ABSTRACT. We prove that vertical projections in the first Heisenberg group preserve, almost surely, the Hausdorff dimension of Borel sets of dimension 3. For sets of dimension $t < 3$ we obtain partial results which improve on the current state-of-the art when $t > 7/3$.

We derive the dimension distortion bounds from a discrete result concerning the vertical projections of $(\delta, t)$-sets of Heisenberg $\delta$-balls, which is sharp for all values $t \in [0, 3]$. We prove that if $B$ is a $(\delta, t)$-set of $\delta$-balls in the Heisenberg group, then most vertical projections of $\cup B$ have Lebesgue measure comparable to $\delta^{3-t}$.

The proofs are based on a point-line duality principle which allows us to transform the Heisenberg problems into point-plate incidence questions in $\mathbb{R}^3$. To solve these questions, we apply a Kakeya inequality for plates in $\mathbb{R}^3$, due to Guth, Wang, and Zhang.

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1. INTRODUCTION

Fix $e \in S^1 \times \{0\} \subset \mathbb{H}$, and consider the vertical plane $\mathbb{W}_e := e^\perp$ in the first Heisenberg group $\mathbb{H}$, see Section 2.1 for the definitions. Every point $p \in \mathbb{H}$ can be uniquely
decomposed as \( p = w \cdot v \), where
\[
w \in \mathbb{W}_e \quad \text{and} \quad v \in \mathbb{L}_e := \text{span}(e).
\]
This decomposition gives rise to the vertical projection \( \pi_e := \pi_{\mathbb{W}_e} : \mathbb{H} \to \mathbb{W}_e \), defined by \( \pi_e(p) := w \). A good way to visualise \( \pi_e \) is to note that the fibres \( \pi_e^{-1}(w), w \in \mathbb{W}_e \), coincide with the horizontal lines \( w \cdot \mathbb{L}_e \). These lines foliate \( \mathbb{H} \), as \( w \) ranges in \( \mathbb{W}_e \), but are not parallel. Thus, the projections \( \pi_e \) are non-linear maps with linear fibres. For example, in the special cases \( e_1 = (1, 0, 0) \) and \( e_2 = (0, 1, 0) \) we have the concrete formulae
\[
\pi_{e_1}(x, y, t) = \left(0, y, t + \frac{xy}{2}\right) \quad \text{and} \quad \pi_{e_2}(x, y, t) = \left(x, 0, t - \frac{xy}{2}\right).
\]  
(1.1)

From the point of view of geometric measure theory in the Heisenberg group, the vertical projections are the Heisenberg analogues of orthogonal projections to \( (d-1) \)-planes in \( \mathbb{R}^d \). One of the fundamental theorems concerning orthogonal projections in \( \mathbb{R}^d \) is the Marstrand-Mattila projection theorem [15, 16]: if \( K \subset \mathbb{R}^d \) is a Borel set, then
\[
\dim_{\mathbb{E}} \pi_V(K) = \min\{ \dim_{\mathbb{E}} K, d-1 \}
\]  
(1.2)

for almost all \( (d-1) \)-planes \( V \subset \mathbb{R}^d \). Here \( \dim_{\mathbb{E}} \) refers to Hausdorff dimension in Euclidean space – in contrast to the notation “\( \dim_{\mathbb{E}} \)” which will refer to Hausdorff dimension in the Heisenberg group. In \( \mathbb{R}^d \), orthogonal projections are Lipschitz maps, so the upper bound in (1.2) is trivial, and the main interest in (1.2) is the lower bound.

The vertical projections \( \pi_e \) are not Lipschitz maps \( \mathbb{H} \to \mathbb{W}_e \) relative to the natural metric \( d_{\mathbb{H}} \) in \( \mathbb{H} \) and \( \mathbb{W}_e \). Indeed, they can increase Hausdorff dimension: an easy example is a horizontal line, which is 1-dimensional to begin with, but gets projected to a 2-dimensional set – a parabola – in almost all directions. For general (sharp) results on how much \( \pi_e \) can increase Hausdorff dimension, see [1, Theorem 1.3]. We note that the vertical planes \( \mathbb{W}_e \) themselves are 3-dimensional, and \( \mathbb{H} \) is 4-dimensional.

Can the vertical projections lower Hausdorff dimension? In some directions they can, and the general (sharp) universal lower bound was already found in [1, Theorem 1.3]:
\[
\dim_{\mathbb{H}} \pi_e(K) \geq \max\{0, \frac{1}{2} (\dim_{\mathbb{H}} K - 1), 2 \dim_{\mathbb{H}} K - 5\}, \quad e \in S^1.
\]

Our main result states that the dimension drop cannot occur in a set of directions of positive measure for sets of dimension 3:

**Theorem 1.3.** Let \( K \subset \mathbb{H} \) be a Borel set with \( \dim_{\mathbb{H}} K \geq 2 \). Then,
\[
\dim_{\mathbb{E}} \pi_e(K) \geq \min\{ \dim_{\mathbb{H}} K - 1, 2 \}
\]  
(1.4)

for \( \mathcal{H}^1 \) almost every \( e \in S^1 \), and
\[
\dim_{\mathbb{H}} \pi_e(K) \geq \min\{2 \dim_{\mathbb{H}} K - 3, 3\}
\]  
(1.5)

for \( \mathcal{H}^1 \) almost every \( e \in S^1 \).

Theorem 1.3 will be deduced from a \( \delta \)-discretised result which may have independent interest. We state here a slightly simplified version (the full version is Theorem 3.11):

**Theorem 1.6.** Let \( 0 \leq t \leq 3 \) and \( \eta > 0 \). Then, the following holds for \( \delta, \epsilon > 0 \) small enough, depending only on \( \eta \). Let \( B \) be a non-empty \( (\delta, t, \delta^{-\epsilon}) \)-set of Heisenberg balls of radius \( \delta \), all contained in \( B_{\mathbb{H}}(1) \). Then, there exists \( e \in S^1 \) such that
\[
\text{Leb}(\pi_e(\cup B)) \geq \delta^{3-t+\eta}.
\]  
(1.7)
Here Leb denotes Lebesgue measure on $\mathbb{W}_e$, identified with $\mathbb{R}^2$. For the definition of $(\delta, t)$-sets of $\delta$-balls, see Definition 3.1.

1.1. Sharpness of the results and related work. The Euclidean bound (1.4) is sharp in the range $\dim_{\mathbb{H}} K \geq 2$. To see this, fix $t \in [2, 3]$ and one vertical plane, say $\mathbb{W}_0 := \mathbb{R} \times \{0\} \times \mathbb{R}$. Then, construct a $t$-dimensional set $K \subset \mathbb{W}_0$ as a union of vertical lines. More precisely, take $K_0 \subset \mathbb{R}$ with Euclidean dimension $\dim_{\mathbb{E}} K_0 = t - 2$, and note that $K := K_0 \times \{0\} \times \mathbb{R} \subset \mathbb{W}_0$ has $\dim_{\mathbb{H}} K = t$. Then, the structure of $\pi_e(K)$ can be described as follows: first project $K_0$ in $\mathbb{R}^2$ orthogonally to the line $e^t$, and denote the resulting set $K_e$. Then, $\pi_e(K) = K_e \times \mathbb{R}$. Finally, $\dim_{\mathbb{E}}(K_e \times \mathbb{R}) \leq \dim_{\mathbb{E}} K_0 + 1 = t - 1$ for all $e \in S^1$.

The Heisenberg lower bound (1.5) is sharp for $\dim_{\mathbb{H}} K = 3$, but is unlikely to be sharp for $\dim_{\mathbb{H}} K < 3$. In fact, (1.5) is simply obtained from the Euclidean estimate (1.4) via a general dimension comparison principle [1, Theorem 2.8] inside the planes $\mathbb{W}_e$. According to [1, Conjecture 1.5], the sharp bound for the Heisenberg dimension of the projections $\pi_e(K)$ is

$$\dim_{\mathbb{H}} \pi_e(K) \geq \min\{\dim_{\mathbb{H}} K, 3\}$$

(1.8)

for $\mathcal{H}^1$ almost every $e \in S^1$.

Unlike Theorem 1.3, Theorem 1.6 is sharp for all values of $t \in [0, 3]$. Indeed, it is possible that $|B| = \delta^{-t}$, and then $\text{Leb}(\pi_e(\cup B)) \lesssim \delta^{3-t}$ for every $e \in S^1$. It also follows from (1.7) that the smallest number of $d_{\mathbb{H}}$-balls of radius $\delta$ needed to cover $\pi_e(\cup B)$ is $\gtrsim \delta^{-t+\eta}$. One might think that this solves Conjecture 1.8 for all $\dim_{\mathbb{H}} K \in [0, 3]$, but we were not able to make this deduction rigorous: the difficulty appears when attempting to $\delta$-discretise Conjecture 1.8, and is caused by the non-Lipschitz behaviour of $\pi_e: (\mathbb{H}, d_{\mathbb{H}}) \to (\mathbb{W}_e, d_{\mathbb{E}})$. This problem will be apparent in the proof of Theorem 1.3 in Section 5. Another, more heuristic, way of understanding the difference between Theorem 1.6 and Conjecture 1.8 is this: $\text{Leb}(\pi_e(K))$ is invariant under left-translating $K$, but $\dim_{\mathbb{H}} \pi_e(K)$ is generally not.

In addition to Theorem 1.3, there have been several partial results towards (1.8). In [1, Theorem 1.4], conjecture (1.8) was verified for Borel sets $K \subset \mathbb{H}$ with $\dim_{\mathbb{H}} K \leq 1$. For Borel sets $K \subset \mathbb{H}$ with $\dim_{\mathbb{H}} K > 1$, non-trivial lower bounds have been obtained by Hovia and the first author [4], and Harris [9, 10]. Most recently, Harris [10] in 2021 used a Fourier-analytic technique to show that

$$\dim_{\mathbb{H}} \pi_e(K) \geq \min\left\{\frac{1 + \dim_{\mathbb{H}} K}{2}, 2\right\}$$

for almost all $e \in S^1$. Our bound (1.5) improves on Harris’ result for $\dim_{\mathbb{H}} K > 7/3$.

The problem has been also studied (and remains open) in higher dimensional Heisenberg groups, see [2, 11]. The paper [11] also contains bounds which relate $\dim_{\mathbb{E}} \pi_e(K)$ to $\dim_{\mathbb{H}} K$. It may seem unnatural to compare $\dim_{\mathbb{E}} \pi_e(K)$ to $\dim_{\mathbb{H}} K$, as we do in Theorem 1.3, but this was motivated by the ensuing sharp result in the case $\dim_{\mathbb{H}} K = 3$.

Finally, we mention that problems concerning the vertical projections $\pi_e$ are closely connected (at least in spirit) to questions about restricted families of orthogonal projections in $\mathbb{R}^3$. The latter have witnessed very substantial progress recently, see [7, 13, 18]. The proof of Theorem 1.3 does not directly overlap with the techniques in these papers, but the argument was certainly inspired by the recent developments.

1.2. Proof outline. The proof of Theorem 1.3 is mainly based on two ingredients. The first one is a point-line duality principle between horizontal lines in $\mathbb{H}$, and $\mathbb{R}^2$. To describe
this principle, let $\mathcal{L}_H$ be the family of all horizontal lines in $\mathbb{H}$, and let $\mathcal{L}_C$ be the family of all lines in $\mathbb{H}$ which are parallel to some line contained in a conical surface $C$. In Section 2.2, we show that there exist maps $\ell: \mathbb{R}^3 \to \mathcal{L}_H$ and $\ell^*: \mathbb{H} \to \mathcal{L}_C$ (whose ranges cover almost all of $\mathcal{L}_H$ and $\mathcal{L}_C$) which preserve incidence relations in the following way:

$$q \in \ell(p) \iff p \in \ell^*(q), \quad p \in \mathbb{R}^3, \ q \in \mathbb{H}.$$  

Thus, informally speaking, incidence-geometric questions between points in $\mathbb{H}$ and lines in $\mathcal{L}_H$ can always be transformed into incidence-geometric questions between points in $\mathbb{R}^3$ and lines in $\mathcal{L}_C$. Less explicitly, the point-line duality principle described here was first used by Liu [14] to study Kakeya sets (formed by horizontal lines) in $\mathbb{H}$. The question about vertical projections in $\mathbb{H}$ can – after suitable discretisation – be interpreted as an incidence geometric problem between points in $\mathbb{H}$ and lines in $\mathcal{L}_H$. It can therefore be transformed into an incidence-geometric problem between points in $\mathbb{R}^3$ and lines in $\mathcal{L}_C$. Which problem is this? It turns out that while the dual $\ell^*(p)$ of a point $p \in \mathbb{H}$ is a line in $\mathcal{L}_C$, the dual $\ell^*(B_\mathbb{H})$ of a Heisenberg $\delta$-ball resembles an $\delta$-plate in $\mathbb{R}^3$ – a rectangle of dimensions $1 \times \delta \times \delta^2$ tangent to $C$. So, the task of proving Theorem 1.6 (hence Theorem 1.3) is (roughly) equivalent to the task of solving an incidence-geometric problem between points in $\mathbb{R}^3$, and family of $\delta$-plates.

