Abstract. The aim of this paper is to study the points and localising subcategories of the topos of \( M \)-sets, for a finite monoid \( M \). We show that the points of this topos can be fully classified using the idempotents of \( M \). We introduce a topology on the iso-classes of these points, which differs from the classical topology introduced in SGA4. Likewise, the localised subcategories of the topos \( M \)-sets correspond to the set of all two-sided idempotent Ideals of \( M \).

1. Introduction

For a topos \( \mathcal{E} \), we denote the category of points of \( \mathcal{E} \) by \( \text{Pts}(\mathcal{E}) \) and the isomorphism classes of \( \text{Pts}(\mathcal{E}) \) is denoted by \( F(\mathcal{E}) \). In SGA4 \([1]\), the authors defined a topological space structure on the set \( F(\mathcal{E}) \), based on the subobjects of the terminal object of \( \mathcal{E} \). In the case when \( \mathcal{E} \) is the topos \( \text{Sh}(X) \) of sheaves on a sober topological space \( X \), the space constructed in \([1]\) is homeomorphic to \( X \). However, this topology is quite trivial in many cases. Particularly, when \( \mathcal{E} = \mathcal{I}_M \) is the topos of right \( M \)-sets. In this case, the obtained topological space has only two open sets, though \( F(M) \) could be quite big. Hence, it is natural to search for other topologies on \( F(\mathcal{E}) \).

In recent years, there was a considerable attention to the problem of understanding the points and possible topological space structures on \( \mathcal{E} = \mathcal{I}_M \). When \( M = \mathbb{N}_+^\times \) is the multiplicative monoid of strictly positive natural numbers there is an interesting topological space structure on the set \( F(\mathcal{I}_M) \) studied by Connes-Consani \([2]\) and Le Bruyn \([10]\), see also a related paper by Hemelaer \([9]\).

On the other hand, there exists the very interesting topological space \( \text{Spec}(M) \), for a commutative monoid \( M \). The elements of \( \text{Spec}(M) \) are prime ideals of \( M \) \([9]\) and the topology is defined exactly as in classical algebraic geometry. This space plays an important role in \( F_1 \)-mathematics and \( K \)-theory, see for example \([5],[3],[4]\).

In our previous papers \([14],[15]\), we reconstructed the topological space \( \text{Spec}(M) \), from the topos \( \mathcal{I}_M \), when \( M \) is finitely generated and commutative. In fact, we constructed a bijections of sets \( \text{Spec}(M) \rightarrow F(\mathcal{I}_M) \) and \( \text{Open}(\text{Spec}(M)) \rightarrow \mathcal{L}(\mathcal{I}_M) \), where \( \mathcal{L}(\mathcal{E}) \) is the set of localised subcategories of a topos \( \mathcal{E} \) and \( \text{Open}(X) \) is the set of all open subsets of a topological space \( X \).

We introduce the following notation: For a localising subcategory \( \mathcal{I} \) of \( \mathcal{E} \) and \( p = (p_*,p^*) : \text{Sets} ightarrow \mathcal{E} \) a topos point of \( \mathcal{E} \), we write \( p \in \mathcal{I} \) if \( p_*(S) \in \mathcal{I} \) for every set \( S \in \text{Sets} \).

The aim of this work is to investigate the points and localising categories of \( \mathcal{I}_M \), when \( M \) is a finite monoid. We show that localising subcategories induces an interesting topology on the set \( F(\mathcal{I}_M) \). In more details, we construct a bijection from the set \( F(\mathcal{I}_M) \) to the
set of $\mathcal{J}$-classes of idempotents, where $\mathcal{J}$ is the Green relation (that is, $e \mathcal{J} f$ if and only if $MeM = MfM$). Since this set has a natural order $\leq_3$ ($e \leq_3 f$ if $MeM \subseteq MfM$), one can consider the order topology on $\mathbf{F}(\mathcal{J}_M)$. Our next result claims that there are bijections from the set of localised subcategories $\mathbf{Loc}(\mathcal{J}_M)$ to the set of all two-sided idempotent ideals of $M$, and also to the set of all open subsets of the ordered topology on $\mathbf{F}(\mathcal{J}_M)$.

2. Preliminaries

2.1. Points and filtered $M$-sets. Recall that a point of a Grothendieck topos, or simply a topos $\mathcal{T}$, is a geometric morphism $p = (p_*, p^*) : S \to \mathcal{T}$ from the topos of sets $S$ to $\mathcal{T}$. The inverse image functor $p_* : S \to \mathbf{F}(\mathcal{T})$ preserves colimits and finite limits. It is also well-known that, conversely, for any functor $f : T \to S$ which preserves colimits and finite limits, one has $f = p_*$ for a uniquely defined point $p$. We let $\mathbf{Pts}(\mathcal{T})$ be the category of points of $\mathcal{T}$ and $\mathbf{F}(\mathcal{T})$ the isomorphism classes of the category $\mathbf{Pts}(\mathcal{T})$.

Let $M$ be a monoid. The category of left (resp. right) $M$-sets is denoted by $\mathbf{M}_S$ (resp. $\mathbf{S}_M$). It is well known that these categories are topoi. Instead of $\mathbf{Pts}(\mathcal{S}_M)$ and $\mathbf{F}(\mathcal{S}_M)$ we write $\mathbf{Pts}(M)$ and $\mathbf{F}(M)$. By Diaconescu’s theorem [11], the category $\mathbf{Pts}(M)$ is equivalent to the category of filtered left $M$-sets. Recall that a left $M$-set $A$ is called filtered, provided the functor

$(-) \otimes_M A : \mathcal{M} \mathcal{J} \to \mathcal{J}$

commutes with finite limits. The topos point of $\mathcal{J}_M$ corresponding to a filtered left $M$-set $A$ is denoted by $p_A = (p_A^*, p_A^*)$. The inverse image functor $p_A^* : \mathcal{J}_M \to \mathcal{J}$ is given by

$p_A^*(X) = X \otimes_M A,$

while the direct image functor $p_A^* : \mathcal{J} \to \mathcal{J}_M$ sends a set $Y$ to $\text{Hom}_\mathcal{J}(A, Y)$. The latter is considered as a right $M$ set via

$(am)(a) := \alpha(ma).$

Here, $\alpha \in \text{Hom}_\mathcal{J}(A, X)$, $a \in A$ and $m \in M$.

