The diffusive instability of kaon condensate in neutron star matter

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The beta equilibrated dense matter with kaon condensate is analysed with respect to extended stability conditions including charge fluctuations. This kind of the diffusive instability, appeared to be common property in the kaon condensation case. Results for three different nuclear models are presented.

I. INTRODUCTION

Following the pioneering work of Kaplan and Nelson [1], the possibility of having kaon condensation in neutron stars has attracted a lot of attention during past few years. We can divide the various proposed models (for a review see [2]) into two groups: those coming from effective chiral theory and those coming from the meson exchange model. The former ones, based on chiral symmetry SU(3)l × SU(3)r, treat kaons as members of the Goldstone boson octet and couples them to baryons according to the prescription appropriate for the nonlinear representation of the chiral symmetry group [3]. For the second group, kaons are not coupled to baryons but only to heavy mesons [4, 5] (ω, ρ, σ) and these in turn interact with the baryons [6, 7]. Although both approaches give similar predictions for the onset of kaon condensation, the matter composition remains uncertain in the chiral approach due to the shape of the nuclear symmetry energy (an important quantity describing the pure nucleon interaction in the isovector channel) not being well-established. Interesting results, among them the prediction of the complete protonization of the stellar core, were presented in [8]. However, there is also a qualitative difference between the chiral and meson exchange models from a thermodynamical point of view which we would like to stress in the present paper.

Since kaon-nucleon interactions are strongly attractive, the presence of a kaon condensate in dense matter makes the EOS softer. Because the K-N coupling strength is not well known, usually a range of values for it are taken into account. For a sufficiently strong, but still reasonable, K-N coupling the pressure-energy relation, \( p(\varepsilon) \), exhibits a region where the compressibility of the matter is negative. This is the signal of phase separation and means that one needs to make a kind of "construction" which ensures that the compressibility is positive or at least zero (as in the Maxwell construction for the van der Walls equation of state). For neutron star matter, the correct approach consists of applying the so-called Gibbs construction. The system may separate into different phases provided that their pressures and all relevant chemical potentials are equal (the Gibbs conditions). The relevant chemical potentials are those connected with quantum numbers which are conserved by the interactions maintaining the equilibrium. In neutron star matter there are two such quantities: the baryon number \( B \) and the charge \( Q \). This means that the corresponding chemical potentials, for neutrons \( \mu_n \) and for electrons \( \mu_e \), must be continuous across the phase boundary

\[
\mu_n^I = \mu_n^{II} \quad \mu_e^I = \mu_e^{II}.
\]

These equalities state that the phases are in chemical equilibrium - there is no flow of baryon number or charge across the phase boundary. Similarly, pressure equality

\[
P^I = P^{II}
\]

states the condition for mechanical equilibrium - the phase boundary stays at rest with respect to the matter. If the Gibbs conditions [12] can be fulfilled, the system forms a mixture of two phases with different densities of charge and baryon number, occupying the volume in different proportions. This scenario, in the kaon condensation case, was successfully presented in [8], where the meson exchange model was used. Mixed phase formation has never been demonstrated for chiral models, although the possibility of phase separation was noticed in [10] where the Maxwell construction was used to remove the negative compressibility region in the EOS. In the case of neutron star matter, the Maxwell construction is usually made under the assumption that the matter is locally neutral. This means that the construction ensures only the equality of \( P \) and \( \mu_n \) whereas \( \mu_e \) is different for the different phases, so that the EOS obtained is not stable. The primary objective of this work was to carry out a correct construction to enforce the Gibbs conditions in the chiral model and get a stable EOS. However, a thorough analysis shows that kaon condensation in the chiral approach leads to an EOS which is completely unstable, independent of the strength of the kaon-nucleon coupling. The same kind of stability considerations were also applied to the meson exchange approach and we found that the stability of an EOS with a kaon condensate depends

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not on the strength of $K$--$N$ interactions but on the manner in which they are realized.

This paper is organized as follows: in Sec. II the more general stability conditions for beta stable matter are formulated. In Sec. III it is shown that it is not possible to make the Gibbs construction for the chiral model and that the stability conditions are not fulfilled. Then, in Sec. IV the same analysis is repeated for two representative meson exchange models, one for which the Gibbs construction had been performed earlier by other authors and one for which that was not possible. In the final section, we discuss the results obtained and the properties of Lagrangians which allow the instability to be avoided.