Moreover: the plates in our problem appear as duals of certain Heisenberg $\delta$-balls, approximating a $t$-dimensional set $K \subset \mathbb{H}$, with $0 \leq t \leq 3$. Consequently, the plates can be assumed to satisfy a $t$-dimensional "non-concentration condition" relative to the metric $d_\mathbb{H}$. In common jargon, the plate family is a $(\delta, t)$-set relative to $d_\mathbb{H}$.

In [8], Guth, Wang, and Zhang proved the sharp (reverse) square function estimate for the cone in $\mathbb{R}^3$. A key component in their proof was a new incidence-geometric ("Kakeya") estimate [8, Lemma 1.4] for points and $\delta$-plates in $\mathbb{R}^3$ (see Section 4 for the details). While this was not relevant in [8], it turns out that the incidence estimate in [8, Lemma 1.4] interacts perfectly with a $(\delta, 3)$-set condition relative to $d_\mathbb{H}$. This allows us to prove, roughly speaking, that the vertical projections of 3-Frostman measures on $\mathbb{H}$ have $L^2$-densities. See Corollary 3.6 for a more precise statement.

For $0 \leq t < 3$, the $(\delta, t)$-set condition relative to $d_\mathbb{H}$ no longer interacts so well with [8, Lemma 1.4]. However, we were able to (roughly speaking) reduce Theorem 1.6 for $(\delta, t)$-sets, $0 \leq t \leq 3$, to the special case $t = 3$. This argument is explained in Section 3, so we omit the discussion here.

2. Preliminaries

2.1. The Heisenberg group.}
The group product gives rise to projection-type mappings onto subgroups that are invariant under Heisenberg dilations. For \( e \in S^1 \), we define the horizontal subgroup
\[
\mathbb{L}_e := \{(se, 0) : s \in \mathbb{R}\}.
\]
The vertical subgroup \( \mathbb{W}_e \) is the Euclidean orthogonal complement of \( \mathbb{L}_e \) in \( \mathbb{R}^3 \); in particular it is a plane containing the vertical axis. Every point \( p \in \mathbb{H} \) can be written in a unique way as a product \( p = p_{\mathbb{W}_e} \cdot p_{\mathbb{L}_e} \) with \( p_{\mathbb{W}_e} \in \mathbb{W}_e \) and \( p_{\mathbb{L}_e} \in \mathbb{L}_e \). The vertical Heisenberg projection onto the vertical plane \( \mathbb{W}_e \) is the map
\[
\pi_e : \mathbb{H} \to \mathbb{W}_e, \quad p = p_{\mathbb{W}_e} \cdot p_{\mathbb{L}_e} \mapsto p_{\mathbb{W}_e}.
\]
The vertical projection to the \( xt \)-plane \( \{(x, 0, t) : x, t \in \mathbb{R}\} \) will play a special role; this projection will be denoted \( \pi_{xt} \), and it has the explicit formula stated in (1.1). Preliminaries about Heisenberg projections can be found for instance in [17, 2, 1]. These mappings have turned out to play an important role in geometric measure theory of the Heisenberg group endowed with a left-invariant non-Euclidean metric. The Korányi metric \( d_{\mathbb{H}} \) is defined by
\[
d_{\mathbb{H}}(p, q) := \|q^{-1} \cdot p\|,
\]
where \( \| \cdot \| \) is the Korányi norm given by
\[
\| (x, y, t) \| = 4\sqrt{(x^2 + y^2)^2 + 16t^2}.
\]
We will use the symbol \( B_{\mathbb{H}}(p, r) \) to denote the ball centered at \( p \) with radius \( r \) with respect to the Korányi metric. Balls centred at the origin are denoted \( B_{\mathbb{H}}(r) \). All vertical planes \( \mathbb{W}_{e_x}, e \in S^1 \), equipped with \( d_{\mathbb{H}} \) are isometric to each other via rotations of \( \mathbb{R}^3 \) about the vertical axis. The Heisenberg dilations are similarities with respect to \( d_{\mathbb{H}} \), and it is easy to see that \( (\mathbb{H}, d_{\mathbb{H}}) \) is a 4-regular space, while the vertical subgroups \( \mathbb{W}_e \) are 3-regular with respect to \( d_{\mathbb{H}} \). Moreover, there exists a constant \( 0 < c < \infty \), independent of \( e \), such that under the obvious identification of \( \mathbb{W}_e \) with \( \mathbb{R}^2 \), the restriction of the 3-dimensional Hausdorff measure \( \mathcal{H}^3 \) to \( \mathbb{W}_e \) agrees with the 2-dimensional Lebesgue measure \( \text{Leb} \) on \( \mathbb{R}^2 \) up to the multiplicative constant \( c \).

Vertical projections are neither group homomorphisms nor Lipschitz mappings with respect \( d_{\mathbb{H}} \). However, they behave well with respect to the Lebesgue measure on vertical planes. Namely, for every Borel set \( E \subset \mathbb{H} \), we have that
\[
\text{Leb} (\pi_e(p \cdot E)) = \text{Leb}(\pi_e(E)), \quad p \in \mathbb{R}^3, \ e \in S^1,
\]
see the formula at the bottom of page 1970 in the proof of [6, Lemma 2.20].

### 2.2. Duality between horizontal lines and \( \mathbb{R}^3 \)

We introduce a notion of duality that associates to points and horizontal lines in \( \mathbb{H} \) certain lines and points in \( \mathbb{R}^3 \). The lines in \( \mathbb{R}^3 \) will be light rays – translates of lines on a fixed conical surface. To define these, we let \( C_0 \) be the vertical cone
\[
C_0 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1^2 + z_2^2 = z_3^2\},
\]
and we denote by \( C \) the (45°) rotated cone
\[
C = R(C_0) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 = 2z_1z_3\},
\]
where \( R(z_1, z_2, z_3) = (z_1 + z_3)/\sqrt{2}, z_2, (-z_1 + z_3)/\sqrt{2} \). The cone \( C \) is foliated by lines
\[
L_y = \text{span}_{\mathbb{R}}(1, -y, y^2/2), \quad y \in \mathbb{R},
\]
Remark 2.6. Finally, (2.13) is equivalent to $p^* = (a, b, c) = (0, x, t - xy/2) + L_y(a)$. (2.13)

Finally, (2.13) is equivalent to $p^* \in \ell^*(p)$. □
Measures on the space of horizontal lines. The duality \( p \mapsto \ell(p) \) between points in \( p \in \mathbb{R}^3 \) and horizontal lines \( \ell(p) \) in Definition 2.7 allows one to push-forward Lebesgue measure "Leb" on \( \mathbb{R}^3 \) to construct a measure "m" on the set of horizontal lines:

\[
m(\mathcal{L}) := (\ell \cdot \text{Leb})(\mathcal{L}) = \text{Leb}(\{p \in \mathbb{R}^3 : \ell(p) \in \mathcal{L}\}).
\]

There is, however, a more commonly used measure on the space of horizontal lines. This measure "h" is discussed extensively for example in [5, Section 2.3]. The measure \( h \) is the unique (up to a multiplicative constant) non-zero left invariant measure on the set of horizontal lines. One possible formula for it is the following:

\[
h(\mathcal{L}) = \int_{S^1} \mathcal{H}^1(\{w \in \mathbb{H} : \pi_e^{-1}\{w\} \in \mathcal{L}\}) \, d\mathcal{H}^1(e). \tag{2.14}
\]

Let \( f \in L^1(\mathbb{H}) \), and consider the weighted measure \( \mu_f := f \, d\text{Leb} \). Then, starting from the definition (2.14), it is easy to check that

\[
\int_{S^1} \|\pi_e \mu_f\|_{L^2}^2 \, d\mathcal{H}^1(e) = \int X f(\ell)^2 \, dh(\ell), \tag{2.15}
\]

where \( X f(\ell) := \int E f \, d\mathcal{H}^1 \).

While the measure \( h \) is mutually absolutely continuous with respect to \( m \), the Radon-Nikodym derivative is not bounded (from above and below): with our current notational conventions, the lines \( \ell(p) \) are never parallel to the \( x \)-axis, and the \( m \)-density of lines making a small angle with the \( x \)-axis is smaller than their \( h \)-density. The problem can be removed by restricting our considerations to lines which make a substantial angle with the \( x \)-axis. For example, let \( \mathcal{L}_\perp \) be the set of horizontal lines which have slope at most 1 relative to the \( y \)-axis; thus

\[\mathcal{L}_\perp = \ell(\{(a, b, c) \in \mathbb{R}^3 : |a| \leq 1\}).\]

Then, \( m(\mathcal{L}) \sim h(\mathcal{L}) \) for all Borel sets \( \mathcal{L} \subset \mathcal{L}_\perp \). The lines in \( \mathcal{L}_\perp \) coincide with pre-images of the form \( \pi_e^{-1}\{w\}, e \in S \subset S^1 \), where \( S \) consists of those vectors making an angle at most 90° with the \( y \)-axis. Now, (2.15) also holds in the following restricted form:

\[
\int_S \|\pi_e \mu_f\|_{L^2}^2 \, d\mathcal{H}^1(e) = \int_{\mathcal{L}_\perp} X f(\ell)^2 \, dh(\ell) \sim \int_{\mathcal{L}_\perp} X f(\ell)^2 \, dm(\ell). \tag{2.16}
\]

This equation will be useful in establishing Theorem 3.2. This will, formally, only prove Theorem 3.2 with "S" in place of "S1", but the original version is easy to deduce from this apparently weaker version.

Ball-plate duality. Recall from (2.9) the definition of the (dual) line set \( \ell^*(P) \) for \( P \subset \mathbb{H} \). What does \( \ell^*(B_{\mathbb{H}}(p, r)) \) look like? The answer is: a plate tangent to the cone \( \mathcal{C} \). Informally speaking, for \( r \in (0, \frac{1}{2}] \), an \( r \)-plate tangent to \( \mathcal{C} \) is a rectangle of dimensions \( \sim (1 \times r \times r^2) \) whose long side is parallel to a light ray, and whose orientation is such that the plate is roughly tangent to \( \mathcal{C} \), see Figure 1. To prove rigorously that \( \ell^*(B_{\mathbb{H}}(p, r)) \) looks like such a plate (inside \( B(1) \)), we need to be more precise with the definitions.

Recall that the cone \( \mathcal{C} \) is a rotation of the "standard" cone \( \mathcal{C}_0 = \{(x, y, z) : z^2 = x^2 + y^2\} \).

The intersection of \( \mathcal{C} \) with the plane \( \{x = 1\} \) is the parabola

\[P = \{(1, -y, y^2/2) : y \in \mathbb{R}\}.\]
For every $r \in (0, \frac{1}{2}]$ and $p \in \mathbb{P}$, choose a rectangle $\mathcal{R} = \mathcal{R}_r(p)$ of dimensions $r \times r^2$ in the plane $\{x = 1\}$, centred at $p$, such that the longer $r$-side is parallel to the tangent line of $\mathbb{P}$ at $p$. Then $\mathbb{P} \cap B(0, cr) \subset \mathcal{R}$ for an absolute constant $c > 0$. Now, the $r$-plate centred at $p$ is the set obtained by sliding the rectangle $\mathcal{R}$ along the light ray containing $p$ inside $\{|x| \leq 1\}$, see Figure 1. We make this even more formal in the next definition.

