Abstract. Penrose’s two-spinor notation for 4-dimensional Lorentzian manifolds is extended to two-component notation for quaternionic manifolds, which is a useful tool for calculation. We can construct a family of quaternionic complexes over unimodular quaternionic manifolds only by elementary calculation. On complex quaternionic manifolds as complexification of quaternionic Kähler manifolds, the existence of these complexes was established by Baston by using twistor transformations and spectral sequences. Unimodular quaternionic manifolds constitute a large nice class of quaternionic manifolds: there exists a very special curvature decomposition; the conformal change of a unimodular quaternionic structure is still unimodular quaternionic; the complexes over such manifolds are conformally invariant. This class of manifolds is the real version of torsion-free QCFs introduced by Bailey and Eastwood. These complexes are elliptic. We also obtain a Weitzenböck formula to establish vanishing of the cohomology groups of these complexes for quaternionic Kähler manifolds with negative scalar curvatures.

1. Introduction

Quaternionic manifolds are important in supersymmetric theory in physics, in particular in nonlinear sigma models. It is well known that the supersymmetric sigma models are deeply related to geometries with complex structures: the target manifold must be a Kähler manifold in $N = 1$ theories; it must be a hyperKähler manifold in rigid $N = 2$ supersymmetric theories; while in local supersymmetric $N = 2$ theories, it must be a quaternionic Kähler manifold of negative curvature, etc. The quaternionic complexes over quaternionic Kähler manifolds were used to investigate $N = 2$ supersymmetric black holes recently [16]. The Baston operator in the these complexes appears in quantization of $N = 2$ supergravity black holes [27]. Physicists are also interested in supersymmetric and superconformal theory over more general quaternionic manifolds [8]. The geometry of quaternionic manifolds is an active direction of research in last four decades (cf. e.g. [6] [14] [20] [24] [32] [33] [38] [39] and references therein).

Recall that an almost quaternionic structure on a manifold $M$ is a rank-3 subbundle of $\text{End}TM$ which is locally spanned by three almost complex structures on $T M$ satisfying the commuting relation of quaternions, i.e. the frame bundle of $M$ reduces to a principal bundle $P$ with structure group $\text{GL}(n, \mathbb{H})\text{Sp}(1) \cong \text{GL}(n, \mathbb{H}) \times \mathbb{Z}_2 \times \text{Sp}(1)$, where $\text{Sp}(1)$ is the Lie group of right multiplying unit quaternions. It is a quaternionic manifold if there exists a torsion-free connection on $P$. It is called quaternionic Kähler if the Levi-Civita connection for the metric preserves the quaternionic structure, i.e. the frame bundle of $M$ reduces to a principal $\text{Sp}(n)\text{Sp}(1)$-bundle with a torsion-free connection. A quaternionic manifold $M$ is called unimodular if the quaternionic connection preserves a volume form on $M$, i.e. the frame bundle of $M$ reduces to a principal $\text{SL}(n, \mathbb{H})\text{Sp}(1)$-bundle with a torsion-free connection.

Given a representation $W$ of $\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)$ (a double covering of $\text{GL}(n, \mathbb{H})\text{Sp}(1)$), choose a lift of $P$ to a principal $\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)$-bundle $\hat{P}$. Then we can define the associated bundle $\hat{P} \times_{\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)} W$. 

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Such a lifting always exists locally, and exists globally when the obstruction to the lifting in \( H^2(M, \mathbb{Z}_2) \) vanishes, e.g. when it is an \( 8n \)-dimensional quaternionic Kähler manifold, (cf. section 2 in [32] and section 2 in [33]). In the sequel, we assume that such a lifting always exists. Taking the standard GL(\( n, \mathbb{H} \))-module \( \mathbb{C}^{2n} \) and Sp(1)-module \( \mathbb{C}^2 \), we have associated vector bundles

\[
E := \tilde{P} \times_{\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)} \mathbb{C}^{2n}, \quad H := \tilde{P} \times_{\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)} \mathbb{C}^{2}, \quad TM \cong \tilde{P} \times_{\text{GL}(n, \mathbb{H}) \times \text{Sp}(1)} \mathbb{H}^n.
\]

A connection on the principal bundle \( P \) is trivially lifted to the principal bundle \( \tilde{P} \), and so induces connections on associated vector bundle \( E \), \( H \) and \( TM \), respectively. It is well known [32] [33] that the complexified tangent bundle of an almost quaternionic manifold \( M \) decomposes as the tensor product

\[
CTM \cong E \otimes H.
\]

Denote by \( \Gamma(V) \) the space of smooth sections of a vector bundle \( V \) over \( M \). In this paper we will discuss the construction of a family of elliptic differential complexes over a unimodular quaternionic manifold \( M \) (\( M \) is right conformally flat when \( \dim \mathbb{C} = 4 \)).

\[
0 \longrightarrow \Gamma (\otimes^kH^*) \xrightarrow{D^{(k)}_0} \Gamma (\Lambda^1E^* \otimes \otimes^kH^*) \xrightarrow{D^{(k)}_1} \cdots \cdots \longrightarrow \Gamma (\Lambda^kE^*) \xrightarrow{D^{(k)}_k} \Gamma (\Lambda^{k+2}E^*) \longrightarrow 0
\]

where \( \Lambda^qE^* \) is the \( q \)-th exterior product of \( E^* \), and \( \otimes^pH^* \) is the \( p \)-th symmetric product of \( H^* \). The first operator \( D^{(k)}_0 \) is called the \( k \)-Cauchy-Fueter operator. \( D^{(k)}_j \)'s are differential operators of the first order except \( D^{(k)}_k \), which is of the second order (cf. Theorem 3.1). By using the twistor transformation and spectral sequences, Baston [6] proved the existence of these complexes over complex quaternionic manifolds. He generalized the result and the complex geometric method of Eastwood, Penrose and Wells [19] for \( n = 1 \). A complex quaternionic manifold \( M \) with \( \dim \mathbb{C} = 4n \) is a complex manifold whose holomorphic tangential bundle decompose as \( T\bar{M} \cong \bar{E} \otimes \bar{H} \), where \( \bar{E} \) and \( \bar{H} \) are holomorphic vector bundles of dimension \( 2n \) and \( 2 \), respectively, and there exists a torsion-free holomorphic connection preserving symplectic forms \( \epsilon \) on \( \bar{E} \) and \( \varepsilon \) on \( \bar{H} \). Then \( \epsilon \otimes \varepsilon \) is a complex Riemannian metric on \( \bar{M} \). Baston used Levi-Civita connection on \( \bar{M} \) to construct these complexes after section 2.6 in his paper [6].

Another motivation to consider these complexes comes from the function theory of several quaternionic variables. We write a vector in the quaternionic space \( \mathbb{H}^n \) as \( \mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_{n-1}) \) with \( \mathbf{q}_l = x_{4l+1} + x_{4l+2}i + x_{4l+3}j + x_{4l+4}k \in \mathbb{H}, \ l = 0, 1, \ldots, n-1 \). The usual Cauchy-Fueter operator is defined as

\[
\partial : C^1(\mathbb{H}^n, \mathbb{H}) \rightarrow C(\mathbb{H}^n, \mathbb{H}^n), \quad \partial f = \begin{pmatrix}
\overline{\partial q}_0 f \\
\vdots \\
\overline{\partial q}_{n-1} f
\end{pmatrix},
\]

for \( f \in C^1(\mathbb{H}^n, \mathbb{H}) \), where \( \overline{\partial q}_l = \partial x_{4l+1} + i\partial x_{4l+2} + j\partial x_{4l+3} + k\partial x_{4l+4} \), \( l = 0, 1, \ldots, n-1 \). A function \( f : \mathbb{H}^n \rightarrow \mathbb{H} \) is called (left) regular if \( \partial f = 0 \) on \( \mathbb{H}^n \). As in the theory of several complex variables, to investigate regular quaternionic functions, it is important to solve the non-homogeneous Cauchy-Fueter equation

\[
\partial f = h
\]
for prescribed $h \in C^2(\Omega, \mathbb{H}^n)$ over a domain $\Omega$. This system is overdetermined, i.e. the number of equations is larger than the number of unknown functions for $n > 1$. So for (1.4) to be solvable, $h$ must satisfy some compatible condition. This condition was found by Adams, Loustaunau, Palamodov and Struppa [1] with the help of computer algebra method, namely,

$$\mathcal{D}_1 h = 0,$$

for some differential operator of second order $\mathcal{D}_1 : C^2(\Omega, \mathbb{H}^n) \to C(\Omega, \Lambda^3 \mathbb{H}^n)$. In fact, there exists a differential complex corresponding to the Dolbeault complex in the theory of several complex variables: (1.5)

$$0 \to \Gamma(\Omega, \mathbb{H}) \overset{\mathcal{D}_1}{\to} \Gamma(\Omega, \mathbb{H}^n) \overset{\mathcal{D}_2}{\to} \Gamma(\Omega, \Lambda^3 \mathbb{H}^n) \to \cdots \to 0,$$

called the Cauchy-Fueter complex (cf. [17] [18] and references therein). It was realized later that the Cauchy-Fueter operator is exactly the 1-Cauchy-Fueter operator and the Cauchy-Fueter complex [10] [17] is equivalent to a sequence obtained by Baston in [6], although Baston’s result is a complexified version. In [45], the author introduced notions of the $k$-Cauchy-Fueter operator on the quaternionic space $\mathbb{H}^n$ and $k$-regular functions annihilated by this operator. In the function theory, (1.3) is called the $k$-Cauchy-Fueter complex. The $k$-Cauchy-Fueter complex over $\mathbb{H}^n$ was explicitly written down by using the twistor transformation and spectral sequences [45] (see also [9]). By solving the non-homogeneous $k$-Cauchy-Fueter equations, we showed the Hartogs’ phenomenon for $k$-regular functions [45]. To develop the function theory over curved manifolds, we need to write down these complexes on manifolds explicitly.

The $k$-Cauchy-Fueter operator over the 1-dimensional quaternionic space $\mathbb{H}$ also has the origin in physics: it is the elliptic version of spin $k/2$ massless field operator [15] [19] [30] [31] over the Minkowski space: $D_0^{(1)} \phi = 0$ corresponds to the Dirac-Weyl equation whose solutions correspond to neutrinos; $D_0^{(2)} \phi = 0$ corresponds to the Maxwell equation whose solutions correspond to photons; $D_0^{(3)} \phi = 0$ corresponds to the Rarita-Schwinger equation; $D_0^{(4)} \phi = 0$ corresponds to linearized Einstein’s equation whose solutions correspond to weak gravitational fields, etc..

Salamon [33] constructed another family of quaternionic complexes over quaternionic manifolds:

$$0 \to \Gamma\left(\mathcal{O}^k H^*\right) \to \Gamma\left(\Lambda^1 E^* \otimes \mathcal{O}^{k+1} H^*\right) \to \cdots \to \Gamma\left(\Lambda^p E^* \otimes \mathcal{O}^{k+p} H^*\right) \to \cdots .$$

The half sequence of the $k$-Cauchy-Fueter complex (1.3), beginning with the operator $D^{(k)}_{k+1}$, is similar to Salamon’s complexes. In last two decades, quaternionic manifolds were also studied from the point of view of parabolic geometry (cf. [11] [12] [14] [36] [17] and reference therein). Several interesting differential complexes over curved quaternionic manifolds have been constructed from BGG-sequences [13] [14]. Recall that for a parabolic subalgebra $\mathfrak{p}$ (resp. subgroup $P$) of a complex semisimple Lie algebra $\mathfrak{g}$ (resp. group $G$), let $E(\lambda)$ be the irreducible $\mathfrak{p}$-module with the lowest weight $-\lambda$. Denote by $\mathcal{O}_\mathfrak{p}(\lambda)$ the sheaf of holomorphic sections of vector bundle associated to $E(\lambda)$ over $G/P$. A general BGG-sequence is an exact sequence

$$0 \to E_\mathfrak{g}(\lambda) \to \mathcal{O}_\mathfrak{p}(\lambda) \xrightarrow{d_0} \bigoplus_{W \in W^\mathfrak{p}, i(w) = 1} \mathcal{O}_\mathfrak{p}(w, \lambda) \xrightarrow{d_1} \bigoplus_{W \in W^\mathfrak{p}, i(w) = 2} \mathcal{O}_\mathfrak{p}(w, \lambda) \to \cdots ,$$

for a dominant weight $\lambda$ of $\mathfrak{g}$, where $W^\mathfrak{p}$ is the Hasse diagram associated to $\mathfrak{p}$ (cf. theorem 8.4.1 in [7]). $E_\mathfrak{g}(\lambda)$ is a finite dimensional irreducible representation of $\mathfrak{g}$. But on the flat space $\mathbb{H}^n$, the $k$-Cauchy-Fueter complex after complexification is a sequence (1.7) with the weight $\lambda$ singular for $\mathfrak{g}$, but dominant for $\mathfrak{p}$ (cf. theorem 11 in [6]). In this case, $E_\mathfrak{g}(\lambda)$ is an infinite dimensional irreducible representation.
of \( \mathfrak{g} \). So it is not a BGG-sequence. Moreover, \( D_{k}^{(k)} \) is an example of non-standard invariant operators (cf. Remark 12 in [2]). In general, it is not easy to construct an exact sequence with singular weights. People usually construct such a sequence from a relative BGG sequence, case by case, by using the twistor method (cf. e.g. [7] [29] [10] and references therein) or the method of cohomology parabolic induction in the representation theory (cf. section 11.3 of [7]). From the point of view of function theory, we are especially interested in differential complexes (1.7) with singular weights, because only in this case “regular functions” as elements of \( \ker d_{0} \cong E_{k}(\lambda) \) are abundant. On the flat space \( \mathbb{H}^{n} \), a generalized Penrose integral formula provides all solutions to the \( k \)-Cauchy-Fueter equation, which is of infinite dimensional (cf. [25]).

The \( 0 \)-Cauchy-Fueter operator \( D_{0}^{(0)} : \Gamma(M, \mathbb{C}) \to \Gamma(M, \Lambda^{2}E^{*}) \) is called the Baston operator. Certain exterior product of this operator gives us the quaternionic Monge-Ampère operator by Alesker [2]. This interpretation together with the second operator \( D_{1}^{(0)} \) allows us to develop pluripotential theory over \( \mathbb{H}^{n} \) [40] [41] [42] [43]. To develop pluripotential theory over curved quaternionic manifolds, in particular to study the quaternionic Calabi-Yau problem on quaternionic manifolds [3] [4], we need to know \( 0 \)-Cauchy-Fueter complex on manifolds explicitly.

Penrose’s two-spinor notation is useful for studying 4-dimensional manifolds [30] [31]. It is generalized to complex quaternionic manifolds by Baston [6] and to more general complex paraconformal manifolds by Bailey and Eastwood [7]. As a real version, we extend this notation to quaternionic manifolds simply by realizing the isomorphism \( \mathbb{C} TM \cong E \otimes H \) in (1.2): for local frames \( \{e_{A}\} \) and \( \{e_{A'}\} \) of \( E \) and \( H \), respectively, we identify \( e_{A} \otimes e_{A'} \) with a complex tangential vector \( Z_{A A'} \) (see section 2.1). The quaternionic connection on \( M \) induces a \( \mathfrak{gl}(2n, \mathbb{C}) \)-connection on \( E \) and a \( \mathfrak{su}(2) \)-connection on \( H \), respectively, and so the curvature of the quaternionic connection has two components

\[
R_{abA}^{B} \quad \text{and} \quad R_{abA'}^{B'},
\]

corresponding to curvatures of the bundles \( E \) and \( H \), respectively. Here we use indices \( A, B \) and \( A', B' \) for components of sections of bundles \( E \) and \( H \), respectively, and indices \( a, b \) for components of the local quaternionic frame of the tangent bundle \( TM \). Furthermore, curvatures of a unimodular quaternionic connection have a very special decomposition (cf. Proposition 2.2), with the help of which we can check that the sequence (1.3) is a complex, i.e. \( D_{j+1}^{(k)} \circ D_{j}^{(k)} = 0 \), by direct calculation in Section 3.1. Two-component notation is a useful tool for calculation over a quaternionic manifold and everything in this paper is based on elementary calculation by this notation. Unimodular quaternionic manifolds constitute a nice class of quaternionic manifolds, because the conformal change of a unimodular quaternionic structure is still unimodular quaternionic, while the conformal change of a quaternionic Kähler structure is usually not quaternionic Kähler (cf. [28]). We also give the conformal transformation formula of these operators \( D_{j}^{(k)} \) in Section 3.1.

