(n, d)-COHERENT RINGS, (n, d)-COSEMIHEREDITARY RINGS
AND (n, d)-V-RINGS

Zhu Zhanmin

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Abstract. Let \( R \) be a ring, \( n \) be a non-negative integer and \( d \) be a positive integer or \( \infty \). A right \( R \)-module \( M \) is called \((n, d)^*\)-projective if \( \text{Ext}_1^R(M, C) = 0 \) for every \( n \)-copresented right \( R \)-module \( C \) of injective dimension \( \leq d \); a ring \( R \) is called right \((n, d)^\ast\)-cocoherent if every \( n \)-copresented right \( R \)-module \( C \) with \( \text{id}(C) \leq d \) is \((n+1)\)-copresented; a ring \( R \) is called right \((n, d)^\ast\)-cosemihereditary if whenever \( 0 \to C \to E \to A \to 0 \) is exact, where \( C \) is \( n \)-copresented with \( \text{id}(C) \leq d \), \( E \) is finitely cogenerated injective, then \( A \) is injective; a ring \( R \) is called right \((n, d)^\ast\)-V-ring if every \( n \)-copresented right \( R \)-module \( C \) with \( \text{id}(C) \leq d \) is injective. Some characterizations of \((n, d)^*\)-projective modules are given, right \((n, d)^\ast\)-cocoherent rings, right \((n, d)^\ast\)-cosemihereditary rings and right \((n, d)^\ast\)-V-rings are characterized by \((n, d)^*\)-projective right \( R \)-modules. \((n, d)^*\)-projective dimensions of modules over right \((n, d)^\ast\)-cocoherent rings are investigated.

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1. Introduction

Throughout this paper, \( R \) is an associative ring with identity and all modules considered are unitary, \( n \) is a non-negative integer, \( d \) is a positive integer or \( \infty \) unless a special note.

In 1982, V. A. Hiremath [4] defined and studied finitely correlated modules. Following [4], a right \( R \)-module \( M \) is said to be finitely correlated if there is a short exact sequence \( 0 \to M \to N \to K \to 0 \) of right \( R \)-modules with \( N \) finitely cogenerated, cofree and \( K \) is finitely cogenerated, where a right \( R \)-module \( N \) is said to be cofree if it is isomorphic to a direct product of the injective hulls of some simple right \( R \)-modules. Finitely correlated modules are also called finitely copresented modules.

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in some literatures such as [7]. Following [12], a right $R$-module $M$ is said to be \textit{FCP-projective} if $\text{Ext}^1_R(M, C) = 0$ for every finitely copresented right $R$-module $C$. In [12], right $V$-rings are characterized by FCP-projective right $R$-modules. We recall also that $R$ is called \textit{right co-semihereditary} [6,8,12] if every finitely cogenerated factor module of a finitely cogenerated injective right $R$-module is injective, $R$ is called \textit{right co-coherent} [12] if every finitely cogenerated factor module of a finitely cogenerated injective right $R$-module is finitely copresented. It is easy to see that right $V$-rings, right co-semihereditary rings and right co-coherent rings are the dual concepts of von Neumann regular rings, right semihereditary rings and right coherent rings. In this paper, right \textit{cocoherent} rings will denote right co-coherent rings in order to facilitate. In [12], right $V$-rings, right co-semihereditary rings are characterized by FCP-projective right $R$-modules, FCP-projective dimensions of right $R$-modules over right cocoherent rings are investigated. For example, we show that a ring $R$ is right co-semihereditary if and only if every submodule of an FCP-projective right $R$-module is FCP-projective if and only if every submodule of a projective right $R$-module is FCP-projective [12, Theorem 3], a ring $R$ is a right $V$-ring if and only if every right $R$-module is FCP-projective [12, Theorem 4].

