Multivariate functional approximations with Stein’s method of exchangeable pairs

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Abstract: We combine the multivariate method of exchangeable pairs with Stein’s method for functional approximation and give a general linearity condition under which an abstract Gaussian approximation theorem for stochastic processes holds. We apply this approach to estimate the distance from a pre-limiting mixture process of a sum of random variables chosen from an array according to a random permutation and prove a functional combinatorial central limit theorem. We also consider a graph-valued process and bound the speed of convergence of the joint distribution of its rescaled edge and two-star counts to a two-dimensional continuous Gaussian process.

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1. Introduction

Stein’s method is a powerful tool for finding bounds on distances between probability distributions. It was first used in [Ste72] as a method for normal approximations. In 1975 Chen published his paper [Che75] on the Poisson approximation, which gave rise to the well-known Chen-Stein method. The monograph by Stein [Ste86] introduced the concept of auxiliary randomisation and shed new light on Stein’s method with his use of exchangeable pairs. Barbour [Bar88] and Götze [Göt91] developed the so-called Markov-Process approach to finding Stein’s equation, which made it possible to adapt the method to many other probability laws. Furthermore, a powerful connection has been found between Stein’s method and Malliavin calculus which has led to many new results concerning primarily normal approximations (see [NP12]). Stein’s method for multivariate approximations, gamma, binomial and other distributions has also been developed. An accessible account of the method can be found, for example, in the surveys [LRS17] and [Ros11] as well as the books [BHJ92] and [CGS11],
which treat the cases of Poisson and normal approximation, respectively, in detail. [Swa16] is a database of information and publications connected to Stein’s method.

Approximations by laws of stochastic processes have not been covered in the Stein’s method literature very widely, with the notable exceptions of [Bar90], [BJ09], [CD13] and recently [Kas17a] and [Kas17b]. [Kas17a] establishes a method for bounding the speed of weak convergence of continuous-time Markov chains satisfying certain assumptions to diffusion processes and so introduces a Stein method approach to a large part of the Stroock-Varadhan theory of diffusion approximation (see [SV79]). [Kas17b], on the other hand, treats multi-dimensional processes represented by scaled sums of random variables with different dependence structures using Stein’s method and establishes bounds on their distances from continuous Gaussian processes.

The aim of Stein’s method is to find a bound of the quantity

\[ |\mathbb{E}_{\nu_n} h - \mathbb{E}_{h} h|, \]

where \( \mu \) is the target (known) distribution, \( \nu_n \) is the approximating law and \( h \) is chosen from a suitable class of real-valued test functions \( \mathcal{H} \). The procedure can be described in terms of three steps. First, an operator \( A \) acting on a class of real-valued functions is sought, such that

\[ (\forall f \in \text{Domain}(A) \quad \mathbb{E}_{\nu_n} Af = 0) \iff \nu = \mu, \]

where \( \mu \) is our target distribution. Then, for a given function \( h \in \mathcal{H} \), the following Stein equation:

\[ Af = h - \mathbb{E}_{\mu} h \]

is solved. Finally, using properties of the solution and various mathematical tools (among which the most popular are Taylor’s expansions in the continuous case, Malliavin calculus, as described in [NP12], and coupling methods), a bound is sought for the quantity \( |\mathbb{E}_{\nu_n} Af h| \).

In this paper we combine Stein’s method for multi-dimensional functional approximations with the multivariate method of exchangeable pairs. The exchangeable-pair approach to Stein’s method was first used for one-dimensional approximations in [Ste86]. Therein, for a centred and scaled random variable \( W \), its copy \( W' \) is constructed in such a way that \( (W, W') \) forms an exchangeable pair and a linear regression condition:

\[ \mathbb{E}^W [W' - W] = -\lambda W \]

is satisfied (where \( \mathbb{E}^W [.] := \mathbb{E} [\cdot | W] \)) for some \( \lambda > 0 \), which, in many cases, simplifies the process of obtaining bounds on the distance of \( W \) from the normal distribution. This approach was extended in [RR97] to examples in which an approximate linear regression condition holds:

\[ \mathbb{E}^W [W' - W] = -\lambda W + R \]

for some remainder \( R \). A multivariate version of the method was first described in [CM08], where distances from the normal distribution of vectors of the type \( W = (W_1, \cdots, W_d) \) are obtained by constructing a random vector...
Let $W' = (W'_1, \ldots, W'_d)$ such that $(W,W')$ form an exchangeable pair and satisfy the condition that, for all $i = 1, \cdots, d$:

$$\mathbb{E}^W (W'_i - W_i) = -\lambda W_i$$  \hspace{1cm} (1.1)

for some fixed $\lambda > 0$.

Our condition and the abstract approximation result described in Theorem 4.1 are, however, in the spirit of those given in [RR09], where, for an exchangeable pair of $d$-dimensional vectors $(W, W')$ the following condition is used:

$$\mathbb{E}^W (W' - W) = -\Lambda W + R$$

for some invertible matrix $\Lambda$ and a remainder term $R$. It is worth noting that this general linear condition allows for a much wider class of examples to be treated with the method of exchangeable pairs than condition (1.1). The approach of [RR09] was further reinterpreted and combined with the approach of [CM08] in [Mec09]. An interesting application of the results of this last paper together with other works on exchangeable pairs is the recent work [NZ17], which provides a new proof of the quantitative fourth moment theorem in one and multiple dimensions.

Our paper is the first attempt to extend the multivariate method of exchangeable pairs to a functional context. We apply our abstract approximation result (Theorem 4.1) in two examples. The first one is a combinatorial functional central limit theorem. The second one considers a two-dimensional process representing edge and two-star counts in a graph-valued process created by unveiling subsequent vertices of a Bernoulli random graph as time progresses.

The former is a functional version of the result proved qualitatively in [HC78] and quantitatively in [CF15]. It also extends the setup considered in [BJ09] to an array $\{X_{i,j} : i, j = 1, \cdots, n\}$ of i.i.d. random variables, which are then used to create a stochastic process:

$$t \mapsto \frac{1}{s_n} \sum_{i=1}^{\lfloor nt \rfloor} X_{i,\pi(i)},$$  \hspace{1cm} (1.2)

where $s_n$ is an appropriate scaling constant and $\pi$ is a uniform random permutation on $\{1, \cdots, n\}$. Similar setups, yet with a deterministic array of numbers, and in a finite-dimensional context have also been considered by other authors (see [WW44] for one of the first works on this topic and [Bol84], [Gol05], [NR12] for quantitative results) and the motivation in many of their works has been provided by permutation tests in non-parametric statistics. We establish a bound of the distance between process (1.2) and a Gaussian, piecewise constant process and a qualitative result showing convergence of process (1.2) to a continuous Gaussian limiting process.

The second example, which considers Bernoulli random graphs, goes back to [JN91] and was first studied using exchangeable pairs in a finite-dimensional context in [RR10], where a distance from a random vector whose components represent statistics corresponding to the number of edges, two-stars and triangles...
to a normal vector was bounded. We consider a functional analogue of this result, concentrating, for simplicity, only on the number of edges and two-stars and bounding the distance between the two-dimensional process representing those and a continuous Gaussian limiting process. Our approach can, however, be also easily extended to encompass the number of triangles. All of those statistics are often of interest in applications, for example, when approximating the clustering coefficient of a network or in conditional uniform graph tests.

The paper is organised as follows. In Section 2 we introduce the spaces of test functions which will be used in the main results, and quote results showing that, under certain assumptions, they determine convergence in distribution under the uniform topology. In Section 3 we setup the Stein equation for approximation by a pre-limiting Gaussian process and provide properties of the solutions. In Section 4 we provide an exchangeable-pair condition and prove an abstract exchangeable-pair-type approximation theorem. Section 5 is devoted to the functional combinatorial central limit theorem example and Section 6 discusses the graph-valued process example.

2. Spaces $M$, $M^1$, $M^2$, $M^0$

The following notation, similar to the one of [Kas17b], is used throughout the paper. For a function $w$ defined on the interval $[0, 1]$ and taking values in a Euclidean space, we define $\|w\| = \sup_{t \in [0, 1]} |w(t)|$, where $|\cdot|$ denotes the Euclidean norm. We also let $D^p = D([0, 1], \mathbb{R}^p)$ be the Skorokhod space of all càdlàg functions on $[0, 1]$ taking values in $\mathbb{R}^p$. In the sequel, for $i = 1, \ldots, p$, $e_i$ will denote the $i$th unit vector of the canonical basis of $\mathbb{R}^p$ and the $i$th component of $x \in \mathbb{R}^p$ will be represented by $x(i)$, i.e. $x = (x(1), \ldots, x(p))$.

Let $p \in \mathbb{N}$. Let us define:

$$\|f\|_L := \sup_{w \in D^p} \frac{|f(w)|}{1 + \|w\|^3},$$

and let $L$ be the Banach space of continuous functions $f : D^p \to \mathbb{R}$ such that $\|f\|_L < \infty$. Following [Bar90], we now let $M \subset L$ consist of the twice Fréchet differentiable functions $f$, such that:

$$\|D^2 f(w + h) - D^2 f(w)\| \leq k_f \|h\|,$$  \hspace{1cm} (2.1)

for some constant $k_f$, uniformly in $w, h \in D^p$. By $D^k f$ we mean the $k$-th Fréchet derivative of $f$ and the $k$-linear norm $B$ on $L$ is defined to be $\|B\| = \sup_{\|h\| = 1} |B[h, \ldots, h]|$. Note the following lemma, which can be proved in an analogous way to that used to show (2.6) and (2.7) of [Bar90]. We omit the proof here.
Lemma 2.1. For every $f \in M$, let:
\[
\|f\|_M := \sup_{w \in D^p} \frac{|f(w)|}{1 + \|w\|^3} + \sup_{w \in D^p} \frac{\|Df(w)\|}{1 + \|w\|^2} + \sup_{w \in D^p} \frac{\|D^2f(w)\|}{1 + \|w\|}
+ \sup_{w, h \in D^p} \frac{\|D^2f(w + h) - D^2f(w)\|}{\|h\|}.
\]
Then, for all $f \in M$, we have $\|f\|_M < \infty$.

For future reference, we let $M^1 \subset M$ be the class of functionals $g \in M$ such that:
\[
\|g\|_{M^1} := \sup_{w \in D^p} \frac{|g(w)|}{1 + \|w\|^3} + \sup_{w \in D^p} \|Dg(w)\| + \sup_{w \in D^p} \|D^2g(w)\|
+ \sup_{w, h \in D^p} \frac{\|D^2f(w + h) - D^2f(w)\|}{\|h\|} < \infty
\tag{2.2}
\]
and $M^2 \subset M$ be the class of functionals $g \in M$ such that:
\[
\|g\|_{M^2} := \sup_{w \in D^p} \frac{|g(w)|}{1 + \|w\|^3} + \sup_{w \in D^p} \|Dg(w)\| + \sup_{w \in D^p} \|D^2g(w)\|
+ \sup_{w, h \in D^p} \frac{\|D^2f(w + h) - D^2f(w)\|}{\|h\|} < \infty.
\tag{2.3}
\]
We also let $M^0$ be the class of functionals $g \in M$ such that:
\[
\|g\|_{M^0} := \sup_{w \in D^p} |g(w)| + \sup_{w \in D^p} \|Dg(w)\| + \sup_{w \in D^p} \|D^2g(w)\|
+ \sup_{w, h \in D^p} \frac{\|D^2f(w + h) - D^2f(w)\|}{\|h\|} < \infty.
\]
We note that $M^0 \subset M^1 \subset M^2 \subset M$. The next proposition is a $p$-dimensional version of [BJ09, Proposition 3.1] and shows conditions, under which convergence of the sequence of expectations of a functional $g$ under the approximating measures to the expectation of $g$ under the target measure for all $g \in M^0$ implies weak convergence of the measures of interest. Its proof can be found in the appendix of [Kas17b].

Definition 2.2. $Y \in D([0, 1], \mathbb{R}^p)$ is piecewise constant if $[0, 1]$ can be divided into intervals of constancy $[a_k, a_{k+1})$ such that the Euclidean norm of $(Y(t_1) - Y(t_2))$ is equal to 0 for all $t_1, t_2 \in [a_k, a_{k+1})$.

Proposition 2.3. Suppose that, for each $n \geq 1$, the random element $Y_n$ of $D^p$ is piecewise constant with intervals of constancy of length at least $r_n$. Let $(Z_n)_{n \geq 1}$ be random elements of $D^p$ converging weakly in $D^p$, with respect to the Skorokhod topology, to a random element $Z \in C([0, 1], \mathbb{R}^p)$. If:
\[
|\mathbb{E}g(Y_n) - \mathbb{E}g(Z_n)| \leq C \tau_n \|g\|_{M^0}
\tag{2.4}
\]
for each $g \in M^0$ and if $\tau_n \log^2(1/r_n) \xrightarrow{n \to \infty} 0$, then $Y_n \Rightarrow Z$ (converges weakly) in $D^p$, in both the uniform and the Skorokhod topology.
3. Setting up Stein’s method for the pre-limiting approximation

The steps of the construction presented in this section will be similar to those used to set up Stein’s method in [Bar90] and [Kas17b]. After defining the process $D_n$ whose distribution will be the target measure in Stein’s method, we will construct a process $(W_n(u) : u \geq 0)$ for which the target measure is stationary. We will then calculate its infinitesimal generator $A_n$ and take it as our Stein operator. Next, we solve the Stein equation $A_n f = g$ using the analysis of [KDV17] and prove some properties of the solution $f_n = \phi_n(g)$, with the most important one being that its second Fréchet derivative is Lipschitz.

3.1. Target measure

Let

$$D_n(t) = \sum_{i_1, \ldots, i_m=1}^{n} \left( \tilde{Z}(i_1, \ldots, i_m) J^{(1)}_{i_1, \ldots, i_m}(t), \ldots, \tilde{Z}(p) J^{(p)}_{i_1, \ldots, i_m}(t) \right), \quad t \in [0, 1],$$

(3.1)

where $\tilde{Z}(k,i_1,\ldots,i_m)$’s for $k = 1, \ldots, p$ are centred Gaussian and:

A) the covariance matrix $\Sigma_n \in \mathbb{R}^{(n^m)p \times (n^m)p}$ of the vector $\tilde{Z}$ is positive definite, where $\tilde{Z} \in \mathbb{R}^{(n^m)p}$ is formed out of the $\tilde{Z}(k,i_1,\ldots,i_m)$’s in such a way that they appear in the lexicographic order with $\tilde{Z}(k,i_1,\ldots,i_m)$ appearing before $\tilde{Z}(k+1,j_1,\ldots,j_m)$ for any $k \in [p-1]$ and $i_1, \ldots, i_m, j_1, \ldots, j_m \in [n]$;

B) $J^{(k)}_{i_1,\ldots,i_m} \in D([0,1], \mathbb{R})$, for $i_1, \ldots, i_m \in [n]$, $k \in [p]$, are independent of the $\tilde{Z}(k,i_1,\ldots,i_m)$’s. A typical example would be $J^{(k)}_{i_1,\ldots,i_m} = \mathbb{I}_{A_{i_1,\ldots,i_m}}$ for some measurable set $A_{i_1,\ldots,i_m}$.

