RAMANUJAN’S MASTER THEOREM FOR RADIAL SECTIONS OF LINE BUNDLE OVER NONCOMPACT SYMMETRIC SPACES

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Abstract. We prove analogues of Ramanujan’s Master theorem for the radial sections of the line bundles over the Poincaré upper half plane $\text{SL}(2,\mathbb{R})/\text{SO}(2)$ and over the complex hyperbolic spaces $\text{SU}(1,n)/\text{S}(\text{U}(1) \times \text{U}(n))$.

1. Introduction

Ramanujan’s Master theorem (\cite{4}) states that if a function $f$ can be expanded around zero in a power series of the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k,$$

then

$$\int_{0}^{\infty} f(x) x^{-\lambda - 1} \, dx = -\frac{\pi}{\sin \pi \lambda} a(\lambda), \quad \text{for } \lambda \in \mathbb{C}.$$  \hspace{1cm} (1.1)

One needs some assumptions on the function $a$, as the theorem is not true for $a(\lambda) = \sin \pi \lambda$.

Hardy provides a rigorous statement of the theorem above as: Let $A, p, \delta$ be real constants such that $A < \pi$ and $0 < \delta \leq 1$. Let $\mathcal{H}(\delta) = \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\delta \}$. Let $\mathcal{H}(A, p, \delta)$ be the collection of all holomorphic functions $a : \mathcal{H}(\delta) \to \mathbb{C}$ such that

$$|a(\lambda)| \leq Ce^{-p(|\Re \lambda| + A|\Im \lambda|)} \text{ for all } \lambda \in \mathcal{H}(\delta),$$

where $\Re \lambda, \Im \lambda$ respectively denote the real and imaginary parts of $\lambda$.

Theorem 1.1 (Ramanujan’s Master theorem, Hardy \cite{4}). Suppose $a \in \mathcal{H}(A, p, \delta)$. Then

1. The power series

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k,$$

converges for $0 < x < e^p$ and defines a real analytic function on that domain.

2. Let $0 < \eta < \delta$. For $0 < x < e^p$ we have

$$f(x) = \frac{1}{2\pi i} \int_{-\eta - i\infty}^{-\eta + i\infty} \frac{-\pi}{\sin \pi \lambda} a(\lambda) x^\lambda \, d\lambda.$$

The integral on the right side of the equation above converges uniformly on compact subsets of $[0, \infty)$ and is independent of $\eta$.

3. Also

$$\int_{0}^{\infty} f(x) x^{-\lambda - 1} \, dx = -\frac{\pi}{\sin \pi \lambda} a(\lambda),$$

holds for the extension of $f$ to $[0, \infty)$ and for all $\lambda \in \mathbb{C}$ with $0 < \Re \lambda < \delta$.

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This theorem can be thought of as an interpolation theorem, which reconstructs the values of \( a(\lambda) \) from its given values at \( a(k), k \in \mathbb{N} \cup \{0\} \). In particular if \( a(k) = 0 \) for all \( k \in \mathbb{N} \cup \{0\} \), then \( a \) is identically 0.

Bertram (in [2]) provides a group theoretic interpretation of the theorem in the following way: consider \( x^\lambda, \lambda \in \mathbb{C} \) and \( x^k, k \in \mathbb{Z} \) as the spherical functions on \( X_G = \mathbb{R}^+ \) and \( X_U = U(1) \) respectively. Both \( X_G \) and \( X_U \) can be realized as the real forms of their complexification \( X_C = \mathbb{C}^* \).

Let \( \tilde{f} \) and \( \tilde{f} \) denote the spherical transformation of \( f \) on \( X_G \) and on \( X_U \) respectively. Then it follows from equation (1.1) that

\[
\tilde{f}(\lambda) = -\frac{\pi}{\sin \pi \lambda} a(\lambda), \quad \tilde{f}(k) = (-1)^k a(k).
\]

Using the duality between \( X_U = U/K \) and \( X_G = G/K \) inside their complexification \( X_C = G_C/K_C \), Bertram proved an analogue of Ramanujan’s Master theorem for Riemannian symmetric spaces of noncompact type with rank one. This theorem was further extended by Ölafsson and Pasquale (see [9]) to higher rank case. Also it was further extended for the hypergeometric Fourier transform associated to root systems by Ölafsson and Pasquale (see [10]).

In this paper we prove analogue of Ramanujan’s Master theorem for the radial sections of line bundles over the Poincaré upper half plane and over complex hyperbolic spaces. More precisely, in the first part of the paper, we work with the group \( G = \text{SL}(2, \mathbb{R}) \) but instead of the \( K \)-biinvariant functions with \( K = \text{SO}(2) \) we consider functions with the property

\[
f(k_\theta g k_\alpha) = e^{i n (\theta + \alpha)} f(g),
\]

where \( n \in \mathbb{Z} \) is fixed and \( k_\theta, k_\alpha \in K \). However, this case offers a fresh challenge, in the sense that one needs to take the discrete series into the consideration for \( n \) positive.

In the second part of this paper, we consider \( G = \text{SU}(1, n), K = S(U(1) \times U(n)) \) and consider one dimensional representations of \( S(U(1) \times U(n)) \). The one dimensional representations of \( S(U(1) \times U(n)) \) are given by \( \chi_l \) for \( l \in \mathbb{Z} \) (see section 3 for precise definition). This is the only case among the class of real rank one semisimple Lie groups for which nontrivial one dimensional representations of \( K \) exists. We prove analogue of Theorem 3.1 for all \( \chi_l \)-radial functions with the restriction that \( |l| < n \) (see Definition 3.1 for the definition of \( \chi_l \)-radial functions). The reason for working under the above restriction on \( l \) is the fact that the mathematical machinery related to Ramanujan’s master theorem is available only for \( l \in \mathbb{Z} \) with \( |l| < n \) (see [5], [6]). We observe that discrete series representations do not arise in the spherical inversion formula for \( \chi_l \)-radial functions with \( |l| < n \) (see [3,6]). We also observe that these two cases (that is, the case of line bundles over the Poincaré upper half plane and over complex hyperbolic spaces) are mutually disjoint.

In section 2 we prove analogue of Ramanujan’s master theorem for the radial sections of line bundles over Poincaré upper half plane and in section 3 we prove the theorem for the radial sections of line bundles over complex hyperbolic spaces.

### 2. Ramanujan’s Master theorem for Poincaré upper half plane

Let \( G = \text{SL}(2, \mathbb{R}) \). Let

\[
k_\theta = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right), \quad a_t = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \quad \text{and} \quad n_\xi = \left( \begin{array}{cc} 1 & \xi \\ 0 & 1 \end{array} \right).
\]

Then \( K = \{ k_\theta \mid \theta \in [0, 2\pi) \}, A = \{ a_t \mid t \in \mathbb{R} \}, N = \{ n_\xi \mid \xi \in \mathbb{R} \} \) are subgroups of \( G \), in which \( K = \text{SO}(2) \) is a maximal compact subgroup of \( G \). Let \( G = KAN \) be an Iwasawa decomposition of \( G \) and for \( x \in G \), let \( x = k_\theta a_t n_\xi \) be the corresponding decomposition. Then we write \( K(x) := k_\theta \), \( H(x) = t \), \( a(x) = a_t \) and \( n(x) = n_\xi \). In fact if \( x = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{R}) \), then \( \theta, t \) and \( \xi \) are given.
by

\[ e^{2t} = a^2 + c^2, \quad e^{i\theta} = \frac{a - ic}{\sqrt{a^2 + c^2}} \quad \text{and} \quad \xi = \frac{ab + cd}{\sqrt{a^2 + c^2}}. \]

The Haar measure \( dx \) of \( G \) splits according to this decomposition as

\[ dx = e^{2t}dk_\theta \ dt \ d\xi, \]

where \( dk_\theta = (2\pi)^{-1} \ d\theta \) is the normalised Haar measure of \( K \) and \( d\xi, \ dt \) are Lebesgue measures on \( \mathbb{R} \). The Cartan decompositon of \( G \) is given by \( G = K A^+ K \) where \( A^+ = \{ a_t \mid t > 0 \} \) and \( x \in G \) can be written by \( x = k_\theta a_t k_\phi \) with \( t \geq 0 \). Also the Haar measure \( dx \) of \( G \) corresponding to that decomposition is given by

\[ dx = \sinh 2t \ d\theta \ d\xi \ d\phi. \]

Let \( \sigma(x) = \sigma(k_\theta a_t k_\phi) := |t| \). In fact \( \sigma(x) = d(xK, eK) \) where \( d \) is the distance function on \( G/K \).

Let \( \hat{K} = \{ e_n \mid n \in \mathbb{Z} \} \) be the set of irreducible unitary representations of \( K \), where \( e_n(k_\theta) = e^{i\theta} \).

**Definition 2.1.** A function \( f \) on \( G \) is said to be of type \( (n, n) \) if

\[ f(k_\theta x k_\phi) = e_n(k_\theta) f(x) e_n(k_\phi), \quad k_\theta, k_\phi \in K, \ x \in G. \]

Let \( M = \{ \pm I \} \) where \( I \) is the \( 2 \times 2 \) identity matrix. The unitary dual of \( M \) is \( \hat{M} = \{ \sigma_+, \sigma_- \} \) where \( \sigma_+ \) is the trivial representation and \( \sigma_- \) is the only nontrivial unitary irreducible representation of \( M \). Let \( \mathbb{Z}^+ \) (respectively \( \mathbb{Z}^- \)) be the set of even (respectively odd) integers.

For \( \sigma \in \hat{M} \) and \( \lambda \in \mathbb{C} \), let \( (\pi_{\sigma, \lambda}, H_\sigma) \) be the principal series representation of \( G \), where \( H_\sigma \) is the subspace of \( L^2(K) \) generated by the orthonormal set \( \{ e_n \mid n \in \mathbb{Z} \} \) is given by

\[ (\pi_{\sigma, \lambda}(x)e_n)(k_\theta) = e^{-(\lambda+1)H(x^{-1}k_\theta^{-1})} e_n(K(x^{-1}k_\theta^{-1})^{-1}). \]

This representation is unitary if and only if \( \lambda \in i \mathbb{R} \). For every \( k \in \mathbb{Z}^* \), the set of nonzero integers, there is a discrete series representation \( \pi_k \), which occur as a subrepresentation of \( \pi_{\sigma, |k|}, k \in \mathbb{Z} \setminus \mathbb{Z}^* \).

