i\mathcal{N}^{(k)}(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{k+1}).$$

In particular, $(\delta_1\mathcal{N}^{(1)})(g_1, g_2)$ corresponds to the left-hand side of Eq. (11). The important property which the coboundary operators satisfy is that they form a chain complex, i.e. $\delta_{k+1} \circ \delta_k = 0$.

We can now formulate the reduction process for symmetries acting on a manifold of arbitrary spatial dimension $d$. At the $k$-th step of the process, we have a set of functionals $\mathcal{N}^{(k)}$ acting on a closed $d - k$-dimensional manifold $C_k$ and indexed by $k$ group elements, satisfying $\delta_k\mathcal{N}^{(k)} = 0$. We then consider restrictions $\mathcal{N}_M^{(k)}$ of these functionals onto the manifold $M_k$, where $M_k$ is a submanifold of $C_k$ with boundary. As $\mathcal{N}_M^{(k)}$ must act the same as $\mathcal{N}^{(k)}$ in the interior of $M_k$, it follows that $\mathcal{N}^{(k+1)} = \delta_k\mathcal{N}^{(k)}$ acts on the boundary $\partial M_k \equiv C_{k+1}$. Furthermore, as $\delta_{k+1} \circ \delta_k = 0$, it follows that $\delta_{k+1}\mathcal{N}^{(k+1)} = 0$. Thus, we
just iterate these reduction steps, terminating when we reach \( \omega = \mathcal{N}(d+1) \), which is simply a mapping from \( d + 1 \) group elements to U(1) satisfying \( \delta_{d+1} \mathcal{N}(d+1) \); this the definition of a U(1) \((d + 1)\)-cocycle. Due to the ambiguity in the choice of restrictions, it follows that \( \omega \) is only defined up to

\[
\omega \sim \omega + \delta_{d+1} \lambda
\]

where \( \lambda \) is some element of U(1) depending on \( d + 1 \) group elements. The group of \((d + 1)\)-cocycles quotiented by the equivalence relation Eq. (14) is, by definition, the cohomology group \( H^{d+1}(G, \text{U}(1)) \). Thus, we recover the cohomological classification of SPT phases in arbitrary dimensions.

Finally, let us discuss the case of symmetry groups that contain anti-unitary operations. It is perhaps unclear in general what is meant by restriction of an anti-unitary operation (although see Ref. 41). Nevertheless, if we consider only symmetries that can be represented as a suitable generalization of Eq. (8), the same arguments as above can be applied with only minor modifications. Specifically, we consider symmetries of the form

\[
U(g) = N(g)S(g)K^{n(g)},
\]

where \( N(g) \) and \( S(g) \) are as before, \( K \) is complex conjugation in the \( \{|\alpha\} \) basis, and \( n(g) \) is 0 for unitary elements of \( G \) and 1 for anti-unitary elements. If we define the action of \( G \) on functionals as \( g\mathcal{F}[\alpha] = (-1)^{n(g)}\mathcal{F}[g^{-1}\alpha] \), all of the steps in the above derivation can be carried through without change, except that there is a residual non-trivial action of \( G \) on

---

\[
\begin{align*}
(\cdot)(g_1) & \quad \mathcal{N}^{(1)} \xrightarrow{\text{Restrict}} \tilde{\mathcal{N}}^{(1)} \\
(\cdot)(g_1, g_2) & \quad 0 \quad \delta_1 \tilde{\mathcal{N}}^{(1)} \xrightarrow{\text{Restrict}} \tilde{\mathcal{N}}^{(2)} \\
(\cdot)(g_1, g_2, g_3) & \quad 0 \quad \delta_2 \tilde{\mathcal{N}}^{(2)} \xrightarrow{\text{Restrict}} \tilde{\mathcal{N}}^{(3)} \\
(\cdot)(g_1, g_2, g_3, g_4) & \quad 0 \quad \delta_3 \tilde{\mathcal{N}}^{(3)} \xrightarrow{\text{Restrict}} \tilde{\mathcal{N}}^{(4)} = \omega
\end{align*}
\]
U(1). Thus, the classification is $H^{d+1}(G, U(1))$, but with U(1) considered as a non-trivial $G$-module, with anti-unitary elements acting by complex conjugation.

II. EXAMPLE: “CHIRAL” SYMMETRY ON THE EDGE OF A (2+1)-D SPT

It was shown in Ref. 42 that the action of the symmetry on the gapless edge of some non-trivial (2+1)-D SPT’s is “chiral”, as expressed (for example) in the fact that it acts differently on the left- and right-moving fields. Let us show how this corresponds to a local but not on-site symmetry and calculate the corresponding 3-cocycle. We will focus on the simplest case where the symmetry is just $\mathbb{Z}_2$, but similar arguments can be made for $\mathbb{Z}_n$ or U(1) symmetries.

We assume the low-energy theory of the (1+1)-D edge is described by a massless boson field $\varphi$ with compactification radius $2\pi$, i.e. a bosonic Luttinger liquid, with Lagrangian density

$$\mathcal{L} = \frac{g}{2\pi} \left[ \frac{1}{v} (\partial_t \varphi)^2 - v (\partial_x \varphi)^2 \right]. \quad (16)$$

We introduce the dual boson field $\theta$ according to $\partial_x \theta = 2\pi \Pi$, where $\Pi$ is the canonical momentum conjugate to $\varphi$. The commutation relation for $\theta$ and $\varphi$ is, therefore,

$$[\varphi(x), \theta(x')] = -2\pi i \Theta(x-x') \quad (17)$$

where $\Theta(x)$ is the unit step function. Note that this definition, together with the fact that total angular momentum is quantized to integers, implies that $\theta$ is also an angular variable defined modulo $2\pi$.

Now, suppose that the fields $\varphi$ and $\theta$ transform under $\mathbb{Z}_2$ according to

$$\varphi \rightarrow \varphi + n\pi, \quad \theta \rightarrow \theta + m\pi, \quad (18)$$

Here $(n, m) = (1, 0)$ corresponds to a normal on-site $\pi$ rotation of the boson field. On the other hand, as we shall see, $(n, m) = (1, 1)$ is the non-on-site symmetry that we would expect at the edge of a non-trivial $\mathbb{Z}_2$ SPT. Also, $m \neq 0$ corresponds to a superficially “chiral” symmetry in the sense that the left- and right-moving fields $\phi_{L,R} = \varphi \pm \theta$ transform differently under $\mathbb{Z}_2$, but in the $\mathbb{Z}_2$ case [though not for $\mathbb{Z}_n$ or U(1)] this chirality is not physically meaningful because $\theta \sim \theta + 2\pi$ so $m$ is actually only defined modulo 2.

From the commutation relations (17), one can show that Eq. (18) is effected by the unitary operator $U = (-1)^{nL+mW} = N^m S^n$, where $L$ is the total angular momentum and $W$ is the total winding number, and we define

$$N = \exp \left( -\frac{i}{2} \int \partial_x \varphi \, dx \right), \quad \quad (19)$$

$$S = \exp \left( -\frac{i}{2} \int \partial_x \theta \, dx \right). \quad \quad (20)$$
We now define the restriction \( U_{[a,b]} = N_{[a,b]}^m S_{[a,b]}^n \) to a finite interval \([a, b]\), where

\[
N_{[a,b]} = \exp \left( -\frac{i}{2} \int_a^b \partial_x \varphi \, dx \right) \tag{21}
\]

\[
S_{[a,b]} = \exp \left( -\frac{i}{2} \int_{a-\epsilon}^{b+\epsilon} \partial_x \theta \, dx \right), \tag{22}
\]

where we have made use of our freedom to redefine the restriction near the boundary of \([a, b]\) to shift the endpoints of the second integral by some small \(\epsilon > 0\). This ensures that \(N_{[a,b]}\) and \(S_{[a,b]}\) commute. Hence, we find that

\[
U^2_{[a,b]} = N_{[a,b]}^2 S_{[a,b]}^2, \tag{23}
\]

\[
N_{[a,b]}^2 = \exp \left( -i \int_a^b \partial_x \varphi \, dx \right) = e^{i\varphi(a)} e^{-i\varphi(b)}, \tag{24}
\]

\[
S_{[a,b]}^2 = \exp \left( -i \int_{a-\epsilon}^{b+\epsilon} \partial_x \theta \, dx \right) = e^{i\theta(a-\epsilon)} e^{-i\theta(b+\epsilon)}. \tag{25}
\]

Thus, as expected, we find that \(\Omega \equiv U_{[a,b]}^2 = [e^{in\theta(a-\epsilon)} e^{im\varphi(a)}][e^{-in\theta(b+\epsilon)} e^{-im\varphi(b)}] \equiv \Omega_a \Omega_b\) still acts non-trivially at the endpoints \(a\) and \(b\) even though \(U^2 = 1\).

In the present example, Eq. (4) takes the form

\[
U_{[a,b]} \Omega U_{[a,b]}^{-1} = \Omega, \tag{27}
\]

and this equality can readily be verified directly from the forms of \(U_{[a,b]}\) and \(\Omega\) given above. On the other hand, the restriction \(\Omega_a\) satisfies this equation in general only up to a phase factor. Indeed, we find

\[
U_{[a,b]} \Omega U_{[a,b]}^{-1} = e^{-in\theta(a-\epsilon)} e^{-im\varphi(a)-im\pi} = (-1)^{mn} \Omega_a. \tag{28}
\]

Hence, we find that the 3-cocycle associated with the realization of \(\mathbb{Z}_2\) is given by \(\omega(X, X, X) = (-1)^{mn}\) and \(\omega(g_1, g_2, g_3) = 1\) for \((g_1, g_2, g_3) \neq (X, X, X)\), where \(X\) is the generator of \(\mathbb{Z}_2\). For \(m = n = 1\) this corresponds to a non-trivial 3-cocycle, and the corresponding representation of \(\mathbb{Z}_2\) would appear at the boundary of a non-trivial \((2+1)\)-D \(\mathbb{Z}_2\) SPT.

### III. PROOF OF SEPARATION OF PHASES IN (2+1)-D.

In this section, we will outline how one can use the ideas given above to prove for \((2+1)\)-D systems that systems characterized by different elements of the cohomology group \(H^3(G, U(1))\) must be separated by a bulk phase transition; the details are left to the appendices. (Unfortunately, the proof cannot be applied in higher dimensions due to the lack of a completely general characterization of anomalous symmetry.)

First, as we want to make statements about bulk properties, we need to reformulate the ideas of Section II in a slightly different way, in terms of properties of the ground state in the bulk rather than the low-energy physics at the edge. We show in Appendix C that, given a general ground state \(|\Psi\rangle\) in some SPT phase in \(d\) spatial dimensions \((d \leq 2)\), and a region \(A\)
in the bulk, one can find a representation $V_{\partial A}(g)$ of the symmetry group, which acts \textit{inside} $A$, but only near the boundary $\partial A$, such that $U_A(g)|\Psi\rangle = V_{\partial A}(g)|\Psi\rangle$. Here $U_A(g)$ is the restriction of the symmetry onto the region $A$ (which can be defined consistently since we are assuming the symmetry is represented on-site in the bulk.) The physical interpretation of this result is simply that, as $|\Psi\rangle$ is invariant under $U(g)$, therefore $U_A(g)|\Psi\rangle$ can differ from $|\Psi\rangle$ only near the boundary $\partial A$. This representation $V_{\partial A}(g)$ can be anomalous in the same way as the representation of the symmetry on a physical edge, and the anomaly can be classified using the method of Section I.

