MODIFICATIONS OF BOHR’S INEQUALITY IN VARIOUS SETTINGS

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Abstract. The concept of Bohr radius for the class of bounded analytic functions was introduced by Harald Bohr in 1914. His initial result received greater interest and was sharpened-refined-generalized by several mathematicians in various settings—which is now called Bohr phenomenon. Various generalization of Bohr’s classical theorem is now an active area of research and has been a source of investigation in numerous other function spaces and including holomorphic functions of several complex variables. Recently, a new generalization of Bohr’s ideas was introduced and investigated by Kayumov et al. In this note, we investigate and refine generalized Bohr’s inequality for the class of quasi-subordinations.

1. Introduction and Preliminaries

Throughout the discussion, we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and $H^\infty$ denote the Banach algebra of all bounded analytic functions $f$ on the unit disk $\mathbb{D}$ with the supremum norm $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$. Also, let $
abla = \{f \in H^\infty : \|f\|_\infty \leq 1\}$.

By the maximum principle, the only members of $\nabla$ that touch the boundary $\partial \mathbb{D}$ of the unit disk are unimodular constant functions. Thus, it is sometimes convenient to exclude constant functions (eg. in the discussion of subordination), but this does not affect our discussion.

In 1914, Bohr [10] observed that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1$$

holds for $0 \leq r \leq 1/6$ and for all $f \in \nabla$. Then M. Riesz, I. Schur and F. W. Wiener independently showed that (1) holds for $0 \leq r \leq 1/3$ and that the constant $1/3$ is best possible. Other proofs and generalizations were also known in the literature, e.g. Rizzonelli [31], Ricci [30], Sidon [37], Tomic [38], and Paulsen et al. [24]. In 1997, Boas and Khavinson [9] showed that a similar phenomenon occurs for polydiscs in $\mathbb{C}^n$. Other multidimensional variants of Bohr-type theorems for bounded complete Reinhardt domains

2010 Mathematics Subject Classification. Primary: 30A10, 30B10 31A05, 30H05, 41A58; Secondary: 30C62, 30C75, 40A30.

Key words and phrases. Analytic functions, harmonic function, quasiconformal mapping, Bohr’s inequality, subordination and quasisubordination.
were obtained by Aizenberg [1]. It is important to mention that Bombieri [11] proved that
\[\sum_{n=0}^{\infty} |a_n| r^n \leq \frac{3 - \sqrt{8(1 - r^2)}}{r} \quad \text{for } 1/3 \leq r \leq 1/\sqrt{2}\]
and for an alternate proof of this inequality, we refer to the recent article [21]. So, it is
natural to ask for the best constant \(C(r) \geq 1\) such that
\[\sum_{n=0}^{\infty} |a_n| r^n \leq C(r)\]
whenever \(f \in B\). Later in [12], Bombieri and Bourgain proved that
\[\sum_{n=0}^{\infty} |a_n| r^n < \frac{1}{\sqrt{1 - r^2}} \quad \text{for } r > 1/\sqrt{2}\]
which in turn shows that \(C(r) \asymp (1 - r^2)^{-1/2}\) as \(r \to 1\). In the same paper they also
obtained a lower bound.

Besides these, several authors investigated the Bohr phenomenon in the recent years. For instance, Kayumov and Ponnusamy [19,20] determined the Bohr radius for the class of
\(p\)-symmetric analytic functions with multiple zeros at the origin, and introduced the notion
of \(p\)-Bohr radius for harmonic functions and obtained the \(p\)-Bohr radius for the class of
odd harmonic bounded functions (see also [4,22]) while in [21] the same authors discussed
powered Bohr radius, originally discussed by Djakov and Ramanujan [14]. Aytuna and
Djakov [6] studied the Bohr property of bases for holomorphic functions, and Ali et al. [3]
discussed the Bohr radius for the class of starlike logarithmic mappings. For further
studies on the Bohr phenomenon, we refer to the survey articles [2,16] and the references
therein. On the other hand, the authors in [7] (see also Queffélec [29]) extended the work
of Bohr to the setting of Dirichlet series.

In order to make the statements of the recent generalization and our present refined
formulation, we need to introduce some basic notations. Let \(\mathcal{F}\) denote the set of all
sequences \(\varphi = \{\varphi_n(r)\}_{n=0}^{\infty}\) of nonnegative continuous functions in \([0, 1)\) such that the
series \(\sum_{n=0}^{\infty} \varphi_n(r)\) converges locally uniformly with respect to \(r \in [0, 1)\). For convenience,
we let \(\Phi_N(r) = \sum_{n=N}^{\infty} \varphi_n(r)\) whenever \(\varphi \in \mathcal{F}\). Also, for \(f(z) = \sum_{n=0}^{\infty} a_n z^n \in B\) and
\(f_0(z) := f(z) - f(0)\), in what follows we let
\[B_N(f, \varphi, r) := \sum_{n=N}^{\infty} |a_n| \varphi_n(r) \quad \text{for } N \geq 0\]
so that \(B_0(f, \varphi, r) = |a_0| \varphi_0(r) + B_1(f, \varphi, r)\). In addition, we also let
\[A(f_0, \varphi, r) := \sum_{n=1}^{\infty} |a_n|^2 \left( \frac{\varphi_{2n}(r)}{1 + |a_0|} + \Phi_{2n+1}(r) \right) \quad \text{and} \quad \|f_0\|_r^2 = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.\]
In particular, when \(\varphi_n(r) = r^n\), the formula for \(A(f_0, \varphi, r)\) takes the following simple form
\[A(f_0, r) := \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r^2,\]
since \(\Phi_{2n+1}(r) = r^{2n+1}/(1 - r)\).
The following generalization is obtained recently by Kayumov et al. [18].

**Theorem A.** (18) Let \( f \in \mathcal{B}, f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( p \in (0, 2) \). If \( \varphi = \{ \varphi_n(r) \}_{n=0}^{\infty} \in \mathcal{F} \) satisfies \( p \varphi_0(r) > 2 \Phi_1(r) \) for \( r \in [0, R) \), where \( R \) is the minimal positive root of the equation \( p \varphi_0(x) = 2 \Phi_1(x) \), then the following sharp inequality holds:

\[
B_f(\varphi, p, r) := |a_0|^p \varphi_0(r) + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) \leq \varphi_0(r) \quad \text{for all } r \leq R.
\]

In the case when \( \varphi_0(x) < (2/p) \sum_{k=1}^{\infty} \varphi_k(x) \) in some interval \((R, R + \varepsilon)\), the number \( R \) cannot be improved. If the functions \( \varphi_k(x) \) \((k \geq 0)\) are smooth functions then the last condition is equivalent to the inequality

\[
\varphi_0'(R) < \frac{2}{p} \sum_{k=1}^{\infty} \varphi_k'(R).
\]

For convenience, we let \( B_f(\varphi, r) := B_f(\varphi, 1, r) \), and it is natural to call this new majorant series as a **generalized majorant series** for \( f \in \mathcal{B} \). We now begin with the discussion with the two basic properties of the new majorant series, which will be used in Section 6.

This lemma is well-known in the fundamental case of \( \varphi_n(r) = r^n \) (cf. [25]).

