SYSTEMATIC ERRORS OF TRANSITION FORM FACTORS
EXTRACTED BY MEANS OF LIGHT-CONE SUM RULES

Wolfgang Lucha
Institute for High Energy Physics, Austrian Academy of Sciences,
Nikolsdorfergasse 18, A-1050, Vienna, Austria

Dmitri Melikhov
Institute for High Energy Physics, Austrian Academy of Sciences,
Nikolsdorfergasse 18, A-1050, Vienna, Austria

and

Nuclear Physics Institute, Moscow State University,
119992, Moscow, Russia

Silvano Simula
INFN, Sezione di Roma III,
Via della Vasca Navale 84, I-00146, Roma, Italy

Abstract

This talk presents results of our study of heavy-to-light transition form factors extracted with the help of light-cone sum rules. We employ a model with scalar particles interacting via massless-boson exchange and study the heavy-to-light correlator, relevant for the extraction of the transition form factor. We calculate this correlator in two different ways: by making use of the Bethe–Salpeter wave function of the light bound state and by making use of the light-cone expansion. This allows us to calculate the full correlator and separately the light-cone contribution to it. In this way we show that the off-light cone contributions are not suppressed compared to the light-cone one by any large parameter. Numerically, the difference between the value of the form factor extracted from the full correlator and from the light-cone contribution to this correlator is found to be about 20–30% in a wide range of masses of the particles involved in the decay process.
In a previous talk\(^1\) (see also Ref. [2] for details) we have shown that the hadron parameters can be extracted from sum rules only with some accuracy, which lies beyond the control of the standard procedure adopted in the method of sum rules, even if the correlator in a limited range of the Borel parameter is known precisely. In the light-cone sum-rule analysis of hadron form factors, the relevant correlator is not known precisely and is obtained as an expansion near the light cone (LC)\(^3\). This entails additional uncertainties in the extraction of hadron parameters, in this case, of the form factors. This talk reports the results of our recent systematic analysis of off-light-cone effects in sum rules for heavy-to-light form factors\(^4\).

The effects are investigated in a model involving scalar constituents. We consider two types of scalar “quarks”, viz., heavy quarks \(Q\) of mass \(m_Q\) and light quarks \(\phi\) of mass \(m\), and study the weak transition of the heavy scalar “meson” \(M_Q(Q\phi)\) to the light “meson” \(M(\phi\phi)\) induced by the weak heavy-to-light \(Q \to \phi\) quark transition. The analysis of this model is technically simpler but allows one to study some essential features of the corresponding QCD case.

For calculating the correlator of interest, we need the Bethe–Salpeter (BS) amplitude of the light meson, defined by

\[
\Psi_{BS}(x, p') = \langle 0 | T \phi(x) \phi(0) | M(p') \rangle = \Psi(x^2, xp', p'^2 = M^2). \tag{1}
\]

As a function of \(xp'\), this amplitude may be represented by the Fourier integral

\[
\Psi_{BS}(x, p') = \int_0^1 d\xi \exp(-i\xi p' x) K(x^2, \xi), \tag{2}
\]

where the \(\xi\)-integration runs from 0 to 1. The kernel \(K(x^2, \xi)\) may be expanded near the light cone \(x^2 = 0\):

\[
K(x^2, \xi) = \phi_0(\xi) + x^2 \phi_1(\xi, \log(-x^2)) + O(x^4). \tag{3}
\]

It is convenient to use the parametrization of \(K(x^2, \xi)\) proposed by Nakanishi\(^5\)

\[
K(x^2, \xi) = \frac{1}{(2\pi)^4} \int_0^\infty \frac{dz}{z} \frac{\exp(-ik'x)}{G(z, \xi) \left[ \frac{1}{z + m^2} - \xi (1 - \xi) M^2 - k'^2 - i0 \right]^3}, \tag{4}
\]

where \(G(z, \xi)\) exhibits no singularities in the integration regions in \(z\) and \(\xi\). The function \(G(z, \xi)\) may be obtained as the solution of an equation obtained from the BS equation for \(\Psi_{BS}(x, p')\). The LC distribution amplitudes \(\phi_i\)
can be expressed in terms of $G(z, \xi)$. For instance, the light-cone distribution amplitude reads

$$
\phi_0(\xi) = \frac{1}{32\pi^2} \int_0^\infty dz \frac{G(z, \xi)}{z + m^2 - \xi(1 - \xi)M^2}.
$$

