SHAPE-INvariance and EXACTLY SOLVABLE PROBLEMS IN QUANTUM MECHANICS

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Algebraic approach to the integrability condition called shape invariance is briefly reviewed. Various applications of shape-invariance available in the literature are listed. A class of shape-invariant bound-state problems which represent two-level systems are examined. These generalize the Jaynes-Cummings Hamiltonian. Coherent states associated with shape-invariant systems are discussed. For the case of quantum harmonic oscillator the decomposition of identity for these coherent states is given. This decomposition of identity utilizes Ramanujan's integral extension of the beta function.

1. Introduction

The technique of factorization is a widely-used method to find eigenvalues and eigenvectors of quantum mechanical Hamiltonians. The factorization method was most recently utilized in the context of supersymmetric quantum mechanics \(^1,2\). In this method the Hamiltonian, after subtracting the ground state energy, is written as the product of an operator \(\hat{A}\) and its Hermitian conjugate, \(\hat{A}^\dagger\):

\[
\hat{H} - E_0 = \hat{A}^\dagger \hat{A},
\]

where \(E_0\) is the ground state energy. With this definition the ground state wavefunction in supersymmetric quantum mechanics is annihilated by the operator \(\hat{A}\):

\[
\hat{A} |\psi_0\rangle = 0.
\]

The Hamiltonian in Eq. (1) is called shape-invariant \(^3\) if the condition

\[
\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)
\]

is satisfied. In Eq. (3) \(a_1, a_2, \cdots\) represent the parameters of the Hamiltonian. (The original Hamiltonian has the parameter \(a_1\), the transformed
Hamiltonian has \(a_2\) and so on. The parameter \(a_2\) is a function of the parameter \(a_1\) and the remainder \(R(a_1)\) is independent of the dynamical variables of the problem.

Shape-invariance problem was formulated in algebraic terms in Ref. [4]. To introduce this formalism we define an operator which transforms the parameters of the potential:

\[
\hat{T}(a_1)(a_1)\hat{T}^{-1}(a_1) = O(a_2).
\]

(4)

Introducing new operators

\[
\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1)
\]

\[
\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1).
\]

(5)

one can show that the Hamiltonian can be written as

\[
\hat{H} - E_0 = \hat{A}^\dagger\hat{A} = \hat{B}_+\hat{B}_-.
\]

(6)

Using the definition given in Eq. (5), the shape-invariance condition of Eq. (3) takes the form

\[
[\hat{B}_- , \hat{B}_+] = R(a_0),
\]

(7)

where \(R(a_0)\) is defined via

\[
R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1).
\]

(8)

One can show that

\[
\hat{B}_- |\psi_0\rangle = 0,
\]

(9)

\[
[\hat{H}, \hat{B}_+] = (R(a_1) + R(a_2) + \cdots + R(a_n))\hat{B}_+, \]

(10)

and

\[
[\hat{H}, \hat{B}_-] = -\hat{B}_+^\dagger(R(a_1) + R(a_2) + \cdots + R(a_n)),
\]

(11)
i.e. \(\hat{B}_+^\dagger|\psi_0\rangle\) is an eigenstate of the Hamiltonian with the eigenvalue \(R(a_1) + R(a_2) + \cdots + R(a_n)\). The normalized wavefunction is

\[
|\psi_n\rangle = \frac{1}{\sqrt{R(a_1) + \cdots + R(a_n)}}\hat{B}_+\cdots\frac{1}{\sqrt{R(a_1) + R(a_2)}}\hat{B}_+\frac{1}{\sqrt{R(a_1)}}\hat{B}_+|\psi_0\rangle.
\]

(12)

The algebra is given by the commutators

\[
[\hat{B}_-, \hat{B}_+] = R(a_0),
\]

(13)

\[
[\hat{B}_+, R(a_0)] = (R(a_1) - R(a_0))\hat{B}_+,
\]

(14)
and

\[ [\hat{B}_+, (R(a_1) - R(a_0))\hat{B}_+] = \{(R(a_2) - R(a_1)) - (R(a_1) - R(a_0))\} \hat{B}_+^2, \tag{15} \]

and so on. In general there are an infinite number of such commutation relations. If the quantities \( R(a_n) \) satisfy certain relations one of the commutators in this series may vanish. For such a situation the commutation relations obtained up to that point plus their complex conjugates form a Lie algebra with a finite number of elements.

