FAST REACTION LIMIT WITH NONMONOTONE REACTION FUNCTION

BENOÎT PERTHAME AND JAKUB SKRZECZKOWSKI

Abstract. We analyse fast reaction limit in the reaction-diffusion system with nonmonotone reaction function and one non-diffusing component. As speed of reaction tends to infinity, the concentration of non-diffusing component exhibits fast oscillations. We identify precisely its Young measure which, as a by-product, proves strong convergence of the diffusing component, a result that is not obvious from a priori estimates. Our work is based on analysis of regularization for forward-backward parabolic equations by Plotnikov. We rewrite his ideas in terms of kinetic functions which clarifies the method, brings new insights, relaxes assumptions on model functions and provides a weak formulation for the evolution of the Young measure.

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1991 Mathematics Subject Classification. 35K57, 35B25, 35B36.

Key words and phrases. reaction-diffusion, cross-diffusion, oscillations, fast reaction limit, forward-backward diffusion, unstable solutions, kinetic formulation, Young measures.

Benoît Perthame has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No 740623).

Jakub Skrzeczkowski was supported by National Science Center, Poland through project no. 2018/30/M/ST1/00423. This work was completed while J.S. was a visitor at Laboratoire Jacques-Louis Lions whose kind hospitality he appreciates.
1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain. We consider the following system of reaction-diffusion equations with Neumann boundary conditions,

\begin{align}
\partial_t u^\varepsilon &= \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}, \\
\partial_t v^\varepsilon &= \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon},
\end{align}

where $t \geq 0$, $x \in \Omega$ and $F : \mathbb{R} \to [0, \infty)$ is a sufficiently smooth function. System (1.1)–(1.2) with a non-monotonic $F$, which is our interest here, is an interesting toy model for studying oscillations in reaction-diffusion systems as they are known to occur in its steady states [26].

**Assumption 1.1 (Initial data).** The system is completed with initial values $u^\varepsilon(0, x) = u_0(x)$, $v^\varepsilon(0, x) = v_0(x)$ satisfying

1. Nonnegativity: $u_0, v_0 \geq 0$.
2. Regularity: $u_0, v_0 \in C^{2+\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.
3. Boundary condition: $u_0, v_0$ satisfy the Neumann boundary condition.

Under appropriate assumptions (see Theorem 3.1), there is a unique classical solution of (1.1)–(1.2) which is bounded and nonnegative. Such systems are usually called *mass conservative* as it is easy to check that the quantity

$$
\int_{\Omega} u(t, x) + v(t, x) \, dx
$$

remains constant. Such equations have been used to model biological and chemical phenomena including cell polarity regularization (asymmetric organization of cellular structures) [31] and they received a lot of mathematical attention [24, 38] in particular for their pattern formation ability related to Turing instability [25, 26]. Moreover, systems with one non-diffusive component are widely studied in the literature, serving as models for early carcinogenesis [21] and also for pattern formation [20, 22].

Our interest lies in the so-called fast reaction limit corresponding to $\varepsilon \to 0$. By now, this problem is fairly classical assuming that reaction function $F$ is monotone [4]. In this spirit, fast reaction limits have been studied for a great variety of reaction-diffusion systems, also with more than two components [5, 10, 28] or reaction-diffusion equation coupled with an ODE [17]. They usually lead to the cross-diffusion systems where the gradient of one quantity induces a flux of another one [16].
a phenomena that is non-negligible for instance in chemistry [42] and is constantly studied from the mathematical point of view, see [6] [7] and references therein. A slightly different type of problem deals with the fast-reaction limit for irreversible reactions which leads to free boundary problems [9] [11]. We refer the reader to [18] [27] and references therein for further details and another limits in reaction-diffusion systems.

To focus our attention, we consider functions $F$ with particular monotonicity profile as plotted in Fig. 1. Our first result asserts that, up to a subsequence,

\[(1.3) \quad F(u^\varepsilon), v^\varepsilon \rightarrow v \text{ in } L^2((0,T) \times \Omega), \quad u^\varepsilon \rightharpoonup u := \sum_{i=1}^{3} \lambda_i S_i(v) \text{ in } L^\infty((0,T) \times \Omega),\]

where the weights $\lambda_1(t,x)$, $\lambda_2(t,x)$ and $\lambda_3(t,x)$ are nonnegative numbers such that $\sum_{i=1}^{3} \lambda_i = 1$ while $S_1$, $S_2$ and $S_3$ are three possible inverses of $F$ defined in Notation 2.1, cf. Fig. 1. Our main result is however to derive a kinetic equation for the weights. On the one hand, strong convergence of $v^\varepsilon$ is surprising as its compactness in time does not seem to be available from a priori estimates.

On the other hand, weak* limit of $u^\varepsilon$ can be interpreted as the weak form of the identity $v = F(u)$ known from the classical fast reaction limits. However, in our case, mass of $u$ splits for three parts associated to the preimages of $v$ under the map $F$. This intuition is made more precise in Theorem 2.3 using the language of Young measures.

Our strategy to prove (1.3) is to combine ideas from kinetic formulations of PDEs [34, 35] and from the insightful work of Plotnikov [36] (see also [13] [30] [32] [39] for similar problems). He considered the following regularization:

\[(1.4) \quad \partial_t w^\varepsilon = \Delta A(w^\varepsilon) + \varepsilon \Delta(\partial_t w^\varepsilon)\]

of the ill-posed problem $\partial_t w = \Delta A(w)$ where $A$ is assumed to have a similar monotonicity profile as in Fig. 1. Plotnikov studied the limit of $w^\varepsilon$ as $\varepsilon \rightarrow 0$. Using the theory of Young measures, he was able to predict oscillations in the limit and obtain the similar characterization of the limit $w$ as $u$ in our result (1.3). We comment more on connection between our work and Plotnikov paper in Section 7.1. We remark that analysis of $\partial_t w^\varepsilon = \Delta A(w^\varepsilon)$ with non-necessarily monotone function $A$ is constantly receiving attention in mathematical community [3] [13] [23] [41].
Unlike the original work of Plotnikov, in the process of limit identification, we exploit kinetic formulation. This is a well-known concept for scalar conservation laws\cite{19,34} that brought some connections with kinetic equations\cite{35} and degenerate parabolic equations\cite{8,14}. Although this is an approach equivalent with Young measures, working directly on functions is simpler as limit identification is based on a certain functional identity cf. Theorem 5.6. This approach results in a PDE satisfied by the kinetic functions cf. (4.3) which provides some information on evolution of weights $\lambda_i$ in (1.3), cf. Section 6. We comment more on connections between our work and Plotnikov’s paper in Section 7.1.

In this paper, we discuss the limit of (1.1)–(1.2) when $\varepsilon \to 0$. First, we present a priori estimates (Section 3). Then, in Section 4, we introduce kinetic formulation which allows to prove (1.3) in Section 5. In Section 6, we use kinetic formulation to derive some formal differential equations for coefficients $\lambda_i$ in (6). The two last subsections are devoted to discuss how our work is related to the Plotnikov’s paper and present some open problems in the field.

We list the main novelties of our work below.

- We rewrite Plotnikov’s method in terms of kinetic functions and identify limits of $u^\varepsilon$ and $v^\varepsilon$ in system (1.1)–(1.2).
We establish the PDE (4.8) satisfied by limiting kinetic functions. The latter can be viewed as a weak formulation for equation (6.3) satisfied, when \( v \) is smooth, by weights \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), see Section 6.

We modify Plotnikov’s method by exploiting natural energy for (1.1)–(1.2) rather than Plotnikov’s variables. Therefore, we do not need to assume \( F'(u) > -1 \), see Section 7.1.

We relax the nondegeneracy condition on the nonlinearity \( F \), see (3) in Assumption 2.2.

2. Assumptions and the main results

Before we start, let us precisely formulate our assumptions and notations for inverses of function \( F \).

Notation 2.1. Let \( S_1(\lambda) \leq S_2(\lambda) \leq S_3(\lambda) \) be the solutions of equation \( F(S_i(\lambda)) = \lambda \) as already introduced in (3) in Assumption 2.2 (see Fig. 1). These are inverses of \( F \) satisfying

\[
S_1 : (-\infty, f_+) \to (-\infty, \alpha_+), \quad S_2 : (f_-, f_+) \to (\alpha_+, \beta_-), \quad S_3 : (f_-, \infty) \to (\beta_-, \infty).
\]

Their role is too focus analysis on parts of the plot of \( F \) where the monotonicity of \( F \) does not change. By a small abuse of notation, we extend functions \( S_i \) by a constant value to the whole of \( \mathbb{R} \).

We usually write, for images of functions \( S_1, S_2, S_3 \) and for their domains

\[
I_1 = (-\infty, \alpha_+), \quad I_2 = (\alpha_+, \beta_-), \quad I_3 = (\beta_-, \infty),
\]

\[
J_1 = (-\infty, f_+), \quad J_2 = (f_-, f_+), \quad J_3 = (f_-, \infty).
\]

Assumption 2.2 (Reaction function \( F \)). We assume that the function \( F(u) \) satisfies:

1. Regularity, nonnegativity: \( F \in C^1(\mathbb{R}; [0, \infty)) \), with \( F(0) = 0 \).
2. Piecewise monotonicity of \( F \): there are \( \alpha_- < \alpha_+ < \beta_- < \beta_+ \) such that \( F(\beta_-) = F(\alpha_-) \), \( F(\alpha_+) = F(\beta_+) \), \( F \) is strictly increasing on \( (-\infty, \alpha_+) \cup (\beta_-, \infty) \) and strictly decreasing on \( (\alpha_+, \beta_-) \) (see Fig. 1). Moreover, \( \lim_{u \to \infty} F(u) = \infty \).
3. Nondegeneracy: in all subintervals of \( (f_-, f_+) \), the vanishing linear combination

\[
\sum_{i=1}^{3} a_i \left( S_i'(r) + 1 \right) = 0
\]

implies \( a_1 + a_2 + a_3 = 0 \).

Let us comment on the nondegeneracy condition (3) that is by no means an innocent assumption. For instance, it holds true if the functions \( 1 + S_1'(r), 1 + S_2'(r), 1 + S_3'(r) \) are linearly independent in each subinterval of \( (f_-, f_+) \) which was the original assumption made by Plotnikov [56]. On the
other hand, this condition excludes piecewise linear functions $F$. Nevertheless, it is usually made in this type of problems \cite{11 30 36}. For sufficiently smooth functions, a typical approach to check nondegeneracy condition is computing Wronskian of these functions \cite[Section 1.3]{2}, see \cite[Proposition 2]{32} for a particular example. We list it as one of the open problems in Section 7.2 to relax the nondegeneracy condition.