**Definition 2.17** ($r$-plate). Let $r \in (0, \frac{1}{2}]$, and let $p = (1, -y, y^2/2) \in \mathbb{P} \subset \mathcal{C}$ with $y \in [-1, 1]$. Let $\mathcal{R}_r(0) := [-r, r] \times [-r^2, r^2]$, and define $\mathcal{R}_r(y) := M_y(\mathcal{R}_r(0)) \subset \mathbb{R}^2$, where

$$M_y = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}.$$  

(The rectangle $\mathcal{R}_r(y)$ is the intersection of an $r$-plate with the plane $\{x = 0\}$.) Define

$$\mathcal{P}_r(p) := \{(0, \tilde{r}) + L_y([-1, 1]) : \tilde{r} \in \mathcal{R}_r(y)\},$$

The set $\mathcal{P}_r(p)$ is called the $r$-plate centred at $p \in \mathbb{P}$. In general, an $r$-plate is any translate of one of the sets $\mathcal{P}_r(p)$, for $p = (1, -y, y^2/2)$ with $y \in [-1, 1]$, and $r \in (0, \frac{1}{2}]$.

**Remark 2.18.** Since we require $y \in [-1, 1]$ in Definition 2.17, it is clear that an $r$-plate contains, and is contained in, a rectangle of dimensions $\sim (1 \times r \times r^2)$. It is instructive to note that the number of "essentially distinct" $r$-plates intersecting $B(0, 1)$ is roughly $r^{-4}$: to see this, take a maximal $r$-separated subset of $\mathbb{P} \subset \mathbb{R}$, and note that for each $p \in \mathbb{P}_r$, the plate $\mathcal{P}_r(p)$ has volume $r^3$. Therefore it takes $\sim r^{-3}$ translates of $\mathcal{P}_r(p)$ to cover $B(0, 1)$. This $r^{-4}$-numerology already suggests that the various $r$-plates might correspond to Heisenberg $r$-balls via duality.

To relate the plates $\mathcal{P}_r$ to Heisenberg balls, we define a slight modification of the plates $\mathcal{P}_r$. Whereas $\mathcal{P}_r$ is a union of (truncated) light rays in one fixed direction, the following "modified" plates contain full light rays in an $r$-arc of directions. These "modified" plates will finally match the duals of Heisenberg balls, see Proposition 2.23.
Definition 2.19 (Modified r-plate). Let \( r \in (0, \frac{1}{2}] \) and \( y \in [-1, 1] \). Let \( \mathcal{R}_r(y_0) \subset \mathbb{R}^2 \) be the rectangle from Definition 2.17. For \((u, v) \in \mathbb{R}^2\), define the modified r-plate

\[
\Pi_r(u, v, y) := (0, u, v) + \{(0, \tilde{r}) + L_{y'} : \tilde{r} \in \mathcal{R}_r(y) \text{ and } |y' - y| \leq r \}.
\] (2.20)

Remark 2.21. The relation between the sets \( \mathcal{P}_r \) and \( \Pi_r \) is that the following holds for some absolute constant \( c > 0 \): if \( r \in (0, \frac{1}{2}] \), \( y \in [-1, 1] \), and \( u, v \in \mathbb{R} \), then

\[
\Pi_{cr}(u, v, y) \cap \{(s, y, z) : |s| \leq 1\} \subset (0, u, v) + \mathcal{P}_r(y) \subset \Pi_r(u, v, y).
\] (2.22)

To see this, it suffices to check the case \( u = 0 = v \). Consider the "slices" of \( \Pi_r(0, 0, y) \) and \( \mathcal{P}_r(y) \) with a fixed plane \((x = s)\) for \(|s| \leq 1\). If \( s = 0 \), both slices coincide with the rectangle \( \mathcal{R}_y(y) \). If \( 0 < |s| \leq 1 \), the slice \( \Pi_r(0, 0, y) \cap \{x = s\} \) can be written as a sum

\[
\Pi_r(0, 0, y) \cap \{x = s\} = \mathcal{R}_r(y) + \{L_{y'}(s) : |y' - y| \leq r\},
\]

whereas \( \mathcal{P}_r(y) \cap \{x = s\} = \mathcal{R}_r(y) + \{L_y(s)\} \). The relationship between these two slices is depicted in Figure 2. After this, we leave it to the reader to verify that \( \Pi_{cr}(0, 0, y) \cap \{x = 0\} \subset \mathcal{P}_r(y) \cap \{x = 1\} \) if \( c > 0 \) is sufficiently small, and for \(|s| \leq 1\).

We record the following consequence of (2.22): \( \Pi_r(u, v, y) \cap \{(s, y, z) : |s| \leq 1\} \) is contained in a tube of width \( r \) around the line \((0, u, v) + L_y\). This is because \( \mathcal{P}_r(y) \) is obviously contained in a tube of width \( -r \) around \( L_y \) (this is a very non-sharp statement, using only that the longer side of \( \mathcal{R}_y(r) \) has length \( r \)).

We then show that the \( \ell^s \)-duals of Heisenberg balls are essentially modified plates:

Proposition 2.23. Let \( p = (u_0, 0, v_0) \cdot (0, y_0, 0), r \in (0, \frac{1}{2}], \) and \( B := B_{\Pi}(p, r) \). Then,

\[
\ell^s(B) \subset \Pi_{2r}(u_0, v_0, y_0) \subset \ell^s(CB),
\] (2.24)

where \( C > 0 \) is an absolute constant, and \( CB = B_{\Pi}(p, Cr) \).

Remark 2.25. To build a geometric intuition, it will be helpful to notice the following. The \( y \)-coordinate of the point \( p = (u_0, 0, v_0) \cdot (0, y_0, 0) = (u_0, y_0, v_0 + \frac{1}{2} u_0 y_0) \) is "\( y_0 \)". On the other hand, while the modified plate \( \Pi_{2r}(u_0, v_0, y_0) \) contains many lines, they are all "close" to the "central" line \((0, u_0, v_0) + L_{y_0} \) (see Definition 2.20). According to the inclusions in (2.24), this means that the "direction" \( L_{y_0} \) of the modified plate containing the dual \( \ell^s(B(p, r)) \) is determined by the \( y \)-coordinate of \( p \). Even less formally: Heisenberg balls whose centres have the same \( y \)-coordinate are dual to parallel plates.

![Figure 2](image-url)
Proof of Proposition 2.23. To prove the inclusion $\ell^s(B) \subset \Pi_{2r}(u_0, v_0, y_0)$, let $q \in B_{\mathbb{H}}(p, r)$, and write $q := (u, 0, v) \cdot (0, y, 0)$ with $(u, v) \in \mathbb{R}^2$ and $y \in \mathbb{R}$. First, we note that

$$|y - y_0| \leq d_{\mathbb{H}}(p, q) \leq r. \quad (2.26)$$

Let $\pi_{xt}$ be the vertical projection to the $xt$-plane $\{(u', 0, v') : (u', v') \in \mathbb{R}^2\}$. Then $(u, 0, v) = \pi_{xt}(q) \in \pi_{xt}(B)$ by the definition of $\pi_{xt}$. We now observe that $B = (u_0, 0, v_0) \cdot B((0, y_0, 0), r)$, so

$$\pi_{xt}(B) = (u_0, 0, v_0) + \pi_{xt}(B_{\mathbb{H}}((0, y_0, 0), r)).$$

We claim that

$$\pi_{xt}(B_{\mathbb{H}}((0, y_0, 0), r)) \subset \{(u', 0, v') : (u', v') \in \mathcal{R}_{2r}(y_0)\}. \quad (2.27)$$

This will prove that

$$(u, 0, v) \in (u_0, 0, v_0) + \{(u', 0, v') : (u', v') \in \mathcal{R}_{2r}(y_0)\}. \quad (2.28)$$

Recalling the definition (2.20), a combination of (2.26) and (2.28) now shows that

$$\ell^s(q) = \ell^s((u, 0, v) \cdot (0, y, 0)) \overset{(2.11)}{=} (0, u, v) + L_y \subset \Pi_{2r}(u_0, v_0, y_0).$$

This will complete the proof of the inclusion $\ell^s(B) \subset \Pi_{2r}(u_0, v_0, y_0)$.

Let us then prove (2.27). Pick $(x, y, t) \in B_{\mathbb{H}}((0, y_0, 0), r)$. Then,

$$\|(x, y - y_0, t + \frac{1}{2}xy_0)\| = d_{\mathbb{H}}((x, y, t), (0, y_0, 0)) \leq r,$$

so

$$|x| \leq r, \quad |y - y_0| \leq r, \quad \text{and} \quad |t + \frac{1}{2}xy_0| \leq r^2. \quad (2.29)$$

Now, to prove (2.27), recall that $\pi_{xt}(x, y, t) = (x, 0, t - \frac{1}{2}xy)$. Thus, we need to show that $(x, t - \frac{1}{2}xy) \in \mathcal{R}_{2r}(y_0) = M_{y_0}(\mathcal{R}_0(2r))$. Equivalently, $M_{y_0}^{-1}(x, t - \frac{1}{2}xy) \in \mathcal{R}_0(2r)$. Recalling the definition of $M_{y_0}$, one checks that

$$M_{y_0}^{-1}(x, t - \frac{1}{2}xy) = \begin{pmatrix} 1 & 0 \\ y_0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t - \frac{1}{2}xy \end{pmatrix} = \begin{pmatrix} x, xy_0 + t - \frac{1}{2}xy \end{pmatrix} = \begin{pmatrix} x, t + \frac{1}{2}xy_0 + \frac{1}{2}x(y_0 - y) \end{pmatrix}.$$

Using (2.29), we finally note that the point on the right lies in the parabolic rectangle $\mathcal{R}_0(2r)$. This concludes the proof of (2.27).

Let us then prove the inclusion $\Pi_r(u_0, v_0, y_0) \subset \ell^s(CB)$. The set $\Pi_r(u_0, v_0, y_0)$ is a union of the lines $(0, u_0, v_0) + (0, \bar{r}) + L_y$, where $\bar{r} \in \mathcal{R}_r(y_0)$ and $|y - y_0| \leq r$. We need to show that every such line can be realised as $\ell^s(q)$ for some $q \in B_{\mathbb{H}}(p, Cr)$. In this task, we are aided by the formula

$$\ell^s((u, 0, v) \cdot (0, y, 0)) = (0, u, v) + L_y$$

observed in (2.8). This formula shows that we need to define $q := (u, 0, v) \cdot (0, y, 0)$, where $(u, v) := (u_0, v_0) + \bar{r}$, and $y$ is as in “$L_y$.” Then we just have to hope that $q \in B_{\mathbb{H}}(p, Cr)$.

Recalling that $p = (u_0, 0, v_0) \cdot (0, y_0, 0)$, one can check by direct computation that

$$d_{\mathbb{H}}(p, q) = \|(u_0 - u, y_0 - y, v_0 - v + y_0(u_0 - u) + \frac{1}{2}(u - u_0)(y_0 - y))\|. \quad (2.30)$$

On the other hand, one may easily check that $(u, v) \in (u_0, v_0) + \mathcal{R}_r(y_0)$ is equivalent to

$$(u - u_0, v - v_0 + y_0(u_0 - u)) \in \mathcal{R}_r(0),$$

and so on.
which implies \(|u - u_0| \leq r\) and \(|v - v_0 + y_0(u - u_0)| \leq r^2\). Since moreover \(|y - y_0| \leq r\) by assumption, it follows from (2.30) and the definition of the norm \(\| \cdot \|\) that \(d_{\mathbb{H}}(p, q) \leq r\). This completes the proof. \(\square\)

We close the section with two additional auxiliary results:

**Proposition 2.31.** Let \(p, q \in \mathbb{H}\) and \(r \in (0, \frac{1}{2}]\), and assume that \(\|p\| \leq 1/10\). Assume moreover that \(\ell^*(p) \cap B(1) \subset \ell^*(B_{\ell^2}(q, r))\). Then \(p \in B_{\ell^2}(q, Cr)\) for some absolute constant \(C > 0\).

