A Strong Law of Large Numbers for Set-Valued Random Variables in $G_{\alpha}$ Space

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Abstract

In this paper, we shall represent a strong law of large numbers (SLLN) for weighted sums of set-valued random variables in the sense of the Hausdorff metric $d_{H}$, based on the result of single-valued random variable obtained by Taylor [11].

Keywords and phrases: set-valued random variable, the laws of large numbers, Hausdorff metric.

1 Introduction

We all know that the limit theories are important in probability and statistics. For single-valued case, many beautiful results for limit theory have been obtained. In [11], there are many results of laws of large numbers at different kinds of conditions and different kinds of spaces. With the development of set-valued random theory, the theory of set-valued random variables and their applications have become one of new and active branches in probability theory. And the theory of set-valued random variables has been developed quite extensively (cf. [1], [2], [6], [7], [8], [10] etc.). In [1], Artstein and Vitale used an embedding theorem to prove a strong law of large numbers for independent and identically distributed set-valued random variables whose basic space is $\mathbb{R}^d$, and Hiai extended it to separable Banach space $X$ in [5]. Taylor and Inoue proved SLLN’s for only independent case in Banach space in [10]. Many other authors such as Giné, Hahn and Zinn [4], Puri and Ralescu [9] discussed SLLN’s under different settings for set-valued random variables where the underlying space is a separable Banach space.

In this paper, what we concerned is the SLLN of set-valued independent random variables in $G_{\alpha}$ space. Here the geometric conditions are imposed on the Banach spaces to obtain SLLN for set-valued random variables. The results are both the extension of the single-valued’s case and the extension of the set-valued’s case.

This paper is organized as follows. In section 2, we shall briefly introduce some definitions and basic results of set-valued random variables. In section 3, we shall prove a strong law of large numbers for set-valued independent random variables in $G_{\alpha}$ space.
2 Preliminaries on set-valued random variables

Throughout this paper, we assume that \((\Omega, \mathcal{A}, \mu)\) is a nonatomic complete probability space, \((X, \|\cdot\|)\) is a real separable Banach space, \(\mathbb{N}\) is the set of natural numbers, \(\mathcal{K}(X)\) is the family of all nonempty closed subsets of \(X\), and \(\mathcal{K}_{bc}(X)\) is the family of all nonempty bounded closed convex subsets of \(X\).

Let \(A\) and \(B\) be two nonempty subsets of \(X\) and \(\lambda \in \mathbb{R}\), the set of all real numbers. We define addition and scalar multiplication as

\[
A + B = \{a + b : a \in A, b \in B\},
\]

\[
\lambda A = \{\lambda a : a \in A\}.
\]

The Hausdorff metric on \(\mathcal{K}(X)\) is defined by

\[
d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},
\]

for \(A, B \in \mathcal{K}(X)\). For an \(A \in \mathcal{K}(X)\), let \(\|A\|_{\mathcal{K}} = d_H(\{0\}, A)\). The metric space \((\mathcal{K}_b(X), d_H)\) is complete, and \(\mathcal{K}_{bc}(X)\) is a closed subset of \((\mathcal{K}_b(X), d_H)\) (cf. [8], Theorems 1.1.2 and 1.1.3).

For more general hyperspaces, more topological properties of hyperspaces, readers may refer to a good book [3].

For each \(A \in \mathcal{K}(X)\), define the support function by

\[
s(x^*, A) = \sup_{a \in A} \langle x^*, a \rangle, \quad x^* \in X^*,
\]

where \(X^*\) is the dual space of \(X\).

Let \(S^*\) denote the unit sphere of \(X^*\), \(C(S^*)\) the all continuous functions of \(S^*\), and the norm is defined as \(\|v\|_{C} = \sup_{x^* \in S^*} |v(x^*)|\).

The following is the equivalent definition of Hausdorff metric.

For each \(A, B \in \mathcal{K}_{bc}(X)\),

\[
d_H(A, B) = \sup\{|s(x^*, A) - s(x^*, A)| : x^* \in S^*\}.
\]

A set-valued mapping \(F : \Omega \to \mathcal{K}(X)\) is called a set-valued random variable (or a random set, or a multifunction) if, for each open subset \(O\) of \(X\), \(F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}\).

