PARABOLIC CONFORMALLY SYMPLECTIC STRUCTURES I;
DEFINITION AND DISTINGUISHED CONNECTIONS

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Abstract. We introduce a class of first order $G$–structures, each of which
has an underlying almost conformally symplectic structure. There is one such
structure for each real simple Lie algebra which is not of type $C_n$ and admits a
contact grading. We show that a structure of each of these types on a smooth
manifold $M$ determines a canonical compatible linear connection on the tangent
bundle $TM$. This connection is characterized by a normalization condition on
its torsion. The algebraic background for this result is proved using Kostant’s
theorem on Lie algebra cohomology.

For each type, we give an explicit description of both the geometric s truc-
ture and the normalization condition. In particular, the torsion of the canonical
connection naturally splits into two components, one of which is exactly the ob-
struction to the underlying structure being conformally symplectic. This article
is the first in a series aiming at a construction of differential complexes naturally
associated to these geometric structures.

1. Introduction

This article is the first part in a series of three. The main motivation for this
series originally came from the work [9] of M.G. Eastwood and H. Goldschmidt
on integral geometry. The main part of that article is devoted to the construction
of a family of differential complexes on complex projective space, and to proving
some results on their cohomology, which then imply results on integral geometry.
The form of these complexes is rather unusual and the construction in [9] does not
explain whether these complexes are associated to a geometric structure on $\mathbb{C}P^n$
and, if yes, what this structure actually is.

An attempt to sort out these questions was made in the first version of the
preprint [10] by M.G. Eastwood and J. Slovák. There the authors define so–called
conformally Fedosov structures and associate a tractor bundle to such a structure.
In the second version of the preprint, which has appeared very recently, this tractor
bundle was used to construct differential complexes of the kind used in [9]. This

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construction starts from twisted de–Rham complexes associated to tractor bundles and is similar to the machinery of BGG–sequences as introduced in \cite{5} and \cite{3}.

The tractor bundle associated to a conformally Fedosov structure in \cite{10} actually looks similar to the standard tractor bundle of a contact projective structure in one more dimension. This leads to the idea of constructing sequences like the ones from \cite{9} via descending usual BGG sequences from a contact projective structure in one higher dimension (which also is in accordance with the length of the complexes that can be traced from the construction in \cite{9}). Now contact projective structures fit into the class of so–called \textit{parabolic contact structures}. These are finite order geometric structures (indeed, parabolic geometries), which carry an underlying contact structure. The available types of parabolic contact structures correspond to simple Lie algebras which admit a so–called contact grading (which is the case for almost all non–compact real simple Lie algebras). Contact projective structures in this picture correspond to Lie algebras of type $C_n$ and are slightly exceptional, see Section 4.2 of \cite{7}.

In the series of articles starting with this one, we carry out the idea of descending BGG complexes to appropriate quotients. This is not only done for contact projective structures but for all parabolic contact structures. It turns out that the quotients in question can be characterized by the fact that they carry certain geometric structures. The current article is devoted to the study of the basic properties of these geometric structures, independently of any realization as a quotient. The exceptional behavior of the $C_n$–type Lie algebras mentioned above also shows up in this setting. Hence we exclude them from the discussion this part of the series. We will discuss conformally Fedosov structures in the framework of contactification in the second part \cite{5} of the series.

To any contact grading on a real simple Lie algebra, which is not of type $C_n$, we associate a first order $G$–structure, which has an underlying almost conformally symplectic structure. We call the resulting structures \textit{parabolic almost conformally symplectic structures} or \textit{PACS–structures}. If the underlying structure is conformally symplectic, then the structure is called a \textit{PCS–structure}. It should be mentioned here that we do not use the classical definition of an (almost) conformally symplectic structure via a representative two–form. Rather than that, we proceed similarly to conformal geometry and view the structures as line subbundles in the bundle of two forms, which leads to several simplifications.

The main result of the article is Corollary \cite{1.3} which states that any PACS–structure on a smooth manifold $M$ determines a unique linear connection on $TM$ whose torsion satisfies a suitable normalization condition. Moreover, the torsion of this canonical connection naturally splits into two components. One of these is a complete obstruction against the underlying almost conformally symplectic structure being conformally symplectic, while the other is an obstruction against integrability of the additional structure.

This main result is proved by showing that the Lie algebra of the structure group of any PACS–structure has vanishing first prolongation (in the standard sense of Sternberg, see \cite{14}). This result is proved via Kostant’s theorem (see \cite{12}) on Lie algebra cohomology, which also leads to an appropriate normalization
condition on the torsion. We also use Kostant’s theorem to show that, except in the $A_n$–case, the Lie algebra of the structure group of a PACS–structure is a maximal subalgebra of the corresponding conformally symplectic Lie algebra. The computations of Lie algebra cohomologies needed for our applications of Kostant’s theorem are available in the literature on parabolic contact structures, see Section 4.2 of [7], which also provides explicit descriptions of the normalization conditions on torsions. In that way, we obtain both an explicit description of all PACS–structures (see Sections 3.2 to 3.5) and explicit descriptions and interpretations of the components of the torsion of the canonical connections (see Sections 4.5 and 4.6).

There is a second important motivation for this article. The PCS–structures we introduce can be viewed as forming the geometric background for the class of special symplectic connections as introduced by M. Cahen and L. Schwachhöfer in [2]. The latter form a class of torsion free linear connections which preserve a symplectic form and satisfy a certain condition on their curvature (which is related to contact gradings of simple Lie algebras), see Section 4.7 for details. This class on the one hand contains all affine connections of exceptional holonomy, which preserve a symplectic form. According to the classification of affine holonomies, see [13], these cover a substantial part of all exceptional holonomies. On the other hand, the Levi–Civita connections of Bochner–Kähler metrics, see [1], and of Bochner–bi–Lagrangian metrics are special symplectic connections. We prove in Theorem 4.7 that, except for connections of Ricci type (which correspond to type $C_n$), any special symplectic connection is the canonical connection of a torsion free PCS–structure. Moreover, if the type is also different from $A_n$, the converse holds, i.e. the canonical connection of a torsion free PCS–structure is automatically a special symplectic connection. In particular, all affine connections with exceptional symplectic holonomy fall into this class.

For completeness, let us briefly describe the contents of the other articles in the series. In the second part [5], we describe the relation between PCS–structures and parabolic contact structures. This builds on the corresponding relation between conformally symplectic structures and contact structures via contactification as discussed in [4]. In this context, we also consider conformally Fedosov structures as introduced in [10] (adapted to the version of conformally symplectic structures that we use). These can be considered as the analogs of PCS–structures corresponding to the contact gradings of the simple Lie algebras of type $C_n$. We show that a quotient of a parabolic contact structure by a transversal infinitesimal automorphism inherits a PCS–structure of the corresponding type. Moreover, locally any PCS–structure arises in this way and the inducing parabolic contact structure is locally unique up to isomorphism.

We also clarify the relation between the canonical connection associated to the PCS–structure on the quotient and distinguished connections of the original parabolic contact structure. Finally, using contactifications, we complete the characterization of special symplectic connections in terms of PCS–structures. This provides new proofs and generalizations to cases of non–trivial torsion for several results from [2].
In the last article [6] of the series, contactifications are used to descend BGG sequences and relative BGG sequences associate to parabolic contact structures to natural sequences of differential operators on manifolds endowed with PCS–structures and study their properties. In many situations this can be used to construct differential complexes intrinsically associated to special symplectic connections and more general PCS–structures.

2. Conformally symplectic structures

We start by looking at (almost) conformally symplectic structures from the point of view of first order $G$–structures.

2.1. Almost conformally symplectic structures. Traditionally, (locally) conformally symplectic structures are defined via representative two forms. For our purposes, it will be more natural to use notions suggested by the theory of conformal structures.

Definition 2.1. Let $M$ be a smooth manifold of even dimension $n = 2m \geq 4$.

(1) An almost conformally symplectic structure on $M$ is a smooth line subbundle $\ell \subset \Lambda^2 T^* M$ in the bundle of two–forms on $M$ such that for each $x \in M$, each non–zero element of $\ell_x$ is non–degenerate as a bilinear form on $T_x M$.

(2) The structure is called conformally symplectic if and only if locally around each $x \in M$ there is a smooth section $\tau$ of $\ell$ which is closed as a two–form and satisfies $\tau(x) \neq 0$.

Observe that in part (2), one may equivalently replace “closed” by “exact”.

Proposition 2.2. Let $M$ be a smooth manifold of even dimension $n = 2m \geq 4$ and let $\ell \subset \Lambda^2 T^* M$ be an almost conformally symplectic structure. Then we have

(i) If $\ell$ is conformally symplectic, then for any local non–vanishing section $\tau$ of $\ell$, we have $d\tau = \varphi \wedge \tau$ for some $\varphi \in \Omega^1(M)$.

(ii) Conversely, if $n > 4$ and for each point $x \in M$, there is a local section $\tau$ of $\ell$ which satisfies the condition in (i) and $\tau(x) \neq 0$, then $\ell$ is conformally symplectic.

(iii) If $\ell$ is conformally symplectic, then local closed sections of $\ell$ are uniquely determined up to a constant factor.

Proof. (i) It suffices to show this locally, so we can assume that $\tau = f \tilde{\tau}$ for a nowhere vanishing closed section $\tilde{\tau}$ of $\ell$ and a non–zero function $f$. But then $d\tau = df \wedge \tilde{\tau} = (df/f) \wedge \tau$.

(ii) If $\tau$ is a nowhere vanishing local section of $\ell$ such that $d\tau = \varphi \wedge \tau$, then we conclude that $0 = d\varphi \wedge \tau$. For $n > 4$, non–degeneracy of $\tau$ implies $d\varphi = 0$. Hence restricting to some smaller subset, we can find a smooth non–zero function $f$ such that $-\varphi = d\log(f) = df/f$ and then $\tilde{\tau} = f\tau$ is closed.

(iii) As in (i), we write $\tau = f \tilde{\tau}$ with $\tilde{\tau}$ closed and non–vanishing. Then $0 = d\tau$ implies $0 = df \wedge \tilde{\tau}$, so non–degeneracy of $\tilde{\tau}$ implies $df = 0$, even if $n = 4$. □

Observe that for $n = 4$, non–degeneracy of a two–form $\tau$ implies that wedging with $\tau$ is an isomorphism from one–forms to three–forms. Hence in this case, the condition from part (i) of the proposition is always satisfied.
2.2. **First order G–structures.** Let us briefly review some basics from the theory of first order G–structures. Consider a finite dimensional real vector space $V$ and a Lie group $G$ endowed with an infinitesimally injective representation on $V$. This means that one has given a homomorphism $G \to GL(V)$ whose derivative is injective, thus identifying the Lie algebra $\mathfrak{g}$ of $G$ with a Lie subalgebra of $L(V,V)$.

Then a first order structure with structure group $G$ on a smooth manifold $M$ with $\dim(M) = \dim(V)$ is defined as a smooth principal fiber bundle $P \to M$ with structure group $G$, that is endowed with a one–form $\theta \in \Omega^1(P,V)$. The form $\theta$ has to be $G$–equivariant and strictly horizontal in the sense that its kernel in each point is the vertical subbundle of $P \to M$. This means that for each point $u \in P$ lying over $x \in M$, the value $\theta(u)$ descends to a linear isomorphism $T_x M \to V$, so one obtains a map to the linear frame bundle of $M$. In particular, $\theta$ gives rise to an identification of the associated bundle $P \times_G V$ with the tangent bundle $TM$.

