The Area Metric Reality Constraint in Classical General Relativity

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Abstract

A classical foundation for an idea of reality condition in the context of spin foams (Barrett-Crane models) is developed. I extract classical real general relativity (all signatures) from complex general relativity by imposing the area metric reality constraint; the area metric is real iff a non-degenerate metric is real or imaginary. First I review the Plebanski theory of complex general relativity starting from a complex vectorial action. Then I modify the theory by adding a Lagrange multiplier to impose the area metric reality condition and derive classical real general relativity. I investigate two types of action: Complex and Real. All the non-trivial solutions of the field equations of the theory with the complex action correspond to real general relativity. Half the non-trivial solutions of the field equations of the theory with the real action correspond to real general relativity. Discretization of the area metric reality constraint in the context of Barrett-Crane theory is discussed. In the context of Barrett-Crane theory the area metric reality condition is equivalent to the condition that the scalar products of the bivectors associated to the triangles of a four simplex be real. The Plebanski formalism for the degenerate case and Palatini formalism are also briefly discussed by including the area metric reality condition.

1 Introduction

1.1 Motivation

The problem of imposing reality conditions is a non-trivial problem in canonical quantum gravity [9]. My research indicates that there is an analogous concept of reality conditions in the context of spin foam models of gravity [5]. My goal in this paper is to discuss the classical foundation of this idea. The quantum application of this idea is dealt with in Ref.[1].

Let me briefly discuss ideas from spin foam models which served as the motivation for this article. Consider the Barrett-Crane models of Lorentzian general relativity [4]. It is developed using the Gelfand-Naimarck unitary representation theory of $SL(2, C)$ [21]. A unitary representation of $SL(2, C)$ is labeled by a
complex number \( \chi = \frac{n}{2} + i\rho \), where \( \rho \) is a real number and \( n \) is an integer. A Hilbert space \( D_\chi \) of a unitary representation of the Lorentz group \( SL(2, C) \) is assigned to each triangle of a simplicial manifold. There are two real Casimirs for \( SL(2, C) \). Upto numerical constants the eigenvalues are \( \rho n \) and \( -\rho^2 + \frac{n^2}{4} \). The \( -\rho^2 + \frac{n^2}{4} \) corresponds to the area spectrum in the Lorentzian Barrett-Crane models. The Barrett-Crane simplicity constraint requires \( \rho n = 0 \). So we are allowed to assign only one of \( \chi = \rho \) and \( \chi = i\frac{n}{2} \) to each triangle.

The two real Casimirs of \( SL(2, C) \) can be written together in a complex form

\[ \hat{\mathcal{C}} = \det \left[ \begin{array}{cc} \hat{X}_3 & \hat{X}_1 - i \hat{X}_2 \\ \hat{X}_1 + i \hat{X}_2 & -\hat{X}_3 \end{array} \right], \]

where \( X_i = F_i + iH_i \in \text{sl}(2, C) \), the \( H_k \) correspond to rotations and the \( F_k \) correspond to boosts. The eigen-value of the complex Casimir in \( D_\chi \) is

\[ \chi^2 - 1 = -\rho^2 + \frac{n^2}{4} - 1 + i\rho n. \]  (1)

The \( \rho n \) is precisely the imaginary part of the Casimir. So if \( \chi^2 - 1 \) is interpreted as the square of the area of a triangle, then \( \rho n = 0 \) simply constrains the square of the area to be real.

The reality of the squares of the areas is better understood from the point of view of the Barrett-Crane model for \( SO(4, C) \) general relativity theory developed in Ref: [1]. The \( SO(4, C) \) Barrett-Crane model can be constructed using the unitary representation theory of the group \( SO(4, C) \) [1]. The unitary representations of \( SO(4, C) \) can be constructed using the relation

\[ SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z^2}. \]  (2)

So similar to the unitary representation theory of \( SO(4, R) \), the unitary representations of \( SO(4, C) \) can be labeled by two \( \chi \)'s: \( \chi_L = \frac{n}{2} + i\rho_L \), \( \chi_R = \frac{n}{2} + i\rho_R \), where each \( \chi \) represents a unitary representation of \( SL(2, C) \) [2]. There are two Casimirs for \( SO(4, C) \) which are essentially the sum and the difference of the Casimirs of the left and the right handed \( SL(2, C) \) parts.

The \( SO(4, C) \) Barrett-Crane simplicity constraint sets one of the \( SO(4, C) \) Casimir’s eigen value \( \chi_L^2 - \chi_R^2 \) to be zero, which in turn sets \( \chi_L = \pm \chi_R \) (=\( \chi \) say). Then the other Casimir’s eigen value is

\[ (\chi_L^2 + \chi_R^2 - 2)/2 = \chi^2 - 1, \]

which corresponds to the square of the area of a triangle. By setting this eigenvalue to be real, we deduce the area quantum number to be assigned to a triangle

\[ \text{Here the } n_L + n_R \text{ must be an even number. Please see appendix B of [1] for details.} \]
of a Lorentzian spin foam. So from the point view of the $SO(4, C)$ Barrett-Crane model, the simplicity condition of Lorentzian general relativity appears to be a reality condition for the squares of the areas.

The Barrett-Crane four simplex amplitude can be formally expressed using a complete set of orthonormal propagators over a homogenous space of the gauge group. The $SO(4, C)$ Barrett-Crane model involves the propagators on the homogenous space $SO(4, C)/SL(2, C)$ which is the complex three sphere $CS^3$ \[1\]. The complex three sphere $CS^3$ is defined in $C^4$ by

$$x^2 + y^2 + z^2 + t^2 = 1,$$

where $x, y, z, t$ are complex coordinates. The propagators can be considered as the eigen functions of the square of the area operator with the complex area eigen values. The homogenous spaces corresponding to real general relativity theories of all signatures are real subspaces of $CS^3$ such that 1) they possess a complete set of orthonormal propagators\(^2\) and 2) the propagators correspond to the real squares of area eigenvalues \[1\]. Then this naturally suggests that the spin foams for real general relativity theories for all signatures are formally related to the $SO(4, C)$ Barrett-Crane model motivated by the reality of the squares of the areas \(^3\).

Even though the above two paragraphs suggests the reality of the square of areas as the reality conditions in the context of spin foams the correct form of the reality conditions will be discussed below.

1.2 Content and Organization

This article aims to develop a classical foundation for the relationship between real general relativity theories and $SO(4, C)$ general relativity through a reality constraint which has application to Barrett-Crane theory \[3\]. The classical continuum analog of the square of area operators of spin foams is the area metric. In the case of non-degenerate general relativity, it will be shown in this article that the reality of the area metric is the necessary and the sufficient condition for real geometry. Since an area metric can be easily expressed in terms of a bivector 2-form field, the area metric reality condition can be naturally combined with the Plebanski theory \[2\] of general relativity using a Lagrange multiplier.

On a simplicial manifold a bivector two form field can be discretized by associating bivectors to the triangles. In the context of the Barrett-Crane theory, it will be shown in this article that the necessary and sufficient condition for the reality of a flat four simplex geometry is condition that the scalar products of the bivectors associated to the triangles be real. This idea in conjunction with the Barrett-Crane constraint can be used to develop unified treatment of the Barrett-Crane models for the four di-

\(^2\)The propagators are complete in the sense that there exists a sum over them that yields a delta function on the homogenous space.

\(^3\)The Barrett-Crane model based on the propagators on the null-cone \[4\] is an exception to this.
Let me briefly discuss the content and organization of this article. In section two of this article I review the Plebanski formulation of $SO(4,C)$ general relativity starting from vectorial actions. In section three I discuss the area metric reality constraint. After solving the Plebanski (simplicity) constraints, I show that, the area metric reality constraint requires the space-time metric to be real or imaginary for the non-denegerate case.

