A Revised Dynamic Hurst Parameter Estimation Method Based on Generalized Exponential Window Function

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Abstract. Dynamic Hurst parameter estimation is very important in the data filtering and system modelling. It is difficult to accurately estimate the dynamic Hurst parameter of long memory processes. In this research, a generalized exponential window function based dynamic Hurst parameter estimation method is provided, and the effective performance of the method is analyzed using fractional Gaussian noise. In the end of this paper, the application of revised estimation method in sleeping electroencephalogram signal is given. The simulation and sleeping electroencephalogram signal analysis results show that the revised estimation method can effectively analyze the random signals with local correlation characteristics. This method can be widely used in biomedical signal analysis, network traffic analysis, vibration signal analysis, statistical data analysis, etc.

Keywords: Dynamic Hurst parameter estimation, Electroencephalogram signal, Generalized exponential window function

1. Introduction

The importance of long-term relevance comes from a series of articles published by Hurst [1]. Since then, people have been paying more and more attention to the long-term correlation of random signals. In the long-term correlation analysis and processing of random signals, Hurst parameters have been widely studied and applied. Hurst parameters include all the information of scale signals. Hurst parameters can also be used to analyze whether random signals have chaotic characteristics [2]. The traditional Hurst parameter is a global parameter, that is, the Hurst parameter of the whole signal is a constant greater than zero and less than 1, which represents a long correlation characteristic of the whole signal. Experts and scholars have analyzed a large number of data with long correlation characteristics and found that Hurst parameters in different stages are different [3]. However, the traditional Hurst parameter estimation method can only describe the long correlation characteristics of the whole signal data, and cannot describe the local mutation information of these signals, so the
analysis of the local correlation characteristics of random signals has gradually attracted the attention of experts and scholars. It is very important to study and analyze the local correlation of fractional random signals for building more accurate system model and data prediction. In 1995, Peltier put forward the concept of Hölder parameter in a research report\textsuperscript{[4]}. Hölder parameter is a dynamic Hurst parameter which can be used to analyze the local correlation and multi-fractal characteristics of signals. Therefore, the dynamic Hurst parameter estimator provides a novel and efficient analysis method for the exploration and research of long memory signals. With the in-depth study of local correlation characteristics of long memory signals, the dynamic Hurst parameter estimation method is widely used in biomedical field. The analysis of the local correlation characteristics of sleep electroencephalogram (EEG) signals has attracted many scholars' attention\textsuperscript{[5]}.

In this research, a generalized exponential window function based dynamic Hurst parameter estimation method is introduced to estimate the local correlation characteristics of signals. In order to verify the accuracy of the algorithm, simulated fractional Gaussian noise (FGN) is analyzed using revised estimation method. Furthermore, the revised estimation method is applied in sleeping electroencephalogram signal to dynamically evaluate sleep quality.

2. Long-range dependence

Long-range dependence refers to the fact that fractional random signals still have correlation in a long time interval\textsuperscript{[6]}. A common definition of long phase correlation is: if fractional random signal is\( \{X_t\} \), then

\[
\gamma_s(k) \sim c_s |k|^{(1-\alpha)}, \quad \alpha \in (0,1)
\] (1)

The auto-covariance function \( y \) of the fractional order random signal \( x \) has a large lag, which is similar to the slow decline of power law. This kind of signals can be called long-range dependence signal. The relation expression between parameter \( \alpha \) and Hurst parameter \( h \) is \( \alpha = 2 - 2H \). Equivalently, it can be defined as the power-law divergence at the beginning of its spectrum, when the power spectral density of fractional order random signal satisfies:

\[
f_x(v) \sim c_f |v|^{-\alpha}
\] (2)

It is said that the random signal has a long correlation property, where \( C_f > 0, \alpha \in (0,1) \). In the discrete time process, the following shall be met:

\[
\gamma_s(0) = \sigma^2_s = \int_{-1/2}^{1/2} f_x(v)dv
\] (3)

Where \( \sigma^2_s \) is the variance of \( \{X_t\} \).

We can use Hurst parameter to estimate the long correlation of random signal. When Hurst parameter is \( 0.5 < H < 1 \), the random signal has a positive long-term correlation, and the future development trend is the same as in the past, and the closer \( H \) is to 1, the stronger the long-term correlation; when Hurst parameter is \( 0 < H < 0.5 \), the random signal has a negative long-term correlation, but the future development trend is contrary to the past, the closer \( H \) value is to 0, the stronger the anti persistence is; when Hurst parameter is used, the random signal has no correlation.

3. Generalized exponential window function based dynamic Hurst parameter estimation method

In this section, a revised dynamic Hurst parameter estimation method which is based on generalized exponential window function is introduced. The revised estimation method is an improved algorithm of Higuchi estimator, which is provided by Higuchi in\textsuperscript{[7]}. The revised estimation method is as follows.

