Modified Coulomb law in a strongly magnetized vacuum

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We study electric potential of a charge placed in a strong magnetic field \(B \gg B_0 = 4.4 \times 10^{13} \text{ G}\), as modified by the vacuum polarization. In such field the electron Larmour radius is much less than its Compton length. At the Larmour distances a scaling law occurs, with the potential determined by a magnetic-field-independent function. The scaling regime implies short-range interaction, expressed by Yukawa law. The electromagnetic interaction regains its long-range character at distances larger than the Compton length, the potential decreasing across \(B\) faster than along. Correction to the nonrelativistic ground-state energy of a hydrogenlike atom is found. In the limit \(B = \infty\), the modified potential becomes the Dirac \(\delta\)-function plus a regular background. With this potential the ground-state energy is finite - the best pronounced effect of the vacuum polarization.

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There is now compelling evidence that many compact astronomical objects (soft gamma-ray repeaters, anomalous X-ray pulsars, and some radio pulsars) identified with neutron stars have surface magnetic fields as high as \(\sim 10^{14} - 10^{15} \text{ G}\). More strong magnetic fields \((B \sim 10^{16} - 10^{17} \text{ G})\) are predicted to exist at the surface of cosmological gamma-ray bursters if they are rotation-powered neutron stars similar to radio pulsars. All these fields, however, are much smaller than the maximum value inherent in quantum electrodynamics [3].

Vacuum in an external magnetic field \(B\) behaves as an anisotropic dielectric medium with spatial and frequency dispersion, \((e.g., 4)\). These properties may become important, provided that the field strength achieves the characteristic value \(B_0 = m^2/e \simeq 4.4 \times 10^{13} \text{ G}\), where \(m\) is the electron mass and \(e\) is its charge. Henceforth, we set \(h = c = 1\) and refer to the Heaviside-Lorentz system of units. Although much work has been devoted to study of electromagnetic wave propagation in the magnetized vacuum, problems of electro- and magneto-statics in this medium did not attract sufficient attention, save Refs. [5], [6], where corrections to the Coulomb law were found when these are small: for \(B/B_0 \ll 1\) in [5], or at large distances from the source for \(1 \ll B/B_0 \ll 3\alpha^{-1}\) in [6] \((\alpha = \epsilon^2/\pi = 1/137)\).

In this Letter, we find that for sufficiently large \(b \equiv B/B_0 \gg 1\) the electric field produced by a pointlike charge at rest may be significantly modified by the vacuum polarization, the modification being determined by the characteristic factor \(\sigma b\). The modified Coulomb potential in the close vicinity of its charge, characterized by the Larmour length \(L_B = (eB)^{-1/2} = (1/\sigma m \sqrt{B})\), goes steeper than the standard one, following a Yukawa law, whereas it obeys a long-range ”anisotropic Coulomb law” far from the source, at distances characterized by the electron Compton length \(m^{-1}, m^{-1} \gg L_B\). (Details of the corresponding derivations can be found in the accompanying preprint [7]). The short-range part of the modified potential tends to the Dirac \(\delta\)-function in the limit \(b \rightarrow \infty\). The modification of the Coulomb law should affect, first of all, the field of an atomic nucleus, placed in a magnetic field. We determine the corresponding correction to the lowest energy level of a hydrogenlike atom. Unlike the famous result of Ref. [8], referred to in many speculations on behavior of matter on the surface of strongly magnetized neutron stars \((e.g., the review [9] and references therein)\), we find that the (vacuum-polarization-corrected) ground state energy remains finite in the limit of infinite magnetic field.

Let the constant and homogeneous magnetic field \(B\) be directed along axis 3 in the frame where the pointlike charge \(q\) is at rest in the origin \(x = \{x_1, x_2, x_3\} = 0\), and no external magnetic field exists. By using the tensor decomposition of the photon propagator over eigenmodes in a magnetic field [10, 11] it is straightforward to show that the electrostatic potential \(A_0\) produced by this charge has the form

\[
A_0(x) = \frac{q}{(2\pi)^3} \int \frac{\exp(-ikx) \, d^3k}{k^2 - \kappa_2 (k_+^2 + k_3^2)},
\]

