Duffin-Kemmer-Petiau equation in Riemannian space-times

J. T. Lunardi, B. M. Pimentel and R. G. Teixeira

Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona 145
01405-900 - São Paulo, S.P.
Brazil

Abstract

In this work we analyze the generalization of Duffin-Kemmer-Petiau equation to the case of Riemannian space-times and show that the usual results for Klein-Gordon and Proca equations in Riemannian space-times can be fully recovered when one selects, respectively, the spin 0 and 1 sectors of Duffin-Kemmer-Petiau theory.

PACS: 11.10.-z, 03.65.Pm

*On leave from Departamento de Matemática e Estatística. Setor de Ciências Exatas e Naturais. Universidade Estadual de Ponta Grossa. Ponta Grossa, PR - Brazil
†E-mail: lunardi@ift.unesp.br
‡E-mail: pimentel@ift.unesp.br
§E-mail: randall@ift.unesp.br
1 Introduction

The Duffin-Kemmer-Petiau (DKP) equation is a convenient relativistic wave equation to describe spin 0 and 1 bosons with the advantage over standard relativistic equations, like Klein-Gordon (KG) and Proca ones, of being of first order in derivatives. As a matter of fact, this equation was developed specifically to fulfill this characteristic, and provide an equation for bosons similar to Dirac spin 1/2 equation.

The first one to propose what is now known as the $16 \times 16$ DKP algebra was Petiau [1], a de Broglie’s student, who took as starting point the former’s work on first order wave equations on $16 \times 16$ matrices that were products of different Dirac matrices spaces. Latter it was showed that this algebra could be decomposed into 5, 10 and 1 degrees irreducible representations, the latter one being trivial [2]. Anyway, Petiau’s work remained unknown to the majority of scientific community so that Kemmer, working independently, wrote Proca’s equation as a set of coupled first order equations as well as the equivalent spin 0 case [3]. Although these set of equations could be written in $10 \times 10$ (spin 1) and $5 \times 5$ (spin 0) matrix forms, it was not clear which algebraic relations these matrices should obey. Based on this work, Duffin was able to put the equations sets in a first order $\beta$ matrix formulation presenting 3 of the 4 commutation relations present in the DKP algebra [4]. This result provided the motivation to Kemmer to complete formalism and present the complete theory of a relativistic wave equation for spin 0 and 1 bosons [5]. These and other facts related with the historical development of DKP theory, as well as a detailed list of references on the subject, can be found in Krajcik and Nieto paper on historical development of Bhabha first order relativistic equations [6].

More recently there have been an increasing interest in DKP theory. Specif-
ically, it has been applied to QCD (large and short distances) by Gribov [7], to covariant Hamiltonian dynamics by Kanatchikov [8] and has been used in its generally relativistic version by Red’kov to study, using a specific representation of the 10 degrees DKP algebra, spin 1 particles in the abelian monopole field [9].

But, although DKP equation is known to be completely equivalent to KG and Proca in the free field case, doubts arise, specially with respect to KG equation, when minimal interaction with electromagnetic field comes into play. This equivalence has been proved recently at classical level [10, 11] and at classical and quantum levels too by Fainberg & Pimentel [12]; but leaves open the question about whether other processes or interactions could distinguish DKP from KG and Proca equations. So, our intention in this work is to analyze the generalization to Riemannian space-times of DKP equation and its equivalence to the Riemannian versions of KG and Proca equations. This demonstration will be performed by showing that one obtains the generalized KG and Proca equations when the spin 0 and 1 sectors of the generalized DKP equation are selected.

In order to make this work more self contained we will dedicate some space to basic results on DKP theory and to the construction of its Riemannian generalization using the tetrad formalism. So, we start by presenting in section 2 the basic results on DKP equation on Minkowski space-time. We shall not enter in full details of the theory but simply quote the most important results and properties, specially about the projectors of physical spin 0 and 1 components of DKP field, necessary to the understanding of this work. For further details we suggest the reader to original works [1, 4, 5] or classic textbooks [13, 14]. In section 3 we will present some basic results on the tetrad formalism and perform the passage from Minkowskian to Riemannian space-times. In section 4 the equivalence between DKP equation and KG and Proca equations in Riemannian space-times will be demonstrated using the projectors of physical spin 0 and 1 sectors of DKP theory.
Finally, we present our conclusions and comments in section 5. Throughout this work we will adopt the signature $(+ - - -)$ to the metric tensors as well as Einstein implicit summation rule, except otherwise stated. Moreover, Latin letters will be used when labelling indexes concerned to the Minkowski space-time or to the Minkowskian manifolds tangent to the Riemannian manifold, while Greek letters will label indexes referred to the Riemannian manifold. Both Latin and Greek indexes will run from 0 to 3, except when we clearly state the opposite.

