1. Introduction

1.1. General background. Let $M$ be a closed connected Riemannian manifold of dimension $n$ with Riemannian volume density $dM$, and denote by $\Delta$ the Laplace-Beltrami operator on $M$ with domain $C^\infty(M)$. One of the central problems of ergodic theory is to study the properties of eigenfunctions of $-\Delta$ in the limit of large eigenvalues. Concretely, let $\{u_j\}$ be an orthonormal basis of $L^2(M)$ of eigenfunctions of $-\Delta$ with respective eigenvalues $\{E_j\}$, repeated according to their multiplicity. As $E_j \to \infty$, one is interested among other things in the pointwise convergence of the $u_j$, bounds of the $L^p$-norms of the $u_j$ for $1 \leq p \leq \infty$, and the weak convergence of the measures $|u_j|^2 dM$.

The last mentioned problem has been studied extensively for chaotic systems, one of the central results being the quantum ergodicity theorem, which goes back to work of Shnirelman \cite{Shnirelman}, Zelditch \cite{Zelditch}, and Colin de Verdière \cite{Colin}. The guiding idea behind is the correspondence principle of semiclassical physics. To explain this in more detail, consider the unit co-sphere bundle $S^* M$, which corresponds to the phase space of a classical free particle moving with constant absolute momentum. Each point in $S^* M$ represents a state of the classical system, its motion being given by the geodesic flow in $S^* M$, and classical observables correspond to functions $a \in C^\infty(S^* M)$. On the other hand, by the Kopenhagen interpretation of quantum mechanics, quantum observables correspond to self-adjoint operators $A$ in...
the Hilbert space $L^2(M)$. The elements $\psi \in L^2(M)$ are interpreted as states of the quantum mechanical system, and the expectation value for measuring the property $A$ while the system is in the state $\psi$ is given by $\langle A\psi, \psi \rangle_{L^2(M)}$. The transition between the classical and the quantum-mechanical picture is given by a quantization map

$$S^l(M) \ni a \mapsto \text{Op}_h(a), \quad l \in \mathbb{R},$$

where $\text{Op}_h(a)$ is a pseudodifferential operator in $L^2(M)$ depending on Planck’s constant $h$ and the particular choice of the map $\text{Op}_h$, and $S^l(M) \subset C^\infty(T^*M)$ denotes a suitable space of symbol functions. The correspondence principle then says that, in the limit of high energies, the quantum mechanical system should behave more and more like the corresponding classical system. Now, consider the distribution $\mu_j : C^\infty(S^*M) \rightarrow \mathbb{C}$, $a \mapsto \langle \text{Op}_h(a)u_j, u_j \rangle_{L^2(M)}$.

If it exists, the distribution limit $\mu = \lim_{j \to \infty} \mu_j$ constitutes a so-called quantum limit for the eigenfunction sequence $\{u_j\}$. Furthermore, the probability measure on $S^*M$ defined by a quantum limit is invariant under the geodesic flow and independent of the choice of $\text{Op}_h$. Since the measure $\mu$ projects to a weak limit $\bar{\mu}$ of the measures $\bar{\mu}_j = |u_j|^2dM$, it is called a microlocal lift of $\bar{\mu}$, and one can reduce the study of the measures $\bar{\mu}$ to the classification of quantum limits. The quantum ergodicity theorem then says that if the geodesic flow on $S^*M$ is ergodic with respect to the Liouville measure $d(S^*M)$, then there exists a subsequence $\{u_{jk}\}_{k \in \mathbb{N}}$ of density 1 such that the $\mu_{jk}$ converge to $d(S^*M)$ as distributions, and consequently the measures $\bar{\mu}_{jk}$ converge weakly to $dM$. Intuitively, the geodesic flow being ergodic means that the geodesics are distributed on $S^*M$ in a sufficiently chaotic way, and this equidistribution of trajectories in the classical system implies asymptotic equidistribution for a density 1 subsequence of states of the corresponding quantum system.

A large class of manifolds whose geodesic flow is ergodic are closed manifolds with negative sectional curvature [15] [4], and one of the main conjectures in ergodic theory is the Rudnick-Sarnak conjecture on quantum unique ergodicity (QUE) [29], which says that if $M$ has negative sectional curvature, the whole sequence $|u_j|^2dM$ converges weakly to the normalized Riemannian measure $(\text{vol}M)^{-1}dM$ as $j \to \infty$. It has been verified in certain arithmetic situations by Lindenstrauss [22], but in general, the conjecture is still very open. Sequences of eigenfunctions with a quantum limit different from the Liouville measure are called exceptional subsequences, and it has been shown by Jacobson and Zelditch [18] that any flow-invariant measure on the unit co-sphere bundle of a standard $n$-sphere occurs as a quantum limit for the Laplacian, showing that the family of exceptional subsequences for the Laplacian can be quite large if the geodesic flow fails to be ergodic. However, it was shown by Faure, Nonnenmacher, and de Bièvre [13] that ergodicity of the geodesic flow alone is not sufficient to rule out the existence of exceptional subsequences for particular elliptic operators.

1.2. Problem and methods. In this article, we will address the problem of determining quantum limits for sequences of eigenfunctions of Schrödinger operators in case that the underlying classical system possesses certain symmetries. Due to the presence of conserved quantities, the corresponding Hamiltonian flow will in parts be integrable, and not totally chaotic, in contrast to the hitherto examined chaotic systems. The question is then how this partially chaotic behavior is reflected in the ergodic properties of the eigenfunctions.

To explain things more precisely, let $G$ be a compact connected Lie group that acts effectively and isometrically on $M$. Note that there might be orbits of different dimensions, and that the orbit space $M/G$ may not be a manifold. Let $\Delta$ be the Laplace-Beltrami operator on $M$, and consider the Schrödinger operator

$$\hat{P}(h) = -h^2\Delta + V, \quad \hat{P}(h) : C^\infty(M) \rightarrow C^\infty(M), \quad h \in \mathbb{R}_{>0},$$

Here one regards $s \in C^\infty(S^*M)$ as an element in $S^l(M) \subset C^\infty(T^*M)$ by extending it $l$-homogeneously to $T^*M$ with the zero-section removed, and then cutting off that extension smoothly near the zero section.
with a $G$-invariant potential $V \in C^\infty(M, \mathbb{R})$. The Schrödinger operator $\hat{P}(h)$ has a unique self-adjoint extension as an unbounded operator in $L^2(M)$

$$P(h) : H^2(M) \to L^2(M),$$

where $H^2(M) \subset L^2(M)$ denotes the second Sobolev space, and one calls $P(h)$ a Schrödinger operator, too. For each $h > 0$, the spectrum of $P(h)$ is discrete and bounded from below, and its eigenvalues form a sequence unbounded towards $+\infty$. When studying the spectral asymptotics of Schrödinger operators, one often uses the semiclassical method. Instead of studying the spectral properties of $P(h)$ for fixed $h = h$ and high energies, one considers fixed energy intervals, allowing $h > 0$ to become small. The two methods are essentially equivalent. In the special case $V \equiv 0$, the Schrödinger operator is just a rescaled version of $-\Delta$ so that the semiclassical method can be used to study the spectral asymptotics of the Laplace-Beltrami operator. Now, since $P(h)$ commutes with the $G$-action, one can use representation theory to study the spectrum of $P(h)$ in a more refined way. Indeed, by the Peter-Weyl theorem, the left-regular representation of $G$ on $L^2(M)$ has an orthogonal decomposition into isotypic components given by

$$L^2(M) = \bigoplus_{\chi \in \hat{G}} L^2_{\chi}(M), \quad L^2_{\chi}(M) = T_{\chi} L^2(M),$$

with associated orthogonal projections $T_{\chi} : L^2(M) \to L^2_{\chi}(M)$. Since $P(h)$ commutes with each $T_{\chi}$, we can consider the restricted operators $P(h)|_{L^2_{\chi}(M)} : L^2_{\chi}(M) \to L^2_{\chi}(M)$, and study their spectral asymptotics. Conversely, each eigenspace of the Schrödinger operator $P(h)$ constitutes a unitary $G$-module, and its decomposition into a direct sum of irreducible $G$-representations represents the so-called fine structure of the spectrum of $P(h)$. Note that, so far, it is a priori irrelevant whether the group action has various different orbit types or not.

On the other hand, the principal symbol of the Schrödinger operator is given by the $G$-invariant symbol function

$$p : T^* M \to \mathbb{R}, \quad (x, \xi) \mapsto \|\xi\|^2_x + V(x),$$

and represents a Hamiltonian on the co-tangent bundle $T^* M$ with canonical symplectic form $\omega$. It defines a Hamiltonian flow $\varphi_t : T^* M \to T^* M$, which in the special case $V \equiv 0$ corresponds to the geodesic flow on $T^* M$. Consider now for a regular value $c$ of $p$ the hypersurface $\Sigma_c := p^{-1}\{c\} \subset T^* M$. It is invariant under the Hamiltonian flow $\varphi_t$, and carries a canonical hypersurface measure $d\Sigma_c$ induced by $\omega$. In the special case $\Sigma_c = S^* M$, $d\Sigma_c = d(S^* M)$ is commonly called the Liouville measure. Now, $\varphi_t$ cannot be ergodic on $(\Sigma_c, d\Sigma_c)$ due to the presence of symmetries. To describe the ergodic properties of the system, it is therefore convenient to divide out the symmetries, which can be done by performing a procedure called symplectic reduction. The latter is based on the fundamental fact that the presence of conserved quantities or first integrals of motion leads to elimination of variables, and reduces the given configuration space with its symmetries to a lower-dimensional one, in which the degeneracies and the conserved quantities have been eliminated. Namely, let $J : T^* M \to \mathfrak{g}$ denote the momentum map of the Hamiltonian $G$-action on $T^* M$, which represents the conserved quantitites of the system, and consider the space $\Omega = J^{-1}\{0\}$, as well as the topological quotient space $\tilde{\Omega} := \Omega / G$. In contrast to the situation encountered in the Peter-Weyl theorem, the orbit structure of the underlying $G$-action on $M$ is not at all irrelevant for the symplectic reduction. Indeed, if the $G$-action is not free the space $\Omega$ need not be a manifold. Nevertheless, $\Omega$ and $\tilde{\Omega}$ are stratified spaces, where each stratum is a smooth manifold that consists of orbits of one particular type. In particular, $\Omega$ and $\tilde{\Omega}$ each have a principal stratum $\Omega_{\text{reg}}$ and $\tilde{\Omega}_{\text{reg}}$, respectively, which is the connected smooth manifold consisting of (the union of) all orbits with minimal isotropy type. Moreover, $\tilde{\Omega}_{\text{reg}}$ carries a canonical symplectic structure, and the Hamiltonian flow on $T^* M$ induces a flow $\tilde{\varphi}_t : \tilde{\Omega}_{\text{reg}} \to \tilde{\Omega}_{\text{reg}}$, which is the Hamiltonian flow associated to the reduced Hamiltonian $\tilde{p} : \tilde{\Omega}_{\text{reg}} \to \mathbb{R}$ induced by $p$. One calls $\tilde{\varphi}_t$ the reduced Hamiltonian flow. Since the orbit projection $\Omega_{\text{reg}} \to \tilde{\Omega}_{\text{reg}}$ is a submersion, $c$ is also a regular value of the reduced symbol function $\tilde{p}$, and we define $\tilde{\Sigma}_c := \tilde{p}^{-1}\{c\} \subset \tilde{\Omega}_{\text{reg}}$. Similarly to $(\Sigma_c, d\Sigma_c)$, the smooth hypersurface
\( \Sigma_c \) carries a measure \( d\Sigma_c \) induced by the symplectic form on \( \Omega_{\text{reg}} \), and one can interpret the measure space \( (\Sigma_c, d\Sigma_c) \) as the symplectic reduction of the measure space \( (\Sigma, d\Sigma) \).

Now, assume that the reduced Hamiltonian flow \( \tilde{\varphi}_t \) is ergodic on \( (\Sigma_c, d\Sigma_c) \). Consider further an orthonormal basis \( \{u_j^\lambda(h)\} \) of a specific isotypic component \( L_\lambda^2(M) \) consisting of eigenfunctions of \( P(h) \). It is then a natural question whether there is a non-trivial family of sets \( \{\Lambda(h)\}_{h>0} \), \( \Lambda(h) \subset \mathbb{N} \), such that the distributions

\[
\mu_j^\lambda(h) : C^\infty_c(\Sigma_c) \rightarrow \mathbb{C}, \quad a \mapsto \langle Op_h(a)u_j^\lambda(h), u_j^\lambda(h) \rangle_{L^2(M)}
\]

converge, as \( j \in \Lambda(h) \) and \( h \rightarrow 0 \), to a distribution limit with density 1, which would answer the corresponding question for the measures \( \|u_j^\lambda(h)\|^2 dM \). In particular, in the special case \( V \equiv 0 \), \( c = 1 \), the problem is equivalent to finding quantum limits for sequences of eigenfunctions of the Laplace-Beltrami operator in each isotypic component of \( L^2(M) \).

The general idea behind our approach can be summarized as follows. The existence of symmetries of a classical Hamiltonian system implies the existence of conserved quantities and partial integrability of the Hamiltonian flow, forcing the system to behave less chaotically. Symplectic reduction divides out the symmetries, and hence, order, and allows to study the symmetry-reduced ergodic properties of the classical system. In our approach, we shall combine well-known methods from semiclassical analysis and symplectic reduction with results on singular equivariant asymptotics recently developed in [27]. In case of the Laplacian, it would also be possible to study the problem via the original high-energy approach of Shnirelman, Zelditch and Colin de Verdière.

1.3. Main results. To describe our results, we need to fix some additional notation. Let \( H \) be a principal isotropy group of the \( G \)-action and \( \kappa \) the dimension of the principal orbits. Throughout the whole paper we shall assume \( \kappa < n = \dim M \). Further, let \( d_\chi \) be the dimension of an irreducible \( G \)-representation of isomorphism class \( \chi \in \hat{G} \) and \( [\pi_\chi]_H : 1 \) the multiplicity of the trivial representation in the \( H \)-representation \( \pi_\chi|_H \). For \( m \in \mathbb{R} \), denote by \( \Psi_0^m(M) \) the set of semiclassical pseudodifferential operators on \( M \) of order \( m \). Our first main result is

**Result 1 (Generalized equivariant semiclassical Weyl law, Theorem 4.1).** Let \( B \in \Psi_0^m(M) \) be a semiclassical pseudodifferential operator with principal symbol \( \sigma(B) = [b] \), and assume that \( b \in S^0(M) \) is independent of \( h \). Let \( c \in \mathbb{R} \) be a regular value of the symbol function \( p \). Then, for each \( \beta \in (0, \frac{1}{2(\kappa+2)}) \) and \( \chi \in \hat{G} \) one has

\[
(2\pi)^{n-\kappa} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, u_j(h) \in L_2^2(M), \ E_\lambda(h) \in [c, c+h^\beta]} \langle Bu_j(h), u_j(h) \rangle_{L^2(M)} = d_\chi [\pi_\chi|_H : 1] \int_{\Sigma_c \cap \Omega_{\text{reg}}} b \frac{d\mu_c}{\text{vol}_\mathcal{O}} + O \left( h^{\frac{-\kappa+\beta}{2}} (\log h^{-1})^{\Lambda-1} \right),
\]

where \( \mu_c \) is the canonical hypersurface measure on \( \Sigma_c \cap \Omega_{\text{reg}} \), \( \text{vol}_\mathcal{O}(\eta) := \text{vol} \mathcal{O}_\eta \) denotes the function which assigns to a point \( \eta \) the Riemannian volume of its orbit \( \mathcal{O}_\eta := G \cdot \eta \), and \( \Lambda \in \mathbb{N} \) is bounded by the number of orbit types.

Result 1 relies on an equivariant trace formula for Schrödinger operators with remainder estimate which is the content of Theorem 3.1. Its proof reduces to the asymptotic description of certain oscillatory integrals that have recently been studied in [27] using resolution of singularities. The involved phase functions are given in terms of the underlying \( G \)-action on \( M \), and if singular orbits occur, the corresponding critical sets are no longer smooth, so that a partial desingularization process has to be implemented in order to obtain asymptotics with remainder estimates via the stationary phase principle. Relying on Result 1 and after studying symmetry-reduced classical ergodicity, we obtain

**Result 2 (Equivariant quantum ergodicity for Schrödinger operators, Theorem 6.4).** Let \( c \in \mathbb{R} \) be a regular value of the symbol function \( p \), and suppose that the reduced flow \( \tilde{\varphi}_t \) is ergodic on
Let \( \chi \in \hat{G} \), \( \beta \in (0, \frac{1}{2(k+2)}) \) be fixed, and set
\[
J^\chi(h) := \{ j \in \mathbb{N} : E_j(h) \in [c, c + h^3], \ u_j(h) \in L^2_1(M) \}.
\]
Then there is a \( h_0 > 0 \) such that for each \( h \in (0, h_0) \) we have a subset \( \Lambda^\chi(h) \subset J^\chi(h) \) satisfying
\[
\lim_{h \to 0} \frac{\# \Lambda^\chi(h)}{\# J^\chi(h)} = 1
\]
and for each \( A \in \Psi_h^0(M) \) with principal symbol \( \sigma(A) = |a| \), given by an \( h \)-independent symbol function \( a \), the following holds. For all \( \varepsilon > 0 \) there is a \( h_\varepsilon \in (0, h_0] \) such that
\[
\left| \langle Au_j(h), u_j(h) \rangle_{L^2_1(M)} - \frac{1}{\text{vol}_{\mathcal{O}}(\Sigma_c \cap \Omega_{\text{reg}})} \int_{\Sigma_c \cap \Omega_{\text{reg}}} \frac{a}{\text{vol}_{\mathcal{O}}} \right| < \varepsilon \quad \forall j \in \Lambda^\chi(h), \ \forall h \in (0, h_\varepsilon).
\]

Let us remark that the remainder estimate in Result 1 and consequently the desingularization process implemented in [27] are crucial in obtaining the sharp energy localization \( E_j(h) \in [c, c + h^3] \) in Result 2. In the special case of the Laplacian, Result 2 becomes an equivariant version of the classical quantum ergodicity theorem of Shnirelman [31], Zelditch [37], and Colin de Verdière [9], yielding

**Result 3 (Equivariant quantum limits for the Laplacian, Theorem 7.1).** Assume that the reduced geodesic flow is ergodic, and choose \( \chi \in \hat{G} \). Let \( \{u_j^\chi\}_{j \in \mathbb{N}} \) be an orthonormal basis of \( L^2_1(M) \) of eigenfunctions of \( -\Delta \). Then, there is a subsequence \( \{u_{j_k}^\chi\}_{k \in \mathbb{N}} \) of density 1 in \( \{u_j^\chi\}_{j \in \mathbb{N}} \) such that for all \( a \in C^\infty(S^*M) \) one has
\[
\langle \text{Op}(a)u_{j_k}^\chi, u_{j_k}^\chi \rangle_{L^2_1(M)} \to \frac{1}{\text{vol}_{\mathcal{O}}(S^*M \cap \Omega_{\text{reg}})} \int_{S^*M \cap \Omega_{\text{reg}}} \frac{a}{\text{vol}_{\mathcal{O}}} \quad \text{as } k \to \infty,
\]
where we wrote Op for Op\(_1\), which is the ordinary non-semiclassical quantization, and \( \mu \) for \( \mu_1 \).

The obtained quantum limits (\( \frac{1}{\text{vol}_{\mathcal{O}}(\Sigma_c \cap \Omega_{\text{reg}})} \int \frac{d\mu}{\text{vol}_{\mathcal{O}}} \)) describe the ergodic properties of the eigenfunctions in the presence of symmetries, and are the answer to our initial question. They are singular measures since they are supported on \( \Sigma_c \cap \Omega_{\text{reg}} \), which is a submanifold of \( \Sigma_c \) of codimension \( \kappa \). In fact, they correspond to Liouville measures on the unit co-sphere bundle of the space of principal orbits in \( M \), see Theorem 7.4.

Projecting from \( S^*M \cap \Omega_{\text{reg}} \) onto \( M \) and generalizing to continuous functions, we immediately get the weak convergence of measures
\[
|u_{j_k}^\chi|^2 dM \to \left( \frac{\text{vol}_{\mathcal{O}}}{\text{vol}_{\mathcal{O}}} \right)^{-1} \frac{dM}{\text{vol}_{\mathcal{O}}} \quad \text{as } k \to \infty,
\]
which describes the asymptotic equidistribution of the eigenfunctions in the presence of symmetries, see Corollary 7.4. Note that the fact that the reduced and the non-reduced flow cannot be simultaneously ergodic is consistent with the QUE conjecture, since otherwise our results would, in principle, imply the existence of exceptional subsequences for ergodic geodesic flows. In this sense, our results can be understood as complementary to the previously known results. From Corollary 7.4 one can deduce a statement on convergence of measures on the topological Hausdorff space \( M/G \), see Corollary 7.3 and applying some elementary representation theory we infer

**Result 4 (Representation-theoretic equidistribution theorem, Theorem 7.1).** Assume that the reduced geodesic flow is ergodic, and let \( \chi \in \hat{G} \). By the spectral theorem, choose an orthogonal decomposition \( L^2_1(M) = \bigoplus_{i \in \mathbb{N}} V_i^\chi \) into irreducible unitary \( G \)-modules of class \( \chi \) such that each \( V_i^\chi \) is contained in some eigenspace of the Laplace-Beltrami operator. For each \( i \in \mathbb{N} \), select some \( v_i \in V_i^\chi \) with \( \|v_i\|_{L^2(M)} = 1 \), and define the orbital integral \( \Theta_i^\chi(\mathcal{O}) := \int_G v_i(g \cdot x_\mathcal{O}) \, dg, \ x_\mathcal{O} \in \mathcal{O} \) arbitrary, which
is independent of the choice of \( v_i \). Then, there is a subsequence \( \{ V_{i_k}^\chi \}_{k \in \mathbb{N}} \) of density 1 in \( \{ V_i^\chi \}_{i \in \mathbb{N}} \) such that we have the weak convergence

\[
\Theta_{i_k} \Delta_{M/G} \to \left( \frac{\text{vol}_{d_{M/G}} M/G}{\text{vol}} \right)^{-1} \frac{d_{M/G}}{\text{vol}},
\]

where \( d_{M/G} := \pi_* dM/G \) is the pushforward measure defined by the orbit projection \( \pi : M \to M/G \) and \( \text{vol} : M/G \to [0, \infty) \) assigns to an orbit its volume.

Note that Result 4 is a statement about limits of representations, or multiplicities, and not eigenfunctions, since it assigns to each \( \chi \)-isotypic \( G \)-module in \( L^2(M) \) a measure on \( M/G \), and then considers the weak convergence of those measures. In essence, it can therefore be regarded as a representation-theoretic statement in which the spectral theory for the Laplacian only enters in choosing a concrete decomposition of each isotypic component. In the case of the trivial group \( G = \{ e \} \), there is only one isotypic component in \( L^2(M) \), associated to the trivial representation, and choosing the family \( \{ V_i^\chi \}_{i \in \mathbb{N}} \) is equivalent to choosing a Hilbert basis of \( L^2(M) \) of eigenfunctions of the Laplace-Beltrami operator. Result 4 then reduces to the classical equidistribution theorem for the Laplacian.

Examples. Section 8 contains a few concrete examples to illustrate our results. They include
- compact locally symmetric spaces \( Y := \Gamma \setminus G/K \), where \( G \) is a connected semisimple Lie group with finite center, \( \Gamma \) a torsion-free co-compact subgroup, and \( K \) a maximal compact subgroup;
- all surfaces of revolution diffeomorphic to the 2-sphere;
- \( S^3 \)-invariant metrics on the 4-sphere.

In the first case, the reduced geodesic flow on \( M = X \) coincides with the geodesic flow on \( Y \) and is ergodic, since \( Y \) has negative sectional curvature. Furthermore, \( K \) acts freely on \( X \) with constant orbit volume. Our results recover the Shnirelman-Zelditch-Colin-de-Verdière theorem for \( L^2(Y) \approx L^2(X)^K \), and generalize it to non-trivial isotypic components of \( L^2(X) \). In the examples of the 2- and 4-dimensional spheres, the considered actions have two fixed points, and the reduced geodesic flow is ergodic for topological reasons, regardless of the choice of invariant Riemannian metric and in spite of the fact that the geodesic flow can be totally integrable. Since the eigenfunctions of the Laplacian on the standard 2-sphere – the spherical harmonics – are well understood, we can independently verify Result 4 in this case.