In 1999, Xue introduced $n$-copresented modules and $n$-cocoherent rings respectively in [9]. According to [9], $M$ is said to be \textit{n-copresented} if there is an exact sequence of right $R$-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$, where each $E_i$ is a finitely cogenerated injective module. It is easy to see that a module $M$ is finitely cogenerated if and only if it is 0-copresented, a module $M$ is finitely copresented if and only if it is 1-copresented. We call any module $(−1)$-copresented. $n$-copresented modules have been studied in [2,9,11]; $R$ is called \textit{right n-cocoherent} [9] in case every n-copresented right $R$-module is $(n + 1)$-copresented. It is easy to see that $R$ is right cocoherent if and only if it is right 1-cocoherent. Following [5], a ring $R$ is called right co-noetherian if every factor module of a finitely cogenerated right $R$-module is finitely cogenerated. By [4, Proposition 17], a ring $R$ is right co-noetherian if and only if it is right 0-cocoherent. In [11], we extend the concepts of FCP-projective modules, cosemihereditary rings and $V$-rings to $(n,d)$-projective modules, $n$-cosemihereditary rings and $n$-$V$-rings respectively, right $n$-$V$-rings and right $n$-cosemihereditary rings are characterized by $(n,0)$-projective right $R$-modules, $(n,0)$-projective dimensions of right $R$-modules over right $n$-cocoherent rings are investigated. Following [11], a right $R$-module $M$ is called $(n,d)$-projective if $\text{Ext}^{d+1}_R(M, A) = 0$ for every $n$-copresented right $R$-module $A$; a ring $R$ is called right $n$-cosemihereditary if every submodule of a projective right
$R$-module is $(n,0)$-projective, a ring $R$ is called a right $n$-$V$-ring if every right $R$-module is $(n,0)$-projective. Clearly, a right $R$-module $M$ is FCP-projective if and only if it is $(1,0)$-projective, a ring $R$ is right cosemihereditary if and only if it is right 1-cosemihereditary, a ring $R$ is a right $V$-ring if and only if it is a right 0-$V$-ring if and only if it is a right 1-$V$-ring. Characterizations of $n$-cosemihereditary rings and $n$-$V$-rings can be found in [11, Theorem 3.7] and [11, Theorem 3.9], respectively.

In this paper, we generalize the concepts of $(n,0)$-projective modules, $n$-cocoherent rings, $n$-cosemihereditary rings, $n$-$V$-rings to $(n,d)^*$-projective modules, $(n,d)^*$-cocoherent rings, $(n,d)^*$-cosemihereditary rings and $(n,d)^*$-$V$-rings respectively. $(n,d)^*$-cosemihereditary rings, $(n,d)^*$-$V$-rings will be characterized by $(n,d)^*$-projective modules, $(n,d)^*$-projective dimensions of modules over $(n,d)^*$-cocoherent rings will be investigated. As corollaries, some new characterizations of right $V$-rings will be given.

2. $(n,d)^*$-Projective modules and $(n,d)^*$-cocoherent rings

We start with the following definition.

Definition 2.1. A right $R$-module $M$ is said to be $(n,d)^*$-projective if $\text{Ext}_R^1(M,C) = 0$ for every $n$-copresented right $R$-module $C$ with $id(C) \leq d$. A right $R$-module $C$ is said to be $(n,d)^*$-injective if $\text{Ext}_R^1(M,C) = 0$ for every $(n,d)^*$-projective right $R$-module $M$.

Remark 2.2. (1) It is easy to see that if a module $M$ is $(n,d)^*$-projective, then it is $(n',d')^*$-projective for any $n' \geq n$ and $d' \leq d$.

(2) A module $M$ is $(n,0)$-projective if and only if it is $(n,\infty)^*$-projective.

Recall that a short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ is said to be $n$-copure [11] if every $n$-copresented module is injective with respect to this sequence.

Definition 2.3. A short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ is said to be $(n,d)$-copure if every $n$-copresented module with injective dimension $\leq d$ is injective with respect to this sequence.

Remark 2.4. A short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ is $n$-copure if and only if it is $(n,\infty)$-copure.

Theorem 2.5. Let $M$ be a right $R$-module. Then the following statements are equivalent:

(1) $M$ is $(n,d)^*$-projective.
(2) $M$ is projective with respect to the exact sequence $0 \to C \to B \to A \to 0$ of right $R$-modules, where $C$ is $n$-copresented and $\text{id}(C) \leq d$.

(3) If $E'$ is an $(n-1)$-copresented factor module of a finitely cogenerated injective right $R$-module $E$ and $\text{id}(E') \leq d-1$, then every right $R$-homomorphism $f$ from $M$ to $E'$ can be lifted to a homomorphism from $M$ to $E$.

(4) Every exact sequence $0 \to M'' \to M' \to M \to 0$ is $(n,d)$-copure.

(5) There exists an $(n,d)$-copure exact sequence $0 \to K \to P \to M \to 0$ of right $R$-modules with $P$ projective.

(6) There exists an $(n,d)$-copure exact sequence $0 \to K \to P \to M \to 0$ of right $R$-modules with $P (n,d)^*\text{-projective}.

