SPECIAL RELATIVITY, EINSTEIN VELOCITY ADDITION, AND
GYROGROUPS: AN INTRODUCTION

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Abstract. In these notes we give an introductory unified treatment to the topics of special relativity, Lorentz transformations and the Lorentz group, Einstein velocity addition, and gyrogroups and gyrovector spaces. An effort has been made to present the material in a manner that is accessible to non-specialists and graduate students, and may even serve as the basis for a graduate course or seminar.

The material for this article had its origin in a graduate seminar taught by the author as a first introduction to the mathematics of special relativity with a particular focus on Einstein velocity addition and its encoding in the language and structure of gyrogroups and gyrovector spaces. The favorable reception of this material encouraged the author to make this material and its shortened and simplified presentation more widely available. Most of the following material can be found in much greater detail and depth in A. A. Ungar’s monograph [3]. A somewhat variant, but overlapping, approach can be found in Chapter 1 of Y. Friedman’s monograph [1]. It is the hope of the author, however, that what follows might be more suitable for a first look at the material or for an introductory seminar.

1. Introduction to Special Relativity

A common understanding among the ancients of physical dynamics was that objects near the earth left to themselves would move themselves as close to the center of the earth as possible. Heavenly objects, on the other hand, were perfectly formed objects that would move in perfect circles around the earth. Close observation, however, revealed that this was not true for the planets, so Ptolemy used orbits described by epicycles (paths obtained from a circular motion around a center, which is also moving in a circular motion) and eccentric circles (with the earth not at the center) to model the movement of heavenly bodies, in particular the planets.
1.1. The Principle of Inertia. Such ideas persisted until the time of Galileo. His experiments with objects rolling down ramps and other physical and mental experiments led him to the conclusion that objects free from external influence would either remain at rest or move in a straight line at a constant speed. This is sometimes known as Galileo’s Principle of Inertia and was popularized as Newton’s first law. We recall from the vector geometry of \( \mathbb{R}^3 \) that an object moving in a straight line at constant speed, or equivalently moving at some constant velocity \( \mathbf{v} \in \mathbb{R}^3 \), has a parametric description of the form \( \mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} \) (the solution of \( \dot{\mathbf{x}}(t) = \mathbf{v}, \mathbf{x}(0) = \mathbf{x}_0 \)), where we may view \( t \) as the time parameter. Such motion is called rectilinear motion (or linear motion).

1.2. Galilean relativity. Further, Galileo and Newton assumed a principle of relativity for dynamics: all observers in a system at rest or moving with a constant velocity will encounter the same laws of dynamics (the physics of objects or masses and their movements). For example, a tennis player would experience no difference playing tennis on land or playing in the depths of a large ocean liner sailing on a smooth sea at a constant velocity.

More formally, we can define an inertial frame, reference frame, or simply frame as a coordinatization of space-time \( \mathbb{R} \times \mathbb{R}^3 \) in which rectilinear motion has the description \( \mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} \) for \( t \in \mathbb{R} \) and \( \mathbf{x}_0, \mathbf{v} \in \mathbb{R}^3 \). We alternatively say that Newton’s first law of motion, i.e., Galileo’s Principle of Inertia, holds. Suppose that the coordinates of a frame \( S' \) can be computed from those of \( S \) by a basic Galilean transformation:

\[
\mathbf{x}' = \mathbf{x} - t\mathbf{v} \text{ and } t' = t,
\]  

(1.1)

where \( \mathbf{x} \in \mathbb{R}^3, t \in \mathbb{R} \) are the coordinates in the first frame \( S \) and \( \mathbf{x}', t' \in \mathbb{R} \) are coordinates in the second \( S' \). Then we say that the inertial frame \( S' \) is moving away from the frame \( S \) at the (constant) velocity \( \mathbf{v} \in \mathbb{R}^3 \). Note that the two coordinate systems agree at time \( t = 0 \).

Problem 1.1. Find the Galilean transformation if a second inertial frame \( S' \) is moving away from the first \( S \) at velocity \( \mathbf{v} \) and the second one has coordinate \( \mathbf{x}_0 \) for the origin of the first at time \( t = 0 \).
Problem 1.2. Show that linear motion is preserved by basic Galilean transformations, and hence one of the coordinate systems is an inertial frame iff the other one is.

Problem 1.3. (i) Suppose $S'$ is computed from $S$ by the Galilean transformation $x' = x - tv$, $t' = t$, and that an object is moving with constant velocity $w$ with respect to the frame $S'$. Find its velocity with respect to $S$.
(ii) Suppose $S'$ is moving away from $S$ at constant velocity $v$ and $S''$ is moving away from $S'$ at constant velocity $w$. How fast is $S''$ moving away from $S$?

Problem 1.4. Show that the inverse of a basic Galilean transformation is a Galilean transformation and that the composition of two is another.

Problem 1.5. Show that a basic Galilean transformation is a linear map from $\mathbb{R}^3 \oplus \mathbb{R} \approx \mathbb{R}^4$ to itself.

Problem 1.6. (i) Show for a basic Galilean transformation that $x_2' - x_1' = x_2 - x_1$ (more precisely, this means that if the transformation carries $(x_1, t)$ to $(x_1', t')$ and $(x_2, t)$ to $(x_2', t')$, then the asserted equality holds). (ii) Think about the previous result long enough for it to be intuitively obvious (but you may quit in 24 hours).
(iii) Formulate a version of (i) for two frames $S'$ and $S''$ moving away from a fixed frame $S$ at velocities $v'$ and $v''$ resp.

The next problem illustrates how the quantities in the Newtonian law of motion $F = ma$ are unaltered by Galilean transformations.

Problem 1.7. Consider two point masses in space of mass $m_1$ and $m_2$ and suppose that the force between them depends only on their separation:

$$F(x_1, x_2) = f(x_1 - x_2) \quad (1.2)$$

when mass $m_1$ is at $x_1 \in \mathbb{R}^3$ and mass $m_2$ is at $x_2 \in \mathbb{R}^3$.
(i) Show that for a basic Galilean transformation from $S$ to $S'$, $F(x_1', x_2') = F(x_1, x_2)$.
(ii) From Newton’s second law $F = ma$, conclude that $a_2 = a_2'$, where $a_2$ resp. $a_2'$ is the acceleration of $m_2$ in the inertial frame $S$ resp. $S'$. (In Newtonian dynamics the inertial mass $m$ is a constant under Galilean transformations.)
1.3. The Einstein postulates. In the nineteenth century physicists postulated the existence of an “ether” in space that would enable the transmission of electromagnetic and light waves in space (based on their experience with sound waves, water waves, vibrating strings, etc.). However, all efforts to experimentally verify this ether, such as the famous Michaelson-Morley experiment, failed completely to detect any such ether. This led physicists to consider further explanations and culminated in the original and penetrating insights of Albert Einstein, who introduced his theory of special relativity in 1905. Einstein based his theory on two basic postulates:

**Postulate 1:** All inertial frames are equivalent with respect to all the laws of physics.

**Postulate 2:** The speed of light in empty space has the same value $c$ in any inertial frame.

The first postulate expanded on the relativity principle of Galileo and Newton by assuming its validity for all laws of physics, in particular those of electricity and magnetism (Maxwell’s equations were not preserved by general Galilean transformations). It is a tribute to the insight of Einstein that a whole new dynamics can be built upon these two brief statements.

One of the startling consequences of Einstein’s postulates was the inference that time was relative, i.e., dependent on one’s frame of reference. Newton has assumed a universal time: “Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external.” Einstein recognized that judgements about time were inextricably tied up with judgments about simultaneity, but defining the latter was problematic if information can only be transmitted at a finite speed.

Einstein used the large but finite speed $c$ of electromagnetic signals (in a vacuum) for relating time measurements. For example, he introduced the following method of synchronizing clocks at different locations in a given inertial frame. If a signal starts at a point $A$ at time $t = t_0$, is reflected back by a mirror at $B$, and returns to $A$ at time $t_1$, then the time at which the signal reached $B$ is defined as $t_0 + (1/2)\Delta(t)$, where $\Delta(t) := t_1 - t_0$. In this manner clocks at all points can be synchronized in the given frame.

1.4. Spacetime. In special (and general) relativity time is relative in the sense that it depends on the inertial frame we choose and varies from frame to frame. We thus
no longer refer to “space-time”, but to “spacetime” since the two are inextricably bound together. Mathematically we may think of spacetime as an “uncoordinatized” four-dimensional space $M$, a manifold. Members of spacetime $M$ are called *events*. We assume that we have a notion of rectilinear motion in $M$, and that there exist *inertial frames*, bijections $S$ from $\mathbb{R} \times \mathbb{R}^3$ to $M$ that endow $M$ with a coordinate system such that rectilinear motion in $M$ corresponds to rectilinear motion in the frame, i.e., has the form $t \mapsto (t, x_0 + tv)$ in the coordinates of the frame $S$. In the coordinates of any reference frame, the first coordinate of the frame is the *time coordinate* and the last three are called the *space coordinates*. We assume that an unhindered light signal, or more generally an electromagnetic signal, moves rectilinearly with speed $c$ in any inertial frame and that this is the maximal attainable speed. A *world line* is a mapping from $\mathbb{R}$ into $M$ which is given by $t \mapsto (t, x(t))$ in the coordinates of a frame, where $x(\cdot) : \mathbb{R} \to \mathbb{R}^3$ is a continuous, piecewise smooth path with speed bounded by $c$.

**Problem 1.8.** Suppose that $\alpha(t) = (t, x_0 + tv)$ describes rectilinear motion in $M$ for some frame $S$. Argue that $\alpha((s + t)/2)$ is the midpoint of $\alpha(t)$ and $\alpha(s)$ in $\mathbb{R} \oplus \mathbb{R}^3$.

**Problem 1.9.** Argue that a coordinate frame composed with the inverse of another coordinate frame preserves rectilinear motion.

**Problem 1.10.** Show that if two coordinate systems have the same spacetime origin, then the map in the preceding problem is a linear one. (Hint: Use Problem 1.8 to show the map is additive, i.e., preserves vector addition.)

1.5. *Minkowski diagrams.* Given two inertial frames moving at constant velocity with respect to each other, we may conveniently recoordinatize the frames by taking as the $x$-axis the direction of the velocity and assuming the coordinate systems agree at time $t = 0$. The Galilean transformation between the systems then simplifies to

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$  \hfill (1.3)

To illustrate graphically various features of special relativity for the two inertial frames, we suppress the $y$ and $z$ coordinates and draw two-dimensional space-time diagrams with the $x$-axis horizontal and the $t$-axis, scaled by a factor of $c$, vertical.
We assume that the $x, t$-coordinates represent the first coordinate system, which is at rest with respect to the second one, which is moving at a velocity $v$ in the $x$-direction.

We draw world lines on the diagram that display the complete history of a one-dimensional motion. We assume that time is synchronized at all points with respect to the given frame, and then the world line of a moving object consists of all positions it occupies together with the time at that position. For example, objects at rest within the given frame of reference have world lines that are vertical lines with the constant $x$-coordinate being their fixed position. If we take any fixed point $B$ at $x_B$ on the $x$-axis and send out a light signal in the positive $x$-direction at time $t = 0$, then the world line of the signal is a line of slope 1 emanating from $x_B$ at $t = 0$.

**Problem 1.11.** Suppose that an object is moving with constant velocity $v$ along the $x$-axis. Show that its world line has slope $c/v \geq 1$.

### 1.6. Simultaneity

Consider three observation stations $A < B < C$ equally spaced along the $x$-axis in a Minkowski diagram for an inertial frame $S$ at points $x_A, x_B, x_C$. Assume first that $A, B, C$ are at rest in this frame. Then their world lines are vertical. Suppose that a light signal is sent out from $B$ at time $t = 0$, traveling at speed $c$ both forward and backward along the $x$-axis.

**Problem 1.12.** (i) Argue that if the clocks at $A$ and $C$ are synchronized in $S$, then the two signals must strike $A$ and $C$ at the same time.

(ii) Show that (i) is equivalent to the point of intersection of the world line for $A$ and the world line for the signal moving to the left having the same $t$-coordinate as the point of intersection of the world line for $C$ and the world line for the signal moving to the right. Illustrate with appropriate world lines in a Minkowski diagram.

In light of (ii) of the preceding problem, we see that a line of simultaneous events is a line for which $t$ is constant, i.e., a horizontal line or line parallel to the $x$-axis.

Now suppose that $A, B, C$ are all moving with speed $v$ along the $x$-axis, i.e., are at rest in the inertial frame $S'$ that is moving with respect to $S$ at a speed $v$ along the $x$-direction. Again a signal is sent from $B$ at time $t = 0 = t'$, and by the preceding reasoning its reaching of $A$ and $C$ must be simultaneous events in the frame $S'$. However in $S$ the signal travels further to the receding $C$ than to the approaching $A$, and hence the two events are no longer simultaneous in $S$. Thus simultaneity of
events depends on the frame of reference, if we demand that the speed of light have the same value $c$ in all reference frames.

Problem 1.13. (i) Draw the world lines for $A$, $B$, $C$, and the signals in the reference frame $S$. A line of simultaneity for $S'$ is given by the (oblique) line connecting the point of intersection of the world line for $A$ and the world line for the signal moving to the left and the point of intersection of the world line for $C$ and the world line for the signal moving to the right.

(ii) The axes for the moving frame $S'$ with respect to the stationary frame $S$ can be found by taking the $t'$-axis to be the line through the origin parallel to the world lines of $A$, $B$, and $C$, and the $x'$-axis the line through the origin parallel to any line of simultaneity. Draw these lines in the Minkowski graph.

2. Lorentz Transformations

Suppose that one observer uses an inertial coordinate frame $S$ with coordinates $(ct, x, y, z)$, and another observer uses another inertial coordinate frame $S'$ given by coordinates $(ct', x', y', z')$. We assume that the frames are in motion with respect to each other so that the $S$-frame observer observes the other to be moving at speed $v$ in the $+x$-direction and the $S'$-frame observer observes the other to be moving at the same speed in the $-x'$-direction, with the $y, z$-coordinates being the same for both observers. We further assume that the space-time origin agrees for the two frames. Let us suppose we know the location of an event according to one observer’s coordinate frame and wish to determine the location according to the other coordinate frame. We transform the coordinates of the event from the one to the other by doing a Lorentz coordinate transformation. A reasonable guess for the Lorentz coordinate transformation equations is to do a slight generalization of the Galilean case, namely multiply by a constant $\gamma$, at least for the space coordinates:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z;$$

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z'.$$  \quad (2.4)

Note by the first equation that the world line of the origin, $x' = 0$, in $S'$ transforms in $S$ to $x = vt$ and conversely $x = 0$ transforms to $x' = -vt$, justifying the choice of $v$ for the $t$-coefficient.
If a light signal originates from the origin in the $x$-direction, then it has world line $x = ct$ in $S$ and $x' = ct'$ in $S'$ by Einstein’s Postulate 2. Substituting these values into (2.4) yields for any point on the world line of the signal

\[
ct' = \gamma(ct - vt) = \gamma(c - v)t \\
ct = \gamma(ct' + vt') = \gamma(c + v)t';
\]

Transposing the second equation, then dividing the two equations and cross-multiplying yields

\[
c^2 = \gamma^2(c^2 - v^2)
\]

Solving for $\gamma$ yields

\[
\gamma_v := \gamma = \frac{1}{(1 - v^2/c^2)^{1/2}} \text{ for } |v| < c. \tag{2.5}
\]

The constant $\gamma_v$ is sometimes called the Lorentz factor.