In the shape-invariant problem the parameters of the Hamiltonian are viewed as auxiliary dynamical variables. One can imagine an alternative approach of classifying some of the dynamical variables as “parameters”. An example of this is provided by the supersymmetric approach to the spherical Nilsson model of single particle states \(^5\). The Nilsson Hamiltonian is given by

\[ H = \sum_i a_i^\dagger a_i - 2kL \cdot S + kνL^2. \tag{16} \]

The superalgebra \( Osp(1/2) \) is the dynamical symmetry algebra of this problem \(^6\). Introducing the odd generator of this superalgebra

\[ F^\dagger = \sum_i \sigma_i a_i^\dagger \tag{17} \]

one can show that the “Hamiltonians”

\[ H_1 = F^\dagger F = \sum_i a_i^\dagger a_i - \sigma \cdot L \tag{18} \]

and

\[ H_2 = FF^\dagger = \sum_i a_i a_i^\dagger + \sigma \cdot L \tag{19} \]

can be considered as supersymmetric partners of each other \(^6\). The shape-invariance condition of Eq. (3) can be written as

\[ FF^\dagger = F^\dagger F + R, \tag{20} \]

where the remainder is

\[ R = \sigma \cdot L - 3/4, \tag{21} \]

i.e. in this example the radial variables are considered as the main dynamical variables and the angular variables are considered as the “auxiliary parameters”.
A number of applications of shape-invariance are available in the literature. These include i) Quantum tunneling through supersymmetric shape-invariant potentials \(^7\); ii) Study of neutrino propagation through shape-invariant electron densities \(^8\); iii) Exploration of the relationship between algebraic techniques of Gaudin developed to deal with many-spin systems, quasi-exactly solvable potentials, and shape-invariance \(^9\); iv) Investigation of coherent states for shape-invariant potentials \(^10,11\); and v) As attempts to devise exactly solvable coupled-channel problems, generalization of Jaynes-Cummings type Hamiltonians to shape-invariant systems \(^12,13\). In this article we focus on the last two applications.

### 2. A Generalized Jaynes-Cummings Hamiltonian For Shape-Invariant Systems

Attempts were made to generalize supersymmetric quantum mechanics and the concept of shape-invariance to coupled-channel problems \(^14,15\). In general it is not easy to find exact solutions to coupled-channels problems. In the coupled-channels case a general shape-invariance is only possible in the limit where the superpotential is separable \(^15\) which corresponds to the well-known sudden approximation in the coupled-channels problem \(^16\).

However it is possible to solve a class of shape-invariant coupled-channels problems which correspond to the generalization of the Jaynes-Cummings Hamiltonian \(^17\) widely used in atomic physics to describe a two-level atom interacting with photons:

\[
\hat{H}_{JC} = \omega_0 \hat{a}^\dagger \hat{a} + \omega \sigma_3 + \Omega \left( \sigma_+ \hat{a}^\dagger + \sigma_- \hat{a} \right). \tag{22}
\]

The shape-invariant generalization of the Jaynes-Cummings Hamiltonian is \(^12\):

\[
\hat{H}_{SUSYJC} = \hat{A}^\dagger \hat{A} + \frac{1}{2} \left[ \hat{A}, \hat{A}^\dagger \right] \left( \sigma_3 + 1 \right) + \sqrt{\hbar \Omega} \left( \sigma_+ \hat{A}^\dagger + \sigma_- \hat{A} \right). \tag{23}
\]

To find the eigenvalues of the Hamiltonian in Eq. (23) we introduce the operator

\[
\hat{S} = \sigma_+ \hat{A} + \sigma_- \hat{A}^\dagger \tag{24}
\]

the square of which can be written as

\[
\hat{S}^2 = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_- \hat{B}_+ & 0 \\ 0 & \hat{B}_+ \hat{B}_- \end{bmatrix} \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix}. \tag{25}
\]

We now introduce the states

\[
| \Psi_m \rangle \pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} | m \rangle \\ | m+1 \rangle \end{bmatrix}, \quad m = 0, 1, 2, \ldots, \tag{26}
\]
where \( | m \rangle \) is the eigenstate of the shape-invariant Hamiltonian \( \hat{A}^\dagger \hat{A} \) with eigenvalue \( \varepsilon_m \). It can be shown that the states in Eq. (26) are the eigenstates of the operator \( \hat{S} \):

\[
\hat{S} | \Psi_m \rangle_\pm = \sqrt{\varepsilon_m + 1} | \Psi_m \rangle_\pm .
\]  

(27)

Since the Hamiltonian of Eq. (23) can be written as

\[
\hat{H}_{\text{SUSYJC}} = \hat{S}^2 + \sqrt{\hbar \Omega} \hat{S},
\]  

(28)

it has the eigenvalue spectrum

\[
\hat{H}_{\text{SUSYJC}} | \Psi_m \rangle_\pm = \left( \varepsilon_m + 1 \pm \sqrt{\hbar \Omega \sqrt{\varepsilon_m + 1}} \right) | \Psi_m \rangle_\pm ,
\]  

(29)

for all states except the ground state which is given by

\[
| \Psi_0 \rangle = \begin{bmatrix} 0 \\ | 0 \rangle \end{bmatrix},
\]  

(30)

where \( | 0 \rangle \) is the ground state of \( \hat{A}^\dagger \hat{A} \). The Hamiltonian \( \hat{H}_{\text{SUSYJC}} \) has an eigenvalue 0 on the state given in Eq. (30). A variant of the usual Jaynes-Cummings Model takes the coupling between matter and the radiation to depend on the intensity of the electromagnetic field. This variant can also be generalized to shape-invariant systems.

