Harmonic forms on asymptotically ADS metrics

Guido Franchetti\(^1,\textsuperscript{*}\) and Raúl Sánchez Galán\(^2\)

\(^1\) Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, Somerset, BA2 7AY, United Kingdom
\(^2\) School of Mathematics and Statistics, Nanjing University of Science and Technology, Xuanwu, Nanjing, Jiangsu 210094, People’s Republic of China

E-mail: gf424@bath.ac.uk

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Abstract
In this paper we study the rotationally invariant harmonic cohomology of a two-parameter family of Einstein metrics g which admits a cohomogeneity one action of SU(2) \times U(1) and has AdS asymptotics. Depending on the values of the parameters, g is either of NUT type, if the fixed-point locus of the U(1) action is zero-dimensional, or of bolt type, if it is two-dimensional. We find that if g is of NUT type then the space of SU(2)-invariant harmonic two-forms is three-dimensional and consists entirely of self-dual forms; if g is of bolt type it is four-dimensional. In both cases we explicitly determine a basis. The pair (g, F) for F a self-dual harmonic two-form is also a solution of the bosonic sector of 4D supergravity. We determine for which choices it is a supersymmetric solution and the amount of preserved supersymmetry.

Keywords: harmonic cohomology, supergravity, Killing spinors

1. Introduction

In this paper we study the harmonic cohomology, that is the space of harmonic \(L^2\) forms, of a certain two-parameter family of Riemannian Einstein manifolds with negative cosmological constant \((\mathcal{M}, g)\) of bi-axial Bianchi IX type. We also determine when the pair \((g, F)\), for \(F\) an (anti) self-dual harmonic form, gives a supersymmetric solution of the supergravity equations and determine the amount of preserved supersymmetry.

A harmonic two-form is a solution of the Maxwell equations of electromagnetism, and solutions of the coupled Einstein–Maxwell system are of obvious mathematical and physical
interest. In four dimensions, the reason for physical interest is twofold: not just as solutions of the general relativity equations in the presence of an electromagnetic field, but also as solutions of the formally equivalent equations for the bosonic sector of 4D supergravity, where the Maxwell field describes the graviphoton.

Solutions with negative cosmological constant are particularly important because of their relevance to the anti de Sitter/conformal field theory (AdS/CFT) correspondence [19]. Initially formulated in the conformally flat case, the AdS/CFT correspondence is believed to extend to curved solutions which are asymptotically locally AdS, that is, which asymptotically approach anti de Sitter space (or a finite quotient of it)—in Riemannian signature the name asymptotically locally hyperbolic would be more appropriate. The possibility of using supersymmetric localisation techniques [16, 28] to perform non-perturbative computations on curved Riemannian manifolds also motivates the study of curved solutions of the supergravity equations in their role of supersymmetric gravity duals of rigid supersymmetric field theories on curved boundaries [7, 14, 20–22, 24].

Reference [21] gives an exhaustive treatment of the gravity dual of rigid supersymmetric field theories defined on a bi-axially squashed three-sphere, that is, a topological three-sphere with a metric admitting an $SU(2) \times U(1)$ isometric action. It would be interesting to understand the gravity dual in the case of boundaries having $SU(2)$, rather than $SU(2) \times U(1)$, symmetry. While dropping the extra axial symmetry may seem a minor modification, it can have a profound effect on the type of solutions allowed. For example, two famous Bianchi IX type hyperkähler metrics are the Taub-NUT and the Atiyah–Hitchin manifolds. While the former has symmetry $SU(2) \times U(1)$, the latter only arises if the symmetry is relaxed to $SU(2)/\mathbb{Z}_2$.

However, finding the relevant supergravity backgrounds, that is global solutions of the Einstein–Maxwell system with $SU(2)$ symmetry and locally AdS asymptotics, already requires a substantial effort, before even being able to consider the amount of supersymmetry these backgrounds have. Therefore, one of the aims of this paper is to test the ground by considering the case where the metric retains the extra $U(1)$ symmetry, but the graviphoton field does not. The results are encouraging: we find harmonic forms which are $SU(2)$- but not $SU(2) \times U(1)$- invariant and which depend in an interesting way on the detailed structure of the metric. We do not obtain new Killing spinors, however it is clear from our computations that this is a consequence of the mismatch in symmetry between the metric and the graviphoton. Once this mismatch is lifted we expect extra solutions, which have no analogue in the axially symmetric case, to arise.

From the mathematical point of view, the study of harmonic objects on a Riemannian manifold is a natural problem, and determining the harmonic cohomology of particular examples gives useful insight in the case of non-compact manifolds, where the usual Hodge decomposition results do not apply. In fact, not much is known about generic harmonic forms on a non-compact manifold, but the situation improves in the case of square-integrable ones [3]. For certain asymptotic behaviours of the metric $g$, the harmonic cohomology of $\mathcal{M}$ can be characterised in terms of a particular compactification [12] of $\mathcal{M}$. Important examples include ALF gravitational instantons, which have finite-dimensional harmonic cohomology in middle dimension [2, 6, 8, 9]. In contrast, asymptotically (locally) hyperbolic metrics, such as the one we are going to consider in this paper, are conformally compact and have infinite dimensional middle dimension harmonic cohomology [23]. However, we will obtain a finite-dimensional space by restricting to forms invariant under a subgroup of the isometry group.

The outline of this paper is as follows. In section 2 we introduce the class of metric that we are going to study: a family $g$ of asymptotically locally hyperbolic Einstein metric of bi-axial Bianchi IX type, that is, admitting a cohomogeneity one action of $SU(2) \times U(1)$. In terms of
the basis ($\eta_i$) of left-invariant one-forms on $SU(2)$, $g$ has the form

$$g = \left( \frac{r^2 - N^2}{\Delta(r)} \right) \text{d}r^2 + (r^2 - N^2)(\eta_1^2 + \eta_2^2) + 4N^2\left( \frac{\Delta(r)}{r^2 - N^2} \right) \eta_3^2,$$

$$\Delta(r) = r^4 + (1 - 6N^2)r^2 - 2Mr + N^2(1 - 3N^2).$$

In (1) we have fixed the overall length scale by setting the cosmological constant to $-3$. The two continuous parameters $M,N$ are thus effective. An additional discrete parameter $p > 0$ arises by considering $\mathbb{Z}_p$ quotients of the $U(1)$ orbits.

The metric (1) is smooth only for particular values of the parameters and we review when this is the case. Two classes of solutions arise, depending on the nature of the fixed point locus of the $U(1)$ action. If it is a point, known as a NUT, one needs to take $p = 1$ and obtains a one-parameter family of topologically trivial solutions known as Taub-NUT-AdS. If it is a two-sphere, known as a bolt, then for every value of $p$ one obtains a one-parameter family of solutions with the topology of the complex line bundle $O(-p) \to \mathbb{C}P^1$. For future use, we also determine when $g$ has an (anti) self-dual Weyl tensor. It turns out that the NUT-type metrics are always self-dual, while for bolt-type metrics we obtain further constraints.

In section 3 we study the harmonic cohomology of the spaces introduced in section 2. It is non-trivial only in middle dimension, where it is infinite dimensional. We reduce to a finite dimensional subspace by considering rotationally invariant, i.e. $SU(2)$-invariant, two-forms. We show, by exhibiting an explicit basis, that the space of $L^2$, $SU(2)$-invariant harmonic two-forms is three-dimensional and consists entirely of self-dual forms for metrics of NUT type; four-dimensional with a three-dimensional self-dual subspace in the case of metrics of bolt-type. Because of the assumed $SU(2)$ symmetry, finding the harmonic forms reduces to solving an ODE in the radial variable. Forms which are also invariant under the extra $U(1)$ isometry were already known [21]. They are given by $\text{d}\xi^g$ and $^{\text{\dagger}}\text{d}\xi^g$ for $\xi$ the Killing vector field generating the extra $U(1)$ isometry and have a particularly simple expression which does not depend on the polynomial $\Delta(r)$ appearing in (1). This is not the case for the non-$U(1)$-invariant forms which depend explicitly on the roots of $\Delta(r)$. The results in this paper extend those in [9] where the case of vanishing cosmological constant is considered.

In section 4 we consider the problem of when the pair $(g,F)$, for $g$ one of the metrics of section 2 and $F$ a harmonic (anti) self-dual two-form of the type studied in section 3, is supersymmetric, that is, admits non-trivial solutions of a certain first order linear PDE known as the Killing spinor equation. Having reviewed some global issues due to the fact that not all the manifolds we consider are spin, and discussed the resulting quantisation conditions on $F$, we proceed to examine the integrability conditions for the Killing spinor equation.

The integrability conditions reproduce the constraint forcing $g$ to be half-conformally-flat (i.e. the associated Weyl tensor is either self-dual or anti self-dual), something which was known a priori from the general arguments in [5]. We then proceed to explicitly solve the Killing spinor equations obtaining, for both NUT and bolt type metrics, a 1/4 BPS solution and a 1/2 BPS one. These supersymmetric solutions only arise if the harmonic form $F$ is $U(1)$-invariant and satisfies a certain normalisation condition which, in the topologically non-trivial case of bolt-type metrics, guarantees that $F$ is the curvature of a $U(1)$ connection.

It is important to point out that the problem studied in section 4 has been solved in [21] for solutions of the Einstein–Maxwell system having $SU(2) \times U(1)$ invariance and hyperbolic asymptotics. Because of the required $SU(2) \times U(1)$ invariance, the two-form $F$ considered in [21] is less general than the one we consider, see the main text for more details. However, since we find that non-trivial Killing spinors only arise if $F$ is $U(1)$-invariant, thus reducing $F$ to the form considered in [21], we end up re-obtaining their solutions.
2. Bi-axial Bianchi IX type metrics

2.1. Conventions

We let \( \{ \eta_1, \eta_2, \eta_3 \} \) be a basis for the space of left-invariant differential one-forms on \( SU(2) \). In terms of the Euler angles \((\theta, \phi, \psi)\) parameterising \( SU(2) \) we have

\[
\begin{align*}
\eta_1 &= \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi, \\
\eta_2 &= \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \\
\eta_3 &= d\psi + \cos \theta \, d\phi.
\end{align*}
\]

We will consider Riemannian orientable manifolds of dimension 4 endowed with a bi-axial Bianchi IX metric. A Bianchi IX metric admits an isometric action of \( SU(2) \) (or its quotient by a finite subgroup) of cohomogeneity one, that is, with generic orbits of codimension one. Such a metric can be locally written [11]

\[
g = f^2 \, dr^2 + a^2 \eta_1^2 + b^2 \eta_2^2 + c^2 \eta_3^2, \tag{3}
\]

where \( f, a, b, c \) are functions of a radial variable \( r \) only.