(5)

For interactions dominated by exchange of a massless boson at small distances, the solution of the BS equation in the ladder approximation takes the form

$$
G(z, \xi) = \delta(z) G(\xi), \quad G(\xi) = \xi(1 - \xi) f(\xi),
$$

(6)

where $f(\xi)$ is nonzero at the end-points. In this case, all distribution amplitudes exhibit the same end-point behaviour, namely,

$$
\phi_0(\xi) \simeq \xi, \quad \phi_1(\xi) \simeq \xi, \quad \ldots.
$$

(7)

Now, to extract the $M_Q \to M$ transition form factor, we analyze the correlator

$$
\Gamma(p^2, q^2) = i \int d^4x \exp(ipx) \langle 0 | T\varphi(x) Q(x) Q(0) | M(p') \rangle.
$$

(8)

We should (i) write this correlator as a dispersion representation in $p^2$

$$
\Gamma_{\text{th}}(p^2, q^2) = \int \frac{ds}{s - p^2 - i0} \Delta_{\text{th}}(s, q^2),
$$

(9)

(ii) perform the Borel transform $p^2 \to \mu^2$ which gives

$$
\Gamma_{\text{th}}(p^2, q^2) \to \hat{\Gamma}_{\text{th}}(\mu^2, q^2) = \int ds \exp\left(-\frac{s}{2\mu^2}\right) \Delta_{\text{th}}(s, q^2),
$$

(10)

and (iii) cut the correlator at an effective continuum threshold $s = s_0$ getting

$$
\hat{\Gamma}_{\text{th}}(\mu^2, q^2, s_0) = \int ds \theta(s < s_0) \exp\left(-\frac{s}{2\mu^2}\right) \Delta_{\text{th}}(s, q^2).
$$

(11)

The form factor is related to the cut correlator by

$$
f_{M_Q} F_{M_Q \to M}(q^2) = \exp\left(M_Q^2/2\mu^2\right) \hat{\Gamma}_{\text{th}}(\mu^2, q^2, s_0(\mu^2, q^2)),
$$

(12)

where $f_{M_Q}$ is the decay constant of the heavy meson $M_Q$ and $s_0(\mu^2, q^2)$ is an effective continuum threshold, dependent on both $q^2$ and $\mu^2$. 
For large $m_Q$ and for $q^2 \ll m_Q^2$, up to terms power-suppressed by $1/m_Q^2$, the correlator reads \[ \Gamma_{th}(p^2, q^2) = \int d^4k \frac{1}{m_Q^2 - k^2 - i0} (0|T\varphi(x)\varphi(0)|M(p')). \] (13)

In order to calculate this correlator, we may proceed along two different lines.

I. Express the correlator in terms of the BS amplitude $\Psi_{BS}$ in momentum space:

\[ \Gamma_{th}(p^2, q^2) = \frac{1}{(2\pi)^4} \int d^4k \Psi_{BS}(k, p') \] (14)

It is then straightforward to calculate $\Delta_{th}(s, q^2)$ in terms of the kernel $G(z, \xi)$.

The corresponding explicit expression for $\Gamma_{th}$ may be found in Ref. [4].

II. Use the light-cone expansion of $\Psi_{BS}(x, p')$:

\[ \Gamma_{th}(p^2, q^2) = \int d^4k d^4x \frac{1}{m_Q^2 - k^2 - i0} \sum_{n=0}^{\infty} (x^2)^n \int_0^1 d\xi e^{-i\xi' \xi} \phi_n(\xi), \] (15)

with the functions $\phi_n(\xi)$ related to $G(z, \xi)$.

