Research article

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Maximum likelihood estimation for sub-fractional Vasicek Model

Abstract: We investigate the asymptotic properties of maximum likelihood estimators of the drift parameters for fractional Vasicek model driven by a sub-fractional Brownian motion.

Keywords and phrases: Sub-fractional Vasicek model; sub-fractional Brownian motion; maximum likelihood estimation.

MSC 2020: Primary 62M09, Secondary 60G22.

1 Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao [21] and Mishura [18]. Processes driven by fBm with Hurst index $H \geq \frac{1}{2}$ have been used for modeling purposes whenever there is long range dependence (cf. Prakasa Rao [21]). However in some applications such as turbulence phenomena in hydromechanics, it was found that fBm is adequate for modeling small increments but it seems to be inadequate for large increments. For this reason, a sub-fractional Brownian motion may be an alternative to fBm for modeling (cf. Mishura and Zili [17]). There has been a recent interest to study inference problems for stochastic processes driven by a sub-fractional Brownian motion. Bojdecki et al. [1] introduced a centered Gaussian process $\zeta^H = \{\zeta^H(t), t \geq 0\}$ called sub-fractional Brownian motion (sub-fBm) with the

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covariance function

\[ C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}] \]

where \( 0 < H < 1 \). The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor [34] introduced a Wiener integral with respect to a sub-fBm. Tudor [31,32,33,34] discussed some properties related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained in Tudor [34]. Diedhiou et al. [4] investigated parametric estimation for a stochastic differential equation (SDE) driven by a sub-fBm. Mendy [16] studied parameter estimation for the sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

\[ dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0 \]

where \( H > \frac{1}{2} \). This is an analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process \( X = \{X_t, t \geq 0\} \) which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a sub-fBm \( \zeta^H = \{\zeta^H, t \geq 0\} \) with Hurst parameter \( H \). Mendy [16] proved that the least squares estimator estimator \( \hat{\theta}_T \) is strongly consistent as \( T \to \infty \). Kuang and Xie [10] studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Kuang and Liu [9] discussed about the \( L^2 \)-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. [40] obtained the Ito’s formula for sub-fractional Brownian motion with Hurst index \( H > \frac{1}{2} \). Shen and Yan [41] studied estimation for the drift of sub-fractional Brownian motion and constructed a class of biased estimators of James-Stein type which dominate the maximum likelihood estimator under the quadratic risk. El Machkouri et al. [6] investigated the asymptotic properties of the least squares estimator for non-ergodic Ornstein-Uhlenbeck process driven by Gaussian processes, in particular, sub-fractional Brownian motion. Es-sebaiy and Es-sebaiy [7] investigated the problem of estimation of drift parameters in a non-ergodic fractional Vasicek model. In a recent paper, we have investigated optimal estimation of a signal perturbed by a sub-fractional Brownian motion in Prakasa Rao [23]. Some maximal and integral inequalities for a sub-fBm were derived in Prakasa Rao [22,26]. Parametric estimation for linear stochastic differential equations driven by a sub-fractional Brownian motion is studied in Prakasa Rao [24]. A Berry-Esseen type bound for the distribution of the maximum likelihood estimator for the drift parameter of a fractional Ornstein-Uhlenbeck type process
driven by a sub-fractional Brownian motion is studied in Prakasa Rao [25]. Nonparametric estimation of trend for stochastic differential equations driven by sub-fractional Brownian motion is investigated in Prakasa Rao [27]. Mishura and Zili [17] gives a survey of stochastic analysis of mixed fractional Gaussian processes including a sub-fractional Brownian motion.

Vasicek [35] introduced an interest rate model

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t, X_0 = x_0 \in R$$

where \( \{W_t, t \geq 0\} \) is the standard Brownian motion for studying the market behaviour due to changes in the interest rates. This model is now known as the Vasicek model and it was generalized to a fractional Vasicek model to study processes with long range dependence which appears in financial mathematics and other areas such as telecommunication networks, turbulence and image processing. Properties of fractional Vasicek model for modeling are investigated in Chronopoulu and Viens [2], Corlay et al. [3], Hao et al. [8], Song and Li [30] and Xiao et al. [37] among others. Maximum likelihood estimation for fractional Vasicek model is investigated in Lohvinenko and Ralchenko [12,13,14,15] and Xiao and Yu [38,39]. Maximum likelihood estimation for a fractional Vasicek model driven by a mixed fractional Brownian motion is studied in Prakasa Rao [28].

Our aim in this paper is to investigate the problem of maximum likelihood estimation of parameters in a sub-fractional Vasicek model for processes driven by a sub-fractional Brownian motion.

## 2 Sub-fractional Brownian motion

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a stochastic basis satisfying the usual conditions and the processes discussed in the following are \((\mathcal{F}_t)\)-adapted. Further the natural filtration of a process is understood as the \(P\)-completion of the filtration generated by this process.

