General quantum-mechanical setting for field-antifield formalism as a hyper-gauge theory

Igor A. Batalin\textsuperscript{(a,b)}\textsuperscript{1}, Peter M. Lavrov\textsuperscript{(b,c)}\textsuperscript{2}

\textsuperscript{(a)} P.N. Lebedev Physical Institute, Leninsky Prospect 53, 119 991 Moscow, Russia
\textsuperscript{(b)} Tomsk State Pedagogical University, Kievskaya St. 60, 634061 Tomsk, Russia
\textsuperscript{(c)} National Research Tomsk State University, Lenin Av. 36, 634050 Tomsk, Russia

Abstract

A general quantum-mechanical setting is proposed for the field-antifield formalism as a unique hyper-gauge theory in the field-antifield space. We formulate a Schrödinger-type equation to describe the quantum evolution in a "current time" purely formal in its nature. The corresponding Hamiltonian is defined in the form of a supercommutator of the delta-operator with a hyper-gauge Fermion. The initial wave function is restricted to be annihilated with the delta-operator. The Schrödinger’s equation is resolved in a closed form of the path integral, whose action contains the symmetric Weyl’s symbol of the Hamiltonian. We take the path integral explicitly in the case of being a hyper-gauge Fermion an arbitrary function rather than an operator.

Keywords: field-antifield formalism, quantum antibracket, Weyl symbol

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\textsuperscript{1}E-mail: batalin@lpi.ru
\textsuperscript{2}E-mail: lavrov@tspu.edu.ru
1 Introduction

The field-antifield (BV) formalism [1, 2] is known as the most powerful method for covariant (Lagrangian) quantization of gauge-field theories of the general kind, with general open gauge algebra, both irreducible or any-stage reducible. On the other hand, the field-antifield formalism was derived directly from the Hamiltonian generalized canonical quantization [3, 4, 5, 6, 7], and thus, the physical unitarity was guaranteed by construction.

The main ingredient of the field-antifield formalism is the quantum master equation formulated in terms of the nilpotent odd Laplacian also known as the delta-operator. The nilpotency of the delta-operator causes the natural arbitrariness [8, 9, 10] for the quantum master action. That quantum arbitrariness is realized in the form of the so-called anticanonical master transformations [10], infinitesimal or finite [1, 10]. These master transformations do generalize comprehensively the famous BRST transformations [11, 12]. Gradually, it was realized that the whole field-antifield formalism has the characteristic features of some unique hyper-gauge theory which lives in the antisymplectic phase space. These ideas take their most symmetric form within the framework of the so-called $W$-$X$ construction proposed recently [13].

In the present paper, we would like to make a new step to unique hyper-gauge theory. Namely, we would like to propose a new quantum-mechanical setting for the field-antifield formalism as a unique hyper-gauge theory. First, we introduce a new "current time", purely formal in its nature. Then, we define the quantum evolution in the new "current time" by means of the Schrödinger equation with a Hamiltonian chosen as a general delta-exact form, which is a supercommutator of the delta-operator, with some hyper-gauge Fermion. That delta-exact form is directly related to the pair of dual quantum antibrackets [14, 15]. Quantum state is described in terms of a wave function living in the field-antifield "configuration space". We restrict the initial wave function to satisfy the quantum master equation, i.e. to be annihilated with the delta-operator. Because of the delta-exactness of the Hamiltonian, it follows immediately that the current state satisfies the quantum master equation, as well. In this way, a hyper-gauge Fermion describes the natural arbitrariness in resolving the quantum master equation [8, 9, 10].

We resolve the Schrödinger equation in a closed form, in terms of the path integral whose action contains the corresponding symbol of the Hamiltonian operator. In the previous paper [16] we have used the normal $Z_P$ symbol, while in the present paper we make use of the symmetric Weyl’s symbol. Of course, it is the simplest possible case when a hyper-gauge Fermion is an arbitrary function, rather than an operator. In that case, in the path integral, the momenta integration yields the delta-functional concentrated on the orbit of the anticanonical transformation generated by the hyper-gauge Fermion. When resolving the delta-functional, the latter yields the corresponding Jacobian. If one uses the normal $Z_P$ symbol [17] then the latter Jacobian equals to one [16], while in the case of the Weyl’s symbol, the corresponding Jacobian is rather nontrivial. The difference between the two cases is caused by the fact that
the boundary conditions, the integration trajectory should satisfy to, appears dependent actually of the type of the symbol chosen. In the present article, we have calculated explicitly the Jacobian yielded by the delta-functional in the case of the symmetric Weyl’s symbol chosen.