**Lemma 1.** Let \( f, g \in \mathcal{B} \) and \( \varphi = \{ \varphi_n(r) \}_{n=0}^{\infty} \in \mathcal{F} \). Then

1. \( B_{f+g}(\varphi, r) \leq B_f(\varphi, r) + B_g(\varphi, r) \) for \( r \in [0, 1) \).
2. \( B_{fg}(\varphi, r) \leq B_f(\varphi, r) B_g(\varphi, r) \) for \( r \in [0, 1) \) provided \( \varphi_k \)'s satisfy the additional condition \( \varphi_{m+n}(r) \leq \varphi_m(r) \varphi_n(r) \) for all \( m, n \geq 0 \) and \( r \in [0, 1) \).

Also, we note the trivial fact \( B_{\alpha f}(\varphi, r) = |\alpha| B_f(\varphi, r) \) for all \( \alpha \in \mathbb{C} \).

**Proof.** The proof of this lemma is easy, but for the sake of completeness, we include the proofs. Clearly,

\[
\sum_{n=0}^{\infty} |a_n + b_n| \varphi_n(r) \leq \sum_{n=0}^{\infty} |a_n| \varphi_n(r) + \sum_{n=0}^{\infty} |b_n| \varphi_n(r).
\]

The product \( fg \) takes the form

\[
\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \left( \sum_{m+j=n} a_m b_j \right) z^n,
\]

which by equating the coefficients of \( z^n \) on both sides gives

\[
c_n = \sum_{m+j=n} a_m b_j \quad \text{for each } n \geq 0.
\]
Applying the triangle inequality to the last relation shows that
\[
\left| c_n \right| \varphi_n(r) \leq \sum_{n=0}^{\infty} \left( \sum_{m+j=n} \left| a_m \right| \left| b_j \right| \right) \varphi_n(r)
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{m+j=n} \left| a_m \right| \varphi_m(r) \left| b_j \right| \varphi_j(r)
\]
\[
= \left( \sum_{m=0}^{\infty} a_m \varphi_m(r) \right) \sum_{j=0}^{\infty} \left| b_j \right| \varphi_j(r),
\]
where in the second inequality above we have used the inequality \( \varphi_m + \varphi_j \leq \varphi_m \varphi_j \).

The proof is complete. \( \square \)

The article is organized as follows. In Section 2, our aim is to improve Theorem A (see Theorem 1) and as a consequence, we establish few corollaries, remarks and examples. In Section 3, we present an application which will provide more examples of Bohr-type inequality in a refined form. Section 4 is dedicated to derive weighted Bohr radius for quasi-subordination family of analytic functions whereas in Section 5 we deal with weighted Bohr type inequality for locally univalent quasiconformal harmonic mappings. Finally in Section 6, using the discussion of earlier sections, we consider the weighted Bohr type inequality for the derivative of Schwarz functions along with few other related results.

2. Weighted Bohr Type Inequality for Functions in \( \mathcal{B} \)

For the proof of our first theorem, we need the following lemma due to Carlson [13].

**Lemma B.** Suppose that \( f \in \mathcal{B} \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Then the following inequalities hold.

(a) \( |a_{2n+1}| \leq 1 - |a_0|^2 - \cdots - |a_n|^2, \quad n = 0, 1, \ldots \)

(b) \( |a_{2n}| \leq 1 - |a_0|^2 - \cdots - |a_{n-1}|^2 - \frac{|a_n|^2}{1+|a_0|}, \quad n = 1, 2, \ldots \)

Further, to have equality in (a) it is necessary that \( f \) is a rational function of the form
\[
f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n + \varepsilon z^{2n+1}}{1 + (a_0 z^{2n} + \cdots + a_0 z^{2n+1}) \varepsilon}, \quad |\varepsilon| = 1,
\]
and to have equality in (b) it is necessary that \( f \) is a rational function of the form
\[
f(z) = \frac{a_0 + a_1 z + \cdots + \frac{a_n}{1+|a_0|} z^n + \varepsilon z^{2n}}{1 + \frac{a_0}{1+|a_0|} z^{2n} + \cdots + \frac{a_0}{1+|a_0|} z^{2n+1} \varepsilon}, \quad |\varepsilon| = 1,
\]
where the term \( a_0 a_n^2 \varepsilon \) is nonnegative real.

**Theorem 1.** Let \( f \in \mathcal{B} \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( p \in (0, 2) \). If \( \{ \varphi_n(r) \}_{n=0}^{\infty} \in \mathcal{F} \) satisfies the inequality
\[
p \varphi_0(r) > 2 \Phi_1(r),
\]

The proof of Theorem 1 is based on Lemma B and the triangle inequality. 

The proof is complete. \( \square \)
then the following sharp inequality holds:

\[(4) \quad |a_0|^p \varphi_0(r) + B_1(f, \varphi, r) + A(f_0, \varphi, r) \leq \varphi_0(r) \text{ for } r \leq R,\]

where \(R\) is the minimal positive root of the equation \(p\varphi_0(x) = 2 \Phi_1(x)\). In the case when \(p\varphi_0(x) < 2 \Phi_1(x)\) in some interval \((R, R + \epsilon)\), the number \(R\) cannot be improved.

**Proof.** Using Carlson’s lemma, we may rewrite \(B_1(f, \varphi, r)\) as

\[
B_1(f, \varphi, r) = \sum_{n=1}^{\infty} |a_{2n}| \varphi_{2n}(r) + \sum_{n=0}^{\infty} |a_{2n+1}| \varphi_{2n+1}(r)
\]

\[
\leq \sum_{n=1}^{\infty} \left[ 1 - \sum_{k=0}^{n-1} |a_k|^2 \frac{|a_n|^2}{1 + |a_0|} \right] \varphi_{2n}(r) + \sum_{n=0}^{\infty} \left[ 1 - \sum_{k=0}^{n} |a_k|^2 \right] \varphi_{2n+1}(r)
\]

\[
= (1 - |a_0|^2) \Phi_1(r) - \sum_{n=1}^{\infty} \frac{|a_n|^2}{1 + |a_0|} \varphi_{2n}(r) - |a_1|^2 \Phi_3(r) - |a_2|^2 \Phi_5(r) - \cdots
\]

\[
= (1 - |a_0|^2) \Phi_1(r) - \sum_{n=1}^{\infty} |a_n|^2 \left[ \frac{\varphi_{2n}(r)}{1 + |a_0|} + \sum_{m=2n+1}^{\infty} \varphi_m(r) \right]
\]

so that

\[
|a_0|^p \varphi_0(r) + B_1(f, \varphi, r) + A(f_0, \varphi, r) \leq \Phi(p, |a_0|, r),
\]

where

\[
\Phi(p, |a_0|, r) = |a_0|^p \varphi_0(r) + (1 - |a_0|^2) \Phi_1(r).
\]

It can be easily shown that for \(0 < p \leq 2\), the inequality \(\Phi(p, |a_0|, r) \leq \varphi_0(r)\) holds for \(r \leq R\), under the condition \((3)\) (see also [18, Theorem 1]). This completes the proof of the inequality \((4)\).

