Bounds on the Average Distance and Distance Enumerator with Applications to Non-Interactive Simulation

Lei Yu and Vincent Y. F. Tan, Senior Member, IEEE

Abstract

We leverage proof techniques in coding theory and Fourier analysis to derive new bounds for the problem of non-interactive simulation of random variables. Previous bounds in the literature were derived by applying data processing inequalities concerning maximal correlation or hypercontractivity. We show that our bounds are sharp in some regimes, and are also tighter than the existing ones in some other regimes. Furthermore, as by-products of our analyses, various new properties of the average distance and distance enumerator of binary block codes are established.

Index Terms

Non-interactive simulation, distance distribution, average distance, distance enumerator, noise stability, Fourier analysis

I. INTRODUCTION

Given a joint distribution $P_{XY}$, assume that $(X, Y) \sim P_{XY}$ (i.e., $(X, Y)$ are $n$ i.i.d. copies of $(X, Y) \sim P_{XY}$) is a pair of correlated memoryless sources and $(U, V)$ are random variables such that $U - X - Y - V$ forms a Markov chain. A natural question arises: What are the possible joint distributions of $(U, V)$? This problem is termed Non-Interactive Simulation of Random Variables [1]. In this paper, we restrict $X, Y, U, V$ to be Boolean random variables taking values in $\{-1, 1\}$ and $P_{XY}$ to be a Boolean symmetric distribution (or doubly symmetric binary source), i.e.,

$$P_{XY} = \begin{bmatrix} -1 & \frac{1+\rho}{2} & \frac{1-\rho}{2} \\ \frac{1+\rho}{2} & 1 & \frac{1-\rho}{2} \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & 1 \end{bmatrix}$$

(1)

with $\rho \in [-1, 1]$. Since the joint distribution of $U$ and $V$ is determined by the triple $(\mathbb{P}(U = 1), \mathbb{P}(V = 1), \mathbb{P}(U = V = 1))$, the closure of the set of the possible joint distributions of $(U, V)$ is determined by the following two quantities.

$$\lim_{n \to \infty} \min_{P(U=1)=a, \mathbb{P}(V=1)=b} \mathbb{P}(U = V = 1)$$

(2)

and

$$\lim_{n \to \infty} \max_{P(U=1)=a, \mathbb{P}(V=1)=b} \mathbb{P}(U = V = 1)$$

(3)

This problem appears to be non-trivial. Indeed, (2) and (3) have been addressed only for the case $a = b = \frac{1}{2}$ or the trivial cases $a \in \{0, 1\}$ or $b \in \{0, 1\}$ in the literature [2], [3]. We make progress for other values of $a$ and $b$ in this paper.

This problem is motivated naturally by several models in cryptography. Suppose two terminals respectively take possession of correlated sources $X$ and $Y$, and want to generate keys $U$ and $V$ respectively, such that $U$ and $V$ agree with as high a probability as possible. Then what is this maximal agreement probability? This Non-Interactive Secret Key Agreement Problem (and also termed Non-Interactive Correlation Distillation Problem [4]–[6]) is implied by the non-interactive simulation problem described above. Therefore, studying the non-interactive simulation problem is not only of theoretical significance, but is also of tremendous applicabilities, especially in cryptography. The non-interactive simulation problem or the non-interactive correlation distillation problem can be also interpreted from the perspectives of noise-stability (or noise-sensitivity); see [5]. For noise-corrupted versions $X^{(i)}, 1 \leq i \leq k$ of a source $X$, under the expectation constraints $\mathbb{E}[f_i(X^{(i)})] = c_i \in \mathbb{R}$, the Boolean functions $f_i$ that maximize the probability $\mathbb{P}(f_1(X^{(1)}) = \ldots = f_k(X^{(k)}))$ are the “most stable” to noise over all Boolean functions with expectations $c_i$. It is known that when $c_i = \frac{1}{2}, 1 \leq i \leq k$ (i.e., the Boolean functions are balanced) and the number of parties $k$ is 2 or 3, the optimal functions are dictator functions (i.e., $j$th-bit functions for $1 \leq j \leq n$) [3], [5]. Furthermore, noise-stability were also studied in related works [7], [8] but with a different definition of “noise-stability”.

The authors are with the Department of Electrical and Computer Engineering, National University of Singapore (Emails: leiyu@nus.edu.sg, vtan@nus.edu.sg). V. Y. F. Tan is also with the Department of Mathematics, National University of Singapore.
It can be verified that the non-interactive simulation problem is equivalent to a coding-theoretic problem. A subset $C$ of $\{-1, 1\}^n$ with size $M$ is called a binary $(n, M)$-code. The average distance of $C$ is defined as the average Hamming distance of every pair of codewords in $C$. Ahlswede and Katona [9] posed the following problem concerning the extremal combinatorics in Hamming space: For every $1 \leq M \leq 2^n$, determine the minimum of the average distance of $C$ over all sets $C \subseteq \{-1, 1\}^n$ of a given cardinality $M$. Kündgen [10] observed that this problem is equivalent to a covering problem in graph theory. Althöfer and Sillke [11], as well as, Fu, Xia, together with other authors [12]–[15] proved various bounds on the minimum average distance problem.

In this paper, we study properties of the average distance and distance enumerator for the case of two codes $A$ and $B$. Then by combining Fourier analysis techniques and coding-theoretic results on the minimum average distance problem, we derive bounds on the distance enumerator. By the equivalence between the distance enumerator and the non-interactive simulation, bounds on the former imply some nontrivial results on the latter. Moreover, they are sharp in certain regimes.

This paper is organized as follows. In Section II, we introduce the definitions of several quantities, including the distance distribution, the average distance, and the distance enumerator. We provide the formulation of the non-interactive simulation problem. In Section III, we study properties of the average distance and the distance enumerator. In Section IV, these properties are applied to derive bounds on the distance enumerator, or equivalently, bounds for the non-interactive simulation problem.

