EXISTENCE AND UNIQUENESS OF VISCOSITY SOLUTIONS TO THE EXTERIOR PROBLEM OF A PARABOLIC MONGE-AMPÈRE EQUATION

SHUYU GONG, ZIWEI ZHOU AND JIGUANG BAO*

School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, China

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Abstract. In this paper, we use the Perron method to prove the existence and uniqueness of the exterior problem for a kind of parabolic Monge-Ampère equation \(-u_t + \log \det D^2 u = f(x)\) with prescribed asymptotic behavior at infinity, where \(f\) is asymptotically close to a radial function at infinity. We generalize the results of both the elliptic exterior problems and the parabolic interior problems for the Monge-Ampère equations.

1. Introduction. Elliptic Monge-Ampère equations, a hot research field in partial differential equation, have many valuable results over the years, which also promote the extensive study of parabolic Monge-Ampère equations. These two kinds of equations have some common research perspectives, such as the liouville property, asymptotic behavior and existence, while the existence and uniqueness is the focus of this paper.

We start with some known significant conclusions of elliptic Monge-Ampère equations and mainly prove an existence theorem of the exterior problem of a kind of parabolic Monge-Ampère equation. In the following statement, unless otherwise specified, we always assume that \(\Omega\) is a smooth, bounded and strictly convex domain in \(\mathbb{R}^n\).

In 1984, Caffarelli, Nirenberg and Spruck [4] obtained the existence and uniqueness of strictly convex smooth solutions for the interior Dirichlet problem

\[
\begin{align*}
\det D^2 u &= f(x) \text{ in } \Omega, \\
u &= \varphi(x) \text{ on } \partial \Omega,
\end{align*}
\]

where \(f \in C^\infty(\overline{\Omega})\) is a positive function, and \(\varphi \in C^\infty(\overline{\Omega})\). The result also holds for viscosity solutions, see [16], [3] and [11].

On this basis, many mathematicians have discussed the existence and uniqueness of exterior problems and global solutions with asymptotics, see [1, 2, 5, 13]. The following theorem is a major representative of the exterior problems.

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* Corresponding author.
Theorem 1.1 ([5]). Let \( n \geq 3, \varphi \in C^2(\partial \Omega) \). Then for any given real \( n \times n \) symmetric positive definite matrix \( A \) with \( \det A = 1, \ b \in \mathbb{R}^n \), there exists some constant \( c_* \) depending only on \( n, \Omega, \varphi, A \) and \( b \), such that for every \( c > c_* \), there exists a unique function \( u \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega) \) that satisfies

\[
\begin{aligned}
\det D^2u &= 1, \quad D^2u > 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \\
u &= \varphi(x) \text{ on } \partial \Omega, \\
l \lim \sup_{|x|\to+\infty} |x|^{-2} |u(x) - \left( \frac{1}{2} x'Ax + b \cdot x + c \right)| &< \infty.
\end{aligned}
\]

The corresponding result for \( n = 2 \) was given by Ferrer, Martínez and Milán in [9, 10] through the complex variable method. In regard to the conclusion of Theorem 1.1, Li and Lu took a further step in [14], discovering the definite relationship between \( c_* \) and the existence and nonexistence of solutions for \( n \geq 3 \).

In addition, under some asymptotic condition, Caffarelli and Li [5] obtained an important existence result about the global solutions of

\[
\det D^2u = f \text{ in } \mathbb{R}^n,
\]

where \( n \geq 3 \), and \( f \) is a continuous function which is constant outside a bounded set and satisfies

\[
0 \leq \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty.
\]

In [1], Bao, Li and Zhang weakened the requirement of \( f \) in (1.2) to a perturbation of 1 and got a similar conclusion. Recently, Bao, Xiong and Zhou solved the two-dimensional case, see [2].

On the other hand, there are many studies about the existence and uniqueness of solutions of different forms of parabolic Monge-Ampère equations, see [12, 17, 19].

Let \( Q_T = \Omega \times (0, T) \), where \( T > 0 \) is a positive constant, and \( \partial_p Q_T \) denote the parabolic boundary of \( Q_T \). By means of approximation and nonlinear perturbation, Wang and Wang [19] proved the existence and uniqueness of viscosity solutions to the initial-boundary value problem

\[
\begin{aligned}
-u_t + \log \det D^2u &= f(x, t) \text{ in } Q_T, \\
u &= \varphi(x, t) \text{ on } \partial_p Q_T,
\end{aligned}
\]

where \( f \) is Lipschitz continuous in \( \overline{Q_T} \) and \( \varphi \in C^{2,2}(\overline{Q_T}) \) is strictly convex in \( x \) for any fixed \( t \in [0, T] \). Moreover, (1.3) has application in geometric problems such as surface deformation as well as Minkowski problems, see [6].

In this paper, we will prove that under certain conditions, there exists a unique viscosity solution to the exterior problem of the parabolic Monge-Ampère equation

\[
\begin{aligned}
-u_t + \log \det D^2u &= f(x) \text{ in } (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T], \\
u &= \varphi(x, t) \text{ on } \partial \Omega \times [0, T], \\
u(x, 0) &= \phi(x) \text{ in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \( n \geq 2 \).

When \( f \equiv -1 \), the result was proved by Dai in [8] with the condition \( \varphi_{x_i, t}(x, t) = 0, \ x \in \partial \Omega, \ 0 \leq t \leq T \). Now we consider the case where \( f \) is asymptotically close to a radial function at infinity and the restriction on \( \varphi \) is removed.

To begin with, let us recall the definition of viscosity solutions of parabolic equations [18, 20]. Denote

\[
Q_r(\bar{x}, \bar{t}) := \{ (x, t) \mid |x - \bar{x}| < r, \bar{t} - r^2 < t \leq \bar{t} \}.
\]
For simplicity, USC is short for upper-semicontinuous and LSC is short for lower-semicontinuous.

**Definition 1.2.** Let \( u \) be a USC (LSC) function in \( (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T] \). \( u \) is called a viscosity subsolution (supersolution) of (1.4), if for any point \((\bar{x}, \bar{t})\) in \( (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T] \) and any function \( h \in C^{2,1}(Q_r(\bar{x}, \bar{t})) \) (with some \( Q_r(\bar{x}, \bar{t}) \subset (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T] \)) and satisfying

\[
 u(x, t) - h(x, t) \leq (\geq) u(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}), \quad \forall (x, t) \in Q_r(\bar{x}, \bar{t}),
\]

we have

\[
 -h_t(\bar{x}, \bar{t}) + \log \det(D^2 h(\bar{x}, \bar{t})) \geq (\leq) f(\bar{x}).
\]

For the supersolution, we also require that \( D^2 h(\bar{x}, \bar{t}) > 0 \) in the matrix sense.

\( u \in C^0((\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T]) \) is called a viscosity solution of (1.4), if it is both a viscosity subsolution and supersolution of (1.4).

**Definition 1.3.** A function \( u \) is called a viscosity subsolution (supersolution) of the problem (1.4), (1.5), (1.6), if \( u \) is a viscosity subsolution (supersolution) of (1.4), \( u(x, t) \leq (\geq) \phi(x, t) \) on \( \partial \Omega \times [0, T] \), and \( u(x, 0) \leq (\geq) \phi(x) \) in \( \mathbb{R}^n \setminus \Omega \).

