The replicator equation in stochastic spatial evolutionary games

Yu-Ting Chen∗†

Abstract

We study the stochastic spatial evolutionary game with death-birth updating. The model is a voter model perturbation. Here, we require perturbation strengths \( w \) such that \( 1/N \ll w \ll 1 \) for typical expanding populations of size \( N \to \infty \); this range is in contrast to \( w = O(1/N) \) for the diffusion approximations of the model. The first main result proves that the density processes of strategy converge to the replicator equation in the limit of a large finite population subject to appropriate, but general, conditions. Multiple strategies and mutations are allowed. Moreover, after normalization, the limiting fluctuations are shown to be Gaussian with a covariance function in the replicator equation’s solution through the Wright–Fisher noise coefficient. In particular, to the degree of using constants that may depend implicitly on space, we prove a conjecture from the biological literature that the expected density processes approximate the replicator equation on many non-regular graphs.

Keywords: Evolutionary games; voter model; coalescence; almost exponentiality of hitting times; the replicator equation; the Wright–Fisher diffusion.

Mathematics Subject Classification (2000): 60K35, 82C22, 60F05, 60J99.

Contents

1 Introduction ......................................................... 2
2 Main results ......................................................... 6
3 Semimartingale dynamics .......................................... 9
4 Decorrelation in the ancestral lineage distributions ............... 13
  4.1 Mixing conditions for pairs of adjacent sites .................. 14
  4.2 Extensions to triplets of sites and other pairs ................. 17
5 Convergence of the vector density processes ....................... 20
  5.1 Asymptotic closure of equation and path regularity ........ 20
  5.2 The replicator equation and the Wright–Fisher fluctuations 30
6 Further properties of coalescing lineage distributions .......... 32
  6.1 A comparison with mutations .................................. 32
  6.2 Full decorrelation on the large random regular graphs ...... 35
7 References .................................................................... 38

∗Department of Mathematics and Statistics, University of Victoria, British Columbia, Canada.
†Email: chenyuting@uvic.ca
1 Introduction

The replicator equation is the most widely applied dynamics in evolutionary game models. It defines the first dynamics in evolutionary game theory \[37\], thereby establishing an important connection to theoretical explanations of animal behaviors \[28\]. For this application, the original derivation of the equation considers a large well-mixed population. The individuals use strategies from a given finite set \(S\) with \(#S \geq 2\). The payoff to \(\sigma \in S\) from playing against \(\sigma' \in S\) is denoted by \(\Pi(\sigma, \sigma')\). In the continuum, the density \(X_\sigma\) of \(\sigma\) evolves with a per capita rate given by the difference between its payoff

\[
F_\sigma(X) = \sum_{\sigma' \in S} \Pi(\sigma, \sigma')X_{\sigma'}
\]

and the average payoff of the population. Hence, the following dynamic is obeyed by \(X_\sigma\):

\[
\dot{X}_\sigma = X_\sigma \left( F_\sigma(X) - \sum_{\sigma'' \in S} F_{\sigma''}(X)X_{\sigma''} \right).
\]

This differential equation is a point of departure for studying connections between the payoff matrix \((\Pi(\sigma, \sigma'))_{\sigma, \sigma' \in S}\) and the equilibrium states of the model by methods from dynamical systems. See \[21, 24\] for an introduction to the analysis, and \[24, 35\] for the relationships to the Lotka–Volterra equation of ecology and Fisher’s fundamental theorem of natural selection.

In this paper, we consider the stochastic evolutionary game dynamics in large finite structured populations. Our goal is to prove that the density processes of strategy converge to the replicator equation. In this direction, one of the major results in the biological literature is the replicator equation on graphs \[31\]. The authors further conjecture that the approximation applies to graphs more general than the large random regular graphs in their derivation \[31\], Section 5\]. To obtain the proofs, we view the model as a perturbation of the voter model. This viewpoint circumvents the non-attractiveness of the model \[18\] and has led to several other mathematical results (e.g. \[18, 8, 16, 11, 4, 12\]). Our starting point here is the methods in \[12\] for proving the diffusion approximation of the model under weak selection. The corresponding perturbation away from the voter model has strengths typically given by \(w = O(1/N)\), where \(N\) is the population size. Nevertheless, the present question has to deal with very different properties. The proof in \[12\] considers the equivalence of probability laws between the evolutionary game model and the voter model in the limit. But now this property breaks down for nontrivial parameters according to the predictions in \[31\]. Distributional limits of the density processes under the evolutionary game and the voter model degenerate to delta distributions of solutions of different differential equations. The convergence to the replicator equation also uses the different range of perturbation strengths \(w\) that satisfy \(1/N \ll w \ll 1\), and thus, are stronger. Due to this change of perturbation strengths, the characterization of the limiting equation is another subject for extending the methods in \[12\].

The evolutionary game model studied in this paper is defined as follows. First, we impose directed weights \(q(x, y)\) for all adjacent sites \(x\) and \(y\) in a given population of size \(N\). We assume that \(q\) is an irreducible probability transition kernel with a zero trace \(\sum_x q(x, x) = 0\). A given perturbation strength \(w > 0\) now defines the selection strength. For an individual at site \(x\) using strategy \(\sigma_x\), interactions with the neighbors determine the fitness as the sum \(1 - w + w \sum_y q(x, y)\Pi(\sigma_x, \sigma_y)\), where \(\sigma_y\) denotes the strategy of the neighbor at \(y\). Under the condition of positive fitness by tuning the selection strength appropriately, the death-birth updating requires that in a transition of state, an individual is chosen to die with rate 1. Then the neighbors compete to fill in the site by reproduction with probability proportional to their fitness. The main results of this paper allow for mutations of...
strategies. But we postpone introducing this additional mechanism and do not take it into account in the rest of this introduction.

The density processes of strategy continue to play a significant role in the biological literature for studying equilibrium states of stochastic spatial evolutionary games. With spatial structure, the key issue for using the densities (and the variations stated below for the main result) arises from the non-closure of the stochastic equations. More specifically, the density processes are projections of the whole systems, and in general, the density functions are not Markov functions in the sense of [34]. For the evolutionary game with death-birth updating introduced above, the local changes in strategies held by pairs of individuals determine the changes in the densities. It is neither clear how to simplify the dynamics analytically by calculating the infinitesimal generators.

One of the main results in [31] predicts that for selection strengths $w \ll 1$, the expected density processes approximate the following extended form of the replicator equation (1.1):

$$
\dot{X}_\sigma = w X_\sigma \left( F_\sigma(X) + \tilde{F}_\sigma(X) - \sum_{\sigma'' \in S} F_{\sigma''}(X) X_{\sigma''} \right), \quad \sigma \in S. \tag{1.3}
$$

The derivation of (1.3) in [31] uses the physics method of pair approximation on the large random $k$-regular graphs, for any integer $k \geq 3$. The functions $F_\sigma(X)$ and $\tilde{F}_\sigma(X)$ for (1.3) are linear in $X$, with coefficients explicit in the payoff matrix and the graph degree $k$. Note that (1.1) underlines the nontrivial effect of spatial structure, since the coefficients are different from those for the replicator equation (1.1) in mean-field populations. Moreover, based on computer simulations from [32], the authors of [31] make the conjecture briefly mentioned at the beginning of this article. It states that the approximate replicator equation in [31] applies to many non-regular graphs, provided that the constant graph degree $k$ in the coefficients of the replicator equation is replaced by the corresponding average degrees. In approaching this conjecture, it is still unclear on how the average degrees enter. The methods in this paper do not extend to this generality either. On the other hand, even within the scope of the large random regular graphs, the constant graph degrees and the locally regular tree-like property appear to be crucial in [31]. The locally tree-like property is also useful for applying pair approximation in general contexts [36]. See [12, Section 1] for further discussions of pair approximation.

In the presence of two strategies, the supplementary information (SI) of [32] shows that the density processes of a fixed strategy approximate the Wright–Fisher diffusion with drift. The derivation therein also applies pair approximation on the large random $k$-regular graphs for $k \geq 3$. But the argument is noticeably different from the argument in [31] for deriving the replicator equation. In particular, [32, SI] considers the slow-fast dynamical system for the density and certain rapidly convergent local density. Although it is not clear how to justify these arguments mathematically, the diffusion approximation of the density processes can be proven on large finite spatial structures subject to appropriate, but general, conditions [12, Theorem 4.6]. Moreover, the asymptotic equivalence of probability laws between the voter model and the evolutionary game model via perturbations holds (not just between the laws of the density processes). This property holds in the form of the convergence of the Radon–Nikodým derivative processes to an exponential martingale of the limiting density processes. More specifically, the convergence results in [12, Theorem 4.6] consider the presence of two strategies. The density processes are subject to time changes given by the expected meeting times of two independent stationary $q$-Markov chains. (Recall that a spatial structure is defined by $(E, q)$. See also the discussion below for the emergence of the meeting times.) These time changes encode the spatial structures and are implicit in most cases. Nevertheless, the large random $k$-regular graphs assumed in the aforementioned applications of pair approximation allows for explicit asymptotics of the expected meeting times as $N(k-1)/[2(k-2)]$ for $N \to \infty$ [13]. Additionally, explicit drift coefficients of the limiting diffusions in [12] are obtained under the payoff matrices for prisoner’s dilemma. See
(2.17) for these matrices and [13, Remark 3.1] for correcting an inaccuracy in [12] on passing limits along the large random regular graphs.

With the particular spatial structures and payoff matrices mentioned above, the diffusion approximation in [32, SI] is accurate to the degree of matching constants if the time changes are formally undone [13]. This standpoint allows us to reconsider the replicator equation on graphs from [31] by formally applying smaller time changes. See the end of Section 2 for details.

The main results of this paper are proven for the convergence of the evolutionary game with death-birth updating. The limit is passed along a sequence of large finite structured populations. Multiple strategies and mutations are allowed. On spatial structures $(E, q)$ where $q$ is not symmetric, the density of a strategy now refers to the variation given by the stationary measure (from $q$) of the strategy in the population. See (2.4). Under this setting, the first of the main results [Theorem 2.2 $(1^o)$] obtains the convergence of the vector density processes of strategy after growing time changes. For the assumptions, we require that the stationary distributions associated with the spatial structures are asymptotically comparable to the uniform distributions (see (2.12)), and these spatial structures allow for suitable time changes and selection strengths (Definition 2.1). In particular, typically for the eligible populations, the time changes can range in $1 \ll \theta \ll N$; the selection strengths are in an inverse relation to these time changes as $1/N \ll w \ll 1$. Then the limiting equation is given by the extended replicator equation as in (1.3), with the small selection strengths $w$ replaced by their scaling limit $w_{\infty}$ to absorb the time changes in use. More importantly, $F_{\sigma}(X)$ and $\bar{F}_{\sigma}(X)$ in (1.3) are determined as the following linear functions:

$$F_{\sigma}(X) = \overline{\kappa}_{0|2|3} \sum_{\sigma' \in S} \Pi(\sigma, \sigma') X_{\sigma'},$$

$$\bar{F}_{\sigma}(X) = (\overline{\kappa}_{(2,3)|0} - \overline{\kappa}_{0|2|3}) \Pi(\sigma, \sigma) + \sum_{\sigma' \in S} \left( \overline{\kappa}_{(0,3)|2} - \overline{\kappa}_{0|2|3} \right) \left[ \Pi(\sigma, \sigma') - \Pi(\sigma', \sigma) \right] X_{\sigma'},$$

where $\overline{\kappa}_{(2,3)|0}, \overline{\kappa}_{(0,3)|2}, \overline{\kappa}_{0|2|3}$ are nonnegative constants and depend only on the spatial structures in the limit.

In contrast to the explicit results in [31, 32], we do not have explicit solutions of the $\overline{\kappa}$-constants in (1.4)–(1.5). On the large random regular graphs, characterizations by random walks on infinite regular trees are possible under the present methods, though. See Sections 4 and 6.2. On the other hand, the first main result has the same spirit of the diffusion approximations of the voter model and the evolutionary game in [9, 10, 12]. The model allows for spatial universality of the scaling limit as the replicator equation. This result does not require convergence of local geometry as in the large discrete tori and the large random regular graphs. The spatial structures can be sparse in the limit, which is in stark contrast to usual assumptions for proving scaling limits of particle systems. The locally tree-like property assumed in using pair approximation is not required either. Based on these properties, the first main result gives an answer in the positive for the conjecture in [31]. The approximations of the density processes by the replicator equation extend to many non-regular graphs.

To establish connections to the approximate Wright–Fisher diffusion from [32, SI], we propose two additional aspects for the convergence of the density processes, hence the other two main results of this paper. Multiple strategies continue to apply. First, the fluctuations of the density processes, after normalization, are proven to converge to a vector Gaussian process [Theorem 2.2 $(2^o)$]. The covariance function has a coefficient given by the Wright–Fisher diffusion coefficient matrix in the solution $X$ of
the limiting replicator equation:

$$\int_0^t X_\sigma(s)[\delta_{\sigma,\sigma'} - X_{\sigma'}(s)]ds, \quad \sigma, \sigma' \in S,$$

where $\delta_{\sigma,\sigma'}$ are the Kronecker deltas. Note that the convergence to the replicator equation and the convergence of the fluctuations do not imply the diffusion approximation of the density processes. The second aspect applies the main theorem to the prisoner dilemma payoff matrices. The result shows that in terms of the formal comparison mentioned above, the replicator equation and the Gaussian process recover precisely the approximate Wright–Fisher diffusion from [32, SI] (Corollary 2.3).

The proof of the first main result leads to all the central arguments of this paper. Our starting point is the methods from [9, 10, 12] for the diffusion approximations of the voter model and the evolutionary game model. However, as pointed out before, an asymptotic equivalence of probability laws between the two models does not exist in the present direction. The first step to resolve this issue focuses on the expected local changes of strategies driving the changes in the game density processes. Duhamel’s principle is applied to deduce approximations by the voter model (Proposition 5.3). The proof then proceeds to consider the time-reversed genealogical structure of the voter model, which are coalescing Markov chains defined on the spatial structures.

The choice of the time changes and the characterization of the limiting equation’s coefficients consists in the main technical features of this article, in addition to the application of Duhamel’s principle mentioned above. The proof considers the following arguments. First, for diffusion approximations, fast mixing ensures approximations of the coalescence times in the ancestral lineages by analogous exponential distributions in mean-field populations. This idea goes back to [15]. Due to time reversal, the coalescence times can be viewed as the first meeting times of Markov chains (defined by the spatial structures). Then the central meeting times are for two independent copies of the stationary Markov chains. The almost exponentiality of Markov chain hitting times [1, 2, 3] thus enters as the driving mechanism for the generalizations [9, 10, 12]. The expected values of these meeting times are also used as the time changes of the density processes. In contrast, the time changes for the convergence to the replicator equation can only grow slower, since the limiting trajectories are less rougher. With these slower time changes, the non-triviality of the associated coalescence times calls for the decorrelation of the meeting time distributions. Informally, this decorrelation occurs after the details of the spatial structures govern the meeting time distributions, and before the exponential approximations for mean-field approximations of the ancestral lineages. Moreover, the limiting distributions of the scaled meeting times is a nontrivial convex combination of the delta distributions at zero and infinity. The possibility of other probability distributions thus arising is ruled out. Then due to the presence of various kinds of local changes that characterize the changes of the densities, there are various kinds of meeting times to deal with. All the relevant meeting time distributions are required to decorrelate simultaneously and smoothly. The $\kappa$-constants in (1.4)–(1.5) thus arise.

For the voter model, the existence of such time changes is obtained in [9]. Now, our goal is to extend those results in [9] to handle several relevant meeting times all at once. We are also interested in finding the best possible range of these time changes. See Section 4 for the results of these time changes. Finally, we remark that the application of these slow time changes is reminiscent of the slow-fast dynamical system in [32, SI] mentioned above.

**Organization.** Section 2 introduces the evolutionary game model and the voter model analytically and discusses the main results (Theorem 2.2 and Corollary 2.3). In Section 3, we define the voter model and the evolutionary game model as semimartingales and briefly explain the role of the coalescing duality. In Section 4, we quantify the time changes used in the main results and characterize the coefficients of the limiting equation. Section 5 is devoted to the proofs of Theorem 2.2 and Corollary 2.3. Finally, Section 6 presents some auxiliary results on coalescing Markov chains.
Acknowledgments. The author would like to thank Lea Popovic for comments on earlier drafts. Support from the Simons Foundation before the author’s present position and from the Natural Science and Engineering Research Council of Canada is gratefully acknowledged.

2 Main results

In this section, we introduce the stochastic spatial evolutionary game with death-birth updating in more detail. A discussion of the main results of this paper then follows. To be consistent with the viewpoint of voter model perturbations and the neutral role of the voter model, strategies will be called types in the rest of this paper. The settings here and in the next section are adapted from those in [9, 10, 12] to the context of evolutionary games with multiple types.

Recall that a discrete spatial structure considered in this paper is given by an irreducible, reversible probability kernel \( q \) on a finite nonempty set \( E \) such that \( \text{tr}(q) = \sum_{x \in E} q(x, x) = 0 \). Write \( N = \# E \) and \( \pi \) for the unique stationary distribution of \( q \). The interactions of individuals are defined by a payoff matrix \( \Pi = (\Pi((\sigma, \sigma'))_{\sigma, \sigma' \in S} \) of real entries. Fix \( \overline{w} \in (0, \infty) \) such that

\[
 w + w \sum_{z \in E} q(y, z)|\Pi((\xi(y), \xi(z))| < 1, \quad \forall w \in [0, \overline{w}], \; y \in E. \tag{2.1}
\]

Then the following perturbed transition probability is used to update types of individuals due to interactions:

\[
 q^w(x, y, \xi) \overset{\text{def}}{=} \frac{q(x, y) \left[ (1 - w) + w \sum_{z \in E} q(y, z)\Pi((\xi(y), \xi(z)) \right]}{\sum_{y' \in E} q(x, y') \left[ (1 - w) + w \sum_{z \in E} q(y', z)\Pi((\xi(y'), \xi(z)) \right]}. \tag{2.2}
\]

With these updates and the updates based on a mutation measure \( \mu \) on \( S \), two types of configurations \( \xi^{x,y}, \xi^{x,\sigma} \in S^E \) result. They are obtained from \( \xi \in S^E \) by changing only the type at \( x \) such that \( \xi^{x,y}(x) = \xi(y) \) and \( \xi^{x,\sigma}(x) = \sigma \). Hence, the evolutionary game \( (\xi_t) \) is a Markov jump process with a generator given by

\[
 L^w H(\xi) = \sum_{x,y \in E} q^w(x, y, \xi)[H(\xi^{x,y}) - H(\xi)] + \sum_{x \in E} \int_S [H(\xi^{x,\sigma}) - H(\xi)]d\mu(\sigma), \quad H : S \to \mathbb{R}. \tag{2.3}
\]

The first sum on the right-hand side of (2.3) governs changes of types due to selection, and the second sum is responsible for mutations. Given \( \xi \in S^E \) and a probability distribution \( \nu \) on \( S^E \) as initial conditions, we write \( \mathbb{P}^w_\xi \) and \( \mathbb{P}^w_\nu \), or \( \mathbb{P}^w_\xi^\nu \) and \( \mathbb{P}^w_\nu^\nu \), under the laws associated with \( L^w \). For \( w = 0 \), the generator \( L^w \) is reduced to the generator \( L \) of the multi-type voter model with mutation, and the notation \( \mathbb{P} \) and \( \mathbb{E} \) is used.

The object in this paper is the vector density processes \( p(\xi_t) = (p_\sigma(\xi_t) ; \sigma \in S) \) for the evolutionary game with death-birth updating. Here, the density function of \( \sigma \in S \) is given by

\[
 p_\sigma(\xi) = \sum_{x \in E} 1_\sigma \circ \xi(x)\pi(x), \tag{2.4}
\]

where \( f \circ \xi(x) = f(\xi(x)) \). Under \( \mathbb{P}^w \), \( p_\sigma(\xi_t) \) admits a semimartingale decomposition:

\[
 p_\sigma(\xi_t) = p_\sigma(\xi_0) + A_\sigma(t) + M_\sigma(t), \tag{2.5}
\]

where \( A_\sigma(t) = \int_0^t L^w p_\sigma(\xi_s)ds \). In the sequel, we study the convergence of the vector density processes and the martingales \( M_\sigma \) separately, along an appropriate sequence of discrete spatial structures \( (E_n, q^{(n)}) \) with \( N_n = \# E_n \to \infty \).
Convention for superscripts and subscripts. Objects associated with \((E_n, q^{(n)})\) will carry either superscripts “\((n)\)” or subscripts “\(n\)”, although additional properties may be assumed so that these objects are not just based on \((E_n, q^{(n)})\). Otherwise, we refer to a fixed spatial structure \((E, q)\). ■

For the main theorem, we choose parameters as time changes for the density processes, mutation measures, and selection strengths. The choice is according to the underlying discrete spatial structures. We use \(\nu_n(1) = \sum_{x \in E_n} \pi^{(n)}(x)^2\) and the first moment \(\gamma_n\) of the first meeting time of two independent stationary rate-1 \(q^{(n)}\)-Markov chains. The other characteristic of the spatial structure considers the mixing time \(t_{\text{mix}}^{(n)}\) of the \(q^{(n)}\)-Markov chains and the spectral gap \(g_n\) as follows. Recall that the semigroup of the continuous-time rate-1 \(q^{(n)}\)-Markov chain is given by \((e^{t(q^{(n)})}; t \geq 0)\). With
\[
d_{E_n}(t) = \max_{x \in E_n} \|e^{t(q^{(n)})-1}(x, \cdot) - \pi^{(n)}\|_{TV}
\] (2.6)
for \(\| \cdot \|_{TV}\) denoting the total variation distance, we choose
\[
t_{\text{mix}}^{(n)} = \inf\{t \geq 0; d_{E_n}(t) \leq (2e)^{-1}\}.
\] (2.7)
The spectral gap \(g_n\) is the distance between the largest and second largest eigenvalues of \(q^{(n)}\).