**Problem 2.1.** Solve the system (2.4) to obtain

\[
ct = \gamma(ct' + vx'/c) \\
ct' = \gamma(ct - vx/c)
\]

Combining the previous results, we can express the Lorentz transformations by

\[
t' = \gamma(t - vx/c^2) \\
x' = \gamma(x - vt), \ y' = y, \ z' = z. \tag{2.7}
\]

and

\[
t = \gamma(t' + vx'/c^2) \\
x = \gamma(x' + vt'), \ y = y', \ z = z'. \tag{2.8}
\]

The previous Lorentz transformations are called a Lorentz boost and often written in matrix form, for example:

\[
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix}
= \begin{bmatrix}
\gamma & -\gamma v/c & 0 & 0 \\
-\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}. \tag{2.9}
\]
Problem 2.2. Show that the limit as $v/c \to 0$ of the Lorentz transformation (2.7) reduces to a Galilean transformation.

Problem 2.3. A frame $S'$ has velocity 0.6$c$ in the $x$-direction of frame $S$ and the origins of the two frames coincide at $t = 0 = t'$.

(i) If an event occurs at $x' = 60$m, $t' = 8 \times 10^{-8}$sec in $S'$, what are the spacetime coordinates of this event in $S$?

(ii) If an event occurs in $S$ at $t = 2 \times 10^{-7}$sec and $x = 50$m, what time does it occur in $S'$?

(iii) If a second event (after (ii)) occurs at $(3 \times 10^{-7}$sec,10m) in $S$, what is the time interval between the events in $S'$?

Problem 2.4. Two events have coordinates in the frame $S$ as follows:

Event 1: $x_1 = x_0$, $t_1 = x_0/c$ ($y_1 = 0 = z_1$)
Event 2: $x_2 = 2x_0$, $t_2 = x_0/2c$ ($y_2 = 0 = z_2$).

(i) There exists a frame in which these two events are simultaneous. Find the velocity $v$ of this frame with respect to $S$.

(ii) What is the value of $t'$ for when both events occur in this $S'$?

Problem 2.5. An observer does not have a complete view of what is happening everywhere in his reference frame at a given instant; he is aware only of what is happening at his location at that instant. Suppose a meter stick pointing in the $x$-direction moves along the $x$-axis with speed 0.8$c$ in frame $S$, with its midpoint passing through the origin 0 at time $t = 0$. Assume that an observer in frame $S$ is located at $x = 0$ and $y = 1$m. Assume $c = 300,000$ km/sec.

(i) In $S$, where are the ends of the meter stick at time $t = 0$? ($\pm 0.3$m)

(ii) When does the observer see the midpoint pass the origin? ($0.33 \times 10^{-8}$ sec)

(iii) Where do the endpoints appear to be at this time? ($0.27$m, -0.34m)

2.1. Time dilation and length contraction. For a coordinate frame $S'$ moving with velocity $v$ along the $x$-axis of a rest frame $S$, we can rewrite the Lorentz transformation and its inverse in terms of coordinate differences to obtain

$$\Delta t' = \gamma (\Delta t - v \Delta x/c^2)$$
$$\Delta x' = \gamma (\Delta x - v \Delta t).$$
and

\[ \Delta t = \gamma (\Delta t' + v\Delta x' / c^2) \]
\[ \Delta x = \gamma (\Delta x' + v\Delta t') . \]

Suppose we have a clock at rest in the system $S$. Two consecutive ticks of this clock are then characterized by $\Delta x = 0$. If we want to know the relation between the times between these ticks as measured in both systems, we can use the first equation and find: $\Delta t' = \gamma \Delta t$ (for events in which $\Delta x = 0$). Since $\gamma = \gamma_v = 1 / \sqrt{1 - v^2 / c^2} > 1$, this shows that the time $\Delta t'$ between the two ticks as seen in the ‘moving’ frame $S'$ is larger than the time $\Delta t$ between these ticks as measured in the rest frame of the clock. This phenomenon is called *time dilation*.

Similarly, suppose we have a measuring rod at rest in the unprimed system $S$. In this system, the length of this rod is written as $\Delta x$. If we want to find the length of this rod as measured in the ‘moving’ system $S'$, we must make sure to measure the distances $x'$ to the end points of the rod simultaneously in the primed frame $S'$. In other words, the measurement is characterized by $\Delta t' = 0$, which we can combine with the fourth equation to find the relation between the lengths $\Delta x$ and $\Delta x'$:

\[ \Delta x' = (1 / \gamma) \Delta x \] for events satisfying $\Delta t' = 0$.

This shows that the length $\Delta x'$ of the rod as measured in the ‘moving’ frame $S'$ is shorter than the length $\Delta x$ in its own rest frame. This phenomenon is called *length contraction* or *Lorentz contraction*. These effects are not merely appearances; they are explicitly related to our way of measuring time intervals between events which occur at the same place in a given coordinate system (called “co-local” events). These time intervals will be different in another coordinate system moving with respect to the first, unless the events are also simultaneous. Similarly, these effects also relate to our measured distances between separated but simultaneous events in a given coordinate system of choice. If these events are not co-local, but are separated by distance (space), they will not occur at the same spacial distance from each other when seen from another moving coordinate system.
Problem 2.6. The nearest star Centauri is 4.2 light years distance from earth. How long would it take a space ship traveling at \((2/3)c\) to reach the star, according to its internal clock.

Problem 2.7. A rocketship of proper length \(l_0\) travels at constant velocity \(v\) in the positive \(x\)-direction relative to frame \(S\). The nose of the ship \(A'\) passes over the point \(A\) on the \(x\)-axis at \(t = 0 = t'\), and at that instant a light signal is sent in the negative \(x\)-direction to the tail of the ship \(B'\).

(i) When by rocketship time \(t'\) does the signal reach the tail \(B'\) of the ship? \((l_0/c)\)

(ii) At what time \(t_1\), measured in \(S\), does the signal reach \(B'\)?

(iii) At what time \(t_2\) in \(S\) does the tail of the ship \(B'\) pass \(A\)?

Problem 2.8. In a reference frame \(S\) a flash of light if emitted at position \(x_1\) on the \(x\)-axis and is absorbed at \(x_2 = x_1 + l\). In a frame \(S'\) moving with velocity \(v = \beta c\) along the \(x\)-axis:

(i) What is the spatial separation \(l'\) between the point of emission and point of absorption of the light?

(ii) How much time elapses (in \(S'\)) between the emission and absorption of the light?

2.2. Lorentz boosts. The preceding considerations generalize in a straightforward way to the setting of two frames \(S\) and \(S'\), both with the same space-time origin. We assume that \(S'\) is moving within frame \(S\) at a velocity \(\mathbf{v}\) in the coordinate system of the frame \(S\), and hence that \(S\) is moving within \(S'\) at velocity \(-\mathbf{v}\) in the coordinate system of the frame \(S'\). Let \(\mathbf{x} \in \mathbb{R}^3\) and let \(x_\parallel\) be the orthogonal projection of \(x\) onto the line \(\mathbb{R}v\) and \(x_\perp\) be the orthogonal projection onto the hyperplane subspace \(v^\perp\) perpendicular to \(v\). Then by a straightforward generalization of the preceding calculations we have for \(\gamma_v := \frac{1}{\sqrt{1 - (v/c)^2}}\) for \(||v|| < c\)

\[
ct' = \gamma_v (ct - \frac{v \cdot x}{c})
\]

\[
x' = x_\perp + \gamma_v (x_\parallel - tv)
\]
By standard vector geometry $x_\parallel = (\mathbf{v} \cdot \mathbf{x})\mathbf{v}/|\mathbf{v}|^2$ and $x_\perp = \mathbf{x} - (\mathbf{v} \cdot \mathbf{x})\mathbf{v}/|\mathbf{v}|^2$; hence we can alternatively write

$$
ct' = \gamma_v (ct - \frac{\mathbf{v} \cdot \mathbf{x}}{c})
$$

$$
x' = \mathbf{x} - \frac{\mathbf{v} \cdot \mathbf{x}}{|\mathbf{v}|^2} \mathbf{v} + \gamma_v \frac{\mathbf{v} \cdot \mathbf{x}}{|\mathbf{v}|^2} (\mathbf{v} - t\mathbf{v})
$$

$$
= \mathbf{x} - \gamma_v t\mathbf{v} + \frac{\gamma_v - 1}{|\mathbf{v}|^2} (\mathbf{v} \cdot \mathbf{x})\mathbf{v}.
$$

(2.11)

Reversing the roles of $S$ and $S'$, we obtain

$$
ct = \gamma_v (ct' + \frac{\mathbf{v} \cdot \mathbf{x}'}{c})
$$

$$
x = \mathbf{x}' + \gamma_v t'\mathbf{v} + \frac{\gamma_v - 1}{|\mathbf{v}|^2} (\mathbf{v} \cdot \mathbf{x}')\mathbf{v}.
$$

(2.12)

The latter coordinate transformation is called the Lorentz boost along $\mathbf{v}$, and can be written in matrix notation (and unprimed coordinates) as

$$
B(\mathbf{v}) \begin{bmatrix} ct \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \gamma_v (ct + \frac{\mathbf{v} \cdot \mathbf{x}}{c}) \\ \mathbf{x} + \gamma_v t\mathbf{v} + \frac{\gamma_v - 1}{|\mathbf{v}|^2} (\mathbf{v} \cdot \mathbf{x})\mathbf{v} \end{bmatrix}
$$

(2.13)

**Problem 2.9.** Suppose that the frame $S'$ is moving within frame $S$ at velocity $\mathbf{v} = \langle 1, 1, 1 \rangle$. Write out the equations for $ct$, $x$, $y$, and $z$ in terms of $ct'$, $x'$, $y'$, $z'$.

**Problem 2.10.** Find a matrix representation for the Lorentz boost $B(\mathbf{v})$.

2.3. **Minkowski spacetime.** In physics and mathematics, Minkowski space (or Minkowski spacetime) is the standard mathematical setting for Einstein’s theory of special relativity. In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional manifold for representing a spacetime. Minkowski space is named after the German mathematician Hermann Minkowski, who introduced it in 1908.

*Minkowski spacetime* $\mathcal{M}$ is defined to be the vector space $\mathbb{R}^4$ equipped with the symmetric bilinear form

$$
\eta \left( \begin{bmatrix} ct \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} ct' \\ \mathbf{x}' \end{bmatrix} \right) = -c^2 tt' + \mathbf{x} \cdot \mathbf{x}', \text{ for } t, t' \in \mathbb{R}, \mathbf{x}, \mathbf{x}' \in \mathbb{R}^3.
$$

(2.14)
If we define

\[
I_{1,3} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

then we may alternatively define \( \eta(a, b) = \langle a, I_{1,3}b \rangle \), where \( \langle a, b \rangle = a \cdot b \), the usual euclidean inner product. We call \( \eta \) the Lorentzian form of the spacetime \( \mathcal{M} \).

A linear transformation \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) is said to preserve the form \( \eta \) if \( \eta(Tx, Ty) = \eta(x, y) \) for all \( x, y \in \mathbb{R}^4 \). It follows from

\[
\eta(Tx, Ty) = \langle Tx, I_{1,3}Ty \rangle = \langle x, T^T I_{1,3} Ty \rangle
\]

that \( T \) preserves \( \eta \) iff \( I_{1,3} = T^T I_{1,3} T \), where \( T^T \) is the adjoint of \( T \). We summarize:

**Proposition 2.11.** A linear transformation \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) preserves \( \eta \) iff \( I_{1,3} = T^T I_{1,3} T \) iff \( I_{1,3} T^{-1} = T^T I_{1,3} \).

**Problem 2.12.** Show that the Lorentz boost

\[
B(v) = \begin{bmatrix}
\gamma_v & \frac{\gamma v^T}{c} \\
\frac{\gamma v}{c} & I + \frac{\gamma - 1}{|v|^2} vv^T
\end{bmatrix}
\]

preserves \( \eta \). (Hint: Use the preceding proposition and the fact that \( (B(v))^{-1} = B(-v) \).)

Thus while Lorentz boosts do not preserve time or distance, they do preserve the form \( \eta \), i.e., \( \eta \) is an invariant for Lorentz boosts. This fact can frequently simplify special relativity calculations.

**Problem 2.13.** Two events occur at the same place and 4 seconds apart in inertial frame \( S \). What is their spatial separation in a frame \( S' \) in which the events are 6 seconds apart?

**Problem 2.14.** Two events occur at the same time in an inertial frame \( S \) and are separated by a distance of 1 km along the \( x \)-axis. What is the time difference between these two events as measured in a frame \( S' \) moving with constant velocity along the \( x \)-axis for which the spatial separation of the two events is measured as 2 km.
2.4. **Minkowski diagrams revisited.** We have seen earlier that in the Minkowski diagram with coordinate axes determined by the frame $S$, a frame $S'$ moving along the $x$-axis at velocity $v$ has one axis $x = vt = (v/c)(ct)$ (corresponding to $x' = 0$) and the other axis the line through the origin parallel to any line of simultaneity (corresponding to $t' = 0$).

**Problem 2.15.** Find an equation for the second axis. Show that the two axes for $S'$ make the same angle with the diagonal, one on each side of it.

In $S'$ the unit along the $x'$-axis has coordinates $x' = 1, t' = 0$. Thus $-(ct')^2 + (x')^2 = 1$. Since the Lorentz transformation to $S$ must preserve the Lorentzian form, it follows that the unit on the $x'$-axis must be the intersection of that axis with the hyperbola $-(ct)^2 + x^2 = 1$ or $x^2 - (ct)^2 = 1$. A similar calculation yields the unit length on the $t'$-axis. Thus to read off the space and time $t', x'$-coordinates of a given point event $P$, draw lines through $P$ parallel to the $x'$- and $t'$-axes, and read off the intercepts.

**Problem 2.16.** Two reference frames $S$ and $S'$ move with speed $c/2$ with respect to each other.

(a) Draw a Minkowski diagram relating the two systems (let the $x$ and $ct$ axes be perpendicular). Draw the calibration hyperbolas that allow you to define distance on the $ct'$ and $x'$ axes.

(b) Plot the following points on the diagram: (1) $x = 1, ct = 1$, (2) $x' = 1, ct' = 1$, (3) $x' = 2, ct' = 0$, (4) $x = 0, ct = 2$.

(c) From your diagram determine the coordinates in the other coordinate system for each point plotted in (b).

3. **The Lorentz Group**

In the preceding section we have considered transformations between certain reference frames moving at a constant velocity with respect to one another. These transformations consist of what we called Lorentz boosts, and we saw that they preserved the Lorentzian form $\eta$. The invertible linear transformations on $\mathbb{R}^4$ that preserve $\eta$ form a group under composition, usually referred to as the generalized orthogonal group $O(1, 3)$. This is also frequently called the Lorentz group, but we prefer to define the *Lorentz group* to be the subgroup of $O(1, 3)$ of time-preserving or
orthochronous $\eta$-preserving invertible linear transformations. One characterization of time preservation is that the vector $(1, 0, 0, 0) \in \mathbb{R}^4$ is carried into a vector with a positive $t$-component. Such transformations we shall call Lorentz transformations or linear isometries of Minkowski space. We denote the Lorentzian group by $O^+(1, 3)$. We extend our notion of inertial or reference frame to include frames arising from the usual coordinates of $\mathbb{R}^4$ by applying a Lorentz transformation.

**Problem 3.1.** Show that a Lorentz boost is a Lorentz transformation in the preceding sense. Show that a transformation of $\mathbb{R}^4$ that leaves the time coordinate fixed and acts as an orthogonal transformation on the space coordinates is a Lorentz transformation.

In physics and mathematics, the Lorentz group is the group of all Lorentz transformations of Minkowski spacetime, the special relativistic setting for all (nongravitational) physical phenomena. The mathematical form of standard physical laws such as the kinematical laws of special relativity, Maxwell’s field equations in the theory of electromagnetism, and Dirac’s equation in the theory of the electron, are each invariant under Lorentz transformations. Therefore the Lorentz group can be said to express a fundamental symmetry of many of the known fundamental laws of nature.

