Extreme mass ratio inspirals with spinning secondary: a detailed study of equatorial circular motion

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Extreme mass-ratio inspirals detectable by the future Laser Interferometer Space Antenna provide a unique way to test general relativity and fundamental physics. Motivated by this possibility, here we study in detail the EMRI dynamics in the presence of a spinning secondary, collecting and extending various results that appeared in previous work and also providing useful intermediate steps and new relations for the first time. We present the results of a frequency-domain code that computes gravitational-wave fluxes and the adiabatic orbital evolution for the case of circular, equatorial orbits with (anti)aligned spins. The spin of the secondary starts affecting the gravitational-wave phase to next-to-leading order in the mass ratio (being thus comparable to the leading-order conservative part and to the second-order dissipative part of the self-force) and introduces a detectable dephasing, which can be used to measure it at $5 - 25\%$ level, depending on individual spins. In a companion paper we discuss the implication of this effect for tests of the Kerr bound.

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I. INTRODUCTION

Extreme mass-ratio inspirals (EMRIs) are among the most interesting gravitational-wave (GW) sources for the future space-based Laser Interferometer Space Antenna (LISA) [1] and for evolved concepts thereof [2]. An EMRI consists of a stellar-size compact object (henceforth dubbed as secondary) orbiting a supermassive object (henceforth dubbed as primary). The mass ratio of the binary is $q = \mu/M \in (10^{-7} - 10^{-4})$ and the secondary makes $O(1/q)$ cycles before plunging. This provides a unique opportunity to map the spacetime of the primary and to study radiation-reaction effects that govern the evolution of the orbit.

While parameter estimation still faces challenging open problems [3, 4], in principle an EMRI detection with LISA can provide exquisite measurements of the properties of the binary [5]. In addition, EMRIs are unique probes of fundamental physics [5, 6]. Probing both the conservative and the dissipative sector of the dynamics, they allow for novel tests of gravity [7, 8] and of the nature of supermassive objects [3, 13, 16].

With these motivations in mind, in this work we provide a detailed study of the EMRI dynamics in the presence of a spinning secondary. The spin of the secondary starts affecting the dynamics at next-to-leading order in the mass ratio, being thus comparable to the leading-order conservative part and to the second-order dissipative part of the self-force [17, 22].

EMRI detection and parameter estimation require accurate models of the waveform up to next-to-next to leading order in the mass ratio [17, 18, 22]. Therefore, no EMRI inspiral and waveform model is complete without including the spin of the secondary, which motivates sev-
eral work on this topic.

Earlier work in perturbation theory mostly focused on the effect of the spin on unbound orbits \cite{23,24}, and the spin of the secondary was taken to be unrealistically large in order to maximize its effect and compensate for the mass-ratio suppression. One of the first work to consider dissipative spin effects on bound orbits is Ref. \cite{30}, which estimated post-Newtonian terms for the fluxes by expanding the Teukolsky equation (see also Ref. \cite{27} for a more recent analysis). A more recent work \cite{19} considered the precession of a gyroscope in Schwarzschild spacetime induced by the conservative self-torque of the particle. The effects of conservative spin-curvature coupling and self-force were studied in Ref. \cite{21} for circular orbits in Schwarzschild, and later on in Ref. \cite{22} for generic orbits. The GW fluxes for circular orbits in Schwarzschild and Kerr spacetimes were computed accurately using a time-domain code \cite{28,29}, comparing also some of the most used choices for the supplementary spin conditions discussed below. Recently, Ref. \cite{22} considered spin dissipative effects with a spinning test particle and derived new flux-balance laws relating the asymptotic fluxes of energy and angular momentum to the adiabatic changes of the orbital parameters, focusing on the case of circular orbits around a Schwarzschild and secondary spin perpendicular to the orbital plane. However, to the best of our knowledge, none of the previous work went on to compute explicitly the adiabatic evolution to the leading order and the corresponding spin-correction to the GW phase, which is crucial to estimate the detectability of the secondary spin. In this work we present a detailed study in this direction for circular, equatorial orbits around a Kerr BH and (anti)aligned spins.

The plan of the paper is as follows. Section II is devoted to an introduction of the motion of a spinning test particle in curved spacetime. In Sec. III the problem is specialized to the case of a primary Kerr metric and, in particular, to circular, equatorial orbits with (anti)aligned spins. The adiabatic approximation used to evolve the orbit and to compute the dephasing is discussed in Sec. IV. Section V is devoted to a brief discussion of the numerical methods used to solve the problem. Results are presented in Sec. VI. Future work is discussed in the conclusion, Sec. VII. In Appendices A and B we provide some details on the Sasaki-Nakamura (SN) equation and on the Teukolsky source term for a spinning particle, collecting and extending various results that appeared in previous work and also providing useful intermediate steps and new relations for the first time. Finally, a comparison with the GW fluxes computed in previous work is presented in Appendix C.

In a companion paper we discuss how measurements of the spin of the secondary can be used to devise model-independent tests of the Kerr bound, i.e. the fact that spinning black holes (BHs) in general relativity cannot spin above a critical value of the angular momentum \cite{31}.

Throughout this work we use geometric units, $G = c = 1$, and define the Riemann tensor as

$$ R_{\mu\nu\sigma} \delta \omega_3 = 2\nabla_{[\mu} \nabla_{\nu]} \omega_3 \ , $$

where $\nabla_\mu$ is the covariant derivative and $\omega_3$ an arbitrary 1-form, while the square brackets denote the antisymmetrization. This is the same notation adopted in the package \textsc{xAct} \cite{32} of the software \textsc{Mathematica}, which we used for all the tensor computations. The metric signature is $(-, +, +, +)$.

II. MULTIPLE MOMENTS AND EMRI DYNAMICS

The dynamical evolution of an EMRI can be suitably studied in the framework of perturbation theory, in which a small (secondary) object perturbs the background metric of a larger (primary) BH. We consider the secondary as a point particle, thus neglecting finite-size effects arising from tidal interactions. If the size of the small body is considerably smaller than the typical scale of the binary, set by the curvature radius of the central object, its stress-energy tensor $T^{\mu \nu}$ allows for a multipolar expansion within the so-called gravitational skeletonization \cite{33,34,35,36}.

For a given worldline $X^\alpha(\tau)$, specified by the secondary proper time $\tau$, the multipole moments in general relativity have the following integral representation \cite{37}

$$ \int_{x^\alpha = \text{const}} T^{\mu \nu} \delta x^\alpha_1 \cdots \delta x^\alpha_n \sqrt{-g} \, d^3x \ , $$

where $\delta x^\alpha = x^\alpha - X^\alpha$ is the deviation from $X^\alpha(\tau)$, defined inside the world-tube of the body, and $g = \det(g_{\mu \nu})$ is the determinant of the metric $g_{\mu \nu}$. Hereafter we consider the pole-dipole approximation, by neglecting all moments of the secondary higher than the first two: the linear momentum $p^\alpha$, and the spin-dipole described by the skew-symmetric tensor $S^{\mu \nu}$:

$$ p^\alpha = \int_{x^\alpha = \text{const}} \sqrt{-g} d^3x T^{\alpha 0} \ , $$

$$ S^\alpha{}^\beta (X^\alpha) = \int_{x^\alpha = \text{const}} \sqrt{-g} d^3x (\delta x^\alpha T^{30} - \delta x^\beta T^{\alpha 0}) \ . $$

The integrals (3)-(4) are computed choosing a coordinate frame such that $\delta x^\alpha = 0$, while $\delta x^\nu$ lie inside the integration region. We refer the reader to Refs. \cite{26,34,38} for a covariant representation of the multipole moments and for a detailed discussion on their properties.

The covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu \nu} = 0$, leads to the Mathisson- Papapetrou-Dixon (MPD) equations of motion for the spinning test body. These equations were first obtained by Mathisson in linearized theory of gravity \cite{39}, and then by Papapetrou in full general relativity \cite{40,41}. A covariant formulation was obtained by Tulczyjew \cite{35} and Dixon \cite{34,36}, who also included the higher-order multipole moments of the secondary. A modern derivation is given in...
Ref. [12]. The MPD equations of motion read:

\[
\frac{dX^\mu}{d\zeta} = v^\mu, \quad (5)
\]

\[
\nabla_v p^\mu = -\frac{1}{2} R^{\rho\alpha\beta\gamma} v^\mu S_{\alpha\beta}, \quad (6)
\]

\[
\nabla_v S^{\mu\nu} = 2\tilde{p}^{[\mu} v^{\nu]}, \quad (7)
\]

\[
m \equiv -p_\mu v^\mu, \quad (8)
\]

where \(\nabla_v \equiv v^\rho \nabla_\rho\), \(v^\mu\) is the tangent vector to the representative worldline, and \(\zeta\) is an affine parameter that can be different from the proper time \(\tau\). Thus, the tangent vector \(v^\mu\) does not need to be the 4-velocity of a physical observer. The timelike condition \(v^2 \equiv v^\mu v_\mu < 0\) is not a priori guaranteed by the MPD equations, i.e., \(v^2\) is not necessarily an integral of motion. The mass \(m\) is the so-called monopole rest-mass, which is related to the energy of the particle as measured in the center of mass frame. The total or dynamical rest mass of the object is given by

\[
m^2 = -p^\mu p_\mu, \quad (9)
\]

and represents the mass measured in a reference frame where the spatial components of \(p^\mu\) vanish. Neither \(m\) nor \(\mu\) are necessarily constants of motion [33]. The spin parameter \(S\) is defined as

\[
S^2 \equiv \frac{1}{2} S^{\mu\nu} S_{\mu\nu}, \quad (10)
\]

which is also not a priori conserved. The 4-velocity and the linear momentum are not aligned since

\[
p^\mu = \frac{1}{v^2} (m v^\mu - v_\sigma \nabla_\sigma S^{\mu\sigma}). \quad (11)
\]

The system of MPD equations is underdetermined, since there are 18 dynamical variables \(\{X^\mu, v^\mu, p^\mu, S^{\mu\nu}\}\) (note that \(S^{\mu\nu}\) is skew-symmetric) and only 15 equations of motion. One therefore needs to specify 3 additional constraints to close the system of equations. These constraints are given by choosing a spin-supplementary condition, which fixes the reference worldline with respect to which the moments are computed. We choose as a reference worldline the body’s center of mass. However, in general relativity the center of mass of a spinning body is observer-dependent, thus it is necessary to specify a reference frame by fixing, for example, the spin-supplementary condition covariantly as\(^\dagger\)

\[
S^{\mu\nu} V_\nu = 0, \quad (12)
\]

and by choosing \(V^\nu\) as the 4-velocity of a physical observer. The representative worldline \(X^\nu(\zeta)\) identifies then the center of mass measured by an observer with timelike 4-velocity \(V^\nu\) (for more details see [44, 45]).

Hereafter we choose the Tulczyjew-Dixon condition:

\[
S^{\mu\nu} p_\nu = 0, \quad (13)
\]

which corresponds to \(V^\mu \equiv p^\mu\), i.e. one requires that the center of mass is measured in the frame where \(p^\mu = 0\). This spin condition fixes a unique worldline, and gives a relation between the 4-velocity \(v^\mu\) and the linear momentum \(p^\mu:\)

\[
v^\mu = \frac{m}{\mu^2} \left( p^\mu + \frac{2S^{\mu\nu} R_{\nu\rho\sigma\beta} p^\rho S^{\sigma\beta}}{4\mu^2 + R_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}} \right). \quad (14)
\]

Moreover, as a consequence of the Tulczyjew-Dixon spin-supplementary condition, the mass \(m\) and the spin \(S\) become constants of motion, unlike the mass term \(m\). To fix the latter, we first need to choose an affine parameter \(\zeta\) for the MPD equations. One possible choice is setting \(\zeta\) equal to the proper time \(\tau\), which guarantees that \(v^\mu v_\mu = -1\) throughout the dynamics. Imposing \(v^\mu v_\mu = -1\) automatically fixes \(m\). Another possibility, first proposed in [46], (see also [47, 48]) consists in rescaling \(\zeta\) such that

\[
p^\mu v_\mu = -\mu \implies \mu = m = \text{const}, \quad (15)
\]

which makes \(m\) constant. In this case however we need to check that \(v^\mu v_\mu < 0\) during the orbital evolution. This choice of the affine parameter will be labeled with \(\zeta = \lambda\), to differentiate it from the generic affine parameter \(\zeta\). It has been numerically shown that, by imposing the same initial conditions, \(\lambda\) and \(\tau\) are equivalent and lead to the same worldline [47]. In the next sections we will also check that the condition \(v^\mu v_\mu < 0\) is always satisfied for all configurations, and that it is equivalent to impose \(v^\mu v_\mu = -1\) and to require that \(m \in \mathbb{R}\). Finally, the conservation of the mass parameter \(\mu\) in the Tulczyjew-Dixon spin-supplementary condition guarantees that the normalization \(\mu^2 = -p^\mu p_\mu\) holds during the dynamical evolution.

Plugging Eq. (14) into Eq. (7), it is easy to see that

\[
\nabla_v S^{\mu\nu} = O(q) . \quad (16)
\]

Thus, the spin tensor is parallel-transported along the worldline to leading order in the mass ratio.

The freedom in the choice of the spin-supplementary condition reflects the physical requirement that in classical theories particles with intrinsic angular momentum must have a finite size, and that any point of the body can be used to fix the representative worldline. Given \(R\) the typical size of the rotating object, it has been shown that \(S/\mu \geq R\) where \(S/\mu\) is the Møller radius [49]. Hence, denoting with \(|R_{\mu\nu\rho\sigma}|\) the magnitude of the Riemann tensor, the MPD equations are valid as long as the condition \(|R_{\mu\nu\rho\sigma}|^{-1} \gg (S/\mu)^2\) is satisfied, i.e if the size of the spinning secondary is much smaller than the curvature radius of the primary. For a Kerr spacetime, the

\(^\dagger\) There are several possible physical spin-supplementary conditions, at least in the pole-dipole approximation. See for example Ref. [37] for a summary of the most common choices used in the literature.
Kretschmann scalar is $48M^2/r^6$ on the equatorial plane, so $|R_{\mu\nu\rho\sigma}| \approx M/r^3$. Thus, the validity condition of the MPD equations for a Kerr background becomes

$$\left(\frac{r}{M}\right)^3 \gg \left(\frac{S}{\mu M}\right)^2.$$  

In the following it will be useful to define the dimensionless spin parameter $\sigma$ as

$$\sigma := \frac{S}{\mu M} = c\chi,$$

where $\chi = S/\mu^2$ is the reduced spin of the secondary. Regardless of the nature of the secondary, in EMRIs it is expected $|\chi| \ll 1/q$, which implies $|\sigma| \ll 1$. This also shows that Eq. (17) is always satisfied in the EMRI limit.

### III. ORBITAL MOTION

In this section we review the orbital motion of a spinning test particle in the Kerr metric, focusing on the case of circular, equatorial orbits and (anti)aligned spins. Along the way we present some useful intermediate steps and novel relations that, to the best of our knowledge, have not been presented anywhere else.

The background spacetime is described by the Kerr metric in Boyer-Lindquist coordinates,

$$ds^2 = -dt^2 + \Sigma (\Delta^{-1}dr^2 + d\theta^2) + (r^2 + a^2) \sin^2 \theta \, ds^2 + 2Mr/\Sigma a \sin^2 \theta - dt)^2,$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$, and $a$ is the spin parameter such that $|a| \leq M$. Without loss of generality, we assume that the specific spin $a$ of the primary is aligned to the $z$-axis, namely $a \geq 0$. The spin $S$ of the secondary is positive (negative) when it is aligned (antialigned) with the primary spin.

The computations in this section are valid for a generic spin parameter $\sigma$, although later on we will be interested mostly in the case $\sigma \ll 1$ which is relevant for EMRIs.

#### A. Field equations in the tetrad formalism and constants of motion

To describe the orbital motion it is convenient to introduce the following orthonormal tetrad frame (in Boyer-Lindquist coordinates)

$$e_\mu^{(0)} = \left(\sqrt{\frac{\Delta}{\Sigma}},0,0,-a\sin^2 \theta \sqrt{\frac{\Delta}{\Sigma}}\right),$$

$$e_\mu^{(1)} = \left(0,\frac{\Sigma}{\Delta},0,0\right),$$

$$e_\mu^{(2)} = \left(0,0,\sqrt{\frac{\Delta}{\Sigma}},0\right),$$

$$e_\mu^{(3)} = \left(-\frac{a}{\sqrt{\Sigma}} \sin \theta,0,0,\frac{r^2 + a^2 \sin^2 \theta}{\sqrt{\Sigma}}\right).$$

We use the notation $e_\mu^{(a)} = (e_\theta^{(a)}, e_\phi^{(a)}, e_r^{(a)}, e_\varphi^{(a)})$, with the Latin indices for the tetrad components, which are raised/lowered using the metric $\gamma_{\mu\nu} = \text{diag}(-1,1,1,1)$.