In section four I modify the vectorial Plebanski actions by adding a Lagrange multiplier to impose the reality constraint. For the complex action all the non-trivial solutions of the field equations correspond to real general relativity. For the case of real action I show that real general relativity emerges for non-degenerate metrics for the following cases 1) the metric is real and the signature type is Riemannian or Kleinian and 2) the metric is imaginary and Lorentzian.

In section five I discuss the discretization of the area metric reality constraint on the simplicial manifolds in the context of the Barrett-Crane theory. I also discuss various possible discrete actions.

In section six I discuss various further considerations: the area metric reality constraint for arbitrary metrics, the Plebanski formulation with the reality constraint for the degenerate case briefly and the Palatini’s formulation with the area metric constraint.

In the appendix I have discussed the spinorial expansion of a tensor with the symmetries of the Riemann curvature tensor.

2 $SO(4,C)$ General Relativity

Plebanski’s work on complex general relativity presents a way of recasting general relativity in terms of bivector 2-form fields instead of tetrad fields or space-time metrics. It helped to reformulate general relativity as a topological field theory called the BF theory with a constraint (for example Reisenberger). Originally Plebanski’s work was formulated using spinors instead of vectors. The vector version of the work can be used to formulate spin foam models of general relativity. Understanding the physics behind this theory simplifies with the use of spinors. Here I would like to review the Plebanski theory for a $SO(4,C)$ general relativity on a four dimensional real manifold starting from vectorial actions.

In the cases of Riemannian and $SO(4,C)$ general relativity the Lie algebra elements are the same as the bivectors. Let me define some notations to be used in this article.

Notation 1 I would like to use the letters $i, j, k, l, m, n$ as $SO(4,C)$ vector indices, the letters $a, b, c, d, e, f, g, h$ as space-time coordinate indices, the letters $A, B, C, D, E, F$ as spinorial indices to do spinorial expansion on the coordinate
indices. On arbitrary bivectors $a^{ij}$ and $b^{ij}$, I define
\[ a \wedge b = \frac{1}{2} \varepsilon_{ijkl} a^{ij} b^{kl} \quad \text{and} \quad a \bullet b = \frac{1}{2} \eta_{ij} a^{ij} b^{kl}. \]

2.1 BF $SO(4, C)$ action

Consider a four dimensional manifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B^{ij}$ a complex bivector valued 2-form on $M$. I would like to restrict myself to non-denegerate general relativity in this and the next section by assuming $b = \frac{1}{4!} \varepsilon^{abcd} B_{ab} \wedge B_{cd} \neq 0$. Let $F$ be the curvature 2-form of the connection $A$. I define real and complex continuum $SO(4, C)$ BF theory actions as follows,

\[ S_{cBF}(A, B_{ij}) = \int_M \varepsilon^{abcd} B_{ab} \wedge F_{cd} \quad \text{and} \quad (3) \]

\[ S_{rBF}(A, B_{ij}, \bar{A}, \bar{B}_{ij}) = \text{Re} \int_M \varepsilon^{abcd} B_{ab} \wedge F_{cd}. \quad (4) \]

The $S_{cBF}$ is considered as a holomorphic functional of its variables. In $S_{rBF}$ the variables $A, B_{ij}$ and their complex conjugates are considered as independent variables. The wedge is defined in the Lie algebra coordinates. The field equations corresponding to the extrema of these actions are

\[ D_{[a} B_{bc]} = 0 \quad \text{and} \quad F_{cd} = 0. \]

BF theories are topological field theories. It is easy to show that the local variations of solutions of the field equations are gauged out under the symmetries of the actions [5]. The spin foam quantization of the BF theory using the real action has been discussed in Ref. [1].

2.2 Actions for $SO(4, C)$ General Relativity

The Plebanski actions for $SO(4, C)$ general relativity is got by adding a constraint term to the BF actions. First let me define a complex action [13],

\[ S_{cGR}(A, B_{ij}, \phi) = \int_M \left[ \varepsilon^{abcd} B_{ab} \wedge F_{cd} + \frac{1}{2} b^{abcd} B_{ab} \wedge B_{cd} \right] d^4x, \quad (5) \]

and a real action

\[ S_{rGR}(A, B_{ij}, \bar{A}, \bar{B}_{ij}, \bar{\phi}) = \text{Re} S_C(A, B_{ij}, \phi). \quad (6) \]

\[ \text{The wedge product in the bivector coordinates plays a critical role in the spin foam models. This is the reason why the } \wedge \text{ is used to denote a bivector product instead of an exterior product.} \]
The complex action is a holomorphic functional of its variables. Here $\phi$ is a complex tensor with the symmetries of the Riemann curvature tensor such that $\phi^{abcd} \epsilon_{abcd} = 0$. The $b$ is inserted to ensure the invariance of the actions under coordinate change.

The field equations corresponding to the extrema of the actions $S_C$ and $S$ are

\[ D_{[a} B^{ij}_{bc]} = 0, \quad (7a) \]

\[ \frac{1}{2} \epsilon^{abcd} F_{cd} = b \phi^{abcd} B_{ij}^{cd} \text{ and,} \quad (7b) \]

\[ B_{ab} \wedge B_{cd} - b \epsilon_{abcd} = 0, \quad (7c) \]

where $D$ is the covariant derivative defined by the connection $A$. The field equations for both the actions are the same.

Let me first discuss the content of equation (7c) called the simplicity constraint. The $B_{ab}$ can be expressed in spinorial form as

\[ B_{ab}^{ij} = B_{AB}^{ij} \epsilon_{\dot{A}\dot{B}} + B_{\dot{A}\dot{B}}^{ij} \epsilon_{AB}, \]

where the spinor $B_{AB}$ and $B_{\dot{A}\dot{B}}$ are considered as independent variables. The tensor

\[ P_{abcd} = B_{ab} \wedge B_{cd} - b \epsilon_{abcd} \]

has the symmetries of the Riemann curvature tensor and its pseudo-scalar component is zero. In appendix A the general ideas related to the spinorial decomposition of a tensor with the symmetries of the Riemann Curvature tensor have been summarized. The spinorial decomposition of $P_{abcd}$ is given by

\[ P_{abcd} = B_{(AB} \wedge B_{CD)} \epsilon_{\dot{A}\dot{B} \dot{C}\dot{D}} + B_{(\dot{A}\dot{B} \wedge B_{CD)} \epsilon_{AB} \epsilon_{\dot{C}\dot{D}} +} + \frac{\hat{b} \delta_{[a} \delta_{b]d}}{6} B_{AB} \wedge B_{\dot{A}\dot{B}} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{D}B} (\epsilon_{\dot{A}\dot{B}} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{D}B} + \epsilon_{AB} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{C}\dot{D}} \epsilon_{\dot{D}B}), \]

where $\hat{b} = B_{AB} \wedge B^{AB} + B_{\dot{A}\dot{B}} \wedge B^{\dot{A}\dot{B}}$. Therefore the spinorial equivalents of the equations (7c) are

\[ B_{(AB} \wedge B_{CD)} = 0, \quad (8a) \]

\[ B_{(\dot{A}\dot{B} \wedge B_{CD)} = 0, \quad (8b) \]

\[ B_{AB} \wedge B^{AB} + B_{\dot{A}\dot{B}} \wedge B^{\dot{A}\dot{B}} = 0 \text{ and} \quad (8c) \]

\[ B_{AB} \wedge B_{\dot{A}\dot{B}} = 0. \quad (8d) \]

These equations have been analyzed by Plebanski \[2\]. The only difference between my work (also Reisenberger \[13\]) and Plebanski’s work is that I have spinorially decomposed on the coordinate indices of $B$ instead of the vector indices. But this does not prevent me from adapting Plebanski’s analysis of these equations as the algebra is the same. From Plebanski’s work, we can conclude that the above equations imply $B_{ab}^{ij} = \theta_a^i \theta_b^j$ where $\theta_a^i$ are a complex tetrad.
Equations (8) are not modified by changing the signs of $B_{AB}$ or $B_{BA}$. These are equivalent to replacing $B_{ab}$ by $-B_{ab}$ or $\pm \frac{1}{2} \epsilon_{ab}^{cd} B_{cd}$ which produce three more solutions of the equations [15], [13].

