Free expansion of fermionic dark solitons in a boson–fermion mixture

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Abstract
We use a time-dependent dynamical mean-field-hydrodynamic model to study the formation of fermionic dark solitons in a trapped degenerate Fermi gas mixed with a Bose–Einstein condensate in a harmonic as well as a periodic optical-lattice potential. The dark soliton with a ‘notch’ in the probability density with a zero at the minimum is simulated numerically as a nonlinear continuation of the first vibrational excitation of the linear mean-field-hydrodynamic equations, as suggested recently for pure bosons. We study the free expansion of these dark solitons as well as the consequent increase in the size of their central notch and discuss the possibility of experimental observation of the notch after free expansion.

(Some figures in this article are in colour only in the electronic version)
This time-dependent mean-field-hydrodynamic model was suggested recently by the present author [13] to study the collapse dynamics of a DFG and is a time-dependent extension of a time-independent model suggested for the stationary states by Capuzzi et al [11, 12] based essentially on a Thomas–Fermi–Weizsäcker approximation.

Zakharov and Shabat [15] have shown that the dimensionless nonlinear Schrödinger (NLS) equation in the repulsive or self-defocusing case [16]

\[ iu_t + u_{xx} - |u|^2 u = 0 \]  

(1.1)
sustains the following dark and grey solitons [17]:

\[ u(x, t) = r(x - ct) \exp[-i\phi(x - ct) - \mu t], \]  

(1.2)

with

\[ r^2(x - ct) = \eta - 2\kappa^2 \text{sech}^2[\kappa(x - ct)], \]  

(1.3)

\[ \phi(x - ct) = \tan^{-1}[-2\kappa/c \tanh[\kappa(x - ct)]], \]  

(1.4)

\[ \kappa = \sqrt{(2\eta - c^2)/2}, \]  

(1.5)

where \( c \) is the velocity of the soliton, \( \mu \) is the parametric energy and \( \eta \) is related to intensity. Soliton (1.2) having a ‘notch’ over a background density is grey in general. It is dark if density \(|u|^2 = 0\) at the minimum. The soliton can move freely with velocity \( c \) and at zero velocity the soliton becomes a dark soliton: \(|u(x, t)| = \sqrt{\eta} \tanh[x \sqrt{\eta}/2].\)

The similarity of the NLS equation (1.1) to the GP equation (2.1) implies the possibility of a dark soliton in a trapped BEC [18]. It has been suggested that the dark soliton of a trapped BEC could be a stationary eigenstate of the GP equation [18, 19] as in the case of the trapless NLS equation. The usual search for a dark soliton in the GP equation proceeds through time evolution starting with the ansatz [20, 21] \( u_{DS} = \tanh(x)u_G(x), \) where \( u_G \) is the ground state of the GP equation and the \( \tanh(x) \) factor is introduced in analogy with the dark soliton in the NLS equation. However, the function \( u_{DS} \) is not an eigenfunction of the GP equation and hence this procedure leads to numerical instability on time evolution [20, 21]. It has been demonstrated [22] that the time evolution of the GP equation with the initial state \( u_{DS} \) leads to a dark soliton which is the lowest vibrational excitation of the system. Exploring this, a stable numerical procedure has been suggested [22] for the simulation of dark soliton which we use in this investigation.

We study the formation of fermionic dark solitons in a DFG mixed with a BEC in a harmonic as well as a periodic optical-lattice trap. There have been experimental [23] and theoretical [24] studies of the formation of dark solitons in a harmonically trapped BEC. In view of this, here we study for the first time the possibility of the formation of a fermionic dark soliton in a DFG using a mean-field model for a mixture of DFG and BEC.

Collective excitations in the form of solitons and vortices in trapped fermions have also been investigated recently by Damski et al [25] and Karpiuk et al [26]. However, they considered isolated ultra-cold fermions and not a realistic mixture of trapped DFG and BEC as in the experiments and as discussed in this paper. Also they did not demonstrate the existence of stable dark solitons with a zero at the centre of the notch as in the present study. They identified grey soliton-like structure with a shallow dip in the isolated fermionic density distribution quite distinct from the stable fermionic dark solitons in a DFG–BEC mixture as noted in this investigation.

