Exact operator expansion of free Green function and asymptotic extension of Grimus-Stockinger formula

S. E. Korenblit, D. V. Taychenachev

Irkutsk State University, 664003, blvd Gagarin, 20, Irkutsk, Russia
e-mail: korenb@ic.isu.ru

Abstract

An exact operator expansion of the free Green function of Schrödinger equation is obtained, which leads to asymptotic extension of the Grimus-Stockinger formula. Its connection with multipole expansion is shown. The simple examples are considered.

1 Introduction

A considerable efforts was made recently [1, 2] to extend so called Grimus-Stockinger theorem [3], which is the main tool of the modern theory of neutrino oscillations [4, 5] and gives the leading asymptotic behavior with $R = |\mathbf{R}| \to \infty$ for the integral:

$$J(\mathbf{R}) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i(\mathbf{q} \cdot \mathbf{R})}}{(q^2 - k^2 - i0)} \Phi(q) \approx \frac{e^{ikR}}{4\pi R} \Phi(-k\mathbf{n}) \left[ 1 + O(R^{-1/2}) \right],$$

where: $\mathbf{R} = R\mathbf{n}$, $\mathbf{n}^2 = 1$, and the function $\Phi(q) \in C^3$ decreases at least like $1/q^2$ together with its first and second derivatives. Here unlike [1, 2] the closed simple formula for all coefficients of asymptotic expansion of $J(\mathbf{R})$ is derived in all orders of $R^{-s}$. Such extension is potentially important for neutrino oscillations problem for example in explanation of recently revealed reactor anomaly [1]. But it may find much more wide implementation in quantum physics and optics, everywhere where the Green function of Helmholtz equation is used.

2 Preliminaries

In order to understand the physical nature of this expansion, we notice that for infinitely differentiable $\Phi(q)$ uniquely representable by its Taylor expansion for any finite $|q| < \infty$:

$$e^{-i(\mathbf{q} \cdot \mathbf{R})} = \Phi(i \nabla_{\mathbf{R}}) e^{-i(\mathbf{q} \cdot \mathbf{R})},$$

and then, formally:

$$J(\mathbf{R}) = \Phi(i \nabla_{\mathbf{R}}) \frac{e^{ikR}}{4\pi R} \Phi(-i \nabla_{\mathbf{x}}) \left. \frac{e^{ik|\mathbf{R} - \mathbf{x}|}}{4\pi |\mathbf{R} - \mathbf{x}|} \right|_{\mathbf{x} = 0},$$
where the differential vector operator in spherical basis \( n, \eta_\theta, \eta_\varphi \) has the following properties:

\[
\nabla_\mathbf{R} = \mathbf{n} \partial_R + \frac{1}{R} \partial_n, \quad \partial_n = \eta_\theta \partial_\theta + \frac{\eta_\varphi}{\sin \vartheta} \partial_\varphi, \quad (\mathbf{n} \cdot \nabla_\mathbf{R}) = \partial_R,
\]

\( n = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \), \( \eta_\theta = \partial_\theta \mathbf{n} \), \( \sin \vartheta \eta_\varphi = \partial_\varphi \mathbf{n} \),

\([n \cdot \partial_n] = 0\), \([\partial_n \cdot n] = 2\), \((n \times \partial_n)^2 = \partial_n^2\), \((n \times \partial_n) = iL_n\),

\( -\partial_n^2 = L_n^2 = 2R(n \cdot \nabla_\mathbf{R}) + R^2 ((n \cdot \nabla_\mathbf{R})^2 - \nabla_\mathbf{R}^2) \), whence,

\[
\text{for } \cos \vartheta = c : \quad L_n^2 = -[\partial_c (1 - c^2) \partial_c + (1 - c^2)^{-1} \partial_\varphi^2] \equiv L_n,
\]

and the well known representation for spherical wave as a free Schrödinger's 3-dimensional Green function [6] is used:

\[
\frac{e^{ik|R-x|}}{4\pi|R-x|} = \int \frac{d^3q}{(2\pi)^3} \frac{e^{i(q \cdot (R-x))}}{q^2 - k^2 - i0},
\]

which thus satisfies the equation:

\[
(-\nabla_\mathbf{R}^2 - k^2) \frac{e^{\pm ikR}}{4\pi R} = \delta_3(\mathbf{R}), \quad \text{with: } \nabla_\mathbf{R}^2 = \partial_R^2 + \frac{2}{R} \partial_R - \frac{L_n}{R^2}, \quad \text{for } R > 0.
\]