Remark 3.1. It is worth noting that processes $D_n$ taking the form (3.1) often approximate interesting continuous Gaussian processes very well. An example is a Gaussian scaled random walk, i.e. $D_n$ of (3.1), where all the $\tilde{Z}(k,i_1,\ldots,i_m)$’s are standard normal and independent, $m = 1$ and $J^{(k)}_{i} = \mathbb{I}_{[i/n,1]}$ for all $k = 1, \ldots, p$ and $i = 1, \ldots, n$. It approximates Brownian Motion. By Proposition 2.3, under several assumptions, proving by Stein’s method that a piece-wise constant process $Y_n$ is close enough to process $D_n$ proves $Y_n$’s convergence in law to the continuous process that $D_n$ approximates.

Now let $\{(X^{(k)}_{i_1,\ldots,i_m}(u), u \geq 0) : i_1, \ldots, i_m = 1, \ldots, n, k = 1, \ldots, p\}$ be an array of i.i.d. Ornstein-Uhlenbeck processes with stationary law $N(0, 1)$, independent of the $J^{(k)}_{i_1,\ldots,i_m}$’s. Consider $\hat{U}(u) = (\Sigma_n)^{1/2} X(u)$, where $X(u) \in \mathbb{R}^{n^m}$ is formed out of the $X^{(k)}_{i_1,\ldots,i_m}(u)$’s in such a way that they appear in the same order as the $\tilde{Z}(k,i_1,\ldots,i_m)$’s appear in $\tilde{Z}$. Write $U^{(k)}_{i_1,\ldots,i_m}(u) = \left( \hat{U}(u) \right)_{l(k,i_1,\ldots,i_m)}$ using the bijection $I : \{(k,i_1,\ldots,i_m) : i_1, \ldots, i_m = 1, \ldots, n, k = 1, \ldots, p\} \rightarrow \{1, \ldots, pm^m\}$.
given by:
\[ I(k, i_1, \ldots, i_m) = (k - 1)n^m + (i_1 - 1)n^{m-1} + \cdots + (i_{m-1} - 1)n + i_m. \] (3.2)

Consider a process:
\[ W_n(t, u) = \left( W_n^{(1)}(t, u), \ldots, W_n^{(p)}(t, u) \right), \quad t \in [0, 1], u \geq 0, \]
where, for all \( k = 1, \ldots, p \):
\[ W_n^{(k)}(t, u) = \sum_{i_1, \ldots, i_m=1}^n U_{i_1, \ldots, i_m}^{(k)}(u) J_{i_1, \ldots, i_m}^{(k)}(t), \quad t \in [0, 1], u \geq 0. \]

It is easy to see that the stationary law of the process \((W_n(\cdot, u))_{u \geq 0}\) is exactly the law of \(D_n\).

### 3.2. Stein equation

By [Kas17b, Propositions 5.1 and 5.3], the following result is immediate:

**Proposition 3.2.** The infinitesimal generator of the process \((W_n(\cdot, u))_{u \geq 0}\) acts on any \(f \in M\) in the following way:
\[ A_n f(w) = -Df(w)[w] + E D^2 f(w) \left[ D_n^{(2)} \right]. \]

Moreover, for any \( g \in M \) such that \( E g(D_n) = 0 \), the Stein equation \( A_n f_n = g \) is solved by:
\[ f_n = \phi_n(g) = -\int_0^\infty T_{n,u} g du, \] (3.3)

where \((T_{n,u}f)(w) = E \left[ f(we^{-u} + \sigma(u)D_n(\cdot)) \right]\). Furthermore, for \( g \in M \):

- A) \( \|D\phi_n(g)(w)\| \leq \|g\|_M \left( 1 + \frac{2}{3}\|w\|^2 + \frac{4}{3}\|E\|\|D_n\|^2 \right) \),
- B) \( \|D^2\phi_n(g)(w)\| \leq \|g\|_M \left( \frac{1}{2} + \frac{\|w\|}{3} + \frac{\|E\|\|D_n\|}{3} \right) \),
- C) \( \|D^2\phi_n(g)(w+h) - D^2\phi_n(g)(w)\| \leq \sup_{w, h \in D^n} \frac{\|D^2(g+c)(w+h) - D^2(g+c)(w)\|}{3\|h\|} \),

for any constant function \( c : D^n \to \mathbb{R} \) and for all \( w, h \in D^n \). Moreover, for all \( g \in M^1 \):

- A) \( \|D\phi_n(g)(w)\| \leq \|g\|_{M^2} \),
- B) \( \|D^2\phi_n(g)(w)\| \leq \frac{1}{2}\|g\|_{M^2} \),

and for all \( g \in M^2 \):
\[ \|D\phi_n(g)(w)\| \leq \|g\|_{M^2}. \]
4. An abstract approximation theorem

We now present a theorem which provides an expression for a bound on the distance between some process $Y_n$ and $D_n$, defined by (3.1), provided that we can find some $Y_n'$ such that $(Y_n, Y_n')$ is an exchangeable pair satisfying an appropriate condition. Our condition (4.1) is similar to that of [RR09, (1.7)], as we explain in Remark 4.4.

**Theorem 4.1.** Assume that $(Y_n, Y_n')$ is an exchangeable pair of $D([0,1], \mathbb{R}^p)$-valued random vectors such that:

$$Df(Y_n)[Y_n] = 2\mathbb{E}Y_n Df(Y_n) [(Y_n - Y_n') \Lambda_n] + R_f,$$

where $\mathbb{E}Y_n[f] := \mathbb{E}[f|Y_n]$, for all $f \in M$, some $\Lambda_n \in \mathbb{R}^{p \times p}$ and some random variable $R_f = R_f(Y_n)$. Let $D_n$ be defined by (3.1). Then, for any $g \in M$:

$$|\mathbb{E}g(Y_n) - \mathbb{E}g(D_n)| \leq R_1 + R_2 + R_3,$$

where $f = \phi_n(g)$, as defined by (3.3), and

$$R_1 = \frac{\|g\|_M}{6} \mathbb{E} \| (Y_n - Y_n') \Lambda_n \| \| Y_n - Y_n' \|^2,$$

$$R_2 = \| \mathbb{E} D^2 f(Y_n) [(Y_n - Y_n') \Lambda_n, Y_n - Y_n'] - \mathbb{E} D^2 f(Y_n) [D_n, D_n] \|,$$

$$R_3 = |\mathbb{E} R_f|.$$

**Remark 4.2.** Condition (4.1) is always satisfied, for example with $\Lambda_n = 0$ and $R_f = Df(Y_n)[Y_n]$ for all $f \in M$. However, for the bound in Theorem 4.1 to be small, we require the expectation of $R_f$ to be small in absolute value.

**Remark 4.3.** The term

$$|\mathbb{E} D^2 f(Y_n) [(Y_n - Y_n') \Lambda_n, Y_n - Y_n'] - \mathbb{E} D^2 f(Y_n) [D_n, D_n]|$$

in the bound obtained in Theorem 4.1 is an analogue of the second condition in [Mec09, Theorem 3]. Therein, a bound on approximation by $\mathcal{N}(0, \Sigma)$ of a $d$-dimensional vector $X$ is obtained by constructing an exchangeable pair $(X, X')$ satisfying:

$$\mathbb{E}^X [X' - X] = \Lambda X + E \quad \text{and} \quad \mathbb{E}^X [(X' - X)(X' - X)^T] = 2\Lambda \Sigma + E'$$

for some invertible matrix $\Lambda$ and some remainder terms $E$ and $E'$. In the same spirit, Theorem 4.1 could be rewritten to assume (4.1) and:

$$\mathbb{E}^Y D^2 f(Y_n) [(Y_n - Y_n') \Lambda_n, Y_n - Y_n'] = D^2 f(Y_n) [D_n, D_n] + R_f.$$

The bound would then take the form:

$$|\mathbb{E}g(Y_n) - \mathbb{E}g(D_n)| \leq \frac{\|g\|_M}{6} \mathbb{E} \| (Y_n - Y_n') \Lambda_n \| \| Y_n - Y_n' \|^2 + |\mathbb{E} R_f| + |\mathbb{E} R_f'|.$$
Remark 4.4. The role of $\Lambda_n$ in condition (4.1) is equivalent to that played by $\Lambda^{-1}$ in [RR09] for $\Lambda$ defined by (1.7) therein. As also observed in [RR09], the condition involving a matrix $\Lambda$ is a generalisation of the condition of [CM08, Theorem 1], where a scalar is used instead. It should be noted that condition (4.1) is more appropriate in the functional setting than a straightforward adaptation of the condition of [RR09]. This is due to the fact that for general processes $Y_n$ the properties of the Frechet derivative do not allow us to treat evaluating the derivative in the direction of $Y_n - Y_n'$ as matrix multiplication and multiplying both sides of the hypothetical condition:

$$-Df(Y_n)[\Lambda Y_n] = \mathbb{E}Y_n Df(Y_n)[Y_n - Y_n']$$

by $\Lambda^{-1}$ does not give:

$$-Df(Y_n)[Y_n] = \mathbb{E}Y_n Df(Y_n)[\Lambda^{-1}(Y_n - Y_n')]$$.

Proof of Theorem 4.1. Our aim is to bound $|\mathbb{E}g(Y_n) - \mathbb{E}g(D_n)|$ by bounding $|\mathbb{E}A_n f(Y_n)|$, where $f$ is the solution to the Stein equation:

$$A_n f = g - \mathbb{E}g(D_n),$$

for $A_n$ defined in Proposition 3.2. Note that, by exchangeability of $(Y_n, Y_n')$ and (4.1):

$$0 = \mathbb{E} (Df(Y_n') + Df(Y_n)) [Y_n - Y_n'] \Lambda_n$$

$$= \mathbb{E} (Df(Y_n') - Df(Y_n)) [Y_n - Y_n'] \Lambda_n + 2 \mathbb{E} \left\{ \mathbb{E}^Y_n Df(Y_n) [(Y_n - Y_n') \Lambda_n] \right\}$$

$$= \mathbb{E} (Df(Y_n') - Df(Y_n)) [Y_n - Y_n'] \Lambda_n + \mathbb{E} Df(Y_n) |Y_n| - \mathbb{E} R_f$$

and so:

$$\mathbb{E} Df(Y_n) |Y_n| = \mathbb{E} (Df(Y_n) - Df(Y_n')) [(Y_n - Y_n') \Lambda_n] + \mathbb{E} R_f.$$  

Therefore:

$$|\mathbb{E} A_n f(Y_n)|$$

$$= |\mathbb{E} Df(Y_n) |Y_n| - \mathbb{E} D^2 f(Y_n) |D_n, D_n|$$

$$= |\mathbb{E} (Df(Y_n) - Df(Y_n')) [(Y_n - Y_n') \Lambda_n] - \mathbb{E} D^2 f(Y_n) |D_n, D_n| + \mathbb{E} R_f|$$

$$\leq |\mathbb{E} D^2 f(Y_n) [(Y_n - Y_n') \Lambda_n, Y_n - Y_n'] - \mathbb{E} D^2 f(Y_n) |D_n, D_n]| + |\mathbb{E} R_f|$$

$$\leq \frac{\|g\|^M}{6} \mathbb{E} ||(Y_n - Y_n') \Lambda_n|| ||Y_n - Y_n'||^2 + |\mathbb{E} R_f|$$

$$+ |\mathbb{E} D^2 f(Y_n) [(Y_n - Y_n') \Lambda_n, Y_n - Y_n'] - \mathbb{E} D^2 f(Y_n) |D_n, D_n|,$$

where the last inequality follows by Taylor’s theorem and Proposition 3.2. □
5. A functional Combinatorial Central Limit Theorem

In this section we consider a functional version of the result proved in [HC78].
Our object of interest is a stochastic process represented by a scaled sum of independent random variables chosen from an \( n \times n \) array. Only one random variable is picked from each row and for row \( i \), the corresponding random variable is picked from column \( \pi(i) \), where \( \pi \) is a random permutation on \([n]\). Theorem 5.1 established a bound on the distance between this process and a pre-limiting Gaussian process and Theorem 5.4 shows convergence of this process, under certain assumptions, to a continuous Gaussian process.

Our analysis in this section is similar to that of [BJ09], where the summands in the scaled sums are chosen from a deterministic array. The authors therein also establish bounds on the approximation by a pre-limit Gaussian process and show convergence to a continuous Gaussian process. Furthermore, they establish a bound on the distance from the continuous Gaussian process for a restricted class of test functions. For random arrays the situation is more involved.

Our setup is analogous to the one considered in [CF15], where a bound on the speed of convergence in the one-dimensional combinatorial central limit theorem is obtained using Stein’s method of exchangeable pairs.

5.1. Introduction

Let \( \mathcal{X} = \{ X_{i,j} : i, j \in [n] \} \) be an \( n \times n \) array of independent \( \mathbb{R} \)-valued random variables, where \( n \geq 2 \), \( \mathbb{E}X_{i,j} = c_{ij} \), \( \text{Var}X_{i,j} = \sigma_{ij}^2 \geq 0 \) and \( \mathbb{E}|X_{i,j}|^3 < \infty \).

Suppose that \( c_{i} = c_{j} = 0 \) where \( c_{i} = \sum_{j=1}^{n} c_{i,j} = \mathbb{E}X_{i\pi(i)} \), \( c_{j} = \sum_{i=1}^{n} c_{i,j} \). Let \( \pi \) be a uniform random permutation of \([n]\), independent of \( \mathcal{X} \) and let

\[
Y_n = \frac{1}{s_n} \sum_{i=1}^{n} X_{i\pi(i)} = \frac{1}{s_n} \sum_{i=1}^{n} X_{i\pi(i)} \mathbb{1}_{[i/n, 1]},
\]

where:

\[
s_n^2 = \frac{1}{n} \sum_{i,j=1}^{n} \sigma_{ij}^2 + \frac{1}{n-1} \sum_{i,j=1}^{n} c_{ij}^2. \tag{5.1}
\]

We note that \( s_n^2 = \text{Var} \left[ \sum_{i=1}^{n} X_{i\pi(i)} \right] \) by the first part of [CF15, Theorem 1.1]. The process \( Y_n \) is similar to the process \( Y \) considered in [BJ09] and defined by (1.4) therein with the most important difference being that we allow the \( X_{i,j} \)'s to be random, whereas the authors in [BJ09] assumed them to be deterministic. Bounds on the distance between one-dimensional distributions of \( Y_n \) and a normal distribution have been obtained via Stein’s method in [CF15, Theorem 1.1].
5.2. Exchangeable pair setup

Select uniformly at random two different indices \( I, J \in [n] \) and let:

\[
Y'_n = Y_n - \frac{1}{s_n} X_{I \pi(I)} \mathbb{1}[I/n, 1] - \frac{1}{s_n} X_{J \pi(J)} \mathbb{1}[J/n, 1] + \frac{1}{s_n} X_{I \pi(J)} \mathbb{1}[J/n, 1] + \frac{1}{s_n} X_{J \pi(I)} \mathbb{1}[J/n, 1].
\]