while its vector potential is zero, \(A_{1,2,3}(x) = 0\). The static charge gives rise to electric field only, as it might be expected. Here \(\kappa_2\) is one (out of three) eigenvalue(s) of the polarization operator \(\Pi_{\mu}^{\nu} : \Pi_{\mu}^{\nu} \gamma_{\nu}^{(a)} = \kappa_a \gamma_{\nu}^{(a)}\), \(a = 1, 2, 3\), \(\mu, \nu = 0, 1, 2, 3\). The eigenvectors \(\gamma_{\nu}^{(a)}\) are 4-potentials of the eigenmodes. The eigenvalues depend on two combinations of the photon momentum components \(k_3^2 - k_2^2\) and \(k_+^2 = k_1^2 + k_2^2\) taken at zero frequency \(k_0 = 0\). Eq. [12] is approximation-independent and axial symmetric: \(A_0(x) = A_0([x_3, x_\perp]), x_\perp = \sqrt{x_1^2 + x_2^2}\). Eq. [12] indicates that only mode-2 photons mediate electrostatic interaction. This fact may be better understood, if we examine electric and magnetic fields intrinsic to the virtual (off-shell) photons of this mode, obtained from its 4-vector potential \(\gamma_{\nu}^{(2)} = (k_3 \, 0 \, 0 \, k_0)_{\mu}\). It may be seen from explicit representation for these fields [11] that in the limit \(k_0 = 0\) the magnetic field of mode-2 photon disappears, while its electric field is collinear with \(k\), \(i.e.,\)
it becomes a purely longitudinal virtual photon. Virtual photons of other modes, in the static case, are carriers of stationary magnetic fields. For instance, mode-1 photons are responsible for the field produced by a constant current, flowing parallel to the external magnetic field.

In the asymptotic regime of high magnetic field, \( eB \gg k_3^2 \) and \( B \gg m^2/e \equiv B_0 \), with the accuracy to terms that only grow with \( B \) as its logarithm or slower, the eigenvalue \( \kappa_2 \), calculated within the one-loop approximation in [10][11], acquires the form \([6][12] \]

\[
\kappa_2(k_3^2, k_1^2) = -\frac{2\alpha bm^2}{\pi} \exp\left(-\frac{k_1^2}{2m^2}\right) T\left(\frac{k_3^2}{4m^2}\right), \quad (2)
\]

\[
T(y) = y \int_0^1 \left(1 - \eta^2\right)^{1/2} \mathrm{d}\eta / \left(1 + y(1 - \eta^2)^{1/2}\right), \quad (3)
\]

Note the properties: \( T(y \rightarrow 0) \approx 2y/3, \quad T(\infty) = 1 \).

The deviation of the potential \([1]\) from the standard Coulomb potential \( A_0^C(x) = q/(4\pi \sqrt{x_1^2 + x_3^2}) \) is

\[
\Delta A_0(x) \equiv A_0^C(x) - A_0(x) = \frac{q}{8\pi^2} \int \infty \int_0^\infty J_0(k_1 x_1) \mathrm{d}k_1^2
\times \int_0^\infty \left(\exp(-ik_3 x_3) - \frac{\exp(-ik_3 x_3)}{k_1^2 + k_3^2 - \kappa_2(k_3^2, k_1^2)}\right) \mathrm{d}k_3. \quad (4)
\]

Here \( J_0 \) is the Bessel function of order zero. This integral defines \( \Delta A_0(x) \) as a finite function of the coordinates in the origin, unlike \( A_0 \) and \( A_0^C \). As the integration variable \( k_3 \) in \([1]\) approaches the large values \( \pm \sqrt{eB} \), one has \( \kappa_2 \ll k_3^2 \). Hence, no essential contribution comes from the integration over the region \( |k_3| > \sqrt{eB} \), wherein eq. \([2]\) is not valid. The expression to be substituted for it is even smaller (since it does not contain the large factor \( b \)). The leading terms of the expansion near the origin \( x_3 = x_1 = 0 \) are

\[
A_0(x) \approx \frac{q}{4\pi} \left(\frac{1}{|x|} - 2mC\right), \quad C = \frac{2\alpha}{qm} \Delta A_0(0) > 0, \quad (5)
\]

where \( C \) is a constant depending on the magnetic field.

