Non-linear constitutive equations for gravitoelectromagnetism

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This paper studies non-linear constitutive equations for gravitoelectromagnetism. Eventually, the problem is solved of finding, for a given particular solution of the gravity-Maxwell equations, the exact form of the corresponding non-linear constitutive equations.

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I. INTRODUCTION

Over the past decade, a description of non-linear classical electrodynamics and Yang-Mills theory has been considered in the literature [1–3], with the hope of being able to extend it to a broader framework, including gauge theories of gravity [4] and quantum gravity [5].

However, no explicit calculation had been performed, and the formulation remained too general for the physics community to be able to appreciate its potentialities. For this purpose, as a first step, we here consider the gravitoelectromagnetism in the weak-field approximation (following, e.g., [6]). Recall the standard Maxwell equations in SI units [7]

\[
\begin{align*}
\text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \\
\text{curl } \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \text{div } \mathbf{D} = \rho,
\end{align*}
\]

(1.1)

where \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic field, \( \rho \) is charge density, \( \mathbf{j} \) is electric current density. In the linear case

\[
\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E},
\]

(1.2)

In the non-linear case these equations can be presented in the form [8]

\[
\begin{align*}
\mathbf{D} &= M(I_1, I_2) \mathbf{B} + \frac{1}{c^2} N(I_1, I_2) \mathbf{E}, \\
\mathbf{H} &= N(I_1, I_2) \mathbf{B} - M(I_1, I_2) \mathbf{E},
\end{align*}
\]

(1.3)

where the invariants are \( F_{\mu\nu} \) being the electromagnetic field tensor, with Hodge dual *\( F^{\mu\nu} \)

\[
I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2, \quad I_2 = -\frac{c}{4} F_{\mu\nu} * F^{\mu\nu} = \mathbf{B} \cdot \mathbf{E}.
\]

(1.4)

Their gravitational analogues in SI units are

\[
\begin{align*}
\text{curl } \mathbf{E}_g &= -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div } \mathbf{B}_g = 0, \\
\text{curl } \mathbf{B}_g &= \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t} + \frac{1}{\varepsilon_0 c^2} \mathbf{j}_g, \quad \text{div } \mathbf{E}_g = \frac{1}{\varepsilon_0} \rho_g,
\end{align*}
\]

(1.5)

(1.6)
where \( E_g \) is the static gravitational field (conventional gravity, also called gravitoelectric for the sake of analogy), \( B_g \) is the gravitomagnetic field, \( \rho_g \) is mass density, \( j_g \) is mass current density, \( G \) is the gravitational constant, \( \varepsilon_g \) is the gravity permittivity (analog of \( \varepsilon_0 \)). Here

\[
\varepsilon_g = -\frac{1}{4\pi G}, \quad \mu_g = -\frac{4\pi G}{c^2},
\]

are the gravitational permittivity and permeability, respectively.

The main idea is to introduce analogues of \( H \) and \( D \) to write (1.5) and (1.6) in the Maxwell form for 4 fields in SI as

\[
curl E_g = -\frac{\partial B_g}{\partial t}, \quad \text{div} B_g = 0, \tag{1.8}
\]

\[
curl H_g = \frac{\partial D_g}{\partial t} + j_g, \quad \text{div} D_g = \rho_g. \tag{1.9}
\]

In the linear-gravity case

\[
D_g = \varepsilon_g E_g, \tag{1.10}
\]

\[
B_g = \mu_g H_g, \tag{1.11}
\]

\[
\varepsilon_g \mu_g = \frac{1}{c^2}. \tag{1.12}
\]

Note now that the linear-gravity case (1.10)–(1.12) corresponds to weak approximation and some special case of gravitational field configuration. We generalize it to non-linear case which can describe other configurations and non-weak fields, as in (1.3), by

\[
D_g = M_g (I_{g1}, I_{g2}) B_g + \frac{1}{c^2} N_g (I_{g1}, I_{g2}) E_g, \tag{1.13}
\]

\[
H_g = N_g (I_{g1}, I_{g2}) B_g - M_g (I_{g1}, I_{g2}) E_g, \tag{1.14}
\]

where the invariants are

\[
I_{g1} = B_g^2 - \frac{1}{c^2} E_g^2, \quad I_{g2} = B_g \cdot E_g. \tag{1.15}
\]

The gravity-Maxwell equations (1.8)–(1.9) together with the non-linear gravity-constitutive equations (1.13)–(1.14) can give a non-linear electrodynamics formulation of gravity (or at least some particular instances of this construction).