(7) $M$ is projective with respect to every exact sequence $0 \to C \to B \to A \to 0$ of right $R$-modules with $C (n,d)^*\text{-injective}.

(8) $M$ is projective with respect to every exact sequence $0 \to C \to E \to A \to 0$ of right $R$-modules with $C (n,d)^*\text{-injective}$ and $E$ injective.

**Proof.** (1) $\Rightarrow$ (2) By the exact sequence
$$\text{Hom}(M, B) \to \text{Hom}(M, A) \to \text{Ext}^1_R(M, C) = 0.$$ (2) $\Rightarrow$ (3) Since $E$ is finitely cogenerated injective and $E'$ is $(n-1)$-copresented with $\text{id}(E') \leq d-1$, the kernel $K$ of the natural epimorphism $E \to E'$ is $n$-copresented and $\text{id}(K) \leq d$. So (3) follows immediately from (2).

(3) $\Rightarrow$ (1) For any $n$-copresented module $C$ with $\text{id}(C) \leq d$, there exists an exact sequence $0 \to C \to E \to E' \to 0$, where $E$ is finitely cogenerated injective, $E'$ is $(n-1)$-copresented, and $\text{id}(E') \leq d-1$. Hence we get an exact sequence $\text{Hom}(M, E) \to \text{Hom}(M, E') \to \text{Ext}^1_R(M, C) \to \text{Ext}^1_R(M, E) = 0$, and thus $\text{Ext}^1_R(M, C) = 0$ by (3).

(1) $\Rightarrow$ (4) Assume (1). Then we have an exact sequence
$$\text{Hom}(M', C) \to \text{Hom}(M'', C) \to \text{Ext}^1_R(M, C) = 0$$
for every $n$-copresented module $C$ with $\text{id}(C) \leq d$, and so (4) follows.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1) By (6), we have an $(n,d)$-copure exact sequence $0 \to K \xrightarrow{f} P \to M \to 0$ of right $R$-modules with $P (n,d)^*\text{-projective}$, and so, for each $n$-copresented module $C$ with $\text{id}(C) \leq d$, we have an exact sequence $\text{Hom}(P, C) \xrightarrow{f^*} \text{Hom}(K, C) \to \text{Ext}^1_R(M, C) \to \text{Ext}^1_R(P, C) = 0$ with $f^*$ epic. This implies that $\text{Ext}^1_R(M, C) = 0$, and therefore (1) follows.

(1) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1) are similar to the proofs of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). $\square$
Definition 2.6. (1) The \((n,d)^*\)-projective dimension of a module \(M_R\) is defined by
\[
(n,d)^* - pd(M_R) = \inf \{ m : \text{Ext}_R^{m+1}(M,C) = 0 \text{ for every } n\text{-copresented module } C \text{ with } id(C) \leq d \}
\]
(2) \(r. (n,d)^*\)-PD(R) is defined by
\[
r. (n,d)^*\text{-PD}(R) = \sup \{ (n,d)^*\text{-pd}(M) : M \text{ is a right } R\text{-module} \}.
\]

Definition 2.7. A ring \(R\) is called right \((n,d)\)-cocoherent, if every \(n\)-copresented right \(R\)-module with injective dimension \(\leq d\) is \((n+1)\)-copresented.

Remark 2.8. (1) It is easy to see that if a ring \(R\) is right \((n,d)\)-cocoherent, then it is right \((n',d')\)-cocoherent for any \(n' \geq n\) and \(d' \leq d\).

(2) Every ring \(R\) is right \((n,1)\)-cocoherent.

(3) A ring \(R\) is right \(n\)-cocoherent if and only if it is right \((n,\infty)\)-cocoherent.

Lemma 2.9. Let \(R\) be a right \((n,d)\)-cocoherent ring and \(M\) a right \(R\)-module. Then the following statements are equivalent:

1. \((n,d)^*\text{-pd}(M) \leq k\).
2. \(\text{Ext}_R^{k+1}(M,C) = 0\) for all \(n\)-copresented modules \(C\) with \(id(C) \leq d\).

Proof. (1) \(\Rightarrow\) (2) Use induction on \(k\). Clear if \((n,d)^*\text{-pd}(M) = k\). If \((n,d)^*\text{-pd}(M) \leq k-1\). Since \(C\) is \(n\)-copresented, there exists an exact sequence \(0 \to C \to E \to E' \to 0\), where \(E\) is finitely cogenerated injective, and \(E'\) is \((n-1)\)-copresented. Since \(id(C) \leq d\), we have \(id(E') \leq d\). But \(R\) is right \((n,d)\)-cocoherent, \(C\) is \((n+1)\)-copresented, so \(E'\) is \(n\)-copresented, and thus \(\text{Ext}_R^{k+1}(M,A) \cong \text{Ext}_R^{k}(M,E') = 0\) by induction hypothesis.

