The M2-M5 Brane System and a Generalized Nahm’s Equation

Anirban Basu\(^1\) and Jeffrey A. Harvey\(^2\)

Enrico Fermi Institute and Department of Physics, University of Chicago, 5640 S. Ellis Avenue, Chicago IL 60637, USA

Abstract

We propose an equation that describes M2-branes ending on M5-branes, and which generalizes the description of the D1-D3 system via Nahm’s equation. The simplest solution to this equation constructs the transverse geometry in terms of a fuzzy three sphere. We show that the solution passes a number of consistency checks including a calculation of the energy of the system, matching to the self-dual string solution in the M5-brane world volume, and a study of simple fluctuations about the ground state configuration. We write down certain terms in the effective action of multiple membranes, which includes a sextic scalar coupling.

\(^1\)email: basu@theory.uchicago.edu
\(^2\)email: harvey@theory.uchicago.edu
1 Introduction

It is possible to have branes ending on branes in both string and M theory [1–3]. In M theory, the boundary of an M2-brane ending on an M5-brane describes a BPS self–dual string soliton—the classical soliton solution to the five brane equations of motion has been constructed in [4]. Our aim is to try to analyze the membrane theory and understand the geometry when multiply coincident M2 branes end on a five brane.

Similar constructions have been made in string theory. One can have D-strings ending on D3-branes, and these BPS saturated Bion solutions have been constructed in [5, 6], as spikes in the world volume theory on the D3-branes. The boundary of $k$ coincident D-strings ending on $N$ D3-branes can be interpreted as a $k$ monopole solution in the $SU(N)$ gauge theory. From the point of view of the D-string world volume theory, the corresponding Bogomolnyi equation turns out to be the Nahm equation [9] for the moduli space of monopoles in the gauge theory [10] (see also [11–13]).

One can construct solutions to the D-string equations of motion which open up into a funnel representing the D3-brane [14], obtaining a match between the D3 and the D1 points of view. Here the transverse coordinates of the D-strings representing the D3 world volume define a noncommutative (fuzzy) two sphere [15]. (D0-branes when placed in a four form field strength background expand into a fuzzy two sphere as well [16].) Similar solutions for D-strings opening into D5-branes has been constructed [17] with the transverse coordinates forming a fuzzy four sphere [18, 19]. (For D-strings opening into D7-branes see [20].)

Our aim is to understand something similar for the case of multiple membranes ending on a fivebrane in M theory. The strategy is to construct “solutions” to the membrane world volume theory which represent this configuration and obtain a funnel–like solution representing a five brane growing out of the membranes. However, the world volume theory of multiple membranes is not known. So we proceed by using various analogies with D-brane systems, matching the solution with the soliton solution in [4] and making some consistency checks.

It should be noted that the matrices we describe here are very different from the ones in Matrix theory [21]. In Matrix theory, the large $N$ limit of the matrices gives a single membrane, while in our case we represent the degrees of freedom by $N$ by $N$ matrices.

It is not entirely clear to us whether these matrices should be thought of as the fun-

\footnote{Similar BPS bounds for dyons have been constructed in [7, 8].}
damental variables of the ultraviolet description of multiple membranes, or some effective
description which happens to capture the behavior of the M2-M5 brane system.

In the next section we remind the reader of Nahm’s equation and the D1-brane descrip-
tion of the D3-D1 system. We then propose a generalization to the M5-M2 system and
subject our proposal to a number of consistency checks, including matching to the self-dual
string solution on the M5-brane, an analysis of the energy of the system, and a study of
a particular mode representing transverse fluctuations of the membranes which we show
to be consistent with the interpretation of this configuration as membranes ending on an
M5-brane. We end with a summary and discussion of open problems.

2 Review of the D1-D3 system

Our proposal for the equation describing self-dual string solutions is based on a general-
ization of Nahm’s equation for magnetic monopoles. This in turn arises as the description
of the D1-D3 brane system from the point of view of the D1 world-volume theory [10]. In
this section we review the main facts about the D1-D3 system that we want to generalize
in our description of the M2-M5 brane system.

We first consider \(N\) coincident D1-branes ending on a single D3-brane. The D3 world-
volume description of this configuration was found in [5], [4], [6]. We take the D3 world-
volume to lie along the directions 0,1,2,3 and the D1 world-volume to lie along 0,9. The
solution corresponding to the stack of \(N\) D1-branes excites a magnetic field on the D3-brane
and one of the transverse scalars with a profile given by

\[
X^9 = N \pi \alpha^i / \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}. \tag{1}
\]

\(\alpha^i\) are the fundamental variables of the ultraviolet description of multiple membranes, or some effective
description which happens to capture the behavior of the M2-M5 brane system.

In the next section we remind the reader of Nahm’s equation and the D1-brane descrip-
tion of the D3-D1 system. We then propose a generalization to the M5-M2 system and
subject our proposal to a number of consistency checks, including matching to the self-dual
string solution on the M5-brane, an analysis of the energy of the system, and a study of
a particular mode representing transverse fluctuations of the membranes which we show
to be consistent with the interpretation of this configuration as membranes ending on an
M5-brane. We end with a summary and discussion of open problems.

2 Review of the D1-D3 system

Our proposal for the equation describing self-dual string solutions is based on a general-
ization of Nahm’s equation for magnetic monopoles. This in turn arises as the description
of the D1-D3 brane system from the point of view of the D1 world-volume theory [10]. In
this section we review the main facts about the D1-D3 system that we want to generalize
in our description of the M2-M5 brane system.