The equations of motion then read

$$\frac{d}{d\lambda} p_\mu^{(a)} = \omega_{(b)}^{(c)}(a) u_b^{(c)} p_c^{(a)} - \frac{1}{2} \sum_{(b)} (b)(d) \chi^{(c)}(d) S_{(c)}^{(d)}.$$

The computations in this section are valid for a generic Killing field $\kappa\mu$ of the background spacetime there exists a first integral of motion

$$C_\kappa = p_\mu \kappa^\mu - \frac{1}{2} \sum_{\nu} \kappa^\nu S^{\mu\nu},$$

which is conserved also when higher multipoles are included [46]. The conserved quantities $C_{\xi} \equiv E$ and $C_{\Sigma} \equiv J_\perp$ are associated with $\xi^\mu$ and $\Sigma^\mu$, respectively [24].

It is convenient to introduce the spin vector

$$s^{(a)} = -\frac{1}{2} \epsilon^{(a)(b)(c)(d)} u_b^{(c)} S_{(c)}^{(d)},$$

where $\epsilon^{(a)(b)(c)(d)}$ is the antisymmetric Levi-Civita tensor ($\epsilon^{(0)(1)(2)(3)} = 1$) and $u^{(a)} = p^{(a)}/\mu$. The spin tensor can be recast in the following form

$$s^{(a)} = \epsilon^{(a)(b)(c)(d)} u_c S_{(d)}.$$

#### B. Equations of motion on the equatorial plane

When the orbit is equatorial, and neglecting radiation-reaction effects, it can be shown that if the spin vector is parallel to the $z$-axis, i.e. $s^\mu = s^0 b_0^\mu$, the spinning particle is constrained on the equatorial plane. In fact, suppose we set $s^\mu = s^0 b_0^\mu$ as initial condition. By construction, $s^\mu p_\mu = 0$, which implies $p^\theta = 0$ and $S^{\mu\theta} = 0$. Thus, using the equations of motion (7):

$$\nabla_s S^{\mu\theta} = 0 \Rightarrow p^{\mu} v^\theta - p^\theta v^\mu = 0 \Rightarrow p^{\mu} v^\theta = 0,$$

which implies the only nontrivial solution $v^\theta = 0$. One also needs to prove that $\theta = \pi/2$ is a solution of the equations of motion. From Eq. (6), we have

$$\nabla_s p^\theta = 0 \Rightarrow 0 = -\frac{1}{2} R^{\mu\nu\rho\sigma} e_\mu S_{\nu\rho} \propto \cos \theta.$$
which shows that \( \theta = \pi/2 \) is a solution. If \( \theta = \pi/2 \) at \( \lambda = 0 \), then the initial condition \( s^\mu = s^\delta \delta^\mu_\delta \) guarantees that \( \theta = \pi/2 \) for any value of the evolution parameter \( \lambda \). Note that this property does not depend on the spin-supplementary condition.

Hereafter, in order to simplify the notation, we introduce the hatted dimensionless quantities as \( \hat{a} = a/M \) and \( \hat{r} = r/M \). We also set \( s^{(2)} = -S \), such that for \( S > 0 \) (resp. \( S < 0 \)) the spin is parallel (resp. antiparallel) to the \( z \)-axis.

Using Eqs. \((14)\) and \((15)\) and the normalization \( u^{(a)}u^{(a)} = -1 \), it is possible to write the velocities \( v^{(a)} \) in terms of the normalized momenta \( u^{(a)} \)

\[
\begin{align*}
    v^{(0)} &= \frac{1}{N} \left( 1 - \frac{\sigma^2}{\hat{r}^2} \right) u^{(0)}, \\
    v^{(1)} &= \frac{1}{N} \left( 1 - \frac{\sigma^2}{\hat{r}^2} \right) u^{(1)}, \\
    v^{(3)} &= \frac{1}{N} \left( 1 + \frac{2\sigma^2}{\hat{r}^2} \right) u^{(3)},
\end{align*}
\]

with \( N = 1 - \frac{\sigma^2}{\hat{r}^2} \left[ 1 + 3(u^{(3)})^2 \right] \). Likewise, the conserved quantities can be written as

\[
\begin{align*}
    \hat{E} &= \frac{\sqrt{\Delta}}{\hat{r}} u^{(0)} + \hat{a} \sigma + \frac{\sigma^2}{\hat{r}^2} u^{(3)}, \\
    \hat{J}_z &= \frac{\sqrt{\Delta}}{\hat{r}} (\hat{a} + \sigma) u^{(0)} + \left[ \frac{\hat{r}^2 + \sigma^2}{\hat{r}} + \frac{\sigma^2}{\hat{r}^2} (1 + \hat{r}) \right] u^{(3)},
\end{align*}
\]

where \( \hat{E} = E/\mu \) and \( \hat{J}_z = J_z/(\mu M) \). Since we assumed \( a \geq 0 \), the orbit is prograde and retrograde for \( \hat{J}_z > 0 \) and \( \hat{J}_z < 0 \), respectively. At infinity the constant of motion \( J_z \) can be interpreted as the total angular momentum on the \( z \)-axis, i.e. the sum \( J_z \approx L_z + S \) of the orbital angular momentum \( L_z \) and of the spin \( S \) of the secondary.

The above relations can be inverted to obtain \( u^{(0)} \) and \( u^{(3)} \) in terms of \( \hat{E} \) and \( \hat{J}_z \):

\[
\begin{align*}
    u^{(0)} &= -\frac{\hat{E} \hat{r}^3 + (\hat{E} \hat{a} - \hat{J}_z) \sigma + \hat{r} \hat{a} [\hat{J}_z - \hat{E}(\hat{a} + \sigma)]}{\Sigma_\sigma \sqrt{\Delta}}, \\
    u^{(3)} &= \frac{\hat{r} [\hat{J}_z - \hat{E}(\hat{a} + \sigma)]}{\Sigma_\sigma},
\end{align*}
\]

where

\[
\Sigma_\sigma = \hat{r}^2 \left( 1 - \frac{\sigma^2}{\hat{r}^2} \right) > 0,
\]

which is positive due to the constraint \((17)\). Using Eqs. \((38)-(39)\) and the relations between the velocities \( v^{(a)} \) and the normalized momenta \( u^{(a)} \) [Eqs. \((33)-(35)\)], we can write the equations of motion in Boyer-Lindquist coordinates as (see also Ref. \cite{21})

\[
\begin{align*}
    \Sigma_\sigma A_\sigma \frac{d\hat{f}}{d\lambda} &= \hat{a} \left( 1 + 3\sigma^2 \frac{\hat{r}}{\Sigma_\sigma^2} \right) [\hat{J}_z - \hat{E}(\hat{a} + \sigma)] + \frac{\hat{r}^2 + \sigma^2}{\Delta} P_\sigma, \\
    (\Sigma_\sigma A_\sigma)^2 \left( \frac{d\hat{r}}{d\lambda} \right)^2 &= R_\sigma^2, \\
    \Sigma_\sigma A_\sigma \frac{d\phi}{d\lambda} &= \left( 1 + 3\sigma^2 \frac{\hat{r}}{\Sigma_\sigma^2} \right) [\hat{J}_z - \hat{E}(\hat{a} + \sigma)] + \frac{\hat{a}}{\Delta} P_\sigma,
\end{align*}
\]

where

\[
\begin{align*}
    A_\sigma &= 1 - 3\sigma^2 \hat{r} [-(\hat{a} + \sigma) \hat{E} + \hat{J}_z]^2, \\
    R_\sigma &= P_\sigma^2 - \Delta \left( \frac{\Sigma_\sigma^2}{\hat{r}^2} + \frac{(-\hat{a} + \sigma) \hat{E} + \hat{J}_z|^2}{\hat{r}^2} \right), \\
    P_\sigma &= \left[ (\hat{r}^2 + \sigma^2) + \frac{\sigma^2}{\hat{r}} (\hat{r} + 1) \right] \hat{E} - \left[ \hat{a} + \sigma \right] \hat{J}_z,
\end{align*}
\]

and \( \frac{1}{\Delta} \Sigma_\sigma A_\sigma N = N \).

As previously discussed, condition \((13)\) does not necessarily imply \( v^{(a)}v^{(a)} < 0 \) and the latter condition must be checked during the dynamics. The norm of \( v^{(a)} \) reads

\[
v^{(a)}v^{(a)} = -\hat{r}^6 + 3\sigma^2 (u^{(3)})^2 \left( 2\hat{r}^3 + \sigma^2 \right) + 2\sigma^2 \hat{r}^3 - \sigma^4 \]

\[
\frac{(\hat{r}^3 N)^2}{(\hat{r}^3 N)^2},
\]

and the constraint \( v^{(a)}v^{(a)} < 0 \) leads to

\[
A_\sigma > \frac{\hat{r}^3 + 2\sigma^2}{2\hat{r}^3 + \sigma^2}.
\]

Equation \((47)\) shows that \( A_\sigma \) must be positive definite, which implies \( N > 0 \). Moreover, for realistic values of \( \sigma \) (recall that \( |\sigma| \ll 1 \) when \( |\chi| \ll 1/q \), see Eq. \((13)\) the constraint \((47)\) reduces to

\[
A_\sigma > \frac{1}{2} \quad \text{for} \quad \sigma \ll 1
\]

and, since \( \hat{E} \) and \( \hat{J}_z \) are usually \( O(1) \) during the dynamics, \( A_\sigma \approx 1 \) for \( \sigma \ll 1 \). Thus Eq. \((47)\) is always satisfied for bound equatorial EMRIs. Finally, we note that choosing the proper time of the object as evolution parameter, the condition \( v^{(a)}v^{(a)} = -1 \) fixes the kinematical mass \( m \) as

\[
m(\hat{r}) = \frac{\hat{r}^3 N}{\sqrt{\hat{r}^6 - 3\sigma^2 (u^{(3)})^2 (2\hat{r}^3 + \sigma^2) - 2\sigma^2 \hat{r}^3 + \sigma^4}}.
\]

Imposing that \( m(\hat{r}) \) is a real number gives again the constraint \((47)\).
C. Effective potential, ISCO, and orbital frequency

For circular orbits, there are two additional constraints on the motion: one enforces zero radial velocity, the other requires zero radial acceleration. The condition \( v = 0 \) implies \( v^{(1)} = 0 \) and, together with Eq. (53), yields \( \rho^{(1)} = 0 \), whereas zero radial acceleration requires \( \frac{d\varphi}{dt}^{(1)} = 0 \). Imposing these constraints is equivalent to ask the orbital radius to be the local minimum of an effective potential.

For a spinning particle moving on the equatorial plane of a Kerr BH, the effective potential depends on the spin-supplementary condition (see Refs. [29][30] for the form of the effective potentials for some common choices of the spin-supplementary conditions). Following Ref. [51] we use

\[
V_{\sigma}(\hat{r}) = \frac{1}{\hat{r}^{2}}(\alpha_{\sigma}E^{2} - 2\beta_{\sigma}E + \gamma_{\sigma}),
\]  

(50)

where

\[
\alpha_{\sigma} = \left[ \hat{r}^{2} + \hat{a}^{2} + \frac{\hat{a}\sigma(\hat{r} + 1)}{\hat{r}} \right]^{2} - \Delta(\hat{a} + \sigma)^{2},
\]  

(51)

\[
\beta_{\sigma} = \left[ \left( \hat{a} + \frac{\sigma}{\hat{r}} \right)(\hat{r}^{2} + \hat{a}^{2} + \frac{\hat{a}\sigma(\hat{r} + 1)}{\hat{r}}) - \Delta(\hat{a} + \sigma) \right] \hat{j}_{z},
\]  

(52)

\[
\gamma_{\sigma} = \left( \hat{a} + \frac{\sigma}{\hat{r}} \right)^{2} \hat{j}_{z}^{2} - \Delta \left[ \hat{r}^{2}\left(1 - \frac{\sigma^{2}}{\hat{r}^{2}}\right) + \hat{j}_{z}^{2} \right].
\]  

(53)

The effective potential reduces to the standard one for a nonspinning particle in Kerr when \( \sigma = 0 \). The condition for a circular orbit with radius \( \hat{r}_{0} \) translates to

\[
V_{\sigma}(\hat{r}_{0}) = 0, \quad \frac{dV_{\sigma}}{d\hat{r}} \bigg|_{\hat{r}=\hat{r}_{0}} = 0,
\]

and stability of such orbits against radial perturbations requires \( \frac{d^{2}V_{\sigma}}{d\hat{r}^{2}} \bigg|_{\hat{r}=\hat{r}_{0}} < 0 \), although the orbit might still be unstable under perturbation in the \( \theta \) direction [52]. The innermost stable circular orbit (ISCO) is obtained by imposing \( \frac{d^{2}V_{\sigma}}{d\hat{r}^{2}} \bigg|_{\hat{r}=\hat{r}_{0}} = 0 \).

In order to compute the GW fluxes, we also need the orbital frequency of a circular equatorial orbit as measured by an observer located at infinity,

\[
\hat{\Omega} = M\Omega = \frac{d\phi}{dt} = \frac{\hat{a}v^{(0)} + \sqrt{\Delta}v^{(3)}}{(\hat{r}^{2} + \hat{a}^{2})v^{(0)} + \hat{a}\sqrt{\Delta}v^{(3)}},
\]

In terms of the momenta \( \hat{\Omega} \) is given by

\[
\hat{\Omega} = \frac{\hat{a}(\hat{r}^{3} - \hat{a}^{2})u^{(0)} + \sqrt{\Delta}(\hat{r}^{3} + 2\hat{a}^{2})u^{(3)}}{(\hat{r}^{2} + \hat{a}^{2})(\hat{r}^{3} - \hat{a}^{2})u^{(0)} + \hat{a}\sqrt{\Delta}(\hat{r}^{3} + 2\hat{a}^{2})u^{(3)}},
\]

(54)

where \( u^{(0)} \) and \( u^{(3)} \) are given in terms of \( \hat{r} \) by solving \( \frac{d\rho^{(1)}}{dx} = 0 \):

\[
u^{(0)} = \frac{1}{\sqrt{1 - U_{\mp}^{2}}}, \quad u^{(3)} = \frac{U_{\mp}}{\sqrt{1 - U_{\mp}^{2}}},
\]  

(55)

where \( U_{\mp} = \frac{u^{(3)}}{u^{(0)}} = -\frac{2\hat{a}\sigma^{3} + 3\sigma\hat{r}^{2} + 4\hat{a}\sigma^{2} \mp \Delta}{2\sqrt{\Delta}(\hat{r}^{3} + 2\sigma^{2})},
\]  

(56)

with

\[
\Delta = \sqrt{4\hat{r}^{2} + 12\hat{a}\sigma^{2}\hat{r}^{5} + 13\sigma^{2}\hat{r}^{4} + 6\hat{a}\sigma^{2}\hat{r}^{2} - 8\sigma^{4}\hat{r} + 9\hat{a}^{2}\sigma^{4}},
\]

(57)

and the \( \mp \) sign corresponding to co-rotating and counter-rotating orbits, respectively. Note that the argument of the square root is not positive definite for generic values of \( \sigma \). Nevertheless, for \( \sigma \ll 1 \), it is easy to see that Eq. (56) is always real. Using Eq. (55), the orbital frequency \( \hat{\Omega} \) can be recast as

\[
\hat{\Omega} = \frac{(2\hat{a} + 3\sigma)v^{3} + 3(2\hat{a}^{2} + \hat{a}\sigma^{2})\hat{r} + 4\hat{a}\sigma^{2} \mp \hat{\rho}D}{2(\hat{a}^{2} + 3\hat{a}\sigma + \sigma^{2})\hat{r}^{3} + 6\sigma(\hat{a} + \sigma)\hat{a}^{2}\hat{r} + 4\hat{a}^{2}\sigma^{2} - 2\hat{r}^{6}},
\]

(58)

This formula agrees with the one shown in Ref. [28].

Plugging Eq. (55) into Eqs. (36)-(37) finally yields the first integrals \( \hat{E} \) and \( \hat{j}_{z} \) for a spinning object in circular equatorial orbit in the Kerr spacetime:

\[
\begin{align*}
\hat{E} &= \frac{\hat{r}\sqrt{\Delta} + (\hat{a} + \sigma)U_{\mp}}{\hat{r}^{2}\sqrt{1 - U_{\mp}^{2}}}, \\
\hat{j}_{z} &= \frac{\sqrt{\Delta}(a + \sigma) + [\hat{r}^{3} + \hat{r}(\hat{a} + \sigma) + \hat{a}\sigma]U_{\mp}}{\hat{r}^{2}\sqrt{1 - U_{\mp}^{2}}},
\end{align*}
\]

(59)

(60)

The minus and plus sign in Eq. (58) correspond to prograde and retrograde orbits, respectively. To our knowledge, the expressions (59) and (60) have not been presented elsewhere in the literature, but will be useful when studying the adiabatic evolution of the orbit.

