MAXIMAL OPERATORS OF WALSH–NÖRLUND MEANS ON THE DYADIC HARDY SPACES

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(Received July 2, 2022; revised November 7, 2022; accepted November 7, 2022)

Abstract. We establish necessary and sufficient conditions for the maximal operator of Walsh–Nörlund means with non-increasing weights to be bounded from the dyadic Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$.

1. Introduction

Móricz and Siddiqi [17] investigated the rate of the approximation by Nörlund means which in turn are related to the Walsh–Fourier series. For Nörlund means with monotone weights they gave a sufficient condition which provides Nörlund means for having convergence in $L_p$ norm ($1 \leq p < \infty$) and $C_W$ norm.

The result of [17] was extended by Fridli, Manchanda and Siddiqi [3] for dyadic martingale Hardy spaces and dyadic homogeneous Banach spaces. Recently, the theorem of Móricz and Siddiqi was generalized for $\Theta$-means of Walsh–Fourier series in $L_p$ spaces ($1 \leq p < \infty$) and $C_W$ [1].

The theorems mentioned above are related to the approximation of the Nörlund means which in turn is related to the uniform boundedness of the corresponding operators of the Nörlund means. The study of almost everywhere convergence of Nörlund means is connected with the study of the boundedness of the maximum operators corresponding to the Nörlund means.

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n$ is due to Fine [2]. Later, Schipp [19] showed that the maximal

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The author is very thankful to United Arab Emirates University (UAEU) for the Start-up Grant 12S100.

Key words and phrases: Walsh system, Nörlund mean, Hardy space, weak type inequality, almost everywhere convergence.

Mathematics Subject Classification: 42C10.
operator of the Walsh–Fejér means is of weak type \((1, 1)\), from which the a.e. convergence follows by standard argument [15]. Schipp’s result also implies by interpolation the boundedness of \(\sup_n |\sigma_n| : L_p \to L_p\) \((1 < p \leq \infty)\). This fails to hold for \(p = 1\) but Fujii [4] proved that \(\sup_n |\sigma_n|\) is bounded from the dyadic Hardy space \(H_1(\mathbb{I})\) to the space \(L_1(\mathbb{I})\) (see also Simon [21]). Fujii’s theorem was extended by Weisz [26]. In particular, Weisz [26] proved that the maximum operator is bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\), when \(p > 1/2\). The essence of the condition \(p > 1/2\) was proved by the author [9]. If \(\{q_k\}\) is an non-decreasing sequence then it can be proved that the following inequality occurs (see Persson, Tephnadze, Wall [18])

\[
\sup_n |t_n| \leq c \sup_n |\sigma_n|,
\]

where \(t_n\) denotes the Nörlund means of Walsh–Fourier series. From (1) it follows that the maximum operator \(\sup_n |t_n|\) is bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\), when \(p > 1/2\).

The situation is different when the sequence \(\{q_k\}\) is decreasing. Let us cite the following two cases:

- say \(q_k = A_k^{\alpha-1}\), \(\alpha \in (0, 1)\), where

\[
A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!},
\]

then it is easy to see that \(\{q_k\}\) is decreasing and at the same time the operator \(\sup_n |\sigma_n^\alpha|\) \((0 < \alpha < 1)\) is bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\), when \(p > 1/(1 + \alpha)\) (see Weisz [27]);

- assume that \(q_k = 1/(k + 1)\). Then the sequence is decreasing, but the maximum operator is not bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\) by any \(p \in (0, 1]\) (see [11]).

Therefore, Nörlund means with non-increasing weights can be divided into two groups:

- Nörlund means with non-increasing weights, whose corresponding maximum operator is bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\) for some \(p \in (0, 1]\);

- Nörlund means with non-increasing weights that are not bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\) by any \(p \in (0, 1]\).

We will prove necessary and sufficient conditions for the maximal operator of Nörlund means with non-increasing weights to be bounded from the Hardy space \(H_p(\mathbb{I})\) to the space \(L_p(\mathbb{I})\). It also follows from the established theorem that the boundedness of the maximal operator of Nörlund means with non-increasing weights from the Hardy space \(H_1(\mathbb{I})\) to the space \(L_1(\mathbb{I})\) is equivalent to the type \((\infty, \infty)\).
2. Walsh functions

We denote the set of non-negative integers by $\mathbb{N}$. By a dyadic interval in $I := [0, 1)$ we mean one of the form $I(l, k) := \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}$, $0 < l \leq 2^k$. Given $k \in \mathbb{N}$ and $x \in [0, 1)$, let $I_k(x)$ denote the dyadic interval of length $2^{-k}$ which contains the point $x$. We also use the notation $I_n := I_n(0)$ ($n \in \mathbb{N}$), $\overline{I}_k(x) := \mathbb{I}\setminus I_k(x)$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1 and if $x$ is a dyadic rational number we choose the expansion which terminates in 0’s.

