THE FELL TOPOLOGY AND THE MODULAR GROMOV-HAUSDORFF PROPINQUITY

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ABSTRACT. Given a unital AF-algebra $A$ equipped with a faithful tracial state, we equip each (norm-closed two-sided) ideal of $A$ with a metrized quantum vector bundle structure, when canonically viewed as a module over $A$, in the sense of Latrémolière using previous work of the first author and Latrémolière. Moreover, we show that convergence of ideals in the Fell topology implies convergence of the associated metrized quantum vector bundles in the modular Gromov-Hausdorff propinquity of Latrémolière. In a similar vein but requiring a different approach, given a compact metric space $(X, d)$, we equip each ideal of $C(X)$ with a metrized quantum vector bundle structure, and show that convergence in the Fell topology implies convergence in the modular Gromov-Hausdorff propinquity.

1. INTRODUCTION AND BACKGROUND

The realm of noncommutative metric geometry introduced by Rieffel \cite{rieffel1, rieffel2} provides a metric framework for continuity of C*-algebras using noncommutative analogues to compact metric spaces and the Gromov-Hausdorff distance. Examples of C*-algebras that vary continuously in certain senses include: continuous fields of C*-algebras and in particular inductive/direct limits of C*-algebras, matrices converging to the sphere and other structures found in the high-energy physics literature, etc \cite{rieffel3}. Rieffel was first to introduce a noncommutative analogue to the Gromov-Hausdorff distance called the Gromov-Hausdorff distance \cite{rieffel2}, and many others developed their own noncommutative analogues to answer various questions left after Rieffel developed his \cite{latremoliere1, latremoliere2, latremoliere3, latremoliere4, latremoliere5}. Our work has led us to work mainly with Latrémolière’s distance, the Gromov-Hausdorff propinquity \cite{latremoliere2, latremoliere3}, due to the advantages it has afforded the first author - in particular, the completeness argument in \cite{latremoliere2} as applied in \cite{latremoliere4, latremoliere5}. Moreover, with Latrémolière, the first author also established convergence of AF-algebras using the Gromov-Hausdorff propinquity.

However, there are many important structures one can associate to C*-algebras such as group actions, Hilbert C*-modules, spectral triples, etc, whose continuity is also of interest, and Latrémolière has developed a suite of quantum distances to address each of these scenarios \cite{latremoliere6, latremoliere7, latremoliere8} and has answered fascinating continuity results of these structures \cite{latremoliere9}. Rieffel has also utilized Latrémolière’s Gromov-Hausdorff propinquity to show convergence of various structures as well \cite{rieffel6, rieffel7}.

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In this article, we focus on convergence of certain Hilbert C*-modules. Although quantum distances require more information than that of the C*-algebras such as L-seminorms (noncommutative analogues to the Lipschitz seminorm), it is natural to consider structures arising purely from the C*-algebraic structure. For instance, when Latremolière introduced the modular Gromov-Hausdorff propinquity, he first proved that convergence of a sequence of quantum compact metric spaces in the Gromov-Hausdorff propinquity implies convergence of free modules of finite rank in the modular Gromov-Hausdorff propinquity, and second, he provided sufficient conditions for when convergence of two sequences of modules implies convergence of their direct sum [14, Section 7 and 8]. Following in this theme, we look at ideals of C*-algebras in this article, which canonically form modules over C*-algebras, and ask if convergence of ideals in the Fell topology provide convergence in the modular Gromov-Hausdorff propinquity. In Section 2, we answer this in the positive for ideals of any unital AF-algebra equipped with a faithful tracial state. In Section 3, we answer this in the positive for any separable unital commutative C*-algebra (note that a unital C*-algebra equipped with a quantum metric is necessarily separable, so this case is as general as possible). For the remainder of this section, we list the necessary definitions and results for the rest of the article. We note that all convergence result in this paper are also true for the dual modular Gromov-Hausdorff propinquity since the modular propinquity dominates it (see [17, Proposition 3.17]).

Convention 1.1. Let \( A \) be a unital C*-algebra. We denote its norm by \( \| \cdot \|_A \) and identity by \( 1_A \) unless otherwise specified. Moreover, by an ideal of a C*-algebra, we mean a norm-closed two-sided ideal, and we denote the set of ideals by \( \text{Ideals}(A) \).

The story begins with quantum compact metric spaces.

Definition 1.2. [14, Definition 2.3] A function \( F : [0, \infty)^4 \to [0, \infty) \) is admissible when for all \((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in [0, \infty)^4 \) such that \( x_j \leq y_j \) for all \( j \in \{1, 2, 3, 4\} \), we have
\[
F(x_1, x_2, x_3, x_4) \leq F(y_1, y_2, y_3, y_4)
\]
and
\[
x_1x_4 + x_2x_3 \leq F(x_1, x_2, x_3, x_4).
\]

Definition 1.3. [14, Definition 2.4] An \( F\text{-Leibniz quantum compact metric space} \( (A, L) \), for some admissible function \( F \), is a unital C*-algebra \( A \) and a seminorm \( L \) defined on a dense Jordan-Lie subalgebra \( \text{dom}(L) \) of \( sa(A) \), such that
\[
(a) \ \{ a \in \text{dom}(L) : L(a) = 0 \} = \mathbb{R}1_A,
\]
(b) the Monge-Kantorovich metric \( mk_L \), defined on the state space \( S(A) \) by setting for all \( \varphi, \psi \in S(A) \)
\[
mk_L(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a \in sa(A), L(a) \leq 1 \},
\]
metrizes the weak* topology on \( S(A) \),
(c) \( L \) is lower semi-continuous, i.e. \( \{ a \in sa(A) : L(a) \leq 1 \} \) is norm closed,
(d) for all \( a, b \in \text{dom}(L) \), we have:
\[
\max\{L(a \circ b), L(\{a, b\})\} \leq F(\|a\|_A, \|b\|_A, L(a), L(b)).
\]

The seminorm \( L \) of a \( F \)-Leibniz quantum compact metric space \((A, L)\) is called an \( L \)-seminorm of type \( F \).