### 3. Coherent States for the Quantum Oscillator and Ramanujan Integrals

#### 3.1. Quantum Oscillator as a Shape-invariant Potential

One class of shape-invariant potentials are reflectionless potentials with an infinite number of bound states, also called self-similar potentials. Shape-invariance of such potentials were studied in detail in Refs. [20] and [21]. For such potentials the parameters are related by a scaling:

\[
a_n = q^{n-1} a_1 .
\]  

(31)

For the simplest case studied in Ref. [21] the remainder of Eq. (3) is given by

\[
R(a_1) = c a_1 ,
\]  

(32)

which corresponds to the quantum harmonic oscillator. Introducing the operators

\[
\hat{S}_+ = \sqrt{\pi} \hat{B}_+ (a_1)^{-1/2}
\]  

(33)
and

\[ \hat{S}_- = (\hat{S}_+)^\dagger = \sqrt{q} R(a_1)^{-1/2} \hat{B}_-, \] (34)

one can write the Hamiltonian of the quantum harmonic oscillator as

\[ \hat{H} - E_0 = R(a_1) \hat{S}_+ \hat{S}_-. \] (35)

This Hamiltonian has the energy eigenvalues

\[ E_n = R(a_1) \frac{1 - q^n}{1 - q}, \] (36)

and the eigenvectors

\[ | n \rangle = \sqrt{(1 - q^n)/(q^n)} (\hat{S}_+)^n | 0 \rangle. \] (37)

In writing down Eq. (37) we used the \( q \)-shifted factorial defined as

\[ (z; q)_n = \frac{1}{(1 - zq^n)}, \quad n = 1, 2, \ldots \] (38)

### 3.2. Coherent States for Shape-Invariant Systems

Coherent states for shape-invariant potentials were introduced in Refs. [9] and [22]. (For a description of an alternative approach see Ref. [23] and references therein). Following the definitions in Eqs. (5) and (6) (with \( E_0 = 0 \)) we introduce the operator

\[ \hat{H}^{-1} \hat{B}_+ = \hat{B}_-^{-1}, \quad (\hat{B}_- \hat{B}_-^{-1} = 1). \] (39)

The coherent state can be defined as

\[ | z \rangle = \sum_{n=0}^{K} \left( zf[R(a_1)] \hat{B}_-^{-1} \right)^n | 0 \rangle, \] (40)

where \( f(t) \) is an arbitrary function. This state can explicitly be written as

\[ | z \rangle = | 0 \rangle + z \frac{f[R(a_1)]}{\sqrt{R(a_1)}} | 1 \rangle + z^2 \frac{f[R(a_1)]f[R(a_2)]}{\sqrt{R(a_2)[R(a_1) + R(a_2)]}} | 2 \rangle + z^3 \frac{f[R(a_1)]f[R(a_2)]f[R(a_3)]}{\sqrt{R(a_3)[R(a_2) + R(a_3)][R(a_3) + R(a_2) + R(a_1)]}} | 3 \rangle + \cdots \] (41)
where we used the normalized eigenstates of the operator $\hat{H}$:
\[ | n \rangle = \left[ \hat{H}^{-1/2} \hat{B}_+ \right]^n | 0 \rangle. \tag{42} \]

In a similar way to the coherent states for the ordinary harmonic oscillator the coherent state in Eq. (40) is an eigenstate of the operator $\hat{B}_-$:
\[ \hat{B}_- | z \rangle = z f[R(a_0)] | z \rangle. \tag{43} \]

3.3. $q$-Coherent States:

To derive the overcompleteness relation of $q$-coherent states here we follow the proof given in Ref. [11]. An alternative, but equivalent, derivation was given in Ref. [24]. To introduce the coherent states for the $q$-oscillator we take the arbitrary function in Eq. (40) to be
\[ f[R(a_n)] = R(a_n). \tag{44} \]

The resulting coherent states are
\[ | z \rangle = \sum_{n=0}^{\infty} \frac{(1-q)^{n/2}}{\sqrt{(q;q)_n}} q^{n(n-1)/4} \sqrt{(R(a_1))^n z^n} | n \rangle. \tag{45} \]

Further introducing the auxiliary variable
\[ \zeta = \frac{\sqrt{(1-q)}}{\sqrt{q}} \sqrt{R(a_1)} z \tag{46} \]

these coherent states take the form
\[ | \zeta \rangle = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4}}{\sqrt{(q;q)_n}} \zeta^n | n \rangle. \tag{47} \]

The overcompleteness of these coherent states can easily be proven using the integral
\[ \int_0^\infty dt \frac{t^n}{(-t; q)_\infty} = \frac{(q;q)_n}{q^{n(n+1)/2} (-\log q)}. \tag{48} \]

This integral was proven by Ramanujan in an attempt to generalize integral definition of the beta function $25$. (An elementary proof is given by Askey in Ref. [26]). Using Eq. (48) the overcompleteness relation of the coherent states in Eq. (47) can be obtained in a straightforward way:
\[ I = \int \frac{d\zeta d\zeta^*}{2\pi i} \frac{1}{(-\log q)(-|\zeta|^2; q)_\infty} | \zeta \rangle \langle \zeta | = 1. \tag{49} \]
This overcompleteness relation could be useful to write down coherent-state path integrals for quantum harmonic oscillator.

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