NONEXISTENCE OF POSITIVE SOLUTIONS FOR HÉNON EQUATION

JORGE GARCÍA-MELIÁN

ABSTRACT. We consider the semilinear elliptic equation
\[-\Delta u = |x|^{\alpha} u^p \text{ in } \mathbb{R}^N,\]
where $N \geq 3$, $\alpha > -2$ and $p > 1$. We show that there are no positive solutions provided that the exponent $p$ additionally verifies
\[1 < p < \frac{N + 2\alpha + 2}{N - 2}.\]
This solves an open problem posed in previous literature, where only the radially symmetric case was fully understood. We also characterize all positive solutions when $p = \frac{N + 2\alpha + 2}{N - 2}$ and $-2 < \alpha < 0$.

1. Introduction

Probably the most well-known nonlinear Liouville theorem in the literature is the one obtained in the celebrated paper [18]. There, nonexistence of positive solutions of the elliptic equation
\[-\Delta u = u^p \text{ in } \mathbb{R}^N\]
is established, provided that $N \geq 3$ and the exponent $p$ verifies the ‘subcriticality’ condition
\[1 < p < \frac{N + 2}{N - 2}.\]
This nonexistence theorem can be complemented with a classification result when $p = \frac{N + 2}{N - 2}$. It was proved in [10] that every positive solution of (1.1) for this value of $p$ is of the form
\[u(x) = (N(N - 2))^{\frac{N - 2}{4}} \left( \frac{\mu}{\mu^2 + |x - x_0|^2} \right)^{\frac{N - 2}{2}},\]
for some $x_0 \in \mathbb{R}^N$ and $\mu > 0$. See also [22] for a simpler proof.

The existence of multiple applications of these results (starting with [19] in the context of a priori bounds) and its intrinsic interest have led the community to the search on one hand for simpler proofs (cf. [8], [11], [27]) and on the other for generalizations (see [6] and [22]).

One of the generalizations corresponds to the sometimes called Hénon equation, namely:
\[-\Delta u = |x|^\alpha u^p \text{ in } \mathbb{R}^N,\]
where $N \geq 3$, $p > 1$ and the parameter $\alpha$ is an arbitrary real number. To begin with, it can always be assumed that $\alpha > -2$, since when $\alpha \leq -2,$
there are no solutions of (1.4) in any punctured neighborhood of \(x = 0\) (see for instance Theorem 2.3 in [13]).

A complete analysis of problem (1.4) does not seem to have been performed, at the best of our knowledge. However, restricting the attention to radially symmetric solutions, it has been shown in [7] that the nonexistence range of positive solutions is exactly

\[(1.5) \quad 1 < p < \frac{N + 2\alpha + 2}{N - 2}.
\]

This has led (see for instance [26]) to the statement of the following:

**Conjecture A.** Assume \(\alpha > -2\) and \(p\) verifies (1.5). Then problem (1.4) does not admit any positive solution.

Conjecture A has been proved in the case \(\alpha < 0\) in [6] and for bounded solutions in dimension \(N = 3\) in [26]. There are also some partial results, aside the just cited works. First of all, nonexistence of solutions with a further restriction on \(p\) is a consequence of the general nonexistence result for supersolutions obtained in [24] (see also Corollary 4.2 in [1]). It is shown there that when \(p\) verifies

\[(1.6) \quad 1 < p \leq \frac{N + \alpha}{N - 2}
\]

then no positive supersolutions of (1.4) exist. On the other hand, the case \(\alpha \geq 2\) is covered in [18], with the further restriction (1.2). The same restriction is found in [8], where a general \(\alpha > -2\) is allowed. The most general result for \(\alpha > 0\) known to us for the moment is the one obtained in [6], where nonexistence of positive solutions was shown to hold provided that

\[
1 < p \leq \frac{N + \alpha + 2}{N - 2}.
\]

But, as far as we know, when \(\alpha > 0\) and solutions are not necessarily bounded, the full range (1.5) is still not covered, hence Conjecture A remains unsolved for the moment.

Another interesting question concerning (1.4) arises when the ‘critical’ case \(p = \frac{N + 2\alpha + 2}{N - 2}\) is considered. It can be checked that, for every \(\mu > 0\), the functions

\[(1.7) \quad u(x) = \left(\frac{4}{(\alpha + 2)^2}\right)^{\frac{N}{N-2}}(N(N - 2))^{\frac{N-2}{2}}\left(\frac{\mu}{\mu^2 + |x|^{2+\alpha}}\right)^{\frac{N-2}{N+\alpha}}
\]

are solutions of (1.4). Moreover, when \(\alpha = 0\), these reduce to the corresponding ones for (1.1) taking \(x_0 = 0\) in (1.3). Of course, when \(\alpha \neq 0\), solutions which are radially symmetric with respect to a point \(x_0 \neq 0\) are not possible, and one may wonder whether there is a similar classification as that in [10] for problem (1.4).

