Asymptotic formulae of two divergent bilateral basic hypergeometric series

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Abstract

We provide new formulae for the degenerations of the bilateral basic hypergeometric function \(1\psi_1(a; b; q, z)\) with using the \(q\)-Borel-Laplace transformation. These are thought of as the first step to construct connection formulae of \(q\)-difference equation for \(1\psi_1(a; b; q, z)\). Moreover, we show that our formulae have the \(q \to 1 - 0\) limit.

1 Introduction

In this paper, we give two asymptotic formulae for the bilateral basic hypergeometric series,\n
\[
1\psi_1(0; b; q, x) := \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} x^n
\]

and

\[
1\psi_0(a; -; q, x) := \sum_{n \in \mathbb{Z}} (a; q)_n \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n
\]

from the viewpoint of the connection problems on \(q\)-difference equations. Here, \((a; q)_n\) is the \(q\)-shifted factorial defined by

\[
(a; q)_n := \begin{cases} 
1, & n = 0, \\
(1 - a)(1 - aq) \ldots (1 - aq^{n-1}), & n \geq 1, \\
[(1 - aq^{-1})(1 - aq^{-2}) \ldots (1 - aq^n)]^{-1}, & n \leq -1.
\end{cases}
\]

Notice that the \(q\)-shifted factorial is the \(q\)-analogue of the shifted factorial

\[
(a)_n = a(a + 1) \ldots (a + (n-1)).
\]

Moreover, \((a; q)_\infty := \lim_{n \to \infty} (a; q)_n\) and we use the shorthand notation

\[
(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \ldots (a_m; q)_\infty.
\]

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Connection problems on $q$-difference equations are originally studied by G. D. Birkhoff [1]. At first, we review the connection problems on second order $q$-difference equations of the form

$$(a_0 + b_0 x)u(q^2 x) + (a_1 + b_1 x)u(q x) + (a_2 + b_2 x)u(x) = 0,$$  \hspace{1cm} (1)$$

where $a_0 a_2 b_0 b_2 \neq 0$. Let $u_1(x)$, $u_2(x)$ be independent solutions of equation (1) around the origin and let $v_1(x)$, $v_2(x)$ be those around infinity. We take suitable analytic continuation of $u_1(x)$ and $u_2(x)$ [2]. Then we obtain the connection formulae in the following matrix form:

$$
\begin{pmatrix}
  u_1(x) \\
u_2(x)
\end{pmatrix} =
\begin{pmatrix}
  C_{11}(x) & C_{12}(x) \\
  C_{21}(x) & C_{22}(x)
\end{pmatrix}
\begin{pmatrix}
v_1(x) \\
v_2(x)
\end{pmatrix}.
$$

Here, functions $C_{jk}(x), (j, k = 1, 2)$ are $q$-periodic and unique valued, namely, elliptic functions. G. N. Watson gave the first example of the connection formula. He showed a connection formula for the (unilateral) basic hypergeometric series

$$2 \varphi_1(a, b; c; q, x) := \sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} x^n$$ \hspace{1cm} (2)$$

The series (2) satisfies the second order $q$-difference equation

$$(c - ab q x)u(q^2 x) - \{ c + q - (a + b) q x \} u(q x) + q(1 - x)u(x) = 0$$ \hspace{1cm} (3)$$

around the origin. Equation (3) also has solutions around infinity as follows:

$$v_1(x) = \frac{(ax, q/ax; q)_\infty}{(x, q/x; q)_\infty} 2 \varphi_1 \left( a, \frac{aq}{c}; b; q, \frac{cq}{abx} \right), \quad v_2(x) = \frac{(bx, q/bx; q)_\infty}{(x, q/x; q)_\infty} 2 \varphi_1 \left( b, \frac{bq}{c}; a; q, \frac{cq}{abx} \right).$$

The connection formula by Watson [3] is

$$u_1(x) = \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty} \frac{(-ax, -aq/ax; q)_\infty}{(-x, -aq/x; q)_\infty} \frac{(x, q/x; q)_\infty}{(ax, q/ax; q)_\infty} v_1(x)$$

$$+ \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty} \frac{(-bx, -bq/bx; q)_\infty}{(-x, -bq/x; q)_\infty} \frac{(x, q/x; q)_\infty}{(bx, q/bx; q)_\infty} v_2(x).$$

We remark that each connection coefficient is given by the elliptic function.