In Section 3.2, we show that the \( k \)-Cauchy-Fueter complex is elliptic, i.e. its symbol complex is a exact sequence of complex vector spaces. Write the \( k \)-Cauchy-Fueter complex as

\[
0 \longrightarrow \Gamma \left( \psi_{0}^{(k)} \right) \xrightarrow{D_{0}^{(k)}} \Gamma \left( \psi_{1}^{(k)} \right) \xrightarrow{D_{1}^{(k)}} \cdots \longrightarrow \Gamma \left( \psi_{2n-1}^{(k)} \right) \longrightarrow 0,
\]

where \( \psi_{j}^{(k)} \) is the \( j \)-th vector space in the sequence (1.3). By the theory of elliptic operators, we know the Hodge-type decomposition and that the \( j \)-th cohomology group

\[
H_{j}^{(k)}(M) = \ker D_{j}^{(k)}/\text{Im} D_{j-1}^{(k)}
\]
of the \( k \)-Cauchy-Fueter complex over a compact unimodular quaternionic manifold (right conformally flat if \( \dim_{\mathbb{R}} M = 4 \)) is finite dimensional, and can be represented by Hodge-type elements.

In Section 4, we prove a Weitzenböck formula for these complexes over a quaternionic Kähler manifold \( M \), and show a vanishing theorem for the cohomologies \( H^j_k(M) \), \( j = 1, \ldots, k-1 \), if its scalar curvature is negative. The Weitzenböck formula and vanishing theorem for Salamon’s complexes over such manifolds with negative scalar curvatures were already given by Horan [22] (see also Homma [21] and Nagatomo-Nitta [26]). The latter one essentially gives us the result for the \( k \)-Cauchy-Fueter complex for \( j \geq k + 3 \).

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2. Unimodular Quaternionic Manifolds and Their Curvatures

2.1. Realization of the isomorphism \( CTM \cong E \otimes H \). Denote by \( GL(n, \mathbb{H}) \) the group of all invertible quaternionic \((n \times n)\)-matrices. \( \text{Sp}(n) := \{ A \in GL(n, \mathbb{H}); A^t A = I_{n \times n} \} \). \( \mathfrak{s}(n, \mathbb{H}) := \{ A \in \mathfrak{gl}(n, \mathbb{H}); \text{Re}(\text{tr}(A)) = 0 \} \). We denote by \( SL(n, \mathbb{H}) \) the connected component containing the identity of the Lie group with Lie algebra \( \mathfrak{s}(n, \mathbb{H}) \).

Let \( A = (A_{jk})_{p \times m} \) be a quaternionic \((l \times m)\)-matrix and write \( A_{jk} = a_{jk}^1 + i a_{jk}^2 + j a_{jk}^3 + k a_{jk}^4 \in \mathbb{H} \). We define \( \tau(A) \) to be the complex \((2p \times 2m)\)-matrix

\[
\tau(A) = \begin{pmatrix}
\tau(A_{00}) & \tau(A_{01}) & \cdots & \tau(A_{0(m-1)}) \\
\tau(A_{10}) & \tau(A_{11}) & \cdots & \tau(A_{1(m-1)}) \\
\vdots & \vdots & \ddots & \vdots \\
\tau(A_{(p-1)0}) & \tau(A_{(p-1)1}) & \cdots & \tau(A_{(p-1)(m-1)})
\end{pmatrix},
\]

where \( \tau(A_{jk}) \) is the complex \((2 \times 2)\)-matrix

\[
\begin{pmatrix}
a_{jk}^1 + i a_{jk}^2 & -a_{jk}^3 - i a_{jk}^4 \\
a_{jk}^3 - i a_{jk}^4 & a_{jk}^1 - i a_{jk}^2
\end{pmatrix}.
\]

This is motivated by the embedding of quaternionic numbers into \( 2 \times 2 \)-matrices. The definition of \( \tau \) above and the following proposition are the conjugate version of those in [33].

**Proposition 2.1.** (Proposition 2.1 in [33]) (1) \( \tau(AB) = \tau(A)\tau(B) \) for a quaternionic \((p \times m)\)-matrix \( A \) and a quaternionic \((m \times l)\)-matrix \( B \). In particular, for \( q' = Aq, q, q' \in \mathbb{H}^n, A \in GL(n, \mathbb{H}) \), we have

\[
\tau(q') = \tau(A)\tau(q)
\]

as complex \((2n \times 2)\)-matrices.

(2) For \( A \in GL(n, \mathbb{H}) \), we have

\[
J\tau(A) = \tau(A)J,
\]

where

\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\vdots & \ddots
\end{pmatrix}.
\]

(3) \( \tau(A^t) = \tau(A)^t \) for a quaternionic \((n \times n)\)-matrix \( A \). If \( A \in \text{Sp}(n) \), \( \tau(A) \) is symplectic, i.e., \( \tau(A)J\tau(A)^t = J \).
Proposition 2.1 (3) implies \( \tau(\text{Sp}(n)) \subset \text{SU}(2n) \). Proposition 2.2 implies \( \tau(\text{sl}(n, \mathbb{H})) \subset \text{sl}(2n, \mathbb{C}) \), and so \( \tau(\text{SL}(n, \mathbb{H})) \subset \text{SL}(2n, \mathbb{C}) \). Given the standard volume form on \( \mathbb{R}^{4n} \), \( \text{SL}(n, \mathbb{H}) \) is the group consisting of elements of \( \text{GL}(n, \mathbb{H}) \) which induce transformations of \( \mathbb{R}^{4n} \) preserving this volume form. Let \( I_1, I_2, I_3 \) be the induced action of \( i, j, k \) on the frame bundle. Then we can choose a frame of the tangent bundle
\[
(X_1, X_1 I_1, X_1 I_2, X_1 I_3, \ldots, X_{4l+1} I_1, X_{4l+1} I_2, X_{4l+1} I_3, \ldots)
\]
called a **local quaternionic frame**. Label this frame as \( (X_1, \ldots, X_{4n}) \).

\( H \otimes E \) is isomorphic to the tangent bundle \( CTM \) as follows. It follows from Proposition 2.1 that \( \mathbb{C}^{2n} \) is a \( \text{GL}(n, \mathbb{H}) \)-module with \( A \in \text{GL}(n, \mathbb{H}) \) acting on \( \mathbb{C}^{2n} \) by \( \tau(A) \), and \( \mathbb{C}^2 \) is a \( \text{Sp}(1) \)-module with \( q \in \text{Sp}(1) \) acting on \( (z_1, z_2) \in \mathbb{C}^2 \) by right multiplying the \( 2 \times 2 \)-matrix \( \tau(q) \). Let \( \{v_A\}_{A=0}^{2n-1}, \{v_A\}_{A=0}^{2n-1} \) and \( \{w_A\}_{a=1}^{2n} \) be the standard bases of \( \mathbb{C}^{2n} \), \( \mathbb{C}^2 \) and \( \mathbb{R}^{4n} \), respectively. Write \( v_{AA'} := v_A \otimes v_{A'} \) in \( \mathbb{C}^{2n} \otimes \mathbb{C}^2 \). The map \( \tau \) provides an isomorphism from \( \mathbb{C} \otimes \mathbb{H}^n \) to \( \mathbb{C}^{2n} \otimes \mathbb{C}^2 \) as \( \text{GL}(n, \mathbb{H}) \times \text{Sp}(1) \)-module. Under this identification of \( \tau \), we have
\[
2w_{4l+1} = v_{(2l)0'} + v_{(2l+1)1'}, \quad 2w_{4l+2} = -iv_{(2l)0'} + iv_{(2l+1)1'},
2w_{4l+3} = -v_{(2l)1'} + v_{(2l+1)0'}, \quad 2w_{4l+4} = iv_{(2l)1'} + iv_{(2l+1)0'},
\]
by definition \((2.1)-(2.2)\) of \( \tau \) for \( m = 1 \). Thus
\[
\begin{pmatrix}
v_{(2l)0'} & v_{(2l+1)1'} \\
v_{(2l+1)0'} & v_{(2l+1)1'}
\end{pmatrix} = \begin{pmatrix}
w_{4l+1} + i w_{4l+2} & -w_{4l+3} - i w_{4l+4} \\
w_{4l+3} - i w_{4l+4} & w_{4l+1} - i w_{4l+2}
\end{pmatrix}.
\]

Now for a local quaternionic frame \( \epsilon = (X_1, \ldots, X_{4n}) \), define local sections
\[
e_A := (\epsilon, v_A), \quad e_{A'} := (\epsilon, v_{A'}), \quad X_a := (\epsilon, w_a)
\]
of \( E, H \) and \( TM \) in \((1.1)\), respectively. Then \((2.6)\) implies that \( Z_{AA'} = \frac{1}{\sqrt{2}} e_A \otimes e_{A'} \) are given by
\[
(2.8)
\]
See \((2.20)\) for the reason to factor \( \sqrt{2} \) here. This frame over the flat quaternionic space \( \mathbb{H}^n \) plays an important role in the investigation of quaternionic analysis \( [25, 10, 44, 45] \).

Let \( \{\omega\} \) be the coframe dual to \( \{X_i\} \) and let \( \{e^{AA'}\} \) be complex 1-forms dual to the two-component local quaternionic frame \( \{Z_{AA'}\} \) in \((2.3)\), i.e. \( e^{AA'}(Z_{BB'}) = \delta^{A}_{B} \delta^{A'}_{B'} \). It is obvious that
\[
\sqrt{2}e^{00'} = \omega^1 - i \omega^2, \quad \sqrt{2}e^{01'} = -\omega^3 + i \omega^4, \quad \sqrt{2}e^{10'} = \omega^3 + i \omega^4, \quad \sqrt{2}e^{11'} = \omega^1 + i \omega^2, \ldots,
\]
by the expression of \( \{Z_{AA'}\} \) in \((2.3)\), and so \( e^{00'} \wedge e^{11'} \wedge (e^{10'} \wedge e^{01'}) = -\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \). Consequently,
\[
(2.9)
\]
A local quaternionic frame \( \{X_1, \ldots, X_{4n}\} \) is called a **local unimodular quaternionic frame** if the volume form of the manifold is locally given by \( \text{vol} \) in \((2.9)\). Note that a local quaternionic frame becomes unimodular simply by multiplying a suitable factor.
2.2. **The two-component notation.** Denote by $e^{A'} := (\epsilon, v^{A'})$ a local section of the dual bundle $H^*$, where $v^{A'}$ is the dual of $v_A$ in $\mathbb{C}^2$. It is similar to define $E^*$ and $e^A$. Consider

$$(2.10)\quad \varepsilon_{A'B'} e^{A'} \otimes e^{B'}$$

where $\varepsilon_{A'B'}$ is antisymmetric with $\varepsilon_{01'} = 1$. Here and in the following, we use the Einstein’s convention of summation over repeated indices. It is a section of the line bundle $\Lambda^2 H^*$ (a symplectic form on $H$ pointwisely). This is because $(2.10)$ is invariant under the action of $\text{Sp}(1)$ by Proposition 2.1 (3). So they can be glued to be a global section. When the manifold is unimodular quaternionic, consider

$$(2.11)\quad \varepsilon_{A_1 \ldots A_{2n}} e^{A_1} \otimes \ldots \otimes e^{A_{2n}}$$

where $\varepsilon_{A_1 \ldots A_{2n}}$ is the sign of the permutation from $A_1, \ldots, A_{2n}$ to $1, \ldots, 2n$. It is a global section of the line bundle $\Lambda^{2n} E^*$, because $(2.11)$ is invariant under the action of $\text{SL}(2n, \mathbb{C})$.

A section $f$ of $\mathbb{T}^l_{q,p} := (\otimes^l H) \otimes (\otimes^q E^*) \otimes (\otimes^p H^*)$ can be written as

$$(2.12)\quad f = f^{A_1' \ldots A_l'} e^{A_1} \otimes \ldots \otimes e^{A_l} \otimes e^{B_1} \otimes \ldots \otimes e^{B_q} \otimes e^{B_1'} \otimes \ldots \otimes e^{B_p'}.$$ 

We can identify this section with the tuple of functions

$$(2.13)\quad \left( \ldots f^{A_1' \ldots A_l'}, f^{A_1' \ldots A_l'} \right).$$

A **contraction** is a map $C : \mathbb{T}^{l+1}_{q,p} \to \mathbb{T}^l_{q,p}$ given by $(C f)^{A_1' \ldots A_l'} := f^{A_1' \ldots A_l' D_1' \ldots D_q' \ldots D_p'}$, where the superscript and subscript $D_j'$ appear in $j$-th and $\hat{j}$-th places, respectively. It is a well defined element of $\mathbb{T}^l_{q,p}$ since $e^{D_j'} \otimes e_{D_j'} = e^{E_j'} \otimes e_{E_j'}$ under the transformation $e^{E_j'} = (M^{-1})_{D_j'} e^{D_j'}$, $e_{E_j'} = M_{E_j'} D_j' e_{D_j'}$ for $(M_{E_j'} D_j') \in \text{Sp}(1)$. We use $\varepsilon_{A'B'}$ to raise or lower primed indices. For example,

$$(2.14)\quad (\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\varepsilon_{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

On a unimodular quaternionic manifold, we can not use $\epsilon$ to raise or lower unprimed indices. This is why we only consider tensors as sections of $\mathbb{T}^l_{q,p}$. But on quaternionic Kähler manifold, we can use $\epsilon_{AB}$ to raise or lower unprimed indices (cf. Section 4).

Recall that a **covariant derivative** of a vector bundle $V$ is a mapping $\nabla : \Gamma(V) \to \Gamma((TM)^* \otimes V)$ satisfying $\nabla(fv) = df \otimes v + f \nabla v$, $\nabla(v_1 + v_2) = \nabla v_1 + \nabla v_2$, for any $v, v_1, v_2 \in \Gamma(V)$ and scalar function $f$. $\nabla$ acts on $V^*$ naturally by duality: $X(v, v^*) = (\nabla_X v, v^*) + (v, \nabla_X v^*)$ for any vector field $X \in TM, v \in \Gamma(V)$ and $v^* \in \Gamma(V^*)$. A covariant derivative can be naturally extended to a map $\nabla : \Gamma((\otimes^k V) \otimes (\otimes^l V^*)) \to \Gamma((TM)^* \otimes (\otimes^k V) \otimes (\otimes^l V^*))$.

The quaternionic connection induces an $\mathfrak{sl}(2n, \mathbb{C})$-connection $\omega'$ on $E$ and an $\mathfrak{su}(2)$-connection $\omega''$ on $H$. When the manifold is unimodular or quaternionic Kähler, $\omega'$ is $\mathfrak{sl}(2n, \mathbb{C})$- or $\mathfrak{sp}(n)$-valued. $\nabla$ is naturally extended to well defined mappings $E \to (CTM)^* \otimes E$ and $H \to (CTM)^* \otimes H$ by $\nabla_{X+Y} := \nabla_X + \nabla_Y$, which induce well defined mappings $E^* \to (CTM)^* \otimes E^*$ and $H^* \to (CTM)^* \otimes H^*$ by duality, and so we get a well defined mapping $\mathbb{T}^l_{q,p} \to (CTM)^* \otimes \mathbb{T}^l_{q,p}$.