**Definition 2.1.** For all \(n \geq 1\), let \(\theta_n \in (0, \infty)\) be a time change, \(\mu_n\) a mutation measure on \(S\), and \(w_n \in [0, \pi]\). The sequence \((\theta_n, \mu_n, w_n)\) is said to be admissible if all of the following conditions hold. First, \((\theta_n)\) satisfies
\[
\lim_{n \to \infty} \theta_n = \infty, \quad \lim_{n \to \infty} \gamma_n \theta_n < \infty, \quad \lim_{n \to \infty} \gamma_n \nu_n(1)e^{-\theta_n} = 0, \quad \forall t \in (0, \infty),
\] (2.8)
and at least one of the two mixing conditions holds:
\[
\lim_{n \to \infty} \gamma_n \nu_n(1)e^{-\theta_n} = 0 \quad \text{or} \quad \lim_{n \to \infty} \frac{t_{\text{mix}}^{(n)}}{\theta_n}[1 + \log^+ (\gamma_n/t_{\text{mix}}^{(n)})] = 0,
\] (2.9)
where \(\log^+ \alpha = \log(\max\{\alpha, 1\})\). Second, we require the following limits for \((\mu_n)\) and \((w_n)\):
\[
\lim_{n \to \infty} \mu_n(\sigma)\theta_n = \mu_\infty(\sigma) < \infty, \quad \forall \sigma \in S; \quad \text{(2.10)}
\]
\[
\lim_{n \to \infty} w_n = 0, \quad \lim_{n \to \infty} \frac{w_n\theta_n}{2\gamma_n \nu_n(1)} = w_\infty < \infty, \quad \limsup_{n \to \infty} w_n \theta_n < \infty.
\] (2.11)

Another condition of the main theorem requires that \(\sup_n N_n \max_{x \in E_n} \pi^{(n)}(x) < \infty\), which implies \(\gamma_n \geq \mathcal{O}(N_n)\) (see (3.22) or [9, (3.21)] for details). In this context, the admissible \(\theta_n\) has the following effects. If \(\lim_n \theta_n/\gamma_n \in (0, \infty)\), the time-changed density processes \(p_t(\xi_{\theta_n t})\) of the voter model converge to the Wright–Fisher diffusion [9, 10]. Moreover, the density processes of the evolutionary game converge to the same diffusion but with a drift [12, Theorem 4.6]. These diffusion approximations hold under the mixing conditions slightly different from those in (2.9). Therefore, assuming \(\gamma_n \nu_n(1)\theta_n = 0\) in (4.2) has the heuristic that the time-changed density processes have paths less rougher in the limit, and so, do not converge to diffusion processes.

The other conditions for the admissible sequences mainly consider the typical case of “transient” spatial structures. The kernels are characterized by the condition \(\sup_n \gamma_n \nu_n(1) < \infty\) [9, Remark 2.4]. In this case, (2.8) can be satisfied by any sequence \((\theta_n)\) such that \(1 \ll \theta_n \ll N_n\), and (2.10) and (2.11) allow for nonzero \(\mu_\infty\) and \(w_\infty\). The somewhat tedious condition in (2.11) simplifies drastically, and we get \(N_n^{-1} \ll w_n \ll 1\) when \(\lim_n w_n \theta_n\) is nonzero. As for the mixing conditions in (2.9), they
can pose severe limitations if the spatial structures are “recurrent” \((\sup_n \gamma_n \nu_n(\mathbb{1}) = \infty)\). In this case, we may not be able to find admissible sequences such that \(w_\infty > 0\), so that the limiting equation to be presented below only allows constant solutions in the absence of mutation. For example, the two-dimensional discrete tori satisfy \(\gamma_n \nu_n(\mathbb{1}) \sim C \log N_n\), \(t^{(n)}_{\text{mix}} \leq O(N_n)\) and \(g_n = O(1/N_n)\). See [15] and [27, Theorem 10.13 on p.133, Theorem 5.5 on p.66 and Section 12.3.1 on p.157]. We can choose \(\theta_n = N_n(\log \log N_n)^2\) to satisfy (2.8) with \(\lim_n \theta_n/\gamma_n = 0\), and the first mixing condition in (2.9). But now the admissible \((w_n)\) only gives \(w_\infty = 0\). We notice that a similar restriction is pointed out in [20] on the low density scaling limits of the biased voter model, where the limit is Feller’s branching diffusion with drift.

From now on, we write \(\pi_{\text{min}} = \min_{x \in E} \pi(x)\) and \(\pi_{\text{max}} = \max_{x \in E} \pi(x)\) for the stationary distribution \(\pi\) of \((E, q)\). The main theorem stated below shows a law of large numbers type convergence for the density processes and a central limit theorem type convergence for the fluctuations. These two results do not combine to give the diffusion approximation of the density processes proven in [12].

**Theorem 2.2.** Let \((E_n, q^{(n)}_\cdot)\) be a sequence of irreducible, reversible probability kernels defined on finite sets with \(N_n = \#E_n \rightarrow \infty\). Assume the following conditions:

(a) Let \(\nu_n\) be a probability measure on \(S^{E_n}\) such that \(\nu_n(\xi; \cdot, p(\xi) \in \cdot)\) converges in distribution to a probability measure \(\nu_{\infty}\) on \([0, 1]^S\).

(b) It holds that

\[
0 < \liminf_{n \rightarrow \infty} N_n \pi^{(n)}_{\text{min}} \leq \limsup_{n \rightarrow \infty} N_n \pi^{(n)}_{\text{max}} < \infty. \tag{2.12}
\]

(c) The limits in (4.5) and (4.6) defining the nonnegative constants \(\overline{\kappa}_{(2,3)|0}, \overline{\kappa}_{(0,3)|2}\) and \(\overline{\kappa}_{0|2|3}\) exist. These constants depend only on space.

(d) We can choose an admissible sequence \((\theta_n, \mu_n, w_n)\) as in Definition 2.1 such that \(\lim_n \theta_n/\gamma_n = 0\).

Then the following convergence in distribution of processes holds:

\(^{(1')}\) The sequence of the vector density processes \((p(\theta_n, \cdot), \mathbb{P}^{w_n}_{\nu_n})\) converges to the solution \(X\) of the following differential equation with the random initial condition \(\mathbb{P}(X_0 \in \cdot) = \nu_{\infty}\):

\[
\dot{X}_\sigma = w_\infty X_\sigma \left( F_\sigma(X) + \bar{F}_\sigma(X) - \sum_{\sigma'' \in S} F_{\sigma''}(X)X_{\sigma''} \right) + \mu_\infty(\sigma)(1 - X_\sigma) - \mu_\infty(S \setminus \{\sigma\})X_\sigma, \quad \sigma \in S,
\]

where \(F_\sigma(X)\) and \(\bar{F}_\sigma(X)\) are linear functions in \(X\):

\[
F_\sigma(X) = \overline{\kappa}_{0|2|3} \sum_{\sigma' \in S} \Pi(\sigma, \sigma')X_{\sigma'}, \tag{2.14}
\]

\[
\bar{F}_\sigma(X) = (\overline{\kappa}_{(2,3)|0} - \overline{\kappa}_{0|2|3})\Pi(\sigma, \sigma) + \sum_{\sigma' \in S} (\overline{\kappa}_{(0,3)|2} - \overline{\kappa}_{0|2|3})\Pi(\sigma, \sigma') - \Pi(\sigma', \sigma)X_{\sigma'} - (\overline{\kappa}_{(2,3)|0} - \overline{\kappa}_{0|2|3}) \sum_{\sigma' \in S} \Pi(\sigma', \sigma')X_{\sigma'} \tag{2.15}
\]

Moreover, the sum of the \(\overline{\kappa}\)-constants in \(F_\sigma(X)\) and \(\bar{F}_\sigma(X)\) is nontrivial:

\[
(\overline{\kappa}_{(2,3)|0} - \overline{\kappa}_{0|2|3}) + \overline{\kappa}_{0|2|3} + (\overline{\kappa}_{(0,3)|2} - \overline{\kappa}_{0|2|3}) \in (0, \infty). \tag{2.16}
\]
(2°) Recall the vector martingale defined by (2.5), and set $M^{(n)}_{\sigma}(t) = (M_{\sigma}(\theta_{nt}); \sigma \in S)$ under $\mathbb{P}^{w_{n}}_{\nu_{n}}$. If, moreover, $\lim_{n} \gamma_{n} \nu_{n}(1)/\theta_{n} = 0$ holds, then $(\gamma_{n}/\theta_{n})^{1/2}M^{(n)}$ converges to a vector Gaussian process with covariance matrix given by $(\int_{0}^{t} X^{(n)}_{\sigma}(s)\left[\delta_{\sigma,\sigma'} - X^{(n)}_{\sigma'}(s)\right]ds; \sigma, \sigma' \in S)$.

We present the proof of Theorem 2.2 in Section 5. The existence of the limits in condition (c) is proven in Proposition 4.4. See Lemma 5.5 for the additional condition in Theorem 2.2 (2°).

To illustrate Theorem 2.2, we consider the generalized prisoner’s dilemma matrix in the rest of this section. The matrix is for games among individuals of two types:

$$\Pi = \begin{pmatrix} 1 & 0 \\ b - c & -c \\ b & 0 \end{pmatrix}$$

(2.17)

for real entries $b, c$. (The usual prisoner’s dilemma matrix requires $b > c > 0$.) The proof of the following corollary also appears in Section 5.

**Corollary 2.3.** Let conditions (a)–(d) of Theorem 2.2 be in force and $\Pi$ be given by (2.17). If, moreover, $q^{(n)}$ are symmetric ($q^{(n)}(x, y) \equiv q^{(n)}(y, x)$) and

$$\lim_{n \to \infty} \gamma_{n} \nu_{n}(1)\pi^{(n)}\{x \in E_{n}; q^{(n)}(x, x) \neq q^{(\infty)}, 2\} = 0$$

(2.18)

for some constant $q^{(\infty)}$, then the differential equation for $X_{1} = 1 - X_{0}$ takes a simpler form:

$$\dot{X}_{1} = w_{\infty}(bq^{(\infty)} - c)X_{1}(1 - X_{1}) + \mu_{\infty}(1)(1 - X_{1}) - \mu_{\infty}(0)X_{1}.$$  (2.19)

Corollary 2.3 applies to the large random $k$-regular graph for a fixed integer $k \geq 3$, with $q^{(\infty)} = 1/k$ and $\gamma_{n}/N_{n} \to (k - 1)/(2(k - 2))$ (see (6.19) and the discussion there). Additionally, $(\theta_{n})$ can be chosen to be any sequence such that $1 < \theta_{n} \ll N_{n}$, and $(w_{n})$ can be any such that $(w_{n}\theta_{n})$ converges in $[0, \infty)$. See [13] and Section 6.2. (More precisely, the application needs to pass limits along subsequences, since these graphs are randomly chosen.) Assume the absence of mutation. Then in this case, one can formally recover the replicator equation (2.19) from the drift term of the approximate Wright–Fisher diffusion in [32, SI] as follows. For the density process $p_{1}(\xi_{t})$ under $\mathbb{P}^{w_{n}}$, that drift term reads

$$w_{n} \cdot \frac{(k - 2)(b - ck)}{k(k - 1)}p_{1}(\xi_{t})[1 - p_{1}(\xi_{t})].$$

(2.20)

Note that $\gamma_{n} \approx N_{n}(k - 1)/(2(k - 2))$ as mentioned above and the choice in (2.11) of $w_{n}$ gives $w_{n} \approx w_{\infty}2\gamma_{n}N_{n}^{-1}/\theta_{n}$. By using these approximations and multiplying the foregoing drift term by $\theta_{n}$ as a time change, we get the approximate drift $w_{\infty}(b/k - c)p_{1}(\xi_{\theta_{n}t})[1 - p_{1}(\xi_{\theta_{n}t})]$ of $p_{1}(\xi_{\theta_{n}t})$. This approximation recovers (2.19). The same formal argument can be used to recover the noise coefficient in Theorem 2.2 (2°). See also [12, Remark 4.10] for the case of diffusion approximations.

### 3 Semimartingale dynamics

In this section, we define the voter model and the evolutionary game model as solutions to stochastic integral equations driven by point processes. Then we view these equations in terms of semimartingales and identify some leading order terms for the forthcoming perturbation argument. We recall the coalescing duality for the voter model briefly at the end of this section.
First, given a triplet \((E, q, \mu)\), an equivalent characterization of the corresponding voter model is given as follows. Introduce independent \((\mathcal{F}_t, \mathcal{P})\)-Poisson processes \(\{\Lambda(x, y); x, y \in E\}\) and \(\{\Lambda^x_t(x); \sigma \in S, x \in E\}\) such that

\[
\Lambda_t(x, y) \quad \text{with rate} \quad \mathbb{E}[\Lambda_1(x, y)] = q(x, y) \quad \text{and} \\
\Lambda^x_t(x) \quad \text{with rate} \quad \mathbb{E}[\Lambda^x_1(x)] = \mu(\sigma), \quad x, y \in E, \; \sigma \in S.
\]

(3.1)

These jump processes are defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Then given an initial condition \(\xi_0 \in S^E\), the \((E, q, \mu)\)-voter model can be defined as the pathwise unique \(S^E\)-valued solution of the following stochastic integral equations [17, 30]: for \(x \in E\) and \(\sigma \in S\),

\[
\mathbb{1}_\sigma \circ \xi_t(x) = \mathbb{1}_\sigma \circ \xi_0(x) + \sum_{y \in E} \int_0^t \mathbb{1}_\sigma \circ \xi_{s-}(y) - \mathbb{1}_\sigma \circ \xi_{s-}(x) d\Lambda_s(x, y) \\
+ \int_0^t \mathbb{1}_{S \setminus \{\sigma\}} \circ \xi_{s-}(x) d\Lambda^\sigma_s(x) - \sum_{\sigma' \in \mathcal{S} \setminus \{\sigma\}} \int_0^t \mathbb{1}_\sigma \circ \xi_{s-}(x) d\Lambda^\sigma_{s'}(x).
\]

(3.2)

Hence, the type at \(x\) is replaced and changed to the type at \(y\) when \(\Lambda(x, y)\) jumps, and the type seen at \(x\) is \(\sigma\) right after \(\Lambda^\sigma(x)\) jumps.

Recall that the rates of the evolutionary game are defined by (2.2). With the choice of \(\overline{w}\) from \((2.1), q^w(x, y, \xi) > 0\) if and only if \(q(x, y) > 0\). Hence, Girsanov’s theorem for point processes [25, Section III.3] can be applied to change the intensities of the Poisson processes \(\Lambda(x, y)\) to \(q^w(x, y, \xi)\) such that under a probability measure \(\mathbb{P}^w\) equivalent to \(\mathbb{P}\) on \(\mathcal{F}_t\) for all \(t \geq 0\),

\[
\hat{\Lambda}_t(x, y) \overset{\text{def}}{=} \Lambda_t(x, y) - \int_0^t q^w(x, y, \xi_s) ds \quad \& \quad \hat{\Lambda}^\sigma_t(x) \overset{\text{def}}{=} \Lambda^\sigma_t(x) - \mu(\sigma)t
\]

(3.3)

are \((\mathcal{F}_t, \mathbb{P}^w)\)-martingales. See [12, Section 2] for the explicit form of \(D^w\) when \(S = \{0, 1\}\). Since all of \(\hat{\Lambda}(x, y)\) and \(\hat{\Lambda}^\sigma(x)\) do not jump simultaneously under \(\mathbb{P}^w\) by the absolute continuity with respect to \(\mathbb{P}\), the product of any distinct two of them has a zero predictable quadratic variation [25, Theorem 4.2, Proposition 4.50, and Theorem 4.52 in Chapter I].

The point processes defined above now allows for straightforward representations of the dynamics of the density processes. By (3.2),

\[
p_\sigma(\xi_t) = p_\sigma(\xi_0) + \sum_{x, y \in E} \pi(x) \int_0^t \mathbb{1}_\sigma \circ \xi_{s-}(y) - \mathbb{1}_\sigma \circ \xi_{s-}(x) d\Lambda_s(x, y) \\
+ \sum_{x \in E} \pi(x) \int_0^t \mathbb{1}_{S \setminus \{\sigma\}} \circ \xi_{s-}(x) d\Lambda^\sigma_s(x) - \sum_{\sigma' \in \mathcal{S} \setminus \{\sigma\}} \sum_{x \in E} \pi(x) \int_0^t \mathbb{1}_\sigma \circ \xi_{s-}(x) d\Lambda^\sigma_{s'}(x).
\]

(3.4)

To obtain the limiting semimartingale for the density processes, we use the foregoing equation to derive the explicit semimartingale decompositions of the density processes.

To obtain these explicit decompositions, first, note that the dynamics of \(p_\sigma(\xi_t)\) under \(\mathbb{P}^w\) relies on various kinds of frequencies and densities as follows. For all \(x \in E, \xi \in S^E\) and \(\sigma, \sigma_1, \sigma_2 \in S\), we set

\[
f_\sigma(x, \xi) = \sum_{y \in E} q(x, y) \mathbb{1}_\sigma \circ \xi(y), \quad f_{\sigma_1 \sigma_2}(x, \xi) = \sum_{y \in E} q(x, y) \mathbb{1}_{\sigma_1}(y) \sum_{z \in E} q(y, z) \mathbb{1}_{\sigma_2} \circ \xi(z), \quad f_{\sigma_1}(x, \xi) = \sum_{y \in E} q(x, y) \sum_{z \in E} q(y, z) \mathbb{1}_{\sigma_1} \circ \xi(z), \quad f(x, \xi) = \sum_{x \in E} \pi(x) f(x, \xi), \quad \mathcal{T}(\xi) = \sum_{x \in E} \mathbb{1}_\sigma \circ \xi(x).
\]

(3.5)
To minimize the use of the summation notation, we also express these functions in terms of stationary
discrete-time $q$-Markov chains $\{U_\ell; \ell \in \mathbb{Z}_+\}$ and $\{U'\ell; \ell \in \mathbb{Z}_+\}$ with $U_0 = U'_0$ such that conditioned on $U_0$, the two chains are independent. Additionally, let $(U, U') \sim \pi \otimes \pi$ and $(V, V')$ be distributed as

$$
\mathbb{P}(V = x, V' = y) = \frac{\nu(x, y)}{\nu(1)}, \quad x, y \in E,
$$

for $\nu(x, y) = (x)^2 q(x, y)$ and $\nu(1) = \sum_{x,y} \nu(x, y) = \sum_x (x)^2$. (When $q$ is symmetric, $\nu(1)$ reduces to $N^{-1}$.) For example, $\prod_{\sigma} = \mathbb{E}[1_{\sigma} \circ \xi(U'_1)1_{\sigma} \circ \xi(U_1)1_{\sigma} \circ \xi(U_2)]$. We also set

$$
p_{\sigma \sigma'}(\xi) = \mathbb{E}[1_{\sigma} \circ \xi(V)1_{\sigma'} \circ \xi(V')].
$$

Second, we turn to algebraic identities that determine the leading order terms for the forthcoming
perturbation arguments. For $w \in [0, \overline{w}]$, the kernel $q^w$ defined by (2.2) can be expanded to the second
order in $w$ as follows:

$$
q^w(x, y, \xi) = q(x, y) 1 - w B(y, \xi) - w A(x, \xi) = q(x, y) + \sum_{i=1}^{\infty} w^i q(x, y)[A(x, \xi) - B(y, \xi)]A(x, \xi)^{i-1}
$$

$$
= q(x, y) + w q(x, y)[A(x, \xi) - B(y, \xi)] + w^2 q(x, y)R^w(x, y, \xi),
$$

where

$$
A(x, \xi) = 1 - \sum_{z \in E} q(z, z') \Pi(\xi(z), \xi(z')), \quad B(y, \xi) = 1 - \sum_{z \in E} q(y, z) \Pi(\xi(y), \xi(z)),
$$

and $R^w$ is uniform bounded in $w \in [0, \overline{w}], x, y, \xi, (E, q)$.