Note that \((Y_n, Y'_n)\) is an exchangeable pair and that for all \( f \in M \):

\[
\mathbb{E}^{Y_n} \{ Df(Y_n) [Y_n - Y'_n] \} = \frac{1}{s_n} \mathbb{E}^{Y_n} \{ Df(Y_n) [X_{I \pi(I)} \mathbb{1}[I/n, 1] + X_{J \pi(J)} \mathbb{1}[J/n, 1] - X_{I \pi(J)} \mathbb{1}[J/n, 1] - X_{J \pi(I)} \mathbb{1}[J/n, 1]] \}
\]

\[
= \frac{1}{n(n-1)s_n} \sum_{i,j=1}^{n} \mathbb{E}^{Y_n} \{ Df(Y_n) [X_{i \pi(i)} \mathbb{1}[i/n, 1] + X_{j \pi(j)} \mathbb{1}[j/n, 1] - X_{i \pi(j)} \mathbb{1}[j/n, 1] - X_{j \pi(i)} \mathbb{1}[i/n, 1]] \}
\]

\[
= \frac{2}{n-1} Df(Y_n) [Y_n] - \frac{2}{n(n-1)s_n} \sum_{i,j=1}^{n} \mathbb{E}^{Y_n} Df(Y_n) [X_{i \pi(j)} \mathbb{1}[i/n, 1]]
\]

Therefore:

\[
\mathbb{E}^{Y_n} \{ Df(Y_n) [Y_n - Y'_n] \} = \frac{2}{n-1} Df(Y_n) \left[ Y_n - \frac{1}{ns_n} \sum_{i,j=1}^{n} \mathbb{E}^{Y_n} [X_{i \pi(j)} \mathbb{1}[i/n, 1]] \right].
\]

So condition \((4.1)\) is satisfied with

\[
\Lambda_n = \frac{n-1}{4} \quad \text{and} \quad R_f = \frac{1}{ns_n} \sum_{i,j=1}^{n} Df(Y_n) \left[ \mathbb{E}^{Y_n} [X_{i \pi(j)} \mathbb{1}[i/n, 1]] \right].
\]

5.3. Pre-limiting process

Now let

\[
A_n = \frac{1}{s_n} \sum_{i=1}^{n} \tilde{Z}_i,
\]

where \( \tilde{Z}_i = \frac{1}{\sqrt{n-1}} \sum_{l=1}^{n} X''_{il} \left( Z_{il} - \frac{1}{n} \sum_{j=1}^{n} Z_{jl} \right) \), for \( \mathbb{X}'' = \{ X''_{ij} : i, j \in [n] \} \) being an independent copy of of \( \mathbb{X} \) and \( Z_{il} \)'s i.i.d. standard normal, independent...
of all the \( X_{il} \)'s and \( X_{ij}'' \)'s. Note that \( \hat{Z}_i \) has mean 0 for all \( i \) and

\[
\mathbb{E} \hat{Z}_i^2 = \frac{1}{n-1} \sum_{l=1}^{n} \mathbb{E} \left[ X_{il}^2 \right] \mathbb{E} \left[ \left( Z_{il} - \frac{1}{n} \sum_{j=1}^{n} Z_{jl} \right)^2 \right]
\]

\[
+ \frac{1}{n-1} \sum_{1 \leq l \neq k \leq n} \mathbb{E} \left[ X_{il}X_{ik} \right] \mathbb{E} \left[ \left( Z_{il} - \frac{1}{n} \sum_{j=1}^{n} Z_{jl} \right) \left( Z_{ik} - \frac{1}{n} \sum_{j=1}^{n} Z_{jk} \right) \right]
\]

\[
= \frac{1}{n-1} \sum_{l=1}^{n} \mathbb{E} X_{il}^2 \left( 1 - \frac{2}{n} + \frac{1}{n} \right)
\]

\[
= \frac{1}{2n^2} \left( 2(n-1) \sum_{l=1}^{n} \mathbb{E} X_{il}^2 + 2 \sum_{i=1}^{n} \mathbb{E} X_{ir}^2 \right)
\]

\[
= \frac{1}{2n^2} \left( \sum_{1 \leq k \neq l \leq n} \mathbb{E} \left[ (X_{ik} - X_{il})^2 \right] + 2 \sum_{1 \leq k \neq l \leq n} \mathbb{E} X_{ik} \mathbb{E} X_{il} + 2 \sum_{r=1}^{n} \mathbb{E} X_{ir}^2 \right)
\]

\[
= \frac{1}{2n^2} \left( \sum_{1 \leq k \neq l \leq n} \mathbb{E} \left[ (X_{ik} - X_{il})^2 \right] + 2 \sum_{r=1}^{n} \sigma_{ir}^2 \right) \quad (5.3)
\]

as \( c_i = 0 \), and, for \( i \neq j \),

\[
\mathbb{E} \hat{Z}_i \hat{Z}_j = \frac{1}{n-1} \sum_{k,l=1}^{n} \mathbb{E} (X_{ik}X_{jl}) \mathbb{E} \left[ \left( Z_{ik} - \frac{1}{n} \sum_{r=1}^{n} Z_{rk} \right) \left( Z_{jl} - \frac{1}{n} \sum_{r=1}^{n} Z_{rl} \right) \right]
\]

\[
= - \frac{1}{n(n-1)} \sum_{k=1}^{n} c_k c_{jk}
\]

\[
= \frac{1}{2n^2(n-1)} \left( 2 \sum_{k=1}^{n} (-\mathbb{E} X_{ik}) \mathbb{E} X_{jk} - 2(n-1) \sum_{k=1}^{n} \mathbb{E} X_{ik} \mathbb{E} X_{jk} \right)
\]

\[
= \frac{1}{2n^2(n-1)} \left( 2 \sum_{1 \leq k \neq l \leq n} \mathbb{E} X_{il} \mathbb{E} X_{jk} - 2 \sum_{1 \leq k \neq l \leq n} \mathbb{E} X_{ik} \mathbb{E} X_{jk} \right)
\]

\[
= \frac{1}{2n^2(n-1)} \sum_{1 \leq k \neq l \leq n} \mathbb{E} (X_{ik} - X_{il})(X_{jl} - X_{jk}). \quad (5.4)
\]

5.4. Pre-limiting approximation

**Theorem 5.1.** For \( Y_n \) defined in Subsection 5.1, \( A_n \) defined in Subsection 5.3 and any \( g \in M^1 \):

\[
|\mathbb{E}g(Y_n) - \mathbb{E}g(A_n)|
\]
\[ \leq \frac{\|g\|_{M^1}}{n^3(n-1)s_n^3} \sum_{1 \leq i,j,k,l,m \leq n} \left\{ 3E|X_{ik}|^3 + 5E|X_{ik}|E|X_{jl}|^2 + 7E|X_{ik}|^2E|X_{jl}| \\
+ 5E|X_{ik}|^2E|X_{jk}| + 16E|X_{ik}|E|X_{ul}|E|X_{jl}| + 2E|X_{ul}|E|X_{ik}|E|X_{jl}| \\
+ 4E|X_{ul}|E|X_{ul}|E|X_{jk}| + 6E|X_{ul}|E|X_{ik}|E|X_{jk}| + 2E|X_{ul}|E|X_{ik}|E|X_{jk} \right\} \\
+ \frac{1}{n} \left( 2E|X_{ik}| + 2E|X_{jk}| + 2E|X_{ul}| \right) \sum_{r=1}^{n} (E|X_{ir}|^2 + |c_{ir}c_{jr}|) \right\} \\
+ \frac{2\|g\|_{M^1}}{\sqrt{n}} + \frac{4\|g\|_{M^1}}{3ns_n^2} \sum_{i,j=1}^{n} \sigma_{i,j}^2. \]

**Remark 5.2.** Assuming that \( s_n = O(\sqrt{n}) \), we obtain that the bound in Theorem 5.1 is of order \( \frac{1}{\sqrt{n}} \).

**Remark 5.3.** If we assume that \( E|X_{ik}|^3 \leq \beta_3 \) for all \( i, k = 1, \ldots, n \) then the bound simplifies in the following way

\[ \|Eg(Y_n) - E(g(A_n))\| \leq \|g\|_{M^1} \left( \frac{5\beta_3 n^2}{(n-1)s_n^3} + \frac{8\beta_3^{1/3}}{n(n-1)s_n^3} \sum_{i,j=1}^{n} |c_{ir}c_{jr}| + \frac{2}{\sqrt{n}} + \frac{4}{3ns_n^2} \sum_{i,j=1}^{n} \sigma_{i,j}^2 \right). \]

We will use Theorem 4.1 to prove Theorem 5.1. In the proof below we first justify why Theorem 4.1 may indeed be used in this case. In Step 2 we bound terms \( R_1 \) and \( R_3 \) coming from Theorem 4.1. Then, in Step 3, we treat the remaining term using a strategy analogous to that of the proof of [BJ09, Theorem 2.1]. Finally, in Step 3 we combine the estimates obtained in the previous steps to obtain the assertion.

**Proof of Theorem 5.1.** We adopt the notation of Subsections 5.1, 5.2 and 5.3.

**Step 1.** We note that \( A_n \) can be expressed in the following way:

\[ A_n = \sum_{i,j=1}^{n} \left( Z_{il} - \frac{1}{n} \sum_{j=1}^{n} Z_{jl} \right) J_{i,j}, \quad \text{where} \quad J_{i,j}(t) = \frac{X''_{il}}{s_n \sqrt{n-1}} \mathbb{1}_{[i/n, 1]}(t), \]

which, together with (5.2), lets us apply Theorem 4.1.

**Step 2.** For the first term in Theorem 4.1, for any \( g \in M^1 \):

\[ R_1 = \frac{\|g\|_{M^1}}{6E\|(Y_n - Y'_n)A_n\|} \|Y_n - Y'_n\|^2 \leq \frac{(n-1)\|g\|_{M^1}}{24E\|Y_n - Y'_n\|^3}. \]
We note that:
\[
\mathbb{E}\|Y_n - Y_n'\|^3 \leq \frac{8}{n^3} \left( \mathbb{E}|X_{i\pi(I)}|^3 + \mathbb{E}|X_{j\pi(J)}|^3 + \mathbb{E}|X_{i\pi(I)}|^3 + \mathbb{E}|X_{j\pi(J)}|^3 \right)
\]
\[
= \frac{8}{n(n-1)s_n^3} \sum_{i \neq j} (\mathbb{E}|X_{i\pi(i)}|^3 + \mathbb{E}|X_{j\pi(j)}|^3 + \mathbb{E}|X_{i\pi(i)}|^3 + \mathbb{E}|X_{j\pi(i)}|^3)
\]
\[
= \frac{16}{n(n-1)s_n^3} \sum_{i \neq j} (\mathbb{E}|X_{i\pi(i)}|^3 + \mathbb{E}|X_{j\pi(j)}|^3)
\]
\[
= \frac{32}{n^2s_n^3} \sum_{i,j=1}^n \mathbb{E}|X_{ij}|^3.
\]
Hence,
\[
R_1 \leq \frac{4\|g\|_{M^1}^3}{3n^3s_n} \sum_{i,j=1}^n \mathbb{E}|X_{ij}|^3. \quad (5.5)
\]
Furthermore, by Proposition 3.2:
\[
R_3 = \left| \frac{1}{ns_n} \sum_{i,j=1}^n \mathbb{E}Df(Y_n) \left[ X_{i,j} \mathbb{I}_{[i/n,1]} \right] \right| \leq \frac{1}{ns_n} \sum_{i,j=1}^n \mathbb{E}Df(Y_n) \left[ X_{i,j} \mathbb{I}_{[i/n,1]} \right]
\]
\[
\leq \|g\|_{M^1} \frac{1}{ns_n} \mathbb{E} \left| \sum_{i,j=1}^n X_{i,j} \right|^2
\]
\[
\leq \left( \frac{2\|g\|_{M^1}}{ns_n} \right) \mathbb{E} \left| \sum_{i,j=1}^n X_{i,j} \right|^2
\]
\[
\leq \left( \frac{2\|g\|_{M^1}}{ns_n} \right) \sum_{i,j=1}^n \sigma_{i,j}^2
\]
\[
\leq \left( \frac{2\|g\|_{M^1}}{\sqrt{n}} \right) \sum_{i,j=1}^n \sigma_{i,j}^2,
\]
where we have used Doob’s $L^2$ inequality in the second inequality and (5.1) in the last one.

**Step 3.** Now define a new permutation $\pi_{ijkl}$ coupled with $\pi$ such that:
\[
\mathcal{L}(\pi_{ijkl}) = \mathcal{L}(\pi|\pi(i) = k, \pi(j) = l).
\]
As noted in [CF15], we can construct it in the following way. For $\tau_{ij}$ denoting the transposition of $i, j$:
\[
\pi_{ijkl} = \begin{cases} 
\pi, & \text{if } l = \pi(j), k = \pi(i) \\
\pi \cdot \tau_{\pi^{-1}(k), i}, & \text{if } l = \pi(j), k \neq \pi(i) \\
\pi \cdot \tau_{\pi^{-1}(l), j}, & \text{if } l \neq \pi(j), k = \pi(i) \\
\pi \cdot \tau_{\pi^{-1}(l), i} \cdot \tau_{\pi^{-1}(k), j} \cdot \tau_{ij}, & \text{if } l \neq \pi(j), k \neq \pi(i).
\end{cases}
\]
We also let
\[ Y_{n,ijkl} = \frac{1}{s_n} \sum_{i'=1}^{n} X_{i''} \pi(i''(v)) \mathbb{1}_{[v/n,1]} ] . \]

Then \( \mathcal{L}(Y_{n,ijkl}) = \mathcal{L}(Y_n | \pi(i) = k, \pi(j) = l) \). Also, for each choice of \( i \neq j, k \neq l \) let \( \mathcal{X}_{ijkl} := \{ X_{i''} : i', j' \in [n] \} \) be the same as \( \mathcal{X} := \{ X_{ij} : i, j \in [n] \} \) except that \{ \( X_{ik}, X_{il}, X_{jk}, X_{jl} \) \} has been replaced by an independent copy \{ \( X'_{ik}, X'_{il}, X'_{jk}, X'_{jl} \) \}. Then let
\[ Y_{n,ijkl}^{ijkl} = \frac{1}{s_n} \sum_{i'=1}^{n} X_{i''} \pi(i''(v)) \mathbb{1}_{[v/n,1]} ] \]
and note that \( Y_{n,ijkl}^{ijkl} \) is independent of \{ \( X_{ik}, X_{il}, X_{jk}, X_{jl} \) \} and \( \mathcal{L}(Y_{n,ijkl}^{ijkl}) = \mathcal{L}(Y_n) \).

This construction will be used below in (5.8) and (5.9) in order to obtain independence and apply Taylor’s theorem.