The answer to this question is negative. It was shown in Theorem 1.6 of [21] that for every positive even integer \(\alpha_0\), there exists a continuum of solutions \(\{(\alpha, u_\alpha)\}\) of (1.4) with \(p = \frac{N + 2\alpha + 2}{N - 2}\) which are not radially symmetric, and bifurcate from \(\{(\alpha_0, U_{\alpha_0})\}\), where \(U_{\alpha_0}\) is a radially symmetric solution of (1.4) with \(\alpha = \alpha_0\) and \(p = \frac{N + 2\alpha_0 + 2}{N - 2}\). Thus it is likely that non radially symmetric solutions exist at least for all large positive values of \(\alpha\).
When \(-2 < \alpha < 0\), however, the situation is expected to be different: it was shown in [14] that positive solutions \(u\) of (1.4) in the critical case are given by (1.7) if \(|x|^{\alpha}u^{p+1} \in L^1(\mathbb{R}^N)\). Thus it makes sense to pose the following

**Question B.** Assume \(-2 < \alpha < 0\) and \(p = \frac{N+2\alpha+2}{N-2}\). Are all positive solutions of (1.4) of the form (1.7)?

We come now to the statement of our results. By a solution of (1.4), we mean a function \(u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)\) verifying the equation in the weak sense. However, it is worthy of mention that with the aid of the results in [26] and [8], it suffices to assume that \(u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})\) verifies the equation in the weak sense in \(\mathbb{R}^N \setminus \{0\}\), together with the condition

\[
\lim_{x \to 0} |x|^{2+\alpha} u(x) = 0.
\]

We will show in our first result that Conjecture A holds in the full regime (1.5) and for all values \(\alpha > -2\), therefore providing a proof which unifies both cases \(\alpha < 0\) and \(\alpha > 0\).

**Theorem 1.** Assume \(N \geq 3\), \(\alpha > -2\) and \(p\) verifies (1.5). Then problem (1.4) does not admit any positive solution.

Moreover, in our second result we also answer affirmatively Question B in the full regime \(-2 < \alpha < 0\).

**Theorem 2.** Assume \(N \geq 3\), \(-2 < \alpha < 0\) and

\[
(1.8)
\]

\[
p = \frac{N + 2\alpha + 2}{N - 2}.
\]

Let \(u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)\) be a positive solution of (1.4). Then \(u\) is of the form (1.7) for some \(\mu > 0\).

The proofs of Theorems 1 and 2 rely on the well-known trick of writing the equation in polar coordinates, and then introducing the Emden-Fowler transformation. An application of the moving planes method as in [9] yields the monotonicity of the function

\[
|x|^{\frac{2+\alpha}{p-1}} u(x)
\]

in \(\mathbb{R}^N\), provided that (1.5) holds. In the case \(\alpha = 0\), the same argument applied with respect to an arbitrary origin then shows that \(u\) has to be constant, which is not possible, thus we obtain a simplified proof of the Liouville theorem in [18]. But this argument does not carry over to deal with \(\alpha \neq 0\). However, the essential point in our proof is to realize that the monotonicity alluded to above shows that

\(u\) is a stable solution of (1.4), in the usual sense that for every \(\phi \in C_0^\infty(\mathbb{R}^N)\) there holds

\[
(1.9)
\]

\[
\int_{\mathbb{R}^N} |\nabla \phi|^2 - p|x|^{\alpha} u^{p-1} \phi^2 \geq 0.
\]
A Liouville theorem for stable solutions of (1.4) is already available (cf. [13] and the previous work [16] for problem (1.1)) and it implies \( u \equiv 0 \) in \( \mathbb{R}^N \), a contradiction. For the reader’s convenience, we include an independent, simplified proof of this theorem.

As for Theorem 2, we obtain as a byproduct of the same moving plane argument that \( u \) behaves at infinity like the fundamental solution of the Laplacian, hence the results in [14] can be used to obtain that \( u \) is of the form (1.7).