2 DKP equation in Minkowski space-time

The DKP equation is given by

$$(i \beta^a \partial_a - m) \psi = 0,$$

where the matrices $\beta^a$ obey the algebraic relations

$$\beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = \beta^a \eta^{bc} + \beta^c \eta^{ba},$$

being $\eta^{ab}$ the metric tensor of Minkowski space-time. This equation is very similar to Dirac’s equation but the algebraic properties of $\beta^a$ matrices, which have no inverses, make it more difficult to deal with. From the algebraic relation above we can obtain (no summation on repeated indexes)

$$(\beta^a)^3 = \eta^{aa} \beta^a,$$

so that we can define the matrices

$$\eta^a = 2 (\beta^a)^2 - \eta^{aa}$$

that satisfy

$$(\eta^a)^2 = 1, \eta^a \eta^b - \eta^b \eta^a = 0$$
\[ \eta^a \beta^b + \beta^b \eta^a = 0 \ (a \neq b), \tag{6} \]
\[ \eta^{aa} \beta^a = \eta^a \beta^a = \beta^a \eta^a. \tag{7} \]

With these results we can write the Lagrangian density for DKP field as
\[ \mathcal{L} = \frac{i}{2} \bar{\psi} \beta^a \leftrightarrow \partial_a \psi - m \bar{\psi} \psi, \tag{8} \]
where \( \bar{\psi} \) is defined as
\[ \bar{\psi} = \psi^\dagger \eta^0 \tag{9} \]

From this Lagrangian we can obtain DKP equation through a variational principle. Moreover, we will choose \( \beta^0 \) to be hermitian and \( \beta^i \ (i = 1, 2, 3) \) anti-hermitian so that the equation for \( \bar{\psi} \) can also be easily obtained by applying hermitian conjugation to equation (9).

Besides that, under a Lorentz transformation \( x'^a = \Lambda^a_b x^b \) we have
\[ \psi \rightarrow \psi' = U (\Lambda) \psi, \tag{10} \]
\[ U^{-1} \beta^a U = \Lambda^a_b \beta^b, \tag{11} \]
which gives, for infinitesimal transformations \( \Lambda^{ab} = \eta^{ab} + \omega^{ab} \ (\omega^{ab} = -\omega^{ba}) \) \[13, \]
\[ U = 1 + \frac{1}{2} \omega^{ab} S_{ab}, \quad S_{ab} = [\beta_a, \beta_b]. \tag{12} \]

If one uses two sets of Dirac matrices \( \gamma^a \) and \( \gamma'^a \) acting on different indexes of a 16 component \( \psi \) wave function it can be verified that the matrices
\[ \beta^a = \frac{1}{2} (\gamma^a I' + I \gamma'^a) \tag{13} \]
satisfy the algebraic relation [2], but these matrices form a reducible representation since it can be shown [5, 13] that this algebra has 3 inequivalent irreducible representations: a trivial 1 degree (\( \beta^a = 0 \)) without physical significance; a 5 degree one, corresponding to a 5 component \( \psi \) that describes a spin 0 boson; and a 10 degree one, corresponding to a 10 component \( \psi \) describing a spin 1 boson.
2.1 The spin 0 sector of DKP theory

The spin 0 sector can be selected from a general representation of $\beta$ matrices through the operators

$$P = - (\beta^0)^2 (\beta^1)^2 (\beta^2)^2 (\beta^3)^2,$$

(14)

which satisfies $P^2 = P$, and

$$P^a = P \beta^a.$$  

(15)

It can be shown [13] that

$$P^a \beta^b = P \eta^{ab}, \ P S^{ab} = S^{ab} P = 0, \ P^a S^{bc} = (\eta^{ab} P^c - \eta^{ac} P^b),$$

(16)

and, as a consequence, under infinitesimal Lorentz transformations [12] we have

$$P U \psi = P \psi,$$

(17)

so that $P \psi$ transforms as a (pseudo)scalar. Similarly

$$P^a U \psi = P^a \psi + \omega^a \psi,$$

(18)

showing that $P^a \psi$ transforms like a (pseudo)vector.

Applying these operators to DKP equation [11] we have

$$\partial_a (P^a \psi) = \frac{m}{i} P \psi,$$

(19)

and

$$P^b \psi = \frac{i}{m} \partial^b (P \psi),$$

(20)

which combined provide

$$\partial_a \partial^a (P \psi) + m^2 (P \psi) = \Box (P \psi) + m^2 (P \psi) = 0.$$  

(21)

These results show that all elements of the column matrix $P \psi$ are scalar fields of mass $m$ obeying KG equation while the elements of $P^a \psi$ are $\frac{i}{m}$ times the
derivative with respect to $x^a$ of the corresponding elements of $P\psi$. Moreover, we can choose a 5 degree irreducible representation of the $\beta^a$ matrices in such a way that

$$P\psi = P \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0_{4 \times 1} \\ \psi_4 \end{pmatrix}, \quad P_a\psi = \begin{pmatrix} 0_{4 \times 1} \\ \psi_a \end{pmatrix},$$

(22)

so that equation (20) results in

$$\psi_a = \frac{i}{m} \partial_a \psi_4$$

(23)

allowing us to make $\psi_4 = \sqrt{m}\varphi$ and obtain

$$\psi = \begin{pmatrix} \frac{i}{\sqrt{m}} \partial_a \varphi \\ \sqrt{m}\varphi \end{pmatrix}, \quad \Box \varphi + m^2 \varphi = 0,$$

(24)

making evident that the DKP equation describes a scalar particle.

