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Discretization effects in the finite element simulation of seismic waves in elastic and elastic-plastic media

Kohei Watanabe · Federico Pisano · Boris Jeremić

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Abstract Presented here is a numerical investigation that (re-) appraises standard rules for space/time discretization in seismic wave propagation analyses. Although the issue is almost off the table of research, situations are often encountered where (established) discretization criteria are not observed and inaccurate results possibly obtained. In particular, a detailed analysis of discretization criteria is carried out for wave propagation through elastic and elastic-plastic media. The establishment of such criteria is especially important when accurate prediction of high-frequency motion is needed and/or in the presence of highly non-linear material models. Current discretization rules for wave problems in solids are critically assessed as a condition sine qua non for improving verification/validation procedures in applied seismology and earthquake engineering. For this purpose, the propagation of shear waves through a 1D stack of 3D finite elements is considered, including the use of wideband input motions in combination with space.

Keywords Wave propagation · Seismic · Discretization · Elastic · Elastic-plastic · Verification

1 Introduction

The study of wave motion is of utmost importance in many applied sciences, as it supports the understanding of transient phenomena in many natural and anthropic dynamic systems. In particular, seismic waves propagating through the earth crust deserve the highest consideration, especially in light of their destructive potential and socio-economic impact.

In the last decades, mathematicians, geophysicists and engineers have devoted massive research efforts to the prediction of seismic motion, based on either analytical [21, 32–34, 39] or numerical methods [2, 53, 63]. When linear elastic wave problems are considered, either time-domain or frequency-domain solutions may be sought, whereas time-domain approaches are usually needed in the presence of non-linearities (constitutive or geometrical). In this respect, it should be remarked that much interest in earthquake engineering is nowadays on non-linear wave phenomena, since they govern (i) the occurrence of natural catastrophes (e.g., landslides and debris flows) induced by soil instabilities, such as liquefaction and strain localization [18, 24, 63]; (ii) the interaction between geomaterials and man-made structures [13, 16, 20, 28, 53, 59].

It is thus apparent that reliable numerical simulations of seismic motion and earthquake-soil-structure interaction can only be performed by means of high-fidelity computational tools, capable of coping with the remarkable complexity of the aforementioned problems. The accuracy of
The ultimate goal of this work is to reopen the debate on the accuracy of wave simulations from a verification/validation perspective, also in the presence of constitutive non-linearities. The results reported provide renovated critical insight into, and review of, traditional discretization rules for practical simulation purposes.

2 FE modeling of 1D seismic wave propagation

1D shear wave problems originate from the ideal situation in which wave propagation is nearly vertical, with no lateral geometrical/material inhomogeneities. In these conditions, all vertical cross-section can be regarded as symmetry planes and the soil deposit undergoes a "double plane-strain" deformation, with both horizontal direct strains prevented by symmetry [10, 49]. As a consequence, all variables only depend on time and vertical elevation (the problem is geometrically one-dimensional), whereas the stress state is still multi-axial [17]. The initial-boundary value problem under consideration is sketched in Fig. 1.

Like in general 3D problems, the numerical analysis of 1D seismic wave propagation requires a suitable computational platform for (i) space/time discretization, (ii) material modeling and (iii) simulation under given initial/boundary conditions. The Real ESSI Simulator has been used here for these purposes.

The Real ESSI Simulator is a software, hardware and documentation system developed specifically for high-fidelity, realistic modeling and simulation of earthquake-soil structure-interaction (ESSI). The Real ESSI program features a number of simple and advanced modeling features. For example, on the finite element side, available are solids elements (8, 20, 27, 8-27 node, dry and saturated bricks), structural elements (trusses, beams, shells), contact elements (frictional slip and gap, dry and saturated), isolator and dissipator elements; on the material side, available are elastic (isotropic, anisotropic, linear and non-linear) and elastic-plastic models (isotropic, anisotropic hardening). The seismic input can be applied using the Domain Reduction Method [7, 61], while sequential and parallel versions of the program are available (the latter is based on the Plastic Domain Decomposition (PDD) method [25]). Recent applications of
2.1 Space discretization and time marching

The Real ESSI program is based on a standard displacement-space described, namely (i) the standard linear elastic material model, (ii) the elastic-plastic von Mises model with linear kinematic hardening [26, 40] and (iii) the bounding surface elastic-plastic model by [48].

2.2.1 Linear elastic model

Discretization issues will be first addressed with reference to linear elastic problems. While relevant concepts in elastodynamics can be found in [21], it is only worth reminding here the relationship between the shear wave velocity \( V \) and the two elastic parameters (Young’s modulus \( E \) and Poisson’s ratio \( \nu \)):

\[
\nu = \frac{1}{2(1+\nu)} \rho \frac{C}{G}
\]

where \( G = \frac{E}{2(1+\nu)} \) is the elastic shear modulus. As for space discretization, the 1D FE model has been built using a stack of properly constrained 3D brick elements—as was previously done, for instance, by [10]. Real ESSI program enables the use of 8-, 20- and 27-node elements, so that several options are given in terms of spatial interpolation order. The well-known Newmark method has been adopted for time marching [43]. The main feature of the integration algorithm relates to the approximate series expansion for displacement and velocity components, \( u \) and \( \dot{u} \), respectively:

\[
\begin{aligned}
\text{space} & \quad V = \frac{E}{\rho} = \beta + \gamma \\
& \text{space(4)}
\end{aligned}
\]

\[
\begin{aligned}
\text{space} & \quad \gamma = \frac{1}{2} (\nu + n) \beta \\
& \text{space(3)}
\end{aligned}
\]

\[
\begin{aligned}
\text{space} & \quad 2 \beta = 4 (\nu + 2)
\end{aligned}
\]

The more sophisticated constitutive relationship recently proposed by [48] will be also used. At variance with the spacecyclically loaded soils [18, 63]. Hereafter, the material

\[
\begin{aligned}
n + 1 & = n + \beta \dot{u} + \beta \ddot{u} + \beta \dddot{u}
\end{aligned}
\]

2.2.2 Elastic-plastic: von Mises kinematic hardening (VMKH) model

The relationship among discretization, accuracy and material non-linearity will be first explored through the elastic-plastic von Mises kinematic hardening (VMKH) model, of the same kind described in [26, 40].
models adopted for wave propagation analyses are briefly

1 Non-linear hardening models should rather be used—see e.g., [9, 10] space

formulation, the PB model can quite accurately reproduce indicating
1. Development of inelastic strains from the very onset of loading. This is reproduced by exploiting the concept of “vanishing yield locus”;
2. Frictional shear strength, i.e., depending on the effective confining pressure;
3. Non-linear hardening, implying a continuous transition from small-strain to failure stiffness;
4. Coupling between deviatoric and volumetric responses;
5. Stiffness degradation and damping under cyclic shear loading.

A remarkable feature of the PBS constitutive formulation is the low number of input parameters required (only seven), which makes the model particularly suitable for practical use:

- Two elastic parameters—$E$ and $\nu$—to characterize the material behavior at vanishing strains;

**Fig. 2** Ormsby wavelet ($f_1=0.1$ Hz, $f_2=1$ Hz, $f_3=18$ Hz, $f_4=20$ Hz)

$\text{space}(\pi f)^2$

- Shear strength parameter—$M$—directly related to the material frictional angle;

$u(t) = A$

$\pi f$:

$\text{sinc} (\pi f t) - \pi f$

$\text{sinc} (\pi f t) - \pi f$

$\text{sinc} (\pi f t)$

- Two parameters—$k$ and $\xi$—governing the development of plastic volumetric strains during shearing;

$\text{sinc} (\pi f t)$

- Two hardening parameters—$h$ and $m$—to be identified on the basis of stiffness degradation and damping cyclic curves.

2.3 Initial/boundary conditions and input motion

All the FE results hereafter presented have been obtained under the following initial and boundary conditions (Fig. 1):

1. The system is initially at rest (nil initial velocities and accelerations);
2. A horizontal $x$-displacement time history is imposed at the bottom boundary to reproduce rigid bedrock conditions;
3. No loads are applied to the top boundary (free surface);
4. The aforementioned “double plane-strain” conditions have been achieved by preventing $y$-displacements throughout the model, as well as imposing master/slave connections to nodes at the same elevation (tied nodes).

