Sunyer-i-Balaguer’s Almost Elliptic Functions and Yosida’s Normal Functions

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Abstract

We study the properties of two classes of meromorphic functions in the complex plane. The first one is the class of almost elliptic functions in the sense of Sunyer-i-Balaguer. This is the class of meromorphic functions \( f \) such that the family \( \{ f(z + h) \}_{h \in \mathbb{C}} \) is normal with respect to the uniform convergence in the whole complex plane. Given two sequences of complex numbers, we provide sufficient conditions for them to be zeros and poles of some almost elliptic function. These conditions enable one to give (for the first time) explicit non-trivial examples of almost elliptic functions.

The second class was introduced by K.Yosida, who called it a class of normal functions of the first category. This is the class of meromorphic functions \( f \) such that the family \( \{ f(z + h) \}_{h \in \mathbb{C}} \) is normal with respect to the uniform convergence on compacta in the complex plane and no limit point of the family is a constant function. We give necessary and sufficient conditions for two sequences of complex numbers to be zeros and poles of some normal function of the first category and obtain a parametric representation for this class in terms of zeros and poles.

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According to M.Bessonoff [2] and M.Favard [7], a meromorphic function \( f \) in the complex plain \( \mathbb{C} \) is called almost elliptic if the following condition holds: for each \( \varepsilon > 0 \) and \( \delta > 0 \), there exists \( L < \infty \) such that every real or pure imaginary interval of the length \( L \) contains points \( \tau \) such that the inequality

\[
|f(z + \tau) - f(z)| < \varepsilon
\]

holds for all points \( z \in \mathbb{C} \) whose distance to the set of poles of \( f \) is larger than \( \delta \).

A subclass of such functions with a uniformly bounded number of poles in all discs of radius 1 was investigated by H.Yoshida [20].

Another definition of an almost elliptic function was suggested by F.Sunyer-i-Balaguer.

**Definition 1** ([17]). A meromorphic function \( f \neq \text{const} \) in the complex plain \( \mathbb{C} \) is called almost elliptic (we will say \( f \in \mathcal{AE} \), or \( f \) is an \( \mathcal{AE} \)-function), if for every \( \varepsilon > 0 \) there exists \( L < \infty \) such that every real or pure imaginary interval of the length \( L \) contains at most one \( \varepsilon \)-almost period of \( f \), i.e., a point \( \tau \) with the property

\[
\rho_S(f(z + \tau), f(z)) < \varepsilon \quad \text{for all} \quad z \in \mathbb{C},
\]

where \( \rho_S \) is the spherical metric in \( \mathbb{C} \).

In his seminal paper [17] Sunyer-i-Balaguer investigated the class \( \mathcal{AE} \). In particular, he proved that \( \alpha \)-points of every \( \mathcal{AE} \)-function have a uniform in a certain sense distribution. Moreover, they are equidistributed for all \( a \in \mathbb{C} \cup \{\infty\} \). He also simplified conditions describing location of \( \varepsilon \)-almost periods and proved Bochner’s criterion.
Theorem SB. The following conditions are equivalent
\( \alpha) f \in \mathcal{AE} \),
\( \beta) \) for each \( \varepsilon > 0 \) there exists \( L < \infty \) such that every disc \( \{ z \in \mathbb{C} : |z - c| < L \} \) contains an \( \varepsilon \)-almost period \( \tau \),
\( \gamma) \) for each sequence \( (h_n) \subset \mathbb{C} \) there exists a subsequence \( (h_{n'}) \) such that \( \rho_S(f(z + h_{n'}), f(z + h_{m'})) \rightarrow 0 \) as \( n', m' \rightarrow \infty \), uniformly in \( \mathbb{C} \).

Next, following to K.Yosida [19] (see also [20], [13], [5]), we introduce the definition of a normal function.

**Definition 2.** A meromorphic function \( f \) is called normal if for each sequence \( (h_n) \subset \mathbb{C} \) there exists a subsequence \( (h_{n'}) \) such that \( \rho_S(f(z + h_{n'}), f(z + h_{m'})) \rightarrow 0 \) as \( n', m' \rightarrow \infty \) uniformly on compacta in \( \mathbb{C} \). A normal meromorphic function \( f \) is of the first category \((f \in \mathcal{N}_1, \) or \( f \) is an \( \mathcal{N}_1 \)-function\)) if the family \( \{ f(z + h) \}_{h \in \mathbb{C}} \) has no constant functions as limit points.

A typical example of \( \mathcal{N}_1 \)-function is \( 1/z + E(z) \), where \( E \) is an arbitrary elliptic function.

The class of normal meromorphic functions is well known and has proved to be useful in complex analysis, and in particular, in the Nevanlinna theory (see [19], [20], [5], [12], [13], [21]).

However, until now, no description of zeros and poles of either normal or \( \mathcal{N}_1 \)-functions was known.

The class \( \mathcal{AE} \) forms a natural subclass of the class of normal functions in \( \mathbb{C} \), moreover, \( \mathcal{AE} \subset \mathcal{N}_1 \). Yet, there were no examples of \( \mathcal{AE} \)-functions except for usual elliptic ones. Note that an appropriate description of the class of normal functions in \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) (i.e., meromorphic functions \( f \) such that the family \( \{ f(\lambda z) : \lambda \in \mathbb{C}^* \} \) is normal in \( \mathbb{C}^* \)) was obtained by A.Ostrowski [15] (see also [14] and [5]).

This paper is organized as follows.

In §1, we prove main properties of \( \mathcal{N}_1 \)-functions and \( \mathcal{AE} \)-functions. Some of these properties are new, others were described in [5], [17], [20]. For reader’s convenience, below we will give complete proofs of these results.

In §2, we obtain necessary and sufficient conditions for a pair of discrete sets with multiplicities (in our definition, a divisor) to be the zero set and the pole set, respectively, of an \( \mathcal{N}_1 \)-function. We also show that any \( \mathcal{N}_1 \)-function is a conditionally convergent meromorphic Weierstrass product of genus 1.

In §3, we introduce a general notion of an almost periodic mapping. Following [16] and [8], we introduce the notion of an almost periodic divisor. We prove that each \( \mathcal{AE} \)-function has an almost periodic divisor. Next, under an additional assumption on the regularity of indexing of the divisor, we investigate properties of \( \arg f \).

In §4, we find necessary and sufficient conditions for an almost periodic divisor with a regular indexing to be a divisor of an \( \mathcal{AE} \)-function \( f \). Moreover, if this is the case, then \( f \) is a conditionally convergent meromorphic Weierstrass product of genus 0.

In §5, we construct two examples of \( \mathcal{AE} \)-functions, which are not elliptic.

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§1. Main properties of \( \mathcal{N}_1 \)-functions and \( \mathcal{AE} \)-functions

Let \( \mathbb{Z} \) be the set of all integers. A mapping \( D : \mathbb{C} \rightarrow \mathbb{Z} \) is a *divisor* in the complex plane \( \mathbb{C} \) if \( \text{supp} D \) is a discrete set; the divisor is *positive* if \( D(\mathbb{C}) \subset \mathbb{Z}^+ \cup \{0\} \).

\(^1\)See also Proposition 4 below.
Let \((a_n)\) be a sequence of points from the set \(D^{-1}\{k \in \mathbb{Z} : k > 0\}\), in which every point \(a\) occurs \(D(a)\) times, and \((b_n)\) be a sequence of points from the set \(D^{-1}\{k \in \mathbb{Z} : k < 0\}\), in which every point \(b\) occurs \(−D(b)\) times. We will write a divisor \(D\) as \(((a_n), (b_n))\). Also, \(((a_n), \emptyset)\) means a positive divisor. Furthermore, \(D = ((a_n), (b_n))\) is the divisor of a meromorphic function \(f\) if \(\{a_n\}\) is the zero set of \(f\), \(\{b_n\}\) is the pole set of \(f\), \(D(a_n)\) are multiplicities of the zeros \(a_n\), and \(−D(b_n)\) are multiplicities of the poles \(b_n\). By \(\mu_D\) we denote the discrete signed measure with masses \(D(a_n)\) at the points \(a_n\) and negative masses \(D(b_n)\) at the points \(b_n\). Let \(m_2(E)\) be the two-dimensional Lebesgue measure of a set \(E \subset \mathbb{C}\), \(m_1(E)\) be the Hausdorff 1-dimensional measure of \(E\). If \(E\) is a rectifiable curve, then \(m_1(E)\) is just the length of \(E\). By \(B(c, R)\) denote the disc \(\{z \in \mathbb{C} : |z − c| < R\}\), by \(\text{card}A\) denote the number of points in a discrete set \(A\), and by \(C\) without indices denote any constant depending only on the divisor \(D\) or the function \(f\).

We begin with some simple properties of \(\mathcal{N}_1\)-functions and \(\mathcal{AE}\)-functions (for \(\mathcal{AE}\)-functions see [17]).

It is easy to check that if \(f\) is \(\mathcal{N}_1\)-function and for a sequence \(\{h_n\}\)
\[
\rho_S(f(z + h_n), f(z + h_m)) \to 0 \quad \text{as} \quad n, m \to \infty
\]
uniformly on compacta in \(\mathbb{C}\), then there exists a function \(g \in \mathcal{N}_1\) such that
\[
\rho_S(f(z + h_n), g(z)) \to 0 \quad \text{as} \quad n \to \infty
\]
uniformly on compacta in \(\mathbb{C}\). Moreover, the classes \(\mathcal{N}_1\) and \(\mathcal{AE}\) are closed with respect to the uniform convergence in \(\mathbb{C}\). It is also clear that every \(\mathcal{N}_1\)-function is a uniformly continuous map of \(\mathbb{C}\) into the Riemann sphere. Next, a composition of a rational function with an \(\mathcal{N}_1\)-function (\(\mathcal{AE}\)-function) is the \(\mathcal{N}_1\)-function (\(\mathcal{AE}\)-function).

**Theorem 1** (20). The divisor of any \(\mathcal{N}_1\)-function \(f\) satisfies the separation condition
\[
\inf_{n,k} |a_n − b_k| = \delta_0 > 0. \quad (1)
\]

**Proof.** If \(f(a_n) = 0\), then \(|f(z)| < 1\) for \(z \in B(a_n, \delta_0)\), where \(\delta_0\) does not depend on \(n\).

**Theorem 2** (20). Suppose \(D = ((a_n), (b_n))\) is the divisor of an \(\mathcal{N}_1\)-function \(f\), and \(\delta > 0\) is an arbitrary real number. Then there exists \(C_\delta < \infty\) such that
\[
|f(z)| \leq C_\delta \quad \forall z \notin \bigcup_n B(b_n, \delta), \quad (2)
\]
\[
|f(z)| \geq 1/C_\delta \quad \forall z \notin \bigcup_n B(a_n, \delta), \quad (3)
\]
and
\[
|f'(z)/f(z)| \leq C_\delta \quad \forall z \notin \bigcup_n B(b_n, \delta) \cup \bigcup_n B(a_n, \delta). \quad (4)
\]

**Proof.** Suppose (3) is false. Take a sequence \((w_k)\) such that \(w_k \notin \bigcup_n B(a_n, \delta)\) for all \(k\) and \(f(w_k) \to 0\) as \(k \to \infty\). Then there is a subsequence \((w_{k'})\) such that \(\rho_S(f(z + w_{k'}), g(z)) \to 0\) uniformly on compacta in \(\mathbb{C}\). In particular, \(g(0) = 0\). Now Hurwitz’ Theorem leads to a contradiction. In the same way we obtain (2). Using (2) and (3) for \(\delta/2\), we get (4).

**Theorem 3** (20). Let \(D = ((a_n), (b_n))\) be the divisor of an \(\mathcal{N}_1\)-function \(f\), and \(|\mu_D|\) be the variation of the measure \(\mu_D\). Then
\[
|\mu_D|(B(c, 1)) = \text{card}\{n : a_n \in B(c, 1)\} + \text{card}\{n : b_n \in B(c, 1)\} < C_0 \quad \forall c \in \mathbb{C}. \quad (5)
\]
Proof. If \( \text{card}\{n : a_n \in B(c_k, \delta_0/3)\} \to \infty \) as \( k \to \infty \), then for a subsequence \((c_{k'})\) we get \( \rho(f(z + c_{k'}), g(z)) \to 0 \) uniformly on compacta in \( \mathbb{C} \), where \( g \) is a nonzero meromorphic function. In view of (1) and (2), we get \( |f(z + c_k)| < C \) for \( |z| > \delta_0/3 \), hence \( |f(z + c_{k'}) - g(z)| \to 0 \) in this disc. Then Hurwitz’ theorem leads to a contradiction. Therefore, \( \text{card}\{n : a_n \in B(c, \delta_0/3)\} < C \) for all \( c \in \mathbb{C} \). The same is valid for poles of \( f \). Thus, we obtain (5).

Corollary 1. Suppose \( D \) is a divisor with property (5); then

\[
|\mu_D|(E) \leq C_0 S(E), \quad \text{for every set } E \subset \mathbb{C}, \tag{6}
\]

where \( S(E) \) is the number of closed discs of radius 1 covering \( E \),

\[
|\mu_D|(B(c, r)) < Cr^2, \quad \forall c \in \mathbb{C}, \quad r > 1, \tag{7}
\]

and

\[
\int_{r<|w-c|} |w-c|^{-k}d|\mu_D|(w) < Cr^{-(k-2)}, \quad \forall c \in \mathbb{C}, \quad r > 1, \quad k > 2. \tag{8}
\]

Proof. Clearly, (7) follows from (6), and implies (8).

Theorem 2, Proposition 1, and Liouville’s Theorem yield

Corollary 2. Every entire \( N_1 \)-function is a constant.