The final result that we need is that the element of $H^3(G, U(1))$ is independent of the choice of region $A$, \textit{even} in the presence of spatial inhomogeneity; this is also proved in Appendix C. (Actually, as discussed in that appendix, we only prove this for certain regions $A$, but that is sufficient for the following discussion.) This allows us to prove that two systems $S$ and $S'$ characterized by different elements of $H^{d+1}(G, U(1))$ must be separated by a phase transition. Indeed, consider two systems connected without a phase transition. Then, without closing the gap, one can create an interpolated system that looks like $S$ on some region $A$ and like $S'$ on another region $A'$ (see Appendix C for a careful proof of this fact.) It therefore follows that the same element of $H^{d+1}(G, U(1))$ must be obtained in both cases. By a similar argument, one also finds that a spatial boundary between two different SPT phases must either be gapless or break the symmetry.

\section{Non-Linear Sigma Models}

It has been found\cite{8,9,44,45} that a quite general way to reproduce the essential features of various SPT phases is through the field theory of a quantum non-linear sigma model (NL\(\sigma\)M), where topological properties of the SPT phase arise out of the bulk theta term included in the action. Here, we will show in such models, the presence of the theta term indeed leads to an obstruction to on-site representation of the symmetry on a spatial edge, in such a way as to allow a straightforward calculation of the corresponding element of the cohomology group.

For example, consider in $D$ space-time dimensions (i.e. $D = d + 1$) a NL\(\sigma\)M for the $(D+1)$-component vector field $n$, constrained to have unit norm, i.e. $n$ lies on a unit $D$-sphere. The (Euclidean) action can be written as the sum of a dynamical contribution $S_{\text{dyn}}$ and a topological contribution $S_{\text{top}}$:

$$S_{\text{bulk}} = S_{\text{dyn}} + S_{\text{top}},$$

$$S_{\text{dyn}} = \frac{1}{\gamma} \int d^Dx \partial_{\mu} n \cdot \partial_{\mu} n,$$

$$S_{\text{top}} = i\Theta \frac{1}{V_D} \int n^*(\omega_V),$$

where $V_D$ is the volume of the unit $D$-sphere, and $n^*(\omega_V)$ is the pullback through the map $n$ of the volume form on the unit $D$-sphere. Written componentwise, this amounts to

$$S_{\text{top}} = i\Theta \frac{1}{V_D} \int d^Dx \epsilon^{a_1, \ldots, a_{D+1}} n^{a_1} \partial_0 n^{a_2} \partial_1 n^{a_2} \cdots \partial_{D-1} n^{a_{D}},$$

where $\epsilon^{a_1, \ldots, a_{D+1}}$ is the $(D+1)$-dimensional Levi-Civita symbol. The theta term $S_{\text{top}}$ measures a topologically invariant “generalized winding number” in $\pi_D(S^D) \cong \mathbb{Z}$, and for space-times without boundary is quantized to integer multiples of $i\Theta$. Hence, we implement the
requirement that SPT phases be trivial in the bulk by setting $\Theta$ to be an integer multiple of $2\pi$, thus ensuring that $S_{\text{top}}^{\text{bulk}}$ makes no contribution to the partition function $\int \mathcal{D}[\mathbf{n}] e^{-S}$. In fact, although we have given a specific form of $S_{\text{top}}^{\text{bulk}}$ for concreteness, it will not be important for our analysis as the topological features of the system are entirely captured by $S_{\text{top}}^{\text{bulk}}$.

Although the inclusion of $S_{\text{top}}^{\text{bulk}}$ has no effect on the partition function in the bulk, it does play a crucial role once we introduce a spatial edge. In that case $S_{\text{top}}^{\text{bulk}}$ depends (mod $2\pi i$) only on the values of $n$ on the boundary (to see this, note that any two extensions into the bulk can be connected at the boundary to give a closed surface, on which $e^{-S_{\text{top}}} = 1$); the action on the boundary is referred to as the Wess-Zumino-Witten action $S_{\text{WZW}}$. Thus, we can integrate out the gapped bulk to give an effective action for the low-energy excitations on the edge of the form

$$\exp(-S_{\text{edge}}) = \exp(-S_{\text{edge}}^{\text{dyn}} - S_{\text{WZW}}), \quad (33)$$

where $S_{\text{edge}}^{\text{dyn}} = \int d^d x L_{\text{edge}}^{\text{dyn}}$ is some unimportant dynamical term derived from $S_{\text{top}}^{\text{bulk}}$. Note that one can then write $S_{\text{WZW}} = \int d^d x L_{\text{WZW}}$ for some local Lagrangian density $L_{\text{WZW}}$ defined on the edge. However, there is no canonical way to do so.

Now let us consider the symmetry group $G$ in the bulk corresponding to some invertible action $n \rightarrow g n$ for $g \in G$. We demand that $S_{\text{top}}^{\text{bulk}}$ and $S_{\text{top}}^{\text{bulk}}$ be be locally invariant under the symmetry, i.e. that the integrands in Eqs. (30) and (31) must be invariant, not just the integral. Then we expect that $S_{\text{edge}}^{\text{dyn}}$ is also locally invariant under the symmetry. $S_{\text{WZW}}$ must also be globally invariant (at least, modulo $2\pi i$) but in general we do not expect it to be locally invariant. Indeed, because there is no canonical choice for $L_{\text{WZW}}$, one expects that the symmetry will transform $L_{\text{WZW}}$ to a different Lagrangian that nevertheless integrates to the same action (modulo $2\pi i$) in a spacetime without boundary.

We will now show that, after quantization, the lack of local invariance of $S_{\text{WZW}}$ implies the non-on-site nature of the unitary representation of the symmetry on the edge. We assume that after quantization the Hilbert space is spanned by a basis of states labeled by spatial configurations of $n$ at a fixed time. We can calculate the imaginary-time propagator $e^{-\beta H}$ (or equivalently, the Hamiltonian $H$) by a path integral

$$\langle \mathbf{n}' | e^{-\beta H} | \mathbf{n} \rangle = \int \mathcal{D}[\mathbf{n}(\tau)] e^{-S_{\text{edge}}^{\text{dyn}}\{0,\beta\}}, \quad (34)$$

where

$$S_{\text{edge}}^{\text{dyn}}\{0,\beta\} = \int d^{D-2} x \int^\beta_0 d\tau \left( L_{\text{edge}}^{\text{dyn}} + L_{\text{WZW}} \right) \quad (35)$$

is the action evaluated on a spacetime with temporal boundaries at $\tau = 0$ and $\tau = \beta$. Now, so far we only know that $S_{\text{WZW}}$ is globally invariant (modulo $2\pi i$) on a space-time manifold without boundary. Since $S_{\text{WZW}}$ is not locally invariant, in the presence of a temporal boundary we can only conclude that it will transform as $S_{\text{WZW}}\{0,\beta\} \rightarrow g S_{\text{WZW}}\{0,\beta\} \left( g \in G \right)$, where the difference can be expressed in terms of the field configurations at the temporal boundaries:

$$g S_{\text{WZW}}\{0,\beta\} - S_{\text{WZW}}\{0,\beta\} = i \mathcal{N}(g)[\mathbf{n}(\tau)] - i \mathcal{N}(g)[\mathbf{n}(0)] \quad (\text{mod } 2\pi i), \quad (36)$$

where $\mathcal{N}(g)$ is a functional of the field configuration at a fixed time.
Eq. (36) implies that the edge Hamiltonian is not invariant under the naive on-site implementation of the symmetry, \( S(g) = \int \mathcal{D}[n] |gn\rangle\langle n| \). Indeed, combined with Eq. (34), we find

\[
\langle n'|S(g)^\dagger e^{-\beta H}S(g)|n\rangle = e^{iN(g)|n'|-iN(g)|n}\langle n'|e^{-\beta H}|n\rangle.
\]

where

\[
N(g) = \int \mathcal{D}[n] e^{iN(g)|n|} |n\rangle\langle n|.
\]

Hence, we see that the correct implementation of the symmetry on the edge, which does commute with the Hamiltonian, is \( U(g) = N(g)S(g) \). In general, there is no reason to expect \( N(g) \) to be on-site, as we shall see. However, as we show in Section IV A, it is necessarily local. Thus, the symmetry on the edge is a local but non-on-site symmetry precisely of the form considered in Section I C, and we can calculate the appropriate element of the cohomology group using the reduction procedure of that section.

We can also consider anti-unitary symmetries by a straightforward extension of the above considerations. Specifically, an anti-unitary symmetry is implemented in the action by \( n \rightarrow gn, i \rightarrow (-1)^{n(g)}i \). Then we find that the representation of the symmetry on the edge is \( U(g) = N(g)S(g)K^{n(g)} \), with \( N(g) \) and \( S(g) \) as before and \( K \) complex conjugation in the \( n \) basis.

A. Calculating the cocycle in nonlinear sigma models using \( U(1) \) cochains on the target manifold

A particularly compact and elegant way of calculating the cocycle for NL\( \sigma \)Ms is by interpreting the theta term in terms of a \( U(1) \) cochain defined on the target manifold \( T = S^D \). First we need to state some definitions. We refer to \( k \)-dimensional oriented integration domains on a manifold \( T \) as \( k \)-chains. Given a \( k \)-chain \( A \), we denote the opposite orientation by \(-A\), and we can also define a sum operation on \( k \)-chains in the natural way, so that the \( k \)-chains can be viewed as an additive group. (If one wanted to be rigorous, one would define \( k \)-chains as formal linear combinations of oriented \( k \)-simplices with integer coefficients.) A \( U(1) k \)-cochain is a linear mapping from \( k \)-chains to \( U(1) \) \([\text{which we here write additively as } \mathbb{R}/(2\pi\mathbb{Z})]\). (Note that we are here referring to topological cochains on a manifold; these should be distinguished from the group cochains that are used to construct the group cohomology of some group \( G \).) In particular, each differential \( k \)-form \( \omega \) induces a \( U(1) k \)-cochain by integration,

\[
\omega(A) = \left( \int_A \omega \right) \text{ mod } 2\pi.
\]

where in an abuse of notation we will denote the \( k \)-form and the \( U(1) k \)-cochain by the same symbol. Any \( U(1) k \)-cochain \( \omega \) on the target manifold \( T \) can be used to define a local \( U(1) \)-valued functional \( F_\omega \) for a \( T \)-valued field \( n \) on a \( k \)-dimensional space(-time) manifold \( M \) via

\[
F_\omega[n] = \omega(n(M)),
\]

where \( n(M) \) is the image of \( M \), viewed as a chain, under the mapping \( n \). If \( \omega \) is derived from a differential \( k \)-form, this is equivalent to defining \( F_\omega \) as the integral of the pullback,
TABLE I. A tabular representation of the reduction process to extract a U(1) group 3-cocycle \( \nu = \omega^{(3)} \) starting from a symmetric topological term in (2+1)-D represented by a topological U(1) cochain \( \omega^{(0)} \). Each cell in the table is specified by a row label \( l \) and a column label \( k \), and corresponds to a set of \( k \)-cochains labeled by \( l \) group elements. Going left in the table corresponds to applying the topological coboundary operator \( d \), whereas going down corresponds to applying the group coboundary operator \( \delta \) defined by Eq. (47). These two operations commute, so the table can be interpreted as a commutative diagram.

\[
\begin{array}{cccc}
\delta & 3 & 2 & 1 \\
0 & \omega^{(0)} & \kappa^{(0)} & 0 \\
1 & 0 & \omega^{(1)} & \kappa^{(1)} \\
2 & 0 & \omega^{(2)} & \kappa^{(2)} \\
3 & 0 & \omega^{(3)} & \\
4 & 0 & \\
\end{array}
\]

\( F_\omega[n] = (\int_M n^*(\omega)) \mod 2\pi \). In particular, the topological theta term action of Eq. (31) is a special case of Eq. (41).

We define the coboundary operator \( d \) which maps \( k \)-cochains to \((k+1)\)-cochains according to

\[
(d\omega)(A) = \omega(\partial A),
\]

where \( \partial A \) is the boundary of \( A \). We call a \( k \)-cochain \( \omega \) exact if it can be written as \( \omega = d\kappa \) for some \((k-1)\)-cochain \( \kappa \). Our central tool is the following result.