Choose a local quaternionic frame $\epsilon = \{ \epsilon_a := X_a \}_{a=1}^{2n}$ of $TM$ and its dual $\{ \epsilon^a \}_{a=1}^{2n}$. Write

$$\nabla \epsilon_A = \Gamma_{AB} e^a \otimes \epsilon_B, \quad \nabla \epsilon_A' = \Gamma_{A'B'} e^a \otimes \epsilon_B'.$$
where $\Gamma_{aA}B = \omega'(X_a)_A^B$ and $\Gamma_{aA'B'} = \omega''(X_a)_A^B$ are connection coefficients. Then by duality, we have 
\[ \nabla e^A = -\Gamma_{aB}^A e^a \otimes e^B, \quad \nabla e^{A'} = -\Gamma_{aB}^{A'} e^a \otimes e^{B'}, \]
which are equivalent to 
\[ \nabla_a f_A = X_a f_A - \Gamma_{aA}^D f_D, \quad \nabla_a f_{A'} = X_a f_{A'} - \Gamma_{aA'}^{D'} f_{D'}. \]
In general, $\nabla f$ for $f$ given by (2.12) – (2.13) is the tuple 
\[ (\ldots, \nabla_a f_{A_1}^{A_2} \cdots, \Gamma_{aA}^D f_D \cdots), \]
as a section of $(\mathcal{T}^1 + \mathcal{T}^1)^* \otimes \mathcal{T}^1_{q,p} \approx \Sigma^{1}_{q+1,p+1}$, by the identification (1.2), where 
\[ \nabla_a f_{B_1}^{A_1} \cdots f_{B_p}^{A_p} := X_a f_{B_1}^{A_1} \cdots f_{B_p}^{A_p} + \Gamma_{aA'}^{A_1'} f_{B_1}^{A_1'} \cdots f_{B_p}^{A_p} - \Gamma_{aB_1}^D f_{B_1}^{A_1} \cdots f_{B_p}^{A_p} \cdots \Gamma_{aB_p}^{D'} f_{B_1}^{A_1} \cdots f_{B_p}^{A_p}. \]
(2.15)

The covariant derivative is invariant after contraction: $\nabla(Cf) = C(\nabla f)$, because by (2.15), we have 
\[ [C(\nabla_a f) - \nabla_a (Cf)]_{B_1}^{A_1} \cdots f_{B_p}^{A_p} = \Gamma_{aA'}^{A_1'} f_{B_1}^{A_1'} \cdots f_{B_p}^{A_p} - \Gamma_{aB_1}^D f_{B_1}^{A_1} \cdots f_{B_p}^{A_p} = 0. \]
We will use the notation 
\[ \nabla_{AA'} := \nabla_{Z_{AA'}} = \frac{1}{\sqrt{2}}(\nabla_{X_a} + i \nabla_{X_b}) \]
if we write $Z_{AA'} = \frac{1}{\sqrt{2}}(X_a + i X_b)$ for some $a, b$ (cf. (2.8)). We also write $\nabla_{AA'}$ as $\nabla_{A'A}$ when it is more convenient. Then 
\[ \nabla_{AA'} e_B = \Gamma_{A'B'}^B e_{C'}, \quad \nabla_{AA'} e_{B'} = \Gamma_{A'B'}^{C'} e_{C'}, \]
where $\Gamma_{A'B'}^B = \frac{1}{\sqrt{2}}(\Gamma_{ab}^B + i \Gamma_{ab}^{B'})$ and $\Gamma_{A'B'}^{C'} = \frac{1}{\sqrt{2}}(\Gamma_{ab}^{C'} + i \Gamma_{ab}^{B'C'})$. The formula (2.15) holds for $a = AA'$. Denote 
\[ 2\nabla_a \nabla_b := \nabla_a \nabla_b - \nabla_b \nabla_a. \]
The torsion is defined as $2\nabla_a \nabla_b \phi = T_{ab} c \nabla_c \phi$ for any scalar function $\phi$. Then by definition $T_{ab} c = \Gamma_{ab} c - \Gamma_{ba} c + C_{ab} c$ where $C_{ab} c$ are given by $[X_a, X_b] = C_{ab} c X_c$. It is direct to check that 
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a)(\phi f_A) = T_{ab} c \nabla_c \phi f_A + \phi (\nabla_a \nabla_b - \nabla_b \nabla_a) f_A \]
for a scalar function $\phi$, by the formula (2.15) for covariant derivatives. So when the connection is torsion-free, $2\nabla_a \nabla_b$ is an endomorphism of $\Gamma(E^*)$ as a $\mathcal{C}^\infty(M)$-module for fixed $a, b$ (similarly for $\Gamma(H^*)$). So we can write 
\[ 2\nabla_a \nabla_b f_A := -R_{abA}^D f_D, \quad 2\nabla_a \nabla_b f_{A'} := -R_{abA'}^{D'} f_{D'}. \]
By (2.15), we see that 
\[ \nabla_a \nabla_b (f_{\mathcal{A}} h_{\mathcal{B}}) = \nabla_a \nabla_b f_{\mathcal{A}} h_{\mathcal{B}} + f_{\mathcal{A}} \nabla_a \nabla_b h_{\mathcal{B}}. \]
In general, we have the generalized Ricci identity. 
\[ \nabla_a \nabla_b f_{A_1}^{A_1} \cdots f_{B_p}^{A_p} := R_{abA_1}^{A_1'} f_{B_1}^{A_1'} \cdots f_{B_p}^{A_p} - R_{abB_1}^D f_{B_1}^{A_1} \cdots f_{B_p}^{A_p} - R_{ab}^{D'} f_{B_1}^{A_1} \cdots f_{B_p}^{A_p}. \]
(2.17)
See Penrose-Rindler [30] or Bailey-Eastwood [5].

If the manifold is unimodular quaternionic, the connection on $E$ preserves the $2n$-form $\epsilon$ (2.11), i.e. 
\[ \nabla a \epsilon_{A_1} \cdots A_{2n} = \Gamma_{aA_1}^D \epsilon_{A_1} \cdots A_{2n} = \Gamma_{aD} \epsilon_{A_1} \cdots A_{2n} = 0. \]
(2.18)
where \( \epsilon_{A_1 \ldots A_{2n}} \) is nonzero only if \((A_1 \ldots A_{2n})\) is a permutation of \((0, \ldots, 2n-1)\) and \((\Gamma_{\alpha A}^P)\) is \(\mathfrak{sl}(2n, \mathbb{C})\)-valued. Similarly, the symplectic form \(\varepsilon_{A'B'}\) on \(H\) is also preserved by the connection on \(H\), i.e.

(2.19) \[ \nabla_a \varepsilon_{A'B'} = 0. \]

Thus when the manifold is quaternionic Kähler, \(\nabla\) is a connection of \(CTM = H \otimes E\) compatible with metric

(2.20) \[ g_{ab} = g(Z_{AA'}, Z_{BB'}) = \varepsilon_{AB} \varepsilon_{A'B'}, \quad a = AA', \quad b = BB', \]

for \(Z_{AA'}\) in (2.2) if we choose local quaternionic frame \(\{X_a\}\) in (2.2) orthonormal.

The notion of unimodular quaternionic structure is a real version of the notion of torsion-free QCF-structure on a complex manifold introduced by Bailey and Eastwood [5]. A quaternionic conformal structure (briefly QCF) on a \(4n\)-dimensional complex quaternionic manifold \(\tilde{M}\) is given by an isomorphism \(T \tilde{M} \cong E \otimes H\) and a fixed isomorphism between \(\Lambda^{2n} \tilde{E}\) and \(\Lambda^2 \tilde{H}\). Given a symplectic form \(\tilde{\varepsilon}\) in \(\Lambda^2 \tilde{H}\), the isomorphism induces a \(2n\)-form \(\varepsilon\) in \(\Lambda^2 \tilde{E}\), and there exists a unique connection \(\nabla\) preserving \(\tilde{\varepsilon}\) and \(\varepsilon\) (cf. theorem 2.4 in Bailey-Eastwood [5]). The QCF-structure is called torsion-free if the induced connection on the holomorphic tangent bundle is torsion-free.

The curvature of the complexified tangent bundle is

\[ R_{AA'B'B'C'C'}^{DD'} f_{CC'}^{CC'} = (\nabla_{AA'} \nabla_{BB'} - \nabla_{BB'} \nabla_{AA'}) f_{DD'}^{CC'}, \]

if we identify \((f_{DD'})\) with a local section of \(CTM\). By the generalized Ricci identity, we see that the curvature has the decomposition:

(2.21) \[ R_{AA'B'B'C'C'}^{DD'} = R_{AA'B'B'C'}^{D} \delta_{C'}^{D'} + R_{AA'B'B'C'}^{D'} \delta_{C'}^{D}. \]

We will use the following notations: for \(a = AA', b = BB',\)

\[ R_{A'B'ABC}^D := R_{AA'B'B'C'}^{D} = R_{ab}^{D}, \quad R_{ABA'B'C'}^{D'} := R_{AA'B'B'C'}^{D'} = R_{ab}^{D'}. \]

Corollary 2.1. On a unimodular quaternionic manifold, we have

(2.22) \[ R_{A'B'ABC}^C = 0, \quad R_{ABA'B'C'}^{C'} = 0. \]

Proof. Note that \( (\nabla_{AA'} \nabla_{BB'} - \nabla_{BB'} \nabla_{AA'}) \epsilon_{A_1 \ldots A_{2n}} = 0 \) by (2.18), which implies that

\[ R_{A'B'ABA_j}^D \epsilon_{A_1 \ldots A_{2n}} = 0 \]

by the generalized Ricci identity again. Noting that \(\epsilon_{A_1 \ldots A_{2n}}\) is nonzero only if \((A_1 \ldots A_{2n})\) is a permutation of \((0, \ldots, 2n-1)\), we find the first trace vanishing. It is similar for the second one.

We will use symmetrisation and antisymmetrisation of indices

\[ f_{\ldots (A_1 \ldots A_k) \ldots} := \frac{1}{k!} \sum_{\sigma \in S_k} f_{\ldots A_{\sigma(1)} \ldots A_{\sigma(k)} \ldots}, \quad f_{\ldots [A_1 \ldots A_k] \ldots} := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) f_{\ldots A_{\sigma(1)} \ldots A_{\sigma(k)} \ldots}, \]

where \(\text{sgn}(\sigma)\) is the sign of the permutation \(\sigma\). It is obvious that

(2.23) \[ f_{\ldots [A_1 \ldots A_k] \ldots} = f_{\ldots ([A_1 \ldots A_k] \ldots)} \]

by definition of antisymmetrisation. We will also use the notation

\[ f_{\ldots (A_1 \ldots A_k \ldots) \ldots}, \]

which means antisymmetrisation of indices \(A_1 \ldots A_k\) except for that in \(\mathcal{A}\). We will use similar notations for symmetrisation of primed indices.
2.3. The curvature decomposition on unimodular quaternionic manifolds.

Proposition 2.2. For a unimodular quaternionic manifold with dimension > 4, the curvatures decompose as

\[ R_{[A'B']ABC}^D = \varepsilon_{A'B'}\left( \Psi_{ABC}^D + 2\delta_{[A} \Lambda_{B)]C} \right) , \]
\[ R_{(A'B')ABC}^D = 2\delta_{B[A'} \Phi_{B\rangle |A'B'} , \]
\[ R_{(AB)A'B'C'}^{D'} = 2\delta_{A'} \Phi_{B'C'AB} , \]
\[ R_{[AB]A'B'C'}^{D'} = 2\Lambda_{AB} \delta_{(A'} \varepsilon_{B')C'} , \]

where the first identity above can viewed as the definition of \( \Psi_{ABC}^D \), and

\[ \varepsilon_{A'B'}\Lambda_{AB} := \frac{1}{3} R_{[AB][A'B']}^C , \quad \Phi_{ABA'B'} := R_{(AB)(A'B')}^C \]

and \( \Phi_{ABA'B'} := \Phi_{ABA'B'} \). Moreover,

\[ \Lambda_{AB} = \Lambda_{[AB]} , \quad \Phi_{ABA'B'} = \Phi_{(AB)(A'B')} , \quad \Psi_{ABC}^D = \Psi_{(ABC)}^D \]

and \( \Psi_{ABC}^D \) are totally trace free: \( \Psi_{ABC}^A = \Psi_{ABC}^B = \Psi_{ABC}^C = 0 \).

When the manifold is 4-dimensional, (2.27) holds except for the last identity replaced by

\[ R_{[AB]A'B'C'}^{D'} = \varepsilon_{AB}\Psi_{A'B'C'}^{D'} + 2\Lambda_{AB} \delta_{(A'} \varepsilon_{B')C'} , \]

with \( \Psi_{A'B'C'}^{D'} = \Psi_{(A'B'C')}^{D'} \) also totally trace free. (2.27) can be viewed as the definition of \( \Psi' \).

In the 4-dimensional case, we will only consider right conformally flat manifolds later, i.e. \( \Psi_{A'B'C'}^{D'} = 0 \) (cf. section 6.9 of [31] for this concept and its necessity for defining massless field equations). See Penrose and Rindler’s book (section 4.6 of [30]) for this curvature decomposition for 4-dimensional manifolds. It is generalized to torsion-free QCFs by Bailey and Eastwood (cf. p.83 in [5]) with a sketched proof. See the Appendix for a detailed proof by only using the first Bianchi identity.

It is well known that a quaternionic Kähler manifold is Einstein. See lemma A.1 and theorem 7.8 in [5] for the proofs of the following proposition for QCF manifolds. See also the Appendix for a detailed proof.

Proposition 2.3. (1) If the manifold is unimodular quaternionic, then we have \( \nabla_{[A} \Lambda_{BC]} = 0 \).

(2) For a quaternionic Kähler manifold, we have

\[ \Phi_{ABA'B'} = 0 , \quad \Lambda_{AB} = \Lambda_{(AB)} , \quad \Lambda = \frac{s_g}{8n(n+2)} , \]

where \( s_g \) is the scalar curvature. Namely, it is Einstein.

3. The quaternionic complexes

3.1. The \( k \)-Cauchy-Fueter complexes. Recall that an element of the symmetric power \( \otimes^p H^* \) is given by a tuple \( (f_{A'_1,\ldots,A'_p}) \), which as an element of \( \otimes^p H^* \) is invariant under the permutation of subscripts \( A'_1,\ldots,A'_p = 0',1' \). An element of the exterior power \( \Lambda^q E^* \) is given by a tuple \( (f_{A_1,\ldots,A_q}) \), which as an element of \( \otimes^q E^* \) is antisymmetric under the permutation of subscripts \( A_1,\ldots,A_q = 0,\ldots,2n - 1 \).

The covariant derivative defines a differential operator \( \nabla : \Gamma(\Sigma_q) \to \Gamma(\Sigma_{q+1,\rho+1}) \) given by

\[ (\nabla f)_{A_0,\ldots,A_q,\ldots,A_0',\ldots,A_p'} = \nabla_{A_0} A_0' f_{A_1,\ldots,A_q,\ldots,A_0'\ldots,A_p'} , \]
Note that for \( f \in \Gamma(\Lambda^q E^* \otimes \omega^p H^*) \), \((\nabla f)_{A_0 \ldots A_q A'_0 \ldots A'_{p}} = \nabla_{A_0} f_{A_1 \ldots A_q} A'_1 \ldots A'_{p} \) is still symmetric in \( A'_1, \ldots, A'_p \) and antisymmetric in \( A'_1, \ldots, A'_p \) by using (3.4). We will also frequently use the following corollary of Proposition 2.1

\[
(\hat{\nabla} f)_{A_0 \ldots A_q A'_0 \ldots A'_{p}} = \nabla_{A_0} f_{A_1 \ldots A_q} A'_1 \ldots A'_{p}.
\]

Let us consider operators a little bit more general than those appearing in the \( k \)-Cauchy-Fueter complexes (1.3): \( \mathcal{D}_q : \Gamma(\Lambda^q E^* \otimes \omega^p H^*) \to \Gamma(\Lambda^q \omega^p H^*) \) given by

\[
(\mathcal{D}_q f)_{A_1 \ldots A_{q-1} A'_1 \ldots A'_{p}} = \nabla_{A_0} f_{A_1 \ldots A_{q-1}} A'_0 \ldots A'_{p},
\]

for a local section \( f \) in \( \Gamma(\Lambda^q E^* \otimes \omega^p H^*) \) (it is well defined since contraction over \( A'_1 \) is well defined), and \( \mathcal{D}_q^p : \Gamma(\Lambda^q E^* \otimes \omega^p H^*) \to \Gamma(\Lambda^q \omega^p H^* \otimes \omega^p H^*) \) given by

\[
(\mathcal{D}_q^p f)_{A_1 \ldots A_{q-1} A'_1 \ldots A'_{p}} = \nabla_{A_0} f_{A_1 \ldots A_{q-1}} A'_0 \ldots A'_{p},
\]

for a local section \( f \) in \( \Gamma(\Lambda^q E^* \otimes \omega^p H^*) \). The Baston operator \( \Delta : \Gamma(\Lambda^k E^*) \to \Gamma(\Lambda^k \omega^p E^*) \) is given by

\[
(\Delta f)_{A_1 \ldots A_{k+2}} := \nabla_{A_0} f_{A_1 \ldots A_{k+2}} + 2 \Lambda_{A_1 A_2} f_{A_3 \ldots A_{k+2}}.
\]

**Theorem 3.1.** Suppose that the manifold \( M \) is unimodular quaternionic and is right conformally flat if \( \dim_{\mathbb{R}} M = 4 \). For \( k = 0, 1, 2, \ldots \), the sequences (1.3) are elliptic differential complexes, where the operators \( D_j^{(k)} = \mathcal{D}_{j-k-j} \) for \( j = 0, \ldots, k-1 \), the operators \( D_j^{(k)} = \mathcal{D}_{j+1-k} \) for \( j = k+1, \ldots, 2n-2 \), and the operator \( D_{k}^{(k)} \) is the Baston operator \( \Delta \).

