Form Factors in Off–Critical Superconformal Models

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September 7, 2018

Abstract

We discuss the determination of the lowest Form Factors relative to the trace operators of $N = 1$ Super Sinh-Gordon Model. Analytic continuations of these Form Factors as functions of the coupling constant allows us to study a series of models in a uniform way, among these the latest model of the Roaming Series and a class of minimal supersymmetric models.

Introduction

A full control of a Quantum Field Theory (QFT) is reached once the complete set of correlation functions of its fields is known. Perturbation theory often proves to be an inadequate approach to this problem therefore more effective and powerful methods need to be developed. In this respect, one of the most promising methods is the Form Factor approach which is applicable to integrable models [1, 2]. This consists in computing exactly all matrix elements of the quantum fields and then using them to obtain the spectral representations of the correlators. In addition to the rich and interesting mathematical structure presented by the Form Factors themselves (which has been investigated in a series of papers, among which [2-11], the resulting spectral series usually show a remarkable convergent behaviour which allows approximation of the correlators (or quantities related to them) within any desired accuracy [11-17].

In this talk, based on the original paper [3], I will briefly discuss some new results obtained for those QFT which are invariant under a supersymmetry transformation which mixes the elementary bosonic and fermionic excitations. Namely, I will present the result relative to the lowest Form Factors of the Super Sinh-Gordon Theory (SShG) and of those models which can be obtained by its analytic continuations. In these models, the degeneracy of the spectrum dictated by the supersymmetry implies the existence of multi–channel scattering processes and the resulting $S$-matrix is necessarily non–diagonal. In this case, the complete determination of the matrix elements of the quantum fields for an arbitrary number of asymptotic particles is still an open problem.

The SShG and Superconformal Models

Scattering theories of integrable super-symmetric theories have been discussed in detail in Schoutens’s paper [18] (see also [19]). We refer to these papers and also to

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for a general discussion of the features of these theories and the relative notation.

Let us concentrate then our attention to the main subjects of this talk, i.e. the Super Sinh-Gordon model and the integrable deformations of the superconformal models. In the euclidean space, the Super Sinh-Gordon model can be defined in terms of its action

\[ A = \int d^2 z d^2 \theta \left[ \frac{1}{2} D \Phi \overline{D} \Phi + i \frac{m}{\lambda^2} \cosh \lambda \Phi \right] , \]  

where the covariant derivatives are defined as \( D = \partial_\theta - \theta \partial_z \) and \( \overline{D} = \partial_{\overline{\theta}} - \overline{\theta} \partial_{\overline{z}} \), and the superfield \( \Phi(z, \overline{z}, \theta, \overline{\theta}) \) has an expansion as

\[ \Phi(z, \overline{z}, \theta, \overline{\theta}) = \phi(z, \overline{z}) + \theta \psi(z, \overline{z}) + \overline{\theta} \overline{\psi}(z, \overline{z}) + \theta \overline{\theta} F(z, \overline{z}) . \]  

The integration on the \( \theta \) variables and also the elimination of the auxiliary field \( F(z, \overline{z}) \) by means of its algebraic equation of motion leads to the Lagrangian

\[ L = \frac{1}{2} (\partial_z \phi \partial_{\overline{z}} \phi + \overline{\psi} \partial_z \overline{\psi} + \psi \partial_{\overline{z}} \psi) + \frac{m^2}{2\lambda^2} \sinh^2 \lambda \phi + i m \overline{\psi} \psi \cosh \lambda \phi . \]  

Of all the different ways of looking at the SSHG model, one of the most convenient is to consider it as a deformation of the superconformal model described by the action

\[ A_r = \frac{1}{2} \int (\partial_z \phi \partial_{\overline{z}} \phi + \overline{\psi} \partial_z \overline{\psi} + \psi \partial_{\overline{z}} \psi) . \]  

This superconformal model has central charge \( C = 3/2 \). At this point, it is useful to briefly remind some properties of the superconformal models and their deformations.

