Exact calculations for the ground state properties of the frustrated 2-dimensional Ising model

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Abstract. We have developed a technique that enables the exact calculation of the ground state properties of the 2-dimensional ±\(J\) Ising model. This paper gives an overview of the methodology and describes the latest results on the correlation function.

1. Introduction
The 2-dimensional Ising model is one of the most widely studied systems in statistical physics largely because it is one of the few non-trivial models that admit an exact solution. The original method of Onsager [1] based on Lie algebras was superseded by a combinatorial approach [2, 3] in which the partition function is expressed in terms of a determinant (or Pfaffian). One can alternatively express the formalism in terms of a Grassmann algebra [4, 5], in which the system takes the form of a non-interacting field theory in anticommuting variables (or an interacting field theory in three dimensions). Although the early formulation considered specifically the perfect Ising lattice, problems of disordered systems can also be addressed using the technique.

Perhaps the most studied disordered system is the so-called spin-glass problem [6] in which the key element is frustration. This is characterized by conflicts in the spin configurations needed to minimize the energy. Frustration can arise from the geometry of the lattice or from a quenched random distribution of positive (ferromagnetic) and negative (antiferromagnetic) bonds. One of the canonical models of frustrated systems is the 2-dimensional ±\(J\) Ising model.

We showed [7] a number of years ago how certain features of the 2D-dimensional ±\(J\) Ising model are particularly transparent when expressed within the determinantal formalism [3], and developed a technique for the exact calculation of the ground state energy and entropy of this system and applied it to the square [8, 9] and triangular [10] lattices. Recently we have extended these ideas to evaluate the ground state correlation functions [11]. This paper reviews the methodology and discusses the latest results.

2. Background
2.1. The model
The Hamiltonian for the system is written as

\[ H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j. \] (1)
where \(i\) and \(j\) label sites and the \(\sigma\) variables take the usual Ising model values of \(\pm1\). The exchange interactions \(J_{ij}\) couple nearest-neighbour only and are quenched random variables of fixed magnitude but random sign \((\pm J)\). The bonds are negative with a probability \(p \in [0, 0.5]\) and we focus here on the square lattice. The model with \(p = 0.5\) on the square lattice has attracted the most interest and it is generally accepted that spin glass behaviour occurs at zero temperature \([12, 13, 14, 15, 16, 17]\), persisting down to a critical probability \(p_c\) of around 0.11 \([9, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]\).

The ground state properties we wish to address are the energy, entropy, and the correlation between spins separated by distance \(R\) which is expected to decay algebraically with a characteristic exponent \(\eta\) according to

\[
[<\sigma_0 \sigma_R^2>]_{av} \sim R^{-\eta}.
\] (2)

The subscript ‘\(av\)’ indicates a statistical average over configurations of bonds. The advantage of our approach is that it gives direct access to the ground state without the need to extrapolate from finite temperatures as is necessary, for example, in Monte Carlo based methods.

2.2. Formalism

The partition function for the nearest neighbour Ising model on an \(N\)-site square lattice at temperature \(T\) can be written generally as

\[
Z = \prod_{<ij>} \cosh(J_{ij}/kT) [\text{Tr} \prod_{<ij>} (1 + t_{ij} \sigma_i \sigma_j)],
\] (3)

where the products are over the \(2N\) pairs of nearest neighbour sites \(<ij>\) and \(t_{ij} = \tanh(J_{ij}/kT)\). The trace is over the \(2^N\) spin configurations. The second factor in Eq. (3) can be written diagrammatically as a sum over closed polygons and this can be expressed in closed form as a determinant \([2, 3]\)

\[
Z = 2^N \prod_{<ij>} \cosh(J_{ij}/kT) |\text{det } D|^{1/2}.
\] (4)

\(D\) is a \(4N \times 4N\) skew-symmetric matrix. Its form can be visualised if we decorate each lattice site with four nodes (as shown in figure 1). \(D\) contains \(N\) blocks, \(\Omega\), down the diagonal; the elements of each block connect nodes associated with a single lattice site.

\[
\Omega = \begin{pmatrix}
0 & -1 & i & i \\
1 & 0 & -i & i \\
-i & i & 0 & -i \\
-i & -i & i & 0
\end{pmatrix}
\] (5)

\(D\) also contains elements that link nodes 1(3) of one site with 2(4) of neighbouring sites along the \(x(y)\) Cartesian directions. For pairs of sites \(<ij>\) these elements are \(\mp it_{ij}\) where the minus and plus signs correspond respectively to the positive and negative Cartesian directions.