We first consider \(N\) coincident D1-branes ending on a single D3-brane. The D3 world-
volume description of this configuration was found in [5], [4], [6]. We take the D3 world-
volume to lie along the directions 0,1,2,3 and the D1 world-volume to lie along 0,9. The
solution corresponding to the stack of \(N\) D1-branes excites a magnetic field on the D3-brane
and one of the transverse scalars with a profile given by

\[
X^9 = N \pi \alpha^i / \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}. \tag{1}
\]

\(\alpha^i\) are the fundamental variables of the ultraviolet description of multiple membranes, or some effective
description which happens to capture the behavior of the M2-M5 brane system.

In the next section we remind the reader of Nahm’s equation and the D1-brane descrip-
tion of the D3-D1 system. We then propose a generalization to the M5-M2 system and
subject our proposal to a number of consistency checks, including matching to the self-dual
string solution on the M5-brane, an analysis of the energy of the system, and a study of
a particular mode representing transverse fluctuations of the membranes which we show
to be consistent with the interpretation of this configuration as membranes ending on an
M5-brane. We end with a summary and discussion of open problems.
Choosing the $\alpha^i$ to be in the $N$-dimensional irreducible representation of $SU(2)$ with quadratic Casimir $C = N^2 - 1$, the solution, at fixed $X^9$, describes a fuzzy or non-commutative two sphere with physical radius

$$R = \sqrt{\frac{(2\pi\alpha')^2}{N}} \sum_i Tr[(X^i)^2] = \frac{(2\pi\alpha') \sqrt{(N^2 - 1)}}{2X^9}. \quad (5)$$

At large $N$ this matches exactly with the result derived from the D3 world-volume point of view.

In [14] other evidence is given that the D1 world-volume description gives a correct description of the D1 ending on a D3-brane. This includes matching of the energy of the configuration, and the existence of fluctuations in the overall transverse directions which fluctuate along the D1 and then out into the D3 and obey the correct higher-dimensional wave equation.

We now turn to our proposed generalization of Nahm’s equations and the description of the M2-M5-brane juncture from the M2 world-volume point of view. A different modification of Nahm’s equation relevant to the D2-D4 system has been studied in [22].

3 The M2-M5 system

3.1 A generalized Nahm’s equation

We first recall the description of $N$ M2-branes ending on an M5-brane from the M5 world-volume point of view. This is given by the self–dual string soliton constructed in [4]. This solution has one collective coordinate on the five brane world volume theory excited–it is the direction along which the membranes extend away from the five brane. If $s$ parametrizes this direction, then

$$s \sim \frac{N}{R^2}, \quad (6)$$

where $R$ is the distance in the five brane world volume. So this collective coordinate represents a “ridge” solution in the M5-brane theory. This solution is valid for large $R$, when the fields in the M5-brane theory are slowly varying. We now construct an ansatz for the membrane world volume theory valid near the core of the ridge (large $s$) which satisfies this condition. We shall later explain the validity of matching the solutions which are naively valid in different regimes.

To generalize Nahm’s equation to this situation we want an equation which has $SO(4)$ symmetry (rather than the $SO(3)$ symmetry of Nahm’s equation), which has translation
symmetry, which constructs the M5-brane using a fuzzy construction of the three-sphere, and which gives the correct result for the physical radius of the M2-M5 brane configuration as determined by the self-dual string soliton solution.

The construction of a fuzzy three-sphere is discussed in Appendix A following the original construction in [31]. The algebra is constructed in terms of representations of $\text{spin}(4) = SU(2) \times SU(2)$, $\mathcal{R}^+, \mathcal{R}^-$ given by the $\left(\frac{n+1}{4}, \frac{n-1}{4}\right)$ and $\left(\frac{n-1}{4}, \frac{n+1}{4}\right)$ representations respectively with $n$ an odd integer. The dimension of $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$ is $N = (n+1)(n+3)/2$.

The coordinates on the fuzzy $S^3$ are $N \times N$ matrices $G^i$ which map $\mathcal{R}^+$ to $\mathcal{R}^-$ and $\mathcal{R}^-$ to $\mathcal{R}^+$. The matrix $G_5$ is given in terms of the difference of projection operators onto these representations,

$$G_5 = \mathcal{P}_{\mathcal{R}^+} - \mathcal{P}_{\mathcal{R}^-}. \quad (7)$$

As discussed in [32], there are three closely related algebras related to the $G^i$. One is the algebra generated by taking arbitrary products of the $G^i$. The second is the algebra of $N$ by $N$ complex matrices, $\text{Mat}_N(C)$, which contains the first algebra as a subalgebra. Finally, there is a projection of the Matrix algebra which in the large $N$ limit agrees with the classical algebra of functions on $S^3$. It is not entirely clear to us which of these three algebras should be used in our construction. For the purposes of this paper we use $\text{Mat}_N(C)$ but keep in mind the possibility that future developments may require a refinement of this choice.

Taking the matrix M2-coordinates to be in $\text{Mat}_N(C)$, our proposed equation is then

$$\frac{dX^i}{ds} + \frac{\lambda M_3^3}{8\pi} \epsilon_{ijkl} \frac{1}{4!}[G_5, X^j, X^k, X^l] = 0, \quad (8)$$

where the quantum Nambu 4-bracket is defined by

$$[A_1, A_2, A_3, A_4] = \sum_{\text{permutations } \sigma} \text{sgn}(\sigma) A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} A_{\sigma_4}, \quad (9)$$

and $\lambda$ is an arbitrary parameter which we fix shortly. This structure has appeared in other attempts to define an odd quantum Nambu bracket [23], [24], [25].

Actually, for the solutions we will discuss, all $4!$ terms in the 4-bracket are equal, and so there are many equivalent ways that one could write the equation. The 4-bracket has the most natural mathematical form, but future developments may prefer some other form of the equation which is equivalent on the solution we describe below.
The equation (8) is manifestly $SO(4)$ invariant as a result of the $SO(4)$ action on the fuzzy $S^3$. It is also clearly invariant under translations taking $X^i \rightarrow X^i + v^i I$ with $I$ the identity matrix.