(2) \(\Rightarrow\) (1) is clear. \(\square\)

Corollary 2.10. Let \(R\) be a right \((n,d)\)-cocoherent ring and let \(M_R\) be \((n,d)^*\)-projective. Then \(\text{Ext}_R^{k}(M,C) = 0\) for all \(n\)-copresented modules \(C\) with \(id(C) \leq d\) and all positive integers \(k\).

Corollary 2.11. Let \(R\) be a right \((n,d)\)-cocoherent ring and let \(M\) be a right \(R\)-module. If the sequence \(0 \to P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_{k-1}} \cdots \to P_0 \to M \to 0\) is exact with \(P_0, \ldots, P_{k-1}\) \((n,d)^*\)-projective, then \(\text{Ext}_R^{k+1}(M,C) \cong \text{Ext}_R^{1}(P_k,C)\) for any \(n\)-copresented modules \(C\) with \(id(C) \leq d\).

Proof. Since \(R\) is right \((n,d)\)-cocoherent and \(P_0, P_1, \ldots, P_{k-1}\) are \((n,d)\)-projective, by Corollary 2.10, we have
\[
\text{Ext}_R^{k+1}(M,C) \cong \text{Ext}_R^k(\text{Ker}(d_0),C) \cong \text{Ext}_R^{k-1}(\text{Ker}(d_1),C) \cong \cdots \cong
\]
Theorem 2.12. Let $R$ be a right $(n, d)$-cocoherent ring and $M$ be a right $R$-module. Then the following statements are equivalent:

1. $(n, d)^*\cdot pd(M_R) \leq k$.
2. $\text{Ext}^{k+1}_R(M, C) = 0$ for all $n$-copresented modules $C$ with $id(C) \leq d$ and all positive integers $l$.
3. $\text{Ext}^{k+1}_R(M, C) = 0$ for all $n$-copresented modules $C$ with $id(C) \leq d$.
4. If the sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ is exact with $P_0, \ldots, P_{k-1}$ $(n,d)^*$-projective, then $P_k$ is also $(n,d)^*$-projective.
5. There exists an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ of right $R$-modules with $P_0, \ldots, P_{k-1}, P_k$ $(n,d)^*$-projective.

Proof. (1) $\Rightarrow$ (2) Assume (1). Then $(n, d) - pd(M_R) \leq k + l - 1$, and so (2) follows from Lemma 2.9.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) by Corollary 2.11. □

3. $(n,d)$-Cosemihereditary rings and $(n,d)$-V-rings

As the beginning of this section, we extend the concept of $n$-cosemihereditary rings as follows.

Definition 3.1. A ring $R$ is called right $(n, d)$-cosemihereditary, if for every finitely cogenerated injective right $R$-module $E$, each $(n-1)$-copresented factor module $E'$ of $E$ with $id(E') \leq d - 1$ is injective. A ring $R$ is called right cohereditary if it is right $(0, \infty)$-cosemihereditary.

Remark 3.2. (1) It is easy to see that if a ring $R$ is right $(n, d)$-cosemihereditary, then it is right $(n', d')$-cosemihereditary for any $n' \geq n$ and $d' \leq d$.

(2) Every ring $R$ is right $(n, 1)$-cosemihereditary.

(3) A ring $R$ is right $n$-cosemihereditary if and only if it is right $(n, \infty)$-cosemihereditary.

(4) A ring $R$ is right cohereditary if and only if every factor module of a finitely cogenerated injective right $R$-module is injective.

(5) A ring $R$ is right cosemihereditary if and only if it is right $(1, \infty)$-cosemihereditary.

Theorem 3.3. The following statements are equivalent for a ring $R$:

1. $R$ is a right $(n, d)$-cosemihereditary ring.
(2) \( R \) is right \((n,d)\)-cocoherent and \( r.(n,d)^*\)-PD(\( R \)) \( \leq 1 \).

(3) \( \text{Ext}^2_R(M,C) = 0 \) for any right \( R \)-module \( M \) and any \( n \)-copresented right \( R \)-module \( C \) with \( \text{id}(C) \leq d \).

