ANALYSIS OF A NON-AUTONOMOUS MUTUALISM MODEL DRIVEN BY LEVY JUMPS

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Abstract. This article is concerned with a mutualism ecological model with Lévy noise. The local existence and uniqueness of a positive solution are obtained with positive initial value, and the asymptotic behavior to the problem is studied. Moreover, we show that the solution is stochastically bounded and stochastic permanence. The sufficient conditions for the system to be extinct are given and the conditions for the system to be persistence in mean are also established.

1. Introduction. Mutualism is an important biological interaction in nature. It occurs when one species provides some benefit in exchange for some benefit, for example, pollinators and flowering plants, the pollinators obtain floral nectar (and in some cases pollen) as a food resource while the plant obtains non-trophic reproductive benefits through pollen dispersal and seed production. Another instance is ants and aphids, in which the ants obtain honeydew food resources excreted by aphids while the aphids obtain increased survival by the non-trophic service of ant defense against natural enemies of the aphids. Lots of authors have discussed these models [1, 5, 7, 10, 12, 13, 14, 24, 36]. One of the simplest models is the classical Lotka-Volterra two-species mutualism model as follows:

\[
\begin{align*}
\dot{x}(t) &= x(t)(a_1 - b_1 x(t) + c_1 y(t)), \\
\dot{y}(t) &= y(t)(a_2 - b_2 y(t) + c_2 x(t)).
\end{align*}
\]

(1)
Among various types mutualistic model, we should specially mention the following model which was proposed by May [32] in 1976:

\[
\begin{align*}
\dot{x}(t) &= x(t)\left(r_1 - \frac{b_1 x(t)}{K_1 + y(t)} - \varepsilon_1 x(t)\right), \\
\dot{y}(t) &= y(t)\left(r_2 - \frac{b_2 y(t)}{K_2 + x(t)} - \varepsilon_2 y(t)\right),
\end{align*}
\]  

where \(x(t), y(t)\) denote population densities of each species at time \(t\), \(r_i, K_i, b_i, \varepsilon_i\) (\(i=1, 2\)) are positive constants, \(r_1, r_2\) denote the intrinsic growth rate of species \(x(t), y(t)\) respectively, \(K_1\) is the capability of species \(x(t)\) being short of \(y(t)\), similarly \(K_2\) is the capability of species \(y(t)\) being short of \(x(t)\). For (2), there are three trivial equilibrium points

\[
E_1 = (0, 0), \quad E_2 = \left(\frac{r_1}{\varepsilon_1 + \frac{b_1}{K_1}}, 0\right), \quad E_3 = (0, \frac{r_2}{\varepsilon_2 + \frac{b_2}{K_2}}),
\]

and a unique positive interior equilibrium point \(E^* = (x^*, y^*)\) satisfying the following equations

\[
\begin{align*}
r_1 - \frac{b_1 x^*}{K_1 + y^*} - \varepsilon_1 x^* &= 0, \\
r_2 - \frac{b_2 y^*}{K_2 + x^*} - \varepsilon_2 y^* &= 0,
\end{align*}
\]

where \(E^*\) is globally asymptotically stable.

In addition, population dynamics is inevitably affected by environmental noises. May [33] pointed out the fact that due to environmental fluctuation, the birth rates, carrying capacity, and other parameters involved in the model system exhibit random fluctuation to a greater or lesser extent. Consequently the equilibrium population distribution fluctuates randomly around some average values. Therefore lots of authors introduced stochastic perturbation into deterministic models to reveal the effect of environmental variability on the population dynamics in mathematical ecology [8, 11, 16, 17, 18, 21, 25, 26, 27, 34, 35]. Li and Gao et al took into account the effect of randomly fluctuating environment in [23], where they considered white noise to each equation of the problem (2). Suppose that parameter \(r_i\) is stochastically perturbed, with

\[
r_i \to r_i + \alpha_i W_i(t), \quad i = 1, 2,
\]

where \(W_1(t), W_2(t)\) are mutually independent Brownian motion, \(\alpha_i, \ i = 1, 2\) represent the intensities of the white noise. Then the corresponding deterministic model system (2) may be described by the Itô problems:

\[
\begin{align*}
dx(t) &= x(t)\left(r_1 - \frac{b_1 x(t)}{K_1 + y(t)} - \varepsilon_1 x(t)\right)dt + \alpha_1 x(t)dW_1(t), \\
dy(t) &= y(t)\left(r_2 - \frac{b_2 y(t)}{K_2 + x(t)} - \varepsilon_2 y(t)\right)dt + \alpha_2 y(t)dW_2(t).
\end{align*}
\]

On the other hand, population systems may suffer abrupt environmental perturbations, such as epidemics, earthquakes, hurricanes, etc. As a consequence, these systems are very complex and their sample paths may not be continuous, which yields the system (4) fail to cope with them. It is recognized that introducing Lévy noise into the underlying population system may be quite suitable to describe such discontinuous systems. There exists some interesting literatures concerned with
non-autonomous system with jumps: SDEs with jumps. We here only mention Bao et al [3, 4], Liu and Liang [22], Liu and Wang [28]. Motivated by those studies, in this paper we consider the following non-autonomous system with jumps:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t^{-}) \left[ r_1(t) - \frac{b_1(t)x(t^{-})}{K_1(t) + y(t^{-})} - \varepsilon_1(t)x(t^{-}) \right] + \alpha_1(t)dW_1(t) \\
& \quad + \int_{\mathcal{Y}} \gamma_1(t, u) \tilde{N}(dt, du), \\
\frac{dy(t)}{dt} &= y(t^{-}) \left[ r_2(t) - \frac{b_2(t)y(t^{-})}{K_2(t) + x(t^{-})} - \varepsilon_2(t)y(t^{-}) \right] + \alpha_2(t)dW_2(t) \\
& \quad + \int_{\mathcal{Y}} \gamma_2(t, u) \tilde{N}(dt, du),
\end{align*}
\]

