New inequalities for $\eta$-quasiconvex functions*

Eze R. Nwaeze and Delfim F. M. Torres

Abstract The class of $\eta$-quasiconvex functions was introduced in 2016. Here we establish novel inequalities of Ostrowski type for functions whose second derivative, in absolute value raised to the power $q \geq 1$, is $\eta$-quasiconvex. Several interesting inequalities are deduced as special cases. Furthermore, we apply our results to the arithmetic, geometric, Harmonic, logarithmic, generalized log and identric means, getting new relations amongst them.

Keywords: Ostrowski inequality, $\eta$-quasiconvexity, Hölder’s inequality.

2010 MSC: 26D15, 26E60 (Primary); 26A51 (Secondary).

1 Introduction

A function $G : I \to \mathbb{R}$ is said to be convex on the interval $I \subset \mathbb{R}$ if

$$G(xu + (1 - x)v) \leq xG(u) + (1 - x)G(v)$$

holds for all $u, v \in I$ and $x \in [0, 1]$. Many interesting inequalities have been established for convex functions. Worthy of mention is the following result proved in 2011 by Sarikaya and Aktan [8].

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Theorem 1 (See [8]) Let $I \subset \mathbb{R}$ be an open interval, $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, $\lambda \in [0, 1]$, and $G : I \to \mathbb{R}$ be a twice differentiable mapping such that $G''$ is integrable. If $|G''|$ is a convex function on $[\alpha, \beta]$, then

$$
\left| (\lambda - 1)G\left( \frac{\alpha + \beta}{2} \right) - \lambda \frac{G(\alpha) + G(\beta)}{2} + \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx \right| \\
\leq \frac{(\beta - \alpha)^2}{16} \left[ \left( \lambda^4 + (1 + \lambda)(1 - \lambda)^3 + \frac{5(1 - \lambda)}{4} \right) |G''(\alpha)| \right. \\
+ \left( \lambda^4 + (2 - \lambda)(1 - \lambda)^3 + \frac{2(1 - \lambda)}{3} \right) |G''(\beta)| \right], \quad \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{48} \left[ |G''(\alpha)| + |G''(\beta)| \right], \quad \text{if } \frac{1}{2} \leq \lambda \leq 1.
$$

In 2015, Liu obtained a related inequality for $s$-convex functions [7]. The notion of $s$-convexity was introduced in 1994 by Hudzik and Maligranda [5]. Let us recall it here.

Definition 2 (See [5]) A function $G : [0, \infty) \to \mathbb{R}$ is said to be $s$-convex if

$$
G(ux + (1 - x)v) \leq x^s G(u) + (1 - x)^s G(v)
$$

holds for all $u, v \in I$, $x \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Evidently, the notion of $s$-convexity given in Definition 2 generalizes the classical concept of convexity. For this class of functions, Liu [7], among other things, established the following result:

Theorem 3 (See [7]) Let $I \subset [0, \infty)$, $G : I \to \mathbb{R}$ be a twice differentiable function on $I^+$ such that $G'' \in L_1[\alpha, \beta]$, where $\alpha, \beta \in I$ with $\alpha < \beta$. If $|G''|^q$ is $s$-convex on $[\alpha, \beta]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then

$$
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G\left( \frac{\alpha + \beta}{2} \right) - \lambda \frac{G(\alpha) + G(\beta)}{2} \right| \\
\leq \frac{(\beta - \alpha)^2}{16} \left[ \left( \lambda^4 + (1 + \lambda)(1 - \lambda)^3 + \frac{5(1 - \lambda)}{4} \right) |G''(\alpha)| \right. \\
+ \left( \lambda^4 + (2 - \lambda)(1 - \lambda)^3 + \frac{2(1 - \lambda)}{3} \right) |G''(\beta)| \right], \quad \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{48} \left[ |G''(\alpha)| + |G''(\beta)| \right], \quad \text{if } \frac{1}{2} \leq \lambda \leq 1.
$$

