DISCRETE LAX PAIRS AND HIERARCHIES OF INTEGRABLE DIFFERENCE SYSTEMS

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Abstract. We introduce a family of order $N \in \mathbb{N}$ Lax matrices that is indexed by the natural number $k \in \{1, \ldots, N-1\}$. For each value of $k$ they serve as strong Lax matrices of a hierarchy of integrable difference systems in edge variables that in turn lead to hierarchies of integrable difference systems in vertex variables or in a combination of edge and vertex variables. Furthermore, the entries of the Lax matrices are considered as elements of a division ring, so we obtain hierarchies of discrete integrable systems extended in the non-commutative domain.

Keywords: Discrete Lax pairs, hierarchies, non-commutative integrable difference systems, Yang-Baxter maps

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1. Introduction

Multi-component versions of scalar integrable evolution equations were introduced by the Russian school during 1970's. Gel'fand and Dikii [1], Manin [2], as well as Drinfeld and Sokolov [3], by studying higher-order spectral problems, which generalize spectral problems of KdV type, they arrived to multi-component hierarchies of integrable evolution equations of KdV type. These studies on higher-order spectral problems were continued by the seminal contributions of Mikhailov [4], Fordy et.al [5, 6] and recently by Adler and Sokolov [7] in the non-commutative setting.

In the discrete scheme, results on multi-component partial difference systems, or scalar partial difference equations defined on higher order stencils, are rather sparse [8, 9, 10, 11, 12, 13, 14].
Moreover, there are just a few results on hierarchies of integrable partial difference equations, for example discrete analogues of the Gel’fand-Dikii hierarchy, of the modified and the Schwarzian Gel’fand-Dikii hierarchies have been introduced in the past. Specifically in [21], a hierarchy of discrete equations which under suitable continuous limits leads to the Gel’fand-Dikii hierarchy was presented and it was referred to as the lattice Gel’fand-Dikii hierarchy. The first two members of the lattice Gel’fand-Dikii hierarchy respectively are the lattice potential KdV equation and the lattice version of the Boussinesq equation. Furthermore, the first two members of the lattice-modified Gel’fand-Dikii hierarchy, namely the lattice-modified KdV and Boussinesq equation were also explicitly presented in [21]. The full explicit presentation of all members of the lattice-modified Gel’fand-Dikii hierarchy had to wait for [12] and [22]. Moreover, the first two members of the lattice-Schwarzian Gel’fand-Dikii hierarchy i.e. the lattice-Schwarzian KdV and lattice-Schwarzian Boussinesq equations, were firstly introduced in [23] and the whole hierarchy was presented in [12]. In addition, an extension of the lattice-modified Gel’fand-Dikii hierarchy to the non-commutative domain was considered in [24, 25].

The results of this paper serve as a contribution to the growing interest of deriving and extending integrable difference systems to the non-commutative setting [26, 27, 28, 29, 30, 31, 24, 32, 25, 33, 34, 35, 36, 37]. Specifically, by introducing a family of discrete Lax matrices of order \( N \in \mathbb{N} \) that we denote as \( L^{N,k} \) (see Section 2.1), with entries elements of a division ring, for each value of the index \( k \in \{1, \ldots, N-1\} \), we obtain a hierarchy of difference systems in non-commutative edge variables that in turn leads to hierarchies of difference systems in non-commutative vertex variables or in a combination of edge and vertex variables. Hierarchies of integrable difference systems that correspond to a Lax matrix with a specific index \( k \), can as well arise as reductions of hierarchies which correspond to a Lax matrix with index \( k' > k \). In that respect, in this paper we present a hierarchy of hierarchies of integrable difference systems in edge and vertex variables.

The outline of this paper is as follows. In Section 2, after introducing the notation and definitions used throughout this paper, we introduce the Lax matrices \( L^{N,k} \) and the discrete spectral problem that they participate. Moreover, we prove that these Lax matrices are strong (see Definition 1) and we provide implicitly the associated family of hierarchies of difference systems in non-commutative edge variables. In Section 3 we derive explicitly a hierarchy of difference systems in edge non-commutative variables associated with the Lax matrix \( L^{N,1} \) for arbitrary \( N \) and we prove integrability. Furthermore, we obtain the associated hierarchies of difference systems in vertex variables. We show that when certain centrality assumptions are imposed, the explicit form of \( 2N \)-parameter extensions of the non-commutative lattice-modified and lattice-Schwarzian Gel’fand-Dikii hierarchies are obtained. Also, we provide explicitly the first two members of the hierarchy of Yang-Baxter maps that correspond to these hierarchies and implicitly the full hierarchy of Yang-Baxter maps. In Section 4 we obtain the explicit form of a hierarchy of difference systems in edge non-commutative variables associated with the Lax matrix \( L^{N,2} \), that results a hierarchy in vertex variables. Furthermore, we show that this hierarchy includes both hierarchies, obtained by the linear problem associated with \( L^{N,1} \), as reductions. We end this paper with Section 5 where conclusions and perspectives for future research are presented.

2. Notation, definitions and the family of Lax matrices \( L^{N,k} \)

The \( \mathbb{Z}^2 \) graph is defined as the graph with set of vertices \( V = \{(m,n)|m,n \in \mathbb{Z}\} \) and set of edges \( E = E_H \cup E_V \), i.e. the disjoint union of horizontal edges \( E_H = \{(m,n),(m+1,n)|m,n \in \mathbb{Z}\} \) and
and the vertical ones $E_V = \{(m,n),(m,n+1)\}|m,n \in \mathbb{Z}\}$ (see Figure 1). It is convenient to

![Figure 1](image-url)

**Figure 1.** (a): The set $V$ where the dependent variables of the vertex equations are assigned. (b): The sets of horizontal $E_H$ (solid lines) and the set of vertical edges $E_V$ (dashed lines) where the dependent variables of the edge equations are assigned.

label with $(m,n) \in \mathbb{Z}^2, (m+1/2,n) \in \mathbb{Z}^2$ and $(m,n+1/2) \in \mathbb{Z}^2$ the elements of $V$, $E_H$ and $E_V$, respectively.

We consider the functions $\phi^i : V \ni (m,n) \mapsto \phi^i_{m,n} \in \mathbb{D}, x^i : E_H \ni (m,n+1/2) \mapsto x^i_{m,n+1/2} \in \mathbb{D}$ and $y^i : E_V \ni (m+1/2,n) \mapsto y^i_{m+1/2,n} \in \mathbb{D}, i = 1,2,\ldots,N$, where $\mathbb{D}$ a non-commutative division ring f.i. an associative algebra over the field of complex numbers $\mathbb{C}$ with a multiplicative identity element denoted by 1 and every non-zero element $x \in \mathbb{D}$ has a unique multiplicative inverse denoted by $x^{-1}$ s.t. $xx^{-1} = x^{-1}x = 1$. We also consider the functions $p^i : E_H \ni (m+1/2,n) \mapsto p^i_{m+1/2} \in \mathbb{D}$ and $q^i : E_V \ni (m,n+1/2) \mapsto q^i_{m+1/2} \in \mathbb{D}, i = 1,2,\ldots,N$, that we assume that they are elements of the center of the algebra $\mathbb{D}$ i.e. they commute with every element of $\mathbb{D}$. In this article, the functions $p^i,q^i$ will be simply referred to as parameters. We simplify the notation above by denoting $\phi^i := \phi^i_{m,n}, \phi^i_2 := \phi^i_{m,n+1}, \phi^i_1 := \phi^i_{m+1,n}, \phi^i_2 := \phi^i_{m+1,n+1}, i = 1,2,\ldots,N$ etc. By slightly abusing this notation, we also denote $x^i := x^i_{m+1/2,n}, y^i := y^i_{m,n+1/2}, y^i_2 := y^i_{m,n+3/2}, i = 1,2,\ldots,N$ etc. (see Figure 2).

A set of equations involving edge variables f.i. $x^i,y^i i = 1,2,\ldots,N$ and a finite number of their shifts, is called system of difference equations in edge variables. A set of equations involving vertex variables f.i. $\phi^i i = 1,2,\ldots,N$ and a finite number of their shifts, is called system of difference equations in vertex variables. Finally, a set of equations involving edge and vertex variables f.i. $x^i,y^i i = 1,2,\ldots,N_1, \phi^i,i = 1,2,\ldots,N_2$, and a finite number of their shifts, is called system of difference equations in edge and vertex variables.
Let us denote with $X$, respectively $Y$, the sets \{\(x_1, \ldots, x_N\)\}, respectively \{\(y_1, \ldots, y_N\)\}, of dependent variables, as well as the sets of parameters \(P := \{p_1, \ldots, p_N\}\) and \(Q := \{q_1, \ldots, q_N\}\), \(N \in \mathbb{N}\). We now proceed to the following definitions.

**Definition 1.** A matrix of order \(N\), \(L(X; P, \lambda)\) is called a Lax matrix of the difference system in edge variables
\[
\begin{align*}
x_i^2 &= F^i(X, Y; P, Q), \quad y_i^1 = G^i(X, Y; P, Q), \quad i = 1, \ldots, N, \quad N \in \mathbb{N} \\
L(X; P, \lambda) L(Y; Q, \lambda) &= L(Y; Q, \lambda) L(X; P, \lambda),
\end{align*}
\]  
where \(F^i, G^i, \quad i = 1, \ldots, N\) are functions of the indicated variables, if (1) implies that (2) holds for all \(\lambda\) where \(\lambda\) the spectral parameter. \(L(X; P, \lambda)\) is called a strong Lax matrix of (1), if the converse also holds.

The matrix equation (2) is referred to as the discrete Lax equation or the discrete zero-curvature condition and arises as the compatibility condition of the following linear system:
\[
\Psi_2 = L(X; P, \lambda) \Psi, \quad \Psi_1 = L(Y; Q, \lambda) \Psi,
\]
where \(\Psi\) stands for an \(N\)–component vector.

**Definition 2.** The difference system in edge variables (1) will be called birational if it implies
\[
\begin{align*}
x_i^1 &= f^i(X_2; Y_1; P, Q), \quad y_i^1 = g^i(X_2; Y_1; P, Q), \quad i = 1, \ldots, N, \quad N \in \mathbb{N},
\end{align*}
\]  

![Figure 2. Dependent variables assigned on the vertices and on the edges of the \(\mathbb{Z}^2\) graph. (a): Standard notation. (b): Notation used in the paper](image-url)
where $f^i, g^i, \ i = 1, \ldots, N$ are functions of the indicated variables. Moreover, a birational system \( f. \) will be called quadrirational if it implies that

\[
x^i = H^i(X_2, Y; P, Q), \quad y^i = K^i(X_2, Y; P, Q), \quad i = 1, \ldots, N, \quad N \in \mathbb{N},
\]

where $H^i, K^i, \ i = 1, \ldots, N$ are functions of the indicated variables and \( f. \) is a birational system. Furthermore, system \( f. \) will be referred to as the companion system of the quadrirational system in edge variables \( f. \)

**Definition 3.** The following change of dependent variables

\[
x^2_i \mapsto (a^i(P)x^2_i + b^i(P)) (c^i(P)x^2_i + d^i(P))^{-1}, \quad x^i \mapsto (a^i(P)x^i + b^i(P)) (c^i(P)x^i + d^i(P))^{-1},
\]

\[
y^1_i \mapsto (a^i(Q)y^1_i + b^i(Q)) (c^i(P)y^1_i + d^i(P))^{-1}, \quad y^i \mapsto (a^i(Q)y^i + b^i(Q)) (c^i(P)y^i + d^i(P))^{-1},
\]

or

\[
x^2_i \mapsto (a^i(P)x^2_i + b^i(P)) (c^i(P)x^2_i + d^i(P))^{-1}, \quad x^i \mapsto (a^i(P)x^i + b^i(P)) (c^i(P)x^i + d^i(P))^2,
\]

\[
y^1_i \mapsto (a^i(Q)y^1_i + b^i(Q)) (c^i(P)y^1_i + d^i(P))^{-1}, \quad y^i \mapsto (a^i(Q)y^i + b^i(Q)) (c^i(P)y^i + d^i(P))^2.
\]

\forall i \in \{1, \ldots, N\}, \text{ where } a^i, b^i, c^i \text{ and } d^i \text{ functions of the indicated parameters, will be called admissible Moebius transformations or simply (Mob)^2 transformations of the difference systems in edge variables \( f. \). Two difference systems in edge variables which are related by an admissible Moebius transformation will be called equivalent.}

Note that admissible Moebius transformations respect the multidimensional compatibility a.k.a integrability of a difference system in edge variables \( f. \)

**Definition 4.** A matrix of order $N$, $L(\Phi_1, \Phi; P, \lambda)$ is called a Lax matrix of the difference system in vertex variables

\[
\phi^i_{12} = H^i(\Phi_1, \Phi_2; P, Q), \quad i = 1, \ldots, M, \quad M \in \mathbb{N}
\]

where $H^i, \ i = 1, \ldots, M$ are functions of the indicated variables, if \( f. \) implies that

\[
L(\Phi_{12}, \Phi_2; P, \lambda) L(\Phi_2, \Phi_1; Q, \lambda) L(\Phi_{12}, \Phi_1; P, \lambda),
\]

holds for all $\lambda$ where $\lambda$ the spectral parameter. $L(\Phi_1, \Phi; \lambda)$ is called a strong Lax matrix of \( f. \), if the converse also holds.