Note that by using \( \varepsilon^{AB'}(2.19)\) to raise primed indices, we have the following commutators

\[
(\nabla^A \nabla^B - \nabla^B \nabla^A) f_{C'} = (\nabla^A \nabla^B - \nabla^B \nabla^A) f_{C'} \varepsilon^{A'B'} = \varepsilon^{A'B'} f_{C'}
\]

(3.4)

\[
= - R_{AB'BC'}^{A'B'} f_{C'} \varepsilon^{A'B'} = - R_{AB'BC'}^{A'B'} f_{C'},
\]

where \( R_{AB'BC'}^{A'B'} := R_{AB'BC'}^{A'B'} \varepsilon^{A'B'} \) by raising indices. We can move \( \varepsilon^{A'B'} \) to the left since the connection \( \nabla \) preserves it. The following formulae for commutators \( \nabla_{[A} \nabla_{B]} \) as curvatures are important in the proof of Theorem 3.1

\[
\nabla_{[A_1} \nabla_{A_2]} h_{C} = \frac{1}{4} \left( \nabla_{A_1} \nabla_{A_2} \nabla_{A_1} A'_2 - \nabla_{A_1} \nabla_{A_2} A'_1 - \nabla_{A_1} \nabla_{A_2} A'_2 + \nabla_{A_1} \nabla_{A_2} A'_1 \right) h_{C'} = - \frac{1}{2} R_{[A_1 A_2]}^{A_1 A_2} C C' h_{D'},
\]

(3.5)

\[
\nabla_{[A_1} \nabla_{A_2]} h_{C} = - \frac{1}{2} R_{[A_1 A_2]}^{A_1 A_2} C C' h_{D'},
\]

by using (3.4). We will also frequently use the following corollary of Proposition 2.2.

**Corollary 3.1.** On a unimodular quaternionic manifold (right conformally flat if it is 4-dimensional), we have

\[
R_{[A_1 A_2]}^{A'B'} C D' = 2 \Lambda_{A_1 A_2} \delta_{C'} (A' \varepsilon^{B'}) D',
\]

(3.6)

in particular,

\[
R_{[A_1 A_2]}^{A'B'} C D' = 0,
\]

(3.7)

and

\[
R_{[A_1 A_2]}^{A'B'} C D' = 2 \delta_{A_1}^B \delta_{A_2}^C (A' \varepsilon^{B'}) D',
\]

(3.8)
Proof. (6) follows from
\[
R_{ABCD}^{A'B'C'} = R_{ABCD}^{A'B'C'} \epsilon \bar{A}' \epsilon \bar{B}' = 2 \Lambda_{A_1 A_2} \delta_{(A'_1 D' \epsilon \bar{B}' C') \epsilon \bar{A}' \epsilon \bar{B}' = 2 \Lambda_{A_1 A_2} \delta_{C'} (A'_1 \epsilon B') D'),
\]
by the last identity in (2.21) and \( \epsilon D A' \) antisymmetric.

The first identity in (6.3) follows from antisymmetrising \([ABC]\) in
\[
(3.9) \quad R_{A_1 A_2 ABC}^{A'B'} = \epsilon A_1 A_2 (\Psi_{ABC} D + 2 \delta_{A} D A_B) + 2 \delta_{A} D \Phi_{B[C} A_1 A_2',
\]
by using (2.24) and symmetry (2.25) of \( \Phi \) and \( \Psi \) in subscripts. For the second identity, we have
\[
R^{(A'B')}_{ABCD} = -R_{A_1 A_2 ABC}^{A'B'} \epsilon A_1' (A'_2 \epsilon B') = -2 \delta_{A} D \Phi_{B[C} A_1 A_2' \epsilon A_1' (A'_2 \epsilon B') = 2 \delta_{A} D \Phi_{B[C} A_1 A_2' \epsilon A_1' (A'_2 \epsilon B') = 0.
\]
by using (3.9) and \( \epsilon A_1' A_2' \epsilon A_1 (A'_2 \epsilon B') = -\epsilon (B' A') = 0 \). \( \square \)

Proof of Theorem 3.7
Case 1: \( j = 0, \ldots, k - 2 \). To check \( D_{j+1}^{(k)} \circ D_j^{(k)} = 0 \), it is sufficient to show \( \partial_{q+1, p-1} \partial_{q, p} f = 0 \). For a local section \( f \in \Gamma(\Lambda^* \otimes \otimes^p H^*) \), we have
\[
(\partial_{q+1, p-1} \partial_{q, p} f)_{A_1 A_2 A_3 \ldots A_p} = \nabla_{A_1} A_2 f_{A_3 \ldots A_p} = \nabla_{A_1} A_2 f_{A_3 \ldots A_p} - \nabla_{A_1} A_2 f_{A_3 \ldots A_p} = \nabla_{A_1} f_{A_2 A_3 \ldots A_p}.
\]
by (2.23). We can symmetrise superscripts \((A'_1 A'_2)\) since \( \nabla_b f \ldots A'_1 A'_2 \ldots A_p \) is symmetric in \( A'_1, \ldots, A'_p \) as we have mentioned under (3.1), and so is \( \nabla_a \nabla_b f \ldots A'_1 A'_2 \ldots A_p \). Apply formula (3.5) for commutators to the above identity and antisymmetrise unprimed indices to get \((\partial_{q+1, p-1} \partial_{q, p} f)_{A_1 A_2 A_3 \ldots A_p} \) equal to
\[
(3.10) \quad -\frac{1}{2} R_{[A_1 A_2]}^{A_1 A_2} D' f_{A_3 \ldots A_p} = -\frac{1}{2} R_{[A_1 A_2]}^{A_1 A_2} D' f_{A_3 \ldots A_p} = 0
\]
by using (2.28). The second term in (3.10) vanishes by (3.8), while the first term in (3.10) is equal to
\[
-\frac{3}{2} \left\{ \frac{\epsilon A_1' D' \Lambda_{[A_1 A_2] f_{A_3 \ldots A_p}]}{A_1 A_2 f_{A_3 \ldots A_p} A_1 A_2} + \frac{\epsilon A_1' D' \Lambda_{[A_1 A_2] f_{A_3 \ldots A_p}}}{A_1 A_2 f_{A_3 \ldots A_p} A_1 A_2} \right\} = 0
\]
by (3.6)–(3.7), since \( f \) is symmetric in primed indices. Consequently, we get \( \partial_{q+1, p-1} \partial_{q, p} f = 0 \).

Case 2: \( j = k + 1, \ldots, 2n - 2 \). To check \( D_j^{(k)} \circ D_j^{(k)} = 0 \), it is sufficient to show \( \partial_{q+1, p-1} \partial_{q, p} f = 0 \). Similarly, we have
\[
(\partial_{q+1, p-1} \partial_{q, p} f)_{A_1 A_2 A_3 \ldots A_p} = \nabla_{A_1} \nabla_{A_2} f_{A_3 \ldots A_p} = \nabla_{A_1} \nabla_{A_2} f_{A_3 \ldots A_p} = \nabla_{[A_1 A_2]} f_{A_3 \ldots A_p}.
\]
by (3.11) \( \text{vanishes by } (3.8) \), again for vanishing of curvatures.

Case 3: \( j = k - 1 \). Recall that \( \Gamma \left( \Lambda^{k-1} E^* \otimes H^* \right) \) \( \Delta_{k-1} \Gamma \left( \Lambda^{k} E^* \right) \rightarrow \Gamma \left( \Lambda^{k+2} E^* \right) \) \( \text{vanishes for } \Delta \left( \Lambda^{k-1} E^* \otimes H^* \right), \) Let us show that for a local section \( f \in \Gamma(\Lambda^{k-1} E^* \otimes H^*), \)
\[
(3.12) \quad (\Delta_{k-1} \partial_{k-1} f)_{A_1 A_2 A_3 \ldots A_p} = \nabla_{A_1} A_2 f_{A_3 \ldots A_p} + 2 \Lambda_{A_1 A_2} \nabla_{A_3 A_4 \ldots A_p} f_{A_3 \ldots A_p} = 0.
\]
Locally we choose a coordinate chart \( U_\alpha \) with trivialization \( E^* |_{U_\alpha} = U_\alpha \times \mathbb{C}^{2n}, \) \( H^* |_{U_\alpha} = U \times \mathbb{C}^{2}, \) and a two-component local quaternionic frame \( \{ Z_{AA'} \} \) such that \( \epsilon \) and \( \epsilon \) are standard. In particular,
Thus, we find that \( \Sigma R \) (here by using (3.5)-(3.6), (3.14)
\[
\varepsilon^{1'o'} = -\varepsilon^{1'1'} = 1. \text{Note that}
\]
\[
\nabla_{A'}[A_1 \nabla_{A_2} ^{A_3} \nabla_{A_{k+2}} ^{f_{A_{k+2}}}] B' = \nabla_0'[A_1 \nabla_0' ^{0'} f_{A_{k+2}}] 0' + \nabla_1'[A_1 \nabla_{A_2} ^{1'} \nabla_{A_{k+2}} ^{f_{A_{k+2}}}] 1' + \nabla_0'[A_1 \nabla_{A_2} ^{0'} \nabla_{A_{k+2}} ^{1'} f_{A_{k+2}}] 1' = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,
\]
where both sides are sections of \( \Sigma_{k+2} \), while the left hand side can be viewed as a section of \( \Sigma_{k+2}^0 \) after contraction of two primed indices. Since
\[
\nabla_{[A} ^{A'} \nabla_{B]} f \ldots = \frac{1}{2} \left( \nabla_{A'} ^{A'} \nabla_{B} - \nabla_{[A} ^{A'} \nabla_{B]} \right) f \ldots,
\]
we have
\[
\Sigma_1 = \nabla_0'[A_1 \nabla_0'^{0'} f_{A_{k+2}}] 0' = \frac{1}{2} \nabla_0'[A_1 \left( -R_{[A_2 A_3]} ^{0'} f_{A_{k+2}} \right) 0' + \nabla_0'[A_1 \nabla_0' ^{1'}] f_{A_{k+2}}] 1' 
\]
\[
= \nabla_0'[A_1 \left( A_2 A_3 f_{A_{k+2}} \right) 1'] = -\Lambda [A_2 A_3 f_{A_{k+2}}] 1',
\]
by using (3.5) and Proposition 2.3 (1). Similarly, we have
\[
\Sigma_4 = \frac{1}{2} \nabla_1'[A_1 \left( -R_{[A_2 A_3]} ^{1'} A' f_{A_{k+2}} \right) 1'] = -\Lambda [A_2 A_3 f_{A_{k+2}}] 1',
\]
To calculate the term \( \Sigma_2 \), applying the trivial identity
\[
(\nabla_0' \nabla_1' f \ldots = \left( \nabla_1' \nabla_0' + (\nabla_0' \nabla_1' - \nabla_0' \nabla_1') \right) f \ldots
\]
and (3.4) for commutators, then antisymmetrising [\( A_1 \ldots A_{k+2} \)], we get
\[
\Sigma_2 = \nabla_1' [A_2 \nabla_1' ^{A_3} \nabla_{A_{k+2}} ^{f_{A_{k+2}}}] 0' = \nabla_1' [A_2 \nabla_0' ^{a'} f_{A_{k+2}}] 0' + \nabla_1' [A_2 \nabla_0' ^{a'} f_{A_{k+2}}] 1' - \nabla_1' [A_2 \nabla_0' ^{1'} A' f_{A_{k+2}}] 1' 
\]
\[
= \nabla_1' [A_2 \nabla_0' ^{a'} f_{A_{k+2}}] 0' + \nabla_1' [A_2 \nabla_0' ^{a'} f_{A_{k+2}}] 1' - \nabla_1' [A_2 \nabla_0' ^{1'} A' f_{A_{k+2}}] 1' 
\]
\[
= \nabla_1' [A_2 \nabla_0' ^{a'} f_{A_{k+2}}] 0' \quad \text{(here } D \text{ is in the } j\text{-th place of } A_3, \ldots, A_{k+2}), \text{since the last two curvature terms cancel by (3.17)}
\]
\[
R_{[A_1 A_2]} ^{1'} A' \varepsilon^{0'o'} = R_{[A_1 A_2]} ^{0'o'} A',
\]
and using (3.6) again. Similarly, we have
\[
\Sigma_3 = -\nabla_1' [A_2 \nabla_1' ^{A_3} \nabla_{A_{k+2}} ^{f_{A_{k+2}}}] 1' = -\nabla_1' [A_2 \nabla_1' ^{a'} f_{A_{k+2}}] 1' + \nabla_1' [A_2 \nabla_1' ^{a'} f_{A_{k+2}}] 1' - \nabla_1' [A_2 \nabla_1' ^{1'} A' f_{A_{k+2}}] 1' 
\]
Thus, we find that \( \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 = -2\Lambda [A_2 A_3 f_{A_{k+2}}] A' \). So (3.12) follows.
Case 4: $j = k$. For $k \geq 1$ and $B' = 0'$, we have
\[
(\mathcal{G}_{\kappa+2}^0 \Delta f)_{A_1 \ldots A_{k+3}}^{B'} = \nabla^B_{A_1} \nabla_A^{|A'|A_2} \nabla_A^{|A'\ldots A_{k+3}} \nabla_A^{|A_{k+3}|} f_{A_1 \ldots A_{k+3}} + 2\nabla^B_{A_1} (\Lambda_{A_2 A_3} f_{A_4 \ldots A_{k+3}})
\]
(3.18)
\[
= \nabla^B_{A_1} \nabla^D_{A_2} f_{A_1 \ldots A_{m+2}} - \nabla^D_{A_1} \nabla^D_{A_2} \nabla^D_{A_3} f_{A_4 \ldots A_{k+3}} + 2\Lambda_{A_2 A_3} \nabla^D_{A_1} f_{A_4 \ldots A_{k+3}}
\]
\[
= 2\nabla^{0'}_{[A_1} \nabla^{0'}_{A_2} \nabla^A_{A_4 \ldots A_{m+2}} + \nabla^{0'}_{[A_1} \nabla^{0'}_{A_2} \nabla^{0'}_{A_3} \nabla^D_{A_4 \ldots |D| A_{m+2}} + 2\Lambda_{A_2 A_3} \nabla^D_{A_1} f_{A_4 \ldots A_{k+3}}
\]
by raising the index $A'$ and applying (3.18) to $\nabla^{0'}_{A_2} \nabla^D_{A_1} f_{A_4 \ldots A_{k+3}}$. The second term in the right hand side vanishes by (3.8), while the first term equals to
\[
R_{[A_1 A_2}^{0'} \nabla^{0'}_{[A_3} \nabla^{0'}_{A_4} f_{A_5 \ldots A_{k+3}]} - R^{0'}_{[A_1 A_2} A_3 A_4} \nabla^{0'}_{A_5} f_{A_6 \ldots A_{k+3}]} = -2\Lambda_{[A_1 A_2} \nabla^{0'}_{A_3} f_{A_4 \ldots A_{k+3]}},
\]
by using Corollary 3.1 again. So $(\mathcal{G}_{\kappa+2}^0 \Delta f)_{B'}$ vanishes for $B' = 0'$. It is similar for $B' = 1'$. If $k = 0$, we can obtain the result by using vanishing of torsions.

The ellipticity will be proved in Subsection 3.2.

Consider conformal transformation
\[
(3.19)
\]
Fix a two-component local quaternionic frame $\{Z_{AA'}\}$ with respect to a volume element $\text{vol}$ in (2.4) and denote $\Upsilon_{AA'} := \Omega^{-1} Z_{AA'} \Omega$. Under the conformal transformation (3.19), define a new connection $\bar{\nabla}$ on the bundles $E^*$ and $H^*$ by
\[
(3.20)
\]
Then it is a quaternionic connection for the unimodular quaternionic structure with respect to the volume $\Omega^{2n+2} \text{vol}$. The curvatures of the unimodular quaternionic connection $\bar{\nabla}_{AA'}$ satisfy
\[
(3.21)
\]
Proposition 3.1. The operators associated to the unimodular quaternionic connection $\bar{\nabla}_{AA'}$ are conformal covariant in the following sense:
\[
(3.22)
\]
(3.22) holds with respect to fixed frame and coframe. The weight factors in (3.22) coincide with that obtained by the representation theory in Proposition 10 in Baston [6]. See Penrose and Rindler [20] section 5.6-5.7 for conformal transformation of spin $k/2$ massless field operator over 4-dimensional manifolds. The conformal change of connections for QCF's was introduced by Bailey and Eastwood (cf. section 2.2 of [5]). See the Appendix for a detailed proof.
3.2. The ellipticity of the $k$-Cauchy-Fueter complex. Locally for a matrix-valued differential operator $L = \sum_{\nu=0}^{n} \sum_{|\nu|=m} A_{\nu} \partial_{x_{\nu_1}} \cdots \partial_{x_{\nu_j}}$ of order $m$ on a $N$-dimensional manifold $M$, its symbol at point $p$ is

$$\sigma(L)(\xi) = \sum_{|\nu|=m} A_{\nu}(p) i\xi_{\nu_1} \cdots i\xi_{\nu_j}, \quad \xi \in \mathbb{R}^N.$$ 

$L$ is called elliptic if $\sigma(L)(\xi)$ invertible for any $0 \neq \xi \in \mathbb{R}^N$. A differential complex is called elliptic if the associated symbol sequence is exact at each point $p$. It is well known that the ellipticity of a differential operator or a differential complex is independent of the choice of the local coordinate charts [19]. For a fixed point $p \in M$, let $\sigma(X_j) = i\xi_j$ and

$$(3.23) \quad (\xi_{AA'}) := \begin{pmatrix} \xi_1 + i\xi_2 & -\xi_3 - i\xi_4 \\ \xi_3 - i\xi_4 & \xi_1 - i\xi_2 \\ \vdots & \vdots \\ \xi_{4n-3} + i\xi_{4n-2} & -\xi_{4n-1} - i\xi_{4n} \\ \xi_{4n-1} - i\xi_{4n} & \xi_{4n-3} - i\xi_{4n-2} \end{pmatrix}.$$ 

In particular, $\sigma(Z_{AA'})(\xi) = \frac{1}{\sqrt{2}} \xi_{AA'}$ by $Z_{AA'}$ in (2.3). For fixed $k$, we use notations

$$\sigma_j(\xi) := \frac{\sqrt{2}}{i} \sigma \left( D_j^{(k)} \right)(\xi), \quad \text{for } j \neq k, \quad \text{and} \quad \sigma_k(\xi) := \frac{2}{i\pi} \sigma \left( D_k^{(k)} \right)(\xi).$$

The ellipticity of the complex (1.3) is given by the following exact sequence of the associated symbols, which can be easily proved by using elementary linear algebra.