2 Operators and symbols

Let $Z^A$ be the complete set of field-antifield antisymplectic variables, and let $P_A$ be their canonically conjugate momenta,

$$P_A := -i\hbar^{-1/2}\partial_A(-1)^{\varepsilon_A}\rho^{1/2}, \quad [Z^A, P_B] = i\hbar\delta^A_B,$$  \hspace{1cm} (2.1)

where the Grassmann parities of the operators in (2.1) are denoted by

$$\varepsilon_A := \varepsilon(Z^A) = \varepsilon(P_A).$$ \hspace{1cm} (2.2)

All these operators are Hermitian with respect to the standard scalar product

$$<\psi|\phi> := \int d\mu(Z)\psi^*(Z)\phi(Z), \quad d\mu(Z) := dZ\rho(Z).$$ \hspace{1cm} (2.3)

Let us proceed with the Cauchy problem for the Schrödinger equation

$$i\hbar\partial_t\Psi(t) = H(Z, P)\Psi(t), \quad \Psi(t = 0) = \Psi_0,$$ \hspace{1cm} (2.4)

where the Hamiltonian is defined as

$$H(Z, P) := (i\hbar)^{-1}[\Delta(Z, P), F(Z, P)],$$ \hspace{1cm} (2.5)

$$(i\hbar)^{-1}\text{ad}(H) = (i\hbar)^{-2/3}(\text{ad}_\Delta(F) + \text{ad}_F(\Delta)), $$ \hspace{1cm} (2.6)

$$[\text{ad}_\Delta(F), \text{ad}_F(\Delta)] = -\frac{1}{4}\text{ad}(\Delta) \text{ad}_\Delta\left(\frac{1}{2}[F, F]\right), $$ \hspace{1cm} (2.7)

with $\text{ad}_\Delta(F)$ being the adjoint action of the $\Delta$-generated quantum $F$-antibracket \[14, 15\], and the nilpotent Fermion Hermitian operator $\Delta$ being defined as

$$\Delta(Z, P) := \frac{1}{2}\rho^{-1/2}P_A\rho E^{AB}P_B\rho^{-1/2}(-1)^{\varepsilon_B} + (i\hbar)^2\nu(Z), $$ \hspace{1cm} (2.8)

with $E^{AB}(Z)$ being the antisymplectic metric and $\nu(Z)$ being the Fermion function \[18, 19, 20\] introduced in order to provide for the measure density $\rho$ to be independent of the antisymplectic metric $E^{AB}(Z)$. In terms of the operator (2.8), the initial state $\Psi_0(Z)$ in the second in (2.4) is restricted to satisfy the quantum master equation

$$\Delta(Z, P)\Psi_0 = 0,$$ \hspace{1cm} (2.9)
which implies the same equation as to the current state $\Psi(t, Z)$,
\begin{equation}
\Delta(Z, P)\Psi(t) = 0,
\end{equation}
due to the property
\begin{equation}
[\Delta(Z, P), H(Z, P)] = 0.
\end{equation}
In turn, we identify
\begin{equation}
\Psi_0(Z) = \exp \left\{ \frac{i}{\hbar} W(Z) \right\}, \quad \Psi(1, Z) = \exp \left\{ \frac{i}{\hbar} W'(Z) \right\}
\end{equation}
with $W(Z)$ and $W'(Z)$ being the original and the new (transformed) master action, respectively.

The operator $F(Z, P)$ in (2.5) is an arbitrary Fermion Hermitian operator. Its arbitrariness describes the one of the field-antifield formalism as a hyper-gauge theory. In that sense, one can consider the condition (2.9)/(2.10) as the one as to define physical states.

Of course, the current time $t$ in (2.4) is purely formal in its nature. However, one is allowed to use it formally in all aspects of quantum description in the usual way. The situation here resembles a bit the one with the proper time of Schwinger/Fock [21, 22]. For instance, by proceeding from the Schrödinger’s picture (2.4) one can easily change for the Heisenberg’s or the Dirac’s picture, if desired for the sake of technical convenience.

With respect to the operator-valued functions of the basic elements $Z^A$ and $P_A$, one can change for the corresponding symbol calculus of Berezin [17], such as $ZP$-normal symbols, or symmetric symbol of Weyl, and so on. In our previous consideration [16], the corresponding formalism has been developed for $ZP$ normal symbol, the simplest one technically. Here, we will consider below the formalism based on the use of symmetric symbols of Weyl.

Given an operator $H(Z, P)$ in the symmetric Weyl’s form,
\begin{equation}
H(Z, P) =: \left. \left( \exp \left\{ Z^A \frac{\partial}{\partial Z^A} + P_A \frac{\partial}{\partial P_A} \right\} H(\bar{Z}, \bar{P}) \right) \right|_{\bar{Z}=0, \bar{P}=0},
\end{equation}
a function $H(\bar{Z}, \bar{P})$ of classical phase variables $\bar{Z}^A, \bar{P}_A$ is called a Weyl’s symbol. In what follows below we will use the short-hand notation for Weyl’s symbols $Z, P$ as functions of classical phase variables $Z^A, P_A$.