In the development that follows it is useful to consider the eigenvalues of \(D\). They occur in pairs, \(\pm \epsilon\), which are real since \(D\) is Hermitian (we can alternatively define \(D\) as real skew symmetric and perform a \(2 \times 2\) block diagonalisation). For the perfect system (\(J_{ij}\) independent of \(<ij>\)), the eigenvalues fall into two bands whose upper and lower bounds are shown as a function of \(t\) in figure 2. The critical temperature is at \(t = \sqrt{2} - 1\) where there is a zero eigenvalue.

One can also calculate the correlation function for spins at any pair of sites 0 and \(R\). It is given by
\[
< \sigma_0 \sigma_R > = \left[ \frac{\det C}{\det D} \right]^{1/2} \prod_{\text{path}} t_{ij},
\]
where the product is over the bonds along an arbitrary path between the two sites, and \( C \) is obtained from \( D \) by replacing \( t_{ij} \) by \( 1/t_{ij} \) for all elements corresponding to bonds along that path. The reason for this is easy to see; to calculate the correlation function we require the quantity

\[
Z' = \left[ \prod_{<ij>} \cosh(J_{ij}/kT) \right][\text{Tr} \prod_{<ij>} \sigma_0 \sigma_R (1 + t_{ij} \sigma_i \sigma_j)],
\]

Since \( \sigma_i^2 = 1 \), we can introduce a term \( (\sigma_2 \sigma_b \cdots) \) for all spins along a path joining \( \sigma_0 \) and \( \sigma_R \). Then grouping these spins with the terms in parenthesis corresponding to links along the path, we can write \( \sigma_i \sigma_j (1 + t_{ij} \sigma_i \sigma_j) = t_{ij} (1 + t_{ij}^{-1} \sigma_i \sigma_j) \), and the result immediately follows.

In the frustrated system we average over many configurations of quenched disorder and prefer to define correlations in the form given in Eq. (2), and so we will base our discussion on the expression

\[
< \sigma_0 \sigma_R >^2 = \frac{\det C}{\det D} \prod_{\text{path}} t_{ij}^2.
\]

**Figure 1.** The labelling of nodes associated with a lattice site.

**Figure 2.** Spectral bounds (positive eigenvalues) for the perfect Ising system.

### 2.3. Simple defects

We consider the introduction of a few negative bonds into an otherwise perfect lattice. Defect eigenvalues appear at \( \epsilon = 0 \) in the \( T \to 0 (t \to 1) \) limit. At small temperatures each eigenvalue can be written

\[
\epsilon = \frac{1}{2} X \exp(-2rJ/kT),
\]
where \( r \) is an integer and \( X \) is a real number. The number of defect eigenvalues is exactly equal to the number of frustrated plaquettes (elementary squares on the lattice containing an odd number of negative bonds).

Two examples will illustrate the behaviour. There are two frustrated plaquettes in figure 3 and two defect eigenvalues \((\pm \epsilon)\); \( \epsilon \) is as in Eq. (9) with \( r = 1 \) and \( X = 1 \). In figure 4 there are four frustrated plaquettes and so two pairs of defect eigenvalues of the form of Eq. (9); one has \( r = 1, X = 1 \) and the other \( r = 3, X = 5 \).

The free energy of the perfect lattice is given by \( F = -2NJ \) (there are \( 2N \) bonds). With negative defects present there are unsatisfied bonds and the free energy can be expressed as \( F = -2NJ + \Delta F \). It is straightforward to show that

\[
\Delta F = 2J \sum_d r_d,
\]

where the sum is over all defect eigenvalues. It can also be shown [8] that the ground state degeneracy, \( M \), or entropy, \( S \), can be written in terms of the factors, \( X \), in Eq. (9)

\[
M = \prod_d X_d,
\]

\[
S = k \sum_d \ln X_d.
\]

\[\text{Figure 3. A single negative bond (denoted by a heavy line) in a lattice of positive bonds. Shading indicates frustrated plaquettes.}\]

\[\text{Figure 4. A more complex defect configuration with four negative bonds. Notation is as in figure 3.}\]