Let \(\zeta^H = \{\zeta^H_t, t \geq 0\}\) be a normalized sub-fractional Brownian motion (sub-fBm) with Hurst parameter \(H \in (0, 1)\), that is, a Gaussian process with continuous sample paths such that \(\zeta^H_0 = 0, E(\zeta^H_t) = 0\) and

$$E(\zeta^H_s \zeta^H_t) = t^{2H} + s^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}], t \geq 0, s \geq 0.$$

Bojdecki et al. [1] noted that the process

$$\frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0,$$
where \(\{W^H(t), -\infty < t < \infty\}\) is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. Let \(D_H(s, t)\) denote the covariance function of a standard fractional Brownian motion with Hurst index \(H\). Note that
\[
D_H(s, t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}).
\]
Bojdecki et al. [1] proved the following result concerning properties of a sub-fBm.

**Theorem 2.1:** Let \(\zeta^H = \{\zeta^H(t), t \geq 0\}\) be a sub-fBm defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)\). Then the following properties hold.

(i) The process \(\zeta^H\) is self-similar, that is, for every \(a > 0\),
\[
\{\zeta^H(at), t \geq 0\} \overset{\Delta}{=} \{a^H \zeta^H(t), t \geq 0\}
\]
in the sense that the processes, on both sides of the equality sign, have the same finite dimensional distributions.

(ii) The process \(\zeta^H\) is not Markov and it is not a semi-martingale.

(iii) For all \(s, t \geq 0\), the covariance function \(C_H(s, t)\) of the process \(\zeta^H\) is positive for all \(s > 0, t > 0\). Furthermore
\[
C_H(s, t) > D_H(s, t) \text{ if } H < \frac{1}{2}
\]
and
\[
C_H(s, t) < D_H(s, t) \text{ if } H > \frac{1}{2}.
\]

(iv) Let \(\beta_H = 2 - 2^{2H-1}\). For all \(s \geq 0, t \geq 0\),
\[
\beta_H(t - s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq (t - s)^{2H}, \text{ if } H > \frac{1}{2}
\]
and
\[
(t - s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq \beta_H(t - s)^{2H}, \text{ if } H < \frac{1}{2}
\]
and the constants in the above inequalities are sharp.

(v) The process \(\zeta^H\) has continuous sample paths almost surely and, for each \(0 < \epsilon < H\) and \(T > 0\), there exists a random variable \(K_{\epsilon, T}\) such that
\[
|\zeta^H(t) - \zeta^H(s)| \leq K_{\epsilon, T}|t - s|^{H-\epsilon}, 0 \leq s, t \leq T.
\]
Let \( f : [0, T] \rightarrow \mathbb{R} \) be a measurable function and \( \alpha > 0 \), and \( \sigma \) and \( \eta \) be real. Define the Erdelyi-Kober-type fractional integral

\[
(I^\alpha_{T, \sigma, \eta} f)(s) = \frac{\sigma s^\sigma}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-\alpha)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, \quad s \in [0, T],
\]

and the function

\[
n_H(t, s) = \frac{\sqrt{\pi}}{2^{H-\frac{1}{2}} \Gamma_{T, 2} \frac{\Gamma_{2, -2H}}{\sqrt{\pi}} (u^{H-\frac{1}{2}}) I_{[0, t]}(s)} = \frac{2^{1-H} \sqrt{\pi}}{\Gamma(H - \frac{1}{2})} s^{\frac{3}{2}-H} \int_0^t (x^2 - s^2)^{H-\frac{3}{2}} dx \ I_{[0, t]}(s).
\]

The following theorem is due to Dzhaparidze and Van Zanten [5] (cf. Tudor [34]).

**Theorem 2.2:** The following representation holds, in distribution, for a sub-fBm \( \zeta^H \):

\[
\zeta^H_t \triangleq c_H^2 \int_0^t n_H(t, s) dW_s, \quad 0 \leq t \leq T
\]

where

\[
\frac{c_H^2}{\pi} = \frac{\sin(\pi H)}{\Gamma(2H + 1) \sin(\pi H)}
\]

and \( \{W_t, t \geq 0\} \) is the standard Brownian motion.

Tudor [34] has defined integration of a non-random function \( f(t) \) with respect to a sub-fBm \( \zeta^H \) on an interval \([0, T]\) and obtained a representation of this integral as a Wiener integral for a suitable transformed function \( \phi_f(t) \) depending on \( H \) and \( T \). For details, see Theorem 3.2 in Tudor [34].

