A ‘Third’ Quantization Constructed for Gauge Theory of Gravity

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In general, a global and unique vacuum state cannot be constructed for a curved space. As a remedy, we introduce a curved space background geometry with a Minkowski metric tensor and locally non-zero curvature and torsion. Based on this geometry, we propose a ‘third’/vacuum quantization model as a consequence of Unruh effect. Accordingly, we introduce a ‘third’ quantization scalar field as a general coordinate transformation of spacetime for the second quantization fields. Then we show that in the classical limit, the ‘third’ quantization fields appear as Riemannian manifolds with an emergent metric on which the second quantization fields are located. This way, the standard model of field theory turns out as an effective theory. Moreover, using the proposed ‘third’ quantization fields, we build a $U(1) \times SU(4)$ Yang-Mills gauge theory for gravity. According to this gravitational model, we indicate that an analytical solution of the presented gravitational model, for the ‘third’ quantum field particle trajectory (such as a star), corresponds to the trajectory of a test particle in the Mannheim-Kazanas space. Furthermore, by using non-perturbative methods and lattice gauge theory results, we render a solution for the potential of the constructed model that can explain the galaxy rotation curves and gravitational lensing without any need to dark matter. We also address the cosmic microwave background phenomenon and the expansion of the universe.

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I. INTRODUCTION

General relativity is an elegant and interesting theory of gravity that is invariant under diffeomorphism – see, e.g., Refs. [1–3] and references therein. Physicists have been interested in expressing it (and, in general, gravitation) as a gauge theory. At the early stages of gauge theory, in Refs. [4, 5], a gauge theory based model of gravity was proposed. Thereafter, by introducing a gauge theory based on the Poincaré group for gravity [6, 7], other types of models have been suggested but their gauge transformation method differs compared to the standard approach [8]. Nowadays, there are many examples of gauge gravitational models built upon gauge gravitation theory (GGT). Most of these models are developed by using differential geometry approach – see Refs. [2, 8] and references therein.

On the other hand, in Ref. [2], a gauge theory of gravity called gauge theory gravity (GTG) has been constructed in the language of geometric algebra [10]. There is an important difference between GGT and GTG. The background space is Riemannian for the former, and Minkowski for the latter. The fact that there is not a unique vacuum state in a curved background space [11] is a problem for GGT based models but not for GTG. The GTG is the first point of inspiration for the work presented in this article. The second point has to do with the second quantization in quantum field theory.

There is a fundamental difference between standard gauge theories such as for quantum electrodynamics (QED) and those proposed for gravity. Consider a fermionic test particle with mass $m$ and charge $e$, in a Riemann-Cartan manifold [1] with connection $\Gamma^\alpha_{\mu\nu}$, subjected to an external electromagnetic gauge field $A^\alpha$. Its autoparallel equation is

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu
u} \dot{x}^\mu \dot{x}^\nu = \frac{e}{m} (A^\nu_{\ -\mu} - A^\nu_{\ \mu\nu})\dot{x}^\nu,$$  \hspace{1cm} (1)

where the Greek letters run from zero to three. Adding a constant value to the electromagnetic gauge field will alter the particle’s velocity but, due to Eq. (1), will keep its acceleration invariant. Based on QED, a fermion cannot be accelerated by a quantum of the second quantization gauge connection field. A statistical set of quanta will be needed. In contrast with the case for gravity, adding a

1 The physical consequences of the existence of torsion have been considered in the literature. For example, the compatibility of torsion with the equivalence principle has been discussed in Ref. [12], and the existence of stability in theories including torsion has been investigated in Refs. [13–15]. Torsion and discussions about the unitarity, the existence of ghost and some issues related to quantum field theory have also been discussed in Refs. [16, 17]. However, considering these types of issues for the proposed model in this study will be investigated in a separate work.

2 In Refs. [13, 14], it has been shown that an inertial state is equivalent to an accelerated state on which a statistical set of the second quantization creation operators is applied. (Of course, conversely in this work, we will show that an accelerated state is equivalent to an inertial state on which a statistical set of the second quantization creation operators is applied.)
constant value to the gravitational connection field will change the particle’s acceleration. Inspired by this comparison, we envisage that a quantum of the gravitational connection field cannot be a quantum of a second quantization field. Instead, the effect of it should be similar to the effect of a statistical set of, or a partition function for, the quanta of the second quantization gauge connection fields.

For a theory of quantum gravity, we require that the effect of a quantum of the gravitational connection field be equivalent to that of a statistical set, or a partition function, of second quantization quanta. Before defining such a field, one must remedy the problem of defining a unique vacuum state. For this issue, in Sec. II, we first introduce a geometry with a Minkowski background metric by making changes to the topology of a Riemann-Cartan manifold (i.e., a manifold with non-zero torsion).

II. MINKOWSKI METRIC WITH NON-ZERO CURVATURE AND TORSION

One of the controversial issues in curved space quantum field theory is about the definition of vacuum state. In general, as the notion of vacuum state in a curved space is highly non-unique \([\ref{11}]\), defining a global vacuum state for it is out of reach. However, given that the concept of vacuum in the Minkowski spacetime is well-defined, if one can somehow replace the background manifold with it, then it might open a way to bypass the problem of vacuum definition there.

