Nested Coordinate Systems in Geometric Algebra

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Abstract
A nested coordinate system is a reassigning of independent variables to take advantage of geometric or symmetry properties of a particular application. Polar, cylindrical and spherical coordinate systems are primary examples of such a regrouping that have proved their importance in the separation of variables method for solving partial differential equations. Geometric algebra offers powerful complimentary algebraic tools that are unavailable in other treatments.

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0 Introduction

Geometric algebra $\mathbb{G}_3$ is the natural generalization of Gibbs-Heaviside vector algebra, but unlike the latter, it can be immediately generalized to higher dimensional geometric algebras $\mathbb{G}_{p,q}$ of a quadratic form. On the other hand, Clifford analysis, the generalization of Hamilton’s quaternions, is also expressed in Clifford’s geometric algebras [1]. The main purpose of this article is to formulate the concept of a nested coordinate system, a generalization of the well-known methods of orthogonal coordinate systems to apply to any coordinate system. We restrict ourselves to the geometric algebra $\mathbb{G}_3$ because of its close relationship to the Gibbs-Heaviside vector calculus [3]. This restriction also draws attention to the clear advantages of geometric algebra over the later, because of its powerful associative algebraic structure.

The idea of a nested rectangular coordinate system arises naturally when studying properties of polar coordinates in the 2 and 3-dimensional Euclidean vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. We begin by discussing the relationship between ordinary polar coordinates and the nested rectangular coordinate system $N_{1,2}$, before going on to the higher dimensional nested coordinate system $N_{1,2,3}$ utilized in the reformulation of cylindrical and spherical coordinates. A detailed
discussion of the geometric algebra $\mathbb{G}_3$ is not given here, but results are often expressed in the closely related well-known Gibbs-Heaviside vector analysis for the benefit of the reader.

1 Polar and nested coordinates systems

Let $\mathbb{G}_2 := \mathbb{G}(\mathbb{R}^2)$ be the geometric algebra of 2-dimensional Euclidean space $\mathbb{R}^2$. An introductory treatment of the geometric algebras $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ is given in [4, 5, 6]. Most important in studying the geometry of the Euclidean plane is the position vector $x := x[\hat{x}, \hat{x}] = x\hat{x}$ (1) expressed here as a product of its Euclidean magnitude $x$ and its unit direction, the unit vector $\hat{x}$. In terms of rectangular coordinates $(x_1, x_2) \in \mathbb{R}^2$,

$$x = x[x_1, x_2] = x_1e_1 + x_2e_2,$$

(2)

for the orthogonal unit vectors $e_1, e_2$ along the $x_1$ and $x_2$ axis, respectively. The advantage of our notation is that it immediately generalizes to 3 and higher dimensional spaces of arbitrary signature $(p, q)$ in any of the definite geometric algebras $\mathbb{G}_{p,q} := \mathbb{G}(\mathbb{R}^{p,q})$ of a quadratic form.

The vector derivative, or gradient in the Euclidean plane is defined by

$$\nabla := e_1\partial_1 + e_2\partial_2,$$

(3)

where $\partial_1 := \frac{\partial}{\partial x_1}$ and $\partial_2 := \frac{\partial}{\partial x_2}$ are partial derivatives [3, p.105]. Clearly,

$$e_1 = \partial_1 x = e_1 \cdot \nabla x, \quad e_2 = \partial_2 x = e_2 \cdot \nabla x.$$

Since $\nabla$ is the usual 2-dimensional gradient, it has the well-known properties

$$\nabla x = 2, \quad \text{and} \quad \nabla x = \hat{x}.$$

With the help of the product rule for differentiation,

$$2 = \nabla x = (\nabla x)\hat{x} + x(\nabla \hat{x}) = \hat{x}^2 + x(\nabla \hat{x}).$$

(4)

Since in geometric algebra $x^2 = x^2$, it follows that $\hat{x}^2 = 1$, so that for $x \in \mathbb{R}^2$,

$$\nabla \hat{x} = \frac{1}{x} \quad \text{and} \quad e_1 \cdot \nabla x = e_1 \cdot \hat{x} = \frac{x_1}{x}, \quad e_2 \cdot \nabla x = e_2 \cdot \hat{x} = \frac{x_2}{x}.$$

(5)

Similarly, $\nabla \hat{x} = \frac{e_1}{x}$ for $x \in \mathbb{R}^n$. This is the first of many demonstrations of the power of geometric algebra over standard vector algebra.

