Billiard representation for pseudo-Euclidean Toda-like systems of cosmological origin

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The pseudo-Euclidean Toda-like system of cosmological origin is considered. When certain restrictions on the parameters of the model are imposed, the dynamics of the model near the “singularity” is reduced to a billiard on the $(n-1)$-dimensional Lobachevsky space $H^{n-1}$. The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the problem to the problem of illumination of $(n-2)$-dimensional sphere $S^{n-2}$ by point-like sources. Some examples are considered.

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1 Introduction

In this paper we consider pseudo-Euclidean Toda-like system described by the following Lagrangian

\[ L = L(z^a, \dot{z}^a, \mathcal{N}) = \frac{1}{2} \mathcal{N}^{-1} \eta_{ab} \dot{z}^a z^b - \mathcal{N} V(z), \quad (1.1) \]

where \( \mathcal{N} > 0 \) is the Lagrange multiplier (modified lapse function), 
\( (\eta_{ab}) = diag(-1,+1,\ldots,+1) \) is matrix of minisuperspace metric, \( a, b = 0, \ldots, n-1 \), and 
\[ V(z) = \sum_{\alpha=1}^{m} A_\alpha \exp(u^a_\alpha z^a) \quad (1.2) \]
is the potential. Here \( A_\alpha \neq 0 \).

Models of such type occur in multidimensional cosmology (see, for example, [1]-[6] and references therein) as well as in multicomponent 4-dimensional cosmology [7]-[10]. The Lagrange systems (1.1), (1.2) are not well studied yet. We note, that the Euclidean Toda-like systems are more or less well studied [12]-[18] (at least for certain sets of parameters, associated with finite-dimensional Lie algebras or affine Lie algebras). There is also a criterion of integrability by quadrature (algebraic integrability) for the Euclidean Toda-like systems established by Adler and van Moerbeke [16]. We note that in gravitational context the Euclidean Toda-like systems were first considered in [19]-[21].

In this paper we consider the behavior of the dynamical system (1.1) for \( n \geq 3 \) in the limit

\[ z^2 \equiv -(z^0)^2 + (\vec{z})^2 \to -\infty, \quad z = (z^0, \vec{z}) \in \mathcal{V}_-, \quad (1.3) \]

where \( \mathcal{V}_- \equiv \{(z^0, \vec{z}) \in \mathbb{R}^n | z^0 < -|\vec{z}|\} \) is the lower light cone. The limit (1.3) implies

\[ z^0 \to -\infty \quad (1.4) \]

and under certain additional assumptions describes the approaching to the singularity in corresponding cosmological models. We impose the following restrictions on the vectors \( u^a = (u^a_0, \vec{u}^a) \) in the potential (1.2)

1) \( A_\alpha > 0 \), if \( (u^a)^2 = -(u^a_0)^2 + (\vec{u}^a)^2 > 0 \); \quad (1.5)
2) \( u^a_0 > 0 \) for all \( \alpha = 1, \ldots, m \). \quad (1.6)

We note that in multidimensional cosmology a special interest is connected with the investigations of oscillatory behavior of scale factors near the singularity [22]-[26]. This direction in higher-dimensional gravity was stimulated by well-known results for "mixmaster" model [7]-[10]. We note, that there is also an elegant explanation for oscillatory behavior of scale factors of Bianchi-IX model suggested by Chitre [9, 10] and recently considered in [27]-[29]. In the Chitre’s approach the Bianchi-IX cosmology near the singularity is reduced to a billiard on the Lobachevsky space \( H^2 \) (see Fig. 4 below). The volume of this billiard is finite. This fact together with the well-known behavior (exponential divergences) of geodesics on the spaces of negative curvature may lead to a stochastic behavior of the dynamical system in the considered regime [30, 31].
Here we consider the generalization of Chitre’s approach [9] to the multidimensional case [26]. In the limit (1.3) the dynamics of the model (1.1), (1.2) (with the restrictions (1.5), (1.6) imposed) is reduced to a billiard on the \((n - 1)\)-dimensional Lobachevsky space \(H^{n-1}\) (Sec. 2). The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the considered problem to the geometrical (or topological) problem of illumination of \((n - 2)\)-dimensional unit sphere \(S^{n-2}\) by \(m_+ \leq m\) point-like sources located outside the sphere [32, 33]. These sources correspond to the components with \((u^a)^2 > 0\) (Sec. 2). When these sources illuminate the sphere then, and only then, the billiard has a finite volume and the corresponding cosmological model possesses an oscillatory behavior near the singularity. In this case from topological requirements we obtain the restriction on the number of components with \((u^a)^2 > 0\): \(m_+ \geq n\), i.e. this number should be no less than the minisuperspace dimension. In Sec. 3 we illustrate the suggested approach using the examples of Bianchi-IX and prototype cosmological models.