The equation (8) should be the Bogomolnyi equation for the membrane theory and should follow from the vanishing of the supersymmetry variation of the fermion fields, that is from the BPS condition.

We will study the equation (8) on the semi-infinite interval \( s \in (0, \infty) \) which is appropriate to semi-infinite M2-branes terminating on an M5-brane. In analogy to Nahm’s equation, it would also be interesting to study this equation on the finite interval \( s \in (-1, 1) \) which would correspond to finite M2-branes suspended between M5-branes. In this case the discussion below suggests that the proper boundary conditions are that

\[
X^i(s) \sim \frac{G^i}{\sqrt{s+1}},
\]

where the $G^i$ are defined in Appendix A.

### 3.2 Comparison to the Self-Dual String Solution

We now construct a solution of (8) that represents $N$ M2-branes ending on a single M5-brane. To do this we make an ansatz for $X^i$ in terms of a radial function and the coordinates on the fuzzy three sphere defined in Appendix A.

\[
X^i(s) = \frac{i\sqrt{2\pi}}{\sqrt{\lambda(n+2)sM_{11}^3}} f(s)G^i,
\]

where $M_{11}$ is the eleven dimensional Planck mass. Then, using the identities in Appendix A, it is easy to see that this solves (8) in the large $N$ limit provided that

\[
f(s) = \frac{1}{\sqrt{s}}.
\]

Sending $X^i \rightarrow G_5X^i$, we get (8) with a relative minus sign between the two terms, describing the solution for anti–self dual strings.

We can also define a distance $\hat{R}$ given by

\[
\hat{R} = \frac{\sqrt{2\pi}}{\sqrt{\lambda(n+2)sM_{11}^3}},
\]
such that $X^i = i\hat{R}G^i$. We also define the physical radius $R$ by

$$R = \sqrt{\frac{1}{N} |\text{Tr} \sum_i (X^i)^2|} = \sqrt{N|\hat{R}|}. \quad (14)$$

We would like the quantities $s$ and $R$ in (11) and (14) to have the same interpretation as the ones in (6). Now (14) yields

$$s = \frac{2\pi N}{\lambda(n+2)M_{11}^3 R^2}, \quad (15)$$

which is not of the form (5) for arbitrary $\lambda$.

We shall soon see that our analysis is valid for large $N$, and in this limit we will have the same scaling with $R$ and $N$ provided that $\lambda^2 N$ is held fixed at large $N$. Since $\lambda$ can be interpreted as the coupling in the theory (which will turn out to be classically marginal from the discussion below), this amounts to an analogue of the ‘t Hooft coupling being held fixed at large $N$.

### 3.3 Energy and Action

Given the Nahm equation, it is natural to define the energy of this static configuration by the expression

$$E = T_2 \int d^2\sigma \text{Tr} \left[ \left( \frac{dX^i}{ds} + \frac{\lambda M_{11}^3}{8\pi} \epsilon_{ijkl} G^5 X^j X^k X^l \right)^2 + \left( 1 - \frac{\lambda M_{11}^3}{16\pi} \epsilon_{ijkl} \left\{ \frac{dX^i}{ds}, G^5 X^j X^k X^l \right\} \right)^2 \right]^{1/2}, \quad (16)$$

where $T_2 = M_{11}^3/(2\pi)^2$ is the membrane tension and we have integrated over the space-like membrane worldvolume directions, where $\sigma_1 = \sigma$ and $\sigma_2 = s$. To make translation invariance explicit this equation should more properly be written in terms of Nambu brackets. However, these reduce to the above for our solution and hence we will simplify the analysis and presentation by assuming from now on that the equations we write apply to configurations for which $\{G^5, X^i\} = 0$.

From the generalized Nahm equation and the energy, we see that the $X^i$s can be interpreted as transverse coordinates to the membranes and (11) represents a funnel solution of the membranes opening into the fivebrane, while $s$ does represent the direction along the membranes and orthogonal to the fivebrane. The classical solution (11) is independent of
the direction along which the self dual string extends in the five brane world volume. In the BPS limit \( \hat{8} \), the energy becomes

\[
E = T_2 \int d^2 \sigma \mathrm{Tr} \left( 1 - \frac{\lambda M_{11}^3}{8 \pi} \epsilon_{ijkl} \frac{dX^i}{ds} G_5 X^j X^k X^l \right),
\]

and the energy density linearizes. The second term in this expression is a boundary term localized on the self dual string. In the large \( N \) limit using the various matrix identities, this yields

\[
E = NT_2 L \int ds + T_5 L \int 2\pi^2 dRR^3,
\]

where \( T_5 = \frac{M_{11}^6}{(2\pi)^5} \) is the five brane tension, and \( L \) is the length of the self dual string. The energy has separated into two contributions—the first term is from the \( N \) membranes while the second term is from the single five brane (both of these terms diverge because of infinite volume, but the energy densities have the correct values). The ansatz for the membrane is valid at the core, but as \( N \to \infty \), it matches the solution even away from the core. The fact that the energy becomes the sum of the two contributions rests on the fact that the BPS condition is obeyed, and is a consistency check for our ansatz.

From the non–linear expression for the energy \( \hat{18} \) (and also from certain terms in the action which we shall write down later), it is clear that a Taylor expansion in powers of \( X^i \) is possible only when \( M_{11}^6 \hat{R}^6 \ll 1 \), i.e., \( R \ll \sqrt{N}M_{11}^{-1} \). Thus, for \( N \to \infty \), \( R \) can be large as well. So in the large \( N \) limit, we have a region of overlap between the five-brane and the membrane descriptions, which justifies matching the solutions for these theories as discussed below \( \hat{3} \).