**II. LINEAR GRAVITO-ELECTROMAGNETIC WAVES**

The gravity-Maxwell equations for gravito-electromagnetic waves (far from sources) are

\[
curl E_g = -\frac{\partial B_g}{\partial t}, \quad \text{div} B_g = 0, \tag{2.1}
\]

\[
curl H_g = \frac{\partial D_g}{\partial t}, \quad \text{div} D_g = 0 \tag{2.2}
\]

with generic values of permittivity and permeability (1.7). Then

\[
curl E_g = -\mu_g \frac{\partial H_g}{\partial t}, \quad \text{div} H_g = 0, \tag{2.3}
\]

\[
curl H_g = \varepsilon_g \frac{\partial E_g}{\partial t}, \quad \text{div} E_g = 0, \tag{2.4}
\]

We differentiate the first equation with respect to time: \( \text{curl} \left( \frac{\partial}{\partial t} E_g \right) = -\mu_g \frac{\partial^2 H_g}{\partial t^2} \Rightarrow \frac{1}{\varepsilon_g} \text{curl} (\text{curl} H_g) = -\mu_g \frac{\partial^2 H_g}{\partial t^2} \).

Since \( \text{curl} (\text{curl} H_g) = \text{grad}(\text{div} H_g) - \Delta H_g = -\Delta H_g \), then

\[
\Delta H_g = \varepsilon_g \mu_g \frac{\partial^2 H_g}{\partial t^2}. \tag{2.5}
\]
By analogy, from the second equation \( \text{curl} \frac{\partial}{\partial t} H_g = \varepsilon_g \frac{\partial^2 E_g}{\partial t^2} \Rightarrow -\frac{1}{\mu_g} \text{curl} \text{curl} E_g = \varepsilon_g \frac{\partial^2 E_g}{\partial t^2} \). Hence we get the wave equation for \( E_g \),

\[
\Delta E_g = \varepsilon_g \mu_g \frac{\partial^2 E_g}{\partial t^2}.
\] (2.6)

### III. NONLINEAR GRAVITO-ELECTROMAGNETIC WAVES

The differences begin with the constitutive equations (1.13)–(1.14). For simplicity put first \( M_g = 0 \). Then

\[
D_g = \frac{N}{c^2} E_g,
\] (3.1)

\[
B_g = \frac{1}{N} H_g
\] (3.2)

where \( N \equiv N_g (I_{g1}, I_{g2}) \). The Maxwell equations become (hereafter the dots denote time derivatives)

\[
\text{curl} E_g = -\left( \frac{1}{N} \right) \cdot H_g - \frac{1}{N} \frac{\partial H_g}{\partial t},
\] (3.3)

\[
\text{div} \left( \frac{1}{N} H_g \right) = H_g \text{grad} \left( \frac{1}{N} \right) + \frac{1}{N} \text{div} (H_g) = 0,
\] (3.4)

\[
\text{curl} H_g = \frac{\dot{N}}{c^2} E_g + \frac{N}{c^2} \frac{\partial E_g}{\partial t},
\] (3.5)

\[
\text{div} \left( \frac{N}{c^2} E_g \right) = E_g \text{grad} \left( \frac{1}{c^2} \right) + \frac{N}{c^2} \text{div} (E_g) = 0.
\] (3.6)

Take derivative of (3.3) with respect to time and get

\[
\text{curl} \frac{\partial}{\partial t} E_g = -\left( \frac{1}{N} \right) \cdot \ddot{H}_g - 2 \left( \frac{1}{N} \right) \cdot \frac{\partial H_g}{\partial t} - \frac{1}{N} \frac{\partial^2 H_g}{\partial t^2}.
\] (3.7)