**Proof.** Write \(p = (u, 0, v) \cdot (0, y, 0)\), so that \(\ell^*(p) = (0, u, v) + L_y\). Since \(\|p\| \leq 1/10\), in particular \(|u| + |v| \leq 1/5\). By the previous proposition, we already know that

\[
((0, u, v) + L_y) \cap B(1) = \ell^*(p) \cap B(1) \subset \Pi_{2r}(u_0, v_0, y_0),
\]

where we have written \(q = (u_0, 0, v_0) \cdot (0, y_0, 0)\). Since \((0, u, v) \in B(1)\), we know that \((0, u, v) \in \ell^*(p) \cap \Pi_{2r}(u_0, v_0, y_0)\). But

\[
\Pi_{2r}(u_0, v_0, y_0) \cap \{x = 0\} = \{(0, u', v') : (u', v') \in (0, v_0) + \mathcal{R}_{y_0}(r)\},
\]

so we may deduce that

\[
(u, v) \in (u_0, v_0) + \mathcal{R}_{y_0}(r). \tag{2.32}
\]

Moreover, in Remark 2.21 we noted that \(\Pi_{2r}(u_0, v_0, y_0) \cap B(1)\) is contained in the \(r\)-neighbourhood \(T\) of the line \((0, u_0, v_0) + L_{y_0}\). Therefore also \((0, u, v) + L_y \cap B(1) \subset T\). This implies that \(\angle(L_y, L_{y_0}) \lesssim r\), and hence \(|y - y_0| \lesssim r\).

Now, we want to use (2.32) and \(|y - y_0| \lesssim r\) to deduce that \(d_{\mathbb{H}}(p, q) \lesssim r\). We first expand

\[
d_{\mathbb{H}}(p, q) = \|((u - u, 0, y - v, 0 - v + y_0(u - u) + \frac{1}{2}(u - u_0)(y_0 - y))\|.
\]

Then, using the definition of \(\mathcal{R}_{y_0}(r) = M_y(\mathcal{R}_0(r))\), we note that (2.32) is equivalent to

\[(u - u_0, v - v_0 + y_0(u - u_0)) \in \mathcal{R}_r(0).
\]

Combined with \(|y - y_0| \lesssim r\), and recalling the definition of \(\| \cdot \|\), this shows that the right hand side of (2.33) is bounded by \(\lesssim r\), as claimed. \(\square\)

We already noted in Remark 2.25 that the (modified) \(2r\)-plaques containing \(\ell^*(B(p_1, r))\) and \(\ell^*(B(p_2, r))\) have (almost) the same direction if the points \(p_1, p_2\) have (almost) the same \(y\)-coordinate. In this case, if \(d_{\mathbb{H}}(p_1, p_2) \geq Cr\), it is natural to expect that \(\ell^*(B(p_1, r))\) and \(\ell^*(B(p_2, r))\) are disjoint, at least inside \(B(1)\). The next lemma verifies this intuition.

**Lemma 2.34.** Let \(p_1 = (u_1, 0, v_1) \cdot (0, y_1, 0) \in B_{\mathbb{H}}(1)\) and \(p_2 = (u_2, 0, v_2) \cdot (0, y_2, 0) \in B_{\mathbb{H}}(1)\) be points with the properties

\[
|y_1 - y_2| \leq r \quad \text{and} \quad \ell^*(B_{\mathbb{H}}(p_1, r)) \cap \ell^*(B_{\mathbb{H}}(p_2, r)) \cap B(1) \neq \emptyset. \tag{2.35}
\]

Then, \(d_{\mathbb{H}}(p_1, p_2) \lesssim r\).

**Proof.** We may reduce to the case \(y_1 = y_2\) by the following argument. Start by choosing a point \(p'_2 \in B_{\mathbb{H}}(p_2, r)\) such that the \(y\)-coordinate of \(p'_2\) equals \(y_1\). This is possible, because \(|y_1 - y_2| \leq r\), and the projection of \(B_{\mathbb{H}}(p_2, r)\) to the \(xy\)-plane is a Euclidean disc of radius \(r\). Then, notice that \(B_{\mathbb{H}}(p_2, r) \subset B_{\mathbb{H}}(p'_2, 2r)\), so

\[
\ell^*(B_{\mathbb{H}}(p_1, 2r)) \cap \ell^*(B_{\mathbb{H}}(p'_2, 2r)) \cap B(1) \neq \emptyset.
\]

Now, if we have already proven the lemma in the case \(y_1 = y_2\) (and for “\(2r\)” in place of “\(r\)”), it follows that \(d_{\mathbb{H}}(p_1, p'_2) \leq r\), and finally \(d_{\mathbb{H}}(p_1, p_2) \leq d_{\mathbb{H}}(p_1, p'_2) + d_{\mathbb{H}}(p'_2, p_2) \lesssim r\).
Let us then assume that \( y_1 = y_2 = y \). It follows from (2.35) and the first inclusion in Proposition 2.23 combined with the first inclusion in (2.22) that
\[
(0, u_1, v_1) + \mathcal{P}_{C_r}(y) \cap (0, u_2, v_2) + \mathcal{P}_{C_r}(y) \neq \emptyset
\]
for some absolute constant \( C > 0 \). Let "\( x \)" be a point in the intersection, and (using the definition of \( \mathcal{P}_{C_r}(y) \)), express \( x \) in the two following ways:
\[
(0, u_1, v_1) + (0, \vec{r}_1) + L_y(s) = x = (0, u_2, v_2) + (0, \vec{r}_2) + L_y(s),
\]
where \( \vec{r}_1 \in \mathcal{R}_{C_r}(y) = M_y(\mathcal{R}_{C_r}(0)) \) and \( \vec{r}_2 \in M_y(\mathcal{R}_r(0)) \), and \( s \in [-1, 1] \). The terms \( L_y(s) \) conveniently cancel out, and we find that
\[
(u_1, v_1) - (u_2, v_2) = \vec{r}_2 - \vec{r}_1 \in M_y(\mathcal{R}_{2C_r}(0)),
\]
or equivalently
\[
(u_1 - u_2, v_1 - v_2 + y(u_1 - u_2)) = M_y^{-1}(u_1 - u_2, v_1 - v_2) \in \mathcal{R}_{2C_r}(0).
\]
(2.36)
We have already computed in (2.33) that
\[
d_H(p_1, p_2) = \| (u_1 - u_2, 0, v_1 - v_2 + y(u_1 - u_2)) \|,
\]
and now it follows immediately from (2.36) that \( d_H(p_1, p_2) \lesssim r \).

3. DISCRETISING THE MAIN THEOREM

The purpose of this section is to reduce the proof of Theorem 1.3 to Theorem 3.2 which concerns \((\delta, t, C)\)-sets. We start by defining these precisely:

**Definition 3.1** \((\delta, t, C)\)-set. Let \((X, d)\) be a metric space, and let \( t \geq 0 \) and \( C, \delta > 0 \). A non-empty bounded set \( P \subset X \) is called a \((\delta, t, C)\)-set if
\[
|P \cap B(x, r)|_{\delta} \leq Cr^t \cdot |P|_{\delta}, \quad x \in X, \quad r \geq \delta.
\]
Here \(|A|_{\delta}\) is the smallest number of balls of radius \( \delta \) needed to cover \( A \). A family of sets \( B \) (typically: disjoint \( \delta \)-balls) is called a \((\delta, t, C)\)-set if \( P := \cup B \) is a \((\delta, t, C)\)-set.

If \( P \subset \mathbb{H} \), or \( B \subset \mathcal{P}(\mathbb{H}) \), the \((\delta, t, C)\)-set condition is always tested relative to the metric \( d_H \). We then state a \( \delta \)-discretised version of Theorem 1.3 for sets of dimension 3:

**Theorem 3.2.** For every \( \eta > 0 \), there exists \( \epsilon > 0 \) and \( \delta_0 > 0 \) such that the following holds for all \( \delta \in (0, \delta_0] \). Let \( \delta \in (0, 1] \), and let \( B \) be a non-empty \((\delta, 3, \delta^{-\epsilon})\)-set of \( \delta \)-balls contained in \( B_{\mathbb{H}}(1) \), with \( \delta \)-separated centres. Let \( \mu = \mu_f \) be the (probability) measure on \( \mathbb{H} \) with density
\[
f := (\delta^4 |B|)^{-1} \sum_{B \in B} \mathbf{1}_B.
\]
(3.3)
Then,
\[
\int_{S^1} \| \pi_e \mu \|^2_{L^2} d\mathcal{H}^1(e) \leq \delta^{-\eta}.
\]

The proof of Theorem 3.2 will be given in Section 4. Deducing Theorem 1.3 from Theorem 3.2 involves two steps. The first one, carried out in Section 5, is to reduce Theorem 1.3 to a \( \delta \)-discretised version, which concerns \((\delta, t)\)-sets with all possible values \( t \in [0, 3] \). This statement is Theorem 3.11 below, a simplified version of which was stated as Theorem 1.6 in the introduction.

The second – and less standard – step, carried out in this section, is to deduce Theorem 3.11 from Theorem 3.2. Heuristically, Theorem 3.2 is nothing but the 3-dimensional
Then, there exists a Borel set \( G \delta \) with \( P p \). Fix \( \eta \) and \( \varepsilon > 0 \) where \( \mu \) and \( \eta \) concern the Lebesgue measure (not the dimension) of \( H \). The dimension of \( H \) is larger than ones of the form \( P p \) of measure \( \mu \). Let \( G \) be determined by an application of Theorem 3.2, but we will require at least that \( \varepsilon \) is small enough, so we may apply Chebychev’s inequality that there exists a set \( G \) such that \( \mu(B_{\mathbb{H}}(x, \delta)) \leq 3 \delta^{-\varepsilon} \) for all \( x \in \mathbb{H} \).

If the constant \( C > 0 \) is irrelevant, a \((\delta, C)\)-measure may also be called a \( \delta \)-measure.

We will use the following notion of \( \delta \)-truncated Riesz energy:

\[
I_{\delta}^s(\mu) := \int d\mu(x) d\mu(y) \frac{1}{d_{\mathbb{H}, \delta}(x, y)^s},
\]

where \( \mu \) is a Radon measure, \( 0 \leq s \leq 4 \), and \( d_{\mathbb{H}, \delta}(x, y) := \max\{d_{\mathbb{H}}(x, y), \delta\} \).

**Corollary 3.6.** For every \( \eta > 0 \), there exists \( \delta_0, \varepsilon_0 > 0 \) such that the following holds for all \( \delta \in (0, \delta_0] \) and \( \varepsilon \in (0, \varepsilon_0] \). Let \( \mu \) be a \((\delta, \delta^{-\varepsilon})\)-probability measure on \( B_{\mathbb{H}}(1) \) with \( I_{\delta}^1(\mu) \leq \delta^{-\varepsilon} \). Then, there exists a Borel set \( G \subset \mathbb{H} \) such that \( \mu(G) \geq 1 - 2\delta^{\varepsilon_0} \) and

\[
\int_{G} \|\pi_e(\mu|G)\|_{L^2} \, d\mathcal{H}^1(e) \leq \delta^{-\eta}.
\]

**Proof.** Fix \( \eta > 0 \), \( \varepsilon \in (0, \varepsilon_0] \), and \( \delta \in (0, \delta_0] \). The dependence of \( \delta_0 \) and \( \varepsilon_0 \) on \( \eta \) will eventually be determined by an application of Theorem 3.2, but we will require at least that \( \varepsilon_0 \leq \eta \).