The fundamental invariants of such structures are obtained via connections. Recall that on any principal bundle there are principal connections and that any principal connection on $P$ induces a linear connection on the associated vector bundle $TM$. This induced linear connection has a torsion which is a section of the bundle $\Lambda^2 T^* M \otimes TM \cong P \times_G (\Lambda^2 V^* \otimes V)$. Now the dependence of the torsion on the choice of connection can be described using linear algebra, via the process of prolongation, see [14].

As noted above, the Lie algebra $\mathfrak{g}$ of $G$ is a subalgebra of $L(V,V) = V^* \otimes V$. Then it is well known that the space of principal connections on $P$ is an affine space modeled on the space of smooth sections of the associated bundle $P \times_G (V^* \otimes \mathfrak{g})$. The change of torsion induced by a change of connection is described by a $G$–equivariant linear map

$$\partial: V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V$$

called the Spencer differential. Explicitly, this is given by first including $V^* \otimes \mathfrak{g}$ into $V^* \otimes V^* \otimes V$ and then alternating in the first two arguments. Alternatively, viewing the domain and target as linear maps and skew symmetric bilinear maps, respectively, one has $\partial \Phi(v,w) := \Phi(v)(w) - \Phi(w)(v)$.

The subspace $\text{im}(\partial) \subset L(\Lambda^2 V,V)$ gives rise to a smooth subbundle in $\Lambda^2 T^* M \otimes TM$, and we denote by $\mathcal{J}$ the quotient of $\Lambda^2 T^* M \otimes TM$ by this subbundle. Then it is clear from the above description that the projection of the torsion of the induced connection on $TM$ to this quotient bundle is the same for all principal connections on $P$. Hence one obtains a section of $\mathcal{J}$ which is an invariant of first order structures with structure group $G$. This is called the *intrinsic torsion* of the structure.

On the other hand, $\mathfrak{g}^{(1)} := \ker(\partial) \subset V^* \otimes \mathfrak{g}$ is called the *first prolongation* of $\mathfrak{g}$. The above discussion shows that for a fixed principal connection $\gamma$ on $P$, the space of all principal connections on $P$ which have the same torsion as $\gamma$ is an affine space modeled on sections of the associated bundle $P \times_G \mathfrak{g}^{(1)}$. In particular, if $\mathfrak{g}^{(1)} = \{0\}$, then any principal connection on $P$ is uniquely determined by its torsion.

The standard way to proceed further is to choose a $G$–invariant linear subspace $N \subset \Lambda^2 V^* \otimes V$, which is complementary to $\text{im}(\partial)$. Usually, one refers to $N$
as a normalization condition. Via associated bundles, \( N \) determines a smooth subbundle \( \mathcal{N} \subset \Lambda^2 T^* M \otimes TM \) and from above we see that there always exist normal principal connections, i.e. ones for which the torsion is a section of \( \mathcal{N} \).

The space of all normal connections is an affine space modeled on \( \mathcal{P} \times_G \mathfrak{g}^{(1)} \).

A classical simple example of this situation is the case \( G = O(V) \), the orthogonal group of a non-degenerate bilinear form on \( V \). In this case, first order structures with structure group \( G \) are equivalent to pseudo-Riemann metrics on manifolds of dimension \( n = \dim(V) \) of the signature of the given bilinear form. One easily verifies that \( \partial \) is a linear isomorphism in this case, which shows that any such geometry admits a unique torsion-free connection.

2.3. To apply this in the case of almost conformally symplectic structures, consider a symplectic vector space \( (V, b) \) of dimension \( n = 2m \) and define

\[
\text{Sp}(V) := \{ A \in GL(V) : b(Av, Aw) = b(v, w) \quad \forall v, w \in V \}
\]

\[
\text{CSp}(V) := \{ A \in GL(V) : \exists \lambda \in \mathbb{R} : b(Av, Aw) = \lambda b(v, w) \quad \forall v, w \in V \},
\]

the symplectic group and the conformally symplectic group of \( V \). These are closed subgroups of \( GL(V) \) and it is well known that the Lie algebra \( \mathfrak{sp}(V) \) is simple, while \( \mathfrak{csp}(V) = \mathbb{R} \oplus \mathfrak{sp}(V) \) is reductive with one-dimensional center. Moreover, as a representation of \( \text{Sp}(V) \), we have \( \mathfrak{sp}(V) \cong S^2 V \), the symmetric square of the standard representation \( V \).

The symplectic inner product \( b \) determines a non-degenerate element in \( \Lambda^2 V^* \), whose inverse is a non-degenerate element \( b^{-1} \in \Lambda^2 V \). This gives rise to a \( \text{Sp}(V) \)-equivariant map \( \Lambda^k V^* \to \Lambda^{k-2} V^* \), which is surjective for \( k \leq m + 1 \). Its kernel is called the tracefree part \( \Lambda^k_0 V^* \subset \Lambda^k V^* \).

Proposition 2.3. Let \( V \) be a real symplectic vector space of (even) dimension \( n \). Then we have:

1. A first order structure with structure group \( G := \text{CSp}(V) \) is equivalent to an almost conformally symplectic structure.
2. For \( g := \mathfrak{csp}(V) \), we obtain \( \mathfrak{g}^{(1)} \cong S^2 V \) and \( (\Lambda^2 V^* \otimes V) / \text{im}(\partial) \cong \Lambda^3_0 V^* \) as representations of \( \text{Sp}(V) \). In particular \( \partial \) is surjective for \( n = 4 \).
3. If \( n > 4 \), then an almost conformally symplectic structure has vanishing intrinsic torsion if and only if it is conformally symplectic.
4. For any torsion free connection compatible with a conformally symplectic structure, the induced connection on \( \ell \subset \Lambda^2 T^* M \) is flat and its local parallel sections are exactly those sections of \( \ell \) which are closed as two-forms on \( M \).

Proof. (1) The line in \( \Lambda^2 V^* \) spanned by \( b \) is by definition invariant under \( G \), so it gives rise to a smooth line subbundle \( \ell \subset \Lambda^2 T^* M \). Non-degeneracy of \( b \) implies that any non-zero element in \( \ell \) is non-degenerate, so it defines an almost conformally symplectic structure.

Conversely, suppose that we are given a smooth line subbundle \( \ell \subset \Lambda^2 T^* M \) in which each non-zero element of \( \ell \) is non-degenerate. Then for each point \( x \in M \), one defines \( \mathcal{P}_x \) to consist of all linear isomorphisms \( V \to T_x M \) such that the induced isomorphism \( \Lambda^2 V^* \to \Lambda^2 T^* M \) maps \( b \) to an element of \( \ell_x \). Of course, there is at least one such isomorphism and fixing this, one obtains a bijection.
between $P_x$ and $G = CSp(V)$. One then defines $P$ to be the disjoint union of the spaces $P_x$, and obtains a natural projection $P \to M$. It is then easy to see that this is a smooth principal $G$–bundle over $M$ which defines a reduction of structure group as required.

(2) We know that $\partial$ is $CSp(V)$ equivariant, and we analyze its kernel and its image as representations of $Sp(V)$. We have noted above that $csp(V) \cong \mathbb{R} \oplus S^2V$. Tensorizing with $V^* \cong V$, it is well known that $S^2V \otimes V$ decomposes as a direct sum of three irreducible representations. Using $b$, one can form a trace $S^2V \otimes V \to V$ and the kernel of this trace decomposes as the direct sum of $S^3V$ and an irreducible representation $W$ (the kernel of the symmetrization map inside the tracefree part).

Likewise, $\Lambda^2V^*$ decomposes as $\Lambda^2_0V^* \oplus \mathbb{R}$. Forming the tensor product with $V \cong V^*$, $\Lambda^2_0V^* \otimes V^*$ again decomposes as a direct sum of three irreducibles if $n > 4$, while there are only two irreducible components for $n = 4$. Using $b^{-1}$, one defines a trace $\Lambda^2_0V^* \otimes V^* \to V^*$, and its kernel decomposes into $\Lambda^2_0V^* \oplus W'$, where the first summand is non–zero only for $n > 4$ and $W'$ is the kernel of the alternation within the tracefree part. It is well known that $W$ and $W'$ are isomorphic irreducible representations of $Sp(V)$.

Since the irreducible representation $S^3V$ does not occur in $\Lambda^2V^* \otimes V$, it must be contained in $csp(V)^{(1)}$. One then easily verifies directly that $\partial$ is injective on the sum of two copies of $V$ contained in $V^* \otimes csp(V)$ and non–zero (and thus injective) on the irreducible subspace $W$. This proves the claim on the first prolongation. Moreover, it also implies that $\text{im}(\partial)$ is isomorphic to $W \oplus V \oplus V$, which immediately implies the second claim.

(3) For an almost conformally symplectic structure $\ell \subset \Lambda^2T^*M$, consider the corresponding first order structure $P \to M$ with structure group $G$. By construction, a linear connection $\nabla$ on $TM$ is induced by a principal connection $\gamma$ on $P$ if and only if the line subbundle $\ell$ is preserved by the induced connection on $\Lambda^2T^*M$.

More explicitly, for a vector field $\xi \in \mathfrak{x}(M)$ and a local non–vanishing section $\tau$ of $\ell$, also $\nabla_\xi \tau$ has to be a section of $\ell$.

For such a section $\tau$, we define $\iota_\tau \tau$ to be the map $(\xi, \eta, \zeta) \mapsto \tau(T(\xi, \eta, \zeta))$. From part (2) we conclude that the intrinsic torsion of our geometry vanishes if and only if the three form obtained by alternating $\iota_\tau \tau$ is a section of the trace part of $\Lambda^3T^*M$, i.e. can be written as $\tau \wedge \varphi$ for some $\varphi \in \Omega^1(M)$. Now it is well known that for a torsion–free connection, $d\tau$ can be computed as the alternation of $\nabla \tau$. In the presence of torsion, this formula has to be modified by adding a non–zero multiple of the alternation of $\iota_\tau \tau$. But since the alternation of $\nabla \tau$ automatically is a section of the trace part, we conclude that vanishing of the intrinsic torsion is equivalent to the fact that $d\tau$ is a section of the trace part. So this means $d\tau = \tau \wedge \varphi$ for some $\varphi \in \Omega^1(M)$ and from Proposition 2.2 we know that, for $n > 4$, this is equivalent to $\ell$ defining a conformally symplectic structure.

(4) From the proof of part (3) we know that for a locally non–vanishing section $\alpha$ of $\ell$ and any connection $\nabla$ compatible with the structure, we have $\nabla \alpha = \varphi \otimes \alpha$ for some one–form $\varphi$ on $M$. If $\nabla$ is torsion free, then the alternation $\varphi \wedge \alpha$ of this coincides with $d\alpha$. So if $d\alpha = 0$, then $0 = \varphi \wedge \alpha$ and we have noted above that by
non-degeneracy of $\alpha$ this implies $\varphi = 0$. Hence any locally closed section of $\ell$ is parallel for $\nabla$, which implies both claims. \hfill \Box

3. PACS– and PCS–structures

In this section, we introduce the geometric structures studied in this article and describes their basic properties. The theory of simple Lie algebras leads to a family of subalgebras of conformally symplectic Lie algebras via so-called contact gradings. Groups with this Lie algebra then give rise to geometric structures, each of which has an underlying almost conformally symplectic structure. Via the classification of simple Lie algebras, one can describe the resulting geometric structures explicitly.