II. DEFINITIONS

A. Distance Distributions

For two subsets of the Boolean hypercube (termed codes) $A, B \subseteq \{-1, 1\}^n$, the distance distribution between $A$ and $B$ is a probability mass function $P(A, B)$ such that for $i \in \{0, 1, \ldots, n\}$,

$$P(A, B)(i) := \frac{1}{|A||B|} |\{(x, x') \in A \times B : d_H(x, x') = i\}|,$$

where $d_H(x, x') := |\{i : x_i \neq x_i'\}|$ denotes the Hamming distance between vectors $x, x'$ (i.e., the number of components of $x$ and $x'$ that differ). It is clear that $P(A, B)(0) = \frac{|A \cap B|}{|A||B|}$, $\sum_{i=0}^{n} P(A, B)(i) = 1$, and $P(A, B)(i) \geq 0$ for $i \in \{0, 1, \ldots, n\}$.

Define the $k$-th distance moment and the distance enumerator between $A, B \subseteq \{-1, 1\}^n$ respectively as

$$D_k(A, B) := \frac{1}{|A||B|} \sum_{x \in A} \sum_{x' \in B} d_H^k(x, x') = \sum_{i=0}^{n} P(A, B)(i) \cdot i^k,$$

and

$$\Gamma_z(A, B) := \frac{1}{|A||B|} \sum_{x \in A} \sum_{x' \in B} z^{d_H(x, x')} = \sum_{i=0}^{n} P(A, B)(i) \cdot z^i, \quad z \geq 0.$$

Clearly, $D_k(A, B)$ and $\Gamma_z(A, B)$ are respectively the $k$-th moment and the generating function of $P(A, B)$. For $k = 1$, $D_1(A, B) =: D(A, B)$ corresponds to the average distance. Furthermore, for $z = 1$, $\Gamma_1(A, B) = 1$.

By Taylor’s expansion, $D_k(A, B)$ and $\Gamma_z(A, B)$ admit the following relationship. For $z \geq 0$,

$$\Gamma_z(A, B) = \frac{1}{|A||B|} \sum_{x \in A} \sum_{x' \in B} e^{d_H(x, x')} \log z$$

$$= \frac{1}{|A||B|} \sum_{x \in A} \sum_{x' \in B} \sum_{k=0}^{\infty} \frac{(d_H(x, x') \log z)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(\log z)^k}{k!} D_k(A, B).$$

Thus there is an intimate relationship between $\Gamma_z(A, B)$ and $D_k(A, B)$ for $k = 0, 1, 2, \ldots$.
The dual distance distribution between codes $A, B$ is defined by

$$Q^{(A,B)}(i) := \frac{1}{|A||B|} \sum_{u \in \{0,1\}^{n}} \left( \sum_{x \in \{0,1\}^{n}} (-1)^{(u,x)} \right)^i \left( \sum_{x \in \{0,1\}^{n}} (-1)^{(u,x)} \right), \quad i = 0, 1, ..., n,$$

(4)

where $w_H(u) := d_H(u, 0)$ denotes the Hamming weight (i.e., the number of nonzero components) of a vector $u$, and $(u,x) := (\sum_{i=1}^{n} u_i x_i) \mod 2$ denotes the inner product of vectors $u, x \in \mathbb{F}_2^n$. It is worth noting that $Q^{(A,B)}(i)$ could be negative for some $i$, and $\sum_{i=0}^{n} Q^{(A,B)}(i)$ could be smaller than 1 or larger than 1. Hence in general, $Q^{(A,B)}$ is not a probability distribution.

The dual distance enumerator between $A, B$ is defined as

$$\Pi_z (A, B) := \sum_{i=0}^{n} Q^{(A,B)}(i) \cdot z^i, \quad z \geq 0.$$

It is easy to verify that the following MacWilliams–Delsarte identities hold.

$$\Pi_z (A, B) = (1+z)^n \Gamma_{1-z \over 1+\frac{z}{2}} (A, B)$$

(5)

$$\Gamma_z (A, B) = \left( \frac{1+z}{2} \right)^n \Pi_{1-z \over 1+\frac{z}{2}} (A, B).$$

(6)

If $A = B$, then $Q^{(A,A)}(i) \geq 0$, and by (5), $\sum_{i=0}^{n} Q^{(A,A)}(i) = 2^n |A|$. Hence for this case, $|A| \over 2^n Q^{(A,A)}(\cdot)$ is a probability distribution. Furthermore, the MacWilliams–Delsarte identities for $A = B$ can be found in [12], [16].

B. Non-Interactive Simulation

Assume $(X, Y)$ is a pair of binary random variables on $\{-1,1\}$ with the distribution given in (1), where $\rho \in [-1,1]$ denotes the correlation between $X$ and $Y$. Assume $(X, Y)$ are $n$ i.i.d. copies of $(X, Y)$. Then we focus on the following non-interactive simulation problem: Given $a, b \in [0,1]$, what is the possible range of the collision probability $P(U = V)$ for all Boolean random variables $U, V$ (or equivalently, conditional Boolean distributions $P_{U|X}, P_{V|Y}$) such that $U - X - Y - V$ and $P(U = V) = a, P(V = 1) = b$? Obviously, $P(U = V) = 1 + 2P(U = V = 1) - a - b$. Hence it suffices to consider the possible range (i.e., the maximum and minimum) of the probability $P(U = V = 1)$. Furthermore, the maximum and minimum $P(U = V = 1)$ can be approximated by replacing the conditional Boolean distributions $P_{U|X}, P_{V|Y}$ with Boolean functions, as shown by the following lemma. For dyadic rationals $a_n := \lfloor 2^n a \rfloor 2^n$ and $b_n := \lfloor 2^n b \rfloor 2^n$ with $[x]$ denoting the maximum integer not larger than $x$, define

$$\Delta_n^+ := \max_{U,V:U-X-Y-V,\ P(U=1)=a_n, \ P(V=1)=b_n} P(f(X) = g(Y) = 1),$$

and define $\Delta_n^-$ as the RHS of (8) with both the maximizations replaced by minimizations.