\( u \in C^0((\mathbb{R}^n \setminus \Omega) \times [0, T]) \) is called a viscosity solution of (1.4), (1.5), (1.6), if \( u \) is a viscosity solution of (1.4), \( u(x, t) = \phi(x, t) \) on \( \partial \Omega \times [0, T] \), and \( u(x, 0) = \phi(x) \) in \( \mathbb{R}^n \setminus \Omega \).

In the problem (1.4), (1.5), (1.6), \( f(x), \phi(x) \) are given functions satisfying the following conditions:

\((F)\) \( f \in C^0(\mathbb{R}^n) \), and there exists some positive constant \( \beta \) such that

\[
e^f(x) = e^{f_0(|x|)} + O(|x|^{-\beta}), \quad |x| \to +\infty
\]

for some \( f_0 \in C^0([0, +\infty)) \) satisfying

\[
e^{f_0(|x|)} = C|x|^{\alpha}, \quad |x| \geq R, \tag{1.7}
\]

where \( C, R \) are positive constants, and \( \alpha, \beta > 0 \) satisfy

\[
\min\{\beta, n\} > 2 - \alpha + \frac{\alpha}{n}.
\]

\((\Phi)\) \( \phi \in C^0(\mathbb{R}^n \setminus \Omega) \) satisfies

\[
\begin{cases}
-\tau + \log \det D^2 \phi = f(x), \quad D^2 \phi > 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \\
\phi = \varphi(x, 0) \text{ on } \partial \Omega,
\end{cases}
\tag{1.8}
\]

in the viscosity sense, where \( \tau \) is a constant, and there exist \( b \in \mathbb{R}^n, c \in \mathbb{R} \) such that

\[
\lim_{|x| \to +\infty} |x|^{\min\{\beta, n\} - 2 + \alpha} \phi(x) - (u_0(|x|) + b \cdot x + c) < \infty, \quad \beta \neq n,
\]

\[
\lim_{|x| \to +\infty} |x|^{n - 2 + \alpha} (\ln |x|)^{-1} \phi(x) - (u_0(|x|) + b \cdot x + c) < \infty, \quad \beta = n. \tag{1.9}
\]

Here

\[
u_0(|x|) = \int_0^{|x|} \left( \int_0^s n s^{n-1} e^{f_0(z) + \tau z} \frac{dz}{z} \right)^{\frac{1}{\tau}} ds
\]

is the solution of \( \det D^2 u_0 = e^{f_0(|x|) + \tau} \) and satisfies \( u_0(0) = 0, \quad u_0'(0) = 0 \).

For a domain \( D \subset \mathbb{R}^{n+1} \), we say a function \( u \in C^{k,j}(D) \) if \( u \) is \( k \)-th continuously differentiable with the spatial variable \( x \in \mathbb{R}^n \) and \( j \)-th continuously differentiable with the time variable \( t \) for \((x, t) \in D \).

Our main theorem is
\textbf{Theorem 1.4.} Let $f$, $\phi$ satisfy (F), (Φ) respectively, $\varphi \in C^{2,1}(\partial \Omega \times [0, T])$. If $c$ is sufficiently large, there exists a unique viscosity solution $u_c \in C^0((\mathbb{R}^n \setminus \Omega) \times [0, T])$ of (1.4), (1.5), (1.6) and

\begin{align}
\text{lim sup } |x|^{\min(\beta,n)-2(\frac{n}{2}+\frac{\beta}{2})} |u_c(x,t)-(\tau t+u_0(|x|)+b \cdot x+c)| &< \infty, \quad \beta \neq n, \\
\text{lim sup } |x|^{n-2(\frac{n}{2}+\frac{\beta}{2})} (\ln |x|)^{-1} |u_c(x,t)-(\tau t+u_0(|x|)+b \cdot x+c)| &< \infty, \quad \beta = n,
\end{align}

for any $t \in [0, T]$.

\textbf{Remark 1.} If $f_0 = 0$, we can easily see that $f(x)$ is a perturbation of 0 at infinity. More precisely, the condition (F) is:

\[(F') \ f \in C^0(\mathbb{R}^n) \text{ satisfies that for the positive constant } \beta > 2, \quad f(x) = O(|x|^{-\beta}), \quad |x| \to +\infty.\]

The condition (Φ) is:

\[(\Phi') \ \varphi \in C^0(\mathbb{R}^n \setminus \Omega) \text{ satisfies (1.8) in the viscosity sense and there exist } b \in \mathbb{R}^n, c \in \mathbb{R} \text{ such that}
\begin{align}
\text{lim sup } |x|^{\min(\beta,n)-2}|\varphi(x) - \left(\frac{\beta}{2} |x|^2 + b \cdot x + c\right)| &< \infty, \quad \beta \neq n, \\
\text{lim sup } |x|^{n-2(\frac{n}{2}+\frac{\beta}{2})}(\ln |x|)^{-1} |\varphi(x) - \left(\frac{\beta}{2} |x|^2 + b \cdot x + c\right)| &< \infty, \quad \beta = n,
\end{align}

where $\tau$ is a constant.

We draw the following conclusion at once.

\textbf{Corollary 1.} Let $f, \phi$ satisfy (F'), (Φ') respectively, $\varphi \in C^{2,1}(\partial \Omega \times [0, T])$. If $c$ is sufficiently large, there exists a unique viscosity solution $u_c \in C^0((\mathbb{R}^n \setminus \Omega) \times [0, T])$ of (1.4), (1.5), (1.6) and

\begin{align}
\text{lim sup } |x|^{\min(\beta,n)-2}|u_c(x,t)-(\tau t + \frac{\beta}{2} |x|^2 + b \cdot x + c)| &< \infty, \quad \beta \neq n, \\
\text{lim sup } |x|^{n-2(\frac{n}{2}+\frac{\beta}{2})}(\ln |x|)^{-1} |u_c(x,t)-(\tau t + \frac{\beta}{2} |x|^2 + b \cdot x + c)| &< \infty, \quad \beta = n,
\end{align}

for any $t \in [0, T]$.

The paper is arranged as follows. In Section 2, we give some useful lemmas. In Section 3, we prove Theorem 1.4 using the Perron method.