Referring back to figure 3, we see that \( r = 1 \) and \( X = 1 \) imply that \( \Delta F = 2J \) (there is one ‘wrong’ bond) and \( M = 1 \) (there is only one spin configuration that achieves the minimum energy). In figure 4, with \( (r = 1, X = 1) \) and \( (r = 3, X = 5) \), we get \( \Delta F = 2J \times (1 + 3) \) (there are four ‘wrong’ bonds), and \( M = 1 \times 5 \). In this case flipping spins on sites \( a, b, c \), or both \( a \) and \( c \) produces new configurations while leaving the ground state energy unchanged. In the next section we shall see how this rather simple structure can be developed to deal with an arbitrary concentration, \( p \), of negative bonds.
3. The $±J$ system (energy and entropy)

There are several features of the procedure described above that allow us to develop a method that can deal with a system with an arbitrary concentration of negative bonds. The first is the form of the eigenvalues, Eq. (9), which suggests $\exp(-2J/kT)$ as a small parameter in the problem, and the use of perturbation theory to lift the degeneracy of the defect eigenvalues thus determining $r$ (which will be the order of perturbation theory at which the degeneracy is lifted), and the associated prefactor, $X$. It should be noted that, although we are using perturbation theory, it is an exact solution for the ground state properties that emerges; once the perturbation expansion has been taken to a sufficiently high level to lift the degeneracy of all $T = 0$ eigenvalues, we have a complete solution.

The second feature, which makes it feasible to perform calculations on rather large lattices, is the fact that the zero temperature eigenstates are completely localised on the frustrated plaquettes. Let us make this statement a little more specific. We introduced a node basis as displayed in figure 1, and denote a basis state associated with a lattice site $(x, y)$ as $|x, y; k\rangle$, where $k = 1 \cdots 4$. It is convenient to work in a bond basis. We form bond basis states as linear combinations of the node states associated with a particular bond. For the bonds connecting sites $(x, y)$ and $(x + 1, y)$ along the $x$-axis, the new basis states are $[|x, y; 1\rangle \pm |x + 1, y; 2\rangle]$. We can associate one of these with the plaquette above the bond and one with the plaquette below. A similar procedure is adopted for the $y$ direction with the linear combination $[|x, y; 3\rangle \pm |x, y + 1; 4\rangle]$.

We can associate four bond basis states with each plaquette and, assuming a judicious choice of assigning bond states to plaquettes is made, we find that the zero eigenvalue defect states (denoted by $|d\rangle$) are completely localised on their respective frustrated plaquette.

The perturbation that we have to deal with in considering small finite temperatures is also highly local. Denoting the $T = 0$ value of $D$ by $D_0$ (by definition, $D_0 |d\rangle = 0$), we can write $D$ as

$$D = D_0 + \delta D_1,$$

where $\delta = 1 - \tanh(J/kT)$. Eq. (13) is exact, but at low temperatures it will also be convenient to use $\delta$ as a small parameter in the perturbation expansion. The matrix elements of $D_1$ are zero except between the two basis states associated with a particular bond. If there are $n$ frustrated plaquettes on an $N$ site lattice, there are $n$ defect states and $4N - n$ non-zero eigenstates which we refer to as continuum states: $D_0 |c\rangle = \epsilon_c |c\rangle$. Note all states occur as positive and negative pairs; we need only discuss the $2N$ positive ones.

3.1. Degenerate-state perturbation theory

The perturbation formalism used to lift the degeneracy of the defect states and determine $r_d$ and $X_d$ is developed in this section. The theory is based on Eq. (13). We write the ground-state defect eigenstates $|d\rangle$ more specifically as $|r, i\rangle$, where $r$ is the order of perturbation theory at which degeneracy is lifted and $i$ labels eigenstates of that order.

The eigenvalue equation is formally written

$$D |\Psi\rangle = \lambda |\Psi\rangle,$$

and $\lambda$ and $|\Psi\rangle$ can be expanded in powers of $\delta$

$$\lambda = \delta^n e_n^j + O(\delta^{n+1})$$

$$|\Psi\rangle = |\Psi_0\rangle + \delta |\Psi_1\rangle + \cdots$$
where the eigenvalue is the \( j \)th defect eigenvalue of order \( n \). \( | \Psi_0 \rangle = | n, j \rangle \) is the leading term for the corresponding eigenvector, that is \( D_0 | n, j \rangle = 0 \).