Lemma 1. We have that

$$\Delta_n^+ \geq \Delta_n^- \geq a - a_n + b - b_n \leq 2^{-(n-1)}.$$  

(8)

In particular, if $n, a, b$ satisfy $a = \frac{M}{2^n}$ and $b = \frac{N}{2^n}$ with some $M, N \in \mathbb{N}$, then $\Delta_n^+ = \Delta_n^- = 0$, i.e., the maximum or minimum $P(U = V = 1)$ is attained by some Boolean functions $f, g : \{-1,1\}^n \rightarrow \{-1,1\}$ such that $U = f(X)$ and $V = g(Y)$.

The proof of this lemma is provided in Appendix A.

From this lemma, we have $\lim_{n \to \infty} \Delta_n^+ = \lim_{n \to \infty} \Delta_n^- = 0$. Hence in this paper, we restrict the maximization of $P(U = V = 1)$ over Markov chains $U - X - Y - V$ to the maximization over pairs of Boolean functions $f, g : \{-1,1\}^n \rightarrow \{-1,1\}$ such that $U = f(X)$ and $V = g(Y)$. That is, we consider the following question: Given $a = \frac{M}{2^n}$ and $b = \frac{N}{2^n}$ for some $M, N \in \mathbb{N}$, what is the maximum and minimum values of the probability $P(f(X) = g(Y) = 1)$ for all Boolean functions $f, g : \{-1,1\}^n \rightarrow \{-1,1\}$ such that $P(f(X) = 1) = a, P(g(Y) = 1) = b$?

Denote $A := \{x : f(x) = 1\}$ and $B := \{x : g(x) = 1\}$. Then for $a = \frac{M}{2^n}$ and $b = \frac{N}{2^n}$ with some $M, N \in \mathbb{N}$, $P(f(X) = 1) = a$ and $P(g(Y) = 1) = b$ imply $|A| = M$ and $|B| = N$. By (5), we have

$$P(f(X) = g(Y) = 1) = \sum_{x \in A} \sum_{x' \in B} \left( \frac{1 - \rho}{4} \right) d_H(x,x') \left( \frac{1 + \rho}{4} \right)^{n-d_H(x,x')} = \left( \frac{1 + \rho}{4} \right)^n \sum_{x \in A} \sum_{x' \in B} \left( \frac{1 - \rho}{1 + \rho} \right)^{d_H(x,x')} = |A||B| \left( \frac{1 + \rho}{4} \right)^n \Gamma_{1-z \over 1+\frac{z}{2}} (A, B)$$

(9)

$$= ab \Pi_z (A, B)$$

(10)
Lemma 2. For $H$, hence we have (12) and (14).

Furthermore, we define $[n] := \{1, 2, ..., n\}$. For a set $A \subseteq \{-1, 1\}^n$, define its complement and componentwise complement respectively as

$$A^c := \{-1, 1\}^n \setminus A,$$

and

$$A^* := \{-x : x \in A\},$$

where $-x = (-x_1, -x_2, ..., -x_n)$.

III. BASIC PROPERTIES OF DISTANCE MOMENTS AND DISTANCE ENUMERATORS

For average distances and distance enumerators, we have the following properties.

Lemma 2. For $A, B \subseteq \{-1, 1\}^n$, the following hold.

$$|A|D(A, B) + |A^c|D(A^c, B) = n2^{n-1};$$

(11)

$$D_k(A^*, B) = \sum_{i=0}^{k} \binom{k}{i} n^{k-i} (-1)^i D_i(A, B);$$

(12)

$$|A|\Gamma_z(A, B) + |A^c|\Gamma_z(A^c, B) = (1 + z)^n;$$

(13)

$$\Gamma_z(A^*, B) = z^n\Gamma_{\frac{1}{z}}(A, B).$$

(14)

Remark 1. For $k = 1$, (12) implies that

$$D(A, B) + D(A^*, B) = n.$$  

(15)

Proof: Equations (11) and (13) can be verified easily.

Since $d_H(-x, x^c) = n - d_H(x, x^c)$, we have the following equalities.

$$D_k(A^*, B) : = \frac{1}{|A||B|} \sum_{x \in A^*} \sum_{x' \in B} d_H^2(x, x')$$

$$= \frac{1}{|A||B|} \sum_{x \in A^*} \sum_{x' \in B} (n - d_H(x, x'))^k$$

$$= \sum_{i=0}^{k} \binom{k}{i} n^{k-i} (-1)^i D_i(A, B),$$

and

$$\Gamma_z(A^*, B) : = \frac{1}{|A||B|} \sum_{x \in A^*} \sum_{x' \in B} z^{d_H(x, x')}$$

$$= \frac{1}{|A||B|} \sum_{x \in A^*} \sum_{x' \in B} z^{n-d_H(x, x')}$$

$$= z^n \frac{1}{|A||B|} \sum_{x \in A^*} \sum_{x' \in B} \left(\frac{1}{z}\right)^{d_H(x, x')}$$

$$= z^n \Gamma_{\frac{1}{z}}(A, B).$$

Hence we have (12) and (14).

Lemma 3. For $A, B \subseteq \{-1, 1\}^n$, the following hold.

$$\left|\frac{n}{2} - D(A, B)\right| \leq \sqrt{\left(\frac{n}{2} - D(A, A)\right)\left(\frac{n}{2} - D(B, B)\right)} \leq \frac{n}{2} - \frac{1}{2}(D(A, A) + D(B, B)).$$

(16)

For $0 \leq z \leq 1$,

$$\Gamma_z(A, B) \leq \sqrt{\Gamma_z(A, A)\Gamma_z(B, B)} \leq \frac{1}{2}(\Gamma_z(A, A) + \Gamma_z(B, B));$$

(17)

and for $z \geq 1$,

$$\Gamma_z(A, B) \leq \sqrt{\Gamma_z(A^*, A)\Gamma_z(B^*, B)} \leq \frac{1}{2}(\Gamma_z(A^*, A) + \Gamma_z(B^*, B)).$$

(18)
The proof of Lemma 3 is given in Appendix B.

In fact, in the proof of Lemma 3, we prove the following inequality: The dual distance distribution between any codes $A$ and $B$ satisfies

$$|Q^{(A,B)}(k)| \leq \sqrt{Q^{(A,A)}(k)Q^{(B,B)}(k)}, \quad 0 \leq k \leq n. \quad (19)$$

Inequality (16) corresponds to inequality (19) with $k = 1$.