(4) Every submodule of an \((n,d)^*\)-projective right \( R \)-module is \((n,d)^*\)-projective.

(5) Every submodule of a projective right \( R \)-module is \((n,d)^*\)-projective.

**Proof.** (1) \( \Rightarrow \) (2) Let \( C \) be an \( n \)-copresented right \( R \)-module with injective dimension \( \leq d \). Then there exists an exact sequence \( 0 \rightarrow C \rightarrow E \rightarrow E' \rightarrow 0 \), where \( E \) is finitely cogenerated injective, \( E' \) is \((n-1)\)-copresented and \( \text{id}(E') \leq d - 1 \). Since \( R \) is right \((n,d)\)-cosemihereditary, \( E' \) is finitely cogenerated injective, and so \( C \) is \((n+1)\)-copresented, it shows that \( R \) is right \((n,d)\)-cocoherent. Now let \( M \) be a right \( R \)-module. Then for any \( n \)-copresented right \( R \)-module \( C \) with \( \text{id}(C) \leq d \), we have an exact sequence \( 0 \rightarrow C \rightarrow E \rightarrow E' \rightarrow 0 \) of right \( R \)-modules, where \( E \) is finitely cogenerated injective, \( E' \) is \((n-1)\)-copresented and \( \text{id}(E') \leq d - 1 \). Since \( R \) is right \((n,d)\)-cosemihereditary, by the above proof, \( E' \) is injective. Thus the exact sequence \( 0 = \text{Ext}^1_R(M,E') \rightarrow \text{Ext}^2_R(M,C) \rightarrow \text{Ext}^2_R(M,E) = 0 \) implies that \( \text{Ext}^2_R(M,C) = 0 \). This follows that \( r.(n,d)^*\)-PD(\( R \)) \( \leq 1 \).

(2) \( \Rightarrow \) (3) It follows from Theorem 2.12.

(3) \( \Rightarrow \) (4) Let \( M \) be an \((n,d)^*\)-projective right \( R \)-module and \( K \) be its submodule. Then for any \( n \)-copresented module \( C \) with \( \text{id}(C) \leq d \), we have an exact sequence \( 0 = \text{Ext}^1_R(M,C) \rightarrow \text{Ext}^1_R(K,C) \rightarrow \text{Ext}^1_R(M/K,C) = 0 \) by (3), it follows that \( \text{Ext}^1_R(K,C) = 0 \), as required.

(4) \( \Rightarrow \) (5) It is obvious.

(5) \( \Rightarrow \) (1) Let \( E' \) be an \((n-1)\)-copresented factor module of a finitely cogenerated injective right \( R \)-module \( E \) and \( \text{id}(E') \leq d - 1 \). Let \( f \) be an epimorphism from \( E \) to \( E' \). Then for any projective right \( R \)-module \( P \) and any submodule \( K \) of \( P \), \( K \) is \((n,d)^*\)-projective by (4). So for any \( n \)-copresented right \( R \)-module \( C \) with \( \text{id}(C) \leq d \), we have an exact sequence \( 0 = \text{Ext}^1_R(K,C) \rightarrow \text{Ext}^1_R(P/K,C) \rightarrow \text{Ext}^1_R(P/C) = 0 \), which implies that \( \text{Ext}^1_R(P/K,C) = 0 \). Note that \( \text{Ker}(f) \) is \( n \)-copresented and \( \text{id}(\text{Ker}(f)) \leq d \), we get an exact sequence \( 0 = \text{Ext}^1_R(P/K,E) \rightarrow \text{Ext}^1_R(P/K,E') \rightarrow \text{Ext}^1_R(P/K,\text{Ker}(f)) = 0 \), and then \( \text{Ext}^1_R(P/K,E') = 0 \), which shows that \( E' \) is \( P_R \)-injective from the exact sequence \( \text{Hom}(P,E') \rightarrow \text{Hom}(K,E') \rightarrow \text{Ext}^1_R(P/K,E') \). Therefore, \( E' \) is injective.

Our following Corollary 3.4 improves [11, Theorem 3.7] partly.

**Corollary 3.4.** The following statements are equivalent for a ring \( R \):

(1) \( R \) is a right \( n \)-cosemihereditary ring.
(2) $R$ is right $n$-coherent and $r.(n,0)\text{-PD}(R) \leq 1$.
(3) $\text{Ext}^2_R(M, C) = 0$ for any right $R$-module $M$ and any $n$-copresented right $R$-module $C$.
(4) Every submodule of an $(n,0)$-projective right $R$-module is $(n,0)$-projective.
(5) Every submodule of a projective right $R$-module is $(n,0)$-projective.