It should be noted that fixing the gauge in (1.1) \(N = 1\) we get Lagrange system

\[
L_1 = \frac{1}{2} \eta_{ab} \dot{z}^a \dot{z}^b - V(z),
\]  

with the zero-energy constraint

\[
E_1 = \frac{1}{2} \eta_{ab} \dot{z}^a \dot{z}^b + V(z) = 0.
\]

For non-zero energies \(E_1\) we are lead to Lagrange systems with the Lagrangians

\[
L_2 = L_1 + \frac{1}{2} \varepsilon (\dot{z}^n)^2
\]  

\(\varepsilon = -\text{sign} E_1\), and the zero-energy constraint \(E_2 = 0\). Thus, for \(E_1 < 0\) we get the pseudo-Euclidean Toda-like system (1.1) with \(n\) replaced by \(n + 1\) and the potential (1.2). In cosmology this corresponds to addition of scalar field. For \(E_1 > 0\) we obtain the Lagrangian system with the kinetic term of signature \((-,-,\ldots,+,-)\). In cosmology this corresponds to the incorporation of a phantom field. We also note that the Euclidean Toda-like systems may be embedded into (1.1) by considering the potentials (1.2) with \(u_0 = 0\).

### 2 Billiard representation

Here we consider the behavior of the dynamical system, described by the Lagrangian (1.1) for \(n \geq 3\) in the limit (1.3). We restrict the Lagrange system (1.1) on \(V_\ast\), i.e. we consider the Lagrangian

\[
L_\ast \equiv L|_{TM_\ast}, \quad M_\ast = V_\ast \times R_+, \tag{2.1}
\]

where \(TM_\ast\) is tangent vector bundle over \(M_\ast\) and \(R_+ \equiv \{N > 0\}\). (Here \(F|_A\) means the restriction of function \(F\) on \(A\).) Introducing an analogue of the Misner-Chitre coordinates in \(V_\ast\) [9, 10]

\[
\begin{align*}
\zeta^0 & = -\exp(-y^0) \frac{1 + y^0}{1 - y^2}, \quad \zeta = -2 \exp(-y^0) \frac{\dot{y}}{1 - y^2}, \tag{2.2, 3.3}
\end{align*}
\]
\(|\vec{y}| < 1\), we get for the Lagrangian (1.1)

\[
L_\pm = \frac{1}{2} \mathcal{N}^{-1} e^{-2y^0} \left[-(\dot{y}^0)^2 + h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j\right] - \mathcal{N}V. \tag{2.4}
\]

Here

\[
h_{ij}(\vec{y}) = 4\delta_{ij}(1 - \vec{y}^2)^{-2}, \tag{2.5}
\]
i, j = 1, \ldots, n - 1, and

\[
V = V(y) = \sum_{\alpha=1}^m A_\alpha \exp \Phi(y, u^\alpha), \tag{2.6}
\]

where

\[
\Phi(y, u) \equiv -e^{-y^0}(1 - \vec{y}^2)^{-1}[u_0(1 + \vec{y}^2) + 2\vec{u}\vec{y}], \tag{2.7}
\]

We note that the \((n - 1)\)-dimensional open unit disk (ball)

\[
D^{n-1} \equiv \{\vec{y} = (y^1, \ldots, y^{n-1})||\vec{y}| < 1\} \subset \mathbb{R}^{n-1} \tag{2.8}
\]

with the metric \(h = h_{ij}(\vec{y}) dy^i \otimes dy^j\) is one of the realization of the \((n - 1)\)-dimensional Lobachevsky space \(H^{n-1}\).