Note that:
\[ R_2 = \left| \mathbb{E} D^2 f(Y_n) [(Y_n - Y'_n) A_n, Y_n - Y'_n] - \mathbb{E} D^2 f(Y_n)[A_n, A_n] \right| \]
\[ = \frac{n - 1}{4} \mathbb{E} D^2 f(Y_n)[Y_n - Y'_n, Y_n - Y'_n] - \mathbb{E} D^2 f(Y_n)[A_n, A_n] \]
\[ = \frac{1}{2n s_n^2} \sum_{i,j=1}^{n} \mathbb{E} \left\{ (X_{i \pi(i)} - X_{i \pi(j)})^2 D^2 f(Y_n) \left[ \mathbb{1}_{i/n,1} \mathbb{1}_{j/n,1} \right] \right\} \]
\[ + \frac{1}{2n s_n^2} \sum_{i,j=1}^{n} \mathbb{E} \left\{ (X_{i \pi(i)} - X_{i \pi(j)}) (X_{j \pi(j)} - X_{j \pi(i)}) D^2 f(Y_n) \left[ \mathbb{1}_{i/n,1} \mathbb{1}_{j/n,1} \right] \right\} \]
\[ - \mathbb{E} D^2 f(Y_n)[A_n, A_n] \]
\[ = \frac{1}{2n^2 (n-1) s_n^2} \sum_{1 \leq i, j, k, l \leq n}^{n} \mathbb{E} \left\{ (X_{ik} - X_{il})^2 \cdot D^2 f(Y_n) \left[ \mathbb{1}_{i/n,1} \mathbb{1}_{l/n,1} \right] \pi(i) = k, \pi(j) = l \right\} \]
\[ + \frac{1}{n (n-1) s_n^2} \sum_{1 \leq i, j, k, l \leq n}^{n} \mathbb{E} \left\{ (X_{ik} - X_{il}) (X_{jl} - X_{jk}) \right\} \]
\[ \cdot D^2 f(Y_n) \left[ \mathbb{1}_{i/n,1} \mathbb{1}_{l/n,1} \right] \pi(i) = k, \pi(j) = l \right\} \]
\[ - \frac{1}{s_n^2} \sum_{1 \leq i, j \leq n}^{n} \mathbb{E} [\hat{Z}_i \hat{Z}_j] \mathbb{E} D^2 f(Y_n)[\mathbb{1}_{i/n,1} \mathbb{1}_{j/n,1}] \]
\[ - \frac{1}{(n-1) s_n^2} \sum_{1 \leq i, j \leq n}^{n} \mathbb{E} [\hat{Z}_i^2] \mathbb{E} D^2 f(Y_n)[\mathbb{1}_{i/n,1} \mathbb{1}_{i/n,1}] \]
\[
\frac{1}{2n^2(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ (X_{ik} - X_{il})^2 \right] D^2 f(Y_{ijkl}) \left[ \mathbb{I}_{[i/n,1]} \mathbb{I}_{[i/n,1]} \right] \right\} \\
+ \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} \right] D^2 f(Y_{ijkl}) \left[ \mathbb{I}_{[i/n,1]}, \mathbb{I}_{[j/n,1]} \right] \right\} \\
- \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E}[\hat{Z}_i \hat{Z}_j] \mathbb{E} D^2 f(Y_{ijkl})[\mathbb{I}_{[i/n,1]}, \mathbb{I}_{[j/n,1]}] \\
- \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E}[\hat{Z}_i^2] \mathbb{E} D^2 f(Y_{ijkl})[\mathbb{I}_{[i/n,1]}, \mathbb{I}_{[i/n,1]}].
\]

(5.7)

Now, taking the absolute value.

\[
R_2 \leq A + B,
\]

(5.8)

where:

\[
A = \left| \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\hat{Z}_i^2}{n-1} \right] \right\} \right|,
\]

\[
B = \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\hat{Z}_i^2}{n-1} \right] D^2 f(Y_{ijkl}) \left[ \mathbb{I}_{[i/n,1]} \mathbb{I}_{[i/n,1]} \right] \right\} \\
+ \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} - \hat{Z}_i \hat{Z}_j \right] \right\} \cdot D^2 f(Y_{ijkl}) \left[ \mathbb{I}_{[i/n,1]}, \mathbb{I}_{[j/n,1]} \right].
\]

(5.9)

Now, unconditioning:

\[
B = \left| \frac{1}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n \atop i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\hat{Z}_i^2}{n-1} \right] D^2 f(Y_{ijkl}) \left[ \mathbb{I}_{[i/n,1]} \mathbb{I}_{[i/n,1]} \right] \right\} \right|.
\]
\begin{align*}
&+ \frac{1}{n(n-1)s_n^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{il} - X_{ij})(X_{jl} - X_{jk})}{2n} - \hat{Z}_i \hat{Z}_j \right] \right\} \\
&\cdot D^2 f \left( Y_n^{ijkl} \right) \left[ \mathbb{1}_{[i/n,1]}, \mathbb{1}_{[j/n,1]} \right] \\
&= \frac{1}{n(n-1)s_n^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left[ \left\{ \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\hat{Z}_i^2}{n-1} \right\} \right] \mathbb{E} \left\{ D^2 f \left( Y_n \right) \left[ \mathbb{1}_{[i/n,1]} \mathbb{1}_{[j/n,1]} \right] \right\} \\
&+ \frac{1}{n(n-1)s_n^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left[ \left\{ \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} - \hat{Z}_i \hat{Z}_j \right\} \right] \\
&\cdot \mathbb{E} \left\{ D^2 f \left( Y_n \right) \left[ \mathbb{1}_{[i/n,1]} \mathbb{1}_{[j/n,1]} \right] \right\} \\
&\leq \frac{\|g\|_{M^1}}{n^2(n-1)s_n^2} \sum_{1 \leq i,j \leq n, i \neq j} \sum_{r=1}^{n} \sigma_{ir}^2 \\
&= \frac{\|g\|_{M^1}}{n^2s_n^2} \sum_{i,j=1}^{n} \sigma_{i,j}^2, \tag{5.10}
\end{align*}

where the second to last inequality follows by (5.3), (5.4) and Proposition 3.2.
Furthermore, for $A$ in (5.8), define index sets $\mathcal{I} = \{i, j, \pi^{-1}(k), \pi^{-1}(l)\}$ and $\mathcal{J} = \{k, l, \pi(i), \pi(j)\}$. Then, letting $S = \frac{1}{s_n} \sum_{i' \in \mathcal{I}} X_{i'\pi(i')} \mathbb{1}_{[i'/n,1]}$, we can write:

$$Y_{n,ijkl} = S + \frac{1}{s_n} \sum_{i' \in \mathcal{I}} X_{i'\pi,ijkl(i')} \mathbb{1}_{[i'/n,1]}, \quad Y_{n,ijkl}^{ij} = S + \frac{1}{s_n} \sum_{i' \in \mathcal{I}} X_{i'\pi(i')} \mathbb{1}_{[i'/n,1]}.$$

Since $S$ depends only on the components of $\mathbb{X}$ outside the square $\mathcal{I} \times \mathcal{J}$ and $\{\pi(i) : i \notin \mathcal{I}\}$, $S$ is independent of:

$$\left\{ X_{il}, X_{jk}, X_{ik}, X_{jl}, \sum_{i' \in \mathcal{I}} X_{i'\pi,ijkl(i')}, \sum_{i' \in \mathcal{I}} X_{i'\pi(i')} \right\},$$

given $\pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)$.

We can write:

\begin{align*}
&\frac{1}{n(n-1)s_n^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\hat{Z}_i^2}{n-1} \right] \right\} \\
&\cdot \left( D^2 f \left( Y_{n,ijkl} \right) - D^2 f \left( Y_{n,ijkl}^{ij} \right) \right) \left[ \mathbb{1}_{[i/n,1]} \mathbb{1}_{[j/n,1]} \right] \\
&+ \frac{1}{n(n-1)s_n^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left\{ \left[ \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} - \hat{Z}_i \hat{Z}_j \right] \right\} \\
&\cdot \left( D^2 f \left( Y_{n,ijkl} \right) - D^2 f \left( Y_{n,ijkl}^{ij} \right) \right) \left[ \mathbb{1}_{[i/n,1]} \mathbb{1}_{[j/n,1]} \right] \\
&\cdot \left( D^2 f \left( Y_{n,ijkl} \right) - D^2 f \left( Y_{n,ijkl}^{ij} \right) \right) \left[ \mathbb{1}_{[i/n,1]} \mathbb{1}_{[j/n,1]} \right] \\
\end{align*}
\[
\leq \frac{\|g\|_{\mathcal{M}}}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \sum_{l \neq k, j \neq l} \mathbb{E} \left\{ \left| Y_{n,ijkl} - Y_{n,ijkl}^{ijl} \right| \left( \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\mathbb{E}\hat{Z}_i^2}{n-1} \right) + \left| \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} - \mathbb{E}(\hat{Z}_i\hat{Z}_j) \right| \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n(n-1)s^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \sum_{l \neq k, j \neq l} \left\{ \left| X_{v,ijkl}(ijkl) - X_{v,ijkl}^{ijkl} \right| \right\}
\sum_{i \neq j, l \neq k} \sum_{l \neq k} \sum_{j \neq l} \left\{ \left| X_{ik} - X_{il} \right| \left( \frac{(X_{ik} - X_{il})^2}{2n} - \frac{\mathbb{E}\hat{Z}_i^2}{n-1} \right) + \left| \frac{(X_{ik} - X_{il})(X_{jl} - X_{jk})}{2n} - \mathbb{E}(\hat{Z}_i\hat{Z}_j) \right| \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{2(n^2+1)^2s^2} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \mathbb{E} \left\{ \left( \left| X_{ik} \right| + \left| X_{il} \right| + \left| X_{jl} \right| + \left| X_{jk} \right| \right) \left( \left| X_{ik} \right|^2 + \left| X_{il} \right|^2 + \left| X_{jl} \right|^2 + \left| X_{jk} \right|^2 + 2 \left| X_{ik}X_{jl} \right| + 2 \left| X_{ik}X_{jk} \right| + 2 \left| X_{il}X_{jk} \right| + 2(n-1) \mathbb{E}(\hat{Z}_i\hat{Z}_j) \right) \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{2n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ \left| X_{ik} \right|^2 + \left| X_{il} \right|^2 + \left| X_{jl} \right|^2 + \left| X_{jk} \right|^2 + 2 \left| X_{ik}X_{jl} \right| + 2 \left| X_{ik}X_{jk} \right| + 2 \left| X_{il}X_{jk} \right| + 2 \left| X_{il}X_{jl} \right| + \frac{2n}{n} \sum_{r=1}^n \mathbb{E}|X_{ir}|^2 \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ \left( \mathbb{E}|X_{il}| + \mathbb{E}|X_{jl}| + \mathbb{E}|X_{ik}| + \mathbb{E}|X_{kl}| \right) \mathbb{E} \left\{ \left| X_{ik} \right|^2 + \left| X_{il} \right|^2 \right\} \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ \left( \mathbb{E}|X_{il}| + \mathbb{E}|X_{jl}| + \mathbb{E}|X_{ik}| + \mathbb{E}|X_{kl}| \right) \mathbb{E} \left\{ \left| X_{ik} \right|^2 + \left| X_{il} \right|^2 \right\} \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ \mathbb{E}|X_{il}| + \mathbb{E}|X_{jl}| + \mathbb{E}|X_{ik}| + \mathbb{E}|X_{kl}| \mathbb{E} \left\{ \left| X_{ik} \right|^2 + \left| X_{il} \right|^2 \right\} \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ 5\mathbb{E}|X_{ik}|^2 + 16\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|^2 + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{kl}|^2 + 4\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ 5\mathbb{E}|X_{ik}|^2 + 16\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|^2 + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{kl}|^2 + 4\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| \right\}
\]

\[
\leq \frac{\|g\|_{\mathcal{M}}}{n^3(n-1)s^3} \sum_{1 \leq i,j,k,l \leq n, i \neq j, k \neq l} \left\{ 5\mathbb{E}|X_{ik}|^2 + 16\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|^2 + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{kl}|^2 + 4\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| + 2\mathbb{E}|X_{ik}|\mathbb{E}|X_{il}|\mathbb{E}|X_{kl}| \right\}
\]
\[ + \frac{1}{n} \left( 2 \mathbb{E} |X_{ik}| + 2 \mathbb{E} |X_{jl}| + 2 \mathbb{E} |X_{uk}| + 2 \mathbb{E} |X_{ul}| \right) \cdot \sum_{r=1}^{n} \left( \mathbb{E} |X_{ir}|^2 + |c_{ir}c_{jr}| \right) \).\] (5.11)

We now use (5.5), (5.6), (5.9), (5.10), (5.11) to obtain the assertion. \(\square\)

### 5.5. Convergence to a continuous Gaussian process

**Theorem 5.4.** Let \(X\) and \(Y_n\) be as defined in Subsection 5.1 and suppose that for all \(u, t \in [0, 1]\):

\[ \frac{1}{s_n^2(n-1)} \sum_{i=1}^{[nt]} \sum_{j=1}^{[nu]} \sum_{k=1}^{n} \mathbb{E} X_{ik} X_{jk} \left( \delta_{i,j} - \frac{1}{n} \right) \xrightarrow{n \to \infty} \sigma(u,t) \] (5.12)

and

\[ \frac{1}{s_n^2(n-1)} \sum_{i=1}^{[nt]} \sum_{j=1}^{[nu]} \sum_{l=1}^{n} \mathbb{E} X_{il} X_{jl} \xrightarrow{n \to \infty} \sigma^{(2)}(u,t) \] (5.13)

for some functions \(\sigma, \sigma^{(2)} : [0,1]^2 \to \mathbb{R}_+\). Suppose furthermore that:

\[ \sup_{n \in \mathbb{N}} \frac{1}{n^2 s_n^4} \sum_{i=1}^{n} \sum_{l=1}^{n} \text{Var} \left( X_{il}^2 \right) < \infty. \] (5.14)

and:

\[ \frac{1}{s_n^2(n-1)} \sum_{i=1}^{[nt]} \left( \sum_{l=1}^{n} X_{il}'' Z_{il} \right)^2 \xrightarrow{P} c(t) \] (5.15)

pointwise for some function \(c : [0,1] \to \mathbb{R}_+\) and:

\[ \lim_{n \to \infty} \frac{1}{s_n \sqrt{n}} \mathbb{E} \left[ \sup_{i=1, \ldots, n} |X_{il}'' Z_{il}| \right] = 0. \] (5.16)

Then \((Y_n(t), t \in [0,1])\) converges weakly in the uniform topology to a Gaussian process \((Z(t), t \in [0,1])\) with the covariance function \(\sigma\).

The proof of Theorem 5.4 will be similar to the proof of [BJ09, Theorem 3.3]. The pre-limiting approximand \(A_n\), defined in Subsection 5.3, will be expressed as a sum of two parts. In Steps 1 and 2 we prove that each of those parts is C-tight (i.e. they are tight and for each of them any convergent subsequence converges to a process with continuous sample paths). In Step 3 we show that the assumptions of Theorem 5.4 trivially imply the convergence of the covariance function of \(A_n\), which together with C-tightness implies the convergence of \(A_n\) to a continuous process. Theorem 6.2 will then be combined with Proposition 2.3 to show convergence of \(Y_n\) to the same limiting process. Finally, the combinatorial central limit theorem for random arrays, proved in [HC78] and analysed in [CF15], will imply that \(Z\) is Gaussian.
Proof of Theorem 5.4. We will use the notation of Subsections 5.1 and 5.3.

**Step 1.** Note that $A_n = A^{(1)}_n + A^{(2)}_n$, where:

$$A^{(1)}_n(t) = \frac{1}{s_n \sqrt{n} - 1} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{l=1}^{n} X''_{il} Z_{il}, \quad A^{(2)}_n(t) = \frac{1}{s_n \sqrt{n} - 1} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{l=1}^{n} X''_{il} \widetilde{Z}_l$$

for $\widetilde{Z}_l = \frac{1}{n} \sum_{j=1}^{n} Z_{jl}$.

Now, note that, by (5.15):

$$\langle A^{(1)}_n \rangle_t \xrightarrow{P} c(t)$$

pointwise, where $\langle \cdot \rangle$ denotes quadratic variation. Therefore, by [EK86, Chapter 7, Theorem 1.4] and using (5.16), we obtain that $A^{(1)}_n$ converges weakly in the Skorokhod topology on $D[0,1]$ to a continuous Gaussian process with independent increments.