The two main examples of quantum metric spaces we focus on in this article are given in the following two theorems.

**Theorem 1.4.** \[13, Proposition 3.6\] Let \((X, d)\) be a compact metric space. If we define
\[
L_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}
\]
for all \( f \in C(X) \), then \((C(X), L_d)\) is an \( F \)-Leibniz compact quantum metric space, where \( F(x_1, x_2, x_3, x_4) = x_1 x_4 + x_2 x_3 \) for all \((x_1, x_2, x_3, x_4) \in [0, \infty)^4\).

**Theorem 1.5.** \[3, Theorem 3.5\] Let \( A = \bigcup_{n \in \mathbb{N}} A_n \) be a unital AF-algebra equipped with a faithful tracial state \( \tau \) such that \( A_0 = \mathbb{C}1_A \). For each \( n \in \mathbb{N} \), let
\[
E_n : A \to A_n
\]
denote the unique \( \tau \)-preserving conditional expectation onto \( A_n \). Let \((\beta(n))_{n \in \mathbb{N}}\) be a sequence in \((0, \infty)\) that converges to \(0\). For each \((x_1, x_2, x_3, x_4) \in [0, \infty)^4\), set
\[
F(x_1, x_2, x_3, x_4) = 2(x_1 x_4 + x_2 x_3).
\]
If we define
\[
L_{\beta}(a) = \sup \left\{ \frac{\|a - E_n(a)\|_A}{\beta(n)} : n \in \mathbb{N} \right\}
\]
for all \( a \in A \), then \((A, L)\) and \((A_n, L)\) for all \( n \in \mathbb{N} \) are \( F \)-Leibniz quantum compact metric spaces such that
\[
\Lambda((A_n, L), (A, L)) \leq \beta(n)
\]
for all \( n \in \mathbb{N} \), and thus
\[
\lim_{n \to \infty} \Lambda((A_n, L), (A, L)) = 0,
\]
where \( \Lambda \) is the quantum Gromov-Hausdorff propinquity of \[13\].

Now, we move on to the module setting.

**Definition 1.6.** \[14, Definition 3.6\] A triple \((F, G, H)\) is admissible when
(a) \( F : [0, \infty)^4 \to [0, \infty) \) is admissible in the sense of Definition 1.2.
(b) \( G : [0, \infty)^3 \to [0, \infty) \) satisfies \( G(x, y, z) \leq G(x', y', z') \) if \( x \leq x', y \leq x', z \leq z' \), while
\[
(x + y)z \leq G(x, y, z),
\]
and
(c) \( H : [0, \infty)^2 \to [0, \infty) \) satisfies \( H(x, y) \leq H(x', y') \) if \( x \leq x', y \leq y' \) while \( 2xy \leq H(x, y) \).

**Notation 1.7.** Let \( a \in A \), where \( A \) is a \( C^* \)-algebra. We denote \( \Re a = \frac{a + a^*}{2} \in sa(A) \) and \( \Im a = \frac{a - a^*}{2i} \in sa(A) \), where \( a = \Re a + i \Im a \).
Definition 1.8. [14, Definition 3.8] An \((F, G, H)\)-metrized quantum vector bundle 
\[ \Omega = (M, \langle \cdot, \cdot \rangle_M, D_M, A, L) \]
for some admissible triple \((F, G, H)\) is given by an \(F\)-Leibniz quantum compact metric space \((A, L)\) as well as a left Hilbert module \((M, \langle \cdot, \cdot \rangle_M)\) over \(A\) (see [14, Definition 3.4] and [22]) and a norm \(D_M\) defined on a dense \(\mathbb{C}\)-subspace \(\text{dom}(D_M)\) of \(M\) such that

(a) \(\| \cdot \|_M \leq D_M\), where \(\| \cdot \|_M = \sqrt{\langle \cdot, \cdot \rangle_M} A\) is the norm induced by \(\langle \cdot, \cdot \rangle_M\),

(b) the set 
\[ \{ \omega \in M : D_M(\omega) \leq 1 \} \]
is compact for \(\| \cdot \|_M\)

(c) for all \(a \in sa(A)\) and for all \(\omega \in M\), we have:
\[ D_M(a \omega) \leq G(\|a\|_A, L(a), D_M(\omega)), \]

(d) for all \(\omega, \eta \in M\), we have
\[ \max\{L(\Re(\omega, \eta)_M), L(\Im(\omega, \eta)_M)\} \leq H(D_M(\omega), D_M(\eta)). \]
The norm \(D_M\) is called the \(D\)-norm of \(\Omega\).

The modular Gromov-Hausdorff propinquity, \(\Lambda^{mod}\), introduced by Latrémiolère in [14] forms a metric on the class of \((F, G, H)\)-metrized quantum vector bundles up to a natural notion of isomorphism (see [14, Definition 3.18, 3.19]). The rest of the article concerns itself by establishing natural convergence results with respect to \(\Lambda^{mod}\). We do not define \(\Lambda^{mod}\) in this article since we do not require its entire construction to achieve our results.

