Status of Sterile Neutrino fits with Global Data

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DPF Meeting
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Outline

• Oscillation Formalism
• Anomalies and Sterile Neutrinos
• Overview of Experiments
• Global Fits
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Standard Model

Quarks:
- u (up) charge 2/3, spin 1/2, mass ≈2.3 MeV/c²
- c (charm) charge 2/3, spin 1/2, mass ≈1.275 GeV/c²
- t (top) charge 2/3, spin 1/2, mass ≈173.07 GeV/c²
- d (down) charge -1/3, spin 1/2, mass ≈4.8 MeV/c²
- s (strange) charge -1/3, spin 1/2, mass ≈95 MeV/c²
- b (bottom) charge -1/3, spin 1/2, mass ≈4.18 GeV/c²

Leptons:
- e (electron) charge -1, spin 1/2, mass 0.511 MeV/c²
- μ (muon) charge -1, spin 1/2, mass 105.7 MeV/c²
- τ (tau) charge -1, spin 1/2, mass 1.777 GeV/c²

Neutrinos:
- νᵽ (electron neutrino) charge 0, spin 1/2, mass <2.2 eV/c²
- νₑ (electron neutrino) charge 0, spin 1/2, mass <0.17 MeV/c²
- νₜ (tau neutrino) charge 0, spin 1/2, mass <15.5 MeV/c²

Bosons:
- γ (photon) mass 0
- Z (Z boson) mass 91.2 GeV/c²
- W (W boson) mass 80.4 GeV/c²
- H (Higgs boson) mass ≈126 GeV/c²
PMNS Matrix

The weak interaction eigenstates ($\nu_e, \nu_\mu, \nu_\tau$) are a linear superposition of mass eigenstates ($\nu_1, \nu_2, \nu_3$), given by the Pontecovo-Maki-Nakagawa-Sakata (PMNS) Matrix.

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
$$

As with any transformation matrix, PMNS is unitary
Two Neutrino Oscillation

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix}
\]

- \( P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2(1.27 \frac{\Delta m^2 [eV^2]}{E_\nu [MeV]} L [m]) \)
- \( P(\nu_e \to \nu_\nu) = 1 - \sin^2(2\theta)\sin^2(1.27 \frac{\Delta m^2 L}{E_\nu}) \)

Where \( \Delta m^2 = 0.002 \, eV^2 \), \( \sin^2(2\theta) = 0.8 \), \( E_\nu = 1 \, GeV \)
And Three Neutrinos

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = 
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
\]

A little more complicated, but the idea is the same

\[
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha \beta} - 4 \sum_{j>i} Re(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}) \sin^2 \left( \frac{1.27 \Delta m_{ij}^2 L}{E} \right)
\]

\[
+ 2 \sum_{j>i} Im(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}) \sin \left( \frac{2.53\Delta m_{ij}^2 L}{E} \right)
\]

The PMNS Matrix is frequently written as

\[
U_{\text{PMNS}} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13} e^{i\delta} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Important Values

\[
\begin{pmatrix}
|U_{e1}| & |U_{e2}| & |U_{e3}| \\
|U_{\mu1}| & |U_{\mu2}| & |U_{\mu3}| \\
|U_{\tau1}| & |U_{\tau2}| & |U_{\tau3}|
\end{pmatrix}
\sim
\begin{pmatrix}
0.82 & 0.55 & 0.15 \\
0.38 & 0.57 & 0.70 \\
0.39 & 0.59 & 0.69
\end{pmatrix}
\]

\[\theta_{12} \approx 34^\circ \quad \sin^2(2\theta_{12}) \approx 0.85\]
\[\theta_{23} \approx 42^\circ \Rightarrow \sin^2(2\theta_{23}) \approx 0.99\]
\[\theta_{13} \approx 8.5^\circ \quad \sin^2(2\theta_{13}) \approx 0.08\]

\[\delta_{cp} = 261^\circ \pm 55 \rightarrow \text{Note that this is within } 2\sigma \text{ of } 360^\circ \text{ (i.e. } 0^\circ)\]

\[\Delta m^2_{21} = (7.5 \pm 0.2) \times 10^{-5} eV^2\]
\[|\Delta m^2_{32}| = (2.52 \pm 0.04) \times 10^{-3} eV^2\]
Important Values