Finally, it is interesting to mention that the approach followed to prove Theorem 1 can be used to deal with other related problems. For instance, when the weight \(|x|^\alpha\) is replaced by \(|x_1|^\alpha\) or even more general functions. In this regard, the problem

\[
(1.10) \quad -\Delta u = x_1^m u^p \quad \text{in} \quad \mathbb{R}^N,
\]

where \( m \) is a positive integer, has been already considered in previous literature. We refer to [2], [23], [15]. However, in all these works only odd integers are allowed. Our methods enable us to obtain a Liouville theorem in the complementary case where \( m \) is an even integer. The proof of the following result is a slight variant of that of Theorem 1 and will not be given.

**Theorem 3.** Assume \( N \geq 3 \), \( m \) is an even integer and

\[
1 < p < \frac{N + 2m + 2}{N - 2}.
\]

Then problem (1.10) does not admit any positive solution.

The rest of the paper is organized as follows: in Section 2 we include some preliminaries related with regularity of solutions, principal eigenvalues in smooth bounded domains with unbounded coefficients and the Liouville theorem for stable solutions of (1.4). Section 3 is dedicated to the proof of Theorems 1 and 2.

**2. Preliminaries**

In this section we will consider some preliminaries on positive solutions of (1.4). Most of them deal with regularity, especially in the case \( \alpha < 0 \). We will also briefly deal with an eigenvalue problem with coefficients which are not bounded and we will include a proof of the nonexistence of stable positive solutions of (1.4) when \( 1 < p \leq \frac{N + 2m + 2}{N - 2} \).

The first result is a consequence of standard regularity theory.

**Lemma 4.** Let \( u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \) be a positive weak solution of (1.4). Then:

1. If \( -2 < \alpha < 0 \), we have \( u \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap W^{2,q}_{\text{loc}}(\mathbb{R}^N) \) for some \( q > \frac{N}{\alpha} \). Moreover, there exists \( \eta \in (0, 1) \) such that \( u \in C^{\eta}(\mathbb{R}^N) \). In addition, there exists \( C > 0 \) such that

\[
|\nabla u(x)| \leq \frac{C}{|x|} \quad \text{if} \quad 0 < |x| < 1.
\]

2. When \( \alpha \geq 0 \), \( u \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C^{2,\eta}(\mathbb{R}^N) \) for some \( \eta \in (0, 1) \).
Proof. The assertion about $C^\infty$ regularity in $\mathbb{R}^N \setminus \{0\}$ is immediate by bootstrapping and the fact that $|x|^\alpha$ is $C^\infty$ there, while $u$ is positive (cf. [20]).

The regularity in the case $\alpha \geq 0$ is a consequence of standard theory: we obtain that $\Delta u$ is locally bounded in $\mathbb{R}^N$, so that $u \in C^1(\mathbb{R}^N)$. This implies in turn that $\Delta u \in C^\eta(\mathbb{R}^N)$ for some $\eta \in (0, 1)$, so that $u \in C^{2,\eta}(\mathbb{R}^N)$. Observe that solutions become more regular the larger $\alpha$ is.

When $-2 < \alpha < 0$, we see that $|x|^\alpha \in L^q_{\text{loc}}(\mathbb{R}^N)$ for every $q < \frac{N}{|\alpha|}$. Thus we may choose and fix such a $q$ additionally verifying $q > \frac{N}{2}$. This implies that $h = |x|^\alpha u^p \in L^q_{\text{loc}}(\mathbb{R}^N)$. Denoting the unit ball of $\mathbb{R}^N$ by $B$, we can use Theorem 9.15 in [20] to guarantee that the problem

$$
\begin{aligned}
-\Delta w &= h \quad \text{in } B \\
w &= u \quad \text{on } \partial B
\end{aligned}
$$

admits a unique strong solution $w \in W^{2,q}(B)$. Since $q > \frac{N}{2}$ it also follows by Sobolev embeddings that $w \in C^\eta(\mathbb{R}^N)$ for some $\eta \in (0, 1)$. We deduce that $\Delta (u - w) = 0$ in $B \setminus \{0\}$, while $u - w$ is bounded in $B$. It is well-known that this implies $u \equiv w$ in $B$. We conclude that $u \in W^{2,q}(\mathbb{R}^N) \cap C^\eta(\mathbb{R}^N)$.