1.4. Previously known results. In case that \( G \) acts on \( M \) with only one orbit type, \( \tilde{M} := M/G \) is a closed smooth manifold with Riemannian metric induced by the \( G \)-invariant Riemannian metric on \( M \). By co-tangent bundle reduction, \( T^* \tilde{M} \) is symplectomorphic to \( J^{-1}(\{0\})/G \), so the ergodicity of the reduced geodesic flow on \( M \) and that of the geodesic flow on \( \tilde{M} \) are equivalent. If this is the case, one can apply the classical Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem to \( \tilde{M} \), yielding an equidistribution statement for the eigenfunctions of the Laplacian \( \Delta_{\tilde{M}} \) on \( \tilde{M} \) in terms of weak convergence of measures on \( \tilde{M} \). On the other hand, one could as well apply Corollary 7.5 to \( M \), yielding also a statement about weak convergence of measures on \( M \), but this time with measures related to eigenfunctions of the Laplacian \( \Delta_M \) on \( M \) in a single isotypic component of \( L^2(M) \). It is then an obvious question how these two results are related. The answer is rather difficult in general, since – in spite of the presence of the group action – the geometry of \( M \) may be much more complicated than that of \( \tilde{M} \). Consequently, the eigenfunctions of \( \Delta_M \), even those in the trivial isotypic component, that is, those that are \( G \)-invariant, may be much harder to understand than the eigenfunctions of \( \Delta_{\tilde{M}} \). Only in case that all orbits are totally geodesic or minimal submanifolds, or, more generally, do all have the same volume, one can show that an eigenfunction of \( \Delta_{\tilde{M}} \) lifts to a unique \( G \)-invariant eigenfunction of \( \Delta_M \). In this particular situation, it is easy to see that the application of the Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem implies our results, but only for the trivial isotypic component. The case of a compact locally symmetric space treated in Section 8.1 is an example of this. In cases where the orbit volume is not constant, we do not know of any significant results about the relation between the eigenfunctions of \( \Delta_{\tilde{M}} \) and \( \Delta_M \).
A second previously studied case is that of a general free $G$-action, when the projection $M \to M/G$ is a Riemannian principal $G$-bundle. Extending work of Schröder and Taylor [30], Zelditch [38] obtained quantum limits for sequences of eigenfunctions of $\Delta_M$ in so-called fuzzy ladders. These are subspaces of $L^2(M)$ contained in a whole sequence of isotypic components, namely those associated to a ray of representations originating from some chosen $\chi \in \hat{G}$. The obtained quantum limits are then directly related to the symplectic orbit reduction $J^{-1}(O_\chi)/G \simeq T^*(M/G)$, where $O_\chi \subset g^*$ is the co-adjoint orbit associated to $\chi$ by the Borel-Weil theorem. They are given by Liouville measures on hypersurfaces in $J^{-1}(O_\chi)/G$, and essentially agree with ours.

Thirdly, significant efforts were recently made towards the understanding of quantum (unique) ergodicity for locally symmetric spaces, which are particular manifolds of negative sectional curvature. As further lines of research, it would be interesting to see whether our results can be generalized to $G$-vector bundles, as well as manifolds with boundary and non-compact situations. Also, in view of Result 2, it might be possible to deepen our understanding of equivariant quantum ergodicity via representation theory. Finally, it seems natural to ask what could be a suitable symmetry-reduced version of the QUE conjecture, and we intend to deal with these questions in a sequel to this paper.

Thus, in all the previously examined cases, neither exceptional nor singular orbits are present.

Finally, Marklof and O’Keefe [23] obtained quantum limits in situations where the geodesic flow is ergodic only in certain regions of phase space. Conceptually, this is both similar and contrary to our approach, since in this case the geodesic flow is partially ergodic as well, but not due to symmetries.

1.5. Comments and outlook. We would like to close this introduction by making some comments, and indicating some possible research lines for the future.

Weaker versions of Results 2 and 3 can be proven by the same methods employed here with a less sharp energy localization in a fixed interval $[c, d]$, $c < d$, instead of the interval $[c, c + h^3]$. The point is that for these weaker statements no remainder estimate in the semiclassical Weyl law is necessary, see Remark 6.3. Thus, in principle, these weaker results could have also been obtained in the late 1970’s using heat kernel methods as in [11] or [6]. In contrast, for the stronger versions of equivariant quantum ergodicity proven in Results 2 and 3, remainder estimates in Weyl’s law, and in particular the results obtained in [27] for general group actions via resolution of singularities, are necessary. However, the weaker versions would still be strong enough to imply Corollary 7.4 and Result 4. Therefore, in principle, the representation-theoretic equidistribution theorem could have been proven already when Shnirelman formulated his theorem more than 40 years ago.

As further lines of research, it would be interesting to see whether our results can be generalized to $G$-vector bundles, as well as manifolds with boundary and non-compact situations. Also, in view of Result 3, it might be possible to deepen our understanding of equivariant quantum ergodicity via representation theory. Finally, it seems natural to ask what could be a suitable symmetry-reduced version of the QUE conjecture, and we intend to deal with these questions in a sequel to this paper. In the particular case of the SO(2)-action on the standard 2-sphere studied in Section 8, we actually show that the representation-theoretic equidistribution theorem for the Laplacian applies to the full sequence of spherical harmonics in a fixed isotypic component, so that equivariant QUE holds in this case.

2. Setup and background

In this section we shall prepare some material needed in the sequel, and fix some global notation.
2.1. Actions of compact Lie groups and symplectic reduction. In what follows, we recall some essential facts from the theory of compact Lie group actions on smooth manifolds. For a detailed introduction, we refer the reader to [24]. Let $X$ be a smooth manifold of dimension $n$ and $G$ a compact Lie group acting locally smoothly on $X$. For $x \in X$, denote by $G_x$ the isotropy group and by $G \cdot x = O_x$ the orbit of $x$ so that

$$G_x = \{ g \in G, \ g \cdot x = x \}, \quad O_x = G \cdot x = \{ g \cdot x \in X, \ g \in G \}.$$  

Note that $G \cdot x$ and $G/G_x$ are homeomorphic. The equivalence class of an orbit $O_x$ under equivariant homeomorphisms, written $[O_x]$, is called its orbit type. The conjugacy class of a stabilizer group $G_x$ is called its isotropy type, and written $\{G_x\}$. If $K_1$ and $K_2$ are closed subgroups of $G$, a partial ordering of orbit and isotropy types is given by

$$[G/K_1] \leq [G/K_2] \iff (K_2) \leq (K_1) \iff K_2 \text{ is conjugate to a subgroup of } K_1.$$  

The set of all orbits is denoted by $X/G$, and equipped with the quotient topology it becomes a compact Hausdorff space [3, Theorem 3.1]. In the following we shall assume that it is connected. One of the central results in the theory of compact group actions is

**Theorem 2.1 (Principal orbit theorem, [3 Theorem IV.3.1]).** There exists a maximum orbit type $[O_{\text{max}}]$ with associated minimal isotropy type $(H)$. The union $X(H)$ of orbits of isotropy type $(H)$ is open and dense in $X$, and its image in $X/G$ is connected.

We call $[O_{\text{max}}]$ the principal orbit type of the $G$-action on $X$ and a representing orbit a principal orbit. Similarly, we call the isotropy type $(H)$ and an isotropy group $G_x \sim H$ principal. Casually, we will identify orbit types with isotropy types and say an orbit of type $(H)$ or even an orbit of type $H$, making no distinction between equivalence classes and their representants. The reduced space $X(H)/G$ is a smooth manifold of dimension $n - \kappa$, where $\kappa$ is the dimension of $O_{\text{max}}$, since $G$ acts with only one orbit type on $X(H)$.

Let now $(X, \omega)$ be a connected symplectic manifold with an action of a Lie group $G$ that leaves $\omega$ invariant. Denote by $\mathfrak{g}$ the Lie algebra of $G$. Note that $G$ acts on $\mathfrak{g}$ via the adjoint action and on $\mathfrak{g}^*$ via the co-adjoint action. The group $G$ is said to act on $X$ in a Hamiltonian fashion, if for each $X \in \mathfrak{g}$ there exists a $C^\infty$-function $\mathcal{J}_X : X \to \mathbb{R}$ depending linearly on $X$ such that the fundamental vector field $\tilde{X}$ on $X$ associated to $X$ is given by the Hamiltonian vector field of $\mathcal{J}_X$. One then has

$$d\mathcal{J}_X = -\tilde{X} \mathcal{J}_X,$$

and one defines the momentum map of the Hamiltonian action as the equivariant map

$$\mathcal{J} : X \to \mathfrak{g}^*, \quad \mathcal{J}(\eta)(X) = \mathcal{J}_X(\eta).$$

It is clear from the definition that a momentum map is unique up to addition of a constant function. Furthermore, for each $X \in \mathfrak{g}$ the function $\mathcal{J}_X$ is a conserved quantity or integral of motion for any $G$-invariant function $p \in C^\infty(X)$. Indeed, let $\{\cdot, \cdot\}$ denote Poisson-bracket on $X$ given by $\omega$. Then

$$\{\mathcal{J}_X, p\} = \omega(\text{s-grad} \mathcal{J}_X, \text{s-grad} p) = d\mathcal{J}_X(\text{s-grad} p) = -\omega(\tilde{X}, \text{s-grad} p) = dp(\tilde{X}) = \tilde{X}(p) = 0.$$  

An important class of examples of Hamiltonian group actions is given by those actions of Lie groups on co-tangent bundles which are induced from Lie group actions on the base manifolds by dualizing the derivatives. Indeed, let $M$ be a smooth manifold with smooth action of $G$ and $\tau : X = T^*M \to M$ the co-tangent bundle, with induced $G$-action and with standard symplectic form $\omega = -d\theta$, where $\theta$ is the tautological or Liouville one-form on $T^*M$. Then there is a momentum map $\mathcal{J}$ given explicitly by the formula

$$T^*M \ni \eta \mapsto \mathcal{J}(\eta)(X) := \eta(\tilde{X}(\tau(\eta))), \quad X \in \mathfrak{g}.$$  

Next, let us briefly recall some central results from the theory of symplectic reduction of Marsden and Weinstein, Sjamaar, Lerman and Bates. It emerged out of classical mechanics and is based on the fundamental fact that the presence of conserved quantities or integrals of motion leads to the elimination of variables. For a detailed exposition of these facts we refer the reader to [24]. Assume
that $(X, \omega)$ carries a global Hamiltonian action of $G$. Let $\mathcal{J} : X \to \mathfrak{g}^*$ be the corresponding momentum map, $\mu$ a value of $\mathcal{J}$, and $G_\mu$ the isotropy group of $\mu$ with respect to the co-adjoint action on $\mathfrak{g}^*$. Consider further an isotropy group $K \subset G$ of the $G$-action on $X$, let $\eta \in \mathcal{J}^{-1}(\mu)$, and $X_K$ be the connected component of $X_K = \{ \zeta \in X : G_\zeta = K \}$ containing $\eta$. We then have the following

**Theorem 2.2** ([25, Theorem 8.1.1]).

1. The set $\mathcal{J}^{-1}(\mu) \cap G_\mu \cdot X_K$ is a smooth submanifold of $X$.
2. The quotient $\tilde{\Omega}^{(K)} : = (\mathcal{J}^{-1}(\mu) \cap G_\mu \cdot X_K) / G_\mu$ possesses a differentiable structure such that the projection $\pi_\mu : \mathcal{J}^{-1}(\mu) \cap G_\mu \cdot X_K \to \tilde{\Omega}^{(K)}$ is a surjective submersion.
3. There exists a unique symplectic form $\omega$ on $\tilde{\Omega}^{(K)}$ such that $(i_\mu(K))^* \omega = (\pi_\mu(K))^*(\omega_\mu)$, where $i_\mu(K) : \mathcal{J}^{-1}(\mu) \cap G_\mu \cdot X_K \to X$ denotes the inclusion.
4. Let $p \in C^\infty(X)$ be a $G$-invariant function, $H_p := \text{grad}_p$ its Hamiltonian vector field, and $\varphi_t$ the corresponding flow. Then $\varphi_t$ leaves invariant the components of $\mathcal{J}^{-1}(\mu) \cap G_\mu \cdot X_K$ and commutes with the $G_\mu$-action, yielding a reduced flow $\tilde{\varphi}_t$ on $\tilde{\Omega}^{(K)}$ given by $\pi_\mu(K) \circ \varphi_t \circ i_\mu(K) = \tilde{\varphi}_t^\mu \circ \pi_\mu(K)$.
5. The reduced flow $\tilde{\varphi}_t^\mu$ on $\tilde{\Omega}^{(K)}$ is Hamiltonian, and its Hamiltonian $p_\mu(K) : \tilde{\Omega}^{(K)} \to \mathbb{R}$ satisfies $p_\mu(K) \circ \pi_\mu(K) = p \circ i_\mu(K)$.

Remark 2.3. With the notation above we have $G \cdot X_K = X(K)$. Indeed, for $x \in X_K$, the isotropy group of $x$ is $K$. If $g' \cdot x = g \cdot x$ for some $g, g' \in G$, then $g^{-1}g' \cdot x = x$, hence $g^{-1}g' \in K$, that is $g' \in (K)$. That shows $G \cdot X_K \subset X(K)$. On the other hand, if $x \in X(K)$, then $(Gx) = (K)$, hence for every $g' \in G$, there is a $k \in K$ and a $g \in G$ such that $g' = kg^{-1}$. But then $kg^{-1} \cdot x = g^{-1} \cdot x$, so that $g^{-1} \cdot x \in X_K$, in particular $G \cdot X_K$.

Let now $M$ be a connected closed Riemannian manifold of dimension $n$, carrying an isometric effective action of a compact connected Lie group $G$. In all what follows, the principal isotropy type of the action will be denoted by $(H)$ and the dimension of the principal orbits in $M$ by $\kappa$. Furthermore, we shall always assume that $\kappa < n$, and write

$$\Omega := \mathcal{J}^{-1}(\{0\}) = \bigsqcup_{x \in M} \text{Ann} T_x(G \cdot x),$$

where $\mathcal{J} : T^*M \to \mathfrak{g}^*$ is the momentum map associated to the $G$-action on $T^*M$. Note that as soon as there are two orbits $G \cdot x, G \cdot x'$ in $M$ of different dimensions, their annihilators $\text{Ann} T_x(G \cdot x)$ and $\text{Ann} T_{x'}(G \cdot x')$ have different dimensions, so that $\Omega$ is not a vector bundle in that case. Further, let

$$M_{\text{reg}} := M(H), \quad \Omega_{\text{reg}} := \Omega \cap (T^*M)(H),$$

where $M(H)$ and $(T^*M)(H)$ denote the union of orbits of principal type in $M$ and $T^*M$, respectively. By Theorem 2.1 $M_{\text{reg}}$ is open in $M$, hence $M_{\text{reg}}$ is a smooth submanifold. We then define

$$\tilde{M}_{\text{reg}} := M_{\text{reg}} / G.$$

$\tilde{M}_{\text{reg}}$ is a smooth manifold, since $G$ acts with only one orbit type on $M_{\text{reg}}$. Moreover, because the Riemannian metric $g$ on $M$ is $G$-invariant, it induces a Riemannian metric $\tilde{g}$ on $\tilde{M}_{\text{reg}}$. On the other hand, by Theorem 2.2 $\Omega_{\text{reg}}$ is a smooth submanifold of $T^*M$, and the quotient

$$\tilde{\Omega}_{\text{reg}} := \Omega_{\text{reg}} / G$$

possesses a unique differentiable structure such that the projection $\pi : \Omega_{\text{reg}} \to \tilde{\Omega}_{\text{reg}}$ is a surjective submersion. Furthermore, there exists a unique symplectic form $\bar{\omega}$ on $\tilde{\Omega}_{\text{reg}}$ such that $\pi^* \bar{\omega} = \omega$, where $\iota : \tilde{\Omega}_{\text{reg}} \hookrightarrow T^*M$ denotes the inclusion and $\omega$ the canonical symplectic form on $T^*M$. Consider now the inclusion $j : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) / G \hookrightarrow \tilde{\Omega}_{\text{reg}}$. The symplectic form $\bar{\omega}$ on $\tilde{\Omega}_{\text{reg}}$ induces a symplectic form
\( j^*\bar{\omega} \) on \((T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G\). The following lemma allows us to understand that induced symplectic structure on \((T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G\) in more concrete terms.

**Lemma 2.4 (Singular co-tangent bundle reduction).** Let \( \bar{\omega} \) denote the canonical symplectic form on the co-tangent bundle \( T^*\tilde{M}_{\text{reg}} \). Then the two \( 2(n-\kappa) \)-dimensional symplectic manifolds \((T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G, j^*\bar{\omega})\) and \((T^*\tilde{M}_{\text{reg}}, \bar{\omega})\) are canonically symplectomorphic.

**Remark 2.5.** If \( M = M_{\text{reg}} \), the previous lemma simply asserts that \( T^*(M/G) \) and \( \tilde{\Omega} \subset (T^*M)/G \) are isomorphic as symplectic manifolds.

**Proof.** First, we apply Theorem 2.2 once to the manifold \( T^*M \) and once to the manifold \( T^*M_{\text{reg}} \). Noting that the momentum map of the \( G \)-action on \( T^*M_{\text{reg}} \) agrees with the restriction of the momentum map of the \( G \)-action on \( T^*M \) to \( T^*M_{\text{reg}} \), we get that \( j^*\bar{\omega} \) is the unique symplectic form on \((T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G\) which fulfills

\[
 i^*\omega = \Pi^*j^*\bar{\omega},
\]

where \( \Pi : T^*M_{\text{reg}} \to T^*M_{\text{reg}}/G \) is the orbit projection, \( i : T^*M_{\text{reg}} \cap \Omega_{\text{reg}} \hookrightarrow T^*M_{\text{reg}} \) is the inclusion, and \( \omega \) is the canonical symplectic form on \( T^*M_{\text{reg}} \). The rest of the proof is now essentially the proof of the standard co-tangent bundle reduction theorem [24, Theorem 2.2.2] for the manifold \( M_{\text{reg}} \). A detailed proof of the present lemma is also given in [21]. □

### 2.2. Semiclassical pseudodifferential operators

In what follows, we shall give a brief overview of the theory of semiclassical pseudodifferential operators. For a detailed introduction, we refer the reader to [39, Chapters 9 and 14] and [10]. Semiclassical analysis developed out of the theory of ordinary pseudodifferential operators, a thorough exposition of which can be found in [22]. Let \( M \) be a smooth manifold of dimension \( n \).

**Definition 2.1.** Let \( m \in \mathbb{R} \) and \( \{(U_\alpha, \gamma_\alpha)\}_{\alpha \in A}, \ \gamma_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n \), be an atlas for \( M \). Then one defines

\[
 S^m(M) := \{ a \in C^\infty(T^*M), (\gamma_\alpha^{-1})^*a \in S^m(V_\alpha) \ \forall \ \alpha \in A \},
\]

where \( T^*U_\alpha \) is identified with \( V_\alpha \times \mathbb{R}^n \), and for an open subset \( E \subset \mathbb{R}^n \) one sets

\[
 S^m(E) := \{ a \in C^\infty(E \times \mathbb{R}^n) : \text{ for all compact } K \subset E \text{ and multiindices } s,t \text{ there exists } C^K_{s,t} > 0 \text{ such that } |\partial_x^s \partial_{\xi}^t a(x,\xi)| \leq C^K_{s,t} |\xi|^{m-|t|} \ \forall x \in K \},
\]

where \( |\xi| := \sqrt{1 + |\xi|^2} \).

The definition is independent of the choice of atlas, and we call an element of \( S^m(M) \) a symbol function. Next, for \( h \gg 0 \), let \( \Psi^m_h(M) \) denote the \( \mathbb{C} \)-linear space of all semiclassical pseudodifferential operators \( P : C^\infty_c(M) \to C^\infty(M) \) of order \( m \). Such operators are locally of the form

\[
 Au(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y,\xi)} a(x,\xi) u(y) dy \, d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y,\xi)} a(x, h\xi) u(y) dy \, d\xi,
\]

where \( a \in S^m(\mathbb{R}^n), u \in C^\infty_c(\mathbb{R}^n), \) and \( d\xi = (2\pi)^{-n} d\xi \). A symbol function may depend on the parameter \( h \gg 0 \). If this is the case, one requires that there is a \( h_0 > 0 \) such that the constants \( C^K_{s,t} \) in (2.3) are independent of \( h \) for \( 0 < h \leq h_0 \). For simplicity of notation we usually do not make a possible \( h \)-dependence explicit when working with symbol functions. The class \( \Psi^m_h(M) \) emerges from the usual class of pseudodifferential operators of order \( m \) by substituting in the amplitude \( \xi \) by \( h\xi \). This quantization is motivated by the fact that the classical Hamiltonian \( H(x,\xi) = \xi^2 \) should correspond to the quantum Laplacian \( -h^2\Delta \). We write \( \Psi^m(M) := \Psi^m_1(M) \) for the linear space of the usual pseudodifferential operators of order \( m \). From the classical theorems about ordinary pseudodifferential operators one infers in particular the following relation between symbol functions and semiclassical pseudodifferential operators.
Theorem 2.6 ([17], page 86, [39], Theorem 14.1]). There is a $\mathbb{C}$-linear map
\begin{equation}
\Psi^m_h(M) \to S^m(M)/(hS^{m-1}(M)), \quad P \mapsto \sigma(P)
\end{equation}
which assigns to a semiclassical pseudodifferential operator its principal symbol. Moreover, for each choice of a covering $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of $M$ and a partition of unity $\{\varphi_\alpha\}_{\alpha \in \mathcal{A}}$ subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{A}}$, there is a $\mathbb{C}$-linear map called quantization, written
\begin{equation}
S^m(M) \to \Psi^m_h(M), \quad s \mapsto \text{Op}_{h,\{U_\alpha,\varphi_\alpha\}_{\alpha \in \mathcal{A}}}(s).
\end{equation}
Any choice of such a map induces the same $\mathbb{C}$-linear bijection
\begin{equation}
\Psi^m_h(M)/(h\Psi^{m-1}_h(M)) \cong S^m(M)/(hS^{m-1}(M)), \quad \text{Op}_h
\end{equation}
which means in particular that the bijection exists and is independent from the choice of covering and partition of unity.

We will write $\text{Op} := \text{Op}_1$ for the usual non-semiclassical quantization map, and we do not make a difference in our notation between the maps on the quotients (2.6) and the maps (2.4), (2.5) obtained by precomposition with the quotient projections. However, we will call an element in a quotient space difference in our notation between the maps on the quotients (2.6) and the maps (2.4), (2.5) obtained.

Any choice of such a map induces the same $\mathbb{C}$-linear bijection
\begin{equation}
\sigma
\end{equation}
which means in particular that the bijection exists and is independent from the choice of covering and partition of unity.

The following theorem says that from the linear isomorphisms of the preceding theorem for various $m \in \mathbb{R}$ we actually get an isomorphism of $\mathbb{R}$-graded algebras.

Theorem 2.7 ([39], Theorem 14.1]). If $A \in \Psi^m_h(M)$ and $B \in \Psi^{m'}_h(M)$ then $AB \in \Psi^{m+m'}_h(M)$ and $\sigma(AB) = \sigma(A)\sigma(B)$.

2.3. Schrödinger operators on closed Riemannian manifolds and symmetries. In the following, let $M$ be a connected closed Riemannian manifold with Riemannian volume density $dM$ and $L^2(M)$ the corresponding space of $L^2$-integrable functions. Consider the Schrödinger operator
\begin{equation}
\tilde{P}(h) : C^\infty(M) \to C^\infty(M), \quad \tilde{P}(h) = -h^2\Delta + V,
\end{equation}
parametrized by $h \in \mathbb{R}_{>0}$, where the symbol $\Delta$ denotes the Laplace-Beltrami operator and $V \in C^\infty(M, \mathbb{R})$ is identified with the operator given by pointwise multiplication with $V$. Note that $\tilde{P}(h)$ is essentially self-adjoint, hence it has a unique self-adjoint extension
\begin{equation}
P(h) : H^2(M) \to L^2(M)
\end{equation}
in the Hilbert space $L^2(M)$, the domain being the second Sobolev space, see [39], Theorem 14.7 ii]. By continuity, $P(h)$ is equivariant. We will call $P(h)$ a Schrödinger operator, too, and use the same notation for $P(h)$ and $\tilde{P}(h)$ whenever the precise meaning can be inferred from the context. Let us collect a list of well known facts about the operator $P(h)$.

Theorem 2.8 ([39], Theorem 14.7 ii]). Fix some $h > 0$. The eigenfunctions of $P(h)$ are in $C^\infty(M)$. There exists a Hilbert basis $\{u_j(h)\}_{j \in \mathbb{N}}$ of $L^2(M)$ such that each $u_j(h)$ is an eigenfunction of $P(h)$. The associated eigenvalues $\{E_j(h)\}_{j \in \mathbb{N}}$ of $P(h)$ fulfill
\begin{equation}
\lim_{j \to \infty} E_j(h) = \infty.
\end{equation}
Note that (2.8) implies that all eigenspaces of $P(h)$ are finite-dimensional since the eigenvalues are repeated in $\{E_j(h)\}_{j \in \mathbb{N}}$ according to their multiplicity.

Theorem 2.9 (Theorems 14.9 and 14.10). Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions, and let $f \in \mathcal{S}(\mathbb{R})$. Then, the operator $f(P(h))$ defined by the spectral theorem for unbounded self-adjoint operators is an element in $\Psi_h^\infty(M) = \bigcap_{m \in \mathbb{Z}} \Psi^m_h(M)$. Furthermore, $f(P(h))$ extends to a bounded operator $f(P(h)) : L^2(M) \to L^2(M)$ of trace class, and the principal symbol of $f(P(h))$ is

\begin{equation}
\sigma(f(P(h))) = f \circ \sigma(P(h)),
\end{equation}

where

\[ \sigma(P(h)) = [p], \quad p = \|\cdot\|_{T^*M}^2 + V \circ \tau, \]

is the principal symbol of $P(h)$, with $\tau : T^*M \to M$ denoting the co-tangent bundle.

