Abstract

We discuss level crossing of the free-energy of TAP solutions under variations of external parameters such as magnetic field or temperature in mean-field spin-glass models that exhibit one-step Replica-Symmetry-Breaking (1RSB). We study the problem through a generalized complexity that describes the density of TAP solutions at a given value of the free-energy and a given value of the extensive quantity conjugate to the external parameter. We show that variations of the external parameter by any finite amount can induce level crossing between groups of TAP states whose free-energies are extensively different. In models with 1RSB, this means strong chaos with respect to the perturbation. The linear-response induced by extensive level crossing is self-averaging and its value matches precisely with the disorder-average of the non self-averaging anomaly computed from the 2nd moment of thermal fluctuations between low-lying, almost degenerate TAP states. We present an analytical recipe to compute the generalized complexity and test the scenario on the spherical multi-\( p \) spin models under variation of temperature.

1 Introduction

Mean-Field Spin-Glass Models like the Sherrington-Kirkpatrick model or the \( p \)-spin models are known to have a very complicated phase space with many metastable states. An important physical consequence is that the system fluctuates not only within each equilibrium state but also among different equilibrium states whose free-energies are sufficiently low and close to each other. The non-trivial nature of the fluctuations among the low-lying states have been fully uncovered by the powerful theoretical tools, i.e. the TAP, replica and cavity methods [1]. The presence of many states leads also to the so called chaos problem, i.e. the question of whether equilibrium states at different values of the external parameters such as magnetic field or temperature are correlated or not [2]. In the present paper we discuss the problem from a TAP perspective. The states are usually identified with solutions of the TAP equations [1]: if a given TAP solution has a non-vanishing Hessian it can be continued analytically upon a change of the external parameter. We will say that two states at different external parameter coincides if one is the analytical continuation of the other. A question related to chaos is to know whether the equilibrium TAP states at a given values of, say, magnetic field \( h \) are the same (in the sense of the analytical continuation) of those at a different value of \( h \). If this is the case chaos is certainly not present. Instead if the states are not the same we derive chaos provided we assume that different states are not correlated; this is surely the case in 1RSB models (because by definition different states are minimally correlated) but can be more complicated for FRSB models. In 1RSB systems the TAP states with free energies below the threshe values have a non-vanishing Hessian therefore each of them can be analytically continued upon changing the external parameters. Therefore the equilibrium states at the new value of the the external parameters must have been already present as some TAP states at the old values and they can be identified considering the evolution of the old TAP states by the variation of the parameters. We show that to describe the evolution of the TAP states we must
consider a generalized complexity which represents the density of TAP states at a given value of free-energy (per spin) \( f \) and of an extensive quantity (per spin) \( y = Y/N \), where \( N \) is the number of spins, conjugate to the external parameter \( h_y \) to be varied, i.e. magnetization for the magnetic field and entropy for the temperature.

In general at fixed values of the external parameters the typical states with a given value of \( f \) have a definite value of \( y \) but there are also TAP states with the same \( f \) and different values of \( y \) although with lower complexity than the typical ones. Thus in general the function \( \Sigma(f, y) \) is non-trivial. Assuming the existence of this function we draw the following conclusions. We prove that variation of the external parameters by any finite amount induces level crossing of the free-energies of TAP states at extensive levels. Thus the equilibrium TAP states at different values of the external parameter are different. Furthermore from the function \( \Sigma(f, y) \) we can compute the induced inter-state linear-response; it turns out to be self-averaging and its value matches precisely with the value predicted by the analysis of spontaneous thermal fluctuations through the FDT. We present an analytical recipe to compute the generalized complexity and present explicit calculation for some specific cases. In particular we show the existence of the function \( \Sigma(f, m) \) for generic FRSB and 1RSB models. We also consider the entropy-free-energy function \( \Sigma(f, s) \) (related to the behavior under temperature changes) in 1RSB spherical \( p \)-spin models. This function exists for 1RSB spherical model with multiple \( p \)-spin interaction implying chaos in temperature while its support shrinks to a single line in the \((f, s)\) plane in the limit of a single \( p \)-spin interaction consistently with the absence of chaos in temperature in this case.

The problem of level-crossing of TAP states has been recognized in earlier works, for instance in Ref. \[4, 5\] and others. In particular level crossing of individual TAP states upon infinitesimal changes in the values of the magnetic field \( (\delta h = O(1/\sqrt{N})) \) was observed. In the present paper we are interested instead with evolution of TAP states under small but finite changes in the external parameters, i.e. changes that induce extensive variation of the free energy. We want to know if the set of equilibrium states at a given value of the external parameters contains as a whole the same set of equilibrium states at different values; note that this does not exclude the possibility of some internal reshuffling of the relative weights of the states. If only the latter happens we would have just some mild, sub-extensive level crossing between the states but no chaos.

More recently Krzakala and Martin (KM) \[6\] studied the level crossing phenomena in an extended version of the random energy model\[7\] in which each state has a random energy and a random extensive variable conjugate to an external parameter, such as temperature. Both random variables are assumed to follow Gaussian distributions. Based on the phenomenological model they provided a very interesting general phenomenology on the chaos problem. The generalized complexity we study in the present paper provides a firmer ground for their picture.