**Lemma 3.1.** For all $\xi \in S^E$ and $\sigma \in S$,

$$
\overline{D}_\sigma(\xi) \overset{\text{def}}{=} \sum_{x, y \in E} \pi(x)[1_{\sigma} \circ \xi(y) - 1_{\sigma} \circ \xi(x)]q(x, y)[A(x, \xi) - B(y, \xi)]
$$

$$
= \sum_{\sigma_0, \sigma_3 \in S, \sigma_0 \neq \sigma} \Pi(\sigma, \sigma_3)\overline{f}_{\sigma_0} \overline{f}_{\sigma_2} (\xi) - \sum_{\sigma_2, \sigma_3 \in S, \sigma_2 \neq \sigma} \Pi(\sigma_2, \sigma_3)\overline{f}_{\sigma} \overline{f}_{\sigma_2} (\xi). \tag{3.10}
$$

In particular, if $\Pi$ is given by (2.17), then

$$
\overline{D}_1(\xi) = b_{10} \overline{f}_{0}(\xi) - c_{10} \overline{f}_0(\xi). \tag{3.11}
$$

**Proof.** By using the reversibility of $q$ and taking $y$ in (3.9) as the state of $U_0$ in the sequence $\{U_\ell\}$
defined above, we can compute $\overline{D}_\sigma$ as

$$
\overline{D}_\sigma(\xi) = - \sum_{x, y \in E} \pi(x)1_{\sigma} \circ \xi(y)q(x, y) \sum_{z \in E, z' \in E} q(z, z') \Pi(\xi(z), \xi(z'))
$$

$$+
\sum_{x, y \in E} \pi(x)1_{\sigma} \circ \xi(y)q(x, y) \sum_{z \in E} q(y, z) \Pi(\xi(y), \xi(z))
$$

$$+
\sum_{x, y \in E} \pi(x)1_{\sigma} \circ \xi(x)q(x, y) \sum_{z \in E} q(y, z) \Pi(\xi(y), \xi(z))
$$

$$-
\sum_{x, y \in E} \pi(x)1_{\sigma} \circ \xi(x)q(x, y) \sum_{z \in E} q(z, z') \Pi(\xi(z), \xi(z')). \tag{3.12}
$$
\[ \begin{align*}
&= -E \left[ \mathbb{f}(U_0) \mathbb{I}(\xi(U_2)) \right] + E \left[ \mathbb{f}(U_2) \mathbb{I}(\xi(U_2)) \right] \\
&= -E \left[ \mathbb{f}(U_0) \mathbb{I}(\xi(U_2)) \right] - E \left[ \mathbb{f}(U_0) \mathbb{I}(\xi(U_2)) \mathbb{I}(\xi(U_3)) \right] + E \left[ \mathbb{f}(U_2) \mathbb{I}(\xi(U_2)) \mathbb{I}(\xi(U_3)) \right] \\
&= E \left[ \mathbb{f}(U_0) \mathbb{I}(\xi(U_2)) \mathbb{I}(\xi(U_3)) \right] - E \left[ \mathbb{f}(U_2) \mathbb{I}(\xi(U_2)) \mathbb{I}(\xi(U_3)) \right].
\end{align*} \]

Here, we use the reversibility of \( q \) with respect to \( \pi \) to cancel the last two terms in (3.12) and write the first term in (3.12) as the first term in (3.13). See [11, Lemma 1 on p.8] for the case of two types.

The proof of (3.11) appears in [12, Lemma 7.1]. Now (3.10) allows for a quick proof: \( \text{Tr}_1(\xi) = (b - c)I_{f_{11}} - cI_{f_{10}} - bI_{f_{1}} \). Then we use the identities \( I_{f_{11}} + I_{f_{10}} = I_{f_{11}} + I_{f_{01}} = I_{f_{1}} \), and \( I_{f_{01}} + I_{f_{01}} = I_{f_{1}} \). This calculation will be used in the proof of Corollary 2.3.

We are ready to state the explicit semimartingale decompositions of the density processes and identify the leading order terms. From (3.4), (3.8) and the martingales in (3.3), we obtain the following decompositions extended from (2.5):

\[ p_\sigma(\xi) = p_\sigma(\xi_0) + A_\sigma(t) + M_\sigma(t) = p_\sigma(\xi_0) + I_\sigma(t) + R_\sigma(t) + M_\sigma(t), \]

where

\[ I_\sigma(t) = w \int_0^t \mathbb{D}_\sigma(\xi_s) \, ds + \int_0^t \left( \mu(\sigma) \sum_{\sigma' \in S \setminus \{\sigma\}} p_{\sigma'}(\xi_s) - \mu(S \setminus \{\sigma\})p_\sigma(\xi_s) \right) \, ds, \]

\[ R_\sigma(t) = w^2 \sum_{x,y \in E} \pi(x) \int_0^t \left[ \mathbb{I}_\sigma \circ \xi_s(y) - \mathbb{I}_\sigma \circ \xi_s(x) \right] q(x,y) R_{\mu}^w(x,y,\xi_s) \, ds, \]

\[ M_\sigma(t) = \sum_{x,y \in E} \pi(x) \int_0^t \left[ \mathbb{I}_s \circ \xi_{s-}(y) - \mathbb{I}_s \circ \xi_{s-}(x) \right] d\hat{\Lambda}_s(x,y) \]
\[ + \sum_{x \in E} \pi(x) \int_0^t \mathbb{I}_{S \setminus \{\sigma\}}(x,\xi_s) d\hat{\Lambda}^w_s(x) - \sum_{\sigma' \in S \setminus \{\sigma\}} \sum_{x \in E} \pi(x) \int_0^t \mathbb{I}_\sigma \circ \xi_{s-}(x) d\hat{\Lambda}^w_{\sigma'}(x). \]

By (3.3), the predictable quadratic variations and covariances of \( M_\sigma \) and \( M_{\sigma'} \), for \( \sigma \neq \sigma' \), are

\[ \langle M_\sigma, M_\sigma \rangle_t \]
\[ = \sum_{x,y \in E} \pi(x)^2 \int_0^t \left\{ \mathbb{I}_\sigma \circ \xi_s(y) \left[ 1 - \mathbb{I}_\sigma \circ \xi_s(x) \right] + \left[ 1 - \mathbb{I}_\sigma \circ \xi_s(y) \right] \mathbb{I}_\sigma \circ \xi_s(x) \right\} q_{\mu}^w(x,y,\xi_s) \, ds, \]

\[ = -\sum_{x,y \in E} \pi(x)^2 \int_0^t \left[ \mathbb{I}_\sigma \circ \xi_s(y) \mathbb{I}_{\sigma'} \circ \xi_s(x) + \mathbb{I}_\sigma \circ \xi_s(y) \mathbb{I}_{\sigma'} \circ \xi_s(x) \right] q_{\mu}^w(x,y,\xi_s) \, ds \]
\[ - \sum_{x \in E} \pi(x)^2 \int_0^t \left[ \mathbb{I}_{S \setminus \{\sigma\}} \circ \xi_{s-}(x) \mathbb{I}_\sigma \circ \xi_{s-}(x) \mu(\sigma) + \mathbb{I}_{S \setminus \{\sigma'\}} \circ \xi_{s-}(x) \mathbb{I}_{\sigma'} \circ \xi_{s-}(x) \mu(\sigma') \right] \, ds. \]
In Section 5, the above equations play the central role in characterizing the limiting density processes. For this study, we apply the coalescing duality between \((E, q, \mu)\)-voter model and the coalescing rate-1 \(q\)-Markov chains \(\{B^x; x \in E\}\), where \(B^0 = x\). These chains move independently before meeting, and for any \(x, y \in E\), \(B^x = B^y\) after their first meeting time \(M_{x,y} = \inf\{t \geq 0; B^x_t = B^y_t\}\). In the absence of mutation, the duality is given by

\[
E \left[ \prod_{i=1}^{n} \mathbf{1}_{s_i} \circ \xi_0(B^x_i) \right] = E_{\xi_0} \left[ \prod_{i=1}^{n} \mathbf{1}_{s_i} \circ \xi_t(x_i) \right] \tag{3.20}
\]

for all \(\xi_0 \in S^E\), \(s_1, \ldots, s_n \in S\), distinct \(x_1, \ldots, x_n \in E\) and \(n \in \mathbb{N}\). See the proof of Proposition 6.1 for the foregoing identity and the extension to the case with mutations.

Without mutation, the density process is a martingale under the voter model by \((3.21)\) and \((3.22)\) that, for any \(\sigma \neq \sigma'\),

\[
E_{\xi}^{0}[p_\sigma(\xi_t)p_{\sigma'}(\xi_t)] = p_\sigma(\xi(p_{\sigma'}(\xi_s)) - \nu(1) \int_0^t E_{\xi}^{0}[p_{\sigma\sigma'}(\xi_s) + p_{\sigma'\sigma}(\xi_s)]ds. \tag{3.21}
\]

For the present problem, the central application of this dual relation is the foregoing identity \([9]\). Let the random variables defined below \((3.5)\) to represent frequencies and densities be independent of the coalescing Markov chains. Then the foregoing equality implies that

\[
P(M_{U,V'} > t) = 1 - \nu(1) - 2\nu(1) \int_0^t P(M_{V,V'} > s)ds, \quad \forall \ t \geq 0. \tag{3.22}
\]

See \([9, \text{Corollary 4.2}]\) and \([3, \text{Section 3.5.3}]\). This identity for meeting times has several important applications to the diffusion approximation of the voter model density processes. See \([9, \text{Sections 3 and 4}]\) and \([10]\).

## 4 Decorrelation in the ancestral lineage distributions

This section is devoted to a study of degenerate limits of meeting time distributions. Here, we consider meeting times defined on a sequence of spatial structures \((E_n, q^{(n)})\) as before. According to the coalescing duality, these distributions are part of the ancestral line distributions of the voter model, and by approximation, the ancestral line distributions of the evolutionary game. On the other hand, these meeting times encode the typical local geometry of the space, but in a rough manner. With the study of these distributions, the main results of this section (Propositions 4.2 and 4.4) extend to the choice of appropriate time scaling constants and the characterization of the limiting density processes. These properties are crucial to the forthcoming limit theorems.

Our direction in this section can be outlined in more detail as follows. Recall the auxiliary random variables defined below \((3.5)\), which are introduced to represent frequencies and densities. Under mild mixing conditions similar to those in \((2.9)\) with \(\gamma_n\) replaced by \(\theta_n\) and the condition \(\nu_n(1) \to 0\), the sequence \(P^{(n)}(M_{V,V'}/\gamma_n \in \cdot)\) is known to converge. The limiting distribution is a convex combination of the delta distribution at zero and an exponential distribution. Moreover, one can choose some \(s_n \to \infty\) such that \(s_n/\gamma_n \to 0\) and the following \(t\)-independent limit exists:

\[
\pi_0 \overset{\text{def}}{=} \lim_{n \to \infty} 2\gamma_n\nu_n(1)P^{(n)}(M_{V,V'} > s_n t), \quad \forall \ t \in (0, \infty) \tag{4.1}
\]

with \(\pi_0 = 1\). See \([9, \text{Corollary 4.2 and Proposition 4.3}]\) for these results. As an extension of this existence result, our first goal in this section is to introduce sufficient conditions for these sequences \((s_n)\). Specifically, we require that the limit \((4.1)\) exists with \(\pi_0 \in (0, \infty)\). See Section 4.1. The following is enough for the existence and the applications in the next section.
Definition 4.1. We say that \((s_n)\) is a slow sequence if
\[
\lim_{n \to \infty} s_n = \infty, \quad \lim_{n \to \infty} \frac{s_n}{n} = 0, \quad \lim_{n \to \infty} \gamma_n \nu_n(1)e^{-t s_n} = 0, \quad \forall \, t \in (0, \infty),
\]
and at least one of the two mixing conditions holds:
\[
\lim_{n \to \infty} \gamma_n \nu_n(1)e^{-s_n s_n} = 0 \quad \text{or} \quad \lim_{n \to \infty} \frac{t(n)}{s_n} \left[1 + \log^+(\gamma_n/t(n))\right] = 0.
\]

Our second goal is to extend the existence of the limit (4.1) to the existence of analogous time-independent limits for other meeting time distributions: for integers \(\ell \geq 1\) and \(\ell_0, \ell_1, \ell_2 \geq 0\) with \(\ell_0, \ell_1, \ell_2\) all distinct,
\[
\mathcal{R}_\ell \overset{\text{def}}{=} \lim_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(M_{U_{\ell_0}, U_{\ell_1}} > s_n t), \quad \forall \, t \in (0, \infty);
\]
\[
\mathcal{R}_{(\ell_0, \ell_1) \ell_2} \overset{\text{def}}{=} \lim_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(M_{U_{\ell_0}, U_{\ell_1}, U_{\ell_2}} > s_n t), \quad \forall \, t \in (0, \infty);
\]
\[
\mathcal{R}_{\ell_0 | \ell_1 | \ell_2} \overset{\text{def}}{=} \lim_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(M_{U_{\ell_0}, U_{\ell_1}, U_{\ell_2}} > s_n t), \quad \forall \, t \in (0, \infty).
\]
The extension to \(\mathcal{R}_1\) is straightforward if we allow passing limits along subsequences. Indeed, it follows from the definition of \(\{U_{\ell}\}\) and \((V, V')\) that
\[
\frac{\pi_{\text{min}}}{\pi_{\text{max}}} \mathbb{P}(M_{V, V'} \in \Gamma) \leq \mathbb{P}(M_{U_{\ell_0}, U_{\ell_1}} \in \Gamma) \leq \frac{\pi_{\text{max}}}{\pi_{\text{min}}} \mathbb{P}(M_{V, V'} \in \Gamma), \quad \forall \, \Gamma \in \mathcal{B}(\mathbb{R}_+).
\]
Hence, by taking a subsequence of \((E_n, g^{(n)})\) if necessary, (4.1) and condition (a) of Theorem 2.2 imply the existence of the limit \(\mathcal{R}_1\).

In Section 4.2, we prove the existence of the other limits \(\mathcal{R}_\ell, \, \ell \geq 2\). More precisely, we prove tightness results as in the case of \(\mathcal{R}_1\) so that the limits may be passed along subsequences. We also prove that the limits \(\mathcal{R}_\ell, \, \ell \geq 2\), are in \((0, \infty)\). Note that in proving these results, we do not impose convergence of local geometry as in the case of discrete tori or random regular graphs.

### 4.1 Mixing conditions for pairs of adjacent sites

To apply mixing conditions to meeting times, first, we recall some basic properties of the spectral gap and the mixing time for the product of the continuous-time \(q\)-Markov chains. Note that by coupling the product chain with initial condition \((x, y)\) after the two coordinates meet, we get the coalescing chain \((B^x, B^y)\) defined before (3.21).

Now, the discrete-time chain for the product chain has a transition matrix such that each of the coordinates is allowed to change with equal probability. Hence, the spectral gap is given by \(\bar{g} = g/2\) [27, Corollary 12.12 on p.161]. If \((\bar{q}_t)\) denotes the semigroup of the product chain, then
\[
\sup_{(x, y) \in E \times E} \left\| \bar{q}_t ((x, y), \cdot) - \pi \otimes \pi \right\|_{\text{TV}} \leq 2d_E(t),
\]
where \(d_E\) is the total variation distance defined by (2.6). Additionally, it follows from the definition the mixing time in (2.7) that
\[
d_E(kt_{\text{mix}}) \leq e^{-k}, \quad \forall \, k \in \mathbb{N}
\]
Taking Laplace transforms of both sides of the last equality, we get, for
\[ \tilde{t}_{\text{mix}} \leq 3t_{\text{mix}}. \] (4.10)

We are ready to prove the first main result of Section 4. Note that under the condition \(\sup_n N_{n}^{(n)} < \infty\) (see the discussion below (2.11)), the first condition in (4.3) implies the first one in (4.11).

**Proposition 4.2.** Suppose that \((s_n)\) satisfies (4.2) and at least one of the following mixing conditions:

\[ \lim_{n \to \infty} g_n s_n = \infty \quad \text{or} \quad \lim_{n \to \infty} \frac{t_{\text{mix}}^{(n)}}{s_n} \left[ 1 + \log^+ (\gamma_n / t_{\text{mix}}^{(n)}) \right] = 0. \] (4.11)

Then (4.1) holds with \(\pi_0 = 1\).

**Proof.** Write \(f_n(t) = \mathbb{P}^{(n)}(M_U > t)\) and \(g_n(t) = \mathbb{P}^{(n)}(M_V > t)\). The required result is proved in two steps.

**Step 1.** We start with a preliminary result: for all \(t_0 \in [0, \infty)\) and \(\mu \in (0, \infty)\),

\[ \lim_{n \to \infty} 2\gamma_n \nu_n(1) \int_0^\infty e^{-\mu t} g_n(s_n(t + t_0)) \, dt = \frac{1}{\mu}. \] (4.12)

To obtain (4.12), first, we derive a representation of the integrals in (4.12) by \(f_n(t)\). Note that (3.22) under the \(q^{(n)}\)-chain takes the following form:

\[ f_n(s_n t) = 1 - \nu_n(1) - 2\nu_n(1) s_n \int_0^t g_n(s_n s) \, ds, \quad t \geq 0. \]

Hence, for any fixed \(0 \leq t_0 < \infty\),

\[ f_n(s_n(t + t_0)) - f_n(s_n t_0) = -2\nu_n(1) s_n \int_0^t g_n(s_n(s + t_0)) \, ds, \quad t \geq 0. \] (4.13)

Taking Laplace transforms of both sides of the last equality, we get, for \(\mu > 0\),

\[
\int_0^\infty e^{-\mu t} \left[ f_n(s_n(t + t_0)) - f_n(s_n t_0) \right] \, dt = -2\nu_n(1) s_n \int_0^\infty e^{-\mu t} \int_0^t g_n(s_n(s + t_0)) \, ds \, dt \\
= -\frac{2\nu_n(1) s_n}{\mu} \int_0^\infty e^{-\mu t} g_n(s_n(t + t_0)) \, dt.
\]

where the last integral coincides with the integral in (4.12).

Next, rewrite the last equality as

\[
2\gamma_n \nu_n(1) \int_0^\infty e^{-\mu t} g_n(s_n(t + t_0)) \, dt \\
= -\frac{\gamma_n}{s_n} \int_0^\infty e^{-\mu t} \left[ f_n(s_n(t + t_0)) - f_n(s_n t_0) \right] \, dt \\
= -\frac{\gamma_n}{s_n} \int_0^\infty e^{-\mu t} \left[ f_n(s_n(t + t_0)) - f_n(s_n t_0) \right] \, dt + \frac{e^{-s_n t_0 / \gamma_n}}{\mu + s_n / \gamma_n}. \] (4.14)
The last term tends to $1/\mu$ since $s_n/\gamma_n \to 0$. To take the limit of the integral term in (4.14), we use the first mixing condition in (4.3). In this case, a bound for exponential approximations of the distributions of $M_{U,U'}$ [3, Proposition 3.23] gives

$$ \left| \frac{\gamma_n}{s_n} \int_0^\infty e^{-\mu t} \left\{ \left[ f_n(s_n(t + t_0)) - f_n(s_n t_0) \right] - [e^{-s_n(t+t_0)/\gamma_n} - e^{-s_n t_0/\gamma_n}] \right\} dt \right| \leq \frac{2}{g_{n}s_n} \to 0 \quad (4.15) $$

now that $g_n = g_n/2$. Alternatively, by a different bound from [1, Theorem 1.4], the foregoing inequality holds with the bound replaced by

$$ \frac{C_{4.16} \overline{t}_n}{s_n} \left[ 1 + \log^+ (\gamma_n/\overline{t}_n) \right] \leq \frac{C_{4.16} \cdot 3 \overline{t}_n}{s_n} \left[ 1 + \log^+ (\gamma_n/3 \overline{t}_n) \right] \quad (4.16) $$

by (4.10), the monotonicity of $x \mapsto x(1 + \log(x^{-1} \lor 1))$ on $(0, \infty)$, where $C_{4.16}$ is independent of the $q(n)$-chains. The last term in (4.16) tends to zero by the second mixing condition in (4.3).

Finally, we apply (4.15) and (4.16) to (4.14). Since the last term in (4.14) tends to $1/\mu$, we have proved (4.12).

**Step 2.** We are ready to prove the existence of the limit in (4.1) and its independence of $t$. First, note that since $g_n$ is decreasing, we have

$$ 2\gamma_n \nu_n(\mathbb{1}) e^{-\mu t} g_n(s_n(t + t_0)) \leq \frac{1}{t} 2\gamma_n \nu_n(\mathbb{1}) \int_0^t e^{-\mu s} g_n(s_n(s + t_0)) ds, \quad \forall t, t_0 \in (0, \infty), $$

whereas the last integral is bounded by the same integral with the upper limit $t$ of integration replaced by $\infty$. By the (4.14) and the convergence proven for it in the preceding step, the last inequality implies that $t \mapsto 2\gamma_n \nu_n(\mathbb{1}) e^{-\mu t} g_n(s_n t), \ n \geq 1,$ are uniformly bounded on $[a, \infty)$, for any $a \in (0, \infty)$. Hence, by Helly’s selection theorem, every subsequence of $\{ t \mapsto 2\gamma_n \nu_n(\mathbb{1}) g_n(s_n t) \}$ has a further subsequence, say indexed by $n_j$, such that for some left-continuous function $g_\infty$ on $(0, \infty)$,

$$ \lim_{j \to \infty} 2\gamma_n \nu_{n_j}(\mathbb{1}) g_{n_j}(s_{n_j} t) = g_\infty(t), \quad \forall t \in (0, \infty). \quad (4.17) $$

Moreover, this convergence holds boundedly on compact subsets of $(0, \infty)$ in $t$.

To find $g_\infty$, note that, as in (4.15) and (4.16), either of the mixing conditions (4.3) implies that for fixed $0 < t_1 < t_2 < \infty$,

$$ \frac{\gamma_n}{s_n} [f_n(s_n t_2) - f_n(s_n t_1)] = \frac{\gamma_n}{s_n} (e^{-s_n t_2/\gamma_n} - e^{-s_n t_1/\gamma_n}) + o(1) = -(t_2 - t_1) + o(1), \quad (4.18) $$

where $o(1)$’s refer to terms tending to $0$ as $n \to \infty$. By the foregoing equality and (4.13), we get

$$ t_2 - t_1 = \lim_{j \to \infty} \int_{t_1}^{t_2} 2\gamma_n \nu_{n_j}(\mathbb{1}) g_{n_j}(s_{n_j} t) dt = \int_{t_1}^{t_2} g_\infty(t) dt, \quad \forall 0 < t_1 < t_2 < \infty, $$

where the last equality follows from (4.17) and dominated convergence. The last equality and the left-continuity of $g_\infty$ give $g_\infty \equiv 1$ on $(0, \infty)$. We obtain (4.1) upon passing limit along $(n_j)$. By the choice of $(n_j)$, the convergence in (4.1) along the whole sequence holds. The proof is complete. 