2. THE AF-ALGEBRA CASE

We now establish that for a given unital AF algebra equipped with faithful tracial state, it holds that convergence of ideals in the Fell topology implies convergence of modules equipped with appropriate \(D\)-norms. The strategy is to use the \(L\)-seminorm of Theorem 1.5 and introduce a quantity that captures the ideal structure. This is done by canonical approximate identities for ideals of AF algebras. This approach allows us establishing finite-dimensional approximations to achieve our results. We also note that have to restrict the sequence \((\beta(n))_{n \in \mathbb{N}}\) of Theorem 1.5 in order to allow for finite-dimensional approximations and note that \((1/(n!))_{n \in \mathbb{N}}\) satisfies the following restriction on \((\beta(n))_{n \in \mathbb{N}}\).

Proposition 2.1. Using the setting of Theorem 1.5 assume furthermore that
\[ \left( \frac{\beta(n)}{\beta(n-1)} \right)_{n \in \mathbb{N}} \]
converges to 0. Let \(I \in \text{Ideals}(A)\). For any \(n \in \mathbb{N}\), put \(I_n = I \cap A_n\). Since \(I_n\) is finite dimensional, let \(1_n\) be the unit in \(I_n\) (note that \((1_n)_{n \in \mathbb{N}}\) is the canonical approximate identity for \(I\)). For any \(\omega \in I\), define
\[ D_I(\omega) = \max \left\{ L_{\beta}(\omega), \|\omega\|_A, \sup_{n \in \mathbb{N}} \left\{ \frac{\|\omega - \omega 1_n\|_A}{\beta(n)} \right\} \right\}. \]
Let $D_{I_n}$ be the restriction of $D_I$ to $I_n$. Let $G(x, y, z) = F(x, y, z, z)$ and $H(x, y) = F(x, x, y, y)$ for all $(x, y, z) \in [0, \infty)^3$ with $F$ as in Theorem \[1.5\]. Then, $(F, G, H)$ is an admissible triple. Define

$$\langle \omega, \nu \rangle_I = \omega \nu^*$$

for all $\omega, \nu \in I$, and let $\langle \cdot, \cdot \rangle_{I_n}$ denote the restriction of $\langle \cdot, \cdot \rangle_I$ to $I_n$.

It holds that $\Omega^\beta_I = (I, \langle \cdot, \cdot \rangle_I, D_I, A, L_{\beta})$ and $\Omega^\beta_{I_n} = (I_n, \langle \cdot, \cdot \rangle_{I_n}, D_{I_n}, A_n, L_{\beta})$ are $(F, G, H)$-metrized quantum vector bundles for any $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \Lambda^{\text{mod}}(\Omega^\beta_{I_n}, \Omega^\beta_I) = 0,$$

where $\Lambda^{\text{mod}}$ is the modular Gromov-Hausdorff propinquity of [14].

In particular, for each $n \in \mathbb{N}$, $n \geq 1$, it holds that

$$\Lambda^{\text{mod}}(\Omega^\beta_{I_n}, \Omega^\beta_I) \leq \max\{\beta(n), x_n - 1\},$$

where $x_n = \max\{1 + \beta(n), x'_n\}$ with $x'_n = \max_{m \leq n} \left\{2\beta(n) + \frac{\beta(n)}{\beta(m)}\right\}$, and $\max\{\beta(n), x_n - 1\}$ converges to $0$ as $n \to \infty$.

**Proof.** For ease of notation in this proof, we use $L$ for $L_{\beta}$, $D$ for $D_I$, $D_n$ for $D_{I_n}$, $\| \cdot \|$ for $\| \cdot \|_A$, $\Omega$ for $\Omega^\beta_I$, and $\Omega_n$ for $\Omega^\beta_{I_n}$. Moreover, let $B = \{\omega \in I : D(\omega) \leq 1\}$ and $B_n = B \cap I_n$. We let

$$\gamma_n = \{\Omega_n, \Omega, A, 1_A, \iota_n, \text{id}_A, \omega_j, (\omega_j)_{j \in J}\}$$

be the bridge (see [14, Definition 4.4]) between $\Omega_n$ and $\Omega$, where $\iota_n : A_n \to A$ is the inclusion map, $\text{id}_A : A \to A$ is the identity map on $A$, and $J = B_n$ and for any $j \in J$, we let $\omega_j = \nu_j = j$. Thus, $\{\omega_j : j \in J\} = \{\nu_j : j \in J\} = B_n$.

We shall break the proof of the proposition into several steps. First, we verify that $\Omega$ and $\Omega_n$ are $(F, G, H)$-metrized quantum vector bundles. Then, we verify that $\lambda(\gamma_n) \to 0$ as $n \to \infty$ by bounding the basic reach $\varrho_0(\gamma_n)$, the height $\varsigma(\gamma_n)$, the modular reach $\varrho^*(\gamma_n)$ and the imprint $\varpi(\gamma_n)$ (see [14] Definition 4.18, 4.10, 4.12, 4.14, and 4.15) for the definitions of $\lambda, \varrho_0, \varsigma, \varrho^*, \varpi$, respectively).

**Step 1: We first show that $\Omega$ and $\Omega_n$ are $(F, G, H)$-metrized quantum vector bundles.**

We show this for $\Omega$ and the result for $\Omega_n$ follows immediately.