We now note that the Skorokhod space equipped with the metric (topologically equivalent to the Skorokhod metric) with respect to which it is complete is also universally measurable by the discussion at the beginning of [Dud02, Chapter 11.5]. Since it is also separable and $A^{(1)}_n \Rightarrow Z_1$, for some continuous process $Z_1$, in the Skorokhod topology, [Dud02, Theorem 11.5.3] implies that $(A^{(1)}_n)_{n \geq 1}$ is C-tight.

**Step 2.** Also, note that for $u > t$ s.t. $|nu| \geq |nt| + 1$,

$$\mathbb{E} \left[ \left| A^{(2)}_n(u) - A^{(2)}_n(t) \right|^2 \right|_{X''_{il}, i, l \in [n]} = \frac{1}{n(n-1)s_n^2} \sum_{i=1}^{\lfloor nu \rfloor} \sum_{i=\lfloor nt \rfloor + 1}^{\lfloor nu \rfloor} X''_{il}^2 \leq \frac{|nu| - |nt|}{n(n-1)s_n^2} \sum_{i=1}^{\lfloor nu \rfloor} \sum_{i=\lfloor nt \rfloor + 1}^{\lfloor nu \rfloor} X''_{il}^2$$

and

$$\mathbb{E} \left[ \left| A^{(2)}_n(u) - A^{(2)}_n(t) \right|^2 \right|_{X''_{il}, i, l \in [n]} = 0, \quad \text{for } u > t \text{ s.t. } |nu| = |nt|.$$ 

Since $(A^{(2)}_n |_{X''_{il}, i, l \in [n]})$ is Gaussian for $u$, such that $|nu| \geq |nt| + 1$,

$$\mathbb{E} \left| A^{(2)}_n(u) - A^{(2)}_n(t) \right|^4 = 3 \mathbb{E} \left( \mathbb{E} \left[ \left| A^{(2)}_n(u) - A^{(2)}_n(t) \right|^2 \right|_{X''_{il}, i, l \in [n]} \right)^2 \right)$$

$$\leq 3 \left( \frac{|nu| - |nt|}{n(n-1)s_n^2} \right)^2 \mathbb{E} \left( \sum_{i=1}^{\lfloor nu \rfloor} \sum_{i=\lfloor nt \rfloor + 1}^{\lfloor nu \rfloor} X''_{il}^2 \right)^2$$
\[ \begin{aligned}
&= 3 \left| \frac{|nu| - |nt|}{n(n-1)s_n^2} \right|^2 \left[ \sum_{i=1}^{n} \frac{|nu|}{i=1} \frac{|nu|}{|nt|+1} \frac{\mathbb{E}X_i^2}{n} \right]^2 + \sum_{i=1}^{n} \sum_{i=|nt|+1}^{n} \left( \mathbb{E}X_i^2 - (\mathbb{E}X_i^2)^2 \right) \\
&\leq C \left( \frac{|nu| - |nt|}{(n-1)} \right)^2
\end{aligned} \tag{5.17} \]

for some constant \( C \), by (5.14). Now, note that:

\[ \text{Cov} \left( A_n^{(2)}(t), A_n^{(2)}(u) \right) = \frac{1}{s_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}X_{ij}X_{jl} n^{-\infty} \sigma(2)(t, u), \]

by (5.13). Consider a mean zero Gaussian process \( Z_2 \) with covariance function \( \mathbb{E}Z_2(t)Z_2(u) = \sigma(2)(t, u) \). The finite dimensional distributions of \( A_n^{(2)} \) converge to those of \( Z_2 \). We can now construct \( A_n^{(2)} \) and \( Z^{(2)} \) on the same probability space and use Skorokhod’s representation theorem, Fatou’s lemma and (6.4) to conclude that:

\[ \mathbb{E} \left( |Z_2(u) - Z_2(t)|^4 \right) \leq \lim_{n \to \infty} \mathbb{E} \left( \left| A_n^{(2)}(u) - A_n^{(2)}(t) \right|^4 \right) \leq C(u - t)^2. \]

By [Bil68, Theorem 12.4], we can assume that \( Z_2 \in C[0,1] \). Now, note that for \( 0 \leq t \leq v \leq u \leq 1 \):

\[ \begin{aligned}
&= \mathbb{E} \left| A_n^{(2)}(v) - A_n^{(2)}(t) \right| \left| A_n^{(2)}(v) - A_n^{(2)}(u) \right|^2 \\
&\leq \sqrt{\mathbb{E} \left| A_n^{(2)}(v) - A_n^{(2)}(t) \right|^2} \mathbb{E} \left| A_n^{(2)}(v) - A_n^{(2)}(u) \right|^2 \\
&\leq C(\frac{|nu| - |nt|}{(n-1)^2}) \tag{6.4} \\
&\leq C(u - t)^2;
\end{aligned} \]

for some constant \( \tilde{C} \). Therefore, by [Bil68, Theorem 15.6], \( A_n^{(2)} \Rightarrow Z_2 \) in the Skorokhod and uniform topologies and so, by [Dud02, Theorem 11.5.3], \( A_n^{(2)} \) is C-tight.

**Step 3.** Since both \( A_n^{(1)} \) and \( A_n^{(2)} \) are C-tight, so is their difference \( A_n \). Now:

\[ \text{Cov}(A_n(t), A_n(u)) = \frac{1}{s_n^2(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k,l=1}^{n} \mathbb{E} \{ X_{ik}X_{jl} (Z_{ik} - \bar{Z}_k) (Z_{jl} - \bar{Z}_l) \} \]

\[ = \frac{1}{s_n^2(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E} \{ X_{ik}X_{jk} (Z_{ik} - \bar{Z}_k) (Z_{jk} - \bar{Z}_k) \} \]

\[ = \frac{1}{s_n^2(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E}X_{ik}X_{jk} \left( \delta_{i,j} - \frac{1}{n} \right) n^{-\infty} \sigma(u, t), \]

for some constant \( \tilde{C} \). Therefore, by [Bil68, Theorem 15.6], \( A_n^{(2)} \Rightarrow Z_2 \) in the Skorokhod and uniform topologies and so, by [Dud02, Theorem 11.5.3], \( A_n^{(2)} \) is C-tight.
by (5.12) and we obtain that $A_n$ converges to a random element $Z \in C[0, 1]$ with covariance function $\sigma$ in distribution with respect to the uniform and Skorokhod topologies.

Proposition 2.3 and Theorem 5.1 therefore imply that $(Y_n(t), t \in [0, 1])$ converges weakly to $(Z(t), t \in [0, 1])$ in the uniform topology. Using, for example, [CF15, Theorem 1.1], we conclude that $Z$ is a Gaussian process.

6. Edge and two-star counts in Bernoulli random graphs

In this section we consider a two-dimensional process whose first coordinate is a properly rescaled number of edges and the second one is a rescaled number of two-stars in a Bernoulli random graph with a fixed edge probability and $\lfloor nt \rfloor$ edges for $t \in [0, 1]$. A similar setup has been considered in [RR10], where the authors established a bound on the distance between a three-dimensional vector consisting of a rescaled number of edges, a rescaled number of two-stars and a rescaled number of triangles in a $G(n, p)$ graph and a three-dimensional Gaussian vector. We first compare our process to a two-dimensional Gaussian pre-limiting Gaussian processes with paths in $D([0, 1])$ and bound the distance between the two in Theorem 6.2. Then, in Theorem 6.4, we bound the distance of our process from a continuous two-dimensional Gaussian process.

It is worth noting that the analysis of this section could easily be extended to one on a three-dimensional process whose coordinates represent the number of edges, the number of triangles and the number of two-stars in a $G(\lfloor nt \rfloor, p)$ graph. The only reason we do not do it here is that it would require some more involved algebraic computations and would make this section rather lengthy.

6.1. Introduction

Let us consider a Bernoulli random graph $G(n, p)$ on $n$ vertices with edge probabilities $p$.

Let $I_{i,j} = I_{j,i}$ be the Bernoulli($p$)-indicator that edge $(i, j)$ is present in this graph. These indicators, for $(i, j) \in \{1, \cdots, n\}^2$ are independent. We will look at a process representing at each $t \in [0, 1]$ the re-scaled total number of edges in the graph formed out of the given Bernoulli random graph by considering only its first $\lfloor nt \rfloor$ vertices and the edges between them:

$$T_n(t) = \frac{\lfloor nt \rfloor - 2}{2n^2} \sum_{i,j=1}^{\lfloor nt \rfloor} I_{i,j} = \frac{\lfloor nt \rfloor - 2}{n^2} \sum_{1 \leq i < j \leq \lfloor nt \rfloor} I_{i,j},$$

and at a process representing a re-scaled statistic related to the number of two-stars in the same graph:

$$V_n(t) = \frac{1}{2n^2} \sum_{1 \leq i,j,k \leq \lfloor nt \rfloor} I_{ij}I_{jk} = \frac{1}{n^2} \sum_{1 \leq i < j < k \leq \lfloor nt \rfloor} (I_{i,j}I_{j,k} + I_{i,j}I_{i,k} + I_{j,k}I_{i,k}).$$
Let $Y_n(t) = (T_n(t) - \mathbb{E}T_n(t), V_n(t) - \mathbb{E}V_n(t))$ for $t \in [0, 1]$.

**Remark 6.1.** Note that, for all $t \in [0, 1]$, $\mathbb{E}T_n(t) = \frac{|nt| - 2}{n^2} \binom{|nt|}{2} p$ and $\mathbb{E}V_n(t) = \frac{3}{n^2} \binom{|nt|}{3} p^2$. Furthermore, note that, by an argument similar to that of [RR10, Section 5], the covariance matrix of $(T_n(t) - \mathbb{E}T_n(t), V_n(t) - \mathbb{E}V_n(t))$ is given by

$$
3 \frac{(|nt| - 2) \binom{|nt|}{2} p (1 - p)}{n^4} \begin{pmatrix} 1 & 2p \\ 2p & 4p^2 \end{pmatrix}.
$$

Hence, the scaling ensures that the covariances are of the same order in $n$.

### 6.2. Exchangeable pair setup

We now construct an exchangeable pair, as in [RR10], by picking $(I, J)$ according to $\mathbb{P}[I = i, J = j] = \frac{1}{\binom{n}{2}}$ for $1 \leq i < j \leq n$. If $I = i, J = j$, we replace $I_{i,j} = I_{j,i}$ by an independent copy $I'_{i,j} = I'_{j,i}$ and put:

$$
T'_n(t) = T_n(t) - \frac{|nt| - 2}{n^2} (I_{I,J} - I'_{I,J}) \mathbb{1}_{[t/n,1]\cap [I/J,1]}(t),
$$

$$
V'_n(t) = V_n(t) - \frac{1}{n^2} \sum_{k:k \neq i,j} (I_{I,J} - I'_{I,J}) (I_{J,k} + I_{I,k}) \mathbb{1}_{[t/n,1]\cap [I/J,1]\cap [k,J,1]}(t).
$$

We also let $Y'_n(t) = (T'_n(t) - \mathbb{E}T_n(t), V'_n(t) - \mathbb{E}V_n(t))$ and note that, for $Y_n = (Y_n(t), t \in [0,1])$ and $Y'_n = (Y'_n(t), t \in [0,1])$, $(Y_n, Y'_n)$ forms an exchangeable pair. Let $e_1 = (1,0)$, $e_2 = (0,1)$. We note that, for any $m = 1,2$ and for $f$ denoting the $g$-solution to the Stein equation:

$$
\mathbb{E}^{Y_n} \left\{ Df(Y_n) \left[ (T'_n - T_n) e_m \right] \right\}
$$

$$
= \mathbb{E}^{Y_n} \left\{ Df(Y_n) \left[ \frac{|nt| - 2}{n^2} (I_{I,J} - I'_{I,J}) \mathbb{1}_{[t/n,1]\cap [I/J,1]}e_m \right] \right\}
$$

$$
= \frac{2}{n^3(n-1)} \sum_{i<j} \mathbb{E}^{Y_n} \left\{ Df(Y_n) \left[ (|nt| - 2) (I_{I,J} - I'_{I,J}) \mathbb{1}_{[t/n,1]\cap [I/J,1]}e_m \right] \right\} I = i, J = j
$$

$$
= - \frac{1}{\binom{n}{2}} Df(Y_n)[T_n e_m] + \frac{2}{n^3(n-1)} \sum_{i<j} Df(Y_n) \left[ (|nt| - 2) \mathbb{1}_{[t/n,1]\cap [I/J,1]}e_m \right]
$$

$$
= - \frac{1}{\binom{n}{2}} Df(Y_n)[(T_n(\cdot) - \mathbb{E}T_n(\cdot)) e_m].
$$
Also:

\[
\mathbb{E}^{Y_n} Df(Y_n) [(V_n - V'_n) e_m]
\]

\[
= \frac{1}{n^2 \binom{n}{2}} \sum_{i < j} \mathbb{E}^{Y_n} \left\{ \sum_{k: k \neq i, j} Df(Y_n) \left[ (I_{i,j} - I'_{i,j}) (I_{j,k} + I_{i,k}) \right. \right.
\]

\[
\cdot \mathbb{I}_{[i/n, 1] \cap [j/n, 1] \cap [k/n, 1] \cap [\Lambda/n, 1]} \left. \right| I = i, J = j \right\}
\]

\[
= \frac{2}{\binom{n}{2}} Df(Y_n) [V_n e_m]
\]

\[
- \frac{p}{n^2 \binom{n}{2}} \sum_{i < j} \mathbb{E}^{Y_n} Df(Y_n) \left[ (I_{i,j} + I_{j,i}) \mathbb{I}_{[i/n, 1] \cap [j/n, 1] \cap [\Lambda/n, 1]} \right]
\]

\[
= \frac{2}{\binom{n}{2}} Df(Y_n) [V_n e_m] - \frac{p}{n^2 \binom{n}{2}} \sum_{1 \leq i, j, k \leq n} \mathbb{E}^{Y_n} Df(Y_n) \left[ I_{i,j} \mathbb{I}_{[i/n, 1] \cap [j/n, 1] \cap [\Lambda/n, 1]} \right]
\]

\[
= \frac{2}{\binom{n}{2}} Df(Y_n) [(V_n - \mathbb{E} V_n) e_m]
\]

\[
- \frac{p}{n^2 \binom{n}{2}} \sum_{1 \leq i, j, k \leq n} \mathbb{E}^{Y_n} Df(Y_n) \left[ (I_{i,j} - p) \mathbb{I}_{[i/n, 1] \cap [j/n, 1] \cap [\Lambda/n, 1]} \right]
\]

\[
= \frac{2}{\binom{n}{2}} Df(Y_n) [(V_n - \mathbb{E} V_n) e_m] - \frac{2p}{\binom{n}{2}} Df(Y_n) [(T_n - \mathbb{E} T_n) e_m].
\]

Therefore, for any \( m = 1, 2 \):

A) \( Df(Y_n) [(T_n - \mathbb{E} T_n) e_m] = \frac{n(n-1)}{2} \mathbb{E}^{Y_n} \left\{ Df(Y_n) [(T_n - T'_n) e_m] \right\} \)