Tudor [32] (cf. Tudor [34], p. 467) obtained the prediction formula for a sub-fBm. For any \( 0 < H < 1 \), and \( 0 < a < t \),

\[
E[\zeta^H_t | \zeta^H_a, 0 \leq s \leq a] = \zeta^H_a + \int_0^a \psi_{a,t}(u) d\zeta^H_u
\]

where

\[
\psi_{a,t}(u) = \frac{2 \sin(\pi (H - \frac{1}{2}))}{\pi} u(a^2 - u^2)^{H-\frac{1}{2}} \int_a^t \frac{(z^2 - a^2)^{H-\frac{1}{2}}}{z^2 - u^2} z^{H-\frac{1}{2}} dz.
\]

Let

\[
M^H_t = d_H \int_0^t s^{\frac{3}{2}-H} dW_s = \int_0^t k_H(t, s) d\zeta^H_s
\]

where

\[
d_H = \frac{2^{H-\frac{1}{2}}}{c_H \Gamma(\frac{1}{2} - H) \sqrt{\pi}},
\]
(2. 10) \[ k_H(t, s) = d_H s^{\frac{1}{2} - H} \psi_H(t, s), \]
and

\[
\psi_H(t, s) = \frac{s^{H - \frac{1}{2}}}{\Gamma\left(\frac{1}{2} - H\right)} \left[t^{H - \frac{3}{2}} (t^2 - s^2)^{\frac{1}{2} - H} - (H - \frac{3}{2}) \int_s^t (x^2 - s^2)^{\frac{1}{2} - H} x^{H - \frac{3}{2}} dx\right] I_{(0, t)}(s).
\]

It can be shown that the process \( M^H = \{M^H_t, 0 \leq t \leq T\} \) is a Gaussian martingale (cf. Tudor [34], Diedhiou et al. [4]) and is called the \textit{sub-fractional fundamental martingale}. The filtration generated by this martingale is the same as the filtration \( \{\mathcal{F}_t, t \geq 0\} \) generated by the sub-fBm \( \zeta^H \) and the quadratic variation \( < M^H >_s \) of the martingale \( M^H \) over the interval \([0, s]\) is equal to \( w^H_s = \frac{d_H}{2 - 2H} s^{2 - 2H} = \lambda_H s^{2 - 2H} \) (say). For any measurable function \( f : [0, T] \to R \) with \( \int_0^T f^2(s) s^{1 - 2H} ds < \infty \), define the probability measure \( Q_f \) by

\[
\frac{dQ_f}{dP}|_{\mathcal{F}_t} = \exp\left(\int_0^t f(s) dM^H_s - \frac{1}{2} \int_0^t f^2(s) d<M^H>_s(\cdot)\right) = \exp\left(\int_0^t f(s) dM^H_s - \frac{d_H^2}{2} \int_0^t f^2(s) s^{1 - 2H} ds\right)
\]

where \( P \) is the underlying probability measure. Let

(2. 11) \[ (\psi_H f)(s) = \frac{1}{\Gamma\left(\frac{3}{2} - H\right)} I_{0, \frac{3}{2} - H} f(s) \]

where, for \( \alpha > 0 \),

(2. 12) \[ (I_0^{\alpha, \sigma, \eta} f)(s) = \frac{\sigma s^{-\sigma(\alpha + \eta)}}{\Gamma(\alpha)} \int_0^s \frac{1}{(t^\sigma - s^\sigma)^{1-\alpha}} f(t) dt, s \in [0, T]. \]

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor [34]).

\textbf{Theorem 2.3}: The process

\[ \zeta^H_t - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T \]

is a sub-fbm with respect to the probability measure \( Q_f \). In particular, choosing the function \( f \equiv a \in R \), it follows that the process \( \{\zeta^H_t - at, 0 \leq t \leq T\} \) is a sub-fBm under the probability measure \( Q_f \) with \( f \equiv a \in R \).
Let $Y = \{Y_t, t \geq 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ and suppose the process $Y$ satisfies the stochastic differential equation
\[(2.13) \quad dY_t = C(t)dt + d\zeta^H_t, t \geq 0\]
where the process $\{C(t), t \geq 0\}$, adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, such that the process
\[(2.14) \quad R_H(t) = \frac{d}{dw^H_t} \int_0^t k_H(t,s)C(s)ds, t \geq 0\]
is well-defined and the derivative is understood in the sense of absolute continuity with respect to the measure generated by the function $w_H$. Differentiation with respect to $w^H_t$ is understood in the sense:
\[dw^H_t = \lambda_H(2 - 2H) t^{1-2H} dt\]
and
\[\frac{df(t)}{dw^H_t} = \frac{df(t)}{dt} \frac{1}{dt}.\]
Suppose the process $\{R_H(t), 0 \leq t \leq T\}$, defined over the interval $[0, T]$ belongs to the space $L^2([0, T], dw^H_t)$. Define
\[(2.15) \quad \Lambda_H(t) = \exp\left\{\int_0^t R_H(s)dM^H_s - \frac{1}{2} \int_0^t [R_H(s)]^2 dw^H_s\right\}\]
with $E[\Lambda_H(T)] = 1$ and the distribution of the process $\{Y_t, 0 \leq t \leq T\}$ with respect to the measure $P^Y = \Lambda_H(t) P$ coincides with the distribution of the process $\{\zeta^H_t, 0 \leq t \leq T\}$ with respect to the measure $P$.

