Radiative magneto-micropolar fluid flow over a stretching/shrinking sheet with slip flow model

Muhammad Kamran and Benchawan Wiwatanapataphee
Department of Mathematics and Statistics, Curtin University of Technology, GPO Box U1987, Perth, Australia
E-mail: muhammad.kamran@curtin.edu.au

Abstract. This paper presents a study of mixed convective magneto-micropolar fluid flow over a porous stretching/shrinking surface under slip boundary condition in the presence of the thermal radiation effect. The impact of the Newtonian heating is assumed in the thermal boundary condition. The proposed governing flow model is transformed and solved by a semi-analytical technique named Homotopy Analysis Method, and the obtained solutions have excellent agreements with the analytical and the numerical results under special cases. The obtained results reveal that when the sheet stretches at a higher magnetic field parameter, the velocity boundary layer thickness becomes shorter with an increase in the thermal radiation parameter as compared to the lower value of the magnetic field parameter. On the other hand, a higher value of the thermal radiation parameter causes to produce a wider thermal boundary layer thickness as the value of the magnetic field parameter enhances. However, at a lower value of the thermal radiation parameter yields a significant change in the temperature of the micropolar fluid flow. MHD radiative micropolar fluid flow may have an important consideration in magnetic resonance imaging (MRI) and in the circulatory system to control the blood flow by considering the slip flow regime.

1. Introduction
The theory of microfluid [1] and the micropolar fluid [2] adequately explain the microscopic characteristics of polymeric fluids, fluids with additives or the animal blood [3] due to their randomly oriented rigid structure and the micro-motion of microelements. Later on, this theory extended to heat conduction and heat dissipation effects [4], some basic fluid flows [5] through concentric cylinders [6, 7] and a variety of theoretical and experimental investigations [8, 9, 10, 11, 12, 13].

The study of magneto-micropolar fluid flow has great attention in heat transfer phenomena due to its bio-engineering and industrial applications such as the analysis of the electrically conducting blood flow [14, 15, 16] in the cardiovascular system and the crystallization [8] of fluids with certain additives. In literature, there are many investigations [17, 18, 19, 20, 21, 22, 23] have been investigated and reported on the magneto-micropolar fluid flow over different geometries and with varying behaviours of flow. But the MHD micropolar fluid flow past a stretching/shrinking surface with Wu’s [24] model for slip flow regime is relatively quite new to consider. With this reported slip flow regime, microelements of the micropolar fluid flow still obey the Navier–Stokes equations [25], in which the microelements of the micropolar fluid flow slip along the surface due to their finite tangential velocity. In recent years, Wu’s [24] slip flow
model under different flow configurations of the micropolar fluid flow [26, 27, 28, 29, 30] has been studied and reported in the literature.

However, in this paper, we extend the work of Ibrahim [30] to the radiative magnetomicro- 
polar fluid flow over a permeable stretching/shrinking sheet under slip flow regime modelled 
by Wu [24] in the presence of the Newtonian heating [31, 32]. A dilute concentration of the 

microelements in the fluid flow is considered in this study. The governing flow behaviour is 
transformed into a coupled system of nonlinear ordinary differential equations along with the 
mixed derivative boundary conditions and then solved by a semi-analytical technique, named 
Homotopy Analysis Method. Up to best of our knowledge, the current results has not been 
reported in the literature before.

2. Problem formulation

It is assumed that a steady state, two dimensional mixed convective, laminar, incompressible 

micropolar fluid flow over a vertical permeable stretching/shrinking sheet with velocity $u = 
u u_w(x)$, where $u_w(x) = a x$, along the $x$–axis direction and $a > 0$ is a dimensional constant. A 
transverse uniform magnetic field is applied along the $y$–axis, as shown in figure 1. Velocity 
boundary layer equation is formulated by a second order slip flow model [24], and thermal 
boundary layer equation represents the Newtonian heating [31] effect in it. Under usual 
Boussinesq approximations, the governing system of equations [26, 30] can be formulated as

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \mu + \chi \frac{(\partial^2 u)}{(\partial y^2)} + g \beta_T (T - T_\infty) + \frac{\chi}{\rho} \left( \frac{\partial \omega}{\partial y} \right) - \frac{\sigma B_0^2}{\rho} u, \]  
\[ u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \frac{\gamma}{\rho j} \left( \frac{\partial^2 \omega}{\partial y^2} \right) - \frac{\chi}{\rho j} \left( 2 \omega + \frac{\partial u}{\partial y} \right), \]  
\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \left( \frac{\partial^2 T}{\partial y^2} \right) - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y}. \]
with boundary conditions,

\[
\begin{align*}
  u &= s u_w(x) + u_{\text{slip}}, \quad v = v_0, \quad \omega = -n \frac{\partial u}{\partial y}, \quad k \frac{\partial T}{\partial y} = -h_s T \text{ at } y = 0, \\
  u &\to 0, \quad \omega \to 0, \quad T \to T_\infty \text{ as } y \to \infty,
\end{align*}
\]