\[
\begin{pmatrix}
|U_{e1}| & |U_{e2}| & |U_{e3}| \\
|U_{\mu1}| & |U_{\mu2}| & |U_{\mu3}| \\
|U_{\tau1}| & |U_{\tau2}| & |U_{\tau3}|
\end{pmatrix} \sim \begin{pmatrix}
0.82 & 0.55 & 0.15 \\
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\[\delta_{cp} = 261^\circ \pm 55 \rightarrow \text{Note that this is within } 2\sigma \text{ of } 360^\circ \text{ (i.e. } 0^\circ)\]

\[\Delta m^2_{21} = (7.5 \pm 0.2) \times 10^{-5} eV^2\]
\[\Delta m^2_{32} = (2.52 \pm 0.04) \times 10^{-3} eV^2 \quad \text{Small!}\]
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Liquid Scintillator Neutrino Detector

- Used accelerator at Los Alamos to impinge a proton beam on a target to create $\pi^+$, which would decay to a $\mu^+$, which also decay

\[
\pi^+ \rightarrow \mu^+ + \nu_\mu \\
\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu
\]

- Searched for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ appearances

- An excess of $87.9 \pm 22.4 \pm 6.0 \bar{\nu}_e$ was found above background.
Another Mass Splitting?

If interpreted as an oscillation from a two neutrino model, the results have a best fit of $\sin^2(2\theta) = 0.003$ and $\Delta m^2 = 1.2$ eV$^2$. This mass splitting is much higher than the two seen above!

A. Aguilar-Arevalo et al. [LSND Collaboration], Phys. Rev. D 64, 112007 (2001) [hep-ex/0104049].
More Neutrinos?

Can there be more than 3 neutrinos?

Consider the Z decay width:

\[ \Gamma_Z = 3 \Gamma_{ll} + \Gamma_{hadrons} + N_\nu \Gamma_{\nu\nu} \]

\[ N_\nu = \frac{\Gamma_Z - 3 \Gamma_{ll} - \Gamma_{hadrons}}{\Gamma_{\nu\nu}^{SM}} \]

\[ N_\nu = 2.9840 \pm 0.0082 \]

Experiments suggests that there only exist 3 neutrinos that interact weakly!
An additional neutrino must then be “sterile” (i.e. does not interact weakly.)
HOW TO ADD MORE NEUTRINOS?
3+1 Model

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_s
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\
U_{s1} & U_{s2} & U_{s3} & U_{s4}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix}
\]
3+1 Model

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_s
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \\
U_{s1} & U_{s2} & U_{s3} & U_{s4}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix}
\]

What we’ve seen before
3+1 Model

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_s
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \\
U_{s1} & U_{s2} & U_{s3} & U_{s4}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix}
\]

What we’ve seen before

New independent parameters, as well as \( \Delta m^2_{41} \)
3+1 Model

\[ \Delta m^2_{\text{sterile}} >> \Delta m^2_{\text{atm}} > \Delta m^2_{\text{solar}} \]

What we’ve seen before

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_s
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} \\
U_{s1} & U_{s2} & U_{s3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix}
\]

New independent parameters, as well as \( \Delta m^2_{41} \)
What about even **MORE** Neutrinos?

- **3+2 model:**

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_{s1} \\
\nu_{s2}
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} \\
U_{s11} & U_{s12} & U_{s13} \\
U_{s21} & U_{s22} & U_{s23}
\end{pmatrix}
\begin{pmatrix}
U_{e4} & U_{e5} \\
U_{\mu4} & U_{\mu5} \\
U_{\tau4} & U_{\tau5} \\
U_{s14} & U_{s15} \\
U_{s24} & U_{s25}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\nu_5
\end{pmatrix}
\]

More independent parameters, with $\Delta m^2_{41}$, $\Delta m^2_{51}$, $\Phi_{41}$

We introduce a CP violating phase!
More Neutrinos: MiniBooNE

- Can go into neutrino or antineutrino mode
- Searched for $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$
- Higher energies than LSND
- Higher backgrounds than LSND 😞
- Systematic not statistics limited
Results

- Neutrino mode saw an excess of $160.0 \pm 47.8$ above the null, primarily under 475 MeV, and almost none above
- Antineutrino mode saw an excess of $78.4 \pm 28.5$ events, both above and below 475 MeV
- LSND saw its excess above an equivalent of 475 MeV, given the MB baseline
Differences in compatibility between neutrino and antineutrino data suggests CP violation, i.e. $3+(>1)$ models
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\[ \bar{\nu}_\mu \rightarrow \bar{\nu}_e \text{ or } \nu_\mu \rightarrow \nu_e \]

