The Goldberg–Kerr Approach to Lorentz Covariant Gravity

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Abstract

In this paper, the approach to asymptotic electromagnetic fields introduced by Goldberg and Kerr (J. Math. Phys. 5 (1964)) is used to study various aspects of Lorentz Covariant Gravity. Retarded multipole moments of the source, the central objects of this study, are defined, and a sequence of conservation equations for these are derived from the conservation of the energy-stress-momentum tensor of the source. The solution of the linearized Einstein field equation is obtained in terms of the retarded moments for a general bound source field, correct to $O(r^{-4})$. This is used to obtain the peeling–off of the linearized field, and to study the geometric optics approximation for the field and for the energy-momentum of the field, given by the Landau–Lifschitz pseudotensor. It is shown that the energy-momentum 4-vector splits into the ‘total radiated 4-momentum’ and the ‘bound 4-momentum of the source’, similar to the case of the electromagnetic field. The role played by the conservation equations in studying the radiative behaviour of the field is stressed throughout. In addition, in the case of a source which has only retarded pole, dipole and quadrupole moments, it is shown how to derive the arbitrary dependence of the field on a null coordinate. This allows comparison with the solutions for linearized gravity obtained by other authors.

I. INTRODUCTION

In previous articles [4,11], the Goldberg–Kerr (GK) [2] approach to the electromagnetic field was used to study various aspects of asymptotic electromagnetic fields due to bounded sources. In particular, the geometric optics approximation and the arbitrary dependence of the field on a null coordinate were established, and the relation of the multipole moments of the field to certain retarded moments of the source (integrals over null hyperplanes) was determined. One would like to be able to determine the same relationship in the case of the gravitational field. However, the non-linear nature of Einstein’s field equations prohibits

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this, and one must resort to approximation techniques to obtain results in this vein. Such techniques form the keystone in the study of gravitational waves [1,19].

The simplest approximation is linearized General Relativity (GR), used in the analysis of weak gravitational waves [4,7]. Here, the gravitational field is the Riemann tensor obtained from a first order perturbation to Minkowski space–time, quantified by writing the metric as

\[ g_{ij} = \eta_{ij} + \gamma_{ij}, \]  

(1.1)

Tensor indices are raised and lowered with the background Minkowski metric \( \eta_{ij} \), and \( \gamma_{ij} \) and its derivatives are taken to be small of first order, so that any products among these terms which arise are neglected. From the metric (1.1) the linearized Riemann and Einstein tensors can be calculated, and the ensuing linear field equations tackled, with or without a non-zero right hand side. Such discussions are usually augmented by a pseudotensor or effective stress tensor description of the energy-momentum of the gravitational field.

The aim of this paper is to study aspects of linearized GR using the GK approach to retarded fields due to bound sources. This involves obtaining the solution to the linearized field equations explicitly in terms of certain integrals over the source. In full GR, the linear approximation certainly does not hold all the way down to the source, but is usually applied in the distant wave zone. Hence, to distinguish these cases, the theory studied here will be referred to as Lorentz Covariant Gravity (LCG). However, since most of the results presented here are of an asymptotic nature, they will apply to the distant wave zone of relativistic sources.

The fundamental object of this study is \( \gamma_{ij} \), and the field equations that we consider may be written

\[ \Box \gamma^{*}_{ij} = -16\pi T_{ij}, \]  

\[ \gamma^{*ij}_{;j} = 0, \]  

(1.2)

(1.3)

where \( \gamma^{*}_{ij} \) is the trace reversed part of \( \gamma_{ij} \), and the trace reversal of any symmetric second order tensor is

\[ f^{*}_{ij} = f_{ij} - \frac{1}{2} \eta_{ij} f, \quad f = \eta^{ij} f_{ij}. \]  

(1.4)

\( \Box \) is the d’Alembertian of flat space–time, and \( T_{ij} \) is the energy-stress–momentum tensor of the source, which is conserved;

\[ T^{*ij}_{;j} = 0. \]  

(1.5)

The gravitational field in this theory is given by

\[ L_{ijkl} = \frac{1}{2}(\gamma_{il,jk} + \gamma_{jk,il} - \gamma_{ik,jl} - \gamma_{jl,ik}), \]  

(1.6)

where \( \gamma_{ij} \) is obtained from the trace reversal of \( \gamma^{*}_{ij} \).

The field due to an isolated, extended, bound source will be considered. Thus the energy-stress–momentum tensor \( T_{ij} \) of the source obeys the following conditions [2].
(1) There exists a time-like world line \( C \) : \( x^i = z^i(\tau) \) with unit tangent \( v^i = dz^i/d\tau \), where \( \tau \) is proper time along \( C \).

(2) There exists a scalar function \( h(\tau) \) such that \( T^{ij}(x) = 0 \) for all points \( x \) such that
\[
(x^i - z^i(\tau))(x_i - z_i(\tau)) \geq \max \left\{ 0, h^2(\tau) - [v_i(x^i - z^i(\tau))]^2 \right\}.
\]

Thus in an instantaneous rest frame (IRF) of \( C \), \( T^{ij} \) vanishes in the infinite region extending to spatial infinity which is bounded by the future and past null cones with apex \( z^i(\tau) \in C \) and the cylinder \( |x - z(\tau)| = h(\tau) \). The conditions (1) and (2) above define the world tube \( W \) which is the support of \( T^{ij} \), and in addition the weak assumption of ‘strong vanishing on the boundary of \( W \)’ is made, that is it is assumed that \( T^{ij} \) and a sufficient number of its derivatives vanish on the boundary of \( W \).

Using the world-line \( C \), coordinates and an associated null tetrad for Minkowski spacetime \( M \) can be set up in the following way (see [18,4]). Let \( x^a \) be any point of \( M \). The intersection of the past null cone \( N \) with vertex \( x^a \) and the world line \( C \) is a unique point, and defines a unique value \( u \) of \( \tau \), in terms of \( x^a \), which is referred to as retarded time. Tensor functions of \( u \) defined on \( C \) then become tensor fields on \( M \) by parallel transport up the null cone \( N \) to \( x^a \); e.g. \( v^a(x) \equiv v^a(\tau)|_{\tau=u} \). \( a^a, b^a, c^a \) respectively are used to denote the second (acceleration), third and fourth retarded time derivatives (denoted by a dot) of \( z^a(u) \). \( k^a \) (null) is defined by
\[
x^a = z^a(u) + rk^a,
\]
where the normalization \( k_av^a = -1 \) is chosen. Defining the stereographic coordinates (see appendix A of [18] for the definition of the polar angles in this case)
\[
\zeta = e^{i\phi} \cot \theta/2,
\]
one can write
\[
k^a = p^{-1}(1 + \zeta \bar{\zeta}, 1 - \zeta \bar{\zeta}, \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta})),
\]
where
\[
p = (1 + \zeta \bar{\zeta})v^0 - (1 - \zeta \bar{\zeta})v^1 - \zeta(v^2 - iv^3) - \bar{\zeta}(v^2 + iv^3).
\]
One can also give
\[
\frac{\partial}{\partial r} = k^a \frac{\partial}{\partial x^a}, \quad \frac{\partial}{\partial u} = v^a \frac{\partial}{\partial x^a} + ra \cdot k \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \zeta} = r \frac{\partial k^a}{\partial \zeta} \frac{\partial}{\partial x^a},
\]
\[
\eta^{ab} = \frac{1}{2}p^2 \left( \frac{\partial k^a}{\partial \zeta} \frac{\partial k^b}{\partial \zeta} + \frac{\partial k^a}{\partial \zeta} \frac{\partial k^b}{\partial \zeta} \right) - k^a v^b - v^a k^b + k^a k^b,
\]
\[
\frac{\partial}{\partial x^j} = -k_j \frac{\partial}{\partial u} - (v_j - (1 + ra \cdot k)k_j) \frac{\partial}{\partial r} + \frac{p^2}{2r} \left( \frac{\partial k_j}{\partial \zeta} \frac{\partial}{\partial \zeta} + \frac{\partial k_j}{\partial \zeta} \frac{\partial}{\partial \zeta} \right),
\]
and

$$\Box = \eta^{ab} \frac{\partial^2}{\partial x^a \partial x^b} = \frac{1}{r^2} \Delta + (1 + 2 r a \cdot k) \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) - \frac{2}{r} \frac{\partial}{\partial u} - 2 \frac{\partial^2}{\partial u \partial r} ,$$

(1.15)

where $\Delta = p^2 \partial^2 / \partial \zeta \partial \bar{\zeta}$ is the Laplacian on the unit sphere.

The retarded distance is $r = -v_a (x^a - z^a(u))$, and the following derivatives obtain:

$$u_{,a} = -k_a ,$$

(1.16a)

$$v_{b,a} = -a_b k_a ,$$

(1.16b)

$$r_{,a} = -v_a + (1 + r a \cdot k) k_a ,$$

(1.16c)

$$k_{b,a} = \frac{1}{r} (\eta_{ab} + v_a k_b + k_a v_b - (1 + r a \cdot k) k_a k_b) .$$

(1.16d)

Here, $a \cdot k = a_a k^a$. From these, it is seen that $k^a$ is tangent to geodesics, with $r$ an affine parameter along its integral curves. One can use (1.16a) to differentiate tensor functions of $u$;

$$T_{b..}(u)_{,a} = -T_{b..}^a k_a .$$

(1.17)

The field equation (1.2) is solved by (see Synge [16], p.407)

$$\gamma^{*}_{ij}(x) = 4 \int_{N \cap W} T_{ij} d\sigma^{*} = 4 \int_{N \cap W} \frac{T_{ij}(y^0, y)}{|x - y|} d_3y ,$$

(1.18)

where $d\sigma$ is the ‘absolute two-content of the three cell on the null cone’ (Synge, [16] p.429) which reduces to the familiar form in an IRF of $C$. (Equations capped with an asterisk hold in the IRF, and boldface letters represent spatial co-ordinates $x^\alpha \ , \ \alpha = 1, 2, 3$.) Events in $W$ are labelled $y^i = (y^0, y)$, and the integration is over the intersection of the past null cone $N$ with vertex $x^i$ with the world tube $W$.