**Proposition 3.2.** For any $0 \neq \xi \in \mathbb{R}^{4n}$,

$$(3.24) \quad 0 \longrightarrow \circ^{k} \mathbb{C}^2 \xrightarrow{\sigma_0(\xi)} \Lambda^1 \mathbb{C}^{2n} \otimes \circ^{k-1} \mathbb{C}^2 \xrightarrow{\sigma_1(\xi)} \cdots \xrightarrow{\sigma_{k-1}(\xi)} \Lambda^k \mathbb{C}^{2n} \otimes \circ^{k+1} \mathbb{C}^2 \xrightarrow{\sigma_k(\xi)} \Lambda^k \mathbb{C}^{2n} \otimes \circ^{k+1} \mathbb{C}^2 \longrightarrow 0$$

is exact. Namely, $\ker \sigma_0(\xi) = \{0\}$,

$$(3.25) \quad \ker \sigma_j(\xi) = \text{range } \sigma_{j-1}(\xi)$$

for $j = 1, \ldots, 2n-2$, and $\sigma_{2n-2}(\xi)$ is surjective.

**Proof.** Case 1. $j < k$. We need to show that

$$(3.26) \quad \Lambda^{j-1} \mathbb{C}^{2n} \otimes \circ^{p+1} \mathbb{C}^2 \xrightarrow{\sigma_{j-1}(\xi)} \Lambda^j \mathbb{C}^{2n} \otimes \circ^{p} \mathbb{C}^2 \xrightarrow{\sigma_j(\xi)} \Lambda^{j+1} \mathbb{C}^{2n} \otimes \circ^{p-1} \mathbb{C}^2$$

is exact for $p = k-j$, where the linear mapping $\sigma_j(\xi)$ is given by

$$(3.27) \quad [\sigma_j(\xi) \vartheta]_{A_0 \ldots A_j A'_2 \ldots A'_p} = \xi_{[A_0} \vartheta_{A_1 \ldots A_j] A'_2 \ldots A'_p} = \frac{1}{j+1} \sum_{s=0}^{j} (-1)^s \xi_{A'_s} \vartheta_{A_0 \ldots A_s A'_2 \ldots A'_p},$$

for $\vartheta \in \Lambda^{j} \mathbb{C}^{2n} \otimes \circ^{p} \mathbb{C}^2$, by definitions of symbols and antisymmetrisation. Then

$$[\sigma_j(\xi) \circ \sigma_{j-1}(\xi) \vartheta]_{A_0 \ldots A_j A'_2 \ldots A'_p} = \xi_{[A_0} \xi_{B'_s} \vartheta_{A_1 \ldots A_j] B'_2 \ldots A'_p} = 0$$

by $\xi_{[A_0} = 0$ and $\vartheta$ symmetric in the primed indices.
We can choose a transformation $M \in \text{GL}(n, \mathbb{H})$ such that its complexification matrix $\tau(M) \in \text{GL}(2n, \mathbb{C})$ transforming $\xi := \left( \begin{array}{c} \xi_{00'} \\ \vdots \\ \xi_{(2n-1)0'} \end{array} \right) \neq 0$ to $\tilde{\xi} := \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ \xi_{(2n-1)1'} \end{array} \right)$. Note that $\left( \begin{array}{c} \xi_{00'} \\ \vdots \\ \xi_{(2n-1)1'} \end{array} \right) = -J \left( \begin{array}{c} \xi_{00'} \\ \vdots \\ \xi_{(2n-1)1'} \end{array} \right)$ by (3.23). It follows from (3.29) (iii) for $2 \leq A, A', A''$, ..., $A_p$. Therefore $\vartheta_{A_1, A_2, \ldots, A_p} := \prod_{i=1}^p \tau(M)_{A_i B_i} \vartheta_{B_1, B_2, \ldots, B_p}$. It is direct to see that

$$\left( \bar{\xi}_{AA'} \right) = \tau(M) (\bar{\xi}_{AA'}) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \end{array} \right).$$

Then $\bar{\xi}_{AA'} = \tau(M)_{A} B_{B'} \bar{\xi}_{B'B''}$ by raising indices. Set $\bar{\vartheta}_{A_1, A_2, \ldots, A_p} := \prod_{i=1}^p \tau(M)_{A_i B_i} \vartheta_{B_1, B_2, \ldots, B_p}$. It is direct to see that $\bar{\vartheta}_{A_1, A_2, \ldots, A_p}$ is a transformation for $2 \leq A_1, A_2, \ldots, A_p$. Therefore $\vartheta \in \ker \sigma_j(\xi)$ for $j \geq 1$ if and only if $\vartheta_{A_1, A_2, \ldots, A_p} = 0$, for $2 \leq A_1, \ldots, A_p$. Since $\bar{\xi}_{AA'} = 0$ for such $A$'s; (ii) for $2 \leq A_1, \ldots, A_j$, $\alpha \leq A_1, \ldots, A_j$,

$$\left[ \sigma_j(\xi) \vartheta \right]_{A_1, A_2, \ldots, A_p} = -\frac{1}{j+1} \vartheta_{A_1, A_2, A_3' \ldots, A'_p}, \quad \left[ \sigma_j(\xi) \vartheta \right]_{A_1, A_2, A'_3 \ldots, A'_p} = \frac{1}{j+1} \vartheta_{A_1, A_2, A'_3 \ldots, A'_p};$$

(iii) for $2 \leq A_2, \ldots, A_j, A_j$

$$\left[ \sigma_j(\xi) \vartheta \right]_{0 A_2, \ldots, A_j A'_2 \ldots, A'_p} = -\frac{1}{j+1} (\vartheta_{0 A_2, \ldots, A_j A'_2 \ldots, A'_p} + \vartheta_{0 A_2, \ldots, A_j A'_2 \ldots, A'_p}).$$

Therefore $\vartheta \in \ker \sigma_j(\xi)$ for $j \geq 1$ if and only if

$$\vartheta_{A_1, A_2, A'_2 \ldots, A'_p} = 0, \quad \vartheta_{A_1, A'_2 \ldots, A'_p} = -\vartheta_{0 A_2, \ldots, A_j A'_2 \ldots, A'_p},$$

for any $A_1', A_2, \ldots, A_p' = 0, 1'$ and $2 \leq A_1 < \cdots < A_j$. While for $j = 0$, $\vartheta \in \ker \sigma_0(\xi)$ if and only if $\vartheta_{A'_1 A'_2 \ldots, A'_p} = 0$. So $\vartheta \in \ker \sigma_0(\xi) = \{0\}$. 

To show the surjectivity of $\sigma_{j-1}(\xi)$, we need to find an element $\Theta \in \Lambda^{-1}C^{2n} \otimes \mathbb{C}^{p+1}C^2$ such that $\sigma_{j-1}(\xi)\Theta = \vartheta$ for $\vartheta$ satisfying (3.32). We define such an element $\Theta$ by

$$\Theta_{A_2...A_j\vartheta\rho',A_p'} := j\vartheta_{A_2...A_j\vartheta\rho'A_p'} = -j\vartheta_{0A_2...A_j\vartheta\rho'A_p'},$$

(3.35) for $2 \leq A_2, \cdots, A_j$, and

$$\Theta_{A_2...A_j\vartheta\rho',A_p'} = -j\vartheta_{0A_2...A_j\vartheta\rho'A_p'},$$

(3.36) and all other kind of entries vanish. Obviously, there exists an element $\Theta$ satisfying conditions (3.35)-(3.36), but such $\Theta$ is not unique. Then by definition (3.34) for the symbol $\sigma_{j-1}(\xi)$ again, as in (3.34), it is easy to see that for any $2 \leq A_1, \cdots, A_j$, we have $[\sigma_{j-1}(\xi)\Theta]_{A_1...A_jA_p}\rho' = 0$, and

$$[\sigma_{j-1}(\xi)\Theta]_{0A_2...A_j0\vartheta\rho',A_p'} = -\frac{1}{j}\Theta_{A_2...A_j0\vartheta\rho'A_p'} = \vartheta_{0A_2...A_j0\vartheta\rho'A_p'},$$

(3.35), and

$$[\sigma_{j-1}(\xi)\Theta]_{1A_2...A_j\vartheta\rho',A_p'} = \frac{1}{j}\Theta_{A_2...A_j\vartheta\rho'A_p'} = \vartheta_{1A_2...A_j\vartheta\rho'A_p'},$$

by (3.35) again, and

$$[\sigma_{j-1}(\xi)\Theta]_{0A_2...A_j0\vartheta\rho',A_p'} = -\frac{1}{j}\left(\Theta_{1A_2...A_j\vartheta\rho'A_p'} + \Theta_{0A_2...A_j0\vartheta\rho'A_p'}\right) = \vartheta_{0A_2...A_j\vartheta\rho'A_p'},$$

by (3.36). Thus $\sigma_{j-1}(\xi)\Theta = \vartheta$, and so the sequence (3.26) is exact.

Case 2. $j = k$. We need to show that

$$\Lambda^{-1}C^{2n} \otimes \mathbb{C}^2 \xrightarrow{\sigma_{k-1}(\xi)} \Lambda^{k+1}C^2 \twoheadrightarrow \Lambda^{k+2}C^{2n}$$

(3.37) is exact. For $\vartheta \in \Lambda^{k+2}C^{2n}$, we have

$$[\sigma_{k}(\xi)\vartheta]_{A_1...A_{k+2}} = \xi_{\rho'}A_k \xi_{\rho}A_{k+1} \vartheta_{A_1...A_{k+2}};$$

(3.38) $\sigma_{k}(\xi) \circ \sigma_{k-1}(\xi) = 0$ holds as before. Without loss of generality, we can assume $\xi$ is given by (3.25) as in case 1. We only need to consider unprimed indices with $A_1 < \cdots < A_{k+2}$. Then for this $\xi$, we have

$$[\sigma_{k}(\xi)\vartheta]_{A_1...A_{k+2}} = \begin{cases} 2^{(k+2)(k+1)}\vartheta_{A_3...A_{k+2}}, & \text{if } A_1 = 0, A_2 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

by $\xi_{\rho'}A_{k+1} \xi_{\rho}A_k = 0$ except for $\xi_{\rho'}[\xi_{\rho}]_{k+1} = 1$. Therefore $\ker\sigma_{k}(\xi)$ consists of $\vartheta \in \Lambda^{k+2}C^{2n}$ with $\vartheta_{B_1...B_k} = 0$ for $2 \leq B_1, \cdots, B_k$. On the other hand, for $\Theta \in \Lambda^{k-1}C^{2n} \otimes \mathbb{C}^2$ and $2 \leq A_1, \cdots, A_k$, we have

$$[\sigma_{k-1}(\xi)\Theta]_{A_1...A_k} = 0,$$

(3.35)

$$[\sigma_{k-1}(\xi)\Theta]_{0A_2...A_k} = -\frac{1}{k}\Theta_{A_2...A_k\vartheta}, \quad [\sigma_{k-1}(\xi)\Theta]_{1A_2...A_k} = \frac{1}{k}\Theta_{A_2...A_k\vartheta},$$

$$[\sigma_{k-1}(\xi)\Theta]_{0A_3...A_k} = -\frac{1}{k}\left(\Theta_{1A_3...A_k\vartheta} + \Theta_{0A_3...A_k\vartheta}\right).$$
Hence as in the case 1, if we set \( \Theta_{A_2 \ldots A_k} := -k \vartheta_{0A_2 \ldots A_k} \), \( \Theta_{A_2 \ldots A_k,0} := k \vartheta_{1A_2 \ldots A_k} \) and \( \Theta_{A_3 \ldots A_k,1} + \Theta_{0A_3 \ldots A_k,1} = -k \vartheta_{01A_3 \ldots A_k} \), for \( A_2, \ldots, A_k \geq 2 \), and all other kind of entries vanish, then we get \( \sigma_{k-1}(\xi) \Theta = 0 \). So (3.37) is exact.

**Case 3.** \( j \geq k + 1 \). It is similar. We omit details.

We can consider the associated Laplacian operators as in the flat case \([45]\), some of which are of order 4. This is because \( D_k^{(k)} \)'s are of order 2. They are all self-adjoint elliptic operators. Then by applying the standard theory of elliptic operators \([49]\), we can obtain the following theorem.

**Theorem 3.2.** Suppose that \( M \) is a compact unimodular quaternionic manifold and is right conformally flat if \( \dim_{\mathbb{R}} M = 4 \). Then we have the Hodge-type orthogonal decomposition:

\[
\Gamma \left( \gamma_j^{(k)} \right) = \text{Image} D_j^{(k)} \oplus \text{Image} D_j^{(k)*} \oplus \mathcal{H}_j^{(k)}(M),
\]

where \( \mathcal{H}_j^{(k)}(M) \cong \{ f \in \Gamma(\gamma_j^{(k)}); D_j^{(k)} f = 0, D_j^{(k)*} f = 0 \} \cong H_j^{(k)}(M) \) are finite dimensional.

See the Appendix for a detailed proof. For domains in \( \mathbb{H}^n \), the cohomology groups of the \( k \)-Cauchy-Fueter complex are much more difficult to study. In \([15] [48]\), we got results for the associated Neumann problem over domains in \( \mathbb{H} \).

4. **Vanishing Theorem over quaternionic Kähler manifolds**

We only consider quaternionic Kähler manifolds in this section.

**Theorem 4.1.** Suppose that \( M \) is a compact quaternionic Kähler manifold (right conformally flat if \( \dim_{\mathbb{R}} M = 4 \)) with negative scalar curvature. Then we have \( H_j^{(k)}(M) = \{ 0 \} \) for \( j = 1, \ldots, k - 1 \).

Horan \([22] [23]\) proved the Weitzenböck formula and the vanishing theorem for the first cohomology group of Salamon’s complex. See also theorem 4.3 of Nagatomo-Nitta \([26]\) for vanishing theorem for Salamon’s complexes when the manifolds have negative scalar curvatures, which essentially implies the vanishing of \( H_j^{(k)}(M) \) of the \( k \)-Cauchy-Fueter complex for \( j \geq k + 3 \). See also \([37]\) for indices of Salamon’s complexes.

It is sufficient to prove the associated Weitzenböck formula. In this section, on a quaternionic Kähler manifold, we will choose coordinate charts \( U_\alpha \) with trivialization \( E^*|_{U_\alpha} = U_\alpha \times \mathbb{C}^{2\alpha} \), \( H^*|_{U_\alpha} = U \times \mathbb{C}^2 \), two-component local orthonormal quaternionic frame \( \{ Z_{A,B} \} \) such that \( g, \varepsilon \) and \( \epsilon \) are standard:

\[
(\epsilon_{AB}) = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 1 \\
-1 & 0 
\end{pmatrix}.
\]

\((\epsilon^{AB})\) is the inverse of \((\epsilon_{AB})\). They are used to raise or lower unprimed indices, e.g. \( \nabla^A_{\bar{A}} = \nabla_{A'B} \epsilon^{BA} \). Define inner product locally as

\[
\langle v, w \rangle := \sum_{B_1, \ldots, B_p} v_{B_1, \ldots, B_p} w_{B_1, \ldots, B_p} \overline{w}_{B_1, \ldots, B_p, B'_1, \ldots, B'_q}
\]
for two \(\mathfrak{F}_{q,p}\)-tensor \(v\) and \(w\), and \(|v|^2 := \langle v, v \rangle\). It is obviously well defined globally since it is invariant under the transformation \(v \rightarrow \tilde{v} (w \rightarrow \tilde{w})\) with
\[
\tilde{v}_{A_1 \ldots A_q A'_1 \ldots A'_p} = \prod_{j=1}^{q} \Phi_{A_j} B_j \prod_{l=1}^{p} \Psi_{A'_l} B'_l v_{B_1 \ldots B_q B'_1 \ldots B'_p},
\]
for \(\Phi \in \text{SU}(2n), \Psi \in \text{SU}(2)\). It is an Hermitian inner product. The covariant derivatives can be extended naturally to \((\otimes^j E) \otimes (\otimes^k H) \otimes (\otimes^p E^*) \otimes (\otimes^s H^*)\). Define \(L^2\)-inner product \((f, h) := \int_M \langle f, h \rangle dV_g\), where \(dV_g\) is the volume form \((2.9)\) of the metric \(g\). Denote \(\|f\| := (f, f)^{1/2}\).