By a nested rectangular coordinate system $N_{1,2}(x_1, x[x_1, x_2])$, we mean

$$x = x\hat{x} = x[x_1, x] = x[x_1, x[x_1, x_2]].$$

Note in geometric algebra, unlike in standard vector analysis, we need not write $\nabla \cdot x = 2$. This has many important consequences in the development of the subject.
The grouping of the variables allows us to consider \( x_1 \) and \( x := \sqrt{x_1^2 + x_2^2} \) to be independent. The partial derivatives with respect to these independent variables is denoted by \( \hat{\partial}_1 := \frac{\partial}{\partial x_1} \) and \( \hat{\partial}_2 := \frac{\partial}{\partial x_2} \), the hat on the partial derivatives indicating the new choice of independent variables.

For polar coordinates \((x, \theta) \in \mathbb{R}^2\), for \( x := \sqrt{x_1^2 + x_2^2} \geq 0, 0 \leq \theta < 2\pi\), and \( \mathbf{x} := x[x, \theta] \),

\[
\mathbf{x} = x \hat{x}[\theta] = x(e_1 \frac{x_1}{x} + e_2 \frac{x_2}{x}) = x(e_1 \cos \theta + e_2 \sin \theta),
\]

where \( \cos \theta := \frac{x_1}{x} \) and \( \sin \theta := \frac{x_2}{x} \). Using (5),

\[
\nabla \hat{x} = \nabla \hat{x}[\theta] = (\nabla \theta) \frac{\partial \hat{x}}{\partial \theta} = \frac{1}{x} \iff \nabla \theta = \frac{1}{x} \frac{\partial \hat{x}}{\partial \theta} \nabla^2 \theta = 0,
\]

since

\[
\nabla \hat{x} = (\nabla \theta) \partial_\theta(e_1 \cos \theta + e_2 \sin \theta) = (\nabla \theta)(-e_1 \sin \theta + e_2 \cos \theta),
\]

and

\[
\nabla^2 \theta = -\hat{x} \frac{\partial}{\partial x} \hat{x} + \frac{1}{x}(\nabla \theta) \partial_\theta \hat{x} = -2 \left( \frac{\hat{x} \cdot (\partial_\theta \hat{x})}{x^2} \right) = 0.
\]

The \( \iff \) follows by multiplying both sides of the first equation by the unit vector \( \partial_\theta \hat{x} \), which is allowable in geometric algebra. Note also the use of the famous geometric algebra identity \( 2a \cdot b = (ab + ba) \) for vectors \( a \) and \( b \) [1, p.26].

The 2-dimensional gradient \( \nabla \),

\[
\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} = e_1 \partial_1 + e_2 \partial_2
\]

already defined in (3), and the Laplacian \( \nabla^2 \) is given by

\[
\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \partial_1^2 + \partial_2^2.
\]

In polar coordinates,

\[
\hat{\nabla} = (\nabla x) \hat{\partial}_x + (\nabla \theta) \hat{\partial}_\theta = \hat{x} \hat{\partial}_x + \frac{1}{x} (\hat{\partial}_\theta x) \hat{\partial}_\theta
\]

for the gradient where \( \hat{\partial}_\theta := \frac{\hat{\partial}}{\partial \theta} \), and since \( \hat{\nabla}^2 \theta = 0 \),