We fix the gauge

\[
\mathcal{N} = \exp(-2y^0) = -z^2. \tag{2.9}
\]

Then, it is not difficult to verify that the Lagrange equations for the Lagrangian (1.1) with the gauge fixing (2.9) are equivalent to the Lagrange equations for the Lagrangian

\[
L_\ast = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V_\ast \tag{2.10}
\]

with the energy constraint imposed

\[
E_\ast = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j + V_\ast = 0. \tag{2.11}
\]

Here

\[
V_\ast = e^{-2y^0}V = \sum_{\alpha=1}^m A_\alpha \exp(\Phi(y, u^\alpha)), \tag{2.12}
\]

where

\[
\Phi(y, u) = -2y^0 + \Phi(y, u). \tag{2.13}
\]

Now we are interested in the behavior of the dynamical system in the limit \(y^0 \to -\infty\) (or, equivalently, in the limit (1.3) implying (1.4)). Using the relations \((u_0 \neq 0)\)

\[
\Phi(y, u) = -u_0 \exp(-y^0) A(\vec{y}, -\vec{u}/u_0) - 2y^0, \tag{2.14}
\]

\[
A(\vec{y}, \vec{v}) \equiv (\vec{y} - \vec{v})^2 - \vec{v}^2 + 1, \tag{2.15}
\]

we get

\[
\lim_{y^0 \to -\infty} \exp \Phi(y, u) = 0 \tag{2.16}
\]
for $u^2 = -u_0^2 + (\vec{u})^2 \leq 0$, $u_0 > 0$ and

$$\lim_{y^0 \to -\infty} \exp \Phi(y, u) = \theta_\infty(-A(\vec{y}, -\vec{u}/u_0))$$

(2.17)

for $u^2 > 0$, $u_0 > 0$. In (2.17) we denote

$$\theta_\infty(x) \equiv + \infty, \quad x \geq 0,\quad 0, \quad x < 0.$$  

(2.18)

Using restrictions (1.5), (1.6) and relations (2.12), (2.16), (2.17) we obtain

$$V_\infty(\vec{y}) \equiv \lim_{y^0 \to -\infty} V_*(y^0, \vec{y}) = \sum_{\alpha \in \Delta_+} \theta_\infty(-A(\vec{y}, -\vec{u}/u_0^\alpha)).$$

(2.19)

Here we denote

$$\Delta_+ \equiv \{ \alpha | (u^\alpha)^2 > 0 \}.$$  

(2.20)

The potential $V_\infty$ may be also written as following

$$V_\infty(\vec{y}) = V(\vec{y}, B) \equiv 0, \quad \vec{y} \in B,\quad +\infty, \quad \vec{y} \in D^{n-1} \setminus B,$$

(2.21)

where

$$B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha) \subset D^{n-1},$$

(2.22)

$$B(u^\alpha) = \left\{ \vec{y} \in D^{n-1} : \left| \vec{y} + \vec{u}^\alpha \right| > \sqrt{\left( \frac{\vec{u}^\alpha}{u_0^\alpha} \right)^2 - 1} \right\},$$

(2.23)

$$\alpha \in \Delta_+. \quad B \text{ is an open domain. Its boundary } \partial B = \tilde{B} \setminus B \text{ is formed by certain parts of } m_+ = |\Delta_+| \quad (m_+ \text{ is the number of elements in } \Delta_+) \text{ of } (n - 2)-\text{dimensional spheres with the centers in the points}$$

$$\vec{v}^\alpha = -\vec{u}^\alpha/u_0^\alpha, \quad \alpha \in \Delta_+,$$

(2.24)

$$|v^\alpha| > 1$$

and radii

$$r_\alpha = \sqrt{(\vec{v}^\alpha)^2 - 1}$$

(2.25)

respectively (for $n = 3$, $m_+ = 1$, see Fig. 1).