**Definition 5.** The order $N$ lower-triangular nilpotent matrices $\nabla^k, k = 1, 2, \ldots, N-1$ are defined by

\[
(\nabla^k)_{ij} := \begin{cases} 0, & i \leq j \\ \delta_{i,j+k}, & i > j \end{cases}
\]

and are said to have level $-k$.

**Definition 6.** The order $N$ upper-triangular nilpotent matrices $\Delta^k, k = 1, 2, \ldots, N-1$ are defined by

\[
(\Delta^k)_{ij} := \begin{cases} \delta_{i+N-k,j}, & i < j \\ 0, & i \geq j \end{cases}
\]

and are said to have level $N - k$.
Remark 2.1. For $1 \leq k, l \leq N - 1$ it holds: 
\[
\nabla^k \nabla^l = \begin{cases} 
\nabla^{k+l}, & k + l \leq N - 1 \\
0, & k + l > N - 1
\end{cases}, \quad \Delta^k \Delta^l = \begin{cases} 
\Delta^{k+l} \pmod{N}, & k + l \geq N + 1 \\
0, & k + l < N + 1
\end{cases}
\]

Let $A$ an order $N$ diagonal matrix. Clearly the matrices $\nabla^k A$ and $\Delta^k A$ have level $-k$ and level $N - k$ respectively.

2.1. The family of Lax matrices $L^{N,k}$. Consider the following family of Lax matrices of order $N \in \mathbb{N}$,
\[
L^{N,k}(X; P, \lambda) := P + X^\nabla + \lambda X^\Delta, \quad N \in \mathbb{N}, \quad k \in \{1, 2, \ldots, N - 1\},
\]
where
\[
X^\nabla := \sum_{i=1}^{k} \nabla^i X^{(i)}, \quad X^\Delta := \sum_{i=1}^{k} \Delta^i X^{(i)},
\]
with $X^{(j)}$, $j = 1, \ldots, k$ the order $N$ diagonal matrices with entries $(X^{(j)})_{i,i} := x^{j,i}$, that stands for the shorthand notation of $x^{j,i}_{m+1/2,n}$, $j = 1, \ldots, k$, $i = 1, \ldots, N$, $m, n \in \mathbb{Z}$. Also with $\lambda$ we denote the spectral parameter and with $P$ the order $N$ diagonal matrix with entries $(P)_{i,j} := p^j$, where, as it was defined earlier, $p^j$ stands for the shorthand notation of $p^j_{m+1/2}$, $m \in \mathbb{Z}$.

Let us also denote with $Q$ the order $N$ diagonal matrix with entries $(Q)_{i,i} := q^i \equiv q^i_{n+1/2}$; $i = 1, \ldots, N$, $n \in \mathbb{Z}$ and with $Y^{(j)}$, $j = 1, 2, \ldots, k$ the order $N$ diagonal matrices with entries $(Y^{(j)})_{i,j} := y^{j,i} \equiv y^{j,i}_{m,n+1/2}$, $j = 1, \ldots, k$, $i = 1, 2, \ldots, N$, $m, n \in \mathbb{Z}$. Note that the dependent variables $x^{j,i}$, $y^{j,i}$ are considered elements of a division ring $\mathbb{D}$ and the parameters $p^j$, $q^i$, as well as the spectral parameter $\lambda$, are assumed central elements of $\mathbb{D}$ i.e. they commute with any element of $\mathbb{D}$.

The discrete Lax equation
\[
L^{N,k}(X_2; P, \lambda)L^{N,k}(Y; Q, \lambda) = L^{N,k}(Y_1; Q, \lambda)L^{N,k}(X; P, \lambda)
\]
reads:
\[
(P + X_2^\nabla) (Q + Y^\nabla) + \lambda ((P + X_2^\nabla) Y^\Delta + X_2^\Delta (Q + Y^\nabla)) + \lambda^2 X_2^\Delta Y^\Delta = (Q + Y_1^\nabla) (P + X^\nabla) + \lambda ((Q + Y_1^\nabla) X^\Delta + Y_1^\Delta (P + X^\nabla)) + \lambda^2 Y_1^\Delta X^\Delta. 
\]
(7)
The requirement that (7) holds for all $\lambda$ implies
\[
(P + X_2^\nabla) (Q + Y^\nabla) = (Q + Y_1^\nabla) (P + X^\nabla), \quad (8)
(P + X_2^\nabla) Y^\Delta + X_2^\Delta (Q + Y^\nabla) = (Q + Y_1^\nabla) X^\Delta + Y_1^\Delta (P + X^\nabla), \quad (9)
X_2^\Delta Y^\Delta = Y_1^\Delta X^\Delta. \quad (10)
\]

Proposition 2.2. The matrix equations (8)-(10) consist of $2kN$ scalar equations in total.

Proof. First we count the number of scalar equations of (8). For $N$ even and $k \leq N/2$, making use of Remark (2.1) we deduce that the non-zero entries of the matrix equation (8) are at levels $-1, -2, \ldots, -2k$ so there are $\sum_{i=1}^{2k} (N - i)$ scalar equations, while for $k > N/2$, (8) is strictly lower triangular so it consists of $\frac{N(N-1)}{2}$ equations. Similarly, when $N$ is odd and $k \leq \frac{N-1}{2}$, equation (8) consists of $\sum_{i=1}^{2k} (N - i)$ scalar equations, while for $k > \frac{N-1}{2}$, it consists of $\frac{N(N-1)}{2}$ equations.
When $N$ is even and $K \leq N/2$, equation (9) has non-zero entries at levels $N - 1, \ldots, N - 2k$ so consists of $k(2k + 1)$ scalar equations, while for $k > N/2$, the non-zero entries exist at the upper triangular part of (9) as well as to the levels $0, -1, \ldots, N - 2k$, so there are $N(N + 1)/2 + \sum_{i=1}^{2k-N} (N - i)$ scalar equations. Similarly, when $N$ is odd and $k \leq N/2$ equation (9) consists of $k(2k + 1)$ scalar equations, while for $k > N/2$, it consists of $N(N + 1)/2 + \sum_{i=1}^{2k-N} (N - i)$ equations.

Finally, equation (10), when $N$ is even and $k \leq N/2$ consists of zero scalar equations since $X^3_Y$ and $Y^3_X$ are both the zero matrices, while for $k > N/2$, it consists of $\sum_{i=1}^{2k-N} i$ equations. When $N$ is odd and $k \leq N/2$, we have zero scalar equations, but for $k > N/2$, we have $\sum_{i=1}^{2k-N} i$ scalar equations.

We summarize the results above in Table 1, where it is evident that the total number of scalar equations in (8)-(10) is always $2kN$.

**Table 1. Total number of scalar equations in (8)-(10)**

|               | $N$ odd | $N$ even |
|---------------|---------|----------|
| $k \leq N - 1/2$ | $2k\sum_{i=1}^{N-i}$ | $N(N-1)/2$ |
| $k > N - 1/2$ | $N(N-1)/2$ | $2k\sum_{i=1}^{N-i}$ |
| $k \leq N/2$ | $k(2k+1)$ | $N(N+1)/2$ |
| $k > N/2$ | $2k(2k+1)$ | $N(N+1)/2 + \sum_{i=1}^{2k-N} (N-i)$ |
| $k \leq N/2$ | $\sum_{i=1}^{2k-N} i$ | $2k-N\sum_{i=1}^{N-i}$ |
| $k > N/2$ | $2k-N\sum_{i=1}^{N-i}$ | $2k-N\sum_{i=1}^{N-i}$ |
| total # of eqs. | $2kN$ | $2kN$ |

Corollary 1. The matrix equations (8)-(10) constitute a difference system of $2kN$ scalar equations on the $2kN$ edge variables $x^{j,i}, j = 1, \ldots, k, i = 1, \ldots, N, k \in \{1, \ldots, N-1\}$.

In the example that follows we present the explicit form of the Lax matrices $L^{2,1}(X; P, \lambda)$, $L^{3,1}(X; P, \lambda)$, $L^{3,2}(X; P, \lambda)$ and $L^{4,2}(X; P, \lambda)$.

**Example 2.3.** The Lax matrices $L^{2,1}(X; P, \lambda)$, $L^{3,1}(X; P, \lambda)$, $L^{3,2}(X; P, \lambda)$ and $L^{4,2}(X; P, \lambda)$, respectively read:

$$L^{2,1}(X; P, \lambda) = P + \nabla^1 X^{(1)} + \lambda \Delta^1 X^{(1)} = \begin{pmatrix} p^1_{x^{1,1}} & \lambda x^{1,2} \\ x^{1,1} & p^2 \end{pmatrix},$$

$$L^{3,1}(X; P, \lambda) = P + \nabla^1 X^{(1)} + \lambda \Delta^1 X^{(1)} = \begin{pmatrix} p^1_{x^{1,1}} & 0 & \lambda x^{1,3} \\ x^{1,1} & p^2 & 0 \\ 0 & x^{1,2} & p^3 \end{pmatrix},$$
\[ L^{3,2}(X; P, \lambda) = P + \nabla^1 X^{(1)} + \nabla^2 X^{(2)} + \lambda \left( \Delta^1 X^{(1)} + \Delta^2 X^{(2)} \right) = \begin{pmatrix} p^1 & \lambda x^{2,2} & \lambda x^{1,3} \\ x^{1,1} & p^2 & \lambda x^{2,3} \\ x^{2,1} & x^{1,2} & p^3 \end{pmatrix}, \]

\[ L^{4,2}(X; P, \lambda) = P + \nabla^1 X^{(1)} + \nabla^2 X^{(2)} + \lambda \left( \Delta^1 X^{(1)} + \Delta^2 X^{(2)} \right) = \begin{pmatrix} p^1 & 0 & \lambda x^{2,3} & \lambda x^{1,4} \\ x^{1,1} & p^2 & \lambda x^{2,4} \\ x^{2,1} & x^{1,2} & p^3 & 0 \\ 0 & x^{2,2} & x^{1,3} & p^4 \end{pmatrix}. \]

3. The Lax matrix \( L^{N,1} \) and integrable hierarchies of difference systems

For \( k = 1 \), the Lax matrix \( L^{N,1}(X; P, \lambda) \) explicitly reads:

\[ L^{N,1}(X; P, \lambda) = P + \nabla^1 X^{(1)} + \lambda \Delta^1 X^{(1)} = \begin{pmatrix} p^1 & 0 & \ldots & 0 & \lambda x^N \\ x^1 & p^2 & \ldots & 0 \\ 0 & x^2 & \ldots & \vdots \\ \vdots & \ldots & p^{N-1} & 0 \\ 0 & 0 & \ldots & x^{N-1} & p^N \end{pmatrix}, \]

where we have written \( x^i \) instead of \( x^{1,i} \) for the entries of the matrix \( X^{(1)} \). The compatibility conditions \((8)-(10)\) read:

\[ x^i y^{i-1} = y^i x^{i-1}, \]

\[ q^i x^i - q^{i+1} x^i = p^i y^i - p^{i+1} y^i, \]

where the superscripts \( i = 1, 2, \ldots, N \) are considered modulo \( N \).

For arbitrary \( N \), system \((12),(13)\) under the change of the dependent variables

\[ (x^i, x^{i}, y^i) \mapsto \left( \frac{q^{i+1} p^{i+1}}{q^i} x^i, p^{i+1} x^{i+1}, \frac{q^{i+1} p^{i+1}}{p^i} y^i, q^{i+1} y^{i+1} \right), \quad \forall i \in \{1, \ldots, N\}, \]

is mapped to the \( N \)-periodic reduction of the so-called non-commutative KP map introduced in \([24, 25]\). The change of variables \((14)\) though is not admissible (see Definition 3 and Example 3.5 for discussion).