To show (2.1), we make use once more of standard regularity. Fix $x \in B \setminus \{0\}$ and consider the ball $B_x$ with center $x$ and radius $\frac{|x|}{2}$. There exists a positive constant $C$ which does not depend on $x$ nor on $u$ such that

$$
|x| |\nabla u(y)| \leq C (|x|^2 \sup_{z \in B_x} |\Delta u(z)| + \sup_{z \in B_x} |u(z)|)
$$

for every $y \in B_x$ (cf. for instance (4.45) in [20]). Observe that $|z| \geq \frac{|x|}{2}$ for every $z \in B_x$, so that

$$
|x|^2 |\Delta u(z)| \leq C |x|^2 |z|^\alpha \leq C |x|^{2+\alpha} \leq C.
$$

Thus taking $y = x$ in (2.2) we obtain (2.1).

Next we consider a special solution of the linearized equation, which is one of the keys to our proofs in Section 3. Also, we need to ‘fine tune’ the regularity of the gradient of the solutions near $x = 0$. Throughout the rest of the paper, we will denote

$$
(2.3) \quad \beta = \frac{2 + \alpha}{p - 1}.
$$

Lemma 5. Assume $u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a positive weak solution of (1.4). Then the function $v(x) = \nabla u(x) \cdot x + \beta u(x)$ belongs to $C^\infty(\mathbb{R}^N \setminus \{0\}) \cap W^{2,q}_{\text{loc}}(\mathbb{R}^N)$ for some $q > \frac{N}{2}$ and verifies

$$
-\Delta v = p|x|^\alpha u^{p-1}v \quad \text{in } \mathbb{R}^N \setminus \{0\}.
$$

Moreover $v \in C^\eta(\mathbb{R}^N)$ for some $\eta \in (0, 1)$ and in particular

$$
(2.4) \quad \lim_{x \to 0} \nabla u(x) \cdot x = 0.
$$
Proof. Let \( v(x) = \nabla u(x) \cdot x + \beta u(x) \). Then it is not hard to see that
\[
-\Delta v = -\nabla (\Delta u) \cdot x - (\beta + 2) \Delta u \\
= (\alpha + \beta + 2)|x|^\alpha u^p + p|x|^\alpha u^{p-1} \nabla u \cdot x \\
= (\alpha + \beta + 2 - \beta p)|x|^\alpha u^p + p|x|^\alpha u^{p-1} v \\
= p|x|^\alpha u^{p-1} v
\]
in \( \mathbb{R}^N \setminus \{0\} \).

On the other hand, it is clear from Lemma 4 that \( v \in C^1(\mathbb{R}^N) \) when \( \alpha \geq 0 \), while \( v \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) if \( \alpha < 0 \) (cf. in particular equation (2.1)). Then
\[
h = p|x|^\alpha u^{p-1} v \in L^q_{\text{loc}}(\mathbb{R}^N)
\]
for some \( q > N/2 \), and reasoning as in the proof of Lemma 4 we deduce \( v \in W^{2,q}_{\text{loc}}(\mathbb{R}^N) \). The \( C^\infty \) regularity in \( \mathbb{R}^N \setminus \{0\} \) is immediate from Lemma 4.

From Sobolev embeddings, we also have \( v \in C^\eta(\mathbb{R}^N) \) for some \( \eta \in (0,1) \). We deduce that \( \nabla u(x) \cdot x \) is continuous at zero, so that the limit
\[
\ell = \lim_{x \to 0} \nabla u(x) \cdot x
\]
exists. Since \( u \) is bounded at zero, this limit has to be zero. This concludes the proof. \( \square \)

It is the turn now to consider an auxiliary eigenvalue problem. In what follows, we deal with a smooth bounded domain \( \Omega \) of \( \mathbb{R}^N \), and a coefficient \( a \in L^q(\Omega) \), where \( q > \frac{N}{2} \). We are interested in the principal eigenvalue of
\[
\begin{align*}
-\Delta u + a(x)u &= \lambda u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]
that is, the first of the eigenvalues, which is associated to a positive eigenfunction. Although we expect the next result to be well-known, we have not been able to find a pertinent reference. We refer the reader to Theorem 1 in [12], where the extension to the \( p \)-Laplacian setting is analyzed.

**Lemma 6.** Assume \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, and let \( a \in L^q(\Omega) \) for some \( q > \frac{N}{2} \). Then problem (2.5) admits a principal eigenvalue \( \lambda_\Omega^1(a) \), which can be variationally characterized as
\[
\lambda_\Omega^1(a) = \inf_{w \in H_0^1(\Omega)} \frac{\int \Omega |\nabla w|^2 + a(x)w^2}{\int \Omega w^2}.
\]
Moreover, there exists an associated positive eigenfunction \( \phi \in H_0^1(\Omega) \cap W^{2,q}(\Omega) \cap C^\eta(\overline{\Omega}) \), for some \( \eta \in (0,1) \).