It follows from (2.13) that the star multiplication for Weyl’s symbols has the form
\begin{equation}
\star =: \exp \left\{ \frac{i\hbar}{2} \left( \frac{\overleftarrow{\partial}}{\partial Z^A} \frac{\overrightarrow{\partial}}{\partial P_A} - \frac{\overrightarrow{\partial}}{\partial P_A} \frac{\overleftarrow{\partial}}{\partial Z^A} (-1)^{\varepsilon_A} \frac{\overrightarrow{\partial}}{\partial Z^A} \right) \right\}.
\end{equation}
In terms of (2.14), the operator valued definition (2.5) and the property (2.11) rewrite for symbols as
\begin{equation}
H(Z, P) = (i\hbar)^{-1} [\Delta(Z, P), F(Z, P)]_\star, \quad [\Delta(Z, P), H(Z, P)]_\star = 0,
\end{equation}
where $[,]_\star$ means the symbol supercommutator
\begin{equation}
[A(Z, P), B(Z, P)]_\star =: A(Z, P) \star B(Z, P) - B(Z, P) \star A(Z, P) (-1)^{\varepsilon(A)\varepsilon(B)}.
\end{equation}
3 Path integral resolution for Schrödinger equation in terms of Weyl’s symbols

Given the Weyl’s symbol $H(Z, P)$ of the Hamiltonian, a formal solution to the Cauchy problem (2.4) is

$$\Psi(1, Z) = (2\pi i)^{-D} \int dY dP \ U \left( 1, \frac{1}{2}(Z + Y), P \right) \exp \left\{ \frac{i}{\hbar} P(Z - Y) \right\} \Psi_0(Y), \tag{3.1}$$

where $D$ is the number of Bosons among $Z^A$, and $U(t, Z, P)$ is the symbol of the evolution operator,

$$i\hbar \partial_t U(t, Z, P) = H(Z, P) \ast U(t, Z, P), \quad U(0, Z, P) = 1, \tag{3.2}$$

with $H(Z, P)$ being the symbol (2.15) of the Hamiltonian. By making use of the standard functional methods [17, 23, 24], one derives the following path integral representation for $U(1, Z, P)$ (see Appendix A for details)

$$U(1, Z, P) = \left\langle \exp \left\{ -\frac{i}{\hbar} P_A(Z^A(1) - Z^A(0)) - \frac{i}{\hbar} \int_0^1 dt \ H(Z(t), P(t)) \right\} \right\rangle, \tag{3.3}$$

where the average is defined by

$$\left\langle (...) \right\rangle = \frac{\int \mathcal{D}V \mathcal{D}P (...) \exp \left\{ \frac{i}{\hbar} \int_0^1 dt P_A \dot{Z}^A \right\}}{\int \mathcal{D}V \mathcal{D}P \exp \left\{ \frac{i}{\hbar} \int_0^1 dt P_A \dot{Z}^A \right\}}, \tag{3.4}$$

the integration trajectory $Z^A(t)$ is restricted to satisfy the Weyl’s boundary condition

$$Z^A(1) + Z^A(0) = 2Z^A, \tag{3.5}$$

which resolves in terms of unrestricted integration velocities $V^A(t)$,

$$Z^A(t) =: Z^A + \int_0^1 dt' \frac{1}{2} \text{sign}(t - t') V^A(t'), \quad \dot{Z}^A(t) = V^A(t). \tag{3.6}$$

It is also worth to mention that the famous Berezin’s formula (3.1) has a nice interpretation within the following basic proposal related directly to the symbol multiplication law,

$$\Psi(1, Z) = \int dY \mathcal{K}(1, Z, Y) \Psi_0(Y), \tag{3.7}$$

$$\mathcal{K}(1, Z, Y) = (2\pi i)^{-D} \int dP \ U(1, Z, P) \ast \exp \left\{ \frac{i}{\hbar} 2P(Z - Y) \right\}. \tag{3.8}$$

By inserting the standard Weyl’s multiplication for the $\ast$ (2.14), we get

$$\mathcal{K}(1, Z, Y) = \int dX (2\pi i)^{-D} dP \ U(1, Z - X, P) \delta(2X - Z + Y) \exp \left\{ \frac{i}{\hbar} 2PX \right\}. \tag{3.9}$$
It follows immediately from (3.9) that the standard Berezin’s formula (3.1) holds,
\[ K(1, Z, Y) = (2\pi i)^{-D} \int dP U \left( 1, \frac{1}{2} (Z + Y), P \right) \exp \left\{ \frac{i}{\hbar} P (Z - Y) \right\}, \] (3.10)
together with its inverse [17],
\[ U(1, Z, P) = \int dX K \left( 1, Z + \frac{1}{2} X, Z - \frac{1}{2} X \right) \exp \left\{ -\frac{i}{\hbar} PX \right\}. \] (3.11)

For details as to how the formula (3.10) does follow from the general definition of the Weyl’s operators, see Appendix B.