3.1. An integrable hierarchy of difference systems in edge variables.

**Proposition 3.1.** Matrix \([11]\) serves as a strong Lax matrix for the hierarchy of difference systems in edge variables \([12),(13)\] that in its solved form reads

\[ x^i = (q^{i+1} p^{i+1} x^i - p^{i+1} y^i) x^{i-1} (q^i x^{i-1} - p^i y^{i-1})^{-1}, \]

\[ y^i = (q^{i+1} p^{i+1} y^i - p^{i+1} y^{i-1}) y^{i-1} (q^i x^{i-1} - p^i y^{i-1})^{-1}, \]

\[ i = 1, 2, \ldots, N. \]

**Proof.** Substituting \((15)\) to the compatibility conditions \((12),(13)\), one can easily check that the latter are satisfied. \( \square \)
Corollary 2. The products $P^i := x^{N+i-1} \cdots x^{k+1} x^k$ and $Q^i := y^{N+i-1} \cdots y^{k+1} y^k$, $i = 1, 2, \ldots, N$ satisfy the relations:

$$
P^i = (q^{i+1} x^i - p^{i+1} y^i) P^{i+1} (q^{i+1} x^i - p^{i+1} y^i)^{-1}, \quad Q^i = (q^{i+1} x^i - p^{i+1} y^i) Q^{i+1} (q^{i+1} x^i - p^{i+1} y^i)^{-1}, \quad i = 1, 2, \ldots, N. \tag{16}
$$

Proof. The direct substitution of (15) into (16) validates the formulae. □

In the commutative setting where all variables are considered elements of the center of the algebra $\mathbb{D}$, the first set of equations of Corollary 2 reduce to $\prod_{i=1}^{N} x^i = \prod_{i=1}^{N} x^i$, that suggests $\prod_{i=1}^{N} x^i$ does not change along the set of vertical edges $E_V$, i.e. it is function that depends only on the independent variable $m$. We denote this function as $p^0$ to the power $N$ for reasons that will become clear later. Similarly we can show that $\prod_{i=1}^{N} y^i$ is function that depends on the independent variable $n$ only that we denote as $(q^0)^N$. Namely we have

$$
\prod_{i=1}^{N} x^i = (p^0)^N = \prod_{i=1}^{N} x^i, \quad \prod_{i=1}^{N} y^i = (q^0)^N = \prod_{i=1}^{N} y^i. \tag{17}
$$

In the non-commutative setting, if we assume that the products $P^i, Q^i$ belong to the center of the algebra $\mathbb{D}$, then from (16) it follows that $P^i$ are functions that depends on the independent variable $m$ only and $Q^i$ are functions that depends on the independent variable $n$ only. We denote these functions as $(p^{i-1})^N$ and $(q^{i-1})^N$ respectively, namely

$$
x^{N+i-1} \cdots x^{k+1} x^k = (p^{i-1})^N, \quad y^{N+i-1} \cdots y^{k+1} y^k = (q^{i-1})^N, \quad i = 1, \ldots, N. \tag{18}
$$

Moreover we have that $p^i = p^j$, $i, j = 0, 1, \ldots, N - 1$, that follows if we eliminate one variable from any pair of products of the first set of equations of (18). Similarly we show that $q^i = q^j$, $i, j = 0, 1, \ldots, N - 1$. From further on, when we refer to the centrality assumption, we refer to the formulas:

$$
x^{N} \cdots x^2 x^1 = (p^0)^N = x^N, \quad y^{N} \cdots y^2 y^1 = (q^0)^N = y^N. \tag{19}
$$

The centrality assumptions were first introduced in [24, 25] for the $N$-periodic reduction of the KP-map and they play here as well a crucial role to the quadrirationality of the hierarchy of difference systems in edge variables (15), as it is shown in the Proposition that follows.

Proposition 3.2. The hierarchy of difference systems (15) is birational. If the centrality assumptions (19) are imposed, then (15) is quadrirational.

Proof. The polynomial form of (15), namely the equations (12), (13), can be solved rationally for $x^i, y^i$ in terms of $x^i_1, y^i_1$ and that proves birationality of the system. Specifically from (12), (13) we obtain:

$$
x^i = (q^{i+1} x^i_1 - p^{i+1} y^i_1)^{-1} x^{i+1}_1 (q^{i+1} y^i_2 - p^i y^i_1), \quad y^i = (q^{i+1} x^{i+1}_2 - p^{i+1} y^{i+1}_1)^{-1} y^{i+1}_1 (q^{i+1} x^{i+1}_2 - p^i y^i_1), \quad i = 1, 2, \ldots, N. \tag{20}
$$

As it was stated in Definition 2, if the birational system (15) can be solved rationally for $x^i_1, y^i_1$ in terms of $x^i, y^i_1$ and the resulting system is birational, then system (15) will be called quadrirational.

\footnote{A less restrictive assumption is that the products $P^i, Q^i$ commute with the elements $q^i x^{i-1} - p^i y^{i-1}, i = 1, \ldots, N.$}
and the resulting birational system will be called *companion system*. Now we are ready to prove that (15) is quadrirational provided that the centrality assumptions (19) are imposed.

From (12) we obtain
\[ x_i^2 = y_i x_i^{-1} (y_i^{-1})^{-1}, \]
(21)

substituting these equations into (13) we get
\[ p^{i+1} y^i y_i^{-1} - (q^{i+1} x_i^i + p^i y_i^i) y_i^{-1} + q^i y_i^i x_i^{-1} = 0. \]
(22)

Multiplying equation (22) respectively with \( y_i^i \), \( y_i^{i-2} \), \( y_i^{i-3} \), \( \cdots \), \( \prod_{k=2}^{N-1} y_i^{-k} \) we have
\[ p^{i+1} y^i y_i^{-1} y_i^{-2} - (q^{i+1} x_i^i + p^i y_i^i) y_i^{-1} y_i^{-2} + q^i y_i^i x_i^{-1} y_i^{-2} = 0, \]
\[ p^{i+1} y^i y_i^{-1} y_i^{-3} - (q^{i+1} x_i^i + p^i y_i^i) y_i^{-1} y_i^{-3} + q^i y_i^i x_i^{-1} y_i^{-3} = 0, \]
\[ \vdots \]
\[ p^{i+1} y^i y_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} - (q^{i+1} x_i^i + p^i y_i^i) y_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} + q^i y_i^i x_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} = 0. \]
(23)

Assuming centrality we have that \( y_i^i y_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} = (q^0)^N \) and the last equation of (23) reads
\[ p^{i+1} (q^0)^N - (q^{i+1} x_i^i + p^i y_i^i) y_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} + q^i y_i^i x_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} = 0. \]
(24)

The products \( y_i^i y_i^{-1} \prod_{k=2}^{N-1} y_i^{-k} \), \( \prod_{k=2}^{N-1} y_i^{-k} \) in (24) can be determined recursively from (22), (23) and they are linear polynomials on \( y_i^{-N-1} \). Letting \( i \) run from 1 to \( N \) we obtain \( y_i^i \) as functions of \( x_i^i, y_1^i \), and together with (21), we have solved (15) rationally for \( x_2^i, y_i^i \) in terms of \( x_1^i, y_1^i \). In exactly similar manner we can solve (15) rationally for \( x_i^i, y_1^i \) in terms of \( x_2^i, y_i^i \) and that completes the proof. \( \square \)

3.1.1. *Multidimensional compatibility*. Under the identifications \( X^i_a := x_i^i, X^i_b := y_i^i, X^i_a := y_1^i, X^i_b := y_1^i \), the hierarchy of difference systems (15) obtains the compact form
\[ X^i_a = \left( p^{i+1,a} X^i_a - p^{i+1,a} X^i_b \right) X^{i-1,a} \left( p^{i+1,a} X^{i-1,a} - p^{i,a} X^{i-1,b} \right)^{-1}, \]
(25)

\( i = 1, \ldots, N, a \neq b \in \{1, 2\} \).

**Lemma 3.3.** It holds
\[ p^{i+1,a} X^i_a - p^{i+1,a} X^i_c = K_{bc}^i \left( p^{i,a} X^{i-1,a} - p^{i,a} X^{i-1,c} \right)^{-1}, \]
(26)

\( i = 1, \ldots, N, a \neq b \neq c \neq a \in \{1, \ldots, n\}, \) where
\[ K_{bc}^i := \left( A \left( p^{i,c} X^{i-1,b} - p^{i,b} X^{i-1,c} \right)^{-1} \right) X^{i-1,a} + B, \]
with
\[ A := p^{i+1,b} p^{i+1,c} X^{i,a} \left( p^{i,c} X^{i-1,b} - p^{i,b} X^{i-1,c} \right) + p^{1+i,a} p^{i,b} p^{1+i,b} X^{i,c} X^{i-1,c} - p^{1+i,a} p^{i,c} p^{1+i,c} X^{i,b} X^{i-1,b}, \]
\[ B := p^{i,a} p^{i+1,a} \left( p^{i+1,c} X^{i,b} - p^{i+1,b} X^{i,c} \right) \left( p^{i,c} (X^{i-1,c})^{-1} - p^{i,b} (X^{i-1,b})^{-1} \right)^{-1}, \]

and the functions \( K^{i,a}_{bc} \) are symmetric under the interchange \( b \leftrightarrow c \) of the discrete shifts i.e. \( K^{i,a}_{bc} = K^{i,a}_{cb} \).

**Proof.** Substituting from (25) the expressions of \( X^{i,a} \) and \( X^{i,b} \) into \( p^{i+1,b} X^{i,a} - p^{i+1,a} X^{i,b} \), upon expansion, recollection of terms and making use of the identity
\[ X^{i-1,b} \left( p^{i,c} X^{i-1,b} - p^{i,b} X^{i-1,c} \right)^{-1} X^{i-1,c} = \left( p^{i,c} (X^{i-1,c})^{-1} - p^{i,b} (X^{i-1,b})^{-1} \right)^{-1}, \]
we obtain (26). The expressions of \( B \) and of \( A \left( p^{i,c} X^{i-1,b} - p^{i,b} X^{i-1,c} \right)^{-1} \) are clearly symmetric under the interchange \( b \leftrightarrow c \), hence there is \( K^{i,a}_{bc} = K^{i,a}_{cb} \). \( \square \)

**Proposition 3.4.** The system of difference equations (22) can be extended in a compatible way to \( n \)-dimensions as follows
\[ X^{i,a}_b = \left( p^{i+1,b} X^{i,a} - p^{i+1,a} X^{i,b} \right) X^{i-1,a} \left( p^{i,b} X^{i-1,a} - p^{i,a} X^{i-1,b} \right)^{-1}, \]
with \( i = 1, \ldots, N, \ a \neq b \in \{1, \ldots, n\} \). The compatibility conditions
\[ X^{i,a}_{bc} = X^{i,a}_{cb}, \quad i = 1, \ldots, N, \ a \neq b \neq c \neq a \in \{1, \ldots, n\} \]
hold.

**Proof.** Shifting (27) at the \( c \)-direction, we obtain
\[ X^{i,a}_{bc} = \left( p^{i+1,b} X^{i,a} - p^{i+1,a} X^{i,b} \right) X^{i-1,a} \left( p^{i,b} X^{i-1,a} - p^{i,a} X^{i-1,b} \right)^{-1}. \]
Substituting from (27) the expression of \( X^{i-1,a}_c, X^{i-1,b}_c, X^{i,a}_c \) and \( X^{i,b}_c \) to the equations above and by making use of Lemma \([33]\) we derive the following compatibility formula
\[ X^{i,a}_{bc} = K^{i,a}_{bc} X^{i-1,a} \left( K^{i-1,a}_{bc} \right)^{-1}, \]
that is clearly symmetric under the interchange \( b \) to \( c \) and that completes the proof. \( \square \)

**Example 3.5 (\( N = 2 \)).** For \( N = 2 \), we have the Lax matrix \( L^{2,1}(x^1, x^2, p^1, p^2, \lambda) \) that reads:
\[ L^{2,1}(x^1, x^2, p^1, p^2, \lambda) = \begin{pmatrix} p^1 & \lambda x^2 \\ x^1 & p^2 \end{pmatrix} \]
and it serves as a strong Lax matrix for the birational difference system \([15]\) with \( N = 2 \). The centrality assumption \([14]\) that now reads
\[ x^2 x^1 = (p^0)^2 = x^2_2 x^1_2, \quad y^2 y^1 = (q^0)^2 = y^2_1 y^1_1, \]
allow us to eliminate $x^2, y^2$ and their respective shifts from \([15]\) with $N = 2$. For simplicity we denote $x := x^1$, $y := y^1$, $p := (p^0)^2$ and $q := (q^0)^2$ to obtain the quadrirational difference system
\[
x_2 = p(q^2x - p^2y)(pq^1y - qp^1x)^{-1}y, \quad y_1 = q(q^2x - p^2y)(pq^1y - qp^1x)^{-1}x,
\]
with strong Lax matrix
\[
L^{2,1}(x^1; p, p^1, p^2; \lambda) = \begin{pmatrix} p^1 \\ x^1 \\ p^2 \\ x^2 \end{pmatrix} \left( \lambda p (x^1)^{-1} \right).
\]
Solving \((28)\) for $x, y_1$ in terms of $x_2, y$ and by applying the following admissible transformation
\[
(x, y, x_2, y_1) \mapsto (x^{-1}, y^{-1}, x_2^{-1}, y_1^{-1}),
\]
for $p^1 = p^2 = q^1 = q^2 = 1$ we obtain:
\[
x = (p)^{-1}y(y + x_2)^{-1}(px_2 + qy), \quad y_1 = (q)^{-1}x_2(y + x_2)^{-1}(px_2 + qy). \tag{29}
\]

The difference system \((27)\) defines a parametric Yang-Baxter map $R^{(p,q)} : \mathbb{D} \times \mathbb{D} \ni (x_2, y) \mapsto (x, y_1) \in \mathbb{D} \times \mathbb{D}$, since it holds
\[
R^{(p,q)}_1 \circ R^{(p,q)}_2 \circ R^{(q,r)}_3 = R^{(q,r)}_3 \circ R^{(p,q)}_2 \circ R^{(p,q)}_1, \tag{30}
\]
where $R^{(p,q)}_i : \mathbb{D} \times \mathbb{D} \times \mathbb{D} \mapsto \mathbb{D} \times \mathbb{D} \times \mathbb{D}$ is defined as the map that as $R^{(p,q)}$ on the first and second component of $\mathbb{D} \times \mathbb{D} \times \mathbb{D}$ and as an identity to the third; Similarly are defined the maps $R^{(p,r)}_i$ and $R^{(q,r)}_i$. The first instances of set-theoretical-solutions of the quantum Yang-Baxter equation \((30)\) appeared in \([40, 41]\) and the term Yang-Baxter maps was coined to these set-theoretical-solutions in \([42, 43]\).

