Construction of $\mathcal{PT}$–asymmetric non-Hermitian Hamiltonians with $\mathcal{CPT}$ symmetry

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Abstract

Within $CPT$—symmetric quantum mechanics the most elementary differential form of the “charge operator” $\mathcal{C}$ is assumed. A closed-form integrability of the related coupled differential self-consistency conditions and a natural embedding of the Hamiltonians in a supersymmetric scheme is achieved. For a particular choice of the interactions the rigorous mathematical consistency of the construction is scrutinized suggesting that quantum systems with non-self-adjoint Hamiltonians may admit probabilistic interpretation even in presence of a manifest breakdown of both $\mathcal{T}$ symmetry (i.e., Hermiticity) and $\mathcal{PT}$ symmetry.

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1 Introduction

The popularity of anharmonic-oscillator models, such as

\[ H^{(\text{AHO})}(f, g) = -\frac{d^2}{dx^2} + x^2 + f x^3 + g x^4 \]

seems to reflect a fortunate combination of physical appeal (the potential is safely confining at \( g > 0 \)) and computational tractability. In this letter, we intend to join the effort of studying these models in a non-self-adjoint regime [1]. While the construction of the solutions becomes fairly easy in perturbative framework [2,3], a certain paradox arises because the perturbative power series (near \( f = 0 \))

\[ E^{(\text{AHO})}(f, g) = \sum_{k=0}^{\infty} f^k E_k^{(\text{AHO})}(0, g) \]

should represent the energies for all the complex couplings which lie in a sufficiently small circle of convergence. A deeper analysis [4,5] revealed that the energies \( E_n^{(\text{AHO})}(f, g) \) should be considered as the infinite sets of the values of a single analytic function of the couplings on different Riemann sheets.

The latter idea has steadily stimulated interest in the manifestly non-Hermitian anharmonic oscillators [6–8]. Finally, a real boom of interest in similar models arose after the seminal letter [9] by Bender and Boettcher, who argued that the reality of the spectra should be related to the symmetries of the Hamiltonians. Indeed, once we re-write Hermiticity, \( H = H^\dagger \), in the form of an involutive time-reversal symmetry, \( \mathcal{T} [10] \),

\[ \mathcal{T} H = H \mathcal{T} , \quad (1) \]

it is quite natural to replace eq. (1) with the constraint

\[ \mathcal{PT} H = H \mathcal{PT} \quad (2) \]
where $\mathcal{P}$ denotes parity. Eq. (2) is valid for Hamiltonians that are invariant under $\mathcal{PT}$, but not necessarily under $\mathcal{P}$ and $\mathcal{T}$ separately.

The expected reality of the energies $E_{n}^{(AHO)}(i\lambda, g)$, with real $\lambda$ and $g$, has been supported, in some cases, by rigorous proofs [11,12]. A further weakening of the standard Hermiticity is possible once we replace $\mathcal{P}$ in Eq. (2) by any other Hermitian operator $\mathcal{F} = \mathcal{F}^\dagger$ [13]. One thus has the new condition [14]

$$\mathcal{F}\mathcal{T}H = H\mathcal{F}^\dagger \iff \mathcal{F}H^\dagger = H\mathcal{F},$$

(3)

which, for $\mathcal{F} = 1$, becomes Hermiticity and for $\mathcal{F} = \mathcal{P}$ becomes $\mathcal{PT}$ symmetry. Equation (3) implies that if $H$ has an eigenvalue $E$, then $E^*$, apart from normalization problems, is also an eigenvalue unless $\mathcal{F}\psi^* = 0$, so eigenvalues are either real, or enter in complex conjugate pairs [15].

One could, generically, construct many operators $\mathcal{F}$ which would be, via eq. (3), compatible with a given Hamiltonian $H$. Among all the possible choices of these (metric) operators, a privileged position is occupied by the positive bounded Hermitian ones, because the corresponding Hamiltonians admit probabilistic interpretation [16].

## 2 CPT–symmetric models

### 2.1 Factorized $\mathcal{F}$

One may demand a factorization of the operator $\mathcal{F} = \mathcal{F}^\dagger$ into a product, say, $\mathcal{F} \equiv \mathcal{C}\mathcal{P}$ where, conventionally, $\mathcal{C}$ can be called a “charge conjugation” operator [1,17]. In principle, this would constitute a $\mathcal{CPT}$-symmetric quantum mechanics, with an obvious ambition of being a zero-dimensional $\mathcal{CPT}$-symmetric field theory.

