New Characterizations for the Multi-output Correlation-Immune Boolean Functions

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Abstract
In stream ciphers, the correlation-immune functions serve as an important metric for measuring a cryptosystem resistance against correlation attacks. In block ciphers, there is an equivalence between multipermutations and correlation-immune functions. In this paper, three new methods are proposed to characterize the multi-output correlation-immune Boolean functions. Compared with the Walsh spectral characterization method, the first method can significantly reduce the computational complexity. The last two characterization methods are in terms of the Fourier spectral characterizations over the complex field. Furthermore, only one point of the Fourier spectra needs to be calculated when characterize a symmetric multi-output correlation-immune Boolean function.

Keywords: Correlation immunity, Resilience, Discrete Fourier transform, Multi-output Boolean function, Generalized Boolean function.

1 Introduction
Multi-output Boolean functions are widely used in cryptography. For a running key generators in stream ciphers, multi-output Boolean functions can be used as a combiner of $n$ linear feedback shift registers (LFSR) or filter functions, increasing the speed of the cipher compare to single-output Boolean function which generate only one bit at each clock cycle. This combining function should then be a correlation-immune function in order to resist Siegenthaler’s correlation attack [18](or ‘divide and conquer attack ’ [17]).
Moreover, multi-output Boolean functions also play an important role in the design and analysis of block ciphers: Schnorr and Vaudenay [16] claimed that all S-boxes should be multipermutations. An multi-output Boolean functions is called a multipermutation if two different \((n + m)\)-tuples of the form \((x, f(x))\) cannot collide in any \(n\) positions [19]. Such a multi-output Boolean function ensures a perfect diffusion, which is a very significant criterion in the design of S-box. Camion and Canteaut [3] showed that a multi-output Boolean function \(f\) is a multipermutation if and only if \(y = f(x)\) is a \(t\)th order correlation-immune Boolean function. Therefore, we can research the multipermutation by studying the correlation immunity of a multi-output Boolean function.

In the single-output case, Xiao and Massey [21] showed the Walsh spectral characterization for a Boolean function, that is, a Boolean function is \(t\)th-order correlation-immune if and only if its Walsh-Hadamard transform vanishes for all points with Hamming weights between 1 and \(t\). They applied spectrum theory to cryptography for the first time and opened up a broad road for further research. Golomb [9, 10] introduced the concept of the invariants of Boolean functions for classification of Boolean functions. But Golomb did not explicitly mention that the motivation to propose invariants of Boolean functions was about correlation attack and invariants are the same concept of Walsh-Hadamard spectral characterization of correlation-immune functions until his paper [11] published in 1999.

In the multi-output case, Gopalakrishnan and Stinson [13] investigated the multi-output correlation-immune Boolean functions and deduced their equivalent characterization, that is, a \(t\)th-order multi-output correlation-immune Boolean function is equivalent to a generalized large set of orthogonal arrays. Next year, Camion and Canteaut [3] gave a Walsh spectral characterization for correlation-immune functions over any finite alphabet endowed with the structure of an Abelian group. That is, a multi-output Boolean function is \(t\)th-order correlation-immune if and only if all the nonzero linear combination of the component functions is \(t\)th-order correlation-immune. At the same time, Feng obtained the same conclusion in his Phd thesis [8]. Later, Chen [6] and Carlet [5] proved it by different methods.

In this paper we firstly gave a simper characterization for a multi-output Boolean function, which reduced the number of calculations from \(2^m - 1\) only to \(m\) times (It needs \(2^m - 1\) times Walsh transforms when we determine if a multi-output Boolean function is correlation-immune by the Walsh spectrum). In paper [20], Wang and Gong investigated discrete Fourier transform over the complex field of single-output Boolean functions, and deduced the equivalent condition for an \(t\)th-order correlation-immune Boolean function. Cyclotomic polynomials and associated polynomials were used as bridges to get the Fourier spectral characterization. Inspired by this, we tried to characterize \(t\)th-order multi-output correlation-immune Boolean functions by using those tools, and obtained two different expressions in terms of the Fourier spectra.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions of the correlation immunity and resilience, associated polynomials, discrete Fourier transform over the complex field of the functions, as well as Walsh transform for a multi-output Boolean function. In Section 3, we present our first characterization for multi-output correlation-immune Boolean functions. In Section 4, we give our second new method of
describing multi-output correlation-immune Boolean functions in terms of discrete Fourier transform over the complex field. In Section 5 another expression for the Fourier spectral characterization is given. Section 6 concludes the paper.