Proposition 4.2 and the convergence in (6.19), extended to the convergence of the first moments, are enough to validate (4.1) and reinforce it to an explicit form on the large random regular graphs. In Section 6.2, we give an alternative proof of these properties of (4.1). In this case, the limit (4.1) holds only by passing subsequential limits. Nevertheless, the use of subsequences is due to the randomness of the graphs.
4.2 Extensions to triplets of sites and other pairs

We start with a basic recursion formula to relate tail distributions of the relevant meeting times \( M_{U_0,U_\ell}, \ell \geq 2 \), to the tail distribution of \( M_{U_0,U_1} \).

**Lemma 4.3.** For any integer \( \ell \geq 1 \), it holds that

\[
\mathbb{P}(M_{U_0,U_\ell} > t) = e^{-2t} \mathbb{P}(U_0 \neq U_\ell) + \int_0^t 2e^{-2(t-s)} \mathbb{P}(M_{U_0,U_{\ell+1}} > s)ds - \int_0^t 2e^{-2(t-s)} \sum_{x,y \in E} \pi(x)q^\ell(x,x)q(x,y)\mathbb{P}(M_{x,y} > s)ds, \quad t \in [0, \infty).
\] (4.19)

**Proof.** Since \( M_{x,x} \equiv 0 \) and \((U_0,U_\ell)\) is independent of the meeting times, conditioning on \((U_0,U_\ell)\) gives \( \mathbb{P}(M_{U_0,U_\ell} > t) = \mathbb{P}(M_{U_0,U_\ell} > t, U_0 \neq U_\ell) \). Conditioning on the first update time of \((B^{U_0},B^{U_\ell})\), which is an exponential variable with mean 1/2, yields

\[
\mathbb{P}(M_{U_0,U_\ell} > t) = e^{-2t} \mathbb{P}(U_0 \neq U_\ell) + \int_0^t 2e^{-2(t-s)} \mathbb{P}(U_0 \neq U_\ell, M_{U_0,U_{\ell+1}} > s)ds. \] (4.20)

Here, the initial condition \((U_0,U_{\ell+1})\) in the last term follows from transferring the first transition of state of \((B^{U_0},B^{U_\ell})\) to the initial condition. We also use the stationarity of \( \{U_\ell; \ell \geq 0\} \) when that first transition is made by \( B^{U_0} \). To rewrite the integral term in (4.20), note that

\[
\mathbb{P}(U_0 \neq U_\ell, U_0 = x, U_{\ell+1} = y) = \mathbb{P}(U_0 = x, U_{\ell+1} = y) - \mathbb{P}(U_0 = U_\ell, U_0 = x, U_{\ell+1} = y) = \pi(x)q^{\ell+1}(x,y) - \pi(x)q^\ell(x,x)q(x,y)
\]

so that

\[
\mathbb{P}(U_0 \neq U_\ell, M_{U_0,U_{\ell+1}} > s) = \mathbb{P}(M_{U_0,U_{\ell+1}} > s) - \sum_{x,y \in E} \pi(x)q^\ell(x,x)q(x,y)\mathbb{P}(M_{x,y} > s).
\] (4.21)

Applying (4.21) to (4.20) yields (4.19). \(\blacksquare\)

We are ready to prove the existence of the limits in (4.4) and (4.5).

**Proposition 4.4.** For any sequence \( (s_n) \) satisfying (4.2), we have the following properties:

1° For any integer \( \ell \geq 2 \), every subsequence of \( (E_n,q^{(n)}) \) contains a further subsequence such that the limit in (4.4) exists in \([\overline{\rho}_1, \ell \overline{\rho}_1]\) and is independent of \( t \in (0, \infty) \).

2° Without taking any subsequence, (4.4) holds for \( \ell = 2 \) with \( \overline{\rho}_2 = \overline{\rho}_1 \).

3° Suppose that (2.18) holds for some constant \( q^{(\infty),2} \). Then without taking any subsequence, (4.4) holds \( \overline{\rho}_3 = (1 + q^{(\infty),2})\overline{\rho}_1 \).

4° For all distinct nonnegative integers \( \ell_0, \ell_1, \ell_2 \), it holds that

\[
\overline{\rho}_{(\ell_1,\ell_2)|\ell_0} + \overline{\rho}_{(\ell_0,\ell_1)|\ell_2} - \overline{\rho}_{(\ell_0,\ell_1)|\ell_2} = \overline{\rho}_{(\ell_2-\ell_0)},
\]

provided that all of the limits defining these constants exist.

**Proof.** (1°) To lighten notation in the rest of this proof but only in this proof, write \( A_\ell = \mathbb{P}(U_0 \neq U_\ell) \), \( J_\ell \) for \( M_{U_0,U_\ell} \),

\[
B_\ell = \sum_{x,y \in E} \pi(x)q^\ell(x,x)q(x,y),
\]

17
and $K_\ell$ for the first meeting time for the pair of coalescing Markov chains where the initial condition is distributed independently as $B^{-1}_\ell \pi(x)q^\ell(x,x)q(x,y)$ provided that $B_\ell \neq 0$. We set $K_\ell$ to be an arbitrary random variable.

Fix an integer $\ell \geq 1$. If $e$ is an independent exponential variable with mean 1, then (4.19) can be written as

$$
\mathbb{P}(J_\ell > t) = A_\ell \mathbb{P}(\frac{1}{2}e > t) + \mathbb{P}(J_{\ell+1} + \frac{1}{2}e > t, \frac{1}{2}e \leq t) - B_\ell \mathbb{P}(K_\ell + \frac{1}{2}e > t, \frac{1}{2}e \leq t).
$$

After rearrangement, the foregoing equality yields

$$
\mathbb{P}(J_{\ell+1} + \frac{1}{2}e > t) = \mathbb{P}(J_\ell > t) + B_\ell \mathbb{P}(K_\ell + \frac{1}{2}e > t) + (1 - A_\ell - B_\ell)\mathbb{P}(\frac{1}{2}e > t).
$$

Hence, for all left-open intervals $\Gamma \subset (0, \infty)$,

$$
\mathbb{P}(J_{\ell+1} + \frac{1}{2}e \in \Gamma) = (A_\ell + B_\ell)\mathbb{P}(\frac{1}{2}e \in \Gamma) = \mathbb{P}(J_\ell \in \Gamma) + B_\ell \mathbb{P}(K_\ell + \frac{1}{2}e \in \Gamma) + \mathbb{P}(\frac{1}{2}e \in \Gamma). 
$$

(4.22)

Since $q^\ell(x, x) \leq 1$, we have $B_\ell \mathbb{P}(K_\ell \in \cdot) \leq \mathbb{P}(J_1 \in \cdot)$, and so, the foregoing identity gives

$$
\mathbb{P}(J_{\ell+1} + \frac{1}{2}e \in \Gamma) + (A_\ell + B_\ell)\mathbb{P}(\frac{1}{2}e \in \Gamma) \leq \mathbb{P}(J_\ell \in \Gamma) + \mathbb{P}(J_1 + \frac{1}{2}e \in \Gamma) + \mathbb{P}(\frac{1}{2}e \in \Gamma). 
$$

(4.23)

We are ready to prove the required result. For any $0 < T_0 < T_1 < \infty$, repeated applications of the first and third limits in (4.2) for all of the next three equalities give

$$
\limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} \in (s_nT_0, s_nT_1]) 
\leq \limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} + \frac{1}{2}e \in (s_nT_0, s_n2T_1]) 
\leq \limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_\ell \in (s_nT_0, s_n2T_1]) 
+ \limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_1 + \frac{1}{2}e \in (s_nT_0, s_n2T_1]) 
\leq \limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_\ell \in (s_nT_0, s_n2T_1]) 
+ \limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_1 \in (s_n2^{-\ell}T_0, s_n2^{2}\ell T_1]) 
\leq (\ell + 1)\limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_1 \in (s_n2^{-\ell}T_0, s_n2^{2}\ell T_1]) = 0,
$$

(4.24)

(4.25)

where (4.24) also uses (4.23), the last inequality follows from induction, and the equality in (4.25) follows from (4.4) with $\ell = 1$. Moreover, by setting $T_0 = t$ and $T_1 = \infty$, a similar argument as in the display for (4.25) shows that (4.4) with $\ell = 1$ gives

$$
\limsup_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} > s_na) \leq (\ell + 1)\overline{\pi}_1, \quad \forall t \in (0, \infty).
$$

(4.26)

On the other hand, since $\mathbb{P}(J_{\ell+1} + \frac{1}{2}e \in \cdot) + |A_\ell + B_\ell - 1|\mathbb{P}(\frac{1}{2}e \in \cdot) \geq \mathbb{P}(J_\ell \in \cdot)$ by (4.22), it follows from (4.4) with $\ell = 1$ and an argument similar to the one leading to (4.25) that

$$
\liminf_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} > s_nT_1) \geq \overline{\pi}_1 > 0, \quad \forall t \in (0, \infty).
$$

(4.27)

Combining (4.26) and (4.27), we deduce that for fixed $t_0 \in (0, \infty)$, any subsequence of the numbers $2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} > s_nT_0)$ has a further subsequence that converges in $[\overline{\pi}_1, (\ell + 1)\overline{\pi}_1]$. By (4.25), this limit extends to the existence of the limit of the corresponding subsequence of $2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(J_{\ell+1} > s_nT)$ for any $t \in (0, \infty)$, and all of these limits for different $t$ are equal. We have proved (4.4).
(2°) Note that \( B_1^{(n)} = 0 \) since \( \text{tr}(q^{(n)}) = 0 \) by assumption. Then an inspection of (4.24) shows the second limit superior on the right-hand side there can be dropped. The rest of the argument in (2°), especially (4.26) and (4.27), can be adapted accordingly to get the required identity.

(3°) The proof is done again by improving the argument for (4.26) and (4.27), but now using (4.22) with \( \ell = 2 \). In doing so, we also use the following implication of (2.18):

\[
\lim_{n \to \infty} \sup_{s \geq 0} \gamma_n \nu_n(1)|B_2^{(n)}\mathbb{P}(K_2 > s) - q^{(\infty)}(1)\mathbb{P}(J_1 > s)| = 0,
\]

which follows since the distributions of \( K_2 \) and \( J_1 \) differ by the initial conditions.

(4°) By the definitions in (4.4)–(4.6), we have

\[
\begin{align*}
(\mathbb{R}(t_1,t_2)_{t_0} \mathbb{R}(t_0,t_1)_{t_2} + (\mathbb{R}(t_0,t_1)_{t_2} - \mathbb{R}(t_0,t_1)_{t_2}) + \mathbb{R}(t_0,t_1)_{t_2}) &= \lim_{n \to \infty} 2^n \nu_n(1)\mathbb{P}(M_{U_{t_0}U_{t_1}} > s_n t, M_{U_{t_1}U_{t_2}} < s_n t, M_{U_{t_0}U_{t_2}} > s_n t) \\
&+ \lim_{n \to \infty} 2^n \nu_n(1)\mathbb{P}(M_{U_{t_0}U_{t_1}} < s_n t, M_{U_{t_1}U_{t_2}} > s_n t, M_{U_{t_0}U_{t_2}} > s_n t) \\
&+ \lim_{n \to \infty} 2^n \nu_n(1)\mathbb{P}(M_{U_{t_0}U_{t_1}} > s_n t, M_{U_{t_1}U_{t_2}} > s_n t, M_{U_{t_0}U_{t_2}} > s_n t) \\
&= \lim_{n \to \infty} 2^n \nu_n(1)\mathbb{P}(M_{U_{t_0}U_{t_2}} > s_n t) = \mathbb{R}(t_2 - t_0).
\end{align*}
\]

Here, the next to the last equality follows since on \( \{M_{U_{t_0}U_{t_2}} > s_n t\} \), we cannot have both \( M_{U_{t_0}U_{t_1}} \leq s_n t \) and \( M_{U_{t_1}U_{t_2}} \leq s_n t \) by the coalescence of the Markov chains, and the last equality follows from the stationarity of the chain \( \{U_t\} \). The proof is complete. \( \blacksquare \)

We close this subsection with another application of Lemma 4.3. It will be used in Section 5.

**Proposition 4.5.** Let \( s_0 \in (2, \infty) \). For all integers \( \ell \geq 1 \) and all \( t \in (0, \infty) \), it holds that

\[
\int_0^t \mathbb{P}(M_{U_{t_0}U_{t}} > s_0 s)ds \leq \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2^{k+1}t})^{-1} \int_0^{2^{j}t} \mathbb{P}(M_{U_{t_0}U_{1}} > s_0 s)ds,
\]

where \( \prod_{k=1}^{j} a_k \equiv 1 \) for \( j < i \).

**Proof.** We prove (4.28) by an induction on \( \ell \geq 1 \). The inequality is obvious for all \( t \in (0, \infty) \) if \( \ell = 1 \). Suppose that for some \( \ell \geq 1 \), (4.28) holds for all \( t \in (0, \infty) \). By (4.19) with \( t \) replaced by \( s_0 r \),

\[
\begin{align*}
&\int_0^r 2s_0 e^{-2s_0 (r-s)} \mathbb{P}(M_{U_{t_0}U_{t_1}} > s_0 s)ds \\
&\leq \mathbb{P}(M_{U_{t_0}U_{t}} > s_0 r) + \int_0^r 2s_0 e^{-2s_0 (r-s)} \mathbb{P}(M_{U_{t_0}U_{1}} > s_0 s)ds.
\end{align*}
\]

The assumption \( s_0 \in (2, \infty) \) gives \( t \leq 2t(1 - s_0^{-1}) \), and so, for \( h \) nonnegative and Borel measurable,

\[
(1 - e^{-4t}) \int_0^t h(s_0 s)ds \leq \int_0^{2t(1 - s_0^{-1})} h(s_0 s)(1 - e^{-4t})ds \\
\leq \int_0^{2t} h(s_0 s)(1 - e^{-2s_0 (2t-s)})ds = \int_0^{2t} \int_0^r 2s_0 e^{-2s_0 (r-s)} h(s_0 s)ds dr \leq \int_0^{2t} h(s_0 s)ds.
\]

19
Integrating both sides of (4.29) over \([0, 2t]\) and applying the first and last inequalities in (4.30) give

\[
(1 - e^{-4t}) \int_0^t \mathbb{P}(M_{U_0,U_{t+1}} > s_0 s) ds \leq \int_0^{2t} \mathbb{P}(M_{U_0,U_{t}} > s_0 s) ds + \int_0^{2t} \mathbb{P}(M_{U_0,U_{1}} > s_0 s) ds
\]

\[
\leq \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2^{k+2}t})^{-1} \int_0^{2^{j+1}t} \mathbb{P}(M_{U_0,U_{1}} > s_0 s) ds + \int_0^{2^\ell} \mathbb{P}(M_{U_0,U_{1}} > s_0 s) ds
\]

\[
\leq \sum_{j=2}^{\ell+1} \prod_{k=2}^{j-1} (1 - e^{-2^{k+1}t})^{-1} \int_0^{2^{j}t} \mathbb{P}(M_{U_0,U_{1}} > s_0 s) ds + \int_0^{2^{\ell+1}t} \mathbb{P}(M_{U_0,U_{1}} > s_0 s) ds,
\]

(4.31)

where the second inequality follows from induction. Dividing both sides of (4.31) by \((1 - e^{-4t})\) proves (4.28) for \(\ell\) replaced by \(\ell + 1\). Hence, (4.28) holds for all \(\ell \geq 1\) by induction.

\[\Box\]

5 Convergence of the vector density processes

We present the proofs of Theorem 2.2 and Corollary 2.3 in this section. The key result is Proposition 5.3 where we reduce the evolutionary game model to the voter model. Throughout this section, conditions (a)–(d) of Theorem 2.2 are in force.

The other settings for this section are as follows. First, we write \(I^{(n)}_{\sigma}(t) = I_{\sigma}(\theta_{n,t})\) for the process \(I_{\sigma}(t)\) defined by (3.15), when the underlying particle system is based on \((E_{\sigma}, q^{(n)})\). This notation extends to the other processes in the decompositions (3.14) by using the same time change. Next, recall that \(S\) denotes the type space. We will mostly consider \((\sigma_0, \sigma_2, \sigma_3) \in S \times S \times S\) such that \(\sigma_0 \neq \sigma_2\). These triplets fit into the context of (3.10), from which we will prove the limiting replicator equation in Theorem 2.2. Additionally, given an admissible sequence \((\theta_{n, \mu_n, w_n})\) such that \(\lim_n \theta_{n, \gamma_n} = 0\), we can choose a slow sequence \((s_n)\) (recall Definition 4.1) such that

\[
\lim_{n \to \infty} \frac{s_n}{\theta_n} = 0.
\]

(5.1)

5.1 Asymptotic closure of equation and path regularity

We begin by showing that the leading order drift term \(I^{(n)}_{\sigma}(t)\) in (3.14) can be asymptotically closed by the vector density process \((p_{\sigma}(\xi_{\theta_{n,t}}); \sigma \in S)\). By (3.15), this term takes the following explicit form:

\[
I^{(n)}_{\sigma}(t) = \int_0^t \mathcal{D}_{\sigma}(\xi_{\theta_{n,s}}) ds + \int_0^t \left( \theta_{n, \mu_n}(\sigma)[1 - p_{\sigma}(\xi_{\theta_{n,s}})] + \theta_{n, \mu_n}(S \setminus \{\sigma\})p_{\sigma}(\xi_{\theta_{n,s}}) \right) ds.
\]

(5.2)

Specifically, in terms of the explicit form of \(\mathcal{D}_{\sigma}\) in (3.10), our goal is to prove that

\[
\lim_{n \to \infty} \sup_{\xi \in S \cap \mathbb{R}^n} \mathbb{E}_{\xi}^{\mu_n} \left[ \int_0^t w_n \theta_{n, \int_{\sigma_0} f_{\sigma_0, \sigma_3}(\xi_{\theta_{n,s}})} - w_\infty Q_{\sigma_0, \sigma_2, \sigma_3}(p(\xi_{\theta_{n,s}})) ds \right] = 0, \quad \forall t \in (0, \infty),
\]

(5.3)

where \(\sigma_0 \neq \sigma_2\), \(w_\infty\) is defined by (2.11), and \(Q_{\sigma_0, \sigma_2, \sigma_3}(X)\) is a polynomial in \(X = (X_{\sigma})_{\sigma \in S}\) defined by

\[
Q_{\sigma_0, \sigma_2, \sigma_3}(X) \overset{\text{def}}{=} \mathbb{1}_{\{\sigma_2 = \sigma_1\}} (\bar{\tau}_{(2,3)}[0 - \bar{\tau}_{0,2}] X_{\sigma_0} X_{\sigma_2} + \bar{\tau}_{0,2} X_{\sigma_0} X_{\sigma_2}) + \mathbb{1}_{\{\sigma_0 = \sigma_3\}} (\bar{\tau}_{(0,3)}[2 - \bar{\tau}_{0,2}] X_{\sigma_0} X_{\sigma_2})
\]

(5.4)

\[
+ \bar{\tau}_{0,2} X_{\sigma_0} X_{\sigma_2} X_{\sigma_3}.
\]
The choice of $Q_{\sigma_0,\sigma_2,\sigma_3}$ is due to the proof of Lemma 5.4.

The proof of (5.3) begins with an inequality central to the proof of [9, Theorem 2.2], which goes back to [19] and is also central to the proof of [10, Lemma 4.2]. This inequality is presented in a general form for future references. In what follows, we write $a \wedge b$ for $\min\{a, b\}$.

**Proposition 5.1.** Given a Polish space $E_0$ and $T \in (0, \infty)$, let $(X_t)_{0 \leq t \leq T}$ be an $E_0$-valued Markov process with càdlàg paths. Let $f$ and $g$ be bounded Borel measurable functions defined on $E_0$. Suppose that $x \mapsto E_x[f(X_t)]$ is Borel measurable, and for some bounded decreasing function $a(t)$,

$$\sup_{x \in E_0} E_x[|f(X_t)|] \leq a(t), \quad \forall t \in [0, T]. \quad (5.5)$$

Then for all $0 < 2\delta < t \leq T$,

$$\sup_{x \in E_0} E_x\left[ \left| \int_0^t (f(X_s) - g(X_s)) \, ds \right| \right] \leq \int_0^\delta a(s) \, ds + \left( 8ta(\delta) \int_0^\delta a(s) \, ds \right)^{1/2} + 3\delta \|g\|_\infty + t \sup_{x \in E_0} |E_x[f(X_\delta)] - g(x)|. \quad (5.6)$$

**Proof.** For $s \geq \delta$, define $H(s) = f(X_s) - E_{X_{s-\delta}}[f(X_\delta)]$. Then

$$E_x\left[ \left| \int_0^t (f(X_s) - g(X_s)) \, ds \right| \right] \leq \int_0^\delta (E_x[|f(X_s)|] + \|g\|_\infty) \, ds + E_x\left[ \left( \int_\delta^t H(s) \, ds \right)^2 \right]^{1/2}$$

$$+ E_x\left[ \int_\delta^t |E_{X_{s-\delta}}[f(X_\delta)] - g(X_{s-\delta})| \, ds \right] + E \left[ \int_\delta^t (g(X_{s-\delta}) - g(X_s)) \, ds \right]$$

$$\leq \int_0^\delta a(s) \, ds + E_x\left[ \left( \int_\delta^t H(s) \, ds \right)^2 \right]^{1/2} + 3\delta \|g\|_\infty + t \sup_{x \in E_0} |E_x[f(X_\delta)] - g(x)|. \quad (5.7)$$

Note that in the last inequality, $2\delta \|g\|_\infty$ is contributed by the integral $\int_\delta^t (g(X_{s-\delta}) - g(X_s)) \, ds$.

