Large deviation principles for SDEs under locally weak monotonicity conditions

Jian Wang\textsuperscript{1*} Hao Yang\textsuperscript{1†} Jianliang Zhai\textsuperscript{1‡} Tusheng Zhang\textsuperscript{2§}

1. School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China.
2. Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK.

Abstract: This paper establishes a Freidlin-Wentzell large deviation principle for stochastic differential equations (SDEs) under locally weak monotonicity conditions and Lyapunov conditions. We illustrate the main result of the paper by showing that it can be applied to SDEs with non-Lipschitzian coefficients, which can not be covered in the existing literature. These include the interesting biological models like stochastic Duffing-van der Pol oscillator model, stochastic SIR model, etc.

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\textsuperscript{*}E-mail: wg1995@mail.ustc.edu.cn
\textsuperscript{†}E-mail: yhaomath@ustc.edu.cn
\textsuperscript{‡}E-mail: zhaijl@ustc.edu.cn
\textsuperscript{§}E-mail: Tusheng.Zhang@manchester.ac.uk
1 Introduction and motivation

The small noise large deviation principle (LDP) for stochastic differential equations (SDEs) has a long history and has been studied by many authors. The general LDP was first formulated by Varadhan [21] in 1966. Then after the pioneering work of the LDP on Markov process [11] and dynamical systems [14] in the 1970s and 1980s, respectively, the LDP has attracted considerable attention as it deeply reveals the rules of the extreme events in risk management, statistical mechanics, informatics, quantum physics and so on.

SDEs with non-Lipschitzian coefficient often resulted in describing important models in physics, engineering, biology etc. They are of constant interest. In this paper, we are concerned with the LDP for SDEs with non-Lipschitzian coefficient satisfying locally weak monotonicity conditions and Lyapunov conditions, see the precise assumptions in Section 2. Our results can be applied to a number of interesting biological models with nonlinear coefficients, such as the stochastic oscillation model, stochastic SIR model, stochastic Lorenz equation etc (see also [8]).

As we are aware of, there are not many results on the Freidlin-Wentzell type LDP for SDEs with non-Lipschitzian coefficients. Let us mention some. In 2005, the fourth named author and Fang [12] used discretization and exponential equivalence to establish the LDP for a class of SDE with log-Lipschitz condition. Based on the weak convergence method, the authors in [21] and [22] also obtained the LDP for SDEs under the log-Lipschitz condition. In 2020, Cheng and Huang [8] studied the LDP for SDEs with the Dini continuity of the drift. However, the diffusion coefficients are required to be non-degenerate condition. We also refer the readers to [6, 17, 25] for related results.

The existing results could not cover the situations where both the drifts and the diffusion coefficients are only Hölder continuous. In this paper, we establish a LDP for a class of SDE with locally weak monotonicity conditions and Lyapunov conditions, see (2.2) and (2.5) for the precise assumptions. Our framework is sufficiently general to include the cases of Hölder continuous coefficients and also can be applied to many interesting biological models, see examples in Section 5.

To obtain our result, we will adopt the weak convergence approach introduced by Budhiraja, Dupuis and Maroulas in [4] and [3]. This is a challenging task because the locally weak monotonicity coefficients can be quite irregular and the Gronwall type equalities are not applicable. One of the difficulties is to prove that the solutions of controlled equations converge to the solution of the skeleton in probability. This is mainly caused by the locally weak monotonicity condition, see (2.2). To overcome the difficulty, we construct some special
control functions, apply stopping time techniques and use a particularly suitable sufficient condition proved in [20] to verify the criteria of Budhiraja-Dupuis-Maroulas.

The organization of the paper is as follows. In Section 2, we introduce the precise framework and the main result. Section 3 and Section 4 are devoted to the proof of the LDP for Eq. (2.1). Some illustrating examples are given in Section 5.

2 The framework and main results

Throughout this paper, we will use the following notation. Let \((\mathbb{R}^d, \langle \cdot, \cdot \rangle, | \cdot |)\) be the \(d\)-dimensional Euclidean space with the inner product \(\langle \cdot, \cdot \rangle\) which induces the norm \(| \cdot |\). The norm \(\| \cdot \|\) stands for the Hilbert-Schmidt norm \(\| \sigma \|^2 := \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2\) for any \(d \times m\)-matrix \(\sigma = (\sigma_{ij}) \in \mathbb{R}^d \otimes \mathbb{R}^m\). \(A^T\) stands for the transpose of the matrix \(A\). \(A \cdot x\) denotes the product of the matrix \(A\) and the vector \(x\).

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions and \(\{B(t)\}_{t \geq 0}\) a \(m\)-dimensional Brownian motion on this probability space. Fix \(T \in (0, \infty)\). Consider the following stochastic differential equation:

\[
 dx(t) = b(t, x(t))dt + \sqrt{\sigma(t, x(t))}dB(t),\quad t \in [0, T],\quad x(0) = x_0,
\]  

where the initial data \(x_0 \in \mathbb{R}^d\). \(\sigma : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto \sigma(t, x) \in \mathbb{R}^d \otimes \mathbb{R}^m\) and \(b : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto b(t, x) \in \mathbb{R}^d\) are Borel measurable functions which are continuous with respect to the space variable \(x\).

Let us now introduce the following assumptions. Let \(f, g\) be nonnegative integrable functions on \([0, T]\).