The following, well-known, fact [12, p.24] is a very useful tool for checking whether a given $M$-set is filtered.

**Lemma 2.1.1.** A left $M$-set $A$ is filtered if and only if the following three conditions hold:

(F1) $A \neq \emptyset$.

(F2) If $m_1, m_2 \in M$ and $a \in A$ satisfies the condition

$m_1a = m_2a,$

there exist $m \in M$ and $\bar{a} \in A$, such that $m \bar{a} = a$ and $m_1m = m_2m$.

(F3) If $a_1, a_2 \in A$, there are $m_1, m_2 \in M$ and $a \in A$, such that $m_1a = a_1$ and $m_2a = a_2$.

**Example 2.1.2.** i) Clearly, $A = M$ is always filtered and corresponds to the canonical point, denoted by $p_M$. Thus, $p_M^*$ is the forgetful functor $\mathcal{J}_M \to \mathcal{J}$.

ii) We take $M = \{1, t\}$, $t^2 = t$. In this case, the singleton, which is a terminal object in $\mathcal{J}_M$, is a filtered $M$-set.
2.2. Points and prime ideals. If \( M \) is commutative and \( p \) is a prime ideal, then the localisation \( M_p \) is filtered \( \mathbb{1} \) and in this way, one obtains an injective map

\[
\text{Spec}(M) \to \mathbb{F}_M.
\]

The inverse image functor corresponding to the point, associated to the filtered \( M \)-set \( M_p \), sends a right \( M \)-set \( X \) to the localisation \( X_p \), considered as a set.

Moreover, if \( M \) is finitely generated, the map is a bijection.

For a monoid \( M \), denote by \( M^\text{com} \) the maximal commutative and by \( M^\text{sl} \) the maximal semilattice quotient respectively. As a semilattice is commutative by definition, we have natural surjective homomorphisms \( M \to M^\text{com} \to M^\text{sl} \).

According to \( \mathbb{1} \), for any commutative monoid \( M \), the induced map

\[
\text{Spec}(M^\text{sl}) \cong \text{Spec}(M)
\]

is bijective and furthermore, there is an injective map \( \mathbb{1} \)

\[
M^\text{sl} \to \text{Spec}(M^\text{sl}) \cong \text{Spec}(M).
\]

This is bijective if \( M \) is commutative and finitely generated. It follows that under these assumptions, we have

\[
|\mathbb{F}_M| = |M^\text{sl}|.
\]

Example 2.2.1. Let \( M = \{1, t\} \), \( t^2 = t \) as in part ii) of Example 2.1.2. Then \( \mathbb{F}_M \) has only two elements, one corresponds to the filtered \( M \)-set \( M \) and another one to the singleton.

2.3. Induced points. We will discuss some functorial properties of \( \mathbb{F}_M \) and its consequences. We start with the following well-known fact:

Lemma 2.3.1. Let \( f : M \to M' \) be a monoid homomorphism. For any filtered left \( M \)-set \( A \), the left \( M' \)-set \( M' \otimes_M A \) is filtered.

The \( M' \)-set constructed in the Lemma is said to be induced from \( A \) via the homomorphism \( f \). In this way, one obtains a functor

\[
\text{Pts}(f) : \text{Pts}(M) \to \text{Pts}(M')
\]

and the induced map

\[
\mathbb{F}_f : \mathbb{F}_M \to \mathbb{F}_{M'}.
\]

Example 2.3.2. Let \( e \) be an idempotent of a monoid \( M \). We have a homomorphism of monoids \( \eta : \{1, t\} \to M \), where \( t^2 = t \) and \( \eta(t) = e \). The singleton is filtered over \( \{1, t\} \), see Example 2.2.1. Thus, it induces a filtered \( M \)-set, which is easily seen to be \( Me \). The fact that an idempotent of \( M \) gives rise to a point of \( \mathcal{S}_M \) was already recently observed in \( \mathbb{1} \).

One of our main result claims that if \( M \) is finite, any filtered \( M \)-set is isomorphic to \( Me \) for an idempotent \( e \in M \); that is, induced from the submonoid \( \{1, e\} \), see Theorem 3.7.1 below.
Denote by \( q : M \rightarrow M^{sl} \) the quotient map. As we said, the induced map
\[
F_q : F_M \rightarrow F_{M^{sl}}
\]
is bijective if \( M \) is finitely generated and commutative. In the noncommutative setting, this induced map is not a bijection in general, even if \( M \) is finite. However, in this case, we will show that \(|F_M|\) is finite, see Theorem 3.3.2 below. We also have the following:

**Proposition 2.3.3.** Let \( M \) be a finite monoid. The canonical homomorphism
\[
F_q : F_M \rightarrow F_{M^{sl}}
\]
is surjective.

**Proof.** Denote by \( \text{Idem}(M) \) the set of all idempotents of \( M \). Then \( \text{Idem}(M) \ni e \mapsto Me \) yields the map \( \text{Idem}(M) \rightarrow F_M \). Since \( \text{Idem}(M^{sl}) = M^{sl} \), the functoriality yields the following commutative diagram
\[
\begin{array}{ccc}
\text{Idem}(M) & \longrightarrow & F_M \\
\downarrow & & \downarrow \\
M^{sl} & \longrightarrow & F_{M^{sl}}.
\end{array}
\]
The bottom arrow is a bijection and the left vertical map is surjective, thanks to [18, Lemma 1.6]. It follows that the left vertical map is surjective as well. \( \square \)

3. Points of \( \mathcal{S}_M \) and idempotents of \( M \)

3.1. **Category \( \mathcal{J}(M) \).** Let \( m \in M \) be an element. We have a natural left ideal \( Mm \), which can also be considered as a left \( M \)-set. We have the following well-known fact:

**Lemma 3.1.1.** Let \( e \in M \) be an idempotent. For any left \( M \)-set \( X \), we have a bijection
\[
eX \cong \text{Hom}_M(Me, X),
\]
which sends an element \( ex \in eX \) to the homomorphism \( \alpha_{ex} : Me \rightarrow X \), given by \( \alpha_{ex}(me) = mex \).