(5)

where \( x(t^{-}) \) and \( y(t^{-}) \) are the left limit of \( x(t) \) and \( y(t) \) respectively, \( r_i(t), b_i(t), K_i(t), \alpha_i(t), i = 1, 2 \) are all positive, continuous and bounded functions on \([0, +\infty)\). \( N \) is a Poisson random measure with compensator \( \tilde{N} \) and characteristic measure \( \mu \) on a measurable subset \( \mathcal{Y} \) of \((0, +\infty)\) with \( \mu(\mathcal{Y}) < +\infty \), \( \tilde{N}(dt, du) = N(dt, du) - \mu(du)dt \), \( \gamma_i : \mathcal{Y} \times \Omega \rightarrow \mathbb{R} \) is bounded and continuous with respect to \( \mu \), and is \( \mathcal{B}(\mathcal{Y}) \times \mathcal{F}_t \)-measurable, \( i = 1, 2 \).

In the next section, the global existence and uniqueness of the positive solution to problem (5) are proved by using comparison theorem for stochastic equations. Sections 3 is devoted to stochastic boundedness. Section 4 deals with stochastic permanence. Section 5 discusses the persistence in mean and extinction, sufficient conditions of persistence in mean and extinction are obtained.

Throughout this paper, we let \((\Omega, F, \{F_t\}_{t \geq 0}, P)\) be a complete probability space with a filtration \( \{F_t\}_{t \geq 0} \) satisfying the usual conditions. For convenience, we assume that \( X(t) = (x(t), y(t)) \) and \( |X(t)| = \sqrt{x^2(t) + y^2(t)} \). \( 1 + \gamma_i(t, u) > 0 \), \( u \in \mathcal{Y}, i = 1, 2 \), there exists a constant \( k_1 > 0, k_2 > 0 \) such that

\[
\int_{\mathcal{Y}} \left| \gamma_i(t, u) \right| \vee \left| \ln(1 + \gamma_i(t, u)) \right|^2 \mu(du) < k_1, \\
\int_{\mathcal{Y}} \left| \ln(1 + \gamma_i(t, u)) \vee \left| \ln(1 + \gamma_i(t, u)) \right|^2 \mu(du) < k_2,
\]

\[
\beta_i(t) = 0.5 \alpha_i^2(t) + \int_{\mathcal{Y}} \left[ \gamma_i(t, u) - \ln(1 + \gamma_i(t, u)) \right] \mu(du),
\]

\[
Q_i(t) = \int_0^t \int_{\mathcal{Y}} \left( \ln(1 + \gamma_i(s, u)) - \gamma_i(t, u) \right) \tilde{N}(ds, du), \quad i = 1, 2,
\]

\[
\tilde{f} = \inf_{t \geq 0} f(t), \quad \bar{f} = \sup_{t \geq 0} f(t).
\]

We end this section by recalling three definitions which we will use in the forthcoming sections.

**Definition 1.1.** [29] If for any \( 0 < \varepsilon < 1 \), there is a constant \( \delta(\varepsilon) > 0 \) such that the solution \( X(t) \) of (5) satisfies

\[
\limsup_{t \to \infty} P\{|X(t)| < \delta\} \geq 1 - \varepsilon,
\]

for any initial value \((x_0, y_0) > (0, 0)\), then we say the solution \( X(t) \) be stochastically ultimate boundedness.
Theorem 2.2. If for arbitrary $0 < \varepsilon < 1$, there are two positive constants $\zeta_1 := \zeta_1(\varepsilon)$ and $\zeta_2 := \zeta_2(\varepsilon)$ such that

$$\liminf_{t \to \infty} P\{x(t) \leq \zeta_1\} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} P\{y(t) \leq \zeta_1\} \geq 1 - \varepsilon.$$  

$$\liminf_{t \to \infty} P\{x(t) \geq \zeta_2\} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} P\{y(t) \geq \zeta_2\} \geq 1 - \varepsilon.$$  

Then solution of problem (5) is said to be stochastically permanent.

**Definition 1.2.** If $x(t), y(t)$ satisfy the following condition

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s)ds > 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t y(s)ds > 0 \quad a.s.$$  

Then the problem of (5) is said to be persistence in mean.

2. **Existence and uniqueness of the positive global solution.** We show that there exists a unique global positive solution of (5).

**Lemma 2.1.** For the given positive initial value $(x_0, y_0)$, there exists a unique global positive solution of (5) a.s. for $t \in [0, \tau)$.