2 Preliminaries

2.1 The Representations of Multi-output Boolean functions

Multi-output Boolean functions $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$ viewed from a cryptographic viewpoint, that is, functions from the vector space $\mathbb{F}_2^n$, to the vector space $\mathbb{F}_2^m$, for some positive integers $n$ and $m$, where $\mathbb{F}_2$ is the finite field with two elements, and $\mathbb{F}_2^n$ is $n$-dimension vector space on $\mathbb{F}_2$. Every component function $f_i$, $1 \leq i \leq m$, is Boolean function. Obviously, these functions include the (single-output) Boolean functions which correspond to the case $m = 1$. Multi-output Boolean functions are also called vectorial Boolean functions. We will refer to a multi-output Boolean function as an $(n, m)$-function for simplicity.

We define a generalized Boolean function to be a function $f$ from $\mathbb{F}_2^n$ to $\mathbb{F}_{2^m}$, where $m \geq 1$. Since there is a natural one-to-one correspondence between vectors in $\mathbb{F}_{2^m}$ and integers in $[0, 2^m - 1]$, which allows us to order the vectors in $\mathbb{F}_{2^m}$ according to their corresponding integers. That is, a generalized Boolean function and a multi-output Boolean function has a one-to-one correspondence.

If $m$ is a divisor of $n$, then any $(n, m)$-function $f(x)$ can be viewed as a function from $\mathbb{F}_{2^n}$ to itself, since $\mathbb{F}_{2^m}$ is a sub-field of $\mathbb{F}_{2^n}$. Thus an $(n, m)$-function $f(x)$ can be represented in the form $\text{tr}_{n/m}(\sum_{j=0}^{2^n-1} \delta_{x_j} x_j)$, where $\text{tr}_{n/m}(x) = x + x^{2^m} + x^{2^{2m}} + x^{2^{3m}} + \cdots + x^{2^{n-m}}$ is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$. But such a representation is not unique.

2.2 Correlation Immunity and Resiliency

The correlation-immune multi-output Boolean functions is defined initially from the perspective of probability theory, which is similar to the definition of correlation-immunity of Boolean functions.

Definition 1. Let $t$ be an integer such that $0 \leq t \leq n$. An $(n, m)$-function $f(x)$ is called $t$th-order correlation immune if its output distribution does not change when at most $t$ coordinates $x_i$ of $x$ are kept constant. In other words,

$$P_r(f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m)|x_{i_j} = a_j, 1 \leq j \leq t) = P_r(f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m))$$

for every $t$-subset $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, n\}$, $a_j \in \mathbb{F}_2(1 \leq j \leq t)$, and $(y_1, y_2, \cdots, y_m) \in \mathbb{F}_{2^m}$.

The $(n, m)$-function $f(x)$ is said to be balanced if every possible output occurs with equal probability $2^{-m}$. Furthermore, if $f$ is $t$th-order correlation-immune and balanced, then $f$ is said to be $t$th-order resilient. That is $f(x)$ stays balanced when at most $t$
coordinates $x_i$ of $x$ are kept constant. We will refer to a $t$th-order correlation-immune $(n, m)$-function as $(n, m, t)$-CI function. A $t$th-order resilient $(n, m)$-function is denoted similarly as $(n, m, t)$-resilient function.

### 2.3 Associated Polynomial

We describe a sequence $f$ of length $2^n$ corresponding to a generalized Boolean function by listing the values taken by $f(x_1, x_2, \cdots, x_n)$ as $(x_1, x_2, \cdots, x_n)$ which ranges over all its $2^n$ values in lexicographic order. In other words, sequence $f$ is defined by

$$f = (f(0), f(1), \cdots, f(2^n - 1)),$$

where $f(k) = f(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_n)$ is the binary representation of the integer $k$ for $0 \leq k \leq 2^n - 1$, i.e., $k = \sum_{i=1}^{n} x_i2^{i-1}$. For example, for $n = 3$ and $m = 2$ we have $3x_1x_2x_3 = (000000023)$ and $2x_1x_2 + 3x_3 + 1 = (10101032)$ respectively.