**Proof.** Take any morphism of \( M \)-sets \( \beta : Me \rightarrow X \). We have \( \beta(e) = \beta(ee) = e\beta(e) \). Thus, \( \beta(e) \in eX \) and \( \beta \mapsto \beta(e) \) defines a map \( \text{Hom}_M(Me, X) \rightarrow eX \), which obviously is inverse of the map \( ex \mapsto \alpha_{ex} \). \( \square \)

**Corollary 3.1.2.** Let \( e, f \in M \) be idempotents. We have \( \text{Hom}_M(Me, Mf) = eMf \).

We define the category \( \mathcal{J}(M) \) as follows: Objects are idempotents of \( M \) and a morphism from \( e \) to \( f \) is an element of \( M \) of the form \( fme, m \in M \). That is
\[
\text{Hom}_{\mathcal{J}(M)}(e, f) = fMe.
\]
The composition is given by the multiplication in \( M \), i.e. \( (gnf) \circ (fme) = gnfme \). That is, the composite of arrows
\[
e \xrightarrow{fme} f \xrightarrow{gnf} g
\]
Lemma 3.3.1. Let \( e \) be equal to \( e^{\alpha_{fm}} g \).

The identity morphism of \( e \) is just \( e \). It is clear that
\[
\mathcal{I}(M^{op}) = (\mathcal{I}(M))^{op}.
\]

Recall the Green relations \( \mathcal{L}, \mathcal{R}, \mathcal{J} \). By definition, we have \( e \mathcal{L} e' \) provided \( Me = Me' \), \( e \mathcal{R} e' \) if \( eM = e'M \) and \( e \mathcal{J} e' \) if \( MeM = Me'M \). We let \( \text{idem}_M \) be the corresponding quotient set, where \( \mathcal{R} \in \{ \mathcal{L}, \mathcal{R}, \mathcal{J} \} \).

It is clear that if \( e \mathcal{L} e' \) and \( f \mathcal{R} f' \), then
\[
\text{Hom}_{\mathcal{J}(M)}(e, f) = \text{Hom}_{\mathcal{J}(M)}(e', f') \quad \text{Hom}_{\mathcal{J}(M)}(e, f) = \text{Hom}_{\mathcal{J}(M)}(e, f').
\]

**Lemma 3.1.3.** The assignment \( e \mapsto Me \) induces a contravariant duality between the category \( \mathcal{I}(M) \), and, the full subcategory of \( _M \mathcal{I} \), consisting of objects of the form \( Me \).

**Proof.** This is a direct consequence of Corollary 3.1.2. \( \square \)

**Lemma 3.1.4.** Idempotents \( e \) and \( f \) are isomorphic in \( \mathcal{I}(M) \) if and only if there are \( a, b \in M \), such that \( ab = e \) and \( ba = f \). This happens if and only if \( e \mathcal{I} f \). Thus, the set of iso-classes of the category \( \mathcal{I}(M) \) is bijective to the set \( \text{idem}_{\mathcal{J}}(M) \).

**Proof.** A consequence of [18, Theorem 1.11]. \( \square \)

### 3.2. \( M \)-congruences

For an equivalence relation \( \rho \) on a set \( S \), we denote by \( q \) the canonical map \( q : S \rightarrow S/\rho \).

Let \( M \) be a monoid and \( A \) a left \( M \)-set. An equivalence relation \( \sim_\rho \) is called an \( M \)-congruence if \( a \sim b \) implies \( ma \sim mb \) for all \( m \in M \). It is clear that in this case, the quotient \( A/\sim_\rho \) has an unique left \( M \)-set structure such that \( q : M \rightarrow M/\sim_\rho \) is an \( M \)-set map.

We will use this terminology, to distinguish between congruences on a monoid \( M \), in the world of monoids, and congruences on \( M \), considered as a left \( M \)-set, using the multiplication in \( M \).

**Lemma 3.2.1.** Let \( \sim_\rho \) be an \( M \)-congruence on a left \( M \)-set \( A \). For any \( a \in A \), the subset
\[
K_\rho(a) = \{ x \in M | a \sim xa \}
\]
is a submonoid of \( M \).

**Proof.** Since \( \sim \) is an equivalence relation, we have \( a \sim a = 1 \cdot a \). Thus, \( 1 \in K \). Assume \( x, y \in K(a) \). That is, \( a \sim ya \) and \( a \sim xa \). Since \( \sim \) is \( M \)-congruence, we have \( xa \sim yxa \). It follows that \( a \sim xa \sim yxa \). Hence, \( xy \in K(a) \). \( \square \)

### 3.3. Finiteness of points

We start with the following observation:

**Lemma 3.3.1.** Let \( A \) be a filtered left \( M \)-set and \( a_1, \ldots, a_k \) elements of \( A \). There are \( m_1, \ldots, m_k \in M \) and \( c \in A \), such that \( a_i = m_i c \) for all \( 1 \leq i \leq k \).

**Proof.** The case \( n = 1 \) is clear and \( n = 2 \) is just condition F3. We proceed by induction. Assume there are \( n_1, \ldots, n_{k-1} \in M \) and \( b \in A \), such that \( a_i = n_ib \) for all \( 1 \leq i \leq n - 1 \). By the case \( n = 2 \), we can choose \( c \in A \) and \( n', m_k \in M \), such that \( n'c = b \) and \( m_k c = a_k \). We put \( m_i = n'n_i \) for \( i = 1, \ldots, k - 1 \) to obtain \( m_i c = a_i \) for all \( 1 \leq i \leq k \). \( \square \)
Theorem 3.3.2. Let $M$ be a finite monoid.

i) Any filtered $M$-set $A$ is cyclic, that is, generated by a single element. In particular, we have $|A| \leq |M|$.

ii) The set $F_M$ is finite.

Proof. i) Assume there are $k$ distinguished elements of $A$, say $a_1, \ldots, a_k$. By Lemma 3.3.1, we can find $m_1, \ldots, m_k \in M$ and $c \in A$, such that $a_i = m_i c$, for all $1 \leq i \leq k$. If $|M| < k$, we see that there are $i \neq j$, such that $m_i = m_j$. It follows that $a_i = m_i c = m_j c = a_j$. This contradicts our assumptions on the $a_i$'s. Hence, $k \leq |M|$. This implies $|A| \leq |M|$.