**Proof.** By using Itô’s formula to $ln x(t), ln y(t)$, then problem (5) can be deduced to the following system

$$dln x(t) = (r_1(t) - \beta_1(t) - \frac{b_1(t)x(t^-)}{K_1(t) + y(t^-)} - \varepsilon_1(t)x(t^-))dt + \alpha_1(t)dW_1(t)$$

$$+ \int_{\mathbb{Y}} ln(1 + \gamma_1(t,u))\tilde{N}(dt,du),$$

$$dln y(t) = (r_2(t) - \beta_2(t) - \frac{b_2(t)y(t^-)}{K_2(t) + x(t^-)} - \varepsilon_2(t)y(t^-))dt + \alpha_2(t)dW_2(t)$$

$$+ \int_{\mathbb{Y}} ln(1 + \gamma_2(t,u))\tilde{N}(dt,du),$$

on $t \geq 0$ with initial value $(ln x_0, ln y_0)$. Obviously, the coefficients of (6) satisfy the local Lipschitz condition, then making use of the theorem [9, 31] about existence and uniqueness for stochastic differential equation there is a unique local solution $(ln x(t), ln y(t))$ on $t \in [0, \tau)$, where $\tau$ is the explosion time. Hence, $X(t) = (x(t), y(t))$ is a unique positive local solution to problem (5) with positive initial value.

Next we need to prove solution is global, that is $\tau = \infty$

**Theorem 2.2.** For any positive initial value $(x_0, y_0)$, there exists a unique global positive solution $(x(t), y(t))$ to problem (5) almost surely, which satisfies

$$\lambda(t) \leq x(t) \leq \Lambda(t), \quad \theta(t) \leq y(t) \leq \Theta(t), \quad t \geq 0, \quad a.s.$$  

where $\lambda(t), \Lambda(t), \theta(t) \text{ and } \Theta(t)$ are defined as (8), (7), (12) and (11).

**Proof.** The reference of [17] was the main source of inspiration for its proof.

Because of $(x(t), y(t))$ is positive, from (5), we can define the following problems

$$d\Lambda(t) = \Lambda(t^-)[(r_1(t) - \varepsilon_1(t)\Lambda(t^-))dt + \alpha_1(t)dW_1(t)$$

$$+ \int_{\mathbb{Y}} \gamma_1(t,u)\tilde{N}(dt,du)],$$

$$\Lambda(0) = x_0,$$  

$$d\Theta(t) = \Theta(t^-)[(r_2(t) - \varepsilon_2(t)\Theta(t^-))dt + \alpha_2(t)dW_2(t)$$

$$+ \int_{\mathbb{Y}} \gamma_2(t,u)\tilde{N}(dt,du)],$$

$$\Theta(0) = y_0.$$  

Then solution of problem (5) is said to be stochastically permanent.
Then for any positive initial value 

By the Theorem 2.1 in [22], we know that \( \Lambda(t) \), \( \lambda(t) \) will not be exploded in any finite time. By the comparison theorem for stochastic equations [15] we can conclude that

\[
\lambda(t) \leq x(t) \leq \Lambda(t), \quad t \in [0, \tau), \ a.s.
\]  

Similarly, we can get

\[
\theta(t) \leq y(t) \leq \Theta(t), \quad t \in [0, \tau), \ a.s.,
\]

where \( \Theta(t), \theta(t) \) satisfies the following problems respectively.

\[
\begin{aligned}
\begin{cases}
\frac{d\Theta(t)}{dt} = \Theta(t^-)[(r_2(t) - \varepsilon_2(t)\Theta(t^-))dt + \alpha_2(t)dW_2(t) + \int_{\mathbb{V}} \gamma_2(t,u)\tilde{N}(dt,du)], \\
\Theta(0) = y_0,
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
\frac{d\theta(t)}{dt} = \theta(t^-)[(r_2(t) - \frac{b_2(t)}{K_2(t)} + \varepsilon_2(t)\theta(t^-))dt + \alpha_2(t)dW_2(t) + \int_{\mathbb{V}} \gamma_2(t,u)\tilde{N}(dt,du)], \\
\theta(0) = y_0.
\end{cases}
\end{aligned}
\]

Moreover \( \Theta(t), \theta(t) \) will not be exploded in any finite time. Combining (9) with (10), that \( X(t) = (x(t), y(t)) \) exists globally almost surely.

3. **Stochastically ultimate boundedness.** In a population dynamical system, the nonexplosion property is often not good enough but the property of ultimate boundedness is more desired. Now, let us present a theorem about the stochastically ultimate boundedness of (5) for any positive initial value.

**Theorem 3.1.** Assume that there exists a constant \( L(q) > 0 \) such that

\[
\int_{\mathbb{V}} |\gamma_i(s,u)|^q \mu(du) \leq L(q), \quad q > 1, \quad i = 1, 2.
\]

Then for any positive initial value \((x_0, y_0)\), the solution \( X(t) \) of problem (5) is stochastically ultimate boundedness.

**Proof.** As the reference of [4] we define a Lyapunov function \( U(x) = x^q \). By the Itô formula:

\[
E(e^tU(x(t))) = U(x_0) + E\int_0^t e^s\{U(x(s^-)) + q[r_1(s) - \frac{b_1(s)x(s^-)}{K_1(s) + y(s^-)} - \varepsilon_1(s)x(s^-)] + \frac{(q-1)\alpha^2(s)}{2} + \int_{\mathbb{V}} [(1 + \gamma_1(s,u))^q - 1 - q\gamma_1(s,u)\mu(du)]U(x(s^-))\}ds
\]
\[
\begin{align*}
\leq & U(x_0) + E \int_0^t e^s \left\{ \left[-qz_1(s)x(s^-) + 1 + qr_1(s) + \frac{q(q-1)\alpha_1(s)^2}{2} \right] + \int_{\gamma} \left[ (1 + \gamma_1(s,u))^q - 1 - q\gamma_1(s,u) \mu(du) \right] U(x(s^-)) \right\} ds \\
& + E \int_0^t e^s [U(x(s)) - U(x(s^-))] ds.
\end{align*}
\]