Let $\omega_i = \exp\left(\frac{2\pi \sqrt{-1}}{2^m}\right)$ be $2^i$, $2^{2i}$, $\cdots$, $2^{m^2}$ primitive root of unity over the complex field respectively, $i \in [1, m]$ is a integer. The polynomials associated with the sequence $f$ defined by generalized Boolean function: $F_2^n \rightarrow F_2$ is given by

$$F(z) = \sum_{k=0}^{2^n-1} \omega_i f(k) z^k. \quad (1)$$

### 2.4 Discrete Fourier Transform

We now introduce the concept of the discrete Fourier transform (DFT) over the complex field of the generalized Boolean function. Note that DFT over the complex field introduced here is the traditional DFT, which is different from the DFT over the finite field [12].

**Definition 2.** Let $\xi = \exp\left(\frac{2\pi \sqrt{-1}}{N}\right)(N = 2^n)$ be an $N$th primitive root of unity over the complex field. The discrete Fourier transform (DFT) of the generalized Boolean function $f(x)$ over the complex field is defined by

$$F_f(j) = \sum_{k=0}^{N-1} \omega_i f(k) \xi^{-kj}, 0 \leq j \leq N - 1. \quad (2)$$

Then the inverse discrete Fourier transform (IDFT) of the generalized Boolean function $f(x)$ is given by

$$\omega_i f(k) = \frac{1}{N} \sum_{j=0}^{N-1} F_f(j) \xi^{kj}, 0 \leq k \leq N - 1. \quad (3)$$

### 2.5 Walsh Transform

We shall call *Walsh transform* of an $(n, m)$-function $f(x)$ the function which maps any ordered pair $(u, v) \in F_2^n \times F_2^m$ (where $F_2^m = F_2^m \setminus \{0\}$) to the value at $u$ of the Walsh transform of the component function $v \cdot f(x)$, that is,

$$\hat{f}(u, v) = \sum_{x \in F_2^n} (-1)^{v \cdot f(x) + u \cdot x} \quad (4)$$
Fact 1. ([4, 5]) An \((n, m)\) function is \((n, m, t)\)-CI function if and only if \(\hat{f}(u, v) = 0\) for \(1 \leq wt(u) \leq t\), where \(wt(u)\) denotes the Hamming weight of \(u\).

3 First Characterization

Note that the essence of Walsh Spectral Characterization for a multi-output correlation-immune Boolean function is every nonzero linear combination of all component functions is correlation-immune. Thus, we have to calculate \(2^m - 1\) times Walsh transforms. In this section, we shall give a simpler method to Characterize multi-output correlation-immune Boolean functions, which reduces the number of calculations only to \(m\) times.

Theorem 1. Let \(f(x_1, x_2, \cdots, x_n)\) be a multi-output Boolean function from \(\mathbb{F}_2^n\) to \(\mathbb{F}_2^m\). Then \(f\) is an \((n, m, t)\)-CI function if and only if
\[
\sum_{x \in \mathbb{F}_2^n} \omega_i f(x) (-1)^{c \cdot x} = 0,
\]
where \(1 \leq wt(c) \leq t\), \(\omega_i\) are \(2^{th}, 2^2^{th}, \cdots, 2^m^{th}\) primitive root of unity in the complex field respectively.

Before prove theorem 1, we first introduce a very important lemma called ‘linear combination lemma ’ and give its simple proof [2].

Lemma 1. The discrete random variable \(Z\) is independent of the \(k\) independent binary random variables \(Y = (Y_1, Y_2, \cdots, Y_k)\) if and only if \(Z\) is independent of the sum \(c \cdot Y = c_1Y_1, c_2Y_2, \cdots, c_kY_k\) for every choice of \(c_1, c_2, \cdots, c_k\), not all zeros, in \(\mathbb{F}_2\).

Proof. The necessity is obvious. We just explain sufficiency.