Let us now assume that $M$ carries certain symmetries in form of a connected compact Lie group $G$ acting isometrically and effectively on $M$. It induces a unitary representation of $G$ on $L^2(M)$ given by

\begin{equation}
G \times L^2(M) \to L^2(M), \quad (g, f) \mapsto (L_g f : x \mapsto f(g^{-1} \cdot x)),
\end{equation}

and called the left-regular representation. We will often just write $gf$ for $L_g f$. Clearly, we can define an action similar to (2.10) also on $C^\infty(M)$. Let $\hat{G}$ be the character ring of $G$, which we identify with the set of isomorphism classes of irreducible representations of $G$. An irreducible character $\chi \in \hat{G}$ defines an equivalence class of irreducible unitary $G$-representations of dimension $d_\chi = \chi(e)$, where $e \in G$ is the unit element. By the Peter-Weyl theorem, the left-regular representation (2.10) of $G$ on $L^2(M)$ decomposes orthogonally into isotypic components according to

\begin{equation}
L^2(M) = \bigoplus_{\chi \in \hat{G}} L^2_\chi(M).
\end{equation}

The projection in $L^2(M)$ onto $L^2_\chi(M)$ is given by

\[ T_\chi : L^2(M) \to L^2_\chi(M), \quad f \mapsto \left( x \mapsto d_\chi \int_G \chi(g)f(g^{-1} \cdot x) \, dg \right), \]

where $dg$ is the normalized Haar measure on $G$. For an operator $A$ in $L^2(M)$, we call

\[ A_\chi := T_\chi AT_\chi \]

the reduced operator associated to the irreducible representation $\chi$. Further, an operator $A$ is called $G$-equivariant if it commutes with the $G$-representation on $L^2(M)$, that is, if $P(gf) = gP(f)$ for all $f \in L^2(M)$ and $g \in G$. If $A$ is $G$-equivariant, it commutes with each projection $T_\chi$, so that $A_\chi = AT_\chi = T_\chi A$. Note that in general, the kernel of $A_\chi$ is much larger than that of $A$ because of the projection onto $L^2_\chi(M)$. However, one can consider the restricted operator

\[ A_{\chi\lambda} := A|_{L^2_\chi(M)} : L^2_\chi(M) \to L^2_\chi(M) \]

which has the advantage over $A_\chi$ that the dimension of the eigenspace of $A_{\chi\lambda}$ associated to an eigenvalue $\lambda$ is not larger than the dimension of the eigenspace of $A$ associated to $\lambda$.

In particular, if in the previous situation the potential $V \in C^\infty(M, \mathbb{R})$ of the considered Schrödinger operators is $G$-invariant, that is, $V(g \cdot x) = V(x)$ for all $g \in G$, $x \in M$, each Schrödinger operator is equivariant with respect to the left-regular $G$-representation on $L^2(M)$, so that $P(h) \circ L_g = L_g \circ P(h)$.
2.4. Measure spaces and group actions. In what follows, we give an overview of the spaces and measures that will be relevant in the sequel. As before, let $M$ be a connected closed Riemannian manifold of dimension $n$ with Riemannian volume density by $dM$, carrying an isometric effective action of a compact connected Lie group $G$ with Haar measure $dg$. Note that if $\dim G > 0$, $dg$ is equivalent to the normalized Riemannian volume density on $G$ associated to any choice of left-invariant Riemannian metric on $G$. If $\dim G = 0$, $dg$ is the normalized counting measure. Consider further $T^*M$ with its canonical symplectic form $\omega$, endowed with the natural Sasaki metric. Then the Riemannian volume density $d(T^*M)$ given by the Sasaki metric coincides with the symplectic volume form $\omega^n/n!$, see [19] page 537. Next, if $\Omega := J^{-1}(\{0\})$ denotes the zero level of the momentum map, we regard $\Omega_{reg} \subset T^*M$ as a Riemannian submanifold with Riemannian metric induced by the Sasaki metric on $T^*M$, and denote the associated Riemannian volume density by $d\Omega_{reg}$. Similarly, let

$$C := \{ (\eta, g) \in \Omega \times G : g \cdot \eta = \eta \}.$$  

As $\Omega$, the space $C$ is not a manifold in general. We consider therefore the space $\text{Reg} \ C$ of all regular points in $C$, that is, all points that have a neighbourhood which is a smooth manifold. $\text{Reg} \ C$ is a smooth, non-compact submanifold of $T^*M \times G$, and it is not difficult to see that

$$\text{Reg} \ C = \{ (\eta, g) \in \Omega \times G, \ g \cdot \eta = \eta, \ \eta \in \Omega_{reg} \},$$

see e.g. [27, (17)]. We then regard $\text{Reg} \ C \subset T^*M \times G$ as a Riemannian submanifold with Riemannian metric induced by the product metric of the Sasaki metric on $T^*M$ and some left-invariant Riemannian metric on $G$, and denote the corresponding Riemannian volume density by $d(\text{Reg} \ C)$. In the same way, if $x \in M$ and $\eta \in T^*M$ are points, the orbits $G \cdot x$ and $G \cdot \eta$ are smooth submanifolds of $M$ and $T^*M$, respectively, and if they have dimension greater than zero, we endow them with the corresponding Riemannian orbit measures, denoted by $d\mu_{G,x}$ and $d\mu_{G,\eta}$, respectively. If the dimension of an orbit is zero, it is a finite collection of isolated points, since $G$ is compact, and we define $d\mu_{G,x}$ and $d\mu_{G,\eta}$ to be the counting measures. We will generally write $\text{vol}$ and $\text{vol}_C$ for the function which assigns to an orbit and to a point in an orbit the volume of the orbit, respectively. Note that by definition we have $\text{vol} > 0$, $\text{vol}_C > 0$ for all orbits, singular or not. An important property of the orbit measures is their relation to the normalized Haar measure on $G$. Namely, for any orbit $G \cdot x$ and any continuous function $f : G \cdot x \to \mathbb{C}$, we have

$$\int_{G \cdot x} f(x') d\mu_{G,x}(x') = \text{vol}(G \cdot x) \int_G f(g \cdot x) dg,$$

and similarly for $G \cdot \eta$ with $\eta \in T^*M$.

We describe now the quotient spaces and measures on them that will be relevant to us. Let $\tilde{g}$ be the Riemannian metric induced on $\tilde{M}_{reg}$ by the $g$-invariant metric $g$ on $M$, and let $\tilde{dM}_{reg}$ be the corresponding Riemannian volume density. Regarding the co-tangent bundle $T^*\tilde{M}_{reg}$, we endow it with the canonical symplectic structure and let $d(T^*\tilde{M}_{reg})$ be the corresponding symplectic volume form. Again, it coincides with the Riemannian volume form given by the natural Sasaki metric on $T^*\tilde{M}_{reg}$. Similarly, the symplectic stratum $\tilde{\Omega}_{reg}$ will be endowed with the canonical symplectic form $\tilde{\omega}$ from Theorem 2.2 and $d\tilde{\Omega}_{reg} = \tilde{\omega}^{(n-\kappa)}/(n-\kappa)!$ will denote the corresponding symplectic volume form. One can then show that $d\tilde{\Omega}_{reg}$ agrees with the Riemannian volume density defined by the Riemannian metric on $\tilde{\Omega}_{reg}$ induced by the Riemannian metric on $\Omega_{reg}$, see Lemma A.1. Since orbit projections on principal strata define fiber bundles [3, Theorem IV.3.3], Lemma A.1 implies that $d\mu_{G,x}$ and $d\mu_{G,\eta}$ are the unique measures on the orbits in $M_{reg}$ and $\Omega_{reg}$ such that

$$\int_{M_{reg}} f(x) dM(x) = \int_{\tilde{M}_{reg}} f(x') d\mu_{G,x}(x') d\tilde{M}_{reg}(G \cdot x) \quad \forall f \in C(M_{reg}),$$

$$\int_{\Omega_{reg}} f(\eta) d(\Omega_{reg})(\eta) = \int_{\tilde{\Omega}_{reg}} f(\eta') d\mu_{G,\eta}(\eta') d\tilde{\Omega}_{reg}(G \cdot \eta) \quad \forall f \in C(\Omega_{reg}).$$
Next, hypersurfaces will be endowed with the measures induced by the measures on the ambient manifold, compare Lemma A.3. Thus, for a smooth proper function \( p : T^*M \to \mathbb{R} \) with regular value \( c \in \mathbb{R} \), there is a canonical measure \( d\Sigma_c \) on the hypersurface \( \Sigma_c := p^{-1}(\{c\}) \), induced by the symplectic volume form on \( T^*M \), which in the case \( \Sigma_c = S^*M \) is called the Liouville measure and denoted by \( d(S^*M) \). Similarly, for \( S^*M_{\text{reg}} \), the unit co-sphere bundle over \( M_{\text{reg}} \), the induced measure \( d(S^*M_{\text{reg}}) \) is the Liouville measure, and for a general hypersurface \( \Sigma_c := \tilde{p}^{-1}(\{c\}) \), where \( \tilde{p} \) is a smooth proper function \( \Omega_{\text{reg}} \to \mathbb{R} \) with regular value \( c \) and \( \Omega_{\text{reg}} \) is endowed with the measure \( d\Omega_{\text{reg}} \), we denote the induced hypersurface measure by \( d\Sigma_c \). Furthermore, since the intersection is transversal, \( \Sigma_c \cap \Omega_{\text{reg}} \) is a smooth hypersurface of \( \Omega_{\text{reg}} \), and therefore carries a measure \( \mu_c \) induced by \( d\Omega_{\text{reg}} \). As the orbit projection \( \Sigma_c \cap \Omega_{\text{reg}} \to \Sigma_c \) is a fiber bundle, \( \mu_c \) fulfills

\[
\Phi(x, \xi, g) := \langle \gamma(x) - \gamma(g \cdot x), \xi \rangle, \quad (x, \xi) \in T^*U, \ g \in G,
\]

which \( \langle \cdot, \cdot \rangle \) denotes the Euclidean scalar product on \( \mathbb{R}^n \). It represents a global analogue of the moment map, and oscillatory integrals with phase function given by the latter have been examined in [27] while studying the spectrum of an invariant elliptic operator. Thus, let \( M \) be a Riemannian manifold carrying a smooth effective action of a connected compact Lie group \( G \). Consider a chart \( \gamma : M \supset U \xrightarrow{\sim} V \subset \mathbb{R}^n \) on \( M \), and write \( (x, \xi) \) for an element in \( T^*U \cong U \times \mathbb{R}^n \) with respect to the canonical trivialization of the co-tangent bundle over the chart domain. Let \( a_\mu \in C^\infty_c(U \times T^*U \times G) \) be an amplitude that might depend on a parameter \( \mu \in \mathbb{R}_{>0} \), and consider the phase function

\[
I(\mu) = \int_{T^*U} e^{i\mu \Phi(x, \xi, g)} a_\mu(g \cdot x, x, \xi, g) \, d\gamma \, d(T^*U)(x, \xi).
\]
immediately be applied to the integrals $I(\mu)$. Nevertheless, it was shown in \cite{27} that by constructing a strong resolution of the set

$$\mathcal{N} = \{(p, g) \in M \times G : g \cdot p = p\}$$

a partial desingularization $Z : \mathcal{X} \to \mathcal{X} := T^* M \times G$ of the critical set $\mathcal{C}$ can be achieved, and applying the stationary phase theorem in the resolution space, an asymptotic description of $I(\mu)$ can be obtained. More precisely, the map $Z$ yields a partial monomialization of the local ideal $I_\Phi = (\Phi)$ generated by the phase function (2.16) according to

$$Z^*(I_\Phi) \cdot \mathcal{E}_{\mathcal{X}, \mathcal{X}} = \prod_j \sigma_j^{l_j} \cdot Z_\ast^{-1}(I_\Phi) \cdot \mathcal{E}_{\mathcal{X}, \mathcal{X}},$$

where $\mathcal{E}_{\mathcal{X}}$ denotes the structure sheaf of rings of $\mathcal{X}$, $\sigma_j$ are local coordinate functions near each $\tilde{x} \in \mathcal{X}$ and $l_j$ natural numbers. As a consequence, the phase function factorizes locally according to $\Phi \circ Z \equiv \prod \sigma_j^{l_j} \cdot \tilde{\Phi}^{wk}$, and one shows that the weak transforms $\tilde{\Phi}^{wk}$ have clean critical sets. Asymptotics for the integrals $I(\mu)$ are then obtained by pulling them back to the resolution space $\mathcal{X}$, and applying the stationary phase theorem to the $\tilde{\Phi}^{wk}$ with the variables $\sigma_j$ as parameters. As a consequence, one obtains

**Theorem 2.10.** \cite{27} Theorem 9.1 \ Let $\Lambda$ be the maximal number of elements of a totally ordered subset of the set of isotropy types of the $G$-action on $M$. Then, as $\mu \to \infty$,

$$\left| I(\mu) - \left( \frac{2\pi}{\mu} \right)^{\kappa} \int_{\text{Reg} \mathcal{C}} \frac{a_\mu(g \cdot x, x, \xi, g)}{|\det \Phi''(x, \xi, g)_{N(x, \xi, g) \text{Reg} \mathcal{C}}|^{1/2}} \frac{d(\text{Reg} \mathcal{C})(x, \xi, g)}{|\det \Phi(x, \xi, g)_{N(x, \xi, g) \text{Reg} \mathcal{C}}|^{1/2}} \right| \leq C \text{vol}(\text{supp } a_\mu) \sup_{l \leq 2\kappa + 3} \|D^l a_\mu\|_\infty \mu^{-\kappa-1} \log(\mu)^{\Lambda-1},$$

where $D^l$ is a differential operator of order $l$. The expression $\Phi''(x, \xi, g)_{N(x, \xi, g) \text{Reg} \mathcal{C}}$ denotes the restriction of the Hessian of $\Phi$ to the normal space of $\text{Reg} \mathcal{C}$ inside $T^* U \times G$ at the point $(x, \xi, g)$. In particular, the integral in (2.18) exists.

The precise form of the remainder estimate follows from the corresponding estimate in the generalized stationary phase theorem, see \cite{27} Theorem 4.1. This precise form will allow us to give remainder estimates also in the case when the amplitude depends on $\mu$. Finally, let us note the following

**Lemma 2.11.** Let $b \in C^\infty_c(\Omega \cap T^* U)$ and $\chi \in \mathcal{G}$. Then

$$\int_{\text{Reg} \mathcal{C}} \frac{\chi(g) b(x, \xi)}{|\det \Phi''(x, \xi, g)_{N(x, \xi, g) \text{Reg} \mathcal{C}}|^{1/2}} \frac{d(\text{Reg} \mathcal{C})(x, \xi, g)}{|\det \Phi(x, \xi, g)_{N(x, \xi, g) \text{Reg} \mathcal{C}}|^{1/2}} = [\pi_{\chi, \mu} : 1] \int_{\Omega_{\text{Reg}}} b(x, \xi) \frac{d\Omega_{\text{Reg}}(x, \xi)}{\text{vol}_G (G \cdot (x, \xi))},$$

where $[\pi_{\chi, \mu} : 1]$ denotes the multiplicity of the trivial representation in the representation of $H$ given by the restriction of the irreducible $G$-representation $\pi_\chi$ to $H$.

**Proof.** By using a partition of unity, the proof essentially reduces to the one of \cite{8} Lemma 7], which involves only local calculations. Furthermore, $b \in C^\infty_c(\Omega_{\text{Reg}})$ is required there. However, similarly as in \cite{27} Lemma 9.3, one can use Fatou’s Lemma to show that it suffices to require only $b \in C^\infty_c(\Omega)$. \qed

3. An equivariant trace formula

In this section, we shall prove a trace formula in the equivariant setting which will be crucial for all what follows. It is a generalization of \cite{36} Theorem 3.1 to compact $G$-manifolds for Schrödinger operators. With the notation and setup as in the previous sections, let $M$ be a connected closed Riemannian manifold with an effective and isometric action of a compact connected Lie group $G$, and consider the Schrödinger operator

$$\hat{P}(h) = -h^2 \Delta + V, \quad V \in C^\infty(M, \mathbb{R}), \quad h \in \mathbb{R}_{>0},$$
$V$ being a $G$-invariant potential, together with its unique self-adjoint extension
\begin{equation}
(3.1) \quad P(h) : H^2(M) \rightarrow L^2(M).
\end{equation}
Its principal symbol is given by the $h$-independent symbol function $p = \|\tau\|_{L^2(M)}^2 + V \circ \tau \in C^\infty(T^*M)$, and as $p$ is $G$-invariant, it defines a reduced symbol function $\tilde{p} \in C^\infty(\Omega_{\text{reg}})$ on the reduced space $\Omega_{\text{reg}}$. Next, recall from (2.11) the Peter-Weyl decomposition and that by the functional calculus for self-adjoint operators we can define for any $\varrho \in C_c^\infty(\mathbb{R})$ the operator $\varrho(P(h))$, which by Theorem 2.9 is a trace class operator in $\Psi^{-\infty}_h(M)$. Furthermore, any $B \in \Psi^0_h(M)$ yields a bounded operator in $L^2(M)$, compare [39] Theorem 14.2. Now, for a function $a \in C(T^*M)$ consider the orbital integral
\begin{equation}
(3.2) \quad \langle a \rangle_G(\eta) := \int_G a(g \cdot \eta) \, dg, \quad \eta \in T^*M,
\end{equation}
and let $(\hat{a})_G \in C(T^*M/G)$ be the induced function on the quotient. We then have

**Theorem 3.1 (Equivariant seminormalized trace formula for Schrödinger operators).** Let $B \in \Psi^0_h(M)$ be a semiclassical pseudodifferential operator with principal symbol $\sigma(B) = [b]$ given by an $h$-independent symbol function $b$. Choose $\chi \in \hat{G}$ and let $\varrho \in C_c^\infty(\mathbb{R})$. Then $(\varrho(P(h)) \circ B)_\chi$ is a trace class operator, and its trace satisfies the asymptotic estimate\footnote{Note that $\varrho \circ p$ is compactly supported.}
\begin{align*}
&|2\pi h|^{-n-\kappa} \text{tr}(\varrho(P(h)) \circ B)_\chi - d_\chi(\pi_{\chi} : \mathbb{1}) \int_{\Omega_{\text{reg}}} (\varrho \circ p) \, b \frac{d\Omega_{\text{reg}}}{\text{vol}_G} \\
&\quad \leq C \sup_{j \leq 2n+3} \|\varrho^{(j)}\|_\infty \, h \log(h^{-1})^{\Lambda-1},
\end{align*}
where $\Lambda$ is a natural number bounded by the number of orbit types, and $\varrho^{(j)}$ denotes the $j$-th derivative of $\varrho$. Furthermore, the integral in the leading term can also be written as $\int_{\Omega_{\text{reg}}} (\varrho \circ \tilde{p}) (\tilde{b})_G d\tilde{\Omega}_{\text{reg}}$.

**Proof.** \textbf{Step 1.} By Theorem 2.8 $P(h)$ has only finitely many eigenvalues $E(h)_1, \ldots, E(h)_{N(h)}$ in supp $\varrho$, and the corresponding eigenspaces are all finite-dimensional. By the spectral theorem,
\begin{equation}
(3.3) \quad \varrho(P(h)) = \sum_{j=1}^{N(h)} \varrho(E_j(h)) \Pi_j,
\end{equation}
where $\Pi_j$ denotes the spectral projection onto the eigenspace $\text{Eig}(P(h), E_j(h))$ of $P(h)$ belonging to the eigenvalue $E_j(h)$. Hence, $\varrho(P(h))$ is a finite sum of projections onto finite-dimensional spaces and, consequently, a finite rank operator, and therefore of trace class. It follows immediately that $T_\chi \circ \varrho(P(h)) \circ B \circ T_\chi = (\varrho(P(h)) \circ B)_\chi$ is trace class, too.

\textbf{Step 2.} To compute the trace of $(\varrho(P(h)) \circ B)_\chi$, recall that $P(h) - z$ is invertible for $z \in \mathbb{C} - \mathbb{R}$, see [39] Lemma 14.6, and that by the Helffer-Sjöstrand formula [39] Theorem 14.8 one has
\begin{equation*}
\varrho(P(h)) = \frac{1}{i\pi} \int_{\mathbb{C}} \partial_z \hat{\varrho}(z)(P(h) - z)^{-1} \, dz, \quad z = s + it \in \mathbb{C},
\end{equation*}
where $dz$ denotes the Lebesgue measure on $\mathbb{C}$ and $\hat{\varrho} : \mathbb{C} \rightarrow \mathbb{C}$ the almost analytic extension of $\varrho$ defined in [39] Theorem 3.6, while $\partial_z = (\partial_s + i\partial_t)/2$. In what follows, we construct an approximation for $(P(h) - z)^{-1}$ in the framework of ordinary pseudodifferential operators following the construction of the parametrix of a hypoelliptic operator in [39] Proof of Theorem 5.1. Notice that our strategy follows the proof of the corresponding statement in the non-equivariant setting [39] Theorem 14.10, the major difference being that we have to consider approximations to $(\hat{P}(h) - z)^{-1}$ up to order $[(\kappa + 1)/2]$. Consider thus a finite atlas $\{(U_\alpha, \gamma_\alpha)\}_{\alpha \in A}$ for $M$ with charts $\gamma_\alpha : M \supset U_\alpha \xrightarrow{\sim} V_\alpha \subset \mathbb{R}^n$, ...
and let \( \{ \varphi_\alpha \}_{\alpha \in A} \) be a partition of unity subordinate to \( \{ U_\alpha \}_{\alpha \in A} \). In the chart \((U_\alpha, \gamma_\alpha)\), the action of \( \hat{P}(h) - z\) on a function \( f \in C^\infty(U_\alpha) \) is given by

\[
(\hat{P}(h) - z)f(x) = A_\alpha(f \circ \gamma_\alpha^{-1})(y) := P_\alpha(f \circ \gamma_\alpha^{-1})(y) - z
\]

where we wrote \( y = \gamma_\alpha(x) \), while \( g_{ij} : V_\alpha \to \mathbb{R} \) are the local coefficients of the metric, \( g = \det(g_{ij}) \), and \((g^{ij})\) denotes the inverse matrix of \((g_{ij})\). As a differential operator on \( V_\alpha \), the local operator \( A_\alpha \in \Psi^2(V_\alpha) \) is a properly supported ordinary pseudodifferential operator with total symbol

\[
\sigma_{A_\alpha}(y, \eta) := -h^2 \sum_{i,j=1}^n \eta_i \eta_j g^{ij}(y) + (V \circ \gamma_\alpha^{-1})(y) - \frac{h^2}{\sqrt{g(y)}} \sum_{i,j=1}^n \frac{\partial}{\partial y^i} \left( \sqrt{g(y)} \gamma_\alpha^{ij}(y) \eta_i - z.\right)
\]

In particular \( A_\alpha \) is elliptic, and therefore a hypoelliptic operator of class \( H^{2,1}_\alpha(V_\alpha) \) \footnote{32 Proposition 5.1}. Assume now \( \text{Im} \, z \neq 0 \) and let \( q^0_\alpha \in H^{-2,-2}_1(U_\alpha \times \mathbb{R}^n) \) be a hypoelliptic symbol satisfying \( q^0_\alpha(y, \eta) = (p_\alpha(y, \eta) - z)^{-1} \) and \( Q^0_\alpha \in H^{-2,-2}_1(V_\alpha) \) a properly supported operator such that \( \sigma_{Q^0_\alpha} - q^0_\alpha \) in \( S^{-\infty}(V_\alpha \times \mathbb{R}^n) \). Then

\[
Q^0_\alpha \circ A_\alpha = 1 - R^1_\alpha, \quad R^1_\alpha \in \Psi^{-1}(V_\alpha),
\]

since by the composition formula \footnote{32 Theorem 3.4} one has

\[
\sigma_{Q^0_\alpha \circ A_\alpha} = \frac{\sigma_{A_\alpha} \circ Q^0_\alpha}{p_\alpha(y, \eta) - z} + \sum_{|\alpha| \geq 1} \frac{1}{\alpha!}(\partial^\alpha p_\alpha - z)^{-1}(y, \eta)((-i\partial_\eta)^\alpha \sigma_{A_\alpha})(y, \eta).
\]

Iteration yields with

\[
R^k_\alpha = (R^1_\alpha)^k \in \Psi^{-k}(V_\alpha), \quad Q^k_\alpha = Q^0_\alpha \circ R^k_\alpha \in \Psi^{-k-2}(V_\alpha),
\]

the relation

\[
\sum_{k=0}^N Q^k_\alpha \circ A_\alpha = 1 - R^{N+1}_\alpha,
\]

compare also \footnote{39 Proof of Theorem 14.9}. In particular, one sees that \( R^1_\alpha \in h\Psi^{-1}(V_\alpha) \) when regarded as a semiclassical pseudodifferential operator. In this way, we obtain a parametrix for the local operator \( A_\alpha \). By considering a collection of functions \( \{ \varphi_\alpha \}_{\alpha \in A} \) with \( \varphi_\alpha \in C^\infty_c(M, [0,1]) \), \( \text{supp} \, \varphi_\alpha \subset U_\alpha \) and \( \varphi_\alpha \equiv 1 \) in a neighborhood of \( \text{supp} \, \varphi_\alpha \), one can construct a global parametrix for \( \hat{P}(h) - z \), see \footnote{32 Theorem 5.1} for details, and we denote the corresponding global operators on \( M \) by

\[
R^k = \sum_\alpha \Phi_\alpha \circ R^k_\alpha \circ \varphi_\alpha, \quad Q^k = \sum_\alpha \Phi_\alpha \circ Q^k_\alpha \circ \varphi_\alpha,
\]

where \( \Phi_\alpha \) and \( \varphi_\alpha \) are the operators of multiplication with \( \phi_\alpha \) and \( \varphi_\alpha \), respectively, so that

\[
\sum_{k=0}^N Q^k \circ (\hat{P}(h) - z) = 1 - R^{N+1}.
\]

Now, the kernels of \( Q^0_\alpha \) and \( \text{Op}(q^0_\alpha) \) differ only outside a neighborhood of the diagonal \footnote{32 Proposition 3.3}. Therefore, the Helffer-Sjöstrand-formula implies that up to terms of order \( h\infty \)

\[
\varrho(P_\alpha) = \text{Op}(\varrho \circ p_\alpha) + \sum_{k=1}^N \tilde{Q}^k_\alpha + R^{N+1}_\alpha,
\]

where

\[
\tilde{Q}^k_\alpha = \frac{1}{i\pi} \int_{\mathbb{C}} \bar{\partial}_z \varrho(z) Q^k_\alpha(z) \, dz, \quad R^{N+1}_\alpha = \frac{1}{i\pi} \int_{\mathbb{C}} \bar{\partial}_z \varrho(z) \, dz, \quad A^{-1}_\alpha \, dz,
\]
since a direct calculation shows that
\[
\frac{1}{i\pi} \int_{\mathcal{C}} \bar{\partial} z \tilde{\varrho}(z) \text{Op}(q^\alpha_0(y,\eta)) \, dz = \text{Op}(\varrho \circ p_\alpha).
\]
Similarly, up to terms of order $h^\infty$,
\begin{equation}
(3.6) \quad \varrho(P(h)) = \text{Op}_h(\varrho \circ p) + \sum_{k=1}^{N} \tilde{Q}^k + \tilde{R}^{N+1},
\end{equation}
where $\tilde{Q}^k$ and $\tilde{R}^{N+1}$ are global operators corresponding to the operators $Q^k_\alpha$ and $R^{N+1}_\alpha$, respectively.