4 Taking the Weyl’s path integral in the simplest case

For the sake of further simplicity, here we choose the Darboux co-ordinates,
\[ E^{AB} = \text{const}(Z), \quad \rho(Z) = 1, \quad \nu(Z) = 0. \] (4.1)

Let us consider the simplest case, when the hyper-gauge Fermion \( F \) is an arbitrary function of \( Z \),
\[ F = F(Z) \Rightarrow H(Z, P) = P_A(F, Z^A). \] (4.2)

Then, the \( P \)-integration yields the delta functional
\[ \delta[\dot{Z}^A - (F, Z^A)] = J^{-1}[Z] \delta[Z^A - Z^A_R], \] (4.3)
concentrated on the orbit \( Z^A_R(t) \) of an anticanonical transformation, with the \( F(Z) \) being a generator,
\[ \dot{Z}^A = (F, Z^A). \] (4.4)

By resolving the latter together with (3.5), we get the solution
\[ Z^A_R(t) =: \frac{\exp \left\{ t \text{ad}(F) \right\}}{\exp \left\{ \text{ad}(F) \right\} + 1} 2Z^A = \frac{\exp \left\{ (t - \frac{1}{2}) \text{ad}(F) \right\}}{\cosh \left( \frac{1}{2} \text{ad}(F) \right)} Z^A, \] (4.5)
where \( \cosh(x) \) is the ordinary hyperbolic cosine. In a short-hand matrix notation, the logarithm of the delta-functional’s Jacobian is expressed as:
\[ \ln J[Z] = -\int_0^1 d\lambda \int_0^1 dt \text{str} [\Gamma(t, t) X(t)], \] (4.6)
where \( \Gamma(t, t') \) satisfies the equation
\[ [\partial_t 1 - \lambda X(t)]\Gamma(t, t') = \delta(t - t')1, \] (4.7)
and the boundary condition
\[ \Gamma(t = 1, t') + \Gamma(t = 0, t') = 0. \tag{4.8} \]

The matrix \( X(t) \) reads
\[ X^A_B(t) = (F, Z^A) \frac{\partial}{\partial B}(Z(t)). \tag{4.9} \]

The solution to the boundary problem \((4.7)/(4.8)\) has the form
\[ \Gamma(t, t') =: \frac{1}{2} U(t) \left[ \text{sign}(t - t') + \frac{1 - U(1)}{1 + U(1)} \right] U^{-1}(t'), \tag{4.10} \]
where the holonomy matrix
\[ U(t) =: T \exp \left\{ \int_0^t dt' \lambda X(t') \right\}, \tag{4.11} \]
is the solution to the Cauchy problem:
\[ \partial_t U(t) = \lambda X(t) U(t), \quad U(t = 0) = 1. \tag{4.12} \]

From \((4.10)\) at coincident arguments, we have
\[ \Gamma(t, t) = \frac{1}{2} U(t) \left[ \frac{1 - U(1)}{1 + U(1)} U^{-1}(t) \right] = \frac{1}{2} \left[ 1 - U(t)(1 + U(1))^{-1} U(1) U^{-1}(t) \right]. \tag{4.13} \]

By inserting \((4.13)\) into \((4.6)\), one obtains
\[ \ln J[Z] = - \int_0^1 d\lambda \int_0^1 dt \text{str} \left[ \frac{1}{2} X(t) - (1 + U(1))^{-1} U(1) U^{-1}(t) X(t) U(t) \right]. \tag{4.14} \]

On the other hand, by differentiating \((4.12)\) with respect to \( \lambda \), we have
\[ \frac{\partial U(1)}{\partial \lambda} = U(1) \int_0^1 dt \ U^{-1}(t) X(t) U(t). \tag{4.15} \]

By substituting this result into the second term in the right-hand side in \((4.14)\), we find
\[ \ln J[Z] = - \frac{1}{2} \int_0^1 dt \ \text{str} \ [X(t)] + \text{str} \ [\ln (1 + U(1)])|_{\lambda=0}^{\lambda=1}. \tag{4.16} \]

Let us restrict the functional \((4.16)\) on the special trajectory \((4.5)\). Because of \((4.4)\) together with \((4.12)\) at \( \lambda = 1 \), we have
\[ U(1)|_{\lambda=0} = 1, \tag{4.17} \]
\[ U(1)|_{\lambda=1} = (Z_R(1) \otimes \partial)(Z_R(0) \otimes \partial)^{-1}. \tag{4.18} \]
By using (4.17), (4.18), finally (4.16) reads
\[
\ln J[Z_R] = E(\text{ad}(F))(\Delta F)(Z_R(0)) - \text{str ln} [Z_R(0) \otimes \partial] + \\
+ \text{str ln} \left[ \frac{1}{2} (Z_R(1) + Z_R(0)) \otimes \partial \right].
\] (4.19)