For \( 1 \leq m \leq n \), equating powers of \( \delta \), we obtain

\[
D_0 | \Psi_m \rangle + D_1 | \Psi_{m-1} \rangle = \delta_{mn} \epsilon_n^j | n, j \rangle \tag{17}
\]

By definition, \( D_0 | r, i \rangle = 0 \), so that

\[
\langle r, i | D_1 | \Psi_{m-1} \rangle = \delta_{mn} \delta_{rn} \delta_{ij} \epsilon_n^j \tag{18}
\]

and the first order states are determined by diagonalization of \( D_1 \) in the defect basis set, that is the localized vectors associated with the frustrated plaquettes. This determines all the \( r = 1 \) states of Eq. (9).

For the zero eigenstates whose degeneracy is not lifted at first order, we proceed order by order until all degeneracy is lifted. At second order we need to consider intermediate states, the continuum states, \( | c \rangle \). We introduce a Green’s function \( G_c(E) \) defined as

\[
G_c(E) = \sum_c | c \rangle \langle E - \epsilon_c | \langle c |, \tag{20}
\]

where \( E \) is an energy parameter and \( \epsilon_c \) are the eigenvalues of the continuum states.

Since we are concerned with the lifting of the degeneracy of the zero eigenvalue states, we require specifically \( G_c(E = 0) \), which henceforward will be designated simply as \( G_c \). By detailed consideration of the matrix elements we can show that \( G_c \) takes a simple form. It can be expressed as

\[
G_c = (1 - P)g_c(1 - P), \tag{21}
\]

where \( P \) is a projection operator,

\[
P = \sum_d | d \rangle \langle d |, \tag{22}
\]

and \( g_c \) is a quantity we term the continuum propagator. Detailed expressions for \( g_c \) within the bond basis are available in ref. [8].

Now, acting with a continuum state on equation (17), we have

\[
\epsilon_c \langle c | \Psi_m \rangle = -\langle c | D_1 | \Psi_{m-1} \rangle \tag{23}
\]

Then using closure

\[
\sum_{r,i} | r, i \rangle \langle r, i | + \sum_c | c \rangle \langle c | = 1, \tag{24}
\]

and the definition of the continuum Green’s function in equation (20), we obtain

\[
| \Psi_m \rangle = G_c D_1 | \Psi_{m-1} \rangle + \sum_{r,i} Z_{r'}^i | r, i \rangle, \tag{25}
\]

where \( Z_{r'}^i \) is equal to \( \langle r, i | \Psi_m \rangle \), but for our purposes these are just coefficients to be determined. Then, using equation (18) and (21), we can write in terms of a new set of coefficients \( Z_{r'}^i \).
\[ | \Psi_m \rangle = g_c D_1 | \Psi_{m-1} \rangle + \sum_{r,i} Z^i_r | r, i \rangle. \]  

(26)

Now, acting on Eq. (26) with \( \langle r, i | D_1 \), and using Eq. (18), we obtain

\[ \delta_{m,n-1} \delta_{rn} \delta_{ij} \epsilon_{n}^{j} = \langle r, i | D_2 | \Psi_{m-1} \rangle + \delta_{r1} Z^i_1 \epsilon_{1}^{i}, \]  

(27)

where

\[ D_2 = D_1 g_c D_1. \]  

(28)

With \( r = n = 2 \) and \( m = 1 \), we obtain

\[ \epsilon_{2}^{j} \delta_{ij} = \langle 2, i | D_2 | 2, j \rangle, \]  

(29)

which completes the analysis as far as second order \((r = 2)\).

For orders higher than 2, we insert \( r = 1 \) into Eq. (27) which gives

\[ Z^i_1 = -\langle 1, i | D_2 | \Psi_{m-1} \rangle / \epsilon_{1}^{i}, \]  

(30)

and hence from Eq. (26)

\[ | \Psi_m \rangle = (1 + G_1 D_1) g_c D_1 | \Psi_{m-1} \rangle + \sum_{r>1,i} Z^i_r | r, i \rangle. \]  

(31)

\( G_1 \) is a first-order Green’s function, which is a special case of the general definition

\[ G_r = -\sum_i | r, i \rangle (1/\epsilon_{r}^{i}) \langle r, i | . \]  

(32)

Acting on Eq. (31) with \( \langle r, i | D_2 \) and using Eq. (27) gives

\[ \delta_{m,n-2} \delta_{rn} \delta_{ij} \epsilon_{n}^{j} = \langle r, i | D_3 | \Psi_{m-1} \rangle + \delta_{r2} Z^i_2 \epsilon_{2}^{i}, \]  

(33)

where

\[ D_3 = D_2 (1 + G_1 D_1) g_c D_1 \]  

(34)

With \( r = n = 3 \) and \( m = 1 \), we obtain

\[ \epsilon_{3}^{j} \delta_{ij} = \langle 3, i | D_3 | 3, j \rangle, \]  

(35)

which completes the analysis as far as third order \((r = 3)\).