Furthermore, the above quantities can be used to derive analytical expressions for the ISCO location and frequency to \( \mathcal{O}(\sigma) \) (see also Ref. [51]). The orbital frequency can be written as

\[
\hat{\Omega}(\hat{r}) = \hat{\Omega}^{0}(\hat{r}) + \sigma\hat{\delta}\hat{\Omega}(\hat{r}) + \mathcal{O}(\sigma^{2}),
\]

(61)

where \( \hat{\Omega}^{0}(\hat{r}) = 1/(\hat{a} \pm \hat{r}^{3/2}) \) is the orbital frequency of a nonspinning particle around Kerr, and

\[
\hat{\delta}\hat{\Omega}(\hat{r}) = -\frac{3}{2}\frac{\sqrt{\hat{r}^{3}} \mp \hat{a}}{\sqrt{\hat{r}^{3} + 2\hat{a}}}.
\]

(62)

The ISCO location can be expanded in the same way and its leading-order spin correction reads

\[
\delta\hat{r}_{\text{ISCO}} = \frac{4\hat{a}}{\hat{r}^{0}_{\text{ISCO}}} \mp \frac{4}{\sqrt{\hat{r}^{3}_{\text{ISCO}}}},
\]

(63)

where \( \hat{r}^{0}_{\text{ISCO}} \) is the (normalized) ISCO location of the Kerr metric for a nonspinning secondary, which is solution to \( \hat{r}^{2} - 6\hat{r} + 8\hat{a}\hat{r}^{1/2} - 3\hat{a}^{2} = 0 \) (its analytical expression as a function of \( \hat{a} \) can be found in Ref. [53]). Using the
above results, the leading-order spin correction to the ISCO orbital frequency is
\[
\delta \hat{\Omega}_{\text{ISCO}} = \frac{9}{2} \left( \frac{\sqrt{r^2_{\text{ISCO}} + \hat{a}}}{\sqrt{r^2_{\text{ISCO}} / (\sqrt{r^2_{\text{ISCO}}})^{-3/2} + \hat{a}}} \right). \tag{64}
\]
This quantity is shown in Fig. 1 as a function of \( \hat{a} \) for prograde orbits (upper sign Eq. [64]). Note that \( \delta \hat{\Omega}_{\text{ISCO}} > 0 \) for any \( \hat{a} \) (being zero in the extremal case), i.e., if the spin of the secondary is aligned to that of the primary the orbital frequency at the ISCO is higher.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Spin correction to the orbital frequency at the ISCO as a function of \( \hat{a} \) for prograde orbits (upper sign Eq. [64]).}
\end{figure}

\section*{IV. RADIATION-REACTION EFFECTS AND BALANCE LAWS}

We study radiation-reaction effects within the \textit{adiabatic approximation}, assuming that the emission timescale is much longer than orbital period, namely
\[
\frac{2\pi}{\Omega} \ll \left| \frac{d\hat{r}}{dt} \right|^{-1}. \tag{65}
\]
In this approximation, changes to the mass terms \( \mu \) and \( M \) and to the spin \( \hat{a} \) are smaller than the leading-order dissipative terms \[54]. The change to the primary mass and spin due to GW absorption at the horizon formally enter at the next-to-leading order, although with a small coefficient \[55].

Thus, for a \textit{nonspinning} object on an equatorial orbit around a Kerr BH
\[
\frac{dE}{dt} = \Omega \frac{dL_z}{dt}. \tag{66}
\]
In the adiabatic approximation, the following balance equations hold:
\[
\left( \frac{dE}{dt} \right)_{\text{GW}} = -\left( \frac{dE}{dt} \right), \quad \left( \frac{dL_z}{dt} \right)_{\text{GW}} = -\left( \frac{dL_z}{dt} \right),
\]
where the brackets denote time-averaging over a time length much longer than the time evolution of the orbital parameters but shorter than the radiation time scales. The gravitational energy and angular momentum luminosities include both the contribution at infinity and at the event horizon, and are calculated by averaging over several wavelengths. Since in an EMRI the GW luminosities are of \( \mathcal{O}(q^2) \), Eq. [65] is satisfied throughout the dynamics at least up to the ISCO, as we have numerically checked a posteriori.

For a spinning particle in Kerr, there is an extra degree of freedom related to the spin of the small object. In general the evolution of the constants of motion can also depend on the secondary spin evolution. However, it was recently shown that the evolution of the \( E \) and \( J_z \) are formally the same as those above to first order in \( \sigma \) [22]. On the other hand, the evolution of the spin tensor \( S_{\mu\nu} \) depends on local metric perturbations and not only on asymptotic fluxes [22]. This evolution determines that of the particle 4-velocity through Eq. [28]. However, as shown in Eq. [16], the spin tensor evolves at \( \mathcal{O}(q) \) and it affects the particle acceleration to higher order in the mass ratio. Likewise, the effect of the secondary spin on the adiabatic changes to \( M \) and \( \hat{a} \) is subleading. Thus – for what concerns the leading-order spin corrections to the dynamics – the evolution of the binary masses and spins can be neglected.

It remains to prove that the equation
\[
\frac{dE}{dt} = \hat{\Omega} \frac{dJ_z}{dt} \tag{68}
\]
holds for a spinning object with the above assumptions. This was shown in Ref. [26] in the case of a spinning object moving in a circular orbit off the equatorial plane at first order in the spin. In fact, for a circular equatorial orbit and (anti)aligned spins, this property can be proved straightforwardly to \( \mathcal{O}(\sigma) \): assuming that the secondary spin remains constant to leading order in \( q \), Eq. [68] is equivalent to
\[
\hat{\Omega} = \frac{\delta \tilde{E}}{\delta J_z} = \tilde{E} \left( \frac{\partial J_z}{\partial \tilde{r}} \right)^{-1}. \tag{69}
\]
Using Eqs. [58], [59], and [60], it is immediate to see that the previous relation is satisfied in our case.

\subsection*{A. GW fluxes in the Teukolsky formalism}

We use the Teukolsky formalism to compute the gravitational wave flux at infinity. Metric perturbations of
the Kerr background are decomposed using the Newman-Penrose tetrad basis, that allows to isolate the nontrivial degrees of freedom of the Riemann tensor. At infinity, the two GW polarizations are both encoded in the $\Psi_4$ Weyl scalar:

$$\Psi_4(r \to \infty) = \frac{1}{2} \frac{\partial^2}{\partial t^2} (h_{+} - i h_{\times}) .$$  \hfill (70)

In the Fourier space,

$$\Psi_4 = \rho \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} d\omega R_{\ell m \omega}(\hat{r}) e^{i (m \phi - \omega t)} ,$$  \hfill (71)

where $\rho = |\hat{r} - i \hat{a} \cos \theta|^{-1}$, and the $s = -2$ spin-weighted orthonormal spheroidal harmonics $e^{i (m \phi - \omega t)}$ and radial function $R(\hat{r})$ obey two decoupled ordinary differential equations. For the angular component:

$$\left[ \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d}{d \theta} \right) - \hat{a}^2 \omega^2 \sin^2 \theta - \left( \frac{m - 2 \cos \theta}{\sin \theta} \right)^2 \right] e^{i(m \phi - \omega t)}$$

$$+ 4 \hat{a} \cos \theta \omega \theta - 2 + 2 \hat{a} m \omega + \lambda_{\ell m \omega} = 0 ,$$  \hfill (72)

where $\lambda_{\ell m \omega} = E_{\ell m \omega} - 2 \hat{a} m \omega + \hat{a}^2 \omega^2 - 2$. The eigenvalues and the eigenfunctions satisfy the following identities: $\lambda_{\ell m - \omega} = \lambda_{\ell m \omega}$ and

$$-2 S_{\ell m \omega}(\theta) = (-1)^l -2 S_{\ell m}^{\omega}(\pi - \theta) ,$$  \hfill (73)

while $-2 S_{\ell m \omega}(\theta)e^{im\phi}$ reduces to the spin-weighted spherical harmonics for $\hat{a} = 0$ or $\omega = 0$. We have employed the numerical routines provided by the BH Perturbation Toolkit \cite{50} to compute $\lambda_{\ell m \omega}$, the spin-weighted spheroidal harmonics, and their derivatives.

The radial Teukolsky equation is given by

$$\Delta^2 \frac{d}{d \hat{r}} \left( \Delta \frac{d}{d \hat{r}} \right) - V(\hat{r}) R_{\ell m \omega}(\hat{r}) = \mathcal{T}_{\ell m \omega} ,$$  \hfill (74)

where the source term $\mathcal{T}_{\ell m \omega}$ is discussed below and the potential $V(\hat{r})$ reads

$$V(\hat{r}) = -K^2 + 4i(\hat{r} - 1) \Delta + 3 \hat{a} \hat{r} + \lambda_{\ell m \omega} ,$$  \hfill (75)

$$K = (\hat{r}^2 + \hat{a}^2) \omega - \hat{a} m .$$  \hfill (76)

The homogeneous Teukolsky equation admits two linearly independent solutions, $R_{\ell m \omega}^{\text{in}}$ and $R_{\ell m \omega}^{\text{up}}$, with the following asymptotic values at horizon $\hat{r}_+$, and at infinity:

$$R_{\ell m \omega}^{\text{in}} \sim \begin{cases} D_{\ell m \omega}^{\text{out}} \hat{r}^3 e^{-i \hat{r}^*} & \hat{r} \to \hat{r}_+ , \\ D_{\ell m \omega}^{\text{in}} \hat{r}^3 e^{i \hat{r}^*} & \hat{r} \to \infty , \end{cases}$$  \hfill (77)

$$R_{\ell m \omega}^{\text{up}} \sim \begin{cases} B_{\ell m \omega}^{\text{out}} \hat{r}^3 e^{-i \hat{r}^*} & \hat{r} \to \hat{r}_+ , \\ B_{\ell m \omega}^{\text{in}} \hat{r}^3 e^{i \hat{r}^*} & \hat{r} \to \infty . \end{cases}$$

$$= \frac{1}{W_\hat{r}} \left\{ R_{\ell m \omega}^{\text{up}}(\hat{r}) \int_{\hat{r}_+}^{\hat{r}_+} d\hat{r}' R_{\ell m \omega}^{\text{in}}(\hat{r}') T_{\ell m \omega}(\hat{r}') \frac{\Delta^2}{\hat{r}^2} \right. + \left. R_{\ell m \omega}^{\text{in}}(\hat{r}) \int_{\hat{r}_-}^{\hat{r}_-} d\hat{r}' R_{\ell m \omega}^{\text{up}}(\hat{r}') T_{\ell m \omega}(\hat{r}') \right\} ,$$  \hfill (80)

The radial Teukolsky equation can be solved through the Green function method \cite{57}. The solution with the correct asymptotics reads

$$R_{\ell m \omega}(\hat{r}) = \frac{1}{W_\hat{r}} \left\{ R_{\ell m \omega}^{\text{up}}(\hat{r}) \int_{\hat{r}_+}^{\hat{r}_+} d\hat{r}' R_{\ell m \omega}^{\text{in}}(\hat{r}') T_{\ell m \omega}(\hat{r}') \frac{\Delta^2}{\hat{r}^2} \right. + \left. R_{\ell m \omega}^{\text{in}}(\hat{r}) \int_{\hat{r}_-}^{\hat{r}_-} d\hat{r}' R_{\ell m \omega}^{\text{up}}(\hat{r}') T_{\ell m \omega}(\hat{r}') \right\} ,$$  \hfill (81)

The solution is purely outgoing at infinity and purely ingoing at the horizon:

$$R_{\ell m \omega}(\hat{r} \to \hat{r}_+) = Z_{\ell m \omega}^{\infty} \Delta^2 e^{-i \hat{r}^*} ,$$  \hfill (82)

$$R_{\ell m \omega}(\hat{r} \to \infty) = Z_{\ell m \omega}^{H} \hat{r}^3 e^{i \hat{r}^*} ,$$  \hfill (83)

with

$$Z_{\ell m \omega}^{\infty} = C_{\ell m \omega}^{\infty} \int_{\hat{r}_+}^{\infty} d\hat{r}' R_{\ell m \omega}^{\text{in}}(\hat{r}') T_{\ell m \omega}(\hat{r}') ,$$  \hfill (84)

$$Z_{\ell m \omega}^{H} = C_{\ell m \omega}^{H} \int_{\hat{r}_-}^{\infty} d\hat{r}' R_{\ell m \omega}^{\text{up}}(\hat{r}') T_{\ell m \omega}(\hat{r}') ,$$  \hfill (85)

and

$$C_{\ell m \omega}^{H} = \frac{1}{2 \hat{a} \omega B_{\ell m \omega}^{\text{in}}} , \quad C_{\ell m \omega}^{\infty} = \frac{B_{\ell m \omega}^{\text{up}}}{2 \hat{a} \omega B_{\ell m \omega}^{\text{in}}} .$$  \hfill (86)

The amplitudes $Z_{\ell m \omega}^{H}$ and $Z_{\ell m \omega}^{\infty}$ fully determine the asymptotic GW fluxes at infinity and at the horizon. The factors $B_{\ell m \omega}^{\text{up}}$ and $D_{\ell m \omega}^{\text{in}}$ are arbitrary, but it is convenient to fix their values as shown in Appendix A. As discussed in Sec. V, we compute $R_{\ell m \omega}^{\text{in}}$ and $R_{\ell m \omega}^{\text{up}}$ using two different methods: the Mano Suzuki Takasugi (MST) method \cite{58,60} and by solving the SN equation (see Appendix A). These methods agree with each others within the numerical accuracy.

The source term $\mathcal{T}_{\ell m \omega}$ of the radial Teukolsky equation is rather cumbersome, even for nonspinning bodies. For generic bound orbits, the source term is given by

$$Z_{\ell m \omega}^{\infty} = C_{\ell m \omega}^{\infty} \int_{-\infty}^{\infty} d\hat{t} e^{i(\hat{t} \hat{r} - m \phi(\hat{t}))} \mathcal{T}_{\ell m \omega}^{\infty} (\hat{r}(\hat{t}), \theta(\hat{t})) ,$$  \hfill (87)

where $\hat{r} = \hat{r} - m \hat{\omega} + \hat{r}_\pm = 1 \pm \sqrt{1 - \hat{a}^2}$, $\hat{\omega} = \hat{a}/(2\hat{r}_+)$, and being $\hat{r}^*$ the tortoise coordinate of the Kerr metric,

$$\hat{r}^* = \hat{r} + \frac{2\hat{r}_+}{\hat{r}_+ - \hat{r}_-} \ln \left( \frac{\hat{r} - \hat{r}_+}{2} \right) - \frac{2\hat{r}_-}{\hat{r}_+ - \hat{r}_-} \ln \left( \frac{\hat{r} - \hat{r}_-}{2} \right) .$$  \hfill (79)
where $\mathcal{I}^{\ell,\infty}[\hat{r}(t), \theta(t)]$ is

$$
\mathcal{I}^{\ell,\infty}[\hat{r}(t), \theta(t)] = \left[ A_0 - (A_1 + B_1) \frac{d}{dt} + (A_2 + B_2) \frac{d^2}{dt^2} - B_3 \frac{d^3}{dt^3} \right] \bigg|_{t=\theta(i), \hat{r} = \hat{r}(i)}.
$$

(88)

Related technical details as well as the explicit form of this term are given in Appendix B [61, 62].

At infinity, Eqs. (71) and (85) lead to the gravitational-wave signal

$$
h_+ - i h_\times \sim -\sum_{\ell m} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \mathcal{Z}_{\ell m,\hat{\omega}}^H e^{i\omega(\hat{r}^* - t) - 2iS_\ell(\theta) t} e^{i\epsilon \phi},
$$

where $\theta$ is the angle between the observer’s line of sight and the spin axis of the primary (here aligned with the z-axis), while $\varphi \equiv \phi(t = 0)$.

For a circular equatorial orbit, the form of the source term greatly simplifies and, since $\phi(t) = \hat{\Omega} t$, Eq. (87) reduces to

$$
\mathcal{Z}_{\ell m,\hat{\omega}}^{H,\infty} = \delta(\hat{\omega} - m\hat{\Omega}) \mathcal{A}_{\ell m,\hat{\omega}}^H\hat{\Omega},
$$

(90)

with $\mathcal{A}_{\ell m,\hat{\omega}}^H = 2\pi \mathcal{Z}_{\ell m,\hat{\omega}}^{H,\infty}(\hat{r}_0, \pi/2)$ computed for a specific orbital radius $\hat{r}_0$. In this case the waveform (89) reduces to

$$
h_+ - i h_\times \sim -\sum_{\ell m} \frac{\mathcal{A}_{\ell m,\hat{\omega}}^H}{(m\hat{\Omega})^2} e^{im\hat{\Omega} t} - 2S_\ell(\theta) e^{i\epsilon \phi},
$$

(91)

and the GW energy fluxes are given by

$$
\left( \frac{d\hat{E}}{dt} \right)_{GW}^\infty = \frac{1}{16\pi} \left( \langle h_+^2 \rangle + \langle h_\times^2 \rangle \right)_{GW},
$$

(92)

$$
= \frac{1}{4\pi^2} \sum_{\ell m} \frac{|\mathcal{A}_{\ell m,\hat{\omega}}^H|^2}{(m\hat{\Omega})^2} \left| -2S_\ell(\theta) \right|^2,
$$

(93)

where the angle brackets here denote averaging over several wavelengths. Using the waveform (91) and the normalization condition of the spin-weighted spheroidal harmonics, the gravitational luminosities are obtained by integrating the fluxes over the solid angle, which yields:

$$
\left( \frac{d\hat{E}}{dt} \right)_{GW} = \sum_{\ell m} \frac{|\mathcal{A}_{\ell m,\hat{\omega}}^H|^2}{2\pi (m\hat{\Omega})^2},
$$

(94)

$$
\left( \frac{d\dot{\hat{z}}}{dt} \right)_{GW} = \sum_{\ell m} \frac{|\mathcal{A}_{\ell m,\hat{\omega}}^H|^2}{2\pi (m\hat{\Omega})^3},
$$

(95)

where the sum over $m$ goes for $m = 1, \ldots, \ell$ since $Z_{\ell m,\hat{\omega}}^{H,\infty} = (-1)^{l}Z_{\ell m,\hat{\omega}}^{H,\infty}$ and the bar denotes complex conjugation.