II. INTRINSIC STABILITY OF A SINGLE PHASE UNDER BETA EQUILIBRIUM

The basic question in all neutron star matter research is the shape of the relationship between pressure and energy density $p = p(\varepsilon)$; usually called the Equation of State. At zero temperature, the state of neutron star matter should be uniquely described by quantities which are conserved by processes leading to the equilibrium. For matter above nuclear density $\rho_0 = 0.16$ fm$^3$ we have the nucleon beta cycles

$$n \rightarrow p + l + \bar{\nu}_l \quad p + l \rightarrow n + \nu_l$$

which are governed by weak interactions. The baryon number $B$ is conserved by this type of process so that the energy density $\varepsilon$ and pressure $P$ should be functions only of the baryon number density $n_B = B/V$. However, this common statement includes the implicit assumption that the matter is electrically neutral and spatially homogeneous. The star as a whole must obviously be electrically neutral but the matter does not need to be locally neutral. There is no reason why the matter needs to be homogeneous - it may separate into two or more phases with different charge densities in such a way that the total volume on a large scale is neutral. This scenario was presented in detail in [7]. The second conserved quantity, electric charge, should then be treated on the same basis as baryon number or, in other words, the thermodynamic state of a given phase is described not by one but two quantities: baryon number $B$ and charge $Q$. Where $Q$ is the sum over all charge carriers $Q = N_p + N_e + N_\mu + \ldots$ Starting from this point, we will formulate correct stability conditions for a system which is not neutral, in general, but may have local charge density different from 0. Consider a given phase with volume $V$: its energy $U$ is the function of $V, B$ and $Q$, i.e. $U = U(V, B, Q)$. To investigate the intrinsic stability of single phase it is more convenient to introduce intensive rather than extensive quantities:

$$u = U/B, \quad v = V/B, \quad q = Q/B$$

and then, the energy per baryon becomes a function of two variables

$$u = u(v, q)$$

and the first law of thermodynamics takes the form

$$du = -P dv + \mu dq$$

where the pressure and electrical chemical potential are:

$$P = -\left(\frac{\partial u}{\partial v}\right)_q, \quad \mu = \left(\frac{\partial u}{\partial q}\right)_v$$

From the principle of minimum energy, it can be deduced that the phase is intrinsically stable (i.e. it does not separate into different phases) if and only if the energy per baryon is a convex function of its variables $v$ and $q$ (for a detailed analysis, see [8]). In terms of its differentials

$$d^2u = \frac{1}{2} u_{vv} dv^2 + u_{vq} dv dq + \frac{1}{2} u_{qq} dq^2 > 0.$$  (5)

The sufficient condition for the quadratic form to be positive is that its determinant must be positive and at least one of derivatives $u_{vv}$ or $u_{qq}$ must be positive:

$$\left|\begin{array}{cc} u_{vv} & u_{vq} \\ u_{vq} & u_{qq} \end{array}\right| > 0 \begin{array}{c} \text{and} \\ \text{or} \end{array} \left(\begin{array}{c} u_{vv} > 0 \\ u_{qq} > 0 \end{array}\right)$$  (6)

Thermodynamical identities applied to the above second derivatives allow us to express the positivity condition in terms of two equivalent pairs of inequalities:

$$-\left(\frac{\partial P}{\partial v}\right)_q > 0 \quad \left(\frac{\partial \mu}{\partial q}\right)_p > 0$$  (7)

or

$$-\left(\frac{\partial P}{\partial v}\right)_\mu > 0 \quad \left(\frac{\partial \mu}{\partial q}\right)_v > 0$$  (8)

The pressure derivatives correspond to the well-known compressibilities of matter, one with constant charge and the other with constant chemical potential:

$$\kappa_q = -\frac{1}{v} \left(\frac{\partial v}{\partial P}\right)_q \quad \kappa_\mu = -\frac{1}{v} \left(\frac{\partial v}{\partial P}\right)_\mu$$  (9)