If \( q > 1 \), we can deduce that there exists constant \( L_1(q) > 0 \) by assumption such that
\[
U(x(s^-)) \left\{ \left[ (1 + qr_1(s) + \frac{q(q-1)\alpha_1(s)^2}{2}) - qz_1(s)x(s^-) \right] + \int_{\gamma} \left[ (1 + \gamma_1(s,u))^q - 1 - q\gamma_1(s,u) \mu(du) \right] \right\} \leq L_1(q).
\]
If \( 0 < q \leq 1 \), using \( (1 + \gamma_1(s,u))^q - 1 - q\gamma_1(s,u) \leq 0 \), then we have
\[
\begin{align*}
U(x(s^-)) \left\{ \left[ (1 + qr_1(s) + \frac{q(q-1)\alpha_1(s)^2}{2}) - qz_1(s)x(s^-) \right] + \int_{\gamma} \left[ (1 + \gamma_1(s,u))^q - 1 - q\gamma_1(s,u) \mu(du) \right] \right\} \\
& \leq U(x(s^-)) [1 + qr_1(s) - qz_1(s)x(s^-)] \leq L_1(q).
\end{align*}
\]

Therefore,
\[
E(e^t U(x(t))) \leq E(U(x_0)) + (L_1(q) + M)(e^t - 1),
\]
where \( M \) is a finite value. Thus,
\[
\limsup_{t \to \infty} E(x(t))^q \leq L_{11}(q), \tag{13}
\]
where \( L_{11}(q) = L_1(q) + M \). Similarly, we have
\[
\limsup_{t \to \infty} E(y(t))^q \leq L_{21}(q), \tag{14}
\]
where \( L_{21}(q) = L_2(q) + M \). We now combine (13), (14) with the formula \( [x(t)^2 + y(t)^2]^\frac{q}{2} \leq 2^\frac{q}{2} [x(t)^q + y(t)^q] \) to yield
\[
\limsup_{t \to \infty} E|X|^q \leq 2^\frac{q}{2} [L_{11}(q) + L_{21}(q)] < +\infty.
\]
By the Chebyshev’s inequality [31] and the above inequality we can complete the proof. 

4. **Stochastic permanence.** In the study of population models, stochastic permanence is one of the most interesting and important topics. We will discuss this property by using the method as in [28] in this section.

**Theorem 4.1.** If \( \min \{ \hat{r}_1 - \hat{\beta}_1, \hat{r}_2 - \hat{\beta}_2 \} > 0 \), then solution of problem (5) is stochastically permanent.

**Proof.** For a positive constant \( 0 < \eta < 1 \), we set a function
\[
Z(x) = \frac{1}{x}, \quad V(x) = e^M Z^\eta(x).
\]
Straightforward computation \( dV(x) \) by Itô's formula shows that

\[
dV(t) = \eta e^{\lambda t} Z^\alpha(t^-) \left\{ - Z^2(t^-) \left[ r_1(t) - \frac{\alpha^2_1(t)}{2} \right] - \int_Y \gamma_1(t, u) \mu(du) - \frac{\eta \alpha_1^2(t)}{2} \right\} dt
\]

\[
- \int_Y \left( \frac{1}{\eta (1 + \gamma_1(t, u))^\eta} - \frac{1}{\eta} \right) \mu(du) - \frac{\lambda}{\eta} \left[ \epsilon_1(t) + \left( \frac{b_1(t^-)}{\mu_1(t) + \gamma(t^-)} \right) \right] dt
\]

\[
- \eta \alpha_1 e^{\lambda t} Z^\eta(t^-) dW_1(t) + e^{\lambda t} Z^\eta(t^-) \int_Y \left( \frac{1}{(1 + \gamma_1(t, u))^\eta} - 1 \right) \tilde{N}(dt, du)
\]

\[
+ \left[ \lambda e^{\lambda t} Z^\eta(t^-) - \lambda e^{\lambda t} Z^\eta(t^-) \right] dt
\]

Due to

\[
- \int_Y \ln(1 + \gamma_1(t, u)) \mu(du) = \lim_{\eta \to 0^+} \left\{ \frac{\eta \alpha_1^2(t)}{2} + \int_Y \frac{1 - (1 + \gamma_1(t, u))^\eta}{\eta (1 + \gamma_1(t, u))^{\eta}} \mu(du) \right\},
\]

then when \( \hat{\gamma}_1 - \tilde{\gamma}_1 > 0 \), we can choose a sufficiently small \( \eta \) to satisfy

\[
\frac{\lambda}{\eta} < r_1(t) - \frac{\alpha^2_1(t)}{2} - \int_Y \gamma_1(t, u) \mu(du) - \left\{ \frac{\eta \alpha_1^2(t)}{2} + \int_Y \frac{1 - (1 + \gamma_1(t, u))^\eta}{\eta (1 + \gamma_1(t, u))^{\eta}} \mu(du) \right\}.
\]

Let us choose \( \lambda > 0 \) sufficiently small to satisfy

\[
\frac{\lambda}{\eta} < r_1(t) - \frac{\alpha^2_1(t)}{2} - \int_Y \gamma_1(t, u) \mu(du) - \left\{ \frac{\eta \alpha_1^2(t)}{2} + \int_Y \frac{1 - (1 + \gamma_1(t, u))^\eta}{\eta (1 + \gamma_1(t, u))^{\eta}} \mu(du) \right\}.
\]