We deal next with a very well-known property of the principal eigenvalue. When the coefficient \( a \) is bounded, \( \lambda_\Omega^1(a) \) can be characterized as:
\[
\lambda_\Omega(a) = \sup \left\{ \lambda > 0 : \begin{array}{l} \text{there exists } v > 0 \text{ in } \overline{\Omega} \text{ such that} \\
-\Delta v + a(x)v \geq \lambda v \text{ a. e. in } \Omega \end{array} \right\},
\]
while the functions \( v \) are taken in \( W^{2,N}(\Omega) \) (see for instance [14] or a more recent account in unbounded domains in [5]). In particular, the existence
of a positive function \( v \in W^{2,N}(\Omega) \) verifying \(-\Delta v + a(x)v \geq 0 \) in \( \Omega \) implies \( \lambda_1^\Omega(a) > 0 \).

We are not aware of any similar property when the coefficient \( a \) is not bounded, or when the function \( v \) is not in \( W^{2,N}(\Omega) \). Thus we obtain one which is sufficient for our purposes in Section 3.

**Lemma 7.** Assume \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain and let \( a \in L^q(\Omega) \) for some \( q > \frac{N}{2} \). If there exists \( v \in W^{2,q}(\Omega) \cap C(\overline{\Omega}) \) such that \( v > 0 \) in \( \Omega \) and \(-\Delta v + a(x)v \geq 0 \ a.e. \ in \Omega \), then
\[
\lambda_1^\Omega(a) > 0.
\]

*Proof.* The proof is based on standard arguments, the only important point being the use of a strong maximum principle for \( W^{2,q} \) functions with \( q > \frac{N}{2} \), available thanks to Theorem 1 in [25] or Corollary 5.1 in [28].

Assume for a contradiction that \( \lambda_1^\Omega(a) \leq 0 \), and let \( \phi \in H^1_0(\Omega) \cap W^{2,q}(\Omega) \cap C(\overline{\Omega}) \) be a positive eigenfunction given by Lemma 6. Since \( v \in C(\Omega) \) is positive, we have
\[
\gamma = \inf_{x \in \Omega} \frac{v(x)}{\phi(x)} > 0.
\]
Consider the function \( z = v - \gamma \phi \). It is clear that \( z \in W^{2,q}(\Omega) \cap C(\overline{\Omega}) \), while \( z \geq 0 \) in \( \Omega \). By continuity, and since \( v > 0 \) on \( \partial \Omega \) while \( \phi = 0 \) there, there exists \( x_0 \in \Omega \) such that \( z(x_0) = 0 \). Moreover,
\[
-\Delta z + a(x)z \geq -\lambda_1^\Omega(a) \gamma \phi \geq 0 \quad \text{in} \ \Omega.
\]

We may use the strong maximum principle to conclude that \( z \equiv 0 \) in \( \Omega \), which is not possible because \( v > 0 \) on \( \partial \Omega \) and \( \phi \) vanishes on \( \partial \Omega \). Therefore \( \lambda_1^\Omega(a) > 0 \), as we wanted to show. \( \Box \)

To conclude the section, we consider the Liouville theorem for stable solutions of (1.4). It is worthy of mention that the nonexistence result in [13] (Theorem 1.2 there) is more general, but we restrict ourselves to the subcritical range of the parameter, which allows us to give a simpler proof.

**Theorem 8.** Assume \( N \geq 3 \), \( \alpha > -2 \) and
\[
1 < p \leq \frac{N + 2\alpha + 2}{N - 2}.
\]

Then the unique stable solution of (1.4) is \( u \equiv 0 \).

*Proof.* Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) be arbitrary. Taking \( \varphi^2u \) as a test function in (1.4) and setting \( \phi = \varphi u \) in (1.3), we obtain
\[
(p - 1) \int |x|^{\alpha} \varphi^2 u^{p+1} \leq \int u^2 |\nabla \varphi|^2.
\]