$$
= \left\{ \begin{array}{ll}
\frac{(\beta - \alpha)^2}{16} \left[ \left( \lambda^4 + (1 + \lambda)(1 - \lambda)^3 + \frac{5(1 - \lambda)}{4} \right) |G''(\alpha)| \right. \\
+ \left( \lambda^4 + (2 - \lambda)(1 - \lambda)^3 + \frac{2(1 - \lambda)}{3} \right) |G''(\beta)| \right], & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{48} \left[ |G''(\alpha)| + |G''(\beta)| \right], & \text{if } \frac{1}{2} \leq \lambda \leq 1.
\end{array} \right.
$$
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for \( 0 \leq \lambda \leq \frac{1}{2} \) and

\[
\frac{1}{\beta - \alpha} \int_\alpha^\beta G(x) \, dx - (1 - \lambda)G\left(\frac{\alpha + \beta}{2}\right) - \lambda \frac{G(\alpha) + G(\beta)}{2}
\]

\[
\leq \frac{(\beta - \alpha)^2}{16} \left(\lambda - \frac{1}{3}\right)^{\frac{1}{4}} \left\{ \left[ 2(s + 3)\lambda - s - 2 \right] \left| G''\left(\frac{\alpha + \beta}{2}\right)\right|^q \right.
\]

\[
+ \frac{2(s + 3)\lambda - 2}{(s + 1)(s + 2)(s + 3)} \left| G''(\alpha)\right|^q \left. \right\} \quad \quad \left[ + \frac{2(s + 3)\lambda - s - 2}{(s + 2)(s + 3)} \left| G''\left(\frac{\alpha + \beta}{2}\right)\right|^q \right.
\]

\[
+ \frac{2(s + 3)\lambda - 2}{(s + 1)(s + 2)(s + 3)} \left| G''(\beta)\right|^q \}
\]

for \( \frac{1}{2} \leq \lambda \leq 1 \).

In 2016, Eshaghi Gordji et al. [4] proposed a larger class of functions called \( \eta \)-quasiconvex.

**Definition 4 (See [4])** A function \( G : I \subset \mathbb{R} \to \mathbb{R} \) is said to be an \( \eta \)-quasiconvex function with respect to \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), if

\[
G(xu + (1 - x)v) \leq \max \{ G(v), G(v) + \eta(G(u), G(v)) \}
\]

for all \( u, v \in I \) and \( x \in [0, 1] \).

An \( \eta \)-quasiconvex function \( G : [\alpha, \beta] \to \mathbb{R} \) is integrable if \( \eta \) is bounded from above on \( G([\alpha, \beta]) \times G([\alpha, \beta]) \) (see [2, Remark 4]). By taking \( \eta(x, y) = x - y \) in Definition 4, one recovers the classical definition of quasiconvexity. It is also important to note that any convex function is \( \eta \)-quasiconvex with respect to \( \eta(x, y) = x - y \). For some results around this recent class of functions, we invite the interested reader to see [6, 3, 1] and references therein.

Motivated by the above results, it is our purpose to generalize Theorems 1 and 3 for the class of \( \eta \)-quasiconvex functions. To the best of our knowledge, the results we prove here (see Theorems 7 and 10) are novel and provide an interesting contribution to the literature of Ostrowski type results. In addition, we apply our results to some special known means of positive real numbers.

The paper is organized as follows. We begin by recalling in Section 2 two results, needed in the sequel. In Section 3, we formulate and prove our main results, that is, Theorems 7 and 10, followed by several interesting corollaries. Section 4 contains applications of our results to special means, in particular to the arithmetic, geometric, harmonic, logarithmic, the generalized log-mean, and identric means (see Propositions 14, 15 and 16). We end with Section 5 of conclusion.
2 Preliminaries

In this section, we recall two results that will be needed in the proof of our main results.