2. Some Useful Lemmas.

\textbf{Lemma 2.1.} Let $n \geq 2$, $\varphi(x,t) \in C^{2,0}(\partial \Omega \times [0, T])$. Then for any positive constant $\tilde{c}$, there exists a positive constant $C_0$, depending on $n, \varphi, \Omega, T, \tilde{c}$, such that for any $\xi \in \partial \Omega, t \in [0, T]$, there exists $\tilde{x}(\xi,t) = -\frac{1}{\tilde{c}} D_\nu \varphi(\xi,t) + \tilde{x}_\nu \nu \in \mathbb{R}^n$ satisfying

\[|\tilde{x}(\xi,t)| \leq C_0,\]

and

\[w_{\xi}(x,t) < \varphi(x,t) \text{ on } (\partial \Omega \times [0, T]) \setminus \{\xi\},\]

where

\[w_{\xi}(x,t) = \varphi(\xi,t) + \frac{\tilde{c}}{2} (|x - \tilde{x}(\xi,t)|^2 - |\xi - \tilde{x}(\xi,t)|^2) \text{ in } \mathbb{R}^n \times [0, T].\]
Proof. Let $\xi \in \partial Q$. Through a translation, scaling and rotation, without loss of generality, we may assume $\xi = 0$, and $\partial \Omega$ is locally represented by
\[
x_n = \rho(x') = O(|x'|^2),
\]
and $\varphi$ has the local expansion
\[
\varphi(x', \rho(x'), t) = \varphi(0, t) + \varphi_{x_1}(0, t)x_1 + \cdots + \varphi_{x_{n-1}}(0, t)x_{n-1} + \varphi_{x_n}(0, t)x_n + O(|x|^2)
\]
where $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, $\mathcal{N}$ is the neighborhood of $x' = 0$. Let
\[
x = -\frac{1}{c}(\varphi_{x_1}(0, t), \varphi_{x_2}(0, t), \cdots, \varphi_{x_{n-1}}(0, t), -c \bar{x}_n),
\]
and
\[
w(x, t) = \varphi(0, t) + \frac{\bar{c}}{2}(|x - \bar{x}|^2 - |\bar{x}|^2)
\]
in $\mathbb{R}^n \times [0, T]$. Then
\[
w(x, t) = \varphi(0, t) + \frac{\bar{c}}{2}|x|^2 + \sum_{i=1}^{n-1} x_i \varphi_{x_i}(0, t) - c \bar{x}_n \bar{x}_n
\]
\[
= \varphi(0, t) + \sum_{i=1}^{n-1} x_i \varphi_{x_i}(0, t) + \frac{\bar{c}}{2}(|x'|^2 + \rho^2(x')) - c \rho(x') \bar{x}_n.
\]
Therefore for $x' \in \mathcal{N}$,
\[
(w - \varphi)(x', \rho(x'), t) = \frac{\bar{c}}{2}(|x'|^2 + \rho^2(x')) - c \rho(x') \bar{x}_n + O(|x'|^2)
\]
\[
\leq C_1(|x'|^2 + \rho^2(x')) - c \rho(x') \bar{x}_n,
\]
where $C_1$ depends on $||\varphi||_{C^{2,\alpha}(\partial \Omega \times [0, T])}$ and $\bar{c}$.

On account of the strict convexity of $\partial \Omega$, there exists a positive constant $\delta$ depending only on $\Omega$, such that for $|x'| < \delta$,
\[
\rho(x') \geq \delta |x'|^2,
\]
then
\[
(w - \varphi)(x', \rho(x'), t) \leq C_1(|x'|^2 + \rho^2(x')) - c \bar{x}_n \delta |x'|^2, |x'| < \delta,
\]
combining with (2.1), for large $x$, (depending on $\delta, ||\varphi||_{C^{2,\alpha}(\partial \Omega \times [0, T])}, \bar{c}$), we have
\[
(w - \varphi)(x', \rho(x'), t) < 0, 0 < |x'| < \delta, 0 \leq t \leq T.
\]

Besides, from (2.2) and the strict convexity of $\partial \Omega$, for any $x \in \partial \Omega \setminus \{(x', \rho(x')) | |x'| < \delta\}$, we have $x_n \geq \delta^3$. Consequently, for any $x \in \partial \Omega \setminus \{(x', \rho(x')) : |x'| < \delta\}, t \in [0, T]$,
\[
w(x, t) \leq \varphi(0, t) + \frac{\bar{c}}{2}|x|^2 + \sum_{i=1}^{n-1} x_i \varphi_{x_i}(0, t) - c \delta^3 \bar{x}_n.
\]
Choosing $\bar{x}_n$ large (depending on $\delta, T, \Omega, ||\varphi||_{C^{2,\alpha}(\partial \Omega \times [0, T])}$), we also have
\[
w(x, t) - \varphi(x, t) < 0, x \in \partial \Omega \setminus \{(x', \rho(x')) | |x'| < \delta\}, t \in [0, T].
\]

We say $D$ is an open set in the parabolic sense if $D = \overline{\Omega} \setminus \partial_p D$. 

\]
Lemma 2.2. Let $Q \subset D$ be two open subsets in $\mathbb{R}^{n+1}$ in the parabolic sense. Suppose $u \in USC(D)$ and $v \in USC(Q)$ are locally convex in $x$ and satisfy
\begin{equation}
- u_t + \log \det D^2 u \geq f(x, t) \text{ in } D, \tag{2.3}
\end{equation}
and
\begin{equation}
- v_t + \log \det D^2 v \geq f(x, t) \text{ in } Q \tag{2.4}
\end{equation}
in the viscosity sense respectively. Furthermore, $u \leq v$ in $Q$, $u = v$ on $\partial Q$. Let
\begin{equation*}
w(x, t) = \begin{cases} 
v(x, t) \text{ in } Q, \\
u(x, t) \text{ in } D \setminus Q.
\end{cases}
\end{equation*}
Then $w \in USC(D)$ is locally convex in $x$ and satisfies
\begin{equation*}
- w_t + \log \det D^2 w \geq f(x, t) \text{ in } D
\end{equation*}
in the viscosity sense.

Proof. Let $h \in C^2(D)$ and $(\bar{x}, \bar{t}) \in D$ satisfying
\begin{equation*}
w(x, t) - h(x, t) \leq w(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}), \text{ } \forall (x, t) \in Q_r(\bar{x}, \bar{t}),
\end{equation*}
for some $Q_r(\bar{x}, \bar{t}) \subset D$.

If $(\bar{x}, \bar{t}) \in Q$, then for some $Q_0(\bar{x}, \bar{t}) \subset Q(\bar{x}, \bar{t}) \cap Q, \forall (x, t) \in Q_r(\bar{x}, \bar{t}),$
\begin{equation*}
v(x, t) - h(x, t) = w(x, t) - h(x, t) \leq w(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}) = v(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}).
\end{equation*}
By (2.4), we have
\begin{equation*}
- h_t(\bar{x}, \bar{t}) + \log \det D^2 h(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).
\end{equation*}
If $(\bar{x}, \bar{t}) \in D \setminus Q$, then $\forall (x, t) \in Q_0(\bar{x}, \bar{t}),$
\begin{equation*}
u(x, t) - h(x, t) \leq w(x, t) - h(x, t) \leq w(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}) = u(\bar{x}, \bar{t}) - h(\bar{x}, \bar{t}).
\end{equation*}
By (2.3), we have
\begin{equation*}
- h_t(\bar{x}, \bar{t}) + \log \det D^2 h(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).
\end{equation*}

Now we give the comparison principle below.