The plan of the paper is the following. In section 2, we review previous works related to the present paper. In section 3, we introduce the generalized complexity. We discuss its evolution under variation of external parameters and explain its physical consequences. In section 4 we present an explicit calculation of the evolution of the generalized complexity of a spherical multi-\( p \) spin model under variation of the temperature. At the end we discuss our results.

2 Intra-state and Inter-state Susceptibility

A well known effect of RSB is the difference between the susceptibility inside a state and the true thermodynamical susceptibility. For example, the magnetic susceptibility inside a state \( \alpha \) in zero magnetic field is given by

\[ \chi_\alpha = \beta(1 - q_{EA}) \] (1)

where \( q_{EA} \) is the Edwards-Anderson order parameter, while the actual magnetic susceptibility of the system is given, according to the Parisi solution \[11\], by:

\[ \chi = \beta(1 - \bar{q}) \] (2)

where \( \bar{q} \) is the average of the overlap between replicas. De Dominicis and Young \[8\] have shown that this is a consequence of the presence of many states, so that in the application of the fluctuation-dissipation-theorem there is a new term which takes into account the fluctuations of the magnetizations over different
states. They assumed that the free energy of the system is given by a sum over all TAP solutions weighted with their free energy:

\[ F = -\frac{1}{\beta N} \sum_{\alpha} e^{-\beta N f_{\alpha}} \]  

\[ \chi_y = \frac{\partial^2}{\partial h_y^2} \frac{1}{\beta N} \ln \sum_{\alpha} e^{-\beta N f_{\alpha}} = -\left\langle \frac{\partial^2 f_{\alpha}}{\partial h_y^2} \right\rangle + \beta N \left[ \langle y_{\alpha}^2 \rangle - \langle y_{\alpha} \rangle^2 \right] \]  

where the square brackets mean Boltzmann average over the states

\[ \langle O_{\alpha} \rangle = \frac{\sum_{\alpha} e^{-\beta N f_{\alpha}} O_{\alpha}}{\sum_{\alpha} e^{-\beta N f_{\alpha}}} \]  

and \( y_{\alpha} \) is the value on state \( \alpha \) of the parameter conjugated to \( h_{\alpha} \) (e.g. magnetization or entropy per spin):

\[ y_{\alpha} = \frac{\partial f_{\alpha}}{\partial h_{y_{\alpha}}} \]  

The first term is the susceptibility of a state while the second term is the fluctuation over the states of the parameter \( y_{\alpha} \). The first term gives a contribution of \( \beta(1 - q_{EA}) \) while the second term can be written as \( \beta(q_{EA} - \overline{q}) \) so that the correct result \( \chi \) for the susceptibility is recovered. Another interesting feature of the susceptibility is that the susceptibility on a given sample, defined in terms of correlation functions of the spontaneous thermal fluctuations is not self-averaging. This has been pointed out by Young, Bray and Moore in Ref. 5, where they studied the magnetic susceptibility, the profile converges to a unique limiting curve in the thermodynamic limit as \( \lim_{N \to \infty} m_J(h) \). So \( m(h) \) is self-averaging and thus the linear-susceptibility defined as \( \chi = \lim_{\Delta h \to 0} \Delta m(h)/\Delta h \) is also self-averaging. It is reasonable to expect that \( \chi \) matches with disorder average of \( \chi_J \). Note that \( \delta h \) and \( \Delta h \) used in the definitions of \( \chi_J \) and \( \chi \) are at completely different scales. While \( \delta h \) must be chosen smaller than the typical spacing between the steps, which is likely to be of order \( O(1/\sqrt{N}) \), \( \Delta h \) can be chosen to be arbitrary small but fixed when the thermodynamic limit \( N \to \infty \) is taken.

According to the following argument by G. Parisi 9, the difference between the susceptibility and the intra-state susceptibility in general implies that the equilibrium states at different values of the external
parameter cannot be the same. Indeed from eq. (2), it follows that the magnetization of the equilibrium
states in presence of a small but finite magnetic field $h$ becomes,

$$m \simeq \beta (1 - q) h.$$  

(8)

On the other hand the analytical continuation of the old equilibrium states would develop a smaller magn-
etization $\beta (1 - q_{EA}) h$ where $q_{EA}$ is the Edwards-Anderson order parameter. Therefore the equilibrium states in presence of a small but finite field $h$ had a non-zero magnetization per spin even in the absence of the field

$$m \simeq \beta (q_{EA} - q) h.$$  

(9)

Therefore the new equilibrium states cannot be the analytical continuation of the old equilibrium states.

Here an important point is that $h$ is chosen arbitrary small but fixed when the thermodynamic limit $N \to \infty$ is taken, i.e. $h$ is at the scale of $\Delta h$ and not of $\delta h$. In particular at this scale we have no problems of lack of self-averageness. As we explained in the introduction, we are interested in extensive level crossing therefore in the following we are going to consider always variations in the external parameter at scale $\Delta h$.