The positivity of the compressibility, usually referred to as mechanical stability, here means that baryon density fluctuations are stable. The physical content of the $\mu$ variation in [7] and [8] becomes clear if we look carefully at their dimensions. As $\mu$ has dimensions of energy/charge, these derivatives have dimensions of energy/charge$^2$ which is the inverse of the unit of electrical capacitance. It is useful to introduce the electrical capacitance of matter: $\chi$. Obviously, we must specify under which constraints it is taken - under constant pressure or constant volume:

$$\chi_p = \left(\frac{\partial q}{\partial \mu}\right)_P \quad \chi_v = \left(\frac{\partial q}{\partial \mu}\right)_v$$  (10)
When $\chi$ is positive, it means the system is diffusively stable, i.e. the charge fluctuations are stable. If $\chi$ changes its sign, the charge fluctuations become infinite which means that the phase can be no longer homogeneous and separates itself into parts with different charge density.

So, finally, it may be concluded that for stable matter, the compressibility and electrical capacitance must be positive, what may be expressed in two ways:

$$
\begin{cases}
\kappa_q > 0 \\
\chi_p > 0
\end{cases}
\quad \text{or} \quad
\begin{cases}
\kappa_\mu > 0 \\
\chi_\nu > 0
\end{cases}
$$

We can notice here some similarities to the one-component system with non-zero temperature, commonly treated in textbooks on thermodynamics. For this kind of system, the internal energy $U$ is a function of volume, entropy and particle number: $U(V, S, N)$. If we make the following replacements:

$$q \to s$$

$$\mu \to T$$

i.e. the charge per baryon plays the role of specific entropy $s = S/N$ and the chemical potential plays the role of temperature, then the electrical capacitance $\chi_i$ corresponds to the specific heat:

$$c_i = T \left( \frac{\partial s}{\partial T} \right)_i , \quad i = v, P$$

Instead of compressibility under fixed $q$ or $\mu$, we here have compressibility under constant $T$ or $s$ and, in the same way, convexity of the internal energy, which is now a function of specific volume and entropy $u(v, s)$, requires the standard compressibility and specific heat to be positive:

$$
\begin{cases}
\kappa_T > 0 \\
\kappa_v > 0
\end{cases}
\quad \text{or} \quad
\begin{cases}
\kappa_s > 0 \\
\kappa_p > 0
\end{cases}
$$

which are well known sufficient conditions for stability of a one component system at non-zero temperature.

In the next section we stress that the positivity requirements on the electric capacitance of matter may not always be ensured. This property is strictly connected with the interactions used for the description of the dense nuclear matter.

### III. THE CHIRAL MODEL

To investigate phase stability in the chiral approach, we focus on the model used in [11] where kaon-nucleon interactions are described by the Lagrangian coming from chiral effective field theory

$$
\mathcal{L}_\chi = \frac{\xi^2}{4} \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \text{Tr} \bar{B} (i\gamma^\mu D_\mu - m_B) B \\
+ F \text{Tr} \bar{B} \gamma^\mu \gamma_5 [A_\mu, B] + D^\mu B \gamma^\mu \gamma_5 [A_\mu, B] \\
+ c \text{Tr} M(U + U^\dagger) + a_1 \text{Tr} B (\xi M^\dagger \xi + \xi^\dagger M \xi^\dagger) B \\
+ a_2 \text{Tr} \bar{B} B (\xi M \xi + \xi^\dagger M^\dagger) \\
+ a_3 \text{Tr} \bar{B} B \text{Tr} (\xi M \xi + \xi^\dagger M^\dagger). 
$$

The only parameter in $\mathcal{L}_\chi$ which is not well determined is $a_3$, which is connected with the kaon-nucleon sigma term $\Sigma_{KN} = \frac{1}{3}m_s \langle N|\bar{u}u + \bar{s}s|N \rangle = -(\frac{1}{2}a_1 + a_2 + 2a_3)m_s$. Reasonable values for $a_3m_s$ are in the range $-310 \text{ MeV} < a_3m_s < -134 \text{ MeV}$ which corresponds to the strangeness content for the proton within the range $0.2 > \langle \bar{s}s \rangle_p / (\bar{u}u + \bar{d}d + \bar{s}s)_p > 0$ [11].