In the tetrad/frame formalism\([\ref{2}]\) for a tangent bundle, the local relation between the tetradic components of metric (e.g., \(g_{ab}\)) and its coordinate ones are

\[
g_{ab} = e^a_\mu e^b_\nu g_{\mu\nu},
\]

where the Latin letters serve to label the tetradic bases. For a Minkowski frame (i.e., \(g_{ab} \equiv \eta_{ab}\)), the metricity \(g_{ab;\mu} = 0\) makes the spin connection to be

\[
\omega^{ab}_\mu = -\omega^{ba}_\mu.
\]

From the topological point of view, the tangent bundle is over the base manifold \(M\). If one exchanges the fiber with the base space in the mentioned case, in general, one will have

\[
\begin{cases}
g_{\mu\nu} \equiv \eta_{\mu\nu} \\
g_{\mu\nu;\alpha} = -\Gamma^{\gamma}_{\mu\alpha} g_{\gamma\nu} - \Gamma^{\gamma}_{\nu\alpha} g_{\mu\gamma} = 0 \rightarrow \\
\omega^{ab}_\mu \neq -\omega^{ba}_\mu \\
\Gamma_{\mu\nu\alpha} = -\Gamma_{\nu\mu\alpha}.
\end{cases}
\]

Besides, the connection \(\Gamma_{\mu\nu\alpha}\) is just the contorsion tensor. In this case, the tangent space (as a Minkowski space with the Greek letters) is considered as the base space, and the manifold \(M\) (with Latin letters) as the fiber one.

For a local infinitesimal transformation \(\Lambda^a_b\) on the manifold \(M\),

\[
\delta e^a_\mu = \Lambda^a_b e^b_\mu,
\]

we obtain

\[
\delta \omega^{a\mu}_c \sim \Lambda^a_{e\mu} + \Lambda^a_{b\mu} \epsilon^b_c - \omega^{b\mu}_c \Lambda^a_e.
\]

This transformation is a local \(GL(4,\mathbb{R})\) gauge transformation that is isomorphic to local \(U(1) \times SU(4)\) gauge transformation. Such a transformation corresponds to diffeomorphism in the Riemann-Cartan geometry related to \([3]\). We refer to the situation of case \([4]\) as a ‘Minkowski-Cartan’ geometry, because the metric is Minkowski but with a non-zero torsion. Such a geometry, in terms of degrees of freedom, corresponds to the Riemann-Cartan geometry with property \([3]\). Accordingly, this ‘Minkowski-Cartan’ geometry corresponds to a principal \(U(1) \times SU(4)\) bundle, with \(\omega^{ab}_\mu\), as a connection on it. \(\epsilon^a_\mu\) is a section of the associated vector bundle, and the symbol ‘;’ is the induced covariant derivative on it.

III. QUANTUM INTERPRETATION OF COORDINATE TRANSFORMATION VIA A ‘THIRD’ QUANTIZATION PROCESS

Inspired by the well-known approach of Ref. \([19]\), from the quantum field theory point of view, we first show

\(3\) Most tensors become simple in this system, and it has the ability to reflect important physical aspects of the spacetime however, it does not alter reality.
that the Unruh effect can lead to a kind of ‘third’ quantization as vacuum quantization. For this issue, we represent a Rindler vacuum as a coordinate transformation of a Minkowski vacuum. This task will be performed by defining a ‘third’ quantization operator in terms of a partition function of the second quantization fields, which acts on the unique Minkowski vacuum constructed in the previous section. Based on this procedure, we obtain a general coordinate transformation of the Minkowski vacuum by applying a ‘third’ quantization scalar field to the Minkowski vacuum. We interpret this procedure as a ‘third’ (or, vacuum) quantization. This approach defines a way for representing gravitational connections in terms of a partition function of the second quantization fields. Hopefully this should pave a way towards a quantum process for gravitation.

In this regard, it is also well-known that in an accelerated reference frame, in the right and left sides of the Rindler wedges with a uniform proper acceleration (say, α) as

\[
\begin{align*}
R : (x, t) &= (\alpha^{-1} e^{\alpha \xi} \cosh \alpha \tau, \alpha^{-1} e^{\alpha \xi} \sinh \alpha \tau) \\
L : (x, t) &= (-\alpha^{-1} e^{\alpha \xi} \cosh \alpha \tau, -\alpha^{-1} e^{\alpha \xi} \sinh \alpha \tau),
\end{align*}
\]

(7)

the Minkowski vacuum appears as a thermal bath, where (ξ, τ) and (x, t) are respectively the Rindler and Minkowski spacetime coordinates. Indeed, in Ref. [19], via the Bogolyubov transformation [20], the Minkowski vacuum has been shown to be equivalent to a gas of the Rindler quanta in a thermal equilibrium.

Accordingly, in two dimensions, the relation between the Minkowski vacuum and the Rindler vacuum with a uniform proper acceleration has been represented as [19]

\[ |0_M > : = \hat{\Psi}_\alpha |0_{R, \alpha} >, \]

(8)

where (in the natural unit c = 1 = h) operator \( \hat{\Psi}_\alpha \), in the Rindler space, is

\[ \hat{\Psi}_\alpha \equiv \exp \left[ iW + \sum_{\omega=0}^{\infty} e^{-\pi \omega / \alpha} \left( \hat{a}_{+\omega}^{R \dagger} \hat{a}_{+\omega}^{L} + \hat{a}_{-\omega}^{R \dagger} \hat{a}_{-\omega}^{L} \right) \right]. \]

Here, the term \( e^{iW} \) is the vacuum persistence amplitude with [19]

\[ 2 \text{Im}(W) = \ln[\text{tr} e^{(-2\pi / \alpha)H}] = (\pi \alpha / 6) \delta(0), \quad \text{Re}(W)|_{\alpha=0} = 0 \quad \text{and} \quad \text{Re}(W)|_{\alpha \neq 0} = \infty. \]

(10)

The argument of the Dirac delta function is energy, \( \hat{a}_{+\omega}^{R \dagger}, \hat{a}_{-\omega}^{R \dagger}, \hat{a}_{+\omega}^{L}, \hat{a}_{-\omega}^{L} \) and \( \hat{a}_{-\omega}^{L} \) are respectively the right and left parts of left (+) and right (−) moving creation operators (with energy \( \omega \)) of a massless scalar field \( \hat{\phi} = \hat{\phi}_a + \hat{\phi}_c \). The corresponding creation field, in the Rindler frame, is

\[ \hat{\phi}(\xi, \tau) \equiv \int \frac{d\omega}{\sqrt{4\pi \omega}} \left( \hat{a}_{+\omega}^{R \dagger} e^{i\omega(\xi - \tau)} + \hat{a}_{-\omega}^{R \dagger} e^{i\omega(\xi - \tau)} \right) \hat{\phi}_a + \hat{\phi}_c. \]