Let us introduce the following quantities: the binding energy of the heavy hadron $\epsilon_Q$ by $M_Q = m_Q + \epsilon_Q$; a new Borel parameter $\beta$ by $\mu^2 = m_Q \beta$; a new effective continuum threshold $\delta$ by $s_0 = (m_Q + \delta)^2$, such that $\epsilon < \delta < \beta$. The parameters $\epsilon$, $\delta$, and $\beta$ remain finite in the limit $m_Q \to \infty$. Hereafter, the light-meson mass is set equal to zero: $M = 0$. We consider the case $q^2 = 0$, and suppress the argument $q^2$ in the correlators.

The uncut Borel image (not related to the form factor of interest) reads

\[ e^{-m_Q^2 \Gamma_{th}(\beta)} = \int_0^{1/\beta} \frac{d\xi}{1-\xi} \left[ \phi_0(\xi) - \frac{1}{\beta} \frac{\phi_1(\xi)}{(1-\xi)^2} + \cdots \right] \exp \left( -\frac{m_Q \xi}{2\beta (1-\xi)} \right). \] (16)

For large $m_Q$, the integral is saturated by region of small $\xi = O(\beta/m_Q)$.

The cut Borel image, i.e. the l.h.s. of (12) which yields the heavy-to-light form factor, takes the form [one should be careful with the surface terms when applying the cut in the dispersion representation, see details in ref.[4]]:

\[ e^{-m_Q^2 \Gamma_{th}(\beta, \delta)} = \int_0^{\epsilon_Q/\beta} \frac{d\xi}{1-\xi} \left[ \phi_0(\xi) - \frac{\phi_1(\xi)}{\beta^2 (1-\xi)^2} + \cdots \right] \exp \left( -\frac{m_Q \xi}{2\beta (1-\xi)} \right) \]

\[ -4 \exp \left( \frac{\epsilon_Q - \delta}{\beta} \right) \left[ \frac{\phi_1(\xi_0)}{m_Q^2} + \frac{\phi_1(\xi_0)}{2m_Q \beta} + \frac{\phi_1(\xi_0)}{m_Q^2} \right] + \cdots, \] (17)
where $\xi_n = 2\delta/m_Q$ and $\cdots$ stand for the contributions of terms corresponding to $n \geq 2$ and of terms power-suppressed for large $m_Q$.

Let us now address an important question: Are the off-LC contributions (which represent one of the higher-twist effects) suppressed compared to the light-cone contribution?

In the uncut correlator, the off-LC terms are suppressed by powers of the parameter $1/\beta$ (but remain of the same order in $1/m_Q$ as the LC contribution).

For the cut correlator, however, the situation is quite different because of the presence of surface terms. We may consider the following cases: $\delta, m \ll \beta$, while $m_Q \to \infty$ and $\delta, m \ll m_Q$, while $\beta \to \infty$. Due to the end-point behaviour of the distribution amplitudes, in both cases the contributions of the terms $n = 0, 1, \ldots$ have the same order. Therefore we conclude that for the realistic case of interactions dominated by massless-boson exchange at short distances, the off-LC contributions are not suppressed compared to the LC contribution by any large parameter.

Next, we give numerical estimates. Fig. 1 shows results for beauty-meson decay, with $M_Q = 5.27$ GeV, $m_Q = 4.8$ GeV, and $m = 150$ MeV. The discussion of the relevant parameter values and further examples may be found in Ref. [4].

Hereafter, the $n = 0$ contribution to the correlator in Eq. (15) is referred to as the light-cone correlator; $\Delta_{LC}(s)$ is the corresponding spectral density.