First, $\|\omega\| \leq D(\omega)$ for all $\omega \in I$ by definition of $D$. Next, By [27, Theorem 1.9] and Theorem [1.5] the set $\{\omega \in A : \|\omega\| \leq 1, L(\omega) \leq 1\}$ is compact. Note that

$$\left\{\omega \in A : \sup_{n \in \mathbb{N}} \left\{\frac{\|\omega - \omega_{1n}\|}{\beta(n)}\right\} \leq 1\right\}$$

is closed, $I$ is closed and the intersections of compact sets with closed sets are compact. Thus, $B$ is compact. Next, let $a \in sa(A)$ and $\omega \in I$. We have

$$L(a\omega) = 2(\|a\|L(\omega) + \|\omega\|L(a)) \leq 2(\|a\|D(\omega) + L(a)D(\omega)) = F(\|a\|, L(a), D(\omega), D(\omega)) = G(\|a\|, L(a), D(\omega)).$$

Moreover,

$$\|a\omega\| \leq \|a\|\|\omega\| \leq \|a\|D(\omega) \leq 2(\|a\| + L(a))D(\omega) = G(\|a\|, L(a), D(\omega))$$
Similarly, for any \( n \in \mathbb{N} \)
\[
\frac{\|a\omega - a\omega_1\|}{\beta(n)} \leq \|a\| \cdot \frac{\|\omega - \omega_1\|}{\beta(n)} \leq \|a\| D(\omega) \leq G(\|a\|, L(a), D(\omega)).
\]
Therefore, we have that \( D(\omega) \leq G(\|a\|, L(a), D(\omega)) \). Next, let \( \omega, \nu \in I \). Since \( L \) is \(*\)-preserving as conditional expectations are positive, we have
\[
L(\langle \omega, \nu \rangle) = L(\omega \nu^\ast) \leq 2(\|\omega\| L(\nu^\ast) + \|\nu^\ast\| L(\omega)) = 2(\|\omega\| L(\nu) + \|\nu\| L(\omega)) \leq 2(D(\omega)D(\nu) + D(\omega)D(\nu)) = H(D(\omega), D(\nu)).
\]
Again, as \( L \) is \(*\)-preserving, we have
\[
\max\{L(\mathfrak{R}(\omega, \nu)), L(\mathfrak{I}(\omega, \nu))\} \leq L(\langle \omega, \nu \rangle) \leq H(D(\omega), D(\nu)).
\]
**Step 2: bounding the height and the basic reach.** The basic bridge (\cite{14} Definition 4.8) for \( \gamma_n \) is
\[
\gamma_{nb} = (A, 1_A, \iota_n, I_A).
\]
However, this is exactly the bridge in the proof of \cite{3} Theorem 3.5. Thus, from that proof, we have \( \varsigma(\gamma) = \varsigma(\gamma_{nb}) = 0 \) and \( \varphi(\gamma_{nb}) \leq \beta(n) \).

**Step 3: bounding the imprint.**

The modular Monge-Kantorvich metric (\cite{14} Definition 3.23) for our \( \Omega \) is
\[
k_{\Omega}(\omega, \nu) = \sup\{\|\omega^\ast \xi - \nu^\ast \xi\| : \xi \in B\}
\]
for any \( \omega, \nu \in I \) (note that the author of \cite{14} uses \( k_D \) and \( k_\Omega \) interchangeably). The modular Monge-Kantorvich metric for our \( \Omega_n \) is
\[
k_{\Omega_n}(\omega, \nu) = \sup\{\|\omega^\ast \xi - \nu^\ast \xi\| : \xi \in B_n\}
\]
for any \( \omega, \nu \in I_n \). Note
\[
\|\omega^\ast \xi - \nu^\ast \xi\| \leq \|\omega - \nu\| \|\xi\| \leq \|\omega - \nu\|
\]
for any \( \xi \in B \). Thus, \( k_{\Omega_n}(\omega, \nu) \leq \|\omega - \nu\| \) and \( k_{\Omega}(\omega, \nu) \leq \|\omega - \nu\| \). The imprint \( \varpi(\gamma_n) \) is
\[
\varpi(\gamma_n) = \max\{\text{Haus}_{k_{\Omega}}(\{\omega_j : j \in J\}, B_n), \text{Haus}_{k_{\Omega}}(\{\nu_j : j \in J\}, B)\}.
\]
Note that
\[
\text{Haus}_{k_{\Omega}}(\{\omega_j : j \in J\}, B_n) = \text{Haus}_{k_{\Omega}}(B_n, B_n) = 0
\]
and
\[
\text{Haus}_{k_{\Omega}}(\{\nu_j : j \in J\}, B_n) = \text{Haus}_{k_{\Omega}}(B_n, B) \leq \text{Haus}_{\|\|}(B_n, B).
\]
Our goal now is to show that
\[
\lim_{n \to \infty} \text{Haus}_{\|\|}(B_n, B) = 0.
\]
Note that \( \sup_{b \in B_n} d(b, B) = 0 \) as \( B_n \subseteq B \). Thus,
\[
\text{Haus}_{\|\|}(B_n, B) = \max\{\sup_{b \in B_n} d(b, B), \sup_{b \in B} d(b, B_n)\} = \sup_{b \in B} d(b, B_n).
\]
Let \( b \in B \), then
\[
\|b\| \leq 1, L(b) \leq 1 \quad \text{and} \quad \sup_{m \in \mathbb{N}} \left\{ \frac{\|b - b_{1m}\|}{\beta(m)} \right\} \leq 1.
\]
Thus, for any $m \in \mathbb{N}$, $\|b-E_m(b)\| \leq \beta(m)$ and $\|b-b_1\| \leq \beta(m)$. Note that $E_m(b_1) = E_m(b)1_m \in I_m$ as $E_m$ is a conditional expectation on $A_m$ and $1_m \in I_m$. Next, we shall bound $D(E_n(b1_n))$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Then
\[
\|E_n(b1_n)\| \leq \|b1_n\| \leq \|b\|\|1_n\| \leq 1
\]
since $1_n$ is a projection. Fix $m \in \mathbb{N}$, if $m \geq n$,
\[
\|E_n(b1_n) - E_m(E_n(b1_n))\| = 0.
\]
If $m < n$,
\[
\|E_n(b1_n) - E_n(E_n(b1_n))\| = \|E_n(b1_n) - E_n(E_m(b1_n))\| \\
= \|E_n(b1_n) - E_m(b1_n)\| \leq \|b1_n - E_m(b1_n)\| \\
\leq \|b - b_1\| + \|b - E_m(b)\| + \|E_m(b) - E_m(b1_n)\| \\
\leq \|b - b_1\| + \|b - E_m(b)\| + \|b - b_1\| \\
\leq 2\beta(n) + \beta(m)
\]
Fix $m \in \mathbb{N}$, if $m \geq n$,
\[
\|E_n(b1_n) - E_n(b)1_n1_m\| = 0.
\]
If $m < n$,
\[
\|E_n(b1_n) - E_n(b)1_n1_m\| = \|E_n(b1_n) - E_n(b)1_m\| \\
\leq \|E_n(b) - E_n(b1_n)\| + \|E_n(b) - E_n(b1_m)\| \\
\leq \|b - b_1\| + \|b - b_1\| \\
\leq \beta(n) + \beta(m) < 2\beta(n) + \beta(m)
\]
Put $x'_n = \max_{m<n} \left\{ \frac{2\beta(n)+\beta(m)}{2\beta(n)} \right\}$ and $x_n = \max\{1+\beta(n), x'_n\}$. Note that $x_n \to 1$ as $n \to \infty$ by Expression (2.1). Moreover, $x_n$ is independent of our choice of $b$. Then, we have that $D(E_n(b1_n)) \leq x_n$. Thus, 
\[
\frac{E_n(b1_n)}{x_n} \in B_n.
\]
Finally,
\[
\left\| b - \frac{E_n(b1_n)}{x_n} \right\| \leq \left\| E_n(b1_n) - \frac{E_n(b1_n)}{x_n} \right\| + \left\| b - E_n(b1_n) \right\| \\
\leq (x_n - 1) + \left\| E_n(b1_n) - E_n(b) \right\| + \left\| E_n(b) - b \right\| \\
\leq (x_n - 1) + \left\| b_1 - b \right\| + \left\| E_n(b) - b \right\| \leq (x_n - 1) + 2\beta(n).
\]
Thus, we have that 
\[
d(b, B_n) \leq (x_n - 1) + 2\beta(n).
\]
And so $\text{Haus}_{\text{kl}}(B_n, B) \leq (x_n - 1) + 2\beta(n)$ which tends to zero as $n \to \infty$. As a result, 
\[
\varpi(\gamma_n) \to 0 \text{ as } n \to \infty.
\]
**Step 4: bounding the modular reach.** Note that for any $j \in J$, we have that $\omega_j = \nu_j$ so the modular reach

