Some asymptotic results for nonlinear Hawkes processes

Fuqing Gao\textsuperscript{1}, Lingjiong Zhu\textsuperscript{2}

November 22, 2017

Abstract

Hawkes process is a class of simple point processes with self-exciting and clustering properties. Hawkes process has been widely applied in finance, neuroscience, social networks, criminology, seismology, and many other fields. In this paper, we study fluctuations, large deviations and moderate deviations nonlinear Hawkes processes in a new asymptotic regime, the large intensity function and the small exciting function regime. It corresponds to the large baseline intensity asymptotics for the linear case, and can also be interpreted as the asymptotics for the mean process of Hawkes processes on a large network.

1 Introduction

Let $N$ be a simple point process on $\mathbb{R}$ and let $\mathcal{F}_t^\infty := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ be an increasing family of $\sigma$-algebras. Any nonnegative $\mathcal{F}_t^\infty$-progressively measurable process $\lambda_t$ with

$$
\mathbb{E} \left[ N(a, b) | \mathcal{F}_a^\infty \right] = \mathbb{E} \left[ \int_a^b \lambda_s ds | \mathcal{F}_a^\infty \right],
$$

a.s. for all intervals $(a, b]$ is called an $\mathcal{F}_t^\infty$-intensity of $N$. We use the notation $N_t := N(0, t]$ to denote the number of points in the interval $(0, t]$.

A Hawkes process is a simple point process $N$ admitting an $\mathcal{F}_t^\infty$-intensity

$$
\lambda_t := \phi \left( \int_{-\infty}^t h(t - s)N(ds) \right),
$$

where $\phi(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is locally integrable, left continuous, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ and locally integrable. In (1.1), $\int_{-\infty}^t h(t - s)N(ds)$ stands for $\sum_{\tau < t} h(t - \tau)$, where $\tau$ are the occurrences of the points before time $t$. In the literature, $h(\cdot)$ and $\phi(\cdot)$ are usually referred to as exciting function (or sometimes kernel function or self-interaction function) and intensity function respectively, see e.g. \textsuperscript{8}. A Hawkes process is linear if the intensity function $\phi(\cdot)$ is linear and it is nonlinear otherwise.

The Hawkes process when $\phi(\cdot)$ is linear was first proposed by Alan Hawkes in 1971 to model earthquakes and their aftershocks \textsuperscript{24}. The nonlinear Hawkes process was first

\textsuperscript{1}School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People’s Republic of China; fggao@whu.edu.cn

\textsuperscript{2}Department of Mathematics, Florida State University, 1017 Academic Way, Tallahassee, FL-32306, United States of America; zhu@math.fsu.edu.
introduced by Brémaud and Massoulié [4]. The Hawkes process naturally generalizes the Poisson process and it captures both the self-exciting property and the clustering effect, and it is a very versatile model for statistical analysis. These explain why it has wide applications in neuroscience, genome analysis, criminology, social networks, healthcare, seismology, insurance, finance and many other fields. For a list of references, we refer to [41].

Most of the asymptotic results for Hawkes processes in the literature are the large time limit theorems. For the linear Hawkes process, the functional law of large numbers and functional central limit theorems were studied in Bacry et al. [1]; the large deviations principle was studied in Bordenave and Torrisi [3]; and the moderate deviation principle was obtained in Zhu [33]. The precise large and moderate deviations are recently studied in Gao and Zhu [19]. For the nonlinear Hawkes process, Zhu [42] studied the functional central limit theorems by using Poisson embeddings and a careful analysis of the decay of the correlations over time. In [44], Zhu obtained a process-level, i.e. level-3 large deviation principle and the rate function is expressed as a variational problem optimizing over a certain entropy function of any simple point process against the underlying nonlinear Hawkes process. When the exciting function is exponential and the process is Markovian, an alternative expression for the rate function for the large deviations was obtained in Zhu [45]. Very recently, using the techniques as a combination of Poisson embeddings, Stein’s method and Malliavin calculus, the quantitative Gaussian and Poisson approximations were studied in Torrisi [38, 39]. The Malliavin calculus for Hawkes processes has also appeared in [37]. In the case of linear Hawkes process, the limit theorems for nearly unstable, also known as, nearly critical case, that is, when $\phi(z) = \nu + z$ and $\|h\|_{L^1} \approx 1$ are studied in Jaisson and Rosenbaum [26] when the exciting function has light tail and in Jaisson and Rosenbaum [27] when the exciting function has heavy tail.