2.2 The spin 1 sector of DKP theory

In order to select the spin 1 sector of DKP equation from a general representation of $\beta$ matrices we can use the operators

$$R^a = (\beta^1)^2 (\beta^2)^2 (\beta^3)^2 \left[ \beta^a \beta^0 - \eta^{a0} \right],$$

(25)

and

$$R^{ab} = R^a \beta^b.$$  

(26)

From the definitions it can be shown [13] that these operators have the following properties

$$R^{ab} = -R^{ba},$$

(27)

$$R^a \beta^b \beta^c = R^{ab} \beta^c = \eta^{bc} R^a - \eta^{ac} R^b,$$

(28)
\[ R^a S^{bc} = \eta^{ab} R^c - \eta^{ac} R^b, \quad S^{bc} R_a = 0, \tag{29} \]

and
\[ R^{ab} S^{cd} = \eta^{bc} R^{ad} - \eta^{ac} R^{bd} - \eta^{bd} R^{ac} + \eta^{ad} R^{bc}. \tag{30} \]

Then, under infinitesimal Lorentz transformations (12), we have
\[ R^a U \psi = R^a \psi + \omega^a_b R^b \psi \tag{31} \]

and
\[ R^{ab} U \psi = R^{ab} \psi + \omega^b_c R^{ac} \psi + \omega^a_c R^{cb} \psi, \tag{32} \]
showing that \( R^a \psi \) transforms like a (pseudo)vector while \( R^{ab} \psi \) transforms like a rank 2 (pseudo)tensor.

The application of these operators to DKP equation results in
\[ \partial_b \left( R^{ab} \psi \right) = \frac{m}{i} R^a \psi, \tag{33} \]
and
\[ R^{ab} \psi = -\frac{i}{m} U^{ab}, \tag{34} \]
where
\[ U^{ab} = \partial^a R^b \psi - \partial^b R^a \psi \tag{35} \]
is the strength tensor of the massive vector field \( R^a \psi \). Combined, these results provide
\[ \partial_b \left( -\frac{i}{m} U^{ab} \right) = \frac{m}{i} R^a \psi \tag{36} \]
\[ \partial_b U^{ba} + m^2 R^a \psi = 0, \tag{37} \]
or equivalently
\[ (\Box + m^2) R^a \psi = 0; \quad \partial_a R^a \psi = 0. \tag{38} \]

So, all elements of the column matrix \( R^a \psi \) are components vector fields of mass \( m \) obeying Proca equation; being the elements of \( R^{ab} \psi \) equal to \( \frac{\omega}{m} \) times
the field strength tensor of the vector field of which the corresponding elements of $R^a \psi$ are components. So, similarly to the spin 0 case, this procedure selects the spin 1 content of DKP theory, making explicitly clear that it describes a massive vectorial particle.

Similarly to the spin 0 sector of DKP theory, we can find a 10 degree irreducible representation of the $\beta^a$ matrices in such a way that

$$R_a \psi = R_a \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_9 \end{pmatrix} = \begin{pmatrix} \psi_a \\ 0_{9 \times 1} \end{pmatrix}.$$  \hspace{1cm} (39)

$$R_{10} \psi = \begin{pmatrix} \psi_7 \\ 0_{9 \times 1} \end{pmatrix}, \quad R_{20} \psi = \begin{pmatrix} \psi_8 \\ 0_{9 \times 1} \end{pmatrix}, \quad R_{30} \psi = \begin{pmatrix} \psi_9 \\ 0_{9 \times 1} \end{pmatrix},$$  \hspace{1cm} (40)

$$R_{12} \psi = \begin{pmatrix} \psi_6 \\ 0_{9 \times 1} \end{pmatrix}, \quad R_{31} \psi = \begin{pmatrix} \psi_5 \\ 0_{9 \times 1} \end{pmatrix}, \quad R_{23} \psi = \begin{pmatrix} \psi_4 \\ 0_{9 \times 1} \end{pmatrix}.$$  \hspace{1cm} (41)

Defining

$$\psi_a = \sqrt{m} B_a,$$  \hspace{1cm} (42)

where $B_a$ is a vector field, we get from equation (35) that

$$U_{ab} = \sqrt{m} \begin{pmatrix} \partial_a B_b - \partial_b B_a \\ 0_{9 \times 1} \end{pmatrix} = \sqrt{m} \begin{pmatrix} K_{ab} \\ 0_{9 \times 1} \end{pmatrix},$$  \hspace{1cm} (43)

where

$$K_{ab} = (\partial_a B_b - \partial_b B_a).$$  \hspace{1cm} (44)

Consequently equation (34) will result in

$$R_{ab} \psi = -\frac{i}{\sqrt{m}} \begin{pmatrix} K_{ab} \\ 0_{9 \times 1} \end{pmatrix}.$$  \hspace{1cm} (45)
which, together with equations (10) and (11), determines the components $\psi_4$ to $\psi_9$ in $\psi$. So equation (37) can now be written as

$$\partial_a (K^{ab}) + m^2 (B^b) = 0,$$

making explicit that the DKP equation describes a massive vector field.

### 3 Passage to Riemannian space-times

Before constructing the DKP equation in Riemannian $\mathcal{R}^4$ space-times we will construct the tensor quantities on $\mathcal{R}^4$ using the tensors defined on a Minkowski manifold tangent to each point of $\mathcal{R}^4$, and for this purpose we will use the standard formalism of “tetrads”. Here we will just mention the necessary fundamental results and point out the reader to reference [15].