We call the process $\Lambda^H$ as the likelihood process or the Radon-Nikodym derivative $\frac{dP^Y}{dP}$ of the measure $P^Y$ with respect to the measure $P$.

Tudor [34] derived the following Girsanov type formula.

**Theorem 2.4:** Suppose the assumptions of Theorem 2.2 hold. Define
\[(2.16) \quad \Lambda_H(T) = \exp\left\{\int_0^T R_H(t) dM^H_t - \frac{1}{2} \int_0^T R^2_H(t) dw^H_t\right\}.\]
Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T) P$ is a probability measure and the probability measure of the process $Y$ under $P^*$ is the same as that of the process $V$ defined by
\[(2.17) \quad V_t = \int_0^t d\zeta^H_s, 0 \leq t \leq T.\]
3 Sub-Fractional Vasicek model

The following model was introduced by Vasicek [35] for modeling interest rates in finance. It is a model of the form

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t, 0 \leq t \leq T$$

where $\alpha, \beta, \gamma$ are positive real numbers and $\{W_t, t \geq 0\}$ is the standard Brownian motion. The parameter $\beta$ corresponds to the speed of recovery, the ratio $\alpha/\beta$ is the long-term average interest rate and the parameter $\gamma$ represents the stochastic volatility. The Vasicek model is used in finance, economics, biology, physics, chemistry, medicine, environmental studies and in other areas for modeling purposes. In a series of papers, Lohvinenko and Ralchenko [12,13,14,15] studied asymptotic properties of the maximum likelihood estimators of the parameters $\alpha$ and $\beta$ in the fractional Vasicek model

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t^H, 0 \leq t \leq T$$
as $T \to \infty$ where $\alpha, \beta, \gamma$ are positive real numbers and $\{W_t^H, t \geq 0\}$ is the standard fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Our aim in this paper is to obtain the asymptotic properties of the maximum likelihood estimator for the parameters $\alpha, \beta$ in a sub-fractional Vasicek model driven by a sub-fractional Brownian motion. For related results on estimation of parameters involved in processes driven by sub-fractional Brownian motion (mFBm), see Mendy [16], Xiao et al. [37], Kuang and Liu [9], Kuang and Xie [10], Yu [41], Prakasa Rao [22,23,24,25,26,27,28] among others.

Let us consider the sub-fractional Vasicek model

(3. 1) $$dX_t = (\alpha - \beta X_t)dt + d\zeta_t^H, t \geq 0$$

with $X_0 = x_0$ where $\alpha, \beta$ are unknown positive parameters with known Hurst index $H \in (1/2, 1)$. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

(3. 2) $$X_t = x_0 + \int_0^t (\alpha - \beta X_s)ds + \zeta_t^H, t \geq 0.$$ 

The process $X = \{X_t, t \geq 0\}$ is termed as the sub-fractional Vasicek process driven by a sub-fractional Brownian motion also called the sub-fractional Vasicek process. This equation has a unique solution given by

(3. 3) $$X_t = x_0 e^{-\beta t} + \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \int_0^t e^{-\beta (t-s)} d\zeta_s^H, t \geq 0$$
where the integral
\[ \int_0^t e^{-\beta(t-s)}d\xi^H_t \]
is interpreted as a Wiener integral with respect to a sub-fractional Brownian motion (cf. Tudor [34]). Let
\[
Q_H(t) = \frac{d}{d < M^H >_t} \int_0^t k_H(s,t)(\alpha - \beta X(s))ds, t \geq 0.
\] (3. 4)

The process \( \{Q_H(t), t \geq 0\} \) is well defined from the results in Tudor [34] and the sample paths of the process \( \{Q_H(t), 0 \leq t \leq T\} \) belong almost surely to \( L^2([0,T], d < M^H >_t) \).