The value of this parameter does not appear to be of great importance for the following analysis of matter stability. The energy density coming from $\mathcal{L}_\chi$ is found by means of Baym’s theorem, which states that $\langle K^- \rangle = \frac{1}{\sqrt{2}} \exp(-\mu_K t)$ (assuming only the s-wave kaon-nucleon interaction):

$$
\varepsilon_{KN} = f^2 \frac{\mu^2_k}{2} \sin^2 \theta + 2(m^2_Kf^2 - n\Sigma_{KN}(x)) \sin \frac{\theta}{2} 
$$

where

$$
\Sigma_{KN}(x) = -(a_1x + a_2 + 2a_3)m_s.
$$

The kaon-nucleon contribution must be properly minimized with respect to $\theta$, and this leads to the following equation for $\theta$

$$
\cos \theta = \frac{1}{f^2 \mu^2_k} \left( m^2_Kf^2 - n\Sigma_{KN}(x) - \frac{1}{2} \mu_K n(1+x) \right).
$$

For the interactions in the pure nucleonic sector we adopt the phenomenological formula [12] for the energy density

$$
\varepsilon_{NN} = \frac{3}{5} E_f(0)n_0(\frac{m_0}{\rho_0})^{5/3} + V(n) + mn + n(1 - 2x)^2E_s(n),
$$

where

$$
E_s(n) = \frac{3}{5} (2^{2/3} - 1)E_f(0)(\frac{m_0}{\rho_0})^{2/3} - F(\frac{m_0}{\rho_0}) + E_s(n_0)F(\frac{m_0}{\rho_0}).
$$

There are two potential contributions in [15] coming from the isoscalar and isovector sectors of the nucleon interaction, $V(n)$ and $E_s(n)$ respectively. $V(n)$ is parametrized to get correct values for the compression modulus at the saturation point $K_0 = 240 \text{ MeV}$ and it mainly determines the stiffness of the EOS. The nuclear symmetry energy $E_s(n)$ at the saturation point takes the value 30 MeV. The potential contribution $F(u)$ is an unknown function which determines the behaviour of $E_s$ at higher densities. The importance of the symmetry energy for the chemical composition of dense matter in the kaon condensate case was presented in [14]. Here we put $F(u) = u$ to mimic typical results from relativistic mean field theory (RMF) since we are going to compare the chiral model to the RMF approach only as regards the difference in the kaon-nucleon interaction. The total energy density is a sum

$$
\varepsilon = \varepsilon_{KN} + \varepsilon_{NN} + \varepsilon_{lep},
$$

where the leptonic contribution is expressed in the standard way by means of chemical potentials for electrons.
and muons

\[ \varepsilon_{lep} = \frac{\mu_e^4}{4\pi^2} + m^4 \mu_e g(\sqrt{\eta(\mu_e^2 - m_e^2)}) / m_\mu, \]

where the function \( g(t) \) is:

\[ g(t) = \frac{1}{8\pi^2} ((2t^3 + t) \sqrt{1 + t^2 - \text{arsinh}^2 t}), \]

\[ \eta(x) = \left\{ \begin{array}{ll} x & ; x \geq 0 \\ 0 & ; x \leq 0 \end{array} \right. \]

From the total energy of the system we are able to find all of the necessary thermodynamic quantities such as the nucleon chemical potentials and the pressure

\[ \mu_n = E_F^{(0)} \left( \frac{n}{m_0} \right)^{2/3} + m + V'(n) + n(1-2x)^2 E'_s(n) \]

\[ + (1-4x^2) E_s(n) - (2\Sigma_{K_n} + \mu_K) \sin^2 \frac{\theta}{2}, \]

\[ \mu_p = E_F^{(0)} \left( \frac{n}{m_0} \right)^{2/3} + m + V'(n) + n(1-2x)^2 E'_s(n) \]

\[ - (1-2x)(3-2x) E_s(n) - (2\Sigma_{K_p} + 2\mu_K) \sin^2 \frac{\theta}{2}, \]

\[ P = -\varepsilon + \sum_i \mu_i n_i = \]

\[ = \frac{2}{5} E_F^{(0)} n_0 \left( \frac{n}{m_0} \right)^{5/3} - V(n) \]

\[ + n \sqrt{V'(n)} + n^2 (1-2x)^2 E'_s(n) \]

\[ + \frac{1}{2} f_2^2 \mu_K^2 \sin^2 \theta - 2m_K^2 f_2^2 \sin^2 \frac{\theta}{2} \]

\[ + \frac{\mu_e^4}{12\pi^2} + m^4 \mu_p \sqrt{(\eta(\mu_e^2 - m_e^2)) / m_\mu}, \]

where for the chemical potentials it is useful to introduce kaon-neutron and kaon-proton sigma terms: \( \Sigma_{K_n} = \Sigma_{KN}(0), \Sigma_{K_p} = \Sigma_{KN}(1) \).