**Lemma 1.** A U(1) \( k \)-cochain \( \omega \) on a manifold \( T \) is exact if and only if \( \omega(C) = 0 \) for all closed (i.e. boundaryless) \( k \)-chains \( C \).

**Proof.** See Appendix [D].

The property that \( \omega(C) = 0 \) for closed \( C \) in turn is equivalent to requiring of the induced functional \( F_\omega \) that it vanish on all closed space-time manifolds. If this is satisfied, then one expects that for a space-time manifold \( M \) with boundary, \( F_\omega[n] \) should depend only on the values of \( n \) on the boundary \( \partial M \). Indeed, given \( \omega = d\kappa \), one finds that

\[
F_\omega[n] = (d\kappa)(n(M))
= \kappa(\partial n(M))
= \kappa(n(\partial M))
\equiv F_\kappa[n(\partial M)].
\]

Given the above considerations, one can show that the procedure for obtaining the edge symmetry from the theta term, and then the cocycle from the edge symmetry, can be reduced to a simple prescription in terms of the U(1) cochains defined on the target manifold, with no reference to the space-time manifold at all, which we now describe.

We start from a topological action \( S_\text{top} \) on a spacetime-manifold \( M \) with \( d \)-dimensional target manifold \( T \), written as \( S_\text{top}[n] = F_{\omega^{(0)}}[n] = \omega^{(0)}(n(M)) \), where \( \omega^{(0)} \) is an exact U(1) \( d \)-cochain on \( T \) which is invariant under the action of the symmetry, \( g\omega^{(0)} = \omega^{(0)} \). Here we defined the action of the symmetry on a cochain by \( g\omega(A) = (-1)^{n(g)}\omega(gA) \), where \( n(g) \) is 1
for anti-unitary elements and 0 for unitary elements, and the action of \( g \) on chains is derived from its action on \( \mathbf{n} \). Hence, we have \( \delta g \omega(0) = g \omega(0) - \omega(0) = 0 \), where we have introduced the \textit{group} coboundary operators \( \delta_k \) (not the same as the topological coboundary operator \( d \) defined above) in the same way as Eq. [13] above, namely:

\[
(\delta_k \omega^{(k)})(g_1, \ldots, g_{k+1}) = g_1 \omega^{(k)}(g_2, \ldots, g_n) + (-1)^{k+1} \omega^{(k)}(g_1, \ldots, g_k) + \sum_{i=1}^{k} (-1)^i \omega^{(k)}(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{k+1}),
\]

(47)

Given a set of exact \((d-k)\)-cochains \( \omega^{(k)} \) indexed by \( k \) group elements which satisfy \( \delta_k \omega^{(k)} = 0 \), we can write \( \omega^{(k)} = d\kappa^{(k)} \) for some set of \((d-k-1)\)-cochains \( \kappa^{(k)} \). Now, \( \delta_k \omega^{(k)} = 0 \) implies that, for closed chains \( C \), \( (\delta_k \kappa^{(k)})(C) = (\delta_k \omega^{(k)})(\partial C) = 0 \). Hence, we can define \( \omega^{(k+1)} = \delta_k \kappa^{(k)} \) which is exact and satisfies \( \delta_{k+1} \omega^{(k+1)} = 0 \). The sequence terminates when we reach \( \omega^{(D)} \), which is a set of 0-cochains indexed by \( k \) group elements. Now a 0-cochain is essentially just a scalar \( \text{U}(1) \) function defined on the target manifold \( T \). But the fact that \( \omega^{(D)} \) evaluates to zero for the closed 0-chain \( a-b \) (where \( a \) and \( b \) are any two points) implies that the \( \omega^{(D)} \) are \textit{constant} \( \text{U}(1) \) functions. Thus, \( \omega^{(D)} \) defines a mapping from \( D \) group elements to \( \text{U}(1) \) satisfying \( \delta_D \omega^{(D)} = 0 \), which defines an element of the group cohomology group \( H^D(G, \text{U}(1)) \).

B. Examples

The possible symmetry transformations that leave the Lagrangian of Eq. (29) invariant in space-time dimensions \( D = 2, 3, 4 \) were constructed in Ref. [15] for a variety of different symmetry groups. Our framework allows in principle for the element of the cohomology group \( H^D(G, \text{U}(1)) \) to be calculated in all of these cases. Let us consider a few examples.

1. \( \mathbb{Z}_2^T \) in \((1+1)\)-D

We write the symmetry group as \( \mathbb{Z}_2^T = \{1, \mathbb{T}\} \). The target manifold is \( S^2 \) and we work in spherical coordinates \( \mathbf{n} = (\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi) \). The action of \( \mathbb{T} \) on \( \mathbf{n} \) is \( \mathbb{T} \mathbf{n} = -\mathbf{n} \), or in terms of the spherical coordinates \( \theta \to \pi - \theta, \varphi \to \varphi + \pi \). The initial \( \text{U}(1) \) cochain can be written in terms of a 2-form

\[
\omega^{(0)} = \Theta \frac{1}{4\pi} \sin \theta (d\theta \wedge d\varphi).
\]

(48)

As \( \omega^{(0)} \) integrates to 0 (mod \( 2\pi \)) over the whole 2-sphere, it follows that it can written as \( \omega^{(0)} = d\kappa^{(0)} \) for some \( \text{U}(1) \) 1-cochain \( \kappa^{(0)} \). We can write \( \kappa^{(0)} \) explicitly as

\[
\kappa^{(0)} = \Theta \frac{1}{4\pi} (1 - \cos \theta) d\varphi
\]

(49)

Treating \( \kappa^{(0)} \) as a differential 1-form and taking the exterior derivative, one recovers Eq. (48). When written as a 1-form, \( \kappa^{(0)} \) appears to have a singularity at \( \theta = \pi \). To show that, as a \( \text{U}(1) \) 1-cochain, \( \kappa^{(0)} \) is actually well-defined and satisfies \( d\kappa^{(0)} = \omega^{(0)} \) globally, it
is sufficient to check that $\int_C \kappa^{(0)} = 0 \pmod{2\pi}$ for an loop $C$ of infinitesimal size encircling the apparent singularity at $\theta = \pi$, which is indeed the case.

Now, following the general prescription of Section IV A, we define $\omega^{(1)} = \delta_0 \kappa^{(0)}$. The only non-trivial component is

$$\omega^{(1)}(\mathbb{T}) = \mathbb{T} \kappa^{(0)} - \kappa^{(0)}$$

$$= -\frac{\Theta}{4\pi} (1 + \cos \theta) - \frac{\Theta}{4\pi} (1 - \cos \theta)$$

$$= -\frac{\Theta}{2\pi} d\varphi,$$

from which we immediately read off that $\omega^{(1)} = d\kappa^{(1)}$, where $\kappa^{(1)} = -\frac{\Theta}{2\pi} \varphi$ (which is well-defined as a $U(1)$ 0-cochain because $\varphi$ is defined modulo $2\pi$). Thus, we can define the cocycle $\nu = \delta_1 \kappa^{(1)}$, and the only non-zero component is

$$\nu(\mathbb{T}, \mathbb{T}) = \mathbb{T} \kappa^{(1)} + \kappa^{(1)}$$

$$= \frac{\Theta}{2\pi} \{ \varphi + \pi - \varphi \}$$

$$= \frac{\Theta}{2}.$$

Thus, if $\Theta$ is an odd multiple of $2\pi$, this 2-cocycle corresponds to a non-trivial SPT phase, with the zero-dimensional boundary transforming projectively under the symmetry, i.e. as a Kramers doublet with $\mathbb{T}^2 = -1$. On the other hand, if $\Theta$ is an even multiple of $2\pi$, we have a trivial SPT phase with the edge transforming as $\mathbb{T}^2 = 1$. Thus, by different choices of $\Theta$ one recovers both elements of the cohomology group $H^2(Z^U_2, U(1)) \cong \mathbb{Z}_2$.

2. $Z_2$ in $(2+1)$-D

We write the symmetry group as $Z_2 = \{1, X\}$. The target manifold is $S^3$ and we work in generalized spherical coordinates $\mathbf{n} = (\cos \theta, \sin \theta \mathbf{n}_2)$, where $\mathbf{n}_2 \in S^2$. The action of $X$ on $\mathbf{n}$ is $X\mathbf{n} = -\mathbf{n}$, or in terms of the generalized spherical coordinates $\theta \to \pi - \theta, \mathbf{n}_2 \to -\mathbf{n}_2$. The initial $U(1)$ cochain is

$$\omega^{(0)} = \Theta \frac{1}{V_3} \sin^2 \theta (d\theta \wedge \omega_{V,2})$$

where $V_3$ is the volume of the 3-sphere, and $\omega_{V,2}$ is the volume form for $\mathbf{n}_2$. We then find that $\omega^{(0)} = d\kappa^{(0)}$, where

$$\kappa^{(0)} = \Theta \frac{1}{V_3} \left( \int_0^\theta \sin^2 x \, dx \right) \omega_{V,2}. $$

We observe that $V_3$ can be expressed as ($V_2 = 4\pi$)

$$V_3 = V_2 \int_0^\pi \sin^2 \theta \, d\theta.$$
From this one can show that $\kappa^{(0)}$ is well-defined despite the apparent singularity at $\theta = \pi$. Now the only non-trivial element of $\omega^{(1)} = d\kappa^{(1)}$ is

$$\omega^{(1)}(X) = X\kappa^{(0)} - \kappa^{(0)}$$

$$= \Theta \frac{1}{\sqrt{3}} \omega_{\nu,2} \int_0^\pi \sin^2 x dx$$

$$= \Theta \frac{1}{4\pi} \omega_{\nu,2}$$

(here we used Eq. (58) and the fact that $\omega_{\nu,2}$ is odd under $n_2 \rightarrow -n_2$.) In fact, this is identical to Eq. (48). The reduction process then proceeds nearly identically to that in Section IV B 1 above and one finds that the only non-zero component of the 3-cocycle is

$$\nu(X, X, X) = \frac{\Theta}{2}.$$ 

Thus, one recovers both elements of $H^3(Z_2, U(1)) \cong Z_2$ for $\Theta$ an odd or even multiple of $2\pi$ respectively.

V. LATTICE MODELS OF SPT PHASES

In Ref. 7, the classification of SPT phases in $d$ spatial dimensions was based on an explicit construction of a field theory for a $d + 1$-dimensional discrete spacetime for each element of the cohomology group $H^{d+1}(G, U(1))$. Although a discrete spacetime is perhaps hard to interpret physically, the construction of Ref. 7 can also be used to derive a ground-state wavefunction on a spatial lattice; a gapped Hamiltonian with this wavefunction as its ground state constitutes an (albeit unrealistic) lattice Hamiltonian realizing the SPT phase. Hence, it is worthwhile to show that the symmetry on the edge of such of a lattice model is indeed classified under our scheme by the same element of the cohomology group that was used to construct the wavefunction. We do this in Appendix E. In particular, this shows that every element of the cohomology group $H^{d+1}(G, U(1))$ can be realized in an explicit lattice model.

VI. BEYOND THE COHOMOLOGICAL CLASSIFICATION

It is now well established that in $(3+1)$-D there exists an SPT phase with respect to time-reversal symmetry that is beyond the standard cohomological classification. The reason why this phase is outside the cohomological classification can be readily understood, as follows. Deriving the cohomological classification using arguments such as those presented in this paper requires at the very least the assumption that the symmetry can be implemented locally on a standalone realization of the edge. We will now argue that the beyond-cohomology phase violates this assumption.