We can simplify the expression for the energy in \( \hat{18} \). We get that (dropping irrelevant factors)

\[
E = T_2 \int d^2 \sigma \mathrm{Tr} \left[ 1 + \left( \frac{dX^i}{ds} \right)^2 - \frac{\lambda^2}{4} Q^{ijk} H^{ijk} + \frac{\lambda^2}{16} \left( \frac{dX^i}{ds}, Q^{ijkl} \right) \right],
\]

where

\[
Q^{ijk} = \left\{ [X^i, X^j], X^k \right\}, \quad H^{ijk} = Q^{ijk} + Q^{kij} + Q^{kji}.
\]

Note that the projection matrix \( G_5 \) has dropped out of the expression. Though it is difficult to write down the complete action for multiple membranes, using \( \hat{19} \) we can write down
some of the terms in the action. Thus we get that

$$S = -T_2 \int d^3 \sigma \text{Tr} \left[ 1 + (\partial_a X^M)^2 - \frac{\lambda^2}{4} Q^{LMN} H^{LMN} + \frac{\lambda^2}{16} [\partial_a X^L, Q^{MNP}] \right. \times \left( [\partial^a X^L, H^{MNP}] + [\partial^a X^M, H^{LPN}] + [\partial^a X^N, H^{LMP}] + [\partial^a X^P, H^{LMN}] \right) + \ldots \right]^{1/2}, \quad (21)$$

where $X^M$ ($M = 1$ to 8) are the transverse coordinates of the membranes, $Q^{LMN} = \{ [X^L, X^M], X^N \}$, $H^{LMN} = Q^{LMN} + Q^{NLM} + Q^{MLN}$, and $\sigma^a$ ($a = 1$ to 3) are the world volume coordinates of the membranes.

### 3.4 Membrane Fluctuations

We now perform an analysis of the simplest possible fluctuations of the membranes. This will enable us to make a consistency check that this system indeed describes membranes ending on a five brane. The fluctuations we consider are the overall transverse fluctuations of the system—these are fluctuations $\delta X^m$ where $m$ runs over the four spatial indices transverse to both the membranes and the five brane. A related analysis of scattering in this system can be found in [26].

In spite of knowing only a few terms in the membrane action (21), we shall be able to make some statements about membrane fluctuations. Consistency checks with the system of D-strings ending on D3-branes will also be helpful. For the D-brane system, suppose that we did not know the exact DBI action for the D-strings, but only knew some of the terms in the action by considering D-strings ending on a D3-brane given in [17]. We could then try to write down a covariant form for these terms (as we have done in going from (19) to (21)), and hope to study linearized fluctuations of the D-strings [14]. Now in the linearized fluctuation analysis, we are essentially looking at the D-brane theory as dimensional reduction of ten dimensional super Yang–Mills, i.e., keeping only the quartic potential term along with the kinetic terms.

Similarly, in our case, we need to consider a generalization of the membrane action keeping the $O(X^6)$ potential term leading to

$$S = -T_2 \int d^3 \sigma \text{Tr} \sqrt{1 + (\partial_a X^M)^2 - \frac{\lambda^2}{4} Q^{LMN} H^{LMN}}. \quad (22)$$

Clearly the first two terms under the square root in the action (22) can be obtained from the expression

$$-T_2 \int d^3 \sigma \text{Tr} \sqrt{-\det(P[G]_{ab})}, \quad (23)$$
i.e., from the determinant of the pullback of the metric to the worldvolume coordinates. This is because, in flat space, in static gauge,

\[ P[G]_{ab} = \eta_{ab} + \partial_a X^M \partial_b X^M, \tag{24} \]

which gives rise to the kinetic terms in (22) when (24) is inserted in (23) (this is essentially the Nambu–Goto part of the membrane action). So we take (23) to generalize the kinetic terms in (22) (we shall discuss the potential term shortly). Using the classical solution \( X^i = \hat{R} G^i \), where \( \hat{R} = 1/\sqrt{s} \), we can compute (23) keeping up to terms quadratic in the fluctuations.

We consider the fluctuation \( \delta X^m(t, s, \sigma) = f^m(t, s, \sigma)1_N \), i.e., fluctuations proportional to the identity. Clearly these are the simplest fluctuations that we can possibly analyze. Here \( \sigma \) is the coordinate along the self-dual string. Then we get that

\[ \sqrt{-\det(P[G]_{ab})} = \sqrt{H - H(\partial_t f^m)^2 + (\partial_s f^m)^2 + H(\partial_\sigma f^m)^2}, \tag{25} \]

where

\[ H(s) = 1 + \frac{\pi N}{2M_1^3 s^3}. \tag{26} \]

(This actually resembles the form of the similar terms in the action for the D string theory.)

Now we consider the contribution of the potential term to the action. The part of it involving \( Q^{ijk} H^{ijk} \) contributes to the classical background and can possibly change the first term in (23), i.e., the coefficient of the \( 1/s^3 \) term in \( H \), but not the \( Hs \) multiplying \( (\partial_t f^m)^2 \) and \( (\partial_\sigma f^m)^2 \). So this will not change the equation of motion for fluctuations and is irrelevant for our purposes. \(^4\) Now let us turn to the contribution of the potential term to the fluctuations. Because \( \delta X^m = f^m 1_N \), we get that (keeping only terms quadratic in the fluctuations)

\[ Q^{LMN} H^{LMN} = 4(f^m)^2[X^i, X^j]^2. \tag{27} \]

Thus we get that

\[ S = -T_2 \int d^3\sigma \text{Tr} \sqrt{H - H(\partial_t f^m)^2 + (\partial_s f^m)^2 + H(\partial_\sigma f^m)^2 - \lambda^2(f^m)^2[X^i, X^j]^2}. \tag{28} \]

(The analogue of the potential term contributing to fluctuations proportional to the identity is absent for the D string theory.) We now proceed to evaluate \( [X^i, X^j]^2 = [G^i, G^j]^2/s^2 = 4(G^{ij})^2/s^2 \).