For \( n \in \mathbb{Z} \) and \( k \in \mathbb{Z} \setminus \mathbb{Z}^* \), let

\[ \Phi^{n,n}_{\sigma, \lambda}(x) = \langle \pi_{\sigma, \lambda}(x)e_n, e_n \rangle, \]

and

\[ \Psi^{n,n}_k(x) = \langle \pi_k(x)e_n, e_n \rangle, \]

be the matrix coefficients of the principal series and the discrete series representations respectively, where \( e_n \) are the renormalised basis and \( \langle \cdot, \cdot \rangle_k \) is the renormalised inner product for \( \pi_k \). Therefore the integral representation of \( \Phi^{n,n}_{\sigma, \lambda} \) is given by

\[ \Phi^{n,n}_{\sigma, \lambda}(x) = \int_K e^{-(\lambda+1)H(x^{-1}k_\theta^{-1})} e_n(K(x^{-1}k_\theta^{-1})^{-1}) e_{-n}(k_\theta) \ d\theta, \]

\[ = \int_K e^{-(\lambda+1)H(x^{-1}k_\theta)}/(x^{-1}k_\theta) \ d\theta. \]

This follows that \( \Phi^{n,n}_{\sigma, \lambda} \) is a \((n, n)\) type function. We observe that \( \Phi^{0,0}_{\sigma^+, \lambda} \) is the elementary spherical function, denoted by \( \phi_\lambda \). Also,

\[ |\Phi^{n,n}_{\sigma, \lambda}(x)| \leq \int_K e^{-(\lambda+1)H(xk_\theta)} \ d\theta = \phi_\lambda(x). \]

It is well known that for \( \lambda \in \mathbb{C} \),

\[ |\phi_\lambda(x)| \leq C(1 + |\sigma(x)|) e^{(|\Re \lambda|-1)|\sigma(x)|}. \]

Therefore for all \( \lambda \in \mathbb{C} \) we have

\[ |\Phi^{n,n}_{\sigma, \lambda}(x)| \leq C(1 + |\sigma(x)|) e^{(|\Re \lambda|-1)|\sigma(x)|}. \]
For \( \sigma \in \widehat{M} \), let

\[
-\sigma = \begin{cases} 
\sigma^- & \text{if } \sigma = \sigma^+,
\sigma^+ & \text{if } \sigma = \sigma^-.
\end{cases}
\]

For \( k \in \mathbb{Z}^* \), let \( \sigma \in \widehat{M} \) be determined by \( k \in \mathbb{Z}^-\sigma \) defined by

\[
\mathbb{Z}(k) = \begin{cases} 
\{ n \in \mathbb{Z}^\sigma \mid n \geq k+1 \} & \text{if } k \geq 1,
\{ n \in \mathbb{Z}^\sigma \mid n \leq k-1 \} & \text{if } k \leq -1.
\end{cases}
\]

Then for \( k \in \mathbb{Z}^* \) and \( n \in \mathbb{Z}(k) \), we have (II Proposition 7.3)

\[
(2.4) \quad \Phi^{n,n}_{\sigma,|k|} = \Psi^{n,n}_{k},
\]

where \( \sigma \in \widehat{M} \) is determined by \( k \in \mathbb{Z}^-\sigma \).

For a \((n,n)\)-type function \( f \) the spherical Fourier transform of \( f \) is defined by

\[
\widehat{f_H}(\sigma, \lambda) = \int_G f(x)\Phi^{n,n}_{\sigma,\lambda}(x^{-1}) \, dx,
\]

and

\[
\widehat{f_B}(k) = \int_G f(x)\Phi^{n,n}_{k}(x^{-1}) \, dx,
\]

where \( \sigma \in \widehat{M} \) is determined by \( n \in \mathbb{Z}^\sigma \) and \( k \in \mathbb{Z}^{-\sigma} \).

For \( \sigma \in \widehat{M} \) and \( n \in \mathbb{Z}^\sigma \), let

\[
L^{n,n}_{\sigma} = \{ k \in \mathbb{Z}^{-\sigma} \mid 0 < k < n \text{ or } n < k < 0 \}.
\]

Then for a nice \((n,n)\)-type function \( f \) the inversion formula is given by (II Theorem 10.2):

\[
f(x) = \frac{1}{4\pi^2} \int_{i\mathbb{R}} \widehat{f_H}(\sigma, \lambda)\Phi^{n,n}_{\sigma,\lambda}(x)\mu(\sigma, \lambda) \, d\lambda + \frac{1}{2\pi} \sum_{k \in L^{n,n}_{\sigma}} \widehat{f_B}(k)\Psi^{n,n}_{k}(x)|k|,
\]

where \( \sigma \in \widehat{M} \) is determined by \( n \in \mathbb{Z}^\sigma \) and \( \mu(\sigma, \lambda) \) is given by

\[
(2.5) \quad \mu(\sigma, \lambda) = \begin{cases} 
\frac{\lambda\pi i}{2} \tan(\frac{n\pi}{2}) & \text{if } \sigma = \sigma^+,
\frac{-\lambda\pi i}{2} \cot(\frac{n\pi}{2}) & \text{if } \sigma = \sigma^-.
\end{cases}
\]

Let \( \mathfrak{g} = \text{sl}(2, \mathbb{R}) \) be the Lie algebra of \( \text{SL}(2, \mathbb{R}) \) and \( \mathcal{U}(\mathfrak{g}) \) be the universal enveloping algebra of \( \mathfrak{g} \). For \( 0 < p \leq 2 \) the \( L^p \)-Schwartz spaces \( C^p(G)_{n,n} \) is the set of all \((n,n)\)-type functions \( f \in C^\infty(G) \) such that

\[
\sup_{x \in G}(1 + |x|)^s \phi_0(x)^{-\frac{2}{p}} |f(D; x; E)| < \infty,
\]

for any \( D, E \in \mathcal{U}(\mathfrak{g}) \) and any integer \( s \geq 0 \), where \( f(D; x; E) \) is defined by

\[
f(D; x; E) = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} f(\exp tD x \exp sE).
\]

The Schwartz space \( C^p(G)_{n,n} \) is topologized by the seminorms

\[
\sigma^p_{D,E,s}(f) = \sup_{x \in G}(1 + |x|)^s \phi_0(x)^{-\frac{2}{p}} |f(D; x; E)|.
\]

Then it follows that \( C^\infty_c(G)_{n,n} \) is dense in \( C^p(G)_{n,n} \) and \( C^p(G)_{n,n} \) is dense in \( L^p(G)_{n,n} \).

Let \( C^2_B(\widehat{G})_{n,n} \) be the set of all functions on \( L^{n,n}_{\sigma} \). Then we have (see II Theorem 16.1)

**Theorem 2.2.** The map \( f \mapsto (\widehat{f_H}, \widehat{f_B}) \) is a topological isomorphism from \( C^2(G)_{n,n} \) onto \( S(i\mathbb{R})_c \times C^2_B(\widehat{G})_{n,n} \).
The complexification of $\mathfrak{g}$ is $\mathfrak{g} + i\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of $G_C = \text{SL}(2, \mathbb{C})$. The Cartan involution for $\mathfrak{g}$ is given by $\theta(A) = -A^T$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition where

$$\mathfrak{k} = \{ A \mid \theta(A) = A \} = \{ A \in \text{sl}(2, \mathbb{R}) \mid A^T = -A \},$$

and

$$\mathfrak{p} = \{ A \mid \theta(A) = -A \} = \{ A \in \text{sl}(2, \mathbb{R}) \mid A^T = A \}.$$

The complexification of $\mathfrak{g}$ is given by

$$\mathfrak{g}_C = \mathfrak{k} \oplus i\mathfrak{p} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^T = -X \}.$$  

The corresponding connected group $G_C$ whose Lie algebra is $\mathfrak{g}_C$ is given by

$$K_C = \text{SO}(2, \mathbb{C}) = \{ A \in \text{GL}(2, \mathbb{C}) \mid A^T A = I = A A^T \}.$$

This is noncompact group.

Now let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Then

$$\mathfrak{u} = \{ A + iB \mid A \in \mathfrak{k}, B \in \mathfrak{p} \} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^* = -X \},$$

and the corresponding connected group whose Lie algebra is $\mathfrak{u}$ is $U = \text{SU}(2)$. Therefore starting with the Riemannian symmetric space of noncompact type $G/K = \text{SL}(2, \mathbb{R})/\text{SO}(2)$ we get a Riemannian symmetric space of compact type $U/K = \text{SU}(2)/\text{SO}(2)$. Such a compact symmetric space $U/K$ is called the compact dual of $G/K$. We observe that

$$G/K \cong \mathbb{H}^2 = \{ x + iy \mid x \in \mathbb{R}, y > 0 \} \text{ and } U/K \cong S^2.$$

It is easy to check that the complexification of $\mathfrak{g}$ is same as complexification of $\mathfrak{u}$, that is $\mathfrak{g}_C = \mathfrak{u}_C$. Also both the Riemannian symmetric spaces $G/K$ and $U/K$ are embedded as a totally real submanifold in the (non-Riemannian) symmetric space $G_C/K_C$.

Let $P_m$ be the space of polynomials in two variables, with complex coefficients and homogeneous of degree $m$. We note that dimension of $P_m$ is $m + 1$. The set of irreducible representations of $U = \text{SU}(2)$ is given by $\pi_m, m \in \mathbb{N} \cup \{0\}$ on $P_m$ defined by

$$\pi_m(g)f(u, v) = f((u, v)g).$$

It is well known that $\pi_m$ has weights, $-m, -m + 2, \ldots, m - 2, m$ with weight vectors (say), $v_m, v_{-m}, \ldots, v_{m-2}, v_m$ respectively (see [3]). We say $\pi_m$ is $K$-spherical if there exists a nonzero function $f \in P_m$ such that

$$\pi_m(k_\theta)f = f \text{ for all } k_\theta \in K.$$  

For a fixed $n \in \mathbb{Z}$, we say $\pi_m$ is $n$-spherical if there exists a nonzero function $f \in P_m$ such that

$$\pi_m(k_\theta)f = e^{in\theta}f \text{ for all } k_\theta \in K,$$

where $e_n(k_\theta) = e^{in\theta}$. Also such a vector $f$ will be called a $n$-spherical vector.

**Proposition 2.3.** For $n \in \mathbb{N} \cup \{0\}$, the $n$-spherical representations are given by $\pi_{2m+n}, m \in \mathbb{N} \cup \{0\}$.

**Proof.** Let $f \in P_1$ be a $n$-spherical vector for $\pi_1$. Since $f \in P_1$ we can write $f$ as

$$f = c_0 f_0 + c_1 f_1 + \cdots + c_m f_m,$$

where $f_j((u, v)) = u_j v^{m-j}$ and $c_j \in \mathbb{C}$ for $j = 0, 1, \ldots, m$. Then

$$\pi_1(k_\theta)f = e^{-in\theta}f \text{ for all } k_\theta \in K,$$

implies that

$$(2.6) \quad f((u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta)) = e^{-in\theta}f((u, v)),$$

for all $u, v$ and $\theta$. That is for all $r, \theta, \phi$ we have,

$$f(r \cos(\theta + \phi), r \sin(\theta + \phi)) = e^{-in\theta}f(r \cos \phi, r \sin \phi).$$
This implies that,
\[ f(\cos \theta, \sin \theta) = e^{-in\theta} f((1,0)). \]
But \( f \) is a homogeneous polynomial of degree \( l \). Hence
\[ f(r \cos \theta, r \sin \theta) = r^l f(\cos \theta, \sin \theta) = f((1,0)) r^l e^{-in\theta} = f((1,0)) r^l (\cos \theta - i \sin \theta)^n. \]
Hence
\[ f(u,v) = f((1,0)) r^l \left( \frac{u}{r} - i \frac{v}{r} \right)^n = f((1,0))(u^2 + v^2)^{\frac{l-n}{2}} (u - iv)^n. \]
This \( f \) is a polynomial of degree \( l \) if and only if \( \frac{l-n}{2} \) is a positive integer. Hence \( \pi_l \) is a \( n \)-spherical representation if \( l = n + 2m \) with \( m \in \mathbb{N} \cup \{0\} \) and in this case
\[ f((u,v)) = (u^2 + v^2)^{\frac{l-n}{2}} (u - iv)^n, \]
is a \( n \)-spherical vector.
Conversely, if \( l - 2m = n \), it is easy to check that
\[ f((u,v)) = (u^2 + v^2)^{\frac{l-n}{2}} (u - iv)^n, \]
is a \( n \)-spherical vector. \( \square \)

**Remark 2.4.** From the proof above it follows that the dimension of \( n \)-spherical vectors for the representation \( \pi_{n+2m} \) is 1.

The proof of the following proposition is similar to that of Proposition 2.3.