Then, there is a positive constant \( L_1 \) satisfying

\[
r_1(t) - \frac{\alpha^2_1(t)}{2} - \int_Y \left[ \gamma_1(t, u) + \frac{1 - (1 + \gamma_1(t, u))^\eta}{\eta (1 + \gamma_1(t, u))^{\eta}} \right] \mu(du)
\]

\[
> \frac{\eta \alpha_1^2(t)}{2} + \frac{\lambda}{\eta} - L_1.
\]

\[
dV(t) =: \eta e^{\lambda t} Z^\alpha(t^-) dt - \eta \alpha_1 e^{\lambda t} Z^\eta(t^-) dW_1(t) + \lambda e^{\lambda t} Z^\eta(t^-) dt
\]

\[
+ e^{\lambda t} Z^\eta(t^-) \int_Y \left( \frac{1}{(1 + \gamma_1(t, u))^\eta} - 1 \right) \tilde{N}(dt, du) - \lambda e^{\lambda t} Z^\eta(t^-) dt,
\]

Denote \( L =: \sup_{z > 0} L(z(t^-)) < +\infty \), then we have the following inequality

\[
dV(t) \leq \eta e^{\lambda t} dt + e^{\lambda t} Z^\eta(t^-) \int_Y \left( \frac{1}{(1 + \gamma_1(t, u))^\eta} - 1 \right) \tilde{N}(dt, du)
\]

\[
+ \lambda e^{\lambda t} Z^\eta(t^-) - Z^\eta(t^-) dt - \eta \alpha_1 e^{\lambda t} Z^\eta(t^-) dW_1(t),
\]

Integrating and then taking expectations yields

\[
E[V(t)] = e^{\lambda t} E[Z^\eta(x)] \leq \left( \frac{1}{x_0} \right)^{\eta} + \frac{\eta L + \lambda N}{\lambda} (e^{\lambda t} - 1),
\]

where \( N \) is the finite value. Therefore,

\[
\limsup_{t \to +\infty} E\left[ \frac{1}{x^n(t)} \right] \leq \frac{\eta L + \lambda N}{\lambda}.
\]

Similarly, when \( \hat{r}_2 - \tilde{\gamma}_2 > 0 \), we have

\[
\limsup_{t \to +\infty} E\left[ \frac{1}{y^n(t)} \right] \leq \frac{\eta L + \lambda N}{\lambda}.
\]
For arbitrary $\varepsilon \in (0,1)$, choosing $\zeta_2(\varepsilon) = \left(\frac{\lambda\varepsilon}{\eta L + \lambda N}\right)^{1/4}$ and using Chebyshev inequality, we yield the following inequalities,

$$P\{x(t) < \zeta_2\} = P\left\{\frac{1}{x^n(t)} > \frac{1}{\zeta_2^2}\right\} \leq \frac{E\left[\frac{1}{x^n(t)}\right]}{\zeta_2^2}.$$

$$P\{y(t) < \zeta_2\} = P\left\{\frac{1}{y^n(t)} > \frac{1}{\zeta_2^2}\right\} \leq \frac{E\left[\frac{1}{y^n(t)}\right]}{\zeta_2^2}.$$

Hence,

$$\limsup_{t \to +\infty} P\{x(t) < \zeta_2\} \leq \varepsilon, \quad \limsup_{t \to +\infty} P\{y(t) < \zeta_2\} \leq \varepsilon.$$

then,

$$\liminf_{t \to +\infty} P\{x(t) \geq \zeta_2\} \geq 1 - \varepsilon, \quad \liminf_{t \to +\infty} P\{y(t) \geq \zeta_2\} \geq 1 - \varepsilon.$$

Combining Chebyshev's inequality with (13), (14), we can prove that for arbitrary $\varepsilon \in (0,1)$, there is a positive constant $\zeta_1$ such that

$$\liminf_{t \to +\infty} P\{x(t) \leq \zeta_1\} \geq 1 - \varepsilon, \quad \liminf_{t \to +\infty} P\{x(t) \leq \zeta_1\} \geq 1 - \varepsilon.$$

This completes the proof. \(\square\)

5. **Persistence in mean and extinction.** In the description of population dynamics, it is critical to discuss the property of persistence in mean and extinction. First, we give a Lemma using the argument as in [25, 28] with suitable modifications.

**Lemma 5.1.** Suppose that $x_n(t) \in C(\Omega \times [0, \tau), R_+)$, $\forall \tau > 0$.

(A) If there exist three positive constants $T, \eta$ and $\eta_0$ such that

$$\ln x_n(t) \leq \eta t - \eta_0 \int_0^t x_n(s)ds + \int_0^t \sigma_i(s)dW_i(s) + Q(t) \quad i = 1 \text{ or } 2$$

for all $t \geq T$, then

$$\limsup_{t \to +\infty} \frac{\int_0^t x_n(s)ds}{t} \leq \frac{\eta}{\eta_0} \quad \text{a.s.}$$

(B) If there exist three positive constants $T, \eta$ and $\eta_0$ such that

$$\ln x_n(t) \geq \eta t - \eta_0 \int_0^t x_n(s)ds + \int_0^t \sigma_i(s)dW_i(s) + Q(t) \quad i = 1 \text{ or } 2$$

for all $t \geq T$, then

$$\liminf_{t \to +\infty} \frac{\int_0^t x_n(s)ds}{t} \geq \frac{\eta}{\eta_0} \quad \text{a.s.}$$

