The 3d Effective Field Theory of the High Temperature Abelian Higgs Model

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Abstract

We study a weakly coupled 3 dimensional $\varphi^4$ type model consisting of $N$ real scalars $\varphi^i$ coupled to an abelian gauge field $A^a$ and one extra scalar field $\rho$. We argue that, below some scale $\Lambda = O(T)$, this is the effective field theory of a 3+1 dimensional abelian Higgs model at a high temperature $T$. The effective theory is sufficient to study the nature of the phase transition in four dimensions. By introducing an auxiliary field $\chi$ we eliminate the explicit $\varphi^4$ term; the new Lagrangian allows for a simple computation of the dominant corrections to the effective scalar potential $V_{eff}$ in the large $N$ limit. We study three cases: a) $6g^2/\lambda \leq O(1)$, b) $6g^2/\lambda \sim O(N)$ and c) $6g^2/\lambda \sim O(N^{3/2})$ where $g$ and $\lambda$ are the 4d gauge and scalar couplings, respectively. For case a) which is the most thoroughly studied we find that the leading $O(N) + O(1)$ result for $V_{eff}$ admits only a second order phase transition. For the other cases we find that b) the leading $O(1)$ result for $V_{eff}$ admits a first order phase transition whose strength is independent of $N$ and c) the leading $O(N^{3/2})$ result for $V_{eff}$ admits only a second order phase transition – the $O(1)$ corrections to this can be interpreted as indicating a first order phase transition whose strength diminishes as $N$ increases.
1. Introduction.

There has been much recent interest in the nature of the electroweak phase transition. In general, for a spontaneously broken $4d \phi^4$ type model, one can attempt to compute the high temperature effective scalar potential $V_{\text{eff}}$. At very high temperature the symmetry is believed to be restored, and at some temperature $T = T_2$ the origin ($|\phi| = 0$) goes from being a local minimum to being a local maximum. If at $T > T_2$ the origin is also the global minimum the theory admits only a second order phase transition which proceeds by a roll-over after the temperature drops below $T_2$. On the other hand, if at some $T_1 > T_2$ the scalar potential has another minimum, degenerate in energy with the one at the origin, the theory can admit a first order phase transition.

The determination of a reliable effective potential has proven difficult because, in generic models, ordinary perturbation theory in $\hbar$ has infrared divergence problems just at values of $\phi^2$ and $T$ where the phase transition occurs. In a $\phi^4$ model with $N$ real scalars, Dolan and Jackiw [1] showed over 18 years ago that the one-loop potential has problems. They summed up an infinite class of diagrams, the “superdaisies” to obtain a reliable result for the effective mass at the origin.

Giving a reliable estimate of $V_{\text{eff}}$ in the case of the standard model is much more involved. The typical approach [2,3,4,5] for the corrections from the bosonic sector of the standard model has been to use not tree-level propagators but propagators corrected by one-loop vacuum polarization effects in a one-loop calculation of the effective potential. This has the effect of summing up an infinite class of diagrams, the so-called “ring” diagrams, with the correct combinatorics except for the two-loop graphs [4,5,6] which are overcounted. When only the leading $T^2$ dependent corrections are used for the vacuum polarizations the overcounting is not serious, but when field dependent vacuum polarizations are used it can result in dangerous $T^3|\phi|$ terms in the effective potential [4,5] which results in the effective mass blowing up as one approaches the origin.

Assuming no such dangerous terms appear, previous studies have concentrated on determining the coefficient of the $|\phi|^3$ term in their estimates of $V_{\text{eff}}$. For small $\phi^2$, the effective potential can be expanded as ($d > 0$)

$$V_{\text{eff}} = c + a\phi^2 - b|\phi|^3 + d\phi^4 + \cdots.$$  \hspace{1cm} (1)

The coefficient $a$ is positive when $T > T_2$ and negative when $T < T_2$. In a general 3+1 dimensional scalar and gauge system the coefficients $a$, $b$ and $d$ can be found in the ring approximation to $V_{\text{eff}}$ as follows [2,3,4,5]. In the background field evaluation of $V_{\text{eff}}$ one expands the action about quantum scalar fields $\hat{\phi}^i$ and quantum gauge fields $A^\mu$. Each quantum degree of freedom has a mass depending on the background field $\phi^2$. At zero external momentum, the leading vacuum polarization effects give additional $T^2$ contributions to the mass of the quantum scalars and the longitudinal gauge field, but not to the transverse gauge fields. Then, when $T$ is sufficiently bigger than $T_2$, the result of the polarization effects is that the scalar and longitudinal gauge boson loops give only $\phi^2$ and $\phi^4$ contributions to eq. (1). In this limit, the transverse gauge fields still contribute
term. It is the existence of this term in the context of the standard model which led the authors of [2,4] to conclude that the electroweak model with sufficiently small self–scalar coupling admits a (weak) first order phase transition.

The analysis we have just described is however incomplete. For example, how are these results modified if one keeps the field dependent contributions to the vacuum polarizations? As already mentioned, the simple ring diagram procedure can lead to spurious $O(|\phi|)$ contributions to $V_{\text{eff}}$, so obviously some care is needed. One would also like to investigate how the momentum dependence of the vacuum polarizations modifies the $|\phi|^3$ result. Finally, we would like to know in what way the ring (or “daisy”) diagram sum can be interpreted as a consistent approximation scheme.

To address these questions and problems, we study as a toy model a 3 dimensional $\phi^4$ type model consisting of $N$ real scalars $\phi^i$ coupled to an abelian gauge field $A^a$ and an extra scalar $\rho$, which we argue is the low energy effective field theory for a 3+1 dimensional Abelian Higgs model at high temperature. The computations in the effective 3d theory are however much easier than in the full four dimensional model, and it is for this reason we feel this approach will prove very useful in studying the much more complicated electroweak model at high temperature. In this paper we denote the 3d scalars $\phi$ and the 4d scalars $\phi$. The 4d gauge and scalar couplings are $g$ and $\lambda$, respectively.

For $6g^2/\lambda \leq O(1)$, by introducing an auxiliary field $\chi$ we eliminate the explicit $\phi^4$ term and develop a systematic $1/N$ expansion for $V_{\text{eff}}$ and compute the leading and next–to–leading corrections in this expansion. By matching linearly divergent integrals in the effective 3d theory with the $O(T^2)$ corrections in the 4d theory we show how to obtain exactly the correct high temperature result of the full 4d model, at least in the $g = 0$ case which is known [6]. The effective 3d model is sufficient to study the nature of the phase transition which occurs at values of $\phi^2$ and $\chi$ much less than $T^2$. In contrast to previous studies, we find no $|\phi|$ term in $V_{\text{eff}}$, nor for $6g^2/\lambda \ll N$ any significant $|\phi|^3$ term in $V_{\text{eff}}$. In fact, to the order we compute, the model admits only a second order phase transition. First order phase transitions are however found for $6g^2/\lambda \sim O(N)$ and $6g^2/\lambda \sim O(N^{3/2})$.

We mention that the high temperature 3+1 dimensional abelian Higgs model has also been investigated in [7] as an effective 3 dimensional theory. However, in that reference, the $\epsilon$–expansion method is used to study the nature of the phase transition and it does not shed any light on the above questions and problems.

In section 2 we show how the pure scalar result can be obtained in our approach and find agreement with an earlier calculation of this author [6]. In section 3 we extend the analysis to the gauged case for $6g^2/\lambda \leq O(1)$. In section 4 we find the leading contributions to $V_{\text{eff}}$ for $6g^2/\lambda \sim O(N)$ and $O(N^{3/2})$. In section 5 we discuss our results and the phase transition and our conclusions appear in section 6. The pure scalar example, although previously studied at finite temperature [6], sets much of the foundation for the analysis in section 3, including in particular the vacuum polarization effects.
2. \( \phi^4 \) Model to Subleading Order.

First consider a model with \( N \) real scalars \( \phi^i \) with the tree level 3+1 dimensional Lagrangian

\[
L[\phi] = \frac{N}{2} \delta_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{N \lambda}{4!} (\phi^2 - v^2)^2.
\]

(2)

\( \lambda \) is a dimensionless coupling and \( v \) is the tree level vev of \( |\phi| \). The space–time metric is \( \eta^{\mu\nu} = \text{diag}(+,-,-,-) \). Furthermore, to avoid problems associated with triviality we will assume \( \lambda > 0 \) and consider the model as an effective low energy model valid below some scale \( \Lambda \) [10].

The systematic \( 1/N \) expansion allows us to calculate (1) as a perturbation in \( 1/N \) near \( \phi = 0 \) [8,9]. Root [9] has evaluated the leading corrections and given formal expressions for the next–to–leading corrections to the zero temperature scalar potential in 4,3,2 and 1 dimensions. The procedure at finite temperature is very similar.

By introducing a dimension two auxiliary field \( \chi \) we replace the Lagrangian (2):

\[
L[\phi,\chi] = L[\phi] + 3N^2 \frac{\lambda}{2\lambda} \left( \chi - \frac{\lambda}{6} (\phi^2 - v^2) \right)^2
\]

\[
= \frac{N}{2} (\partial_\mu \phi^i)^2 + 3N^2 \frac{\lambda}{2\lambda} \chi^2 - \frac{N}{2} (\phi^2 - v^2) \chi.
\]

(3)

The auxiliary field has eliminated the \( \phi^4 \) term; the original form of (3) is easily recovered by use of the equation of motion for \( \chi \).

To calculate the effective potential \( V_{\text{eff}}(\phi) \) one proceeds as follows. First, using the background field method one computes the effective potential as a function of backgrounds of \( \phi \) and \( \chi \). Then, the background of \( \chi \) is eliminated by its equation of motion.

The systematic \( 1/N \) expansion first requires expanding the Lagrangian \( L[\phi,\sigma] \) about real backgrounds \( \phi \) and \( \chi \) thus [8,9,11]:

\[
\phi^i \to \phi^i + \frac{\hat{\phi}^i}{\sqrt{N}}, \quad \chi \to \chi + \frac{\hat{\chi}}{\sqrt{N}},
\]

(4)

and then deleting terms linear in the quantum fields \( \hat{\phi}, \hat{\chi} \). This procedure defines a quantum Lagrangian \( L[\hat{\phi},\hat{\chi}] \), the sum of whose one–particle irreducible (1PI) diagrams give what we call the effective action. Since we are interested in the effective potential we assume the backgrounds are space–time constants. We also assume non-negative \( \chi \).