The Yang-Baxter map $R^{(p,q)}$ provided by \([24]\), serves as the non-commutative extension of the $H_{III}^{A}$ Yang-Baxter map that was introduced in \([39]\) in the commutative setting. Note that \([12], [13]\) under the non-admissible change of variables \([14]\) coincides with the $N$–periodic reduction of the KP map \([24, 25]\), that for $N = 2$ and under the centrality assumptions, results to non-commutative extension \([25]\) of the $H_{III}^{B}$ Yang-Baxter map that was introduced in \([39]\) in the commutative setting. The Yang-Baxter maps $H_{III}^{A}$ and $H_{III}^{B}$ are non-equivalent up to $(\text{Mob})^2$ transformations \([39]\), and that justifies why we consider that the $N$–periodic reduction of the KP map is not equivalent to the difference system \([12], [13]\).

Example 3.6 $(N = 3)$. For $N = 3$, we have the following strong Lax matrix
\[
L^{3,1}(x^1, x^2, x^3; p^1, p^2, p^3; \lambda) = \begin{pmatrix} p^1 \\ x^1 \\ p^2 \\ x^2 \\ p^3 \\ x^3 \end{pmatrix} \left( \lambda x^3 \right),
\]
that serves as a Lax matrix for the birational difference system \([15]\) with $N = 3$. The centrality assumption \([19]\) that now reads
\[
x^3x^2x^1 = (p^0)^3 = x_2^3x_2^2x_2, \quad y^3y^2y^1 = (q^0)^3 = y_1^3y_1^2y_1,
\]
allow us to eliminate $x^3, y^3$ and their respective shifts from \([15]\) with $N = 3$ and obtain the quadrirational difference system
\[
x_2 = (q^{i+1}x^i - p^{i+1}y^i)x^{i-1}(q^ix^{i-1} - p^iy^{i-1})^{-1}, \quad
y_1 = (q^{i+1}x^i - p^{i+1}y^i)y^{i-1}(q^ix^{i-1} - p^iy^{i-1})^{-1}, \tag{31}
\]
where the index $i$ is considered modulo 3 and take the values $i = 1, 2, 3$. Let $x^3 = (p_0^3 x^1)^{-1} (x^2)^{-1}$ and $y^3 = (q_0^3 y^1)^{-1} (y^2)^{-1}$. The quadrirational system (37) has as strong Lax matrix the matrix

$$L^{3,1}(x^1, x^2, p^0, p^1, p^2, p^3, \lambda) = \begin{pmatrix} p^1 & 0 & \lambda (p_0^3 x^1)^{-1} (x^2)^{-1} \\ p^1 x^1 & p^2 & 0 \\ 0 & x^2 & p^3 \end{pmatrix}.$$ 

Following the method introduced in the proof of Proposition 3.2, we solve (37) for $x_1^2, x_2^2, y_1^1, y_2^1$ to obtain the companion system:

$$x_1^2 = (p^1 y_1^1 x_1^{i+2} S^{i+1} + q^1 i q_1^{i+2} (p_0^3)) (q_1^{i+2} x_1^{i+2} + p_1^{i+2} y_1^{i+1} p_1^{i+2} y_1^{i+1})^{-1} \quad (32)$$

$$y_1^1 = (p_1^{i+2} S^{i+2} y_1^{i+1} + q_1^{i+2} i q_1^{i+2} x_1^{i+2} x_1^{i+1})^{-1} (q_1^{i+1} + p_1^{i+2} q_1^{i+2}(q_0^3)^3),$$

where the index $i$ is considered modulo 3 and take the values $i = 1, 2, 3$. Let $x^3 = (p_0^3 x^1)^{-1} (x^2)^{-1}$, $y^3 = (q_0^3 y^1)^{-1} (y^2)^{-1}$ and the expressions $S^j$ are defined by $S^j := q_1^{i+1} x^j + p_1^{i+1} y_1^j$.

The difference system (32) defines a Yang-Baxter map

$$R : \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{D} \ni (x^1, x^2, y_1^1, y_2^1) \mapsto (x_1^2, x_2^2, y_1^1, y_2^1) \in \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{D},$$

that in the simple case where $p_1 = p_2 = p_3 = 1 = q_1 = q_2 = q_3$, can be considered as the second member of the hierarchy of $H_{11}$ Yang-Baxter maps extended in the non-commutative domain.

### 3.2 Integrable hierarchies of difference systems in vertex variables

A procedure to obtain integrable difference systems in vertex variables from integrable systems in edge (or even face variables) was introduced in [44]. Nowadays the name potentialisation is coined to this procedure and it is widely applied [45 46 47 48]. In order to obtain integrable difference systems in vertex variables associated with the integrable difference system (13), we follow the potentialisation procedure as it was used in [24 48].

#### 3.2.1. Multiplicative potentials and the lattice-modified Gel’fand-Dikii hierarchy

The hierarchy of difference systems (13) in its polynomial form constitutes of the sets of equations (12), (13). Equations (12) are identically satisfied if we set

$$x^i = p^0 \phi^i (\phi^i)^{-1}, \quad y^i = q^0 \phi^i (\phi^i)^{-1}, \quad i = 1, \ldots, N. \quad (33)$$

The functions $\phi^i, i = 1, \ldots, N$ can be considered as potential functions. In terms of these potential functions equations (12) are identically satisfied, while equations (13) read

$$q^0 \left( p_0^{i+1} \phi_2^i (\phi^i)^{-1} - p_0^{i} \phi_2^{i+1} (\phi_1^i)^{-1} \right) - p^0 \left( q_0^{i+1} \phi_2^i (\phi^i)^{-1} - q_0^{i} \phi_2^{i+1} (\phi_2^{i+1})^{-1} \right) = 0, \quad (34)$$

$i = 1, \ldots, N$ and constitute a hierarchy of difference systems in vertex variables.

**Proposition 3.7.** The hierarchy of difference systems in vertex variables (34), after rearranging terms reads

$$\left( q^0 p_0^{i+1} \phi_2^i - p_0^0 q_0^{i+1} \phi_2^i \right) (\phi^i)^{-1} = \phi_2^i \left( q^0 p_2^i (\phi_1^i)^{-1} - p_0^0 q_2^i (\phi_2^{i+1})^{-1} \right), \quad i = 1, \ldots, N. \quad (35)$$
(1) arises as the compatibility conditions of the Lax equation \((7)\), associated with the strong Lax matrix

\[
L(\Phi_1, \Phi; P, p^0, \lambda) = \begin{pmatrix}
p^1 & 0 & \ldots & 0 & \lambda p^0 \phi_N^{-1} (\phi_N^{-1})^{-1} \\
p^0 \phi_1^{-1} & p^2 & 0 & \ldots & 0 \\
0 & p^0 \phi_1^{-1} & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & p^{N-1} & 0 \\
0 & 0 & \ldots & p^0 \phi_1^{N-1} (\phi_N^{-2})^{-1} & p^N
\end{pmatrix};
\]

(2) it is multidimensional consistent;
(3) it is invariant under the following permutations of the dependent variables

\[
\tau : (\phi^i, \phi_1^i, \phi_2^i, \phi_{12}^i, p^0, q^0, p^i, q^i) \mapsto (\phi^i, \phi_2^i, \phi_1^i, \phi_{12}^i, q^0, p^0, q^i, p^i),
\]
\[
\sigma : (\phi^i, \phi_1^i, \phi_2^i, \phi_{12}^i, p^0, q^0, p^i, q^j) \mapsto (\phi_{12}^i, \phi_2^i, \phi_1^i, q^0, p^0, q^i, p^j),
\]
for all \(j \in \{1, \ldots, N\}\), i.e. it respects the rombic symmetry;
(4) it is an integrable hierarchy of difference systems in vertex variables.

Proof. (1) Substituting the expressions of the potential functions \((33)\) into \((11)\) we obtain the Lax matrix presented in this Proposition.

(2) This is a direct consequence of the multidimensional compatibility (see Proposition \((3.4)\)) of the underlying difference system in edge variables \((13)\). Specifically, for the system \((27)\) we have proved that the compatibility conditions

\[
X_{bc}^{i,a} = X_{cb}^{i,a}, \quad i = 1, \ldots, N, \quad a \neq b \neq c \neq a \in \{1, \ldots, n\},
\]

hold. For the potential functions \(\phi^i\) we have

\[
X^{i,a} = p^{0,a} \phi_a^i (\phi^{-1})^{-1}, \quad i = 1, \ldots, N, \quad a \in \{1, \ldots, n\}.
\]

From \((37)\) and \((36)\) we obtain

\[
\phi_{abc}^i = \phi_{acb}^i, \quad i = 1, \ldots, N, \quad a \neq b \neq c \neq a \in \{1, \ldots, n\},
\]

that proves the multidimensional consistency of the hierarchy.

(3) It is easy to see that \((35)\) it is invariant under \(\tau\), while acting with \(\sigma\) on \((35)\) we obtain

\[
(p^0 q^{i+1} \phi_2^i - q^0 p^{i+1} \phi_1^i) (\phi_{12}^{-1})^{-1} = \phi_i^i \left( p^0 q^i (\phi_1^{-1})^{-1} - q^0 p^i (\phi_2^{-1})^{-1} \right), \quad i = 1, \ldots, N,
\]

which is \((35)\) in disguise. Indeed, by acting on \((35)\) with \(T_{-1} T_{-2}\) we obtain \((38)\). Here with \(T_r\) we denote the forward shift operator on the \(r\)-th direction and with \(T_s\) we denote the backward shift operator on the \(s\)-th direction i.e.

\[
T_1 : \phi_{m,n}^i \mapsto \phi_{m+1,n}^i \equiv \phi_{1}^i, \quad T_{-1} : \phi_{m,n}^i \mapsto \phi_{m-1,n}^i \equiv \phi_{-1}^i, \quad T_{-2} : \phi_{m,n}^i \mapsto \phi_{m,n-1}^i \equiv \phi_{-2}^i, \quad \text{etc.}
\]

(4) Due to statements (1) – (3), \((35)\) constitutes an integrable hierarchy of difference systems in vertex variables defined on the black-white lattice \([50, 51, 52]\).

\[\square\]
In the commutative setting where all variables are considered as elements of the center of the algebra $D$, relations (17) in terms of the multiplicative potential functions (33), lead to

$$\prod_{i=1}^{N} \phi_i^j = \prod_{i=1}^{N} \phi^j, \quad \prod_{i=1}^{N} \phi_2^j = \prod_{i=1}^{N} \phi^j,$$

so the potential functions are not independent but they satisfy

$$\prod_{i=1}^{N} \phi^j = c, \quad (39)$$

where $c$ is a constant that can be scaled to 1. From (35) together with (39) one can solve rationally for any of the corner variable sets $\{\phi^1, \ldots, \phi^N\}$, $\{\phi_1^1, \ldots, \phi_1^N\}$, $\{\phi_2^1, \ldots, \phi_2^N\}$, $\{\phi_{12}^1, \ldots, \phi_{12}^N\}$, that is a reminiscence of the associated quadrirational difference system in edge variables (15), (19).