The analysis to arbitrary order follows through in a similar fashion with the \( r \)th order eigenvalue being given by

\[ \epsilon_{r}^{j} \delta_{ij} = \langle r, i | D_r | r, j \rangle, \]  

(36)

where

\[ D_r = D_{r-1} (1 + G_{r-2} D_{r-2}) \cdots (1 + G_1 D_1) g_c D_1 \]  

(37)
3.2. Results
We have performed numerical calculations [8, 9] based on the above theory to obtain the full set of $r$ and $X$ values for large finite lattices of size $L \times L$. The largest value of $r$ (order of perturbation theory necessary to complete the calculation) varies between configurations of disorder but is seldom larger than 12. Averages are performed over many configurations of disorder. Calculations were performed with $L = 64, 128$ and 256 and the energy and entropy extrapolated to an infinite lattice using the forms

$$F(L) = F + \alpha L^{-1},$$

(38)

$$S(L) = S + \beta L^{-1},$$

(39)

where $F(L)$ and $S(L)$ are the values for an $L \times L$ lattice and $F$ and $L$ are the $L \to \infty$ limits. The results obtained for a selection of values of $p$ (the concentration of negative bonds) is shown in Table 1. Between 300 and 1000 samples are used in each configurational average. The perfect Ising ferromagnet corresponds to $p = 0$ and has values of $F$ and $S$ of −2 and 0 respectively.

Table 1. Ground state energy $F$ (in units of $J$) and entropy $S$ (in units of $k$) per spin for selected values of $p$.

| $p$ (%) | $F$ | $S$ |
|---------|-----|-----|
| 5       | -1.8024±0.0005 | 0.0103±0.0002 |
| 8       | -1.6924±0.0005  | 0.0252±0.0003  |
| 10      | -1.6274±0.0003  | 0.0363±0.0004  |
| 11      | -1.5994±0.0003  | 0.0405±0.0004  |
| 12      | -1.5750±0.0003  | 0.0440±0.0004  |
| 15      | -1.5182±0.0003  | 0.0528±0.0005  |
| 20      | -1.4590±0.0005  | 0.0620±0.0003  |
| 50      | -1.4021±0.0002  | 0.0709±0.0004  |

3.3. Critical value of $p$

The perfect Ising ferromagnet has $p = 0$ while $p = 50\%$ for the canonical $\pm J$ spin-glass. In between there is a critical concentration, $p_c$, that marks the transition from pure Ising to spin-glass behaviour. There is no singular behaviour in either $F$ or $S$ to flag this transition so we take a different approach.

As noted earlier the eigenvectors associated with the zero eigenvalue states are localised on the respective frustrated plaquettes. Lifting the degeneracy as described produces eigenvectors that are linear combinations of these basis states. As frustration is increased by introducing more negative bonds these eigenvectors become more extended. This suggests that $p_c$ might be marked by a transition from localised to extended states indicating the demarcation of a regime in which frustration is a local perturbation to one in which frustration dominates the whole lattice.

We denote the linear size of an eigenvector by $l$ and the number of states on an $L \times L$ lattice whose size is larger than $l$ by $N(l)$. The procedure for obtaining $l$ numerically is described elsewhere [8, 9]. We find that, if $p > p_c$, $N(l)$ fits rather well to a power law

$$N(l) \sim L^2 l^{-p}.$$ 

(40)
The number of defect eigenvectors whose size is comparable to that of the lattice is \( \sim L^{(2 - \rho)} \). This approaches zero or infinity as \( L \to \infty \) according to whether \( \rho > \rho_c \) or \( \rho < \rho_c \), where \( \rho_c = 2 \). We thus take \( \rho < \rho_c \) as the criterion for the presence of extended states.

The results for a number of values of \( p \) are shown in figures 5 and 6. The fall-off in the data at higher values of \( l \) is due to the finite size of the samples. We find good straight-line behaviour over the concentration range \( 11.5\% \leq p \leq 50\% \) with a value of the exponent of \( \rho = 1.66 \pm 0.02 \) over the entire range. At lower concentrations (\( p \leq 10.5\% \)), as shown in figure 6, a straight-line fit to the data is no longer feasible, and clearly there is a faster fall-off in \( N(l) \) with increasing \( l \) than a power law indicating that the effects of frustration are extremely localised. This is strong confirmation that \( p_c = 11.0 \pm 0.5\% \).