At finite temperature the model is formally equivalent to a euclidean field theory with one compact dimension. When the backgrounds \( \phi^2 \) and \( \chi \) are much below \( T^2 \) it is sufficient to study an effective three dimensional theory, a fact that we exploit below. Thus, at sufficiently high temperature, our results for four dimensions should be similar to those of a three dimensional euclidean field theory with a dimensionful \( \phi^4 \) coupling [7]. Indeed, in three dimensions and zero temperature, the leading \( O(N) \) potential has long
been known [8]. It has exactly the same form as the sum of finite temperature superdaisy graphs that were computed by Dolan and Jackiw [1] for a four dimensional \( \phi^4 \) theory. An important point in our approach is that the effect of introducing an auxiliary field \( \chi \) is to shift the \( \phi \) mass term and as a result there are no infrared divergences in this formalism.

To perform the dimensional reduction from 4 to 3 dimensions, one can follow [7]. One decomposes the fields as follows,

\[
\hat{\phi}^i(x^\mu) = \sum \hat{\phi}^i_n(\vec{x})\psi_n(\tau), \quad \hat{\chi}(x^\mu) = \sum \hat{\chi}_n(\vec{x})\psi_n(\tau),
\]

where \( \psi_n(\tau) \) are a complete set of periodic functions on the circle. For what we are interested in only the zero modes \( n = 0 \), for which \( \psi_0(\tau) = \psi_0(0) \), are important in the effective 3d model. This is because for \( n \neq 0 \) the fields \( \hat{\phi}^i_n \) pick up nonzero masses of \( O(T^2) \) when the compact dimension is integrated out in the action. Therefore, truncating the spectrum to keep only the zero modes and integrating out the compact dimension gives the effective 3d euclidean Lagrangian (\( \beta = 1/T \))

\[
\frac{1}{\beta} L[\hat{\phi}_0, \hat{\chi}_0] = -\frac{3N^2}{2\lambda} \chi^2 + \frac{N}{2} \chi(\phi^2 - v^2)
\]

\[
+ \frac{1}{2} \hat{\phi}_0^i[\delta_{ij}(\vec{\nabla}^2 + \chi)]\hat{\phi}_0^j + \frac{\hat{\phi}_0^2 \hat{\chi}_0}{2\sqrt{N}} + \phi^i \hat{\phi}_0^i \hat{\chi}_0 - \frac{3N^2 \chi^2}{2\lambda}.
\]

(6)

Recall that \( \phi \) and \( \chi \) are space–time constants. Defining the three dimensional quantities,

\[
\varphi^i = \sqrt{\beta} \phi^i, \quad \hat{\varphi}^i = \sqrt{\beta} \hat{\phi}^i_0, \quad \hat{\lambda} = \lambda/\beta, \quad \hat{v}^2 = \beta v^2,
\]

we finally obtain the 3d quantum Lagrangian

\[
L[\hat{\varphi}, \hat{\chi}] = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \chi(\varphi^2 - \hat{v}^2)
\]

\[
+ \frac{1}{2} \hat{\varphi}^i[\delta_{ij}(\vec{\nabla}^2 + \chi)]\hat{\varphi}^j + \frac{\hat{\varphi}^2 \hat{\chi}}{2\sqrt{N}} + \varphi^i \hat{\varphi}_i \hat{\chi} - \frac{3N \chi^2}{2\lambda}.
\]

(8)

Here we have dropped the zero mode subscript on \( \hat{\chi}_0 \).

Before proceeding we would like to stress some important points. As shown in [7], if one integrates out the nonzero modes \( n \neq 0 \) at the quantum level rather than just truncating the spectrum, there is a finite \( O(T^2) \) correction to the mass of \( \hat{\phi}_0^i \). This correction is very important, in fact it is the term that gives symmetry restoration at high enough temperature. We will obtain this term another way, in analogy with what one does in effective low energy theories of the strongly interacting standard model or effective four dimensional supergravity models inspired by string theory [12]. The three dimensional field theory will be divergent. Here we will regulate all the divergent integrals by introducing the same scale \( \Lambda \). In fact, the only divergent integral that will be important is linearly divergent and we will regulate it by simply using a sharp momentum cutoff.\(^a\)

\(^a\)A different scheme, such as Pauli–Villars, will in general give a different coefficient for the \( O(\Lambda) \) result. However, the precise coefficient will not be important in what follows [12].
then give a physical interpretation to $\Lambda$, i.e. the scale at which the full four dimensional physics becomes important. Thus, $\Lambda$ is proportional to $T$. This is in complete analogy with, for example, the effective 4d theories where the regulating scale is taken to be of order the compactification scale. In our case the identification can be made precise because the corresponding 4d correction is well known. We will see that the identification $\Lambda = \pi^2 T/6$ will reproduce exactly the $T^2$ results from the four dimensional integrals. In addition, since the four dimensional model (2) is only valid up to a scale $\bar{\Lambda}$ we must require $\Lambda < \bar{\Lambda}$, i.e. that $T$ is sufficiently small. Finally we note that the 4d effective potential can be obtained from the three dimensional one by dividing by $\beta$ and using (7).

To compute the 3d effective potential we must sum the one–particle irreducible (1PI) diagrams of (8). For readability, and for later use in the abelian Higgs model, we briefly describe the steps that lead to the next–to–leading result for $V_{eff}$, rather than just stating the result. Throughout, $\text{Tr}$ stands for momentum and internal space traces.

Integrating out the $\hat{\phi}^i$ in this model gives the $O(N)$ corrections to $V_{eff}$ to all orders in $\hbar$ and defines an effective Lagrangian which only has quantum dependence on $\hat{\chi}$. The part of this effective Lagrangian which is quadratic in $\hat{\chi}$ includes the tree level part and a one–loop vacuum polarization part. The one–loop $\hat{\chi}$ corrections give the next–to–leading corrections in $N$, also to all orders in $\hbar$.

Because (8) is quadratic in $\hat{\phi}$, the $O(\hat{\phi} \hat{\chi})$ mixing term can be removed by making a change of field variables [6]. This has the effect of replacing the mixing term by

$$+ \varphi^i \hat{\phi}^i \hat{\chi} \rightarrow -\frac{\hat{\chi}^2}{2} \frac{\varphi^2}{(-\vec{\partial}^2 + \chi + \hat{\chi}/\sqrt{N})} \rightarrow -\frac{\hat{\chi}^2}{2} \frac{\varphi^2}{(-\vec{\partial}^2 + \chi)},$$

where the last replacement holds to next–to–leading order in the $1/N$ expansion.

The gaussian integral over $\hat{\phi}$ is now straightforward and its vacuum polarization contribution to the $\hat{\chi}$ propagator at nonzero external momentum is known [8,9] in 3d,

$$\Pi_{\hat{\chi}}(\vec{k}) = \frac{\lambda}{6} \int \frac{1}{[p^2 + \chi][(p + \vec{k})^2 + \chi]} = \frac{\lambda}{24\pi} \frac{1}{\sqrt{k^2}} \sin^{-1} \left( \frac{1}{\sqrt{1 + 4\chi/k^2}} \right).$$

The final result for the leading + next–to–leading potential is [8,9,6]

$$V_{eff} = \frac{-3N}{2\lambda} \chi^2 + \frac{N}{2} \chi(\varphi^2 - \bar{\varphi}^2) + \frac{N}{2} \text{Tr} \ln(-\vec{\partial}^2 + \chi) + V_{N\mu},$$

where the next–to–leading contribution from the gaussian integral over $\hat{\chi}$ is

$$V_{N\mu} = \frac{1}{2} \text{Tr} \ln \left[ 1 + \frac{\lambda}{24\pi} \frac{1}{\sqrt{-\vec{\partial}^2}} \sin^{-1} \left( \frac{1}{\sqrt{1 - 4\chi/\vec{\partial}^2}} \right) + \frac{\hat{\lambda} \varphi^2/3}{-\vec{\partial}^2 + \chi} \right].$$

Root [9] has shown, without explicit calculation, that the 3d field theory is renormalizable to this order in $N$, as it should be. Although we will give a physical interpretation
to our regulating scale, renormalizability of the 3d theory places strong constraints on
the type of linearly divergent contributions we can obtain. For example, a contribution
of $O(\Lambda \varphi^2)$ is by itself not renormalizable in our formalism. How this enters will be seen
in the explicit calculations below.

Although we did not find a simple expression for $V_{N\nu}$, we found that the dominant
contribution comes from the large external momentum limit of (10), $-\vec{\partial}^2 \gg \chi$, not the
zero external momentum limit. Although this seems entirely reasonable in the limit of
vanishing $\chi$, why it should be so for sufficiently large $\chi$ is not apparent. Indeed, we
would like to stress the importance of vacuum polarization effects at nonzero external
momentum. If in (12) we had kept only the zero external momentum part of the $\hat{\chi}$ field
vacuum polarization, $\Pi_{\hat{\chi}\hat{\chi}}(0) = \lambda / (48\pi \sqrt{\chi})$ then we would have obtained an incorrect (and
nonrenormalizable) answer. $\Pi_{\hat{\chi}\hat{\chi}}(0)$ when used for the whole momentum integral from 0
to $\Lambda$ gives a dangerous $O(\lambda \Lambda^3 / \sqrt{\chi})$ contribution in the expansion of the log in (12). In
fact, the leading field dependent contributions seem to arise from the most ultraviolet
divergent field dependent term in the high momentum expansion of $\Pi_{\hat{\chi}\hat{\chi}}$. One can check
this assertion when $\chi$ is sufficiently large by the following method. In this limit, the
arcsine term in (12) is small for all values of the momentum, and we expand the ln using
$\ln[1 + x] = x + ...$. The arcsine possesses two different expansions depending on whe
ther $-\vec{\partial}^2$ is bigger or smaller than $4\chi$, and accordingly the momentum integral implied by the
Tr must be broken up into two different regions. The different contributions can then be
evaluated with the result that very little error is made is using only the large momentum
limit of (10) everywhere. This effect will also be very important in the next section.