In §2, the basic variables of the study, the retarded multipole moments of the source are defined. A sequence of conservation equations for these moments are derived using (1.5), and their space-time derivatives are calculated. The solution $\gamma^{*}_{ij}$ is obtained to $O(r^{-4})$ for a completely general source.

In §3, the linearized curvature tensor is calculated. This allows verification of Sachs’ peeling–off theorem for linearized gravity [14], but for the more satisfactory case of a field due to an extended bounded source, rather than one with its source confined to a time-like world line [8]. The role played by the conservation equations of §2 in determining the Petrov type of the coefficients of the field is stressed, particularly for the leading order (radiative) part of the field. The results of this section are given in part in the Newman–Penrose (NP) formalism.

The geometric optics approximation to different aspects of the field is considered in §4. First, it is demonstrated how the leading order term in the curvature tensor may be considered an approximate solution to the vacuum field equations which obeys the usual geometric conditions of a radiation field (algebraic type N with shear-free twist-free geodesic rays). Second, the Landau–Lifshitz pseudotensor $t^{ij}$ is used to discuss the energy–momentum of the field. As in the case of the electromagnetic energy tensor [4,18], it is shown that the energy-momentum pseudotensor may be split into two parts, each separately conserved, $\text{rad} t^{ij}$ and
rem \tau_{ij} = \tau_{ij} - \text{rad} \tau_{ij}. The first term \text{rad} \tau_{ij} has the form associated with a geometric optics field. From it, the ‘radiative 4-momentum’ \text{rad} P^i is constructed. This is associated with a ‘cloud of gravitons’ which get radiated away to \mathcal{J}^+ while the corresponding rem P^i is shown to determine a bound 4-momentum for the field [18]. In the course of this discussion the Sommerfeld outgoing radiation conditions are discussed, and an interesting interplay between these and the first of the series of conservation equations for the retarded moments of the source is pointed out.

As mentioned above, it is possible to relate the multipole moments of the field to the retarded moments of the source in the case of electromagnetism. One expects the same to be true in the case of LCG, and in §5 a decomposition of the field, in the case of a pole-dipole-quadrupole source, is obtained. This decomposition would allow one to obtain the stated relation, and also brings out the expected arbitrary dependence of the field on a null coordinate [21]. The resulting alternative form for the solution \gamma_{ij} of (1.2) allows comparison with the solutions obtained by other authors [13,15].

II. MATHEMATICAL PRELIMINARIES

In this section, the retarded multipole moments of the source \tau_{ij} are defined, and the conservation equations which can be deduced from (1.5) are given. The space–time derivatives of the first few moments, which will be of use in the following sections, are derived.

The Goldberg–Kerr formalism is based on the intuitive notion that as the field point moves infinitely far away from the source, the domain of integration in (1.18) is asymptotically planar; the portion of the past null cone with apex \( x^a \) which intersects \( W \) flattens out. Thus the central feature of the GK analysis is the use of retarded multipole moments of the current 4–vector, defined by

\[
\mathcal{T}_{ij} = \int_P \tau_{ij} d\Sigma, \tag{2.1a}
\]

\[
\mathcal{T}_{ij:k_1\ldots:k_n} = \int_P \tau_{ij} \xi_{k_1} \cdots \xi_{k_n} d\Sigma, \quad n = 1, 2, 3, \ldots \tag{2.1b}
\]

Eq. (2.1a) gives the retarded monopole moment, and (2.1b) the retarded 2^n–pole moment. Here,

\[
\xi_i = y_i - z_i(u), \tag{2.2}
\]

and on \( P \),

\[
\xi^i = \begin{pmatrix} \xi & \mathbf{k} \cdot \xi \\ k^0 \end{pmatrix}^* = (\xi, \mathbf{k} \cdot \xi), \tag{2.3}
\]

or equivalently,

\[
k_i \xi^i = 0. \tag{2.4}
\]

The domain of integration \( P \) is the null hyperplane which contains the null geodesic from the field point \( x \) to \( C \), and \( d\Sigma \) is the invariant volume element on \( P \), obeying
\[ d\Sigma \doteq d_3y, \quad (2.5) \]

and it can be shown that

\[ d\Sigma_i = a \cdot kk_i d\Sigma. \quad (2.6) \]

Clearly, \( T_{ij; k_1 \ldots k_n} \) is symmetric on all its indices after the colon, and by its construction, obeys

\[ k^i T_{ij; l_1 \ldots l_n} = 0. \quad (2.7) \]

Integrals of the form \( (2.1b) \) arise naturally in this context in the following way. Points \( y^i \) in the domain of integration of \( (1.18) \) lie on the past null cone with vertex \( x^i \), and so

\[ y^0 = x^0 - |x - y| \\
= z^0 + rk^0 - |x - y| \\
= z^0 + k \cdot \xi - \sigma, \quad (2.8) \]

where

\[ |x - y| \doteq r - k \cdot \xi + \sigma, \quad (2.9) \]

and

\[ \sigma = \sum_{n=1}^{\infty} b_n r^{-n}. \quad (2.10) \]

The coefficients \( b_n \) may be obtained from a power series expansion of \( |x - y| = |r k - \xi| \). Then if \( \sigma = 0 \),

\[ y^0 \doteq z^0 + k \cdot \xi \\
= z^0 + \frac{1}{k_0} k \cdot \xi, \quad (2.11) \]

so that according to \( (2.4) \), \( y^i \) lies on \( P \).

Introducing the notation used in GK,

\[ [F] = F(z^0 + \frac{1}{k_0} k \cdot \xi, y), \]

it follows by applying the chain rule that

\[ [F]_a = [F]_a + \frac{k_0}{k} [F], \quad (2.12) \]

where \([F]_a\) means ‘put \( y^0 = z^0 + \frac{k_0}{k} \xi \), and then differentiate w.r.t. \( y^a\)’ while \([F]_a\) means ‘differentiate w.r.t. \( y^a\) and then put \( y^0 = z^0 + \frac{k_0}{k} \xi \).’

The conservation equation for the source \( (1.5) \) is used to obtain a sequence of conservation equations for the retarded moments as follows. In \( (2.12) \), we take \( F \) to be respectively
\( F = T^{ij} \) and \( F = T^{ij} \xi^k \cdots \xi^n \), integrate over the intersection of the null plane \( P \) with the world tube \( W \) and use (1.3) and the strong vanishing on the boundary to obtain

\[ k^j D T_{ij} = 0, \quad \text{(2.13a)} \]

\[ k^j D T_{ij;k_1 \ldots k_n} + n \left\{ T_{i(k_1; k_2 \ldots k_n)} + k^j T_{ij; k_2 \ldots k_n v_{k_1}} \right\} = 0. \quad \text{(2.13b)} \]

To obtain the derivatives of the retarded moments, the procedure given by Hogan and Ellis (HE) [4] is followed. The calculation for the monopole moment is given in full detail; derivatives of the higher moments are obtained in a similar way. From (2.1a) and (2.6), it follows that

\[ k^j D T_{ij,k} = \int_P \frac{\partial}{\partial x^k} [T_{ij}] + a \cdot k k^l [T_{ij}] d\Sigma \]

\[ = a \cdot k T_{ij,k} + \int_P \frac{\partial}{\partial x^k} [T_{ij}] d\Sigma. \quad \text{(2.14)} \]

The latter term is given by

\[ \int_P \frac{\partial}{\partial x^k} [T_{ij}] d\Sigma = \int_P \left[ T_{ij,(1)} \frac{\partial y^0}{\partial x^k} d\Sigma \right. \]

\[ \left. + \int_P \left[ T_{ij,(1)} \right] \left\{ -k_k + \frac{1}{r} (\xi_k - (k \cdot \xi) k_k) \right\} d_3 y. \quad \text{(2.15)} \]

To write this in terms of the retarded moments, note first that

\[ k \cdot \xi = v^i \xi_i. \]

In general,

\[ [F,0] = k^0 \left\{ \frac{\partial}{\partial u} [F] - \left[ \frac{\partial F}{\partial u} \right] \right\}, \quad \text{(2.16)} \]

and using (2.6),

\[ \int_P \frac{\partial}{\partial u} [F] d\Sigma = D \int_P [F] d\Sigma. \quad \text{(2.17)} \]

Combining these results, one obtains

\[ \int_P \frac{\partial}{\partial x^k} [T_{ij}] = -k_k D T_{ij} + \frac{1}{r} (D T_{ij;k} + v^i k_k D T_{ij;i}) , \quad \text{(2.18)} \]

and so

\[ T_{ij,k} = -\dot{T}_{ij,k} + \frac{1}{r} (D T_{ij;k} + v^i k_k D T_{ij;i}). \quad \text{(2.19a)} \]

Similarly, it is found that

\[ T_{ij;k,l} = -\dot{T}_{ij;k} k_l + \frac{1}{r} (v_k (T_{ij;l} + v^m k_l T_{ij;m}) + D T_{ij;k,l} + v^m k_l D T_{ij;k,m}) , \quad \text{(2.19b)} \]
and
\[ T_{ij,kl,m} = -\dot{T}_{ij,kl}k_m + \frac{1}{r} \left( v_k (T_{ij,lm} + v^p k_m T_{ij,lp}) + v_l (T_{ij,km} + v^p k_m T_{ij,kp}) \right. \]
\[ \left. + D T_{ij,klm} + k_m v^p D T_{ij,klp} \right). \] (2.19c)

