PULLBACK ATTRACTOR AND INVARIANT MEASURES FOR
THE THREE-DIMENSIONAL REGULARIZED MHD EQUATIONS

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Abstract. This article studies the three-dimensional regularized Magnetohydrodynamics (MHD) equations. Using the approach of energy equations, the authors prove that the associated process possesses a pullback attractor. Then they establish the unique existence of the family of invariant Borel probability measures which is supported by the pullback attractor.

1. Introduction. In this article, we consider the following non-autonomous three-dimensional (3D) regularized Magnetohydrodynamics (MHD) equations

\[
\begin{align*}
\partial_t (u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p &= f(t), \\
\partial_t (b - \beta^2 \Delta b) - \mu \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u &= g(t), \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(x, \tau) = u_\tau(x), &\quad b(x, \tau) = b_\tau(x), \quad x \in \Omega,
\end{align*}
\]

where \( \tau \in \mathbb{R} \) and \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth enough boundary \( \partial \Omega \), the velocity field \( u = (u_1, u_2, u_3) \), the magnetic field \( b = (b_1, b_2, b_3) \) and the total pressure \( p \) are the unknown terms, \( \nu \) is the kinematic viscosity and \( \mu \) is the constant magnetic resistivity, \( f \) represents volume force applied to the fluid, \( g \) is usually zero when Maxwell’s displacement currents are ignored. We will assume the constants \( \nu, \mu, \alpha \) and \( \beta \) are all positive.

Equations (1.1)-(1.2) are regularization in both the velocity and the magnetic field of the following large eddy simulation model for the turbulent flow of a magnetofluid (see [7]):

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p &= f(t), \\
\partial_t b - \mu \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u &= g(t).
\end{align*}
\]

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Equations (1.6)-(1.7) (or the relevant equations) have been widely studied, see e.g. [8, 24] for the existence and uniqueness of the solutions and [2, 27, 35, 36] for the regularity criteria. Compared to equations (1.6) and (1.7), equations (1.1) and (1.2) contain the extra regularizing terms \((-\alpha^2 \partial_t \Delta u)\) and \((-\beta^2 \partial_t \Delta b)\), respectively. These two terms regularize the equations in a way that the 3D equations (1.1)-(1.5) become now globally wellposed (see [7]). Also, equations (1.1)-(1.5) have been deeply studied, see e.g., Catania and Secchi [6, 8], Catania [7], Larios and Titi [14, 15], Levant, Ramos and Titi [16], Zhao and Li [32].

The first aim of this article is to prove the existence of the pullback attractors associated to equations (1.1)-(1.5). Our idea originates from article [11]. In [11], García-Luengo et al. established a sufficient condition for the existence of the minimal pullback attractor. Then they proved the global wellposedness and the existence of the minimal pullback attractor for 3D Navier-Stokes-Voigt (NSV) equations. Different to the NSV equations studied in [11], equations (1.1)-(1.2) contain the coupled Maxwell’s equations which rule the magnetic field. Observing the coupled structure of the addressed equations, we extend the energy method of Ball (see e.g. [1]). We find that the coupled structure plays an important role when we investigate the pullback asymptotic compactness of the generated process \(\{U(t, \tau)\}_{t \geq \tau}\) in space \(W\) (see notation in Section 2).

The idea of energy equation (or enstrophy equation) was initially introduced by Ball (see e.g. [1]) and developed later by Moise et al. [21, 22] in a systematic and abstract framework. In fact, such idea has been well extended and widely used in verifying the asymptotic compactness of the semiflow or process associated with the partial differential equations on unbounded domain (see e.g. [3, 12, 17, 23, 25, 28, 29, 30]). Recently, the approach of enstrophy equation was used in a concise form in [11] to investigate the existence and tempered behavior of the pullback attractor for the two-dimensional (2D) Navier-Stokes equations. This concise form was used in [31, 33] to investigate the existence of the pullback attractor for the 2D non-Newtonian fluid equations and 2D non-autonomous micropolar fluid flows with infinite delays.

The second purpose of this article is to verify the unique existence of a family of invariant Borel probability measures which are supported by the obtained attractor. The invariant measures and statistical solutions have proven to be very useful in the understanding of turbulence (see Foias et al. [10]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Later, the invariant measures and statistical properties of dissipative systems were studied in a series of references. For instance, Wang investigated the upper semi-continuity of stationary statistical properties of dissipative systems in [26]. Łukaszewicz, Real and Robinson [19] used the generalized Banach limit to construct the invariant measures for general continuous dynamical systems on metric spaces. Also, Chekroun and Glatt-Holtz [9] used the generalized Banach limit to construct the invariant measures for a broad class of dissipative semigroups, which generalized and simplified the proofs of [19] and [26].

Very recently, Łukaszewicz and Robinson [20] used the techniques developed in the articles Łukaszewicz [18] and Łukaszewicz et al. [19], which were in turn based on works of Foias et al. [10] and Wang [26], to provide a construction of invariant measures for non-autonomous systems with minimal assumptions on the underlying dynamical process. The results of Łukaszewicz and Robinson ([20, Theorem 3.1])
show that a continuous process \( \{\mathcal{U}(t, \tau)\}_{t \geq \tau} \) on a complete metric space \( X \) possesses a unique family of Borel invariant probability measures in \( X \) if \( \{\mathcal{U}(t, \tau)\}_{t \geq \tau} \) satisfies

(i) the process \( \{\mathcal{U}(t, \tau)\}_{t \geq \tau} \) possesses a pullback attractor in \( X \); and

(ii) for every \( u_0 \in X \) and every \( t \in \mathbb{R} \), the \( X \)-valued function \( \tau \mapsto \mathcal{U}(t, \tau)u_0 \) is continuous and bounded on \( (-\infty, t] \).

Zhao and Yang used this theory to construct the invariant Borel probability measures for the non-autonomously globally modified Navier-Stokes equations in [34]. Here we will also borrow this result to obtain the unique existence of the family of invariant Borel probability measures which are supported by the obtained pullback attractor. The key step is to estimate the difference between two solutions \((u^{(1)}, b^{(1)})\) and \((u^{(2)}, b^{(2)})\) of equations (1.1)-(1.5). When estimating the following nonlinear terms

\[
\langle (u^{(1)} \cdot \nabla)u^{(1)} - (u^{(2)} \cdot \nabla)u^{(2)}, u^{(1)} - u^{(2)} \rangle,
\langle (u^{(1)} \cdot \nabla)b^{(1)} - (u^{(2)} \cdot \nabla)b^{(2)}, b^{(1)} - b^{(2)} \rangle,
\langle (b^{(1)} \cdot \nabla)b^{(1)} - (b^{(2)} \cdot \nabla)b^{(2)}, u^{(1)} - u^{(2)} \rangle,
\langle (b^{(1)} \cdot \nabla)u^{(1)} - (b^{(2)} \cdot \nabla)u^{(2)}, b^{(1)} - b^{(2)} \rangle,
\]

we also find that the coupled structure of the addressed equations plays a key role.

The rest of this article is arranged as follows. The next section is preliminaries concerning some notations and operators, as well as the existence and uniqueness of solutions to the 3D regularized MHD equations. Section 3 is devoted to establishing the existence of pullback attractor for the associated process in space \( W \) via the approach of energy equations. In the last section, we prove that there exists a unique family of invariant Borel probability measures on the pullback attractor.

2. Existence and uniqueness of the solutions. In this section, we first introduce some notations and operators. Then we present the existence and uniqueness of solutions to equations (1.1)-(1.5).