**Proposition 2.5.** For any negative integer \( n \), the \( n \)-spherical representations are given by \( \pi_{2m+n}, m \in \mathbb{N} \cup \{0\} \) such that \( n + m \geq 0 \). In this case
\[ f((u,v)) = (u^2 + v^2)^m (u - iv)^n, \]
is a \( n \)-spherical vector.

A function \( f \) on \( U \) is said to be of type \((n,n)\) if it satisfies the same condition of (2.2) with \( x \in G \) replaced by \( x \in U \).

We now define the \( n \)-spherical function \( \psi_{2m+n,n} \) (associated to \( \pi_{2m+n} \)) by
\[ \psi_{2m+n,n}(x) = \frac{1}{\|f'\|^2} \langle \pi_{2m+n}(x^{-1}) f', f' \rangle, \]
where \( f' \) is a \( n \)-fixed vector for \( \pi_{2m+n} \). It is easy to check that \( \psi_{2m+n,n}(e) = 1 \) and \( \psi_{2m+n,n} \) is a \((n,n)\)-type function.

For \( f \in L^1(U) \), the Fourier coefficients of \( f \) are defined by
\[ \hat{f}(m) = \int_U f(g) \pi_m(g^{-1}) \, dg, \ \ m = 0, 1, 2, \ldots. \]
For a function \( f \in L^2(U) \), the Fourier series of \( f \) is
\[ f(g) = \sum_{m=0}^{\infty} (m+1) \text{Tr} \left( \hat{f}(m) \pi_m(g) \right). \]
It is now easy to check that for a \((n,n)\)-type function \( f \) the possible nonzero Fourier coefficients are given by
\[ \hat{f}(2m+n) = \int_U f(g) \psi_{n+2m,n}(g^{-1}) \, dg, \ \ m = 0, 1, 2, \ldots. \]
Also for a \((n,n)\)-type function, \( n \) nonnegative integer, the Fourier series reduces to
\[ f(g) = \sum_{m=0}^{\infty} (2m+n+1) \hat{f}(2m+n) \psi_{2m+n,n}(g). \]
For a \((n, n)\)-type function, \(n\) negative integer, the Fourier series reduces to

\[
f(g) = \sum_{m=\left\lfloor \frac{|m|}{2} \right\rfloor}^{\infty} (2m + n + 1) f(2m + n) \psi_{2m+n,n}(g).
\]

We will need the following Polar decomposition of \(U\):

\[U = SO(2) B SO(2),\]

where

\[B = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}.
\]

We now have the following relation between the \(n\)-spherical functions on \(G\) and \(U\).

**Theorem 2.6.** For \(m, n \in \mathbb{N} \cup \{0\}\), the function \(\psi_{2m+n,n}\) admits a holomorphic extension to \(U_C\) and

\[\psi_{2m+n,n} |_{G} = \Phi_{\sigma,2m+n+1}^{n,n} \text{ with } n \in \mathbb{Z}^\sigma.
\]

**Proof.** The proof is similar to [6, Lemma 4.6]. Since \(\pi_{n+2m}\) is a representation of the compact group \(U\), it extends holomorphically to \(U_C = SL(2, \mathbb{C})\). We denote the extended representation again by \(\pi_{n+2m}\). Let \(n\) be fixed non-negative integer.

We define \(P_K^n\) on \(P_{n+2m}\) by

\[P_K^n(f) = \int_K e_n(k_\theta) \pi_{n+2m}(k_\theta) f dk_\theta.
\]

Then it is easy to check that

\[\pi_{n+2m}(k_\theta) P_K^n(f) = e_n(k_\theta^{-1}) P_K^n(f),
\]

that is \(P_K^n(f)\) is an \(n\) spherical vector. Also \(P_K^n\) is an orthogonal projection and self adjoint (that is, \((P_K^n)^* = P_K^n\)). Let \(f'\) be a \(n\)-spherical vector for \(\pi_{n+2m}\). Since the space of \(n\)-spherical vectors is of dimension 1,

\[P_K^n(f) = \frac{1}{\|f'\|^2} \langle f, f' \rangle f'.
\]

Let \(g\) be the highest weight vector for \(\pi_{n+2m}\) with \(\langle g, f' \rangle = 1\). We claim that:

\[\psi_{n+2m,n}(x) = \frac{1}{\|f'\|} \int_K e_n(k_\theta) \langle \pi_{n+2m}(x^{-1}k_\theta) g, f' \rangle dk_\theta.
\]

We have

\[P_K^n(g) = \frac{f'}{\|f'\|}.
\]

Hence

\[f' = \|f'\| P_K^n(g) = \|f'\| \int_K e_n(k_\theta) \pi_{n+2m}(k_\theta) g dk_\theta.
\]

This implies that

\[\psi_{n+2m,n}(x) = \frac{1}{\|f'\|^2} \langle \pi_{n+2m}(x^{-1}) f', f' \rangle = \frac{1}{\|f'\|^2} \int_K e_n(k_\theta) \langle \pi_{n+2m}(x^{-1}k_\theta) g, f' \rangle dk_\theta.
\]

This is true for all \(x \in U\) and hence true for all \(x \in U_C = SL(2, \mathbb{C})\).

For \(x \in SL(2, \mathbb{R})\),

\[
\langle \pi_{n+2m}(x^{-1}k_\theta) g, f' \rangle = \langle \pi_{n+2m} (K(x^{-1}k_\theta)a(x^{-1}k_\theta)n(x^{-1}k_\theta)) g, f' \rangle
\]

\[= \langle \pi_{n+2m} (a(x^{-1}k_\theta)n(x^{-1}k_\theta)) g, \pi_{n+2m}(K(x^{-1}k_\theta)^{-1}) f' \rangle.
\]
Since \( g \) is a highest weight vector for \( \pi_{n+2m} \), we have
\[
\pi_{n+2m}(n(x^{-1}k_\theta))g = g,
\]
and
\[
\pi_{n+2m}(a(x^{-1}k_\theta))g = e^{(n+2m)H(x^{-1}k_\theta)}g.
\]
Also since \( f' \) is \( n \)-spherical vector, we have
\[
\pi_{n+2m}(K(x^{-1}k_\theta)^{-1})f' = e_n(K(x^{-1}k_\theta))f'.
\]
Therefore,
\[
\langle \pi_{n+2m}(x^{-1}k_\theta)g, f' \rangle = \langle \pi_{n+2m}(a(x^{-1}k_\theta))g, \pi_{n+2m}(K(x^{-1}k_\theta)^{-1})f' \rangle
\]
\[
= \langle e^{(n+2m)H(x^{-1}k_\theta)}g, e_n(K(x^{-1}k_\theta))f' \rangle
\]
\[
= e^{(n+2m)H(x^{-1}k_\theta)}e_{-n}(K(x^{-1}k_\theta))\langle ||f'|| \rangle.
\]
Hence for \( x \in \text{SL}(2, \mathbb{R}) \),
\[
\psi_{n+2m,n}(x) = \int_{K} e^{(n+2m)H(x^{-1}k_\theta)}e_{-n}(K(x^{-1}k_\theta))e_{n}(k_\theta) \, dk_\theta = \Phi_{\sigma,n+2m+1}^{n,n}(x).
\]
This completes the proof. \( \square \)

The proof of the following theorem is similar.

**Theorem 2.7.** Let \( n \) be a negative integer. Then for integers \( m \) such that \( n + m \geq 0 \), \( \psi_{2m+n,n} \) admits a holomorphic extension to \( U_\mathbb{C} \) and
\[
\psi_{2m+n,n} \big|_G = \Phi_{\sigma,2m+n+1}^{n,n} \text{ with } n \in \mathbb{Z}^+.
\]

We need to estimate the \( n \)-spherical function \( \psi_{2m+n,n} \) on \( U_\mathbb{C} \). The following decomposition will help us in this regard.

**Proposition 2.8.** \( G_\mathbb{C} = \text{SL}(2, \mathbb{C}) \) has a unique decomposition
\[
\text{SL}(2, \mathbb{C}) = \text{SU}(2) \exp \mathfrak{a}^\perp \text{SO}(2, \mathbb{C}),
\]
where \( \mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R} \right\} \).

**Proof.** Let \( \theta(X) = -X^* \) be the Cartan involution for \( \mathfrak{g}_\mathbb{C} = \text{sl}(2, \mathbb{C}) \). Then
\[
\mathfrak{h}_1 := \mathfrak{g}_\mathbb{C}^\theta = \{ X \in \text{sl}(2, \mathbb{C}) \mid \theta(X) = X \} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^* = -X \},
\]
\[
\mathfrak{p}_1 := \{ X \in \text{sl}(2, \mathbb{C}) \mid d\theta(X) = -X \} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^* = X \},
\]
so that
\[
\mathfrak{g}_\mathbb{C} = \mathfrak{h}_1 \oplus \mathfrak{p}_1.
\]

Let us consider another involution \( \sigma \) of \( \mathfrak{g}_\mathbb{C} \) by
\[
\sigma(X) = -X^T.
\]
Then
\[
\mathfrak{h} := \mathfrak{g}_\mathbb{C}^\sigma = \{ X \in \text{sl}(2, \mathbb{C}) \mid \sigma(X) = X \} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^T = -X \} = \text{so}(2, \mathbb{C}),
\]
and
\[
\mathfrak{q} := \{ X \in \text{sl}(2, \mathbb{C}) \mid \sigma(X) = -X \} = \{ X \in \text{sl}(2, \mathbb{C}) \mid X^T = X \},
\]
so that
\[
\mathfrak{g}_\mathbb{C} = \mathfrak{h} \oplus \mathfrak{q}.
\]
Then
\[ p_1 \cap q = \{ X \in sl(2, \mathbb{C}) \mid X^T = X \} = \{ X \in sl(2, \mathbb{R}) \mid X^T = X \} = p_{sl(2, \mathbb{R})}, \]
and
\[ p_1 \cap h = \{ X \in sl(2, \mathbb{C}) \mid X^T = -X^T \} = \left\{ \begin{pmatrix} 0 & ib \\ -ib & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\}. \]
Therefore (see [5, Proposition 2.2, p. 106]), any element \( g \in SL(2, \mathbb{C}) \) can uniquely be written as
\[ g = k \exp X \exp Y \]
for some \( k \in SU(2), X \in p_1 \cap q, Y \in p_1 \cap h. \)
Also since \( \exp X \in SL(2, \mathbb{R}) \), Cartan decomposition implies
\[ \exp X = k_\theta \exp Z k_\phi \]
for some \( k_\theta, k_\phi \in SO(2) \) and for some unique \( Z \in \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \geq 0 \right\}. \)
Hence
\[ g = (kk_\theta) \exp Z(k_\phi \exp Y) \]
where \( kk_\theta \in SU(2), Z \in \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \geq 0 \right\}, \)
and \( k_\phi \exp Y \in SO(2, \mathbb{C}). \)

The following theorem gives the required estimate of the spherical function \( \psi_{2m+n,n} \) on \( U_C \) (see [8, Proposition 12] for corresponding result on \( X_C \)).

