A New Exponential Factor-Type Estimator for Population Distribution Function Using Dual Auxiliary Variables under Stratified Random Sampling

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In this paper, we propose a generalized class of exponential-type estimators for estimating the finite population distribution function using dual auxiliary variables under stratified sampling. The biases and mean squared errors (MSEs) of the proposed class of estimators are derived up to the first order of approximation. The empirical and theoretical study of comparisons is discussed. Four populations are taken for the support of the theoretical findings. It is observed that the proposed class of estimators performs better as compared to all other considered estimators in stratified sampling.

1. Introduction

In survey sampling, the auxiliary information is often used to increase the precision of an estimator of population parameter(s), such as population mean, median, distribution function, quantiles, and standard deviation, etc., exist in the literature, which need single or two auxiliary information.

Our primary goal is to enhance the precision of the estimator; for this reason, we use stratified random sampling. If the population of interest is homogeneous, then simple random sampling performs good. But there is a situation when the population of interest is heterogeneous, in such situation, it is advisable to use the stratified random sampling instead of simple random sampling. In stratified random sampling, we split the whole aggregate into number of nonoverlapping groups or subgroups called strata. These groups are homogeneous entirely and sample is drawn independently from each stratum separately. To obtain the maximum benefit from stratification, the values of the Nh must be known. When the strata have been determined, a sample is drawn from each, and the drawings being made independently. In stratified sampling, every stratum is handled as separate population, and consequently samples are drawn independently from every stratum.

In other words, if SRS is used in each stratum for the selection of the sample, then the corresponding sample is called a stratified random sample. For good stratification, it requires that each stratum should be internally homogeneous but should externally differ from one another. Stratification may often produce gains in the precision of estimates. In stratified random sampling, the given population is divided into several strata. Then, from each stratum, a simple random sample is selected depending upon the size of the stratum. Estimators are first drawn from each stratum and then combined into a precise estimate of the population parameter.
A lot of work has been done on the estimation of the population mean. Some important references on the population mean estimation using auxiliary information include Diana [1], Kadilar and Cingi [2, 3], Shabbir and Gupta [4], Shabbir and Gupta [5], Haq and Shabbir [6], Aladag and Cingi [7], Singh and Khalid [8], Malik and Singh [9], Muneer et al. [10], Shabbir and Gupta [11], Haq et al. [12], Kaur et al. [13], Ahmad and Shabbir [14], Singh and Khalid [15], Al-Marzouki et al. [16], Ahmad et al. [17], Ahmad et al. [18], and Ahmad et al. [19]. In these works, the authors have suggested improved ratio, product, and regression-type estimators for estimating the finite population mean. They introduced estimators which used the auxiliary information to estimate the population mean and total.

In the literature of sampling, the authors have estimated the DF using information on one or more auxiliary variable. Chambers and Dunstan [20] suggested an estimator for estimating the DF that requires information both on the study and auxiliary variables. Similarly, Rao et al. [21] and Rao [22] suggested ratio and difference/regression estimators for estimating the DF under a general sampling design. Kuk [23] suggested a kernel method for estimating the DF using the auxiliary information. Ahmed and Abu-Dayyeh [24] estimated the DF using information on multiple auxiliary variables. A calibration approach was used by Rueda et al. [25] to devise an estimator for estimating the DF. Singh et al. [26] considered the problem of estimating the DF and quantiles with the use of auxiliary information at the estimation stage of a survey. Moreover, Yaqub and Shabbir [27], Hussain et al. [28], and Hussain et al. [29] considered a generalized class of estimators for estimating the DF in the presence of non-response, while Hussain et al. [30] proposed two new families of estimators using dual auxiliary information under simple and stratified random sampling. Furthermore, Ahmad et al. [31] suggested a new estimator of DF using auxiliary information.

In this paper, we propose a new estimator for estimating the DF using information on the distribution function and mean of the auxiliary variable. The biases and mean squared errors (MSEs) of the existing and proposed estimators of the DF are derived under the first order of approximation. From theoretical and numerical comparisons, we can say that the proposed estimator is more precise than the existing adapted estimators when estimating the DF.

The rest of the paper is organized as follows. In Section 2, some notations are given. In Section 3, some existing estimators of the finite population mean for estimating the finite DF are studied. The proposed estimator is given in Section 4. In Sections 5 and 6, theoretical and numerical comparisons are made, respectively. In Section 7, interpretation of the results in tables is deliberated. Finally, conclusions are drawn in Section 8.

2. Notation

Consider a finite population \( \Omega = \{1, 2, \ldots, N\} \) of \( N \) distinct units, which is divided into \( L \) homogeneous strata, where the size of the \( h \)th stratum is \( N_h \), for \( h = 1, 2, \ldots, L \) such that \( \sum_{h=1}^{L} N_h = N \). Let \( Y \) and \( X \) be the study and auxiliary variables which take values \( y_k \) and \( x_{k,h} \), respectively, where \( i = 1, 2, \ldots, N_h \) and \( h = 1, 2, \ldots, L \); for estimating finite population distribution function, assume that a sample of size \( n_h \) is drawn from the \( h \)th stratum using simple random sampling without replacement, such that \( \sum_{h=1}^{L} n_h = N \), where \( n \) is the sample size.