Lemma 5 (See [7]) Let \( I \subset \mathbb{R} \) and \( G : I \to \mathbb{R} \) be a twice differentiable function on \( I \) such that \( G'' \in L^1[\alpha, \beta] \), where \( \alpha, \beta \in I \) with \( \alpha < \beta \). Then,

\[
\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G\left(\frac{\alpha + \beta}{2}\right) - \lambda G(\alpha) + \frac{G(\beta)}{2} = \frac{(\beta - \alpha)^2}{16} \left[ \int_{0}^{1} (x^2 - 2\lambda x)G'' \left(\frac{x \alpha + \beta}{2} + (1 - x)\alpha\right) \, dx \right]
\]

holds for any \( \lambda \in [0, 1] \).

Lemma 6 (See [8]) Let \( I \subset \mathbb{R} \) and \( G : I \to \mathbb{R} \) be a twice differentiable function on \( I \) such that \( G'' \in L^1[\alpha, \beta] \), where \( \alpha, \beta \in I \) with \( \alpha < \beta \). Then,

\[
\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G\left(\frac{\alpha + \beta}{2}\right) - \lambda G(\alpha) + \frac{G(\beta)}{2} = \frac{(\beta - \alpha)^2}{16} \int_{0}^{1} p(x)G''(x \alpha + (1 - x)\beta) \, dx
\]

holds for any \( \lambda \in [0, 1] \), where

\[
p(x) = \begin{cases} \frac{1}{2}(x - \lambda), & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}(1 - x)(1 - \lambda - x), & \frac{1}{2} \leq x \leq 1. \end{cases}
\]

3 Main results

We now state and prove our first main result.

Theorem 7 Let \( I \subset [0, \infty) \) and \( G : [\alpha, \beta] \subset I \to \mathbb{R} \) be a twice differentiable function on \( (\alpha, \beta) \) with \( \alpha < \beta \). If \( |G''|^{q}, \quad q \geq 1 \), is \( \eta \)-quasiconvex on \( [\alpha, \beta] \) and \( \eta \)-bounded from above on \( |G''|^{q}([\alpha, \beta]) \times |G''|^{q}([\alpha, \beta]) \), then

\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G\left(\frac{\alpha + \beta}{2}\right) - \lambda G(\alpha) + \frac{G(\beta)}{2} \right|
\]
Using Lemma 5, (2), (3) and Hölder’s inequality, one obtains that

\[
\begin{aligned}
&\leq \begin{cases} 
\frac{(\beta - \alpha)^2}{16} (\frac{8\lambda^3 - 3\lambda + 1}{3}) (\mathcal{M}_{q, \eta} + 2), & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{(\beta - \alpha)^2}{16} (\lambda - \frac{1}{2}) (\mathcal{M}_{q, \eta} + 2), & \text{if } \frac{1}{2} \leq \lambda \leq 1,
\end{cases}
\end{aligned}
\]

holds, where

\[
\mathcal{M}_{q, \eta} := \max \left\{ |G''(\alpha)|^q, |G''(\alpha)|^q + \eta \left( \left| G'' \left( \frac{\alpha + \beta}{2} \right) \right|^q, |G''(\alpha)|^q \right) \right\}
\]

and

\[
\mathcal{N}_{q, \eta} := \max \left\{ |G''(\beta)|^q, |G''(\beta)|^q + \eta \left( \left| G'' \left( \frac{\alpha + \beta}{2} \right) \right|^q, |G''(\beta)|^q \right) \right\}.
\]

**Proof.** The hypothesis that function \(|G''|^q, q \geq 1\), is \(\eta\)-quasiconvex on \([\alpha, \beta]\), implies that for \(x \in [0, 1]\) we have

\[
\left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\alpha \right) \right|^q \leq \mathcal{M}_{q, \eta}
\]

(2)

and

\[
\left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\beta \right) \right|^q \leq \mathcal{N}_{q, \eta}.
\]

(3)