In the non-commutative setting, (35) can be solved only for the corner variable sets $\{\phi^1, \ldots, \phi^N\}$ and $\{\phi_{12}, \ldots, \phi_{12}^{N}\}$. If we impose the centrality assumptions (40), that in terms of the potential functions (33) take the form

$$\phi_1^N (\phi^{N-1})^{-1} \cdots \phi_1^j (\phi^1)^{-1} \phi_1^j (\phi^N)^{-1} = \phi_1^{N+1} (\phi_{12}^{N+1})^{-1} \cdots \phi_{12}^{j} (\phi_{12}^1)^{-1} \phi_{12}^j (\phi_{12}^N)^{-1},$$

$$\phi_2^N (\phi^{N-1})^{-1} \cdots \phi_2^j (\phi^1)^{-1} \phi_2^j (\phi^N)^{-1} = \phi_{12}^{N+1} (\phi_{12}^{N+1})^{-1} \cdots \phi_{12}^{j} (\phi_{12}^1)^{-1} \phi_{12}^j (\phi_{12}^N)^{-1}, \quad (40)$$

then (35) can be solved for any corner variable set as in the commutative case.

For arbitrary $N$ in the non-commutative setting, a point equivalent form of (35) with $p^i = q^i = 1$, $\forall i \in \{1, \ldots, N\}$ was introduced in [24], [25] and it was shown that it serves as the non-commutative lattice-modified Gel’fand-Dikii hierarchy. In the commutative setting, the same system appeared first in [22] and afterwards in [48].

Note that the point transformation $\phi_{m+k,n+l}^i \rightarrow (p^{j+1})^{k} (q^{j+1})^{l} \phi_{m+k,n+l}^{i} \phi_{m+k,n+l}^{k}, k, l \in \mathbb{Z}$ that scales off $p^i, q^i, i = 1, 2, \ldots, N$ in (35), re-written in terms of $x^i, y^i$ by using (33), results to a non-admissible Mobiüs transformation. So, hierarchy (35) together with (40) can be considered as a $2N$-parameter extension of the non-commutative lattice-modified Gel’fand-Dikii hierarchy.

**Example 3.8 (N = 2).** For $N = 2$, we have the Lax matrix $L(\Phi_1, \Phi; P, P^0, \lambda)$ that reads:

$$L(\Phi_1, \Phi; P, P^0, \lambda) = \begin{pmatrix} p^1 & \lambda p^0 \phi_1^1 (\phi^1)^{-1} \\ p^0 \phi_1^1 (\phi^2)^{-1} & p^2 \end{pmatrix}$$

and it serves as a strong Lax matrix for the integrable difference system in vertex variables (34) with $N = 2$, namely

$$q^0 (p^2 \phi_2^2 (\phi^2)^{-1} - p^1 \phi_{12}^2 (\phi_2^2)^{-1}) - p^0 (q^2 \phi_1^2 (\phi^2)^{-1} - q^1 \phi_{12}^1 (\phi_2^2)^{-1}) = 0, \quad (41)$$

$$q^0 (p^1 \phi_2^2 (\phi^1)^{-1} - p^2 \phi_{12}^2 (\phi_2^1)^{-1}) - p^0 (q^1 \phi_1^2 (\phi^1)^{-1} - q^2 \phi_{12}^1 (\phi_2^1)^{-1}) = 0. \quad (42)$$

The centrality assumption (40) that now reads

$$\phi_1^1 (\phi^1)^{-1} \phi_1^1 (\phi^2)^{-1} = 1 = \phi_{12}^1 (\phi_1^1)^{-1} \phi_{12}^2 (\phi_2^1)^{-1}, \quad \phi_2^2 (\phi^1)^{-1} \phi_2^2 (\phi^2)^{-1} = 1 = \phi_{12}^2 (\phi_1^1)^{-1} \phi_{12}^2 (\phi_2^1)^{-1},$$
allow us to re-write the difference system in terms of one potential function only. Indeed, from the centrality assumption we have

\[
\begin{align*}
\phi_2^2 (\phi_1) & = \phi_2^2 (\phi_0^2) = \phi_2 (\phi_2) = \phi_2^2 (\phi_0^2), \\
\phi_0^2 (\phi_1) & = \phi_0^2 (\phi_0^2), \\
\phi_2 (\phi_1) & = \phi_2 (\phi_2), \\
\phi_2^2 (\phi_1) & = \phi_2 (\phi_2), \\
\phi_0^2 (\phi_1) & = \phi_0^2 (\phi_0^2), \\
\phi_2 (\phi_1) & = \phi_2 (\phi_2),
\end{align*}
\]

that allow us to eliminate \(\phi^2, \phi_1^2, \phi_2^2\), from (42) to obtain

\[
q^0 \left( p^1 \phi_2^2 (\phi_1) - p^2 \phi_1 (\phi_0^2) \right) - p^0 \left( q^1 \phi_2^2 (\phi_1) - q^2 \phi_2 (\phi_0^2) \right) = 0, \tag{43}
\]

where for simplicity we have denoted \(\phi := \phi^1, \phi_1 := \phi_0^1\), etc. When \(p^1 = p^2 = 1 = q^1 = q^2\) and in the commutative setting, equation (43) coincides with the lattice potential modified KdV equation \[\text{[29]}\] or up to a point transformation to \(H_3^0\) in \[\text{[56]}\]. An one-parameter extension of the lattice potential modified KdV equation (again in the commutative setting) was introduced in \[\text{[55]}\], whereas a two-parameter extension of the same equation was introduced in \[\text{[57]}\].

The non-commutative version of the lattice potential modified KdV equation was introduced in \[\text{[29]}\] and it includes two parameters. The extended non-commutative lattice potential modified KdV equation that was introduced in this Section includes six parameters. So (43) can be considered as a non-commutative version of the 4-parameter extension of the lattice potential modified KdV equation.

**Example 3.9 (\(N \geq 3\)).** For \(N = 3\), we have the Lax matrix \(L(\Phi_1, \Phi; P, p^0, \lambda)\) that reads:

\[
L(\Phi_1, \Phi; P, p^0, \lambda) = \begin{pmatrix}
p^1 & 0 & \lambda p^0 \phi_1^2 (\phi^2)^{-1} \\
p^0 \phi_1^2 (\phi^2)^{-1} & p^2 & 0 \\
0 & p^0 \phi_1^2 (\phi^2)^{-1} & p^3
\end{pmatrix}
\]

and it serves as a strong Lax matrix for the integrable difference system in vertex variables \[\text{[34]}\] with \(N = 3\). Explicitly it reads

\[
\begin{align*}
q^0 \left( p^1 \phi_2^3 (\phi^2)^{-1} - p^1 \phi_2^3 (\phi_2) \right) - p^0 \left( q^1 \phi_1^3 (\phi^2)^{-1} - q^1 \phi_1^3 (\phi_2) \right) &= 0, \tag{44} \\
q^0 \left( p^2 \phi_2^3 (\phi^2)^{-1} - p^2 \phi_2^3 (\phi_2) \right) - p^0 \left( q^2 \phi_1^3 (\phi^2)^{-1} - q^2 \phi_1^3 (\phi_2) \right) &= 0, \tag{45} \\
q^0 \left( p^3 \phi_2^3 (\phi^2)^{-1} - p^3 \phi_2^3 (\phi_2) \right) - p^0 \left( q^3 \phi_1^3 (\phi^2)^{-1} - q^3 \phi_1^3 (\phi_2) \right) &= 0. \tag{46}
\end{align*}
\]

From the centrality assumption \[\text{[40]}\] with \(N = 3\) we have

\[
\phi_2^3 (\phi^2)^{-1} = \phi_2^3 (\phi_2)^{-1}, \quad \phi_2^3 (\phi_2)^{-1} = \phi_2^3 (\phi_2)^{-1}, \\
\phi_2 (\phi_2)^{-1} = \phi_2 (\phi_2)^{-1}, \quad \phi_2 (\phi_2)^{-1} = \phi_2 (\phi_2)^{-1},
\]

that allow us to eliminate \(\phi_2^3, \phi_2^3, \phi_2^3\), from (46) that together with (45) we obtain

\[
\begin{align*}
q^0 \left( p^1 \phi_2^3 (\phi^2)^{-1} - p^1 \phi_2^3 (\phi_2) \right) - p^0 \left( q^1 \phi_2^3 (\phi^2)^{-1} - q^1 \phi_2^3 (\phi_2) \right) &= 0, \\
q^0 \left( p^2 \phi_2^3 (\phi^2)^{-1} - p^2 \phi_2^3 (\phi_2) \right) - p^0 \left( q^2 \phi_2^3 (\phi^2)^{-1} - q^2 \phi_2^3 (\phi_2) \right) &= 0, \\
q^0 \left( p^3 \phi_2^3 (\phi^2)^{-1} - p^3 \phi_2^3 (\phi_2) \right) - p^0 \left( q^3 \phi_2^3 (\phi^2)^{-1} - q^3 \phi_2^3 (\phi_2) \right) &= 0.
\end{align*}
\]
When \( p^1 = p^2 = p^3 = 1 = q^1 = q^2 = q^3 \) and in the commutative setting, equation (47) serves as the lattice-modified Boussinesq equation \([21, 12, 22]\), that in the non-commutative setting it was introduced in \([24]\). So (47) serves as a non-commutative version of the 6–parameter extension of the lattice-modified Boussinesq equation.

In order to recover exactly the form of the lattice-modified Boussinesq equation introduced in \([12, 22]\) but in the non-commutative setting, instead of the centrality assumption (40) with \( N = 3 \), one should consider (mimicking the commutative setting) the restricted assumption

\[
\phi_3 \phi_2 \phi_1 = 1
\]

and perform the re-potentialization

\[
(\phi^3, \phi^2, \phi^1) \mapsto ((\phi^2)^{-1}, \phi^2(\phi^1)^{-1}, \phi^1).
\]

(48)

Then, (44)-(46) together with (48) lead exactly to the desired form.

For arbitrary \( N \), the hierarchy of difference systems in vertex variables \([23]\) is a non-commutative integrable hierarchy that when the centrality assumptions (40) are imposed, it serves as the non-commutative version of a \( 2N \)–parameter extension of the lattice-modified Gel’fand-Dikii hierarchy. In the commutative setting the lattice-modified Gel’fand-Dikii hierarchy was introduced implicitly in \([21]\) (the first two members of the hierarchy were presented explicitly), the whole hierarchy was explicitly presented in \([12, 22]\). Moreover it includes two parameters. Note that the non-commutative version of the lattice-modified Gel’fand-Dikii hierarchy that was introduced in \([24]\) includes two parameters as well. The extended non-commutative lattice-modified Gel’fand-Dikii hierarchy that was introduced in this Section includes \( 2N + 2 \) parameters. Furthermore, if instead of the centrality assumptions (40) we consider \( \phi_N \cdots \phi_2 \phi_1 = 1 \) and perform the re-potentialization

\[
(\phi^N, \phi^{N-1}, \ldots, \phi^2, \phi^1) \mapsto ((\phi^{N-1})^{-1}, \phi^{N-1}(\phi^{N-2})^{-1}, \ldots, \phi^2(\phi^1)^{-1}, \phi^1),
\]

(49)

then form (34) together with (49), we obtain a form of the non-commutative lattice-modified Gel’fand-Dikii hierarchy that for \( p^i = q^i = 1, \forall i \in \{1, \ldots, N\} \), coincides with the form introduced in \([24]\).

3.2.2. Additive potentials and the lattice-Schwarzian Gel’fand-Dikii hierarchy. The hierarchy of difference systems (15) in its polynomial form constitutes of the sets of equations (12), (13).

Equations (13) are identically satisfied if we set

\[
x^i = p^i\chi^i - p^{i+1}\chi^i \quad y^i = q^i\chi^i - q^{i+1}\chi^i, \quad i = 1, \ldots, N.
\]

(50)

The functions \( \chi^i, i = 1, \ldots, N \) can be considered as potential functions. In terms of these potential functions, equations (13) are identically satisfied, while equations (12) read

\[
(p^i\chi_{12} - p^{i+1}\chi_2) (q^{i-1}\chi_{2} - q^i\chi_{i-1}) - (q^i\chi_{12} - q^{i+1}\chi_1) (p^{i-1}\chi_{1} - p^i\chi_{i-1}) = 0,
\]

(51)

\( i = 1, \ldots, N \) and constitute a hierarchy of difference systems in vertex variables.