Define
\[
Z_t = \int_0^t k_H(s,t)dX_s, t \geq 0.
\] (3. 5)

Then the process \( Z = \{Z_t, t \geq 0\} \) is an \((\mathcal{F}_t)\)-semimartingale with the decomposition
\[
Z_t = \int_0^t Q_H(s)d < M^H >_s + M^H_t, t \geq 0
\] (3. 6)

where \( M^H \) is the fundamental Gaussian martingale. Let \( P^T_\theta \) be the probability measure induced by the process \( \{X_t, 0 \leq t \leq T\} \) when \( \theta = (\alpha, \beta) \) is the true parameter. We write \( Q_{H,\theta}(t) \) for \( Q_H(t) \) hereafter when \( \theta \) is the true parameter. Following Theorem 2.4, we get that the Radon-Nikodym derivative of \( P^T_\theta \) with respect to \( P^T_0 \) is given by
\[
L_T(\theta) \equiv \frac{dP^T_\theta}{dP^T_0} = \exp\left[ \int_0^T Q_{H,\theta}(s)dZ_s - \frac{1}{2} \int_0^T [Q_{H,\theta}(s)]^2d < M^H >_s \right].
\] (3. 7)

We now consider the problem of estimation of the parameter \( \theta = (\alpha, \beta) \) based on the observation of the process \( X = \{X_t, 0 \leq t \leq T\} \) or equivalently \( \{Z_t, 0 \leq t \leq T\} \) and study the asymptotic properties of such estimators as \( T \to \infty \). Let \( \Theta = \mathbb{R}_+^2 \).

Maximum Likelihood Estimation:

Maximum likelihood estimator (MLE) \( \hat{\theta}_T \) is defined by the relation
\[
L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta).
\] (3. 8)

We assume that there exists a measurable MLE. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao [19]). Observe that
\[
\Lambda_T(\theta) = \log L_T(\theta) = \int_0^T Q_{H,\theta}(s)dZ_s - \frac{1}{2} \int_0^T [Q_{H,\theta}(s)]^2d < M^H >_s
\] (3. 9)
\[
\begin{align*}
\alpha Z_T - \beta \int_0^T P_{H}(t) dZ_t - \frac{1}{2} \alpha^2 M^H_T > T \\
+ \alpha \beta \int_0^T P_{H}(t) d < M^H >_t - \frac{1}{2} \beta^2 \int_0^T [P_{H}(t)]^2 d < M^H >_t
\end{align*}
\]

where

\[P_{H}(t) = \frac{d}{d < M^H >_t} \int_0^t k_{H}(s,t) X_s ds.\]

**Theorem 3.1:** Suppose the parameter \(\beta\) is known. Then the MLE \(\hat{\alpha}_T\) for \(\alpha\) is

\[\hat{\alpha}_T = \frac{Z_T + \beta \int_0^T P_{H}(t) d < M^H >_t}{< M^H >_T}.\]

The MLE \(\hat{\alpha}_T\) is unbiased, strongly consistent as \(T \to \infty\). Furthermore the random variable \(T^{1-H}(\hat{\alpha}_T - \alpha)\) has a normal distribution with mean zero and variance \(\lambda_H^{-1}\) depending on the Hurst index \(H\).

**Proof:** Maximizing the log-likelihood \(\Lambda_T(\theta)\), lead to the equations

\[
\frac{\partial \Lambda_T(\theta)}{\partial \alpha} = Z_T - \alpha < M^H >_T + \beta \int_0^T P_{H}(t) d < M^H >_t
\]

and

\[
\frac{\partial^2 \Lambda_T(\theta)}{\partial \alpha^2} = - < M^H >_T.
\]

The equations given above imply that the MLE \(\hat{\alpha}_T\) of \(\alpha\) is given by the equation (3.11). By Theorem 2.1, the process \(Z\) has the representation

\[Z_T = \alpha < M^H >_T - \beta \int_0^T P_{H}(t) d < M^H >_t + M^H_T.\]

Applying this representation, it is easy to check that

\[\hat{\alpha}_T = \alpha + \frac{M^H_T}{< M^H >_T}.\]

Observe that the process \(M^H\) is a martingale with the quadratic variation \(< M^H >_t = \frac{d\lambda_H}{2} t^{2-2H} = \lambda_H t^{2-2H}\) (say). Since \(H < 1\), it follows that the function \(< M^H >_T\) tends to infinity as \(T \to \infty\). Hence, by the strong law of large numbers for martingales (cf. Liptser and Shiryaev [11], Theorem 2.6.10, Prakasa Rao [20]), it follows that

\[\frac{M^H_T}{< M^H >_T} \to 0\]
almost surely as $T \to \infty$. Hence $\hat{\alpha}_T \to \alpha$ almost surely as $T \to \infty$. Since the process $M^H$ is a Gaussian martingale with the quadratic variance $< M^H >$, it follows that the random variable

$$\frac{M^H}{\sqrt{< M^H >_T}}$$

has the standard normal distribution for any fixed $T > 0$. This in turn proves that the random variable

$$\lambda_H^{1/2}T^{1-H}(\hat{\alpha}_T - \alpha)$$

has the standard normal distribution. Hence

$$T^{1-H}(\hat{\alpha}_T - \alpha)$$

has the normal distribution with mean zero and variance $\lambda_H^{-1}$.