**Lemma 4.** For $1 \leq M \leq 2^n$,

$$\min_{A,B:|A|=|B|=M} D(A,B) = \min_{A:|A|=M} D(A,A), \quad (20)$$

$$\max_{A,B:|A|=|B|=M} D(A,B) = n - \min_{A:|A|=M} D(A,A). \quad (21)$$

For $0 \leq z \leq 1$,

$$\max_{A,B:|A|=|B|=M} \Gamma_z(A,B) = \max_{A:|A|=M} \Gamma_z(A,A); \quad (22)$$

and for $z \geq 1$,

$$\max_{A,B:|A|=|B|=M} \Gamma_z(A,B) = \max_{A:|A|=M} \Gamma_z(A^*,A). \quad (23)$$

**Proof:** Equations (20) and (21) follow by (15). Equations (22) and (23) respectively follow from (17) and (18). \hfill \blacksquare

Fu, Wei, and Yeung [15, Thm. 4] considered the average distance between codewords in the same set (i.e., restricting $A$ and $B$ to be identical), showed that for $M \leq 2^{n-1}$,

$$\min_{A:|A|=M} D(A,A) \geq \frac{n}{2} - \frac{2^{n-2}}{M}, \quad (24)$$

and equality in (24) holds for $M = 2^{n-1}$ or $2^{n-2}$ by setting $A$ to be a subcube (e.g., $A = \{1\} \times \{-1,1\}^{n-1}$ for $M = 2^{n-1}$ and $A = \{1\}^2 \times \{-1,1\}^{n-2}$ for $M = 2^{n-2}$). Combining this result with Lemmas 2 and 4 yields the following bounds on the average distance between two possibly non-identical sets.

**Lemma 5.** For $1 \leq M \leq 2^n$, $N = M$ or $2^n - M$, we have

$$\min_{A,B:|A|=M,|B|=N} D(A,B) \geq \frac{n}{2} - \frac{\min\{a,1-a\}}{4ab}, \quad (25)$$

$$\max_{A,B:|A|=M,|B|=N} D(A,B) \leq \frac{n}{2} + \frac{\min\{a,1-a\}}{4ab}, \quad (26)$$

where $a := \frac{M}{2^\rho}$ and $b := \frac{N}{2^\rho}$. Equality in (25) (resp. (26)) holds for $M = 2^{n-1}$ or $2^{n-2}$ by setting $A$ to a subcube and $B = A$ (resp. $A^*$).

**Proof:** The case of $M = N \leq 2^{n-1}$ follows by combining (20), (21), and (6). Combining the results for this case with (11) yields the results in (25) and (26) for the case of $N = 2^n - M$. Furthermore, by (11) we have

$$|A||B|D(A,B) - |A^c||B^c|D(A^c,B^c) = (|B| - |A^c|) n 2^{n-1}. \quad (27)$$

Combining the results for the case of $M = N \leq 2^{n-1}$ with (27) yields the results in (25) and (26) for the case of $M = N > 2^{n-1}$. \hfill \blacksquare

**IV. Non-Interactive Simulation and Bounds on Distance Enumerators**

In this section, we derive bounds on $\Gamma_z(A,B)$ and $\Pi_z(A,B)$, $z \geq 0$ for $A,B$ such that $|A| = M$ and $|B| = N$. By (10), this is equivalent to bounding $P(f(X) = g(Y) = 1)$ for Boolean functions $f,g$ such that $P(f(X) = 1) = a, P(g(Y) = 1) = b$ where $a = \frac{M}{2^n}$ and $b = \frac{N}{2^n}$. Without loss of generality, we may assume $0 \leq a \leq b \leq \frac{1}{2}$ (or $0 \leq M \leq N \leq 2^{n-1}$ and $\rho \in [0,1]$). This is because that if $\rho < 0$, then we can then set $-X$ to $X$; if $a > \frac{1}{2}$ or $b > \frac{1}{2}$, set $-f$ to $f$ or $-g$ to $g$. For Boolean functions $f,g$ such that $P(f(X) = 1) = a, P(g(Y) = 1) = b$, define

$$q := P(f(X) = g(Y) = 1).$$

Then $q$ can be bounded as follows. The proof is given in Appendix C.

**Theorem 1 (Bounds on $q$).**

$$\max\{\Upsilon_1^{LB}, \Upsilon_2^{LB}\} \leq q \leq \min\{\Upsilon_1^{UB}, \Upsilon_2^{UB}\},$$
we conjecture that $f$ by the functions $\Theta_m$ is also sharp and attained by the functions $f$ (29) is also sharp and attained by the functions

$$
\Theta_m := \max \left\{ \frac{a+b}{4} \rho - \frac{ab + \sqrt{ab} \rho}{2} \right\}
$$

$$
\Theta^L_m := \max \left\{ \frac{a+b}{4} \rho - \left( \frac{a+b}{4} - ab \right) \rho^2 \right\}
$$

$$
\Theta^L_2 := \sqrt{\theta^+(a)\theta^+(b)}
$$

$$
\Theta^L_2 := \sqrt{\theta^+(a)\theta^+(b)}
$$

with

$$
\theta^+(t) := \min \left\{ t, t^2 + \frac{t}{2} \rho + \left( \frac{t}{2} - \rho^2 \right) \rho^2 \right\}.
$$

Remark 2. Since $q$ and the distance enumerator (resp. dual distance enumerator) admit the relationship in (9) (resp. (10)), Theorem 1 also provides bounds for the distance enumerator (resp. dual distance enumerator) between $A$ and $B$.