In the large momentum limit, we have

$$(-\vec{\partial}^2 + \chi)\Pi_{\hat{\chi}\hat{\chi}}(-\vec{\partial}^2) \rightarrow \frac{\lambda \chi}{48} - \frac{\lambda \sqrt{\chi}}{12\pi}. \quad (13)$$

We then write eq. (12) as

$$V_{N\nu} \approx \frac{1}{2} \Tr \ln \left[ -\vec{\partial}^2 + \chi + \frac{\lambda}{48} \sqrt{-\vec{\partial}^2} + \frac{\lambda}{3} \left( \varphi^2 - \sqrt{\chi} \right) \right] - \frac{1}{2} \Tr \ln \left[ -\vec{\partial}^2 + \chi \right]. \quad (14)$$

It can now be argued that if $\lambda / 48 \ll \Lambda$ and $\chi$ is sufficiently large the $O(\sqrt{-\vec{\partial}^2})$ term in
the log can be dropped [13]. This is what we will assume.

We can now give an explicit expression for $V_{\text{eff}}$, up to the approximations we have
made by using the following result for three dimensions ($\chi \geq 0$):

$$\Tr \ln(-\vec{\partial}^2 + \chi) \rightarrow \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln(k^2 + \chi) = \frac{\Lambda \chi}{2\pi^2} - \frac{\chi^3}{6\pi} + \text{const.} \quad (15)$$

A sharp momentum cutoff was used to evaluate the field dependent divergent part. Adding
everything up we obtain, up to a constant,

$$V_{\text{eff}} = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \chi (\varphi^2 - \bar{v}^2) + N \left[ \frac{\Lambda \chi}{4\pi^2} - \frac{\chi^3}{12\pi} \right] + \frac{\Lambda \lambda}{12\pi^2} \left( \varphi^2 - \sqrt{\chi} \right)$$
\[-\frac{1}{12\pi} \left( \chi + \frac{\lambda}{3} \varphi^2 - \frac{\bar{\lambda}}{12\pi} \sqrt{\chi} \right)^\frac{3}{2} - \chi^\frac{3}{2} \right). \tag{16}\]

To write the effective potential for $\varphi^2$ alone we use the equations of motion for $\chi$ to eliminate it. To the order we are working it is consistent to use the leading order equation for $\chi$ to eliminate it \[9\]. $\partial V_{\text{eff}}/\partial \chi = 0$ from the terms proportional to $N$ gives:

$$\chi = \frac{\bar{\lambda}}{6} \left( \varphi^2 - v^2 \right) + \frac{\bar{\lambda}}{6} \left( \frac{A}{2\pi^2} - \frac{\sqrt{\chi}}{4\pi} \right). \tag{17}$$

If we rewrite the $O(1)$ terms in $V_{\text{eff}}$ using this equation, and ignore $\bar{\lambda}/(12\pi)$ in comparison with $\sqrt{\chi}$ then we obtain up to a constant,

$$V_{\text{eff}} = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \chi (\varphi^2 - v^2) + (N+2) \frac{A\chi}{4\pi^2} - \frac{1}{12\pi} \left( \chi + \frac{\bar{\lambda}}{3} \varphi^2 \right)^\frac{3}{2} + (N-1)\chi^\frac{3}{2} \right). \tag{18}$$

This is the leading + next–to–leading result for $V_{\text{eff}}$ for a pure scalar theory in 3d when $\sqrt{\chi}$ is large enough. To obtain the high temperature four dimensional result, we recall the high temperature 4d leading order mass–gap equation \[1,6\]:

$$\chi = \frac{\lambda}{6} \left( \varphi^2 - v^2 \right) + \frac{\lambda}{6} \left( \frac{1}{12\beta^2} - \frac{\sqrt{\chi}}{4\beta} \right) + \ldots. \tag{19}$$

Using (7), we see that the identification

$$\Lambda = \pi^2 T/6 \tag{20}$$

in (17) reproduces exactly the four dimensional result (19). In fact, with this identification, (18) reproduces exactly the leading and next–to–leading 4d finite temperature scalar potential found in [6]. The computations here have been much easier. This potential was studied in [6] and admits no first order phase transition for the range of $\varphi^2$ and $T$ where it is valid (i.e. not too close to the origin when the temperature is near $T_\chi$). Similar conclusions appear in [7,14]. Our result is essentially due to the fact that $V_{\text{eff}}$ contains no $O(|\varphi|^3)$ terms for $\lambda \varphi^2 < 3\chi$.

The potential (18) is valid when $\bar{\lambda}/(12\pi)$ is ignorable compared to $\sqrt{\chi}$. We will give a more extensive discussion of the phase transition in the more general gauged case, in the last section.
3. Abelian Higgs Model, $\frac{6g^2}{\lambda} \leq O(1)$.

Now consider the 3+1 dimensional gauge invariant scalar QED Lagrangian

$$L[\phi, A] = \frac{N}{2} \delta_{ij} \partial_{\mu} \phi^i \partial^\mu \phi^j - \frac{N \lambda}{4!} (\phi^2 - v^2)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - g \epsilon_{ij} (\partial_{\mu} \phi^i) \phi^j A^\mu + \frac{1}{2} g^2 \phi^2 A^2.$$  (21)

Here, $g$ is the gauge coupling, $\phi^i$, $i = 1...N$, are real as before and $\epsilon_{ij}$ is antisymmetric with the nonzero components $\epsilon_{12}, \epsilon_{21}, \epsilon_{34}, \epsilon_{43}, ...$ all having magnitude 1. The field strength and covariant derivative $D_{\mu}$ are given in terms of the gauge field $A_{\mu}$ by

$$D_{\mu} \phi^i = \partial_{\mu} \phi^i - g \epsilon_{ij} \phi^j A_{\mu},$$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}.$$  (22)

This model is nothing but $N/2$ copies of the simple scalar QED Lagrangian discussed in many textbooks, and also in ref. [1], but with only one gauge vector field.

We would like to calculate the leading + next-to-leading corrections to $V_{\text{eff}}$ in the $1/N$ expansion at high temperature. As in section 2, we will do this by first writing an effective 3d theory. This three dimensional theory will turn out to be an abelian Higgs model with an extra scalar. This can be understood intuitively because in 4 dimensions the gauge field has three massive components, while in three dimensions it only has two. Hence, to get the same light degrees of freedom we need an additional scalar.

To proceed, we introduce an auxiliary field, as in eq. (3), to obtain

$$L[\phi, \chi, A] = \frac{N}{2} (\partial \phi)^2 + \frac{3N}{2\lambda} \chi^2 - \frac{N}{2} (\phi^2 - v^2) \chi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - g \epsilon_{ij} (\partial_{\mu} \phi^i) \phi^j A^\mu + \frac{1}{2} g^2 \phi^2 A^2.$$  (23)

We then expand this using (4), and delete all terms linear in $\hat{\phi}$ and $\hat{\chi}$. Since we are interested in the scalar potential we need not keep a background for the gauge fields.

We also need to gauge fix. We add the gauge fixing term

$$L_{\text{g.f.}} = -\frac{1}{2\alpha} (\partial_{\mu} A^\mu + \alpha g \epsilon_{ij} \hat{\phi}^i \hat{\phi}^j)^2.$$  (24)

Here, $\alpha$ is an arbitrary parameter. Although the calculations simplify in the Landau gauge $\alpha \to 0$, this extra parameter is useful in checking that physical quantities such as the critical temperature are gauge fixing independent. The ghost term for this gauge fixing is [1]

$$L_{\text{gh}} = \bar{\theta} (\partial^2 + \alpha g^2 \phi^2) \theta,$$  (25)

where $\theta, \bar{\theta}$ are Grassmanian ghost fields. We assume non-negative $g^2, \alpha$ and $\alpha \leq O(1)$ [4].

After performing all these operations, the total quantum Lagrangian is given by

$$L[\hat{\phi}, \hat{\chi}, A, \theta] = L[\hat{\phi}, \hat{\chi}] + \frac{1}{2} A_{\nu} \partial^2 A^\nu + \frac{1}{2} (1 - \frac{1}{\alpha}) (\partial A)^2$$

$$+ \frac{1}{2} g^2 \phi^2 + 2 \hat{\phi} \hat{\phi}^i / \sqrt{N} \hat{\phi}^2 / N) A^2 - \frac{g}{\sqrt{N}} \epsilon_{ij} (\partial_{\mu} \hat{\phi}^i) \phi^j A^\mu$$

$$- \frac{1}{2} \alpha g^2 \hat{\phi} \hat{\phi}^k \epsilon_{ij} \epsilon_{kl} \phi^i \phi^l + \bar{\theta} (\partial^2 + \alpha g^2 \phi^2) \theta.$$  (26)
We dropped all total divergences and assumed the backgrounds are space–time constants. There is no $O(\hat{\phi}A)$ term because of our gauge choice. Finally, $L[\hat{\phi}, \hat{\chi}]$ is the pure scalar part of the quantum Lagrangian.

To write the 3d effective field theory for the high temperature limit of this model we follow exactly the approach described in section two. We simply truncate the spectrum to keep the zero modes of the compact dimension of the euclidean theory and then give a physical interpretation to the scale used to regulate divergent integrals. This truncation means $\partial_0 \rightarrow 0$. Then, with the definitions (7) as well as $(a=1,2,3)$

$$A^a = \sqrt{\beta} A^a, \quad \rho = i \sqrt{\beta} A^0, \quad \vartheta = \sqrt{\beta} \theta, \quad \tilde{g} = g/\sqrt{\beta},$$  

we write the effective 3d quantum Lagrangian as

$$L[\hat{\phi}, \hat{\chi}, A, \rho, \vartheta] = L[\hat{\phi}, \hat{\chi}] - \frac{1}{2} A_a \partial^a A^a - \frac{1}{2} (1 - \frac{1}{\alpha})(\partial A^2) - \frac{1}{2} \rho \partial^2 \rho$$

$$+ \frac{1}{2} g^2 (\varphi^2 + 2 \varphi \hat{\varphi}^i / \sqrt{N} + \hat{\varphi}^2 / N)(A^2 + \rho^2) - \frac{\tilde{g}}{\sqrt{N}} \epsilon_{ij} (\partial_a \hat{\varphi}^i) \hat{\varphi}^j A^a$$

$$+ \frac{1}{2} \alpha \tilde{g}^2 \hat{\varphi}^i \hat{\varphi}^j \epsilon_{ij} \epsilon_{kl} \hat{\varphi}^i \hat{\varphi}^l + \tilde{\vartheta} (- \partial^2 + \alpha \tilde{g}^2 \varphi^2) \vartheta.$$  

(28)

$L[\hat{\phi}, \hat{\chi}]$ is given by eq. (8), and $A_a = A^a$, $\partial_a = \partial^a$.