Corollary 3. Suppose \( f_1, f_2 \) are \( N_1 \)-functions with the same divisor; then \( f_1 = K f_2 \) with a constant \( K \in \mathbb{C} \).

Note that Theorem 2, Theorem 3 and Corollaries 1 – 3 for \( \mathcal{A} \mathcal{E} \)-functions were proved in [17].

The following simple proposition are needed for the sequel.

Proposition 1. Suppose \( D \) is a divisor with property (5) and \( \delta < 1/(2C_0) \). Then the diameter of every connected component \( A \) of the set \( A(\delta) = (\bigcup_n B(a_n, \delta)) \cup (\bigcup_n B(b_n, \delta)) \) is at most \( 2\delta C_0 \).

Proof. It follows from (5) that any circle \( \{z : |z - c| = 1\}, c \in A \), has no common point with \( A \). Therefore the disc \( B(c, 1) \) contains \( A \). The statement now follows from (5).

Theorem 4 ([19]). For any \( f \in N_1 \) with a divisor \( D = ((a_n), (b_n)) \) there is \( R_0 \) such that

\[
B(w, R_0) \cap \{a_n\}_{n \in \mathbb{N}} \neq \emptyset, \quad B(w, R_0) \cap \{b_n\}_{n \in \mathbb{N}} \neq \emptyset, \quad \forall w \in \mathbb{C}. \tag{9}
\]

Proof. Let \( B(w_k, R_k) \) be discs without poles of \( f \) such that \( R_k \to \infty \). Using (2) with \( \delta = 1 \), we see that \( |f(w_k + z)| < C \) for \( |z| < R_k - 1, k = 1, \ldots \). Furthermore, \( f(z + w_k') \to g(z) \) uniformly on compacta in \( \mathbb{C} \) for a subsequence \( (w_k') \subset (w_k') \). Hence we get \( g(z) \equiv \text{const} \), which is impossible. Since \( 1/f \in N_1 \), we obtain the similar result for zeros of \( f \).

Theorem 5 ([20]). A meromorphic function \( f \) is an \( N_1 \)-function if and only if the following conditions are fulfilled:

a) for any \( \delta > 0 \)

\[
1/C_\delta \leq |f(z)| \leq C_\delta, \quad \forall z \notin A(\delta) = \bigcup_n B(a_n, \delta) \cup \bigcup_n B(b_n, \delta), \tag{10}
\]

b) zeros and poles of \( f \) satisfies (1), (3), and (4).

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Proof. For any $\mathcal{N}_1$-function $f$ properties a) and b) follows from Theorems 1, 3, 2, and 4.

Let a meromorphic function $f$ satisfy a) and b). If $\delta_0$ is the same as in (1), then any connected component $A$ of the set $A(\delta_0/3)$ can not contain zeros and poles simultaneously. If it does not contain poles of $f$, then inequality (10) and the Maximum Modulus Principle yield $|f(z)| < C$ for $z \in A$. Similarly, if $A$ does not contain zeros of $f$, then $|f(z)| > 1/C$ for $z \in A$. Take a disc $B(z_0, \delta_0/3)$. We see that for each sequence $(w_k)$ there is a subsequence $(w_{k'})$ such that at points of the disc either $|f(z + w_{k'})| < C$ for all $k'$, or $|1/f(z + w_{k'})| < C$ for all $k'$. Hence, in both cases there is a subsequence $(w_{k''}) \subset (w_{k'})$ and a meromorphic function $g(z)$ in the disc $B(z_0, \delta_0/3)$ such that uniformly in $z \in B(z_0, \delta_0/4)$

$$
\rho_S(f(z + w_{k''}), g(z)) \to 0 \quad \text{as} \quad k'' \to \infty.
$$

Let $\{B(z_n, \delta_0/4)\}$ be a denumerable covering of $\mathbb{C}$. Using the diagonal procedure, we obtain the subsequence $(w_{k''}) \subset (w_{k'})$ such that (11) holds uniformly on compacta $\in \mathbb{C}$.

In view of (9), we get

$$
\rho_S(f(z + w_{k''}), g(z)) \to 0 \quad \text{as} \quad k'' \to \infty.
$$

Let $\rho_S(f(z + w_{k''}), g(z)) \to 0$ as $k'' \to \infty$,

uniformly on compacta in $\mathbb{C}$. Let $U$ be the union of the discs $B(b_j, \delta)$ over all poles $b_j$ of $g_1$ and $g_2$. Using Theorem 2, we obtain that uniformly on compacta in $\mathbb{C} \setminus U$

$$
(f_1(z + w_{k''}) - g_1(z) \to 0, \quad f_2(z + w_{k''}) - g_2(z) \to 0,
$$

$$
(f_1f_2)(z + w_{k''}) - (g_1g_2)(z) \to 0, \quad k'' \to \infty.
$$

Suppose that the distances between zeros and poles of the function $f_1f_2$ are at least $\varepsilon$. Taking into account Proposition 1, we see that for sufficiently small $\delta$, the diameter of any connected component $A$ of the set $U$ is less than $\varepsilon$. Hence, $A$ does not contain simultaneously poles and zeros of the function $(f_1f_2)(z + w_{k''})$. If $A$ does not contain poles of $(f_1f_2)(z + w_{k''})$ for a subsequence $(w_{k''}) \subset (w_{k'})$, then the Maximum Modulus Principle and (12) imply the convergence of the functions $(f_1f_2)(z + w_{k''})$ to $(g_1g_2)(z)$ uniformly in $z \in A$. If $A$ does not contain zeros of $(f_1f_2)(z + w_{k''})$, then the same argument shows that

$$
\rho_S(f_1f_2(z + w_{k''}), g_1g_2(z)) \to 0 \quad \text{as} \quad k'' \to \infty
$$

uniformly in $z \in A$. Consequently,

$$
\rho_S(f_1f_2(z + w_{k''}), g_1g_2(z)) \to 0 \quad \text{as} \quad k'' \to \infty
$$

uniformly on compacta in $\mathbb{C}$. In view of (9), we get $g_1g_2 \neq \text{const}$ and $f_1f_2 \in \mathcal{N}_1$. Likewise, if $f_1, f_2 \in \mathcal{A}\mathcal{E}$ and the divisor of $f_1f_2$ satisfies (11) and (9), then $f_1f_2 \in \mathcal{A}\mathcal{E}$. 

Theorem 6. If a meromorphic function $f$ satisfies (11) for any $\delta > 0$, and its divisor satisfies (7) and (5), then $f$ is a normal function.

Theorem 7. The product of $f_1, f_2 \in \mathcal{N}_1$ is a $\mathcal{N}_1$-function if and only if the divisor of the function $f_1f_2$ satisfies (11) and (5). A similar assertion holds for the class $\mathcal{A}\mathcal{E}$.

Proof. Necessity follows from Theorems 1 and 4. To prove sufficiency, take a sequence $(w_k) \subset \mathbb{C}$. Then there exist functions $g_1, g_2 \in \mathcal{N}_1$ and a subsequence $(w_{k'})$ such that

$$
\rho_S(f_1(z + w_{k'}), g_1(z)) \to 0, \quad \rho_S(f_2(z + w_{k'}), g_2(z)) \to 0 \quad \text{as} \quad k' \to \infty,
$$

uniformly on compacta in $\mathbb{C}$. Let $U$ be the union of the discs $B(b_j, \delta)$ over all poles $b_j$ of $g_1$ and $g_2$. Using Theorem 2, we obtain that uniformly on compacta in $\mathbb{C} \setminus U$

$$
(f_1(z + w_{k'}) - g_1(z) \to 0, \quad f_2(z + w_{k'}) - g_2(z) \to 0,
$$

$$
(f_1f_2)(z + w_{k'}) - (g_1g_2)(z) \to 0, \quad k' \to \infty.
$$

Suppose that the distances between zeros and poles of the function $f_1f_2$ are at least $\varepsilon$. Taking into account Proposition 1, we see that for sufficiently small $\delta$, the diameter of any connected component $A$ of the set $U$ is less than $\varepsilon$. Hence, $A$ does not contain simultaneously poles and zeros of the function $(f_1f_2)(z + w_{k'})$. If $A$ does not contain poles of $(f_1f_2)(z + w_{k'})$ for a subsequence $(w_{k'}) \subset (w_{k'})$, then the Maximum Modulus Principle and (12) imply the convergence of the functions $(f_1f_2)(z + w_{k'})$ to $(g_1g_2)(z)$ uniformly in $z \in A$. If $A$ does not contain zeros of $(f_1f_2)(z + w_{k'})$, then the same argument shows that

$$
1/(f_1f_2)(z + w_{k'}) - 1/(g_1g_2)(z) \to 0 \quad \text{as} \quad k' \to \infty
$$

uniformly in $z \in A$. Consequently,

$$
\rho_S(f_1f_2(z + w_{k'}), g_1g_2(z)) \to 0 \quad \text{as} \quad k' \to \infty
$$

uniformly on compacta in $\mathbb{C}$. In view of (9), we get $g_1g_2 \neq \text{const}$ and $f_1f_2 \in \mathcal{N}_1$. Likewise, if $f_1, f_2 \in \mathcal{A}\mathcal{E}$ and the divisor of $f_1f_2$ satisfies (11) and (9), then $f_1f_2 \in \mathcal{A}\mathcal{E}$. 

Theorem 8 ([5]). For any $\mathcal{N}_1$-function $f$ there is a constant $C < \infty$ such that

$$
\int_0^{2\pi} |\log |f(c + re^{i\theta})||d\theta < C,
$$

for all $c \in \mathbb{C}$ and $r > 1$. 

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Proof. Let \( D = ((a_n), (b_n)) \) be the divisor of \( f \), and \( A_1, \ldots, A_N \) the connected components of the set \( A(\delta) = \bigcup A(i) B(a_n, \delta) \cup \bigcup B(b_n, \delta) \), which have nonempty intersection with the circle \( z = c + r e^{i\varphi} \). Put \( \delta = \min \{ \delta_j/(2C_0), 1/(2C_0) \} \). By Proposition 11 it follows that \( \text{diam} A_j < \min \{ 1, \delta_0 \} \) for every \( j = 1, \ldots, N \). From (11) it follows that each \( A_j \) does not contain poles and zeros of \( f \) simultaneously. From (6) it also follows that the number \( k_j \) of zeros or poles in the set \( A_j \) does not exceed \( C_0 \). Since the annulus \( r - 1 < |w| < r + 1 \) is covered by \( 6\pi r \) discs of radius 1, we have

\[
\sum_{j=1}^{N} k_j < 6\pi C_0 r. \tag{14}
\]

Furthermore, using (3) and (2), we get

\[
\int_{0}^{2\pi} |\log |f(c + r e^{i\varphi})||d\theta \leq \sum_{j=1}^{N} \int_{E_j} |\log |f(c + r e^{i\varphi})||d\theta + 2\pi \log C, \tag{15}
\]

where \( E_j = \{ \theta : c + r e^{i\varphi} \in A_j \} \). Suppose a component \( A_j \) contains zeros of \( f \). Take \( P_j(z) = \delta^{-k_j} \prod_{a_n \in A_j} (z - a_n) \). Since \( \text{diam} A_j < \delta_0 \), we get

\[
1 \leq |P_j(z)| \leq (\text{diam} A_j/\delta)^{k_j} < (2C_0)^{C_0} \quad \forall z \in \partial A_j.
\]

Therefore, \( |\log |f(z)/P_j(z)|| \leq C \) for \( z \in \partial A_j \). The Maximum Modulus Principle yields the same inequality for all \( z \in A_j \). Hence,

\[
\int_{E_j} |\log |f(c + r e^{i\varphi})||d\theta \leq \int_{E_j} |\log |P_j(c + r e^{i\varphi})||d\theta + C m_1(E_j). \tag{16}
\]

Next, we have

\[
\int_{E_j} \log^+ |P_j(c + r e^{i\varphi})|d\theta \leq \sum_{a_n \in A_j} \int_{|r e^{i\varphi} - a_n| < \delta} \log^+ \frac{\delta}{r e^{i\varphi} + c - a_n} d\theta. \tag{17}
\]

and

\[
\int_{E_j} \log^+ \left| \frac{1}{P_j(c + r e^{i\varphi})} \right| d\theta \leq \sum_{a_n \in A_j} \int_{|r e^{i\varphi} - a_n| < \delta} \log^+ \frac{\delta}{r e^{i\varphi} + c - a_n} d\theta. \tag{18}
\]

Note that for every \( te^{i\varphi} \in \mathbb{C} \), \( t \geq 0 \), we have

\[
\int_{\theta:|r e^{i\varphi} - te^{i\varphi}| < \delta} \log^+ \frac{\delta}{r e^{i\varphi} - te^{i\varphi}} d\theta \leq \int_{|\theta - \varphi| < \pi \delta/2r} \log^+ \frac{\delta/r}{(\theta - \varphi)^2/2\pi} d\theta < \delta \pi/2r.
\]

Combining (16), (17), and (18), we obtain

\[
\int_{E_j} |\log^+ |f(c + r e^{i\varphi})||d\theta < C (m_1(E_j) + k_j/r).
\]

A similar bound is valid for \( A_j \) containing poles of \( f \). Therefore, (15) and (14) imply (13).

\[\blacksquare\]

**Theorem 9.** Suppose \( f \in \mathcal{N}_1 \) has the divisor \( D = ((a_n), (b_n)) \); then

\[
\left| \int_0^r \text{card} \{ n : |a_n| < t \} - \text{card} \{ n : |b_n| < t \} \right| dt < C.
\]

The constant \( C \) is the same for all \( r > 1 \) and all shifts of \( f \).
Theorem 10. Suppose \( f \in \mathcal{N}_1 \) has the divisor \( D = ((a_n), (b_n)) \); then
\[
\left| \frac{1}{r} \int_1^r \left( \sum_{n: |a_n| < t} a_n - \sum_{n: |b_n| < t} b_n \right) \frac{dt}{t} \right| < C \quad \forall r > 1. \tag{19}
\]
The constant \( C \) is the same for all shifts of \( f \).