\(^4\)In the D-brane case, there is actually no change as this occurs as an overall factor—presumably the same is true here, which can be checked if the complete membrane action is known.
Using the definition (10), it is easy to show that \([(G^{ij})^2, G^{kl}] = 0\), and using the invariance of \((G^{ij})^2\) under chirality flip \(\Gamma_5 \rightarrow -\Gamma_5\), it follows that
\[
[G^i, G^j]^2 = 2G^i G^j G^i G^j - 2N^2 (P_{R+} + P_{R-}) \sim (P_{R+} + P_{R-}),
\]
(29)
where we have used (50). So we have to evaluate
\[
G^i G^j G^i G^j P_{R+} = \sum_{r,s,t,u=1}^n \rho_r(\Gamma^i P_-) \rho_s(\Gamma^j P_+) \rho_t(\Gamma^i P_-) \rho_u(\Gamma^j P_+) P_{R+}.
\]
(30)
In addition to the relations listed in (44), we also need the relations
\[
\sum_i (\Gamma^i \otimes \Gamma^i)(P_+ \otimes P_+)_{\text{sym}} = \sum_i (\Gamma^i \otimes \Gamma^i)(P_- \otimes P_-)_{\text{sym}} = 0.
\]
(31)
Thus we obtain that
\[
G^i G^j G^i G^j P_{R+} = (n+1)(n^2 - 2n - 3)P_{R+},
\]
(32)
finally leading to
\[
[G^i, G^j]^2 = -\frac{(n+1)(n^3 + 3n^2 + 23n + 21)}{2}.
\]
(33)
Inserting (33) into the action in (28) and keeping up to terms quadratic in the fluctuations, we obtain the equation for linearized fluctuations
\[
(H \partial_t^2 - \partial_s^2 - H \partial_{\sigma}^2) f^m(t, s, \sigma) + \lambda \frac{(n+1)(n^3 + 3n^2 + 23n + 21)}{2s^2} f^m(t, s, \sigma) = 0.
\]
(34)
Now our configuration should have two distinct interpretations in two distinct limits—as \(s \rightarrow \infty\), it should represent multiple membranes while as \(s \rightarrow 0\), it should represent the fivebrane, with the self-dual string in its world volume. We show that this structure is consistent with (34). As \(s \rightarrow \infty\), (34) trivially reduces to
\[
(-\partial_t^2 + \partial_s^2 + \partial_{\sigma}^2) f^m = 0,
\]
(35)
representing free plane waves propagating in the membrane world volume having \(SO(2,1)\) symmetry.

As in similar treatments of other brane systems [5,14], the \(s \rightarrow 0\) limit is strictly speaking outside the range of validity of our approximations, but nonetheless correctly matches on to the free wave equation we expect from the M5-brane point of view. To see this, we note that as \(s \rightarrow 0\) (i.e., as \(R \rightarrow \infty\)), the \(1/s^3\) term in \(H\) dominates and (34) yields
\[
(-\partial_t^2 + \partial_s^2) f^m + R^{-3} \frac{\partial}{\partial R} \left( R^3 \frac{\partial f^m}{\partial R} \right) = 0.
\]
(36)
This precisely represents free plane waves propagating in the five-brane world volume having $SO(1, 1) \times SO(4)$ symmetry due to the presence of the self-dual string.\(^5\) Thus the structure of the action, which was crucial in obtaining (34), is consistent with the interpretation of the system as membranes ending on a five brane.

Ideally one would like to do a general analysis of fluctuations of the fuzzy three sphere. However, apart from the fuzzy two sphere, it is difficult to do so for all other cases. For the fuzzy two sphere, the analysis was done by [27] where generic fluctuations were characterized by symmetric traceless polynomials in the equivalents of the $G^i$s (which satisfy the Lie algebra of $SU(2)$ for the fuzzy two sphere). However, for all other cases, the $G^i$s by themselves do not close to form a Lie algebra, and one has to add other matrices transforming in other representations as well to make the algebra close, and generic fluctuations involve polynomials built from all the matrices.

### 4 Discussion and Open Problems

We have proposed an equation that describes an M2-brane ending on an M5-brane from the M2 world-volume point of view. This generalizes Nahm’s equation which describes a D1-brane ending on a D3-brane.

In this equation the membrane degrees of freedom are represented by $N$ by $N$ matrices. On the other hand, there are various indications that $N$ membranes in M-theory have $N^{3/2}$ degrees of freedom. How can we reconcile this with our proposal?

The D1-brane theory is free in the ultraviolet and strongly coupled in the infrared. The derivation of Nahm’s equation as a BPS condition for configurations of the D1-brane follows from analysis of the classical action, and hence is strictly speaking only valid in the ultraviolet, that is for very short D1-branes, or equivalently for light monopoles (from the D3-brane point of view). However, the non-renormalization theorems of $N = 4$ supersymmetry guarantee that many results derived from the analysis of Nahm’s equation will continue to hold even in the infrared, that is for heavy monopoles.

We believe that something similar may be at work in the M2-brane system. The counting of $N^{3/2}$ degrees of freedom \cite{28,29} is a counting of the infrared degrees of freedom of the theory. This same result holds for D2-branes (see e.g. the discussion in section 6.1 of \cite{30}), and they clearly have a UV description in terms of matrix degrees of freedom. Similarly,

\(^5\)Note that the potential term in (34) drops out in both the $s \to \infty$ and the $s \to 0$ limits.
there may be a description of multiple M2-branes by matrices in the UV which flows to a superconformal theory in the IR with $N^{3/2}$ degrees of freedom. It is also possible that our description by matrices obeying the algebra of the fuzzy three-sphere is only some approximation to the correct description.