3.1. Contact gradings of simple Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$. A contact grading on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (with $\mathfrak{g}_\ell = \{0\}$ if $|\ell| > 2$), and such that $\dim_K(\mathfrak{g}_{-2}) = 1$ and the Lie bracket of $\mathfrak{g}$ restricts to a non-degenerate bilinear map $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Since $\mathfrak{g}_{-2}$ is one-dimensional, this bracket gives rise to a well defined one-dimensional subspace in $\Lambda^2(\mathfrak{g}_{-1})^*$, in which each non-zero element is non-degenerate.

Gradings of this kind are used in the theory of quaternionic and para-quaternionic symmetric spaces and in the theory of parabolic contact structures, the latter will be discussed briefly in the second part of this series. It is well known that any complex simple Lie algebra admits (up to isomorphism) a unique grading of this type. Moreover, this grading can be restricted to almost all non-compact real forms, see Section 3.2.4 and Example 3.2.10 of [7] for a complete discussion.

From the grading property, it is clear that $\mathfrak{g}_0$ acts on each $\mathfrak{g}_i$ via the restriction of the adjoint action. It is well known that the resulting representation $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is faithful. Since the adjoint action preserves the Lie bracket, we conclude that it preserves the line in $\Lambda^2(\mathfrak{g}_{-1})^*$ constructed above, so we actually get an inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$.

Next, we have to choose a group $G_0$ with Lie algebra $\mathfrak{g}_0$. For the general discussion of PACS–structures, we only have to assume that the representation $\mathfrak{g}_0 \to \mathfrak{csp}(\mathfrak{g}_{-1})$ integrates to a representation $G_0 \to CSp(\mathfrak{g}_{-1})$, which then is infinitesimally faithful by construction. As usual in the theory of first order geometric structures, different choices of groups leads to very similar geometries. We will describe the geometries for one natural choice of group below and not go into further details on possible other choices of groups. When dealing with contactifications in the second part of this series of articles, we will have to restrict the choice of groups a bit.

Having chosen $G_0$, we can consider first order $G_0$–structures on smooth manifolds of dimension $\dim(\mathfrak{g}_{-1})$, and by construction any such structure will have an underlying almost conformally symplectic structure. There is a particular case here, that we have to exclude from the further discussion. Namely, if one takes $\mathfrak{g}$ of type $C_n$, i.e. if $\mathfrak{g}$ is a symplectic Lie algebra, then the resulting contact grading has
the property that $\mathfrak{g}_0$ is the full Lie algebra $\mathfrak{csp}(\mathfrak{g}_{-1})$. Hence in this case, a reduction to a structure group with Lie algebra $\mathfrak{g}_0$ is essentially only an almost conformally symplectic structure, and there is no additional structure there. Therefore, in the rest of this article, we will always assume that $\mathfrak{g}$ is not of type $C_n$.

**Definition 3.1.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ which admits a contact grading, let $\mathfrak{g}_0 \hookrightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$ be the corresponding representation. Let $G_0$ be a Lie group with Lie algebra $\mathfrak{g}_0$ such that there is a corresponding homomorphism $G_0 \to CSp(\mathfrak{g}_{-1})$.

1. The first order structures with structure group $G_0$ on manifolds of dimension $\dim(\mathfrak{g}_{-1})$ coming from the representation $G_0 \to CSp(\mathfrak{g}_{-1})$ are called **parabolic almost conformally symplectic structures** or PACS–structures (associated to $G_0$).

2. A **parabolic conformally symplectic structure** or PCS–structure is a PACS–structure for which the underlying almost conformally symplectic structure is conformally symplectic.

**Remark 3.2.** (i) We will refine the terminology on PACS– and PCS–structures in the discussion of the individual examples (according to the classification of simple Lie algebras admitting a contact grading) in the rest of this section.

(ii) The Lie subalgebra $\mathfrak{g}_0$ is known to be reductive, with center of dimension 2 in the $A_n$–case and dimension 1 in all other cases. There is a natural codimension–one subalgebra $\mathfrak{g}_0^0 \subset \mathfrak{g}_0$, consisting of those elements of $\mathfrak{g}_0$ which act trivially on $\mathfrak{g}_{\pm 2}$ under the adjoint action. By construction, this coincides with $\mathfrak{g}_0 \cap \mathfrak{sp}(\mathfrak{g}_{-1})$, and, apart from the $A_n$–case, it also coincides with the semisimple part of $\mathfrak{g}_0$. The resulting subalgebras of symplectic Lie algebras are exactly the **special symplectic subalgebras** as introduced by M. Cahen and L. Schwachhöfer in [2].

The main notion introduced in that reference is the one of a **special symplectic connection**, a torsion free connection, which preserves a symplectic form and whose curvature lies in a certain space $\mathcal{R}_{\mathfrak{g}_0^0}$ associated to a special symplectic subalgebra $\mathfrak{g}_0^0$. The family of special symplectic connections in particular includes all affine connections having exceptional holonomy of symplectic type. A detailed description of $\mathcal{R}_{\mathfrak{g}_0^0}$ will be given in Section 4.7, where we also discuss the relation between PCS–structures and special symplectic connections.

### 3.2. PACS–structures of Kähler and para–Kähler type.

To obtain more explicit descriptions of PACS–structures, we have to go through the list of contact gradings of simple Lie algebras. From their use in the theory of parabolic contact structures, algebraic descriptions of the resulting subalgebras in conformally symplectic algebras are available. This can be directly converted into information on the corresponding first order structure.

We start this discussion with simple Lie algebras of type $A_n$, i.e. real forms of $\mathfrak{sl}(n, \mathbb{C})$. It is well known that there are three different types of real forms of this algebra, namely $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sl}(n/2, \mathbb{H})$ (for even $n$) and $\mathfrak{su}(p, q)$ with $p + q = n$. Contact gradings are available on $\mathfrak{sl}(n, \mathbb{R})$ and on $\mathfrak{su}(p, q)$ if both $p$ and $q$ are non–zero.

The contact grading of $\mathfrak{sl}(n + 2, \mathbb{R})$ is described in Section 4.2.3 of [7]. The grading component $\mathfrak{g}_{-1} \cong \mathbb{R}^{2n}$ splits as a direct sum $\mathfrak{g}_1^\mathbb{R} \oplus \mathfrak{g}_{-1}^\mathbb{R}$ of two subspaces
of dimension $n$, which both are isotropic for the Lie bracket. The subalgebra $\mathfrak{g}_0 \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ consists of those maps which preserve this decomposition. A natural choice for a group $G_0$ thus is the subgroup of $\mathcal{CSp}(\mathfrak{g}_{-1})$ consisting of all maps which preserve that decomposition of $\mathfrak{g}_{-1}$. Motivated by the description in the following Proposition, we will refer to the corresponding geometric structure as a PACS–structure of para–Kähler type.

The contact grading of $\mathfrak{su}(p+1, q+1)$ for $p + q = n$ is discussed in Section 4.2.4 of [7]. Here $\mathfrak{g}_{-1}$ is a complex vector space and the bracket has the property that $[iX, iY] = [X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$. Moreover, $\mathfrak{g}_0$ consists of all maps in $\mathfrak{csp}(\mathfrak{g}_{-1})$ which are complex linear. This implies that $[,]$ is the imaginary part of a $\mathfrak{g}_{-2}$–valued Hermitian form, which turns out to have signature $(p, q)$. A natural choice for a group $G_0$ thus is the subgroup of complex linear maps in $\mathcal{CSp}(\mathfrak{g}_{-1})$, so $G_0 \cong \mathbb{C}U(p, q)$ is a conformal pseudo–unitary group. We will call the corresponding geometric structure a PACS–structure of Kähler type.

To formulate the description of these types of PACS–structures, let us recall some concepts. An almost Hermitian metric on a smooth manifold of even dimension $2n$ is given by an almost complex structure $J$ on $M$ and a pseudo–Riemannian metric $g$ which is Hermitian with respect to $J$, i.e. such that $g(J\xi, J\eta) = g(\xi, \eta)$ for all tangent vectors $\xi$ and $\eta$. Given such a structure, any conformal rescaling of $g$ defines an almost Hermitian metric on $M$, too, so it is no problem to talk about a conformal class of almost Hermitian metrics. The basic properties of almost Hermitian metrics are analyzed in the seminal article [11] of Gray and Hervella. The main ingredient there is the fundamental two–form of $g$ defined by $\omega(\xi, \eta) := -g(J\xi, \eta)$, so this is non–degenerate in each point. In particular, $(g, J)$ is called an almost Kähler metric if and only if $\omega$ is closed.

Similarly, an almost para–Hermitian metric on a smooth manifold $M$ of even dimension $2n$ is defined as a decomposition $TM = E \oplus F$ into two sub–bundles of rank $n$ and a pseudo–Riemannian metric $g$ on $M$, for which both $E$ and $F$ are isotropic. This implies that $g$ has split–signature $(n, n)$. As above, it is no problem to consider conformal classes of almost para–Hermitian metrics. The similarity to the Hermitian case becomes evident if one describes the decomposition as an almost para–complex structure $\mathcal{J} : TM \to TM$, where $\mathcal{J}$ acts as the identity on $E$ and as minus the identity on $F$, so $\mathcal{J}^2 = \text{id}$. Then the compatibility of $g$ with the decomposition is equivalent to the fact that $g(\mathcal{J}\xi, \eta) = -g(\xi, \mathcal{J}\eta)$, and one defines a fundamental two–form $\omega$ as above. The metric is then called almost para–Kähler if and only if $\omega$ is closed.

**Proposition 3.3.** (1) A PACS–structure of Kähler type of signature $(p, q)$ on a smooth manifold $M$ of real dimension $2(p + q) \geq 4$ is equivalent to a conformal class of almost Hermitian metrics $(g, J)$ on $M$.

(2) A PACS–structure of para–Kähler type on a smooth manifold $M$ of real dimension $2n \geq 4$ is given by a conformal class of para–Hermitian metrics $(g, TM = E \oplus F)$ on $M$.

(3) In both cases, the underlying almost conformally symplectic structure $\ell \subset \Lambda^2 T^* M$ is spanned by the fundamental two–forms of the metrics in the class. In
particular, the structure is PCS if and only if the conformal class locally contains almost (para–)Kähler metrics, which then are unique up to a constant factor.

Proof. (1) From the description of $G_0$ it is clear that a first order structure corresponding to this group on $M$ is equivalent to an almost conformally symplectic structure $\ell \subset \Lambda^2 T^* M$ and an almost complex structure $J$ on $M$ such that for each $x \in M$ any element of $\ell_x$ is Hermitian for $J_x$. Now given a non–zero element $\omega \in \ell_x$, one defines $g_x : T_x M \times T_x M \to \mathbb{R}$ by $g_x(\xi, \eta) = \omega(\xi, J\eta)$, which by construction is Hermitian and of signature $(p, q)$. Different elements in $\ell_x$ lead to proportional metrics, so we obtain a conformal class of almost–Hermitian metrics. Conversely, given such a class, one takes the almost complex structure $J$ together with the line $\ell$ spanned by the fundamental two–forms to obtain a first order structure with group $G_0$.

(2) is proved in the same way as (1).