Let \(P_{Y|Z}(y|z)\) be the conditional density of \(Y\), and let \(S(c|z)\) be its Walsh-Hadamard transform. Then for every fixed \(c \in \mathbb{F}_2^k\)
\[
S(c|z) = \sum (-1)^{c \cdot y} P_{Y|Z}(y|z) = E_{Z=x} [(-1)^{cY}] = E[(-1)^{cY}] = S(c)
\]
since each sum \(c \cdot Y\) is supposed to be independent of \(Z\). Inverting the transform gives that
\[
P_{Y|Z}(y|z) = \frac{1}{2^n} \sum (-1)^{c \cdot y} S(c|z) = \frac{1}{2^n} \sum (-1)^{c \cdot y} S(c) = P_Y(y)
\]
and thus yields independence. \(\square\)

Now, we shall prove Theorem 1 by using Lemma 1.

Proof. The equation (5) can be divided into two parts. One is for \(c \cdot x = 0\), and the other is for \(c \cdot x = 1\).
\[
\sum_{c \cdot x = 0} \omega_i f(x) - \sum_{c \cdot x = 1} \omega_i f(x) = 0.
\]
We denote that
\[
a_\alpha = \#\{x : f(x) = \alpha, c \cdot x = 0\},
\]
and 

\[ b_\alpha = \#\{x : f(x) = \alpha, c \cdot x = 1\}, \]

where \( \alpha \in [0, 2^m - 1] \). Hence, equation (6) is equivalent to

\[ \sum_{\alpha=0}^{2^m-1} a_\alpha \omega_i^\alpha - \sum_{\alpha=0}^{2^m-1} b_\alpha \omega_i^\alpha = 0 \iff \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) \omega_i^\alpha = 0. \]

Since \( \omega_i \) are \( 2\text{th}, 2^2\text{th}, \cdots, 2^m\text{th} \) primitive root of unity in the complex field, We have \( \Phi_2(z) = z + 1, \Phi_4(z) = z^2 + 1, \cdots, \Phi_{2^m}(z) = z^{2^{m-1}} + 1 \) all divide \( \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) z^\alpha \). In addition, \( \Phi_2(z), \Phi_4(z), \cdots, \Phi_{2^m}(z) \) are all irreducible in the integer ring.

\[ g(z) = (z + 1)(z^2 + 1) \cdots (z^{2^{m-1}} + 1) = \frac{z^{2^m} - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{2^m-1}. \]

Note that \( \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) z^\alpha \) must be a multiple of \( g(z) \), we obtain that

\[ a_1 - b_1 = a_2 - b_2 = \cdots = a_{2^m-1} - b_{2^m-1} = c, \]

where \( c \) is a constant. Since

\[ \sum_{\alpha=0}^{2^m-1} a_\alpha = \sum_{\alpha=0}^{2^m-1} b_\alpha = 2^n - 1 \Rightarrow \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) = 0, \]

we obtain that

\[ a_\alpha - b_\alpha = 0 \Rightarrow a_\alpha = b_\alpha. \]

Thus, \( f(x) \) is independent of \( c \cdot x \). Then we get \( f(x) \) is independent of \( c_{i_1}, c_{i_2}, \cdots, c_{i_t} \) according to lemma 1. In other words,

\[ P_r(f(x) = \alpha|x_{i_1}, x_{i_2} \cdots x_{i_t}) = P_r(f(x) = \alpha). \]

which is exactly the definition of the \((n, m, t)\)-CI function.

\[ \square \]

4 Second Characterization

Let \( \pi \) be a permutation of symbols \( \{1, 2, \cdots, n\} \), \( f_\pi = f(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)}) \) a function obtained by permuting the variables in \( f(x_1, x_2, \cdots, x_n) \), and \( F_\pi(z) \) the polynomial associated with the function \( f_\pi \).

For any integer \( d \), let \( \Phi_d(z) \) denote the \( d \)th cyclotomic polynomial [14]. Then \( \Phi_d(z) \) is a monic polynomial with integer coefficients that is the minimal polynomial over the rational field of any primitive \( d \)th-root of unity. The second characterization for multi-output correlation-immune Boolean function is given as follows.
**Theorem 2.** Let $f(x)$ be a multi-output Boolean function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$. Then $f$ is $(n, m, t)$-CI function if and only if

$$\Phi_{2^t}(z)|\sum_{k=0}^{2^n-1} \omega_t^{f_x(k)} z^k = F_\pi(z),$$

for all permutation $\pi$, where $\omega_i$ are $2^{th}$, $2^{2th}$, $\cdots$, $2^{mth}$ primitive root of unity in the complex field respectively.