For each set-valued random variable \(F\), the expectation of \(F\), denoted by \(E[F]\), is defined as

\[
E[F] = \left\{ \int_{\Omega} f d\mu : f \in S_F \right\},
\]

where \(\int_{\Omega} f d\mu\) is the usual Bochner integral in \(L^1[\Omega, X]\), the family of integrable \(X\)-valued random variables, and \(S_F = \{f \in L^1[\Omega; X] : f(\omega) \in F(\omega), \text{a.e.}(\mu)\}\). This integral was first introduced by Aumann [2], called Aumann integral in literature.
3. Main Results

In this section, we will give the limit theorems for independent set-valued random variables in $G_\alpha$ space. The following definition and lemma are from [11], which will be used later.

**Definition 3.1** A Banach space $X$ is said to satisfy the condition $G_\alpha$ for some $0 < \alpha \leq 1$, if there exists a mapping $G : X \to X^*$ such that

1. $\|G(x)\| = \|x\|^{1+\alpha}$;
2. $G(x)x = \|x\|^{1+\alpha}$;
3. $\|G(x) - G(y)\| \leq A\|x - y\|^\alpha$ for all $x, y \in X$ and some positive constant $A$.

Note that Hilbert spaces are $G_1$ with constant $A = 1$ and identity mapping $G$.

**Lemma 3.2** Let $X$ be a separable Banach space which is $G_\alpha$ for some $0 < \alpha \leq 1$ and let $\{V_1, V_2, \cdots, V_n\}$ be single-valued independent random elements in $X$ such that $E[V_k] = 0$ and $E[\|V_k\|^{1+\alpha}] < \infty$ for each $k = 1, 2, \cdots, n$. Then

$$E[\|V_1 + \cdots + V_n\|^{1+\alpha}] \leq A \sum_{k=1}^n E[\|V_k\|^{1+\alpha}]$$

where $A$ is the positive constant in (iii).

**Theorem 3.2** Let $X$ be a separable Banach space which is $G_\alpha$ for some $0 < \alpha \leq 1$. Let $\{F_n : n \geq 1\}$ be a sequence of independent set-valued random variables in $K_{bc}(X)$, such that $E[F_n] = \{0\}$ for each $n$. If

$$\sum_{j=1}^\infty E[\|F_j\|^{1+\alpha}] < \infty$$

where $\phi_0(t) = t^{1+\alpha}$ for $0 \leq t \leq 1$ and $\phi_0(t) = t$ for $t \geq 1$, then $\sum_{j=1}^\infty F_j$ converges with probability 1 in the sense of $d_H$.

**Proof.** Define

$U_j = F_j I\{\|F_j\|_K \leq 1\}$ and $W_j = F_j I\{\|F_j\|_K > 1\}$.

Note that $F_j = U_j + W_j$ for each $j$ and that both $\{U_j : j \geq 1\}$ and $\{W_j : j \geq 1\}$ are independent sequences of set-valued random variables. Next, for each $m$ and $n$

$$E\left[\left\|\sum_{j=n}^m W_j\right\|_K\right] \leq \sum_{j=n}^m E\left[\|W_j\|_K\right] \leq \sum_{j=n}^m E\left[\phi_0(\|F_j\|_K)\right].$$

That means $\{E\left[\left\|\sum_{j=1}^m W_j\right\|_K\right] : m \geq 1\}$ is a Cauchy sequence and hence

$$E\left[\left\|\sum_{j=1}^\infty W_j\right\|_K\right]$$

converges

as $m \to \infty$. Since convergence in the mean implied convergence in probability, Ito and Nisio’s(1968) result for independent random elements(rf. Section 4.5) provides that

$$\left\|\sum_{j=1}^\infty W_j\right\|_K$$

converges in probability 1 as $n \to \infty$.  