\[
\nabla^2 = \hat{\nabla} (\hat{x} \hat{\partial}_x + (\nabla \theta) \hat{\partial}_\theta) = \left( \hat{\nabla} \hat{x} + \hat{x} \cdot \hat{\nabla} \right) \hat{\partial}_x + \left( \nabla^2 \theta + (\nabla \theta) \cdot \nabla \right) \hat{\partial}_\theta
\]

\[
= \partial_1^2 + \frac{1}{x} \partial_2 + \frac{1}{x^2} \partial_\theta^2.
\]

for the Laplacian. The decomposition of the Laplacian (11), directly implies that Laplace’s differential equation is separable in polar coordinates.
When expressed in nested rectangular coordinates $N_{1,2}(x_1, x)$, the gradient $\nabla \equiv \hat{\nabla}$ takes the form

$$\hat{\nabla} := (\nabla x_1) \frac{\hat{\partial}}{\partial x_1} + (\nabla x) \frac{\hat{\partial}}{\partial x} = e_1 \hat{\partial}_1 + \hat{x} \hat{\partial}_x.$$  \hspace{1cm} (12)

Dotting equations (8) and (12) on the left by $e_1$ and $\hat{x}$ gives the transformation rules

$$\partial_1 = \hat{\partial}_1 + \frac{x_1}{x} \hat{\partial}_x, \quad \hat{x} \cdot \hat{\nabla} = \frac{x_1}{x} \hat{\partial}_1 + \hat{\partial}_x = \cos \theta \hat{\partial}_1 + \sin \theta \hat{\partial}_2.$$  

Using these formulas the nested Laplacian takes the form

$$\hat{\nabla}^2 = \hat{\partial}_1^2 + 2 \frac{x_1}{x} \hat{\partial}_x \hat{\partial}_1 + \frac{1}{x} \hat{\partial}_x^2 + \frac{1}{x} \frac{\partial}{\partial x} = -\hat{\partial}_1^2 + 2 \hat{\partial}_1 \hat{\partial}_1 + \frac{1}{x} \frac{\partial}{\partial x}.$$  \hspace{1cm} (13)

The unusual feature of the nested Laplacian is that it is defined in terms of both the ordinary partial derivative $\partial_1$ and the nested partial derivative $\hat{\partial}_1$. Whereas partial derivatives generally commute, partial derivatives of different types do not. For example, it is easily verified that

$$\partial_1 \hat{\partial}_1 x^2 = 2x_1, \quad \text{whereas} \quad \hat{\partial}_1 \hat{\partial}_1 x^2 = 4x_1.$$  

Because the mixed partial derivatives $\hat{\partial}_x \hat{\partial}_1$ occurs in (13), Laplace’s differential equation in the real rectangular coordinate system $N_{1,2}(x_1, x)$ is not, in general, separable. Indeed, suppose that a harmonic function $F$ is separable, so that $F = X_1 X$ for $X_1 = X_1[x_1], X = X[x]$. Using (13),

$$\frac{\hat{\nabla}^2 F}{X_1 X} = \frac{\hat{\partial}_1^2 X_1}{X_1} + \left( \frac{\hat{\partial}_x^2 X + \frac{1}{x} \hat{\partial}_x}{X} \right) \frac{X}{X} + 2 \left( \frac{x_1 \hat{\partial}_1 X_1}{X_1} \right) \frac{\hat{\partial}_x X}{x X} = 0.$$  \hspace{1cm} (14)

The last term on the prevents $F$ in general from being separable. However, it is easily checked that $F = k \frac{X}{x}$ is harmonic and a solution of (13). When $X_1[x_1] = kx_1$, it is easily checked that $\frac{x_1 \hat{\partial}_1 X_1}{X_1} = 1$. Letting $F = kx_1 X[x]$, and requiring $\hat{\nabla}^2 F = 0$, leads to the differential equation for $X[x]$.

$$3 \hat{\partial}_x X + x \hat{\partial}_x^2 X = 0,$$

with the solution $X[x] = c_1 \frac{1}{x} + c_2$. The simplest example of a harmonic function $F = X_1 X$ is when $X_1 = x_1$ and $X = \frac{1}{x^2}$. A graph of this function is shown in Figure 1.