As for the input displacement, the Ormsby wavelet [52] fits the authors’ intent:

$sinc(\pi f t)$

where $t$ denotes the physical time and $A$ the signal amplitude, $sinc(x) = (\sin x)/x$ is the cardinal sine function, $f(i = 1, 2, 3, 4)$ stand for the low-cut, low-pass, high-cut and high-pass frequencies, respectively. The meaning of the $f_i$ frequencies can be grasped from Fig. 2b, illustrating the amplitude Fourier spectrum of function (5). In particular, the suitability of the Ormsby wavelet has a twofold motivation:

1. Function (5) has a number of sign reversals and will induce several loading/unloading cycles into the soil undergoing wave motion (Fig. 2a);
2. The peculiar flat branch in the amplitude Fourier spectrum (Fig. 2b) is convenient for frequency domain analysis (see next section).

The above features of the Ormsby wavelet will enable the analysis of discretization effects over frequency ranges of choice. Although most seismic energy relates to frequencies lower than 20 Hz, ensuring accuracy at higher frequencies may be relevant when seismic serviceability analyses are to be performed for structures, systems and
components (SSCs) related to nuclear power plants and other industrial objects.

2.4 Misfit criteria

The analysis of discretization effects requires objective criteria to quantify the discrepancy (misfit) between different numerical solutions. In numerical seismology, the difference between the numerical solution and a reliable reference solution is often adopted for this purpose, although it only enables visual/qualitative observations; simple integral criteria (e.g., root mean square misfit) can provide some quantitative insight, but still with no distinction of amplitude or phase errors.

A significant improvement in this area was introduced by [36], who compared seismograms on the basis of the time-frequency representation (TFR) obtained through continuous wavelet transformation [22]. The TFR of signal misfit allows to extract the time evolution of the spectral content, and thus to define the following local time-frequency envelope difference:

3 spaceLinear elastic wave simulations

In this section, the influence of discretization on accuracy is first discussed for linear elastic problems. As for space/time discretization, [41] stated that “the accuracy of the finite element method depends on the ratio obtained by dividing the length of the side of the largest element by the minimum wavelength of elastic waves propagating in the system. For accurate results this ratio should be smaller than 1/12”. Since then, it has been believed that approximately ten nodes per wavelength are appropriate in most cases, whereas fewer than ten nodes are likely to result in undesired numerical attenuation/dispersion.

Accordingly, suitable space-time discretization is usually determined by considering the minimum relevant wavelength (or highest frequency $f_{\text{max}}$) in the input signal [28]:

\begin{equation}
\text{space} E(t,f) = |W(t,f) - W_e(t,f)| \leq |W(t,f)|
\end{equation}

and time-frequency phase difference:

\begin{equation}
P(t,f) = \arg W_e(t,f)
\end{equation}

space\arg W(t,f)

The selection of appropriate grid spacing and time-step size is usually based on very simple rules. As for space discretization, [41] stated that “the accuracy of the finite element method depends on the ratio obtained by dividing the length of the side of the largest element by the minimum wavelength of elastic waves propagating in the system. For accurate results this ratio should be smaller than 1/12”. Since then, it has been believed that approximately ten nodes per wavelength are appropriate in most cases, whereas fewer than ten nodes are likely to result in undesired numerical attenuation/dispersion.

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3.1 Standard rules for space/time discretization

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\end{equation}

space\arg W(t,f)
space

3.2 Model parameters

The geometrical/mechanical parameters adopted for elastic wave simulations are here reported. A uniform soil layer has been considered, having thickness $H = 1$ km and made of an elastic material with $p = 2000$ kg/m$^3$, $V = 1000$ m/s and $v = 0.3$ (corresponding to $G = 2$ GPa). No Rayleigh damping has been introduced.

---

2 Henceforth, t., □ □ □ □ x will always denote the vertical node spacing, coinciding with the element thickness in the case of 8-node bricks.

### Table 1

| Case # | $f_{\text{max}}$ (Hz) | $u_{\text{x_ad}}$ (m) | $u_{\text{z_ad}}$ (s) | $u_x$ (m) |
|--------|------------------------|----------------------|----------------------|---------|
|        |                        | Brick type           |                      |         |
|        |                        | EL1                  | 20                   | 5       |
|        |                        | EL2                  | 20                   | 5       |
|        |                        | EL3                  | 50                   | 2       |
|        |                        | EL4                  | 50                   | 2       |
|        |                        | EL5                  | 20                   | 5       |
|        |                        | EL6                  | 20                   | 5       |
|        |                        | EL7                  | 20                   | 5       |
|        |                        | EL8                  | 50                   | 2       |
|        |                        | EL9                  | 50                   | 2       |
|        |                        | EL10                 | 20                   | 5       |

---

As for the input motion, two different Ormsby wavelets have been employed, corresponding with the following input parameters in Eq. (5):

- input 1: $f = \frac{\omega}{2 \pi}$ Hz (plotted in Figs. 3a–3c, 4b, 5b, 6b),
- input 2: $f = \frac{\omega}{2 \pi}$ Hz,

---

3.3 Discussion of numerical results

The influence of grid spacing and time-step size is discussed separately for the sake of clarity. Since the Real ESSI program is based on a displacement FE formulation, displacement components are the most reliable output; however, some attention is also paid to accelerations, post-calculated through second-order central differentiation.

Table 1 provides a list of the comparative simulations performed for linear problems. Each case is denoted by: (i) maximum frequency $f_{\text{max}}$ in the input wavelet (in [9]; (ii) grid spacing $u_x$ and (iii) time-step size $u_t$ from standard discretization rules (10) (11); (iv) $u_x$ and (v) $u_t$ actually used; (vi) type of brick elements adopted.

The results being presented aim to assess the quality of standard discretization rules, as well as the improvements attainable through refined discretization. For this purpose, the numerical results are discussed in both time and frequency domains—the Fourier spectra of considered time histories are plotted in terms of (i) amplitude and (ii) phase difference with respect to the analytical solution (known at the free surface). Additional quantitative insight is also gained through the EM and PM misfit criteria introduced in Sect. 2.4. Unless differently stated, numerical outputs at the top of the soil layer are considered.

3.3.1 Influence of grid spacing

Grid spacing effects at the top of the FE model are illustrated in Figs. 3, 4, 5, 6 for the cases EL1–EL5 (Table 1) in terms of: (a–b) displacement time history; (c) Fourier amplitude and (d) phase difference at the surface; (e) EM and PM misfit (for each numerical solution, misfits are calculated with respect to the exact analytical solution). Starting from Fig. 4, displacement time histories are not compared with the input motion (as done in Fig. 3a) for the sake of brevity, whereas only a reduced time window around the output motion is displayed for clearer visualisation (e.g., as in Fig. 3b).

Figs. 3, 4, 5, 6 suggest the following observations (some of which expected):

- even though $u_x$ is set on the basis of the maximum frequency $f_{\text{max}}$ its suitability is not uniform over the input spectrum. Indeed, increasing inaccuracies in the frequency domain are clearly visible as $f_{\text{max}}$ is...
approached (check for instance the Fourier amplitudes compared in Figs. 3c and 4, 5, 6b). Grid spacing affects output Fourier spectra both in amplitude and phase;

- in all cases, envelope and phase misfits, EM and PM, are quantitatively very similar (Figs. 1e and 4, 5, 6d); reducing $u_x$ below $u_{x_c}$ is beneficial only if $u_r$ is also lower than $u_{r_c}$. This is apparent in Fig. 3e, where an increase in EM and PM is observed as $u_x$ gets lower than $u_{x_c}$. Conversely, monotonic EM/PM trends are shown in Figs. 4, 5d;

- at given grid spacing $u_x$, reducing the time-step improves the numerical solution mostly in terms of Fourier phase, not amplitude (comparers Figs. 3c–d, 4b–c). It may be generally stated that, when $u_x$ is not appropriate, reducing the time-step size does not produce substantive improvements;

(a) Displacement time history (0.0–4.0 s)

(b) Displacement time history (2.2–3.8 s)

![Displacement time history diagrams](image)

The above conclusions apply to 8-node brick elements, while Fig. 6 shows that "ten elements per wavelength" are still suitable when higher-order elements (here 27-node bricks\(^3\)) are employed. However, this lighter requirement for grid spacing seems to perform well in combination with

$\frac{\Delta t}{\Delta x} \leq \frac{\Delta V}{\Delta x} = \frac{\Delta t}{\Delta x} = \frac{\Delta t}{\Delta x} = \frac{\Delta t}{\Delta x}$. These enhanced discretization rules hold for low-order FEs (8-node brick elements) but are not affected by the frequency bandwidth of the input signal. In the latter respect, Figs. 4, 5d show quantitatively similar EM-PM trends for $f_{\text{max}}$ equal to 20 Hz and 50 Hz. Also, minimum misfits are attained in the EL2 case (Fig. 4d), where a smaller $\frac{\Delta t}{\Delta x}$ ratio has been purposely set.