In this article, \( \mathbb{R} \) denotes the set of real numbers and \( c(\cdot, \cdot) \) stands for the generic constant (depending essentially on the quantities in the brackets) that can take different values in different places. \( L^p(\Omega) = (L^p(\Omega))^3 \) is the 3D Lebesgue space with norm \( \| \cdot \|_{L^p(\Omega)} \), and \( \| \cdot \|_{L^2(\Omega)} \) is \( \| \cdot \| \) for brevity. At the same time, \( H^m(\Omega) \) is the 3D Sobolev space \( \{ \phi \in L^2(\Omega) : |\nabla^k \phi \in L^2(\Omega), k \leq m \} \) with norm \( \| \cdot \|_{H^m(\Omega)} \), and \( \mathbb{H}^{1}_{0}(\Omega) \) stands for the closure of \( \{ \phi \in (C^\infty_0(\Omega))^3 \} \) in \( H^1(\Omega) \) with norm \( \| \cdot \|_{H^1(\Omega)} \). Furthermore, we set

\[
\mathcal{V} = \{ \phi \in (C^\infty_0(\Omega))^3 : |\nabla \cdot \phi = 0 \};
\]

\( H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega) \text{ with norm } \| \cdot \| \text{ and dual space } H'; \)

\( V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega) \text{ with norm } \| \cdot \|_V = \| \cdot \|_{H^1(\Omega)} \text{ and dual space } V'; \)

\( W = V \times V'. \)

We will use \( (\cdot, \cdot) \) for the inner product in \( H \) and \( (\cdot, \cdot) \) for the dual pairing between \( V \) and \( V' \). We also use the norm in \( W \) as

\[
\|(u, b)\|_W \triangleq (\|u\|^2 + \|b\|^2 + \alpha^2\|\nabla u\|^2 + \beta^2\|\nabla b\|^2)^{1/2}, \quad \forall (u, b) \in W,
\]

where \( \alpha \) and \( \beta \) come from (1.1) and (1.2), respectively. By the Poincaré inequality, we see that above norm is equivalent to the usual norm

\[
\|(u, b)\|_W = (\|u\|_V^2 + \|b\|_V^2)^{1/2}.
\]
We consider the usual operators in the theory of Navier-Stokes equations. Let $P$ be the Leray-Helmholtz projection from $L^2(\Omega)$ onto $H$ and $A : V \mapsto V'$ be the linear operator defined as

$$
\langle Au, v \rangle \triangleq (\nabla u, \nabla v), \quad u, v \in V.
$$

(2.1)

We denote $D(A) = \{ u \in V | Au \in H \}$. Since the smoothness of $\partial \Omega$, we have $D(A) = H^2(\Omega) \cap V$ and $Au = -P\Delta u, \forall u \in D(A)$. We also define a continuous trilinear form

$$
\langle B(u, v), w \rangle \triangleq \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u, v, w \in H^1_0(\Omega),
$$

and set $\langle B(u), w \rangle = \langle B(u, u), w \rangle$ for short. Note that $V \subseteq H^1_0(\Omega)$ is a closed subspace. Thus $B(u, v) : V \times V \mapsto V'$ is continuous. For above introduced operators $A$ and $B$, we have the following classical results.

**Lemma 2.1.** ([4, 13]) There exist two positive constants $c_1$ and $c_2$ depending only on $\Omega$ such that

$$
|\langle B(u, v), w \rangle| \leq c_1 \|u\|_V \|v\|_V \|w\|_V, \quad \forall u, v, w \in V, \quad (2.2)
$$

$$
|\langle B(u, v), w \rangle| \leq c_2 \|Au\|_V \|v\|_V \|w\|_V, \quad \forall u \in D(A), v \in V, w \in H, \quad (2.3)
$$

$$
|\langle B(u, v), w \rangle| \leq c_2 \|u\|_V^{1/4} \|Au\|_V^{3/4} \|v\|_V \|w\|_V, \quad \forall u \in D(A), v \in V, w \in H, \quad (2.4)
$$

$$
\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \langle B(u, v), v \rangle = 0, \quad \forall u, v, w \in V. \quad (2.5)
$$

As a consequence of (2.2), we have

$$
\|B(u)\|_{V'} \leq c_1 \|u\|_V^2, \quad \forall u \in V. \quad (2.6)
$$

With the above notations, excluding the pressure $p$, we can express the weak version of equations (1.1)-(1.5) in the solenoidal vector field as follows (see e.g. [32]):

$$
\partial_t (u + \alpha^2 Au) + \nu Au + B(u) - B(b) = f(t) \text{ in } D'(\tau, +\infty; V'), \quad (2.7)
$$

$$
\partial_t (b + \beta^2 Ab) + \mu Ab + B(u, b) - B(b, u) = g(t) \text{ in } D'(\tau, +\infty; V'), \quad (2.8)
$$

$$
u(x, \tau) = u_\tau, \quad b(x, \tau) = b_\tau, \quad x \in \Omega. \quad (2.9)
$$

Assume that $(u_\tau, b_\tau) \in W$ and both $f$ and $g$ belong to $L^1_{\text{loc}}(\mathbb{R}; V')$. We next specify the definition of weak solutions to (2.7)-(2.9).

**Definition 2.2.** It is said $(u, b)$ be a weak solution to (2.7)-(2.9) if $(u, b) \in L^2(\tau, T; W)$ for all $T > \tau$ and satisfies (2.7)-(2.9).

We next show that the initial condition $(u_\tau, b_\tau) \in W = V \times V$ makes sense. In fact, from the Lax-Milgram lemma, we see that for each $h \in V'$, there exists a unique $\phi_h \in V$ such that

$$
\phi_h + \alpha^2 A\phi_h = h. \quad (2.10)
$$

The mapping $C_\alpha : \phi \in V \mapsto \phi + \alpha^2 \Delta \phi \in V'$ is linear and bijective with $C_\alpha^{-1} h = \phi_h$. By (2.10) we have $\|\phi_h\|^2 + \alpha^2 \|\phi_h\|_{V'}^2 \leq \|\phi_h\|_V \|h\|_{V'}$, and in particular $\|\phi_h\|_V \leq \alpha^{-2} \|h\|_{V'}$, that is

$$
\|C_\alpha^{-1} h\|_V \leq \alpha^{-2} \|h\|_{V'}, \quad \forall h \in V'. \quad (2.11)
$$
Similarly, the mapping $C_\beta : \phi \in V \mapsto \phi + \beta^2 \Delta \phi \in V$ is linear and bijective with $C_\beta^{-1}h = \phi_h$ and
\[\|C_\beta^{-1}h\|_V \leq \beta^{-2}\|h\|_{V'}, \quad \forall h \in V'.\] (2.12)

Now, if $(u, b)$ is a weak solution of (2.7)-(2.9), then both $\eta(t) = u(t) + \alpha^2 u(t)$ and $\zeta(t) = b(t) + \beta^2 Ab(t)$ belong to $L^2(\tau, T; V')$ for all $T > \tau$. By (2.6), both $\eta'(t) = \frac{\partial u}{\partial t}$ and $\zeta'(t) = \frac{\partial b}{\partial t} \in L^1(\tau, T; V')$ for all $T > \tau$. Consequently, both $\eta(t)$ and $\zeta(t)$ belong to $C([\tau, +\infty); V')$. Therefore, by (2.11) and (2.12) we see
\[u(t), b(t) \in C([\tau, +\infty); V).\] (2.13)

In particular, (2.9) has a sense.

For the existence and uniqueness of solutions to (2.7)-(2.9), we have the following result.

**Lemma 2.3.** Let $f(t), g(t) \in L^2_{\text{loc}}(\mathbb{R}; V')$. Then for each $\tau \in \mathbb{R}$ and any $(u_\tau, b_\tau) \in W$, there exists a unique weak solution $(u, b) = (u(\cdot, \tau), b(\cdot, \tau))$ of (2.7)-(2.9). Moreover, if $f$ and $g$ belong to $L^2_{\text{loc}}(\mathbb{R}; H)$ and $(u_\tau, b_\tau) \in D(A) \times D(A)$, then the weak solution satisfies $(u, b) \in C([\tau, +\infty); D(A) \times D(A))$, $(u', b') \in L^2([\tau, T); D(A) \times D(A))$ for all $T > \tau$. Furthermore,
\[\|(u, b)\|^2_W + \varepsilon \int_\tau^t e^{\sigma(s-t)}(\|\nabla u(s)\|^2 + \|\nabla b(s)\|^2)ds \leq e^{\sigma(t-\tau)}\|(u_\tau, b_\tau)\|^2_W + \frac{1}{\varepsilon} \int_\tau^t e^{\sigma(s-t)}(\|f(s)\|^2_2 + \|g(s)\|^2_2)ds, \quad \forall t > \tau,\] (2.14)

where
\[\|(u, b)\|^2_W \triangleq \|u\|^2 + \|b\|^2 + \alpha^2\|\nabla u\|^2 + \beta^2\|\nabla b\|^2,\] (2.15)
\[0 < \sigma < \min \left\{2\nu \left(\frac{1}{\lambda_1} + \alpha^2\right)^{-1}, 2\mu \left(\frac{1}{\lambda_1} + \beta^2\right)^{-1}\right\},\]
\[\varepsilon \triangleq \min \left\{\nu - \frac{\sigma}{2} \left(\frac{1}{\lambda_1} + \alpha^2\right), \mu - \frac{\sigma}{2} \left(\frac{1}{\lambda_1} + \beta^2\right)\right\} > 0,\] (2.17)

and $\lambda_1$ is the first eigenvalue of the operator $A$.