**Theorem 3.2:** Suppose the parameter $\alpha$ is known. Suppose that

$$\int_0^T [P_H(t)]^2 d < M^H >_t \to \infty$$

in probability as $T \to \infty$ where $P_H(t)$ is as defined by the equation (3.10). Then the MLE $\hat{\beta}_T$ for the parameter $\beta$ is

(3.12) \[ \hat{\beta}_T = \frac{\int_0^T P_H(t) d < M^H >_t - \int_0^T P_H(t) dZ_t}{\int_0^T [P_H(t)]^2 d < M^H >_t}. \]

The estimator $\hat{\beta}_T$ is strongly consistent as $T \to \infty$. Furthermore the random variable

$$\sqrt{\int_0^T [P_H(t)]^2 d < M^H >_t} (\hat{\beta}_T - \beta)$$

is asymptotically standard normal as $T \to \infty$.

**Proof:** Maximizing the log-likelihood $\Lambda_T(\theta)$, lead to the equations

$$\frac{\partial \Lambda_T(\theta)}{\partial \beta} = -\int_0^T P_H(t) dZ_t + \alpha \int_0^T P_H(t) d < M^H >_t - \beta \int_0^T [P_H(t)]^2 d < M^H >_t$$

and

$$\frac{\partial^2 \Lambda_T(\theta)}{\partial \beta^2} = -\int_0^T [P_H(t)]^2 d < M^H >_t.$$

which proves that the MLE $\hat{\beta}$ is given by the equation (3.12). Note that

(3.13) \[ dZ_t = \alpha d < M^H >_t - \beta P_H(t) d < M^H >_t + dM^H_t. \]
\[ \int_0^T P_H(t) dZ_t = \alpha \int_0^T P_H(t) d < M^H >_t - \beta \int_0^T [P_H(t)]^2 d < M^H >_t + \int_0^T P_H(t) d < M^H >_t. \] 

(3.14)

Hence
\[ \hat{\beta}_T - \beta = \frac{\int_0^T P_H(t) dM_t^H}{\int_0^T [P_H(t)]^2 d < M^H >_t}. \]

(3.15)

Since the process \( M^H \) is a martingale with quadratic variation \( < M^H > \), the process
\[ \left\{ \int_0^T P_H(t) dM_t^H, T \geq 0 \right\} \]

is a local martingale with the quadratic variation
\[ \left\{ \int_0^T [P_H(t)]^2 d < M^H >_t, T \geq 0 \right\}. \]

Observe that the process
\[ \left\{ \int_0^T [P_H(t)]^2 d < M^H >_t, T \geq 0 \right\}. \]

is monotone increasing to infinity in probability as \( T \to \infty \). Applying the strong law of large numbers for local martingales (cf. Liptser and Shiryayev [11], Theorem 2.6.10, Prakasa Rao [20]), it follows that \( \hat{\beta}_T \) converges almost surely to \( \beta \) as \( T \to \infty \). Furthermore
\[ \sqrt{\int_0^T [P_H(t)]^2 d < M^H >_t} (\hat{\beta}_T - \beta) = -\frac{\int_0^T P_H(t) dM_t^H}{\sqrt{\int_0^T [P_H(t)]^2 d < M^H >_t}} \]

and the term on the right side of the above equation tends to the standard normal distribution as \( T \to \infty \) by the central limit theorem for local martingales (cf. Prakasa Rao [20]).

**Theorem 3.3:** Suppose both the parameters \( \alpha \) and \( \beta \) are unknown. Then the MLEs of \( \alpha \) and \( \beta \) are given by

\[ \hat{\alpha}_T = \frac{\int_0^T P_H(t) dZ_t \int_0^T P_H(t) d < M^H >_t - Z_T \int_0^T [P_H(t)]^2 d < M^H >_t}{\left[ \int_0^T P_H(t) d < M^H >_t \right]^2 - < M^H >_T \int_0^T [P_H(t)]^2 d < M^H >_t} \]

and
\[ \hat{\beta}_T = \frac{< M^H >_T \int_0^T P_H(t) dZ_t - Z_T \int_0^T P_H(t) d < M^H >_t}{\left[ \int_0^T P_H(t) d < M^H >_t \right]^2 - < M^H >_T \int_0^T [P_H(t)]^2 d < M^H >_t}. \]

**Proof:** Maximizing the log-likelihood \( \Lambda_T(\theta) \) with respect to the parameter \( \alpha \) and \( \beta \) simultaneously lead to the equations
\[ \frac{\partial \Lambda_T(\theta)}{\partial \alpha} = Z_T - \alpha < M^H >_T + \beta \int_0^T P_H(t) d < M^H >_t = 0 \]