For our purposes, however, the involutory property $\mathcal{C}^2 = 1$ of the charge, or
of the so called quasi-parity [18], in some exactly solvable one-dimensional examples

\[ H = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R} \]  

(4)

is by far not necessary. Our interest will naturally be focused on the possibility that any system in \(\mathcal{CPT}\)-symmetric quantum mechanics may violate both the \(\mathcal{PT}\) and \(\mathcal{T}\) symmetries.

We proceed constructively and, for the sake of definiteness, we select the class of operators

\[ \mathcal{C} = \frac{d}{dx} + w(x) \]

in conjunction with the above Hamiltonians (4).

In our specific Ansatz for \(\mathcal{C}\), in order to enforce Hermiticity of \(\mathcal{F}\), keeping in mind that \(\mathcal{P} = \mathcal{P}^\dagger\) and \(\mathcal{P}^2 = 1\), we have to postulate definite spatial symmetry properties of the complex function \(w(x) = \sigma(x) + i \alpha(x)\) with

\[ \Re w(x) = \sigma(x) = \sigma(-x), \quad \Im w(x) = \alpha(x) = -\alpha(-x), \quad x \in \mathbb{R} \]  

(5)

### 2.2 Compatibility conditions and their integrability

Our Hamiltonian, \(H\), is assumed compatible with the \(\mathcal{CPT}\) symmetry condition, i.e. Eq. (3) with \(\mathcal{F} = \mathcal{CP}\). A verification of this condition will be the core of our present construction. It necessitates a decomposition of our Hamiltonian in the sum

\[ H = -\frac{d^2}{dx^2} + \Sigma(x) + K(x) + i S(x) + i D(x), \]  

(6)

where the separate components of the complex potential \(V(x)\) may be chosen to exhibit definite parities,

\[ \Sigma(x) = \Sigma(-x), \quad K(x) = -K(-x), \quad S(x) = S(-x), \quad D(x) = -D(-x). \]
This simplifies the form of $H^\dagger$ and our main compatibility constraint (3). In principle, it should be a linear differential operator of the third order but once we re-write it in the form of a product, $[HC - C(PH^\dagger P)]P$, we immediately see that the coefficients of the third and of the second derivative are identically zero. A condition of the vanishing of the coefficient of the first derivative remains nontrivial and relates the potentials $K$ and $S$ to the choice of $C$,

$$K(x) = \frac{d}{dx} \sigma(x), \quad S(x) = \frac{d}{dx} \alpha(x).$$  \tag{7}

In this way we are left with the condition which connects the two complex functions $V(x)$ and $w(x)$. Its separation into real and imaginary part proves encouragingly simple

$$\begin{aligned}
\frac{d}{dx} \Sigma(x) &= 2 \sigma(x) \frac{d}{dx} \sigma(x) - 2 \alpha(x) \frac{d}{dx} \alpha(x), \\
\frac{d}{dx} D(x) &= 2 \sigma(x) \frac{d}{dx} \alpha(x) + 2 \alpha(x) \frac{d}{dx} \sigma(x),
\end{aligned}$$  \tag{8}

luckily admitting an entirely elementary integration, with just a single real integration constant, $\omega$

$$\Sigma(x) = \sigma^2(x) - \alpha^2(x) + \omega, \quad D(x) = 2 \sigma(x) \alpha(x),$$  \tag{9}

This means that we may contemplate a family of anharmonic-oscillator examples with $\sigma(x) = \sigma_n(x) = \mu_n x^{2n}$ and $\alpha(x) = \alpha_n(x) = \nu_n x^{2n-1}$ for illustration, with real $\mu_n$, $\nu_n$ and with any choice of the index $n = 1, 2, \ldots$. 
3 Discussion

3.1 Supersymmetric picture

One can rewrite equations (9) supersymmetrically [19]. It is not difficult to check that $H$ of eq. (6) with $\omega = 0$ satisfies

$$H = \mathcal{F}\mathcal{F}^*, \quad H^\dagger = \mathcal{F}^*\mathcal{F},$$

where $\mathcal{F}^*$ is the complex conjugate of $\mathcal{F}$. We introduce the super-charges

$$Q \equiv \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}, \quad \tilde{Q} \equiv \begin{pmatrix} 0 & 0 \\ \mathcal{F}^* & 0 \end{pmatrix}. \quad (11)$$

It is easy to check that $Q \neq \tilde{Q}^\dagger$, insofar as $\mathcal{F}^\dagger \neq \mathcal{F}^*$, $Q^2 = \tilde{Q}^\dagger = 0$, and $[Q, H] = [\tilde{Q}, H] = 0$. The super-Hamiltonian reads

$$\mathcal{H} \equiv \{Q, \tilde{Q}\} = \begin{pmatrix} \mathcal{F}\mathcal{F}^* & 0 \\ 0 & \mathcal{F}^*\mathcal{F} \end{pmatrix}. \quad (12)$$

It is worth noting that in a way characteristic for non-Hermitian supersymmetric examples [20] the operator $\mathcal{H}$ is not necessarily positive.