It is also important to evaluate grid spacing effects on acceleration components, as they will affect the inertial forces transmitted to man-made structures on the ground surface. Since acceleration time histories are dominated by high frequencies, the poorer performance of standard discretization rules at high frequencies becomes more evident. In Figs. 7 and 8, grid spacing plays qualitatively

![Amplitude and phase difference diagrams](image)

3 For a given number of nodes per wavelength, the size $u_x$ of 27-node elements along the propagation direction is double than for 8-node bricks.
as in Figs. 3, 4, 5, although the EM/PM trends—similar in shape—are shifted upwards. This means that, in the presence of low-order elements, more severe discretization requirements should be fulfilled if very accurate accelerations are needed.

3.3.2 Influence of time-step size

For given grid spacings, the influence of \( \Delta t \) has been studied by varying the time-step size with respect to the limit space

\[
\Delta t = \frac{x}{V}.
\]

Time discretization effects are illustrated in Figs. 9, 10, 11, 12, 13, 14 and suggest the following inferences:

- space as observed in the previous subsection, \( \Delta t \) mainly affects the Fourier phase, with comparable EM and PM values in all cases. Phase differences with respect to
(a) Displacement time history (2.2–3.8 s)
(b) Amplitude of displacement Fourier spectrum

(c) Acceleration time history (2.2–3.8 s)
(d) EM/PM misfits (ref. solution: analytical)

Fig. 6 Influence of grid spacing: displacement plot, case EL5
($f_{\text{max}} = 20)$ Hz, $u_{\text{x}} = 5$, $u_{\text{t}} = 0.005$ s, $u_{\text{x}} = 2, 5, 10$ m, $u_{\text{t}} = 0.002$ s, 27-node brick)

(a) Acceleration time history (2.2–2.8 s)
(b) EM/PM misfits (ref. solution: analytical)
(c) Acceleration time history (2.2–2.8 s)
(d) EM/PM misfits (ref. solution: analytical)

Fig. 7 Influence of grid spacing, acceleration plot, cases (a–b) EL1 ($f_{\text{max}} = 20$ Hz, $u_{\text{x}} = 5$, $u_{\text{t}} = 0.005$ s, $u_{\text{x}} = 2, 5, 10$ m, $u_{\text{t}} = 0.002$ s, 8-node brick) and (c–d) EL2 ($f_{\text{max}} = 20$ Hz, $u_{\text{x}} = 5$, $u_{\text{t}} = 0.005$ s, $u_{\text{x}} = 2, 5, 10$ m, $u_{\text{t}} = 0.002$ s, 8-node brick)

Fig. 8 Influence of grid spacing, acceleration plot, cases (a–b) EL3 ($f_{\text{max}} = 50$ Hz, $u_{\text{x}} = 2$, $u_{\text{t}} = 0.002$ s, $u_{\text{x}} = 0.8, 2, 4$ m, $u_{\text{t}} = 0.002$ s, 8-node brick) and (c–d) EL4 ($f_{\text{max}} = 50$ Hz, $u_{\text{x}} = 2$, $u_{\text{t}} = 0.002$ s, $u_{\text{x}} = 0.8, 2, 4$ m, $u_{\text{t}} = 0.001$ s, 8-node brick)
(c) Phase difference of displacement Fourier spectrum
(d) EM/PM misfits (ref: solution: analytical)

Fig. 9 Influence of time-step size, displacement plot, case EL6
($f_{max} = 20$ Hz, $u_{x_{std}} = 5$ m, $u_{t_{std}} = 0.005$ s, $u_x = 5$ m, $u_t = 0.002$, $0.005$, $1.10$ s, 8-node brick)

- when 27-node bricks are used, the use of $u_x = u_{x_{std}}$ and $\phi t \leq \phi t_{std}/2$ is still an appropriate option, giving rise to EM and PM lower than 5% (Fig. 12). Even in this case, discretization errors are still governed by space phase differences, while excellent performance in terms of Fourier amplitude is observed;
- Figs. 13 and 14 show that the above findings apply qualitative to acceleration time histories as well. However,

Fig. 10 Influence of grid spacing, displacement plot, case EL7
($f_{max} = 20$ Hz, $u_{x_{std}} = 5$ m, $u_{t_{std}} = 0.005$ s, $u_x = 2$ m, $u_t = 0.001$, $0.002$, $0.005$, $1.10$ s, 8-node brick)

(a) Displacement time history (2.2–2.8 s)
(b) Amplitude of displacement Fourier spectrum
(c) Phase difference of displacement Fourier spectrum
(d) EM/PM misfits (ref: solution: analytical)

Fig. 11 Influence of time-step size, displacement plot, case EL9
($f_{\text{max}} = 50$ Hz, $u_{\text{x std}} = 2$ m, $u_{\text{t std}} = 0.002$ s, $u_{\text{x}} = 0.8$ m, $u_{\text{t}} \leq 0.0005$, 0.001, 1.2 s, 8-node brick)

space EM and PM values are quite high (significantly larger than 10%) when $u_t \geq u_{t \text{std}}$, regardless of the grid spacing ratio. Accuracy is quickly regained when $u_t$ is reduced and $\phi_x < \phi_x / 2$.

space While the above conclusions have been all drawn on the basis of the first incoming wave, many reflected waves may in reality hit the ground surface because of soil layering. In the present elastic case (no energy dissipation),

Fig. 12 Influence of time-step size, displacement plot, case EL10
($f_{\text{max}} = 20$ Hz, $u_{\text{x std}} = 5$ m, $u_{\text{t std}} = 0.005$ s, $u_{\text{x}} = 5$ m, $u_{\text{t}} = 0.002$, 0.005, 1.10 s, 27-node brick)
perfect reflections occur at the lower rigid bedrock and never-ending wave motion is established. It is thus interesting to check how discretization errors propagate in time at the free surface, as is shown in Fig. 13. Cumulative wave dispersion is observed. Even though satisfactory accuracy is achieved on the first arrival, an increase in high-frequency phase difference is detected in Fig. 13d, with negligible variation in space.

**Fig. 13** Influence of time-step size, acceleration plot, cases EL6 (f_{max} = 20 Hz, \( \Delta t_{std} = 3 \) m, \( \Delta t_{std} = 0.005 \) s, \( \Delta t = 0.002 \), 0.005, 0.10 s, 8-node brick) and EL7 (f_{max} = 20 Hz, \( \Delta t_{std} = 5 \) m, \( \Delta t_{std} = 0.005 \) s, \( \Delta t = 2 \) m, \( \Delta t = 0.001 \), 0.002, 0.005 s, 8-node brick).

**Fig. 14** Influence of time-step size, acceleration plot, cases EL1a (f_{max} = 50 Hz, \( \Delta t_{std} = 2 \) m, \( \Delta t_{std} = 0.002 \) s, \( \Delta t = 0.001 \), 0.002), 0.003 s, 8-node brick) and EL9 (f_{max} = 50 Hz, \( \Delta t_{std} = 2 \) m, \( \Delta t_{std} = 0.002 \) s, \( \Delta t = 0.001 \), 0.002, 0.005 s, 8-node brick).

**Fig. 15** Time evolution of wave dispersion, displacement plot, case EL7 (f_{max} = 20 Hz, \( \Delta t_{std} = 5 \) m, \( \Delta t_{std} = 0.005 \) s, \( \Delta t = 2 \) m, \( \Delta t = 0.0005 \), 0.001, 0.002, 0.005 s, 8-node brick).