Now let us prove that \(R\) is an optimal number. We consider the function

\[
f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} z^n, \quad a \in [0, 1).
\]

For this function, with \(a_0 = a\) and \(a_n = -(1 - a^2)a^{n-1}\), straightforward calculation shows that
\[ |a_0|^p \varphi_0(r) + B_1(f, \varphi, r) + A(f_0, \varphi, r) \]

\[ = a^p \varphi_0(r) + (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) + (1 - a^2)^2 \sum_{n=1}^{\infty} a^{2n-2} \left[ \frac{\varphi_{2n}(r)}{1 + a} + \Phi_{2n+1}(r) \right] \]

\[ = \varphi_0(r) + (1 - a) \left[ 2 \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) - p \varphi_0(r) \right] \]

\[ - (1 - a) \left[ (1 - a) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) + \left( \frac{1 - a^p}{1 - a} - p \right) \varphi_0(r) \right] \]

\[ + (1 - a^2)^2 \sum_{n=1}^{\infty} a^{2n-2} \left[ \frac{\varphi_{2n}(r)}{1 + a} + \Phi_{2n+1}(r) \right] \]

\[ = \varphi_0(r) + (1 - a) \left[ 2 \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) - p \varphi_0(r) \right] + O((1 - a)^2) \]

as \( a \to 1^- \). Now it is easy to see that the right hand side number in the above expression is \( > \varphi_0(r) \) when \( a \) is close to 1. The proof of the theorem is complete. \( \square \)

In what follows, \( [x] \) denotes the largest integer no more than \( x \), where \( x \) is a real number.

**Corollary 1.** Let \( f \in B, f(z) = \sum_{k=0}^{\infty} a_k z^k \), and \( p \in (0, 2] \). Then for each \( n \in \mathbb{N} \) with \( n \geq 3 \), the following inequality holds:

\[ |a_0|^p + \sum_{k=1}^{n} |a_k| r^k + \frac{1}{1 + |a_0|} \sum_{k=1}^{s} |a_k|^2 r^{2k} + \sum_{k=1}^{t} |a_k|^2 r^{2k+1} \left( \frac{1 - r^{n-2k}}{1 - r} \right) \leq 1 \]

for \( r \leq R_n(p) \), where \( s = \lfloor n/2 \rfloor \), \( t = \lfloor (n - 1)/2 \rfloor \) and \( R_n(p) \) is the minimal positive root of the equation

\[ p(1 - r) - 2r(1 - r^n) = 0, \text{ or } p = 2 \sum_{k=1}^{n} r^k. \]

**Proof.** Set \( \varphi_k(r) = r^k \) for \( 0 \leq k \leq n \), and \( \varphi_k(r) = 0 \) for \( k > n \). First we remark that for the case \( n = 1 \), (1) reduces to

\[ |a_0|^p + |a_1| r \leq 1 \text{ for } r \leq R_1(p) = \frac{p}{2}, \]

and, for the case \( n = 2 \), (1) becomes

\[ |a_0|^p + |a_1| r + |a_2| r^2 + |a_1|^2 \frac{r^2}{1 + |a_0|} \leq 1 \text{ for } r \leq R_2(p) = \frac{-1 + \sqrt{1 + 2p}}{2}, \]

where \( R_2(p) \) is the minimal positive root of the equation \( p = 2r(1 + r) \). Next we let \( n \geq 3 \). Then (1) is equivalent to

\[ |a_0|^p + \sum_{k=1}^{n} |a_k| r^k + I \leq 1 \text{ for all } r \leq R_n(p), \]

for \( p \in (0, 2] \).
where $R_n(p)$ is the minimal positive root of the equation $p = 2 \sum_{k=1}^{n} r^k$ and

$$I := \sum_{k=1}^{\infty} |a_k|^2 \left[ \frac{r^{2k}}{1 + |a_0|} + r^{2k+1} + \cdots + r^n \right].$$

Now, the proof is divided into two cases. For the even values of $n \in \mathbb{N}$, we set $n = 2m$ $(m \geq 2)$. It follows easily that

$$I = |a_1|^2 \left( \frac{r^2}{1 + |a_0|} + r^3 + \cdots + r^{2m} \right) + \cdots + |a_{m-1}|^2 \left( \frac{r^{2(m-1)}}{1 + |a_0|} + r^{2m-1} + r^2m + r^{2m+1} \right) + |a_m|^2 \left( \frac{r^{2m}}{1 + |a_0|} + r^{2m+1} \right)$$

$$= \frac{1}{1 + |a_0|} \sum_{k=1}^{m} |a_k|^2 r^{2k} + \sum_{k=1}^{m-1} |a_k|^2 \left( r^{2k+1} + \cdots + r^{2m} \right)$$

so that (5) gives the desired inequality when $n \geq 4$ is even. For the odd values of $n \in \mathbb{N}$, we set $n = 2m + 1$ $(m \geq 1)$. It follows that

$$I = |a_1|^2 \left( \frac{r^2}{1 + |a_0|} + r^3 + \cdots + r^{2m+1} \right) + \cdots + |a_{m-2}|^2 \left( \frac{r^{2(m-1)}}{1 + |a_0|} + r^{2m-1} + r^{2m} \right) + |a_m|^2 \left( \frac{r^{2m}}{1 + |a_0|} + r^{2m+1} \right)$$

$$= \frac{1}{1 + |a_0|} \sum_{k=1}^{m} |a_k|^2 r^{2k} + \sum_{k=1}^{m-1} |a_k|^2 \left( r^{2k+1} + \cdots + r^{2m} \right)$$

Again, (5) gives the desired inequality when $n \geq 3$ is odd. Combining the last two cases concludes the proof. \qed

Allowing $n \to \infty$, we obtain the following.

Example 1. Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $p \in (0,2]$. Then Theorem 1 gives the following: For $\varphi_k(r) = r^k$ $(k \geq 0)$, we easily have

$$|a_0|^p + \sum_{k=1}^{\infty} |a_k|^2 r^k + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq 1$$

for $r \leq R(p) = \frac{p}{2 + p}$ and the constant $R(p)$ cannot be improved. The case $p = 2$ is obtained in [27, Theorem 2], the general case is remarked in [28, Remark 1], and the inequality in this case does play a special role. The above inequality is a refined version of the classical Bohr inequality.
The values of $R_n(1)$ and $R_n(2)$ from Corollary [1] for certain choices of $n$ are listed in Table [1].

| $n$ | $R_n(1)$ | $n$ | $R_n(1)$ | $n$ | $R_n(2)$ | $n$ | $R_n(2)$ |
|-----|----------|-----|----------|-----|----------|-----|----------|
| 2   | 0.366025 | 3   | 0.342508 | 2   | 0.618034 | 3   | 0.543689 |
| 4   | 0.336197 | 5   | 0.334263 | 4   | 0.51879  | 5   | 0.50866  |
| 6   | 0.33364  | 7   | 0.333435 | 6   | 0.504138 | 7   | 0.502017 |
| 8   | 0.33337  | 9   | 0.333345 | 8   | 0.500994 | 9   | 0.500493 |
| 10  | 0.33333  | 15  | 0.333333 | 10  | 0.500245 | 15  | 0.500008 |
| 20  | 0.33333  | 25  | 0.333333 | 20  | 0.5      | 25  | 0.5      |
| 30  | 0.33333  | 35  | 0.333333 | 30  | 0.5      | 35  | 0.5      |

Table 1. $R_n(p)$ is the unique root of the equation $p(1-r) - 2r(1-r^n) = 0$ in $(0,1)$.