Though the present dark solitons are numerically stable in the mean-field formulation, they could be unstable physically due to quantum fluctuations [27]. The effect of quantum fluctuations is lost in the mean-free model and can only be studied in a field-theoretic approach.
Moreover, being an excited state, they are thermodynamically unstable. There have been suggestions about how to excite a dark soliton by phase imprinting method [25, 26]. The dark soliton is the lowest vibrational excitation of the BEC [22] and there have also been investigations about how to attain such excited states [28]. Nevertheless, despite different suggestions about how to excite a dark soliton [25, 26, 28], experiments to date have not yet generated a stationary dark soliton. Experimentally, so far the dark solitons have been unstable [18, 19]. Although we cannot make definite suggestion(s) for the formation of stable fermionic dark solitons, considering that they are stationary excitations of the mean-field equation their creation seems possible at least as a non-stationary dark soliton which may turn grey and oscillate before decaying due to quantum fluctuations and thermodynamic effects.

Lately, the periodic optical-lattice potential has played an essential role in many theoretical and experimental studies of Bose–Einstein condensation, e.g., in the study of Josephson oscillation [29] and its disruption [30], interference of matter wave [31], BEC dynamics on periodic trap [32], etc. The periodic optical-lattice confinement generated experimentally by a standing-wave laser field creates a BEC in an entirely different shape and trapping condition form a conventional harmonic oscillator trapping. In view of this, we study the possibility of the formation of a fermionic dark soliton in an optical-lattice potential. The formation of a bosonic dark soliton in an optical-lattice potential has already been investigated [21].

The central notch is the earmark of a dark soliton. Experimentally, a dark soliton is identified after removing the traps so that a free expansion of the DFG allows the notch to widen and be photographed clearly. In view of this, we study a free expansion of dark solitons in a DFG–BEC mixture and study the possibility of detection of a fermionic dark soliton in the laboratory.

In section 2, we present an account of the time-dependent mean-field model consisting of a set of coupled partial differential equations involving a BEC and a DFG. In the case of a cigar-shaped system with stronger radial trapping, the above model is reduced to an effective one-dimensional form appropriate for the study of dark solitons. In section 3, we present our results for stationary fermionic dark solitons as well as a study of their free expansion in a boson–fermion mixture. Finally, a summary of our findings is given in section 4.

2. Nonlinear mean-field-hydrodynamic model

The time-dependent BEC wavefunction $\Psi(r, t)$ at position $r$ and time $t$ may be described by the following mean-field nonlinear GP equation [14]:

$$
\left[ -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m_B} + V_B(r) + g_{BB} n_B \right] \Psi_B(r, t) = 0,
$$

(2.1)

with normalization $\int \mathrm{d}r |\Psi_B(r, t)|^2 = N_B$. Here, $m_B$ is the mass and $N_B$ is the number of bosonic atoms in the condensate, $n_B \equiv |\Psi_B(r, t)|^2$ is the boson probability density and $g_{BB} = 4\pi\hbar^2 a_{BB}/m_B$ is the strength of inter-atomic interaction, with $a_{BB}$ the boson–boson scattering length. The trap potential with axial symmetry may be written as $V_B(r) = \frac{1}{2} m_B \omega^2 (\rho^2 + \nu^2 z^2)$, where $\omega$ and $\nu \omega$ are the angular frequencies in the radial ($\rho$) and axial ($z$) directions with $\nu$ the anisotropy parameter. The probability density $n_F$ of an isolated DFG in the Thomas–Fermi approximation is given by [10]

$$
n_F = \frac{\mathrm{max}(0, [\epsilon_F - V_F(r)])^{3/2}}{A^{3/2}},
$$

(2.2)

where $A = \hbar^2(6\pi^2)^{2/3}/(2m_F)$, $\epsilon_F$ is the Fermi energy, $m_F$ is the fermionic mass and the function max denotes the larger of the arguments. The confining trap potential $V_F(r)$ has
an axial symmetry. The number of fermionic atoms $N_F$ is given by the normalization $\int d\mathbf{r} n_F = N_F$.