Similarly, the GW luminosities at the horizon read

$$
\left( \frac{d\hat{E}}{dt} \right)_{GW} = \sum_{\ell m} \frac{|\mathcal{A}_{\ell m,\hat{\omega}}^H|^2}{2\pi (m\hat{\Omega})^2},
$$

(96)

$$
\left( \frac{d\dot{\hat{z}}}{dt} \right)_{GW} = \sum_{\ell m} \frac{m |\mathcal{A}_{\ell m,\hat{\omega}}^H|^2}{2\pi (m\hat{\Omega})^3},
$$

(97)

where

$$
\alpha_{\ell m} = 256(2\hat{\gamma}_\perp^2 + 4\hat{\epsilon})(4\hat{\gamma}_\perp^2 + 16\hat{\epsilon})(m\hat{\Omega})^3
$$

with $\epsilon = \sqrt{1 - \hat{a}^2}/(4\hat{\gamma}_\perp)$, and

$$
|C_{\ell m}|^2 = [(\lambda_{\ell m,\hat{\omega}}^2 + 2)^2 + 4\hat{\epsilon}(m\hat{\Omega}) - 4\hat{\epsilon}^2(m\hat{\Omega})^2]
	\times [\lambda_{\ell m,\hat{\omega}}^2 + 36m\hat{\omega}^2(m\hat{\Omega}) - 36\hat{\epsilon}^2(m\hat{\Omega})^2]
+ (2\lambda_{\ell m,\hat{\omega}}^2 + 3)|\hat{\omega}_0^2(m\hat{\Omega})^2 - 48m\hat{\omega}^2(m\hat{\Omega})^2]
+ 144(m\hat{\Omega})^2(1 - \hat{a}^2).
$$

(98)

B. Orbital evolution and GW phase

To compute the overall orbital phase $\Phi$ accumulated during the EMRI, it is necessary to calculate the total energy luminosities (from now on also called “fluxes,” with a slightly abuse of terminology):

$$
\mathcal{F} = \frac{1}{q} \left[ \left( \frac{d\hat{E}}{dt} \right)_{GW} + \left( \frac{d\hat{E}}{dt} \right)_{\infty} \right].
$$

(99)

All fluxes were calculated in normalized units, and they were rescaled by the mass ratio $q$. $\mathcal{F}_{\ell m}$ denotes the flux for the harmonic indexes $l$ and $m$. We remind that $\hat{E} = E/\mu$. Since $\hat{E} \propto q^2$ to the leading order, the normalized flux $\mathcal{F}$ does not depend on $q$.

With the fluxes $\mathcal{F}$ at hand, it is possible to calculate the adiabatic evolution of the orbital radius $\hat{r}(\hat{t})$ and phase $\Phi(\hat{t})$ due to radiation losses as follows:

$$
\frac{d\hat{r}}{d\hat{t}} = -q\mathcal{F}(\hat{r})\left( \frac{d\hat{E}}{d\hat{r}} \right)^{-1} \frac{d\Phi}{d\hat{t}} = \hat{\Omega}(\hat{r}(\hat{t})),
$$

(100)

with $\hat{E}$ given by Eq. (59).

Finally, for the dominant mode, the GW phase is related to the orbital phase by $\Phi_{GW} = 2\Phi$.

V. NUMERICAL METHODS

The solutions $R_{\ell m,\hat{\omega}}^{in}$ and $R_{\ell m,\hat{\omega}}^{up}$ to the homogeneous Teukolsky equation were calculated in two different ways:

- through the MST method [59, 60], as implemented in the MATHEMATICA packages of the BH Perturbation Toolkit [59].
• by first solving the SN equation and then transforming the obtained solution to $R^{in}_{\ell m \hat{\omega}}$ and $R^{up}_{\ell m \hat{\omega}}$ (see Appendix A).

Both methods require arbitrary precision arithmetics, and the MST method is usually faster and more accurate than solving directly the SN equation. Unfortunately, the implementation of the MST method of [56] has one limitation: the precision of $R^{in}_{\ell m \hat{\omega}}$ and $R^{up}_{\ell m \hat{\omega}}$ crucially depends on the gravitational frequency $m \hat{\omega}$. As $m \hat{\omega}$ increases, the precision of the input parameters should drastically increase as well, in order for the computed $R^{in}_{\ell m \hat{\omega}}$ and $R^{up}_{\ell m \hat{\omega}}$ to have enough significant figures. Thus, the MST method tends to become slower for large values of $\ell$ and when $\hat{r}$ approaches the ISCO.

We, therefore, took the best of the two methods and implemented both in a MATHEMATICA code. We checked that the methods agree with each other within numerical accuracy in the entire parameter space.

Our algorithm is the following:

• Choose the parameters $\hat{a}$ and $\chi$;
• Loop on the harmonic index $\ell$, starting with $\ell = 2$ until $\ell_{\text{max}}$. We typically used $\ell_{\text{max}} = 20$, see discussion below;
• If $\ell \leq 8$, loop on the index $m = 1, \ldots, \ell$ starting with $m = 1$. For larger values of $\ell$, we only considered the $m = \ell$ and $m = \ell - 1$, since the others are negligibly small.\(^4\)
• Loop on the values of an array of orbital radii $\hat{r}$, starting from $\hat{r}_{\text{start}}$. The starting point $\hat{r}_{\text{start}}$ is calculated in such a way that all the spinning test objects start the inspiral with the same frequency of a nonspinning object (i.e $\chi = 0$) at the reference value $\hat{r} = 10.1$;
• Compute the energy fluxes $\mathcal{F}$, using the MST method as implemented in [56] to obtain $R^{in}_{\ell m \hat{\omega}}$ and $R^{up}_{\ell m \hat{\omega}}$.
• The above point is performed within a certain precision threshold. If the MST method fails to give the fluxes with prescribed precision (for increasing number of figures in the input parameters; the number depends on $\ell$), switch to the SN method. To solve the SN equation, we employed the boundary conditions described in Appendix A, keeping 10 and 13 terms for the series at the horizon and infinity, respectively.

• Stop the $\hat{r}$ loop at the ISCO. Interpolate the fluxes in the range $\hat{r} \in (\hat{r}_{\text{ISCO}}, \hat{r}_{\text{start}})$;
• Using the interpolated fluxes, solve Eq. (100) to compute the orbital phase.

All the fluxes were calculated for prograde stable orbits. The parameters chosen for the numerical simulations are the following:

• $\hat{a} = (0, 0.1, 0.2, \ldots 0.9, 0.95, 0.97, 0.990, 0.995)$
• $\chi \in (-2, 2)$ with steps $\delta \chi = 0.2$
• $\mu = 30 M_\odot$ and $M = 10^6 M_\odot$, hence $q = 3 \times 10^{-5}$.

To estimate the maximum truncation errors of our code, we computed the fluxes at the ISCO for a spinning particle with $\chi = 2$ for $\ell = 21$ and $\ell = 22$ and compared with the corresponding fluxes summed up $\ell_{\text{max}} = 20$. Choosing $\chi = 2$ as a reference is just for convenience: the truncation error is practically independent of the spin of the secondary, but it is greatly affected by $\hat{a}$ and by the orbital radius. In Table I we report the fractional truncation error $\Delta^\ell(\mathcal{F})$ obtained by comparing, for $\chi = 2$ and $q = 3 \times 10^{-5}$, the fluxes at the ISCO truncated at $\ell = 20$ with the fluxes including the $\ell = 21$ and $\ell = 22$ contributions.

| $\hat{a}$   | $\Delta^\ell(\mathcal{F})$ |
|------------|----------------------------|
| 0          | $3.5 \times 10^{-14}$      |
| 0.3        | $4.5 \times 10^{-10}$      |
| 0.5        | $3.7 \times 10^{-9}$       |
| 0.8        | $3.4 \times 10^{-7}$       |
| 0.9        | $3.8 \times 10^{-6}$       |
| 0.97       | $6.1 \times 10^{-5}$       |
| 0.995      | $5.0 \times 10^{-4}$       |

**TABLE I.** Fractional truncation error $\Delta^\ell(\mathcal{F})$, obtained by taking $\chi = 2$ and $q = 3 \times 10^{-5}$ as reference. The error were estimated at the ISCO by comparing the fluxes truncated at $\ell_{\text{max}} = 20$ with the ones truncated at $\ell_{\text{max}} = 22$.

In Appendix C we compare our results for the fluxes with previous work, overall finding excellent agreement.
VI. RESULTS

A. Spin corrections to fluxes and GW phase

Due to the small mass ratio, the GW fluxes $F$ can be expanded as

$$F = F^0 + \sigma \delta F^\sigma + \mathcal{O}(\sigma^2),$$  \hspace{0.5cm} (101)

where $F^0$ are the fluxes for a nonspinning secondary around a Kerr primary and $\delta F^\sigma$ are the linear spin corrections. The coefficients $\delta F^\sigma$ were obtained by fitting the fluxes $F$ with a cubic polynomial in $\sigma$ and then retaining only the linear terms. Such fitting procedure was repeated for each value of $\hat{r}$ at which we computed the fluxes. Figure 2 shows the linear spin corrections

$$\delta F^\ell = \sum_{m=-\ell}^{\ell} \delta F^\ell_{\ell m},$$  \hspace{0.5cm} (102)

for $\ell = 2, 3, 4$ and summing up to all values of $m$ such that $|m| \leq \ell$. An analogous plot for the total flux, $\delta F^\sigma = \sum_{\ell=2}^{4} \delta F^\ell$ (summing up to $\ell = 20$) is presented in Ref. [31].

Having computed the fluxes, we can now proceed to determine the adiabatic orbital evolution and the orbital phase by solving Eqs. (100). We consider an inspiral starting at $\dot{r} = \dot{r}_{\text{start}}$ (such that the initial frequency is the same for any $\sigma$, see Sec. V). Ideally, one would like to evolve the inspiral up to the ISCO for a given value of $\hat{r}$. However, since the latter depends on $\sigma$, so it does the duration of the inspiral, also for a fixed value of $\hat{a}$. It would therefore be complicated to compare the phase evolution for different spins of the secondary. Thus, we chose to evolve the inspiral up to a reference end time $t_{\text{ref}} = t_{\text{end}} - 1/2$ day, where $t_{\text{end}}$ is the time to reach the ISCO for a nonspinning secondary for a given value of $\hat{a}$. The offset of 1/2 day is chosen so that the evolution stops before the ISCO for any value of $\hat{a}$ and $\chi$.

Throughout the inspiral, the phase $\Phi(t)$ can be written as

$$\Phi(t) = \Phi^0(t) + \frac{\sigma}{q} \delta \Phi^\sigma(t) + \mathcal{O}(\sigma^2/q),$$  \hspace{0.5cm} (103)

where $\Phi^0(t)$ is the phase for a nonspinning secondary and $\delta \Phi^\sigma(t)$ is the change due to the $\mathcal{O}(\sigma)$ contribution. Note that, since $\sigma = q\chi$, the linear spin correction is independent of $q$ to the leading order, and it is therefore suppressed by a factor $q$ relative to $\Phi^0(t) = \mathcal{O}(1/q)$. The coefficients $\delta \Phi^\sigma(t)$ were obtained by interpolating $\Phi(t) - \Phi^0(t)$ with a cubic polynomial in $\chi$ as follows

$$\Phi(t) - \Phi^0(t) = a_0 + \chi a_1 + q \chi^2 a_2 + q^2 \chi^3 a_3,$$  \hspace{0.5cm} (104)

where $a_i$ are the fit coefficients, with $a_0 \approx 0$. The reported values of $a_1$ are the fit coefficients, with $a_0 \approx 0$. The orbital phase $\Phi(t)$ is then related to the GW phase of the dominant mode by $\Phi_{\text{GW}}(t) = 2\Phi(t)$. The GW phase as a function of time is shown in Fig. 3 for various values of $\hat{a}$. Figure 4 also shows the phase difference $\Phi_{\text{GW}}(t_{\text{ref}}) - \Phi^0_{\text{GW}}(t_{\text{ref}})$ computed at $t_{\text{ref}}$ as a function of the spin $\chi$, showing that it is linear to excellent accuracy. Although we only present the range $|\chi| \leq 2$, the phase difference is linear provided $|\sigma| \ll 1$, i.e. $|\chi| \ll 1/q$, as expected.

| $\hat{a}$ | $\delta \Phi^\sigma_{\text{GW}}(t_{\text{ref}})$ [rad] | $\Delta \chi$ |
|----------|-------------------------------|-----------|
| 0        | -2.416                        | -0.414    |
| 0.1      | -2.962                        | -0.338    |
| 0.2      | -3.606                        | -0.277    |
| 0.3      | -4.367                        | -0.229    |
| 0.4      | -5.277                        | -0.189    |
| 0.5      | -6.379                        | -0.157    |
| 0.6      | -7.748                        | -0.129    |
| 0.7      | -9.522                        | -0.105    |
| 0.8      | -12.013                       | -0.0832   |
| 0.9      | -16.215                       | -0.0617   |
| 0.95     | -20.328                       | -0.0492   |
| 0.97     | -23.271                       | -0.0430   |
| 0.990    | -29.201                       | -0.0342   |
| 0.995    | -32.570                       | -0.0307   |

TABLE II. Spin corrections to the phase $\delta \Phi^\sigma_{\text{GW}}(t_{\text{ref}})$ and its inverse (which gives the resolution on a measurement of $\chi$ according to criterion (107) with $\alpha = 1$) for different values of $\hat{a}$.

The values of $\delta \Phi^\sigma_{\text{GW}}(t_{\text{ref}})$ (i.e., the slope of the lines shown in Fig. 4) for different values of $\hat{a}$ are given in Table II and plotted in Ref. [31]. We fitted these data with two different fits. The first one is

$$\delta \Phi^\sigma_{\text{GW}}(t_{\text{ref}}) = \sum_{i=0}^{3} b_i (1 - \hat{a}^2)^{i/2} + b_4 \hat{a},$$  \hspace{0.5cm} (105)

where $b_0 = 38.44, b_1 = -90.36, b_2 = 99.43, b_3 = -44.95, b_4 = 1.91$. This fit is accurate within 5% in the whole range $\hat{a} \in [0, 0.995]$, with better accuracy at large $\hat{a}$. The second fit is

$$\delta \Phi^\sigma_{\text{GW}}(t_{\text{ref}}) = \left\{ \begin{array}{ll}
\sum_{i=0}^{3} d_i \hat{a}^i & \text{if } \hat{a} \leq 0.7 \\
\sum_{i=0}^{3} e_i (1 - \hat{a}^2)^{i/2} & \text{if } 0.7 \geq a < 0.995
\end{array} \right. ,$$  \hspace{0.5cm} (106)

where $d_0 = -2.40, d_1 = -5.70, d_2 = 0.13, d_3 = -9.25$, and $e_0 = -41.42, e_1/e_0 = -2.49, e_2/e_0 = 3.30, e_3/e_0 = -2.47$. This piecewise fit is accurate within 1% in the whole range $\hat{a} \in [0, 0.995]$.

B. Minimum resolvable spin of the secondary

In a companion paper [31] we briefly discussed how the above results can be used to place a constraint on the spin of the secondary in a model-independent fashion, i.e.
without assuming any property of the secondary other than its mass and spin. Here we extend that discussion.

Measuring the binary parameters from an EMRI signal is a challenging and open problem [3][4], which requires developing accurate waveform models, performing a statistical analysis that can account for correlations among the waveform parameters, and also taking into account that the EMRI events in LISA might overlap with several (possibly louder) simultaneous signals from supermassive BH coalescences and other sources [1][4][63].

FIG. 2. The spin-correction coefficient $\delta F^\sigma_\ell$ [see Eqs. (101) and (102)] as a function of the orbital radius (up to the ISCO) for different values of the spin $\hat{a}$ of the primary and for $\ell = 2, 3, 4$ (from left to right), summing up to all values of $m$ such that $|m| \leq \ell$. An analogous plot for the total spin-correction $\delta F^\sigma = \sum_{\ell=2}^{40} \delta F^\sigma_\ell$ is presented in a companion paper [31]. Data for the fluxes are available online [62] and on the BH Perturbation Toolkit webpage [60]. Note that, for nearly-extremal primary ($\hat{a} \gtrsim 0.99$), $\delta F^\sigma_\ell$ is nonmonotonic near the ISCO, although near extremality $\ell = 2$ is not the dominant spin correction to the flux [63] and the total correction $\delta F^\sigma$ is monotonic [31].

FIG. 3. Time evolution of the linear spin corrections to the GW phase $\delta \Phi^\sigma_GW(t)$ for different values of $\hat{a}$.