Non-linear elastic-plastic wave simulations

This section concerns discretization effects in presence of material non-linearity. As most commonly done in
...Geomechanics [63], the non-linear cyclic response of geomaterials can be described in the framework of elastoplasticity, and here the VMKH and PM models described in spacemarching rule may be regarded as an upper bound for non-linear problems (instead of (11)):

\[ \delta x_i \leq \delta x_i \]

Sect. 2.2 have been adopted. Prior to presenting numerical results, some preliminary remarks should be made:

\[ \delta x_i \leq \delta x_i \]

1) the non-linear problem under consideration cannot be solved analytically. Therefore, the quality of discretization settings may only be assessed by evaluating the converging behavior of numerical solutions upon \( x \rightarrow \delta x_i \) refinement;

2) with no analytical solution at hand, one needs engineering judgement to establish when the (unknown) exact solution is reasonably approached. In this respect, light is shed on several expected pitfalls, all relevant to the global verification process [3, 45, 51];

3) the accuracy of non-linear computations is highly affected by the input amplitude. This governs the amount of non-linearity mobilized by wave motion and, as a consequence, the accuracy of numerical solutions at varying discretization.

In non-linear (elastic-plastic) problems, discretization is not only responsible for the numerical representation of waves (dissipation, dispersion, stability), but also governs the accuracy of constitutive integration [8, 54]. For instance, changes in time-step size will affect the strain size driving the constitutive integration algorithm and, in turn, the final simulation results. This dependence of the constitutive response (material model and constitutive integration algorithm) on the dynamic step size precludes direct development of automatic criteria for discretization. However, as tangent elastic-plastic response can be established for any stress-strain combination, (lowest) elastic-plastic (shear) stiffness may be used to develop suitable discretization via Equation 4. Apparently, this approach assumes that the stress-strain response is already known, as is not the case when discretization is being set. This means that an iterative approach is in principle needed, whereby one will first design discretization based on an estimate of the strain level, run the dynamic simulation, and record the actual stress-strain response. After few iterations, a stable discretization will be usually achieved.

In this study, VMKH and PM constitutive equations have been integrated via the standard forward Euler, explicit algorithm [11, 15]. Although implicit algorithms may possess better accuracy/stability properties, explicit integration is often preferred for advanced constitutive formulations and cyclic loading [27]. There is also wide consensus on the poor performance in elastic-plastic computations of time-step sizes derived through elastic parameters and Equation (11), especially in combination with explicit stress-remap algorithms. For this reason, the following time spacelin the following, rules (10) and (12) will be assumed as starting discretization criteria and critically assessed. For shorter discussion, only input 1 (\( f_{\text{max}} = 20 \) Hz) and 8-node brick elements are employed for non-linear simulations.

4.1 VMKH model

4.1.1 Model parameters and parametric analysis

A heterogeneous 1 km thick soil deposit has been considered, formed by a 200 m thick VMKH sub-layer resting on an elastic stratum (remaining 800 m). At the surface, a thin layer (5 m) of elastic material has been added to prevent numerical problems with very strong motions and the so-called whip effect. The following constitutive parameters (see Sect. 2.2.2) have been set (same elastic parameters for both the VMKH and the elastic sub-layers), with no algorithmic nor Rayleigh damping introduced in numerical computations:

- mass density and elastic properties: \( \rho = 2000 \) kg/m\(^3\), \( E = 5.2 \) GPa and 0.3, whence the elastic shear wave velocity \( V_s = 1000 \) m/s results (same elastic parameters employed for both the elastic and the VMKH sub-layers);
- yielding parameter (radius of the von Mises cylinder): \( k = 10.4 \) kPa;
- different \( h \) values (hardening parameter) have been set: \( h = 0.5E, 0.05E, 0.01E \).

In the analysis of VMKH cases, the influence of the hardening parameter \( h \) and the input amplitude \( A \) has been also considered, as they affect the material elastic-plastic stiffness and the amount of plasticity mobilized. The VMKH simulation programme is reported in Table 2, where \( t_i \) has been determined through Equation (12) (i.e., \( t_i = \delta x_i / 10V \)).

4.1.2 Influence of grid spacing and time-step size

The results in Figs. 16 and 17 exemplify the role played by space discretization in elastic-plastic simulations. These results have been obtained by employing a time-step smaller than \( t_i \) (cases VMKH1–2 in Table 2), a low input amplitude \( A = 0.1 \) mm corresponds with a peak
ground acceleration approximately equal to 0.05g), and two different values of the hardening parameter \( h = 0.5E \) and \( h = 0.05E \). The following observations arise from the two figures:

- Table 2 List of VMKH space simulations

| Case # | \( u_{x,\text{rad}} \) (m) | \( u_{i,\text{rad}} \) (s) | \( u_{x} \) (m) | \( h \) (s) |
|--------|----------------|----------------|--------------|-----------|
| VMKH1  | 5             | 0.0005         | -            | 0.0005    |
| VMKH2  | 5             | 0.0005         | -            | 0.0005    |
| VMKH3  | 5             | 0.0005         | -            | 0.0005    |
| VMKH4  | 5             | 0.0005         | -            | 0.0005    |
| VMKH5  | 5             | 0.0005         | -            | 0.0005    |
| VMKH6  | 5             | 0.0005         | -            | 0.0005    |
| VMKH7  | 5             | 0.0005         | -            | 0.0005    |
| VMKH8  | 5             | 0.0005         | -            | 0.0005    |
| VMKH9  | 5             | 0.0005         | -            | 0.0005    |
| VMKH10 | 5             | 0.0005         | -            | 0.0005    |

**Table 2** List of VMKH space simulations

- Spacing: Influence of grid spacing, displacement plot, case VMKH1 (\( u_{x,\text{rad}} = 5 \text{ m}, u_{i,\text{rad}} = 0.0005 \text{ s}, u_{x} = 1 \text{ m}, \sigma_{x} = 0.0001 \)), which leads to irreversible displacement (Fig. 17c), which affects substantial internal decay of grid effects, and also influences the final time integration. Since the effects of \( u_{x} \) reduction are quite small in both time and frequency domains (for a given \( \sigma_{x} \)), it is reasonable to suggest that the influence should be actually appropriate in common practical situations, as long as no soil failure mechanisms are triggered - as for example in seismic loading.

- Steady irreversible deformations are associated with prominent static components (at nil frequency) in the Fourier amplitude spectrum (Figs. 16, 17a), not present in the input Ormsby wavelet (Fig. 2b); the numerical representation of wavelengths is dominated by soil plasticity, producing more deviation from the input waveform than variations in grid spacing. For the breaks, only two \( u_{x} \) values have been used in this subsection for illustrative purposes, whereas EM/PM plots have been deemed not necessary; the influence of \( u_{x} \) seems slightly magnified when lower \( h \) values, and thus lower elastic-plastic stiffness, are used (see Fig. 17). It is indeed not surprising that wave propagation in softer media may be more affected by space discretization, as in linear problems. However, it should be noted that \( u_{x} \) mainly influences the final irreversibility (Fig. 17b, c), which leads to substantial internal decay of grid effects, and also influences the final time integration. Since the effects of \( u_{x} \) reduction are quite small in both time and frequency domains (for a given \( \sigma_{x} \)), it is reasonable to suggest that the influence should be actually appropriate in common practical situations, as long as no soil failure mechanisms are triggered – as for example in seismic loading.

**Fig. 16** Influence of grid spacing, displacement plot, case VMKH1 (\( u_{x,\text{rad}} = 5 \text{ m}, u_{i,\text{rad}} = 0.0005 \text{ s}, u_{x} = 1 \text{ m}, \sigma_{x} = 0.0001 \)). Spacing through a dissipative elastic-plastic material alters significantly the shape of the input signal. All plots display significant wave attenuation/distortion, while final unrecoverable displacements are produced by soil plastifications (Figs. (a) Displacement time history (0.0–4.0 s) (b) Displacement time history (2.2–3.8 s)
Fig. 17 Influence of grid spacing, displacement plot, case VMKH2 (ux std = 5 m, ut std = 0.0005 s, ux = 1 m, 5 m, ur = 0.0001 s, h = 0.05E, A = 0.1 mm)

In addition, Fig. 18 illustrates the shear stress-strain VMKH response at the deepest integration (Gauss) point of the VMKH sub-layer. The material response is bilinear (elastic and elastic-plastic), with the elastic stiffness recovered upon stress reversal until new yielding occurs [40]. As mentioned above, the observable (small) differences in stress-strain response at different space/ux may not be straightforwardly attributed to grid spacing deficiencies, but rather to the coupled influence of discretization in space and time on the global dynamics of the system.