For simplicity, we have chosen a massless scalar field.

\[ + \hat{a}_{+\omega}^{L} e^{i\omega(\xi - \tau)} \hat{\phi}_a + \hat{a}_{-\omega}^{L} e^{i\omega(\xi - \tau)} \hat{\phi}_a - \hat{a}_{+\omega}^{L} e^{i\omega(\xi + \tau)} \hat{\phi}_a + \hat{a}_{-\omega}^{L} e^{i\omega(\xi + \tau)} \hat{\phi}_a \right), \]

(11)

where \( \theta(x) \) is the Heaviside step function. The corresponding annihilation field, \( \hat{\varphi}_a = \hat{\phi}_a^\dagger \), and the Hamiltonian is given by

\[ \hat{H} = \sum_{\omega=0}^{\infty} \omega \left( \hat{a}_{+\omega}^{R \dagger} \hat{a}_{+\omega}^{R} + \hat{a}_{-\omega}^{R \dagger} \hat{a}_{-\omega}^{R} + \hat{a}_{+\omega}^{L \dagger} \hat{a}_{+\omega}^{L} + \hat{a}_{-\omega}^{L \dagger} \hat{a}_{-\omega}^{L} \right); \]

(12)

where the zero-point energy has been omitted. Moreover, the second quantization independent operators for the Minkowski vacuum (say, \( \hat{b}_{\pm\omega} \) and its Hermitian adjoint) have also been defined [18] in terms of the second quantization operators of the Rindler frame with a uniform proper acceleration,

\[ \hat{b}_{-\omega} \equiv \frac{1}{\sqrt{1 - e^{-2\pi \omega / \alpha}}} \begin{pmatrix} 1 & e^{-\pi \omega / \alpha} \\ e^{\pi \omega / \alpha} & -e^{-\pi \omega / \alpha} \end{pmatrix} \hat{a}_{-\omega}^{R \dagger} \hat{a}_{+\omega}^{L \dagger} \right), \]

(13)

Now, we intend to find out the inverse relation of [8], i.e. getting the Rindler vacuum with a uniform proper acceleration from the Minkowski vacuum. Accordingly, similar to the technique used in Ref. [19] in obtaining relation [20], we indicate that the Rindler vacuum can be achieved in terms of the Minkowski-Fock space in the form of

\[ |0_{R, \alpha} > = \hat{\Phi}_\alpha^\dagger |0_M >, \]

(14)

with

\[ \hat{\Phi}_\alpha^\dagger \equiv \exp \left[ -iW - \sum_{\omega=0}^{\infty} e^{-\pi \omega / \alpha} \left( \hat{b}_{+\omega} \hat{b}_{-\omega}^\dagger \right) \right] \]

(15)

and the corresponding Hamiltonian

\[ \hat{H} = \sum_{\omega=0}^{\infty} \omega \left( \hat{b}_{+\omega}^\dagger \hat{b}_{+\omega} + \hat{b}_{-\omega}^\dagger \hat{b}_{-\omega} \right), \]

(16)

where again the zero-point energy has been omitted. To prove the claim of relation (14), it would be sufficient to show that the act of any of the corresponding annihilation operators on this Rindler vacuum vanishes. For this task and without loss of generality, for instance, we perform the procedure for the annihilation operator \( \hat{a}_{+\omega}^{R \dagger} \), wherein, by transformation [18], we have

\[ \hat{\Phi}_\alpha^\dagger |0_{R, \alpha} > = \hat{b}_{-\omega} + \hat{b}_{+\omega} e^{-\pi \omega / \alpha} |0_M >. \]

(17)

Then, by employing definition [15], it is straightforward to show that relation (17) vanishes.

On the other hand, in Ref. [19], it has been indicated that the Minkowski- and Rindler-Fock spaces are unitarily inequivalent. We furthermore prove that every two
Rindler’s vacua with different uniform proper accelerations are also orthogonal to each other. For this purpose, via relations (11), (13) and (14), we achieve
\[
<0_{R,\alpha'}|0_{R,\alpha}> = e^{i\Re(W-W') - \Im(W'+W)} \text{tr} \left[ e^{-\pi(1/\alpha' + 1/\alpha)H} \right].
\] (18)

Then, by using the simple relations
\[
\text{tr} e^{(-2\pi/\alpha)H} = \prod_{\omega=0}^{\infty} \frac{1}{1 - e^{-2\pi\omega/\alpha}}\]
(19)
and
\[
\left[ \sqrt{(1 - e^{-2\pi\omega/\alpha})(1 - e^{-2\pi\omega/\alpha'})} \right]_{\alpha \neq \alpha'} < 1,
\] (20)
and relations (14), relation (18) reads
\[
<0_{R,\alpha'}|0_{R,\alpha}> = \delta_{\alpha'\alpha}.
\] (21)
Moreover, due to relation (11) and definition (15), it is clear that
\[
<0_{M}|\hat{\Phi}_\alpha^+,\hat{\Phi}_\alpha|0_{M}> = e^{i\Re(W-W') - \Im(W'+W')} = 0.
\] (22)

Therefore, the Minkowski vacuum and the Rindler vacua with different uniform proper accelerations, not only are perpendicular to each other, but each of the latter ones (as ‘third’ quantization states) can also be obtained via the defined operator (consisted of a statistical distribution function of the second quantization operators), which acts on the Minkowski vacuum. Thus, these vacua form a set of orthogonal bases for their corresponding Fock space with operators $\hat{\Phi}_\alpha^+$ and $\hat{\Phi}_\alpha$ as the creation and annihilation operators with relation
\[
<0_{M}|[\hat{\Phi}_\alpha,\hat{\Phi}_\alpha^+]|0_{M}> = \delta_{\alpha\alpha'}.
\] (23)
Accordingly, we have established a kind of ‘third’ (or, vacuum) quantization procedure.

