FURTHER GENERALIZATIONS OF THE PARALLELOGRAM LAW

ANTONIO M. OLLER-MARCÉN

Abstract. In a recent work of Alessandro Fonda, a generalization of the parallelogram law in any dimension $N \geq 2$ was given by considering the ratio of the quadratic mean of the measures of the $(N-1)$-dimensional diagonals to the quadratic mean of the measures of the faces of a parallelotope. In this paper, we provide a further generalization considering not only $(N-1)$-dimensional diagonals and faces, but the $k$-dimensional ones for every $1 \leq k \leq N-1$.

1. Introduction

If we consider the usual Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, the well-known identity

\begin{equation}
\|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2)
\end{equation}

is called the parallelogram law.

This identity can be extended to higher dimensions in several ways. For example, it is straightforward to see that

\begin{equation}
\|a+b+c\|^2 + \|a+b-c\|^2 + \|a-b+c\|^2 + \|a-b-c\|^2 = 4(\|a\|^2 + \|b\|^2 + \|c\|^2)
\end{equation}

with the subsequent analogous identities arising inductively. There are many works devoted to provide generalizations of (1.1) in many different contexts [1, 3, 4].

Note that if we rewrite (1.1) as

\begin{equation}
\frac{\|a + b\|^2 + \|a - b\|^2}{2} = 2 \left(\|a\|^2 + \|b\|^2 + \|a\|^2 + \|b\|^2\right)
\end{equation}

this means that in any parallelogram, the ratio of the quadratic mean of the lengths of its diagonals to the quadratic mean of the lengths of its sides equals $\sqrt{2}$. With this interpretation in mind, Alessandro Fonda [2] recently proved the following interesting generalization.
Theorem 1.1. Given linearly independent vectors \( a_1, \ldots, a_N \in \mathbb{R}^n \), it holds that
\[
\sum_{i<j} \left( \left\| \left( a_i + a_j \right) \wedge \bigwedge_{k \neq i,j} a_k \right\|^2 + \left\| \left( a_i - a_j \right) \wedge \bigwedge_{k \neq i,j} a_k \right\|^2 \right) = (N-1) \sum_{k=1}^N \left\| a_1 \wedge \cdots \wedge \tilde{a}_k \wedge \cdots \wedge a_N \right\|^2.
\]
In other words, for any \( N \)-dimensional parallelootope, the ratio of the quadratic mean of the \((N-1)\)-dimensional measures of its diagonals to the quadratic mean of the \((N-1)\)-dimensional measures of its faces is equal to \( \sqrt{2} \).

In this work we extend this result to faces of dimension \( k \) for every \( 1 \leq k \leq N-1 \) and to a suitable definition of the \( k \)-dimensional diagonal of a parallelootope. Then Theorem 1.1 will be a particular case of our result for \( k = N-1 \). Indeed, our result can be stated as follows.

Theorem 1.2. Let us consider an \( N \)-dimensional parallelootope and let \( 1 \leq k \leq N-1 \). The ratio of the quadratic mean of the \( k \)-dimensional measures of its \( k \)-dimensional diagonals to the quadratic mean of the \( k \)-dimensional measures of its \( k \)-dimensional faces is equal to \( \sqrt{N-k+1} \).

In fact, our generalization follows in line with the work [3] but instead considers the definition of a diagonal face given in [2].

2. Notation and preliminaries

In this section, we introduce some notation and present some basic facts that will be useful in the sequel. Let us consider a parallelootope \( \mathcal{P} \) generated by a family of linearly independent vectors \( \mathcal{B} = \{ a_1, a_2, \ldots, a_N \} \subseteq \mathbb{R}^n \). This means that
\[
\mathcal{P} = \left\{ \sum_{i=1}^N \alpha_i a_i : \alpha_i \in [0,1] \right\}.
\]

Let us fix \( 1 \leq k \leq N-1 \). Given \( k \) different vectors \( \mathcal{S} = \{ a_{i_1}, \ldots, a_{i_k} \} \subseteq \mathcal{B} \), we can consider the face generated by them:
\[
\mathcal{F}(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v v : \alpha_v \in [0,1] \right\}.
\]
This face can now be translated by one or more of the remaining vectors thus obtaining a face
\[
\mathcal{F}^I(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v a_v + \sum_{w \in \mathcal{B} \setminus \mathcal{S}} \alpha_w w : \alpha_w \in \mathcal{P} \cap \{0,1\} \right\},
\]
where \( I = (\alpha_v)_{v \in S} \in \{0,1\}^{N-k} \). Since each choice of a set \( S \subseteq B \) and a vector \( I \in \{0,1\}^{N-k} \) leads to a different face and every face can be obtained in this way, it follows that \( P \) has exactly \( 2^{N-k} \binom{N}{k} \) \( k \)-dimensional faces. Moreover, it is clear that all the \( 2^{N-k} \) different faces \( \mathcal{F}(S) \) are congruent to the set generated by \( S, \mathcal{F}(S) \).

