On the exact solubility in momentum space of the trigonometric Rosen–Morse potential

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Abstract
The Schrödinger equation with the trigonometric Rosen–Morse potential in a flat three-dimensional Euclidean space, $E_3$, and its exact solutions are shown to be exactly Fourier transformable to momentum space, though the resulting equation is purely algebraic and cannot be cast into the canonical form of an integral Lippmann–Schwinger equation. This is because the cotangent function does not allow for an exact Fourier transform in $E_3$. In addition, we recall that the above potential can also be viewed as an angular function of the second polar angle parametrizing the three-dimensional spherical surface, $S^3$, of a constant radius, in which case the cotangent function would allow for an exact integral transform to momentum space. On that basis, we obtain a momentum space Lippmann–Schwinger-type equation, though the corresponding wavefunctions have to be obtained numerically.

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1. Introduction

Constructing the phase spaces of quantum mechanical systems is one of the key issues in their description. The knowledge of the potential and the wavefunctions in momentum space is indispensable both in relativistic and nonlinear dynamics frameworks. In particular, the trigonometric Rosen–Morse potential (Rosen–Morse I), being of a finite range, is becoming of interest in the design of confinement phenomena ranging form electrons in quantum dots [1] to quarks in hadrons [2], a reason for which studying its momentum space representation appears timely.

In a flat three-dimensional (3D) Euclidean position space, $E_3$, the Schrödinger equation,

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi (\mathbf{r}) + V (\mathbf{r}) \psi (\mathbf{r}) = E \psi (\mathbf{r}),$$

(1)
provides a suitable tool for the description of systems interacting via central potentials, many of which result exactly solvable [4]. The momentum space counterpart to equation (1) is obtained by means of a Fourier transform as [5]

\[ \frac{p^2}{2\mu} \phi(p) + \frac{1}{(2\pi)^2} \int e^{i\bar{p} \cdot r} V(r) \psi(r) \, d^3r = E \phi(p), \]  

(2)

with \( \phi(p) \) being the Fourier transform of \( \psi(r) \),

\[ \phi(p) = \frac{1}{(2\pi)^2} \int e^{i\bar{p} \cdot r} \psi(r) \, d^3r, \]  

(3)

and supposed to exist. In the cases where the Fourier transform of the potential too exists, i.e.

\[ \frac{1}{(2\pi)^2} \int e^{i\bar{q} \cdot r} V(r) \, d^3r = V(q), \]  

(4)

the momentum space Schrödinger equation can be cast into the canonical form

\[ \frac{p^2}{2\mu} \phi(p) + \frac{1}{(2\pi)^2} \int V(q) \phi(p') \, d^3p' = E \phi(p), \quad q = p - p'. \]  

(5)

The latter equation is frequently referred to as the Lippmann–Schwinger equation. From now onwards, we introduce \( F(p) \) as a special symbol for the integral appearing in equation (2), i.e. we define

\[ F(p) = \frac{1}{(2\pi)^2} \int e^{i\bar{p} \cdot r} V(r) \psi(r) \, d^3r. \]  

(6)

Provided that the integral in equation (4) exists, an equivalent representation of equation (6) is obtained as

\[ F(p) = \frac{1}{(2\pi)^2} \int V(q) \phi(p') \, d^3p'. \]  

(7)

In case the integrals in equations (3) and (6) can be taken in close forms, while the integral in equation (4) does not exist, equation (2) is purely algebraical. It is the goal of the present study to show that this precisely is the case of the trigonometrical Rosen–Morse potential (Rosen–Morse I) when considered as a central potential in \( E_3 \). The resulting momentum space is then conjugate to the infinite flat position space.

Various phase spaces can be designed in considering the argument of Rosen–Morse I as the arc of \( S^3 \), the minimal geodesic distance on a three-dimensional space of constant positive curvature. In particular, we consider the arc in its projection on the equatorial disk, a 3D space of finite volume, and show that for this very case the momentum space Schrödinger equation is no longer algebraic but takes a form similar to the integral equation (5).