We developed a set of practical time-dependent mean-field-hydrodynamic equations for the interacting boson–fermion mixture starting from the following Lagrangian density [13]:

$$
\mathcal{L} = \frac{i}{2} \left[ \Psi_B^* \frac{\partial \Psi_B^*}{\partial t} - \Psi_B \frac{\partial \Psi_B}{\partial t} \right] + \frac{i}{2} \left[ \sqrt{n_F} \frac{\partial \sqrt{n_F}^*}{\partial t} - \sqrt{n_F}^* \frac{\partial \sqrt{n_F}}{\partial t} \right] + \left( \frac{\hbar^2}{2m_B} \nabla^2 \Psi_B^* - V_B |\Psi_B|^2 \right) + \frac{1}{2} g_{BF} |\Psi_B|^4 + g_{BF} n_F |\Psi_B|^2, \tag{2.3}
$$

where $g_{BF} = 2\pi \hbar^2 a_{BF}/m_R$, with the boson–fermion reduced mass $m_R = m_B m_F/(m_B + m_F)$, where $a_{BF}$ is the boson–fermion scattering length.

It may not be entirely proper to define an average fermionic wavefunction $\Psi_F = \sqrt{n_F}$ in a DFG like in a BEC. The correct fermionic wavefunction is to be calculated from a Slater determinant Schrödinger equation for degenerate fermions [9]. However, the probability density $n_F$ of a DFG calculated in this fashion should lead to reasonable results [9] and has led to proper probability distribution for a DFG [12, 13] as well as results for collapse of a DFG [10, 12, 13] in agreement with the experiment. This approach has also been used successfully to predict a fermionic bright soliton in a boson–fermion mixture [33].

The terms in the first parentheses on the right-hand side of (2.3) are the standard Gross–Pitaevskii terms for the bosons and are related to a Schrödinger-like equation [14]. However, terms in the second parentheses, although bear a resemblance with the first, are derived from the hydrodynamic equation of motion of the fermions including a Weizsäcker kinetic energy and are not related to a Schrödinger-like equation [11]. Hence, the second kinetic energy term has a different mass factor $6m_F$ and not the conventional Schrödinger mass factor $2m_B$ as in the first term. Finally, the last term in this equation corresponds to an interaction between bosons and fermions. The interaction between bosons and between bosons and fermions are described by contact potentials parametrized by coupling constants $g_{BB}$ and $g_{BF}$ defined above. The interaction between fermions in spin-polarized state is highly suppressed due to Pauli blocking and has been neglected in (2.3) and will be neglected throughout this paper.

Recently, Jezek et al [34] used the Thomas–Fermi–Weizsäcker kinetic energy term $\epsilon_F$ of fermions in their formulation which, in our notation, will correspond to a fermionic kinetic energy of $\hbar^2 |\nabla_F \sqrt{n_F}|^2/(6m_F)$ in (2.3) in place of the present term $\hbar^2 |\nabla_F \sqrt{n_F}|^2/(6m_F)$. This kinetic energy term contributes little to this problem compared to the dominating $3A n_F^{5/3}/5$ term in (2.3) and is usually neglected in the Thomas–Fermi approximation. However, its inclusion leads to a probability density which is a smooth and analytic function of the space variable [34]. For a discussion of these two fermionic kinetic energy terms, we refer the reader to [11, 34, 35].