**Proof.** For the existence and regularity of the solutions, we can proceed using the Galerkin scheme as that as in [11]. Here we omit the details. We next prove the energy inequality (2.14).

Multiplying $u$ with (2.7) and $b$ with (2.8), respectively, and then adding the resulting equalities give
\[\frac{1}{2} \frac{d}{dt}(\|u(t)\|^2 + \|b(t)\|^2 + \alpha^2\|\nabla u(t)\|^2 + \beta^2\|\nabla b(t)\|^2) + \nu \|\nabla u(t)\|^2 + \mu \|\nabla b(t)\|^2 = (f(t), u(t)) + (g(t), b(t)).\] (2.18)

Let $\sigma$ be some constant satisfying (2.16). By (2.18), we have, using the Poincaré inequality and the Hölder inequality,
\[\frac{d}{dt}(e^{\sigma(t)}(\|u(t)\|^2 + \|b(t)\|^2 + \alpha^2\|\nabla u(t)\|^2 + \beta^2\|\nabla b(t)\|^2)) = \sigma e^{\sigma(t)}(\|u(t)\|^2 + \|b(t)\|^2 + \alpha^2\|\nabla u(t)\|^2 + \beta^2\|\nabla b(t)\|^2)\]
+ \varepsilon^t (2\sigma f(t), u(t)) + 2\langle g(t), b(t) \rangle - 2\nu \|\nabla u(t)\|^2 - 2\mu \|\nabla b(t)\|^2 \\
\leq \varepsilon^t \left\{ \left( \frac{1}{\lambda_1} + \alpha^2 \right) - 2\nu \varepsilon + \varepsilon \right\} \|\nabla u(t)\|^2 + \left( \frac{1}{\lambda_1} + \beta^2 \right) - 2\mu \varepsilon \|\nabla b(t)\|^2 \\
+ \frac{\varepsilon^t}{\varepsilon} \left( \|f(t)\|^2_{V'} + \|g(t)\|^2_{V'} \right), \quad \text{a.e. } t > \tau, \quad (2.19)

where the constant \varepsilon is given by \eqref{2.17}. Hence \eqref{2.19} yields

\frac{d}{dt} [\varepsilon^t (\|u(t)\|^2 + \|b(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2 + \beta^2 \|\nabla b(t)\|^2)] \\
+ \varepsilon^2 \|\nabla u(t)\|^2 + \|\nabla b(t)\|^2 \leq \frac{\varepsilon^t}{\varepsilon} (\|f(t)\|^2_{V'} + \|g(t)\|^2_{V'}), \quad \text{a.e. } t > \tau. \quad (2.20)

Integrating \eqref{2.20}, we get \eqref{2.14}.

We next establish the uniqueness of the solutions. Let \((u^{(1)}, b^{(1)})\) and \((u^{(2)}, b^{(2)})\) be two solutions of \eqref{2.7}-\eqref{2.9} corresponding to the same data \(f, g, \tau\) and \((u_\tau, b_\tau)\). We denote \(\phi = u^{(1)} + \alpha^2 Au^{(1)}\), then by \eqref{2.6} and \eqref{2.7}, \(\phi' = \frac{\partial \phi}{\partial s} \in L^2(\tau, T; V')\) for all \(T > \tau\), and hence, as \((u^{(1)})' = C^{-1}_\tau \phi'\), we conclude that \((u^{(1)})' \in L^2(\tau, T; V)\) for all \(T > \tau\). Combining this consideration and the definition of the weak solutions of \eqref{2.7}-\eqref{2.9}, we have for \(i = 1, 2\) and for all \(t \geq \tau\) that

\begin{align*}
\dot{u}^{(i)}(t) + \alpha^2 Au^{(i)}(t) + \int_\tau^t \left( \nu Au^{(i)}(s) + B(u^{(i)}(s)) - B(b^{(i)}(s)) \right) ds \\
= u_\tau + \alpha^2 Au_\tau + \int_\tau^t f(s) ds, \quad \text{in } V', \quad (2.21)
\end{align*}

\begin{align*}
\dot{b}^{(i)}(t) + \beta^2 Ab^{(i)}(t) + \int_\tau^t \left( \mu Ab^{(i)}(s) + B(u^{(i)}(s), b^{(i)}(s)) - B(b^{(i)}(s), u^{(i)}(s)) \right) ds \\
= b_\tau + \beta^2 Ab_\tau + \int_\tau^t g(s) ds, \quad \text{in } V'. \quad (2.22)
\end{align*}

Set

\begin{align*}
\left\{ \begin{array}{l}
\ddot{u} = u^{(1)} - u^{(2)} \\
\dot{b} = b^{(1)} - b^{(2)}
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
\ddot{\nu} = \ddot{u} + \alpha^2 A\ddot{\nu} \\
\dot{\xi} = \dot{b} + \beta^2 A\dot{\xi}
\end{array} \right.
\end{align*}

Then both \(\ddot{\nu}\) and \(\dot{\xi}\) belong to \(C([\tau, +\infty); V')\) with

\begin{align*}
\ddot{\nu}(t) = -\nu \int_\tau^t A\ddot{u}(s) ds - \int_\tau^t \left( B(u^{(1)}(s), u^{(1)}(s)) - B(u^{(2)}(s), u^{(2)}(s)) \right) ds \\
+ \int_\tau^t \left( B(b^{(1)}(s), b^{(1)}(s)) - B(b^{(2)}(s), b^{(2)}(s)) \right) ds, \quad (2.23)
\end{align*}

\begin{align*}
\dot{\xi}(t) = -\mu \int_\tau^t A\ddot{b}(s) ds - \int_\tau^t \left( B(u^{(1)}(s), b^{(1)}(s)) - B(u^{(2)}(s), b^{(2)}(s)) \right) ds \\
+ \int_\tau^t \left( B(b^{(1)}(s), u^{(1)}(s)) - B(b^{(2)}(s), u^{(2)}(s)) \right) ds. \quad (2.24)
\end{align*}

By \eqref{2.2}, we get

\begin{align*}
\|B(u^{(1)}(s), u^{(1)}(s)) - B(u^{(2)}(s), u^{(2)}(s))\|_{V'} \\
= \sup_{w \in V, \|w\|_{V'} = 1} \left| (B(u^{(1)}(s), u^{(1)}(s), w) - B(u^{(2)}(s), u^{(2)}(s), w)) - (B(u^{(1)}(s), u^{(1)}(s), \psi) - B(u^{(2)}(s), u^{(2)}(s), \psi)) \right| \\
\leq c_1 (\|u^{(1)}(s)\|_V + \|u^{(2)}(s)\|_V) \|u^{(1)}(s) - u^{(2)}(s)\|_V.
\end{align*}
Similarly,
\[
\|B(b^{(1)}(s), b^{(1)}(s)) - B(b^{(2)}(s), b^{(2)}(s))\|_V' \\
\leq c_1 (\|b^{(1)}(s)\|_V + \|b^{(2)}(s)\|_V) \|b^{(1)}(s) - b^{(2)}(s)\|_V.
\]
\[
\|B(u^{(1)}(s), b^{(1)}(s)) - B(u^{(2)}(s), b^{(2)}(s))\|_V' \\
\leq c_1 \|b^{(1)}(s)\|_V \|u^{(1)}(s) - u^{(2)}(s)\|_V + c_1 \|u^{(2)}(s)\|_V \|b^{(1)}(s) - b^{(2)}(s)\|_V.
\]
\[
\|B(b^{(1)}(s), u^{(1)}(s)) - B(b^{(2)}(s), u^{(2)}(s))\|_V' \leq c_1 \|u^{(1)}(s)\|_V \|b^{(1)}(s) - b^{(2)}(s)\|_V \\
+ c_1 \|b^{(2)}(s)\|_V \|u^{(1)}(s) - u^{(2)}(s)\|_V.
\]