Note that this argument can be applied whenever the susceptibility to a given field $h_y$ is different from the intrastate susceptibility, i.e. whenever the fluctuation of the conjugated parameter $y$, i.e. the second term in eq. (4), is not zero. The fluctuations obviously vanish if there is only one state.

### 3 Extensive Level Crossings

The presence of metastable states with extensive non-zero magnetization in zero field may appear rather
counter-intuitive, however in the TAP context their number can be computed and one can show that there
is an exponential number of solutions with non zero magnetization, although with a smaller complexity
with respect to the solutions with zero magnetization. In order to have a deeper look into the evolution of
the phase space we consider a generalized complexity i.e. the logarithm of the number of TAP solutions
with given values of the free energy and of the magnetization:

$$\Sigma(f, m) = \frac{1}{N} \ln \sum_\alpha \delta(m_\alpha - m) \delta(f - f_\alpha).$$

(10)

We want to study the evolution of the curve $\Sigma(f, m)$ under the application of a magnetic field.

![Figure 1: Complexity of the TAP solution as a function of the free energy and magnetization per spin](image)

In figure 1 we show a schematic plot of the function $\Sigma(f, m)$ near the lower-band edge where the
equilibrium states are located. Near the equilibrium states and below the critical temperature, the function
can be expanded as:

$$\Sigma(f, m) = \beta x f - a m^2$$

(11)

where $x$ is the Parisi parameter and $a$ is some parameter to be determined, and we have shifted the free
energy so that the equilibrium free energy in zero field is zero. Let us emphasize that the function $\Sigma(f, m)$
is, by definition, an extensive self-averaging quantity. Now we want to consider the evolution of the states on this curve when the small field \( h \) is switch on. The free energy of a state will be modified according to:

\[
f'_\alpha = f_\alpha + \frac{\partial f_\alpha}{\partial h} h + \frac{1}{2} \frac{\partial^2 f_\alpha}{\partial h^2} h^2 + O(h^3) ;
\]

by definition we have \( m_\alpha = -\frac{\partial f_\alpha}{\partial h} \) therefore the new free energy of the set of TAP solutions with given values of \( f \) and \( m \) is given by:

\[
f' = f - mh + \frac{1}{2} \frac{\partial^2 f}{\partial h^2} h^2 + O(h^3)
\]

their magnetization is given by:

\[
m' = m + \frac{\partial^2 f_\alpha}{\partial h^2} h + O(h^2)
\]

Expressing through eq. (11) the free energy in terms of the complexity \( c \) and the magnetic field we get:

\[
f = \frac{c}{\beta x} + bm^2
\]

where we defined

\[
b = \frac{a}{\beta x}.
\]

Putting this expression into eq. (13), we obtain the new value (after the field is switched on) of the free energy of the states that in zero field had complexity \( c \) and magnetization \( m \):

\[
f' = bm^2 - mh + \frac{1}{2} \frac{\partial^2 f_\alpha}{\partial h^2} h^2 + O(h^3) + \frac{c}{\beta x}
\]

The new equilibrium states are those that minimize \( f' \). First of all we note that the minimum with respect to \( c \) is obtained for \( c = 0 \), this is consistent with the fact that the equilibrium states under any circumstance below the critical temperature should always have zero complexity, and therefore the zero-field TAP states that are candidate to become equilibrium states in a field must have zero complexity. Thus we are interested in the evolution of the equilibrium states along the zero-complexity line \( f = bm^2 \). In order to minimize \( f' \) with respect to \( m \) we note that the third term in eq. (17) in principle depends on \( m \), but for values of \( m \) of order \( O(h) \) this variation is basically a third order effect, therefore at second order in \( h \) it can be considered as a constant. Then we obtain:

\[
\frac{df'}{dm} = 2bm - h = 0 \quad \rightarrow \quad m = \frac{h}{2b}
\]

Thus the equilibrium states in presence of a field are the states that had a non-zero magnetization \( m = h/2b \) in zero field and the evolution of the TAP states is driven by extensive level crossing, indeed the free energy difference between these states was \( \Delta f = h^2/4b \) in zero field while it becomes negative \( \Delta f = -h^2/4b \) in presence of a field. This is the same result obtained above: in presence of a field the TAP solutions with lowest free energy are not the continuation of the TAP solutions with lowest free energy in zero-field. Accordingly the magnetization is given by:

\[
m' = \frac{h}{2b} - \frac{d^2 f}{dh^2} h + O(h^2)
\]

and the full linear-susceptibility is given by:

\[
\chi = \frac{1}{2b} - \frac{d^2 f}{dh^2}
\]

In 1RSB models TAP states with extensive difference in the free-energy must have zero overlap with respect to each other, i.e. they are totally uncorrelated. Thus extensive level-crossing automatically means strong chaos in 1RSB systems.