For the time series \( X(1), X(2), \cdots, X(N) \) of length \( N \), a sliding exponential time window function with length \( \Delta t - 1 \) is used to intercept the data with length \( \Delta t - 1 \). The exponential time window function is\textsuperscript{[8]}.
\[ w_s = w_0 \exp(-\frac{s}{\theta}), \forall s \in \{0,1,2,\ldots,\Delta t - 1\} \] (4)

where

\[ w_0(\alpha) = \frac{1-e^{-\alpha}}{1-e^{-\alpha \Delta t}} \] (5)

\( \theta \) is the weight characteristic time and \( \alpha = \frac{1}{\theta} \) is the attenuation factor. The exponential weighting of random signal \( X(t) \) over time window \([t - \Delta t + 1, t]\) can be defined as

\[ \langle X \rangle_w(t) = \sum_{j=0}^{\Delta t-1} w_j X(x_{i+j}) \] (6)

For the time series \( \langle X \rangle_w(t), \langle X \rangle_w(t+1), \ldots, \langle X \rangle_w(t - \Delta t + 1) \) of length \( \Delta t - 1 \), the reconstruction time series is defined as

\[ \langle X \rangle_{w_k}^m : \langle X \rangle_w(m), \langle X \rangle_w(m+k), \langle X \rangle_w(m+2k), \ldots, \langle X \rangle_w(m + \left\lfloor \frac{\Delta t - 1 - m}{k} \right\rfloor k), m = 1, 2, \ldots, k \] (7)

where \( m \) is the initial time and \( k \) is the interval time, then a new sequence can be constructed. The length of each new time series is

\[ L_n(k) = \frac{\Delta t - 2}{m} \sum_{j=1}^{\Delta t - 1} \left\lfloor \frac{\Delta t - 1 - j}{m} \right\rfloor \sum_{j=1}^{\left\lfloor \frac{\Delta t - 2 + j}{m} \right\rfloor} \langle X \rangle_{w_{i+j}} - \langle X \rangle_{w_{i+j-1}} \] (8)

Local correlation time series satisfy the relation

\[ E(L_n(k)) \propto Cm^{H-2} \] (9)

Calculate the logarithm of both sides of equation (9), and the parameter can be calculated from

\[ \ln(L_n(m)) \sim (H - 2) \ln(m) + \ln(C) \] (10)

A straight line with slope \( \beta = H - 2 \) can be obtained by linear fitting the scatter points on the double logarithmic coordinates with the least square algorithm, and the Hurst parameter \( H = 2 + \beta \) can be obtained. Compared with other algorithms, the difference of the revised method is the generalized exponential window function. Such operation makes the algorithm more accurate, but the cost is that the amount of computation also increases. The flow chart of the method is shown in Figure 1.
Figure 1. Flow chart of the revised estimation method  

Figure 2. Simulated FGN with $H=0.75$

4. Simulation analysis results of the revised estimation method

4.1. Dynamic Hurst parameter estimation of FGN time series

In order to verify the accuracy of the algorithm, simulated FGN is analyzed using revised estimation method. The simulated FGN is generated using Fast Fourier transform algorithm of power spectrum [9]. The FGN with $H=0.75$ is shown in Figure 2.

For the same FGN time series, the dynamic Hurst parameter estimated by the revised estimation method. In order to accurately estimate the correlation of fractional random signals, this section first analyzes the selection of the improved exponential window size. Let the FGN time series as shown in Figure 2. The length of FGN time series is 100000, wsize = 4000, wsize = 6000, wsize = 8000, and wsize = 10000, respectively. Under different window sizes, the estimation results of the revised estimation method are shown in Figure 3.

Figure 3. Dynamic Hurst parameter estimation results under different window sizes

It can be seen from Figure 3 that the revise Hurst parameter estimation method can effectively estimate the dynamic parameter $H(t)$, and the $H(t)$ value can fluctuate around 0.75. When the window is too small, the estimation curve of $H(t)$ presents obvious fluctuation. As the window increases gradually, the estimation curve fluctuation tends to be smooth and stable. In order to
compare the influence of the window size on dynamic parameter $H(t)$ more clearly, the mean square error of the revised Hurst parameter estimation results of four different length windows is calculated as shown in Table 1. Through the mean square error, it can be concluded that the mean square error decreases with the increase of window size. But it's not that the larger the window is, the better, because when the window is too large, a lot of information at the end of the time series will be ignored, so we can't estimate the correlation of this part of the time series. In a comprehensive consideration, we can set the window size of the index window as 10000.

**Table 1.** Mean square error of the revise estimation method under different window size

| Window size | Mean square error |
|-------------|-------------------|
| 4000        | 0.0011            |
| 6000        | 0.0014            |
| 8000        | 0.000952          |
| 10000       | 0.000877          |

4.2. **Dynamic Hurst parameter estimation of multi-fractional time series**

In reality, we often encounter fractional random signals with different local correlation characteristics, such as EEG, network traffic, stock volatility and so on. The effective analysis of the local correlation of these signals is of great significance to the study of the essential characteristics of local correlation signals. In this section, the revised dynamic Hurst parameter estimation method is evaluated using multi-fractional time series. A FGN time series with a length of 120000 and $H = 0.65 + 0.75 + 0.85$ is generated, as shown in Figure 4.