A tetrad is constituted by a set of four vector fields $e^{a}_\mu (x)$ that satisfy, at each point $x$ of $\mathcal{R}^4$, the relations

$$\eta^{ab} = e^{a}_\mu (x) e^{b}_\nu (x) g^{\mu \nu} (x),$$

$$g_{\mu \nu} (x) = e^{a}_\mu (x) e^{b}_\nu (x) \eta_{ab},$$

and

$$\eta_{ab} = e^{a}_\mu (x) e^{b}_\nu (x) g_{\mu \nu} (x),$$

$$g^{\mu \nu} (x) = e^{a}_\mu (x) e^{b}_\nu (x) \eta^{ab},$$

where

$$e^{a}_\mu (x) = g^{\mu \nu} (x) \eta_{ab} e^{b}_\nu (x),$$

the Latin indexes being raised and lowered by the Minkowski metric $\eta^{ab}$ and the Greek ones by the metric $g^{\mu \nu}$ of the manifold $\mathcal{R}^4$. 
The components $B^{ab}$ in $\mathcal{M}^4$ of a tensor $B^{\mu\nu}$ defined on $\mathcal{R}^4$ are given by

$$B^{ab} = \epsilon^a_\mu \epsilon^b_\nu B^{\mu\nu}, \quad B_{ab} = \epsilon^\mu_a \epsilon^\nu_b B_{\mu\nu},$$

or inversely

$$B^{\mu\nu} = \epsilon^\mu_a \epsilon^\nu_b B^{ab}, \quad B_{\mu\nu} = \epsilon^\mu_a \epsilon^\nu_b B_{ab},$$

and it is easy to see that $A_\mu B^\mu = A_a B^a$.

The Lorentz covariant derivative $D_\mu$ is defined as

$$D_\mu B^a = \partial_\mu B^a + \omega_\mu^{\; ab} (x) B^b,$$

such that $D_\mu B^a$ transforms like a vector under local Lorentz transformations $x^a = \Lambda^a_b (x) x^b$ on the $\mathcal{M}^4$ tangent manifold, where the connection $\omega_\mu^{\; ab} (x)$ on $\mathcal{M}^4$ satisfies the transformation rule

$$\omega'_\mu^{\; ab} = \Lambda^a_c \omega_\mu^{\; bc} \left( \Lambda^{-1} \right)^{lb} - (\partial_\mu \Lambda)^a_c \left( \Lambda^{-1} \right)^{cb}.$$

Requiring $D_\mu (B^a A_a) = \partial_\mu (B^a A_a)$, since $B^a A_a$ is a scalar, one gets

$$D_\mu B_a = \partial_\mu B_a - \omega_\mu^{\; b a} B_b.$$

The covariant derivative $\nabla_\mu$ of an object $B_\nu$ defined in the Riemannian manifold is given, as usual, as

$$\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma_\mu^{\; \nu \alpha} B_\alpha,$$

where $\Gamma_\mu^{\; \nu \alpha}$ is the connection on the $\mathcal{R}^4$ manifold. Then the total covariant derivative $\nabla_\mu$ of a quantity $B_\nu^a$, with Lorentzian and Riemannian indexes, will be given by

$$\nabla_\mu B_\nu^a = D_\mu B_\nu^a - \Gamma_\mu^{\; \nu \beta} B_\beta^a,$$

or

$$\nabla_\mu B^{\nu a} = D_\mu B^{\nu a} + \Gamma_\mu^{\; \nu \alpha} B^{\alpha a}.$$
The relation between connections $\omega_{\mu}^{ab}$ and $\Gamma_{\mu\nu}^{\alpha}$ can be found by requirement that $\nabla_{\mu}e_{\nu}^{a} = 0$, which implies that

$$\omega_{\mu}^{ab} = e_{\alpha}^{a}e_{\nu}^{b}\Gamma_{\mu\nu}^{\alpha} - e_{\nu}^{b}\partial_{\mu}e_{\nu}^{a}. \quad (60)$$

Moreover, the metricity condition $\nabla_{\mu}g^{\nu\alpha} = 0$ will imply that $\omega_{\mu}^{ab} = -\omega_{\mu}^{ba}$. Besides this, it is easy to see that $\nabla_{\mu}A^{\mu} = \nabla_{a}A^{a}$, where $\nabla_{a} = e_{\mu}^{a}\nabla_{\mu}$, and that $e_{\nu}^{a}D_{\mu}B^{a} = \nabla_{\mu}B^{\nu} = \nabla_{\mu}(e_{\nu}^{a}B^{a}) = e_{\nu}^{a}\nabla_{\mu}B^{a}$.

### 3.1 Generalized DKP equation

The procedure to generalize DKP equation is very similar to the case of Dirac one \[15\]. First we will generalize the matrices $\beta^{a}$ defined on flat Minkowski manifold $M^{4}$ obeying equation (2) to matrices $\beta^{\mu}$ defined on Riemannian manifold $R^{4}$ by

$$\beta^{\mu} = e_{\mu}^{a}\beta^{a}, \quad (61)$$

that will satisfy

$$\beta^{\mu}\beta^{\nu}\beta^{\alpha} + \beta^{\alpha}\beta^{\nu}\beta^{\mu} = \beta^{\mu}g^{\nu\alpha} + \beta^{\alpha}g^{\nu\mu}, \quad (62)$$

as it can be shown using the properties of $\beta^{a}$ and equation \((50)\).