**Proof.** (A) Denote $M_i(t) = \int_0^t \alpha_i(s)dW_i(s), Q_i(t) = \int_0^t \int_\mathbb{Y} \ln(1 + \gamma_i(s,u))\tilde{N}(ds,du)$, then $M_i(t), Q_i(t), i = 1, 2$ are real valued local martingales vanishing at $t = 0$. One can see that the quadratic variations of $M_i(t)$ and $Q_i(t)$ are

$$\langle M_i(t), M_i(t) \rangle = \int_0^t \alpha_i^2(s)ds \leq \bar{\alpha}_i^2 t,$$

$$\langle Q_i(t), Q_i(t) \rangle = \int_0^t \int_\mathbb{Y} (\ln(1 + \gamma_i(s,u)))^2 \mu(du)ds \leq kt, \quad i = 1, 2$$

where $\langle M, M \rangle$ is Meyer's angle bracket process, and

$$\rho_M(t) = \int_0^t \frac{d\langle M, M \rangle(s)}{(1 + s)^2} < \max\{k, \bar{\alpha}_1^2\} \int_0^t \frac{ds}{(1 + s)^2} < \infty.$$
By the strong law of large numbers for local martingales [30], we have
\[
\lim_{t \to \infty} \frac{\int_0^t \alpha_i(s) dW_i(s)}{t} = 0,
\]
and
\[
\lim_{t \to \infty} \frac{Q_i(t)}{t} = 0, \text{ a.s. } i = 1, 2.
\]
Then for arbitrary \( \varepsilon > 0 \), there exists a \( T_1 > 0 \) such that for \( t > T_1 \)
\[
-\varepsilon t < \int_0^t \alpha_i(s) dW_i(s) + Q_i(t) < \varepsilon t.
\]
Set \( g_n(t) = \int_0^t x_n(s) ds \) for all \( t > T_2 \), then we have
\[
\ln \frac{dg_n(t)}{dt} \leq (\eta + \varepsilon)t - \eta_0 g_n(t), \quad t \geq T = \max\{T_1, T_2\}.
\]
That is to say: for \( t \geq T, e^{\eta_0 g_n(t)} \frac{dg_n(t)}{dt} \leq \eta + \varepsilon \), integrating this inequality from \( T \) to \( t \), we can get
\[
g_n(t) \leq \frac{\ln(e^{\eta_0 g_n(T)} + \frac{\eta_0}{\eta + \varepsilon} (e^{(\eta + \varepsilon)t} - e^{(\eta + \varepsilon)T}))}{\eta_0}.
\]
Therefore
\[
\limsup_{t \to +\infty} \frac{\int_0^t x_n(s) ds}{t} \leq \limsup_{t \to +\infty} \frac{g_n(t)}{t} \leq \limsup_{t \to +\infty} \frac{\ln(e^{(\eta + \varepsilon)t})}{\eta_0 t} = \frac{\eta + \varepsilon}{\eta_0}.
\]
Using the arbitrariness of \( \varepsilon \) we have the assertion.

The proof of (B) is similar to (A). The proof is completed. \( \square \)

Remark 1. The solution \( x(t) \) of (5) has a left limit and is right-continuous, it is measurable and bounded, hence it is integrable. By the Theorem 5.5.1 in [37], there is a \( \{x_n(t)\} \) which is continuous satisfying \( \lim_{n \to \infty} x_n(t) = x(t) \). Therefore, the conclusions in Lemma 5.1 still hold for \( x(t) \).

Using Lemma 5.1 and Remark 1, we have following theorems.

Theorem 5.2. Suppose that \( \tilde{r}_i > \bar{\beta}_i \), \( i = 1, 2 \), \( X(t) \) is the positive solution to (5) with positive initial value \( (x_0, y_0) \), then the problem (5) is persistent in mean.

Proof. The method is similar to [22]. We first deduce
\[
\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq 0, \quad \limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq 0, \text{ a.s.}
\]
Making use of Itô’s formula to \( e^t \ln x \), we deduce
\[
e^t \ln x(t) - \ln x_0 = \int_0^t e^s \ln x(s) + r_1(s) - \beta_1(s) - \varepsilon_1(s)x(s^-) - \frac{b_1(s)x(s^-)}{K_1(s) + y(s^-)} ds + L_1(t) + L_2(t),
\]
where \( L_1(t) = \int_0^t e^s \alpha_1(s) dW_1(s), \) \( L_2(t) = \int_0^t e^s \int_\gamma \ln(1 + \gamma_1(s, u)) \bar{N}(ds, du) \) are martingales with the quadratic forms
\[
(L_1(t), L_1(t)) = \int_0^t e^{2s} \alpha_1^2(s) ds,
\]
\[
(L_2(t), L_2(t)) = \int_0^t e^{2s} \int_\gamma (\ln(1 + \gamma_1(s, u)))^2 \mu(du) ds \leq k \int_0^t e^{2s} ds.
\]
By the exponential martingale inequality [31], for any positive constants \( k, \gamma, \delta \), we can get that

\[
P\left\{ \sup_{0 \leq t \leq \gamma k} \left[ L_i(t) - 0.5e^{-\gamma k}(L_i(t), N_i(t)) \right] \geq \delta e^{\gamma k} \ln k \right\} \leq k^{-\delta},
\]

it follows from the Borel-Cantelli lemma that for almost all \( \omega \), there is \( k_0(\omega) \) such that for each \( k \geq k_0(\omega) \),

\[
L_i(t) \leq 0.5e^{-\gamma k}(L_i(t), L_i(t)) + \delta e^{\gamma k} \ln k, \quad 0 \leq t \leq \gamma k.
\]