FIG. 4. Phase difference $\Phi^\sigma_GW(t_{\text{ref}}) - \Phi^\sigma_GW(0)$ between a spinning and nonspinning secondary as a function of $\chi$, calculated at $t_{\text{ref}} = t_{\text{end}} - 1/2$ day, where $t_{\text{end}}$ is the time to reach the ISCO for a nonspinning secondary. Note that the curves are linear to an excellent accuracy, showing that $\Phi^\sigma_GW(t_{\text{ref}}) - \Phi^\sigma_GW(0) \propto \chi$.

Postponing a data-analysis study for a follow-up work, here we estimate the minimum resolvable $\chi$ by computing the uncertainty on $\chi$ which would lead to a total GW dephasing $\approx 1$ rad. A larger dephasing would substantially impact a matched-filter search, leading to a significant loss of detected events and potentially to systematics in
It is interesting to compare such resolution with typical values of $\chi$ for known astrophysical objects. If the secondary is a Kerr BH, then $|\chi| \leq 1$. For the fastest millisecond pulsars, $\chi \approx 0.3$. However, $\chi$ can be much larger than unity for other objects. For example, a ball of radius 1 cm and mass 1 kg making one rotation per second has $\chi \approx 1 \times 10^{17}$. Astrophysical objects do not reach such extreme values, but can have $\chi \gg 1$ [67]. For example, $\chi \approx 140$ for Earth, and $\chi \approx 10$ for the fastest white dwarfs. The above reference values are shown in Fig. 5 by horizontal lines.

Note that $|\Delta \chi| < 1$ in all cases, and therefore our simplified analysis suggests that a spin of a rapidly spinning Kerr secondary could be measured with an accuracy greater than 100%.

C. Model-independent constraints on “superspinars”

Compact dark objects which exceed the Kerr bound $|\chi| \leq 1$ (so-called “superspinars”) were suggested to arise generically in high-energy modifications to general relativity such as string theories [68]. Our results of Fig. 5 show that the typical resolution on $\chi$ achievable with an EMRI detection can be used to rule out (or detect) superspinars in a large region of the parameter space [31]. For example, if $\chi \approx 0.5 - 0.7$, a measurement with absolute error $\Delta \chi$ would exclude $\chi > 1$ at 3$\sigma$ confidence level. This is particularly interesting in light of the fact that no theoretical upper bound is expected for superspinars, besides, possibly, those coming from the ergoregion instability [69–72]. A measurement of $\chi$ at the level reported above can thus potentially probe a vast region of the parameter space for superspinars [31].

In the context of our study one could wonder whether it is theoretically consistent to study a secondary superspinar around a primary Kerr BH. This is indeed the case in two scenarios (see Ref. [73] for a review): a) if superspinars arise within general relativity in the presence of exotic matter fields, in such case both Kerr BHs and superspinars can co-exist in the spectrum of solutions of the theory; b) if superspinars arise in high-energy modified theories of gravity such as string theories, as originally proposed [68]. In the latter case it is natural to expect that high-energy corrections which are relevant for the secondary might be negligible for the primary. Indeed, in an effective-field-theory approach high-energy corrections to general relativity modify the Einstein-Hilbert action with the inclusion of higher-order curvature terms of the form [5, 74]

$$ R + \ldots + \gamma (R_{abcd})^n + \ldots, \quad n > 1 \tag{108} $$

where $R$ is the Ricci scalar, $R_{abcd}$ schematically denotes terms that depend on the Riemann tensor, and $\gamma$ is a coupling constant with dimensions of a (length)$^{2(n-1)}$. In these theories relative corrections to the metric of a
compact object of size $\sim L$ are of the order of \[ \frac{\gamma}{L^{2(n-1)}} \tag{109} \]
or some power thereof. Thus, the difference between the high-curvature corrections of the secondary relative to those of the primary scales as

\[ \sim \frac{M^2(n-1)}{\mu^2(n-1)} = q^{2(1-n)} \gg 1. \tag{110} \]

This heuristically shows the obvious fact that in an EMRI the secondary is much more affected by the high-curvature corrections than the primary, especially for high-order terms (i.e., higher values of $n$).

In certain high-curvature corrections to general relativity, the secondary might also be charged under new fundamental fields, in which case there is also extra emission (in particular there could be dipolar, $\ell = 1$, fluxes) \[9, 12, 76].\]

VII. CONCLUSION AND FUTURE WORK

We have studied the GW fluxes and the adiabatic evolution of a spinning point particle in circular, equatorial motion around the Kerr background and with spin (anti)aligned to that of the central BH. Our results for the fluxes agree with those previously appeared in the literature, whereas the computation of the GW phase is novel.

Since the EMRI dynamics does not depend on the nature of the secondary but only on its multiple moments, the GW signal can be used to derive model-independent constraints on the secondary, for example to measure the spin of a Kerr secondary, or to distinguish whether the secondary is a fastly spinning BH or a slowly-spinning neutron star, or also whether the secondary satisfies the Kerr bound or is a superspinar \[31].\]

This work represents a first step in the analysis of the impact of the secondary spin on EMRI’s evolution, in parallel with recent work along related directions. Future work will include extensions to generic orbits (e.g., along the lines of Ref. \[77]), misaligned spins (which introduce precession \[19, 26, 78, 79\]), and the development of data analysis approaches \[4\] to assess the detectability of such effects. A complete account of dissipative effects in the case of a spinning secondary would also require to consider the spin evolution due to self-force effects, which is a more challenging problem, especially for generic orbits \[22\].

Another interesting extension is to include the quadrupole moment of the secondary \[50, 80, 81\]. Compared to the spin, this effect is suppressed by a further power of the mass ratio and is probably negligible for EMRI detection with LISA, although a rigorous study is required to assess whether neglecting this term can affect parameter estimation for the loudest events. Furthermore, since the quadrupole moment of a Kerr BH is uniquely determined in terms of its mass a spin, measuring the quadrupole of the secondary would allow for model-independent tests of the BH no-hair theorem.

Finally, more theoretical related work includes nonintegrability and chaotic motion for generic values of the spin \[67, 82, 83\], although these effects might require extremely high values for the spin of the secondary and should not be directly relevant for the phenomenology of EMRI signals detectable with LISA.

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Appendix A: Sasaki-Nakamura equation

In this and in the following appendix we provide further technical details on the formalisms that we use in Secs. IV-V to compute the GW fluxes.

The homogeneous Teukolsky equation is an example of stiff differential problem, with the solutions \[77, 78\] rapidly diverging at infinity due to the long-range character of the potential. High accuracy solutions require therefore time-consuming numerical integrations. A substantial improvement in this direction has been achieved by Sasaki and Nakamura, finding a suitable transformation which maps the homogeneous Teukolsky equation to an equivalent form with a short-range potential that is easier to solve numerically \[81\]. The SN equation is given by (we remind that hatted quantities are dimensionless)

\[
[f(\hat{r})]^2 \frac{d^2}{d\hat{r}^2} + f(\hat{r}) \left( \frac{df(\hat{r})}{d\hat{r}} - F(\hat{r}) \right) \frac{d}{d\hat{r}} - U(\hat{r}) \right] \chi_{\ell m 2} = 0,
\]

(A1)

with $f(\hat{r}) = \frac{d\hat{r}}{d\hat{r}} = \frac{\Delta}{\hat{r}^2 + \hat{a}^2}$. The coefficient $F(\hat{r})$ is defined as

\[
F(\hat{r}) = \frac{\eta(\hat{r})}{\eta(\hat{r})} \frac{\Delta}{\hat{r}^2 + \hat{a}^2},
\]

(A2)

where $\ddot{r}$ denotes the derivative with respect to $\hat{r}$ and

\[
\eta(\hat{r}) = c_0 + \frac{c_1}{\hat{r}} + \frac{c_2}{\hat{r}^2} + \frac{c_3}{\hat{r}^3} + \frac{c_4}{\hat{r}^4},
\]

(A3)
with
\[ c_0 = -12i\omega + \lambda_{\ell m}(\lambda_{\ell m} + 2) - 12\omega(\omega - m), \]  \tag{A4}
\[ c_1 = 8i\omega[3\omega - \lambda_{\ell m}(\omega - m)], \]  \tag{A5}
\[ c_2 = -24i\omega(\omega - m) + 12\omega^2[1 - 2(\omega - m)^2], \]  \tag{A6}
\[ c_3 = 24i\omega^3(\omega - m) - 24\omega^2, \]  \tag{A7}
\[ c_4 = 12\omega^4. \]  \tag{A8}

The function \( U(\tilde{r}) \) in Eq. (A1) reads
\[ U(\tilde{r}) = \frac{\Delta U_1(\tilde{r})}{(\tilde{r}^2 + \tilde{a}^2)^2} + G(\tilde{r})^2 + \frac{\Delta G(\tilde{r})}{\tilde{r}^2 + \tilde{a}^2} - F(\tilde{r})G(\tilde{r}), \]  \tag{A9}
where
\[ G(\tilde{r}) = -\frac{2(\tilde{r} - 1)}{\tilde{r}^2 + \tilde{a}^2} + \frac{\tilde{r}\Delta}{(\tilde{r}^2 + \tilde{a}^2)^2}, \]  \tag{A10}
\[ U_1(\tilde{r}) = V(\tilde{r}) + \frac{\Delta^2}{\beta} \left[ (2\alpha + \beta, \tilde{r}) - \frac{\eta(\tilde{r})}{\eta(\tilde{r})} \right] \left( 1 + \frac{\beta, \tilde{r}}{\Delta} \right), \]  \tag{A11}
\[ \alpha = -iK(\tilde{r}) \frac{\beta}{\Delta^2} + 3iK(\tilde{r})\tilde{r} + \lambda_{\ell m} + \frac{6\Delta}{\tilde{r}^2}, \]  \tag{A12}
\[ \beta = 2\Delta \left[ -iK(\tilde{r}) + \tilde{r} - 1 - \frac{2\Delta}{\tilde{r}} \right]. \]  \tag{A13}

The two functions \( K(\tilde{r}) \) and \( V(\tilde{r}) \) are the same introduced for the Teukolsky radial equation \( (74) \).

The SN equation admits two linearly independent solutions, \( X^{in}_{\ell m\omega} \) and \( X^{up}_{\ell m\omega} \), which behave asymptotically as
\[ X^{in}_{\ell m\omega} \sim \begin{cases} e^{-im\tilde{r}} & \tilde{r} \to \tilde{r}_+, \\ A^{out}_{\ell m\omega}e^{im\tilde{r}} + A^{in}_{\ell m\omega}e^{-im\tilde{r}} & \tilde{r} \to \infty, \end{cases} \]  \tag{A14}
\[ X^{up}_{\ell m\omega} \sim \begin{cases} C^{out}_{\ell m\omega}e^{im\tilde{r}} + C^{in}_{\ell m\omega}e^{-im\tilde{r}} & r \to r_+, \\ \tilde{r} \to \infty, \end{cases} \]  \tag{A15}

The solutions of the Teukolsky and SN equations are related by:
\[ P^{in,up}_{\ell m\omega}(\tilde{r}) = \frac{1}{\eta} \left( \alpha + \beta, \tilde{r} \right) Y^{in,up}_{\ell m\omega} - \frac{\beta}{\Delta} Y^{in,up}_{\ell m\omega}, \]  \tag{A16}
\[ Y^{in,up}_{\ell m\omega} = \frac{\Delta}{\sqrt{\tilde{r}^2 + \tilde{a}^2}} X^{in,up}_{\ell m\omega}. \]  \tag{A17}

With the above normalization of the solutions \( X^{in}_{\ell m\omega} \), \( X^{up}_{\ell m\omega} \), these transformations allow to fix the arbitrary constants \( D^{\text{tran}}_{\ell m\omega} \) and \( B^{\text{tran}}_{\ell m\omega} \) (cf. Eq. [86]) as \[ \frac{4\omega^2}{c_0} \] and \[ \frac{1}{d_{\ell m\omega}} \] respectively.

The coefficient \( c_0 \) is given in Eq. (A4).

The numerical values of \( X^{in}_{\ell m\omega} \) (resp. \( X^{up}_{\ell m\omega} \)) are obtained by integrating Eq. (A1) from \( \tilde{r}_+ \) (resp. \( \infty \)) up to \( \infty \) (resp. \( \tilde{r}_+ \)) using the boundary conditions (A14) (resp. (A15)). In this work we have derived the boundary conditions for the homogeneous SN equation in terms of explicit recursion relations which can be truncated at arbitrary order (see Sec. A4). We finally transform back \( X^{in}_{\ell m\omega}, X^{up}_{\ell m\omega} \) to the Teukolsky solutions using Eq. (A16). The amplitude \( B^{\text{in}}_{\ell m\omega} \) can be obtained from the Wronskian \( W_{\tilde{r}} \) at a given orbital separation.

1. Boundary conditions for the SN equation in terms of recursion relations

We have derived accurate boundary conditions by looking for series expansions of the master equation at the outer horizon \( \tilde{r}_+ \) and at infinity. To this aim we have studied the singularities on the real axis of Eq. (A1), which can be recast in the form
\[ \Delta^2 \frac{d^2X_{\ell m\omega}}{d\tilde{r}^2} + \Delta F(\tilde{r}) \frac{dX_{\ell m\omega}}{d\tilde{r}} + U(\tilde{r})X_{\ell m\omega} = 0, \]  \tag{A20}
where
\[ F(\tilde{r}) = (\tilde{r}^2 + \tilde{a}^2) \left( \frac{df(\tilde{r})}{d\tilde{r}} - F(\tilde{r}) \right), \]  \tag{A21}
\[ U(\tilde{r}) = -\left( \tilde{r}^2 + \tilde{a}^2 \right)^2 U(\tilde{r}). \]  \tag{A22}

Moreover
\[ F(\tilde{r}_\pm) = 0, \quad F(\tilde{r}) \xrightarrow{\tilde{r} \to \infty} 0, \]  \tag{A23}
\[ U(\tilde{r}_+) = -\tilde{h}^2, \quad U(\tilde{r}) \xrightarrow{\tilde{r} \to \infty} -\omega^2. \]  \tag{A24}

Since the functions \( F(\tilde{r}) \) and \( U(\tilde{r}) \) are analytic on the positive real axis, it turns out that the Eq. (A1) has three singularities: two at the horizons \( \tilde{r}_- = \tilde{r}_+ \) and \( \tilde{r} = \tilde{r}_+ \), both of which are regular singularities, and one at \( \tilde{r} = \infty \) which is an irregular singularity of rank 1. By Frobenius theorem, the solutions of the SN equation around \( \tilde{r}_+ \) can be written as Frobenius series, with radius of convergence
\[ \tilde{r}_+ - \tilde{r}_- = 2\sqrt{1 - \tilde{a}^2}. \]  \tag{A25}

For \( \tilde{r} = \infty \) or \( \tilde{a} = 1 \) (for which \( \tilde{r}_+ = \tilde{r}_- \)) the boundary conditions can be written in terms of asymptotic expansions.

a. Boundary condition at the horizon

To compute the boundary conditions at the outer horizon \( \tilde{r}_+ \), it is convenient to recast the SN equation as
\[ (\tilde{r} - \tilde{r}_+)^2 \frac{d^2X_{\ell m\omega}}{d\tilde{r}^2} + (\tilde{r} - \tilde{r}_+) p_H(\tilde{r}) \frac{dX_{\ell m\omega}}{d\tilde{r}} + q_H(\tilde{r})X_{\ell m\omega} = 0 \]  \tag{A26}
where
\[
P_H(\hat{r}) = \left(\frac{\hat{r}^2 + \hat{a}^2}{\hat{r} - \hat{r}_-}\right)\left[\frac{df(\hat{r})}{d\hat{r}} - F(\hat{r})\right], \quad (A27)
\]
\[
q_H(\hat{r}) = -\left(\frac{\hat{r}^2 + \hat{a}^2}{\hat{r} - \hat{r}_-}\right)^2 U(\hat{r}). \quad (A28)
\]
Following the Frobenius method we look for a power series solution of the form
\[
X_{\ell m \omega} = (\hat{r} - \hat{r}_+)^d \sum_{n=0}^{\infty} a_n (\hat{r} - \hat{r}_+)^n, \quad (A29)
\]
where \(d\) is one of the solutions of the indicial equation
\[
I(d) = d(d - 1) + p_H(\hat{r}_+) d + q_H(\hat{r}_+) = 0. \quad (A30)
\]
For Eq. (A1), the latter corresponds to
\[
I(d) = d^2 + \nu^2 \left(\frac{2\hat{r}_+}{\hat{r}_+ - \hat{r}_-}\right)^2 = 0, \quad \hat{k} = \hat{\omega} - \frac{m\hat{a}}{2\hat{r}_+}. \quad (A31)
\]
Given \((d_1, d_2)\) two solutions of the above equation, their difference \(d_1 - d_2\) is neither zero nor an integer. We have therefore two linearly independent solutions such that
\[
X_{\ell m \omega} = \exp\left\{\pm i\hat{k} \frac{2\hat{r}_+}{\hat{r}_+ - \hat{r}_-} \log(\hat{r} - \hat{r}_+)\right\} \sum_{n=0}^{\infty} a_n (r - r_+)^n. \quad (A32)
\]
The recursion relation for the coefficients \(a_n\) is (setting \(a_0 = 1\))
\[
a_n = -\frac{1}{I(d + n)} \sum_{k=0}^{n-1} (k + d) p_H^{(n-k)}(\hat{r}_+) + q_H^{(n-k)}(r_+) \frac{a_k}{(n-k)!}, \quad (A33)
\]
where \(p_H^{(k)}(\hat{r}_+)\) and \(q_H^{(k)}(\hat{r}_+)\) are the \(k\)-th derivatives of the coefficients \(p_H(\hat{r})\) and \(q_H(\hat{r})\) with respect to \(\hat{r}\), and calculated at \(\hat{r}_+\). For \(\hat{a} \leq 0.9\), the boundary conditions at the horizon have been calculated at \(\hat{r}_m = \hat{r}_+ + \epsilon\) with \(\epsilon = 10^{-3}\), while for higher spins we have fixed \(\epsilon = 10^{-5}\). To increase precision, we truncate compute the series coefficients up to \(n = 10\).