Taking into account that the end-point region is essential for the transition form factors, we can without loss of generality take the kernel of the form $G(z, \xi) = m^2\delta(z)(1 - \xi)$. It is then straightforward to calculate the spectral densities $\Delta_{th}$ and $\Delta_{LC}$ [cf. Fig. 1]. It is important that the thresholds in $\Delta_{th}$ and $\Delta_{LC}$ do not coincide: in the light-cone spectral density the threshold is $m_Q^2$ whereas in the full spectral density it is $(m_Q + m)^2$. The region near the threshold provides the main contribution to the cut Borel-transformed correlator. The mismatch of the thresholds is responsible for the nonvanishing of the off-light-cone effects in the cut correlator.

The effective continuum threshold $\delta$ is the quantity which determines to a great extent the values of hadron observables extracted from the sum rule. We fix $\delta$ by a standard procedure: we require that, for both LC and full spectral densities,

$$\langle s(\beta, \delta) \rangle = M_Q^2.$$  \hspace{1cm} (18)

This equation may be used as the definition of the implicit function $\delta(\beta)$. We,
Figure 1: Plots for the parameters corresponding to beauty-meson decay $m_Q = 4.8$ GeV, $m = 150$ MeV, $\delta_{LC} = 0.96$ GeV, and $\delta_{th} = 0.79$ GeV. **Upper left panel**: Spectral densities $m_Q^2 \Delta_{th}(s)$ (solid red line) and $m_Q^2 \Delta_{LC}(s)$ (dashed blue line). **Upper right panel**: $\sqrt{\langle s \rangle_{th}}$ (solid red line) and $\sqrt{\langle s \rangle_{LC}}$ vs. $\beta$ (dashed blue line). The horizontal (green) line locates $M_Q = 5.27$ GeV. **Lower left panel**: $\tilde{\Gamma}(\beta, \delta) = m_Q^2 \exp \left( M_Q^2 / (2 \mu^2) \right) \tilde{\Gamma}(\mu^2, s_0)$ vs. $\beta$: $\Gamma_{th}(\beta, \delta_{th})$ (solid red line) and $\Gamma_{LC}(\beta, \delta_{LC})$ (dashed blue line). **Lower right panel**: The ratio $\tilde{\Gamma}_{th}(\beta, \delta_{th}) / \tilde{\Gamma}_{LC}(\beta, \delta_{LC})$ vs. $\beta$.

however, proceed in a different way: we do not consider the $\beta$-dependent $\delta_{th}$ and $\delta_{LC}$, but determine constant values $\delta_{th}$ and $\delta_{LC}$ such that relation (18) is satisfied for some specific value of $\beta$. Here, $\delta$ is fixed from

$$\sqrt{\langle s \rangle_{LC}} = \sqrt{\langle s \rangle_{th}} = M_Q$$

for $\beta = 0.5$ GeV; this gives $\delta_{LC} = 0.96$ GeV and $\delta_{th} = 0.79$ GeV.

As can be seen from the plots, the light-cone contribution to the correlator considerably exceeds the full correlator. Obviously, the difference between these two quantities is just the contribution of the off-LC terms in the LC expansion of the correlator. This difference is to a large extent of pure “kinematical” origin, related to the mismatch between the thresholds in $\Delta_{th}$ and $\Delta_{LC}$.


The main results of the present analysis may be summarized as follows:

1. The difference between the cut full correlator and the LC contribution to the latter is always nonvanishing, since the off-LC contributions are not suppressed by any large parameter compared to the LC one. In heavy-to-light decays, there exists no rigorous theoretical limit in which the cut LC correlator coincides with the cut full correlator.

2. The light-cone contribution provides numerically the bulk of the cut full correlator, the contribution of the off-LC terms being always negative. Thus, the light-cone correlator systematically overestimates the full correlator, the difference at small $q^2$ being $20 \div 30\%$.

3. The Borel curves for the full and the LC correlators turn out to be of similar shapes. Such a similarity of the Borel curves implies that the systematic difference between the correlators cannot be diminished by any relevant choice of the criterion for extracting the heavy-to-light form factor.