Next, it is demonstrated how (1.18) is evaluated. Again, the procedure of HE is followed. A power series expansion at infinity in the variable \( y^0 \) yields
\[ T_{ij}(y^0, y) = \sum_{n=0}^{\infty} \frac{(-\sigma)^n}{n!} [T_{ij,(n)}], \]
where the subscript \((n)\) means differentiate \( n \) times with respect to \( y^0 \). Writing
\[ H_n = \frac{(-\sigma)^n}{n!} \frac{1}{|\mathbf{x} - \mathbf{y}|}, \quad n = 0, 1, 2, \ldots, \] (2.20)
(1.18) becomes (in the IRF)
\[ \gamma^*_{ij} = 4 \sum_{n=0}^{\infty} \int_{\Sigma} H_n \left[ T_{ij,(n)} \right] d_3 y, \] (2.21)
This can be written as a power series in \( r^{-1} \), with the coefficients identified as IRF values of certain combinations of the retarded moments. For illustrative purposes, the calculation for the first two coefficients is given.
With the \( b_n \) defined by (2.10), it is found that
\[ b_1 = \frac{1}{2} (|\xi|^2 - (\mathbf{k} \cdot \xi)^2), \] (2.22)
whence
\[ H_0 = \frac{1}{r} + \frac{1}{r^2} \mathbf{k} \cdot \xi + O(r^{-3}), \] (2.23)
\[ H_1 = -\frac{1}{r^2} (|\xi|^2 - (\mathbf{k} \cdot \xi)^2) + O(r^{-3}), \] (2.24)
\[ H_2 = O(r^{-3}). \] (2.25)
Thus
\[ \gamma^*_{ij} = \frac{1}{r} \left\{ 4 \int_{\Sigma} [T_{ij}] d_3 y \right\} \]
\[ + \frac{2}{r^2} \left\{ \int_{\Sigma} 2 \mathbf{k} \cdot \xi [T_{ij}] - (|\xi|^2 - (\mathbf{k} \cdot \xi)^2) [T_{ij,(1)}] d_3 y \right\} + O(r^{-3}). \] (2.26)
The terms in the integrand here obey
\[ \int_{\Sigma} [T_{ij}] d_3 y \equiv \int_{\Sigma} [T_{ij}] d\Sigma = T_{ij}, \] (2.27)
\[ \int_{\Sigma} \mathbf{k} \cdot \xi [T_{ij}] d_3 y \equiv -\int_{\Sigma} v^k \xi_k [T_{ij}] d\Sigma = -T_{ij,k} v^k, \] (2.28)
\[ (|\xi|^2 - (\mathbf{k} \cdot \xi)^2) \equiv |\xi|^2 - (\xi^0)^2 = \xi_i \xi^i, \] (2.29)
and so
\[ \int_P \left( |\xi|^2 - (k \cdot \xi)^2 \right) T_{ij,(1)} \, d^3y = \int_P T_{ij,(1)} \xi^k \xi^k d^3y = DT_{ij,k}^k, \tag{2.30} \]
where (2.16), (2.17) and \( \partial [\xi^i] / \partial u = 0 \) have been used. Then combining these results, one obtains
\[
\gamma^*_{ij} = \frac{4}{r} T_{ij} - \frac{2}{r^2}(2T_{ij,k}v^k + DT_{ij,k}^k) + O(r^{-3}). \tag{2.31}
\]

Using the same procedure, the next term in the power series solution may be calculated to give
\[
\gamma^*_{ij} = \frac{4}{r} T_{ij} - \frac{2}{r^2}(2T_{ij,k}v^k + DT_{ij,k}^k) + \frac{2}{r^3}(2T_{ij,kl}v^k v^l - T_{ij,k} v^l) \\
+ 2DT_{ij,k}^k v^l + T_{ij,k}^k Dv^l + \frac{1}{4}(D + a \cdot k)DT_{ij,kl}^k + O(r^{-4}). \tag{2.32}
\]

To end this section, it is pointed out that no assumptions, other than strong vanishing at the boundary, have been made about the source in order to obtain (2.32). In particular, the conservation equations (2.13) have not been imposed, nor has a restriction to a source possessing only a finite number of non-zero retarded multipole moments been made. Finally, it should be pointed out that (1.3) is required to obtain the solution (1.18), so that the comments below regarding withholding the conservation equations only strictly apply to quantities \( \gamma^*_{ij} \) formally defined by retarded integrals of the form (1.18). This will not affect the interpretation of the rôle played by the conservation equation (2.13a).

### III. STRUCTURE OF THE CURVATURE TENSOR

Having seen in the previous section how to evaluate the solution \( \gamma^*_{ij} \) of (1.4), the curvature tensor associated with this solution is now evaluated and its structure examined. Sachs [14] established the peeling–off theorem for linear fields with sources confined to a time-like world line; the same can now be done for the more realistic case of fields due to extended bound sources (see [8]). This is interesting in its own right, but in addition will show up the important rôle played by the conservation equations in the determination of the peeling–off behaviour.

Thus the main object of this section is to calculate the linearized curvature tensor (1.6) using the solution (2.32) correct to \( O(r^{-4}) \). Clearly, this would be an extremely lengthy procedure, so in order to establish the main results with the minimum fuss, the second derivatives of \( \gamma^*_{ij} \) are calculated and then the NP Weyl tensor components are evaluated. This will give the following limited peeling–off result;

\[
\Psi_4 = \frac{\Psi_4^{(0)}}{r} + O(r^{-2}), \tag{3.1a}
\]
\[
\Psi_3 = \frac{\Psi_3^{(0)}}{r^2} + O(r^{-3}), \tag{3.1b}
\]
\[
\Psi_2 = \frac{\Psi_2^{(0)}}{r^3} + O(r^{-4}). \tag{3.1c}
\]
\[ \Psi_1 = O(r^{-4}), \]
\[ \Psi_0 = O(r^{-4}), \]  

(3.1d)

(3.1e)

where the \( \Psi_A, A = 0 - 4 \) are calculated from the linearized curvature tensor \( (1.6) \). See e.g. [6] for the definitions. These components will be evaluated on the null tetrad \( NT = \{k^i, n^i, s^i, \bar{s}^i\} \) where

\[
n_i = v_i - \frac{1}{2} k_i, \quad s_i = \frac{1}{\sqrt{2}} p \frac{\partial k_i}{\partial \zeta}, \quad \bar{s}_i = \frac{1}{\sqrt{2}} p \frac{\partial k_i}{\partial \zeta},
\]

(3.2)

and so the only non-vanishing inner products are

\[ k_i n^i = -1, \quad s_i \bar{s}^i = 1. \]  

(3.3)

The formulae of \( \S1 \) and the expressions above for the first derivatives of the retarded moments are used to obtain the following expressions. (Note the use of \( (1.13) \) in these calculations, and \( \dot{p} = -a \cdot kp \), which follows immediately from \( (1.11) \).)

\[
\dot{T}_{ij,k} = -\ddot{T}_{ij} k_l + \frac{1}{r} (\delta^l_k + v^l k_k) D^2 T_{ij;l},
\]

(3.4)

\[
\dot{T}_{ij,k,l} = -\ddot{T}_{ij} k_l l + \frac{1}{r} (\delta^m_l + v^m k_l) (v_k D T_{ij;m} + a_k T_{ij;m} \cdot D^2 T_{ij;km}),
\]

(3.5)

\[
\ddot{T}_{ij;m} + \dot{T}_{ij;m} k_l = -\ddot{T}_{ij;m} l + \frac{1}{r} (\delta^p_l + v^p k_l) (2 v^m D T_{ij;mp} + 2 a^m T_{ij;mp} + D^2 T_{ij;m}^m_{mp}),
\]

(3.6)

\[
\dddot{T}_{ij;m} + \ddot{T}_{ij;m} k_l = -\dddot{T}_{ij;m} l + \frac{1}{r} (\delta^p_l + v^p k_l) (2 D^2 (v^m T_{ij;mp}) + D^3 T_{ij;m}^m_{mp}).
\]

(3.7)

Using these and the expressions for the first derivatives given in \( \S2 \), the second derivative of the (trace reversed) metric perturbation can be calculated. The resulting lengthy expressions are given in Appendix A. Using these, the complete tensorial expression for \( L_{ijkl} \) may be given, correct to \( O(r^{-4}) \). However, the main results may be expressed more succinctly in the NP formalism.

Foremost of these results is the verification of \( (3.1) \), which is done using the expressions \( (A6)-(A10) \) of Appendix A, \( (1.6) \) and the relation \( (1.4) \) between \( \gamma_{ij} \) and \( \gamma^{*}_{ij} \). In the general case, it is found that

\[
\Psi_4^{(0)} = -2(D + a \cdot k) D T_{ij} s^i s^j,
\]

(3.8)

\[
\Psi_3^{(0)} = 4a \cdot \bar{s} k^i n^j T_{ij} + a \cdot \bar{s} T^i_{ij} - 2 \bar{s}^i (a^i + a \cdot k v^j) T_{ij}
- 2 \bar{s}^i (D + a \cdot k) D T_{jm}^{m} + \bar{s}^j (D + a \cdot k) D T_{ij}^{i}. 
\]

(3.9)

An examination of Appendix A will convince the reader that \( \Psi_2^{(0)} \) is too long to merit inclusion. For simple illustrative purposes it may be calculated for the case of a monopole particle, that is for a source for which

\[ T_{ij} \neq 0, \quad T_{ijkl...k_n} = 0 \quad n \geq 1. \]

In this case, it is found that
\[ \Psi_2 = -\frac{m}{r^3}, \]  
(3.10)

where \( m \) is a constant, which is the usual result.