For a generic superconformal model, the supersymmetric charges can be represented by the differential operators \( Q = \partial_\theta + \theta \partial_z ; \overline{Q} = \partial_{\overline{\theta}} + \overline{\theta} \partial_{\overline{z}} \). The analytic part of the stress-energy tensor \( T(z) \) and the current \( G(z) \) which generates the supersymmetry combine themselves into the analytic superfield \( W(z, \theta) = G(z) + \theta T(z) \), which is called the super stress-energy tensor. For the anti-analytic sector we have correspondingly \( \overline{W}(\overline{z}) = \overline{G}(\overline{z}) + \overline{\theta} \overline{T}(\overline{z}) \). These fields are mapped one into the other by means of the super-charges. As it is well known \[23\], reducible unitary representations of the \( N = 1 \) superconformal symmetry occurs for the discrete values of the central charge

\[ C = \frac{3}{2} - \frac{12}{m(m+2)} . \]  

At these values, realizations of the \( N = 1 \) superconformal algebra are given in terms of a finite number of superfields in the Neveu-Schwartz sector and a finite number of ordinary conformal primary fields in the Ramond sector. Their conformal dimensions are given by

\[ \Delta_{p,q} = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{1}{32} [1 - (-1)^{p-q}] , \]  

where \( p - q \) even corresponds to the primary Neveu-Schwartz superfields \( N^{(m)}_{p,q}(z, \theta) \) and \( p - q \) odd to the primary Ramond fields \( R^{(m)}_{p,q}(z) \). These fields enter the so-called superconformal minimal models \( SM_m \).

The Witten index \( Tr(-1)^F \) of the superconformal models can be computed by initially defining them on a cylinder \[24\]: the Hamiltonian on the cylinder is
given by \( H = Q^2 = L_0 - C/24 \), where as usual \( C/24 \) is the Casimir energy on the cylinder and \( L_0 = \frac{1}{2\pi i} \oint dz zT(z) \). Considering that for any conformal state \( | a \rangle \) with \( \Delta > C/24 \) there is the companion state \( Q | a \rangle \) of opposite fermionic parity, their contributions cancel each other in \( Tr(-1)^F \) and therefore only the ground states with \( \Delta = C/24 \) (which are not necessarily paired) enter the final expression of the Witten index. For the minimal models, there is a non-zero Witten index only for \( m \) even. Therefore the lowest superconformal minimal model with a non-zero Witten index is the one with \( m = 4 \), which has a central charge \( C = 1 \) and corresponds to the class of universality of the critical Ashkin-Teller model. The superconformal theory with \( C = 3/2 \) made of free bosonic and fermionic fields also has a non-zero Witten index, because an unpaired Ramond field \( R(z) \) is explicitly given by the spin field \( \sigma(z) \) of Majorana fermion \( \psi(z) \) with conformal dimension \( \Delta = 1/16 \), i.e. by the magnetization operator of the Ising model.

The above observations become important in the understanding the off-critical laws are given by

\[
\begin{align*}
T(z) &= -\frac{1}{2} \left[ (\partial_z \varphi)^2 - \psi \partial \psi \right] ; \\
G(z) &= i\psi \partial_z \varphi ,
\end{align*}
\]

and they satisfy the conservation laws \( \partial_z T(z) = \partial_z G(z) = 0 \). Once this superconformal model is deformed according to the Lagrangian \( \mathcal{L}_1 \), the new conservation laws are given by \( \partial_z T(z, \bar{z}) = \partial_z \Theta(z, \bar{z}) ; \partial_z G(z, \bar{z}) = \partial_z \chi(z, \bar{z}) \), where

\[
\begin{align*}
\Theta(z, \bar{z}) &= \frac{m^2}{2\lambda^2} \sinh^2 \lambda \varphi + im\bar{\psi} \psi \cosh \lambda \varphi , \\
\chi(z, \bar{z}) &= \frac{m}{\lambda} \bar{\psi} \psi \sinh \lambda \varphi .
\end{align*}
\]

\( ^1 \)This is in contrast of what happens in the ordinary conformal minimal models, where the deformation of the conformal action by means of the operator \( \phi_{1,3} \) induces a massless flow between two next neighbor food models.