(3) From the construction in the proof of (1) it is clear that $\ell \subset \Lambda^2 T^* M$ is spanned by the fundamental two–forms of the metrics in the class. But then by definition, a local section of $\ell$ is closed if and only if the corresponding local metric is almost (para–)Kähler. \hfill \Box

3.3. PACS–structures of Grassmannian type. We next move to real forms of the complex orthogonal Lie algebras $\mathfrak{so}(n, \mathbb{C})$, i.e. to types $B_m$ and $D_m$ in the classification of simple Lie algebras. In view of the isomorphisms $B_2 \cong C_3$ for $D_3 \cong A_3$, we only have to look at case $n \geq 7$ here. The obvious real forms in this case are the orthogonal Lie algebras $\mathfrak{so}(p, q)$ with $p + q = n$, and these admit a contact grading provided that $p, q \geq 2$. For the even orthogonal Lie algebras, there is a second real form which is discussed in Section 3.4 below.

So we have to consider real Lie algebras of the form $\mathfrak{so}(p + 2, q + 2)$ with $p + q = n \geq 3$ and their contact gradings are described in Section 4.2.5 of \cite{7}. In this case $\mathfrak{g}_{-1}$ is a space of matrices, $\mathfrak{g}_{-1} \cong M_{n,2}(\mathbb{R}) = \mathbb{R}^{2n} \oplus \mathbb{R}^n$ where $n = p + q$. Correspondingly, there is a natural inclusion of $\mathfrak{s}(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{gl}(n, \mathbb{R}))$ into $\mathfrak{gl}(\mathfrak{g}_{-1})$ and $\mathfrak{g}_0$ is contained in that subalgebra and hence acts by maps preserving the tensor product decomposition. Hence, as a representation of $\mathfrak{g}_0$, we get

$$
\Lambda^2(\mathfrak{g}_{-1})^* = \Lambda^2(\mathbb{R}^2 \otimes \mathbb{R}^n)^* \cong (\Lambda^2 \mathbb{R}^2 \otimes \Lambda^2 \mathbb{R}^n)^* \oplus (\Lambda^2 \mathbb{R}^2 \otimes S^2 \mathbb{R}^n)^*.
$$

and the line corresponding to the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is contained in the second summand. Hence it gives rise to an inner product on $\mathbb{R}^n$ defined up to scale, and this has signature $(p, q)$. Given this line, the algebra $\mathfrak{g}_0$ consists of those elements of $\mathfrak{csp}(\mathfrak{g}_{-1})$ which are compatible with the tensor product decomposition.

Hence we obtain a natural choice of group $G_0$ by intersecting $\mathcal{CSp}(\mathfrak{g}_{-1})$ with the subgroup of $GL(\mathfrak{g}_{-1})$ consisting of those maps which preserve the tensor product decomposition. The latter group is the image of the natural homomorphism $GL(2, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(\mathfrak{g}_{-1})$ obtained by multiplying with matrices from both sides. Up to a covering, $G_0$ is isomorphic to $GL(2, \mathbb{R}) \times SO(p, q)$.

To describe the corresponding PACS–structures, let us recall a bit of background. Suppose that $M$ is a manifold of real dimension $2n$. Then an almost
Grassmannian structure of type \((2, n)\) on \(M\) is given by two auxiliary vector bundles \(E\) and \(F\) over \(M\) of rank two and \(n\), respectively, together with isomorphisms \(TM \cong L(E, F) = E^* \otimes F\) and \(\Lambda^2 E \cong \Lambda^n F\). Equivalently, these can be described as first order structures corresponding to subgroup of \(GL(M_{n,2}(\mathbb{R}))\) described above.

For \(n = 2\), an almost Grassmannian structure is equivalent to a split signature conformal structure, and also for \(n > 2\), almost Grassmannian structures are similar to conformal structures in several respects. This is related to the fact that they can be equivalently described by a canonical Cartan connection with homogeneous model the Grassmannian of two–planes in \(\mathbb{R}^{n+2}\) viewed as a homogeneous space of \(SL(n+2, \mathbb{R})\), see Section 4.1.3 of [7]. Hence they fall into the class of so–called AHS–structures, a subclass of parabolic geometries which also contains conformal structures.

Given an almost Grassmannian structure \(TM \cong E^* \otimes F\) on \(M\), one obtains an isomorphism \(\Lambda^2 T^* M \cong (S^2 E \otimes \Lambda^2 F^*) \oplus (\Lambda^2 E \otimes S^2 F^*)\), so there are two different types of two–forms. In particular, one calls a two–form Hermitian (in the Grassmannian sense) if it lies in \(\Lambda^2 E \otimes S^2 F^*\). Since \(\Lambda^2 E\) is a line bundle, a Hermitian two–form induces a symmetric bilinear form on the bundle \(F\) determined up to scale. In particular, in the non–degenerate case, one can associate to such a two–form a well defined signature \((p, q)\) with \(p + q = n\).

In view of this discussion, it is natural to call the PACS–structures corresponding to the contact gradings of real orthogonal algebras of Grassmannian type and the following result is evident.

**Proposition 3.4.** A PACS–structure of Grassmannian type on a smooth manifold \(M\) of dimension \(2(p + q) \geq 6\) is given by an almost Grassmannian structure \(TM \cong E^* \otimes F\) of type \((2, p + q)\) on \(M\) together with an almost conformally symplectic structure \(\ell \subset \Lambda^2 T^* M\), which is Hermitian in the Grassmannian sense, i.e. contained in the subbundle \(\Lambda^2 E \otimes S^2 F^*\), and has signature \((p, q)\).

**Remark 3.5.** For comparison with the quaternionic case to be discussed below, we note that these almost Grassmannian structures admit an alternative description. Given an almost Grassmannian structure \(TM \cong E^* \otimes F\) of \(E\) induces an endomorphism of \(TM\). For each \(x \in M\), the space \(L(T_x M, T_x M)\) naturally is an associative algebra under composition, and we obtain an inclusion of \(L(E_x, E_x) \cong M_2(\mathbb{R})\) as a subalgebra. These clearly fit together to define a bundle of subalgebras in the locally trivial bundle \(L(TM, TM)\) of (associative) algebras.

Since the algebra \(M_2(\mathbb{R})\) is the unique 4–dimensional normed real algebra with indefinite quadratic form (given by the determinant), it is also called the algebra of split quaternions. Conversely, given such a bundle of subalgebras, one can use the fact that \(M_2(\mathbb{R})\) is a simple algebra to recover a tensor product decomposition of \(TM\). Hence almost Grassmannian structures of type \((2, n)\) are also called almost split quaternionic structures.

### 3.4. PACS–structures of quaternionic type.

As mentioned in Section 3.3, there is another real form of the even orthogonal Lie algebras \(\mathfrak{so}(2n, \mathbb{C})\), which admits a contact grading. This is usually denoted by \(\mathfrak{so}^*(2n)\) and is related to...
skew–Hermitian forms on quaternionic vector spaces. We will discuss this rather briefly here and more thoroughly in the second part of this series in the context of contactification.

Recall that the concept of Hermitian forms makes sense over the quaternions. A (quaternionically) Hermitian form on a (right) vector space $V$ over the skew–field $\mathbb{H}$ of quaternions is defined as a real bilinear map $b : V \times V \to \mathbb{H}$ such that for $v, w \in V$ and $q \in \mathbb{H}$, one has $b(vq, w) = \overline{q} b(v, w)$, $b(v, wq) = b(v, w)q$ and $b(w, v) = \overline{b(v, w)}$. Here the bar denotes the usual conjugation on $\mathbb{H}$. Likewise, a (quaternionically) skew–Hermitian form on $V$ is defined a real bilinear map $\omega : V \times V \to \mathbb{H}$ such that $\omega(vq, w) = \overline{q} \omega(v, w)$, $\omega(v, wq) = \omega(v, w)q$ and $\omega(w, v) = -\overline{\omega(v, w)}$. (The analog of this concept also exists over the complex numbers, but it is not studied there, since multiplication by $i$ defines an isomorphism between Hermitian and skew–Hermitian forms in the complex case.)

Similarly as for quaternionically Hermitian forms, a skew–Hermitian form on $V$ can be recovered from its real part, which now is a skew–symmetric real valued bilinear form on $V$ which is preserved by multiplication by all unit quaternions. Conversely, any such bilinear form can be uniquely extended to a quaternionically skew–Hermitian form.

It turns out that on any finite dimensional quaternionic vector space there is a unique (up to quaternionically linear automorphisms) non–degenerate quaternionically skew–Hermitian form. Hence we can choose one such form $\omega$ on $\mathbb{H}^n$ and define $SO^*(2n)$ as the group of all quaternionically linear automorphisms $A$ of $\mathbb{H}^n$ such that $\omega(Av, Aw) = \omega(v, w)$ for all $v, w \in \mathbb{H}^n$. This clearly is a closed subgroup of $GL(n, \mathbb{H})$ and hence a Lie group, and we denote by $sl^*(2n)$ its Lie algebra. In the definition of $SO^*(2n)$, one may equivalently replace $\omega$ by its real part, which shows that, $SO^*(2n)$ is the intersection in $GL(4n, \mathbb{R})$ of $GL(n, \mathbb{H})$ with the stabilizer of the real part of $\omega$, which is isomorphic to $Sp(4n, \mathbb{R})$.

It is easy to verify explicitly that $so^*(2n)$ is a real form of the complex orthogonal algebra $so(2n, \mathbb{C})$. While for $n = 2, 3, 4$, this is isomorphic to real forms as discussed in Section 3.3 above, one obtains a genuinely different real form for $n \geq 5$. The following description of the associated PACS–structure is also valid for $n = 2, 3, 4$.

The situation here is closely parallel to the Grassmannian case treated in Section 3.3. Starting from $g = so^*(2n+2)$, we obtain $g_{-1} \cong \mathbb{H}^n$ and the bracket $[\ , \ ] : g_{-1} \times g_{-1} \to g_{-2}$ is the real part of a quaternionically skew–Hermitian form. The Lie subalgebra $g_0 \subset csp(g_{-1})$ is isomorphic to $\mathbb{H} \oplus so^*(g_{-1})$ with the first factor acting via quaternionic scalar multiplications (which are not quaternionically linear since $\mathbb{H}$ is non–commutative). As a natural choice for $G_0$ we can then use the subgroup $H^* \cdot SO^*(g_{-1}) \subset CSp(g_{-1})$ with the first factor acting by scalar multiplications.

Now it is well known that a reduction of structure group of a manifold of dimension $4n$ to the subgroup $H^* \cdot GL(n, \mathbb{H}) \subset GL(4n, \mathbb{R})$ which is generated by quaternionic scalar multiplications and quaternionically linear automorphisms of $\mathbb{H}^n$ is equivalent to an almost quaternionic structure. An almost quaternionic structure can be equivalently described as a bundle of subalgebras in the bundle $L(TM, TM)$ of associative algebras with modeling algebra $\mathbb{H}$. A more traditional
A PACS–structure of quaternionic type on a smooth manifold $M$ of dimension $4n \geq 8$ is given by an almost quaternionic structure on $M$ together with an almost conformally symplectic structure $\ell \subset \Lambda^2 T^* M$, such that each non–zero element of $\ell$ is the real part of a quaternionically skew–Hermitian form.

Example 3.7. Let us give an example of a homogeneous torsion–free PCS–structure of quaternionic type. This example will be discussed in more detail in the context of contactification in the next part of this series.