$$\varrho^\#(\gamma_n) = \max \{dn_\gamma(\omega_j, \nu_j) \} = 0$$

(for $dn_\gamma$ see [14 Definition 4.13]). Thus, from [14 Definition 4.18 and 4.16],

$$\lambda(\gamma_n) = \max \{\varsigma(\gamma_n), \varrho(\gamma_n) \} = \max \{\varsigma(\gamma_n), \varrho_b(\gamma_n), \varrho^\# + \varpi(\gamma_n) \}$$

$$= \max \{\varrho_b(\gamma_n), \varpi(\gamma_n) \} \leq \max \{\beta(n), x_n - 1 \}$$

Thus, $\lambda(\gamma_n) \to 0$ as $n \to \infty$. Note that

$$\Lambda^{mod}(\Omega_n, \Omega) \leq \lambda(\gamma_n)$$

by [14 Definition 5.6] since any bridge is a trek. Thus, $\Lambda^{mod}(\Omega_n, \Omega) \to 0$ as $n \to \infty$. □

Given these finite-dimensional approximations, we now prove that for unital AF-algebras equipped with a faithful tracial state, convergence in Fell topology implies convergence in the modular propinquity equipped with the metrized quantum vector bundle structure of the previous proposition.

**Theorem 2.2.** Using the setting of Proposition 2.1, we have that the map

$$I \in \text{Ideals}(A) \mapsto \Omega^B_I$$

is continuous with respect to the Fell topology and the topology induced by the modular Gromov-Hausdorff propinquity.

**Proof.** Since the Fell topology is compact metrizable for separable unital C*-algebras (see [1 Lemma 3.19]), we may prove continuity by sequential continuity. Denote $\mathbb{N} = \mathbb{N} \cup \{\infty\}$. Let $(I^n)_{n \in \mathbb{N}}$ be a sequence in $\text{Ideals}(A)$ that converges in the Fell topology to $I^\infty$. Let $\epsilon > 0$. By Proposition 2.1 there exists $N \in \mathbb{N}$ such that $\max \{\beta(N), x_N - 1 \} < \epsilon$, and thus

$$\Lambda^{mod}(\Omega^B_{I^n}, \Omega^B_{I^\infty}) \leq \max \{\beta(N), x_N - 1 \} < \epsilon$$

for all $n \in \mathbb{N}$. Next, by [1 Lemma 3.24], there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$

$$I^n_N = I^\infty_N.$$

Let $n \geq N'$. Hence, by construction of $D_{I^n_N}$ and $D_{I^\infty_N}$, we have that $\Omega^B_{I^n_N} = \Omega^B_{I^\infty_N}$. Thus, by the triangle inequality for $\Lambda^{mod}$, we have