3.1. **Spacetime intervals and causality.** The spacetime interval $s^2$ between two events is defined to be $s^2 = -(c\Delta t)^2 + (\Delta x)^2$, where $\Delta x$ is the distance between the space coordinates. Note that the spacetime interval is invariant under any Lorentz transformation. If $s^2 < 0$, then the interval is said to be **time-like** and the **proper time** of the interval is defined to be $\sqrt{(\Delta t)^2 - (\Delta x/c)^2}$. If $s^2 = 0$, the interval is said to be **light-like**, and if $s^2 > 0$, the interval is said to be **space-like** and the **proper distance** is defined to be $\sqrt{(\Delta x)^2 - (c\Delta t)^2}$. Similarly for an individual element $u \in \mathcal{M}$, the element is called time-like, resp. light-like, resp. space-like depending on whether $\eta(u, u)$ is less than 0, resp. equal to 0, resp. greater than 0.

We note that the light-like elements form a double cone, each cone having a circular cross section. The time-like elements consist of two connected components, one making up the interior of the top light cone and the other the bottom. The time-like and light-like elements together with $t$-coordinate greater than or equal to 0 make up a closed convex cone $K$ with dense interior $\text{Int}(K)$ made up by the time-like vectors. Such cones are called Lorentzian cones.
The cone $K$ induces an order on spacetime $\mathcal{M}$ defined by $u \leq v$ if $v - u \in K$. This order is a partial order called the \textit{causal order}. We write $u < v$ if $u \neq v$ and $u \leq v$ and $u \prec v$ if $v - u \in \text{Int}(K)$. If $u \leq v$, then we say that $u$ has a \textit{potential causal connection} with $v$.

\textbf{Problem 3.2.} Show that the causal order is a partial order.

\textbf{Problem 3.3.} Show that the sets of time-like, space-like, and light-like vectors respectively are preserved by members of $O(1, 3)$.

\textbf{Problem 3.4.} Argue that the time-like vectors consist of two connected components. Argue that an $\eta$-preserving linear transformation is orthochronous iff it preserves the interior of $K$.

\textbf{Problem 3.5.} Show that $K$ and $\text{Int}(K)$ are preserved by Lorentz transformations. Show that the causal order and the order $\prec$ are preserved by Lorentz transformations.

\textbf{Problem 3.6.} Argue that if $u < v$, then in any reference frame the $t$-coordinate of $u$ is less than the $t$-coordinate of $v$.

\textbf{Problem 3.7.} An object is moving with a constant (admissible) velocity $v$ in a reference frame $S$. Argue that events on its world line are getting larger in the causal order as $t$ grows.

\textbf{Problem 3.8.} Answer both questions for the two events in each part: Is there a potential causal connection between the events? Is there a frame in which the two events are simultaneous?

(a) $(2 \times 10^{-9}\text{sec}, 0.3m, 0.5m, 0)$ and $(3 \times 10^{-9}\text{sec}, 0.4m, 0.7m, 0)$.

(b) $(5 \times 10^{-9}\text{sec}, 0.7m, 0.5m, 0)$ and $(4 \times 10^{-9}\text{sec}, 0.4m, 0.6m, 0)$.

\textbf{3.2. Symmetric matrices.} Throughout this section and the next all matrices are assumed to be square matrices with real entries. A matrix $A$ is \textit{symmetric} if $A = A^T$, where $A^T$ denotes the transpose. Let $\text{Sym}(n, \mathbb{R})$, or simply $\text{Sym}$ when $n$ is understood, be the vector space of all $n \times n$ symmetric matrices. For $A \in \text{Sym}$, we recall that $A$ is positive semidefinite, denoted $0 \leq A$, if $x^T A x = \langle x, Ax \rangle \geq 0$ for all $x \in \mathbb{R}^n$, where $\langle \cdot , \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. Similarly $A$ is positive definite, denoted $0 < A$, if it is positive semidefinite and invertible, or equivalently if $x^T A x = \langle x, Ax \rangle > 0$ for
all non-zero $x$. We denote the set of positive definite (semidefinite) matrices by $\text{Sym}^{>0}$ ($\text{Sym}^{\geq 0}$).

The following “internal” characterization of a positive definite matrix involves orthogonal matrices, matrices $U$ such that $U^{-1} = U^T$, and is a standard linear algebra result.

**Proposition 3.9.** A symmetric matrix $A$ is positive definite (semidefinite) if and only if $A = U^T DU$ for some orthogonal matrix $U$ and some diagonal matrix $D$ with positive (non-negative) diagonal entries if and only if $A$ has all eigenvalues positive (non-negative).

By standard matrix spectral theory, we can write $A \in \text{Sym}(n, \mathbb{R})$ uniquely as $A = \sum_{i=1}^{r} \lambda_i E_i$, where $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are distinct, each $E_i$ is a non-zero orthogonal projection, and the collection $\{E_i\}_{1 \leq i \leq r}$ satisfies $\sum_{i=1}^{r} E_i = I$ and $E_i E_j = 0$ for $i \neq j$. Indeed the existence follows by choosing the $\lambda_i$ to be the distinct eigenvalues and each $E_i$ to be the orthogonal projection onto the eigenspace of $\lambda_i$. The uniqueness, on the other hand, follows by (i) observing that for any such decomposition the $\lambda_i$, $i = 1, \ldots, r$, must be eigenvalues and the range of $E_i$ must consist of eigenvectors for $\lambda_i$, (ii) using the equality $\sum_{i=1}^{r} E_i = I$ and the orthogonality $E_i E_j = 0$ for $i \neq j$ to argue that $\mathbb{R}^n$ is the direct sum of the ranges of the $E_i$, and then (iii) deducing that the ranges of the $E_i$ must exhaust the eigenspaces and thus that the $\lambda_i$ must exhaust the eigenvalues. We call $\{\lambda_1, \ldots, \lambda_r\}$ the spectrum of $A$ and $\sum_{i=1}^{r} \lambda_i E_i$ the spectral decomposition.

**Problem 3.10.** Use the spectral decomposition to construct a proof of the representation of a positive definite matrix given in Proposition 3.9. (Hint: Pick an orthonormal basis of eigenvectors and consider a change of coordinates between it and the standard basis.)

For a bijection $f : M_1 \to M_2$, where $M_1, M_2 \subseteq \mathbb{R}$, we define a function on all $A \in \text{Sym}$ with spectrum contained in $M_1$ by $f(A) = \sum_{i=1}^{r} f(\lambda_i) E_i$, where $A = \sum_{i=1}^{r} \lambda_i E_i$ is the spectral decomposition (functions constructed in this way are called matrix functions and provide a simple example of the functional calculus). Note (from uniqueness of spectral decomposition) that $f$ is well-defined and defines a bijection.
from all symmetric matrices with spectrum contained in $M_1$ to all symmetric matrices with spectrum contained in $M_2$ (with inverse defined from $f^{-1} : M_2 \to M_1$).

Extending $\exp : \mathbb{R} \to (0, \infty)$ to Sym, we obtain (in light of Proposition 3.9) that

**Proposition 3.11.** The exponential function $\exp : \text{Sym} \to \text{Sym}^{>0}$ given by

$$\exp A = \exp \left( \sum_{i=1}^{r} \lambda_i E_i \right) = \sum_{i=1}^{r} e^{\lambda_i} E_i$$

is a bijection. In particular, a symmetric matrix is positive definite if and only if it is the exponential of a symmetric matrix.

We recall that the matrix exponential function is more commonly defined by the power series $e^A = \sum_{n=0}^{\infty} A^n / n!$, which converges for all $A$. Since for $x$ in the eigenspace of an eigenvalue $\lambda_i$ of $A$, we have $e^{A}(x) = e^{\lambda_i} x = \exp A(x)$, the matrix operators $e^A$ and $\exp A$ agree on the eigenspaces of $A$, and hence $e^A = \exp A$.

Using the same methods as those employed for Proposition 3.11 we obtain by extending the bijection $f(x) = x^2$ on $(0, \infty)$ (respectively, $[0, \infty)$ ) to the matrices with spectrum contained in $(0, \infty)$ (respectively, $[0, \infty)$), that is, the positive definite (respectively, semidefinite) matrices, the following

**Proposition 3.12.** If $A > 0$ (respectively, $A \geq 0$), then $A$ has a unique positive definite (respectively, semidefinite) square root, denoted $A^{1/2}$.

For any $A > 0$, by Proposition 3.11 there exists a unique symmetric matrix $\log A$ such that $\exp(\log A) = A$. We can thus define $A^r$ for any $A > 0$ by $A^r = \exp(r \log A)$. Since $\exp(A + B) = \exp(A) \exp(B)$ if $AB = BA$, the one-parameter group $\{ A^r : r \in \mathbb{R} \}$ satisfies the standard laws of exponents. This leads to the

**Proposition 3.13.** The map $\exp$ satisfies

$$A^{t+s} = \exp((t + s)X) = \exp(tX + sX) = \exp(tX) \exp(sX) = A^t A^s$$

for any $X \in \text{Sym}$ and $A \in \text{Sym}^{>0}$ satisfying $A = \exp X$ (equivalently, $X = \log A$).

For $A$ positive definite and $r = 1/2$, the preceding definition for $A^{1/2}$ agrees with that of Proposition 3.12 since $\exp((1/2) \log A)$ is a positive definite square root of $A$ and this square root is unique.
Problem 3.14. Use the fact that \( \exp : \text{Sym} \to \text{Sym}^{>0} \) is bijective and a homomorphism on each one-dimensional subspace to show that each member of \( \text{Sym}^{>0} \) has a unique \( n \)th root in \( \text{Sym}^{>0} \).

3.3. Polar decompositions. A polar decomposition for an invertible matrix \( A \) is a factorization \( A = PU \), where \( P \) is a positive definite matrix and \( U \) is an orthogonal matrix.

Proposition 3.15. (Polar Decomposition) Each invertible matrix \( A \) has a unique polar decomposition \( A = PU \). Furthermore, \( P = \sqrt{AA^T} \) is the unique positive definite square root of \( AA^T \).

Proof. Since \( A \) is invertible, \( \langle x, AA^T x \rangle = \langle A^T x, A^T x \rangle > 0 \) for all \( x \neq 0 \), and hence \( AA^T = (AA^T)^T \) is positive definite. By Proposition 3.12, \( AA^T \) has a unique positive definite square root \( P \). Set \( U := P^{-1}A \). Clearly \( PU = A \). Furthermore,

\[ UU^T = P^{-1}AA^T(P^{-1})^T = P^{-1}P^2P^{-1} = I, \]

so \( U \) is orthogonal.

Suppose \( A = QV \) is another polar decomposition. Then \( AA^T = QVV^TQ = Q^2 \). Since positive definite square roots are unique, \( Q = P \), and hence \( V = Q^{-1}A = P^{-1}A = U \).

We want to show next that if \( A \) is a Lorentz transformation, then so are the polar factors. First, a lemma.

Lemma 3.16. Let \( P \) be positive definite matrix preserving the Lorentzian form. Then \( P \) and all of its powers \( P^t, t \in \mathbb{R}, \) are Lorentz transformations.

Proof. By Proposition 3.9 there exists an orthogonal \( U \) such that \( P = U^T DU \), where \( D \) is a diagonal matrix with positive entries down the diagonal. By Proposition 2.11, \( PI_{1,3}P = I_{1,3} \). Thus

\[ D(UI_{1,3}U^T)D = UPU^TU_{1,3}U^TUPU^T = UPI_{1,3}PU^T = UI_{1,3}U^T. \]

If we set \( B = UI_{1,3}U^T \), then \( DBD = B \). The only way this can happen for a diagonal matrix \( D \) is that \( d_{i,i}d_{j,j} = 1 \) whenever \( b_{i,j} \neq 0 \). It then follows for \( t \in \mathbb{R} \) that \( d_{i,i}^td_{j,j}^t = 1 \) whenever \( b_{i,j} \neq 0 \), and thus \( D^tBD^t = B \). Reversing our earlier argument,
we conclude that $U^T D^t U I_{1,3} U^T D^t U = I_{1,3}$. It is straightforward that $U^T D^t U$ provides an alternative way for computing $P^t$, and thus $P^t$ preserves the Lorentzian form.

Let $t$ denote the unit vector in the $ct$-direction in Minkowski space. Since all $P^t$ preserve the Lorentzian form, they must carry the vector $t$ into the open positive cone of time-like vectors or its negative. Since the map from $\mathbb{R}$ to $\mathbb{R}$ given by $t \mapsto \pi_{ct}(P^t(t))$ is continuous, where $\pi_{ct}$ is projection into the $ct$-coordinate, takes the value $0$ at $t = 0$, and can’t assume the value $0$, by the Intermediate Value Theorem, we conclude that it takes on only positive values. Thus all $P^t$ are Lorentz transformations. \hfill $\square$

**Problem 3.17.** Show in the previous proof that $U^T D^t U$ gives the power $P^t$, as defined in Section 3.2.

**Lemma 3.18.** If $A \in O(1,3)$, so is $A^T$.

**Proof.** If $A^T I_{1,3} = I_{1,3}$, then taking inverses we obtain $A^{-1} I_{1,3} (A^T)^{-1} = I_{1,3}$. Since $A^{-1} I_{1,3} (A^T)^{-1} = ((A^{-1})^T)^T I_{1,3} (A^{-1})^T$, we conclude that $(A^{-1})^T$ preserves the Lorentzian form. Now if $A \in O(1,3)$, so also is $A^{-1}$, and applying the preceding to $A^{-1}$, we conclude that $A^T \in O(1,3)$. \hfill $\square$

**Proposition 3.19.** The polar factors $P, U$ of the polar decomposition $A = PU$ of a Lorentz transformation $A$ are also Lorentz transformations.

**Proof.** By the preceding lemma, $A^T \in O(1,3)$, so $AA^T \in O(1,3)$. Since $AA^T$ is positive definite, it follows from Lemma 3.16 that it and $P = (AA^T)^{1/2}$ (from Proposition 3.15) are Lorentz transformations. Thus $U = P^{-1} A$ is a Lorentz transformation. \hfill $\square$

We close this section by characterizing those orthogonal matrices that are Lorentz transformations.

**Proposition 3.20.** An orthogonal matrix $U$ is a Lorentz transformation iff it has the block form

$$U = \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix},$$

where $S \in O(3)$, i.e., is an orthogonal transformation on $\mathbb{R}^3$.

**Proof.** The proof follows directly from the equation $U^T I_{1,3} U = I_{1,3}$, or equivalently, $I_{1,3} U = U I_{1,3}$. The 1,2-blocks on each side of the latter equation are negatives of
each other and hence must be 0, and ditto for the 2,1-blocks. Hence $U$ is block diagonal with each block having its transpose for its inverse. Since $U$ is a Lorentz transformation, it follows that the 1,1-entry must be positive and its own inverse, hence 1. The block $S$ must also have its transpose being its inverse, hence must be orthogonal.

3.4. **Positive definite Lorentz transformations.** The goal of this section to to show that the positive definite Lorentz transformations are precisely the Lorentz boosts. It follows from Problems 2.12 and 3.1 that Lorentz boosts are symmetric Lorentz transformations, so we need to establish the converse.

We write an arbitrary positive definite Lorentz transformation in the form

$$A = \begin{bmatrix} \tau & x^T \\ x & S \end{bmatrix},$$

where $\tau$ is a positive scalar (since $A$ is a Lorentz transformation), $S$ is a $3 \times 3$ symmetric matrix (since $A$ is symmetric), and $x \in \mathbb{R}^3$ is a column vector (by the symmetric property its transpose row vector $x^T$ must appear after $\tau$ as the remainder of the first row). Since $A$ is positive definite, we have for $0 \neq y \in \mathbb{R}^3$,

$$0 < \begin{bmatrix} 0 & y^T \\ y & S \end{bmatrix} \begin{bmatrix} \tau & x^T \\ x & S \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = y^T S y,$$

from which we conclude that $S$ is positive definite.