Corollary 3.5. The following statements are equivalent for a ring $R$:

(1) $R$ is a right cosemihereditary ring.
(2) $R$ is right cocoherent and $r.\text{FCP-PD}(R) \leq 1$.
(3) $\text{Ext}^2_R(M, C) = 0$ for any right $R$-module $M$ and any finitely copresented right $R$-module $C$.
(4) Every submodule of an FCP-projective right $R$-module is FCP-projective.
(5) Every submodule of a projective right $R$-module is FCP-projective.

Corollary 3.6. The following statements are equivalent for a ring $R$:

(1) $R$ is a right cohereditary ring.
(2) $R$ is right co-noetherian and $r.\text{FCG-PD}(R) \leq 1$.
(3) $\text{Ext}^2_R(M, C) = 0$ for any right $R$-module $M$ and any finitely cogenerated right $R$-module $C$.
(4) Every submodule of an FCG-projective right $R$-module is FCG-projective.
(5) Every submodule of a projective right $R$-module is FCG-projective.

Next we extend the concept of right $n$-$V$-rings as follows.

Definition 3.7. A ring $R$ is called right $(n,d)$-$V$-ring if every right $R$-module is $(n,d)^*$-projective.

Remark 3.8. (1) It is easy to see that if $n' \geq n$ and $d' \leq d$, then a right $(n,d)$-$V$-ring is a right $(n',d')$-$V$-ring.
(2) A ring $R$ is a right $n$-$V$-ring if and only if it is a right $(n,\infty)$-$V$-ring.

Now we give some characterizations of right $(n,d)$-$V$-rings.

Theorem 3.9. The following conditions are equivalent for a ring $R$:

(1) $R$ is a right $(n,d)$-$V$-ring.
(2) Every $(n-1)$-copresented right $R$-module with injective dimension $\leq d-1$ is $(n,d)^*$-projective.
(3) $R$ is right $(n,d)$-cosemihereditary and $E(S)$ is $(n,d)^*$-projective for every simple right $R$-module $S$. 


(4) $R$ is right $(n,d)$-cocoherent and for every finitely cogenerated injective right $R$-module $E$, every $n$-copresented factor module $E'$ of $E$ with $id(E') \leq d−1$ is $(n,d)^*$-projective.

(5) For every finitely cogenerated injective right $R$-module $E$, every $(n−1)$-copresented factor module $E'$ of $E$ with $id(E') \leq d−1$ is $(n,d)^*$-projective.

(6) Every $n$-copresented right $R$-module with injective dimension $\leq d$ is injective.

Proof. (1) $\Rightarrow$ (2) and (6) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3) Assume (2). Then it is clear that $E(S)$ is $(n,d)^*$-projective for every simple right $R$-module $S$. Let $E$ be a finitely cogenerated injective module and $E'$ an $(n−1)$-copresented factor module of $E$ with $id(E') \leq d−1$. By (2), $E'$ is $(n,d)^*$-projective, so by Theorem 2.5(3), we have that $E'$ is isomorphic to a direct summand of $E$ and hence $E'$ is injective. Therefore, $R$ is right $(n,d)$-cosemihereditary.

(3) $\Rightarrow$ (4) Assume (3). Since $R$ is right $(n,d)$-cosemihereditary, it is right $(n,d)$-cocoherent by Theorem 3.3. Now let $E$ be a finitely cogenerated injective right $R$-module and $E'$ an $(n−1)$-copresented factor module of $E$ with $id(E') \leq d−1$. Since $R$ is right $(n,d)$-cocoherent, $E'$ is $n$-copresented and hence finitely cogenerated. Thus, the injective envelope $E(E')$ of $E'$ is a finitely cogenerated injective module, and so $E(E') \cong \oplus_{i=1}^{k} E(S_i)$ for some simple modules $E_i, i = 1, 2, \ldots, k$. Since each $E_i$ is $(n,d)^*$-projective by (3), $E(E')$ is also $(n,d)^*$-projective. Observing that $R$ is right $(n,d)$-cosemihereditary, by Theorem 3.3, $E'$ is also $(n,d)^*$-projective.

(4) $\Rightarrow$ (5) Let $E$ be a finitely cogenerated injective module and $E'$ an $(n−1)$-copresented factor module of $E$ with $id(E') \leq d−1$. Since $R$ is right $(n,d)$-cocoherent, $E'$ is $n$-copresented. By (4), $E'$ is $(n,d)^*$-projective.