\( ^2 \)As well known, the lowest model with \( C = 7/10 \) corresponds to the class of universality of the Tricritical Ising Model.
For the anti-analytic part of the super stress-energy tensor we have
\[ \partial_z \bar{T}(z, \bar{z}) = \partial_{\bar{z}} \Theta(z, \bar{z}) \]
\[ \partial_{\bar{z}} \bar{T}(z, \bar{z}) = \partial_z \chi(z, \bar{z}) \]
where \( \Theta(z, \bar{z}) \) is as before and the other fields are
given by
\[ G(z, \bar{z}) = -\frac{i}{\psi} \partial_{\bar{z}} \phi \]
\[ \chi(z, \bar{z}) = \frac{m}{\lambda} \psi \sinh \lambda \phi . \]
The operators \( \Theta(z, \bar{z}), \chi(z, \bar{z}) \) and \( \chi(z, \bar{z}) \) belong to the trace of the supersymmetric stress-energy tensor and they are related each other by
\[ \Theta(z, \bar{z}) = \{ \chi(z, \bar{z}), Q \} \]
\[ \partial_z \chi(z, \bar{z}) = [\Theta(z, \bar{z}), Q] ; \]
\[ \Theta(z, \bar{z}) = \{ \chi(z, \bar{z}), Q \} \]
\[ \partial_{\bar{z}} \chi(z, \bar{z}) = [\Theta(z, \bar{z}), Q] ; \]
where the charges of supersymmetry are expressed by
\[ Q = \int G(z, \bar{z}) \, dz + \chi(z, \bar{z}) \, d\bar{z} ; \]
\[ \bar{Q} = \int \bar{G}(z, \bar{z}) \, d\bar{z} + \bar{\chi}(z, \bar{z}) \, dz . \]
In addition to the above conservation laws, the SSHG model possesses higher integrals of motion which were explicitly determined in [21]. Therefore its scattering processes are purely elastic and factorizable. and its two-body \( S \)-matrix is discussed in the next section.

**The \( S \)-matrix of the SSHG**

The exact \( S \)-matrix of the SSHG model has been determined in [22]. It is given by
\[ S(\beta) = Y(\beta) \begin{pmatrix} 1 - \frac{2i \sin \pi \alpha}{\sinh \beta} & -\frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & 0 & 0 \\ -\frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & -1 - \frac{2i \sin \pi \alpha}{\sinh \beta} & 0 & 0 \\ 0 & 0 & -\frac{i \sin \pi \alpha}{\sinh \frac{\beta}{2}} & 1 \\ 0 & 0 & 1 & -\frac{i \sin \pi \alpha}{\sinh \frac{\beta}{2}} \end{pmatrix} \] (12)

where
\[ Y(\beta) = \frac{\sinh \frac{\beta}{2}}{\sinh \frac{\beta}{2} + i \sin \pi \alpha} U(\beta, \alpha) , \]
and the function \( U(\beta) \) is given by
\[ U(\beta) = \exp \left[ i \int_0^\infty dt \frac{\sin \alpha t \sinh(1 - \alpha) t}{\cosh^2 \frac{\beta}{2} \cosh t} \sin \frac{\beta t}{\pi} \right] . \]

The angle \( \alpha \) is a positive quantity, related to the coupling constant \( \lambda \) of the model by
\[ \alpha = \frac{1}{4\pi} \frac{\lambda^2}{1 + \frac{\lambda^2}{4\pi}} . \]

This equation implies that the SSHG is a quantum field theory invariant under the strong-weak duality \( \lambda \to \frac{4\pi}{\lambda} \). It is now interesting to analyse several analytic continuations of the above \( S \)-matrix in the parameter \( \alpha \).

Under the analytic continuation \( \alpha \to -\alpha \), the SSHG model goes into the Super Sine-Gordon model and the \( S \)-matrix [12] describes in this case the scattering of the
lowest breather states of the latter model. At the particular value \(\alpha = \pi/3\) we can consistently truncate the theory at the Super Sine-Gordon breather sector only. At this value, the pole at \(\beta = 2\pi i/3\) of the \(S\)-matrix can be regarded as due to the bosonic and fermionic one-particle states \(|b(\beta)\rangle\) and \(|f(\beta)\rangle\). These particles are therefore bound states of themselves, in the channels \(bb \rightarrow b \rightarrow bb\), \(ff \rightarrow b \rightarrow ff\), for the bosonic particle \(b\), and in the channels \(bf \rightarrow f \rightarrow fb\), \(fb \rightarrow f \rightarrow bf\), for the fermionic particle \(f\). There is of course a price to pay for this truncation: this means that the residues of the \(S\)-matrix at these poles will be purely imaginary. Such a model therefore would be the supersymmetric analogous of the Yang-Lee model for the ordinary Sine-Gordon model \cite{20}. It was considered originally by Schoutens \cite{18} and it has been identified with the off-critical supersymmetric deformation of the non-unitary superconformal minimal model with central charge \(C = -21/4\).