b. Boundary condition at infinity

Ordinary differential equations with irregular singularities of rank 1, like the SN equation, admit general expressions for asymptotic expansions around such singularities (see Refs. [85] [86] and especially Ref. [87] for more details). To calculate the boundary conditions at infinity we rewrite the SN equation as
\[
\frac{d^2 X_{\ell m \omega}}{d\hat{r}^2} + p_\omega(\hat{r}) \frac{dX_{\ell m \omega}}{d\hat{r}} + q_\omega(\hat{r}) X_{\ell m \omega} = 0, \quad (A34)
\]
where
\[
p_\omega(\hat{r}) = \left(\frac{\hat{r}^2 + \hat{a}^2}{\Delta}\right) \left[\frac{df(\hat{r})}{d\hat{r}} - F(\hat{r})\right], \quad (A35)
\]
\[
q_\omega(\hat{r}) = -\left(\frac{\hat{r}^2 + \hat{a}^2}{\Delta}\right)^2 U(\hat{r}). \quad (A36)
\]
The functions \(p_\omega(\hat{r})\) and \(q_\omega(\hat{r})\) are analytic on the positive real axis, so the series
\[
p_\omega(\hat{r}) = \sum_{n=0}^{\infty} \frac{p_\omega^{(n)}(\hat{r})}{\hat{r}^n}, \quad q_\omega(\hat{r}) = \sum_{n=0}^{\infty} \frac{q_\omega^{(n)}(\hat{r})}{\hat{r}^n}, \quad (A37)
\]
converge, with \(p_\omega^{(n)}(\hat{r})\) and \(q_\omega^{(n)}(\hat{r})\) being the \(n\)-th derivatives of the coefficients \(p_\omega\) and \(q_\omega\) with respect to \(\hat{r}\). If at least one of \(p_\omega\), \(q_\omega\) or \(q_\omega^{(1)}\) is nonzero, the formal solution is given by
\[
X_{\ell m \omega} = e^{\gamma \hat{r}^\nu} \sum_{n=0}^{\infty} \frac{b_n}{\hat{r}^n}, \quad (A38)
\]
where \(\gamma\) is one of the solutions of the characteristic equation
\[
\gamma^2 + p_\omega(0) \gamma + q_\omega(0) = 0, \quad (A39)
\]
while
\[
\xi = \frac{p_\omega(1) \gamma + q_\omega(1)}{p_\omega(0) + 2\gamma}. \quad (A40)
\]
For the SN equation
\[
p_\omega(0) = 0 = p_\omega(1), \quad q_\omega(0) = \omega^2, \quad q_\omega(1) = 4\hat{\omega}^2, \quad (A41)
\]
\[
\gamma^2 + \omega^2 = 0, \quad \xi = \frac{q_\omega(1)}{2\gamma} = \pm 2i\hat{\omega}. \quad (A42)
\]
Therefore, we have two series solutions
\[
X_{\ell m \omega} = \exp\{\pm i\omega [\hat{r} + 2 \log(\hat{r})]\} \sum_{n=0}^{\infty} \frac{b_n}{\hat{r}^n}. \quad (A43)
\]
The general recursion relation for the coefficients \(b_n\) is (we set again \(b_0 = 1\))
\[
(p_\omega(0) + 2\gamma) nb_n = (n - \xi)(n - 1 - \xi)b_{n-1} + \sum_{k=1}^{n} \left[\gamma p_\omega^{(k+1)} + q_\omega^{(k+1)} - (n - k - \xi) p_\omega^{(k)}\right] b_{n-k}. \quad (A44)
\]
It can be proved that the series solutions constructed in this way diverge, and they have to be considered as asymptotic expansions. However, these solutions are unique and linearly independent. We computed the series coefficients up to \(n = 13\).

c. Cross check of the boundary conditions with Ref. [88]

We compared our boundary conditions with the ones used in Ref. [88], which are in form
\[
e^{\pm i\omega \hat{r}} \sum_{n=0}^{\infty} a_n \hat{r}^n, \quad (A44)
\]
\[
e^{\pm \hat{a} \omega \hat{r}} \sum_{n=0}^{\infty} \frac{1}{(\hat{\omega})^n}. \quad (A45)
\]
First, we notice that the tortoise coordinate \( \hat{r}^* (\hat{r}) \) at the boundaries can be written as

\[
\hat{r}^* (\hat{r}) \sim \hat{r} + 2 \ln(\hat{r}) - 2 \ln(2), \tag{A46}
\]

\[
\hat{r}^* \left( \hat{r} \right) \sim \frac{2\hat{r}^+}{\hat{r}^+ - \hat{r}^-} - \ln(\hat{r} + \hat{r}^+) + \delta \hat{r}^+(\hat{r}^+) \hat{r}^+, \tag{A47}
\]

at \( \hat{r} \to \infty \) and \( \hat{r} \to \hat{r}^+ \), respectively, and where we defined

\[
\delta \hat{r}^+(\hat{r}^+) \equiv -2\ln(2) - \frac{2\hat{r}^-}{\hat{r}^+ - \hat{r}^-} - \ln(\hat{r}^+ - \hat{r}^-) + \hat{r}^+. \tag{A48}
\]

If we multiply Eq. \( \text{A32} \) by the phase factor \( \exp \{ i k \delta \hat{r}^+(\hat{r}^+) \} \) and Eq. \( \text{A42} \) by \( \exp \{ -i \omega \ln(2) \} \), our boundary conditions have the same modulus and phase as those in Ref. \( \text{S8} \) for all the values of the parameters space we have considered, up to numerical error. In the worst case, for \( \tilde{a} = 0.995 \) and \( \ell = 20 \) at the ISCO, the fractional difference in both modulus and phase is at most of one part in \( 10^{10} \), and typically much smaller.

Since the solutions by means of series expansion of an ordinary differential equation are uniquely determined a part for a complex constant factor, the boundary conditions \( \text{A32} \) and \( \text{A42} \) are consistent with the ones of Ref. \( \text{S8} \).

Appendix B: Teukolsky source term

1. Spinning particle on a general bound orbit

The source term of the Teukolsky equation reads

\[
\mathcal{T}_{\ell m} = 4 \int d\theta d\phi \sin \theta \sin \phi \left\{ B_2' + B_2'^* \right\} \frac{1}{\rho \hat{\rho}^3} S_{\ell m} \exp \left( i (m \phi + \omega \tau) \right), \tag{B1}
\]

where the functions \( B_2' \) and \( B_2'^* \) are defined as

\[
B_2' = \frac{1}{\rho \hat{\rho}^3} \mathcal{L}_s \left( L_{\ell 0} \frac{T_{\ell m}}{\rho^3} \right) + \frac{1}{2} \mathcal{L}_s \mathcal{L}_o \left( \frac{2}{\rho^3} \frac{L_{\ell m}}{\rho^3} \right), \tag{B2}
\]

\[
B_2'^* = -\frac{1}{4} \mathcal{L}_s \mathcal{L}_o \left( \frac{2}{\rho^3} \frac{L_{\ell m}}{\rho^3} \right) + \frac{1}{2} \mathcal{L}_s \mathcal{L}_o \left( \frac{2}{\rho^3} \frac{L_{\ell m}}{\rho^3} \right), \tag{B3}
\]

with \( J_+ = \frac{\partial}{\partial \phi} + \frac{\mathcal{K}}{4} \) and

\[
\rho = \frac{1}{\hat{r}^+ - i \tilde{a} \cos(\theta)}, \quad \hat{r} = \frac{1}{\hat{r}^+ + i \tilde{a} \cos(\theta)}, \tag{B4}
\]

\[
\mathcal{L}_s = \frac{\partial}{\partial \theta} + \frac{m}{\sin(\theta)} \sin(\theta) + s \cot(\theta), \tag{B5}
\]

\[
\mathcal{L}_o = \frac{\partial}{\partial \theta} - \frac{m}{\sin(\theta)} \cos(\theta) + s \cot(\theta). \tag{B6}
\]

The components \( T_{\ell n}, T_{\ell m}, \) and \( T_{\ell mm} \) are the projections of the stress-energy tensor with respect to the Newman-Penrose (NP) tetrad:

\[
\begin{align*}
\mu^\nu &= \sqrt{\frac{\Sigma}{\Delta}} \left( e^\mu_{(0)} + e^\mu_{(1)} \right), \quad n^\mu &= \frac{1}{2} \sqrt{\frac{\Delta}{\Sigma}} \left( e^\mu_{(0)} - e^\mu_{(1)} \right), \tag{B7} \\
m^\mu &= \tilde{\rho} \sqrt{\frac{\Sigma}{2}} \left( e^\mu_{(2)} + ie^\mu_{(3)} \right), \quad \bar{m}^\mu &= \rho \sqrt{\frac{\Sigma}{2}} \left( e^\mu_{(2)} - ie^\mu_{(3)} \right), \tag{B8}
\end{align*}
\]

where, for example, \( T_{\ell n} = n^\mu n^\nu T_{\mu \nu} \). Henceforth we use the notation \( S_{\ell m}^{\omega \hat{\omega}} \) instead of \( S_{\ell m}^{\omega \hat{\omega}} \) for the spin-weighted spheroidal harmonics to reduce clutter in the notation.

All \( \theta \)-derivatives in \( T_{\ell n}, T_{\ell m}, \) and \( T_{\ell mm} \) can be removed by repeated integrations by parts and by making use of the following identity

\[
\int_0^\pi h(\theta) \mathcal{L}_s \left[ g(\theta) \right] \sin(\theta) d\theta = -\int_0^\pi g(\theta) \mathcal{L}_s \left[ h(\theta) \right] \sin(\theta) d\theta, \tag{B9}
\]

with \( h(\theta) \) and \( g(\theta) \) regular functions. It is thus possible to write

\[
\mathcal{T}_{\ell m \omega} = \int d\theta d\phi \Delta^2 \mathcal{E}^{(i \omega - m \phi)} (T_{n n} + T_{\ell m} + T_{\ell mm}), \tag{B10}
\]

with

\[
T_{\ell m} = -\frac{2}{\Delta^2 \rho^2 \hat{\rho}^3} \mathcal{L}_s \left[ \frac{1}{\rho^2} \mathcal{L}_o \left[ \frac{2}{\rho^3} S_{\ell m}^{\omega \hat{\omega}} \right] \sin(\theta) T_{n n} \right], \tag{B11}
\]

\[
T_{\ell m n} = \frac{4}{\sqrt{2}} \frac{\mathcal{L}_s}{\mathcal{L}_o} \left[ \frac{2}{\rho^3} S_{\ell m}^{\omega \hat{\omega}} \mathcal{J}_+ \left[ \frac{T_{\ell m}}{\Delta \rho^2 \hat{\rho}^3} \right] \sin(\theta) + \frac{1}{\sqrt{2}} \frac{1}{\rho^2 \hat{\rho}^2 \Delta \mathcal{L}_s} \left[ \frac{3}{\rho^3} S_{\ell m}^{\omega \hat{\omega}} \frac{d}{d\rho} \left( \frac{\rho^2}{\rho^3} \right) \right] \sin(\theta) T_{\ell m} \right], \tag{B12}
\]

\[
T_{\ell mm} = -\frac{2}{\Delta^2 \rho^3 \hat{\rho}^3} \mathcal{L}_s \left[ \frac{2}{\rho^3} S_{\ell m}^{\omega \hat{\omega}} \mathcal{J}_+ \left[ \frac{T_{\ell m}}{\Delta \rho^3 \hat{\rho}^3} \right] \sin(\theta) \right]. \tag{B13}
\]

It is convenient to expand the previous terms in order to isolate the derivatives of the projected stress-energy tensor with respect to \( \hat{r} \) and the derivative of \( S_{\ell m}^{\omega \hat{\omega}} \) with respect to \( \theta \). After some algebra, we get

\[
T_{n n} = -\frac{2}{\Delta^2 \hat{\rho}^2} \left[ \mathcal{L}_s \mathcal{L}_o \left[ \frac{\mathcal{L}_s}{\mathcal{L}_o} \left[ \frac{2}{\rho^3} S_{\ell m}^{\omega \hat{\omega}} T_{n n} \right] \right] \right], \tag{B14}
\]
\( T_{\mu\nu} = 4\sin(\theta) \left\{ \partial_\nu \left[ \left( L^1_{\mu\nu} + i\bar{\rho} \sin \theta (\tilde{\rho} - \rho) \bar{S}_{\mu\nu}^\alpha \right) \frac{T_{\mu\nu}}{\rho^2 \Delta} \right] \right\} \)

\[ + \left[ \left( \frac{iK}{\Delta} + \rho + \bar{\rho} \right) L_{\nu}^1 \bar{S}_{\mu\nu}^\alpha \right] \left[ - \tilde{\rho} \sin(\theta) \frac{K}{\Delta} (\tilde{\rho} - \rho) \bar{S}_{\mu\nu}^\alpha \right] \frac{T_{\mu\nu}}{\rho^2 \Delta} \right\}, \]

\[ \tag{B15} \]

\[ T_{\mu\mu} = \left\{ - \rho^2 \left( \frac{\tilde{\rho}}{\rho^2} - 2\rho \frac{iK}{\Delta} \frac{T_{\mu\mu}}{\rho^2 \Delta} \right) - 2\rho \left( \frac{d}{d\rho} \frac{iK}{\Delta} + \frac{K^2}{\Delta^2} \right) \bar{T}_{\mu\mu} \right\} \sin(\theta) \bar{S}_{\mu\nu}^\alpha \bar{S}_{\mu\nu}^\alpha . \]

The stress-energy tensor for a spinning object is given by \[26]\]

\[ T^{\mu\nu} = \frac{q}{\sqrt{-g}} \left[ \partial^{(4)}_{x,z(\lambda)} (u^{(a)}_\mu v^{(b)}_\nu) - \nabla_\sigma \left( S^{\sigma\mu\nu}_\rho \partial^{(4)}_{x,z(\lambda)} \sqrt{-g} \right) \right], \]

where \( \partial^{(4)}_{x,z(\lambda)} = \prod_{\nu=0}^1 \delta(x^\nu - z^\nu(\hat{\lambda})) \) and indices within parenthesis denote symmetrization. The tetrad components are \[26]\]

\[ T^{(a)(b)} = q \int \frac{d\hat{\lambda}}{\sqrt{-g}} \left[ \frac{u^{(a)}_\mu v^{(b)}_\nu}{\sqrt{-g}} \right]_{x,z(\lambda)} \phi^{(4)}_{x,z(\lambda)} - e^{(a)}_\mu e^{(b)}_\nu \partial_\sigma \left( S^{\sigma\mu\nu}_\rho \phi^{(4)}_{x,z(\lambda)} \right) \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

The above equation can be written as

\[ T^{(a)(b)} = q \int \frac{d\hat{\lambda}}{\sqrt{-g}} \left[ \phi^{(4)}_{x,z(\lambda)} (u^{(a)}_\mu v^{(b)}_\nu) + \right. \]

\[ + \omega_{(d)(c)} (a)_l (b)_l \partial^{(4)}_{x,z(\lambda)} (a)_l (b)_l v^{(c)} + \]

\[ - \partial_\sigma \left( S^{(a)}_{(b)} \phi^{(4)}_{x,z(\lambda)} \right) \left[ - \phi \cdot \phi(\hat{\lambda}) \right] \left[ - \phi \cdot \phi(\hat{\lambda}) \right], \]

\[ \tag{B19} \]

For bound orbits, it is useful to rewrite the energy-momentum tensor as

\[ T^{(a)(b)} = \frac{1}{\sqrt{-g}} \left[ \phi^{(4)}_{x,z(\lambda)} (P^{(a)(b)} - S^{(a)(b)} \partial_\mu) + \right. \]

\[ + \frac{1}{\sqrt{-g}} \left. \partial_\mu \left( S^{(a)}_{(b)} \partial^{(4)}_{x,z(\lambda)} \right) \right] \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

\[ \tag{B20} \]

where \( i = \{ r, \theta, \phi \} \), \( \phi^{(3)}_{x,z(\lambda)} = \delta(r - \hat{r}(\hat{\lambda})) \delta(\theta - \theta(\hat{\lambda})) \delta(\phi - \phi(\hat{\lambda})) \), and we defined

\[ P^{(a)(b)} := \frac{d}{d\lambda} \left[ \frac{d}{d\lambda} \right]^{-1} \left[ \phi^{(a)}_{x,z(\lambda)} + \omega_{(d)(c)} (a)_l (b)_l \partial^{(4)}_{x,z(\lambda)} \right] \]

\[ - \omega_{(d)(c)} (a)_l (b)_l \partial^{(4)}_{x,z(\lambda)} v^{(c)} \right] \left[ - \phi \cdot \phi(\hat{\lambda}) \right] \left[ - \phi \cdot \phi(\hat{\lambda}) \right], \]

\[ \tag{B21} \]

\[ S^{(a)(b)} := -q \left[ \frac{d}{d\lambda} \right]^{-1} \left[ \phi^{(a)}_{x,z(\lambda)} \right] \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

To rewrite the stress-energy tensor we used the well-known property of the derivative of a Dirac delta:

\[ \int_{-\infty}^{x_h(x)} \frac{d\lambda}{dx} \delta(x - x_0) = - \frac{d\lambda}{dx} \bigg|_{x=x_0} . \]

\[ \tag{B23} \]

In this way, the stress-energy tensor can be interpreted as a linear differential operator that acts on the smooth functions inside of the Teukolsky source term.