From (3.5) it follows \( \frac{\partial E_g}{\partial t} = \frac{c^2}{N} \text{curl} H_g - \frac{\dot{N}}{N} E_g \). Then we get

\[
\text{curl} \left( \frac{c^2}{N} \text{curl} H_g - \frac{\dot{N}}{N} E_g \right) = -\left( \frac{1}{N} \right) \cdot \ddot{H}_g - 2 \left( \frac{1}{N} \right) \cdot \frac{\partial H_g}{\partial t} - \frac{1}{N} \frac{\partial^2 H_g}{\partial t^2}.
\] (3.8)

The left-hand side here is

\[
\text{curl} \left( \frac{c^2}{N} \text{curl} H_g - \frac{\dot{N}}{N} E_g \right)
\]

\[
= \text{grad} \frac{c^2}{N} \text{curl} H_g + \frac{c^2}{N} \text{grad} \text{div} H_g - \frac{c^2}{N} \Delta H_g - \frac{\dot{N}}{N} \text{curl} E_g - \text{grad} \frac{\dot{N}}{N} \times E_g.
\]

From (3.4) we get \( \text{div} (H_g) = -NH_g \text{grad} \left( \frac{1}{N} \right) \neq 0 \). Thus, the non-linear analogue of the wave equation is

\[
\text{grad} \frac{c^2}{N} \times \text{curl} H_g + \frac{c^2}{N} \text{grad} \text{div} H_g - \frac{c^2}{N} \Delta H_g - \frac{\dot{N}}{N} \text{curl} E_g - \text{grad} \frac{\dot{N}}{N} \times E_g
\]

\[
= -\left( \frac{1}{N} \right) \cdot \ddot{H}_g - 2 \left( \frac{1}{N} \right) \cdot \frac{\partial H_g}{\partial t} - \frac{1}{N} \frac{\partial^2 H_g}{\partial t^2}.
\] (3.9)

Note that if \( N = \text{const} \), then we obtain the usual wave equation

\[
\Delta H_g = \frac{1}{c^2} \frac{\partial^2 H_g}{\partial t^2}.
\] (3.10)
Take now the constitutive equations in the form
\[ \mathbf{D}_g = M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g, \]
\[ \mathbf{H}_g = N \mathbf{B}_g - M \mathbf{E}_g, \]
where \(N, M\) are constants. In absence of sources, the Maxwell equations become
\[
\text{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div} \mathbf{B}_g = 0, \quad (3.14)
\]
\[
\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \quad \text{div} \mathbf{D}_g = 0. \quad (3.15)
\]
If we express the Maxwell equations through \(\mathbf{E}_g\) and \(\mathbf{B}_g\), the second pair of equations become
\[
\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t} \implies N \text{curl} \mathbf{B}_g - M \text{curl} \mathbf{E}_g = M \frac{\partial \mathbf{B}_g}{\partial t} + \frac{N}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (3.16)
\]
Since \(\text{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}\), we get
\[
\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (3.17)
\]
The second equation, \(\text{div} \mathbf{D}_g = 0\), reduces to \(M \text{div} \mathbf{B}_g + \frac{N}{c^2} \text{div} \mathbf{E}_g = 0\). Since \(\text{div} \mathbf{B}_g = 0\), we get
\[
\text{div} \mathbf{E}_g = 0. \quad (3.18)
\]
Thus, using constitutive equations with constant \(M\) and \(N\) we have Maxwell equations in terms of \(\mathbf{B}_g\) and \(\mathbf{E}_g\), i.e.
\[
\text{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div} \mathbf{B}_g = 0, \quad (3.19)
\]
\[
\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}, \quad \text{div} \mathbf{E}_g = 0. \quad (3.20)
\]
At this stage, we get the wave equations in the standard way. The time derivative of the first equation yields
\[
\text{curl} \frac{\partial}{\partial t} \mathbf{E}_g = -\frac{\partial^2 \mathbf{B}_g}{\partial t^2} \implies c^2 \text{curl} (\text{curl} \mathbf{B}_g) = -\frac{\partial^2 \mathbf{B}_g}{\partial t^2}. \quad (3.21)
\]
Since \(\text{curl} (\text{curl} \mathbf{B}_g) = \text{grad} (\text{div} \mathbf{B}_g) - \Delta \mathbf{B}_g = -\Delta \mathbf{B}_g\), then
\[
\Delta \mathbf{B}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}_g}{\partial t^2}. \quad (3.21)
\]
By analogy \(\frac{\partial}{\partial t} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2} \implies -\text{curl} (\text{curl} \mathbf{E}_g) = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2}\), and we get the wave equation for \(\mathbf{E}_g\),
\[
\Delta \mathbf{E}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2}. \quad (3.22)
\]
Thus, the gravi-electromagnetic waves \(\mathbf{E}_g\) and \(\mathbf{B}_g\) have speed \(c\) and do not depend on the constants \(M\) and \(N\).