For any given $n \in \mathbb{N}$ it is possible to write $n$ uniquely as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) 2^k,$$

where $\varepsilon_k(n) = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of $n$ and the numbers $\varepsilon_k(n)$ will be called the binary coefficients of $n$. Let us introduce for $1 \leq n \in \mathbb{N}$ the notation $|n| := \max\{j \in \mathbb{N} : \varepsilon_j(n) \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

Let us set the $n$th ($n \in \mathbb{N}$) Walsh–Paley function at point $x \in \mathbb{I}$ as:

$$w_n(x) = (-1)^{\sum_{j=0}^{\infty} \varepsilon_j(n)x_j}.$$

Let us denote the logical addition on $\mathbb{I}$ by $\dot{+}$. That is, for any $x, y \in \mathbb{I}$

$$x \dot{+} y := \sum_{n=0}^{\infty} |x_n - y_n|2^{-(n+1)}.$$  

The $n$th Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that [14,20]

$$D_{2^n}(x) = 2^n \mathbf{1}_{I_n}(x),$$

where $\mathbf{1}_E$ is the characteristic function of the set $E$.  

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As usual, denote by \( L_1(I) \) the set of measurable functions defined on \( I \), for which

\[
\|f\|_1 := \int_I |f(t)| \, dt < \infty.
\]

Let \( f \in L_1(I) \). The partial sums of the Walsh–Fourier series are defined as follows:

\[
S_M(f; x) := \sum_{i=0}^{M-1} \hat{f}(i) w_i(x),
\]

where the number

\[
\hat{f}(i) = \int_I f(t) w_i(t) \, dt
\]

is said to be the \( i \)th Walsh–Fourier coefficient of the function \( f \). Let us set \( E_n(f; x) = S_{2^n}(f; x) \). The maximal function is defined by

\[
E^*(f; x) = \sup_{n \in \mathbb{N}} |E_n(f; x)|.
\]

### 3. Walsh–Nörlund means

We set \( \{q_k : k \geq 0\} \) as a sequence of non-negative numbers. We define the \( n \)th Nörlund mean of the Walsh–Fourier series by

\[
t_n(f; x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f; x),
\]

where \( Q_n := \sum_{k=0}^{n-1} q_k \) (\( n \geq 1 \)). It is always assumed that \( q_0 > 0 \) and \( \lim_{n \to \infty} Q_n = \infty \). In this case, the summability method generated by the sequence \( \{q_k : k \geq 0\} \) is regular (see [17]) if and only if

\[
\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0.
\]

The Nörlund kernels are defined by

\[
F_n(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t).
\]

The Fejér means and kernels are

\[
\sigma_n(f, x) := \frac{1}{n} \sum_{k=1}^{n} S_k(f, x), \quad K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t).
\]
It is easily to see that the means $t_n(f)$ and $\sigma_n(f)$ can be obtained by convolution of $f$ with the kernels $F_n$ and $K_n$. That is,

$$t_n(f, x) = \int_G f(x + t) F_n(t) \, dt = (f * F_n)(x),$$

$$\sigma_n(f, x) = \int_G f(x + t) K_n(t) \, dt = (f * K_n)(x).$$

It is known that $L_1$ norm of Fejér kernels is uniformly bounded, that is

$$\|K_n\|_1 \leq c \text{ for all } n \in \mathbb{N}. \quad (5)$$

Yano [28] estimated the value of $c$ and obtained $c = 2$. Recently, in a paper (see [24]) it was shown that the exact value of $c$ is $\frac{17}{15}$.

4. Auxiliary propositions

To prove the main results, we need the following theorems.

**Theorem SWS** [20]. Let $n \in \mathbb{N}$ and $e_j := 2^{-j-1}$. Then

$$K_{2^n}(x) = \frac{1}{2} \left( 2^{-n} D_{2^n}(x) + \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x + e_j) \right). \quad (6)$$

**Theorem GN1** [13]. Let $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$ with $n_1 > n_2 > \cdots > n_r \geq 0$. Let us set $n^{(0)} := n$ and $n^{(i)} := n^{(i-1)} - 2^{n_i}$ ($i = 1, \ldots, r - 1$), $n^{(r)} := 0$. Then the following decomposition holds.

$$F_n = \frac{w_n}{Q_n} \sum_{j=1}^{r} Q_{n^{(j-1)}} w_{2^{n_j}} D_{2^{n_j}}$$

$$- \frac{w_n}{Q_n} \sum_{j=1}^{r} w_{n^{(j-1)}} w_{2^{n_j-1}} \sum_{k=1}^{2^{n_j-1}} q_{k+n^{(j)}} D_k =: F_{n,1} - F_{n,2}. \quad (7)$$

**Theorem GN2** [13]. Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-negative numbers. If the sequence $\{q_k : k \in \mathbb{N}\}$ is monotone non-increasing (in sign $q_k \downarrow$). Then

$$\|F_n\|_1 \sim \frac{1}{Q_n} \sum_{k=1}^{\lfloor n \rfloor} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k}. \quad (8)$$
Using Abel transformation we have

\[
\sum_{k=1}^{2^n j - 1} q_{k+n(j)} D_k = \sum_{k=1}^{2^n j - 2} (q_{k+n(j)} - q_{k+n(j)+1}) k K_k \\
+ q_{n(j-1)-1}(2^n j - 1) K_{2^n j - 1}, \quad j = 1, 2, \ldots, r.
\]

