Tautological classes and symmetry in Khovanov-Rozansky homology

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For any link $L$, Khovanov and Rozansky defined **triply graded homology** $\text{HHH}_{i,j,k}(L)$ categorifying the unreduced HOMFLY-PT polynomial

$$P(a, q) = \sum_{i,j,k} a^i q^j (-1)^k \dim \text{HHH}_{i,j,k}(L).$$

For knots, there is a reduced version $\text{HHH}_{i,j,k}(K)$ which yields a finite dimensional vector space.

**Theorem (GHM)**

*For any knot $K$, the reduced Khovanov-Rozansky homology is symmetric:*

$$\text{HHH}_{i,-2j,k}(K) = \text{HHH}_{i,2j,k+2j}(K).$$
This was conjectured by Gukov, Dunfield and Rasmussen in 2005:

\[ \begin{array}{c}
\text{0} & \text{2} & \text{4} & \text{4} & \text{6} \\
\text{3} & \text{5} & \text{5} & \text{7} & \text{7} \\
\text{8} & & & & \\
\end{array} \]

**Figure 3.7.** Differentials for \( T_{3,4} \). The bottom row of dots has \( a \)-grading 6. The leftmost dot on that row has \( q \)-grading \(-6\), which you can determine by noting that the vertical axis of symmetry corresponds to the line \( q = 0 \).
Other approaches:

- Oblomkov, Rozansky: use matrix factorizations over Hilbert schemes of points on the plane.
- Galashin, Lam (for knots related to Richardson varieties): use graded Koszul duality for category $\mathcal{O}$ developed by Bezrukavnikov-Yun.

Both approaches use very heavy machinery of geometric representation theory. Our proof is more straightforward, and generalizes to links.
Problems for links:

- No good reduced homology
- There is an action of a polynomial ring $\mathbb{C}[x_1, \ldots, x_c]$ on the unreduced homology $\text{HHH}(L)$, where $c$ is the number of components of $L$
- The symmetry does not preserve the degrees of $x_i$.

Solution: use “$y$-ified” homology $\text{HY}(L)$ defined by G.-Hogancamp. It is naturally a module over $\mathbb{C}[x_1, \ldots, x_c, y_1, \ldots, y_c]$. The symmetry would exchange $x_i$ with $y_i$.

Theorem (G.,Hogancamp)

For all $n, k \geq 0$ the homology of the $(n, kn)$ torus link with $n$ components is given by:

$$\text{HY}(T(n, kn)) = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j, \theta_i - \theta_j)^k \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n] = \text{HY}(\text{unlink}).$$
Main result

**Theorem (GHM)**

For any link $L$, there is an action of operators $F_k$ on $\text{HY}(L)$ satisfying the following relations:

$$[F_k, F_m] = 0, \quad [F_k, x_i] = 0, \quad [F_k, y_i] = kx_i^{k-1}.$$ 

Furthermore, $F_2$ satisfies “hard Lefshetz property”:

$$F_2^j : \text{HY}_{i,-2j,k}(L) \to \text{HY}_{i,2j,k+2j}(L)$$

is an isomorphism, and $F_2$ extends to an action of $\mathfrak{sl}_2$ on $\text{HY}(L)$.

For knots, $\text{HY}(K) = \overline{\text{HHH}}(K) \otimes \mathbb{C}[x, y]$ and the symmetry of $\text{HY}(K)$ implies the symmetry of $\overline{\text{HHH}}(K)$. 

Let $R = \mathbb{C}[x_1, \ldots, x_n]$.  

- Define the $R - R$-bimodules $B_i = R \otimes_{R(i, i+1)} R$.  
- To a simple crossing, associate Rouquier complexes

  $$T_i = [B_i \to R], \quad T_i^{-1} = [R \to B_i]$$

- To any braid $\beta$, associate the product $T_\beta$ of $T_i, T_i^{-1}$. It is naturally a complex of $R - R$ bimodules.  
- The braid closure corresponds to the Hochschild homology $\text{HH}(T_\beta)$.