The four solutions and their physical nature were discussed in the context of Riemannian general relativity by Reisenberger [13]. It can be shown that equation (13a) is equivalent to the zero torsion condition. Then $A$ must be the complex Levi-Civita connection of the complex metric $g_{ab} = \delta_{ij} \theta_i^a \theta_j^b$ on $M$. Because of this the curvature tensor $F_{cd}^{ab} = F_{ij}^{ab} \theta_i^c \theta_j^d$ satisfies the Bianchi identities. This makes $F$ to be the $SO(4, C)$ Riemann curvature tensor. Using the metric $g_{ab}$ and its inverse $g^{ab}$ we can lower and raise coordinate indices.

We can define the dualization operation on an arbitrary antisymmetric tensor $S_{ab}$ as

$$\ast S_{ab} = \frac{1}{2} g_{ca} g_{db} \epsilon^{cdef} S_{ef},$$

(9)

where $\epsilon_{abcd}$ is the undensitized epsilon tensor. It can be verified that $\ast \ast S_{ab} = g S_{ab}$. To differentiate between the dual operations on the suffices and the prefixes let me define two new notations:

$$\bar{S}_{ab} = \ast S_{ab},$$

$$\overline{S}^{ab} = g^{ae} g^{bd} \ast (g_{ec} g_{fd} S_{ef}).$$

Let me assume I have solved the simplicity constraint, and $dB = 0$. Substitute in the action $S$ the solutions $B_{ab}^{ij} = \pm \frac{1}{2} \epsilon_{ab}^{cd} B_{cd}$ and $A$ the Levi-Civita connection for a complex metric $g_{ab} = \theta_a \cdot \theta_b$. This results in a reduced action which is a function of the metric only,

$$S(\theta) = \mp \int d^4 x \epsilon^{abcd} F_{abcd},$$

where $F$ is the scalar curvature $F_{ab}^{cd}$ and $b^2 = \det(g_{ab})$. This is simply the Einstein-Hilbert action for $SO(4, C)$ general relativity.

The solutions $\pm \frac{1}{2} \epsilon_{ab}^{cd} B_{cd}$ do not correspond to general relativity [13]. If $B_{ab}^{ij} = \pm \frac{1}{2} \epsilon_{ab}^{cd} B_{cd}$, we obtain a new reduced action,

$$S(\theta) = \mp \Re \int d^4 x \epsilon^{abcd} F_{abcd},$$

which is zero because of the Bianchi identity $\epsilon^{abcd} F_{abcd} = 0$. So there is no other field equation other than the Bianchi identities.

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[5] For a proof please see footnote-7 in Ref. [13].
2.3 Analysis of the field equations

To extract the content of equation \(7c\), let me discuss the spinorial expansion\(^6\) of \(\phi_{ab}^{cd}\) and \(F_{ab}^{cd} = F_{ab}^{ij} \theta_i^a \theta_j^b\).

\[
F_{ab}^{cd} = F_{AB}^{CD} \epsilon_{AB}^{cd} \epsilon_{CD}^{AB} + \frac{S}{12} \epsilon_{ab}^{cd} + \frac{F}{12} \delta_{[a}^{cd} \delta_{b]}^{cd}\]  
(10)

\[
F_{ab}^{cd} = F_{AB}^{CD} \epsilon_{AB}^{cd} \epsilon_{CD}^{AB} + \frac{F}{12} \epsilon_{ab}^{cd} + \frac{S}{12} \delta_{[a}^{cd} \delta_{b]}^{cd}\]  
(11)

\[
F_{ab}^{cd} = F_{AB}^{CD} \epsilon_{AB}^{cd} \epsilon_{CD}^{AB} + \frac{S}{12} \epsilon_{ab}^{cd} + \frac{F}{12} \delta_{[a}^{cd} \delta_{b]}^{cd}\]  
(12)

where \(F = F_{ab}^{cd}\) and \(S = \frac{1}{2} e^{cd} F_{ab}^{cd} \). Please notice that in \(F_{ab}^{cd}\), the \(F\) and the \(S\) have exchanged positions due to the dualization. The pseudo scalar \(S\) is zero since the connection is torsion free.

\[
\phi_{ab}^{cd} = \phi_{AB}^{CD} \epsilon_{AB}^{cd} \epsilon_{CD}^{AB} + \phi_{AB}^{CD} \epsilon_{AB}^{cd} \epsilon_{CD}^{AB} + \frac{\phi}{12} \delta_{[a}^{cd} \delta_{b]}^{cd}\]  
(13)

where \(\phi = \phi_{ab}^{cd}\). The pseudoscalar \(\alpha = \frac{1}{2} e^{cd} \phi_{ab}^{cd}\) is absent, because it is zero by definition.

**Case 1:** \(B_{ab}^{ij} = \pm \theta_i^a \theta_j^b\); In this case equation \(7b\) implies

\[
L_{ab}^{cd} = b \phi_{ab}^{cd}.
\]

Using the spinor expansions in equations \(11\) and \(13\) we find that the scalar curvature \(F = \alpha = 0\). By equating the mixed spinor terms and using the exchange symmetry \(F_{AB}^{CD} = F_{CD}^{AB}\), we find the trace free Ricci curvature \(F_{AB}^{CD}\) is zero. Since the scalar curvature and the trace-free Ricci tensor are the free components of the Einstein tensor, we have the Einstein’s equations satisfied.

**Case 2:** \(B_{ab}^{ij} = \pm \frac{1}{2} e^{cd} \theta_i^a \theta_j^b\). In this case equation \(7b\) implies

\[
L_{ab}^{cd} = b \phi_{ab}^{cd}.
\]

Using the spinor expansions we find that there is no restriction on the curvature tensor \(F_{ab}^{cd}\) apart from the Bianchi identities.

\(^6\) A suitable soldering form and a variable spinorial basis need to be defined to map between coordinate and spinor space.
3 Reality Constraint for $b \neq 0$

Let the bivector 2-form field $B^{ij}_{ab} = \pm \theta^i_a \theta^j_b$ and the space-time metric $g_{ab} = \delta_{ij} \theta^a_i \theta^b_j$. Then, the area metric is defined by

$$A_{abcd} = B_{ab} \bullet B_{cd}$$

(14a)

$$= \frac{1}{2} \eta_{ik} \eta_{jl} B^{ij}_{ab} B^{kl}_{cd}$$

(14b)

$$= g_{a[c} g_{d]b}.$$  

(14c)

Consider an infinitesimal triangle with two sides as real coordinate vectors $X^a$ and $Y^b$. Its area $A$ can be calculated in terms of the coordinate bivector $Q_{ab} = \frac{1}{2} [X^a Y^b]$ as follows

$$A^2 = A_{abcd} Q^{ab} Q^{cd}.$$  

In general $A_{abcd}$ defines a metric on coordinate bivector fields: $< \alpha, \beta > = A_{abcd} \alpha^{ab} \beta^{cd}$ where $\alpha^{ab}$ and $\beta^{cd}$ are arbitrary bivector fields.

Consider a bivector 2-form field $B^{ij}_{ab} = \pm \theta^i_a \theta^j_b$ on the real manifold $M$ defined in the last section. Let $\theta^i_a$ be non-degenerate complex tetrads. Let $g_{ab} = g^R_{ab} + ig^I_{ab}$, where $g^R_{ab}$ and $g^I_{ab}$ are the real and the imaginary parts of $g_{ab} = \theta^a_i \bullet \theta^b_j$.

**Theorem 1** The area metric being real

$$\text{Im}(A_{abcd}) = 0,$$

(15)

is the necessary and the sufficient condition for the non-degenerate metric to be real or imaginary.