Lemma 2.3 (Comparison Principle). Let $D$ be a bounded open set in $\mathbb{R}^{n+1}$ in the parabolic sense. Let $u \in USC(\overline{D})$ and $v \in LSC(\overline{D})$ satisfy
\begin{equation*}
- u_t + \log \det D^2 u \geq f(x, t) \text{ in } D \text{ and } - v_t + \log \det D^2 v \leq f(x, t) \text{ in } D
\end{equation*}
in the viscosity sense, respectively. Then we have
\begin{equation}
\sup_D (u - v) \leq \sup_{\partial_p D} (u - v). \tag{2.5}
\end{equation}

Under the assumptions $u, v \in C^0(\overline{D})$ and $D$ being a cylindrical domain, the lemma was proved by Wang and Wang in [17]. Based on their result, we can then directly obtain our comparison principle.

Definition 2.4. Let $g \in C^0(\Omega)$ be a positive function, and $u \in C^0(\Omega)$ a locally convex function. We say that $u$ is a viscosity subsolution of
\begin{equation}
\det D^2 u = g \text{ in } \Omega \tag{2.6}
\end{equation}
or a viscosity solution of
\begin{equation*}
\det D^2 u \geq g \text{ in } \Omega
\end{equation*}
if for every $\bar{x} \in \Omega$ and every convex $\varphi \in C^2(\Omega)$ satisfying $\varphi \geq u$ in $\Omega$ and $\varphi(\bar{x}) = u(\bar{x})$, we have
\begin{equation*}
\det D^2 \varphi(\bar{x}) \geq g(\bar{x}).
\end{equation*}
Similarly, \( u \) is a viscosity supersolution of (2.6) if for every \( \bar{x} \in \Omega \) and every convex \( \phi \in C^2(\Omega) \) satisfying \( \phi \leq u \) in \( \Omega \) and \( \phi(\bar{x}) = u(\bar{x}) \), we have
\[
\det D^2 \phi(\bar{x}) \leq g(\bar{x}).
\]

\( u \) is a viscosity solution of (2.6) if \( u \) is both a viscosity subsolution and a viscosity supersolution of (2.6).

**Lemma 2.5.** [7] Let \( n \geq 2 \), \( u \in USC(\Omega) \) and \( v \in LSC(\Omega) \) be the viscosity sub- and supersolution of
\[
\det D^2 u = f(x) \text{ in } \Omega,
\]
respectively, where \( f(x) \in C^0(\Omega) \) is a positive function. Furthermore,
\[
u(x) \leq v(x) \text{ on } \partial \Omega.
\]
Then
\[
\sup_\Omega (u - v) \leq \sup_{\partial \Omega} (u - v).
\]

To introduce the Perron method for parabolic equations, we first define viscosity solutions which do not satisfy (semi) continuous properties.

**Definition 2.6.** We say a function \( u \) is a weak viscosity subsolution of (1.4) if the USC envelope of \( u \), namely,
\[
 u^*(x, t) = \lim_{r \to 0} \sup_{(y, \tau) \in B_r(x, t)} u(y, \tau)
\]
is finite and a viscosity subsolution, where
\[
 B_r(x, t) := \{(y, \tau) \mid |x - y|^4 + |t - \tau|^2 < r^2\} \subset (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T].
\]

Similarly, one uses LSC envelope \( u_* = -(\cdot)^* \) for supersolutions. If \( u \) is a weak viscosity sub- and supersolution, we call \( u \) a weak viscosity solution.

We can also define weak viscosity solutions of the problem (1.4), (1.5), (1.6) by giving the boundary condition like Definition 1.3.

Similar to the process by Y. Zhan [20], we have the two lemmas below. We give the proof here for completeness.

**Lemma 2.7.** Let \( D \) be an open set in \( \mathbb{R}^{n+1} \) in the parabolic sense. Let \( S \) denote the nonempty set of weak viscosity subsolutions of
\[
- v_t + \log \det D^2 v = f(x, t) \text{ in } D.
\]
Set
\[
u(x, t) = \sup \{v(x, t) \mid v \in S\} \text{ for } (x, t) \in D.
\]
Suppose \( u^*(x, t) < \infty \) for \( (x, t) \in D \), then \( u \) is a weak viscosity subsolution of (2.8).

**Proof.** By the definition of weak viscosity subsolutions, we need to prove that for all functions \( \varphi \in C^{2,1}(D) \), if there exists \((\bar{x}, \bar{t}) \in D\) such that
\[
\max_{Q_r} (u^* - \varphi) = (u^* - \varphi)(\bar{x}, \bar{t}),
\]
for some \( Q_r := Q_r(\bar{x}, \bar{t}) \), then
\[
-\varphi_t(\bar{x}, \bar{t}) + \log \det D^2 \varphi(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).
\]

Without loss of generality, we can assume that \((u^* - \varphi)(\bar{x}, \bar{t}) = 0\). Set
\[
\psi(x, t) = \varphi(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2,
\]
then \( u^* - \psi \) attains its strict maximum in \( Q_r \) at \((\bar{x}, \bar{t})\). So in \( Q_r \),
\[
(u^* - \psi)(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2 = (u^* - \varphi)(x, t) \leq 0 = (u^* - \psi)(\bar{x}, \bar{t}),
\]
and
\[
(u^* - \psi)(x, t) \leq -|x - \bar{x}|^4 - |t - \bar{t}|^2.
\]
By the definition of \( u \), for any \( k \), there is a \( v_k \in S \) such that
\[
u(\bar{x}, \bar{t}) - \frac{1}{k} < v_k(\bar{x}, \bar{t}).
\]
Since \( v_k^* - \psi \in USC(D) \), it attains its maximum at \((y_k, s_k)\) in some compact neighborhood \( B \subset Q_r \) of \((\bar{x}, \bar{t})\). Noting that in \( Q_r \),
\[
(v_k^* - \psi)(x, t) \leq (u^* - \psi)(x, t) \leq -|x - \bar{x}|^4 - |t - \bar{t}|^2,
\]
we have
\[
- \frac{1}{k} = u^*(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t}) - \frac{1}{k} \leq v_k^*(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t})
\leq v_k^*(y_k, s_k) - \psi(y_k, s_k) \leq -|y_k - \bar{x}|^4 - |s_k - \bar{t}|^2 \leq 0.
\]
Let \( k \to \infty \), we have
\[
\lim_{k \to \infty} (y_k, s_k) = (\bar{x}, \bar{t}).
\]
Since \( v_k^* \) is a viscosity subsolution of (2.8), and \( v_k^* - \psi \) attains its local maximum at \((y_k, s_k)\), then
\[
- \psi_t + \log \det D^2 \psi(y_k, s_k) \geq f(y_k, s_k),
\]
let \( k \to \infty \), we have
\[
- \psi_t + \log \det D^2 \psi(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}),
\]
Since at \((\bar{x}, \bar{t})\), \( \psi_t = \varphi_t \), \( D_x \psi = D_x \varphi \), and \( D^2_{xx} \psi = D^2_{xx} \varphi \), we have
\[
- \varphi_t(\bar{x}, \bar{t}) + \log \det D^2 \varphi(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).
\]

**Lemma 2.8 ([7]).** Let \( \mathcal{F} \) denote the nonempty set of weak viscosity subsolutions of (2.7). Set
\[
w(x) = \sup\{u(x) \mid u \in \mathcal{F}\} \text{ for } x \in \Omega.
\]
Suppose \( w^*(x) < \infty \) for \( x \in \Omega \), then \( w \) is a weak viscosity subsolution of (2.7).