3.2 The problem of invertibility

Whenever our operator $\mathcal{F}$ is unbounded [16], it may still be invertible and the inverse operator may be bounded. If $\mathcal{F}^{-1}$ exists, one can derive algebraically the following relation

$$H^\dagger\mathcal{F}^{-1} = \mathcal{F}^{-1}H.$$
Supersymmetrically, one can define new conserved charges

\[
Q^{-1} = \begin{pmatrix} 0 & 0 \\ \mathcal{F}^{-1} & 0 \end{pmatrix}, \quad \tilde{Q}^{-1} = \begin{pmatrix} 0 & \mathcal{F}^{*-1} \\ 0 & 0 \end{pmatrix}.
\]

(14)

such that

\[
[Q^{-1}, H] = [	ilde{Q}^{-1}, H] = 0,
\]

(15)

with \( Q^{-1} \) being not the standard inverse operator, but satisfying

\[
\{Q, Q^{-1}\} = \mathcal{I}, \quad \{\tilde{Q}, \tilde{Q}^{-1}\} = \mathcal{I},
\]

(16)

with \( \mathcal{I} \) the identity operator, in analogy with the anticommutation relations of fermion creators and annihilators.

Let us now examine in more detail the case \( n = 1 \), with \( \sigma(x) = \mu_1 x^2 \), where \( \mu_1 \neq 0 \), and \( \alpha(x) = \nu_1 x \). \( \mathcal{F} \) is not bounded and not positive; however, it is invertible in \( L^2(\mathbb{R}) \) and Eq. (13) holds. In fact, as an operator in \( L^2(\mathbb{R}) \),

\[
\mathcal{C} = \frac{d}{dx} + \mu_1 x^2 + i\nu_1 x
\]

is unitarily equivalent to

\[
\mathcal{C}_1 = \frac{d}{dx} + \mu_1 x^2 + \nu_1^2/(4\mu_1),
\]

via the translation \( x \rightarrow x - i\nu_1/(2\mu_1) \). In turn, \( \mathcal{C}_1 \) is unitarily equivalent to

\[
\mathcal{C}_2 = -\mu_1 d^2/dx^2 + ix + \nu_1^2/(4\mu_1),
\]

via the Fourier transformation. As is well known [21], \( \mathcal{C}_2 \) has empty spectrum and is thus invertible; as a consequence, \( \mathcal{C} \) is invertible, too; the same holds for \( \mathcal{F} = \mathcal{C}\mathcal{P} \) (see also Ref. [22]), which is therefore invertible in \( L^2(\mathbb{R}) \) with bounded inverse, \( \mathcal{F}^{-1} \), defined on the whole \( L^2(\mathbb{R}) \).

### 3.3 The problem of positivity

While \( \mathcal{F}^{-1} \) for \( n = 1 \) is a bounded Hermitian operator acting on \( L^2(\mathbb{R}) \) [22], the positivity requirement presents problems, in general; however, evaluation of matrix elements of Eq. (13) between eigenstates of the Hermitian operator
$\mathcal{F}^{-1}$ yields

\[
\langle j \mid H^\dagger \mathcal{F}^{-1} \mid k \rangle = \langle j \mid \mathcal{F}^{-1} H \mid k \rangle, \tag{17}
\]
\[
\lambda_k \langle j \mid H^\dagger_1 \mid k \rangle = \lambda_j \langle j \mid H_1 \mid k \rangle, \tag{18}
\]

where $H_1$ reads

\[
H_1 = -\frac{d^2}{dx^2} + \mu_1^2 x^4 - \nu_1^2 x^2 + 2\mu_1 x + i\nu_1 + 2i\mu_1\nu_1 x^3 \tag{19}
\]
\[
\equiv H_R + i\nu_1 + 2i\mu_1\nu_1 x^3 = H_R + i\nu_1 V_I. \]