For a bi-axial Bianchi IX metric \( a = b \) in (3), so that

\[
g = f^2 \, dr^2 + a^2 (\eta_1^2 + \eta_2^2) + c^2 \eta_3^2 \tag{4}
\]

and there is an additional \( U(1) \) isometric action generated by the vector field \( \partial_\psi \) dual to \( \eta_3 \). Note that

\[
d\Omega^2 = \eta_1^2 + \eta_2^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \tag{5}
\]

is the round metric on \( S^2 \). Fixed point sets of the \( U(1) \) action are either points, which are called NUTs, or two-surfaces, which are called bolts [10]. Away from fixed points of the \( U(1) \) action a hypersurface of fixed \( r \) has the topology of a circle fibration over \( S^2 \). The usual spherical coordinates \( \theta \in [0, \pi], \phi \in [0, 2\pi) \) parameterise the base \( S^2 \) and the \( S^1 \) fibre is parameterised by \( \psi \). We take \( \psi \) to have period

\[
T = \frac{4\pi}{p} \tag{6}
\]

with \( p \) a positive integer. For \( p = 1 \) the circle fibration is the standard Hopf fibration \( S^3 \to S^2 \).

For \( p > 1 \) we have a lens space \( S^1 \to \mathbb{S}^3 / \mathbb{Z}_p \to S^2 \) with \( \mathbb{Z}_p \) acting on the \( S^1 \) fibre.

We are going to consider the following family of bi-axial Bianchi IX metrics,

\[
g = f^2 \, dr^2 + a^2 (\eta_1^2 + \eta_2^2) + c^2 \eta_3^2, \tag{7}
\]

\[
a = \sqrt{r^2 - N^2}, \quad f = -\sqrt{r^2 - N^2 \Delta(r) \over \Delta(r)}, \quad c = 2N \sqrt{r^2 - N^2 \Delta(r) \over \Delta(r)},
\]

where \( \Delta(r) \) is the 4th order polynomial

\[
\Delta(r) = -\Lambda_3 r^4 + (1 + 2\Lambda N^2) r^2 - 2Mr + N^2(1 + \Lambda N^2). \tag{8}
\]

On occasion, we will work with the orthonormal coframe

\[
e^1 = a \eta_1, \quad e^2 = a \eta_2, \quad e^3 = c \eta_3, \quad e^4 = -f \, dr. \tag{9}
\]
The corresponding volume element is,

\[
\text{vol} = e^1 \wedge e^2 \wedge e^3 \wedge e^4 = -2Na^2 \sin \theta \, dr \wedge d\theta \wedge d\phi \wedge d\psi.
\]  

(10)

The metric (7) arises as a special case of the construction in [26]. It is Einstein with cosmological constant \( \Lambda \). The three parameters appearing in (7) and (8) are:

- The NUT parameter \( N \),
- The Komar mass \( M \),
- The cosmological constant \( \Lambda \), which sets a length scale \( l \) via

\[
l = \sqrt{\frac{3}{\abs{\Lambda}}},
\]  

(11)

For \( \Lambda = 0 \) one obtains the Ricci-flat metrics studied in [9]. For generic values of \( M \) and \( N \), these metrics have conical singularities while special values give the self-dual Taub-NUT metric, Taub-bolt, Euclidean Schwarzschild and Eguchi–Hanson. For \( \Lambda > 0 \) one obtains the family of metrics studied by Page [25]. For generic values of \( M, N \) these metrics have conical singularities while special values give the product metric on \( S^2 \times S^2 \), a \( \mathbb{Z}_2 \) quotient of the round metric on \( S^4 \), the Fubini–Study metric on \( \mathbb{C}P^2 \) and Page’s metric on \( \mathbb{C}P^2 \).

In this paper we are going to consider the case \( \Lambda < 0 \). Only two of the three parameters \( \{ \Lambda, M, N \} \) are effective as, up to a global rescaling of the metric, one of them can always be absorbed by a redefinition of the coordinates, so from now on we set

\[
\Lambda = -3.
\]  

(12)

Since (7) and (8) only depend on \( N^2 \), we take \( N > 0 \), which by (10) is equivalent to fixing the orientation of the underlying manifold. Suitably taken, the limits \( N \to 0 \) and \( N \to \infty \) also yield interesting metrics, the Schwarzschild–anti de Sitter one in the former case, and the Eguchi–Hanson–anti de Sitter one in the latter, but we will not considered them here. The Komar mass \( M \) is for now allowed to take any real value.

Asymptotically (7) approaches the metric

\[
g_{r \gg 1} \simeq \frac{dr^2}{r^2} + r^2 \left( 4N^2 \eta_3^2 + d\Omega^2 \right).
\]  

(13)

The Riemann tensor of (13) is that of a space of constant sectional curvature \(-1\) up to corrections of order \( O(r^{-3}) \). For \( M = 0, N = 1/2 \), (7) is exactly the metric of \( H^4 \).

It is interesting to compare the asymptotic behaviour (13) with that of the Ricci-flat case \( \Lambda = 0 \) [9]. If \( \Lambda = 0 \) the coefficients of the \( \eta_1 \) and \( \eta_2 \) terms scale with \( r^2 \) while that of \( \eta_3 \) is asymptotically constant. Therefore the volume of the base of the asymptotic circle fibration grows unboundedly with \( r \) while the fibres approach a fixed length. More formally, one could say that asymptotically the volume of a geodesic ball grows like the cube of its radius—to be contrasted with the fourth power behaviour of Euclidean four-space. For historical reasons, spaces exhibiting this behaviour are called ALF. In contrast, if \( \Lambda < 0 \) all the \( \eta_i \) terms scale exponentially with respect to geodesic distance, and we are dealing with a family of conformally compact metrics [1].


2.2. Particular cases of bi-axial Bianchi IX metrics

The metric (7) and (8) is smooth only for some particular values of the parameters $M, N$ which we will now review. It can be checked (e.g. by calculating the $L^2$-norm of the Riemann tensor) that (7) has a curvature singularity at $r = N$ unless (8) has a double zero there. Hence if $r_1$ is the largest real root of $\Delta(r)$, in order to get a complete non-singular metric we have two possibilities:

(a) $r_1 = N$ is a double root of $\Delta(r)$ and $r \in [N, \infty)$.
(b) $r_1 > N$ is a single root and $r \in [r_1, \infty)$.

We do not need to consider the case of $r$ ranging from $-\infty$ to the smallest root of $\Delta(r)$ since (7) is invariant under the simultaneous change $r \rightarrow -r$, $M \rightarrow -M$.

2.2.1. Solutions of NUT type. Consider first the case where $\Delta(r)$ has a double root at $r = N$.

Imposing

$$\Delta(N) = 0 = \Delta'(N)$$

(14)
gives $M = M_+$ where

$$M_+ = N \left(1 - 4N^2\right).$$

(15)
The polynomial (8) factorises as $(r - N)^2(r - x_+)(r - y_+)$, where

$$x_+ = -N + \sqrt{4N^2 - 1}, \quad y_+ = -N - \sqrt{4N^2 - 1}.$$  

(16)
The largest root is $r = N > 0$, so we take $r \in [N, \infty)$. The resulting metric is the Taub-NUT-AdS (TN-AdS) metric [27],

$$g_+ = \left(\frac{r^2 - N^2}{\Delta_+(r)}\right)dr^2 + (r^2 - N^2)d\Omega^2 + 4N^2 \left(\frac{\Delta_+(r)}{r^2 - N^2}\right)\eta_3^2,$$

(17)

$$\Delta_+(r) = (r - N)^2(r - x_+)(r - y_+).$$

Note that $N$ is allowed to take any value in $(0, \infty)$. The $+$ subscript in $g_+$ and other symbols is due to the fact that the associated Weyl tensor is self-dual, a fact which will play a role in the analysis of Killing spinors of section 4.

The coefficients of $d\Omega^2$ and $\eta_3$ in (17) both vanish at $r = N$ which is thus a NUT. Regularity at the NUT can be checked by substituting $r = N = \rho^2/(8N)$ and Taylor expanding (17) near $\rho = 0$, getting

$$g = d\rho^2 + \frac{\rho^2}{4}(\eta_1^2 + \eta_2^2 + \eta_3^2),$$

(18)

which is the Euclidean metric on $\mathbb{R}^4$. Hence the geometry near the NUT is that of flat space, just as in the case of ordinary Taub-NUT, which is obtained as $\Lambda \rightarrow 0$. TN-AdS is thus a smooth complete Einstein metric on a manifold with the topology of $\mathbb{R}^4$. For the special value $N = 1/2$ (15) gives $M = 0$, $\Delta_+(r)$ simplifies to $(r^2 - \frac{1}{4})^2$ and (17) becomes the metric of hyperbolic space $H^4$. 


2.2.2. Solutions of bolt type. Suppose now the largest real root $r_1$ of (8) satisfies $r_1 > N$. The coefficient of $d^2U^2$ in (7) is non-vanishing at $r_1$, so $\{ r = r_1 \}$ is a bolt with the topology of a two-sphere. We need to ensure that the metric has no conical singularities around the bolt.