(5) $\Rightarrow$ (6) Let $C$ be an $n$-copresented right $R$-module with $id(C) \leq d$. Then there exists an exact sequence $0 \rightarrow C \rightarrow E \rightarrow E' \rightarrow 0$ of right $R$-modules, where $E$ is finitely cogenerated injective, $E'$ is $(n−1)$-copresented and $id(E') \leq d−1$. By (5), $E'$ is $(n,d)^*$-projective, so $E'$ is projective respect to this exact sequence by Theorem 2.5(3). This follows that $C$ is isomorphic to a direct summand of $E$, and therefore $C$ is injective.

Recall that a right $R$-module $M$ is called $FCG$-projective [11] if $\text{Ext}^1_R(M, A) = 0$ for every finitely cogenerated right $R$-module $A$. By Remark 2.2, a right $R$-module is FCG-projective if and only if it is $(0, \infty)^*$-projective, a right $R$-module is FCP-projective if and only if it is $(1, \infty)^*$-projective, every FCG-projective module is FCP-projective.
Corollary 3.10. The following conditions are equivalent for a ring $R$:

1. $R$ is a right $V$-ring.
2. $R$ is a right $(0, \infty)$-$V$-ring.
3. $R$ is a right $(1, \infty)$-$V$-ring.
4. Every right $R$-module is FCG-projective.
5. $R$ is right cohereditary and $E(S)$ is FCG-projective for every simple right $R$-module $S$.
6. $R$ is right co-noetherian and for every finitely cogenerated injective right $R$-module $E$, every finitely cogenerated factor module $E'$ of $E$ is FCG-projective.
7. For every finitely cogenerated injective right $R$-module $E$, every factor module $E'$ of $E$ is FCG-projective.
8. Every finitely cogenerated right $R$-module is injective.
9. Every finitely cogenerated right $R$-module is FCP-projective.
10. $R$ is right cosemihereditary and $E(S)$ is FCP-projective for every simple right $R$-module $S$.
11. $R$ is right cocoherent and for every finitely cogenerated injective right $R$-module $E$, every finitely copresented factor module $E'$ of $E$ is FCP-projective.
12. For every finitely cogenerated injective right $R$-module $E$, every finitely copresented factor module $E'$ of $E$ is FCP-projective.
13. Every finitely copresented right $R$-module is injective.

Proof. (2) $\Rightarrow$ (3) is obvious. By Theorem 3.9, we have

$(2) \iff (4) \iff (5) \iff (6) \iff (7) \iff (8)$; and $(3) \iff (9) \iff (10) \iff (11) \iff (12) \iff (13)$.

(1) $\Rightarrow$ (8) Let $R$ be a right $V$-ring. Then every simple right $R$-module is injective. For any finitely cogenerated right $R$-module $M$, we have $E(M) \cong E(S_1) + \cdots + E(S_n)$ for some finite set $S_1, \ldots, S_n$ of simple modules by [1, Proposition 18.18], so $E(M) \cong S_1 + \cdots + S_n$ is semisimple. Thus $M$ is a direct summand of $E(M)$, and therefore $M$ is injective.

(13) $\Rightarrow$ (1) Let $S$ be any simple right $R$-module. Suppose $S$ is not injective. Let $x \in E(S) \setminus S$ and let $A$ be a submodule of $E(S)$ maximal with respect to $S \subseteq A$ and $x \notin A$, then $0 \neq x + A \in \cap \{K \leq E(S)/A \mid K \neq 0\}$, which implies that $E(S)/A$ is finitely cogenerated and whence $A$ is finitely copresented. By (13), $A$ is injective. It follows that $A = E(S)$, which contradicts the fact that $x \notin A$. Hence $S$ is injective and so $R$ is a right $V$-ring. $\square$
Recall that a right $R$-module $M$ is called $n$-presented [3] if there is an exact sequence of right $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ where each $F_i$ is a finitely generated free, equivalently projective right $R$-module; a left $R$-module $M$ is called $(n, 0)$-flat [10] if $\text{Tor}_1^R(A, M) = 0$ for every $n$-presented right $R$-module $A$. A ring $R$ is called right $n$-regular [10] if every $n$-presented right $R$-module is projective. By [10, Theorem 3.9], a ring $R$ is right $n$-regular if and only if every left $R$-module $M$ is $(n, 0)$-flat.