More exactly the singular behavior near the origin will be presented if we note that it is provided by the integration over large \( k_3 \) (and \( k_\perp \)) in \([1]\), where we may set \( T(k_3^2/4m^2) = T(\infty) = 1 \) and perform \( k_3 \)-integration in Eq.\([1]\) by calculating residues. Then

\[
A_0(x) \approx \frac{\bar{A}_0(\bar{x})}{L_B} = \frac{q}{4\pi L_B} \int_0^\infty \int_0^\infty J_0(\bar{k}_1 \bar{x}_\perp) \bar{k}_\perp
\times \frac{\exp\left(-|\bar{x}_3| \sqrt{k_1^2 + (2\alpha/\pi) \exp\left(-\bar{k}_3^2/2\right)}\right)}{\sqrt{k_1^2 + (2\alpha/\pi) \exp\left(-\bar{k}_3^2/2\right)}} \mathrm{d}\bar{k}_\perp. \quad (6)
\]

Here \( \bar{A}_0(\bar{x}) \) is a dimensionless, external-field-independent function of the arguments \( \bar{x}_3 = x_3/L_B, \quad \bar{x}_\perp = x_\perp/L_B, \) and the integration variables \( \bar{k}_3 = k_3 L_B, \quad \bar{k}_\perp = k_\perp L_B \) are used. We refer to Eq. \([6]\) as establishing a scaling regime that describes the potential measured in inverse Larmour units as a universal function of coordinates measured in Larmour units. It holds for \( |x| \ll (2m)^{-1} \).

The simple representation \([6]\) can be further simplified if \( x_3 \) or \( x_\perp \) are large in the Larmour scale: \( |\bar{x}_3| \gg 1 \), or \( |\bar{x}_\perp| \gg 1 \) (but remain small in the Compton scale). In this case the integration in \([6]\) is restricted to the domain \( \bar{k}_3^2 \ll 1 \) where the exponential \( \exp(-\bar{k}_3^2/2) \) should be taken as unity. Then \([6]\) is reduced to the isotropic Yukawa law

\[
A_0(x) \approx \frac{q}{4\pi L_B} \frac{\exp\left(\frac{-2\alpha/\pi}{\sqrt{x_1^2 + x_3^2}}\right)}{\sqrt{x_1^2 + x_3^2}}. \quad (7)
\]

This can be established by tracing \([6]\) back to \([1]\) with

\[
- \kappa_2(\infty, 0) = \frac{2\alpha}{\pi L_B} = \frac{2ob}{\pi} m^2 = M^2 \quad (8)
\]

substituted for \( -\kappa_2(k_3^2, k_1^2) \) in the denominator. Here \( M \) is the "effective photon mass" noted in Ref. \([13]\). The Yukawa law \([7]\) establishes the short-range character of the static electromagnetic forces in the Larmour scale. Stress, however, that the genuine photon mass understood as its rest energy is always strictly equal to zero as a consequence of the gauge invariance reflected in the approximation-independent relation \( \kappa_2(0, 0) = 0 \) respected by \([2]\). Hence, the potential, produced by a static charge, should be long-range for sufficiently large distances. This is the case, indeed. One can see by inspecting the curves of Fig.1, computed using Eq. \([1]\) for \( x_\perp = 0 \), that the scaling regime \([6], [7] \) does fail for sufficiently large values of \( x_3: \sim 0.1/(2m) = 50L_B \) for \( b = 10^6 \), \( \sim 0.2/(2m) = 33L_B \) for \( b = 10^6 \), \( \sim 0.3/(2m) = 15L_B \) for \( b = 10^4 \) - the larger, the smaller the field [for these distances Eqs. \([6]\) and \([7]\) are already the same]. Starting with these values, the potential curves approach their envelope, that can be fitted as \( A_0(x_3, 0) \approx (q/4\pi)1.41/(x_3 + 1.04/2m) \), unlike the scaling curves \([6]\), and \([7]\) that tend fast to zero. It is in this place that the abruptly falling - short-range - potential turns into a slowly decreasing - along the envelope curve - long-range potential. An analogous change from the short- to long-range behavior is observed in Fig. 2, where the electron energy in the field \([1]\) is plotted against the transverse distance from the charge at \( x_3 = 0 \). At small distances all the dashed curves in Figs. \([1]\) and \([2]\) approach the thick solid Coulomb curve in accord with \([13]\).