So, in the limit $y^0 \to -\infty$ we are led to the dynamical system

$$L_\infty = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V_\infty(\vec{y}),$$
$$E_\infty = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_\infty(\vec{y}) = 0,$$

(2.26)

(2.27)

which after the separating of $y^0$ variable
Figure 1: An example of billiard for \( n = 3, m_+ = 1 \).

\[
y^0 = \omega(t - t_0), \quad (2.28)
\]

\((\omega \neq 0, t_0 \text{ are constants})\) is reduced to the Lagrange system with the Lagrangian

\[
L_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B). \quad (2.29)
\]

Due to (2.28)

\[
E_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j + V(\vec{y}, B) = \frac{\omega^2}{2}. \quad (2.30)
\]

We put \( \omega > 0 \), then the limit \( t \to -\infty \) corresponds to (1.3). When the set (2.20) is empty \( (\Delta_+ = \emptyset) \), we have \( B = D^{n-1} \) and the Lagrangian (2.29) describes the geodesic flow on the Lobachevsky space \( H^{n-1} = (D^{n-1}, h_{ij} dy^i \otimes dy^j) \). In this case there are two families of non-trivial geodesic solutions (i.e. \( y(t) \neq \text{const} \)):

1. \( \vec{y}(t) = \vec{n}_1 \sqrt{v^2 - 1} \cos \varphi(t) - v + \vec{n}_2 \sqrt{v^2 - 1} \sin \varphi(t), \quad (2.31) \)

\[
\varphi(t) = 2 \arctan[(v - \sqrt{v^2 - 1}) \tanh(\frac{1}{2} \omega(t - t_1))], \quad (2.32)
\]

2. \( \vec{y}(t) = \vec{n}_2 \tanh(\frac{1}{2} \omega(t - t_1)). \quad (2.33) \)

Here \( \vec{n}_1^2 = \vec{n}_2^2 = 1, \vec{n}_1 \vec{n}_2 = 0, v > 1, \omega > 0, t_1 = \text{const} \).

Graphically the first solution corresponds to the arc of the circle with the center at point \((-v\vec{n}_1)\) and the radius \( \sqrt{v^2 - 1} \). This circle belongs to the plane spanned by vectors \( \vec{n}_1 \) and \( \vec{n}_2 \) (the centers of the circle and the ball \( D^{n-1} \) also belong to this plane). We note, that the solution (2.31)-(2.32) in the limit \( v \to \infty \) coincides with the solution (2.33).

When \( \Delta_+ \neq \emptyset \) the Lagrangian (2.33) describes the motion of the particle of unit mass, moving in the \((n - 1)\)-dimensional generalized billiard \( B \subset D^{n-1} \) (see (2.22). The geodesic motion in \( B \) (2.31)-(2.33) corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to the change of Kasner epochs. For \( n = 3 \) some examples of (2-dimensional) billiards are depicted in Figs. 2-4.
Figure 2: Billiard with infinite volume for $n = 3$, $m_+ = 3$.

Figure 3: Compact billiard for $n = 3$, $m_+ = 3$. 
We note that the boundary of the billiard $\partial B$ is formed by geodesics. For some billiards this fact may be used for "gluing" certain parts of boundaries.

The billiard $B$ in Fig. 2. has an infinite volume: $\text{vol} B = +\infty$. In this case there are three open zones at the infinite circle $|\vec{y}| = 1$. After a finite number of reflections from the boundary the particle (in general position) moves toward one of these open zones. For corresponding cosmological model we get the "Kasner-like" behavior in the limit $t \to -\infty$.