\( Y \): the study variable,
\( X \): the auxiliary variable.

Let

\[
U_i = I(Y_{ih} \leq t_{yh})
\]

\[
= \begin{cases} 
1 & \text{if } Y_{ih} \leq t_{yh}, \\
0 & \text{if } Y_{ih} > t_{yh},
\end{cases}
\]

and

\[
V_i = I(X_{ih} \leq t_{xh}) = \begin{cases} 
1 & \text{if } X_{ih} \leq t_{xh}, \\
0 & \text{if } X_{ih} > t_{xh}.
\end{cases}
\]

\( I(Y_{ih} \leq t_{yh}) \): indicator variable based on \( Y \),
\( I(X_{ih} \leq t_{xh}) \): indicator variable based on \( X \),
\( \overline{U}_h = \sum_{i=1}^{n_h} U_{ih}/n_h = \sum_{i=1}^{N_h} I(Y_{ih} \leq t_{yh})/N_h \): the population distribution function of \( Y \) for the \( h \)th stratum,

\( \overline{V}_h = \sum_{i=1}^{n_h} V_{ih}/n_h = \sum_{i=1}^{N_h} I(X_{ih} \leq t_{xh})/n_h \): the population distribution function of \( X \) for the \( h \)th stratum,

\( \overline{X}_h = \sum_{i=1}^{n_h} X_{ih}/n_h \): the population mean of \( X \) for the \( h \)th stratum,

\( \overline{Y}_h = \sum_{i=1}^{n_h} Y_{ih}/n_h \): the population mean of \( Y \) for the \( h \)th stratum.

In this paper, we propose a new estimator of DF using information on the distribution function and mean of the auxiliary variable. The biases and mean squared errors (MSEs) of the existing and proposed estimators of the DF are derived under the first order of approximation. From theoretical and numerical comparisons, we can say that the proposed estimator is more precise than the existing adapted estimators when estimating the DF.
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\[ S_{xvh} = \sum_{i=1}^{N_h} [(U_{ih} - \bar{U}_h)(V_{ih} - \bar{V}_h)]/(N_h - 1) \]: the population covariance between \( U \) and \( V \), for the \( h \)th stratum,

\[ S_{xvh} = \sum_{i=1}^{N_h} [(U_{ih} - \bar{U}_h)(X_{ih} - \bar{X}_h)]/(N_h - 1) \]: the population covariance between \( U \) and \( X \), for the \( h \)th stratum,

\[ S_{xvh} = \sum_{i=1}^{N_h} [(V_{ih} - \bar{V}_h)(X_{ih} - \bar{X}_h)]/(N_h - 1) \]: the population covariance between \( V \) and \( X \), for the \( h \)th stratum,

\[ R_{xvh} = S_{xvh}/(S_{uh}S_{vh}) \): the population correlation coefficient between \( U \) and \( V \) for the \( h \)th stratum,

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\[ R_{xvh} = S_{xvh}/(S_{uh}S_{vh}) \): the population correlation coefficient between \( V \) and \( X \) for the \( h \)th stratum,

\[ R_{xvh}^2 = (R_{xuh}^2 + R_{xvh}^2 - 2R_{xuh}R_{xvh}/(1 - R_{xvh}^2)) \]: multiple correlation coefficient of \( U \) on \( V \) and \( X \).

In order to obtain the biases and mean squared errors (MSEs) of the adapted and proposed estimators of \( F(y) \), we consider the following relative error terms.

Let \( e_i = \bar{U} - \bar{U}/\bar{U} \), \( e_1 = \bar{V} - \bar{V}/\bar{V} \), and \( e_2 = \bar{X} - \bar{X}/\bar{X} \) such that \( E(e_i) = 0 \) for \( i = 0, 1, 2 \), where \( E(\cdot) \) is the mathematical expectation of (\( \cdot \)). Let \( E(e_i) = 0 \), \( i = 0, 1, 2 \),

\[
\begin{align*}
E(e_{iuh}) &= \lambda_k C_{uh}, \\
E(e_{ih}) &= \lambda_k C_{uh}, \\
E(e_{iuh}) &= \lambda_k C_{uh}, \\
E(e_{iuh}) &= \lambda_k R_{xuh}C_{uh}C_{vh}, \\
E(e_{iuh}) &= \lambda_k R_{xuh}C_{uh}C_{vh}, \\
E(e_{iuh}) &= \lambda_k C_{uh}C_{vh}.
\end{align*}
\]

(3) Murthy [33] suggested the usual product estimator of \( \bar{U} \), which is given by

\[
\bar{U}_{p,h} = \sum_{h=1}^{L} W_h \bar{U}_h \left( \frac{\bar{V}_h}{\bar{V}_h} \right).
\]

The bias and MSE of \( \bar{U}_{p,h} \) to first order of approximation, are given by

\[
\begin{align*}
&\text{Bias(} \bar{U}_{p,h} \text{)} = \sum_{h=1}^{L} W_h \lambda_k \bar{U}_h (R_{xuh}C_{uh}C_{vh}), \\
&\text{MSE(} \bar{U}_{p,h} \text{)} = \sum_{h=1}^{L} W_h^2 \lambda_k^2 \bar{U}_h^2 (C_{uh}^2 + C_{vh}^2 + 2R_{xuh}C_{uh}C_{vh}).
\end{align*}
\]