Corollary 1. If $a = b$, then

$$
\theta^-(a) \leq q \leq \theta^+(a),
$$

where

$$
\theta^-(t) := \max \left\{ 0, t^2 - \frac{t}{2} \rho - \left( \frac{t}{2} - \rho^2 \right) \rho^2 \right\}.
$$

In particular, for $a = b = \frac{1}{2}$,

$$
\frac{1 - \rho}{4} \leq q \leq \frac{1 + \rho}{4}, \tag{28}
$$

and for $a = b = \frac{1}{4}$,

$$
\frac{1 - 2\rho - \rho^2}{16} \leq q \leq \left( \frac{1 + \rho}{4} \right)^2. \tag{29}
$$

Note that the bounds in (28) are not new; see [3]. However, the bounds in (29) are new. The lower and upper bounds in (28) are sharp since the upper bound is attained by the functions $f(x) = g(x)$ and $f(x) = 1$ if $x_1 = 1, -1$ otherwise, and the lower bound is attained by the functions $f(x) = g(-x)$ and $f(x) = 1$ if $x_1 = 1, -1$ otherwise. For $a = b = \frac{1}{2}$, the upper bound in (29) is also sharp and attained by the functions $f(x) = g(x)$ and $f(x) = 1$ if $x_1 = x_2 = 1$ otherwise. For $a = b = \frac{1}{4}$, we conjecture that $q$ is in fact lower bounded by $\left( \frac{1 - \rho}{4} \right)^2$. If this conjecture is true, then this lower bound is sharp since it is attained by the functions $f(x) = g(-x)$ and $f(x) = 1$ if $x_1 = x_2 = 1, -1$ otherwise. More generally, for $a = b = 2^{-i}$, $i = 1, 2, \ldots$, we conjecture that $\left( \frac{1 - \rho}{4} \right)^{i} \leq q \leq \left( \frac{1 + \rho}{4} \right)^{i}$. Similarly, if this conjecture is true, then both the upper and lower bounds are sharp. The upper bounds is attained by the symmetric subcube functions $f_i(x) = g_i(x)$ and $f_i(x) = 1$ if $x_1 = x_2 = \ldots = x_i = 1; -1$ otherwise, and the lower bounds is attained by the anti-symmetric subcube functions $f_i(x) = g_i(-x)$ and $f_i(x) = 1$ if $x_1 = x_2 = \ldots = x_i = 1; -1$ otherwise.

A. Comparisons to Other Bounds

Define the maximal correlation between two random variables $X, Y$ as

$$
\rho_m (X; Y) := \sup_{f,g} \mathbb{E} \left[ f(X)g(Y) \right],
$$

where the supremum is taken over all real-valued Borel-measurable functions $f$ and $g$ such that $\mathbb{E} [f(X)] = \mathbb{E} [g(Y)] = 0$ and $\mathbb{E} [f^2(X)] = \mathbb{E} [g^2(Y)] = 1$. When specialized to the binary case,

$$
\rho_m^2 (X; Y) = \sum_{x,y \in \{0,1\}} P^2 \chi_{XY}(x,y) P(x)P(y) - 1.
$$

This quantity was first introduced by Hirschfeld [17] and Gebelein [18], then studied by Rényi [19], and it has been exploited to provide a necessary condition for the non-interactive simulation problem [1], [3]. Non-interactive simulation of $(U, V) \sim P_{UV}$ using $(X, Y) \sim P_{XY}$ is possible only if $\rho_m (U; V) \leq \rho_m (X; Y)$. When specialized to the binary case, it leads to the following (necessary) condition on $q$:

$$
\frac{q^2}{ab} + \frac{(a-q)^2}{ab} + \frac{(b-q)^2}{ab} + \frac{(1+q-a-b)^2}{ab} \leq 1.
$$
Simplifying this inequality, we obtain the following bounds on \( q \).

**Proposition 1** (Maximal Correlation Bounds),

\[
ab - \sqrt{ab} \leq q \leq ab + \sqrt{ab}.
\]

By comparing the maximal correlation upper bound above to \( \Upsilon_1^{UB} \), we get \( \Upsilon_1^{UB} \leq ab + \sqrt{ab} \). Equality here holds only when \( a = b = \frac{1}{2} \). Hence our upper bound \( \Upsilon_1^{UB} \) is tighter than the maximal correlation upper bound. On the other hand, comparing the maximal correlation lower bound above to \( \Upsilon_2^{LB} \), we get that for \( a = b \) (i.e., the symmetric case), \( \Upsilon_2^{LB} \geq ab - \sqrt{ab} \), i.e., our bound \( \Upsilon_2^{LB} \) is tighter than the maximal correlation lower bound. For the case \( a \leq b = \frac{1}{2} \), \( \Upsilon_2^{LB} \leq ab - \sqrt{ab} \), i.e., our bound \( \Upsilon_2^{LB} \) is tighter than the maximal correlation lower bound. Hence, in general, our lower bound \( \Upsilon_2^{LB} \) is neither tighter nor looser than the maximal correlation lower bound.

Furthermore, hypercontractivity (together with its reverse version) was used to prove the following bounds for the non-interaction simulation problem [1].

**Proposition 2** (Hypercontractivity Bounds), [1, Equations (28) and (29)]

\[
\sup_{s,t > 0, (s-1)(t-1)(\kappa - 1) < 0} \varphi(s, t, \kappa) \leq q \leq \inf_{s,t > 0, (s-1)(t-1)(\kappa - 1) > 0} \varphi(s, t, \kappa),
\]

where

\[
\varphi(s, t, \kappa) := \left( \frac{\kappa'}{s - 1} \right)^{\frac{1}{\kappa'}} \frac{t^{\frac{1}{\kappa'}} - 1}{(t - 1) (t - 1)} - \frac{a}{s - 1} - \frac{b}{t - 1}
\]

with \( \kappa' := 1 + \frac{\kappa^2}{\kappa - 1} \).

Our bounds in Corollary 1, the maximal correlation bounds in Proposition 1, and the hypercontractivity bounds in Proposition 2 are plotted in Fig. 1 for the symmetric case in which \( a = b \). We set \( \rho = 0.1, 0.5, \) and \( 0.9 \). These numerical results show that the hypercontractivity lower bound is either tighter than our lower bound in general, and the hypercontractivity upper bound is also neither tighter nor looser than our upper bound in general. More specifically, the hypercontractivity bounds are tighter than ours for smaller \( a \), while our bounds are tighter than the hypercontractivity ones for larger \( a \) (and they coincide for \( a = 1/2 \)). These figures also verify that both our bounds and the hypercontractivity bounds are uniformly tighter than the maximal correlation bounds for the symmetric case. The fact that for the symmetric case, the hypercontractivity bounds are uniformly tighter than the maximal correlation bounds, has been rigorously proven in [1, Corollary 1].