**LSND:**  
\[ \bar{\nu}_\mu \rightarrow \bar{\nu}_e \]

**KARMEN:**  
\[ \bar{\nu}_\mu \leftrightarrow \bar{\nu}_e \]

**MiniBooNE:**  
- **BNB:**  
  \[ \nu_\mu \rightarrow \nu_e \]
  \[ \bar{\nu}_\mu \rightarrow \bar{\nu}_e \]

- **NuMI:**  
  \[ \nu_\mu \leftrightarrow \nu_e \]

**NOMAD:**  
\[ \nu_\mu \leftrightarrow \nu_e \]
$\nu_\mu \rightarrow \nu_\mu$ or $\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$

SciBooNE-MB: $\nu_\mu \leftrightarrow \nu_\mu$
$\bar{\nu}_\mu \leftrightarrow \bar{\nu}_\mu$

CCFR84: $\nu_\mu \leftrightarrow \nu_\mu$

CDHS: $\nu_\mu \leftrightarrow \nu_\mu$

MINOS: $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_\mu$

ATM: $\nu_\mu \leftrightarrow \nu_\mu$
\[ \nu_e \rightarrow \nu_e \text{ or } \bar{\nu}_e \rightarrow \bar{\nu}_e \]

**KARMEN/LSND:** \(\nu_e \leftrightarrow \nu_e\)

**BUGEY:** \(\bar{\nu}_e \rightarrow \bar{\nu}_e\)

**Gallium:** \(\nu_e \rightarrow \nu_e\)
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3+1: All Data

Best fit:

\[ \Delta m_{41}^2 = 1.75 \text{ eV}^2 \]
\[ |U_{e4}| = 0.163 \]
\[ |U_{\mu 4}| = 0.117 \]

\[ \sin(2\theta) = 1.45 \times 10^{-3} \]

\[ \chi^2 = 306.81 \text{ (318 dof)} \]
\[ \chi^2_{null} = 359.81 \text{ (315 dof)} \]

\[ \Delta \chi^2_{null-min} \text{ (dof)} = 52.34 \text{ (3)} \]
3+1 Appearance/Disappearance Tension

No overlap between appearance and disappearance data
Similarly, there is tension between the neutrino and antineutrino data
3+2 allows us to introduce a CP violating phase

Best fit of both mass differences like $<1 \text{ eV}^2$, but with a large range of possible solutions. The CP violating phase pretty much covers every possible value, and doesn’t improve fit by much

$$\Delta \chi^2_{null-min} (dof) = 56.99 \ (7)$$
Take away

• Tension between neutrino/antineutrino data, as well as appearance/disappearance in 3+1 data

• Introduction of CP violating phase in 3+2 model doesn’t improve things much

• A best fit of $\Delta m^2 \approx 2 \text{ eV}$ places MicroBooNE (L = 470 m, E = 700 MeV) at a better position for oscillation maximum:

• $\sin \left( 1.27 \frac{\Delta m^2 L}{E} \right) \approx \sin(1.7) \approx \sin(\frac{\pi}{2})$
FUTURE?
MicroBooNE Updates

• Blind analysis on the low energy excess is proceeding! (see Yates, Thurs afternoon, https://indico.fnal.gov/contributionDisplay.py?contribId=316&confId=11999)

![MicroBooNE Diagram]

Example simulated event, 1 electron and 1 proton:

Very high resolution events that allow excellent background rejection
MiniBooNE Updates

While MicroBooNE has taken data, MiniBooNE has been running
Total POT collected since 2015: 6E20 POT
Original data set, neutrino mode: 6.5E20 POT

Two analyses planned:
• Identical procedure and cuts as the original data set.
  • Current plan is to present a separate result for new runs, as well as a combined result.
• Introduce new information
  • Beam-bucket timing requirement- made possible by upgrades
MiniBooNE Updates

Beam is delivered in 81 RF “buckets” separated by 19.2 ns

MB can determine bucket max to <1ns

Measured bucket width: 1.5 ns (preliminary)

By requiring events to be in a bucket, we can cut photon backgrounds
Thank you. Questions?
Background Cut with Beam Timing

MB can measure the time from the Resistive Wall Monitor signal to the appearance of the event vertex.