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Then for \( n, m \geq 1, m > n \), by triangular inequality we have

\[
d_H\left( \sum_{j=1}^{n} W_j, \sum_{j=1}^{m} W_j \right) = d_H\left( \sum_{j=1}^{n} W_j, \sum_{j=1}^{n} W_j + \sum_{j=n+1}^{m} W_j \right) \leq d_H\left( \{0\}, \sum_{j=n+1}^{m} W_j \right) = \left\| \sum_{j=n+1}^{m} W_j \right\|_{K} \to 0, \text{ a.e.. as } n, m \to \infty.
\]

By the completeness of \((K_0(\mathcal{X}), d_H)\), we can have \( \sum_{j=1}^{n} W_j \) converges almost everywhere in the sense of \( d_H \).

Since by equivalent definition of Hausdorff metric, we have

\[
E[\| \sum_{j=n}^{m} U_j \|_{K}^{1+\alpha}] = E\left[ d_H\left( \sum_{j=n}^{m} U_j, \{0\} \right) \right]^{1+\alpha} = E\left[ \sup_{x^* \in S^*} |s(x^*, \sum_{j=n}^{m} U_j)| \right]^{1+\alpha}.
\]

For any fixed \( n, m \), there exists a sequence \( x_k^* \in S^* \), such that

\[
\lim_{k \to \infty} |s(x_k^*, \sum_{j=n}^{m} U_j)| = \sup_{x^* \in S^*} |s(x^*, \sum_{j=n}^{m} U_j)|.
\]

Then by dominated convergence theorem, Minkowski inequality and Lemma ??, we have

\[
E\left[ \| \sum_{j=n}^{m} U_j \|_{K}^{1+\alpha} \right] = E\left[ \lim_{k \to \infty} |s(x_k^*, \sum_{j=n}^{m} U_j)|^{1+\alpha} \right] = \lim_{k \to \infty} E\left[ |s(x_k^*, \sum_{j=n}^{m} U_j)|^{1+\alpha} \right] \leq \lim_{k \to \infty} E\left[ \sup_{j=n}^{m} \left| s(x_k^*, U_j) - s(x_k^*, E[U_j]) \right| \left| E(s(x_k^*, \sum_{j=n}^{m} U_j)) \right| \right]^{1+\alpha} + \lim_{k \to \infty} \left[ \sum_{j=n}^{m} E[|s(x_k^*, W_j)|] \right]^{1+\alpha}
\]

\[
\leq 2^{1+\alpha} \left\{ A \lim_{k \to \infty} \sum_{j=n}^{m} E\left[ \sup_{x^* \in S^*} |s(x^*, U_j)| \right]^{1+\alpha} + \lim_{k \to \infty} \left[ \sum_{j=n}^{m} E\left[ \sup_{x^* \in S^*} |s(x^*, W_j)| \right] \right]^{1+\alpha} \right\}
\]

\[
\leq 2^{1+\alpha} \left\{ A2^{2+\alpha} \sum_{j=n}^{m} E\left[ \sup_{x^* \in S^*} |s(x^*, U_j)| \right]^{1+\alpha} + \left[ \sum_{j=n}^{m} E\left[ \sup_{x^* \in S^*} |s(x^*, W_j)| \right] \right]^{1+\alpha} \right\}
\]

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\[
\leq 2^{1+\alpha} \left\{ A^2^{2+\alpha} \sum_{j=n}^{m} E[\phi_0(\|F_j\|_K)] + \left[ \sum_{j=n}^{m} E[\phi_0(\|F_j\|_K)] \right]^{1+\alpha} \right\}
\]

for each \( n \) and \( m \). Thus, \( \{ E[\| \sum_{j=1}^{m} U_j \|_K] : m \geq 1 \} \) is a Cauchy sequence, and hence converges.

Hence, by the similar way as above to prove \( \sum_{j=1}^{\infty} W_j \) converges with probability one in the sense of \( d_H \). We also can prove that

\[
\sum_{j=1}^{\infty} U_j
\]

with probability one in the sense of \( d_H \). The result was proved. \( \Box \)

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