Eq. (28) is nothing but a three dimensional abelian Higgs model with an extra massless scalar field $\rho$. It has a simple canonical kinetic term and its tree level coupling to the scalars is simply $+\tilde{g}^2 \varphi^2 \rho^2 / 2$. We now compute the 1PI diagrams of this 3d Lagrangian to leading and next–to–leading order in the $1/N$ expansion. It is clear that the gauge sector contributes only at next–to–leading order; there are no Feynman diagrams involving the gauge fields that contribute at $O(N)$. The correction from the 3d ghosts involves only a simple one–loop calculation. For the rest, we can either first integrate out the $A$ and $\rho$ at the quantum level, or integrate out the $\hat{\varphi}$. We have done both, and present only the latter computation.

To integrate out the $\hat{\varphi}$ we must first make a field redefinition to eliminate terms in (28) which are linear in $\hat{\varphi}$. For $\tilde{g} = 0$ this is just the replacement (9) to the required order. For $\tilde{g} \neq 0$ we have to shift $\hat{\varphi}$ in such a way as to also eliminate the $O(\tilde{g}^2)$ linear terms in $\hat{\varphi}$. When this is done we found no contributions from the $O(\tilde{g}^2 \hat{\varphi}^i)$ terms that were important at next–to–leading order. Another way of saying this is that there are no Feynman diagrams involving the $O(\tilde{g}^2 \hat{\varphi}^i)$ terms that contribute at next–to–leading order. Hence we just forget these terms in the Lagrangian.

In the pure scalar case we saw that integrating out the scalars gave the leading order result and generated a vacuum polarization term $\Pi_{\hat{\chi} \hat{\chi}}$ for $\hat{\chi}$. Here, something similar happens. In addition to $\Pi_{\hat{\chi} \hat{\chi}}$ we will generate a vacuum polarization $\Pi_{\rho \rho}$ for $\rho$ and a vacuum polarization matrix $\Pi^{ab}$ for the 3d gauge fields. This result can be derived very simply as follows. The scalar loop integral in (28) contributes a piece

$$\frac{1}{2} \text{Tr} \ln \left[ \Delta^{-1}_{ij} + \delta_{ij} \frac{\hat{\chi}}{\sqrt{N}} + \delta_{ij} \frac{\tilde{g}^2}{N} (A^2 + \rho^2) - 2 \frac{\tilde{g}}{\sqrt{N}} \epsilon_{ij} A_a \partial^a \right]$$  

(29)
where the inverse scalar propagator is given by the matrix

\[
\Delta^{-1}_{ij} = \delta_{ij}(-\partial^2 + \chi) + \alpha \tilde{g}^2 \epsilon_{ii'} \epsilon_{jj'} \varphi^i \varphi^j.
\]  

(30)

One can expand (29) as a power series in \(A\); the terms linear in \(A\) in this expansion vanish because \(\text{tr} \epsilon_{ij} = 0\) and only the terms quadratic in \(A\) and \(\rho\) contribute at \(O(1)\) in the \(1/N\) expansion using \(\text{tr} \epsilon_{ij} \epsilon_{jk} = -N\). At zero external momentum the expansion is easy and gives to \(O(1)\)

\[
\frac{1}{2} \text{Tr} \ln(\Delta^{-1}_{ij} + \delta_{ij} \hat{\chi}/\sqrt{N}) + \frac{1}{2} g^2 (A^2 + \rho^2) \text{Tr}(-\partial^2 + \chi)^{-1} + \tilde{g}^2 A_a A_b \text{Tr} \partial^a \partial^b / (-\partial^2 + \chi)^2.
\]  

(31)

The \(A^2, \rho^2\) terms are clearly vacuum polarization corrections to the tree level kinetic terms of these fields. The vacuum polarization for \(\hat{\chi}\) is contained in the first term above, and is in fact the same as in the pure scalar case at \(O(1)\). Since we have seen that vacuum polarization effects at nonzero external momentum are very important even in the pure scalar case we will also keep them here. This can be done by carefully accounting for the 3-space dependence of all the quantum fields in (29) and the result for the full \(A, \rho\) vacuum polarizations is nothing other than the one–loop result given for scalar QED in many standard textbooks [15]. Altogether, one has

\[
\begin{align*}
\Pi_{\chi\chi}(\vec{k}) &= \frac{\lambda}{6} \int \frac{1}{\vec{p}^2 + \chi} \\
\Pi_{\rho\rho}(\vec{k}) &= \frac{g^2}{\vec{p}^2 + \chi} \\
\Pi^{ab}(\vec{k}) &= \frac{\delta^{ab}}{\vec{p}^2 + \chi} - \frac{1}{2} (2p + k)^a (2p + k)^b \left[ \frac{\delta^{ab}}{\vec{p}^2 + \chi} \right].
\end{align*}
\]  

(32)

\(V_{eff}\) then is given by the tree–level piece, a contribution from the scalars loops, \(\frac{1}{2} \text{Tr} \ln \Delta^{-1}_{ij}\), a simple one–loop contribution from ghosts \(\vartheta\), and the contributions from quadratic integrals over \(A, \rho, \chi\) with propogators modified by the vacuum polarization effects(32). The result is:

\[
V_{eff} = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \chi (\varphi^2 - \tilde{v}^2) + \frac{1}{2} \text{Tr} \ln \Delta^{-1}_{ij} + V_{Nt} - \text{Tr} \ln(-\partial^2 + \alpha \tilde{g}^2 \varphi^2) + \frac{1}{2} \text{Tr} \ln(-\partial^2 + \tilde{g}^2 \varphi^2 + \Pi_{\rho\rho}) + \frac{1}{2} \text{Tr} \ln(\Delta^{-1}_{ab} + \Pi^{ab}).
\]  

(33)

The tree–level gauge kinetic term is given by

\[
\Delta^{-1}_{ab} = -\delta_{ab} \partial^2 + (1 - \alpha^{-1}) \partial_a \partial_b + \tilde{g}^2 \varphi^2 \delta_{ab}.
\]  

(34)

The contribution from \(\hat{\chi}\) is the same as in the pure scalar case, eq. (12), because \(\Pi_{\chi\chi}\) did not change to \(O(1)\) in \(1/N\).
A few remarks are in order:

- In the Landau gauge $\alpha \to 0$, the 3d gauge contributions are given only by the last term in (33). In this gauge, the $\rho$ and $A_\alpha$ contributions are similar to the 4d gauge loop ring sum worked out in refs. [2,4]. As mentioned in the introduction, their procedure overcounts the $O(\hbar^2)$ graphs. For the $\tilde{g}$ dependent $O(\hbar^3)$ graphs the overcounting is due to an additional contribution from the scalars because for the scalar loop contribution they use propagators with $\tilde{g}$ dependent vacuum polarization effects. Our method of computing $V_{eff}$ automatically avoids this overcounting. In addition our contribution from (33) includes much more than the ring sum in [2,4] because our vacuum polarization effects depend on $\chi$ which is a self consistent solution of $\partial V_{eff}/\partial \chi = 0$ to all orders in $\hbar$, whereas the vacuum polarization used in [2,4] is of $O(\hbar)$ only.

- In a general gauge, $\alpha \neq 0$, the contributions to $V_{eff}$ due to nonzero $\Pi_{ab}$ are independent of $\alpha$. This is a result due to gauge invariance, i.e. $k^a \Pi_{ab}(k) = 0$ formally holds for (32) – as it also should for a properly regulated $\Pi_{ab}$. We have $\Delta^{-1}_{ab} + \Pi_{ab} = \delta_{ab}(-\tilde{\partial}^2 + \tilde{g}^2 \varphi^2) + [(1 - \alpha^{-1}) \partial_a \partial_b + \Pi_{ab}]$. The log of this can formally be expanded as a power series in the term in square brackets. Since $\Pi_{ab}$ is transverse, $\text{tr}[(1 - \alpha^{-1}) \partial_a \partial_b + \Pi_{ab}]^n$ factorizes so that there are no mixed $\alpha$-dependent and $\Pi$ dependent terms. Altogether, this means

$$\text{Tr} \ln[\Delta^{-1}_{ab} + \Pi_{ab}] = \text{Tr} \ln \Delta^{-1}_{ab} + 2 \text{Tr} \ln(-\tilde{\partial}^2 + \tilde{g}^2 \varphi^2 + \Pi) - 2 \text{Tr} \ln(-\tilde{\partial}^2 + \tilde{g}^2 \varphi^2)$$

$$= \text{Tr} \ln(-\tilde{\partial}^2 + \alpha \tilde{g}^2 \varphi^2) + 2 \text{Tr} \ln(-\tilde{\partial}^2 + \tilde{g}^2 \varphi^2 + \Pi).$$

Here, $\Pi_{ab} = (\delta_{ab} - k_a k_b / k^2) \Pi$ which is true due to $k^a \Pi_{ab} = 0$ and 3d euclidean invariance.

- Renormalizability of the model places strong constraints on the type of linearly divergent corrections to $V_{eff}$. The reason is the same as in the pure scalar case and as in section 2 we cannot obtain, by itself, a $O(\Lambda \varphi^2)$ contribution to $V_{eff}$.

- As already described we will give a physical interpretation to the regulating scale for divergent integrals. In general, one should introduce arbitrary parameters for the different divergent terms [12]. In the pure scalar case we did not do this because it turned out to be unnecessary. In our computations for this section it turned out that the match between the linearly divergent 3d integrals and the finite $T^2$ corrections of the full theory required that the linearly divergent part of $\Pi_{\rho \rho}$ be regulated not with $\Lambda \to \pi^2 T / 3$ but with $\Lambda' \to \pi^2 T / 3$. In general, we should also use unspecified sharp momentum cutoffs for the linear divergences from the final $\rho$ and $A_\alpha$ loop contributions. However, the same cutoff $\Lambda \to \pi^2 T / 6$ that gave the $O(T^2)$ corrections in the scalar case also worked here. This should be kept in mind in what follows.