**Proposition 2.9.** The \( n \)-spherical function \( \psi_{2m+n,n} \) satisfies the following estimate:
\[ |\psi_{2m+n,n}(g)| \leq C e^{(2m+n)t} \|e_n(h^{-1})\|, \]
where \( g = k \exp a(g)h \) with \( k \in SU(2), h \in SO(2, \mathbb{C}) \) and \( a(g) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}^+. \)

**Proof.** The following steps will lead to the proof of the theorem.

**Step-1:** Let \( \pi : U \to B(V) \) be a unitary representation of a compact group \( U \). Then \( d\pi : u \to B(V) \)
is a representation of the Lie algebra \( u \) and since \( \pi \) is unitary so \( d\pi(X)^* = -d\pi(X) \). We extend \( d\pi \)
to \( u_C \) by \( (d\pi)^C : u + iu \to B(V) \) as
\[ (d\pi)^C(X + iY) = d\pi(X) + i d\pi(Y). \]
Then
\[ (d\pi)^C(iY)^* = (id\pi(Y))^* = i d\pi(Y). \]
We define \( \pi^C : U_C \to B(V) \) as \( \pi^C(\exp(X + iY)) = \exp(d\pi^C(X + iY)) \). Then
\[ \pi^C(\exp(iY)) \pi^C(\exp(iY)) = \exp d\pi^C(iY)^* \exp d\pi^C(iY) = \exp (2i d\pi(Y)). \]

**Step-2:** Let \( f' \) be a \( n \)-spherical vector for \( \pi_{n+2m} \) (with respect to \( K \)). Therefore it follows that \( f' \)
is a \( n \)-spherical vector for \( \pi^C_{n+2m} \) (with respect to \( K_C = SO(2, \mathbb{C}) \)).

**Step-3:** We consider an orthonormal basis of \( P_{n+2m} \) taking one element as \( f^0 = \frac{f'}{\|f'\|} \), where \( f' \) is a
\( n \)-spherical vector and let the basis be \( \{f^0, f^1, \cdots, f^{n+2m}\} \). We claim that
\[ \sum_{j=0}^{2m+n} \left| \langle \pi^C_{n+2m}(g) f^0, f^j \rangle \right|^2 \leq C e^{(4m+2n)t} \|e_n(h^{-1})\|^2, \]
where $g = k \exp a(g)h$ and $a(g) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. As a consequence it will follow that

$$|\psi_{n+2m,n}(k \exp a(g)h)| \leq e^{(2m+n)t} |e_n(h^{-1})|.$$

The projection $P^n_K$ satisfies

$$\pi_{n+2m}(k)P^n_K(f) = e_n(k^{-1})P^n_K(f).$$

Using the properties that $P^n_K$ is an orthogonal projection, self adjoint and the space of $n$-spherical vectors is of dimension 1 it follows that

$$\sum_{j=0}^{2m+n} |\langle \pi_{n+2m}^C(k \exp a(g)h) f^0, f^j \rangle |^2 = \sum_{j=0}^{2m+n} |\langle \pi_{n+2m}^C(k \exp a(g))e_n(h^{-1}) f^0, f^j \rangle |^2$$

$$= \sum_{j=0}^{2m+n} |e_n(h^{-1})|^2 |\langle \pi_{n+2m}^C(k \exp a(g))P^n_K f^i, f^j \rangle |^2$$

$$= |e_n(h^{-1})|^2 \|\pi_{2m+n}^C(k \exp a(g))P^n_K\|_{HS}^2.$$

The expression above is equal to

$$|e_n(h^{-1})|^2 \text{Tr} \left( (P^n_K)^* \circ \pi_{2m+n}^C(k \exp a(g)) \circ \pi_{2m+n}^C(k \exp a(g)) \circ P^n_K \right).$$

By Step-1 and the fact that $\pi_n$ has weights, $-m, -m + 2, \ldots, m - 2, m$ with weight vectors, $v_{-m}, v_{-m+2}, \ldots, v_{m-2}, v_m$ respectively we have

$$\sum_{j=0}^{2m+n} |\langle \pi_{n+2m}^C(k \exp a(g)h) f^0, f^j \rangle |^2$$

$$= |e_n(h^{-1})|^2 \text{Tr} \left( P^n_K \circ \pi_{2m+n}^C(\exp 2a(g)) \right)$$

$$= |e_n(h^{-1})|^2 \sum_{j=0}^{n+2m} |\langle P^n_K \circ \pi_{n+2m}^C(\exp 2a(g))v_{-2m-n+2j}, v_{-2m-n+2j} \rangle |^2$$

$$= |e_n(h^{-1})|^2 \sum_{j=0}^{n+2m} |\langle \pi_{n+2m}^C(\exp 2a(g))v_{-2m-n+2j}, P^n_K v_{-2m-n+2j} \rangle |^2$$

$$= |e_n(h^{-1})|^2 \sum_{j=0}^{n+2m} (e^{2(-2m-n+2j)t} v_{-2m-n+2j}, P^n_K v_{-2m-n+2j} \rangle |^2$$

$$\leq |e_n(h^{-1})|^2 e^{(4m+2n)t} \text{Tr}(P^n_K).$$

The last inequality follows because

$$\langle v_{-2m-n+2j}, P^n_K v_{-2m-n+2j} \rangle = \|P^n_K v_{-2m-n+2j}\|^2 \geq 0.$$

But

$$\text{Tr}(P^n_K) = \sum_{i=0}^{n+2m} \langle P^n_K f^i, f^i \rangle = 1.$$

This implies that

$$\sum_{j=0}^{2m+n} |\langle \pi_{n+2m}^C(k \exp a(g)h) f^0, f^j \rangle |^2 \leq |e_n(h^{-1})|^2 e^{(4m+2n)t}.$$ 

This completes the proof. □

**Corollary 2.10.** For $t \geq 0$, we have

$$|\psi_{2m+n,n}(\exp \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix})| \leq C e^{(2m+n)t}.$$
This follows because

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
e^t & 0 \\
0 & e^{-t}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.
\]

We recall that

\[
\Phi_{\sigma,\lambda}^{n,n}(a_t) = \int_K e^{-\lambda(1+1)H(a_t)k_\theta)} e^{-n}(K(a_t)k_\theta)) e_n(k_\theta) dk_\theta.
\]

We now extend holomorphically this \(\Phi_{\sigma,\lambda}^{n,n}\) to a subset of \(A_C\) (where \(A_C\) is the complexification of \(A\) inside \(G_C\)) containing \(A\) in the following way:

We have

\[
a_t + is k_\theta = \begin{pmatrix} e^{t+is} \cos \theta & e^{t+is} \sin \theta \\ -e^{-t-is} \sin \theta & e^{-t-is} \cos \theta \end{pmatrix}.
\]

Then from (2.1) it follows that the function \(t \mapsto e^{-(\lambda+1)H(a_t)k_\theta}\) extends holomorphically to a certain neighborhood of \(A\) as

\[
e^{-(\lambda+1)H(a_t)k_\theta)} = \left( e^{2(t+is)} \cos^2 \theta + e^{-2(t+is)} \sin^2 \theta \right)^{\frac{\lambda+1}{2}},
\]

and the function \(t \mapsto e_m(K(a_t)k_\theta)\) extends holomorphically to a certain neighborhood of \(A\) as

\[
e_m(K(a_t)k_\theta)) = \left( \frac{e^{t+is} \cos \theta + ie^{-t-is} \sin \theta}{\sqrt{e^{2(t+is)} \cos^2 \theta + e^{-2(t+is)} \sin^2 \theta}} \right)^{-m}.
\]

We now extend \(\Phi_{\sigma,\lambda}^{n,n}(a_t)\) in the following way:

\[
\Phi_{\sigma,\lambda}^{n,n}(a_t + is) = \int_K e^{-\lambda(1+1)H(a_t - is k_\theta)} e^{-n}(K(a_t - is k_\theta)) e_n(k_\theta) dk_\theta.
\]

This function is holomorphic for \(|s| < \delta_1\) for some \(\delta_1 > 0\). The following proposition gives an estimate of \(n\)-spherical function on a certain neighborhood of \(A\).

**Proposition 2.11.** For \(\sigma \in \widehat{M}\),

\[
|\Phi_{\sigma,\lambda_1 + i\lambda_2}^{n,n}(a_t + is)| \leq C e^{2|n||t||} e^{\lambda_1 ||t|| + \lambda_2 |s|},
\]

for all \(t, s \in \mathbb{R}\) with \(|s| < \delta\) for some \(\delta > 0\).

**Proof.** **Case-1:** Let \(n\) be a non-negative integer. We have (see [7, (4.17)]) that for \(t \in \mathbb{R}\)

\[
\Phi_{\sigma,\lambda}^{n,n}(a_t) = \left( \cosh(t) \right)^{2n} \phi_{\lambda}^{(0,2n)}(t),
\]

where \(\phi_{\lambda}^{(0,2n)}\) is the Jacobi function given by

\[
\phi_{\lambda}^{(0,2n)}(t) = F_1\left(n + \frac{1 - \lambda}{2}, n + \frac{1 - \lambda}{2}, 1, -\sinh^2 t\right).
\]

The hypergeometric function associated to the root system \(BC_1 = \{-\alpha, \alpha, -2\alpha, 2\alpha\}\) is the classical Gauss hypergeometric functions \(\phi_{\lambda}^{(a,b)}\) with \(a = \frac{m_\alpha + m_{2\alpha} - 1}{2}\) and \(b = \frac{m_{2\alpha} - 1}{2}\). It is known by Opdam [11, Theorem 3.15] that for each \(\lambda \in \mathbb{C}\), the function \(\phi_{\lambda}^{(0,2n)}\) has a holomorphic extension to \(\{t + is \in \mathbb{C} \mid |s| < \delta_2\}\) for some \(\delta_2 > 0\). Therefore, there exists \(\delta > 0\) such that for \(t \in \mathbb{R}\) and \(|s| < \delta\),

\[
\Phi_{\sigma,\lambda}^{n,n}(a_t + is) = \left( \cosh(t + is) \right)^{2n} \phi_{\lambda}^{(0,2n)}(t + is).
\]

Since, \(m_{2\alpha} \geq 0\) and \(m_\alpha + m_{2\alpha} \geq 0\) (i.e. \(a \geq -\frac{1}{2}, b \geq -\frac{1}{2}\)) by Ho and Ölafsson (see [6, Prop. A.6]) we have

\[
|\phi_{\lambda_1 + i\lambda_2}^{(0,2n)}(t + is)| \leq Ce^{\lambda_2 ||s|| + \lambda_1 ||t||}.
\]
Hence we have
\[ |\Phi_{\sigma,\lambda_1+i\lambda_2}^{n,n}(a_{t+is})| \leq e^{2nt}e^{\lambda_1|t|+\lambda_2|s|}.\]

**Case-2:** Let \( n \) be a negative integer. It follows from (2.8), (2.9) that
\[ e^{-(\lambda+1)H(a_{t+is}k_0)} = e^{-(\lambda+1)H(a_{-i\bar{s}}k_0)},\]
and
\[ e_m(K(a_{t+is}k_0)) = e_m(K(a_{t-i\bar{s}}k_0)).\]
Hence
\[ \Phi_{\sigma,\lambda}^{n,n}(a_{t+is}) = \Phi_{\sigma,\lambda}^{-n,-n}(a_{t-is}).\]
Therefore from Case-1, it follows that
\[ |\Phi_{\sigma,\lambda}^{n,n}(a_{t+is})| = |\Phi_{\sigma,\lambda}^{n,n}(a_{t-is})| = |\Phi_{\sigma,\lambda}^{-n,-n}(a_{t-is})| \leq e^{2nt}e^{\lambda_1|t|+\lambda_2|s|}.\]
\[ \square\]

Let \( A, p, \delta \) be real numbers such that \( A < \frac{\pi}{2}, p > 0 \) and \( 0 < \delta \leq 1 \). Let
\[ \mathcal{H}(\delta) = \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\delta \},\]
and
\[ \mathcal{H}(A, p, \delta) = \{ a : \mathcal{H}(\delta) \to \mathbb{C} \text{ holomorphic} \mid |a(\lambda)| \leq Ce^{-p(\Re \lambda)+A|\Im \lambda|} \text{ for all } \lambda \in \mathcal{H}(\delta) \}.\]
We now present an analogue of Ramanujan’s Master theorem alluded to introduction.