Using Lemma 5, (2), (3) and Hölder’s inequality, one obtains that

\[
\begin{aligned}
&\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G \left( \frac{\alpha + \beta}{2} \right) - \lambda \frac{G(\alpha) + G(\beta)}{2} \right| \\
&\leq \frac{(\beta - \alpha)^2}{16} \left[ \int_{0}^{1} |x^2 - 2\lambda x| \left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\alpha \right) \right| \, dx \\
&\quad + \int_{0}^{1} |x^2 - 2\lambda x| \left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\beta \right) \right| \, dx \right] \\
&\leq \frac{(\beta - \alpha)^2}{16} \left[ \left( \int_{0}^{1} |x^2 - 2\lambda x| \, dx \right)^{1 - \frac{1}{q}} \\
&\quad \times \left( \int_{0}^{1} |x^2 - 2\lambda x| \left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\alpha \right) \right|^q \, dx \right)^{\frac{1}{q}} \\
&\quad + \left( \int_{0}^{1} |x^2 - 2\lambda x| \, dx \right)^{1 - \frac{1}{q}} \left( \int_{0}^{1} |x^2 - 2\lambda x| \left| G'' \left( x \frac{\alpha + \beta}{2} + (1-x)\beta \right) \right|^q \, dx \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(\beta - \alpha)^2}{16} \left[ \left( \int_{0}^{1} |x^2 - 2\lambda x| \, dx \right)^{1 - \frac{1}{q}} \left( \int_{0}^{1} |x^2 - 2\lambda x| \mathcal{M}_{q, \eta} \, dx \right)^{\frac{1}{q}} \right].
\end{aligned}
\]
Corollary 8

Let $I \subset [0, \infty)$ and $G : [\alpha, \beta] \subset I \to \mathbb{R}$ be a twice differentiable function on $(\alpha, \beta)$ with $\alpha < \beta$. If $|G''|$ is $\eta$-quasiconvex on $[\alpha, \beta]$ and $\eta$ bounded from above on $|G''|([\alpha, \beta]) \times |G''|([\alpha, \beta])$, then the inequality

$$
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda)G \left( \frac{\alpha + \beta}{2} \right) - \frac{\lambda (G(\alpha) + G(\beta))}{2} \right| 
\leq \begin{cases} 
\frac{(\beta - \alpha)^2}{16} \left( \frac{8\lambda^3 - 3\lambda + 1}{3} \right) (\mathcal{M}_\eta + \mathcal{M}_\eta), & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{(\beta - \alpha)^2}{16} \left( \lambda - \frac{1}{2} \right) (\mathcal{M}_\eta + \mathcal{M}_\eta), & \text{if } \frac{1}{2} \leq \lambda \leq 1,
\end{cases}
$$

holds, where

$$
\mathcal{M}_\eta := \max \left\{ |G''(\alpha)|, |G''(\beta)| + \eta \left( G'' \left( \frac{\alpha + \beta}{2} \right), |G''(\alpha)| \right) \right\},
$$

and

$$
\mathcal{M}_\eta := \max \left\{ |G''(\beta)|, |G''(\beta)| + \eta \left( G'' \left( \frac{\alpha + \beta}{2} \right), |G''(\beta)| \right) \right\}.
$$

To finish the proof, we need to evaluate $\int_{0}^{1} |x^2 - 2\lambda x| \, dx$. For this, we consider two cases.

**Case I:** $0 \leq \lambda \leq \frac{1}{2}$. We get $0 \leq 2\lambda \leq 1$ and

$$
\int_{0}^{1} |x^2 - 2\lambda x| \, dx = \int_{0}^{2\lambda} |x^2 - 2\lambda x| \, dx + \int_{2\lambda}^{1} |x^2 - 2\lambda x| \, dx 
= \int_{0}^{2\lambda} (2\lambda x - x^2) \, dx + \int_{2\lambda}^{1} (x^2 - 2\lambda x) \, dx 
= 8\lambda^3 - 3\lambda + 1.
$$

**Case II:** $\frac{1}{2} \leq \lambda \leq 1$. We get $2\lambda \geq 1$ and $x^2 \leq 2\lambda x^2 \leq 2\lambda x$, because $x \in [0, 1]$. It follows that

$$
\int_{0}^{1} |x^2 - 2\lambda x| \, dx = \int_{0}^{1} (2\lambda x - x^2) \, dx 
= \lambda - \frac{1}{3}.
$$

The desired inequalities are obtained by using (5) and (6) in inequality (4).
Proof. The proof follows by setting \( q = 1 \) in Theorem 7.