2 Special harmonic functions in nested coordinates

Consider the real nested rectangular coordinate system $(x_1, x_p, x)$, defined by

$$N_{1,2,3} := \{(x_1, x_p, x) | x = x \hat{x} = x_1 e_1 + x_p \hat{x}_p + x \hat{x}\},$$
Figure 1: The harmonic 2-dimensional function $F = \frac{x_1}{x_1^2 + x_2^2}$ is shown.

where $x_p = \sqrt{x_1^2 + x_2^2} \geq 0$, $x = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$. In nested coordinates, the gradient $\nabla = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3$ takes the form

$$
\hat{\nabla} = (\hat{\nabla} x_1) \hat{\partial}_1 + (\hat{\nabla} x_p) \hat{\partial}_p + (\hat{\nabla} x) \hat{\partial}_x = e_1 \hat{\partial}_1 + \hat{x}_p \hat{\partial}_p + \hat{x} \hat{\partial}_x,
$$

(15)

where $\hat{\partial}_p := \frac{\partial}{\partial x_p}$. Formulas relating the gradients $\nabla$ and $\hat{\nabla}$ easily follow:

$$
\partial_1 = \hat{\partial}_1 + e_1 \cdot \hat{x}_p \hat{\partial}_p + e_1 \cdot \hat{x} \hat{\partial}_x = \hat{\partial}_1 + \frac{x_1}{x_p} \hat{\partial}_p + \frac{x_1}{x} \hat{\partial}_x
$$

(16)

$$
\partial_2 = e_2 \cdot \hat{x}_p \hat{\partial}_p + e_2 \cdot \hat{x} \hat{\partial}_x = \frac{x_2}{x_p} \hat{\partial}_p + \frac{x_2}{x} \hat{\partial}_x
$$

(17)

and

$$
\partial_3 = e_3 \cdot \hat{x} \hat{\partial}_x = \frac{x_3}{x} \hat{\partial}_x.
$$

(18)

For the Laplacian $\nabla^2$ in nested coordinates, with the help of (15),

$$
\hat{\nabla}^2 = \hat{\nabla}(e_1 \hat{\partial}_1 + \hat{x}_p \hat{\partial}_p + \hat{x} \hat{\partial}_x) = e_1 \cdot \hat{\nabla} \hat{\partial}_1 + \hat{\nabla} \cdot \hat{x}_p \hat{\partial}_p + \hat{\nabla} \cdot \hat{x} \hat{\partial}_x
$$
We now calculate the interesting expression

\[ e_1 \cdot \nabla \hat{\partial}_1 + \hat{\nabla} \cdot \hat{x}_p \hat{\partial}_p + \hat{\nabla} \cdot \hat{x} \hat{\partial}_x \]

\[ = (\hat{\partial}_1 + \frac{x_1}{x_p} \hat{\partial}_p + \frac{x_1}{x} \hat{\partial}_x)\hat{\partial}_1 + \left( \frac{1}{x_p} + \frac{x_1}{x_p} \hat{\partial}_1 + \hat{\partial}_p + \frac{x_1}{x} \hat{\partial}_p \right)\hat{\partial}_p \]

\[ + \left( \frac{2}{x} + \frac{x_1}{x} \hat{\partial}_1 + \frac{x_1}{x} \hat{\partial}_p + \hat{\partial}_x \right)\hat{\partial}_x \]

\[ = \hat{\partial}_1^2 + \hat{\partial}_p^2 + \hat{\partial}_x^2 + 2 \left( \frac{x_1}{x_p} \hat{\partial}_1 \hat{\partial}_p + \frac{x_1}{x} \hat{\partial}_1 \hat{\partial}_x + \frac{x_1}{x} \hat{\partial}_p \hat{\partial}_x \right) \]

\[ + \frac{1}{x_p} \hat{\partial}_p + \frac{2}{x} \hat{\partial}_x. \quad (19) \]