Note that when $p = 1$ and $n \geq 16$, the approximate value of the root has no change in the 6th decimal place and we truncated the remaining digits. As $n \to \infty$, the roots converge to 1/3. Similarly, when $p = 2$ and $n \geq 20$, the approximate value of the root has no change in number up to the 6th decimal place and we truncated the remaining digits. As $n \to \infty$, the roots converge to 1/2. Moreover,

\[
R_2(1) = \frac{1}{2}(-1 + \sqrt{3}) \approx 0.366025 \\
R_3(1) = \frac{1}{3}\left(-1 - \frac{2 \times 2^{2/3}}{(41 + 3\sqrt{201})^{1/3}} + \frac{(41 + 3\sqrt{201})^{1/3}}{2^{2/3}}\right) \approx 0.342508 \\
R_2(2) = \frac{1}{2}(-1 + \sqrt{5}) \approx 0.618034 \\
R_3(2) = \frac{1}{3}\left(-1 - \frac{2}{(17 + 3\sqrt{33})^{1/3}} + (17 + 3\sqrt{33})^{1/3}\right) \approx 0.543689.
\]

Remark 1. Moreover if $f \in \mathcal{B}$, then Corollary [4] for $p = 1$ shows that for each $n \geq 3$, the partial sum $S_n(z,f) = \sum_{k=0}^{n} a_k z^k$ satisfies the sharp inequality

\[
|S_n(z,f)| + \frac{1}{1 + |a_0|} \sum_{k=1}^{s} |a_k|^2 r^{2k} + \sum_{k=1}^{t} |a_k|^2 r^{2k+1} \left( \frac{1 - r^{n-2k}}{1-r} \right) \leq 1 \text{ for } |z| = r \leq R_n(1),
\]

where $s = \lfloor n/2 \rfloor$, $t = \lfloor (n-1)/2 \rfloor$ and $R_n(1)$ is the positive root of the equation $$(1-r) - 2r(1-r^n) = 0,$$ or $\sum_{k=1}^{n} r^k = \frac{1}{2}.$$

Recall from the previous corollary that $R_1(1) = \frac{1}{2}$ and $R_\infty(1) = \frac{1}{3}$. Indeed, as we have to deal the cases $n = 1$ and $n = 2$ separately, we have

\[
|S_1(z,f)| \leq |a_0| + |a_1| r \leq 1 \text{ for } r \leq R_1(1) = \frac{1}{2}.
\]
and

\[ |S_2(z,f)| + |a_1|^2 \frac{r^2}{1 + |a_0|} \leq |a_0| + |a_1|r + |a_2|r^2 + |a_1|^2 \frac{r^2}{1 + |a_0|} \leq 1 \text{ for } r \leq R_2(1) = \frac{-1 + \sqrt{3}}{2}.

We see that \( R_n(1) \) is a decreasing function of \( n \) from \( \frac{1}{2} \) to \( \frac{1}{3} \). At this point it is worth noting that if \( f \in \mathcal{B} \) then, according to Rogosinski [34] (see also [23, 36]),

\[ |S_n(z,f)| \leq 1 \text{ for } |z| \leq \frac{1}{2}. \]

3. An application of Theorem 1

Here is a simple application of Theorem 1. One can apply Theorem 1 with \( \varphi_n(r) = b_n r^n \) and \( b_n \geq 0 \) for \( n \geq 0 \). For instance, the fractional derivative of \( f \) of order \( \alpha \in \mathbb{R} \) is defined by

\[ D^\alpha f(z) = \sum_{n=0}^{\infty} (n+1)^\alpha a_n z^n = (f * g)(z), \quad z \in D, \]

where \( g(z) = \sum_{n=0}^{\infty} (n+1)^\alpha z^n \). Note that \( D^1 f(z) = (zf)'(z) \) and \( D^{-1} f(z) = \frac{1}{z} f(z) dt \). Then \( (f_0 * g_0)(z) = (f * g)(z) - a_0 \) and thus, by applying Theorem 1 with \( \varphi_0(r) = 1 \) and \( \varphi_n(r) = (n+1)^\alpha r^n \) for \( n \geq 1 \), we obtain the following result.

**Theorem 2.** Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( \mathcal{B} \), \( 0 < p \leq 2 \), and \( \Phi_N(r) = \sum_{n=N}^{\infty} (n+1)^\alpha r^n \) with \( \alpha \in \mathbb{R} \). Then

\[ |a_0|^p + \sum_{n=1}^{\infty} (n+1)^\alpha |a_n|^r^n + \sum_{n=1}^{\infty} |a_n|^2 \left[ \frac{(2n+1)^\alpha r^{2n}}{1 + |a_0|} + \Phi_{2n+1}(r) \right] \leq 1 \text{ for } r \leq R, \]

where \( R \) is the minimal positive root of the equation \( \Phi_1(r) = p/2 \).

**Example 2.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B} \) and \( 0 < p \leq 2 \), then

\[ (1) \quad |a_0|^p + \sum_{n=1}^{\infty} (n+1)^\alpha |a_n|^r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} (2n+1)^\alpha |a_n|^2 r^{2n} \\
+ \frac{r}{(1 - r)^2} \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1 \text{ for } r \leq R_1(p) = 1 - \sqrt{\frac{2}{2 + p}}. \]

Indeed this follows from Theorem 2 by setting \( \varphi_0(r) = 1 \), \( \varphi_n(r) = (n+1)^\alpha r^n \) for \( n \geq 1 \) and noting that

\[ \Phi_{2n+1}(r) = \sum_{k=2n+1}^{\infty} (k+1)^\alpha r^k = r^{2n+1} \sum_{m=0}^{\infty} (m+1 + (2n+1)) r^m \\
= r^{2n+1} \left( \frac{1}{(1 - r)^2} + \frac{2n+1}{1 - r} \right). \]

\[ (2) \quad |a_0|^p + \sum_{n=1}^{\infty} \frac{|a_n|}{n+1} r^n + \sum_{n=1}^{\infty} |a_n|^2 \left[ \frac{r^{2n}}{(2n+1)(1 + |a_0|)} + \frac{1}{r} \int_0^r t^{2n+1} dt \right] \leq 1 \text{ for } r \leq R_2(p), \]
where $R_2(p)$ is the unique positive root of the equation

$$-\frac{\log(1 - r)}{r} = \frac{p + 2}{2}, \quad r \in (0, 1).$$

Indeed, by setting $\alpha = -1$, $\varphi_0(r) = 1$ and $\varphi_n(r) = r^n/(n + 1)$ for $n \geq 1$, we have at first

$$\Phi_1(r) = \sum_{n=1}^{\infty} \frac{r^n}{n + 1} = -\frac{\log(1 - r)}{r} - 1,$$

which is an increasing function of $r \in [0, 1)$, and increases from 0 to $\infty$. Secondly, it follows that

$$\Phi_{2n+1}(r) = \sum_{k=2n+1}^{\infty} \frac{r^k}{k + 1} = \sum_{m=1}^{\infty} \frac{r^{m+2n}}{m + 2n + 1} = \frac{1}{r} \int_0^r \frac{t^{2n+1}}{1 - t} dt.$$

The desired conclusion follows from Theorem 2. By a standard computation (e.g., using Mathematica), we find that $R_2(1) \approx 0.582812$ and $R_2(2) \approx 0.796812$.