A first result of this paper reads:

**Theorem 2.3** (Limits for $v^\varepsilon$, $u^\varepsilon$). Let $T > 0$ and $(u^\varepsilon, v^\varepsilon)$ be the solution of \eqref{1.1}–\eqref{1.2}. Then, up to a subsequence, $u^\varepsilon \rightharpoonup^\ast u$ weakly$^\ast$ in $L^\infty((0, T) \times \Omega)$ and $F(u^\varepsilon), v^\varepsilon \to v$ strongly in $L^2((0, T) \times \Omega)$. Moreover, the Young measure generated by $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ is a convex combination of Dirac masses

\begin{equation}
\mu_{t,x} = \lambda_1(t,x) \delta_{S_1(v(t,x))} + \lambda_2(t,x) \delta_{S_2(v(t,x))} + \lambda_3(t,x) \delta_{S_3(v(t,x))},
\end{equation}

where $S_1$, $S_2$ and $S_3$ are the inverses of $F$ defined in Notation 2.1 while $\lambda_1$, $\lambda_2$, $\lambda_3$ are nonnegative numbers such that $\sum_{i=1}^{3} \lambda_i = 1$.

The proof is presented in Section 5. Loosely speaking, representation \eqref{2.1} means that for small values of $\varepsilon$, the function $u(t,x)$ should oscillate between (at most) three values. This is observed by numerical simulations in Fig. 2. In fact the unstable state is reached only during the transient.

The connection between $u$ and $v$ in Theorem 2.3 is formulated in the language of Young measures and reader not familiar with this topic is referred to \cite{12} for a concise introduction with applications. Briefly speaking, Young measures allow to represent weak limits of nonlinear functions. More precisely, let $\{\mu_{t,x}\}_{t,x}$ and $\{\nu_{t,x}\}_{t,x}$ be the Young measures generated by sequences $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ and $\{v^\varepsilon\}_{\varepsilon \in (0,1)}$ respectively. Then, for any bounded function $G : \mathbb{R} \to \mathbb{R}$ we have (up to a subsequence and for a.e. $(t,x) \in (0,T) \times \Omega$)

\begin{align*}
G(u^\varepsilon) &\rightharpoonup^\ast \int_{\mathbb{R}} G(\lambda) \, d\mu_{t,x}(\lambda) := \langle G, \mu_{t,x} \rangle, \\
G(v^\varepsilon) &\rightharpoonup^\ast \int_{\mathbb{R}} G(\lambda) \, d\nu_{t,x}(\lambda) := \langle G, \nu_{t,x} \rangle.
\end{align*}

The proof of Theorem 2.3 goes as follows. One rewrites equations \eqref{1.1}–\eqref{1.2} in terms of kinetic functions. Using compensated compactness \cite{29 30}, we obtain Lemma 4.4 and the functional identity for kinetic functions \eqref{4.11} from which we deduce the kinetic function shape for $v(t,x)$ in Section 5. This implies that the Young measure generated by $\{v^\varepsilon\}_{\varepsilon \in (0,1)}$ is a Dirac mass which proves \eqref{2.1}.

The crucial step in the proof of Theorem 2.3 which is a new result by its own, is a PDE satisfied by the kinetic functions generated by $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ and $\{v^\varepsilon\}_{\varepsilon \in (0,1)}$. 
Theorem 2.4 (Kinetic PDE). Let \( p, q : (0, T) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be the \( L^\infty \)-weak* limits (up to extraction of subsequences) as below:

\[
p(t, x, \xi) = \lim_{\varepsilon \to 0} \mathbf{1}_{0 \leq \xi \leq \nu^\varepsilon(t, x)}(\xi), \quad q(t, x, \xi) = \lim_{\varepsilon \to 0} \mathbf{1}_{0 \leq \xi \leq \nu^\varepsilon(t, x)}(\xi).
\]

Then, there is a bounded nonnegative measure \( n \) on \((0, T) \times \Omega \times \mathbb{R}\) such that equation

\[
\partial_t \int_{\mathbb{R}} p(t, x, \lambda) \delta_{\xi = F(\lambda)}(\xi) d\lambda + q(t, x, \xi) - \Delta_x q(t, x, \xi) = \partial_\xi n(t, x, \xi)
\]

holds in the sense of distributions.

Theorem 2.4 is proved in Section 4.1 (part of Theorem 4.3). As a consequence, we can formulate equations for evolution of weights \( \{\lambda_i(t, x)\}_{i=1,2,3} \) proved in Section 6.

Theorem 2.5 (Equations for weights). Let \( \{\lambda_i(t, x)\}_{i=1,2,3} \) be as in (2.1). We set

\[
\kappa_1(t, x) = 1 - \lambda_1(t, x), \quad \kappa_2(t, x) = \lambda_3(t, x).
\]

(1) Suppose additionally that sequences \( \{\partial_\varepsilon v^\varepsilon\}_{\varepsilon \in (0,1)} \) and \( \{\Delta v^\varepsilon\}_{\varepsilon \in (0,1)} \) are uniformly bounded in \( L^2((0, T) \times \Omega) \). Then, we have \( \partial_t \lambda_i(t, x) = 0 \) for \( i = 1, 2, 3 \) and \((t, x) \in \mathcal{O}\) where \( \mathcal{O} \subset (0, T) \times \Omega \) is any open set where \( v(t, x) \) is continuous. In particular, no splitting of mass may occur.

(2) In general, if

\[
\mathcal{O} \subset \{(t, x, \xi) : f_- < v(t, x) < \xi_0 < \xi < f_+\}
\]

is an open set for some \( \xi_0 \), we have for \((t, x, \xi) \in \mathcal{O}\)

\[
\partial_t \int_{S_1(v(t, x))} \kappa_1(t, x) \delta_{\xi = F(\lambda)}(\xi) d\lambda = \partial_\xi n(t, x, \xi),
\]

where \( n \) is a nonnegative measure from Theorem 2.4. Similarly, if

\[
\mathcal{O} \subset \{(t, x, \xi) : f_- < \xi < \xi_0 < v(t, x) < f_+\}
\]

is an open set for some \( \xi_0 \), we have for \((t, x, \xi) \in \mathcal{O}\)

\[
\partial_t \int_{S_2(v(t, x))} \delta_{\xi = F(\lambda)}(\xi) d\lambda + \partial_t \int_{S_2(v(t, x))} \kappa_2(t, x) \delta_{\xi = F(\lambda)}(\xi) d\lambda = \partial_\xi n(t, x, \xi).
\]

Part [1] of Theorem 2.5 implies that if \( u^\varepsilon \) oscillates between two states in some subset, function \( v^\varepsilon \) should form a discontinuity there. This phenomenon is presented in Fig. 2.
Figure 2. Evolution of $u^\varepsilon$ (continuous line) and $v^\varepsilon$ (dash-dotted line) solving (1.1)–(1.2) in one space dimension with fixed and small value of $\varepsilon > 0$. Two time shots are presented to show dependence between oscillations of $u^\varepsilon$ and $v^\varepsilon$. When $u^\varepsilon$ oscillates, $v^\varepsilon$ also exhibits oscillatory behaviour. However, when the weights in the equation (2.1) stabilize and only one of them is not vanishing, oscillations of $v^\varepsilon$ disappear.

The main tool to prove Theorem 2.4 is the following energy equality. Given a smooth test function $\phi : \mathbb{R} \to \mathbb{R}$, we define

$$
\Psi(\lambda) := \int_0^\lambda \phi(F(\tau)) \, d\tau, \quad \Phi(\lambda) := \int_0^\lambda \phi(\tau) \, d\tau.
$$

Multiplying equation (1.1) with $\phi(F(u^\varepsilon))$ and equation (1.2) with $\phi(v^\varepsilon)$ we obtain

$$
\partial_t \Psi(u^\varepsilon) = \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon} \phi(F(u^\varepsilon)),
$$
$$
\partial_t \Phi(v^\varepsilon) = \Delta \Phi(v^\varepsilon) - \phi'(v^\varepsilon) |\nabla v^\varepsilon|^2 + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon} \phi(v^\varepsilon).
$$

Summing up these equations we deduce

$$
\partial_t \Psi(u^\varepsilon) + \partial_t \Phi(v^\varepsilon) = \Delta \Phi(v^\varepsilon) - \phi'(v^\varepsilon) |\nabla v^\varepsilon|^2 - \frac{(v^\varepsilon - F(u^\varepsilon))}{\varepsilon} \left( \phi(v^\varepsilon) - \phi(F(u^\varepsilon)) \right).
$$

This energy equality provides a priori estimates stated in Theorem 3.1, the PDE for the kinetic functions in Theorem 4.3 and compensated compactness results (Lemma 4.4) necessary to derive the functional identity (4.11).
3. A priori estimates

As long as $\varepsilon$ is positive, solutions of system (1.1)–(1.2) are smooth, however some estimates, which are instrumental for studying the oscillatory limit, are uniform in $\varepsilon$.

**Theorem 3.1.** There exists the unique classical solution $u^{\varepsilon}, v^{\varepsilon}: [0, \infty) \times \Omega \to \mathbb{R}$ of (1.1)–(1.2) which is nonnegative and has regularity

$$u^{\varepsilon} \in C^{\alpha, 1+\alpha/2}([0, \infty) \times \Omega), \quad v^{\varepsilon} \in C^{2+\alpha, 1+\alpha/2}([0, \infty) \times \Omega).$$

Moreover, we have

1. $0 \leq u^{\varepsilon} \leq M, \ 0 \leq v^{\varepsilon} \leq M$ with $M = \max(\|F(u_0)\|_{\infty}, \|u_0\|_{\infty}, \|v_0\|_{\infty}, f_+, \beta_+),$
2. $\{\nabla v^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ is uniformly bounded in $L^2((0, \infty) \times \Omega),$
3. $\{\frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\sqrt{\varepsilon}}\}_{\varepsilon \in (0, 1)}$ and $\{\sqrt{\varepsilon} \Delta v^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ are uniformly bounded in $L^2((0, \infty) \times \Omega).$

**Proof.** Because the right hand sides are locally Lipschitz continuous, local existence, nonnegativity and uniqueness follows from standard theory. To prove global existence, we need to establish uniform bounds as in (1). We consider smooth and nondecreasing test function $\phi: \mathbb{R} \to \mathbb{R}$ as well as $\Psi$ and $\Phi$ defined by (2.5). Then, it follows from (2.6) that

$$\partial_t \Psi(u^{\varepsilon}) + \partial_t \Phi(v^{\varepsilon}) \leq \Delta \Phi(v^{\varepsilon})$$

thanks to the monotonicity of $\phi$. Therefore, the nonnegative map

$$(3.1) \quad t \mapsto \int_{\Omega} \left[ \Psi(u^{\varepsilon}(t, x)) + \Phi(v^{\varepsilon}(t, x)) \right] dx$$

is nonincreasing. We choose $\phi$ such that $\phi = 0$ on $[0, M]$ and $\phi' > 0$ on $(M, \infty)$. Then, the map in (3.1) vanishes at $t = 0$ and so, it has to vanish for all $t \geq 0$. This proves uniform bounds on $\{u^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ and $\{v^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ in $L^\infty((0, \infty) \times \Omega)$ and concludes the proof of global existence.