**Lemma 2.9.** Let \( g \) be a weak viscosity supersolution of (2.8). Let
\[
S_g := \{v \mid v \text{ is a weak viscosity subsolution of (2.8) and } v \leq g\}
\]
and
\[
u(x, t) := \sup\{v(x, t) \mid v \in S_g\}.
\]
If \( S_g \) is not empty, then \( u \) is a weak viscosity solution of (2.8).

**Proof.** By Lemma 2.7, \( u \) is a weak viscosity subsolution. If \( u \) is not a weak viscosity supersolution, there exists a function \( \varphi \in C^{2,1}(D) \), and a point \((\bar{x}, \bar{t}) \in D \) such that
\[
\min_{Q_r} (u_* - \varphi) = (u_* - \varphi)(\bar{x}, \bar{t}) = 0
\]
for some \( Q_r := Q_r(\bar{x}, \bar{t}) \), and
\[
- \varphi_t(\bar{x}, \bar{t}) + \log \det D^2 \varphi(\bar{x}, \bar{t}) > 0.
\]
We may assume here
\[
(u_* - \varphi)(x, t) \geq |x - \bar{x}|^4 + |t - \bar{t}|^2
\]
for \((x, t) \in Q_r \) since the function \( \varphi \) can be modified as \( \varphi - |x - \bar{x}|^4 - |t - \bar{t}|^2 \) if necessary.
Clearly $\varphi \leq u_* \leq g_*$ in $Q_\rho$, so $u_*(\bar{x}, \bar{t}) = \varphi(\bar{x}, \bar{t}) < g_*(\bar{x}, \bar{t})$, otherwise it would contradict the fact that $g$ is a weak viscosity supersolution of (2.8).

Since $F$ and $\varphi$ are continuous, for $\delta > 0$ small enough, we have

$$-\varphi_1(x, t) + \log \det D^2 \varphi(x, t) \geq f(x, t), \quad (2.9)$$

$$\varphi(x, t) + \delta^2 \leq g_*(x, t)$$

for $(x, t) \in B_{2\delta} = B_{2\delta}(\bar{x}, \bar{t}) \subset Q_\rho$, and $B_\delta(x, t) = \{(y, s) \in Q_\rho \mid |x - y|^4 + |t - s|^2 < \delta^2\}$.

(2.9) indicates that the function $\varphi(x, t) + \delta^2$ is a subsolution in $B_{2\delta}$. Furthermore, we have

$$u(x, t) \geq u_*(x, t) \geq \varphi(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2 \geq \varphi(x, t) + \delta^2 \text{ in } B_{2\delta} \setminus B_\delta.$$

Define $w(x, t)$ by

$$w(x, t) = \begin{cases} \max\{\varphi(x, t) + \delta^2, u(x, t)\} \text{ in } B_\delta, \\ u(x, t) \text{ in } D \setminus B_\delta, \end{cases}$$

then

$$w(x, t) = \max\{\varphi(x, t) + \delta^2, u(x, t)\} \text{ in } B_{2\delta}.$$ 

By Lemma 2.2, $w$ is a weak viscosity subsolution of (2.8) over $D$. Since $w \leq g$, $w \in S_g$. By the definition of $u$, we have $u \geq w$.

On the other hand, since

$$0 = (u_* - \varphi)(\bar{x}, \bar{t}) = \lim_{l \to 0} \inf_{(y, s) \in B_l} (u - \varphi)(y, s),$$

there is a point $(z, s) \in B_\delta$ such that $u(z, s) - \varphi(z, s) < \delta^2$ and $u(z, s) < w(z, s)$, which leads to a contradiction. \hfill \Box

3. **Proof of Theorem 1.4.** The uniqueness can be directly obtained by the asymptotic behavior and the comparison principle. Now we prove the existence. By an affine transformation, we only need to prove the case $b = 0$.

Firstly, we give the definition of the Perron solution $u_\epsilon$ to (1.4), (1.5), (1.6).

Let $S_\epsilon$ denote the set of functions $v$ which are locally convex in $x$ in $\mathbb{R}^n \setminus \Omega \times [0, T]$, where $v$ is the weak viscosity subsolution of (1.4), (1.5), (1.6) with an upper bound $u_\epsilon$ which is a supersolution of (1.4), (1.5), (1.6) satisfying (111).

Define the Perron solution to (1.4), (1.5), (1.6) as

$$u_\epsilon(x, t) = \sup\{w(x, t) \mid v \in S_\epsilon\} \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0, T].$$

To ensure the definition of the upper bound meaningful, we need the set $S_\epsilon$ to be nonempty, that is we need to find an appropriate function satisfying all the requirements for the functions in $S_\epsilon$ at the same time. In fact, the function $u_\epsilon$ we construct is a viscosity subsolution of (1.4), (1.5), (1.6) satisfying (111). Let us start doing it now.

Choose $R_1 > 2$. Suppose $B_2(0) \subset \subset \Omega \subset \subset B_{R_1}(0)$, and then choose $R_3 > 3R_1$. By Lemma 2.1, for any $\xi \in \partial \Omega$, $t \in [0, T]$, there exists $\bar{x}(\xi, t) = -\frac{1}{c_\epsilon}D_\nu \varphi(\xi, t) + \bar{x}_n \nu \in \mathbb{R}^n$ satisfying

$$|\bar{x}(\xi, t)| < \infty, \quad \text{and} \quad w_\xi(x, t) < \varphi(x, t) \text{ on } (\partial \Omega \times [0, T]) \setminus \{(\xi, t)\},$$

where

$$w_\xi(x, t) = \varphi(\xi, t) + \frac{c_\epsilon}{2} \left(|x - \bar{x}(\xi, t)|^2 - |\xi - \bar{x}(\xi, t)|^2\right) \text{ in } \mathbb{R}^n \times [0, T].$$
and \( c_* \) is sufficiently large to make sure the following two inequalities hold simultaneously:

\[-(w_\xi)_t + \log \det D^2 w_\xi = -\varphi_t + (\xi - x) \cdot D_\xi \varphi_t(\xi, t) + \log c_* \geq \max_{B_{R_2}(0)} f \geq f \text{ in } (B_{R_2}(0) \setminus \overline{\Omega}) \times (0, T),\]

\[\det(D^2 w_\xi(x, 0)) = c_*^n \geq e^{f(x)+\tau} \text{ in } B_{R_2}(0) \setminus \overline{\Omega}.\]

Define

\[w(x, t) = \sup_{\xi \in \partial \Omega} w_\xi(x, t), \ (x, t) \in \mathbb{R}^n \times [0, T],\]

then \( w \) is a locally Lipschitz function in \( \mathbb{R}^{n+1} \), and

\[w(x, t) = \varphi(x, t) \text{ on } \partial \Omega \times [0, T].\]

By Lemma 2.7 and Lemma 2.8,

\[-w_t + \log \det D^2 w \geq f \text{ in } (B_{R_2}(0) \setminus \overline{\Omega}) \times (0, T),\]

\[\det(D^2 w(x, 0)) \geq e^{f(x)+\tau} \text{ in } B_{R_2}(0) \setminus \overline{\Omega}.\]  