Now choose \( \xi \in C_0^\infty(B_2) \) such that \( 0 \leq \xi \leq 1 \) and \( \xi \equiv 1 \) in \( B_1 \). Take \( R > 0 \) and choose
\[
\phi(x) = \xi \left( \frac{x}{R} \right)^{\frac{p+1}{p-1}}
\]
in (2.8). It is easily seen that this implies
\[
\int_{B_{2R}} |x|^{\alpha} \varphi^2 u^{p+1} \leq \frac{C}{R^2} \int_{B_{2R} \setminus B_R} \varphi^{\frac{p+1}{p-1}} u^2,
\]
for some \( C > 0 \) (from now on, we are using the letter \( C \) to denote different constants not depending on \( R \)). Using Hölder’s inequality in the last integral with conjugate exponents \( \frac{p}{p+1} \) and \( \frac{p+1}{p} \) yields

\[
\int_{B_{2R}} |x|^{\alpha} \varphi^2 u^{p+1} \leq CR^{2+\frac{N-2}{p+1}} \left( \int_{B_{2R}\setminus B_R} \varphi^2 u^{p+1} \right)^{\frac{2}{p+1}} \\
\leq CR^{-2-\frac{2\alpha}{p+1} + \frac{N-2}{p+1}} \left( \int_{B_{2R}\setminus B_R} |x|^\alpha \varphi^2 u^{p+1} \right)^{\frac{2}{p+1}}.
\]

As a consequence we arrive at

\[
\int_{B_R} |x|^\alpha u^{p+1} \leq \int_{B_{2R}} |x|^\alpha \varphi^2 u^{p+1} \leq CR^{\frac{N(p-1)-2(p+1)-2\alpha}{p-1}}.
\]

It is not hard to check that, when the second inequality in (2.7) is strict, the exponent of \( R \) in (2.10) is negative. Hence letting \( R \to +\infty \) we see that \( u \equiv 0 \).

In the case where \( p = \frac{N+2\alpha+2}{N-2} \), inequality (2.10) gives \( |x|^\alpha u^{p+1} \in L^1(\mathbb{R}^N) \). Thus letting \( R \to +\infty \) in (2.9) we also obtain \( u \equiv 0 \) in \( \mathbb{R}^N \). The proof is concluded. \( \square \)

3. Proof of the main results

This section is devoted to prove Theorems 1 and 2. As we have mentioned in the Introduction, the fundamental step in both theorems is to obtain a monotonicity property of the solutions. It is worthy of mention that in this case, the restriction

\[
(3.1) \quad p > \frac{N + \alpha}{N - 2}
\]

is important, and does not imply any loss in generality. The idea for the proof of the next result comes from [9]. Recall our definition (2.3).

**Theorem 9.** Assume \( \alpha > -2 \) and \( p \) verifies (3.1). Let \( u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^{\infty}_{\text{loc}}(\mathbb{R}^N) \) be a positive weak solution of (1.4). Then:

(a) When (1.5) holds, the function \( |x|^\beta u(x) \) is nondecreasing in \( |x| \).

(b) If (1.8) holds, either \( |x|^\beta u(x) \) is nondecreasing in \( |x| \) or there exists \( \mu > 0 \) such that

\[
(3.2) \quad u(x) = \mu^{N-2}|x|^{2-N} u \left( \mu^{\frac{2}{|x|^2}} \right), \quad x \in \mathbb{R}^N \setminus \{0\},
\]

that is, \( u \) coincides with its Kelvin transform with respect to some ball centered at the origin.

**Proof.** To start with, we consider polar coordinates, and write \( u(x) = v(r, \theta) \), where \( r = |x| \) and \( \theta = \frac{x}{|x|} \in S^{N-1} \). It is not hard to see that \( v \) verifies

\[
v_{rr} + \frac{N - 1}{r} v_r + \frac{1}{r^2} \Delta_{\theta} v + r^\alpha v^p = 0 \quad \text{in} \ (0, +\infty) \times S^{N-1},
\]
in the classical sense, where $\Delta_\theta$ stands for the Laplace-Beltrami operator on $S^{N-1}$. We now introduce the Emden-Fowler transformation,

$$w(t, \theta) = r^\beta v(r, \theta),$$

where $t = \log r$.

After a straightforward calculation we see that $w$ verifies

$$w_{tt} + aw_t + \Delta_\theta w - bw + w^p = 0 \quad \text{in } \mathbb{R} \times S^{N-1},$$

where

$$a = 2\beta - (N - 2) \geq 0$$
$$b = \beta(N - 2 - \beta) > 0.$$

Our proof reduces to show that $w$ is nondecreasing in the variable $t$ in $\mathbb{R}$. For this sake we employ the moving planes method (cf. [17], [3]).