We have tried to reduce our system directly to the usual Nahm equation by reducing the M2-M5 system to the D1-D3 system through compactification and T-duality. Unfortunately we have not been able to obtain the Nahm equation in a straightforward way. It may be that the relationship between the UV description of M2 and D2-branes is more subtle than a direct identification of the two matrix descriptions.

It would be interesting to try to use our generalization of the Nahm equation for self-dual strings to study their moduli space, as has been done for monopoles. The Nahm data also allow one to reconstruct the monopole solution on the D3-brane world-volume. It would be interesting to see if our generalization could be used to give some clues as to the form of the non-Abelian tensor theory which governs M5-brane dynamics. As mentioned earlier, it would also be interesting to explore solutions to our equation on a finite interval which should represent finite length M2-branes suspended between M5-branes.

Acknowledgements

We would like to thank D. Berman, D. Kutasov and S. Sethi for useful discussions and N. Lambert for helpful correspondence. We would also like to thank D. Berman and N. Copland for pointing out an error in the earlier version of the paper. This work was supported in part by NSF Grant No. PHY-0204608.

A The Fuzzy Three Sphere

By analogy to the string theory constructions, it is natural to think that the four transverse coordinates to the self dual string parametrizing the five brane world volume form a fuzzy three sphere. The fuzzy three sphere has been constructed in [31] in the context of solving the equations of motion for fields in the world volume theory of non-BPS D0 branes in a background with non-vanishing five form flux in the IIB theory. The three sphere construction has been generalized for odd fuzzy spheres in [32, 33].

We review and derive a few results for the fuzzy three sphere which will be useful later. We follow the notation in [31–33]. Consider the $N \times N$ matrices $G^i$ ($i = 1$ to 4) where
\[ N = \frac{(n+1)(n+3)}{2}, \text{ and } n \text{ is an odd integer.} \]

The matrices are given by

\[ G^i = \mathcal{P}_{\mathcal{R}^+} \sum_{s=1}^{n} \rho_s (\Gamma^i P_-) \mathcal{P}_{\mathcal{R}^-} + \mathcal{P}_{\mathcal{R}^-} \sum_{s=1}^{n} \rho_s (\Gamma^i P_+) \mathcal{P}_{\mathcal{R}^+}. \tag{37} \]

Here

\[ \sum_{s=1}^{n} \rho_s (\Gamma^i) = (\Gamma^i \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes \Gamma^i)_{\text{sym}}, \tag{38} \]

where \( \text{sym} \) stands for the completely symmetrized \( n \)-fold tensor product representation of \( \text{spin}(4) \). Also \( P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5) \), and \( \mathcal{P}_{\mathcal{R}^+}, \mathcal{P}_{\mathcal{R}^-} \) are projection operators onto the irreducible representations \( \mathcal{R}^+, \mathcal{R}^- \) respectively of \( \text{spin}(4) \). Using \( \text{spin}(4) = SU(2) \times SU(2) \), \( \mathcal{R}^+, \mathcal{R}^- \) are given by the \( (\frac{n-1}{4}, \frac{n+1}{4}) \) and \( (\frac{n+1}{4}, \frac{n-1}{4}) \) representations respectively. Also consider the matrix \( G_5 \) given by

\[ G_5 = \mathcal{P}_{\mathcal{R}^+} - \mathcal{P}_{\mathcal{R}^-}. \tag{39} \]

(For \( n = 1 \), the matrices \( G^i \) and \( G_5 \) become \( \Gamma^i \) and \( \Gamma_5 \) respectively.) So the generators of \( \text{spin}(4) \) are given by

\[ G^{ij} = \frac{1}{2} [G^i, G^j] = \mathcal{P}_{\mathcal{R}^+} \left( \sum_r \rho_r (\Gamma^{ij} P_+ \rho_s (\Gamma^{ji} P_+) \mathcal{P}_{\mathcal{R}^+} + \mathcal{P}_{\mathcal{R}^-} \left( \sum_r \rho_r (\Gamma^{ij} P_- \rho_s (\Gamma^{ji} P_-) \mathcal{P}_{\mathcal{R}^-} \right) \right) \right) \tag{40} \]

In contrast to the case of even fuzzy spheres, for odd fuzzy spheres we must deal with a reducible representation \( \mathcal{R} = \mathcal{R}^+ \otimes \mathcal{R}^- \). The \( G^i \) are elements of \( \text{End}(\mathcal{R}) \). We can write \( G^i = G^i_+ + G^i_- \) with \( G^i_\pm = \frac{1}{2}(1 \pm G_5)G^i \) and then \( G^i_\pm \) act as homomorphisms from \( \mathcal{R}_\pm \) to \( \mathcal{R}_\mp \).

Using the above definitions, it follows that

\[ \epsilon_{ijkl}[G_5 G^i G^j G^k G^l, G^{mn}] = 0. \tag{41} \]

So, \( \epsilon_{ijkl}G_5 G^i G^j G^k G^l \) is proportional to the identity operator in each irreducible representation, and using the symmetry under chirality flip \( (\Gamma_5 \rightarrow -\Gamma_5) \), it follows that \( \epsilon_{ijkl}G_5 G^i G^j G^k G^l \) is proportional to the identity operator, i.e.