**Assumption 2.1** For arbitrary \(R > 0\),

\[
 \int_0^T \sup_{|x| \leq R} (|b(s, x)| + \|\sigma(s, x)\|^2)ds < \infty.
\]

**Assumption 2.2** There exists \(\epsilon_0 \in (0, 1)\) such that for arbitrary \(R > 0\), if \(|x| \vee |y| \leq R\), \(|x - y| \leq \epsilon_0\), the following locally weak monotonicity condition holds

\[
 2\langle x - y, b(s, x) - b(s, y) \rangle + \|\sigma(s, x) - \sigma(s, y)\|^2 \leq g(s)\eta_R(|x - y|^2), \quad \forall s \in [0, T],
\]

where \(\eta_R : [0, 1) \to \mathbb{R}_+\) is an increasing, continuous function satisfying

\[
  \eta_R(0) = 0, \quad \int_{0^+} \frac{dx}{\eta_R(x)} = +\infty.
\]

**Assumption 2.3** There exists \(V \in C^2(\mathbb{R}^d; \mathbb{R}_+)\) and \(\delta > 0\), \(\eta > 0\) such that

\[
  \lim_{|x| \to +\infty} V(x) = +\infty,
\]

\[
  \lim_{|x| \to +\infty} V(x) = +\infty,
\]
\[
\langle b(s, x), V_x(x) \rangle + \frac{\delta}{2} \text{trace}(V_{xx}(x)\sigma(s, x)\sigma^T(s, x)) + \frac{|\langle \sigma(s, x), V_x(x) \rangle |^2}{\eta V(x)} \leq f(s)(1 + \gamma(V(x))),
\]
(2.5)
and
\[
\text{trace}(V_{xx}(x)\sigma(s, x)\sigma^T(s, x)) \geq 0,
\]
(2.6)
here \( \gamma : [0, +\infty) \to \mathbb{R}_+ \) is a continuous, increasing function satisfying
\[
\int_0^{+\infty} \frac{1}{\gamma(s) + 1} ds = +\infty.
\]
(2.7)
Here \( V_x \) and \( V_{xx} \) stand for the first derivative and second derivative respectively.

**Assumption 2.4** For any \( 0 \leq c \leq 1 \),
\[
\sup_{s \in [0, \varepsilon_0]} c\eta_R(s) < \infty, \quad \sup_{s \in [0, \infty)} c\gamma(s) < \infty.
\]
Here \( \varepsilon_0 \) is the constant appearing in Assumption 2.2.

**Remark 2.1** The examples of the function \( \eta_R(s) \) in Assumption 2.2 include \( R(s) \log \frac{1}{s} \) and the examples of the function \( \gamma(s) \) in Assumption 2.3 include \( s \log s + 1 \), etc.

The following proposition gives the solution of the perturbation Eq. (2.1). The proof is similar to that of Theorem 1.1 in [18].

**Proposition 2.1** For any \( 0 < \epsilon < 1 \), under Assumptions 2.1, 2.2 and 2.3, there exists a unique strong solution of Eq. (2.1).

For each \( h \in L^2([0, T], \mathbb{R}^m) \), consider the so called skeleton equation:
\[
dx^h(t) = b(t, x^h(t)) dt + \sigma(t, x^h(t)) \cdot h(t) dt,
\]
(2.8)
with the initial data \( x^h(0) = x_0 \). We have the following result:

**Proposition 2.2** Under Assumptions 2.1, 2.2 and 2.3, there exists a unique solution to Eq. (2.8).

The proof of this proposition is similar to the proof of Theorem 1.1 in [18], so we omit it here.

We now formulate the main result in this paper.

**Theorem 2.1** For \( \epsilon > 0 \), let \( X^\epsilon \) be the solution to Eq. (2.1). Suppose Assumptions 2.1, 2.2, 2.3 and 2.4 are satisfied, then the family \( \{X^\epsilon\}_{\epsilon > 0} \) satisfies a large deviation principle on the space \( C([0, T], \mathbb{R}^d) \) with the rate function \( I : C([0, T], \mathbb{R}^d) \to [0, \infty] \), where
\[
I(y) = \inf_{\{h \in L^2([0, T], \mathbb{R}^m) : y = x^h\}} \left\{ \frac{1}{2} \int_0^T |h(s)|^2 ds \right\},
\]
(2.9)
with the convention \( \inf\{} = \infty \), here \( x^h \in C([0, T], \mathbb{R}^d) \) solves Eq. (2.8).
According to Proposition 2.2, there exists a measurable mapping $\Gamma^0(\cdot) : C([0, T], \mathbb{R}^m) \to C([0, T], \mathbb{R}^d)$ such that $x^h = \Gamma^0(\int_0^h (s) ds)$ for $h \in L^2([0, T], \mathbb{R}^m)$.

Set $S^N := \{h \in L^2([0, T], \mathbb{R}^m) : |h|^2_{L^2([0, T], \mathbb{R}^m)} \leq N\}$, and

\[ \tilde{S}^N := \{\phi : \phi \text{ is } \mathbb{R}^m\text{-valued } \mathcal{F}_t\text{-predictable process such that } \phi(\omega) \in S^N, \mathbb{P}\text{-a.s.}\} . \]

Throughout this paper, $S^N$ is endowed with the weak topology on $L^2([0, T], \mathbb{R}^m)$, under which $S^N$ is a compact Polish space.

By the Yamada-Watanabe theorem, the existence of a unique strong solution of Eq. (2.1) implies that for every $\epsilon > 0$, there exists a measurable mapping $\Gamma^\epsilon(\cdot) : C([0, T], \mathbb{R}^m) \to C([0, T], \mathbb{R}^d)$ such that $X^\epsilon = \Gamma^\epsilon(B(\cdot))$, and applying the Girsanov theorem, for any $N > 0$ and $h^\epsilon \in \tilde{S}^N$,

\[ Y^\epsilon := \Gamma^\epsilon(B(\cdot) + 1/\sqrt{\epsilon} \int_0^\epsilon h^\epsilon(s) ds) \] (2.10)

is the solution of the following SDE

\[ Y^\epsilon(t) = x + \int_0^t b(s, Y^\epsilon(s)) ds + \int_0^t \sigma(s, Y^\epsilon(s)) \cdot h^\epsilon(s) ds + \sqrt{\epsilon} \int_0^t \sigma(s, Y^\epsilon(s)) dB(s). \] (2.11)

According to Theorem 3.2 in \cite{20}, Theorem 2.1 is established once we have proved:

(i) for every $N < +\infty$ and any family $\{h_n, n \in \mathbb{N}\} \subset S^N$ converging weakly to some element $h$ as $n \to \infty$, $\Gamma^0(\int_0^h (s) ds)$ converges to $\Gamma^0(\int_0^h (s) ds)$ in the space $C([0, T], \mathbb{R}^d)$.