(4) The conventional difference estimator of \( \bar{U} \) is

\[
\bar{U}_{reg,h} = \sum_{h=1}^{L} W_h \bar{U}_h \left[ 1 + m \left( \bar{V}_h - \bar{V}_h \right) \right],
\]

where \( m \) is an unknown constant. \( \bar{U}_{reg,h} \) is an unbiased estimator of \( \bar{U} \). The simplified minimum variance of \( \bar{U}_{reg,h} \) at the optimum value of \( m_{(\text{opt})} = R_{xuv} \delta_1/\delta_2 \) is

\[
\text{Var}_{\text{min}}(\bar{U}_{reg,h}) = \sum_{h=1}^{L} W_h^2 \lambda_k^2 \bar{U}_h^2 [C_{uh}^2 - C_{uh}^2 - C_{vh}^2].
\]

(5) Rao [37] suggested an improved difference-type estimator of \( \bar{U} \), which is given by
\[ \hat{U}_{R,D,h} = \sum_{h=1}^{L} W_k \hat{U}_h \left[ m_1 + m_2 \left( \nabla_h - \hat{\nabla}_h \right) \right], \]  

where \( m_1 \) and \( m_2 \) are unknown constants. The bias and MSE of \( \hat{U}_{R,D,h} \) to the first order of approximation, respectively, are

\[
\text{Bias} \left( \hat{U}_{R,D} \right) = \sum_{h=1}^{L} W_k \lambda h \left[ (m_1 - 1) + C \exp \left( \frac{\lambda h C}{1 + C} \right) \right],
\]

\[
\text{MSE} \left( \hat{U}_{R,D} \right) = \sum_{h=1}^{L} W_k^2 \lambda h^2 \left[ \left( (m_1 - 1)^2 + C^2 m_1^2 \right) + \left( \frac{\lambda h}{1 + C} \right) \right]
\]

\[- \sum_{h=1}^{L} W_k^2 \lambda h \left[ 2 \left( \nabla_h R_{u,v} C \right) \right]. \]

The optimum values of \( m_1 \) and \( m_2 \) are

\[
m_{1(\text{opt})} = \frac{1}{1 + \lambda h C^2 (1 - R^2_{u,v})} \quad \text{and} \quad m_{2(\text{opt})} = \frac{L}{1 + \lambda h C^2 (1 - R^2_{u,v})} \]

The biases and MSEs of \( \hat{U}_{BT,R,h} \) and \( \hat{U}_{BT,P,h} \), to first order of approximation, respectively, are

\[
\text{Bias} \left( \hat{U}_{BT,R} \right) = \sum_{h=1}^{L} W_k \lambda h \left[ \frac{3 C^2}{8} - \left( \frac{R_{u,v} C}{2} \right) \right],
\]

\[
\text{MSE} \left( \hat{U}_{BT,R} \right) = \sum_{h=1}^{L} W_k^2 \lambda h^2 \left[ \frac{1}{4} \left( 4C^2 - 2R_{u,v} C \right) \right],
\]

\[
\text{Bias} \left( \hat{U}_{BT,P} \right) = \sum_{h=1}^{L} W_k \lambda h \left[ \frac{R_{u,v} C}{2} - \frac{C^2}{8} \right],
\]

\[
\text{MSE} \left( \hat{U}_{BT,P} \right) = \sum_{h=1}^{L} W_k^2 \lambda h^2 \left[ \frac{1}{4} \left( 4C^2 + 2R_{u,v} C \right) \right].
\]

(7) Grover and Kaur [35] suggested a generalized class of ratio-type exponential estimators, which is given by

\[
\hat{U}_{G,K,h} = \sum_{h=1}^{L} W_k \left[ m_3 + m_4 \left( \nabla_h - \hat{\nabla}_h \right) \right] \exp \left( \frac{\nabla_h - \hat{\nabla}_h}{\nabla_h + \hat{\nabla}_h} \right),
\]

where \( m_3 \) and \( m_4 \) are unknown constants. The bias and MSE of \( \hat{U}_{G,K,h} \) to the first order of approximation, respectively, are

\[
\text{Bias} \left( \hat{U}_{G,K} \right) = \sum_{h=1}^{L} W_k \lambda h \left[ 1 - m_3 + \frac{3}{8} m_3^2 C^2 + \frac{1}{2} m_4 \left( \nabla C^2 \right) \right]
\]

\[- \sum_{h=1}^{L} W_k^2 \lambda h \left[ \frac{1}{2} m_3 \left( \nabla C^2 \right) \right]. \]

The optimum values of \( m_3 \) and \( m_4 \) determined by minimizing (24) are
values of \( m_3 \) and \( m_4 \) is given by

\[
\text{MSE}_{\text{min}} \left( \widehat{U}_{G,K,h} \right) = \sum_{h=1}^{L} W_h^2 \lambda_h^2 \left[ \frac{\{64C_{nh}^2 (1 - R_{12}^2) - \lambda_h^4 C_{nh}^4 - 16\lambda_h^2 C_{nh}^2 c_{nh} (1 - R_{12}^2)\}}{64\{1 + \lambda_h C_{nh} (1 - R_{12}^2)\}} \right].
\]  

(27)

**4. Proposed Class of Estimators**

The precision of an estimator surges by using the appropriate secondary information at the estimation stage. In previous studies, the sample distribution function of the auxiliary variable was used to expand the productivities of the prevailing distribution function estimators. In a recent study, Hussain et al. [30] recommended to use ranks of the auxiliary variable as an additional auxiliary variable to increase the precision of an estimator of the population distribution function. Similarly, we use additional auxiliary information on sample mean and sample distribution function of the auxiliary variable along with the sample distribution function of study variable to estimate the finite CDF.