Estimates (2) and (3) follow directly from (2.6) with $\phi(\lambda) = \lambda$. Finally, the estimate on $\Delta v^{\varepsilon}$ is deduced by multiplying the equation for $v^{\varepsilon}$ by $\varepsilon \Delta v^{\varepsilon}$. □

**Corollary 3.2.** Let $u^{\varepsilon}, v^{\varepsilon}$ be the solution of system (1.1)–(1.2). Then, $F(u^{\varepsilon}) - v^{\varepsilon} \to 0$ strongly in $L^2((0, \infty) \times \Omega)$.

Recall that we write $\{\mu_{t,x}\}_{t,x}$ and $\{\nu_{t,x}\}_{t,x}$ for Young measures generated by sequences $\{u^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ and $\{v^{\varepsilon}\}_{\varepsilon \in (0, 1)}$ respectively. We make an elementary observation.
Lemma 3.3. Sequence \( \{F(u^\varepsilon)\}_{\varepsilon \in (0,1)} \) generates Young measure \( \{F^\# \mu_{t,x}\}_{t,x} \) (i.e. push-forward of \( \mu_{t,x} \) along map \( F \)). Moreover, for a.e. \((t, x) \in (0, \infty) \times \Omega\) we have
\[
F^\# \mu_{t,x} = \nu_{t,x}.
\]

Proof. Let \( \Psi : \mathbb{R} \to \mathbb{R} \) be a bounded function and let \( \{\rho_{t,x}\}_{t,x} \) be the Young measure generated by sequence \( \{F(u^\varepsilon)\}_{\varepsilon \in (0,1)} \). Then, up to a subsequence, weak* limit of \( \Psi(F(u^\varepsilon)) \) can be written as
\[
\int_{\mathbb{R}} \Psi(\lambda) \, d\rho_{t,x}(\lambda) = \int_{\mathbb{R}} \Psi(F(\lambda)) \, d\mu_{t,x}(\lambda).
\]
Therefore, \( \rho_{t,x} = F^\# \mu_{t,x} \) holds for a.e. \((t, x) \) as desired. Moreover, Corollary 3.2 shows that \( F(u^\varepsilon) - v^\varepsilon \to 0 \) strongly in \( L^2((0,T) \times \Omega) \). Hence, Young measures generated by these sequences coincide \cite[Lemma 6.3]{33} and the proof is concluded. \( \square \)

4. Kinetic formulation

4.1. Kinetic functions and the kinetic PDE. To understand the behaviour of sequences \( \{u^\varepsilon\}_{\varepsilon \in (0,1)} \) and \( \{v^\varepsilon\}_{\varepsilon \in (0,1)} \), we introduce kinetic function for \( \alpha \geq 0 \),
\[
\chi_\alpha(\xi) = 1_{0 < \xi \leq \alpha}.
\]
As Young measures, it is a way to represent nonlinear functions \( \varphi : \mathbb{R} \to \mathbb{R} \) since we have a fundamental identity
\[
\int_{\mathbb{R}} \chi_\alpha(\xi) \varphi'(\xi) \, d\xi = \int_{0}^{\alpha} \varphi'(\xi) \, d\xi = \varphi(\alpha) - \varphi(0).
\]
We let
\[
p^\varepsilon(t, x, \xi) = \chi_{u^\varepsilon(t,x)}(\xi) \quad q^\varepsilon(t, x, \xi) = \chi_{v^\varepsilon(t,x)}(\xi).
\]
so for any differentiable and bounded \( \Psi : \mathbb{R} \to \mathbb{R} \) we have by (4.2)
\[
\Psi(u^\varepsilon(t, x)) = \Psi(0) + \int_{\mathbb{R}} p^\varepsilon(t, x, \xi) \Psi'(\xi) \, d\xi, \quad \Psi(v^\varepsilon(t, x)) = \Psi(0) + \int_{\mathbb{R}} q^\varepsilon(t, x, \xi) \Psi'(\xi) \, d\xi.
\]
After extraction of a weakly* converging subsequence in \( L^\infty((0,T) \times \Omega \times \mathbb{R}) \), we may assume that \( p^\varepsilon \rightharpoonup p \) and \( q^\varepsilon \rightharpoonup q \), i.e.,
\[
p(t, x, \xi) = \lim_{\varepsilon \to 0} p^\varepsilon(t, x, \xi), \quad q(t, x, \xi) = \lim_{\varepsilon \to 0} q^\varepsilon(t, x, \xi).
\]
Connection between \( p \) and \( q \) will be explored in Lemma 4.1. We usually say that kinetic functions \( p \) and \( q \) are generated by sequences \( \{u^\varepsilon\}_{\varepsilon \in (0,1)} \) and \( \{v^\varepsilon\}_{\varepsilon \in (0,1)} \) respectively. Basic properties of \( p \) and \( q \) are recorded below.
Lemma 4.1 (Properties of \( p \) and \( q \)). Let \( p \) and \( q \) be given by (4.5). Then, for a.e. \((t, x) \in (0, T) \times \Omega\),

1. we have \( 0 \leq p(t, x, \xi) \leq 1 \) and \( 0 \leq q(t, x, \xi) \leq 1 \);
2. the maps \( p(t, x, \xi) \) and \( q(t, x, \xi) \) are supported in \((0, M)\) with \( M \) defined in Theorem 3.1;
3. the maps \( \xi \mapsto p(t, x, \xi) \) and \( \xi \mapsto q(t, x, \xi) \) are non-increasing;
4. we have
   \[
   q(t, x, \xi) = \int_{\mathbb{R}} p(t, x, \lambda) F'(\lambda) \delta_{\xi = F(\lambda)}(\xi) \, d\lambda
   \]
   in the sense of distributions.

Proof. Property 1 follows from the fact that the sequences are nonnegative and this property is preserved under weak limits. Property 2 follows from uniform boundedness of sequences \( \{u^\varepsilon\}_{\varepsilon \in (0, 1)} \) and \( \{v^\varepsilon\}_{\varepsilon \in (0, 1)} \). Property 3 is a consequence of the same for \( p^\varepsilon \) and \( q^\varepsilon \). To see 4, we fix a smooth test function \( \psi(\xi) \) with \( \Psi(\xi) = \int_{\xi} \psi(\eta) \, d\eta \) and we consider the map
   \[
   \mathbb{R} \ni w \mapsto \int_{\mathbb{R}} \psi(\lambda) \chi_{F(w)}(\lambda) \, d\lambda = \Psi(F(w)).
   \]

With the change of variable \( \xi = F(\lambda) \), (4.2) implies the identity
   \[
   \int_{\mathbb{R}} \psi(\xi) \chi_{F(w)}(\xi) \, d\xi = \int_{\mathbb{R}} \psi(F(\lambda)) \chi_w(\lambda) F'(\lambda) \, d\lambda = \int_{\mathbb{R}} \psi(\xi) \int_{\mathbb{R}} \chi_w(\lambda) F'(\lambda) \delta_{\xi = F(\lambda)}(\xi) \, d\lambda \, d\xi.
   \]

We plug \( w = u^\varepsilon(t, x) \) and deduce
   \[
   (4.6) \quad \int_{\mathbb{R}} \psi(\lambda) \chi_{F(u^\varepsilon)}(\lambda) \, d\lambda = \int_{\mathbb{R}} \psi(\xi) \int_{\mathbb{R}} p^\varepsilon(t, x, \xi) F'(\lambda) \delta_{\xi = F(\lambda)}(\xi) \, d\lambda \, d\xi.
   \]

Now, to identify the weak* limit on the (LHS) of (4.6), we note that
   \[
   \left| \int_{\mathbb{R}} \psi(\xi) \chi_{F(u^\varepsilon)}(\xi) \, d\xi - \int_{\mathbb{R}} \psi(\xi) \chi_{v^\varepsilon}(\xi) \, d\xi \right| = |\Psi(F(u^\varepsilon)) - \Psi(v^\varepsilon)| \rightarrow 0 \text{ in } L^2((0, T) \times \Omega)
   \]
due to Corollary 3.2. Therefore, sending \( \varepsilon \rightarrow 0 \) in (4.6), we obtain 4. \( \square \)

In view of Lemma 3.3 let us also connect kinetic functions with Young measures.

Lemma 4.2 (Young measures vs kinetic formulation). Let \( p \) and \( q \) be given by (4.5) and let \( \{\mu_{t,x}\}_{t,x} \) and \( \{\nu_{t,x}\}_{t,x} \) be the Young measures generated by \( \{u^\varepsilon\}_{\varepsilon \in (0, 1)} \) and \( \{v^\varepsilon\}_{\varepsilon \in (0, 1)} \) respectively. Then, for a.e. \((t, x) \in (0, T) \times \Omega\) we have, in the sense of distributions,
   \[
   \partial_\xi p(t, x, \xi) = \delta_0 - \mu_{t,x}, \quad \partial_\xi q(t, x, \xi) = \delta_0 - \nu_{t,x} = \delta_0 - F^\# \mu_{t,x}.
   \]
Proof. We only prove the first formula as the second follows from a similar reasoning combined with Lemma 3.3. Let \( \Psi : \mathbb{R} \to \mathbb{R} \) be an arbitrary smooth and bounded function. Passing to the limit \( \varepsilon \to 0 \) in (4.4) we obtain
\[
\int_\mathbb{R} \Psi(\lambda) \, d\mu_t(x)(\lambda) = \Psi(0) + \int_\mathbb{R} p(t, x, \xi) \, \Psi'(\xi) \, d\xi
\]
which proves the first identity. \( \square \)

We conclude this subsection with a distributional PDE that will be exploited in the compactness result (Lemma 4.4).