Indeed, by setting $\alpha = 2$, $\varphi_0(r) = 1$ and $\varphi_n(r) = (n + 1)^2 r^n$ for $n \geq 1$, we have

$$\Phi_1(r) = \sum_{n=1}^{\infty} (n + 1)^2 r^n = \frac{1 + r}{(1 - r)^3} - 1,$$

and so the minimal positive root of the equation

$$\frac{1 + r}{(1 - r)^3} = \frac{p + 2}{2}$$

gives the number $R_3(p)$. Again, it follows from the choices of $\varphi_n(r)$’s that

$$\Phi_{2n+1}(r) = \sum_{k=2n+1}^{\infty} (k + 1)^2 r^k = r^{2n+1} \sum_{m=0}^{\infty} [(m + 1)^2 + 2(m + 1)(2n + 1) + (2n + 1)^2] r^m$$

$$= r^{2n+1} \sum_{m=0}^{\infty} [1 + r + \frac{2(2n + 1)}{(1 - r)^2} + \frac{(2n + 1)^2}{1 - r}] r^m$$

The desired conclusion follows from Theorem 2. It is easy to find that $R_3(1) = \frac{4 - \sqrt{13}}{3} \approx 0.13148$ and $R_3(2) = \frac{5 - \sqrt{17}}{4} \approx 0.21922$. 
4. Weighted Bohr Radius for Quasi-Subordination Family

Throughout this section, we let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathbb{D}$. Unless otherwise stated, when we write $f, g \in \mathcal{A}$, we always assume the following power series representation:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k \quad \text{for} \quad z \in \mathbb{D}. \quad (6)$$

**Definition 4.1.** [32, 33] For any two analytic functions $f, g \in \mathbb{D}$, we say that the function $f$ is quasi-subordinate to $g$ (relative to $W$), denoted by $f \prec_W g$, if there exist two functions $W \in \mathcal{B}$, $\omega \in \mathcal{B}$ with $\omega(0) = 0$ such that $f(z) = W(z)g(\omega(z))$.

**Theorem 3.** Assume that $\varphi = \{\varphi_n(r)\}_{n=0}^{\infty}$ belongs to $\mathcal{F}$ such that $\varphi_0(r) = 1$ and

$$\varphi_{m+n}(r) \leq \varphi_m(r)\varphi_n(r) \quad \text{for all} \quad m, n \geq 0 \quad \text{and} \quad r \in [0, 1). \quad (7)$$

If $f, g \in \mathcal{A}$ are given by (6) and $f \prec_W g$ in $\mathbb{D}$, then we have

$$\sum_{k=0}^{\infty} |a_k|\varphi_k(r) \leq \sum_{k=0}^{\infty} |b_k|\varphi_k(r) \quad \text{for all} \quad r \leq R,$$

where $R$ is the minimal positive root of the equation $1 = 2\Phi_1(x)$, $\Phi_1(x) = \sum_{n=1}^{\infty} \varphi_n(x)$.

**Proof.** We remark that this theorem was proved in [5, Theorem 2.1] for $\varphi_k(r) = r^k \ (k \geq 0)$. We follow the method of proof of [5, Theorem 2.1]. Suppose that $f \prec_W g$. Then there exist two analytic functions $W$ and $\omega$ satisfying $\omega(0) = 0$, $|\omega(z)| \leq 1$ and $|W(z)| \leq 1$ for all $z \in \mathbb{D}$ such that

$$f(z) = W(z)g(\omega(z)). \quad (8)$$

Now for the analytic function $\omega(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, the $k$-th power of $\omega$, where $k \in \mathbb{N}$, can be written as

$$\omega^k(z) = \sum_{n=k}^{\infty} \alpha_n^{(k)} z^n = \sum_{m=0}^{k} \sum_{m+k}^{\infty} \alpha_m^{(k)} z^m. \quad (9)$$

For $k = 0$, we set $\omega^0(z) = 1$ so that $\alpha_0^{(0)} = 1$ and $\alpha_n^{(0)} = 0$ for $n \geq 1$. As $\omega \in \mathcal{B}$ with $\omega(0) = 0$, we may write $\omega^k(z) = z^k\omega_1(z)$ with $\omega_1 \in \mathcal{B}$. Applying Theorem 1 to $\omega_1$ with $p = 1$, it follows in particular that

$$\sum_{n=k}^{\infty} |\alpha_n^{(k)}|\varphi_{n-k}(r) \leq \varphi_0(r) = 1 \quad \text{for all} \quad r \leq R. \quad (10)$$

For the analytic function $W(z)$, we may write $W(z) = \sum_{m=0}^{\infty} w_m z^m$ and thus, by Theorem 1, we have

$$\sum_{m=0}^{\infty} |w_m|\varphi_m(r) \leq 1 \quad \text{for all} \quad r \leq R. \quad (11)$$
The relation (8) with the help of (9) takes the form (cf. [5, Theorem 2.1])
\[
\sum_{k=0}^{\infty} a_k z^k = \sum_{m=0}^{\infty} w_m z^m \sum_{k=0}^{\infty} B_k z^k = \sum_{k=0}^{\infty} \left( \sum_{m+j=k} w_m B_j \right) z^k,
\]
which by equating the coefficients of \( z^k \) on both sides gives
\[
(12) \quad a_k = \sum_{m+j=k} w_m B_j \quad \text{for each } k \geq 0,
\]
where \( B_k = \sum_{n=0}^{k} b_n \alpha_k^{(n)} \). Applying the triangle inequality to the last relation shows that
\[
\sum_{k=0}^{\infty} |a_k| \varphi_k(r) \leq \sum_{k=0}^{\infty} \left( \sum_{m+j=k} \left| w_m \right| \left| B_j \right| \right) \varphi_k(r)
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{m+j=k} \left| w_m \right| \varphi_m(r) \left| B_j \right| \varphi_j(r)
\]
\[
= \left( \sum_{m=0}^{\infty} \left| w_m \right| \varphi_m(r) \right) \sum_{k=0}^{\infty} \left| B_k \right| \varphi_k(r)
\]
\[
\leq \sum_{k=0}^{\infty} \left| B_k \right| \varphi_k(r) \quad \text{for all } r \leq R \quad \text{(by (11))},
\]
where in the second inequality above we have used the inequality (7). Also, because
\[
\left| B_k \right| \leq \sum_{n=0}^{k} \left| b_n \right| \left| \alpha_k^{(n)} \right|,
\]
we obtain that
\[
\sum_{k=0}^{\infty} \left| B_k \right| \varphi_k(r) \leq \sum_{k=0}^{\infty} \left| b_k \right| \sum_{n=k}^{\infty} \left| \alpha_n^{(k)} \right| \varphi_n(r)
\]
\[
\leq \sum_{k=0}^{\infty} \left| b_k \right| \left( \sum_{n=k}^{\infty} \left| \alpha_n^{(k)} \right| \varphi_{n-k}(r) \right) \varphi_k(r) \quad \text{(by (7))}
\]
\[
\leq \sum_{k=0}^{\infty} \left| b_k \right| \varphi_k(r) \quad \text{for all } r \leq R \quad \text{(by (10))}
\]
and hence, we deduce that
\[
\sum_{k=0}^{\infty} |a_k| \varphi_k(r) \leq \sum_{k=0}^{\infty} \left| B_k \right| \varphi_k(r) \leq \sum_{k=0}^{\infty} \left| b_k \right| \varphi_k(r) \quad \text{for all } r \leq R.
\]
The proof of Theorem 3 is complete. \( \square \)