Finally, let us point out the following: Although the model discussed here differs, in many aspects, from QCD, it mimics correctly those features which are essential for the effects discussed. Therefore, many of the results derived in this work hold also for QCD. In particular, the following relationship between the light-cone and the full correlators for large values of $m_Q$ and $\mu$ is valid in QCD:

$$\frac{\hat{\Gamma}_{\text{th}}(\mu^2, q^2 = 0, \delta)}{\Gamma_{\text{LC}}(\mu^2, q^2 = 0, \delta)} = 1 - O \left( \frac{\Lambda_{\text{QCD}}}{\delta} \right).$$

(20)

For numerical estimates, we used parameter values relevant for $B$ and $D$ decays. We therefore believe that also the numerical estimates for off-LC effects (one of the higher-twist effects) obtained in this work provide a realistic estimate for higher-twist effects in QCD.

Thus, our analysis suggests a sizeable contribution to heavy-to-light correlators, related to higher-twist effects in QCD. This contribution is hard to control in the method of light-cone sum rules because higher-twist distribution amplitudes are not known with sufficient accuracy. Therefore, one might expect sizeable errors in the heavy-to-light form factors, related to higher-twist effects. [These errors arise in addition to the systematic errors related to the procedure of extracting hadron observables from a correlator discussed in our first talk [II].] The effect is larger for decays of heavy mesons containing the strange quark, i.e., of $B_s$ and $D_s$, than for the decays of $B$ and $D$ mesons.
The off-LC and other higher-twist effects in weak decays of heavy mesons in QCD deserve a detailed investigation: for the method of light-cone sum rules the corresponding distribution amplitudes are “external” objects and should be provided by other nonperturbative methods. In particular, the combination of light-cone sum rules with approaches based on the constituent quark picture \(^7\), which successfully describe heavy-meson decays, might be fruitful. Moreover, it seems promising to apply different versions of QCD sum rules to transition form factors \(^8\); this may be helpful in understanding the genuine uncertainties of the form factors extracted from the light-cone sum rules.

Acknowledgements. D. M. would like to thank the Austrian Science Fund (FWF) for support under project P17692.

References

1. W. Lucha, D. Melikhov, and S. Simula, “Systematic errors of bound-state parameters extracted by means of SVZ sum rules”, [arXiv:0712.0177](http://arxiv.org/abs/0712.0177).

2. W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D \textbf{76}, 036002 (2007); Phys. Lett. B \textbf{657}, 148 (2007); W. Lucha and D. Melikhov, Phys. Rev. D \textbf{73}, 054009 (2006); Phys. Atom. Nucl. \textbf{70}, 891 (2007).

3. I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, Nucl. Phys. B \textbf{312}, 509 (1989); V. M. Braun and I. Filyanov, Z. Phys. C \textbf{44}, 157 (1989); V. I. Chernyak and I. R. Zhitnitsky, Nucl. Phys. B \textbf{345}, 137 (1990); P. Ball and V. M. Braun, Phys. Rev. D \textbf{58}, 094016 (1998).

4. W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D \textbf{75}, 096002 (2007).

5. N. Nakanishi, Phys. Rev. \textbf{130}, 1230 (1963).

6. V. A. Karmanov and J. Carbonell, Eur. Phys. J. A \textbf{27}, 1 (2006).

7. D. Melikhov, Phys. Rev. D \textbf{53}, 2460 (1996); Phys. Rev. D \textbf{56}, 7089 (1997); Eur. Phys. J. direct \textbf{C4}, 2 (2002) [hep-ph/0110087]; D. Melikhov and S. Simula, Eur. Phys. J. C \textbf{37}, 437 (2004); D. Melikhov and B. Stech, Phys. Rev. D \textbf{62}, 014006 (2000).

8. V. Braguta, W. Lucha, and D. Melikhov, [arXiv:0710.5461](http://arxiv.org/abs/0710.5461) [hep-ph].