**Proof.** Equation (15) is equivalent to the following:

$$g_{ac} g^I_{db} = g^R_{ad} g^I_{cb}.$$  

(16)

From equation (16) the necessary part of our theorem is trivially satisfied. Let $g$, $g^R$ and $g^I$ be the determinants of $g_{ab}$, $g^R_{ab}$ and $g^I_{ab}$ respectively. The consequence of equation (16) is that $g = g^R + g^I$. Since $g \neq 0$, one of $g^R$ and $g^I$ is non-zero. Let me assume $g^R \neq 0$ and $g^I_{ac}$ is the inverse of $g^R_{ab}$. Let me multiply both the sides of equation (16) by $g^I_{ac}$ and sum on the repeated indices. We get $g^I_{ab} g^I_{db} = g_{ab} g^I_{db}$, which implies $g^I_{ab} g^I_{db} = 0$. Similarly we can show that $g^I \neq 0$ implies $g^I_{ab} = 0$. So we have shown that the metric is either real or imaginary if the area metric is real. 

Since an imaginary metric essentially defines a real geometry, we have shown that the area metric being real is the necessary and the sufficient condition for real geometry (non-degenerate) on the real manifold $M$. In the last section of this article I discuss this for any dimensions and rank of the space-time metric.
4 Extracting Real General Relativity

To understand the nature of the four volume after imposing the area metric reality constraint, consider the determinant of both the sides of the equation $g_{ab} = \theta_a \cdot \theta_b$.

$$g = b^2,$$

where $b = \frac{1}{4} \epsilon^{abcd} B_{ab} \wedge B_{cd} \neq 0$. From this equation we can deduce that $b$ is not sensitive to the fact that the metric is real or imaginary. But $b$ is imaginary if the metric is Lorentzian (signature $++--$ or $---+$) and it is real if the metric is Riemannian or Kleinian $(+++, -- -, --+, +++)$.

The signature of the metric is directly related to the signature of the area metric $A_{abcd} = g_{a[c} g_{d]b}$. It can be easily shown that for Riemannian, Kleinian and Lorentzian geometries the signatures type of $A_{abcd}$ are $(6,0)$, $(4,2)$ and $(3,3)$ respectively.

Consider the dualizing operator defined in (9) for complex metrics. Then for real or imaginary metrics it can be verified that

$$**B_{ab} = g B_{ab},$$

where $g = b^2$ is the determinant of the metric.

Consider the Levi-Civita connection

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} [\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}],$$

defined in terms of the metric. From the expression for the connection we can clearly see that it is real even if the metric is imaginary. Similarly the Riemann curvature tensor

$$F^a_{bcd} = \partial_c \Gamma^a_{bd} + \Gamma^e_{bc} \Gamma^a_{de}$$

is real since it is a function of $\Gamma^a_{bc}$ only. But $F_{bcd} = g^{de} F^a_{ec}$ and the scalar curvature are real or imaginary depending on the metric.

In background independent quantum general relativity models, areas are fundamental physical quantities. In fact the area metric contains the full information about the metric up to a sign\(^7\). If $B_{ab}^R$ and $B_{ab}^L$ (vectorial indices suppressed) are the self-dual and the anti-self dual parts of an arbitrary $B_{ij}^{ij}$, one can calculate the left and right area metrics as

$$A_{abcd}^L = B_{ab}^L \cdot B_{cd}^L - \frac{1}{4!} \epsilon^{efgh} B_{ef}^L \cdot B_{gh}^L \epsilon_{abcd}$$

and

$$A_{abcd}^R = B_{ab}^R \cdot B_{cd}^R + \frac{1}{4!} \epsilon^{efgh} B_{ef}^R \cdot B_{gh}^R \epsilon_{abcd}$$

respectively\(^13\). These metrics are pseudo-scalar component free. Reisenberger has derived Riemannian general relativity by imposing the constraint that the

\(^7\)For example, please see the proof of theorem 1 of Ref.:\(^{16}\).
left and right area metrics be equal to each other [13]. This constraint is equivalent to the Plebanski constraint \( B_{ab} \wedge B_{cd} - \delta_{abcd} = 0 \). I would like to take this one step further by utilizing the area metric to impose reality constraints on \( SO(4, C) \) general relativity.

Next, I would like to proceed to modify \( SO(4, C) \) general relativity actions defined before to incorporate the area metric reality constraint. The new actions are defined as follows:

\[
S_c(A, B, \tilde{B}, \phi, q) = \int_M \varepsilon^{abcd} B_{ab} \wedge F_{cd} d^4x + C_S + C_R, \tag{17}
\]

and

\[
S_r(A, B, \tilde{A}, \tilde{B}, \phi, \tilde{\phi}, q) = \Re S(A, B, \tilde{B}, \phi, q),
\]

where

\[
C_S = \int_{M_c} \frac{b}{2} \phi^{abcd} B_{ab} \wedge B_{cd} d^4x \tag{18}
\]

and

\[
C_R = \int_{M_c} \frac{|b|}{2} q^{abcd} \Im (B_{ab} \bullet B_{cd}) d^4x. \tag{19}
\]

The field \( \phi^{abcd} \) is the same as in the last section. The field \( q^{abcd} \) is real with the symmetries of the Riemann curvature tensor. The \( C_R \) is the Lagrange multiplier term introduced to impose the area metric reality constraint.

The field equations corresponding to the extrema of the actions under the \( A \) and \( \phi \) variations are the same as given in section two. They impose the condition \( B_{ij}^{ab} = \pm \theta_i^a \theta_j^b \) or \( \pm * \theta_i^a \theta_j^b \) and \( A \) be the Levi-Civita connection for the complex metric. The field equations corresponding to the extrema of the actions under the \( q^{abcd} \) variations are \( \Im (B_{ab} \bullet B_{cd}) = 0 \). This, as we discussed before, imposes the condition that the metric \( g_{ab} = \theta_a \bullet \theta_b \) be real or imaginary.

Let me assume I have solved the simplicity constraint, the reality constraint and \( dB = 0 \). Substitute the solutions \( B_{ij}^{ab} = \pm \theta_i^a \theta_j^b \) and \( A \) the Levi-Civita connection for a real or imaginary metric \( g_{ab} = \theta_a \bullet \theta_b \) in the action \( S \). This results in a reduced action which is a function of the tetrad \( \theta_i^a \) only,

\[
S(\theta) = \pm \Re \int d^4x \varepsilon^{abcd} F_{ab}.
\]

where \( F \) is the scalar curvature \( F_{ab} \). Recall that \( F \) is real or imaginary depending on the metric. This action reduces to Einstein-Hilbert action if both the metric and space-time density are simultaneously real or imaginary. If not, it is zero and there is no field equation involving the curvature \( F_{cd}^{ab} \) tensor other than the Bianchi identities.

If \( B_{ij}^{ab} = \pm * \theta_i^a \theta_j^b \), we get a new reduced action,

\[
S(\theta) = \pm \Re \int d^4x \varepsilon^{abcd} F_{ab}, \tag{20}
\]

\[\text{Also for } B_{ij}^{ab} = \pm * \theta_i^a \theta_j^b, \text{ it can be verified that the reality constraint implies that the metric } g_{ab} = \theta_a \bullet \theta_b \text{ be real or imaginary.}\]
which is zero because of the Bianchi identity \( \varepsilon^{abcd} F_{abcd} = 0 \). So there is no other field equation other than the Bianchi identities.

### 4.1 Understanding the Field equations

The field equations corresponding to the extrema of action \( S_r \) under the \( B \) and \( \bar{B} \) variations about \( B^{ij}_{ab} = \pm \theta^{[i}_a \theta^{j]}_b \) are

\[
\varepsilon^{abcd} F^{ij}_{cd} \varepsilon^{ijkl} = 0, \quad (21a)
\]

\[
\Rightarrow \frac{1}{b} F^{ef}_{ab} = \phi^{ef}_{ab} - \frac{1}{4} i \bar{q}^{cd}_{ab} |b|. \quad (21b)
\]

Here, the star corresponds to dualization on the coordinate variables.