It should be noted that in deriving eqs. (3.11) and the coefficients above, the conservation equations (2.13) have been used repeatedly. A more careful examination of the leading order term of the field demonstrates what role these equations play. It is found that

\[ L_{ijkl} = N_{ijkl}r^{-1} + O(r^{-2}), \]  
(3.11)

where

\[ N_{ijkl} = N_{iljk}k_jk_k + N_{jkli}k_jk_l - N_{iklj}k_lk_l - N_{ijkl}k_ik_i, \]  
(3.12)

and

\[ N_{ij} = (D + a \cdot k)D T_{ij} - \frac{1}{2}n_{ij}(D + a \cdot k)D T_m^m. \]  
(3.13)

Imposing the conservation equation (2.13a), it is found that

\[ N_{ijkl}k_lk_l = 0, \]  
(3.15)

Thus, to leading order the field is Petrov type N, with degenerate principal null direction \( k^i \). This vector has geodesic, twist–free, shear–free and expanding integral curves, the null rays of the radiation field. This will allow in §4 a discussion of the geometric optics approximation to the field, by considering the field

\[ L_{ijkl} = \frac{N_{ijkl}}{r}, \]  
(3.16)

and working along the same lines as §4 of [4].

It should be pointed out that in general, i.e. without imposing the conservation equations (2.13), the leading term in the field (3.11) is not type N. Indeed

\[ N_{ijkl}k_l = D(k^l D T_{il})k_jk_k - D(k^l D T_{jl})k_ik_k, \]

which will be non–zero in general. However, it is straightforward to show that

\[ N_{ijkl}k_m|k^l = 0, \]  
(3.17)

so that the leading order term is Petrov type III, with degenerate principal null direction \( k^i \), which type of field is still characteristic of gravitational radiation. Thus it is seen how a fundamental property of the source, the monopole conservation equation, has consequences for the dynamics of the field. These consequences of conservation will also manifest themselves when the flux of energy-momentum of the field is considered.
IV. THE GEOMETRIC OPTICS APPROXIMATION

In this section, the geometric optics approximation to the Lorentz covariant gravitational field is discussed. Two related aspects of this problem are considered. Firstly, it is shown that there exists a linearized curvature tensor which may be considered as an approximate solution to the field equations, and which has the geometric structure of a radiation field. Secondly, the energy momentum of the full field is examined, using the Landau–Lifschitz (LL) pseudotensor \[ T_{ij} \]. This is shown to split into a geometric optics part, which gets radiated away to \( J^+ \) along null geodesics, and a remainder term which in the way set out below, remains bound to the source.

The LL pseudotensor appropriate to linearized gravity is \[ T_{ij} = \frac{1}{16\pi} \left\{ 2\Gamma^{nk}_{i} \Gamma^p_{np} - \Gamma^i_{n} \Gamma^p_{nk} - \Gamma^i_{n} \Gamma^p_{nk} \right\} \eta^{ik} \], which is conserved, \[ T_{ij,.j} = 0 \]. (4.2)

Obviously, this is not constructed from the field \( L_{ijkl} \), but from the ‘field strengths’, \( \Gamma^i_{jk} \). So unlike the situation in electromagnetic theory, where the geometric optics part of the energy–momentum tensor of the field is the energy–momentum tensor of the geometric optics part of the field, the geometric optics approximations must be defined separately. The relation between the two will be clear. A different approach to the geometric optics approximation in linearized gravity involves a WKB analysis of the field equation (1.2) \[ \Box \phi = 0 \].

The first part of the discussion involves an analysis of the leading order term in the field, \[ L_{ijkl} = \frac{1}{r} N_{ijkl} \], (4.3)

where \( N_{ijkl} \) is defined in (3.12) and (3.13).

To show that (4.3) may be considered an approximate solution of the linearized field equations, for large values of \( r \), it must be shown to be (approximately) a linearized Riemann tensor with vanishing Ricci tensor. Away from the source, the linearized Ricci tensor constructed from (4.3) is \[ R_{ij} = L_{ijk}^j = 0 \]. (4.4)

Now \( L_{ijkl} \) clearly has the symmetries of a Riemann tensor, so it remains to show that it obeys (approximately) the linearized Bianchi identities, \[ L_{ijkl,m} = 0 \]. (4.5)

Using some results given in §3, one can calculate
\[ N_{ij,k}^* = -2(D - a \cdot k)(D + a \cdot k) DT_{ij}k_k \]
\[ + \frac{2}{r} \left\{ (\delta^i_k + v^i k_k)(D + 2a \cdot k)(D + a \cdot k) DT_{ij}l + 3(a_k + a \cdot k(v_k - k_k)) \hat{T}_{ij} \right\} . \] (4.6)

This is the central equation used to obtain
\[ L_{ij[kl,m]} = \frac{1}{r^2} C_{ij[kl,m]} , \] (4.7)
where
\[ C_{ijkl} = 8B_{ijkl} , \] (4.8)
\[ B_{ijkl} = \left\{ (D + 2a \cdot k)(D + a \cdot k) DT_{ik,l}^* + 3(a_l + a \cdot k v_l) \hat{T}_{ik}^* \right\} k_j + (D + a \cdot k) DT_{ik}^*(\eta_{jl} + 2k_j v_l) . \] (4.9)

A rest frame calculation along the lines of §2 shows that
\[ B_{ijkl} \hat{T}_{ik}^* = k_j \int P_{T_{ik,(3)}^*} d3y + (\eta_{jl} - 2k_j \delta^i_l) \int P_{T_{ik,(2)}^*} d3y . \] (4.10)

The domain of integration here is the sphere
\[ |\zeta| \leq h(u) , \]
where \( h \) is the ‘radius’ of the source given in the introduction. Also, since \( T_{ij}(x) \) is, in each integral, compactly supported, there exist scalars \( M_n(u), n = 0, 1, 2, \ldots \) such that
\[ |T_{ij,(n)}^*| \leq M_n(u) , \quad n = 0, 1, 2, \ldots \] (4.11)
Using this fact, the following bound can be obtained for (4.10):
\[ |B_{ijkl}| \leq 4\pi h^3 M_2 + \frac{4}{3} \pi h^4 M_3 , \] (4.12)
which leads to a similar bound for \( C_{ijkl} \).

Thus equations (4.4), (4.7) and (4.12) show that \( L_{ijkl} \) is an approximate solution of the vacuum linearized Einstein equations. The accuracy of the approximation is determined by the bounds \( M_2 \) and \( M_3 \).

It has already been shown that \( L_{ijkl} \) is Petrov type N, with \( k^i \) as repeated p.n.d. Therefore, the field picks out a unique null direction whose integral curves form a shear-free, twist-free and expanding congruence of null geodesics. This ray geometry is characteristic of a geometric optics field, and so \( L_{ijkl} \) is referred to as the geometric optics approximation to the linearized gravitational field. One usually thinks of the field as ‘propagating in the direction \( k^i \)’, and so it is natural now to consider the question of energy-momentum transfer.

For this, the LL pseudotensor (4.1) is used. As pointed out above, this is not a functional of the field (curvature tensor), and so a separate but related geometric optics approximation is assumed. A detailed examination of the first two terms (coefficients of \( r^{-2} \) and \( r^{-3} \)) of \( t^ij \) is required, to obtain which one can calculate
\( \gamma_{ij,k} = -\frac{4}{r} D \mathcal{T}_{ij} k_k + \frac{4}{r^2} \{ \mathcal{T}_{ij}(v_k - k_k) + D \mathcal{T}_{ij,k} \} + (v^l(2D + a \cdot k) \mathcal{T}_{ij,l} + a^l \mathcal{T}_{ij,l} + \frac{1}{2}(D + a \cdot k) D \mathcal{T}_{ij,l} k_k \} + O(r^{-3}) \), \hspace{1cm} (4.13)

which leads to

\[
\Gamma_{ijk} = -\frac{2}{r}(D \mathcal{T}^*_{kl} k_j + D \mathcal{T}^*_{ji} k_k - D \mathcal{T}^*_{jk} i_l)
\]

\[
+ \frac{1}{r^2}(D \mathcal{T}^*_{kl} k_j + D \mathcal{T}^*_{ji} k_k - D \mathcal{T}^*_{jk} i_l) v_j + \mathcal{T}_{ji} v_k - \mathcal{T}_{jk} v_l + a_{kl} k_j + a_{ji} k_k - a_{jk} k_l
\]

\[+ O(r^{-3}), \hspace{1cm} (4.14)\]

where

\[ a_{ij} = \frac{1}{2}(D + a \cdot k)(D \mathcal{T}^*_{ij,l} + v^l(2D + a \cdot k) \mathcal{T}^*_{ij,l} + a^l \mathcal{T}^*_{ij,l} - \mathcal{T}^*_{ij}). \hspace{1cm} (4.15)\]

Trace reversals are taken over indices before the colon. One then finds

\[ a_{ij} k^i = -D \mathcal{T}^*_{ij,l} - \mathcal{T}_{ij} v^l \]

\[-(\frac{1}{4}(D + a \cdot k) D \mathcal{T}^*_{m,l} + v^l(D + \frac{1}{2} a \cdot k) \mathcal{T}^*_{m,l} + \frac{1}{2} a^l \mathcal{T}^*_{m,l} - \frac{1}{2} \mathcal{T}^*_{m} k_l). \hspace{1cm} (4.16)\]