So we have

\[\det(D^2 w(x, 0)) \geq \det(D^2 \phi(x)) \text{ in } B_{R_2}(0) \setminus \overline{\Omega}.\]

Let \( \overline{f}(|x|), \ f(|x|) \) be two positive continuous functions in \([0, +\infty)\) satisfying

\[\overline{f}(|x|) \geq e^{f(x)+\tau} \geq f(|x|) \text{ in } \mathbb{R}^n \setminus B_2(0),\]

and there exists a positive constant \( \bar{R} \) such that

\[\frac{\overline{f}(|x|)}{f(|x|)} = e^{(e^{\alpha(|x|)} - c_1|x|^{-\beta})}, \ |x| \geq \bar{R},\]

\[\frac{\overline{f}(|x|)}{f(|x|)} = e^{(e^{\alpha(|x|)} + c_2|x|^{-\beta})}, \ |x| \geq \bar{R},\]

where \( c_1, c_2 \) are positive constants.

For \( \alpha > 0 \), define

\[u_-(x, t) = \tau t + \inf_{B_{R_1}(0) \times [0, T]} w + \int_{2R_1} |x| \left( \int_1^{R_1} n z^{-n-1} \overline{f}(z) dz + a \right)^{1/2} ds \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times [0, T],\]

\[u_+(x, t) = \tau t + \sup_{B_{R_1}(0) \times [0, T]} w + \int_{2} |x| \left( \int_1^{R_1} n z^{-n-1} f(z) dz + a \right)^{1/2} ds \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times [0, T],\]

then

\[u_-(x, t) \leq u_+(x, t) \text{ on } \partial B_2(0) \times [0, T],\]

\[-(u_-, t) + \log \det D^2 u_- = -\tau + \log \overline{f} \geq f \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times (0, T),\]

\[-(u_+, t) + \log \det D^2 u_+ = -\tau + \log f \leq f \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times (0, T),\]

\[\det(D^2 u_-) = f \geq e^{f+\tau} \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times [0, T],\]

\[\det(D^2 u_+) = \overline{f} \leq e^{f+\tau} \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times [0, T],\]

Choose \( a_0 > 0 \) such that for \( \alpha \geq a_0 \), the following four inequalities hold at the same time

\[u_-(x, t) \leq \tau t + \inf_{B_{R_1}(0) \times [0, T]} w + \int_{2R_1} |x| \left( \int_1^{R_1} n z^{-n-1} \overline{f}(z) dz + a \right)^{1/2} ds \leq w(x, t), \ |x| \leq R_1, \ 0 \leq t \leq T,\]  

(3.3)
u_-(x, t) = \tau t + \inf_{B_{R_1}(0) \times [0, T]} w \quad u_+(x, t) = \tau t + \sup_{B_{R_1}(0) \times [0, T]} w

\begin{align*}
&\geq w(x, t), \quad |x| = R_2, \quad 0 \leq t \leq T, \\
u_+(x, t) \geq w(x, t), \quad |x| = R_2, \quad 0 \leq t \leq T, \\
u_+(x, t) \geq w(x, t) = \varphi(x, t), \quad x \in \partial \Omega, \quad 0 \leq t \leq T.
\end{align*}

For \((x, t) \in (\mathbb{R}^n \setminus B_2(0)) \times [0, T], \)
\[ u_-(x, t) = \tau t + \inf_{B_{R_1}(0) \times [0, T]} w \]

\begin{align*}
&\quad + \int_{2R_1}^\infty \left[ \left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} - \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} \right] ds \\
&\quad - \int_{|x|}^\infty \left[ \left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} - \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} \right] ds \\
&\quad + \int_0^1 \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} ds - \int_0^{2R_1} \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} ds \\
&= \tau t + u_0(|x|) + \mu_1(a) \\
&\quad - \int_{|x|}^\infty \left[ \left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} - \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} \right] ds,
\end{align*}

where \(u_0(|x|)\) is defined in (1.10), and
\[ \mu_1(a) = \int_{2R_1}^\infty \left[ \left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} - \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} \right] ds \\
\quad - u_0(2R_1) + \inf_{B_{R_1}(0) \times [0, T]} w.
\]

It is easy to see that \(\mu_1(a)\) is increasing in \((0, +\infty),\) and
\[ \lim_{a \to +\infty} \mu_1(a) = +\infty. \]

Now we compute the asymptotic behavior of
\[ \int_{|x|}^\infty \left[ \left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} - \left( \int_0^s n z^{-1} e^{f_0(z)+\tau} dz \right)^{\frac{1}{\gamma}} \right] ds.
\]
Set
\[ h(s) = \int_0^s n z^{-1} e^{f_0(z)+\tau} dz.
\]
As \(s \to +\infty,\)
\begin{align*}
\left( \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} \\
= \left( \int_1^{R} n z^{-1} f(z) dz + \int_1^s n z^{-1} f(z) dz + a \right)^{\frac{1}{\gamma}} \\
= \left( \int_1^{R} n z^{-1} f(z) dz + h(s) \right) + \int_{R}^{s} n c_2 e^{\tau z^{-1} - \beta} dz + a \right)^{\frac{1}{\gamma}}
\end{align*}
where

\[ c_3 = \int_1^\bar{R} n s^{-1}f(z)dz - h(\bar{R}) - \frac{n c_2 e^\gamma}{n - \beta} \bar{R}^{n - \beta} + a, \quad c_4 = \frac{n c_2 e^\gamma}{n - \beta}, \]

\[ \bar{c}_3 = \int_1^\bar{R} n s^{-1}f(z)dz - h(\bar{R}) - n c_2 e^\gamma \ln \bar{R} + a, \quad \bar{c}_4 = n c_2 e^\gamma. \]

We can deduce \( \alpha + n > 0 \) from \( -n(\min(\beta, n) - 2) < \alpha. \) By (1.7), we have

\[ \lim_{s \to \infty} \frac{h(s)}{s^{n + \alpha}} = \lim_{s \to \infty} \frac{\int_0^s n s^{-1} e^{f_0(z) + \tau} dz}{s^{n + \alpha}} = \lim_{s \to \infty} \frac{n s^{n - 1} e^{f_0(s) + \tau}}{(n + \alpha)s^{n + \alpha - 1}} = N, \]

where \( N \) is a constant. So there exists a constant \( N_1 > 0 \) such that

\( h(s) = N s^{n + \alpha}, \ s \geq N_1. \)