(ii) for every $N < +\infty$ and any family $\{h^\epsilon, \epsilon > 0\} \subset \tilde{S}^N$ and any $\delta > 0$,

\[ \lim_{\epsilon \to 0} \mathbb{P}(\rho(Y^\epsilon, Z^\epsilon) > \delta) = 0, \]

where $Z^\epsilon = \Gamma^0(\int_0^\epsilon h^\epsilon(s) ds)$ and $\rho(\cdot, \cdot)$ stands for the uniform metric in the space $C([0, T], \mathbb{R}^d)$.

Statement (i) is verified in Proposition 3.1 in Section 3 and statement (ii) is established in Proposition 4.1 in Section 4. \qed

In the sequel, the symbol $C$ will denote a positive generic constant whose value may change from place to place.

3 Proof of statement (i)

In this section, we will prove the following result.
Proposition 3.1 Suppose Assumptions 2.1, 2.2, 2.3 and 2.4 are satisfied and assume that $h_n, n \in \mathbb{N}$, $h \in S^N$, $\lim_{n \to \infty} h_n = h$ in the weak topology of $L^2([0, T], \mathbb{R}^m)$. Then $\lim_{n \to \infty} x^{h_n} = x^h$ in the space $C([0, T], \mathbb{R}^d)$, where $x^h$ solve Eq. (2.8), and $x^{h_n}$ solve Eq. (2.8) with $h$ replaced by $h_n$.

Proof The proof is divided into two steps.

Step 1: We prove that the family $\{x^{h_n}, n \in \mathbb{N}\}$ is tight in $C([0, T], \mathbb{R}^d)$.

Define $\varphi(y) = \int_0^y \frac{1}{\gamma(s)} + ds$ and $w^{h_n}(t) = e^{-\eta \int_0^t |h_n(s)|^2} V(x^{h_n}(t))$, where $\eta > 0$, $\gamma$, and $V$ are in Assumption 2.3. Apply the chain rule to get

$$
\varphi(w^{h_n}(t)) = \varphi(V(x_0)) + \int_0^t \varphi'(w^{h_n}(s)) \cdot e^{-\eta \int_0^s |h_n(r)|^2} \cdot \left[ -\eta |h_n(s)|^2 V(x^{h_n}(s)) + \langle V_x(x^{h_n}(s)), b(s, x^{h_n}(s)) \rangle + \langle V_x(x^{h_n}(s)), \sigma(s, x^{h_n}(s)) \cdot h_n(s) \rangle \right] ds \\
\leq \varphi(V(x_0)) + \int_0^t \varphi'(w^{h_n}(s)) \cdot e^{-\eta \int_0^s |h_n(r)|^2} \cdot \left[ \frac{\langle \sigma(s, x^{h_n}(s)), V_x(x^{h_n}(s)) \rangle}{\eta V(x^{h_n}(s))} \right] ds \\
\leq \varphi(V(x_0)) + C \int_0^t f(s) ds.
$$

Assumption 2.4 has been used in getting the last inequality. The above inequality (3.1) yields

$$
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} V(x^{h_n}(t)) < \infty.
$$

By the condition (2.4) on the function $V$, we deduce that

$$
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |x^{h_n}(t)| \leq L
$$

for some constant $L > 0$.

From the proof of (3.2) we also see that

$$
\sup_{\chi \in \mathcal{S}^N} \sup_{t \in [0, T]} |x^\chi(t)| \leq L_1
$$

for some constant $L_1$. This fact will be used in next section.

For $0 \leq s < t \leq T$,

$$
x^{h_n}(t) - x^{h_n}(s) = \int_s^t b(r, x^{h_n}(r)) dr + \int_s^t \sigma(r, x^{h_n}(r)) \cdot h_n(r) dr.
$$
By the Hölder inequality and (3.2) we have

\[ |x^h_n(t) - x^h_n(s)| \leq \int_s^t \sup_{|x| \leq L} |b(r, x)| dr + (\int_s^t \sup_{|x| \leq L} \|\sigma(r, x)\|^2 dr)^{1/2} (\int_s^t |h_n(r)|^2 dr)^{1/2} \]
\[ \leq \int_s^t \sup_{|x| \leq L} |b(r, x)| dr + (\int_s^t \sup_{|x| \leq L} \|\sigma(r, x)\|^2 dr)^{1/2} N^{1/2}. \]

By Assumption 2.1 it follows that, for any \( \kappa > 0 \), there exists \( \theta > 0 \) such that for any \( 0 \leq s < t \leq T \) with \( t-s \leq \theta \),

\[ \sup_{n \in \mathbb{N}} |x^h_n(t) - x^h_n(s)| \leq \kappa. \quad (3.4) \]

By (3.2), (3.4), and the Arzela-Ascoli theorem, \( \{x^h_n, n \in \mathbb{N}\} \) is tight in the space \( C([0, T], \mathbb{R}^d) \). Hence there exists a subsequence of \( x^h_n \) (still denoted as \( x^h_n \)) and \( \tilde{x} \in C([0, T], \mathbb{R}^d) \) satisfying

\[ x^h_n \to \tilde{x} \quad \text{in} \quad C([0, T], \mathbb{R}^d). \quad (3.5) \]