Now, the total energy and pressure are functions of five variables (the number densities \( n_n, n_p, \mu_K, \mu_e, \mu_p \) corresponding to the five species of particles \( p, n, K^-, e, \mu \) present in the system). By the use of the beta equilibrium conditions

\[ \mu_K = \mu_e = \mu_p = \mu_n - \mu_p \]

we may reduce these to two independent variables as mentioned in section 11. The above equations also allow us to show that

\[ \mu_e = \left( \frac{\partial u}{\partial q} \right)_v \]

which justifies identifying \( \mu_e \) with the charge chemical potential appearing in 11

\[ \mu \equiv \mu_e. \]

Which two of the five variables are chosen is a matter of convenience. For instance, in order to make the Gibbs construction we need the pressure as function of the chemical potentials \( P(\mu_n, \mu) \) for finding a state preserving the Gibbs conditions [12] for the two-phase coexistence. Usually, the compressibility of the neutral matter is first tested \( \frac{\partial P}{\partial \mu} q=0 > 0 \), and following this procedure, for the full stability considerations we have chosen the pair of inequalities expressed by 17. For the \( \mu \) derivative in 17 it is useful to express the pressure as a function of \( \mu \) and \( q \), where

\[ q = -n x + j^2 \mu \sin^2 \theta + n(1 + x) \sin^2 \frac{\theta}{2} + \varepsilon_{lep} \]

and then to look at the pressure contours in the \( q-\mu \) plot. Using the condition of beta equilibrium [23], the pressure in Eq. (22) can be written as a function of \( n, x, \theta, \mu \). These quantities were found numerically for given \( P \) and \( q \) by solving Eqs (14, 22, 24), with the proton fraction under beta equilibrium being

\[ x = \frac{1}{2} - \frac{\mu(1 + \cos \theta) + 2a_1 n_s (1 - \cos \theta)}{16 E_s(n)}. \]

IV. PHASE STABILITY IN CHIRAL MODEL

For the kaon condensation presented in 11, the pressure-density relation revealed a region of negative compressibility when the matter was locally neutral \( q = 0 \). This unstable region was removed by means of the Maxwell construction which, as mentioned in the introduction, does not preserve continuity of the charge chemical potential. By abandoning the local neutrality of matter one may to try to perform the Gibbs construction as presented in Fig. 1, where the pressure is shown as a function of the baryon density and the charge chemical potential and also \( \mu \) discontinuity is shown in the case of the Maxwell construction. From this figure one can see that the pressure of the kaon phase is always lower than that of the normal phase. The only points where the two pressures are equal is on the \( \theta = 0 \) line, where the condensate starts to grow. However, at this line these two surfaces are tangent, or the pressure gradients for both phases are equal:

\[ \left( \frac{\partial P'}{\partial n} \right)_{\mu_n} = \left( \frac{\partial P'}{\partial \mu} \right)_{\mu_n} \]

These pressure gradients are just the charge densities, according to the thermodynamical relation

\[ dP = n \, d\mu_n + n_q \, d\mu \]