For fixed $(\theta, \phi)$, the geometry near the bolt is that of a two-dimensional disk parameterized by $(r, \psi)$. For a smooth metric, as $r$ approaches $r_1$ the geometry becomes increasingly Euclidean. Therefore the ratio $dL/d\epsilon$, where $L$ is the length of the small curve consisting of points at geodesic distance $\epsilon$ from the two-sphere $r = r_1$, converges as $\epsilon \to 0$ to its Euclidean value of $2\pi$. In general, consider a two-dimensional metric of the form

$$ds^2 = F^2(r)dr^2 + C^2(r)d\psi^2,$$

with a radial coordinate $r \geq r_1 > 0$ and an angular coordinate $\psi$ with period $T$. Then

$$d\epsilon = F dr, \quad L = \int_0^T C(r(\epsilon))d\psi = T C(r(\epsilon)),$$

hence the smoothness condition at $r = r_1$ is

$$2\pi = T \lim_{r \to r_1} dC/dr \frac{dr}{d\epsilon} = T \lim_{r \to r_1} \frac{C'}{F}.$$

Substituting in (20) the expressions for $F$ and $C$ from (7), $F = -f = \sqrt{r_2 - N^2}/\Delta(r)$, $C = 2N/F$, and the period $T = 4\pi/p$ of $\psi$, we get

$$\frac{\Delta'(r_1)}{r_1^2 - N^2} = \frac{p}{2N}.$$

Expressing $M$ in terms of $r_1$ from the condition $\Delta(r_1) = 0$, substituting in (21) and solving for $r_1$ gives two solutions,

$$\rho_1 = \frac{p - \sqrt{p^2 - 48N^2 + 144N^4}}{12N}, \quad \rho_2 = \frac{p + \sqrt{p^2 - 48N^2 + 144N^4}}{12N}.$$

The solution $\rho_1$ satisfies $\rho_1 > N$ only for

$$p = 1 \quad \text{and} \quad 0 < N \leq \frac{1}{2} \sqrt{\frac{2 - \sqrt{3}}{3}}.$$

Solving $\Delta(\rho_1) = 0$ for $M$ then gives

$$M_1 = \frac{1}{864N^3} \left(1 - (1 + 24N^2 - 288N^4)\sqrt{1 - 48N^2 + 144N^4}\right).$$

The condition $\rho_2 > N$ instead admits solutions for any positive integer $p$,

$$\begin{cases} p = 1 \quad \text{and} \quad 0 < N \leq \frac{1}{2} \sqrt{\frac{2 - \sqrt{3}}{3}}, \\ p = 2 \quad \text{and} \quad 0 < N < \frac{1}{\sqrt{6}}, \\ p \geq 3 \quad \text{and} \quad 0 < N < \infty. \end{cases}$$
The value of $M$ corresponding to $\rho_2$ is

$$M_2^p = \frac{1}{864N^5} \left( p^3 + (p^2 + 24N^2 - 288N^3) \sqrt{p^2 - 48N^2 + 144N^4} \right).$$  \(26\)

These bolt-type solutions are smooth complete Einstein metric with the topology of the total space of the complex line bundle $\mathcal{O}(-p) \to \mathbb{C}P^1$. The case $p = 1$, with $N$ constrained by either (23) or (25) and with the corresponding value of $M$, is known as the Taub-bolt-AdS (TB-AdS) metric [4]. Reinstating the cosmological constant $\Lambda$ one notes that $\lim_{\Lambda \to 0} \rho_1 = 2N$, twice the radius of the bolt in the Ricci-flat Taub-bolt space, while $\rho_2$ diverges in this limit. Thus, as already noted in [4], only the solution with $r_1 = \rho_1$ converges to the Ricci-flat Taub-bolt space as $\Lambda \to 0$.

2.2.3. Half-conformally-flat solutions. In section 4 we will need to know for which values of the parameters $M, N$ the metric (7) and (8) is half-conformally-flat, that is has self-dual or anti self-dual Weyl tensor. It can be checked that, for the choice of orientation (10), the Weyl tensor of (7) is self-dual if and only if

$$M_+ = N(1 - 4N^2),$$  \(27\)

which gives the TN-AdS solution (17), and is anti self-dual if and only if $M$ takes the opposite value

$$M_- = -N(1 - 4N^2).$$  \(28\)

We write $\Delta_-(r)$ for (8) with $M = M_-$,

$$\Delta_-(r) = (r + N)^2(r - x_+)(r - y_-),$$  \(29\)

where

$$x_+ = N + \sqrt{4N^2 - 1}, \quad y_- = N - \sqrt{4N^2 - 1}$$  \(30\)

and $x_+$ is the largest root.

There are no simultaneous solutions of (27) or (28) and (24) or (26) with $p = 1$, thus TB-AdS is not half-conformally-flat for any value of $N$. However for $p > 1$ both (28) and (26) are satisfied if $N = N_p$, having defined

$$N_p = \frac{p}{4\sqrt{p - 1}}, \quad p \geqslant 3.$$  \(31\)

We need to take $p \geqslant 3$ in (31) in order to satisfy (25). The values of $M$ and $r_1$ corresponding to (31) are

$$M_p = -\frac{p(p - 2)^2}{16(p - 1)^{3/2}}, \quad (r_1)_p = \frac{3p - 4}{4\sqrt{p - 1}}.$$  \(32\)

The metric (7) and (8) with $N = N_p, M = M_p$ and $p \geqslant 3$ is the quaternionic Eguchi–Hanson solution [13, 18].
\[ g = \left( \frac{r^2 - N^2}{\Delta_p(r)} \right) dr^2 + (r^2 - N^2_p) d\Omega^2 + 4N^2_p \left( \frac{\Delta_p(r)}{r^2 - N^2_p} \right) \eta^2, \]
\[
\Delta_p(r) = \left( r + \frac{p}{4\sqrt{p - 1}} \right)^2 \left( r + \frac{4 - 3p}{4\sqrt{p - 1}} \right) \left( r + \frac{4 - p}{4\sqrt{p - 1}} \right), \quad p \geq 3.
\]

(33)

Note that the parameters in (32) are well-defined and real for \( p = 2 \). The corresponding metric is that of \( H^4/\mathbb{Z}_2 \) which is conformally flat, but has a conical singularity at \( r = 0 \) because of the \( \mathbb{Z}_2 \) quotient. As we have already seen, the smooth \( H^4 \) metric can be obtained for \( p = 1 \) as a special case of (17).

To summarise, for \( M \) given by (27) we have the one-parameter family (17) of self-dual solutions of NUT type. For \( M \) given by (28) we have the infinite discrete family (33) of anti-self-dual solutions of bolt type. The self-dual and anti self-dual branches join at \( M = 0 \), which gives the conformally flat metric of \( H^4 \).

3. Spherically symmetric harmonic two-forms

We are going to study the harmonic cohomology, that is the space of harmonic square-integrable differential forms, of the Riemannian manifold \( M \) equipped with the metric (7) and (8). We recall that depending on the values of the parameters \( M, N \) and \( p, M \) is either diffeomorphic to \( \mathbb{R}^4 \) or to the total space of the line bundle \( \mathcal{O}(-p) \to \mathbb{C}P^1 \). We denote by \( L^2(M) \) the space of square-integrable forms on \( M \). Since \( M \) is complete, the harmonic forms in \( L^2(M) \) are precisely the closed and co-closed forms in \( L^2(M) \).

Since \( M \) is conformally compact, harmonic cohomology is trivial except in middle dimension where it is infinite dimensional [23]. We will restrict our analysis to harmonic two-forms in \( L^2(M) \) which are invariant under the \( SU(2) \) subgroup of the isometry group of \( M \). For \( \beta \in \Omega^2(M) \) this invariance conditions reads
\[ L_{Z_i} \beta = 0, \quad i = 1, 2, 3, \]

(34)

where \( \{ Z \} \) are the right-invariant vector fields on \( SU(2) \),
\[
Z_1 = - \sin \phi \partial_\theta - \frac{\cos \phi}{\sin \theta} (\cos \theta \partial_\phi - \partial_\psi),
\]
\[
Z_2 = \cos \phi \partial_\theta - \frac{\sin \phi}{\sin \theta} (\cos \theta \partial_\phi - \partial_\psi),
\]
\[
Z_3 = \partial_\phi.
\]

(35)

If \( \beta \) is harmonic then it is closed so locally we can write
\[ \beta = d\alpha \]

(36)

for some one-form \( \alpha \). Since \( L_{Z_i} \eta_j = 0 \), (34) is solved by taking
\[ \alpha = \alpha_1(r) \eta_1 + \alpha_2(r) \eta_2 + \alpha_3(r) \eta_3, \]

(37)

where the \( \alpha_i \) are arbitrary functions of \( r \) only. We write
\[ \beta_i = d(\alpha_i, \eta_i). \]

(38)
We could also include an exact term $\alpha_4(r)dr$ in (37), but since we are only interested in $\beta = d\alpha$ we can safely drop it. Observe that if $\beta$ is also required to be invariant under the $U(1)$-isometry generated by $\partial_\psi$, then necessarily $\alpha_1 = \alpha_2 = 0$.