**Theorem 4.3 (PDE satisfied by kinetic functions).** Let \( p \) and \( q \) be given by (4.5). Then, there is a uniformly bounded sequence of nonnegative measures \( \{n^\varepsilon\}_{\varepsilon \in (0,1)} \) on \( (0, T) \times \Omega \times \mathbb{R} \) such that
\[
\partial_t \left[ \int_\mathbb{R} p(t, x, \lambda) \, \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + q(t, x, \xi) \right] - \Delta_x q(t, x, \xi) = \partial_\xi n(t, x, \xi)
\]
in the sense of distributions. In particular, there is a bounded nonnegative measure \( n \) on \( (0, T) \times \Omega \times \mathbb{R} \) such that
\[
\partial_t \left[ \int_\mathbb{R} p(t, x, \lambda) \, \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + q(t, x, \xi) \right] - \Delta_x q(t, x, \xi) = \partial_\xi n(t, x, \xi).
\]

Proof. We consider smooth test function \( \phi : \mathbb{R} \to \mathbb{R} \) as well as \( \Psi \) and \( \Phi \) defined by (2.5). From (2.6) we know that
\[
\partial_t \Psi(u^\varepsilon) + \partial_t \Phi(v^\varepsilon) = \Delta \Phi(v^\varepsilon) - \phi'(v^\varepsilon) |\nabla v^\varepsilon|^2 - \frac{(v^\varepsilon - F(u^\varepsilon)) (\phi(v^\varepsilon) - \phi(F(u^\varepsilon)))}{\varepsilon}.
\]
Now, using kinetic functions we can write \( \partial_t \Psi(u^\varepsilon) \) as
\[
\partial_t \Psi(u^\varepsilon) = \partial_t \int_\mathbb{R} p^\varepsilon(t, x, \lambda) \phi(F(\lambda)) \, d\lambda = \partial_t \int_\mathbb{R} \phi(\xi) \int_\mathbb{R} p^\varepsilon(t, x, \lambda) \delta_{\xi=F(\lambda)}(\xi) \, d\xi \, d\lambda,
\]
while \( \partial_t \Phi(v^\varepsilon) \) and \( \Delta \Phi(v^\varepsilon) \) as
\[
\partial_t \Phi(v^\varepsilon) = \partial_t \int_\mathbb{R} \phi(\xi) q^\varepsilon(t, x, \xi) \, d\xi, \quad \Delta \Phi(v^\varepsilon) = \Delta \int_\mathbb{R} \phi(\xi) q^\varepsilon(t, x, \xi) \, d\xi.
\]
Now, term \( - \phi'(v^\varepsilon) |\nabla v^\varepsilon|^2 \) can be interpreted as a derivative of a nonnegative measure \( n^\varepsilon(t, x, \xi) \):
\[
- \phi'(v^\varepsilon) |\nabla v^\varepsilon|^2 = \langle \phi(\xi), \partial_\xi \delta_{\xi=v^\varepsilon(t,x)}(\xi) |\nabla v^\varepsilon(t, x)|^2 \rangle := \langle \phi(\xi), \partial_\xi n^\varepsilon(t, x, \xi) \rangle.
\]
We note that the sequence \{n^\varepsilon\}_{\varepsilon \in (0,1)} is uniformly bounded in the space of measures according to estimate (2) in Theorem 3.1. Similarly, using
\[
\phi(v^\varepsilon) - \phi(F(u^\varepsilon)) = \int_0^1 \frac{d}{ds} \phi(s v^\varepsilon + (1-s) F(u^\varepsilon)) \, ds = \\
= (v^\varepsilon - F(u^\varepsilon)) \int_0^1 \phi'(s v^\varepsilon + (1-s) F(u^\varepsilon)) \, ds,
\]
we can write
\[
- \frac{\varepsilon}{\phi(v^\varepsilon) - \phi(F(u^\varepsilon))} (\phi(v^\varepsilon) - \phi(F(u^\varepsilon))) = \frac{(v^\varepsilon - F(u^\varepsilon))^2}{\varepsilon} \left( \phi(\xi), \partial_\xi \int_0^1 \delta_{s v^\varepsilon + (1-s) F(u^\varepsilon)}(\xi) \, ds \right)
\]
where the last term can be interpreted as a Bochner integral in the space of measures. We let
\[
n_2^\varepsilon(t, x, \xi) = \frac{(v^\varepsilon - F(u^\varepsilon))^2}{\varepsilon} \int_0^1 \delta_{s v^\varepsilon + (1-s) F(u^\varepsilon)}(\xi) \, ds
\]
which is a uniformly bounded sequence of nonnegative measures on \((0, T) \times \Omega \times \mathbb{R}\) due to (3) in Theorem 3.1. Therefore, it has a subsequence converging weakly*, i.e. \(n_2^\varepsilon \rightharpoonup n_2\). Collecting all the terms, we obtain distributional identity (4.7). Passing to the weak* limit with \(\varepsilon \to 0\), we deduce (4.8) and Theorem 4.3 is proved.

4.2. A kinetic identity satisfied by \(p\) and \(q\). Now, following [30] for Young measures, we formulate a distributional identity that will be used to identify kinetic functions \(p\) and \(q\). We start with the result of compensated compactness type.

**Lemma 4.4** (compensated compactness). Let \(p\) and \(q\) be given by (4.5). Then,
\[
q^\varepsilon(t, x, \xi) \left[ \int_\mathbb{R} p^\varepsilon(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) \, d\lambda + q^\varepsilon(t, x, \eta) \right] \to q(t, x, \xi) \left[ \int_\mathbb{R} p(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) \, d\lambda + q(t, x, \eta) \right]
\]
in the sense of distributions. More precisely, for all smooth and compactly supported test functions \(\phi, \psi : \mathbb{R} \to \mathbb{R}\) and \(\varphi : (0, T) \times \Omega \to \mathbb{R}\) we have
\[
\int_{(0,T) \times \Omega} \varphi(t, x) \int_\mathbb{R} \phi(\xi) q^\varepsilon(t, x, \xi) \, d\xi \int_\mathbb{R} \psi(\eta) \left[ \int_\mathbb{R} p^\varepsilon(t, x, \tau) \delta_{\eta=F(\tau)}(\eta) \, d\tau + q^\varepsilon(t, x, \eta) \right] \, d\eta \, dx \, dt \to \\
\int_{(0,T) \times \Omega} \varphi(t, x) \int_\mathbb{R} \phi(\xi) q(t, x, \xi) \, d\xi \int_\mathbb{R} \psi(\eta) \left[ \int_\mathbb{R} p(t, x, \tau) \delta_{\eta=F(\tau)}(\eta) \, d\tau + q(t, x, \eta) \right] \, d\eta \, dx \, dt.
\]

**Proof.** Let
\[
\mathcal{P}_{\phi}^\varepsilon(t, x) = \int_\mathbb{R} \psi(\eta) \left[ \int_\mathbb{R} p^\varepsilon(t, x, \xi)(t, x, \tau) \delta_{\eta=F(\tau)}(\eta) \, d\tau + q(t, x, \eta) \right] \, d\eta,
\]
\[
\mathcal{Q}_{\phi}^\varepsilon(t, x) = \int_\mathbb{R} \phi(\xi) q^\varepsilon(t, x, \xi) \, d\xi.
\]
Setting $\mathcal{P}_\psi^\epsilon \rightharpoonup \mathcal{P}_\psi$ and $Q_\phi^\epsilon \rightharpoonup Q_\phi$, we have to prove that
\[
\int_{(0,T)\times\Omega} \varphi(t,x) \mathcal{P}_\psi^\epsilon(t,x) \ Q_\phi^\epsilon(t,x) \ dt \ dx \rightarrow \int_{(0,T)\times\Omega} \varphi(t,x) \mathcal{P}_\psi(t,x) \ Q_\phi(t,x) \ dt \ dx.
\]

Consider operator $T : L^2(\Omega) \rightarrow H^2(\Omega)$ defined as the solution of the Neumann problem
\[
(4.10) \quad -\Delta [T(f)] = \mu [T(f)] + f \text{ in } \Omega, \quad \frac{\partial}{\partial n} [T(f)] = 0 \text{ on } \partial\Omega,
\]
where $\mu > 0$ is some fixed number from the resolvent of $-\Delta$ with Neumann boundary conditions. By elliptic regularity theory, $\|T(f)\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$ for a.e. $t \in (0,T)$. In particular, up to a subsequence, uniqueness of solutions to (4.10) implies
\[
\text{that } T(\mathcal{P}_\psi^\epsilon) \rightarrow T(\mathcal{P}_\psi) \text{ in } L^2(0,T;H^1(\Omega)).
\]
Using the operator $T$ we can write
\[
\int_{(0,T)\times\Omega} \varphi \mathcal{P}_\psi^\epsilon \ Q_\phi^\epsilon \ dt \ dx = \int_{(0,T)\times\Omega} \nabla T(\mathcal{P}_\psi^\epsilon) \cdot (\varphi \ Q_\phi^\epsilon) \ dt \ dx - \mu \int_{(0,T)\times\Omega} T(\mathcal{P}_\psi^\epsilon) \ (\varphi \ Q_\phi^\epsilon) \ dt \ dx.
\]

Clearly, up to a subsequence, $Q_\phi^\epsilon \rightharpoonup Q_\phi$ in $L^2(0,T;H^1(\Omega))$ because $Q_\phi^\epsilon = \Phi(v^\epsilon)$ (where $\Phi' = \phi$) and $\{v^\epsilon\}_{\epsilon > 0}$ is bounded in $L^2(0,T;H^1(\Omega))$ \cite{2} in Theorem 3.1. Therefore, it is sufficient to prove that $T(\mathcal{P}_\psi^\epsilon) \rightarrow T(\mathcal{P}_\psi)$ strongly in $L^2(0,T;H^1(\Omega))$.

We want to apply Aubin-Lions Lemma for the case where time derivative is a measure cf. \cite{34} Corollary 7.9. In view of the regularity estimate
\[
\|T(\mathcal{P}_\psi^\epsilon)(t,x)\|_{L^2(0,T;H^2(\Omega))} \leq C \left\|\mathcal{P}_\psi^\epsilon(t,x)\right\|_{L^2(0,T;L^2(\Omega))},
\]
we only have to prove that the sequences of distributional time derivatives \(\left\{\frac{\partial}{\partial t} T(\mathcal{P}_\psi^\epsilon)\right\}_{\epsilon > 0}\) and \(\left\{\frac{\partial}{\partial t} \nabla T(\mathcal{P}_\psi^\epsilon)\right\}_{\epsilon > 0}\) are bounded in $C(0,T;X)^*$ for some separable Banach space $X$ such that $L^2(\Omega) \subset X^*$.

To this end, we note that equation (4.10) implies that sequence \(\left\{\frac{\partial}{\partial t} \mathcal{P}_\psi^\epsilon\right\}_{\epsilon > 0}\) is bounded in space $[C(0,T;H^k(\Omega))]^*$ for some $k > d$ so that $H^k(\Omega)$ embeds continuously into $L^\infty(\Omega)$ cf. \cite{15} Corollary 7.11]. Then, we claim that equation (4.10) implies that
\[
\left\{\frac{\partial}{\partial t} T(\mathcal{P}_\psi^\epsilon)\right\}_{\epsilon > 0}\text{ is bounded in } [C(0,T;H^{k-2}(\Omega))]^*,
\]
\[
\left\{\frac{\partial}{\partial t} \nabla T(\mathcal{P}_\psi^\epsilon)\right\}_{\epsilon > 0}\text{ is bounded in } [C(0,T;H^{k-1}(\Omega))]^*.
\]
Indeed, if $\zeta(t, x)$ is a (vector-valued) smooth and compactly supported test function we have

$$
\int_{(0,T) \times \Omega} \nabla T \left( \mathcal{P}_\psi^\varepsilon \right) (t, x) \cdot \partial_t \zeta(t, x) \, dt \, dx =
$$

$$
= - \int_{(0,T) \times \Omega} \nabla T \left( \mathcal{P}_\psi^\varepsilon \right) (t, x) \cdot \left[ \Delta T \left( \partial_t \zeta \right) (t, x) + \mu T \left( \partial_t \zeta \right) (t, x) \right] \, dt \, dx
$$

$$
= - \int_{(0,T) \times \Omega} \mathcal{P}_\psi^\varepsilon \partial_t \left( \text{div} T(\zeta) \right) \, dt \, dx \leq C \| \text{div} T(\zeta) \|_{C(0,T;H^k(\Omega))}
$$

$$
\leq C \| T(\zeta) \|_{C(0,T;H^{k+1}(\Omega))} \leq C \| \zeta \|_{C(0,T;H^{k-1}(\Omega))}.
$$

Similar computation can be performed for the term $\int_{(0,T) \times \Omega} T \left( \mathcal{P}_\psi^\varepsilon \right) (t, x) \partial_t \zeta(t, x) \, dt \, dx$. This concludes the proof. \( \square \)

We are in position to formulate a functional identity relating the kinetic functions $p$ and $q$.