**Step 3.** We examine now the $h$-dependence of the introduced operators. Thus, let $r^k_\alpha(z)$ be the symbol of the local operator $R^k_\alpha$. Then (3.4) implies that for each $N \in \mathbb{N}$, $N \geq k$, there is an expansion
\[
r^k_\alpha(z) = \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} c^{k}_{\alpha,l,j}(p_\alpha(z) - z)^{-j} \right) + O_{S_{-k-N-1}(V_\alpha)}(h^{2(N+1)}), \quad \text{Im} \, z \neq 0,
\]
with explicitly given coefficients $c^{k}_{\alpha,l,j} \in \mathbb{C}(V_\alpha \times \mathbb{R}^n)$ that are independent of $z$ and $h$. With similar arguments as above, we obtain for the local symbol $q^k_\alpha(z)$ of the operator $Q^k_\alpha$
\[
q^k_\alpha(z) = \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(p_\alpha(z) - z)^{-j} \right) + O_{S_{-k-N-1}(V_\alpha)}(h^{2(N+1)}), \quad \text{Im} \, z \neq 0,
\]
where, again, the coefficients $d^{k}_{\alpha,l,j} \in \mathbb{C}(V_\alpha \times \mathbb{R}^n)$ are independent of $z$ and $h$. The complex number $z$ is just a parameter, that is, a constant with respect to the occurring derivatives. Note that for each $(y,\eta) \in V_\alpha \times \mathbb{R}^n$ and $j \in \mathbb{N}$, the function $\mathbb{C} - \mathbb{R} \to \mathbb{C}$, $z \mapsto (p_\alpha(y,\eta) - z)^{-j}$, is analytic since $p_\alpha(y,\eta)$ is real. As a consequence, we get – by the linearity of the symbol calculus – for the symbol $\tilde{q}^k_\alpha$ of the local operator $\tilde{Q}^k_\alpha$ for each $N \geq k$ the expression
\[
\tilde{q}^k_\alpha(y,\eta) = \frac{1}{i\pi} \int_{\mathcal{C}} \bar{\partial} z \tilde{\varrho}(z) \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(y,\eta)(p_\alpha(y,\eta) - z)^{-j} \right) \, dz + O_{S_{-k-N-1}(V_\alpha)}(h^{2(N+1)}).
\]
As in [39] Proof of Theorem 14.8], we can evaluate the complex integrals explicitly using Stokes’ theorem and the Cauchy integral formula. Thus,
\[
\frac{1}{i\pi} \int_{\mathcal{C}} \bar{\partial} z \left[ \tilde{\varrho}(z) \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(y,\eta)(p_\alpha(y,\eta) - z)^{-j} \right) \right] \, dz
\]
\[
= \frac{1}{i\pi} \lim_{\varepsilon \to 0} \int_{\mathcal{C} - B_{\varepsilon}(p_\alpha(y,\eta))} \bar{\partial} z \left[ \tilde{\varrho}(z) \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(y,\eta)(p_\alpha(y,\eta) - z)^{-j} \right) \right] \, dz
\]
\[
= \frac{1}{2i\pi} \lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(p_\alpha(y,\eta))} \bar{\partial} z \sum_{l=k}^{N} h^{2l} \left( \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(y,\eta)(p_\alpha(y,\eta) - z)^{-j} \right) \, dz
\]
\[
= \frac{1}{2i\pi} \sum_{l=k}^{N} h^{2l} \sum_{j=1}^{l+1} d^{k}_{\alpha,l,j}(y,\eta) \lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(p_\alpha(y,\eta))} \left( (p_\alpha(z) + O(\varepsilon))(p_\alpha(y,\eta) - z)^{-j} \right) \, dz
\]
\[
= \sum_{l=k}^{N} h^{2l} d^{k}_{\alpha,l,1}(y,\eta)(\varrho \circ p_\alpha)(y,\eta),
\]
where $B_z(t) \subset \mathbb{C}$ denotes the disk with center $t$ and radius $\varepsilon$. Setting $d^k_{\alpha,l} := d^k_{\alpha,l,1}$ we conclude that for each $N \geq k$

$$(3.7) \quad \hat{q}^k_{\alpha} = q \circ p_{\alpha} \sum_{l=k}^{N} h^2 d^k_{\alpha,l} + O_{S_{-k-2-N-1}(V_\alpha)}(h^{2N+2}), \quad d^k_{\alpha,l} \in C^\infty(V_\alpha \times \mathbb{R}^n).$$

In particular, since $q \circ p_{\alpha}$ has compact support in $\eta$, one sees that $\hat{q}^k_{\alpha}$ is rapidly falling in $\eta$. Thus, for each $k \in \mathbb{N}$, the operator $\tilde{Q}^k$ is an element of $\Psi^{-\infty}(M)$. Regarding the operator $\tilde{R}^{N+1}$, note that by the symmetry of $P(h)$ one has $\| (P(h) - z) u \|_{L^2} \geq |\text{Im } z| \| u \|_{L^2}$ uniformly in $h$ for $u \in C^\infty(M)$ [39, Proof of Lemma 14.6]. On the other hand, $\tilde{R}^{N+1}$ is a bounded, compact operator in $L^2(M)$, compare [32, Theorem 6.5]. With the expansion for $r^k_{\alpha}(z)$ we therefore see that

$$\| R^{N+1} \circ (P(h) - z)^{-1} \|_{L^2 \to L^2} = O(h^{2N+2} |\text{Im } z|^{-k})$$

for some $k \in \mathbb{N}$. Since $|\tilde{\partial}_z \tilde{q}(z)| \leq C_l |\text{Im } z|^l$ for any $l \in \mathbb{N}$, we conclude that

$$\| \tilde{R}^{N+1} \|_{L^2 \to L^2} = O(h^{2N+2}).$$

The same reasoning applies to the contributions to $\tilde{Q}^k$ coming from the remainder terms of order $h^{2N+2}$ in (3.7). Let us now examine the composition of $\theta(P(h))$ with $B$. If $B_{loc} \in \Psi^0(V_\alpha)$ denotes a properly supported ordinary pseudodifferential operator, (3.5) yields

$$\theta(P_\alpha) \circ B_{loc} = \sum_{k=0}^{N} \tilde{Q}^k_{\alpha} \circ B_{loc} + \tilde{R}^{N+1}_{\alpha} \circ B_{loc},$$

where

$$\tilde{Q}^k_{\alpha} \circ B_{loc} = \frac{1}{i\pi} \int_C \tilde{\partial}_z \tilde{q}(z) \tilde{Q}^k_{\alpha}(z) \circ B_{loc} \, dz, \quad \tilde{R}^{N+1}_{\alpha} \circ B_{loc} = \frac{1}{i\pi} \int_C \tilde{\partial}_z \tilde{q}(z) \tilde{R}^{N+1}_{\alpha}(z) \circ A^{-1}_{\alpha} \circ B_{loc} \, dz.$$

By the composition formula, the symbol of $\tilde{Q}^k_{\alpha}(z) \circ B_{loc}$ has an analogous expansion to that of $\tilde{q}^k_{\alpha}$, and we can repeat all our previous considerations. In particular,

$$\| R^{N+1} \circ (P(h) - z)^{-1} \|_{L^2 \to L^2} = O(h^{2N+2} |\text{Im } z|^{-k}),$$

$B$ being a bounded operator in $L^2(M)$. Collecting everything together we have established that up to terms of order $h^{2N}$ we have

$$(3.8) \quad \theta(P(h)) \circ B = \text{Op}_h((\theta \circ p) b) + \sum_{k=1}^{N} h^{2k} S^k + T^{N+1}$$

for each $N$, where the $T^{N+1}$ are bounded operators satisfying

$$\| T^{N+1} \|_{L^2 \to L^2} = O(h^{2N+2}),$$

and the $S^k \in \Psi^{-\infty}(M)$ are ordinary pseudodifferential operators on $M$ with local symbols of compact support and independent of $h$ and $\theta$. The operator $\text{Op}_h((\theta \circ p) b)$ is an element of $\Psi^{-\infty}(M)$, and by Theorems 2.7 and 2.9 we have $\theta(P(h)) \circ B \in \Psi^{-\infty}(M)$, so that $T^{N+1} \in \Psi^{-\infty}(M)$. Thus, for arbitrary $N \in \mathbb{N}$ and $f \in C^\infty(M)$ we have that up to terms of order $h^{2(N+1)}$

$$(3.9) \quad [\Phi_\alpha \circ \theta(P(h)) \circ B \circ \Phi_\alpha] f(x) = \frac{\pi}{(2\pi h)^n} \int_{T^* U_\alpha} e^{\frac{i}{h} \pi \gamma_\alpha(x) - \gamma_\alpha(y, \xi)} s_\alpha(x, \xi) \varphi_\alpha(y) f(y) \, d(T^* U_\alpha)(y, \xi),$$

where

$$s_\alpha = (\theta \circ p) \left( b + \sum_{k=1}^{N} h^{2k} s^k_\alpha \right), \quad s^k_\alpha \in C^\infty(V_\alpha \times \mathbb{R}^n),$$

each $s^k_\alpha$ being independent of $h$ and $\theta$. Note that $s_\alpha$ is compactly supported in $\xi$, since $\theta \circ p$ is.
Step 4. We can now compute the trace of the reduction of the localized operator \( \overline{\Phi}_\alpha \circ \varrho \circ (P(h)) \circ B \circ \Phi_\alpha \) to the \( \chi \)-isotypic component. Using (2.12) and (3.9) we obtain for \( f \in C^\infty(M) \) up to terms of order \( h^{2(N+1)} \)

\[
(3.10) \quad \left[ T_\chi \circ \overline{\Phi}_\alpha \circ \varrho(P(h)) \circ B \circ \Phi_\alpha f \right](x) = \frac{d_\chi}{(2\pi h)^n} \int_G \int_{T^*U_\alpha} \chi(g) e^{i \gamma_\alpha(g^{-1} \cdot x) - \gamma_\alpha(y,\xi)} - (\varrho \circ p)(g^{-1} \cdot x,\xi) \left( b(g^{-1} \cdot x,\xi) + \sum_{k=1}^N h^{2k} s^k \varphi_\alpha(y) \right) f(y) \, d(T^*U_\alpha)(y,\xi) \, dg.
\]

Note that the integrands are compactly supported and that the remainder operator \( T^{N+1} \) in (3.8) has a smooth Schwartz kernel, since we observed that \( T^{N+1} \in \Psi^{-\infty}(M) \). Thus, if \( K_\alpha(h)(x,y) \) denotes the Schwartz kernel of the operator \( T_\chi \circ \overline{\Phi}_\alpha \circ \varrho(P(h)) \circ B \circ \Phi_\alpha \), we infer from this that

\[
\text{tr} \left( T_\chi \circ \overline{\Phi}_\alpha \circ \varrho(P(h)) \circ B \circ \Phi_\alpha \right) = \int_M K_\chi_\alpha(h)(x,x) \, dM(x)
\]

\[
\quad = \frac{d_\chi}{(2\pi h)^n} \int_M \int_G \chi(g) e^{i \gamma_\alpha(g^{-1} \cdot x) - \gamma_\alpha(x,\xi)} \varphi_\alpha(x) \varphi_\alpha(g \cdot x) \, dg \, dM(x) + O(h^{2(N+1)})
\]

\[
\quad = \frac{d_\chi}{(2\pi h)^n} \int_{T^*U_\alpha} e^{i \gamma_\alpha(x) - \gamma_\alpha(g \cdot x,\xi)} J_\alpha(g \cdot x) \varphi_\alpha(x) \varphi_\alpha(g \cdot x) \, dg + O(h^{2(N+1)}),
\]

where \( J_\alpha(x,g) \) is the Jacobian of the substitution \( x = g \cdot x' \), and \( N \in \mathbb{N} \) is arbitrary. We are now prepared to use our last major tool, Theorem 2.10. For this, define \( u^{\chi,N}_{h,\alpha} \in C^\infty(U_\alpha \times T^*U_\alpha \times G) \) by

\[
(3.12) \quad u^{\chi,N}_{h,\alpha}(g \cdot x, x, \xi, g) = J_\alpha(x,g) \chi(g) \varrho(p(x,\xi)) \left( b(x,\xi) + \sum_{k=1}^N h^{2k} s^k (x,\xi) \right) \varphi_\alpha(x) \varphi_\alpha(g \cdot x).
\]

Then (3.11) and Theorem 2.10 imply for each \( N \in \mathbb{N} \) the estimate

\[
(2\pi h)^n \text{tr} \left( T_\chi \overline{\Phi}_\alpha \varrho(P(h)) B \Phi_\alpha \right) - d_\chi (2\pi h)^n \int_{\text{Reg}C_\alpha} \frac{u^{\chi,N}_{h,\alpha}(g \cdot x, x, \xi, g)}{|\det \Phi''(x,\xi, g)|^{1/2} N(x,\xi,g)} \, d(\text{Reg}C)(x,\xi, g) \leq C_{\alpha,N} \left( \sup_{l \leq 2k+3} \| D^l u^{\chi,N}_{h,\alpha} \|_\infty h^{k+1} \log \left( h^{-1} \Lambda^{-1} + h^{2(N+1)} \right) \right),
\]

where \( \text{Reg}C_\alpha = \{(x,\xi, g) \in (\Omega \cap T^*U_\alpha) \times G, \, g(x,\xi) = (x,\xi), \, x \in M(H)\} \), \( D^l \) is a differential operator of order \( l \), and \( \Phi''(x,\xi, g)|_{\text{Reg}C_\alpha} \) denotes the restriction of the Hessian of \( \Phi(x,\xi, g) = \gamma_\alpha(x) - \gamma_\alpha(g \cdot x,\xi) \) to the normal space of \( \text{Reg}C_\alpha \) inside \( T^*U_\alpha \times G \) at the point \( (x,\xi, g) \). Note that the domain of integration contains only such \( g \) and \( x \) for which we have \( g \cdot x = x \), so that the integral simplifies to

\[
(3.14) \quad A_\alpha = \int_{\text{Reg}C} \frac{\chi(g) \varrho(p(x,\xi)) \left( b(x,\xi) + \sum_{k=1}^N h^{2k} s^k (x,\xi) \right) \varphi_\alpha(x)}{|\det \Phi''(x,\xi, g)|^{1/2} N(x,\xi,g)} \, d(\text{Reg}C)(x,\xi, g).
\]
Here we used that $J_{\alpha}(x,g) = 1$ in the domain of integration, since the substitution $x' = g \cdot x$ is the identity when $g \cdot x = x$, and that $\Phi_{\alpha} = 1$ on $\text{supp } \varphi_{\alpha}$. By Lemma 2.11 this simplifies further to

$$\mathfrak{A}_\alpha = \left[ \pi_{\chi_{A}} : 1 \right] \int_{\Omega_{\text{reg}}} \varrho(p(x,\xi)) \left( b(x,\xi) + \sum_{k=1}^{N} h^{2k} s_{\alpha}^{k}(x,\xi) \right) \varphi_{\alpha}(x) \frac{d\Omega_{\text{reg}}(x,\xi)}{\text{vol } (G \cdot (x,\xi))}. \tag{3.15}$$

**Step 5.** To finally calculate the trace of $(\varrho(P(h)) \circ B)_{\chi}$, note that by definition $\text{supp } (1 - \Phi_{\alpha}) \cap \text{supp } \varphi_{\alpha} = \emptyset$. As already noted, a fundamental property of a semiclassical pseudodifferential operator is that outside the diagonal its kernel is smooth and the supremum norm of the kernel is rapidly decreasing in $h$. Therefore,

$$T_{\chi} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} = T_{\chi} \circ \Phi_{\alpha} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} + T_{\chi} \circ R_{\alpha} \circ T_{\chi}, \tag{3.16}$$

where $R_{\alpha} \in h^\infty \Psi^{-\infty}_{\text{c}}(M)$. The remainder operator $T_{\chi} \circ R_{\alpha} \circ T_{\chi}$ in (3.16) is trace class since it has a smooth kernel, and its trace is of order $h^\infty$. Consequently, for each $N \in \mathbb{N}$ we have

$$\text{tr } (\varrho(P(h)) \circ B)_{\chi} = \text{tr } \left( T_{\chi} \circ \varrho(P(h)) \circ B \circ \left( \sum_{\alpha \in A} \Phi_{\alpha} \right) \circ T_{\chi} \right) = \sum_{\alpha \in A} \text{tr } \left( T_{\chi} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} \right) \tag{3.17}$$

$$= \sum_{\alpha \in A} \text{tr } \left( T_{\chi} \circ \Phi_{\alpha} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} \right) + O(h^\infty).$$

Note that since $T_{\chi}^2 = T_{\chi}$, one has

$$\text{tr } \left( T_{\chi} \circ \Phi_{\alpha} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} \right) = \text{tr } \left( T_{\chi} \circ \Phi_{\alpha} \circ \varrho(P(h)) \circ B \circ \Phi_{\alpha} \circ T_{\chi} \right),$$

by invariance of the operator trace under cyclic permutations. Therefore, (3.13) and (3.17) together lead to

$$\left| (2\pi h)^{n} \text{tr } (\varrho(P(h)) \circ B)_{\chi} - d_{\chi} (2\pi h)^{n} \sum_{\alpha \in A} \mathfrak{A}_{\alpha} \right| \leq \sum_{\alpha \in A} C_{\alpha,N} \left( \sup_{l \leq 2\kappa + 3} \| D^{l} u_{h,\alpha} \|_{\infty} h^{\kappa+1} \log (h^{-1})^{\lambda-1} + h^{2(N+1)} \right). \tag{3.18}$$

Since the functions $s_{\alpha}^{k}$ do not depend on any derivatives of $\varrho$, we have

$$\sup_{l \leq 2\kappa + 3} \| D^{l} u_{h,\alpha} \|_{\infty} \leq c_{1} \sup_{l \leq 2\kappa + 3} \| D^{l} (\varrho \circ p) \|_{\infty} \leq c_{2} \sup_{l \leq 2\kappa + 3} \| \varrho^{(l)} \|_{\infty} \tag{3.19}$$

for suitable constants $c_{1} > 0$. From (3.18) and (3.19) we conclude that

$$\left| (2\pi h)^{n} \text{tr } (\varrho(P(h)) \circ B)_{\chi} - d_{\chi} (2\pi h)^{n} \sum_{\alpha \in A} \mathfrak{A}_{\alpha} \right| \leq C_{N} \left( \sup_{l \leq 2\kappa + 3} \| \varrho^{(l)} \|_{\infty} h^{\kappa+1} \log (h^{-1})^{\lambda-1} + h^{2(N+1)} \right).$$

Finally, (3.15) implies that for each $N \in \mathbb{N}$

$$\sum_{\alpha \in A} \mathfrak{A}_{\alpha} = \left[ \pi_{\chi_{A}} : 1 \right] \int_{\Omega_{\text{reg}}} \varrho(p(x,\xi)) b(x,\xi) \left( \sum_{\alpha \in A} \varphi_{\alpha}(x) \right) \frac{d\Omega_{\text{reg}}(x,\xi)}{\text{vol } (G \cdot (x,\xi))} + \left[ \pi_{\chi_{A}} : 1 \right] \sum_{\alpha \in A} h^{2k} \sum_{k=1}^{N} \int_{\Omega_{\text{reg}}} \varrho(p(x,\xi)) s_{\alpha}^{k}(x,\xi) \varphi_{\alpha}(x) \frac{d\Omega_{\text{reg}}(x,\xi)}{\text{vol } (G \cdot (x,\xi))}.$$
Absorbing the second term on the right hand side into the remainder estimate we finally obtain with
\[ N = \left(\frac{\kappa + 1/2}{2}\right) \text{ as } h \to 0 \]
\[
\left| (2\pi h)^{n-\kappa} \text{ tr } (\rho \circ B(x)) - d_x \left[ \pi_x|_{H^1} : 1 \right] \int_{\Omega_{\text{reg}}} \rho(p(x, \xi)) b(x, \xi) \frac{d\Omega_{\text{reg}}(x, \xi)}{\text{vol} (G \cdot (x, \xi))} \right|
\leq C \sup_{\ell \leq 2\kappa + 3} \left\| \phi^{(\ell)} \right\|_{\infty} h \log (h^{-1})^{\Lambda-1}.
\]
Taking into account (2.12) and (2.14), the G-invariance of \( p \) implies
\[
\int_{\Omega_{\text{reg}}} \frac{1}{\text{vol} (G \cdot (x, \xi))} \int_{G \cdot (x, \xi)} \rho(p(x', \xi')) b(x', \xi') \, d\mu_{G \cdot (x, \xi)}(x', \xi') \, d(\tilde{\Omega}_{\text{reg}})(G \cdot (x, \xi))
= \int_{\Omega_{\text{reg}}} \int_G \rho(p \cdot (x', \xi')) b(g \cdot (x', \xi')) \, dg \, d(\tilde{\Omega}_{\text{reg}})(G \cdot (x, \xi)) = \int_{\Omega_{\text{reg}}} \rho(\tilde{p}(O)) \langle \tilde{b} \rangle G(O) \, d(\tilde{\Omega}_{\text{reg}})(O),
\]
and the assertion of the theorem follows. \( \square \)

4. Generalized equivariant semiclassical Weyl law

We are now in the position to state and prove a generalized semiclassical Weyl law for Schrödinger operators in the equivariant setting.

**Theorem 4.1 (Generalized equivariant semiclassical Weyl law).** Consider a compact connected Lie group \( G \) acting effectively and isometrically on a closed connected Riemannian manifold \( M \). Let \( \{u_j(h)\}_{j \in \mathbb{N}} \) be a Hilbert basis in \( L^2(M) \) of eigenfunctions of a Schrödinger operator \( P(h) \) with \( G \)-invariant symbol function \( p \), and denote the eigenvalue associated to \( u_j(h) \) by \( E_j(h) \). Let \( B \in \Psi^0_h(M) \) be a semiclassical pseudodifferential operator with principal symbol \( \sigma(B) = [b] \), and assume that \( b \in S^0(M) \) is independent of \( h \). Let \( c \in \mathbb{R} \) be a regular value of the symbol function \( p \). Then, for each \( \beta \in (0, \frac{1}{2+\kappa+2}) \) and \( \chi \in \hat{G} \) one has

\[
(2\pi)^{n-\kappa} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, u_j(h) \in L^2(M), \quad E_j(h) \in [c, c + h^\beta]} \langle B u_j(h), u_j(h) \rangle_{L^2(M)} = d_x \left[ \pi_x|_{H^1} : 1 \right] \int_{\Sigma_c \cap \Omega_{\text{reg}}} b \frac{d\mu_c}{\text{vol}(O)} + O \left( h^{\frac{3}{2}+\kappa+\beta} (\log h^{-1})^{\Lambda-1} \right),
\]

where \( \left[ \pi_x|_{H^1} : 1 \right] \) denotes the multiplicity of the trivial representation in the \( H \)-representation \( \pi_x|_{H^1} \) and \( \Lambda \in \mathbb{N} \) is bounded by the number of orbit types. Furthermore, the integral in the leading term equals 
\[ \int_{\Sigma_c} \langle \tilde{b} \rangle_O \, d\Sigma_c \).