Now, we focus on the \( k \)-dimensional diagonals which will be defined following the ideas in [2]. Let us consider \( N-k+1 \) different vectors \( T = \{a_1, \ldots, a_{N-k+1}\} \subseteq B \) and let \( T = T_1 \cup T_2 \) be a decomposition of \( T \) into two disjoint sets (either \( T_1 \) or \( T_2 \) could be empty). Then, the following set

\[
\mathcal{D}_T(T_1, T_2) = \left\{ \alpha \sum_{v \in T_1} v + (1 - \alpha) \sum_{v \in T_2} v + \sum_{w \in B \setminus T} \alpha w : \alpha, \alpha w \in [0,1] \right\}
\]

is called the \( k \)-dimensional diagonal associated to \((T, T_1, T_2)\). Clearly each choice of a set \( T \subseteq B \) and a decomposition \( T = T_1 \cup T_2 \) allows us to define a diagonal. Since it is clear that \( \mathcal{D}_T(T_1, T_2) = \mathcal{D}_T(T_2, T_1) \), it readily follows that \( P \) has exactly \( 2^{N-k} \binom{N}{N-k+1} \) different \( k \)-dimensional diagonals. Moreover, if we define the vector

\[
V_T(T_1, T_2) = \sum_{v \in T_1} v - \sum_{v \in T_2} v,
\]

we have that

\[
\mathcal{D}_T(T_1, T_2) = \left\{ \alpha V_T(T_1, T_2) + \sum_{v \in T_2} v + \sum_{w \in B \setminus T} \alpha w : \alpha, \alpha w \in [0,1] \right\}
\]

and consequently, it is clear that the diagonal \( \mathcal{D}_T(T_1, T_2) \) is a translation of the set generated by \( \{V_T(T_1, T_2), w : w \in B \setminus T \} \) and hence it is congruent to it.

**Example.** Let us see how the definition of \( \mathcal{D}(T_1, T_2) \) applies in the case of lower dimensions; i.e., if \( N = 2, 3 \).

In the case \( N = 2 \), we only consider \( k = 1 \). If we consider the parallelogram \( P \) generated by \( B = \{a_1, a_2\} \subseteq \mathbb{R}^N \), then clearly \( T = B \) (because \( k = 1 \)) and \( P \) has two different diagonals which are defined by the two possible decompositions \( T = \{a_1\} \cup \{a_2\} \) and \( T = T \cup \emptyset \). In fact,

\[
\mathcal{D}_B(\{a_1\}, \{a_2\}) = \{ \alpha a_1 + (1 - \alpha) a_2 : \alpha \in [0,1] \}
\]

\[
= a_2 + \{\alpha(a_1 - a_2) : \alpha \in [0,1]\},
\]

\[
\mathcal{D}_B(B, \emptyset) = \{\alpha(a_1 + a_2) : \alpha \in [0,1]\}.
\]

Figure 1 shows how we obtain the two diagonals of the parallelogram. Note that, in this case, \( V_B(B, \emptyset) = a_1 + a_2 \) and \( V_B(\{a_1\}, \{a_2\}) = a_1 - a_2 \).
Now, if \( N = 3 \) and \( k = 1 \), let us consider the parallelepiped \( \mathcal{P} \) generated by \( \mathcal{B} = \{a_1, a_2, a_3\} \subseteq \mathbb{R}^N \). Again, \( \mathcal{T} = \mathcal{B} \) but in this case there are four different 1-dimensional diagonals which are defined by the decompositions \( \mathcal{T} = \{a_1, a_2\} \cup \{a_3\} \), \( \mathcal{T} = \{a_1, a_3\} \cup \{a_2\} \), \( \mathcal{T} = \{a_2, a_3\} \cup \{a_1\} \), and \( \mathcal{T} = \mathcal{T} \cup \emptyset \). In fact,

\[
\begin{align*}
\mathcal{D}_\mathcal{B}(\{a_1, a_2\}, \{a_3\}) &= \{\alpha(a_1 + a_2) + (1 - \alpha)a_3 : \alpha \in [0, 1]\} \\
&= a_3 + \{\alpha(a_1 + a_2 - a_3) : \alpha \in [0, 1]\}, \\
\mathcal{D}_\mathcal{B}(\{a_1, a_3\}, \{a_2\}) &= \{\alpha(a_1 + a_3) + (1 - \alpha)a_2 : \alpha \in [0, 1]\} \\
&= a_2 + \{\alpha(a_1 - a_2 + a_3) : \alpha \in [0, 1]\}, \\
\mathcal{D}_\mathcal{B}(\{a_2, a_3\}, \{a_1\}) &= \{\alpha(a_2 + a_3) + (1 - \alpha)a_1 : \alpha \in [0, 1]\} \\
&= a_1 + \{\alpha(-a_1 + a_2 + a_3) : \alpha \in [0, 1]\}, \\
\mathcal{D}_\mathcal{B}(\mathcal{B}, \emptyset) &= \{\alpha(a_1 + a_2 + a_3) : \alpha \in [0, 1]\}.
\end{align*}
\]