Indeed, one possible surface termination for the beyond-cohomology phase is a gapped “three-fermion” topological phase $\mathcal{F}$ in which all three non-trivial particle sectors are fermions. Any purely $(2+1)$-D realization of this phase is necessarily chiral; that is, its conjugate $\overline{\mathcal{F}}$ under time reversal cannot be connected to $\mathcal{F}$ without a phase transition. (One way to see this is to note that $\mathcal{F}$ and $\overline{\mathcal{F}}$ have opposite edge chiral central charges $c_+ = \pm 4$ and hence a spatial boundary between them must be gapless. If we make the
spatial variation from $\mathcal{F}$ to $\overline{\mathcal{F}}$ sufficiently slow, this gapless spatial boundary must be interpreted as a bulk phase transition.) Suppose that a state $|\Psi\rangle$ within the phase $\mathcal{F}$ could be invariant under a local anti-unitary operation $T$. Then one can always write $T = UT$, where $T$ is the normal on-site representation of time-reversal, and $U$ is a local unitary. But then, since $T|\Psi\rangle$ is in the conjugate phase $\overline{\mathcal{F}}$, we see that $U$ is a local unitary connecting $\mathcal{F}$ and $\overline{\mathcal{F}}$, which is a contradiction.

VII. FERMIONIC SYSTEMS

The restriction arguments given in Section IV are quite general and therefore can be equally well applied to fermionic systems, at least in $(2+1)$-D. (Generalizing to higher dimensions would require one to find an appropriate fermionic equivalent of the special form of the symmetry considered in Section IV C.) Here we will discuss in general terms the issues arising which result in the fermionic classification differing from the bosonic one, with reference to a particular example of a Fermion SPT protected by a $\mathbb{Z}_2$ symmetry. As the general classification is somewhat complicated, we we leave the details to Appendix E. It would be interesting to see whether it can be related to the “supercohomology” classification proposed in Ref. 19.

The first issue that needs to be considered is the privileged role of fermion parity. Any local fermionic system must be invariant under the fermion parity $(-1)^F$, where $F$ is the total fermion number. Therefore, the fermionic symmetry group $G_f$ characterizing a fermion SPT always contains fermion parity. This must commute with all the other elements of $G_f$ if they describe local symmetries. If we now consider the $(1+1)$-D edge of a $(2+1)$-D SPT, by assumption it is realizable as a strictly $(1+1)$-D local fermion system. As this $(1+1)$-D system must always be invariant under the fermion parity of the edge, we expect that, in the realization of $G_f$ on the $(1+1)$-D edge, the parity element is represented as the actual fermion parity of the edge. (This can be verified by using the techniques of Appendix A to construct the edge representation.) That is, by contrast to the bosonic case, the fermionic symmetry group contains an element that is always realized on-site on the boundary. Furthermore, even when we restrict and consider the action of the symmetry on a finite interval, the restricted operations must be local, and therefore must still commute with the fermion parity (whereas there is no analogous requirement in the bosonic case.)

The other main difference from the fermionic case occurs when defining the restriction of the obstruction operator $\Omega(g_1, g_2)$, which acts on a pair of points $a$ and $b$, to a single point $a$. At this point, one encounters a subtlety that was glossed over in the bosonic treatment. $\Omega(g_1, g_2)$ is clearly local in the sense (“locality preserving”) that it maps local operators (including fermion creation and annihilation operators) to local operators under Heisenberg evolution. (We can deduce this from the fact that it is true for the $U_M(g)$’s and that the locality preserving property is invariant under multiplication.) This does not necessarily imply that it is a local unitary in the sense (“locally generated”) that it can be written as the time evolution of a local fermionic Hamiltonian in a domain containing only the two points $a$ and $b$. In other words, we might not be able to write $\Omega(g_1, g_2) = \Omega_a(g_1, g_2)\Omega_b(g_1, g_2)$, where $\Omega_a$ and $\Omega_b$ are fermionic local unitaries acting only near the points $a$ and $b$. For example, the following unitary is locality preserving but not locally generated:

$$\Omega = (c_a + c_a^\dagger)(c_b + c_b^\dagger),$$

where $c_{a,b}$ are the annihilation operators for fermions at points $a$ and $b$ respectively. If $\Omega$ is
However, this is an example of the possibility discussed above, of the operators $\Omega$ that $\Omega \equiv \omega U$ one aspect of the non-trivial fermionic cocycle. The other aspect comes from the relation

$$\Omega(g_1, g_2) = \Omega_a(g_1, g_2) \Omega_b(g_1, g_2),$$

where $\Omega_a(g_1, g_2)$ is either a fermionic local unitary acting near the point $a$, or it is such a local unitary multiplied by $c_a + c_a^\dagger$ (and similarly for $\Omega_b$). In the latter case, however, the restricted operations $\Omega_a(g_1, g_2)$ and $\Omega_b(g_1, g_2)$ will anti-commute rather than commute. This anti-commutation leads to fermionic corrections to the 3-cocycle condition [Eq. (6)], to the equivalence relation [Eq. (7)], and to the product rule for “stacking” SPT phases; see Appendix F for more details.

A. Example: Fermionic SPT with $Z_2$ symmetry

In order to illustrate the ideas discussed above, let us consider a (1+1)-D field theory which we expect to describe the edge of a (2+1)-D fermionic SPT protected by a $Z_2$ symmetry. (This $Z_2$ is in addition to the always-present fermion parity; thus the full fermionic symmetry group is $G_f = Z_2 \times Z_2^f$.) This theory is the fermionic analogue of the bosonic edge we considered in Section I. The low-energy physics is described by a gapless Dirac point (which can emerge, for example, from a microscopic lattice model of non-interacting electrons with a Fermi surface.) Thus, we define the fermionic fields $\Psi_R(x)$ and $\Psi_L(x)$ corresponding to left- and right-moving fermions (in terms of the original lattice operators, these will be local on a length scale set by the energy cutoff.) We can define the corresponding number operators $N_{L,R} = \int \Psi_{L,R}^\dagger \Psi_{L,R} dx$. The Hamiltonian is

$$H = J(N_L + N_R).$$

(64)

The fermion parity is $(-1)^{N_R+N_L}$ and sends $\Psi_L \rightarrow -\Psi_L, \Psi_R \rightarrow -\Psi_R$. We assume that the additional $Z_2$ symmetry is given by $U = (-1)^{N_L}$; thus, it acts only on the right-movers and sends $\Psi_L \rightarrow \Psi_L, \Psi_R \rightarrow -\Psi_R$. This forbids perturbations like $\Psi_R^\dagger \Psi_L^\dagger$ which would open up a gap, suggesting that the gapless edge is protected by the symmetry. Indeed, we will show that the symmetry corresponds to a non-trivial fermionic cocycle.

We can define the restriction of the $Z_2$ symmetry to a finite interval $[a, b]$ according to

$$U_{[a,b]} = \exp \left( -i\pi \int_a^b \Psi_R^\dagger \Psi_R dx \right).$$

(65)

If we invoke the bosonization correspondences $\Psi_R^\dagger \Psi_R \sim \partial_x \phi_R(x)/(2\pi), \Psi_R(x) \sim e^{i\phi_R}$, we see that $\Omega \equiv U_{[a,b]}^2 \sim \Psi_R(a) \Psi_R^\dagger(b) \equiv \Omega_a \Omega_b$. Thus, $U_{[a,b]}^2$ acts only on the endpoints as expected. However, this is an example of the possibility discussed above, of the operators $\Omega_a$ carrying non-trivial fermion parity.

The parity of $\Omega_a$, which we call $\sigma$ ($\sigma = -1$ in the current calculation) constitutes one aspect of the non-trivial fermionic cocycle. The other aspect comes from the relation $U_{[a,b]}^2 \Omega \Omega_{[a,b]}^\dagger = \Omega$. The restricted operations $\Omega_a$ satisfy this relation only up to a phase factor $\omega$. To calculate this phase factor we need to regularize the integral Eq. (65) by introducing
a soft cutoff; that is, we replace Eq. (65) by
\[ U_{[a,b]} = \exp \left( -i\pi \int_{-\infty}^{\infty} f(x)\Psi_R^\dagger \Psi_R dx \right), \]  
where \( f \) is a smooth function such that \( f(x) = 1 \) for \( x \in [a + \epsilon, b - \epsilon] \), and \( f(x) = 0 \) for \( x < (a - \epsilon) \) or \( x > (b + \epsilon) \). Using the bosonization correspondence to express \( U_{[a,b]}^2 \) in terms of \( \partial_x \phi_R \) and integrating by parts gives
\[ \Omega_a \sim \exp \left( i \int_{a-\epsilon}^{a+\epsilon} f'(x)\phi_R dx \right). \]  
Using the fact that \( \phi_R \rightarrow \phi_R + \pi \) under \( U_{[a,b]} \) gives
\[ \omega \equiv U_{[a,b]} \Omega_a U_{[a,b]}^\dagger \Omega_a^\dagger = \exp \left( i\pi \int_{a-\epsilon}^{a+\epsilon} f'(x)f(x)dx \right) \]
\[ = \exp \left( i\pi \frac{1}{2} \int_{a-\epsilon}^{a+\epsilon} \frac{d}{dx} [f(x)]^2 \right) \]
\[ = i. \]

The numbers \((\omega, \sigma) = (i, -1)\) constitute the fermionic 3-cocycle for the \( Z_2 \) symmetry. We see that taking four copies of the same edge leads to a trivial fermionic 3-cocycle (in agreement with the results of Ref. 10 showing that four copies of the theory under consideration can be gapped out without breaking the symmetry.) Furthermore, if one applies the fermionic 3-cocycle condition [see Appendix F] one sees that the only allowable values of the fermionic 3-cocycle are the ones obtained by taking copies in this way, namely \((1, 1), (i, -1), (-1, 1)\) and \((-i, -1)\). Thus, we have recovered all the elements of a \( Z_4 \) classification for fermionic SPT’s with \( G_f = Z_2 \times Z_2^f \) (which is the same result obtained from supercohomology\(^{19}\)). By contrast, Refs. 30–32, and 50 obtained a \( Z_8 \) classification for the same \( G_f \). The odd-numbered phases in this classification have an odd number of gapless Majorana modes at the edge, each of which is “half” of the gapless Dirac mode considered here. The explanation for the discrepancy in the classification is that the symmetry in these odd-numbered phases does not act locally on the edge, and hence they are not captured by our approach.

**VIII. CONCLUSIONS**

Suppose we have a system whose bulk ground state is invariant under a group \( G \) of symmetries that commute with the Hamiltonian. Let us further suppose that there is an energy gap to all bulk excitations and a concomitant finite correlation length and that we can solve the Hamiltonian (with a sufficiently powerful computer, for instance) for systems much larger than the correlation length. Armed with this information, we wish to determine if the system is in a symmetry-protected topological phase and, if so, which one. In a 1D system on a finite interval, we can identify an SPT phase by the presence of gapless excitations at the ends of the system that transform under a projective representation of the symmetry (or, alternatively, the presence of such states in the bipartite entanglement spectrum\(^{15,18}\). But how do we identify an SPT phase in higher dimensions? One approach is to gauge
the symmetry $G^{17,22,51,52}$. In 2D, the resulting theory has anyonic excitations in the bulk\cite{17}. By determining the statistics of these excitations, one can deduce the SPT phase of the ungauged system. In 3D, the gauged theory has anyonic excitations on its surface\cite{52} and extended excitations (e.g. vortex lines) in its bulk\cite{22}. The topological properties of surface and bulk excitations of the gauged model can be used to identify the underlying SPT phase. But this approach involves modifying the system drastically, and it cannot be used if all that we are given are the low-energy eigenstates of the original Hamiltonian. Moreover, it may be more difficult, as a practical matter, to solve the gauged model and deduce its quasiparticles’ topological properties than it is to solve the original Hamiltonian.