Theorem 3.2 is a consequence of the Hodge-type decomposition in Theorem 3.2 and the following Weitzenböck formula, because the right hand side of (4.2) is negative if the scalar curvature is negative.

**Proposition 4.2.** Suppose that \(f \in \Gamma(N^p E^* \otimes \otimes^{k-j} H^*)\) satisfying \(D_j f = 0\), \(D_j f = 0\) on a compact quaternionic Kähler manifold \(M\) of dimension \(4n\) (it is right conformally flat if \(\text{dim}_{\mathbb{R}} M = 4\)), \(j = 1, \ldots, k - 1\). Then we have
\[
\left\| \tilde{\nabla} f \right\|^2 + c_j \left\| \nabla^* f \right\|^2 = \frac{2n - 1 + (-1)^j}{(j + 1)} (p + 2) \int_M \Lambda |f|^2 dV_g,
\]
where \(p = k - j\), \(c_j = \frac{1}{2} (-1)^j (p+1) \geq 0\), \(\Lambda = \frac{s_g}{s_{n(n+2)}}\), and \(s_g\) is the scalar curvature.

**Remark 4.1.** On quaternionic Kähler manifold, Semmelmann and Weingart [34] derived Weitzenböck formulae only involving twisted Dirac and twistor operators, but not for our operators \(D_j\)'s. They [35] obtained universal Weitzenböck formula for all irreducible non-symmetric holonomy groups, and an recursive procedure for obtaining coefficients of Weitzenböck formulas for the holonomy groups \(\text{SO}(n), \text{G}_2\) and \(\text{Spin}(7)\). But the recursive formula for coefficients of Weitzenböck formulae for the holonomy group \(\text{Sp}(n)\text{Sp}(1)\) was not given there. Even if we obtain the recursive formula and know the concrete Weitzenböck formulae, we also need combine several Weitzenböck formulae to obtain the identity (4.2). It will be a tedious algebraic calculation to derive (4.2) from universal Weitzenböck formula in [35]. See also Homma [21] for Weitzenböck formulae over quaternionic Kähler manifolds.

### 4.1. The formal adjoint operators

By two-component notation, we can derive the formal adjoint operators of \(\nabla\) and \(\mathfrak{F}_{q,p}\) explicitly, while there is no proof for them in Horan [22, 24].

**Proposition 4.4.** On a quaternionic Kähler manifold, we have
\[
Z^{AA'} = \overline{Z_{AA'}};
\]
and
\[
\Gamma^{AA'B'}_{AA'B'} = -\Gamma^{AA'}_{B'C'} - \Gamma^{AA'}_{C'B'}.
\]
Moreover, the formal adjoint of \(Z_{AA'}\) is
\[
Z^{AA'} = -Z^{AA'} + \Gamma^{AA'}_{D'C'} A' + \Gamma^{AA'}_{B'C'} B'.
\]

**Proof.** (1) By definition, \(Z^{AA'} := Z_{BB'} \varepsilon^{BA} \varepsilon^{B'A'}\) and \(\epsilon^{10} = -\epsilon^{01} = 1, \epsilon^{1'0'} = -\epsilon^{0'1'} = 1\). It is direct from definition of \(Z_{AA'}\)'s in (2.8) to see that
\[
\overline{Z_{00'}} = -Z_{11'}, \quad \overline{Z_{10'}} = -Z_{01'}, \quad \overline{Z_{11'}} = -Z_{00'}, \quad \overline{Z_{01'}} = -Z_{10'}, \quad \overline{Z_{11'}} = Z_{00'}.\]
(2) On a quaternionic Kähler manifold \( M \), we know that for any \( X \in TM \), the connection \( \omega^i(X) \) on the bundle \( \mathcal{E} \) is \( \mathfrak{su}(2n) \)-valued. If we write \( Z_{AA'} = X + iY \) for some \( X,Y \in TM \), we have

\[
\Gamma_{AA'B}^C = \omega^j(X + iY)_B^C = \omega^j(X)_B^C - i\omega^j(Y)_B^C = -i\omega^j([Z_{AA'}])_C^B = -\Gamma_{AA'}^B C.
\]

Here we have used \( Z_{AA'} = Z^{AA'} \) in \((4.3)\). It is similar for \( \Gamma_{AA'B'}^C \).

(3) Recall that the formal adjoint \( Z_{AA'}^* \) of \( Z_{AA'} \) satisfies

\[
(4.6) \quad \int_M Z_{AA'} f \cdot \overline{\eta} dV_g = \int_M f \cdot \overline{Z_{AA'}}^* \eta dV_g
\]

for any compactly supported scalar functions \( f \) and \( \eta \). Note that \( Xf = i_X df \), where \( i_X \) is the interior operator. By Stokes’ formula, we have

\[
(4.7) \quad \int_M Z_{AA'} (f \overline{\eta}) dV_g = \int_M (f \overline{\eta}) \wedge i_{Z_{AA'}} dV_g = -\int_M f \overline{\eta} \cdot d(i_{Z_{AA'}} dV_g),
\]

where the volume form \( dV_g \) is given by \((2.9)\), \( \{e^{BB'}\} \) is dual to \( \{Z_{BB'}\} \) and

\[
(4.8) \quad d(i_{Z_{AA'}} dV_g) = \sum_{(B,B') \neq (A,A')} de^{BB'} \wedge i_{Z_{BB'}} i_{Z_{AA'}} dV_g.
\]

By the standard exterior differentiation formula, \( d\varphi(X,Y) = \frac{i}{2}[(\nabla_X \varphi)(Y) - (\nabla_Y \varphi)(X) + \varphi(\tau_{X,Y})] \) for \( 1 \)-form \( \varphi \in \Omega^1(M) \) and the torsion \( \tau_{X,Y} = 0 \) since the connection is torsion-free, we find that

\[
2de^{BB'}(Z_{CC'}, Z_{DD'}) = \left(-\Gamma_{CC'E} e^{EB'} - \Gamma_{CC'E'} e^{EB'} - \Gamma_{DD'E} e^{EB'} \right)(Z_{DD'}) = -2de^{BB'}(Z_{CC'}, Z_{DD'}).
\]

Using \( \omega_1 \wedge \omega_2(X,Y) := \frac{i}{2}([\omega_1(X) \omega_2(Y) - \omega_1(Y) \omega_2(X)]), \) we get the Cartan formula

\[
(4.9) \quad \omega_1 \wedge \omega_2 = (\Gamma_{CC'E} e^{EB'} + \Gamma_{CC'E'} e^{EB'}) e^{CC'} \leftrightarrow e^{DD'}.
\]

Note that for fixed \( AA', BB' \), we can write \( dV_g = e^{AA'} \wedge e^{BB'} \wedge \overline{\omega} \) with the \( (4n-2) \)-form \( \omega = i_{Z_{BB'}} i_{Z_{AA'}} dV_g \). Substitute it and \((4.9)\) into \((4.8)\) to get

\[
(4.10) \quad d(i_{Z_{AA'}} dV_g) = -\sum_{(B,B') \neq (A,A')} \left( \Gamma_{AA'B}^B + \Gamma_{AA'B'}^B - \Gamma_{BB'A}^B + \Gamma_{BB'A'}^B \right) i_{Z_{BB'}} i_{Z_{AA'}} dV_g.
\]

Now substituting the conjugate of \((4.10)\) into the right hand side of \((4.7)\), we find that for \( A \) and \( A' \) fixed,

\[
(4.11) \quad Z_{AA'} = -\overline{Z_{AA'}} - \sum_{(B,B') \neq (A,A')} \left( \Gamma_{AA'B}^B + \Gamma_{AA'B'}^B \right) + \sum_{B \neq A} \Gamma_{BB'} A^B + \sum_{B' \neq A'} \Gamma_{BB'} A'_{B'}.
\]

by using definition \((4.10)\) and \( Z_{AA'} = Z_{AA'}^* \) in \((4.3)\). But for \( A \) and \( A' \) fixed, we have

\[
(4.12) \quad \sum_{(B,B') \neq (A,A')} \Gamma_{AA'B}^B = \Gamma_{AA'}^A A + 2 \sum_{B \neq A} \Gamma_{AA'}^B A'_{B'} - \Gamma_{BB'} A^B_{B'}
\]

by \( \text{tr} \Gamma_{a*} = 0 \) since it is also \( \mathfrak{su}(2n, \mathbb{C}) \)-valued, and similarly

\[
(4.13) \quad \sum_{(B,B') \neq (A,A')} \Gamma_{AA'B'}^B = \sum_{B' \neq A'} \Gamma_{AA'}^B A'_{B'} + (2n - 1) \sum_{B'} \Gamma_{AA'}^B B' = -\Gamma_{AA'} A'.
\]

by \( \text{tr} \Gamma_{a*} = 0 \). Now \((4.5)\) follows from substituting \((4.12)\) - \((4.13)\) into \((4.11)\). \(\square\)
Proposition 4.3. On a quaternionic Kähler manifold, we have

1. The formal adjoint $\nabla^* : \Gamma(\mathcal{Q}_{q,p+1}) \to \Gamma(\mathcal{Q}_{q,p})$ of $\nabla$ is given by

$$(\nabla^*)_{A_1...A_qA'_1...A'_p} = -\nabla^{AA'} f_{AA_1...A_qA'_1...A'_p};$$

2. the formal adjoint $\mathcal{D}_{q,p}^* : \Gamma(\Lambda^{q+1}E^* \otimes \Omega^pH^*) \to \Gamma(\Lambda^{q}E^* \otimes \Omega^{p-1}H^*)$ of $\mathcal{D}_{q,p}$ is given by

$$(\mathcal{D}_{q,p}^*)_{A_1...A_qA'_1...A'_p} = \nabla^{A_1}_A f_{AA_1...A_qA'_1...A'_p}.$$ 

Proof. (1) For any local section $h \in \Gamma(\mathcal{Q}_{q,p})$, we have

$$\nabla h, f = \int \nabla_{BB'} h_{A_1...A_qA'_1...A'_p} f_{BA_1...A_qB'A'_1...A'_p} dV_g$$

$$= \int \left[ Z_{BB'} h_{A_1...A_qA'_1...A'_p} - \Gamma_{BB'A_1}^D h_{A_1...D...A_qA'_1...A'_p}^D \right] f_{BA_1...A_qB'A'_1...A'_p} dV_g$$

$$= \int h_{A_1...A_qA'_1...A'_p} \left[ Z_{BB'} - \Gamma_{BB'A_1}^D \right] f_{BA_1...A_qB'A'_1...A'_p} + \int h_{A_1...A_qA'_1...A'_p} \Gamma_{BB'A_1}^D f_{BA_1...A_qB'A'_1...A'_p} dV_g$$

by using (4.5) for the expression of the formal adjoint $Z_{BB'}$ and (4.4) for the connection coefficients. Then we can relabel indices so that the right hand side can be written as $(h, \nabla^* f)$ with

$$(\nabla^*)_{A_1...A_qA'_1...A'_p} = -Z_{A_0A'_0} f_{A_0A_1...A_qA'_1...A'_p} + \Gamma_{A_0A'_0}^{A_0} E_{A_0} f_{A_0A_1...A_qA'_1...A'_p} + \Gamma_{A_0A'_0}^{E'} f_{A_0A_1...A_qA'_1...A'_p}$$

(2) For any local sections $h \in \Gamma(\Lambda^{q+1}E^* \otimes \Omega^pH^*)$ and $f \in \Gamma(\Lambda^{q}E^* \otimes \Omega^{p-1}H^*)$, we have

$$\mathcal{D}_{q,p} h, f = \int \nabla_{A_1}^{A'_1} h_{A_1...A_qA'_1...A'_p} f_{AA_1...A_qA'_1...A'_p} dV_g$$

$$= \int h_{A_1...A_qA'_1...A'_p} \nabla_{BB'}^{A'} f_{BA_1...A_qB'A'_1...A'_p} + \int h_{A_1...A_qA'_1...A'_p} \nabla_{BB'} f_{BA_1...A_qB'A'_1...A'_p} dV_g$$

by using (1) and $\varepsilon^{B'A'} = -\varepsilon_{B'A'}$. Here we drop antisymmetrisation in the second identity by

$$\sum_{B_0,...,B_q} \left( g_{[B_0...B_q]|\mathcal{B}'}, \tilde{g}_{[B_0...B_q]|\mathcal{B}'} \right) = \sum_{B_0,...,B_q} \left( g_{B_0,...,B_q|\mathcal{B}'}, \tilde{g}_{B_0,...,B_q|\mathcal{B}'} \right)$$

for any local sections $g, \tilde{g} \in \Gamma(\otimes^{q+1}E^* \otimes (\otimes^pH^*))$, and add symmetrisation in the last identity by

$$\sum_{B_0,...,B_q} \left( h_{\mathcal{B}'(B_0...B_q)}, \tilde{h}_{\mathcal{B}'(B_0...B_q)} \right) = \sum_{B_0,...,B_q} \left( h_{\mathcal{B}'(B_0...B_q)}, \tilde{h}_{\mathcal{B}'(B_0...B_q)} \right)$$

for any local sections $h, \tilde{h} \in \Gamma(\otimes^{q}E^* \otimes (\otimes^pH^*))$. \(\diamond\)
4.2. A Weitzenböck formula. The Weitzenböck formula for the $k$-Cauchy-Fueter complex is much more complicated than that for the De Rham complex, since we have not only exterior forms, but also symmetric forms. Note that $D_{j-1}^{(k)}f = 0$, i.e. $\mathcal{D}_{q-1,p+1}f = 0$ for $q = j$, $p = k - j$, implies that
\begin{equation}
\nabla_{(B'_1, f|AB_1...B_{q-1}|B'_2...B'_{p+1})}^A = 0.
\end{equation}
by Proposition 1.3 (2). The following is another version of lemma 2.1 of Horan [22] without proof.

**Lemma 4.1.** For a tensor $f \in \Gamma(\Sigma_q)$ such that
\begin{equation}
f_{A_1', A_2', ... A'_r} = f_{A_1'(A_2', ... A'_r)} \quad \text{and} \quad f_{A_1'(A_2', ... A'_r)} = 0,
\end{equation}
we have
\begin{equation}
f_{A_1', A_2', ... A'_r} = -\frac{r - 1}{r} \varepsilon_{A_1'(A_2' f_{A_1'(A_2', ... A'_r)C'})}
\end{equation}
Proof. Suppose that $A_1' + \cdots + A'_r = l$, i.e. there are $l$'s $1'$ in $\{A_1', \ldots, A'_r\}$. Note that (4.18) implies that $f_{A_1'(A_2', ... A'_r)} = \cdots f_{A_1'(A_2', ... A'_r)} + \cdots = 0$, and so
\begin{equation}
(r - l)f_{A_1'(A_2', ... A'_r) + 1} = 0.
\end{equation}
Assume that $A_1' = 0'$ in (4.18). Then
\begin{equation}
\varepsilon_{A_2'(A_2' f_{A_1'(A_2', ... A'_r)C'})} = \frac{l}{r - 1} \varepsilon_{A_1'(A_2' f_{A_1'(A_2', ... A'_r)C'})} = \frac{l}{r - 1} (-f_{A_1'(A_2', ... A'_r) + 1} + f_{A_1'(A_2', ... A'_r) + 1})
\end{equation}
by using
\begin{equation}
f_{A_1'(A_2', ... A'_r) + 1} = 0.
\end{equation}
in the second identity and (4.19) in the last identity. It is similar for $A_1' = 1'$.