**Theorem (Khovanov, Rozansky)**

The output of this construction is a topological invariant of the closure of $\beta$.  


Dg algebra $\mathcal{A}$

Let

$$B = \frac{\mathbb{C}[x_1, \ldots, x_n, x'_1, \ldots, x'_n]}{f(x_1, \ldots, x_n) = f(x'_1, \ldots, x'_n)} \text{ for any symmetric function } f.$$ 

We define a dg algebra $\mathcal{A}$ as follows:

$$\mathcal{A} = B[\xi_1, \ldots, \xi_n, u_1, \ldots, u_n], \quad d(\xi_i) = x_i - x'_i, \quad d(u_k) = \sum_{i=1}^n h_{k-1}(x_i, x'_i)\xi_i$$

Here $h_{k-1}(x_i, x'_i) = x_i^{k-1} + \ldots + (x'_i)^{k-1}$ is the complete symmetric function. Note that

$$d^2(u_k) = \sum_{i=1}^n h_{k-1}(x_i, x'_i)(x_i - x'_i) = \sum x_i^k - \sum (x'_i)^k = 0.$$
Dg algebra $\mathcal{A}$: properties

**Theorem**

The dg algebra $\mathcal{A}$ is a free resolution of $R$ as a $B$-module.

**Theorem**

The dg algebra $\mathcal{A}$ has a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$ defined by the equations

$$
\begin{align*}
\Delta(x_i) &= x_i \otimes 1, \\
\Delta(x'_i) &= 1 \otimes x'_i, \\
\Delta(\xi_i) &= \xi_i \otimes 1 + 1 \otimes \xi_i, \\
\Delta(u_k) &= u_k \otimes 1 + 1 \otimes u_k + \sum_{i=1}^{n} h_{k-2}(x_i, x'_i, x''_i) \xi_i \otimes \xi_i.
\end{align*}
$$

This coproduct is coassociative up to homotopy.
The dg algebra $\mathcal{A}$ acts on simple crossings $T_i, T_i^{-1}$ as follows: $\xi_i$ are explicit “dot sliding homotopies” and $u_k$ act by 0. By using the coproduct on $\mathcal{A}$, we can extend the action to arbitrary Rouquier complexes and obtain the following:

**Theorem**

There is an action of $\mathcal{A}$ on the Rouquier complex $T_\beta$ associated to an arbitrary braid $\beta$. This action is invariant under braid relations and unique up to homotopy.
We can use the action of $\xi_i$ to deform the differential on $T_\beta$:

$$D = d + \sum \xi_i y_i$$

The differential $D$ does not square to zero, but $D^2$ vanishes after closing the braid (that is, applying HH), and we can define

$$HY(\beta) = H(HH(T_\beta) \otimes \mathbb{C}[y_1, \ldots, y_c], D)$$

**Theorem (G., Hogancamp)**

The “$y$-ified” homology $HY(\beta)$ is the topological invariant of the closure of $\beta$. 
Construction of $F_k$

The action of $u_k$ can be used to define the operators

$$F_k = h_{k-1}(x_i, x'_i) \frac{\partial}{\partial y_i} + u_k.$$ 

One can check that $[D, F_k] = 0$ and hence $F_k$ yield well-defined operators on $HY(\beta)$.

**Theorem (GHM)**

*The action of $F_k$ on $HY(\beta)$ is a topological invariant of the closure of $\beta$.***
To prove that $F_2$ satisfies “hard Lefshetz property”, we use the objects $K_{i,j} = R \xrightarrow{x_i - x_j} R$ which are $\mathcal{A}$-modules as well. The maps in the “skein exact triangle”

$$T_i \rightarrow T_i^{-1} \rightarrow K_{i,i+1}$$

agree with the action of $\mathcal{A}$ (up to homotopy). This allows us to reduce a complicated Rouquier complex to the ones for unlinks, possibly multiplied by products of $K_{ij}$. We explicitly compute the action of all $F_k$ for such complexes, and verify the hard Lefshetz property for them.
Let me comment on geometric motivation behind the construction of $A$ and the coproduct. Let $G = GL(n)$, for any symmetric function $Q(x_1, \ldots, x_n)$ of degree $d$ one can construct the following differential forms:

$$\Phi_1(Q) \in \Omega^{2d-1}(G), \quad \Phi_2(Q) \in \Omega^{2d-2}(G \times G), \quad \ldots \quad \Phi_d(Q) \in \Omega^d(G^d)$$

satisfying equations

$$d\Phi_1(Q) = 0, \quad d\Phi_2(Q) = \Phi_1(Q) \otimes 1 + 1 \otimes \Phi_1(Q) - m^*(\Phi_1(Q)), \quad \ldots$$

where $m : G \times G \to G$ is the multiplication map on $G$. For example, for $Q = \sum x_i^2$ we get a 3-form on $G$ and a 2-form on $G \times G$. 
Atiyah, Bott, Jeffrey and others used these forms to construct interesting cohomology classes on character varieties.

Suppose that $f : X \to G$ and $g : Y \to G$ are matrix-valued functions such that $f^* \Phi_1(Q) = d\omega_X$ and $g^* \Phi_1(Q) = d\omega_Y$ for some forms $\omega_X$ and $\omega_Y$, then we can define

$$f \cdot g : X \times Y \to G, \quad \omega_{X \times Y} = \omega_X \otimes 1 + 1 \otimes \omega_Y + (f \times g)^* \Phi_2(Q)$$

such that

$$d(\omega_{X \times Y}) = (f \cdot g)^*(\Phi_1(Q)).$$

This is very similar to our coproduct, where $\omega_X$ play the role of $u_k$. Note that even if $\omega_X = \omega_Y = 0$, $\omega_{X \times Y}$ could be nontrivial.
Geometric motivation: braid varieties

Given a positive braid $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, Mellit defined the braid variety $X(\beta) = \{z_1, \ldots, z_k : B_{i_1}(z_1) \cdots B_{i_k}(z_k) \text{ is upper-triangular}\}$ where $B_{i}(z)$ are certain explicit matrices.

**Theorem (Mellit)**

There is an algebraic closed 2-form $\omega$ on $X(\beta)$. The cup product with $\omega$ satisfies “curious hard Lefshetz property” with respect to the weight filtration in $H^*(X(\beta))$.

The form $\omega$ corresponds to the symmetric function $Q = \sum x_i^2$.

**Theorem (Casals,G.,M.Gorsky,Simental)**

If $\beta = \gamma \Delta^2$ then $X(\beta)$ is smooth and $\omega$ is holomorphic symplectic.
Theorem (Galashin, Lam)

The braid variety for the \((m, n)\) torus knot is isomorphic to the open positroid stratum in \(Gr(m, m+n)\), up to a free action of a certain torus.

For example, \((3, 4)\) torus knot corresponds to the open positroid stratum in \(Gr(3, 7)\), also known as \(E_6\) cluster variety. The weight filtration in its cohomology was computed by Lam and Speyer:

\[
\begin{array}{c|ccccccc}
E_6: & k - p = 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
& 1 & 0 & 1 & 0 & 1 & 1 &
\end{array}
\]

The 2-form \(\omega\) generates the second cohomology group of this variety, and lifts to an action of \(\mathfrak{sl}_2\). The symmetric function \(Q = \sum x_i^3\) corresponds to an interesting generator of \(H^4\).
One can compare this with the bottom row of the HOMFLY-PT homology:

\[ E_6 : \begin{array}{c|cccccc}
   k - p = 0 & H^0 & H^1 & H^2 & H^3 & H^4 & H^5 & H^6 \\
   1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
   1 & & & & & & &
\end{array} \]

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1. N. Dunfield, S. Gukov, J. Rasmussen. The superpolynomial for knot homologies. Experiment. Math. 15 (2006), no. 2, 129–159.
2. T. Lam, D. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. 1604.06843
Thank you