We recall from Proposition 2.11 (and the fact that $A$ is symmetric) that

$$\begin{bmatrix} \tau & x^T \\ x & S \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tau & x^T \\ x & S \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}.$$

If we multiply out the left-hand side and set the 1,1-entries equal on both sides of the equation, we obtain $-\tau^2 + x^T x = -1$, which implies $\tau = \sqrt{1 + x^T x}$. We set

$$v := \frac{c}{\sqrt{1 + x^T x}} x.$$

Then

$$\frac{|v|^2}{c^2} = \frac{x^T x}{1 + x^T x} < 1; \text{ hence } |v|^2 < c^2, \text{ i.e., } |v| < c.$$
We next compute
\[
\gamma = \gamma_v = \frac{1}{\sqrt{1 - \frac{v^T v}{c^2}}} = \frac{1}{\sqrt{1 - \frac{x^T x}{1 + x^T x}}} = \sqrt{1 + x^T x} = \tau. \quad (3.17)
\]
Thus we obtain
\[
A = \begin{bmatrix}
\gamma_v & \frac{\gamma_v v^T}{c} \\
\frac{\gamma_v v}{c} & S
\end{bmatrix}
\]

By equating the 2, 2-entries in the equation \( A I_{1,3} A = I_{1,3} \), we obtain the following string of equivalent equalities:
\[
-\frac{\gamma^2}{c^2} vv^T + S^2 = I
\]
\[
S^2 = I + \frac{\gamma^2}{c^2} vv^T
\]
\[
S = \sqrt{I + \frac{\gamma^2}{c^2} vv^T}. \quad (3.18)
\]
The last equation follows from the fact that \( S \) is positive definite, so \( S^2 \) is, and thus has unique positive square root \( S \) (Proposition 3.12).

We record the following useful identity, which can be directly verified from the definition of \( \gamma_v \), and which we use in the last step of the following calculation.
\[
\gamma_v^2 - 1 = \frac{|v|^2}{c^2} \gamma_v. \quad (3.19)
\]

We now calculate
\[
\left( I + \frac{\gamma_v - 1}{|v|^2} vv^T \right)^2 = I + 2 \frac{\gamma_v - 1}{|v|^2} vv^T + \frac{(\gamma_v - 1)^2}{|v|^4} vv^T vv^T
\]
\[
= I + 2 \frac{\gamma_v - 2}{|v|^2} vv^T + \frac{\gamma_v^2 - 2 \gamma_v + 1}{|v|^2} vv^T
\]
\[
= I + \frac{\gamma_v^2 - 1}{|v|^2} vv^T
\]
\[
= I + \frac{\gamma_v^2}{c^2} vv^T
\]
It follows from this calculation and Equation (3.18) that
\[
A = \begin{bmatrix}
\gamma_v & \frac{\gamma_v v^T}{c} \\
\frac{\gamma_v v}{c} & S
\end{bmatrix} = \begin{bmatrix}
\gamma_v & \frac{\gamma_v v^T}{c} \\
\frac{\gamma_v v}{c} & I + \frac{\gamma_v - 1}{|v|^2} vv^T
\end{bmatrix}
\]
We have thus established the
Proposition 3.21. A positive definite Lorentz transformation is a Lorentz boost.

4. Einstein velocity addition

The notion of velocity addition arises in at least two obvious contexts in special relativity. The first is the problem of finding the velocity of an object in a frame $S'$, given that it is moving at some constant velocity in a reference frame $S$. The second is the problem of finding the velocity at which $S''$ is moving with respect to $S$, given the knowledge of how fast $S'$ is moving with respect to $S$ and $S''$ is moving with respect to $S'$. The problems are more-or-less interchangeable since one can pass from an object to a frame in which it is stationary, and from a frame to an object stationary in that frame. There is the caveat, however, that for an object moving at constant velocity in one frame, there are more than one such frames in which it is stationary (for any such frame, consider the new frame obtained by a rotation of its space coordinates).

We suppose first that the frame $S'$ is moving with velocity $v$ in the $x$-direction in the reference frame $S$. We set $\gamma = \gamma_v = (1 - v^2/c^2)^{-1/2}$. We then have the Lorentz boost given by the equations

$$t = \gamma(t' + vx'/c^2), \quad x = \gamma(x' + vt'), \quad y = y', \quad z = z'.$$

Suppose that an object has velocity components $u'_x, u'_y, u'_z$ as measured in frame $S'$. This means by definition that

$$u'_x = \frac{dx'}{dt'}, \quad u'_y = \frac{dy'}{dt'}, \quad u'_z = \frac{dz'}{dt'}.$$

Differentiating the equations of the Lorentz boost, we obtain

$$u_x = \frac{dx}{dt} = \frac{dx/\gamma dt'}{dt'/\gamma} = \frac{\gamma(dx'/dt' + v)}{\gamma(1 + v(dx'/dt')/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2},$$

$$u_y = \frac{dy}{dt} = \frac{dy/\gamma dt'}{dt'/\gamma} = \frac{u'_y/\gamma}{1 + vu'_x/c^2},$$

$$u_z = \frac{dz}{dt} = \frac{dz/\gamma dt'}{dt'/\gamma} = \frac{u'_z/\gamma}{1 + vu'_x/c^2}.$$  \hspace{1cm} (4.20)

Problem 4.1. Suppose that $v = u'_x = 0.5c$ and $u'_y = u'_z = 0$. What is $u_x$? What would it be in Newtonian mechanics?
In a completely analogous way one obtains
\[ u'_x = \frac{u_x - v}{1 - vu_x/c^2}, \quad u'_y = \frac{u_y/\gamma}{1 - vu_x/c^2}, \quad u'_z = \frac{u_z/\gamma}{1 - vu_x/c^2}. \] (4.21)

4.1. **The one-dimensional case.** The simplest case to consider is the case that the velocity \( v \) and the velocity \( u \) are both in the direction of the \( x \)-axis. We consider a frame \( S' \) moving with velocity \( v \) along the \( x \)-axis with respect to a frame \( S \) and a frame \( S'' \) moving with velocity \( u \) along the \( x \)-axis with respect to \( S' \). We assume that all three frames share a common origin. We may assume that \( S'' \) is the frame at which some object moving with velocity \( u \) in \( S' \) is at rest. Then by equation (4.20) we have that \( S'' \) is moving with respect to \( S \) with velocity \( w := (v + u)/(1 + vu/c^2) \).

We therefore define the Einstein velocity addition of \( v \) and \( u \) for \( u, v \in \mathbb{R}_c = \{ u \in \mathbb{R} : |u| < c \} \) by
\[ v \oplus u = \frac{v + u}{1 + \frac{vu}{c^2}}. \] (4.22)

Note that the formula gives the velocity \( v \oplus u \) with which a third frame or object is traveling with respect to a first frame or object, given that a second frame or object is traveling with velocity \( v \) with respect to a first frame or object, and a third with velocity \( u \) with respect to the second.

**Problem 4.2.** Suppose that rockets A and B are speeding toward each other at speeds 0.8\( c \) for A and 0.9\( c \) for B, both speeds calculated in reference frame \( S \). Suppose that rocket A fires a missile toward rocket B at a velocity 0.7\( c \) with respect to the frame of A. How fast is the missile traveling in the original reference frame and in the reference frame of B?

**Problem 4.3.** A \( K^0 \) meson at rest decays into a \( \pi^+ \) meson and a \( \pi^- \) meson, each having a speed of 0.85\( c \). If a \( K^0 \) meson traveling at a speed of 0.9\( c \) in frame \( S \) decays, what is the greatest speed that one of the \( \pi \) mesons can have (again in \( S \))? What is the least speed?

**Problem 4.4.** Show that the map \( f : \mathbb{R} \to \mathbb{R}_c \) defined by \( f(x) = ctanh(x) \) is an isomorphism from \((\mathbb{R}, +)\) to \((\mathbb{R}_c, \oplus)\).
4.2. A general definition of velocity addition. In this section we turn to the
general definition of Einstein velocity addition for velocities in the open ball \( \mathbb{R}^3_c \) of
radius \( c \) in \( \mathbb{R}^3 \). Recall that the world line of an object A moving with respect to a
reference frame \( S' \) with constant velocity \( \mathbf{v} \) is given by
\[
\left\{ \begin{bmatrix} ct' \\ t' \mathbf{v} \end{bmatrix} : t' \in \mathbb{R} \right\}.
\]
We suppose further that the frame \( S' \) is moving at constant velocity \( \mathbf{u} \) with respect
to the frame \( S \). We can calculate the equation of the world line of A in frame \( S \) via
the Lorentz boost for change of coordinates from \( S' \) to \( S \):
\[
\begin{bmatrix} \gamma_u & \frac{\gamma_u u^T}{c} \\ \frac{\gamma_u u}{c} & I + \frac{\gamma_u - 1}{|u|^2} uu^T \end{bmatrix} 
\begin{bmatrix} ct' \\ t' \mathbf{v} \end{bmatrix} = 
\begin{bmatrix} \gamma_u (ct' + \frac{t'u^T v}{c}) \\ \gamma_u t'u + t'v + \frac{\gamma_u - 1}{|u|^2} t'u u^T \mathbf{v} \end{bmatrix} = 
\begin{bmatrix} ct'\gamma_u (1 + \frac{u^T v}{c^2}) \\ t'(\gamma_u \mathbf{u} + v + \frac{\gamma_u - 1}{|u|^2} (u^T \mathbf{v}) \mathbf{u}) \end{bmatrix}
\]
We conclude that the image of the world line of A under the Lorentz boost is the
world line
\[
\left\{ \begin{bmatrix} ct \\ t(\mathbf{u} \oplus \mathbf{v}) \end{bmatrix} : t \in \mathbb{R} \right\}
\]
where
\[
t = \gamma_u (1 + \frac{u^T v}{c^2}) t'
\]
and
\[
\mathbf{u} \oplus \mathbf{v} := \frac{1}{1 + \frac{u^T v}{c^2}} \left( \mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} + \frac{\gamma_u - 1}{\gamma_u |\mathbf{u}|^2} (u^T \mathbf{v}) \mathbf{u} \right)
\]
Using the equation \( c^2(\gamma^2 - 1) = \gamma^2 |\mathbf{u}|^2 \) (equation (3.19)), where \( \gamma = \gamma_u \), we note that
\[
\frac{c^2(1 + \gamma)}{\gamma} = \frac{c^2(1 + \gamma)}{\gamma} \cdot \frac{\gamma - 1}{\gamma - 1} = \frac{c^2(\gamma^2 - 1)}{\gamma(\gamma - 1)} = \frac{\gamma^2 |\mathbf{u}|^2}{\gamma(\gamma - 1)} = \frac{\gamma |\mathbf{u}|^2}{\gamma - 1}.
\]
Inverting, we conclude
\[
\frac{\gamma_u}{c^2(1 + \gamma_u)} = \frac{\gamma_u - 1}{\gamma_u |\mathbf{u}|^2}.
\]
This allows us to rewrite the definition of Einstein velocity addition in the form
\[
\mathbf{u} \oplus \mathbf{v} := \frac{1}{1 + \frac{u^T v}{c^2}} \left( \mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} + \frac{\gamma_u}{c^2(1 + \gamma_u)} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right)
\]
In the case that \( \mathbf{u} \) and \( \mathbf{v} \) are parallel (i.e., one is a scalar multiple of the other), Einstein addition reduces to

\[
\mathbf{u} \oplus \mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}}
\]  

(4.26)

**Problem 4.5.** Prove equation (4.26). (Hint: Let \( \mathbf{u} = r \mathbf{w}, \mathbf{v} = s \mathbf{w} \) for some \( \mathbf{w} \), apply equation (4.23) and reduce.)

**Problem 4.6.** Alternatively prove for \(-c < r, s < c\) that

\[
\frac{r \mathbf{u}}{|\mathbf{u}|} \oplus \frac{s \mathbf{u}}{|\mathbf{u}|} = \frac{(r + s) \mathbf{u}}{(1 + \frac{rs}{c^2})|\mathbf{u}|}.
\]

**Problem 4.7.** Prove that \( F : (\mathbb{R}, +) \rightarrow (\mathbb{R}_c(x/|x|), \oplus) \) defined by \( F(x) = c \tanh(x) \) is an isomorphism. Hence \( \oplus \) restricted to any \( \mathbb{R}_c x \) is a group operation isomorphic to (\( \mathbb{R}, + \)).

**Problem 4.8.** Show that if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal, then \( \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \gamma_u^{-1} \mathbf{v} \). In particular, the operation \( \oplus \) is not commutative.

4.3. **Einstein addition and Lorentz boosts.** This is a very close and useful connection between Einstein velocity addition, Lorentz boosts, and polar decompositions, which we develop in this section.

**Proposition 4.9.** For \( \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3 \), \( B(\mathbf{u})B(\mathbf{v}) = B(\mathbf{u} \oplus \mathbf{v})h(\mathbf{u}, \mathbf{v}) \), where the right hand side is the polar decomposition of the left hand side in the Lorentz group \( O^+(1, 3) \).

**Proof.** We calculate that

\[
B(\mathbf{u})B(\mathbf{v}) = \begin{bmatrix}
\gamma_u & \frac{\gamma_u \mathbf{u}^T}{c} \\
\frac{\gamma_u \mathbf{u}}{c} & I + \frac{\gamma_u - 1}{|\mathbf{u}|^2} \mathbf{uu}^T
\end{bmatrix}
\begin{bmatrix}
\gamma_v & \frac{\gamma_v \mathbf{v}^T}{c} \\
\frac{\gamma_v \mathbf{v}}{c} & \frac{\gamma_v - 1}{|\mathbf{v}|^2} \mathbf{vv}^T
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\gamma_u \gamma_v (1 + \frac{\mathbf{u}^T \mathbf{v}}{c^2}) & * \\
\frac{\gamma_u \gamma_v u}{c} + \frac{\gamma_v \mathbf{v}}{c} + \frac{\gamma_u - 1}{|\mathbf{u}|^2} \mathbf{uu}^T \mathbf{v} & *
\end{bmatrix}
\]

We note from Proposition 3.20 that the left column of the product must be the left column of the Lorentz boost \( P \) in the polar factorization \( P \Theta \) of \( B(\mathbf{u})B(\mathbf{v}) \). Since a Lorentz boost matrix \( P = \begin{bmatrix}
\gamma & \mathbf{x}^T \\
\mathbf{x} & S
\end{bmatrix} \) is the Lorentz boost for the vector \( \mathbf{w} = (c/\gamma) \mathbf{x} \),
we conclude that the Lorentz boost $P$ in the polar factorization of $B(u)B(v)$ is the Lorentz boost for the vector
\[
\frac{c}{\gamma_u \gamma_v (1 + \frac{u \cdot v}{c^2})} \left( \frac{\gamma_u \gamma_v u}{c} + \frac{\gamma_v v}{c} + \left( \frac{\gamma_u - 1}{\gamma_u} \right) \frac{\gamma_v uu^T v}{c|u|^2} \right) = \frac{1}{1 + \frac{u \cdot v}{c^2}} (u + \frac{1}{\gamma_u} v + \frac{\gamma_u - 1}{\gamma_u |u|^2} (u \cdot v) u) = u \oplus v.
\]
Thus $B(u)B(v) = P \Theta = B(u \oplus v) h(u, v)$, where we define $h(u, v)$ to be $\Theta$.  

**Problem 4.10.** Show that if $P = \begin{bmatrix} \gamma & * \\ x & * \end{bmatrix}$ is a Lorentz boost, then $P = B(w)$ for $w = (c/\gamma) x$ and $\gamma_w = \gamma$.

**Problem 4.11.** Argue from the proof of Proposition 4.9 and the preceding problem that
\[
\gamma_{u \oplus v} = \gamma u \gamma v \left(1 + \frac{u \cdot v}{c^2}\right). \tag{4.27}
\]
5. Gyrogroups

Definition 5.1. (Groupoids or Magmas, and Automorphism Groups of Groupoids) A groupoid or a magma is a nonempty set with a binary operation. An automorphism of the groupoid \((S, \ast)\) is a bijection of \(S\) that respects the binary operation \(\ast\) in \(S\). The set of all automorphisms of \((S, \ast)\) forms a group under composition, denoted by \(\text{Aut}(S, \ast)\).

An important subcategory of the category of groupoids is the category of loops.