If \( \beta \neq n, \)

\[
\int_{|x|}^\infty \left[ \left( \int_1^s n s^{-1}f(z)dz + a \right)^{\frac{1}{n}} - \left( \int_0^s n s^{-1} e^{f_0(z) + \tau} dz \right)^{\frac{1}{n}} \right] ds
\]

\[
= \int_{|x|}^\infty \left[ \left( h(s) + c_3 + c_4 s^{-\beta} \right)^{\frac{1}{n}} - \left( h(s) \right)^{\frac{1}{n}} \right] ds
\]

\[
= \int_{|x|}^\infty \left( h(s) \right)^{\frac{1}{n}} \left[ \left( 1 + \frac{c_3}{h(s)} + c_4 s^{-\beta} \right)^{\frac{1}{n}} - 1 \right] ds
\]

\[
= \int_{|x|}^\infty O(s^{1+\frac{\alpha}{n}}) \left[ 1 + O(s^{-n-\alpha}) + O(s^{-\beta-\alpha}) - 1 \right] ds
\]

\[ = O(|x|^{2-\alpha+\frac{\beta}{n}-\min(\beta, n)}), \ \ |x| \to +\infty. \]

If \( \beta = n, \)

\[
\int_{|x|}^\infty \left[ \left( \int_1^s n s^{-1}f(z)dz + a \right)^{\frac{1}{n}} - \left( \int_0^s n s^{-1} e^{f_0(z) + \tau} dz \right)^{\frac{1}{n}} \right] ds
\]

\[
= \int_{|x|}^\infty \left[ \left( h(s) + \bar{c}_3 + \bar{c}_4 \ln s \right)^{\frac{1}{n}} - \left( h(s) \right)^{\frac{1}{n}} \right] ds
\]

\[
= \int_{|x|}^\infty \left( h(s) \right)^{\frac{1}{n}} \left[ \left( 1 + \frac{\bar{c}_3}{h(s)} + \bar{c}_4 \ln s \right)^{\frac{1}{n}} - 1 \right] ds
\]

\[
= \int_{|x|}^\infty O(s^{1+\frac{\alpha}{n}}) \left[ 1 + O(s^{-n-\alpha} \ln s) - 1 \right] ds
\]

\[ = O(|x|^{2-\alpha+\frac{\beta}{n}-\alpha \ln |x|}), \ \ |x| \to +\infty. \]

As \( |x| \to +\infty, \)

\[ u_-(x, t) = \begin{cases} \tau t + u_0(|x|) + \mu_1(a) + O(|x|^{2-\alpha+\frac{\beta}{n}-\min(\beta, n)}), & \beta \neq n, \\ \tau t + u_0(|x|) + \mu_1(a) + O(|x|^{2-\alpha+\frac{\beta}{n}-\alpha \ln |x|}), & \beta = n. \end{cases} \]

Similarly, we have

\[ u_+(x, t) = \tau t + u_0(|x|) + \mu_2(a) \]

\[
- \int_{|x|}^\infty \left[ \left( \int_1^s n s^{-1} f(z)dz + a \right)^{\frac{1}{n}} - \left( \int_0^s n z^{n-1} e^{f_0(z) + \tau} dz \right)^{\frac{1}{n}} \right] ds,
\]
We can also obtain that as $|x| \to +\infty$, 
$$
u_+(x, t) = \begin{cases} 
\tau t + u_0(|x|) + \mu_2(a) + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta \neq n, \\
\tau t + u_0(|x|) + \mu_2(a) + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta = n.
\end{cases}
$$

For sufficiently large $c$ in (1.9), there exist $a_1(c)$, $a_2(c)$ satisfying 
$$
\mu_1(a_1(c)) = \mu_2(a_2(c)) = c.
$$

Then as $|x| \to +\infty$, 
$$
u_-(x, t) = u_+(x, t) = \begin{cases} 
\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta \neq n, \\
\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta = n.
\end{cases}
$$

Therefore 
$$
\lim_{|x| \to +\infty} (\nu_-(x, t) - u_+(x, t)) = 0, \ 0 \leq t \leq T. \tag{3.13}
$$

By (3.3), (3.6), (3.7), (3.13) and Lemma 2.5, 
$$
u_-(x, 0) \leq u_+(x, 0) \text{ in } \mathbb{R}^n \setminus B_2(0). \tag{3.14}
$$

By (3.3), (3.4), (3.5), (3.13), (3.14) and Lemma 2.3, 
$$
u_-(x, t) \leq u_+(x, t) \text{ in } (\mathbb{R}^n \setminus B_2(0)) \times [0, T]. \tag{3.15}
$$

For $a \geq a_0$, we define 
$$
\nu(x, t) = \begin{cases} 
\max\{w(x, t), \nu_-(x, t)\} \text{ in } (B_{R_2}(0) \setminus \Omega) \times [0, T], \\
\nu_-(x, t) \text{ in } (\mathbb{R}^n \setminus B_{R_2}(0)) \times [0, T].
\end{cases}
$$

We know $\nu \in C^0((\mathbb{R}^n \setminus \Omega) \times [0, T])$ from (3.9). By Lemma 2.2, $\nu$ satisfy
$$
-\nu + \log \det D^2 \nu \geq f \text{ in } (\mathbb{R}^n \setminus \Omega) \times (0, T)
$$
in the viscosity sense. We also know that, from (3.2) and (3.6),
$$
\det D^2 \nu(x, 0) \geq e^{f(x)+\tau} = \det D^2 \phi \text{ in } \mathbb{R}^n \setminus \overline{\Omega}.
$$

By (3.12), as $|x| \to +\infty$, 
$$
\nu(x, t) = \begin{cases} 
\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta \neq n, \\
\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\beta}{n}} \ln |x|), & \beta = n.
\end{cases}
$$

we have 
$$
\limsup_{|x| \to +\infty} (\nu(x, 0) - \phi(x)) = 0.
$$

By (3.8), we obtain that 
$$
\nu(x, 0) = w(x, 0) = \varphi(x, 0) = \phi(x) \text{ on } \partial \Omega.
Then combining with Lemma 2.5,
\[ \underline{u}(x,0) \leq \phi(x) \text{ in } \mathbb{R}^n \setminus \Omega. \]

Using (3.8) again, we have
\[ \underline{u}(x,t) = w(x,t) = \varphi(x,t) \text{ on } \partial \Omega \times [0,T]. \] (3.17)

Thus, \( \underline{u} \) is a viscosity subsolution of (1.4), (1.5), (1.6) satisfying (1.11).

By (3.2), (3.7), (3.10), (3.11) and Lemma 2.5,
\[ w(x,0) \leq u_+(x,0) \text{ in } B_{R_2}(0) \setminus \Omega. \] (3.18)

Then from (3.1), (3.5), (3.10), (3.11), (3.18) and Lemma 2.3, we have
\[ w(x,t) \leq u_+(x,t), \text{ in } B_{R_2}(0) \setminus \Omega \times [0,T]. \]

Combining with (3.15),
\[ \underline{u}(x,t) \leq u_+(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T]. \]

Therefore, \( \underline{u} \in S_c \) and \( S_c \) is nonempty.