With the Lagrangian density (2.3), the Euler–Lagrange equations of motion become [13]

$$
\left[ -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m_B} + V_B(\mathbf{r}) + g_{BB} n_B + g_{BF} n_F \right] \Psi_B(\mathbf{r}, t) = 0, \tag{2.4}
$$

$$
\left[ -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{6m_F} + V_F(\mathbf{r}) + A n_F^{2/3} + g_{BF} n_B \right] \sqrt{n_F}(\mathbf{r}, t) = 0. \tag{2.5}
$$

When the nonlinearity in (2.5) is very large, the kinetic energy term in this equation can be neglected and the time-independent stationary form of this equation becomes

$$
n_F = \left[ \max(0, [\epsilon_F - V_F(\mathbf{r}) - g_{BF} n_B]) \right]^{1/2}, \tag{2.6}
$$
which is the generalization of (2.2) in the presence of boson–fermion coupling. Equation (2.6) has been used by Modugno et al [10] for an analysis of a BEC coupled to a DFG. We shall see in the following that in actual experimental condition the nonlinearity in (2.5) is quite large and (2.6) is a good approximation. We note that the Lagrangian density of the formulation of Jezek et al [34] reduces to the present Lagrangian density in this approximation upon the neglect of the fermionic kinetic energy term.

The solution of the coupled three-dimensional equations above for studying dark solitons in a boson–fermion mixture is a formidable task. Hence, we shall reduce (2.4) and (2.5) to the minimal one-dimensional form suitable for the study of dark solitons in a cigar-shaped geometry for \( \nu \ll 1 \). We perform this reduction below where we take \( V_B(r) = V_F(r) = \frac{1}{2} m_B \omega^2 (\rho^2 + v^2 z^2) + V_0 \sin^2(2\pi z/\lambda) \), which corresponds to a suppression of \( \omega \) and \( v \omega \) for fermions by a factor \( \sqrt{m_B/m_F} \) as in the studies by Modugno et al [10] and Jezek et al [34]. In the confining potential, we also include the following optical-lattice potential: \( V_0 \sin^2(2\pi z/\lambda) \) [36]. Here, \( V_0 \) is the strength of the optical-lattice potential and \( \lambda \) is the wavelength of the laser.

For \( \nu \ll 1 \), (2.4) and (2.5) can be reduced to an effective one-dimensional form by considering solutions of the type \( \Psi_B(r, t) = \phi_B(z, t) \psi_B^{(0)}(\rho) \) and \( \sqrt{m_F}(r, t) = \phi_F(z, t) \psi_F^{(0)}(\rho) \), where

\[
|\psi^{(0)}_i(\rho)|^2 \equiv \frac{M_i \omega}{\pi \hbar} \exp\left(-\frac{M_i \omega \rho^2}{\hbar}\right), \quad i = B, F
\]

(2.7)
corresponds to the respective circularly symmetric ground-state wavefunction in the absence of nonlinear interactions and satisfies

\[
-\frac{\hbar^2}{2 m_B} \frac{\partial^2}{\partial \rho^2} \psi_B^{(0)} + \frac{1}{2} m_B \omega^2 \rho^2 \psi_B^{(0)} = \hbar \omega \psi_B^{(0)},
\]

(2.8)

\[
-\frac{\hbar^2}{6 m_F} \frac{\partial^2}{\partial \rho^2} \psi_F^{(0)} + \frac{1}{2} m_F \nu^2 \rho^2 \psi_F^{(0)} = \sqrt{\frac{m_B}{3m_F}} \hbar \omega \psi_F^{(0)},
\]

(2.9)

with normalization \( 2\pi \int_0^\infty |\psi_i^{(0)}(\rho)|^2 \rho \, d\rho = 1 \). Now the dynamics is carried by \( \phi_i(z, t) \) and the radial dependence is frozen in the ground state \( \psi_i^{(0)}(\rho) \). The separation of the variables is suggested by the structure of (2.4) and (2.5).

Averaging over the radial mode \( \psi_i^{(0)}(\rho) \), i.e., multiplying (2.4) and (2.5) by \( \psi_i^{(0)*}\rho \) and integrating over \( \rho \), we obtain the following one-dimensional dynamical equations [37]:

\[
\begin{bmatrix}
-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2 m_B} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m_B \nu^2 \omega^2 z^2 + V_0 \sin^2\left(\frac{2\pi z}{\lambda}\right) + F_{BB} |\phi_B|^2 + F_{BF} |\phi_F|^2
\end{bmatrix} \phi_B(z, t) = 0,
\]