**Proof.** The proof is an adaptation of the proof of [39] Theorem 15.3 to our situation, but with a sharper energy localization. For simplicity, we shall also write \( u_j(h) = u_j \) and \( E_j(h) = E_j \) in the following, but it is essential to keep in mind that these objects depend on \( h \). To begin, let \( h > 0 \) and \( 0 < \varepsilon < \frac{1}{h} \). Choose \( f_\varepsilon, g_\varepsilon \in C^\infty_c(\mathbb{R}, [0, 1]) \) such that \( \text{supp } f_\varepsilon \subset [-\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon] \), \( f_\varepsilon \equiv 1 \) on \([-\frac{1}{2} + 3\varepsilon, \frac{1}{2} - 3\varepsilon] \), \( \text{supp } g_\varepsilon \subset [-\frac{1}{2} - 3\varepsilon, \frac{1}{2} + 3\varepsilon] \), \( g_\varepsilon \equiv 1 \) on \([-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \), and

\[
|\partial_y^j f_\varepsilon(y)| \leq C_j \varepsilon^{-j}, \quad |\partial_y^j g_\varepsilon(y)| \leq C_j \varepsilon^{-j},
\]

compare [16] Theorem 1.4.1 and (1.4.2). Let \( c(h) := ch^{-\beta} + \frac{1}{2} \), so that \( x \mapsto h^{-\beta} x - c(h) \) defines a diffeomorphism from \([-1/2, 1/2] \) to \([c, c + h^\beta] \), and set \( f_{\varepsilon, h^{-\beta}}(x) := f_\varepsilon(h^{-\beta} x - c(h)) \), \( g_{\varepsilon, h^{-\beta}}(x) := g_\varepsilon(h^{-\beta} x - c(h)) \).
$g_{\varepsilon}(h^{-\beta}c - c(h))$. Let $\Pi_\chi$ be the projection onto the span of $\{u_j \in L^2(\Omega) : E_j \in [c, c + \varepsilon h^\beta]\}$. Then, for sufficiently small $\varepsilon > 0$,

\begin{equation}
\begin{aligned}
f_{\varepsilon, h^{-\beta}}(P(h))_\chi \circ \Pi_\chi &= \Pi_\chi \circ f_{\varepsilon, h^{-\beta}}(P(h))_\chi = f_{\varepsilon, h^{-\beta}}(P(h))_\chi, \\
g_{\varepsilon, h^{-\beta}}(P(h))_\chi \circ \Pi_\chi &= \Pi_\chi \circ g_{\varepsilon, h^{-\beta}}(P(h))_\chi = \Pi_\chi.
\end{aligned}
\end{equation}

(4.3)

Note that the operators $f_{\varepsilon, h^{-\beta}}(P(h))$, $g_{\varepsilon, h^{-\beta}}(P(h))$, $\Pi_\chi$ and arbitrary, multiple compositions of these operators with $T_\chi$ or $B$ are finite rank operators. For that elementary reason, all operators we consider in the following are trace class. In particular, by (4.3) we have

$$\sum_{j \in \mathbb{N}, u_j \in L^2(M)} \langle Bu_j, u_j \rangle_{L^2(M)} = \text{tr } \Pi_\chi \circ B \circ \Pi_\chi = \text{tr } \Pi_\chi \circ B \circ \Pi_\chi$$

(4.4)

Next, we estimate $\Re_{\varepsilon, h^{-\beta}}$ using the trace norm $\|\cdot\|_{L^1(M)}$. It is defined for a compact operator $A \in B(L^2(M))$ by $\|A\|_{L^1(M)} = \sum_{k \in \mathbb{N}} s_k$, where $\{s_k\}_{k \in \mathbb{N}}$ are the eigenvalues of the self-adjoint operator $A^*A$. Now, if $L \in L^1(M)$ and $M \in B(L^2(M))$, then $\|LM\|_{L^1(M)} \leq \|L\|_{L^1(M)} \|M\|_{B(L^2(M))}$, see e.g. [39] p. 337. By the functional calculus this implies

\begin{equation}
\begin{aligned}
|\Re_{\varepsilon, h^{-\beta}}| &\leq \left\| \Pi_\chi \circ (g_{\varepsilon, h^{-\beta}}(P(h)) \circ (1 - f_{\varepsilon, h^{-\beta}}(P(h))) \circ B) \right\|_{L^1(M)} \\
&\leq \left\| (g_{\varepsilon, h^{-\beta}}(P(h)) \circ (1 - f_{\varepsilon, h^{-\beta}}(P(h))) \circ B) \right\|_{L^1(M)} \|B\|_{B(L^2(M))}
\end{aligned}
\end{equation}

(4.5)

where we set $v_{\varepsilon, h^{-\beta}} = g_{\varepsilon, h^{-\beta}}(1 - f_{\varepsilon, h^{-\beta}}) \in C^\infty_c([0, 1])$. In particular, $v_{\varepsilon, h^{-\beta}}$ is non-negative. By the spectral theorem, $v_{\varepsilon, h^{-\beta}}(P(h))$ is a positive operator. $T_\chi$ is a projection, hence positive as well. It follows that $v_{\varepsilon, h^{-\beta}}(P(h))_\chi$ is positive as the composition of positive operators. For a positive operator, the trace norm is identical to the trace. Therefore (4.5) implies

\begin{equation}
|\Re_{\varepsilon, h^{-\beta}}| \leq \|B\|_{B(L^2(M))} \text{ tr } v_{\varepsilon, h^{-\beta}}(P(h))_\chi.
\end{equation}

(4.6)

From our knowledge about the supports of $f_{\varepsilon}$ and $g_{\varepsilon}$, we conclude that

\begin{equation}
\supp v_{\varepsilon, h^{-\beta}} \subset [c - 3\varepsilon h^\beta, c + 3\varepsilon h^\beta] \cup [c + \varepsilon h^\beta, c + h^\beta + 3\varepsilon h^\beta].
\end{equation}

(4.7)

Now, by Theorem 3.1 with $B = \text{id}_{L^2(M)}$ and (4.2) we conclude

\begin{equation}
\left| (2\pi h)^{n-\kappa} \text{ tr } v_{\varepsilon, h^{-\beta}}(P(h))_\chi - d_\chi [\pi_{\chi(h)} : 1] \int_{\Omega_{\text{reg}}} (v_{\varepsilon, h^{-\beta}} \circ \overline{\mu})_\chi d(\overline{\mu}_{\text{reg}})(\Omega) \right|
\end{equation}

\begin{equation}
\leq C\varepsilon^{-2\kappa-3} h^{1-\beta(2\kappa+3)} (\log h^{-1})^{\Lambda-1}.
\end{equation}

(4.8)

On the other hand, applying Theorem 3.1 to the first summand on the right hand side of (4.4) yields

\begin{equation}
\left| (2\pi h)^{n-\kappa} \text{ tr } (f_{\varepsilon, h^{-\beta}}(P(h))_\chi \circ B) - d_\chi [\pi_{\chi(h)} : 1] \int_{\Omega_{\text{reg}}} (f_{\varepsilon, h^{-\beta}} \circ \overline{\mu})_\chi d(\overline{\mu}_{\text{reg}})(\Omega) \right|
\end{equation}

\begin{equation}
\leq C\varepsilon^{-2\kappa-3} h^{1-\beta(2\kappa+3)} (\log h^{-1})^{\Lambda-1}.
\end{equation}

(4.9)
Combining this with (4.4) leads to
\[(2\pi h)^{n-\kappa} \sum_{j \in \mathbb{N}, u_j \in L^2(M), \ E_j \in [c, c+h^\beta]} (Bu_j, u_j)_{L^2(M)} = d_x [\pi_{\chi_H} : 1] \int_{\Omega_{\text{reg}}} (f_{\varepsilon, h^{-\beta}} \circ \tilde{p})(\Omega) (\tilde{b})_G(\Omega) d(\tilde{\Omega}_{\text{reg}})(\Omega) \]
(4.9)
\[+ (2\pi h)^{n-\kappa} \Omega_{\varepsilon, h^{-\beta}} + O(\varepsilon^{-2n-3} h^{1-\beta(2n+3)}) (\log h^{-1})^{n-1}. \]
Furthermore, by (4.6), and (4.8),
\[(4.10) \]
\[|(2\pi h)^{n-\kappa} \Omega_{\varepsilon, h^{-\beta}}| \leq \|B\|_{B(L^2(M))} d_x [\pi_{\chi_H} : 1] \int_{\Omega_{\text{reg}}} (v_{\varepsilon, h^{-\beta}} \circ \tilde{p})(\Omega) d(\tilde{\Omega}_{\text{reg}})(\Omega) \]
\[+ O(\varepsilon^{-2n-3} h^{1-\beta(2n+3)}) (\log h^{-1})^{n-1}. \]
Now, let \(a, b \in \mathbb{R}\). Then \(p^{-1}((a, b]) \subset T^*M\) is closed and bounded, hence compact. On the other hand, \(\tilde{p}^{-1}((a, b])\) is a closed subset of \(p^{-1}((a, b])\)/\(G\). The latter space carries the quotient topology, and in particular is compact. It follows that \(\tilde{p}^{-1}((a, b])\) is a compact subspace of \(T^*M/G\). Since \(\Omega_{\text{reg}} \subset T^*M\) carries the subspace topology, also \(\tilde{\Omega}_{\text{reg}} \subset T^*M/G\) carries the subspace topology. Therefore, \(\tilde{p}^{-1}((a, b])\) is in fact a compact subspace of \(\tilde{\Omega}_{\text{reg}}\). Thus, we have established that the reduced symbol function \(\tilde{p}\) is a proper map. Next, note that by Lemma 4.8 we have
\[\int_{\tilde{\Omega}_{\text{reg}}} (v_{\varepsilon, h^{-\beta}} \circ \tilde{p})(\Omega) d(\tilde{\Omega}_{\text{reg}})(\Omega) = O(\varepsilon h^\beta), \]
and with \(\tilde{\Sigma}_c = \tilde{p}^{-1}(\{c\})\)
\[\int_{\tilde{\Omega}_{\text{reg}}} (f_{\varepsilon, h^{-\beta}} \circ \tilde{p})(\Omega) (\tilde{b})_G(\Omega) d(\tilde{\Omega}_{\text{reg}})(\Omega) \]
\[= \text{vol} \left( [c + \varepsilon h^\beta, c + h^\beta - \varepsilon h^\beta] \right) \left( \int_{\tilde{\Sigma}_c} (\tilde{b})_G(\Omega) d\tilde{\Sigma}_c(\Omega) + O(\varepsilon h^\beta) \right). \]
Using the last two estimates together with (4.10) in (4.9) yields
\[(2\pi h)^{n-\kappa} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, u_j \in L^2(M), \ E_j \in [c, c+h^\beta]} (Bu_j, u_j)_{L^2(M)} - d_x [\pi_{\chi_H} : 1] \int_{\tilde{\Sigma}_c} (\tilde{b})_G(\Omega) d\tilde{\Sigma}_c(\Omega) \]
\[= O(\varepsilon) + O(\varepsilon^{-2n-3} h^{1-\beta(2n+4)}) (\log h^{-1})^{n-1}. \]
If we now put \(\varepsilon = h^{\frac{1}{2n+1}}-\beta\), and take into account (2.12) and (2.15), the assertion of the theorem follows. \qed

As consequence of the previous theorem we obtain in particular

**Theorem 4.2 (Equivariant semiclassical Weyl law).** Let \(\chi \in \hat{G}, \ \beta \in (0, \frac{1}{2n+1})\) and let \(c \in \mathbb{R}\) be a regular value of \(p\). Then
\[(4.11) \quad (2\pi h)^{n-\kappa} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, \ E_j(h) \in [c, c+h^\beta]} \text{mult}_\chi(h) = [\pi_{\chi_H} : 1] \text{vol} \tilde{\Sigma}_c + O \left( h^{\frac{1}{2n+1}}-\beta} (\log h^{-1})^{n-1} \right). \]
\[\square\]
We close this section with the following elementary

3 In fact, we could get an even sharper estimate if we also took into account that \(\text{vol} supp v_{\varepsilon, h^{-\beta}} \sim \varepsilon\).
Corollary 4.3. Let \( c \in \mathbb{R} \) be as in Theorem 4.2, and \( \beta \in (0, \frac{1}{2(k+1)}) \). Then, for each \( \chi \in \hat{G} \) there is a \( h_\chi > 0 \) such that for all \( h \in (0, h_\chi) \)

\[
(4.12) \bigcup_{j \in \mathbb{N}, E_j(h) \in [c, c+h^\beta]} \text{Eig}(P(h), E_j(h)) \cap L^2_\chi(M) \neq \{0\}.
\]

Proof. Since \( c \) is a regular value of \( \bar{p}, \Sigma_c = \bar{p}^{-1}(\{c\}) \) is a non-degenerate hypersurface of \( \hat{G}_{\text{reg}} \), which implies that \( \text{vol}_{\delta\Sigma_c} \Sigma_c > 0 \). Consequently, the leading term on the right hand side of (4.11) is non-zero. If the claim did not hold, we could find a sequence \( (h_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}_{>0} \), converging to zero, such that the left hand side of (4.11) would be zero for all \( k \in \mathbb{N} \), a contradiction. \( \square \)

5. Symmetry-reduced classical ergodicity

This section is devoted to the study of classical ergodicity in the presence of symmetries within the framework of symplectic reduction. As we already mentioned, the latter is based on the fundamental fact that the presence of conserved quantities or first integrals of motion leads to the elimination of variables, and reduces the given configuration space with its symmetries to a lower-dimensional one, in which the degeneracies and the conserved quantitites have been eliminated. In particular, the Hamiltonian flows associated to \( G \)-invariant Hamiltonians give rise to corresponding reduced Hamiltonian flows on the different symplectic strata of the reduction. Therefore, the concept of ergodicity can be studied naturally in the context of symplectic reduction, leading to a symmetry-reduced notion of ergodicity.

Recall that, in general, a measure-preserving transformation \( T : X \to X \) on a finite measure space \( (X, \mu) \) is called ergodic if \( T^{-1}(A) = A \) implies \( \mu(A) \in \{0, \mu(X)\} \) for every measurable set \( A \subseteq X \). Consider now a connected, symplectic manifold \( (X, \omega) \) with a global Hamiltonian action of a Lie group \( G \), and let \( J : X \to g^* \), \( J(\eta)(X) = \mathbb{J}X(\eta) \) be the corresponding momentum map. As already noted in Section 2.1, for each \( X \in g \) the function \( J_X \) is a conserved quantity for any \( G \)-invariant function \( p \in C^\infty(X, \mathbb{R}) \), so that \( \{J_X, p\} = 0 \). This implies that for any value \( \mu \) of \( J \), the fiber \( J^{-1}(\{\mu\}) \) is invariant under the Hamiltonian flow of \( p \), which means that \( J \) fulfills Noether's condition. In particular, the preimage under \( J \) of any open proper subset in \( \mathbb{J}(X) \) will be an open proper subset in \( X \) that is invariant under the Hamiltonian flow of \( p \), so the latter cannot be ergodic with respect to the induced Liouville measure on \( \Sigma_c := p^{-1}(\{c\}) \) for any \( c \in \mathbb{R} \) being a regular value of \( p \).

Let now \( p \) and \( \mu \) be fixed, \( K \subset G \) an isotropy group of the G-action on \( X \), and \( \eta \in J^{-1}(\{\mu\}) \). With the notation as in Theorem 2.2, let \( c \in \mathbb{R} \), and put \( \Sigma_{\mu,c}^{(K)} := (p^{(K)}_{\mu})^{-1}(\{c\}) \). Let \( \bar{g} \) be a Riemannian metric on \( \bar{\Omega}_{\mu,c}^{(K)} \) and \( J : T\bar{\Omega}_{\mu,c}^{(K)} \to T\bar{\Omega}_{\mu,c}^{(K)} \) the almost complex structure determined by \( \bar{\omega}_{\mu,c}^{(K)} \) and \( \bar{g} \), so that \((\bar{\Omega}_{\mu,c}^{(K)}, J, \bar{g})\) becomes an almost Hermitian manifold. We then make the following

Assumption 1. \( c \) is a regular value of \( p^{(K)}_{\mu} \).

Note that this assumption is implied by the condition that for all \( \xi \in J^{-1}(\{\mu\}) \cap G_{\mu} \cdot X_K \cap \Sigma_c \) one has

\[ H_{\mu}(\xi) \notin g_{\mu} \cdot \xi. \]

Indeed, assume that there exists some \((\xi) \in \Sigma_{\mu,c}^{(K)} \) such that \( \text{grad } p^{(K)}_{\mu}(\xi) = 0 \). Since

\[ \bar{\omega}_{\mu,c}^{(K)}(\text{grad } p^{(K)}_{\mu}, X) = dp^{(K)}_{\mu}(X) = \bar{g}(\text{grad } p^{(K)}_{\mu}, X), \]

we infer that \( H_{\mu,c}(\xi) = \text{grad } p^{(K)}_{\mu}(\xi) = 0 \), which means that \((\xi) \in \Sigma_{\mu,c}^{(K)} \) is a stationary point for the reduced flow, so that \( \varphi^{(K)}_{\mu}(\xi) = \xi \) for all \( t \in \mathbb{R} \). By the fourth assertion in Theorem 2.2, this is equivalent to

\[ \pi_{\mu,c}^{(K)} \circ \varphi^{(K)}_{\mu}(\xi) = \varphi^{(K)}_{\mu}(\xi) \quad \forall \ t \in \mathbb{R}, \ \xi' \in G_{\mu} \cdot \xi, \]

which in turn is equivalent to \( \varphi^{(K)}_{\mu}(\xi') \in G_{\mu} \cdot \xi' \). Thus, there exists a \( G_{\mu} \)-orbit in \( J^{-1}(\{\mu\}) \cap G_{\mu} \cdot X_K \cap \Sigma_c \) which is invariant under \( \varphi^{(K)}_{\mu} \). In particular one has \( H_{\mu}(\xi') \notin g_{\mu} \cdot \xi' \) for all \( \xi' \in G_{\mu} \cdot \xi. \)
Assumption 1 ensures that \( \tilde{\Sigma}^{(K)}_{\mu,c} \) is a smooth submanifold of \( \tilde{\Omega}^{(K)}_{\mu} \). Equipping \( \tilde{\Omega}^{(K)}_{\mu} \) with the symplectic volume measure defined by the unique symplectic form on \( \tilde{\Omega}^{(K)}_{\mu} \) described in Theorem 2.2, Lemma A.8 says that there is a unique hypersurface measure \( \nu^{(K)}_{\mu,c} \) on \( \tilde{\Sigma}^{(K)}_{\mu,c} \). Moreover, \( \nu^{(K)}_{\mu,c} \) is invariant under the reduced flow \( \tilde{\varphi}^t \), since the latter constitutes a symplectomorphism due to Cartan’s homotopy formula. It is then natural to make the following

**Definition 5.1.** The reduced flow \( \tilde{\varphi}^t \) is called ergodic on \( \tilde{\Sigma}^{(K)}_{\mu,c} \) if for each connected component \( S \) of \( \tilde{\Sigma}^{(K)}_{\mu,c} \) and any measurable set \( E \subset S \) with \( \tilde{\varphi}^t(E) = E \), one has

\[
\nu^{(K)}_{\mu,c}(E) = 0 \quad \text{or} \quad \nu^{(K)}_{\mu,c}(E) = \nu^{(K)}_{\mu,c}(S).
\]

We can now formulate

**Theorem 5.1 (Symmetry-reduced mean ergodic theorem).** Let Assumption 1 above be fulfilled, and suppose that the reduced flow \( \tilde{\varphi}^t \) is ergodic on \( \tilde{\Sigma}^{(K)}_{\mu,c} \). Then, for each \( f \in L^2(\tilde{\Sigma}^{(K)}_{\mu,c}, \nu^{(K)}_{\mu,c}) \) we have

\[
\langle f \rangle_T \to T \to \infty \frac{1}{\nu^{(K)}_{\mu,c}(\tilde{\Sigma}^{(K)}_{\mu,c})} \int f \, d\nu^{(K)}_{\mu,c}(\tilde{\Sigma}^{(K)}_{\mu,c})
\]

with respect to the norm topology of \( L^2(\tilde{\Sigma}^{(K)}_{\mu,c}, \nu^{(K)}_{\mu,c}) \), where

\[
\langle f \rangle_T ([\mu]) := \frac{1}{T} \int_0^T f(\tilde{\varphi}^t([\mu])) \, dt, \quad [\mu] \in \tilde{\Sigma}^{(K)}_{\mu,c}.
\]

**Proof.** The proof is completely analogous to the existing proofs of the classical mean ergodic theorem, compare e.g. [39] Theorem 15.1. \( \square \)

In all what follows, we shall be interested mainly in the case where \( X = T^*M \) is the co-tangent bundle of a \( G \)-manifold \( M \), \( \mu = 0 \), and \( K = H \) is given by a principal isotropy group. We shall therefore use the simpler notation

\[
\tilde{\Omega}_{\text{reg}} = \tilde{\Omega}_0^{(H)}, \quad \tilde{\varphi} = \tilde{\varphi}^0, \quad \tilde{\Sigma}_c = \tilde{\Sigma}_c^{(H)}, \quad d\tilde{\Sigma}_c = d\nu_0^{(H)}, \quad \tilde{p} = \tilde{p}_0^{(H)}.
\]

As a special case of Theorem 5.1 we get the following

**Theorem 5.2.** Let \( c \in \mathbb{R} \) be a regular value of the symbol function \( p \). Suppose that the reduced flow \( \tilde{\varphi}_t \) is ergodic on \( (\tilde{\Sigma}_c, d\tilde{\Sigma}_c) \). Then for each \( f \in L^2(\tilde{\Sigma}_c, d\tilde{\Sigma}_c) \),

\[
\lim_{T \to \infty} \int_{\tilde{\Sigma}_c} \left( \langle f \rangle_T - \frac{1}{\tilde{\Sigma}_c} \int_{\tilde{\Sigma}_c} f \, d\tilde{\Sigma}_c \right)^2 \, d\tilde{\Sigma}_c = 0.
\]

\( \square \)

Next, we examine the relation between classical time evolution and symmetry reduction. Let \( \mu \in C^\infty(T^*M) \). For a \( G \)-equivariant diffeomorphism \( \Phi: T^*M \to T^*M \), we have

\[
\langle a \circ \Phi \rangle_G (\eta) = \int_G a(\Phi(g \cdot \eta)) \, dg = \int_G a(g \cdot \Phi(\eta)) \, dg = \langle a \rangle_G(\Phi(\eta)),
\]

so that \( \langle a \circ \Phi \rangle_G = \langle a \rangle_G \circ \Phi \) and consequently \( \langle a \circ \Phi \rangle_G = \langle (a \circ \Phi) \rangle_G \) holds. Now, we apply this result to the case \( \Phi = \varphi_t \), where \( \varphi_t \) is the Hamiltonian flow associated to the symbol function \( p \) of the Schrödinger operator. If \( i : \tilde{\Omega}_{\text{reg}} \hookrightarrow T^*M \) denotes the inclusion and \( \pi : \tilde{\Omega}_{\text{reg}} \to \tilde{\Omega}_{\text{reg}} \) the projection onto the \( G \)-orbit space, Theorem 2.2 says that \( \pi \circ \varphi_t \circ i = \tilde{\varphi}_t \circ \pi \). Since

\[
(a)_G \circ \varphi_t \circ i = (a)_G \circ \varphi_t \circ \pi, \quad (a)_G \circ i = (a)_G \circ \pi,
\]

we get

\[
(a)_G \circ \varphi_t \circ i = (a)_G \circ \varphi_t \circ \pi = (a)_G \circ i, \quad (a)_G \circ i = (a)_G \circ \varphi_t \circ i = (a)_G \circ \varphi_t \circ \pi,
\]

where we used that \( i \circ \varphi_t \circ i = \varphi_t \circ i \) holds as \( \varphi_t \) leaves \( \tilde{\Omega}_{\text{reg}} \) invariant. Since \( \pi \) is surjective, we have shown
Lemma 5.3. Let \( a \in C^\infty(T^*M) \) and \( \varphi_t \) be the flow on \( T^*M \) associated to the Hamiltonian \( p \). Let \( \tilde{\varphi}_t \) be the reduced flow on \( \tilde{\Omega}_{reg} \) associated to \( \tilde{p} \). Then time evolution and reduction commute:

\[
\langle (a)_G \circ \varphi_t \rangle = \langle (a)_G \circ \tilde{\varphi}_t \rangle.
\]

\( \square \)

6. Equivariant quantum ergodicity

We are now ready to formulate our first quantum ergodic theorem based on symmetry reduction.

Theorem 6.1 (Integrated equivariant quantum ergodicity). Consider a compact connected Lie group \( G \) acting effectively and isometrically on a closed connected Riemannian manifold \( M \). Let \( \{ u_j(h) \}_{j \in \mathbb{N}} \) be a Hilbert basis in \( L^2(M) \) of eigenfunctions of a Schrödinger operator \( P(h) \) with \( G \)-invariant symbol function \( p \), and denote the eigenvalue associated to \( u_j(h) \) by \( E_j(h) \). Let \( c \in \mathbb{R} \) be a regular value of \( p \), and suppose that the reduced flow \( \tilde{\varphi}_t \) corresponding to the reduced symbol function \( \tilde{p} \) is ergodic on \( \tilde{\Sigma}_c := \tilde{p}^{-1}(\{ c \}) \). Let \( A \in \Psi^h(M) \) be a semiclassical pseudodifferential operator with principal symbol \( \sigma(A) = [a] \), such that \( a \) is independent of \( h \). Then, for each \( \chi \in \tilde{G} \) and \( \beta \in (0, \frac{1}{2(n+2)}) \),

\[
\lim_{h \to 0} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, u_j(h) \in L^2(M), \ E_j(h) \in [c,c+h^\beta]} \left| \langle Au_j(h), u_j(h) \rangle_{L^2(M)} - \int_{\tilde{\Sigma}_c \cap \Omega_{reg}} a \frac{d\mu_c}{vol_G} \right|^2 = 0.
\]

The integral in (6.1) can also we written as \( \int_{\tilde{\Sigma}_c} \tilde{(a)}_G d\tilde{\Sigma}_c \).