To bound the second term in (5.7), we note that for $\delta \leq r < s - \delta$,

$$E_x[H(s)H(r)] = E_x[f(X_s)] \left( f(X_r) - E_{X_{r-\delta}}[f(X_\delta)] \right)$$

$$- E_x\left[ E_{X_{s-\delta}}[f(X_\delta)] \left( f(X_r) - E_{X_{r-\delta}}[f(X_\delta)] \right) \right] = 0,$$

where the last equality follows by applying the Markov property at time $s - \delta$ to the first expectation on the right-hand side of the first equality. Hence,

$$E_x\left[ \left( \int_\delta^t H(s) \, ds \right)^2 \right] = 2 \int_\delta^t \int_r^{t \wedge (r + \delta)} E_x[H(r)H(s)] \, ds \, dr, \quad (5.8)$$

whereas for $r \leq s \leq r + \delta$,

$$E_x[H(r)H(s)] = E_x[f(X_r)f(X_s)] - E_x[f(X_r)E_{X_{s-\delta}}[f(X_\delta)]]$$

$$- E_x\left[ E_{X_{s-\delta}}[f(X_\delta)]f(X_s) \right] + E_x\left[ E_{X_{r-\delta}}[f(X_\delta)]E_{X_{s-\delta}}[f(X_\delta)] \right]$$

$$- E_x\left[ E_{X_{s-\delta}}[f(X_\delta)]f(X_s) \right] + E_x\left[ E_{X_{r-\delta}}[f(X_\delta)]E_{X_{s-\delta}}[f(X_\delta)] \right]$$

$$= E_x[f(X_r)f(X_s)] - E_x[f(X_r)E_{X_{s-\delta}}[f(X_\delta)]]$$

$$- E_x\left[ E_{X_{s-\delta}}[f(X_\delta)]f(X_s) \right] + E_x\left[ E_{X_{r-\delta}}[f(X_\delta)]E_{X_{s-\delta}}[f(X_\delta)] \right].$$
by (5.5) and the Markov property. Since $a$ is decreasing, integrating the terms in the last line yields

$$2 \int_{\delta}^{t} \int_{r}^{t\wedge(r+\delta)} (a(r)a(s-r) + a(r)a(\delta) + a(\delta)a(s) + a(\delta)^2) \, ds \, dr$$

$$\leq 2ta(\delta) \int_{0}^{\delta} a(s) \, ds + 2a(\delta)^2 t\delta + 2ta(\delta) \int_{0}^{\delta} a(s) \, ds + 2a(\delta)^2 t\delta \leq 8ta(\delta) \int_{0}^{\delta} a(s) \, ds.$$  \hspace{1cm} (5.10)

Applying (5.8)–(5.10) to (5.7), we get (5.6).

To prove (5.3), we apply Proposition 5.1 with the following choice:

\[\begin{align*}
X_t &= \xi_{\theta_n t} \text{ under } \mathbb{P}_{\xi}, \\
\delta &= \delta_n = 2s_n/\theta_n, \\
f &= f_n = w_n\theta_n \sigma_0 \sigma_2 \sigma_3, \\
g &= g_n = w_0 Q_{\sigma_0, \sigma_2 \sigma_3} \circ p.
\end{align*}\] \hspace{1cm} (5.11)

The next two results are used to identify the appropriate $a(t) = a_n(t)$ that satisfies (5.5). We recall for the last time that conditions (a)–(d) of Theorem 2.2 are in force throughout Section 5.

**Lemma 5.2.** Let $s_0 \in (2, \infty)$. Then for any $t \in (0, \infty)$ and integer $\ell \geq 1$,

$$2\gamma_n \nu_n(1) \int_{0}^{t} \mathbb{P}((M_{U_0,U_\ell} > s_0 s) \, ds \leq C_{5.12} \left( \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2k+1}t) \right) \left( \frac{\pi_{\max}^{(n)}}{\pi_{\min}^{(n)}} \right)$$

$$\times \left( \ell t + \min \left\{ \frac{1}{g_{n}s_0}, \frac{t_{\text{mix}}^{(n)}}{s_0} [1 + \log^+(\gamma_n/t_{\text{mix}}^{(n)})] \right\} \right),$$

\hspace{1cm} (5.12)

where $C_{5.12}$ is a universal constant.

**Proof.** By (4.7) and Proposition 4.5, we obtain the following inequality:

$$2\gamma_n \nu_n(1) \int_{0}^{t} \mathbb{P}((M_{U_0,U_\ell} > s_0 s) \, ds$$

$$\leq \frac{\gamma_n}{s_0} \cdot \left( \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2k+1}t) \right) \left( \frac{\pi_{\max}^{(n)}}{\pi_{\min}^{(n)}} \right) 2s_0 \nu_n(1) \int_{0}^{\ell t} \mathbb{P}((M_{U,U'} > s_0 s) \, ds$$

$$= \frac{\gamma_n}{s_0} \cdot \left( \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2k+1}t) \right) \left( \frac{\pi_{\max}^{(n)}}{\pi_{\min}^{(n)}} \right) \left[ \mathbb{P}((M_{U,U'} > 0) - \mathbb{P}((M_{U,U'} > 2t_{s_0} s_0 t) \right]$$

$$\leq C_{5.14} \frac{\gamma_n}{s_0} \cdot \left( \sum_{j=1}^{\ell} \prod_{k=1}^{j-1} (1 - e^{-2k+1}t) \right) \left( \frac{\pi_{\max}^{(n)}}{\pi_{\min}^{(n)}} \right)$$

$$\times \left( 1 - e^{-2s_0 t/\gamma_n} \right) + \min \left\{ \frac{2}{g_n \gamma_n}, \frac{t_{\text{mix}}^{(n)}}{\gamma_n} [1 + \log^+(\gamma_n/t_{\text{mix}}^{(n)})] \right\} \right)$$

\hspace{1cm} (5.14)

for a universal constant $C_{5.14}$. Here, (5.13) follows from (3.22), and (5.14) follows from the exponential approximation of $M_{U,U'}$ as in the proof of Proposition 4.2. Recall the reduction of mixing of products.
chains to mixing of the coordinates as used in that proposition, and the inequality $1 - e^{-x} \leq x$ holds for all $x \geq 0$. Hence, we obtain (5.12) from (5.14). The proof is complete. ■

**Proposition 5.3.** Fix $(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in S \times S \times S$ such that $\sigma_0 \neq \sigma_2$ and $\sigma_0 \neq \sigma_1$.

(1°) For any $w \in [0, \overline{w}]$ and $t \in (0, \infty)$, the following estimates of the evolutionary game by the voter model holds: for some constant $C_{5.15}$ depending only on $\Pi$,

$$
\sup_{\xi \in S^E} \left| E^w_{\xi} \left[ \mathcal{J}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t) \right] - E^0_{\xi} \left[ \mathcal{J}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t) \right] \right| 
\leq C_{5.15} w \int_0^t \mathbb{P}(M_{U_0, U_2} > r)ds + C_{5.15} w \mu(\mathbb{1}) \int_0^t \int_0^s \mathbb{P}(M_{U_0, U_2} > r)drds; \quad (5.15)
$$

$$
\sup_{\xi \in S^E} \left| E^w_{\xi} \left[ \mathcal{J}_{\sigma_0 \sigma_1}(\xi_t) \right] - E^0_{\xi} \left[ \mathcal{J}_{\sigma_0 \sigma_1}(\xi_t) \right] \right| 
\leq C_{5.15} w \int_0^t \mathbb{P}(M_{U_0, U_1} > s)ds + C_{5.15} w \mu(\mathbb{1}) \int_0^t \int_0^s \mathbb{P}(M_{U_0, U_1} > r)drds. \quad (5.16)
$$

(2°) For any admissible sequence $(\theta_n, \mu_n, w_n)$, it holds that

$$
\lim_{n \to \infty} \int_0^T \sup_{\xi \in S^E_n} \left| E^w_{\xi} \left[ w_n \theta_n f_{\sigma_0 \sigma_2 \sigma_3}(\xi_{2st_n}) \right] - E^0_{\xi} \left[ w_n \theta_n f_{\sigma_0 \sigma_2 \sigma_3}(\xi_{2st_n}) \right] \right| dt = 0, \quad \forall \ T \in (0, \infty). \quad (5.17)
$$

**Proof.** (1°) Recall that the generator of $L^w$ of the evolutionary game is given by (2.3), and $L = L^0$ denotes the generator of the voter model. By Duhamel’s principle [22, (2.15) in Chapter 1],

$$
e^{tL}H = e^{tL}H + \int_0^t e^{(t-s)L} (L^w - L)e^{sL}Hds. \quad (5.18)
$$

Here, it follows from (2.3) that

$$(L^w - L)H_1(\xi) = \sum_{x,y \in E} \left[ q^w(x, y, \xi) - q(x, y) \right] [H_1(\xi^{x,y}) - H_1(\xi)]. \quad (5.19)
$$

To apply (5.18) and (5.19), we choose $H = f_{\sigma_0 \sigma_2 \sigma_3}$ and $H_1 = e^{sL}f_{\sigma_0 \sigma_2 \sigma_3}$. The following bound will be proved in Proposition 6.1 (2°):

$$
\sup_{\xi \in S^E} \left| e^{sL}f_{\sigma_0 \sigma_2 \sigma_3}(\xi) - e^{sL}f_{\sigma_0 \sigma_2 \sigma_3}(\xi) \right|
\leq \sum_{\ell \in \{0,2,3\}} 4\mathbb{P}(M_{U_0, U_2} > s, B_{U_s}^{U_1} = x) + \sum_{\ell \in \{0,2,3\}} 4\mu(\mathbb{1}) \int_0^s \mathbb{P}(M_{U_0, U_2} > r, B_{U_s}^{U_1} = x)dr. \quad (5.20)
$$

To bound $(L^w - L)e^{sL}H = (L^w - L)H_1$ in the expansion (5.18), notice that

$$
|q^w(x, y, \xi) - q(x, y)| \leq C_{5.21} wq(x, y) \quad (5.21)
$$

by (3.8) for some $C_{5.21}$ depending only on $\Pi$. Putting (5.19), (5.20) and (5.21) together, we get

$$
\sup_{\xi \in S^E} |(L^w - L)e^{sL}f_{\sigma_0 \sigma_2 \sigma_3}(\xi)| \leq C_{5.21} 4w \sum_{\ell \in \{0,2,3\}} \sum_{x,y \in E} q(x, y)\mathbb{P}(M_{U_0, U_2} > s, B_{U_s}^{U_1} = x)
$$

23
\[ + C_{5.21} 4w \mu(1) \sum_{t \in \{0, 2, 3\}} \sum_{x,y \in E} q(x, y) \int_0^s \mathbb{P}(M_{U_0, U_2} > r, B_{s_t}^U = x) \, dr \]
\[ \leq C_{5.21} 12w \mathbb{P}(M_{U_0, U_2} > s) + C_{5.21} 12w \mu(1) \int_0^s \mathbb{P}(M_{U_0, U_2} > r) \, dr. \]

Since \( e^{(t-s)\lambda^w} \) is a probability, the required inequality in (5.17) follows upon applying the foregoing inequality to (5.18). We have proved (5.15). The proof of (5.16) is almost the same if we use Proposition 6.1 (3') instead of Proposition 6.1 (2'). The details are omitted.

(2') By the first limit in (2.11) and (5.15), it is enough to show that all of the following limits hold:
\[ \lim_{n \to \infty} w_n(2s_n) \cdot 2\gamma_n \nu_n(1) \int_0^T \mathbb{P}^{(n)}(M_{U_0, U_2} > 2s_n s) \, ds = 0; \]  
\[ \lim_{n \to \infty} [w_n(2s_n) + 1] \cdot \mu_n(1) (2s_n) \cdot 2\gamma_n \nu_n(1) \int_0^T \mathbb{P}^{(n)}(M_{U_0, U_2} > 2s_n s) \, ds = 0. \]  

(The limit (5.23) is stronger than needed but is convenient for the other proofs below.)

To get (5.22), first, note that by (4.3), (5.1) and the limit superior in (2.11),
\[ \lim_{n \to \infty} w_n(2s_n) \cdot \left[ T + \min \left\{ \frac{1}{g_n(2s_n)} \frac{t_{\text{mix}}^{(n)}}{2s_n} [1 + \log^+(\gamma_n / t_{\text{mix}}^{(n)})] \right\} \right] = 0. \]  

We get (5.22) from applying (2.12) and (5.24) to (5.12) with \( s_0 = 2s_n \). For (5.23), \( \lim_{n \to \infty} \mu_n(1)(2s_n) = 0 \) by (2.10) and (5.1). The limit superior in (2.11) and (5.1) give \( \lim \sup_n w_n(2s_n) < \infty \). These two properties are enough for (5.23). The proof is complete.

To satisfy (5.5) under the setting of (5.11), we consider the sum of the right-hand side of (5.15), with \( t \) replaced by \( \theta_n t \), and \( \sup_{\xi \in S_{E_n}} \mathbb{E}^\xi_{\theta_n} \left[ w_n \theta_n f_{\sigma_0 \sigma_2} (\xi_{\theta_n t}) \right] \). Moreover, this supremum can be bounded by using (6.1) and \( \mathbb{P}(M_{U_0, U_2} > \theta_n t) \), thanks to duality and the choice \( \sigma_0 \neq \sigma_2 \). Therefore, given \( T \in (0, \infty) \), we set \( a(t) = a_n(t) = \sum_{\ell=1}^3 a_{n, \ell}(t) \) for \( t \in [0, T] \), where
\[ a_{n, \ell}(t) \overset{\text{def}}{=} C_{5.25} \cdot \frac{w_n \theta_n}{2\gamma_n \nu_n(1)} \cdot w_n \theta_n \cdot 2\gamma_n \nu_n(1) \int_0^T \mathbb{P}^{(n)}(M_{U_0, U_2} > \theta_n s) \, ds \]
\[ + C_{5.25} \cdot \frac{w_n \theta_n}{2\gamma_n \nu_n(1)} \cdot 2\gamma_n \nu_n(1) \mathbb{P}^{(n)}(M_{U_0, U_2} > \theta_n t) \]
\[ + C_{5.25} \cdot \frac{w_n \theta_n}{2\gamma_n \nu_n(1)} \cdot (w_n \theta_n + 1) \cdot \mu_n(1) \theta_n \cdot 2\gamma_n \nu_n(1) \int_0^T \mathbb{P}^{(n)}(M_{U_0, U_2} > \theta_n s) \, ds \]

and \( C_{5.25} \) depends only on \( (\Pi, T) \). For any \( n \geq 1 \), \( t \mapsto a_n(t) \) is bounded and decreasing on \( [0, T] \), and
\[ \sup_{\xi \in S_{E_n}} \mathbb{E}^\xi_{\theta_n} \left[ w_n \theta_n f_{\sigma_0 \sigma_2} (\xi_{\theta_n t}) \right] \leq a_n(t), \quad \forall t \in [0, T]. \]

Hence, the conditions of \( a_n(t) \) required in Proposition 5.1 hold.

For the proof of (5.3), the next step is to show that under the setting of (5.11) and the above choice of \( a(t) = a_n(t) \), the right-hand side of (5.6) vanishes as \( n \to \infty \). For the first term on the right-hand side of (5.6), proving \( \int_0^s a_n(t) \, dt \) amounts to proving \( \int_0^s a_n(t) \, dt \to 0 \) for all \( 1 \leq \ell \leq 3 \). For the latter limits, note that \( \delta_n \to 0 \) by (5.1). Also, a slight modification of the proofs of (5.22)–(5.23) shows that

\[ \int_0^s a_n(t) \, dt \to 0. \]
for the right-hand side of (5.25), the first and last terms in are bounded in $n$, and the second term satisfies
\[
\lim_{n \to \infty} 2\gamma_n \nu_n(1) \int_0^{\delta_n} \mathbb{P}^{(n)}(M_{U_0,U_t} > \theta_n s)ds = \lim_{n \to \infty} \frac{2s_n}{\theta_n} \cdot 2\gamma_n \nu_n(1) \int_0^1 \mathbb{P}^{(n)}(M_{U_0,U_t} > 2s_n s)ds = 0.
\]
For the second term in (5.6), it is enough to show that $a_n(\delta_n)$’s are bounded. From the above argument for the first term in (5.6), this property follows if we use the second limit in (2.11) and note that
\[
2\gamma_n \nu_n(1) \mathbb{P}^{(n)}(M_{U_0,U_t} > \theta_n t)|t=\delta_n = 2\gamma_n \nu_n(1)\mathbb{P}^{(n)}(M_{U_0,U_t} > 2s_n) \xrightarrow{n \to \infty} \pi_t,
\]
where the limit follows from Proposition 4.2 and Proposition 4.4. To use these propositions precisely, passing the foregoing limit actually requires that given any subsequence of $(E_n, q^{(n)})$, a suitable further subsequence is used. To lighten the exposition, we continue to suppress similar uses of subsequential limits.

For the third term in (5.6), note that $\delta_n \to 0$ by (5.1), and the $g_n$’s in (5.11) are uniformly bounded in $n$. The last term in (5.6) is the major term. By (5.11) and Proposition 5.3 (2°), it remains to prove
\[
\lim_{n \to \infty} \sup_{\xi \in S} \left| \mathbb{E}^\xi_0[w_n \theta_n \mathcal{J}_{\sigma_0 \sigma_2 \sigma_3}(\xi_{2s_n})] - w_\infty Q_{\sigma_0 \sigma_2 \sigma_3}(p(\xi)) \right| = 0. \tag{5.26}
\]
For the next lemma, recall that the total variation distance $d_E$ and the spectral gap $\gamma$ are defined at the beginning of Section 2. Also, here and in what follows, we use the shorthand notation $\mathbb{E}[Z; A] = \mathbb{E}[Z 1_A]$. 

**Lemma 5.4.** Fix $(\sigma_0, \sigma_2, \sigma_3) \in S \times S \times S$ such that $\sigma_0 \neq \sigma_2$.

1°) Given any $0 < s < t < \infty$, the following estimate of the voter model by the coalescing Markov chains holds:
\[
\sup_{\xi \in S} \left| \mathbb{E}^\xi_0[\mathcal{F}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t)] - \mathbb{E}^\xi_0[\mathcal{F}_{\sigma_0 \sigma_2}(\xi_t)] \right| \leq C_{5.27} \sum_{\ell=1}^3 \Gamma_\ell(s,t),
\]
where $C_{5.27}$ is a universal constant and
\[
\Gamma_\ell(s,t) \overset{def}{=} \mathbb{P}(M_{U_0,U_t} \in (s,t]) + \min \left\{ \sqrt{\frac{\pi_{\max}}{\mu(1)}} e^{-g(t-s)} \mathbb{P}(M_{U_0,U_t} > s)d_E(t-s) \right\} + (1 - e^{-2\mu(1)t})\mathbb{P}(M_{U_0,U_t} > t) + \mu(1) \int_0^t \mathbb{P}(M_{U_0,U_t} > r)dr. \tag{5.28}
\]
2°) The limit in (5.26) holds.

**Proof.** (1°) First, we consider the case that there is no mutation. Roughly speaking, the method of this proof is to express $\mathbb{E}^\xi_0[\mathcal{F}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t)]$ in terms of coalescing Markov chains before any two coalesce. This way we can express the coalescing Markov chains as independent Markov chains and compute
the asymptotics of \( \mathbb{E}^0\left[ f_{\sigma_0} f_{\sigma_2} (\xi_t) \right] \) by the \( \kappa \)-constants defined in (4.4)–(4.6). This idea goes back to [9, Proposition 6.1].

Now, by duality and the assumption \( \sigma_0 \neq \sigma_2 \), it holds that

\[
\mathbb{E}^0\left[ f_{\sigma_0} f_{\sigma_2} (\xi_t) \right] = \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) \mathbb{1}_{\sigma_3} \circ \xi \left( B_{t}^{U_3} \right) \right]
\]

\[
= \mathbb{1}_{\{\sigma_2 = \sigma_3\}} \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) \mathbb{1}_{\sigma_3} \circ \xi \left( B_{t}^{U_3} \right) \right] + \mathbb{1}_{\{\sigma_0 = \sigma_3\}} \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) \mathbb{1}_{\sigma_3} \circ \xi \left( B_{t}^{U_3} \right) \right] + \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) \mathbb{1}_{\sigma_3} \circ \xi \left( B_{t}^{U_3} \right) \right] \]

\[
= \mathbb{1}_{\{\sigma_2 = \sigma_3\}} I + \mathbb{1}_{\{\sigma_0 = \sigma_3\}} II + III. \quad (5.29)
\]

We can further write I and II as

\[
I = \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) ; M_{U_0, U_2} > t, M_{U_0, U_3} > t \right]
\]

\[
- \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) ; M_{U_0, U_2} > t, M_{U_2, U_3} > t, M_{U_0, U_3} > t \right] = I' - I'', \quad (5.30)
\]

\[
II = \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) ; M_{U_0, U_2} > t, M_{U_2, U_3} > t \right]
\]

\[
- \mathbb{E}\left[ \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) ; M_{U_0, U_2} > t, M_{U_2, U_3} > t, M_{U_0, U_3} > t \right] = II' - II''. \quad (5.31)
\]

We estimate \( I', I'', II' \) and III below, using the property that the coalescing Markov chains move independently before meeting.