We now need to project \( T^{ab} \) with respect to the NP null tetrad. In the following, we will employ a reduced version of the NP tetrad:

\[ \tilde{\rho} = \left( e_{(0)}^\mu + e_{(1)}^\mu \right), \quad \tilde{\rho} = \frac{1}{2} \left( e_{(0)}^\mu + e_{(1)}^\mu \right), \]

\[ \tilde{n}^\mu = \frac{1}{\sqrt{2}} \left( e_{(2)}^\mu + ie_{(3)}^\mu \right), \quad \tilde{k}^\mu = \frac{1}{\sqrt{2}} \left( e_{(2)}^\mu + ie_{(3)}^\mu \right), \]

\[ \tag{B24} \]

\[ \tag{B25} \]

where \( \tilde{k}^\mu \) is the complex conjugate of \( \tilde{n}^\mu \). Taking into account that the \( i \) and \( \phi \) coordinates in the Teukolsky source term are only present in the exponential, and using the definitions \( T_{nn} = n^\rho n^\mu e_{(a)}(e_b) T^{(a)(b)} \) and so on, the projected components read

\[ T_{nn} = \delta^{(3)}_{x,z(\lambda)} (\partial)^{\tilde{n}}_{\rho} \left[ \tilde{n} \{ n \} \right] \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

\[ \tag{B26} \]

\[ T_{mn} = \delta^{(3)}_{x,z(\lambda)} (\partial)^{\tilde{n}}_{\rho} \left[ \tilde{n} \{ n \} \right] \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

\[ \tag{B27} \]

\[ T_{mm} = \delta^{(3)}_{x,z(\lambda)} (\partial)^{\tilde{n}}_{\rho} \left[ \tilde{n} \{ n \} \right] \sin(\theta) S^\alpha_{x,z(\lambda)} . \]

\[ \tag{B28} \]

with

\[ N_{nn} = \frac{\Delta}{\sqrt{-g} \Sigma}, \quad N_{nm} = \frac{\sqrt{2} \rho}{\sqrt{-g}}, \quad N_{mm} = \frac{\Sigma_{\rho}^2}{\sqrt{-g}} \]

and where we define the following linear operators acting on a generic smooth function \( h(\hat{\tau}, \hat{\theta}) \):

\[ D_{\tilde{n} \tilde{n}} [N_{nn} h(\hat{\tau}, \hat{\theta})] \equiv (P_{\tilde{n} \tilde{n}} - i\Sigma_{\rho}^2 \tilde{n} \{ n \} \partial_\rho) \left( \frac{\Sigma_{\rho}^2}{\sqrt{-g}} h(\hat{\tau}, \hat{\theta}) \right), \]

\[ \tag{B29} \]

\[ D_{\tilde{n} \tilde{k}} [N_{kn} h(\hat{\tau}, \hat{\theta})] \equiv (P_{\tilde{n} \tilde{k}} - i\Sigma_{\rho}^2 \tilde{n} \{ n \} \partial_\rho) \left( \frac{\sqrt{2} \rho}{\sqrt{-g}} h(\hat{\tau}, \hat{\theta}) \right), \]

\[ \tag{B30} \]

\[ D_{\tilde{k} \tilde{k}} [N_{mm} h(\hat{\tau}, \hat{\theta})] \equiv (P_{\tilde{k} \tilde{k}} - i\Sigma_{\rho}^2 \tilde{n} \{ n \} \partial_\rho) \left( \frac{\Sigma_{\rho}^2}{\sqrt{-g}} h(\hat{\tau}, \hat{\theta}) \right) . \]

\[ \tag{B31} \]

\[ \tag{B32} \]

Using the relations \[B26], \[B27] and \[B28\], we can now rewrite the terms \( T_{nn}, T_{mn}, T_{mm} \), obtaining

\[ T_{nn} = \left[ \delta^{(3)}_{x,z(\lambda)} (\partial)^{\tilde{n}}_{\rho} \left[ \tilde{n} \{ n \} \right] \right] f_{nn}^{(0)}, \]

\[ \tag{B33} \]
\[
\begin{align*}
\eta^{(0)}_{mn} &:= -\frac{2}{\Delta} \overline{\rho} \left( \mathcal{L} - 2i \rho \sin(\theta) \right) \mathcal{L} \overline{S}_{\ell m}^\theta \\
\mathcal{T}_{mn} &:= \left[ \mathcal{D}_{k}^{(3)} \mathcal{D}_{k}^{(3)} + \partial_{t} \left( \mathcal{S}_{k}^{\theta} \mathcal{D}_{k}^{(3)} \right) \right] f_{mn}(t) + \\
&+ \partial_{t} \left[ \left( \mathcal{D}_{k}^{(3)} \mathcal{D}_{k}^{(3)} + \partial_{t} \left( \mathcal{S}_{k}^{\theta} \mathcal{D}_{k}^{(3)} \right) \right) f_{mn}(t) \right], \\
\eta^{(1)}_{mn} &:= \frac{4}{\sqrt{2} \rho \sqrt{\Delta}} \left( \frac{iK}{\Delta} + \rho + \overline{\rho} \right) \mathcal{L} \mathcal{L} \overline{S}_{\ell m}^\theta \\
&- \overline{\eta} \sin \theta \frac{K}{\Delta} (\overline{\rho} - \rho) \overline{S}_{\ell m}^\theta, \\
\eta^{(2)}_{mn} &:= \frac{\overline{\rho}}{\rho} \overline{S}_{\ell m}^\theta.
\end{align*}
\]

We now have all the necessary ingredients to rewrite the inhomogeneous solutions of the Teukolsky equation in a form suitable to exploit the possible quasi-periodicities in the bound orbits. First of all, by plugging the terms \([B33, B35]\) and \([B38]\) into Eq. \([B10]\), integrating over the angles and using the \(\delta(\theta - \theta(t))\delta(\phi - \phi(t))\) function, the Teukolsky source term becomes

\[
\mathcal{T}_{\ell m\hat{\omega}} = \int d\hat{t} e^{i(\hat{\omega} - m\phi(t))} \Delta^2 \left\{ \mathcal{T}_{D}^{(0)} \delta_{r, r(t)} + \\
+ \partial_{t} \left( \mathcal{T}_{D}^{(0)} \delta_{r, r(t)} \right) + \partial_{t}^{2} \left( \mathcal{T}_{D}^{(0)} \delta_{r, r(t)} \right) + \\
+ \partial_{t}^{2} \left( \mathcal{T}_{S}^{(2)} \delta_{r, r(t)} \right) \right\}_{\theta = \theta(t)},
\]

when \(\delta_{r, r(t)} := \delta(\hat{r} - \hat{r}(t))\), and we have rearranged the previous terms, defining

\[
\begin{align*}
\mathcal{T}_{D}^{(0)} &= \mathcal{D}_{n n} f_{n n}^{(0)} + \mathcal{D}_{k n} f_{m n}^{(0)} + \mathcal{D}_{k k} f_{m n}^{(0)}, \\
\mathcal{T}_{D}^{(1)} &= \mathcal{D}_{k n} f_{m n}^{(1)} + \mathcal{D}_{k k} f_{m n}^{(1)}, \\
\mathcal{T}_{D}^{(2)} &= \mathcal{D}_{k k} f_{m n}^{(2)},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{T}_{S}^{(0)} &= \partial_{t} \left[ \mathcal{S}_{k n}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(0)} + \partial_{t} \left[ \mathcal{S}_{k n}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(0)} + \\
&+ \partial_{t} \left[ \mathcal{S}_{k k}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(0)}, \\
\mathcal{T}_{S}^{(1)} &= \partial_{t} \left[ \mathcal{S}_{k n}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(1)} + \partial_{t} \left[ \mathcal{S}_{k k}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(1)}, \\
\mathcal{T}_{S}^{(2)} &= \partial_{t} \left[ \mathcal{S}_{k k}^{\theta} \mathcal{D}_{m n} \right] f_{m n}^{(2)}.
\end{align*}
\]

To obtain the asymptotic fluxes, we need to calculate the amplitudes \([B4], [B5]\), namely

\[
\mathcal{Z}_{\ell m\hat{\omega}} = \mathcal{C}_{\ell m\hat{\omega}} \int_{\hat{r}+} d\hat{t} \frac{R_{\ell m\hat{\omega}}^{\alpha\beta}(\hat{r}')}{\Delta^2} \mathcal{T}_{\ell m\hat{\omega}}(\hat{r}').
\]

By changing the order of integration between \(\hat{r}'\) and \(\hat{t}\), we get

\[
\mathcal{Z}_{\ell m\hat{\omega}} = \mathcal{C}_{\ell m\hat{\omega}} \int_{\hat{r}+} \int_{\hat{t}} d\hat{t} \left[ \mathcal{T}_{D}^{(0)} \mathcal{T}_{D}^{(1)} \mathcal{T}_{D}^{(2)} \right] R_{\ell m\hat{\omega}}^{\alpha\beta}(\hat{r}', \hat{t}) \mathcal{T}_{\ell m\hat{\omega}}(\hat{r}', \hat{t}),
\]

which is calculated at \(\theta = \theta(t)\). In the integral on the first line we have used the \(\delta(\hat{r} - \hat{r}(t))\) function. The double integral on the second line can be simplified with multiple integrations by parts, obtaining the general expression

\[
\mathcal{Z}_{\ell m\hat{\omega}} = \mathcal{C}_{\ell m\hat{\omega}} \int_{\hat{r}+} \int_{\hat{t}} d\hat{t} e^{i(\hat{\omega} - m\phi(t))} \left\{ A_0(\hat{t}) - (A_1 + B_1) \frac{d}{d\hat{t}} + (A_2 + B_2) \frac{d^2}{d\hat{t}^2} - B_3 \frac{d^3}{d\hat{t}^3} \right\} R_{\ell m\hat{\omega}}^{\alpha\beta}(\hat{r}', \hat{t})
\]

where

\[
\begin{align*}
A_0 &:= \mathcal{O}_{\bar{n}} \mathcal{f}_{\bar{n}}^{(0)} + \mathcal{O}_{\bar{k}} \mathcal{f}_{\bar{m}}^{(0)} + \mathcal{O}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(0)}; \\
A_1 &:= \mathcal{O}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(1)}; \\
A_2 &:= \mathcal{O}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(2)};
\end{align*}
\]

and

\[
\begin{align*}
B_1 &:= \mathcal{S}_{\bar{n}} \mathcal{f}_{\bar{n}}^{(0)} + \mathcal{S}_{\bar{k}} \mathcal{f}_{\bar{m}}^{(0)} + \mathcal{S}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(0)}; \\
B_2 &:= \mathcal{S}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(1)}; \\
B_3 &:= \mathcal{S}_{\bar{k}^2} \mathcal{f}_{\bar{m}}^{(2)};
\end{align*}
\]

with the operators \(\mathcal{O}_{\bar{n}}, \mathcal{O}_{\bar{k}}, \mathcal{O}_{\bar{k}^2}\) being defined as

\[
\begin{align*}
\mathcal{O}_{\bar{n}} &:= \mathcal{P}_{\bar{n}} - i \hat{\omega} \mathcal{S}_{\bar{n}}^{\theta} + \text{im} \mathcal{S}_{\bar{n}}^{\varphi} - \mathcal{S}_{\bar{n}} \partial_{\theta} - \mathcal{S}_{\bar{n}} \partial_{\varphi}; \\
\mathcal{O}_{\bar{k}} &:= \mathcal{P}_{\bar{k}} - i \hat{\omega} \mathcal{S}_{\bar{k}}^{\theta} + \text{im} \mathcal{S}_{\bar{k}}^{\varphi} - \mathcal{S}_{\bar{k}} \partial_{\theta} - \mathcal{S}_{\bar{k}} \partial_{\varphi}; \\
\mathcal{O}_{\bar{k}^2} &:= \mathcal{P}_{\bar{k}^2} - i \hat{\omega} \mathcal{S}_{\bar{k}^2}^{\theta} + \text{im} \mathcal{S}_{\bar{k}^2}^{\varphi} - \mathcal{S}_{\bar{k}^2} \partial_{\theta} - \mathcal{S}_{\bar{k}^2} \partial_{\varphi};
\end{align*}
\]
with \( \mathcal{P}_{\hat{n}\hat{n}} = \hat{n}^\mu \hat{n}_\nu \epsilon_{\mu(a)} \epsilon_{\nu(b)} \mathcal{P}^{(a)(b)} \), while \( \mathcal{S}^n_{\hat{n}\hat{n}} = \hat{n}^\mu \hat{n}_\nu \epsilon_{\mu(a)} \epsilon_{\nu(b)} \mathcal{S}^{\sigma(a)(b)} \) and so on. The terms \( f_{\hat{n}\hat{n}}^{(i)} \), \( f_{\hat{m}\hat{n}}^{(i)} \), \( f_{\hat{m}\hat{m}}^{(i)} \) (with \( i = 0, 1, 2 \)) are defined in Eqs. (B34)–(B41).

We remark that Eq. (B51) is general: it is valid for any bound orbit for a spinning test particle in Kerr spacetime.

2. Circular equatorial orbits

On the equatorial plane, \( \theta = \pi/2 \), the Teukolsky source term drastically simplifies. First of all, some terms of the previous equations vanish, namely

\[
\mathcal{S}^\theta_{\hat{n}\hat{n}} = \mathcal{S}^\theta_{\hat{k}\hat{n}} = \mathcal{S}^\theta_{\hat{k}\hat{k}} = 0, \quad (B61)
\]

for \( \theta = \pi/2 \). Furthermore, we can write

\[
f_{\hat{n}\hat{n}}^{(0)} = -\frac{4}{\Delta} \hat{S}(r), \quad (B62)
\]

\[
f_{\hat{m}\hat{n}}^{(0)} = \frac{4}{\sqrt{2}} \frac{\hat{S}}{\sqrt{\Delta}} \left( iK \frac{\Delta}{r} + 2 \frac{\hat{\omega}}{r} \right), \quad (B63)
\]

\[
f_{\hat{m}\hat{m}}^{(1)} = \frac{4}{\sqrt{2}} \frac{\hat{S}}{\sqrt{\Delta}}, \quad (B64)
\]

where we applied the angular Teukolsky equation, with

\[
\hat{S} := \frac{d\hat{S}^{\hat{n}\hat{m}}}{d\theta} \bigg|_{\theta=\pi/2} + (\hat{\omega} - m) \hat{S}^{\hat{n}\hat{m}}_{\hat{m}\hat{n}}(\pi/2), \quad (B65)
\]

\[
\hat{S}(\hat{r}) := \left( \hat{\omega} - m \left( \frac{2}{r} \right) \right) \hat{S} = \frac{\lambda \hat{\omega} m}{2} \hat{S}^{\hat{n}\hat{m}}_{\hat{m}\hat{n}}(\pi/2). \quad (B66)
\]

Moreover

\[
f_{\hat{m}\hat{m}}^{(0)} = \left( \frac{d}{d\hat{r}} \left[ iK \frac{\Delta}{\hat{r}} - 2 \frac{K^2}{\Delta^2} \right] \hat{S}^{\hat{n}\hat{m}}_{\hat{m}\hat{n}}(\pi/2), \quad (B67)
\]

\[
f_{\hat{m}\hat{m}}^{(1)} = -2 \left( \frac{1}{\hat{r}} + iK \frac{\Delta}{\Delta^2} \right) \hat{S}^{\hat{n}\hat{m}}_{\hat{m}\hat{n}}(\pi/2), \quad (B68)
\]

\[
f_{\hat{m}\hat{m}}^{(2)} = -\hat{S}^{\hat{n}\hat{m}}_{\hat{m}\hat{n}}(\pi/2). \quad (B69)
\]