Definition 5.2. (Loops) A loop is a magma \((S, \cdot)\) with an identity element in which each of the two equations \(a \cdot x = b\) and \(y \cdot a = b\) with unknowns \(x\) and \(y\) possesses a unique solution. As customary, we frequently denote the product \(a \cdot b\) by juxtaposition \(ab\).

Problem 5.3. Show that \((G, \cdot)\) is a group iff it is an associative loop.

Being nonassociative, the Einstein velocity addition on the set of relativistically admissible velocities in the special theory of relativity is not a group operation. However, it does possess group-like properties that have been axiomatized by A. A. Ungar as structures called “gyrogroups,” and studied in detail in his book *Analytic Hyperbolic Geometry and Albert Einstein’s Special Theory of Relativity*. The gyrogroup concept abstracts both Einstein’s velocity addition and the corresponding Thomas precession. The abstract Thomas precession is called the Thomas gyration and suggests the prefix “gyro” for many of the concepts of the theory.

Definition 5.4. (Gyrogroups) The magma \((G, \oplus)\) is a gyrogroup if its binary operation satisfies the following axioms.

\[(\gamma 1)\] There exists in \(G\) some element, \(0\), called a left identity, satisfying for all \(a \in G\):
\[
0 \oplus a = a \quad \text{(Left Identity)}
\]

\[(\gamma 2)\] For each \(a \in G\) there is an \(x \in G\), called a left inverse of \(a\), satisfying
\[
x \oplus a = 0 \quad \text{(Left Inverse)}
\]

\[(\gamma 3)\] For any \(a, b, z \in G\) there exists a unique element \(\text{gyr}[a, b]z \in G\) such that
\[
a \oplus (b \oplus z) = (a \oplus b) \oplus \text{gyr}[a, b]z \quad \text{(Left Gyroassociative Law)}
\]
(γ4) If \(\text{gyr}[a,b]\) denotes the map \(\text{gyr}[a,b]: G \to G\) given by \(z \mapsto \text{gyr}[a,b]z\) then

\[\text{gyr}[a,b] \in \text{Aut}(G, \oplus)\]  
(Gyroautomorphism)

and \(\text{gyr}[a,b]\) is called the Thomas gyration, or the gyroautomorphism of \(G\), generated by \(a, b \in G\).

(γ5) The gyroautomorphism \(\text{gyr}[a,b]\) generated by any \(a, b \in G\) satisfies

\[\text{gyr}[a,b] = \text{gyr}[a \oplus b,b]\]  
(Left Loop Property)

**Problem 5.5.** For a gyrogroup \((G, \oplus)\) establish the following properties.

1. \(a \oplus b = a \oplus c \Rightarrow b = c\) (left cancellation).
2. \(\text{gyr}[0,a] = I\), the identity map on \(G\).
3. \(\text{gyr}[x,a] = I\) if \(x \oplus a = 0\).
4. \(\text{gyr}[a,a] = I\).
5. \(a \oplus 0 = a\), i.e., 0 is an identity.
6. There is only one left identity.
7. Every left inverse is a right inverse.
8. The left inverse, denoted \(\ominus a\), is unique, and \(\ominus(\ominus a) = a\).
9. \(\ominus a \oplus (a \oplus b) = b\).
10. \(\text{gyr}[a,b]x = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x))\).
11. \(\text{gyr}[a,b]0 = 0\).
12. \(\text{gyr}[a,b](\ominus x) = \ominus \text{gyr}[a,b]x\).
13. \(\text{gyr}[a,0] = \text{gyr}[0,b] = I\).

The preceding list of axioms is minimal in nature. We typically work with the more extensive, but equivalent, set of axioms.

**Definition 5.6.** (Gyrogroups-Alternative Definition) The magma \((G, \oplus)\) is a gyrogroup if its binary operation satisfies the following axioms.

(G1) There exists in \(G\) a unique identity element element \(0\) satisfying for all \(a \in G\):

\[0 \oplus a = a \oplus 0 = a\]  
(Identity)

(G2) For each \(a \in G\), there exists a unique inverse \(\ominus a \in G\) satisfying

\[\ominus a \oplus a = a \oplus (\ominus a) = 0\]  
(Inverse)
For all \(a, b \in G\), the map \(\text{gyr}[a, b]\) of \(G\) into itself given by the equation

\[
\text{gyr}[a, b]z = \ominus((a \oplus b) \oplus (a \oplus (b \oplus z)))
\]

for all \(z \in G\), satisfies the following axioms:

(G3) \(\text{gyr}[a, b] \in \text{Aut}(G, \oplus)\), the gyroautomorphism group.

(G4)
\[
(a \oplus (b \oplus c)) = (a \oplus b) \oplus \text{gyr}[a, b]c
\]

(Left Gyroassociative Law)
\[
(a \oplus (b \oplus c)) = a \oplus (b \oplus \text{gyr}[b, a]c)
\]

(Right Gyroassociative Law)

(G5)
\[
\text{gyr}[a, b] = \text{gyr}[a \oplus b, b]
\]

(Left Loop Property)
\[
\text{gyr}[a, b] = \text{gyr}[a, b \oplus a]
\]

(Right Loop Property)

(G6) A gyrogroup is called gyrocommutative if it satisfies
\[
a \oplus b = \text{gyr}[a, b](b \oplus a)
\]

(Gyrocommutative Law)

**Problem 5.7.** Derive the second set of axioms (except gyrocommutativity) for a gyrogroup from the first set.

5.1. **Involutive groups and gyrogroups.** We work in this section in the setting of an involutive group \(G\), a group equipped with an involutive automorphism \(\tau\) such that \(\tau \circ \tau\) is the identity. We set \(g^* = \tau(g^{-1}) = (\tau(g))^{-1}\) and note that \(g \mapsto g^* : G \to G\) is an involutive antiautomorphism. Let

\[
G^\tau := \{x \in G : \tau(x) = x\}, \quad P_G := \{xx^* : x \in G\} \subseteq G_\tau := \{g \in G : g = g^*\}.
\]

(That \((xx^*)^* = x**x^* = xx^*\) shows \(P_G \subseteq G_\tau\).)

A subset \(B\) of a group \(G\) is called a twisted subgroup if the identity \(e\) is in \(B\), \(B\) is closed under inversion, and \(xyx \in B\) whenever \(x, y \in B\).

**Lemma 5.8.** The sets \(P_G\) and \(G_\tau\) are twisted subgroups.

**Proof.** Since \(\tau(e^{-1}) = \tau(e) = e\), we have \(e \in P_G \subseteq G_\tau\). Since \((g^*)^{-1} = \tau(g) = (g^{-1})^*\), we have that \(g = g^*\) implies \((g^{-1})^* = g^{-1}\). Also \((gg^*)^{-1} = (g^{-1})^*g^{-1} = (g^{-1})(g^{-1})^*\). Thus \(G_\tau\) and \(P_G\) are closed under inversion.

Let \(gg^*, hh^* \in P_G\). Then \(gg^*hh^*gg^* = gg^*h(gg^*h)^* \in P_G\). Thus \(P_G\) is a twisted subgroup. A similar argument holds for \(G_\tau\). \(\square\)
Problem 5.9. Show that if \( x, y \in G_\tau \), then \( xyx \in G_\tau \).

We recall other basic terminology. Let \( G \) be a group with subgroup \( H \). A subset \( L \) of \( G \) is said to be transversal to \( H \) if the identity \( e \in L \) and \( L \) intersects each coset \( gH \) of \( H \) in precisely one point. One sees readily that a subset \( L \) containing \( e \) is a transversal to \( H \) if and only if the map \( (x, h) \mapsto xh : L \times H \to G \) is a bijection.

In the case of an involutive group \((G, \tau)\), if \( L \subseteq \{ g \in G : g = g^* \} \), then the map \( (x, k) \mapsto xk : L \times G^\tau \to G \) is called a polar map. Hence \( L \) containing \( e \) is transversal to \( G^\tau \) if and only if the polar map is a bijection. If it is a bijection, then the pair \((L, G^\tau)\) is called a polar decomposition for \((G, \tau)\).

We now come to the study of involutive groups with polar decomposition. A twisted subgroup is uniquely \(2\)-divisible if each member of \( P \) has a unique square root in \( P \).

Proposition 5.10. Let \((G, \tau)\) be an involutive group, \( P = \{ gg^* | g \in G \} \). The following are equivalent:

1. \( P \) is a uniquely \(2\)-divisible twisted subgroup.
2. \( P \) is transversal to \( G^\tau \), i.e., the map \( (x, g) \mapsto xg : P \times G^\tau \to G \) is bijective.
3. Every element \( g \in G \) has a unique polar decomposition \( g = xk \in PG^\tau \), \( x \in P \), \( k \in G^\tau \). where \( x = (gg^*)^{1/2} \).

Proof. \(1\)\(\Rightarrow\)(3): If \( g = x_1k_1 = x_2k_2 \in PG^\tau \), then \( gg^* = (x_1)^2 = (x_2)^2 \). Hence \( x_1 = x_2 \), and then \( k_1 = k_2 \). Thus factorizations, when they exist, are unique.

For \( g \in G \), set \( x := (gg^*)^{1/2} \in P \). Choose \( k \in G \) so that \( g = xk \). We are finished if we show that \( k \in G^\tau \). We have

\[
k^* = (x^{-1}g)(x^{-1}g)^* = (gg^*)^{-1/2}gg^*(gg^*)^{-1/2} = e,
\]

and thus \( k^* = k^{-1} \), i.e., \( k \in G^\tau \).

\(3\)\(\Rightarrow\)(2): Immediate.

\(2\)\(\Rightarrow\)(3): For \( gg^* \in P \), let \( g = xk \in PG^\tau \). Then \( gg^* = xk(xk)^* = x^2 \), so \( x \in P \) is a square root of \( gg^* \). If \( y \in P \) were another, then one verifies that \( y(y^{-1}g) \) would give another decomposition of \( g \), since

\[
y^{-1}g = y^{-1}gg^*(g^*)^{-1} = y^{-1}y^2\tau(g) = y\tau(g) = \tau(y^{-1}g).
\]

\(\square\)
Theorem 5.11. Let \((G, \tau)\) be an involutive group, \(P = \{gg^* | g \in G\}\). If \(P\) is uniquely 2-divisible, then \(P\) is a gyrocommutative gyrogroup for the operation \(x \oplus y = (xy^2x)^{1/2}\). The gyration automorphisms given by \(\text{gyr}[a, b]x = h(a, b)xh(a, b)^{-1}\), inner automorphism by \(h(a, b)\) where \(ab = (a \oplus b)h(a, b)\), is the polar decomposition of \(ab\).

Proof. By Lemma 5.8 \(P\) is a twisted subgroup, then by Proposition 5.10 \(P\) is transversal to \(G\), and hence each element of \(G\) has a unique polar decomposition. Furthermore, for each \(a, b \in P\), the \(P\)-factor of the polar decomposition of \(ab\) is given by

\[
((ab)(ab)^*)^{1/2} = (ab^2a)^{1/2} = a \oplus b,
\]

where the second equality is true by definition. Thus

\[
ab = (a \oplus b)h(a, b) \in PG^\tau, \quad (5.29)
\]

where \(h(a, b)\) is defined to be the \(G^\tau\)-factor in the polar decomposition of \(ab\).

Directly from the definition \(a \oplus b = (ab^2a)^{1/2}\), we conclude that \(e \oplus a = a = a \oplus e\) and \(a \oplus a^{-1} = e = a^{-1} \oplus a\). Hence Axioms \((\gamma 1)\) and \((\gamma 2)\) are satisfied, and \(e = 0\) and \(a^{-1} = \ominus a\) in the gyrogroup terminology.

To verify \((\gamma 3)\), we have on the one hand that

\[
(ab)c = (a \oplus b)h(a, b)c = (a \oplus b)h(a, b)c(h(a, b)^{-1}h(a, b)) = (a \oplus b) \text{gyr}[a, b]c(h(a, b)) = ((a \oplus b) \oplus \text{gyr}[a, b]c)h(a \oplus b, \text{gyr}[a, b]c)h(a, b).
\]

and on the other hand that

\[
a(bc) = a(b \oplus c)h(b, c) = a \oplus (b \oplus c)h(a, b \oplus c)h(b, c).
\]

Axiom \((\gamma 3)\) now follows from uniqueness of decomposition.

Let \(k \in G^\tau\). Then \(k(gg^*)^{-1}k = kgk^{-1}k^*g^{-1} = (kgk^*)(kgk^*)^*\), and thus \(k(gg^*)^{-1}k \in P\). It follows that \(P\) is invariant under inner automorphism by any member of \(G^\tau\). Denote \(kak^{-1}\) by \(a^k\). It is straightforward to verify that \((ab^2a)^k = a^k(b^k)^2a^k\) and hence that \(((ab^2a)^{1/2})^k = (a^k(b^k)^2a^k)^{1/2}\). Since \(h(a, b) \in G^\tau\) and \(\text{gyr}[a, b]x = x^{h(a, b)}\),
we conclude that each \( \text{gyr}[a, b] \) is an automorphism of \((P, \oplus)\), and thus Axiom \((\gamma 4)\) is satisfied.

To establish that \((P, \oplus)\) is gyrocommutative, we consider the equations
\[
(ab)^* = ((a \oplus b)h(a, b))^* = h(a, b)^{-1}(a \oplus b)
\]
and
\[
(ab)^* = ba = (b \oplus a)h(b, a).
\]
From these equations we conclude that
\[
h(a, b)^{-1}(a \oplus b) = (b \oplus a)h(b, a)
\]
and hence that
\[
a \oplus b = h(a, b)(b \oplus a)h(a, b)^{-1}h(a, b)h(b, a) = \text{gyr}[a, b](b \oplus a)h(a, b)h(b, a).
\]
From uniqueness of the polar decomposition, we conclude \(a \oplus b = \text{gyr}[a, b](b \oplus a)\) (gyrocommutativity) and \(h(a, b)^{-1} = h(b, a)\).

Finally, to establish \((\gamma 5)\), we observe
\[
b(ab) = b(a \oplus b)h(a, b) = (b \oplus (a \oplus b))h(b, a \oplus b)h(a, b).
\]
It follows again from the uniqueness of the polar decomposition that \(h(b, a \oplus b)^{-1} = h(a, b)\). From the last of the preceding paragraph we see that \(h(b, a \oplus b)^{-1} = h(a \oplus b, b)\), and we conclude
\[
h(a \oplus b, b) = h(b, a \oplus b)^{-1} = h(a, b).
\]
We then have directly from the definition of the gyroautomorphisms that \(\text{gyr}[a \oplus b, b] = \text{gyr}[a, b]\).

**Remark 5.12.** We remark that there is a converse to the preceding theorem, namely that every uniquely 2-divisible gyrocommutative gyrogroup can be realized (up to isomorphism) as \((P, \oplus)\) for some involutive group satisfying the hypotheses of the preceding theorem.

There is a weaker converse for general gyrogroups, which we present in the next problem.
Problem 5.13. Let \((G, \oplus)\) be a gyrogroup and let \(\mathcal{A}\) be a subgroup of its automorphism group containing all automorphisms \(\text{gyr}[a, b], a, b \in G\). Then \(G \times \mathcal{A}\) is a group with respect to the operation
\[(a, A)(b, B) = (a + Ab, \text{gyr}[a, Ab]AB).\] (5.30)

Problem 5.14. Show that for \(a \in G\), the map \(n \mapsto (a, I)\) is an isomorphism from the subgyrogroup of \(G\) generated by \(a\) to the subgroup of \(G \times \mathcal{A}\) generated by \((a, I)\), where \((n + 1)a = na \oplus a\). In particular, a gyrogroup is power associative.

5.2. The Einstein gyrogroup. The main goal of this section is to show that the set of admissible velocities \(\mathbb{R}^3_c\) endowed with the Einstein addition is a gyrocommutative gyrogroup.