In terms of (37), the co-closure condition $d^*\beta = 0$ becomes

$$\alpha_i'' - \frac{2f'}{f} \alpha_i' - \frac{f^4}{4N^2} \alpha_i = 0, \quad i = 1, 2, \tag{39}$$

$$\alpha_3'' + \frac{2a'}{a} \alpha_3' - \frac{4N^2}{a^3} \alpha_3 = 0. \tag{40}$$

Equation (40) can be integrated explicitly, obtaining

$$\alpha_3 = C_1 \left( \frac{r^2 + N^2}{r^2 - N^2} \right) - C_2 \left( \frac{2Nr}{r^2 - N^2} \right), \tag{41}$$

with $C_1$, $C_2$ arbitrary constants. The form $\beta_3 = d(\alpha_3 \eta_3)$ is given by

$$\beta_3 = -\frac{1}{(r^2 - N^2)^2} \left[ (C_1(r^2 + N^2) - 2C_2rN) e^1 \wedge e^2 + (C_2(r^2 + N^2) - 2C_1rN) e^3 \wedge e^4 \right]. \tag{42}$$

Clearly (42) is self-dual (respectively anti self-dual) if and only if $C_1 = C_2$ ($C_1 = -C_2$). We denote by $\beta_3^+$, the self-dual form obtained by setting $C_1 = C_2 = -1$,

$$\beta_3^+ = \frac{1}{(r+N)^2} (e^1 \wedge e^2 + e^3 \wedge e^4). \tag{43}$$

and by $\beta_3^-$ the anti-self-dual form obtained by setting $C_1 = -C_2 = -1$,

$$\beta_3^- = \frac{1}{(r-N)^2} (e^1 \wedge e^2 - e^3 \wedge e^4). \tag{44}$$

The form $\beta_3$ can then be written as

$$\beta_3 = \left( \frac{C_1 + C_2}{2} \right) \beta_3^+ + \left( \frac{C_1 - C_2}{2} \right) \beta_3^- . \tag{45}$$

We also define

$$\alpha_3^\pm = \left( \frac{r-N}{r+N} \right)^{\pm1} \tag{46}$$

so that $\beta_3^\pm = d(\alpha_3^\pm \eta_3)$.

Consider now the ODE (39). Taking the ansatz

$$\alpha_i(r) = \exp(l(r)), \tag{47}$$

and substituting in (39) one finds the condition

$$f' = \pm \frac{1}{2N} f^2 = \pm \frac{1}{2N} \left( \frac{r^2 - N^2}{\Delta(r)} \right). \tag{48}$$
Defining
\[ \varphi(r) = -\frac{1}{2N} \int \left( \frac{r^2 - N^2}{\Delta(r)} \right) dr, \] (49)
the general solution of (39) is
\[ \alpha_i(r) = \left( \frac{K_i - L_i}{2} \right) \exp(\varphi(r)) + \left( \frac{K_i + L_i}{2} \right) \exp(-\varphi(r)), \] (50)
with \( K_1, K_2, L_1, L_2 \) arbitrary constants. For \( \alpha_i \) given by (50), \( \beta_i \) is self-dual (respectively anti self-dual) if and only if \( K_i = L_i \) (\( K_i = -L_i \)). Thus defining
\[ \alpha_i^\pm = \exp(\mp \varphi), \quad \beta_i^\pm = d(\alpha_i^\pm \eta_i), \quad i = 1, 2, \] (51)
we have
\[ \alpha_i = \left( \frac{K_i - L_i}{2} \right) \alpha_i^- + \left( \frac{K_i + L_i}{2} \right) \alpha_i^+, \] (52)
and similarly for \( \beta_i \).

The precise form of \( \varphi \) depends on whether any of the roots \( \{r_1, r_2, r_3, r_4\} \) of \( \Delta(r) \) are repeated. We will write down an explicit expression in the case of TN-AdS, where \( r_1 = r_2 = N \) and \( r_3, r_4 \) are distinct, see section 3.1, and in the generic case for bolt-type solutions, where \( \{r_i\} \) are all distinct, see section 3.2. The other cases can be handled similarly.

In order to check the \( L^2 \) condition we will make use of the following expression, valid for any exact two-form \( \beta = d\alpha \) on a dense subset of a space with metric (7),
\[ \beta \wedge \ast \beta = \left[ \left( \frac{\alpha'}{f} \right)^2 + \left( \frac{\alpha f}{2N} \right)^2 \right] \frac{\text{vol}}{a^2}. \] (53)
In particular, the form (42) has norm
\[ \|\beta_3\| = \int_M \beta_3 \wedge \ast \beta_3 = 8\piTN \lim_{r \to r_1} \frac{r}{(r^2 - N^2)^{3/2}} \left( (C_1^2 + C_2^2)(r^2 + N^2) - 4C_1C_2Nr \right), \] (54)
for \( r_1 \) the largest root of \( \Delta(r) \) and \( T \) the period of \( \psi \).

3.1. Spherically symmetric harmonic two-forms on solutions of NUT type

Let \( \mathcal{M} \) be TN-AdS. Consider \( \beta_3 \) first. Since \( r_1 = N \), by (42) \( \beta_3 \) is well defined at \( r = N \) if and only if it is self-dual, so we need to take
\[ \alpha_3 = C\alpha_3^+. \] (55)
The form \( \beta_3^+ \) is in \( L^2(\mathcal{M}) \) as, by (54) with \( T = 4\pi, r_1 = N \),
\[ \|\beta_3^+\|^2 = 16\pi^2. \] (56)
Since \( \alpha_3^+(N) = 0, \alpha_3^+ \eta_3 \) is a globally defined one-form. Thus \( \beta_3^+ \) is exact, which had to be the case since TN-AdS is topologically trivial.
Consider now the forms $\beta_i$ for $i = 1, 2$. $\Delta_+(r)$ has a double zero at $r = N$ and factorises as

$$\Delta_+(r) = (r - N)^2(r - x_+)(r - y_+), \quad x_+ = -N + \sqrt{4N^2 - 1}, \quad y_+ = -N - \sqrt{4N^2 - 1}.$$  \hfill (57)

The integral in (49) thus gives

$$\varphi_n(r) = \frac{N^2 - x_+^2}{2N\Delta_+(x_+)} \log(r - x_+) + \frac{N^2 - y_+^2}{2N\Delta_+(y_+)} \log(r - y_+) - \log(r - N). \hfill (58)$$

Suppose $x_+ = y_+$ first, so that we are dealing with hyperbolic four-space. Then $\varphi_n = -\log(r - N)$ and

$$\alpha_i = \left(\frac{K_i - L_i}{2}\right) \frac{1}{r - N} + \left(\frac{K_i + L_i}{2}\right) (r - N). \hfill (59)$$

It is easily checked that $\beta_i$ does not belong to $L^2(M)$.

Assume now $x_+ \neq y_+$. From (51) and (58) we see that $\lim_{r \rightarrow N^+} \alpha_i^+ = 0$, $\lim_{r \rightarrow N^+} \alpha_i^- = \infty$. Thus to have an $L^2$ form we need to take

$$\alpha_i = K_i \alpha_i^+. \hfill (60)$$

Asymptotically, to leading order,

$$\alpha_i^+(r) = e^{-\varphi_n(r)} \sim r^{-\frac{1}{2}} \left(\frac{N^2 - x_+^2}{\Delta_+(x_+)} + \frac{N^2 - y_+^2}{\Delta_+(y_+)} - 2N\right) = 1, \hfill (61)$$

having used

$$\frac{N^2 - x_+^2}{\Delta_+(x_+)} + \frac{N^2 - y_+^2}{\Delta_+(y_+)} - 2N = 0. \hfill (62)$$

We can calculate the $L^2$-norm using Stokes’ theorem,

$$\|\beta_i^+\|^2 = -\int_{\partial M} (\alpha_i^+)^2 \eta_1 \wedge \eta_2 \wedge \eta_3 = 16\pi^2 \lim_{r \rightarrow \infty} (\alpha_i^+)^2 = 16\pi^2. \hfill (63)$$

Both $\beta_1^+$ and $\beta_2^+$ are globally defined exact two-forms. It can be checked that none of $\{\beta_1^+, \beta_2^+, \beta_3^+\}$ is $L^2$-exact.

Summarising, the space of square-integrable $SU(2)$-invariant harmonic two-forms on TN-AdS consists entirely of self-dual exact but not $L^2$-exact forms. For $N \in (0, 1/2) \cup (1/2, \infty)$ it is three-dimensional and spanned by $(\beta_1^+, \beta_2^+, \beta_3^+)$, which for TN-AdS take the form

$$\beta_1^+ = d[e^{-i\pi \eta_1}], \quad \beta_2^+ = d[e^{-i\pi \eta_2}], \quad \beta_3^+ = d\left[\frac{r - N}{r + N}\right] \eta_3, \hfill (64)$$

with $\varphi_n$ given by (58). For $N = 1/2$ we obtain $H^4$ and the space of square-integrable $SU(2)$-invariant harmonic two-forms is one-dimensional and spanned by $\beta_3^+$ only.
3.2. Spherically symmetric harmonic two-forms on solutions of bolt type

Let $M$ be a solution of bolt type. The form $\beta_3$ is again given by (42). This time since $r_1 > N$, (54) is finite for any value of $C_1$, $C_2$ so $\beta_3 \in L^2(M)$. The two-form $\beta_3$ is globally defined, however since $\psi$ is not defined at the bolt $\{r = r_1\}$, the one-form $\alpha_3 \eta_3$ is not globally defined unless $\alpha_3(r_1) = 0$. We can write

$$\alpha_3 = \tilde{C}_1 \left( \frac{2N r_1}{r_1^2 + N^2} \frac{r^2 + N^2}{r^2 - N^2} - \frac{2N r}{r^2 - N^2} \right) + \tilde{C}_2 \left( \frac{2N r_1}{r_1^2 + N^2} \frac{r^2 + N^2}{r^2 - N^2} + \frac{2N r}{r^2 - N^2} \right),$$  

(65)

with

$$\tilde{C}_1 = \frac{C_2}{2} + \frac{C_1}{2} \left( \frac{r_1^2 + N^2}{2N r_1} \right), \quad \tilde{C}_2 = -\frac{C_2}{2} + \frac{C_1}{2} \left( \frac{r_1^2 + N^2}{2N r_1} \right).$$  

(66)

We thus have $\alpha_3(r_1) = 0$ if and only if $\tilde{C}_2 = 0$, or equivalently

$$C_2 = \left( \frac{r_1^2 + N^2}{2N r_1} \right) C_1.$$  

(67)