**Theorem 3.11.** Let $R$ be a commutative ring. Then every $(n, 0)$-projective module is $(n, 0)$-flat.

**Proof.** Let $M$ be an $(n, 0)$-projective module. To prove $M$ is $(n, 0)$-flat, we need prove $\text{Tor}_1^R(A, M) = 0$ for every $n$-presented $R$-module $A$. Since $A$ is $n$-presented, $\text{Hom}_R(A, E(S))$ is $n$-copresented for any simple module $S$. Let $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be an exact sequence of $R$-modules with $P$ projective. Then by Theorem 2.5, this exact sequence is $n$-copure. And so we get an exact sequence of $R$-modules

$$0 \rightarrow \text{Hom}_R(M, \text{Hom}_R(A, E(S))) \rightarrow \text{Hom}_R(P, \text{Hom}_R(A, E(S))) \rightarrow \text{Hom}_R(K, \text{Hom}_R(A, E(S))) \rightarrow 0.$$

By [1, Proposition 20.6, Proposition 20.7], this induces an exact sequence

$$0 \rightarrow \text{Hom}_R(M \otimes_R A, E(S)) \rightarrow \text{Hom}_R(P \otimes_R A, E(S)) \rightarrow \text{Hom}_R(K \otimes_R A, E(S)) \rightarrow 0.$$

Let $\mathcal{S}_0$ denote an irredundant set of representatives of the simple $R$-modules and let $C = \prod_{S \in \mathcal{S}_0} E(S)$. Then by [1, Corollary 18.16], $C$ is a cogenerator. And we have an exact sequence of $R$-modules

$$0 \rightarrow \text{Hom}_R(M \otimes_R A, C) \rightarrow \text{Hom}_R(P \otimes_R A, C) \rightarrow \text{Hom}_R(K \otimes_R A, C) \rightarrow 0.$$

So, by [1, Proposition 18.14], the sequence

$$0 \rightarrow K \otimes_R A \rightarrow P \otimes_R A \rightarrow M \otimes_R A \rightarrow 0$$

of $R$-modules is exact. This shows that $\text{Tor}_1^R(A, M) = 0$, as required. \qed

**Corollary 3.12.** Let $R$ be a commutative $n$-$V$-ring. Then it is an $n$-regular ring.

The following result is well-known.

**Corollary 3.13.** Let $R$ be a commutative $V$-ring. Then it is a regular ring.

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References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, 2nd ed., Graduate Texts in Mathematics, 13, Springer-Verlag, New York, 1992.
[2] D. Bennis, H. Bouzraa and A.-Q. Kaed, On $n$-copresented modules and $n$-coherent rings, Int. Electron. J. Algebra, 12 (2012), 162-174.
[3] D. L. Costa, Parameterizing families of non-noetherian rings, Comm. Algebra, 22(10) (1994), 3997-4011.
[4] V. A. Hiremath, Cofinitely generated and cofinitely related modules, Acta Math. Acad. Sci. Hungar., 39(1-3) (1982), 1-9.
[5] J. P. Jans, On co-noetherian rings, J. London Math. Soc., 1(2) (1969), 588-590.
[6] R. W. Miller and D. R. Turnidge, Factors of cofinitely generated injective modules, Comm. Algebra, 4(3) (1976), 233-243.
[7] R. Wisbauer, Foundations of Module and Ring Theory, Algebra, Logic and Applications, 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.
[8] W. M. Xue, On co-semihereditary rings, Sci. China Ser. A., 40(7) (1997), 673-679.
[9] W. M. Xue, On $n$-presented modules and almost excellent extensions, Comm. Algebra, 27(3) (1999), 1091-1102.
[10] Z. M. Zhu, On $n$-coherent rings, $n$-hereditary rings and $n$-regular rings, Bull. Iranian Math. Soc., 37(4) (2011), 251-267.
[11] Z. M. Zhu, $n$-cocoherent rings, $n$-cosemihereditary rings and $n$-V-rings, Bull. Iranian Math. Soc., 40(4) (2014), 809-822.
[12] Z. M. Zhu and J. L. Chen, FCP-projective modules and some rings, J. Zhejiang Univ. Sci. Ed., 37(2) (2010), 126-130.

Zhu Zhanmin
Department of Mathematics
College of Mathematics Physice and Information Engineering
Jiaxing University
Jiaxing, Zhejiang Province, 314001, P.R.China
e-mail: zhuzhanminjxu@hotmail.com