Using the above idea on the lines of Shukla et al. [36], we suggest a general class of exponential factor-type estimators which contains many stable and efficient estimators. By combining the idea of Bahl and Tuteja and Shukla et al. [34, 36], the first estimator is given by

\[
\widehat{U}_{\text{prop}_1} = \sum_{h=1}^{L} W_h \widehat{U}_h \left[ \exp \left( \frac{S_{ih} - M_{ih}}{S_{ih} + M_{ih}} \right) \exp \left( \frac{S_{2ih} - M_{2ih}}{S_{2ih} + M_{2ih}} \right) \right].
\]

(28)

\[
\widehat{U}_{\text{prop}_1} = \sum_{h=1}^{L} W_h \widehat{U}_h \left[ \left(1 + e_{ih}\right) \left(1 + \frac{1}{2} \theta_{ih} e_{ih} - \frac{1}{4} \theta_{ih}^2 \theta_{2ih} e_{ih}^2 + \frac{1}{8} \theta_{ih}^2 e_{ih}^2 + \cdots \right) \right],
\]

(30)

where
Table 1: Some members of the proposed class of family of estimators $\hat{\mathcal{U}}_{\text{prop}}$

| S.No | $K_{1h}$ | $K_{2h}$ | Estimators |
|------|--------|--------|------------|
| 1    | 1      | 1      | $\hat{\mathcal{U}}_{\text{prop}1h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 2    | 1      | 2      | $\hat{\mathcal{U}}_{\text{prop}2h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 3    | 1      | 3      | $\hat{\mathcal{U}}_{\text{prop}3h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 4    | 1      | 4      | $\hat{\mathcal{U}}_{\text{prop}4h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 5    | 2      | 1      | $\hat{\mathcal{U}}_{\text{prop}5h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 6    | 2      | 2      | $\hat{\mathcal{U}}_{\text{prop}6h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 7    | 2      | 3      | $\hat{\mathcal{U}}_{\text{prop}7h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 8    | 2      | 4      | $\hat{\mathcal{U}}_{\text{prop}8h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{V_h - \hat{\mathcal{V}}_h}{V_h + \hat{\mathcal{V}}_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 9    | 3      | 1      | $\hat{\mathcal{U}}_{\text{prop}9h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 10   | 3      | 2      | $\hat{\mathcal{U}}_{\text{prop}10h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 11   | 3      | 3      | $\hat{\mathcal{U}}_{\text{prop}11h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 12   | 3      | 4      | $\hat{\mathcal{U}}_{\text{prop}12h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 13   | 4      | 1      | $\hat{\mathcal{U}}_{\text{prop}13h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 14   | 4      | 2      | $\hat{\mathcal{U}}_{\text{prop}14h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 15   | 4      | 3      | $\hat{\mathcal{U}}_{\text{prop}15h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |
| 16   | 4      | 4      | $\hat{\mathcal{U}}_{\text{prop}16h} = \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \exp \left( \frac{X_h - X_h}{X_h + X_h} \right) \exp \left( \frac{X_h - X_h}{X_h + X_h} \right)$ |

\[
\theta_{1h} = \frac{f_h B_{1h} - C_{1h}}{A_{1h} + f_h B_{1h} + C_{1h}} \quad \text{(31)}
\]

\[
\Theta_{1h} = \frac{f_h B_{1h} + C_{1h}}{A_{1h} + f_h B_{1h} + C_{1h}}
\]

\[
\theta_{2h} = \frac{f_h B_{2h} - C_{2h}}{A_{2h} + f_h B_{2h} + C_{2h}} \quad \text{(32)}
\]

\[
\Theta_{2h} = \frac{f_h B_{2h} + C_{2h}}{A_{2h} + f_h B_{2h} + C_{2h}}
\]

\[
\left( \hat{\mathcal{U}}_{\text{prop}} - \mathcal{U}_h \right) \equiv \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \left[ e_{0h} + \frac{1}{2} \theta_{1h} e_{1h} + \frac{1}{4} \theta_{2h} e_{2h} + \frac{1}{2} \theta_{1h} e_{1h} + \frac{1}{4} \theta_{2h} e_{2h} \right]
\]

\[
\left( \hat{\mathcal{U}}_{\text{prop}} - \mathcal{U}_h \right) \equiv \sum_{h=1}^{L} W_h \hat{\mathcal{U}}_h \left[ e_{0h} + \frac{1}{4} \Theta_{1h} e_{1h} + \frac{1}{8} \Theta_{2h} e_{2h} + \frac{1}{8} \Theta_{1h} e_{1h} + \frac{1}{8} \Theta_{2h} e_{2h} \right]
\]