**Theorem 4.5** (Functional identity for $p$ and $q$). Let $p$ and $q$ be given by \[4.5\]. Then, the following identity is satisfied in the sense of distributions

$$
q(t, x, \xi) \int \frac{\partial p(t, x, \lambda)}{\partial \lambda} \delta_{\eta=F(\lambda)}(\eta) \, d\lambda = \chi_\eta(t, x) \int p(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) \, d\lambda 
$$

$$
+ \chi_\eta(t, x) q(t, x, \xi) + \chi_\eta(t, x, \xi) q(t, x, \eta) - q(t, x, \eta) q(t, x, \xi)
$$

$$
+ \int \int \int p(t, x, \lambda) \chi_\lambda(\tau) \delta_{\eta=F(\tau)}(\eta) \delta_{\xi=F(\lambda)}(\xi) F'(\lambda) \, d\lambda \, d\tau.
$$

**Proof.** Consider two smooth test functions $\phi, \varphi : \mathbb{R} \to \mathbb{R}$ and define

$$
\Psi(\lambda) := \int_0^\lambda \phi(F(\tau)) \, d\tau, \quad \Phi(\lambda) := \int_0^\lambda \phi(\tau) \, d\tau, \quad \Theta(\lambda) := \int_0^\lambda \varphi(\tau) \, d\tau.
$$

Now, the plan is to consider the limit as $\varepsilon \to 0$ of the expression

$$
[\Psi(u^\varepsilon) + \Phi(v^\varepsilon)] \Theta(v^\varepsilon).
$$

From the kinetic representation \[4.2\], we can write

$$
\Psi(u^\varepsilon) = \int \phi(F(\lambda)) p^\varepsilon(t, x, \lambda) \, d\lambda = \int \phi(\eta) \delta_{\eta=F(\lambda)}(\eta) p^\varepsilon(t, x, \lambda) \, d\eta \, d\lambda,
$$

$$
\Phi(v^\varepsilon) = \int \phi(\eta) q^\varepsilon(t, x, \eta) \, d\eta,
$$

$$
\Theta(v^\varepsilon) = \int \varphi(\xi) q^\varepsilon(t, x, \xi) \, d\xi.
$$

On the other hand, using Lemma \[4.4\] we obtain

$$
[\Psi(u^\varepsilon) + \Phi(v^\varepsilon)] \Theta(v^\varepsilon) \to \int \int \phi(\eta) \varphi(\xi) q(t, x, \xi) \left[ \int p(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) \, d\lambda + q(t, x, \eta) \right] \, d\eta \, d\xi.
$$
On the other hand, we can replace the term $[\Psi(u^\varepsilon) + \Phi(v^\varepsilon)]\Theta(v^\varepsilon)$ with $[\Psi(u^\varepsilon) + \Phi(F(u^\varepsilon))]\Theta(F(u^\varepsilon))$ because $v^\varepsilon - F(u^\varepsilon) \to 0$ strongly. Therefore, we can use the kinetic representation

$$\int_{\mathbb{R}} p^\varepsilon(t, x, \lambda) \left[ (\Psi(\lambda) + \Phi(F(\lambda))) \Theta(F(\lambda)) \right]' \, d\lambda =: A^\varepsilon + B^\varepsilon + C^\varepsilon$$

where these three terms come from the differentiation of the product. We have

$$A^\varepsilon = \int_{\mathbb{R}} p^\varepsilon(t, x, \lambda) \Theta(F(\lambda)) \left( \Phi(\lambda) + \Phi'(F(\lambda)) F'(\lambda) \right) \, d\lambda$$

$$B^\varepsilon = \int_{\mathbb{R}} p^\varepsilon(t, x, \lambda) \Psi(\lambda) \Theta'(F(\lambda)) F'(\lambda) \, d\lambda$$

and

$$C^\varepsilon = \int_{\mathbb{R}} p^\varepsilon(t, x, \lambda) \Phi(F(\lambda)) \Theta'(F(\lambda)) F'(\lambda) \, d\lambda$$

Passing to the weak\(^*\) limit $\varepsilon \to 0$ in these three terms, we obtain

$$q(t, x, \xi) \int_{\mathbb{R}} p(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) \, d\lambda + q(t, x, \eta) q(t, x, \xi) =$$

$$= \chi(\xi) \int_{\mathbb{R}} p(t, x, \lambda) \delta_{\xi=F(\lambda)}(\xi) F'(\lambda) \, d\lambda + \chi_{\eta}(\xi) \int_{\mathbb{R}} p(t, x, \lambda) \delta_{\eta=F(\lambda)}(\eta) (1 + F'(\lambda)) \, d\lambda$$

$$+ \int_{\mathbb{R}} p(t, x, \lambda) \chi_{\lambda}(\tau) \delta_{\eta=F(\tau)}(\eta) \delta_{\xi=F(\lambda)}(\xi) F'(\lambda) \, d\lambda \, d\tau$$

understood in the sense of distributions. Using \([4]\) in Lemma \([1]\) we simplify \((4.12)\) to \((4.11)\). \(\square\)
5. Proof of the Young measure representation

We are in position to prove the representation of \( u \) by Young measures as stated in Theorem 2.3. It turns out, that equations (4) in Lemma 4.1 and (4.11) completely characterize the kinetic function \( q \) (and \( p \)). The proof is based on the identity (4.11) with fixed \( (t, x) \in (0, T) \times \Omega \) so we simplify notations.

**Notation 5.1.** In this section, we fix \( (t, x) \in (0, T) \times \Omega \) and write \( p(\xi) \) and \( q(\xi) \) for \( p(t, x, \xi) \) and \( q(t, x, \xi) \) respectively.

In order to translate distributional identity (4.11) into a functional one, we use the following

**Lemma 5.2** (Adjoint distribution to \( \delta_{\xi=F(\tau)}(\xi) \)). There exists a distribution

\[
\delta^*_{\xi=F(\tau)}(\tau) := \sum_{i=1}^{3} \delta_{\tau=S_i(\xi)}(\tau) |S'_i(\xi)|
\]

with \( S_1, S_2, S_3 \) defined in Notation 2.1, such that, for all test functions \( \psi, \Theta \),

\[
\int_{\mathbb{R}} \Theta(\tau) \int_{\mathbb{R}} \Psi(\xi) \delta_{\xi=F(\tau)}(\xi) d\xi d\tau = \int_{\mathbb{R}} \Theta(\tau) \delta^*_{\xi=F(\tau)}(\tau) d\tau d\xi.
\]

**Proof.** By definition

\[
\int_{\mathbb{R}} \Theta(\tau) \int_{\mathbb{R}} \Psi(\xi) \delta_{\xi=F(\tau)}(\xi) d\xi d\tau = \int_{\mathbb{R}} \Theta(\tau) \Psi(F(\tau)) d\tau
\]

Let \( \{I_i\}_{i=1}^{3} \) and \( \{J_i\}_{i=1}^{3} \) be as in Notation 2.1. Then, we can integrate by substitution

\[
\int_{\mathbb{R}} \Theta(\tau) \Psi(F(\tau)) d\tau = \sum_{i=1}^{3} \int_{I_i} \Theta(\tau) \Psi(F(\tau)) d\tau = \sum_{i=1}^{3} \int_{J_i} \Theta(S_i(\xi)) \Psi(\xi) |S'_i(\xi)| d\xi.
\]

As inverses are extended by a constant to the whole of \( \mathbb{R} \) we can write

\[
\int_{\mathbb{R}} \Theta(\tau) \Psi(F(\tau)) d\tau = \sum_{i=1}^{3} \int_{\mathbb{R}} \Theta(S_i(\xi)) \Psi(\xi) |S'_i(\xi)| d\xi = \int_{\mathbb{R}} \Psi(\xi) \int_{\mathbb{R}} \Theta(\tau) \delta^*_{\xi=F(\tau)}(\tau) d\tau d\xi.
\]

\( \square \)

**Corollary 5.3.** Identity (4) in Lemma 4.1 can be also written explicitly as

(5.1) \[
q(\xi) = \sum_{i=1}^{3} (-1)^{i+1} p(S_i(\xi)) \mathbb{1}_{J_i}(\xi).
\]

This follows from Lemma 5.2 and an observation that \( F'(S_i(\lambda)) |S'_i(\lambda)| = (-1)^{i+1} \mathbb{1}_{J_i}(\lambda) \) where \( \{J_i\}_{i=3}^{3} \) are defined in Notation 2.1.
Lemma 5.4 (Explicit formulation of the kinetic identity). The functional identity (4.11) can be written explicitly as

\begin{equation}
[q(\xi) - \chi_\eta(\xi)] S(\eta) = \chi_\xi(\eta) q(\xi) + \chi_\eta(\xi) q(\eta) - q(\eta) q(\xi) + R(\eta, \xi),
\end{equation}

where

\begin{equation}
S(\eta) = \sum_{i=1}^{3} p(S_i(\eta)) |S'_i(\eta)|, \quad R(\eta, \xi) = \sum_{i=1}^{3} (-1)^{i+1} p(S_i(\xi)) \mathbb{1}_{J_i}(\xi) \sum_{j=1}^{3} \chi_{S_i(\xi)}(S_j(\eta)) |S'_j(\eta)|.
\end{equation}

Proof. Using adjoint distribution from Lemma 5.2, distributional identity (4.11) can be reformulated as follows. For a.e. \( \eta, \xi > 0 \), it holds

\[
q(\xi) \int_{\mathbb{R}} p(\lambda) \delta_{\eta=F(\lambda)}(\lambda) \, d\lambda = \chi_\eta(\xi) \int_{\mathbb{R}} p(\lambda) \delta_{\eta=F(\lambda)}(\lambda) \, d\lambda + \int_{\mathbb{R}} \int_{\mathbb{R}} p(\lambda) \chi_\lambda(\tau) F'(\lambda) \delta_{\eta=F(\tau)}(\tau) \delta_{\xi=F(\lambda)}(\lambda) \, d\tau \, d\lambda + \chi_\xi(\eta) q(\xi) + \chi_\eta(\xi) q(\eta) - q(\eta) q(\xi).
\]

If we define

\[
S(\eta) = \int_{\mathbb{R}} p(\lambda) \delta_{F'(\tau)=\eta}(\tau) \, d\tau, \quad R(\eta, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}} p(\lambda) \chi_\lambda(\tau) F'(\lambda) \delta_{F'(\tau)=\eta}(\tau) \delta_{F(\lambda)=\xi}(\lambda) \, d\tau \, d\lambda,
\]

we obtain (5.3) and we just have to prove the claimed formulas for \( S(\eta) \) and \( R(\eta, \xi) \) as in (5.3).