The one-dimensional subspace of $\{\beta_3 : (\tilde{C}_1, \tilde{C}_2) \in \mathbb{R}^2\}$ obtained by setting $\tilde{C}_2 = 0$ is exact. The complementary subspace obtained by setting $\tilde{C}_1 = 0$ generates $H^1_{\text{GR}}(M) \simeq \mathbb{R}$. Since $r_1 > N$, the conditions $C_2 = \pm C_1$, $C_2 = \pm \frac{r_1^2 + N^2}{2N r_1} C_1$ are incompatible for any choice of signs; in particular if $\beta_3 \neq 0$ then it cannot be (anti) self-dual and exact. In fact writing $\beta_3 = \tilde{C}_1 \beta_3^\text{exact} + \tilde{C}_2 \beta_2^\text{exact}$ we find

$$\beta_3^\text{exact} = \frac{1}{4N r_1} (r_1 + N)^2 \beta_3^+ - (r_1 - N)^2 \beta_3^-, \quad \beta_2^\text{exact} = \frac{1}{4N r_1} (r_1 + N)^2 \beta_2^+ - (r_1 - N)^2 \beta_2^-.$$  

(68)

The form $\beta_3^\text{exact}$ is exact but not $L^2$-exact. In fact, $\beta_3^\text{exact} = d(\alpha_3^\text{exact} \eta_3 + \gamma)$ where $\alpha_3^\text{exact}$ is (41) with $C_2 = 0$ and $\gamma$ is any closed one-form. Since $H^1_{\text{GR}}(M) = 0$ we can write $\gamma = df_3$ for some smooth function $f_3$. Then

$$\|\alpha_3^\text{exact} \eta_3 + df_3\|^2 = \|(\alpha_3^\text{exact})^2 \eta_3\|^2 + \|df_3\|^2 + 2\langle \alpha_3^\text{exact} \eta_3, df_3 \rangle.$$  

(69)

We have

$$\|(\alpha_3^\text{exact})^2 \eta_3\|^2 = 16\pi^2 N \int_{r_1}^{\infty} \frac{(r^2 - N^2)^2}{\Delta(r)} (\alpha_3^\text{exact})^2 \, dr,$$  

(70)

which diverges since the integrand is asymptotically a non-zero constant. Hence in order for (69) to be finite we would need $df_3$ to be asymptotically equal to $-\alpha_3^\text{exact} \eta_3$, which is not possible as the latter is not asymptotically exact.

Let $\{r_1, r_2, r_3, r_4\}$ be the roots of the polynomial $\Delta(r)$, labelled so that $r_1 > N$ is the largest positive root, and assume they are all distinct. Integrating (48) then gives

$$\varphi_3(r) = \frac{1}{2N} \sum_{i=1}^{4} N^2 - r_i^2 \frac{\log(r - r_i)}{\Delta(r)}.$$  

(71)
It is convenient at this point to recall that if
\[ p(r) = \prod_{i=1}^{n} (r - r_i) \]  
(72)
is a monic polynomial of degree \( n \geq 4 \) with distinct roots \( r_i \) then
\[ \sum_{i=1}^{n} \frac{1}{p'(r_i)} = 0, \]  
(73)
\[ \sum_{i=1}^{n} \frac{r_i}{p'(r_i)} = 0, \]  
(74)
\[ \sum_{i=1}^{n} \frac{r_i^2}{p'(r_i)} = 0. \]  
(75)
These identities follow from the more general one, valid for any function \( h \),
\[ \sum_{i=1}^{k} \frac{h(r_i)}{p'(r_i)} = h[r_1, \ldots, r_k], \]  
(76)
with \( h[r_1, \ldots, r_k] \) the \( k \)th divided difference of \( h \). Taking \( h(x) = 1, k \geq 2 \) gives (73); taking \( h(x) = x, k \geq 3 \) gives (74) and taking \( h(x) = x^2, k \geq 4 \) gives (75).

We now check square-integrability. Since \( r_1 > N > 0 \) and \( \Delta'(r_1) > 0 \),
\[ \lim_{r \to r_1^+} \phi_0(r) = +\infty. \]  
(77)
Therefore by (50), \( \alpha_1^2 \) is unbounded in a neighbourhood of the bolt \( \{ r = r_1 \} \) unless \( K_i = L_i \) so that \( \alpha_i = K_i \alpha_i^1 \). Since \( \lim_{r \to r_1^+} (\alpha_i^1)^2 = 0 \), we obtain
\[ \| \beta_1^+ \|^2 = \int_{\partial M} (\alpha_i^1)^2 \eta_1 \wedge \eta_2 \wedge \eta_3 = 4\pi T \lim_{r \to \infty} (\alpha_i^1)^2. \]  
(78)
To evaluate the limit at infinity note that by (73) and (75), to leading order,
\[ \alpha_i^+ \simeq e^{\frac{1}{r_1^4} \sum_{i=1}^{n} \frac{r_i^2 - r_1^2}{2}} = 1. \]  
(79)
Therefore,
\[ \| \beta_1^+ \|^2 = \| \beta_2^+ \|^2 = \frac{16\pi^2}{4} \frac{1}{p} \]  
(80)
having substituted \( T = 4\pi/p \) for the period of \( \psi \). Since \( \alpha_i^+(r_1) = 0 \), the one-form \( \alpha_i^+ \eta_i, i = 1, 2, \) is globally defined and \( \beta_i^+ \) is exact, but, essentially by the same argument used for \( \beta_1^+ \), not \( L^2 \)-exact.

To summarise, the space of square-integrable \( SU(2) \)-invariant harmonic two-forms on bolt-type solutions is four-dimensional and spanned by \( (\beta_1^+, \beta_2^+, \beta_3^+, \beta_3^-) \), which for bolt-type
solutions take the form
\[
\beta^+_1 = d[e^{-\varphi_b} t], \quad \beta^+_2 = d[e^{-\varphi_b} s], \\
\beta^+_3 = d\left[\frac{r-N}{r+N}\right] t, \quad \beta^-_3 = d\left[\frac{r+N}{r-N}\right] s.
\]
with \(\varphi_b\) given by (71). The three-dimensional subspace spanned by \((\beta^+_1, \beta^+_2, \beta^+_3)\) comprises of self-dual forms. The two-space spanned by \((\beta^+_1, \beta^-_3)\) is exact but not \(L^2\)-exact. The two-space spanned by \((\beta^+_3, \beta^-_3)\) can be split into a one-dimensional space of exact but not \(L^2\)-exact forms and a cohomologically non-trivial one-space generating \(H^{2\text{dR}}(M) = \mathbb{R}\).

4. Killing spinors

A triple \((\mathcal{M}, g, F)\), for \((\mathcal{M}, g)\) a Riemannian manifold, \(F\) the curvature of a \(U(1)\) connection on \(\mathcal{M}\), solves the Einstein–Maxwell system with non-zero cosmological constant \(\Lambda\) if
\[
d \ast F = 0, \quad R_{\mu\nu} - \Lambda g_{\mu\nu} = 2\left(F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right).
\]
The two-form \(F\) describing the electromagnetic field is thus a closed and coclosed form of the type studied in section 3. The condition \(F \in L^2(\mathcal{M})\), which physically corresponds to finite electromagnetic energy, is often also required but we will not impose it here. A generic solution of the Maxwell equation (82) on an Einstein manifold \((\mathcal{M}, g)\) in general will not solve (83). However, if \(F\) is self-dual or anti self-dual then the right-hand side of (83) vanishes so \((\mathcal{M}, g, F)\) is a solution of the coupled Einstein–Maxwell system.

Solutions of (82) and (83) coincide with solutions of the bosonic sector of 4D, \(N = 2\) minimal gauged supergravity. Supersymmetric solutions, that is admitting non-trivial Killing spinors, are of particular interest. Killing spinors are solutions of a certain first-order equation known as the Killing spinor equation, and the dimension of the space of solutions divided by the rank of the spinor bundle is known as the fraction of supersymmetry preserved. With respect to a local orthonormal frame \((e_\mu)\) and gauge potential \(A\), the Killing spinor equation charged by the gauge potential \(A\) takes the form
\[
\left(\nabla_{e_\mu} - iA(e_\mu) + \frac{1}{2} \Gamma_\mu + iF_{\nu\rho} \Gamma^{\nu\rho} \right) \epsilon = 0.
\]
Here \(\epsilon\) is a four-component spinor field, the curvature \(F\) of the \(U(1)\) connection satisfies \(F = dA\) and the matrices \(\{\Gamma_{\mu}\}\) generate a four-dimensional representation of the Euclidean Clifford algebra \(Cl(0,4)\). We follow the convention
\[
\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu},
\]
so that
\[
\nabla_{e_\mu}(\epsilon) = e_\mu(\epsilon) + \frac{1}{4} \omega_{\rho\sigma}(e_\mu) \Gamma^{\rho\sigma} \epsilon
\]
for \(\omega_{\rho\sigma}\) the Levi-Civita connection local one-form associated to the frame \((e_\mu)\).

We are going to determine for which values of the parameters \(M, N\) and choice of an (anti) self-dual curvature two-form \(F\) the metric (7) and (8) admits Killing spinors. This problem has
been studied in [21] for general solutions of the Einstein–Maxwell system (82) and (83) with $H^4$ asymptotic and $SU(2) \times U(1)$ symmetry. The class of metrics studied in [21] is thus broader than (7) and (8) as it allows for non-Einstein manifolds. Explicitly $g$ is still given by (7) but with (8) replaced by

$$\frac{\Lambda}{3} r^4 + (1 + 2\Lambda N^2)r^2 - 2Mr + N^2(1 + \Lambda N^2) + P^2 - Q^2. \quad (87)$$

The polynomial (87) differs from (8) by the addition of the constant term $P^2 - Q^2$. The resulting metric is Einstein if and only if $P = \pm Q$. The harmonic two-form $F$ considered in [21] is

$$\left(\frac{P + Q}{2}\right) \beta^+ + \left(\frac{P - Q}{2}\right) \beta^-, \quad (88)$$

where $\beta^+_3, \beta^-_3$ are given by (43) and (44) and $P, Q$ are the same constants as those appearing in (87). The two-form (88) is the most general $SU(2) \times U(1)$-invariant harmonic two-form on $\mathcal{M}$, but, as shown in section 3, if we relax the requirement to $SU(2)$-invariance only other possibilities arise. We are thus going to study the problem for a narrower class of metrics but a broader class of two-forms. Unfortunately, we will conclude that considering a wider class of harmonic forms leads to no solutions other than those found in [21].