where \( n_q = Q/V \), so there is no difference in charge densiyt between the normal and kaon phases - the transition from the normal phase to the kaon phase is of second order. The lack of charge difference between the phases makes it impossible to have formation of the mixed phase along \( \theta = 0 \). There is no other region in the \( (\mu_n, \mu) \) plane where the pressures for the two phases might be equal.
This is a signal of inconsistency between pressure and energy for the kaon phase. Thermodynamic principles state that the system prefers the phase having greater pressure for given chemical potentials ("pressure maximization") and simultaneously smaller energy for given particle numbers ("energy minimization"). Going along the neutrality line with increasing density, after the unstable region is passed, the line should have entered into a region where the mixed phase disappears and the pure kaon phase is again preferable. Although for all densities the kaon phase has lower energy, suggesting that it is the preferred phase, its pressure $P_K$ never exceeds $P_N$. The fundamental reason for this is that the kaon phase does not represent a stable system at all. This conclusion is confirmed by testing the second derivative of the energy, $\chi_P$, which is shown by means of constant pressure contours in the $q$-$\mu$ plot, in Fig. 2. The isobars are represented keeping in mind the physical constraints: $0 < x < 1$ and $-1 < \cos \theta < 1$, and so usually they do not cover the whole area of the plot. As one can see, the isobar slope for the kaon phase is negative for the neutral phase $q = 0$. Moreover, a negative $\chi_P$ is also observed for almost all values of $q$. For comparison, the isobars for the normal phase are also plotted and they always exhibit a positive value of $\chi_P$ which means that the normal phase is always stable. Such behaviour is not an effect of the kaon-nucleon coupling being too strong. A negative $\chi_P$ is also observed for the minimal kaon-nucleon coupling corresponding to the vanishing strangeness content of proton where $a_3m_s = -134$ MeV. The plots for this minimal coupling are presented in Fig. 3. The pressure-density relation for the neutral matter does not reveal any region of negative compressibility, as seen in the left panel in Fig. 3. Nevertheless the electrical capacity $\chi_P$ is still negative for $q = 0$. This means that the matter, although being mechanically stable, does not fulfil the diffusive stability condition. Also, for most values of $q$, kaonic matter is diffusively unstable except in a small
region for low values of pressure and only positive values of \( q \). At this point we conclude that, within the framework of the chiral model, matter with a kaon condensate is intrinsically unstable with respect to charge fluctuations for any values of \( K-N \) coupling.

V. STABILITY IN THE MESON EXCHANGE MODEL

Having discussed the stability considerations for the chiral model we now want to address the same question for the second class of models used for treating kaon condensation - the meson exchange models. The common feature of this class of models is that kaons are coupled to vector or scalar mesons but not directly to baryons. The way of making this coupling is not unique. Let us focus on two different ways of coupling presented by Knorrein, Prakash and Ellis in [9] and by Glendenning and Schaffner-Bielich in [10]. In the following, we refer to them as the KPE and GS approaches. For the GS model, the Gibbs construction was successfully carried out whereas for KPE it was not possible to do this as was later mentioned in [14]. The two approaches differ regarding the pure nucleon interactions as well as the kaon-nucleon interactions. In order to underline the role played by the difference in the kaon-nucleon sector we have formulated two models which are identical in their nucleon part but different in the kaon part. Hence, for both of them we keep the same Lagrangian for the non-strange sector

\[
\mathcal{L}_N = \bar{N} \left( i\gamma_\mu \partial^\mu - m_N + g_\sigma N \sigma - g_\omega N \gamma_\mu V_\mu - g_\rho N \gamma_\nu \bar{\tau}_N R_\mu \right) N + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - U(\sigma)
\]

\[
- \frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m_\omega^2 V_\mu V^\mu
\]

\[
- \frac{1}{4} \bar{R}_\mu \bar{R}^{\mu\nu} + \frac{1}{2} m_\rho^2 \bar{R}_\mu \bar{R}^\mu,
\]

where \( N \) denotes nucleons and \( \sigma, V, \bar{R} \) are the scalar, vector and isovector meson fields (for details see [8]). In the kaonic sector we used two different Lagrangians:

\[
\mathcal{L}_{K}^{GS} = D_\mu^* K \tilde{D}_\mu^* K - m_K^2 K^2 (29)
\]

\[
\mathcal{L}_{K}^{KPE} = \partial_\mu K^+ \partial^\mu K^- - m_K^2 K^+ K^- + i (\gamma_\omega K^+ V_\mu + g_\rho R^\mu_\sigma) (K^+ \partial^\mu K^- - K^- \partial^\mu K^+) (30)
\]

where the covariant derivative is

\[
D_\mu = \partial_\mu + i g_\omega K V_\mu + i g_\rho \bar{\tau}_K \bar{R}_\mu
\]

The two Lagrangians have different mass terms:

\[
m_{K}^{GS} = m_K - g_\sigma K^\sigma (32)
\]

\[
m_{K}^{KPE} = \sqrt{m_K^2 - g_\sigma K m_K^\sigma} (33)
\]