**APPENDIX A**

**Proof of Lemma 1**

First, we claim that for arbitrary \( n \in \mathbb{N} \) and \( a, b \in [0, 1] \), the maximum or minimum \( \mathbb{P}(U = V = 1) \) over all Boolean random variables \( U, V \) such that \( U - X = Y - V \) and \( \mathbb{P}(U = 1) = a, \mathbb{P}(V = 1) = b \) is attained by some conditional Boolean distributions \( P_{U|X}, P_{V|Y} \) such that there exists at most one \( x_0 \in \{-1, 1\}^n \) such that \( 0 < P_{U|X}(1|x_0) < 1 \) and there exists at most one \( y_0 \in \{-1, 1\}^n \) such that \( 0 < P_{V|Y}(1|y_0) < 1 \). This claim follows from the following argument. Fix \( P_{V|Y} \) and consider the maximization or minimization of \( \mathbb{P}(U = V = 1) \) as a linear programming problem with the vector \( P_{U|X}(1|x) : x \in \{-1, 1\}^n \) considered as the optimization variable. The vector \( P_{U|X}(1|x) : x \in \{-1, 1\}^n \) satisfies \( 0 \leq P_{U|X}(1|x) \leq 1 \) and \( \mathbb{P}(U = 1) = \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} P_{U|X}(1|x) = a \). These constraints correspond to the intersection of the hyperplane \( \left\{ t \in \mathbb{R}^2 : \frac{1}{2^n} \sum_{i=1}^n t_i = a \right\} \) and the hypercube \([0, 1]^2\). This intersection is a convex polygon. Hence, by the fundamental theorem of linear programming, the maximum and minimum \( \mathbb{P}(U = V = 1) \) are attained at the corners of this polygon. The corners of this region are the points \( P_{U|X}(1|x) \) such that \( 0 \leq P_{U|X}(1|x_0) \leq 1 \) for some \( x_0 \) and \( P_{U|X}(1|x) = 0 \) or \( 1 \) for all \( x \neq x_0 \). Hence we establish the claim above.

Furthermore, for arbitrary \( n \in \mathbb{N} \) and \( a, b \in [0, 1] \), observe that the contribution of \( P_{U|X}(1|x_0) \) to the probability \( \mathbb{P}(U = 1) \) is \( P_{U|X}(1|x_0)P_X(x_0) = a - a_n \). The contribution of \( P_{U|X}(1|x_0) \) to the probability \( \mathbb{P}(U = V = 1) \) is \( P_{U|X}(1|x_0)P_X(x_0)P_{V|X}(1|x_0) \) which is lower bounded by \( 0 \) and upper bounded by \( P_{U|X}(1|x_0)P_X(x_0) = a - a_n \). Apply the same argument above to \( P_{V|Y} \). Then we obtain Lemma 1.

**APPENDIX B**

**Proof of Lemma 3**

Given the basis \( \chi_S(x) := \prod_{i \in S} x_i \) for \( S \subseteq [n] \), for a Boolean function \( f : \{-1, 1\}^n \rightarrow \{-1, 1\} \), define its Fourier coefficients as

\[
\hat{f}_S := \mathbb{E}_{x \sim \text{Unif}\{-1, 1\}^n}[f(x)\chi_S(x)], \quad S \subseteq [n].
\]
Fig. 1. Illustrations of our bounds in Corollary 1, the maximal correlation bounds in Proposition 1, and the hypercontractivity bounds in Proposition 2, where \( a = b \) and the figures from top to bottom respectively correspond to the cases of \( \rho = 0.1, 0.5, \) and \( 0.9 \). The higher set of curves correspond to upper bounds, while the lower set of curves correspond to lower bounds. The symmetric subcube schemes correspond to the mappings \( \{ (f_i, g_i) \}_{i=1}^{\infty} \) such that \( f_i(x) = g_i(x) \) and \( f_i(x) = 1 \) if \( x_1 = x_2 = \ldots = x_i = 1; -1 \) otherwise. The anti-symmetric subcube schemes correspond to the mappings \( \{ (f_i, g_i) \}_{i=1}^{\infty} \) such that \( f_i(x) = g_i(-x) \) and \( f_i(x) = 1 \) if \( x_1 = x_2 = \ldots = x_i = 1; -1 \) otherwise.
Then the Fourier expansions of any Boolean functions $f$ and $g$ (cf. [20, (1.6)]) are respectively

$$f(x) = \sum_{S \subseteq [n]} \hat{f}_S x^S(x)$$

$$g(x) = \sum_{S \subseteq [n]} \hat{g}_S x^S(x).$$

Define

$$a := \frac{1 + \hat{f}_0}{2} = \mathbb{P}(f(X) = 1),$$

$$b := \frac{1 + \hat{g}_0}{2} = \mathbb{P}(g(X) = 1),$$

and

$$\theta_\rho := \frac{1}{4} \sum_{S \subseteq [n]: |S| \geq 1} \hat{f}_S \hat{g}_S \rho^{|S|}.$$  \hspace{1cm} (31)

In analogy to [20, Plancherel’s Theorem and Proposition 1.9], the inner product between $f(X)$ and $g(Y)$ satisfies

$$\mathbb{E}[f(X)g(Y)] = \hat{f}_0 \hat{g}_0 + 4\theta_\rho = 1 - 2\mathbb{P}(f(X) = g(Y)).$$