The influence of the time-step size is illustrated for cases VMKH3–5 (Table 2) in Figs. 19, 20, encompassing three h values (0.5E, 0.05E and 0.01E) and also including EM/PM plots (Fig. 19d). In the lack of analytical solutions, misfits have been determined on the basis of a “sufficiently accurate” reference solution, here obtained numerically by setting $\xi = \xi_t / 5 = 0.0001$ s. For a relatively small input amplitude ($A = 0.1$ mm), convergence seems overall quite fast, and even $ur = ut$ results in both EM and PM values.
4.1.3 Influence of input motion

In non-linear problems, it is hard to draw general conclusions on the interaction between space/time discretization and input amplitude. The effect of soil non-linearity mobilized stiffness, in turn affecting the requirements for accurate constitutive integration.

In Fig. 21, the parametric study in Figs. 22 and 23, illustrating the results obtained for $u_x = \xi_x$, $h$ equal to 0.5E, 0.05E and 0.01E (cases VMKH8-10 in Table 2); EM/PM plots come from the numerical reference solution corresponding with $\phi = 0.0001 s$.

The comparison of Figs. 21 and 22 suggests that, even with a much larger input amplitude, $u_x = \xi_x$ is still an appropriate grid spacing for elastic-plastic problems, as long as $u_x$ is substantially reduced to comply with (explicit) constitutive integration requirements. This inference is supported by the following observations:

- $u_x$ affects not only the residual component of displacement time histories (as in Fig. 21), but also their maximum/minimum transient values – i.e., the numerical space.

Both findings are likely related to constitutive integration on the residual displacement than on other response variables: accumulated displacement mainly affects the output signal, not its phase attributes. In fact, variations in the stress-strain response at the bottom of the VMKH sub-layer (Fig. 20), exhibiting little sensitiveness to the time-step size. Some additional comments stem from the EM/PM trends do not depend monotonically on the hardening parameter $h$. For PM values at $h = 0.05E$ are obtained for $h = 0.5E$ and $h$.


\[ t = 0.5 E \]

\[ \text{strain} \]

\[ s - s \]

\[ V_{\text{MKH}}^4 (h = 0.05E) \text{ and VMKH}^5 \]

\[ \text{response} \]

\[ a \]

\[ \text{at the bottom of the } \]

\[ V_{\text{MKH}} \]

\[ \text{sub-layer}, \text{ cases } \]

\[ V_{\text{MKH}}^3 \]

\[ (h) \]
**Fig. 20** Influnce of time-step size, displacement plot, cases VMKH6 \(h = 0.5E\) (2.2–3.8 s) and VMKH7 \(h = 0.05E\) (2.2–3.8 s) for acceptable constitutive integration and overall accuracy in elastic-plastic simulations. However, this heuristic conclusion may be altered by the use of different material models (see next section) and stress-point algorithms.

This set of results suggests that \(\tau_r\) should be at least in the order of \(\sqrt{\Delta t/20}\) for acceptable constitutive integration and overall accuracy in elastic-plastic simulations. However, this heuristic conclusion may be altered by the use of different material models (see next section) and stress-point algorithms.

**4.2 PBS model**

**4.2.1 Model parameters and parametric analysis**

The influence of space/time discretization is now explored in combination with the non-linear PBS soil model introduced in Sect. 2.2.3 [48]. As in real geomaterials, the PBS model features an elastic-plastic response since the very onset of loading (vanishing yield locus), with the stiffness smoothly evolving space from small-strain elastic behavior to failure (nil stiffness).

The results presented hereafter concern a 500 m thick soil layer, whose upper 100 m are made of a non-linear PBS soil resting on a 400 m elastic sub-layer. As done for
the VMKH simulations, a thin layer (2.5 m) of elastic material has been added to prevent numerical problems with very strong motions and the whip effect at the ground surface. Input 1 with \( A = 1 \) mm has been exclusively considered, along with the following set of PBS parameters [48] (the same elastic parameters for both the PBS and the elastic sub-layers have been set):

- \( \rho = 2000 \) kg/m\(^3\), \( E = 1.3 \) GPa and \( \mu = 0.3 \), implying an elastic shear wave velocity \( V_s = 500 \) m/s;
- shear strength parameter: \( M = 1.2 \), corresponding with friction angle equal to 30 deg under triaxial compression;
- dilatancy parameters: \( k = 0.0 \) and \( \xi = 0.04 \);
- hardening parameters: \( h = 300 \) and \( 1 \).

The list of PBS simulations is reported in Table 3, while the next figures will also illustrate the good performance of the PBS model in reproducing the cyclic soil behavior.

---

Table 3 List of PBS simulations

| Case# | \( u_{x_{std}} \) (m) | \( u_{t_{std}} \) (s) | \( u_x \) (m) | \( u_t \) (s) | \( A \) (mm) |
|-------|-------------------|-------------------|-------------|-------------|-------------|
| PBS1  | 2.5               | 0.0005            | 0.5, 2.5   | 0.0001      | 1           |
| PBS2  | 2.5               | 0.0005            | 0.1, 0.5   | 0.00002     | 1           |
4.2.2 Influence of grid spacing and time-step size

Most of the issues observed in VMKH simulations appear to be magnified by the more complex PBS model. A summary of the main inferences drawn on the basis of Figs. 24, 25, 26, 27, 28, 29, 30 is provided below.

Grid spacing turns out to be influential again (Figs. 24, 26), as a consequence of more severe variations (than in VMKH cases) in shear stiffness during cyclic loading. In fact, one would have to follow the stiffness reduction space

(a) Displacement time history (0.0–4.0 s)
(b) Displacement time history (2.2–3.8 s)

Fig. 24 Influence of grid spacing, displacement plot, case PBS1 ($u_{x \text{std}} = 2.5 \text{ m}, u_{t \text{std}} = 0.0005 \text{ s}, u_x = 0.5, 2.5 \text{ m}, u_t = 0.0001 \text{ s}, A = 1 \text{ mm}$)

(a) At the top of the layer
(b) At the bottom of the layer

Fig. 25 Influence of grid spacing, shear stress-strain response in the PBS sub-layer, case PBS1 ($u_{x \text{std}} = 2.5 \text{ m}, u_{t \text{std}} = 0.0005 \text{ s}, u_x = 0.5, 2.5 \text{ m}, u_t = 0.0001 \text{ s}, A = 1 \text{ mm}$)

(a) Displacement time history (0.0–4.0 s)
(b) Displacement time history (2.2–3.8 s)

Fig. 27 Influence of time-step size, displacement plot, case PBS3 ($u_{x \text{std}} = 2.5 \text{ m}, u_{t \text{std}} = 0.0005 \text{ s}, u_x = 2.5 \text{ m}, u_t = 0.0002, 0.0005, 0.0001 \text{ s}, A = 1 \text{ mm}$)
Figure 2

Influence of footstep parameters in the PbS bulb, as PB3 (\text{std} = 2.5 \text{m}, \text{mut} = 0.0005 s, \text{ux} = 2.5 \text{m}).
Fig. 29 Influence of time-step size, displacement plot, case PBS4 ($u_x_{\text{std}} = 2.5\text{ m}, u_t_{\text{std}} = 0.0005\text{ s}, u_x = 2.5\text{ m}, u_t = 0.00001, 0.00002, 0.0001\text{ s}, A = 1\text{ mm})$.