In equation (1), if we assume that $a_0 a_2 b_0 b_2 = 0$, some power series which appear in formal solutions may be divergent. Therefore, we should take a suitable resummation of a divergent series. J.-P. Ramis and C. Zhang introduced the $q$-Borel-Laplace resummation method to study the connection problems. The $q$-Borel-Laplace transformation is given as follows:

1. We assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is a formal power series. The $q$-Borel transformation of the first kind $B_q^+$ is given by

$$\left( B_q^+ f \right)(\xi) := \sum_{n \geq 0} a_n q^{n(n-1)/2} \xi^n.$$ \hspace{1cm} (4)$$

We denote $\varphi_f(\xi) = \left( B_q^+ f \right)(\xi)$. If $f(x)$ is a convergent series, then $\varphi_f(\xi)$ is an entire function.
2. We fix $\lambda \in \mathbb{C}^* \setminus q\mathbb{Z}$. For any entire function $\varphi(\xi)$, the $q$-Laplace transformation of the first kind $L^+_{q,\lambda}$ [4] is given by

$$
(L^+_{q,\lambda}(\varphi))(x) := \frac{1}{1 - q} \int_0^{\lambda \infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q \xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)},
$$

where

$$
\int_0^{\lambda \infty} f(t) d_q t := (1 - q) \lambda \sum_{n \in \mathbb{Z}} f(\lambda q^n) q^n
$$

is Jackson’s $q$-integral on $(0, \lambda \infty)$ [2].

Thanks to these resummation methods, Zhang gave a connection formula for a divergent series $2\varphi_0(a, b; -; q, x)$ [4]. Morita also gave connection formulae for a divergent series $2\varphi_0(0, 0; -; q, x)$ [5] and some unilateral divergent series [6] by these transformations.

But when we consider the connection formulae for the bilateral series, the connection problems are not so clear. Though L. J. Slater gave a relation between the bilateral series $r_H^s$ in [7], other relations are not known well. The main aim of this paper is to give (connection) formulae between divergent bilateral series and convergent unilateral series by $q$-Borel-Laplace transformations in Section 3. In the last section, we also give the classical limit $q \to 1 - 0$ of our new formulæ.

We are closing this section with commenting on relationship with physics. We proved summation formulæ of $1_H^s$ in [8] which are politic generalizations of the ones found from the study of physics called Abelian mirror symmetry in three-dimensional supersymmetric gauge theories on the unorientable manifold $\mathbb{R}P^2 \times S^1$ [9, 10]. We hope our new formulæ here could open up the class of connection formulæ and provide prominent insight into physics.

2 Notation

In this section, we declare basic notation. In the following, we assume that $0 < |q| < 1$. The bilateral basic hypergeometric series with the base $q$ is defined by

$$
r_H^s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) := \sum_{n \in \mathbb{Z}} (a_1, \ldots, a_r; q)_n \left(\frac{b_1, \ldots, b_s; q}{b_1, \ldots, b_s; q}_n\right) \left(\frac{1}{a_1, \ldots, a_r; q}_n\right)^{s-r} z^n.
$$

This series diverges for $z \neq 0$ if $s < r$ and converges for $|b_1 \ldots b_s/a_1 \ldots a_r| < |z| < 1$ if $r = s$ (refer to [2] for more details). Further, the series (6) is the $q$-analogue of the bilateral hypergeometric function

$$
r_H^s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; z) := \sum_{n \in \mathbb{Z}} \frac{(\alpha_1, \ldots, \alpha_r)_n}{(\beta_1, \ldots, \beta_s)_n} z^n,
$$
which is the bilateral extension of the generalized hypergeometric function

\[ rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; z) := \sum_{n \geq 0} \frac{(\alpha_1, \ldots, \alpha_r)_n}{(\beta_1, \ldots, \beta_s)_n} \frac{z^n}{n!}. \]