IV. WAVES AND CONSTITUTIVE EQUATIONS FOR LINEAR CONSTITUTIVE FUNCTIONS

Let us consider the constitutive equations \([1.13] - [1.14]\) as linear functions of the invariants, i.e.
\[
M = M_g (I_{g1}, I_{g2}) = a_m I_{g1} + b_m I_{g2}, \quad (4.1)
\]
\[
N = N_g (I_{g1}, I_{g2}) = c^2 \varepsilon_g + a_n I_{g1} + b_n I_{g2}, \quad (4.2)
\]
a\(_m\), \(b\(_m\), \(a\(_n\), \(b\(_n\) being some constants. From all the Maxwell equations in material media, and in the absence of sources one finds \(\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}\), \(\text{curl} (N \mathbf{B}_g - M \mathbf{E}_g) = \frac{\partial}{\partial t} \left(M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g\right)\), and \(N \text{curl} \mathbf{B}_g - M \text{curl} \mathbf{E}_g = M \frac{\partial \mathbf{B}_g}{\partial t} + \frac{N}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}\).

Since \(\text{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}\), from the last equation one gets
\[
\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (4.3)
\]
The second equation, \( \text{div} \, \mathbf{D}_g = 0 \), reduces to \( \text{div} \left( M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g \right) = 0 \), or \( M \text{div} \mathbf{B}_g + \frac{N}{c^2} \text{div} \mathbf{E}_g = 0 \). Since \( \text{div} \mathbf{B}_g = 0 \), one gets
\[
\text{div} \mathbf{E}_g = 0. \tag{4.4}
\]

V. INVERSE PROBLEM OF NON-LINEAR GRAVITO-ELECTROMAGNETISM

In electrodynamics the direct solution of the Maxwell equations together with the non-linear constitutive equations is a non-trivial and complicated task even for simple systems \([1, 2]\). In previous sections we presented some very special cases of the non-linear functions \( N \) and \( M \). Here we formulate the following inverse problem: if we have some particular solution of the gravity-Maxwell equations (1.8)–(1.9), can we then find the exact form of the corresponding non-linear gravity-constitutive equations (1.13)–(1.14)?

It is natural to consider the case of plane gravitational waves, when the fields have only one space coordinate. We will show that even in this case one can have a non-trivial non-linearity. Let us choose \( \mathbf{E}_g \) and \( \mathbf{B}_g \) mutually orthogonal and perpendicular to the direction of motion
\[
\mathbf{E}_g = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_g = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}, \tag{5.1}
\]
where \( E \equiv E(t, y) \), \( B \equiv B(t, y) \). Now the invariants (1.15) become
\[
I_{g1} = B^2 - \frac{1}{c^2} E^2 \equiv I, \tag{5.2}
\]
\[
I_{g2} = 0. \tag{5.3}
\]

The use of the non-linear gravity-constitutive equations (1.13)–(1.14) gives for the other fields
\[
\mathbf{D}_g = \begin{pmatrix} \frac{1}{c^2} N E \\ 0 \\ M B \end{pmatrix}, \quad \mathbf{H}_g = \begin{pmatrix} -M E \\ 0 \\ N B \end{pmatrix}, \tag{5.4}
\]
where \( N \equiv N(I), M \equiv M(I) \) are the sought for gravity-constitutive functions. They depend on \( I \) only, because of Lorentz invariance (see \([1, 2]\)). Inserting the fields (5.1) and (5.4) into the gravity-Maxwell equations (1.8)–(1.9) without sources gives us 3 equations (hereafter, a prime with the corresponding subscript denotes the first partial derivative with respect to the variable in the subscript, while dot denotes time derivative)
\[
E_y' = B, \tag{5.5}
\]
\[
(NB)'_y = \frac{1}{c^2} (NE)', \tag{5.6}
\]
\[
(ME)'_y = (MB)', \tag{5.7}
\]