Due to the boundary condition (3.5) the third term in the right-hand side in (4.19) equals to zero. Thus, it follows from (3.3), (4.19) that
\[
U(1, Z, P) = \exp \left\{ -\frac{i}{\hbar} P_A (Z_R^A(1) - Z_R^A(0)) \right\} \times \\
\times \exp \left\{ -E(\text{ad}(F))(\Delta F)(Z_R(0)) + \text{str ln} [Z_R(0) \otimes \partial] \right\},
\] (4.20)

where
\[
Z_R^A(0) = \exp \left\{ -\frac{1}{2} \text{ad}(F) \right\} \frac{1}{\cosh \left( \frac{1}{2} \text{ad}(F) \right)} Z^A.
\] (4.21)

Notice also that
\[
E(\text{ad}(F))(\Delta F)(Z_R(0)) = \frac{\sinh \left( \frac{1}{2} \text{ad}(F) \right)}{\frac{1}{2} \text{ad}(F)} (\Delta F) \left( \text{cosh}^{-1} \left( \frac{1}{2} \text{ad}(F) \right) Z \right) = \\
= -\frac{1}{2} \text{str ln} [Z_R(1) \otimes \partial]_{Z_R(0)},
\] (4.22)

where \(\sinh(x)\) is the ordinary hyperbolic sine, and
\[
Z_R^A(1) = \frac{\exp \left\{ \frac{1}{2} \text{ad}(F) \right\}}{\cosh \left( \frac{1}{2} \text{ad}(F) \right)} Z^A = Z_R^A(0) \big|_{F \to -F} = \exp \{\text{ad}(F)\} Z_R^A(0).
\] (4.23)

Due to (4.22), the formulae (4.19), (4.20) become
\[
\ln J[Z_R] = -\frac{1}{2} \text{str ln} [Z_R(1) \otimes \partial] - \frac{1}{2} \text{str ln} [Z_R(0) \otimes \partial] = \\
= -\frac{1}{2} \text{str ln} [Z_R(0) \otimes \partial] + (F \to -F) = \\
= -\frac{1}{2} \text{str ln} [Z_R(1) \otimes \partial] + (F \to -F),
\] (4.24)

\[
U(1, Z, P) = J^{-1}[Z_R] \exp \left\{ -\frac{i}{\hbar} P_A (Z_R^A(1) - Z_R^A(0)) \right\}.
\] (4.25)

By inserting (3.3) into (3.1) one gets
\[
\Psi(1, Z) = \left\langle \exp \left\{ -\frac{i}{\hbar} \int_0^1 dt H(Z'(t), P(t)) \right\} \Psi_0(Z'(0)) \right\rangle,
\] (4.26)

\footnote{In (4.19) and below, \(\Delta\) means the standard odd Laplacian in the Darboux co-ordinates (4.1), \(\Delta = \frac{1}{2}(-1)^C \partial_A E^{AB} \partial_B\).}
where
\[ Z'^A(t = 1) = Z^A, \quad Z'^A(t) = Z^A - \int^t_1 dt' V^A(t'). \] (4.27)

Here in (4.27), the set of integration trajectories is the same as the one specific for the case of \( ZP \)-normal symbols [16], although the \( H(Z, P) \) in (4.26) is just the \(^*\)Weyl’s symbol\(^*\) given by (4.2). It is also worth to mention that the path integral (4.26) is regularized by the condition
\[ \theta(0) = \frac{1}{2}, \] (4.28)
appropriate to the case of Weyl’s symbols. Thus, by comparing (4.27) to (3.5), one realizes that boundary condition for integration trajectories may depend actually both on the type of symbol chosen, and on the type of specific quantity which the path-integral representation is defined for.

Due to (4.2), the \( P \)-integration yields the delta functional (4.3) with \( Z_R' \) standing for \( Z_R \), where
\[ Z'^R_A(t) = \exp \{(t - 1) \text{ad}(F)\} Z^A, \quad Z'^R_A(0) = \exp \{- \text{ad}(F)\} Z^A. \] (4.29)

The corresponding Jacobian was calculated in [16], formula (A.14), up to the regularization (4.28),
\[ - \ln J[Z'_R] = -\theta(0) \int^1_0 dt \text{str} [(F, Z) \otimes \partial_t' \cdot (Z'_R(t))] = E(\text{ad}(-F)) \Delta F. \] (4.30)

By inserting (4.3) with \( Z'_R \) standing for \( Z_R \), together with (4.29), (4.30), into (4.26), we reproduce the standard formula
\[ \Psi(1, Z) = \exp \{E(\text{ad}(-F)) \Delta F\} \Psi_0(\exp \{- \text{ad}(F)\} Z). \] (4.31)