Our search is immediately constrained by the following result [5]. Let $(\mathcal{M}, g)$ be a Riemannian four-manifold, $F$ a self-dual (respectively anti self-dual) harmonic two-form. Then in order for the Killing spinor equation (84) to admit non-trivial solutions, $(\mathcal{M}, g)$ must be Einstein with self-dual (respectively anti self-dual) Weyl tensor. The self-dual case is realised by the TN-AdS metric (17) with $F$ a linear combination of $\{\beta^+_3, \beta^+_1, \beta^+_2\}$. The anti self-dual case by the bolt-type metric (33) with $F$ a linear combination of $\{\beta^-_1, \beta^-_2, \beta^-_3\}$. Thus we will consider the form

$$F = P \beta^+_3 + K_1 \beta^+_1 + K_2 \beta^+_2 \quad (89)$$

with $P, K_1, K_2$ arbitrary constants. Locally $F = dA$ with

$$A = P \left(\frac{r - N}{r + N}\right)^{\pm 1} \eta_3 + e^{\mp \varphi}(K_1 \eta_1 + K_2 \eta_2) \quad (90)$$

and $\varphi$ given by (49). The sign choice in (89) and (90) corresponds to $F = \pm \ast F$ with the upper sign relevant for TN-AdS and the lower sign for the anti self-dual bolt-type solutions. For $K_1 = K_2 = 0$ (89) is equal to (88) with $Q = \pm P$.

Before attacking the local equation (84), let us pause to consider some global issues. In order to admit spinors, let alone Killing ones, the manifold $\mathcal{M}$ must be spin. This is the case for TN-AdS, which is topologically trivial. Bolt-type solutions instead have the topology of the complex line bundle $\mathcal{O}(-p) \to \mathbb{C}P^1$ which is spin only for $p$ even. If $\mathcal{M}$ is not spin we can equip it with a Spin$^c$ structure by tensoring the spinor bundle on $\mathcal{M}$ with a complex line bundle $L$ having half-integer Chern number. Separately, neither the spinor bundle on $\mathcal{M}$ nor $L$ are globally well-defined, but their tensor product is [17]. Sections of the bundle so constructed give globally defined Spin$^c$ spinors which are charged under the gauge potential $A$ of a connection on $L$. If $\mathcal{M}$ is spin we obtain charged spinors by tensoring the spinor bundle with a line bundle $L$, which in this case has an integer Chern number.

In order for $F$ to be the curvature of a connection on the relevant line bundle $L \to \mathcal{M}$, it has to satisfy certain quantisation conditions. Since TN-AdS is topologically trivial, no conditions
arise. If $\mathcal{M}$ is a bolt-type solution with even $p$, then it is spin, $L$ has integer Chern number and $F$ has to satisfy the familiar quantisation condition
\begin{equation}
\frac{1}{2\pi} \int_{\text{bolt}} F = k \in \mathbb{Z}. \tag{91}
\end{equation}
If $\mathcal{M}$ is a bolt-type solution with odd $p$ then $L$ has half-integer Chern number and $F$ has to satisfy the quantisation condition
\begin{equation}
\frac{1}{2\pi} \int_{\text{bolt}} F = k + \frac{1}{2}, \quad k \in \mathbb{Z}. \tag{92}
\end{equation}

Let $F$ be given by (89) and $\mathcal{M}$ be a solution of bolt type. Being exact, the forms $\beta_1, \beta_2$ integrate to 0 so
\begin{equation}
\frac{1}{2\pi} \int_{\text{bolt}} F = \frac{P}{2\pi} \int_{\text{bolt}} \beta_3 = 2P \left( \frac{r_1 + N}{r_1 - N} \right). \tag{93}
\end{equation}

Substituting (31) and (32) for $N, r_1$ we thus get the condition
\begin{equation}
4P \left( \frac{p - 1}{p - 2} \right) = k + \frac{1}{2}(p \mod 2). \tag{94}
\end{equation}

We finally proceed to study (84) for $A$ given by (90), where $K_1, K_2$ are arbitrary constants and $P$ is arbitrary in the case of TN-AdS and constrained by (94) for the anti self-dual bolt-type solutions. We mostly follow the treatment in [21]. We already know that Killing spinors can only arise in the half-conformally-flat cases (17) and (33), with $M = M_0$ in the former case and $M = M_-$ in the latter, but since it does not require much additional effort we carry out the computation without imposing any condition on $M, N$.

The Killing spinor equation can be viewed as a parallelism condition with respect to the super-covariant derivative
\begin{equation}
\mathcal{D}_\mu = \nabla_\mu - iA(\varepsilon_\mu) + \frac{1}{2} F_{\mu\rho} \Gamma_{\rho\sigma} \Gamma_\sigma. \tag{95}
\end{equation}
A necessary condition for the integrability of (84) is thus
\begin{equation}
\mathcal{I}_{\mu\nu} \varepsilon = [\mathcal{D}_\mu, \mathcal{D}_\nu] \varepsilon = 0. \tag{96}
\end{equation}

One calculates
\begin{equation}
\mathcal{I}_{\mu\nu} = \frac{1}{4} F_{\mu\nu} \omega_{\alpha\beta} \Gamma_{\alpha\beta} + \frac{1}{2} \Gamma_{\mu\nu} - iF_{\mu\nu} I_4 + \frac{i}{2} \nabla_{[\mu} F_{\nu]\rho\sigma \Gamma_{\rho\sigma} \Gamma_\nu] + \frac{i}{4} F_{\rho\sigma} \Gamma_{[\mu} \Gamma_{\rho\sigma} \Gamma_{\nu]} \\
- \frac{1}{16} F_{\alpha\beta} F_{\rho\sigma}[\Gamma^\gamma_{\alpha\beta} \Gamma_{\mu}, \Gamma^\sigma_{\rho\sigma} \Gamma_\nu] + \frac{1}{4} F_{\rho\sigma} \Gamma^\gamma_{\rho\sigma} \Gamma_{\mu\nu}. \tag{97}
\end{equation}

In order for (96) to admit non-trivial solutions
\begin{equation}
\det(\mathcal{I}_{\mu\nu}) = 0 \tag{98}
\end{equation}
needs to hold for all values of $\mu, \nu$. Taking the gamma matrices in the chiral representation
\begin{equation}
\Gamma_i = \left( \begin{array}{cc} 0 & \sigma_i \\ \sigma_i & 0 \end{array} \right), \quad i = 1, 2, 3, \quad \Gamma_4 = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right), \tag{99}
\end{equation}
where \( \{ \sigma_i \} \) are the Pauli matrices, we calculate
\[
\det(I_{12}) = \det(I_{34}) = I + (K_1^2 + K_2^2)H,
\]
\[
\det(I_{13}) = \det(I_{24}) = \frac{I}{16} + \left( K_1^2 + \frac{K_2^2}{4} \right) H,
\]
\[
\det(I_{14}) = \det(I_{23}) = \frac{I}{16} + \left( K_2^2 + \frac{K_1^2}{4} \right) H,
\]
where
\[
I = -\left( \frac{D^2 r^2 + D(B_+ - B_-)r - B_+ B_-}{(r^2 - N^2)^P} \right),
\]
\[
H = -\frac{1}{16 N^2 P^2} \left( \frac{D^2 e^{2z} \phi}{r + N^2 \Delta(r)} \right),
\]
and, with the sign choice referring to \( F = \pm \ast F \),
\[
B_1 = (M \pm NP)^2 - N^2(P + 1 - 4N^2)^2,
\]
\[
B_2 = (M \mp NP)^2 - N^2(P - 1 + 4N^2)^2,
\]
\[
D = 2P(M \mp N \pm 4N^3).
\]
Therefore we get the necessary conditions
\[
D = 0, \quad B_1 B_2 = 0.
\]
Since \( P \neq 0, D = 0 \) is equivalent to
\[
M = \pm N(1 - 4N^2),
\]
which is exactly the (anti) self-duality condition for the Weyl tensor, see (27) and (28), which we already knew to be a necessary condition for the existence of non-trivial Killing spinors. From now on we assume that (104) is satisfied. More precisely, in the self-dual case we have \( F = \ast F, M = M_+ \) and in the anti self-dual case \( F = -\ast F, M = M_- \). Substituting (104) into the expressions for \( B_1, B_2 \) we see that
\[
B_1 = B_2 = 0
\]
automatically holds. Despite having considered a more general two-form \( F \), the conditions (103) are exactly the same as those found in [21]. However, because of the constraint (96), we find that the system (111) and (112) below admits no non-trivial solutions unless
\[
K_1 = K_2 = 0,
\]
which is the case studied in [21]. Since the treatment in [21] is mostly geared towards the non-self-dual case, for reference we reproduce their solutions in the self-dual case and provide some more details on the computation.

Write
\[
\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad \epsilon_1 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.
\]
For $K_1 = K_2 = 0$, (96) gives

$$
\mu_1 = i \left[ \frac{\Delta_+(r)}{r^2 - N^2} \frac{1}{r \pm NP(1 - 4N^2 + P)} \right]^{\pm 1} \lambda_1, \quad (108)
$$

$$
\mu_2 = i \left[ \frac{\Delta_+(r)}{r^2 - N^2} \frac{1}{r \mp NP(1 - 4N^2 - P)} \right]^{\pm 1} \lambda_2. \quad (109)
$$

The sign choice corresponds to the self-dual or anti self-dual case and

$$
\Delta_\pm(r) = (r \mp N)^2 (r - x_\pm)(r - y_\pm), \quad (110)
$$

where $x_\pm, y_\pm$ are given by (16) and $x_-, y_-$ by (30).