$$\Lambda^{mod}(\Omega^B_{I^n}, \Omega^B_{I^\infty}) \leq \Lambda^{mod}(\Omega^B_{I^n_N}, \Omega^B_{I^\infty_N}) + \Lambda^{mod}(\Omega^B_{I^n_N}, \Omega^B_{I^\infty_N}) + \Lambda^{mod}(\Omega^B_{I^\infty_N}, \Omega^B_{I^\infty})$$

$$< \epsilon + 0 + \epsilon = 2\epsilon.$$ □

### 3. The Commutative Case

Considering the commutative case is natural due to the desire to see how the modular propinquity behaves with respect to C*-algebraic structure. However, there’s another result that motivated this pursuit. Indeed, given a compact metric space $(X, d)$, denote the set of closed subsets of $X$ by $cl(X)$. It holds that the map

$$F \in cl(X) \mapsto (C(F), L_d)$$

(3.1)
is continuous with respect to the topology induced by the Hausdorff distance and the Gromov-Hausdorff distance of Latrémolière (see [13] Theorem 6.6) for a more general result). On the other hand, the map

\[ F \in cl(X) \mapsto I_F := \{ f \in C(X) : \forall x \in F, f(x) = 0 \} \in Ideals(C(X)) \]

is homeomorphism onto \( Ideals(C(X)) \) with respect to the topology induced by the Hausdorff distance and the Fell topology (this is a classical result, but we provide a proof of this in the following proof). Thus, the following result somewhat complements the continuity result of Expression (3.1), and this is also displayed in the following proof since we are essentially approximating families continuous of functions that vanish on different closed sets rather than the families of continuous functions defined on those different closed sets.

**Theorem 3.1.** Let \((X, d)\) be a compact metric space. Let \(I \in Ideals(C(X))\). For each \(f \in I\), we define

\[ D_I(f) = \max\{\|f\|_{C(X)}, L_d(\Re f), L_d(\Im f)\} \]

using the setting of Theorem 1.4. For each \(f, g \in I\), we define

\[ \langle f, g \rangle = fg^*. \]

Let \(G(x, y, z) = F(x, z, y, z)\) and \(H(x, y) = F(x, y, x, y)\) for all \((x, y, z) \in [0, \infty)^3\) with \(F\) as in Theorem 1.4. Then, \((F, G, H)\) is an admissible triple.

It holds that \(\Omega^d_I = (I, \langle \cdot, \cdot \rangle, D_I^d, C(X), L_d)\) is an \((F, G, H)\)-metrized quantum vector bundle. Moreover, the map

\[ I \in Ideals(C(X)) \mapsto \Omega^d_I \]

is continuous with respect to the Fell topology and the topology induced by the modular Gromov-Hausdorff propinquity. Furthermore, using notation introduced before the theorem, we have that

\[ F \in cl(X) \mapsto \Omega^d_{I_F} \]

is continuous with respect to the topology induced by the Hausdorff distance and the topology induced by the modular Gromov-Hausdorff propinquity.

**Proof.** In this proof, we denote \(\|\cdot\|_{C(X)}\) by \(\|\cdot\|\), \(L_d\) by \(L\), and \(D_I^d\) by \(D\), which causes no confusion since each of these quantities are defined on all of \(C(X)\).

**Step 1:** for all \(I \in Ideals(C(X))\), we show that \(\Omega^d_I\) is a \((F, G, H)\)-metrized quantum vector bundle. Let \(I \in Ideals(C(X))\). First, \(\|\omega\| \leq D(\omega)\) for all \(\omega \in I\) by definition of \(D\). Next, by [27] Theorem 1.9 or the Arzelà-Ascoli Theorem, the set \(\{ \omega \in C(X) : \|\omega\| \leq 1, L(\omega) \leq 1 \}\) is compact. Note that \(I\) is closed and the intersection of compact sets with closed sets are compact. Thus, \(\{ f \in I : D(f) \leq 1 \}\) is compact. First, we note that \((F, G, H)\) is admissible by definition. Next, let \(a \in sa(A)\) and \(\omega \in I\). We have since \(L\) is *-preserving,

\[ \max\{L(\Re a\omega), L(\Im a\omega)\} \leq L(a\omega) \leq F(\|a\|, \|\omega\|, L(a), L(\omega)) \leq F(\|a\|, D(\omega), L(a), D(\omega)) = G(\|a\|, L(a), D(\omega)) \]

since \(L\) satisfies the Leibniz rule. Moreover,

\[ \|a\omega\| \leq \|a\| \|\omega\| \leq \|a\| D(\omega) \leq (\|a\| + L(a))D(\omega) = G(\|a\|, L(a), D(\omega)) \]
Therefore, we have that $D(a\omega) \leq G(\|a\|, L(a), D(\omega))$. Next, let $\omega, \nu \in I$. Since $L$ is $*$-preserving, we have
\[
L(\langle \omega, \nu \rangle) = L(\omega^{*}\nu) \leq \|\omega\|L(\nu^{*}) + \|\nu^{*}\|L(\omega) = \|\omega\|L(\nu) + \|\nu\|L(\omega)
\]
\[
\leq D(\omega)D(\nu) + D(\omega)D(\nu) = H(D(\omega), D(\nu))
\]
Again, as $L$ is $*$-preserving, we have $\max\{L(\Re(\omega, \nu)), L(\Im(\omega, \nu))\} \leq L(\langle \omega, \nu \rangle) \leq H(D(\omega), D(\nu))$.