**Step 2:** We verify that \( \tilde{x} = x^h \), completing the proof.

Recall that \( x^h_n \) is the unique solution of

\[ x^h_n(t) = x + \int_0^t b(s, x^h_n(s)) ds + \int_0^t \sigma(s, x^h_n(s)) \cdot h_n(s) ds, \quad t \in [0, T]. \quad (3.6) \]

By the dominated convergence theorem and the continuity of \( b \) with respect to the second variable,

\[ \lim_{n \to \infty} \int_0^t |b(s, x^h_n(s)) - b(s, \tilde{x}(s))| ds = 0. \quad (3.7) \]

Next, we prove

\[ \lim_{n \to \infty} \int_0^t \left[ \sigma(s, x^h_n(s)) \cdot h_n(s) - \sigma(s, \tilde{x}(s)) \cdot h(s) \right] ds = 0. \quad (3.8) \]

Note that

\[ |\int_0^t \left[ \sigma(s, x^h_n(s)) \cdot h_n(s) - \sigma(s, \tilde{x}(s)) \cdot h(s) \right] ds| \]
\[ \leq |\int_0^t (\sigma(s, x^h_n(s)) - \sigma(s, \tilde{x}(s))) \cdot h_n(s) ds| + |\int_0^t \sigma(s, \tilde{x}(s)) \cdot (h_n(s) - h(s)) ds| \]
\[ = I_1 + I_2. \quad (3.9) \]

By the Hölder inequality, the dominated convergence theorem and (3.5),

\[ I_1 \leq (\int_0^t \|\sigma(s, x^h_n(s)) - \sigma(s, \tilde{x}(s))\|^2 ds)^{1/2} \left(\int_0^t |h_n(s)|^2 ds\right)^{1/2} \]
\[ \leq (\int_0^t \|\sigma(s, x^h_n(s)) - \sigma(s, \tilde{x}(s))\|^2 ds)^{1/2} N^{1/2} \to 0, \quad \text{as} \ n \to \infty. \quad (3.10) \]
Let $e_i$, $1 \leq i \leq d$, be the canonical basis of $\mathbb{R}^d$. Since $h_n \to h$ weakly in $L^2([0,T], \mathbb{R}^d)$, we derive that for each $1 \leq i \leq d$,

$$\langle \int_0^t \sigma(s, \tilde{x}(s)) \cdot (h_n(s) - h(s))ds, e_i \rangle \to 0,$$

which further implies that $\lim_{n \to \infty} I_2 = 0$. Combining (3.9) and (3.10), we obtain (3.8).

Now Let $n \to \infty$ in (3.6) to see that $\tilde{x}$ is the solution of the equation:

$$\tilde{x}(t) = x + \int_0^t b(s, \tilde{x}(s))ds + \int_0^t \sigma(s, \tilde{x}(s)) \cdot h(s)ds, \quad t \in [0,T]. \quad (3.11)$$

By the uniqueness, we have $\tilde{x} = x^h$, completing the proof.

$$\square$$

## 4 Proof of statement (ii)

Let $Y^\epsilon, Z^\epsilon$ be defined as in Section 2. We have the following result.

**Proposition 4.1** Suppose the Assumptions 2.1, 2.2, 2.3, and 2.4 are satisfied, then $\rho(Y^\epsilon, Z^\epsilon) \to 0$ in probability.

**Proof** Recall that $\epsilon_0$ is the constant appeared in Assumption 2.2. For $R > 0, 0 < p \leq \epsilon_0$, define

$$\tau_R^\epsilon = \inf\{t \geq 0 : |Y^\epsilon(t)| \geq R\}, \quad \tau_p^\epsilon = \inf\{t \geq 0 : |Y^\epsilon(t) - Z^\epsilon(t)|^2 \geq p\}.$$

By (3.3), there exists a constant $L > 0$ such that

$$\sup_{\epsilon > 0} \sup_{t \in [0,T]} |Z^\epsilon(t)| \leq L.$$

We first prove that, for $R > 2L$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\tau_p^\epsilon \leq \tau_R^\epsilon \wedge T\right) = 0. \quad (4.1)$$

Set $\phi_\delta(y) = \int_0^y \frac{1}{yR(s)^2}ds$. Applying the Itô formula gives

$$e^{-\int_0^t |h^\epsilon(r)|^2dr} \mathbb{E}^\mathbb{P}\left[|Y^\epsilon(t) - Z^\epsilon(t)|^2 \right] \quad (4.2)$$

where

$$= \int_0^t e^{-\int_0^r |h^\epsilon(s)|^2ds} \left\{ - |h^\epsilon(s)|^2|Y^\epsilon(s) - Z^\epsilon(s)|^2 + 2|Y^\epsilon(s) - Z^\epsilon(s), b(s, Y^\epsilon(s)) - b(s, Z^\epsilon(s))| + 2\langle Y^\epsilon(s) - Z^\epsilon(s), \sigma(s, Y^\epsilon(s)) - \sigma(s, Z^\epsilon(s)) \rangle \right\}ds$$

$$+ 2\sqrt{\epsilon} \int_0^t e^{-\int_0^r |h^\epsilon(s)|^2ds} \langle \sigma(s, Y^\epsilon(s)), Y^\epsilon(s) - Z^\epsilon(s) \rangle dB(s).$$