Finally, for a circular equatorial orbit the projected components of \( \mathcal{P}^{(a)(b)} \) and \( \mathcal{S}^{\sigma(a)(b)} \) onto the reduced NP basis are

\[
\mathcal{P}_{\hat{n}\hat{n}} = -\frac{q}{4} \frac{P_\sigma}{\Sigma_\sigma I_+} \left( (\hat{r}^3 + 2\sigma^2) \Delta \hat{x} \sigma - \hat{r} \Sigma_\sigma \left[ 2\hat{x} \sigma (\hat{r} - \hat{a}^2) + P_\sigma (\hat{r}^2 - \sigma \hat{a}) \right] \right), \quad (B70)
\]

\[
\mathcal{P}_{\hat{k}\hat{n}} = -\frac{iq}{4\sqrt{2}} \frac{\sqrt{\Delta}}{\Sigma_\sigma I_+} \left( -\hat{r} (\hat{r}^3 + 2\sigma^2) \hat{\Sigma} \sigma (\hat{r} - \hat{a}^2) + P_\sigma (\hat{r}^2 + \sigma \hat{a}) \right) - \hat{r} \Sigma_\sigma \left[ \hat{r}^2 \hat{x} + \sigma (3\hat{x} \hat{a} + P_\sigma) \right], \quad (B71)
\]

\[
\mathcal{P}_{\hat{k}\hat{k}} = \frac{q}{2} \frac{1}{\Sigma_\sigma I_-} \left[ \Delta \left( \sigma (P_\sigma + 2\hat{x} \hat{a}) + \hat{x} \hat{r}^2 \right) (\hat{r}^3 + 2\sigma^2) + \hat{a} \sigma \hat{r} \Sigma_\sigma P_\sigma^2 \right] = \quad (B72)
\]

with \( \hat{x} := \hat{J}_z - (\hat{a} + \sigma) \hat{E}, \quad \Gamma_\pm := 3\hat{x} \hat{a} \sigma^2 \Delta \pm \hat{r} \Sigma_\sigma \left[ P_\sigma (\hat{r}^2 + \hat{a}^2) + \hat{x} \Delta \right] \), and

\[
\mathcal{S}^\mu_{\hat{n}\hat{n}} = \frac{1}{4} q \sigma \hat{r}^2 \hat{P}_\sigma \left( \hat{a} P_\sigma + \hat{x} (\hat{r}^2 + \hat{a}^2), -\frac{\Delta \hat{x}}{\Gamma_-}, 0, -\frac{\Delta \hat{x} + P_\sigma}{\Gamma_-} \right), \quad (B73)
\]

\[
\mathcal{S}^\mu_{\hat{k}\hat{n}} = \frac{iq \sigma}{4\sqrt{2}} \frac{\hat{r} \sqrt{\Delta}}{\Sigma_\sigma I_+} \left( (\hat{r}^3 + 2\sigma^2) \left[ \hat{a} P_\sigma + \hat{x} (\hat{r}^2 + \hat{a}^2) \right], \hat{x} \Delta (\hat{r}^3 + 2\sigma^2) + \frac{\hat{r}}{\hat{x}} \Sigma_\sigma P_\sigma^2, 0, (\hat{a} \hat{x} + P_\sigma) (\hat{r}^3 + 2\sigma^2) \right), \quad (B74)
\]

\[
\mathcal{S}^\mu_{\hat{k}\hat{k}} = \frac{1}{4} q \sigma \hat{r} \hat{P}_\sigma \left( 0, \Delta \hat{x} (\hat{r}^3 + 2\sigma^2), 0, 0 \right). \quad (B75)
\]

In Ref. [26] the Teukolsky source was calculated at first order in the spin. Our results for the source term are general and, when truncated at \( \mathcal{O}(\sigma) \), agree with those in Ref. [26], except for a factor \( 1/\sqrt{2} \) in their \( \tilde{Z}_{\ell m}^{m\alpha} \) term. This is probably a typo in their source term, since with our source term we can reproduce previous results for the fluxes of a nonspinning particle (see also Appendix [C]).
Appendix C: Comparisons of the GW fluxes with previous work

We have tested our code by comparing the GW fluxes against results already published in the literature. In this section we provide a detailed comparison in order to assess the accuracy of our method.

1. Comparison with Harms et al.

The GW fluxes at infinity for a spinning particle have been calculated in Ref. [28] by solving the Teukolsky equation in the time domain and assuming \( q = 1 \), so that \( \sigma = q \chi \) is not small when \( \chi = \mathcal{O}(1) \). To make the comparison, we also set \( q = 1 \). We remark that we use the same spin supplementary conditions and the same orbital dynamics as in Ref. [28].

Tables III and IV of Ref. [28] list the results obtained in Table II and III, of Ref. [28] for the \( \ell = 2, 3 \) modes. The fluxes are normalized with respect to the leading Post-Newtonian order. Here the normalized fluxes are denoted as follows:

\[
\hat{F}_{\ell m} = \frac{F_{\ell m}}{k_{\ell m}},
\]

where

\[
k_{22} = \frac{32}{5} \hat{\Omega}_{\ell m}^{22}, \quad k_{21} = \frac{8}{45} \hat{\Omega}_{\ell m}^{21}, \quad k_{31} = \frac{243}{28} \hat{\Omega}_{\ell m}^{31}
\]

and \( \hat{F}_{\ell m} \) includes only the fluxes at infinity, assuming \( q = 1 \), and therefore \( \sigma = \chi \). Moreover, we define

\[
\Delta_{\ell m} = 100 \left| 1 - \frac{\hat{F}_{\ell m}}{\hat{F}_{\ell m}} \right|
\]

where \( \hat{F}_{\ell m} \) given in [28]. Note that Ref. [28] assumed \( J_z > 0 \), distinguishing prograde and retrograde orbits on the base of the sign of \( \hat{\omega} \). In our work we consider the opposite convention: we fix \( \hat{\omega} \geq 0 \), while \( J_z \) is positive (negative) for corotating (counter-rotating) orbits. Therefore, for retrograde orbits we compare our fluxes for \( \sigma > 0 \) with the results \( \sigma < 0 \) of Ref. [28] and vice versa.

Tables III and IV show that our results are in good agreement with those of Ref. [28], with relative errors of the order of percent or below for all the considered configurations. For the \( \ell = m = 2 \) and \( \ell = m = 3 \) modes the fractional difference is always less than 0.5%.

This picture does not change for \( \Delta_{21} \) except for fast spinning bodies with \( \hat{\omega} = 0.9 \): in this case retrograde and prograde orbits lead to maximum discrepancies of 1.3% and 16%, respectively. We believe that the last value may be given by numerical rounding, since the corresponding flux is given in Ref. [28] with only one significant figure.

Finally, in Fig. 6 we plot \( \hat{F}_{22} \) for prograde orbits with \( \hat{\omega} = 0.9 \) and \( \hat{\omega} = 3 \) as a function of \( \chi \). Owing to the fact that \( q = 1 \) (and therefore \( \sigma \) is not small), the fluxes depend on the spin of the secondary in a nonlinear fashion when \( \chi = \mathcal{O}(1) \).
\(a = 0.9\) retrograde orbits

\[
\begin{array}{cccccc}
\hat{r} & \sigma & \mathcal{F}_2^{\mathcal{F}_o} & \Delta_{21}^\text{[\%]} & \mathcal{F}_1 & \Delta_{31}^\text{[\%]} \\
5 & -0.9 & 1.2361 & 0.2 & 5.6616 & 0.4 & 1.0827 & 0.3 \\
 & -0.5 & 1.6251 & 0.2 & 6.6959 & 0.3 & 1.5729 & 0.3 \\
 & 0.5 & 3.3150 & 0.2 & 10.789 & 0.3 & 3.9783 & 0.3 \\
 & 0.9 & 4.4462 & 0.2 & 13.255 & 0.3 & 5.7567 & 0.3 \\
6 & -0.9 & 1.0335 & 0.2 & 4.6842 & 0.4 & 0.8937 & 0.3 \\
 & -0.5 & 1.2023 & 0.2 & 4.8148 & 0.4 & 1.1143 & 0.3 \\
 & 0.5 & 1.7181 & 0.2 & 4.8963 & 0.4 & 1.8635 & 0.3 \\
 & 0.9 & 1.9563 & 0.2 & 4.7247 & 0.3 & 2.2404 & 0.3 \\
8 & -0.9 & 0.9123 & 0.2 & 3.7900 & 0.4 & 0.7911 & 0.3 \\
 & -0.5 & 0.9784 & 0.2 & 3.5167 & 0.4 & 0.8842 & 0.3 \\
 & 0.5 & 1.1510 & 0.2 & 2.6978 & 0.4 & 1.1499 & 0.3 \\
 & 0.9 & 1.2208 & 0.2 & 2.3159 & 0.3 & 1.2679 & 0.3 \\
10 & -0.9 & 0.8816 & 0.2 & 3.3399 & 0.4 & 0.7727 & 0.3 \\
 & -0.5 & 0.9193 & 0.2 & 2.9873 & 0.4 & 0.8286 & 0.3 \\
 & 0.5 & 1.0142 & 0.2 & 2.0862 & 0.4 & 0.9799 & 0.3 \\
 & 0.9 & 1.0519 & 0.2 & 1.7269 & 0.3 & 1.0446 & 0.3 \\
20 & -0.9 & 0.8875 & 0.3 & 2.4826 & 0.7 & 0.8130 & 1.2 \\
 & -0.5 & 0.8969 & 0.1 & 2.1581 & 0.6 & 0.8290 & 0.4 \\
 & 0.5 & 0.9202 & 0.2 & 1.4249 & 0.3 & 0.8699 & 0.0 \\
 & 0.9 & 0.9294 & 0.2 & 1.1662 & 1.3 & 0.8866 & 0.3 \\
\end{array}
\]

TABLE IV. Normalized fluxes and fractional differences with the fluxes in Table III of Ref. [28] in the case \(a = 0.9\), retrograde orbits. The fluxes \(\mathcal{F}_2\), with \(\sigma < 0\) have to be compared with the fluxes \(\mathcal{F}_{\text{Stoc}}\) with \(\sigma > 0\) and vice versa.

2. Comparison with Akcay et al.

Recently, a new flux balance law relating the local changes of energy of a spinning particle in Kerr spacetime with the asymptotic fluxes of energy and angular momentum was obtained in Ref. [22]. This procedure has been applied to particles with spin perpendicular to the orbital plane on circular orbits in the Schwarzschild spacetime, computing the linear spin corrections to the fluxes. Table [VI] provides our spin corrections to the flux and the fractional difference with respect to the sum of the spin’s contributions at horizon and infinity given in Table I of Ref. [22]. The errors show a very good agreement between the two results.

3. Comparison with Taracchini et al.

Reference [89] computed high-precision GW fluxes for nonspinning particles orbiting around Schwarzschild and Kerr BHs solving the Teukolsky equation in the frequency domain. We have checked our code against both their set-up. The relative errors are shown in Tables [VII,X] for the values of the GW fluxes computed at the ISCO and at a different orbital separations \(\hat{r}\), as a function of the primary spin. Note that in Ref. [89] the sum over the harmonic index \(\ell\) was truncated at a certain value \(\ell_{\text{max}}\) such that the fractional error between the flux at \(\ell_{\text{max}}\) and \(\ell_{\text{max}} - 1\) was less than \(10^{-14}\). To achieve this accuracy the required \(\ell_{\text{max}}\) is in general very large: at the ISCO, for example, \(\ell_{\text{max}} = 30\) for \(a = 0\), and \(\ell_{\text{max}} = 66\) for \(a = 0.99\). In our calculations we fixed \(\ell_{\text{max}} = 20\). Nonetheless, the agreement between our results and those computed in Ref. [89] is extremely good. Even for the fastest spinning BH considered (with \(a = 0.9\)), we find a relative difference smaller than \(10^{-5}\).
TABLE VII. Fluxes for a nonspinning objects around Kerr BHs $F^0$ at the ISCO and fractional difference $\Delta^{\text{iso}}(F^0)$ compared to the results of Ref. [89].

| $a$ | ISCO | $F^0$ | $\Delta^{\text{iso}}(F^0)$ |
|-----|------|-------|-----------------------------|
| 0.1 | 5.669| $1.203797640 \times 10^{-3}$ | $8.5 \times 10^{-11}$ |
| 0.3 | 4.979| $2.10037308 \times 10^{-3}$ | $1.4 \times 10^{-9}$ |
| 0.5 | 4.233| $4.11717449 \times 10^{-3}$ | $6.9 \times 10^{-10}$ |
| 0.8 | 2.907| $1.71190 \times 10^{-2}$   | $4.4 \times 10^{-7}$  |
| 0.9 | 2.321| $3.5223 \times 10^{-2}$   | $5.4 \times 10^{-6}$  |

4. Comparison with Gralla et al.

Finally, we tested our code in the case of a nonspinning secondary and fastly spinning primary BHs with $\hat{a} > 0.9$. In this case we use the data obtained in Ref. [63] using the Teukolsky formalism in the frequency domain and assuming $\ell_{\text{max}} = 30$ [63]. The comparison is shown in Table X for $\hat{a} = 0.99$ and $\hat{a} = 0.995$ for orbital radii equal to and larger than the ISCO. The discrepancy between our results and those of Ref. [63] increases for larger spins and smaller orbital separation. However, in the worst case scenario, the fluxes differ at most by one part over $10^3$. 
\begin{table}[h]
\centering
\begin{tabular}{c|c|c|c|c|c|c|c}
\hline
$\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ & $\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ & $\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ \\
\hline
10 & $6.1563167846 \times 10^{-5}$ & $1.8 \times 10^{-14}$ & 0.3 & $5.72185605812 \times 10^{-5}$ & $1.1 \times 10^{-12}$ & 0.5 & $5.4706016232 \times 10^{-5}$ & $3.0 \times 10^{-12}$ \\
8 & $1.961045485836 \times 10^{-4}$ & $1.6 \times 10^{-14}$ & 1.0 & $1.757401400491 \times 10^{-4}$ & $2.4 \times 10^{-14}$ & 1.0 & $1.64390512713 \times 10^{-4}$ & $7.2 \times 10^{-13}$ \\
6 & $9.40339356 \times 10^{-4}$ & $3.8 \times 10^{-11}$ & 1.0 & $7.7105423521 \times 10^{-4}$ & $1.2 \times 10^{-11}$ & 1.0 & $6.8651481394 \times 10^{-4}$ & $7.1 \times 10^{-12}$ \\
\hline
\end{tabular}
\caption{Fluxes for a non spinning object $F^0$ and fractional difference $\Delta^{\text{rel}}(F^0)$ with respect to the fluxes listed in [56] for fast rotating BHs with $\hat{a} = (0, 0.3, 0.5)$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{c|c|c|c|c|c|c|c|c}
\hline
$\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ & $\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ \\
\hline
10 & $5.13763911701 \times 10^{-5}$ & $4.3 \times 10^{-13}$ & 0.8 & $5.036802531 \times 10^{-5}$ & $1.4 \times 10^{-12}$ \\
8 & $1.4997326131 \times 10^{-4}$ & $2.6 \times 10^{-13}$ & 0.9 & $1.4574909234 \times 10^{-4}$ & $9.5 \times 10^{-13}$ \\
6 & $5.8851295900 \times 10^{-4}$ & $2.7 \times 10^{-12}$ & 0.8651481394 & $5.6168859157 \times 10^{-4}$ & $1.5 \times 10^{-12}$ \\
4 & $3.9084751 \times 10^{-3}$ & $2.2 \times 10^{-9}$ & 0.9 & $3.53976293 \times 10^{-3}$ & $1.4 \times 10^{-9}$ \\
\hline
\end{tabular}
\caption{Fluxes for a non spinning object $F^0$ and fractional difference $\Delta^{\text{rel}}(F^0)$ with respect to the fluxes listed in [56] for fast rotating BHs with $\hat{a} = (0.8, 0.9)$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{c|c|c|c|c|c|c|c|c}
\hline
$\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ & $\hat{a}$ & $F^0$ & $\Delta^{\text{rel}}(F^0)$ \\
\hline
10 & $4.9500572776 \times 10^{-5}$ & $2.7 \times 10^{-12}$ & 0.990 & $4.9453383948 \times 10^{-5}$ & $3.4 \times 10^{-12}$ \\
8 & $1.421652170 \times 10^{-4}$ & $1.5 \times 10^{-11}$ & 0.995 & $1.419678387 \times 10^{-4}$ & $1.4 \times 10^{-11}$ \\
6 & $5.395577551 \times 10^{-4}$ & $6.6 \times 10^{-11}$ & 0.990 & $5.38379633 \times 10^{-4}$ & $6.6 \times 10^{-11}$ \\
4 & $3.26013974 \times 10^{-3}$ & $1.3 \times 10^{-9}$ & 0.995 & $3.24583765 \times 10^{-3}$ & $1.3 \times 10^{-9}$ \\
2 & $4.301 \times 10^{-2}$ & $1.1 \times 10^{-5}$ & 0.990 & $4.221 \times 10^{-2}$ & $1.0 \times 10^{-5}$ \\
ISCO & $9.17 \times 10^{-2}$ & $5.0 \times 10^{-4}$ & 0.995 & $9.5 \times 10^{-2}$ & $1.0 \times 10^{-3}$ \\
\hline
\end{tabular}
\caption{Fluxes for a nonspinning object $F^0$ and fractional difference $\Delta^{\text{rel}}(F^0)$ with respect to the fluxes listed in [56]. The ISCO is at $\hat{r} = 1.454$ and $\hat{r} = 1.341$ for $\hat{a} = 0.990$ and $\hat{a} = 0.995$ respectively.}
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