Using this, it is found that

\[ \Gamma_{k, i} = \gamma^{*k,i} = 0, \hspace{1cm} (4.17)\]

\[ \Gamma_{ik} = \frac{2}{r} D \mathcal{T}^*_{li} k_j - \frac{2}{r^2}(D \mathcal{T}^*_{lj} + T^*_{lj} v_j - a^l k_j), \hspace{1cm} (4.18)\]

\[ \Gamma_{ijk} = \frac{2}{r^2} (-T_{ji} + \frac{1}{2} \eta_{ji} + b_{[ji]}) + O(r^{-3}), \hspace{1cm} (4.19)\]

\[ \Gamma_{ijkl} = \frac{2}{r} D \mathcal{T}^*_{li} k_j k_k - \frac{2}{r^2} \left\{ T_{jk} + \frac{1}{2} \eta_{jk} T^*_{li} + \frac{1}{2} \mathcal{T}^*_{li} (v_j k_k + v_k k_j) \right\}
\]

\[+ \frac{1}{2} D \mathcal{T}^*_{m} k_j + \frac{1}{2} D \mathcal{T}^*_{m,k} k_j + (D \mathcal{T}^*_{kl} + T^*_{kl} v^l) k_j + (D \mathcal{T}^*_{ji,l} + T^*_{ji} v^l) k_k
\]

\[+ (\frac{1}{2}(D + a \cdot k) D \mathcal{T}^*_{m,l} + v^l(2D + a \cdot k) \mathcal{T}^*_{m,l} + a^l \mathcal{T}^*_{m,l} - \mathcal{T}^*_{m}) k_j k_k \} + O(r^{-3}). \hspace{1cm} (4.20)\]

where

\[ b_{jl} = (\mathcal{T}^*_{m} v_l + D \mathcal{T}^*_{m,l} - 2D \mathcal{T}^*_{m,l} - 2 \mathcal{T}^*_{m,v^m}) k_j, \hspace{1cm} (4.21a)\]

which gives

\[ b_{jl} k^j = 0 \hspace{1cm} b_{ji} k^i = \mathcal{T}^*_{m} k_j. \hspace{1cm} (4.21b)\]

It will be convenient to write

\[ T^{ij} = \sum_{n=1}^{\infty} (\frac{a}{r^{n+1}})^{ij}. \hspace{1cm} (4.22)\]

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where
\[ \frac{\partial (n)^{tij}}{\partial r} = 0, \quad n \geq 1, \] (4.23)
and to define
\[ t^{ij} = \text{rad} t^{ij} + \text{rem} t^{ij}, \] (4.24)
where
\[ \text{rad} t^{ij} = \frac{(1) t^{ij}}{r^2}. \] (4.25)

The ‘rad’ and ‘rem’ here stand for ‘radiation’ and ‘remainder’ respectively. Using (4.1) and (4.14) - (4.21), it is found that
\[ \text{rad} t^{ij,j} = 0, \] (4.27)
which is the usual form for the energy–momentum of a geometric optics or radiation field. It is straightforward to show that this obeys
\[ \text{rem} t^{ij,j} = 0. \] (4.28)

Now a long calculation using (4.1) and (4.14) - (4.21) yields the following important result,
\[ t^{ij} k_j = O(r^{-4}), \] (4.29)
which from (4.26) is equivalent to
\[ (2) t^{ij} k_j = 0. \] (4.30)

Hogan and Ellis showed that the energy–momentum tensor of the electromagnetic field due to a bound source obeys a set of equations similar to (4.26) - (4.30) above. Thus a theorem of theirs can be applied to the linearized gravitational field:

**Theorem 1** Let \( K^{ijk} = U^{ik} k^j - U^{ij} k^k \), where
\[ U^{ik} = \frac{1}{r^2 (2)} t^{ik} + \sum_{n=3}^{\infty} \frac{1}{r^n} \left( \frac{(n) t^{ik}}{n-1} - \frac{(n) t^{il} k^l k^k}{(n-1)(n-2)} \right). \]

Then one can write
\[ K^{ijk} k_k = \sum_{n=2}^{\infty} \frac{(n) t^{ij}}{r^{n+1}} = \text{rem} t^{ij}. \] (4.31)
Proof: See [4], p.202.

To examine the transfer of energy–momentum by the field, one considers the flux of 4–momentum across the following fundamental surfaces. This follows exactly the procedure of HE for the electromagnetic case.

Consider the 4-volume $V$ of Minkowskian space time bounded by two time–like 3-surfaces $r = r_1$ and $r = r_2 > r_1$ and by the null cones $u = u_1$ and $u = u_2 > u_1$. $r_1$ is taken to be large enough so that we are outside the source, and so away from the world-line $C$ given by $r = 0$. Thus $t^{ij}$ is non-singular in $V$. The fluxes of 4-momentum across the boundaries of $V$ (in the directions of increasing $r, u$ on the appropriate surfaces) are given by

$$(^A)P^i = r_A^2 \int_{u_1}^{u_2} du \int t^{ij} r_j d\omega,$$  \hspace{1cm} (4.32)

across $r = r_A, (A = 1, 2)$ and

$$(^A)Q^i = - \int_{r_1}^{r_2} r^2 dr \int t^{ij} k_j d\omega,$$  \hspace{1cm} (4.33)

across $u = u_A$. In these integrals, $d\omega$ is the area element on the unit 2-sphere. The conservation equation $t^{ij,j} = 0$ gives the conservation law

$$(^1)P^i + (^1)Q^i = (^2)P^i + (^2)Q^i,$$  \hspace{1cm} (4.34)

which says that the total 4-momentum entering $V$ is equal to the total 4-momentum leaving $V$.

Using the decomposition of the energy-momentum pseudotensor (4.24), one can write

$P^i = \text{rad} P^i + \text{rem} P^i,$  \hspace{1cm} (4.35)

on $r =$constant, and

$Q^i = \text{rad} Q^i + \text{rem} Q^i,$  \hspace{1cm} (4.36)

on $u =$constant, where the first terms on the right hand sides are obtained by using $\text{rad} t^{ij}$ in the definitions (4.32) and (4.33), and the second terms on the right hand side by using $\text{rem} t^{ij}$.

From (4.26), one finds that

$$\text{rad} P^i = \frac{1}{2\pi} \int_{u_1}^{u_2} \int (D\tau_{mn} D\tau^{mn} - \frac{1}{2} (D\tau_m^m)^2) k^i d\omega,$$  \hspace{1cm} (4.37)

and

$$\text{rad} Q^i = 0.$$  \hspace{1cm} (4.38)

Noticing that $\text{rad} P^i$ is independent of $r$, these results give trivially the conservation law

$$(^1)P^i + (^1)Q^i = (^2)P^i + (^2)Q^i,$$  \hspace{1cm} (4.39)

which also follows from (1.27). This also means that $\text{rad} P^i$ may be used to calculate the flux of 4-momentum at $J^+ (r \to \infty, u \text{ finite})$. Indeed, since $k^i$ is future pointing and the coefficient
\[ DT_{mn} DT^{mn} - \frac{1}{2} (DT_m^m)^2 \geq 0 \]

(see below), one sees that \( \text{rad} P^i \) is future pointing, and so this vector is referred to as the \textit{total radiated 4-momentum of the field} across \( r = \text{constant} \) in the retarded time \( u_2 - u_1 \).

At this stage, the question of outgoing radiation conditions may be addressed. Trautman has expressed Sommerfeld’s outgoing radiation conditions as follows [20,3]: there exist coordinate systems and a tensor \( f_{ij} = O(r^{-1}) \) such that

\[ g_{ij} = \eta_{ij} + O(r^{-1}) , \tag{4.40a} \]
\[ g_{ij,k} = f_{ij}k^k + O(r^{-2}) , \tag{4.40b} \]
\[ f^*_{ij}k^j = O(r^{-2}) , \tag{4.40c} \]

with \( f^*_{ij} \) obtained from (1.4).

The energy–momentum pseudotensor calculated from (4.40) turns out to obey

\[ \varepsilon^{ij} = \frac{1}{32\pi} \left\{ f_{mn} f^{*mn} - \frac{1}{2} (f^{*m}_m)^2 \right\} k^i k^j + O(r^{-3}) , \tag{4.41} \]

and the coefficient of \( k^i k^j \) here is positive. Indeed, given any symmetric second order tensor \( A_{ij} \) on \( \mathcal{M} \) obeying \( A_{ij}k^j = 0 \), one can write

\[ A_{ij} = \alpha k^i k_j + \beta (k_i s_j + k_j s_i) + \beta (k_i \bar{s}_j + k_j \bar{s}_i) \\
+ \gamma s_i s_j + \gamma \bar{s}_i \bar{s}_j + \delta (s_i \bar{s}_j + s_j \bar{s}_i) , \]

where \( \alpha \) and \( \delta \) are real valued and the null tetrad of §3 has been used. Then \( A^i_i = 2\delta \) and \( A_{ij} A^{ij} = 2\gamma \bar{\gamma} \) so that

\[ A_{ij} A^{ij} - \frac{1}{2} (A^i_i)^2 = 2\gamma \bar{\gamma} , \]

which is positive. This is applied to \( f_{ij}^* \), using (4.40c), to show that coefficient of \( k^i \) in the integrand in (4.37) is positive, so that the integral leads to a positive outward flux of 4–momentum of the field.

Now from (1.13), it is seen that (4.40a) and (1.40b) will be satisfied for the field by taking

\[ f_{ij} = -(4DT_{ij} - 2\eta_{ij} DT^k_k) r^{-1} . \tag{4.42} \]

Then the key equation (4.40d) which is needed for the procedure above is

\[ k^j DT_{ij} = 0 , \tag{4.43} \]

which is exactly (2.13a), the monopole conservation equation of §2.

This may be compared with the result obtained by Hogan for a moving monopole particle [3]. Consider then a source for which

\[ T_{ij} \neq 0 , \quad T_{ij:k1...kn} = 0 , n \geq 1 . \tag{4.44} \]

From (2.13b) with \( n = 1 \), one obtains

From (2.13b) with \( n = 1 \), one obtains
\[ T_{ij} = m v_i v_j , \quad (4.45) \]

for some scalar \( m \). Then the outgoing radiation conditions, i.e. \((5.3)\) give

\[ (T_{ij} k^j) = -\dot{m} v_i - m a_i = 0 , \quad (4.46) \]

from which (contract with \( v^j \), then \( a^i \), using \( v_i a^i = 0 \)) it is found \( \dot{m} = 0 \) and \( a^i = 0 \). This is exactly the result obtained by Hogan; the outgoing radiation conditions force the acceleration of the particle to be zero. What is seen here is how this relates to the conservation of energy momentum of the source, i.e. of the particle. In general, this shows how a fundamental property of the source, its conservation, has important consequences for the dynamics of the field.