For the action \( S_c \), only the field equations corresponding to its extrema under \( B \) variations are the same as Eq. (21). The field equations corresponding to \( \bar{B} \) variations are

\[
q^{abcd} B^{kl}_{cd} = 0, \quad (22)
\]

which imply \( q^{abcd} = 0 \) if \( b \neq 0 \).

#### 4.1.1 The Field Equations of Action \( S_c \)

Consider the field equation corresponding to the extrema action \( S_c \) under the variations of it’s variables. Since \( q^{abcd} = 0 \) \((b \neq 0)\), equation (23) is the same as equation \( (21) \). So Einstein’s equations are satisfied. Since the metric is essentially real, the field theory of action \( S_c \) corresponds to real general relativity.

Please recall that the \( b \) is imaginary if the metric is Lorentzian and is real if the metric is Riemannian or Kleinian. Thus, it is noticed that the reduced action \( S_c \) after the reality constraint imposed is real if both the metric and the space-time density are simultaneously real or imaginary. If not, the action is imaginary.

#### 4.1.2 The Field Equations of Action \( S_r \)

Let me analyze the field equations for action \( S_r \). Here \( q^{abcd} \) need not be zero.

Let me assume \( B^{ij}_{ab} = \pm \theta^{[i}_a \theta^{j]}_b \), then let me rewrite equation \( (21b) \) below,

\[
\frac{1}{b} F^{ef}_{ab} = \phi^{ef}_{ab} - \frac{1}{4} i \bar{q}^{cd}_{ab} |b|. \quad (23)
\]

There are two different cases now.

**Case 1:** The metric and the space-time density \( b \) are simultaneously real or imaginary.

Consider the real part of equation \( (23) \)

\[
\frac{1}{b} F_{ab}^{cd} = \text{Re} \phi^{cd}_{ab}.
\]
This equation is the same as equation (7b) with both the sides being real. There is no other restriction on $F_{ab}^{cd}$ other than the Bianchi identities. So Einstein’s equations are satisfied. Since $b$ is real, this case corresponds to Riemannian or Kleinian general relativity.

**Case 2:** The metric and the space-time density are not simultaneously real or imaginary.

For this case, the imaginary part of equation is

$$\frac{1}{b} F_{ab}^{cd} = \text{Im} \phi_{ab}^{cd} \pm \frac{1}{4} \bar{q}_{ab}^{cd},$$

with all the terms real. The $q_{ab}^{cd}$ is arbitrary apart from the constraint imposed by this equation. Therefore we find that there is no restriction on $F_{ab}^{cd}$ except for the Bianchi identities. So this case does not correspond to real general relativity.

Let $B_{ij}^{ab} = \pm \theta_i^a \theta_j^b$. In this case the field equations corresponding to the extrema of $S_r$ under $B_{ab}$ variations are

$$F_{ab}^{cd} = \phi_{ab}^{cd} - i|b| \bar{q}_{ab}^{cd}.$$

This situation is the same as in case (2) of section two, where $F_{ab}^{cd}$ is unrestricted except for the constraints due to Bianchi identities. So this case does not correspond to general relativity.

## 5 Discretization

### 5.1 BF theory

Consider that a continuum manifold is triangulated with four simplices. The discrete equivalent of a bivector two-form field is the assignment of a bivector $B_{ij}^b$ to each triangle $b$ of the triangulation. Also the equivalent of a connection one-form is the assignment of a parallel propagator $g_{ej}$ to each tetrahedron $e$. Using the bivectors and parallel propagators assigned to the simplices, the actions for general relativity and BF theory can be rewritten in a discrete form [6]. The real $SO(4, C)$ BF action can be discretized as follows [19]:

$$S(B_b, g_e) = \text{Re} \sum_b B_{ij}^b \ln H_{bij}. \quad (24)$$

The $H_b$ is the holonomy associated to the triangle $b$. It can be quantized to get an spin foam model [1] as done by Ooguri.

### 5.2 Barrett–Crane Constraints

The bivectors $B_i$ associated with the ten triangles of a four simplex in a flat Riemannian space satisfy the following properties called the Barrett-Crane constraints [3]:

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1. The bivector changes sign if the orientation of the triangle is changed.

2. Each bivector is simple.

3. If two triangles share a common edge, then the sum of the bivectors is also simple.

4. The sum of the bivectors corresponding to the edges of any tetrahedron is zero. This sum is calculated taking into account the orientations of the bivectors with respect to the tetrahedron.

5. The six bivectors of a four simplex sharing the same vertex are linearly independent.

6. The volume of a tetrahedron calculated from the bivectors is real and non-zero.

The items two and three can be summarized as follows:

\[ B_i \wedge B_j = 0 \quad \forall i, j, \]

where \( A \wedge B = \varepsilon_{IJKL} A^{I} B^{KL} \) and the \( i, j \) represents the triangles of a tetrahedron. If \( i = j \), it is referred to as the simplicity constraint. If \( i \neq j \) it is referred as the cross-simplicity constraints.

Barrett and Crane have shown that these constraints are sufficient to restrict a general set of ten bivectors \( E_b \) so that they correspond to the triangles of a geometric four simplex up to translations and rotations in a four dimensional flat Riemannian space \([3]\).

The Barrett-Crane constraints theory can be easily extended to the \( SO(4, \mathbb{C}) \) general relativity. In this case the bivectors are complex and so the volume calculated for the sixth constraint is complex. So we need to relax the condition of the reality of the volume.

We would like to combine the area metric reality constraint with the Barrett-Crane Constraints. For this we must find the discrete equivalent of the area metric reality condition. For this let me next discuss the area metric reality condition in the context of three simplices and four simplices. I would like to show that the discretized area metric reality constraint combined with the Barrett-Constraint constraint requires the complex bivectors associated to a three or four simplex to describe real flat geometries.

### 5.2.1 Three Simplex

Consider a tetrahedron \( t \). Let the numbers 0 to 3 denote the vertices of the tetrahedron. Let me choose the 0 as the origin of the tetrahedron. Let \( B_{ij} \) be the complex bivector associated with the triangle \( 0ij \) where \( i \) and \( j \) denote one of the vertices other than the origin and \( i < j \). Let \( B_0 \) be the complex bivector...
associated with the triangle 123. Then similar to Riemannian general relativity\textsuperscript{3}, the Barrett-Crane constraints\textsuperscript{9} for $SO(4, C)$ general relativity imply that
\begin{align*}
B_{ij} &= a_i \land a_j, \quad \text{(25a)} \\
B_0 &= -B_{12} - B_{23} - B_{34}, \quad \text{(25b)}
\end{align*}
where $a_i, i = 1 \text{ to } 3$ are linearly independent complex four vectors associated to the links $0i$ of the three simplex. Let me choose the vectors $a_i, i = 1 \text{ to } 3$ to be the complex vector basis inside the tetrahedron. Then the complex 3D metric inside the tetrahedron is
\begin{equation}
g_{ij} = a_i \cdot a_j, \quad \text{(26)}
\end{equation}
where the dot is the scalar product on the vectors. This describes a flat complex three dimensional geometry inside the tetrahedron. The area metric is given by
\begin{equation}
A_{ijkl} = g_{ij} g_{kl}.
\end{equation}
The coordinates of the vectors $a_i$ are simply
\begin{align*}
a_1 &= (1, 0, 0), \\
a_2 &= (0, 1, 0), \\
a_3 &= (0, 0, 1).
\end{align*}
Because of this all of the six possible scalar products made out of the bivectors $B_{ij}$ are simply the elements of the area metric. From the discussion of the last section the reality of the area metric simply requires that the metric $g_{ij}$ be real or imaginary. Since $B_0$ is also defined by equation (25b) its inner product with itself and other bivectors are real. Thus in the context of a three simplex, the discrete equivalent of the area metric reality constraint is that the all possible scalar products of bivectors associated with the triangles of a three simplex be real.