Therefore, for any given $T > \tau$, we set
\[
R_T \triangleq c_1 \max_{s \in [\tau, T]} \left( \|u^{(1)}(s)\|_V + \|u^{(2)}(s)\|_V + \|b^{(1)}(s)\|_V + \|b^{(2)}(s)\|_V \right),
\]
and obtain for any $s \in [\tau, T]$ that
\[
\|B(u^{(1)}(s), u^{(1)}(s)) - B(u^{(2)}(s), u^{(2)}(s))\|_V' \leq R_T \|\tilde{u}(s)\|_V, \tag{2.25}
\]
\[
\|B(b^{(1)}(s), b^{(1)}(s)) - B(b^{(2)}(s), b^{(2)}(s))\|_V' \leq R_T \|\tilde{b}(s)\|_V, \tag{2.26}
\]
\[
\|B(u^{(1)}(s), b^{(1)}(s)) - B(u^{(2)}(s), b^{(2)}(s))\|_V' \leq R_T (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V), \tag{2.27}
\]
\[
\|B(b^{(1)}(s), b^{(1)}(s)) - B(b^{(2)}(s), u^{(2)}(s))\|_V' \leq R_T (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V). \tag{2.28}
\]

Since $\|A\tilde{u}(s)\|_V' = \|\tilde{u}(s)\|_V$ and $\|A\tilde{b}(s)\|_V' = \|\tilde{b}(s)\|_V$, we deduce from (2.23)-(2.28) that
\[
\|\tilde{v}(t)\|_V' \leq \nu \int_\tau^t \|\tilde{u}(s)\|_V ds + R_T \int_\tau^t (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V) ds,
\]
\[
\|\tilde{\xi}(t)\|_V' \leq \mu \int_\tau^t \|\tilde{b}(s)\|_V ds + 2 R_T \int_\tau^t (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V) ds.
\]

Hence, by (2.11) and (2.12),
\[
\|\tilde{u}(t)\|_V \leq \nu \alpha^{-2} \int_\tau^t \|\tilde{u}(s)\|_V ds + \alpha^{-2} R_T \int_\tau^t (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V) ds,
\]
\[
\|\tilde{b}(t)\|_V \leq \mu \beta^{-2} \int_\tau^t \|\tilde{b}(s)\|_V ds + 2 \beta^{-2} R_T \int_\tau^t (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V) ds,
\]
and thus
\[
\|\tilde{u}(t)\|_V + \|\tilde{b}(t)\|_V \\
\leq (\nu \alpha^{-2} + \beta^{-2} \mu + (\alpha^{-2} + 2 \beta^{-2}) R_T) \int_\tau^t (\|\tilde{u}(s)\|_V + \|\tilde{b}(s)\|_V) ds,
\]
for all $t \in [\tau, T]$. From this inequality and the Gronwall inequality we obtain
\[
\|\tilde{u}(s)\|_V = \|\tilde{b}(s)\|_V = 0
\]
for all $s \in [\tau, T]$. The uniqueness of the weak solutions is proved. \qed

Lemma 2.3 shows the globally existence and uniqueness of the weak solutions to (2.7)-(2.9). In Section 4, we will prove that the solutions are continuous with respect to the initial data (see Lemma 4.4). Therefore, we can define the solution operators $U(t, \tau)$ as
\[
U(t, \tau) : (u_\tau, b_\tau) \in W \mapsto U(t, \tau)(u_\tau, b_\tau) = (u(t; \tau, u_\tau), b(t; \tau, b_\tau)) \in W, \ \forall t \geq \tau,
\]
which generates a continuous process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( W \), where \( (u(t; \tau, u_{\tau}), b(t; \tau, b_{\tau})) \) denotes the solution of (2.7)-(2.8) corresponding to the initial datum \( (u_{\tau}, b_{\tau}) \) at initial time \( \tau \). Moreover, we see from (2.13) that \( (u, b) \in C([\tau, T]; W) \) for all \( T > \tau \), which implies
\[
\begin{cases}
\text{for given } \tau \in \mathbb{R} \text{ and } (u_{\tau}, b_{\tau}) \in W, \text{ the function } \\
\quad \tau < t \mapsto U(t, \tau)(u_{\tau}, b_{\tau}) \text{ is continuous with values in } W.
\end{cases}
\tag{2.29}
\]

3. **Existence of the pullback attractors.** In this section, we first introduce some definitions concerning the pullback attractors. Then we establish the existence of pullback attractors for \( \{U(t, \tau)\}_{t \geq \tau} \) in \( W \).

In the sequel, we use \( \mathcal{P}(W) \) to denote the family of all nonempty subsets of \( W \), and consider a family of nonempty sets \( \hat{D}_{0} = \{D_{0}(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(W) \). Let \( D \) be a given nonempty class of families parameterized in time \( \hat{D} = \{D(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(W) \). The class \( D \) will be called a universe in \( \mathcal{P}(W) \).

**Definition 3.1.** It is said that \( \hat{D}_{0} = \{D_{0}(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(W) \) is pullback \( D \)-absorbing for the process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( W \) if for any \( t \in \mathbb{R} \) and any \( D = \{D(t) | t \in \mathbb{R}\} \in D \), there exists a \( \tau_{0}(t, \hat{D}) \leq t \) such that \( U(t, \tau)D(\tau) \subseteq D_{0}(t) \) for all \( \tau \leq \tau_{0}(t, \hat{D}) \).

**Definition 3.2.** The process \( \{U(t, \tau)\}_{t \geq \tau} \) is said to be pullback \( \hat{D}_{0} \)-asymptotically compact if for any \( t \in \mathbb{R} \) and any sequences \( \{\tau_{n}\} \subseteq (-\infty, t] \) and \( \{(x_{n}, y_{n})\} \subseteq W \) satisfying \( \tau_{n} \to -\infty \) and \( (x_{n}, y_{n}) \in D_{0}(\tau_{n}) \) for all \( n \), the sequence \( \{U(t, \tau_{n})(x_{n}, y_{n})\} \) is relatively compact in \( W \). \( \{U(t, \tau)\}_{t \geq \tau} \) is said to be pullback \( D \)-asymptotically compact if it is \( \hat{D} \)-asymptotically compact for any \( \hat{D} \in D \).

**Definition 3.3.** A family \( \mathcal{A}_{D} = \{A_{D}(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(W) \) is said to be a pullback \( D \)-attractor for the process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( W \) if it has the following properties:

(a) **Compactness:** for each \( t \in \mathbb{R} \), \( A_{D}(t) \) is a nonempty compact subset of \( W \);

(b) **Invariance:** \( U(t, \tau)A_{D}(\tau) = A_{D}(t), \quad \forall \tau \leq t \);

(c) **Pullback attracting:** \( A_{D} \) is pullback \( D \)-attracting in the following sense
\[
\lim_{\tau \to -\infty} \operatorname{dist}_{W}(U(t, \tau)D(\tau), A_{D}(t)) = 0, \quad \forall \hat{D} = \{D(t) | t \in \mathbb{R}\} \in D, \quad t \in \mathbb{R}.
\]

References [3, 11] proved the general existence and minimality results of a pullback attractor and its property for general process. For example, García-Luengo, Marín-Rubio and Real in [11] pointed out that the family \( \mathcal{A}_{D} \) is minimal in the sense that if \( \hat{C} = \{C(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(W) \) is a family of closed sets such that for any \( \hat{D} = \{D(t) | t \in \mathbb{R}\} \in D \),
\[
\lim_{\tau \to -\infty} \operatorname{dist}_{W}(U(t, \tau)D(\tau), C(t)) = 0,
\]
then \( A_{D}(t) \subseteq C(t) \).