Many of the traces in (34) are straightforward to compute. Using (15) we find up to field independent terms,

$$\text{Tr} \ln \Delta^{-1}_{ij} = N \left[ \frac{\Lambda \chi}{2 \pi^2} - \frac{\chi^2}{6 \pi} \right] + \frac{\alpha \tilde{g}^2 \Lambda \varphi^2}{2 \pi^2} - \frac{1}{6 \pi} \left[ (\chi + \alpha \tilde{g}^2 \varphi^2)^{\frac{3}{2}} - \chi^2 \right],$$

$$\text{Tr} \ln(-\tilde{\partial}^2 + \alpha \tilde{g}^2 \varphi^2) = \frac{\alpha \tilde{g}^2 \Lambda \varphi^2}{2 \pi^2} - \frac{\alpha \tilde{g}^3 |\varphi|^3}{6 \pi}.$$
\[
\text{Tr} \ln(-\partial^2 + \bar{g}^2 \varphi^2 + \Pi_{\rho \rho}) = \frac{\bar{g}^2 \Lambda (\varphi^2 - \frac{\sqrt{\chi}}{4\pi})}{2\pi^2} - \frac{\bar{g}^3 \left( \frac{\varphi^2}{2\pi^2} + \varphi^2 - \frac{\sqrt{\chi}}{4\pi} \right)^{\frac{3}{2}}}{6\pi}. \tag{36}
\]

Notice that the contribution from the 3d scalar \(\rho\) occurs only in the combination \(\varphi^2 - \sqrt{\chi}/4\pi\), a fact that is demanded by multiplicative renormalizability of the 3d theory.

We cannot compute the last term in (33) without first evaluating \(\Pi_{ab}\) at nonzero external momentum. This can be explicitly done by following [15]. Gauge invariance forbids a linearly divergent gauge mass correction and in dimensional regularization the vacuum polarization displays no poles at \(d = 3\). Since \(\Pi_{ab} = (\delta_{ab} - k_a k_b/k^2)\Pi\), we simply compute \(\Pi = \frac{1}{2} \delta^{ab}\Pi_{ab}\). We find

\[
\Pi = \frac{\bar{g}^2}{8\pi} \left[ -\sqrt{\chi} + \frac{2\chi + \frac{1}{2} k^2}{\sqrt{k^2}} \sin^{-1} \left( \frac{1}{\sqrt{1 + 4\chi/k^2}} \right) \right]. \tag{37}
\]

This has the zero momentum value

\[
\Pi = 0, \tag{38}
\]

and the high momentum limit

\[
\Pi \sim -\frac{\bar{g}^2 \sqrt{\chi}}{4\pi} + \frac{\bar{g}^2 \sqrt{k^2}}{32}. \tag{39}
\]

In the zero momentum limit \(\Pi_{ab} = 0\), and if this is wrongly used in the whole momentum integral in (33) we will obtain a nonrenormalizable answer for the 3d potential. This is most easily seen in the Landau gauge for which all linearly divergent corrections from the 3d scalar sector come in a combination proportional to \(\varphi^2 - \frac{\sqrt{\chi}}{4\pi}\). The 3d gauge loop contributions for \(\Pi_{ab} = 0\) however contribute \(O(\bar{g}^2 \Lambda \varphi^2)\) corrections which are not renormalizable. Thus, keeping only the zero momentum limit of \(\Pi_{ab}\) is incorrect.

We believe that the dominant contributions to \(V_{\text{eff}}\) from the 3d gauge loop integral arise from the high momentum limit of \(\Pi_{ab}\); we justify this as follows. For \(\bar{g}^2/12\pi\) ignorable in comparison to \(\sqrt{\chi}\) it follows that \(\Pi(-\partial^2)\) is small compared to \(-\partial^2\) for all values of \(-\partial^2\). We then expand \(\text{Tr} \ln(-\partial^2 + \bar{g}^2 \varphi^2 + \Pi)\) as a power series in \(\Pi\). The momentum integrals implied by the Tr must be broken up into the regions \(-\partial^2 > 4\chi\) and \(-\partial^2 < 4\chi\) in accordance with the different expansions \(\Pi\) possesses in these two regions. At \(O(\Pi)\) the different contributions can be computed with the result that only a small error is made in using (39) for the full momentum integral [17].

Note that due to our aim of calculating the high temperature \(V_{\text{eff}}\), we identify \(\rho\) with the longitudinal 4d gauge field so that a bare mass for \(\rho\) is neither necessary nor allowed.

In 3d this last statement can be understood as the requirement that there should be no nonlocal ultraviolet counterterms [16].

This agrees with the results given for the transverse components of the photon in [5].
When $\Lambda \gg \varphi^2 - \sqrt{\chi}/4\pi \gg \tilde{g}^2/(64)^2$ [13] we find, using eq. (35), a sharp momentum cutoff $\Lambda$ and ignoring constants,

$$\text{Tr} \ln(\Delta^{-1}_{ab} + \Pi_{ab}) = \frac{\alpha \tilde{g}^2 \Lambda \varphi^2}{2\pi^2} - \frac{\alpha \tilde{g}^3 |\varphi|^3}{6\pi} + \frac{\tilde{g}^2 \left( \varphi^2 - \frac{\sqrt{\chi}}{4\pi} \right) \Lambda}{\pi^2} - \frac{\tilde{g}^3 \left( \varphi^2 - \frac{\sqrt{\chi}}{4\pi} \right)^{\frac{3}{2}}}{3\pi}.$$  

(40)

Adding everything up, we obtain for the 3d $V_{eff}$,

$$V_{eff} = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \chi (\varphi^2 - \tilde{v}^2) + \frac{\Lambda \chi}{4\pi^2} (N + 2 + \frac{18\tilde{g}^2}{\lambda})$$

$$- \frac{1}{12\pi} \left[ \left( 3\chi + \frac{\tilde{\lambda}}{3}(\tilde{v}^2 - \Lambda/2\pi^2) \right)^{\frac{3}{2}} + (N - 1)\chi^{\frac{3}{2}} \right]$$

$$- \frac{(6\tilde{g}^2/\tilde{\lambda})^{\frac{3}{2}}}{12\pi} \left[ \left( \chi + \frac{\tilde{\lambda} \tilde{v}^2}{6} + \frac{\tilde{\lambda}}{12\pi^2}(\Lambda' - \Lambda) \right)^{\frac{3}{2}} + 2 \left( \chi + \frac{\tilde{\lambda} \tilde{v}^2}{6} - \frac{\tilde{\lambda}}{12\pi^2}\Lambda \right)^{\frac{3}{2}} \right]$$

$$+ V_{\alpha} + \text{neglected terms} + O(1/N) \text{ terms.}$$  

(41)

$V_{\alpha}$ is the pure gauge fixing dependent part,

$$V_{\alpha} = -\frac{1}{12\pi} \left[ (\chi + \alpha \tilde{g}^2 \varphi^2)^{\frac{3}{2}} - \chi^{\frac{3}{2}} - \alpha \tilde{g}^3 |\varphi|^3 \right],$$  

(42)

and satisfies $V_{\alpha}(\chi = 0) = V_{\alpha}(\varphi^2 = 0) = 0$. The “neglected terms” in (41) refers to the (small) errors that were made in only keeping the large external momentum limit of the vacuum polarizations, as well as other approximations. Here, we also used the leading order result for $\chi$, eq. (17), to rewrite the next–to–leading terms in $V_{eff}$ – a substitution that only changes the $O(1/N)$ terms.

To get the high temperature 4d $V_{eff}$ from (41) we must match the $\Lambda$, $\Lambda'$ terms with the 4d result for $T^2$ finite pieces as well as multiplying the 3d potential by $T$ and using eqs. (7), (27). For the case $\tilde{g} = 0$ we already found $\Lambda = \pi^2 T/6$ reproduces the correct 4d result. In fact, this identification also works in the $\tilde{g} \neq 0$ case. This can be checked by examining the $O(\tilde{h})$ (i.e. one loop in usual perturbation theory) result for which the third term in (41) reduces to [6] $\Lambda \varphi^2 (N\tilde{\lambda} + 2\lambda + 18\tilde{g}^2)/24\pi^2$. The identification $\Lambda = \pi^2 T/6$ reproduces the one–loop result for scalar QED given by Dolan and Jackiw [1]. Finally, $\Lambda' = \pi^2 T/3$ reproduces the $O(T^2)$ scalar QED longitudinal gauge boson polarization given in refs. [2,4,5]. To arrive at (41) we made the assumptions $\Lambda \gg \sqrt{\chi} \gg \tilde{\lambda}/12\pi, \tilde{g}^2/12\pi, \Lambda \gg \varphi^2 - \sqrt{\chi}/4\pi \gg \tilde{g}^2/(64)^2$ [13].

To compare we must rescale $\varphi^2 \rightarrow \varphi^2/N, \tilde{g}^2 \rightarrow \tilde{g}^2 N$ and set $N = 2$. 

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*To compare we must rescale $\varphi^2 \rightarrow \varphi^2/N, \tilde{g}^2 \rightarrow \tilde{g}^2 N$ and set $N = 2$. 

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4. Abelian Higgs Model, $6g^2/\lambda \sim O(N)$ and $O(N^2/\lambda^2)$.