5.2.2 Four Simplex

In the case of a four simplex $s$ there are six bivectors $B_{ij}$. There are four $B_0$ type bivectors. Let $B_i$ denote the bivector associated to the triangle made by connecting the vertices other than the origin and vertex $i$. The Barrett-Crane constraints imply equation (25a) with $i, j = 1 \text{ to } 4$. There is one equation for each $B_i$ similar to equation (25b). Now the metric $g_{ij} = a_i \cdot a_j$ describes a complex four dimensional flat geometry inside the four simplex $s$. Now assuming we are dealing with non-degenerate geometry, the reality of the geometry requires the reality of the area metric. Similar to the three dimensional case, the components of the area metric are all of the possible scalar products made out of the bivectors $B_{ij}$. The scalar products of the bivectors $B_i$ among themselves or with $B_{ij}$’s are simple real linear combinations of the scalar products made from

\textsuperscript{9}We do not require to use the fifth Barrett-Crane constraint since we are only considering one tetrahedron of a four simplex.
B_{ij}’s. So one can propose that the discrete equivalent of the area metric reality constraint is simply the condition that the scalar product of these bivectors be real. Let me refer to the later condition as the bivector scalar product reality constraint.

**Theorem 2** The necessary and sufficient conditions for a four simplex with real non-degenerate flat geometry are 1) The SO(4, C) Barrett-Crane constraints\(^{10}\) and 2) The reality of all possible bivector scalar products.

**Proof.** The necessary condition can be shown to be true by straight forward generalization of the arguments given by Barrett and Crane [3] and application of the discussions in the last paragraph. The sufficiency of the conditions follow from the discussion in the last paragraph. ■

The quantization of a four simplex using the SO(4, C) Barrett-Crane constraints and the bivector scalar product reality constraint has been argued in Ref.[11]

### 5.3 Actions for Simplicial General Relativity

Here we would like define actions for general relativity which has application for the Barrett-Crane models [3]. [11].

The discrete BF theory described in equation (24) can be further modified by imposing the SO(4, C) Barrett-Crane constraints on it to get the SO(4, C) Barrett-Crane model [11]. [6]. The resulting model can be considered as a path-integral quantization of the simplicial version of the action in equation (17),

\[
S_{GR}(B_b, g_e; \phi) = \sum_b B_b^i \ln H_{b_{ij}} + \frac{1}{2} \sum_{bb} \phi_{bb} B_b \wedge B_b',
\]

where \(\phi_{bb}\) are to impose the Barrett-Crane constraints (2) and (3) on \(B_b\). There is one \(\phi_{bb}\) for every pair of triangles \(bb\) such that either they are the same or they intersect at a link.

A proposal for an action for real general relativity is a modified form of equation (17) that includes extra Lagrange multipliers to impose the bivector

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\(^{10}\)The SO(4, C) Barrett-Crane constraints differ from the real Barrett-Crane constraints by the following:

1. The bivectors are complex, and
2. The condition for the reality of the volume of tetrahedron is not required.
scalar product conditions\textsuperscript{11}:

\[ S_{rGR}(B_b, g_e, \phi, q) = \text{Re} \sum_b B_b^{ij} \ln H_{bij} \]
\[ + \frac{1}{2} \text{Re} \sum_{bb} \phi_{bb} B_{bij} \wedge B_b^{ij} \]
\[ + \frac{1}{2} \sum_{bb} q_{bb} \text{Im}(B_b \circ B_b), \]

where there is one real \( q_{bb} \) for every pair of triangles \( bb \) such that either they are same or they intersect at a link. The Lagrange multipliers \( q_{bb} \) helps impose the conditions that

- the scalar product of a bivector \( B_b \) with itself is real and
- the scalar product of a bivectors associated to triangles which intersect at a link is real.

Above we have ignored to impose reality of the scalar products of the bivectors associated to any two triangles of the same four simplex which intersect at only at one vertex. This is because these constraints appears not to be needed for a formal extraction \textsuperscript{11} of the Barrett-Crane models of real general relativity from that of \( SO(4, C) \) general relativity. Imposing these constraints may not be required because of the enormous redundancy in the bivector scalar product reality constraints defined in the last section\textsuperscript{12}. This issue need to be carefully investigated.

An alternative discrete action for general relativity is that of Regge \textsuperscript{22}. In any dimension \( n \), given a simplicial geometry, the Regge action is

\[ S_{\text{Reg}} = \sum_b A_b \xi_b. \]

The asymptotic limit of the \( SO(4, C) \) Barrett-Crane model recovers \( SO(4, C) \) Regge Calculus and the bivectors that satisfy the Barrett-Crane constraints \textsuperscript{11}. This is also true for models of real general relativity theories for various signatures as they are simple restrictions of \( SO(4, C) \) ideas \textsuperscript{11}.

\textsuperscript{11}The square of area reality conditions state that,

- the square of the area of the triangle calculated as scalar product of the associated bivector is real.
- the square of area calculated as scalar product of sum of the bivectors associated with two triangle of a tetrahedron is real.

Assume the first constraint is imposed on each of any two triangles of a tetrahedron. Then the second constraint is equivalent to the condition that the scalar product of the bivectors associated to these triangle is real.

\textsuperscript{12}Please notice that only about ten independent conditions are required to reduce a complex four metric to a real four metric.
Above, the $A_b$ are the areas of the triangles expressed as functions of link lengths of the four simplex. The link lengths are the free variables of the Regge theory. The $\varepsilon_b$ is the deficit around a bone $b$. This action can be easily generalized to $SO(4, C)$ general relativity. Similar to the action in equation (28) the reality constraints can be combined with the Regge Calculus:

\[ S_{R \text{Reg}} = \sum_b A_b \varepsilon_b + \frac{1}{2} \sum_{bb} q_{bb} \text{Im}(B_b \circ B_b), \quad (29) \]

where the $B_b$, $A_b$ and $\varepsilon_b$ can be considered as the functions of complex vectors associated to the links of the triangulation. The link vectors can be considered as the free variables of this theory.

In a discrete general relativity theory on the simplicial manifolds we do not require the continuity of the metric a priori. This means that the flat geometry associated to each four simplex can be of any signature. This means that the actions (28) and (29) describe a multi-signature discrete general relativity where the geometry of each simplex has a different signature.

6 Further Considerations

6.1 Reality Constraint for Arbitrary Metrics

Here we analyze the area metric reality constraint for a metric $g_{ac}$ of arbitrary rank in arbitrary dimensions, with the area metric defined as $A_{abcd} = g_{a[c}g_{d]b}$. Let the rank of $g_{ac}$ be $r$.

If the rank $r = 1$ then $g_{ab}$ is of form $\lambda_a \lambda_b$ for some complex non-zero co-vector $\lambda_a$. This implies that the area metric is zero and therefore not an interesting case.

Let me prove the following theorem.

**Theorem 3** If the rank $r$ of $g_{ac}$ is $\geq 2$, then the area metric reality constraint implies the metric is real or imaginary. If the rank $r$ of $g_{ac}$ is equal to 1, then the area metric reality constraint implies $g_{ac} = \eta \alpha_a \alpha_b$ for some complex $\eta \neq 0$ and real non-zero co-vector $\alpha_a$.