Remark 9 By choosing different values of \( \lambda \in [0, 1] \) in the inequality of Corollary 8, we obtain different results for \( \eta \)-quasiconvex functions. For example,

1. for \( \lambda = 0 \), we get a midpoint type inequality:
\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x)\,dx - G \left( \frac{\alpha + \beta}{2} \right) \right| \leq \frac{(\beta - \alpha)^2}{48} (\mathcal{M}_\eta + \mathcal{M}_\eta); \tag{7}
\]

2. for \( \lambda = \frac{1}{3} \), we get a Simpson type inequality:
\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x)\,dx - \frac{2}{3} G \left( \frac{\alpha + \beta}{2} \right) - \frac{G(\alpha) + G(\beta)}{6} \right| \leq \frac{(\beta - \alpha)^2}{162} (\mathcal{M}_\eta + \mathcal{M}_\eta); \tag{8}
\]

3. for \( \lambda = \frac{1}{2} \), we obtain a midpoint-trapezoid type inequality:
\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x)\,dx - \frac{1}{2} G \left( \frac{\alpha + \beta}{2} \right) - \frac{G(\alpha) + G(\beta)}{4} \right| \leq \frac{(\beta - \alpha)^2}{96} (\mathcal{M}_\eta + \mathcal{M}_\eta); \tag{9}
\]

4. for \( \lambda = 1 \), we have a trapezoid type inequality:
\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x)\,dx - \frac{G(\alpha) + G(\beta)}{2} \right| \leq \frac{(\beta - \alpha)^2}{24} (\mathcal{M}_\eta + \mathcal{M}_\eta). \tag{10}
\]

Follows the second main result of our paper.

Theorem 10 Let \( I \subset [0, \infty) \) and \( G : [\alpha, \beta] \subset I \to \mathbb{R} \) be a twice differentiable function on \( (\alpha, \beta) \) with \( \alpha < \beta \). If \( |G''|^q, q \geq 1 \), is \( \eta \)-quasiconvex on \( [\alpha, \beta] \) and \( \eta \) bounded from above on \( |G''|([\alpha, \beta]) \times |G''|([\alpha, \beta]) \), then the inequality
\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x)\,dx - (1 - \lambda) G \left( \frac{\alpha + \beta}{2} \right) - \lambda G(\alpha) + G(\beta) \right| \leq \begin{cases} \frac{(\beta - \alpha)^2}{8} \left( \frac{3\lambda^2 - 3\lambda + 1}{5} \right) \mathcal{U}_{q, \eta}, & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\ \frac{(\beta - \alpha)^2}{8} \left( \frac{3\lambda - 1}{3} \right) \mathcal{U}_{q, \eta}, & \text{if } \frac{1}{2} \leq \lambda \leq 1, \end{cases}
\]
holds, where
\[\mathcal{U}_{q, \eta} := \max \left\{ |G''(\beta)|^q, |G''(\beta)|^q + \eta \left( |G''(\alpha)|^q, |G''(\beta)|^q \right) \right\}.\]

Proof. Since \( |G''|^q \) is \( \eta \)-quasiconvex on \( [\alpha, \beta] \), the inequality
\[
|G''(x\alpha + (1 - x)\beta)|^q \leq \mathcal{U}_{q, \eta} \tag{11}
\]
holds for \( x \in [0, 1] \). From the definition of \( p(x) \) given by (1), we observe that for \( 0 \leq \lambda \leq \frac{1}{2} \) one has

\[
\int_{0}^{1} |p(x)| \, dx = \int_{0}^{\frac{1}{2}} \frac{1}{2} |x - \lambda| \, dx + \int_{\frac{1}{2}}^{1} \frac{1}{2} (1 - x) (1 - \lambda - x) \, dx \\
= \frac{1}{2} \left[ \int_{0}^{\lambda} x (\lambda - x) \, dx + \int_{\lambda}^{1} x (\lambda - x) \, dx + \int_{\frac{1}{2}}^{1 - \lambda} (1 - x) (1 - \lambda - x) \, dx \\
+ \int_{1 - \lambda}^{1} (1 - x) (x - 1 + \lambda) \, dx \right] \\
= \frac{8\lambda^3 - 3\lambda + 1}{24}.
\]