Moreover, by taking $a_1, \ldots, a_k$ to be all the elements of $A$, we see that $A$ is generated by $c$ and hence, it is cyclic.

ii) This is an obvious consequence of i). □

3.4. $F$-monoids and $F$-submonoids. We call a monoid $K$ an $F$-monoid, if for any $m, n \in K$, there exits an $x \in K$, such that $mx = nx$.

Clearly, $K = \{1, e\}$ is an $F$-monoid, where $e^2 = e$. More generally, if $M$ is a semilattice, then $M$ is an $F$-monoid (we can take $x = mn$).

Another class of $F$-monoids (which is a generalisation of semilattices in the finite case) are monoids that have a right zero. That is, an element $\varrho$, such that $x \varrho = \varrho$ for all $x \in K$. Clearly, if such a $\varrho$ exists, it is unique and an idempotent. Our next goal is to show that the converse is also true, if $M$ is finite. We will need the following lemma.

Lemma 3.4.1. Let $K$ be an $F$-monoid. For any finite collection of elements $m_1, \ldots, m_k$ of $K$, there exists an $x \in K$, such that $m_i x = x$ for all $i = 1, \ldots, k$.

Proof. We proceed by induction. Let $k = 1$. In this case, the assertion follows directly from the definition of an $F$-monoid, by taking $m = m_1$ and $n = 1$. Next, consider the case $k = 2$. Since the assertion is true for $k = 1$, we can find $y_i$, $i = 1, 2$, such that $m_i y_i = y_i$, $i = 1, 2$. As $K$ is an $F$-monoid, there exits a $z \in K$, such that $y_1 z = y_2 z$. Now, for $x = y_1 z = y_2 z$, we have $m_i x = m_i y_i z = y_1 z = x$, $i = 1, 2$.

Let $k > 2$. By the induction assumption, there exists a $y \in K$, for which $m_i y = y$, for all $i = 1, \cdots, k - 1$. Since the result is also true for $k = 2$, we can apply it for $y$ and $m_k$, to conclude that there exists a $x \in K$, for which $m_k x = x$ and $yx = x$. We have $m_i x = m_i y x = yx = x$, $i < k$ and thus, for all $1 \leq i \leq k$. This finishes the proof. □

Corollary 3.4.2. A finite monoid is an $F$-monoid if and only if it has a right zero element.

A submonoid $K$ of a monoid $M$ is called an $F$-submonoid if $K$ is an $F$-monoid.
3.5. **Saturated submonoids.** A submonoid $K$ of a monoid $M$ is called *saturated*, if for any $m \in M$, for which $mx = x$ for some $x \in K$, one has $m \in K$. It is clear, that to any submonoid $K$, there is a smallest saturated submonoid containing $K$. This is the intersection of all saturated submonoids containing $K$. (The fact that the intersection of saturated submonoids is again saturated is readily checked). We denote this associated saturated submonoid by $\hat{K}$ and sometimes refer to it as the *saturation* of $K$.

**Example 3.5.1.** Let $M = \{1, 0, a, b, ab\}$, where $a^2 = a, b^2 = b$ and $ba = 0$. The only saturated $F$-submonoids of $M$ are $\{1\}, \{1, a\}, \{1, b\}$ and $M$ itself. On the other hand, $\{1, 0\}, \{1, 0, a\}, \{1, 0, b\}$ and $\{1, 0, ab\}$ are non-saturated $F$-submonoids.

3.6. **Quotient by $F$-submonoids.** Let $H \subseteq A$ be a submonoid of a monoid $A$. We can define a relation $\sim$ on $A$ by $a \sim_H b$ if and only if there exist $x, y \in H$, such that $ax = by$. This relation, however, need not be an equivalence relation, as the transitivity property need not hold. If it does, however, than it is an $M$-congruence and the quotient, denoted by $M/K$, is a natural $M$-set.

**Lemma 3.6.1.** Let $K$ be an $F$-submonoid of $M$ and $m, n \in M$. Then its induced relation $\sim_K$ is an $M$-congruence and $m \sim_K n$ if and only if there exists an $x \in K$, such that $mx = nx$.

**Proof.** First, we show that the two relations are the same. One side is clear, by letting $x = y$. For the other side, let $x, y \in K$ be elements, such that $mx = ny$. Since $K$ is an $F$-submonoid, we can find a $z \in K$, such that $xz = yz$. As $mx = ny \Rightarrow mxz = nyz$, the result follows.

To see that $\sim_K$ is an $M$-congruence, let $m \sim_K n$ and $n \sim_K k$. That is, we have $x, y \in K$, such that $mx = mx$ and $ny = ky$. As $K$ is an $F$-submonoid, there exists a $z \in K$, such that $xz = yz$. We now have

$$m(xz) = (mx)z = (nx)z = n(xz) = n(yz) = (ny)z = (ky)z = k(yz).$$

Hence, $m \sim_K k$ with the first definition and so, transitivity holds.

In general, the $M$-congruence $\sim_K$ is not a congruence (that is, the quotient $M/K$ need not be a monoid), even if $K$ is an $F$-submonoid. In fact, take again $M = \{1, 0, a, b, ab\}$, where $a^2 = a, b^2 = b$ and $ba = 0$. We take $K = \{1, b\}$. In this case, we have three equivalence classes $1 \sim_K b$, $a \sim_K ab$ and $0$. Since $1 \cdot a = a \not\sim_K ba = 0$, we see that $\sim_K$ is not a congruence.

We have shown that $M/K$ is a left $M$-set. More is true, however.