To prove the existence of the pullback attractor for \( \{U(t, \tau)\}_{t \geq \tau} \) in \( W \), we need the following assumption on the external force functions \( f(t) \) and \( g(t) \).

**Assumption \((H_{1})\).** Assume \( f, g \in L_{loc}^{2}(\mathbb{R}; V') \) and
\[
\int_{-\infty}^{t} e^{\sigma s}(\|f(s)\|_{V'}^{2} + \|g(s)\|_{V'}^{2}) \, ds < +\infty, \quad \text{for all } t \in \mathbb{R},
\tag{3.1}
\]
for some constant \( \sigma \) satisfying (2.16).
From now on, we denote by $\mathcal{D}_\sigma$ the class of all families of nonempty subsets $\hat{D}(t) = \{D(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(W)$ such that
\[
\lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{(u,b) \in D(\tau)} \|u, b\|_W^2 \right) = 0. \tag{3.2}
\]

**Lemma 3.4.** Let assumption $(H_1)$ hold. Then the process $\{U(t, \tau)\}_{t \geq \tau}$ possesses a pullback $\mathcal{D}_\sigma$-absorbing set in space $W$.

**Proof.** Set
\[
R_\sigma(t) \equiv 1 + \frac{e^{-\sigma t}}{\varepsilon} \int_{-\infty}^{t} e^{\sigma s} \left( \|f(s)\|_{V'}^2 + \|g(s)\|_{V'}^2 \right) ds \tag{3.3}
\]
and
\[
\hat{D}_\sigma(t) = \{D_\sigma(t)|t \in \mathbb{R}\} \text{ with } D_\sigma(t) = \mathcal{B}_W(0, R_\sigma^{1/2}(t)), \quad t \in \mathbb{R}, \tag{3.4}
\]
where $\mathcal{B}_W(0, R_\sigma^{1/2}(t))$ denotes the family of closed balls in space $W$ centered at zero and with radius $R_\sigma^{1/2}(t)$. Then from (2.14) we see that the family $\hat{D}_\sigma(t)$ defined by (3.4) is the desired pullback $\mathcal{D}_\sigma$-absorbing set for $\{U(t, \tau)\}_{t \geq \tau}$ in $W$.

We next use the approach of energy equation to investigate the pullback asymptotic compactness of the process $\{U(t, \tau)\}_{t \geq \tau}$ in $W$ for the universe $\mathcal{D}_\sigma$. To this end, we first introduce two projection operators and then give some convergent relations for the solutions. The projection operators $\Pi_1$ and $\Pi_2$ are defined as
\[
\Pi_1 : W = V \times V \mapsto V, \quad \Pi_1(u, b) = u, \quad \forall (u, b) \in W; \tag{3.5}
\]
\[
\Pi_2 : W = V \times V \mapsto V, \quad \Pi_2(u, b) = b, \quad \forall (u, b) \in W. \tag{3.6}
\]

**Lemma 3.5.** Let assumption $(H_1)$ hold and $\tau < t$ be given. Consider a sequence $\{(u_{\tau,n}, b_{\tau,n})\} \subset W$ weakly converging to $(u_, b_\tau)$ in $W$ as $n \to \infty$. Then the following convergent relations hold:
\[
\Pi_1U(\cdot, s)(u_{\tau,n}, b_{\tau,n}) \rightarrow \Pi_1U(\cdot, s)(u_\tau, b_\tau) \quad \text{weakly in } L^2(\tau, t; V), \tag{3.7}
\]
\[
\Pi_2U(\cdot, s)(u_{\tau,n}, b_{\tau,n}) \rightarrow \Pi_2U(\cdot, s)(u_\tau, b_\tau) \quad \text{weakly in } L^2(\tau, t; V), \tag{3.8}
\]
\[
U(t, \tau)(u_{\tau,n}, b_{\tau,n}) \rightarrow U(t, \tau)(u_\tau, b_\tau) \quad \text{weakly in } W. \tag{3.9}
\]

**Proof.** The proof of (3.7)-(3.8) can be done similarly to that of [11, Theorem 4, (2.26)], and the proof of (3.9) can be done analogously to that of [11, Theorem 6], since the a priori estimates follow the similar. The fact that the whole sequence meets the above convergent relations is a consequence of the uniqueness of the weak solution for (2.7)-(2.9).

**Lemma 3.6.** Let assumption $(H_1)$ hold. Then the process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}_\sigma$-asymptotically compact in space $W$.

**Proof.** Let $t \in \mathbb{R}$, $\{\tau_n\} \subseteq (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{(u_{\tau_n}, b_{\tau_n})\} \subseteq D_\sigma(\tau_n)$ for all $n$ be given. We shall prove that the sequence $\{U(t, \tau_n)(u_{\tau_n}, b_{\tau_n})\}_{n \geq 1}$ is relatively compact in $W$.

From Lemma 3.4, we see that for each integer $k \geq 0$, there exists some $\tau(\hat{D}_\sigma, k) \leq t - k$ such that $U(t - k, \tau)D_\sigma(\tau) \subseteq D_\sigma(t - k)$ for all $\tau \leq \tau(\hat{D}_\sigma, k)$. Note that each $D_\sigma(t)$ defined by (3.4) is a bounded set in $W$. By this fact and the diagonal argument, we can extract a subsequence $\{(u_{\tau_n'}, b_{\tau_n'})\}$ of $\{(u_{\tau_n}, b_{\tau_n})\}$ such that
\[
U(t - k, \tau_n')(u_{\tau_n'}, b_{\tau_n'}) \rightarrow (\phi_k, \psi_k) \quad \text{weakly in } W, \quad \text{for each } k \geq 0. \tag{3.10}
\]
where \((\phi_k, \psi_k) \in D_\sigma(t-k)\). For brevity, we use “\(W\)-weak” to denote the weak limit in \(W\) and set \(U(t-k)(\phi_k, \psi_k) = (\phi_*, \psi_*)\). Then applying Lemma 3.5 on each fixed interval \([t-k, t]\), as well as the continuous property of the process \(\{U(t, \tau)\}_{t \geq \tau}\), we obtain

\[
U(t, t-k)(\phi_k, \psi_k) = U(t, t-k)[W - \text{weak } \lim_{n' \to \infty} U(t-k, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})]
\]

\[
= W - \text{weak } \lim_{n' \to \infty} U(t-k)(U(t-k, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}}))
\]

\[
= W - \text{weak } \lim_{n' \to \infty} U(t, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})
\]

\[
= (\phi_*, \psi_*) \tag{3.11}
\]

By the lower semicontinuity of the norm, we have

\[
||((\phi_*, \psi_*))||_W \leq \liminf_{n' \to \infty} ||U(t, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})||_W. \tag{3.12}
\]

We next aim to establish

\[
||((\phi_*, \psi_*))||_W \geq \limsup_{n' \to \infty} ||U(t, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})||_W. \tag{3.13}
\]

By (2.18), we have

\[
\frac{d}{dt}(e^{\sigma t}(\|u(t)\|^2 + \|b(t)\|^2 + \alpha^2\|\nabla u(t)\|^2 + \beta^2\|\nabla b(t)\|^2))
\]

\[
= e^{\sigma t}(\sigma\|u(t)\|^2 + (\sigma\alpha^2 - 2\nu\|\nabla u(t)\|^2 + \sigma\|b(t)\|^2 + (\sigma\beta^2 - 2\mu\|\nabla b(t)\|^2)
\]

\[
+ 2e^{\sigma t}((f(t), u(t)) + (g(t), b(t))), \text{ a.e. } t > \tau. \tag{3.14}
\]

We now define \([\cdot, \cdot]_1 : V \times V \to \mathbb{R}\) and \([\cdot, \cdot]_2 : V \times V \to \mathbb{R}\) as

\[
[u, v]_1 = (2\nu - \sigma\alpha^2)(\nabla u, \nabla v) - \sigma(u, v), \quad \forall u, v \in V,
\]

\[
[b, \xi]_2 = (2\mu - \sigma\beta^2)(\nabla b, \nabla \xi) - \sigma(b, \xi), \quad \forall b, \xi \in V.
\]