Proof. Consider the distribution \( \Delta(u \log |f(w)|) \). Since the function \( (\log |f(w)|)_u \) is locally integrable over \( \mathbb{C} \), we see that the function \( u \log |f(w)| \) up to a harmonic function is a logarithmic potential of the measure \( u \cdot 2\pi \mu_D(w) + 2(\log |f(w)|)_u m_2(w) \) in the disc \( B(0, R) \). Applying Jensen–Privalov’s formula for the annulus \( 1 \leq |w| \leq r \), we get
\[
\int_1^r \left[ \int_{B(0,t)} u \, d\mu_D(w) + \int_{B(0,t)} 2(\log |f(w)|)_u \, dm_2 \right] \frac{dt}{t}.
\]
Note that
\[
\int_{B(0,t)} (\log |f(w)|)_u \, dm_2(w) = \int_{-1}^t (\log |f(\sqrt{t^2 - v^2} + iv)| - \log |f(-\sqrt{t^2 - v^2} + iv)|) \, dv
\]
\[
= \int_0^{2\pi} \log |f(t \cos \theta + it \sin \theta)| t \cos \theta \, d\theta.
\]
Taking into account (13), we obtain
\[
\left| \frac{1}{r} \int_1^r \left[ \int_{B(0,t)} u \, d\mu_D(w) \right] \frac{dt}{t} \right| < C.
\]
The same bound is valid for the measures \( v \cdot \mu_D(w) \) and \( (u + iv) \cdot \mu_D(w) \). Hence, we obtain (19). Obviously, the constant \( C \) does not depend on shifts of \( f \).

Theorem 11. Let \( D = ((a_n), (b_n)) \) be the divisor of an \( \mathcal{N}_1 \)-function \( f \). Then for any simply connected bounded domain \( E \subset \mathbb{C} \) such that \( m_1(\partial E) < \infty \), and any holomorphic function \( g \) in \( E \) we have
\[
\left| \int_E g(w) \, d\mu_D(w) \right| \leq C \sup_E |g(w)|(m_1(\partial E) + 1). \tag{20}
\]
In particular, putting \( g(w) \equiv 1 \), we obtain
\[
|\mu_D(E)| \leq C(m_1(\partial E) + 1). \tag{21}
\]
Next, for any \( k \in \mathbb{Z} \) and \( 1 < r < R < \infty \)
\[
\int_{|w| < R} w^k \, d\mu_D(w) < C(r^{k+1} + R^{k+1}). \tag{22}
\]
The constants in (20), (21), and (22) do not depend on \( g \) and shifts of \( f \).
We need the following simple lemma.

**Lemma 1.** Suppose a compact set \( F \subset \mathbb{C} \) consists of \( M \) connected components and \( m_1(F) < \infty \). Then there are at most \( M(6m_1(F)+1) \) discs \( B(z_k, 1) \) such that \( \bigcup_k B(z_k, 1) \supset F \).

**Proof.** Let \( F \) be a connected set. If \( F \subset B(z_0, 1) \) for some \( z_0 \in F \), there is nothing to prove. Otherwise, by Besicovitch’s covering principle [1], also see [10, Lemma 3.2], there are discs \( B(z_k, 1), z_k \in F, k = 1, \ldots, S \), such that \( \bigcup_k B(z_k, 1) \supset F \) and every point of \( \mathbb{C} \) belongs to at most 6 discs. Clearly, for each disc we have \( m_1(F \cap B(z_k, 1)) \geq 1 \). Therefore, \( S \leq \sum_k m_1(F \cap B(z_k, 1)) \leq 6m_1(F) \).

For the general case, one should apply this result to each connected component of \( F \). \( \blacksquare \)

**Proof of the Theorem.** Using Lemma 1 take a covering of \( \partial E \) by discs \( B(z_k, 1), k = 1, \ldots, S, S \leq 6(m_1(\partial E) + 1) \). Put \( E_1 = E \setminus \bigcup_k B(z_k, 1) \). We have

\[
    m_1(\partial E_1) \leq 2\pi S \leq 12\pi(m_1(\partial E) + 1).
\]

Next, using (1), we get

\[
    |\mu_D|(E \setminus E_1) \leq 6C_0(m_1(\partial E) + 1).
\]  \hspace{1cm} (23)

Take \( \delta = 1/(2C_0) \). Let \( A_1, \ldots, A_N \) be all connected components of the set \( A(\delta) = \bigcup_n B(a_n, \delta) \cup \bigcup_n B(b_n, \delta) \) with nonempty intersection with \( \partial E_1 \). Clearly, the diameter of every connected component is less than 1. Therefore, \( \bigcup_j A_j \subset \bigcup_k B(z_k, 2) \). Using (1), we obtain

\[
    |\mu_D|(\bigcup_j A_j) \leq |\mu_D|(\bigcup_k B(z_k, 2)) \leq C(m_1(\partial E) + 1).
\]  \hspace{1cm} (24)

Moreover,

\[
    m_1(\bigcup_j \partial A_j) \leq 2\pi \delta |\mu_D|(\bigcup_j A_j) \leq C(m_1(\partial E) + 1).
\]

Put \( E_2 = E_1 \setminus \bigcup_j A_j \). Obviously, \( E_2 \) is a finite union of domains in \( \mathbb{C} \), and \( \partial E_2 \) is a finite union of circular arcs such that

\[
    m_1(\partial E_2) \leq m_1(\partial E_1) + m_1(\bigcup_j \partial A_j) \leq C(m_1(\partial E) + 1).
\]

In addition, \( \partial E_2 \cap A(\delta) = \emptyset \). Using (1), we obtain that right-hand side of the equality

\[
    \int_{E_2} g(w) \, d\mu_D(w) = \frac{1}{2\pi i} \int_{\partial E_2} \frac{g(w)f'(w)}{f(w)} \, dw
\]  \hspace{1cm} (25)

does not exceed \( C \max_{\partial E_2} |g(w)|(m_1(\partial E) + 1) \). This bound together with (23) and (21) proves (20).

To prove (22), we apply the above argument to the doubly-connected domain \( E = \{w : r < |w| \leq R\} \) with \( g(w) = w^k \). The integral in the right-hand side of (25) over \( \partial E_2 \cap \{w : |w| < r+3\} \) is bounded by \( Cr^{k+1} \) and the integral over \( \partial E_2 \cap \{w : |w| > R-3\} \) is bounded by \( CR^{k+1} \). \( \blacksquare \)

Putting \( k = -2 \) and \( k = -1 \) in (22), we obtain

**Corollary 4.** For every \( \mathcal{N}_1 \)-function with the divisor \( D = ((a_n), (b_n)) \) there is a finite limit

\[
    \lim_{r \to \infty} \sum_{1 \leq |a_n| < r} 1/a_n^2 - \sum_{1 \leq |b_n| < r} 1/b_n^2 = \lim_{r \to \infty} \sum_{1 \leq |a_n| < r} 1/a_n - \sum_{1 \leq |b_n| < r} 1/b_n < C.
\]

The constant \( C \) does not depend on shifts of \( D \).

\(^2m_1(E) \) is an additive function on Borel sets, see [6], p.191
Remark. In fact, we have proved all theorems of this section (with the exception of Theorems 3 and 5) for normal functions $f$ such that $g(z) \equiv 0$ and $g(z) \equiv \infty$ are not limiting functions for the family of shifts of $f$.

§2. Representation for $N_1$-functions and description of zero sets and pole sets

The main result of this section is

**Theorem 12.** A divisor $D = ((a_n), (b_n))$, $a_n \neq 0, b_n \neq 0$ for all $n$ is the divisor of $f \in N_1$, if and only if the following conditions are fulfilled:

1. $\inf_{n,k} |a_n - b_k| > 0$,
2. $\text{card} \{n : |a_n| \leq 1\} + \text{card} \{n : |b_n| \leq 1\} < C$, uniformly with respect to shifts of $f$,
3. there exists a radius $R < \infty$ such that every disc $B(c, R), c \in \mathbb{C}$, intersects with $\{a_n\}$ and $\{b_n\}$, simultaneously,
4. if $r > R$ for all $t \geq 0$, then
5. \[ \left| \frac{1}{r} \int_1^r \text{card} \{n : |a_n| < t\} - \text{card} \{n : |b_n| < t\} \frac{dt}{t} \right| < C \]
   uniformly with respect to shifts of $f$,
6. if $r > 1$ uniformly with respect to shifts of $f$,
7. there exists a finite limit
   \[ \lim_{r \to \infty} \sum_{1 \leq |a_n| < r} \frac{1}{a_n^2} - \sum_{1 \leq |b_n| < r} \frac{1}{b_n^2}. \]

Moreover, each $N_1$-function with the divisor $D = ((a_n), (b_n))$ up to a constant factor has the form

\[ f(z) = e^{\alpha z} \lim_{r \to \infty} \prod_{n: |a_n| < r} (1 - z/a_n)e^{z/a_n} \prod_{n: |b_n| < r} (1 - z/b_n)e^{z/b_n}. \] (26)

Here the limit exists uniformly on compacta in $\mathbb{C}$ and

\[ \alpha = \lim_{r \to \infty} \sum_{n: |b_n| < r} \left( \frac{1}{b_n} - \frac{b_n/|b_n|^2}{r^2} \right) - \sum_{n: |a_n| < r} \left( \frac{1}{a_n} - \frac{\overline{a_n}/|a_n|^2}{r^2} \right). \] (27)

**Proof.** For a function $f \in N_1$, conditions a), b), c), d), e), and f) follow from Theorems 1, 3, 4, 9, 10 and Corollary 4 respectively.

Let us prove sufficiency of these conditions. Put

\[ \alpha(r) = \int_0^R (1/|w|^2 - 1/r^2)\overline{w}d\mu_D(w), \quad \beta(t) = \int_{B(0,t)} \overline{w}d\mu_D(w), \quad \gamma(t) = \int_0^t (\beta(s)/s)ds. \]

Note that $\text{supp}\mu_D \cap B(0, t) = \emptyset$ for sufficiently small $t$. Integrating by parts, we get

\[ \alpha(R) - \alpha(r) = 2 \int_0^R \beta(t) dt = \frac{2\gamma(R)}{R^2} - \frac{2\gamma(r)}{r^2} + 4 \int_r^R \gamma(t) t^3 dt. \]

Using e), we get $|\gamma(t)| < Ct$. Therefore, the limit $\alpha = -\lim_{r \to \infty} \alpha(r)$ in (27) exists.

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Furthermore, condition b) yields (6) and (7). Conditions (7) and c) imply that both sequences \((a_n)\) and \((b_n)\) have the genus 2. Using f), we obtain that the function \(f\) in (26) is well defined. Let us show that it belongs to the class \(N_1\).

It follows from (27) that we can rewrite (26) in the form

\[
f(z) = \lim_{r \to \infty} f_r(z),
\]

where

\[
f_r(z) = \frac{\prod_{n:|a_n|<r} (1-z/a_n)e^{a_n z/r^2}}{\prod_{n:|b_n|<r} (1-z/b_n)e^{b_n z/r^2}}.
\]

Integrating by parts, we get

\[
\log |f_r(z)| = \int_{|w|<r} \log \left| \frac{z-w}{r} \right| d\mu_D(w) - \int_{|w|<r} \log \left| \frac{w}{r} \right| d\mu_D(w) + \text{Re} \left\{ \int_{|w|<r} \frac{z w}{r^2} d\mu_D(w) \right\}
\]

\[
= \int_0^r \mu_D(B(0,t)) - \mu_D(B(z,t)) \frac{dt}{t} + \int_{|w|<r,|w-z|\geq r} \log \left| \frac{w-z}{r} \right| d\mu_D(w) + \text{Re} \left\{ \int_{|w|<r} \frac{z w}{r^2} d\mu_D(w) \right\}.
\]

Let \(s(w)\) be a nonnegative number such that \(|w-s(w)z| = r\). If \(|w| < r\) and \(|w-z| \geq r\), then \(s(w) \in (0,1]\) and

\[
\log \left| \frac{w-z}{r} \right| = \text{Re} \frac{z(s(w) - 1)}{(w-s(w)z)} + O(|z|^2/r^2) = \text{Re} \frac{z w (s(w) - 1)}{r^2} + O(|z|^2/r^2) \quad \text{as} \quad r \to \infty.
\]

Since \(\mu_D\) satisfies (6), we see that

\[
|\mu_D|\{\{|w| < r, |w-z| \geq r\}\} \leq Cr(|z|+1).
\]

Hence,

\[
\int_{|w|<r,|w-z|\geq r} \log \left| \frac{w-z}{r} \right| d\mu_D(w) = \text{Re} \left\{ \frac{z}{r^2} \int_{|w|<r,|w-z|\geq r} (s(w) - 1)w d\mu_D(w) \right\} + o(1)
\]

as \(r \to \infty\). An application of the same argument shows that the sum of the last three integrals in (28) is equal to the real part of the sum

\[
\frac{z}{r^2} \left[ \int_{|w-z|<r} w d\mu_D(w) + \int_{|w|<r,|w-z|\geq r} s(w)w d\mu_D(w) - \int_{|w|\geq r,|w-z|<r} s(w)w d\mu_D(w) \right]
\]

up to the term \(o(1)\) as \(r \to \infty\). Note that every point \(w \in B(0,r) \setminus B(z,r)\) belongs to \(B(sz,r)\) only for \(s \in [0,s(w)]\) and every point \(w \in B(z,r) \setminus B(0,r)\) belongs to \(B(sz,r)\) only for \(s \in (s(w),1]\). Therefore, (29) is equal to the integral

\[
\frac{z}{r^2} \int_0^1 \int_{|w-sz|<r} w d\mu_D(w) \, ds.
\]
Hence we obtain

$$\log |f(z)| = \lim_{r \to \infty} \int_0^r \frac{\mu_D(B(0,t)) - \mu_D(B(z,t))}{t} dt + \Re \left( \int_0^1 \int_{|w-sz|<r} \frac{z\overline{w}}{r^2} d\mu_D(w) ds \right).$$