To obtain event time incoming $\nu$ path assumed to be direct

External photons will have taken longer to reach detector
Another example is that a $\pi^0$ can appear late if it takes a diagonal path.
\[ P(\nu_\alpha \to \nu_\beta) \simeq -4 |U_{\alpha 5}| |U_{\beta 5}| |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{54} \sin^2(1.27 \Delta m_{54}^2 L/E) \\
-4 |U_{\alpha 6}| |U_{\beta 6}| |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{64} \sin^2(1.27 \Delta m_{64}^2 L/E) \\
-4 |U_{\alpha 5}| |U_{\beta 5}| |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{65} \sin^2(1.27 \Delta m_{65}^2 L/E) \\
+4(|U_{\alpha 4}| |U_{\beta 4}| + |U_{\alpha 5}| |U_{\beta 5}| \cos \phi_{54} + |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{64} |U_{\alpha 4}| |U_{\beta 4}| \sin^2(1.27 \Delta m_{41}^2 L/E) \\
+4(|U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{54} + |U_{\alpha 5}| |U_{\beta 5}| + |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{65} |U_{\alpha 5}| |U_{\beta 5}| \sin^2(1.27 \Delta m_{51}^2 L/E) \\
+4(|U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{64} + |U_{\alpha 5}| |U_{\beta 5}| \cos \phi_{65} + |U_{\alpha 6}| |U_{\beta 6}|) |U_{\alpha 4}| |U_{\beta 4}| \sin^2(1.27 \Delta m_{61}^2 L/E) \\
+2|U_{\beta 5}| |U_{\alpha 5}| |U_{\beta 4}| |U_{\alpha 4}| \sin \phi_{54} \sin(2.53 \Delta m_{54}^2 L/E) \\
+2|U_{\beta 6}| |U_{\alpha 6}| |U_{\beta 4}| |U_{\alpha 4}| \sin \phi_{64} \sin(2.53 \Delta m_{64}^2 L/E) \\
+2|U_{\beta 6}| |U_{\alpha 6}| |U_{\beta 5}| |U_{\alpha 5}| \sin \phi_{65} \sin(2.53 \Delta m_{65}^2 L/E) \\
+2(|U_{\alpha 5}| |U_{\beta 5}| \sin \phi_{54} + |U_{\alpha 6}| |U_{\beta 6}| \sin \phi_{64}) |U_{\alpha 4}| |U_{\beta 4}| \sin(2.53 \Delta m_{41}^2 L/E) \\
+2(-|U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{54} + |U_{\alpha 6}| |U_{\beta 6}| \sin \phi_{65}) |U_{\alpha 5}| |U_{\beta 5}| \sin(2.53 \Delta m_{51}^2 L/E) \\
+2(-|U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{64} - |U_{\alpha 5}| |U_{\beta 5}| \sin \phi_{65}) |U_{\alpha 6}| |U_{\beta 6}| \sin(2.53 \Delta m_{61}^2 L/E) . \]
\[ P(\nu_\alpha \to \nu_\beta) \approx \frac{4 |U_{\alpha 5}| |U_{\beta 5}| |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{54} \sin^2(1.27 \Delta m_{54}^2 L/E)}{4 |U_{\alpha 6}| |U_{\beta 6}| |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{64} \sin^2(1.27 \Delta m_{64}^2 L/E)} - \frac{4 |U_{\alpha 5}| |U_{\beta 5}| |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{65} \sin^2(1.27 \Delta m_{65}^2 L/E)}{4 |U_{\alpha 4}| |U_{\beta 4}| + 4 |U_{\alpha 5}| |U_{\beta 5}| \cos \phi_{54} + 4 |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{64} |U_{\alpha 4}| |U_{\beta 4}| \sin^2(1.27 \Delta m_{41}^2 L/E)} + 4 |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{54} + 4 |U_{\alpha 5}| |U_{\beta 5}| \cos \phi_{64} + 4 |U_{\alpha 6}| |U_{\beta 6}| \cos \phi_{65} |U_{\alpha 