 Proof of Proposition 4.1 Since $D_{j-1}^{(k)}f = 0$, i.e. $\mathcal{D}_{q,p}f = 0$ for $q = j$, $p = k - j$, we have
\begin{equation}
0 = (\mathcal{D}_{q,p}f, \mathcal{D}_{q,p}f) = (\mathcal{D}_{q,p}^* \mathcal{D}_{q,p}f, f).
\end{equation}
Then by choosing local orthonormal quaternionic frame as before, locally we have $(\mathcal{D}_{q,p}^* \mathcal{D}_{q,p}f)_{B_1...B_q|B'_1...B'_{p}} = \nabla_{(B'_1, f|AB_1...B_q|A'|B'_2...B'_{p})}^A$. To calculate
\begin{equation}
\left(\nabla_{(B'_1, f|AB_1...B_q|A'|B'_2...B'_{p})}^A, \nabla_{(B_1...B_q|A'|B'_2...B'_{p})}^A\right),
\end{equation}
we only need to calculate the term without symmetrisation by (4.18), i.e.
\begin{equation}
\nabla_{(B'_1, f|AB_1...B_q|A'|B'_2...B'_{p})}^A = \varepsilon_{B' B'_1} \left(\nabla_{(A'|B'_2...B'_{p})}^A + \nabla_{(B'_1|A)|A'|B'_2...B'_{p})}^A\right) f_{B_1...B_q|A'|B'_2...B'_{p}}.
\end{equation}
For the first term in the right hand side, note that
\begin{equation}
\nabla_{(A'|B'_1)}^A = \frac{\varepsilon_{B' B'_1}}{2} \left(\nabla_{A'|0'}^A \nabla_{C|D'}^0 - \nabla_{A'|D'}^0 \nabla_{C|0'}^0\right) = -\frac{\varepsilon_{B' B'_1}}{2} \left(\nabla_{A'|0'} \nabla_{C|0'} + \nabla_{A'|D'} \nabla_{C|D'}\right),
\end{equation}
since by Corollary 3.1 and Proposition 2.2. Note that

Now we get 0 = (S1f + S2f + S3f + S4f + S5f) Bj1...Bj′1...Bj′p.

Recall that \( \Phi = 0 \) for quaternionic Kähler manifold, and so we have

and Proposition 4.3, we have

(4.23)

:= (S1f + S2f + S3f + S4f + S5f) Bj1...Bj′1...Bj′p.

Now we get 0 = (S1f, f) + \ldots + (S5f, f) by (4.21). The reason we use the expansion above is that \( S2f \)
and \( S3f \) are commutators of the form \( \nabla_{[A}^{B'} \nabla_{B]}^{B''} \), which are curvature terms. Obviously by using (4.14) and Proposition 4.3 we have

(4.24)

by antisymmetrisation (4.14) in the second identity.

Recall that \( \Phi = 0 \) for quaternionic Kähler manifold, and so we have

(4.25)

by Corollary 3.1 and Proposition 2.2.

Note that

(4.26)

by using (3.3) for the commutator, Proposition 2.3 for \( \Lambda_{AB} \) and (4.25), and for \( j = 2, \ldots, p, \)

since \( \epsilon^{B'0D'} \) is antisymmetric and \( f_{...} \) is symmetric in primed indices, while for \( j = 0, \)

since \( \epsilon^{B'0D'} \) is antisymmetric and \( f_{...} \) is symmetric in primed indices, while for \( j = 0, \)

(4.27)

by using (4.20) we get

(4.27)
and similarly, applying (4.26) to \( S_3 f \), we get

\[
(S_3 f)_{B_1 \ldots B_s B'_1 \ldots B'_s} = 2 \sum_{s=1}^{q} \frac{(-1)^{s} \varepsilon^A}{q + 1} \varepsilon^{B'_1 B'_s} \nabla^{(B'_1} \nabla^{B'_s)} f_{B_1 \ldots B_s B'_1 \ldots B'_s} = \sum_{s=1}^{p + 2} \frac{p + 2}{q + 1} (-1)^{s} \varepsilon^A \varepsilon C B_s \Lambda f_{B_1 \ldots B_s B'_1 \ldots B'_s} = \frac{1 - (-1)^{p} + 2}{q + 1} \Lambda f_{B_1 \ldots B_s B'_1 \ldots B'_s}.
\]

Hence

\[
(S_2 f, f) + (S_3 f, f) = -\frac{2n + 1 + (-1)^{q}}{2(q + 1)} (p + 2) (\Lambda f, f).
\]

Similarly we have

\[
(S_4 f)_{B_1 \ldots B_s B'_1 \ldots B'_s} = \sum_{s=1}^{q} \frac{1}{2(q + 1)} \nabla^{A'_1} \nabla^{A} f_{A'B_1 \ldots B_s A'B'_1 \ldots B'_s} = \frac{1}{2(q + 1)} \sum_{s=1}^{q} (\nabla^{*} f)_{B_1 \ldots B_s B'_1 \ldots B'_s}
\]

by \( f \) antisymmetric in unprimed indices and the expression of \( \nabla^{*} \) in Proposition 4.3 and

\[
(S_5 f)_{B_1 \ldots B_s B'_1 \ldots B'_s} = -\sum_{s=1}^{q} \frac{1}{2(q + 1)} \nabla^{A'_1} \nabla^{A} f_{A'B_1 \ldots B_s A'B'_1 \ldots B'_s} = \sum_{s=1}^{q} \frac{1}{2(q + 1)} \frac{1}{p + 1} \nabla^{A'_1} \nabla^{A} f_{A'B_1 \ldots B_s A'B'_1 \ldots B'_s} = \sum_{s=1}^{q} \frac{1}{2(q + 1)} (\nabla^{*} f)_{B_1 \ldots B_s B'_1 \ldots B'_s} + \tilde{S}_5 f
\]

where in the second identity we apply Lemma 4.1 to \( \nabla^{A'_1} \nabla^{A} f_{A'B_1 \ldots B_s A'B'_1 \ldots B'_s} \) with \( r = p + 1 \), since the condition of this lemma is satisfied by (4.10). Note that \( (\tilde{S}_5 f, f) = 0 \) by \( f \) symmetric but \( \varepsilon B'_1 B'_s \) antisymmetric in \( B'_1, B'_s \). Therefore

\[
(S_4 f, f) + (S_5 f, f) = \frac{(p + 2)}{2(q + 1)(p + 1)} \sum_{s=1}^{q} (-1)^{s-1} (\nabla^{*} f, \nabla^{*} f) = \frac{(1 - (-1)^{q})(p + 2)}{4(q + 1)(p + 1)} (\nabla^{*} f, \nabla^{*} f).
\]

Substituting (4.24), (4.28) and (4.29) into \( 0 = (S_1 f, f) + \ldots + (S_5 f, f) \), we get the identity (4.2).

\[\square\]

**Appendix A. Proof of Some Propositions**

At first, the traces of curvatures vanish when we antisymmetrise primed (or unprimed) indices and symmetrise unprimed (or primed) indices.

**Proposition A.1.** The curvatures of a unimodular quaternionic manifold satisfy

\[
R_{[A'B'\ldots C]}^{C} = 0, \quad R_{[A'B']C}^{C} = 0, \quad R_{[A'B]C}^{C} = 0, \quad R_{[A'B'C]}^{C} = 0.
\]


Proof. The first Bianchi identity
\[(A.2) \quad R_{abc}^d = 0,\]
and the antisymmetry of $R_{abc}^d$ in $a$ and $b$ implies the cyclicity:
\[(A.3) \quad R_{AA'BB'CC'}^D + R_{BB'CC'AA'}^D + R_{CC'AA'BB'}^D = 0\]
for all $A, A', \ldots$. Take trace over $C'-D'$ in (A.3) to get
\[(A.4) \quad 2R_{A'B'ABC}^D + R_{B'A'BCA}^D + R_{B'A'CAB}^D + R_{BCB'C'A'}^D \delta_A^D + R_{CAC'B'C'}^D \delta_A^B = 0.\]
by the decomposition (2.21) of curvatures. Here and in the sequel we use vanishing of traces in (2.22) repeatedly.

Taking trace over $C-D$ in (A.4), we get
\[(A.5) \quad R_{B'A'BCA}^C + R_{B'A'CAB}^C + R_{BAB'C'A'}^C + R_{BAC'A'B'}^C = 0,\]
by using (2.22) again. Antisymmetrising $[A'B']$ and symmetrising $(AB)$ in (A.5), we get
\[(A.6) \quad R_{[B'A']C(AB)}^C + R_{(AB)C'[A'B']}^C = 0,\]
by using (A.7)
\[(A.7) \quad R_{[B'A']BCA}^D = R_{[B'A']CBA}^D \quad \text{and} \quad R_{(BA)B'C'A'}^D = -R_{(BA)C'B'A'}^D\]
which follows from antisymmetry
\[(A.8) \quad R_{A'B'BAC}^D = -R_{B'A'BCA}^D, \quad R_{ABA'B'C'}^D = -R_{BAB'A'C'}^D,\]
i.e., $R_{abC}D = -R_{baC}D$ and $R_{abC}D' = -R_{baC}D'$. Here (A.7) means that $R_{[B'A']BCA}^D$ is symmetric in $B,C$, while $R_{(BA)B'C'A'}^D$ is antisymmetric in $B',C'$. These identities will be used frequently to change the order of indices.

Taking trace over $B-D$ in (A.4), we get
\[(A.9) \quad 2R_{A'B'ABC}^B + R_{B'A'BCA}^B + R_{ACB'C'A'}^C + 2nR_{CAC'A'B'}^C = 0.\]
Antisymmetrising $[A'B']$ and symmetrising $(AC)$ in (A.9), we see that
\[(A.10) \quad R_{[A'B']B(AC)}^B + (2n+1)R_{(AC)C'[A'B']}^C = 0,\]
by (A.7). Now (A.6) and (A.10) imply that the first and the last identities in (A.1).

Symmetrising $(A'B')$ and antisymmetrising $(AB)$ in (A.5), we get
\[(A.11) \quad R_{[B'A']C(AB)}^C + R_{(AB)C'[B'A']}^C = 0.\]
On the other hand, we take trace over $C-D$ in (A.3) to get
\[(A.12) \quad R_{B'C'BCA}^C \delta_A^{D'} + R_{C'A'CAB}^C \delta_B^{D'} + 2nR_{ABA'B'C'B'}^D + R_{BAB'C'A'}^D + R_{BAC'A'B'}^D = 0.\]
Taking trace over $B'-D'$ in (A.12), we get
\[(A.13) \quad R_{A'C'BCA}^C + 2R_{C'A'CAB}^C + 2nR_{ABA'B'C'B'}^B + R_{BAB'C'A'}^B = 0.\]
Then symmetrising $(A'C')$ and antisymmetrising $(AB)$ in (A.13), we get
\[3R_{(A'C')C(AB)}^C + (2n-1)R_{[AB]B'(A'C')}^B = 0,\]
by using (A.7) again. This together with (A.11) implies that the second and the third identity in (A.1). The proposition is proved. □
Proof of Proposition 2.2. (1) The first identity in (2.24) follows directly from the definition of $\Psi$ in (2.25).

(2) Now antisymmetrising $[A'B']$ and $[AC]$ in (A.9), we see that

$$3R_{[A'B'][AC]}^B - (2n + 1)R_{[AC]C'[A'B']}^{C'} = 0$$

by using (A.7) again. Namely

$$R_{[A'B'][AC]}^C = (2n + 1)\Lambda_{ABC} \varepsilon_{A'B'}$$

by the definition of $\Lambda$ in (2.25). On the other hand symmetrising $(A'B')$ and $[AC]$ in (A.9), we get

$$- R_{(A'B')B(AC)}^B + (2n - 1)R_{(AC)C'(A'B')}^{C'} = 0.$$  

Then by the definition of $\Phi$ in (2.25), we get

$$R_{(A'B')C(AB)}^{C'} = (2n - 1)\Phi_{ABA'B'}.$$ 

(3) Antisymmetrising $[A'B']$ and symmetrising $(AB)$ in (A.4), we get

$$2R_{[A'B'][ABC]}^{D} - 2R_{[A'B'][C(AB)]}^{D} + R_{[CB'C][A'B']}^{C'} \delta_A^D + R_{[CAC][A'B']}^{C'} \delta_B^D = 0,$$

by using the antisymmetry (A.8). It follows from the last identity in (A.1) and the definition of $\Lambda$ in (2.25) that

$$R_{[CB'C][A'B']}^{C'} \delta_A^D = R_{[CB'C][A'B']}^{C'} \delta_A^D = 3\Lambda_{CDB} \delta_A^D \varepsilon_{A'B'}.$$

Thus the sum of the last two terms in (A.16) is equal to $6\Lambda_{C(D\delta_A)^D}$. Then by the trivial identity $2\Lambda_{C(D\delta_A)^D} = \delta_{(C^D\Lambda_A)B} + \delta_{(C^D\Lambda_B)A}$ and the definition of $\Psi$ in (2.25), we see that (A.16) is equivalent to

$$\Psi_{ABC}^{D} = \Psi_{C(AB)}^{D}.$$

Then $\Psi_{ABC}^{D} = \Psi_{C(AB)}^{D}$ since $\Psi_{ABC}^{D}$ is symmetric in $A, B$ by definition. $\Psi_{ABC}^{A} = 0$ by symmetrising $(BC)$ in the definition (2.25) of $\Psi$ and using the first identity in (A.1).

(4) Symmetrise $(A'B')$ and antisymmetrise $[AB]$ in (A.4) to get

$$2R_{(A'B')ABC}^{D} + 2R_{(A'B')C(AB)}^{D} - R_{[CB'C][A'B']}^{C'} \delta_A^D + R_{[CAC][A'B']}^{C'} \delta_B^D = 0.$$

Apply (A.1) and the definition of $\Phi$ in (2.25) to the last two terms above to get

$$R_{(A'B')ABC}^{D} + R_{(A'B')C(AB)}^{D} - \delta_{[A^D}\Phi_B|^C_{A'B'} = 0.$$  

Now antisymmetrising $[ABC]$ above, we find that

$$R_{(A'B')[ABC]}^{D} = 0,$$

which is equivalent to

$$R_{(A'B')ABC}^{D} + 2R_{(A'B')[CAB]}^{D} = 0,$$

by $R_{(A'B')ABC}^{D}$ antisymmetric in $A, B$. Substitute this into (A.17) to get the second identity in (2.24).

(5) To show the last identity in (2.24), antisymmetrise $[AB]$ in (A.12) to get

$$- R_{C'B'C[BA]}^{C} \delta_A^{D'} + R_{C'A'C[AB]}^{C} \delta_B^{D'} + 2nR_{[AB][C'A'B']}^{D'} - 2R_{[AB][C'[A'B']]^{D'} = 0}$$
by using (A.7) to the first term. Then we have
\begin{equation}
(2n + 1)\Lambda_{AB} \left( \varepsilon_{C'B'}\delta_{A'D'} + \varepsilon_{C'A'}\delta_{B'D'} \right) + 2nR_{[AB]A'B'C'}^{D'} - 2R_{[AB]C'(A'B')}^{D'} = 0,
\end{equation}
by (A.1) and (A.14). Symmetrising \((A'B'C')\) in the above identity, we get
\begin{equation}
(2n - 1)R_{[AB]A'B'C'}^{D'} = 0.
\end{equation}
If \(n \neq 1\), it is equivalent to
\begin{equation}
R_{[AB]A'B'C'}^{D'} + 2R_{[AB]C'(A'B')}^{D'} = 0,
\end{equation}
by \(R_{[AB]A'B'C'}^{D'}\) symmetric in \(A'\) and \(B'\). Substitute this into (A.19) to get the last identity in (2.24). If \(n = 1\), (2.27) follows from the definition of \(\Psi_{A'B'C'}^{D'}\). We have \(\Psi_{A'B'C'}^{D'} = \Psi_{(A'B'C')}^{D'}\) as in part (3) by exchanging the primed and unprimed indices.

(6) At last, we symmetrise \((AB)\) in (A.12) to get
\[ -R_{C'B'C(AB)}^{C} \delta_{A'D'} + R_{C'A'C(AB)}^{C} \delta_{B'D'} + 2nR_{(AB)A'B'C'}^{D'} - R_{(AB)C'B'A'}^{D'} + R_{(AB)C'A'B'}^{D'} = 0, \]
by using (A.7) again. By (A.1), (A.15) and the second identity in (2.24), we find that
\begin{equation}
(2n - 1) \left( -\Phi_{ABC'B'}\delta_{A'D'} + \Phi_{ABC'A}\delta_{B'D'} \right) + 2nR_{(AB)A'B'C'}^{D'} + 2R_{(AB)C'[A'B']}^{D'} = 0.
\end{equation}
Antisymmetrise \([A'B'C']\) above to get
\begin{equation}
(2n + 2)R_{(AB)[A'B'C']}^{D'} = 0.
\end{equation}
Namely
\begin{equation}
R_{(AB)A'B'C'}^{D'} + 2R_{(AB)C'[A'B']}^{D'} = 0.
\end{equation}
Substitute this into (A.22) to get
\[ R_{(AB)A'B'C'}^{D'} = \Phi_{ABC'B'}\delta_{A'D'} - \Phi_{ABC'A}\delta_{B'D'}. \]
The third identity in (2.24) is obtained. The proposition is proved.