There have been some progress made in the direction of asymptotic results other than the large time limits. For instance, when the exciting function is exponential, the intensity process and the pair $(N_t, \lambda_t)$ are Markovian. In Gao and Zhu [20], they studied the functional central limit theorems for the linear Hawkes process when the initial intensity is large, and they further studied the large deviations and applied their results to insurance and queueing systems in [21]. For the more general linear and non-Markovian case, Gao and Zhu [22] considered the large baseline intensity asymptotic results and studied the applications to queueing systems.

In recent years, the mean-field limit for high dimensional Hawkes processes has also been studied, and it first appeared in Delattre et al. [12]. They showed that under a certain setting, the mean-field limit is an inhomogeneous Poisson process. Other mean-field limit works include Chevallier [8] who studied a generalized Hawkes process model with an inclusion of the dependence on the age of the process, and Delattre and Fournier [11] who studied the mean-field limit for Hawkes processes on a graph with two nodes whether or not influence each other modeled by i.i.d. Bernoulli random variables.

In this paper, we are interested in studying a new asymptotic regime for the nonlinear
Hawkes process starting from empty past history, in which the intensity function is large and the exciting function is small. More precisely, we introduce the small parameter $\epsilon > 0$ and consider the nonlinear Hawkes process $N_\epsilon^t$ with intensity:

$$\lambda_\epsilon^t = \frac{1}{\epsilon} \phi \left( \int_0^{t^-} eh(t - s)dN_s^\epsilon \right).$$  \hspace{1cm} (1.2)

In this asymptotic regime, the pair of the intensity function and the exciting function has the transformation $(\phi, h) \mapsto \left( \frac{1}{\epsilon} \phi, \epsilon h \right)$.

Now, let us explain why this asymptotic regime is natural and also point out that such a regime has been studied extensively in many similar settings in the literature.

When the Hawkes process is linear, say $\phi(z) = \nu + z$, $h(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$, where $\nu$ is the baseline intensity, we have

$$\lambda_\epsilon^t = \frac{\nu}{\epsilon} + \int_0^{t^-} h(t - s)dN_s^\epsilon.$$  \hspace{1cm} (1.3)

This gives the intensity of a linear Hawkes process with exciting function $h$, and a large baseline intensity $\frac{\nu}{\epsilon}$. Therefore, the asymptotic regime considered in this paper corresponds to the large baseline intensity regime that is studied in Gao and Zhu [22].

The asymptotic regime studied in this paper for the univariate Hawkes process is also equivalent for the asymptotics for the mean process for the high-dimensional multivariate Hawkes process. Our work is related to the mean-field limit for high-dimensional Hawkes processes in [12, 8, 11]. To see the connection of our work with the mean-field limit literature of Hawkes processes, let us first define a multivariate Hawkes process as follows. An $N$-dimensional Hawkes process $(Z_1^t, \ldots, Z_n^t)$ is an $N$-dimensional point process admitting an $\mathcal{F}_t$-intensity $(\lambda_1^t, \ldots, \lambda_N^t)$ such that

$$\lambda_i^t := \phi_i \left( \sum_{j=1}^{N} \int_0^{t^-} h_{ij}(t - s)dZ_s^j \right),$$  \hspace{1cm} (1.4)