Proof. We shall adapt the existing proofs of quantum ergodicity to the equivariant situation, following mainly [39, Theorem 15.4] and also [12, Theorem 5 in Appendix D]. For simplicity, we shall again write \( u_j(h) = u_j \) and \( E_j(h) = E_j \). Let \( \varrho \in C^\infty_c(\mathbb{R},[0,1]) \) be such that \( \varrho \equiv 1 \) in a neighbourhood of \( c \). Without loss of generality we may assume for the rest of the proof that \( h \) is small enough so that \( \varrho \equiv 1 \) on \( [c,c+h^\beta] \). Set

\[
B := \varrho(P(h)) \circ (A - \alpha \mathbf{1}_{L^2(M)}), \quad \alpha := \int_{\tilde{\Sigma}_c \cap \Omega_{reg}} a \frac{d\mu_c}{vol_G} = \int_{\tilde{\Sigma}_c} \tilde{(a)}_G d\tilde{\Sigma}_c,
\]

where \( \tilde{(a)}_G \) was defined in (3.2). Note that by Theorems 2.9 and 2.7 we have \( B \in \Psi^h(M) \). Furthermore, by Theorem 2.7

\[
\sigma(B) = \sigma(\varrho \circ P(h)) \sigma(A - \alpha \mathbf{1}_{L^2(M)}) = [(\varrho \circ p)(a - \alpha \mathbf{1}_{T^*M})] \in S^{-\infty}(M)/hS^{-\infty}(M).
\]

Let us write \( b := (\varrho \circ p)(a - \alpha \mathbf{1}_{T^*M}) \), so that \( \sigma(B) = [b] \). Clearly,

\[
\tilde{(\varrho)_G} = (\varrho \circ p)(\langle a \rangle_G - \alpha \mathbf{1}_{T^*M}) \equiv (\varrho \circ \tilde{p})(\langle a \rangle_G - \alpha \mathbf{1}_{T^*M}) = (\varrho \circ \tilde{p})(\tilde{\langle a \rangle}_G - \alpha \mathbf{1}_{\tilde{\Omega}_{reg}}).
\]

Next, we define

\[
\varepsilon(h) := (2\pi)^{n-\kappa} h^{n-\kappa-\beta} \sum_{j \in \mathbb{N}, u_j \in L^2(M), \ E_j \in [c,c+h^\beta]} \left| \langle Bu_j, u_j \rangle_{L^2(M)} \right|^2.
\]

By the spectral theorem, \( \varrho(P(h))u_j = u_j \) for \( E_j \in [c,c+h^\beta] \), since \( \varrho \equiv 1 \) on \( [c,c+h^\beta] \). Taking into account the self-adjointness of \( \varrho(P(h)) \) one sees that

\[
\langle Bu_j, u_j \rangle_{L^2(M)} = \langle Au_j, u_j \rangle_{L^2(M)} - \alpha.
\]

Consequently, we will be done with the proof if we can show that

\[
\lim_{h \to 0} \varepsilon(h) = 0.
\]

We proceed by defining the time evolution operator

\[
F^h(t) : L^2(M) \to L^2(M), \quad F^h(t) := e^{-itP(h)/h}, \quad t \in \mathbb{R},
\]
which by Stone’s theorem is a well-defined bounded operator. One then sets
\[ B(t) := F^h(t)^{-1} B F^h(t). \]
In order to make use of classical ergodicity, one notes that the expectation value
\[
\langle Bu_j, u_j \rangle_{L^2(M)} = \left\langle B e^{-itE_j/h} u_j, e^{-itE_j/h} u_j \right\rangle_{L^2(M)} = \left\langle B e^{-itP(h)/h} u_j, e^{-itP(h)/h} u_j \right\rangle_{L^2(M)}
\]
\[
= \langle B(t) u_j, u_j \rangle_{L^2(M)} \quad \forall \ t \in [0, T]
\]
is actually time-independent. This implies
\[
\langle Bu_j, u_j \rangle_{L^2(M)} = \langle \langle B \rangle_T u_j, u_j \rangle_{L^2(M)},
\]
where we set \( \langle B \rangle_T = \frac{1}{T} \int_0^T B(t) dt \in \Psi_h^{-\infty}(M) \). Taking into account \( \|u_j\|_{L^2(M)}^2 = 1 \) and the Cauchy-Schwarz inequality one arrives at
\[
\left| \langle Bu_j, u_j \rangle_{L^2(M)} \right|^2 \leq \|\langle B \rangle_T u_j\|_{L^2(M)}^2.
\]
We therefore conclude from (6.4) that
\[
\varepsilon(h) \leq (2\pi)^{n-k} h^{n-k-\beta} \sum_{j \in \mathbb{N}, u_j \in L^2(M), E_j \in [c,c+h^\beta]} \left| \langle B^* \rangle_T \langle B \rangle_T u_j, u_j \rangle_{L^2(M)} \right|.
\]
Next, let \( \overline{B}(t) \) be a representant of the equivalence class \( \text{Op}_h(\sigma(B) \circ \varphi_t) \). By the weak Egorov theorem [39, Theorem 15.2] one has
\[
\|B(t) - \overline{B}(t)\|_{B(L^2(M))} = O(h) \quad \text{uniformly for} \ t \in [0, T],
\]
which implies
\[
\langle B \rangle_T = \langle \overline{B} \rangle_T + O_T^0_{B(L^2(M))}(h).
\]
From the definition of \( \overline{B} \) we get
\[
\sigma \left( \langle \overline{B} \rangle_T \right) = \left[ \frac{1}{T} \int_0^T b \circ \varphi_t dt \right].
\]
Furthermore, the symbol map is a \( * \)-algebra homomorphism from \( \Psi_h^{-\infty}(M) \) to \( S^{-\infty}(M)/hS^{-\infty}(M) \), with involution given by the adjoint operation and pointwise complex conjugation on representants, respectively. That leads to
\[
\sigma \left( \langle \overline{B}^* \rangle_T \langle \overline{B} \rangle_T \right) = \left[ \frac{1}{T} \int_0^T b \circ \varphi_t dt \right]^2.
\]
Now, note that by Lemma 5.3
\[
(2\pi)^{n-k} h^{n-k-\beta} \sum_{j \in \mathbb{N}, u_j \in L^2(M), E_j \in [c,c+h^\beta]} \left| \langle \overline{B}^* \rangle_T \langle \overline{B} \rangle_T u_j, u_j \right|_{L^2(M)}
\]
\[
= d_x[\pi\chi|\eta \cdot 1] \int_{S_x} |\langle \overline{b} \rangle_{G^T}|^2 dS_x + O(h^{1/2\pi} e^{-\beta}(\log h)^{\Lambda - 1}).
\]
From (6.3) we see that over $\Sigma_c = \tilde{\rho}^{-1}(\{c\})$ we have $\langle \tilde{b}_c \rangle_{\Sigma_c} = \langle \tilde{a}_c \rangle_{\Sigma_c} - \alpha \cdot 1_{\Sigma_c} =: \tilde{b}_c$. With (6.7), (6.8) and (6.10) we deduce
\[
\varepsilon(h) \leq d_\chi[\pi_h] : 1 \int_{\Sigma_c} |\tilde{b}_c|_T|^2 \, d\Sigma_c \leq O(h^{\frac{1}{1+n-\kappa}})(\log h^{-1})^{\Lambda-1} \\
+ (2\pi)^{n-\kappa} h^{n-\kappa-\beta} \#\{j \in \mathbb{N}, E_j \in [c, c+h], u_j \in L^2_x(M)\} \cdot O(h).
\]
By the equivariant semiclassical Weyl law, Theorem 4.2, the factor in front of the $O(h)$-estimate is convergent and therefore bounded as $h \to 0$. Thus,
\[
(6.11) \quad \lim\sup_{h \to 0} \varepsilon(h) \leq d_\chi[\pi_h] : 1 \int_{\Sigma_c} |\tilde{b}_c|_T|^2 \, d\Sigma_c.
\]
This is now the point where symmetry-reduced classical ergodicity is used. Since $\tilde{b}_c$ fulfills $\int_{\Sigma_c} \tilde{b}_c \, d\Sigma_c = 0$, Theorem 5.2 yields $\lim_{T \to \infty} \int_{\Sigma_c} |\tilde{b}_c|_T|^2 \, d\Sigma_c = 0$. Because the left hand side of (6.11) is independent of $T$, it follows that it must be zero, yielding (6.6).

**Remark 6.2.** Note that one could have still exhibited the Weyl remainder estimate in (6.11). But since the rate of convergence in Theorem 5.2 is unknown in general, it is not possible to give a remainder estimate in Theorem 6.1 with the methods employed here.

**Remark 6.3.** A weaker version of Theorem 6.1 can be proven with a less sharp energy localization in an interval $[r, s]$ with $r < s$ by the same methods employed here. In fact, under the additional assumption that the mean value $\alpha$ introduced in (6.2) is the same for all $c \in [r, s]$ and all considered $c$ are regular values of $p$, the reduced flow being ergodic on each of the contemplated hypersurfaces $\Sigma_c$, one can show that
\[
\lim_{h \to 0} h^{n-\kappa} \sum_{j \in \mathbb{N}, u_j(h) \in L^2_x(M), E_j(h) \in [r, s]} \left| \langle Au_j(h), u_j(h) \rangle_{L^2(M)} - \int_{p^{-1}([r,s]) \cap \Omega_{\text{reg}}} a \frac{d\Omega_{\text{reg}}}{\text{vol}} \right|^2 = 0.
\]
The proof of this relies on a corresponding semiclassical Weyl law for the interval $[r, s]$
\[
(2\pi h)^{n-\kappa} \sum_{j \in \mathbb{N}, u_j(h) \in L^2_x(M), E_j(h) \in [r, s]} \langle Bu_j(h), u_j(h) \rangle_{L^2(M)} = d_\chi[\pi_h] : 1 \int_{p^{-1}([r,s]) \cap \Omega_{\text{reg}}} \frac{b}{\text{vol}} \, d\Omega_{\text{reg}} + O(h^{\frac{1}{1+n-\kappa}})(\log h^{-1})^{\Lambda-1},
\]
which is proven analogously to Theorem 4.1. The point is that for this weaker statement no remainder estimate in Weyl’s law is necessary, since the rate of convergence $h^{n-\kappa}$ is the one of the leading term in Weyl’s law. Thus, in principle, this weaker result could have also been obtained using heat kernel methods as in [11] or [6]. Nevertheless, for the stronger version of equivariant quantum ergodicity proven in Theorem 6.1 remainder estimates in Weyl’s law, and in particular the results of [27], are necessary due to the lower rate of convergence $h^{n-\kappa-\beta}$.

In what follows, we shall use our previous results to prove a symmetry-reduced quantum ergodicity theorem for Schrödinger operators. Again, remainder estimates in Weyl’s law, and in particular the results of [27], are necessary.

**Theorem 6.4 (Equivariant quantum ergodicity for Schrödinger operators).** With the notation and assumptions as in Theorem 6.1, let $\chi \in \mathcal{G}$, $\beta \in (0, \frac{1}{2(n+2)})$ be fixed, and set
\[
J^\chi(h) := \{ j \in \mathbb{N} : E_j(h) \in [c, c+h], u_j(h) \in L^2_x(M) \}.
\]
Then there is a $h_0 > 0$ such that for each $h \in (0, h_0]$ we have a subset $\Lambda^\chi(h) \subset J^\chi(h)$ satisfying
\[
(6.12) \quad \lim_{h \to 0} \frac{\#\Lambda^\chi(h)}{\#J^\chi(h)} = 1.
\]
such that for each semiclassical pseudodifferential operator $A \in \Psi^0_h(M)$ with principal symbol $\sigma(A) = [a]$ given by an $h$-independent symbol function $a$ the following holds. For all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ such that

$$
(6.13) \quad \left| \langle Au_j(h), u_j(h) \rangle_{L^2(M)} - \int_{\Sigma_c \cap \Omega_{reg}} a \frac{d\mu_c}{vol_G} \right| < \varepsilon \quad \forall j \in \Lambda^\chi(h), \forall h \in (0, h_\varepsilon],
$$

where the integral in (6.13) equals $\int_{\Sigma_c} \langle \hat{a} \rangle_G d\tilde{\Sigma}_c$.

Remark 6.5. Note that the limit in (6.13) does not contain any representation theoretic quantity, contrasting with the form of the leading term in the equivariant Weyl law of Theorem 4.1.

Proof. For simplicity, we shall again write $u_j(h) = u_j$ and $E_j(h) = E_j$ in the following. By Corollary 4.3, we can choose a $h_0 := h \chi > 0$ such that $J^\chi(h) \neq \emptyset$ for all $h \in (0, h_0)$, and in what follows we shall suppose that $h \in (0, h_0)$. With the notation as in (3.2), we set for any smooth function $s$ on $T^*M$

$$
\alpha(s) := \int_{\Sigma_c} \langle \hat{s} \rangle_G d\tilde{\Sigma}_c.
$$

Let $\tau \in C^\infty_c([0, 1])$ be such that $\tau \equiv 1$ in a neighborhood of $c$. Without loss of generality, we assume for the rest of the proof that $h_0$ is small enough so that $\tau \equiv 1$ on $[c, c + h^\beta_0]$. Now, for any operator $A$ as in the statement of the theorem set

$$
B := A - \alpha(a) \tau (P(h)).
$$

From Theorem 2.9 we know that the principal symbol of $B$ is given by $\sigma(B) = [b]$ with $b := a - \alpha(a) \tau \circ \theta \circ \rho$. Clearly, $\alpha(b) = 0$, since $\tau \circ \rho \equiv 1$ on $\tilde{\Sigma}_c$. Let us now assume that the statement of the theorem holds for all operators $A$ with $\alpha(a) = 0$. Then, there is a sequence of subsets $\Lambda^\chi(h)$ of density 1 such that for all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0)$ such that

$$
(6.14) \quad \left| \langle Bu_j, u_j \rangle_{L^2(M)} \right| < \varepsilon \quad \forall h \in (0, h_\varepsilon), \forall j \in \Lambda^\chi(h).
$$

Due to the choice of the function $\tau$ we have $\tau(P(h))(u_j) = u_j$ for all $u_j$ with $E_j \in [c, c + h^\beta]$. Consequently, (6.14) implies that for all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0)$ such that

$$
\left| \langle Au_j, u_j \rangle_{L^2(M)} - \alpha(a) \right| < \varepsilon \quad \forall h \in (0, h_\varepsilon), \forall j \in \Lambda^\chi(h),
$$

and we obtain the statement of the theorem for general $A$. We are therefore left with the task of proving (6.14) for arbitrary operators $B$ with $\alpha(b) = 0$, and shall proceed in a similar fashion to parts 1 - 5 of the proof of [39, Theorem 15.5], pointing out only the main arguments. By Theorem 6.1 we have for fixed $B$

$$
|Bu_j, u_j)_{L^2(M)}|^2 \geq r(h) \rightarrow 0 \quad h \rightarrow 0
$$

as $h \rightarrow 0$. One then defines the $B$-dependent subsets $\Lambda^\chi(h) = J^\chi(h) - \{j \in J^\chi(h) : |\langle Bu_j, u_j \rangle_{L^2(M)}|^2 \geq \sqrt{r(h)}\}$, and verifies that (6.14) is fulfilled for these particular $\Lambda^\chi(h)$ and $B$ by taking into account Theorem 4.2. Consider now a family $\{A_k\}_{k \in \mathbb{N}}$ of semiclassical pseudodifferential operators in $\Psi^0_h(M)$ whose principal symbols are given by $h$-independent symbol functions. By our previous considerations, for each $k$ there is a sequence of subsets $\Lambda^\chi_k(h) \subseteq J^\chi(h)$ such that (6.12) and (6.13) hold for each particular $A_k$ and $\Lambda^\chi_k(h)$. One then shows that for sufficiently small $h$ there is a sequence of subsets $\Lambda^\chi_k(h) \subseteq J^\chi(h)$ of density 1 such that $\Lambda^\chi_k(h) \subseteq \Lambda^\chi_k(h)$. Hence, the theorem is true for countable families of operators. To obtain it in general, one constructs a sequence of operators $\{A_k\}_{k \in \mathbb{N}}$ whose symbols $\sigma(A_k) = [a_k]$ are given by $h$-independent symbol functions $a_k$, and which is dense in $\mathcal{P}$

$$
\mathcal{P} := \{A \in \Psi^\infty_h(M) : \sigma(A) = [a], a \text{ is } h\text{-independent}\}
$$

Note that we do not need Zworski’s technical condition that the value of the integral $\int_{\Sigma_c} \hat{a} d\tilde{\Sigma}_c$ must stay the same when varying $c$ in some interval, which slightly simplifies the proof.
in the sense that for given $A \in \Psi_h^{-\infty}(M)$ and $\varepsilon > 0$ there exists a $k$ such that
\[
\|A - A_k\|_{L^2(M)} < \varepsilon,
\]
for sufficiently small $h$. To construct the sequence $\{A_k\}_{k \in \mathbb{N}}$, note that for two symbol functions $a$ and $b$ and the corresponding quantizations $A$ and $B$, one has
\[
\|A - B\|_{L^2(M)} \leq \|a - b\|_{L^\infty(M)} + C\sqrt{h},
\]
Consequently, we only need to find a sequence in (6.15) such that for each symbol function $C$ whose equivalence class in $S^{-\infty}(M)/(hS^{-\infty}(M))$ is the principal symbol of a pseudodifferential operator in $\mathcal{P}$, we have for each $\varepsilon > 0$ an index $k \in \mathbb{N}$ satisfying
\[
\|a - a_k\|_{L^\infty(M)} < \varepsilon.
\]
To define such a sequence, note that $C^\infty_c(T^*M)$ is dense in the Banach space $C_0(T^*M) \supset S^{-\infty}(M)$ of continuous functions vanishing at infinity. Since $C^\infty_c(T^*M) \subset S^{-\infty}(M)$ is separable, we can find a dense sequence $\{a_k\}_{k \in \mathbb{N}} \subset C^\infty_c(T^*M)$ with respect to the supremum norm, which is the desired sequence. This proves the theorem for operators $A \in \Psi_h^{-\infty}(M)$. Finally, if $A \in \Psi_h^0(M)$ is a general operator, one multiplies $A$ with the smoothing operator $\varrho(P(h))$, where $\varrho \in C^\infty_c(\mathbb{R})$ equals 1 near $c$. This completes the proof of the theorem.

Remark 6.6. Theorem 6.4 generalizes easily to the case where finitely many isotypic components are considered all at once. Replacing $L^2(M)$ by $L^2_G(M) := \bigoplus_{\chi \in \mathcal{G}} L^2_{\chi}(M)$ for some finite $\mathcal{G} \subset \hat{G}$, one obtains sets $\Lambda \mathcal{G}(h)$, $\mathcal{J} \mathcal{G}(h)$ for which statements analogous to (6.12) and (6.13) hold. The more general version follows directly from Theorem 6.4 and the observation that the union of a finite family of density $1$ subsequences $\{u_{n_k}\}_{k \in \mathbb{N}}, \ldots, \{u_{n_k}\}_{k \in \mathbb{N}}$ has density $1$ in $\bigcup_{n=1}^{\infty} \{u_{n_k}\}_{k \in \mathbb{N}}$. For details, see [21].

7. Equivariant quantum limits for the Laplace-Beltrami operator

We shall now apply the semiclassical results from the previous section to study the distribution of eigenfunctions of the Laplace-Beltrami operator on a closed connected Riemannian $G$-manifold $M$ in the limit of large eigenvalues, $G$ being a compact connected Lie group acting isometrically and effectively on $M$. In what follows, let $\Delta$ be the unique self-adjoint extension of the Laplace-Beltrami operator $\hat{\Delta}$ on $M$, and choose an orthonormal basis $\{u_j\}_{j \in \mathbb{N}}$ of $L^2(M)$ of eigenfunctions of $-\Delta$ with corresponding eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$, repeated according to their multiplicity. Consider further the Schrödinger operator $\hat{P}(h) = -\hbar^2 \hat{\Delta} + V$ with $V \equiv 0$ and principal symbol defined by the symbol function $p = \|\cdot\|_{T^*M}^2$. Clearly, each $c > 0$ is a regular value of $p$. The self-adjoint extension of $\hat{P}(h)$ is given by $P(h) = -\hbar^2 \Delta$ and each $u_j$ is an eigenfunction of $P(h)$ for each $h > 0$ with eigenvalue $E_j(h) = \hbar^2 E_j$. Furthermore, under the identification $T^*M \simeq TM$ given by the Riemannian metric, the Hamiltonian flow $\varphi_t$ induced by $p$ corresponds to the geodesic flow of $M$. Since $V \equiv 0$, the dynamics of the reduced geodesic flow $\tilde{\varphi}_t$ are equivalent on any two hypersurfaces $\Sigma_c$ and $\tilde{\Sigma}_c$. In the following, we therefore choose $c = 1$ without loss of generality. That means we will call the reduced geodesic flow ergodic if it is ergodic on $\tilde{\Sigma}_1 = \tilde{\varphi}^{-1}(\{1\})$.

We are now prepared to state and prove an equivariant version of the classical Shnirelman-Zelditch-Colin-de-Verdière quantum ergodicity theorem [31, 37, 9].

Theorem 7.1 (Equivariant quantum limits for the Laplacian). With the notation as above, assume that the reduced geodesic flow is ergodic. Choose $\chi \in \hat{G}$ and let $\{u_j^\chi\}_{j \in \mathbb{N}}$ be an orthonormal basis in $L^2_{\chi}(M)$ of eigenfunctions of $-\Delta$. Then, there is a subsequence $\{u_{n_k}^\chi\}_{k \in \mathbb{N}}$ of density $1$ in $\{u_j^\chi\}_{j \in \mathbb{N}}$ such that for all $s \in C^\infty(S^* M)$ one has
\[
\langle \text{Op}(s)u_{n_k}^\chi, u_{n_j}^\chi \rangle_{L^2(M)} \rightarrow \int_{S^* M \cap \Omega_{reg}} s \frac{d\mu}{\text{vol}G} \quad \text{as} \quad k \rightarrow \infty,
\]
where we wrote \( \mu \) for \( \mu_1 \).

**Remark 7.2.** The integral in (7.1) can also be written as \( \int_{S^*\tilde{M}_{\text{reg}}} s' \, d(S^*\tilde{M}_{\text{reg}}) \), where \( s' \in C(S^*\tilde{M}_{\text{reg}}) \) is the function corresponding to \( \langle s \rangle_G \) under the diffeomorphism \( \Sigma_1 \simeq S^*\tilde{M}_{\text{reg}} \) up to a null set, and \( d(S^*\tilde{M}_{\text{reg}}) \) is the Liouville measure on the unit co-sphere bundle. The expression of density 1 means
\[
\lim_{m \to \infty} \frac{\# \{ k, j_k \leq m \}}{m} = 1.
\]

**Proof.** First, we extend \( s \) to a function \( \tilde{s} \in S^0(M) \subset C^\infty(T^*M) \) with \( \tilde{s}|_{S^*M} = s \) as follows. Set \( \tilde{s}(x, \xi) := s(x, \xi/\|\xi\|_x) \) for \( x \in M, \xi \in T^*_xM - \{0\} \). Choose a small \( \delta > 0 \) and a smooth cut-off function \( \varphi : T^*(M) \to [0, 1] \) with
\[
\varphi(x, \xi) = \begin{cases} 1 & \forall x \in M, \forall \xi \in T^*_xM \text{ with } \|\xi\|_x \geq 1 - \delta, \\ 0 & \forall x \in M, \forall \xi \in T^*_xM \text{ with } \|\xi\|_x \leq \delta. \end{cases}
\]

Now set \( \tilde{s}(x, \xi) := \varphi(x, \xi) \tilde{s}(x, \xi) \) for \( \xi \in T^*_xM - \{0\} \) and \( \tilde{s}(x, 0) := 0 \). Then \( \text{Op}(\tilde{s}) \) is a pseudodifferential operator in \( \Psi^0(M) \). Because \( \tilde{s} \) is polynomoidal of degree 0 and therefore independent of \( \|\xi\| \) for large \( \xi \), the ordinary non-semiclassical quantization \( \text{Op}(\tilde{s}) \) differs only by an operator in \( h^\infty \Psi^0(M) \) from the semiclassical pseudodifferential operator \( \text{Op}_h(\tilde{s}) \in \Psi^0_h(M) \) with principal symbol \( \sigma(\text{Op}_h(\tilde{s})) = [\tilde{s}] \). Thus, we can apply Theorem 6.4 to \( P(h) = -h^2\Delta \) and we are allowed to replace \( \text{Op}_h(\tilde{s}) \) by \( \text{Op}(\tilde{s}) \) in the results. As in Theorem 6.4 fix some \( \beta > 0 \), and let \( \{E^h_j\}_{j \in \mathbb{N}} \) be the eigenvalues associated to the eigenfunctions \( \{u^h_j\}_{j \in \mathbb{N}} \). Set
\[
J^\chi(h) = \left\{ j \in \mathbb{N}; h^2E^h_j \in [1, 1 + h^2] \right\} = \left\{ j \in \mathbb{N}; E^h_j \in \left[ \frac{1}{h^2}, \frac{1}{h^2} + \frac{1}{h^2 - \beta} \right] \right\}.
\]

By Theorem 6.4 there is a number \( h_0 > 0 \) together with subsets \( \Lambda^\chi(h) \subset J^\chi(h) \), where \( h \in (0, h_0] \), satisfying
\[
(7.2) \quad \lim_{h \to 0} \frac{\# \Lambda^\chi(h)}{\# J^\chi(h)} = 1,
\]
and for each \( s \in C^\infty(S^*M) \) and arbitrary \( \varepsilon > 0 \) there is a \( h_\varepsilon \in (0, h_0] \) such that
\[
(7.3) \quad \left| \langle \text{Op}(\tilde{s})u^h_j, u^h_{j'} \rangle_{L^2(M)} - \int_{\Sigma_1} \langle \tilde{s} \rangle_G d\tilde{\Sigma}_1 \right| < \varepsilon \quad \forall j \in \Lambda^\chi(h), \forall h \in (0, h_\varepsilon].
\]