Thus, we get

\[
F_{n,2} = \frac{w_n}{Q_n} \sum_{j=1}^r \sum_{k=1}^{2^n j - 2} w_{n(j-1)} w_{2^n j - 1} (q_{k+n(j)} - q_{k+n(j)+1}) k K_k \\
+ \frac{w_n}{Q_n} \sum_{j=1}^r w_{n(j-1)} w_{2^n j - 1} q_{n(j-1)-1}(2^n j - 1) K_{2^n j - 1} =: F_{n,2}^{(1)} + F_{n,2}^{(2)}.
\]

**Lemma 1.** Let \( p \in \left( \frac{1}{2}, 1 \right] \). Then

\[
\int \sup_{1 \leq n \leq 2^N} (n|K_n|)^p \leq c_p 2^{N(2p-1)}.
\]

**Proof.** Let \( p = 1 \). Since (see [20])

\[
n |K_n(x)| \leq c \sum_{s=0}^{2^n} 2^s K_{2^s}(x)
\]

from (5) we have

\[
\int \sup_{1 \leq n \leq 2^N} (n|K_n(x)|) \ dx \leq c \sum_{s=0}^{N} 2^s \int K_{2^s}(x) \ dx \leq c 2^N.
\]

Let \( 1/2 < p < 1 \). Applying the inequality

\[
\left( \sum_{k=0}^{\infty} a_k \right)^p \leq \sum_{k=0}^{\infty} a_k^p \quad (a_k \geq 0, \ 0 < p \leq 1)
\]

and (see [8])

\[
\int (2^s K_{2^s}(x))^p \ dx \leq c_p 2^{s(2p-1)}, \quad 1/2 < p < 1
\]
we get

\[
\int \sup_{1 \leq n \leq 2^N} (n|K_n(x)|)^p \, dx \leq c_p \sum_{s=0}^{N} \int_{I} (2^s K_{2^s}(x))^p \, dx \leq c_p 2^{N(2p-1)}. \quad \square
\]

5. Dyadic Hardy spaces

The norm (or quasinorm) of the space \( L_p(\mathbb{I}) \) is defined by

\[
\|f\|_p := \left( \int_{\mathbb{I}} |f(x)|^p \, dx \right)^{1/p} \quad (0 < p < +\infty).
\]

In case \( p = \infty \), by \( L_p(\mathbb{I}) \) we mean \( L_\infty(\mathbb{I}) \), endowed with the supremum norm.

The space weak-\( L_1(\mathbb{I}) \) consists of all measurable functions \( f \) for which

\[
\|f\|_{\text{weak}-L_1(\mathbb{I})} := \sup_{|\lambda|>0} \lambda \left( |f| > \lambda \right) < +\infty.
\]

Let \( f \in L_1(\mathbb{I}) \). For \( 0 < p < \infty \) the Hardy space \( H_p(\mathbb{I}) \) consists of all functions for which

\[
\|f\|_{H_p} := \|E^*(f)\|_p < \infty.
\]

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a dyadic interval \( I \), such that

a) \( \int_I a = 0 \);

b) \( \|a\|_\infty \leq |I|^{-1/p} \);

c) \( \text{supp } a \subset I \).

An operator \( T \) is called \( p \)-quasi-local if there exist a constant \( c_p > 0 \) such that for every \( p \)-atom \( a \)

\[
\int_{\mathbb{I} \setminus I} |Ta|^p \leq c_p < \infty,
\]

where \( I \) is the support of the atom. We shall need the following

**Theorem W1** [25]. Suppose that the operator \( T \) is \( \sigma \)-sublinear and \( p \)-quasi-local for each \( 0 < p \leq 1 \). If \( T \) is bounded from \( L_\infty(\mathbb{I}) \) to \( L_\infty(\mathbb{I}) \), then

\[
\|Tf\|_p \leq c_p \|f\|_p \quad (f \in H_p(\mathbb{I}))
\]

for every \( 0 < p < \infty \). In particular for \( f \in L_1(\mathbb{I}) \), it holds

\[
\|Tf\|_{\text{weak}-L_1(\mathbb{I})} \leq C \|f\|_1.
\]
Theorem W2 [25]. If a sublinear operator is bounded from $H_{p_0}(\mathbb{I})$ to $L_{p_0}(\mathbb{I})$ and from $L_{p_1}(\mathbb{I})$ to $L_{p_1}(\mathbb{I})$ ($p_0 \leq 1 < p_1 \leq \infty$) then it is also bounded from $H_p(\mathbb{I})$ to $L_p(\mathbb{I})$ if $p_0 < p < p_1$.

6. Maximal operators of Walsh–Nörlund means

The aim of this section is to study the boundedness of the maximal operators of Walsh–Nörlund means on dyadic Hardy spaces. More precisely, to find necessary and sufficient conditions for the maximal operator of Walsh–Nörlund means to be bounded from the Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$ for fixed $p \in (0, 1]$.

Let us first prove that if the condition

\[(11) \sup_{n \in \mathbb{N}} \frac{1}{Q_n} \sum_{k=1}^{[n]} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k} = \infty \]

is fulfilled, then the boundedness of the maximum operator from the Hardy space $H_1(\mathbb{I})$ to the space $L_1(\mathbb{I})$ does not occur. Moreover, we prove that the following is valid:

**Theorem 1.** Let $\{m_A : A \in \mathbb{N}\} \subset \mathbb{N}$ be a subsequence for which the condition

\[
\sup_{A \in \mathbb{N}} \frac{1}{Q_{m_A}} \sum_{k=1}^{[m_A]} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| Q_{2^k} = \infty
\]

holds. The operator $t_{m_A}(f)$ is not uniformly bounded from the dyadic Hardy spaces $H_1(\mathbb{I})$ to the space $L_1(\mathbb{I})$.