On the left hand side of Figure 2, we can see the above four 1-dimensional diagonals of \( \mathcal{P} \) (in red, purple, green, and blue, respectively). Note that, in this case, \( V_\mathcal{B}(\mathcal{B}, \emptyset) = a_1 + a_2 + a_3 \), \( V_\mathcal{B}(\{a_1, a_2\}, \{a_3\}) = a_1 + a_2 - a_3 \), \( V_\mathcal{B}(\{a_1, a_3\}, \{a_2\}) = a_1 - a_2 + a_3 \), and \( V_\mathcal{B}(\{a_2, a_3\}, \{a_1\}) = -a_1 + a_2 + a_3 \).
In the same way, if \( N = 3 \) and \( k = 2 \), we could define the six 2-dimensional diagonals of \( P \). On the right hand side of Figure 2 we see, for instance, \( D_{\{a_1,a_3\}}(\{a_1\},\{a_3\}) \) in red and \( D_{\{a_2,a_3\}}(\{a_2\},\{a_3\}) \) in blue.

3. PROOF OF THEOREM 1.2

After introducing the notation and the basic objects involved in this work, we are now ready to prove the main result of the paper.

Let \( P \) be a parallelotope generated by \( B = \{a_1, a_2, \ldots, a_N\} \subseteq \mathbb{R}^n \). We first compute the quadratic mean of the \( k \)-dimensional measures of its \( k \)-dimensional faces. We first note that for every \( S = \{a_{i_1}, \ldots, a_{i_k}\} \subseteq B \), the \( k \)-dimensional measure of the face \( F(S) \) is \( \|a_{i_1} \wedge \cdots \wedge a_{i_k}\| \). In the previous section we have seen that \( P \) has exactly \( 2^{N-k} \binom{N}{k} \) \( k \)-dimensional faces and moreover, there are exactly \( 2^{N-k} \) copies of each face \( F(S) \). Consequently, the quadratic mean of the \( k \)-dimensional measures of the \( k \)-dimensional faces of \( P \) is:

\[
\left(3.1\right) \quad \sqrt{\frac{2^{N-k} \sum \|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2}{2^{N-k} \binom{N}{k}}}.
\]

Now we have to compute the quadratic mean of the \( k \)-dimensional measures of the \( k \)-dimensional diagonals of \( P \). First of all, recall that \( P \) has exactly \( 2^{N-k} \binom{N}{N-k+1} \) different \( k \)-dimensional diagonals. Each of them is a translation of the set generated by \( \{V_T(T_1, T_2), w : w \in B \setminus T\} \) for exactly one choice of \( (T, T_1, T_2) \). The \( k \)-dimensional measure of this latter set is \( \|V_T(T_1, T_2) \wedge \bigwedge_{w \in B \setminus T} w\| \). Consequently, the quadratic mean of the \( k \)-dimensional measures of the \( k \)-dimensional diagonals of \( P \) is:

\[
\left(3.2\right) \quad \sqrt{\sum_{T, T_1, T_2} \frac{\|V_T(T_1, T_2) \wedge \bigwedge_{w \in B \setminus T} w\|^2}{2^{N-k} \binom{N}{N-k+1}}}.
\]

Using the bilinearity of the scalar product and taking into account the definition of \( V_T(T_1, T_2) \), it can be easily seen that when we vary \( (T, T_1, T_2) \), we get the term \( \|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2 \) exactly \( 2^{N-k}k \) times for every possible choice of \( \{a_{i_1}, \ldots, a_{i_k}\} \subseteq B \). This implies that the quadratic mean of the \( k \)-dimensional measures of the \( k \)-dimensional diagonals of \( P \) \( \left(3.2\right) \) can be written as:

\[
\left(3.3\right) \quad \sqrt{\frac{2^{N-k}k \sum \|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2}{2^{N-k} \binom{N}{N-k+1}}}.
\]
Finally to obtain Theorem 1.2, it is enough to divide (3.3) by (3.1) to get

\[ \sqrt{\frac{k \binom{N}{k}}{\binom{N}{N-k+1}}} = \sqrt{N-k+1}. \]

References

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Centro Universitario de la Defensa de Zaragoza
Ctra. Huesca s/n, 50090 Zaragoza, Spain
E-mail address: oller@unizar.es