First, for \( 0 < s < t < \infty \), it follows from the Markov property of independent \( q \)-Markov chains at time \( s \) that \( I' \) in (5.30) can be estimated as follows:

\[
|I' - \mathbb{P}(M_{U_0, U_2} > s, M_{U_0, U_3} > s)p_{\sigma_0}(\xi)p_{\sigma_2}(\xi)|
\]

\[
\leq \mathbb{E}\left[ e^{(t-s)(q-1)} \mathbb{1}_{\sigma_0} \circ \xi \left( B_{t}^{U_0} \right) e^{(t-s)(q-1)} \mathbb{1}_{\sigma_2} \circ \xi \left( B_{t}^{U_2} \right) - p_{\sigma_0}(\xi)p_{\sigma_2}(\xi) ; M_{U_0, U_2} > s, M_{U_0, U_3} > s \right]
\]

\[
+ \mathbb{P}(M_{U_0, U_2} \in (s, t)) + \mathbb{P}(M_{U_0, U_3} \in (s, t)). \quad (5.32)
\]

On the event \( \{M_{U_0, U_2} > s, M_{U_0, U_3} > s\} \), \( (B_{t}^{U_0})_{0 \leq r \leq s} \) and \( (B_{t}^{U_2})_{0 \leq r \leq s} \) are independent \( q \)-Markov chains and each chain is stationary by the assumption on \( \{U_\ell\} \). Since \( p_\sigma(\xi) = \sum_x \mathbb{1}_x \circ \xi(x)\pi(x) \), the expectation in (5.32) can be estimate as in the proof of [9, Proposition 6.1]. We get

\[
|I' - \mathbb{P}(M_{U_0, U_2} > s, M_{U_0, U_3} > s)p_{\sigma_0}(\xi)p_{\sigma_2}(\xi)|
\]

\[
\leq \min \left\{ 2 \sqrt{\frac{\pi_{\max}}{\nu(1)}} e^{-(t-s)^q}, 4\mathbb{P}(M_{U_0, U_2} > s, M_{U_0, U_3} > s)d_E(t-s) \right\}
\]

\[
+ \mathbb{P}(M_{U_0, U_2} \in (s, t)) + \mathbb{P}(M_{U_0, U_3} \in (s, t)).
\]

Similar estimates apply to the other terms \( I'', II' \) and III in (5.29), (5.30) and (5.31).

Applying all of these estimates to (5.29) proves (5.27) when there is no mutation. The additional terms in (5.27) arise when we include mutation and use (6.1) again.

(2°) Recall the second limit in (2.11) and (5.23). Then by (5.27) and (5.28) with \( t = 2s_n \) and \( s = s_n \), it suffices to show all of the following limits:

\[
\lim_{n \to \infty} 2\gamma_n \nu_n(1)\mathbb{P}(n)(M_{U_0, U_\ell} \in (s_n, 2s_n]) = 0, \quad 1 \leq \ell \leq 3; \quad (5.33)
\]

\[
\lim_{n \to \infty} (1 - e^{-2\gamma_n(1)\cdot(2s_n)\cdot s_n}) \cdot 2\gamma_n \nu_n(1)\mathbb{P}(n)(M_{U_0, U_\ell} > 2s_n), \quad 1 \leq \ell \leq 3; \quad (5.34)
\]

26
\[ \lim_{n \to \infty} \Gamma_{n,\ell} = 0, \quad 1 \leq \ell \leq 3, \]

(5.35)

where \( \Gamma_{n,\ell} \) is given by the minimum of the following two terms:

\[ \gamma_n \nu_n(I) + \sqrt{\pi_{\max}^{(n)}} e^{-x_n s_n}, \quad 2\gamma_n \nu_n(I) \mathbb{P}(M_{U_0,U_\ell} > s_n) dE_n(s_n). \]

(5.36)

To see (5.33), we simply use Propositions 4.2 and 4.4. The limit in (5.34) follows from the same propositions, in addition to (2.10) and (5.1). For (5.35), we consider the following two cases. When \( \Gamma_n \) is given by the first term in (5.36), the required limit holds by (2.12) and the first limit in (4.3). When \( \Gamma_n \) is given by the other term in (5.36), we first use Propositions 4.2 and 4.4. Then note that the second limit in (4.3) implies \( \lim_n t(n) \mu(n)/s_n = 0 \), and so, \( \lim_n dE_n(s_n) = 0 \) by (4.9). We have proved (5.35). The proof is complete.

Up to this point, we have proved the asymptotic closure of equation in the sense of (5.3). Note that under (5.11), the convergence of the last term in (5.6) also contributes to asymptotic path regularity of the density processes.

The next lemma proves the asymptotic path regularity more explicitly as tightness in the convergence results of Theorem 2.2. The limit of the normalized martingale terms in Theorem 2.2 (2°) is also proven. Here, recall that the density processes satisfy the decompositions in (2.5). From now on, \( \prod_{n \to \infty} \) refers to convergence in distribution as \( n \to \infty \).

Lemma 5.5. Fix \( \sigma \in S \).

1°) The sequence of laws of \( I^{(n)}_{\sigma} \) as continuous processes under \( \mathbb{P}^{w_n}_{\nu_n} \) is tight.

2°) The sequence of laws of \( I_{\{w_n > 0\} w_n^{-1}} R^{(n)}_{\sigma} \) as continuous processes under \( \mathbb{P}^{w_n}_{\nu_n} \) is tight.

3°) The sequence of laws of \( M^{(n)}_{\sigma} \) as continuous processes under \( \mathbb{P}^{w_n}_{\nu_n} \) converges to zero in distribution.

If, in addition, \( \lim_n \gamma_n \nu_n(I)/\theta_n = 0 \), then the following holds.

4°) The sequence of laws of

\[ \left( \frac{\gamma_n}{\theta_n} \right)^{1/2} M^{(n)}_{\sigma}(t), \quad \frac{\gamma_n}{\theta_n} \left( M^{(n)}_{\sigma}, M^{(n)}_{\sigma'} \right)_t - \int_0^t p_\sigma(\xi_{\theta_n,s}) [1 - p_\sigma(\xi_{\theta_n,s})] ds; \quad t \geq 0 \]

as processes with càdlàg paths under \( \mathbb{P}^{w_n}_{\nu_n} \) is C-tight, and the second coordinates converge to zero in distribution as processes. Moreover, for all \( T \in (0, \infty) \),

\[ \sup_{n \geq 1} \sup_{t \in [0,T]} \sup_{\xi \in \mathcal{S}_{\mathcal{E}_n}} \frac{\gamma_n}{\theta_n} \mathbb{E}^{w_n}_{\pi} \left[ M^{(n)}_{\sigma}(t)^2 \right] < \infty. \]

(5.38)

For any \( \sigma' \in S \) with \( \sigma' \neq \sigma \), the sequence

\[ \left( \frac{\gamma_n}{\theta_n} \left( M^{(n)}_{\sigma}, M^{(n)}_{\sigma'} \right)_t + \int_0^t p_\sigma(\xi_{\theta_n,s}) p_{\sigma'}(\xi_{\theta_n,s}) ds; \quad t \geq 0 \right) \]

under \( \mathbb{P}^{w_n}_{\nu_n} \) converges to zero in distribution as processes.
Proof. (1°) First, we show a bound for \( \sup_{\xi \in S} \mathbb{E}_\xi^{w_n} [ |I^{(n)}_{\sigma}(\theta)| ] \) explicitly in \( \theta \). By (3.9),

\[
|D_{\sigma}(\xi)| \leq C_{5.40} \sum_{\sigma' \in S, \sigma' \neq \sigma} \overline{f}_{\sigma \sigma'}(\xi)
\]

(5.40)

for some constant \( C_{5.43} \) depending only on \( \Pi \) and \# \( S \). Indeed, for \( q(x, y) > 0 \), \( \mathbb{I}_\sigma \circ \xi(y) - \mathbb{I}_\sigma \circ \xi(x) \neq 0 \) implies that either \( \xi(x) \) or \( \xi(y) \) is \( \sigma \) but not both. By (5.2) and (5.40),

\[
\sup_{\xi \in S} \mathbb{E}_\xi^{w_n} [ |I^{(n)}_{\sigma}(\theta)| ] \leq C_{5.40} \sum_{\sigma' \in S, \sigma' \neq \sigma} \int_0^\theta \sup_{\xi \in S} \mathbb{E}_\xi^{w_n} \left[ w_n \theta_n \overline{f}_{\sigma \sigma'}(\xi_{\theta_n s}) \right] ds + 2 \mu_n(\mathbb{I}) \theta_n \cdot \theta.
\]

(5.41)

Furthermore, we can bound the expectations on the right-hand side of (5.41) by using an analogue of the \( a_n(t) \) in (5.25), but now involving only the meeting time \( M_{V,V'} \). Specifically, given \( T \in (0, \infty) \), the following inequality holds for all \( t \in [0, T] \):

\[
\mathbb{E}_\xi^{w_n} \left[ w_n \theta_n \overline{f}_{\sigma \sigma'}(\xi_{\theta_n t}) \right] \leq C_{5.42} \cdot \frac{w_n \theta_n}{2 \gamma_n \nu_n(\mathbb{I})} \cdot w_n \theta_n \cdot 2 \gamma_n \nu_n(\mathbb{I}) \int_0^T \mathbb{P}^{(n)}(M_{V,V'} > \theta_n s) ds \\
+ C_{5.42} \cdot \frac{w_n \theta_n}{2 \gamma_n \nu_n(\mathbb{I})} \cdot 2 \gamma_n \nu_n(\mathbb{I}) \mathbb{P}^{(n)}(M_{V,V'} > \theta_n t) \\
+ C_{5.42} \cdot \frac{w_n \theta_n}{2 \gamma_n \nu_n(\mathbb{I})} \cdot (w_n \theta_n + 1) \cdot \mu_n(\mathbb{I}) \theta_n \cdot 2 \gamma_n \nu_n(\mathbb{I}) \int_0^T \mathbb{P}^{(n)}(M_{V,V'} > \theta_n s) ds
\]

(5.42)

and \( C_{5.42} \) depends only on \( \Pi, T, \sup_n \mu^{(n)}_\sigma / \pi^{(n)}_\sigma \). To see (5.42), we combine (5.16) and [10, Proposition 3.2] and then use (2.12) and (4.7) to reduce probabilities of \( M_{U_0, U_1} \) to probabilities of \( M_{V,V'} \).

Next, we show that

\[
\lim_{\theta \searrow 0} \lim_{n \to \infty} \sup_{\xi \in S} \mathbb{E}_\xi^{w_n} [ |I^{(n)}_{\sigma}(\theta)| ] = 0.
\]

(5.43)

First, (2.10) readily gives the required limit of the last term of (5.41). We focus on the sum of integrals on the right-hand side of (2.10). For each of these integrals, note that the first and last terms in (5.42) are uniformly bounded in \( n \) as in the case of (5.25). The integral over \( t \in [0, \theta] \) of the second term in (5.42) satisfies

\[
\lim_{\theta \searrow 0} \limsup_{n \to \infty} \frac{w_n \theta_n}{2 \gamma_n \nu_n(\mathbb{I})} \cdot 2 \gamma_n \nu_n(\mathbb{I}) \int_0^{\theta \theta_n} \mathbb{P}^{(n)}(M_{V,V'} > s) ds = 0
\]

(5.44)

by the second limit of (2.11), (3.22), and (4.18) with \( s_n \) replaced by \( \theta_n \) and with \( t_2 = \theta \) and \( t_1 = 0 \) since \( (\theta_n) \) is also a slow sequence. (The use of \( M_{V,V'} \) allows us to circumvent Lemma 5.2 due to the explosion of the bound in (5.12) as \( t \to 0 \)). We have proved (5.43).

Finally, the required tightness follows from (5.43), the strong Markov property of the particle system and Aldous’s criterion for tightness [25, Proposition VI.4.5 on p.356]. The detail is similar to the proof of [9, Theorem 5.1 (1)].

(2°) Recall the equation (3.16) of \( R_{\sigma} \), and the explicit form of \( R^w \) can be read from (3.8). Then by the same reason for (5.40), the coefficient of \( R_{\sigma} \) satisfies

\[
\sum_{x,y \in E} \pi(x) \mathbb{I}_\sigma \circ \xi(y) - \mathbb{I}_\sigma \circ \xi(x) q(x,y) |R^w_{\sigma}(x,y,\xi)| \leq C_{5.45} \sum_{\sigma' \in S} \overline{f}_{\sigma \sigma'}(\xi),
\]

(5.45)
where $C_{5.45}$ depends only on $\Pi$ and $\#S$. From (5.45), the argument in (1°) applies again.

(3°) The proof follows from a slight modification of the proof of (4°) below even without the additional assumption $\lim_n \gamma_n \nu_n(1)/\theta_n = 0$.

(4°) We start with the convergence of the second coordinate in (5.37). Define a density function $\tilde{p}_\sigma(\xi)$ on $S^{E_n}$ such that the stationary weights $\pi^{(n)}(x)$ in $p_\sigma(\xi)$ are replaced by $\pi^{(n)}(x)^2/\nu_n(1)$. From (3.18), the following equality holds under $P^{\xi_n}_\sigma$ for all $\xi \in S^{E_n}$:

$$
\frac{\gamma_n}{\theta_n}(M^{(n)}_{\sigma}, M^{(n)}_{\sigma})_t = \gamma_n \nu_n(1) \int_0^t \sum_{\sigma' \in S \setminus \{\sigma\}} [p_{\sigma'\sigma}(\xi_{\theta_n,s}) + p_{\sigma\sigma'}(\xi_{\theta_n,s})]d\sigma \\
+ \gamma_n \nu_n(1) \int_0^t \left( \left[1 - \tilde{p}_\sigma(\xi_{\theta_n,s})\right] \mu_n(\sigma) + \tilde{p}_\sigma(\xi_{\theta_n,s}) \mu_n(S \setminus \{\sigma\}) \right) d\sigma \\
+ \frac{\gamma_n \nu_n(1)}{\theta_n} \cdot w_n \theta_n \int_0^t \tilde{R}^{(n)}(\xi_{\theta_n,s}) d\sigma.
$$

(5.46)

Here, $\tilde{R}^{(n)}_{\xi_n}$ can be bounded in the same way as (5.40). Note that due to the use of $\tilde{R}^{(n)}_{\xi_n}$, we only involve the first term $q(x,y)$ in the expansion (3.8) of $q^{\xi_n}(x,y,\xi)$.

Let us explain how the required convergence of the second coordinate in (5.37) follows (5.46). First, since $\lim_n \gamma_n \nu_n(1)/\theta_n = 0$ by assumption, the bound for $\tilde{R}^{(n)}_{\xi_n}$ mentioned above and the proof of (1°) show that the continuous process defined by the last integral of (5.46) converges to zero in distribution. Next, for all $0 \leq T_0 < T_1 < \infty$, it holds that

$$
\gamma_n \nu_n(1) \int_{T_0}^{T_1} \left( \sum_{\sigma' \in S \setminus \{\sigma\}} \tilde{p}_\sigma(\xi_{\theta_n,s}) \mu_n(\sigma) + \tilde{p}_\sigma(\xi_{\theta_n,s}) \mu_n(S \setminus \{\sigma\}) \right) d\sigma \\
\leq \frac{\gamma_n \nu_n(1)}{\theta_n} \cdot \mu_n(1) \theta_n \cdot (T_1 - T_0) \xrightarrow{n \to \infty} 0
$$

by (2.10) and the assumption $\lim_n \gamma_n \nu_n(1)/\theta_n = 0$. Hence, the second integral on the right-hand side of (5.46) converges to zero in distribution as a continuous process.

The sequence of laws of the first integrals on the right-hand side of (5.46) is tight for a reason similar to (5.44). Moreover, since $\bar{\kappa}_0$ in (4.1) is 1 by Proposition 4.2, a slight modification of the proof of (5.3) shows that

$$
\gamma_n \nu_n(1) \int_0^t \sum_{\sigma' \in S \setminus \{\sigma\}} [p_{\sigma'\sigma}(\xi_{\theta_n,s}) + p_{\sigma\sigma'}(\xi_{\theta_n,s})]d\sigma - \int_0^t p_\sigma(\xi_{\theta_n,s}) [1 - p_\sigma(\xi_{\theta_n,s})]d\sigma \xrightarrow{n \to \infty} 0
$$

as processes. See (5.16) and [9, Proposition 6.1]. By this convergence and the explanation in the preceding paragraph, (5.46) thus shows that the sequence of laws of the second coordinates in (5.37) as processes converges to zero in distribution.

To get the $C$-tightness of the sequence of laws of the first coordinates in (4.1), first, note that the tightness readily follows from the $C$-tightness of $(\gamma_n/\theta_n)(M^{(n)}_{\sigma}, M^{(n)}_{\sigma})$ proven above [25, Theorem VI.4.13 on p.358]. For the stronger $C$-tightness, note that the jump sizes of $p_\sigma(\xi_t)$ are given by $\pi(x)$’s. Hence,

$$
(\gamma_n/\theta_n)^{1/2} \sup_{t \geq 0} |\Delta M^{(n)}_{\sigma}(t)| \leq (\gamma_n/\theta_n)^{1/2} \pi^{(n)}_{\max},
$$

(5.47)
where the right-hand side tends to zero by (2.12) and the assumption \( \lim_{n} \gamma_n \nu_n(1)/\theta_n = 0 \). The required \( C \)-tightness now follows from [25, Proposition 3.26 in Chapter VI on p.351].

Finally, the above argument for the tightness of \( (\gamma_n/\theta_n)^{1/2}M^{(n)}_\sigma \) also proves (5.38).

\( 5^\circ \) The proof is almost identical to the proof of \( 4^\circ \) for the second coordinate of (5.37), if we start with (3.19). The details are omitted. ■

### 5.2 The replicator equation and the Wright–Fisher fluctuations

In this subsection, we complete the proof of Theorem 2.2 and give the proof of Corollary 2.3.

**Completion of the proof of Theorem 2.2.** By (3.10), (5.2), (5.3) and Lemma 5.5 \( (1^\circ) - (3^\circ) \), we have proved that the following vector process converges to zero in distribution:

\[
p_\sigma(\xi_{n,t}) = \int_{0}^{t} w_\infty \left( \sum_{\sigma_0, \sigma_3 \in S} \Pi(\sigma, \sigma_3) Q_{\sigma_0, \sigma_3, \sigma_3}(p(\xi_{0,s})) \right) - \sum_{\sigma_2, \sigma_3 \in S} \Pi(\sigma_2, \sigma_3) Q_{\sigma_2, \sigma_3}(p(\xi_{0,s})) \right) ds
\]

\[
- \int_{0}^{t} \left( \mu_\infty(\sigma) [1 - p_\sigma(\xi_{0,s})] - \mu_\infty(S \setminus \{\sigma\}) p_\sigma(\xi_{0,s}) \right) ds, \quad \sigma \in S,
\]

where the polynomials \( Q_{\sigma_0, \sigma_2, \sigma_3} \) are defined in (5.4). Hence, the sequence of laws of \( p(\xi_{0,t}) \) is \( C \)-tight, and \( p(\xi_{0,t}) \) converges in distribution to \( X(t) \) as processes, where \( X \) is the unique solution to the following system:

\[
\dot{X}_\sigma = w_\infty Q_\sigma(X) + \mu_\infty(\sigma)(1 - X_\sigma) - \mu_\infty(S \setminus \{\sigma\}) X_\sigma, \quad \sigma \in S,
\]

and the polynomial \( Q_\sigma(X) \) in (5.48) is given by

\[
Q_\sigma(X) = \sum_{\sigma_0, \sigma_3 \in S} \Pi(\sigma, \sigma_3) Q_{\sigma_0, \sigma_3, \sigma_3}(X) - \sum_{\sigma_2, \sigma_3 \in S} \Pi(\sigma_2, \sigma_3) Q_{\sigma_2, \sigma_3, \sigma_3}(X). \tag{5.49}
\]

To simplify (5.49) to the required form in (2.13), note that the constraints \( \sigma_0 \neq \sigma \) and \( \sigma_2 \neq \sigma \) in (5.49) can be removed from the definition of \( Q_\sigma(X) \) by cancelling repeating terms. In doing so, we extend the definition \( Q_{\sigma_0, \sigma_2, \sigma_3}(X) \) to \( \sigma_0 = \sigma_2 \) by the same formula in (5.4), but only in this proof. We also lighten notation by the following: \( A = \mathcal{K}(2,3)0 - \mathcal{K}_0(2,3) \) and \( B = \mathcal{K}(2,3)2 - \mathcal{K}_0(2,3) \) and \( C = \mathcal{K}_0(2,3). \) Then by (5.49),

\[
Q_\sigma(X) = \sum_{\sigma_0, \sigma_3 \in S} \Pi(\sigma, \sigma_3) \left( \mathbb{1}_{\{\sigma_2 = \sigma_3\}} A X_{\sigma_0} X_{\sigma_2} + \mathbb{1}_{\{\sigma_0 = \sigma_3\}} B X_{\sigma_0} X_{\sigma_2, \sigma_3} + C X_{\sigma_0} X_{\sigma_2, \sigma_2} \right) \bigg|_{\sigma_2 = \sigma}
\]

\[
- \sum_{\sigma_2, \sigma_3 \in S} \Pi(\sigma_2, \sigma_3) \left( \mathbb{1}_{\{\sigma_2 = \sigma_3\}} A X_{\sigma_0} X_{\sigma_2} + \mathbb{1}_{\{\sigma_0 = \sigma_3\}} B X_{\sigma_0} X_{\sigma_2} + C X_{\sigma_0} X_{\sigma_2, \sigma_3} \right) \bigg|_{\sigma_0 = \sigma}
\]

\[
= X_\sigma \sum_{\sigma_0 \in S} A \Pi(\sigma, \sigma) X_{\sigma_0} + X_\sigma \sum_{\sigma_0 \in S} B \Pi(\sigma, \sigma_0) X_{\sigma_0} + X_\sigma \sum_{\sigma_3 \in S} C \Pi(\sigma, \sigma_3) X_{\sigma_3}
\]

\[
- X_\sigma \sum_{\sigma_2 \in S} A \Pi(\sigma, \sigma_2) X_{\sigma_2} - X_\sigma \sum_{\sigma_2 \in S} B \Pi(\sigma, \sigma_2) X_{\sigma_2} - X_\sigma \sum_{\sigma_3 \in S} \left( \sum_{\sigma_3 \in S} C \Pi(\sigma, \sigma_3) X_{\sigma_3} \right) X_{\sigma_2}
\]

\[30\]
\[ X_\sigma \left( A\Pi(\sigma, \sigma) + \sum_{\sigma' \in S} B[\Pi(\sigma, \sigma') - \Pi(\sigma', \sigma)]X_{\sigma'} + \sum_{\sigma' \in S} C\Pi(\sigma, \sigma')X_{\sigma'} \right) \\
- X_\sigma \sum_{\sigma' \in S} \left( A\Pi(\sigma', \sigma') + \sum_{\sigma'' \in S} C\Pi(\sigma', \sigma'')X_{\sigma''} \right)X_{\sigma'}. \]

Note that we have used the property \( \sum_\sigma X_\sigma = 1 \) in the last two equalities. The last equality is enough for the required form in (2.13) upon recalling (5.48) and involving the polynomials \( F_\sigma(X) \) and \( \tilde{F}_\sigma(X) \) in (2.14) and (2.15). Moreover, (2.16) holds by Proposition 4.2, (4.7), and Proposition 4.4 (1°) and (4°).