- As in VMKH simulations, grid spacing mainly affects residual displacements. This is clearly shown by the space curves arising from the constitutive response, and use minimum stiffness to decide on space discretization;
- As in VMKH simulations, grid spacing mainly affects residual displacements. This is clearly shown by the

(a) Displacement time history (2.2–3.8 s)

(b) EM/PM misfits (ref. solution: $\Delta t = 0.00001\text{ s}$)
Fig. 30 Influence of time-step size, shear stress-strain response at the bottom of the PBS sub-layer, case PBS4 ($u_{xstd} = 2.5\, \text{m}, u_{tstd} = 0.0005\, \text{s}, u_x = 2.5\, \text{m}, u_t = 0.00001, 0.00002, 0.0001\, \text{s}, A = 1\, \text{mm}$)

Fig. 26b, where EM errors larger than 10% arise even when a very small time-step size is used ($\Delta t = \Delta t_{std}/25 = 0.00002\, \text{s}$); conversely, phase misfits are less affected by residual displacements and thus always quite limited. In presence of high non-linearity, it seems safer to use $U_x \approx 4 \div 5$ times smaller than $u_{xstd} = V/10f_{max}$;

- the combination of explicit constitutive integration and high non-linearity makes time-stepping effects quite prominent, as is shown by Figs. 27 and 28. Further, Fig. 29 leads to conclude that $\Delta t = \Delta t_{std}/50$ may be needed to obtain EM errors lower than 10% (Figs. 29, 30). Apparently, analysts have to compromise on accuracy and computational costs in these situations;

- as expected, the shear stress-strain cycles in Figs. 25 and 28 show that the sensitivity to discretization builds up as increasing non-linearity is mobilized. This is the case for instance at the top of the PBS layer, where cycles are more dissipative than at the bottom due to lower overburden stresses and dynamic amplification.

Since displacement components result from strains through spatial integration, the displacement performance can be well-predicted on condition that strains are accurately computed all along the soil domain. For the same reason, the discretization requirements for displacement space

Fig. 31 Influence of grid spacing at different locations along the PBS layer, displacement plot, PBS2 case ($u_{xstd} = 2.5\, \text{m}, u_{tstd} = 0.0005\, \text{s}, u_x = 0.1, 0.5, 1\, \text{m}, u_t = 0.00002\, \text{s}, A = 1\, \text{mm}$)
space convergence are not uniform along the soil deposit. Figs. 31 and 32 illustrate in the time-domain the displacements simulated at different depths in the nonlinear sub-layer (the vertical x axis points upward—Fig. 1) and at different locations along the PBS layer, displacement plot, PBS4 case ($\Delta x = 2.5$ m, $u_t = 0.0005$ s, $\Delta x = 2.5$ m, $\Delta t = 0.00001, 0.00002, 0.0001$ s, $A = 1$ mm)

$\Delta x$ and $u_t$. These figures clearly point out that accuracy space requirements may be more or less hard to satisfy depending on the specific spatial location. In 1D wave propagation problems, faster convergence is attained far from the ground surface, since it requires satisfactory accuracy in a lower number of nodes and integration points.

Fig. 32 Influence of time-step size at different locations along the PBS layer, displacement plot, PBS4 case ($\Delta x = 2.5$ m, $u_t = 0.0005$ s, $\Delta x = 2.5$ m, $\Delta t = 0.00001, 0.00002, 0.0001$ s, $A = 1$ mm)

(a) $x = 500$ m
(b) $x = 480$ m
(c) $x = 460$ m
(d) $x = 440$ m

Fig. 33 Influence of grid spacing at different locations along the PBS layer, acceleration plot, PBS2 case ($\Delta x = 2.5$ m, $u_t = 0.0005$ s, $\Delta x = 0.1, 0.5, 1$ m, $\Delta t = 0.00002$ s, $A = 1$ mm)

(a) $x = 500$ m
(b) $x = 480$ m
(c) $x = 460$ m
(d) $x = 440$ m
Conclusions inferred are hereafter summarized:  

- The main material models (referred to as VMKH and PBS) have been used at increasing level of complexity. The initial linear computations, two different non-linear high-frequency motion and exploring the relationship has been numerically simulated, with focus on capturing propagation of seismic shear waves. In Computational Dynamics. The 1D procedures in Computational Dynamics. The 1D

0.4 spaceplane. When linear elements (8-node bricks) are used, \( \psi = \psi x / 2 \) and \( \psi_t = \psi t / 2 \) seem to ensure sufficient accuracy over the whole frequency range (both in amplitude and phase); higher-order elements (e.g., 27-node bricks) will allow the use of \( \psi x = \psi x_0 \) still in combination with \( \psi t = \psi t / 2 \). Preserving accuracy in simulations with large domains and/or time durations seems intrinsically more difficult, since attenuation/dispersion phenomena are cumulative.

0.6 Elastic-plastic simulations Conclusion criteria for elastic-plastic problems can be hardly established, as space/time discretization also interferes with the integration of non-linear constitutive equations. In this respect, different outcomes may be found depending on (i) kind of non-linear-erity associated with the material model (stiffness variations during straining), (ii) stress-point integration algorithm (e.g., explicit or implicit), (iii) input motion amplitude. The experience gained through the use of the PBS model (explicitly integrated in 8-node brick elements) suggests that \( \psi x = \psi x_0 \) and \( \psi t = \psi t / 10 \) may need to be reduced by factors up to 4 ÷ 5 and 50, respectively, in the presence of strong input motions and severe stiffness variations. Importantly, these conclusions also depend on which output component is considered and where within the computational domain.

0.8 The present study is, however, not conclusive, especially when it comes to non-linear elastic-plastic problems. There are in fact several aspects that will deserve in the future further consideration, such as the implications space of using higher-order finite elements. The same comment applies to geometrical effects (e.g., wave scattering) in 2D/3D problems, whose influence on discretization criteria for elastic-plastic simulations would be per se a whole research topic.

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5 Concluding remarks

Previously established criteria for space/time discretization in wave propagation FE simulations have been reappraised and critically discussed to strengthen verification procedures in Computational Dynamics. The 1D propagation of seismic shear waves (Ormsby wavelets) through both linear and non-linear (elastic-plastic) media has been numerically simulated, with focus on capturing high-frequency motion and exploring the relationship between material response and discretization effects. After initial linear computations, two different non-linear material models (referred to as VMKH and PBS) have been used at increasing level of complexity. The main conclusions inferred are hereafter summarized:

- Elastic simulations Setting grid spacing (element size) and time-step size as per standard rules (\( x_{ad} = V / 10 f_{max} \) and \( t_{ad} = V / 10 f_{max} \)) has proven not always appropriate, especially to reproduce high-frequency motion components (this can be clearly visualized in the Fourier phase spaceplane).

- Space Conversely, the close relationship between plastic strains and residual displacements has slender influence on acceleration components. In this respect, Figs. 33 and 34 show that, as long as reasonable grid spacing is set (possibly in the order of \( x_{ad} / 2 = V / 20 f_{max} \)), the sensitivity of acceleration components to \( t / 2 \) is much weaker than for residual displacements.

- Space of using higher-order finite elements. The same comment applies to geometrical effects (e.g., wave scattering) in 2D/3D problems, whose influence on discretization criteria for elastic-plastic simulations would be per se a whole research topic.

References
Abell J (2016) Earthquake-soil-structure interaction modeling of nuclear power plants for near-field events. PhD thesis, University of California Davis

Argyris JH, Mlejnek HP (1991) Dynamics of structures. North-Holland, Amsterdam

Babuška I, Oden JT (2004) Verification and validation in computational engineering and science: basic concepts. Comput Methds Appl Mech Eng 193(36):4057–4066

Bao H, Bielak J, Chantzos O, Kallivokas LF, O’Halloran DR, Shewchuk JR, Xu J (1998) Large-scale simulation of elastic wave propagation in heterogeneous media on parallel computers. Comput Methods Appl Mech Eng 152(1–2):85–102

Bayliss A, Goldstein CJ, Turkel E (1985) On accuracy conditions for the numerical computation of waves. J Comput Phys 59(3):396–404

Benjemaa M, Glinsky-Olivier N, Cruz-Atienza V, Virieux J, Piperno S (2007) Dynamic non-planar crack rupture by a finite volume method. Geophys J Int 171(1):271–285

Bielak J, Loukakis K, Hisada Y, Yoshimura C (2003) Domain reduction method for three-dimensional earthquake modeling in localized regions. Part I: theory. Bull Seismol Soc Am 93(2):817–824

Borja RI (2013) Plasticity modeling and computation. Springer, Berlin

Borja RI, Amies AP (1994) Multiaxial cyclic plasticity model for clays. J Geotech Engrg 120(6):1051–1070

Borja RI, Chao H-Y, Montàns FJ, Lin C-H (1999) Nonlinear ground response at lotung site. J Geotech Geoenvironmental Eng 125(3):187–197

Chen WF, Han FY (1988) Plasticity for structural engineers.