The paper is organized as follows. In the next section, we present the cotangent interaction from the perspective of potential theory. Section 3 contains the algebraic momentum space equation. The integral momentum space equation is given in section 4. The paper closes with brief conclusions.

2. The cotangent interaction from the perspective of potential theory

The trigonometric Rosen–Morse potential (up to additive constants) is given by

\[ V(\chi) = -2B \cot \chi + \frac{\hbar^2}{2\mu d^2} l(l + 1) \csc^2 \chi. \]  

(8)
where \( \chi \) is a dimensionless angular variable, \( \mu \) has the dimensionality of mass, and \( d \) is a matching length parameter. It has the virtue to be one of the few potentials that can be introduced along the line of potential theory [6]. According to potential theory, functions which solve the Laplace–Beltrami equation on a manifold of interest, the so-called harmonic functions, have the remarkable property, if employed as interactions, to preserve the symmetry of the free geodesic motion [6]. It was observed long ago by Schrödinger in [7] that \( \cot \chi \) is a harmonic function on the three-dimensional (3D) surface of constant positive curvature, the hypersphere \( S^3 \) embedded in a flat Euclidean space of four dimensions, \( E_4 \), where \( \chi \) can be viewed as the second polar angle. The isometry group of \( S^3 \) is \( SO(4) \) and the cotangent function, being harmonic there, respects this very symmetry [8]. The latter is best understood by first noting that the Schrödinger equation with the csc\(^2\) potential is closely related to the eigenvalue problem of the 4D angular momentum on \( S^3 \). Through the paper we consider an ordinary Euclidean flat space, \( E_3 \), embedded in a 4D Euclidean space, \( E_4 \), and parametrize the 3D spherical surface \( S^3 \) as \( x_2^2 + r^2 = R^2 \) with \( x_4 = R \cos \chi \), and \( |r| = R \sin \chi \). Here, \( \chi \in [0, \pi] \) is the second polar angle in \( E_4 \), while \( R \) stands for the constant hyper-radius of \( S^3 \). The 4D Laplace–Beltrami operator, \( \nabla^2 \), is proportional to the operator of the squared 4D angular momentum, \( K^2 \), for constant radius, and is given by

\[
\nabla^2 = \frac{-1}{R^2} K^2. \tag{9}
\]

The analogue on the 2D sphere, \( S^2 \), of constant radius \( |r| = a \), is the well-known relation \( \nabla^2 = -\frac{1}{a^2} L^2 \). Consequently, the Schrödinger equation for free geodesic motion on \( S^3 \) becomes

\[
\left[ \frac{\hbar^2}{2\mu R^2} K^2 - E \right] \psi(\chi) = 0. \tag{10}
\]

The \( K^2 \) eigenvalue-problem reads [9]

\[
K^2|Klm\rangle = K(K + 2)|Klm\rangle, \quad |Klm\rangle \in \left( \begin{array}{c} K \\ 2 \\ K \\ 2 \end{array} \right), \tag{11}
\]

and the \( |Klm\rangle\)-levels belong to irreducible \( SO(4) \) representations of the type

\[
|Klm\rangle \in \left( \begin{array}{c} K \\ 2 \\ K \\ 2 \end{array} \right). \tag{12}
\]

The quantum numbers, \( K, l, \) and \( m \) define the eigenvalues of the respective four-, three- and two-dimensional angular momentum operators upon the state. These quantum numbers correspond to the

\[
SO(4) \supset SO(3) \supset SO(2) \tag{13}
\]

reduction chain and satisfy the branching rules,

\[
K = 0, 1, 2, \ldots, \infty, \quad l = 0, 1, 2, \ldots, K, \quad m = -l, \ldots, +l. \tag{14}
\]

Therefore, the spectrum of equation (10) is given by

\[
E = \frac{\hbar^2}{2\mu R^2} K(K + 2) = \frac{\hbar^2}{2\mu R^2} (K + 1)^2 - \frac{\hbar^2}{2\mu R^2}. \tag{15}
\]