5}| |U_{\beta 5}| \sin^2(1.27 \Delta m_{51}^2 L/E)} + 4 |U_{\alpha 4}| |U_{\beta 4}| \cos \phi_{64} + 4 |U_{\alpha 5}| |U_{\beta 5}| \cos \phi_{65} + 4 |U_{\alpha 6}| |U_{\beta 6}| \sin \phi_{64} |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{54} \sin(2.53 \Delta m_{34}^2 L/E)} + 2 |U_{\beta 5}| |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{54} \sin(2.53 \Delta m_{65}^2 L/E)} + 2 |U_{\beta 6}| |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{64} \sin(2.53 \Delta m_{64}^2 L/E)} + 2 |U_{\beta 6}| |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{65} \sin(2.53 \Delta m_{51}^2 L/E)} + 2 |U_{\alpha 5}| |U_{\beta 5}| \sin \phi_{54} + 2 |U_{\alpha 6}| |U_{\beta 6}| \sin \phi_{64} |U_{\alpha 4}| |U_{\beta 4}| \sin(2.53 \Delta m_{41}^2 L/E)} + 2 |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{54} + 2 |U_{\alpha 6}| |U_{\beta 6}| \sin \phi_{65} |U_{\alpha 5}| |U_{\beta 5}| \sin(2.53 \Delta m_{51}^2 L/E)} + 2 |U_{\alpha 4}| |U_{\beta 4}| \sin \phi_{64} + 2 |U_{\alpha 5}| |U_{\beta 5}| \sin \phi_{65} |U_{\alpha 6}| |U_{\beta 6}| \sin(2.53 \Delta m_{61}^2 L/E) \]
\[ P(\nu_\alpha \rightarrow \nu_\beta) \approx -4|U_{\alpha 5}||U_{\beta 5}||U_{\alpha 4}||U_{\beta 4}| \cos \phi_{54} \sin^2(1.27\Delta m_{54}^2 L/E) \]
\[ -4|U_{\alpha 6}||U_{\beta 6}||U_{\alpha 4}||U_{\beta 4}| \cos \phi_{64} \sin^2(1.27\Delta m_{64}^2 L/E) \]
\[ -4|U_{\alpha 5}||U_{\beta 5}||U_{\alpha 6}||U_{\beta 6}| \cos \phi_{65} \sin^2(1.27\Delta m_{65}^2 L/E) \]
\[ +4(|U_{\alpha 4}||U_{\beta 4}| + |U_{\alpha 5}||U_{\beta 5}| |U_{\alpha 4}||U_{\beta 4}| \cos \phi_{54} + |U_{\alpha 6}||U_{\beta 6}| \cos \phi_{64})|U_{\alpha 4}||U_{\beta 4}| \sin^2(1.27\Delta m_{41}^2 L/E) \]
\[ +4(|U_{\alpha 4}||U_{\beta 4}| \cos \phi_{54} + |U_{\alpha 5}||U_{\beta 5}| + |U_{\alpha 6}||U_{\beta 6}| \cos \phi_{65})|U_{\alpha 5}||U_{\beta 5}| \sin^2(1.27\Delta m_{51}^2 L/E) \]
\[ +4(|U_{\alpha 4}||U_{\beta 4}| \cos \phi_{54} + |U_{\alpha 5}||U_{\beta 5}| \cos \phi_{64} + |U_{\alpha 6}||U_{\beta 6}| \cos \phi_{65})|U_{\alpha 6}||U_{\beta 6}| \sin^2(1.27\Delta m_{61}^2 L/E) \]
\[ +2|U_{\beta 5}||U_{\alpha 5}||U_{\beta 4}||U_{\alpha 4}| \sin \phi_{54} \sin(2.53\Delta m_{54}^2 L/E) \]
\[ +2|U_{\beta 6}||U_{\alpha 6}||U_{\beta 4}||U_{\alpha 4}| \sin \phi_{64} \sin(2.53\Delta m_{64}^2 L/E) \]
\[ +2|U_{\beta 6}||U_{\alpha 6}||U_{\beta 5}|\sin \phi_{65} \sin(2.53\Delta m_{65}^2 L/E) \]
\[ +2(|U_{\alpha 5}||U_{\beta 5}| \sin \phi_{54} + |U_{\alpha 6}||U_{\beta 6}| \sin \phi_{64})|U_{\alpha 4}||U_{\beta 4}| \sin(2.53\Delta m_{41}^2 L/E) \]
\[ +2(\!-\!|U_{\alpha 4}||U_{\beta 4}| \sin \phi_{54} \!+\! |U_{\alpha 6}||U_{\beta 6}| \sin \phi_{65})|U_{\alpha 5}||U_{\beta 5}| \sin(2.53\Delta m_{51}^2 L/E) \]
\[ +2(\!-\!|U_{\alpha 4}||U_{\beta 4}| \sin \phi_{64} \!-\! |U_{\alpha 5}||U_{\beta 5}| \sin \phi_{65})|U_{\alpha 6}||U_{\beta 6}| \sin(2.53\Delta m_{61}^2 L/E) \).