On the other hand, one could insert the primed trajectory (4.27)/(4.29), generated by the formula (3.1), directly into the first line in (4.24). As the quantities
\[ Z'_R(1) = Z, \quad Z'_R(0) = \exp \{- \text{ad}(F)\} Z, \] (4.32)
are not related to each other with the \( F \)-inversion as mapping \( F \) to \(-F\), the second and the third equality in (4.24) does not hold for the primed trajectory (4.29). As to the first line in (4.24), the first term is zero, while the second term yields
\[ J^{-1}[Z'_R] = \exp \{E(\text{ad}(F)) \Delta F(Z)\}, \] (4.33)
which coincides exactly with the first exponential in (4.31).
5 Conclusion

In the present article we have formulated the general quantum-mechanical setting for the field-antifield BV formalism as a hyper-gauge theory based on the Schrödinger equation (2.4) with the Hamiltonian (2.5). In terms of symmetric, Weyl’s, symbols, we have resolved the equation (3.2) for the symbol of the evolution operator in the form of a functional path integral (3.3) with specific, Weyl’s, boundary conditions (3.5) for integration trajectories. By making use of Berezin’s formula (3.1), we derive then the path-integral representation (4.26) for the wave function, in the form of a modified path integral with the modified boundary conditions (4.27) for primed integration trajectories.

In the simplest case of a hyper-gauge Fermion (4.2) being a function rather than an actual operator, we have taken the path integral (3.3) as reduced with the delta-functional (4.3) concentrated on the orbit (4.4) of an anticanonical transformation. We have calculated explicitly the delta-functional’s Jacobian (4.24) by closed resolving the corresponding boundary problem (4.7), (4.8) in the form (4.10). We have performed a similar reduction procedure as applied to the modified path integral (4.26). In this way, we have reproduced the standard formula (4.31) as describing, together with the identification (2.12), a canonical part of the arbitrariness in resolving the quantum master equation (2.9) / (2.10). We have shown that an anticanonical transformation encoded in (4.31) comes directly from the classic orbit equation (4.4), while the corresponding measure, the first exponential in (4.31), comes from the Jacobian (4.24). In contrast to that, in our previous article [16], where we did use the normal $ZP$-symbols, in the course of a similar reduction procedure, the encoded anticanonical transformation did come together with the corresponding measure from the classical orbit equation, while the delta-functional’s Jacobian was equal to one.

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Appendix A. Resolving equation (3.2) for symbol of evolution operator

The basic observation is that the equation (3.2) for the symbol $U$ of the evolution operator allows for a resolution in the variation-derivative form with respect to the external sources $J^A(t), K^A(t)$,

$$U(1, Z, P) = \exp \left\{ -\frac{i}{\hbar} \int_0^1 dt \left( i\hbar \delta \frac{\delta}{\delta J^A}, i\hbar \delta \frac{\delta}{\delta K^A} \right) \right\} X(1, Z, P)|_{J^A=0, K^A=0}, \quad (A.1)$$
where \( X(t, Z, P) \) resolves the following Cauchy problem

\[
i \hbar \frac{\partial}{\partial t} X = \left[ J_A(t) \left( Z^A + \frac{i \hbar}{2} \frac{\partial}{\partial P_A} \right) + K^A(t) \left( P_A - \frac{i \hbar}{2} \frac{\partial}{\partial Z^A} (-1)^{\varepsilon_A} \right) \right] X, \quad X(0, Z, P) = 1. \tag{A.2}\]

The latter Cauchy problem resolves explicitly

\[
X(1, Z, P) = \exp \left\{ -\frac{i}{\hbar} \int_0^1 dt \, J_A(t) \left[ Z^A + \frac{1}{2} \int_0^1 dt' \, \text{sign}(t - t') K^A(t')(-1)^{\varepsilon_A} \right] - \frac{i}{\hbar} \int_0^1 dt \, K^A(t) P_A \right\}. \tag{A.3}
\]

By inserting the unity

\[
1 = \text{const} \int \mathcal{D}V \mathcal{D}P \exp \left\{ \frac{i}{\hbar} \int_0^1 dt \, P_A(t) [V^A(t) - K^A(t)(-1)^{\varepsilon_A}] \right\}, \tag{A.4}
\]

into the right-hand side in (A.1), to the right of the first exponential, we get

\[
U(1, Z, P) = \left\langle \exp \left\{ -\frac{i}{\hbar} P_A(Z^A(1) - Z^A(0)) - \frac{i}{\hbar} \int_0^1 dt \, H(Z(t), P(t)) \right\} \right\rangle, \tag{A.5}
\]

where the average and the \( Z^A(t) \) is defined in (3.4) and (3.6), respectively.