For billiards depicted in Figs. 3 and 4 we have $\text{vol} B < +\infty$. In the first case (Fig. 3) the closure of the billiard $\bar{B}$ (in the topology of $D^{n-1}$) is compact (in the topology of $D^{n-1}$) and in the second case (Fig. 4) $\bar{B}$ is non-compact. In these two cases we have an "oscillatory-like" motion of the particle.

Analogous arguments may be applied to the case $n > 3$. So, we are interested in the configurations with finite volume of $B$.

We propose a simple geometric criterion for the finiteness of the volume of $B$ and compactness of $\bar{B}$ in terms of the positions of the points (2.24) with respect to the $(n-2)$-dimensional unit sphere $S^{n-2}$ ($n \geq 3$).

**Proposition 1.** The billiard $B$ (2.22) has a finite volume if and only if the point-like sources of light located at the points $v^{\alpha}$ (2.24) illuminate the unit sphere $S^{n-2}$. The closure of the billiard $\bar{B}$ is compact (in the topology of $D^{n-1} \simeq H^{n-1}$) if and only if the sources at points (2.24) strongly illuminate $S^{n-2}$.

**Definition.** Here the point is called strongly illuminated if it belongs to the interior (open) part of the illuminated region. At Fig. 1 the source $P$ strongly illuminates the open arc $(P_1, P_2)$.

**Proof.** We consider the set $\partial c B \equiv B^c \setminus \bar{B}$, where $B^c$ is the completion of $B$ (or, equivalently, the closure of $B$ in the topology of $R^{n-1}$). We recall that $\bar{B}$ is the closure of $B$...
in the topology of $D^{n-1}$. Clearly, that $\partial\mathfrak{c}B$ is a closed subset of $S^{n-2}$, consisting of all those points that are not strongly illuminated by sources (2.24). There are three possibilities: i) $\partial\mathfrak{c}B$ is empty; ii) $\partial\mathfrak{c}B$ contains some interior point (i.e. the point belonging to $\partial\mathfrak{c}B$ with some open neighborhood); iii) $\partial\mathfrak{c}B$ is non-empty finite set, i.e. $\partial\mathfrak{c}B = \{\bar{y}_1, \ldots, \bar{y}_l\}$. The first case i) takes place if and only if $\bar{B}$ is compact in the topology of $D^{n-1}$. Only in this case the sphere $S^{n-2}$ is strongly illuminated by the sources (2.24). Thus the second part of proposition is proved. In the case i) $\text{vol}B$ is finite. For the volume we have

$$\text{vol}B = \int_B d^{n-1}\bar{y} \sqrt{h} = \int_0^1 dr (1-r^2)^{1-n} S_r. \quad (2.34)$$

The ”area” $S_r \to C > 0$ as $r \to 1$ in the case ii) and, hence, the integral (2.34) is divergent. In the case iii)

$$S_r \sim C_1 (1-r)^{2(n-2)} \text{ as } r \to 1 \quad (2.35)$$

($C_1 > 0$) and, so, the integral (2.34) is convergent. Indeed, in the case iii), when $r \to 1$, the ”area” $S_r$ is the sum of $l$ terms. Each of these terms is the $(n-2)$-dimensional ”area” of a transverse side of a deformed pyramid with a top at some point $\bar{y}_k$, $k = 1, \ldots, l$. This multidimensional pyramid is formed by certain parts of spheres orthogonal to $S^{n-2}$ in the point of their intersection $\bar{y}_k$. Hence, all lengths of the transverse section $r = \text{const}$ of the ”pyramid” behaves like $(1-r)^2$, when $r \to 1$, that justifies (2.35). But the unit sphere $S^{n-2}$ is illuminated by the sources (2.24) only in the cases i) and iii). This completes the proof.

To our knowledge, the problem of illumination of convex body in a vector space by point-like sources for the first time was considered in [32, 33]. For the case of $S^{n-2}$ this problem is equivalent to the problem of covering the spheres with spheres [34, 35]. There exists a topological bound on the number of point-like sources $m_+$ illuminating the sphere $S^{n-2}$ [33]:

$$m_+ \geq n. \quad (2.36)$$

Thus, we are led to the following proposition.