Lemma 5.15. The Lorentz group endowed with the involution for which \(A^* = A^T\) satisfies the hypotheses of Theorem 5.11 with \(P\) equal to the set of Lorentz boosts and \(O^+(1, 3)^\tau\) equal to the subgroup of orthogonal Lorentz transformations.

Proof. In the Lorentz group \(O^+(1, 3)\) it is straightforward to verify that \(\tau(A) = (A^{-1})^T\) is an involution with \(A^* = A^T\). We verify that \(P = \{AA^T : A \in O^+(1, 3)\}\) is the set of Lorentz boosts. On the one hand, for any \(A \in O^+(1, 3)\), we have directly that \(AA^T\) is a positive definite Lorentz transformation, hence a Lorentz boost by Proposition 3.21. Conversely let \(A\) be a Lorentz boost, say \(B(v)\). It follows from Problem 4.7 that there exists a velocity \(u\) such that \(u \oplus u = v\). By Proposition 3.19 we obtain \(B(u)B(u) = B(u \oplus u)h(u, u)\). But \(B(u)B(u) = B(u)B(u)^*\) is in \(P\) already, so the factor \(h(u, u)\) is the identity, and
\[B(v) = B(u \oplus u) = B(u)B(u) = B(u)B(u)^* \in P.\]

If \((U^T)^{-1} = \tau(U) = U\), then taking inverses we obtain \(U^T = U^{-1}\) if and only if \(U\) is orthogonal. Hence the Lorentz transformations fixed by \(\tau\) are precisely the orthogonal ones. \(\square\)

Theorem 5.16. The correspondence \(v \mapsto B(v)\) defines an isomorphism between the gyrocommutative gyrogroups \((\mathbb{R}^3_c, \oplus)\) of admissible velocities under Einstein velocity addition and the set \(P = \{B(u) : u \in \mathbb{R}^3_c\}\) of Lorentz boosts under the operation \(B(u) \oplus B(v) = (B(u)B(v)^2B(u))^{1/2}\).
Proof. By Proposition 4.9 for \( u, v \in \mathbb{R}^3 \), \( B(u)B(v) = B(u \oplus v)h(u, v) \), where the right hand side is the polar decomposition of the left hand side in the Lorentz group \( O^+(1, 3) \). By Lemma 5.15 and the first paragraph of the proof of Theorem 5.11, we have \( B(u)B(v) = (B(u) \oplus B(v))h(B(u), B(v)) \). By uniqueness of the polar decomposition in \( O^+(1, 3) \), we have \( B(u) \oplus B(v) = B(u \oplus v) \), which shows that \( u \to B(u) \) is an isomorphism (since essentially by definition it is a bijection). It is straightforward to verify that an isomorphism of groupoids (magmas) preserves all the properties of Definition 5.6, and hence one of the systems is a gyrocommutative gyrogroup if and only if the other is. \( \square \)

5.3. Basic theory of gyrogroups. We assume throughout this section the axioms of Definition 5.6 for a gyrogroup. We first list some basic properties of the gyrations; see [3, Chapter 2].

**Proposition 5.17.** Let \((G, \oplus)\) be a gyrogroup. Then for all \( a, b \in G \) the gyrations satisfy the following properties:

(i) \( \text{gyr}[\ominus a, \ominus b] = \text{gyr}[a, b] \);
(ii) \( \text{gyr}[b, a] = \text{gyr}^{-1}[a, b] \), the inverse of \( \text{gyr}[a, b] \);
(iii) \( \text{gyr}[a \oplus b, \ominus a] = \text{gyr}[a, b] \);
(iv) \( \text{gyr}[na, ma] = I \) for all integers \( m, n \).

**Problem 5.18.** Using Problem 5.13 and the fact that \((a, A)^{-1} = (A^{-1}(\ominus a), A^{-1})\), invert both sides of the equation \((a, I)(b, I) = (a \oplus b, \text{gyr}[a, b])\) to show that \( \text{gyr}[\ominus b, \ominus a] = \text{gyr}^{-1}[a, b] \) and \( \ominus (a \oplus b) = \text{gyr}[a, b](\ominus b \ominus a) \).

A magma is called **left power alternative** if for all \( a, b \) and all integers \( m, n \),

\[
ma \oplus (na \oplus b) = (m + n)a \oplus b.
\]

**Corollary 5.19.** A gyrogroup is left power alternative.

**Problem 5.20.** Use Proposition 5.17(iv) to prove the preceding corollary.

**Proposition 5.21.** A gyrogroup satisfies the left Bol identity

\[
a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c.
\] (5.31)
Proof. We have

\[
a \oplus (b \oplus (a \oplus c)) = a \oplus ((b \oplus a) \oplus \text{gyr}[b, a]c)
= (a \oplus (b \oplus a)) \oplus \text{gyr}[a, b \oplus a] \text{gyr}[b, a]c.
\]

Noting from Proposition 5.17 that \(\text{gyr}[a, b \oplus a] \text{gyr}[b, a] = \text{gyr}[a, b] \text{gyr}[b, a] = I\), we obtain the result. \(\square\)

In the next two propositions we show that a gyrogroup is a loop.

Proposition 5.22. In a gyrogroup \((G, \oplus)\), the equation \(a \oplus x = b\) in unknown \(x\) has the unique solution \(x = \ominus a \oplus b\).

Problem 5.23. Prove Proposition 5.22

Definition 5.24. In a gyrogroup \((G, \oplus)\), we define the coaddition \(\boxplus\) by

\[
a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b.
\]

We set

\[
a \boxdot b := a \boxplus (\ominus b) = a \oplus \text{gyr}[a, b](\ominus b) = a \ominus \text{gyr}[a, b]b.
\]

Proposition 5.25. In a gyrogroup \((G, \oplus)\) the equation \(x \oplus a = b\) has the unique solution \(x = b \boxdot a\).

Proof. If \(x \oplus a = b\), then

\[
x = x \oplus (a \ominus a)
= (x \oplus a) \ominus \text{gyr}[x, a](\ominus a)
= (x \oplus a) \ominus \text{gyr}[x, a]a
= (x \oplus a) \ominus \text{gyr}[x + a, a]a
= b \ominus \text{gyr}[b, a]a
= b \boxdot a.
\]
Conversely we first note that

\[ b \bowtie a = b \oplus \text{gyr}[b, a](\ominus a) = \lambda_b \lambda_{(b \ominus a)} \lambda_b \lambda_a(\ominus a) = \lambda_b \lambda_{(b \ominus a)} \lambda_b(0) = b \oplus (\ominus (b \oplus a) \ominus b). \]

Therefore

\[(b \bowtie a) \oplus a = (b \oplus (\ominus (b \oplus a) \ominus b)) \oplus a = b \oplus (\ominus (b \oplus a) \ominus (b \oplus a)) = b.\]

where the second equality follows from the left Bol identity. \(\square\)

**Problem 5.26.** Show for a gyrogroup \((G, \oplus)\) that \((G, \boxplus)\) is a loop with the same identity and inverses as \((G, \oplus)\).

We consider a basic alternative characterization of gyrocommutative gyrogroups.

**Proposition 5.27.** A gyrogroup \((G, \oplus)\) is gyrocommutative if and only if it satisfies the automorphic inverse property

\[ \ominus (a \oplus b) = \ominus a \ominus b. \]

**Proof.** Assume that \(G\) is gyrocommutative. By Problem 5.18

\[ \ominus (a \oplus b) = \text{gyr}[a, b](\ominus b \oplus (\ominus a)) = \text{gyr}[a, b] \text{gyr}[\ominus b, \ominus a](\ominus a \ominus b) = \ominus a \ominus b. \]

Conversely, assume that \(G\) satisfies the automorphic inverse property. Then using Proposition 5.17(i) and gyroassociativity, we obtain

\[ \ominus (a \oplus b) \oplus \text{gyr}[a, b](b \oplus a) = (\ominus a \oplus (\ominus b)) \oplus \text{gyr}[\ominus a, \ominus b](b \oplus a) = \ominus a \oplus (\ominus b \oplus (b \oplus a)) = \ominus a \oplus a = 0. \]

It follows that \(\text{gyr}[a, b](b \oplus a)\) is the inverse of \(\ominus (a \oplus b)\), and hence must equal \(a \oplus b\). \(\square\)
6. The Einstein gyrovector space

In this section we enrich the structure of the Einstein velocity gyrogroup. Our ultimate goal is to equip it with enough structure to carry out an analytic hyperbolic geometry.

6.1. Gyrovector spaces. In this section let \((G, \oplus)\) be a gyrocommutative gyrogroup.

**Definition 6.1.** A gyrovector space consists of a gyrocommutative group \((G, \oplus)\) such that for every \(0 \neq x \in G\), there exists a unique injective homomorphism \(\alpha_x\) of \((\mathbb{R}, +)\) into \((G, \oplus)\) such that \(\alpha_x(1) = x\). In this case we define a scalar multiplication from \(\mathbb{R} \times G\) to \(G\) by \(r.x = \alpha_x(r)\). We sometimes write \(x.r\) for \(r.x\) to avoid proliferation of parentheses.

**Lemma 6.2.** In a gyrovector space scalar multiplication satisfies
(i) \(1.x = x\) and \((-1).x = \ominus x\), \(0.x = 0\);
(ii) \((s + t).x = s.x \oplus t.x\);
(iii) \(s.t.x = (st).x\)
(iv) \(m.x = \bigoplus_{i=1}^{m} x\) for any positive integer \(m\).

**Proof.** For (iii) let \(\alpha_x : \mathbb{R} \to G\) be a homomorphism with \(\alpha_x(1) = x\). Then \(t.x = \alpha_x(t)\). Define \(\beta : \mathbb{R} \to G\) by \(\beta(r) = \alpha_x(rt)\). Then \(\beta\) is a homomorphism (since \(\alpha_x\) is) and \(\beta(1) = \alpha_x(t) = t.x\). Thus by definition \(\beta(s) = s.t.x\). But also \(\beta(s) = \alpha_x(st) = (st).x\), and hence the two are equal. \[\square\]

**Problem 6.3.** Verify the other conclusions of the preceding lemma.

**Problem 6.4.** Suppose that one is given a scalar multiplication \((r, x) \mapsto r.x\) satisfying the conditions of the preceding lemma. Show that if one defines \(\alpha_x(t) = t.x\) for each nonzero \(x\), then one obtains a gyrovector space. What is a minimal set of the properties in Lemma 6.2 needed to derive this result?

**Definition 6.5.** A topological gyrovector space is a gyrovector space \(G\) equipped with a Hausdorff topology such that \(\oplus : G \times G \to G\) and \(t.x : \mathbb{R} \times G \to G\) are continuous.

**Definition 6.6.** An exponential function for a topological gyrovector space \(G\) is a homeomorphism \(\exp : V \to G\) from a real topological vector space \(V\) to \(G\) such that the restriction to any one-dimensional subspace is an additive homomorphism into \(G\).
Lemma 6.7. The continuity of the scalar multiplication \( t.x : \mathbb{R} \times G \to G \) in a topological gyrovector space follows from the existence of an exponential function.

Proof. The scalar multiplication can be written as the continuous composition \((t, x) \mapsto \exp(t \cdot \log x)\). \qed

6.2. The exponential for gyroboosts and admissible velocities. We begin with a lemma which we have essentially proved, but never formally stated.

Lemma 6.8. The following are equivalent for \( A \in O^+(1, 3) \):

1. \( A \) is a Lorentz boost.
2. \( A \) is positive definite.
3. \( A = BB^* \) for some \( B \in O^+(1, 3) \).

Proof. That (3) implies (2) is immediate and (2) implies (1) is the content of Proposition 3.21. Suppose that \( A = B(u) \) is a Lorentz boost. From the fact that one-dimensional subspaces intersect \( \mathbb{R}^3 \) in one-dimensional subgroups isomorphic to \((\mathbb{R}, +)\), we have that \( u = tu \oplus tu \) for some \( t \). Since \( B(tu)B(tu) = B(tu)B(tu)^T \in P \), we have that the \( O(3) \) component of the polar decomposition of \( B(tu)B(tu) \) is the identity and hence

\[ B(tu)B(tu) = B(tu) \oplus t \mathbf{u} = B(u). \]

Hence \( B(u) \in P \). \qed

Let \( P \) be the subset of the Lorentzian group \( O^+(1, 3) \) consisting of all Lorentzian boosts. The Lie algebra \( \mathfrak{o}^+(1, 3) = \{ X \in M_4(\mathbb{R}) : \forall t \in \mathbb{R}, \exp(tX) \in O^+(1, 3) \} \) of \( O^+(1, 3) \) is computed in the standard way for Lie groups defined by preserving a bilinear form and is given by

\[ \mathfrak{o}^+(1, 3) = \{ X \in M_4(\mathbb{R}) : I_{1,3}X + XI_{1,3} = 0 \} \]

Our first goal is to compute the tangent space \( \mathfrak{p} \) of \( P \).

Lemma 6.9. The tangent space \( \mathfrak{p} = \{ X \in \mathfrak{o}^+(1, 3) : \exp(tX) \in P \text{ for all } t \in \mathbb{R} \} \) of \( P \) is given by

\[ \mathfrak{p} = \{ X \in \mathfrak{o}^+(1, 3) : X = \begin{bmatrix} 0 & u^T \\ u & 0 \end{bmatrix} \text{ for some } u \in \mathbb{R}^3 \}. \]
Proof. If \( \exp(tA) = e^{tA} \) is symmetric for all \( t \), then

\[
A = \frac{d}{dt} e^{tA}|_{t=0} = \lim_{t \to 0} \frac{e^{tA} - I}{t}
\]

is symmetric. It is straightforward to verify that the conditions of symmetry and \( I_{1,3}X + XI_{1,3} = 0 \) imply that \( X \) must be of the form specified in the lemma.

For the converse direction, consider \( X = \begin{bmatrix} 0 & u^T \\ u & 0 \end{bmatrix} \). By direct computation one verifies that \( X^{2n} = |u|^{2n-2} \begin{bmatrix} u^T & 0 \\ 0 & uu^T \end{bmatrix} \) and \( X^{2n+1} = |u|^{2n}X \). It follows that

\[
\exp(tX) = \begin{bmatrix} \cosh(t|u|) & \frac{\sinh(t|u|)}{|u|} u^T \\ \frac{\sinh(t|u|)}{|u|} u & I + \frac{\cosh(t|u|)-1}{|u|^2} uu^T \end{bmatrix}.
\]

One verifies directly that the preceding matrix satisfies the conditions of equation 3.18 and hence is a Lorentz boost by Proposition 3.21.

\[ \square \]

Problem 6.10. Verify in detail that the powers of \( X \) and \( \exp(tX) \) are indeed as asserted in the previous proof. Verify in detail that \( \exp(tX) \) is a Lorentz boost.

Proposition 6.11. The exponential map from \( \mathfrak{p} \) to \( (P, \oplus) \) is an exponential map from \( \mathfrak{p} \) to the topological gyrovector space of Lorentz boosts.

Proof. We have seen in Section 3.2 that \( \exp : \text{Sym} \to \text{Sym}^{>0} \) is a homeomorphism and a group homomorphism on each one-dimensional subspace of \( \text{Sym} \). Since \( P \) is the intersection of the set of \( O^+(1, 3) \) and the positive definite matrices (Lemma 6.8) and the latter two sets are closed in the general linear group \( GL_4(\mathbb{R}) \), it follows that \( \exp^{-1}(P) \) is closed in \( \text{Sym} \). It follows easily from Problem 4.7 that each element of \( (\mathbb{R}^3_+, \oplus) \) has an \( n \)-th root for each positive integer \( n \). By Theorem 5.16 the isomorphic gyrogroup of Lorentz boosts must have \( n \)-th roots for each element. But for a positive definite element \( A \) these roots are unique and are given by \( \exp((1/n)(\log A)) \). Hence \( B_n = (1/n)(\log A) \) is in \( \exp^{-1}(P) \) for each \( n \). Then for each integer \( m \), \( \exp((m/n)B_n) = (B_n^m) \in P \), since the product is positive definite and in \( O^+(1, 3) \). It follows that \( \exp^{-1}(P) \) contains a dense subset of \( \mathbb{R} \log A \), and by its closeness must therefore contain \( \mathbb{R} \log A \). It follows that \( \mathbb{R} \log A \subseteq \mathfrak{p} \), in particular \( \log A \in \mathfrak{p} \). Thus \( \log P \subseteq \mathfrak{p} \), or applying \( \exp \), we obtain \( P \subseteq \exp \mathfrak{p} \). The reverse inclusion is
immediate. Thus \( \exp: \mathfrak{p} \to P \) is a homeomorphism that is group homomorphism on one-dimensional subspaces into the multiplicative structure of \( P \). It follows from Theorem 5.16 that the multiplication agrees with gyroaddition on commutative subgroups of \( P \), in particular on the image of one-parameter subgroups. Hence \( \exp \) restricted to any one-dimensional subspace of \( \mathfrak{p} \) is a homomorphism into \((P, \oplus)\).