To first-order approximation, we have
Taking squaring and expectation of (33) to first order of approximation, we get the bias and MSE:

\[
\text{Bias}(\hat{\theta}_{\text{prop}}) \approx \sum_{i=1}^{L} W_h \theta_h \left[ \frac{1}{2} \theta_{1h} R_{uh} C_{uh} C_{vh} + \frac{1}{2} \theta_{2h} R_{uh} C_{uh} C_{xh} \right],
\]

(34)

\[
\text{MSE}(\hat{\theta}_{\text{prop}}) \approx \sum_{i=1}^{L} W_h \theta_h \left[ C_{uh}^2 + \frac{1}{4} \theta_{1h}^2 C_{uh}^2 + \frac{1}{4} \theta_{2h}^2 C_{xh}^2 + \theta_{1h} R_{uh} C_{uh} C_{xh} \right].
\]

(35)

Differentiate (35) with respect to \( \theta_{1h} \) and \( \theta_{2h} \), and we get the optimum values of \( \theta_{1h} \) and \( \theta_{2h} \), i.e.,

\[
\theta_{1h(\text{opt})} = \frac{2C_{uh} (R_{uh} - R_{xuh} R_{xuhh})}{C_{vh} (R_{xuhh} - 1)}
\]

and

\[
\theta_{2h(\text{opt})} = \frac{2C_{uh} (R_{xuhh} - R_{xuh} R_{xuh})}{C_{xh} (R_{xuhh} - 1)}.
\]

(36)

Substituting the optimum values of \( \theta_{1h(\text{opt})} \) and \( \theta_{2h(\text{opt})} \) in (35), we get minimum MSE of \( \hat{\theta}_{\text{prop}} \) which is given by

\[
\text{MSE}_{\text{min}}(\hat{\theta}_{\text{prop}}) \approx \sum_{i=1}^{L} W_h \theta_h \left[ 1 - R_{xuhh}^2 \right].
\]

(37)

where

\[
\text{Bias}(\hat{\theta}_{\text{prop1h}}) \approx \sum_{h=1}^{L} W_h \theta_h \left[ \frac{3}{8} C_{vh}^2 + \frac{3}{8} C_{xh}^2 - \frac{3}{2} R_{vh} C_{uh} C_{vh} - \frac{1}{2} R_{xuh} C_{uh} C_{xh} \right]
\]

(40)

\[
\text{MSE}(\hat{\theta}_{\text{prop1h}}) \approx \sum_{h=1}^{L} W_h \theta_h \left[ C_{uh}^2 + \frac{1}{4} C_{vh}^2 + \frac{1}{4} C_{xh}^2 - R_{vh} C_{uh} C_{vh} - R_{xuh} C_{uh} C_{xh} \right].
\]

(41)

(2) For \( K_{1h} = 1 \) and \( K_{2h} = 2 \),

\[
\hat{\theta}_{\text{prop2h}} = \sum_{h=1}^{L} W_h \theta_h \left[ \frac{\nabla h - \nabla h}{\nabla h + \nabla h} \right] \exp \left( \frac{\nabla h - \nabla h}{\nabla h + \nabla h} \right).
\]

(42)

The bias and MSE of \( \hat{\theta}_{\text{prop2h}} \) are given by

\[
\text{Bias}(\hat{\theta}_{\text{prop2h}}) \approx \sum_{h=1}^{L} W_h \theta_h \left[ -\frac{1}{8} C_{vh}^2 - \frac{1}{8} C_{xh}^2 - \frac{1}{2} R_{vh} C_{uh} C_{vh} \right].
\]

(43)
\[
\text{MSE}(\hat{U}_{\text{prop2}}) \equiv \sum_{h=1}^{L} \frac{1}{\lambda_h} \hat{U}_h^T \left[ C_{uh}^2 + \frac{1}{4} C_{vh}^2 + \frac{1}{4} C_{xh}^2 - \frac{1}{2} R_{uvh} C_{uh} C_{vh} + \frac{1}{2} R_{uxh} C_{uh} C_{xh} \right].
\]

(3) For \( K_{1h} = 1 \) and \( K_{2h} = 3 \),
\[
\hat{U}_{\text{prop3h}} = \sum_{h=1}^{L} W_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{n(X_h - \bar{X}_h)}{2N\bar{X}_h - n(\bar{X}_h + \bar{X}_h)} \right).
\]

The bias and MSE of \( \hat{U}_{\text{prop3h}} \) are given by
\[
\text{Bias}(\hat{U}_{\text{prop3h}}) \equiv \sum_{h=1}^{L} W_h \lambda_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{n(X_h - \bar{X}_h)}{2N\bar{X}_h - n(\bar{X}_h + \bar{X}_h)} \right) \left[ \frac{3}{8} C_{vh}^2 + \frac{1}{8} C_{xh}^2 - \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \right]^2 C_{xh}^2 - \frac{1}{2} R_{uvh} C_{uh} C_{vh} \right].
\]

(4) For \( K_{1h} = 2 \) and \( K_{2h} = 1 \),
\[
\hat{U}_{\text{prop5h}} = \sum_{h=1}^{L} W_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{X_h - \bar{X}_h}{X_h + \bar{X}_h} \right).
\]