Using Lemma 5.2, we have

\[
S(\eta) = \sum_{i=1}^{3} p(S_i(\eta)) |S'_i(\eta)|.
\]

For \( R(\eta, \xi) \) we additionally note that \( F'(S_i(\xi)) |S'_i(\xi)| = (-1)^{i+1} \mathbb{1}_{J_i}(\xi) \). Hence,

\[
R(\eta, \xi) = \sum_{i=1}^{3} p(S_i(\xi)) F'(S_i(\xi)) |S'_i(\xi)| \sum_{j=1}^{3} \chi_{S_i(\xi)}(S_j(\eta)) |S'_j(\eta)| = \sum_{i=1}^{3} (-1)^{i+1} p(S_i(\xi)) \mathbb{1}_{J_i}(\xi) \sum_{j=1}^{3} \chi_{S_i(\xi)}(S_j(\eta)) |S'_j(\eta)|.
\]

Formula for \( R(\eta, \xi) \) seems to be complicated. We compute its value for \( \eta \) and \( \xi \) in unstable region below.

Lemma 5.5 (\( R(\eta, \xi) \) in the unstable region). Let \( R(\eta, \xi) \) be defined with (5.3). Moreover, let \( \xi_1, \eta \) and \( \xi_2 \) be such that \( f_- < \xi_1 < \eta < \xi_2 < f_+ \). Then,

\begin{equation}
R(\eta, \xi_1) = [p(S_3(\xi_1)) - p(S_2(\xi_1))] \left[ S'_1(\eta) - S'_2(\eta) \right],
\end{equation}

where

\[
\mathbb{1}_{J_1}(\xi) = \chi_{S_1(\xi)}(S_1(\eta)) |S'_1(\eta)|.
\]
Hence, (5.3) and Remark 5.3 imply

Therefore, we find

Similarly, Assumption 2.2-(2), see also Fig. 1, implies that

Let (5.5)

Therefore,

Hence, (5.3) implies

Similarly, Assumption 2.2-(2), see also Fig. 1 implies that

Therefore, we find

Hence, (5.3) and Remark 5.3 imply

Finally, we prove the following characterization result based on the nondegeneracy condition (3) in Assumption 2.2.

**Theorem 5.6** (Strong convergence of \( v^\varepsilon \) for non-degenerate \( F \)). Let \( F \) be as in Assumption 2.2 (in particular, it satisfies (3) in this assumption). Let \( p \) and \( q \) be given by (4.5). Then, there exists \( \alpha(t, x) \geq 0 \) such that \( q(t, x, \xi) = \chi_{\alpha(t, x)}(\xi) \), cf. equation (4.1). Consequently \( \alpha(t, x) = v(t, x) \) and \( v^\varepsilon(t, x) \) converges strongly in \( L^2((0, T) \times \Omega) \).
Proof. We know that \( p(\xi) \) and \( q(\xi) \) are bounded, nonnegative and compactly supported. Moreover, they vanish for \( \xi < 0 \). Consider the support of \( q \) denoted with \( \text{supp} \ q \). The proof is divided for three parts where we systematically increase possible support of \( q \).

Case 1: \( \text{supp} \ q \subset [0, f_-) \). Consider \( \eta \in (0, f_-) \) and \( \xi \) such that \( \xi > \eta \). We want to use (5.2). Notice that \( S'_1(\eta) > 0 \) and \( S'_2(\eta) = S'_3(\eta) = 0 \). Using (5.3) and Remark 5.3 we write
\[
S(\eta) = p(S'_1(\eta)) S'_1(\eta) = q(\eta) S'_1(\eta).
\]
Moreover, \( \chi_\eta(\xi) = 0 \). When it comes to \( R(\eta, \xi) \) we observe that there is only one \( \tau \) such that \( F(\tau) = \eta \), namely \( \tau = S_1(\eta) \). Moreover, as \( \xi > \eta \), \( S_1(\eta) < S_i(\xi) \) for \( i = 1, 2, 3 \). Therefore, (5.3) implies
\[
R(\eta, \xi) = S'_1(\eta) \sum_{i=1}^{3} (-1)^{i+1} p(S_i(\xi)) \mathbb{1}_{J_i}(\xi) = S'_1(\eta) q(\xi).
\]
Hence, (5.2) simplifies to
\[
q(\xi) q(\eta) S'_1(\eta) = q(\xi) S'_1(\eta) + q(\xi) - q(\xi) q(\eta)
\]
which can be rearranged to
\[
(q(\xi) q(\eta) - q(\xi)) (S'_1(\eta) + 1) = 0.
\]
As \( S'_1(\eta) > 0 \), this implies that for \( \eta \in (0, f_-) \) and \( \xi > \eta \) we have
\[
q(\xi) (q(\eta) - 1) = 0.
\]
Since \( q \leq 1 \) is non-increasing, it follows that \( q(\xi) = 1 \) with at most one jump from 1 to 0. If there is a jump, the result is proved and thus we now continue with the case \( q = 1 \) on \( (0, f_-) \).

Case 2: \( [0, f_-) \subset \text{supp} \ q \subset [0, f_+) \). We consider three points \( \xi_1, \eta \) and \( \xi_2 \) such that \( f_- < \xi_1 < \eta < \xi_2 < f_+ \). The proof in this case will be concluded if we demonstrate
\[
q(\xi_2) \neq 0 \implies q(\xi_1) = 1.
\]
Using (5.2) with \( (\xi_1, \eta) \) and \( (\xi_2, \eta) \) we obtain two equations
\[
[g(\xi_1) - 1] S(\eta) = R(\eta, \xi_1) + q(\eta) - q(\eta) q(\xi_1) = R(\eta, \xi_1) + q(\eta) (1 - q(\xi_1)),
\]
\[
q(\xi_2) S(\eta) = R(\eta, \xi_2) + q(\xi_2) - q(\eta) q(\xi_2).
\]
We multiply \((5.7)\) with \(q(\xi_2)\) and combine it with \((5.8)\) to deduce
\[
(q(\xi_1) - 1) \left( R(q, \xi_2) + q(\xi_2) \right) = R(q, \xi_1) q(\xi_2).
\]
Now, we use Lemma \(5.5\). Namely, we plug \((5.4)\) and \((5.5)\) above to discover identity
\[
(q(\xi_1) - 1) \left[ q(\xi_2) S'_1(q) + (S_3(q) - S'_3(q)) + q(\xi_2) \right] =
q(\xi_2) (p(S_3(q)) - p(S_2(q))) (S'_1(q) - S'_2(q)).
\]
It can be rewritten as
\[
(q(\xi_1) - 1) \left[ q(\xi_2) (S'_1(q) + 1) + (S_3(q) - 1 - S'_3(q)) \right] =
q(\xi_2) (p(S_3(q)) - p(S_2(q))) (S'_1(q) + 1 - S'_2(q) - 1).
\]
This can be seen as a linear equation for functions \(S'_1(q) + 1, S'_2(q) + 1\) and \(S'_3(q) + 1\) satisfied for \(q \in (\xi_1, \xi_2)\). Using \([3]\) in Assumption \(2.2\) we obtain that sum of the coefficients standing next to these functions vanish. Hence,
\[
(q(\xi_1) - 1) q(\xi_2) = 0 \quad (\xi_1 < \xi_2)
\]
which proves \((5.6)\).

Case 3: \([0, f_+) \subset \text{supp} \, q\). This is very similar to the first case. We consider \(q \in (f_+, \infty)\) and arbitrary \(\xi \in (0, \eta)\). Note that \(\chi_\alpha(\xi) = 1\). Moreover, \(S'_1(q) = S'_2(q) = 0\) and so,
\[
S(q) = p(S'_1(q)) S'_1(q) = q(q) S'_1(q).
\]
Finally, \(S_i(\xi) < S_3(\eta)\) for \(i = 1, 2, 3\). Therefore \(R(q, \xi) = 0\) and so, \((5.2)\) simplifies to
\[
q(\xi) q(\eta) S'_3(q) = S'_3(q) q(q) + q(q) - q(\xi) q(q).
\]
Since \(S'_3(q) > 0\), we deduce that for all \(q \in (f_+, \infty)\) and \(\xi < \eta\) we have
\[
q(q) (q(\xi) - 1) = 0.
\]
We conclude as in Case 1 and find that \(q(\xi) = \chi_\alpha(\xi)\).

Once we know that, it is easy to derive strong convergence. Since
\[
v(t, x) \overset{\star}{\rightarrow} v^*(t, x) = \int \chi_{\star\nu(t,x)}(\xi) d\xi \overset{\star}{\rightarrow} \int R q(t, x; \xi) d\xi = \int R \chi_{\star\alpha(t,x)}(\xi) d\xi = \alpha(t, x),
\]
we find that $\alpha(t, x) = v(t, x)$. Then, we have
\[(v^c(t, x))^2 = 2 \int \xi \, \chi_{v^c(t, x)}(\xi) \, d\xi \leq 2 \int \xi \, \chi_{v(t, x)}(\xi) \, d\xi = (v(t, x))^2\]
which implies strong convergence.

Now, we may conclude the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Using (4) in Lemma 4.1 we may write
\[\chi_{v(t,x)}(\xi) = \int p(t, x, \tau) \frac{\partial}{\partial \tau} \chi_{F(\tau)}(\xi) \, d\tau = -\int \frac{\partial p}{\partial \tau}(t, x, \tau) \chi_{F(\tau)}(\xi) \, d\tau.\]
Differentiating with respect to $\xi$ we obtain
\[\delta_{v(t,x)}(\xi) = -\int \frac{\partial p}{\partial \tau}(t, x, \tau) \delta_{\xi=F(\tau)}(\xi) \, d\tau.\]
Recalling that $\frac{\partial p}{\partial \tau}$ is nonpositive, for $F(\tau) \neq v(t, x)$ we conclude that $\frac{\partial p}{\partial \tau}(t, x, \tau) = 0$. In other words, $p(t, x, \tau)$ can only have non-increasing jumps at the three roots of $F(\tau) = v(t, x)$, i.e., $S_1(v(t, x))$, $S_2(v(t, x))$ and $S_3(v(t, x))$, see Fig. 1. Finally, because $p(\tau)$ decreases from 1 to 0, the three weights $\{\lambda_i(t, x)\}_{i=1,2,3}$ have to sum-up to 1 and the representation formula for $u$ in Theorem 2.3 is proved.

6. **Equation satisfied by weights $\lambda_1(t, x)$, $\lambda_2(t, x)$ and $\lambda_3(t, x)$**

In order to prove Theorem 2.5 we first connect the Young measure representation 2.1 from Theorem 2.3 with the kinetic function $p(t, x)$, function $p(t, x, \xi)$ has four jumps (see Fig. 3):
- from 0 to 1 at $\xi = 0$,
- from 1 to $\kappa_1(t, x)$ at $\xi = S_1(v(t, x))$ for some $\kappa_1(t, x) \in (0, 1)$,
- from $\kappa_1(t, x)$ to $\kappa_2(t, x)$ at $\xi = S_2(v(t, x))$ for some $\kappa_2(t, x) \in (0, \kappa_1(t, x))$,
- from $\kappa_2(t, x)$ to 0 at $\xi = S_3(v(t, x))$.