Another analytic continuation of the \(S\)-matrix gives rise to the so-called Roaming Models. Notice, in fact, that the \(S\)-matrix (12) of the SShG model has zeros in the physical strip located at \(\alpha_1 = i\pi\alpha\) and \(\alpha_2 = i\pi(1 - \alpha)\). By varying the coupling constant \(\lambda\), they move along the imaginary axis and they finally meet at the point \(i\pi/2\), at the self-dual value of the coupling constant \(\lambda^2 = 4\pi\). If we further increase the value of the coupling constant, they simply swap positions. But there is a more interesting possibility: as first proposed by Al. Zamolodchikov for the analogous case of the ordinary Sinh-Gordon model \cite{25}, once the two zeros meet at \(i\pi\), they can enter the physical strip by taking complex values of the coupling constant. In this way, the location of the two zeros are given by \(\alpha_\pm = \frac{1}{2} \pm i\alpha_0\). From the analytic \(S\)-matrix theory, the existence of complex zeros in the physical strip implies the presence of resonances in the system. By analysing the finite-size behaviour of the theory by means of the Thermodynamic Bethe Ansatz \cite{26}, the interesting result is that the net effect of these resonances consists in an infinite cascade of massless Renormalization Group flows generated by the Neveu-Schwarz fields \(N_{1,3}^{m}\) and passing through all minimal superconformal model \(SM_{m}\) with non-zero Witten index. As previously discussed, the ending point of this infinite-nested RG flow should describe the \(N_{1,3}\) deformation of the superconformal model \(SM_{4}\). Is this really the case? By taking the limit \(\alpha \rightarrow i\infty\) into the \(S\)-matrix (12), it is easy to see that it reduces the \(S\)-matrix of the Sine-Gordon model at \(\xi = 2\pi\). Since this \(S\)-matrix describes a massive deformation of the \(C = 1\) model, in order to confirm the above roaming trajectory scenario the only thing that remains to check is the comparison of the anomalous dimension of deforming field. In the Sine-Gordon model at \(\xi = 2\pi\), the anomalous dimension of the deforming field is \(\Delta = 2/3\), which is indeed the conformal dimension of the top component of the superfield \(N_{1,3}\) in the model \(SM_{4}\).

**Form Factors of the Trace Operators of the SShG Model**

For integrable quantum field theories, the knowledge of the \(S\)-matrix is very often the starting point for a complete solution of quantum field dynamics in terms of an explicit construction of the correlation functions of all fields of the theory. This result can be obtained by computing first the matrix elements of the operators on the asymptotic states (the so-called Form Factors) \cite{1,2} and then inserting them.

\[^{3}\text{The general class of these models is obtained by taking } \alpha = \pi/(2N + 1), \ N = 1,2,\ldots \ \text{and they correspond to the supersymmetric deformation of the non-unitary minimal superconformal models with central charge } C = -3N(4N + 3)/(2N + 2).\]
into the the spectral representation of the correlators. For instance, in the case of the two-point correlation function of a generic operator $O(z, \bar{z})$ we have

$$
\mathcal{G}(z, \bar{z}) = \langle 0 \mid O(z, \bar{z}) O(0, 0) \mid 0 \rangle = \int_0^\infty da^2 \rho(a^2) K_0(a \sqrt{zz}) ,
$$

where $K_0(x)$ is the usual Bessel function. The spectral density $\rho(a^2)$ is given in this case by

$$
\rho(a^2) = \sum_{n=0}^\infty \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \delta(a - \sum_i m \cosh \beta_i) \delta(\sum_i \sinh \beta_i) \times
$$

$$
| \langle 0 \mid O(0, 0) \mid A_1(\beta_1) \cdots A_n(\beta_n) \rangle |^2 .
$$