Provided \( \Re(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_r) > 1 \), D’Alembert’s ratio test could verify that \( rH_r \) converges only for \( |z| = 1 \) [7]. For later use, we would like to write down Ramanujan’s summation formula given by S. Ramanujan [11],

\[ _1\psi_1(a; b; q, z) = \sum_{n \in \mathbb{Z}} \frac{(a)_n}{(b)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} \]  

with \( |b/a| < |z| < 1 \). Ramanujan’s summation formula is considered as the bilateral extension of the \( q \)-binomial theorem [2]

\[ \sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \]  

for \( |z| < 1 \). The \( q \)-binomial theorem was derived by Cauchy [12], Heine [13], and many mathematicians.

One of the \( q \)-exponential functions is defined as

\[ E_q(z) := \varphi_0(-; -; q, -z) = \sum_{n \geq 0} \frac{1}{(q; q)_n} (-1)^n q^{n(n-1)/2} (-z)^n, \]  

which can be rewritten by the infinite product expression

\[ E_q(z) = (-z; q)_\infty \]  

with \( |z| < 1 \). We note that the limit \( q \to 1 - 0 \) of this \( q \)-exponential is actually the standard exponential

\[ \lim_{q \to 1 - 0} E_q(z(1 - q)) = e^z. \]  

The \( q \)-gamma function \( \Gamma_q(z) \) is given by

\[ \Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{-1 - z}. \]  

The \( q \to 1 - 0 \) limit of \( \Gamma_q(z) \) reproduces the gamma function [2]

\[ \lim_{q \to 1 - 0} \Gamma_q(z) = \Gamma(z). \]  

The theta function of Jacobi with the base \( q \) which we will use is given by

\[ \theta_q(z) := \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} z^n, \quad \forall z \in \mathbb{C}^*. \]
With this definition, Jacobi’s triple product identity is expressed by
\[ \theta_q(z) = (q, -z, -q/z; q)_\infty. \]  
(15)

The theta function has the inversion formula
\[ \theta_q(z) = \theta_q(q/z), \]  
(16)

and satisfies the \( q \)-difference equation
\[ \theta_q(zq^k) = z^{-k}q^{-\frac{k(k-1)}{2}}\theta_q(z). \]  
(17)

In our study, the following proposition about the theta function [14, 15] is useful to consider the \( q \to 1 - \) limit of our formulae in Section 4.

**Proposition 1.** For any \( z \in \mathbb{C}^*(\pi < \arg{z} < \pi) \), we have
\[ \lim_{q \to 1-0} \frac{\theta_q(q^\alpha z)}{\theta_q(q^\beta z)} = z^{\alpha-\beta}. \]  
(18)

and
\[ \lim_{q \to 1-0} \frac{\theta_q \left( \frac{q^\alpha z}{1-q} \right)}{\theta_q \left( \frac{q^\beta z}{1-q} \right)} (1 - q)^{\beta-\alpha} = z^{\beta-\alpha}. \]  
(19)

We also use the following limiting formula of the ratio of the \( q \)-shifted factorial [2]:
\[ \lim_{q \to 1-0} \frac{(zq^\alpha; q)_\infty}{(z; q)_\infty} = (1 - z)^{-\alpha}, \quad |z| < 1. \]  
(20)

The crucial ingredients for our new formulae are the \( q \)-Borel-Laplace transformation. In the rest of the paper, we concentrate on the sequence of the action of the \( q \)-Borel-Laplace transformation on degenerations of the divergent series \( 1 \psi_1^{\text{deg}} \),
\[ 1 \psi_1^{\text{deg}}(x) \xrightarrow{B_q^+} \psi(\xi) \xrightarrow{L^+_{q, \lambda}} \widetilde{\psi}_\lambda(x). \]

We remark that \( \lambda \)-dependence on \( L^+_{q, \lambda} \circ B_q^+ f \) vanishes, i.e., \( L^+_{q, \lambda} \circ B_q^+ f = f \) if \( f(x) \) is a convergent series.