Once again from Lemma 4.2 we deduce that
\[\lambda_1(t, x) = 1 - \kappa_1(t, x), \quad \lambda_2(t, x) = \kappa_1(t, x) - \kappa_2(t, x), \quad \lambda_3(t, x) = \kappa_2(t, x).\]

Hence, to understand dynamics of weights $\lambda_1(t, x)$, $\lambda_2(t, x)$ and $\lambda_3(t, x)$, it is sufficient to study coefficients $\kappa_1(t, x)$ and $\kappa_2(t, x)$. Moreover, we have representation
\[(6.1) \quad p(t, x, \tau) = 1_{[0,S_1(v(t,x))]}(\tau) + \kappa_1(t, x) 1_{[S_1(v(t,x)),S_2(v(t,x))]}(\tau) + \kappa_2(t, x) 1_{[S_2(v(t,x)),S_3(v(t,x))]}(\tau).\]
we can write
\[ \| \text{from equation (1.2)} \text{ that} \]

Since sequences \( S_1(v(t,x)) \) satisfy PDE (4.8), we deduce
\[ \int_{\mathbb{R}} \delta_{\tau=S_1(v(t,x))}(\tau) \delta_{\xi=F(\tau)}(\xi) d\tau = \delta_{\xi=v(t,x)}(\xi) \]
because \( F(S_1(v(t,x))) = v(t,x) \). Therefore, plugging (6.2) into (4.8), we deduce
\[ (\Delta - \partial_t) q(t,x,\xi) + \delta_{\xi=v(t,x)}(\xi) \partial_t v(t,x) S'_1(v(t,x)) \]
\[ + \delta_{\xi=v(t,x)}(\xi) \sum_{i=1}^2 (S'_{i+1}(v(t,x)) - S'_i(v(t,x))) \kappa_i(t,x). \]

Since sequences \( \{ \partial_t v^\varepsilon \}_{\varepsilon \in (0,1)} \) and \( \{ \Delta v^\varepsilon \}_{\varepsilon \in (0,1)} \) are uniformly bounded in \( L^2((0,T) \times \Omega) \), we deduce from equation (1.2) that
\[ \| v^\varepsilon - F(u^\varepsilon) \|_{L^2((0,T) \times \Omega)} \leq C \varepsilon \]
for some constant $C$. Therefore, term $n_{2}^{2}(t,x,\xi)$ defined with (4.9) converges to 0 as $\varepsilon \to 0$ and measure $n(t,x,\xi)$ equals
\begin{equation}
(6.5) \quad n(t,x,\xi) = \delta_{v(t,x)}(\xi) |\nabla v(t,x)|^2.
\end{equation}

We claim now that (6.2) implies
\begin{equation}
(6.6) \quad \partial_{t}\kappa_{1}(t,x) = 0, \quad \partial_{t}\kappa_{2}(t,x) = 0
\end{equation}

which is equivalent to the assertion because $\lambda_{1} + \lambda_{2} + \lambda_{3} = 1$. To see (6.6), fix $(t_{0},x_{0})$ such that $v(t_{0},x_{0}) \in (f_{-},f_{+})$ and open neighbourhood $O_{t_{0},x_{0}}$ such that $v(t,x) < \xi_{0} < f_{+}$ in $O_{t_{0},x_{0}}$ for some $\xi_{0}$. Then, (6.3) for $(t,x,\xi) \in O_{t_{0},x_{0}} \times (\xi_{0},f_{+})$ boils down to
\begin{equation*}
\left[ \int_{S_{2}(v(t,x))} \delta_{\xi = F(\tau)}(\xi) \, d\tau \right] \partial_{t}\kappa_{1}(t,x) + \left[ \int_{S_{2}(v(t,x))} \delta_{\xi = F(\tau)}(\xi) \, d\tau \right] \partial_{t}\kappa_{2}(t,x) = 0
\end{equation*}

because $q(t,x,\xi) = \chi_{v(t,x)}(\xi)$. However, as $v(t,x) < \xi$, the second term vanishes because

\begin{equation*}
S_{2}(\xi) < S_{2}(v(t,x)) < S_{3}(v(t,x)) < S_{3}(\xi),
\end{equation*}

cf. Fig. 1. Hence, we deduce $\partial_{t}\kappa_{1}(t,x) = 0$ in $O_{t_{0},x_{0}}$. Equality $\partial_{t}\kappa_{3}(t,x) = 0$ follows from the similar reasoning - this time we need to localize equation so that $f_{-} < \xi < v(t,x)$.

\begin{remark}
One can prove (1) in Theorem 2.5 under weaker assumption on $\{\Delta v^{\varepsilon}\}_{\varepsilon \in (0,1)}$, namely that the sequence $\{\varepsilon^{1/2-\delta} \Delta v^{\varepsilon}\}_{\varepsilon \in (0,1)}$ is bounded in $L^{2}((0,T) \times \Omega)$ for some $\delta > 0$. Indeed, in this case we obtain
\begin{equation*}
\|v^{\varepsilon} - F(u^{\varepsilon})\|_{L^{2}((0,T) \times \Omega)} \leq C \varepsilon^{1/2+\delta}
\end{equation*}

instead of (6.4) and the same conclusion as in (6.5) follows concerning form of the measure $n(t,x,\xi)$. The assumption on $\{\partial_{t}v^{\varepsilon}\}_{\varepsilon \in (0,1)}$ is still necessary to guarantee existence of $\partial_{t}v$.
\end{remark}

We remark that identity (6.3) was obtained in [13, Appendix A]. We also note that derivation of (6.3) requires some smoothness of $v(t,x)$ and so, our PDE for kinetic functions (4.8) may be seen as a weak formulation of (6.3).

\begin{proof}[Proof of (2) in Theorem 2.5]
This time we need to be more careful as $\partial_{t}v$ can be understood only in the sense of distributions and computation (6.2) is no longer valid. Still we can write
\begin{equation}
(6.7) \quad \partial_{t}p(t,x,\tau) = \partial_{t} \mathbf{1}_{[0,S_{1}(v(t,x))]}(\tau) + \partial_{t} \{ \kappa_{1}(t,x) \mathbf{1}_{[S_{1}(v(t,x)),S_{2}(v(t,x))]}(\tau) \}
\end{equation}

\begin{equation*}
+ \partial_{t} \{ \kappa_{2}(t,x) \mathbf{1}_{[S_{2}(v(t,x)),S_{3}(v(t,x))]}(\tau) \}.
\end{equation*}

Therefore, plugging (6.7) into (4.8), we deduce
\[
\partial_t \int_{S_1(v(t,x))} \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + \partial_t \int_{S_2(v(t,x))} \kappa_1(t,x) \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + \\
+ \partial_t \int_{S_3(v(t,x))} \kappa_2(t,x) \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + \partial_t q(t,x,\xi) - \Delta x q(t,x,\xi) = \partial_\xi n(t,x,\xi).
\]

As in the proof of (1) above, we localize around \((t,x,\xi)\) such that \(v(t,x) < \xi\). In this case,
\[
S_1(v(t,x)) < S_1(\xi) < S_2(\xi) < S_2(v(t,x)) < S_3(v(t,x)) < S_3(\xi)
\]
so that \(\int_{S_1(v(t,x))} \delta_{\xi=F(\lambda)}(\xi) \, d\lambda = 0\) and \(\int_{S_3(v(t,x))} \kappa_2(t,x) \delta_{\xi=F(\lambda)}(\xi) \, d\lambda = 0\). Therefore,
\[
\partial_t \int_{S_2(v(t,x))} \kappa_1(t,x) \delta_{\xi=F(\lambda)}(\xi) \, d\lambda = \partial_\xi n(t,x,\xi)
\]
holds in any open set where \(f_- < v(t,x) < \xi_0 < \xi < f_+\) for some \(\xi_0\). This proves (2.3). In a similar way we localize around \(\xi < v(t,x)\) and deduce (2.4).

Understanding dynamics of weights and deducing more information from (2.3) and (2.4) is one of the three open problems discussed in Section 7.2.

7. Concluding remarks

7.1. Connection with Plotnikov’s approach. Our work uses ideas from a seminal paper by Plotnikov [36] who studied another regularization
\[
\partial_t w^\varepsilon = \Delta A(w^\varepsilon) + \varepsilon \Delta (\partial_t w^\varepsilon)
\]
of the forward-backward problems \(\partial_t w = \Delta A(w)\) where \(A\) has a similar monotonicity profile as presented in Fig. 1 for function \(F\). Below we summarize his argument adapted to our system (1.1)–(1.2) and emphasize the differences between his and our approach. We remark that Plotnikov worked with Young measures and obtained identities for measures of arbitrary sets rather than functional identities as (4.11). Nevertheless, we believe that the equivalent approach of kinetic formulation allows to simplify the reasoning and bring new information.

Here, following [36], we assume additionally that \(F'(u) > -1\) and we define functions
\[
I(u) := u + F(u), \quad A(w) = F(I^{-1}(w)).
\]
Note that $I$ is bijective so function $A$ has the same monotonicity profile as function $F$. If we let $w^\varepsilon = u^\varepsilon + v^\varepsilon$, we deduce from (1.1)–(1.2) that

\begin{equation}
\partial_t w^\varepsilon = \Delta v^\varepsilon
\end{equation}

and the connection between (7.1) and (7.3) comes from an observation that $v^\varepsilon - A(w^\varepsilon) \to 0$ in $L^2((0,T) \times \Omega)$. Indeed, it is sufficient to write

\begin{equation}
v^\varepsilon - A(w^\varepsilon) = v^\varepsilon - A(u^\varepsilon + F(u^\varepsilon)) + A(u^\varepsilon + F(u^\varepsilon)) - A(w^\varepsilon)
\end{equation}

and use the strong convergence of $v^\varepsilon - F(u^\varepsilon) \to 0$ from Corollary 3.2.