Moreover, under a local infinitesimal Lorentz transformation on $M^{4}$, the multicomponent field $\psi$ will transform as given by equations \((10)\) and \((12)\) so that its variation will be

$$\delta\psi = \frac{1}{2}\omega_{ab}S^{ab}\psi \quad (63)$$

and the variation of the tetrad vectors will be

$$\delta e_{\mu}^{a} = \omega_{\nu}^{a}e_{\mu}^{b}. \quad (64)$$

If we consider two nearby points $x_{1}$ and $x_{2}$ with the local tetrads $e_{\mu}^{a}(x_{1})$ and $e_{\mu}^{a}(x_{2})$ then $\psi(x_{1})$ and $\psi(x_{2})$ are the field $\psi$ referred to these tetrads,
respectively. Performing a parallel displacement from $x_1$ to $x_2$ on the tetrad $e^a_\mu(x_1)$ we get a new one denoted by $e^b_\mu(x_2)$ and the field $\psi$ at $x_2$ with respect to this new tetrad will be $\psi(x_2)$. Then, the covariant differential of the field can be defined as

$$D\psi = dx^a \nabla_a \psi = dx^\mu \nabla_\mu \psi = \psi'(x_2) - \psi(x_1), \quad (65)$$

or

$$D\psi = \psi(x_2) - \psi(x_1) - [\psi(x_2) - \psi'(x_2)], \quad (66)$$

where we separated the translation term $\psi(x_2) - \psi(x_1)$ from the local Lorentz transformation term $\psi'(x_2) - \psi(x_2)$. Explicitly we have

$$\psi(x_2) - \psi(x_1) = dx^a \partial_a \psi = dx^\mu \partial_\mu \psi, \quad (67)$$

and

$$\psi(x_2) - \psi'(x_2) = -\frac{1}{2} \omega_{ab} S^{ab}, \quad (68)$$

so that

$$D\psi = dx^a \partial_a \psi + \frac{1}{2} \omega_{ab} S^{ab} \psi. \quad (69)$$

From the expression (56) for the Lorentz covariant derivative we see that the variation of the tetrad vector $e^a_\mu$ under Lorentz transformations on the $\mathcal{M}^4$ manifold is

$$\delta e^a_\mu = \omega^a_{\nu b} e^b_\mu dx^\nu, \quad (70)$$

so, comparing with equation (64) we can identify

$$\omega^{ab} = \omega^a_{\nu b} dx^\nu \quad (71)$$

and rewrite (59) as

$$D\psi = dx^\mu \left( \partial_\mu + \frac{1}{2} \omega^a_{\mu ab} S^{ab} \right) \psi. \quad (72)$$
From this expression we can obtain the Lorentz covariant derivative \( D_\mu \) of field \( \psi \) and, since \( \psi \) has no Riemannian index, this derivative will be equal to the total covariant derivative \( \nabla_\mu \) of \( \psi \); so we have that

\[
\nabla_\mu \psi = D_\mu \psi = \left( \partial_\mu + \frac{1}{2} \omega_{\mu ab} S^{ab} \right) \psi .
\]

Applying the hermitian conjugation to this expression and using the definition of \( \overline{\psi} \) we get

\[
\nabla_\mu \overline{\psi} = \partial_\mu \overline{\psi} - \frac{1}{2} \omega_{\mu ab} \overline{\psi} S^{ab}.
\]

Now we can write the generalized expression for the \( \psi \) field Lagrangian

\[
\mathcal{L} = \sqrt{-g} \left[ \frac{i}{2} \left( \overline{\psi} \beta^\mu \nabla_\mu \psi - \nabla_\mu \overline{\psi} \beta^\mu \psi \right) - m \overline{\psi} \psi \right],
\]

that can be written explicitly as

\[
\mathcal{L} = \sqrt{-g} \left[ \frac{i}{2} \left( \overline{\psi} \beta^\mu \left( \partial_\mu \psi + \frac{1}{2} \omega_{\mu ab} S^{ab} \psi \right) \right) - \beta^\mu \psi \right] .
\]

So we have

\[
\frac{\partial \mathcal{L}}{\partial \psi} = \sqrt{-g} \left[ \frac{i}{2} \left( \beta^\mu \partial_\mu \psi + \omega_{\mu ab} \beta^\mu S^{ab} \psi - \omega_{\mu b} \beta^b \psi \right) - m \psi \right] ,
\]

and

\[
\frac{\partial \mathcal{L}}{\partial \left( \partial_\nu \psi \right)} = -\frac{i}{2} \left[ \partial_\nu \left( \sqrt{-g} \beta^\nu \right) \psi + \sqrt{-g} \beta^\nu \partial_\nu \psi \right] ,
\]

\[
\frac{\partial \mathcal{L}}{\partial \left( \partial_\nu \psi \right)} = -\frac{i}{2} \left[ \sqrt{-g} \omega_{\mu a} \beta^a \psi + \sqrt{-g} \beta^a \partial_\nu \psi \right] ,
\]

where we used the result \[15\]

\[
\partial_\nu \left( \sqrt{-g} e^\nu_a \right) = \sqrt{-g} \omega^b_a = \sqrt{-g} \omega^\mu_a
\]
valid for a Riemannian manifold.

Combining these results we get the generalized DKP equation of motion for Riemannian space-times

$$i\beta^\mu \nabla_\mu \psi - m\psi = 0. \quad (81)$$

4 The equivalence with KG and Proca equations

Now we will analyze the equivalence between DKP and KG and Proca theories in Riemannian space-times. In order to do this we will generalize the operators defined in Section 2 to select the spin 0 and 1 sectors and apply them to the generalized DKP equation. Then we can compare the obtained results with KG and Proca equations in Riemannian space-times, in a way analogous to what was done in the Minkowskian free field case.

