Accretion limits the compactness of static stars

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ABSTRACT

General relativity limits the compactness of static stars. If the pressure of the fluid is positive and the density decreases with distance from the center, the value of the circumferential radius of the star must be greater than \((9/4)GM/c^2\), or equivalently the redshift of the surface must be less than two. If constraints on the equation of state of the material are relaxed, general relativity alone does not restrict the redshift of a static stellar surface. However, because black hole candidates in the universe generally accrete material from their environs, the process of accretion provides upper limits on the redshift of a astrophysical black-hole candidates.

Subject headings: stars: black hole

1. Introduction

The Tolman-Oppenheimer-Volkoff (Tolman 1934; Oppenheimer & Volkoff 1939) equations restrict the radius \(R\) of a static star of mass \(M\) to be greater than \((9/4)GM/c^2\) (Weinberg 1972; Shapiro & Teukolsky 1983). There are two physically motivated assumptions behind this conclusion. First, the density of the material must decrease or remain constant with increasing radius. Second, the pressure of the fluid must be positive throughout. If the properties of the material are less constrained, this limit may be evaded, but important limits may still be obtained for astrophysical objects, because these objects interact with their surroundings. As a compact object accretes matter, its configuration must adjust to support the additional material. This adjustment cannot occur more quickly than the light-crossing time for the object, so one can obtain a natural limit to the accretion rate onto a static star or conversely a limit to the compactness of the star.

The letter examines two stellar models which evade the normal compactness limit and derives more liberal limits to the value of \(M/R\). Although the resulting limits are much less constraining than the standard result, they do have important consequences on the possible physical processes that could halt the collapse of a black-hole progenitor.

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2. Calculations

Outside a spherical static star, Birkhoff’s theorem restricts the spacetime to be Schwarzschild. Because the focus will be solutions without horizons, the standard Schwarzschild coordinates are adequate. The redshift of the stellar surface is given by $1 + z = (1 - 2M/R)^{-1/2}$, in units with $G = c = 1$. An infinite redshift relative to infinity is achieved where the metric component $g_{00}$ vanishes, yielding an event horizon and a classical Schwarzschild black hole.

The ultracompact star may be modelled either as a thin shell with a radius $R > 2M$ which surrounds vacuum or material at a constant density with $p = -\rho$. The properties of the material in the shell become singular as the thickness of the shell vanishes and can be determined from the Israel (1966, 1967) matching conditions or by evaluating Einstein’s equation directly. The first model (case A) is an extreme counterexample to the assumption that the density must decrease with radius. Here the density inside is zero and suddenly increases at the inner surface of the shell. The second model (case B) exhibits negative internal pressure, in fact the maximal negative pressure which still satisfies the weak-energy condition.

A convenient parameter to characterize the configurations is $\epsilon = g_{00,\text{surf}} = 1 - 2M/R$. In both cases the exterior metric is given by the Schwarzschild solution. In case A, the interior metric is

$$ds^2 = \left(1 - \frac{2M}{R}\right)dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(1)

A simple redefinition of the time coordinate reveals that the interior spacetime is flat. The gravitational redshift relative to infinity is constant inside the shell.

In case B, the metric is slightly more complicated

$$ds^2 = \left(1 - \frac{2M(r)}{r}\right)dt^2 - \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

(2)

where $M(r) \propto r^3$ is the mass enclosed within the radius $r$. Inside the star, the redshift relative to infinity decreases from its value at the surface to zero at the center of the star. The configuration discussed in detail by Mazur & Mottola (2001) falls into this latter class.

An important characteristic of these solutions is the coordinate time (the time measured at infinity) for light to go from the surface of the star to the center. This is the shortest time that the star can adjust its configuration without violating causality. For the mass shell (case A), $\tau = \epsilon^{-1/2}R$. If the accretion is perfectly spherically symmetric, the shell could in principle adjust on a faster timescale of $\tau \sim \epsilon^{1/2}R$, the time for a signal to propagate from the surface of the shell to where the Schwarzschild radius would be. However, the accretion
is unlikely to be perfectly symmetric, so different parts of the shell would have to be in causal contact for the entire shell to adjust, and the longer timescale is appropriate.