Then by the choice of the constant \(\sigma\) (see (2.16)), we find

\[
(2\nu - \sigma\alpha^2 - \frac{\sigma}{\lambda_1})\|\nabla u\|^2 \leq [u, u]_1 = (2\nu - \sigma\alpha^2)\|\nabla u\|^2 - \sigma\|u\|^2
\]

\[
\leq (2\nu - \sigma\alpha^2)\|\nabla u\|^2, \quad \forall u \in V. \tag{3.15}
\]

Similarly,

\[
(2\mu - \sigma\beta^2 - \frac{\sigma}{\lambda_1})\|\nabla b\|^2 \leq [b, b]_2 = (2\mu - \sigma\beta^2)\|\nabla b\|^2, \quad \forall b \in V. \tag{3.16}
\]

(3.15) and (3.16) imply that both \([\cdot, \cdot]_1\) and \([\cdot, \cdot]_2\) define equivalent norms of \(\cdot \|V\|.\)

Hence, (3.14) can be written as the following energy differential equality

\[
\frac{d}{dt}(e^{\sigma t|[u, b]_W^2}) = e^{\sigma t}([u]^2 + [b]_2^2) + 2e^{\sigma t}((f, u) + (g, b)), \text{ a.e. } t > \tau. \tag{3.17}
\]

We integrate the above differential equality in the interval \([t-k, t]\) for the solutions \(U(\cdot, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})\) with \(\tau_{n'} \leq t - k\) and obtain

\[
||U(t, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})||_W^2
\]

\[
= ||U(t-k, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})||_W^2
\]

\[
= e^{-k||U(t-k, \tau_{n'})(u_{\tau_{n'}}, b_{\tau_{n'}})||_W^2}
\]
At the same time, we see that
\[ L \text{ continuity of the norm that } \| \| \text{ for each } k \geq 0. \]
Thus, we conclude from (3.18)-(3.19) and the lower semi-
\[ \sigma \text{ for each } k \geq 0. \]
where the operators \( \Pi_1 \) and \( \Pi_2 \) are defined by (3.5) and (3.6). Now, by (3.7) and (3.10), we see that
\[ \Pi_1 U(t, t-k)U(t-k, \tau_{n'}) \to U(t-k) \phi_k, \psi_k \] (3.19)
weakly in \( L^2(t-k, t; V) \) for each \( k \geq 0 \). Also by (3.8) and (3.10),
\[ \Pi_2 U(t, t-k)U(t-k, \tau_{n'}) \to U(t-k) \phi_k, \psi_k \] (3.20)
weakly in \( L^2(t-k, t; V) \) for each \( k \geq 0 \). Since both \( \sigma^{(s-t)} f(\cdot) \) and \( \sigma^{(s-t)} g(\cdot) \) belong to \( L^2(t-k, t; V') \), we get
\[ \lim_{n' \to \infty} \int_{t-k}^t \sigma^{(s-t)} (f(s), \Pi_1 U(s, t-k)U(t-k, \tau_{n'}) (u_{r_n}, b_{r_n})) ds \]
\[ = \int_{t-k}^t \sigma^{(s-t)} (f(s), \Pi_1 U(s, t-k) \phi_k, \psi_k)) ds, \] (3.21)
\[ \lim_{n' \to \infty} \int_{t-k}^t \sigma^{(s-t)} (g(s), \Pi_2 U(s, t-k)U(t-k, \tau_{n'}) (u_{r_n}, b_{r_n})) ds \]
\[ = \int_{t-k}^t \sigma^{(s-t)} (g(s), \Pi_2 U(s, t-k) \phi_k, \psi_k) ds. \] (3.22)
At the same time, we see that \( (\int_{t-k}^t \sigma^{(s-t)} [f(s)]^2 ds)^{1/2} \) and \( (\int_{t-k}^t \sigma^{(s-t)} [g(s)]^2 ds)^{1/2} \) define equivalent norms of \( \| \cdot \|_{L^2((t-k), t; V)} \), since both \( [\cdot]_1 \) and \( [\cdot]_2 \) define equivalent norms of \( \| \cdot \|_{V} \). Therefore, we conclude from (3.18)-(3.19) and the lower semi-
\[ \sigma \text{ continuity of the norm that } \| \| \text{ for each } k \geq 0. \]
Taking Lemma 3.4, (3.18), (3.21)-(3.24) into account, we get
\[ \lim_{n' \to \infty} \| U(t, \tau_{n'}) (u_{r_n}, b_{r_n}) \|_{W}^2 \]
\[ \leq e^{-\sigma k} R_{\sigma} (t-k) - \int_{t-k}^t \sigma^{(s-t)} [\Pi_1 U(s, t-k) \phi_k, \psi_k) ds \]
Comparing (3.25) and (3.26), we see that

$$- \int_{t-k}^{t} \sigma(s-t)[\Pi_{2}U(s, t-k)(\phi_{k}, \psi_{k})]^{2}ds$$

$$+ 2 \int_{t-k}^{t} \sigma(s-t)\langle f(s), \Pi_{1}U(s, t-k)(\phi_{k}, \psi_{k})\rangle ds$$

$$+ 2 \int_{t-k}^{t} \sigma(s-t)\langle g(s), \Pi_{2}U(s, t-k)(\phi_{k}, \psi_{k})\rangle ds. \quad (3.25)$$

Since $(\phi_{*}, \psi_{*}) = U(t, t-k)(\phi_{k}, \psi_{k})$, integrating again the differential equality (3.16), we have

$$\|\phi_{*}, \psi_{*}\|_{W} = e^{-\sigma k}\|\phi_{k}, \psi_{k}\|_{W} - \int_{t-k}^{t} \sigma(s-t)[\Pi_{2}U(\phi_{k}, \psi_{k})]^{2}ds$$

$$- \int_{t-k}^{t} \sigma(s-t)[\Pi_{2}U(s, t-k)(\phi_{k}, \psi_{k})]^{2}ds$$

$$+ 2 \int_{t-k}^{t} \sigma(s-t)\langle f(s), \Pi_{1}U(s, t-k)(\phi_{k}, \psi_{k})\rangle ds$$

$$+ 2 \int_{t-k}^{t} \sigma(s-t)\langle g(s), \Pi_{2}U(s, t-k)(\phi_{k}, \psi_{k})\rangle ds. \quad (3.26)$$

Comparing (3.25) and (3.26), we see that

$$\lim_{n' \to \infty} \|U(t, \tau_{n})(u_{\tau_{n}}, b_{\tau_{n}})\|_{W}^{2} \leq e^{-\sigma k}R_{\sigma}(t-k) + \|\phi_{*}, \psi_{*}\|_{W}^{2}, \quad \forall k \geq 0. \quad (3.27)$$

By assumption (H$_4$) and (3.2), we have $\lim_{k \to \infty} e^{-\sigma k}R_{\sigma}(t-k) = 0$. Then (3.27) implies that (3.13) holds true. It follows from (3.12) and (3.13) that

$$\lim_{n' \to \infty} \|U(t, \tau_{n})(u_{\tau_{n}}, b_{\tau_{n}})\|_{W} = \|\phi_{*}, \psi_{*}\|_{W},$$

which together with (3.10) yields

$$U(t, \tau_{n})(u_{\tau_{n}}, b_{\tau_{n}}) \to (\phi_{*}, \psi_{*}) \text{ strongly in } W.$$ 

The proof of Lemma 3.6 is completed. \qed

At this stage, we combine Lemma 3.4 and Lemma 3.6 with García-Luengo et al. [11, Theorem 3.11, Corollary 3.13] to get the main result of this section as follows.