It is no more but the spectrum of the 4D rigid rotor. In terms of \( \chi \) equation (10) takes the explicit form

\[
\frac{\hbar^2}{2\mu R^2} \left[ \frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} - \frac{L^2(\theta, \phi)}{\sin^2 \chi} \right] \psi(\chi) - E \psi(\chi) = 0. \tag{16}
\]
Multiplying equation (16) by \((- \sin^2 \chi)\) and changing the variable \(\psi(\chi)\) to \(\sin \chi S(\chi)\) results in the following Schrödinger equation:

\[ -\frac{\hbar^2}{2\mu R^2} \frac{d^2}{d\chi^2} + U_i(\chi) \] \(S(\chi) = \mathcal{E}S(\chi)\),

with \(U_i(\chi, \kappa)\) now having the meaning of centrifugal barrier on \(S^3\). As a different reading to equations (10) and (17), one can say that the \(\csc^2\) potential, in representing the centrifugal barrier on the 3D hypersphere, has \(SO(4)\) as potential algebra. The solutions to equation (17) are determined by the Gegenbauer polynomials, \(C_m^n(\cos \chi)\), as

\[ S(\chi) = N_K l_{-1}(\cos \chi) \]

where \(N_K\) is a normalization constant.

Introducing now the \(S^3\) harmonic function \((-2G/R \cot \chi)\) as an interaction into equation (16) results in

\[ S^3 : \frac{\hbar^2}{2\mu R^2} \left[ \frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} - \frac{L^2(\theta, \varphi)}{\sin^2 \chi} \right] \psi(\chi) = 2G \frac{\cot \chi}{R} \psi(\chi) - E\psi(\chi) = 0 \]

The extension of equation (17) by same interaction amounts to

\[ S^3 : \frac{\hbar^2}{2\mu R^2} \left[ \frac{d^2}{d\chi^2} + \frac{\hbar^2}{2\mu R^2} l(l + 1) \csc^2 \chi - 2G \frac{\cot \chi}{R} \right] \psi(\chi) = E\psi(\chi) \]

and describes \(SO(4)\) symmetric one-particle motion on \(S^3\) within a cotangent field of a source placed on the ‘Northern pole’ of the hypersphere. This equation is exactly solvable and the solutions are either expressed in terms of Jacobi polynomials of a purely imaginary argument and complex parameters that are conjugate to each other [10, 11] or, alternatively, in terms of the framework of the real Romanovski polynomials, \(C_m^n(\cos \chi)\), as

\[ S(\chi) = \frac{N_K}{\sin \chi} C_{K-1}^n(\cos \chi) \]

From the latter formula, one reads off that in the limit of an infinite radius, i.e. \(\lim R \to \infty\), corresponding to the expansion of the hypersphere towards the \(E_3\) space, in which the contribution of the curved centrifugal barrier tends to zero, the spectrum becomes hydrogen-like [14]. This is the virtue of having introduced the cotangent potential in equation (20) as inversely proportional to \(R\). Note, however, that a complete \(E_4\) wave equation with an angular potential introduced in this manner would not be separable in polar coordinates.

It is furthermore important to note that the trigonometric Rosen–Morse potential describes a system put on a finite volume and thereby a confinement phenomenon, as visible through the fact that its spectrum is exclusively discrete. Yet, in the limit of a vanishing curvature, \(\lim \kappa \to 0\), with \(\kappa = 1/R^2\), the volume increases towards infinity and the space flattens. In the more specific limit of

\[ \lim \frac{(K + 1)}{R} \to \infty \Rightarrow k \to \infty, \quad k = \text{const} : \lim E \to \frac{\hbar^2 k^2}{2\mu} \]

the energies of the high-lying bound excitations start approaching the energies of the scattering continuum states of the Coulomb potential [14]. In effect, the process of shutting down the
curvature of Rosen–Morse I acquires features of a deconfinement transition. This particular property of Rosen–Morse I makes it attractive for the description of confinement phenomena such as quantum dots [1] and quark confinement [2].