The Form-Factor approach has proved to be extremely successful for theories with scalar $S$-matrix, leading to an explicit solution of models of statistical mechanics interest such as the Ising model [1, 2, 3], the Yang-Lee model [13] or quantum field theories defined by a lagrangian, like the Sinh-Gordon model [7, 9]. On the contrary, for theories with a non-scalar $S$-matrix the functional equations satisfied by the Form Factors are generally quite difficult to tackle and a part from the Sine-Gordon model or theories which can be brought back to it [1, 2, 4], there is presently no mathematical technique available for solving them in their full generality. Also in this case, however, the situation is not as impracticable as it might seem at first sight. The reason consists in the fast convergent behaviour of the spectral representation series which approximates the correlation functions with a high level of accuracy even if truncated at the first available matrix elements [11–17]. In the light of this fact, we will compute the lowest matrix elements of two of the most important operators of the theory, namely the trace operators $\Theta(z, \bar{z})$, $\chi(z, \bar{z})$ and $\bar{\chi}(z, \bar{z})$ of the supersymmetric stress-energy tensor of the SShG model. For the operator $\Theta(0, 0)$, they are given by

$$
F_{bb}^{\Theta}(\beta) = \langle 0 \mid \Theta(0, 0) \mid b(\beta_1)b(\beta_2) \rangle ;
$$

$$
F_{ff}^{\Theta}(\beta) = \langle 0 \mid \Theta(0) \mid f(\beta_1)f(\beta_2) \rangle ,
$$

whereas for the operators $\chi(0, 0)$ we have instead

$$
F_{bf}^{\chi}(\beta_1, \beta_2) = \langle 0 \mid \chi(0, 0) \mid b(\beta_1)f(\beta_2) \rangle ;
$$

$$
F_{fb}^{\chi}(\beta_1, \beta_2) = \langle 0 \mid \chi(0, 0) \mid f(\beta_1)b(\beta_2) \rangle ,
$$

(with an analogous result for the lowest Form Factors of the operator $\bar{\chi}(0, 0)$). Since the operators $\Theta$, $\chi$ and $\bar{\chi}$ are related each other by supersymmetry, as consequence of eqs. (14) we have

$$
F_{bb}^{\Theta}(\beta) = \omega \left( e^{\beta_1/2} F_{fb}^{\chi} + e^{\beta_2/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{ff}^{\Theta}(\beta) = -\bar{\omega} \left( e^{\beta_2/2} F_{fb}^{\chi} - e^{\beta_1/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{bb}^{\chi}(\beta) = \bar{\omega} \left( e^{-\beta_1/2} F_{fb}^{\chi} + e^{-\beta_2/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{ff}^{\chi}(\beta) = -\omega \left( e^{-\beta_2/2} F_{fb}^{\chi} - e^{-\beta_1/2} F_{bf}^{\chi} \right) .
$$

---

4They depend on the difference of rapidities $\beta = \beta_1 - \beta_2$ since $\Theta(0, 0)$ is a scalar operator.
It is therefore sufficient to compute the two-particle Form Factors of the operator \( \Theta(z, \bar{z}) \) for determining those of \( \chi(z, \bar{z}) \) and \( \chi(z, \bar{z}) \). Let us discuss the functional equations satisfied by \( F_{bb}^\Theta(\beta) \) and \( F_{ff}^\Theta(\beta) \). The first set of equations (called the unitarity equations) rules the monodromy properties of the matrix elements as dictated by the \( S \)-matrix amplitudes

\[
F_{bb}^\Theta(\beta) = S_{bb}^\Theta(\beta) F_{bb}^\Theta(-\beta) + S_{ff}^\Theta(\beta) F_{ff}^\Theta(-\beta) ;
\]
\[
F_{ff}^\Theta(\beta) = S_{bb}^\Theta(\beta) F_{bb}^\Theta(-\beta) + S_{ff}^\Theta(\beta) F_{ff}^\Theta(-\beta) ,
\]

where \( S_{bb}^\Theta(\beta) \) is the scattering amplitude of two bosons into two bosons and similarly for the other amplitudes. The second set of equations (called crossing equations) express the locality of the operator \( \Theta(z, \bar{z}) \)

\[
F_{bb}^\Theta(\beta + 2\pi i) = F_{bb}^\Theta(-\beta) ;
F_{ff}^\Theta(\beta + 2\pi i) = F_{ff}^\Theta(-\beta) .
\]