**Proposition 3.6.2.** Let $K$ be an $F$-submonoid of $M$. The quotient $M/K$ is a filtered left $M$-set.

**Proof.** The fact that $M/K$ is a left $M$-set is just Lemma 3.6.1. To see that $M/K$ is in addition filtered, consider the canonical surjective map $q : M \to M/K$. Since $M$ is a monoid, it is not empty. Hence, $M/K$ is also non-empty and condition F1 holds.
To show F2, assume $m_1q(a) = m_2q(a)$, with $a, m_1, m_2 \in M$. We have to find $m, \tilde{a} \in M$, such that $m_1m = m_2m$ and $mq(\tilde{a}) = q(a)$. To this end, observe that

$$q(m_1a) = m_1q(a) = m_2q(a) = q(m_2a).$$

Hence, $m_1ax = m_2ax$ for an element $x \in K$. Since $K$ is an $F$-submonoid, there exists an element $y \in K$, such that $xy = y$. So, we can take $m = ax$ and $\tilde{a} = q(1)$. We have $m_1m = m_1ax = m_2ax = m_2m$ and $mq(1) = q(ax) = q(a)$, because $axy = ay$ and $ax \sim_K a$.

To show F3, take $q(a_1), q(a_2) \in M/K$. Then $a_1q(1) = q(a_1)$ and $a_2q(1) = q(a_2)$. Thus, F3 holds with $a = q(1)$. \hfill \Box

We have the following “inverse statement” to the above:

**Proposition 3.6.3.** Let $\rho$ be an $M$-congruence on $M$, such that $M/\sim_\rho$ is a filtered left $M$-set and define

$$K_\rho = \{m \in M | 1 \sim_\rho m\}.$$

i) The subset $K_\rho$ is a saturated $F$-submonoid of $M$.

ii) We have $M/\sim_\rho \simeq M/K_\rho$.

**Proof.** i) By Lemma \[3.2.1\] $K_\rho$ is a submonoid of $M$. Take $x, y \in K_\rho$. We have $x \sim_\rho 1 \sim_\rho y$, by the definition of $K_\rho$. Thus $q(x) = q(y)$, where $q : M \to M/\sim_\rho$ is the canonical map. It follows that $xq(1) = yq(1)$. Since $M/\sim_\rho$ is filtered, there are $m, \tilde{a} \in M$ such that $xm = ym$ and $mq(\tilde{a}) = q(1)$. The last condition implies that $z := m\tilde{a} \in K$. We have $xz = xm\tilde{a} = ym\tilde{a} = yz$. This shows that $K_\rho$ is an $F$-submonoid.

Take $m \in M$ and $x \in K_\rho$, such that $mx = x$. Since $1 \sim_\rho x$, it follows that $m \sim_\rho mx = x \sim_\rho 1$. Thus, $m \in K_\rho$ and $K_\rho$ is saturated.

ii) Assume $m \sim_\rho n$. It follows that $mq(1) = nq(1)$. By F2, there are $x, a \in M$, such that $mx = nx$ and $xq(a) = q(1)$. The last condition implies $y = xa \in K$. Since $my = mxa = nxa = ny$, we see that $m \sim_K n$. Conversely, let $m \sim_K n$. That is $mx = nx$ for $x \in K$. We have $q(mx) = mq(x) = mq(1) = q(m)$ and $q(nx) = q(n)$. This yields $q(m) = q(n)$, which implies $m \sim_\rho n$. This finishes the proof. \hfill \Box

**Corollary 3.6.4.** Let $K$ be an $F$-submonoid of $M$ and $\rho$ be the congruence $\sim_K$, corresponding to $K$. Then

i) $K_\rho = \{m \in M | mx = x, \text{ for an element } x \in K\} = \hat{K}$, where $\hat{K}$ is the saturation of $K$,

ii) $\hat{K}$ is an $F$-submonoid,

iii) if $K$ is saturated, we have $\hat{K} = K$.

**Proof.** i) By definition, $m \in K_\rho$ if and only if $1 \sim_\rho m$. This happens exactly if there exists an $x \in K$, such that $mx = x$. This proves the first equality. Next, take $m \in K$. Since $K$ is an $F$-monoid, there exists an $x \in K$, such that $mx = x$. Hence, $1 \sim_\rho m$ and $m \in K_\rho$. This shows that $K \subseteq K_\rho$. Since $K_\rho$ is saturated by Proposition \[3.6.3\], we have $\hat{K} \subseteq K_\rho$.

To show the opposite inclusion, take any $m \in K_\rho$. Thus, $1 \sim_\rho m$. So, $m = mx$ for some $x \in K$. Since $\hat{K}$ is saturated and $x \in K \subseteq \hat{K}$, we see that $m \in \hat{K}$. It follows that $K_\rho = \hat{K}$.

ii) Follows from part i) by virtue Proposition \[3.6.3\] and iii) is holds by definition. \hfill \Box
Theorem 3.6.5. Let $M$ be a finite monoid. Any filtered left $M$-set is isomorphic to $M/K$, for a suitable saturated $F$-submonoid $K$ of $M$.

Below (see Theorem 3.7.1), we will prove a much stronger result.

Proof. Any filtered $M$-set is cyclic by Theorem 3.3.2. Hence, it is isomorphic to $M/\rho$, where $\rho$ is an $M$-congruence. Part ii) of Proposition 3.6.3 says that $K = K_\rho$ is a saturated $F$-submodule of $M$. It remains to show that $M/\sim_\rho \simeq M/K_\rho$, which we already did in Proposition 3.6.3. □

Corollary 3.6.6. Let $K$ and $L$ be $F$-submonoids of $M$. Then $M/K \simeq M/L$ if and only if $\hat{K} = \hat{L}$. In particular, $M/K \simeq M/\hat{K}$.

3.7. Some examples. i) Assume $M$ is an $F$-monoid and take $K = M$. In this case, $M/M$ is a singleton. So, the terminal object of $\mathcal{S}_M$ is filtered. Conversely, if the terminal object of $\mathcal{S}_M$ is filtered, then $M$ is an $F$-monoid. In fact, the terminal object is $M/\sim_\rho$, where $x \sim_\rho y$ for all $x, y \in M$. In this case, $K = M$ and hence, $M$ is an $F$-monoid, thanks to the proof of Theorem 3.6.5. In particular, the single element set is filtered if $M$ has a right zero or $M$ is a semilattice.

ii) Let $e \in M$ be an idempotent. Then $K = \{1, e\}$ is an $F$-submonoid. In this case, $\sim_K$ is the equivalence relation on $M$, defined by $a \sim_K b$ if and only if $ae = be$. The left $M$-set $M/K \simeq Me$ is filtered.