It follows from \( I_{\delta}^1(\mu) \leq \delta^{-\varepsilon} \) and Chebychev’s inequality that there exists a set \( G_0 \subset \mathbb{H} \) of measure \( \mu(G_0) \geq 1 - 3\delta^{\varepsilon_0} \) such that \( \mu(B_{\mathbb{H}}(x, r)) \leq \delta^{1-\varepsilon_0} \leq \delta^{-2\varepsilon_0} \) for all \( x \in G_0 \) and \( r \geq \delta \). Now, for dyadic rationals \( 0 < \alpha \leq \delta^{-2\varepsilon_0} \leq \delta^2 \), let

\[
G_{0,\alpha} := \{ x \in G_0 : \frac{\alpha}{2} \leq \mu(B_{\mathbb{H}}(x, \delta)) \leq \alpha \}.
\]

We discard immediately the sets \( G_{0,\alpha} \) with \( \alpha \leq \delta^{10} \): the union of these sets has measure \( \leq \delta^{5} \leq \delta^{\varepsilon_0} \) for \( \delta > 0 \) small enough, so \( \mu(G_1) \geq 1 - 2\delta^{\varepsilon_0} \), where

\[
G_1 := G_0 \setminus \bigcup_{\alpha \leq \delta^{10}} G_{0,\alpha}.
\]

Now, \( G_1 \) is covered by the sets \( G_{0,\alpha} \) with \( \delta^{10} \leq \alpha \leq \delta^2 \), and the number of such sets is \( m \leq \log(1/\delta) \). We let \( \alpha_1, \ldots, \alpha_m \) be an enumeration of these values of \( \alpha \), and we abbreviate \( G^j := G_{0,\alpha_j} \). We note that the union of the sets \( G^j \) with \( \mu(G^j) \leq \delta^{2\varepsilon_0} \) has measure at most \( m \cdot \delta^{2\varepsilon_0} \leq \delta^{\varepsilon_0} \) (for \( \delta > 0 \) small), so finally

\[
G := G_1 \setminus \{ G^j : 1 \leq j \leq m \text{ and } \mu(G^j) \leq \delta^{2\varepsilon_0} \}.
\]
has measure \( \mu(G) \geq 1 - 2\delta^\alpha - \delta^\alpha \geq 1 - \delta^\alpha \). Moreover, \( G \) is covered by the sets \( G^j \) with \( \mu(G^j) \geq \delta^{2\alpha} \). Re-indexing if necessary, we now assume that \( \mu(G^j) \geq \delta^{2\alpha} \) for all \( 1 \leq j \leq m \).

For \( 1 \leq j \leq m \) fixed, let \( B_j \) be a finitely overlapping (Vitali) cover of \( G^j \) by balls of radius \( \delta \), centred at \( G^j \). Using the facts \( G^j \subset G_0 \) and \( \mu(G^j) \geq \delta^{2\alpha} \), and the uniform lower bound \( \mu(B_{\mathbb{H}}(x, \delta)) \geq \alpha_j/2 \) for \( x \in G^j \), it is easy to check that each \( B_j \) is a \((\delta, 3, \delta^{-C\alpha})\)-set with

\[
|B_j| \lesssim \alpha_j^{-1}. \tag{3.8}
\]

Thus, writing

\[
f_j := (\delta^4 |B_j|^{-1}) \sum_{B \in B_j} 1_B \quad \text{and} \quad \mu_j := \mu_f,
\]

and assuming that \( \delta_0, \epsilon_0 > 0 \) are sufficiently small in terms of \( \eta \), we may deduce from Theorem 3.2 that

\[
\int_{S^1} \|\pi_e(\mu_j)\|_{L^2}^2 d\mathcal{H}^1(e) \lesssim \delta^{-\eta}, \quad 1 \leq j \leq m.
\]

Finally, it follows from the \((\delta, \delta^{-\epsilon})\)-property of \( \mu \)

\[
\mu(x) \lesssim \delta^{-\epsilon} \cdot \frac{\mu(B_{\mathbb{H}}(x, \delta))}{\delta^4} \leq \delta^{-\epsilon} \cdot \frac{\alpha_j}{\delta^4} \lesssim \delta^{-\epsilon} \cdot \mu_j(x), \quad x \in G^j.
\]

Thus, also the density of \( \pi_e(\mu_{G^j}) \) is bounded from above by the density of \( \pi_e(\mu_j) \):

\[
\int_{S^1} \|\pi_e(\mu_{G^j})\|_{L^2}^2 \, d\mathcal{H}^1(e) \lesssim \delta^{-\epsilon} \sum_{j=1}^m \int_{S^1} \|\pi_e(\mu_j)\|_{L^2}^2 \, d\mathcal{H}^1(e) \lesssim \log(1/\delta) \cdot \delta^{-\eta - \epsilon} \lesssim \delta^{-3\eta}.
\]

This completes the proof of (3.7) (with “3\(\eta\)” in place of “\(\eta\)”).

The concrete \( \delta \)-measures we will consider have the form \( \eta *_{\mathbb{H}} \mu \), where \( \mu = \mu_f \) has a density of the form (3.3) (these are almost trivially \( \delta \)-measures), and \( \eta \) is a (discrete) probability measure. The notation \( \eta *_{\mathbb{H}} \mu \) refers to the (non-commutative!) Heisenberg convolution of \( \eta \) and \( \mu \), that is, the push-forward of \( \eta \times \mu \) under the group product \( (p, q) \mapsto p \cdot q \). Let us verify that such measures \( \eta *_{\mathbb{H}} \mu \) are also \( \delta \)-measures:

**Lemma 3.9.** Let \( \mu \) be \((\delta, C)\) measure, and let \( \eta \) be an arbitrary Borel probability measure on \( \mathbb{H} \). Then \( \eta *_{\mathbb{H}} \mu \) is again a \((\delta, C)\)-measure.

**Proof.** Recall that a \((\delta, C)\) measure is absolutely continuous by definition, so the notation “\( \mu(p) \)” is well-defined for Lebesgue almost every \( p \in \mathbb{H} \). The following formulae are valid, and easy to check, for Lebesgue almost every \( p \in \mathbb{H} \):

\[
(\eta *_{\mathbb{H}} \mu)(p) = \int \mu(q^{-1} \cdot p) \, d\eta(q)
\]

and

\[
(\eta *_{\mathbb{H}} \mu)(B_{\mathbb{H}}(p, r)) = \int_{\text{Leb}(B_{\mathbb{H}}(p, r))} \mu(B_{\mathbb{H}}(q^{-1} \cdot p, r)) \, d\eta(q). \tag{3.10}
\]

Now, if one applies the \( \delta \)-measure assumption to the formula on the left hand side, one obtains

\[
(\eta *_{\mathbb{H}} \mu)(p) \leq C \int_{\text{Leb}(B_{\mathbb{H}}(q^{-1} \cdot p, \delta))} \mu(B_{\mathbb{H}}(q^{-1} \cdot p, \delta)) \, d\eta(q).
\]
Lebesgue measure is invariant under left translations, so
\[
\text{Leb}(B_{\mathbb{H}}(q^{-1} \cdot p, \delta)) = \text{Leb}(B_{\mathbb{H}}(p, \delta)).
\]
Therefore, it follows from equation (3.10) that
\[
(\eta \ast_{\mathbb{H}} \mu)(p) \leq C \cdot \frac{(\eta \ast_{\mathbb{H}} \mu)(B_{\mathbb{H}}(p, \delta))}{\text{Leb}(B_{\mathbb{H}}(p, \delta))}
\]
for Lebesgue almost every \(p \in \mathbb{H}\). This is what we claimed. \(\square\)

We are then ready to state and prove the \(\delta\)-discretised counterpart of Theorem 1.3.

**Theorem 3.11.** Let \(0 < s < t < 3\). Then, there exist \(\epsilon, \delta_0 > 0\), depending only on \(s, t\), such that the following holds for all \(\delta \in (0, \delta_0]\). Let \(B \neq \emptyset\) be a \((\delta, t, \delta^{-s})\)-set of \(\delta\)-balls with \(\delta\)-separated centres, all contained in \(B_{\mathbb{H}}(1)\), and let \(S \subset S^1\) be a Borel set of length \(\mathcal{H}^1(S) \geq \delta^s\). Then, there exists \(e \in S\) such that the following holds: if \(B' \subset B\) is any sub-family with \(|B'| \geq \delta^s|B|\), then
\[
\text{Leb}(\pi_e(\cup B')) \geq \delta^{3-s}.
\]
In particular, \(\pi_e(\cup B')\) cannot be covered by fewer than \(\delta^{-s}\) parabolic balls of radius \(\delta\).

**Proof.** To reach a contradiction, assume that there exists a \((\delta, t, \delta^{-s})\)-set \(B\) of \(\delta\)-balls with \(\delta\)-separated centres, contained in \(B_{\mathbb{H}}(1)\), and violating the conclusion of Theorem 3.11: there exists \(s < t\), and for every \(e \in S\) (Borel subset of \(S^1\) of length \(\mathcal{H}^1(S) \geq \delta^s\)), there exists a subset \(B_e \subset B\) with \(|B_e| \geq \delta^s|B|\) with the property
\[
\text{Leb}(\pi_e(\cup B_e)) \leq \delta^{3-s}.
\]
We aim for a contradiction if \(\epsilon, \delta\) are sufficiently small. We fix an auxiliary parameter \(0 < \eta < (t - s)/2\). Then, we apply Corollary 3.6 to find the constant \(\epsilon_0 > 0\) which depends only on \(\eta\). Finally, we will assume, presently, that \(\epsilon < \epsilon_0/2\), and \(\eta + 3\epsilon < t - s\).

Let \(\mu\) be the uniformly distributed probability measure on \(\cup B\); in particular \(\mu\) is a \(\delta\)-measure (with absolute constant), and \(I^1_{\delta}(\mu) \lesssim \delta^{-\epsilon}\). Apply Proposition A.1 to find a set \(H \subset B_{\mathbb{H}}(1)\) of cardinality \(|H| \leq \delta^{-3}\) such that \(I^2_{\delta}(\tau \ast_{\mathbb{H}} \mu) \lesssim \delta^{-\epsilon}\), where \(\tau\) is the uniformly distributed probability measure on \(H\). Write \(\nu := \tau \ast_{\mathbb{H}} \mu\), so \(\nu\) is a \(\delta\)-probability measure by Lemma 3.9. Since \(\epsilon < \epsilon_0/2\) and \(I^2_{\delta}(\nu) \lesssim \delta^{-\epsilon}\), it follows from Corollary 3.6 that there exists a set \(G \subset \mathbb{H}\) of measure \(\nu(G) \geq 1 - \delta^{\epsilon_0}\) such that
\[
\frac{1}{\mathcal{H}^1(S)} \int_S \|\pi_e(\nu)\|_{L_2}^2 \, d\mathcal{H}^1(e) \leq \frac{1}{\mathcal{H}^1(S)} \int_{S^1} \|\pi_e(\nu|_G)\|_{L_2}^2 \, d\mathcal{H}^1(e) \leq \delta^{-\eta-\epsilon}.
\]
Finally, write \(B_e := H \cdot (\cup B_e)\) for all \(e \in S^1\), and note that \(\nu(B_e) \geq \delta^s\) for all \(e \in S\) (this is a consequence of the general inequality \((\mu_1 \ast_{\mathbb{H}} \mu_2)(A \cdot B) \geq (\mu_1 \times \mu_2)(A \times B)\). Consequently, also \(\nu(G \cap B_e) \geq \nu(G) + \nu(B_e) - 1 \geq \delta^s - \delta^{\epsilon_0} \geq \delta^s/2\), using \(\epsilon < \epsilon_0/2\). Therefore,
\[
\delta^{2\epsilon}/4 \leq \|\pi_e(\nu|_{G \cap B_e})\|_{L_2}^2 \leq \text{Leb}(\pi_e(B_e)) \cdot \|\pi_e(\nu|_G)\|_{L_2}^2,\quad e \in S^1,
\]
using Cauchy-Schwarz, and it follows from (3.13) that \(\text{Leb}(\pi_e(B_e)) \geq \delta^{9+3\epsilon}\) for at least one vector \(e \in S\). On the other hand, note that \(B_e = H \cdot (\cup B_e)\) is a union of \(\leq \delta^{-3}\) left translates of \(\cup B_e\), and recall from (2.1) that
\[
\text{Leb}(\pi_e(p \cdot B_e)) = \text{Leb}(\pi_e(B_e)), \quad p \in \mathbb{H}, B_e \subset \mathbb{H}.
\]
Therefore, we have the upper bound
\[
\text{Leb}(\pi_e(B_e)) = \text{Leb}(\pi_e(H \cdot (\cup B_e))) \overset{(3.12)}{\leq} \delta^{t-3} \cdot \delta^{3-s} = \delta^{t-s}, \quad e \in S^1.
\]
Since $\eta + 3\varepsilon < t - s$ by assumption, the previous lower and upper bounds for $\text{Leb}(\pi_{\varepsilon}(B_{\varepsilon}))$ are not compatible for $\delta > 0$ small enough. A contradiction has been reached. \qed

4. Kakeya Estimate of Guth, Wang, and Zhang

The purpose of this section is to prove Theorem 3.2. This will be based on the duality between horizontal lines and light rays developed in Section 2.2, and an application of a (reverse) square function inequality for the cone, due to Guth, Wang, and Zhang [8]. To be precise, we will not need the full power of this “oscillatory” statement, but rather only a Kakeya inequality for plates in [8, Lemma 1.4]. To introduce the statement, we need to recap some of the terminology and notation in [8]. This discussion follows [8, Section 1], but we prefer a different scaling: more precisely, in our discussion the geometric objects (plates and rectangles) of [8] are dilated by ”$R$” on the frequency side and (consequently) by $R^{-1}$ on the spatial side.