It is well known that the complex Grassmannian $M := \text{Gr}(2, \mathbb{C}^{n+2})$, viewed as a homogeneous space of $SU(n+2)$ is a very remarkable example of a symmetric space. This is due to the fact that it carries an invariant complex structure $J$ and an invariant quaternionic structure $Q$, with $J$ not being contained in $Q$. Moreover, there is an invariant Riemannian metric $g$ on $M$ which is Kähler with respect to $J$ and quaternion–Kähler with respect to $Q$. From Proposition 3.3 we know that the Kähler metric $g$ determines a PCS–structure of Kähler type on $M$. It turns out that $\omega$ defines a second PCS–structure on $M$.

Corollary 3.8. Let $M$ be the symmetric space $SU(n+2)/S(U(n) \times U(2))$, let $Q$ be the invariant quaternionic structure on $M$ and let $\omega$ be the Kähler form of the invariant Kähler metric on $M$. Then $\omega$ defines a PCS–structure of quaternionic type on $(M, Q)$.

Proof. We use the description of invariant structures of homogeneous spaces as developed in Section 1.4 of [7]. The symmetric decomposition of $\mathfrak{g} = \mathfrak{su}(n+2)$ as $\mathfrak{h} \oplus \mathfrak{m}$ can be seen by representing matrices in block form with blocks of size 2 and $n$ as $\begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$. Here $A \in \mathfrak{u}(2)$ and $B \in \mathfrak{u}(n)$ are such that $\text{tr}(A) + \text{tr}(B) = 0$ and form the $\mathfrak{h}$–part, while $X$ is a complex $(n \times 2)$–matrix, which forms the $\mathfrak{m}$–part. The action of the subgroup $H \cong S(U(n) \times U(2))$ corresponding to $\mathfrak{h}$ comes from multiplying $X$ by unitary matrices from both sides. Since this action is complex linear for the usual multiplication by $i$, this multiplication gives rise to an invariant almost complex structure $J$ on $M$.

On the other hand, multiplication from the right by elements of $\mathfrak{su}(2)$ defines a three–dimensional subspace $Q \subset L_{\mathbb{C}}(\mathfrak{m}, \mathfrak{m})$, which is $H$–invariant. The induced action of $H$ on $Q$ is via the adjoint action of $SU(2) \subset H$ on $\mathfrak{su}(2) \subset \mathfrak{h}$. Hence $Q$ gives rise to a three dimensional smooth subbundle $Q \subset L(TM, TM)$. It is well known that $\mathfrak{su}(2)$ is isomorphic to the imaginary quaternions, so $Q$ defines invariant almost quaternionic structure on $M$.

Now the standard Hermitian inner product on the complex vector space $\mathfrak{m}$ can be written as $(X,Y) \mapsto \text{tr}(X^*Y)$. This is Hermitian with respect to multiplication by $i$, so its real part extends to an invariant Riemannian metric $g$ on $M$, which is Hermitian with respect to $J$. Since by construction both $J$ and $g$ are parallel for the canonical connection on $M$, this defines an invariant Kähler structure whose Kähler form $\omega$ is induced by the imaginary part of the Hermitian inner product.
But now the imaginary unit quaternions correspond to those elements \( A \in \mathfrak{su}(2) \) for which \( A^* = A^{-1} = -A \) or equivalently \( A^2 = -\text{id} \). But for such an element, the Hermitian inner product satisfies \( (AX,AY) \mapsto \text{tr}(X^*A^{-1}AY) = \text{tr}(X^*Y) \). Hence the real part of this inner product is Hermitian in the quaternionic sense, so the metric \( g \) is quaternion-Kähler with respect to \( Q \). Likewise, the imaginary part is Hermitian in the quaternionic sense, so \( \omega \) defines a PCS-structure of quaternionic type with respect to \( Q \). \( \square \)

3.5. Exceptional PACS–structures. The contact gradings of real forms of the exceptional Lie algebras are discussed in Section 4.2.8 of [7]. In each case, one obtains a reductive Lie algebra \( \mathfrak{g}_0 \) with one–dimensional center together with an irreducible representation on \( \mathfrak{g}_{-1} \). All these representations have the property that their second exterior square contains a unique one–dimensional invariant subspace (and in fact only one more irreducible component). Hence any reductions of structure group of a manifold of dimension \( \dim(\mathfrak{g}_{-1}) \) automatically gives rise to an almost conformally symplectic structure (corresponding to the one–dimensional summand). In the exceptional cases it thus seems less appropriate to us to view a PACS–structure as an almost conformally symplectic structure plus some additional data. Rather than that one should view this as first order structures which carry an underlying almost conformally symplectic structure. The resulting algebras and representations are collected in the following table.

| type | \( \mathfrak{g}_0 \) | \( \mathfrak{g}_{-1} \) | \( \dim(\mathfrak{g}_{-1}) \) | further real forms |
|------|-----------------|-----------------|-----------------|-----------------|
| \( G_2 \) | \( \mathfrak{gl}(2,\mathbb{R}) \) | \( S^3\mathbb{R}^2 \) | 4 | – |
| \( F_4 \) | \( \mathfrak{csp}(6,\mathbb{R}) \) | \( \Lambda^3 \mathbb{R}^6 \) | 14 | – |
| \( E_6 \) | \( \mathfrak{gl}(6,\mathbb{R}) \) | \( \Lambda^3 \mathbb{R}^6 \) | 20 | \( \mathfrak{cu}(3,3), \mathfrak{cu}(5,1) \) |
| \( E_7 \) | \( \mathfrak{cso}(6,6) \) | \( \text{Spin} \) | 32 | \( \mathfrak{cu}(4,8), \mathfrak{cu}^*(12) \) |
| \( E_8 \) | split \( \mathfrak{e}_7 \) | 56 | one more |

For the \( F_4 \)–case, one has to view \( \mathbb{R}^6 \) as a symplectic vector space and then \( \Lambda^3 \mathbb{R}^6 \) denotes the tracefree part in the third exterior power. For the \( F_4 \)– and \( E_6 \)–cases, the underlying almost conformally symplectic structure comes from the wedge product \( \Lambda^3 \mathbb{R}^6 \times \Lambda^3 \mathbb{R}^6 \to \Lambda^6 \mathbb{R}^6 \). For the two conformally unitary algebras showing up in the \( E_6 \)–case, one has to use appropriate real subrepresentations in \( \Lambda^3 \mathbb{C}^6 \), and likewise in the \( E_7 \)–case, one needs real spin representations (which restricts the available signatures). To our knowledge, there is no established name for the 56–dimensional representation of \( \mathfrak{e}_7 \), it is the irreducible representation of lowest dimension of this algebra. The two real forms which show up here are the ones for which the complex representation of dimension 56 admits a real form.

In the cases associated to \( G_2 \), \( F_4 \) and \( E_6 \), one can describe the reductions of structure group more explicitly. For the PACS–structure determined by \( G_2 \) one has a manifold of dimension 4, together with an identification \( TM \cong S^3\mathcal{E} \) for an auxiliary rank–two bundle \( \mathcal{E} \). Likewise the PACS–structures corresponding to \( F_4 \) and \( E_8 \) can be described in terms of an auxiliary bundle \( \mathcal{E} \) of rank 6 (endowed with a symplectic form in the \( F_4 \)–case). For the remaining two cases, finding a similar description would first require a characterization of the \( \mathfrak{g}_0 \)–representations in question.
4. Canonical Connections

We will next analyze the PACS-structures introduced in Section 3 from the point of view of compatible connections on G-structures as discussed in Section 2.2. We will prove that for all the subalgebras \( g_0 \subset \mathfrak{gl}(g_{-1}) \) coming from contact gradings of simple Lie algebras of type different from \( C_n \), the first prolongation (in the sense of Sternberg as introduced in Section 2.2) vanishes. Moreover, there is a natural choice of a normalization condition, so that any PACS-structure gives rise to a canonical linear connection on the tangent bundle. The form of the resulting intrinsic torsion can be described explicitly for each of the structures. The essential tool for proving all these results is Kostant’s theorem (see [12] or the discussion in Section 3.3 of [2]) and its applications to parabolic contact structures.

4.1. Kostant’s Theorem. Let \( \mathfrak{g} \) be a simple Lie algebra endowed with a contact grading \( \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) as introduced in Section 3.1. Then \( \mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \) is a nilpotent subalgebra of \( \mathfrak{g} \) (a Heisenberg algebra), so \( \mathfrak{g} \) is naturally a module over \( \mathfrak{g}_- \). Then there is the standard complex for computing the Lie algebra cohomology spaces \( H^*(\mathfrak{g}_-, \mathfrak{g}) \), consisting of the chain spaces \( C^k(\mathfrak{g}_-, \mathfrak{g}) := \Lambda^k(\mathfrak{g}_-)^* \otimes \mathfrak{g} \) and differentials \( \partial_K : C^K(\mathfrak{g}_-, \mathfrak{g}) \to C^{K+1}(\mathfrak{g}_-, \mathfrak{g}) \). Viewing \( C^k(\mathfrak{g}_-, \mathfrak{g}) \) as the space of alternating multilinear maps, this differential is given by

\[
\partial_K \varphi(X_0, \ldots, X_k) := \sum_{i=0}^k (-1)^i \left[ X_i, \varphi(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \right] + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),
\]

where the Lie brackets are in \( \mathfrak{g} \). (The subscript is chosen to distinguish this “Kostant-differential” from the Spencer differential introduced in Section 2.2 which we will denote by \( \partial_S \) in what follows.)

Observe that there is a natural notion of homogeneity for multilinear maps mapping \( \mathfrak{g}_- \) to \( \mathfrak{g} \), and \( \partial_K \) preserves homogeneities (since the Lie brackets do). Consequently, one can split the whole complex \( (C^*(\mathfrak{g}_-, \mathfrak{g}), \partial_K) \) and thus also its cohomology according to homogeneity. Moreover, the grading property implies that the restriction of the adjoint action to \( \mathfrak{g}_0 \) preserves each grading component \( \mathfrak{g}_i \). Hence all the spaces \( C^k(\mathfrak{g}_-, \mathfrak{g}) \) are \( \mathfrak{g}_0 \)-modules and one easily verifies that \( \partial_K \) is \( \mathfrak{g}_0 \)-equivariant. Thus, each of the cohomology spaces naturally is a representation of \( \mathfrak{g}_0 \), and this structure is crucial for the description of the cohomology given in Kostant’s theorem.

The first step towards proving Kostant’s theorem is the construction of an adjoint \( \partial^*: C^k(\mathfrak{g}_-, \mathfrak{g}) \to C^{k-1}(\mathfrak{g}_-, \mathfrak{g}) \) to \( \partial_K \), usually called the Kostant codifferential. Using this, one defines the Kostant Laplacian \( \square := \partial^* \partial_K + \partial_K \partial^* \) which maps \( C^k(\mathfrak{g}_-, \mathfrak{g}) \) to itself. The adjointness of \( \partial^* \) and \( \partial_K \) leads to an algebraic Hodge decomposition, which in particular implies that \( H^k(\mathfrak{g}_-, \mathfrak{g}) \cong \ker(\square) \subset C^k(\mathfrak{g}_-, \mathfrak{g}) \) as a \( \mathfrak{g}_0 \)-module and elements in \( \ker(\square) \) are referred to as “harmonic”. Now Kostant’s theorem describes \( \ker(\square) \) as a representation of \( \mathfrak{g}_0 \). It turns out that this representations is simply reducible, i.e. a direct sum of pairwise non-isomorphic irreducible
representations. The highest weights of each of these irreducible components as well as the cohomology degree in which it is contained can be computed in terms of the Weyl group of \( \mathfrak{g} \). This computation is completely algorithmic. We will not discuss the general result but just use the available information on the cohomology groups as it is needed.