We follow the standard notation:

$$\Sigma_{\lambda} = (-\infty, \lambda) \times S^{N-1}$$
$$T_{\lambda} = \{\lambda\} \times S^{N-1}$$
$$t^\lambda = 2\lambda - t \quad \text{if } t < \lambda,$$

and let

$$z^\lambda(t, \theta) = w(t^\lambda, \theta) - w(t, \theta), \quad (t, \theta) \in \Sigma_{\lambda}.$$

By the mean value theorem, there exists $\xi_{\lambda} = \xi_{\lambda}(t, \theta)$ such that $w(t^\lambda, \theta)^p - w(t, \theta)^p = p\xi_{\lambda}^{p-1} z^\lambda$. Then the function $z^\lambda$ verifies the equation

$$z^\lambda_{tt} - az^\lambda_t + \Delta_\theta z^\lambda - bz^\lambda + p\xi_{\lambda}^{p-1} z^\lambda = -2aw_t \quad \text{in } \Sigma_{\lambda}.$$

We next observe that the boundedness of $u$ at $x = 0$ implies

$$\lim_{t \to -\infty} \inf_{\theta \in S^{N-1}} w(t, \theta) = 0 \quad \text{and} \quad \lim_{t \to -\infty} \inf_{\theta \in S^{N-1}} z^\lambda(t, \theta) \geq 0.$$

Moreover,

$$w_t = r^\beta(\beta u + ru_t),$$

so that from (2.4) in Lemma 5 we see that there exists $\bar{t} \ll -1$ such that $w_t > 0$ in $(-\infty, \bar{t}) \times S^{N-1}$. Diminishing $\bar{t}$ if necessary, we can also achieve by (3.3) that

$$-b + pw^{p-1} \leq -\frac{b}{2} < 0 \quad \text{for } t \leq \bar{t}.$$

**Claim:** $z^\lambda \geq 0$ in $\Sigma_{\lambda}$ when $\lambda \leq \bar{t}$.

Indeed, assume on the contrary that

$$\inf_{\Sigma_{\lambda}} z^\lambda < 0$$

for some $\lambda \leq \bar{t}$. By (3.4) and since $z^\lambda = 0$ on $T_{\lambda}$, we deduce the existence of a point $(t_0, \theta_0) \in \Sigma_{\lambda}$ such that the infimum of $z^\lambda$ is achieved. Thus $z^\lambda(t_0, \theta_0) = 0$, $z^\lambda_t(t_0, \theta_0) \geq 0$, $\Delta_\theta z^\lambda(t_0, \theta_0) \geq 0$ and

$$-b + p\xi_{\lambda}(t_0, \theta_0)^{p-1} \leq -b + pw(t_0, \theta_0)^{p-1} \leq -\frac{b}{2}$$

by our choice of $\bar{t}$. Then, using (3.4) and recalling (3.3):

$$(-b + p\xi_{\lambda}(t_0, \theta_0)^{p-1}) z^\lambda(t_0, \theta_0) \leq -2aw_t(t_0, \theta_0) \leq 0,$$

a contradiction. This shows the claim.
Next define
\[ \lambda_0 = \sup \{ \lambda \in \mathbb{R} : \ z^\mu \geq 0 \text{ in } \Sigma_\mu, \ \text{for every } \mu \in (-\infty, \lambda) \}. \]

Two situations are possible:

1. \( \lambda_0 = +\infty; \)
2. \( \lambda_0 < +\infty. \)

In case (i), we would obtain that \( z^\lambda \geq 0 \) in \( \Sigma_\lambda \) for every \( \lambda \in \mathbb{R} \), which implies that \( w \) is nondecreasing in the \( t \) variable.

Therefore we only have to deal with case (ii). By the strong maximum principle we deduce that either \( z^{\lambda_0} \equiv 0 \) in \( \Sigma_{\lambda_0} \) or \( z^{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \) with \( z_t < 0 \) on \( \{\lambda_0\} \times S^{N-1}. \)

The second situation can be easily discarded. By the definition of \( \lambda_0 \), there exist sequences \( \lambda_n \to \lambda_0^+, (t_n, \theta_n) \in \Sigma_{\lambda_n} \) such that
\[ z^{\lambda_n}(t_n, \theta_n) < 0. \]

We claim that we can always assume \( t_n \geq \bar{t}. \) Otherwise, we would have \( z^{\lambda_n} \geq 0 \) in \( \Sigma_{\lambda_n} \setminus \Sigma_{\bar{t}}. \) Reasoning as in the claim above this would yield \( z^{\lambda_n} \geq 0 \) in \( \Sigma_{\lambda_n}, \) a contradiction.