(5)

where \( \mu \) is the dynamic viscosity, \( \gamma \) is the spin gradient viscosity, \( \rho \) is the fluid density, \( T \) is the fluid temperature, \( T_\infty \) is the temperature of the ambient fluid, \( \omega \) is the angular velocity (microrotation vector), \( k \) is the thermal conductivity of the sheet, \( g \) is gravitational acceleration, \( \chi \) is the vortex viscosity (microrotation viscosity), \( \beta_T \) is the coefficient of thermal expansion, \( c_p \) is the specific heat at constant pressure, \( B_o \) is the magnetic field parameter, \( \sigma \) is the electrical conductivity, \( q_r \) is the radiative heat flux, and \( h_s \) is the surface heat transfer coefficient, respectively.

Equation (3) defined the \( \gamma = (\mu + \chi/2) j = \mu (1 + K/2) j \) for the material parameter \( K = \chi/\mu \) and the microinertia density \( j = \nu/a \) [9]. Moreover, the details of equation (5) are reported in [27]. Furthermore, the slip velocity \( u_{\text{slip}} \) of the stretching \((s > 0)/\text{shrinking} \((s < 0)\) sheet is given by Wu [24] as

\[
  u_{\text{slip}} = \frac{2}{3} \left( \frac{3 - \epsilon l^3}{\epsilon} - \frac{3}{2} \frac{1 - l^2}{k_n} \right) \frac{\partial u}{\partial y} - \frac{1}{4} \left( t^4 + \frac{2}{k_n} (1 - l^2) \right) \frac{\partial^2 u}{\partial y^2} = A \frac{\partial u}{\partial y} + B \frac{\partial^2 u}{\partial y^2},
\]

(6)

where \( l \) is the chosen as \( \min(1/k_n, 1) \) for any value of the Knudsen \((k_n)\) number and \( \epsilon \) is the momentum accommodation coefficient whose value between 0 to 1. The molecular mean free path \( d \) is always positive and \( B \) is consequently negative in magnitude. Radiative heat flux has a simplified form with the help of Rosseland approximation [33] as follows:

\[
  q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y},
\]

(7)

where \( \sigma^* \) depicts the Stefan-Boltzmann constant and \( k^* \) represents the mean absorption coefficient. In radiative heat flux, to express the \( T^4 \) as a linear function of temperature, an expansion by Taylor’s series is considered about \( T_\infty \) and neglecting higher order terms which yields

\[
  T^4 \approx 4T^3_\infty T - 3T^4_\infty.
\]

(8)

The following similarity variables with velocity components \( u = ax f'(\eta) \) and \( v = -(a\nu)^{1/2} f(\eta) \) along with

\[
  \eta = \sqrt{a/\nu y}, \quad \omega = \sqrt{a/\nu axh(\eta)} \quad \text{and} \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty},
\]

(9)

the system of equations (1)–(5) reduce to the following nonlinear set of equations:

\[
\begin{align*}
  (1 + K) f''' - (f')^2 + f f'' + Kh' + \lambda \theta - M f' &= 0, \\
  (1 + K/2) h'' - f'h + f h' - K(2h + f'') &= 0, \\
  (1 + R) \theta'' + Pr f \theta' &= 0,
\end{align*}
\]

(10)–(12)

with the boundary conditions

\[
\begin{align*}
  f &= f_w, \quad f' = s + \alpha f'' + \beta f''' , \quad h = -n f'', \quad \theta' = -\delta(1 + \theta) \text{ at } \eta = 0, \\
  f' &\to 0, \quad h \to 0, \quad \theta \to 0 \text{ as } \eta \to \infty.
\end{align*}
\]

(13)
where (‘) denotes the derivative with respect to the similarity variable (η). In the system of equations (10)–(12), the non-dimensional physical parameters are defined as buoyancy parameter $\lambda = Gr_x/Re_x$ for local Grashof number $Gr_x = g\beta T_\infty x/\nu$, magnetic field parameter $M = \sigma B^2_0/\nu$, thermal radiation parameter $R = (16\sigma T^3_\infty k^2)^{-1}$ and Prandtl number $Pr = \mu c_p/k$. In equation (13), the suction/injection parameter is $f_w = -(av)^{-\frac{1}{2}}v_0$, the first and second order slip flow parameters are $\alpha = A \sqrt{a/\nu} > 0$ and $\beta = Ba/\nu < 0$, and the Newtonian heating parameter is $\delta = h_0 k^{-1}(\nu/a)^{1/2}$.