4.1 The spin 0 case: Equivalence with KG

From the $P^a$ operators defined in Section 2 we can construct the generalized projectors $P^\mu$ as

$$P^\mu = e^\mu_a P^a = e^\mu_a P\beta^a = P\beta^\mu, \quad (82)$$

where $P$ is given by equation (14) in terms of the matrices $\beta^a$. Using equations (16) and (50) it is easy to verify that

$$P^\mu \beta^\nu = Pg^{\mu\nu}, \quad PS^{\mu\nu} = S^{\mu\nu}P = 0, \quad P^\mu S^{\alpha\nu} = (g^{\mu\alpha}P^\nu - g^{\mu\nu}P^\alpha). \quad (83)$$

As each component of $P\psi$ was shown to be a scalar under Lorentz transformation on the Minkowski tangent manifold $\mathcal{M}^4$ they will also be scalars under general coordinate transformations on the Riemannian manifold $\mathcal{R}^4$. Similarly,
each component of $P^a \psi$ was shown to be a vector under Lorentz transformations so that the components of $P^\mu \psi$ will also be vectors under general coordinate transformations $[15]$. So, one should expect that the total covariant derivative $\nabla_\mu$ of $P \psi$ and $P^\mu \psi$ should be that of a scalar and a vector, respectively. As a matter of fact, we have that

$$\nabla_\mu (P \psi) = D_\mu (P \psi) = \partial_\mu (P \psi) + \frac{1}{2} \omega_{\mu ab} S^{ab} P \psi = \partial_\mu (P \psi)$$

(84)

and

$$\nabla_\mu (P^\nu \psi) = e^\nu_c \nabla_\mu (P^c \psi) = e^\nu_c D_\mu (P^c \psi)$$

$$= e^\nu_c \left[ \partial_\mu (P^c \psi) + \frac{1}{2} \omega_{\mu ab} S^{ab} P^c \psi + \omega_\mu^c b P^b \psi \right],$$

(85)

$$\nabla_\mu (P^\nu \psi) = e^\nu_c \left[ \partial_\mu (P^c \psi) + \omega_\mu^c b P^b \psi \right],$$

(86)

since $S^{ab} P^c = S^{ab} P^c = 0$, so that the use of equation (60) results in

$$\nabla_\mu (P^\nu \psi) = \partial_\mu (P^\nu \psi) + \Gamma_{\mu \beta}^\nu P^\beta \psi.$$ (87)

These results show that we can calculate the total covariant derivatives $\nabla_\mu$ of $P \psi$ and $P^\mu \psi$ by applying the derivative to each of their components as if they were, respectively, a scalar and a vector on the Riemannian manifold $\mathcal{R}^4$ and neglect its matrix character. Similarly, $P \psi$ and $P^a \psi$ can also be treated as scalar and vector, respectively, when calculating the Lorentz covariant derivative $D_\mu$.

Moreover, we can also see that

$$P \nabla_\mu \psi = \partial_\mu (P \psi) + \frac{1}{2} \omega_{\mu ab} P S^{ab} \psi = \partial_\mu (P \psi)$$

(88)

and

$$P^\nu \nabla_\mu \psi = e^\nu_c P^c \nabla_\mu \psi = e^\nu_c \left[ \partial_\mu (P^c \psi) + \frac{1}{2} \omega_{\mu ab} P^c S^{ab} \psi \right],$$

(89)

$$P^\nu \nabla_\mu \psi = e^\nu_c \left[ \partial_\mu (P^c \psi) + \omega_\mu^c b P^b \psi \right] = \nabla_\mu (P^\nu \psi),$$

(90)
so that it becomes obvious that $P \nabla_\mu \psi = \nabla_\mu (P \psi)$ and $P^\nu \nabla_\mu \psi = \nabla_\mu (P^\nu \psi)$.

Now, applying the operators $P^\mu$ and $P$ to the generalized DKP equation (81), we get

$$P^\mu \psi = \frac{i}{m} \nabla^\mu (P \psi) = \frac{i}{m} \partial^\mu (P \psi),$$

and

$$\nabla_\mu (P^\mu \psi) = \frac{m}{i} (P \psi).$$

Combining equations (91) and (92) we obtain the generalized KG equation

$$\nabla_\mu \nabla^\mu (P \psi) + m^2 (P \psi) = \nabla_a \nabla^a (P \psi) + m^2 (P \psi) = 0.$$ (93)

These results make clear that when we select the spin 0 sector of the generalized DKP equation (81), describing a scalar particle on a Riemannian manifold, we get a complete equivalence with the generalized KG equation.

Once more we can make this equivalence more explicit using the specific choice of the matrices $\beta^a$ that satisfies condition (22). Then we get as result

$$P \psi = \begin{pmatrix} 0_{1 \times 1} \\ \psi_4 \end{pmatrix}, \quad P^\mu \psi = \epsilon^{\mu a} P_a \psi = \epsilon^{\mu a} \left( \begin{array}{c} 0_{4 \times 1} \\ \psi_a \end{array} \right),$$

$$\psi = \begin{pmatrix} \frac{i}{\sqrt{m}} \nabla_a \varphi \\ \sqrt{m} \varphi \end{pmatrix}, \quad \nabla_\mu \nabla^\mu \varphi + m^2 \varphi = 0,$$

and, finally, we note that it is possible to make a change in the representation of $\beta$ matrices in such a way that the form of DKP field $\psi$ becomes

$$\psi \to \psi_R = \begin{pmatrix} \frac{i}{\sqrt{m}} \nabla_\mu \varphi \\ \sqrt{m} \varphi \end{pmatrix}.$$ (96)
4.2 The spin 1 case: Equivalence with Proca