As every second quantization operator is usually indexed with a momentum in a certain direction, these ‘third’ quantization operators $\hat{\Phi}_\alpha^+$ and $\hat{\Phi}_\alpha$ have also been indexed with a uniform proper acceleration in its corresponding direction. Analogously, a vacuum state, in which each point has a different acceleration, can be obtained via the act of an operator on the Minkowski vacuum. In two dimensions, such a Hermitian operator can also be formed from the Fourier transformation of operators $\hat{\Phi}_\alpha^+$ and $\hat{\Phi}_\alpha$ as
\[
\hat{\Theta}(x,t) = \sum_{\alpha=-\infty}^{\infty} \left[ \hat{\Phi}_\alpha^+ e^{i(\alpha x-qt)/c^2} + \hat{\Phi}_\alpha e^{-i(\alpha x-qt)/c^2} \right],
\] (24)
where $\alpha/c^2$ and $q/c^2$, in this presented ‘third’ quantization, are analogous with the wave number and the angular frequency in the second quantization fields, respectively. In this way, we are able to define a general coordinate transformation of the Minkowski vacuum using these scalar ‘third’ quantization operators in terms of a statistical set of the second quantization fields, which can apply the desired acceleration to a particle of the second quantization fields. Indeed, this transformation of coordinates deforms spacetime for the second quantization fields. Since the act of operator (24) on the Minkowski vacuum produce a scalar field, such a field does not cause any torsion or curvature. Of course, it is clear that the relations of the ‘third’ quantization fields with each other are similar to the relations of the second quantization fields with each other. However, in the next section, while introducing a constant $\hbar$ (analogous with the Dirac constant in the second quantization fields), we indicate that the states obtained by acting (24) on the Minkowski vacuum, in the classical limit $\hbar \to 0$, play the role of a Riemannian manifold on which the second quantization fields are located.

To generalize the presented formulation to four dimensional spaces, one can simply use the procedure of the quantum field theory given in Ref. (15). In this regard, consider spaces with coordinates $(x, x, t, \xi, x, \tau)$ and $(\xi, x, x, t)$ as
\[
\begin{align*}
R & : (x, x, t, \xi, x, \tau) = (\alpha^{-1} e^{i\xi} \cos \alpha \tau, x, \alpha^{-1} e^{i\xi} \sin \alpha \tau) \\
L & : (x, x, t, \xi, x, \tau) = (-\alpha^{-1} e^{i\xi} \cos \alpha \tau, x, -\alpha^{-1} e^{i\xi} \sin \alpha \tau),
\end{align*}
\] (25)
where $\xi$ is a spatial dimension in the direction of proper acceleration and the components of $x$ denote the other two spatial dimensions. Hence, relations (11), (13) and (14) will respectively change to
\[
\begin{align*}
\hat{\varphi}(\xi, \tau, x, \tau) & = \int d\omega \left[ a_{\omega k}^+ v_{\omega k}^+ \theta(x-t) \\
& + a_{\omega k} v_{\omega k}^+ \theta(t-x) + a_{\omega k}^+ v_{\omega k} \theta(x+t) \\
& + a_{\omega k} v_{\omega k}^+ \theta(-x-t) \right],
\end{align*}
\] (26)
and
\[
\begin{align*}
\hat{\varphi}_R^+ & = \frac{1}{\sqrt{1 - e^{-2\pi\omega/\alpha}}} \begin{pmatrix} 1 & e^{-\pi\omega/\alpha} \\ e^{-\pi\omega/\alpha} & 1 \end{pmatrix} \hat{\varphi}^+ \\
\hat{\varphi}_L^+ & = \exp[-iW^* - \sum_{\omega=0}^{\infty} e^{-\pi\omega/\alpha} (\hat{b}^+_{\omega k} \hat{b}^+_{\omega k})].
\end{align*}
\] (27)
Here
\[
v_{\pm k} & \equiv \frac{1}{4} \left[ \sinh(\pi\omega/\alpha) \right]^{1/2} \frac{1}{\pi \alpha} K_{i\omega/\alpha}(\alpha \xi) e^{i(x, x, \pm\omega t)}
\] (29)
with $K_{i\omega/\alpha}(\mu)$ as the modified Bessel function, and $k_\perp$ as the corresponding wave vector (15).

\footnote{Obviously, the usual unit of $q$ is (length)$^2$/time$^3$.}
IV. RELATION BETWEEN THE ‘THIRD’ AND SECOND QUANTIZATION FIELDS

Consider a statistical set of particles of the second quantization fields, which is specified in the form of a partition function similar to \( \langle \rangle \) as a scalar ‘third’ quantization field with the creation and annihilation operators \((\hat{b}, \hat{b}^\dagger)\) and the action

\[
S \propto \sum_{\omega=0}^{\infty} \left( \hat{b}_+^{\dagger} \hat{b}_- + \hat{b}_-^{\dagger} \hat{b}_+ \right).
\]

(30)

If an extra particle of the second quantization fields is added to this set, action \( \langle \rangle \) will slightly alter by \( \delta S \). Given the similarity of \( \exp (-iS/\hbar) \) with operator \( \langle \rangle \), the relation between a particle of the second quantization fields and a scalar ‘third’ quantization field can be compared to the relation between \( \delta S \) and the pilot wave \( \exp (-iS/\hbar) \) in the ‘pilot wave theory’ \[22, 21\]. To clarify some aspects of this comparison, let us consider a well-known example in the acoustic black hole topic as follows.