The constant density (case B), \( p = -\rho \) sphere has a crossing time of

\[
\tau = \frac{\tanh^{-1} \sqrt{1 - \epsilon}}{\sqrt{1 - \epsilon} R}. \tag{3}
\]

As \( \epsilon \) approaches zero, this expression diverges as \( \ln \epsilon \), more gently than the mass-shell solution. From the structure of the Tolman-Oppenheimer-Volkoff (Tolman 1934; Oppenheimer & Volkoff 1939) equations, it is apparent that this solution has the minimal light-crossing time for a configuration with a given surface redshift, while satisfying the weak-energy condition.

As material falls onto or through the surface of the star, the value of \( \epsilon \) will change. The rate of change in \( \epsilon \) is proportional to the rate of increase of the gravitational mass of the star

\[
\frac{d\epsilon}{dt} = -\frac{2}{R} \frac{dM}{dt} \left( 1 - \frac{1}{3} \frac{\bar{\rho}}{\rho} \right) \tag{4}
\]

where \( \rho \) is the density at which the material accumulates and \( \bar{\rho} \) is the mean density of the star defined by \( M/(4/3\pi R^3) \). If the accreted material simply sinks through the surface and does not accumulate there, the effective ratio \( \bar{\rho}/\rho \) vanishes. In the subsequent constraints \( \rho \) is taken to equal \( \bar{\rho} \).

For a particular configuration, if \( \epsilon + \tau d\epsilon/dt \) is negative, a trapped surface will form at the stellar surface. In case A the star must collapse to form a singularity (Hawking & Penrose 1970). In case B, the formation of a singularity may be avoided (Farhi et al. 1990). In both cases, the object will appear like a classical black hole to outside observers. Essentially, for sufficiently fast accretion rates, the star cannot adjust its configuration quickly enough to avoid the formation of a horizon.

3. Astrophysical Implications

3.1. Eddington-Limited Accretion

Black-hole candidates have been discovered with masses ranging from a few solar masses to a few billion solar masses. Furthermore, black-hole candidates across this entire range of masses have been observed to accrete up to the Eddington accretion rate. At this rate, the outgoing photon flux is sufficient to quench additional accretion. In geometrized units it is given by \( \dot{M} = 3 \times 10^{-22}\gamma^{-1} M/M_\odot \) where \( \gamma \) is the efficiency of the energy release \( \sim 0.1 \).
Because only a small fraction of the rest-mass energy of the infalling material is radiated to infinity, the gravitational mass of the black-hole candidate must increase, specifically, $dM/dt = (1 - \gamma)\dot{M}$. At the Eddington accretion rate, $d\epsilon/dt = 1.6 \times 10^{-27}(1 - \gamma)/\gamma$ cm$^{-1}$.

This yields an upper limit on the value of $\epsilon$ which is a function of the mass of the black-hole candidate and the spacetime in its interior. For case A (the mass shell),

$$\epsilon > 2.5 \times 10^{-14} \left( \frac{M}{M_\odot} \right)^{2/3}$$

and for case B (the constant density interior)

$$\epsilon > 10^{-18} \frac{M}{M_\odot} \left[ 1 - \frac{1}{50} \ln \left( \frac{M}{M_\odot} \right) \right]$$

where $\gamma = 0.1$ and $\rho = \bar{\rho}$. Although these limits may not appear particularly stringent, the proper distance, $l$, that the surface of the star must move to lie within its horizon is much greater than the Planck length. It is typically $1(M/M_\odot)^{4/3}$ mm for the mass shell and $6(M/M_\odot)^{3/2}$ $\mu$m for the negative-pressure star.

### 3.2. Formative Accretion

Although the details of the formation of supermassive black holes are hazy, black holes of stellar mass form in supernovae. Numerical simulations (MacFadyen & Woosley 1999) indicate that during a supernova that results in a black hole that the mass accretion rate can exceed $0.1 M_\odot s^{-1}$ or $5 \times 10^{-7}$ in geometrized units. This is fourteen orders of magnitude larger than the Eddington accretion rate for a ten-solar-mass object. The system evades the Eddington limit because the accreting material is sufficiently hot that neutrinos rather than photons are the dominant radiative channel (Narayan et al. 2001). In both cases, if the objects formed in supernova are not black holes but static stars, the limiting value of $\epsilon$ is much larger than for the Eddington accretion rate achieved later in the object’s lifetime. For case A $\epsilon > 7 \times 10^{-5}$, and for case B $\epsilon > 5 \times 10^{-6}$.