The case $6g^2/\lambda \sim O(N)$. In section 3 we considered $\lambda$ fixed as $N$ increases. In this case we must assume $(\lambda N)$ fixed as $N$ increases. For a weakly coupled model this means the scalar self–coupling is very weak. First let us consider the pure scalar case of section 2. We again work in 3d. To obtain the leading corrections we make the replacements $\tilde{\chi} = \chi/N$, $\tilde{\chi} = \tilde{\chi}/N$ in (8) and (11). Then $V_{NL}$ is at most of $O(1/N)$; the remainder of (11) is of $O(1)$ and $O(1/\sqrt{N})$. Keeping only the $O(1)$ and $O(1/\sqrt{N})$ terms the answer can be exactly computed. For non-negative $\chi$, (11) gives:

$$V_{eff} = -\frac{3}{2N\lambda} \chi^2 + \frac{1}{2} \chi (\varphi^2 - \tilde{v}^2) + \frac{\Lambda \chi}{4\pi^2} - \frac{\chi^2}{12\pi \sqrt{N}} + \text{const.}$$  \hspace{1cm} (43)

$\chi$ should be eliminated by its leading order equation from the the first three terms above. For $g \neq 0$ the computation of all $O(1/\sqrt{N})$ terms is nontrivial. The replacements $\tilde{\lambda} = \tilde{\chi}/N$, $\chi = \chi/N$ in (33) does not give all $O(1/\sqrt{N})$ terms. The $O(1)$ terms can however be found in this way. The final expression is

$$V_{eff} = -\frac{3}{2N\lambda} \chi^2 + \frac{1}{2} \chi (\varphi^2 - \tilde{v}^2) + \frac{\Lambda \chi}{4\pi^2} + \frac{3\Lambda g^2}{4\pi^2} \varphi^2 - \frac{g^3}{12\pi} \left[(\Lambda'/2\pi^2 + \varphi^2)^2 + 2|\varphi|^3\right] + \text{const.}$$  \hspace{1cm} (44)

$\chi$ is given by the same solution as in the $\tilde{g} = 0$ case. Eq. (44) is $\alpha$ independent to this order and is valid for non-negative $\chi$, $\sqrt{N}$ and $\Lambda \gg \chi, \varphi^2$ and $\varphi^2 \gg \tilde{g}^2/(64)^2$. The matching conditions for the linear divergences is the same as in section 3.

The case $6g^2/\lambda \sim O(N^{1/2})$. Here we must assume $(\lambda N^{1/2})$ fixed as $N$ increases. We can compute the leading corrections by using the replacements $\tilde{\chi} = \chi/N^{1/2}$, $\chi = \chi/N^{1/2}$ and $\tilde{\chi} = \tilde{\chi}/N^{1/2}$ in the results of section 2 and 3. For the pure scalar case, $V_{NL}$ in eq. (11) gives at most $O(N^{-1/2})$ corrections. The rest of eq. (11) gives $O(N^{1/2})$ and $O(1)$ corrections. For $g \neq 0$, the gauge-loop ring sum included in (33) gives $O(1)$ and also more subdominant corrections. Thus the $O(N^{1/2})+O(1)$ corrections can then be easily found in this way. The final expression is

$$V_{eff} = -\frac{3N^{1/2}}{2N\sqrt{\lambda}} \chi^2 + \frac{N^{1/2}}{2} \chi (\varphi^2 - \tilde{v}^2) + \frac{N^{1/2} \Lambda \chi}{4\pi^2} - \frac{\chi^{3/2}}{12\pi} + \frac{3\Lambda g^2}{4\pi^2} \varphi^2 - \frac{g^3}{12\pi} \left[(\Lambda'/2\pi^2 + \varphi^2)^2 + 2|\varphi|^3\right] + \text{const.}$$  \hspace{1cm} (45)

$\chi$ is found from its $O(N^{1/2})$ equation of motion, and the comments after (44) apply. For negative $\chi$ we have to give meaning to eq. (15). One possibility is to simply use $\text{Tr} \ln(-\tilde{g}^2 + \chi) = \Lambda \chi/2\pi^2 + \text{const.}$ This leads to a real potential, but we have no rigorous way of justifying this prescription.
5. The Phase Transition.

We first discuss our results generally before investigating the phase transition. As mentioned in the introduction, ordinary 3+1d perturbation theory in \( \hbar \) is not reliable for high temperature studies of spontaneously broken theories because it suffers from infrared divergence problems just near the value of the temperature at which the phase transition from the symmetric phase to the broken symmetry phase occurs. One can try to circumvent this problem either by trying to resum certain infinite classes of Feynman diagrams to all orders in \( \hbar \) or by performing perturbation theory in another parameter, say \( 1/N \). In order to extract reliable results such a new perturbation series, if it is calculable to any given order in \( 1/N \), must be renormalizable and avoid infrared divergence problems to any given order in \( 1/N \).

For a weakly coupled 3+1d abelian Higgs model at high \( T \) with \( 6g^2/\lambda \leq O(1) \) we introduced an auxiliary field \( \chi \) in the Lagrangian so as to develop a systematic expansion for \( V_{eff} \) in \( 1/N \), where \( N \) is the number of real scalar fields. This was the content of section three. The \( O(N) \) and \( O(1) \) corrections in this expansion are renormalizable and avoid the infrared divergence problems of perturbation theory in \( \hbar \). The result can be expanded to all orders in \( \hbar \) and in fact corresponds to summing certain infinite classes of ordinary Feynman diagrams in \( \hbar \). The \( O(N) \) result for \( V_{eff} \) incorporates the sum of all superdaisy graphs of Dolan and Jackiw [1] at \( O(N) \); the next-to-leading order result incorporates much more, i.e. the sum of all Feynman diagrams of \( O(1) \) as well as \( O(1) \) contributions from the superdaisies. The \( O(1) \) gauge contributions are given by a gauge loop ring sum (but not quite the simple one of [4]). The beauty of our systematic approach over trying to add up some infinite classes of diagrams by hand is that there is no overcounting of \( O(\hbar^2) \) - or any other - Feynman diagrams and in addition the constraint of renormalizability helps to locate all relevant contributions to \( V_{eff} \) in a consistent computation.

For weak \( \lambda \) and strong \( g \) it is not possible to define an expansion in \( 1/N \), and for a weakly coupled theory it is only possible to set \( 6g^2/\lambda \sim O(N) \) if the scalar self-coupling is not just weak but very weak. In this case the expansion of section 3 is unreliable because the dominant contribution is no longer from the pure scalar fields; this is demonstrated soon. For a very weakly coupled 4d pure scalar field theory at high temperature with \( \lambda N \sim O(1) \) it is possible to write \( V_{eff} \) as the sum of \( O(1) \) and \( O(1/\sqrt{N}) \) terms, with subdominant terms of \( O(1/N) \). In section 4 these were found to give just the daisy sum (not superdaisy sum) of Dolan and Jackiw [1]. If computed consistently to \( O(1/\sqrt{N}) \) the daisy sum cannot lead to spurious field dependent terms that come from overcounting of Feynman diagrams in \( \hbar \). The daisy sum is of course renormalizable and avoids infrared divergence problems. In the gauged case with very weak \( \lambda \) and \( 6g^2/\lambda \sim O(N) \) a calculation of \( V_{eff} \) to \( O(1/\sqrt{N}) \) is complicated because the contribution from the gauge sector involves more than just the gauge–loop ring sum of section 3. In fact, if one just adds the pure scalar daisy sum result to \( O(1/\sqrt{N}) \) with the gauge–loop ring sum to \( O(1/\sqrt{N}) \) then the result is multiplicatively nonrenormalizable in the context of the 3d field theory. We did not find a simple expression for the \( O(1/\sqrt{N}) \) corrections involving gauge fields and do not know if these corrections have infrared divergence problems or not. The \( O(1) \) corrections
however have no such problem and in addition give a gauge fixing independent effective potential to leading order.

For $6g^2/\lambda \sim O(N^{3.5})$ it is possible to easily determine the $O(N^{3.5})$ and $O(1)$ contributions to $V_{eff}$, with subdominant contributions of $O(N^{-3.5})$. These were found in section 4 and have the interpretation of the daisy sum of the pure scalar case plus a gauge–loop ring sum taken in the limit of vanishing (field dependent) scalar mass. The effective potential to $O(1)$ has no infrared divergence problems, is renormalizable and is gauge fixing parameter independent.

We now study the implications of our results for the phase transition in the weakly coupled abelian Higgs model. As noted in the introduction, we derived our results by working with an effective 3d field theory (given by eq (28)) and matching linearly divergent terms in the effective theory with finite $O(T^2)$ corrections in the full 4d model. The “neglected terms” therefore incorporate subleading terms that were dropped due to the approximations made in calculating the 3d effective potential, as well as finite terms that the 3d effective theory cannot account for and that arise when the full 4d model is used. We stress that when $\chi \ll T^2$ and $\phi^2 \ll T^2$, the 3d effective theory is sufficient to calculate the leading (i.e. $T$ dependent) finite corrections. This is of course the region of interest for the phase transition.

**The case** $6g^2/\lambda \leq O(1)$. Our main result is the leading + next-to-leading order high temperature $V_{eff}(\phi^2)$ for a 3+1d abelian Higgs model with $N$ real scalars, computed in the systematic $1/N$ expansion. For the tree Lagrangian given by (21) we obtained

$$V_{eff} = -\frac{3N}{2\lambda}\chi^2 + \frac{N}{2}\chi(\phi^2 - v^2) + \frac{T^2\chi}{24}(N + 2 + \frac{18g^2}{\lambda})$$

$$- \frac{T}{12\pi} \left( \left( 3\chi + \frac{\lambda}{3}(v^2 - T^2/12) \right)^{1/2} + (N - 1)\chi^{1/2} \right)$$

$$- \frac{T(6g^2/\lambda)^{3/2}}{12\pi} \left[ \left( \chi + \frac{\lambda}{6}(v^2 + T^2/12) \right)^{3/2} + 2 \left( \chi + \frac{\lambda}{6}(v^2 - T^2/12) \right)^{3/2} \right]$$

$$+ V_\alpha + \text{neglected terms} + O(1/N) \text{ terms}. \quad (46)$$

where $V_\alpha$ is

$$V_\alpha = -\frac{T}{12\pi} \left[ (\chi + \alpha g^2 \phi^2)^{3/2} - \chi^{3/2} - \alpha^{3/2} g^3 |\phi|^3 \right], \quad (47)$$

and $\chi(\phi)$ is the solution of the equation $\partial V_{eff}/\partial \chi = 0$. To next–to–leading order it is sufficient [9] to use not the full solution but the $O(N)$ solution of (19),

$$\sqrt{\chi} = \frac{\lambda T}{48\pi} \left[ 1 + \frac{32\pi^2}{\lambda} \left( \frac{12\phi^2}{T^2} - \frac{12v^2}{T^2} + 1 \right) - 1 \right]. \quad (48)$$

There is another solution of $\sqrt{\chi}$ which comes with an overall minus sign, but this is unphysical [6,8,9]. Our $V_{eff}$ is valid for $T \gg \sqrt{\chi} > \lambda T/12\pi, g^2 T/12\pi$ and $T^2 \gg \phi^2 -
\( T \sqrt{\chi}/4\pi \gg g^2 T^2/(64)^2 \). Also, eq. (48) was used to rewrite the \( O(1) \) terms in (46), which adds \( \lambda(\partial V_1/\partial \chi)^2/3N \) to the \( O(1/N) \) terms, where \( V_1 \) is the \( O(1) \) part of (46). In the Landau gauge, and near \( T = \sqrt{12}v \), this particular \( O(1/N) \) term is negligible for \( \chi/T^2 > (\lambda + 9g^2)(1 + (6g^2/\lambda)^2/2)/3N^2 \).