Hence

\[
e^t \ln x - \ln x_0 \leq \int_0^t e^s \left[ \ln x(s) + r_1(s) - \beta_1(s) - \varepsilon_1(s) x(s^-) - \frac{b_1(s) x(s^-)}{K_1(s) + y(s^-)} \right] ds
\]

\[+ \frac{1}{2} e^{-\gamma k} \int_0^t e^{2s} \alpha_1^2(s) ds + \frac{1}{2} k e^{-\gamma k} \int_0^t e^{2s} ds + 2\delta e^{\gamma k} \ln k
\]

\[\leq \int_0^t e^s \left[ \ln x(s) + r_1(s) \right] ds + \int_0^t \left( |\gamma_1(s, u)| + |\ln (1 + \gamma_1(s, u))| \right) \mu(du)
\]

\[+ \frac{1}{2} \alpha_1^2(s) \left[ 1 - e^{s-\gamma k} \right] + \frac{1}{2} k e^{s-\gamma k} ds + 2\delta e^{\gamma k} \ln k.
\]

Obviously, for any \( 0 \leq s \leq \gamma k \) and \( x > 0 \), there is a constant \( A \) which is independent of \( k \) such that

\[
\ln x(s) + r_1(s) + \int_\gamma \left( |\gamma_1(s, u)| + |\ln (1 + \gamma_1(s, u))| \right) \mu(du)
\]

\[+ 0.5 \alpha_1^2(s) \left[ 1 - e^{s-\gamma k} \right] + 0.5 k e^{s-\gamma k} \leq A.
\]

Then for \( 0 \leq t \leq \gamma k, k > k_0(\omega) \), we derive

\[
e^t \ln x - \ln x_0 \leq A[e^t - 1] + 2\delta e^{\gamma k} \ln k.
\]

That is

\[
\ln x(t) \leq e^{-t} \ln x_0 + A[1 - e^{-t}] + 2e^{-t} \delta e^{\gamma k} \ln k.
\]

Letting \( t \to \infty \), we have

\[
\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq 0.
\]

Similarly, we get

\[
\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq 0.
\]

On the other hand, applying Itô’s formula to (8) we have:

\[
d\ln \lambda(t) = [r_1(t) - \beta_1(t) - (\varepsilon_1(t) + \frac{b_1(t)}{K_1(t)})] \ln x(t^-) dt + \alpha_1(t) dW_1(t)
\]

\[+ \int_\gamma \ln (1 + \gamma_1(t, u)) \tilde{N}(dt, du).
\]
That is
\[
\ln \lambda(t) = \ln x(0) + \int_0^t \left( r_1(s) - \beta_1(s) - (\varepsilon_1(s) + \frac{b_1(s)}{K_1(s)}) \lambda(s^-) \right) ds
+ \int_0^t \alpha_1(s)W_1(s)ds + Q_1(t).
\]
For \( t \geq T \), we have
\[
\ln \lambda(t) \geq (\hat{r}_1 - \hat{\beta}_1 - \varepsilon)t - (\hat{\varepsilon}_1 + \frac{\hat{b}_1}{\hat{K}_1}) \int_0^{t} \lambda(s^-) ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]
Therefore, we can deduce that from the properties for càdlàg functions [2]
\[
\ln \lambda(t) \geq (\hat{r}_1 - \hat{\beta}_1 - \varepsilon)t - (\hat{\varepsilon}_1 + \frac{\hat{b}_1}{\hat{K}_1}) \int_0^{t} \lambda(s^-) ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]
Let \( \varepsilon \) be sufficiently small such that \( \hat{r}_1 - \hat{\beta}_1 - \varepsilon > 0 \), then applying Lemma 5.1 and Remark 1 to above two inequalities, we get
\[
\frac{\hat{K}_1(\hat{r}_1 - \hat{\beta}_1 - \varepsilon)}{\hat{b}_1 + \hat{\varepsilon}_1 \hat{K}_1} \leq \lim_\rightarrow \inf \int_0^t \frac{\lambda(s^-) ds}{t} \leq \lim_\rightarrow \sup \int_0^t \frac{\lambda(s^-) ds}{t}.
\]
Making use of the arbitrariness of \( \varepsilon \) we get
\[
\lim_\rightarrow \sup \int_0^t \frac{\lambda(s^-) ds}{t} \geq \frac{\hat{K}_1(\hat{r}_1 - \hat{\beta}_1)}{\hat{b}_1 + \hat{\varepsilon}_1 \hat{K}_1}.
\]
Then
\[
\lim_\rightarrow \sup \frac{\ln \lambda(t)}{t} \geq 0, \ a.s.
\]
Therefore
\[
\lim_\rightarrow \sup \frac{\ln x(t)}{t} \geq \lim_\rightarrow \sup \frac{\ln \lambda(t)}{t} \geq 0, \ a.s.
\]
To sum up, we have
\[
\lim_\rightarrow \sup \frac{\ln x(t)}{t} = 0.
\]
Similarly, we yield that
\[
\lim_\rightarrow \sup \frac{\ln y(t)}{t} = 0.
\]
Applying Itô’s formula to \( \ln x(s) \), we yield
\[
\int_0^t \frac{b_1(s)x(s^-)}{K_1(s) + y(s^-)} ds = -\int_0^t d\ln x(s) + \int_0^t (r_1(s) - \beta_1(s))ds + \int_0^t \alpha_1(s)W_1(s) + Q_1(t) - \int_0^t \varepsilon_1(s)x(s^-)ds.
\]
Because of \( \int_0^t b_1(s)x(s^-)ds \geq \int_0^t \frac{K_1(s)b_1(s)x(s^-)}{K_1(s) + y(s^-)} ds \), we obtain
\[
\frac{1}{t} \int_0^t b_1(s)x(s^-)ds \geq \frac{\hat{K}_1}{t} \left[ \left( \ln x(t) - \ln x_0 \right) + \int_0^t (r_1(s) - \beta_1(s))ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t) - \int_0^t \varepsilon_1(s)x(s^-)ds \right].
\]
which is
\[
\frac{1}{t} \int_0^t (b_1(s) + K_1 \varepsilon_1(s)) x(s^-) ds \geq \frac{\hat{K}_1}{t} \left[ -\ln x(t) - \ln x_0 \right] + (\hat{r}_1 - \hat{\beta}_1) t + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]