Next consider the remainder terms in \((4.35)\) and \((4.36)\). Using \((4.31)\) and Stokes’ theorem, one can write

\[ \text{rem} P^i = \left[ r^2 \int K^{ijk} k_j v_k \, d\omega \right]_{u = u_2} - \left[ r^2 \int K^{ijk} k_j v_k \, d\omega \right]_{u = u_1} , \quad (4.47) \]

and

\[ \text{rem} Q^i = -\left[ r^2 \int K^{ijk} k_j v_k \, d\omega \right]_{r = r_2} + \left[ r^2 \int K^{ijk} k_j v_k \, d\omega \right]_{r = r_1} . \quad (4.48) \]

Using the definitions of the theorem and \((4.29)\), one can write

\[ r^2 \int K^{ijk} k_j v_k \, d\omega = \sum_{n=3}^{\infty} f_n^i(u) \frac{r^n - 2}{r^n - 2} , \quad (4.49) \]

where

\[ f_n^i(u) = \int (n) k^{ij} k_j \, d\omega , \quad n = 3, 4, 5, \ldots \quad (4.50) \]

This allows one to write

\[ \begin{align*}
\text{rem} P^i &= \sum_{n=3}^{\infty} f_n^i(u_2) - f_n^i(u_1) \frac{r_2^n - 2}{r_2^n - 2} , \quad (4.51a) \\
\text{rem} P^i &= \sum_{n=3}^{\infty} f_n^i(u_2) - f_n^i(u_1) \frac{r_1^n - 2}{r_1^n - 2} , \quad (4.51b)
\end{align*} \]

and

\[ \begin{align*}
\text{rem} Q^i &= \sum_{n=3}^{\infty} f_n^i(u_1) \left( \frac{1}{r_1^n - 2} - \frac{1}{r_2^n - 2} \right) , \quad (4.52a) \\
\text{rem} Q^i &= \sum_{n=3}^{\infty} f_n^i(u_2) \left( \frac{1}{r_1^n - 2} - \frac{1}{r_2^n - 2} \right) , \quad (4.52b)
\end{align*} \]

which lead to the expected conservation law,

\[ \begin{align*}
\text{rem} P^i + \text{rem} Q^i &= \text{rem} P^i + \text{rem} Q^i \quad (4.53)
\end{align*} \]
Unlike the case for $\text{rad} P^i$, the fluxes in (4.51) depend on $r$. Eq. (4.52) indicates a leakage of 4-momentum across null cones, so that the remainder 4-momentum is not purely radiative.

One can make the same observations which Hogan and Ellis made for the electromagnetic field. The total flux of 4-momentum across $r = \text{constant}$ in a proper time $u_2 - u_1$ is

$$\text{rem} P^i = p^i(u_2) - p^i(u_1),$$

with

$$p^i(u) = \sum_{n=3}^{\infty} \frac{f^i_n(u)}{r^{n-2}(n-2)}.$$  

This vanishes in the limit $r \to \infty$ and so is non-radiative in character (no flux at $J^+\). This qualifies the characterization of $\text{rad} P^i$ as being the total radiated 4-momentum. Eq.(4.54) may be said to describe ‘particle-like behaviour’, in that the flux of 4-momentum between $u = u_1$ and $u = u_2$ is simply the difference in the values of the ‘4-momentum’ $p^i(u)$ at the two times. For this reason we refer to $p^i(u)$ as the bound 4-momentum of the source [4,18]. Thus Teitelboim’s idea of the bound 4-momentum of the electromagnetic field [17] may be generalised to (at least) the linearized gravitational field.

The splitting of the flux of 4-momentum across $r = \text{constant}$ surfaces defines the geometric optics approximation for the energy–momentum of the field. The relation to the geometric optics part of the field (curvature tensor) itself is clear; only terms which are involved in the construction of $\mathcal{L}_{ijkl}$ are involved in the construction of $\text{rad} t^{ij}$. Thus associated with $\mathcal{L}_{ijkl}$ is the total radiated 4-momentum, which detaches itself from the source and radiates away to $J^+$ at the speed of light. The energy-momentum which is left behind remains bound to the source.

V. ARBITRARY DEPENDENCE ON A NULL COORDINATE

In this section, results are obtained which can be used to determine the relation between the retarded multipole moments of the source and the multipole moments of the field. This will involve determining a decomposition of the field into arbitrary functions of $u$ (which is a null coordinate), and known functions of angle. As in the electromagnetic case, radiative linearized gravitational fields must contain arbitrary functions of a null coordinate in order to convey information [21]. The decomposition obtained will allow a rewriting of the solution $\gamma_{ij}$ in a form which allows direct comparison with other approaches to linearized gravity [15,13].

In order to keep the calculations to a reasonable length, attention is restricted to a source for which only the retarded pole, dipole and quadrupole moments are non–zero. Such a source will suffice to display generic behaviour. Thus

$$\mathcal{T}_{ij,k_1...k_n} = 0, \quad n \geq 3.$$  

The decomposition of the retarded pole, dipole and quadrupole moments is obtained by analysing the conservation equations (2.13).

Defining the tensors
\[ m_i = T_{ij}k^i, \quad m_{i;k_1...k_n} = k^i T_{ij;k_1...k_n}, \] 

(5.2)

the first few conservation laws may be rewritten as

\[ \dot{m}_i = 0, \] 

(5.3)

\[ \dot{m}_{ij} + T_{ij} + m_i v_j = 0, \] 

(5.4)

\[ \dot{m}_{ij;k} + T_{ij;k} + T_{ik;j} + m_{i;j} v_k + m_{i;k} v_j = 0, \] 

(5.5)

\[ \dot{m}_{ijkl} + T_{ij;kl} + T_{ik;jl} + T_{il;jk} + m_{i;j} v_l + m_{i;k} v_l + m_{i;i} v_k = 0. \] 

(5.6)

The first of these may be rewritten as

\[ T_{ij} = -\frac{1}{2}(\dot{m}_{ij} + \dot{m}_{ji}) - \frac{1}{2}(m_i v_j + m_j v_i). \] 

(5.7)

From (5.4), one can deduce that there exists a tensor \( Q_{ijk} \) such that

\[ T_{ij;k} = -m_{i;j} v_k - \frac{1}{2} \dot{m}_{ijk} - Q_{ijk}, \] 

(5.8)

where \( Q_{ijk} \) obeys

\[ Q_{ijk} = -Q_{ikj}, \] 

(5.9)

\[ Q_{ijk} k^k = -Q_{ikj} k^k = m_{i;j}. \] 

(5.10)

Similarly, using (5.4) along with (5.1), one finds that there exists a tensor \( Q_{ijkl} \) such that

\[ T_{ij;kl} = -\frac{1}{2}(v_k m_{j;il} + v_l m_{j;ik}) + Q_{ijkl}, \] 

(5.11)

and \( Q_{ijkl} \) obeys

\[ Q_{ijkl} = Q_{ijlk}, \quad Q_{ijkl} + Q_{iklj} + Q_{iljk} = 0, \] 

(5.12)

\[ Q_{ijkl} k^l = -\frac{1}{2} m_{i;jk}, \quad Q_{ijkl} k^j = m_{i;kl}, \] 

(5.13)

\[ Q_{ijkl} k^j k^k = Q_{ijkl} k^k k^l = Q_{ijkl} k^k k^l = 0. \] 

(5.14)

The procedure runs as follows: using the formulas for the derivatives of the retarded moments, \( \zeta \)-derivatives of \( Q_{ijk} \) and \( Q_{ijkl} \) are obtained. These give a pair of first order simultaneous differential equations, which are in an integrable form. The ‘constants of integration’ yield the required arbitrary functions of \( u \).

To begin, recall that
\[
\frac{\partial}{\partial \zeta} = r \frac{\partial k^i}{\partial \zeta} \frac{\partial}{\partial x^i}.
\]

Then using the derivatives (2.19) one finds

\[
\frac{\partial m_{i:j;k}}{\partial \zeta} = -(\tau_{ij:kl} + \tau_{ik:jl}) \frac{\partial k^l}{\partial \zeta}, \tag{5.15}
\]

and so

\[
\frac{\partial \dot{m}_{i:j;k}}{\partial \zeta} = -(D\tau_{ij:kl} + D\tau_{ik:jl}) \frac{\partial k^l}{\partial \zeta}. \tag{5.16}
\]

Using (2.19) and (2.19) one obtains respectively

\[
\frac{\partial \tau_{ij:k}}{\partial \zeta} = (v_k \tau_{ij:l} + D\tau_{ij:kl}) \frac{\partial k^l}{\partial \zeta}, \tag{5.17}
\]

and

\[
\frac{\partial m_{i:j}}{\partial \zeta} = -\tau_{ij:l} \frac{\partial k^l}{\partial \zeta}. \tag{5.18}
\]

Using these derivatives and the definition (5.8) of \(Q_{ijk}\) gives

\[
\frac{\partial Q_{ijk}}{\partial \zeta} = -\frac{1}{2} \frac{\partial}{\partial u} \left\{ (\tau_{ij:kl} - \tau_{ik:jl}) \frac{\partial k^l}{\partial \zeta} \right\}. \tag{5.19}
\]

Similarly, one can use (2.19c) and (2.19) in the definition (5.11) (remember \(v^i = v^i(u)\)) to obtain

\[
\frac{\partial Q_{jikl}}{\partial \zeta} = \frac{1}{2} \left\{ (\tau_{ji:lm} - \tau_{jl:im}) v_k + (\tau_{ji:km} - \tau_{jk:im}) v_l \right\} \frac{\partial k^m}{\partial \zeta}. \tag{5.20}
\]