Theorem 3.7.1. Let $M$ be a finite monoid. Any filtered $M$-set is of the form $Me$, for an idempotent $e \in M$. Thus, $e \mapsto Me$ yields an equivalence of categories

$$(\mathcal{I}(M))^{op} \simeq \text{Pts}(M).$$

Proof. We have already proven (see Theorem 3.6.5) that any filtered $M$-set is of the form $M/K$, where $K$ is a saturated $F$-submonoind of $M$. By Corollary 3.4.2, $K$ has a right zero $\varrho$. Take $L = \{1, \varrho\}$. We have $L \subseteq K$ and $K \subseteq \hat{L}$. Since $K$ is saturated, we have $K = \hat{L}$. By Corollary 3.6.6, we have $\sim_K = \sim_L$. Thus, we can take $e = \varrho$. □

Corollary 3.7.2. Let $M$ be a finite monoid, then

i) $|\text{Pts}(M)| = |\text{Pts}(M^{op})|.$

ii) If $p$ is a point corresponding to an idempotent $e$, one has an isomorphism of monoids

$$\text{End}(p) \cong (eMe)^{op}.$$ 

iii) There is a bijection

$$F_{M} \cong \text{Idem}_{3}(M).$$

4. Lattice of localising subcategories in $\mathcal{S}_M$

4.1. General facts on localising subcategories. Recall that a localising subcategory $\mathcal{T}$ of a Grothendieck topos $\mathcal{E}$ is a full subcategory of $\mathcal{E}$, such that the following conditions hold:

i) If $x$ belongs to $\mathcal{T}$ and $y \in \mathcal{E}$ is isomorphic to $x$, then $y$ belongs to $\mathcal{T}$. 


ii) The inclusion \( \iota : \mathcal{T} \to \mathcal{E} \) has a left adjoint \( \rho : \mathcal{E} \to \mathcal{T} \), called the localisation.

iii) The localisation \( \rho \) respects finite limits.

It is well-known that in this case, \( \mathcal{T} \) is also a Grothendieck topos. We denote by \( \text{Loc}(\mathcal{E}) \) the poset of all localising subcategories of \( \mathcal{E} \).

For a localising subcategory \( \mathcal{T} \) of \( \mathcal{E} \) and \( p = (p_*, p^*) : \text{Sets} \to \mathcal{E} \) a topos point of \( \mathcal{E} \), we write \( p \pitchfork \mathcal{T} \) if \( p_*(S) \in \mathcal{T} \) for every set \( S \in \text{Sets} \).

We now consider the case when \( \mathcal{T} = \mathcal{S}^\mathcal{M} \) is the category of right \( \mathcal{M} \)-sets, for a monoid \( \mathcal{M} \). It is well-known (see for example Lemma 2.4.1 [15]) that there is an order reversing bijection between the set of localising subcategories of \( \text{Loc}(\mathcal{S}^\mathcal{M}) \) and all Grothendieck topologies defined on the one object category associated to \( \mathcal{M} \). Under this bijection, the localisation subcategory corresponding to a topology \( \mathfrak{a} \), is the category of sheaves on \( \mathfrak{a} \).

Let us recall the notion of a Grothendieck topology on a monoid and that of a sheaf over said Grothendieck topology. For this, we introduce the following notation: For a right ideal \( \mathfrak{a} \) and an element \( m \in \mathcal{M} \), we set \( (\mathfrak{a} : m) = \{ x \in \mathcal{M} | mx \in \mathfrak{a} \} \).

For a monoid \( \mathcal{M} \), a Grothendieck topology (or simply topology) on a monoid \( \mathcal{M} \), is a collection of right ideals \( \mathfrak{a} \), such that

- (T1) \( M \in \mathfrak{a} \),
- (T2) If \( \mathfrak{a} \subseteq \mathfrak{b} \) and \( m \in \mathcal{M} \), then \( (\mathfrak{a} : m) \subseteq \mathfrak{b} \),
- (T3) If \( \mathfrak{b} \subseteq \mathfrak{a} \) and \( \mathfrak{a} \) is a right ideal of \( \mathcal{M} \), such that \( (\mathfrak{a} : \mathfrak{b}) \subseteq \mathfrak{a} \) for some \( \mathfrak{b} \in \mathfrak{b} \), then \( \mathfrak{a} \subseteq \mathfrak{b} \).

From these conditions we get the following results (see for example [20], [15]):

(i) If \( \mathfrak{a} \subseteq \mathfrak{b} \) are right ideals and \( \mathfrak{a} \subseteq \mathfrak{a} \), then \( \mathfrak{b} \subseteq \mathfrak{a} \).
(ii) If \( \mathfrak{a} \subseteq \mathfrak{b} \), then \( \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a} \).
(iii) If \( \mathfrak{a} \subseteq \mathfrak{b} \), then \( \mathfrak{a} \mathfrak{b} \subseteq \mathfrak{a} \).

A right \( \mathcal{M} \)-set \( A \) is called an \( \mathfrak{a} \)-sheaf if the restriction map

\[
A \to \text{Hom}_\mathcal{M}(\mathfrak{a}, A), \quad a \mapsto f_a
\]

is a bijection for every \( a \in \mathfrak{a} \). Here, \( a \in A \) and \( f_a \in \text{Hom}_\mathcal{M}(\mathfrak{a}, A) \), where \( f_a(x) = xa, x \in \mathfrak{a} \).

We let \( \text{Sh}(\mathfrak{a}) \) denote the full subcategory of \( \mathfrak{a} \)-sheaves.

**Lemma 4.1.1.** Let \( \mathfrak{a} \) be a topology on a monoid \( \mathcal{M} \) and \( A \) a filtered left \( \mathcal{M} \)-set. Then

\( p_A \pitchfork \text{Sh}(\mathfrak{a}) \)

if and only if for any \( a \in \mathfrak{a} \), the canonical map \( a \otimes_\mathcal{M} A \to A \) is an isomorphism. Here, \( p_A \) denotes the point of \( \mathcal{S}_\mathcal{M} \) corresponding to \( A \).