The area metric reality constraint implies

\[ g^{R}_{ac} g_{db} = g^{R}_{ad} g^{I}_{cb}. \quad (30) \]

Let $g_{AC}$ be a $r$ by $r$ submatrix of $g_{ac}$ with a non-zero determinant, where the capitalised indices are restricted to vary over the elements of $g_{AC}$ only. Now we have

\[ g^{R}_{AC} g^{I}_{DB} = g^{R}_{AD} g^{I}_{CB}. \quad (31) \]

From the definition of the determinant and the above equation we have

\[ \det(g_{AC}) = \det(g^{R}_{AC}) + \det(g^{I}_{AC}). \]
Since \( \det(g_{AC}) \neq 0 \) we have either \( \det(g^{R}_{AC}) \) or \( \det(g^{I}_{AC}) \) not equal to zero. Let me assume \( g^{R}_{AC} \neq 0 \). Then contracting both the sides of equation (31) with the inverse of \( g^{R}_{AC} \) we find \( g^{I}_{DB} = 0 \). Now from equation (31) we have

\[
g^{R}_{AC}g^{I}_{DB} = g^{R}_{Ad}g^{I}_{CB} = 0.
\]

Since the Rank of \( g^{R}_{AC} \geq 2 \) we can always find a \( g^{R}_{AC} \neq 0 \) for some fixed \( A \) and \( C \). Using this in equation (32) we find \( g^{I}_{DB} = 0 \). So we have shown that if \( g^{R}_{AC} \neq 0 \) then \( g^{I}_{DB} = 0 \). Similarly if we can show that if \( g^{I}_{AC} \neq 0 \) then \( g^{R}_{DB} = 0 \).

### 6.2 The Plebanski Formulation for \( b = 0 \)

The degenerate case corresponding to \( b = 0 \) has been analyzed in the context of Riemannian general relativity by Reisenberger [13]. In his analysis the simplicity constraint yields

\[
B^{I}_{L} = T_{j}^{I}B^{j}_{R},
\]

where \( B^{I}_{L} \) and \( B^{j}_{R} \) are the left handed and the right handed components of the real bivector valued two-form \( B_{ij} \), the integers \( I, J \) are the Lie algebra indices and \( T_{j}^{I} \) is an \( SO(3,R) \) matrix. If the action is gauge invariant under \( SO(4,R) \), in a proper gauge \( B^{I}_{L} = T_{j}^{I}B^{j}_{R} \) reduces to \( B^{I}_{L} = B^{I}_{R} \). Let me denote \( B^{I}_{L} = B^{I}_{R} \) simply by \( \Sigma^{I} \). Reisenberger starts from the Riemannian version of the action in equation (6) and finally ends up with the following reduced actions:

\[
S_{DG}(\Sigma^{I}, A_{R}, A_{L}) = \int_{M} \delta_{IJ}\Sigma^{I}(F^{I}_{R} \pm F^{I}_{L}),
\]

where \( \Sigma^{I} \) is a \( SU(2) \) Lie-algebra valued two form, \( A_{R}(A_{L}) \) is a right (left) handed \( SU(2) \) connection and \( F_{R} \) (\( F_{L} \)) are their curvature two forms. This action and the analysis that led to this action as carried out done in Ref. [13] can be easily generalized to \( SO(4,C) \) general relativity by replacing \( SU(2) \) with \( SL(2,C) \).

Now, in case the of \( b = 0 \) the area metric defined in terms of \( B^{ij}_{ab} \) is

\[
A = \frac{1}{2} \eta_{ik}\eta_{jl}B^{ij} \otimes B^{kl} = \delta_{IJ}B^{I}_{R} \otimes B^{j}_{R} + \delta_{IJ}B^{I}_{L} \otimes B^{j}_{L} = 2\delta_{IJ}\Sigma^{I} \otimes \Sigma^{J}.
\]

Now the \( B \) field is no longer related to a tetrad, which means we do not have a space-time metric defined. But it can be clearly seen that the area metric is still defined.
The reduced versions of actions $S_r$ and $S_c$ for $b = 0$ with simplicity constraint imposed are,

$$S_{rDG}(A_R, A_L, \Sigma, \bar{\Sigma}) = \int_{M_{\tau}} \epsilon^{abcd} \delta_{IJ} \Sigma^I_{ab} (F^J_{cdR} \pm F^J_{cdL}) + \int q^{abcd} \text{Im}(\delta_{IJ} \Sigma^I_{ab} \Sigma^J_{cd}) \text{ and }$$

$$S_{cDG}(\Sigma, A_R, A_L, \bar{\Sigma}, \bar{A}_R, \bar{A}_L) = \text{Re} S_{DG}(\Sigma, A_R, A_L, \bar{\Sigma}).$$

The field equations relating to $S_{rDG}$ extrema are

$$D_R \Sigma^I = D_L \Sigma^I = 0,$$

$$\frac{1}{2} \epsilon^{abcd} F^J_{cdR} = q^{abcd} \Sigma^I_{cd},$$

$$\frac{1}{2} \epsilon^{abcd} F^J_{cdL} = -q^{abcd} \Sigma^I_{cd} \text{ and }$$

$$\text{Im}(\delta_{IJ} \Sigma^I_{ab} \Sigma^J_{cd}) = 0.$$

For $S_{cDG}$, we have additional equations

$$q^{abcd} \Sigma^I_{cd} = 0,$$

which imply

$$F^J_{cdR} = F^J_{cdL} = 0.$$

The reality constraint requires $A = 2\delta_{IJ} \Sigma^I \otimes \Sigma^J$ to be real. Such expression allows for assigning a real square of area values to the two surfaces of the manifold. The spin foam quantization of the theory of $S_{rDG}$ without the reality constraint in the case of Riemannian general relativity has been studied by Perez [20]. The spin foam quantization of the $SO(4, C)$ theory with the reality constraint needs to be studied.

### 6.3 Palatini Formalism with the Reality Constraint

Consider alternative actions of Palatini’s form [14] which use the co-tetrads $\theta^i$ instead of the bivector 2-form field as a basic variable. The Palatini actions with the reality constraint included are

$$S_{cPT}[A, \theta^i, \bar{\theta}^i, q^{abcd}] = \int \epsilon_{ijkl} \theta^i \theta^j F^{kl} + q^{abcd} \text{Im}(B_{ab} \bullet B_{cd}) \text{ and }$$

$$S_{rPT}[\theta^i, \bar{\theta}^i, A, \bar{A}, q^{abcd}] = \text{Re} S[A, \theta^i, \bar{\theta}^i, q^{abcd}],$$

where $F^{ij}$ is the curvature 2-form corresponding to the $SO(4, C)$ connection $A$ and $B_{ab} = \theta_a \wedge \theta_b$. The equations of motion for the theory of $S_{rPT}$ are

$$D(\theta^k \theta^l) = 0, \quad (34)$$

$$\epsilon^{abcd} \epsilon_{ijkl} \theta^i \epsilon_{kl} = i8q^{abcd} (g_{bd} \theta^i), \quad (35)$$

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\[ \text{Im}(B_{ab} \bullet B_{cd}) = 0, \] (36)

and for \( S_{\text{PT}} \) we have additional equations \( g_{ab}q^{abcd} = 0 \). Equation (35) simply requires the \( A \) to be the Levi-Civita connection of the metric \( g_{ab} = \theta_a \bullet \theta_b \).

Transforming equation (35) we get

\[ b(F_{ga} - \frac{1}{2}\delta_a^f F) = -2i\eta^{fbc} (A_{abcd}), \] (37)

where the left hand side is the Einstein tensor multiplied by \( b = \det(\theta_a^i) \). In the case of \( S_{c,\text{PT}} \) the right hand side is zero, so the Einstein’s equations are satisfied.

Let me discuss the field equations of \( S_{r,\text{PT}} \). The interpretation of equation (37) is similar to that of the various cases discussed for the Plebanski action with the reality constraint. The right hand side is purely imaginary because of the reality constraint. The left side is real if 1) the metric is real and the signature is Riemannian or Kleinian, 2) the metric is imaginary and the signature is Lorentzian. So for these cases the Einstein tensor must vanish if \( b \neq 0 \). So they correspond to general relativity. For all the other combinations and also for \( b = 0 \) the Einstein tensor need not vanish.