Since $\phi_\delta(x)$ is a concave function on the interval $[0, \epsilon_0)$ and $\lim_{x \to 0} \phi_\delta'(x) = \frac{1}{\delta}$, there exists a concave extension of $\phi_\delta(x)$ on the real line denoted by $\bar{\phi}_\delta(x)$ satisfying $\bar{\phi}_\delta(x) = \phi_\delta(x)$
on $[0, \epsilon_0)$. The second order derivative $\phi''_\delta(x)$ of $\phi_\delta(x)$ in the sense of distributions is a non-positive Radon measure. Applying the Itô-Tanaka formula to (4.2) gives

$$
\bar{\phi}_\delta(e^{-\int_0^t |h^r(s)|^2 ds} |Y^\epsilon(t) - Z^\epsilon(t)|^2)
= \int_0^t \bar{\phi}_\delta(e^{-\int_0^s |h^r(r)|^2 dr} |Y^\epsilon(s) - Z^\epsilon(s)|^2) \cdot e^{-\int_0^s |h^r(r)|^2 dr} \cdot \{ -|h^r(s)|^2 |Y^\epsilon(s) - Z^\epsilon(s)|^2 \\
+ 2(Y^\epsilon(s) - Z^\epsilon(s), b(s, Y^\epsilon(s)) - b(s, Z^\epsilon(s)))
+ 2(Y^\epsilon(s) - Z^\epsilon(s), [\sigma(s, Y^\epsilon(s)) - \sigma(s, Z^\epsilon(s))] \cdot h^r(s)) + \epsilon||\sigma(s, Y^\epsilon(s))||^2 \} ds
+ \sqrt{\epsilon} \int_0^t \left[ \bar{\phi}'_\delta(e^{-\int_0^r |h^r(r)|^2 dr} |Y^\epsilon(s) - Z^\epsilon(s)|^2) \cdot e^{-\int_0^r |h^r(r)|^2 dr} \cdot \{ \langle \sigma(s, Y^\epsilon(s)), Y^\epsilon(s) - Z^\epsilon(s) \rangle \} dB(s)
\right.
+ \int_{-\infty}^{+\infty} \Lambda_t(x) \bar{\phi}'_\delta(dx),
$$

where the nonnegative random field $\Lambda = \{ \Lambda_t(x, \omega); (t, x, \omega) \in [0, +\infty) \times \mathbb{R} \times \Omega \}$ is the local time of the semimartingale $e^{-\int_0^t |h^r(s)|^2 ds} |Y^\epsilon(\cdot) - Z^\epsilon(\cdot)|^2$. Noticing

$$
\int_{-\infty}^{+\infty} \Lambda_t(x) \bar{\phi}'_\delta(dx) \leq 0,
$$

we derive that, for the stopping time $\hat{\tau}^\epsilon := T \wedge \tau_R^\epsilon \wedge \tau_p^\epsilon$,

$$
\bar{\phi}_\delta(e^{-\int_0^{\hat{\tau}^\epsilon} |h^r(s)|^2 ds} |Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)|^2)
\leq \int_0^{\hat{\tau}^\epsilon} \bar{\phi}'_\delta(e^{-\int_0^s |h^r(r)|^2 dr} |Y^\epsilon(s) - Z^\epsilon(s)|^2) \cdot e^{-\int_0^s |h^r(r)|^2 dr} \cdot \{ \epsilon||\sigma(s, Y^\epsilon(s))||^2 \} ds
+ \int_0^{\hat{\tau}^\epsilon} \left[ \bar{\phi}_\delta(e^{-\int_0^r |h^r(r)|^2 dr} |Y^\epsilon(s) - Z^\epsilon(s)|^2) \cdot e^{-\int_0^r |h^r(r)|^2 dr} \cdot \{ \langle \sigma(s, Y^\epsilon(s)), Y^\epsilon(s) - Z^\epsilon(s) \rangle \} dB(s)
\right.
+ \int_{-\infty}^{+\infty} \Lambda_t(x) \bar{\phi}_\delta(dx),
$$

Taking expectation we get

$$
\mathbb{E}\bar{\phi}_\delta(e^{-\int_0^{\hat{\tau}^\epsilon} |h^r(s)|^2 ds} |Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)|^2)
\leq \mathbb{E} \int_0^{\hat{\tau}^\epsilon} \bar{\phi}_\delta(e^{-\int_0^s |h^r(r)|^2 dr} |Y^\epsilon(s) - Z^\epsilon(s)|^2) \cdot e^{-\int_0^s |h^r(r)|^2 dr} \cdot \{ \epsilon||\sigma(s, Y^\epsilon(s))||^2 \} ds
\right.
+ \int_{-\infty}^{+\infty} \Lambda_t(x) \bar{\phi}_\delta(dx),
$$

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\[ + g(s)\eta_R(\|Y^\epsilon(s) - Z^\epsilon(s)\|^2)ds \]
\[ \leq C \int_0^T g(s)ds + \frac{\epsilon}{\delta} \mathbb{E} \int_0^\tau_s \|\sigma(s, Y^\epsilon(s))\|^2ds. \] (4.3)

The last inequality follows from the property of \( \eta_R \) stated in Assumption 2.4. Hence by taking \( \epsilon \to 0 \), we obtain

\[ \limsup_{\epsilon \to 0} \mathbb{E} \phi_C(\int_0^\tau_s |h^\epsilon(s)|^2ds | Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)|^2) \leq C \int_0^T g(s)ds, \]

that is,

\[ \limsup_{\epsilon \to 0} \mathbb{E} \int_0^\tau_s e^{-\frac{\tau_s}{\eta(u)}} |h^\epsilon(s)|^2ds | Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)|^2 \leq C \int_0^T g(s)ds. \] (4.4)

Since for \( \eta \leq \epsilon_0 \),

\[ \int_0^{e^{-N\eta}} \frac{1}{\eta(u) + \delta} du \leq C \int_0^T g(s)ds, \]

we assert, by (2.3) and (4.4), that

\[ \limsup_{\epsilon \to 0} \mathbb{P}(\|Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)\|^2 \geq \eta) \leq \lim_{\delta \to 0} \frac{C \int_0^{e^{-N\eta}} g(s)ds}{\int_0^{e^{-N\eta}} \frac{1}{\eta(s) + \delta} ds} = 0. \] (4.6)