Also, for \( \frac{1}{2} \leq \lambda \leq 1 \), we get

\[
\int_{0}^{1} |p(x)| \, dx = \int_{0}^{\frac{1}{2}} \frac{1}{2} |x - \lambda| \, dx + \int_{\frac{1}{2}}^{1} \frac{1}{2} (1 - x) (1 - \lambda - x) \, dx \\
= \frac{1}{2} \left[ \int_{0}^{\frac{1}{2}} x (\lambda - x) \, dx + \int_{\frac{1}{2}}^{1} (1 - x) (\lambda + x - 1) \, dx \right] \\
= \frac{3\lambda - 1}{24}.
\]

Now, using Lemma 6, the Hölder inequality and (11), we obtain that

\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda) G \left( \frac{\alpha + \beta}{2} \right) - \lambda \frac{G(\alpha) + G(\beta)}{2} \right| \\
\leq (\beta - \alpha)^2 \int_{0}^{1} |p(x)| \left| G''(x \alpha + (1 - x) \beta) \right| \, dx \\
\leq (\beta - \alpha)^2 \left( \int_{0}^{1} |p(x)| \, dx \right)^{\frac{1}{\eta}} \left( \int_{0}^{1} |G''(x \alpha + (1 - x) \beta)|^{\eta} \, dx \right)^{\frac{1}{\eta}} \\
\leq (\beta - \alpha)^2 \mathbb{V}_{q, \eta}^{\frac{1}{\eta}} \int_{0}^{1} |p(x)| \, dx.
\]

We get the intended result by using (12) and (13).

**Corollary 11** Let \( I \subset [0, \infty) \) and \( G : [\alpha, \beta] \subset I \rightarrow \mathbb{R} \) be a twice differentiable function on \( (\alpha, \beta) \) with \( \alpha < \beta \). If \( |G''| \) is \( \eta \)-quasiconvex on \( [\alpha, \beta] \) and \( \eta \) bounded from above on \( |G''|([\alpha, \beta]) \times |G''|([\alpha, \beta]) \), then the inequality

\[
\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(x) \, dx - (1 - \lambda) G \left( \frac{\alpha + \beta}{2} \right) - \lambda \frac{G(\alpha) + G(\beta)}{2} \right|
\]
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\[
\begin{align*}
\left( \frac{\beta - \alpha}{8} \right)^2 \left( \frac{8\lambda^3 - 3\lambda + 1}{3} \right) \eta, & \quad \text{if } 0 \leq \lambda \leq 1/2, \\
\left( \frac{\beta - \alpha}{8} \right)^2 \left( \frac{3\lambda - 1}{3} \right) \eta, & \quad \text{if } 1/2 \leq \lambda \leq 1,
\end{align*}
\]
holds, where 

\[
\eta := \max \left\{ |G''(\beta)|, |G''(\beta)| + \eta \left( |G''(\alpha)|, |G''(\beta)| \right) \right\}.
\]

**Proof.** Let $q = 1$ in Theorem 10.

**Remark 12** Choosing different values of $\lambda \in [0, 1]$, we obtain, from Corollary 11, the succeeding results:

1. for $\lambda = 0$, we get

\[
\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta G(x) dx - G \left( \frac{\alpha + \beta}{2} \right) \right| \leq \left( \frac{\beta - \alpha}{24} \right) \eta; \quad (14)
\]