Now we take into account that the gravity-constitutive functions \( N, M \) depend only on the invariant \( I \) and present (5.6)–(5.7) as the differential equations for them
\[
N'_Y \left( BI_y - \frac{1}{c^2} E I \right) + N \left( B'_Y - \frac{1}{c^2} E I \right) = 0, \tag{5.8}
\]
\[
M'_Y \left( EI_y - B I \right) = 0, \tag{5.9}
\]
where we have exploited the identities
\[
N'_y = N'_Y I_y', \quad M'_y = M'_Y I_y', \tag{5.10}
\]
\[
\dot{N} = N'_Y \dot{I}, \quad \dot{M} = M'_Y \dot{I}. \tag{5.11}
\]
The second equation (5.9) can be immediately solved by
\[ M (I) = \begin{cases} \frac{M_0}{I} = \text{const}, & \text{if } EI_y' = B\dot{I}, \\ \text{arbitrary}, & \text{if } EI_y' = B\dot{I}. \end{cases} \] (5.12)

The first equation (5.8) can be solved if
\[ \lambda = \frac{(B_y' - \frac{E}{c^2})}{(BI_y' - \frac{E}{c^2})} \] (5.13)
depends only on \( I \), which is a very special case. One then has the differential equation
\[ N'_I + \lambda (I) N = 0, \] (5.14)
and its solution is
\[ N (I) = N_0 e^{-\int \lambda (I) dI}. \] (5.15)

Otherwise, by using the expressions for \( I_y' \) and \( \dot{I} \) from (5.2), i.e.
\[ I_y' = 2BB_y' - \frac{2EE_y'}{c^2}, \quad \dot{I} = 2B\dot{B} - \frac{2E\dot{E}}{c^2}, \] (5.16)
we obtain
\[ 2N'_I \left( B^2B_y' + \frac{1}{c^2}E^2\dot{E} - \frac{2}{c^2}EBE_y' \right) + N \left( B_y' - \frac{1}{c^2}\dot{E} \right) = 0, \] (5.17)
where the sum of terms in brackets is not a function of \( I \) in general.

Usually, in the wave solutions the dependence of fields on frequency \( \omega \) and wave number \( k \) is the same, and therefore we can consider the concrete choice
\[ E (t, y) = f (\varepsilon\omega t + ky) \equiv f (X (t, y)), \quad B (t, y) = g (\varepsilon\omega t + ky) \equiv g (X (t, y)), \] (5.18)
where \( \varepsilon \equiv \pm 1 \), with \( f \) and \( g \) arbitrary smooth nonvanishing functions. Bearing in mind that
\[ E' = f'_XX' = k f'_X, \quad B'_y = g'_X X'_y = k g'_X, \]
\[ \dot{E} = f'_X \dot{X} = \varepsilon\omega f'_X, \quad \dot{B} = g'_X \dot{X} = \varepsilon\omega g'_X, \]
our Eq. (5.5) yields
\[ \varepsilon\omega g'_X = k f'_X. \] (5.19)
Therefore
\[ g (X) = \frac{k}{\varepsilon\omega} f (X) + \alpha, \] (5.20)
where \( \alpha \) is a constant, so that both \( E \) and \( B \) can be expressed through one function only, i.e. \( f \), and the invariant \( I \) reads eventually as
\[ I = \frac{1}{\omega^2} \left( k^2 - \frac{\omega^2}{c^2} \right) f^2 + 2\frac{k}{\varepsilon\omega} \alpha f + \alpha^2. \] (5.21)