**Theorem 2.12.** Let \( n \) be a positive integer and \( a \in \mathcal{H}(A, p, \delta) \) with \( A, p, \delta \) as above. Then

1. The \((n,n)\)-type spherical Fourier series
\[
(2.11) \quad f(x) = \sum_{m=0}^{\infty} (-1)^{m+1}(2m+n+1)a(2m+1+n)\psi_{n+2m,n}(x),
\]
converges on a compact subset of \( \Omega_p := \text{SO}(2) \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right\} \text{SO}(2) \) and defines a holomorphic function on a neighborhood
\[ \Omega'_p = \text{SU}(2) \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right\} \text{SO}(2,\mathbb{C}) \text{ of } \Omega_p \text{ in } G_\mathbb{C}.\]

2. Let \( \eta \in \mathbb{R} \) with \( 0 \leq \eta < \delta \). Then
\[
(2.12) \quad f(a_t) = \frac{1}{2} \int_{-\eta-i\infty}^{-\eta+i\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma,\lambda}^{(n,n)}(a_t)d\lambda
\]
\[ + \sum_{k \in L_{n,n}^A} (-1)^{k-n+1}k a(k)\Psi_k(a_t),\]
for \( t \in \mathbb{R} \) with \( |t| < p \), where \( b(\lambda) \) is defined by
\[ b(\lambda)\mu(\sigma,\lambda) = \begin{cases} \left( i4 \right)^{-\frac{\lambda}{2}} + 1 \cos \frac{\lambda \pi}{2} & \text{if } n \text{ is even}, \\ \left( i4 \right)^{-\frac{\lambda}{2}} + 1 \sin \frac{\lambda \pi}{2} & \text{if } n \text{ is odd}. \end{cases} \]
The integral above is independent of \( \eta \) and converges uniformly on compact subsets of \{\( a_t \mid t \in \mathbb{R} \}\} and extends as a \((n,n)\)-type function on a neighborhood of \( G \) in \( G_\mathbb{C} \).
(3) The extension of $f$ to $G$ satisfies the following:
\[
\frac{1}{4\pi^2} \int_G f(x) \Phi_{\sigma,\lambda}^{n,n}(x^{-1}) \, dx = \frac{1}{2} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)), \quad \lambda \in i\mathbb{R},
\]
and for $k \in L_{\sigma}^{n,n}$
\[
\frac{1}{2\pi} \int_G f(x) \Psi_{k}^{n,n}(x^{-1}) \, dx = (-1)^{\frac{k-p-1}{2}} a(k).
\]

To prove the theorem we need the following elementary estimates:

**Lemma 2.13.**
1. $|\cos \frac{\pi}{2}(s + ir)| \geq e^{\frac{\pi}{2}r}$ if $r$ is away from 0.
2. $|\cos \frac{\pi}{2}(2k + is)| \geq e^{\frac{\pi}{2}s}$ for all $s$.
3. $|\cos \frac{\pi}{2}(s - i2k)| \geq e^{k\pi}$ for $k \geq 1$.
4. On a small neighborhood of $i\mathbb{R}$, $|\cos \frac{\pi}{2}(s + ir)| \geq e^{\frac{\pi}{2}|r|}$ if $r$ is away from 0.

**Proof of Theorem 2.12:** We first assume that $n$ is an even positive integer. In this case, proof of (1) follows straight away from the Corollary 2.10.

For the proof of (2), we note that the function $b(\lambda)\mu(\sigma^+ , \lambda) = \frac{\lambda}{\cos \frac{\pi}{2}}$, where $c = \left( \frac{1}{2}(-1)^{-\frac{k-1}{2}} \right)$ has simple poles at $\lambda = \pm 1, \pm 3, \pm 5, \cdots$ and it is easy to check that
\[
\text{Res}_{\lambda=\pm 2j+1} (b(\lambda)\mu(\sigma^+ , \lambda)) = (-1)^{j+1} \frac{2c}{\pi} (2j + 1).
\]
Also, it is easy to see from Lemma 2.13 that for $\lambda$ on imaginary axis,
\[
|b(\lambda)\mu(\sigma^+ , \lambda)| \leq |\lambda|e^{-\frac{\pi}{2}|3\lambda|}.
\]

It follows from (2.5), that for $\lambda \in \mathbb{C}$
\[
b(\lambda) = -\frac{2ci}{\pi} \frac{1}{\sin\left(\frac{\pi \lambda}{2}\right)}.
\]

Clearly, this function has simple poles at $\lambda = 0, \pm 2, \pm 4, \cdots$.

We now consider the integral
\[
I = \int_{-i\infty}^{+i\infty} a(\lambda)b(\lambda)\Phi_{\sigma^+,\lambda}^{n,n}(a_t)\mu(\sigma^+, \lambda) \, d\lambda.
\]

By choosing the rectangular path $\gamma_k$ joining the vertices $(0, 2ki), (2k, 2ki), (2k, -2ki), (0, -2ki)$ and observing that the poles of the integrand are situated at the points $\lambda = 1, 3, 5, \cdots, 2k - 1$ inside the path $\gamma_k$ it follows that
\[
\int_{\gamma_k} a(\lambda)b(\lambda)\Phi_{\sigma^+,\lambda}^{n,n}(a_t)\mu(\sigma^+, \lambda) \, d\lambda = 2\pi i \sum_{m=0}^{k-1} a(2m + 1)\Phi_{\sigma^+,2m+1}^{n,n}(a_t) \text{Res}_{\lambda=2m+1} (b(\lambda)\mu(\sigma^+ , \lambda))
\]
\[
= 2\pi i \sum_{m=0}^{k-1} (-1)^{m+1} \frac{2c}{\pi} (2m + 1) a(2m + 1)\Phi_{\sigma^+,2m+1}^{n,n}(a_t).
\]

For $t \in \mathbb{R}$ with $|t| < p$, we have
\[
\int_{2ki}^{2k+2ki} |a(\lambda)| |b(\lambda)| |\Phi_{\sigma^+,\lambda}^{n,n}(a_t)| |\mu(\sigma^+, \lambda) \, d\lambda
\]
\[
= \int_0^{2k} |a(s + 2ki)| |\Phi_{\sigma^+,s+2ki}^{n,n}(a_t)| |b(s + 2ki)| |\mu(\sigma^+, s + 2ki) \, ds
\]
\[
\leq \int_0^{2k} e^{-\frac{\pi}{2}(s+1)(s+2k)}e^{-k\pi} ds (\text{because of (2.3) and Lemma 2.13})
\]
\[
\leq Ck^2 e^{(A - \frac{3}{2})2k}.
\]
We know that for $m$ where

\begin{equation}
(2.13)
\int_{2k + i2k}^{2k - i2k} |a(\lambda)| |b(\lambda)| |\Phi_{\sigma^+, \lambda}^{(n, n)}(a_t)| \mu(\sigma^+, \lambda) \, d\lambda.
\end{equation}

Let $t \geq 0$ with $t < p$. Now,

\[
\int_{2k - i2k}^{2k + i2k} |a(\lambda)| |b(\lambda)| |\Phi_{\sigma^+, \lambda}^{(n, n)}(a_t)| \mu(\sigma^+, \lambda) \, d\lambda
\]

\[
= \int_{-2k}^{2k} |a(2k + is)| |\Phi_{\sigma^+, 2k + is}^{(n, n)}(a_t)| |b(2k + is)| \mu(\sigma^+, 2k + is) \, ds
\]

\[
\leq \int_{-2k}^{2k} e^{-2pA|s|} e^{(2k - 1)t} |2k + |s|| e^{-\frac{s}{2}} |s| \, ds \quad \text{(because of (2.3) and Lemma 2.13)}
\]

\[
= 2 \int_{0}^{2k} e^{-2pA|s|} e^{(2k - 1)t} (2k + |s|) e^{-\frac{s}{2}} |s| \, ds
\]

\[
\leq Cke^{-2kt} \int_{0}^{2k} e^{(A - \frac{s}{2})|s|} \, ds.
\]

The last quantity goes to 0 as $k \to \infty$. Hence the integral in (2.13) goes to zero as $k \to \infty$, for $t \in \mathbb{R}$ with $|t| < p$.

Next,

\[
\int_{2k - i2k}^{2k + i2k} |a(\lambda)| |b(\lambda)| |\Phi_{\sigma^+, \lambda}^{(n, n)}(a_t)| \mu(\sigma^+, \lambda) \, d\lambda
\]

\[
= \int_{0}^{2k} |a(s - i2k)| |b(s - i2k)| |\Phi_{\sigma^+, s - i2k}^{(n, n)}(a_t)| \mu(\sigma^+, s - i2k) \, ds
\]

\[
\leq \int_{2k}^{0} e^{-ps + A2k} e^{(s - 1)t} (|s| + 2k) e^{-k|s|} |s| \, ds \quad \text{(because of (2.3) and Lemma 2.13)}
\]

\[
\leq Cke^{(A - \frac{s}{2})2k}.
\]

This last quantity goes to zero as $k$ goes to infinity. Hence we have

\[
I = 2\pi i \sum_{m=0}^{\infty} (-1)^{m+1} \frac{2c}{\pi} (2m + 1) a(2m + 1) \Phi_{\sigma^+, 2m+1}^{(n, n)}(a_t).
\]

We now write

\[
I = I_1 + I_2,
\]

where

\[
I_1 = 2\pi i \sum_{m=0}^{\infty} (-1)^{m+1} \frac{2c}{\pi} (2m + 1) a(2m + 1) \Phi_{\sigma^+, 2m+1}^{(n, n)}(a_t),
\]

\[
I_2 = 2\pi i \sum_{m=0}^{\infty} (-1)^{m+1} \frac{2c}{\pi} (2m + 1) a(2m + 1) \Phi_{\sigma^+, 2m+1}^{(n, n)}(a_t).
\]

We know that for $m \leq \frac{n-2}{2}$ that is, for $2m+1 < n$ (see (2.4)),

\[
\Phi_{\sigma^+, 2m+1}^{(n, n)}(a_t) = \Psi_{2m+1}(a_t).
\]

Therefore

\[
I_1 = 2\pi i \sum_{m=0}^{\infty} (-1)^{m+1} \frac{2c}{\pi} (2m + 1) a(2m + 1) \Psi_{2m+1}(a_t)
\]

\[
= \sum_{m=0}^{\infty} (-1)^{m+1} 4ci(2m + 1) a(2m + 1) \Psi_{2m+1}(a_t).
\]
We recall from Theorem \[2.6\] that for \(m \geq 0\),
\[
\Phi_{\sigma^+, n + 2m + 1}(a_t) = \psi_{n + 2m, n}(a_t).
\]

Therefore
\[
I_2 = 2\pi i \sum_{m=0}^{\infty} (-1)^{m+1+\frac{n}{2}} 4\pi (2m + 1 + n) a(n + 2m + 1) \psi_{n + 2m, n}(a_t)
\]
\[
= \sum_{m=0}^{\infty} (-1)^{m+1+\frac{n}{2}} 4ci(2m + 1 + n) a(n + 2m + 1) \psi_{n + 2m, n}(a_t).
\]

We have chosen \(c\) such that \(4ci(-1)^{\frac{n}{2}} = 1\). Therefore, for \(t \in \mathbb{R}\) with \(|t| < p\),
\[
f(a_t) = \frac{1}{2} \int_{-\infty}^{+\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma^+, \lambda}(a_t) \mu(\sigma^+, \lambda) d\lambda
\]
\[
+ \sum_{j=0}^{n-2} (-1)^j \frac{n}{2} (2j + 1) a(2j + 1) \Phi_{\sigma^+, \lambda}(a_t).
\]

From Proposition \[2.11\] and Lemma \[2.13\] and using the fact that \(A < \frac{\pi}{2}\), it is clear that the integral
\[
\int_{-\infty}^{+\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma^+, \lambda}(a_t) \mu(\sigma^+, \lambda) d\lambda,
\]
exists and holomorphic for all \(t \in \mathbb{C}\) with \(|3t| < \delta_1\) for some \(\delta_1 > 0\). Therefore using \[2.4\] and \[2.10\] it follows that \(f(a_t)\) has a holomorphic extension for all \(t \in \mathbb{C}\) with \(|3t| < \delta_2\) for some \(\delta_2 > 0\). Hence
\[
f(a_t) = \frac{1}{2} \int_{-\infty}^{+\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma^+, \lambda}(a_t) \mu(\sigma^+, \lambda) d\lambda
\]
\[
+ \sum_{j=0}^{n-2} (-1)^j \frac{n}{2} (2j + 1) a(2j + 1) \Phi_{\sigma^+, \lambda}(a_t),
\]
for \(t \in \mathbb{C}, \ |3t| < \delta_2\).