**Appendix B. Derivation of Berezin’s formula for kernel (3.10)**

Let \( \hat{Z}^A \) and \( \hat{P}_A \) be co-ordinate and momentum operators

\[
\hat{Z}^A =: X^A, \quad \hat{P}_A =: -i \hbar \frac{\partial}{\partial X^A} (-1)^{\varepsilon_A}, \quad [\hat{Z}^A, \hat{P}_B] = i \hbar \delta^{AB}, \quad \varepsilon(\hat{Z}^A) = \varepsilon(\hat{P}_A) = \varepsilon_A, \tag{B.1}
\]

as to apply to functions \( \Psi = \Psi(X) \). We define an arbitrary Weyl’s operator as

\[
\hat{A} = \exp \left\{ \hat{Z}^A \frac{\partial}{\partial Z^A} + \hat{P}_A \frac{\partial}{\partial P_A} \right\} A(Z, P) \bigg|_{Z=0, P=0}, \tag{B.2}
\]

with \( A(Z, P) \) being the Weyl’s symbol of the operator \( \hat{A} \). By definition, the operator \( \hat{A} \) applies to the functions \( \Psi(X) \) by the rule

\[
(\hat{A}\Psi)(X) = \int dY \mathcal{K}(X, Y) \Psi(Y), \tag{B.3}
\]

in terms of the kernel \( \mathcal{K}(X, Y) \). It follows from (B.1) - (B.3) that

\[
\mathcal{K}(X, Y) = \exp \left\{ \hat{Z}^A \frac{\partial}{\partial Z^A} + \hat{P}_A \frac{\partial}{\partial P_A} \right\} A(Z, P) \bigg|_{Z=0, P=0} \delta(X - Y) = \exp \left\{ -i \hbar \frac{\partial}{\partial P_A} X^A \frac{\partial}{\partial Z^A} \right\} A(Z, P) \delta(X - Y) \bigg|_{Z=0, P=0}. \tag{B.4}
\]
Due to the Baker-Campbell-Hausdorff formula, the (B.4) rewrites as
\[
\mathcal{K}(X, Y) = \exp \left\{ -i\hbar \frac{\partial}{\partial P_A} \frac{\partial}{\partial X^A} \right\} \exp \left\{ X^A \frac{\partial}{\partial Z^A} \right\} \exp \left\{ i\hbar \frac{\partial}{2 \partial P_A} \frac{\partial}{\partial Z^A} \right\} \times A(Z, P) \delta(X - Y) \bigg|_{Z=0, P=0}.
\] (B.5)

As the second exponential in (B.5) applies to the right by the shift \( Z \rightarrow Z + X \), the (B.5) rewrites as
\[
\mathcal{K}(X, Y) = \exp \left\{ -i\hbar \frac{\partial}{\partial P_A} \frac{\partial}{\partial X^A} \right\} \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} A(Z + Y, P) \delta(X - Y) \bigg|_{Z=0, P=0}. \tag{B.6}
\]
with the presence of the delta-function, \( \delta(X - Y) \), taken into account. Consider the Fourier-integral representation for the delta-function,
\[
\delta(X - Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\}, \tag{B.7}
\]
with \( D \) being the number of Bosons among \( X^A \). By using then the relation
\[
\exp \left\{ -i\hbar \frac{\partial}{\partial P_A} \frac{\partial}{\partial X^A} \right\} \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} = \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} \exp \left\{ K_A \frac{\partial}{\partial P_A} \right\}, \tag{B.8}
\]
and the fact that the rightmost exponential in (B.8) applies to the right by the shift \( P_A \rightarrow P_A + K_A \), we rewrite the (B.6) in the form
\[
\mathcal{K}(X, Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} \times \exp \left\{ \frac{i\hbar}{2 \partial P_A} \frac{\partial}{\partial Z^A} \right\} A(Z + Y, P + K) \bigg|_{Z=0, P=0}. \tag{B.9}
\]
Due to the symmetric dependence of \( A(Z + Y, P + K) \) on \( Z, Y \) and \( P, K \), we have
\[
\exp \left\{ \frac{i\hbar}{2 \partial P_A} \frac{\partial}{\partial Z^A} \right\} A(Z + Y, P + K) \bigg|_{Z=0, P=0} = \exp \left\{ \frac{i\hbar}{2 \partial K_A} \frac{\partial}{\partial Y^A} \right\} A(Y, K), \tag{B.10}
\]
and
\[
\mathcal{K}(X, Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} \exp \left\{ \frac{i\hbar}{2 \partial K_A} \frac{\partial}{\partial Y^A} \right\} A(Y, K). \tag{B.11}
\]
By integrating in (B.11) by parts, we get
\[
\mathcal{K}(X, Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} \exp \left\{ -\frac{i\hbar}{2 \partial K_A} (-1)^{\varepsilon_A} \frac{\partial}{\partial Y^A} \right\} A(Y, K), \tag{B.12}
\]
or, equivalently,
\[
\mathcal{K}(X, Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} \exp \left\{ \frac{1}{2} (X^A - Y^A) \frac{\partial}{\partial Y^A} \right\} A(Y, K), \tag{B.13}
\]
where in (B.13) the $Y$-derivative in the second exponential applies only to $A(Y,K)$. Thus, we arrive at the famous Berezin’s formula [17]