Here, we take a different approach, which identifies an SPT directly from the realization of the symmetry group $G$ on its boundary states. We consider $d$-dimensional SPTs for which the restriction of $G$ to the low-energy Hilbert space has a local action on the $(d-1)$-dimensional boundary of the system. In such a phase, there may be an obstruction to restricting the action of the symmetry to a $(d-1)$-dimensional proper submanifold of the boundary. To analyze such an obstruction, we construct a new functional of two group elements by taking a suitably defined coboundary of the restriction. This localizes the obstruction to the $(d-2)$-dimensional boundary of the $(d-1)$-dimensional proper submanifold of the boundary. We then continue in the same fashion, either restricting a functional of $k$ group elements on a closed $(d-k)$-dimensional manifold to a $(d-k)$-dimensional submanifold with boundary or constructing the coboundary of a functional of $k$ group elements on a $(d-k)$-dimensional submanifold with boundary, thereby obtaining a functional of $k+1$ group elements on a $(d-k-1)$-dimensional closed manifold. These functionals are operators that act on the local Hilbert spaces of the corresponding submanifolds. The resulting sequence of maps between functionals terminates after we reach functionals of $d$ group elements acting on a single point; the coboundary of such a functional must be an ordinary phase. Equivalence classes of such sequences are classified by the cohomology group $H^{d+1}(G,U(1))$ in $d = 1, 2$ and, with an additional assumption, in $d \geq 3$. Consequently, given the low-energy states of the boundary (or large eigenvalue eigenstates of the reduced density matrix for a bipartition of a system without a real boundary), we can, in principle, determine the corresponding element of $H^{d+1}(G,U(1))$. The Hamiltonian need not take any special form – in fact, it is not even necessary to know the Hamiltonian. As we have shown, this procedure gives the expected results when applied to discrete\cite{7} and continuous\cite{8,9,44,45} non-linear sigma models.

The obstructions classified by these arguments prevent a model from being continuously deformed into a model in which the symmetry is realized on the boundary in an on-site manner. (By assumption, the symmetry can be realized in an on-site manner in the full bulk theory – by grouping multiple sites into a single site, for instance.) As a result of the incorrigibly non-on-site nature of the symmetry, if we try to gauge it, the resulting gauge theory will be anomalous\cite{20,37,38,39,53,54}. Only the action of the symmetry on the whole system, bulk and edge together, can be gauged in an anomaly-free fashion. A simple example is a 2+1-dimensional $U(1)$ SPT, which is very similar to the $\mathbb{Z}_2$ case discussed in Section\cite{11}. Such a state is a bosonic integer quantum Hall state\cite{44}. If the theory is gauged, the edge effective Lagrangian takes the form $L_{\text{edge}} = \frac{\eta}{2\pi}(\partial_\mu \varphi - nA_\mu)^2 + \frac{m}{2\pi}A_\mu \epsilon_{\mu\nu} \partial_\nu \varphi$. Charge is no longer conserved at the edge since an electric field along the edge will cause charge to flow from the bulk to the edge. Following Laughlin\cite{55}, we can understand this in an annular geometry. By adiabatically increasing the flux through the center of the annulus by $2\pi$, the charge at the outer edge is increased by $2nm$, the integer (necessarily even in a bosonic SPT) that characterizes the Hall conductance. The 3-cocycle obtained by our construction reflects this
charge pumped to the edge, as may be seen by noting that a $U(1)$ transformation applied to a finite interval along the edge is equivalent to applying equal and opposite gauge fields at the ends of the interval. Since they are equal and opposite, such gauge fields cannot increase the total charge on the edge, but if we focus on the charge to the left of an arbitrary point in the middle of the interval, then this increases by $2nm$ when the gauge field winds by $2\pi$. The restriction $\Omega_\alpha$ defined in Eq. 3 measures such a charge. Meanwhile, $U_M$ applies gauge fields at the ends of the interval. Then, according to the definition (5), the cocycle measures accumulated charge in response to this change in gauge field. We expect that similar reasoning can relate our constructions to anomalies in higher dimensions and for discrete symmetries.

In this paper, we have confined our attention to “internal” symmetries. It would be interesting to extend them to space group symmetries. States of free fermions protected by inversion symmetry, time-reversal symmetry combined with a point group symmetry, or a rotational symmetry alone have been classified. With the methods described here, it might be possible to extend these ideas to interacting fermion systems and to bosonic systems in which a space group symmetry, projected to the low-energy boundary theory, maps sites to sites and then has an additional “internal” action that is non-on-site. However, care must be taken to consider a boundary that respects the space group symmetry and to consider a sequence of submanifolds (which are, presumably, not connected manifolds) that also respect the symmetry.

We have given a prescription that, in principle, allows one to identify an SPT phase, given its ground state wavefunction, and we have shown how to apply it to some long-wavelength effective field theories and exactly soluble lattice models. But how useful can this prescription be in practice, given an arbitrary – perhaps experimentally-motivated model? This remains to be seen. However, ground state wavefunctions with tensor network descriptions are natural candidates for the reduction procedure. A numerical implementation would open an important avenue for future research.

As noted above, our construction leads to $H^{d+1}(G, U(1))$ in $d \geq 3$ provided we make an additional assumption: there exists a local basis for the Hilbert space of the $(d-1)$-dimensional boundary in which the symmetry acts on the boundary in an on-site manner, except for a diagonal part which cannot be made on-site. This assumption holds in a system that is described by a $d$-dimensional non-linear sigma model with $\theta$-term at long wavelengths, since the symmetry acts in an on-site manner on all gradient energy terms in the $(d-1)$-dimensional boundary effective action and non-on-site only on the Wess-Zumino term, which only enters the phase of the ground state wavefunction. However, it remains an interesting open question whether there are SPT phases in three dimensions that violate this assumption and, consequently, realize the aforementioned non-trivial sequence but in a manner that is not classified by group cohomology. Such an exception to a cohomological classification, if it exists, would be distinct from the so-called “beyond cohomology” SPT phases, which occur due to the violation of a different assumption – that the symmetry is realized locally (but not necessarily on-site) at the boundary of the system. In “beyond cohomology” SPT phases, the symmetry is realized in an inherently non-local manner at the boundary of the system. Our methods do not enable us to classify such phases; once the condition of locality is relaxed, a very different approach may be necessary.

This comment also applies to the most famous SPT phase, the 3D time-reversal-invariant topological insulator, where time-reversal acts in an inherently non-local manner at a 2D surface. However, there are fermionic SPT phases in which the symmetry is realized locally
on the boundary, and these can be classified along the lines discussed in Section VII. Carrying out this classification to completion and relating it to the notion of “supercohomology” is an important goal for future work.

Finally, we note that symmetry-enriched topological (SET) phases generalize SPT phases to systems with topological order. In SET phases, symmetry realization interacts non-trivially with the fusion and braiding properties of anyons, as already occurs in topological phases at the (2+1)-D boundary of a (3+1)-D SPT. The possible symmetry fractionalization patterns in (2+1)-D correspond to different projective representations of the anyons and are classified by $H^2(G, A)$, where $A$ is the group of Abelian anyons. It is possible that an extension of our methods can also be applied to the analysis of symmetry fractionalization in (3+1)-D SET phases which have topological excitations occupying closed loops.

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Appendix A: Explicit construction of the edge theory

In this Appendix we will give an explicit proof of the property that the edge theory of an SPT in $d$ spatial dimensions can always be realized in a strictly $(d-1)$-dimensional system and show given certain assumptions how to construct the representation of the symmetry in this realization.

By definition, the ground state of an SPT phase is gapped and not topologically ordered, which means it can be connected to a product state by a local unitary. Indeed, let $D$ be the local unitary which turns the bulk ground state $|\Psi_{gr}\rangle$ on a boundaryless spatial region product state $|\phi\rangle^\otimes N$, and let $\tilde{D}$ be the restriction of $D$ to a spatial region with boundary. Any low-energy state $|\Psi\rangle$ associated with the boundary must be identical to $|\Psi_{gr}\rangle$ far from the boundary. It follows that $\tilde{D}|\Psi\rangle$ must be identical to $|\phi\rangle^\otimes N$ far from the boundary. Hence, $\tilde{D}|\Psi\rangle$ is simply a direct product with copies of $|\phi\rangle$ in the bulk of some state $|\Psi_B\rangle$ defined on a strip $B$ near the boundary:

$$\tilde{D}|\Psi\rangle = |\Psi_B\rangle \otimes |\Phi_{B^c}\rangle$$  \hspace{1cm} (A1)

where $B^c$ is the complement of $B$ and $|\Phi_{B^c}\rangle$ is a product state of $|\phi\rangle$ on every site in $B^c$. Thus, the states $\{|\Psi_B\rangle : |\Psi\rangle$ a low-energy boundary state $\}$ constitute a $(d-1)$-dimensional realization of the boundary theory. One can also apply the mapping $\tilde{D}$ to the original Hamiltonian for the system with boundary in order to obtain a Hamiltonian for this realization of the boundary theory.

Now suppose that the bulk ground state is invariant under an on-site representation $U(g)$ of the symmetry. As the local unitary $D$ is not required to have any particular properties with respect to the symmetry, in general it might not be easy to determine how the symmetry acts on the boundary theory. However, the task becomes easier if we make the following simplifying assumption: we assume that $D$ can be chosen to commute with $U(g)$ in the absence of boundary. (We emphasize that this does not necessarily imply that we are considering a trivial SPT phase. That would be only be true if we made the stronger
assumption that \( D \) can be continuously connected to the identity by a path which everywhere commutes with the symmetry.). In fact, this assumption is always true in any SPT phase described by the cohomological classification, because, in particular, it is true for the ground states constructed via the discrete topological term construction of Ref. 7 (see Section V above.) This implies that it is also true for any other ground state in the same SPT phase, since, by definition, any two ground states in the same SPT phase can be related by a symmetry-respecting local unitary.

Given this assumption, one can explicitly construct the realization of the symmetry on the edge, as follows. That \( U(g) \) and \( D \) commute in the absence of boundary implies that their restrictions \( \tilde{U}(g) \) and \( \tilde{D} \) to a region with boundary must commute up to boundary terms. Thus,

\[
\tilde{D}\tilde{U}(g)|\Psi\rangle = W_B(g)U_{B^c}(g) |\Psi\rangle_B \otimes |\Phi\rangle_{B^c}
\]

To get to the last line, we used the fact that \( U_{B^c}(g)|\Phi\rangle_{B^c} = |\Phi\rangle_{B^c} \). This follows from the fact that, without boundary, \( |\phi\rangle \otimes \mathbb{N} \) is invariant under \( U(g) \), since it is obtained from \( |\Psi_{gr}\rangle \) [which is certainly invariant under \( U(g) \)] by \( D \) which by assumption commutes with \( U(g) \).

Comparing Eq. (A4) with Eq. (A1), we see that \( W_B(g) \) is the representation of the symmetry on the stand-alone realization of the boundary.

Appendix B: The (2+1)-D reduction procedure

Here we will prove the two key properties of \( \omega(g_1, g_2, g_3) \) defined by Eq. (5) in Section I B; firstly, that it must be a 3-cocycle, and secondly, that up to equivalence it is independent of the choice of restriction \( U(g) \rightarrow U_M(g) \).