(30)

Take $\delta \in (0, 1)$ such that $0 \not\in A(\delta)$. In view of b), the integral

$$\int_0^1 \frac{\mu_D(B(0,t)) - \mu_D(B(z,t))}{t} dt$$

is uniformly bounded in $z \in \mathbb{C} \setminus A(\delta)$. Also, by d), the integral

$$\int_1^r \frac{\mu_D(B(0,t)) - \mu_D(B(z,t))}{t} dt$$

is uniformly bounded in $z \in \mathbb{C}$ and $r > 1$ as well. Furthermore, since bound e) does not depend on shifts of $\mu_D$, we get for all $z \in \mathbb{C}$ and $r < R$,

$$\left| \int_r^R \frac{1}{t} \int_{B(z,t)} w - z \ d\mu_D(w) \ dt \right| < C(r + R).$$

In view of d), we get

$$\left| \int_r^R \frac{1}{t} \int_{B(z,t)} d\mu_D(w) \ dt \right| < C.$$

Therefore,

$$\Re \left( \frac{z}{|z|} \int_r^R \frac{1}{t} \int_{B(z,t)} \overline{w} d\mu_D(w) \ dt \right) \leq \left| \int_r^R \frac{1}{t} \int_{B(z,t)} \overline{w} d\mu_D(w) \ dt \right| < C(r + R + |z|).$$

Replace $z$ by $sz$, $R$ by $r + |z|$, and integrate over $s$ from 0 to 1. We get

$$\frac{1}{|z|} \int_r^{r+|z|} \Re \left( \frac{z}{r} \int_0^1 \int_{B(sz,t)} \overline{w} d\mu_D(w) \ ds \right) dt < C(r + |z|).$$

Therefore, for some $r' \in (r, r + |z|)$

$$\Re \left( \frac{z}{r'} \int_0^1 \int_{B(sz,r')} \overline{w} d\mu_D(w) \ ds \right) < C(r + |z|).$$

Hence for a sequence $r' \to \infty$

$$\Re \left( \frac{z}{(r')^2} \int_0^1 \int_{|w-sz|<r'} \overline{w} d\mu_D(w) \ ds \right) \leq 2C.$$

Similarly, for some sequence $r'' \to \infty$

$$\int_0^1 \Re \left( \frac{z}{(r'')^2} \int_{|w-sz|<r''} \overline{w} d\mu_D(w) \ ds \right) \geq -2C.$$

Taking into account (30) and d), we get (10). Now, by Theorem 5 $f \in \mathcal{N}_1$. The last assertion of the theorem follows from Corollary 3.
Corollary 5. For every \( f \in \mathcal{N}_1 \) with the divisor \( D = ((a_n), (b_n)) \) and \( z_0 \neq a_n, z_0 \neq b_n \) for all \( n \) we have the representation

\[
f(z) = f(z_0)e^{\alpha(z-z_0)} \lim_{r \to \infty} \prod_{n:|a_n|<r} \frac{(z-a_n)/(z_0-a_n)e^{(z-z_0)/(a_n-z_0)}}{\prod_{n:|b_n|<r} \frac{(z-b_n)/(z_0-b_n)e^{(z-z_0)/(b_n-z_0)}}}.
\] (31)

Proof. Note that for \( |z_0| < r/4 \) and \( |z| < r/4 \) we have

\[
\left| \log \prod_{n:|a_n|<r} (1 - z/a_n) \prod_{n:|b_n+z_0|<r} (1 - z/b_n) \right|
\leq \sum_{r-|z_0| \leq |a_n| \leq r+|z_0|} |\log(1 - z/a_n) + z/a_n| + \sum_{r-|z_0| \leq |b_n| \leq r+|z_0|} |\log(1 - z/b_n) + z/b_n|
\leq \frac{C|z|^2}{(r - |z_0|)^2} \left[ \text{card}\{n : r - |z_0| \leq |a_n| \leq r + |z_0|\} + \text{card}\{n : r - |z_0| \leq |b_n| \leq r + |z_0|\} \right].
\]

It follows from (6) that \( |\mu_D|((\{ w : r - |z_0| < |w| < r + |z_0| \}) = O(r) \) as \( r \to \infty \). Hence the right-hand side of (32) tends to 0 as \( r \to \infty \) uniformly on compacta in \( \mathbb{C} \). Applying (26) with \( D = ((a_n-z_0), (b_n-z_0)) \) to \( f(z+z_0) \), we obtain (31). □

Using Theorem 6 instead of 5, we obtain the following result:

Theorem 13. Suppose a divisor \( D = ((a_n), (b_n)) \), \( a_n \neq 0, b_n \neq 0 \) for all \( n \), satisfies conditions a), b), d), e), and f). Then \( D \) is the divisor of the normal function (26).

§3. Special properties of almost elliptic functions

In what follows we need some properties of almost periodic mappings and divisors in the complex plain \( \mathbb{C} \).

Definition 3. A set \( E \subset \mathbb{C} \) is called relatively dense if there exists \( L < \infty \) such that every disc of radius \( L \) has a nonempty intersection with \( E \).

Definition 4. Let \( g \) be a continuous mapping from \( \mathbb{C} \) to a metric space \((Y, d)\). A number \( \tau \in \mathbb{C} \) is called an \( \varepsilon \)-almost period of \( g \) if

\[
d(g(z - \tau), g(z)) < \varepsilon \quad \text{for all} \quad z \in \mathbb{C}.
\] (33)

The mapping \( g \) is called almost periodic if for each \( \varepsilon > 0 \) the set of \( \varepsilon \)-almost periods of \( g \) is relatively dense in \( \mathbb{C} \).

The following results are well known for almost periodic functions in the real axis (see, for example, [3], [4]). One can easily carry over their proofs to our case.

Proposition 2. a) An almost periodic mapping is bounded and uniformly continuous,

\[ b) \text{if a sequence of almost periodic mappings converges uniformly in } \mathbb{C}, \text{then its limit is also an almost periodic mapping.} \]

Proposition 3. Suppose \( f : \mathbb{C} \to \mathbb{C} \) is almost periodic function such that \( \inf_{\mathbb{C}} |f(z)| > 0 \); then we have

\[
f(z) = e^{g(z)+i(\beta'x+\beta''y)}, \quad \beta', \beta'' \in \mathbb{R}, \quad z = x + iy,
\]

where \( g \) is an almost periodic function in \( \mathbb{C} \).

Furthermore, the following proposition is valid.
Proposition 4. Suppose $g : \mathbb{C} \to Y$ is a continuous mapping; then the following conditions are equivalent:

a) $g$ is almost periodic,

b) for each $\varepsilon > 0$ there exists $L < \infty$ such that every interval $(a, a + L)$ of the real axis and every interval $(ib, ib + L)$ of the imaginary axis contains a point $\tau$ satisfying (33),

c) for each sequence $(h_n) \subset \mathbb{C}$ there exists a subsequence $(h_{n'})$ such that $d(g(z + h_n), g(z + h_{n'})) \to 0$ as $n' \to \infty$ uniformly on $\mathbb{C}$,

In addition, if $(Y, d)$ is the space $\mathbb{C}$ with the Euclidean metric, then a) – c) are equivalent to the condition

d) there is a sequence of finite exponential sums

$$S_k(z) = \sum_j c_{j,k} e^{i(\lambda_{j,k} x + \lambda'_{j,k} y)}, \quad \lambda_{j,k}, \lambda'_{j,k} \in \mathbb{R}, \quad z = x + iy,$$

such that $S_k(z) - g(z) \to 0$ as $k \to \infty$ uniformly on $\mathbb{C}$.

Proof. The equivalence of a), c), and d) is well known for almost periodic functions in the real axis $\mathbb{R}$ (see, for example, [3], [4]). In the same way, one can easily prove a similar result in our case. Furthermore, the sum of two $\varepsilon$-almost periods is a $2\varepsilon$-almost period, hence b) implies that there is an $2\varepsilon$-almost period in every disc of radius $\sqrt{2}L$. Therefore, b) implies a). On the other hand, let $G : x \to g(x + iy)$ be the mapping from $\mathbb{R}$ to the space $\tilde{Y}$ of continuous bounded functions $r(y)$, $y \in \mathbb{R}$, with the distance $\tilde{d}(r_1, r_2) = \sup_{y \in \mathbb{R}} \tilde{d}(r_1(y), r_2(y))$. Suppose that a mapping $g$ satisfies condition c); then for each sequence $(h_n) \subset \mathbb{R}$ there is a subsequence $(h_{n'})$ such that $\tilde{d}(G(x + h_n), G(x + h_{n'})) \to 0$ as $n' \to \infty$ uniformly in $\mathbb{R}$. Consequently, $G$ is an almost periodic mapping. Hence for each $\varepsilon > 0$ there exists $L' < \infty$ such that every interval $(a, a + L') \subset \mathbb{R}$ contains a point $\tau$ with the property

$$\sup_{z \in \mathbb{C}} \tilde{d}(g(z + \tau), g(z)) = \sup_{x \in \mathbb{R}} \tilde{d}(G(x + \tau), G(x)) < \varepsilon.$$

For the same reason, there exists $L'' < \infty$ such that every interval $(ib, ib + L'')$ of the imaginary axis contains $\tau$ with property (33). Hence b) is valid for $L = \max\{L', L''\}$. ■

The class $\mathcal{AE}$ is just the set of all nonconstant meromorphic almost periodic mappings from $\mathbb{C}$ to the Riemann sphere. Hence we have just proved Theorem SB.

Definition 5. A number $\tau \in \mathbb{C}$ is an $\varepsilon$-almost period of a divisor $D = ((a_n), (b_n))$ if there exist bijections $\sigma : \mathbb{N} \to \mathbb{N}$ and $\sigma' : \mathbb{N} \to \mathbb{N}$ such that

$$|a_n + \tau - a_{\sigma(n)}| < \varepsilon, \quad |b_n + \tau - b_{\sigma'(n)}| < \varepsilon \quad \forall \quad n \in \mathbb{N}.$$ 

The divisor is almost periodic if for each $\varepsilon > 0$ there exists $L = L(\varepsilon) < \infty$ such that every disc $B(z, L)$ contains $\varepsilon$-almost period $\tau$. If, in addition, (35) holds with $\sigma(n) \equiv \sigma'(n)$, we say that the divisor $D$ is almost periodic with a regular indexing.

For the case of positive divisors, Definition 5 is very close to the definition of an almost periodic zero set in a strip (see [11], Appendix VI, [15], [16], and [8]).

Remark. Note that all previous definitions and statements are stable under any renumeration of $a_n$ and $b_n$. The same is true for the property of a divisor to be almost periodic, because such change means the replacing $D = ((a_n), (b_n))$ by $\tilde{D} = ((s(a_n)), (b_{\sigma'(n)}))$, where $s$, $s'$ are in general different bijections $\mathbb{N} \to \mathbb{N}$. However, it is not hard to see that in the general case the property of a divisor to have a regular indexing may violate, although it survives in the case $s = s'$.
**Proposition 5.** Suppose a divisor $D = ((a_n), (b_n))$ is almost periodic; then (35) holds. If, in addition, $D = ((a_n), (b_n))$ has a regular indexing, then

$$d_0 = \sup_n |a_n - b_n| < \infty. \quad (36)$$

For almost periodic positive divisors in a strip this assertion was obtained in [8].

**Proof.** There is $L < \infty$ such that any disc $B(c, L)$ contains an 1-almost period $\tau$ of the divisor $D$. If (35) holds for bijections $\sigma, \sigma'$, then for any $a_n \in B(c, 1)$ we get $a_{\sigma(n)} \in B(0, 2 + L)$. Hence,

$$\text{card}\{n : |a_n - c| \leq 1\} \leq \text{card}\{n : a_n \in B(0, L + 2)\}.$$  

In the same way, we bound $\text{card}\{n : |b_n - c| \leq 1\}$. Now suppose that the divisor $D$ has a regular indexing. Set

$$M = \max_{n : |a_n| < L + 1} |a_n - b_n|.$$  

For a term $a_{n'}$ take an 1-almost period $\tau \in B(-a_{n'}, L)$. Using (35) with $\varepsilon = 1$, we get $|a_{\sigma(n')}| < L + 1$. Hence, $|b_{n'} - a_{n'}| < |b_{\sigma(n')} - a_{\sigma(n')}| + 2 \leq M + 2$. 