Apart from the usual Weinberg-Tomozawa term (the second term in \( \mathcal{L}_{K}^{KPE} \)), the covariant derivative introduces additional terms which are quadratic in the meson fields \( V_\mu \) and \( \bar{R}_\mu \). These additional terms modify the equation of motion for the vector mesons in such a way that it couples to the conserved current

\[
\partial_\mu V^{\mu\nu} + m_\omega V^\mu = g_\omega N \bar{\gamma}^\mu N - g_\omega K J^\mu_K (34)
\]

where the kaon current \( J^\mu_K \), coming from the symmetry \( K^\pm \to \exp^{\pm i\alpha} K^\pm \), is

\[
J^\mu_K = iK^- \partial^\mu K^+ - i\partial_\mu K^- K^+ - 2g_\omega K V_\mu K^- K^+ - 2g_\rho K \bar{\tau}_K \bar{R}_\mu K^- K^+ (35)
\]

In this scheme, the divergence of the vector field vanishes and so it is possible to fulfill the Ward identity for a vector field in a medium, as pointed out by Schaffner and Mishustin in [13]. In the KPE Lagrangian, the coupling to the scalar \( \sigma \) is only linear whereas for the GS Lagrangian a quadratic term in \( \sigma \) is also present.

For both models presented here, we use the same parametrization for the nucleon part (28) which gives the properties of the saturation point as follows: the binding energy at \( n_0 = 0.153 \text{ fm}^{-3} \) is \( E/A = -16.3 \text{ MeV} \), the symmetry energy \( E_s(n_0) = 32.5 \text{ MeV} \), the compressibility \( K_0 = 240 \text{ MeV} \), and the nucleon effective mass \( m^*/m = 0.78 \). As shown in [10], the density dependence of the symmetry energy \( E_s \) in the RMF models is dominated by a linear increase with density which is in agreement with the \( E_s(n) \) introduced in the chiral approach [14]. In order to keep the same strength for the \( K-N \) interactions in both models, one may compare the optical potential in symmetric nuclear matter (\( R_{0.3} = 0 \)) coming from [30] and [29]

\[
U_{K}^{GS} = -g_\sigma K m_\sigma - g_\omega K V_0 + \frac{(g_\omega K V_0)^2 + (g_\omega K m_\sigma)^2}{2m_K} (36)
\]

\[
U_{K}^{KPE} = \frac{1}{2} g_\sigma K m_\sigma - g_\omega K V_0 (37)
\]

At nuclear density \( n_0 \), where \( U_K \) is fixed, the quadratic terms in \( U_{GS} \) contribute no more than a few percent to the total and so we neglect them, as was also done in [8]. The only difference is then in the scalar coupling constant \( g_\sigma \) which is twice as large in the KPE parametrization as in the GS one. The value of the optical potential may also be used to relate the \( K-N \) coupling strength to the chiral model for which \( U_K \) is

\[
U_K = -\frac{\Sigma_{KN}(\frac{1}{2}) n_s}{2m_K f^2_\pi} - \frac{3n}{8f^2_\pi} (38)
\]

A strangeness content of the proton within the range 0-0.2 then corresponds to \( U_{K} \) being between -73 and -120 MeV. To keep contact with the results of Glendenning and Schaffner, we use the values -80 and -120 MeV in our further analysis.
The models formulated above are treated in the spirit of RMF theory, taking into account that the kaon mean field value is $K^\pm = \frac{\epsilon_0}{\sqrt{2}} \exp^{\pm i \mu t}$. In order to find the pressure contours in the $q$-$\mu$ plot, one has to solve the relevant equation of motion including the beta stable matter (see [3]) for given $P$ and charge per baryon $q$, where

$$q = \frac{-n x + n_{lep} + n_K}{n}$$  \hspace{1cm} (39)$$

and the kaon number density is

$$n_K = 2(\mu + g_{\omega K} V_0 + g_{\mu K} R_{0,3}) K^- K^+$$  \hspace{1cm} (40)$$