In the self-dual case the radial component of the Killing spinor equation (84) is

$$
\partial_r \epsilon_1 = -i \frac{r^2 - N^2}{2 \Delta_+(r)} \epsilon_2, \quad (111)
$$

$$
\partial_r \epsilon_2 = +i \frac{r^2 - N^2}{2 \Delta_-(r)} \left[ I_2 + \left( \frac{2P}{(r + N)^2} \right) \sigma_3 \right] \epsilon_1, \quad (112)
$$

and in the anti self-dual case

$$
\partial_r \epsilon_1 = -i \frac{r^2 - N^2}{2 \Delta_-(r)} \left[ I_2 + \left( \frac{2P}{(r - N)^2} \right) \sigma_3 \right] \epsilon_2, \quad (113)
$$

$$
\partial_r \epsilon_2 = +i \frac{r^2 - N^2}{2 \Delta_+(r)} \epsilon_1. \quad (114)
$$

In the self-dual case substituting (108) in (111) and integrating the resulting ODE gives

$$
\lambda_1 = \sqrt{r + N + \frac{N}{P}(1 - 4N^2)} \upsilon_1, \quad (115)
$$

$$
\lambda_2 = \sqrt{r + N - \frac{N}{P}(1 - 4N^2)} \upsilon_2, \quad (116)
$$

with $\upsilon_1, \upsilon_2$ at this point arbitrary functions of the angular coordinates. Substituting (115) and (116) in (108) and (109) gives

$$
\mu_1 = i \frac{\Delta_+(r)}{r^2 - N^2} \frac{1}{\sqrt{r + N + NP(1 - 4N^2)}} \upsilon_1, \quad (117)
$$

$$
\mu_2 = i \frac{\Delta_+(r)}{r^2 - N^2} \frac{1}{\sqrt{r + N - NP(1 - 4N^2)}} \upsilon_2. \quad (118)
$$

Equations (117) and (118) are compatible with the solution of (112) with $\lambda_1, \lambda_2$ given by (115) and (116) if and only if

$$
P^2 = N^2(4N^2 - 1), \quad (119)$$
or
\[
\left( P = \frac{4N^2 - 1}{2} \text{ and } \lambda_2 = \mu_2 = 0 \right) \quad \text{or} \quad \left( P = \frac{1 - 4N^2}{2} \text{ and } \lambda_1 = \mu_1 = 0 \right). \tag{120}
\]

Proceeding similarly, in the anti self-dual case we find
\[
\begin{align*}
\mu_1 &= \sqrt{r - N - \frac{N}{P} (1 - 4N^2)} \, v_1, \tag{121} \\
\mu_2 &= \sqrt{r - N + \frac{N}{P} (1 - 4N^2)} \, v_2, \tag{122} \\
\lambda_1 &= -i \sqrt{\frac{\Delta (r)}{r^2 - N^2}} \frac{1}{\sqrt{r - N - NP(1 - 4N^2)}} \, v_1, \tag{123} \\
\lambda_2 &= -i \sqrt{\frac{\Delta (r)}{r^2 - N^2}} \frac{1}{\sqrt{r - N + NP(1 - 4N^2)}} \, v_2. \tag{124}
\end{align*}
\]

with the same compatibility conditions (119) and (120) found in the self-dual case.

The sign of \( P \) can be changed by a coordinate redefinition, so without loss of generality we restrict to the two cases
\[
\begin{align*}
P &= -N \sqrt{4N^2 - 1}, \tag{125} \\
P &= \frac{1 - 4N^2}{2}. \tag{126}
\end{align*}
\]

Note that, for anti self-dual bolt-type solutions, \( N \) is given by (31), so (126) becomes
\[
P = -\frac{1}{8} \left( \frac{p - 2}{p - 1} \right)^2, \tag{127}
\]
and (125) becomes
\[
P = -\frac{p}{8} \left( \frac{p - 2}{p - 1} \right). \tag{128}
\]

Substituting (127), (128) in (94) we see that the quantisation condition arising in the case of anti self-dual bolt-type solutions is automatically satisfied provided that \( P \) satisfies (125) or (126).

We now examine the angular part of the Killing spinor. Asymptotically, to leading order in \( r \),
\[
g \simeq \frac{dr^2}{r} + r^2 g^{(3)}, \quad g^{(3)} = \eta_1^2 + \eta_2^2 + 4N^2 \eta_3^2. \tag{129}
\]

Let \((e^i)\) be the orthonormal coframe (9), \((e^{(3)})^1 = \eta_1, (e^{(3)})^2 = \eta_2, (e^{(3)})^3 = 2N\eta_3\). Denote by \(\omega_{ij}, \omega_{ij}^{(3)}\) the Levi-Civita connection one-forms associated to \(g\) and \(g^{(3)}\) with respect to \((e_i)\) and
We have
\[ \omega_{12} = \omega_{12}^{(3)} = (2N^2 - 1)\eta_3, \]
\[ \omega_{13} = \omega_{13}^{(3)} = N\eta_2, \]
\[ \omega_{23} = \omega_{23}^{(3)} = -N\eta_1, \]
\[ \omega_{ik} = e_i, \quad i = 1, 2, 3. \]  

Equations (130)

It follows
\[ \nabla_a \epsilon = \epsilon_a (\epsilon) + \frac{1}{4} \omega_i (e_a) \Gamma^i \epsilon = \frac{1}{r} \left[ X_a (\epsilon) + \frac{1}{4} \omega_i (X_a) \Gamma^i \right] \epsilon + \frac{1}{2} \Gamma_a \Gamma \epsilon, \]  

where \( (X_i) \) are the left-invariant vector fields on \( SU(2) \) dual to \( (2), \)
\[ X_1 = \sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} (\cos \theta \partial_\psi - \partial_\phi), \]
\[ X_2 = \cos \psi \partial_\theta - \frac{\sin \psi}{\sin \theta} (\cos \theta \partial_\psi - \partial_\phi), \]
\[ X_3 = \partial_\psi. \]  

Using the representation (99) for the gamma matrices and noticing that \( (\sigma_a) \) generate the Clifford algebra \( Cl(0, 3) \) we get
\[ \nabla_1 \epsilon = \frac{1}{r} \left[ e_1^{(3)} (\epsilon_1) + \frac{1}{4} \omega_i (X_a) \sigma_i \sigma_j \right] \epsilon_1 - \frac{i}{2} \sigma_a \epsilon_1 = \frac{1}{r} \nabla_1^{(3)} \epsilon_1 = \frac{i}{2} \sigma_a \epsilon_1, \]
\[ \nabla_2 \epsilon = \frac{1}{r} \left[ e_2^{(3)} (\epsilon_2) + \frac{1}{4} \omega_i (X_a) \sigma_i \sigma_j \right] \epsilon_2 + \frac{i}{2} \sigma_a \epsilon_2 = \frac{1}{r} \nabla_2^{(3)} \epsilon_2 + \frac{i}{2} \sigma_a \epsilon_2. \]  

Equations (133)

We also define
\[ A_a^{(3)} = \lim_{r \to \infty} A (e_a^{(3)}). \]  

Equation (134)

The components \( F_{ij} \) of \( F = dA \) with respect to the orthonormal frame \( (e_i) \) are \( O(r^{-2}) \) so to the leading order in \( r \) they can be neglected. Thus asymptotically (84) becomes
\[ (\nabla_a^{(3)} - iA_a^{(3)}) \epsilon_1 + \frac{r}{2} \sigma_a (\epsilon_2 - i\epsilon_1) = 0, \]  

Equation (135)
\[ (\nabla_a^{(3)} - iA_a^{(3)}) \epsilon_2 + \frac{ir}{2} \sigma_a (\epsilon_2 - i\epsilon_1) = 0. \]  

Equation (136)

We now solve (135) and (136) for \( P \) given by (126) or (125). First consider the case
\[ P = -N \sqrt{4N^2 - 1}. \]  

Equation (137)
in the self-dual case. One has
\[
\epsilon_1 = \begin{pmatrix}
\sqrt{r + N + \sqrt{4N^2 - 1}} & \nu_1 \\
\sqrt{r + N - \sqrt{4N^2 - 1}} & \nu_2
\end{pmatrix},
\]
\[
\epsilon_2 = i \begin{pmatrix}
\frac{\Delta(r)}{\sqrt{r^2 - N^2} \sqrt{r + N + \sqrt{4N^2 - 1}}} & \nu_1 \\
\frac{\Delta(r)}{\sqrt{r^2 - N^2} \sqrt{r + N - \sqrt{4N^2 - 1}}} & \nu_2
\end{pmatrix}.
\]
(138)

Asymptotically, setting
\[
\nu = \begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix},
\]
(139)
we get
\[
\epsilon_2 - i\epsilon_1 \simeq -i \frac{1}{\sqrt{r}} (NI_2 + \sqrt{4N^2 - 1} \sigma_3) \nu.
\]
(140)
Both (135) and (136) thus reduce to the same equation
\[
\left( \nabla^{(3)}_a - iA^{(3)}_a - \frac{i}{2} N\sigma_a - \frac{i}{2} \sqrt{4N^2 - 1} \sigma_a \sigma_3 \right) \nu = 0.
\]
(141)
A similar computation shows that the anti self-dual case also leads to equation (141).

The general solution to (141) is [15, 21]
\[
v = \begin{pmatrix}
\cos \frac{\theta}{2} e^{\frac{i}{2}(\phi + \psi)} & -\sin \frac{\theta}{2} e^{\frac{i}{2}(\phi - \psi)} \\
\gamma \sin \frac{\theta}{2} e^{-\frac{i}{2}(\phi - \psi)} & \gamma \cos \frac{\theta}{2} e^{-\frac{i}{2}(\phi + \psi)}
\end{pmatrix} \nu_0,
\]
(142)
with \(\nu_0\) a constant two-spinor and
\[
\gamma = i \left(2N + \sqrt{4N^2 - 1}\right).
\]
(143)
One can check that the spinor (138), or the corresponding expression in the anti self-dual case, with \(\nu\) given by (142) solves all the components of the Killing spinor equation (84), thus giving a 1/2 BPS solution.