**Proposition 6.** Let $D$ be an almost periodic divisor with a regular indexing; then for every sequence $(h_k) \subset \mathbb{C}$ there is a subsequence $(h'_k) \subset (h_k)$, a divisor $\tilde{D} = ((\tilde{a}_n) (\tilde{b}_n))$ with a regular indexing, and bijections $\tilde{\sigma}(k, \cdot) : \mathbb{N} \to \mathbb{N}$ such that

$$\sup_n |a_{\tilde{\sigma}(k,n)} + h'_k - \tilde{a}_n| \to 0, \quad \sup_n |b_{\tilde{\sigma}(k,n)} + h'_k - \tilde{b}_n| \to 0 \text{ as } k \to \infty. \quad (37)$$

**Proof.** Take $\varepsilon > 0$. By $E(\varepsilon/4)$ denote the union of discs $B(\tau, \varepsilon/4)$ over all $\varepsilon/4$-almost periods of $D$. It follows from Definition 5 that each set $h_k + E(\varepsilon/4)$ intersects with the disc $B(0, L(\varepsilon/4))$, moreover, the $m_2$-measure of the intersection is at least $\pi(\varepsilon/4)^2/4$. We get

$$m_2\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} [B(0, L(\varepsilon/4)) \cap (h_k + E(\varepsilon/4))]) > 0. \right.$$  

Hence there is a point $z'$, which belongs to every set $h'_k + E(\varepsilon/4)$ for a subsequence $(h'_k) \subset (h_k)$. Therefore for each two terms $h'_k, h'_s$ there exist $\varepsilon/4$-almost periods $\tau, \tau'$ such that $|h'_k - h'_s| - (\tau - \tau')| < \varepsilon/2$. Since $\tau - \tau'$ is an $\varepsilon/2$-almost period of $D$, we obtain

$$\sup_n |a_n + h'_k - a_{\sigma(k,s,n)} - h'_s| < \varepsilon, \quad \sup_n |b_n + h'_k - b_{\sigma(k,s,n)} - h'_s| < \varepsilon$$

for some bijections $\sigma(k, s, \cdot) : \mathbb{N} \to \mathbb{N}$. Using the diagonal process and passing on to a subsequence if necessary, we get

$$\sup_n |a_n + h'_k - a_{\sigma(k,s,n)} - h'_s| < 2^{-k}, \quad \sup_n |b_n + h'_k - b_{\sigma(k,s,n)} - h'_s| < 2^{-k} \quad \forall k \in \mathbb{N}, s > k. \quad \forall k \in \mathbb{N}, s > k.$$  

By definition, put

$$\tilde{\sigma}(1, \cdot) = \sigma(1, 2, \cdot), \quad \tilde{\sigma}(2, \cdot) = \sigma(2, 3, \cdot) \circ \sigma(1, 2, \cdot), \quad \tilde{\sigma}(3, \cdot) = \sigma(3, 4, \cdot) \circ \sigma(2, 3, \cdot) \circ \sigma(1, 2, \cdot), \ldots$$

Since

$$|a_{\tilde{\sigma}(k,n)} + h'_k - a_{\tilde{\sigma}(k+1,n)} - h'_{k+1}| < 2^{-k}, \quad |b_{\tilde{\sigma}(k,n)} + h'_k - b_{\tilde{\sigma}(k+1,n)} - h'_{k+1}| < 2^{-k} \quad \forall n, k \in \mathbb{N},$$

we see that there exist limits

$$\tilde{a}_n = \lim_{k \to \infty} (a_{\tilde{\sigma}(k,n)} + h'_k), \quad \tilde{b}_n = \lim_{k \to \infty} (b_{\tilde{\sigma}(k,n)} + h'_k), \quad \forall n \in \mathbb{N}.$$  

It can be easily checked that the divisor $\tilde{D} = ((\tilde{a}_n) (\tilde{b}_n))$ is almost periodic, has a regular indexing, and satisfies (37).
Theorem 14. The divisor $D$ of $f \in \mathcal{AE}$ is almost periodic. Moreover, for any $\varepsilon > 0$ there exists a relatively dense set of common $\varepsilon$-almost periods of $f$ and $D$.

Proof. Take $\varepsilon > 0$ and put $\delta = \min\{\delta_0, \varepsilon\}/C_0$, where $\delta_0$ is from (11), and $C_0$ from (5). Let $A$ be a connected component of $A(\delta)$, and $\bar{A}$ the union of $A$ and all bounded connected components of $C \setminus A$. Then $\text{diam} \bar{A} \leq \varepsilon$ and $\bar{A}$ does not contain zeros and poles of $f$ simultaneously. Now, by Proposition 5 $1/C \leq |f(z)| \leq C$ for $z \notin A(\delta)$. Clearly, there is $\eta = \eta(C)$ such that for any $\eta$-almost period $\tau$ of $f$ we obtain

$$1/(2C) \leq |f(z + \tau)| \leq 2C, \quad |f(z + \tau)/f(z) - 1| < 1/2 \quad \forall z \notin A(\delta).$$

Hence the increment of $\text{arg } f(z + \tau)$ along $\partial \bar{A}$ coincides with the increment of $\arg f(z)$, and the functions $f(z + \tau)$ and $f(z)$ have the same numbers of zeros (or poles) in $\bar{A}$. Consequently, there exist bijections $\sigma$ and $\sigma'$ of $\mathbb{N}$ to $\mathbb{N}$ such that (35) holds and $\tau$ is a common $\max\{\varepsilon, \eta\}$-almost period of $f$ and $D$.

Theorem 15. For each $\mathcal{AE}$-function with the divisor $D = ((a_n), (b_n))$ there is an indexing of zeros and poles with property (36).

Proof. The proof is based on a partition of the complex plane into quadrilaterals subordinating to $\varphi \in \mathcal{AE}$. The idea of partition belongs to F. Sunyer-i-Balaguer [17], but his construction contains a small inaccuracy (he does not consider the case when the projections of the zero set and the pole set to the real and imaginary axes are dense). We shall give a complete proof here.

Let $f$ be an $\mathcal{AE}$-function with the divisor $D = ((a_n), (b_n))$, $a_n \neq 0$, $b_n \neq 0$ for all $n$. Set $r = \min\{|a_n|/4, |b_n|/4, n = 1, 2, \ldots\}$, and take $\delta < \min\{r, \varepsilon_0, 1\}/(2C_0)$ such that

$$\delta \neq |\text{Re } a_n|, \delta \neq |\text{Re } b_n|, \delta \neq |\text{Im } a_n|, \delta \neq |\text{Im } b_n| \quad \forall n,$$

$$2\delta \neq |a_n - a_k|, 2\delta \neq |b_n - b_k|, 2\delta \neq |a_n - b_k| \quad \forall n, k.$$

Let $\bigcup_{k=1}^\infty A_k$ be the decomposition of the set $A(\delta) = \cup_k B(a_n, \delta) \cup \cup_k B(b_n, \delta)$ into connected components. Note that $\overline{A_k} \cap \overline{A_k'} = \emptyset$ for all $k \neq k'$ and any disc $B(a_n, \delta)$ or $B(b_n, \delta)$ is not tangent to the real or imaginary axis. Also, by Proposition 1 $\text{diam} A_k < \varepsilon_0$ for all $k$, therefore each $A_k$ does not contain zeros and poles simultaneously.

Furthermore, let $A_{k_1}$ be the component with the minimal index that intersects with the real axis $l$ and $(\alpha_1, \beta_1)$ be the minimal interval of $l$ containing this intersection. Replace in $l$ the interval $(\alpha_1, \beta_1)$ by a Jordan curve $L_1 \subset \partial A_{k_1}$ with the same endpoints. Note that the length of $L_1$ does not exceed $2\pi\delta \text{card}\{n : a_n \in \text{ or } b_n \in A_{k_1}\}$. Take the component $A_{k_2}$ with the minimal index that intersects with $l \setminus (\alpha_1, \beta_1)$ and repeat the above construction. Continuing like this, we obtain a Jordan curve $l_x \subset \{z : |y| < r\}$, $0 \in l_x$, such that $l_x \cap A(\delta) = \emptyset$. Note that if $A_k$ intersects with a segment $[\alpha, \beta]$ of real axis, then $A_k$ is contained in the rectangle $[\alpha - 1, \beta + 1] \times [-1, 1]$. In view of (6), the number of terms $a_n, b_n$ in this rectangle is at most $12 \max\{\beta - \alpha, 1\} C_0$, therefore the length of $l_x$ inside the rectangle $\{z : \alpha < x < \beta, |y| < r\}$ is at most $C' \max\{\beta - \alpha, 1\}$, where $C'$ depends on $\delta$ and $C_0$.

Similarly, we obtain a Jordan curve $l_y \subset \{z : |x| < r\}$, $0 \in l_y$, such that $l_y \cap A(\delta) = \emptyset$ and the length of $l_y$ inside any rectangle $\{z : |x| < r, \alpha < y < \beta\}$ is at most $C' \max\{\beta - \alpha, 1\}$.

In view of (10), there is $C_\delta < \infty$ such that

$$1/C_\delta < |f(z)| < C_\delta \quad \forall z \notin A(\delta).$$

Take $\varepsilon = \varepsilon(\delta)$ such that

$$1/(2C_\delta) \leq |f(z + \tau)| \leq 2C_\delta, \quad |f(z + \tau)/f(z) - 1| < 1/2,$$

(38)
whenever $\rho_2(f(z + \tau), f(z)) < 2\varepsilon$ and $z \notin A(\delta)$. Using Definition \[\text{take } L > 4r \text{ and pick up a sequence of } \varepsilon\text{-almost periods } (\tau_p)_{p \in \mathbb{Z}} \text{ in the real axis and another one } (i\tau'_q)_{q \in \mathbb{Z}} \text{ in the imaginary axis such that}

\[L < \tau_{p+1} - \tau_p < 3L, \quad L < \tau'_{q+1} - \tau'_q < 3L.\]

Since $B(0, 3r) \cap A(\delta) = \emptyset$, we see that (38) implies $B(\tau, 2r) \cap A(\delta) = \emptyset$. Taking into account embedding $l_x \subset \{z : |y| < r\}$, $l_y \subset \{z : |x| < r\}$, we obtain

\[B(\tau_p, 2r) \cap l_x = B(\tau_p, 2r) \cap \{z : \text{Im } z = 0\} \quad \forall p \in \mathbb{Z},
\]

\[B(i\tau'_q, 2r) \cap l_y = B(i\tau'_q, 2r) \cap \{z : \text{Re } z = 0\} \quad \forall q \in \mathbb{Z}.
\]

Hence, the bounded domain $R_{p,q}$ formed by the lines $l_x + i\tau'_q$, $l_y + \tau_p$, $l_x + i\tau'_{q+1}$, $l_y + \tau_{p+1}$, is a quadrilateral. Using (38), we obtain that the difference between the increments of continuous branches of $\text{arg } f(z)$ along opposite sides of each quadrilateral $R_{p,q}$ is less than $\pi$. Now the Argument Principle yields the equality

\[\text{card}\{n : a_n \in R_{p,q}\} = \text{card}\{n : b_n \in R_{p,q}\}.
\]

for all $p, q \in \mathbb{Z}$. Since $\text{diam}R_{p,q} < 15L$ for all $p, q$, we obtain (36). \[\blacksquare\]

By definition, put

\[G_n(z) = \log \left(\frac{b_n(a_n - z)}{a_n(b_n - z)}\right), \quad n \in \mathbb{N}, \quad a_n, b_n \neq 0.
\]

The function $G_n(z)$ is well defined in the complex plane with discontinuity in $[a_n, b_n]$ under the condition $G_n(0) = 0$. Also, put $\text{Im } G_n(z) = \pi + \text{Im } G_n(\infty)$ for $z \in [a_n, b_n]$. Suppose $f$ is an $\mathcal{AE}$-function with a divisor $D = ((a_n), (b_n))$, $a_n \neq 0$, $b_n \neq 0$. By Theorem 12, $f$ has the form (26). Using property (36), we obtain that the function

\[\text{arg } f(z) = \text{Im } \alpha z + \sum_n \left(\text{Im } G_n(z) + \text{Im } (z/a_n - z/b_n)\right)
\]

is well defined in the complex plane, and so is $\log f(z)$ outside zeros and poles of $f$.

**Proposition 7.** Suppose an $\mathcal{AE}$-function $f$ has the divisor $D = ((a_n), (b_n))$ with a regular indexing and property (36). Then there exists a constant $\beta \in \mathbb{C}$ such that

\[\text{arg } f(z) = \text{Im } (\beta z) + O(1).
\]

**Proof.** Let $\varphi(z)$ be a nonnegative smooth function in $\mathbb{C}$ such that $\varphi(z) = 1$ if $\text{dist}\{z, [0, 1]\} < 1/2$ and $\varphi(z) = 0$ if $\text{dist}\{z, [0, 1]\} > 1$. By definition, put

\[H(z) = \exp\left\{\sum_n \varphi\left(z - \frac{a_n}{b_n - a_n}\right) G_n(z)\right\}.
\]

Also, put $\delta = \min\{1/2, \varepsilon_0\}/(5C_0)$. Take $\varepsilon > 0$, and let $\tau$ be a common $\varepsilon$-almost period of $f$ and $D$. It follows from (14) and (36) that every $z \in \mathbb{C}$ belongs to supports of at most $C_0(2d_0 + 6)^2$ terms of the sum in (40). Besides, the divisor $D$ is almost periodic with a regular indexing. Therefore, $1/C < |H(z)| < C$ and $|H(z + \tau) - H(z)| < C\varepsilon$ for $z \in \mathbb{C} \setminus [A(\delta) \cup (A(\delta) - \tau)]$. In view of (10), we have $1/C\delta < |f(z)| < C\delta$ and $|f(z + \tau) - f(z)| < C\varepsilon$ as well. Hence,

\[1/C < \left|\frac{f(z)}{H(z)}\right| < C, \quad \left|\frac{f(z + \tau)}{H(z + \tau)} - \frac{f(z)}{H(z)}\right| < C\varepsilon, \quad \forall z \notin A(\delta) \cup (A(\delta) - \tau).
\]
Let $A$ be a connected component of $A(\delta)$. Since $\text{diam}A < \min\{1/2, \varepsilon_0\}$, we see that $A$ contains either zeros, or poles of $f$. For example, suppose that $A$ contains zeros of $f$. By definition, put 

$$H_A(z) = \exp \left\{ \sum_{n:a_n \in A} \varphi \left( \frac{z - a_n}{b_n - a_n} \right) G_n(z) \right\}.$$ 

We see that $H_A(z)$ is holomorphic and has the same zeros as $f(z)$ in $A$. On the other hand, $1/C < |H(z)/H_A(z)| < C$ for $z \in \overline{A}$. Applying the Maximum Modulus Principle to the function $f(z)/H_A(z)$, we get the bounds $\lvert \arg f(z) \rvert < \lvert \arg f(z) \rvert$ for $z \in A$, therefore, for all $z \in \mathbb{C}$.