(3.19)
and

(3.20) \[
\frac{\partial \Lambda_T(\theta)}{\partial \beta} = -\int_0^T P_H(t)dZ_T + \alpha \int_0^T P_H(t)d < M^H >_t - \beta \int_0^T [P_H(t)]^2 d < M^H >_t = 0.
\]

Solving these equations, we obtain the estimators \(\hat{\alpha}_T\) and \(\hat{\beta}_T\) as given by the equations (3.17) and (3.18) respectively. Observe that

\[
\frac{\partial^2 \Lambda_T(\theta)}{\partial \alpha^2} = -\frac{M^H}{T} < M^H >_T < 0,
\]

\[
\frac{\partial^2 \Lambda_T(\theta)}{\partial \beta^2} = -\int_0^T [P_H(t)]^2 d < M^H >_t < 0,
\]

and

\[
\frac{\partial^2 \Lambda_T(\theta)}{\partial \alpha^2} \frac{\partial^2 \Lambda_T(\theta)}{\partial \beta^2} - \left[ \frac{\partial^2 \Lambda_T(\theta)}{\partial \alpha \partial \beta} \right]^2 = < M^H >_T \int_0^T [P_H(t)]^2 d < M^H >_t - \left[ \int_0^T P_H(t) d < M^H >_t \right]^2 < 0
\]

by the Cauchy-Schwartz inequality which implies that the estimators \(\hat{\alpha}_T\) and \(\hat{\beta}_T\) maximize the likelihood and hence are the MLEs of \(\alpha\) and \(\beta\) respectively. An application of the representation of the process \(\{Z_t, 0 \leq t \leq T\}\) given by Theorem 2.1 implies that

(3.21) \[
\hat{\alpha}_T - \alpha = \frac{\int_0^T P_H(t)dM^H_T - \int_0^T P_H(t)d < M^H >_t - M^H_T \int_0^T [P_H(t)]^2 d < M^H >_t}{\left[ \int_0^T P_H(t) d < M^H >_t \right]^2 - M^H_T \int_0^T (P_H(t))^2 d < M^H >_t}
\]

and

(3.22) \[
\hat{\beta}_T - \beta = \frac{< M^H >_T \int_0^T P_H(t)dM^H - M^H_T \int_0^T P_H(t)d < M^H >_t}{\left[ \int_0^T P_H(t) d < M^H >_t \right]^2 - M^H_T \int_0^T (P_H(t))^2 d < M^H >_t}.
\]

**Remarks:** Even though the form of the function \(k_H(t,s)\) is known, due to its complicated form, it is not possible to use the methods in Lohvinenko and Ralchenko [12,13,14,15] to study the asymptotic distribution of \((\hat{\alpha}_T, \hat{\beta}_T)\) or the asymptotic marginal distributions of \(\hat{\alpha}_T\) and \(\hat{\beta}_T\) after suitable scaling as \(T \to \infty\). However, following ideas in Lohvinenko and Ralchenko [13], we will transform the problem to the study of maximum likelihood estimation for the parameters of sub-fractional Vasicek model to that of estimation of parameters for a sub-fractional Ornstein-Uhlenbeck process (Mendy [16], Yu [41], Es-Sebaiy-Es-sebaiy [7], Xiao et al. [37] ) and derive the asymptotic properties of the corresponding MLE. We consider the case \(H > \frac{1}{2}\).
4 Alternate approach

Consider the following process

\[ U_t = \int_0^t e^{-\beta(t-s)} d\zeta^H_t, \quad t \geq 0. \]  

(4.1)

Then the process \( \{U_t, t \geq 0\} \) is a sub-fractional Ornstein-Uhlenbeck process and it is the solution of the equation

\[ dU_t = -\beta U_t dt + d\zeta^H_t, \quad U_0 = 0 \]  

(4.2)

The sub-fractional Vasicek model defined by the stochastic differential equation

\[ dX_t = (\alpha - \beta X_t)dt + d\zeta^H_t, \quad t \geq 0, \quad X_0 = x_0 \]  

(4.3)

can be rewritten in the form

\[ X_t = \frac{\alpha}{\beta} t + (x_0 - \frac{\alpha}{\beta}) e^{-\beta t} + U_t. \]  

(4.4)

Observe that

\[ P_H(t) = \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) X_s ds \]

\[ = \frac{d}{d < M^H >_t} \int_0^t k_H(t,s)\left(\frac{\alpha}{\beta} + (x_0 - \frac{\alpha}{\beta}) e^{-\beta s} + U_s\right) ds \]

\[ = \frac{\alpha}{\beta} \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) ds + (x_0 - \frac{\alpha}{\beta}) \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) e^{-\beta s} ds \]

\[ + \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) U_s ds \]

\[ = \frac{\alpha}{\beta} J(t) + (x_0 - \frac{\alpha}{\beta}) V_H(t) + \tilde{P}_H(t) \]  