Since for \( R > 0 \) the right hand side of this equation is zero, for spherically symmetric case one immediately obtains formally from (3) an exact answer, which takes place at least for the function \( \Psi(k) \) regular and not increases in upper half plane of complex variable \( k \). If for \( |q| = q \):

\[
\Phi(q) \mapsto \Psi(q), \quad \text{then: } \mathcal{J}(\mathbf{R}) \mapsto \Psi(k) e^{ikR}/(4\pi R).
\]

Thus according to our previous work [2], the higher order corrections originate only by asymmetry of the function \( \Phi(q) \). Indeed remember that to obtain them in [2] we supposed the \( \Phi(q) \) and its first and second derivatives are represented by Fourier-transforms as:

\[
\Phi(q) = \int d^3x e^{i(q \cdot x)} \phi(x), \quad \nabla_q \Phi(q) = i \int d^3x e^{i(q \cdot x)} x \phi(x), \quad \text{and so on.}
\]

Then by interchanging the order of integration and using the Eq. (9) we found the representation:

\[
\mathcal{J}(R) = \int d^3x \frac{e^{ik|R-x|}}{4\pi|R-x|} \phi(x).
\]

Substituting here the expansion, which in the exponential should always contain one additional order with respect to the ones in denominator, for:

\[
|R - x| = R \left[ 1 - \frac{2(n \cdot x)}{R} + \frac{x^2}{R^2} \right]^{1/2} = R \left[ \left( 1 - \frac{(n \cdot x)}{R} \right)^2 + \frac{(n \times x)^2}{R^2} \right]^{1/2},
\]

\[
|R - x| = R - (n \cdot x) + \frac{x^2 - (n \cdot x)^2}{2R} + \ldots, \quad \text{where } x^2 - (n \cdot x)^2 \mapsto (n \times x)^2,
\]
we had the corresponding expansion of integral (13) up to \( O(R^{-2}) \):

\[
J(R) = \frac{e^{ikR}}{4\pi R} \int d^3x e^{-ik(n \cdot x)} \phi(x) \left[ 1 + \frac{(n \cdot x)}{R} + \frac{ik}{2R} (x^2 - (n \cdot x)^2) + \ldots \right],
\]

that by making use of (12) was transcribed as:

\[
J(R) = \frac{e^{ikR}}{4\pi R} \left[ 1 - \frac{i}{R} (n \cdot \nabla_q) + \frac{ik}{2R} ((n \cdot \nabla_q)^2 - \nabla_q^2) + \ldots \right] \Phi(q) \bigg|_{q = -k n},
\]

with:

\[
(n \cdot \nabla_q) \Phi(q) \bigg|_{q = -k n} = -(n \cdot \nabla_k) \Phi(-k) = -\partial_k \Phi(-k n), \quad \text{and so on.}
\]

The next useful observation is that due to (16), with \( k = k n \), for \( q \Rightarrow -k, \nabla_q \Rightarrow -\nabla_k \) from (7), (8) with \( R \Rightarrow k \) one has: \( 2k(n \cdot \nabla_k) + k^2 ((n \cdot \nabla_k)^2 - \nabla_k^2) = \mathcal{L}_n \), and expression (15) takes the following simple form:

\[
J(R) = \frac{e^{ikR}}{4\pi R} \left[ 1 - \frac{i}{R} (n \cdot \nabla_k) + \frac{ik}{2R} ((n \cdot \nabla_k)^2 - \nabla_k^2) + \ldots \right] \Phi(-k n) = \Phi(-k n).
\]

In the next section this result is generalized to all orders by another way. It should be noted that in spite of the one and the same final results following from both the Eqs. (15) and (17), the substitution like (14): \( \nabla_q^2 - (n \cdot \nabla_q)^2 \Rightarrow (n \times \nabla_q)^2 \), which is correct for expressions like (15) in [1, 2], becomes incorrect for noncommutative operators \( n \) and \( \nabla_k \) in expressions like (17). This partially explains the essential difference of expressions (30), (31) or (35) below from the formulas (43), (44) of [1].