Since $\Phi_{2^t}(z)$ is $2^t$th cyclotomic polynomial, which is the minimal polynomial of $\xi^{-2^n-t}$ with respect to polynomial ring with rational coefficients, where $\xi = exp(\frac{2\pi\sqrt{-1}}{2^n})$ in Definition 2. Then $\Phi_{2^m}(z)|F_\pi(z)$ if and only if $F_\pi(\xi^{-2^n-m}) = 0$. Recall the definition of DFT of the function, we have $F_{f_x}(2^{n-m}) = F_\pi(z = \xi^{-2^n-m})$. Fourier spectrum characterization of the multi-output correlation-immune Boolean function is obtained.

**Corollary 1.** Let $f(x)$ be a multi-output function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$. Then $f$ is $(n, m, t)$-CI function if and only if

$$F_{f_x}(2^{n-t}) = 0,$$

for all permutation $\pi$.

Before giving a proof for Theorem 2, we study the cyclotomic polynomial $\Phi_{2^t}(z)$ first. It is obvious from the definition of cyclotomic polynomial that

$$\Phi_{2^t}(z) = \prod \{(z - \xi^j) : 0 \leq j \leq 2^n - 1, \gcd(j, 2^n) = 2^{n-t}\},$$

where gcd denotes the great common divisor. We have

$$\Phi_{2^t}(z) = z^{2^{t-1}} + 1.$$

For ease of illustration, we first consider permutation $\pi$ to be identity, and describe the connection between $\Phi_{2^t}(z)|F_\pi(z)$ and probabilistic expression.

**Lemma 2.** Let $f(x_1, x_2, \cdots, x_n)$ be a multi-output Boolean function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$. Then $\Phi_{2^t}(z)|F_\pi(z)$ if and only if

$$X_m \rightarrow X_1, X_2, \cdots, X_{t-1} \rightarrow f(X_1, X_2, \cdots, X_n)$$

is a markov chain, or alternatively, for $\forall \alpha \in [0, 2^{m-1}]$,

$$P_r(f(x) = \alpha|x_1, \cdots, x_{t-1}, x_t) = P_r(f(x) = \alpha|x_1, \cdots, x_{t-1}).$$

**Proof.** Since

$$F(z) = \sum_{k=0}^{2^n-1} \omega_{i}^{f_x(k)} z^k = \sum_x \omega_{i}^{f(x)} \prod_{i=1}^{n} (z^{2^{t-1}})^{x_i},$$

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we have
\[ \Phi_{2^t}(z) \mid F(z) \iff F(z) \equiv 0 \pmod{\Phi_{2^t}(z)} \iff \sum_{x} \omega_i^{f(x)} \prod_{i=1}^{t} (z^{2i-1})^x_i \equiv 0 \pmod{\Phi_{2^t}(z)}. \]

From the definition of the cyclotomic polynomial, we know
\[ \Phi_{2^t}(z) \mid z^{2^t} - 1, \text{ for } \forall i \geq t, \]
so
\[ \Phi_{2^t}(z) \mid F_i(z) \iff \sum_{x} \omega_i^{f(x)} \prod_{i=1}^{t} (z^{2i-1})^x_i \equiv 0 \pmod{z^{2i-1} + 1}. \quad (7) \]

Then the summation in (7) can be divided into two parts, where the first part is for \( x_t = 0 \) and the second part is for \( x_t = 1 \). Hence \( \Phi_{2^t}(z) \mid F(z) \) is equivalent to
\[
\sum_{x_1, \ldots, x_{t-1}, x_t=0} \omega_i^{f(x)} \prod_{i=1}^{t-1} (z^{2i-1})^x_i - \sum_{x_1, \ldots, x_{t-1}, x_t=1} \omega_i^{f(x)} \prod_{i=1}^{t-1} (z^{2i-1})^x_i = 0,
\]
Combining like terms about \( z \), the above condition is equivalent to
\[
\sum_{x_1, \ldots, x_{t-1}} \left( \omega_i^{f(x)} - \sum_{x_t=1} \omega_i^{f(x)} \right) (z^{2i-1})^{x_t} = 0,
\]
so the coefficients of \((z^{2i-1})^{x_t}\) are
\[ \sum_{x_t=0} \omega_i^{f(x)} - \sum_{x_t=1} \omega_i^{f(x)} = 0. \quad (8) \]