Using Cauchy integral formula and using the estimates of \(\Phi_{\sigma^+, \lambda}, b(\lambda)\mu(\sigma, \lambda)\) and \(a(\lambda)\) we can prove that for \(0 < \eta < \delta(\leq 1)\),
\[
\frac{1}{2} \int_{-\infty}^{+\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma^+, \lambda}(a_t) \mu(\sigma^+, \lambda) d\lambda
\]
\[
= \frac{1}{2} \int_{-\eta}^{+\eta} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma^+, \lambda}(a_t) \mu(\sigma^+, \lambda) d\lambda.
\]

For the proof of \((3)\), we first observe that \(a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)\) is holomorphic in \(|\Re \lambda| < \delta\), as \(a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) = (a(\lambda) - a(-\lambda)) b(\lambda)\) and \((a(\lambda) - a(-\lambda))\) has a zero at \(\lambda = 0\) and \(a(\lambda)\) is holomorphic there. Now for \(s\) away from 0,
\[
|a(r + is)b(r + is)| \leq Ce^{-\frac{\pi r + A|s|}{2}e^{-\frac{x |s|}}.}
\]
This estimate, together with Cauchy integral formula shows that
\[
\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) \in \mathcal{S}(i\mathbb{R})e.
\]
Therefore, using the inversion formula we get the desired formula. This completes the proof for the case when \(n\) is an even nonnegative integer.

Proof for the case when \(n\) is odd is similar. \(\square\)

We now state the theorem for \(n\) negative integer and we observe that there is no discrete series representation involves in this case.
Theorem 2.14. Let \( n \) be a negative integer and \( a \in \mathcal{H}(A, p, \delta) \). Then

1. The spherical Fourier series
\[
f(x) = \sum_{m=|n|}^{\infty} (-1)^{m+1} a(2m + 1 + n) \psi_{n+2m,n}(x),
\]
converges on a compact subset of \( \Omega_p := \text{SO}(2) \left\{ \left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right) \mid t \in \mathbb{R} \text{ with } |t| < p \right\} \text{SO}(2)
and defines a holomorphic function on a neighborhood
\[
\Omega_p' = \text{SU}(2) \left\{ \left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right) \mid t \in \mathbb{C} \text{ with } |t| < p \right\} \text{SO}(2, \mathbb{C}), \text{ of } \Omega_p \text{ in } X_{\mathbb{C}}.
\]

2. Let \( \eta \in \mathbb{R} \) with \( 0 \leq \eta < \delta \). Then
\[
f(a_t) = \frac{1}{2} \int_{-\eta - i\infty}^{\eta + i\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \Phi_{\sigma, \lambda}^{(n,n)}(a_t) \mu(\sigma, \lambda) d\lambda,
\]
for \( t \in \mathbb{R} \) with \( |t| < p \), where \( b(\lambda) \) is defined by
\[
b(\lambda) \mu(\sigma, \lambda) = \begin{cases} 
\left( \frac{i}{4} (-1)^{n+1} \right) \frac{\lambda}{\cos \frac{\pi \lambda}{2}} & \text{if } n \text{ is even}, \\
\left( \frac{1}{4} (-1)^{-n+1} \right) \frac{\lambda}{\sin \frac{\pi \lambda}{2}} & \text{if } n \text{ is odd}.
\end{cases}
\]
The integral above is independent of \( \eta \) and converges uniformly on compact subsets of \( \{a_t \mid t \in \mathbb{R}\} \) and extends as a \((n,n)\)-type function on a neighborhood of \( G \) in \( G_{\mathbb{C}} \).

3. The extension of \( f \) to \( G \) satisfies the following:
\[
(2.14) \quad \frac{1}{4\pi^2} \int_G f(x) \Phi_{\sigma, \lambda}^{(n,n)}(x^{-1}) \ dx = \frac{1}{2} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)), \lambda \in i\mathbb{R}.
\]

Remark 2.15. It can be easily seen that the decay condition on the function \( a \in \mathcal{H}(A, p, \delta) \) (that is, \( A < \frac{\pi}{2} \)) is optimum. In deed, Theorem 2.12 is not true for \( a(\lambda) \), where
\[
a(\lambda) = \begin{cases} 
\lambda \sin \frac{\pi \lambda}{2} & \text{if } n \text{ is odd}, \\
\lambda \cos \frac{\pi \lambda}{2} & \text{if } n \text{ is even}.
\end{cases}
\]
Because for the case \( n \) even, it follows from (2.11) that \( f(x) = 0 \) on \( \Omega_p' \). In particular \( f(I) = 0 \) where \( I \) is the identity matrix. Therefore the integral in the right side of (2.12) should vanishes at \( t = 0 \) but from the expression of \( a(\lambda) \) given above it follows that the integral in the right side of (2.12) does not exist. The reason for the case \( n \) odd is similar. It can be easily checked that \( A < \frac{\pi}{2} \) is optimum in \( [2][6] \). This difference occurs due to the parametrization of \( a, a^* \).

3. Ramanujan’s Master theorem for \( \chi_l \)-radial functions on \( SU(1,n) \)

In this section we shall prove an analogue of Ramanujan’s Master theorem for the \( \chi_l \)-radial functions on \( SU(1,n) \). We refer to [6] and references therein for the preliminaries.

Let \( G = SU(1,n) = \{ g \in \text{GL}_n(\mathbb{C}) \mid g^*I_{1,n}g = I_{1,n}, \det g = 1 \} \) where \( g^* \) is the complex conjugate and
\[
I_{1,n} = \left( \begin{array}{cc} 1 & 0_{1 \times n} \\
0_{n \times 1} & -I_{n \times n} \end{array} \right).
\]
Then the Lie algebra \( \mathfrak{g} \) of \( G \) is given by
\[
\mathfrak{g} = \left\{ \left( \begin{array}{ccc} -\text{Tr}(C) & A & C_{n \times n} \\
A^* & C_{n \times n} & 0_{n \times n} \end{array} \right) \mid C_{n \times n} = -C_{n \times n} \right\}.
\]
If \( \theta(X) = -X^* \) denotes the Cartan involution on \( \mathfrak{g} \), then
\[
\mathfrak{t} = \mathfrak{g}^\theta = \{ X \in \mathfrak{g} \mid \theta(X) = X \} = \left\{ \begin{pmatrix} -\operatorname{Tr}(C) & 0 \\ 0 & C \end{pmatrix} \mid C^* = -C \right\}.
\]
Then \( \mathfrak{t} \) is the Lie algebra of the compact group
\[
K = \left\{ \begin{pmatrix} (\det U)^{-1} & 0 \\ 0 & U \end{pmatrix} \mid U \in \mathfrak{U}(n) \right\} = \mathfrak{S}(\mathfrak{U}(1) \times \mathfrak{U}(n)).
\]
For
\[
p = \{ X \in \mathfrak{g} \mid \theta(X) = -X \} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \mid B \in \mathfrak{M}_{1 \times n}(\mathbb{C}) \right\},
\]
we have
\[
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}.
\]
Let
\[
(3.1) \quad H = \begin{pmatrix} 0 & 0_{1 \times n-1} & 0_{1 \times 1} \\ 0_{n-1 \times 1} & 0_{n-1 \times n-1} & 1_{1 \times 1} \\ 0_{1 \times n-1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix} \in \mathfrak{p},
\]
and let \( \mathfrak{a} = \mathbb{R} H. \) Then \( \mathfrak{a} \) is a maximal abelian subspace of \( \mathfrak{p} \). Let \( \beta \in \mathfrak{a}^* \) be such that \( \beta(H) = 1 \).
Then
\[
\Sigma(\mathfrak{g}, \mathfrak{a}) = \{ \beta, 2\beta \},
\]
\[
m_\beta := \dim \mathfrak{g}_\beta = 2(n-1), m_2 \beta := \dim \mathfrak{g}_{2\beta} = 1 \text{ and } \rho = \frac{1}{2}(m_\beta + 2m_2 \beta) = n.
\]
Let
\[
\mathfrak{u} = \mathfrak{t} \oplus i \mathfrak{p} = \left\{ \begin{pmatrix} -\operatorname{Tr}(C) & iB^* \\ iB & C \end{pmatrix} \mid C^* = -C, B \in \mathfrak{M}_{1 \times n}(\mathbb{C}) \right\}.
\]
This is the set of \((n+1) \times (n+1)\) skew hermitian matrices with trace 0. Then \( \mathfrak{u} \) is Lie algebra of the compact group \( \mathfrak{U} = \mathfrak{S}(n+1) \). It follows that \( \mathfrak{U} = \mathfrak{S}(n+1) \) is the compact dual of \( \mathfrak{G} = \mathfrak{S}(1,n) \) and \( \mathfrak{g}_\mathbb{C} = \mathfrak{u}_\mathbb{C}. \) Let \( \mathfrak{G}_\mathbb{C} \) be the analytic subgroup of \( \mathfrak{GL}(n+1, \mathbb{C}) \) with Lie algebra \( \mathfrak{g}_\mathbb{C}. \) We define, \( \mathfrak{U}_\mathbb{C} = \mathfrak{G}_\mathbb{C}. \)

All one dimensional representations of \( K \) are parametrized by \( l \in \mathbb{Z} \) and given by
\[
\chi_l \left( \begin{pmatrix} (\det U)^{-1} & 0 \\ 0 & U \end{pmatrix} \right) = (\det U)^l.
\]

**Definition 3.1.** A function \( f \) on \( G \) (respectively on \( U \)) is said to be \( \chi_l \)-radial if
\[
f(k_1 x k_2) = \chi_l(k_1^{-1}) f(x) \chi_l(k_2^{-1}), \quad k_1, k_2 \in K,
\]
x \( \in G \) (respectively for \( x \in U \)).

A representation \( \pi \) of \( U \) on a Hilbert space \( V \) is said to be spherical if
\[
V^K_\pi := \{ v \in V \mid \pi(k)v = v \text{ for all } k \in K \},
\]
is non empty and in this case \( \dim V^K_\pi = 1 \). It is known that the spherical representations of \( U \) are parametrized by \( 2m, m \in \mathbb{N} \cup \{0\} \). Analogously, a representation \( \pi \) of \( U \) on a Hilbert space \( V \) is said to be \( \chi_l \)-spherical if
\[
V^l_\pi := \{ v \in V \mid \pi(k)v = \chi_l(k)v \text{ for all } k \in K \},
\]
is non empty. It then follows that \( \dim V^l_\pi = 1 \). It is known that the \( \chi_l \) spherical representations of \( U \) are parametrized by \( 2m + |l|, m \in \mathbb{N} \cup \{0\} \). Let the representation space for \( \pi_{2m+|l|} \) be \( V_{2m+|l|}. \) Also let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be the weights for \( \pi_{2m+|l|} \) with weight vectors say \( v_1, v_2, \ldots, v_r \) respectively. Then
\[
(3.2) \quad \lambda_i(H) \leq 2m + |l|, i = 1, 2, \ldots, r,
\]
as $2m + |l|$ is the highest weight.