We first make a general remark: the structure described in Section I B is known in the mathematics literature as a crossed module extension. Recall that a projective representation of a group \( G \) corresponds to a central extension, which is an exact sequence

\[
1 \rightarrow U(1) \rightarrow H \rightarrow G \rightarrow 1
\]

such that the image of \( U(1) \) is in the center of \( H \). Similarly, a crossed module extension is an exact sequence

\[
1 \rightarrow U(1) \rightarrow K \xrightarrow{\varphi} H \rightarrow G \rightarrow 1
\]

along with a left-action of \( H \) on \( K \), represented by \( k \mapsto h_k \), such that \( \varphi(k)k' = kk'^{-1} \) for all \( k, k' \in K \). It is a well-known theorem in the mathematics literature\(^{70-73}\) that the crossed module extensions of \( G \) by \( U(1) \) are classified by \( H^3(G, U(1)) \). The procedure described in Section I B for obtaining the 3-cocycle \( \omega(g_1, g_2, g_3) \), as well as the proofs of the properties of \( \omega \) given below, are adapted from the proof of this classification theorem given in Ref. 73. However, the reader does not need to understand the connection to crossed module extensions in order to follow these proofs.
To prove that $\omega$ is a 3-cocycle, we calculate $\Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3) \Omega_a(g_1g_2g_3, g_4)$ in two different ways. Firstly,

$$
\Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3) \Omega_a(g_1g_2g_3, g_4)
= \omega(g_1g_2, g_3, g_4) \times \Omega_a(g_1, g_2)^{U_M(g_1g_2)} \Omega_a(g_3, g_4) \Omega_a(g_1g_2, g_3g_4)
$$

(B3)

$$
= \omega(g_1g_2, g_3, g_4) \times \Omega_a(g_3, g_4) \Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3g_4)
$$

(B4)

$$
= \omega(g_1g_2, g_3, g_4) \times \Omega_a(g_3, g_4) \Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3g_4)
$$

(B5)

$$
= \omega(g_1g_2, g_3, g_4) \times U_M(g_1) U_M(g_2) \Omega_a(g_3, g_4) \Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3g_4)
$$

(B6)

$$
= \omega(g_1g_2, g_3, g_4) \times U_M(g_1) U_M(g_2) \Omega_a(g_3, g_4) \Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3g_4)
$$

(B7)

$$
\omega(g_1g_2, g_3, g_4) \times U_M(g_1) U_M(g_2) \Omega_a(g_3, g_4) \Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3g_4),
$$

(B8)

where we applied Eq. (5) twice. To get from Eq. (B5) to Eq. (B7) we used Eq. (3). To get from Eq. (B5) to Eq. (B6), we used the fact that $\Omega(g, g')$ can be written as a product of contributions near $a$ and contributions near $b$, which commute; it follows that for any operator $X_a$ localized near $a$,

$$
\Omega(g_1, g_2) X_a = \Omega(g_1, g_2) X_a.
$$

(B9)

Proceeding in a different way, we also have

$$
\Omega_a(g_1, g_2) \Omega_a(g_1g_2, g_3) \Omega_a(g_1g_2g_3, g_4)
= \omega(g_1, g_2, g_3) \times U_M(g_1) \Omega_a(g_2, g_3) \Omega_a(g_1, g_2g_3) \Omega_a(g_1g_2g_3, g_4)
$$

(B10)

$$
= \omega(g_1, g_2, g_3) \times U_M(g_1) \Omega_a(g_2, g_3) \Omega_a(g_1, g_2g_3) \Omega_a(g_1g_2g_3, g_4)
$$

(B11)

$$
= \omega(g_1, g_2, g_3) \times U_M(g_1) \{ \Omega_a(g_2, g_3) \Omega_a(g_2g_3, g_4) \} \Omega_a(g_1, g_2g_3)
$$

(B12)

$$
= \omega(g_1, g_2, g_3) \times U_M(g_1) \{ \Omega_a(g_2, g_3) \Omega_a(g_2g_3, g_4) \} \Omega_a(g_1, g_2g_3)
$$

(B13)

$$
= \omega(g_1, g_2, g_3) \times U_M(g_1) \{ \Omega_a(g_2, g_3) \Omega_a(g_2g_3, g_4) \} \Omega_a(g_1, g_2g_3)
$$

(B14)

Comparing Eq. (B15) with Eq. (B8) we see that $\omega$ must obey the 3-cocycle condition

$$
\omega(g_1g_2, g_3, g_4) \omega(g_1, g_2g_3) = \omega(g_1, g_2, g_3) \omega(g_1g_2, g_3, g_4) \omega(g_2, g_3, g_4).
$$

(B16)

Next we prove independence from the choice of restriction $U(g) \rightarrow U_M(g)$. Indeed, consider two restrictions $U_M(g)$ and $\tilde{U}_M(g) = \Sigma(g) U_M(g)$, where $\Sigma(g)$ is a local unitary acting near $\partial M = \{a, b\}$. Then we find that

$$
\tilde{U}_M(g) \tilde{U}_M(g') = \tilde{\Omega}(g, g') \tilde{U}_M(gg'),
$$

(B17)

where

$$
\tilde{\Omega}(g, g') = \Sigma(g) U_M(g) \Sigma(g') \Omega(g, g') \Sigma(gg')^{-1}.
$$

(B18)

It is obvious that the equivalence class of the 3-cocycle is independent of the choice of restriction $\Omega \rightarrow \Omega_a$, so we are free to choose a restriction of $\tilde{\Omega}$ such that

$$
\tilde{\Omega}_a(g, g') = \Sigma_a(g) U_M(g) \Sigma_a(g') \Omega_a(g, g') \Sigma_a(gg')^{-1},
$$

(B19)
where Σₐ(g) is the restriction of Σ(g) to the point a. Now we calculate

\[
\tilde{\Omega}_a(g_1, g_2) \tilde{\Omega}_a(g_1 g_2, g_3) \Sigma_a(g_1 g_2 g_3)
\]  \hspace{1cm} (B20)

\[
\tilde{\Omega}_a(g_1, g_2) ^{U_M(g_1 g_2)} \Sigma_a(g_3) \Omega(g_1 g_2, g_3)
\]  \hspace{1cm} (B21)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) \Omega(g_1, g_2) ^{U_M(g_1 g_2)} \Sigma_a(g_3) \Omega(g_1, g_2) \Omega(g_1 g_2, g_3)
\]  \hspace{1cm} (B22)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) ^{\Omega_a(g_1, g_2)} ^{U_M(g_1 g_2)} \Sigma_a(g_3) \Omega_a(g_1, g_2) \Omega_a(g_1 g_2, g_3)
\]  \hspace{1cm} (B23)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) ^{\Omega(g_1, g_2)} ^{U_M(g_1 g_2)} \Sigma_a(g_3) \Omega_a(g_1, g_2) \Omega_a(g_1 g_2, g_3)
\]  \hspace{1cm} (B24)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) ^{U_M(g_1)} ^{U_M(g_2)} \Sigma_a(g_3) \Omega_a(g_1, g_2) \Omega_a(g_1 g_2, g_3)
\]  \hspace{1cm} (B25)

where we applied Eq. (B19) several times, and we also used, again, Eq. (B9) to go from Eq. (B23) to Eq. (B24). The final line used the definition of ω, Eq. (5). On the other hand:

\[
U_M(g_1) \tilde{\Omega}_a(g_2, g_3) \tilde{\Omega}_a(g_1, g_2 g_3) \Sigma_a(g_1 g_2 g_3)
\]  \hspace{1cm} (B26)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \tilde{\Omega}_a(g_2, g_3) ^{-1} \tilde{\Omega}_a(g_1, g_2 g_3) \Omega(g_1, g_2 g_3)
\]  \hspace{1cm} (B27)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \tilde{\Omega}_a(g_2, g_3) ^{U_M(g_1)} \Sigma_a(g_2 g_3) \Omega(g_1, g_2 g_3)
\]  \hspace{1cm} (B28)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \left\{ \tilde{\Omega}_a(g_2, g_3) \Sigma_a(g_2 g_3) \right\} \Omega_a(g_1, g_2 g_3)
\]  \hspace{1cm} (B29)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \left\{ \Sigma_a(g_2) ^{U_M(g_2)} \Sigma_a(g_3) \Omega_a(g_2, g_3) \right\} \Omega_a(g_1, g_2 g_3)
\]  \hspace{1cm} (B30)

\[
\Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) ^{U_M(g_1) U_M(g_2)} \Sigma_a(g_3) ^{U_M(g_1)} \Omega_a(g_2, g_3) \Omega_a(g_1, g_2 g_3)
\]  \hspace{1cm} (B31)

\[
\omega(g_1, g_2, g_3) \Sigma_a(g_1) ^{U_M(g_1)} \Sigma_a(g_2) ^{U_M(g_1) U_M(g_2)} \Sigma_a(g_3) \Omega_a(g_1, g_2) \Omega_a(g_1 g_2, g_3)
\]  \hspace{1cm} (B32)

Comparing Eqs. (B32) and (B25), we find that

\[
U_M(g_1) \tilde{\Omega}_a(g_2, g_3) \tilde{\Omega}_a(g_1, g_2 g_3) = \omega(g_1, g_2, g_3) \tilde{\Omega}_a(g_1, g_2) \tilde{\Omega}_a(g_1 g_2, g_3).
\]  \hspace{1cm} (B33)

On the other hand, by definition, ω also satisfies Eq. (B33) with Ωₐ and \( \tilde{U}_M \) replaced by Ωₐ and \( U_M \). Thus, it does not matter whether we use the restriction \( U_M \) or \( \tilde{U}_M \); one obtains the same 3-cocycle ω.

**Appendix C: Completing the proof of separation of phases in (2+1)-D**

In this section, we will fill in the details of the proof outlined in Section III showing that (2+1)-D SPT phases corresponding to different elements of the cohomology group \( H^3(G, U(1)) \) are necessarily separated by a phase transition. Although we will write the arguments in lemma-proof form, we emphasize that we do not aim for mathematical rigor in our treatment of locality; rigorous proofs could potentially be constructed based on the arguments sketched here, but would require much more careful estimates of how relevant quantities decay at large distances.

The first result that needed to be proved was

**Lemma 2.** Let \( |\Psi\rangle \) be the ground state of an SPT phase in \( d \) spatial dimensions captured by the cohomological classification, and let \( U(g) \) be a local unitary representation of a group \( G \) which leaves \( |\Psi\rangle \) invariant. Let \( U_A(g) \) be the restriction of \( U(g) \) to a region \( A \). Then there exists a representation \( U_{\Omega A}(g) \) acting only in a strip near the boundary \( \partial A \) (and only within the region \( A \)) such that \( U_A(g)|\Psi\rangle = U_{\Omega A}(g)|\Psi\rangle. \)
Proof. We use the fact, discussed in the Appendix A above, that the state $|\Psi\rangle$ can be transformed into a product state $|\Phi\rangle = |\phi\rangle^\otimes N$ through a local unitary $D$ that commutes with $U(g)$. Hence, as in Appendix A, if we define the restriction $\mathcal{D}_A$, it follows that $\mathcal{D}_A U_A(g) \mathcal{D}^\dagger_A = W_B(g) U_{A\setminus B}(g)$, where $W_B(g)$ acts within a strip $B$ near the boundary, and $U_A(g)$ and $U_{A\setminus B}(g)$ are the restriction of $U(g)$ to the respective regions (and $A \setminus B$ is the region $A$ with the strip $B$ excluded.) It is clear (by a similar argument to the one given above in Appendix A) that $U_{A\setminus B}(g) |\Phi\rangle = |\Phi\rangle$, and hence we find that

$$U_A(g) |\Psi\rangle = \mathcal{D}^\dagger_A \mathcal{D}_A U_A(g) \mathcal{D}^\dagger_A \mathcal{D}_A |\Psi\rangle = \mathcal{D}^\dagger_A W_B(g) U_{A\setminus B}(g) \mathcal{D}_A |\Psi\rangle,$$

$$= \mathcal{D}^\dagger_A W_B(g) \mathcal{D}_A |\Psi\rangle$$

where in going from Eq. (C2) to Eq. (C3) we used the fact that $\mathcal{D}_A |\Psi\rangle$ looks like $|\Phi\rangle$ on $A \setminus B$, and therefore $U_{A\setminus B}(g) \mathcal{D}_A |\Psi\rangle = \mathcal{D}_A |\Psi\rangle$. Hence, defining $V_{\partial A}(g) = \mathcal{D}^\dagger_A W_B(g) \mathcal{D}_A$, we have the desired result. □

To proceed further we will also require the following lemma, which states that a “trivial” state cannot be invariant under an “anomalous” symmetry representation.