The bias and MSE of \( \hat{U}_{\text{prop5h}} \) are given by
\[
\text{Bias}(\hat{U}_{\text{prop5h}}) \equiv \sum_{h=1}^{L} W_h \lambda_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{X_h - \bar{X}_h}{X_h + \bar{X}_h} \right) \left[ \frac{1}{8} C_{vh}^2 + \frac{3}{8} C_{xh}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \right]^2 C_{xh}^2 - \frac{1}{2} R_{uvh} C_{uh} C_{vh} \right].
\]

(5) For \( K_{1h} = 2 \) and \( K_{2h} = 2 \),
\[
\hat{U}_{\text{prop6h}} = \sum_{h=1}^{L} W_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{X_h - \bar{X}_h}{X_h + \bar{X}_h} \right).
\]

The bias and MSE of \( \hat{U}_{\text{prop6h}} \) are given by
\[
\text{Bias}(\hat{U}_{\text{prop6h}}) \equiv \sum_{h=1}^{L} W_h \lambda_h \hat{U}_h \exp\left( \frac{V_h - \bar{V}_h}{V_h + \bar{V}_h} \right) \exp\left( \frac{X_h - \bar{X}_h}{X_h + \bar{X}_h} \right) \left[ \frac{1}{8} C_{vh}^2 + \frac{3}{8} C_{xh}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \right]^2 C_{xh}^2 - \frac{1}{2} R_{uvh} C_{uh} C_{vh} \right].
\]
\[
\text{Bias}(\hat{U}_{\text{prop6}}) = \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ -\frac{1}{8} C_{2, h} - \frac{1}{8} C_{2, \bar{h}} + \frac{1}{2} \Delta_{y, h} C_{\text{ab}} C_{\text{rv}} + \frac{1}{2} R_{\text{uxh}} C_{\text{ab}} C_{\text{rvh}} \right], \tag{52}
\]

\[
\text{MSE}(\hat{U}_{\text{prop6}}) = \sum_{h=1}^{L} W_h^2 \lambda_h \mathcal{U}^2_h \left[ C_{2, \text{ab}}^2 + \frac{1}{4} C_{2, \text{rv}} + \frac{1}{4} C_{2, \text{rvh}} + R_{\text{uxh}} C_{\text{ab}} C_{\text{rv}} + R_{\text{uxh}} C_{\text{ab}} C_{\text{rvh}} + \frac{1}{2} R_{\text{uxh}} C_{\text{rvh}} C_{\text{rvh}} \right]. \tag{53}
\]

(6) For \(K_{1h} = 2\) and \(K_{2h} = 3\),

\[
\hat{U}_{\text{prop7}} = \sum_{h=1}^{L} W_h \mathcal{U}_h \exp\left( \bar{n} (\bar{X}_h - \bar{X}_h) \right) \cdot \left( \frac{n(\bar{X}_h - \bar{X}_h)}{2N\bar{X}_h - n(\bar{X}_h + \bar{X}_h)} \right). \tag{54}
\]

\[
\text{Bias}(\hat{U}_{\text{prop7}}) = \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ -\frac{1}{8} C_{2, h} - \frac{1}{8} \left( \frac{f_h}{1 - f_h} \right) C_{2, \bar{h}} + \frac{1}{2} \Delta_{y, h} C_{\text{ab}} C_{\text{rv}} \right], \tag{55}
\]

\[
\text{MSE}(\hat{U}_{\text{prop7}}) = \sum_{h=1}^{L} W_h^2 \lambda_h \mathcal{U}^2_h \left[ C_{2, \text{ab}}^2 + \frac{1}{4} C_{2, \text{rv}} + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{2, \text{rvh}} + R_{\text{uxh}} C_{\text{ab}} C_{\text{rv}} \right]. \tag{56}
\]

(7) For \(K_{1h} = 3\) and \(K_{2h} = 1\),

\[
\hat{U}_{\text{prop9}} = \sum_{h=1}^{L} W_h \mathcal{U}_h \exp\left( \bar{n} (\bar{X}_h - \bar{X}_h) \right) \cdot \left( \frac{n(\bar{X}_h - \bar{X}_h)}{2N\bar{X}_h - n(\bar{X}_h + \bar{X}_h)} \right). \tag{57}
\]

The bias and MSE of \(\hat{U}_{\text{prop9}}\) are given by

\[
\text{Bias}(\hat{U}_{\text{prop9}}) = \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ -\frac{1}{8} \left( \frac{f_h}{1 - f_h} \right)^2 C_{2, h} + \frac{3}{8} C_{2, \bar{h}} + \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) R_{\text{uxh}} C_{\text{ab}} C_{\text{rv}} \right], \tag{58}
\]

\[
\text{MSE}(\hat{U}_{\text{prop9}}) = \sum_{h=1}^{L} W_h^2 \lambda_h \mathcal{U}^2_h \left[ C_{2, \text{ab}}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{2, \text{rv}} + \frac{1}{4} C_{2, \text{rvh}} - \left( \frac{f_h}{1 - f_h} \right) R_{\text{uxh}} C_{\text{ab}} C_{\text{rv}} \right]. \tag{59}
\]
\( (8) \) For \( K_{1h} = 3 \) and \( K_{2h} = 2, \)

\[
\begin{align*}
\tilde{U}_{\text{prop10h}} &= \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{n(\nabla_h - \overline{\nabla}_h)}{2N\nabla_h - n(\overline{\nabla}_h + \overline{\nabla}_h)} \right) \exp \left( \frac{\overline{X}_h - \overline{X}_h}{\overline{X}_h + \overline{X}_h} \right).
\end{align*}
\]