4.2. Maximality. The first result we deduce is at least implicitly in [2] and it is based on the description of \( H^1(\mathfrak{g}_-, \mathfrak{g}) \) obtained from Kostant’s theorem.

**Proposition 4.1.** Suppose that \( \mathfrak{g} \) is not of type \( A_n \) or \( C_n \). Then the subalgebra \( \mathfrak{g}_0 \subset \mathfrak{csp}(\mathfrak{g}_{-1}) \) coming from the contact grading of \( \mathfrak{g} \) is a maximal subalgebra. Indeed, any \( \mathfrak{g}_0 \)-invariant subspace of \( \mathfrak{csp}(\mathfrak{g}_{-1}) \) strictly containing \( \mathfrak{g}_0 \) coincides with \( \mathfrak{csp}(\mathfrak{g}_{-1}) \).

*Proof.* If \( \mathfrak{g} \) is not of type \( A_n \), then the parabolic subalgebra \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) is well known to be maximal (see Section 3.4.2 of [7]). This implies that there is only one simple reflection in the corresponding Hasse–diagram, so by Kostant’s theorem \( H^1(\mathfrak{g}_-, \mathfrak{g}) \) is an irreducible representation. Moreover, if \( \mathfrak{g} \) is not of type \( C_n \), then Propositions 3.3.7 and 4.3.1 of [7] imply that this irreducible representation must be contained in homogeneity zero.

So let us assume that \( \varphi : \mathfrak{g}_- \to \mathfrak{g} \) is a linear map which is homogeneous of degree zero, i.e. consists of components \( \varphi_i : \mathfrak{g}_i \to \mathfrak{g}_i \) for \( i = -1, -2 \). By definition, \( \partial_K \varphi = 0 \) is equivalent to \( \varphi_{-2}([X,Y]) = [\varphi_{-1}(X),Y] - [Y,\varphi_{-1}(X)] \). But this exactly means that \( \varphi_{-1} \in \mathfrak{csp}(\mathfrak{g}_{-1}) \) and then it uniquely determines \( \varphi_{-2} \). On the other hand, the homogeneity–zero part of \( \partial_K : \mathfrak{g} \to C^1(\mathfrak{g}_-, \mathfrak{g}_-) \) is just \( \text{ad} : \mathfrak{g}_0 \to \mathbb{L}(\mathfrak{g}_-, \mathfrak{g}_-) \).

Thus we conclude that \( H^1(\mathfrak{g}_-, \mathfrak{g}) = \mathfrak{csp}(\mathfrak{g}_{-1})/\mathfrak{g}_0 \) as a \( \mathfrak{g}_0 \)-module. Since any \( \mathfrak{g}_0 \)-invariant subspace in \( \mathfrak{csp}(\mathfrak{g}_{-1}) \) strictly containing \( \mathfrak{g}_0 \) descends to a non–zero \( \mathfrak{g}_0 \)-invariant subspace of \( \mathfrak{csp}(\mathfrak{g}_{-1})/\mathfrak{g}_0 \), irreducibility of \( H^1(\mathfrak{g}_-, \mathfrak{g}) \) implies the claim. \( \square \)

The main point about this result is that it adds to the perspective on PACS–structures. Since an almost conformally symplectic structure certainly does not determine a unique linear connection on the tangent bundle, it tells us that there are no intermediate structures between an almost conformally symplectic structure and a PACS–structure, which could already determine a canonical connection.

From that point of view, it is also interesting to see what happens in the \( A_n \)-case. Here the result depends on the real form under consideration. For \( \mathfrak{su}(p+1, q+1) \) it turns out that \( H^1(\mathfrak{g}_-, \mathfrak{g}) \) is also irreducible (deducing this from Kostant’s theorem is slightly more complicated, since one has to look at the complexification for which the cohomology splits into two irreducibles). Hence in this case, one again obtains a maximal subalgebra.

On the other hand, for the real form \( \mathfrak{sp}(n+2, \mathbb{R}) \) the subalgebra \( \mathfrak{g}_0 \) is really non–maximal. As we have seen in Section 3.2 \( \mathfrak{g}_0 \subset \mathfrak{csp}(\mathfrak{g}_{-1}) \) consists of those maps which preserve a decomposition of \( \mathfrak{g}_{-1} \) into a direct sum of two Lagrangean subspaces. Now there are two intermediate subalgebras, which consist of the maps preserving just one of these two subspaces. The geometric structure these subalgebras correspond to is of course a conformally symplectic structure together with a distinguished Lagrangean distribution. But such a structure can certainly not
determine a canonical linear connection on the tangent bundle of the manifold. Starting from any smooth manifold \( N \), the cotangent bundle \( M := T^*N \) carries such a structure coming from the canonical symplectic form and the vertical distribution. Now any diffeomorphism of \( N \) induces a diffeomorphism of \( M \), which preserves both the canonical symplectic structure and the vertical subbundle. Of course, the infinite dimensional group of diffeomorphisms of \( N \) cannot preserve a linear connection on \( TM \).

4.3. **Vanishing of the first prolongation.** To proceed towards the main result of this section, we need a few more facts related to Kostant’s theorem. For this step, the main tool is the homogeneity–one–component of \( H^2(g_-, g) \). For \( C^2(g_-, g) \), the part of homogeneity one is the direct sum of the spaces \( \Lambda^2 g^*_1 \otimes g_- \) and \( g^*_1 \otimes g^*_1 \otimes g_- \). On the other hand, Kostant’s theorem implies that the kernel of \( \Box \) in homogeneity one is contained in \( \Lambda^2 g^*_1 \otimes g_- \), see Lemma 4.2.2 in \cite{7}. On the other hand, the Lie bracket identifies \( g_- \) with \( g^*_1 \otimes g_- \), and hence we can naturally view \( \Lambda^3 g^*_1 \otimes g_- \) as a subspace in \( \Lambda^2 g^*_1 \otimes g_- \). The same then applies to the tracefree part, and using this, we can now formulate

**Theorem 4.2.** Let \( g \) be a simple Lie algebra, which is not of type \( C_n \) and admits a contact grading, and let \( g_0 \subset \mathfrak{sp}(g_-) \subset \mathfrak{gl}(g_-) \) be the corresponding inclusion.

Then the Spencer differential \( \partial_S : g^*_1 \otimes g_0 \rightarrow \Lambda^2 g^*_1 \otimes g_- \) is injective. Moreover, the subspaces \( \ker(\Box) \) and \( \Lambda^3 g^*_1 \otimes g_- \) of \( \Lambda^2 g^*_1 \otimes g_- \) have zero intersection and their direct sum is a linear complement to \( \text{im}(\partial_S) \).

**Proof.** The homogeneity–one part of \( (C^*(g_-, g), \partial_K) \) can be decomposed as

\[
\begin{array}{cccccc}
\text{g}_1 & \longrightarrow & g^*_1 \otimes g_0 & \overset{\partial_S}{\longrightarrow} & \Lambda^2 g^*_1 \otimes g_- & \overset{\gamma}{\longrightarrow} & \Lambda^3 g^*_1 \otimes g_- \\
\alpha & \downarrow & \beta & \downarrow & i & \downarrow & \\
\text{g}^*_2 \otimes g_- & \longrightarrow & g^*_1 \otimes g^*_1 \otimes g_- & & & \end{array}
\]

Here the first column corresponds to \( C^0 \), the sum of the two spaces in the next column is \( C^1 \), and so on. Moreover, we have split \( \partial_K \) into maps between the individual direct summands. The map \( \alpha \) by definition maps \( Z \in \text{g}_1 \) to \( \text{ad}_Z : g_- \rightarrow g_- \), which easily implies that \( \alpha \) is a linear isomorphism. The map \( \beta \) is just a tensor product of the linear isomorphism \( g_- \cong g^*_1 \otimes g_- \) (defined by the bracket) with an identity map, so it is a linear isomorphism, too. Next, \( g^*_1 \otimes g^*_1 \otimes g_- \) naturally includes as the trace part into \( \Lambda^3 g^*_1 \) and \( i \) is just the tensor product of this inclusion with the identity map and thus is injective. Finally, from the definition of \( \partial_K \), one immediately concludes that \( \gamma \) is the composition of the alternation with the obvious isomorphism \( \Lambda^2 g^*_1 \otimes g_- \cong \Lambda^2 g^*_1 \otimes g^*_1 \otimes g_- \).

Now suppose that \( \varphi \in g^*_1 \otimes g_0 \) satisfies \( \partial_S(\varphi) = 0 \). Viewing \( \varphi \) as an element of \( C^1(g_-, g) \), we have \( \partial_K(\varphi) \in g^*_1 \otimes g^*_2 \otimes g_- \) and hence \( 0 = \partial_K\partial_K \varphi = i(\partial_K \varphi) \). Since \( i \) is injective, we have \( \partial_K \varphi = 0 \). As we have noted in Section 4.1 already, Kostant’s theorem implies that \( H^1(g_-, g) \) is concentrated in homogeneity zero, so there must be an element \( Z \in g_1 \) such that \( \varphi = \partial_K(Z) \). But this implies that \( \alpha(Z) = 0 \) and hence \( Z = 0 \) and thus \( \varphi = 0 \), so injectivity of \( \partial_S \) follows.
Next consider \( \ker(\Box) \subset \Lambda^2 g^*_{-1} \otimes g_{-1} \). By Kostant’s theorem, each irreducible component of \( \ker(\Box) \) occurs with multiplicity one in \( C^*(g_-, g) \). Hence we conclude that each such component must have zero intersection with \( \text{im}(\partial_S) \), which is a sum of irreducible representations contained in \( C^1(g_-, g) \). Likewise, it has to have zero intersection with the subspace \( \Lambda^3 g^*_{-1} \otimes g_{-2} \), which is a sum of irreducible representations contained in \( C^3(g_-, g) \).

Now consider \( \varphi \in g^*_{-1} \otimes g_0 \). Then \( 0 = \partial_K \partial_K \varphi = \gamma(\partial_S \varphi) + i(\partial_K \varphi - \partial_S \varphi) \), which implies that \( \gamma(\partial_S \varphi) \) lies in the trace part of \( \Lambda^3 g^*_{-1} \otimes g_{-2} \). This shows that \( \text{im}(\partial_S) \) has zero intersection with \( \Lambda^3 g^*_{-1} \otimes g_{-2} \) and hence also with the direct sum of that space and \( \ker(\Box) \).

To complete the proof, it thus suffices to show that the three subspaces \( \text{im}(\partial_S) \), \( \ker(\Box) \), and \( \Lambda^3 g^*_{-1} \otimes g_{-2} \) span \( \Lambda^2 g^*_{-1} \otimes g_{-1} \). Given \( \psi \in \Lambda^2 g^*_{-1} \otimes g_{-1} \), there is a unique element \( \psi_3 \in \Lambda^3 g^*_{-1} \otimes g_{-2} \subset \Lambda^2 g^*_{-1} \otimes g_{-1} \) such that \( \gamma(\psi_3) \) coincides with the trace-free part of \( \gamma(\psi) \). Hence \( \gamma(\psi - \psi_3) \) is pure trace, so there is a unique element \( \hat{\psi} \in g^*_{-1} \otimes g^*_{-2} \otimes g_{-2} \) such that \( i(\hat{\psi}) = \gamma(\psi - \psi_3) \) and hence \( \partial_K (\psi - \hat{\psi} - \psi_3) = 0 \). Now Kostant’s theorem implies that \( \ker(\Box) \) is a linear complement to \( \text{im}(\partial_K) \) in \( \ker(\partial_K) \). Hence there are elements \( \psi_2 \in \ker(\Box) \) and \( \hat{\psi} \in C^1(g_-, g) \) such that \( \psi - \hat{\psi} - \psi_3 = \psi_2 + \partial_K \hat{\psi} \).