3. Homotopy analysis method

Liao [34] introduced a semi-analytical method named Homotopy Analysis Method (HAM) which is quite useful to solve the system of coupled linear and nonlinear ordinary differential equations even with mixed derivative boundary conditions. Using the standard basic features of the HAM for equations (10)–(13) with the set of base functions $\{\eta^i e^{-\eta}|r \geq 0, i \geq 0\}$ and the initial set of solutions is in the following form:

$$
\begin{align*}
 f_0(\eta) &= f_w + \frac{s(1 - e^{-\eta})}{1 + \alpha - \beta}, \\
 h_0(\eta) &= \frac{sne^{-\eta}}{1 + \alpha - \beta}, \\
 \theta_0(\eta) &= \frac{\delta e^{-\eta}}{1 - \delta}; \delta \neq 1,
\end{align*}
$$

with the linear operators $L_f = \frac{d^2}{d\eta^2} - \frac{d}{d\eta}$ for the velocity equation, $L_h = \frac{d^2}{d\eta^2} + \frac{d}{d\eta}$ for the microrotational velocity equation and $L_\theta = \frac{d^2}{d\eta^2} + \frac{d}{d\eta}$ for the temperature equation satisfying the properties $L_f[F_1 + F_2 e^\eta + F_3 e^{-\eta}] = 0$, $L_h[F_4 + F_5 e^{-\eta}] = 0$ and $L_\theta[F_6 + F_7 e^{-\eta}] = 0$, respectively for arbitrary constants $F_1, F_2, F_3, \ldots$ and $F_7$. Moreover, auxiliary functions corresponding to velocity, microrotational velocity and temperature equations are defined as $H_f(\eta) = H_h(\eta) = H_\theta(\eta) = 1$ and corresponding auxiliary parameters (convergence control parameters) are set to be $h_f = h_h = h_\theta = h$ during the entire computation.

3.1. Convergence control region

In HAM, a non-zero auxiliary parameter ($h$) has a great influence on the convergence of the series solution. Its value is selected from the horizontal line segment of the so-called $h$–curves as shown in figure 2. At $20^{th}$–order of approximation of the HAM, the admissible range for the auxiliary parameter is defined as $-0.3 \leq h_f < 0$, $-0.3 \leq h_h < 0$, $-0.28 \leq h_\theta < 0$ at $K = \lambda = f_w = n = \delta = 0.5$, $\alpha = M = R = 1$, $\beta = -1$, $Pr = 5$. Here, in this sketch it is worthy to mention that $\eta = 0$ is used for $s = 1$ and $\eta = 10$ for $s = -1$. Moreover, the validation and the analysis of the current results is carried out within these ranges.
4. Validation of the results
The current HAM results are validated by comparing with exact and numerical results. Firstly, the obtained results are equated with the exact solutions reported by Fang et al. [25] in a special case. In this comparison, forced convective boundary layer flow over a porous sheet \((s = -1)\) is taken into account with slip model at \(K = M = n = R = 0\). There is an excellent agreement between the Fang et al. [25] and the current results for \(f_w = 2\) and \(\alpha = 1\) as shown in table 1.

Table 1. Result validation for shrinking sheet.

| \(\beta\) | \(f''(0)\)     | \(f''(0)\)     | \(\bar{h}\)   |
|----------|----------------|----------------|--------------|
| -0.01    | 0.63606982     | 0.63606984     | -0.171916    |
| -0.1     | 0.5780061      | 0.57800608     | -0.141216    |
| -0.5     | 0.4043172      | 0.40431709     | -0.2579      |
| -1.0     | 0.2905478      | 0.29054709     | -0.20675     |
| -2.0     | 0.1846568      | 0.18465658     | -0.16638     |

Secondly, we focus on the flow problem over an impermeable stretching sheet \((s = 1)\) taking into account the effect of forced convective magneto-micropolar fluid flow in the absence of thermal radiation effect using \(\alpha = M = 1, \beta = -1\) and \(n = 0.5\). To investigate the impact of the concentration of microelements \((K)\), two values of \(K\) are chosen to be 0 and 2 along with the \(\bar{h} = -0.015\) and \(-0.1\), respectively. The current results (solid lines) in figure 3 are compared with those (dotted lines) obtained by Ibrahim [30].