For the proof of (2°), notice that by (1°) and Lemma 5.5 (4°)–(5°), the following convergence of matrix processes holds:

\[ \left( \frac{\gamma_n}{\theta_n} \langle M_{\sigma}^{(n)}, M_{\sigma'}^{(n)} \rangle_t \right)_{\sigma, \sigma' \in S} \xrightarrow{(d) \ n \to \infty} \left( \int_0^t X_\sigma(s)[\delta_{\sigma, \sigma'} - X_{\sigma'}(s)]ds \right)_{\sigma, \sigma' \in S}. \]

By this convergence and (5.38), the standard martingale problem argument shows that every weakly convergent subsequence of \( ((\gamma_n/\theta_n)^{1/2}M_{\sigma}^{(n)}; \sigma \in S) \) converges to a continuous vector \( L_2 \)-martingale \( (M_{\sigma}^{(\infty)}; \sigma \in S) \) with a quadratic variation matrix given by \( (\int_0^t X_\sigma(s)[\delta_{\sigma, \sigma'} - X_{\sigma'}(s)]ds; \sigma, \sigma' \in S) \). See [25, Proposition 1.12 in Chapter IX on p.525] and the proof of [33, Theorem 1.10 in Chapter XIII on pp.519–520]. Hence, the limiting vector martingale \( (M_{\sigma}^{(\infty)}; \sigma \in S) \) is a Gaussian process with covariance matrix \( (\int_0^t X_\sigma(s)[\delta_{\sigma, \sigma'} - X_{\sigma'}(s)]ds; \sigma, \sigma' \in S) \) [33, Exercise (1.14) in Chapter V on p.186]. Moreover, by uniqueness in law of this Gaussian process, the convergence holds along the whole sequence of the vector martingale \( (\gamma_n/\theta_n)^{1/2}M^{(n)} \).

**Proof of Corollary 2.3.** Write \( u \) for \( X_1 \). In this case, \( X_0 = 1 - u \) and the polynomial \( Q_1(X) \) defined by (5.49) simplifies to

\[ Q_1 = (b - c)Q_{0,11} - cQ_{0,10} - bQ_{1,01} = (Q_{0,11} - Q_{1,01})b - (Q_{0,11} + Q_{0,10})c. \]

By (5.4), the coefficient of \( c \) is given by

\[ -Q_{0,11}(X) - Q_{0,10}(X) = -\left( \bar{\kappa}(2,3)|0 - \bar{\kappa}_0|2;3 \right)(1 - u)u - \bar{\kappa}_0|2;3(1 - u)^2u \]

\[ - \left( \bar{\kappa}(0,3)|2 - \bar{\kappa}_0|2;3 \right)(1 - u)u - \bar{\kappa}_0|2;3(1 - u)^2u, \]

and the coefficient of \( b \) is

\[ Q_{0,11}(X) - Q_{1,01}(X) = \left( \bar{\kappa}(2,3)|0 - \bar{\kappa}_0|2;3 \right)(1 - u)u + \bar{\kappa}_0|2;3u^2(1 - u)^2u \]

\[ - \left( \bar{\kappa}(0,3)|2 - \bar{\kappa}_0|2;3 \right)(1 - u)u - \bar{\kappa}_0|2;3(1 - u)^2u. \]

These two coefficients can be simplified by using the definition of \( Q_{\sigma_0, \sigma_2 \sigma_3} \) and Proposition 4.4 (4°), if we follow the algebra in the proof of Lemma 3.1 that simplifies (3.10) to (3.11). For example, a similar argument as in the proof of Lemma 5.4 shows that (5.26) holds with

\[ Q_{0,01}(X) = \left( \bar{\kappa}(0,2)|3 - \bar{\kappa}_0|2;3 \right)(1 - u)u + \bar{\kappa}_0|2;3(1 - u)^2u, \]

and so

\[ Q_{0,11}(X) + Q_{0,01}(X) = \left( \bar{\kappa}(2,3)|0 - \bar{\kappa}_0|2;3 \right)(1 - u)u + \bar{\kappa}_0|2;3(1 - u)^2u \]

\[ + \left( \bar{\kappa}(0,2)|3 - \bar{\kappa}_0|2;3 \right)(1 - u)u + \bar{\kappa}_0|2;3(1 - u)^2u. \]
where the last equality follows from Proposition 4.4 (4°). In this way, we can obtain from (5.50) and (5.51) that $Q_1(X) = [\varpi_3 - \varpi_1]b - \varpi_2c(1 - u)u$. Moreover, by Proposition 4.4, we can pass limit along the whole sequence to get this limiting polynomial $Q_1(X)$.

\section{Further properties of coalescing lineage distributions}

\subsection{A comparison with mutations}

In this section, we prove some auxiliary results for the proof of Theorem 2.2. The next proposition estimates the voter model $(\xi_t)$ under $\mathbb{P}^0$ by its selection mechanism, that is, by the updates from $\{\Lambda(x, y) : x, y \in E\}$. The proof extends [10, Proposition 3.2]. Recall the notation in Section 3 for the coalescing Markov chains.

**Proposition 6.1.** (1°) Let $f : S \times S \times S \rightarrow [-1, 1]$ be a function such that $f(\sigma, \sigma, \cdot) = 0$ for all $\sigma \in S$. Then for all $t \in (0, \infty)$,

\begin{align*}
\sup_{\xi \in S^E} & \left| \mathbb{E}_{\xi}^0 [f(\xi_t(x), \xi_t(y), \xi_t(z))] - \mathbb{E} [f(\xi(B_t^\ell), \xi(B_t^{\mu_1}), \xi(B_t^{\mu_2}))] \right| \\
\leq & \left(1 - e^{-2\mu(1)}\right) \mathbb{P}(M_{x,y} > t) + \mathbb{1}_{x\neq y}(1 - e^{-\mu(1)}) \mathbb{P}(M_{x,z} \land M_{y,z} > t) \\
& + 2\mu(1) \int_0^t \mathbb{P}(M_{x,y} > s)ds + \mathbb{1}_{x\neq y}\mu(1) \int_0^t \mathbb{P}(M_{x,z} \land M_{y,z} > s)ds, \quad \forall x, y, z \in E. \quad (6.1)
\end{align*}

(2°) For all $(\sigma_0, \sigma_2, \sigma_3) \in S \times S \times S$ with $\sigma_0 \neq \sigma_2$, $t \in (0, \infty)$ and $x \in E$,

\begin{align*}
\sup_{\xi \in S^E} & \left| \mathbb{E}_{\xi}^0 \left[ \mathbb{F}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t) \right] - \mathbb{E}_{\xi}^0 \left[ \mathbb{F}_{\sigma_0 \sigma_2 \sigma_3}(\xi_t) \right] \right| \\
\leq & \sum_{\ell \in \{0,2,3\}} 4\mathbb{P}(M_{U_0, U_2} > t, B_t^{U_\ell} = x) + \sum_{\ell \in \{0,2,3\}} 4\mu(1) \int_0^t \mathbb{P}(M_{U_0, U_2} > s, B_t^{U_\ell} = x)ds. \quad (6.2)
\end{align*}

(3°) For all $(\sigma_0, \sigma_1) \in S \times S$ with $\sigma_0 \neq \sigma_1$, $t \in (0, \infty)$ and $x \in E$,

\begin{align*}
\sup_{\xi \in S^E} & \left| \mathbb{E}_{\xi}^0 \left[ \mathbb{F}_{\sigma_0 \sigma_1}(\xi_t) \right] - \mathbb{E}_{\xi}^0 \left[ \mathbb{F}_{\sigma_0 \sigma_1}(\xi_t) \right] \right| \\
\leq & \sum_{\ell \in \{0,1\}} 4\mathbb{P}(M_{U_0, U_1} > t, B_t^{U_\ell} = x) + \sum_{\ell \in \{0,1\}} 4\mu(1) \int_0^t \mathbb{P}(M_{U_0, U_1} > s, B_t^{U_\ell} = x)ds. \quad (6.3)
\end{align*}

The proof of this proposition extends the proof of [10, Proposition 3.2] and is based on the pathwise duality between the voter model and the coalescing Markov chains. The relation follows from time reversal of the stochastic integral equations in Section 2 of the voter model. More specifically, for fixed $t \in (0, \infty)$, we define a system of coalescing $q$-Markov chains $\{B^{a,t} : a \in E\}$ such that in the absence of mutation, $B^{a,t}$ traces out the time-reversed ancestral line that determines the type at $(a, t)$ under the voter model. For example, if $s$ is the last jump time of $\{\Lambda_r(a, b) : b \in E, r \in (0, t)\}$ and $\Lambda(a, c)$ causes this jump, the state of $B^{a,t}$ stays at $a$ before transitioning to $B^{a,t}_{t-s} = c$. Similarly, with the Poisson
processes $\Lambda^\sigma$ driving the mutations, we can define $e(a, t)$ and $M(a, t)$ for the time and the type from the first mutation event on the trajectory of $B^{a,t}$, with $e(a, t) = \infty$ if there is no mutation. Since $e(a, t) > t$ if and only if $e(a, t) = \infty$, we have

$$
\xi_t(a) = M(a, t)1_{\{e(a, t) \leq t\}} + \xi(B_t^{a,t})1_{\{e(a, t) > t\}}, \quad \forall a \in E, \quad \mathbb{P}^0\text{-a.s.} \quad (6.4)
$$

More details can be seen by modifying the description in [10, Section 6.1]. In the absence of mutation, this relation between the duality and the stochastic integral equations is known in [30].

We also observe two identities for the probability distributions of the mutation times $e(a, t)$'s when we condition on $\mathcal{G} = \sigma(\Lambda(a, b); a, b \in E)$. Let $x, y \in E$. Write $0 = J_0 < J_1 < \cdots < J_N < J_{N+1} = t$ such that $J_1, J_2, \cdots, J_N$ are the jump times of the bivariate chain $(B^{x,t}, B^{y,t})$. Hence, $B_t^{x,t} = x_k$ and $B_t^{y,t} = y_k$, for all $r \in [J_k, J_{k+1})$ and $0 \leq k \leq N$. First, fix $s \in [0, \infty)$, and note that conditioned on $\mathcal{G}$, whether mutation occurs along the trajectory of $B^{x,t}$ over $[J_k, s \wedge J_{k+1})$ depends on whether $\sum_{\sigma \in S}[\Lambda^\sigma_{(t-J_k)} - \Lambda^\sigma_{(t-s\wedge J_{k+1})}] \geq 1$. Note that $s' \mapsto \sum_{\sigma \in S} [\Lambda^\sigma_f(x_k) - \Lambda^\sigma_{(t-s')} \wedge (x_k)]$ is a Poisson process with rate $\mu(\mathbb{1})$. Hence, summing over $\sum_{\sigma \in S} [\Lambda^\sigma_f(x_k) - \Lambda^\sigma_{(t-s\wedge J_{k+1})}]$ in $k$, we deduce

$$
\mathbb{P}(e(x, t) \leq s | \mathcal{G}) = 1 - e^{-\mu(\mathbb{1})s}, \quad s \geq 0. \quad (6.5)
$$

Second, note that $x_k \neq y_k$ for all $k$ such that $J_{k+1} \leq M_{x,y} \wedge s$. In this case, mutations in $[J_k, J_{k+1})$ along the trajectories of $B^{x,t}$ and $B^{y,t}$ are determined by two disjoint subsets of the Poisson processes $\{\Lambda^\sigma(a); \sigma \in S, a \in E\}$. Hence, (6.5) generalizes to the following identity:

$$
\mathbb{P}(e(x, t) \wedge e(y, t) \leq s \wedge M_{x,y} | \mathcal{G}) = 1 - e^{-2\mu(\mathbb{1})(s \wedge M_{x,y})}, \quad s \geq 0. \quad (6.6)
$$

**Proof of Proposition 6.1.** (1') Set a partition $\{A_j\}_{1 \leq j \leq 4}$ as follows:

$$
A_1 = \{e(x, t) \wedge e(y, t) \leq t < M_{x,y}\},
A_2 = \{e(x, t) \wedge e(y, t) \leq M_{x,y} \leq t\},
A_3 = \{M_{x,y} < e(x, t) \wedge e(y, t) \leq t\},
A_4 = \{e(x, t) \wedge e(y, t) > t\}. \quad (6.7)
$$

Then consider the corresponding differences for the left-hand side of (6.1):

$$
\Delta_j = \mathbb{E}_x^0[f(\xi_t(x), \xi_t(y), \xi_t(z)); A_j] - \mathbb{E}[f(\xi(B_t^x), \xi(B_t^y), \xi(B_t^z)); A_j], \quad 1 \leq j \leq 4. \quad (6.8)
$$

Let $e_1$ and $e_2$ be i.i.d. exponential random variables with mean $1/\mu(\mathbb{1})$. It follows from (6.6) and the independence between selection and mutation that

$$
|\Delta_1| \leq \mathbb{P}(e_1 \wedge e_2 \leq t)\mathbb{P}(M_{x,y} > t) = (1 - e^{-2\mu(\mathbb{1})t})\mathbb{P}(M_{x,y} > t), \quad (6.9)
$$

$$
|\Delta_2| \leq \int_0^t \mathbb{P}(t > M_{x,y} > s)\mathbb{P}(e_1 \wedge e_2 \in ds) \leq 2\mu(\mathbb{1}) \int_0^t \mathbb{P}(M_{x,y} > s)ds. \quad (6.10)
$$

On $A_3$, $B_t^{x,t} = B_t^{y,t}$ by coalescence, and hence, $\xi_t(x) = \xi_t(y)$ by (6.4). It follows from the assumption on $f$ that both of the expectations defining $\Delta_3$ are zero.

To bound $\Delta_4$, fix $z \in E$ and partition $A_4$ into the following four sets:

$$
A_{41} = \{e(x, t) \wedge e(y, t) > t, e(z, t) \leq t < M_{x,z} \wedge M_{y,z}\},
A_{42} = \{e(x, t) \wedge e(y, t) > t, e(z, t) \leq M_{x,z} \wedge M_{y,z} \leq t\},
A_{43} = \{e(x, t) \wedge e(y, t) > t, M_{x,z} \wedge M_{y,z} < e(z, t) \leq t\},
$$

33
\[ A_{44} = \{ e(x, t) \land e(y, t) > t, e(z, t) > t \}. \]

Then define \( \Delta_{4k} \) for \( 1 \leq k \leq 4 \) as in (6.8) by replacing \( A_j \) with \( A_{4k} \). By (6.5) and similar arguments for (6.9) and (6.10), we get

\[
|\Delta_{41}| \leq 1_{x \neq y} (1 - e^{-\mu(1)t}) \mathbb{P}(M_{x,z} \land M_{y,z} > t), \quad |\Delta_{42}| \leq 1_{x \neq y} \mu(t) \int_0^t \mathbb{P}(M_{x,z} \land M_{y,z} > s) ds, \quad (6.11)
\]

where the use of the indicator function \( 1_{x \neq y} \) follows from the assumption of \( f \). For \( \Delta_{43} \), it is zero because \( A_{43} = \emptyset \). Indeed, on \( \{ M_{x,z} \land M_{y,z} < e(z, t) \leq t \} \), either \( e(x, t) \leq t \) or \( e(y, t) \leq t \) since either \( e(x, t) = e(z, t) \) (if \( M_{x,z} \land M_{y,z} = M_{x,z} \)) or \( e(y, t) = e(z, t) \) (if \( M_{x,z} \land M_{y,z} = M_{y,z} \)). Hence, \( \{ M_{x,z} \land M_{y,z} < e(z, t) \leq t \} \) does not intersect \( \{ e(x, t) \land e(y, t) > t \} \).

Finally, \( \Delta_{44} = 0 \) by (6.4) now that the random variables being taken expectation are actually equal.

In summary, we have proved that \( \Delta_3 = \Delta_{43} = \Delta_{44} = 0 \). In addition, \( \Delta_1, \Delta_2, \Delta_{41} \) and \( \Delta_{42} \) satisfy (6.9), (6.10) and (6.11). We have proved (6.1).

(2°) For the left-hand side of (6.2), we use (6.4) to write

\[
\mathbb{E}_{\xi_x} \left[ \int_{\sigma_0} \int_{\sigma_2} \sigma_3 (\xi_t) \right] - \mathbb{E}_{\xi} \left[ \int_{\sigma_0} \int_{\sigma_2} \sigma_3 (\xi_t) \right] = \mathbb{E} \left[ \prod_{j \in \{0, 2, 3\}} 1_{\sigma_j} \left( M(U_j, t) \mathbb{1}_{\{ e(U_j, t) \leq t \}} + \xi(\Delta U_{t,j}, t) \mathbb{1}_{\{ e(U_j, t) \leq t \}} \right) \right] - \mathbb{E} \left[ \prod_{j \in \{0, 2, 3\}} 1_{\sigma_j} \left( M(U_j, t) \mathbb{1}_{\{ e(U_j, t) \leq t \}} + \xi(\Delta U_{t,j}, t) \mathbb{1}_{\{ e(U_j, t) \leq t \}} \right) \right]. \quad (6.12)
\]

Mutation neglects the role of the initial condition. Hence, to get a nonzero value for the difference inside the foregoing expectation, we cannot have \( e(U_j, t) \leq t \) for all \( j \in \{0, 2, 3\} \). In this case, at least one of the sums \( 1_{\sigma_j} \circ \xi(\Delta U_{t,j}, t) \) has to be nonzero. We must have \( \Delta U_{t,j} = x \) for some \( j \in \{0, 2, 3\} \). By bounding the indicator functions associated with \( \sigma_3 \) by 1, we obtain from (6.12) that

\[
|\mathbb{E}_{\xi_x} \left[ \int_{\sigma_0} \int_{\sigma_2} \sigma_3 (\xi_t) \right] - \mathbb{E}_{\xi} \left[ \int_{\sigma_0} \int_{\sigma_2} \sigma_3 (\xi_t) \right]| \leq \sum_{\ell \in \{0, 2, 3\}} \left( \mathbb{E}_{\xi_x} + \mathbb{E}_{\xi} \right) \left[ \prod_{j \in \{0, 2\}} 1_{\sigma_j} \circ \xi(U_j); \Delta U_{t,j} = x \right], \quad \forall \ x \in E. \quad (6.13)
\]

The method in (1°) now enters to remove mutations in each of the two expectations in the \( \ell \)-th summand of (6.13). We consider

\[
\mathbb{E}_\eta \left[ \prod_{j \in \{0, 2\}} 1_{\sigma_j} \circ \xi(U_j); \Delta U_{t,j} = x \right] - \mathbb{E}_\eta \left[ \prod_{j \in \{0, 2\}} 1_{\sigma_j} \circ \eta(\Delta U_{t,j}); \Delta U_{t,j} = x \right], \quad \eta \in S^E, \quad (6.14)
\]

and use only the partition in (6.7) with \( x = U_0 \) and \( y = U_2 \). In this case, on \( A_1 \), the two products of the indicator functions in (6.14) are equal. Since \( \sigma_0 \neq \sigma_2 \) ensures that the second expectation in (6.14) can be bounded by \( \mathbb{P}(M_{U_0, U_2} > t, \Delta U_{t,j} = x) \), (6.14) and a slight extension of (6.9) and (6.10) give

\[
\mathbb{E}_\eta \left[ \prod_{j \in \{0, 2\}} 1_{\sigma_j} \circ \xi(U_j); \Delta U_{t,j} = x \right] = \mathbb{E}_\eta \left[ \prod_{j \in \{0, 2\}} 1_{\sigma_j} \circ \xi(U_j); \Delta U_{t,j} = x \right].
\]

34
\[
\begin{align*}
&\leq \mathbb{P}(M_{U_0,U_2} > t, B_t^{U_1} = x) + (1 - e^{-2\mu(1)t})\mathbb{P}(M_{U_0,U_2} > t, B_t^{U_1} = x) \\
&+ 2\mu(1) \int_0^t \mathbb{P}(M_{U_0,U_2} > s, B_s^{U_1} = x)ds, \quad \forall \eta \in S^E, \ x \in E.
\end{align*}
\] (6.15)

The required inequality (6.2) now follows from (6.13) and (6.15).