Back to the 3D curved manifold under consideration, we recall that the $\chi$ angle is measured in terms of the arc, $\tilde{r}$, along the geodesic distance on $S^3$, as

$$S^3 : \chi = \frac{\tilde{r}}{R}. \quad (23)$$

Flat three-dimensional position spaces can be approached through various parametrizations of $\chi$. In one of the possibilities, the $\chi$ angle can be parametrized in terms of the radial distance, $r$, from origin within the finite volume 3D flat space of the equatorial disk, $D_3$, of the hypersphere, according to

$$D_3 : \chi = \sin^{-1} \left( \frac{r}{R} \right), \quad r \in [0, R], \quad r \in D_3 \subset S^3. \quad (24)$$

This is the parametrization used in the Robertson–Walker metric of a closed universe for which equation (19) takes the form

$$D_3 : \left[ -\frac{\hbar^2}{2\mu R^2} \sqrt{1 - \frac{r^2}{R^2}} \frac{d}{dr} \sqrt{1 - \frac{r^2}{R^2}} \frac{dr}{d\tilde{r}} + \frac{\hbar^2}{2\mu} \frac{l(l + 1)}{r^2} - 2G \sqrt{1 - \frac{r^2}{R^2}} \right] \psi \left( \frac{r}{R} \right) = E \psi \left( \frac{r}{R} \right),$$

$$\psi \left( \frac{r}{R} \right) := \Psi \left( \sin^{-1} \left( \frac{r}{R} \right) \right). \quad (25)$$

In the physics of quantum dots mentioned above, another parametrization, $\chi = \tan^{-1} r/R$, has been used [1] which projects the surface of the hypersphere on the infinite 3D flat plane tangential to the North pole. Also in this case, the position space Schrödinger equation contains, beyond the standard kinetic piece, several gradient terms needed to recover the finite range confinement, as also happens in equation (25).

Instead, the parametrization used by Schrödinger [7] himself and subsequently adopted by SUSYQM [4] is

$$E_3 : \chi = \frac{r}{R} \pi = \frac{r}{d}, \quad d = \frac{R}{\pi}, \quad r \in [0, 1]. \quad (26)$$

In the latter parametrization, the $(\cot + \csc^2)$ interaction formally acquires features of a central $E_3$ flat space potential, and the respective Schrödinger equation obtained now from equation (20) takes the well-known standard form of wide spread,

$$E_3 : \left[ -\frac{\hbar^2}{2\mu d^2} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} l(l + 1) \csc^2 \left( \frac{r}{d} \right) - 2B \cot \left( \frac{r}{d} \right) \right] \psi \left( \frac{r}{d} \right) = E \psi \left( \frac{r}{d} \right),$$

$$B = \frac{G}{d}. \quad (27)$$

It is the goal of the present study to transform the above equations (27) and (25) to momentum space. We hope that the reader will not be confused by the repeated use of the same symbol $\psi$ as a wavefunction but will figure out its significance from the relevant context. Before moving
3. The algebraic momentum space equation

Our first point is that the exact solutions of equation (27) constructed earlier by us in [3, 12] allow for exact Fourier transforms to momentum space. As a reminder, as well as for the sake of self-sufficiency of the presentation and illustrative purposes, we here bring these solutions for the first three $S$ wave states, displayed in figure 1.

Subsequently, the solutions of equation (27) will be specified by the quantum numbers defined in equations (12), (13) and (14) as $\Psi_{Klm}(z)$ with $z = r/d$. The following expressions hold valid:

$$\Psi_{0,0,0}(z) = N_0 \frac{\sin z}{z} e^{-b z}, \quad z = \frac{r}{d}, \quad b = \frac{B_2 \mu d^2}{\hbar^2},$$

(28)

$$\Psi_{1,0,0}(z) = N_1 \left( \cos \frac{z}{2} - \sin \frac{z}{2} \right) \frac{\sin z}{z} e^{-b z},$$