### 3.3. Mergers

Supermassive black holes grow through a succession of mergers as their parent galaxies merge (Begelman et al. 1980; Menou et al. 2001). Unlike accretion during the formation of a stellar-mass black hole, this process is neither quasistatic nor quasispherical. However, the general arguments do apply in this case. Specifically, to prevent the formation of a horizon,
the two objects must adjust their configurations during the process of the merger, and this requires at least a light-crossing time. Because the final plunge of the merging objects is approximately head-on at nearly the speed of light (Brandt et al. 2000), the effective accretion rate is \( \sim \frac{dM}{dt} \approx M_2/(2M_1) \), where \( M_1 \) and \( M_2 \) are the masses of the primary and secondary respectively. During the final stage of an equal-mass merger, using the same formulae as in the earlier calculations, unless the radius of each object is larger than \( \approx 4M \), a horizon will form around the coalescing pair in less than internal dynamical time. Detailed simulations of merging black holes support this conclusion (Brandt et al. 2000; Alcubierre et al. 2001).

4. Discussion

If the mass of a protoneutron star exceeds several solar masses, is collapse to a black hole inevitable as the star cools? Or to ask in another way, must black-hole candidates with masses greater than several solar masses indeed be black holes? As a black-hole or neutron-star progenitor collapses, the comoving density of the material increases. If the mass of the object is less three solar-masses or so, the collapse will halt and a neutron star will form. The typical density of a neutron star is \( \sim 10^{15} \text{ g/cm}^3 \) (Shapiro & Teukolsky 1983). The mean density of a ten-solar-mass collapsing star as it passes through its Schwarzchild radius is an order of magnitude smaller than this. The existence of neutron stars indicates that nothing extraordinary happens to the pressure of the material in the collapsing star as its surface passes through its Schwarzchild radius.

Abramowicz et al. (2002) argue that observations of material orbiting and falling toward black-hole candidates cannot determine with certainty that the central object is indeed a black hole. However, if general relativity is correct, the process of accretion in itself provides important limits on what the central object could be. Specifically, if our understanding of how stellar-mass black holes form is correct, the redshift of the surface of a black-hole candidate must not exceed \( 10^3 \). Admittedly radiation from such a surface would be difficult to detect, but this maximal redshift is much less than the value \( 10^{19} \) considered by Abramowicz et al. (2002).

Within classical general relativity, nothing marks the moment the stellar surface passes through its Schwarzchild radius. Quantum mechanically the Schwarzchild radius may be singular due the the back reaction of the Hawking radiation on the spacetime (Zurek & Page 1984; ’t Hooft 1998). Mazur and Mottola (Mazur & Mottola 2001) argue that this backreaction may result in a phase transition resulting in the formation of a static object whose radius is slightly larger than \( 2M \). In the particular model of Mazur and
Mottola, the interior of the object consists of a fluid with \( p = -\rho \), so the compactness of the configuration must be limited, otherwise the object will be unstable as it accretes. Using the most conservative estimate for the maximum value of \( M/R \) discussed in § 3.1, the local energy-density in the quantum-backreaction field is only \( \sim (0.3(M/M_\odot)^{-3/2} \text{ev})^4 \). If the required energy density to induce the phase transition is larger than this (as one would expect), the static star that forms would collapse if it accreted material at the Eddington rate.

The dynamics of accretion onto black-hole candidates combined with the condition that any static star must adjust in a causal manner to changes in its parameters requires either that the redshift of the surfaces of black-hole candidates be limited, that they are not static or consist of material which does not satisfy the weak-energy condition. Although the limit on the redshift of the surfaces is not stringent enough to allow detection of the surfaces of stellar-mass black holes (with \( z < 10^3 \)), it does restrict the physical processes that could halt the collapse of the black hole progenitor to low energies and densities. If supermassive black holes mass grow by mergers, the redshift of their surfaces must be less than 0.2 or they must lack surfaces entirely. Such a surface could easily be detected, so it is unlikely that supermassive black holes have static surfaces.

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