4. The \( \pm J \) system (correlation function)

We now turn to recent developments [11] of our approach to address spin correlations. To our knowledge the only direct computations of spin correlation functions in the ground state are those of Ozeki [29]. This work used a numerical transfer matrix method with long thin samples of circumference \( L \) wrapped around a cylinder of length \( 9L \) with open ends. The maximum possible circumference for this study was limited to \( L = 12 \), due to the transfer matrix computational requirements scaling exponentially with \( L \).

All other attempts to study spin correlations in the ground state have been indirect, involving extrapolation from finite temperature. Monte Carlo techniques have been applied to obtain results for quite low temperatures, for example \( 0.86J \) [30] and \( 0.63J \) [31]. However, the reliability of the extrapolations to zero temperature is hard to assess.

The transfer matrix method has also been used to obtain spin correlations functions at finite temperature [23, 24, 32], although the sample shape restrictions remain severe with cylindrical
circumferences limited to $L \leq 20$. A related approach where the computational requirements scale as $L^3$, rather than exponentially, is the network model [26]. This has been successful in studying systems without severe shape restrictions, but only at finite temperature. The zero temperature limit remains inaccessible.

4.1. Development of the formalism

We need to consider the matrix $C$ that appears in Eq. (8) for the correlation function. At zero temperature $C = D_0$ since $t = 1$ at $T = 0$. At finite temperature we again consider an expansion in powers of $\delta$. For path bonds only we expand $t^{-1}$ while for other bonds it is $t$ as before. The expansion is given by

$$C = D_0 + \delta(D_1 + 2V) + (\delta^2 + \delta^3 + \cdots) V,$$

where $V = -D_1$ across a path bond. All other elements of $V$ are zero.

Unlike $D$ in equation (13) which is written exactly in terms of a perturbation linear in $\delta$, $C$ is expressed as a series in powers of $\delta$, at least as far as the path bonds are concerned. At first sight, it appears that lifting the ground state eigenvalue degeneracy while keeping track of the powers of $\delta$ in a perturbation expansion is going to be an intractable task. However there exists a remarkable property of the continuum propagator, $g_c$, that was introduced in Eq. (21) that does allow us to efficiently achieve this goal. It can be written as the sum of two terms

$$g_c = g_{c1} + g_{c2}. \quad (42)$$

Like many of the other quantities the occur in the formalism, $g_c$ is spatially very localised. $g_{c1}$ is $4 \times 4$ block diagonal in the four bond basis states associated within a plaquette and $g_{c2}$ is $2 \times 2$ block diagonal in the two bond basis states associated with a bond. We can now state an important theorem concerning the determination of $\det C$.

**Theorem. In the ground state limit, the solution for $\det C$ using**

$$C = D_0 + \delta(D_1 + 2V) + (\delta^2 + \delta^3 + \cdots) V$$

$$g_c = g_{c1} + g_{c2} \quad (43)$$

**is exactly the same as the solution using**

$$C = D_0 + \delta(D_1 + 2V)$$

$$g_c = g_{c1} \quad (44)$$

Basically, the effects of $g_{c2}$ exactly cancel the awkward nonlinear terms in $\delta$. We are able to simply rerun the perturbation theory for $D_1 + 2V$ instead of $D_1$, noting that this only involves sign changes across path bonds (since $V = -D_1$). The proof of this theorem arises within an extension of the procedure described in Section 3.1 and the full details are given in ref. [11].

In ref. [8] it was proven that $\det D$ is proportional to the ground state degeneracy (an equivalent statement is Eq. (12)). In analogy, we can consider $\det C$ as proportional to an effective degeneracy for the reciprocal defect system. In fact we can usually write

$$<\sigma_0 \sigma_R>^2 = \exp(2(S_C(R) - S_D)) \quad (45)$$

where $S_D$ is the actual system entropy and $S_C$ is the analogue for the reciprocal defect system. This result is valid if the effective energy of the reciprocal defect system is equal to actual energy. Otherwise the correlation function is zero. We are unaware of previous discussions of this result elsewhere in the literature.
Figure 7. The spin correlation function on $9L \times L$ lattices with $L = 12(+), 16(\times), 32(*)$ and 64($\Box$). The curves are a best fit to Eq. (46).