**Proof.** Set

\[f_A := D_{2^{[m_A]}+1} - D_{2^{[m_A]}}.\]

Then it is easy to see that

\[\sup_{n \in \mathbb{N}} |S_{2^n}(f_A)| = D_{2^{[m_A]}}\]

and consequently,

\[\|f_A\|_{H_1} = \left\| \sup_{n \in \mathbb{N}} |S_{2^n}(f_A)| \right\|_1 = \|D_{2^{[m_A]}}\|_1 = 1.\]

Set

\[m_A = 2^{[m_A]} + r_A,\]
where
\[ r_A := \sum_{j=0}^{\lfloor |m_A| - 1 \rfloor} \varepsilon_j(m_A)2^j. \]

Then we can write
\[ t_{m_A}(f_A) = \frac{1}{Q_{m_A}} \sum_{k=2^{|m_A|}+1}^{2^{|m_A|}+r_A} q_{m_A-k}S_k(f_A). \]

It is easy to see that
\[ S_k(f_A) = S_k(D_{2^{|m_A|}+1} - D_{2^{|m_A|}}) \]
\[ = S_{2^{|m_A|}+1}(D_k) - S_k(D_{2^{|m_A|}}) = D_k - D_{2^{|m_A|}}, \quad 2^{|m_A|} < k \leq m_A. \]

Hence, we have
\[
t_{m_A}(f_A) = \frac{1}{Q_{m_A}} \sum_{k=2^{|m_A|}+1}^{r_A} q_{m_A-k}(D_k - D_{2^{|m_A|}}) \\
= \frac{1}{Q_{m_A}} \sum_{k=1}^{r_A} q_{r_A-k}(D_{k+2^{|m_A|}} - D_{2^{|m_A|}}) = \frac{w_{2^{|m_A|}}}{Q_{m_A}} \sum_{k=1}^{r_A} q_{r_A-k}D_k.
\]

By (8) we have
\[
\|t_{m_A}(f_A)\|_1 = \frac{1}{Q_{m_A}} \left\| \sum_{k=1}^{r_A} q_{r_A-k}D_k \right\|_1 \sim \frac{1}{Q_{m_A}} \sum_{k=1}^{r_A} |\varepsilon_k(r_A) - \varepsilon_{k+1}(r_A)|Q_{2^k}.
\]

Since
\[ \varepsilon_k(m_A) = \begin{cases} \varepsilon_k(r_A), & k < |m_A|, \\ 1, & k = |m_A|, \end{cases} \]
from (12), we can write
\[
\|t_{m_A}(f_A)\|_1 \sim \frac{1}{Q_{m_A}} \sum_{k=1}^{r_A} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)|Q_{2^k} + c.
\]

Consequently, from the condition of Theorem 1 we conclude that
\[ \sup_{A \in \mathbb{N}} \|t_{m_A}(f_A)\|_1 = \infty. \]
Now, we prove that the maximal operator of Walsh–Nörlund means with non-increasing weights can not be bounded from the Hardy space $H_{1/2}(\mathbb{I})$ to the space $L_{1/2}(\mathbb{I})$. Based on the interpolation Theorem W2, the maximum operator of Walsh–Nörlund means with non-increasing weights can not be bounded from the Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$ when $p < 1/2$ (see Persson, Tephnadze, Wall [18]).

**Theorem 2.** The maximal operator of Walsh–Nörlund means with non-increasing weights can not be bounded from the Hardy space $H_{1/2}(\mathbb{I})$ to the space $L_{1/2}(\mathbb{I})$.

**Proof.** Set $f_n := D_{2^{n+1}} - D_{2^n}$. Then it is easy to see that

$$\sup_{m \in \mathbb{N}} |S_{2^n}(f_n)| = D_{2^n}$$

and consequently,

$$\|f_n\|_{H_p} = \left\| \sup_{m \in \mathbb{N}} |S_{2^n}(f_n)| \right\|_p = \|D_{2^n}\|_p = 2^{n(1-1/p)}.$$

Let $s < n$. Then we can write

$$t_{2^{n+2^s}}(f_n) = \frac{1}{Q_{2^n+2^s}} \sum_{j=1}^{2^n+2^s} q_{2^n+2^s-j} S_j(f_n)$$

$$= \frac{1}{Q_{2^n+2^s}} \sum_{j=2^n+1}^{2^n+2^s} q_{2^n+2^s-j} S_j(D_{2^{n+1}} - D_{2^n})$$

$$= \frac{1}{Q_{2^n+2^s}} \sum_{j=2^n+1}^{2^n+2^s} q_{2^n+2^s-j} (S_{2^{n+1}}(D_j) - S_j(D_{2^n}))$$

$$= \frac{1}{Q_{2^n+2^s}} \sum_{j=2^n+1}^{2^n+2^s} q_{2^n+2^s-j} (D_j - D_{2^n})$$

$$= \frac{1}{Q_{2^n+2^s}} \sum_{j=1}^{2^s} q_{2^s-j} (D_{j+2^n} - D_{2^n}) = \frac{w_{2^n}}{Q_{2^n+2^s}} \sum_{j=1}^{2^s} q_{2^s-j} D_j.$$