(4.5)

where

\[ J(t) = \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) ds, \]  

(4.6)

\[ \tilde{P}_H(t) = \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) U_s ds \]  

(4.7)

and

\[ V_H(t) = \frac{d}{d < M^H >_t} \int_0^t k_H(t,s) e^{-\beta s} ds. \]  

(4.8)

Suppose that

\[ \frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d < M^H >_t \to C_{H,\beta} \]  

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in probability as $T \to \infty$ for some positive constant $C_{H,\beta}$. This in turn implies that

$$\int_0^T [\tilde{P}_H(t)]^2 d < M^H >_t \to \infty$$

in probability as $T \to \infty$ and

$$\frac{1}{\sqrt{T}} \int_0^T \tilde{P}_H(t) dM^H_t \to N(0, C_{H,\beta})$$

in distribution as $T \to \infty$ by the central limit theorem for local martingales (cf. Prakasa Rao (1999b)) for some positive constant $C_{H,\beta}$ depending on $H$ and $\beta$. Let $\beta_T^*$ be the maximum likelihood estimator of $\beta$. It can be checked that

$$(4. 9) \quad \beta_T^* - \beta = \frac{\int_0^T \tilde{P}_H(t) dM^H_t}{\int_0^T [\tilde{P}_H(t)]^2 d < M^H >_t}.$$ 

Furthermore

$$(4. 10) \quad \sqrt{T}(\beta_T^* - \beta) \to N(0, C_{H,\beta}^{-1})$$

in distribution as $T \to \infty$.

**Theorem 4.1:** Suppose that

$$\frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d < M^H >_t \to C_{H,\beta}$$

in probability as $T \to \infty$ for some positive constant $C_{H,\beta}$ where $\tilde{P}_H(t)$ is as defined by the equation (3.29). Let $\beta_T^*$ be the maximum likelihood estimator of $\beta$. Then

$$(4. 11) \quad \sqrt{T}(\beta_T^* - \beta) \to N(0, C_{H,\beta}^{-1})$$

in distribution as $T \to \infty$. Further suppose that

$$\lim_{T \to \infty} E\left[\frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d < M^H >_t \right]^p < \infty, p \geq 1.$$ 

Then

$$(4. 12) \quad E[(\sqrt{T}(\beta_T^* - \beta))^p] \to E[\sqrt{C_{H,\beta}^{-1}}Z]^p]$$

as $T \to \infty$ where $Z$ is $N(0,1)$ holds for all integers $p \geq 1$.

**Proof:** Asymptotic normality of the estimator $\beta_T^*$ as $T \to \infty$ was proved by the arguments given above. Note that

$$(4. 13)$$
\[
\frac{1}{<MH>_T} \int_0^T P_H(t) d <MH>_t = \frac{1}{<MH>_T} \int_0^T \left\{ \frac{\alpha}{\beta} J(T) + (x_0 - \frac{\alpha}{\beta}) V_H(t) + \tilde{P}_H(t) \right\} d <MH>_t \\
= \frac{\alpha}{\beta} J(T) + (x_0 - \frac{\alpha}{\beta}) \frac{1}{<MH>_T} \int_0^T V_H(t) d <MH>_t \\
+ \frac{1}{<MH>_T} \int_0^T \tilde{P}_H(t) d <MH>_t.
\]

Note that the convergence of moments

(4.14) \[ E[(\sqrt{T}(\beta_T - \beta)^p)] \to E[(\sqrt{C_{H,\beta}}Z)^p] \]

as \( T \to \infty \) where \( Z \) is \( N(0,1) \) holds for all integers \( p \geq 1 \) if the family \( (\sqrt{T}(\beta_T - \beta))^p \) is uniformly integrable over \( T \) for all integers \( p \geq 1 \). Observe that

(4.15) \[ (E[|\sqrt{T}(\beta_T - \beta)|^p])^2 \leq E\left[ \frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d <MH>_t \right] \leq E\left[ \frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d <MH>_t \right] \leq \left( \frac{1}{C_p} \right) E\left[ \frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d <MH>_t \right] \]

where the last bound holds by the Burkholder-Davis-Gundy inequality with an absolute constant \( C_p \). Hence the limit in the equation (4.14) holds by the de la Vallee Poussin theorem since

\[ \lim_{T \to \infty} E\left[ \frac{1}{T} \int_0^T [\tilde{P}_H(t)]^2 d <MH>_t \right] < \infty \]

for all integers \( p \geq 1 \) by hypothesis.

**Acknowledgment:** Work in this paper was supported under the scheme ‘INSA Senior Scientist” at the CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad, India.

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