3 Degenerations of the bilateral basic hypergeometric series

In this section, we will provide new convergent series obtained by acting the \( q \)-Borel-Laplace transformation on degenerations of the bilateral basic hypergeometric series \( 1 \psi_1(a; b; q, x) \) which satisfies the first order \( q \)-difference equation
\[ \left( \frac{b}{q} - ax \right) u(qx) + (x - 1)u(x) = 0. \]  
(21)

We consider two different degenerations of equation (21).
1. Degeneration A

In the equation (21), if we take the limit \( a \to 0 \), we obtain the following equation:

\[
\frac{b}{q} \tilde{u}(x) + (x - 1) \tilde{u}(x) = 0. \tag{22}
\]

The bilateral series solution is given by

\[
\tilde{u}(x) = \psi_1(0; b; q; x). \tag{23}
\]

We remark that the series is a divergent series around the origin. We can also find a unilateral series solution around the origin as follows:

\[
\tilde{v}(x) = q \left( \frac{x}{b} \right) \psi_0(0; -q; x). \tag{24}
\]

2. Degeneration B

In the equation (21), if we put \( x \mapsto bx \) and take the limit \( b \to \infty \), we obtain another equation as follows:

\[
\left( \frac{1}{q} - ax \right) \tilde{u}(qx) + x \tilde{u}(x) = 0. \tag{25}
\]

The bilateral series solution is

\[
\tilde{u}(x) = \psi_0(a; -q; x). \tag{26}
\]

We remark that the solution \( \tilde{u}(x) \) contains a divergent series around the origin. We also find the unilateral basic hypergeometric series solution around infinity is

\[
\tilde{v}(x) = \frac{\theta_q(ax)}{\theta_q(qx)} \psi_0 \left( 0; -q, \frac{1}{ax} \right). \tag{27}
\]

In the following subsection, we consider asymptotic formulae for these two divergent series.

3.1 Degeneration A

We here deal with the divergent series (23),

\[
\psi_1(0; b; q; x) = \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} x^n. \tag{28}
\]

As preparation, let us show the following formula:

**Lemma 1.** For any \( x \in \mathbb{C}^* \), we have

\[
0 \psi_1(-; b; q; x) = (q; q)^\infty \frac{\theta_q(-x)}{(b; q)^\infty} \theta_q(-\frac{qx}{b}) \left( \frac{qx}{b}; q \right)^\infty. \tag{29}
\]
Proof. Firstly, we scale $x \to x/a$ in the bilateral basic hypergeometric function,

$$\psi_1(a; b; q, x/a) = \sum_{n \in \mathbb{Z}} \frac{(a; q)_n x^n}{(b; q)_n a^n},$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} \frac{(1 - a)(1 - aq) \cdots (1 - aq^n)}{a^n} x^n$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} \left( \frac{1}{a} - 1 \right) \left( \frac{1}{a} - q \right) \cdots \left( \frac{1}{a} - q^n \right) x^n.$$

Then, taking the limit $a \to \infty$ gives

$$\lim_{a \to \infty} \psi_1(a; b; q, x/a) = \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} (-1)^n q^{n(n-1)} x^n = \psi_1(-; b; q, x).$$

On the other hand, we repeat the same process into Ramanujan’s summation formula (7),

$$\lim_{a \to \infty} \psi_1(a; b; q, x/a) = \lim_{a \to \infty} \psi_1(b; a; q, x, q; q)_\infty = \psi_1(b, q; q, q)_\infty.$$

Therefore,

$$0\psi_1(-; b; q, x) = \frac{(q, x, q^{-1}; q)_\infty}{(b, \frac{b}{x}; q)_\infty}$$

$$= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{(x, q^{-1}; q)_\infty}{(b, \frac{b}{x}; q)_\infty} \frac{(qx; q)_\infty}{(b; q)_\infty}$$

$$= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(-x)}{\theta_q\left(-\frac{bx}{b}\right)} \frac{(qx; b; q)_\infty}{(b; q)_\infty},$$

which we actually would like to show. \qed

The main purpose is to apply the $q$-Borel-Laplace transformation to the divergent series $\psi_1(0; b; q, x)$. As a result, we can find the following convergent series for the degeneration $a \to 0$ of $\psi_1$:

**Theorem 1.** For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have

$$\psi_1^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q^{-1}}{b}\right)} \phi_0(0; -; q, x),$$

where $\psi_1^A(x) := \left( \mathcal{L}_{q, \lambda}^+ \circ \psi_1(0; b; q, x) \right)$.

**Proof.** The $q$-Borel transformation (4) of $\psi_1(0; b; q, x)$ provides

$$\psi_1^A(x) := \left( \mathcal{B}_{q}^{+} \psi_1(0; b; q, x) \right)$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} q^{n(n-1)} x^n$$

$$= 0\psi_1(-; b; q, -\xi).$$
Actually, \( \omega \psi_1 \) has the degenerated version of the Ramanujan’s summation formula shown in Lemma 1, that is,

\[
\psi_{\Lambda}(\xi) = \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\xi)}{\theta_q\left(\frac{q \xi}{b}\right)} \left( - \frac{q \xi}{b}; q \right)_\infty
\]

\[
= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\xi)}{\theta_q\left(\frac{q \xi}{b}\right)} E_q \left( \frac{q \xi}{b} \right).
\]

Then, we take the \( q \)-Laplace transformation (5) to \( \psi_{\Lambda}(\xi) \),

\[
\tilde{\psi}_\Lambda^A(x) := \left( L_{q, \Lambda}^+ \psi_{\Lambda} \right)(\xi)
\]

\[
= \sum_{n \in \mathbb{Z}} \omega \psi_1(-; b; q, -\lambda q^n)
\]

\[
\tilde{\psi}_\Lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta_q(\lambda q^n)}{\theta_q\left(\frac{q \lambda q^n}{b}\right)} \frac{\theta_q(\lambda q^n)}{\theta_q\left(\frac{q \lambda q^n}{b}\right)} \sum_{m \geq 0} \frac{(-1)^{m+1} q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left( - \frac{q \lambda q^n}{b} \right)^m
\]

\[
\tilde{\psi}_\Lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q \lambda q^n}{b}\right)} \theta_q\left(\frac{q \lambda q^n}{b}\right) \sum_{m \geq 0} \frac{(-1)^{m+1} q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left( - \frac{q \lambda q^n}{b} \right)^m
\]

\[
\tilde{\psi}_\Lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \theta_q\left(\frac{q \lambda q^n}{b}\right) \sum_{m \geq 0} \frac{1}{(q; q)_m} x^m
\]

\[
\tilde{\psi}_\Lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \theta_q\left(\frac{q \lambda q^n}{b}\right) \sum_{m \geq 0} \frac{1}{(q; q)_m} x^m
\]

The final expression is obliviously a convergent series with \( x \in \mathbb{C}^* \setminus [-\lambda; q] \) as expected. □

**Corollary 1.** By Theorem 1 and (24), we obtain the following relation:

\[
\tilde{\psi}_\Lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \theta_q(\lambda) \theta_q\left(\frac{q \lambda q^n}{b}\right) \theta_q\left(\frac{q \lambda q^n}{b}\right) \tilde{\nu}(x) =: C_\Lambda(x)\tilde{\nu}(x).
\]

We remark that the function \( C_\Lambda(x) \) is an elliptic function, namely, \( q \)-periodic and unique valued.

### 3.2 Degeneration B

Let us turn to divergent series (26),

\[
1\psi_0(a; -; q, x) = \sum_{n \in \mathbb{Z}} (a; q)_n (-1)^n q^{\frac{n(n-1)}{2}} x^n.
\]  

Applying the \( q \)-Borel-Laplace transformation to \( 1\psi_0(a; -; q, x) \) can bring us to the following conclusion:
We remark that the function \( \psi_\lambda(x) \) is \( \lambda \)-periodic and unique valued.