For fixed $m \in \mathbb{N} \cup \{0\}, l \in \mathbb{Z}$, let $f_0$ be the unique $\chi_l$-spherical vector for $\pi_{2m+|l|}$ with $\|f_0\| = 1$. We define the elementary spherical function of type $\chi_l$ on $U$ by

$$\psi_{2m+|l|}(x) := \langle \pi_{2m+|l|}(x)f_0, f_0 \rangle.$$ 

Clearly this function is $\chi_l$-radial on $U$. Let $\mathfrak{n} = \mathfrak{g}_\beta \oplus \mathfrak{g}_\beta$ and let $N$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{n}$ and let $A = \exp \mathfrak{a}$. Then we have the Iwasawa decomposition $G = KAN$ and any element of $g \in G$ can be uniquely written as

$$g = ka_1n$$

for some $k \in K, t \in \mathbb{R}$ and $n \in N$.

Here $a_t = \exp(tH)$ and we define $K(g) = k, H(g) = t$. We also have the following Cartan decomposition: $G = K\mathbb{A}^\tau K$ where $\mathbb{A}^\tau = \{a_t \mid t \geq 0\}$.

The elementary spherical function of type $\chi_l$ on $G$ with spectral parameter $\lambda$ is defined by

$$\phi_{\lambda,l}(g) = \int_K e^{-(\lambda + \rho)H(gk)}\chi_l(K(gk)^{-1}k)\,dk.$$ 

The function $\phi_{\lambda,l}$ satisfies the following relations:

1. $\phi_{\lambda,l} = \phi_{\mu,l}$ if and only if $\lambda = \pm \mu$.
2. $\phi_{\lambda,l}(g) = \phi_{-\lambda,-l}(g^{-1}) = \phi_{-\lambda,l}(g)$.

The function $\phi_{\lambda,l}$ which is analytic on $G$ extends to a holomorphic function on $G_C$ and by restriction to $U$ gives an elementary spherical function on $U$ if and only if $\lambda \in \pm(2\mathbb{N} \cup \{0\} + n + |l|)$. That is,

$$(3.3) \quad \phi_{2m+|l|+n,l} \big|_U = \psi_{2m+|l|} m \in \mathbb{N} \cup \{0\}.$$

For a $\chi_l$-radial function $f$ on $U$, the Fourier series is defined by

$$f(x) = \sum_{m=0}^{\infty} d_{\pi_{2m+|l|}} \widehat{f}(2m + |l|)\psi_{2m+|l|}(x),$$

where $d_{\pi_{2m+|l|}}$ is the dimension of $V_{2m+|l|}$ and

$$\widehat{f}(2m + |l|) = \int_U f(x)\psi_{2m+|l|}(x^{-1})\,dx.$$ 

For a $\chi_l$-type radial function $f$ on $G$, the spherical Fourier transform of $f$ is defined by

$$\widehat{f}(\lambda) = \int_G f(x)\phi_{\lambda,l}(x^{-1})\,dx.$$ 

From now on we will assume that $\|l\| < n$. We define

$$c(\lambda, m) = 2^{n+\lambda} \frac{\Gamma(m + m + 1) \Gamma(\lambda)}{\Gamma(\frac{m}{2} + \frac{m}{2} + \frac{m}{2}) \Gamma(\frac{m}{2} + \frac{m}{2} + \frac{1}{2})}.$$ 

It follows that $c(\rho, m) = 1$. We define a new multiplicity function $m_+(l)$ on $\Sigma$ by

$$(3.4) \quad m_+(l) = (2(n - 1) - 2|l|, 1 + 2|l|).$$

Then

$$(3.5) \quad c(\lambda, m_+(l)) = 2^{n+|l|+\lambda} \frac{\Gamma(n) \Gamma(\lambda)}{\Gamma(\frac{n+|l|}{2}) \Gamma(\frac{n+|l|}{2})},$$

and $\rho(m_+(l)) = n + |l|$.

For a $\chi_l$ radial function $f$ on $G$, the following inversion formula holds:

$$(3.6) \quad f(x) = \frac{1}{2 \cdot 4^n |l|} \int_{\mathbb{R}} \widehat{f}(\lambda)\phi_{\lambda,l}(x)|c(\lambda, m_+(l))|^{-2}\,d\lambda.$$

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We now have the following proposition which is an analogue of Proposition 2.8.

**Proposition 3.2.** Any \( x \in G_C \) has a unique decomposition \( x = u \exp tHk \) for some \( u \in SU(n+1), k \in K_C, \) and \( t \geq 0 \) and \( H \) is as \((3.7)\).

**Proof.** Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( K \) is the maximal compact subgroup of \( G \) with Lie algebra of \( K \) is \( \mathfrak{k} \). Let \( \theta(X) = -X^* \) be the Cartan involution for \( \mathfrak{g}_C \). Then
\[
\mathfrak{g}_C = \mathfrak{t}_1 \oplus \mathfrak{p}_1,
\]
where
\[
\mathfrak{t}_1 = \{ X \in \mathfrak{g}_C \mid \theta(X) = X \} = \{ X \in \mathfrak{g}_C \mid X^* = -X \},
\]
and
\[
\mathfrak{p}_1 = \{ X \in \mathfrak{g}_C \mid \theta(X) = -X \} = \{ X \in \mathfrak{g}_C \mid X^* = X \}.
\]

Let \( K_1 \) be the analytic subgroup whose Lie algebra is \( \mathfrak{t}_1 \). Therefore \( K_1 = SU(n+1) \). Let \( \sigma : \mathfrak{g}_C \to \mathfrak{g}_C \) be an involution given by \( \sigma(X) = -X^T \). Then again
\[
\mathfrak{g}_C = \mathfrak{h} \oplus \mathfrak{q},
\]
where
\[
\mathfrak{h} = \{ X \in \mathfrak{g}_C \mid \sigma(X) = X \} = \{ X \in \mathfrak{g}_C \mid X^T = -X \},
\]
and
\[
\mathfrak{q} = \{ X \in \mathfrak{g}_C \mid \sigma(X) = -X \} = \{ X \in \mathfrak{g}_C \mid X^T = X \}.
\]

Therefore (see [5, Proposition 2.2, p. 106]), any element \( g \in G_C \) can uniquely be written as
\[
g = u \exp X \exp Y, \quad \text{for some } u \in K_1, X \in \mathfrak{p} \cap \mathfrak{q}, Y \in \mathfrak{p} \cap \mathfrak{h}.
\]

It is now easy to check that \( Y \in \mathfrak{p} \cap \mathfrak{h} \) implies that \( Y \) is purely imaginary \((n+1) \times (n+1)\) skew symmetric matrix and hence \( Y \in \mathfrak{t}_C \). Also \( X \in \mathfrak{p} \cap \mathfrak{q} \) implies that \( X \) is real \((n+1) \times (n+1)\) symmetric matrix. Hence \( X \in \mathfrak{p}_g \) and therefore by Cartan decomposition we have \( \exp X = k_1 \exp tHk_2 \) for some \( k_1, k_2 \in K \) and \( t \geq 0 \).

Hence we have
\[
g = uk_1 \exp tHk_2 \exp Y,
\]
where \( uk_1 \in U \) and \( k_2 \exp Y \in K_C \).

We now prove the required estimate of \( \psi_{2m+|l|} \) which is an analogue of Theorem 2.9.

**Theorem 3.3.** For \( x \in G_C \), we have
\[
|\psi_{2m+|l|}(x)| \leq Ce^{(2m+|l|)t}|\chi_l(k^{-1})|,
\]
where \( x = u \exp tHk, u \in U, t \geq 0, k \in K_C \).

**Proof.** Let \( f_0 \) be the \( \chi_l \) spherical vector for \( \pi_{2m+|l|} \), that is,
\[
\pi_{2m+|l|}(k)f_0 = \chi_l(k)f_0.
\]

Then it is easy to check that
\[
\pi_{2m+|l|}^C(k_1)f_0 = \chi_l(k_1)f_0 \quad \text{for all } k_1 \in K_C,
\]
where \( \pi_{2m+|l|}^C \) is the complexification of \( \pi_{2m+|l|} \). Let \( \{ f_0, f_1, \ldots, f_r \} \) be an orthonormal basis of \( V_{2m+|l|} \) such that \( f_0 \) is \( \chi_l \) spherical vector for \( \pi_{2m+|l|} \) and \( ||f_0|| = 1 \). We claim that
\[
\sum_{j=0}^{r} \left| \langle \pi_{2m+|l|}^C(x)f_0, f_j \rangle \right|^2 \leq C|\chi_l(k^{-1})|^2e^{2t(2m+|l|)},
\]
where \( x = u \exp tHk, u \in U, t \geq 0, k \in K_C \) and as a consequence we get

\[
|\psi_{2m+|l|}(x)| \leq Ce^{(2m+|l|)t}|\chi_l(k^{-1})|.
\]

We define \( P_l \) on \( V_{2m+|l|} \) by

\[
P_l(f)(x) = \int_K \chi_l(k)\pi_{2m+|l|}(kx)f \, dk.
\]

It is an orthogonal projection and \( P_l^* = P_l \). Now

\[
\sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tHk)f_0, f_j \rangle \right|^2 = |\chi_l(k^{-1})|^2 \sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tH))f_0, f_j \rangle \right|^2
\]

\[
= |\chi_l(k^{-1})|^2 \sum_{i,j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tH)P_l f_i, f_j \rangle \right|^2
\]

\[
= |\chi_l(k^{-1})|^2 \left\| \pi^C_{2m+|l|}(u \exp tH)^2 P_l \right\|_{HS}^2.
\]

The expression above is equal to

\[
(3.7) \quad |\chi_l(k^{-1})|^2 \text{Tr} \left( P_l^* \circ \pi^C_{2m+|l|}(u \exp tH)^* \circ \pi^C_{2m+|l|}(u \exp tH) \circ P_l \right).
\]

It can be shown using (2.7) that

\[
(3.8) \quad \pi^C_{2m+|l|}(\exp tH)^* \circ \pi^C_{2m+|l|}(\exp tH) = \pi^C_{2m+|l|}(\exp 2tH).
\]

Therefore it follows from (3.7), (3.8) that

\[
\sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tHk)f_0, f_j \rangle \right|^2 = |\chi_l(k^{-1})|^2 \text{Tr} \left( P_l \circ \pi^C_{2m+|l|}(\exp 2H) \right).
\]

Recall that, \( v_0, \ldots, v_r \) are the weight vectors with weights \( \lambda_0, \ldots, \lambda_r \) respectively and let \( \lambda_r = 2m + |l| \) be the highest weight. It then follows that

\[
\lambda_i(H) \leq 2m + |l|, 0 \leq i \leq r.
\]

Hence we have,

\[
\sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tHk)f_0, f_j \rangle \right|^2 = |\chi_l(k^{-1})|^2 \sum_{i=0}^r \langle P_l \circ \pi^C_{2m+|l|}(\exp 2H)v_i, v_i \rangle
\]

\[
= |\chi_l(k^{-1})|^2 \sum_{i=0}^r \langle e^{2t\lambda_i(H)}v_i, P_l^* v_i \rangle.
\]

Since, \( \langle v_i, P_l^* v_i \rangle = \| P_l v_i \|^2 \) is non negative we have,

\[
\sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tHk)f_0, f_j \rangle \right|^2 \leq |\chi_l(k^{-1})|^2 e^{2t(2m+|l|)} \sum_{i=0}^r \langle v_i, P_l^* v_i \rangle
\]

\[
= |\chi_l(k^{-1})|^2 e^{2t(2m+|l|)} \text{Tr}(P_l).
\]

But

\[
\text{Tr}(P_l) = \sum_{i=0}^r \langle P_l f_i, f_i \rangle = \langle P_l f_0, f_0 \rangle = \langle f_0, f_0 \rangle = 1.
\]

This implies that

\[
\sum_{j=0}^r \left| \langle \pi^C_{2m+|l|}(u \exp tHk)f_0, f_j \rangle \right|^2 \leq |\chi_l(k^{-1})|^2 e^{2t(2m+|l|)}.
\]

This completes the proof. \( \square \)
The following Proposition gives an estimate of $\chi_l$-spherical function on a certain neighborhood of $A$ in $G_C$.