Another expression for the Laplacian in mixed coordinates is obtained with the help of \([16]\),

\[ \hat{\nabla}^2 = -\hat{\partial}_1^2 + \hat{\partial}_p^2 + \hat{\partial}_x^2 + 2 \left( \hat{\partial}_1 \hat{\partial}_1 + \frac{1}{x_p} \hat{\partial}_p \hat{\partial}_p \right) \]

\[ + \frac{2}{x} \hat{\partial}_1 + \frac{2}{x} \hat{\partial}_x. \quad (20) \]

Suppose \( F = F[x_1, x_p, x]. \) In order for \( F \) to be harmonic, \( \hat{\nabla}^2 F = 0. \) Assuming that \( F \) is separable, \( F = X_1[x_1]X_p[x_p]X_x[x], \) and applying the Laplacian \([20]\) to \( F \) gives

\[ \hat{\nabla}^2 F = (\hat{\partial}_1^2 X_1)X_p X_x + X_1 \left( (\hat{\partial}_p^2 + \frac{1}{x_p} \hat{\partial}_p) X_p \right) X_x + X_1 X_p \left( \frac{2}{x} \hat{\partial}_x X_x \right) \]

\[ + 2 \left( \frac{x_p \partial_p X_p}{x} \right) \frac{1}{x} \hat{\partial}_1 X_1 + \left( \frac{1}{x_p} \hat{\partial}_p X_x + X_p \hat{\partial}_1 X_x \right). \quad (21) \]

We now calculate the interesting expression

\[ \frac{\left( x_p \partial_p X_p \right) \left( \frac{1}{x} \hat{\partial}_1 X_1 + \left( \frac{1}{x_p} \hat{\partial}_p X_x + X_p \hat{\partial}_1 X_x \right) \right)}{X_1 X_p X_x} \]

\[ = \left( x_p \left( \partial_p \log X_p \right) \right) \left( \frac{1}{x} \hat{\partial}_1 \log X_x \right) + \frac{\hat{\partial}_1 \log (X_p X_x)}{X_1}. \]

In general, because of the last term in \([21]\), a function \( F = X_1 X_p X_x \) will not be separable. However, just as in the two dimensional case, there are 3-dimensional harmonic solutions of the form \( F = x_1^k x_p^m x^n. \) Taking the Laplacian \([19]\) of \( F, \) with the help of \([7]\), gives

\[ \hat{\nabla}^2 F = (2km + m^2) x^n x_1^k x_p^{m-2} + (-k + k^2) x^n x_1^{k-2} x_p^m \]

\[ + (2km + mn + n(1 + n)) x_1^{n-2} x_p^m = 0. \]

This last expression vanishes when the system of three equations,

\[ \{2km + m^2 = 0, \quad -k + k^2 = 0, \quad \text{and} \quad 2kn + mn + n(1 + n) = 0\}. \]

All of the distinct non-trivial harmonic solutions \( F = x_1^k x_p^m x^n \) are listed in the following Table.
Cylindrical and spherical coordinates

Cylindrical and spherical coordinates are examples of nested coordinates $N_{1,2}(\mathbb{R})$, and $N_{2,3}(\mathbb{R})$, respectively. For the first,

$$x = x[x_p, \theta, x_3] = x_p[x_p, \theta] + x_3[x_3],$$

where $x_p = x_p \hat{x}_p[\theta]$, $x_p = \sqrt{x_1^2 + x_2^2}$, and $x_3 = x_3 e_3$. Cylindrical coordinates $(x_p, \theta, x_3) \in \mathbb{R}^3 = \mathbb{R}^2 \cup \mathbb{R}^1$ are a decomposition of $\mathbb{R}^3$ into the polar coordinates $(x_p, \theta) \in \mathbb{R}^2$, already studied in Section 1, and $x_3 \in \mathbb{R}^1$. For spherical coordinates, $x_p = x_p \hat{x}_p[\theta]$ the same as in cylindrical and polar coordinates, and

$$x = x[x, \theta, \varphi] = x \hat{x}[\theta, \varphi]) = x(e_3 \cos \varphi + \hat{x}_p[\theta] \sin \varphi),$$

where

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \hat{x}[\theta, \varphi] = e_3 \cos \varphi + \hat{x}_p[\theta] \sin \varphi, \hat{x}_p[\theta] = e_1 \cos \theta + e_2 \sin \theta.$$