Now from (5.19) one can deduce the existence of tensors \(R_{ijk}\) and \(\dot{Q}_{ijk}(u)\), both having the same symmetries as \(Q_{ijk}\), and \(\dot{Q}_{ijk}\) depending only on \(u\), such that

\[
Q_{ijk} = \dot{R}_{ijk} + \dot{Q}_{ijk}, \tag{5.21}
\]

and

\[
\frac{\partial R_{ijk}}{\partial \zeta} = -\frac{1}{2} (\tau_{ij:km} - \tau_{ik:jm}) \frac{\partial k^m}{\partial \zeta}. \tag{5.22}
\]

Comparing this with (5.20), one can write

\[
Q_{jikl} = -R_{jil} v_k - R_{ijk} v_l + \dot{Q}_{jikl}(u), \tag{5.23}
\]

where \(\dot{Q}_{jikl}\) has the same symmetries as \(Q_{jikl}\) and depends only on \(u\).
\[ m_{jik} = -2R_{jik} + 2v_k R_{jil} k^l - 2\hat{Q}_{jikl} k^l, \] 

(5.24)

which on using (5.23) and (5.11) gives

\[ \mathcal{T}_{ijkl} = -2R_{ijkm} k^m v_l + \hat{Q}_{ijkm} k^m v_l + \hat{Q}_{ijlm} k^m v_k + \hat{Q}_{ijkl}. \] 

(5.25)

Then

\[
(\mathcal{T}_{ijkl} - \mathcal{T}_{ikjl}) \frac{\partial k^l}{\partial \zeta} = (\hat{Q}_{ijkl} + \hat{Q}_{ijlm} k^m v_k - \hat{Q}_{ikjl} - \hat{Q}_{iklm} k^m v_j) \frac{\partial k^l}{\partial \zeta} \\
= \frac{\partial}{\partial \zeta} \left( \hat{Q}_{ijlm} (\delta^m_k + \frac{1}{2} k^m v_k) k^l - \hat{Q}_{iklm} (\delta^m_j + \frac{1}{2} k^m v_j) k^l \right). 
\]

(5.26)

Comparing this with (5.22), one can integrate and give

\[ R_{ijk} = -\frac{1}{2} \left\{ \hat{Q}_{ijlm} (\delta^m_k + \frac{1}{2} k^m v_k) k^l - \hat{Q}_{iklm} (\delta^m_j + \frac{1}{2} k^m v_j) k^l \right\} + \hat{Q}_{ijk}, \]

(5.27)

The constant of integration which arises here can be absorbed into \( \hat{Q}_{ijk} \) and \( \hat{Q}_{ijkl} \), and can thus be set equal to zero without changing \( Q_{ijk} \) or \( Q_{ijkl} \). From (5.21) and (5.23),

\[ Q_{ijk} = -\frac{1}{2} \frac{\partial}{\partial u} \left\{ \hat{Q}_{ijpm} (\delta^m_k + \frac{1}{2} k^m v_k) k^p - \hat{Q}_{jkpm} (\delta^m_i + \frac{1}{2} k^m v_i) k^p \right\} + \hat{Q}_{ijk}, \]

(5.28)

and

\[ Q_{ijkl} = \frac{1}{2} v_l \left\{ \hat{Q}_{ijpm} (\delta^m_k + \frac{1}{2} k^m v_k) k^p - \hat{Q}_{jkpm} (\delta^m_i + \frac{1}{2} k^m v_i) k^p \right\} + \hat{Q}_{ijkl}(u). \]

(5.29)

The tensors \( \hat{Q}_{ijk} \) and \( \hat{Q}_{ijkl} \) will be specified by the source via these last two equations and (5.8) and (5.11). The linear transformation \( \hat{Q} = \hat{Q}(Q) \) implicit in (5.23) is not invertible, that is \( Q_{ijkl} = 0 \) does not imply \( \hat{Q}_{ijkl} = 0 \), but one will certainly be able to determine the non–pure gauge parts of \( \hat{Q}_{ijkl} \) from this equation in terms of \( Q_{ijkl} \), and so in terms of the source. This will be seen more explicitly when the solution \( \gamma_{ij} \) is written in terms of these tensors.

To do this, the combinations of retarded moments which arise in the solution (2.32) are obtained in terms of \( m_i, \hat{Q}_{ijk} \) and \( \hat{Q}_{ijkl} \). For a pole–dipole–quadrupole source described by (5.1), this solution can be written exactly as

\[ \gamma^*_{ij} = \frac{2}{r^2} \mathcal{T}_{ij} - \frac{1}{r^2} \left( 2 \mathcal{T}_{ijk} v^k + D \mathcal{T}_{ijkl} k^l \right) + \frac{1}{r^3} \left( 2 \mathcal{T}_{ijkl} v^k v^l - \mathcal{T}_{ijk} k^l \right). \]

(5.30)

From (5.29) and (5.11), one has

\[ \mathcal{T}_{ijkl} = \hat{Q}_{ijpm} k^p k^m v_k v_l + \hat{Q}_{ijpk} k^p v_k + \hat{Q}_{ijlk} k^p v_l + \hat{Q}_{ijkl}, \]

(5.31)

so that
A direct calculation shows that

\[ T_{ij:k} = -\dot{Q}_{ijkl}k^k k^l + 2\dot{Q}_{ijkl}v^k k^l + \dot{Q}_{ijkl}^k, \quad (5.32) \]

and

\[ T_{ijkl} v^k v^l = \dot{Q}_{ijkl} k^k k^l - 2\dot{Q}_{ijlp} v^l v^p + \dot{Q}_{ijkl} v^k v^l. \quad (5.33) \]

Using (5.28) and (5.21), one can write

\[ T_{ij:k} = -2\dot{R}_{ijkl} v^k - a\cdot k R_{ijkl} - R_{ijkl} a_k + k^l D\dot{Q}_{ijkl} - \dot{Q}_{ijkl} v_k - \dot{Q}_{ijk}, \quad (5.34) \]

and so

\[ T_{ijkl} v^k = 2\dot{R}_{ijkl} + a\cdot k R_{ijkl} + (\dot{Q}_{ijkl} k^l) v^k + \dot{Q}_{ijkl} k^l - \dot{Q}_{ijk} v^k, \quad (5.35) \]

and from (5.11),

\[ m_{ijk} = \dot{R}_{ijkl} k^k + \dot{Q}_{ijkl} k^k. \quad (5.36) \]

Then a straightforward calculation gives

\[ T_{ij} = \frac{1}{2} \dot{Q}_{ijkl} k^k k^l + 3\tilde{a} k \dot{Q}_{ijkl} k^k k^l + \frac{1}{2}(b \cdot k + 3(\tilde{a} k)^2) \dot{Q}_{ijkl} k^k k^l \]

\[ -\dot{Q}_{ijkl} - a \cdot \dot{k} Q_{ijkl} - m_{ij}, \quad (5.37a) \]

\[ 2 T_{ij:k} v^k + DT_{ij:k} = \dot{Q}_{ijkl}(\eta^k l + 4 v^k k^l - 3k^k k^l) \]

\[ + \dot{Q}_{ijkl}(a \cdot k (\eta^k l + 6 v^k k^l - 6k^k k^l) + 2a^k k^l) \]

\[ + 2 \dot{Q}_{ijkl}(l - v^l), \quad (5.37b) \]

\[ 2 T_{ijkl} v^k v^l - T_{ijkl} = \dot{Q}_{ijkl}(-\eta^k l + 3k^k k^l - 6v^k k^l + 2v^k v^l). \quad (5.37c) \]

Thus from these expressions and (5.30), it is seen that one can express the solution in terms of arbitrary functions of \( u \), namely \( m_{ij} \) (which is in fact constant), \( \dot{Q}_{ijk}(u) \) and \( \dot{Q}_{ijkl}(u) \). One can go further than this, and show that these tensors act as potentials for the solution. A direct calculation shows that

\[ \left( \frac{\dot{Q}_{ijkl}}{r} \right)_j = -\frac{1}{r} (\dot{Q}_{ijkl} k^l + a \cdot k \dot{Q}_{ijkl} k^l) + \frac{1}{r^2} \dot{Q}_{ij} (v_l - k_l), \quad (5.38) \]

\[ \left( \frac{\dot{Q}_{ijkl}}{r} \right)_{kl} = \frac{1}{r} (\dot{Q}_{ijkl} + 3a \cdot k \dot{Q}_{ijkl} + (b \cdot k + 3(\tilde{a} k)^2) \dot{Q}_{ijkl}) k^k k^l \]

\[ + \frac{1}{r^2} \left( \dot{Q}_{ijkl} (-\eta^l - 4v^l k_l + 3k^k k^l) \right. \]

\[ + \dot{Q}_{ijkl} (-a \cdot k (\eta^l + 6v^l k_l - 6k^l k_l) - 2a^l k_l) \]

\[ \left. + \frac{1}{r^3} \left( \dot{Q}_{ijkl} (-\eta^l - 6v^l k_l + 3k^k k_l + 2v^l v_l) \right) \right). \quad (5.39) \]
Comparing these with (5.37) and (5.30) above, it has been shown that
\[
\gamma^{*ij} = -2 \frac{m^i v^j}{r} + 2 \left( \frac{\hat{Q}^{ijl}}{r} \right)_l + \left( \frac{\hat{Q}^{ijkl}}{r} \right)_{kl}.
\]
(5.40)

This form of the solution can be used to show that \(\hat{Q}^{ijk}\) and \(\hat{Q}^{ijkl}\) contain the same information as \(Q^{ijk}\) and \(Q^{ijkl}\). To see this, notice that
\[
\left( \frac{Q^{ijk}}{r} \right)_k = \left( \frac{\hat{Q}^{ijk}}{r} \right)_k + \left( \frac{\dot{R}^{ijk}}{r} \right)_k,
\]
(5.41)

and
\[
\left( \frac{Q^{ijkl}}{r} \right)_{kl} = \left( \frac{\hat{Q}^{ijkl}}{r} \right)_{kl} - 2 \left( \frac{\dot{R}^{ijk} v^l}{r} \right)_k = \left( \frac{\hat{Q}^{ijkl}}{r} \right)_{kl} - 2 \left( \frac{\dot{R}^{ijk} v^l}{r} \right)_k,
\]
(5.42)

where \((v^i/r)_i = 0\) and \(R^{ijk} = R^{ijk}(u, \zeta, \bar{\zeta})\) have been used. Thus
\[
2 \left( \frac{Q^{ijk}}{r} \right)_k + \left( \frac{Q^{ijkl}}{r} \right)_{kl} = 2 \left( \frac{\hat{Q}^{ijk}}{r} \right)_k + \left( \frac{\hat{Q}^{ijkl}}{r} \right)_{kl},
\]
(5.43)

and so (5.41) may equally be written without carats on the tensors \(\hat{Q}^{ijk}\) and \(\hat{Q}^{ijkl}\). This validates the comments after Eq. (5.29); the kernel of the transformation only yields terms which do not contribute to \(\gamma^{*ij}\), i.e. pure gauge terms.