where $\phi_i(\cdot): \mathbb{R} \to \mathbb{R}^+$ is locally integrable, left continuous, $h_{ij}(\cdot): \mathbb{R}^+ \to \mathbb{R}$ and we always assume that $\|h_{ij}\|_{L_1} = \int_0^\infty h_{ij}(t)dt < \infty$. For the multivariate Hawkes process, a jump in one component will not only increase the intensity of future jumps of its own component, known as the self-exciting property, but also increase the intensity of the future jumps of or the other components that are connected to its own component, which is known as the mutually-exciting property. By using the Poisson embeddings, see e.g [4, 12], we can express the Hawkes process $(Z_1^t, \ldots, Z_n^t)$ as the solution of a Poisson driven SDE:

$$Z_i^t = \int_0^t \int_0^\infty 1_{\{s \leq \phi_i\left( \sum_{j=1}^{N} \int_0^{t^-} h_{ij}(t - s)dZ_s^j \right) \}} \pi^i(ds dz), \quad 1 \leq i \leq N,$$  \hspace{1cm} (1.5)

where $\{\pi^i(ds dz), i \geq 1\}$ are a sequence of i.i.d. Poisson measures with common intensity measure $d^i ds dz$ on $[0, \infty) \times [0, \infty)$. As a special case, for each $N \geq 1$, we let $h_{ij} = \frac{1}{N} h$ and
\( \phi_i = \phi \) and we consider the Hawkes process \((Z_t^{N,1}, \ldots, Z_t^{N,N})_{t \geq 0}\) which can be expressed as

\[
Z_t^{N,i} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \phi\left(N^{-1} \sum_{j=1}^N \int_0^s h(s-u)dz_u^N\right)\}} \pi^i(ds\,dz), \tag{1.6}
\]

The mean process of the Hawkes processes is defined by \((Z_t^{N,1}, \ldots, Z_t^{N,N})_{t \geq 0}\):

\[
\overline{Z}_t^N = \frac{1}{N} \sum_{i=1}^N Z_t^{N,i}, \quad t \geq 0. \tag{1.7}
\]

It follows from (1.6) that

\[
\overline{Z}_t^N = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \phi\left(\int_0^s h(s-u)\overline{Z}_u^N\right)\}} \frac{1}{N} \sum_{i=1}^N \pi^i(ds\,dz), \tag{1.8}
\]

where \(\sum_{i=1}^N \pi^i(ds\,dz)\) is a Poisson measure on \([0, \infty) \times [0, \infty)\) with intensity \(N\).

On the other hand, let us recall that the nonlinear Hawkes process \(N_t^\epsilon\) with the intensity function \(\frac{\phi}{\epsilon}\) and the exciting function \(\epsilon h\) can be expressed via Poisson embedding as the unique strong solution to the following equation:

\[
N_t^\epsilon = \int_0^t \int_0^\infty \mathbf{1}_{[0, \frac{1}{\epsilon} \phi(f_0^s h(s-u)dN_u)]}(z) \pi(ds\,dz), \tag{1.9}
\]

where \(\pi(dz\,ds)\) is a Poisson random measure on \([0, \infty) \times [0, \infty)\) with intensity \(1\). In this paper, we are interested in the asymptotics for \(Z_t^\epsilon := \epsilon N_t^\epsilon\), which satisfies the dynamics:

\[
Z_t^\epsilon := \epsilon \int_0^t \int_0^\infty \mathbf{1}_{[0, \frac{1}{\epsilon} \phi(f_0^s h(s-u)dN_u)]}(z) \pi(ds\,dz) = \int_0^t \int_0^\infty \mathbf{1}_{[0, \phi(f_0^s h(s-u)dZ_u)]}(z) \epsilon \pi^{\epsilon^{-1}}(dz\,ds). \tag{1.10}
\]

where \(\pi^{\epsilon^{-1}}(dz\,ds)\) is a Poisson random measure on \([0, \infty) \times [0, \infty)\) with intensity \(\epsilon^{-1}\).