**Theorem 3.7.** Let assumption (H$_1$) hold. Then there exists the minimal pullback $\mathcal{D}_{\sigma}$-attractor $\tilde{\mathcal{A}}_{\mathcal{D}_{\sigma}} = \{\mathcal{A}_{\mathcal{D}_{\sigma}}|t \in \mathbb{R}\}$ for the process $\{U(t, \tau)\}_{\mathcal{T}}$ in $W$, which satisfies:

(a) **Compactness:** for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_{\sigma}}(t)$ is a nonempty compact subset of $W$;

(b) **Invariance:** $U(t, \tau)\mathcal{A}_{\mathcal{D}_{\sigma}}(\tau) = \mathcal{A}_{\mathcal{D}_{\sigma}}(t)$, $\forall \tau \leq t$;

(c) **Pullback attracting:** $\tilde{\mathcal{A}}_{\mathcal{D}_{\sigma}}$ is pullback $\mathcal{D}_{\sigma}$-attracting in the following sense

$$\lim_{\tau \to -\infty} \text{dist}_{W}(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma}}(t)) = 0, \quad \forall D = \{D(t)|t \in \mathbb{R}\} \in \mathcal{D}_{\sigma}, t \in \mathbb{R}.$$ 

4. **Invariant measures on the pullback attractor.** The aim of this section is to apply the theory of Lukaszewicz and Robinson [20] to prove the unique existence of invariant Borel probability measures on the pullback $\mathcal{D}_{\sigma}$-attractor $\tilde{\mathcal{A}}_{\mathcal{D}_{\sigma}}$.

We first cite two definitions. For the basic properties of generalized Banach limits we refer the reader to [9, 10, 19].

...
Definition 4.1. ([10, 20]) A generalized Banach limit is any linear functional, which we denote by $\text{LIM}_{T \to \infty}$, defined on the space of all bounded real-valued functions on $[0, +\infty)$ that satisfies

(i) $\text{LIM}_{T \to \infty} \chi(T) \geq 0$ for nonnegative functions $\chi$;

(ii) $\text{LIM}_{T \to \infty} \chi(T) = \lim_{T \to \infty} \chi(T)$ if the usual limit $\lim_{T \to \infty} \chi(T)$ exists.

Definition 4.2. ([20]) A process $\{U(t, \tau)\}_{t \geq \tau}$ is said to be $\tau$-continuous on a metric space $X$ if for every $u_0 \in X$ and every $t \in \mathbb{R}$, the X-valued function $\tau \mapsto U(t, \tau)u_0$ is continuous and bounded on $(-\infty, t]$.

The following result was proved by Łukaszewicz and Robinson in [20].

Lemma 4.3. ([20]) Let $\{U(t, \tau)\}_{t \geq \tau}$ be a $\tau$-continuous evolutionary process in a complete metric space $X$ that has a pullback $D$-attractor $A(\cdot)$. Fix a generalized Banach limit $\text{LIM}_{T \to \infty}$ and let $\gamma : \mathbb{R} \rightarrow X$ be a continuous map such that $\gamma(\cdot) \in D$. Then there exists a family of Borel probability measures $\{\mu_t\}_{t \in \mathbb{R}}$ in $X$ such that the support of the measure $\mu_t$ is contained in $A(t)$ and

$$\text{LIM}_{T \to \infty} \frac{1}{t - \tau} \int_{\tau}^{t} \varphi(U(t, \gamma(s)))ds = \int_{A(t)} \varphi(v)d\mu_t(v)$$

for any real-valued continuous functional $\varphi$ on $X$. In addition, $\mu_t$ is invariant in the sense that

$$\int_{A(t)} \varphi(v)d\mu_t(v) = \int_{A(t)} \varphi(U(t, \tau)v)d\mu_\tau(v), \quad t \geq \tau.$$ 

In order to employ the above result to the pullback $D_\sigma$-attractor $\hat{A}_{D_\sigma}$ obtained in Theorem 3.7, we need check the $\tau$-continuous property of the process $\{U(t, \tau)\}_{t \geq \tau}$ in space $W$. We begin with the following estimation.

Lemma 4.4. Let $(u^{(1)}, b^{(1)})$ and $(u^{(2)}, b^{(2)})$ be two solutions of (2.7)-(2.9) corresponding to the initial data $(u_{1\tau}, b_{1\tau})$ and $(u_{2\tau}, b_{2\tau}) \in W$, respectively. Then there exists some positive constant $c(\nu, \mu, \alpha, \beta, \Lambda_1)$ such that

$$\| (u^{(1)} - u^{(2)}, b^{(1)} - b^{(2)}) \|_W \leq \| (u_{1\tau} - u_{2\tau}, b_{1\tau} - b_{2\tau}) \|_W^2 \times \exp \left\{ c(\nu, \mu, \alpha, \beta, \lambda_1, \Omega) \int_{\tau}^{t} \left( \| u^{(1)}(s) \|_V^2 + \| b^{(1)}(s) \|_V^2 \right) ds \right\}. \tag{4.1}$$

Proof. Let $(u^{(1)}, b^{(1)})$ and $(u^{(2)}, b^{(2)})$ be two solutions of (2.7)-(2.9) corresponding to the initial data $(u_{1\tau}, b_{1\tau})$, $(u_{2\tau}, b_{2\tau}) \in W$, respectively. Set $\tilde{u} = u^{(1)} - u^{(2)}$ and $\tilde{b} = b^{(1)} - b^{(2)}$, then we have

$$\frac{d}{dt} (\tilde{u} - \alpha^2 \Delta \tilde{u}) - \nu \Delta \tilde{u} + B(u^{(1)}) - B(u^{(2)}) - B(b^{(1)}) + B(b^{(2)}) = 0, \tag{4.2}$$

$$\frac{d}{dt} (\tilde{b} - \alpha^2 \Delta \tilde{b}) - \mu \Delta \tilde{b} + B(u^{(1)}, b^{(1)}) - B(u^{(2)}, b^{(2)}) - B(b^{(1)}, u^{(1)}) + B(b^{(2)}, u^{(2)}) = 0. \tag{4.3}$$

Using $\tilde{u}$ to multiply (4.2) gives

$$\frac{1}{2} \frac{d}{dt} (\| \tilde{u} \|^2 + \alpha^2 \| \nabla \tilde{u} \|^2) + \nu \| \nabla \tilde{u} \|^2$$
Similarly, multiplying (4.3) by \(\tilde{b}\) yields
\[
\frac{1}{2} \frac{d}{dt} (||\tilde{u}||^2 + \beta^2 ||\nabla \tilde{u}||^2) + \mu ||\nabla \tilde{b}||^2 + (B(u^{(1)}, b^{(1)})) - B(u^{(2)}, b^{(2)}), \tilde{b})
- (B(b^{(1)}, u^{(1)})) - B(b^{(2)}, u^{(2)}), \tilde{b}) = 0.
\tag{4.6}
\]

Now, using (2.5) we obtain
\[
(B(u^{(1)}, u^{(1)})) - B(u^{(2)}, u^{(2)}), \tilde{u}) = (B(\tilde{u}, u^{(1)}), \tilde{u}),
\tag{4.7}
\]
\[
(B(u^{(1)}, b^{(1)})) - B(u^{(2)}, b^{(2)}), \tilde{b}) = (B(\tilde{u}, b^{(1)}), \tilde{b}),
\tag{4.8}
\]
\[
(B(b^{(1)}, b^{(1)})) - B(b^{(2)}, b^{(2)}), \tilde{u}) = (B(\tilde{b}, b^{(1)}), \tilde{u}) + (B(b^{(2)}, \tilde{b}), \tilde{u}),
\tag{4.9}
\]
\[
(B(b^{(1)}, u^{(1)})) - B(b^{(2)}, u^{(2)}), \tilde{b}) = (B(\tilde{b}, u^{(1)}), \tilde{b}) + (B(b^{(2)}, \tilde{b}), \tilde{b}).
\tag{4.10}
\]

It follows from (4.5)-(4.10) that
\[
\frac{1}{2} \frac{d}{dt} (||\tilde{u}||^2 + \alpha^2 ||\nabla \tilde{u}||^2 + ||\tilde{b}||^2 + \beta^2 ||\nabla \tilde{u}||^2) + \nu ||\nabla \tilde{u}||^2 + \mu ||\nabla \tilde{b}||^2
- (B(\tilde{u}, u^{(1)}), \tilde{u}) - (B(\tilde{u}, b^{(1)}), \tilde{b})
+ (B(\tilde{b}, b^{(1)}), \tilde{u}) + (B(\tilde{b}, u^{(1)}), \tilde{b}).
\tag{4.11}
\]