In our approach vacuum polarization effects at large external momentum played an important role in the same way as they did for the previously studied \([6]\) \( g = 0 \) case.

It is known \([6]\) that at order \( N \), the potential (46) admits no 1st order phase transition. The \( O(N) \) potential is exact in the small \( \chi \) limit (up to 4d corrections). To \( O(N) \), \( dV_{\text{eff}}/d\phi^2 = \partial V_{\text{eff}}/\partial \phi^2 = N \chi/2 \). At \( \phi^2 = 0 \) this vanishes at \( T^2_2 = 12v^2 \). For \( T > \sqrt{12}v \) the origin is a global minimum, and at \( T = T_2 \) the potential grows away from the origin. For \( T > T_2 \) the point \( \chi = 0 \) is away from the origin and this has the interpretation \([8,9]\) as the symmetry breaking minimum below \( T_2 \). For the case \( g = 0 \), Root \([9]\) has shown for the 3d case that the point \( \chi = 0 \) remains a local minimum at next–to–leading order. This analysis did not require a computation of \( V_{\text{eff}} \). Root examined \( dV/d\phi^2 \) in the limit of vanishing \( \chi \) and showed that the leading order gap–equation for \( \chi \) (in our case eqs. (17) and (19)) was sufficient to show that \( V_{\text{eff}}/d\phi^2 = 0 \) still has a solution at \( \chi = 0 \) at next–to–leading order. His analysis can be extended to our gauged case. We will not present a detailed analysis here but instead refer the reader to Root and also ref. \([18]\) where, following Root, an analysis of the small \( \chi \) limit in the full 4d high temperature abelian Higgs model has been performed with the result that \( \chi = 0 \) remains a point of vanishing \( dV_{\text{eff}}/d\phi^2 \) at next–to–leading order.

We believe the fact that \( \chi = 0 \) remains a point of vanishing \( dV_{\text{eff}}/d\phi^2 \) is however insufficient to deduce a second order phase transition to this order. As mentioned by Root, for sufficiently large \( N \) the \( 1/N \) corrections cannot overwhelm the leading order conclusion of a second order phase transition. For \( N \) not arbitrarily large we would like to know the global properties of \( V_{\text{eff}} \) away from the point \( \chi = 0 \), and in particular if there is a point away from the origin at \( T = \sqrt{12}v \) where the \( 1/N \) corrections can produce a new minimum and possibly even result in the breakdown of the \( 1/N \) expansion. In addition, for \( T > \sqrt{12}v \) the point \( \chi = 0 \) never occurs, so the analysis of \([9]\) is by itself insufficient to deduce \( \phi^2 = 0 \) remains a global minimum for \( N \) not arbitrarily large. Our computation of \( V_{\text{eff}} \) sufficiently away from \( \chi = 0 \) gives global information that Root’s analysis cannot give. In addition, (46) appears to characterize the correct behavior in the limit of vanishing \( \chi \). Assuming no pathological behaviour occurs in an exact computation of \( V_{\text{eff}} \) at next–to–leading order for \( \sqrt{\chi} < \lambda T/12\pi, g^2 T/12\pi \) we might expect that (46) gives a good description all the way down to \( \sqrt{\chi} = 0 \) and also \( \phi^2 = 0 \) at \( \sqrt{\chi} = 0 \).

With these points in mind we now investigate if, more generally, (46) can exhibit a first order phase transition for \( 6g^2/\lambda \sim O(1) \).

The critical temperature \( T_2 \) is given by the vanishing of \( dV/d\phi^2 \) at \( \phi^2 = 0 \). Let us write \( V_{\text{eff}} = V_N + V_1 + V_\alpha \), where \( V_N \) is the \( O(N) \) potential. Then since \( \partial V_N/\partial \chi = 0 \) we
\[
\frac{dV}{d\phi^2} = \left[ \frac{\partial V_1}{\partial \chi} + \frac{\partial V_\alpha}{\partial \chi} \right] \frac{\partial \chi}{\partial \phi^2} + \frac{\partial V_{\text{eff}}}{\partial \phi^2}
\]
\[
= \left[ \frac{\partial V_1}{\partial \chi} + \frac{\partial V_\alpha}{\partial \chi} \right] \left( \frac{6}{\lambda} + \frac{T}{8\pi \sqrt{\chi}} \right)^{-1} + \frac{N \chi}{2} - \frac{T}{8\pi} \left[ \alpha g^2 \sqrt{\chi + \alpha g^2 \phi^2} - \alpha^2 g^3 |\phi| \right]
\] (49)

As indicated by our preliminary remarks, this vanishes at \( \chi = 0 \) and at the origin this translates to the gauge fixing independent result
\[
T_2^2 = 12v^2.
\] (50)

This is in fact the leading order result again (i.e. as if \( V_1 \) were absent). The reason is that in a general model the \( O(1/N) \) terms in \( V_{\text{eff}} \) are important in determining the \( O(1/N) \) corrections to \( T_2^2 \) (this is related to the fact that \( \chi = 0 \) remains a point of vanishing \( dV_{\text{eff}}/d\phi^2 \) at \( O(1) \)).

For \( T > T_2 \), \( \phi^2 = 0 \) is the local minimum. At \( \phi^2 = 0 \), \( V_\alpha = 0 \) so \( V_{\text{eff}} \) is gauge parameter independent at this minimum. For there to be a first order phase transition, eq. (49) must have zeros for \( T > T_2 \) and, at \( T = T_2 \), \( dV/d\phi^2 \) should be negative away from \( \phi^2 = 0 \). Let us work in the Landau gauge, \( \alpha \to 0 \) (in an arbitrary gauge \( \alpha \) should be not more than \( O(1) \) [4]). Then at \( T = T_2 \),
\[
\frac{\partial V_1}{\partial \chi} = \frac{T_2^2}{12} \left( 1 + \frac{9g^2}{\lambda} - \frac{\sqrt{6}g}{4\pi} (6g^2/\lambda) \right) - \frac{T_2}{8\pi} \sqrt{\lambda} \left( 3\sqrt{3} - 1 + 2(6g^2/\lambda)^2 \right).
\] (51)

\( \chi \) increases with \( \phi^2 \), hence a necessary condition for \( dV_{\text{eff}}/d\phi^2 \) to become negative at \( T = T_2 \) is that (51) becomes negative. At \( g = 0 \), (51) vanishes only when \( \sqrt{\lambda} \sim \pi T_2/6 \) by which time the effective theory is no longer valid and the positive contribution to \( dV/d\phi^2 \) from the \( O(N) \) term in (49) already dominates. Hence for \( g = 0 \) no first order phase transition is possible by \( T = T_2 \). For \( 6g^2/\lambda \sim O(1) \), \( g^2 \ll 2\pi \), the same conclusion can be reached, namely that within a consistent \( 1/N \) approximation the leading+next-to-leading \( V_{\text{eff}} \) for our weakly coupled abelian Higgs model does not admit a first order phase transition. (Note the absence of an \( O(|\phi|^3) \) term at \( T = T_2 \) – see eq. (48).)

If \( 6g^2/\lambda \) is of \( O(N) \) then (51) can become strongly negative, indicating the possible breakdown of the \( 1/N \) expansion. Note also that the “next–to–leading” corrections to the coefficient of \( \chi \) in (46) actually compete with the \( O(N) \) coefficient and therefore our results cannot be used in this case. 

**The case \( 6g^2/\lambda \sim O(N) \).** Our main result is the high \( T \) contribution to \( V_{\text{eff}} \) which survives in the limit of arbitrarily large \( N \). For the tree Lagrangian given by (21) we obtained (after eliminating \( \chi \)):
\[
V_{\text{eff}} = \frac{\lambda N}{4!} \phi^2 (\phi^2 - 2v^2 + T^2/6) + \frac{g^2 T^2}{8} \phi^2 - \frac{g^3 T}{12\pi} \left( T^2/6 + \phi^2 \right)^{\frac{3}{2}} + 2|\phi|^3 + O(1\sqrt{N}) + \text{neglected terms.}
\] (52)

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Note that for weak coupling one must keep \( g \) and \( (\lambda N) \) fixed as \( N \) increases. As mentioned, the potential above is \( \alpha \) independent to \( O(1) \) in the way we computed it. One can also check that it is gauge fixing parameter independent with the slightly different gauge fixing considered in [4]. Eq. (52) is strictly valid for \( T^2 \gg \phi^2 \gg T^2 g^2/(64)^2 \). The critical temperature \( T_2 \) determined from (52) is given by

\[
12v^2 = T_2^2 \left[ 1 + 3(6g^2/\lambda N) - \frac{g\sqrt{3}}{\pi \sqrt{2}}(6g^2/\lambda N) \right].
\]

(53)

We assume \( g \ll 2\pi \) so that \( T_2 \) here is lower than the leading order critical temperature for the \( 6g^2/\lambda \leq O(N) \) case, \( \sqrt{12v} \). At \( T = T_2 \) the minimum is no longer at the origin and occurs at the point

\[
\frac{|\phi|_{\text{min}}}{T_2} = \frac{1}{2} g(6g^2/\lambda N),
\]

(54)

indicating a first order phase transition whose strength grows with the gauge coupling.

Since we did not compute subleading corrections to \( V_{\text{eff}} \) all we can say is that the model exhibits a first order phase transition for sufficiently large \( N \). This is in contrast to our results in the \( 6g^2/\lambda \leq O(1) \) case.