Similarly to the spin 0 case we can also generalize the operators $R^a$ and $R^{ab}$ defined in Section 2 as

$$R^a = e^a_c R^c, \quad R^{ab} = g^{ac} e^c_d R^{cd},$$  \hspace{1cm} (97)$$

which can be seen to satisfy the relations

$$R^{\mu\nu} = -R^{\nu\mu},$$  \hspace{1cm} (98)$$

$$R^{\mu} \beta^\nu \beta^\rho = R^{\mu
u} \beta^\rho = g^{\nu\rho} R^{\mu} - g^{\mu\rho} R^{\nu},$$ \hspace{1cm} (99)$$

$$R^{\mu} S^{\nu\rho} = g^{\mu
u} R^{\rho} - g^{\mu\rho} R^{\nu}, \quad S^{\nu\rho} R^{\mu} = 0,$$ \hspace{1cm} (100)$$

and

$$R^{\mu\nu} S^{\alpha\beta} = g^{\nu\alpha} R^{\mu\beta} - g^{\nu\beta} R^{\mu\alpha} + g^{\mu\beta} R^{\nu\alpha}.$$ \hspace{1cm} (101)$$

Analogously to the case of $P\psi$ and $P^\mu \psi$, we would expect the total covariant derivatives $\nabla_\mu$ of $R^{\mu\psi}$ and $R^{\mu\nu\psi}$ to be, respectively, those of a vector and a tensor since $R^a \psi$ is a vector and $R^{ab} \psi$ a tensor on the Minkowski tangent manifold $M^4$. This turns out to be true since

$$\nabla_\mu (R^{\nu\psi}) = e^\nu_c \nabla_\mu (R^c \psi) = e^\nu_c \left[ \partial_\mu (R^c \psi) + \frac{1}{2} \omega_{\mu ab} R^{ab} c \right]$$ \hspace{1cm} (102)$$

$$\nabla_\mu (R^{\nu\psi}) = e^\nu_c \left[ \partial_\mu (R^c \psi) + \omega_{\mu c} R^b \psi \right],$$ \hspace{1cm} (103)$$

and

$$\nabla_\mu (R^{\alpha\nu\psi}) = e^\alpha_c e^\nu_d \nabla_\mu (R^{cd} \psi),$$ \hspace{1cm} (104)$$

$$\nabla_\mu (R^{\alpha\nu\psi}) = e^\alpha_c e^\nu_d \left[ \partial_\mu (R^{cd} \psi) + \frac{1}{2} \omega_{\mu ab} R^{ab} c \right]$$ \hspace{1cm} (105)$$

$$\nabla_\mu (R^{\alpha\nu\psi}) = e^\alpha_c e^\nu_d \left[ \partial_\mu (R^{cd} \psi) + \omega_{\mu c} R^b \psi \right],$$ \hspace{1cm} (106)$$
where we used $S^{ab} R^{cd} = S^{ab} R^{cd} = 0$, so that using equation (60) in equations (103) and (106) results in
\[
\nabla^\mu (R^\nu \psi) = e^\nu_c \left[ \partial_\mu (R^c \psi) + \Gamma_{\mu\beta}^{\nu} R^\beta \psi \right],
\]
and
\[
\nabla^\mu (R^{\alpha\nu} \psi) = e^\alpha_c e^\nu_d \left[ \partial_\mu (R^{cd} \psi) + \Gamma_{\mu\beta}^{\alpha} R^{\beta\nu} \psi + \Gamma_{\mu\beta}^{\nu} R^{\alpha\beta} \psi \right].
\]
So we can neglect the matrix character of $R^\mu \psi$ and $R^{\mu\nu} \psi$ when calculating their total covariant derivative and proceed by applying the derivative to each of their components as if they were usual vector and tensor, respectively, on the Riemannian manifold $\mathcal{R}^4$. Moreover we can also show that
\[
R^\nu \nabla_\mu \psi = e^\nu_c R^c \nabla_\mu \psi = e^\nu_c R^c \left[ \partial_\mu (\psi) + \frac{1}{2} \omega^a_{\mu ab} S^{ab} \psi \right],
\]
\[
R^\nu \nabla_\mu \psi = e^\nu_c \left[ \partial_\mu (R^c \psi) + \frac{1}{2} \omega^a_{\mu ab} R^c S^{ab} \psi \right],
\]
\[
R^{\alpha\nu} \nabla_\mu \psi = e^\alpha_c e^\nu_d \left[ \partial_\mu (R^{cd} \psi) + \omega^{\mu d}_{\nu b} R^{d b} \psi \right],
\]
and
\[
R^{\alpha\nu} \nabla_\mu \psi = e^\alpha_c e^\nu_d \left[ \partial_\mu (R^{cd} \psi) + \omega^{\mu d}_{\nu b} R^{d b} \psi + \omega^{\mu c}_{\nu b} R^{cd} \psi \right],
\]
so that it becomes clear that $R^\nu \nabla_\mu \psi = \nabla_\mu (R^\nu \psi)$ and $R^{\alpha\nu} \nabla_\mu \psi = \nabla_\mu (R^{\alpha\nu} \psi)$.

Now we can use the operators $R^\mu$ and $R^{\mu\nu}$ on the generalized DKP equation (81), obtaining
\[
\nabla^\lambda (R^{\mu\lambda} \psi) = \frac{m}{i} R^\mu \psi
\]
and
\[
R^{\mu\nu} \psi = -\frac{i}{m} U^{\mu\nu},
\]
where now we have a covariant strength tensor

\[ U^{\mu\nu} = \nabla^\mu R^\nu \psi - \nabla^\nu R^\mu \psi. \]  

(116)

Combining equations (114) and (115) we get the generalized Proca equation

\[ \nabla_\lambda (U^{\lambda\mu}) + m^2 R^\mu \psi = 0. \]  

(117)

This makes clear that the spin 1 sector of the generalized DKP equation (51), describing a massive vectorial particle on a Riemannian manifold, is completely equivalent to the generalized Proca equation.