Analogous with the Gross-Pitaevskii equation \[22, 23\], we envisage that the ground state of a quantum system of identical bosons is described as

\[
\left( i\hbar \partial_t + b^2 \frac{\nabla^2}{2m} - b|\Theta|^2 \right) \Theta(x,t) = 0,
\]

(31)

where \( \Theta \) is a scalar ‘third’ quantization boson field, \( m \) is the mass of field \( \Theta \) and \( b \) is representative of the self-interaction power of \( \Theta \) with the usual unit \( (\text{length})^2/(\text{time})^3 \). Afterwards, let us purposely set \( \Theta \equiv \sqrt{\rho} \exp (-iS/\hbar) \) and perturb it as \( \rho \rightarrow \rho_0 + \varepsilon \rho_1 \) and \( S \rightarrow S_0 + \varepsilon S \). Hence, by substituting these perturbations into Eq. (31), while using the ‘pilot wave theory’ and working in the classical limit \( \hbar \rightarrow 0 \) (i.e., neglecting the quantum potential term), in the first approximation, we achieve the corresponding Hamilton-Jacobi and the continuity equations, namely

\[
\partial_t S_1 - \frac{p_\mu \cdot \nabla S_1}{m} - b\rho_1 = 0
\]

(32)

and

\[
\partial_t \rho_1 - \nabla \cdot (\rho_0 \nabla S_1 + \rho_1 \rho_0 \mathbf{p}) / m = 0.
\]

(33)

Then, by substituting Eq. (32) into Eq. (33) it gives

\[
\partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu S_1) = 0,
\]

(34)

where the emerged metric is

\[
\sqrt{-g} g^{\mu \nu} \equiv b^{-1} \left( \frac{1}{-p^i/m} - (\rho_0 b/m) \delta^{ij} + p^i p^j / m^2 \right)
\]

(35)

and \( p = \nabla S_0 \). This result reveals that a perturbation in a scalar ‘third’ quantization field (e.g., field \( \Theta \)), as a second quantization field, is located on the emergent of a Riemannian manifold \[22, 21\]. The maximum possible speed (i.e., the usual \( c \) of such second quantization field turned out to be \( c^2 \equiv \rho_0 b/m \), as the realization of interaction properties of a scalar ‘third’ quantization field. This issue is similar to the sound subject \[22 \] in which the speed of sound is the realization of the interaction properties of the second quantization fields \[30\]. Indeed, the properties (such as temperature and mass) of the second quantization fields constitute the properties of sound propagation environment.

Also, it is known that the properties of elastic environments and their sound waves (as perturbation of the second quantization fields) are the realization of the properties of the second quantization fields in a scale of energy. Accordingly, while considering the last two sections, analogous with the sound waves and through the presented ‘third’ quantization point of view, we envisage that the standard model of particle physics and its parameters would emerge from the properties of the ‘third’ quantization fields in the corresponding scale of energy \[12\].

V. MODELING ‘THIRD’ QUANTIZATION FIELDS

In Sec. II, we have introduced a ‘third’ quantization scalar field, which due to the scalar nature of it does not cause any curvature and/or torsion. Now in this section, we intend to make a gravitational model for the ‘third’ quantization fields in such a way that it would have the following properties. First, we require that it possesses the ‘Minkowski-Cartan’ geometry presented in Sec. II. Second, due to the success of the Weyl gravity as a renormalizable gravity theory \[37\] in explaining solar \[38\], galactic \[39, 40\], extra-galactic and cluster scales \[41\] gravitational phenomena, we want to have a structure and solutions similar to the Weyl-Cartan theory.

We also want the model to indicate similarities between a rotating black hole or a rotating star with fermionic
elementary particles. Indeed, there is a not-so-new hypothesis that black holes and elementary particles are comparable. In the older viewpoints, e.g. Refs. [29], efforts were made to show that elementary particles are some kind of black holes. However more recently, alternative views have emerged that hypothesize black holes behave like elementary particles via the double copy theory. Finally, we intend to have a kind of symmetry similar to the one introduced in Ref. [48], proposing that elementary particles are microstructures of cosmic structure as a granular medium. Accordingly, while assuming that stars constitute a statistical set of particles of fermion type, we choose a Lagrangian akin to those as ‘third’ quantization fermionic fields. In addition, as stars constitute a statistical set of particles of fermion type, it is plausible to assume that as ‘third’ quantization fermionic fields. In this way, the similarity between a fermionic fundamental particle and a star can be guaranteed. On the other hand, as stars constitute a statistical set of particles of fermion type, it is plausible to assume those as ‘third’ quantization fermionic fields. In addition, since we have chosen a gauge theory for the model, we presume that a gauge field (as a gravitational field) can change the speed and movement of stars.

At this stage, given that we want a model similar to the Weyl gravity, by considering the structural similarity between the (general) Yang-Mills theory and the Weyl-Cartan gravity [53–56], we choose a Lagrangian akin to the Yang-Mills theory based on symmetry group $U(1) \times SU(4)$ as

$$\mathcal{L}_{YM} = C \bar{\psi}_a (i \hbar \gamma^\mu D_\mu a - m C \delta^a_b) \psi^b - \frac{1}{4} G_{\mu \nu}^{ab} G^{\mu \nu} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}. \quad (36)$$

In this Lagrangian, $\psi^a$ is a spinor fermionic ‘third’ quantization field and $\bar{\psi}_a$ is its conjugate (both as the field of stars, analogous with the quark-like particles) with completeness relation

$$\psi^a \bar{\psi}_a / \psi^2 = 1/4, \quad (37)$$

which lead to

$$\bar{\psi}_a \gamma_\nu \gamma_\mu \psi^a / \psi^2 = \eta_{\mu \nu}, \quad (38)$$

where $\psi = \sqrt{\psi_a \bar{\psi}_a}$. Also, $G_{\mu \nu}^{ab}$ and $F_{\mu \nu}$ represent the curvature or strength field tensors, namely

$$G_{\mu \nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + i \frac{g}{\hbar C}(A_\nu^a A_\mu^{cb} - A_\mu^a A_\nu^{cb}),$$

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (39)$$

where $i g A_\mu^{ab} / (\hbar C)$ is a ‘third’ quantization connection gauge field with its related $SU(4)$ group symmetry indices $a$ and $b$, $i \epsilon A_\mu / (\hbar C)$ is a ‘third’ quantization $U(1)$ vector gauge field, $g$ and $\epsilon$ are coupling constants (i.e., $SU(4)$ and $U(1)$ charges, respectively). Besides,