The basic quantities that define both cylindrical and spherical coordinates are shown in Figure 2.

The gradient $\hat{\nabla}$ and Laplacian $\hat{\nabla}^2$ for cylindrical coordinates are easily calculated. With the help of (7), (10), and (11),

$$\hat{\nabla} = (\hat{\nabla} x_p) \hat{\partial}_p + (\hat{\nabla} \theta) \hat{\partial}_\theta + (\hat{\nabla} x_3) \hat{\partial}_3$$

for the cylindrical gradient, and

$$\hat{\nabla}^2 = \hat{\nabla} \left( \hat{x}_p \hat{\partial}_p + (\hat{\nabla} \theta) \hat{\partial}_\theta + e_3 \hat{\partial}_3 \right) = \hat{\partial}_p^2 + \frac{1}{x_p} \hat{\partial}_p + \frac{1}{x_p^2} \hat{\partial}_\theta^2 + \hat{\partial}_3^2$$

for the cylindrical Laplacian. Letting $F[x] = X_p[x]X_\theta[\theta]X_3[x_3]$, the resulting equation is easily separated and solved by standard methods, resulting in three second order differential equations with solutions,

$$X_p[x_p] = k_1 J_n[\beta x_p] + k_2 Y_n[\beta x_p]$$

$$X_\theta[\theta] = k_3 \cos \theta + k_4 \sin \theta$$

$$X_3[x_3] = k_5 \cosh(\alpha(m - x_3)) + k_6 \sinh(\alpha(m - x_3)).$$
Figure 2: For cylindrical coordinates, \( \mathbf{x} = x_p \hat{x}_p[\theta] + x_3 \mathbf{e}_3 \). For spherical coordinates, \( \mathbf{x} = x(e_3 \cos \varphi + \hat{x}_p[\theta] \sin \varphi) \).

where \( J_n \) and \( Y_n \) are Bessel functions of the first and second kind. The constants are determined by the various boundary conditions that must be satisfied in different applications [8, p.254].

Turning to spherical coordinates \( (x, \theta, \phi) \in \mathbb{R}^3 \), the spherical gradient

\[
\hat{\nabla} = (\hat{\nabla}_x) \hat{x}_p + (\hat{\nabla}_\theta) \hat{\theta}_p + (\hat{\nabla}_\phi) \hat{\phi}_p = \hat{x} \hat{x}_p + \frac{1}{x} (\hat{\theta}_p \hat{x}_p) \hat{\theta}_p + \frac{1}{x} (\hat{\phi}_p \hat{x}_p) \hat{\phi}_p,
\] (26)

where from previous calculations for polar and cylindrical coordinates,

\[
(\hat{\nabla}_\theta) = \frac{1}{x_p} (\hat{\theta}_p \hat{x}_p), \quad (\hat{\nabla} \theta)^2 = \frac{1}{x^2}, \quad \hat{\nabla}^2 \theta = 0, \quad \hat{\nabla}_\phi = \frac{1}{x} \hat{\phi}_x, \quad (\hat{\nabla} \phi)^2 = \frac{1}{x^2},
\] (27)