Using the tensors \(\hat{Q}^{ijk}\) and \(\hat{Q}^{ijkl}\), one could calculate what Janis and Newman [6] call the multipole moments of the field (the coefficients of inverse powers of \(r\) of the NP components of the curvature tensor). These would be given in terms of \(v^i\), the \(\hat{Q}\) tensors and known functions of angle (from the null tetrad of \(\S 2\)). This lengthy calculation would result in combinations of our arbitrary functions of \(u\) and such spin–weighted spherical harmonics as are predicted by the general theory [10,11].

The solution (5.40) is equivalent to those given by Sachs and Bergmann [13] and Pirani [13] for ‘multipole particles’, but has the advantage of being derived from a realistic extended source rather than one confined to a time–like world–line. These authors make use of symmetric trace–free tensors which are functions of the retarded time, and their solutions contain infinite series of such tensors, corresponding to a particle possessing an arbitrary number of multipole moments. It is clear then that for a general source for which arbitrarily many of the retarded multipole moments (2.1) are non-zero, the solution (5.40) would contain an infinite series of tensors \(\hat{Q}^{k_1 \ldots k_n}(u)\) which are arbitrary functions of \(u\). Each of these would be present at the \(r^{-1}\) level in \(\gamma^{ij}\), and would lead to an infinite series of spin–weighted spherical harmonics appearing in the fully decomposed expression for \(T^{ij}\). As seen in the previous section, in some situations, and especially those involving a discussion of the radiation field, it is advantageous to use the retarded moments (2.1).
VI. CONCLUSIONS

The similarity between the electromagnetic and linearized gravitational fields has been exploited in this paper to derive some results about the latter using the Goldberg–Kerr approach to bound source fields. The results reflect those obtained by Hogan and Ellis [4] for the electromagnetic field. Well established results (e.g. peeling-off) have been confirmed, relying on the use of the retarded multipole moments of the source. The advantage of using these can be seen in, for example, Eq.(4.37), where the total radiated 4-momentum is determined by a single integral over the source, the retarded monopole moment. The results of §5 show how this relates to analogous formulae of [19], which involve infinite series of moments of the source. In addition, use of the retarded moments of the source to describe the exterior field has made the consequences of the conservation of the source easy to identify.

It comes as no surprise that a bound source linearized gravitational field possesses a bound 4-momentum. The obvious question is: does this notion extend to the full non-linear gravitational field in the case of asymptotic flatness? The Landau-Lifschitz pseudotensor can be used to construct the 4-momentum for such fields (and the calculations would be carried out in an asymptotically Minkowskian coordinate system, c.f. §20.2 of [9]), and indeed this 4-momentum coincides with the covariant (Bondi–Sachs) constructions [12]. Hence one might expect that some or all of such fields do possess a bound 4-momentum, which could be constructed in a manner similar to that above.

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APPENDIX A: SECOND DERIVATIVES OF THE METRIC

In order to obtain the second derivatives of the metric perturbation $\gamma_{ij}$ correct to $O(r^{-4})$, the solution (2.32) is written as

$$\gamma^*_{ij} = \frac{A_{ij}}{r} + \frac{B_{ij}}{r^2} + C_{ij} + O(r^{-4}), \quad (A1)$$

where the coefficients may be read off from (2.32). For convenience, the decomposition

$$B_{ij} = \alpha_{ij} + \beta_{ij} + \mu_{ij}, \quad (A2)$$

will be used, where

$$\begin{align*}
\alpha_{ij} &= -4T_{ij;m}v^m, \\
\beta_{ij} &= -2\tilde{T}_{ij;m}, \\
\mu_{ij} &= -2a\cdot k\tilde{T}_{ij;m}.
\end{align*} \quad (A3, A4, A5)$$

Then all the information required to construct the field (to $O(r^{-4})$) is contained in the second derivatives of the terms $A_{ij}$, $B_{ij}$, $\alpha_{ij}$, $\beta_{ij}$ and $\mu_{ij}$. These derivatives are obtained using results given in §§3-7. It is found that

$$\left(\frac{A_{ij}}{r}\right)_{,kl} = \frac{4}{r^3} \left\{ (D + a\cdot k)(D + a\cdot k) \right\} T_{ij;m}k_kk_l$$

$$-\frac{4}{r^3} \left\{ 2(D + a\cdot k)D T_{ij;m}(\delta^m_{(k} + v^m_{k(l)}k_l) + DT_{ij;m}(a^m + a\cdot kv^m)k_lk_k \right\}$$

$$+ DT_{ij}(\eta_{kl} + 4v_{(k}k_l) - 3kk_l) + a\cdot kT_{ij}(2v_{(k}k_l) - 3kk_l) + 2T_{ij}(a_{(k}k_l)) \right\}$$

$$+ \frac{4}{r^3} \left\{ T_{ij}(-\eta_{kl} + 2v_{k}v_l - 6v_{(k}k_l) + 3kk_l) + v^m DT_{ij;m}(\eta_{kl} + 2v_{(k}k_l) - k_lk_k) + 2T_{ij;m}a_{(k}(\delta^m_{l)} + v^m_{l)}k_l) + 2(2D + a\cdot k)T_{ij;m}(\delta^m_{(l} + v^m_{k(l)})(v_k) - k_l)) \right\}$$

$$+(D + a\cdot k)D T_{ij;mp}(\delta^m_{k} + v^m_{k}k_l)(\delta^p_{l} + v^p_{k}k_l) \right\}, \quad (A6)$$

$$\left(\frac{\alpha_{ij}}{r^2}\right)_{,kl} = -\frac{4}{r^3} \left\{ (D + 2a\cdot k)(D + a\cdot k)(v^m T_{ij;m}) \right\} k_kk_l$$

$$+ \frac{4}{r^3} \left\{ (v^m T_{ij;m} + a^m T_{ij;m})(\eta_{kl} + 3v_{k}k_l + 3v_{l}k_k - 5k_kk_l) \right\}$$

$$+ 2v^m T_{ij;m}(a_{k}k_l + a_{l}k_k + a\cdot k(\eta_{kl} + 4v_{k}k_l + 4v_{l}k_k - 8k_kk_l))$$

$$+ (- (D + 2a\cdot k)T_{ij;p} + v^m(D + 2a\cdot k)D T_{ij;mp} + a^m T_{ij;mp}) \times$$

$$((\delta^l_{p} + v^l_{p}k_l)k_l + (\delta^p_{l} + v^p_{k}k_l)k_l)$$

$$+ (a^p + a\cdot kv^p)(-T_{ij;p} + DT_{ij;mp})k_kk_l \right\} + O(r^{-4}). \quad (A7)$$

$$\left(\frac{\beta_{ij}}{r^2}\right)_{,kl} = -\frac{2}{r^3}(D + 2a\cdot k)(D + a\cdot k)\tilde{T}_{ij;m}k_kk_l$$

$$+ \frac{2}{r^3} \left\{ \tilde{T}_{ij;m}^m(\eta_{kl} + 3v_{k}k_l + 3v_{l}k_k - 5k_kk_l) \right\}$$
\[
+ 2 \dot{T}_{ij:m}^m (a_k k_l + a_l k_k + a \cdot k (\eta_{kl} + 4v_k k_l + 4v_l k_k - 8k_k k_l))
+ (2(D + 2a \cdot k)D(v^m T_{ij:mp}) + (D + 2a \cdot k)D^2 T_{ij:pm}^m) \times
(\delta_{i}^{p} + v^p k_i)k_k + (\delta_{k}^{p} + v^p k_k)k_l
+ (a^p + a \cdot k v^p)(2v^m D T_{ij:mp} + 2a^m T_{ij:mp} + D^2 T_{ij:pm}^m)k_k k_l \}
+ O(r^{-4}) \quad (A8)
\]

\[
\left( \frac{\mu_{ij}}{r^2} \right)_{,kl} = -\frac{2}{r^2} (D + 2a \cdot k)(D + a \cdot k)(a \cdot k T_{ij:m}^m)k_k k_l
+ \frac{2}{r^3} \{ 2(b \cdot k + 3a \cdot k^2)(2v^m T_{ij:mp} + DT_{ij:pm}^m)(\delta_{i}^{p} + v^p k_i)k_k
+ 2a \cdot k(2v^m D T_{ij:mp} + 2a^m T_{ij:mp} + D^2 T_{ij:pm}^m)(\delta_{i}^{p} + v^p k_i)k_k
+ a \cdot k(2v^m T_{ij:mp} + DT_{ij:pm}^m)(a^p + a \cdot k v^p)k_k k_l \} + O(r^{-4}) \quad (A9)
\]

\[
\left( \frac{C_{ij}}{r^3} \right)_{,kl} = \frac{1}{r^3} (D + 3a \cdot k)(D + 2a \cdot k)C_{ij} k_k k_l + O(r^{-4}) \quad (A10)
\]
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