The scaling regime \([6], [7]\) depends on the fact that the eigenvalue \([2]\) grows linearly with the magnetic field. If this linearity, supposedly, retains in higher-loop approximations, one may conjecture that the calculations of the latter may be reduced to finding \( \alpha^n \)-corrections to the mass \([8]\). Potential \([10]\) implies the suppression of
electrostatic force by the linearly growing term in the denominator of (1) at large distances in the Larmor scale from the charge, but not close to the charge, where it has the same singularity as the Coulomb law.

For larger distances $x_3, \perp$, not small in the Compton scale, one may expect that only integration over small $k_3, \perp$ is important in (1) (for a more thorough analysis see [7]). In this limit (2) behaves as $\kappa_2(k_\perp^2 0) \approx -\frac{2}{\pi^2} k_\perp^2$. With this substitution the integral in (1) is

$$A_0(x_3, x_\perp) \approx \frac{1}{4\pi} \frac{q}{\sqrt{(x_\perp')^2 + x_3^2}},$$

where $x_\perp' = \beta x_\perp$, $\beta \equiv (1 + \alpha b / 3\pi)^{1/2}$, $x_\perp' > x_\perp$. For small $\alpha b / 3\pi$, Eq. (9) coincides with the result of [6].

Eq. (9) is an ”anisotropic Coulomb law”, according to which the attraction force decreases with distance from the source along the transverse direction faster than along the magnetic field, but remains longrange. In accord with (9), the curves $A_0(x_3, 0)$ in Fig 1 all approach at large distances $|x_3|$ the Coulomb law $q/(4\pi |x_3|)$, whereas each curve $A_0(0, x_\perp)$ in Fig. 2 reaches at large $x_\perp$ the asymptote $A_0(0, x_\perp) \approx q/(4\pi x_\perp') = A_0'(0, x_\perp)/\beta$, different for each field, i.e., the potential is anisotropic. Again, the same as for short distances considered above, we face - now anisotropic - suppression of the Coulomb force due to the linearly growing term in (2). The equipotential surface is an ellipsoid stretched along the magnetic field. The electric field of the charge $E = -\nabla A_0(x_3, x_\perp)$ is a vector with the components $(q/2\pi)(x_3^2 + \beta^2 x_\perp^2)^{-3/2}(x_3, x_3, x_\perp)$. It is not directed towards the charge, but makes an angle $\phi$ with the radius-vector $r$, cos $\phi = (x_3^2 + \beta^2 x_\perp^2)(x_3^2 + \beta^2 x_\perp^2)^{-1/2}(x_3^2 + x_\perp^2)^{-1/2}$. In the limit of infinite magnetic field, $\beta = \infty$, the electric field of the point charge is directed normally to the axis $x_3$.

The regime (9) corresponds to the approximation, where only quadratic terms in powers of the photon momentum are kept in $\kappa_2$. Within this scope the dielectric permeability of the vacuum is independent of the frequency, and the refractive index depends only upon the angle in the space [10].

Consider an impact of the Coulomb potential modification on the nonrelativistic hydrogenlike atom ground state energy. The wave function $\Psi(x_3)$ is subject to one-dimensional Schrödinger equation [8] with respect to $x_3$, valid in the region $|x_3| \geq L_B$. The transverse coordinate $x_\perp$ in the argument of the potential is replaced by the momentum of the transverse center-of-mass motion [17] and should be set equal to zero for the ground state.

For the fields in the range $1 \leq b \leq 2\pi/\alpha \approx 10^3$ the correction (1) to the Coulomb potential may be treated as perturbation. Keeping only the first-order term in $\kappa_2$ in (1) we may derive the magnetic analog of the Uehling potential [14]. At $x_\perp = 0$ it is

$$\Delta A_0(x_3, 0) \approx \frac{q b m}{8\pi^2} \int_0^{\pi/2} e^{-2b|m|x_3|/\cos \phi \cos^2 \phi} \cos^2 \phi \, d\phi.$$ (10)

This correction is of the order of $\alpha b$, i.e., is much larger than $\alpha$. Eq. (10) implies $C \approx \alpha b/16$. Calculating its matrix element with Loudon’s [8] wave functions we obtain the positive correction ($q = Ze$)

$$\frac{2Z^2 \alpha^3 b m}{3\pi} \ln \frac{b}{4\alpha^2} = Z^2 b^2 \left( \frac{\ln b}{8.454} + 1 \right) \times 0.356 \text{ eV}$$ (11)

to the negative Loudon-Elliott ground state energy

$$E_0 = -2Z^2 \alpha^2 m \ln \frac{\sqrt{b}}{2\alpha}.$$ (12)