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - i (g A_\mu^{ab} + \epsilon \delta^{ab} A_\mu / \hbar C) \quad (40)$$

is the gauge covariant derivative and $\gamma^\mu$’s are the Dirac matrices. Indeed, this Yang-Mills theory is a principal $U(1) \times SU(4)$ bundle with $i g A_\mu^{ab} / (\hbar C)$ and $i \epsilon A_\mu / (\hbar C)$ as the connections on it, $\psi^a$ as a section of an associated spinor bundle and $\gamma^\mu D_\mu^{ab}$ as the induced Dirac operator of the induced covariant derivative on the associated bundle. Now, if we make a map between the section $\psi^a$ of the spinor bundle in this Yang-Mills theory and the section $e^a_\mu$ of the vector bundle in the ‘Minkowski-Cartan’ geometry, we can state that the presented Yang-Mills theory also has the ‘Minkowski-Cartan’ geometry.

In this regard, we employ an auxiliary constant fermionic Dirac spinor $(\bar{\psi}, \psi)$ and its conjugate, $\bar{\psi}$, with the conditions

$$\bar{\psi} \gamma_4 = 4 \quad \text{and} \quad \bar{\psi} \gamma_4 \psi = 0. \quad (41)$$

Accordingly, if we consider

$$e^a_\mu \equiv \bar{\psi} \gamma_\mu \psi / \psi \quad \text{and} \quad e_{\alpha \mu} \equiv \bar{\psi} \gamma_\mu \psi / \psi \quad (42)$$

respectively as a complex tetrad field and its conjugate, then, by using [13], [57], [58] and the Fierz identities [54], those will lead to

$$\Re(e^a_\mu e_{\alpha \mu}) = \eta_{\alpha \mu} \quad \text{and} \quad \bar{\psi}_a \epsilon_\alpha \bar{\psi} = \bar{\psi}_b \psi^b. \quad (43)$$

Hence, the corresponding torsion field turns out to be

$$T^a_{\alpha \mu} = D_\alpha e^a_\mu - D_\mu e^a_\alpha. \quad (44)$$

Thus, we have established a map between the section $\psi^a$ of the spinor bundle in the presented Yang-Mills theory and the section $e^a_\mu$ of the vector bundle in the ‘Minkowski-Cartan’ geometry.

Before we continue, let us highlight that Lagrangian [36] is similar to that mentioned in Refs. [53–56] for the Weyl-Cartan theory. For this purpose, we make a map between the fields of Lagrangian [36] and the fields of the Weyl-Cartan theory as

$$i \frac{g}{\hbar C} A_\mu^{ab} \rightarrow \omega^{ab}_\mu,$$

$$i \frac{\epsilon}{\hbar C} A_\mu \rightarrow K_\mu, \quad (45)$$

13 In Ref. [29], we have shown that the spin wave of microstructures of an elastic medium behaves similar to a fermion wave. Hence, the rotation of a microstructure corresponds to the spin of a fermion particle. Here, in comparison, we have considered that the spin of stars and planets correspond to the spin of fermion particles.

14 Given the $U(1)$ gauge symmetry in the presented model, relation (43) under the $U(1)$ gauge transformation can also be written as

$$T^a_{\alpha \mu} = \psi^{-1} \bar{\psi} (D_\alpha e^a_\mu - D_\mu e^a_\alpha) \psi\psi.$$
The difference between the presented Yang-Mills theory and the Weyl-Cartan theory is that the former possesses the ‘Minkowski-Cartan’ geometry whose base space is the Minkowski space and its fiber space is a general manifold with $U(1) \times SU(4)$ symmetry. Whereas, the latter has the Weyl-Cartan geometry whose base space is a general manifold with torsion and the Weyl connection, and its fiber space is the Minkowski space. Topologically, these two equations are more comprehensive than the Einstein-Cartan equations. This comprehensiveness is due to the fact that these two equations also include the Weyl connection. In addition, the connection in the Einstein-Cartan gravity is a spin connection, whereas in these equations, the connection is a gauge connection of the $SU(4)$ symmetry group. Thus, the solutions of the Einstein-Cartan gravity equations in this particular example can be regarded as a special case of the solutions of these two equations.

15 The difference between Lagrangian $^{[15]}$ and the corresponding one mentioned in Ref. $^{[14]}$ is that in Lagrangian $^{[14]}$ the spin replaced by the curvature of the special conformal transformation vector and the square of the spin (as the matter part) has been added to it. In addition, the symmetric group of the gravitational connection in Ref. $^{[14]}$ is $SO(1,3)$ while, in Lagrangian $^{[15]}$, it is $U(1) \times SU(4)$.

16 Note that, every solution of Eq. $^{[15]}$ is also a solution of the obtained Dirac equation, whereas any solution of the Dirac equation is not necessarily a solution of Eq. $^{[15]}$.\n
\[ \Omega_{\mu \nu} = \frac{i}{\hbar} F_{\mu \nu} \rightarrow \partial_\mu K_\nu - \partial_\nu K_\mu. \]  

\[ \Omega_{\mu \nu} = \frac{\iota}{\hbar} F_{\mu \nu} \rightarrow \partial_\mu K_\nu - \partial_\nu K_\mu. \]  

\[ \text{where } \omega^{ab}_\mu \text{ is the spin connection as the Lorentz gauge group. } \]  

\[ \text{and } \frac{\Omega_{\mu \nu}}{\hbar} = \frac{\iota}{\hbar} \text{ (in addition, the symmetric group of the gravitational connection in Ref. }^{[14]} \text{ is } SO(1,3) \text{ while, in Lagrangian }^{[15]} \text{, it is } U(1) \times SU(4)). } \]  

\[ \text{Note that, every solution of Eq. }^{[15]} \text{ is also a solution of the obtained Dirac equation, whereas any solution of the Dirac equation is not necessarily a solution of Eq. }^{[15]} \]
the total space of the presented Yang-Mills theory corresponds to the total space in the Weyl-Cartan gravity, where its fiber and base spaces have been displaced.