The basic assumption of this derivation is the existence of the zero-complexity curve \( f = bm^2 \), which follows from the existence of the function \( \Sigma(f, m) \). Once the existence of this function is assumed the
non-trivial result is that the evolution of the TAP states under a change in the magnetic field is driven by extensive level crossing. As such the previous derivation can be extended to any couple \((h_y, y)\) representing an external field and its conjugated extensive variable, e.g. temperature and entropy, provided the zero-complexity curve \(f = b y^2\) exists. This assumption is equivalent to the assumption of the previous section that the parameter \(y_a\) fluctuates over the states. The connection with the result of the previous section can be established also at a quantitative level by showing that the two expressions for the susceptibility eq. (20) and eq. (4) are equivalent. In order to do that we introduce the function:

\[
\Phi(\lambda_y) = \frac{1}{N} \ln \sum_{\alpha} e^{-\beta N f_\alpha + \lambda_y N y_a}
\]

(21)

This is a summation over all TAP states with a weight which depends also on the value of \(Y = N y\), when \(\lambda_y = 0\) it reduces to the Boltzmann weight such that \(\Phi(0)\) is minus the free energy. From the definition follows that

\[
\frac{\partial^2 \Phi}{\partial \lambda_y^2} \bigg|_{\lambda_y=0} = N \left( \langle y_a^2 \rangle - \langle y_a \rangle^2 \right)
\]

(22)

On the other hand using the generalized complexity through eq. (15) we can write

\[
\Phi(\lambda) = \max_{c,y} \left( c - \beta (b y^2 + \frac{c}{\beta x}) + \lambda_y y \right).
\]

(23)

Again the maximum is at \(c = 0\) and the maximization with respect to \(y\) gives

\[
\frac{\partial \Phi}{\partial \lambda} = \langle y \rangle_{\lambda_y} = \frac{\lambda_y}{2b\beta}
\]

(24)

which is linear with respect to \(\lambda_y\). Using eqs. (21) and (22) we get:

\[
\beta \frac{\partial^2 \Phi}{\partial \lambda^2} \bigg|_{\lambda_y=0} = \frac{1}{2b}.
\]

(25)

This equation together with eq. (22) prove the equivalence between eqs. (20) and (4) for the susceptibility, that can be written as:

\[
\chi_y = \chi_{y\alpha} + \beta \frac{\partial^2 \Phi}{\partial \lambda_y^2} \bigg|_{\lambda_y=0}
\]

(26)

where the first term is the generalized susceptibility inside a state, e.g. the specific heat if \(h_y\) is the temperature and \(y\) is the entropy. Notice that we do not need to compute the intra-state susceptibility to infer the picture, it is sufficient to check the existence of the zero-complexity line.

In appendix A we report the general method to compute the function \(\Phi(\lambda)\) for a generic model. In particular in the case of the magnetic field we can show that the second derivative of \(\Phi(\lambda)\) has the correct value needed to recover the right TAP susceptibility in either FRSB and 1RSB models:

\[
\frac{\partial^2 \Phi}{\partial \lambda_m^2} \bigg|_{\lambda_m=0} = q_{EA} - \frac{1}{4}
\]

(27)

Note that the derivation of this section assumes that the zero complexity curve \(f = b m^2\) is a self-averaging smooth function. Of course at any finite \(N\) this curve is actually made of points therefore on sufficiently small \(m\) scale (i.e. scales that go to zero with some proper power of \(1/N\)) we expect it to have rapid sample-to-sample fluctuations around its sample-independent average. These fluctuations and the corresponding lack of self-averaging in the r.h.s. of eq. (22) are irrelevant at the much larger scales which we consider and to which the derivation of the present section applies.

4 Spherical \(p\)-spin Models

In this section we show that picture of the previous section applies to 1RSB spherical \(p\)-spin models. In particular the presence of chaos in temperature can be univoquely associated to the behavior of the
zero-complexity line as a function of the free energy and of the entropy. Following [13] we consider the Hamiltonian:

\[ H = - \sum_{i_1 < \ldots < i_p}^N J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} - \epsilon \sum_{l_1 < \ldots < l_r}^N K_{l_1 \ldots l_r} \sigma_{l_1} \ldots \sigma_{l_r} , \]  

(28)

where the spins \( \sigma_i \) are subject to the spherical constraint \( \sum_i \sigma_i^2 = N \), and the Gaussian random couplings \( J_{i_1 \ldots i_p} \) and \( K_{l_1 \ldots l_r} \) have variance \( p^2 / 2N^{p-1} \) and \( r^2 / 2N^{r-1} \). The \( p + r \) spherical models may display a nontrivial thermodynamic behavior when \( p \geq 3 \) and \( r = 2 \): in that case there is a transition between a 1RSB thermodynamic phase (low \( \epsilon \)), to a FRSB phase (large \( \epsilon \)) [13].