(29)

$$\Psi_{2,0,0}(z) = N_2 \left( 3 \cos^2 z - 2 b \sin z \cos z + \left( 2 \left( \frac{b}{3} \right)^2 - 1 \right) \sin^2 z \right) \frac{\sin z}{z} e^{-b z}.$$  

(30)

The normalization constant is given by

$$N_K = (-1)^K \left( \frac{1}{\pi d^3} \right)^{1/2} \left( \frac{b}{K + 1} \left( \frac{b}{K + 1} \right)^2 + (K + 1)^2 \right)^{1/2},$$

(31)

for $K = 0, 1, 2, \ldots$. In introducing the short-hand $\bar{p}$ as

$$\bar{p} = \frac{p d}{\hbar}, \quad p = |\mathbf{p}|,$$

(32)
the following Fourier transforms of the above wavefunctions are now obtained in a close form:

\[
\phi_{0,0,0}(\vec{p}) = \left(\frac{2bd}{\hbar}\right)^{\frac{1}{2}} \left(\frac{b^2 + 1}{\pi}\right)^{\frac{1}{2}} \frac{1}{\vec{p}^4 + 2(b^2 - 1)\vec{p}^2 + 2b^4 + 1},
\]

\[
\phi_{1,0,0}(\vec{p}) = -\frac{2\frac{i}{\pi}}{\left(\frac{2bd}{\hbar}\right)^{\frac{1}{2}}} \left(\left(\frac{b}{2}\right)^2 + 1\right)^{\frac{1}{2}} \times \frac{\vec{p}^2 - \left(\frac{b}{2}\right)^2 - 2}{\vec{p}^6 + 3\left(\frac{b}{2}\right)^2 - 8} \vec{p}^4 + 3\left(\frac{b}{2}\right)^4 + 16 \vec{p}^2 + \left(\frac{b}{2}\right)^4 + 8\left(\frac{b}{2}\right)^2 + 16\left(\frac{b}{2}\right)^2. (34)
\]

\[
\phi_{2,0,0}(\vec{p}) = \frac{\sqrt{3}}{\pi} \left(\frac{2bd}{\hbar}\right)^{\frac{1}{2}} \left(\left(\frac{b}{3}\right)^2 + 1\right)^{\frac{1}{2}} \left(\vec{p}^4 - \left(\frac{b}{3}\right)^2 + 22\left(\frac{b}{3}\right)^4 + 22\left(\frac{b}{3}\right)^2 + 19\right) \times \left[\vec{p}^8 + 4\left(\left(\frac{b}{3}\right)^2 - 5\right) \vec{p}^6 + 2\left(\left(\frac{b}{3}\right)^4 - 10\left(\frac{b}{3}\right)^2 + 59\right) \vec{p}^4 + 4\left(\left(\frac{b}{3}\right)^8 + 5\left(\frac{b}{3}\right)^4 + 23\left(\frac{b}{3}\right)^2 + 45\right) \vec{p}^2 + \left(\left(\frac{b}{3}\right)^8 + 20\left(\frac{b}{3}\right)^6 + 118\left(\frac{b}{3}\right)^4 + 20\left(\frac{b}{3}\right)^2 + 81\right)\right]^{-1}. (35)
\]

In employing symbolic programs, the existence of exact Fourier transforms for the other states can be checked. The same applies to the related integrals in equation (6). As a consequence, the following algebraic equation emerges:

\[
\left(E - \frac{\vec{p}^2}{2\mu}\right) \phi_{Klm}(\vec{p}) = F_{Klm}(\vec{p}). \tag{36}
\]

\[
F_{Klm}(\vec{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{i\vec{p}\cdot\vec{r}} \left(-\frac{2G}{d}\right) \cot\left(\frac{r}{d}\right) \Psi_{Klm} \left(\frac{r}{d}\right) Y_l^m(\theta, \phi) d^3r.
\]