Proof of Proposition 2.3 (1) By the second Bianchi identity
\begin{equation}
\nabla_{[a}R_{bc]d}^{c} = 0.
\end{equation}
for \(a = AA', b = BA', c = CA'\) and \(d = DD', e = DA'\), and taking summation over repeated indices, we find that
\[ \nabla_{A'[A}R_{[A'B']|BC]D}^{D} \delta_{D}' A' + \nabla_{A'[A}R_{BC][A'B']D}^{D} \delta_{D}' A' = 0. \]
The first term vanishes as a trace by (2.22), while the second term is \(2\nabla_{A'[A}A_{BC]}\varepsilon_{A'D'}\delta_{D}^{D'}\) by the last identity of (2.24) and (2.27) for the 4-dimensional case.

(2) Note that \(\nabla_{BB'}\epsilon_{CD} = 0\) implies \(R_{(A'B')ABC}^{E} \epsilon_{ED} + R_{(A'B')ABD}^{E} \epsilon_{CE} = 0\), or equivalently
\[ -\epsilon_{D[A}\Phi_{B]C'A'B'} + \epsilon_{C[A}\Phi_{B]D'A'B'} = 0. \]
by Proposition 2.2 If we antisymmetrise \([DAB]\), we get \(\epsilon_{[DA}\Phi_{B]CA'B'} = 0\), which can hold only if \(\Phi_{ABA'B'} = 0\).
The Ricci curvature of the connection on tangent bundle is given by
\[ R_{ac} = R_{abcd}g^{bd} = R_{AA'B'B'C'C'D'D'}\epsilon^{BD}\epsilon^{D'} = R_{AA'B'B'C'C'B'B'} \]
\[ = R_{A'B'B'C'A'B'C}'\delta^C_B + R_{ABA'B'B'C}'\delta^C_B = 2(n+2)\Lambda_{AC}\epsilon^A'C', \]
by the curvature decomposition in Proposition 2.22 if we choose the local orthonormal quaternionic frame \( Z_{AA'} \) so that the metric is \( g_{ab} \) in (2.20). Then
\[ s_g = R_{ac}g^{ac} = 2(n+2)\Lambda_{AC}\epsilon^A'C'\epsilon^{A'C'} = 8n(n+2)\Lambda. \]

It remains to show \( \Lambda_{AB} = \Lambda\epsilon_{AB} \). Recall that for any given section \( \phi_{AB} \) in \( \Gamma(\Lambda^2E^*) \) satisfying \( \nabla a\phi_{AB} = 0 \), we must have \( \phi_{AB} = \alpha\Lambda_{AB} \) for some constant \( \alpha \) when the tensor \( \Lambda_{AB} \neq 0 \) (this is lemma 7.7 of [5]). This is because \( \nabla a\phi_{CD} = 0 \) implies \( R_{[A'B']AB'C'}^E\phi_{ED} + R_{[A'B']ABD}^E\phi_{CE} = 0 \), i.e.
\[ (A.24) \]
\[ \Psi_{ABC}^E\phi_{ED} + \Psi_{ABD}^E\phi_{CE} - 2\Lambda_{C(A\phi_{BD})D} + 2\Lambda_{D(A\phi_{BD})C} = 0 \]
by the curvatures decomposition in Proposition 2.22. Symmetrising \( (ABC) \) followed by antisymmetrising \( [CD] \) kills the \( \Lambda \) terms, but simply multiplies the \( \Psi \) terms by \( \frac{1}{4} \). Thus \(-2\Lambda_{C(A\phi_{BD})D} + 2\Lambda_{D(A\phi_{BD})C} = 0 \), which is essentially just the wedge product of two exterior forms. It is easy to see that it vanishes if and only if \( \phi_{AB} \) is a multiple of \( \Lambda_{AB} \).

**Proof of (3.21).** Locally we choose a coordinate chart \( U_\alpha \) with trivialization \( E^*|U_\alpha = U_\alpha \times \mathbb{C}^2 \), and a two-component local quaternionic frame \( \{ Z_{AA'} \} \) such that \( \varepsilon \) and \( \epsilon \) are standard. In particular, \( \varepsilon_{0'1'} = \epsilon^{0'1'} = 1 \). By using (3.20) repeatedly, we get
\[ \nabla_{AA'}\nabla_{BB'}f_{C'} = \nabla_{AA'}(\nabla_{BB'}f_{C'} - \Upsilon_{BC'}f_{B'}) \]
\[ = \nabla_{AA'}\nabla_{BB'}f_{C'} - \nabla_{AA'}\Upsilon_{BC'}f_{B'} - \Upsilon_{BC'}\nabla_{AA'}f_{B'} \]
\[ - \Upsilon_{BA'}\nabla_{AB'}f_{C'} - \Upsilon_{AB'}\nabla_{BA'}f_{C'} - \Upsilon_{AC'}\nabla_{BB'}f_{A'} \]
\[ + \Upsilon_{BA'}\Upsilon_{AC'}f_{B'} + \Upsilon_{AC'}\Upsilon_{BA'}f_{B'} + \Upsilon_{AB'}\Upsilon_{BC'}f_{C'}. \]

Note that when \( A \) and \( B \), \( A' \) and \( B' \) are exchanged, the sum of the 3rd and 6th terms, the sum of 4th and 5th terms, and the sum of 7th and 9th terms in the right hand side above are all invariant. The above identity minus the one with \( A \) and \( B \), \( A' \) and \( B' \) exchanged becomes
\[ (A.25) \]
\[ -\widetilde{R}_{ABA'B'C'}D' = -R_{ABA'B'C'}D' - \nabla_{AA'}\Upsilon_{BC'}\delta_{B'D'} + \nabla_{BB'}\Upsilon_{AC'}\delta_{A'D'} \]
\[ + \Upsilon_{AC'}\Upsilon_{BA'}\delta_{B'D'} - \Upsilon_{BC'}\Upsilon_{AB'}\delta_{A'D'}. \]

Note that \( \Lambda_{AB} = \frac{1}{3}R_{[AB]}^C'[0'1'], \Omega\Lambda_{AB} = \frac{1}{3}\widetilde{R}_{[AB]}^C'[0'1'], \) by definition of \( \Lambda \) in (2.25) and
\[ (A.26) \]
\[ f_{0'1'...}B'... = f_{0'1'...}A'... - f_{0'1'...}A'... = -f_{0'1'...}B'... \]
Antisymmetrise \([AB] \) and \([B'C'] = [0'1'] \) and take trace over \( A'-D' \) in (A.25) to get the transformation formula for \( \Lambda \). We have used, for example, \( \nabla_{0'[A}\Upsilon_{B][} - \nabla_{[A}\Upsilon_{B][} = \nabla_{A'[A}\Upsilon_{B'][} \).

By definition of \( \Psi \) in (2.25), symmetrise \((B'C') \) and \((AB) \) and take trace over \( A'-D' \) in (A.25) to get the transformation formula for \( \Psi \) in (3.21).

Similarly as (A.25) by exchanging primed and unprimed indices, we get
\[ -\widetilde{R}_{ABA'B'C'}D = -R_{ABA'B'C'}D - \nabla_{AA'}\Upsilon_{CB'}\delta_{B'D} + \nabla_{BB'}\Upsilon_{CA'}\delta_{A'D} + \Upsilon_{CA'}\Upsilon_{AB'}\delta_{B'D} - \Upsilon_{CB'}\Upsilon_{BA'}\delta_{A'D}, \]
from which we get the invariance of \( \Psi \) in (3.21) by antisymmetrising \([A'B'] = [0'1'] \) and using the transformation formula for \( \Lambda \) and the definition of \( \Psi \) in (2.25). □
Proof of Proposition 3.1. The new connection $\tilde{\nabla}$ in (3.20) induces the covariant derivatives for $E$ and $H$ by duality:

$$\tilde{\nabla}_{AA'}f^{B'} = \nabla_{AA'}f^{B'} + \Theta_{AA'D'}f^{D'} = \nabla_{AA'}f^{B'} + \delta_{A'A'}\nabla_{AD'}f^{D'}.$$  

(A.27)

If we raise unprimed indices by $\tilde{\varepsilon}^{A'B'} = \Omega^{-1}\varepsilon^{A'B'}$ (note that $\nabla\tilde{\varepsilon}^{A'B'} = 0$), we get

$$\tilde{\nabla}_A^{A'}f^{B'} = \nabla_{AC'}f^{B'}\tilde{\varepsilon}^{C'A'} = \Omega^{-1}\left(\nabla_A^{A'}f^{B'} - \varepsilon^{A'B'}\nabla_{AD'}f^{D'}\right),$$

(A.28)

Applying these formulas of covariant derivatives repeatedly, we get

$$\left(\tilde{\theta}_{q}(\Omega^{-q}f)\right)_{AA'...A_{q+1}} = -q\Omega^{-q-1}\tilde{\nabla}^{(A'_{1}}\Omega \cdot f_{A_{2}...A_{q+1})} + \Omega^{-q}\tilde{\nabla}^{(A'_{1}}f_{A_{2}...A_{q+1})}$$

(A.29)

Here $\tilde{\nabla}_{A'_{1}}\Omega = \tilde{\nabla}_{A'_{1}}\Omega \cdot \tilde{\varepsilon}^{A'\Omega}_{A'_{1}} = Z_{A'_{1}}\varepsilon^{A'_{1}}\Omega = \Omega^{-1}Z_{A'_{1}}A'_{1}$, and the third term in the bracket above vanishes automatically. While for the fourth term in the bracket, we have

$$\nabla^{(A'_{1}}|_{A_{1}}f_{A_{2}...A_{q+1)}) = \nabla^{(A'_{1}}|_{A_{1}}f_{A_{2}...A_{q+1})}, \quad j = 2, \ldots, q+1,$$

(A.30)

and so the first and fourth terms in (A.29) cancel. The first identity in (3.22) follows. Similarly, we have

$$\left(\tilde{\theta}_{q,p}(\Omega^{-q-1}f)\right)_{A_{1}...A_{q+2}} = -\left((q+1)\Omega^{-q-2}\nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}' + \Omega^{-q-1}\tilde{\nabla}^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}'\right.$$  

$$=\left\{-(q+1)\nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}' + \nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}' - \nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}'\right.$$  

$$- \nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+1})}A_{1}'A_{2}'...A_{p}'\right\}.$$  

The last term in the bracket above vanishes by (A.20) and $f$ symmetric in primed indices. We also use (A.20) to raise and lower $A'$ in the 4th term. The result follows from an identity similar to (A.30) that the sum of the third and 4th terms in the bracket cancels the first term. We get the second identity in (3.22). At last

$$\tilde{\nabla}_{A'|_{A_{1}}A_{2}'}(\Omega^{-q-1}f)_{A_{3}...A_{q+2}}$$  

$$= \tilde{\nabla}_{A'|_{A_{1}}}\left\{\Omega^{-q-2}\left(-(q+1)\nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+2}}) + \nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+2}}) - \nabla^{A'|_{A_{1}}f_{A_{2}...A_{q+2}})\right)\right\}$$  

$$= \tilde{\nabla}_{A'|_{A_{1}}}\left\{\Omega^{-q-2}\left(-\nabla^{A'|_{A_{1}}A_{2}'}f_{A_{3}...A_{q+2}}\right)\right\}$$  

$$= \Omega^{-q-2}\left\{(q+2)\nabla^{A'|_{A_{1}}}A_{2}' - (q+2)\nabla^{A'|_{A_{1}}}A_{2}' - \nabla^{A'|_{A_{1}}}A_{2}' + \nabla^{A'|_{A_{1}}}A_{2}' - \nabla^{A'|_{A_{1}}}A_{2}' + \nabla^{A'|_{A_{1}}}A_{2}'\right)$$  

$$+ \Omega^{-q-2}\left(\nabla^{A'|_{A_{1}}}A_{2}' - \nabla^{A'|_{A_{1}}}A_{2}'\right)\right\}.$$
where $m = 3, \ldots, q + 2$, $j = 2, \ldots, q + 2$. Here we use the identity as (A.30) in the second identity. Note that terms of the form $\Upsilon_{A'[A_1]} \nabla_{A_2} A' f_{\ldots}$ cancel each other by $- \Upsilon_{A'[A_1]} \nabla_{A_2} A' f_{\ldots} = - \Upsilon_{A'[A_1]} \nabla_{A_2} A'$ by using (A.20) to raise and lower primed indices, and only one of terms of the form $- \Upsilon_{A'[A_1]} \nabla_{A_2} A' f_{\ldots}$ remains. So

$$\tilde{\nabla}_{A'[A_1]} \tilde{\nabla}_{A_2} (\Omega - 1)_{A_3 \ldots A_{q+2}} = \Omega - 1_{2} ( - \Upsilon_{A'[A_1]} \Upsilon_{A_2} f_{\ldots} - \nabla_{A'[A_1]} \nabla_{A_2} A') f_{A_3 \ldots A_{q+2}}$$

$$= \Omega - 1_{2} ( - 2 \Omega_{A_3 A_4} + 2 \Lambda_{A_1 A_2} + \nabla_{A'[A_1]} \nabla_{A_2} A') f_{A_3 \ldots A_{q+2}},$$

by conformal transformation formula for $\Lambda$ in (3.21). We get the last identity in (3.22).

We need the following theorem to prove Theorem 3.2.

**Theorem A.1.** (Theorem 4.12 in [49]) Let $L$ be a self-adjoint and elliptic differential operator of order $m$ on a vector bundle $F$ over a compact manifold. Then there exist linear mappings $H_L, G_L : C^\infty(X, F) \to C^\infty(X, F)$ so that

1. $H_L(C^\infty(X, F)) = \delta_L(E)$ and $\dim \delta_L(E) < \infty$.
2. $L \circ G_L + H_L = G_L \circ L + H_L$ is the identity on $C^\infty(X, F)$.
3. $H_L$ and $G_L$ extend to bounded linear operators on $L^2(X, F)$.
4. $C^\infty(X, F) = \delta_L(F) \oplus L(\delta_L(C^\infty(X, F)) = \delta_L(F) \oplus G_L \circ L(C^\infty(X, F))$ is an orthogonal decomposition with respect to the inner product in $L^2(X, F)$. In particular, we have $\delta_L(F) = \ker L$.

**Proof of Theorem A.2.** We only give the proof of case $j = k$. Other cases are similar (see [48] [45] for the flat case).

We will omit the superscript $(k)$ for simplicity. Applying Theorem A.1 to the elliptic differential operators $L = \Box_k$ of 4-th order, we see that that there exists a partial inverse operator $G_k : \Gamma (\mathcal{V}_k) \to \Gamma (\mathcal{V}_k)$ such that

$$G_k \Box_k = \Box_k G_k = \text{id} \quad \text{on} \quad (\ker \Box_k)'.

To see $\ker \Box_k = \mathcal{V}_k(M)$, note that $(\Box_k f, f) = 0$ implies that $(D_k f, D_k f) = 0$ and $(D_k - 1 D_k^* f, D_k - 1 D_k^* f) = 0$, i.e. $D_k f = 0$ and $D_k - 1 D_k^* f = 0$. Then we have $D_k - 1 f = 0$ by

$$(D_k - 1 D_k - 1 f, f) = (D_k - 1 f, D_k - 1 f) = 0.

Now suppose that $f \in \Gamma (\mathcal{V}_k)$ such that $f \perp \text{Image } D_k^* \oplus \mathcal{V}_k(M)$. Then obviously $f \in (\ker \Box_k)$ since $\ker \Box_k = \mathcal{V}_k(M)$, and $D_k f = 0$ by $(D_k f, h) = (f, D_k h) = 0$ for any $h \in \Gamma (\mathcal{V}_k)$.

The decomposition (3.39) holds if we can show $f \in \text{Image } D_k$.

If set

$$u := D_k - 1 D_k - 1 D_k^* - 1 G_k f \in \Gamma (\mathcal{V}_k),$$

we have

$$D_k - 1 u = (D_k - 1 D_k - 1)^2 G_k f = ((D_k - 1 D_k - 1)^2 + D_k^* D_k) G_k f = \Box_k G_k f = f,

by (A.31) in the last identity and

$$D_k^* D_k G_k f = 0.

This is because

$$(D_k^* D_k G_k f, D_k^* D_k G_k f) = (D_k G_k f, D_k D_k^* D_k G_k f) = (D_k G_k f, \Box_k G_k f) = 0

by (A.32)

$$D_k (D_k^* D_k + (D_k - 1 D_k - 1)^2) = D_k D_k^* D_k = ((D_k - 1 D_k - 1)^2 + D_k D_k^*) D_k.

$$
and so

\[ \Box_{k+1} D_k G_k f = D_k \Box_k G_k f = D_k f = 0. \]

The commutating relation \((A.33)\) follows from \(D_k D_{k-1} = D_{k+1} D_k = 0\). The result follows. \(\square\)

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