3 Asymptotic and multipole expansions

Lemma 1: For \( x = r \mathbf{v}, \mathbf{v} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta), R = R \mathbf{n}, |x| = r < R \):

\[
\frac{e^{ik|R-x|}}{4\pi |R-x|} = \frac{e^{ikR}}{4\pi R} \left\{ 1 + \sum_{s=1}^{\infty} \frac{\prod_{\mu=1}^{s} [\mathcal{L}_n - \mu(\mu - 1)]}{s!(-2ikR)^s} \right\} e^{-ik(n \cdot x)},
\]

with the operator \( \mathcal{L}_n = L_n^2 \) defined by Eq. (8).

Proof: The usual multipole expansion of the Green function (9) for \( R > r \) and the corresponding
expansion of the plane wave \([6], [7]\) (8.533), (8.534) reads:

\[
e^{\pm ik|\mathbf{R} - x|} = \frac{1}{4\pi|\mathbf{R} - x|} i^{2l} \chi_l(\pm ikR) \psi_{l0}(kr) \sum_{m=-l}^{l} Y_l^m(n_Y)^* Y_l^m(v),
\]

(20)

\[
e^{\pm i(k \cdot x)} = \frac{4\pi}{kr} i^{2l} \psi_{l0}(kr) \sum_{m=-l}^{l} Y_l^m(n_Y)^* Y_l^m(v),
\]

(21)

where the spherical functions \(Y_l^m(n) \equiv \langle n|m \rangle\) and Legendre polynomials \(P_l(c)\), as eigenfunctions of selfadjoint operator (7), (8) on the unit sphere, satisfy to orthogonality and completeness conditions:

\[
L_n^2 Y_l^m(n) = l(l+1)Y_l^m(n), \quad L_n^2 P_l(c) = l(l+1)P_l(c), \quad \text{with:} \quad c = c, \quad \text{or} \quad \tilde{c} = (n \cdot v),
\]

(22)

\[
\int d\Omega(n) Y_l^m(n) Y_l^M(n) = \delta_{ll} \delta_{mm}, \quad \sum_{m=-l}^{l} Y_l^m(n) Y_l^m(v) = \frac{(2l+1)}{4\pi} P_l((n \cdot v)),
\]

(23)

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(n) Y_l^m(v) = \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} P_l((n \cdot v)) = \delta_{\Omega(n,v)},
\]

(24)

where \(\delta_{\Omega(n,v)}\) is delta function on the unit sphere \([6], [8]\). The solutions of free radial Schrödinger equations are defined \([6]\) by McDonald and Bessel functions \(K_\nu(z)\) and \(J_\nu(y)\) \([7]\) and for half integer \(\nu = l + 1/2\) are reduced to elementary functions:

\[
\chi_l(bR) \equiv \left(\frac{2bR}{\pi}\right)^{1/2} K_{l+\frac{1}{2}}(bR), \quad \chi_l(bR) \rightarrow \sum_{l=\text{int}}^{l} \frac{e^{-bR(l+s)!}}{s!(l-s)!(2bR)^s},
\]

(25)

\[
\psi_{l0}(kr) \equiv \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+\frac{1}{2}}(kr) = \frac{i^{-l} \chi_l(-ikr) - i^{l} \chi_l(ikr)}{2i},
\]

(26)

\[
r^2 \left(\frac{1}{r} \partial_r^2 r + k^2\right) \psi_{l0}(kr) = l(l+1) \frac{\psi_{l0}(kr)}{r}.
\]

(27)

Substitution of finite sums (23), (25) and interchange of the order of summations transforms (20) to:

\[
e^{ik|\mathbf{R} - x|} = \frac{e^{ikR}}{4\pi R} \sum_{s=0}^{\infty} \frac{1}{s!(2iR)^s} \frac{1}{k^{l-s}} \sum_{l=s}^{\infty} \frac{i^{-l(l+s)!}}{(l-s)!} \psi_{l0}(kr)(2l+1)P_l((n \cdot v)),
\]