Noting that

\[ \{ \sup_{s \leq T \wedge \tau_R \wedge \tau_p^\epsilon} |Y^\epsilon(s) - Z^\epsilon(s)|^2 \geq p \} \subset \{ |Y^\epsilon(\hat{\tau}^\epsilon) - Z^\epsilon(\hat{\tau}^\epsilon)|^2 \geq p \}, \]

it follows from (4.6) that

\[ \limsup_{\epsilon \to 0} \mathbb{P}(\sup_{s \leq T \wedge \tau_R \wedge \tau_p^\epsilon} |Y^\epsilon(s) - Z^\epsilon(s)|^2 \geq p) = 0. \] (4.7)

Now (4.1) follows from the inclusion:

\[ \{ \tau_p^\epsilon \leq T \wedge \tau_R^\epsilon \} \subset \{ \sup_{s \leq T \wedge \tau_R \wedge \tau_p^\epsilon} |Y^\epsilon(s) - Z^\epsilon(s)|^2 \geq p \}. \]

Next, we prove

\[ \lim_{R \to \infty} \sup_{\epsilon \in (0,1)} \mathbb{P}(\tau_p^\epsilon \leq T \wedge \tau_R^\epsilon) = 0. \] (4.8)

Let \( \varphi(x) \) be defined as that in Proposition 3.1 and denote \( W^{h^\epsilon}(t) = e^{-\eta \int_0^t |h^\epsilon(s)|^2ds} V(Y^\epsilon(t)) \).

Similar to (4.3), applying the Itô-Tanaka formula gives

\[ \mathbb{E} \varphi(W^{h^\epsilon}(T \wedge \tau_R^\epsilon \wedge \tau_p^\epsilon)) \]
\[ \leq \varphi(V(x_0)) + \mathbb{E} \int_0^{T \wedge \tau_R^\epsilon \wedge \tau_p^\epsilon} \varphi'(W^{h^\epsilon}(s)) \cdot e^{-\eta \int_0^s |h^\epsilon(r)|^2dr} \cdot [ - \eta |h^\epsilon(s)|^2 V(Y^\epsilon(s)) \]
\begin{align*}
&+ \langle V_x(Y^e(s)), b(s, Y^e(s)) \rangle + \langle V_x(Y^e(s)), \sigma(s, Y^e(s)) \cdot h^e(s) \rangle \\
&+ \epsilon \cdot \text{trace} \left[ V_{xx}(Y^e(s)) \sigma(s, Y^e(s)) \sigma^T(s, Y^e(s)) \right] ds \\
\leq & \quad \varphi(V(x_0)) + E \int_0^{T \land \tau^e_R \land \tau^e_p} \varphi'(W^{h^e}(s)) \cdot e^{-\eta \int_0^s |h^e(r)|^2 dr \cdot \left[ -\eta |h^e(s)|^2 V(Y^e(s)) \right.} \\
& \quad + \frac{||\sigma(s, Y^e(s)), V_x(Y^e(s))||}{\sqrt{\eta V(Y^e(s))}} \cdot \sqrt{\eta V(Y^e(s)) |h^e(s)|} ds \\
&+ \epsilon \cdot \text{trace} \left[ V_{xx}(Y^e(s)) \sigma(s, Y^e(s)) \sigma^T(s, Y^e(s)) \right] ds \\
\leq & \quad \varphi(V(x_0)) + E \int_0^{T \land \tau^e_R \land \tau^e_p} \varphi'(W^{h^e}(s)) \cdot e^{-\eta \int_0^s |h^e(r)|^2 dr \cdot \left[ \langle V_x(Y^e(s)), b(s, Y^e(s)) \rangle \right.} \\
& \quad + \frac{||\sigma(s, Y^e(s)), V_x(Y^e(s))||^2}{\eta V(Y^e(s))} + \epsilon \cdot \text{trace} \left[ V_{xx}(Y^e(s)) \sigma(s, Y^e(s)) \sigma^T(s, Y^e(s)) \right] ds \\
\leq & \quad \varphi(V(x_0)) + E \int_0^{T \land \tau^e_R \land \tau^e_p} \varphi'(W^{h^e}(s)) \cdot e^{-\eta \int_0^s |h^e(r)|^2 dr \cdot \left. f(s) (1 + \gamma(V(Y^e(s)))) \right]} ds \\
\leq & \quad \varphi(V(x_0)) + C \int_0^T f(s) ds. \quad (4.9)
\end{align*}

The last inequality follows from the property of $\gamma$ in Assumption 2.4. By (4.9) and the definition of $\varphi$, we deduce that

$$
\mathbb{P} \left( \tau^e_R \leq T \land \tau^e_p \right) \leq \frac{\varphi(V(x_0)) + C \int_0^T f(s) ds}{\int_{e^{-\eta N \cdot V(R)}}^{1} \frac{1}{\gamma(s) + 1} ds}.
$$

Finally, by (2.4) and (2.7), letting $R \to \infty$, we obtain (1.8).