Remark 2. Clearly, the conclusion of Theorem 3 continues to hold if the assumption \( f \ll f_g \) is replaced by either \( f \ll g \) or the majorization condition \( \left| f(z) \right| \leq \left| g(z) \right| \) in \( \mathbb{D} \).

The following theorem is due to Rogosinski [35].

**Theorem C** (Rogosinski’s Theorem). Suppose that \( f, g \in \mathcal{A} \) are given by (6), \( A_n = \sum_{k=1}^{n} |a_k|^2 \) and \( B_n = \sum_{k=1}^{n} |b_k|^2 \). If \( f \ll g \) in \( \mathbb{D} \), then \( A_n \leq B_n \) for each \( n \geq 1 \).
Goluzin [17] observed that Rogosinski’s theorem (with $\psi_k(r) = r^k$) can be applied to obtain the following more general result, which has some interesting consequences. Here we state a general result which can be applied to a variety of situations.

**Theorem 4** (Simple generalization of Goluzin’s Lemma). Suppose that $f, g \in \mathcal{A}$ are given by (6), $f \prec g$ in $\mathbb{D}$, and $\{\psi_n(r)\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative functions in $[0, r_\psi)$. Then

$$\sum_{k=1}^{\infty} |a_k|^2 \psi_k(r) \leq \sum_{k=1}^{\infty} |b_k|^2 \psi_k(r) \quad \text{for} \quad r \in [0, r_\psi).$$

**Proof.** Following the method of proof of [15, Theorem 6.3] and the notation of Rogosinski’s Theorem C (i.e. $A_n \leq B_n$ for each $n \geq 1$), a summation by parts gives the desired inequality. So, we omit the details. □

**Theorem 5.** Assume the hypotheses of Theorem 4 and, in addition, suppose that $\{\psi_k(r)\}_{k \geq 1}$ is a decreasing sequence of non-negative functions defined in $[0, r_\psi)$. Then we have

$$\sum_{k=0}^{\infty} |a_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |a_k|^2 \psi_k(r) \leq \sum_{k=0}^{\infty} |b_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 \psi_k(r)$$

holds for all $r \leq \min\{R, r_\psi\}$, where $R$ is as in Theorem 4 and $\lambda(r)$ is a non-negative function of $r$ defined in $[0, 1]$.

**Proof.** According to Theorem 4 we obtain from the assumptions that

(13) $$\sum_{k=0}^{\infty} |a_k| \varphi_k(r) \leq \sum_{k=0}^{\infty} |b_k| \varphi_k(r) \quad \text{for all} \quad r \leq R.$$ 

Finally, by (8), it is also clear that

$$|f(z)|^2 \leq |g(\omega(z))|^2 \quad \text{for} \quad z \in \mathbb{D}$$

and thus, as in the proof of Rogosinski’s Theorem [35], we can easily obtain that (see Robertson [32] and Pommerenke [26, Theorem 2.2])

(14) $$\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2 \quad \text{for} \quad n = 1, 2, \ldots.$$ 

Applying Theorem 4 we obtain that

(15) $$\sum_{k=1}^{\infty} |a_k|^2 \psi_k(r) \leq \sum_{k=1}^{\infty} |b_k|^2 \psi_k(r) \quad \text{for} \quad r \in [0, r_\psi).$$

The desired inequality follows from (13) and (15). □

**Remark 3.** Theorem 3 holds for $f \prec g$ in $\mathbb{D}$ (instead of $f \prec_q g$ in $\mathbb{D}$). In this case, Theorem 3 with $\varphi_k(r) = r^k (k \geq 0)$, $\psi_k(r) = r^{2k} (k \geq 1)$ and $\lambda(r) = \frac{1}{1 + |a_0|} + \Phi_1(r)$, is well-known from [28, Lemma 2]. In this choice the value of $\min\{R, r_\psi\}$ turns out to be $1/3$. It is worth remarking that the weight function $\lambda(r) = \frac{1}{1 + |a_0|} + \Phi_1(r)$ appears very
naturally in some specific situation which refines the classical Bohr inequality and is due to Carlson lemma (see Lemma B). For details, revisit Example 1 and [27, 28].

**Theorem 6.** Under the hypothesis of Theorem 5, we have the following inequality
\[
\sum_{k=0}^{\infty} |a_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |a_k|^2 k r^{2k} \leq \sum_{k=0}^{\infty} |b_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 k r^{2k}
\]
holds for all \( r \leq \min\{R, \frac{1}{\sqrt{2}}\} \).

In particular, for \( f \prec g \) and \( \varphi_k(r) = r^k (k \geq 0) \), the inequality
\[
\sum_{k=0}^{\infty} |a_k| r^k + \lambda(r) \sum_{k=1}^{\infty} |a_k|^2 k r^{2k} \leq \sum_{k=0}^{\infty} |b_k| r^k + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 k r^{2k}
\]
holds for all \( r \leq 1/3 \).

**Proof.** We set \( \psi_k = kr^{2k} \) for \( k \in \mathbb{N} \). Then the sequence \( \{kr^{2k}\} \) is non-increasing if and only if
\[
r \leq \left( \frac{k}{k+1} \right)^{1/2}, \quad k = 1, 2, \ldots,
\]
which holds for all \( k \) if it holds for \( k = 1 \). This gives the condition \( r \leq r_\psi = \frac{1}{\sqrt{2}} \). The desired inequality follows from the method of proof Theorem 5. \( \square \)

**Corollary 2.** Under the hypothesis of Theorem 5, we have the following inequality
\[
\sum_{k=0}^{\infty} |a_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |a_k|^2 k^2 r^{2(k-1)} \leq \sum_{k=0}^{\infty} |b_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 k^2 r^{2(k-1)}
\]
holds for all \( r \leq \min\{R, \frac{1}{2}\} \), and \( R \) is as in Theorem 5.

For \( f \prec g \) and \( \varphi_k(r) = r^k (k \geq 0) \), the inequality
\[
\sum_{k=0}^{\infty} |a_k| r^k + \lambda(r) \sum_{k=1}^{\infty} |a_k|^2 k^2 r^{2(k-1)} \leq \sum_{k=0}^{\infty} |b_k| r^k + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 k^2 r^{2(k-1)}
\]
holds for all \( r \leq 1/3 \).