B) \( Df(Y_n) [(V_n - \mathbb{E} V_n) e_m] \)

\[
= \frac{n(n-1)}{4} \mathbb{E}^{Y_n} \left\{ Df(Y_n) [(V_n - V'_n) e_m] + p Df(Y_n) [(T_n - \mathbb{E} T_n) e_m] \right\}
\]

\[
= \frac{n(n-1)}{4} \mathbb{E}^{Y_n} \left\{ Df(Y_n) [(2p(T_n - T'_n) + V_n - V'_n) e_m] \right\}
\]

and so:

\[
Df(Y_n) [Y_n] = 2 \mathbb{E}^{Y_n} Df(Y_n) [(Y_n - Y'_n) \Lambda_n],
\]

where:

\[
\Lambda_n = \frac{n(n-1)}{8} \begin{pmatrix} 2 & 2p \\ 0 & 1 \end{pmatrix}.
\]
6.3. A pre-limiting process

Let \( A_n = (A_n^{(1)}, A_n^{(2)}) \), where \( A_n^{(2)} = A_n^{(2,1)} + A_n^{(2,2)} \), be defined in the following way:

\[
A_n^{(1)}(t) = ([nt] - 2) \sum_{i,j=1}^{[nt]} Z_{i,j}^{(1)}, \quad t \in [0,1]
\]

\[
A_n^{(2,1)}(t) = ([nt] - 2) \sum_{i,j=1}^{[nt]} Z_{i,j}^{(2,1)}, \quad t \in [0,1]
\]

\[
A_n^{(2,2)}(t) = \sum_{i,j,k=1}^{[nt]} Z_{i,j,k}^{(2,2)}, \quad t \in [0,1]
\]

where \( Z_{i,j}^{(1)} = 0 \) for all \( i \) and \( Z_{i,j}^{(2,2)} = 0 \) if \( i = j \) or \( i = k \) of \( j = k \). Furthermore, assume that the collection \{\( Z_{i,j}^{(1)} : i,j \in [n], i \neq j \)\} \cup \{\( Z_{i,j}^{(2)} : i,j \in [n], i \neq j \neq k \neq i \)\} is jointly centred Gaussian with the following covariance structure:

\[
\mathbb{E}Z_{i,j}^{(1)}Z_{k,l}^{(1)} = \begin{cases} \frac{p(1-p)}{2n^2}, & i = k, j = l, i \neq j \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mathbb{E}Z_{i,j}^{(2,1)}Z_{k,l}^{(2,1)} = \begin{cases} \frac{p^2(1-p)}{4n^2}, & i = k, j = l, i \neq j \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mathbb{E}Z_{i,j,k}^{(2,2)}Z_{l,m}^{(1)} = \begin{cases} \frac{3p^2(1-p)}{4n^2}, & i = l, j = m, i \neq j \neq k \neq i \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mathbb{E}Z_{i,j,k}^{(2,2)}Z_{l,m}^{(2,1)} = \begin{cases} \frac{p^3(1-p)}{2n^4}, & i = l, j = m, i \neq j \neq k \neq i \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mathbb{E}Z_{i,j,k}^{(2,2)}Z_{r,s,t}^{(2,2)} = \begin{cases} \frac{p^2(1-p^2)}{2n^4}, & i = r, j = s, k = t, i \neq j \neq k \neq i \\ \frac{p^2(1-p)}{n^2}, & i = r, j = s, k \neq t, i \neq j \neq k \neq i, i \neq j \neq t \neq i \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mathbb{E}Z_{i,j}^{(2,1)}Z_{k,l}^{(2,2)} = \begin{cases} \frac{1}{n^2}, & i = k, j = l, i \neq j \\ 0, & \text{otherwise} \end{cases}
\]

It will become clear in Remark 6.3 why we have chosen this covariance structure.

6.4. Distance from the pre-limiting process

**Theorem 6.2.** Let \( Y_n \) be defined as in Section 6.1 and \( A_n \) be defined as in Section 6.3. Then, for any \( g \in M^2 \):

\[
\mathbb{E}g(Y_n) - \mathbb{E}g(A_n) \leq 12\|g\|_{M^2}n^{-1}.
\]
Proof. We adopt the notation of sections 6.1, 6.2, 6.3. We will apply Theorem 4.1.

Step 1. First note that, for $R_1$ in Theorem 4.1,
\[
| (Y_n - Y'_n) \Lambda_n | \leq \| \Lambda_n \|_2 | Y_n - Y'_n |,
\]
where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$ and $\| \cdot \|_2$ is the induced operator 2-norm. Furthermore, for $\| \cdot \|_F$ denoting the Frobenius norm
\[
\| \Lambda_n \|_2 \leq \| \Lambda_n \|_F = \frac{n(n-1)}{8} \sqrt{2^2 + (2p)^2 + 0^2 + 1^2} \leq \frac{3n(n-1)}{8}.
\]
Therefore:
\[
\mathbb{E}[(Y_n - Y'_n) \Lambda_n] \| Y_n - Y'_n \|^2 \leq \frac{3n(n-1)}{8} \mathbb{E} \| Y_n - Y'_n \|^3
\]
\[
\leq \frac{3n(n-1)}{8} \mathbb{E} \left[ \frac{(n-2)^2}{n^4} (I_{I,I'} - I_{I,J})^2 + \frac{1}{n^4} \left( \sum_{k,k \neq I,J} (I_{I,J} - I_{I',J}) (I_{I,k} + I_{J,k}) \right) \right]^{3/2}
\]
\[
\leq \frac{3n(n-1)}{8} \left[ \frac{(n-2)^2}{n^4} + \frac{(2(n-2))^2}{n^4} \right]^{3/2}
\]
\[
\leq \frac{5}{n}, \tag{6.1}
\]
where the third inequality follows because $|I_{I,I'} - I_{I,J}| \leq 1$ and $|I_{I,k} + I_{J,k}| \leq 2$ for all $k$.

Step 2. For $R_2$ in Theorem 4.1, we wish to bound:
\[
| \mathbb{E} D^2 f(Y_n) \left[ (Y_n - Y'_n) \Lambda_n, Y_n - Y'_n \right] - \mathbb{E} D^2 f(Y_n) [\Lambda_n, \Lambda_n] |
\]
\[
= \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left[ (T_n - T'_n, 2p(T_n - T'_n) + (V_n - V'_n)), (T_n - T'_n, V_n - V'_n) \right]
\]
\[
- \mathbb{E} D^2 f(Y_n) [\Lambda_n, \Lambda_n]
\]
\[
\leq S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 \tag{6.2}
\]
where:
\[
S_1 = \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left[ (T_n - T'_n)(2, 0), (T_n - T'_n)(1, 0) \right] - \mathbb{E} D^2 f(Y_n) \left[ (A_n^{(1)}, 0), (A_n^{(1)}, 0) \right]
\]
\[
S_2 = \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left[ (T_n - T'_n)(0, 2p), (T_n - T'_n)(1, 0) \right] - 2 \mathbb{E} D^2 f(Y_n) \left[ (0, A_n^{(2,1)}), (A_n^{(1)}, 0) \right]
\]
\[
S_3 = \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left[ (T_n - T'_n)(2, 0), (V_n - V'_n)(0, 1) \right] - \frac{4}{3} \mathbb{E} D^2 f(Y_n) \left[ (A_n^{(1)}, 0), (0, A_n^{(2,2)}) \right]
\]
\[
S_4 = \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left[ (T_n - T'_n)(0, 2p), (V_n - V'_n)(0, 1) \right] - 2 \mathbb{E} D^2 f(Y_n) \left[ (0, A_n^{(2,1)}), (0, A_n^{(2,2)}) \right]
\]
For $S_1$, for fixed $i, j \in \{1, \ldots, n\}$, let $Y_{n,i}^{ij}$ be equal to $Y_n$ except for the fact that $I_{ij}$ is replaced by an independent copy, i.e. for all $t \in [0,1]$ let:

$$T_{n,i}^{ij}(t) = T_n(t) - \frac{nt}{n^2} (I_{ij} - I_{ij}') \mathbb{I}_{[i/n,1]\cap[j/n,1]}(t)$$

$$V_{n,i}^{ij}(t) = V_n(t) - \frac{1}{n^2} \sum_{k \neq i,j} (I_{ij} - I_{ij}') (I_{jk} + I_{ik}) \mathbb{I}_{[i/n,1]\cap[j/n,1]\cap[k/n,1]}(t)$$

and let $Y_{n,i}^{ij}(t) = (T_{n,i}^{ij}(t) - \mathbb{E} T_n(t), V_{n,i}^{ij}(t) - \mathbb{E} V_n(t))$.

By noting that the mean zero $Z_i^{(1)}$ and $Z_j^{(1)}$ are independent for $i \neq j$, we obtain:

$$S_5 = \left\lvert \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left( (V_n - V_n')(0,1), (T_n - T_n')(1,0) \right) - \frac{2}{3} \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,2)}), (A_n^{(1)}), 0 \right) \right\rvert$$

$$S_6 = \left\lvert \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left( (V_n - V_n')(0,1), (V_n - V_n')(0,1) \right) - \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,2)}), (0, A_n^{(2,2)}) \right) \right\rvert$$

$$S_7 = \left\lvert \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,1)}), (0, A_n^{(2,1)}) \right) \right\rvert .$$

$$\拈取:$$

$$S_5 = \left\lvert \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left( (V_n - V_n')(0,1), (T_n - T_n')(1,0) \right) - \frac{2}{3} \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,2)}), (A_n^{(1)}), 0 \right) \right\rvert$$

$$S_6 = \left\lvert \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) \left( (V_n - V_n')(0,1), (V_n - V_n')(0,1) \right) - \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,2)}), (0, A_n^{(2,2)}) \right) \right\rvert$$

$$S_7 = \left\lvert \mathbb{E} D^2 f(Y_n) \left( (0, A_n^{(2,1)}), (0, A_n^{(2,1)}) \right) \right\rvert .$$
I where the last inequality holds because \(|I_{ij}| = 1|∀ij\rangle\) and so, by (6.2) \(\leq \|D^2 f(Y_n) - D^2 f(Y_n^i)\|\), where (12.3) follows from Proposition 3.2. Now,

\[
\|Y_n - Y_n^i\| \leq \frac{1}{n^2} \left( (n-2)^2 |I_{ij} - I_n^i|^2 + \sum_{k:k \neq i,j} |I_{ij} - I_n^i| \cdot |I_{jk} + I_{ik}| \right) ^{\frac{1}{2}}
\]

and so, by (6.3),

\[
\frac{n(n-1)}{8} E D^2 f(Y_n) \left[ (T_n, T_n^i)(2, 0), (T_n, T_n^i)(1, 0) \right]
\]

\[
-2 E D^2 f(Y_n) \left[ \left( A_n^{(1)} \right), \left( A_n^{(1)} \right) \right]
\]

\[
\leq \frac{\|g\|^2_{M^2}}{12n^3} \sum_{1 \leq i \neq j \leq n} \sum_{1 \leq i \neq j \leq n} E \left\{ |I_{i,j} - 2pI_{i,j} + p| \cdot \sqrt{(I_{i,j} - I_n^i)^2 + (|I_{i,j} - I_n^i| \cdot (I_{jk} + I_{ik}))^2} \right\}
\]

\[
\leq \frac{\sqrt{5}\|g\|^2_{M^2}}{12n}, \quad (6.4)
\]

where the last inequality holds because \(|I_{ij} - 2pI_{ij} + p| \leq 1, |I_{ij} - I_n^i| \leq 1 \) and \(I_{jk} + I_{ik} \leq 2 \) for all \(k \in \{1, \cdots, n\}\).

Similarly, for \(S_2\):

\[
\frac{n(n-1)}{8} E D^2 f(Y_n) \left[ (T_n - T_n') (0, 2p), (T_n - T_n') (1, 0) \right]
\]

\[
-2 E D^2 f(Y_n) \left[ \left( 0, A_n^{(2,1)} \right), \left( A_n^{(1)} \right) \right]
\]

\[
\frac{n(n-1)}{8} E D^2 f(Y_n) \left[ (T_n - T_n') (0, 2p), (T_n - T_n') (1, 0) \right]
\]

\[
-2 \sum_{j,k=1}^{n} E D^2 f(Y_n) \sum_{i=1}^{n} Z_{i,k}^{(2,1)} \left( |n| - 2 \right) \mathbb{I}_{[j/n, 1] \cap [k/n, 1]}; (0, 1),
\]
\[
\sum_{i,j=1}^{n} Z_{i,j}^{(1)}([n.] - 2) \mathbb{1}_{[i/n, 1]\cap [j/n, 1]}(1, 0) \\
= \frac{p}{4n^4} \sum_{1 \leq i \neq j \leq n} \mathbb{E}(I_{i,j} - pI_{i,j} + p) D^2 f(Y_n) \left([(n.] - 2) \mathbb{1}_{[i/n, 1]\cap [j/n, 1]}(0, 1),
\right.
\left.((n.] - 2) \mathbb{1}_{[i/n, 1]\cap [j/n, 1]}(1, 0) - \sum_{1 \leq i \neq j \leq n} \mathbb{E}\left(Z_{i,j}^{(1)} Z_{i,j}^{(2,1)}\right)\right)
\leq \frac{\sqrt{5p}\|g\|_{\mathcal{L}^2}}{12n}, \\
\leq \frac{\sqrt{5}\|g\|_{\mathcal{L}^2}}{12n}.
\]

For \( S_t \), let \( Y_{n}^{ij} \) equal to \( Y_n \) except for the fact that \( I_{ij}, I_{jk}, I_{ik} \) are replaced by \( I_{ij}', I_{jk}', I_{ik}' \), i.e. for all \( t \in [0, 1] \) let
\[
T_{n}^{ij}(t) = T_n(t) - \frac{|n| - 2}{n^2} \left( (I_{ij} - I_{ij}') \mathbb{1}_{[i/n, 1]\cap [j/n, 1]}(t)
\right.
\left. + (I_{jk} - I_{jk}') \mathbb{1}_{[j/n, 1]\cap [k/n, 1]}(t) + (I_{ik} - I_{ik}') \mathbb{1}_{[i/n, 1]\cap [k/n, 1]}(t)\right)
\]
\[
V_{n}^{ij}(t) = V_n(t) - \frac{1}{n^2} \sum_{l \neq i,j,k} \left( (I_{ij} - I_{ij}') (I_{jl} + I_{kl}) \mathbb{1}_{[j/n, 1]\cap [l/n, 1]}(t)
\right.
\left. + (I_{ik} - I_{ik}') (I_{jl} + I_{kl}) \mathbb{1}_{[i/n, 1]\cap [l/n, 1]}(t)
\right.
\left. + (I_{jk} - I_{jk}') (I_{jl} + I_{kl}) \mathbb{1}_{[j/n, 1]\cap [l/n, 1]}(t)\right)
\]
\[
= \frac{1}{n^2} \left( (I_{ij} - I_{ij}') (I_{jk} - I_{jk}') (I_{ik} - I_{ik}') \mathbb{1}_{[i/n, 1]\cap [j/n, 1]\cap [k/n, 1]}(t). \right.
\]