Eq. (11) may be used for $Z \leq 6$. For instance, for $Z = 1$ and $b = 200$ it makes about 100 eV, while $E_0 \approx -2.5$ eV.
keV. Although the proton has a finite size $R \sim 10^{-13}$ cm, the Coulomb $1/|x_3|$ part of the potential remains the same within the definition range of the Schrödinger equation, since $R < L_B$ within the range of $b$ considered in this paragraph (moreover, $R \ll L_B$). On the contrary, the vacuum-polarization part does depend on $R$, like it does in no-magnetic-field case [13]. However, the finite-size correction to [11] is $\sim (R/L_B)^2$ to be neglected within the present scope of accuracy.

The singularity $1/|x_3|$ of the Coulomb potential in the origin is known to lead to the energy spectrum unbounded from below [8]: as this singularity is cut off at the Larmour length, $L_B = (m\sqrt{b})^{-1}$, the ground state energy tends to $\infty$ with the growth of the magnetic field, when $L_B \to 0$. This feature is cured by the vacuum polarization. As $b \to \infty$ the region in Fig.1 between the potential curve and the abcissa below the point $-eV \approx -1.4 \times 2\alpha Z m$, where it is crossed by the envelope, becomes infinitely deep and thin. The area

$$S = \int_{L_B}^\infty A_0(x_3, 0)dx_3$$

is found from the equation $A_0(x_3, 0) = V$, has a finite limit if and only if $C$ grows proportionally to $1/L_B$. This is the case: when calculated following Eq. [6], $C \approx 0.9594\sqrt{ab}/2\pi$. Thus, the dominating behavior of the modified Coulomb potential in the origin becomes the $\delta$-function. Combining it with the fit for the envelope we come to the limiting form of the electron potential energy at $b = \infty$ (here $|x_3|$ does not exceed a few $(2m)^{-1}$)

$$-eA(x_3, 0) = -2\alpha Z \left( S \delta(x_3) + \frac{1.4m}{2m|x_3|} + 1.04 \right) ,$$

where $S = \ln(\sqrt{2\alpha/0.96} - 1) + 0.96\sqrt{2\alpha/\pi} = 1.79$. A more rigorous calculation done with the use of eqs. [6] or [7] for the short-range part of the potential instead of $eV$ results in $S = 2.18 \approx -E_i(-2\alpha/\pi)^{1/2}$. $E_i$ is the exponential integral. With the $1/|x|$-singularity replaced by the $\delta$-function, the ground energy level is certainly finite. Applying the formula for the ground energy

$$-2m\left( \int_{L_B}^\infty A_0(x_3, 0)dx_3 \right)^2$$

valid in a shallow well potential [10], with eq. (13) integrated up to the point $x_3 = 2.6/2m$, where the fitted curve crosses the Coulomb law, and the latter taken as the integral for larger $x_3$ up to the Bohr radius $a_B = (\alpha a)^{-1}$, we estimate the finite limiting value for the ground energy as

$$E_{\text{lim}} = -2mZ^2a^2 \approx 73.8 = -Z^2 \times 4 \text{ keV}.$$ (14)

The Loudon-Elliott energy [12] would overrun the limiting energy already for the magnetic field as large as $b = 6600$, when yet $R \ll L_B$. The ground level reaches 92% of its limiting value for $b = 5 \times 10^4$. After the magnetic field reaches the value $b = 1.5 \times 10^5$, when $R$ and $L_B$ equalize, the Coulomb potential is cut off at the proton size $x_3 = R$. Setting $L_B = R$ in [12] we would get the minimum value for the Loudon-Elliott energy ($Z = 1$) to be $-5.0 \text{ keV}$, which is essentially lower than [13].

The significant modification of the Coulomb potential of an electric charge by the vacuum polarization in external constant magnetic field $B \gg B_0$, shown to eliminate the unboundedness from below of the nonrelativistic hydrogen spectrum, is apt of having more implementation as far as other situations where electrostatic fields are important are concerned, e.g. properties of matter on surfaces of extremely magnetized neutron stars.

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