Plotnikov works in variables $(w^\varepsilon, A(w^\varepsilon))$ rather than with $(u^\varepsilon, v^\varepsilon)$ as in this paper. Let

\[ k^\varepsilon(t, x, \xi) = \chi_{w^\varepsilon(t, x)}(\xi), \quad l^\varepsilon(t, x, \xi) = \chi_{A(w^\varepsilon(t, x))}(\xi). \]

To obtain a PDE satisfied by the weak* limits of the kinetic functions $k^\varepsilon$ and $l^\varepsilon$ as in Theorem 4.3, Plotnikov introduces functions

\begin{equation}
G(\lambda) = \int_0^\lambda g(A(\tau)) \, d\tau, \quad H(\lambda) = \int_0^{A(\lambda)} h(\tau) \, d\tau.
\end{equation}

where $g$ and $h$ are smooth test functions. Using chain rule and (7.3), we obtain

\[ \partial_t G(w^\varepsilon) = g(A(w^\varepsilon)) \Delta v^\varepsilon = g(v^\varepsilon) \Delta v^\varepsilon + (g(A(w^\varepsilon)) - g(v^\varepsilon)) \Delta v^\varepsilon = \Delta_x \tilde{G}(v^\varepsilon) - g'(v^\varepsilon) |\nabla v^\varepsilon|^2 + (g(A(w^\varepsilon)) - g(v^\varepsilon)) \Delta v^\varepsilon. \]

where $\tilde{G}$ is a primitive function of $g$. This leads to PDE

\begin{equation}
\partial_t \int_\mathbb{R} k(t, x, \tau) \delta_{\xi = A(\tau)}(\xi) \, d\tau - \Delta_x l(t, x, \xi) = \partial_x m(t, x, \xi).
\end{equation}

where $k$ and $l$ are weak* limits of functions $k^\varepsilon$ and $l^\varepsilon$ while $m$ is a weak* limit of the sequence

\[ m^\varepsilon(t, x, \xi) = \delta_{\xi = w^\varepsilon(t, x)}(\xi) |\nabla v^\varepsilon(t, x)|^2 + \int_0^1 \delta_{\xi = s A(w^\varepsilon + (1-s)w^\varepsilon)}(\xi) \, ds \, (v^\varepsilon - A(w^\varepsilon)) \Delta v^\varepsilon. \]

It is a little bit surprising that it is not clear what is the sign of measure $m$ while the measure $n$ from PDE (4.8) is nonnegative. It is even more mysterious if we realize that left-hand sides of limiting equations (4.8) and (7.6) are exactly the same. This is the content of the following lemma.

**Lemma 7.1.** Let $k$, $l$, $p$ and $q$ be the weak* limits of kinetic functions $k^\varepsilon(t, x, \xi) = \chi_{w^\varepsilon(t, x)}(\xi)$, $l^\varepsilon(t, x, \xi) = \chi_{A(w^\varepsilon(t, x))}(\xi)$, $p^\varepsilon(t, x, \xi) = \chi_{w^\varepsilon(t, x)}(\xi)$ and $q^\varepsilon(t, x, \xi) = \chi_{v^\varepsilon(t, x)}(\xi)$. Then,

1. $l(t, x, \xi) = q(t, x, \xi)$,
observe that

\[ k(t, x, \xi) = p(t, x, I^{-1}(\xi)), \]

(2) \[ \int_R k(t, x, \tau) \delta_{\xi=\lambda}(\xi) \, d\tau = \int_R p(t, x, \lambda) \delta_{\xi=F(\lambda)}(\xi) \, d\lambda + q(t, x, \xi). \]

In particular, (1) and (3) imply that left-hand sides of PDEs (4.8) and (7.6) coincide.

\[ \text{Proof.} \] Property (1) follows from strong convergence of \( v^\varepsilon \) — \( A(w^\varepsilon) \to 0 \) cf. (7.4). To see (2), we first observe that \( I(u^\varepsilon) - w^\varepsilon \to 0 \) strongly. Hence,

\[ k(t, x, \xi) = \int_R q(t, x, \lambda) I'(\lambda) \delta_{\xi=I(\lambda)}(\xi) \, d\lambda \]

in the sense of distributions. Note that \( I \) is assumed to be invertible so we may transform this distributional identity into the pointwise one. For any test function \( \psi(\xi) \) we have

\[ \int_R \psi(\xi) k(t, x, \xi) \, d\xi = \int_R q(t, x, \lambda) I'(\lambda) \psi(I(\lambda)) \, d\lambda = \int_R q(t, x, I^{-1}(\xi)) I'(I^{-1}(\xi)) \psi(\xi) (I^{-1})'(\xi) \, d\xi = \int_R \psi(\xi) q(t, x, I^{-1}(\xi)) \, d\xi. \]

To prove (3), we note that function \( A \) has three inverses \( R_i(\xi) := I(S_i(\xi)) \) where \( 1 \leq i \leq 3 \). By chain rule,

\[ R'_i(\xi) = I'(S_i(\xi)) S'_i(\xi) = (1 + F'(S_i(\xi))) S'_i(\xi) = S'_i(\xi) + 1_{J_i}(\xi) \]

because \( S_i \) are inverses of \( F \) on \( J_i \) and so, \( F'(S_i(\xi)) S'_i(\xi) = 1_{J_i}(\xi) \). Moreover, by virtue of Lemma 5.2 and Corollary 5.3 we have

\[ l(t, x, \xi) = \sum_{i=1}^3 (-1)^{i+1} k(t, x, R_i(\xi)) 1_{J_i}(\xi), \quad \int_R k(t, x, \tau) \delta_{\xi=A(\tau)}(\xi) \, d\tau = \sum_{i=1}^3 k(t, x, R_i(\xi)) |R'_i(\xi)|. \]

Using these facts, we write

\[ \int_R k(t, x, \tau) \delta_{\xi=A(\tau)}(\xi) \, d\tau = \sum_{i=1}^3 (-1)^{i+1} k(t, x, R_i(\xi)) R'_i(\xi) = \sum_{i=1}^3 (-1)^{i+1} k(t, x, R_i(\xi)) S'_i(\xi) + \sum_{i=1}^3 (-1)^{i+1} k(t, x, R_i(\xi)) 1_{J_i}(\xi) = \sum_{i=1}^3 (-1)^{i+1} p(t, x, I^{-1}(R_i(\xi))) S'_i(\xi) + l(t, x, \xi) = \sum_{i=1}^3 (-1)^{i+1} p(t, x, S_i(\xi)) S'_i(\xi) + q(t, x, \xi) = \int_R p(t, x, \tau) \delta_{\xi=F(\tau)}(\xi) \, d\tau + q(t, x, \xi). \]
Lemma 7.1 implies that the next steps in identification of the limit in our paper and in the work of Plotnikov are equivalent. However, working directly with \((u^\varepsilon, v^\varepsilon)\) allows to formulate equation for kinetic function with nonnegative measure. This is useful to gain more information on weights \(\lambda_1(t, x), \lambda_2(t, x)\) and \(\lambda_3(t, x)\) from equations (2.3) and (2.4). Another advantage of our approach is that it does not require assumption \(F'(u) > -1\).

7.2. Open problems and further perspectives. We list here three problems connected to our work. For now, their treatment seems to be unavailable for us.

Problem 1: fast reaction limit for general reaction-diffusion system. System (1.1)–(1.2) studied in this paper is a special case of

\begin{align}
\partial_t u^\varepsilon &= d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}, \\
\partial_t v^\varepsilon &= d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}
\end{align}

for some \(d_1, d_2 \geq 0\). Using refined energy estimates from [25], fast reaction limit was established in [26, Theorem 2.9] for two special cases

\begin{align}
d_2 &\geq d_1, \quad F'(u) + \frac{d_1}{d_2} > 0, \quad \text{or} \quad d_1 > d_2, \quad F'(u) + \frac{d_2}{d_1} > 0.
\end{align}

More precisely, it was proved that \(w^\varepsilon := u^\varepsilon + v^\varepsilon\) converges strongly to the solution of

\begin{align}
\partial_t w - \Delta A(w) = 0, \quad \frac{\partial}{\partial n} w = 0
\end{align}

where

\[ A(w) = d_1 u + d_2 F(u) \quad \text{with} \quad w = u + F(u). \]

Function \(A\) is well-defined because conditions (7.9) imply that \(F'(u) > -1\). Limiting equation (7.10) is a consequence of summing up (7.7)–(7.8) together with a priori estimates that gives strong convergence of \(u^\varepsilon + v^\varepsilon \to u + v\) and \(v^\varepsilon - F(u^\varepsilon) \to 0\). However, if one only assumes \(F'(u) > -1\) without (7.9), the only available energy estimate is

\[
\frac{d}{dt} \int_\Omega \left[ \tilde{F}(u^\varepsilon) + \frac{1}{2} (v^\varepsilon)^2 + \varepsilon d_1 |\nabla u^\varepsilon|^2 + \frac{d_2^2 + d_1 d_2}{2 (d_2 - d_1)^2} (w^\varepsilon)^2 \right] dx =
\]

\[
= -\varepsilon \int_\Omega \left( d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon} \right)^2 dx - \frac{1}{d_2 - d_1} \int_\Omega |d_1 \nabla u^\varepsilon + d_2 \nabla v^\varepsilon|^2 dx
\]
where $\tilde{F}$ is a primitive function of $F$. This equality is too weak to deduce any strong convergence. The only result we can prove in that case is that, setting $w^\varepsilon = u^\varepsilon + v^\varepsilon$ and $z^\varepsilon = d_1 u^\varepsilon + d_2 v^\varepsilon$, we have
\[ w^\varepsilon \rightharpoonup w := u + v, \quad z^\varepsilon \rightharpoonup z := d_1 u + d_2 v, \quad v^\varepsilon - F(u^\varepsilon) \rightharpoonup 0, \quad w_t = \Delta z \]
but it is not clear at all what is the coupling between functions $w$ and $z$.

Problem 2: nondegeneracy condition. Strong convergence $v^\varepsilon \to v$ in our work is rather unavailable to be obtained from a priori estimates. It is a consequence of careful analysis of kinetic function (or Young measure) in Theorem 5.6 and nondegeneracy condition (3) in Assumption 2.2. This technical assumption excludes piecewise affine functions $F$ and is hard to verify for particular examples as one needs to know inverses of $F$ explicitly. On the other hand, this type of condition is a common assumption in papers concerning mostly regularization of forward-backward parabolic problems [30, 36] but also some hyperbolic equations with nonmonotone model functions [1]. We would like to know whether nondegeneracy assumption can be waived and if not, what happens with solutions to (1.1)–(1.2) in the case of piecewise affine function $F$.

Problem 3: understanding equation on weights $\lambda_1$, $\lambda_2$ and $\lambda_3$. In Section 6 we proved equations (2.3) and (2.4) that carry some information on the weights in the decomposition (2.1). It is not clear what is the information hidden in this equality. For instance, is it possible to determine asymptotic values of $\{\lambda_i(t, x)\}_{i=1, 2, 3}$? Some information can be gained from the sign of $n$. For example, for $\psi(\xi) \geq 0$, we can test (2.3) with $\Psi(\xi) = \int_0^\xi \psi(\eta) \, d\eta$ to deduce
\[ \partial_t \left[ \kappa_1(t, x) \int_{S_1(v(t,x))}^{S_2(v(t,x))} \Psi(F(\lambda)) \, d\lambda \right] = - \int_{\mathbb{R}} \psi(\xi) \, dm(t, x, \xi) \leq 0 \]
so that the function $t \mapsto \kappa_1(t, x) \int_{S_1(v(t,x))}^{S_2(v(t,x))} \Psi(F(\lambda)) \, d\lambda$ is nonincreasing. However, it is not clear how $\kappa_1(t, x)$ interacts with $\int_{S_1(v(t,x))}^{S_2(v(t,x))} \Psi(F(\lambda)) \, d\lambda$ to gain more information from that.

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