**Lemma 3.** Let $|\Psi\rangle$ be some short-ranged entangled state, and let $U(g)$ be some local unitary representation of a symmetry group $G$ on a closed 1-dimensional subregion $C$ of the space on which $|\Psi\rangle$ is defined, such that $U(g) |\Psi\rangle = |\Psi\rangle$. Then the element of the cohomology group $H^3(G, U(1))$ obtained via the reduction procedure of Section 1B is necessarily trivial.

**Proof.** Consider the restriction $U_M(g)$ to a subregion $M$ of $C$ with boundary. Then $U_M(g) |\Psi\rangle$ looks like $|\Psi\rangle$ away from the boundary $\partial M$. This implies that, for short-ranged entangled states $|\Psi\rangle$ it must be the case that $U_M(g) |\Psi\rangle = \Sigma_{\partial M}(g) |\Psi\rangle$ for some set of local unitaries $\Sigma_{\partial M}(g)$ acting near the boundary [17]. (To show this, one first establishes it for a product state, from which one can show that it must also apply for any state that can be turned into a product state by a local unitary.) However, the restriction $U(g) \rightarrow U_M(g)$ was only defined up to unitaries at the boundary anyway, so we are free to set $U_M(g) |\Psi\rangle = |\Psi\rangle$. Then, defining $\Omega(g_1, g_2)$ via

$$U_M(g_1) U_M(g_2) = \Omega(g_1, g_2) U_M(g_1 g_2)$$

we can deduce that $\Omega(g_1, g_2) |\Psi\rangle = |\Psi\rangle$. Assuming that $\partial M = \{a, b\}$ where $a$ and $b$ are two points, we can choose the restriction $\Omega_a(g_1, g_2)$ such that $\Omega_a(g_1, g_2) |\Psi\rangle = |\Psi\rangle$. Then, given the definition [Eq. (4)] in Section 1B of the 3-cocycle $\omega(g_1, g_2, g_3)$, one finds that $\omega(g_1, g_2, g_3) |\Psi\rangle = |\Psi\rangle$, and hence $\omega(g_1, g_2, g_3) = 1$. □

Now, let us consider a (possibly spatially inhomogeneous) ground state $|\Psi\rangle$ in an SPT phase, invariant under an on-site symmetry representation $U(g)$ of $G$. We choose two regions $A$ and $A'$, separated by a quasi-one-dimensional buffer region $K$, as depicted in Figure 1. We assume that the combined region $A \cup K \cup A'$ has no boundary. We define $V_{\partial A}(g)$ and $V_{\partial A'}(g)$ according to $U_A(g) |\Psi\rangle = V_{\partial A} |\Psi\rangle$ as per Lemma 2. Denote the corresponding classes of 3-cocycles as $[\omega], [\omega'] \in H^3(G, U(1))$. We want to show that $[\omega] = [\omega']$. First, we need to discuss an important subtlety involved in the definitions of $[\omega]$ and $[\omega']$. Specifically, the reduction procedure of Section 1 implicitly depends on an orientation for the one-dimensional space in which the local unitaries act, in order to provide a consistent convention for reducing from $\partial M = \{a, b\}$ to a single point $a$. Opposite orientations give
FIG. 4. The regions $A$ and $A'$ on which we can prove that the anomalous symmetry on the boundary is classified by the same element of $H^3(G, U(1))$. The orientations of the boundaries $\partial A$ and $\partial A'$ are depicted with arrows.

rise to inverse cocycles. We will take the orientations of $\partial A$ and $\partial A'$ to be derived from that of $A$ and $A'$; if we choose $A$ and $A'$ to have the same orientation (e.g. both specified by normal vectors pointing out of the page), then $\partial A$ and $\partial A'$ have opposite orientations, as depicted by the arrows in Figure 4.

We now observe that $|\Psi\rangle$ is invariant under $V_{\text{sum}}(g) = U_K(g)V_{\partial A}(g)V_{\partial A'}(g)$. Now, it can readily be verified [using the fact that the three components of $V_{\text{sum}}(g)$ all commute with each other] that the element of $H^3(G, U(1))$ characterizing $V_{\text{sum}}(g)$ is equal to $[\omega][\omega']^{-1}$. (The inverse comes from the fact that we have defined the orientations of $\partial A$ and $\partial A'$ to be opposite.) On the other hand, by Lemma 3, this product must be trivial; hence we have established that $[\omega] = [\omega']$.

Finally, let us justify the following claim we made in Section III:

**Lemma 4.** Consider two gapped systems $S$ and $S'$ connected without a phase transition, and two well-separated regions $A$ and $A'$. Then one can construct an interpolating system such that the ground state looks like that of $S$ on $A$ and like that of $S'$ on $A'$.

**Proof.** Let $|\Psi\rangle$ and $|\Psi'\rangle$ be the corresponding ground states. Then there must exist a local unitary $D$ such that $D|\Psi\rangle = |\Psi'\rangle$. Define the restriction $D_{\tilde{A}'}$ of $D$ to some region $\tilde{A}'$ that contains $A'$ well inside, but is also well-separated from $A$. Then applying $D_{\tilde{A}'}$ to $S$ gives a system with the desired properties. 

**Appendix D: Proof of Lemma 1.**

Here we will give a proof of Lemma 1 which we stated in Section IV A. The proof is based on a result called the universal coefficient theorem. Let us first state some definitions. For a manifold $T$, we define $C^k(T, U(1))$ to be the group of closed $U(1)$ $k$-cochains, i.e. those cochains $\omega$ for which $d\omega = 0$, and $B^k(T, U(1))$ to be the group of exact cochains, i.e. those which can be written as $\omega = dk$ for some $\kappa$. We define the cohomology group $H^k(T, U(1)) \equiv C^k(T, U(1))/B^k(T, U(1))$. Similarly, we define $C_k(T)$ and $M_k(T)$ to be the group of closed (i.e. boundaryless) and exact (can be expressed as a boundary) $k$-chains respectively; and the homology group $H_k(T) = C_k(T)/M_k(T)$.

We observe that there is a natural homomorphism $\gamma : H_k(T, U(1)) \to \text{Hom}(H^k(T), U(1))$ defined according to $[\omega] \mapsto ([\sigma] \mapsto \omega(\sigma))$ (where $[\cdot \cdot \cdot]$ denotes equivalence classes in cohomol-
ogy or homology.) The universal coefficient theorem states that $\gamma$ is in fact an isomorphism. [In general, replacing $U(1)$ with an arbitrary abelian group $A$, the universal coefficient theorem states that the homomorphism is surjective and its kernel is isomorphic to $\text{Ext}(G, A)$. But $U(1)$ is divisible and it follows that $\text{Ext}(G, U(1)) = 0$.]

Hence we can prove

**Lemma** A $U(1)$ $k$-cochain $\omega$ on a manifold $T$ is exact if and only if $\omega(C) = 0$ for all closed (i.e. boundaryless) $k$-chains $C$.

**Proof.** If $\omega$ is exact, then $\omega = d\kappa$, and hence $\omega(C) = \kappa(\partial C) = 0$ for any closed $k$-chain $C$.

Conversely, let $\omega$ be some $k$-cochain such that $\omega(C) = 0$ for all closed $A$. Then $\omega$ is closed because $(d\omega)(A) = \omega(\partial A) = 0$. Also, $\gamma([\omega]) = 0$. Since $\gamma$ is an isomorphism, it follows that the equivalence class $[\omega] = 0$. Hence, $\omega$ is exact.

---

**Appendix E: Calculating the element of the cohomology group for discrete non-linear sigma models**

The action for the field theories of Ref. is a discrete analogue of the topological theta term that appeared in the continuous NL$\sigma$M’s in Section IV. Recall from Section IV A that the theta term is derived from a $U(1)$ cochain defined on the target manifold $T$. The same is true in the discrete case, except that the target space $T$ is now discrete, and so the interpretation of the “chains” which are the arguments of the cochains needs to be revised. Specifically, we define a $k$-chain on $T$ to be a formal linear combination (with integer coefficients) of “$k$-simplices”, which are simply ordered $k$-tuples $\Delta = (\Delta_1, \cdots, \Delta_k) \in T^{\times k}$. Then we can define the “boundary” operator $\partial$ acting linearly on $k$-chains by specifying its action on $k$-simplices:

$$\partial(\Delta_1, \cdots, \Delta_k) = \sum_{j=1}^{k} (-1)^{j-1}(\Delta_1, \cdots, \Delta_{j-1}, \Delta_{j+1}, \cdots, \Delta_k). \tag{E1}$$

To construct the discrete topological term corresponding to a $U(1)$ cochain on $T$, one considers a triangulation of a $D$-dimensional spacetime manifold $M$; that is, we build $M$ up out of $D$-simplices. The degrees of freedom of the field theory will live on the vertices of the simplices. We can represent the $D$-simplices in spacetime in terms of their vertices $(x_1, \cdots, x_D)$ [the abstract definition of boundary given in Eq. (E1) then agrees with the geometrical definition]. Thus, we interpret the manifold $M$ as a formal $D$-chain $M = \sum_{\Delta}(\Delta_1, \cdots, \Delta_D)$. Given a $U(1)$ $D$-cochain $\omega$ on $M$, and a function $\alpha$ assigning a value of the target space to each vertex, we can define the action

$$S_{\text{top}} = \omega(\alpha(M)) \equiv \sum_{\Delta} \omega(\alpha(\Delta_1), \cdots, \alpha(\Delta_D)) \tag{E2}$$

We ensure that this action will vanish for closed space-time manifolds $M$ by requiring that $\omega(C) = 0$ for any closed chain $C$. By Lemma in Section IV A above (which holds equally well for discrete cochains), this is equivalent to requiring that $\omega$ be exact.

If we have an action of a group $G$ on the target space $T$, then for each symmetric $D$-cochain $\omega$ one can derive an element of the group cohomology group $H^D(G, U(1))$ by
following the exact same reduction procedure as we did in the continuous case in Section IV A. One might, however, object that the physical significance of this is not clear unless we specify some way to quantize a field theory defined in discrete time. For this reason, we want to reinterpret discrete field theories like those of IV A as prescriptions for constructing lattice models.

Specifically, consider a triangulation of a closed $d$-dimensional spatial manifold $M$ ($d = D - 1$). At each vertex, we put a quantum particle whose basis states are labeled by the elements of the target space $T$. (Hence, each basis state of the whole system is labeled by a function $\alpha$ mapping vertices into $T$.) We can define a quantum state for the system by "imaginary-time evolution" of a $(d + 1)$-dimensional discrete topological action derived from an exact U(1) $(d + 1)$-cochain on $T$. This state is given by

$$|\Psi\rangle = \sum_\alpha \Psi(\alpha)|\alpha\rangle,$$

$$\Psi(\alpha) = \exp\left[i\kappa(\alpha(M))\right],$$

where $\kappa$ is a U(1) $d$-cochain such that $d\kappa = \omega$, and we define $\kappa(\alpha(M))$ in the analogous way to Eq. (E2) above. This wavefunction is invariant under the on-site representation of the symmetry,

$$S(g) = \sum_\alpha |ga\rangle\langle\alpha|.$$ (E6)

Once we have the wavefunction, it is easy to construct a corresponding local Hamiltonian for which it is the gapped ground state. For example, if we let $V = \sum_\alpha \Psi(\alpha)|\alpha\rangle\langle\alpha|$ be the local unitary which creates $|\Psi\rangle$ from the trivial product state $|\Psi_{\text{prod}}\rangle = \sum_\alpha |\alpha\rangle$, then, starting from a local Hamiltonian $H_{\text{prod}}$ which has $|\Psi_{\text{prod}}\rangle$ as its gapped ground state, we can define $H = VH_{\text{prod}}V^\dagger$.