**Proposition 3.10.** For the hierarchy of difference systems in vertex variables (51) it holds:
arises as the compatibility conditions of the Lax equation \( \mathcal{L} \), associated with the strong Lax matrix
\[
L(\Phi_1, \Phi; P, \lambda) = \begin{pmatrix}
p^1 & 0 & \cdots & 0 & \lambda (p^N \chi_1^N - p^1 \chi^N) \\
p^1 \chi_1^1 - p^2 \chi^1 & p^2 & 0 & \cdots & 0 \\
0 & p^2 \chi_1^2 - p^3 \chi^2 & \ddots & \vdots & \vdots \\
0 & \vdots & \ddots & p^{N-1} \chi_1^{N-1} - p^N \chi^{N-1} & 0 \\
0 & 0 & \cdots & p^N - \chi_1^N - p^N \chi^{N-1} & p^N
\end{pmatrix}
\]

(2) it is multidimensional consistent;
(3) it is invariant under the following permutations of the dependent variables
\[
\tau: (\phi^1, \phi^1_1, \phi^1_2, \phi^2, p^0, q^0, p^j, q^j) \mapsto (\phi^1, \phi^1_1, \phi^1_2, \phi^2, p^0, q^0, p^j, q^j),
\]
\[
\sigma: (\phi^1, \phi^1_1, \phi^1_2, \phi^2, p^0, q^0, p^j, q^j) \mapsto (\phi^2, \phi^1_2, \phi^1_1, \phi^1, p^0, q^0, p^j, q^j),
\]
\[
\forall j \in \{1, \ldots, N\},
\]
i.e. it respects the rhombic symmetry;

(4) it is an integrable hierarchy of difference systems in vertex variables.

**Proof.** The proof follows similarly to the proof of Proposition 3.7. 

In the non-commutative setting, \( 51 \) can be solved only for the corner variable sets \( \{\chi^1, \ldots, \chi^N\} \) and \( \{\chi^1_{12}, \ldots, \chi^N_{12}\} \). If we impose the centrality assumptions \( 19 \), that in terms of the potential functions \( 50 \) take the form
\[
(\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) \cdots (\chi^1 \chi^N_{12} - p^3 \chi^N_{12}) (\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) = (p^0)^N,
\]
\[
(\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) \cdots (\chi^1 \chi^N_{12} - p^3 \chi^N_{12}) (\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) = (p^0)^N,
\]
\[
(\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) \cdots (\chi^1 \chi^N_{12} - p^3 \chi^N_{12}) (\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) = (q^0)^N,
\]
\[
(\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) \cdots (\chi^1 \chi^N_{12} - p^3 \chi^N_{12}) (\chi^1 \chi^N_{12} - p^1 \chi^N_{12}) = (p^0)^N,
\]
then \( 51 \) can be solved for any corner variable set.

The point transformation \( \chi_{m+k,n+l} \mapsto \left( \frac{p_{-1}-1}{p_1} \right)^k \left( \frac{q_{-1}}{q_1} \right)^l \chi_{m+k,n+l} \), \( k, l \in \mathbb{Z} \) that scales off \( p^i, q^j \), \( i = 1, 2, \ldots, N \) in \( 51 \), re-written in terms of \( x^i, y^i \) by using \( 50 \), results to a non-admissible M"{o}bi"{u}s transformation. So, for arbitrary \( N \) the system of equations \( 51 \) together with the centrality assumptions \( 52 \), can be considered as a \( 2N \)-parameter extension of non-commutative lattice Schwarzian Gel’fand-Dikii hierarchy and that is apparent in the examples that follow. In the commutative case the lattice-Schwarzian Gel’fand-Dikii hierarchy was introduced in \( 12 \).

**Example 3.11 (N = 2).** For \( N = 2 \), we have the Lax matrix
\[
L(\Phi_1, \Phi; P, \lambda) = \begin{pmatrix}
p^1 & \lambda (p^2 \chi^2 - p^1 \chi^1) \\
p^1 \chi^1 - p^2 \chi^1 & p^2
\end{pmatrix}
\]
and it serves as a strong Lax matrix for the integrable difference system in vertex variables \( 57 \) with \( N = 2 \), namely
\[
(p^1 \chi_{12} - p^2 \chi^1_2) (q^2 \chi^1_2 - q^1 \chi^1_2) - (q^1 \chi_{12} - q^2 \chi^1_2) (p^2 \chi^1_2 - p^1 \chi^1_2) = 0,
\]
\[
(p^2 \chi_{12} - p^1 \chi^2_2) (q^1 \chi^2_2 - q^2 \chi^1_2) - (q^2 \chi_{12} - q^1 \chi^2_2) (p^1 \chi^1_2 - p^2 \chi^1_2) = 0.
\]
The centrality assumption (52) that now reads
\[
(p^2 \chi_1^2 - p^1 \chi^2) (p^1 \chi_1^1 - p^2 \chi^1) = (p^0)^2 = (p^2 \chi_1^2 - p^1 \chi^2) (p^1 \chi_1^1 - p^2 \chi^1),
\]
\[
(q^2 \chi_2^2 - q^1 \chi^2) (q^1 \chi_2^1 - q^2 \chi^1) = (q^0)^2 = (q^2 \chi_2^2 - q^1 \chi^2) (q^1 \chi_2^1 - q^2 \chi^1),
\]
allow us to re-write the difference system in terms of one potential function only. Indeed, from (53) using (55) we eliminate \( \chi^2 \) and its shifts to obtain:
\[
(q^0)^2 (p^1 \chi_1 - p^2 \chi_2) (q^1 \chi_1 - q^2 \chi) - 1 = (p^0)^2 (q^1 \chi_1 - q^2 \chi_1) (p^1 \chi_1 - p^2 \chi) - 1,
\]
where for simplicity we have denoted \( \chi := \chi^1, \chi_1 := \chi_1^1, \) etc. When \( p^1 = p^2 = 1 = q^1 = q^2 \) and in the commutative setting, equation (56) coincides with the lattice-Schwarzian KdV equation (58), that is also known as the cross-ratio equation or Q10 in (58). For arbitrary \( q^i, p^i, i = 1, 2 \), equation (58) was introduced in (59). The non-commutative version of the lattice-Schwarzian KdV equation was introduced in (29) and it includes two parameters. The extended non-commutative lattice-Schwarzian KdV equation that was introduced in this Section includes six parameters. So (58) serves as the non-commutative version of a 4–parameter extension of the lattice-Schwarzian KdV equation.

Example 3.12 \((N \geq 3)\). For \( N = 3 \), we have the Lax matrix
\[
L(\Phi_1, \Phi; P, \lambda) = \begin{pmatrix}
  p^1 & 0 & \lambda (p^3 \chi_3 - p^1 \chi^3) \\
  p^2 & 0 & p^3 \\
  0 & p^3 & p^2 \chi_2 - p^3 \chi^2
\end{pmatrix}
\]
and it serves as a strong Lax matrix for the integrable difference system in vertex variables (61) with \( N = 3 \), namely
\[
(p^1 \chi_1 - p^2 \chi_2) (q^3 \chi_2^3 - q^1 \chi^3) - (q^1 \chi_1 - q^2 \chi_1^1) (p^3 \chi_3 - p^1 \chi^3) = 0,
\]
\[
(p^2 \chi_2 - p^3 \chi_2^3) (q^1 \chi_1 - q^2 \chi) - (q^2 \chi - q^3 \chi_2) (p^1 \chi_1 - p^2 \chi) = 0,
\]
\[
(p^3 \chi_3 - p^1 \chi^3) (q^2 \chi_2^3 - q^3 \chi_2) = 0.
\]
The centrality assumption (52) that now reads
\[
(p^3 \chi_3 - p^1 \chi^3) (p^2 \chi_1^2 - p^3 \chi^2) (p^1 \chi_1^1 - p^2 \chi^1) = (p^0)^3,
\]
\[
(p^3 \chi_2^3 - p^1 \chi^3) (p^2 \chi_1^2 - p^3 \chi^2) (p^1 \chi_1^1 - p^2 \chi^1) = (p^0)^3,
\]
\[
(q^3 \chi_3 - q^1 \chi^3) (q^2 \chi_2^3 - q^3 \chi_2) (q^1 \chi_1^1 - q^2 \chi^1) = (q^0)^3,
\]
allow us to re-write the difference system in terms of potential functions \( \chi^1 \) and \( \chi^2 \) only. Indeed, by using (60), we eliminate \( \chi^3 \) and its shifts from (57) and (59) to obtain:
\[
(q^0)^3 (p^1 \chi_1 - p^2 \chi_2) (q^1 \chi_1^1 - q^2 \chi^1) - 1 (q^2 \chi_2 - q^3 \chi^2) - 1
\]
\[
= (p^0)^3 (q^1 \chi_1 - q^2 \chi_1^1) (p^1 \chi_1 - p^2 \chi_1^1) - 1 (p^2 \chi_1^1 - p^3 \chi_1^1) - 1,
\]
\[
(p^0)^3 (p^1 \chi_1 - p^2 \chi_2) - 1 (p^2 \chi_1^1 - p^3 \chi^1) - 1 (q^2 \chi_2 - q^3 \chi^2)
\]
\[
= (q^0)^3 (q^1 \chi_1 - q^2 \chi_1^1)^{-1} (q^2 \chi_1^1 - q^3 \chi^1)^{-1} (p^1 \chi_1^1 - p^2 \chi_1^1)^{-1} (p^2 \chi_1^1 - p^3 \chi_1^1)^{-1} (p^2 \chi_1^1 - p^3 \chi^1)^{-1} (q^2 \chi_2 - q^3 \chi^2).
\]
When \( p^1 = p^2 = 1 = q^1 = q^2 \) and in the commutative setting, equation (61) coincides with the lattice-Schwarzian Boussinesq equation [23]. So (61) stands for the non-commutative 4-parameter extension of the lattice-Schwarzian Boussinesq equation.

For arbitrary \( N \), the hierarchy of difference systems in vertex variables (51) is a non-commutative integrable hierarchy that when the centrality assumptions (52) are imposed, it can be considered as the non-commutative version of a \( 2N \)-parameter extension of the lattice-Schwarzian Gel’fand-Dikii hierarchy. In the commutative setting the lattice-Schwarzian Gel’fand-Dikii hierarchy was introduced in [12]. Moreover it includes two parameters. The extended non-commutative lattice-Schwarzian Gel’fand-Dikii hierarchy that was introduced in this Section includes \( 2N + 2 \) parameters.

4. The Lax matrix \( L^{N,2} \) and integrable hierarchies of difference systems

For \( k = 2, N \geq 3 \) we have the the Lax matrix

\[
L^{N,2}(X; P, \lambda) := P + \nabla^1 X^{(1)} + \nabla^2 X^{(2)} + \lambda \left( \Delta^1 X^{(1)} + \Delta^2 X^{(2)} \right),
\]

that explicitly reads:

\[
L^{N,2}(X; P, \lambda) = \begin{pmatrix}
p^1 & 0 & \cdots & 0 & \lambda x^{2,N-1} & \lambda x^{1,N} \\
x^{1,1} & p^2 & 0 & \cdots & 0 & \lambda x^{2,N} \\
x^{2,1} & x^{1,2} & \cdots & \cdots & 0 & \\
0 & \ddots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & x^{2,N-2} & x^{1,N-1} & p^{N-1} & 0 & \cdots & p^N
\end{pmatrix},
\]

(62)

The compatibility conditions (8)-(10) read:

\[
x_2^{2,i} y_1^{2,i+1} = y_1^{2,i} x_2^{2,i+1},
\]
\[
q^i x_2^{2,i} - q^{i-1} x_2^{2,i} + x_2^{1,i+1} y_1^{1,i} = p^i y_1^{1,i} - p^{i-1} y_1^{2,i} + y_1^{1,i+1} x_2^{1,i},
\]
\[
x_2^{2,i} y_1^{1,i-1} + x_2^{1,i+1} y_2^{2,i-1} = y_1^{2,i} x_2^{1,i-1} + y_1^{1,i+1} x_2^{2,i-1},
\]
\[
q^i x_2^{1,i} - q^{i+1} x_2^{1,i} = p^i y_1^{1,i} - p^{i+1} y_1^{1,i},
\]

(63)-(66)

where the superscript \( i = 1, 2, \ldots, N \), is considered modulo \( N \).

