Pseudohermitian Hamiltonians, time-reversal invariance and Kramers degeneracy

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Abstract

A necessary and sufficient condition in order that a (diagonalizable) pseudohermitian operator admits an antilinear symmetry $\mathcal{T}$ such that $\mathcal{T}^2 = -1$ is proven. This result can be used as a quick test on the $T$-invariance properties of pseudohermitian Hamiltonians, and such test is indeed applied, as an example, to the Mashhoon-Papini Hamiltonian.

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1 Introduction

Non Hermitian Hamiltonians are usually taken into account in order to describe dissipative systems or decay processes. In particular, in the last few years, a great attention has been devoted to the study of $\mathcal{PT}$-symmetric quantum systems [1], whose Hamiltonians (though non Hermitian) possess real spectra, and in this context the interest rose on the class of pseudohermitian operators [2], i.e., those operators which satisfy

$$\eta H \eta^{-1} = H^\dagger$$

(1)

with $\eta = \eta^\dagger$ (of course, Hermiticity constitutes a particular case of pseudohermiticity, corresponding to $\eta = 1$).

When one considers diagonalizable operators with a discrete spectrum, one can prove that $H$ is pseudohermitian if and only if its eigenvalues are either real or come in complex-conjugate pairs (with the same multiplicity) [3]; furthermore, this result has been generalized to all the (possibly non diagonalizable) matrix Hamiltonians [4], and to the class of all the $\mathcal{PT}$-symmetric standard Hamiltonians having $\mathbb{R}$ as their configuration space [5] (which suggests that it may be valid under more general conditions).

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Another physical reason for studying pseudohermitian operators is the remark that any $T$-invariant (diagonalizable) Hamiltonian must belong to their class. The converse does not hold in general. Indeed, whereas one can prove that to any pseudohermitian operator is associated an antilinear symmetry, in general one cannot interpret it as the time-reversal operator $T$; furthermore, in case of fermionic systems, it is well known that

$$T^2 = -1,$$

and the above-mentioned theorems do not ensure the existence of such a symmetry.

In order to deepen this point, we will prove in Sect. 2 that a Kramers-like degeneracy is a necessary and sufficient condition so as a diagonalizable pseudohermitian operator admits an antilinear symmetry which satisfies condition (2).

Next, as an example, we will apply in Sect. 3 the above result to the study of a non Hermitian Hamiltonian which has been recently proposed to interpret (by a $T$-violating spin-rotation coupling) a discrepancy between experimental and theoretical values of the muon’s $g - 2$ factor, and we will able to state precisely the parameters values associated with the $T$-violation.

## 2 A theorem on pseudohermitian operators

As in [3][6][7], we consider here only diagonalizable operators $H$ with a discrete spectrum. Then, a complete biorthonormal eigenbasis $\{ |\psi_{n,a}\rangle, |\phi_{n,a}\rangle \}$ exists, i.e., a basis such that

$$H |\psi_{n,a}\rangle = E_n |\psi_{n,a}\rangle, \quad H^\dagger |\phi_{n,a}\rangle = E_n^* |\phi_{n,a}\rangle,$$

$$\langle \phi_{m,b} |\psi_{n,a}\rangle = \delta_{mn}\delta_{ab},$$

$$\sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle \langle \phi_{n,a}| = \sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle \langle \psi_{n,a}| = 1,$$

where $a,b$ are degeneracy labels and $d_n$ denotes the degeneracy of $E_n$; hence, the operator $H$ can be written in the form

$$H = \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle E_n \langle \phi_{n,a}|.$$
Proof. Let us assume that condition \( i \) holds; then, \( H \) is pseudohermitian (see [3], Prop. 3 and Prop. 1), hence its eigenvalues are either real or come in complex-conjugate pairs (with the same multiplicity). We will use in the following the subscript '0' to denote real eigenvalues and the corresponding eigenvectors, and the subscript '±' to denote the complex eigenvalues with positive or negative imaginary part, respectively, and the corresponding eigenvectors.