(3°) The proof of (6.3) is almost the same as the proof of (6.2) and is omitted. \hfill \blacksquare

### 6.2 Full decorrelation on the large random regular graphs

In this subsection, we give a different proof of the explicit form of (4.1) by using the graphs’ local convergence. Throughout the rest of this subsection, we use the graph-theoretic terminologies from [5, 14].

We start with the definition of the random regular graphs. Fix an integer \(k \geq 3\). Choose a sequence \(\{N_n\}\) of positive integers such that \(N_n \to \infty\) and \(k\)-regular graphs (without loops and multiple edges) on \(N_n\) vertices exist. The existence of \(\{N_n\}\) follows from the Erdős–Gallai necessary and sufficient condition. Then the random \(k\)-regular graph on \(N_n\) vertices is the graph \(G_n\) chosen uniformly from the set of \(k\)-regular graphs with \(N_n\) vertices. We assume that the randomness defining the graphs is collectively subject to the probability \(\mathbb{P}\) and the expectation \(\mathbb{E}\).

For applications to the evolutionary dynamics, we need two properties of random walks on the random graphs. See [13, Section 3] and the references there for more details. First, the random walks are asymptotically irreducible in the following sense:

\[
\mathbb{P}(G_n \text{ has only one connected component}) \to 1 \quad \text{as } n \to \infty.
\] (6.16)

This property follows since the \(\mathbb{P}\)-probability that \(G_n\) has a nonzero spectral gap tends to one [23, 7]. See [14, Lemma 1.7 (d) on pp.6–7] for connections between graph spectral gaps and numbers of connected components. Second, \(G_n\) for large \(n\) is locally like the infinite \(k\)-regular tree \(G_\infty\) in the following sense. Write \(q^{(n)}(x,y)\) for the \(\ell\)-step transition probability of random walk on \(G_n\). For any \(n, r \in \mathbb{N}\), write \(T_n(r)\) for the set of vertices \(x\) in \(G_n\) such that the subgraph induced by vertices \(y\) with \(d(x,y) < r\) does not have a cycle, where \(d\) denotes the graph distance on \(G_n\). Then a standard result for the random graphs \(\{G_n\}\) [6, Section 2.4] gives

\[
\frac{N_n - \#T_n(\ell)}{N_n} \xrightarrow{n \to \infty} 0, \quad \forall \ \ell \geq 1,
\] (6.17)

where \(\xrightarrow{n \to \infty}\) refers to convergence in \(\mathbb{P}\)-probability. Note that \(\pi^{(n)}\) is uniform on \(G_n\). Consequently, if \(q^{(\infty),(\ell)}\) stands for the \(\ell\)-step transition probability of random walk on \(G_\infty\), then (6.17) implies

\[
\pi^{(n)}\{x \in T_n(2L); q^{(n),(\ell)}(x,y) = q^{(\infty),(\ell)}(x,y), \forall y, \forall \ell \in \{1, 2, \cdots, L\}\} \xrightarrow{n \to \infty} 1, \quad \forall L \in \mathbb{N}.
\] (6.18)

Below we write \(\mathbb{P}^{(n)}\) and \(\mathbb{E}^{(n)}\) for the random walk probability and the expectation under \(q^{(n)} = q^{(n),1}\) for \(n \in \mathbb{N} \cup \{\infty\}\). Notations for meeting times, random walks, and related objects associated with \(q^{(n)}\) extend to \(G_\infty\).

Recall the random variables \(U, U', V, V'\) defined at the beginning of Section 3. Now we recall some main results for the limiting distributions of \(M_{U,U'}\) and \(M_{V,V'}\) on the random regular graphs \(\{G_n\}\).
First, every (nonrandom) subsequence \( \{G_{n_j}\} \) contains a further subsequence \( \{G_{n_{ij}}\} \) such that the following properties hold \( \mathbb{P} \)-a.s.: \( G_{n_{ij}} \) are connected graphs for all (randomly) large \( j \) and

\[
\mathcal{L} \left( \frac{M_{V,V'}}{N_{n_{ij}}} \right) \xrightarrow{\text{(d)}} \mathcal{L} \left( \frac{1}{2} \frac{k-1}{k-2} e \right) \quad (6.19)
\]

[13, Remark 3.1]. Here and in what follows, a meeting time scaled by a constant indexed by \( n \) is under \( \mathbb{P}^{(n)} \), \( e \) is exponential with mean 1, and \( \mathcal{L}(X) \) denotes the distribution of \( X \). Moreover, the convergence (6.19) extends to the convergence of all moments \([13, \text{Theorem 3.3}]. \) By (6.19) and (3.22),

\[
\mathcal{L} \left( \frac{M_{V,V'}}{s_n} \right) \xrightarrow{n \to \infty} \frac{1}{k-1} \delta_0 + \frac{k-2}{k-1} \mathcal{L} \left( \frac{1}{2} \frac{k-1}{k-2} e \right) \quad \mathbb{P}\text{-a.s.} \quad (6.20)
\]

See [9, Section 4] for details. We work with \( \{G_{n_{ij}}\} \) and write this subsequence as \( \{G_n\} \) in the rest of this section.

**Proposition 6.2.** By taking a subsequence of \( \{G_n\} \) if necessary,

\[
\mathcal{L} \left( \frac{M_{V,V'}}{s_n} \right) \xrightarrow{n \to \infty} \frac{1}{k-1} \delta_0 + \frac{k-2}{k-1} \delta_\infty \quad \mathbb{P}\text{-a.s.} \quad (6.21)
\]

for any sequence \( (s_n) \) such that

\[
\lim_{n \to \infty} s_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{s_n}{N_n} = 0. \quad (6.22)
\]

**Proof.** We fix two adjacent vertices \( a \) and \( b \) in \( G_\infty \) and give the proof in a few steps.

**Step 1.** We claim that by taking a subsequence of \( \{G_n\} \) if necessary, it is possible to choose an auxiliary sequence \( (s'_n) \) of constants such that

\[
\lim_{n \to \infty} s'_n = \infty, \quad \lim_{n \to \infty} \frac{s'_n}{N_n} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{T_n(s'_n)}{N_n} = 1 \quad \mathbb{P}\text{-a.s.} \quad (6.23)
\]

To find this sequence, first, note that by (6.17), we can choose a sequence \( (\ell_n) \) such that \( \ell_n \to \infty \) and \( \frac{T_n(\ell_n)}{N_n} \to 1 \) in \( \mathbb{P} \)-probability. Fix a sequence \( (s'_n) \) such that \( s'_n \leq \ell_n \), \( s'_n/N_n \to 0 \) and \( s'_n \to \infty \). Since \( r \mapsto \#T_n(r) \) is decreasing, \( \#T_n(s'_n)/N_n \to 1 \) in \( \mathbb{P} \)-probability as well. We have proved the existence of \( (s'_n) \) satisfying (6.23) such that the third limit holds in the sense of convergence in \( \mathbb{P} \)-probability. Hence, by using a subsequence of \( \{G_n\} \) if necessary, (6.23) holds.

**Step 2.** With respect to the sequence \( (s'_n) \) chosen in **Step 1**, let \( (s_n) \) be any slower sequence such that

\[
\lim_{n \to \infty} s_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{s'_n}{s_n} = \infty. \quad (6.24)
\]

By the second limits in (6.23) and (6.24), \( s_n/N_n \to 0 \) so (6.22) holds. In the next paragraph of this step, we show that

\[
\mathbb{P}^{(n)} \left( M_{V,V'}/s_n \in \cdot \right) \xrightarrow{n \to \infty} \mathbb{P}^{(\infty)}(M_{a,b} < \infty) \delta_0 + \mathbb{P}^{(\infty)}(M_{a,b} = \infty) \delta_\infty \quad \mathbb{P}\text{-a.s.} \quad (6.25)
\]

**Step 3** will show that the limits in (6.21) and (6.25) coincide. Additionally, we will include in **Step 4** the other sequences \( (s_n) \) that satisfy (6.22) but fail to satisfy the second limit in (6.24).

36
Write $J_m$ for the $m$-th jump time of $(B^V, B^V')$ on $G_n$. For all $t \in [0, J_m]$, both $d(V, B^V_t)$ and $d(V', B^V_t')$ are bounded by $m$. Hence, on $\{(V, V') \in T_n(s'_n) \times T_n(s''_n)\}$, the law of $\{(B^V_t, B^V_t'); 0 \leq t \leq J_{[s'_n]/2}\}$ under $\mathbb{P}^{(n)}$ equals the law of $\{(B^0_t, B^0_{t'}); 0 \leq t \leq J_{[s''_n]/2}\}$ under $\mathbb{P}^{(\infty)}$. It follows that

$$\mathbb{E}^{(n)}[e^{-\lambda M_{V,V'}/s_n}; M_{V,V'} \leq J_{[s'_n]/2}] - \mathbb{E}^{(\infty)}[e^{-\lambda M_{a,b}/s_n}; M_{a,b} \leq J_{[s'_n]/2}]$$

$$= \sum_{(x,y) \in [T_n(s'_n) \times T_n(s''_n)]} \mathbb{P}((V, V') = (x, y)) \times \left( \mathbb{E}^{(n)}[e^{-\lambda M_{x,y}/s_n}; M_{x,y} \leq J_{[s'_n]/2}] - \mathbb{E}^{(\infty)}[e^{-\lambda M_{a,b}/s_n}; M_{a,b} \leq J_{[s'_n]/2}] \right),$$

and so

$$\left| \mathbb{E}^{(n)}[e^{-\lambda M_{V,V'}/s_n}] - \mathbb{E}^{(\infty)}[e^{-\lambda M_{a,b}/s_n}] \right| \leq 2 \frac{N_n - \#T_n(s'_n)}{N_n} + 2 \mathbb{E}[e^{-\lambda J_{[s''_n]/2}/s_n}]. \quad (6.26)$$

On the right-hand side of (6.26), the first term converges to zero $\mathbb{P}$-a.s. by the third limit in (6.23), and the choice of $(s_n)$ from (6.24) gives $\mathbb{E}[e^{-\lambda J_{[s''_n]/2}/s_n}] = (2/(2 + \lambda/s_n))^{[s''_n]/2} \to 0$. The right-hand side of (6.26) thus tends to zero $\mathbb{P}$-a.s. Additionally, $\mathbb{E}^{(\infty)}[e^{-\lambda M_{a,b}/s_n}] \to \mathbb{P}^{(\infty)}(M_{a,b} < \infty)$ by the first limit in (6.24). We have proved (6.25).

**Step 3.** In this step, we show that

$$\mathbb{P}^{(\infty)}(M_{a,b} < \infty) = (k - 1)^{-1} \quad (6.27)$$

and only consider random walks on $G_\infty$.

By symmetry, the hitting time $H_{a,b}$ of $b$ by $B^a$ has the same distribution as $2M_{a,b}$. Hence,

$$\mathbb{P}^{(\infty)}(M_{a,b} < \infty) = \mathbb{P}^{(\infty)}(H_{a,b} < \infty) = \mathbb{E}^{(\infty)} \left[ \int_0^\infty \mathbb{1}_{\{b\}}(B^a_t)dt \bigg/ \mathbb{E}^{(\infty)} \left[ \int_0^\infty \mathbb{1}_{\{b\}}(B^b_t)dt \right] \right],$$

where the second equality follows from a standard Green function decomposition for hitting times of points. The Green functions in the last equality satisfy

$$\frac{k - 1}{k - 2} = \mathbb{E}^{(\infty)} \left[ \int_0^\infty \mathbb{1}_{\{b\}}(B^a_t)dt \right] = 1 + \mathbb{E}^{(\infty)} \left[ \int_0^\infty \mathbb{1}_{\{b\}}(B^b_t)dt \right].$$

Here, the first equality is implied by the Kesten–McKay law for the spectral measure of $G_\infty$ (see [13, (3.3)] and the references there); the second equality uses the strong Markov property at the first jump time of $B^b$ and the symmetry of $G_\infty$. The identity in (6.27) follows from the last two displays.

**Step 4.** To complete the proof, we extend (6.25) to all the faster sequences $(s_n)$ such that (6.22) holds, but now, $\lim \inf_{n \to \infty} s'_n/s_n < \infty$. Fix any sequence $(s''_n)$ satisfying $s''_n \to \infty$ and $s'_n/s''_n \to \infty$ as in Step 2. Then it is enough to show (6.25) for all sequences $(s_n)$ satisfying (6.22) and $s_n \geq cs''_n$ for some constant $c \in (0, \infty)$.

As recalled above, the convergence in (6.19) extends to the convergence of all moments. Hence,

$$\lim_{n \to \infty} \frac{2\mathbb{E}^{(n)}[M_{U,U'}]}{N_n} = \frac{k - 1}{k - 2} \quad \mathbb{P}$-a.s. \quad (6.28)$$

Additionally, by (6.25) and (6.27), it holds that

$$\lim_{n \to \infty} \frac{2\mathbb{E}^{(n)}[M_{U,U'}]}{N_n} \mathbb{P}^{(n)}(M_{V,V'} > s''_nt) = 1, \quad \forall \, t \in (0, \infty); \quad \mathbb{P}$-a.s. \quad (6.29)\)
and so, by (6.19) and [9, Proposition 4.3 (2)], (6.29) with \( s''_n \) replaced by \( s_n \) holds. We obtain (6.22) from this limit and (6.28). The proof is complete. □

**Remark 6.3.** McKay [29, Theorem 1.1] derives the limiting spectral measures of large random regular graphs. There the randomness of graphs only plays the role of inducing asymptotically deterministic properties. For the present case, we could have worked with given sequences of \( k \)-regular graphs and obtained the same limit if the graphs have spectral gaps bounded away from zero and are locally tree-like. (Dropping the locally tree-like assumption calls for a different evaluation of the limit.) We choose to work with the above context to explain how the randomness of graphs should be handled for the convergence of the evolutionary game model. □

7 References

[1] Aldous, D. J. (1982). Markov chains with almost exponential hitting times. *Stochastic Processes and their Applications* **13**, 305–310. doi:10.1016/0304-4149(82)90016-3

[2] Aldous, D. J. and Brown, M. (1992). Inequalities for Rare Events in Time-Reversible Markov Chains I. Lecture Notes-Monograph Series **22**, 1–16. doi:10.1214/lmns/1215461937

[3] Aldous, D. J. and Fill, J. A. (2002). *Reversible Markov Chains and Random Walks on Graphs*. Unfinished monograph. Available at https://www.stat.berkeley.edu/users/aldous/RWG/book.pdf

[4] Allen, B., Lippner, G., Chen, Y.-T., Fotouhi, B., Momeni, N., Yau, S.-T. and Nowak, M. A. (2017). Evolutionary dynamics on any population structure. *Nature* **544**, 227–230. doi:10.1038/nature21723

[5] Bollobás, B. (1979). *Graph Theory: An Introductory Course*. Graduate Texts in Mathematics **63**, Springer Verlag, New York. doi:10.1007/978-1-4612-9967-7

[6] Bollobás, B. (2001). *Random Graphs*, 2nd edition. Cambridge Studies in Advanced Mathematics **73**, Cambridge University Press. doi:10.1017/CBO9780511814068

[7] Bordenave, C. (2019). A new proof of Friedman’s second eigenvalue theorem and its extension to random lifts. To appear in *Annales scientifiques de l’École Normale Supérieure*. Available at arXiv:1502.04482

[8] Chen, Y.-T. (2016). Sharp benefit-to-cost rules for the evolution of cooperation on regular graphs. *Annals of Applied Probability* **23**, 637–664. doi:10.1214/12-AAP849

[9] Chen, Y.-T., Choi, J. and Cox, J. T. (2016). On the convergence of densities of finite voter models to the Wright–Fisher diffusion. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* **52**, 286–322. doi:10.1214/14-AIHP639

[10] Chen, Y.-T. and Cox, J. T. (2018). Weak atomic convergence of finite voter models toward Fleming–Viot processes. *Stochastic Processes and their Applications* **128**, 2463–2488. doi:10.1016/j.spa.2017.09.015

[11] Chen, Y.-T., McAvoy, A. and Nowak, M. A. (2016). Fixation probabilities for any configuration of two types on regular graphs. *Scientific Reports* **6**, 39181. doi:10.1038/srep39181.

[12] Chen, Y.-T. (2018). Wright–Fisher diffusions in stochastic spatial evolutionary games with death-birth updating. *Annals of Fisher Probability* **28**, 3418–3490. doi:10.1214/18-AAP1390

[13] Chen, Y.-T. (2020). Meeting times for the voter model on large random regular graphs. arXiv:1711.00127
[14] Chung, F. R. K. (1997). *Spectral Graph Theory*. CBMS Regional Conference Series in Mathematics 92. American Mathematical Society. doi:10.1090/cbms/092

[15] Cox, J. T. (1989). Coalescing random walks and voter model consensus times on the torus in $\mathbb{Z}^d$. *Annals of Probability* 17, 1333–1366. doi:10.1214/aop/1176991158

[16] Cox, J. T. and Durrett, R. (2016). Evolutionary games on the torus with weak selection. *Stochastic Processes and their Applications* 126, 2388–2409. doi:10.1016/j.spa.2016.02.004

[17] Cox, J. T., Durrett, R. and Perkins, E. A. (2000). Rescaled voter models converge to super-Brownian motion. *Annals of Probability* 28, 185–234. doi:10.1214/aop/1019160117

[18] Cox, J. T., Durrett, R. and Perkins, E. A. (2013). Voter model perturbations and reaction diffusion equations. *Astérisque* 349. Société Mathématique de France.

[19] Cox, J. T., Merle, M. and Perkins, E. A. (2010). Coexistence in a two-dimensional Lotka–Volterra model. *Electronic Journal of Probability* 15, 1190–1266. doi:10.1214/EJP.v15-795

[20] Cox, J. T. (2017). Densities of biased voter models on finite sets converge to Feller’s branching diffusion. *Markov Processes and Related Fields* 23, 421–444.

[21] Cressman, R. (2003). *Evolutionary Dynamics and Extensive Form Games*. The MIT Press. doi:10.7551/mitpress/2884.001.0001

[22] Ethier, S. N. and Kurtz, T. G. (2005). *Markov Processes: Characterization and Convergence*, 2nd edition. Wiley Series in Probability and Statistics. Wiley-Interscience.

[23] Friedman, J. (2008). A proof of Alon’s second eigenvalue conjecture and related problems. *Memoirs of the American Mathematical Society* 195. doi:10.1090/memo/0910

[24] Hofbauer, J. and Sigmund, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge University Press. doi:10.1017/CBO9781139173179

[25] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edition. Grundlehren der mathematischen Wissenschaften 288. Springer-Verlag, Berlin. doi:10.1007/978-3-662-05265-5

[26] Liggett, T. M. (2005). *Interacting Particle Systems*. Reprint of the 1985 edition with a new postface. Classics in Mathematics 276. Springer, Berlin. doi:10.1007/b138374

[27] Levin, D. A., Peres, Y. and Wilmer, E. L. (2009). *Markov Chains and Mixing Times*. American Mathematical Society, Providence.

[28] Maynard Smith, J. and Price, G. R. (1973). The logic of animal conflict. *Nature* 246, 15–18. doi:10.1038/246015a0

[29] McKay, B. D. (1981). The expected eigenvalue distribution of a large regular graph. *Linear Algebra and its Applications* 40, 203–216. doi:10.1016/0024-3795(81)90150-6

[30] Müller, C. and Tribe, R. (1995). Stochastic p.d.e.’s arising from the long range contact and long range voter processes. *Probability Theory and Related Fields* 102, 519–545. doi:10.1007/BF01198848

[31] Ohtsuki, H. and Nowak, M. A. (2006). The replicator equation on graphs. *Journal of Theoretical Biology* 243, 86–97. doi:10.1016/j.jtbi.2006.06.004

[32] Ohtsuki, H., Hauert, C., Lieberman, E. and Nowak, M. A. (2006). A simple rule for the evolution of cooperation on graphs and social networks. *Nature* 441, 502–505. doi:10.1038/nature04605
[33] Revuz, D. and Yor, M. (2005). *Continuous Martingales and Brownian Motions*, corrected 3rd edition. Grundlehren der mathematischen Wissenschaften 293. Springer-Verlag, Berlin. doi:10.1007/978-3-662-06400-9

[34] Rogers, L. C. G. and Pitman, J. W. (2005). Markov functions. *Annals of Probability* **9**, 573–582. doi:10.1214/aop/1176994363

[35] Schuster, P. and Sigmund, K. (1983). Replicator dynamics. *Journal of Theoretical Biology* **100**, 533–538. doi:10.1016/0022-5193(83)90445-9

[36] Szabó, G. and Fáth, G. (2007). Evolutionary games on graphs. *Physics Reports* **446**, 97–216. doi:10.1016/j.physrep.2007.04.004

[37] Taylor, P. D. and Jonker, L. B. (1978). Evolutionary stable strategies and game dynamics. *Mathematical Biosciences* **40**, 145–156. doi:10.1016/0025-5564(78)90077-9