Let \( Y_{n}^{ij}(t) = (T_{n}^{ij}(t) - \mathbb{E}T_n(t), V_{n}^{ij}(t) - \mathbb{EV}_n(t)) \) for all \( t \in [0, 1] \). Note that
\[
\left| \frac{n(n-1)}{8} \mathbb{E}D^2 f(Y_n) [(T_n - T_n')(2, 0), (V_n - V_n')(0, 1)]
\right.
\left. - \frac{4}{3} \mathbb{E}D^2 f(Y_n) \left( \left( A_{(1)}^{(1)}, 0 \right), \left( 0, A_{(2,2)}^{(2,2)} \right) \right) \right|
\]
\[
= \frac{n(n-1)}{8} \mathbb{E}D^2 f(Y_n) [(T_n - T_n')(2, 0), (V_n - V_n')(0, 1)]
\]
\[
- \frac{4}{3} \mathbb{E}D^2 f(Y_n) \left[ \sum_{i,j=1}^{n} Z_{i,j}^{(1)}([-n.] - 2)(1, 0) \mathbb{1}_{[i/n, 1]\cap [j/n, 1]}, \sum_{i,j,k=1}^{n} Z_{i,j,k}^{(2,2)}(0, 1) \mathbb{1}_{[i/n, 1]\cap [j/n, 1]\cap [k/n, 1]} \right]
\]
\[
= \frac{1}{4n^4} \sum_{1 \leq i,j,k \leq n} \mathbb{E}(I_{ij} - I_{ij}')^2 (I_{jk} + I_{ik})
\]
\[ D^2f(Y_n) \left[ ((n\cdot - 2)1_{[i/n, 1] \cap [j/n, 1]}(1, 0), 1_{[i/n, 1] \cap [j/n, 1]}(0, 1) \right] \]

\[ - \frac{4}{3} \sum_{1 \leq i, j, k \leq n} \mathbb{E} Z_{i, j, k}^{(1)} Z_{i, j, k}^{(2, 2)} D^2f(Y_n) \left[ ([n\cdot - 2]1_{[i/n, 1] \cap [j/n, 1]}(1, 0), 1_{[i/n, 1] \cap [j/n, 1]}(0, 1) \right] \]

\[ = \sum_{1 \leq i, j, k \leq n} \mathbb{E} \left\{ \left( \frac{1}{4n^3}(I_{ij} - 2pI_{ij} + p)(I_{jk} + I_{ik}) - \frac{4}{3} \mathbb{E} Z_{i, j, k}^{(1)} Z_{i, j, k}^{(2, 2)} \right) \right\} \]

\[ \cdot D^2f(Y_n) \left[ ([n\cdot - 2]1_{[i/n, 1] \cap [j/n, 1]}(1, 0), 1_{[i/n, 1] \cap [j/n, 1]}(0, 1) \right] \]

\[ = \sum_{1 \leq i, j, k \leq n} \mathbb{E} \left\{ \left( \frac{1}{4n^3}(I_{ij} - 2pI_{ij} + p)(I_{jk} + I_{ik}) \right) \right\} \]

\[ \cdot (D^2f(Y_n) - D^2f(Y_n') \right) \left[ ([n\cdot - 2]1_{[i/n, 1] \cap [j/n, 1]}(1, 0), 1_{[i/n, 1] \cap [j/n, 1]}(0, 1) \right] \right\} \]

\[ \leq \left\| g \right\|_{M_2} \sum_{1 \leq i, j, k \leq n} \mathbb{E} (I_{ij} - 2pI_{ij} + p)(I_{jk} + I_{ik}) \left\| Y_n - Y_n' \right\|^2 \tag{6.7} \]

Now, by (6.6), we note that:

\[ \left\| Y_n - Y_n' \right\| \leq \frac{1}{n^2} \left\{ (n - 2)^2 |I_{ij} - I_{ij}'| + |I_{jk} - I_{jk}'| + |I_{ik} - I_{ik}'| \right\}^2 \]

\[ + \left\{ \sum_{l \neq i, j, k} (|I_{ij} - I_{ij}'|(|I_{jl} - I_{jl}'| + |I_{kl} - I_{kl}'|) + |I_{ik} - I_{ik}'|(|I_{jl} - I_{jl}'| + |I_{ik} - I_{ik}'|) \]

\[ + |I_{ij} - I_{ij}'|(|I_{jl} - I_{jl}'| + |I_{ik} - I_{ik}'|) + |I_{ij} - I_{ij}'|(|I_{jl} - I_{jl}'| + |I_{ik} - I_{ik}'|) \right\}^{1/2} \]

\[ \leq \frac{1}{n^2} \sqrt{9(n - 2)^2 + (8(n - 3) + 3)^2} \]

\[ = \frac{\sqrt{73n^2 - 372n + 477}}{n^2}, \]

where the second inequality follows from the fact that for all \( a, b, c \in \{1, \cdots, n\}, |I_{ab} - I_{ab}'| \leq 1, (I_{ab} + I_{bc}) \leq 2 \) and \( |I_{ab}I_{bc} - I_{ab}'I_{bc}'| \leq 1 \). Also, \( (I_{jk} + I_{ik}) \leq 2 \) and \( I_{ij} - 2pI_{ij} + p \leq 1 \). Therefore, by (6.7):

\[ \left\| n(n - 1) \mathbb{E} D^2f(Y_n) \right[ (T_n - T_n')(2, 0), (V_n - V_n')(0, 1) \right] \]

\[ - \frac{4}{3} \mathbb{E} D^2f(Y_n) \left[ \left( A_n^{(1)}(1), 0 \right), \left( 0, A_n^{(2, 2)} \right) \right] \right\| \]

\[ \leq \left\| g \right\|_{M_2}(n(n - 1)(n - 2)\sqrt{73n^2 - 372n + 477}} \]

\[ 6n^5 \]
Similarly, for $S_4$:

\[
\frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(T_n - T_n')(0, 2p), (V_n - V_n')(0, 1)] - 2 \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,1)} \right), \left( 0, A_n^{(2,2)} \right) \right] \leq \frac{\sqrt{178} \|g\|_{L^2}}{6n} \leq \frac{\sqrt{178} \|g\|_{L^2}}{6n}. \tag{6.8}
\]

and, for $S_5$:

\[
\frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(V_n - V_n')(0, 1), (T_n - T_n')(1, 0)] - \frac{2}{3} \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,1)} \right), \left( 0, A_n^{(2,2)} \right) \right] \\
= \frac{1}{2} \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(T_n - T_n')(2, 0), (V_n - V_n')(0, 1)] - \frac{4}{3} \mathbb{E} D^2 f(Y_n) \left[ \left( A_n^{(1)} \right), \left( 0, A_n^{(2,2)} \right) \right] \\
\leq \frac{\sqrt{178} \|g\|_{L^2}}{12n}. \tag{6.9}
\]

Now, for $S_6$, let $Y_n^{ijkl}$ be equal to $Y_n$ except that $I_{ij}, I_{ik}, I_{il}, I_{jk}, I_{jl}, I_{kl}$ are replaced with independent copies $I_{ij}', I_{ik}', I_{il}', I_{jk}', I_{jl}', I_{kl}'$, i.e. for all $t \in [0, 1]$ let

\[
T_n^{ijkl}(t) = T_n(t) - \frac{|nt|}{n^2} \sum_{m:\{m \neq i,j,k,l\}} [(I_{ij} - I_{ij}') (I_{im} + I_{jm})]_{[i/n,1]\cap[j/n,1]\cap[m/n,1]}(t) \\
+ (I_{il} - I_{il}') (I_{im} + I_{jm})]_{[i/n,1]\cap[l/n,1]\cap[m/n,1]}(t) \\
+ (I_{jk} - I_{jk}') (I_{jm} + I_{jm})]_{[j/n,1]\cap[k/n,1]\cap[m/n,1]}(t) \\
+ (I_{kl} - I_{kl}') (I_{km} + I_{km})]_{[k/n,1]\cap[l/n,1]\cap[m/n,1]}(t) \\
- \frac{1}{n^2} [(I_{ij} I_{jk} - I_{ij} I_{jk}') + (I_{ij} I_{ik} - I_{ij} I_{ik}') + (I_{ik} I_{jk} - I_{ik} I_{jk}')]_{[i/n,1]\cap[j/n,1]\cap[k/n,1]}(t) \\
- \frac{1}{n^2} [(I_{ij} I_{jl} - I_{ij} I_{jl}') + (I_{ij} I_{il} - I_{ij} I_{il}') + (I_{il} I_{jl} - I_{il} I_{jl}')]_{[i/n,1]\cap[j/n,1]\cap[l/n,1]}(t) \\
- \frac{1}{n^2} [(I_{ik} I_{kl} - I_{ik} I_{kl}') + (I_{ik} I_{il} - I_{ik} I_{il}') + (I_{il} I_{kl} - I_{il} I_{kl}')]_{[i/n,1]\cap[k/n,1]\cap[l/n,1]}(t) \\
- \frac{1}{n^2} [(I_{jk} I_{jl} - I_{jk} I_{jl}') + (I_{jl} I_{kl} - I_{jl} I_{kl}') + (I_{kl} I_{jk} - I_{kl} I_{jk}')]_{[j/n,1]\cap[k/n,1]\cap[l/n,1]}(t) \tag{6.10}
\]

and for all $t \in [0, 1]$ let $Y_n^{ijkl}(t) = (T_n^{ijkl}(t) - \mathbb{E} T_n, V_n^{ijkl}(t) - \mathbb{E} V_n(t))$. Note
that:

\[
\frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(V_n - V'_n)(0,1), (V_n - V'_n)(0,1)] - \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,2)} \right), \left( 0, A_n^{(2,2)} \right) \right]
\]

\[
\leq \frac{1}{8n^4} \sum_{1 \leq i \neq j \leq n} \sum_{k \neq i,j \neq i,j} \mathbb{E} (I_{ij} - I'_{ij})^2 (I_{jk} + I_{ik})(I_{jl} + I_{il})
\]

\[
\cdot D^2 f(Y_n) \left[ \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1), \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1) \right]
\]

\[
- \mathbb{E} D^2 f(Y_n) \left[ \sum_{i,j,k=1} Z^{(2,2)}_{i,j,k} \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1), \sum_{i,j,k=1} Z^{(2,2)}_{i,j,k} \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1) \right]
\]

\[
\leq \sum_{1 \leq i \neq j \leq n} \sum_{k \neq i,j \neq i,j} \mathbb{E} \left[ \frac{1}{8n^4} (I_{ij} - I'_{ij})^2 (I_{jk} + I_{ik})(I_{jl} + I_{il}) - Z^{(2,2)}_{ijl} \right]
\]

\[
\cdot D^2 f(Y_n) \left[ \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1), \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1) \right]
\]

\[
= \sum_{1 \leq i \neq j \leq n} \sum_{k \neq i,j \neq i,j} \mathbb{E} \left[ \frac{1}{8n^4} (I_{ij} - I'_{ij})^2 (I_{jk} + I_{ik})(I_{jl} + I_{il}) \right]
\]

\[
\cdot \left[ D^2 f(Y_n) - D^2 f(Y^{ijkl}_n) \right] \left[ \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1), \mathbb{1}_{[i/n,1]} \cap [j/n,1] \cap [k/n,1](0,1) \right]
\]

\[
\leq \frac{\|g\|_{M^2}}{24n^4} \sum_{1 \leq i,j,k,l \leq n} \mathbb{E} \left\{ \left( I_{ij} - 2pI_{ij} + p \right)(I_{jk} + I_{ik})(I_{jl} + I_{il}) \right\} \|Y_n - Y^{ijkl}_n\| \}
\]

(6.12)

Now, by (6.11), note that:

\[
\|Y_n - Y^{ijkl}_n\| \leq \frac{1}{n^2} \left\{ (n-2)^2 \left| I_{ij} - I'_{ij} \right| + \left| I_{ik} - I'_{ik} \right| + \left| I_{il} - I'_{il} \right| + \left| I_{jk} - I'_{jk} \right| + \left| I_{jl} - I'_{jl} \right| + \left| I_{kl} - I'_{kl} \right| + \left| I_{ij} - I'_{ij} \right| (I_{im} + I_{jm}) + \left| I_{ik} - I'_{ik} \right| (I_{im} + I_{km}) + \left| I_{il} - I'_{il} \right| (I_{im} + I_{lm}) + \left| I_{jk} - I'_{jk} \right| (I_{jm} + I_{km}) + \left| I_{jl} - I'_{jl} \right| (I_{jm} + I_{lm}) + \left| I_{kl} - I'_{kl} \right| + \left| I_{ij} - I'_{ij} \right| (I_{ij} + I_{kl}) + \left| I_{ik} - I'_{ik} \right| (I_{ij} + I_{kl}) + \left| I_{il} - I'_{il} \right| (I_{ij} + I_{kl}) + \left| I_{jk} - I'_{jk} \right| (I_{ij} + I_{kl}) + \left| I_{jl} - I'_{jl} \right| (I_{ij} + I_{kl}) + \left| I_{kl} - I'_{kl} \right| + \left| I_{ij} - I'_{ij} \right| (I_{ij} + I_{kl}) + \left| I_{ik} - I'_{ik} \right| (I_{ij} + I_{kl}) + \left| I_{il} - I'_{il} \right| (I_{ij} + I_{kl}) + \left| I_{jk} - I'_{jk} \right| (I_{ij} + I_{kl}) + \left| I_{jl} - I'_{jl} \right| (I_{ij} + I_{kl}) + \left| I_{kl} - I'_{kl} \right| \right\}^{1/2}
\]

\[
\leq \sqrt{\frac{36(n-2)^2 + (12(n-4) + 12)^2}{n^2}}
\]

\[
= \sqrt{180n^2 - 1008n + 14400}
\]
Therefore, by (6.12):
\[
\frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(V_n - V'_n)(0,1), (V_n - V'_n)(0,1)] - \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,2)} \right), \left( 0, A_n^{(2,2)} \right) \right] \\
\leq \|g\|_{M^2} \cdot \frac{4 \sqrt{180n^2 - 1008n + 1440}}{24n^2} \leq \frac{\sqrt{612} \|g\|_{M^2}}{6n}.
\] (6.13)

Furthermore, for \(S_7\), note that:
\[
\left| \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,1)} \right), \left( 0, A_n^{(2,1)} \right) \right] \right| \\
= \sum_{i,j=1}^{n} \mathbb{E} \left| Z_{i,j}^{(2,1)} \right|^2 D^2 f(Y_n) \left[ (0, ([n \cdot] - 2) \mathbb{1}_{[i/n, 1 \cap [j/n, 1]}), (0, ([n \cdot] - 2) \mathbb{1}_{[i/n, 1 \cap [j/n, 1]}) \right] \\
\leq \|g\|_{M^2} n^{-1}.
\] (6.14)

Note that, by (6.2) and (6.4), (6.5), (6.8), (6.9), (6.10), (6.13), (6.14):
\[
\left| \mathbb{E} D^2 f(Y_n) [(Y_n - Y'_n)A_n, Y_n - Y'_n] - \mathbb{E} D^2 f(Y_n) [A_n, A_n] \right| \\
= \frac{n(n-1)}{8} \mathbb{E} D^2 f(Y_n) [(T_n - T'_n)(2, 2p) + (V_n - V'_n)(0, 1), (T_n - T'_n)(1, 0) + (V_n - V'_n)(0, 1)] \\
\leq 11 \|g\|_{M^2} n^{-1}.
\] (6.15)

Using Theorem 4.1 together with (6.15) and (6.1) gives the desired result. \(\square\)

**Remark 6.3.** The reasons for the covariance structure of \(A_n\) taking the particular form described in Section (6.3) become clear when we look at (6.4). The processes we compare are two-dimensional. The \(A_n^{(1)}\) part of the pre-limiting process \(A_n\) corresponds to the contribution of \(T_n - T'_n\) to the first coordinate in processes \((Y_n - Y'_n)A_n\) and \(Y_n - Y'_n\). Similarly, \(A_n^{(2,1)}\) corresponds to the contribution of \(T_n - T'_n\) to the second coordinate and \(A_n^{(2,2)}\) corresponds to the contribution of \(V_n - V'_n\) to the second coordinate.