(2.10)

\[
\begin{bmatrix}
-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{6 m_F} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m_F \nu^2 \omega^2 z^2 + V_0 \sin^2\left(\frac{2\pi z}{\lambda}\right) + F_{FF} |\phi_F|^4/3 + F_{BF} |\phi_B|^2
\end{bmatrix} \phi_F(z, t) = 0,
\]

(2.11)

where

\[
F_{BB} = \frac{g_{BB}}{2\pi \hbar} \int_0^\infty |\psi_B^{(0)}|^4 \rho \, d\rho = \frac{m_B \omega}{2 \pi \hbar},
\]

(2.12)
\[
F_{BF} = g_{BF} \int_{0}^{\infty} \left| \psi_{(0)}^{(0)} \right|^{2} \rho \, d\rho = g_{BF} \frac{M_{BF} \omega}{\pi \hbar}, \tag{2.13}
\]

\[
F_{FF} = A \int_{0}^{\infty} \left| \psi_{F}^{(0)} \right|^{2} \rho \, d\rho = \frac{3A}{5} \left( \frac{M_{F} \omega}{\pi \hbar} \right)^{2/3}. \tag{2.14}
\]

In (2.10) and (2.11), we have included the optical-lattice potential and the normalization there is given by \(\int_{-\infty}^{\infty} |\phi(z, t)|^2 \, dz = N_i\).

For calculational purpose, it is convenient to reduce the sets (2.10) and (2.11) to dimensionless form by introducing convenient dimensionless variables. Although the algebra is quite straightforward, the expressions become messy with different factors of masses. As we shall not be interested in a particular boson–fermion system in this paper, but be concerned with the formation of fermionic dark solitons in general, we take in the rest of this paper \(m_B = 3m_F = m^{(39)}Rb\), whence \(m_B = 3m_F/4\), \(M_B = M_F = m_B\) and \(M_{BF} = m_B/2\), where \(m^{(39)}Rb\) is the mass of the Rb atom. In the two experimental situations of [4, 5], \(m_B \approx 3m_F\).

In (2.10) and (2.11), \(y \ll 1\) and we consider the dimensionless variables \(\tau = t \nu \omega/2\), \(y = z/\ell_z\), \(\chi_i = \sqrt{\ell_z/N_i} \phi_i\), with \(\ell_z = \sqrt{\hbar/(\nu m_B)}\), so that

\[
\begin{align*}
\left[ -i \frac{\partial}{\partial \tau} - \frac{d^2}{dy^2} + y^2 + v_0 \sin^2 \left( \frac{2\pi y}{\lambda_0} \right) + N_{BB}|\chi_B|^2 + N_{BF}|\chi_B|^2 \right] \chi_B(y, \tau) &= 0, \tag{2.15} \\
\left[ -i \frac{\partial}{\partial \tau} - \frac{d^2}{dy^2} + y^2 + v_0 \sin^2 \left( \frac{2\pi y}{\lambda_0} \right) + N_{FB}|\chi_B|^2 + N_{FF}|\chi_F|^4 \right] \chi_F(y, \tau) &= 0 \tag{2.16}
\end{align*}
\]

where \(N_{BB} = (4/\nu)a_{BB} N_B/\ell_z, N_{BF} = (8/\nu)a_{BF} N_B/\ell_z, N_{FB} = (8/\nu)a_{BF} N_B/\ell_z\) and \(N_{FF} = 9(6\pi N_F/\nu)^{2/5}/5\). In (2.15) and (2.16), the normalization condition is given by \(\int_{-\infty}^{\infty} |\chi_i(y, \tau)|^2 \, dy = 1\), \(v_0 \equiv 2V_0/(R \omega \nu)\) is the reduced strength of the optical-lattice potential and \(\lambda_0 \equiv \lambda/\ell_z\) is the dimensionless wavelength.

3. Numerical results

The coupled mean-field-hydrodynamic equations (2.15) and (2.16) for dark solitons are solved numerically using a time-iteration method based on the Crank–Nicholson discretization scheme elaborated in [38]. We discretize the mean-field-hydrodynamic equations using time step 0.0005 and space step 0.025.