5. Results and discussion
The current HAM results are presented in graphical illustrations for the fixed values of the physical parameters, i.e. \(Pr = 5, K = \lambda = n = \delta = f_w = 0.5, M = R = \alpha = 1\) and \(\beta = -1\) for both stretching and shrinking sheets or otherwise reported in the illustration. Influence of the concentration of the microelements is displayed in figure 4 and figure 5 for \(\bar{h} = -0.1\) and \(-0.01\), respectively. It can be seen that an increase in the concentration of the microelements raises the velocity component \((f')\) and the microrotational velocity \((\bar{h})\) for stretching sheet \((s = 1)\) and decreases the prescribed velocities for shrinking sheet \((s = -1)\). In figure 4, from the
increasing values of the $K$, it can also be noted that the boundary layer thickness increases 78% for stretching sheet and decreases 46% for shrinking sheet at $\eta = 0$.

![Figure 4. Effect of $K$ on $f'$.](image1)

![Figure 5. Effect of $K$ on $h$.](image2)

Effect of Prandtl number ($Pr$) with the increasing similarity variable $\eta$ on the temperature distribution is investigated by using the $h$ values from $\{-0.01, -0.01, -0.01, -0.001, -0.001, -0.001, -0.0006, -0.0004, -0.0001\}$ and $\{-0.009, -0.007, -0.005, -0.005, -0.001, -0.001, -0.0006, -0.0004, -0.0001\}$ for $s = 1$ and $s = -1$, respectively. The results as shown in figure 6 indicate that an increase in the $Pr$ values reduces the micropolar fluid temperature and the thickness of the thermal boundary layer. It is also noted that thermal boundary thickness of the stretching sheet decreases faster than that of the shrinking sheet.

Figures 7–10 show the impacts of the magnetic field parameter ($M$) on the velocity component and the thermal boundary layer thickness for two values of the thermal radiation parameter $R$, i.e. $R = 1$ and 7. It is observed that an increase in the magnetic field parameter yields a decrease in the velocity component at the lower value of the thermal radiation parameter for $s = 1$. At the same time when sheet shrinks, the increasing value of the magnetic field parameter produces a higher velocity component of the fluid flow at $R = 7$ in comparison to the velocity component at $R = 1$. When the sheet stretches, in figure 9, the temperature of the micropolar fluid flow dramatically increases with the higher values of the magnetic field parameter at $R = 7$. This is maybe because the Lorentz force is reinforced by radiative heat flux which increases the thermal boundary thickness. However, figure 10 shows a reverse behavior for shrinking of the sheet.

![Figure 6. Effect of $Pr$ on $\theta$.](image3)

![Figure 7. Effect of $M$ on $f'$.](image4)

Increasing value of the thermal radiation parameter produces a higher impact on the velocity component and reduces the thermal boundary layer thickness at the lower value of the magnetic field parameter for stretching sheet as shown in figure 11 and figure 13. In a model with higher
value of the magnetic field parameter \((M = 7)\), an increase in the thermal radiation parameter produces a higher velocity component as compared to that obtained from a lower value of magnetic field parameter \((M = 1)\) as depicted in the figure 12. On the other hand, a reverse behaviour is noted for the temperature profile at \(s = -1\) as illustrated in figure 14. Figure 15 is sketched for various values of the suction and injection parameters at \(h = -0.01\) and \(h = -0.008\) for \(s = 1\) and \(s = -1\), respectively. It can be seen that increasing values of the suction parameter reduces the velocity component. Higher the value of the injection parameter gives higher value of the velocity component for both stretching and shrinking cases. When sheet stretches, a raise in the first and second order slip flow parameters yields a decrease in the \(f'\). However, a reverse behaviour is noted for shrinking of the sheet as shown in figure 16 and figure 17.
Furthermore, stretching sheet at the higher value of the magnetic field parameter shows that an increase in the Newtonian heating parameter raises the micropolar fluid temperature. However, when sheet shrinks at $M = 1$, higher values of the Newtonian heating parameter enhances the temperature profile as illustrated in figure 18 and figure 19.
6. Conclusion
In this investigation, we investigated a mixed convective incompressible magneto-micropolar fluid flow over a permeable stretching/shrinking sheet under the influence of the thermal radiation parameter. In addition, the Newtonian heating effect was considered with the slip flow impact. Transformed nonlinear fluid flow model revealed that when sheet stretched; higher the magnetic field parameter, lower the velocity and thermal boundary layer thicknesses at $R = 1$. On the other hand, velocity component and the thermal boundary layer thickness increased at higher value of the thermal radiation parameter for $M = 1$. Furthermore, when sheet stretched, the temperature of the micropolar fluid flow raised significantly with the increased value of the magnetic field parameter at $R = 7$ and when the sheet shrunk, the temperature of the fluid reduced, but it was noticeably higher than that at $R = 1$. It would be a futuristic approach to analysis this reported flow phenomenon over and inside of the permeable or impermeable cylinder.

6.1. Acknowledgments
The first author acknowledges financial support funded by, Spk:319299-Ver 1, 6353 Curtin University, Bentley 6102 Western Australia.

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