(28)

where the multiplier:

\[
\frac{(l+s)!}{(l-s)!} = \prod_{\mu=1}^{s} (l-\mu+1)(l+\mu) = \prod_{\mu=1}^{s} [l(l+1) - \mu(\mu - 1)],
\]

(29)
equals to zero for all missing summands with \(0 \leq l \leq s - 1\) automatically and due to Eqs. (22) may be factored out from the sum over \(l\) as operator product in Eq. (19). Then the formal addition of all that missing in fact zero summands with \(0 \leq l \leq s - 1\) completes the sum over \(l\) into the plane wave (21) and converts the expression (28) into (19).

Remark: Due to Eq. (27) the operator \(L_n\) in (19) with the same success may be replaced by operator in the right hand side of Eq. (27). But that is not convenient for the further manipulations.

**Theorem 1:** Let \(\Phi(q) \in S(\mathbb{R}^3)\), the space of functions infinitely differentiable \(\forall q \in \mathbb{R}^3\), decrease faster than any power of \(1/|q|\) together with all its derivatives. Then integral (1) admits the following infinite asymptotic expansion, which is exact if \(\phi(x)\) has a finite support:

\[
J(R) \approx \frac{e^{ikR}}{4\pi R} \Phi(-kn) \left[ 1 + \sum_{s=1}^{\infty} \frac{C_s(k, n)}{(-2ikR)^s} \right], \quad \text{with:} \quad C_s(k, n) = \frac{1}{s!} \prod_{\mu=1}^{s} \left[ L_n - \mu(\mu - 1) \right] \Phi(-kn),
\]

and which is equivalent to its multipole expansion:

\[
J(R) \approx \frac{1}{4\pi R} \sum_{j=0}^{\infty} \chi_j(-ikR) \sum_{m=-j}^{j} B_j^m(k) Y_j^m(n),
\]

with: \(\Phi(-kn)C_s(k, n) = \frac{1}{s!} \sum_{j=s}^{\infty} \frac{(j+s)!}{(j-s)!} \sum_{m=-j}^{j} B_j^m(k) Y_j^m(n)\),

for: \(\Phi(-kn) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} B_j^m(k) Y_j^m(n)\).

**Proof:** Since Fourier transformation (12) maps the space \(S(\mathbb{R}^3)\) into itself [9], also the function \(\phi(x) \in S(\mathbb{R}^3)\) and is represented by the inverse Fourier transform (38). Let suppose at first that \(\phi(x)\) has a finite support at \(|x| \leq r_0\). Then for \(R > r_0\) we can directly substitute the expansion (19) into representation (13) of \(J(R)\) with the following result after easily justified interchange of the order of summation, differentiation and integration and the use of Eq. (12):

\[
J(R) = \frac{e^{ikR}}{4\pi R} \left[ 1 + \sum_{s=1}^{\infty} \frac{\prod_{\mu=1}^{s} [L_n - \mu(\mu - 1)]}{s!(-2ikR)^s} \Phi(-kn) \right].
\]
This means the Eq. (30) as an exact expansion with the coefficients defined by Eq. (31). However that is not the case for the function $\phi(x)$ with infinite support. Estimating this function for $r > R$ as $|\phi(x)| < C/r^N$ with some $N \gg 1$, the two type of appeared corrections are easy estimated to:

$$
\Delta J^m(x) = \int d^3x \int_{r > R} e^{ik|x|/x} \phi(x), \quad |\Delta J^m(x)| < \frac{C}{R^{N-2}}.
$$

(36)

Since $N \to \infty$ for $\phi(x) \in S(\mathbb{R}^3)$ the expansion (35) acquire the asymptotic meaning of Eq. (30). Their terms with $s = 0, 1$ exactly reproduce the results (11) and (18) of previous section.