**Step 2: We now prove continuity.** Since the Fell topology is compact metrizable for separable unital C*-algebras (see [1] Lemma 3.19), we may prove continuity by sequential continuity. Let $(I_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Ideals}(C(X))$ that converges with respect to the Fell topology to some $I \in \text{Ideals}(C(X))$.

Let $n \in \mathbb{N}$. Put $B_n = \{f \in I_n : D(f) \leq 1\}$ and $B = \{f \in I : D(f) \leq 1\}$. Let
\[
\gamma_n = \{\Omega_n, \Omega, C(X), 1_{C(X)}, id_{C(X)}, \{\omega_j\}_{j \in J}, \{\nu_j\}_{j \in J}\}
\]
be a bridge from $\Omega_n$ to $\Omega$ where $id_{C(X)} : C(X) \to C(X)$ is the identity map from $C(X)$ to itself, $J = B_n$ and $\omega_j = j = \nu_j$ for $j \in B_n$.

By construction, the height and the basic reach are zero, as $I_n$ and $I$ are modules of the same C*-algebra. Next, the modular reach is
\[
\max\{d_{n, \gamma}((\omega_j, \nu_j) : j \in J) = 0.
\]

as $\omega_j = \nu_j$ for $j \in J$. Thus, we only need to bound the imprint
\[
\varpi(\gamma_n) = \max\{\text{Haus}_{k_D}(\{\omega_j : j \in J\}, B_n), \text{Haus}_{k_\Omega}(\{\nu_j : j \in J\}, B)\}
\]
(note that the author of [12] uses $k_D$ and $k_\Omega$ interchangeably). Note that
\[
\{\omega_j : j \in J\} = \{\nu_j : j \in J\} = B_n.
\]
Thus, $\varpi(\gamma_n) = \text{Haus}_{k_D}(B_n, B) \leq \text{Haus}_{\|\cdot\|}(B_n, B)$.

Let $F_n$ be the closed subset of $X$ that corresponds to $I_n$ i.e
\[
F_n = \{x \in X : f(x) = 0 \text{ for all } f \in I_n\}
\]
where $I_n = \{f \in C(X) : f(x) = 0 \text{ for all } x \in F_n\}$, and let $F$ be the closed subset of $X$ that corresponds to $I$, and note that $(I_n)_{n \in \mathbb{N}}$ converges to $I$ in Fell topology if and only if $(F_n)_{n \in \mathbb{N}}$ converges to $F$ in the Hausdorff distance with respect to $d$. Indeed, given any unital C*-algebra $A$, the Fell topology on $\text{Ideals}(A)$ of [6] is given by the Fell topology on the closed subsets of the primitive ideal space on $A$ equipped with the Jacobson topology along with the canonical one-to-one correspondence between $\text{Ideals}(A)$ and closed subsets of primitive ideals in the Jacobson topology. Moreover, in our current setting, the primitive ideals of $C(X)$ are of the form $I_{\{x\}} = \{f \in C(X) : f(x) = 0\}$ for all $x \in X$, and the set of primitive ideals $\{I_{\{x\}} \subseteq C(X) : x \in X\}$ equipped with the Jacobson topology is homeomorphic to $(X, d)$ (this is a well-known result, which follows from [21] Proposition 4.3.3). Finally, the Fell topology of [7] on the closed subsets of $(X, d)$ is homeomorphic to the topology on the closed subsets of $(X, d)$ induced by the Hausdorff distance by [5] Corollary 5.1.11].
Thus, let \( \epsilon > 0 \) and without loss of generality assume that \( \epsilon < 1 \), then there exists \( N \in \mathbb{N} \) such that for any \( n \geq N \), we have
\[
\text{Haus}(F_n, F) < \epsilon^2.
\]
Let \( n \geq N \). Let \( f \in B_n \). Then, \( f(x) = 0 \) for \( x \in F_n \), \( L(\Re f) \leq 1 \), \( L(\Im f) \leq 1 \), and \( \|f\| \leq 1 \). Define
\[
G_n = \{x \in X : d(x, F_n \cup F) < \epsilon\}.
\]
Then \( G_n^c \) is closed. Note that \( (F \cup F_n) \cap G_n^c = 0 \), and thus we may define a real-valued function \( f_1 \) on \( F \cup F_n \cup G_n^c \) by
\[
f_1 = \frac{1}{1 + \epsilon} \Re f \cdot \chi_{G_n^c}
\]
where \( \sup_{x \in F \cup F_n \cup G_n^c} |f(x)| \leq \|f\| \leq 1 \).

**Step 2a: we show \( L(f_1) \leq 1 \).** Let us discuss the following three cases. First, \( x, y \in F \cup F_n \). By the definition of \( f_1 \), \( f_1(x) = f_1(y) = 0 \), so \( \frac{|f_1(x) - f_1(y)|}{d(x,y)} = 0 \).