On the contrary, if both \( p \) and \( r \) are strictly larger than two, the model is expected to have a normal 1RSB thermodynamic behavior. This is the case we will analyze. In particular we have studied numerically the case \( p = 3 \) and \( r = 4 \). The TAP free energy density is [13]:

\[ \beta f_{TAP} = - \frac{1}{\beta N} \sum_{i_1 < \ldots < i_p}^N J_{i_1 \ldots i_p} m_{i_1} \ldots m_{i_p} - \epsilon \frac{1}{\beta N} \sum_{l_1 < \ldots < l_r}^N K_{l_1 \ldots l_r} m_{l_1} \ldots m_{l_r} - \frac{1}{2} \log(1 - q) \]

\[ - \frac{\beta^2}{4} \left[ (p-1)q^p - pq^{p-1} + 1 \right] - \epsilon^2 \frac{\beta^2}{4} \left[ (r-1)q^r - rq^{r-1} + 1 \right] , \]

(29)

where \( m_i = \langle \sigma_i \rangle \) are the local magnetizations, and \( q \) is the self-overlap of a state, \( q = \sum_i m_i^2 / N \). In the case of the single \( p \)-spin interaction [15] [17] it is straightforward to see that there is no chaos in temperature. Indeed by writing \( m_i = q^{1/2} \tilde{s}_i \) where \( \tilde{s}_i \) is the vector of the angular variables normalized to one, we see that the TAP equations for the angular variables do not depend on the temperature, therefore the ordering of the states does not change in temperature. The decomposition of the free energy in angular part and overlap part breaks down if the model have more than a single \( p \)-spin interaction and this could lead to chaos in temperature. In particular in [13] the dynamical evolution under temperature changes of the TAP states was considered between the dynamical and the critical temperature. We note that with some modification the present picture of extensive level crossing can extended also in this region of temperatures where the complexity of the equilibrium states is finite. On the other hand chaos in temperature in the \( p + r \) model below \( T_c \) can be proven considering the free energy shift between two real replicas forced to have a given value of the overlap [19]. Here we want to show that this result can be recovered through the study of the entropic zero-complexity line.

The computation of the complexity of the model [20] can be done through standard methods like those sketched in the previous section and was presented (up to order \( \epsilon^3 \)) in [13]. The complexity at fixed value of the free energy can be obtained extremising the following effective action with respect to the parameter \( B, T, q \) and \( u \):

\[ \hat{S} = \beta u \left[ g(q) + \epsilon^2 h(q) - f \right] + \left( B^2 - T^2 \right) \left[ \frac{1}{4} p(p-1) \beta^2 q^{p-2} + \epsilon^2 \frac{1}{4} r(r-1) \beta^2 q^{r-2} \right] \]

\[ - \frac{1}{2} \log \left( \frac{1}{2} \beta^2 pq^{p-2} + \frac{1}{2} \epsilon^2 \beta^2 r^{r-2} \right) - \log T + \frac{1}{4} \beta^2 u^2 (q^p + \epsilon^2 q^r) - \frac{1}{2} \]

\[ + \frac{1}{4} \beta^2 B^2 \left( pq^{p-2} + \epsilon^2 r^{r-2} \right) + \beta (B + T) \left[ A(q) + \epsilon^2 C(q) \right] + \frac{1}{2} \beta^2 uB \left( pq^{p-1} + \epsilon^2 r^{r-1} \right) , \]

(30)

and where we used the following definitions,

\[ g(q) = - \frac{1}{2} \log(1 - q) - \frac{1}{4} \left[ (p-1)q^p - pq^{p-1} + 1 \right] \]

(31)

\[ h(q) = - \frac{1}{4} \left[ (r-1)q^r - rq^{r-1} + 1 \right] \]

(32)

\[ \frac{\partial g}{\partial m_i} = A(q) m_i \]

(33)

\[ \frac{\partial h}{\partial m_i} = C(q) m_i \]

(34)
In order to compute the complexity at given value of the free energy \( f \) and of the entropy \( s \) we must add to \( (30) \) a term \( \lambda_s s - \lambda_s s(q, \beta) \) and extremise with respect to \( \lambda_s \). The function \( s(q, \beta) \) is the complexity of a given solution which can be obtained from eq. 29

\[
\frac{d\Sigma}{d\epsilon} = \frac{1}{2} \log(1 - q) - \frac{\beta^2}{4} [(p - 1)q^p - pq^{p-1} + 1] - \epsilon^2 \frac{\beta^2}{4} [(r - 1)q^r - rq^{r-1} + 1],
\]

The corresponding saddle point equations can be solved numerically. As noted in [13] there are two solutions of the saddle point equations, one that is BRST (Becchi-Rouet-Stora-Tyutin) symmetric and another that is not. The lower band edge is described by the BRST solution. Numerically we start from this solution and consider the complexity of states with entropy different from the equilibrium one. Solving the SP equations with respect to \( B, T, q, u, \lambda_s \) with extra constraint that the complexity is zero yields the zero-complexity curve.