Now, we denote that
\[ a_\alpha = \#\{x : f(x) = \alpha, x_t = 0, x_{t+1}, \ldots, x_n\}, \]
and
\[ b_\alpha = \#\{x : f(x) = \alpha, x_t = 1, x_{t+1}, \ldots, x_n\}, \]
where \( \alpha \in [0, 2^m - 1] \). Hence, equation (8) is equivalent to
\[
\sum_{\alpha=0}^{2^m-1} a_\alpha \omega_i^\alpha - \sum_{\alpha=0}^{2^m-1} b_\alpha \omega_i^\alpha = 0 \iff \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) \omega_i^\alpha = 0.
\]

Since \( \omega_i \) are \( 2th, 2^2th, \ldots, 2^mth \) primitive root of unity in the complex field, We have \( \Phi_2(z) = z + 1, \Phi_4(z) = z^2 + 1, \ldots, \Phi_{2^m}(z) = z^{2^m-1} + 1 \) all divide \( \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) z^\alpha \). In addition, \( \Phi_2(z), \Phi_4(z), \ldots, \Phi_{2^m}(z) \) are all irreducible in the integer ring.
\[
g(z) = (z + 1)(z^2 + 1) \cdots (z^{2^{m-1}} + 1) = \frac{z^{2^m} - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{2^{m-1}}.
\]
Note that $\sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) z^\alpha$ must be a multiple of $g(z)$, we obtain that

$$a_1 - b_1 = a_2 - b_2 = \cdots = a_{2^m-1} - b_{2^m-1} = c,$$

where $c$ is a constant. Since

$$\sum_{\alpha=0}^{2^m-1} a_\alpha = \sum_{\alpha=0}^{2^m-1} b_\alpha = 2^{n-t} \Rightarrow \sum_{\alpha=0}^{2^m-1} (a_\alpha - b_\alpha) = 0,$$

we obtain that $a_\alpha - b_\alpha = 0$. In other words,

$$P_r(\alpha|x_t = 0, x_1, \cdots, x_{t-1}) = P_r(\alpha|x_t = 1, x_1, \cdots, x_{t-1})$$

for $\forall t$ and $\forall \alpha$, i.e.,

$$P_r(\alpha|x_1, \cdots, x_{t-1}, x_t) = P_r(\alpha|x_1, \cdots, x_{t-1}),$$

which complete the proof.

We now prove Theorem 2 by applying permutation $\pi$ and Lemma 2.

**Proof.** From Lemma 2, we know that $\Phi_{2^t}(z)|F_i(z)$ is equivalent to

$$P_r(f(x) = \alpha|x_t = 0, x_1, \cdots, x_{t-1}) = P_r(f(x) = \alpha|x_t = 1, x_1, \cdots, x_{t-1}).$$

For $1 \leq s \leq t-1$, $\Phi_{2^t}(z)|F_{\pi}(z)$ for all $\pi = (s, t)$ is equivalent to that $P_r(f(x) = \alpha)$ does not depend on the values of $x_1, x_2, \cdots x_t$, i.e,

$$P_r(f(x) = \alpha|x_1, x_2 \cdots x_t) = P_r(f(x) = \alpha).$$

Then consider all the permutation $\pi$, we obtain

$$P_r(f(x) = \alpha|x_{\pi(1)}, x_{\pi(2)} \cdots, x_{\pi(t)}) = P_r(f(x) = \alpha),$$

which is exactly the definition of the $(n, m, t)$-CI function.

The concept of a resilient function was introduced independently by Chor *et al* [7] and Robert *et al* [1]. Areas where resilient functions find their applications include fault-tolerant distributed computing [7], quantum cryptographic key distribution [1], and random sequence generation for stream ciphers [15]. A resilient function is usually chosen so that the statistical dependence between the plaintext and the ciphertext can be avoided in a cryptosystem.