Consequently,

$$\int_I \left( \sup_{1 \leq s < n} |t_{2^n+2^s}(f_n)| \right)^p \geq \sum_{s=0}^{n-1} \int_{I_s \setminus I_{s+1}} |t_{2^n+2^s}(f_n)|^p$$

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Combine (14) and (15) we get
\[ \left| \sum_{j=1}^{2^s} q_{2^s-j} D_j \right|^p = \sum_{s=0}^{n-1} \frac{1}{2^{s+1} Q_{2^n}^p} \left| \sum_{j=1}^{2^s} q_{2^s-j} \right|^p. \]

Since \( q_k \) is non-increasing we can write
\[ Q_{2^n} = \sum_{k=0}^{2^{n-1}-1} q_k + \sum_{k=2^{n-1}}^{2^n-1} q_k \leq 2 \sum_{k=0}^{2^{n-1}-1} q_k = 2Q_{2^{n-1}} \leq \cdots \leq 2^{n-s} Q_{2^s} \]
and
(15) \[ \frac{Q_{2^s}}{2^s} \geq \frac{Q_{2^n}}{2^n} \quad (s \leq n). \]

Combine (14) and (15) we get \((p = 1/2)\)
\[
\begin{align*}
\int \left( \sup_{1 \leq s < n} \left| t_{2^n+2^s} (f_n) \right| \right)^{1/2} &\geq \sum_{s=0}^{n-1} \frac{1}{2^{s+1} Q_{2^n}^{1/2}} \left| \sum_{j=1}^{2^s} q_{2^s-j} \right|^{1/2} \\
&\geq c \sum_{s=0}^{n-1} \frac{2^s \sqrt{Q}_{2^n}^{1/2}}{2^s \sqrt{Q}_{2^n}^{1/2}} \geq c \sum_{s=0}^{n-1} \frac{1}{2^s/2^n} \left( \frac{2^s}{2^n} \right)^{1/2} = \frac{cn}{2^{n/2}}.
\end{align*}
\]

Hence,
\[
\frac{\| t^* (f_n) \|_{1/2}^{1/2}}{\| f_n \|_{H_{1/2}}^{1/2}} \geq \frac{c n}{\sqrt{2}} \quad \text{as} \quad n \to \infty. \quad \square
\]

Finally, we formulate a basic problem: let \( \{q_k\} \) be a non-increasing and positive sequence. Suppose that the operator \( t^* (f) \) is bounded from \( L_{\infty} (\mathbb{I}) \) to \( L_{\infty} (\mathbb{I}) \). Find the necessary and sufficient conditions for the sequence \( \{q_k\} \) such that the maximum operator \( t^* (f) \) may be bounded from the Hardy space \( H_p (\mathbb{I}) \) to the space \( L_p (\mathbb{I}) \), when \( p \in (1/2, 1] \).

The following theorem gives a complete answer to this problem.

**Theorem 3.** Let \( \{q_k\} \) be a non-increasing and positive sequence. Suppose that the operator \( t^* (f) \) is bounded from \( L_{\infty} (\mathbb{I}) \) to \( L_{\infty} (\mathbb{I}) \) and \( p \in (1/2, 1] \). The necessary and sufficient condition for the boundedness of the maximal operator \( t^* (f) \) from the Hardy space \( H_p (\mathbb{I}) \) to the space \( L_p (\mathbb{I}) \) is
\[
\sup_{N} \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^{N} Q_{2j}^p 2^{j(p-1)} < \infty.
\]
Proof. Necessity. We assume that

\begin{equation}
\sup_N \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N Q_j^p 2^{j(p-1)} = \infty.
\end{equation}

From (13) and (14) we have

\begin{align*}
\| t^* (f_n) \|^p_{L_p(I)} &\geq 2^{n(1-p)} \int \left( \sup_{1 \leq s < n} |t_{2^n+2^s} (f_n)| \right)^p \\
&\geq 2^{n(1-p)} \sum_{s=0}^{n-1} \frac{1}{2^{s+1}Q_{2^n+2^s}^p} \sum_{j=1}^{2^s} |q_{2^s-j}^j|^p \\
&\geq c_p \frac{2^{n(1-p)}}{Q_{2^n}^p} \sum_{s=0}^{n-1} \frac{1}{2^s} \sum_{j=2^{s+1}+1}^{2^s} |q_{2^s-j}^j|^p \\
&\geq c_p \frac{2^{n(1-p)}}{Q_{2^n}^p} \sum_{s=0}^{n-1} 2^{s(p-1)} Q_{2^s}^p.
\end{align*}

Then from (16) we get

\[ \sup_{n \in \mathbb{N}} \frac{\| t^* (f_n) \|^p_{L_p(I)}}{\| f_n \|^p_{H_p}} = \infty \]

and consequently, the operator \( t^* \) is not bounded from the Hardy space \( H_p(\mathbb{I}) \) to the space \( L_p(\mathbb{I}) \).