In Appendix A, we give a general discussion of how to decouple a bulk theory from its boundary in order to find the form of the edge symmetry. Applying the method of Appendix A to the situation at hand, one finds that the edge symmetry takes the form $U_{\text{edge}}(g) = N(g)S(g)$, where $S(g)$ is as in Eq. (E6) (but acting only on the degrees of freedom at the edge), and

$$N(g) = \sum_\alpha \exp\{i\kappa(2)^{(g)}(\alpha(\partial M))\}|\alpha\rangle\langle\alpha|,$$ (E7)

where $\kappa(2)$ is the U(1) $d - 1$-cochain such that $d\kappa(2) = g\kappa - \kappa$. This is precisely the discrete version of Eq. (39). It is now easy to see that the general reduction procedure of Section IC will produce the same result as naively applying the method of Section IV A for the discrete case.

It turns out that the reduction procedure can actually be done explicitly starting from an arbitrary exact U(1) $(d + 1)$-cochain $\omega$ on a target space $T$ symmetric under the action of a group $G$. To see this, choose some arbitrary fixed $t_s \in T$, and define

$$\kappa^{(k)}(g_1, \cdots, g_k)(\Delta_1, \cdots, \Delta_{d-k}) = \omega(g_k^{-1} \cdots g_1^{-1} t_s, \cdots, g_1^{-1} t_s, t_s, \Delta_1, \cdots, \Delta_{d-k})$$ (E8)

and $\omega^{(k)} = d\kappa^{(k)}$. Using the fact that $\omega$ is invariant under the symmetry and $\omega(C) = 0$ for closed chains $C$, one can show that (a) $\omega^{(0)} = \omega$; and (b) $\delta_\kappa \kappa^{(k)} = \omega^{(k+1)}$. Thus, we have explicitly constructed the reduction sequence of Section IV A and we find that the
resulting element of the group cohomology group $H^{d+1}(G, \mathrm{U}(1))$ is the equivalence class of the following $\mathrm{U}(1)$ $(d+1)$-group cocycle:

$$
\nu(g_1, \ldots, g_{d+1}) = \omega(g_1^{-1} \cdots g_{d+1}^{-1} t_s, \ldots, g_1^{-1} t_s, t_s).
$$

(E9)

In particular, following Ref. [7], we can consider the case where the target space $T$ is the symmetry group $G$ itself, with $G$ acting on itself by left multiplication. In that case, it is easy to see that Eq. (E9) actually defines a one-to-one mapping between symmetric exact “topological” cochains on the right-hand side and group cocycles on the left-hand side. Thus, for every element of the group cohomology group $H^{d+1}(G, \mathrm{U}(1))$, one can construct a discrete topological term in $d+1$ space-time dimensions via Eq. (E9), and applying our general reduction procedure returns the same element of $H^{d+1}(G, \mathrm{U}(1))$.

**Appendix F: Classification of (2+1)-D fermionic SPT’s**

Here we will implement the ideas discussed in Section VII in order to give a classification of (2+1)-D fermion SPT’s. Consider a (2+1)-D fermionic SPT with fermionic symmetry group $G_f$ (represented on-site), including an element $\pi$ corresponding to the fermion parity. All the symmetries are assumed to be local, so $\pi$ must commute with all the elements of $G_f$. The fermion parity plays a such a key role in the following argument that we find it convenient to write elements of $G_f$ in the form $\varpi(g)\pi^m$, where $m = 0$ or $1$, $g \in G_b \equiv G_f/\mathbb{Z}_2$, and $\varpi$ is an arbitrary identification of elements of $G_b$ with coset representatives in $G_f$, such that $\varpi(g_1)\varpi(g_2) = \pi^{\lambda(g_1,g_2)}\varpi(g_1,g_2)$. Here $\lambda(g_1, g_2)$ takes values $0$ or $1$, and associativity implies that it must be a $\mathbb{Z}_2$ 2-cocycle, i.e. $\delta\lambda = 0 \pmod{2}$, where $\delta$ is the coboundary operator

$$(\delta\lambda)(g_1, g_2) = \lambda(g_1, g_2) + \lambda(g_1g_2, g_3) + \lambda(g_2, g_3) + \lambda(g_1, g_2g_3) \quad (F1)$$

Now, we assume that the edge of this SPT can be realized in a strictly (1+1)-D local fermionic system and be invariant under a local unitary (but not necessarily on-site) representation of $G_f$. The fermion parity is still represented as $\Pi \equiv (-1)^F$ on the edge. We write the fermionic local unitary operator implementing $\varpi(g)\pi^m$ on the edge as $U(g)\Pi^m$. Then $U(g)$ must satisfy

$$
U(g_1)U(g_2) = \Pi^{\lambda(g_1,g_2)}U(g_1g_2).
$$

(F2)

If we restrict the symmetry action to an interval $M = [a, b]$, then the restricted unitaries must satisfy Eq. (F2) up to a boundary term $\Omega(g_1, g_2)$:

$$
U_M(g_1)U_M(g_2) = \Omega(g_1, g_2)\Pi^{\lambda(g_1,g_2)}U_M(g_1g_2).
$$

(F3)

Using the associativity of the $U_M$’s, combined with $\delta\lambda = 0$ and the fact that the $U_M$’s commute with $\Pi$, we see that the $\Omega$’s must satisfy an identical equation to the bosonic case:

$$
\Omega(g_1, g_2)\Omega(g_1g_2, g_3) = U_M(g_1)\Omega(g_2, g_3)\Omega(g_1, g_2g_3),
$$

(F4)

As discussed in Section VII in defining the restriction $\Omega \rightarrow \Omega_a$ we might obtain an operator carrying non-trivial charge under fermion parity. We define the function $\sigma(g_1, g_2)$ to be $0$ if $\Omega_a$ is a fermionic local unitary (no charge under fermion partiy) acting at the point $a$, and $1$ if it is equal to such a local unitary, multiplied by $(c_a + c_a^\dagger)$.
The restriction \( \Omega_a(g_1, g_2) \) must satisfy Eq. (F4) up to a phase factor:

\[
U_{\alpha}(g_1) \Omega_a(g_2, g_3) \Omega_a(g_1, g_2 g_3) = \omega(g_1, g_2, g_3) \Omega_a(g_1, g_2) \Omega_a(g_1 g_2, g_3),
\]

where \( \omega \) is a U(1)-valued function. The pair of functions \((\omega, \sigma)\) constitutes the fermionic 3-cocycle. From Eq. (F5) we immediately see that \( \sigma \) must be a \( \mathbb{Z}_2 \) cocycle, i.e.

\[
\delta \sigma = 0.
\]

Following a similar derivation to the one in Appendix B that gave the bosonic 3-cocycle condition \(\delta \sigma = 0\), we also also find that \( \omega \) must obey

\[
(\delta \omega)(g_1, g_2, g_3, g_4) = (-1)^{[\sigma(g_1, g_2)+\lambda(g_1, g_2)]\sigma(g_3, g_4)}.
\]

where

\[
(\delta \omega)(g_1, g_2, g_3, g_4) = \omega(g_1, g_2, g_3) \omega(g_1 g_2, g_3, g_4)^{-1} \omega(g_1, g_2 g_3, g_4) \omega(g_1, g_2, g_3 g_4)^{-1} \omega(g_2, g_3, g_4).
\]

Eqs. (F6) and (F7) constitute the condition for \((\omega, \sigma)\) to be a fermionic 3-cocycle.

Furthermore, the freedom to redefine the restriction \( U \rightarrow U_M \) and \( \Omega \rightarrow \Omega_a \) implies (again following similar arguments as in Appendix B) that we must identify fermionic 3-cocycles that differ by the transformation

\[
\sigma(g_1, g_2) \rightarrow \sigma(g_1, g_2) + (d \mu)(g_1)
\]

\[
\omega(g_1, g_2, g_3) \rightarrow \omega(g_1, g_2, g_3)(-1)^{[\sigma(g_1, g_2)+\lambda(g_1, g_2)]\mu(g_3) + \mu(g_1)\sigma(g_2, g_3) + (d \mu)(g_2, g_3)}(\delta \beta)(g_1, g_2, g_3)
\]

where

\[
(\delta \beta)(g_1, g_2, g_3) = \beta(g_1, g_2) \beta(g_1 g_2, g_3) \beta(g_2, g_3)^{-1} \beta(g_1, g_2 g_3)^{-1},
\]

and \( \beta \) and \( \mu \) take values in U(1) and \( \{0, 1\} \) respectively. [The numbers \( \mu(g) \) correspond to the fermion parity of the restriction \( \Sigma_a(g) \) of the operator \( \Sigma(g) \) that implements the redefinition \( U_M(g) \rightarrow \Sigma(g) U_M(g) \).] If \((\omega, \sigma)\) is not equivalent to the trivial fermionic 3-cocycle according to the above equivalence relation, then we expect that the edge must correspond to the boundary of a non-trivial SPT phase. This is because such an anomalous non-trivial symmetry precludes a gapped ground state unless the symmetry is spontaneously broken. (This can be derived in a similar way to the equivalent bosonic result, Lemma 3 in Appendix C.) Similarly, two SPT phases characterized by fermionic 3-cocycles not related by the above equivalence relation must be separated by a phase transition.

We also can define a product rule for fermionic 3-cocycles. Physically, the product rule corresponds to “stacking” two SPT phases on top of each other. If the edges of the two systems are characterized by \((\sigma, \omega)\) and \((\sigma', \omega')\), then one can show that the edge of the combined system will be described by the fermionic 3-cocycle \((\sigma_{\text{prod}}, \omega_{\text{prod}})\), where

\[
\sigma_{\text{prod}} = \sigma + \sigma'
\]

\[
\omega_{\text{prod}}(g_1, g_2, g_3) = (-1)^{\sigma'(g_2, g_3)\sigma(g_1, g_2 g_3) + \sigma'(g_1, g_2)\sigma(g_1 g_2, g_3)} \omega(g_1, g_2, g_3) \omega'(g_1, g_2, g_3).
\]

Finally, let us remark that, if we set \( \sigma = 0 \), then the fermionic 3-cocycles reduce to ordinary 3-cocycles for the “bosonic” symmetry group \( G_b = G_f / \mathbb{Z}_2^f \). This reflects the fact that bosonic SPT’s can be realized in a fermion system by pairing fermions to form bosons.
However, according to the equivalence relation Eq. (F9), when $\lambda \neq 0$ (i.e. the fermionic symmetry group is not simply a direct product $G_f = G_b \times Z^f_2$), a non-trivial bosonic 3-cocycle might still be trivial as a fermionic 3-cocycle. Thus, there is the possibility that a bosonic SPT phase could become trivial in the presence of fermions. Examples of this phenomenon (albeit for symmetry groups including anti-unitary symmetries, which we have not considered here) can be found in Ref. 10.

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By restricting the rotation to an interval, we cannot introduce a net twist, but we can separate equal and opposite twists and focus on the vicinity of just one of them.

The precise relation is slightly subtle, however, because \( \Omega_a \) measures the charge to the left of the endpoint \( a \), the very place at which charge is accumulating due to the Hall effect. The precise manner in which \( \Omega_a \) takes into account the charge located at the endpoint itself depends on the restriction \( U \rightarrow U_M \). However, as long as we use the same restriction both to define \( \Omega_a \) and to implement the winding, the value of the measured “charge” is independent of the choice of restriction. Perhaps the easiest case to interpret is where \( U_M \) is defined so that the winding of the phase interpolates linearly over a transition region between no winding outside the interval and the desired winding in the interior of the interval. Then \( \Omega_a \) measures the charge to the left of a point \( x \), averaged over all \( x \) in the transition region. This is \( n \), rather than the charge \( 2n \) measured to the left of a point in the interior of the interval.
Specifically, the difference from the bosonic case comes from in going from Eq. (B5) to Eq. (B6) [due to the potential for $\Omega_b(g_1, g_2)$ and $\Omega_a(g_3, g_4)$ to anticommute], and in going from Eq. (B6) to Eq. (B7) [due to the factor of $\Pi^{\lambda(g_1, g_2)}$ appearing in Eq. (F3).]