Now we are in the position to complete the proof of the theorem. For arbitrary $\delta_1 > 0$, we have

$$
\mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1 \right) \\
= \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1, \tau^e_R \land \tau^e_{\delta_1} > T \right) \\
+ \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1, \tau^e_R \land \tau^e_{\delta_1} \leq T \right) \\
= \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1, \tau^e_R \land \tau^e_{\delta_1} > T \right) \\
+ \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1, \tau^e_R \leq T \land \tau^e_{\delta_1} \right) \\
+ \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1, \tau^e_{\delta_1} \leq \tau^e_R \land T \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq T \land \tau^e_R \land \tau^e_{\delta_1}} |Y^e(s) - Z^e(s)|^2 \geq \delta_1^2 \right) \\
+ \mathbb{P} \left( \tau^e_R \leq T \land \tau^e_{\delta_1} \right) + \mathbb{P} \left( \tau^e_{\delta_1} \leq \tau^e_R \land T \right).
$$

(1.1) and (4.7) (with $p = \delta_1^2$) imply that, for any $R > 2L$,\n
$$
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{0 \leq s \leq T} |Y^e(s) - Z^e(s)| \geq \delta_1 \right) \leq \sup_{\epsilon \in (0,1)} \mathbb{P} \left( \tau^e_R \leq T \land \tau^e_{\delta_1} \right).
$$
Let $R \to \infty$ to get
\[
\lim_{\epsilon \to 0} P\left( \sup_{0 \leq s \leq T} |Y^\epsilon(s) - Z^\epsilon(s)| \geq \delta_1 \right) = 0.
\]
The proof is complete. \qed

5 Applications and examples

In this part, we present some applications. The two Assumptions 2.1 and 2.4 are all satisfied by the following examples. Therefore, the large deviation principle holds for the models.

**Example 5.1.** The following one-dimensional SDE is also considered in [18]:
\[
dx(t) = -x^{\frac{1}{3}}(t)dt + x^{\frac{2}{3}}(t)dB(t).
\]
Note that both the drift term and the diffusion term are Hölder continuous. Indeed,
\[
2\langle x - y, b(x) - b(y) \rangle + \|\sigma(x) - \sigma(y)\|^2
\]
\[
= (x^{\frac{1}{3}} - y^{\frac{1}{3}})^2 - 2(x - y)(x^{\frac{1}{3}} - y^{\frac{1}{3}})
\]
\[
= (x^{\frac{1}{3}} - y^{\frac{1}{3}})^2[(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2 - 2(x^{\frac{1}{3}} + y^{\frac{1}{3}}) + y^{\frac{2}{3}}]
\]
\[
= -(x^{\frac{1}{3}} - y^{\frac{1}{3}})^2(x^{\frac{2}{3}} + y^{\frac{2}{3}})
\]
\[
\leq \eta_R(|x - y|^2).
\]
If we take $\delta = 1$, $\eta = 4$ and $V(x) = x^2$ in Assumption 2.3, then
\[
\langle b(s, x), V_x(x) \rangle + \frac{\delta}{2} {\text{trace}}\left( V_{xx}(x)\sigma(s, x)\sigma^T(s, x) \right) + \frac{\|\sigma(s, x), V_x(x)\|^2}{\eta V(x)} = 0 \leq \gamma(|x|^2).
\]
So Assumption 2.2, 2.3 holds. Hence, Theorem 2.1 holds.

**Example 5.2.** The following multi-dimensional SDEs are also considered in [23, example 171]:
\[
dx(t) = -x(t)|x(t)|^{-\alpha}dt + \sigma(x(t))dB(t),
\]
where $\alpha \in (0, 1)$, $\sigma$ is local Lipschitz continuous. It is easy to check that the function $x|x|^{-\alpha}$ is not locally Lipschitz. However it satisfies
\[
\langle x - y, -x|x|^{-\alpha} + y|y|^{-\alpha} \rangle = -|x|^{2-\alpha} + \langle y, x|x|^{-\alpha} \rangle + \langle x, y|y|^{-\alpha} \rangle - |y|^{2-\alpha}
\]
\[
\leq -|x|^{2-\alpha} - |y|^{2-\alpha} + |y||x|^{1-\alpha} + |x||y|^{1-\alpha}
\]
\[
= (|x| - |y|)(|y|^{1-\alpha} - |x|^{1-\alpha}) \leq 0.
\]
Thus, if we consider an equation of the form (5.3) with any locally Lipschitz coefficient $\sigma$ which satisfies Assumption 2.3 then all the conditions in Theorem 2.1 are ensuerd.
Example 5.3. (Stochastic Duffing-van der Pol oscillator model) The Duffing-van der Pol equation unifies both the Duffing equation and the van der Pol equation and has been used for example in certain flow-induced structural vibration problems [16]. Cox & Hutzenthaler & Jentzen has considered the following more general stochastic model in [9]:

\[
\begin{align*}
\dot{X}_t^{x,1} &= \alpha_2 \dot{X}_t^{x,1} - \alpha_1 X_t^{x,1} - \alpha_3 (X_t^{x,1})^2 \dot{X}_t^{x,1} - (X_t^{x,1})^3 + g(X_t^{x,1}) \dot{W}_t, \\
X_0^{x,1} &= x_1, \quad \dot{X}_0^{x,1} = x_2,
\end{align*}
\]

where \(\alpha_1, \alpha_2, \alpha_3 \in (0, \infty)\). Here we assume \(|g(x)|^2 \leq \eta_0 + \eta_1 |x|^4, \eta_0, \eta_1 > 0\).

By defining \(\dot{X}_t^{x,1} = X_t^{x,2}\), then the above equation can be transformed equivalently into the following SDEs:

\[
\begin{align*}
&dX_t^{x,1} = X_t^{x,2} dt, \\
&dX_t^{x,2} = \left[\alpha_2 X_t^{x,2} - \alpha_1 X_t^{x,1} - \alpha_3 (X_t^{x,1})^2 X_t^{x,2} - (X_t^{x,1})^3\right] dt + g(X_t^{x,1}) dW_t, \\
&X_0^{x,1} = x_1, \quad X_0^{x,2} = x_2.
\end{align*}
\]