Since gyroaddition in \( P \) is multiplication followed by projection into the \( P \)-factor of the product, it is continuous. By Lemma 6.7 the scalar multiplication is continuous.

Problem 6.12. The commutator product of two \( n \times n \)-matrices \( X \) and \( Y \) is defined by \([X, Y] = XY - YX\). Show that \( o^+(1, 3) \) is closed under commutator product (and is hence a Lie algebra) and \( \mathfrak{p} \) is closed under the triple product \( \langle X, Y, Z \rangle := [X, [Y, Z]] \).

Let \((\mathbb{R}^3, \oplus)\) be the gyrogroup of admissible velocities. We define an exponential function \( \exp: \mathbb{R}^3 \to \mathbb{R}^3_c \) by \( \exp(u) := c \tanh(|u|)(u/|u|) \).

Proposition 6.13. Each of the maps in the following diagram is a diffeomorphism (smooth homeomorphism) and the diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\beta} & \mathfrak{p} \\
\downarrow{\exp} & & \downarrow{\exp} \\
\mathbb{R}^3_c & \xrightarrow{B} & P.
\end{array}
\]

In the diagram \( B(u) \) is the Lorentz boost for \( u \) and \( \beta(u) = \begin{bmatrix} 0 & u^T \\ u & 0 \end{bmatrix} \).

Proof. The horizontal maps are coordinatewise smooth, hence smooth. The matrix exponential function is a power series map, hence smooth. From commutativity of the diagram and the bijectivity of each map, it follows that the exponential on \( \mathbb{R}^3 \) is smooth.

Problem 6.14. Verify that all the maps are bijections and that the diagram commutes.

From the preceding proposition we easily obtain the
Corollary 6.15. The map \( \exp : \mathbb{R}^3 \to \mathbb{R}^3 \) defined above is an exponential map for \((\mathbb{R}^3, \oplus)\) and hence \((\mathbb{R}^3, \oplus)\) is a topological gyrovector space.

6.3. Rooted vectors and gyrolines. In this section we consider some basic notions of the vector geometry of gyrovector spaces. We begin by adding two additional axioms to our notion of a gyrovector vector space.

Definition 6.16. A gyrovector space is a gyrocommutative gyrogroup \( V \) equipped with a scalar multiplication \((t, x) \mapsto t.x \) from \( \mathbb{R} \times V \to V \) that satisfies:

1. \( 1.x = x; \) \(-1.x = \ominus x; \) \( 0.x = t.0 = 0; \)
2. \((s + t).x = s.x \oplus t.x; \)
3. \(s.t.x = (st).x.; \)
4. \(\text{gyr}[a, b](s.x) = s.\text{gyr}[a, b]x; \)
5. \(\text{gyr}[s.a, t.a] = I.\)

In a gyrovector space \((V, \oplus, .)\) it is convenient to think of members of \( V \) in two distinct ways: as (geometric) points \( P, Q, R \) and as vectors \( u, v, w \) emanating from the origin 0. A rooted gyrovector is viewed as a vector emanating from other points in addition to the origin. More formally, a rooted gyrovector is a pair \((P, v) \in V \times V\).

We can alternatively write a rooted gyrovector in the equivalent form \( \overrightarrow{PQ} \), where \( v = \ominus P \oplus Q \). The rooted vector \( \overrightarrow{PQ} \) has head \( P \) and tail \( Q \).

Lemma 6.17. In a gyrogroup \((G, \oplus)\),
\[ \ominus(a \oplus b) \oplus (a \oplus c) = \text{gyr}[a, b](\ominus b \oplus c). \]

Proof. The proof is a straightforward application of the fact that \(\text{gyr}[a, b] = \lambda_{\ominus(a \oplus b)}/\lambda_a \lambda_b.\)

Proposition 6.18. In a gyrovector space \( V \), the following are equivalent for \( P, Q, P', Q' \in V \).

1. For some \( v \in V \), \( Q = P \oplus v \) and \( Q' = P' \oplus v. \)
2. \( \ominus P \oplus Q = \ominus P' \oplus Q'. \)
3. For some \( u \in V \), \( P' = \text{gyr}[P, u](u \oplus P) \) and \( Q' = \text{gyr}[P, u](u \oplus Q). \)

In case (3) the vector \( u \) is unique and given by \( u = \ominus P \oplus P'. \)
Proof. (1)⇔(2): From $Q = P \oplus v$, we deduce that $\ominus P \oplus Q = v$ and similarly $\ominus P' \oplus Q' = v$. In the converse case simply set $v = \ominus P \oplus Q = \ominus P' \oplus Q'$.

(2)⇒(3): Set $u = \ominus P \oplus P'$. Then

\[ \text{gyr}[P, u](u \oplus P) = P \oplus u = P \oplus (\ominus P \oplus P') = P', \]

where the first equality follows from gyrocommutativity. Since $\ominus P \oplus Q = \ominus P' \oplus Q'$, we have

\[
\begin{align*}
Q' &= P' \oplus (\ominus P \oplus Q) = \text{gyr}[P, u](u \oplus P) \oplus (\ominus P \oplus Q) \\
&= \text{gyr}[P, u][(u \oplus P) \oplus (\text{gyr}[u, P](\ominus P \oplus Q))] \\
&= \text{gyr}[P, u](u \oplus (P \oplus (\ominus P \oplus Q))) \\
&= \text{gyr}[P, u](u \oplus Q)
\end{align*}
\]

(3)⇒(2): We have by gyrocommutativity, $P' = \text{gyr}[P, u](u \oplus P)$ and $Q' = \text{gyr}[P, u](u \oplus Q)$, so by Lemma 6.17

\[
\text{gyr}[u, P](\ominus P \oplus Q) = \ominus(u \oplus P) \oplus (u \oplus Q).
\]

Applying $\text{gyr}[P, u]$ to both sides yields

\[
\ominus P \oplus Q = \text{gyr}[P, u](\ominus(u \oplus P)) \oplus \text{gyr}[P, u](u \oplus Q) = \ominus P' \oplus Q'.
\]

The uniqueness of $u$ in condition (3) follows from the fact that

\[
\ominus P \oplus P' = \ominus P \oplus \text{gyr}[P, u](u \oplus P) = \ominus P \oplus (P \oplus u) = u.
\]

□

Two rooted gyrovectors $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ are equivalent if they satisfy the equivalent conditions of Proposition 6.18.

Definition 6.19. A gyrol ine in a gyrovector space $V$ is a set of the form

\[
\{P \oplus t \cdot v : t \in \mathbb{R}, \ P, v \in V, \ v \neq 0\}.
\]

The specific gyrol ine is called the gyrol ine through $P$ in direction $v$. The map $t \mapsto P \oplus t \cdot v$ is a linear parameterization of the gyrol ine.

We list some elementary properties of gyrolines.
Lemma 6.20. Given two distinct points $P$ and $Q$ in a gyrovector space $V$, there exists a gyroline containing the two points. The parameterization $t \mapsto P \oplus t(\ominus P \oplus Q)$ is a gyrolinear parameterization taking on the value $P$ at 0 and $Q$ at 1.

Proof. By definition $\{P \oplus t(\ominus P \oplus Q) : t \in \mathbb{R}\}$ is a gyroline, and one sees directly that the given parametrization is gyrolinear and takes on the values $P$ and $Q$ at 0 and 1 resp. □

The notion of a gyroline allows a geometric visualization of a gyrovector $\overrightarrow{PQ}$.

Namely we consider the segment of the gyroline determined by $P$ and $Q$ that lies between $P$ and $Q$, namely $\{P \oplus t(\ominus P \oplus Q) : 0 \leq t \leq 1\}$, directed in the direction from $P$ to $Q$.

Lemma 6.21. The left translation of a gyroline is a gyroline.

Proof. We note that $P \oplus (Q \oplus t.v) = (P \oplus Q) \oplus \text{gyr}[P,Q](t.v) = (P \oplus Q) \oplus t.\text{gyr}[P,Q]v.$ Thus the left translation by $P$ of the gyroline through $Q$ in the direction $v$ is the gyroline through $P + Q$ in the direction $\text{gyr}[P,Q]v$. □

Lemma 6.22. Any gyroline $\ell$ through 0 has a parametrization of the form $\alpha_v(t) = t.v$ for some $v \neq 0$ in $\ell$, which is an injective gyrovector space homomorphism.

Proof. Let $\ell = \{P + t.v : t \in \mathbb{R}\}$ contain 0, i.e., $0 = P \oplus r.v$ for some $r \in \mathbb{R}$. Then $r.v = \ominus P$ or $P = (-r).v$. Set $\alpha_v(t) = t.v$. Then

$$\alpha_v(t) = t.v = (-r).v \oplus (t + r).v = P \oplus (t + r)v.$$ is a gyrolinear parameterization of $\ell$, a gyrovector space homomorphism by properties (2) and (3) of Definition 6.16 and injective by property (1). □

Corollary 6.23. Given two distinct points of a gyrovector space, there is a unique gyroline containing the two points.

Proof. Let $P, Q$ be distinct points. By Lemma 6.20 there exists a gyroline $\ell$ containing the two. Consider the special case that $Q = 0$ (hence $P \neq 0$). By Lemma 6.22 there exists $v \neq 0$ such that $\ell = \{t.v : t \in \mathbb{R}\}$. Let $P = s.v$; then $s \neq 0$. Since $t.v = (t/s).P$ for all $t \in \mathbb{R}$, it follows that $\ell = \{t.P : t \in \mathbb{R}\}$. Since $\ell$ was an arbitrary gyroline containing 0 and $P$, it follows that $\{t.P : t \in \mathbb{R}\}$ is the unique gyroline containing 0.
and \( P \). One can now use Lemma \([6.21]\) to argue that there is a unique gyroline through any two distinct points. \( \square \)

**Remark 6.24.** It follows from Corollary \([6.23]\) and Lemma \([6.20]\) that \( t \mapsto P \oplus t.(\ominus P \oplus Q) \) is a gyrolinear parameterization of the unique gyroline through any two distinct points \( P, Q \).

### 6.4. Gyrovectors spaces with inner product

To talk about length of gyrovectors and angles between gyrovectors, we introduce an inner product.

**Definition 6.25.** A real inner product gyrovector space consists of three components:

1. A gyrovector space \((G, \oplus, .)\) defined on some open ball of a real inner product vector space.
2. The set \( \|G\| := \{\pm\|v\| : v \in G\} \) is equipped with a gyroaddition and scalar multiplication making it a gyrovector space. Here the norm is the one induced by the inner product.
3. The gyrovector space structure is connected to the inner product through the following laws:
   
   \((i)\) \( \text{gyr}[u, v]\{a \cdot \text{gyr}[u, v]b = a \cdot b \), i.e., gyrations preserve inner product.
   
   \((ii)\) \( \frac{\|r\|a}{\|r.a\|} = \frac{a}{\|a\|} \).

   \((iii)\) \( \|r.a\| = |r|\|a\| \).

   \((iv)\) \( \|a \oplus b\| \leq \|a\| \oplus \|b\| \).

**Remark 6.26.** The zero of the gyrogroup \( G \) is equal to 0 in the real inner product vector space.

**Proof.** We have

\[ 0 = \|0\| = | -1| \|0\| = \| -1.0\| = \| \ominus 0\|. \]

It follows that \( \ominus 0 = 0 \). Adding 0 to both sides, we obtain \( Z = 0 \oplus 0 = 2.0 \), where \( Z \) is the gyrogroup additive identity. Multiplying both sides by 1/2 yields \( Z = (1/2).Z = 0 \). \( \square \)

### 6.5. The gyrodistance

We assume in this section we are working in an inner product gyrovector space \( G \).
**Definition 6.27.** The gyrodistance function $d_{\oplus}(a, b)$ is defined by

$$d_{\oplus}(a, b) = \| \ominus a \oplus b \|.$$

**Proposition 6.28.** The gyrodistance function satisfies the standard axioms for a metric, with addition in the triangle inequality replaced by gyroaddition.

*Proof.* From Remark 6.26, we have that $\|v\| = 0$ if and only if $v = 0$, the gyroaddition identity. If $\| \ominus a \oplus b \| = 0$, then $\ominus a \oplus b = 0$ and hence $a = b$.

Since a gyrovector space is gyrocommutative, we have

$$\ominus a \oplus b = \ominus (a \ominus b) = \ominus \text{gyr}[a, \ominus b](\ominus b \oplus a).$$

Since $\| \ominus c \| = \|(-1).c\| = | - 1|\|c\| = \|c\|$, we have

$$\| \ominus a \oplus b \| = \| \ominus \text{gyr}[a, \ominus b](\ominus b \oplus a) \| = \| \text{gyr}[a, \ominus b](\ominus b \oplus a) \| = \| \ominus b \oplus a \|.$$

where the last equality follows from the fact that $\text{gyr}[a, \ominus b]$ preserves the norm since it preserves the inner product. It follows that $d_{\oplus}(a, b) = d_{\oplus}(b, a)$.

By Lemma 6.17

$$\ominus (\ominus a \oplus b) \oplus (\ominus a \oplus c) = \text{gyr}[\ominus a, \ominus b](\ominus b \oplus c),$$

and hence

$$\ominus a \oplus c = (\ominus a \oplus b) \oplus \text{gyr}[\ominus a, \ominus b](\ominus b \oplus c).$$

By the Gyrotriangle Inequality Axiom,

$$\| \ominus a \oplus c \| \leq \| \ominus a \oplus b \| \oplus \| \text{gyr}[\ominus a, \ominus b](\ominus b \oplus c) \| = \| \ominus a \oplus b \| \oplus \| \ominus b \oplus c \|. $$

Hence $d_{\oplus}(a, c) \leq d_{\oplus}(a, b) \oplus d_{\oplus}(b, c)$.

□

**Proposition 6.29.** The distance $d_{\oplus}$ is invariant under gyrations and left translations.

*Proof.* We first note by the invariance of the inner product under gyrations that

$$\| \text{gyr}[a, b]u \| = (\text{gyr}[a, b]u \cdot \text{gyr}[a, b]u)^{1/2} = (u \cdot u)^{1/2} = \|u\|.$$
Hence
\[ d_\oplus(\text{gyr}[a, b]u, \text{gyr}[a, b]v) = \| \ominus \text{gyr}[a, b]u \ominus \text{gyr}[a, b]v \| = \| \text{gyr}[a, b](\ominus u \ominus v) \| = \| \ominus u \ominus v \| = d_\oplus(u, v). \]

For the second assertion, we first note by Lemma 6.17 that
\[ \ominus(a \ominus u) \ominus (a \ominus v) = \text{gyr}[a, u](\ominus u \ominus v), \]
and hence
\[ d_\oplus(a \ominus u, a \ominus v) = \| \ominus (a \ominus u) \ominus (a \ominus v) \| = \| \text{gyr}[a, u](\ominus u \ominus v) \| = \| \ominus u \ominus v \| = d_\oplus(u, v). \]

\[ \square \]

6.6. The Einstein gyrovector space inner product. We have considered previously the gyrogroup $\mathbb{R}^3_c$ of admissible velocities under Einstein velocity addition. We have also introduced the exponential map $\exp : \mathbb{R}^3 \to \mathbb{R}^3_c$ defined by $\exp(u) = ctanh(|u|)(u/|u|)$. Via the exponential map and its inverse log, we can define the scalar multiplication by $t.v := \exp(t \log v)$. The fact that $\exp$ restricted to the one-dimensional subspaces of $\mathbb{R}^3$ is a homomorphism preserving scalar multiplication yields the axioms of a gyrovector space (see Section 6.1).