We can rewrite Eq. (18) as

\[
\lambda_k \left( H_R^{jk} - i\nu_1 V_I^{jk} \right) = \lambda_j \left( H_R^{jk} + i\nu_1 V_I^{jk} \right), \tag{20}
\]

where $H_R^{jk} = A + iB$ and $V_I^{jk} = C + iD$ are complex numbers. Thus, by equating real and imaginary parts of both sides of Eq. (20), one gets

\[
\frac{\lambda_k}{\lambda_j} = \frac{A^2 - \nu_1^2 D^2}{(A + \nu_1 D)^2} = \frac{B^2 - \nu_1 C^2}{(B - \nu_1 C)^2}, \tag{21}
\]

and, if $\lambda_k/\lambda_j < 0$, one can argue that the $H_R^{jk}$’s are strongly suppressed for small values of $\nu_1$. This may lead to a practical decoupling of the two sectors of positive and negative eigenvalues, thus supporting $\mathcal{F}^{-1}$ as a metric operator candidate, since, physically, it is not so important that $\mathcal{F}^{-1}$ is positive, but it is crucial that the Hamiltonian connects only weakly the sectors of positive and negative eigenvalues.

Coming now to the properties of Hamiltonian (19), one can separate the $\mathcal{PT}$-symmetric and antisymmetric parts as

\[
H_1 = H_1^{\mathcal{PT}} + 2\mu_1 x + i\nu_1, \tag{22}
\]

where

\[
H_1^{\mathcal{PT}} = -\frac{d^2}{dx^2} + \mu_1^2 x^4 - \nu_1^2 x^2 + 2i\mu_1\nu_1 x^3. \tag{23}
\]
$H_T^{PT}$ is well controlled from a mathematical point of view, so that our proposal opens a way to study some additional Hamiltonians enriching the class of the recent popular non-Hermitian versions by addition of the non-$\mathcal{PT}$-symmetric Stark-like term. It is worthwhile to point out that, performing a shift $x \rightarrow x - i\nu_1/(2\mu_1)$, one can show that Hamiltonian (22) has real spectrum [23].

In general, for all $\mu, \nu \in \mathbb{R}$, let $H(\mu, \nu)$ denote the Schrödinger operator in $L^2(\mathbb{R})$ defined by

$$H(\mu, \nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + 2i\mu\nu x^3 - \nu^2 x^2 + 2\mu x \quad (24)$$

on the domain $D = H^2(\mathbb{R}) \cap L_4^2(\mathbb{R})$. Then $H(\mu, \nu)$ is a closed operator with compact resolvents and, therefore, discrete spectrum. In fact, the operator $-\frac{d^2}{dx^2} + \mu^2 x^4$ enjoys such properties (see [24]), which extend to the analytic family

$$H_g(\mu, \nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + g(2i\mu\nu x^3 - \nu^2 x^2 + 2\mu x) \quad (25)$$

for $g \in \mathbb{C}$ and in particular to the original operator $H(\mu, \nu)$ for $g = 1$ (for more details on the theory of analytic families of operators see [3] or [5]).

If we now introduce a further perturbation parameter $\gamma \in \mathbb{C}$ only in the linear term:

$$H_\gamma(\mu, \nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + 2i\mu\nu x^3 - \nu^2 x^2 + 2\mu\gamma x, \quad (26)$$

then $H_{\gamma=0}(\mu, \nu)$ is $\mathcal{PT}$-symmetric with real spectrum [12], while for finite non-zero values of $\gamma$ the spectrum is complex [23].

The spectral analysis for the complete operator $H_\gamma(\mu, \nu)$ for $\gamma \in \mathbb{R}$ can be performed in the framework of perturbation theory around $\gamma = 0$. More precisely, referring to results in Ref. [25], it is possible to prove that for fixed $\mu$ and $\nu$ there exists $\delta > 0$ such that the eigenvalues of $H_\gamma(\mu, \nu)$ are real and represent a sequence of analytic functions $E_\gamma(\gamma)$ for $\gamma \in \mathbb{R}$. For such values of $\gamma$
each eigenvalue $E_n(\gamma)$ is the sum of the corresponding Rayleigh-Schrödinger perturbation expansion around the eigenvalue $E_n(0)$ of $H_0(\mu, \nu)$.

Yet to be explored is the usefulness of second and higher order derivatives in the ansatz for the $C$ operator, with the possibility of non-linear algebraic structures [23,26].

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