Second, \( x, y \in G_n^c \).
\[
\frac{|f_1(x) - f_1(y)|}{d(x,y)} = \frac{1}{1 + \epsilon} \cdot \frac{|\Re f(x) - \Re f(y)|}{d(x,y)} \leq 1.
\]

Third, \( x \in F \cup F_n \) and \( y \in G_n^c \). By the definition of \( G_n \), \( d(x,y) \geq \epsilon \). As \( \text{Haus}_d(F_n, F) < \epsilon \), there exists \( z \in F_n \) such that \( d(x, z) \leq \epsilon^2 \), and so \( \frac{d(x,z)}{d(x,y)} \leq \epsilon \). As \( L(\Re f) \leq 1 \), we have
\[
\frac{|\Re f(y)|}{d(y,z)} = \frac{|\Re f(y) - \Re f(z)|}{d(y,z)} \leq 1.
\]
Thus, \( |\Re f(y)| \leq d(y,z) \). Therefore,
\[
\frac{|f_1(x) - f_1(y)|}{d(x,y)} = \frac{|f_1(y)|}{d(x,y)} = \frac{1}{1 + \epsilon} \cdot \frac{|\Re f(y)|}{d(x,y)} \leq \frac{1}{1 + \epsilon} \cdot \frac{d(y,z)}{d(x,y)} \leq \frac{1}{1 + \epsilon} \cdot \left( 1 + \frac{d(x,z)}{d(x,y)} \right) \leq 1.
\]

Note that \( f_1 \) is Lipschitz function such that \( L(f_1) \leq 1 \) and is defined on the closed subset \( F \cup F_n \cup G_n^c \). Thus, we can apply [20, Theorem 1] and [20, Corollary 2] to extend \( f_1 \) to a real-valued function \( g_1 \) defined on \( X \) that has the same Lipschitz constant and norm as \( f \) such that: \( g_1 \) is zero on \( F \cup F_n \), \( g_1 = \frac{1}{1 + \epsilon} \Re f \) on \( G_n^c \), \( L(g_1) \leq 1 \), \( \|g_1\| \leq \|\Re f\| \).

Similarly, we can define \( f_2 \) on \( F \cup F_n \cup G_n^c \) as
\[
f_2 = \frac{1}{1 + \epsilon} \Im f \cdot \chi_{G_n^c}
\]
such that \( L(f_2) \leq 1 \) and obtain a real-valued extension \( g_2 \) of \( f_2 \) to \( X \) such that \( g_2 \) is zero on \( F \cup F_n \), \( g_2 = \frac{1}{1 + \epsilon} \Im f \) on \( G_n^c \), \( L(g_2) \leq 1 \), and \( \|g_2\| \leq \|\Im f\| \). Put \( h = g_1 + ig_2 \).

**Step 2b: we show \( h \in B \).** By the definition of \( h \), \( L(\Re h) \leq 1 \) and \( L(\Im h) \leq 1 \). Moreover
\[
\|h\| \leq \sqrt{\|g_1\|^2 + \|g_2\|^2} \leq \sqrt{\|\Re f\|^2 + \|\Im f\|^2} = \|f\| \leq 1.
\]
Therefore, \( h \) is Lipschitz function on \( X \) and \( \|h\| \leq 1 \), so \( h \in B \).
Step 2c: we show \( \|f - h\| \leq 8\epsilon \). First, assume \( x \in G_n^c \). Thus
\[
|\Re f(x) - f_1(x)| = \left| \Re f(x) - \frac{1}{1 + \epsilon} \Re f(x) \right| \leq \frac{\epsilon}{1 + \epsilon} |f(x)| \leq \epsilon
\]
and similarly \( |\Im f(x) - f_2(x)| \leq \epsilon \). Then,
\[
|f(x) - h(x)| = (|\Re f(x) - f_1(x)|^2 + |\Im f(x) - f_2(x)|^2)^{1/2} \leq \sqrt{2\epsilon} < 8\epsilon.
\]
Second, assume \( x \in G_n \). Since \( d(x, F \cup F_n) < \epsilon \), there exists \( y \in F \cup F_n \) such that
\[
d(x, y) < \epsilon.
\]
Since \( \text{Haus}_d(F_n, F) < \epsilon^2 < \epsilon \), there exists \( z \in F_n \) such that \( d(y, z) < \epsilon \). Hence \( d(x, z) < 2\epsilon \). Hence as \( \Re f(z) = 0 \), we have
\[
|\Re f(x)| = d(x, z) \cdot \frac{|\Re f(x) - \Re f(z)|}{d(x, z)} \leq d(x, z) \cdot 1 < 2\epsilon.
\]
A symmetric argument shows that \( |g_1(x)| < 2\epsilon \). Thus, by the triangle inequality, \( |\Re f(x) - g_1(x)| < 4\epsilon \). Similarly, we have \( |\Re f(x) - g_2(x)| < 4\epsilon \). Thus,
\[
|f(x) - h(x)| = (|\Re f(x) - g_1(x)|^2 + |\Im f(x) - g_2(x)|^2)^{1/2} \leq 4\sqrt{2}\epsilon < 8\epsilon.
\]
Therefore, \( |f(x) - h(x)| < 8\epsilon \) for all \( x \in X \), which completes this step.

Now, we complete Step 2. From Step 2b and 2c, we have \( d(f, B) \leq 8\epsilon \). Similarly, we can show that for any \( f \in B \), \( d(f, B_n) \leq 8\epsilon \). Thus, \( \text{Haus}_{\|\|}(B_n, B) \leq 8\epsilon \). As \( k_D(f) \leq \|f\| \) for any \( f \in C(X) \), we have that \( \text{Haus}_{k_D}(B_n, B) \leq 8\epsilon \). Thus, \( \lambda(\gamma_n) \to 0 \) as \( n \to \infty \). Note that
\[
\Lambda^{\text{mod}}(\Omega_{I_n}^d, \Omega_I^d) \leq \lambda(\gamma_n)
\]
by [14], Definition 5.6) since any bridge is a trek. Thus, \( \Lambda^{\text{mod}}(\Omega_{I_n}^d, \Omega_I^d) \to 0 \) as \( n \to \infty \) which establishes the first continuity result in the statement of the theorem. The last statement of the theorem is established by the homeomorphism discussed above between \( cl(X) \) equipped with the topology induced by the Hausdorff distance and the Fell topology on \( \text{Ideals}(C(X)) \).

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