Thus, the function $F(z)/H(z)$ is almost periodic in $\mathbb{C}$. Applying Proposition 3, we get 

$$f(z)/H(z) = \exp \{g(z) + i\text{Im} (\beta z)\} \quad \text{for some} \quad \beta \in \mathbb{C}.$$ 

Since the functions $g(z)$ and $\text{Im} [\sum_n \varphi(\frac{a_n - b_n}{a_n - b_n})G_n(z)]$ are uniformly bounded in $\mathbb{C}$, we obtain \([39]\). \hfill \blacksquare

For an arbitrary divisor $D$ with properties \([5]\) and \([36]\) denote by $\nu_D$ the discrete measure with the complex masses $a_n - b_n$ at the points $c_n = (a_n + b_n)/2$. Clearly,

$$|\nu_D| (\overline{B(c, 1)}) < C \quad \forall c \in \mathbb{C}. \quad (42)$$

**Proposition 8.** Suppose $f$ is an $\mathcal{AE}$-function, \([R_{p,q}]\) is the above partition of the plane into quadrilaterals, and the divisor $D = (a_n), (b_n)$ of $f$ has a regular indexing such that \(\{n: a_n \in R_{p,q}\} = \{n: b_n \in R_{p,q}\}\). Then for any simply connected bounded domain $E \subset \mathbb{C}$ we have 

$$|\nu_D(E) - \beta m_2(E)/2\pi| \leq C(m_1(\partial E) + 1). \quad (43)$$

**Proof.** Set 

$$I_1 = \{(p, q): \overline{R}_{p,q} \cap E \neq \emptyset\}, \quad I_2 = \{(p, q): \overline{R}_{p,q} \subset E\}, \quad R = \cup_{(p,q) \in I_2} \overline{R}_{p,q}.$$ 

It follows from the definition of $R_{p,q}$ that for every $p, q$ there is $z_{p,q}$ such that $B(z_{p,q}, 1) \subset G_{p,q} \subset B(z_{p,q}, 4L)$ and each point $z \in \mathbb{C}$ is contained in at most $C(L)$ discs $B(z_{p,q}, 5L)$. Since $\partial E$ is connected, we obtain that either $\partial E$ is contained in the unique disc $B(z_{p,q}, 5L)$, or $m_1(\partial E \cap B(z_{p,q}, 5L)) \geq L$ for all $(p, q) \in I_1 \setminus I_2$. In both cases we obtain 

$$\text{card}(I_1 \setminus I_2) \leq C(m_1(\partial E) + 1), \quad m_1(\partial R) \leq C(m_1(\partial E) + 1),$$

$$(m_2 + |\nu_D|) (E \setminus R) \leq (m_2 + |\nu_D|) (\cup_{(p,q) \in I_1 \setminus I_2} \overline{R}_{p,q}) \leq C(m_1(\partial E) + 1). \quad (44)$$

Furthermore, using \([36]\), we get 

$$\text{card} \left\{ \{n: c_n \in R, a_n, b_n \notin R\} \cup \{n: c_n \notin R, a_n, b_n \in R\} \right\} \leq C m_1(\partial R).$$

Therefore,

$$\left| \nu_D(R) - \sum_{a_n, b_n \in R} (a_n - b_n) \right| \leq C m_1(\partial R). \quad (45)$$

Define a continuous branch $\tilde{\arg} f(z)$ of the argument $f$ in the set $\cup_{p,q} \partial R_{p,q}$ by the condition $\tilde{\arg} f(0) = \arg f(0)$. The increment of the argument of $f$ along $\partial R_{p,q}$ equals 0 for all $p, q$, therefore the branch is well defined. The both ends of each segment $[a_n, b_n]$ belong to the same quadrilateral, hence the sum of jumps of $\tilde{\arg} f(z)$ along each side of $R_{p,q}$ is zero. Therefore, $\tilde{\arg} f(\tau_p + i\tau_{p,q}) = \arg f(\tau_p + i\tau_{p,q})$ for all $p, q$. Note that $|\tilde{\arg} f^\prime(z)| = |\tilde{\arg} f^\prime(z)| \leq |f^\prime(z)/f(z)| < C$ in $\cup_{p,q} \partial R_{p,q}$. Using \([39]\), we get 

$$|\tilde{\arg} f(z) - \text{Im} (\beta z)| < C \quad \forall z \in \cup_{p,q} \partial R_{p,q}.$$
Integrating by parts, we obtain
\[ \sum_{a_n, b_n \in \mathbb{R}} (a_n - b_n) = \frac{1}{2\pi i} \int_{\partial R} z f'(z) \, dz = -\frac{1}{2\pi} \int_{\partial R} \arg f(z) \, dz + O(m_1(\partial R)) \]
\[ = -\frac{1}{2\pi} \int_{\partial R} \Im(\beta z) \, dz + O(m_1(\partial R)) = \beta m_2(R)/2\pi + O(m_1(\partial R)). \]

Now (44) and (45) yield the assertion of the proposition.

§ 4. Almost elliptic functions with divisors having regular indexing.

Let \( \mu^\tau \) be the translation of a measure \( \mu \) in \( \mathbb{C} \), i.e., \( \mu^\tau(E) = \mu(E + \tau) \) for each Borel set \( E \subset \mathbb{C} \).

Here we shall prove the following theorem:

**Theorem 16.** An almost periodic divisor \( D = ((a_n), (b_n)), a_n \neq 0, b_n \neq 0 \) for all \( n \), with a regular indexing is the divisor of \( f \in \mathcal{AE} \) if and only if the following conditions are fulfilled:

a) the divisor \( D \) obeys (47),

b) the measure \( \nu_D \) satisfies (43) for every convex bounded subset of \( \mathbb{C} \),

c) the measure \( \nu_D \) satisfies the bound
\[ \limsup_{r \to \infty} \left| \int_{|w| \leq r} \frac{d\nu_D(w) - d\nu_D(w)}{w} \right| < C \quad \forall z \in \mathbb{C}. \]  

(46)

In this case the function \( f \) has the form
\[ f(z) = e^{\beta z/2} \lim_{r \to \infty} \prod_{n: |c_n| < r} e^{G_n(z)} \]

(47)

up to a constant factor.

We begin with some lemmas:

**Lemma 2.** Suppose an almost periodic divisor \( D = ((a_n), (b_n)) \) has a regular indexing; then we have
\[ \left| \int_{E^+(r)} (1/w - \overline{w}/r^2) d\nu(w) \right| + \left| \int_{E^-(r)} (1/w - \overline{w}/r^2) d\nu(w) \right| < C \frac{1 + |z|^2}{r}, \quad r > 1, \]

(48)

where

\[ E^+(r) = \{ w : |w| \leq r, |w + z| > r \}, \quad E^-(r) = \{ w : |w + z| \leq r, |w| > r \}. \]

Moreover, if b) holds for \( \nu_D \) and \( \lambda = \nu_D - (\beta/2\pi)m_2 \), then we have
\[ \left| \int_{|w| \leq r} w \, d\lambda(w) \right| \leq Cr^2, \quad \left| \int_{|w| \leq r} \overline{w} \, d\lambda(w) \right| \leq Cr^2, \quad r > 1, \]

(49)

\[ \left| \int_{1 \leq |w| \leq r} d\lambda(w)/w \right| \leq C(1 + \log r), \quad r > 1, \]

(50)

It follows from (52) that we can check this condition only at points of a relatively dense subset.
\[
\left| \int_{r \leq |w| \leq R} \frac{d\lambda(w)/w^2}{w} \right| \leq Cr^{-1}, \quad R > r > 1. \tag{51}
\]
\[
\left| \int_{1 \leq |w| \leq r} \frac{d\lambda^2(w) - d\lambda(w)}{w} \right| \leq C(1 + \log^+ |z|), \quad r > |z|^2 + 1, \tag{52}
\]
\[
\left| \int_{1 < |w| \leq r, 1 < |w-z|} \left( \frac{1}{w-z} - \frac{1}{w} \right) d\nu_D(w) - \int_{1 \leq |w| \leq r} \frac{d\nu_D^2(w) - d\nu_D(w)}{w} + \frac{\beta z}{2} \right| < C. \tag{53}
\]

**Proof.** Using Proposition \([3]\) we obtain \([5]\) and \([36]\). Hence the measure \(\nu\) is well defined and satisfies \([12]\). Hence, we obtain the bound \(|\nu|(E^+(r) \cup E^-(r)) < C(1 + |z|)r\). Since \(|1/w - w/r^2| < C|w(r + |z|)/r^3\) for \(w \in (E^+(r) \cup E^-(r))\), we get \([48]\).

Next, put \(w = u + iv\), \(\alpha_1(t) = \lambda\{w : |w| \leq r, u < t\}\). We have
\[
\int_{|w| \leq r} u \ d\lambda(w) = \int_{-r}^{r} t \ d\alpha_1(t) = r\alpha_1(r) - \int_{-r}^{r} \alpha_1(t) \ dt. \tag{54}
\]
Since \(|\alpha_1(t)| \leq Cr\), we see that the module of \([54]\) has the bound \(Cr^2\). Clearly, \(\int_{|w| \leq r} v \ d\lambda(w)\) has the same bound \(Cr^2\). Hence, we get \([19]\).

Furthermore, integrating by parts, we get
\[
\int_{1 \leq |w| \leq r} \frac{d\lambda(w)}{w} = \int_{1 < |w| \leq r} \frac{w \ d\lambda(w)}{|w|^2} = \frac{1}{r^2} \int_{1 < |w| \leq r} \frac{w \ d\lambda(w)}{w} + \int_{1 \leq |w| \leq t} \left( \int_{1 < |w| \leq t} \frac{w \ d\lambda(w)}{w} \right) \ dt. \tag{55}
\]

Now \([19]\) implies \([51]\).

To prove \([51]\), consider the integral
\[
\int_{r < u^2 + v^2 \leq R, u > 0} \frac{u^2}{(u^2 + v^2)^2} d\lambda(w) = \int_{r}^{\infty} \frac{1}{t^2} d\alpha_2(t) = 2 \int_{r}^{\infty} \alpha_2(t) / t^3 \ dt, \tag{55}
\]
where \(\alpha_2(t) = \lambda(B(t/2, t/2) \cap [B(0, R) \setminus B(0, r)]\). Using \([13]\), we get
\[
|\alpha_2(t)| \leq |\lambda(B(0, R) \cap B(t/2, t/2))| + |\lambda(B(0, r) \cap B(t/2, t/2))| \leq Ct.
\]
Hence the modulus of integral \([55]\) does not exceed \(Cr/r\). Clearly, the same bound is valid for the integral
\[
\int_{r < |w| \leq R} \Re{(1/w^2) d\lambda(w)} = \int_{r < |w| \leq R} \frac{u^2}{(u^2 + v^2)^2} d\lambda(w) - \int_{r < |w| \leq R} \frac{v^2}{(u^2 + v^2)^2} d\lambda(w). \tag{56}
\]
The orthogonal transformation of coordinates \(u' = (u + v)/\sqrt{2}, \ v' = (u - v)/\sqrt{2}\) reduces the integral
\[
\int_{r < |w| \leq R} \Im{(1/w^2) d\lambda(w)} = \int_{r < |w| \leq R} \frac{(2uv)/(u^2 + v^2)^2} d\lambda(w)
\]
to \([56]\), so \([51]\) follows.

To prove bound \([52]\), put \(r_1 = (|z| + 1)^2\). Decompose integral in \([52]\) into the sum
\[
\int_{1 \leq |w| \leq r_1} \frac{d\lambda^2(w) - d\lambda(w)}{w} + z \int_{r_1 < |w| \leq r} \frac{d\lambda(w)}{u^2} + z^2 \int_{r_1 < |w| \leq r} \frac{d\lambda(w)}{w^2(u^2 + v^2)} + \int_{E^-(r_1)} \frac{d\lambda^2(w)}{w}
\]


\[- \int_{E^+(r_1)} \frac{d\lambda^z(w)}{w} + \int_{E^+(r)} \frac{d\lambda^z(w)}{w} - \int_{E^{-}(r)} \frac{d\lambda^z(w)}{w} = I_1 + I_2 + I_3 + I_4 - I_5 + I_6 - I_7.\]

In view of (50) and (51), we have \(|I_1| < C(1 + \log r_1)\) and \(|I_2| < C|z|/r_1\). Using (58) and inequality \(|z| < r_1/2 \leq t/2\), we get

\[|I_3| < 2|z| \int_{r_1}^{r} \frac{d|\lambda|(B(0, t))}{t^3} < C|z|^2/r_1.\]

Next,

\[I_6 - I_7 = \int_{E^+(r)} \left( \frac{1}{w} - \frac{\overline{w}}{r^2} \right) d\lambda^z(w) - \int_{E^{-}(r)} \left( \frac{1}{w} - \frac{\overline{w}}{r^2} \right) d\lambda^z(w) + \int_{|w| \leq r} \frac{\overline{w}}{r^2} d\lambda^z(w) - \int_{|w| \leq r} \frac{\overline{w}}{r^2} d\lambda^z(w) = I_8 - I_9 + I_{10} - I_{11} + I_{12}.\]

Applying (18) with the measure \(\lambda^z\), we get \(|I_8| + |I_9| < C\). Taking into account (49) and (43), we get \(|I_{10}| < C, |I_{11}| < C\), and \(|I_{12}| < C|z|/r < C\). In the same way, \(|I_4 - I_5| < C\).