Finally, there is an element \( Z \in g_1 \) such that \( \alpha(Z) \) coincides with the component of \( \hat{\psi} \) in \( g^*_{-2} \otimes g_{-1} \), and we put \( \varphi = \hat{\psi} - \partial_K Z \in g^*_{-1} \otimes g_0 \). Then by construction, we have \( \partial_K \varphi = \partial_K \hat{\psi} \), so we conclude that \( \psi - \hat{\psi} - \psi_3 = \psi_2 + \partial_K \varphi \).

Now the component of \( \partial_K \varphi \) in \( \Lambda^2 g^*_{-1} \otimes g_{-1} \) equals \( \partial_S \varphi \) by construction, so looking at the components in that space, we get \( \psi = \partial_S \varphi + \psi_2 + \psi_3 \) and the proof is complete.

4.4. Canonical connections for PACS-structures. Converting the algebraic results of Theorem 4.2 to geometry now is an easy task. Consider a contact grading on \( g \) (which is not of type \( C_n \)) and a corresponding group \( G_0 \). Then by definition a PACS-structure of type \( G_0 \) on a smooth manifold \( M \) is given by a reduction of structure group of the linear frame bundle to \( G_0 \). We denote by \( G_0 \to M \) the corresponding principal bundle. Via associated bundles, any representation of \( G_0 \) gives rise to a natural vector bundle on each such manifold. By construction, for the representation \( g_{-1} \), one obtains \( G_0 \times G_0 g_{-1} \cong TM \). Likewise, the dual map to the Lie bracket includes \( g^*_{-2} \) as the distinguished line into \( \Lambda^2 g^*_{-1} \), so \( G_0 \times G_0 g^*_{-2} \cong \ell \subset \Lambda^2 T^* M \), the almost conformally symplectic structure underlying the PACS-structure.

Next, the representation \( \Lambda^2 g^*_{-1} \otimes g_{-1} \) induces the bundle \( \Lambda^2 T^* M \otimes TM \) of tangent–bundle–valued two–forms. The \( G_0 \)–invariant subspace \( \ker(\Box) \) corresponds to a smooth subbundle \( \ker(\Box) \subset \Lambda^2 T^* M \otimes TM \), whose elements will be called algebraically harmonic. Likewise, there is an inclusion \( \Lambda^3 g^*_{-1} \otimes \ell \to \Lambda^2 T^* M \otimes TM \) corresponding to the algebraic inclusion observed before Theorem 4.2. This inclusion depends only on the underlying almost conformally symplectic structure. Indeed, non–degeneracy of \( \ell \) implies that \( \ell \otimes TM \cong T^* M \), so \( TM \cong T^* M \otimes \ell^* \). Hence \( \Lambda^2 T^* M \otimes TM \cong \Lambda^2 T^* M \otimes T^* M \otimes \ell^* \), so the above inclusion is obviously there.
Corollary 4.3. Suppose that $M$ is endowed with a PACS–structure corresponding to a contact grading on $\mathfrak{g}$ (which is not of type $C_n$).

(i) There is a unique linear connection on $TM$ which is compatible with the PACS–structure and whose torsion $T$ lies in the subspace $\ker(\Box) \oplus (\Lambda^3 T^*M \otimes \ell^*)$.

(ii) Decomposing $T = T_h \oplus T_a$ according to the direct sum decomposition in (i), the component $T_a$ is the intrinsic torsion of the conformally almost symplectic structure underlying the PACS–structure. In particular, in the case of a PCS–structure, the canonical connection has algebraically harmonic torsion.

(iii) In the case of a PCS–structure, the connection on $\ell$ induced by the canonical connection $\nabla$ is flat and its local parallel sections are exactly those which are closed as two–forms.

Proof. (i) follows immediately from Theorem 4.2 and the general relation between the Spencer differential and compatible connections as discussed in Section 2.2.

For (ii), recall the description of the intrinsic torsion of the almost conformally symplectic structure from the proof of Proposition 2.3. This shows that, viewing the value of the torsion in a point as $\psi \in \Lambda^2 g^*_1 \otimes g^*_1$, the intrinsic part of the torsion exactly corresponds to the tracefree part of $\gamma(\psi)$, where $\gamma$ is the map from the proof of Theorem 4.2. Since $\gamma$ vanishes on $\ker(\Box)$ and restricts to an injection on $\Lambda^2 T^*M \otimes \ell \subset \Lambda^2 T^*M \otimes TM$, this implies the result.

(iii) By assumption $T_a = 0$ and hence $T = T_h$ is algebraically harmonic. But then the fact that $\ker(\Box) \subset \ker(\gamma)$ implies that for any section $\tau \in \Gamma(\ell)$, the bundle map $i_T \tau$ used in the proof Proposition 2.3 has vanishing alternation. Hence $d\tau$ can be computed as the alternation of $\nabla \tau$ and the proof of part (4) of Proposition 2.3 applies. 

We will refer to the components $T_h$ and $T_a$ from the second part as the harmonic torsion and the acs–torsion of a PACS–structure.

Remark 4.4. It may happen that both $\ker(\Box)$ and $\Lambda^3 g^*_1 \otimes g^*_{-2}$ are not irreducible representations of $G_0$ but decompose into a direct sum of irreducibles. If this is the case, then one obtains corresponding decompositions of $T_h$ and/or $T_a$ and there are notions of “semi–integrability” or “semi–torsion–freeness” available. We will discuss this in examples below.

4.5. Example: (para–)Kähler type. The form of $\ker(\Box) \subset \Lambda^2 T^*M \otimes TM$ can be deduced from Kostant’s theorem, and in most cases, a detailed description is available in the literature on parabolic contact structures. We will discuss the PACS–structures of Kähler type in more detail, since they have the strongest connections to well studied structures, and only briefly comment on the other types.

Suppose that $M$ carries a PACS–structure of Kähler type, and let $J$ be the corresponding almost complex structure on $M$. The harmonic part $\ker(\Box) \subset \Lambda^2 T^*M \otimes TM$ is determined in Section 4.2.4 of 

It consists of those skew symmetric bilinear maps, which are of type (0, 2), i.e. which are conjugate linear (with respect to $J$) in both arguments. Since this subbundle is induced by an irreducible representation, there is no finer decomposition of the harmonic torsion $T_h$ available.
On the other hand, the bundle $\Lambda^3_0 T^* M \otimes \ell^*$ decomposes into the sum of two subbundles according to $(p, q)$–types. (Since we are dealing with real valued forms, there are just two summands, whose complexifications split into the sums of types $(3, 0)$ and $(1, 2)$ respectively $(2, 1)$ and $(0, 3)$.) So there is a corresponding decomposition of the acs–torsion into two components.

**Proposition 4.5.** Consider a PACS–structure of Kähler type on $M$ corresponding to a conformal class of almost Hermitian metrics $(g, J)$ on $M$. Then the harmonic torsion $T_h$ of the geometry coincides (up to a non–zero constant factor) with the Nijenhuis tensor of the almost complex structure $J$.

In particular, the canonical connection $\nabla$ of the structure is torsion free if and only if we deal with a PCS–structure and $J$ is an integrable complex structure. This is equivalent to the conformal class locally containing (pseudo–)Kähler metrics of the given signature, which then are unique up to a constant factor. In this case, the canonical connection coincides with the Levi–Civita connections of the local Kähler metrics in the conformal class.

**Proof.** Compatibility of a linear connection $\nabla$ on $TM$ with the PACS–structure in particular implies that $J$ is parallel with respect to the induced connection. But it is well known that this implies that the $(0, 2)$–part of the torsion of $\nabla$ is a non–zero multiple of the Nijenhuis tensor of $J$. In view of the description of $\ker(\Box)$ given above, this implies the claim on $T_h$ and that $\nabla$ is torsion free if and only if the structure is PCS and $J$ is integrable.

For the last claim, we have already seen in Proposition 3.3 that the PCS–condition is equivalent to the existence of local almost Kähler metrics in the conformal class. But an almost Kähler metric is well known to be Kähler if and only if the corresponding almost complex structure is integrable. We have also seen there that these local metrics are unique up to a constant factor, so they all have the same Levi–Civita connection. Since the Levi–Civita connection also preserves $J$, it preserves the PCS–structure and thus coincides with $\nabla$. □

The case of PACS–structures of para–Kähler type can be analyzed in a very similar fashion. Apart from $\ell \subset \Lambda^2 T^* M$, we have a decomposition $TM = E \oplus F$ into a direct sum of two rank–$n$–distributions which are isotropic for $\ell$ in this case. This gives rise to an isomorphism $\ell \otimes E \cong F^*$. The harmonic part $\ker(\Box) \subset \Lambda^2 (E \oplus F)^* \otimes (E \oplus F)$ is determined in Section 4.3.1 of [7]. It is the direct sum of two bundles induced by irreducible representations of $G_0$ which correspond to the highest weight bits in $\Lambda^2 E^* \otimes F$ and $\Lambda^2 F^* \otimes E$, respectively. Accordingly, the harmonic torsion decomposes into two pieces $T_h = T^{E}_h + T^{F}_h$. A linear connection $\nabla$ on $TM$ which is compatible with the PACS–structure in particular preserves the subbundles $E$ and $F$. Using this, one easily verifies that for $\xi, \eta \in \Gamma(E)$, $T^{E}_h(\xi, \eta)$ is obtained by projecting $[\xi, \eta]$ to $F$. So this is exactly the obstruction to involutivity of the distribution $E$, and similarly for $T^{F}_h$.

In particular, we readily see that torsion freeness of the canonical connection $\nabla$ is equivalent to the structure being PCS and both $E$ and $F$ being involutive. In this case, $\nabla$ locally coincides with the Levi–Civita connection of a para–Kähler
metric. The acs–torsion in this case splits into four components according to the analog of $(p,q)$–types.

### 4.6. Other examples.

For a PACS–structure of Grassmannian type, we have an almost Grassmannian structure $TM \cong E^* \otimes F$, where $E$ and $F$ are auxiliary bundles of ranks 2 and $n$, respectively, and an almost conformally symplectic structure $\ell \subset \Lambda^2 E \otimes S^2 F^* \subset \Lambda^2 T^*M$. As we have noted in Section 3.3, this induces a non–degenerate symmetric bilinear form $b$ on $F$ determined up to scale, thus defining a signature $(p,q)$.

The harmonic part $\ker(\Box) \subset \Lambda^2 T^*M \otimes TM$ for this case is determined in Section 4.2.5 of [7]. It is a direct sum of two subbundles, both of which are induced by irreducible representations. The first of these is isomorphic to $\Lambda^2 E \otimes E^* \otimes S^3_0 F^*$, where $S^3_0 F^*$ denotes the tracefree part of the symmetric cube of $F^*$, which, via $b$, sits inside $S^2 F^* \otimes F$. We denote the corresponding component by $T^h_\ell$.