Sufficiency. We suppose that \( f \in H_p(\mathbb{I}) \). Let the function \( a \) be an \( H_p \) atom. It means that either \( a \) is constant or there is an interval \( I_N(u) \) such that \( \text{supp}(a) \subset I_N(u) \), \( \|a\|_\infty \leq 2^{N/p} \) and \( \int a = 0 \). Without loss of generality we can suppose that \( u = 0 \). Consequently, for any function \( g \) which is \( A_N \)-measurable we have that \( \int ag = 0 \). We prove that the operator \( \sup_{n>2^N} (f * F_n)(x) \) is \( H_p \)-quasi local. That is,

\begin{equation}
\int_{I_N} \left( \sup_{n>2^N} |a * F_n| \right)^p \leq c_p.
\end{equation}

Let \( x \in I_N \). Then from (7) we can write

\begin{align*}
|a * F_n| &= \left| \int_{I_N} a(t) F_n(x + t) \, dt \right| \leq 2^{N/p} \int_{I_N} |F_n(x + t)| \, dt \\
&= 2^{N/p} \int_{I_N} |F_{n,1}(x + t)| \, dt + 2^{N/p} \int_{I_N} |F_{n,2}(x + t)| \, dt.
\end{align*}
We have
\[
\int_{I_N} |F_{n,1}(x + t)| \, dt \leq \frac{1}{Q_n} \sum_{j=1, n_j > N}^{r} Q_{n(j-1)} \int_{I_N} D_{2^{n_j}}(x + t) \, dt
\]
\[
+ \frac{1}{Q_n} \sum_{j=1, n_j \leq N}^{r} Q_{n(j-1)} \int_{I_N} D_{2^{n_j}}(x + t) \, dt.
\]
Since \( t \in I_N \) and \( x \not\in I_N \) we have that \( x + t \not\in I_N \) and consequently by (2) we get \( D_{2^{n_j}}(x + t) = 0 \) for \( n_j > N \). On the other hand, \( \int_{I_N} D_{2^{n_j}}(x + t) \, dt = \frac{1}{2^{n_j}} D_{2^{n_j}}(x) \) for \( n_j \leq N \). Hence, we obtain
\[
\int_{I_N} |F_{n,1}(x + t)| \, dt \leq \frac{1}{Q_n} \sum_{j=1, n_j \leq N}^{r} Q_{n(j-1)} \int_{I_N} D_{2^{n_j}}(x + t) \, dt
\]
\[
= \frac{1}{2^{r}Q_n} \sum_{j=1, n_j \leq N}^{r} Q_{n(j-1)} D_{2^{n_j}}(x) \leq \frac{1}{2^{r}Q_{2^{N}}} \sum_{j=1}^{N} Q_{2^{j}}, D_{2^{j}}(x).
\]
Consequently, from the condition of the theorem we get
\[
\int_{I_N} \sup_{n > 2^{N}} \left( \frac{2^{N/p}}{\int_{I_N} |F_{n,1}(x + t)| \, dt} \right)^{p} \, dx
\]
\[
\leq \frac{c_p 2^{N}}{2^{N/p} Q_{2^{N}}} \sum_{j=1}^{N} Q_{2^{j}}^{p} \int_{I_N} D_{2^{j}}^{p}(x) \, dx = \frac{c_p 2^{N(1-p)}}{Q_{2^{N}}^{p}} \sum_{j=1}^{N} Q_{2^{j}}^{p} 2^{j(p-1)} \leq c_p < \infty.
\]
From (9) we get
\[
\int_{I_N} |F_{n,2}(x + t)| \, dt \leq \int_{I_N} |F_{n,2}^{(1)}(x + t)| \, dt + \int_{I_N} |F_{n,2}^{(2)}(x + t)| \, dt.
\]
We can write
\[
F_{n,2}^{(1)} \leq \frac{1}{Q_n} \sum_{j=1}^{r} \sum_{k=1}^{2^{n_j - 1}} (q_{k+n(j)} - q_{k+n(j)+1}) k |K_k|
\]
\[
= \frac{1}{Q_n} \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{k=2^{m-1}}^{2^{m-1}} (q_{k+n(j)} - q_{k+n(j)+1}) k |K_k|
\]
\[
= \frac{1}{Q_n} \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{k=2^{m-1}}^{2^{m-1}} (q_{k+n(j)} - q_{k+n(j)+1}) k |K_k|
\]
Consequently, we have

\[ \int_{I_N} |F_{n,2}(x + t)| \, dt \leq \frac{2}{2^N Q_n} \sum_{j=1}^{N} q_{j}^{2j-1} \sum_{m=1}^{2^j-1} \sup_{m \leq 2^j-1 \leq k < 2^m} (k|K_k(x)|) \]

\[ + \frac{2}{2^N Q_n} \sum_{j=N+1}^{n_1} q_{j-1}^{2j-1} \sum_{m=1}^{N} \sup_{m \leq 2^j-1 \leq k < 2^m} (k|K_k(x)|) \]

\[ + \frac{2}{Q_n} \sum_{j=N+1}^{n_1} q_{j-1} \sum_{m=1}^{j} \int_{I_N} \sup_{m \leq 2^j-1 \leq k < 2^m} (k|K_k(x + t)|) \, dt \]

\[ =: J_1(n) + J_2(n) + J_3(n). \]