It is known that one of the major advantages of the (general) Yang-Mills theory over the Weyl-Cartan theory and other gravitational theories is that, in general, it is a well-tested theory and its phenomenological behaviors are well-known, see e.g., Refs. [63–64]. Indeed, the result of calculations in this theory corresponds to the result of observations with good accuracy [65–66]. In this respect, in the next section, we use the methods of the (general) Yang-Mills theory to obtain the effective potential function for the fields of the presented Yang-Mills theory to explain the trajectory of stars in the gravitational fields of galaxies and the gravitational lensing.

VI. GALAXY ROTATION CURVES AND GRAVITATIONAL LENSING

In this section, we first address the gravitational potential obtained from the Yang-Mills theory presented in the previous section. Due to Eqs. (53) and (54), the trajectory of a fermion (as a star) in the presented Yang-Mills theory can correspond to the trajectory of a test particle in the Mannheim-Kazanas space. In this regard, the static symmetrical metric of the Mannheim-Kazanas solution (as an analytical solution of the Weyl conformal gravity) for a point particle is

\[ ds^2 = f(r) dt^2 - f^{-1}(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(55)

with the gravitational potential \( g_{00} \) as

\[ f(r) = V_0 - \frac{\beta}{r} + \gamma r - \lambda r^2. \]  

(56)

Here \( V_0, \beta, \gamma \) and \( \lambda \) are some constants of integration. It has been shown that such a potential can explain the gravitational phenomena from solar system to cluster scale without using the concept of dark matter [57–58]. However, in a Yang-Mills theory, the screening and confinement effects are important aspects of the theory [59]. In what follows, we address the deviation of the effective potential function from the analytical solution [58] and/or the Cornell potential (see [62]).

In the literature, for obtaining a suitable potential for the (general) Yang-Mills theory, the non-perturbative methods of Yang-Mills theory (such as lattice field theory) have been employed. Actually, in the non-relativistic limit (note that in our case, such a limit should be relative to the speed \( c \)), one possible method is to display the potential as a Fourier transform of the propagator of the connection field in the real time formalism [67–68]. Another method, while using the Wilson loop [69–70] in the lattice field theory, is to display the potential as

\[ V(r) = \lim_{t \to \infty} \frac{\partial_t W(r, t)}{W(r, t)}. \]  

(57)

wherein the Wilson loop is defined as

\[ W(r, t) \equiv \langle P \exp \left[ \oint dx^\mu A_\mu \right] \rangle, \]  

(58)

with \( P \) as the path-ordering operator.

As a few examples (that can also be applied to the presented theory as a Yang-Mills one), in addition to some interesting anisotropic solutions [67], the real part of some isotropic potential functions of the non-Abelian Yang-Mills theory (that has been obtained through the calculations of the methods mentioned in Refs. [57–58]) are

\[ \mathbb{R}(V(r)) = C - \frac{\alpha}{r} e^{-\mu r} + \frac{\sigma}{\mu}(1 - e^{-\mu r}), \]  

(59)

\[ \mathbb{R}(V(r)) = C - \frac{\alpha}{r} e^{-\mu r} + \frac{2\sigma}{\mu}(1 - e^{-\mu r}) - \frac{2\sigma}{\mu^2} e^{-\mu r}, \]  

(60)

and

\[ \mathbb{R}(V(r)) = C - \frac{\alpha - 2\sigma/\mu^2}{r} e^{-\mu r} - \frac{2\sigma/\mu^2}{r^2}. \]  

(61)

with temperature dependence parameters \( C, \alpha, \mu \) and \( \sigma \). Besides, the parameter \( \mu \) has been known as the Debye mass, which is the result of the Debye screening phenomenon [71]. The Debye mass, at a temperature below a certain temperature (or at a radius less than a certain radius), is equal to zero, in which case the potential function of (59), (60) and (61) becomes the so-called Cornell (or funnel) potential (as one of the most popular potential models)

\[ V(r) = C' - \frac{\alpha'}{r} + \sigma' r. \]  

(62)

However, in the other areas, the potential changes linearly with temperature [67–72]. Nevertheless, with very small amounts of the Debye mass, the shape of their potential functions is in the form of (61).

The imaginary part of the potential function of (59), (60) and (61) has been attributed to the Landau damping phenomenon [72], which—while assuming that the above potentials are valid in the presented Yang-Mills theory—its qualitative consequences analogously cause formation or destruction of stars in our case. However, in general, the value of the imaginary part of the potential is less than the real part of it. In addition, at short distances, its value is close to zero and increases with increasing
distance \[67, 73\]. In comparison to the presented theory, such a result indicates that no star formation would occur near the center of a galaxy, and it should take place farther away from the center of galaxy. Actually, it has been observed that the H II regions (wherein star formation takes place) are in the arms of spiral galaxies or around irregular galaxies \[72, 73\]. It is noticeable that these results are not limited to the \(SU(3)\) symmetry group \[77, 79\].