In figure 2 we plot the entropic zero-complexity line for a 3 + 4 model at temperature \( T = .35 < T_c \) and at values \( \epsilon = .1 \) and \( \epsilon = .2 \). Numerically the second derivative of \( f(s) \) in \( s = s_{eq} \) diverges as \( 1/\epsilon^2 \) for \( \epsilon \to 0 \). In this limit, the angular variables can be factorized and the entropy of the states is univocally determined by their free energy; correspondingly the two branches of the zero complexity curve join on a single line \( s = s_{top}(f) \), that is the typical complexity of the states with free energy \( f \). Note that since the divergence is proportional to \( \epsilon^2 \) it is consistent to consider the action \( (30) \) which is valid at \( O(\epsilon^2) \). From the existence of the zero-complexity curve follows that the dominant TAP states at different temperatures are different. This implies chaos in temperature because in a 1RSB system different states have vanishing mutual overlap. In this context the disappearance of chaos in the limit \( \epsilon \to 0 \) is determined by the divergence of the second derivative of the zero-complexity line.

![Figure 2: Zero-complexity line of the free energy as a function of the entropy for the 3 + 4 spherical model with \( \epsilon = .1 \) (dashed line) and \( \epsilon = .2 \) (continuous line), the second derivative of \( f(s) \) in \( s = s_{eq} \) diverges as \( 1/\epsilon^2 \) for \( \epsilon \to 0 \), in this limit the model has a single \( p \)-spin interaction and chaos in temperature disappears.](image)

5 Discussion

Our approach applies to all situation in which TAP states at a given value of some external parameter \( h_y \) (e.g. temperature or magnetic field) can be continued analytically at different values of \( h_y \). If this is the case the knowledge of the states at given value of \( h_y \) is sufficient to determine the equilibrium values of extensive variable \( y = Y/N \) in a definite range of values. We have shown that this can be done studying the generalized complexity \( \Sigma(f, y) \). In particular our approach applies to 1RSB models because the Hessian of the equilibrium TAP states is non-vanishing. In 1RSB models we could also establish a connection between level-crossings and chaos. Thus our results provides firmer grounds for the phenomenological picture proposed by Krzakala and Martin in [4]. In this context absence of chaos with respect to the external parameter \( h_y \) (magnetic field or temperature) appears when the support of the function \( \Sigma(f, y) \) \( (y \) is the parameter conjugated to \( h_y ) \) shrinks to a single line in the \( (f, y) \) plane.

The application of our approach to FRSB is complicated by the fact that the equilibrium TAP states are marginal so in principle we cannot be sure that they can be continued. However one could study the
zero-complexity line $\Sigma(f, y) = 0$. In the case of magnetic field this curve certainly exists and have the correct slope $q_{EA} - \overrightarrow{\tau}$. It would be interesting to check the existence of the zero-complexity line for the entropy. This is a further motivation to obtain the quenched solution for the complexity in FRSB model. Provided the zero-complexity lines exists in FRSB model for various perturbations it would be interesting to know the stability of the corresponding states, we suspect that they are not-marginal. However in FRSB the connection with chaos is less clear. In this respect, it would be interesting to check the existence or not of the entropic zero-complexity curve of the FRSB spherical model which is the only FRSB model known to be non-chaotic in temperature [23] at variance with the SK model [25].

Starting from the function $\Sigma(f, y)$ we could obtain the total linear-susceptibility using the level crossing argument. Indeed to obtain a description of the evolution of the states at first order in $h_y$ it was sufficient to consider $\Sigma(f, y)$. To obtain the next order we must consider the complexity $\Sigma(f, y, \chi_y)$, where $\chi_y$ is the intrastate susceptibility associated to the field $h_y$. Then the associated zero-complexity line $f = f(y, \chi_y)$ must be used in eq. (17). Extremising with respect to $y$ and $\chi_y$ we can obtain the value of third derivative of the TAP free energy with respect to the external field $h_y$. Higher orders are obtained in the same way, in general to obtain the $k$-th derivative of the TAP free energy we need $\Sigma(f, y, \chi, \ldots, \chi^{(k-1)})$, i.e. the complexity as a function of the intrastate susceptibilities up to order $k - 1$.

In the present paper we focused on the evolution of density of TAP states under variation of external parameters over a small but finite range $\Delta h$. As discussed in section 2, if we go down to scale of $\delta h \sim O(1/\sqrt{N})$, we will observe individual level-crossings whose characters are strongly non-self-averaging. Presumably this is relevant for problems of heterogenous thermal fluctuations and responses at mesoscopic scales [20] [10] [21], some of which have now become accessible experimentally. Further investigation of the intermediate scales between $\delta h$ and $\Delta h$ will be interesting in this respect [11].