Let now \( |\psi_{n_0,a}\rangle \) be an eigenvector of \( H \); then, \( \mathfrak{T} |\psi_{n_0,a}\rangle \) too is an eigenvector of \( H \), corresponding to the same eigenvalue \( E_{n_0} \), and linearly independent from \( |\psi_{n_0,a}\rangle \). (Indeed, would be \( \mathfrak{T} |\psi_{n_0,a}\rangle = \alpha |\psi_{n_0,a}\rangle \) for some \( \alpha \in \mathbb{C} \), applying again \( \mathfrak{T} \) to the previous relation we would obtain \( |\psi_{n_0,a}\rangle = -|\alpha|^2 |\psi_{n_0,a}\rangle \), which is absurd.)

If \( |\psi_{n_0,b}\rangle \) is another eigenvector of \( H \), linearly independent from \( |\psi_{n_0,a}\rangle \) and \( \mathfrak{T} |\psi_{n_0,a}\rangle \), also \( \mathfrak{T} |\psi_{n_0,b}\rangle \) is linearly independent from all three, otherwise, applying once again \( \mathfrak{T} \) to the relation

\[
\alpha |\psi_{n_0,a}\rangle + \beta \mathfrak{T} |\psi_{n_0,a}\rangle + \gamma |\psi_{n_0,b}\rangle + \delta \mathfrak{T} |\psi_{n_0,b}\rangle = 0
\]

we could eliminate, for instance, \( \mathfrak{T} |\psi_{n_0,b}\rangle \) obtaining so a linear dependence between \( |\psi_{n_0,a}\rangle \), \( \mathfrak{T} |\psi_{n_0,a}\rangle \) and \( |\psi_{n_0,b}\rangle \), contrary to the previous hypothesis.

We can conclude, iterating this procedure, that \( d_{n_0} \) must be necessarily even.

Conversely, let condition \( ii \) hold, and let \( \mathfrak{T} \) denote the following antilinear operator:

\[
\mathfrak{T} = \sum_{n_0} \sum_{a=1}^{d_{n_0}/2} \left( |\psi_{n_0,a}\rangle K \langle \phi_{n_0,a+d_{n_0}/2} | - |\psi_{n_0,a+d_{n_0}/2}\rangle K \langle \phi_{n_0,a} | \right),
\]

(7)

where the operator \( K \) acts transforming each complex number on the right into its complex-conjugate. Then, one immediately obtains, by inspection, that \( [H, \mathfrak{T}] = 0 \) and \( \mathfrak{T}^2 = -\mathbb{1} \).

The implication \( i \Rightarrow ii \) we proved above generalizes from various point of view the celebrated Kramers theorem on the degeneracy of any fermionic (Hermitian) Hamiltonian. Indeed, it applies to a larger class than that of the Hermitian operators (concerning their real eigenvalues only): moreover, it does not require a physical interpretation of the antilinear operator \( \mathfrak{T} \) as a time-reversal operator. However, by an abuse of language, we will continue to denote as "Kramers degeneracy" this feature of pseudohermitian operators admitting a symmetry like \( \mathfrak{T} \).

We stress once more that the Kramers degeneracy is a necessary but not a sufficient condition for the \( T \)-invariance.
3 Time-reversal violation in the spin-rotation coupling

On the basis of the previous discussions, we can quickly test the $T$-invariance properties of pseudohermitian Hamiltonians. To illustrate this point with an example, we chose a pseudohermitian Hamiltonian which has been recently introduced to interpret a discrepancy between experimental and standard model values of the muon’s anomalous $g$ factor.

In this model, a spin-rotation coupling, which involves small violations of the conservation of $P$ and $T$, is considered. In particular, the spin-rotation effect described by Mashhoon \[10\] attributes an energy $-\hbar^2 \omega^2 \rightarrow -\sigma \omega$ to a spin-$\frac{1}{2}$ particle in a frame rotating with angular velocity $\omega$ relative to an inertial frame. In the modified Mashhoon model \[8\] one assumes a different coupling of rotation to the right and left helicity states of the muon, $|\psi_+\rangle$ and $|\psi_-\rangle$. Hence, the total effective Hamiltonian is

$$H_{\text{eff}} = \begin{pmatrix} E & i(k_1 \omega^2 - \mu B) \\ -i(k_2 \omega^2 - \mu B) & E \end{pmatrix}, \quad (8)$$

where $\mu$ represents the total magnetic moment of the muon, $B$ is the magnetic field, $k_1, k_2$ reflects the different coupling of rotation to the two helicity states.