Defining $\mathbb{T} := 1 - t$ for $t \in [0, 1]$, we can write

$$\mathbb{P}(f(X) = g(Y) = 1) = ab + \theta_\rho \hspace{1cm} (32)$$

$$\mathbb{P}(f(X) = -g(Y) = 1) = ab - \theta_\rho \hspace{1cm} (33)$$

$$\mathbb{P}(-f(X) = g(Y) = 1) = ab - \theta_\rho \hspace{1cm} (34)$$

$$\mathbb{P}(f(X) = g(Y) = -1) = ab + \theta_\rho. \hspace{1cm} (35)$$

Denote $A := \{x : f(x) = 1\}$ and $B := \{x : g(x) = 1\}$. Combining (32) and (10) yields the following identity:

$$|A||B| \left( \frac{1 + \rho}{4} \right)^n \Gamma_{\frac{n}{1 + \rho}}(A, B) = ab \Pi_\rho(A, B) = ab + \frac{1}{4} \sum_{S: |S| \geq 1} \hat{f}_S \hat{g}_S \rho^{|S|}, \hspace{1cm} (36)$$

which further yields the following relationship between the dual distance distribution $Q^{(A,B)}$ and the Fourier coefficients $\hat{f}_S, \hat{g}_S$:

$$Q^{(A,B)}(k) = \begin{cases} 1, & k = 0 \\ \frac{1}{4} \sum_{S: |S| = k} \hat{f}_S \hat{g}_S, & 1 \leq k \leq n. \end{cases}$$

By the Cauchy–Schwarz inequality, we have for any $0 \leq k \leq n$,

$$\left| \sum_{S: |S| = k} \hat{f}_S \hat{g}_S \right| \leq \sqrt{\left( \sum_{S: |S| = k} \hat{f}_S^2 \right) \left( \sum_{S: |S| = k} \hat{g}_S^2 \right)}, \hspace{1cm} (37)$$

which is equivalent to that

$$|Q^{(A,B)}(k)| \leq \sqrt{Q^{(A,A)}(k)Q^{(B,B)}(k)}.$$

Now consider the case of $k = 1$. For any $i \in [n]$,

$$\hat{f}_{(i)} = \frac{1}{2^m} \left( \sum_{x \in A} x_i - \sum_{x \in A^c} x_i \right) = \frac{2}{2^n} \sum_{x \in A} x_i, \hspace{1cm} (38)$$

since $\sum_{x \in A} x_i + \sum_{x \in A^c} x_i = 0$. Similarly, for any $i \in [n]$,

$$\hat{g}_{(i)} = \frac{2}{2^n} \sum_{x \in B} x_i. \hspace{1cm} (39)$$
Therefore,

\[ c := \sum_{S: |S| = 1} f_S g_S \]

(40)

\[ = \sum_{i=1}^{n} \frac{1}{4^{n-1}} \sum_{x \in A} x_i \sum_{x' \in B} x'_i \]

\[ = \frac{1}{4^{n-1}} \sum_{x \in A} \sum_{x' \in B} \sum_{i=1}^{n} x_i x'_i \]

\[ = \frac{1}{4^{n-1}} \sum_{x \in A} \sum_{x' \in B} (n - 2d_H(x, x')) \]

\[ = 4ab(n - 2D(A, B)) \]

(41)

where \( a = \frac{|A|}{2^n} \) and \( b = \frac{|B|}{2^n} \). Combining (37) with (41) yields

\[ \left| \frac{n}{2} - D(A, B) \right| \leq \sqrt{\left( \frac{n}{2} - D(A, A) \right) \left( \frac{n}{2} - D(B, B) \right)} \leq \frac{n}{2} - \frac{1}{2} (D(A, A) + D(B, B)). \]

Now we prove inequality (17). By the Cauchy–Schwarz inequality, we obtain that for \( \rho \in [0, 1] \),

\[ ab + \frac{1}{4} \sum_{S: |S| \geq 1} f_S g_S \rho^{|S|} \leq \sqrt{\left( a^2 + \frac{1}{4} \sum_{S: |S| \geq 1} f_S^2 \rho^{|S|} \right) \left( b^2 + \frac{1}{4} \sum_{S: |S| \geq 1} g_S^2 \rho^{|S|} \right)}. \]

(42)

Combining (36) with (42) yields

\[ \Gamma_{\frac{1-\rho}{1+\rho}}(A, B) \leq \sqrt{\Gamma_{\frac{1-\rho}{1+\rho}}(A, A) \Gamma_{\frac{1-\rho}{1+\rho}}(B, B)} \leq \frac{1}{2} \left( \Gamma_{\frac{1-\rho}{1+\rho}}(A, A) + \Gamma_{\frac{1-\rho}{1+\rho}}(B, B) \right). \]

Setting \( z := \frac{1-\rho}{1+\rho} \in [0, 1] \), we obtain that for \( 0 \leq z \leq 1 \),

\[ \Gamma_z(A, B) \leq \sqrt{\Gamma_z(A, A) \Gamma_z(B, B)} \leq \frac{1}{2} (\Gamma_z(A, A) + \Gamma_z(B, B)). \]

Inequality (18) follows by combining (17) with (14).

**APPENDIX C**

**PROOF OF THEOREM 1**

By the nonnegativity of the probabilities given in (32)-(35),

\[ -ab \leq \theta_\rho \leq ab. \]

(43)

As in Appendix B, we denote \( A := \{ x : f(x) = 1 \} \) and \( B := \{ x : g(x) = 1 \} \). Then by (16) in Lemma 3 and (25) in Lemma 5, we obtain that

\[ \left| \frac{n}{2} - D(A, B) \right| \leq \frac{n}{2} - \frac{1}{2} (D(A, A) + D(B, B)) \leq \frac{1}{4a} + \frac{1}{4b}, \]

(44)

since \( a, b \leq \frac{1}{2} \) as assumed. Hence \( c \) as defined in (40) satisfies

\[ |c| \leq 4ab \left( \frac{1}{4a} + \frac{1}{4b} \right) = a + b. \]

(45)