Coefficients of multipole expansion (34) may be defined by the following two ways:

$$
B_i^m(k) = \int d\Omega(n) \Phi(-kn)y^m_i(n) = 2\pi \int d^3x \psi_i(kx) \psi_i(qr) = \frac{\pi}{2} \delta(q - k), \quad (-1)^l \psi^m_i(n) = \psi^m_i(-n).
$$

(37)

Substitution into integral (13) the Green’s function multipole expansion (20) by making use of the definition (37) after the same steps and under the same conditions for $\phi(x)$ as above leads to the multipole expansion (32), which with the help of expansion (25) transcribes again as asymptotic expansion (30) with the coefficients given by Eq. (33). But the same expression (33) is obtained by direct substitution of (34) into the definition (31) by means of (22), (29). This confirms the equivalence of expansions (30) and (32).

**Corollary:** From (30) for the squared absolute value follows:

$$
(4\pi R)^2 |J^m(R)|^2 = |\phi(-kn)|^2 \left\{ 1 + \sum_{s=1}^{\infty} \frac{i^s[C_s + (-1)^sC^*_s]}{(2kR)^s} + \sum_{\zeta=2}^{\infty} \frac{i^\zeta}{(2kR)^\zeta} \sum_{s=1}^{\zeta-1} (-1)^{\zeta-s} C_s C^*_{\zeta-s} \right\}. \quad (39)
$$

For $\Phi(-k) = \Phi(-k)$: $C_s = C^*_s$, so $s \to 2n$ at the first sum over $s$. But the second internal sum over $1 \leq s \leq \zeta - 1$ in (39) is equal to itself with the multiplier $(-1)^\zeta$ and thus $\zeta \to 2n$ also. Dividing
further this sum over $1 \leq s \leq 2n-1$ into two parts: with $1 \leq s \leq n$ and $n + 1 \leq s \leq 2n-1$, and putting for the second sum $s = 2n - \sigma$, one finds:

$$
(4\pi R)^2 |J(R)|^2 \approx \Phi^2(-kn) \left\{ 1 + \sum_{n=1}^{\infty} \frac{\Upsilon_n}{(2kR)^{2n}} \right\} = \Phi^2(-kn) \left\{ 1 + \frac{C_1^2 - 2C_2}{(2kR)^2} + \ldots \right\}. \quad (40)
$$

$$
\Upsilon_n = (-1)^n \left[ 2C_{2n} + \sum_{s=1}^{2n-1} (-1)^s C_s C_{2n-s} \right] = 2(-1)^n \left[ C_{2n} + \sum_{s=1}^{n} (-1)^s C_s C_{2n-s} \right] - C_n^2, \quad (41)
$$

what due to “homogeneity” over $C_n$ coincides with the relations obtained in [1] for dimensional analogs of coefficients $C_n, \Upsilon_n$.

4 Simple examples

As was claimed in [1] the sign of $\Upsilon_1$ is important for experimental observation of inverse-square law violation in quantum field theory of neutrino oscillations. Let consider its dependence from the function $\Phi(q)$. It is a simple matter to see that for the case with definite momenta $j$:

$$
\Phi(-kn) \mapsto \sum_{m=-j}^{j} B_j^m(k) Y_j^m(n), \quad \text{it follows: } \Upsilon_1 = C_1^2 - 2C_2 \mapsto 2j(j+1) > 0. \quad (42)
$$

In fact the function $\Phi(q)$ in (1) is a product of convolutions of the wave packets functions [4, 5] for different particles. Since the general principles of quantum field theory imply the wave packets belong to $S'(\mathbb{R}^3)$ [10], the same is truth for their convolution and for $\Phi(q)$ as was supposed above in Theorem 1. Following to [5] in our recent work [11] the following form of the wave packet in momentum representation was advocated as unique possible:

$$
\varphi^\sigma(q, p_a) = N_\sigma(m_a, \zeta_{a}^2) e^{-q_\zeta a}, \quad \zeta_a(p_a, \sigma_a) = p_a g_1(m_a, \sigma_a) + s_a g_2(m_a, \sigma_a), \quad (q_\zeta a) > 0, \quad (43)
$$