4.1. An integrable hierarchy of difference systems in edge variables

**Proposition 4.1.** For the hierarchy of difference systems in edge variables (63)-(66), it holds:

1. matrix (62), serves as its strong Lax matrix;
2. it is birational;

**Proof.** (1) The compatibility conditions (8)-(10) for the Lax matrix (62) are exactly the hierarchy (63)-(66).
(2) In matrix form the hierarchy of difference systems (63)-(66) reads:

\[
\begin{pmatrix}
Q & \mathbf{Y}^{(1)} & Y^{(2)} & 0_N \\
0_N & \mathbf{Q} & Y^{(1)} & Y^{(2)} \\
-P & -\mathbf{X}^{(1)} & -X^{(2)} & 0_N \\
0_N & -\mathbf{P} & -X^{(1)} & -X^{(2)}
\end{pmatrix}
= \begin{pmatrix} b^1 \\ b^2 \\ 0 \\ 0 \end{pmatrix},
\]

where \(0_N\) stands for the order \(N\) zero matrix, while \(0\) stands for the \(N\)-component zero row vector. In the formulae above participate as well the following \(N\)-component vectors:

\[
x_2 := (x_2^{1,1}, x_2^{1,2}, \ldots, x_2^{1,N}),\quad x_2 := (x_2^{2,1}, x_2^{2,2}, \ldots, x_2^{2,N-1}),\quad y_1 := (y_1^{1,1}, y_1^{1,2}, \ldots, y_1^{1,N})\quad \text{and}\quad y_2 := (y_1^{2,1}, y_1^{2,2}, \ldots, y_1^{2,N-1}).
\]

We also have the diagonal order \(N\) matrices:

\[
Q := \text{diag}(q^1, q^2, \ldots, q^N),\quad \mathbf{Q} := \text{diag}(q^N, q^1, \ldots, q^{N-1}),
\]

\[
X^{(j)} := \text{diag}(x^{j,1}, x^{j,2}, \ldots, x^{j,N}),\quad X^{(j)} := \text{diag}(x^{j,N}, x^{j,1}, \ldots, x^{j,N-1}),
\]

\(j = 1, 2\) and similarly are defined the diagonal matrices \(P, Y^{(j)}, \mathbf{Y}^{(j)}\) and \(\mathbf{X}^{(j)}\). Finally, the \(N\)-component vectors:

\[
b^1 := (q^2 x^{1,1} - p^2 y^{1,1}, q^3 x^{1,2} - p^3 y^{1,2}, \ldots, q^N x^{1,N} - p^N y^{1,N}),
\]

\[
b^2 := (q^1 x^{2,1} - p^1 y^{2,1}, q^2 x^{2,2} - p^2 y^{2,2}, \ldots, q^{N} x^{2,N-1} - p^{N} y^{2,N-1}),
\]

also participate in (67).

The Schur block matrix inversion formula, states that for the block matrix

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

its inverse matrix \(M^{-1}\) reads

\[
M^{-1} = \begin{pmatrix} (M/D)^{-1} & (M/B)^{-1} \\ (M/C)^{-1} & (M/A)^{-1} \end{pmatrix}, \quad M/A := D - CA^{-1}B, \quad M/B := C - DB^{-1}A,
\]

\[
M/C := B - AC^{-1}D, \quad M/D := A - BD^{-1}C.
\]

By using this inversion formula in (67), we obtain (63)-(66) in solved form. Working similar, we can obtain the inverse solved form of (63)-(66) and that proves bi-rationality of the latter.

\[ \square \]

The following remarks are in order. First, by setting

\[
x^{2,i} = y^{2,i} = x^{2,i}_1 = y^{2,i}_1 = 0, \quad \forall i \in \{1, \ldots, N\},
\]

equations (63), (65) of the hierarchy of difference systems in Proposition 4.1 vanish, while equations (64), (66) respectively read

\[
x^{2,i+1} y^{1,i} = y^{1,i+1} x^{1,i}, \quad q^{i} x^{1,i} - q^{i+1} x^{1,i} = p^{i} y^{1,i} - p^{i+1} y^{1,i},
\]

which coincide respectively with (13) and (12). So we have obtained a reduction from the hierarchy of difference systems (63)-(66) to the hierarchy of difference systems (12), (13). Similarly by setting

\[
x^{1,i} = y^{1,i} = x^{1,i}_1 = y^{1,i}_1 = 0, \quad \forall i \in \{1, \ldots, N\},
\]
respectively read:

\[ x_2^{2,i} y^{2,i+1} = y_1^{2,i} x^{2,i+1}, \quad q^i x_2^{2,i} - q^{i-1} x^{2,i} = p^i y_1^{2,i} - p^{i-1} y^{2,i}, \]  

(69)

\( i = 1, \ldots, N. \) Note that (69) is mapped to (68) via \((x_2^{2,i}, y_1^{2,i}, x^{2,i}, y^{2,i}) \mapsto (x_2^{1,i}, y_1^{1,i}, x_1^{1,i-2}, y_1^{1,i-2})\). A second remark is that for the hierarchy of difference systems (63)–(66) that is introduced in Proposition 4.1, we have \( i = 1, \ldots, N \geq 3. \) Nevertheless, this system makes perfectly sense even for \( N = 2 \) and although we do not have yet a Lax matrix for this case, we anticipate integrability.

Finally, in the commutative setting it holds \( \prod_{i=1}^{N} x_2^{2,i} = \prod_{i=1}^{N} x^{2,i} \) and \( \prod_{i=1}^{N} y_1^{2,i} = \prod_{i=1}^{N} y^{2,i} \) that leads to

\[ \prod_{i=1}^{N} x_2^{2,i} = (p^0)^N = \prod_{i=1}^{N} x^{2,i}, \quad \prod_{i=1}^{N} y_1^{2,i} = (q^0)^N = \prod_{i=1}^{N} y^{2,i}, \]  

(70)

where \( p^0 \), respectively \( q^0 \), are considered functions of \( m \), respectively \( n \), only. Nevertheless, considering (70) together with (63)–(64), quadrirationality is not assured as it was the case for the hierarchy of difference systems (12), (13). Further restrictions on the dependent variables are required for quadrirationality to be achieved.

### 4.2. An integrable hierarchy of difference systems in vertex variables

The hierarchy of difference systems of Proposition 4.1 constitutes of the sets of equations (63)–(66). The sets of equations (63) and (66) are identically satisfied if we respectively set

\[ x_2^{2,i} = p^0 \phi_2^i (\phi^{i+1})^{-1}, \quad y_1^{2,i} = q^0 \phi_2^i (\phi^{i+1})^{-1}, \]

\[ x_1^{i,i} = p^0 \chi_1^i - p^{i+1} \chi_1^i, \quad y_1^{i,i} = q^0 \chi_2^i - q^{i+1} \chi_2^i, \]  

(71)

The functions \( \phi^i, \chi^i, i = 1, \ldots, N \) can be considered as potential functions. In terms of these potential functions equations (63) and (66) are identically satisfied, while equations (64) and (65) respectively read:

\[ p^0 \left( q^i \phi_{12}^i (\phi_2^{i+1})^{-1} - q^{i-1} \phi_2^i (\phi^{i+1})^{-1} \right) - q^0 \left( p^i \phi_{12}^i (\phi_1^{i+1})^{-1} - p^{i-1} \phi_2^i (\phi^{i+1})^{-1} \right) = (q^{i+1} \chi_{12}^i - q^{i+2} \chi_1^{i+1}) \left( p^i \chi_1^i - p^{i+1} \chi_1^i \right) - (p^{i+1} \chi_{12}^i - p^{i+2} \chi_2^{i+1}) \left( q^i \chi_2^i - q^{i+1} \chi_2^i \right), \]

\[ p^0 \phi_{12}^i (\phi_1^{i+1})^{-1} \left( q^{i-1} \chi_2^i - q^{i+1} \chi_2^{i+1} \right) + q^0 \left( p^{i+1} \chi_{12}^i - p^{i+2} \chi_2^{i+1} \right) \phi_2^i (\phi^{i+1})^{-1} = q^0 \phi_{12}^i (\phi_1^{i+1})^{-1} \left( p^{i-1} \chi_1^i - p^{i+1} \chi_1^{i-1} \right) + q^0 \left( q^{i+1} \chi_{12}^i - q^{i+2} \chi_1^{i+1} \right) \phi_1^i (\phi^{i+1})^{-1}, \]  

(72), (73)

\( i = 1, \ldots, N \) and constitute a hierarchy of difference systems in vertex variables.

**Proposition 4.2.** For the hierarchy of difference equations in vertex variables (72), (73) it holds
(1) arises as the compatibility conditions of the Lax equation (7), associated with the strong Lax matrix

\[ L = \begin{pmatrix}
  p^1 & 0 & \cdots & 0 & \lambda x^{2,N-1} & \lambda x^{1,N} \\
  0 & p^2 & \cdots & 0 & \lambda x^{2,N} & \lambda x^{1,N} \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  x_{2,1} & x_{1,2} & \cdots & \cdots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & \cdots & x_{2,N-2} & x_{1,N-1} & p^N \\
\end{pmatrix}, \]

where \( x^{2,i} := p^0 \phi_1^{\prime}(\phi^{i+1})^{-1}, y^{2,i} := q^0 \phi_2^{\prime}(\phi^{i+1})^{-1}, x^{1,i} := p^i \chi_1^{\prime} - p^{i+1} \chi^i, y^{1,i} := q^i \chi_2^{\prime} - q^{i+1} \chi^i, \)

\( i = 1, \ldots, N; \)

(2) it is invariant under the following permutations of the dependent variables

\[ \tau : (\phi^j, \phi_1^j, \phi_2^j, \phi_1^{i,1}, \chi^1, \chi_2^{i,1}, \chi_2^{j,1}, p^0, q^0, p^j, q^j) \mapsto (\phi^j, \phi_2^j, \phi_1^j, \phi_1^{i,1}, \chi^1, \chi_2^{j,1}, \chi_2^{i,1}, q^0, p^0, q^j, p^j), \]

\[ \sigma : (\phi^j, \phi_1^j, \phi_2^j, \phi_1^{i,1}, \chi^1, \chi_2^{i,1}, \chi_2^{j,1}, p^0, q^0, p^j, q^j) \mapsto (\phi_2^{i,1}, \phi_1^{i,1}, \phi_1^j, \phi_1^j, \chi_2^{i,1}, \chi_2^{j,1}, \chi_2^{i,1}, p^0, q^j, p^0, q^j), \]

\( \forall j \in \{1, \ldots, N\}, \)

i.e. it respects the rhombic symmetry;

(3) it is an integrable hierarchy of difference equations in vertex variables.

**Proof.** The proof follows similarly to the proof of Proposition 3.7.  

The following remarks are in order. First, By setting

\[ \chi^i = \chi_1^i = \chi_2^i = \chi_{12} = 0, \quad \forall i \in \{1, \ldots, N\}, \]

the set of equations (73) vanishes while (72) reads:

\[ p^0 \left( q^{i\prime} \phi_{12}^{\prime} (\phi^{i+1})^{-1} - q^{i\prime-1} \phi_1^{\prime} (\phi^{i+1})^{-1} \right) = q^0 \left( p^{i\prime} \phi_2^{\prime} (\phi^{i+1})^{-1} - p^{i\prime-1} \phi_2^{\prime} (\phi^{i+1})^{-1} \right), \]

\( i = 1, \ldots, N, \) that is mapped to (54) via \((\phi^i, \phi_1^{i+1}, \phi_2^{i+1}, \phi_{12}^{i+1}) \mapsto (\phi^i, \phi_1^{i-1}, \phi_2^{i-1}, \phi_{12}^{i-1}), \) \( \forall i \in \{1, \ldots, N\}. \)

Furthermore, if the centrality assumption (40) are imposed, we arrive to the non-commutative lattice-modified Gel’fand-Dikii hierarchy. To recapitulate, we have obtained a reduction of the hierarchy (72), (73) to the non-commutative lattice-modified Gel’fand-Dikii hierarchy. Second, by considering \( p^0 = q^0 = 0 \) to the sets of equations (72), (73), equations (73) vanishes while equations (72) read:

\[ (q^{i\prime+1} \chi_{12}^{i+1} - q^{i\prime+2} \chi_{12}^{i+1}) \left( p^i \chi_1^i - p^{i+1} \chi^i \right) = (p^{i+1} \chi_{12}^{i+1} - p^{i+2} \chi_{12}^{i+1}) \left( q^i \chi_2^i - q^{i+1} \chi^i \right), \]

\( i = 1, \ldots, N \) that coincide with (51). Furthermore, if the centrality assumptions (52) are imposed, we arrive to the non-commutative lattice-Schwarzian Gel’fand-Dikii hierarchy. To recapitulate, we have obtained a reduction of the hierarchy (72), (73) to the non-commutative lattice-Schwarzian Gel’fand-Dikii hierarchy.
5. Conclusions

In this article we have introduced a family of Lax matrices $L^{N,k}$ that participate to the linear problem:

$$\Psi_2 = L^{N,k}(X; P, \lambda) \Psi, \quad \Psi_1 = L^{N,k}(Y; Q, \lambda) \Psi.$$  \hspace{1cm} (74)

For $k = 1, 2$ we derived the corresponding integrable hierarchies of difference systems in non-commuting edge variables and the associated integrable difference hierarchies in vertex variables together with their Lax matrices. The lattice-modified Gel’fand-Dikii hierarchy and the lattice-Schwarzian Gel’fand-Dikii hierarchy, both in non-commuting variables, together with the underlying integrable difference system in edge variables, were obtained for $k = 1$. For $k = 2$, we have obtained a seemingly novel hierarchy of difference systems in edge and vertex variables, that includes both the lattice-modified and the lattice-Schwarzian Gel’fand-Dikii hierarchies, since the latter are obtained by appropriate reductions. This contribution clearly raises many new questions to be addressed. Let us conclude this article by mentioning a few of them.