**Proof.** Since the direct image functor \( \mathcal{T} \to \mathcal{S}_\mathcal{M} \) corresponding to the point \( p_A \) is given by \( S \mapsto \text{Hom}_{\mathcal{T}}(A, S) \), we see that \( p_A \pitchfork \text{Sh}(\mathfrak{a}) \) if and only if \( \text{Hom}_{\mathcal{T}}(A, S) \) is a \( \mathfrak{a} \)-sheaf. This means that, for all \( a \in \mathfrak{a} \), the canonical map

\[
\text{Hom}_{\mathcal{T}}(A, S) \to \text{Hom}_{\mathcal{S}_\mathcal{M}}(a, \text{Hom}_{\mathcal{T}}(A, S))
\]
is an isomorphism. The map in question is the same as
\[ \text{Hom}_\mathcal{S}(A, S) \to \text{Hom}_\mathcal{S}(a \otimes M A, S). \]
Since \( S \) is any set, Yoneda’s lemma implies that, this happens if and only if \( a \otimes M A \to A \)
is an isomorphism. \( \square \)

4.2. Idempotent ideals.

Lemma 4.2.1. Let \( m \) be a two-sided ideal of a monoid \( M \), such that \( m = m^2 \). The set
\[ F_m = \{ a \mid m \subseteq a \} \]
of right ideals containing \( m \) is a Grothendieck topology on \( M \).

Proof. The condition (T1) is obvious. For (T2), assume \( m \subseteq a \) and \( m \in M \). For any \( x \in m \), \( mx \in Mm = m \subseteq a \). Hence \( x \in (a : m) \). It follows that \( m \subseteq (a : m) \) and \( (a : m) \in F_m \). So (T2) holds. For (T3), take \( b \in F_m \). Assume \( a \) is a right ideal, such that \( (a : b) \in F_m \), for any \( b \in b \). By assumption, \( m \subseteq b \) and \( m \subseteq (a : b) \) for all \( b \in b \). Take any \( x \in m \). Since \( m = m^2 \), we can write \( x = yz \), where \( y, z \in m \). Since \( z \in m \subseteq (a : y) \), we see that \( x = yz \in a \). Thus, \( m \subseteq a \) and \( a \in a_m \), from which (T3) follows. \( \square \)

Proposition 4.2.2. Let \( F \) be a Grothendieck topology on a finite monoid \( M \). There exists a two-sided ideal \( m \), such that \( m = m^2 \) and \( F = F_m \).

Proof. As \( M \) is finite, \( F \) is finite as well. We can also see that \( F \) contains a smallest element \( m \), since \( F \) is closed with respect to finite intersection. Take \( x \in M \). Since \( (m : x) \in F \), it follows that \( m \subseteq (m : x) \). Equivalently, \( zm \subseteq m \), which implies that \( m \) is a two-sided ideal of \( M \). We also know that \( F \) is closed with respect to the product. Hence \( m^2 \in F \). By minimality of \( m \), we have \( m \subseteq m^2 \) and hence, \( m^2 = m \). \( \square \)

We will use the following, well-known, fact, see [18, Proposition 1.23].

Lemma 4.2.3. Any two-sided idempotent ideal of a finite monoid \( M \) has the form
\[ \bigcup_{i \in I} Me_i M, \]
for a (finite) family of idempotents \((e_i)_{i \in I}\).

Lemma 4.2.4. Let \( e \) be an idempotent of \( M \) and \( m \subseteq M \) a two-sided ideal, such that \( m = m^2 \). Then
\[ p_{Me} \cap \text{Sh}(F_m) \]
if and only if \( e \in m \).

Proof. Assume \( p_{Me} \cap \text{Sh}(F_m) \). The canonical map
\[ \mu_m : m \otimes_M Me \to Me \]
is an isomorphism by Lemma 4.1.1. The surjectivity of this map implies that \( e = xme \), for some \( x \in m \), \( m \in M \). So, \( e \in mM = m \). Conversely, assume \( e \in m \). We have to show that for any right ideal \( a \), containing \( m \), the canonical map \( \mu_a : a \otimes_M Me \to Me \) is an
isomorphism. For any \( m \in M \), we have \( me \in m \subseteq a \). Since \( \mu(me \otimes e) = me \), we see that \( \mu \) is surjective. To show that \( \mu \) is injective, we first observe that any element of \( a \otimes_M Me \), can be written as \( a \otimes e \), with \( a \in a \). Assume \( \mu(a \otimes e) = \mu(b \otimes e) \), \( a, b \in a \). Thus \( ae = be \). We have

\[
a \otimes e = a \otimes ee = ae \otimes e = be \otimes e = b \otimes ee = b \otimes e
\]

proving the injectivity of \( \mu \).

Example 4.2.5. i) Let \( M \) once again be the monoid \( \{1, 0, a, b, ab\} \), \( a^2 = 1 = b^2 \) and \( ba = 0 \). It has 4 idempotents \( 1, 0, a, b \). Since \( M1 = M, M0 = \{0\}, Ma = \{0, a\} \) and \( Mb = \{0, b, ab\} \), we see that they all define non-isomorphic idempotents. Thus, \( M \) has 4 non-isomorphic points corresponding to the filtered left \( M \)-sets \( \{0\}, \{0, a\}, \{0, b, ab\} \) and \( M \). Since \( M1M = M, M0M = \{0\}, MaM = \{0, a, ab\}, MbM = \{0, b, ab\} \), we see that there are 6 idempotent ideals

\[\emptyset, \{0\}, \{0, a, ab\}, \{0, b, ab\}, \{0, a, b, ab\}, M.\]

ii) Let \( M = T_3 \) be the monoid of all endomorphic maps \( \{1, 2, 3\} \rightarrow \{1, 2, 3\} \). It has \( 3^3 = 27 \) elements. If \( f \) is a such map, the cardinality of the image of \( f \) is known as the rank of \( F \) and is denoted by \( rk(f) \). There are 3 maps of rank 1 (i.e. constant maps) and all of them are idempotents. There are 6 idempotent elements of rank 2 and only one idempotent of rank 3. All together, we have 10 idempotents. However, \( \text{Idem}_3(M) \) has only three elements, as two idempotents are \( J \)-equivalent if and only if they have the same rank. All two-sided ideals of \( T_3 \) are idempotent and they are

\[\emptyset \subseteq I_1 \subseteq I_2 \subseteq I_3 = M,\]

where \( I_k = \{ f \in T_3 | rk(f) \leq k \} \), \( k = 1, 2, 3 \).