Therefore \( t_n \in [\bar{t}, \lambda_n]. \) Passing to subsequences we may assume \( t_n \to t^* \in [\bar{t}, \lambda_0], (t_n, \theta_n) \to (\theta^* \in S^{N-1}. Then \( z^{\lambda_0}(t^*, \theta^*) = 0. \) Let us see that this is impossible.

If \( t^* < \lambda_0 \) we have an immediate contradiction with \( z^{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0}. \) When \( t^* = \lambda_0, \) we can select points \( s_n \in (t_n, \lambda_n) \) such that \( z_t(s_n, \theta_n) \geq 0. \) Passing to the limit this would yield \( z_t(\lambda_0, \theta^*) \geq 0, \) which is also a contradiction.

To summarize, we have shown that when \( \lambda_0 < +\infty \) we always have \( z^{\lambda_0} \equiv 0 \) in \( \Sigma_{\lambda_0}, \) that is, \( w \) is symmetric with respect to \( \lambda_0. \) When \( (i) \) holds we have that the coefficient \( a \) in \( (3.3) \) is strictly positive. The symmetry of \( w \) would imply from \( (3.1) \) that \( w_t = 0 \) in \( \mathbb{R} \times S^{N-1}, \) which is equivalent to \( w = w_0(\theta) \) for some positive function \( w_0. \) Then \( u(x) = w_0(\frac{x}{|x|})|x|^{-\beta}, \) which contradicts the boundedness of \( u \) near \( x = 0. \) This shows that, with the subcriticality assumption \( (1.3) \) we always have \( \lambda_0 = +\infty, \) and (a) is proved.

When \( p \) verifies \( (1.8), \) both \( \lambda_0 = +\infty \) and \( \lambda_0 < +\infty \) are possible. In the second case, the function \( u \) is symmetric with respect to \( \lambda_0. \) Setting \( \mu^2 = e^{2\lambda_0} \) and rewriting the symmetry property of \( w \) in terms of the original function \( u \) we obtain \( (3.2). \) This concludes the proof of (b).

After all these preliminaries, we are in a position to prove our two main results, Theorems [1] and [2].

**Proof of Theorem [7]** Because of the already mentioned results in [1] and [24], we may assume that \( p \) verifies \( (3.1). \) We claim that \( u \) is stable in \( \mathbb{R}^N. \) To see this, we first apply Theorem [9] to obtain that \( |x|^\beta u(x) \) is nondecreasing as a function of \( |x|. \) Since \( u \) is smooth in \( \mathbb{R}^N \setminus \{0\} \) by Lemma [3], this implies that
\[ v = \nabla u(x) \cdot x + \beta u(x) \geq 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \]
Moreover, from Lemma 5 we see that $v \in W^{2,q}_{\text{loc}}(\mathbb{R}^N)$ is a solution of the linearized equation

$$-\Delta v = p|x|^{\alpha}u^{p-1}v \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$ 

The strong maximum principle gives either $v > 0$ in $\mathbb{R}^N$ or $v \equiv 0$ in $\mathbb{R}^N$. However, this last option may not occur, since it would imply that $u$ is homogeneous of degree $\beta$, contradicting its boundedness at zero.

Therefore $v > 0$ in $\mathbb{R}^N$. By Lemma 7, this implies $\lambda^0_1(-p|x|^{\alpha}u^{p-1}) > 0$, for every smooth bounded domain $\Omega \subset \mathbb{R}^N$. Using the variational characterization (2.4), we arrive at

$$\int_{\Omega} |\nabla \phi|^2 - p|x|^{\alpha}u^{p-1}\phi^2 \geq \lambda^0_1(-p|x|^{\alpha}u^{p-1}) \int_{\Omega} \phi^2 \geq 0,$$

for every $\phi \in C^\infty_0(\Omega)$. Since $\Omega$ is arbitrary, the stability of $u$ is established. To conclude the proof, we use Theorem 8, which implies that $u \equiv 0$, a contradiction.

**Proof of Theorem 2.** By Theorem 9 and the proof of Theorem 1, we see that (3.2) holds for some $\mu > 0$. Observe that this implies

$$\lim_{x \to +\infty} |x|^{N-2}u(x) = \mu^{N-2}u(0).$$

Since $-2 < \alpha < 0$, we may use the moving plane method directly on $u$ to obtain that $u$ is radially symmetric. Then the conclusion follows from Appendix A in [18].

An alternative proof is to observe that (3.6) implies

$$\int_{\mathbb{R}^N} |x|^{\alpha}u(x)^{p+1}dx < +\infty,$$

Theorem 1.2 in [14] implies that $u$ is of the form (1.7) for some $\mu > 0$. □

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