Fix $R \geq 1$, and let

$$\Gamma := \Gamma_R := C \cap \{R/2 \leq |\xi| \leq R\}. \tag{4.1}$$

Let $\Gamma(1)$ be the 1-neighbourhood of $\Gamma$, and let $\Theta$ be a finitely overlapping cover of $\Gamma(1)$ by rectangles of dimensions $R \times R^{-1/2} \times 1$, whose longest side is parallel to a light ray. The statements in [8] are not affected by the particular construction of $\Theta$, but in our application, the relevant rectangles are translates of dual rectangles of the $\delta$-plates in Definition 2.17, with $\delta = R^{-1/2}$. Indeed, $\delta$-plates are rectangles of dimensions $\sim \delta^2 \times \delta \times 1$ tangent to $C$, so their dual rectangles are plates of dimensions $\sim R \times R^{1/2} \times 1$, also tangent to $C$ (this is because $C$ has opening angle $\pi/2$, see Figure 3). For concreteness, we will use translated duals of $R^{-1/2}$-plates (as in Definition 2.17) to form the collection $\Theta$.

For each $\theta \in \Theta$, let $f_\theta \in L^2(\mathbb{R}^3)$ be a function with $\text{spt} f_\theta \subset \theta$, and consider the square function

$$Sf := \left( \sum_{\theta \in \Theta} |f_\theta|^2 \right)^{1/2}.$$  \(\text{(4.2)}\)

Then, [8, Lemma 1.4] contains an inequality of the following form:

$$\int_{\mathbb{R}^3} |Sf|^4 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{e} \sum_{U \in [e]} \text{Leb}(U)^{-1} \|S_Uf\|_{L^4}^4. \tag{4.2}$$

To understand the meaning of the "partial" square functions $S_U$ we need to introduce more terminology from [8]. Fix a dyadic number $s \in [R^{-1/2}, 1]$ (an "angular" parameter), and write $R' := s^2 R \in [1, R]$. The 1-neighbourhood of the truncated cone $\Gamma_{R'} = C \cap \{ |\xi| \sim R' \}$ can be covered by a finitely overlapping family $\Theta_{R'}$ of rectangles of dimensions $R' \times (R')^{1/2} \times 1 = s^2 R \times s R^{1/2} \times 1$.

Consequently, the $(R')^{-1}$-neighbourhood of $\Gamma_R$ is covered by the rescaled rectangles $\mathcal{T}_s := \{ s^{-2} \theta : \theta \in \Theta_{R'} \}$ of dimensions $R \times s^{-1} R^{1/2} \times s^{-2}$. Note that the family $\mathcal{T}_1$ coincides with $\Theta_{R}$ (at least if it is defined appropriately), whereas $\mathcal{T}_{R^{-1/2}}$ consists of $\sim 1$ balls of radius $R$.

For every $s \in [R^{-1/2}, 1]$, the rectangles in $\mathcal{T}_s$ are at least as large as those in $\Theta_{R'}$, so we may assume that every $\theta \in \Theta_{R}$ is contained in at least one rectangle $\tau \in \mathcal{T}_s$. For $\theta \in \Theta_{R}$
and \( \tau \in \mathcal{T}_s \), let \( \theta^* \) and \( \tau^* \) be the dual rectangles of \( \theta \) and \( \tau \). Then both \( \theta^* \) and \( \tau^* \) are rectangles centred at the origin, with dimensions

\[
R^{-1} \times R^{-1/2} \times 1 \quad \text{and} \quad R^{-1} \times sR^{-1/2} \times s^2,
\]

respectively. The longest sides of both \( \theta^* \) and \( \tau^* \) remain parallel to a light ray on \( C \): this is again the convenient property of the "standard" cone \( C \) with opening angle \( \pi/2 \), see Figure 3. Of course, \( \theta^* \) is an \( R^{-1/2} \)-plate in the sense of Definition 2.17, since the elements \( \theta \in \Theta \) were defined as (translates of) duals or \( R^{-1} \)-plates.

![Figure 3](image-url)

**Figure 3.** On the left: the truncated cone \( \Gamma \) and one of the plates \( \theta \). On the right: the cone \( C \) and the dual plate \( \theta^* \).

The set \( \tau^* \) turns out to be (essentially) a dilate of an \( (s^2R)^{-1/2} \)-plate. For every \( \tau \in \mathcal{T}_s \), consider \( U_\tau := s^{-2} \tau^* \), which is a rectangle of dimensions

\[
s^{-2}R^{-1} \times s^{-1}R^{-1/2} \times 1 = (s^2R)^{-1} \times (s^2R)^{-1/2} \times 1.
\]

In particular, \( U_\tau \) is an \( (s^2R)^{-1/2} \)-plate, and hence larger than (or at least as large as) \( \theta^* \): if \( \theta \subset \tau \), then every translate of \( \theta^* \) is contained in some translate of \( 10U_\tau \). We let \( \mathcal{U}_\tau \) be a tiling of \( \mathbb{R}^3 \) by rectangles parallel to \( U_\tau \). Now we may finally define the "partial" square function \( S_{U\tau}f \):

\[
S_{U\tau}f := \left( \sum_{\theta \subset \tau} |f_\theta|^2 \right)^{1/2} \cdot 1_U, \quad U \in \mathcal{U}_\tau.
\]  \( \text{(4.3)} \)

With this notation, we are prepared to prove Theorem 3.2.

**Proof of Theorem 3.2.** Let \( \delta \in (0, \frac{1}{2}] \), and let \( B \) be a \( (\delta, 3, \delta^{-t}) \)-set of \( \delta \)-balls with \( \delta \)-separated centres. In the statement of Theorem 3.2, it was assumed that \( \cup B \subset B_{\mathbb{H}}(1) \), but for slight technical convenience we strengthen this (with no loss of generality) to \( \cup B \subset B_{\mathbb{H}}(c) \) for a small absolute constant \( c > 0 \). As in the statement of Theorem 3.2, let \( \mu \) be the measure on \( \mathbb{H} \) with density

\[
f := (\delta^4|B|)^{-1} \sum_{B \in B} 1_B.
\]
Following the discussion Section 2.3, and in particular recalling equation (2.16), Theorem 3.2 will be proven if we manage to establish that

$$\int_{\mathcal{L}_\perp} Xf(\ell)^2 \, dm(\ell) \leqslant \delta^{-\eta},$$  \hspace{1cm} (4.4)$$
assuming that $\epsilon, \delta > 0$ are small enough, depending on $\eta$. Recall that $\mathcal{L}_\perp = \ell(\{(a, b, c) : |a| \leqslant 1\})$. To estimate the quantity in (4.4), notice first that

$$Xf(\ell) = \int_\ell f \, dH^1 \lesssim (\delta^3 |\mathcal{B}|)^{-1} \cdot |\{B \in \mathcal{B} : \ell \cap B \neq \emptyset\}|, \quad \ell \in \mathcal{L}_\perp,$$  \hspace{1cm} (4.5)$$
because $H^1(B \cap \ell) \lesssim \delta$ for all $B \in \mathcal{B}$. Write $N(\ell) := |\{B \in \mathcal{B} : \ell \cap B \neq \emptyset\}|$. Then, as we just saw,

$$\int_{\mathcal{L}_\perp} Xf(\ell)^2 \, dm(\ell) \lesssim (\delta^3 |\mathcal{B}|)^{-2} \int_{\mathcal{L}_\perp} N(\ell)^2 \, dm(\ell) \leqslant (\delta^3 |\mathcal{B}|)^{-2} \int_{\mathcal{B}(1)} N(\ell(p))^2 \, d\text{Leb}(p).$$

The second inequality is based on (a) the definition of the measure $\mathfrak{m} = \ell_1 \text{Leb}$, and (b) the observation that if $\ell(p) \in \mathcal{L}_\perp$ and $N(\ell(p)) \neq 0$, then $\ell(p) \cap \mathcal{B}(c) \neq \emptyset$, and this forces $p \in \mathcal{B}(1)$ (if $c > 0$ was taken small enough). Finally, by Lemma 2.12, we have

$$N(\ell(p)) \leqslant |\{B \in \mathcal{B} : p \in \ell^s(B)\}| = \sum_{B \in \mathcal{B}} 1_{\ell^s(B)}(p).$$

Indeed, whenever $\ell(p) \cap B \neq \emptyset$ for some $B \in \mathcal{B}$, there exists a point $q \in \ell(p) \cap B$, and then Lemma 2.12 implies that $p \in \ell^s(q) \subset \ell^s(B)$. Therefore, combining (4.4)-(4.5), it will suffice to show that for $\eta > 0$ fixed, the inequality

$$\int_{\mathcal{B}(1)} \left( \sum_{B \in \mathcal{B}} 1_{\ell^s(B)} \right)^2 \leqslant \delta^{-\eta} \cdot (\delta^3 |\mathcal{B}|)^2$$  \hspace{1cm} (4.6)$$
holds assuming that we have picked $\epsilon > 0$ in the $(\delta, 3, \delta^{-\epsilon})$-set hypothesis for $\mathcal{B}$ sufficiently small, depending on $\eta$.

By the discussion in Section 2.4, the intersections $\ell^s(B) \cap \mathcal{B}(1)$ are essentially $\delta$-plates – rectangles of dimensions $1 \times \delta \times \delta^2$ tangent to $\mathcal{C}$. More precisely, for every $B \in \mathcal{B}$, let $\mathcal{P}_B \subset \mathbb{R}^3$ be a $C\delta$-plate (as in Definition 2.17) with the property

$$\ell^s(B) \cap \mathcal{B}(1) \subset \mathcal{P}_B.$$  

This is possible by first applying Proposition 2.23 (which yields a modified $2\delta$-plate containing $\ell^s(B)$), and then the first inclusion in (2.22), which shows that the intersection of the modified $2\delta$-plate with $\mathcal{B}(1)$ is contained in a $C\delta$-plate $\mathcal{P}_B$. Now, we will prove (4.6) by establishing that

$$\int \left( \sum_{B \in \mathcal{B}} 1_{\mathcal{P}_B} \right)^2 \leqslant \delta^{-\eta} \cdot (\delta^3 |\mathcal{B}|)^2.$$  \hspace{1cm} (4.7)$$
Every plate $\mathcal{P}_B$ has a direction, denoted $\theta(\mathcal{P}_B)$: this is the direction of the longest axis of $\mathcal{P}_B$, or more formally the real number $\theta = \ddot{y} \in [-1, 1]$ associated to the line $"L_y"$ in Definitions 2.17. By enlarging the plates $\mathcal{P}_B$ slightly (if necessary), we may assume that their directions lie in the set $\Theta := (\delta \mathbb{Z}) \cap [-1, 1]$: this is because if two plates coincide in all
other parameters, and differ in direction by \( \leq \delta \), both are contained in constant enlargements of the other (this is not hard to check). The reason why we may restrict attention to \([-1, 1]\) is that all the plates \( P_B \) were associated to the balls \( B \subset B_R(c) \), and in fact the \( y \)-coordinate of the centre of \( B \) determines the direction of \( P_B \) (see (2.11)).

We next sort the family \( \{ P_B \}_{B \in B} \) according to their directions:

\[
\{ P_B : B \in B \} = \bigcup_{\theta \in \Theta} P(\theta),
\]

where \( P(\theta) := \{ P_B : \theta(P_B) = \theta \} \). Thus, for \( \theta \in \Theta \) fixed, the plates in \( P(\theta) \) are all translates of each other. Also, the plates in \( P(\theta) \) for a fixed \( \theta \) have bounded overlap: this follows from the assumption that the balls in \( B \) have \( \delta \)-separated centres, and uses Lemma 2.34 (the plates with a fixed direction correspond precisely to Heisenberg balls whose \( y \)-coordinates are, all, within "\( \delta \)" of each other).