Let us study in detail some properties of $H_{\text{eff}}$. A biorthonormal eigenbasis $\{ |\psi_{1,2}\rangle, |\phi_{1,2}\rangle \}$ of $H_{\text{eff}}$ is given by

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{2}} [\pm i\chi^\frac{1}{2} |\psi_+\rangle + |\psi_-\rangle],$$

$$|\phi_{1,2}\rangle = \frac{1}{\sqrt{2}} [\pm i\chi^{-\frac{1}{2}} |\psi_+\rangle + |\psi_-\rangle],$$

where $\chi = \frac{k_1 \omega^2 - 2\mu B}{k_2 \omega^2 - 2\mu B}$. Its eigenvalues are

$$E_{1,2} = E \pm R,$$

where

$$R = \sqrt{(k_1 \omega^2 - \mu B)(k_2 \omega^2 - \mu B)},$$

therefore $E_{1,2}$ either are real or complex-conjugates. This peculiarity of its spectrum ensures us that $H_{eff}$ is a pseudohermitian Hamiltonian \[3\], and indeed an Hermitian operator $\eta$ exists which transform $H_{eff}$ into $H_{eff}^\dagger$ (see Eq.(1)). In the case of real spectrum, for instance, $\eta$ assumes the form \[3\]:

$$\eta = |\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2| = \begin{pmatrix} 1 & 0 \\ \bar{\chi} & 1 \end{pmatrix}. $$

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According to [8], a violation of \((P\text{ and } T)\) in \(H_{\text{eff}}\) would arise though \(k_2 - k_1 \neq 0\). We can improve the discussion on the \(T\)-violating parameters values, by means of the Theorem in Sect. 2. Indeed \(H_{\text{eff}}\) cannot be \(T\)-invariant for all the values of \(k_1\) and \(k_2\) which satisfy the condition

\[
(k_1 \frac{\omega_2}{2} - \mu B)(k_2 \frac{\omega_2}{2} - \mu B) > 0
\]  

(9)
since in this case \(H_{\text{eff}}\) has a real, non degenerate spectrum. (Note that by a suitable choice of \(B\), condition (9) can be verified for all \(k_1, k_2\).)

Let us indeed evaluate the (non unitary) evolution operator \(U(t)\). This is given by [9]

\[
U(t) = |\psi_1\rangle e^{-iE_1 t} \langle \phi_1 | + |\psi_2\rangle e^{-iE_2 t} \langle \phi_2 |
\]

\[
= e^{-iE_1 t} + e^{-iE_2 t} \begin{pmatrix}
\chi & i\chi^2(e^{-iE_1 t} - e^{-iE_2 t}) \\
-i\chi^2(e^{-iE_1 t} - e^{-iE_2 t}) & e^{-iE_1 t} + e^{-iE_2 t}
\end{pmatrix}
\]

Then, assuming the initial condition \(|\psi(0)\rangle = |\psi_-\rangle\), the muon’s state at the time \(t\) is

\[
|\psi(t)\rangle = \frac{1}{2} i\chi^2(e^{-iE_1 t} - e^{-iE_2 t})|\psi_+\rangle + (e^{-iE_1 t} + e^{-iE_2 t})|\psi_-\rangle
\]

The spin-flip probability is therefore

\[
P(t)_{\psi_- \rightarrow \psi_+} = |\langle \psi_+ |\psi(t)\rangle|^2 = \chi^2 \left[1 - \cos 2Rt \right],
\]

(11)

which agrees with the analogous calculation in [8] (where, however, also the width \(\Gamma\) of the muon is taken into account).

Note that the above probability do not depend on the sign of the time; this feature occurs whenever (in a two level system) a transition probability between orthogonal states is considered, and disappears when a different choice of the states is made. Actually, evaluating for instance the transition probability between the states \(|\psi_-\rangle\) and \(|\varphi\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle\) one obtains

\[
P(t)_{\psi_- \rightarrow \varphi} = |\langle \varphi |\psi(t)\rangle|^2 = \frac{1}{2}(\cos Rt + \chi^2 \sin Rt)^2,
\]

(12)

and \(P(t)_{\psi_- \rightarrow \varphi} - P(-t)_{\psi_- \rightarrow \varphi} = \chi^2 \sin 2Rt \neq 0\), which explicitly shows that \(H_{\text{eff}}\) is a \(T\)-violating Hamiltonian (even if \(k_1 = k_2\), in agreement with our Theorem.

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