Two Quick Proofs of a Catalan Lemma Needed by Lisa Sauermann and Yuval Wigderson

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In memory of Robin Chapman (1963-2020), a problem-solving maestro and a great Catalanist

In a recent beautiful article [SW], the authors needed the following identity (Claim 3.8 there)

$$\sum_{i=0}^{s} (-1)^i C_i \binom{i+1}{s-i} = 0 \quad ,$$

(1)

where \(C_i\) are the Catalan numbers. Let’s give two shorter proofs of (1).

**Proof 1** (by SBE): Go to Maple, and type [Note that one does not need Zeilberger, Gosper suffices].

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sum((-1)**i*binomial(2*i,i)/(i+1)*binomial(i+1,s-i),i=0..s); □
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**Proof 2** (by DZ) The Catalan number \(C_i\) famously (inter alia), counts the number of (complete) binary trees with \(i + 1\) leaves. \(\binom{i+1}{s-i}\) counts the number of words with \(s - i\) twos and \(2i + 1 - s\) ones, hence the total number of words in the alphabet \{1, 2\} with \(i + 1\) letters whose sum is \(s + 1\).

Hence \(\sum_{i=0}^{s} C_i \binom{i+1}{s-i}\) counts all binary trees with \(\leq s + 1\) leaves where the leaves have labels 1 or 2 that add-up to \(s + 1\). The lemma is equivalent to the fact that the number of such creatures with an even number of leaves equals the number of those with an odd number of leaves. The following involution provides the needed bijection.

*Scan the leaves from left to right until you either encounter a leaf labeled 2, in which case you make it into an internal vertex and make it give birth to two new leaves each labeled 1, or you encounter a leaf labeled 1 whose sister is also labeled 1, in which case you remove them both and make their mother a new leaf labeled 2.* □

The reason they needed (1) was to prove the following rather hairy identity

$$\sum_{m_1, \ldots, m_t \geq 1 \atop m_1 + \ldots + m_t = l} (-1)^t \binom{l-m_1}{m_1-1} \binom{l-m_2}{m_2} \cdots \binom{l-m_t}{m_t} = (-1)^l C_{l-1} \quad .$$

(2)

Rather than using (1) we will present two direct, shorter proofs. In fact we will prove the more general identity, for \(l \geq m\),

$$\sum_{m_1, \ldots, m_t \geq 1 \atop m_1 + \ldots + m_t = m} (-1)^t \binom{l-m_1}{m_1-1} \binom{l-m_2}{m_2} \cdots \binom{l-m_t}{m_t} = (-1)^m C_{m-1} \quad .$$

(2')

Identity (2) is the special case \(l = m\) of (2').
Calling the left side of \((2')\) \(A(l,m)\), separating the case \(t = 1\) (that yields \(-\binom{l-m}{m-1}\)) and summing over \(m\) (let’s call it \(k\)), we readily get the recurrence

\[
A(l,m) = -\binom{l-m}{m-1} - \sum_{k=1}^{m-1} \binom{l-k}{k} A(l,m-k) .
\]

It would then follow by induction on \(m\) that \(A(l,m)\) equals \((-1)^m C_{m-1}\), if the latter satisfies the same recurrence. But this is equivalent to the identity

\[
\sum_{k=0}^{m} (-1)^k \binom{l-m+k}{m-k} C_k = \binom{l-m-1}{m} . \tag{3}
\]

**Proof 1** (by SBE): Dividing by the right hand side, this is, in turn, equivalent to

\[
\sum_{k=0}^{m} (-1)^k \frac{(l-m+k)}{(l-m-1)} C_k = 1 .
\]

Calling the left side \(f(m)\), go into Maple and type

\[
\text{SumTools}[\text{Hypergeometric}][\text{ZeilbergerRecurrence}](\text{(-1)**k*binomial(2*k,k)/(k+1)*binomial(l-m+k,m-k)/binomial(l-m-1,m),m,k,f,0..m);}
\]

and in one nano-second you would get that \(f(m)\) satisfies the following recurrence

\[
-(m+1) f(m) + (m+2) f(m+1) = 1 ,
\]

Since \(f(1) = 1\) (check!), it follows by induction that \(f(m) = 1\) for every \(m\) (and of course every \(l \geq m\)). □

**Proof 2** (by DZ): The left side of (3) without the \((-1)^k\) is the number of pairs \((T,w)\) where \(T\) is a binary tree with \(\leq m+1\) leaves and \(w\) is a word in \(\{1,2\}\) of length \(l-m-1\) longer than the number of leaves of \(T\), whose sum is \(l\). Hence the left side of (3) is the difference between the number of such creatures where \(T\) has an odd number of leaves and those that have an even number of leaves. The above involution that proved (1) (where the leaves of \(T\) are labeled by the corresponding prefix of \(w\)) is still valid, but now the survivors are the pairs \(.w\) where \(w\) is the one-leaf tree, and \(w\) is a word in \(\{1,2\}\) of length \(l-m-1\) whose sum is \(l-1\). □

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Reference

[SW] Lisa Sauermann and Yuval Wigderson, *Polynomials that vanish to high order on most of the hypercube*, arXiv:2010.00077v1 [math. C0] 30 Sep 2020

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