By (10) and (6), we have \((m > N)\)

\[ \int_{I_N} \sup_{2^m-1 \leq k < 2^m} (k|K_k(x + t)|) \, dt \leq \sum_{s=0}^{m} 2^s \int_{I_N} K_2^s(x + t) \, dt \]
\[\begin{align*}
&= \frac{1}{2N} \sum_{s=0}^{N} 2^s K_{2^s}(x) + \sum_{s=N+1}^{m} 2^s \int_{I_N} K_{2^s}(x + t) \, dt \\
&\leq \frac{1}{2N} \sum_{s=0}^{N} 2^s K_{2^s}(x) + \sum_{s=N+1}^{m} \sum_{l=0}^{s} 2^l \int_{I_N} D_{2^s}(x + t + e_l) \, dt.
\end{align*}\]

Since \(x + t \not\in I_N\), from (2) it is easy to see that
\[\int_{I_N} \sup_{2^{m-1} \leq k < 2^m} (k|K_k(x + t)|) \, dt \leq \frac{1}{2N} \sum_{s=0}^{N} 2^s K_{2^s}(x) + \frac{2m}{2N} \sum_{l=0}^{N-1} 2^l 1_{I_N(e_l)}(x).\]

We can write
\[J_3(n) \leq \frac{2}{2NQ_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}}(j - N) \sum_{s=0}^{N} 2^s K_{2^s}(x)\]
\[\quad + \frac{1}{2NQ_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}} 2^{j+1} \sum_{l=0}^{N-1} 2^l 1_{I_N(e_l)}(x)\]
\[= \frac{1}{2^{2N}Q_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}} 2^{j+1} (j - N) \sum_{s=0}^{N} 2^s K_{2^s}(x)\]
\[\quad + \frac{1}{2NQ_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}} 2^{j+1} \sum_{l=0}^{N-1} 2^l 1_{I_N(e_l)}(x).\]

Since
\[Q_n \geq \sum_{j=1}^{2^{n_1} - 1} q_j \geq \sum_{r=1}^{2^{r-1}} \sum_{j=2^{r-1}}^{n_1} q_j \geq \sum_{r=1}^{n_1} q_{2^r} 2^{r-1},\]
we obtain that

\[
J_3(n) \leq \frac{c}{2^{2N}} \sum_{s=0}^{N} 2^s K_{2^s}(x) + \frac{c}{2^{2N}} \sum_{l=0}^{N-1} 2^l 1_{I_N(e_l)}(x),
\]

(21)

\[
J_2(n) \leq \frac{c}{2^{2N}Q_n} \sum_{j=N+1}^{n_1} 2^j q_{2^{j-1}} \sum_{m=1}^{N} \sup_{2^{m-1} \leq k < 2^m} (k|K_k(x)|)
\]

\[
\leq \frac{c}{2^{2N}} \sum_{m=1}^{N} \sup_{2^{m-1} \leq k < 2^m} (k|K_k(x)|).
\]

By Lemma 1 and from the condition of the theorem we can write

\[
\int_{I_N} \sup_{n>2^N} \left( 2^{N/p} \int_{I_N} |F_{n,2}^{(1)}(x+t)| \, dt \right)^p \, dx
\]

\[
\leq \int_{I_N} \sup_{n>2^N} \left( 2^{N/p} J_1(n) \right)^p + \int_{I_N} \sup_{n>2^N} \left( 2^{N/p} J_2(n) \right)^p + \int_{I_N} \sup_{n>2^N} \left( 2^{N/p} J_3(n) \right)^p
\]

\[
\leq \frac{c^p 2^{N(1-p)}}{Q_n^p} \sum_{j=1}^{N} q_{2^{j-1}}^{p} \sum_{m=1}^{j} \int_{I_N} \sup_{2^{m-1} \leq k < 2^m} (k|K_k(x)|)^p \, dx
\]

\[
+ c_p 2^{N(1-2p)} \sum_{s=0}^{N-1} \int_{I_N} \left( 2^s K_{2^s}(x) \right)^p \, dx + c_p 2^{N(1-2p)} \sum_{l=0}^{N-1} 2^{lp} \int_{I_N} 1_{I_N(e_l)}(x) \, dx
\]

\[
\leq \frac{c^p 2^{N(1-p)}}{Q_n^p} \sum_{j=1}^{N} q_{2^{j-1}}^{p} 2^{j(2p-1)} + c_p 2^{N(1-2p)} \sum_{s=0}^{N-1} 2^{s(2p-1)} + c_p 2^{N(1-p)} \sum_{l=0}^{N-1} 2^{lp}
\]

\[
\leq \frac{c^p 2^{N(1-p)}}{Q_n^p} \sum_{j=1}^{N} \left( q_{2^{j-1}} 2^j \right)^p 2^{j(p-1)} + c_p
\]

\[
\leq c_p \sup_{n \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_n^p} \sum_{j=1}^{N} Q_{2^j}^{p} 2^{j(p-1)} + c_p \leq c_p < \infty.
\]

Analogously, we can prove that

\[
\int_{I_N} \sup_{n>2^N} \left( 2^{N/p} \int_{I_N} |F_{n,2}^{(2)}(x+t)| \, dt \right)^p \, dx \leq c_p < \infty.
\]

(24)
Combine (18), (19), (23) and (24) we complete the proof of Theorem 3.