**Theorem 2.** For any \( x \in \mathbb{C}^* \setminus [-\lambda; q] \), we have

\[
\widetilde{\psi}_\lambda^B(x) = \frac{(q; q)_\infty}{(a; q)_\infty} \frac{\theta_q(a\lambda)\theta_q(a)}{\theta_q(\lambda)\theta_q(\frac{a\lambda}{x})} \varphi_0 \left( 0; -q, \frac{1}{ax} \right),
\]

where \( \widetilde{\psi}_\lambda^B(x) := \left( \mathcal{L}_{q, \lambda}^+ \circ \mathcal{B}_{q}^+ \psi_0 \right)(a; -q, x) \).

**Proof.** We perform at first the \( q \)-Borel transformation (4),

\[
\psi_B(\xi) := (\mathcal{B}_{q}^+ \psi_0)(a; -q, x) = \sum_{n \in \mathbb{Z}} (a; q)_n (-\xi)^n = \psi_1(a; 0; q, -\xi).
\]

Then, it is clear that Ramanujan’s summation formula directly works on \( \psi_B(\xi) \) as

\[
\psi_B(\xi) = \frac{(q, -a\xi, -\frac{q}{a} q; q)_\infty}{(\frac{q}{a}; -\xi; q)_\infty} = \frac{(q; q)_\infty \theta_q(a\xi)}{(\frac{q}{a}; q)_\infty \theta_q(\xi)} E_q \left( \frac{q}{\xi} \right).
\]

Finally, the \( q \)-Laplace transformation (5) of \( \psi_B(\xi) \) results in

\[
\widetilde{\psi}_\lambda^B(x) := \left( \mathcal{L}_{q, \lambda}^+ \psi_B(\xi) \right)(\xi)
\]

\[
= \sum_{n \in \mathbb{Z}} \frac{\psi_1(a; 0; q, -\lambda q^n)}{\theta_q \left( \frac{\lambda q^n}{x} \right)}
\]

\[
= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta_q(a\lambda q^n)}{\theta_q(\lambda q^n)} \frac{1}{\theta_q(\frac{\lambda q^n}{x})} \sum_{m \geq 0} \frac{(-1)^m q^{m(m-1)} q^n}{(q; q)_m} \left( -\frac{q}{\lambda q^n} \right)^m
\]

\[
= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(a\lambda)^{-n} q^{-\frac{n(n-1)}{2}} \theta_q(a\lambda)}{\lambda^{-n} q^{-\frac{n(n-1)}{2}} \theta_q(\lambda)} \frac{1}{\theta_q(\frac{\lambda}{ax})} \sum_{m \geq 0} \frac{(-1)^m q^{m(m-1)} q^n}{(q; q)_m} \left( \frac{1}{ax} \right)^m
\]

\[
= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \sum_{n \in \mathbb{Z}, m \geq 0} \frac{\lambda}{ax} \frac{q^{m(n-m-1)} q^n}{(q; q)_m} \left( \frac{1}{ax} \right)^m
\]

where in the last line the inversion formula (16) of the theta function is used. Therefore, we find the desired convergent series with \( x \in \mathbb{C}^* \setminus [-\lambda; q] \). \( \square \)

**Corollary 2.** By Theorem 2 and (27), we obtain the following relation

\[
\widetilde{\psi}_\lambda^B(x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta_q(a\lambda)\theta_q(a)}{\theta_q(\lambda)\theta_q(\frac{a\lambda}{x})} \varphi_0 \left( 0; -q, \frac{1}{ax} \right) =: C_B(x) \tilde{v}(x).
\]

We remark that the function \( C_B(x) \) is an elliptic function, namely, \( q \)-periodic and unique valued.
4 The $q \to 1 - 0$ limit of the degenerations