**Corollary 2.** Let $f(x)$ be a multi-output Boolean function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$. Then $f$ is $(n, m, t)$-resilient function if and only if

$$\mathcal{F}_f(0) = 0 \text{ and } \mathcal{F}_f(2^{n-t}) = 0,$$

for all permutation $\pi$. 

Definition 3. A function $f$ is called a symmetric function if permuting its variables $(x_1, x_2, \cdots, x_n)$ leads to itself.

For symmetric function $f$, since $f = f_\pi$ for all permutation $\pi$, the second characterization for the multi-output correlation-immune Boolean functions is much simpler.

Corollary 3. Let $f(x_1, x_2, \cdots, x_n) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ be a symmetric function. Then $f$ is $t$th-order correlation-immune if and only if

$$\mathcal{F}_f(2^{n-t}) = 0.$$ 

5 Third Characterization

The first two characterizations are to consider the problem from the perspective of generalized Boolean functions. In this section, we give another Fourier spectral characterization for multi-output correlation-immune Boolean functions in terms of component functions.

Lemma 3. [6] A multi-output Boolean functions $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$ is an $(n, m, t)$-CI function if and only if every nonzero linear combination

$$f'(x) = \bigoplus_{i=1}^{m} c_i f_i(x),$$

of the component functions of $f(x)$ is an $(n, 1, t)$-CI function, where $c_i \in \mathbb{F}_2$ and not all zeroes.

From Lemma 3, we immediately get the third characterization for a symmetric function.

Theorem 3. An $(n, m)$ symmetric function is an $(n, m, t)$-CI function if and only if

$$\mathcal{F}_f(2^{n-t}) = \sum_{k=0}^{N-1} (-1)^{f(k)} \xi^{-2^{n-1}k} = 0,$$

where $f(k) = f'(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_n)$ is the binary representation of the integer $k$ for $0 \leq k \leq 2^n - 1$.

6 Conclusions

In this paper, we study the three kinds of characterizations for multi-output correlation-immune Boolean functions. The essence of Walsh spectral characterization for an $(n, m, t)$-CI function is that every nonzero linear combination of all component functions is $t$th-order correlation-immune. Our first method is simpler than Walsh spectral characterization for an $(n, m, t)$-CI function, which reduced the the number of calculations from $2^m - 1$ only to $m$ times. That is, $f$ is an $(n, m, t)$-CI function if and only if

$$\sum_{x \in \mathbb{F}_2^n} \omega_x^f f(x) (-1)^{e \cdot x} = 0,$$
where $1 \leq wt(c) \leq t$, $\omega_i$ are $2^{th}, 2^2th, \cdots, 2^mth$ primitive root of unity in the complex field respectively. The second characterization is in terms of the Fourier spectral characterization, that is, a function $f(x)$ is an $(n, m, t)$-CI function if and only if its Fourier spectrum vanished at some specified locations for all permutation $\pi$. This kind of characterization is much simpler when characterize symmetric functions, only one point should be calculated, that is, $f$ is an $(n, m, t)$-CI function if and only if

$$
\mathcal{F}_f(2^n-t) = \sum_{k=0}^{N-1} \omega_i^{f(k)} \xi^{-2^{n-1}k} = 0,
$$

where $f(k) = f(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_n)$ is the binary representation of the integer $k$ for $0 \leq k \leq 2^n - 1$. We also give the characterization for resilient functions, that is, a function $f(x)$ is $(n, m, t)$-resilient function if and only if

$$
\mathcal{F}_f(0) = 0 \text{ and } \mathcal{F}_f(2^n-t) = 0,
$$

for all permutation $\pi$. Our third characterization is about another expression for the Fourier spectral characterization, that is, an $(n, m)$ symmetric function is an $(n, m, t)$-CI function if and only if

$$
\mathcal{F}_f(2^n-t) = \sum_{k=0}^{N-1} (-1)^{f(k)} \xi^{-2^{n-1}k} = 0,
$$

where $f(k) = f'(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_n)$ is the binary representation of the integer $k$ for $0 \leq k \leq 2^n - 1$.

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