**Proposition 3.4. (see [6] Prop. A.3, Prop. A.6)** There exists $\delta > 0$ such that, the function

$$t + is \mapsto \phi_{\lambda_1 + i\lambda_2, l}(\exp(t + is)H),$$

is holomorphic for all $t, s \in \mathbb{R}$ with $|s| < \delta$ and satisfies the following estimates

$$|\phi_{\lambda_1 + i\lambda_2, l}(\exp(t + is)H)| \leq C e^{\|t\| \|s\| |\lambda_1| + |\lambda_2| |s|},$$

in that domain.

It follows from (3.5) that

$$\frac{1}{c(\lambda, m_+(l)) c(-\lambda, m_+(l))} = 2^{-2(n+|l|)} (\Gamma(n))^{-2} \frac{\Gamma\left(\frac{\lambda+n+|l|}{2}\right) \Gamma\left(\frac{\lambda+n-|l|}{2}\right) \Gamma\left(\frac{\lambda-n+|l|}{2}\right) \Gamma\left(\frac{\lambda-n-|l|}{2}\right)}{\Gamma(\lambda) \Gamma(-\lambda)}. $$

Using the following formulas

$$\Gamma(2z) = (\pi)^{-\frac{1}{2}} 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}), \Gamma(z) \Gamma(-z) = -\frac{\pi}{z \sin \pi z}, \Gamma(z + \frac{1}{2}) \Gamma(-z + \frac{1}{2}) = \frac{\pi}{\cos \pi z},$$

and (3.9) it is easy to check the following:

$$\frac{1}{c(\lambda, m_+(l)) c(-\lambda, m_+(l))} = \gamma_{n,l} p_{n,l}(\lambda) q_{n,l}(\lambda),$$

where $\gamma_{n,l}$ is a constant given by

$$\gamma_{n,l} = \begin{cases} -\pi 2^{-4n-2|l|+3} (\Gamma(n))^{-2} & \text{if } l \text{ is even}, \\ \pi 2^{-4n-2|l|+3} (\Gamma(n))^{-2} & \text{if } l \text{ is odd}, \end{cases}$$

and $p_{n,l}$ is a polynomial in $\lambda$ given by

$$p_{n,l}(\lambda) = \begin{cases} \lambda \prod_{j=1}^{n-|l|-1} (\lambda^2 - (n-|l| - 2j)^2) \prod_{j=1}^{n+|l|-1} (\lambda^2 - (n+|l| - 2j)^2) & \text{if } n \text{ odd}, l \text{ even}, \\ \lambda^3 \prod_{j=1}^{n+|l|-1} (\lambda^2 - (n+|l| - 2j)^2) \prod_{j=1}^{n-|l|-1} (\lambda^2 - (n-|l| - 2j)^2) & \text{if } n \text{ even}, l \text{ even}, \\ \lambda^3 \prod_{j=1}^{n-|l|-1} (\lambda^2 - (n-|l| - 2j)^2) \prod_{j=1}^{n+|l|-1} (\lambda^2 - (n+|l| - 2j)^2) & \text{if } n \text{ odd}, l \text{ odd}, \\ \lambda \prod_{j=1}^{n+|l|-1} (\lambda^2 - (n+|l| + 2j)^2) \prod_{j=1}^{n-|l|-3} (\lambda^2 - (n-|l| + 2j)^2) & \text{if } n \text{ even}, l \text{ odd}, \end{cases}$$

and $q_{n,l}$ is given by

$$q_{n,l}(\lambda) = \begin{cases} \tan \left(\frac{\pi \lambda}{2}\right) & \text{if } n \text{ odd}, l \text{ even or if } n \text{ even}, l \text{ odd}, \\ \cot \left(\frac{\pi \lambda}{2}\right) & \text{if } n \text{ even}, l \text{ even or if } n \text{ odd}, l \text{ odd}. \end{cases}$$

We also have the following relation between the dimension $d_{\pi_{2m+|l|}}$ of $V_{\pi_{2m+|l|}}$ and the $c$-function ([3] Prop. 5.2.10, p. 78)

$$d_{\pi_{2m+|l|}} = \alpha \frac{c(\rho(m_+), m_+(l)) c(-\rho(m_+), m_+(l))}{c(2m + \rho(m_+), m_+(l)) c(-2m - \rho(m_+), m_+(l))},$$

where $\alpha$ is a fixed constant. We recall that $\rho(m_+) = n + |l|$. 

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Mark: 3.4
This follows from the relation (see [5] for unexplained notation)
\[
\frac{\overline{c}(\rho(m_+), m_+(l))}{\overline{c}(\rho(m_+), m_+(l))} = \frac{\overline{c}(\rho(m_+), m_+(l))}{\overline{c}(\rho(m_+), m_+(l))}, \quad \mu \in P^+.
\]

Let us consider the case \( n \) odd, \( l \) even. In this case it follows from (3.10) and (3.11) that
\[
d_{2m+|l|} = \alpha \frac{p_{n,l}(2m+n+|l|)}{p_{n,l}(m+|l|)} \frac{q_{n,l}(2m+n+|l|)}{q_{n,l}(m+|l|)}
\]
\[
= \alpha \frac{p_{n,l}(2m+n+|l|)}{p_{n,l}(n+|l|)}. \]

We are now in a position to state our final theorem.

**Theorem 3.5.** Let \( n \) odd, \( l \) even and \( \mathcal{H}(A,p,\delta) \) as in Theorem 2.12. Suppose \( a \in \mathcal{H}(A,p,\delta) \) and let \( b \) be the meromorphic function on \( \mathbb{C} \) defined by
\[
b(\lambda) = \frac{c(\lambda_m, m_+(l)) c(-\lambda, m_+(l))}{\alpha \frac{p_{n,l}(\lambda)}{p_{n,l}(n+|l|)} \sin \frac{\pi}{2} (\lambda - n - |l|)},
\]
where \( \alpha \) is a constant given in (3.11). Then

1. The \( \chi_l \)-spherical Fourier series
   \[
f(x) = \sum_{m=0}^{\infty} (-1)^m d_{2m+|l|} a(2m+|l|+n) \psi_{2m+|l|}(x),
   \]
   converges on a compact subset of \( \Omega_p := K \{ \exp(tH) \mid t \in \mathbb{R}, |t| < p \} K \) and defines a holomorphic function on a neighborhood
   \[
   \Omega_p := U \{ \exp(tH) \mid t \in \mathbb{C}, \Re t < p \} K \mathbb{C}, \text{ of } \Omega_p \text{ in } G_\mathbb{C}.
   \]
2. Let \( \eta \in \mathbb{R} \) with \( 0 \leq \eta < n \delta \). Then
   \[
f(a_t) = \int_{-\eta-i\infty}^{-\eta+i\infty} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \phi_{\lambda,l}(a_t)|c(\lambda, m_+(l))|^{-2} d\lambda,
   \]
   for \( t \in \mathbb{R} \) with \( |t| < p \).
   The integral above converges uniformly on compact subsets of \( \{a_t \mid t \in \mathbb{R}\} \) and extends as a \( \chi_l \)-radial function on a neighborhood of \( G \) in \( G_\mathbb{C} \).
3. The extension of \( f \) to \( G \) satisfies:
   \[
   \frac{1}{4\pi^2} \int_G f(x) \phi_{\lambda,l}(x^{-1}) dx = (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)), \; \lambda \in i\mathbb{R}.
   \]

The following elementary estimates will be used in the proof.

**Lemma 3.6.**

1. \(|\sin \frac{\pi}{2}(s+ir)| \geq e^{\frac{r}{2}} \) if \( r \) is away from zero.
2. \(|\sin \frac{\pi}{2}(2k + is)| \geq e^{\frac{r}{2}} \) for all \( s \).
3. \(|\sin \frac{\pi}{2}(s - i2k)| \geq e^{kr} \) for \( k \geq 1 \).
4. On a small neighborhood of \( i\mathbb{R} \), \(|\sin \frac{\pi}{2}(s+ir)| \geq e^{\frac{r}{2}|r|} \) if \( r \) is away from zero.

**Proof of Theorem 3.5.** We first observe that the function
\[
b(\lambda) = \frac{c(\lambda, m_+(l)) c(-\lambda, m_+(l))}{\alpha \frac{p_{n,l}(\lambda)}{p_{n,l}(n+|l|)} \sin \frac{\pi}{2} (\lambda - n - |l|)},
\]
has simple poles at \( \lambda = n + l, n + l + 2, \ldots \) in the domain \( \{ z \in \mathbb{C} \mid \Re z \geq -n\delta \} \).
From (3.10) it follows that
\[ b(\lambda) = \left( \frac{-i}{4} \right) (-1)^{\frac{n-1}{2}} \left( \frac{\alpha}{\pi} \right) 2^{4n+2|l|-3} (\Gamma(n))^2 \frac{1}{p_{n,l}(n+|l|)} \frac{1}{\sin \left( \frac{2\lambda}{\pi} \right)}. \]
This is an odd function and has a simple pole at \( \lambda = 0 \). Therefore the function
\[ \lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda), \]
is holomorphic in a small neighborhood of \( i\mathbb{R} \). Using the same argument as in the proof of the Theorem 2.12 it follows that
\[ \lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) \in \mathcal{S}(i\mathbb{R}). \]
Therefore (3) will follow from (2) by the inversion formula (3.6). Rest of the proof is similar to the proof of Theorem 2.12 using Cauchy’s formula and Lemma 3.6.

\[ \square \]

**Remark 3.7.** We define \( b(\lambda) \) in the other cases by:
\begin{align*}
\frac{b(\lambda)}{c(\lambda, m_+ (l))c(-\lambda, m_+ (l))} &= \begin{cases} 
\left( \frac{-i}{4} \right) (-\alpha) \frac{p_{n,l}(\lambda)}{p_{n,l}(n+|l|)} \frac{1}{\sin \left( \frac{1}{2}(\lambda-n-|l|) \right)} & \text{if } n \text{ even}, l \text{ even}, \\
\left( \frac{-i}{4} \right) \alpha \frac{p_{n,l}(\lambda)}{p_{n,l}(n+|l|)} \frac{1}{\sin \left( \frac{1}{2}(\lambda-n-|l|) \right)} & \text{if } n \text{ odd}, l \text{ even}, \\
\left( \frac{-i}{4} \right) (-\alpha) \frac{p_{n,l}(\lambda)}{p_{n,l}(n+|l|)} \frac{1}{\sin \left( \frac{1}{2}(\lambda-n-|l|) \right)} & \text{if } n \text{ odd}, l \text{ odd}.
\end{cases}
\end{align*}

With this \( b(\lambda) \), analogue of Theorem 3.5 holds true for the cases above and the proof will be exactly same. It is also easy to see that the constant \( A < \frac{\pi}{2} \) is optimum.

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