The bias and MSE of \( \tilde{U}_{\text{prop10h}} \) are given by

\[
\text{Bias}(\tilde{U}_{\text{prop10h}}) = \sum_{h=1}^{L} W_h h \tilde{U}_h \left[ \frac{1}{8} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{vh}}^2 - \frac{1}{8} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{xh}}^2 - \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) R_{uvh} C_{\text{vwh}} C_{\text{vwh}} \right],
\]

\[
\text{MSE}(\tilde{U}_{\text{prop10h}}) = \sum_{h=1}^{L} W_h h \tilde{U}_h ^2 \left[ C_{\text{vwh}}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{vwh}}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{xh}}^2 - \left( \frac{f_h}{1 - f_h} \right) R_{uvh} C_{\text{vwh}} C_{\text{vwh}} \right].
\]

\( (9) \) For \( K_{1h} = 3 \) and \( K_{2h} = 3, \)

\[
\tilde{U}_{\text{prop11h}} = \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{n(\nabla_h - \overline{\nabla}_h)}{2N\nabla_h - n(\overline{\nabla}_h + \overline{\nabla}_h)} \right) \exp \left( \frac{n(\overline{X}_h - \overline{X}_h)}{2N\overline{X}_h - n(\overline{X}_h + \overline{X}_h)} \right).
\]

The bias and MSE of \( \tilde{U}_{\text{prop11h}} \) are given by

\[
\text{Bias}(\tilde{U}_{\text{prop11h}}) = \sum_{h=1}^{L} W_h h \tilde{U}_h \left[ \frac{1}{8} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{vwh}}^2 - \frac{1}{8} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{xh}}^2 - \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) R_{uvh} C_{\text{vwh}} C_{\text{vwh}} \right],
\]

\[
\text{MSE}(\tilde{U}_{\text{prop11h}}) = \sum_{h=1}^{L} W_h h \tilde{U}_h ^2 \left[ C_{\text{vwh}}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{vwh}}^2 + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right)^2 C_{\text{xh}}^2 - \left( \frac{f_h}{1 - f_h} \right) R_{uvh} C_{\text{vwh}} C_{\text{vwh}} \right].
\]

\( (10) \) For \( K_{1h} = 3 \) and \( K_{2h} = 4, \)

\[
\tilde{U}_{\text{prop12h}} = \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{n(\nabla_h - \overline{\nabla}_h)}{2N\nabla_h - n(\overline{\nabla}_h + \overline{\nabla}_h)} \right).
\]
The bias and MSE of $\tilde{U}_{\text{prop}12h}$ are given by

$$
\text{Bias}(\tilde{U}_{\text{prop}12h}) \equiv \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ \frac{1}{8} \left( \frac{f_h}{1-f_h} \right)^2 C_{x_h}^2 - \frac{1}{2} \left( \frac{f_h}{1-f_h} \right) R_{x_h} C_{x_h} \right],
$$

$$
\text{MSE}(\tilde{U}_{\text{prop}12h}) \equiv \sum_{h=1}^{L} W_h^2 \lambda_h^2 \mathcal{U}_h^2 \left[ C_{x_h}^2 + \frac{1}{4} \left( \frac{f_h}{1-f_h} \right)^2 C_{x_h}^2 - \left( \frac{f_h}{1-f_h} \right) R_{x_h} C_{x_h} \right].
$$

(11) For $K_{1h} = 4$ and $K_{2h} = 1$,

$$
\tilde{U}_{\text{prop}13h} = \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{X_h - \tilde{X}_h}{X_h + \tilde{X}_h} \right).
$$

The bias and MSE of $\tilde{U}_{\text{prop}13h}$ are given by

$$
\text{Bias}(\tilde{U}_{\text{prop}13h}) \equiv \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ \frac{1}{2} C_{x_h}^2 - \frac{1}{2} R_{x_h} C_{x_h} \right],
$$

$$
\text{MSE}(\tilde{U}_{\text{prop}13h}) \equiv \sum_{h=1}^{L} W_h^2 \lambda_h^2 \mathcal{U}_h^2 \left[ C_{x_h}^2 + \frac{1}{2} C_{x_h}^2 - R_{x_h} C_{x_h} \right].
$$

(12) For $K_{1h} = 4$ and $K_{2h} = 2$,

$$
\tilde{U}_{\text{prop}14h} = \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{\tilde{X}_h - \tilde{X}_h}{\tilde{X}_h + \tilde{X}_h} \right).
$$

The bias and MSE of $\tilde{U}_{\text{prop}14h}$ are given by

$$
\text{Bias}(\tilde{U}_{\text{prop}14h}) \equiv \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ \frac{1}{2} C_{x_h}^2 + \frac{1}{2} R_{x_h} C_{x_h} \right],
$$

$$
\text{MSE}(\tilde{U}_{\text{prop}14h}) \equiv \sum_{h=1}^{L} W_h^2 \lambda_h^2 \mathcal{U}_h^2 \left[ C_{x_h}^2 + \frac{1}{2} C_{x_h}^2 + R_{x_h} C_{x_h} \right].
$$