Now, due to the discreteness of the set \( \{E^h_j\}_{j \in \mathbb{N}} \) in \( \mathbb{R} \), it is possible to find a strictly decreasing sequence \( \{h_i\}_{i \in \mathbb{N}} \subset (0, h_0] \) including \( h_0 \) such that
\[
(7.4) \quad J^\chi(h_i) \cap J^\chi(h_{i'}) = \emptyset \quad \text{for } i \neq i', \quad J^\chi := \left\{ j \in \mathbb{N}, E^h_j \geq \frac{1}{h_0^2} \right\},
\]
where we set
\[
J^\chi := \bigcup_{i=1}^{\infty} J^\chi(h_i), \quad \Lambda^\chi := \bigcup_{i=1}^{\infty} \Lambda^\chi(h_i).
\]

Writing \( \{j_k\}_{k \in \mathbb{N}} := \Lambda^\chi \) we deduce from (7.2) that
\[
(7.5) \quad \lim_{i \to \infty} \frac{\# \left\{ k : \frac{1}{h_i^2} \leq E^h_k \leq \frac{1}{h_i^2} + \frac{1}{h_i^2 - \beta} \right\}}{\# \left\{ j : \frac{1}{h_i^2} \leq E^h_j \leq \frac{1}{h_i^2} + \frac{1}{h_i^2 - \beta} \right\}} = 1.
\]

We now have the following
Lemma 7.3. Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be sequences of real numbers such that \(0 < a_n \leq b_n\) for all \(n\), and \(\liminf_{n \to \infty} b_n > 0\), \(\lim_{n \to \infty} \frac{a_n}{b_n} = 1\). Then
\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} a_n}{\sum_{n=1}^{N} b_n} = 1.
\]

Proof. Let \(\varepsilon > 0\) be arbitrary and choose \(N_\varepsilon \in \mathbb{N}\) such that \(\frac{a_n}{b_n} \geq 1 - \varepsilon\) for each \(n \geq N_\varepsilon\). Then one computes for \(N > N_\varepsilon\)
\[
\sum_{n=1}^{N} \frac{a_n}{b_n} = \frac{\sum_{n=1}^{N_\varepsilon-1} a_n + \sum_{n=N_\varepsilon}^{N} a_n}{\sum_{n=1}^{N_\varepsilon-1} b_n} \geq \frac{\sum_{n=1}^{N_\varepsilon-1} a_n + (1 - \varepsilon) \sum_{n=N_\varepsilon}^{N} b_n}{\sum_{n=1}^{N_\varepsilon-1} b_n + \sum_{n=N_\varepsilon}^{N} b_n} \to 1 - \varepsilon \quad \text{as} \quad N \to \infty.
\]

We therefore conclude
\[
1 \geq \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} a_n}{\sum_{n=1}^{N} b_n} \geq \liminf_{N \to \infty} \frac{\sum_{n=1}^{N} a_n}{\sum_{n=1}^{N} b_n} = 1 - \varepsilon,
\]
and the lemma follows. \(\square\)

With the previous lemma we deduce from (7.5) the equality
\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} \# \{ k : \frac{1}{h_k} \leq E_{j_k} \leq \frac{1}{h_k} + \frac{1}{h_k^2 - \sigma} \}}{\sum_{n=1}^{N} \# \{ j : \frac{1}{h_j} \leq E_j \leq \frac{1}{h_j} + \frac{1}{h_j^2 - \sigma} \}} = 1.
\]

In view of (7.4), this is equivalent to
\[
\lim_{h \to 0} \frac{\# \{ k : \frac{1}{h_k} \leq E_{j_k} \leq \frac{1}{h_k} + \frac{1}{h_k^2 - \sigma} \}}{\# \{ j : \frac{1}{h_j} \leq E_j \leq \frac{1}{h_j} + \frac{1}{h_j^2 - \sigma} \}} = 1,
\]
and directly implies
\[
\lim_{E \to \infty} \frac{\# \{ k : E_{j_k} \leq E \}}{\# \{ j : E_j \leq E \}} = 1, \quad \lim_{j \to \infty} \frac{\# \{ k : j_k \leq j \}}{j} = 1,
\]
where we took into account that \(\{E_{j_k}\}_{j \in \mathbb{N}}\) and \(\{E_j^X\}_{k \in \mathbb{N}}\) are unbounded increasing sequences. Now, by construction \(\widehat{\langle s \rangle}_G|_{\Sigma_1} = \langle \widehat{s} \rangle_G\), and by Lemma A.10 we have
\[
\int_{\Sigma_1} \langle \widehat{s} \rangle_G d\Sigma_1 = \int_{S^* \widehat{M}_{\text{reg}}} s^* d(S^* \widehat{M}_{\text{reg}}).
\]

From Theorem 6.4 and (7.3) we therefore conclude that the sequence \(\{u_{j_k}\}_{k \in \mathbb{N}}\) associated to \(\{E_{j_k}\}_{k \in \mathbb{N}}\) fulfills (7.1), completing the proof of Theorem 7.1. \(\square\)

Recall that a sequence of measures \(\mu_j\) on a metric space \(X\) is said to converge weakly to a measure \(\mu\), if for all bounded and continuous functions \(f\) on \(X\) one has
\[
\int_X f d\mu_j \to \int_X f d\mu \quad \text{as} \quad j \to \infty.
\]
Projecting from \(S^* M \cap \Omega_{\text{reg}}\) onto \(M\) we now obtain
Corollary 7.4 (Equidistribution of eigenfunctions of the Laplacian). In the situation of Theorem 7.1, we have the weak convergence of measures
\[ (7.6) \quad \langle |u_{jk}^x|^2 \rangle_G \xrightarrow{k \to \infty} \left( \frac{\text{vol}_{M/G}}{\text{vol}_O} M/G \right)^{-1} \frac{dM/G}{\text{vol}_O} \quad \text{as } k \to \infty. \]

Proof. First, we show that the integrals corresponding to the measures converge for smooth functions. Let \( \pi : T^*M \to M \) be the co-tangent bundle projection and consider for \( f \in C^\infty(M) \) the pseudodifferential operator \( \text{Op}(f \circ \pi) \), which corresponds to pointwise multiplication with \( f \) up to lower order terms. Since the Sasaki metric on \( T^*M \) projects onto the Riemannian metric on \( M \) and is fiber-wise just the Euclidean metric, and the Sasaki metric induces \( d\mu \), we have
\[ \int_{S^*M \cap \Omega} f \circ \pi \frac{d\mu}{\text{vol}_O} = \int_{M} f \frac{dM}{\text{vol}_O}, \]
because the orbit volume functions on \( M \) and \( S^*M \cap \Omega \) are compatible, see [21]. Consequently, the assertion follows for smooth functions directly from Theorem 7.1. It remains to show (7.6) for functions which are only continuous. Thus, let \( r \in C(M) \), and choose some \( \varepsilon > 0 \). By the density of \( C^\infty(M) \) in \( C(M) \), we can find a function \( s \in C^\infty(M) \) such that \( \|r - s\|_{\infty} < \varepsilon \). Then for all \( j \in \mathbb{N} \) we have
\[ \left| \int_{M} r(x)|u_{jk}^x|^2 \, dM(x) - \int_{M} s(x)|u_{jk}^x|^2 \, dM(x) \right| \leq \int_{M} |r(x) - s(x)||u_{jk}^x|^2 \, dM(x) \leq \|r - s\|_{\infty} \|u_{jk}^x\|_{L^2} < \varepsilon, \]
as well as \[ \left| \int_{M} r \frac{dM}{\text{vol}_O} - \int_{M} s \frac{dM}{\text{vol}_O} \right| \leq \|r - s\|_{\infty} < \varepsilon. \]
Since \( \varepsilon \) was arbitrary, we are done. \( \square \)

Corollary 7.4 leads to a statement about weak convergence of measures on the topological Hausdorff space \( M/G \), and we obtain

Corollary 7.5. In the situation of Theorem 7.1, there is the weak convergence of measures on \( M/G \)
\[ (7.7) \quad \langle |u_{jk}^x|^2 \rangle_G \xrightarrow{k \to \infty} \left( \frac{\text{vol}_{M/G}}{\text{vol}_{O}} M/G \right)^{-1} \frac{dM/G}{\text{vol}_O} \quad \text{as } k \to \infty. \]

Proof. Let \( f \in C(M/G) \) and let \( \pi : M \to M/G \) be the canonical projection. Then \( f \) lifts to the \( G \)-invariant function \( \widetilde{f} := f \circ \pi \in C(M) \) which by Lemma A.2 fulfills
\[ \int_{M} \widetilde{f}(x)|u_{jk}^x(x)|^2 \, dM(x) = \int_{M_{\text{reg}}} \widetilde{f}(x)|u_{jk}^x(x)|^2 \, dM(x) = \int_{M_{\text{reg}}} \int_{G \cdot x} \widetilde{f}(x')|u_{jk}^x(x')|^2 \, d\mu_{G \cdot x}(x') \, d\widetilde{M}_{\text{reg}}(G \cdot x) \]
\[ = \int_{M_{\text{reg}}} f(G \cdot x) \int_{G \cdot x} |u_{jk}^x(x')|^2 \, d\mu_{G \cdot x}(x') \, d\widetilde{M}_{\text{reg}}(G \cdot x) \]
\[ = \int_{M_{\text{reg}}} f(G \cdot x) \text{vol}(G \cdot x) \int_{G} |u_{jk}^x(g \cdot x)|^2 \, dg \, d\widetilde{M}_{\text{reg}}(G \cdot x), \quad \text{by (2.12)} \]
\[ = \int_{M_{\text{reg}}} f(G \cdot x) \langle |u_{jk}^x|_G^2 \rangle_G (G \cdot x) \, d\widetilde{M}_{M/G}(G \cdot x), \quad \text{by Lemma A.6} \]
Moreover, we have \( \int_{M} \frac{dM}{\text{vol}_O} = \int_{M_{\text{reg}}} f \, d\widetilde{M}_{\text{reg}} = \int_{M_{\text{reg}}} f \, d\widetilde{M}_{\text{reg}} = \int_{M_{\text{reg}}} f \, d\widetilde{M}_{\text{reg}} \) by Lemmas A.2, A.6 and Corollary A.3.

The claim now follows from Corollary 7.4.

We can understand the result of Corollary 7.5 better using elementary representation theory.
Lemma 7.6. Let $V \subset L^2(M)$ be an irreducible $G$-module of class $\chi \in \check{G}$. Let further $\{v_1, \ldots, v_{d_\chi}\}$ denote an $L^2$-orthonormal basis of $V$, and $a \in V \cap C^\infty(M)$ have $L^2$-norm equal to 1. Then, for any $x \in M,$

\begin{equation}
(7.8) \quad \langle |a|^2 \rangle_G(x) = d_\chi^{-1} \sum_{k=1}^{d_\chi} |v_k(x)|^2.
\end{equation}

In particular, the function

$$\Theta_V : M \to \mathbb{R}, \quad x \mapsto d_\chi^{-1} \sum_{k=1}^{d_\chi} |v_k(x)|^2,$$

is a $G$-invariant element of $C^\infty(M)$ that is independent of the choice of orthonormal basis, and the left hand side of $(7.8)$ is independent of the choice of $a$.

Proof. Since the left hand side of $(7.8)$ is clearly $G$-invariant, smooth, and independent of the choice of orthonormal basis, it suffices to prove $(7.8)$. Now, one has $a = \sum_{j=1}^{d_\chi} a_j v_j$ with $a_j \in \mathbb{C}$, $\sum_{j=1}^{d_\chi} |a_j|^2 = 1$, and

$$(L_g a)(x) = a(g^{-1} \cdot x) = \sum_{j=1}^{d_\chi} a_j v_j(g^{-1} \cdot x) = \sum_{j,k=1}^{d_\chi} a_j c_{jk}(g)v_k(x), \quad g \in G, \; x \in M,$$

where $\{c_{jk}\}_{1 \leq j, k \leq d_\chi}$ denote the matrix coefficients of the $G$-representation on $V$. This yields

$$\int_G |a(g^{-1} \cdot x)|^2 dg = \int_G a(g^{-1} \cdot x)\overline{a}(g^{-1} \cdot x) dg = \int_G \left( \sum_{j,k=1}^{d_\chi} a_j c_{jk}(g)v_k(x) \right) \left( \sum_{l,m=1}^{d_\chi} \overline{a_l} \overline{c_{lm}}(g)v_m(x) \right) dg,$$

and we obtain $(7.8)$ by taking into account the Schur orthogonality relations $[20$, Corollary 1.10]

$$\int_G c_{jk}(g) \overline{c_{lm}}(g) dg = d_\chi^{-1} \delta_{jl} \delta_{km},$$

and the fact that the substitution $g \mapsto g^{-1}$ leaves the Haar measure invariant. \qed

We can now restate Corollary 7.5 in representation-theoretic terms.

Theorem 7.7 (Representation-theoretic equidistribution theorem). Let $G$ be a compact connected Lie group, and $M$ a closed connected Riemannian manifold on which $G$ acts effectively by isometries. Assume that the reduced geodesic flow is ergodic and choose $\chi \in \check{G}$. By the spectral theorem, choose an orthogonal decomposition $L^2_\chi(M) = \bigoplus_{i\in\mathbb{N}} V^\chi_i$ into irreducible unitary $G$-modules of class $\chi$ such that each $V^\chi_i$ is contained in some eigenspace of the Laplace-Beltrami operator. As in Lemma 7.6 assign to each $V^\chi_i$ the $G$-invariant function $\Theta^\chi_i := \Theta_{V^\chi_i} : M \to [0, \infty)$, and regard it as a function on $M/G$. Then there is a subsequence $\{V^\chi_{i_k}\}_{k\in\mathbb{N}}$ of density 1 in $\{V^\chi_i\}_{i\in\mathbb{N}}$ such that one has the weak convergence of measures on $M/G$

\begin{equation}
(7.9) \quad \Theta^\chi_{i_k} d_{M/G} \xrightarrow{k \to \infty} \left( \frac{\text{vol}_{M/G}}{\text{vol}} \right)^{-1} \frac{d_{M/G}}{\text{vol}}, \quad \text{as } k \to \infty.
\end{equation}

Proof. Let $\{u^\chi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2_\chi(M)$ of eigenfunctions of $-\Delta$ chosen such that $V^\chi_i = \text{span} \{u^\chi_j : j \in J^\chi_i\}$, where $J^\chi_i := \{i d_\chi, \ldots, (i+1)d_\chi - 1\}$. By Corollary 7.5 there is a subsequence $\{u^\chi_{j_k}\}_{k\in\mathbb{N}}$ of density 1 in $\{u^\chi_j\}_{j\in\mathbb{N}}$ such that we have the weak convergence

\begin{equation}
(7.10) \quad \langle |u^\chi_{j_k}|^2 \rangle_{G} d_{M/G} \longrightarrow \left( \frac{\text{vol}_{M/G}}{\text{vol}} \right)^{-1} \frac{d_{M/G}}{\text{vol}}, \quad \text{as } k \to \infty,
\end{equation}

and by Lemma 7.6

\begin{equation}
(7.11) \quad \langle |u^\chi_{j_k}|^2 \rangle_{G} = \Theta^\chi_k \quad \text{if } u^\chi_{j_k} \in V^\chi_i, \; k \in \mathbb{N}.
\end{equation}
Let now \( \{ i_k \}_{k \in \mathbb{N}} \) be the sequence of those indices \( i \) occurring in (7.11), without repetitions. Then

\[
d_x \# \{ k, i_k \leq m \} \geq \# \{ k, j_k \leq d_x m \}.
\]

Passing to the limit \( m \to \infty \) we obtain

\[
1 \geq \limsup_{m \to \infty} \frac{\# \{ k, i_k \leq m \}}{\# \{ k, j_k \leq m \}} \geq \liminf_{m \to \infty} \frac{\# \{ k, i_k \leq m \}}{\# \{ k, j_k \leq d_x m \}} = 1,
\]

where the final equality holds because \( \{ u_{j_k}^\chi \}_{k \in \mathbb{N}} \) has density 1 in \( \{ u_j^\chi \}_{j \in \mathbb{N}} \). This concludes the proof of the theorem. \( \square \)

Note that Theorem 7.7 is a statement about limits of representations, or multiplicities, in the sense that it assigns to each irreducible \( \chi \)-isotypic \( G \)-module in \( L^2(M) \) a measure on \( M/G \), and then considers the limiting measure. Remarkably, no more explicit mention is made to eigenfunctions of the Laplacian. Of course, one can also formulate a version of Theorem 7.7 which explicitly involves the eigenfunctions of the Laplace–Beltrami operator by taking for \( \Theta_i^\chi \) the function

\[
\Theta_i^\chi : M \to \mathbb{R}, \quad x \mapsto d_x^{-1} \sum_{j \in J^i} |u_j^\chi(x)|^2,
\]

where the sum runs over the eigenfunctions of \(-\Delta\) spanning the \( G \)-module \( V_i^\chi \).

8. Applications

In what follows, we apply our results to a few concrete situations where a closed connected Riemannian manifold carries an effective isometric action of a compact connected Lie group such that the principal orbits are of lower dimension than the manifold, and the reduced geodesic flow is ergodic.

8.1. Compact locally symmetric spaces. Let \( G \) be a connected semisimple Lie group with finite center and \( \Gamma \) a discrete uniform subgroup. Consider a maximal compact subgroup \( K \) of \( G \), and choose a left-invariant metric on \( G \) given by an \( \text{Ad}(K) \)-invariant bilinear form on the Lie algebra \( \mathfrak{g} \) of \( G \).

Since \( \Gamma \) is a uniform lattice, \( M = X := \Gamma \backslash G \) is a compact manifold, and by requiring that the projection \( G \to X \) is a Riemannian submersion, we obtain a Riemannian structure on \( X \). \( \Gamma \) acts on \( G \) and on \( X \) from the right in an isometric and effective way, and the isotropy group of a point \( \Gamma g \in X \) is conjugate to the finite group \( gKg^{-1} \cap \Gamma \). Hence, all \( K \)-orbits in \( X \) are either principal or exceptional. Since the maximal compact subgroups of \( G \) are precisely the conjugates of \( K \), exceptional \( K \)-orbits arise from elements in \( \Gamma \) of finite order. Let us now assume that \( \Gamma \) has no torsion, meaning that no non-trivial element \( \gamma \in \Gamma \) is conjugate in \( G \) to an element of \( K \). In this case, there are no exceptional orbits, the action of \( \Gamma \) on \( G/K \) is free, and \( Y := \Gamma \backslash G/K \) becomes a smooth manifold of dimension \( n - d \), where \( n = \dim X \) and \( d = \dim K \).

Let \( \mathcal{J} : T^*X \to T^* \) be the momentum map of the \( K \)-action on \( \Gamma \backslash G \). The orbit space \( Y \) is a closed manifold with negative sectional curvature inherited from \( G/K \). Consequently, its geodesic flow is ergodic. Furthermore, by co-tangent bundle reduction \( T^*Y \) is symplectomorphic to \( \mathcal{J}^{-1}(\{0\})/K \), compare Lemma 2.1. Since the measures on these spaces are given by the corresponding symplectic forms, the reduced geodesic flow on \( \mathcal{J}^{-1}(\{0\})/K \) is ergodic, and our results apply. Indeed, Theorem 7.1 and Corollaries 7.4, 7.5 imply

**Proposition 8.1.** Let \( \Delta \) be the Laplace–Beltrami operator on \( X \), \( \chi \in \hat{K} \), and let \( \{ u_j^\chi \}_{j \in \mathbb{N}} \) be an orthonormal basis of \( L^2(X) \) of eigenfunctions of \(-\Delta\). Then there is a subsequence \( \{ u_{j_k}^\chi \}_{k \in \mathbb{N}} \) of density 1 in \( \{ u_j^\chi \}_{j \in \mathbb{N}} \) such that for all \( s \in C^\infty(S^*X) \) one has

\[
\langle \text{Op}(s)u_{j_k}^\chi, u_{j_k}^\chi \rangle_{L^2(X)} \xrightarrow{k \to \infty} \int_{S^*X \cap \mathcal{J}^{-1}(\{0\})} s \frac{d\mu}{\text{vol}_C},
\]

as well as

\[
|u_{j_k}^\chi|^2 dX \xrightarrow{k \to \infty} \left( \frac{\text{vol}_C}{\text{vol}_X} X \right)^{-1} \frac{dX}{\text{vol}_C}, \quad \langle |u_{j_k}^\chi|^2 \rangle_G d\gamma \xrightarrow{k \to \infty} \left( \frac{\text{vol}_C}{\text{vol}_Y} Y \right)^{-1} \frac{d\gamma}{\text{vol}_Y}.
\]
Since the orbit volume function is constant in this case, eigenfunctions of the Laplacian $\Delta_Y$ on $Y$ correspond to $K$-invariant eigenfunctions of $\Delta$ on $X$, see Section 1.4. Furthermore, by Lemma A.6, $d_Y \equiv d_Y$ since the orbit volume can be normalized to 1. Consequently, in the special case that $\chi$ corresponds to the trivial representation, Proposition 8.1 yields

**Corollary 8.2 (Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem for $Y$).** Let $\{v_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(Y)$ of eigenfunctions of $-\Delta_Y$. Then there is a subsequence $\{v_{j_k}\}_{k \in \mathbb{N}}$ of density 1 in $\{v_j\}_{j \in \mathbb{N}}$ such that we have the weak convergence of measures $|v_{j_k}|^2 dY \xrightarrow{k \to \infty} (\text{vol}_d d_Y Y)^{-1} dY$.

8.2. **Spheres in dimensions 2 and 4.** In contrast to genuinely chaotic cases, it can happen that the reduced geodesic flow is ergodic simply for topological reasons. Namely, when the singular symplectic reduction of the co-sphere-bundle is just 1-dimensional, a single closed orbit of the reduced flow can have full measure. Although non-generic, this situation is topologically invariant, so that if it occurs for some particular $G$-space, it occurs for any choice of $G$-invariant Riemannian metric on that space, leading to a whole class of examples which might well be complicated geometrically.

In what follows, we will show that the spheres in dimensions 2 and 4, with appropriate group actions and invariant Riemannian metrics, are examples of the form just described. The reason why we consider only the dimensions 2 and 4 is that, in general, the $n$-sphere is topologically the suspension of the $(n-1)$-sphere, but only for $n \in \{2, 4\}$, the $(n-1)$-sphere has the structure of a compact connected Lie group. Thus, let $G$ be a compact connected Lie group. The suspension of $G$ is the quotient space

$$SG := ([{-1, 1}] \times G)/\{(1) \times \{g_1\} \sim\{-1\} \times \{g_2\}, \{1\} \times \{h_1\} \sim\{1\} \times \{h_2\}\}$$

$SG$ is a compact connected Hausdorff space that carries an effective $G$-action induced by the $G$-action on $G$ by left-multiplication and the trivial action on $[-1, 1]$. We will call this induced action the *suspension of the $G$-action*. It has exactly two fixed points $N := \{(1) \times G\}$ and $S := \{(-1) \times G\}$ which we may call *north* and *south pole*. Now, in general, $SG$ does not possess a differentiable structure. However, if $G$ is an $n$-sphere, then $SG$ is homeomorphic to the $(n+1)$-sphere, and consequently carries a canonical smooth structure making it diffeomorphic to the standard $(n+1)$-sphere. As is well-known, the only connected Lie groups that are spheres are $SO(2) \cong S^1$ and $SU(2) \cong S^3$.

Note that $S^2$, with the $S^3$-action given by the suspension of left-multiplication on $S^1$ and equipped with an $S^3$-invariant Riemannian metric, is just a surface of revolution diffeomorphic to the 2-sphere. Similarly, for $G = S^3$, we equip the suspension $S^4 \cong SS^3$ with the $S^3$-action given by the suspension of left-multiplication on $S^3$ and an $S^3$-invariant Riemannian metric, obtaining a class of 4-dimensional examples. We now have the following

**Proposition 8.3.** For $n \in \{2, 4\}$, equip the $n$-sphere $S^n \cong SS^{n-1}$ with the $S^{n-1}$-action given by the suspension of left-multiplication on $S^{n-1}$. Then the reduced geodesic flow with respect to any $S^{n-1}$-invariant Riemannian metric on $S^n$ is ergodic.