The case \( 6g^2/\lambda \sim O(N^{\frac{2}{3}}) \). This case is the most interesting because, unlike the case \( 6g^2/\lambda \sim O(N) \), the next-to-leading corrections are easily determined. The high temperature potential is

\[
V_{\text{eff}} = \frac{\lambda N}{4!} \phi^2(\phi^2 - 2v^2 + T^2/6) + \frac{g^2 T^2}{8} \phi^2 - \frac{\chi^2}{12\pi} \theta(\chi)
\]

\[- \frac{g^2 T}{12\pi} \left[ (T^2/6 + \phi^2)^{\frac{3}{2}} + 2|\phi|^3 \right] + O(N^{-\frac{4}{3}}) + \text{neglected terms.}
\]

(55)

Here,

\[
\chi = \left( \frac{\lambda N^{\frac{2}{3}}}{6} \right) (\phi^2 - v^2 + T^2/12).
\]

(56)

We note that for weak coupling, \( \lambda N^{\frac{2}{3}} \) must be held fixed as \( N \) increases and that the result for \( V_{\text{eff}} \) is gauge fixing parameter independent to \( O(1) \). Finally, note the appearance of the step function \( \theta(\chi) = 1 \) for \( \chi > 0 \), 0 otherwise, in the expression for \( V_{\text{eff}} \). As described in section 4 there is an uncertainty involved in defining the gaussian integral over the quantum scalar fields when \( \chi \) is negative; we have chosen an answer that in light of the results for the cases \( 6g^2/\lambda \leq O(1) \) and \( 6g^2/\lambda \sim O(N) \) gives a physically appealing
interpretation. Therefore, our results here are somewhat speculative. We assume \( g \ll 2\pi \) in what follows. Also \( V_{\text{eff}} \) is strictly valid for \( T^2 \gg \phi^2, \chi \) and \( \phi^2 \gg g^2 T^2/(64)^2 \).

The leading order critical temperature is the same as in the case \( 6g^2/\lambda \leq O(1) \), i.e. \( T_2 = \sqrt{12v} \). The \( O(N^{\frac{4}{3}}) \), potential given by the first term in (55), does not admit a first order phase transition. The next–to–leading order \( T_2 \) is modified. We find the solution \( dV_{\text{eff}}/d\phi^2 \) at \( \phi^2 = 0 \) occurs at negative \( \chi \) and it is again given by relation (53) (but note that it reduces to \( 12v^2 = T_2^2 \) for \( N \to \infty \) here). At \( T = T_2 \) the global minimum is away from the origin and at (54). For \( 6g^2/\lambda \sim O(N^{\frac{2}{3}}) \) we have

\[
\frac{\phi|_{\text{min}}}{T_2} \to 0 \quad \text{as} \quad N \to \infty,
\]

indicating a first order transition that gets weaker and weaker as \( N \) increases.

We end this section with the following discussion. If we chose a prescription without the step function in (55) the potential is complex at or just above \( T_2 \) given by (53) and close enough to the origin. Since we have all terms to \( O(1) \) the imaginary part cannot be removed at sufficiently large \( N \) by subdominant corrections. We do not know the meaning of this result; neither do we know in cases a) and b) if subdominant corrections we have not computed can lead to the possibility of a complex potential at \( T \) just above when the phase transition is supposed to take place. However, one can check that in the case here even without the step function in (55) the expression for \( T_2^2 \) is real to \( O(N^{-\frac{1}{3}}) \) and given by (53) and also that the value of \( V_{\text{eff}} \) at the minimum (54) is real to \( O(1) \). Our physical results, to the order we are working in, are therefore not really so speculative.

6. Conclusions.

We have found that for large \( N \) the weakly coupled abelian Higgs model can exhibit either first or second order phase transitions with first order behaviour possible for \( 6g^2/\lambda \geq O(N) \). For \( 6g^2/\lambda \sim O(N^{\frac{2}{3}}) \) the model exhibits a weak first order phase transition at large \( N \). Intuitively, this only says that the gauge coupling must be large enough so that the gauge contributions can dominate or at least compete with the pure scalar contributions to \( V_{\text{eff}} \). For the three cases a) \( 6g^2/\lambda \leq O(1) \), b) \( 6g^2/\lambda \sim O(N) \) and c) \( 6g^2/\lambda \sim O(N^{\frac{2}{3}}) \) we estimate that our results are reliable whenever the parameter a) \( N^{-1} \), b) \( N^{-\frac{1}{2}} \) and c) \( N^{-\frac{1}{3}} \) is sufficiently small. An inspection of our results for case a) shows that even for \( N = 4 \) the next to leading corrections to the coefficient of \( \phi^2 \) in \( V_{\text{eff}} \) compete with the leading correction to the coefficient of \( \phi^2 \) in \( V_{\text{eff}} \). We do not believe any of our results can be directly applied to the \( N = 4 \) case; however they can give some indication of what happens at \( N = 4 \) in a very weakly coupled theory with \( g \ll 1, \lambda \ll 1 \). In this limit, cases b) and c) admit only a very weak first order phase transition. For case a) it can be argued on the basis of the 3d field theory (and dimensionality of the 3d couplings) that the coefficient of \( \phi^2 \) in \( V_{\text{eff}} \) to lowest order in the couplings and to all orders in \( 1/N \) is given by the one–loop \( O(h) \) graphs. This yields a critical temperature given by [1]

\[
T_2^2 (1 + 2/N + 18g^2/\lambda N) = 12v^2.
\]

(58)
For $N = 4$ and $6g^2/\lambda \sim 1$ this gives a critical temperature $T_2$ much below $\sqrt{12v}$ and demonstrates that our analysis of case a) is insufficient to rule out a significant first order phase transition. Assuming that $O(1/N)$ corrections to (46) are negligible it is possible to use (46) to study the phase transition to temperatures below $T = \sqrt{12v}$. The leading order solution $\sqrt{\chi}$ of (48) becomes negative and eventually complex near the origin as we lower the temperature below $T = \sqrt{12v}$. To avoid this problem one can use the full solution for $\chi$ from (46), not just the leading order one. This approach is not entirely consistent but the result of the analysis can be shown to be again a second order phase transition to $T \geq T_2$ given by (58). This only demonstrates that any first order behaviour is contained in $O(1/N)$ corrections to $V_{\text{eff}}$ which we can expect to be small even at $N = 4$. Hence our result for all cases is that for a very weakly coupled model the phase transition of the exact model, even at $N = 4$, will be for all practical purposes second order.

For $g \sim 1$ and $N = 4$, we believe that a reliable determination of the nature of the phase transition cannot be given by the $O(N) + O(1)$ corrections in case a), nor by the daisy and ring sum results of cases b) and c). In all cases more subleading contributions must be computed than those that we have computed or currently exist in the literature. These extra contributions may be negligible even for $g \sim 1$ and $N = 4$ but this is not a priori clear. However if the exact $g = 0$ model admits only a second order phase transition then we can expect that for $g \neq 0$ and $6g^2/\lambda \leq O(1)$ the exact model will admit at best quite weak first order behaviour.

Our physical results are in all cases gauge fixing parameter independent, they do not overcount any Feynman diagrams and do not suffer from any infrared problems. We found no terms of $O(T^3|\phi|)$ in $V_{\text{eff}}$ in any of the cases. One can argue on the basis of the 3d field theory, dimensionality of the 3d couplings and renormalizability that any such terms must be very small in a very weakly coupled model. More generally, any such terms must be vanishingly small in the limit of large $N$.

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Note Added. As we completed this work we recieved a paper by J. March–Russell, LBL–32540, PUPT–92–1328, which also considers 2nd order phase transitions in gauge theories, but using $\epsilon$–expansion techniques.
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[10] W. A. Bardeen and M. Moshe, Phys. Rev. D28: 1372 (1983) and D34: 1229 (1986).

[11] M. Lüscher, unpublished notes. There is a technical point here. We used the form of the auxiliary field addition in (3) appropriate for the Minkowski space Lagrangian. If one works in euclidean space then the 1/N expansion can be justified at next–to–leading order by taking an imaginary background $\chi$ and a real fluctuation $\hat{\chi}$. Then, the gaussian integral over $\hat{\chi}$ is convergent for values of $\chi$ which we will be interested in. With our form (3) we consider the background real and the fluctuations imaginary.

[12] M. K. Gaillard, in Particle Physics: Cargèse 1987 (NATO ASI, Series B: Physics, Vol. 173), ed. by M. Levy, et al., Plenum, New York, 1988. See also M. K. Gaillard and T. R. Taylor, LBL–31464 (1992). The precise answer for the divergent terms in the effective theory depends on the regularization prescription used. Since we identify the ultraviolet regularization scale with the scale at which the full theory becomes important this ambiguity only says that the effective theory cannot account for the precise way in which full theory comes into play at high energy; i.e. the effective theory could be the low energy limit of several different models. Thus, each of the divergent terms should be given arbitrary coefficients (consistent with all symmetries). This arbitrariness can be removed by matching the divergent terms in the effective theory with the results of known calculations in the full model. In our case, the linear divergences in the effective 3d theory are related to quadratic divergences – and hence $O(\delta^2)$ corrections – in the full theory. These are well known and so the match is easy. In hindsight we found that for the pure scalar case that the same matching condition worked for all the linear divergences, while in the gauged case we needed two conditions.

[13] As pointed out by P. Weisz, $\text{Tr} \ln(-\vec{\partial}^2 + a\sqrt{-\partial^2 + b})$ can be evaluated exactly in 3d. With a sharp momentum cutoff $A$, and the limits $\Lambda^2 \gg b \gg a^2$ one obtains the approximate solution $\text{Tr} \ln(-\vec{\partial}^2 + a\sqrt{-\partial^2 + b}) \approx \text{Tr} \ln(-\vec{\partial}^2 + b) - ab\ln(\Lambda^2/b)/4\pi^2$. The log divergent term can be ignored in comparison with the linearly divergent term whenever $(a/\Lambda)\ln(\Lambda^2/b) \ll 1$, which is clearly true with our assumptions. The log divergent piece can be ignored in comparison to the finite piece from $\text{Tr} \ln(-\vec{\partial}^2 + b)$ whenever $(a/\sqrt{b})\ln(\Lambda^2/b) \ll 1$, which is an extra assumption we will make.

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