A first question concerns the continuous limit of the discrete hierarchy (72), (73). We do not know yet the continuous hierarchy of equations that corresponds to (72), (73). Furthermore, the discrete hierarchies obtained from (74) for $k > 2$ await to be studied and their continuous counterparts to be identified. For example, when $k = 3$ the hierarchy of difference systems in edge variables associated with $L^{N,3}$ is given implicitly by equations (8)-(10) (with $k = 3$) and explicitly reads:

$$\begin{align*}
x_2^{3,i} y_2^{3,i+1} &= y_1^{3,i} x_2^{3,i+1}, \\
q^i x_2^{1,i} - q^{i+1} x_1^{1,i} &= p^i y_1^{1,i} - p^{i+1} y_1^{1,i}, \\
q^i x_2^{2,i} - q^{i+2} x_2^{2,i} + x_2^{1,i+1} y_1^{1,i} &= p^i y_1^{2,i} - p^{i+2} y_2^{2,i} + y_1^{1,i+1} x_1^{1,i}, \\
x_2^{3,i} y_1^{1,i+1} + x_2^{2,i+1} y_2^{2,i+2} + x_2^{1,i+2} y_3^{2,i+2} + x_2^{1,i+3} y_3^{2,i+3} &= y_1^{3,i} x_1^{i+3} + y_1^{2,i+1} x_1^{i+3} + y_1^{1,i+2} x_1^{i+3} + y_1^{1,i+3} x_1^{i+3}, \\
x_2^{3,i} y_2^{2,i+2} + x_2^{2,i+1} y_3^{2,i+2} &= y_1^{3,i} x_2^{2,i} + y_1^{2,i+1} x_3^{2,i+2},
\end{align*}$$

(75)
i = 1, \ldots, N. The first two sets of equations of (75) are identically satisfied by setting

$$\begin{align*}
x^{3,i} &= p^0 \phi^i (\phi^{i+1})^{-1}, \\
y^{3,i} &= q^0 \phi^i (\phi^{i+1})^{-1}, \\
x^{1,i} &= p^i \chi^i - p^{i+1} \chi^i, \\
y^{1,i} &= q^i \chi^i - q^{i+1} \chi^i,
\end{align*}$$
i = 1, \ldots, N,$n

and in terms of the potential functions $\phi^i, \chi^i, i = 1, \ldots, N$, the remaining sets of equations of (75) constitute a hierarchy of difference systems in edge and vertex variables.

For the Lax matrix $L^{N,1}$ we have implicitly presented the hierarchy of associated Yang-Baxter maps (see Proposition 3.2). The fist member of this hierarchy of maps is a 4-parametric extension of the $H^{AdS}_{II}$ Yang-Baxter map, extended in the non-commutative domain and it is presented in Example 3.3, whereas the second member of this hierarchy is explicitly presented in Example 3.6. We anticipate to present the explicit form of the whole hierarchy of Yang-Baxter maps elsewhere as well as the hierarchy of entwining maps associated with this hierarchy. Furthermore, an open question is to obtain explicitly the hierarchies of Yang-Baxter maps associated with the Lax matrices $L^{N,k}$ for $k \geq 2$. 

The linear problem (74) falls into a class of more general linear problems, namely

$$\Psi_2 = L^{N,k_1,a}(X;P,\lambda)\Psi, \quad \Psi_1 = M^{N,k_2,b}(Y;Q,\lambda)\Psi,$$

where

$$L^{N,k_1,a}(X;P,\lambda) := \sum_{i=0}^{k_1} \nabla^i X^{(i)} + \lambda \sum_{i=0}^{k_1} \Delta^i X^{(i)},$$

$$M^{N,k_2,b}(Y;Q,\lambda) := \sum_{i=0}^{k_2} \nabla^i Y^{(i)} + \lambda \sum_{i=0}^{k_2} \Delta^i Y^{(i)},$$

with $N \in \mathbb{N}$, $k_1, k_2, a, b \in \{0, 1, \ldots, N - 1\}$, $k_1 \geq a$, $k_2 \geq b$ and $\nabla^0 = \Delta^0 = I_N$, where $I_N$ the order $N$ identity matrix. Also $X^{(a)} \equiv P$ the diagonal order $N$ matrix with entries the parameters $(X^{(a)})_{i,i} := p^i$; $Y^{(a)} \equiv Q$ the diagonal order $N$ matrix with entries the parameters $(Y^{(a)})_{i,i} := q^i$ and $X^{(j)}$, $Y^{(j)}$, $j = 0, 1, \ldots, N - 1$, $j \neq a, b$, the order $N$ diagonal matrices defined on Section 2.1.

Note that (74) arises as the special case of (76), (77), when $k_1 = k_2 = k$, $a = b = 0$. A classification of the linear problems (76), (77), awaits to be addressed.

Finally, in the seminal articles [24, 25], the geometric interpretation of the KP map in terms of Desargues maps is provided. The geometric interpretation of the difference systems in edge variables that correspond to Lax matrices $L^{N,k}$ with $k \geq 2$ and when we assume that $N \in \mathbb{Z}$, is an open question.

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References

[1] I.M. Gel’fand and L.A. Dikii. Fractional powers of operators and Hamiltonian systems. Funct. Anal. Appl., 10(4):259–273, 1976.
[2] Yu.I. Manin. Algebraic aspects of nonlinear differential equations. J. Sov. Math., 11:1–122, 1979.
[3] V.G. Drinfeld and V.V. Sokolov. Equations of Korteweg-de Vries type, and simple Lie algebras. Dokl. Akad. Nauk SSSR, 258(1):1–122, 1981.
[4] A.V. Mikhailov. Integrability of a two-dimensional generalization of the toda chain. Pis’ma Zh. Eksp. Teor. Fiz., 30(7):443–448, 1979.
[5] A.P. Fordy and J. Gibbons. Integrable nonlinear Klein-Gordon equations and Toda lattices. Commun. Math. Phys., 77:21–30, 1980.
[6] M. Antonowicz and A.P. Fordy. Multicomponent Schwarzian KdV hierarchies. Rep. Mod. Phys., 32:223–233, 1993.
[7] V.E. Adler and V.V. Sokolov. Non-Abelian evolution systems with conservation laws. Math. Phys. Anal. Geom., 24(7), 2021.
[8] F.W. Nijhoff and V.G. Papageorgiou. On some integrable discrete-time systems associated with the Bogoyavlensky lattices. Physica A, 228:172–188, 1996.
[9] A. Tongas and F. Nijhoff. The Boussinesq integrable system: compatible lattice and continuum structures. Glasgow Math. J., 47A:205–219, 2004.
[10] K. Maruno and K. Kajiwara. The discrete potential Boussinesq equation and its multisoliton solutions. Appl. Anal., 89:593–609, 2010.
[11] J. Hietarinta. Boussinesq-like multi-component lattice equations and multi-dimensional consistency. J. Phys. A: Math. Theor., 44:165204, 2011.
[12] J. Atkinson. Integrable lattice equations: connection to the M"obius group, B"acklund transformations and solutions. PhD thesis, University of Leeds, 2008. [http://etheses.whiterose.ac.uk/9081/].
[13] C. Scimiterna, M. Hay, and D. Levi. On the integrability of a new lattice equation found by multiple scale analysis. *J. Phys. A: Math. Theor.*, 47:265204, 2014.
[14] A.V. Mikhailov, G. Papamikos, and J.P. Wang. Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere. *Lett. Math. Phys.*, 106:973–996, 2016.
[15] Nolan M. Joshi N., Lobb S. Constructing initial value spaces of lattice equations. [arXiv:1807.06162 [nlin]], 2018.
[16] A.P. Kels. Extended Z-invariance for integrable vector and face models and multi-component integrable quad equations. *J. Stat. Phys.*, 176:1375–1408, 2019.
[17] A.P. Kels. Two-component Yang-Baxter maps associated to integrable quad equations. [arXiv:1910.05562v5 [math-ph]], 2019.
[18] P. Kassotakis, M. Nieszporski, V. Papageorgiou, and A. Tongas. Integrable two-component difference systems of equations. *Proc. R. Soc. A.*, 476:20190668, 2020.
[19] D. Zhang, P.H van der Kamp, and D.-J. Zhang. Multi-component extension of CAC systems. *SIGMA*, 16(060):30pages, 2020.
[20] J. Hietarinta and D.-J. Zhang. Discrete Boussinesq-type equations. [arXiv:2012.00495 [nlin.SI]], 2020.
[21] F.W. Nijhoff, V.G. Papageorgiou, H.W. Capel, and G.R.W. Quispel. The lattice Gel’fand-Dikii hierarchy. *Inverse Problems*, 8(4):597–621, aug 1992.
[22] A. Bobenko and Yu. B. Suris. Integrable noncommutative mappings. *J. Phys. A: Math. Theor.*, 46(20):205202, 2013.
[23] A. Doliwa. Non-commutative lattice-modified Gel’fand-Dikii systems. *J. Phys. A: Math. Theor.*, 49(30):30LT03, 2016.
[24] S. Konstantinou-Rizos and T.E. Kouloukas. A noncommutative discrete potential KdV lift. *J. Math. Phys.*, 60:123503, 2019.
[42] V.M. Bukhshtaber. Yang-Baxter mappings. *Uspekhi Mat. Nauk*, 53:241–242, 1998.
[43] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett. A*, 314:214–221, 2003.
[44] A. Doliwa and P.M. Santini. The symmetric, D-invariant and Egorov reductions of the quadrilateral lattice. *Journal of Geometry and Physics*, 36(1):60–102, 2000.
[45] P. Kassotakis and M. Nieszporski. Families of integrable equations. *SIGMA*, 7(100):14pp, 2011.
[46] P. Kassotakis and M. Nieszporski. On non-multiaffine consistent-around-the-cube lattice equations. *Phys. Lett. A*, 376(45):3135–3140, 2012. [arXiv:1106.0435].
[47] P. Kassotakis and M. Nieszporski. Difference systems in bond and face variables and non-potential versions of discrete integrable systems. *J. Phys. A: Math. Theor.*, 51(38):385203, 2018.
[48] A.P. Fordy and P. Xenitidis. $\mathbb{Z}^N$ graded discrete Lax pairs and integrable difference equations. *J. Phys. A: Math. Theor.*, 50(16):165205, 2017.
[49] M. Nieszporski and P. Kassotakis. Systems of difference equations on a vector valued function that admits 3d vector space of scalar potentials. [arXiv:1908.01706[math], 2019].
[50] V.G Papageorgiou and P.D Xenitidis. Symmetries and integrability of discrete equations defined on a black-white lattice. *J. Phys. A: Math. Theor.*, 42:454025, 2009.
[51] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Discrete nonlinear hyperbolic equations. Classification of integrable cases. *Funct. Anal. Appl.*, 43(1):3–17, 2009.
[52] R. Boll. Classification of 3D consistent quad-equations. *J. Nonlinear Math. Phys.*, 18(3):337–365, 2011.
[53] L. Bianchi. *Lessioni di geometrica differenziale*. Enrico Spoerri, 1894.
[54] R. Hirota. Nonlinear partial difference equations iii; discrete sine-gordon equation. *J. Phys. Soc. Jpn.*, 43:2079–2086, 1977.
[55] F.W. Nijhoff, G.R.W. Quispel, and H.W. Capel. Direct linearization of nonlinear difference-difference equations. *Phys. Lett. A*, 97:125–128, 1983.
[56] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.*, 233(3):513–543, 2003.
[57] T.E. Kouloukas and V.G. Papageorgiou. 3D compatible ternary systems and Yang-Baxter maps. *J. Phys. A: Math. Theor.*, 45(34):345204, 2012.
[58] F. Nijhoff and H. Capel. The discrete Korteweg-de Vries equation. *Acta Applicandae Mathematica*, 39(1):133–158, 1995.
[59] T.E. Kouloukas and D.T. Tran. Poisson structures for lifts and periodic reductions of integrable lattice equations. *J. Phys. A: Math. Theor.*, 48(7):075202, jan 2015.
[60] P Kassotakis. Invariants in separated variables: Yang-Baxter, entwining and transfer maps. *SIGMA*, 15(048):36pp, 2019.

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