**Proof.** Following the method of proof Theorem 5, we set \( \psi_k = k^2 r^{2(k-1)} \) for \( k \in \mathbb{N} \). It can be easily seen that \( \psi_k \) is decreasing if and only if \( r \leq k/(k+1) \) and thus, the sequence \( \{\psi_k\} \) is decreasing for all \( k \in \mathbb{N} \), provided \( r \leq 1/2 \). The conclusion follows. \( \square \)

We would like to emphasize that for a fixed \( g \) in the assumption \( f \prec_g g \), fixed \( \lambda(r), \varphi_k(r) \) and \( \psi_k(r) \), \( (k \geq 0) \), it is possible to find an interval for \( r \in (0, 1) \) such that \( \sum_{k=1}^{\infty} |b_k| \varphi_k(r) + \lambda(r) \sum_{k=1}^{\infty} |b_k|^2 \psi_k(r) \leq 1 \). By doing so, the largest value of \( r \) satisfying the last inequality gives the Bohr radius with Bohr-type inequality in a more general setting. Our approach in the above theorems and corollaries is to provide a method of obtaining more such Bohr-type inequalities. However the past known examples are obtained by fixing \( \varphi_k(r) = r^k \).
5. Bohr radius for locally univalent harmonic mappings

A sense-preserving harmonic mappings $f$ of the form $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$, is said to be $K$-quasiconformal if $|g'(z)| \leq k|h'(z)|$ in the unit disk, for $k = \frac{k-1}{K+1} \in [0, 1]$. See [22] for discussion on Bohr radius for quasiconformal harmonic mappings.

**Lemma 2.** Let $\{\psi_n(r)\}_{n=1}^{\infty}$ be a decreasing sequence of nonnegative functions in $[0, r_\psi)$, and $g, h \in \mathcal{A}$ such that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and for some $k \in [0, 1]$, where $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\sum_{n=1}^{\infty} |b_n|^2 \psi_n(r) \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 \psi_n(r) \text{ for } r \in [0, r_\psi).$$

**Proof.** Since $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and thus, as in the proof of Rogosinski’s Theorem [35], we can easily obtain that

$$\sum_{m=1}^{n} m^2 |b_m|^2 r^{2(m-1)} \leq k^2 \sum_{m=1}^{n} m^2 |a_m|^2 r^{2(m-1)} \text{ for } n = 1, 2, \ldots.$$

We integrate the last inequality with respect to $r^2$ and obtain

$$\sum_{m=1}^{n} m |b_m|^2 r^{2m} \leq k^2 \sum_{m=1}^{n} m |a_m|^2 r^{2m} \text{ for } n = 1, 2, \ldots.$$

One more integration (after dividing by $r^2$) gives

$$\sum_{m=1}^{n} |b_m|^2 r^{2m} \leq k^2 \sum_{m=1}^{n} |a_m|^2 r^{2m} \text{ for } n = 1, 2, \ldots.$$

By letting $r$ tends to 1, one has

$$\sum_{m=1}^{n} |b_m|^2 \leq k^2 \sum_{m=1}^{n} |a_m|^2 \text{ for } n = 1, 2, \ldots.$$

Now, applying Goluzin lemma (see Theorem 4) for the given set $\{\psi_n(r)\}$ gives the desired result. \hfill \Box

**Theorem 7.** Assume that $g, h \in \mathcal{A}$ such that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and for some $k \in [0, 1]$, where $g(z) = \sum_{n=0}^{\infty} b_n z^n$, and $h(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies the condition $\text{Re } h(z) \leq 1$ in $\mathbb{D}$ and $h(0) = a_0$ is positive. If $\{\varphi_n(r)\}_{n=1}^{\infty} \in \mathcal{F}$ is a decreasing sequence, where $\Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r)$, and satisfies the inequality

$$1 > \frac{2}{p} (1 + k) \Phi_1(r),$$

for some $p \in (0, 1]$, then the following sharp inequality holds:

$$a_0^p + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq 1 \text{ for all } r \leq R,$$
where $0 < p \leq 1$ and $R$ is the minimal positive root of the equation

$$1 = \frac{2}{p}(1 + k)\Phi_1(x).$$

In the case when $1 < \frac{2}{p}(1 + k)\Phi_1(x)$ in some interval $(R, R + \epsilon)$, the number $R$ cannot be improved.

Proof. We recall that if $P(z) = \sum_{n=0}^{\infty} p_n z^n$ is analytic in $\mathbb{D}$ such that $\text{Re} p(z) > 0$ in $\mathbb{D}$, then $|p_n| \leq 2\text{Re} p_0$ for all $n \geq 1$. Applying this result to $p(z) = 1 - h(z)$ leads to $|a_n| \leq 2(1 - a_0)$ for all $n \geq 1$. Thus, as in the proof of Theorem 1, we can easily obtain from Lemma 2 that

$$\sum_{n=1}^{\infty} |b_n|^2 \phi_n(r) \leq 2(1 - a_0) \sum_{n=1}^{\infty} \phi_n(r) = 2(1 - a_0)^2 \Phi_1(r).$$

Consequently, it follows from the classical Schwarz inequality that

$$\sum_{n=1}^{\infty} |b_n|^2 \phi_n(r) \leq \frac{2(1 - a_0)^2}{2(1 + k)\Phi_1(r)} \sum_{n=1}^{\infty} \phi_n(r) = \frac{4(1 - a_0)^2}{2(1 + k)\Phi_1(r)} \sum_{n=1}^{\infty} \phi_n(r),$$

so that

$$a_0^p + \sum_{n=1}^{\infty} |a_n| \phi_n(r) + \sum_{n=1}^{\infty} |b_n| \phi_n(r) \leq a_0^p + 2(1 - a_0)(1 + k)\Phi_1(r)$$

$$= 1 + (1 - a_0) \left[ \frac{2(1 + k)\Phi_1(r) - \left( \frac{1 - a_0^p}{1 - a_0} \right)}{2(1 + k)\Phi_1(r)} \right]$$

$$\leq 1, \quad \text{by Eqn. (10)},$$

for all $r \leq R$, by the definition of $R$. In the last step, we have used the fact that the function

$$B(x) = \frac{1 - x^p}{1 - x}, \quad x \in [0, 1)$$

is a decreasing function of $x \in [0, 1)$ so that

$$B(x) \geq \lim_{x \to 1^-} \frac{1 - x^p}{1 - x} = p.$$

This proves the desired inequality (17). Moreover, sharpness can be seen by considering functions as in Theorem 1.

Example 3. Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving $K$-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h$ satisfies the condition $\text{Re} h(z) \leq 1$ in $\mathbb{D}$ and $h(0) = a_0$ is positive. Then, by choosing $\phi_k(r) = r^k (k \geq 0)$ in Theorem 1 and $p = 1$, we obtain the following sharp inequality (see [22, Theorem 1.3])

$$a_0 + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \quad \text{for all} \quad r \leq \frac{K + 1}{5K + 1}.$$
6. **Weighted Bohr for the derivative of Schwarz functions**

The following three theorems extend the work of Bhowmik and Das [8, Theorems 1, 2 and 3] in our general setting.