Cheng Z, Jeremić B (2009) Numerical modeling and simulation of plate in liguefiable soil. Soil Dyn Earthq Eng 29(11):1405–1416

Chopra AK (2000) Dynamics of structures: theory and applications to earthquake engineering, 2nd edn. Prentice Hall, Englewood Cliffs

De Basabe JD, Sen MK (2007) Grid dispersion and stability criteria of some common finite-element methods for acoustic and elastic wave equations. Geophysics 72(6):T81–T95

Desai CS, Sirivardane HJ (1984) Constitutive laws for engineering materials with emphasis on geologic materials, Prentice-Hall, Englewood Cliffs, p 07632

space

spaced Prisco C, Pisanò F (2011) Seismic response of rigid shallow footings. Eur J Environ Civil Eng 15(sup1):185–221

di Prisco C, Pastor M, Pisanò F (2012) Shear wave propagation along infinite slopes: a theoretically based numerical study. Int J Numer Anal Methods Geomech 36(3):619–642

di Prisco CG, Wood DM (2012) Mechanical behaviour of soils under environmentally induced cyclic loads, vol 534. Springer, Berlin

Fichtner A, Igel H (2008) Efficient numerical surface wave propagation through the optimization of discrete crustal model technique based on non-linear dispersion curve matching (DCM). Geophys J Int 173(2):519–533

Gazetas G, Mylonakis G (1998) Seismic soil-structure interaction: new evidence and emerging issues. In: Geotechnical earth-quake engineering and soil dynamics III, ASCE, pp 1119–1174

Graaff KF (1975) Wave motion in elastic solids. Courier Dover Publications, York

Holsech, J (1995) Wavelets: an analysis tool. Oxford Science Publications, Oxford

Hughes TJ (2012) The finite element method: linear static and dynamic finite element analysis. Courier Dover Publications, New York

Ishihara K (2006) Soil behaviour in earthquake geotechnics. Oxford Universi y Press, Oxford

B, Jie G (2008) Parallel soil–foundation–structure com-putations. In: Papadrakakis NLM, Charmpis DC, Tsonopoulos Y (eds) Progress in computational dynamics and earthquake engi-neering, Taylor and Francis, London

B, Yang Z, Cheng Z, Jie G, K Sett K, Taiebat M, Preisig M, Tafazzoli N, Tasipooulou P, Mena JAA, Pisanò F, Watanabe K, Karapiperis K (2004) Lecture notes on computational geomechanics. In: Elastic finite elements for pressure sensitive materials. Terahminal Report UCD-CompGeoMech-M01–2004, University of California, Davis, 1989-2015

Jeremić B, Cheng Z, Taiebat M, Daafallas Y (2008) Numerical simulation of fully saturated porous materials. Int J Numer Anal Metho s Geomech 32(13):1635–1660

Jeremić B, Jie G, Preisig M, Tafazzoli N (2009) Time domain simulation of soil-foundation-structure interaction in non-uniform soil. Earthq Eng Struct Dyn 38(5):699–718

Jeremić B, Roche-Rivera R, Kammerer A, Tafazzoli J, Abell N, Kamranimoghaddam B, Pisanò F, Jeong C, Aldridge B (2013a) The nrc essi simulator program: current status. In: Proceedings of the structural mechanics in reactor technology (SMiRT) 2013

Jeremić B, N, Blahoianu A (2013b) Seismic behavior of npp structures subjected to realistic seismic events, in liquefiable soil, on surface or embedded foundations. Nucl Eng Design 265:85–94

Kaiser M, Hermann V, de la Puente J (2008) Quantitative accuracy analysis of the discontinuous galerkin method for seismic wave propagation. Geophys J Int 173:990–999

Kausel E (2006) Fundamental solutions in elastodynamics: a compendium. Cambridge University Press, Cambridge

Kausel E, Manolis G (2000) Wave motion in earthquake engi-neering. Wit Press, Southampton

Kolsky H (1963) Stress waves in solids, vol 1098. Courier Dover Publications, New York

Kramer S (1996) Geotechnical earthquake engineering. Prentice Hall, Upper Saddle River, NJ

Kristekova M, Kristek J, Moczo P, Day SM (2006) Misfit criteria for quantitative comparison of seismograms. Bullentin of the Seismological Society of America 96(5):1836–1850

Kristekova M, Kristek J, Moczo P (2009) Time-frequency misfit and goodness-of-fit criteria for quantitative comparison of time signals. Geophys J Int 178:813–825

Kuhlemeier R, Lysmer J (1973) Finite element method accuracy for wave propagation problems. J Soil Mech Found Div, p 99 (Tech Rpt)

Lai CG, Wilmanski K (2005) Surface waves in geomechan-ics: direct and inverse modelling for soils and rocks, vol 481. Springer, Berlin

Lemaître J, Chaboche (1990) Mechanics of solid materials. Cambridge university press, Cambridge

Lysmer J, Kuhlemeier R (1969) Finite dynamic model for infinite media. ASCE, J Eng Mech Div 95(EM4):859–877

Moczo P, Kristek J, Galis M, Pazak P, Balazovjcek M (2007) The finite-difference and finite-element modeling of seismic wave propagation and earthquake motion. Acta Phys Slovaca 57(2):177–406

Newmark NM (1959) A method of computation for structural dynamics. J Eng Mech Div 85(3):67–94

Novak R (2012) Soil mechanics. Wiley, New York

Oberkampf WL, Trucano TG, Hirsch C (2004) Verification, validation, and predictive capability in computational engineering and physics. Appl Mech Rev 57(5):345–384

Orbey N, Jeremi B, Abell J, Luo C, Kennedy RR, Blaihoanu A (2015) Use of non-linear, time domain analysis for design. In: Proceedings of the structural mechanics in reactor technology (SMiRT) 2015 Conference, Manchester, 10–14 Aug 2015

Pérez-Ruiz J, Luzón F, García-Jerez A (2007) Scattering of elastic waves in cracked media using a finite difference method. Stud s Geod 51(1):59–88

Jeremić
2.1 Space discretization and time marching

The Real ESSI program is based on a standard displacement FE formulation, where displacement components are taken as unknown variables in the numerical approximation. As for space discretization, the 1D FE model has been built using a stack of properly constrained 3D brick elements—as was previously done, for instance, by [10]. The ESSI program enables the use of 8-, 20- and 27-nod
elements, so that several options are given in terms of space interpolation order.

The well-known Newmark method has been adopted for time marching [43]. The main feature of the integration algorithm relates to the approximate series expansion for displacement and velocity components, \( u \) and \( \dot{u} \), respectively:

\[
\begin{align*}
    u^{n+1} & = u^n + \Delta t \dot{u}^{n} + \Delta^2 t \left( \frac{1}{2} - \beta \right) \dddot{u}^n + \beta u^{n+1} \\
    \dot{u}^{n+1} & = \dot{u}^n + \Delta t \left[ (1 - \gamma) \dddot{u}^n + \gamma \dot{u}^{n+1} \right]
\end{align*}
\]

between two subsequent time-steps \( n \) and \( n+1 \). Importantly, the expansion uses two parameters, \( \beta \) and \( \gamma \), governing the accuracy and stability properties of the algorithm. It is worth reminding that the algorithm is unconditionally stable as long as [23]:

\[
\gamma \geq \frac{1}{2}, \quad \beta = \frac{1}{4} \left( \gamma + \frac{1}{2} \right)^2
\]

\( \gamma = 1/2 \) is required for second-order accuracy, whereas any \( \gamma \) value larger than \( 1/2 \) introduces numerical attenuation (damping). In this study, the pair \( \gamma = 1/2 \) and \( \beta = 1/4 \) (no algorithmic dissipation) is exclusively considered.