The way to solve these equations was found in [3] and passes by the definition of two auxiliary functions \( H_\pm(\beta) \) which are related to the scattering amplitudes the pure fermionic sector of the theory. They can be used as building blocks for constructing the matrix elements \( F_{bb}^\Theta(\beta) \) and \( F_{ff}^\Theta(\beta) \). For lacking of space, we give here only their final expressions,

\[
F_{bb}^\Theta(\beta) = 2\pi m^2 \frac{\tilde{F}_{bb}(\beta)}{\tilde{F}_{bb}(i\pi)} ;
\]
\[
F_{ff}^\Theta(\beta) = 2\pi m^2 \frac{\tilde{F}_{ff}(\beta)}{\tilde{F}_{ff}(i\pi)} ,
\]

where

\[
\tilde{F}_{bb}(\beta) = \left[ \cosh^2 \frac{\beta}{4} H_+(\beta) - \sinh^2 \frac{\beta}{4} H_-(\beta) \right] G(\beta) ,
\]
\[
\tilde{F}_{ff}(\beta) = \sinh \frac{\beta}{2} \left[ H_+(\beta) + H_-(\beta) \right] G(\beta) .
\]

The functions \( G(\beta) \) and \( H_\pm(\beta) \) are given by their integral representation

\[
G(\beta) = \exp \left[ - \int_0^\infty \frac{dt}{t} \frac{\sinh \alpha t}{\cosh^2 \frac{t}{2} \sinh t \cosh t} \sin^2 \left( \frac{i\pi - \beta}{2\pi} t \right) \right] .
\]
\[
H_+(\beta) = \exp \left[ 4 \int_0^\infty \frac{dt}{t} \frac{\sinh \alpha t}{\cosh t \sinh 2t} \sin^2 \left( \frac{\beta - 2\pi i}{2\pi} t \right) \right] ;
\]
\[
H_-(\beta) = \exp \left[ 4 \int_0^\infty \frac{dt}{t} \frac{\sinh \alpha t}{\cosh t \sinh 2t} \sin^2 \frac{\beta}{2\pi} t \right] .
\]

These matrix elements are normalised as \( F_{bb}(i\pi) = F_{ff}(i\pi) = 2\pi m^2 \). Notice that for large values of \( \beta \), \( F_{bb}^\Theta(\beta) \) tends to a constant whereas \( F_{ff}^\Theta(\beta) \simeq e^{\beta/2} \), both behaviour in agreement with Weinberg’s power counting theorem of the Feynman diagrams.
The Form Factors (23) can now be used to estimate the correlation function
\( C(r) = \langle \Theta(r)\Theta(0) \rangle \) by means of formulas (16) and (17). In the free limit, the
correlator is simply expressed in terms of Bessel functions,
\[
C(r) = m^4 \left( K_2^1(mr) + K_0^2(mr) \right).
\tag{27}
\]
For a finite value of \( \alpha \), a numerical integration of (16) produces the graphs shown
in Figure 1.

![Figure 1. Logarithm of the correlation function](image)

As it was expected, in the ultraviolet limit the curve relative to a finite value
of \( \alpha \) is steeper than the curve relative to the free case whereas it decreases slower
at large values of \( mr \). This curve is expected to correctly capture the long distance
behaviour of the correlator and to provide a reasonable estimate of their short
distance singularity. However, for the exact estimation of the power law singularity
at the origin one would need the knowledge of all higher particle Form Factors.

**C-theorem Sum Rule**

The SSHG model can be seen as a massive deformation of the superconformal
model with the central charge \( C = \frac{3}{2} \). While this fixed point rules the ultraviolet
properties of the model, its large distance behaviour is controlled by a purely massive
theory with \( C = 0 \). The variation of the central charge in this RG flow is dictated
by the C-theorem of Zamolodchikov \cite{Zamolodchikov}, which can be formulated in terms of a
sum rule \cite{Zamolodchikov}
\[
\Delta C = \frac{3}{4\pi} \int d^2 x \ | x \|^2 \langle 0 \ | \ \Theta(x)\Theta(0) \ | \ 0 \rangle_{\text{conn}} = \int_{0}^\infty d\mu \ c(\mu),
\tag{28}
\]
where \( c(\mu) \) is given by
\[
c(\mu) = \frac{6}{\pi^2} \frac{1}{\mu^3} \text{Im} \ G(p^2 = -\mu^2),
\]
\[
G(p^2) = \int d^2 x e^{-ipx} \langle 0 \ | \ \Theta(x)\Theta(0) \ | \ 0 \rangle_{\text{conn}}.
\tag{29}
\]
Inserting a complete set of in-state into (29), the spectral function $c(\mu)$ can be expressed as a sum on the FF’s

$$c(\mu) = \frac{12}{\mu^3} \sum_{n=1}^{\infty} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \delta(\mu - \sum m \cosh \beta_i) \delta(\sum m \sinh \beta_i) \times$$

$$\times \langle 0 | \Theta(0,0) | A_1(\beta_1) \cdots A_n(\beta_n) \rangle^2 .$$