### 5 Conclusions and comments

In this work we analyzed the generalization of DKP theory to Riemannian spacetimes. We followed the standard procedure to perform the generalization by constructing the Riemannian quantities from the flat space-time ones through the use of tetrad formalism. We have obtained a first order relativistic wave equation that describes spin 0 and 1 particles coupled to the gravitational field.

Then we used the operators that select the spin 0 and 1 sectors of the theory and obtained, respectively, the generalized KG and Proca equations for spin 0 and 1 bosons in Riemannian space-times, proving the equivalence between the theories.

It should be also mentioned that the form (55) for DKP field when the matrices $\beta$ satisfy condition (54) is, in principle, analogous to the result obtained when we consider the interaction with electromagnetic field: the “gradient” components of free field case (the first four components) are replaced by the covariant derivatives [12].

This analogy is not clearly manifested in the case of the spin 0 sector because the covariant derivatives in a curved space-time will, of course, simply reduce to
the usual derivatives due to the scalar character of $\varphi$. But if we perform for the
spin 1 sector of DKP field in Riemannian space-times the same construction made
in equations (39) to (45) in Minkowski space-time through a specific choice of a
10 degree representation of the $\beta$ matrices, we will find that the usual derivatives
will be replaced by covariant ones in those equations, changing the forms of the
components $\psi_4$ to $\psi_9$ of $\psi$ field. This procedure shows, together with the result
of [12], that care must be taken in account when using specific representations
of the $\beta$ matrices to obtain the components of DKP field $\psi$: the forms of the
components obtained in the free field case may not be usable to interacting fields,
requiring a new construction.

The perspectives for future developments are diverse and some of them are
currently under our study. We could mention the application of the DKP theory
to the electromagnetic field in Riemannian space-times. Moreover, it is interesting
to look for the construction of a generalization similar to the one proposed by
Dirac to its electron equation [16]. Subsequently, it comes the generalizations
to Riemann-Cartan space-times [15] and to teleparallel description of gravity
[17, 18, 19] as a step to compare the question of coupling of DKP fields to torsion
in both theories, similarly to what has been done to usual spin 0 and 1 formalisms
[20, 21]. Independently, it will be analyzed the quantic processes in DKP theory
with gravitation viewed as an external field, as it has already been done with
DKP field in an external electromagnetic field [14].

6 Acknowledgments

J.T.L. and B.M.P. would like to thank CAPES’s PICDT program and CNPq,
respectively, for partial support. R.G.T. thanks CAPES for full support. The
authors also wish to thank Professor V. Ya. Fainberg by his critical reading of
our manuscript.

References

[1] G. Petiau, University of Paris thesis (1936). Published in *Acad. Roy. de Belg., Classe Sci., Mem in 8°* 16 (1936), No. 2.

[2] J. Géhéniau, *Acad. Roy. de Belg., Classe Sci., Mem in 8°* 18 (1938), No. 1.

[3] N. Kemmer, *Proc. Roy. Soc.* **A166** (1938), 127.

[4] R. J. Duffin, *Phys. Rev.* **54** (1938), 1114.

[5] N. Kemmer, *Proc. Roy. Soc.* **A173** (1939), 91.

[6] R. A. Krajcik and M. M. Nieto, *Am. J. Phys.* **45** (1977), 818.

[7] V. Gribov, *Eur. Phys. J.* **C10** (1999), 71. Also available as [hep-ph/9807224](http://arxiv.org/abs/hep-ph/9807224).

[8] I. V. Kanatchikov, [hep-th/9911175](http://arxiv.org/abs/hep-th/9911175).

[9] V. M. Red’kov, [quant-ph/9812007](http://arxiv.org/abs/quant-ph/9812007).

[10] M. Nowakowski, *Phys. Lett.* **A244** (1998), 329.

[11] V. Ya. Fainberg and B. M. Pimentel, [hep-th/9911219](http://arxiv.org/abs/hep-th/9911219). To appear in *Theoretical and Mathematical Physics*.

[12] J. T. Lunardi, B. M.Pimentel, R. G. Teixeira and J. S. Valverde, [hep-th/9911254](http://arxiv.org/abs/hep-th/9911254). To appear in *Physics Letters A*.

[13] H. Umezawa, “Quantum Field Theory,” North-Holland, 1956.
[14] A. I. Akhiezer and V. B. Berestetskii, “Quantum Electrodynamics,” Inter-
science, 1965.

[15] V. de Sabbata and M. Gasperini, “Introduction to Gravitation,” World Sci-
entific, 1985.

[16] P. A. M. Dirac, “Max Planck Festschrift,” page 339, Veb Deutscher Verlag
der Wissenschafter, 1958.

[17] K. Hayashi and T. Shirafuji, Phys. Rev. D19 (1979), 3524.

[18] V. C. de Andrade and J. G. Pereira, Phys. Rev. D56 (1998), 4689.

[19] J. W. Maluf and J. F. da Rocha-Neto, gr-qc/0002059.

[20] V. C. de Andrade and J. G. Pereira, Gen. Rel. Grav. 30 (1998), 263.

[21] V. C. de Andrade and J. G. Pereira, Int. J. Mod. Phys. D8 (1999), 141.