2.2 Material modeling
\[ \Delta E(t,f) = |W(t,f)| - |W_{REF}(t,f)| \]
and time-frequency phase difference:
\[ \Delta P(t,f) = |W_{REF}(t,f)| \frac{\arg[W(t,f)] - \arg[W_{REF}(t,f)]}{\pi} \]
where \(W(t,f)\) and \(W_{REF}(t,f)\) are the TFR (wavelet transform) of the signal "under evaluation" and the reference seismogram, respectively. As explained by [36], it is possible to obtain purely time- or frequency-dependent misfit measures by projecting \(\Delta E\) and \(\Delta P\) onto one of the two domains. In particular, the following single-valued misfit measures for envelope misfit (EM)
\[ EM = \sqrt{\frac{\sum_f \sum_t |\Delta E(t,f)|^2}{\sum_f \sum_t |W_{REF}(t,f)|^2}} \]
and phase misfit (PM)
\[ PM = \sqrt{\frac{\sum_f \sum_t |\Delta P(t,f)|^2}{\sum_f \sum_t |W_{REF}(t,f)|^2}} \]

\[ \tau(t) = A \left[ \frac{(\pi f_4)^2}{\pi f_4 - \pi f_3} \text{sinc}^2(\pi f_4 t) - \frac{(\pi f_3)^2}{\pi f_4 - \pi f_3} \text{sinc} \right] - \left[ \frac{(\pi f_2)^2}{\pi f_2 - \pi f_1} \text{sinc}^2(\pi f_2 t) - \frac{(\pi f_1)^2}{\pi f_2 - \pi f_1} \text{sinc} \right] \]
Table 1  List of elastic simulations

| Case # | \( f_{\text{max}} \) |
|--------|---------------------|
| EL1    | 20                  |
| EL2    | 20                  |
| EL3    | 50                  |
| EL4    | 50                  |
| EL5    | 20                  |
| EL6    | 20                  |
| EL7    | 20                  |
| EL8    | 50                  |
| EL9    | 50                  |
| EL10   | 20                  |

As for the input motion, two different Orr have been employed, corresponding with input parameters in Eq. (5):

- input 1: \( f_1 = 0.1 \text{ Hz}, f_2 = 1 \text{ Hz}, f_3 = 18 \text{ Hz} \) (plotted in Fig. 2);
- input 2: \( f_1 = 0.1 \text{ Hz}, f_2 = 1 \text{ Hz}, f_3 = 45 \text{ Hz} \)
- the amplitude parameter \( A \) has been alwa duce at the bottom of the layer a maxim ment of 1 mm.
Fig. 3 Influence of grid spacing, displacement plot, (c) 8-node brick

- based on these initial examples, a grid in the order of $V_s/20f_{\text{max}} = \Delta x_{\text{std}}/2$ ensures accuracy (EM and PM < 10%) in combination with $\Delta t = \Delta x/2V_s = \Delta t_{\text{std}}/2$. These enhanced rules hold for low-order FEs (8-node brick) but are not affected by the frequency band of the input signal. In the latter respect, Figs. 4, 5 depict qualitatively similar EM, PM trends for $f_0$ as 10
Fig. 4 Influence of grid spacing, displacement plot, c 8-node brick
(a) Displacement time history (2.2-)

(c) Phase difference of displacement Fourier

Fig. 6 Influence of grid spacing, displacement plot, cs 7-node brick

(a) Acceleration time history (2.2–3)
Fig. 8 Influence of grid spacing, acceleration plot, cases, 8-node brick) and (c–d) EL4 ($f_{\text{max}} = 50 \text{ Hz}$, $\Delta x_{\text{std}} = \ldots$)
Fig. 10 Influence of grid spacing, displacement plot, 0.005 s, 8-node brick
(a) Displacement time history (2.2–)

(c) Phase difference of displacement Fourier

Fig. 12 Influence of time-step size, displacement plot 0.010 s, 27-node brick)

(a) Acceleration time history (2.2–3
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\[ \Delta t \leq \frac{\Delta x}{10V_s} \]

Table 2 List of VMKH simulations

| Case #  | \( \Delta x_{\text{std}} \) (m) |
|---------|---------------------------------|
| VMKH1   | 5                               |
| VMKH2   | 5                               |
| VMKH3   | 5                               |
| VMKH4   | 5                               |
| VMKH5   | 5                               |
| VMKH6   | 5                               |
| VMKH7   | 5                               |
| VMKH8   | 5                               |
| VMKH9   | 5                               |
| VMKH10  | 5                               |

Fig. 14 Influence of time-step size, acceleration plot, 0.005 s, 8-node brick and EL9 \( f_{\text{max}} = 50 \text{ Hz}, \Delta x_{\text{std}} = \) 2m.

(a) Acceleration time history (2.2–2)

(c) Acceleration time history (2.2–2)

Fig. 16 Influence of grid spacing, displacement plot, case VMKH10, \( A = 0.1 \text{ mm} \).

(a) Displacement time history (0.0–4.0 s)

(c) Amplitude of displacement Fourier spectrum
Fig. 17 Influence of grid spacing, displacement plot, \( A = 0.1 \text{ mm} \)

Fig. 18 Influence of grid spacing, shear stress-strain:
\( h = 0.05E \), \( \Delta x_{\text{sid}} = 5 \text{ m}, \Delta t_{\text{sid}} = 0.0005 \text{ s}, \Delta t = 0.0002, 0.0005 \text{ s} \)

Fig. 19 Influence of time-step size, displacement plot, case:
\( \Delta x_{\text{sid}} = 5 \text{ m}, \Delta t_{\text{sid}} = 0.0005 \text{ s}, \Delta x = 5 \text{ m}, \Delta t = 0.0002, 0.0005 \text{ s} \)
Fig. 20 Influence of time-step size, shear stress-strain response at the bottom of the VMKH sub-layer, cases VMKH3 ($h = 0.5E$), VMKH4 ($h = 0.05E$) and VMKH5 ($h = 0.01E$). ($\Delta x_{std} = 5 \text{ m}$, $\Delta t_{std} = 0.0005 \text{ s}$, $\Delta x = 5 \text{ m}$, $A = 0.1 \text{ mm}$)

(a) Displacement time history, $h = 0.5E$ (2.2–3.1)

(c) Displacement time history, $h = 0.01E$ (2.2–3.8)

Fig. 22 Influence of time-step size, displacement plot, case (2.2–3.8)
Table 3: List of PBS simulations

| Case# | $\Delta x_{\text{std}}$ (m) | $\Delta t_{\text{std}}$ (s) | $\Delta x$ (m) | $\Delta t$ (s) |
|-------|---------------------------|---------------------------|---------------|---------------|
| PBS1  | 2.5                       | 0.0005                    | 0.5, 2.5      | 0.0001        |
| PBS2  | 2.5                       | 0.0005                    | 0.1, 0.5, 1   | 0.00002       |
| PBS3  | 2.5                       | 0.0005                    | 2.5           | 0.0002, 0.00  |
| PBS4  | 2.5                       | 0.0005                    | 2.5           | 0.00001, 0.0  |

Fig. 24: Influence of grid spacing, displacement plot,

(a) Displacement time history (0.0

Fig. 25: Influence of grid spacing, shear stress-strain m, $\Delta t = 0.0001$ s, $A = 1$ mm)

(a) At the top of the layer
(a) Displacement time history (0.0)

Fig. 27  Influence of time-step size, displacement $\Delta t$, $A = 1$ mm

Fig. 28  Influence of time-step size, shear stress-strain response in the PBS sub-layer, case PBS3 ($\Delta x_{\text{std}} = 2.5$ m, $\Delta t_{\text{std}} = 0.0005$ s, $\Delta x = 2.5$ m, $\Delta t = 0.0002$, 0.0005, 0.001 s, $A = 1$ mm)
**Fig. 30** Influence of time-step size, shear stress-strain n the bottom of the PBS sub-layer, case PBS4 ($\Delta \tau = 2.5$, 0.0005 s, $\Delta x = 2.5$ m, $\Delta t = 0.00001$, 0.00002, 0.0001 s, $A$):

**Fig. 31** Influence of grid spacing at different locations al: $\Delta x = 0.1, 0.5, 1$ m, $\Delta \tau = 0.00002$ s, $A = 1$ mm)
Fig. 32 Influence of time-step size at different locations along 
$\Delta x = 2.5 \text{ m}, \Delta t = 0.00001, 0.00002, 0.0001 \text{ s}, A = 1 \text{ mm}$

Fig. 33 Influence of grid spacing at different locations along 
$\Delta x = 0.1, 0.5, 1 \text{ m}, \Delta t = 0.00002 \text{ s}, A = 1 \text{ mm}$

Fig. 34 Influence of time-step size at different locations along the PBS
$\Delta x = 2.5 \text{ m}, \Delta t = 0.00001, 0.00002, 0.0001 \text{ s}, A = 1 \text{ mm}$