Since the term $|x|^2$ present in (28) suppresses the ultraviolet singularity of the two-point correlator of $\Theta$, the sum rule (28) is expected to be saturated by the first terms of the series (30). For the SShG model the first approximation to the sum rule (28) is given by the contributions of the two-particle states

$$\Delta C^{(2)} = \frac{3}{8\pi^2 m^4} \int_0^\infty \frac{d\beta}{\cosh^4 \beta} \left[ | F_{bb}^{\Theta}(2\beta) |^2 + | F_{ff}^{\Theta}(2\beta) |^2 \right] .$$

Analogous expression is obtained by a fermionic version of the $c$–theorem [3]. The numerical data relative to the above integral for different values of the coupling constant $\alpha$ is reported in the above table. They are remarkably close to the theoretical value $\Delta C = \frac{3}{2}$, even for the largest possible value of the coupling constant, which is the self-dual point $\lambda = \sqrt{4\pi}$. In addition to this satisfactory check, a more interesting result is obtained by analysing the application of the $c$-theorem sum rule to models which are obtained as analytic continuation of the SShG.

For the roaming model, by taking the analytic continuation

$$\alpha \rightarrow \frac{1}{2} + i\alpha_0$$

and the limit $\alpha_0 \rightarrow \infty$, the result is $\Delta C^{(2)} = 0.9924\ldots$, i.e. a saturation within few percent of the exact value $\Delta C = 1$ relative to this case.

In the analytic continuation $\alpha \rightarrow -\alpha$, the $S$-matrix develops a pole in the physical strip located at $\beta = 2\pi\alpha i$. For this model, the operator $\Theta(z, \bar{z})$ has also a one-particle Form Factor $F_b^\Theta$, which can be easily determined to be $F_b^\Theta = -1.6719(3) i$. 



| $\alpha$ | $\frac{\lambda^2}{4\pi}$ | $\Delta C^{(2)}$ | precision% |
|---------|----------------|-----------------|------------|
| $\frac{1}{10}$ | $\frac{1}{50}$ | 1.49968 | 0.0213 |
| $\frac{1}{10}$ | $\frac{1}{50}$ | 1.49741 | 0.1726 |
| $\frac{1}{20}$ | $\frac{1}{10}$ | 1.49349 | 0.4340 |
| $\frac{1}{20}$ | $\frac{1}{10}$ | 1.47955 | 1.3633 |
| $\frac{3}{20}$ | $\frac{1}{10}$ | 1.46333 | 2.4446 |
| $\frac{3}{20}$ | $\frac{1}{10}$ | 1.44742 | 3.5053 |
| $\frac{3}{20}$ | $\frac{1}{10}$ | 1.42109 | 5.2606 |
| $\frac{3}{20}$ | $\frac{1}{10}$ | 1.40480 | 6.3466 |
| $\frac{3}{20}$ | $\frac{1}{10}$ | 1.39935 | 6.7100 |
The series of the sum rule has alternating sign, with the first contribution given by the one-particle Form Factor \( \Delta C^{(1)} = \frac{2}{3} (P_b^\Theta)^2 = -5.3387(4) \). This quantity differs for a 1.6% from the theoretical value \( \Delta C = -\frac{21}{4} = -5.25 \). By also including the positive contribution of the two-particle FF, computed numerically \( \Delta C^{(2)} = 0.09050(8) \), the estimate of the central charge of the model further improves, \( C = -5.2482(4) \), with a difference from the exact value of just 0.033%.

The Form Factors above determined for the trace operator \( \Theta(x) \) can be used also in this case to estimate its two-point correlation function. The graph of this function is shown in Figure 2: note that this function diverges at the origin, in agreement with the positive conformal dimension \( \Delta = 1/4 \) of this operator, but it presents a non-monotonous behaviour for the alternating sign of its spectral series.

![Figure 2.](image-url)

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