We equip $\mathbb{R}^3_c$ with the usual euclidean inner product on $\mathbb{R}^3$ restricted to $\mathbb{R}^3_c$. We need some preparation to establish the appropriate axioms for the inner product. We recall from Proposition 4.9 that for $u, v \in \mathbb{R}^3_c$ and the corresponding Lorentz boosts $B(u), B(v)$,
\[ B(u)B(v) = B(u \oplus v)h(u, v), \text{ where } h(u, v) \in O^+(1, 3). \]

We note that actually $h(u, v) \in SO^+(1, 3)$, since in the preceding equation it must have a positive determinant for equality to hold, since each Lorentz boost has a positive determinant.
**Proposition 6.30.** For \( u, v \in \mathbb{R}_c^3 \), \( \text{gyr}[u, v] = S(u, v) \in SO^+(1, 3) \), where \( S(u, v) \) is the \( 3 \times 3 \)-block matrix in the block diagonal matrix \( h(u, v) \).

**Proof.** For \( u, v, w \in \mathbb{R}_c^3 \), we have

\[
B(u)(B(v)B(w) = B(u)B(v \oplus w)h(v, w) = B(u \oplus (v \oplus w))h(u, v \oplus w)h(v, w)
\]

and associating the other way

\[
B(u)B(v)B(w) = B(u \oplus v)h(u, v)B(w) = B(u \oplus v)B(w)^{h(u, v)}h(u, v),
\]

where \( B(w)^{h(u, v)} = h(u, v)B(w)h(u, v)^{-1} \). Since \( h(u, v) \in SO^+(1, 3) \), by Proposition 3.20 it has a block diagonal form with diagonal entries 1, \( S = S(u, v) \in SO(3) \). Hence

\[
B(w)^{h(u, v)} = \begin{bmatrix} 1 & 0 \\ 0 & S \\ \gamma & * \\ \frac{2c}{\gamma}w & * \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^{-1} \end{bmatrix} = B(Sw),
\]

where \( \gamma = \gamma_w \) and where the last equality holds since the conjugation of \( B(w) \) must again be a Lorentz boost, and the first column then determines it. We now continue our earlier computation:

\[
B(u \oplus v)B(w)^{h(v, w)}h(u, v) = B((u \oplus v)\oplus Sw)h(u, v) \]

Comparing with our first computation, we conclude \( B(u \oplus (v \oplus w)) = B((u \oplus v)\oplus Sw) \).

Since \( B \) is bijective,

\[
(u \oplus v) \oplus \text{gyr}[u, v]w = u \oplus (v \oplus w) = (u \oplus v) \oplus Sw,
\]

and by left cancellation \( \text{gyr}[u, v] = S = S(u, v) \). \( \square \)

From Proposition 6.30 we have immediately the

**Corollary 6.31.** The gyrations \( \text{gyr}[u, v] \) of \( \mathbb{R}_c^3 \) preserve the inner product.
Rapidities and norm axioms. The notion of the rapidity
\[ \phi_v = \tanh^{-1} \frac{\|v\|}{c} \]
of an admissible velocity \( v \) was introduced very early in the development of special relativity. Rapidities satisfy a number of useful properties.

Lemma 6.32. Let \( v \in \mathbb{R}^3_c \) be an admissible velocity vector.

1. \( c \tanh(\phi_v) = \|v\| \).
2. \( v = \exp \frac{\phi_v v}{\|v\|} \); hence \( \log v = \frac{\phi_v v}{\|v\|} \).
3. \( \cosh \phi_v = \gamma_v \).
4. \( \sinh \phi_v = \gamma_v \frac{\|v\|}{c} \).

Proof. Item (1) follows directly from the definition of the rapidity. Applying the definition \( \exp(x) = c \tanh(|x|)(u/\|u\|) \) to \( \phi_v v / \|v\| \) and noting that the latter’s norm is \( \phi_v \), we have
\[ \exp \frac{\phi_v v}{\|v\|} = c \tanh(\phi_v) \frac{v}{\|v\|} = v, \]
where the last equality follows from (1).

For (3), we have that
\[ \gamma_v = (1 - \frac{\|v\|^2}{c^2})^{-1/2} = (1 - \tanh^2 \phi_v)^{-1/2} = (\text{sech}^2 \phi_v)^{-1/2} = \cosh \phi_v. \]

For (4), \( \sinh \phi_v = \cosh \phi_v \tan \phi_v = \gamma_v \frac{\|v\|}{c} \).

The exponential \( \exp : \mathbb{R} \to \mathbb{R}_c \) is given by \( \exp(x) = c \tanh(|x|)(x/|x|) \), where the last equality holds for \( x \neq 0 \) and follows from the fact \( \tanh(x) \) is an odd function.

Lemma 6.33. For \( r \in \mathbb{R} \), \( v \in \mathbb{R}_c^3 \), \( v \neq 0 \), \( r v = (r \|v\|)(v/\|v\|) \).

Proof. Applying our previous results, we obtain for \( r \neq 0 \),
\[ r v = \exp(r \log v) = \exp \left( r \frac{\phi_v v}{\|v\|} \right) \]
\[ = c \tanh(|r| \phi_v) \frac{r \phi_v v}{|r| \phi_v \|v\|} \]
\[ = \exp(r \phi_v) \frac{v}{\|v\|}. \]
By Lemma 6.32(1) $\|v\| = c \tanh(\phi_v) = \exp \phi_v$, so $\log \|v\| = \phi_v$. Applying this to the previous equalities yields

$$r. v = \exp(r \phi_v) \frac{v}{\|v\|} = \exp(r \log \|v\|) \frac{v}{\|v\|} = (r \|v\|) \frac{v}{\|v\|}. $$

The case $r = 0$ is trivial.

We now verify further axioms of an inner product gyrovector space.

**Lemma 6.34.** For $r \in \mathbb{R}$, $v \in \mathbb{R}^3$, $\|r.v\| = |r| \|v\|$.

**Proof.** Equality trivially holds for the cases $r = 0$ or $v = 0$. So we assume both are not 0. By the preceding lemma

$$\|r.v\| = \|(r \|v\|) \frac{v}{\|v\|}\| = |(r \|v\|)|. $$

If $r > 0$, then $|r \|v\| | = |r| \|v\|$ and we are done. If $r < 0$, then $r \|v\| < 0$, so $|r \|v\| | = -r \|v\| = |r| \|v\|$. □

**Lemma 6.35.** For $0 \neq r \in \mathbb{R}$ and $0 \neq v \in \mathbb{R}^3$, $\frac{|r| v}{\|r.v\|} = \frac{v}{\|v\|}$.

**Proof.** By Lemmas 6.33 and 6.34

$$\frac{|r| v}{\|r.v\|} = \frac{(|r| \|v\|) v}{\|(r \|v\|) \|v\|} = \frac{v}{\|v\|}. $$

The final axiom that we need to verify is the triangular inequality.

**Proposition 6.36.** For $u, v \in \mathbb{R}^3$, $\|u \oplus v\| \leq \|u\| \oplus \|v\|$.

**Proof.** From Problem 4.11 equation 4.27 we have

$$\gamma_{\|u\| \oplus \|v\|} = \gamma_u \gamma_v \left(1 + \frac{\|u\| \|v\|}{c^2}\right) \geq \gamma_u \gamma_v \left(1 + \frac{u \cdot v}{c^2}\right) = \gamma_u \oplus \gamma_v = \gamma_{\|u \oplus v\|}. $$

Since $\gamma_x = \gamma_{\|x\|}$ is a monotonically increasing function of $\|x\|$, it follows that

$$\|u \oplus v\| \leq \|u\| \oplus \|v\|. $$

□
We have thus shown

**Theorem 6.37.** The Einstein gyrovector space \((\mathbb{R}^3_c, \oplus, .)\) is a real inner product gyrovector space.

### 7. A Little Hyperbolic Geometry

We have seen how to define distance and length in a real inner product gyrovector space, although it might better be called a “gyrolength” since it takes values not in the nonnegative reals, but in the nonnegative members of \((\mathbb{R}_c, \oplus, .)\). In this section we consider briefly how to extend other basic aspects of vector analysis in euclidean spaces to the hyperbolic setting of real inner product gyrovector spaces.

In addition to lengths we can also measure angles from the formula

\[
\cos \alpha := \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}
\]

where \(\alpha\) is the measure of the angle at 0 between the vectors \(\mathbf{u}\) and \(\mathbf{v}\). Note that the preceding equation can be rewritten in the familiar form

\[
\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \alpha.
\]

More generally, if \(A, B, C\) are noncollinear points (i.e., don’t lie on a gyroline), we calculate the measure of the gyroangle \(\angle ABC\) determined by the rooted gyrovectors \(\overrightarrow{BA}\) and \(\overrightarrow{BC}\) from the formula

\[
\cos \alpha := \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\| \|\overrightarrow{BC}\|} = \frac{\oplus B \oplus A}{\|\oplus B \oplus A\|} \cdot \frac{\oplus B \oplus C}{\|\oplus B \oplus C\|}.
\]

**Definition 7.1.** We say two segments of gyrolines are congruent if the endpoints \(P, Q\) of the first segment are the same distance apart as those \(P', Q'\) of the second segment, i.e., \(d_\oplus(P, Q) = d_\oplus(P', Q')\). We write \(PQ \cong P'Q'\). We say two gyroangles \(\angle BAC\) and \(\angle B'A'C'\) are congruent if their measures are equal and write \(\angle BAC \cong \angle B'A'C'\). Two triangles are congruent if all corresponding sides and gyroangles are congruent.

**Proposition 7.2.** Let \(A, B, C\) be noncollinear points.

(i) If \(A' = \text{gyr}[\mathbf{u}, \mathbf{v}]A, B' = \text{gyr}[\mathbf{u}, \mathbf{v}]B,\) and \(C' = \text{gyr}[\mathbf{u}, \mathbf{v}]C,\) then \(\angle ABC \cong \angle A'B'C',\) i.e., the measurement of the gyroangles are equal.

(ii) If \(A' = P \oplus A, B' = P \oplus B,\) and \(C' = P \oplus C,\) then \(\angle ABC \cong \angle A'B'C'.\)
Problem 7.3. Prove Proposition 7.2. (Hint: Modify the proof of Proposition 6.29.)

We turn now to the consideration of triangles. Let $A, B, C$ be noncollinear points and the vertices of a triangle. We then have gyroangles at $A, B, C$ denoted $\angle A, \angle B,$ and $\angle C$ resp. We orient the sides of $\triangle ABC$ as rooted gyrovectors $\overrightarrow{AB}, \overrightarrow{CB},$ and $\overrightarrow{CA}$. We set $a = \ominus C \oplus B, b = \ominus C \oplus A,$ and $c = \ominus A \oplus B,$ the sides opposite $\angle A, \angle B,$ and $\angle C$ resp. We denote their lengths by $a = \parallel a \parallel = \parallel \overrightarrow{CB} \parallel = \parallel \ominus C \oplus B \parallel = d_{\ominus}(C, B) = d_{\ominus}(B, C)$

$b = \parallel b \parallel = \parallel \overrightarrow{CA} \parallel = \parallel \ominus C \oplus A \parallel = d_{\ominus}(C, A) = d_{\ominus}(A, C)$

$c = \parallel c \parallel = \parallel \overrightarrow{AB} \parallel = \parallel \ominus A \oplus B \parallel = d_{\ominus}(A, B) = d_{\ominus}(B, A)$.

In particular, the lengths $a, b, c$ are independent of the orientation chosen.

Problem 7.4. Show that for the given orientation

\[ \| \ominus a \oplus b \| = \| \ominus (\ominus C \oplus B) \oplus (\ominus C \oplus A) \| = \| \ominus B \oplus A \| = \| c \|. \]

Problem 7.5. Show that

\[ \cos \angle C = \frac{a \cdot b}{ab}, \tag{7.32} \]

in full analogy to the euclidean case.

7.1. Relativistic hyperbolic geometry. In this section we work in the Einstein gyrovector space $\mathbb{R}_{s}^{3},$ the open ball of radius $s,$ where we switch from $c$ to $s$ to avoid notational confusion. We recall equation 4.27, the gamma identity,

\[ \gamma_{u \oplus v} = \gamma_{u} \gamma_{v} \left( 1 + \frac{u \cdot v}{s^2} \right). \tag{7.33} \]

from Problem 4.11. Since $\gamma_{v} = \gamma_{\|v\|},$ we have from Problem 7.4 that

\[ \gamma_{c} = \gamma_{\ominus a \oplus b} \]

in triangle $\triangle ABC$ of the preceding section. It follows that

\[ \gamma_{c} = \gamma_{c} = \gamma_{\ominus a \oplus b} = \gamma_{\ominus a \oplus b} \left( 1 + \frac{a \cdot b}{s^2} \right) = \gamma_{a} \gamma_{b} \left( 1 - \frac{ab \cos \angle C}{s^2} \right), \]

since $\ominus a = -a.$

Problem 7.6. Show that $\ominus a = -a$ in the Einstein gyrovector space.
We slightly change the notation and record the preceding equation as the relativistic law of cosines.

**Proposition 7.7.** In $\Delta ABC$, we have

$$\gamma_a = \gamma_b \gamma_c \left(1 - \frac{bc}{s^2} \cos \angle A\right),$$

(7.34)

where $a, b, c$ are the lengths of the sides opposite $\angle A, \angle B, \angle C$, resp.

**Problem 7.8.** Show that

$$\frac{a^2}{s^2} = \frac{\gamma_a^2 - 1}{\gamma_a^2}.$$

**Problem 7.9.** Use the preceding problem and equation (7.34) to show

$$\cos \angle A = \frac{\gamma_b \gamma_c - \gamma_a}{\sqrt{\gamma_b^2 - 1} \sqrt{\gamma_c^2 - 1}}.$$

Note the right-hand side of the preceding equation allows one to calculate $\cos \angle A$ from the lengths $a, b, c$ of the sides of $\Delta ABC$. Thus the radian measure of $\angle A$ is uniquely determined in $(0, \pi)$. We thus obtain

**Theorem 7.10.** (SSS) Two triangles are congruent if their corresponding sides are congruent.

**Problem 7.11.** Use the preceding theory to deduce the SAS theorem in the geometry of $\mathbb{R}^3$.

For a gyroangle $\angle A$ define $\sin \angle A := (1 - \cos^2 \angle A)^{1/2}$. One can establish the equality

$$\gamma_a = \frac{\cos \angle A + (\cos \angle B) \cos \angle C}{\sin \angle B \sin \angle C},$$

(7.35)

by direct computation by using the equation of Problem 7.9 to establish the variant form

$$\gamma_a^2 = \frac{(\cos \angle A + \cos \angle B \cos \angle C)^2}{(1 - \cos^2 \angle B)(1 - \cos^2 \angle C)}.$$

One can also derive the relativistic law of sines:

$$\frac{\sin \angle A}{\gamma_a a} = \frac{\sin \angle B}{\gamma_b b} = \frac{\sin \angle C}{\gamma_c c},$$

(7.36)

**Problem 7.12.** Derive one of the two preceding equations. (If you derive the second, you may assume the first.)
Problem 7.13. Show that an equilateral triangle with $\angle A = \angle B = \angle C = \theta$ has sides with length $\sqrt{2 \cos \theta - 1}/\cos \theta$. (Be aware that $\theta < 60^\circ$ in the hyperbolic setting.)

Note that the angle determines the side in this setting, which is certainly not the case in euclidean geometry.

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