The revised Hurst parameter estimation method is used to analyze the multi-fractional Gaussian noise time series in Figure 4. The simulation result of the revised estimation method is shown in Figure 5. The blue solid line in the figure is the estimation result of the revised estimation method, and the green horizontal dotted line is the true value.

**Figure 4.** Multi-fractional Gaussian signal

**Figure 5.** Simulation result of the revised method

Through the experimental result, we can see that the revised estimation method can accurately estimate the local correlation characteristics of multi-fractional time series. By calculating the mean square error, it can be concluded that the mean square error of the revised estimation method is 0.000520. Therefore, the revised estimation method can effectively analyze the signals with local correlation characteristics.

5. **Application of the revised estimation method in sleep EEG**

The correlation of EEG signals is a very important part of EEG data analysis. According to a large number of studies by experts and scholars, EEG signal has obvious local correlation characteristics [5]. By introducing dynamic Hurst parameter into the analysis and detection of EEG signal, we can make a more dynamic evaluation of different stages of EEG signal, reflecting the local correlation characteristics of EEG signal. Human sleep is divided into rapid eye movement sleep (REM) and non rapid eye movement sleep (NREM). In non rapid eye movement sleep (NREM), it is divided into three stages: sleep stage 1, sleep stage 2 and sleep stage 3. The sleep EEG signals in the experiment were collected in MIT-BIH polysomnography database. In MIT-BIH polysomnography database, 16 men's sleep EEG signals (age: 32-56 years old, weight: 89kg-152kg) were collected. The average collection
time of sleep EEG signals was 5h, sampling frequency was 250Hz, digital 12 bits / sample. The research purpose of this section is to use revised estimation method analyze the local correlation characteristics of human EEG signals. In order to describe the correlation characteristics of each sleep stage in detail, we chose to record slp03, because the slp03 includes awake stage, non rapid eye movement sleep stage (sleep stage 1, sleep stage 2, sleep stage 3) and rapid eye movement sleep stage, and there is no such phase Any sleep disorder. In slp03, there are also notes of different sleep stages. Each note records the state of sleep EEG data for 30 seconds.

The duration of sleep EEG signal in Experiment 1 is 30 minutes (i.e. data points), and the EEG data is shown in Figure 6. According to slp03's annotation, we can know that the 30 minute sleep EEG signal can be divided into four stages: awake stage (0s < t0 ≤ 210s), sleep stage 1 (210s < T1 ≤ 420s), sleep stage 2 (420s < T2 ≤ 1230s) and sleep stage 3 (1230s < T3 ≤ 1800s).

![Figure 6. 30-minutes sleep EEG signal](image)

![Figure 7. Analysis of 30-minute sleep EEG signals](image)

The estimated results of 30 minute sleep EEG signal are shown in Figure 7. From the graph, we can see that the dynamic Hurst parameter indicates that human sleep EEG signal has positive correlation, and dynamic Hurst parameter of sleep EEG signal in different stages is obviously different, indicating that human sleep EEG signal has multi-fractional characteristics, that is, different sleep stages have different correlation characteristics, and the correlation degree is expressed by dynamic Hurst parameter. In the 30 minute EEG data, the range of Hurst parameter is 0.8095-0.9649. In the awake state, the range of parameter is 0.8393-0.9397. The main reason is that the large brain activity is relatively complex and active in the awake stage, and the movement of limbs needs the control and coordination of brain. In the sleep stage, 1 The fluctuation range of parameter is 0.9108-0.9649. Compared with the awake stage, the variation range of parameter in sleepy sleep stage 1 is more stable, which is related to the gradual slowing of eye movement and the beginning of light sleep; in sleep stage 2, the fluctuation range of parameter is 0.8095-0.9161, entering light sleep; in sleep stage 3 The fluctuation range of parameter is between 0.8257 and 0.9164. When people enter deep sleep stage, eye movement is more and more slow, and the fluctuation range of parameter is the smallest. From the parameter curve of 30 minute sleep EEG signal, we can conclude that the local correlation of sleep EEG signal in different stages of non REM sleep has multi-fractional characteristics.

6. Conclusion
In this research, the dynamic Hurst parameter estimation method is studied. A generalized exponential window function based dynamic Hurst parameter estimation method is introduced. The revised estimation method is an improved algorithm of Higuchi estimator. It can be proved that the revised Hurst estimation method can effectively analyze the local correlation characteristics of the signal. The analysis results of sleep EEG signal indicate that the dynamic Hurst parameter can be used to evaluate sleep quality. The estimation method provided in this research has great potential in biomedical signal analysis.

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