First, we present the stability considerations for the GS model. In Fig.3 one may compare the results for the two values of the optical potential. For the $U_K = -80$ MeV case, when the neutral kaon phase appears its compressibility is positive at all densities (the left-hand plot). The kaon phase is also diffusively stable, its isobars having a positive slope so that that $\chi_P > 0$. For the $U_K = -120$ MeV case, just after the onset of condensation the compressibility of the neutral phase and $\chi_P$ are both negative. To be more explicit, the isobar $P = 89$ MeV/fm$^3$, corresponding the critical density $n_c$ is plotted to show that it keeps the negative slope for $q = 0$. The diffusive instability means that the charge fluctuations increase infinitely leading to phase separation. The charge is unequally distributed on the micro-scale. Within a sufficiently large volume $V$, one may observe droplets of one phase immersed in the other. The volume fraction $\alpha = V^K / V$ for the two phases is found from the requirement of the global neutrality:

$$\alpha \frac{Q^K}{V^K} + (1-\alpha) \frac{Q^N}{V^N} = \alpha q^K n^K + (1-\alpha) q^N n^K = 0.$$  \hspace{1cm} (42)$$

We may then define the average baryon number density for the mixed phase region

$$\bar{n} = \alpha n^K + (1-\alpha) n^N$$  \hspace{1cm} (43)$$

and represent the other thermodynamical quantities ($P, \mu, \varepsilon$ etc.) as functions of the one variable $\bar{n}$ (see the dotted lines in the left-hand panel in Fig.4). The above analysis confirms that for the GS model the intrinsic stability conditions for a given phase [4] and the mutual stability conditions for the mixed phase [12] are simultaneously fulfilled. As we will see in the following, this

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Kaon phase instability for minimal $K$-$N$ coupling.}
\end{figure}
FIG. 4: Kaon phase stability for the GS model. The left-hand panels show quantities for neutral phase, the dotted lines representing the values for the mixed phase region. In the right-hand panels, isobars for the normal and kaon phases are plotted. For the $U_K = -120$ case, the phase separation is also shown.

does not happen for other meson exchange models like the KPE one. The results for the KPE model are very similar to those for the chiral model. Results for the two cases $U_K = -120, -90$ are shown in Fig. In the first case, negative compressibility appears for neutral phase just after the critical density. It is impossible to find a region in the $q$-$\mu$ plane where the kaon phase would be diffusively stable. Even at higher densities, corresponding to negative values of $\mu$, when the compressibility of $q = 0$ phase again becomes positive the kaon phase is still diffusively unstable. As in the chiral case, this property of the KPE model remains unchanged also for smaller values of the kaon-nucleon coupling. As one may see in Fig for $U_K = -90$ (we took this value instead of -80 so that the effect is more visible), in spite of positive $\kappa_q$ for the whole range of densities considered, the electrical capacitance $\chi_P$ is always negative apart from a small range of $q$ around 0.6 which corresponds very small values of the pressure.

VI. SUMMARY AND DISCUSSION

In this paper we have focused on the diffusive stability of kaon condensates within the framework of three different models of strong interactions: one chiral and two meson-exchange based models (here referred to as GS and KPE). From this analysis, one can understand why the Gibbs construction, which can be made in the GS model, is not applicable to the other models presented here. From a thermodynamical point of view, the chiral and KPE Lagrangians lead to matter which does not ful-
fil one of the fundamental conditions for intrinsic phase stability. This happens independently of the value of the kaon-nucleon coupling strength and must be an intrinsic property of the model. At this point it is difficult to say definitely what are the requirements which should be imposed on a Lagrangian in order to get complete stability. In the work by Pons et. al [13], the authors considered several meson exchange models and found that making the Gibbs construction is possible for some of them. One may suppose that the existence of a mixed phase existence means that the kaon phase is intrinsically stable, but in principle, this should be checked independently for the models presented. Looking at the properties of the Lagrangian for the admissible models, one may suppose that nature of the kaon to vector-meson coupling is the most relevant property. Making the Gibbs construction was possible only for models with the covariant derivative for the kaons given by eq. [51], which leads to vanishing divergence of the vector field. However this condition is not sufficient; as was shown in [13], there was also a model with this property for which the Gibbs construction was not possible. It seems that the kaon to scalar meson coupling is also important. To reach a more solid conclusion on this will require further analysis.

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[16] "Heavy mesons" in the language of chiral symmetry means mesonic degrees of freedom which are multiplets of the spontaneously unbroken flavour subgroup of the full chiral group.
[17] This quantity $Q$, which is introduced with the opposite sign from usual, is positive for negative charge carriers and so the chemical potential $\mu$ corresponding to $Q$ is just the electron chemical potential $\mu_e$ - a fundamental quantity in beta equilibrium matter.