By the Hölder inequality, the Gagliardo-Nirenberg inequality and the Poincaré inequality,
\[
|\langle B(\tilde{u}, u^{(1)}), \tilde{u} \rangle| \leq ||\tilde{u}||^2_{L^\infty(\Omega)} ||\nabla u^{(1)}|| \leq c(\Omega)||\tilde{u}||^{3/2}||\nabla \tilde{u}||^{1/2}||u^{(1)}||_V
\leq c(\lambda_1, \Omega)||\tilde{u}||||\nabla \tilde{u}||^{1/2}||u^{(1)}||_V
\leq \nu \frac{\nu}{6} ||\nabla \tilde{u}||^2 + c(\nu, \lambda_1, \Omega)||\tilde{u}||^2 ||u^{(1)}||_V^2.
\tag{4.12}
\]

Analogously,
\[
|\langle B(\tilde{u}, b^{(1)}), \tilde{b} \rangle| \leq \nu \frac{\nu}{6} ||\nabla \tilde{u}||^2 + c(\nu, \mu, \lambda_1, \Omega)(||\tilde{u}||^2 + ||\tilde{b}||^2)||b^{(1)}||_V^2,
\tag{4.13}
\]
\[
|\langle B(\tilde{b}, b^{(1)}), \tilde{u} \rangle| \leq \nu \frac{\nu}{6} ||\nabla \tilde{u}||^2 + c(\nu, \mu, \lambda_1, \Omega)(||\tilde{u}||^2 + ||\tilde{b}||^2)||b^{(1)}||_V^2,
\tag{4.14}
\]
\[
|\langle B(\tilde{b}, u^{(1)}), \tilde{b} \rangle| \leq \mu \frac{\nu}{6} ||\nabla \tilde{b}||^2 + c(\mu, \lambda_1, \Omega)||\tilde{b}||^2 ||u^{(1)}||_V^2.
\tag{4.15}
\]

Inserting the estimations (4.12)-(4.15) into (4.11) and integrating the resulting inequality on \([\tau, t]\), we obtain, using \(c(\nu, \mu)(||\tilde{u}||^2 + ||\tilde{b}||^2) \leq c(\nu, \mu, \alpha, \beta, \lambda_1)(||\tilde{u}||^2 + ||\tilde{b}||^2)|W_\omega|^2_{W}||u^{(1)}||_V^2 + ||u^{(1)}||_V^2 + ||b^{(1)}||_V^2\),
\[
||u(t), b(t)||^2_W \leq ||(u(\tau), b(t))||^2_W
+ c(\nu, \mu, \alpha, \beta, \lambda_1, \Omega) \int_\tau^t ||(\tilde{u}(s), \tilde{b}(s))||^2_W (||u^{(1)}(s)||_V^2 + ||b^{(1)}(s)||_V^2)ds.
\tag{4.16}
\]

Applying the Gronwall inequality to (4.16) gives (4.1). The proof of Lemma 4.4 is completed.

As pointed out in Section 2, Lemma 4.4 shows that the solutions of (2.7)-(2.9) depend continuously on the initial data in the strong topology of space \(W\).

**Lemma 4.5.** Let assumption \((H_1)\) hold. Then for every \((u_*, b_*) \in W\) and every \(t \in \mathbb{R}\), the \(W\)-valued function \(\tau \mapsto U(t, \tau)(u_*, b_*)\) is continuous and bounded on \((-\infty, t]\).
Proof. Consider any \((u_*, b_*) \in W\) and \(t \in \mathbb{R}\). We shall prove that for any \(\epsilon > 0\) there exists some \(\delta = \delta(\epsilon) > 0\), such that if \(r < t, s < t\) and \(|r - s| < \delta\), then \(\|U(t, r)(u_*, b_*) - U(t, s)(u_*, b_*)\|_W < \epsilon\). We assume that \(r < s\) without loss of generality. Employing Lemma 4.4 and the property of the continuous process, we have

\[
\begin{align*}
\|U(t, r)(u_*, b_*) - U(t, s)(u_*, b_*)\|_W^2 &= \|U(t, s)U(s, r)(u_*, b_*) - U(t, s)U(r, r)(u_*, b_*)\|_W^2 \\
&\leqslant e^\nu \|U(s, r)(u_*, b_*)\|_W^2 + \|U(r, r)(u_*, b_*)\|_W^2 \\
&\quad \times \exp \left\{ \int_t^s \sum_{i=1}^2 \|\Pi_i U(\theta, r)(u_*, b_*)\|_V^2 \, d\theta \right\}.
\end{align*}
\]

(4.17)

Notice that the solutions of (2.7)-(2.9) belong to \(L^2_{\text{loc}}(\mathbb{R}; W)\), hence

\[
\int_t^s \sum_{i=1}^2 \|\Pi_i U(\theta, r)(u_*, b_*)\|_V^2 \, d\theta < +\infty.
\]

From (2.29) we conclude that the right hand side of inequality (4.17) is as small as needed if \(|r - s|\) is small enough. Therefore, the \(W\)-valued function \(\tau \mapsto U(t, \tau)(u_*, b_*)\) is continuous in space \(W\). Now, by (2.14) and assumption (H1) we have for any \(\tau \in \mathbb{R}\) and \(t > \tau\) that

\[
\begin{align*}
\|U(t, \tau)(u_*, b_*)\|_W^2 &\leqslant e^{\sigma(t-\tau)} \|(u_*, b_*)\|_W^2 + \frac{1}{\epsilon} \int_\tau^t e^{\sigma(s-\tau)} (\|f(s)\|_V^2 + \|g(s)\|_V^2) \, ds \\
&\leqslant \|(u_*, b_*)\|_W^2 + \frac{\epsilon t}{\epsilon} \int_\tau^t e^{\sigma(s)} (\|f(s)\|_V^2 + \|g(s)\|_V^2) \, ds.
\end{align*}
\]

(4.18)

(4.18) shows that for every \((u_*, b_*) \in W\) and every \(t \in \mathbb{R}\), the \(W\)-valued function \(\tau \mapsto U(t, \tau)(u_*, b_*)\) is bounded on \((-\infty, t]\). The proof is completed. \qed

At this stage, we take Theorem 3.7, Lemma 4.3 and Lemma 4.5 into account and obtain the following result.

Theorem 4.6. Suppose assumption (H1) hold. Let \(\{U(t, \tau)\}_{\tau \geqslant t}\) be the process associated to the solution operators of (2.7)-(2.9) and \(\tilde{\mathcal{A}}_{\mathcal{D}}\) be the pullback \(\mathcal{D}\)-attractor obtained in Theorem 3.7. Fix a generalized Banach limit \(\text{LIM}_{T \to \infty}\) and let \(\psi : \mathbb{R} \to W\) be a continuous map such that \(\psi(\cdot) \in \mathcal{D}_{\mathcal{G}}\). Then there exists a unique family of Borel probability measures \(\{m_t\}_{t \in \mathbb{R}}\) in space \(W\) such that the support of the measure \(m_t\) is contained in \(\tilde{\mathcal{A}}_{\mathcal{D}_{\mathcal{G}}}(t)\) and

\[
\text{LIM}_{\tau \to -\infty} \frac{1}{t-\tau} \int_\tau^t \phi(U(t, s) \psi(s)) \, ds = \int_{\tilde{\mathcal{A}}_{\mathcal{D}_{\mathcal{G}}}(t)} \phi(w) \, dm_t(w)
\]

for any real-valued continuous functional \(\phi\) on \(W\). In addition, \(m_t\) is invariant in the sense that

\[
\int_{\tilde{\mathcal{A}}_{\mathcal{D}_{\mathcal{G}}}(t)} \phi(w) \, dm_t(w) = \int_{\tilde{\mathcal{A}}_{\mathcal{D}_{\mathcal{G}}}(\tau)} \phi(U(t, \tau) w) \, dm_\tau(w), \quad t \geqslant \tau.
\]

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