7 Conclusion

In this article we have established a classical foundation for a concept of reality conditions in the context of spin foam models. At the classical continuum level it is the condition that the area metric be real. In the context of Barrett-Crane theory this takes the form of the reality of the scalar products of the bivectors associated with the triangles of a four simplex or three simplex. At the quantum level this idea brings together the Barrett-Crane spin foam models of real and \( SO(4, C) \) general relativity theories in four dimensions in a unified perspective. In Ref. two generalizations of real general relativity Barrett models have been proposed. One of them puts together two Lorentzian Barrett-Crane models to get a more general model called the mixed Lorentzian Barrett-Crane model. Another model was defined by putting together the mixed Lorentzian model and the Barrett-Crane models for all other signatures to get a multi-signature model. The theory defined by the real action in equation (15) for \( SO(4, C) \) general relativity with the reality constraint contains the general relativity for all signatures. So this theory must be related to the multi-signature model. The precise details of this idea need to be analyzed further. The continuum and semiclassical limits of the various actions proposed in this article need to be analyzed. Physical usefulness need to be investigated.

8 Acknowledgement

I thank Allen Janis and George Sparling for the correspondences.
A Spinorial Expansion Calculations

Consider a tensor $R_{abcd}$ which has the symmetries of the indices of the Riemann Curvature tensor. In this appendix I would like to briefly summarize the spinorial decomposition of $R_{abcd}$. The expansion of $R_{abcd}$ in terms of the left handed and the right handed spinorial free components is

$$R_{abcd} = R_{ABCD} \epsilon_{\dot{A}\dot{B}} \epsilon_{\dot{C}\dot{D}} + R_{\dot{A}\dot{B}\dot{C}\dot{D}} \epsilon_{ABCD} + R_{\dot{A}\dot{B}C\dot{D}} \epsilon_{AB\dot{C}\dot{D}} + R_{\dot{A}\dot{B}C\dot{D}} \epsilon_{\dot{A}B\dot{C}\dot{D}}.$$  

(38)

The $R_{ABCD}$ and $R_{\dot{A}\dot{B}\dot{C}\dot{D}}$ are independent of each other and $R_{\dot{A}\dot{B}C\dot{D}} = R_{\dot{A}B\dot{C}\dot{D}}$ because of the exchange symmetry. The first and last terms can be expanded into a spin two and spin zero tensors as follows:

$$R_{ABCD} = \frac{1}{24} R_{(ABCD)} + X (\epsilon_{AC} \epsilon_{BD} + \epsilon_{BC} \epsilon_{AD})$$  

and  

$$R_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{1}{24} R_{(\dot{A}\dot{B}\dot{C}\dot{D})} + Y (\epsilon_{\dot{A}\dot{C}} \epsilon_{\dot{B}\dot{D}} + \epsilon_{\dot{B}\dot{C}} \epsilon_{\dot{A}\dot{D}}),$$  

(39)

(40)

where $X = \frac{1}{6} R_{ABAB}$ and $Y = \frac{1}{6} R_{\dot{A}\dot{B}\dot{A}\dot{B}}$. Let me define $C_{ABCD} = \frac{1}{24} R_{(ABCD)}$ and $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{1}{24} R_{(\dot{A}\dot{B}\dot{C}\dot{D})}$. Let me define the two tensors

$$\Pi^{cd}_{ab} = \frac{1}{2} \epsilon^{cd}_{ab} \text{ and }$$  

$$\Delta^{cd}_{ab} = \frac{1}{2} \delta^{[c}_{a} \delta^{d]}_{b}.$$  

The tensor $\Delta$ is a scalar. The $\Pi^{cd}_{ab}$ is a pseudo scalar and the dualizing operator under it’s action on bivectors ($B_{ab} = \Pi^{cd}_{ab} B_{cd}$ or simply $\dot{B} = \Pi B$). Let me define two operations on $\Pi$ and $\Delta$: the product, for example $(\Pi \Delta)^{cd}_{ab} = \Pi^{ef}_{cd} \Delta_{ef}$, and the trace, for example $tr(\Delta) = \Delta^{ab}_{ab}$. We can verify the following properties of $\Pi$ and $\Delta$ which are

$$tr(\Pi) = 0,$$

$$tr(\Delta) = 6,$$

$$\Pi \Delta = \Delta \Pi = \Pi,$$

$$\Pi \Pi = \Delta \text{ and }$$

$$\Delta B = B,$$

where $B$ is an arbitrary bivector. The above properties help in the analysis of tensor with the tensor $R_{abcd}$. The spinorial expansion of $\Pi$ and $\Delta$ are

$$\Pi^{cd}_{ab} = \epsilon_{\dot{A}} c_{\dot{A}} \epsilon_{\dot{B}} d_{\dot{B}} - \epsilon_{\dot{A}} d_{\dot{A}} \epsilon_{\dot{B}} c_{\dot{B}} \frac{1}{2}$$  

and

$$\Delta^{cd}_{ab} = \epsilon_{\dot{A}} d_{\dot{A}} \epsilon_{\dot{B}} c_{\dot{B}} - \epsilon_{\dot{A}} c_{\dot{A}} \epsilon_{\dot{B}} d_{\dot{B}} \frac{1}{2}.$$

\[13\] A suitable soldering form and a variable spinorial basis need to be defined to map between coordinate and spinor space.
respectively. Using equation (39) and equation (40) in equation (38), the result can be simplified using the spinorial expansions of Π and ∆, and the identity
\[ \varepsilon_{[A|B|C|D]} = 0; \]

\[ R_{ab}^{cd} = C_{AB}^{CD} \varepsilon_{AB}^{CD} + C_{AB}^{CD} \varepsilon_{AB}^{CD} + \frac{R}{6} \Delta_{ab}^{cd} + \frac{S}{6} \Pi_{ab}^{cd} + R_{AB}^{CD} \varepsilon_{AB}^{CD} + R_{AB}^{CD} \varepsilon_{AB}^{CD}, \]

where \( R = Tr(R\Delta) = 2(R^{AB} + R_{AB}^{CD}) \) and \( S = Tr(R\Pi) = 2(R^{AB} - R_{AB}^{CD}) \).

If \( R_{ab}^{cd} \) is a general spatial curvature tensor then \( C_{AB}^{CD} \) and \( C_{AB}^{CD} \) are the left handed and the right handed spinorial parts of the Weyl tensor:

\[ C_{ab}^{cd} = C_{AB}^{CD} \varepsilon_{AB}^{CD} + C_{AB}^{CD} \varepsilon_{AB}^{CD}. \]

Each of \( C_{AB}^{CD} \) and \( C_{AB}^{CD} \) has five free components. The \( R_{DBB'}^{CD} = -\frac{1}{2} R_{bd} \) is the trace free Ricci tensor \( R_{bd} = g^{ac} R_{abcd} - \frac{1}{2} g_{bd} R \), which has nine free components. The \( R = Tr(R\Delta) \) is the scalar curvature. The \( S = Tr(R\Pi) \) can be referred to as the pseudo scalar curvature because it changes sign under the change of orientation of space-time. It vanishes for the Riemann curvature tensor as it corresponds to a torsion free connection. For an arbitrary curvature tensor, in terms of the torsion the pseudo-scalar component is

\[ S = Tr(\Pi DT), \]  

(41)

where \( D \) is the exterior space-time covariant derivative, \( T \) is the torsion written as a 3-form with all of it’s indices lowered and anti-symmetrized.

Under the action of dual operation \( R = \Pi R \) we have

\[ R_{ab}^{cd} = C_{AB}^{CD} \varepsilon_{AB}^{CD} - C_{AB}^{CD} \varepsilon_{AB}^{CD} + \frac{R}{6} \Delta_{ab}^{cd} + \frac{S}{6} \Pi_{ab}^{cd} + R_{AB}^{CD} \varepsilon_{AB}^{CD} - R_{AB}^{CD} \varepsilon_{AB}^{CD}. \]

Notice that the \( R_{AB}^{CD} \) and \( R_{AB}^{CD} \) terms have different signs, \( R \) and \( S \) exchanged positions. These properties are crucial for interpreting the field equation (7b) of the Plebanski formulation of general relativity.
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