4.3. Distributivity of the lattice \( \text{Loc}(\mathcal{I}_M) \). Denote by \( \mathcal{I}_M(M) \) the set of two-sided idempotent ideals of a monoid \( M \). Since the union of two-sided idempotent ideals of \( M \) is again a two-sided idempotent ideal, we see that \( \mathcal{I}_M(M) \) is a join-semilattice, with \( I \vee J = I \cup J \). Its greatest element is \( M \) and least element is \( \emptyset \).

In general, the intersection of two-sided idempotent ideals is not an idempotent ideal (see i) of Example 4.2.5). This leads us to introduce the following notion: We will say that monoid is \( III \)-closed (idempotent ideal intersection), if for any two-sided idempotent ideals \( I \) and \( J \), the two-sided ideal \( I \cap J \) is also an idempotent ideal.

Lemma 4.3.1. i) Any finite regular monoid is \( III \)-closed.

ii) Any finite commutative monoid is \( III \)-closed.

Proof. i) This follows from the fact that any two-sided ideal in a finite regular monoid is an idempotent, see [18, Corollary 1.25].

ii) Let \( I \) and \( J \) be idempotent ideals of a finite commutative monoid \( M \). By Lemma 4.2.3, we can assume that \( I \) is generated by \( e_1, \ldots, e_m \) and \( J \) is generated by \( f_1, \ldots, f_n \), where \( e_i, f_j \) are idempotents. Denote by \( K \) the ideal generated by \( e_if_j \). Clearly, \( K \subseteq I \cap J \).
Conversely, take \( x \in I \cap J \). We can write \( x = ae_i = bf_j \), for some \( a, b \in M \) and \( 1 \leq i \leq m, 1 \leq j \leq n \). We have
\[
xf_j = ae_i f_j = bf_j^2 = bf_j = x.
\]
As such, \( x \in K \) and hence, \( K = I \cap J \) is also generated by idempotents. \( \square \)

Our interest in \( \mathcal{II}(M) \) comes from the bijection
\[
\text{Loc}(\S M) \cong \mathcal{II}(M),
\]
which is true for all finite \( M \). This follows from Lemmas 4.2.1 and 4.2.2. In particular, \(|\text{Loc}(\S M)|\) is finite and
\[
|\text{Loc}(\S M)| = |\text{Loc}(\S M)_{\text{op}}|
\]
holds.

It is well-known, that any finite join-semilattice \( L \) with greatest element, is a lattice, where
\[
a \land b = \bigvee_x x,
\]
with \( a, b \in L \) and \( x \) running through all the elements of the set
\[
\{x \in L | x \leq a \text{ and } x \leq b \}.
\]
It follows that for finite \( M \), the set \( \mathcal{II}(M) \) is a lattice.

**Lemma 4.3.2.** Let \( I \) and \( J \) be two-sided idempotent ideals. Then
\[
I \land J = \bigcup_e eMe,
\]
where \( e \) runs through all the idempotents of \( I \cap J \).

**Proof.** The RHS is a two-sided ideal, generated by idempotents. Hence, it is an element of \( \mathcal{II}(M) \). If \( e \in I \cap J \), it follows that \( MeM \subseteq I \) and \( MeM \subseteq J \). Thus, RHS \( \subseteq \) LHS. Conversely, assume \( K \) is a two-sided idempotent ideal, such that \( K \subseteq I \) and \( K \subseteq J \). By Lemma 4.2.3, we can write \( K = \bigcup_{i \in I} e_i M \) for some idempotents \( e_i \). We have \( e_i \in I \cap J \) by assumption, which implies that RHS is also a subset of RHS. The result follows. \( \square \)

**Corollary 4.3.3.** If \( M \) is finite, then \( \mathcal{II}(M) \) is a distributive lattice.

**Proof.** Take \( I, J, K \in \mathcal{II}(M) \). We need to show that
\[
I \land (J \cup K) = (I \land J) \cup (I \land K)
\]
By Lemma 4.3.2, we see that LHS is generated as a two-sided ideal by the idempotent elements of \( I \cap (J \cup K) \), while the RHS is generated (as a two-sided ideal) by the idempotent elements of \( I \cap J \) and \( I \cap K \). The result follows. \( \square \)
4.4. **Topology on $F_M$.** We start by recalling the well-known relationship between (finite) distributive lattices, posets and topologies.

Any poset $P$ has a natural topology, called the *order topology*, where a subset $S \subseteq P$ is open if $y \in P$ and $x \leq y$ imply $x \in P$. Thus, $\text{Open}(P)$ is a distributive lattice and it is finite if $P$ is finite. It is well-known, that any finite distributive lattice $L$ is of this form, for a uniquely defined $P$ (see for example [17, p.106]). Specifically, for $P = \text{Irr}(L)$, the subset of irreducible elements of $L$ (an element $x \in L$ is irreducible if $x = y \lor z$ implies $x = y$ or $x = z$).

The poset of our interest is $F_M$, the iso-classes of the topos points of $M$, where $M$ is finite. According to Corollary 3.7.2, we have $F_M \cong \text{Idem}_3(M)$.

This allows to work with $\text{Idem}_3(M)$ instead. This set has a canonical order $e \preceq_3 f$ if $MeM \subseteq mfM$.

Thus, we have a canonical order topology on $\text{Idem}_3(M)$ and as such, on $F_M$.

**Proposition 4.4.1.** One has a bijection $\mathcal{I}(M) \rightarrow \text{Open}(\text{Idem}_3(M))$.

**Proof.** It suffices to show that irreducible elements of $\mathcal{I}(M)$ are exactly ideals of the form $MeM$, where $e$ is an idempotent. This follows form the facts that for any element $I \in \mathcal{I}(M)$, one has $I = \bigcup_{e \in I} MeM$, where $e$ is an idempotent. If $MeM = J \cup K$, then $e \in K$, or $e \in J$. We get that $MeM = J$, or $MeM = K$. \hfill \square

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