□

For $p = 1$, Theorem 3 implies that the following two conditions are equivalent:

- The maximal operator $t^*$ is bounded from the dyadic Hardy space $H_1(\mathbb{I})$ to the space $L_1(\mathbb{I})$;
- $\sup_{N \in \mathbb{N}} \frac{1}{Q_{2N}} \sum_{j=1}^{N} Q_{2^j} < \infty$.

On the other hand, in [13] it is proved that the following two conditions are equivalent:

- The maximal operator $t^*$ is bounded from the space $L_\infty(\mathbb{I})$ to the space $L_\infty(\mathbb{I})$;
- $\sup_{N \in \mathbb{N}} \frac{1}{Q_{2N}} \sum_{j=1}^{N} Q_{2^j} < \infty$.

Hence, we can conclude that the following.

**Theorem 4.** The following three conditions are equivalent:

- The maximal operator $t^*(f)$ is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$;
- The maximal operator $t^*(f)$ is bounded from $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$;
- $\sup_{n \in \mathbb{N}} \frac{1}{Q_{n}} \sum_{k=1}^{[n]} Q_{2^k} < \infty$.

7. Applications to various summability methods

Since the Nörlund mean is a generalization of many other well-known means with a wide range of literature, in the last section we give applications of our results.

**Example 1. Fejér means:** Let $q_j = 1$. then it is easy to see that $Q_j \sim j$ and we have

$$\frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^{N} Q_{2^j}^p 2^{j(p-1)} = 2^{N(1-p)} \sum_{j=1}^{N} 2^{jp} 2^{j(p-1)} = 2^{N(1-2p)} \sum_{j=1}^{N} 2^{j(2p-1)}.$$

Hence,

$$\left( \sup_{N \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^{N} Q_{2^j}^p 2^{j(p-1)} < \infty \right) \iff (p > 1/2)$$

and we have that the following two conditions are equivalent:

- The maximal operator $\sup_{n \in \mathbb{N}} |\sigma_n(f)|$ is bounded from the dyadic Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$;
- $p > 1/2$.

Let $p > 1/2$. The boundedness of maximal operator $\sup_{n \in \mathbb{N}} |\sigma_n(f)|$ was proved by Weisz [26], and the essence of condition $p > 1/2$ was proved by the author [9].
Example 2. $(C, \alpha)$-means: Let $q_j := A_j^{\alpha-1}, \alpha \in (0, 1)$. It is easy to see that $Q_{2j} \sim 2^{j\alpha}$. Since

$$\frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N Q_{2j}^p 2^j(p-1) = \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N 2^{j(p\alpha)} 2^j(p-1) = \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N 2^j(p(\alpha+1)-1)$$

we conclude that

$$\sup_{N \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N Q_{2j}^p 2^j(p-1) < \infty \iff (p > \frac{1}{1+\alpha}).$$

Consequently, we have the following two conditions are equivalent:

- The maximal operator $\sup_{n \in \mathbb{N}} |\sigma_n^\alpha(f)|$ is bounded from the dyadic Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$;
- $p > \frac{1}{1+\alpha}$.

Let $p > \frac{1}{1+\alpha}$. The boundedness of maximal operator $\sup_{n \in \mathbb{N}} |\sigma_n^\alpha(f)|$ was proved by Weisz [26], and the importance of condition $p > \frac{1}{1+\alpha}$ was proved by the author [10].

Example 3. Let $q_j := j^{\alpha-1}, \alpha \in [0, 1)$. First, we consider the case when $\alpha = 0$. Then the Nörlund means coincide to the Nörlund logarithmic means

$$t_n(f; x) := \frac{1}{Q_n} \sum_{k=1}^{n-1} S_k(f; x) / n - k.$$

Nörlund’s logarithmic means with respect to the trigonometric system was studied by Tkebuchava [22,23]. The convergence and divergence of this means with respect to the Walsh systems was discussed in [5–7,12,16]. Since

$$\sup_{n \in \mathbb{N}} \frac{1}{Q_n} \sum_{k=1}^{n-1} Q_{2k} \sim \sup_{n \in \mathbb{N}} \frac{|n|^2}{\log (n+1)} \sim \sup_{n \in \mathbb{N}} \log(n+1) = \infty$$

from Theorem we conclude that the maximal operator $\sup_{n \in \mathbb{N}} |t_n(f)|$ is not bounded from $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$ and consequently, by interpolation theorem can not be bounded from $H_p(\mathbb{I})$ to $L_p(\mathbb{I})$, when $p < 1$.

Now, we suppose that $\alpha \in (0, 1)$. It is easy to see that

$$\frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N Q_{2j}^p 2^j(p-1) = \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N 2^j(p(\alpha+1)-1)$$
and

\[ \lim_{n \to \infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} (n-k)^{\alpha-1} S_k(f; x) = f(x) \quad \text{for a.e. } x \in \mathbb{I}. \]

We have the following two conditions are equivalent:

- The maximal operator \( \sup_{n \in \mathbb{N}} \frac{1}{n^{\alpha}} \left| \sum_{k=1}^{n} (n-k)^{\alpha-1} S_k(f; x) \right| \) is bounded from the dyadic Hardy space \( H_p(\mathbb{I}) \) to the space \( L_p(\mathbb{I}) \);
- \( p > \frac{1}{1+\alpha} \).

**Acknowledgement.** The author would like to thank the referees for careful reading of the paper and valuable remarks and suggestions.

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