**Proposition 2.** When $m_+ < n$, i.e. the number of components with $(u^{\alpha})^2 > 0$ is less than the minisuperspace dimension, the billiard $B$ (2.22) has infinite volume: $\text{vol}B = +\infty$.

In this case there exist an open zones on the sphere $S^{n-2}$ and the oscillatory behaviour near the singularity for cosmological models is absent (we get a Kasner-like behaviour for $t \to -\infty$ [4, 38]).

Remark 1. Let the points (2.24) form an open convex polyhedron $P \subset R^{n-1}$. Then the sources at (2.24) illuminate $S^{n-2}$, if $D^{n-1} \subset P$, and strongly illuminate $S^{n-2}$, if $\overline{D^{n-1}} \subset P$.

### 3 Some examples

#### 3.1 Bianchi-IX cosmology

Here we consider the well-known “mixmaster” model [7, 8] with the metric

$$g_{\text{mix}} \equiv -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^3 \exp[2x^i(t)]e^i \otimes e^i, \quad (3.1)$$
where 1-forms $e^i = e^i_\nu(\zeta)d\zeta^\nu$ satisfy the relations

$$de^i = \frac{1}{2}\varepsilon_{ijk}e^j \wedge e^k,$$

(3.2)

$i, j, k = 1, 2, 3$. The forms $e^i$ are defined on $S^3 \simeq SU(2)$ and are the components of the Morera-Cartan form (see, for example [36]) on $SU(2)$. The Einstein equations for the metric (3.1) are equivalent to the Lagrange equations for the Lagrangian

$$L = L(\gamma, x, \dot{x}) = \frac{1}{2}\mathcal{N}^{-1}G_{ij}\dot{x}^i\dot{x}^j - \mathcal{N}V(x).$$

(3.3)

Here

$$G_{ij} = \delta_{ij} - 1$$

(3.4)

are the components of the minisuperspace metric,

$$V = V_{mix} \equiv \frac{1}{4}(e^{4x^1} + e^{4x^2} + e^{4x^3} - 2e^{2x^1+2x^2} - 2e^{2x^2+2x^3} - 2e^{2x^1+2x^3})$$

(3.5)

is the potential and the Lagrange multiplier is

$$\mathcal{N} = \exp(\gamma - \gamma_0(x)) > 0,$$

(3.6)

where

$$\gamma_0 = \sum_{i=1}^n x^i$$

(3.7)

and $n = 3$.

In $z$-variables [1,2]

$$z^0 = q^{-1}\sum_{i=1}^n x^i, \quad q = [n/(n-1)]^{1/2},$$

(3.8)

$$z^a = [1/(n-a+1)(n-a)]^{1/2}\sum_{j=a+1}^n (x^j - x^a),$$

(3.9)

$a = 1, \ldots, n-1$, where here $n = 3$, we get the Lagrangian (1.1) with 3-vectors

$$u^1 = \frac{4}{\sqrt{6}}(1, 1, -\sqrt{3}), \quad u^2 = \frac{4}{\sqrt{6}}(1, 1, +\sqrt{3}), \quad u^3 = \frac{4}{\sqrt{6}}(1, -2, 0),$$

(3.10)

$$u^4 = \frac{1}{2}(u^1 + u^2), \quad u^5 = \frac{1}{2}(u^1 + u^3), \quad u^6 = \frac{1}{2}(u^2 + u^3),$$

(3.11)

and, consequently,

$$(u^a)^2 = 8, \quad (u^{3+a})^2 = 0,$$

(3.12)
\( \alpha = 1, 2, 3 \). Thus, the conditions (1.5), (1.6) are satisfied. The components with \( \alpha = 4, 5, 6 \) do not "survive" in the approaching to the singularity (see (2.16)). For the vectors (2.24) we have
\[ \vec{v}^1 = (1, -\sqrt{3}), \quad \vec{v}^2 = (1, +\sqrt{3}), \quad \vec{v}^3 = (-2, 0), \]
which is a triangle from Fig. 4 (see also [27]). In this case the circle \( S^1 \) is illuminated by sources located at points \( \vec{v}^i, i = 1, 2, 3 \), but is not strongly illuminated. In agreement with Proposition 1 the billiard \( B \) has a finite volume, but \( \bar{B} \) is not compact.