The covariances are chosen so that at any time points \(s, t \in [0, 1]\),
\[
\text{Cov}(A_n(s), A_n(t)) \text{ is close to } \text{Cov}((Y_n - Y'_n)A_n(s), (Y_n - Y'_n)(t)).
\]
This makes the bounds in (6.4), (6.5), (6.8), (6.9), (6.10), (6.13) and (6.14) small. Specifically, the only contribution to
\[
\text{Cov}(A_n(s), A_n(t)) - \text{Cov}((Y_n - Y'_n)A_n(s), (Y_n - Y'_n)(t))
\]
for \(s, t \in [0, 1]\) comes from the covariance of \(A_n^{(1)}\) and this is achieved by choosing specific values for \(\text{Cov} \left( A_n^{(2)}(s), A_n^{(2)}(t) \right)\) and \(\text{Cov} \left( A_n^{(1)}(s), A_n^{(2)}(t) \right)\) for \(s, t \in [0, 1]\).

The covariance structure of \(A_n^{(1)}\) is chosen so that
\[
\left| \mathbb{E} D^2 f(Y_n) \left[ \left( 0, A_n^{(2,1)} \right), \left( 0, A_n^{(2,1)} \right) \right] \right|
\]
is small and this choice is made in an arbitrary way.
6.4.1. Distance from a continuous process

We now establish a bound on the speed of convergence of $Y_n$ to a continuous Gaussian process whose covariance is the limit of the covariance of $A_n$. We do this by bounding the distance between $A_n$ and the continuous process via the Brownian modulus of continuity and using Theorem 6.2.

**Theorem 6.4.** Let $Y_n$ be defined as in Subsection 6.1 and let $Z = (Z^{(1)}, Z^{(2)})$ be defined by:

\[
\begin{align*}
Z^{(1)}(t) &= \frac{\sqrt{p(1-p)}}{\sqrt{2 + 8p^2}} tB_1(t^2) + \frac{p\sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_2(t^2), \\
Z^{(2)}(t) &= \frac{p\sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_1(t^2) + \frac{2p^2\sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_2(t^2),
\end{align*}
\]

where $B_1, B_2$ are independent standard Brownian Motions. Then, for any $g \in M^2$:

\[|Eg(Y_n) - Eg(Z)| \leq \|g\|_{M^2} \left(913n^{-1/2} \sqrt{\log n} + 112n^{-1/2}\right).\]

**Proof of Theorem 6.4.** Let $B_1, B_2, B_3, B_4, B_5$ be i.i.d. standard Brownian Motions and let $Z_n = \left(Z_n^{(1)}, Z_n^{(2)}\right)$ be defined by:

\[
\begin{align*}
A_n^{(1)}(u) &= ((|n| - 2)|n|(|n| - 1)) \\
&\quad \frac{\sqrt{p(1-p)}}{n^2 \sqrt{2 + 8p^2}} B_1(|n|(|n| - 1)) \\
&\quad + \frac{|n| - 2}{n^2 \sqrt{2 + 8p^2}} B_1(|n|(|n| - 1)); \\
A_n^{(2)}(u) &= ((|n| - 2)|n|(|n| - 1)) \\
&\quad \frac{\sqrt{2p(1-p)}}{n^2 \sqrt{1 + 4p^2}} B_1(|n|(|n| - 1)) \\
&\quad + \frac{2p^2\sqrt{2p(1-p)}}{n^2 \sqrt{1 + 4p^2}} B_1(|n|(|n| - 1)) \\
&\quad + \frac{\sqrt{2p^3(1-p)}}{n^2} B_5(1).
\end{align*}
\]

Now, note that $(A_n^{(1)}, A_n^{(2)}) \overset{D}{=} (Z_n^{(1)}, Z_n^{(2)})$. To see this, note that:

\[
\begin{align*}
A_n^{(1)}(u) &= ([n(t \wedge u)] - 2)|n(t \wedge u)|(|n(t \wedge u)| - 1) \frac{p(1-p)}{2n^4} \\
&= E[Z_n^{(1)}(u)Z_n^{(1)}(u)]; \\
A_n^{(2)}(u) &= E[A_n^{(2,1)}(u)A_n^{(2,1)}(u) + E[A_n^{(2,1)}(u)A_n^{(2,2)}(u) + E[A_n^{(2,2)}(u)A_n^{(2,1)}(u) + E[A_n^{(2,2)}(u)A_n^{(2,2)}(u)]]
\end{align*}
\]
\[(nt) - 2)(|nu| - 2) \sum_{1 \leq i,j \leq [n(t \wedge u)]} \mathbb{E} \left[ (Z_{ij}^{(2,1)})^2 \right] + \]
\[+ (nt - 2)(|nu| - 2) \sum_{1 \leq i,j \leq [n(t \wedge u)]} \sum_{1 \leq k \leq [nu]}_{i \neq j, k \neq j} \mathbb{E} Z_{i,j,k}^{(2,1)} Z_{i,j,k}^{(2,2)} \]
\[+ (|nu| - 2) \sum_{1 \leq i,j \leq [n(t \wedge u)]} \sum_{1 \leq k \leq [nt]}_{i \neq j, k \neq j} \mathbb{E} Z_{i,j,k}^{(2,1)} Z_{i,j,k}^{(2,2)} \]
\[+ \sum_{1 \leq i,j,k \leq [n(u \wedge t)]} \sum_{i,j,k \text{ distinct}} \mathbb{E} \left[ (Z_{i,j,k}^{(2,2)})^2 \right] \]
\[+ \sum_{1 \leq i,j \leq [n(u \wedge t)]} \sum_{1 \leq k \leq [nu], 1 \leq \leq [nt]}_{i,j,k,l \text{ distinct}} \mathbb{E} Z_{i,j,k}^{(2,2)} Z_{i,j,l}^{(2,2)} \]
\[= \frac{(nt - 2)(|nu| - 2)|n(t \wedge u)||[n(t \wedge u)] - 1}{n^5} \]
\[+ \frac{(nt - 2)(|nu| - 2)|n(t \wedge u)||[n(t \wedge u)] - 1}{n^4} \frac{p^3(1 - p)}{n^4} \]
\[+ \frac{|n(t \wedge u)||[n(t \wedge u)] - 1}{n^4} \frac{p^2(1 - p^2)}{2n^4} \]
\[+ \frac{|n(t \wedge u)||[n(t \wedge u)] - 1}{n^4} \frac{2p^3(1 - p)}{n^4} \]
\[= \frac{(nt - 2)(|nu| - 2)|n(t \wedge u)||[n(t \wedge u)] - 1}{n^5} \frac{2p^3(1 - p)}{n^4} \]
\[+ \frac{|n(t \wedge u)||[n(t \wedge u)] - 1}{n^4} \frac{p^3(1 - p)}{n^4} \]
\[\mathbb{E} Z_n^{(2)}(u); \]

\[C) \quad A_n^{(1)}(t) A_n^{(2)}(u) = (nt - 2)(|nu| - 2)|n(t \wedge u)||[n(t \wedge u)] - 1| \frac{p^2(1 - p)}{n^4} \]
\[\mathbb{E} Z_n^{(1)}(t) Z_n^{(2)}(u). \quad (6.16) \]

Then:
\[\mathbb{E} \| Z_n - Z \| \]
\[\leq \frac{\sqrt{p(1 - p)}}{\sqrt{2 + 8p^2}} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt - 2}{n} B_1 \left( \frac{|nt||nt| - 2}{n^2} \right) - tB_1(t^2) \right| \right] \]
\[+ p \frac{\sqrt{2p(1 - p)}}{\sqrt{1 + 4p^2}} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt - 2}{n} B_2 \left( \frac{|nt||nt| - 2}{n^2} \right) - tB_2(t^2) \right| \right] \]
\[
+ \frac{p \sqrt{2p(1-p)}}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt}{n} - 2 B_1 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) - tB_1(t^2) \right| \right] \\
+ \frac{2p^2 \sqrt{2p(1-p)}}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt}{n} - 2 B_2 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) - tB_2(t^2) \right| \right] \\
+ \frac{1}{n^{1/2}} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt}{n} - 2 B_3 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) \right| \right] \\
+ \frac{p(1-p)}{\sqrt{2n^{1/2}}} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| B_4 \left( \frac{\lfloor nt \rfloor}{n^3} - 1 \right) \right| \right] + \frac{\sqrt{2p^4(1-p)}}{n^2} \mathbb{E} |B_5(1)| \\
\leq (1 + 4p + 4p^2) \sqrt{p(1-p)} \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt}{n} - 2 B_1 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) - tB_1(t^2) \right| \right] \\
+ \frac{2}{n^{1/2}} + \frac{\sqrt{2p(1-p)}}{n^{1/2}} + \frac{\sqrt{p^3(1-p)}}{\sqrt{n^{1/2}}} \\
\leq \frac{(1 + 4p + 4p^2) \sqrt{p(1-p)}}{\sqrt{2 + 8p^2}} \left( \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{nt}{n} - t \right| B_1(t^2) \right] \right) \\
+ \mathbb{E} \left[ \sup_{t \in [0,1]} B_1 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) - B_1(t^2) \right] + \frac{2 + \sqrt{p}(1-p)}{n^{1/2}} + \frac{2 \sqrt{p^3(1-p)}}{\sqrt{n^{1/2}}} \\
\leq \frac{(1 + 4p + 4p^2) \sqrt{p(1-p)}}{\sqrt{2 + 8p^2}} \left( \frac{6}{n} + \frac{30 \sqrt{3 \log n}}{n^{1/2} \sqrt{\pi \log(2)}} \right) + \frac{2 + \sqrt{p}(1-p)}{n^{1/2}} + \frac{2 \sqrt{p^3(1-p)}}{\sqrt{n^{1/2}}} \\
\leq \frac{12}{n^{1/2}} + \frac{51 \sqrt{\log n}}{\sqrt{n}}, \quad (6.17)
\]

where the second inequality follows from the fact that, by Doob’s $L^2$ inequality:

A) \[
\mathbb{E} \left[ \sup_{t \in [0,1]} B_3 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) \right] \leq 2 \sqrt{\mathbb{E} \left[ B_3 \left( \frac{n(n-1)}{n^2} \right)^2 \right]} \leq 2
\]

B) \[
\mathbb{E} \left[ \sup_{t \in [0,1]} B_4 \left( \frac{\lfloor nt \rfloor}{n^3} - 1 \right) \right] \leq 2 \sqrt{\mathbb{E} \left[ B_4 \left( \frac{n^2(n-1)}{n^3} \right)^2 \right]} \leq 2
\]

and the fourth inequality follows from the fact that \( \frac{\lfloor nt \rfloor}{n} - t \leq \frac{3}{n} \) for all \( t \in [0,1] \), \( \mathbb{E} \left[ \sup_{t \in [0,1]} B_1(t^2) \right] \leq 2 \) by Doob’s $L^2$ inequality and

\[
\mathbb{E} \left[ \sup_{t \in [0,1]} B_1 \left( \frac{\lfloor nt \rfloor}{n^2} - 1 \right) - B_1(t^2) \right] \leq \frac{30 \sqrt{3 \log (\frac{\pi}{4})}}{n^{1/2} \sqrt{\pi \log(2)}},
\]

which follows from [FN10, Lemma 3] and the fact that

\[
\left| \frac{\lfloor nt \rfloor}{n^2} - 1 \right| - t^2 \leq \left| \frac{(nt - \lfloor nt \rfloor)(nt + \lfloor nt \rfloor)}{n^2} \right| + \frac{1}{n^2} \leq \frac{3}{n},
\]
Similarly, using Doob's $L^2$ inequality and [FN10, Lemma 3],

$$
\mathbb{E}[|Z_n|] \leq \frac{2p(1-p)}{2 + 8p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_1 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_1(t^2) \right]^2 + 2 \frac{2p^3(1-p)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_2 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_2(t^2) \right]^2
$$

$$
+ 5 \frac{2p^3(1-p)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_1 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_1(t^2) \right]^2 + 5 \frac{2p^3(1-p)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_2 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_2(t^2) \right]^2 + 5 \frac{2p^3(1-p)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_3 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) \right]^2 + 5 \frac{2p^3(1-p)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_4 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) \right]^2
$$

$$
\leq \frac{p(1-p)(1 + 14p^2 + 40p^4)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_1 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_1(t^2) \right]^2 + 20 + 10p^2(1-p)^2 + \frac{2p^3(1-p)}{n^4}
$$

$$
\leq \frac{p(1-p)(1 + 14p^2 + 40p^4)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} \frac{|nt| - n}{n} B_1 \left( \frac{|nt|(|nt| - 1)}{n^2} \right) - tB_1(t^2) \right]^2 + 36 \frac{270 \log n}{n^2 + n \log 2} + 20 + 10p^2(1-p)^2 + \frac{2p^3(1-p)}{n^4}
$$

$$
\leq \frac{121}{n} + \frac{743 \log n}{n}.
$$

(6.18)

Furthermore, by Doob's $L^2$ inequality,

$$
\mathbb{E}[|Z|^2] \leq \mathbb{E} \left[ \sup_{t \in [0,1]} \left( \sqrt{p(1-p)} \sqrt{2 + 8p^2} B_1(t^2) + \frac{p \sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_2(t^2) \right)^2 \right] + \mathbb{E} \left[ \sup_{t \in [0,1]} \left( \frac{p \sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_1(t^2) + \frac{2p \sqrt{2p(1-p)}}{\sqrt{1 + 4p^2}} tB_2(t^2) \right)^2 \right]
$$

$$
\leq \frac{p(1-p)(1 + 8p^2 + 16p^4)}{1 + 4p^2} \mathbb{E} \left[ \sup_{t \in [0,1]} |B_1(t^2)|^2 \right]
$$
\[
\leq \frac{4p(1-p)(1+8p^2+16p^4)}{1+4p^2} \leq 5. \tag{6.19}
\]
We note that \(\|Dg(w)\| \leq \|g\|_{M^2}(1+\|w\|)\) and therefore, by (6.17), (6.18) and (6.19):
\[
\|E[g(Z) - E[g(A_n)]\| \leq \mathbb{E} \left[ \sup_{c \in [0,1]} \|Dg(Z + c(Z_n - Z))\| \|Z - Z_n\| \right] \\
\leq \|g\|_{M^2} \mathbb{E} \left[ \sup_{c \in [0,1]} (1+\|Z + c(Z_n - Z)\|) \|Z - Z_n\| \right] \\
\leq \|g\|_{M^2} \mathbb{E} [\|Z - Z_n\| + \|Z\| \|Z - Z_n\| + \|Z_n\|^2] \\
\leq \|g\|_{M^2} \left( \mathbb{E}[\|Z - Z_n\| + \sqrt{\mathbb{E}[\|Z\|^2] \mathbb{E}[\|Z - Z_n\|^2] + \mathbb{E}[\|Z - Z_n\|^2]} \right] \\
\leq \|g\|_{M^2} \left( 901n^{-1/2} + 112n^{-1/2} \sqrt{\log n} \right),
\]
which, together with Theorem 6.2 gives the desired result.

\[\blacksquare\]

**Remark 6.5.** The representation of \(Z\) in terms of two independent Brownian Motions comes from a careful analysis of the limiting covariance of \(A_n\). Indeed, (6.16) provides an explicit derivation of the covariance, which converges to the covariance of \(Z\).

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