Furthermore, since \( \hat{x} = \hat{x}[\theta, \phi] = e_3 \cos \varphi + \hat{x}_p[\theta] \sin \varphi \)

\[
\frac{2}{x} \hat{\nabla} \hat{x} = (\hat{\nabla} \theta)(\hat{\theta}_p \hat{x}_p) + (\hat{\nabla} \phi)(\hat{\phi}_p \hat{x}_p) = \frac{x_p}{x} (\hat{\theta}_p \hat{x}_p)(\hat{\theta}_p \hat{x}_p) + \frac{1}{x},
\]

it follows that

\[
(\hat{\theta}_p \hat{x}_p)(\hat{\phi}_p \hat{x}_p) = \frac{x_p}{x} \sin \varphi, \quad \text{and} \quad \hat{\nabla}^2 \varphi = \frac{x_3}{x^2 x_p}.
\]

That \( \hat{\nabla}^2 \varphi = \frac{x_p}{x^2 x_p} \) follows using (26) and (27),

\[
\hat{\nabla}^2 \varphi = \hat{\nabla} \left( \frac{1}{x} \hat{\phi}_p \hat{x}_p \right) = \left( \frac{x_p}{x} \hat{\nabla} + \frac{1}{x} \hat{\nabla} \right) \hat{\phi}_p \hat{x}_p
\]

8
\[
\begin{align*}
\mathbf{\nabla}^2 &= \mathbf{\nabla} \left( \mathbf{\hat{x}} \mathbf{\hat{x}} + (\mathbf{\hat{\theta}} \mathbf{\hat{\theta}} + (\mathbf{\hat{\varphi}} \mathbf{\hat{\varphi}}) \right) \\
&= \left( \mathbf{\hat{x}} + \mathbf{\hat{r}} \right) \mathbf{\hat{r}} + (\mathbf{\hat{\theta}} \cdot \mathbf{\nabla}) \mathbf{\hat{\varphi}} + (\mathbf{\nabla}^2 \mathbf{\hat{\varphi}} + (\mathbf{\nabla} \mathbf{\hat{\varphi}}) \cdot \mathbf{\nabla}) \mathbf{\hat{\varphi}} \\
&= \left( \mathbf{\hat{x}} + \frac{2}{x} \right) \mathbf{\hat{r}} x + \frac{1}{x^2} \mathbf{\hat{\theta}} x^2 + \frac{1}{x^2} \mathbf{\hat{\varphi}} x^2 \mathbf{\hat{\varphi}},
\end{align*}
\]
equivalent to the usual expression for the Laplacian in spherical coordinates [8, p.256].

Just as in cylindrical coordinates, the solution of Laplace’s equation in spherical coordinates is separable, \( F = X_x[x]X_\theta[\theta]X_\varphi[\varphi] \), resulting in three second order differential equations with solutions

\[
\begin{align*}
X_x[x] &= k_1 x^\beta + k_2 x^{-(\beta+1)}, \\
X_\theta[\theta] &= k_3 \cos n\theta + k_4 \sin n\theta, \\
X_\varphi[\varphi] &= k_5 P_n^m(\cos \varphi) + k_6 Q_n^m(\cos \varphi),
\end{align*}
\]
where \( P_n^m \) and \( Q_n^m \) are the Legendre functions of the first and second kind, respectively [8, p.258].

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References

[1] R. Ablamowicz, G. Sobczyk, Editors: \textit{Lectures on Clifford (Geometric) Algebras and Applications}, Birkhäuser, Boston 2003.

[2] E. Hobson, 1931, \textit{The theory of spherical and ellipsoidal harmonics}, Cambridge.

[3] J.E. Marsden, A.J. Tromba, \textit{Vector Calculus} 2nd Ed., Freeman and Company, San Francisco 1980.
[4] G. Sobczyk, Matrix Gateway to Geometric Algebra, Spacetime and Spinors, Independent Publisher November 2019. https://www.garretstar.com

[5] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, New York 2013.

[6] G. Sobczyk. Many early versions of my work can be found on arXiv, or on my website: https://www.garretstar.com

[7] S. Wolfram, *Mathematica*.

[8] Tyn Myint-U, *Partial Differential Equations of Mathematical Physics* 2nd Ed., North Holland, NY 1980.