We performed the time evolution of the set of equations (2.15) and (2.16) setting \(N_{BB} = N_{BF} = N_{FB} = N_{FF} = v_0 = 0\) and starting with the eigenfunction of the lowest excited state of the linear harmonic oscillator problem as suggested recently [22], e.g., with \(\chi_B(y, \tau) = \chi_F(y, \tau) = \sqrt{2\pi}^{-1/4} y \exp(-y^2/2) \exp(-3i\tau)\). During the course of time evolution, the nonlinear terms are switched on very slowly and the resultant solutions iterated (about 50 000 times) until convergence was obtained. If converged solutions are obtained, they correspond to the required dark solitons. In the present approach, the time evolution starts with and proceeds through successive eigenfunctions of the coupled mean-field equations (2.15) and (2.16). Hence, it leads to stable numerical results. The usual numerical procedure for the calculation of the dark solitons starts with an approximate solution of the mean-field equation and hence leads to numerical instability on time evolution [20–22].

We solve (2.15) and (2.16) for dark solitons. In this case, the nonlinearity \(N_{BF}\) could be very large for \(N_F > 100\), which may require special care for obtaining accurate numerical
solutions. In our calculation, we use $v = 0.1$, $v_0 = 0$, $v_\omega = 2\pi \times 100$ Hz and $m_B$ to be the mass of $^{87}$Rb. Consequently, $l \approx 1$ \textmu m and unit of time $\tau = 2/(v_\omega)$ is 3 ms. We also take $N_B = 100$, $N_F = 100$, $l = 1$ \textmu m and $a_{BB} = a_{BF} = 5$ nm. With these parameters, the nonlinearities in (2.15) and (2.16) are $N_{BB} = 20$, $N_{BF} = 40$, $N_{FB} = 40$ and $N_{FF} = 1275$.

The results of the present study on dark solitons are presented in figure 1. In figure 1(a), we plot the stationary functions $|\phi(z,t)|$ of the boson and fermion dark solitons. Both functions have a notch in the middle. The notch in the bosonic function is wider than that in the fermionic function. In this case, the fermionic nonlinearity (1275) is large. Consequently, the extension of the fermionic function in figure 1(a) is also much larger than the bosonic function.

One way to observe the dark solitons experimentally is to allow them to expand freely while the central notch will become wider in size to be visible and photographed. With this in mind, we study the free expansion of the boson–fermion mixture. The snapshots of the bosonic and fermionic functions at regular intervals of time during this expansion are shown in figures 1(b) and (c), respectively. After expansion, the central notch in the bosonic dark soliton expands as we find from figure 1(b), but the notch in the fermionic dark soliton in figure 1(c) does not expand enough to be visible. This will make the fermionic dark soliton more difficult to observe experimentally. However, this behaviour is quite expected in a freely expanding fermionic dark soliton with a large nonlinear repulsion. In the fermionic equation, both the

Figure 1. (a) The stationary function $|\phi(z,t)|$ for bosonic and fermionic dark solitons versus $z$ with $N_B = N_F = 100$, $\nu = 0.1$ and $a_{BB} = a_{BF} = 5$ nm. We show in (b) and (c) the profiles of the bosonic and fermionic functions $|\phi_B(z,t)|$ and $|\phi_F(z,t)|$, respectively, of the degenerate mixture (a) during free expansion at regular intervals of time. The nonlinearities are $N_{BB} = 20$, $N_{BF} = 40$, $N_{FB} = 40$, $N_{FF} = 1275$. 

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\[ \text{Fermionic dark solitons in a boson–fermion mixture} \]

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fermion–fermion and fermion–boson interactions are highly repulsive. The dark soliton has a notch (hollow region) at the centre. Because of the very strong repulsion, the DFG tends to expand in all directions including the radially inward direction to fill the central hollow space as well as outward directions. This inward repulsive force balances partially the outward kinetic pressure and does not allow the central notch to expand substantially during free expansion so as to be easily observable. This is not the case for a moderately repulsive or attractive pure single-component BEC, where the outward kinetic pressure overcomes any repulsion among the atoms and the BEC expands only in the radially outward direction with a widening of the notch. In the bosonic wavefunction of figure 1(b), the notch expands reasonably during the expansion of the BEC. However, during the expansion in figures 1(b) and (c), the central notch of both the bosonic and fermionic condensates has always a zero at the origin.