The other component is a bit more complicated to describe. It is contained in $(S^2 E^* \otimes E)_0 \otimes (\Lambda^2 F^* \otimes F)_0$, where the subscripts indicate tracefree parts. Now the first factor already corresponds to an irreducible representation of $g_0$, but the second factor can be included into $\Lambda^2 F^* \otimes F^*$ via $b$, and we have to take the kernel of the resulting alternation map to $\Lambda^3 F^*$. Let us write $T^G_h$ for the corresponding component of the harmonic torsion.

Now an interesting feature of this case is that, in the case $n > 2$, which we always consider here, an almost Grassmannian structure has an intrinsic torsion in its own right, see Section 4.1.3 of [7]. This intrinsic torsion corresponds to the component in $(S^2 E^* \otimes E)_0 \otimes (\Lambda^2 F^* \otimes F)_0$, which thus is the same for all linear connections compatible with the almost Grassmannian structure. In particular, we conclude that the component $T^G_h$ of the harmonic torsion depends only on the underlying almost Grassmannian structure and not on $\ell$, and it vanishes if and only if the structure is Grassmannian. On the other hand, the component $T^h_\ell$ of the harmonic torsion is a basic invariant for an almost conformally symplectic structure which is Hermitian with respect to an almost Grassmannian structure. There is also a finer decomposition of the acs–torsion $T_a$ available in the Grassmannian case, but we do not go into this.

The case of PACS–structures of quaternionic type can be dealt with similarly, with quaternionic linearity and anti–linearity respectively hermiticity and anti–hermiticity properties replacing the decompositions coming from the tensor product structure. In particular, one again obtains one component in the harmonic torsion which only depends on the underlying almost quaternionic structure and whose vanishing is equivalent to the structure being quaternionic.

For the exceptional PACS–structures, we just give a general description of $\ker(\Box)$. Making things more explicit is a question of representation theory. As mentioned in Section 4.2.8 of [7], a case–by–case inspection shows that $\ker(\Box)$ always is induced by an irreducible representation, so there is no finer decomposition of the harmonic torsion available in the exceptional cases. In terms of representation theory, this component can be easily characterized as the Cartan product (i.e. the irreducible component of maximal highest weight in the tensor product)
of $\Lambda^2(\mathfrak{g}_{-1})^*$ and $\mathfrak{g}_{-1}$. Similarly, the decomposition of the acs–torsion is a question of representation theory in these cases, and we do not go into details here.

4.7. Relation to special symplectic connections. To conclude the discussion of canonical connections associated to PACS–structures, we briefly discuss their relation to the theory of special symplectic connections developed in [2]. Consider a real simple Lie algebra $\mathfrak{g}$, which is not of type $C_\infty$ and admits a contact grading as discussed in Section 3.1. Then we get an inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{sp}(\mathfrak{g}_{-1})$. As noted in Section 3.1, the associated special symplectic subalgebra in the sense of Cahen–Schwachhöfer is $\mathfrak{g}_0^0 := \mathfrak{g}_0 \cap \mathfrak{sp}(\mathfrak{g}_{-1})$.

A crucial ingredient in [2] is that the special symplectic subalgebra $\mathfrak{g}_0^0$ determines two spaces of curvature tensors. First, as for any Lie algebra of matrices, there is the space $K(\mathfrak{g}_0^0)$ of formal curvature tensors having values in $\mathfrak{g}_0^0$, which is defined as

$$\{ R \in \Lambda^2(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0^0 : R(X,Y)(Z) + R(Z,X)(Y) + R(Y,Z)(X) = 0 \ \forall X, Y, Z \in \mathfrak{g}_{-1}\}.$$ 

So except for the condition on the values, one just requires the first Bianchi–identity to hold. Observe that $K(\mathfrak{g}_0^0)$ naturally is a $\mathfrak{g}_0^0$–module. For a special symplectic subalgebra $\mathfrak{g}_0^0$, there is a distinguished submodule $\mathcal{R}_{\mathfrak{g}_0^0} \subset K(\mathfrak{g}_0^0)$. In terms of the contact grading of $\mathfrak{g}$, this can be easily described as follows. Choose a non–zero element $\psi \in \mathfrak{g}_2$ and then for $A \in \mathfrak{g}_0^0$ define $R_A : \Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_0$ by

$$R_A(X,Y) := [X, [\psi, [A, Y]]] - [Y, [\psi, [A, X]]].$$

Using the Jacobi–identity, one immediately verifies that since $A \in \mathfrak{g}_0^0$, $R_A(X,Y)$ always acts trivially on $\mathfrak{g}_{-2}$ so the values actually lie in $\mathfrak{g}_0^0$. The Jacobi–identity also implies that, for $A \in \mathfrak{g}_0^0$, $R_A$ satisfies the first Bianchi identity. Hence we have obtained a map $\mathfrak{g}_0^0 \to K(\mathfrak{g}_0^0)$, and we denote by $\mathcal{R}_{\mathfrak{g}_0^0}$ the image of this mapping. In [2] it is proved that the Ricci–type contraction maps $\mathcal{R}_{\mathfrak{g}_0^0}$ to $\mathfrak{g}_0^0$ and that this contraction vanishes on $R_A$ if and only if $A = 0$. Hence $\mathcal{R}_{\mathfrak{g}_0^0}$ is isomorphic to $\mathfrak{g}_0^0$.

By construction, $\mathcal{R}_{\mathfrak{g}_0^0} \subset K(\mathfrak{g}_0^0) \subset K(\mathfrak{sp}(\mathfrak{g}_{-1}))$, so given a symplectic manifold $M$ of dimension $\dim(\mathfrak{g}_{-1})$, the space $\mathcal{R}_{\mathfrak{g}_0^0}$ corresponds to smooth subbundle in $\Lambda^2 T^* M \otimes \mathfrak{sp}(TM)$. Then Cahen–Schwachhöfer define a special symplectic connection as a torsion free connection on a symplectic manifold, whose curvature has values in this subbundle for some special symplectic subalgebra.

The final ingredient we need for our discussion is a result on the structure of $K(\mathfrak{g}_0^0)$ which is proved in Theorem 2.11 of [2]. The form of the result is closely related to the occurrence of Lie algebra cohomology in degree two of homogeneity zero. This suggests that the result can be deduced from Kostant’s theorem in a way similar to our proof of Theorem 4.12 but since this is not directly related to the topic of this article, we do not do this but just quote the result.

**Lemma 4.6.** If $\mathfrak{g}$ is not of type $A_\infty$ or $C_\infty$ (and hence not of type $B_2 \cong C_2$), then $K(\mathfrak{g}_0^0) = \mathcal{R}_{\mathfrak{g}_0^0}$.

Using this, we can now relate special symplectic connections to PCS–structures.
Theorem 4.7. Let $\mathfrak{g}_0$ be a special symplectic subalgebra corresponding to a real simple Lie algebra $\mathfrak{g}$, which is not of type $C_n$. Then any special symplectic connection is the canonical connection associated to a PCS–structure with vanishing harmonic torsion.

For PCS–structures of Grassmannian, quaternionic, and exceptional type, the converse holds, i.e. the canonical connection associated to a torsion–free geometry of that type is a special symplectic connection. Indeed, these connections are exactly those having special symplectic holonomy.

For PCS–structures of Kähler or para–Kähler type, there is an additional condition on the curvature of the canonical connection associated to a torsion–free PCS–structure that has to be satisfied to obtain a special symplectic connection.

Proof. By the classical Ambrose–Singer theorem, a torsion–free connection with curvature contained in $K(\mathfrak{g}_0)$ can be obtained from a reduction of structure group to a group with Lie algebra $\mathfrak{g}_0$. Via the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}_0$, one can extend this to a group $G_0$ with Lie algebra $\mathfrak{g}_0$, and the connection still is compatible with the corresponding reduction. Thus we have obtained a PACS–structure which admits a compatible torsion–free connection. Hence the structure has to be PCS with vanishing harmonic torsion, and the connection has to coincide with the canonical one.

For the converse, part (iii) of Corollary 4.3 implies that the curvature of the canonical connection of any PCS–structure has values in $\mathfrak{g}_0 \subset \mathfrak{g}_0$, so it lies in $K(\mathfrak{g}_0)$. By Lemma 4.6 we have $K(\mathfrak{g}_0) = \mathcal{R}_{\mathfrak{g}_0}$ in the cases of Grassmannian, quaternionic, and exceptional type. Thus in these cases, the connection is special symplectic provided that it is torsion–free. In the remaining cases, there is a natural $\mathfrak{g}_0$–invariant complement $\mathcal{W}_{\mathfrak{g}_0}$ to $\mathcal{R}_{\mathfrak{g}_0} \subset K(\mathfrak{g}_0)$, and to obtain a special symplectic connection, one has to require torsion–freeness and vanishing of the curvature component in $\mathcal{W}_{\mathfrak{g}_0}$. □

As shown in [2], the additional curvature condition in the Kähler and para–Kähler cases turns out to be vanishing of the so–called Bochner–curvature. Thus, special symplectic connections in these cases are Levi–Civita connections of Bochner–Kähler metrics respectively Bochner–bi–Langrangean metrics rather than just of Kähler metrics respectively para–Kähler metrics. We will discuss an alternative characterization of special symplectic connections among the canonical connections of PCS–structures of (para–)Kähler type using contactifications in [5].

References

[1] R. L. Bryant, Bochner–Kähler metrics, J. Amer. Math. Soc. 14 (2001), no. 3, 623–715, DOI 10.1090/S0894-0347-01-00366-6. MR1824987
[2] M. Cahen and L. J. Schwachhöfer, Special symplectic connections, J. Differential Geom. 83 (2009), no. 2, 229–271. MR2577468 (2011b:53045)
[3] D. M. J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, J. Reine Angew. Math. 537 (2001), 67–103, DOI 10.1515/crll.2001.059. MR1856258 (2002k:58048)
[4] A. Čap and T. Salač, *Pushing down the Rumin complex to conformally symplectic quotients*, Differential Geom. Appl. 35 (2014), no. suppl., 255–265, DOI 10.1016/j.difgeo.2014.05.004. MR3254307

[5] A. Čap and T. Salač, *Parabolic conformally symplectic structures II; parabolic contactification*, available at arXiv:1605.01897.

[6] A. Čap and T. Salač, *Parabolic conformally symplectic structures III; Invariant differential operators and complexes*, available at arXiv:1701.01305.

[7] A. Čap and J. Slovák, *Parabolic geometries. I*, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, Providence, RI, 2009. Background and general theory. MR2532439 (2010j:53037)

[8] A. Čap, J. Slovák, and V. Souček, *Bernstein-Gelfand-Gelfand sequences*, Ann. of Math. (2) 154 (2001), no. 1, 97–113, DOI 10.2307/3062111. MR1847589 (2002h:58034)

[9] M. Eastwood and H. Goldschmidt, *Zero-energy fields on complex projective space*, J. Differential Geom. 94 (2013), no. 1, 129–157. MR3031862

[10] M. G. Eastwood and J. Slovák, *Conformally Fedosov manifolds*, available at arXiv:1210.5597.

[11] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) 123 (1980), 35–58, DOI 10.1007/BF01796539. MR581924

[12] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) 74 (1961), 329–387. MR0142696 (26 #265)

[13] S. Merkulov and L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. (2) 150 (1999), no. 1, 77–149, DOI 10.2307/121098. MR1715321

[14] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. MR0193578 (33 #1797)

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