(13) For $K_{1h} = 4$ and $K_{2h} = 3$,

$$
\tilde{U}_{\text{prop}15h} = \sum_{h=1}^{L} W_h \tilde{U}_h \exp \left( \frac{n(X_h - \tilde{X}_h)}{2N_X - n(\tilde{X}_h + \tilde{X}_h)} \right).
$$

The bias and MSE of $\tilde{U}_{\text{prop}15h}$ are given by

$$
\text{Bias}(\tilde{U}_{\text{prop}15h}) \equiv \sum_{h=1}^{L} W_h \lambda_h \mathcal{U}_h \left[ -\frac{1}{8} \left( \frac{f_h}{1-f_h} \right)^2 C_{x_h}^2 - \frac{1}{2} \left( \frac{f_h}{1-f_h} \right) R_{x_h} C_{x_h} \right],
$$

$$
\text{MSE}(\tilde{U}_{\text{prop}15h}) \equiv \sum_{h=1}^{L} W_h^2 \lambda_h^2 \mathcal{U}_h^2 \left[ C_{x_h}^2 + \frac{1}{4} \left( \frac{f_h}{1-f_h} \right)^2 C_{x_h}^2 - \left( \frac{f_h}{1-f_h} \right) R_{x_h} C_{x_h} \right].
$$

5. Theoretical Comparison

(i) From (5) and (37),

$$
\text{MSE}_{\min}(\tilde{U}_{\text{prop}}) < \text{Var}(\tilde{U}_a) \quad \text{if} \quad \text{Var}(\tilde{U}_a) - \text{MSE}_{\min}(\tilde{U}_{\text{prop}}) > 0.
$$

(ii) From (8) and (37),

$$
\text{MSE}_{\min}(\tilde{U}_{\text{prop}}) < \text{MSE}(\tilde{U}_{h}) \quad \text{if} \quad \text{MSE}(\tilde{U}_{h}) - \text{MSE}_{\min}(\tilde{U}_{\text{prop}}) > 0.
$$

(iii) From (11) and (37),

$$
\text{MSE}(\tilde{U}_{\text{prop}13h}) < \text{MSE}(\tilde{U}_{p,h}) \quad \text{if} \quad \text{MSE}(\tilde{U}_{p,h}) - \text{MSE}(\tilde{U}_{\text{prop}13h}) > 0.
$$

(iv) From (13) and (37),

$$
\text{MSE}_{\min}(\tilde{U}_{\text{prop}15h}) < \text{MSE}(\tilde{U}_{\text{R}_p,h}) \quad \text{if} \quad \text{MSE}(\tilde{U}_{\text{R}_p,h}) - \text{MSE}_{\min}(\tilde{U}_{\text{prop}15h}) > 0.
$$

(v) From (17) and (37),

$$
\text{MSE}(\tilde{U}_{\text{prop}}) < \text{MSE}(\tilde{U}_{p,h}) \quad \text{if} \quad \text{MSE}(\tilde{U}_{p,h}) - \text{MSE}(\tilde{U}_{\text{prop}}) > 0.
$$
6. Empirical Study

In this portion, we conduct a numerical study to judge the performances of the existing and proposed DF estimators. For this purpose, two datasets are taken. The summary statistics of these datasets are reported in Tables 2 and 3. The PRE of an estimator \( \hat{\theta}_{hi} \) with respect to \( \hat{\theta}_{st} \) is

\[
\text{PRE}(\hat{\theta}_{hi}, \hat{\theta}_{st}) = \frac{\text{Var}(\hat{\theta}_{hi})}{\text{MSE}_{\text{min}}(\hat{\theta}_{hi})} \times 100,
\]

where \( i = Rh, Ph, \ldots, Gk, h \).

The PREs of DF estimators, computed from five populations, are given in Tables 4 and 5.

Population 1 (Source: 22)

Y: the number of teachers.
X: the number of students in both primary and secondary schools in Turkey in 2007 for 923 districts in six regions.
Population II (Source: 22)

Y: the number of teachers.
X: the number of classes in both primary and secondary schools in Turkey in 2007 for 923 districts in six regions.

7. Interpretation of Results

As mention above, we used two datasets for numerical illustration. The proposed estimator and the existing estimators were compared between each other with respect to their MSE and PRE values. The results of PREs are presented in Tables 4 and 5. In Tables 2 and 3, we see the summary statistics about the populations. It is further noted that the proposed estimator is more precise than the existing distribution function estimators of Cochran [32], Murthy [33], Rao [37], and Grover and Kaur [38], in terms of MSEs and PREs.

8. Conclusion

In this paper, we proposed an improved class of estimators of finite population DF by utilizing real-life datasets on dual auxiliary variables in stratified random sampling (StRS) scheme. Bias and MSE expressions of a proposed class of estimators $\widehat{U}_{\text{proph}}$ are acquired up to first order of approximation. Based on the theoretical and numerical results, the proposed class of estimators performs better than the existing estimators considered under stratified random sampling. From these findings, we suggest the utilization of the proposed estimators for efficient estimation of population distribution function in the presence of the auxiliary information under stratified random sampling.

Data Availability

All the data used for this study can be found inside the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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