Next, we study the stability of the solitons illustrated in figure 1 under a small perturbation inflicted by a sudden change in the boson–fermion scattering length $a_{BF}$. After the solitons of figure 1 are formed, we increase $a_{BF}$ by 10% at $t = 0$ so that the nonlinearities $N_{BF}$ and $N_{FB}$ are suddenly increased from 40 to 44. This can be performed experimentally by varying a background magnetic field near a Feshbach resonance in the boson–fermion system [39]. The solitons then execute small breathing oscillation around a mean position. The snapshots of the soliton profiles shown in figures 2(a) and (b) under this perturbation demonstrate their stability.

Finally, we calculate the dark solitons in the boson–fermion mixture with the parameters of figure 1(a) in the presence of an optical-lattice potential with $v_0 = \lambda_0 = 5$. We plot in figure 3 the function $\phi(z, t)$ for the dark soliton in this case. The presence of the optical-lattice potential creates modulations in both fermionic and bosonic functions. Because of the very strong repulsion, the fermions tend to occupy the whole available region in space and do not allow the formation of pronounced notches with hollow region inside at each optical-lattice site. Consequently, the modulation in the fermionic function is less pronounced than the modulation in the bosonic function. The bosonic function of figure 3 has pronounced notches compared to the smooth function in figure 1(a), whereas the fermionic function of figure 3 with small modulations is qualitatively more similar to the fermionic component of figure 1(a). We also studied free expansion of the dark solitons in this case. The central notch in the fermionic dark soliton does not also expand in this case. No new interesting physics emerges and the results are not shown here.
4. Summary

We use a coupled set of time-dependent mean-field-hydrodynamic equations for a trapped degenerate boson–fermion mixture to study the formation of a fermionic dark soliton (vibrational excitation) in a DFG as stationary states. We calculate the stationary functions with a notch at the centre for fermionic dark soliton of the boson–fermion mixture. The existence of a central notch in the wavefunction for a dark soliton is typical of vibrational excitation. We perform numerical simulation of the dark solitons for a harmonic as well as a harmonic plus optical-lattice traps. The simulation is started with a time evolution of the mean-field equations with the eigenfunction of the lowest excited state of the linear oscillator problem setting all the nonlinearities to zero. The nonlinearities are introduced slowly during time evolution and the iteration continued until convergence is obtained. We demonstrate the stability of the dark soliton upon the application of a perturbation while the soliton executes small breathing oscillation.

The present time-dependent formulation also permits us to study non-equilibrium free expansion of the coupled degenerate boson–fermion mixture. One way to observe the notch experimentally is to allow the dark soliton to undergo free expansion. We find that for a repulsive boson–fermion interaction, after a free expansion the notch in the fermion function for a relatively small fermion number of 100 does not increase in size significantly so as to be easily observable.

In the present investigation, we used a set of mean-field equations for the DFG–BEC mixture. A proper treatment of the DFG should be performed using a fully antisymmetrized many-body Slater determinant wavefunction [9]. Although we believe the present conclusion about fermionic dark soliton to be true in general, it would be of interest to establish its existence using such a fully antisymmetrized fermionic many-body wavefunction in the future. Also, a dark soliton features a particular spatial phase distribution, i.e. a step of $\pi$ phase. In a BEC, this phase profile is supported due to the macroscopic phase of the condensate. Although a macroscopic phase of the fermionic component should emerge in the present mean-field model, to the best of our knowledge its existence in a fermionic many-body wavefunction has not been established rigorously. However, such a study is beyond the scope of this paper and would be a work of future interest. Nevertheless, it would be proper to call the fermionic excitation of the present paper by the term dark soliton due to its appropriate density distribution with a central notch.
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