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Appendix A

In this appendix we show how to compute the function $\Phi(\lambda)$ which is basically the Legendre transform of the zero-complexity curve $f = by^2$. In the following we assume that there exist a local function $y(m_j, q_{EA})$ such that the parameter $y$ can be expressed as $y_\alpha = \sum_j y(m_j^\alpha, q_{EA})/N$. This includes the case of the magnetization and of the entropy. The computation of the function $\Phi(\lambda)$ can be done following standard techniques for computing averages over TAP solutions, we present the result in the case of the SK model and skip the details of the derivation, which are largely described in the literature (see e.g. Ref. [12] [23]). In order to further simplify the presentation we report the expression of $\Phi(\lambda, 0)$ defined as:

$$\Phi(\lambda_y, 0) = \frac{1}{N} \ln \rho \ ; \ \rho \equiv \sum_{\alpha} e^{\lambda_y N y_\alpha}$$

The quenched disorder average of $\Phi(\lambda_y, 0)$ can be computed through the replica method:

$$\overline{\Phi(\lambda_y, 0)} = \lim_{n \to 0} \frac{1}{n} \ln \overline{\rho^n}$$

Using the supersymmetric formulation of Ref. [12] the disorder average of $\overline{\rho^n}$ can be expressed as an integral over eight macroscopic bosonic and fermionic variables $\Theta \equiv \{r_{ab}, \delta_{ab}, \lambda_{ab}, \theta_{ab}, \rho_{ab}, \overline{\tau}_{ab}, \mu_{ab}\}$

$$\overline{\rho^n} = \int d\Theta \exp[N\Sigma_1^{(n)} + N\Sigma_2^{(n)}]$$

Where the action is specified by:

$$\Sigma_1^{(n)} = -\lambda_{ab} q_{ab} - \frac{r_{ab}^2}{2 \beta^2} + \frac{t_{ab}^2}{2 \beta^2} + \overline{\tau}_{ab} \mu_{ab} + 2 \overline{\tau}_{ab} \rho_{ab} + \overline{\tau}_{ab} \rho_{ab}$$

$$\Sigma_2^{(n)} = -\frac{1}{2} \lambda_{ab} q_{ab} + \frac{1}{2} \overline{\tau}_{ab} \rho_{ab} + \overline{\tau}_{ab} \rho_{ab}$$
and

\[ \Sigma_2^{(n)} = \log \left[ \int \prod_a dm_a dx_a d\psi_a d\bar{\psi}_a \exp \left[ x_a \phi_1(q_{aa}, m_a) + \bar{\psi}_a \psi_a \phi_2(q_{aa}, m_a) + \frac{q_{ab} \beta^2 x_a x_b}{2} + r_{ab} m_a x_b + t_{ab} \bar{\psi}_a \psi_b + \lambda_{ab} m_a m_b + \mu_{ab} \beta m_a \psi_b - \bar{\psi}_a \psi_a \gamma_{ab} \beta - \rho_{ab} \beta x_a \psi_b - \bar{\psi}_a x_b \gamma_{ab} \beta + \lambda_y y(m_a, q_{aa}) \right] \right] \]  

(40)

and the functions \( \phi_1(q, m) \) and \( \phi_2(q, m) \) are given by:

\[ \phi_1(q, m) = \beta^2 (1 - q) m + \tanh^{-1}(m), \]  

(41)

\[ \phi_2(q, m) = \beta^2 (1 - q) + \frac{1}{1 - m^2}. \]  

(42)

Note that the only modification with respect to the standard computation (i.e. \( \lambda_y = 0 \)) is in the presence of the term \( \lambda_y y(m_a, q_{aa}) \) in the integral in \( \Sigma_2^{(n)} \). The second derivative of \( \Phi(\lambda_y, 0) \) at \( \lambda_y = 0 \) is given by:

\[ \frac{n \partial^2 \Phi}{\partial \lambda_y^2} = \left\langle \frac{\partial^2 \Sigma_2^{(n)}}{\partial \lambda_y^2} \right\rangle + N \left[ \left\langle \left( \frac{\partial \Sigma_2^{(n)}}{\partial \lambda_y} \right)^2 \right\rangle - \left\langle \frac{\partial \Sigma_2^{(n)}}{\partial \lambda_y} \right\rangle^2 \right\rangle \]  

(43)

Where the square brackets mean average with respect to the action eq. (35) and:

\[ \frac{\partial \Sigma_2^{(n)}}{\partial \lambda_y} = \langle \sum_a y(m_a, q_{aa}) \rangle \]  

(44)

\[ \frac{\partial^2 \Sigma_2^{(n)}}{\partial \lambda_y^2} = \langle \sum_{ab} y(m_a, q_{aa}) y(m_b, q_{ab}) \rangle - \langle \sum_a y(m_a, q_{aa}) \rangle^2 \]  

(45)

Where the double brackets mean average performed with respect to the integrand in the definition of \( \Sigma_2^{(n)} \). The previous averages must be evaluated at \( \lambda_y = 0 \). The action (35) can be evaluated through a saddle-point method. Note that in general to evaluate the the second term in eq. (44) we need to study the Hessian of the saddle-point which in general is very complicated. However in the case of magnetic field perturbation in zero field we have \( y(m_a, q_{aa}) = m_a \) and \( \partial \Sigma_2^{(n)} / \partial \lambda_y \) at \( \lambda_y = 0 \) is identically zero for symmetry reasons, therefore only the first term survives and we don’t need to compute the Hessian of the SP. Thus only the first term in the r.h.s. of eq. (43) contributes to the second derivative of \( \Phi(\lambda_y, 0) \) and we recover the result

\[ \frac{\partial^2 \Phi}{\partial \lambda_y^2} \bigg|_{\lambda_y=0} = \lim_{n \to 0} \frac{1}{n} \left( \langle \sum_{ab} m_a m_b \rangle \right) = q_{EA} - \overline{q} \]  