$\zeta_{a}, \zeta_{a}^0 > 0$, where momentum 4- vectors $q = (E_q, q)$, $p_a = (E_{p_a}, p_a)$ are both on the mass shell with $E_q = \sqrt{q^2 + m_a^2}$, and so on, $s_a$ is a spin 4- vector, $g_1 \gg |g_2|$ are some real functions of mass $m_a$ and width $\sigma_a$, and $N_\sigma = N(\tau) m_a^{-2\sigma}$ is normalization constant with fixed asymptotic behavior as a function of dimensionless invariant $\tau = m_a \sqrt{\zeta_{a}^2(p_a, \sigma_a)}$ at $\tau \to \infty$ ($\sigma_a \to 0$), and $\tau \to 0$ ($\sigma_a \to \infty$):

$$
N(\tau) \mapsto 2(2\pi)^{3/2} \tau^{-3/2} e^{\tau}, \quad \text{and } \ N(\tau) \mapsto N(0) \neq 0. \quad (44)
$$

Since it is physically not meaningless to leave $\Phi(q)$ only with one wave packet (43), we come to the following simple model of this function for $p_a = |p_a| \rho$, with real $\lambda = -k |p_a| g_1 < 0$, and the unit
3-vector $\rho$:

$$\Phi(-kn) \mapsto e^{\lambda \xi}, \quad \xi = (\rho \cdot n) = \sum_{m=-1}^{1} \rho^{(m)} Y_{1}^{m}(n), \quad \mathcal{L}_{n} \xi = 2\xi, \quad \rho^{2} = 1,$$

leading to: $C_{1} \mapsto 2\lambda \xi - \lambda^{2}(1 - \xi^{2})$, $\Upsilon_{1} \mapsto -4\lambda^{2} \left[1 - (2 + \lambda \xi)(1 - \xi^{2})\right]$, so: $\Upsilon_{1} < 0$, $\forall |\lambda| < \infty$, for which: $\lambda \frac{1}{\xi} \left[\frac{1}{1 - \xi^{2}} - 2\right]$.

Since $\lambda < 0$ it is enough to have $1/2 \leq \xi^{2} \leq 1$, or with $\xi = \cos \vartheta$: $0 \leq \vartheta \leq \pi/4$. Such wide region is interesting from experimental point of view [1]. The case $\lambda > 0$ results in the cone around the vector $\rho$ narrowing with $\lambda \to +\infty$ as $1 - \xi \simeq 1/(2\lambda)$.

5 Conclusions

By the same way as above the obtained in Lemma 1 exact operator expansion (19) for the Green function of Helmholtz equation in three dimension space may be easy generalized for the Helmholtz equation in arbitrary dimension Euclidean space. The same concerns assertion of Theorem 1 about the asymptotic expansion (35). We will show that in the forthcoming work.

The following comments are in order. Since for $m_{a} > 0$ the wave packet (43) in momentum representation $\phi^{\sigma}(q, p_{a}) \in S(\mathbb{R}^{3})$ the same takes place [10] for its coordinate representation as a solution of Klein-Gordon equation with the fixed $x_{a}$ and $x^{0} = t$, defined now [5, 11] via Wightman function, which is analytically continued into the future tube $V^{+}$: $\zeta_{a}^{2}, \zeta_{a}^{0} > 0$ [10]:

$$F_{p_{a}x_{a}}(x) = e^{-i(p_{a}x_{a})} \int \frac{d^{3}q}{(2\pi)^{3}2E_{q}} \phi^{\sigma}(q, p_{a}) e^{i(q(x_{a} - x))} = e^{-i(p_{a}x_{a})} \frac{1}{t} D_{m_{a}}(x - x_{a} - i\zeta_{a}(p_{a}, \sigma_{a})).$$

This form agrees to general principles of quantum field theory [10] and due to (44) admits adequate description of both the states localized in position and momentum 3 - dimension space [11]. So $F_{p_{a}x_{a}}(t, x) \in S(\mathbb{R}_{3}^{3})$ obviously with infinite support and the same should be truth for $\phi(x)$. Thus expansion (35) acquires really asymptotic nature with both the corrections (36) are non zero, what may change drastically the behavior of $\mathcal{J}(\mathbb{R})$ at microscopically big but macroscopically small distances discussed in [1].

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