\[ \epsilon_{ijkl}G_5 G^i G^j G^k G^l \sim (\mathcal{P}_{\mathcal{R}^+} + \mathcal{P}_{\mathcal{R}^-}). \tag{42} \]
We now proceed to calculate the proportionality constant (which is a function of \( n \)). We have that
\[
\epsilon_{ijkl}G_iG^jG^kG^l P_{R^+} = \epsilon_{ijkl} \sum_{r,s,t,u=1}^n \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j P_+)\rho_t(\Gamma^k P_-)\rho_u(\Gamma^l P_+) P_{R^+}. \tag{43}
\]
The various terms in (43) can be simplified using the relations listed below. (A similar analysis was done to evaluate \((G^i)^2\) in [33].) The relevant relations are given by
\[
\begin{align*}
\sum_{r} \rho_r(P_+) P_{R^+} &= \frac{(n+1)}{2} P_{R^+}, \\
\sum_{r \neq s} \rho_r(P_-)\rho_s(P_+) P_{R^+} &= \frac{(n+1)(n-1)}{4} P_{R^+}, \\
\sum_{r \neq s} \rho_r(P_+)\rho_s(P_+) P_{R^+} &= \frac{(n+1)(n-1)}{4} P_{R^+}, \\
\sum_{r \neq s \neq t} \rho_r(P_-)\rho_s(P_-)\rho_t(P_+) P_{R^+} &= \frac{(n+1)(n-1)^2}{8} P_{R^+}, \\
\sum_{r \neq s \neq t} \rho_r(P_-)\rho_s(P_-)\rho_t(P_+) P_{R^+} &= \frac{(n+1)(n-1)(n-3)}{8} P_{R^+}, \\
\sum_{i} (\Gamma^i \otimes \Gamma^i)(P_+ \otimes P_-)_{sym} &= 2(P_+ \otimes P_+)_{sym}, \\
\sum_{ij} (\Gamma^{ij} \otimes \Gamma^{ij})(P_+ \otimes P_+)_{sym} &= 4(P_+ \otimes P_+)_{sym}, \\
\sum_{ij} (\Gamma^{ij} \otimes \Gamma^{ij} \otimes \Gamma^{ij})(P_- \otimes P_+ \otimes P_+)_{sym} &= -2(P_+ \otimes P_+ \otimes P_-)_{sym}, \\
\sum_{ij} (\Gamma^{ij} \otimes \Gamma^{ij} \otimes \Gamma^{ij})(P_- \otimes P_- \otimes P_+)_{sym} &= 2(P_+ \otimes P_- \otimes P_-)_{sym}. \tag{44}
\end{align*}
\]
Using (44), the only relevant non-trivial relations are
\[
\begin{align*}
\epsilon_{ijkl} \sum_{r} \rho_r(\Gamma^i \Gamma^j \Gamma^k \Gamma^l P_+) P_{R^+} &= 12(n+1) P_{R^+}, \\
\epsilon_{ijkl} \sum_{r \neq s} \rho_r(\Gamma^i \Gamma^j \Gamma^k P_-)\rho_s(\Gamma^i P_+) P_{R^+} &= 3(n+1)(n-1) P_{R^+}, \\
\epsilon_{ijkl} \sum_{r \neq s} \rho_r(\Gamma^i P_-)\rho_s(\Gamma^j \Gamma^k \Gamma^l P_+) P_{R^+} &= 3(n+1)(n-1) P_{R^+}, \\
\epsilon_{ijkl} \sum_{r \neq s} \rho_r(\Gamma^{ij} P_+)\rho_s(\Gamma^{kl} P_+) P_{R^+} &= 2(n+1)(n-1) P_{R^+}, \\
\epsilon_{ijkl} \sum_{r \neq s \neq t} \rho_r(\Gamma^i P_-)[\rho_s(\Gamma^j \Gamma^k P_-)\rho_t(\Gamma^l P_+) + \rho_s(\Gamma^j P_+)\rho_t(\Gamma^k \Gamma^l P_+)] P_{R^+} &= (n-2)(n-1)(n+1) P_{R^+}. \tag{45}
\end{align*}
\]
Plugging the expressions in (45) into the various terms in (43) finally gives us

$$\epsilon_{ijkl} G_5 G^i G^j G^k G^l = (n + 1)(n + 2)(n + 3)(P_{R+} + P_{R-}). \quad (46)$$

Now the operators act on the space $\mathcal{R}$ given by the direct sum $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$. Let $P_{\mathcal{R}}$ be the corresponding projection operator given by $P_{\mathcal{R}} = P_{R+} + P_{R-}$. Thus (44) leads to

$$\epsilon_{ijkl} G_5 G^i G^j G^k G^l P_{\mathcal{R}} = (n + 1)(n + 2)(n + 3)P_{\mathcal{R}}. \quad (47)$$

So we write

$$\epsilon_{ijkl} G_5 G^i G^j G^k G^l = (n + 1)(n + 2)(n + 3), \quad (48)$$

where the right hand side means the identity operator, which we shall not be writing explicitly in such cases. This helps us to deduce the structure of the operator $\epsilon_{ijkl} G_5 G^i G^j G^k G^l$ which will be useful to us later. On general grounds, the only possible structure of this operator can be

$$\epsilon_{ijkl} G_5 G^i G^j G^k G^l = f(n) G^i + g(n) G_5 G^i. \quad (49)$$

(Note that $g(1) = 0$.) Contracting both sides with $G^i$ and using the relation $[32, 33]$

$$(G^i)^2 = \frac{(n + 1)(n + 3)}{2} = N, \quad (50)$$

we get that

$$\{f(n) + 2(n + 2)\} = g(n) G_5, \quad (51)$$

from which it follows that

$$f(n) = -2(n + 2), \quad g(n) = 0, \quad (52)$$

using the linear independence of $P_{R+} + P_{R-}$ and $P_{R+} - P_{R-}$. Thus we obtain the equation

$$G^i + \frac{1}{2(n + 2)} \epsilon_{ijkl} G_5 G^j G^k G^l = 0. \quad (53)$$

Such a solution to the matrix equation (53) has been observed in [25].

**References**

[1] A. Strominger, “Open p-branes,” *Phys. Lett.* B383 (1996) 44–47, [hep-th/9512059](http://arxiv.org/abs/hep-th/9512059).