We consider the limit $q \to 1 - 0$ of our formulae obtained in the previous section. Before showing those limiting formula, we also focus on the limit $q \to 1 - 0$ of the $q$-binomial theorem in the following linear sum form [9]:

$$
1\varphi_0(a; -; q, x) = 2\varphi_1(a, aq; q^2; x^2) + x\frac{(a, a^3; q^2)_{\infty}}{(aq^2, q^2)_{\infty}}2\varphi_1(aq, aq^2; q^3; q^2, x^2). \quad (31)
$$

**Proposition 2.** We put $a = q^a$ in (31). Then, for any $x \in \mathbb{C}^*$, we have

$$
1F_0(a; -; x) = 2F_1 \left( \frac{\alpha}{2}; \frac{x}{2}; x^2 \right) + \alpha x_2 F_1 \left( \frac{\alpha}{2} + 1, \frac{x}{2}; x^2 \right).
$$

We can straightforwardly derive this sum expression and also verify this directly by decomposing the series expansion of $1F_0(a; -; x)$ into the summations over even and odd integers. In addition, we consider the limit $q \to 1 - 0$ of the special case $a = 0$ of (31):

$$
1\varphi_0(0; -; q, x) = 2\varphi_1(0, 0; q^2; x^2) + x\frac{(q^3; q^2)_{\infty}}{(q; q^2)_{\infty}}2\varphi_1(0, 0; q^3; q^2, x^2). \quad (32)
$$

**Proposition 3.** For any $x \in \mathbb{C}^*$, we have

$$
0F_0(-; -; 2x) = 0F_1 \left( -\frac{1}{2}; x^2 \right) + 2x_0 F_1 \left( -\frac{3}{2}; x^2 \right).
$$

**Proof.** We put $x \mapsto (1 - q^2)x$ in (32) and then implement the limit $q \to 1 - 0$. The left-hand side of (32) converges to the exponential function,

$$
\lim_{q \to 1^-} \sum_{n \geq 0} \frac{(1 - q^2)^n}{(q; q)_n} x^n = \lim_{q \to 1^-} \sum_{n \geq 0} \frac{(1 - q)^n}{(q; q)_n} ((1 + q)x)^n = 0F_0(-; -; 2x) = e^{2x}.
$$

On the other hand, the limit $q \to 1 - 0$ of the right-hand side with $x \mapsto (1 - q^2)x$ reduces to

$$
\lim_{q \to 1^-} \left\{ \sum_{n \geq 0} \frac{(1 - q^2)^{2n}}{(q; q^2)_n(q^2; q^2)_n} x^{2n} + (1 - q^2)x \frac{(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 0} \frac{(1 - q^2)^{2n}}{(q^3; q^2)_n(q^2; q^2)_n} x^{2n} \right\}
$$

$$
= \lim_{q \to 1^-} \sum_{n \geq 0} \frac{(1 - q^2)^n(1 - q^2)^{n}}{(q; q^2)_n(q^2; q^2)_n} x^{2n}
$$

$$
+ x \lim_{q \to 1^-} \frac{(q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (1 - q^2)^{\frac{1}{2}} \times \frac{(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (1 - q^2)^{\frac{1}{2}} \sum_{n \geq 0} \frac{(1 - q^2)^n(1 - q^2)^{n}}{(q^3; q^2)_n(q^2; q^2)_n} x^{2n}
$$

$$
= \lim_{q \to 1^-} \sum_{n \geq 0} \frac{(1 - q^2)^n(1 - q^2)^{n}}{(q; q^2)_n(q^2; q^2)_n} x^{2n} + x \lim_{q \to 1^-} \frac{\Gamma q^2 \left( \frac{1}{2} \right)}{\Gamma q^2 \left( \frac{3}{2} \right)} \sum_{n \geq 0} \frac{(1 - q^2)^n(1 - q^2)^{n}}{(q^3; q^2)_n(q^2; q^2)_n} x^{2n}
$$

$$
= \sum_{n \geq 0} \frac{1}{n!} x^{2n} + x \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \sum_{n \geq 0} \frac{1}{n!} x^{2n}
$$

$$
= 0F_1 \left( -\frac{1}{2}; x^2 \right) + 2x_0 F_1 \left( -\frac{3}{2}; x^2 \right).
$$

Therefore, we obtain the conclusion. \qed
4.1 On the degeneration A

Let us consider the limit \( q \to 1 - 0 \) of Theorem 1. We can find the well-defined limiting formula by scaling \( x \mapsto (1 - q)x \).