4.2. Results
We have calculated correlation functions with paths of length $R$, in units of the lattice spacing, parallel to the axes of long thin cylinders of dimensions $9L \times L$ with $L = 12$, 16, 32 and 64. The results are displayed in figure 7. The error bars indicate statistical uncertainties equal to two standard deviations, meaning that we have 95.4 per cent confidence intervals. The configurational averages are over $10^5$ random samples, except for $L = 64$ where only $10^4$ samples were obtained. The uncertainties are approximately independent of either $L$ or $R$, being only strongly dependent on the number of samples. With $10^5$ samples, they are all in the range 0.0021 to 0.0026. Consistently, the corresponding interval with $10^4$ samples is 0.0073 to 0.0086, about a factor of $\sqrt{10}$ larger. The curves are fits of the data to the expected scaling form

$$[<\sigma_0\sigma_R>^2]_{av} \sim R^{-\eta} \exp(-R/\xi),$$

where the exponential factor is included to deal with the effects of finite sample size. The $\chi^2$ nonlinear fits were done according to the Levenberg-Marquardt method [33] with statistical uncertainties fully incorporated.

It is found that both the coefficient $A$ and the exponent $\eta$ are approximately independent of $L$. The values of $\eta$ are 0.12(6), 0.14(3), 0.15(1) and 0.14(2). In contrast, the correlation length $\xi$ does not vary like $L$ as would be expected from finite size scaling theory. In fact we find that it varies much like $L^{3/2}$. Indeed, the respective values of $\xi/L^{3/2}$ are 0.41(6), 0.41(4), 0.40(3) and 0.47(10). It is also found that the finite size corrections to the entropy for the $9L \times L$ system scale like $L^{-3/2}$ (scaling as Eq. (39) applies with the $L \times L$ lattice). It is clear that the correlation functions remain intimately associated with these corrections to the entropy as is highlighted in equation (45). It is interesting to note that Lukic et al [34] also observe finite size corrections to the ground state entropy scaling like $L^{-3/2}$, although the system geometry is different.

Figure 8 shows the same data plotted against $L^{-3/2}$ for $R = 4$, 8, 16 and 32 and provides good confirmation of the behaviour of the correlation length. The $\chi^2$ fits are to the form $\alpha(R) \exp(-\beta(R)L^{-3/2})$ and were done initially with linearized data before polishing with nonlinear fits which gave the uncertainties for the fitting parameters $\alpha(R)$ and $\beta(R)$. A
subsequent fit of $\alpha(R)$ to a power function $R^{-\eta}$ gave $\eta = 0.14(1)$ where the error estimate is that accumulated from both stages.

Our best estimate of the exponent $\eta$ is $0.14 \pm 0.01$. A wide range of values for $\eta$ have been published in the literature over a number of years. Most of these indicate a value around 0.2, somewhat higher than ours, although in most cases extrapolated Monte Carlo data was used rather than working directly on the ground state. It is also interesting to add that a recent calculation [17] (which involved extrapolated Monte Carlo data) reported $\eta = 0.138 \pm 0.005$ in apparent agreement with our result.

5. Conclusions

The formalism at the level expressed in Eqs. (4) and (6) has, of course, very general validity. The development we have described from that starting point is very specific to the the $\pm J$ model, and it is one of the very few approaches available that enables one to access directly the ground state without the uncertainties of an extrapolation from finite temperatures. This is particularly important for the calculation of the spin correlations. Monitoring the behaviour of the defect states also gives one a very direct insight into the underlying physics of the problem because of the way these states reflect very closely the spatial range over which the frustration exerts an influence.

We have also used the method to study the $\pm J$ model on a triangular lattice [10]. Unlike the square lattice for which the concentration region $0.5 \leq p \leq 1$ is identical to the $0 \leq p \leq 0.5$ range via a gauge transformation, the whole range of $p$ is distinct for the triangular lattice. At $p = 0$ we have a pure ferromagnet, while $p = 1$ results in an antiferromagnet that is fully frustrated. We find that there are two critical concentrations between which we have a spin-glass regime that has much in common to the system described in this paper; one can, for example, study the extension of the defect states in the intermediate regime and extract an exponent $\rho$ that is essentially identical in value to that discussed here, implying a common universality class. One can also identify the higher critical concentration as marking a percolation threshold for frustrated plaquettes.
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