[2] P. K. Townsend, “D-branes from M-branes,” *Phys. Lett.* B373 (1996) 68–75, [hep-th/9512062](http://arxiv.org/abs/hep-th/9512062).
[3] P. K. Townsend, “Brane surgery,” *Nucl. Phys. Proc. Suppl.* 58 (1997) 163–175, [hep-th/9609217](http://arxiv.org/abs/hep-th/9609217).

[4] P. S. Howe, N. D. Lambert, and P. C. West, “The self-dual string soliton,” *Nucl. Phys.* B515 (1998) 203–216, [hep-th/9709014](http://arxiv.org/abs/hep-th/9709014).

[5] C. G. Callan and J. M. Maldacena, “Brane dynamics from the Born-Infeld action,” *Nucl. Phys.* B513 (1998) 198–212, [hep-th/9708147](http://arxiv.org/abs/hep-th/9708147).

[6] G. W. Gibbons, “Born-Infeld particles and Dirichlet p-branes,” *Nucl. Phys.* B514 (1998) 603–639, [hep-th/9709027](http://arxiv.org/abs/hep-th/9709027).

[7] J. P. Gauntlett, J. Gomis, and P. K. Townsend, “BPS bounds for worldvolume branes,” *JHEP* 01 (1998) 003, [hep-th/9711205](http://arxiv.org/abs/hep-th/9711205).

[8] D. Brecher, “BPS states of the non-Abelian Born-Infeld action,” *Phys. Lett.* B442 (1998) 117–124, [hep-th/9804180](http://arxiv.org/abs/hep-th/9804180).

[9] W. Nahm, “A Simple Formalism for the BPS Monopole,” *Phys. Lett.* B90 (1980) 413.

[10] D.-E. Diaconescu, “D-branes, monopoles and Nahm equations,” *Nucl. Phys.* B503 (1997) 220–238, [hep-th/9608163](http://arxiv.org/abs/hep-th/9608163).

[11] A. Kapustin and S. Sethi, “The Higgs branch of impurity theories,” *Adv. Theor. Math. Phys.* 2 (1998) 571–591, [hep-th/9804027](http://arxiv.org/abs/hep-th/9804027).

[12] D. Tsimpis, “Nahm equations and boundary conditions,” *Phys. Lett.* B433 (1998) 287–290, [hep-th/9804081](http://arxiv.org/abs/hep-th/9804081).

[13] A. Giveon and D. Kutasov, “Brane dynamics and gauge theory,” *Rev. Mod. Phys.* 71 (1999) 983–1084, [hep-th/9802067](http://arxiv.org/abs/hep-th/9802067).

[14] N. R. Constable, R. C. Myers, and O. Tafjord, “The noncommutative bion core,” *Phys. Rev.* D61 (2000) 106009, [hep-th/9911136](http://arxiv.org/abs/hep-th/9911136).

[15] J. Madore, “The fuzzy sphere,” *Class. Quant. Grav.* 9 (1992) 69–88.

[16] R. C. Myers, “Dielectric-branes,” *JHEP* 12 (1999) 022, [hep-th/9910053](http://arxiv.org/abs/hep-th/9910053).
[17] N. R. Constable, R. C. Myers, and O. Tafjord, “Non-Abelian brane intersections,” *JHEP* **06** (2001) 023, hep-th/0102080.

[18] H. Grosse, C. Klimcik, and P. Presnajder, “Finite quantum field theory in noncommutative geometry,” *Commun. Math. Phys.* **180** (1996) 429–438, hep-th/9602115.

[19] J. Castelino, S.-M. Lee, and I. Taylor, Washington, “Longitudinal 5-branes as 4-spheres in matrix theory,” *Nucl. Phys.* **B526** (1998) 334–350, hep-th/9712105.

[20] P. Cook, R. de Mello Koch, and J. Murugan, “Non-Abelian Blonic brane intersections,” *Phys. Rev.* **D68** (2003) 126007, hep-th/0306250.

[21] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, “M theory as a matrix model: A conjecture,” *Phys. Rev.* **D55** (1997) 5112–5128, hep-th/9610043.

[22] V. L. Campos, G. Ferretti and P. Salomonson, “The non-Abelian self dual string on the light cone,” *JHEP* **0012** (2000) 011, hep-th/0011271.

[23] Y. Nambu, “Generalized Hamiltonian dynamics,” *Phys. Rev.* **D7** (1973) 2405–2414.

[24] T. L. Curtright and C. K. Zachos, “Branes, strings, and odd quantum Nambu brackets,” hep-th/0312048.

[25] M. M. Sheikh-Jabbari, “Tiny gravitton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture,” hep-th/0406214.

[26] N. S. Deger and A. Kaya, “World-volume description of M2-branes ending on an M5-brane and holography,” *Phys. Lett.* **B538** (2002) 164, hep-th/0203239.

[27] B. de Wit, J. Hoppe, and H. Nicolai, “On the quantum mechanics of supermembranes,” *Nucl. Phys.* **B305** (1988) 545.

[28] I. R. Klebanov and A. A. Tseytlin, “Entropy of Near-Extremal Black p-branes,” *Nucl. Phys.* **B475** (1996) 164–178, hep-th/9604089.

[29] I. R. Klebanov, “World-volume approach to absorption by non-dilatonic branes,” *Nucl. Phys.* **B496** (1997) 231–242, hep-th/9702076.
[30] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.

[31] Z. Guralnik and S. Ramgoolam, “On the polarization of unstable D0-branes into non-commutative odd spheres,” *JHEP* **02** (2001) 032, hep-th/0101001.

[32] S. Ramgoolam, “On spherical harmonics for fuzzy spheres in diverse dimensions,” *Nucl. Phys. B* **610** (2001) 461–488, hep-th/0105006.

[33] S. Ramgoolam, “Higher dimensional geometries related to fuzzy odd-dimensional spheres,” *JHEP* **10** (2002) 064, hep-th/0207111.