The equations for the gravity-constitutive functions take therefore the form
\[ N'_I \left[ 2f \left( k^2 - \frac{\omega^2}{c^2} \right) + 2\frac{\omega^2}{c^2} \alpha^2 \right] + N \left( k^2 - \frac{\omega^2}{c^2} \right) = 0, \] (5.22)
\[ M'_I f'_X \left[ 2f \left( k^2 - \frac{\omega^2}{c^2} \right) + 2k\alpha \right] \alpha = 0, \] (5.23)
having exploited the identities

\[ gI_y' - \frac{f}{c^2} = \left( gI_y' - \frac{f}{c^2} \right) \left[ \frac{2f}{\omega^2} \left( k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \right], \tag{5.24} \]

\[ gf_y' - \frac{f}{c^2} = \frac{f}{\varepsilon \omega} \left[ f \left( k^2 - \frac{\omega^2}{c^2} \right) + k \varepsilon \omega \alpha \right], \tag{5.25} \]

and, after some cancellations,

\[ \left[ f \left( k^2 - \frac{\omega^2}{c^2} \right) + k \varepsilon \omega \alpha \right] \left[ \frac{2f}{\omega^2} \left( k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \alpha \right] = 2 \left( k^2 - \frac{\omega^2}{c^2} \right) I + 2 \frac{\omega^2}{c^2} \alpha^2, \tag{5.26} \]

while

\[ ff_y' - gI = (ff_y' - g\dot{f}) \left[ \frac{2f}{\omega^2} \left( k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \alpha \right], \tag{5.27} \]

\[ ff_y' - g\dot{f} = fX(kf - \varepsilon \omega g) = -\varepsilon \omega fX \alpha. \tag{5.28} \]

The results of our analysis now depend on whether or not \( \alpha \) vanishes. Indeed, if \( \alpha = 0 \), \( M \) is arbitrary and hence we obtain the equation

\[ \left( k^2 - \frac{\omega^2}{c^2} \right) (2IN' + N) = 0, \tag{5.29} \]

which implies that either the dispersion relation

\[ k^2 - \frac{\omega^2}{c^2} = 0 \tag{5.30} \]

holds, with \( N \) kept arbitrary, or such a dispersion relation is not fulfilled, while \( N \) is found from the differential equation

\[ 2IN' + N = 0, \tag{5.31} \]

which is solved by

\[ N(I) = \frac{N_0}{\sqrt{I}}. \tag{5.32} \]

By contrast, if \( \alpha \) does not vanish, \( M \) equals a constant \( M_0 \), while \( N \) solves the more complicated equation \( (5.22) \). At this stage, to be consistent with the dependence of \( N \) on \( I \) only, we have to require again that the dispersion relation \( (5.30) \) should hold, jointly with \( N' = 0 \), which implies the constancy of \( N \): \( N = N_0 \).

VI. CONCLUDING REMARKS

We have brought ‘down to earth’ the general program of considering non-linear constitutive equations for gravito-electromagnetism, by solving the problem of finding, for a given solution of the gravity-Maxwell equations, the exact form of non-linear constitutive equations. We look forward to being able to construct other relevant examples, as well as being able to re-express our models in the language of differential forms, which turned out to be very powerful for general relativity \[ 9–11 \].
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[1] G. A. Goldin and V. M. Shtelen, Phys. Lett. A 279 (2001) 321.
[2] G. A. Goldin and V. M. Shtelen, J. Phys. A: Math. Gen. 37 (2004) 10711.
[3] S. Duplij, G. A. Goldin and V. M. Shtelen, J. Phys. A: Math. Gen. 41 (2008) 304007.
[4] Y. Ne’eman, Acta Phys. Pol. 29 (1998) 827.
[5] G. Esposito, arXiv:1108.3269, in EOLSS Encyclopedia, UNESCO (2011).
[6] S. J. Clark and R. W. Tucker, Class. Quantum Grav. 17 (2000) 4125.
[7] J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1999).
[8] W. I. Fushchich, V. M. Shtelen, and N. I. Serov, Symmetry Analysis and Exact Solutions of Equations of Non-Linear Mathematical Physics (Kluwer, Dordrecht, 1993).
[9] J. F. Plebanski, J. Math. Phys. 18 (1977) 2511.
[10] K. Krasnov, Gen. Rel. Grav. 43 (2011) 1.
[11] R. Capovilla, J. Dell and T. Jacobson, Class. Quantum Grav. 8 (1991) 59.