(46)

Where we have used that SP equations \( q_{ab} = \langle \sum_{ab} m_a m_b \rangle \)

**Appendix B**

In this appendix we show how to compute the sample-to-sample fluctuation of the TAP susceptibility eq. (1) following the similar computation for the true thermodynamic susceptibility. The first term is the intrastate susceptibility and does not fluctuate with the disorder, analytically this is a consequence of the fact that it is a single replica quantity (15). The second term is the fluctuation of the total magnetization over all TAP solutions \( N^{-1} \langle \sum_{ij} (m_i m_j) - \langle m_i \rangle \langle m_j \rangle \rangle \), in order to check if it is self-averaging we compute the average of its square. The computation can be done along the lines of the same replica computation of the thermodynamic susceptibility fluctuations (5). The objects one needs to compute are averages of the form \( \langle m_i,1 m_j,1 \rangle_{TAP} \langle m_i,2 \rangle_{TAP} \langle m_j,3 \rangle_{TAP} \) where 1, 2, 3 are different replicas with the same realization of
the disorder where the square brackets mean summation over all TAP states with the Boltzmann weight.

Introducing source fields $\lambda_i$ in the definition of $\rho \equiv \sum e^{-\beta E_a + \lambda m_{i,a}}$ this can be written as:

$$\langle m_{i,m_{j,1}} \rangle_{\text{TAP}} \langle m_{i,2} \rangle_{\text{TAP}} \langle m_{j,3} \rangle_{\text{TAP}} = \frac{1}{\rho_1^4 \rho_2 \rho_3} \left( \frac{\partial^4}{\partial \lambda_{i,1} \partial \lambda_{j,1} \partial \lambda_{i,2} \partial \lambda_{i,3}} \right) \rho_1^{-1} \rho_2^{-1} \rho_3^{-1}$$  \hspace{1cm} (47)

Now we multiply the quantity in the above disorder average by a factor $\rho^n$ and divide the whole average by $\rho^n$; taking the limit $n \to 0$ the result does not change, therefore we can write:

$$\langle m_{i,m_{j,1}} \rangle_{\text{TAP}} \langle m_{i,2} \rangle_{\text{TAP}} \langle m_{j,3} \rangle_{\text{TAP}} = \lim_{n \to 0} \frac{\partial^4}{\partial \lambda_{i,1} \partial \lambda_{j,1} \partial \lambda_{i,2} \partial \lambda_{i,3}} \ln \rho^n$$  \hspace{1cm} (48)

the expression of $\rho^n$ in presence of the source field can be computed as in appendix A, the result is:

$$\rho^n = \int d\Theta \exp[N\Sigma_{i}^{(n)} + N\Sigma_{2}^{(n)}] \frac{1}{\langle \langle e^{\lambda_{i,1}m_{i,1} + \lambda_{i,3}m_{i,3}} \rangle \rangle_{\langle \langle e^{\lambda_{j,1}m_{j,1} + \lambda_{j,2}m_{j,2}} \rangle \rangle}}$$  \hspace{1cm} (49)

the derivative is:

$$\langle m_{i,m_{j,1}} \rangle_{\text{TAP}} \langle m_{i,2} \rangle_{\text{TAP}} \langle m_{j,3} \rangle_{\text{TAP}} = \lim_{n \to 0} \left( \langle \langle m_{i,m_{j,1}} \rangle \rangle_{\langle \langle m_{j,1}m_{j,2} \rangle \rangle} \right)$$  \hspace{1cm} (50)

Where the meaning of the double square brackets and of the square bracket is the same in appendix A. In the thermodynamic limit this quantities can be averaged by the saddle point method, in particular using the saddle point equation with respect to $\lambda_{ab}$ we get:

$$\langle m_{i,m_{j,1}} \rangle_{\text{TAP}} \langle m_{i,2} \rangle_{\text{TAP}} \langle m_{j,3} \rangle_{\text{TAP}} = q_{13}q_{12}$$  \hspace{1cm} (51)

This must be summed over the different SP, instead we can evaluate on a single SP the same object under all possible permutations of the replica indices:

$$\langle m_{i,m_{j,1}} \rangle_{\text{TAP}} \langle m_{i,2} \rangle_{\text{TAP}} \langle m_{j,3} \rangle_{\text{TAP}} = \frac{1}{n(n-1)(n-2)} \sum_{(a,b,c)} q_{ab}q_{ac}$$  \hspace{1cm} (52)

All the various terms can be evaluated with this method and at the end it turns out that the r.h.s. of eq. 49 is not self-averaging. Furthermore, as shown in [5], at the lower band edge the matrix $q_{ab}$ of the TAP computation coincide with the Parisi solution and one can show that its disorder variance is equal to that of the thermodynamic susceptibility computed in [3].

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