**Theorem 3.** For any \( x \in \mathbb{C}^* \),

\[
\lim_{q \to 1-0} \psi^A_x((1 - q)x) = \Gamma(\beta)x^{1-\beta}e^x. \tag{33}
\]

**Proof.** We put \( x \mapsto (1 - q)x \) and \( b \mapsto q^b \) in Theorem 1 to obtain the limit. Then, we have

\[
\psi^A_x((1 - q)x) = \frac{(q:q)_{\infty}}{(q^b:q)_{\infty}} \frac{\theta_q(\lambda)}{\theta_q(q^{1-\beta}\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} x^n
\]

\[
= \frac{(q:q)_{\infty}}{(q^b:q)_{\infty}}(1 - q)^{1-\beta} \frac{\theta_q(\lambda)}{\theta_q(q^{1-\beta}\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} x^n
\]

\[
= \frac{\Gamma_q(\beta)}{\theta_q(q^{1-\beta}\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} x^n
\]

Now, applying (19) in Proposition 1 to equation (34), we have

\[
\lim_{q \to 0} \psi^A_x((1 - q)x) = \Gamma(\beta) \times (\lambda)^{1-\beta} \times \left(\frac{\lambda}{x}\right)^{\beta-1} \sum_{n \geq 0} \frac{1}{n!} x^n = \Gamma(\beta)x^{1-\beta}e^x,
\]

which is the consistent limiting formula of Theorem 1 as \( q \to 1 - 0 \). \( \square \)

4.2 On the degeneration B

Finally, we treat with Theorem 2 in the limit \( q \to 1 - 0 \). With rescaling \( x \mapsto x/(1 - q) \), the limit \( q \to 1 - 0 \) of Theorem 2 turns out to be the following formula:

**Theorem 4.** For any \( x \in \mathbb{C}^* \),

\[
\lim_{q \to 1-0} \psi^B_x\left(\frac{x}{1-q}\right) = \Gamma(1 - \alpha)x^{-\alpha}e^x. \tag{35}
\]

**Proof.** We put \( x \mapsto x/(1 - q) \) and \( a \mapsto q^a \) in Theorem 2 to consider the limit \( q \to 1 - 0 \). Then, we have

\[
\psi^B_x\left(\frac{x}{1-q}\right) = \frac{(q:q)_{\infty}}{(q^{1-\alpha}:q)_{\infty}} \frac{\theta_q(q^a\lambda)}{\theta_q(\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} \left(\frac{1}{q^a x}\right)^n
\]

\[
= \frac{(q:q)_{\infty}}{(q^{1-\alpha}:q)_{\infty}}(1 - q)^{-\alpha} \frac{\theta_q(q^a\lambda)}{\theta_q(q^{1-\alpha}\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} \left(\frac{1}{q^a x}\right)^n
\]

\[
= \Gamma_q(1 - \alpha) \frac{\theta_q(q^a\lambda)}{\theta_q(\lambda)} \sum_{n \geq 0} \frac{(1 - q)^n}{(q;q)_n} \left(\frac{1}{q^a x}\right)^n. \tag{36}
\]
By the limiting formula (19) in Proposition 1, we can also obtain the well-defined limit $q \to 1 - 0$ of (36),

$$
\lim_{q \to 1 - 0} \psi_B^\lambda \left( \frac{x}{1 - q} \right) = \Gamma(1 - \alpha) \times (\lambda)^{-\alpha} \times \left( \frac{x}{\lambda} \right)^{-\alpha} \sum_{n \geq 0} \frac{1}{n!} \left( \frac{1}{x} \right)^n = \Gamma(1 - \alpha)x^{-\alpha} e^{\frac{x}{x}}.
$$

Therefore, we obtain the limit $q \to 1 - 0$ of our new formulae $\psi_B^\lambda(x)$.  

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