Below we bring the explicit expressions for \(F_{Klm}\) for the two lowest S states, displayed in figure 2, for illustrative purposes,

\[
F_{0,0,0}(\vec{p}) = \frac{1}{\pi} \left(\frac{2bd}{\hbar}\right)^{\frac{1}{2}} \left(\frac{b^2 + 1}{\pi}\right)^{\frac{1}{2}} \frac{1 - b^2 - \vec{p}^2}{\vec{p}^4 + 2(b^2 - 1)\vec{p}^2 + 2b^4 + 1} (37)
\]

\[
F_{1,0,0}(\vec{p}) = -\frac{2\frac{i}{\pi}}{\left(\frac{2bd}{\hbar}\right)^{\frac{1}{2}}} \left(\left(\frac{b}{2}\right)^2 + 1\right)^{\frac{1}{2}} \times \frac{(\frac{b}{2})^2 - 2(\frac{b}{2})^2 - 2 + 6\vec{p}^2 - \vec{p}^2}{\vec{p}^6 + 3(\frac{b}{2})^2 - 8} \vec{p}^4 + 3(\frac{b}{2})^4 + 16 \vec{p}^2 + (\frac{b}{2})^4 + 8(\frac{b}{2})^2 + 16(\frac{b}{2})^2. (38)
\]

4. The integral momentum space equation

Equation (36) from the previous section is quite transparent indeed; however, it does not make the momentum space potential explicit. This may be perceived as a shortcoming by T-matrix-based spectroscopic approaches [15]. Below we shall show that such a shortcoming is not
Figure 2. Momentum dependence of the $F_{Klm}$ quantity in equation (36) for the lowest S wave states. The solid, dashed-dotted, and dotted lines refer to $K = 0$, $K = 1$, and $K = 2$, respectively.

suffered by the Fourier transformed equation (25), obtained from the master equation (20) via the change of variable according to equation (24). In introducing the short hand,

$$x = \frac{r}{R}, \quad r = |r|,$$

equation (25) equivalently rewrites to

$$-\frac{\sqrt{1-x^2}}{x^2} \frac{d}{dx} x^2 \sqrt{1-x^2} \frac{d}{dx} \psi(x) + \frac{l(l+1)}{x^2} \psi(x) - 2b \frac{\sqrt{1-x^2}}{x} \psi(x) - \epsilon \psi(x) = 0,$$

where we in addition have divided by $\frac{\hbar^2}{2\mu R^2}$ and switched to dimensionless constants by introducing the two new notations

$$b = \frac{G}{R} \left( \frac{\hbar^2}{2\mu R^2} \right)^{-1}, \quad \epsilon = E \left( \frac{\hbar^2}{2\mu R^2} \right)^{-1}.$$ (41)

After some algebraic manipulations equation (40) takes the form

$$-\left( \frac{\nabla^2}{x^2} - \frac{1}{x^2} \frac{d}{dx} x^2 \frac{d}{dx} + \frac{3x}{dx} + 6 \right) \psi(x) - 2b \frac{\sqrt{1-x^2}}{x} \psi(x) - \epsilon \psi(x) = 0.$$ (42)

This is the equation which we shall Fourier transform to momentum space. We first obtain the integral transform of the interaction piece of the Hamiltonian as

$$-\frac{2b}{(2\pi)^2} \int d^3k \, e^{ik \cdot x} \sqrt{1-x^2} \psi(x)
= -\frac{1}{2(2\pi)^2} \int d^3k' \int d^3k'' \Pi(|q|) \frac{(J_0(|k''|) - 2J_1(|k''|))}{(k'')^2} \phi(k'),$$

$q = k - k' - k''$. (43)

Here, use has been made of

$$\sqrt{1-x^2} = -\frac{1}{(2\pi)^2} \int d^3k'' e^{-ik'' \cdot x} \sqrt{\frac{\pi}{2}} \frac{J_0(|k''|) - 2J_1(|k''|)}{(k'')^2}, \quad x = |x| \leq 1.$$ (44)
\[
\frac{1}{x} = \frac{1}{(2\pi)^2} \int d^3q \, e^{-i\mathbf{q} \cdot \mathbf{x}} \frac{\sin^2 \left( \frac{|\mathbf{q}|}{2} \right)}{\left( \frac{|\mathbf{q}|}{2} \right)^2} = \frac{1}{(2\pi)^2} \int d^3q \, e^{-i\mathbf{q} \cdot \mathbf{x}} \frac{1}{(-b)} \Pi(|\mathbf{q}|), \quad x = |\mathbf{x}| \leq 1.
\]