So (52) is proved.

To prove (53), note that

\[\left| \int_{B(z, 1) \setminus B(0, 1)} \frac{d\nu_D(w)}{w} \right| + \left| \int_{B(0, 1) \setminus B(z, 1)} \frac{d\nu_D(w)}{w - z} \right| < C/(1 + |z|).\]

Next,

\[\int_{1 < |w| \leq r, 1 < |w - z|} \frac{d\nu_D(w)}{w - z} = \int_{1 < |w| \leq r, 1 < |w + z|} \frac{d\nu_D(w)}{w} + \frac{1}{r^2} \int_{|w + z| \leq r} \overline{w} d\nu_D^z(w)
- \frac{1}{r^2} \int_{|w| \leq r} \overline{w} d\nu_D^z(w) + \int_{E^+(r)} \left( \frac{1}{w} - \frac{\overline{w}}{r^2} \right) d\nu_D^z(w) - \int_{E^{-}(r)} \left( \frac{1}{w} - \frac{\overline{w}}{r^2} \right) d\nu_D^z(w).\]

In view of (48), the difference of last two integrals is uniformly bounded. Furthermore,

\[\int_{|w + z| \leq r} \overline{w} d\nu_D^z(w) = \int_{|w| \leq r} \overline{w} d\nu_D^z(w) = \int_{|w| \leq r} \overline{w} d\lambda(w) = \int_{|w| \leq r} \overline{w} d\lambda^z(w)
- \overline{z} \left( \int_{|w| \leq r} d\lambda(w) + \frac{\beta}{2\pi} \int_{|w| \leq r} dm_2(w) \right).\]

Applying (49) to the measures \(\lambda\) and \(\lambda^z\), and (43) to \(\lambda(B(0, r))\), we obtain (53). The proof is complete.

\[\textbf{Lemma 3. Suppose an almost periodic divisor} D = ((a_n), (b_n)), a_n \neq 0, b_n \neq 0, \text{ with a regular indexing satisfies } b). \text{ Then the limit}

\[G(z) = \lim_{r \to \infty} \sum_{n:|a_n|<r} G_n(z) \]

\[\text{exists uniformly on compacta in } \mathbb{C}. \text{ Moreover, for every } \delta > 0 \text{ there exists a constant } C_\delta \text{ such that the inequality}

\[\limsup_{r \to \infty} \left| G(z) + \frac{\beta \overline{z}}{2} - \int_{1 \leq |w| \leq r} \frac{d\nu_D^z(w) - d\nu_D(w)}{w} \right| < C_\delta \]

\[\text{is valid for all } z \in \mathbb{C} \setminus A(\delta).\]
Proof. If \(|c_n - z| > 3d_0\) and \(|c_n| > 3d_0\), then we get
\[
G_n(z) = \log \frac{a_n - z}{b_n - z} - \log \frac{a_n}{b_n},
\]
\[
|\log \left( \frac{a_n - z}{b_n - z} \right) - \frac{a_n - b_n}{b_n - z} + \frac{(a_n - b_n)^2}{2(b_n - z)^2} | < C \left( \frac{|a_n - b_n|^3}{|b_n - z|^3} \right),
\]
and
\[
a_n - b_n - \frac{(a_n - b_n)^2}{2(b_n - z)^2} = \frac{\nu_D(\{c_n\})}{c_n - z} - \frac{(a_n - b_n)^3}{4(b_n - z)^2(c_n - z)}.\]
Therefore,
\[
|G_n(z) - \nu_D(\{c_n\})/(c_n - z) + \nu_D(\{c_n\})/c_n| < C(|a_n - z|^{-3} + |a_n|^{-3}). \tag{59}
\]

Take \(R > r > 2|z|\). Proposition \([5]\) implies \([5]\). Therefore the sum \(\sum_{|a_n - z| > 1} |a_n - z|^{-3}\) converges uniformly in \(C\). Hence the sum \(\sum_{|a_n| < |C_n| \leq R} G_n(z)\) equals the integral
\[
\int_{r < |w| \leq R} \left( \frac{1}{w - z} - \frac{1}{w} \right) d\nu_D(w) = \int_{r < |w| \leq R} \frac{d\nu_D(w)}{w^2} + \int_{r < |w| \leq R} \frac{d\nu_D(w)}{2w(w - z)} \tag{60}
\]
up to the term tending to zero as \(r \to \infty\). Taking into account \([51]\) and equality \(\int_{r < |w| \leq R} d\nu_D/2w^2 = 0\), we see that the first integral in the right-hand side of \(\tag{60}\) tends to 0 as \(r \to \infty\). Using \(\tag{58}\), we get
\[
\left| \int_{r < |w| \leq R} \frac{d\nu_D(w)}{w^2(w - z)} \right| < \int_R d|\nu_D|B(0, t) \frac{(t - |z|)t^2}{(t + |z|)t^2} \to 0
\]
as \(r \to \infty\). Hence, the limit in \([67]\) exists.

Thus, using \([59]\), we get
\[
\sum_{n: 3d_0 < |c_n| < r, |c_n - z| > 3d_0} G_n(z) - \int_{3d_0 < |w| \leq r, 3d_0 < |w - z|} \left( \frac{1}{w - z} - \frac{1}{w} \right) d\nu_D(w) < C. \tag{61}
\]
Taking into account \([42]\), replace \(3d_0\) by 1. Combining \([53]\) and \([61]\), we get \([58]\). \(\blacksquare\)

Proof of the Theorem. Necessity.

It follows from Theorem \([1]\) and Propositions \([5, 8]\) that the divisor \(D = ((a_n), (b_n))\) of \(\mathcal{A}\mathcal{E}\)-function \(f\) is almost periodic and satisfies conditions \([1]\) and \([5]\). Moreover, since \(D\) has a regular indexation, we get b) and \([36]\). Next, for every \(\delta > 0\) bounds \([10]\) and \([39]\) imply the estimate
\[
|\log |f(z)| + i \arg f(z) - i \text{Im}(\beta z)| < C_\delta \quad \forall z \in \mathbb{C} \setminus A(\delta). \tag{62}
\]
By Lemma \([3]\) the function \(G(z)\) is well defined. Taking into account the definition of \(\arg f(z)\), we see that the function
\[
E(z) = \log f(z) - G(z) - \beta z/2
\]
is holomorphic in \(C\). Using \([58]\) and \([62]\), we get
\[
\limsup_{r \to \infty} \left| E(z) + \int_{1 \leq |w| \leq r} \frac{d\nu_D^z(w) - d\nu_D^z(w)}{w} \right| < C_\delta
\]
for a sufficiently small \(\delta\) and \(z \in \mathbb{C} \setminus A(\delta)\). It follows from \([52]\) that \(|E(z)| < C(1 + \log^+ |z|)\) in this set. Therefore, \(E(z) \equiv \text{const.}\), and we obtain \([47]\) and \([46]\).
Sufficiency. Let $D = ((a_n), (b_n))$ be a divisor satisfying conditions of the Theorem. Let us show that the function $f$ belongs to the class $\mathcal{AE}$.

First, it follows from (58) and (46) that $f$ satisfies (10). The divisor $D$ is almost periodic with a regular indexing, hence Proposition 5 implies (5) and (9). By Theorem 5 we get $f \in \mathcal{N}_1$.

Take $\delta_1 = \min\{\delta_0/6C_0, 1/3\}$. Put $A(D, \delta_1) = \cup_n B(a_n, \delta_1) \cup \cup_n B(b_n, \delta_1)$ and

$$
\alpha_1(f) = \sup \{\log |f(z)| : z \in \mathbb{C} \setminus A(D, \delta_1)\},
\alpha_2(f) = \inf \{\log |f(z)| : z \in \mathbb{C} \setminus A(D, \delta_1)\},
$$

Note that all zeros of $f$ belong to the same connected component of $\mathbb{C} \setminus \cup_n B(b_n, \delta_1)$, and all poles of $f$ belong to the same connected component of $\mathbb{C} \setminus \cup_n B(a_n, \delta_1)$. The function $g(z) = \text{Im} G(z) - \text{Im} \beta z/2$ is harmonic in $\mathbb{C} \setminus \cup_n [a_n, b_n]$ and has the jump $2\pi$ in each segment $[a_n, b_n]$.

It follows from (58) and (46) that $g(z)$ is uniformly bounded in $\mathbb{C}$. Let $(h_k)$ be an arbitrary sequence. It follows from Proposition 6 that there are an almost periodic divisor with a regular indexing $\tilde{D} = ((\tilde{a}_n), (\tilde{b}_n))$ and a subsequence $(h_k')$, which satisfy (37). Passing on to a subsequence, we can suppose that the functions $g(z - h_k')$ converge uniformly on compacta in $\mathbb{C} \setminus \cup_n [\tilde{a}_n, \tilde{b}_n]$ to a function $\tilde{g}$. Clearly, $\tilde{g}$ is harmonic in $\mathbb{C} \setminus \cup_n [\tilde{a}_n, \tilde{b}_n]$ and has the jump $2\pi$ in each segment $[\tilde{a}_n, \tilde{b}_n]$. Moreover, we have

$$
\alpha_3(\tilde{g}) \leq \alpha_3(g), \quad \alpha_4(\tilde{g}) \geq \alpha_4(g).
$$

Furthermore, since $f \in \mathcal{N}_1$, we can pass on to a subsequence to obtain

$$
\rho_S(f(z - h_k'), \tilde{f}(z)) \to 0 \quad \text{as} \quad k \to \infty
$$

uniformly on compacta in $\mathbb{C}$. Clearly, $\tilde{f}(z)$ has the divisor $\tilde{D}$ and the sequence $(\log |f(z - h_k')|)$ converges to $\log |\tilde{f}(z)|$ uniformly on compacta in $\mathbb{C} \setminus A(\tilde{D}, \delta_1)$. Hence,

$$
\alpha_1(\tilde{f}) \leq \alpha_1(f), \quad \alpha_2(\tilde{f}) \geq \alpha_2(f).
$$

Since $f(z) = |f(z)| \exp\{ig(z) + i\text{Im} \beta z\}$, we get

$$
\tilde{f}(z) = |\tilde{f}(z)| e^{i\tilde{g}(z) + i\text{Im} \beta z} e^{i\tilde{g}}, \quad \text{where} \quad e^{i\tilde{g}} = \lim_{k \to \infty} e^{-i\text{Im} (\beta h_k')},
$$

Let us check that inequalities (63) and (64) are in fact equalities. Rewrite (37) in the form

$$
\sup_n |\tilde{a}_{\sigma(n)} - h_k' - a_n| \to 0, \quad \sup_n |\tilde{b}_{\sigma(n)} - h_k' - b_n| \to 0 \quad \text{as} \quad k \to \infty.
$$

For some subsequence $(h_k'') \subset (h_k')$ the functions $\tilde{g}(z + h_k'')$ converge uniformly on compacta in $\mathbb{C} \setminus \cup_n [a_n, b_n]$ to a function $\tilde{g}$, which is harmonic in $\mathbb{C} \setminus \cup_n [a_n, b_n]$ and has the jump $2\pi$ in each segment $[a_n, b_n]$. As above, for some $\mathcal{N}_1$-function $\tilde{f}$ with the divisor $D$ we have

$$
\rho_S(\tilde{f}(z + h_k''), \tilde{f}(z)) \to 0 \quad \text{as} \quad k \to \infty
$$

uniformly on compacta in $\mathbb{C}$. Therefore, the function $f(z)/\tilde{f}(z)$ is an entire function without zeros, which satisfies the inequality $1/C \leq |f(z)/\tilde{f}(z)| \leq C$ in the set $\mathbb{C} \setminus A(D, \delta_1)$. Hence, $\tilde{f}(z) \equiv K f(z)$, $K \in \mathbb{C}$. Now the inequalities

$$
\alpha_1(\tilde{f}) \leq \alpha_1(f) \leq \alpha_1(f), \quad \alpha_2(\tilde{f}) \geq \alpha_2(f) \geq \alpha_2(f)
$$
imply that \( |K| = 1 \), and equalities (64) prevail. Furthermore, using (65), we get
\[
\hat{f}(z) = \left| f(z) \right| e^{i\hat{g}(z)} + \text{Im} \beta z e^{i\hat{\alpha}} e^{i\hat{\phi}}, \quad \text{where} \quad e^{i\hat{\alpha}} = \lim_{k \to \infty} e^{i\text{Im}(\beta v_k^o)}.
\]
Since \( e^{i\hat{\alpha}} = e^{-i\hat{\phi}} \) and \( \hat{f} = e^{i\arg K} f \), we get \( \tilde{g}(z) = \arg K + 2\pi l + g(z), \ l \in \mathbb{Z} \). Hence the inequalities
\[
\alpha_3(\tilde{g}) \leq \alpha_3(\hat{g}) \leq \alpha_3(g), \quad \alpha_4(\tilde{g}) \geq \alpha_4(g),
\]
imply that \( K = 1 \), and equalities (63) prevail.

To prove \( f \in \mathcal{AE} \) we shall show that \( \sup_{z \in \mathbb{C}} \rho_S(f(z - h_k'), \tilde{f}(z)) \to 0 \) as \( k \to \infty \).