2. for $\lambda = 1/3$, we obtain

\[
\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta G(x) dx - \frac{2}{3} G \left( \frac{\alpha + \beta}{2} \right) \right| \leq \left( \frac{\beta - \alpha}{81} \right) \eta; \quad (15)
\]

3. for $\lambda = 1/2$, we have

\[
\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta G(x) dx - \frac{1}{2} G \left( \frac{\alpha + \beta}{2} \right) \right| \leq \left( \frac{\beta - \alpha}{48} \right) \eta; \quad (16)
\]

4. for $\lambda = 1$, we get

\[
\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta G(x) dx - \frac{G(\alpha) + G(\beta)}{2} \right| \leq \left( \frac{\beta - \alpha}{12} \right) \eta. \quad (17)
\]

**Remark 13** Let $0 < \alpha < \beta$. By setting $G(x) = \ln x$ with $x \in [\alpha, \beta]$ and $\eta(x, y) = x - y$ in inequalities (14)–(17), one gets [7, Proposition 3].

### 4 Application to special means

In this section, we apply our results to the following special means of arbitrary positive numbers $\mu$ and $\nu$ with $\mu \neq \nu$:

1. the arithmetic mean

\[
A(\mu, \nu) = \frac{\mu + \nu}{2};
\]
2. the geometric mean  
\[ G(\mu, \nu) = \sqrt{\mu \nu}; \]
3. the harmonic mean  
\[ H(\mu, \nu) = \frac{2 \mu \nu}{\mu + \nu}; \]
4. the logarithmic mean  
\[ L(\mu, \nu) = \frac{\nu - \mu}{\ln \nu - \ln \mu}; \]
5. the generalized log-mean  
\[ L_p(\mu, \nu) = \left[ \frac{\nu^{p+1} - \mu^{p+1}}{(p+1)(\nu - \mu)} \right]^{\frac{1}{p}}, \quad p \neq -1, 0; \]
6. the identric mean  
\[ I(\mu, \nu) = \frac{1}{e} \left( \frac{\nu^\nu}{\mu^\mu} \right) \frac{1}{\nu - \mu}. \]

We now state our findings in the following propositions.

**Proposition 14** Let \( \mu \) and \( \nu \) be two positive numbers, \( \mu < \nu \). The following inequalities hold:

1. \[ |L_2^2(\mu, \nu) - A^2(\mu, \nu)| \leq \frac{(\nu - \mu)^2}{12}; \]
2. \[ |L_2^2(\mu, \nu) - \frac{2A^2(\mu, \nu) + A(\mu^2, \nu^2)}{3}| \leq \frac{2(\nu - \mu)^2}{81}; \]
3. \[ |L_2^2(\mu, \nu) - \frac{A^2(\mu, \nu) + A(\mu^2, \nu^2)}{2}| \leq \frac{(\nu - \mu)^2}{24}; \]
4. \[ |L_2^2(\mu, \nu) - A(\mu^2, \nu^2)| \leq \frac{(\nu - \mu)^2}{6}. \]

**Proof.** The desired inequalities follow by employing (7)–(10) to function \( G(x) = x^2 \) defined on the interval \([\mu, \nu]\). In this case, \( G''(x) = 2 \). By taking \( \eta(x, y) = x - y \), we easily see that \( G''(x) \) is \( \eta \)-quasiconvex. Moreover, \( \mathcal{M}_\eta = \mathcal{N}_\eta = 2 \).