In the latter equation we worked in the momentum-space potential, \( \Pi(|\mathbf{q}|) \), obtained earlier by us in \([16]\) as

\[
\Pi(|\mathbf{q}|) = (-b) \frac{\sin^2 \left( \frac{|\mathbf{q}|}{2} \right)}{\left( \frac{|\mathbf{q}|}{2} \right)^2},
\]

the \( E_4 \) Fourier transform of \((-2b \cot \chi)\) for the elastic scattering case, \( q_4 = 0 \), which we calculated using the \( S^3 \) integration volume according to

\[
4\pi \Pi(|\mathbf{q}|) = -2b \int_0^\infty d|x| |x| \delta(|x| - 1) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \\
\times \int_{0/\pi}^{2/\pi} d\chi \sin^2 \chi \, e^{i|\mathbf{q}| \sin \chi} \cot \chi.
\]

Finally, transforming the pieces of the kinetic part in equation (42) amounts to

\[
\frac{1}{(2\pi)^2} \int d^3x \, e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d\psi(x)}{dx} = -3\phi(k) - \mathbf{k} \cdot \nabla_k \phi(k),
\]

\[
\frac{1}{(2\pi)^2} \int d^3x \, e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d}{dx} \frac{d}{dx} \psi(x) = \mathbf{k} \cdot \nabla_k \mathbf{k} \cdot \nabla_k \phi(k) + 6 \mathbf{k} \cdot \nabla_k \phi(k) + 9\phi(k),
\]

respectively.

Putting it all together, the integral momentum space equation becomes

\[
(k^2 + \mathbf{k} \cdot \nabla_k \mathbf{k} \cdot \nabla_k + 4 \mathbf{k} \cdot \nabla_k + 3)\phi(k) = \frac{1}{2(2\pi)^2} \int d^3k' \phi(k') \\
\times \int d^3k'' \Pi(|\mathbf{q}|) \frac{J_0(|k'x|)}{|k'|^2} - 2J_1(|k'x|) (k')^2 = \epsilon \phi(k).
\]

This equation is satisfied by the numerical Fourier transforms of the wavefunctions from position space.

5. Conclusions

We recalled that the most natural view on the trigonometric Rosen–Morse potential appears within the context of a particle moving in the cotangent field of a source placed at one of the poles of the three-dimensional spherical surface, \( S^3 \), of a constant radius, embedded within a four-dimensional Euclidean space, \( E_4 \). Within this context, the trigonometric Rosen–Morse potential has been considered as an angular function of the arc of the minimal geodesic distance on \( S^3 \) and the related Schrödinger equation has been presented in equation (20). This general view on Rosen–Morse I has the virtue of making its \( SO(4) \) symmetry obvious. Indeed, within this context, the \( \csc^2 \chi \) part plays the part of a centrifugal barrier on \( S^3 \) whose isometry group is \( SO(4) \), a symmetry respected by the \( \cot \chi \) piece, known to be a harmonic function on this manifold. We then considered two different parametrizations of the arc in equation (23) and presented them in equations (24) and (26). We wrote down the related position space
Schrödinger equations in equations (25) and (27), respectively. We noted that equation (27) coincides with the Schrödinger equation with Rosen–Morse managed by SUSYQM, where the potential is treated as central in the ordinary three-dimensional flat Euclidean space, $E_3$. We subjected equations (25) and (27) to ordinary 3D Fourier transformations to momentum space. We showed that in the first case one finds a genuine integral and in the second an exactly solvable algebraic equation.

We expect our findings to be of use in momentum space many-body frameworks based on the $T$-matrix.

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