For \(x = (x_1, x_2) \in \mathbb{R}^2\), set \(b(s, x) = (x_2, \alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3)\) and \(\sigma(s, x) = (0, g(x_1))^T\). Since \(b\) and \(\sigma\) are local Lipschitz continuous, Assumption 2.2 holds if \(\eta_R(s) = L_R s\), where \(L_R\) is a constant only dependent on \(R\). Define \(V(x) = \frac{(x_1)^4}{4} + \alpha_1 (x_1)^2 + (x_2)^2\) and set \(\delta = \eta = 1\) in Assumption 2.3 then

\[
\begin{align*}
\langle b(s, x), V_x(x) \rangle + \frac{1}{2} \text{trace}(V_{xx}(x)\sigma(s, x)\sigma^T(s, x)) &+ \frac{|\langle \sigma(s, x), V_x(x) \rangle|^2}{V(x)} \\
&= x_2 (2(x_1)^3 + 2\alpha_1 x_1) + 2x_2 (2\alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3) + |g(x_1)|^2 \\
&\quad + \frac{(x_1)^4}{4} + \alpha_1 (x_1)^2 + (x_2)^2 \\
&= 2\alpha_2 (x_2)^2 - 2\alpha_3 (x_1)^2 (x_2)^2 + |g(x_1)|^2 + \frac{(x_1)^4}{4} + \alpha_1 (x_1)^2 + (x_2)^2 \\
&\leq \eta_0 + 2\alpha_2 (x_2)^2 + \eta_1 (x_1)^4 + \frac{4\eta_0 (x_2)^2 + 4\eta_1 (x_1)^4 (x_2)^2}{(x_1)^2 + \alpha_1 (x_1)^2 + (x_2)^2} \\
&\leq 5\eta_0 + (8\eta_1 + 2\alpha_2)(x_2)^2 + \eta_1 (x_1)^4 \\
&\leq K(1 + V(x)),
\end{align*}
\]

where \(K = 5\eta_0 + 10\eta_1 + 2\alpha_2\). Hence Assumption 2.3 also holds by taking \(\gamma(s) = s\). Then, Theorem 2.1 holds.

Example 5.4. (Stochastic SIR model) The SIR model from epidemiology for the total number of susceptible, infected and recovered individuals has been introduced by Anderson & May [2]. Here we consider the following stochastic SIR model:

\[
\begin{align*}
&dX_t^{x,1} = (-\alpha X_t^{x,1} X_t^{x,2} - \kappa X_t^{x,1} + \kappa) dt - \beta X_t^{x,1} X_t^{x,2} dW_t,
\end{align*}
\]
Here, we consider the three-dimensional Stratonovich stochastic competitive Lotka-Volterra (LV) systems: systems play an important role in population dynamics, game theory, and so on (see [15]).

**Example 5.5.** For $x = (x_1, x_2, x_3) ∈ [0, ∞)^3$, the well-known Lotka-Volterra systems:

$$\begin{align*}
    dX_t^{x,2} &= (\alpha X_t^{x,1} X_t^{x,2} - (\gamma + \kappa) X_t^{x,2}) dt + \beta X_t^{x,1} X_t^{x,2} dW_t^2, \\
    dX_t^{x,3} &= (\gamma X_t^{x,2} - \kappa X_t^{x,3}) dt, \\
    X_0^{x,1} &= x_1, \quad X_0^{x,2} = x_2, \quad X_0^{x,3} = x_3,
\end{align*}$$

where $\alpha, \beta, \gamma, \kappa ∈ (0, ∞)$ and $x = (x_1, x_2, x_3) ∈ [0, ∞)^3$.

For $x = (x_1, x_2, x_3) ∈ [0, ∞)^3$, set $b(s, x) = (-\kappa x_1 x_2 - \kappa x_1 + \kappa, \alpha x_1 x_2 - (\gamma + \kappa) x_2, \gamma x_2 - \kappa x_3)^T$ and $σ(s, x) = (-β x_1 x_2, β x_1 x_2, 0)^T$. Since $b$ and $σ$ are local Lipschitz continuous, Assumption 2.2 holds if $η_R(s) = L_RS$. So to verify the Theorem 2.1 we need only to verify Assumption 2.3.

Define $V(x) = (x_1 + x_2 - 1)^2$ and let $δ$ and $η$ be any positive constants in Assumption 2.3. A direct calculation gives that:

$$\langle b(s, x), V_x(x) \rangle + \frac{δ}{2} \text{trace}(V_{xx}(x)σ(s, x)σ^T(s, x)) + \frac{|⟨σ(s, x), V_x(x)⟩|^2}{ηV(x)} \leq \frac{γ}{2}.$$ 

So Assumption 2.3 holds.

**Example 5.5.** (Stochastic Lotka-Volterra (LV) systems) The well-known Lotka-Volterra systems play an important role in population dynamics, game theory, and so on (see [15]).

Here, we consider the three-dimensional Stratonovich stochastic competitive Lotka-Volterra systems:

$$\begin{align*}
    dy_1 &= y_1(r - a_{11} y_1 - a_{12} y_2 - a_{13} y_3) dt + σ ∘ y_1 dB(t), \\
    dy_2 &= y_2(r - a_{21} y_1 - a_{22} y_2 - a_{23} y_3) dt + σ ∘ y_2 dB(t), \\
    dy_3 &= y_3(r - a_{31} y_1 - a_{32} y_2 - a_{33} y_3) dt + σ ∘ y_3 dB(t),
\end{align*}$$

where $r > 0$, $a_{ij} > 0$, $i, j = 1, 2, 3$ and initial data $(y_1(0), y_2(0), y_3(0)) ∈ (0, +∞)^3$. According to [7], Theorem 3.2, we obtain $y(t) = (y_1(t), y_2(t), y_3(t)) ∈ (0, +∞)^3$ for all $t > 0$.

The coefficients of above equation are local Lipschitz continuous, so Assumption 2.2 holds. If $\min\{a_{11}, a_{22}, a_{33}\} > \frac{γ}{2}$, then Assumption 2.3 holds with $γ(s) = s$ and $V(y) = |y|^2$. So, all the conditions in Theorem 2.1 are satisfied.

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