**Theorem 8.** Let \( f \) be a Schwarz function, i.e. \( f \in B \) such that \( f(0) \). If \( \varphi = \{ \varphi_n(r) \}_{n=0}^{\infty} \in \mathcal{F} \) such that

\[
(18) \quad \varphi_0(r) \geq 2 \sum_{n=1}^{\infty} (n + 1) \varphi_n(r),
\]

then the following sharp inequality holds:

\[
(19) \quad B_f'(\varphi, r) \leq \varphi_0(r) \text{ for all } r \leq R_0,
\]

where \( R_0 \) is the minimal positive root of the equation

\[
\varphi_0(x) = 2 \sum_{n=1}^{\infty} (n + 1) \varphi_n(x).
\]

In the case when \( \varphi_0(x) < 2 \sum_{n=1}^{\infty} (n + 1) \varphi_n(x) \) in some interval \((R_0, R_0 + \epsilon)\), the number \( R_0 \) cannot be improved.

**Proof.** Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} a_{n+1} z^n \). Then, \( f'(z) = \sum_{n=0}^{\infty} (n + 1) a_{n+1} z^n \) and

\[
B_f'(\varphi, r) = \sum_{n=0}^{\infty} (n + 1) |a_{n+1}| \varphi_n(r).
\]

Using Wiener’s estimates \( |a_{n+1}| \leq 1 - |a_1|^2 \leq 2(1 - |a_1|) \) for \( n \geq 1 \), we obtain that

\[
B_f'(\varphi, r) \leq \varphi_0(r) + (1 - |a_1|) \left[ 2 \sum_{n=1}^{\infty} (n + 1) \varphi_n(r) - \varphi_0(r) \right]
\]

\[
\leq \varphi_0(r), \quad \text{by Eqn. (18)},
\]

for all \( r \leq R_0 \), by the definition of \( R_0 \). This completes the proof of the inequality (19).

Now let us prove that \( R_0 \) is an optimal number. We consider the function \( f = \varphi_a \) given by

\[
\varphi_a(z) = z \left( \frac{a - z}{1 - a z} \right) = a z - (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} z^{n+1}, \quad z \in \mathbb{D},
\]

where \( a \in (0, 1) \). For this function, straightforward calculations show that

\[
B_f'(\varphi, r) = \varphi_0(r) + (1 - a) \left[ 2 \sum_{n=1}^{\infty} a^{n-1} (n + 1) \varphi_n(r) - \varphi_0(r) \right]
\]

\[
- (1 - a)^2 \left[ \sum_{n=1}^{\infty} a^{n-1} (n + 1) \varphi_n(r) \right]
\]

\[
= \varphi_0(r) + (1 - a) \left[ 2 \sum_{n=1}^{\infty} a^{n-1} (n + 1) \varphi_n(r) - \varphi_0(r) \right] + O((1 - a)^2)
\]
as \( a \to 1^- \). Now it is easy to see that the right hand side is \( > 1 \) when \( a \) is close to 1. The proof of the theorem is complete. \( \square \)

**Remark 4.** The choice \( \varphi_k(r) = r^k \) (\( k \geq 0 \)) in Theorem 8 gives Theorem 1, where the corresponding value of \( R_0 \) is \( 1 - \sqrt{2/3} \).

**Theorem 9.** Assume that \( \varphi_k \)'s are defined as in Theorem 8. Also, let \( f, g \in A \). Then we have the following:

(a) if \( f < g \) in \( \mathbb{D} \) then \( B_f(\varphi, r) \leq B_g(\varphi, r) \) for \( r \leq r_0 = \min\{R, R_0\} \).

(b) if \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{D} \), and \( g(0) = 0 \) then \( B_f(\varphi, r) \leq B_g(\varphi, r) \) for \( r \leq R_0 \), where \( R \) and \( R_0 \) are as in Theorems 8 and 8 respectively.

**Proof.** To prove the first part, we let \( f < g \). Then \( f(z) = g(w(z)) \) for \( z \in \mathbb{D} \), where \( w \) is a Schwarz function. Since \( f'(z) = w'(z)g'(w(z)) \), it follows that

\[
B_f(\varphi, r) \leq B_{g w}(\varphi, r)B_{g w}(\varphi, r).
\]

As \( g' \circ w < g' \), the remark followed by Theorem 8 implies that

\[
B_{g w}(\varphi, r) \leq B_{g'}(\varphi, r) \quad \text{for} \quad r \leq R,
\]

and from Theorem 8 we have \( B_{g'}(\varphi, r) \leq 1 \) for \( r \leq R_0 \). Combining the last two observations gives the desired inequality (a).

To prove the second inequality (b), it suffices to assume next that \( |f(z)| < |g(z)| \) for all \( z \in \mathbb{D} \). Then there exists an analytic self map \( h \) of \( \mathbb{D} \) such that \( f(z) = h(z)g(z) \) for \( z \in \mathbb{D} \). Now \( f'(z) = h'(z)g(z) + h(z)g'(z) \). Further, observing that \( B_{g/z}(\varphi, r) \leq B_{g'}(\varphi, r) \) provided \( g(0) = 0 \), we have

\[
B_f(\varphi, r) \leq (B_{zh}(\varphi, r) + B_h(\varphi, r))B_{g'}(\varphi, r).
\]

Following similar lines of calculations as in the proof of Theorem 8 it can be shown that

\[
B_{zh}(\varphi, r) + B_h(\varphi, r) \leq 1 \quad \text{for} \quad r \leq R_0,
\]

and hence \( B_{f'}(\varphi, r) \leq B_{g'}(\varphi, r) \) for \( r \leq R_0 \). \( \square \)

**Remark 5.** If we choose \( \varphi_k(r) = r^k \) (\( k \geq 0 \)), then Theorem 9 gives Theorem 2, where the corresponding value of \( R_0 \) is \( 1 - \sqrt{2/3} \).

**Theorem 10.** Assume that \( \varphi_k \)'s are defined as in Theorem 8. Let \( f(z) = \sum_{n=0}^\infty a_{2n+1}z^{2n+1} \) and \( g(z) = \sum_{n=0}^\infty b_{2n+1}z^{2n+1} \) be two analytic functions defined on \( \mathbb{D} \) such that \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{D} \). If

\[
1 > 2 \sum_{n=1}^\infty \varphi_{2n}(r),
\]

then the following inequality holds:

(20) \[ B_f(\varphi, r) \leq B_g(\varphi, r) \quad \text{for} \quad r \leq R, \]

where \( R \) is the minimal positive root of the equation \( 1 = 2 \sum_{n=1}^\infty \varphi_{2n}(x) \).
Bohr’s Inequality

Proof. As the method of proof is based on the proofs of Theorem 9 and 8, Theorem 3], we omit the details.

□

Remark 6. If we choose \( \varphi_k(r) = r^k \) \((k \geq 0)\), then Theorem 10 gives Theorem 3, where the corresponding value of \( R \) is \( \sqrt{1/3} \).

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