Write \( R := \delta^{-2} \), thus \( \delta = R^{-1/2} \), and recall the truncated cone \( \Gamma = \Gamma_R \) from (4.1). Since the plates \( P \in P(\theta) \) are translates of each other, they all have a common dual rectangle \( P^*_\theta \) of dimensions \( \sim R \times R^{1/2} \times 1 \). The rectangle \( P^*_\theta \) is centred at 0, but we may translate it by \( \sim R \) in the direction of its longest \( R \)-side (a light ray depending on \( \theta \)) so that the translate lies in the \( O(1) \)-neighbourhood of \( \Gamma_R \). Committing a serious abuse of notation, we will denote this translated dual rectangle again by "\( \theta \)" and the collection of all these \( \theta \) is denoted \( \Theta \). This notation coincides with the discussion below (4.1). There is a 1-to-1 correspondence between the directions \( \theta \in \Theta = \delta \mathbb{Z} \cap [-1, 1] \) and the rectangles \( \theta \in \Theta \) defined just above, so the notational inconsistency should not cause confusion.

We gradually move towards applying the inequality (4.2) of Guth, Wang, and Zhang. The next task is to define the functions \( f_0 \) and \( f = \sum_{\theta \in \Theta} f_\theta \). Fix \( \theta \in \Theta \), \( P \in P(\theta) \), and let \( \varphi_P \in C^\infty_c(\mathbb{R}^3) \) be a non-negative function with the properties

1. \( 1_P \leq \varphi_P \leq 1 \),
2. \( \varphi_P \) has rapid decay outside \( P \),
3. \( \varphi_P \subset P^*_\theta \).

Here "rapid decay outside \( P \) has" the usual meaning: if \( \lambda P \) denotes a \( \lambda \)-times dilated, concentric, version of \( P \), then \( \varphi(x) \lesssim_N \lambda^{-N} \) for all \( x \in \mathbb{R}^3 \setminus \lambda P \) (and for any \( N \in \mathbb{N} \)). Then, define the function

\[
f_\theta := \sum_{P \in P(\theta)} e_\theta \cdot \varphi_P.
\]

Here \( e_\theta \) is a modulation, depending only on \( \theta \), such that

\[
e_\theta \cdot \varphi_P \subset \theta.
\]

Now the function \( f = \sum_{\theta \in \Theta} f_\theta \) satisfies all the assumptions of the inequality (4.2), so

\[
\int_{\mathbb{R}^3} \left( \sum_{B \in B} 1_{P_B} \right)^2 \leq \int_{\mathbb{R}^3} \left( \sum_{\theta \in \Theta} \varphi_P \right)^2 \leq \int_{\mathbb{R}^3} \left( \sum_{\theta \in \Theta} \sum_{P \in P(\theta)} e_\theta \cdot \varphi_P \right)^2 \leq \int_{\mathbb{R}^3} |Sf|^4 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum U \sum U \sum \text{Leb}(U)^{-1} \|SUf\|_L^4. \tag{4.8}
\]
Recalling the notation on the right hand side, in particular that \( \delta = R^{-1/2} \leq s \leq 1 \) only runs over dyadic rationals, and the definition of the “partial” square function \( S_\delta f \) from (4.3). The rectangles \( U \) are \( \Delta \)-plates with \( \Delta = (s^2 R)^{-1/2} = s^{-1} \delta \). In particular, every \( U \) is essentially the \( \ell^s \)-dual of a Heisenberg \( \Delta \)-ball: this will allow us to control \( \|S_\delta f\|_{L^2} \) by applying the 3-dimensional non-concentration condition of \( B \) between scales \( \delta \) and 1.

By definition,
\[
\|S_\delta f\|_2^2 = \int_U \sum_{\theta \subset \tau} |f_\theta|^2 = \int_U \sum_{\theta \subset \tau} \left( \sum_{P \in \mathcal{P}(\theta)} \varphi_P \right)^2 \lesssim \int_U \sum_{\theta \subset \tau} \sum_{P \in \mathcal{P}(\theta)} \varphi_P. \tag{4.9}
\]
Above, and in the sequel, the notation \( A \lesssim B \) means that for every \( \rho > 0 \), there exists a constant \( C_\rho > 0 \) such that \( A \leq C_\rho \delta^{-\rho} B \). In (4.9), the final “\( \lesssim \)” inequality follows easily from the rapid decay of the functions \( \varphi_P \), and the bounded overlap of the plates \( P \in \mathcal{P}(\theta) \) for \( \theta \in \Theta \) fixed.

For \( \theta \subset \tau \), each plate \( P \in \mathcal{P}(\theta) \) is contained in some translate of \( 10U_\tau \) (this was discussed above (4.3)), but this translate may not be \( U \). Let \( U \supset U \) be an \((R^s \Delta)\)-plate which is concentric with \( U \). We then decompose the right hand side of (4.9) as
\[
\int_U \sum_{\theta \subset \tau} \sum_{P \in \mathcal{P}(\theta)} \varphi_P \leq \int_U \sum_{\theta \subset \tau} \sum_{P \in \mathcal{P}(\theta)} \varphi_P + \int_U \sum_{\theta \subset \tau} \sum_{P \in \mathcal{P}(\theta)} \varphi_P. \tag{4.10}
\]
Since each \( P \in \mathcal{P}(\theta) \) is contained in element of the tiling \( U_\tau \) (consisting of translates of \( U \)) every plate \( \mathcal{P}(\theta) \) with \( P \not\subset U \) is far away from \( U \): more precisely, \( R^{s/2} P \cap U = \emptyset \). By the rapid decay of \( \varphi_P \) outside \( P \), this implies that \( \varphi_P \lesssim \delta^{100} \) on \( U \), and therefore the second term of (4.10) is bounded by, say, \( \lesssim \delta^{50} \).

We then focus on the first term of (4.10), and we first note that
\[
\int \sum_{\theta \subset \tau} \sum_{P \in \mathcal{P}(\theta)} \varphi_P \lesssim \delta^3 \cdot |\{ P : P \subset U \}|, \tag{4.11}
\]
since \( \|\varphi_P\|_{L^1} \sim \text{Leb}(P) \sim \delta^3 \). So, we need to find out how many \( \delta \)-plates \( P \) are contained in \( U \). Since \( U \) is an \((R^s \Delta)\)-plate, it follows from the second inclusion (2.22), combined with the second inclusion in Proposition 2.23, that
\[
U \subset \ell^s(B_{2\ell}(p_U, CR^s \Delta)) =: \ell^s(B_U).
\]
for some \( p_U \in \mathbb{H} \), and for some absolute constant \( C > 0 \). On the other hand, the plates \( P = P_B, B \in B \), were initially chosen in such a way that \( \ell^s(B) \cap \{ (s, y, z) : |s| \leq 1 \} \subset \mathcal{P}_B \).

Thus, whenever \( P_B \subset U \), we have
\[
\ell^s(B) \cap \{ (s, y, z) : |s| \leq 1 \} \subset \mathcal{P}_B \subset U \subset \ell^s(B_U).
\]
This implies by Proposition 2.31 that \( B \subset B_U \), where possibly \( B_U \) was inflated by another constant factor. Thus,
\[
|\{ P : P \subset U \}| \lesssim |\{ B \in B : B \subset B_U \}|.
\]
Recalling that \( B \) is a \( (\delta, 3) \)-set, this will easily yield useful upper bounds for \( |\{ P : P \subset U \}| \).

To make this precise, we sort the sets “\( U \)” appearing in (4.8) according to the “richness”
\[
\rho(U) := |\{ B \in B : B \subset B_U \}| \leq \delta^{-s}(R^s \Delta)^3 \cdot |B|. \tag{4.12}
\]
The inequality follows from the $(\delta, 3, \delta^{-c})$-hypothesis on $B$. For $s \in [R^{-1/2}, 1]$ fixed, we choose a (dyadic) value $\rho = \rho_s$ such that
\[
\sum_{d(\tau) = s} \sum_{U \in \U} \text{Leb}(U)^{-1} \| S_U f \|_{L^2}^4 \lesssim \sum_{d(\tau) = s} \sum_{U \in \U} \text{Leb}(U)^{-1} \| S_U f \|_{L^2}^4.
\] (4.13)

Let $\U(\rho)$ be the collection of sets "$U" appearing on the right hand side, and let $\mathcal{B}' \subset B$ be the subset of the original $\delta$-balls which are contained in some ball $B_U, U \in \U(\rho)$. Then, evidently,
\[
|\mathcal{B}'| \lesssim \rho \cdot |\U(\rho)| \lesssim R^{C_\varepsilon} |\mathcal{B}'|.
\] (4.14)

The factor "$R^{C_\varepsilon}$" arises from the fact that while distinct sets "$U" are the duals of essentially disjoint Heisenberg $\Delta$-balls, the inflated balls $B_U$ only have bounded overlap, depending on the inflation factor $R^\varepsilon$.

Now, for $U \in \U(\rho)$, we may estimate (4.11) as follows:
\[
\| S_U f \|_{L^2}^2 \lesssim \int \sum_{\theta \in \varepsilon} \sum_{p \in \mathcal{P}(\theta)} \varphi_p \lesssim \delta^3 \cdot \rho \lesssim \delta^3 \cdot R^{C_\varepsilon} \cdot \frac{|\mathcal{B}'|}{|\U(\rho)|}.
\]
(In this estimate, we have omitted the term "$\delta^{50}\varepsilon$" from the second part of (4.10), because this term will soon turn out to be much smaller than the best bounds for what remains.)

Plugging this estimate into (4.13), and observing that $\text{Leb}(U) = \Delta^3$, we obtain
\[
\sum_{d(\tau) = s} \sum_{U \in \U} \text{Leb}(U)^{-1} \| S_U f \|_{L^2}^4 \lesssim |\U(\rho)| \cdot \Delta^{-3} \cdot \left( \delta^3 \cdot R^{C_\varepsilon} \cdot \frac{|\mathcal{B}'|}{|\U(\rho)|} \right)^2
\]
\[
= \Delta^{-3} \cdot \delta^6 \cdot R^{2C_\varepsilon} \cdot \frac{|\mathcal{B}'|^2}{|\U(\rho)|}.
\] (4.12)\&(4.14)
\[
\lesssim \delta^{6-\varepsilon} \cdot R^{3C_\varepsilon} \cdot |\mathcal{B}'| |\mathcal{B}| \lesssim R^{4C_\varepsilon} \cdot (\delta^3 |\mathcal{B}|)^2.
\]

Notably, this estimate is independent of "$\Delta" and the parameter "$s", so we may finally deduce from (4.8) that
\[
\int_{\mathbb{H}^3} \left( \sum_{\mathcal{B} \in \mathcal{B}} 1_{\mathcal{P}_B} \right)^2 \lesssim \varepsilon \cdot R^{4C_\varepsilon} \cdot (\delta^3 |\mathcal{B}|)^2.
\]

This completes the proof of (4.7) (choose $\varepsilon < \eta/(8C)$, and then $\delta > 0$ small enough depending on $\varepsilon$), and consequently the proof of Theorem 3.2. \qed

5. PROOF OF THE MAIN THEOREM

We recall the statement:

**Theorem 5.1.** Let $K \subset \mathbb{H}$ be a Borel set with $\dim_{\mathbb{H}} K = t \in [2, 3]$. Then, $\dim_{\mathbb{H}} \pi_e(K) \geq t - 1$ for $\mathcal{H}^1$ almost every $e \in S^1$. Consequently, $\dim_{\mathbb{H}} \pi_e(K) \geq 2t - 3$ for $\mathcal{H}^1$ almost every $e \in S^1$.

**Proof.** The lower bound for $\dim_{\mathbb{H}} \pi_e(K)$ follows immediately from the lower bound for $\dim_{\mathbb{E}}(K)$, combined with a general inequality between Hausdorff dimensions relative to Euclidean and Heisenberg metrics of subsets of $\mathbb{W}_e$, see [1, Theorem 2.8]. So, we focus on proving that $\dim_{\mathbb{H}}(K) \geq t - 1$ for $\mathcal{H}^1$ almost every $e \in S^1$. 