**Proof.** First, we prove the result for $S^2$. It will then become clear that the situation is entirely analogous for $S^4$. Thus, let $G = S^1 \cong SO(2)$. Then, for any choice of $SO(2)$-invariant metric on $M := SS^1$, we can identify $M$ with a surface of revolution in $R^3$ diffeomorphic to the 2-sphere and endowed with the induced metric from $R^3$. We assume that the *poles* are given by the points $N = (0, 0, 1)$ and $S = (0, 0, -1)$. The corresponding meridians are orthogonal to the $SO(2)$-orbits, and since the metric is $SO(2)$-invariant, each meridian is a closed geodesic. Now, for $(x, \xi) \in T^* M$, set $p(x, \xi) := |\xi|^2_{T_x M}$. Let $c > 0$ and put $\Sigma_c := p^{-1}(\{c\})$ and $\tilde{\Sigma}_c := \tilde{p}^{-1}(\{c\})$, where $\tilde{p} \in C^\infty(\tilde{\Omega}_{\text{reg}})$ is the function induced by $p|_{\tilde{\Omega}_{\text{reg}}}$. Clearly, $c$ is a regular value of $p$. To examine whether the reduced geodesic flow is ergodic on $\tilde{\Sigma}_c$, note that with the identification $T^* M \simeq TM$ given by the Riemannian metric one has

$$\Omega = J^{-1}(\{0\}) \simeq \bigsqcup_{x \in M} T_x (G \cdot x)^1,$$

(8.1)
so that
\[ \Omega_{\text{reg}} \simeq \left( \bigcup_{x \in M_{\text{reg}}} \{x\} \times T_x(G \cdot x)^{\perp} \right) \cup \left( \{N\} \times (T_N M \setminus \{0\}) \right) \cup \left( \{S\} \times (T_S M \setminus \{0\}) \right), \]
\[ \bar{\Omega}_{\text{reg}} \simeq \left( (-1, 1) \times \mathbb{R} \right) \cup \left( \{1\} \times (0, \infty) \right) \cup \left( \{-1\} \times (0, \infty) \right) \simeq \mathbb{R}^2 \setminus \{(0, 1), (0, -1)\}, \]
where \( M_{\text{reg}} = M \setminus \{N, S\} \). The diffeomorphism \( \bar{\Omega}_{\text{reg}} \simeq \mathbb{R}^2 \setminus \{(0, 1), (0, -1)\} \) is illustrated in Figures 8.1 and 8.2 for \( S^2 \) with the round metric, which is the generic case since \( M \) is \( \text{SO}(2) \)-equivariantly diffeomorphic to it. Under the diffeomorphism \( \bar{\Omega}_{\text{reg}} \simeq \mathbb{R}^2 \setminus \{(0, 1), (0, -1)\} \),

\[ \varphi_t : \mathbb{R} \to \mathbb{R}^2 \]
\[ \varphi_t(x, \xi) = (x, \xi + t \cdot \xi), \]

the hypersurface \( \Sigma_c \) corresponds to an ellipse with radii determined by \( c \), as illustrated in Figure 8.2 Let now \( G \cdot (x, \xi) \in \Sigma_c \). Since \( \xi \in T_x(G \cdot x)^{\perp} \), the geodesic flow \( \varphi_t \) transports \( (x, \xi) \) around curves in \( T^* M \) that project onto meridians through \( N \) and \( S \), so that the reduced geodesic flow \( \bar{\varphi}_t(G \cdot (x, \xi)) \equiv G \cdot \varphi_t(x, \xi) \) through \( G \cdot (x, \xi) \) corresponds to a periodic flow around the ellipse \( \Sigma_c \). Consequently, the only subsets of \( \Sigma_c \) which are invariant under \( \bar{\varphi}_t \) are the whole ellipse and the empty set, implying that the reduced flow \( \bar{\varphi}_t \) on \( \Sigma_c \) is ergodic for arbitrary \( c > 0 \).

Next, let us check what happens for a general compact connected Lie group \( G \). Due to the definition of \( SG \) and its \( G \)-action, it is clear that \( SG/G \) is homeomorphic to \([-1, 1]\) and, due to [8.1], that \( \bar{\Omega}_{\text{reg}} \) is diffeomorphic to \( \mathbb{R}^2 \setminus \{(0, 1), (0, -1)\} \) whenever \( SG \) is a smooth manifold, so that we always obtain not only an analogous but essentially the same picture as depicted in Figure 8.2 Hence, for \( G = S^3 \), the reduced geodesic flow is given by a periodic flow around an ellipse, and therefore ergodic.

We shall now apply our results to a surface of revolution diffeomorphic to the 2-sphere. Thus, let \( M \subset \mathbb{R}^3 \) be given by rotating a suitable smooth curve \( \gamma : [0, L] \to \mathbb{R}_{x \geq 0}^2 \) in the \( xz \)-half plane around the \( z \)-axis in \( \mathbb{R}^3 \). We assume that \( \gamma(0) = (0, -1) \) and \( \gamma(1) = (0, 1) \), and that \( \gamma \) is parametrized by arc length, so that \( \gamma : [0, L] \ni \theta \mapsto (R(\theta), z(\theta)) \), where \( R : [0, L] \to [0, \infty) \), \( R(0) = R(L) = 0 \), \( R(\theta) > 0 \)
for \( \theta \in (0, L) \) corresponds to the distance to the z-axis, and \( z : [0, L] \to \mathbb{R} \) is smooth. This leads to a parametrization of \( M \) according to

\[
M = \{(R(\theta) \cos \phi, R(\theta) \sin \phi, z(\theta)), \ \theta \in [0, L], \ \phi \in [0, 2\pi)\}.
\]

Now, let \( M \) be endowed with the induced metric on \( \mathbb{R}^3 \). The Laplace-Beltrami operator \( \Delta \) on \( M \) commutes with \( \partial_\phi \), so that separation of variables leads to a Hilbert basis of \( L^2(M) \) of joint eigenfunctions of both operators of the form

\[
ed_{l,m}(\phi, \theta) = f_{l,m}(\theta)e^{im\phi}, \quad (l, m) \in \mathcal{I} \subset \mathbb{Z} \times \mathbb{Z}.
\]

The irreducible representations of \( \text{SO}(2) \) are \( S^1 = \{e^{i\varphi}, \ \varphi \in [0, 2\pi)\} \subset \mathbb{C} \) are all 1-dimensional, and given by the characters \( \chi_k(e^{i\varphi}) = e^{-ik\varphi}, \ k \in \mathbb{Z} \). Thus, each subspace \( \mathbb{C} \cdot e_{l,m} \) corresponds to an irreducible representation of \( \text{SO}(2) \), and \( \{e_{l,m}\}_{l,m \in \mathcal{I}} \) is a Hilbert basis of \( L^2_{\chi_{l,m}}(M) \). Furthermore, \( |e_{l,m}|^2 \) is manifestly \( \text{SO}(2) \) invariant. Theorem 7.4 then yields for each \( m \in \mathbb{Z} \simeq \text{SO}(2) \) a subsequence \( \{e_{i_k,m}\}_{k \in \mathbb{N}} \) of density 1 in \( \{e_{l,m}\}_{l,m \in \mathcal{I}} \) such that for all \( f \in C(M/\text{SO}(2)) \)

\[
\int_{M/\text{SO}(2)} |f|e_{i_k,m}|^2 d_{M/\text{SO}(2)} \xrightarrow{k \to \infty} \left( \int_{M/\text{SO}(2)} \frac{d_{M/\text{SO}(2)}}{\text{vol}} \right)^{-1} \int_{M/\text{SO}(2)} f \frac{d_{M/\text{SO}(2)}}{\text{vol}}.
\]

Let us write (8.3) more explicitly. An \( \text{SO}(2) \)-orbit of a point \( x \in M \) with coordinates \( (\phi, \theta) \) is of the form \( \{(\phi', \theta) : 0 < \phi' < 2\pi\} \), up to a set of measure zero with respect to the induced orbit measure \( d\mu_{\text{SO}(2),x} = R(\theta) d\phi \), and we obtain \( \text{vol}(\text{SO}(2) \cdot x) = \int_0^{2\pi} R(\theta) d\phi = 2\pi R(\theta) \). Furthermore, \( M/\text{SO}(2) \) is homeomorphic to the closed interval \([0, L] \subset \mathbb{R}\), and the pushforward measure on \( M/\text{SO}(2) \) is given by \( d_{M/\text{SO}(2)}(\theta) \equiv 2\pi R(\theta) d\theta \), where we identified \( \text{SO}(2) \cdot x \) and \( \theta \). Summing up, (8.3) yields

\[
2\pi \int_0^L a(\theta)|f_{i_k,m}|^2(\theta)R(\theta) d\theta \xrightarrow{k \to \infty} \frac{1}{L} \int_0^L a(\theta) d\theta, \quad a \in C([0, L]),
\]

which is a result about weak convergence of measures on \( M/\text{SO}(2) \simeq [0, L] \). Formulated on \( M \), Corollary 7.4 yields that for each \( m \) there is a subsequence \( \{f_{i_k,m}\}_{k \in \mathbb{N}} \) of density 1 in \( \{f_{l,m}\}_{l,m \in \mathcal{I}} \) such that one has the weak convergence of measures

\[
|f_{i_k,m}|^2 dM \xrightarrow{k \to \infty} \frac{1}{2\pi L} \frac{dM}{R}.
\]

Here, \( \frac{dM}{R} \) is to be understood as the extension by zero of the smooth measure \( dM = dM(\phi, \theta) / R(\phi, \theta) \) from \( \{(\phi, \theta), \ \theta \in (0, L)\} \) to \( \{(\phi, \theta), \ \theta \in [0, L]\} \), and we used that \( \text{vol}_{M/\text{SO}(2)} M = L \). In particular, the obtained quantum limit on \( M \) is, up to a constant, related to the Riemannian volume density on \( M \) by the reciprocal of the distance function \( R \), which tends to infinity towards the poles. This is illustrated in Figure 8.3, where the function \( 1/R \) is plotted on a surface of revolution.

**Figure 8.3.** A quantum limit on a surface of revolution.
Physically, one can interprete this result as follows. For each symmetry type, corresponding to an isotypic component of $L^2(M)$, there is a sequence of quantum states such that the corresponding sequence of probability densities on $M$ converges weakly and with density 1 in the high-energy limit to the probability density of finding within a certain surface element of $M$ a classical particle with known energy and zero angular momentum with respect to the $z$-axis, but unknown momentum.

We do not know whether the results \[8.3\] and \[8.5\] are known for general surfaces of revolution. In the simplest case of the standard 2-sphere with the round metric, the eigenfunctions are explicitly known, and we show in the following that \[8.4\] is in agreement with the classical theory of spherical harmonics. In fact, we will see that one does not need to pass to a subsequence of density 1. The eigenvalues of $-\Delta$ on $S^2$ are given by the numbers $l(l + 1)$, $l = 0, 1, 2, \ldots$, and the corresponding eigenspaces $E_l$ are of dimension $2l + 1$. They are spanned by the spherical harmonics

\[
Y_{l,m}(\phi, \theta) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P_{l,m}(\cos \theta)e^{im\phi}, \quad 0 \leq \phi < 2\pi, \ 0 \leq \theta < \pi,
\]

where $m \in \mathbb{Z}$, $|m| \leq l$, and $P_{l,m}$ are the associated Legendre polynomials

\[
P_{l,m}(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l,
\]

compare [8.2]. Each subspace $C \cdot Y_{l,m}$ corresponds to an irreducible representation of $SO(2)$, and each irreducible representation $\chi_k$ with $|k| \leq l$ occurs in the eigenspace $E_l$ with multiplicity 1. The situation is illustrated by the following table. The columns of the table represent the eigenspaces, whereas the $k$-th row represents the isotypic component corresponding to $\chi_k$.

| $m$ | 0 | 1 | 2 | 3 | \ldots |
|-----|---|---|---|---|-----|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 3 | $\chi_{3,3}$ | $\chi_{3,2}$ | $\chi_{3,1}$ | $\chi_{3,0}$ | $\chi_{3,\ldots}$ |
| 2 | $\chi_{2,2}$ | $\chi_{2,1}$ | $\chi_{2,0}$ | $\chi_{2,\ldots}$ | $\vdots$ |
| 1 | $\chi_{1,1}$ | $\chi_{1,0}$ | $\chi_{1,\ldots}$ | $\vdots$ | $\vdots$ |
| 0 | $\chi_{0,0}$ | $\chi_{0,1}$ | $\chi_{0,2}$ | $\chi_{0,3}$ | $\chi_{0,\ldots}$ |
| $-1$ | $\chi_{-1,-1}$ | $\chi_{-1,-2}$ | $\chi_{-1,\ldots}$ | $\vdots$ | $\vdots$ |
| $-2$ | $\chi_{-2,-1}$ | $\chi_{-2,-2}$ | $\chi_{-2,\ldots}$ | $\vdots$ | $\vdots$ |
| $-3$ | $\chi_{-3,-1}$ | $\chi_{-3,-2}$ | $\chi_{-3,\ldots}$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 1: The $k$-th row spans $L^2_{\chi_k}(S^2)$.

The result \[8.4\] now turns into the following result about Legendre polynomials:

\[
(8.8) \quad \frac{2l_k + 1}{2} \frac{(l_k - m)!}{(l_k + m)!} \int_0^\pi a(\theta) \sin(\theta)|P_{l_k,m}(\cos \theta)|^2 d\theta \underset{k \to \infty}{\longrightarrow} \frac{1}{\pi} \int_0^\pi a(\theta) d\theta \quad \forall \ a \in C([0, \pi]).
\]

**Proposition 8.4.** For fixed $m$, \[8.8\] holds still if $l_k$ is replaced by $l$ and “$k \to \infty$” is replaced by “$l \to \infty$”.

**Proof.** Let us begin by recalling the following classical result about the asymptotic behavior of Legendre polynomials [14] page 303]. For fixed $m \in \mathbb{Z}$ and each small $\varepsilon > 0$ one has

\[
(8.9) \quad \frac{1}{lm} P_{l,m}(\cos \theta) = \left(\frac{2}{l\pi \sin \theta}\right)^{1/2} \cos \left(\left(l + \frac{1}{2}\right) \theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) + O\left(l^{-3/2}\right)
\]
as \( l \to \infty \) uniformly in \( \theta \in (\varepsilon, \pi - \varepsilon) \). From (8.6) and (8.9) we therefore obtain

\[
|\hat{Y}_{l,m}(\theta)|^2 = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} P_{l,m}(|\cos \theta|)}^2 = \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \left| \frac{1}{l\pi} P_{l,m}(\cos \theta) \right|^2
\]

\[
= \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \left| \frac{2}{l\pi \sin \theta} \right|^{l^2 m} \cos \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) + O \left( l^{-3/2} \right)
\]

\[
= \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \left| \frac{2}{l\pi \sin \theta} \right|^{l^2 m} \cos \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) + O \left( l^{-2} \right).
\]

The asymptotic relation

\[
(l - m)!/(l + m)! \sim l^{-2m} \quad \text{as} \; l \to \infty
\]

implies that \( \frac{(l - m)!}{(l + m)!} l^{2m} \) is bounded in \( l \), so we can use the simple relation \( \frac{2l + 1}{l^2} = 2 + O(l^{-1}) \) to obtain

\[
|\hat{Y}_{l,m}(\theta)|^2 = \frac{(l - m)!}{(l + m)!} l^{2m} \frac{1}{\pi^2 \sin^2 \theta} \cos^2 \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) + O \left( l^{-1} \right),
\]

uniformly for \( \theta \in (\varepsilon, \pi - \varepsilon) \) and each small \( \varepsilon > 0 \). Now let \( f \in C([0, \pi], \mathbb{R}) \) and choose \( \varepsilon > 0 \). Due to the uniform estimate (8.11) and boundedness of the integration domain we get

\[
2\pi \int_\varepsilon^{\pi-\varepsilon} f(\theta) |\hat{Y}_{l,m}(\theta)|^2 \sin \theta d\theta
\]

\[
= 2\pi \int_\varepsilon^{\pi-\varepsilon} f(\theta) \frac{(l - m)!}{(l + m)!} l^{2m} \frac{1}{\pi^2 \sin^2 \theta} \cos^2 \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) \sin(\theta) d\theta + O \left( l^{-1} \right)
\]

\[
= \frac{2}{\pi} \frac{(l - m)!}{(l + m)!} l^{2m} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2 \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) d\theta + O \left( l^{-1} \right).
\]

The oscillatory integral in (8.12) has the limit

\[
\lim_{l \to \infty} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2 \left( \left( l + \frac{1}{2} \right) \theta - \frac{\pi}{4} + \frac{m\pi}{2} \right) d\theta = \frac{\pi}{\varepsilon} \int_\varepsilon^{\pi-\varepsilon} f(\theta) d\theta,
\]

where the final equality is true because \( \lim_{l \to \infty} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2 (l\theta) d\theta = \lim_{l \to \infty} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \sin^2 (l\theta) d\theta \)
and \( \sin^2 + \cos^2 = 1 \). Using (8.13) and (8.10) we conclude from (8.12) for each small \( \varepsilon > 0 \) that

\[
\lim_{l \to \infty} 2\pi \int_\varepsilon^{\pi-\varepsilon} f(\theta) |\hat{Y}_{l,m}(\theta)|^2 \sin(\theta) d\theta = \frac{\pi}{\varepsilon} \int_\varepsilon^{\pi-\varepsilon} f(\theta) d\theta.
\]

Noting that \( \lim_{x \to \infty} \cos^2(x) \leq 1 \) and \( \lim_{x \to \infty} \cos^2(x) \leq 1 \) exist, the \( \varepsilon = 0 \) version of (8.14) now follows from (8.11) and (8.14) using Fatou’s Lemma. For the details of this, see [21].

**Appendix A.**

In this appendix, we shall collect a few important facts related to the spaces and measures introduced in Section 2.4. With the notation introduced there we have

**Lemma A.1.** The measure \( d\Omega_{\text{reg}} \) agrees with the Riemannian volume density defined by the Riemannian metric on \( \Omega_{\text{reg}} \) that is induced by the Sasaki metric on \( T^*M \).
Proof. By [2 Theorem 4.6] all metrics on \( \tilde{\Omega}_{\text{reg}} \) which are associated to the symplectic form \( \tilde{\omega} \) by an almost complex structure define the same Riemannian volume density, and that density agrees with the one defined by the symplectic form \( \tilde{\omega} \). Hence, it suffices to show that the Riemannian metric on \( \tilde{\Omega}_{\text{reg}} \) induced by the \( G \)-invariant Sasaki metric on \( T^*M \) is associated to \( \tilde{\omega} \) by an almost complex structure. Now, the Sasaki metric \( g_S \) on \( T^*M \) is associated to the canonical symplectic form \( \omega \) on \( T^*M \) by an almost complex structure \( J : TT^*M \rightarrow TT^*M \). Consequently, the Riemannian metric \( i^*g_S \) on \( \Omega_{\text{reg}} \) is associated to the symplectic form \( i^*\omega \) by the almost complex structure \( i^*J \), where \( i : \Omega_{\text{reg}} \rightarrow T^*M \) is the inclusion. Since both \( i^*g_S \) and \( i^*\omega \) are \( G \)-invariant, \( i^*J : T\Omega_{\text{reg}} \rightarrow T\Omega_{\text{reg}} \) is \( G \)-equivariant, and therefore induces an almost complex structure \( i^*\tilde{J} : T\tilde{\Omega}_{\text{reg}} \rightarrow T\tilde{\Omega}_{\text{reg}} \) which associates the metric induced by \( i^*g_S \) on \( \tilde{\Omega}_{\text{reg}} \) with \( \tilde{\omega} \).

Lemma A.2. \( M - M_{\text{reg}} \) is a null set in \( (M,dM) \), and \( \Omega_{\text{reg}} - (T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) \) is a null set in \( (\Omega_{\text{reg}},d\Omega_{\text{reg}}) \).

Proof. The proof is completely analogous to the proof of [5 Lemma 3].

Corollary A.3. \( M/G - \tilde{M}_{\text{reg}} \) is a null set in \( (M/G,dM/G) \), and \( \Omega_{\text{reg}} - (T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) \cap G \) is a null set in \( (\tilde{\Omega}_{\text{reg}},d\tilde{\Omega}_{\text{reg}}) \).

Proof. The first claim is true by definition of the measure \( d_{M/G} \) and Lemma A.2. Concerning the second claim, note that

\[
(\Omega_{\text{reg}} - (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})) \cap G = \tilde{\Omega}_{\text{reg}} - (T^*\tilde{M}_{\text{reg}} \cap \Omega_{\text{reg}}) \cap G.
\]

Consequently, (2.14) and Lemma A.2 together yield

\[
\text{vol} \left( \tilde{\Omega}_{\text{reg}} - (T^*\tilde{M}_{\text{reg}} \cap \Omega_{\text{reg}}) / G \right) = \int_{\Omega_{\text{reg}} - (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})} \frac{1}{\text{vol}(G \cdot \eta)} d\Omega_{\text{reg}}(\eta) = 0.
\]

Lemma A.4. The orbit volume function \( \text{vol}_\mathcal{O}|_{M_{\text{reg}}} : M_{\text{reg}} \rightarrow \mathbb{R}, x \mapsto \text{vol}(G \cdot x) \), is smooth. Moreover, if the dimension of the principal orbits is at least 1, the function \( \text{vol}_\mathcal{O}|_{M_{\text{reg}}} \) can be extended by zero to a continuous function \( \overline{\text{vol}}_{\mathcal{O}} : M \rightarrow \mathbb{R} \).

Proof. See [26 Proposition 1].

Remark A.5. The function \( \overline{\text{vol}}_{\mathcal{O}} : M \rightarrow \mathbb{R} \) from the previous lemma is in general different from the original orbit volume function \( \text{vol}_\mathcal{O} : M \rightarrow \mathbb{R}, x \mapsto \text{vol}(G \cdot x) \). The latter function is by definition nowhere zero and not continuous if there are some orbits of dimension 0 and some of dimension > 0.

Lemma A.6. The orbit normalized measure on \( M/G \) fulfills

\[
\frac{d_{M/G}}{\text{vol}}|_{\tilde{M}_{\text{reg}}} = d\tilde{M}_{\text{reg}}, \quad \frac{d_{M/G}}{\text{vol}}|_{M/G - \tilde{M}_{\text{reg}}} \equiv 0.
\]

Proof. Considering (2.13), (2.12), and the first statement of Corollary A.3, the claimed relations are obvious.

Corollary A.7. The following two measures on \( (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \) agree:

1. the measure \( j^*d\Omega_{\text{reg}} \), where \( j \) is the inclusion \( j : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \hookrightarrow \tilde{\Omega}_{\text{reg}} \) and \( d\tilde{\Omega}_{\text{reg}} \) the symplectic volume form on \( \tilde{\Omega}_{\text{reg}} \);
2. the measure \( \Phi^*d(T^*\tilde{M}_{\text{reg}}) \), where \( \Phi : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \rightarrow T^*\tilde{M}_{\text{reg}} \) is the canonical symplectomorphism from Lemma 2.4 and \( d(T^*\tilde{M}_{\text{reg}}) \) the symplectic volume form on \( T^*\tilde{M}_{\text{reg}} \).

Proof. The measures \( d\tilde{\Omega}_{\text{reg}} \) and \( d(T^*\tilde{M}_{\text{reg}}) \) are defined by the volume forms \( \tilde{\omega}^{n-\kappa}/(n-\kappa)! \) and \( \tilde{\omega}^n/(n-\kappa)! \), respectively, which implies that the measures \( j^*d\Omega_{\text{reg}} \) and \( \Phi^*d(T^*\tilde{M}_{\text{reg}}) \) are defined
by the volume forms $j^*\tilde{\omega}^n/(n-\kappa)!$ and $\Phi^*\tilde{\omega}^n/(n-\kappa)!$, respectively. Using compatibility of the wedge product with pullbacks and Lemma 2.4 we obtain
\[
j^*\tilde{\omega}^n = (j^*\tilde{\omega})^n = (\Phi^*\tilde{\omega})^n = \Phi^*(\tilde{\omega}^n).
\]

**Lemma A.8.** Let $X$ be a smooth manifold with a smooth volume density $dX$. Consider $c \in \mathbb{R}$, and let $f : X \to \mathbb{R}$ be a smooth, proper function for which $c$ is a regular value. For each $\delta > 0$, let $I_\delta \subset [c-\delta, c+\delta]$ be a non-empty interval. Then, for all $a \in C^\infty(X)$ the limit
\[
\lim_{\delta \to 0} \frac{1}{\text{vol}_\mathbb{R}(I_\delta)} \int_{f^{-1}(I_\delta)} a(x) dX(x) =: \int_{f^{-1}(\{c\})} a(x) \mu_c(x)
\]
exists, and uniquely defines a measure $\mu_c$ on the hypersurface $f^{-1}(\{c\})$. Furthermore, in the limit $\delta \to 0$, we have the estimate
\[
\frac{1}{\text{vol}_\mathbb{R}(I_\delta)} \int_{f^{-1}(I_\delta)} a(x) dX(x) - \int_{f^{-1}(\{c\})} a(x) d\mu_c(x) = O(\delta).
\]

**Proof.** This is essentially a consequence of the theorems of Ehresmann and Fubini, see [34, pp. 161] and [21].

**Remark A.9.** The hypersurface measure obtained in Lemma A.8 from the proper function $T^*\tilde{M}_{\text{reg}} \to \mathbb{R}$, $n \mapsto ||\eta||_g$ at the regular value 1 agrees with the Liouville measure $d(S^*\tilde{M}_{\text{reg}})$.

Finally, we observe that Corollary A.7 allows us to replace integrals over some particular hypersurfaces of $\hat{\Omega}_{\text{reg}}$ by integrals over the co-sphere bundle on $\tilde{M}_{\text{reg}}$.

**Lemma A.10.** Let $p : T^*M \to \mathbb{R}$ be the $G$-invariant map given by $p(\eta) = ||\eta||_g^2$, inducing $\tilde{p} \in C^\infty(\hat{\Omega}_{\text{reg}})$. Set $\Sigma_1 := \tilde{p}^{-1}(\{1\})$. Then the symplectomorphism $(T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) / G \simeq T^*\tilde{M}_{\text{reg}}$ from Lemma 2.4 induces a measure-preserving diffeomorphism of measure spaces
\[
(\Sigma_1, d\Sigma_1) \simeq (S^*\tilde{M}_{\text{reg}}, d(S^*\tilde{M}_{\text{reg}}))
\]
up to null sets.

**Proof.** By definition of the Riemannian metric on $\tilde{M}_{\text{reg}}$, the symplectomorphism $(T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) / G \simeq T^*\tilde{M}_{\text{reg}}$ maps $(T^*M_{\text{reg}}(G) \cap \Sigma_1)$ onto $S^*\tilde{M}_{\text{reg}}$. The claim follows now from Corollary A.3 and Remark A.9, and the fact that the hypersurface measure obtained in Lemma A.8 is unique if the initial smooth measure is fixed.

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