**Proposition 15** Let \( \mu \) and \( \nu \) be two positive numbers, \( \mu < \nu \). The following inequalities hold:

1. \[ |A^{-1}(\mu, \nu) - L^{-1}(\mu, \nu)| \leq \frac{(\nu - \mu)^2}{48} \left[ \max \left\{ \frac{2}{\mu^2}, \frac{16}{(\mu + \nu)^2} \right\} + \max \left\{ \frac{2}{\nu^2}, \frac{16}{(\mu + \nu)^2} \right\} \right]; \]
2. \[ |2A^{-1}(\mu, \nu) + H^{-1}(\mu, \nu) - L^{-1}(\mu, \nu)| \leq \frac{(\nu - \mu)^2}{162} \left[ \max \left\{ \frac{2}{\mu^2}, \frac{16}{(\mu + \nu)^2} \right\} + \max \left\{ \frac{2}{\nu^2}, \frac{16}{(\mu + \nu)^2} \right\} \right]; \]
3. \[ \left| \frac{A^{-1}(\mu, \nu) + H^{-1}(\mu, \nu)}{2} - L^{-1}(\mu, \nu) \right| \leq \frac{(\nu - \mu)^2}{96} \left[ \max \left\{ \frac{2}{\mu^2}, \frac{16}{(\mu + \nu)^2} \right\} + \max \left\{ \frac{2}{\nu^2}, \frac{16}{(\mu + \nu)^2} \right\} \right]; \]
4. \(|H^{-1}(\mu, v) - L^{-1}(\mu, v)| \leq \frac{(\nu - \mu)^2}{24} \left[ \max \left\{ \frac{2}{\mu^3}, \frac{16}{(\mu + v)^3} \right\} + \max \left\{ \frac{2}{\nu^3}, \frac{16}{(\mu + v)^3} \right\} \right] \).

**Proof.** We apply inequalities (7)–(10) to the function \(G : [\mu, v] \to \mathbb{R}\) defined by \(G(x) = \frac{x}{\nu}\). For this, we observe that \(G''(x) = \frac{2}{x^3}\) is convex on \([\mu, v]\) and so \(\eta\)-quasiconvex with respect to \(\eta(x, y) = x - y\).

We end with more four new inequalities.

**Proposition 16** Let \(\mu\) and \(v\) be two positive numbers with \(\mu < v\). Then the following inequalities hold:

1. \(|\ln A(\mu, v) - \ln I(\mu, v)| \leq \frac{(\nu - \mu)^2}{48} \left[ \max \left\{ \frac{1}{\mu^2}, \frac{4}{(\mu + v)^2} \right\} + \max \left\{ \frac{1}{v^2}, \frac{4}{v(\mu + v)^2} \right\} \right] \);

2. \(|2\ln A(\mu, v) + \ln G(\mu, v) - \ln I(\mu, v)| \leq \frac{(\nu - \mu)^2}{96} \left[ \max \left\{ \frac{1}{\mu^2}, \frac{4}{(\mu + v)^2} \right\} + \max \left\{ \frac{1}{v^2}, \frac{4}{v(\mu + v)^2} \right\} \right] \);

3. \(|\ln A(\mu, v) + \ln G(\mu, v) - \ln I(\mu, v)| \leq \frac{(\nu - \mu)^2}{96} \left[ \max \left\{ \frac{1}{\mu^2}, \frac{4}{(\mu + v)^2} \right\} + \max \left\{ \frac{1}{v^2}, \frac{4}{v(\mu + v)^2} \right\} \right] \);

4. \(|\ln G(\mu, v) - \ln I(\mu, v)| \leq \frac{(\nu - \mu)^2}{24} \left[ \max \left\{ \frac{1}{\mu^2}, \frac{4}{(\mu + v)^2} \right\} + \max \left\{ \frac{1}{v^2}, \frac{4}{v(\mu + v)^2} \right\} \right] \).

**Proof.** Result follows by applying (7)–(10) to the function \(G(x) = \ln x, x \in [\mu, v]\), taking \(\eta(x, y) = x - y\) and noting that \(|G''(x)| = \frac{1}{x^2}\) is \(\eta\)-quasiconvex.

**5 Conclusion**

We proved two main theorems that establish Ostrowski type inequalities in terms of a parameter \(\lambda \in [0, 1]\). By choosing \(\lambda = 0, 1/3, 1/2, 1\), we deduced midpoint, Simpson, midpoint-trapezoid and trapezoid type inequalities, respectively. Thereafter, we illustrated the importance of our results by applying them to special means of positive real numbers.

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