Let us now examine a specific case in almost more details. As mentioned above, the potential of the presented Yang-Mills fields behaves like the analytical solution \[66\] and/or the Cornell potential in a small radius. In connection with the quantitative explanation of the gravitational lensing phenomenon on a larger scale than clusters (although this potential is somewhat successful \[80, 83\]), in Ref. \[10\], it has been shown that its calculated value is slightly higher than its observed value. This result means that as distance increases, the potential function deviates from the analytical solution \[66\] and/or the Cornell potential. Indeed, it can be considered that, with increasing distance (and hence with increasing potential), the effect of vacuum polarization appears and the potential behavior deviates from the analytical solution \[66\] and/or the Cornell potential. Such an effect of vacuum polarization would become apparent when the vacuum non-Abelian permittivity value is different from one \[67, 70, 84–86\]. It has experimentally been determined that this potential is capable \[80–83\], in its simplest case, is the static limit \(\Pi \rightarrow k \rightarrow 0\), in the momentum space (as a spatial case of solution \[66\]), then the Fourier transform of \(V\) in relation \[66\] will be equal to \[67\]. On the other hand, in this case, solution \[61\] is the same as the potential used in the scalar-tensor-vector theory (MOG) \[67\]. Therefore, for the effective potential in the momentum space, one makes \[67, 70, 84, 86\]

\[
V(k) \rightarrow \tilde{V}(k) = \frac{V(k)}{\varepsilon(k, T)} \tag{63}
\]

where \(T\) is temperature, \(k\) is momentum, \(V(k)\) is the analytical solution of potential, and the non-Abelian permittivity is \[84, 86\]

\[
\varepsilon(k, T) = 1 + \frac{\Pi_L(0, k, T)}{k^2} \tag{64}
\]

An interesting aspect of the presented Yang-Mills theory is that the connection fields can form a bound state on their own (something like glueballs in the QCD) without any need for fermionic particles (stars). Thus, the connection fields in the presented Yang-Mills theory can be a good alternative to dark matter. For more explanation, we refer to an example in the subject of QCD. As an example of a phenomenon described via the (general) Yang-Mills theory, one can refer to protons. It has experimentally been determined that each hadron consists of a number of valence quarks and a large number of seaquarks that are float in the viscous sap of gluons \[63, 64\]. In a proton as a hadron, most of the intra-proton pressure is generated by the field of gluons (as connection fields), and sea quarks have a much smaller role in generating intra-proton pressure \[67\]. Given the similarity between the presented Yang-Mills theory and the (general) Yang-Mills theory, by qualitatively comparing a galaxy and its stars with a hadron and sea quarks within it, such a property of intra-proton pressure would be similar to property of dark matter, which contains the largest share of mass in a galaxy and causes the gravity rotation curves.

**VII. CONCLUSIONS AND OUTLOOK**

We have introduced a new geometry with non-zero local curvature and torsion with a Minkowski metric tensor. This is named ‘Minkowski-Cartan’ geometry. The tangent bundle over the Riemannian-Cartan manifold \(M\) has been replaced with an \(M\) bundle over the Minkowski space. In this way, the total space has not changed but the spin connection is not necessarily antisymmetric and the connection becomes a contorsion tensor. Since the base space is Minkowski, then a unique and well-defined vacuum can be constructed.

We have shown that the quantum of gravitational connection fields cannot be realized via second quantization. Due to this issue, we have defined a field in such a way that its quantum is equivalent to a statistical set or a partition function of second quantization fields’ quanta. This result is then used to show that the Unruh effect leads to a ‘third’ quantization as vacuum quantization. In this regard, we have indicated that the Minkowski vacuum and the Rindler vacua with different uniform proper accelerations form an orthogonal basis of a Fock space, which are related to each other by scalar ‘third’ quantization creation and annihilation operators (i.e., \(\hat{\Phi}_\alpha\) and \(\hat{\Phi}^{\dagger}_\alpha\)). In this way, we are able to define a general coordinate transformation of the Minkowski vacuum via acting the Fourier transform of \(\Phi^\dagger\) and \(\Phi\) on the Minkowski vacuum. Using the ‘pilot wave theory method’ and working in the classical limit (i.e., neglecting the quantum potential term), the ‘third’ quantization scalar fields play the role of Riemannian manifold on which the second quantization fields are located.

Given that these ‘third’ quantization scalar fields lack curvature and torsion, we build an \(U(1) \times SU(4)\) Yang-Mills model of gravity. This model corresponds to the Weyl-Cartan gravity based on general covariance. An analytical solution for the ‘third’ quantum field’s particle (such as a star) trajectory of the model corresponds to the trajectory of a test particle in the Mannheim-Kazanas space and/or the Cornell potential. Solutions for a potential of the model addressing large scales could be obtained by using loop corrections and non-perturbative, lattice gauge theory, results.
pond to modified gravity models capable of explaining galaxy rotation curves and gravitational lensing.

**Perspectives on Cosmology**—A ‘third’ quantization field is represented by a statistical set of the second quantization fields or a partition function of those. Analogously, the universe could be represented by a partition function of the ‘third’ quantization fields with a Lagrangian (of stars and the gravitational field among those), say $\mathcal{L}_{\text{TQF}}$, such that

$$\Psi_{\text{universe}} \propto \exp \left( -i \int_{t_0}^{t} dt \mathcal{L}_{\text{TQF}} \right). \tag{65}$$

In this manner, and since the CMB radiation is a global image of the statistical universe, then this radiation would also be a statistical state. Indeed, it is well-known that the CMB intensity in terms of frequency is equivalent to the chart of a black-body radiation. On the other hand, the number density of CMB as a black-body radiation and the number density of a gas of the Rindler quanta at thermal equilibrium are quantitatively equivalent. As mentioned in Sec. III, the Rindler quanta are partition functions of the second quantization fields. Therefore, the CMB should also be a partition function of the second quantization fields (i.e., photons).

The CMB fluctuations would be indicative of microstates and microstructures of the ‘third’ quantization fields. Based on (65), the universe can either be static for the case of a completely real $\mathcal{L}_{\text{TQF}}$ or dynamic (expanding or collapsing) for a complex Lagrangian, as

$$\mathcal{L}_{\text{TQF}} \rightarrow \mathcal{L}_{\text{TQF}} + i V_{\text{Im}} |\Psi_{\text{universe}}| \propto \exp \left( \int_{t_0}^{t} dt V_{\text{Im}} \right). \tag{66}$$

The Hubble law confirms an expanding universe at the present era. Now, as mentioned in Sec. IV, the ‘third’ quantization fields play the role of a Riemannian manifold for the second quantization fields, such as photons. Therefore, the expansion of the universe implies the expansion of the Riemannian manifold for photons of the CMB, which leads to a decrease in the CMB temperature.

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