$$K(X,Y) = (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K A(X^A - Y^A) \right\} A \left( \frac{1}{2}(X + Y), K \right), \quad (B.14)$$

whose inverse reads

$$A(Z,P) = \int dX K \left( Z + \frac{1}{2} X, Z - \frac{1}{2} X \right) \exp \left\{ -\frac{i}{\hbar} P A X^A \right\}. \quad (B.15)$$

The star-product, $A \star B$, is defined by the formula

$$\int dY \ K_A(X,Y) \ K_B(Y,Z) = K_{A \star B}(X,Z), \quad (B.16)$$

where in (B.16), we have denoted by $K_A, K_B, K_{A \star B}$ the kernel corresponding, in the sense of (B.14), to the symbol $A, B, A \star B$, respectively. It follows then from (B.15), (B.16) immediately that the star-product is given by

$$(A \star B)(Z,P) = \int dX K_{A \star B} \left( Z + \frac{1}{2} X, Z - \frac{1}{2} X \right) \exp \left\{ -\frac{i}{\hbar} P X \right\} = \int dX dY \ K_A \left( Z + \frac{1}{2} X, Y \right) K_B \left( Y, Z - \frac{1}{2} X \right) \exp \left\{ -\frac{i}{\hbar} P X \right\} = (2\pi i)^{-2D} \int dX dY dQ dQ' dP dP' dP'' \delta \left( Q' - \frac{1}{2} \left( Z + \frac{1}{2} X + Y \right) \right) \times \delta \left( Q'' - \frac{1}{2} \left( Y + Z - \frac{1}{2} X \right) \right) A(Q', P') B(Q'', P'') \times \exp \left\{ \frac{i}{\hbar} \left[ P' \left( Z + \frac{1}{2} X - Y \right) + P'' \left( Y - Z + \frac{1}{2} X \right) - P X \right] \right\}, \quad (B.17)$$

which is equivalent exactly [17] to the standard formula (2.14) for the $\star$. Indeed, in (B.17), one removes both delta-functions by taking $X$ and $Y$-integral, and substituting

$$X = 2(Q' - Q''), \quad Y = Q' + Q'' - Z, \quad (B.18)$$

so that the star-product becomes

$$(A \star B)(Z,P) = (2\pi i)^{-2D} \int dQ' dQ'' dP' dP'' A(Q', P') B(Q'', P'') \times \exp \left\{ \frac{2i}{\hbar} \left[ (P' - P'') Z - P' Q'' + P'' Q' - P(Q' - Q'') \right] \right\}, \quad (B.19)$$

which is just an integral counterpart to the bi-differential operator (2.14).

Finally, we present here a generalization to the Berezin’s formula (B.14) and to its inverse (B.15),

$$K(X,Y) =: (2\pi i)^{-D} \int dK \exp \left\{ \frac{i}{\hbar} K_A(X^A - Y^A) \right\} A(\alpha X + \beta Y, K), \quad (B.20)$$
\[ A(Z, P) = \int dX K(Z + \beta X, Z - \alpha X) \exp \left\{ -\frac{i}{\hbar} P_A X^A \right\}, \quad (B.21) \]

where in (B.20), (B.21), parameters \( \alpha, \beta \) are restricted to satisfy the condition

\[ \alpha + \beta = 1. \quad (B.22) \]

The corresponding interpolating operator generalizes (B.2) in the form,

\[ \hat{A} =: \exp \left\{ \hat{Z}^A \frac{\partial}{\partial \hat{Z}^A} + \hat{P}_A \frac{\partial}{\partial \hat{P}_A} + \frac{i\hbar}{2} (\alpha - \beta) \frac{\partial}{\partial \hat{P}_A} \frac{\partial}{\partial \hat{Z}^A} \right\} A(Z, P) \bigg|_{Z=0, P=0}. \quad (B.23) \]

For particular values of parameters we have

- \( \alpha = 1, \beta = 0 \) : \( ZP \)- normal form;
- \( \alpha = 0, \beta = 1 \) : \( PZ \)- normal form;
- \( \alpha = \frac{1}{2}, \beta = \frac{1}{2} \) : symmetric (Weyl’s) form.

By making use of the same method as we did when deriving the (B.19), one can derive from (B.20), (B.21) the corresponding star-product for symbols.

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