Similarly, we yield
\[
\lim_{t \to \infty} \frac{\int_0^t \alpha_1(s) dW_1(s)}{t} = 0, \quad \lim_{t \to \infty} \frac{Q_1(t)}{t} = 0,
\]
and
\[
\lim_{t \to \infty} \frac{\ln x(t)}{t} = 0,
\]
we get
\[
\lim_{t \to \infty} \frac{\int_0^t x(s) ds}{t} \geq \lim_{t \to \infty} \frac{\int_0^t x(s^-) ds}{t} \geq \frac{\hat{r}_1 - \hat{\beta}_1}{b_1 + \varepsilon_1 \hat{K}_1} > 0, \text{ a.s.}
\]
Similarly, we yield
\[
\lim_{t \to \infty} \frac{\int_0^t y(s) ds}{t} \geq \frac{\hat{r}_2 - \hat{\beta}_2}{b_2 + \varepsilon_2 \hat{K}_2} > 0, \text{ a.s.}
\]
This completes the proof. \(\square\)

**Theorem 5.3.** Let \(X(t)\) be a positive solution of (5) with positive initial value \(X(0)\), then
(A) If \(\hat{r}_1 < \hat{\beta}_1, \ \hat{r}_2 < \hat{\beta}_2\), then \(x(t), y(t)\) be extinction.
(B) If \(\hat{r}_1 > \hat{\beta}_1, \ \hat{r}_2 < \hat{\beta}_2\), then \(y(t)\) is extinction, \(x(t)\) is persistent in mean.
(C) If \(\hat{r}_1 < \hat{\beta}_1, \ \hat{r}_2 > \hat{\beta}_2\), then \(x(t)\) is extinction, \(y(t)\) is persistent in mean.

**Proof.** We first prove Case (A) of the theorem. Making use of Itô’s formula to \(\ln x(t), x \in [0, +\infty)\) yields
\[
\ln x(t) - \ln x(0) \leq \int_0^t (r_1(s) - \beta_1(s)) ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]
Because of
\[
\lim_{t \to \infty} \frac{\int_0^t \alpha_1(s) dW_1(s)}{t} = 0, \quad \lim_{t \to \infty} \frac{Q_1(t)}{t} = 0, \text{ a.s.}
\]
and \(\hat{r}_1 - \hat{\beta}_1 < 0\), we can deduce
\[
\lim_{t \to \infty} x(t) = 0, \text{ a.s.}
\]
Similarly
\[
\lim_{t \to \infty} y(t) = 0, \text{ a.s.}
\]
**Case (B).** Since that \(\hat{r}_2 < \hat{\beta}_2\), we have \(\lim_{t \to \infty} y(t) = 0, \text{ a.s.}\) On the other hand,
\[
\ln x(t) - \ln x(0) \geq (\hat{r}_1 - \hat{\beta}_1) t - \varepsilon_1 \int_0^t x(s^-) ds - b_1 \int_0^t \frac{x(s^-)}{\hat{K}_1} ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]
We have that
\[
\ln x(t) - \ln x(0) \geq (\hat{r}_1 - \hat{\beta}_1) t - \varepsilon_1 \int_0^t x(s) ds - b_1 \int_0^t \frac{x(s)}{\hat{K}_1} ds + \int_0^t \alpha_1(s) dW_1(s) + Q_1(t).
\]
Making use of Lemma 5.1 and Remark 1, we obtain

\[
\frac{\hat{K}_1(\hat{r}_1 - \hat{\beta}_1)}{\hat{\varepsilon}_1 K_1 + \hat{b}_1} \leq \lim_{t \to \infty} \inf_{t_0 \leq s \leq t} \frac{1}{t} \int_{t_0}^{t} x(s) \, ds, \quad \text{a.s.}
\]

Hence, we get

\[
\lim_{t \to \infty} \inf_{t_0 \leq s \leq t} \frac{1}{t} \int_{t_0}^{t} x(s) \, ds \geq \frac{\hat{K}_1(\hat{r}_1 - \hat{\beta}_1)}{\hat{\varepsilon}_1 K_1 + \hat{b}_1} > 0.
\]

**Case (C).** Similar to the arguments in Case (A) and (B), it is easy to find that: \(x(t)\) is extinction, \(y(t)\) is persistent in mean, if \(\hat{r}_1 < \hat{\beta}_1, \hat{r}_2 > \hat{\beta}_2\).

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