The first steps of the proof are standard; similar arguments have appeared, for example the deduction of [12, Theorem 2] from [12, Theorem 1]. So we only sketch the first part of the proof, and provide full details where they are non-standard. First, we may assume that $K \subset B_{\mathbb{H}}(1)$, and we may assume, applying Frostman’s lemma, that $K = \text{spt}(\mu)$ for some Borel probability measure $\mu$ satisfying $\mu(B_{\mathbb{H}}(p, r)) \lesssim r^t$ for all $p \in \mathbb{H}$ and $r > 0$.

We make the counter assumption that there exists $s < t$ such that

$$\mathcal{H}^1(\{e \in S^1 : \dim_H \pi_e(K) \leq s - 1\}) > 0.$$  

By several applications of the pigeonhole principle, this assumption can be applied to find the following objects for any $\epsilon > 0$, and for arbitrarily small $\delta > 0$:

1. A Borel subset $S' \subset S^1$ of length $\mathcal{H}^1(S') \geq \delta^{t/2}$.
2. For every $e \in S'$ a collection of $\lesssim \delta^{1-s}$ Euclidean $\delta$-discs $W_e$, contained in $\mathbb{W}_e$.
3. If $W_e := \cup W_e$ and $e \in S'$, then

$$\mu(\pi_e^{-1}(W_e)) \geq \delta^{t/2}. \quad (5.2)$$

We claim that (1)-(3) violate Theorem 3.11 if $\delta, \epsilon > 0$ are small enough. To this end, we first need to construct a relevant $(\delta, t, \delta^{-t})$-set of (Heisenberg) $\delta$-balls $B$ contained in $B_{\mathbb{H}}(1)$. Morally, this collection is a $\delta$-approximation of $K = \text{spt}(\mu)$. More precisely, we need to decompose $K$ to the following subsets:

$$K_\alpha := \{x \in K : \frac{\alpha}{2} \leq \mu(B_{\mathbb{H}}(p, \delta)) \leq \alpha\},$$

where $\alpha > 0$ runs over dyadic rationals with $\alpha \lesssim \delta^t$. By one final application of the pigeonhole principle, and recalling (5.2), one can find a fixed index $\alpha \in 2^{-\mathbb{N}}$ such that

$$\mu(\pi_e^{-1}(W_e) \cap K_\alpha) \geq \delta^{t} \quad (5.3)$$

for all $e \in S \subset S'$, where $\mathcal{H}^1(S) \geq \delta^t$. In particular, $\mu(K_\alpha) \geq \delta^t$. Then, we let $B$ be a (Vitali) cover of $K_\alpha$ by finitely overlapping Heisenberg $\delta$-balls with $(\delta/5)$-separated centres. Note that $\delta \alpha^{-1} \lesssim |B| \lesssim \alpha^{-1}$. Using the definition of $K_\alpha$, and the Frostman condition for $\mu$, it is now easy to check that $B$ is a $(\delta, t, C\delta^{-t})$-set of $\delta$-balls, where $C$ is roughly the Frostman constant of $\mu$.

Finally, from (5.3) and $\alpha \lesssim |B|^{-1}$, we deduce that if $e \in S$, then $\pi_e^{-1}(W_e)$ intersects $\gtrsim \delta^t|B|$ elements of $B$, since

$$\delta^t \leq \mu(\pi_e^{-1}(W_e) \cap K_\alpha) \leq \alpha \cdot |\{B \in B : \pi_e^{-1}(W_e) \cap B \neq \emptyset\}|, \quad e \in S.$$

Write $B_e := \{B \in B : \pi_e^{-1}(W_e) \cap B \neq \emptyset\}$, thus $|B_e| \gtrsim \delta^t|B|$. We now arrive at the point where it is crucial that the elements of $W_e$ are Euclidean $\delta$-discs. Namely, if $B \in B_e$ and $\pi_e^{-1}(W_e) \cap B \neq \emptyset$, then $\pi_e^{-1}(D) \cap B \neq \emptyset$ for some $D \in W_e$. Then, because $D$ is a Euclidean $\delta$-disc, and the Euclidean diameter of $\pi_e(B)$ is $\lesssim \delta$, we may conclude that $\pi_e(B) \subset 2D$. This could seriously fail if $D$ were a disc in the metric $d_{\mathbb{H}}$. Now, however, we see that

$$\pi_e(\cup B_e) \subset \cup\{2D : D \in W_e\},$$

and in particular $\text{Leb}(\pi_e(\cup B_e)) \lesssim \delta^2 \cdot |W_e| \lesssim \delta^{3-s}$ for all $e \in S$. This violates the conclusion of Theorem 3.11, and the proof of Theorem 5.1 is complete. \qed
In this section, we use the following notation for the $\delta$-truncated $s$-dimensional Riesz energy of a Radon measure $\nu$ on $\mathbb{H}$:

$$I_\delta^s(\nu) := \int d\nu(x) d\nu(y) d_{\mathbb{H},\delta}(x,y)^{s+t},$$

where $d_{\mathbb{H},\delta}(x,y) := \max\{d_{\mathbb{H}}(x,y), \delta\}$. We also recall that $\mu \ast \nu$ is the Heisenberg convolution of $\mu$ and $\nu$, that is, the push-forward of $\mu \times \nu$ under the group operation $(p,q) \rightarrow p \cdot q$.

**Proposition A.1.** Let $0 \leq s, t \leq 3$ with $s + t \leq 3$, and let $\delta \in (0, \frac{1}{2}]$. Let $\mu$ be a Borel probability measure on $B_{\mathbb{H}}(1)$ with $I_1^s(\mu) \leq C$. Then, there exists a set $H \subset B_{\mathbb{H}}(1)$ with $|H| \leq \delta^{-s}$ such that the uniformly distributed (discrete) measure $\eta$ on $H$ satisfies

$$I_{s+t}^\delta(\eta \ast \mu) \leq C',$$

where $C' \leq C \log(1/\delta)^C \cdot C$ for some absolute constant $C > 0$.

**Proof.** Let $Z := \delta \cdot \mathbb{Z}^3 \cap B_{\mathbb{H}}(1)$ be a grid of Euclidean $\delta$-separated lattice points in $B_{\mathbb{H}}(1)$. Then $|Z| \sim \delta^{-3}$. Let $H_\omega \subset Z$ be a random set, where each point of $Z$ is included independently with probability $\delta^{-s}/(2|Z|)$. In particular, $\mathbb{E}_\omega|H_\omega| = \delta^{-s}/2$. While we use the symbol "$\omega$" to index the elements in the underlying probability space, no explicit reference to this space will be needed. Let $\eta_\omega$ be the random measure

$$\eta_\omega := \delta^s \sum_{p \in H_\omega} \delta_p = \delta^s \sum_{p \in Z} 1_{H_\omega}(p) \cdot \delta_p.$$

We claim that

$$\mathbb{E}_\omega \left(I_{s+t}^\delta(\eta_\omega \ast \mu)\right) = \int \mathbb{E}_\omega \left(\int \frac{d\eta_\omega(p)d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p,q)^{s+t}} d\mu(x) d\mu(y)\right) \leq C', \tag{A.2}$$

for some $C' \preceq C$. In this argument, the notation "$\preceq$" hides a constant of the form $C \log(1/\delta)^C$. The inequality (A.2) will complete the proof of the proposition, because $|H_\omega| \leq \delta^{-s}$ with probability $\geq \frac{1}{2}$ (for $\delta > 0$ small enough), and therefore, by Chebychev’s inequality, $I_{s+t}^\delta(\eta_\omega \ast \mu) \preceq C'$ for some "$\omega"$ with $|H_\omega| \leq \delta^{-s}$.

To prove (A.2), it clearly suffices to establish that

$$\mathbb{E}_\omega \left(\int \frac{d\eta_\omega(p)d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p,q)^{s+t}}\right) \preceq \frac{1}{d_{\mathbb{H},\delta}(x,y)^t}, \quad x, y \in \text{spt}(\mu) \subset B_{\mathbb{H}}(1). \tag{A.3}$$

By definition of $\eta_\omega$, we have

$$\int \frac{d\eta_\omega(p)d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p,q)^{s+t}} = \delta^{2s} \sum_{p \in Z} \frac{1_{H_\omega}(p)1_{H_\omega}(q)}{d_{\mathbb{H},\delta}(p,q)^{s+t}} = \delta^{2s} \sum_{p \in Z} \frac{1_{H_\omega}(p)1_{H_\omega}(q)}{d_{\mathbb{H},\delta}(x,y)^{s+t}} + \delta^{2s} \sum_{p \neq q} \frac{1_{H_\omega}(p)1_{H_\omega}(q)}{d_{\mathbb{H},\delta}(p,q)^{s+t}} =: \Sigma_1(\omega) + \Sigma_2(\omega).$$

We consider the expectations of $\Sigma_1(\omega)$ and $\Sigma_2(\omega)$ separately. The former one is simple, using that $\mathbb{E}_\omega(1_{H_\omega}(p)) = \mathbb{P}_\omega\{p \in H_\omega\} = \delta^{-s}/(2|Z|) \sim \delta^{3-s}$:

$$\mathbb{E}_\omega \Sigma_1(\omega) \sim \delta^{3-s} \sum_{p \in Z} \frac{1}{d_{\mathbb{H},\delta}(x,y)^{s+t}} \leq \frac{\delta^{s}}{d_{\mathbb{H},\delta}(x,y)^t} \leq \frac{1}{d_{\mathbb{H},\delta}(x,y)^t}. $$

This completes the proof of Proposition A.1.
recalling that $|Z| \lesssim \delta^{-3}$. To handle the expectation of $\Sigma_2(\omega)$, we note that $\{p \in H_\omega\}$ and $\{q \in H_\omega\}$ are independent events for $p \neq q$, hence

$$E_\omega \Sigma_2(\omega) \sim \delta^{2s} \sum_{p,q \in Z \atop p \neq q} \sum_{\delta \in \mathbb{Z}} \sum_{r \in [1]} \delta^{6-2s} d_{H,\delta}(\mu(p \cdot x, q \cdot y))^{s+t} \sim \delta^{6} \sum_{p \in Z} \sum_{\delta \leq r \leq 1} r^{-s-t} |\{q \in Z : d_{H,\delta}(\mu(p \cdot x, q \cdot y)) \sim r\}|,$$

where "$r$" runs over dyadic rationals. Since the product "$\cdot$" is not commutative, in general $d_{H,\delta}(\mu(p \cdot x, q \cdot y)) \neq d_{H,\delta}(\mu(p \cdot y^{-1}, q))$, so the set $\{q \in Z : d_{H,\delta}(\mu(p \cdot x, q \cdot y)) \sim r\}$ is not contained in a $\mathbb{H}$-ball of radius $r$ around $p \cdot x \cdot y^{-1}$. This is the key inefficiency in the argument, and causes the restriction $s+t \leq 3$: under this restriction, it actually suffices to note that $\{q \in Z : d_{H,\delta}(\mu(p \cdot x, q \cdot y)) \sim r\}$ is contained in a Euclidean $Cr$-ball. To see this, note that if $q \in Z$ satisfies $d_{H,\delta}(\mu(p \cdot x, q \cdot y)) \lesssim r$ with $r \geq \delta$, then

$q \in B_{\mathbb{H}}(p \cdot x, Cr) \cdot y^{-1}.$

Here $B_{\mathbb{H}}(p \cdot x, Cr)$ is contained in a Euclidean ball of radius $\lesssim r$ (using $r \leq 1$). The same remains true after the right translation by $y^{-1}$, because $|y| \lesssim 1$ (by assumption), and the right translation $z \mapsto z \cdot y^{-1}$ is Euclidean Lipschitz with constant depending only on $|y|$.

Now, since a Euclidean $r$-ball contains $\lesssim (r/\delta)^3$ points of $Z$, we see that

$$E_\omega \Sigma_2(\omega) \lesssim \delta^3 \sum_{p \in Z} \sum_{\delta \leq r \leq 1} r^{3-s-t} \lesssim 1 \lesssim \frac{1}{d_{H,\delta}(\mu(x, y))^{s+t}},$$

where in the final inequality we used again that $x, y \in \text{spt}(\mu) \subset B_{\mathbb{H}}(1)$. This completes the proof of (A.3), and therefore the proof of the proposition. □

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