As in Appendix B, we denote the Fourier coefficients of \( f \) and \( g \) as \( \hat{f}_S \) and \( \hat{g}_S \), respectively; see (30). Inspired by the proof idea in [21], we partition the set \( \{ S \subseteq [n] : |S| \geq 2 \} \) into two parts: \( \mathcal{P} := \{ S \subseteq [n] : |S| \geq 2, \hat{f}_S \hat{g}_S \geq 0 \} \) and \( \mathcal{N} := \{ S \subseteq [n] : |S| \geq 2, \hat{f}_S \hat{g}_S < 0 \} \). Then we define

\[ \tau^+ := \frac{1}{4} \sum_{S \in \mathcal{P}} \hat{f}_S \hat{g}_S, \quad \tau^- := \frac{1}{4} \sum_{S \in \mathcal{N}} \hat{f}_S \hat{g}_S, \]
and apply the Cauchy–Schwarz inequality to show
\[
\frac{1}{4}c + \frac{1}{4} \sum_{S: |S| \geq 2} |\hat{f}_S| |\hat{g}_S| \leq \frac{1}{4} \sum_{S: |S| \geq 1} |\hat{f}_S| |\hat{g}_S|
\]
\[
\leq \frac{1}{4} \left( \sum_{S: |S| \geq 1} \hat{f}_S^2 \right) \left( \sum_{S: |S| \geq 1} \hat{g}_S^2 \right)
\]
\[
= \frac{1}{4} \sqrt{ \left( 1 - \hat{f}_\emptyset^2 \right) \left( 1 - \hat{g}_\emptyset^2 \right) }
\]
\[
= \sqrt{aabb},
\]
where (46) follows from Parseval’s Theorem [20].

Hence
\[
\tau^+ - \tau^- = \frac{1}{4} \sum_{S: |S| \geq 2} |\hat{f}_S| |\hat{g}_S| \leq \sqrt{aabb} - \frac{1}{4} c. \tag{47}
\]

Since \( \theta_1 = \frac{1}{4}c + \tau^+ + \tau^- \) and \(-ab \leq \theta_1 \leq a\bar{b} \) (see (43)), we have
\[
-a b - \frac{1}{4} c \leq \tau^+ + \tau^- \leq a\bar{b} - \frac{1}{4} c. \tag{48}
\]

We combine (47) and (48) to obtain
\[
\tau^+ \leq \frac{a\bar{b} + \sqrt{aabb}}{2} - \frac{1}{4} c, \quad \tau^- \geq -\frac{a\bar{b} + \sqrt{aabb}}{2}.
\]

By definition, \( \frac{1}{4}c\rho + \rho^2 \tau^- \leq \theta_\rho \leq \frac{1}{4}c\rho + \rho^2 \tau^+ \), and hence,
\[
\theta_\rho \in [\hat{\theta}_\rho^-, \hat{\theta}_\rho^+] \subseteq [\theta_\rho^-, \theta_\rho^+], \tag{49}
\]
where, from (45),
\[
\hat{\theta}_\rho^- := \max \left\{ -ab, \frac{1}{4}c\rho - \frac{ab + \sqrt{aabb}}{2} \rho^2 \right\} \geq \max \left\{ -ab, -a + b \frac{\rho}{4} - \frac{ab + \sqrt{aabb}}{2} \rho^2 \right\} =: \theta_\rho^-,
\]
\[
\hat{\theta}_\rho^+ := \min \left\{ a\bar{b}, \frac{1}{4}c\rho + \left( \frac{a\bar{b} + \sqrt{aabb}}{2} - \frac{1}{4} c \right) \rho^2 \right\} \leq \min \left\{ a\bar{b}, a + b \frac{\rho}{4} + \left( \frac{a\bar{b} + \sqrt{aabb}}{2} - a + b \frac{\rho}{4} \right) \rho^2 \right\} =: \theta_\rho^+.
\]

Combining (49) with (32) gives us the lower bound \( \Upsilon_1^{LB} \) and upper bound \( \Upsilon_1^{UB} \).

Now we prove that \( q \) is lower bounded by \( \Upsilon_2^{LB} \). Starting from the definition of \( \theta_\rho \) in (31), we obtain that
\[
\theta_\rho \geq \frac{1}{4} c\rho - \frac{1}{4} \sum_{i=2}^{n} \left| \sum_{S: |S| = i} \hat{f}_S \hat{g}_S \right| |\rho|^i
\]
\[
\geq \frac{1}{4} c|\rho| - \frac{1}{4} \left( \sum_{i=2}^{n} \left| \sum_{S: |S| = i} \hat{f}_S \hat{g}_S \right| \rho^2 \right), \tag{50}
\]
On the other hand,
\[
(2a - 1) (2b - 1) + |c| + \sum_{i=2}^n \sum_{|S| = i} \hat{f}_S \hat{g}_S
\]
\[
= \sum_{i=0}^n \left( \sum_{|S| = i} \hat{f}_S \hat{g}_S \right)
\leq \sum_{|S| : |S| \geq 0} \hat{f}_S |\hat{g}_S|
\leq \sqrt{\left( \sum_{|S| : |S| \geq 0} \hat{f}_S^2 \right) \left( \sum_{|S| : |S| \geq 0} \hat{g}_S^2 \right)}
\leq 1.
\]

Hence
\[
\sum_{i=2}^n \sum_{|S| : |S| = i} \hat{f}_S \hat{g}_S \leq 1 - (2a - 1) (2b - 1) - |c|. \tag{51}
\]

Substituting (51) into (50), we obtain
\[
\theta_\rho \geq -\frac{1}{4} |c| \rho - \frac{1}{4} \left( 1 - (2a - 1) (2b - 1) - |c| \right) \rho^2
\geq -\frac{a + b}{4} \rho - \frac{1}{4} \left( 1 - (2a - 1) (2b - 1) - a + b \right) \rho^2
\geq -\frac{a + b}{4} \rho - \left( \frac{a + b}{4} - ab \right) \rho^2, \tag{52}
\]
where (52) follows since \(|c| \leq a + b|/see (45). On the other hand, \(\theta_\rho \geq -ab\). Hence,
\[
\theta_\rho \geq \max \left\{ -ab, -\frac{a + b}{4} \rho - \left( \frac{a + b}{4} - ab \right) \rho^2 \right\}.
\]

We have the lower bound \(\Upsilon_2^{LB}\).

Furthermore, by combining upper bound \(\Upsilon_1^{UB}\) with the alternative expression of \(q\) given in (9) and the inequality (17), we obtain the upper bound \(\Upsilon_2^{UB}\).

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