Take now
\[
P = \frac{1 - 4N^2}{2}.
\]
(144)
Then in the self-dual case
\[
\epsilon_1 = \begin{pmatrix}
\sqrt{r + 3N} & \nu_1 \\
\sqrt{r - N} & \nu_2
\end{pmatrix},
\]
\[
\epsilon_2 = i \begin{pmatrix}
\frac{\Delta(r)}{\sqrt{r^2 - N^2} \sqrt{r + 3N}} & \nu_1 \\
\frac{\Delta(r)}{\sqrt{r^2 - N^2} \sqrt{r - N}} & \nu_2
\end{pmatrix}.
\]
(145)
Asymptotically
\[
\epsilon_2 - i \epsilon_1 \simeq \frac{iN}{\sqrt{r}} \begin{pmatrix} -3 \nu_1 \\ \nu_2 \end{pmatrix},
\]
so both (135) and (136) reduce to
\[
\left[ \nabla_a^{(3)} - i A_a^{(3)} \right] \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + \frac{iN}{2} \sigma_a \begin{pmatrix} -3 \nu_1 \\ \nu_2 \end{pmatrix} = 0.
\]
(146)

Again, the anti self-dual case leads to the same equation.

We have
\[
A_a^{(3)} = \left( \frac{P}{2N} \right) \delta_a^3 = \left( \frac{1 - 4N^2}{4N} \right) \delta_a^3,
\]
and, for \( \nu = (\nu_1, \nu_2)^T \),
\[
\begin{align*}
\nabla_1 \nu &= X_1(\nu) - \frac{iN}{2} \sigma_1 \nu, \\
\nabla_2 \nu &= X_2(\nu) - \frac{iN}{2} \sigma_2 \nu, \\
\nabla_3 \nu &= \frac{1}{2N} X_3(\nu) + \frac{i(2N^2 - 1)}{4N} \sigma_3 \nu.
\end{align*}
\]
(147)

Therefore (147) is equivalent to the system
\[
\begin{align*}
X_1(\nu_1) &= X_2(\nu_1) = X_3(\nu_2) = 0, \\
X_1(\nu_2) &= 2iN \nu_1, \\
X_2(\nu_2) &= -2N \nu_1, \\
X_3(\nu_1) &= i \nu_1,
\end{align*}
\]
(148)

which has general solution
\[
\nu = \begin{pmatrix} 0 \\ \nu_0 \end{pmatrix},
\]
(149)

for \( \nu_0 \) a constant. One can check that the spinor (145), or the corresponding expression in the anti self-dual case, with \( \nu \) given by (151) solves all the components of the Killing spinor equation (84), thus giving a 1/4 BPS solution.

To summarise, supersymmetry is preserved only if the Weyl tensor of the metric \( g \) and the two-form \( F \) are both either self-dual or anti self-dual. In the self-dual case, \( F = P \beta_3^+ \), with \( \beta_3^+ \) given by (43), and \( g \) is the metric of TN-AdS, see (17), depending on one continuous parameter \( N \in (0, \infty) \). Despite the underlying manifold being topologically trivial, Killing spinors only arise if \( P \) satisfies (125), leading to a 1/2 BPS solution, or (126), leading to a 1/4 BPS solution. In the anti self-dual case, \( F = P \beta_3^- \), with \( \beta_3^- \) given by (44), and the metric is of bolt-type, see (33), depending on one discrete parameter \( p \in \mathbb{Z}, p \geq 3 \). The value of \( P \) is again constrained by (125) and (126), leading to 1/2 and 1/4 BPS solutions as in the self-dual case. In this case the underlying manifold has a non-trivial topology and (125) and (126) imply the quantisation conditions arising from the requirement that \( F \) is the curvature of a \( U(1) \) connection for a spin structure (for \( p \) even) or spin\(^C \) structure (for \( p \) odd).
5. Conclusions

In this paper we have determined a basis for the space of SU(2)-invariant $L^2$ harmonic two-forms on a two-parameter family of bi-axial Bianchi IX Einstein metrics with negative cosmological constant $\Lambda$. In the case of NUT-type metrics, this space is three-dimensional, comprising entirely of self-dual forms. Since the underlying manifold is topologically trivial, these forms are necessarily exact. They are not, however, $L^2$-exact. In the case of bolt-type solutions, the space is four-dimensional, with a one-dimensional cohomologically non-trivial subspace and a three-dimensional self-dual subspace.

These results can be viewed as a generalisation of those in [9], which considers the Ricci-flat case $\Lambda = 0$. A negative cosmological constant results in a very different asymptotic behaviour of the metric, which is asymptotically hyperbolic rather than asymptotically locally flat. As a consequence, the space of $L^2$ harmonic two-forms is infinite dimensional. Even the finite-dimensional subspace obtained by focusing on SU(2)-invariant forms is bigger than the space of $L^2$ harmonic two-forms obtained in [9] for $\Lambda = 0$. The latter is one-dimensional (respectively two-dimensional) in the case of NUT type (bolt type) metrics. A basis for it can be obtained by taking the $\Lambda \to 0$ limit of the forms $\beta_{1}^\pm$, $\beta_{2}^\pm$ (respectively $\beta_{3}^\pm$ and $\beta_{5}^\pm$) found in this paper. The forms $\beta_{1}^\pm$, $\beta_{2}^\pm$ diverge in the limit.

A self-dual or anti self-dual two-form $F$ on an Einstein manifold with cosmological constant provides a solution of the bosonic sector of 4D, $\mathcal{N} = 2$ minimal gauged supergravity. It is therefore natural to ask whether any of the two-forms that we have found gives raise to supersymmetric solutions. While we have shown that the question has a positive answer, all the solutions that we have found had been obtained before [21]. In fact, having started with a more general form for $F$, which is only requested to be SU(2)-invariant rather than SU(2) × U(1)-invariant, we have found that Killing spinors only arise if $F$ is also U(1)-invariant, thus reducing the problem to the one studied in [21].

The lack of new solutions arises because of the mismatch between the SU(2) invariance of the two-form and the SU(2) × U(1) invariance of the metric. It would therefore be interesting to consider the case in which both the two-form and the metric are SU(2)-invariant only.

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No new data were created or analysed in this study.

ORCID iDs

Guido Franchetti https://orcid.org/0000-0002-1511-6204

References

[1] Anderson M T 2005 Topics in conformally compact Einstein metrics (arXiv:0503243)
[2] Baird T J and Kunduri H 2021 Abelian instantons over the Chen–Teo AF geometry J. Geom. Phys. 168 104310
[3] Carron G 2007 $L^2$ harmonic forms on non-compact Riemannian manifolds (arXiv:0704.3194)
[4] Chamblin A, Emparan R, Johnson C V and Myers R C 1999 Large $N$ phases, gravitational instantons, and the nuts and bolts of AdS holography Phys. Rev. D 59 064010
[5] Dunajski M, Gutowski J, Sabra W and Tod P 2010 Cosmological Einstein–Maxwell instantons and Euclidean supersymmetry; anti-self-dual solutions Class. Quantum Grav. 28 025007
[6] Etesi G and Jardim M 2008 Moduli spaces of self-dual connections over asymptotically locally flat gravitational instantons Commun. Math. Phys. 280 285–313
[7] Farquet D, Lorenzen J, Martelli D and Sparks J 2016 Gravity duals of supersymmetric gauge theories on three-manifolds J. High Energy Phys. JHEP08(2016)080
[8] Franchetti G 2014 Harmonic forms on ALF gravitational instantons J. High Energy Phys. JHEP12(2014)075
[9] Franchetti G 2019 Harmonic forms and spinors on the Taub-bolt space J. Geom. Phys. 141 11–28
[10] Gibbons G W and Hawking S W 1979 Classification of gravitational instanton symmetries Commun. Math. Phys. 66 291–310
[11] Gibbons G W and Pope C N 1979 The positive action conjecture and asymptotically Euclidean metrics in quantum gravity Commun. Math. Phys. 66 267–90
[12] Hausel T, Hunsicker E and Mazzeo R 2004 Hodge cohomology of gravitational instantons Duke Math. J. 122 485–548
[13] Hitchin N J 1995 Twistor spaces, Einstein metrics and isomonodromic deformations J. Diff. Geom. 42 30–112
[14] Huang X, Rey S-J and Zhou Y 2014 Three-dimensional SCFT on conic space as hologram of charged topological black hole J. High Energy Phys. 2014 127
[15] Inamur Y and Yokoyama D 2012 $N = 2$ supersymmetric theories on squashed three-sphere Phys. Rev. D 85 025015
[16] Kapustin A, Willett B and Yaakov I 2010 Exact results for Wilson loops in superconformal Chern–Simons theories with matter J. High Energy Phys. JHEP03(2010)089
[17] Lawson H B and Michelsohn M L 1989 Spin Geometry (Princeton, NJ: Princeton University Press)
[18] LeBrun C 1988 Counter-examples to the generalized positive action conjecture Commun. Math. Phys. 118 591–6
[19] Maldacena J 1998 The large $N$ limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231–52
[20] Martelli D, Passias A and Sparks J 2012 The gravity dual of supersymmetric gauge theories on a squashed three-sphere Nucl. Phys. B 864 840–68
[21] Martelli D, Passias A and Sparks J 2013 The supersymmetric NUTs and bolts of holography Nucl. Phys. B 876 810–70
[22] Martelli D and Sparks J 2013 The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere Nucl. Phys. B 866 72–85
[23] Mazzeo R 1988 The Hodge cohomology of a conformally compact metric J. Diff. Geom. 28 309–39
[24] Nishioka T 2014 The gravity dual of supersymmetric Renyi entropy J. High Energy Phys. JHEP07(2014)061
[25] Page D N 1978 TAUB-NUT instanton with an horizon Phys. Lett. B 78 249–51
[26] Page D N and Pope C N 1987 Inhomogeneous Einstein metrics on complex line bundles Class. Quantum Grav. 4 213–25
[27] Pedersen H 1986 Einstein metrics, spinning top motions and monopoles Math. Ann. 274 35–59
[28] Pestun V 2012 Localization of gauge theory on a four-sphere and supersymmetric Wilson loops Commun. Math. Phys. 313 71–129