Assume the contrary. Then there exists an \( \varepsilon_0 > 0 \) and a sequence \( (z_k) \) such that
\[
\rho_S(f(z_k - h_k'), \tilde{f}(z_k)) > \varepsilon_0. \tag{66}
\]
Using Proposition 5 take a subsequence \( (t_{k'}) \) of \( (h_k') \) with the following properties:
\[
\sup_n |a_{\sigma^*(k',n)} + t_{k'} - z_{k'} - a_{\sigma}^*| \to 0, \quad \sup_n |b_{\sigma^*(k',n)} + t_{k'} - z_{k'} - b_{\sigma}^*| \to 0 \quad \text{as} \quad k' \to \infty, \quad \tag{67}
\]
\[
\sup_n |\tilde{a}_{\sigma^*(k',n)} - z_{k'} - a_{\sigma}^*| \to 0, \quad \sup_n |\tilde{b}_{\sigma^*(k',n)} - z_{k'} - b_{\sigma}^*| \to 0 \quad \text{as} \quad k' \to \infty, \quad \tag{68}
\]
uniformly on compacta in \( \mathbb{C} \setminus \bigcup_n [a_{\sigma}^*, b_{\sigma}^*] \),
\[
\tilde{g}(z + z_{k'}) \to g^\sigma(\infty) \quad \text{as} \quad k' \to \infty
\]
uniformly on compacta in \( \mathbb{C} \setminus \bigcup_n [a_{\sigma}^*, b_{\sigma}^*] \),
\[
\rho_S(f(z - t_{k'} + z_{k'}), f^\sigma(z)) \to 0, \quad \rho_S(\tilde{f}(z + z_{k'}), f^{**}(z)) \to 0 \quad \text{as} \quad k' \to \infty \quad \tag{69}
\]
uniformly on compacta in \( \mathbb{C} \).

Here \( D^* = ((a_{n}, b_{n}^*)), D^{**} = ((a_{n}^{**}, b_{n}^{**})), \sigma^*(k', \cdot) \) and \( \sigma^{**}(k', \cdot) \) are bijections \( \mathbb{N} \to \mathbb{N} \), \( f^* \) is an \( N_1 \)-function in \( \mathbb{C} \), with the divisor \( D^* \), \( f^{**} \) is an \( N_1 \)-function in \( \mathbb{C} \) with the divisor \( D^{**} \), \( g^\sigma \) is a harmonic function in \( \mathbb{C} \setminus \bigcup_n [a_{\sigma}^*, b_{\sigma}^*] \), which has the jump \( 2\pi \) in each segment \( [a_{\sigma}^*, b_{\sigma}^*] \), and \( g^{**} \) is a harmonic function in \( \mathbb{C} \setminus \bigcup_n [a_{\sigma}^{**}, b_{\sigma}^{**}] \), which has the jump \( 2\pi \) in each segment \( [a_{\sigma}^{**}, b_{\sigma}^{**}] \).

Combining (67), (67), and (68), we get
\[
\sup_n |a_{\tilde{\sigma}(k',n)} - a_{\sigma}^*| \to 0, \quad \sup_n |b_{\tilde{\sigma}(k',n)} - b_{\sigma}^*| \to 0 \quad \text{as} \quad k' \to \infty,
\]
where \( \tilde{\sigma}(k', \cdot) = (\sigma^*)^{-1}(k', \cdot) \circ \sigma(k', \cdot) \circ \sigma^{**}(k', \cdot) \). The sequence \( (a_{\sigma}^*) \) has no limit points, hence for each \( n \) the number \( \tilde{\sigma}(k', n) \) is the same for all \( k' > k_n \). Since \( \tilde{\sigma}(k', n) \) are bijections, we obtain that \( \tilde{\sigma}(n) = \lim_{k' \to \infty} \tilde{\sigma}(k', n) \) is a bijection as well. Therefore,
\[
a_{\tilde{\sigma}(n)}^* = a_{\sigma}^{**}, \quad b_{\tilde{\sigma}(n)}^* = b_{\sigma}^{**} \quad \forall n.
\]
Hence, \( D^{**} = D^* \) up to the same rearrangements of \( (a_{\sigma}^*) \) and \( (b_{\sigma}^*) \).

Furthermore, (67) and (65) yield
\[
f^*(z) = |f^*(z)| e^{i\hat{g}^*(z)} e^{i\hat{\alpha}^*}, \quad \text{where} \quad e^{i\hat{\alpha}^*} = \lim_{k \to \infty} e^{-i\text{Im}(\beta v_k^o)}.
\]
\[
f^{**}(z) = |f^{**}(z)| e^{i\hat{g}^{**}(z)} e^{i\hat{\alpha}^{**}}, \quad \text{where} \quad e^{i\hat{\alpha}^{**}} = \lim_{k \to \infty} e^{i\text{Im}(\beta v_k^o)}.
\]
As before, the function \( f^*(z)/f^{**}(z) \) is bounded on the set \( \mathbb{C} \setminus A(D^*, \delta_1) \) and has no zeros and poles in \( \mathbb{C} \). Hence, \( f^*(z) = K f^{**}(z) \). The equalities \( \alpha_j(f^*, D^*) = \alpha_j(f, D) = \alpha_j(\tilde{f}, D) = \alpha_j(f^{**}, D^{**}) \), \( j = 1, 2 \) show that \( |K| = 1 \). Since \( e^{i\hat{\alpha}^*} = e^{i\hat{\phi}^*} \), we get
\[
g^*(z) = g^{**}(z) + \arg K + 2\pi l, \ l \in \mathbb{Z}.
\]
The equalities \( \alpha_j(g^*, D^*) = \alpha_j(g, D) = \alpha_j(\tilde{g}, D) = \alpha_j(g^{**}, D^{**}), \ j = 3, 4 \) yield \( g^*(z) \equiv g^{**}(z) \) and \( f^*(z) \equiv f^{**}(z) \). The last equality contradicts to (69) and (69). Hence, \( f \in \mathcal{AE} \).
§5. Examples of almost elliptic functions

Suppose that \( Q(z) \) is a finite exponential sum \([34]\) (or a uniform limit of a sequence of sums \([34]\)) with the additional condition

\[
\lambda_j \cdot k + n' + \lambda_j' \cdot k + n'' \notin 2\pi \mathbb{Z} \quad \text{for all} \quad n = n' + in'' \in \mathbb{Z}^2.
\]

(70)

Consider the almost periodic mapping \( F(z) = (Q(z); \exp 2\pi i x; \exp 2\pi i y) \) from \( \mathbb{C} \) to \( \mathbb{C}^3 \) with the Euclidean metric. It can be easily checked that \( F(z) \) satisfies condition b) of Proposition \([1]\). Therefore, \( F(z) \) is almost periodic. Then, for each \( \varepsilon > 0 \), every disc \( B(c, L) \) with \( L > L(\varepsilon) \) contains a point \( \tau = \tau' + i\tau'' \) such that the function \( Q(z) : \mathbb{C} \rightarrow \mathbb{C} \) satisfies \([33]\). Besides, we get \( |1 - \exp 2\pi i \tau'| < \varepsilon, |1 - \exp 2\pi i \tau''| < \varepsilon \), hence each disc \( B(\tau, \varepsilon/8) \) intersects with \( \mathbb{Z}^2 \). Since \( Q(z) \) is uniformly continuous in \( \mathbb{C} \), we see that for every \( \eta > 0 \) there exists a relatively dense set \( E \subset \mathbb{Z}^2 \) such that \( |Q(z + \tau) - Q(z)| < \eta \) for all \( z \in \mathbb{C} \) and \( \tau \in E \). Set \( q(n) = Q(n) - Q(n - 1) - Q(n - i) + Q(n - 1 - i) \), \( n \in \mathbb{Z}^2 \subset \mathbb{C} \).

In view of \([70]\), the function \( q(n) \) has no period \( \tau \in \mathbb{Z}^2 \). Also, take a complex number \( p \in \mathbb{B}(0, 1/6) \setminus \{0\} \) and a positive number \( \gamma < |p|/\sup_{n \in \mathbb{Z}^2} |q(n)|^{-1} \).

**Example 1.** Put \( a_n = n + p + \gamma q(n) \), \( b_n = n - p - \gamma q(n) \), \( n \in \mathbb{Z}^2 \subset \mathbb{C} \). Let us show that the divisor \( D = ((a_n), (b_n)) \) satisfies the conditions of Theorem \([34]\). Clearly, the divisor \( D \) is almost periodic with a regular indexing.

Condition \([1]\) follows from the inequalities \( |a_n - b_m| > 1/3 \) for \( n \neq m \) and \( |a_n - b_m| > 2|p| - 2\gamma \sup_{n \in \mathbb{Z}^2} |q(n)| \) for all \( n \).

Furthermore, let \( E \) be a convex bounded set. Since \( (a_n + b_n)/2 = n \) for all \( n \in \mathbb{Z}^2 \) and \( v_D(n) = 2p + 2\gamma q(n) \), we get

\[
\nu_D(E) = 2p \text{card } E \cap \mathbb{Z}^2 + 2\gamma \sum_{n \in E} (Q(n) - Q(n - 1) - Q(n - i) + Q(n - 1 - i)).
\]

Clearly, \( |\text{card } E \cap \mathbb{Z}^2 - m_2(E)| < C(m_1(\partial E) + 1) \). Also,

\[
\left| \sum_{n \in E} Q(n - 1) - \sum_{n \in E} Q(n) \right| < C m_1(\partial E).
\]

The same is true for the difference \( \sum_{n \in E} Q(n - i) - \sum_{n \in E} Q(n - 1 - i) \). Hence, we obtain \([35]\) with \( \beta = 4\pi \bar{p} \).

Furthermore, for any \( k \in \mathbb{Z}^2 \) we have

\[
\left| \sum_{1 \leq |n| < r} \frac{Q(n - 1 - k)}{n} - \sum_{2 \leq |n| < r} \frac{Q(n - k)}{n + 1} \right| < 8 \sup_{n \in \mathbb{Z}^2} |Q(n)|
\]

\[
+ \sup_{n \in \mathbb{Z}^2} |Q(n)| \left( \text{card} \{ n : |n| < r, |n - 1| \geq r \} + \text{card} \{ n : |n - 1| < r, |n| \geq r \} \right).
\]

Obviously, the right-hand side of this inequality is bounded uniformly in \( k \in \mathbb{Z}^2 \) and \( r > 3 \).

By the same argument,

\[
\left| \sum_{1 \leq |n| < r} \frac{Q(n - i - k)}{n} - \sum_{2 \leq |n| < r} \frac{Q(n - k)}{n + i} \right| < C
\]
and
\[ \left| \sum_{1 \leq |n| < r} \frac{Q(n - 1 - i - k)}{n} - \sum_{2 < |n| < r} \frac{Q(n - k)}{n + 1 + i} \right| < C. \]

Therefore, the sum
\[ \sum_{1 \leq |n| < r} \frac{q(n - k)}{n} = \sum_{1 \leq |n| < r} \frac{Q(n - k) - Q(n - 1 - k) - Q(n - i - k) + Q(n - 1 - i - k)}{n} \]
up to a bounded term has the form
\[ \sum_{2 < |n| < r} Q(n - k) \left[ \frac{1}{n} - \frac{1}{n + 1} - \frac{1}{n + i} + \frac{1}{n + 1 + i} \right]. \]

Note that the absolute value of the expression in the square brackets does not exceed \( C|n|^{-3} \). Taking into account the equality \( \nu^k_D(\{n\}) - \nu^k_D(\{n\}) = 2\gamma(q(n - k) - q(n)) \), we obtain that \( \Vert P \Vert \) is satisfied for all \( z \in \mathbb{Z}^2 \subset \mathbb{C} \). So that formula \( 47 \) with the divisor \( D = ((a_n), (b_n)) \) and \( \beta = 4\pi\bar{p} \) defines an \( A\delta \)-function.

**Example 2.** Put \( a_n = n + p + \gamma q(n) \), \( b_n = n - p + \gamma q(n) \), \( n \in \mathbb{Z}^2 \). The divisor \( D = ((a_n), (b_n)) \) is almost periodic with a regular indexing. Also, we obtain \( 46 \). Next, \( (a_n + b_n)/2 = n + \gamma q(n) \), \( a_n - b_n = 2p \), and for every convex bounded set \( E \) we have
\[ |\nu^p_D(E) - 2p \text{ card}(E \cap \mathbb{Z}^2)| < |\nu^p_D((\partial E)_{1/6}) + 2|p| \text{ card}((\partial E)_{1/6} \cap \mathbb{Z}^2) < C(m_1(\partial E) + 1), \]
where \( (\partial E)_{1/6} \) is the \( (1/6) \)-neighborhood of \( \partial E \). So, we obtain \( 48 \). Finally, the difference
\[ \sum_{1 \leq |n + \gamma q(n)| < r} \frac{\nu^p_D(n + \gamma q(n))}{n + \gamma q(n)} - \sum_{1 \leq |n + \gamma q(n-k)| < r} \frac{\nu^p_D(n + \gamma q(n-k))}{n + \gamma q(n-k)} \]
up to a bounded term is equal to the sum
\[ \sum_{2 < |n| < r} 2p\gamma \left[ \frac{q(n-k) - q(n)}{(n + \gamma q(n))(n + \gamma q(n-k))} \right] \quad (71) \]

Arguing as above, we see that the sum \( \sum_{2 < |n| < r} q(n-k)/n^2 \) is uniformly bounded for \( r > 3 \) and \( k \in \mathbb{Z}^2 \). Hence, the same is true for \( 71 \). Therefore, condition \( 46 \) holds for all \( z \in \mathbb{Z}^2 \), so that formula \( 47 \) with \( \beta = 4\pi\bar{p} \) defines an \( A\delta \)-function.

**Question.** If there exists an almost periodic divisor of an \( A\delta \)-function, which has no regular indexing?

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