### 3.2 Prototype model

Now, we consider the model, described by Lagrangian (3.3) with minisupermetric (3.4), \( i, j = 1, \ldots n \), and the potential
\[ V = \frac{1}{4} \sum_{(i,j,k) \in S} \exp \left[ 2 \sum_{i=1}^{n} x^i + 2(x^i - x^j - x^k) \right] \]
where \( S = \{(i, j, k)|i, j, k = 1, \ldots, n; j \neq i \neq k, j < k\}, \ n > 2 \). This model may be considered as a prototype cosmological model describing the behaviour of the solutions to the Einstein equations in the dimension \( D = 1 + n \) near the singularity [22, 23]. The sum in (??) contains \( n(n-1)(n-2)/2 \) terms. In \( z \)-coordinates (3.8), (3.9) we get the Lagrangian (1.1) with \((u^\alpha)^2 = 8 > 0 \) and \( u^\alpha_0 = 2/q > 0 \) for any component \( \alpha \in S \). Thus, the restrictions (1.5) and (1.6) are satisfied.

The corresponding billiard is not compact for all dimensions, and has a finite volume for \( n < 10 \) and infinite volume for \( n \geq 10 \) [38]. This proposition follows from Proposition 1 formulated in terms of inequalities on Kasner parameters [38] and analysis of this inequalities performed in Ref. [23]. For \( n = 4 \) the billiard was studied recently in [37].

### 3.3 Scalar field generalization

Let us consider the Lagrangian (1.1) with \( a, b = 0, \ldots, n \) and \( u^\alpha_0 = 0 \), i.e. the potential does not depend upon the "scalar field" \( \varphi = z^n \). In this case the corresponding billiard has an infinite volume, since at least two points \((0, \ldots, 0, \pm 1)\) ("North and South Poles") on the "Kasner sphere" \( S^{n-1} \) are not illuminated by the sources at the points \( \vec{v}^\alpha \) with \( v^\alpha_0 = 0 \). Thus the oscillatory behaviour in this case is absent. We may expect that the points belonging to the domain of shadow on the sphere \( S^{n-1} \) are integrals of motion in this case. Due to (1.9) this analysis may be also applied to pseudo-Euclidean Lagrange systems (1.7) in the region of negative energy \( E_1 < 0 \).

### 4 Discussions

Thus, we obtained the "billiard representation" for pseudo-Euclidean Toda-like system (1.1), (1.2) and proved the geometrical criterion for the finiteness of the billiard volume and the
compactness of the billiard (Proposition 1, Sec. 2). This criterion may be used as a rather
effective (and universal) tool for the selection of the cosmological models with an oscillatory
behavior near the singularity.

Remark 2. It may be shown that the condition (1.5) may be weakened by the following one
\[ u_0^\alpha > 0, \quad \text{if} \quad (u^\alpha)^2 \leq 0. \tag{4.1} \]

In this case there exists a certain generalization of the set \( B(u^\alpha) \) from (2.23) for arbitrary
\( u_0^\alpha \ ((u^\alpha)^2 > 0) \). The Proposition 1 (Sec. 2) should be modified by including into consideration
the sources at infinity (for \( u_0^\alpha = 0 \)) and ”anti-sources” (for \( u_0^\alpha < 0 \)). For ”anti-source” the
shadowed domain coincides with the illuminated domain for the usual source (with \( u_0^\alpha > 0 \)).
In this case we deal with the kinematics of tachyons (\( \vec{v}^\alpha \) are the velocities of tachions).

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