Secondly, we prove that \( u_c \) satisfy the asymptotic behavior at infinity. By the definition of \( u_c \), we have
\[ u_c(x,t) \leq u_+(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T]. \]

And since \( \underline{u} \in S_c \), by (3.12) and (3.16), as \(|x| \to +\infty|\),
\[ u_c(x,t) = \begin{cases} \tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{2}{n}-\min(\beta,n)}), & \beta \neq n, \\ \tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{2}{n}-\alpha \ln |x|}), & \beta = n. \end{cases} \]

Next comes the proof of \( u_c(0) = \phi(x) \text{ in } \mathbb{R}^n \setminus \Omega \text{ and } u_c(x,t) = \varphi(x,t) \text{ on } \partial \Omega \times [0,T]. \)

Since \( \varphi \in C^{2,1}(\partial \Omega \times [0,T]) \), we may assume
\[ m_1 \leq \varphi_1(x,t) \leq m_2 \text{ on } \partial \Omega \times [0,T], \]

where \( m_1, m_2 \) are two constants. Choose two constants \( p_1, p_2 \) satisfy
\[ p_1 m_1 \leq \min\{\tau, m_1\}, \quad p_2 m_2 \geq \max\{\tau, m_2\}. \]

Set
\[ A(x,t) = p_1 m_1 t + \phi(x) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T], \]
\[ B(x,t) = p_2 m_2 t + \phi(x) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T]. \]

If \( (x,t) \in (\mathbb{R}^n \setminus \Omega) \times (0,T) \),
\[ -A_t + \log \det D^2 A = -p_1 m_1 t + \log \det D^2 \phi = -p_1 m_1 t + \log e^{f + \tau} = -p_1 m_1 t + f + \tau \geq f, \]
\[ -B_t + \log \det D^2 B = -p_2 m_2 t + \log \det D^2 \phi = -p_2 m_2 t + \log e^{f + \tau} = -p_2 m_2 t + f + \tau \leq f. \]

If \( (x,t) \in \partial \Omega \times [0,T] \),
\[ A(x,t) = p_1 m_1 t + \phi(x) = p_1 m_1 t + \varphi(x,0) \leq m_1 t + \varphi(x,0) \leq \varphi(x,t), \]
\[ B(x,t) = p_2 m_2 t + \phi(x) = p_2 m_2 t + \varphi(x,0) \geq m_2 t + \varphi(x,0) \geq \varphi(x,t). \]

If \( x \in \mathbb{R}^n \setminus \Omega \),
\[ A(x,0) = B(x,0) = \phi(x). \]

Thus, \( A \) and \( B \) are a viscosity subsolution and supersolution of (1.4), (1.5), (1.6), respectively.

If \( (x,t) \in (\mathbb{R}^n \setminus \Omega) \times [0,T] \),
\[ A(x,t) - u_+(x,t) \leq \phi(x) - \sup_{B_{R_1}(0) \times [0,T]} \sup_{\partial \Omega \times [0,T]} w \leq \phi(x) - \sup_{\partial \Omega \times [0,T]} w \]
\[ \leq \phi(x) - \phi(x) = 0. \]
we have, in the viscosity sense,
\[ u \leq \phi - \phi(x) \leq \phi(x) - \phi(x,t) = 0. \]

So \( A(x,t) \leq u(x,t) \) in \((\mathbb{R}^n \setminus \Omega) \times [0,T]\). Then \( A \in S_c \), and
\[
A(x,t) \leq u_c(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T].
\] (3.19)

For any \( v \in S_c \), we have
\[
v(x,0) \leq \phi(x) = B(x,0) \text{ in } \mathbb{R}^n \setminus \Omega,
v(x,t) \leq \phi(x(t),t) \leq B(x(t),t) \text{ on } \partial \Omega \times [0,T],
v(x,t) \leq B(x(t),t), \quad |x| \to +\infty, \quad t \in [0,T].
\]

By Lemma 2.3,
\[
v(x,t) \leq B(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T].
\]

It follows from the definition of \( u_c \) that
\[
u_c(x,t) \leq B(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T].
\]

Combining with (3.19), we know that
\[
A(x,t) \leq u_c(x,t) \leq B(x,t) \text{ in } (\mathbb{R}^n \setminus \Omega) \times [0,T].
\]

Thus, we obtain that
\[
u_c(x,0) = \phi(x) \text{ in } \mathbb{R}^n \setminus \Omega.
\]

For any \((\tilde{\xi}, \tilde{\tau}) \in \partial \Omega \times [0,T]\), since \( u \in S_c \), by (3.17),
\[
\liminf_{(x,t) \to (\tilde{\xi}, \tilde{\tau})} u_c(x,t) \geq \lim_{(x,t) \to (\tilde{\xi}, \tilde{\tau})} v(x,t) = \varphi(\tilde{\xi}, \tilde{\tau}).
\]

So we only need to prove
\[
\limsup_{(x,t) \to (\tilde{\xi}, \tilde{\tau})} u_c(x,t) \leq \varphi(\tilde{\xi}, \tilde{\tau}).
\]

In fact, let \( B_{R_1}^T = (B_{R_1} \setminus \Omega) \times (0,T] \), \( \partial_p B_{R_1}^T = ((\overline{B_{R_1}} \setminus \Omega) \times \{t = 0\}) \cup (\partial \Omega \times [0,T]) \cup (\partial B_{R_1} \times [0,T]) \). For every \( v \in S_c \), by the definition of viscosity solutions, we have, in the viscosity sense,
\[
\begin{cases}
-v_t^* + \Delta v^* \geq 0 \text{ in } B_{R_1}^T, \\
v^* \leq \varphi \text{ on } \partial \Omega \times [0,T], \\
v^* \leq B \text{ on } \partial_p B_{R_1}^T \setminus (\partial \Omega \times [0,T]).
\end{cases}
\]

Let \( w^+ \in C^{2,1}(B_{R_1}^T) \cap C^0((\overline{B_{R_1}} \setminus \Omega) \times [0,T]) \) satisfy [15]
\[
\begin{cases}
-w_t^+ + \Delta w^+ = 0 \text{ in } B_{R_1}^T, \\
w^+ = \varphi \text{ on } \partial \Omega \times [0,T], \\
w^+ = B \text{ on } \partial_p B_{R_1}^T \setminus (\partial \Omega \times [0,T]).
\end{cases}
\]

By the comparison principle of the heat conduction equation, which can be directly proved by the definition of viscosity solutions, we have, \( v \leq v^* \leq w^+ \text{ in } B_{R_1}^T \), then
\[
u_c \leq w^+ \text{ in } B_{R_1}^T,
\]
thus
\[
\limsup_{(x,t) \to (\tilde{\xi}, \tilde{\tau})} u_c(x,t) \leq \lim_{(x,t) \to (\tilde{\xi}, \tilde{\tau})} w^+(x,t) = \varphi(\tilde{\xi}, \tilde{\tau}).
\]

From the discussions above, it only remains to prove that \( u_c \) is a viscosity solution of (1.4). In fact, from the definition of \( u_c \) and Lemma 2.9, it can be obtained directly that \( u_c \) is a weak viscosity solution of (1.4). By Lemma 2.3 and the asymptotic
behavior, $u^*_c \leq u_{c*}$. From the definition of $u^*_c$ and $u_{c*}$, we know that $u^*_c \geq u_{c*}$. Then we have $u^*_c = u_{c*} = u_c$. Thus, $u_c$ is continuous and a viscosity solution of (1.4).

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E-mail address: jgbao@bnu.edu.cn
E-mail address: gongshuyu@mail.bnu.edu.cn
E-mail address: zhouziwei@mail.bnu.edu.cn