By comparing (1.8) with (1.10), it becomes clear that the mean process of an \(N\)-dimensional Hawkes process defined in (1.6) has the same dynamics as a univariate Hawkes process with \(N = \frac{1}{\epsilon}\). All the asymptotic results we are going to derive in this paper for the \(Z_t^\epsilon\) process automatically hold for the mean process \(\overline{Z}_t^N\). We will go back to this in Section 3.

The asymptotic results for the mean process for a high-dimensional Hawkes process in Section 3 can shed some lights for the applications of high-dimensional Hawkes processes in various context. Hawkes processes have been applied to the study of neuroscience, see e.g. neuroscience, see e.g. [31, 32, 33, 35]. More recently, mean-field limits for extended Hawkes processes have been used to model the neural networks in e.g. [8, 15, 9]. The large
deviations results in Section 3 can be used to estimate the probability of rare events in a neural network. The moderate deviations results in Section 3 can be used to fill in the gap between the second-order fluctuations and the large deviations regime. We can also use the multivariate Hawkes process of dimension $N$ to represent the loss process for $N$ firms in a large portfolio. The results in Section 3 can be used to provide estimates for the tail probabilities for the loss of a large portfolio. We refer to [10, 13, 23] for the works of large portfolio losses in finance. Note that the results we obtained in Section 3 are for the standard multivariate nonlinear Hawkes processes. In order to apply our results to neural networks in neuroscience, large portfolio losses in finance, and many other contexts, one needs to extend our results for the generalized Hawkes processes suitable for the applications in various contexts. Since there are many different ways to generalize the standard multivariate nonlinear Hawkes processes for the purpose of applications, we restrict the study in this paper to the most standard nonlinear Hawkes processes. Nevertheless, the methodology presented in this paper should be applicable for various extensions.

The scalings in (1.10) for stochastic equations with Poisson noise have been widely studied in the literature, see e.g. Budhiraja et al. [6], Budhiraja et al. [7], Budhiraja et al. [5]. The large deviations and moderate deviations for stochastic equations with jumps can be established usually using the variational representation in [6]. However, in our case, the coefficient of the dynamics (1.10) is a indicator function with path-dependency, which is not continuous. As a result, we cannot apply the results from [6, 7] directly. Instead of pursuing a modification of the variational representation approach in [6, 7], we will adopt a more direct approach to establish large and moderate deviations in our paper.

We organize this paper as follows. In Section 2 we introduce the main results of the paper. We will study fluctuations in Section 2.1, large deviations in Section 2.2 and moderate deviations in Section 2.3. The asymptotic results for the mean process for a high-dimensional Hawkes process are presented in Section 3. Finally, all the proofs will be given in Section 4.

2 Main Results

Before we proceed, let us summarize here a list of key assumptions that will be used throughout the paper.

**Assumption 1.** $\phi(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is $\alpha$-Lipschitz for some $0 < \alpha < \infty$. $h(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally integrable and locally bounded.

**Assumption 2.** $h$ is differentiable and $|h'|$ is locally integrable.

**Assumption 3.** $\phi(\cdot)$ is $\alpha$-Lipschitz and $\alpha \|h\|_{L^1[0,T]} = \alpha \int_0^T |h(t)| dt < 1$.

**Assumption 4.** $\phi(\cdot)$ is twice differentiable and $\|\phi''\|_{L^\infty} = \sup_{x \geq 0} |\phi''(x)| < \infty$. 

5
Assumption 5. \( \inf_{x \geq 0} \phi(x) > 0, \ h \text{ is differentiable and } \| h' \|_{L^\infty[0,T]} = \sup_{t \in [0,T]} |h'(t)| < \infty. \)

We collect here a set of notations that will be used throughout the paper.

- \( C[0, T] \) is the space of real-valued continuous functions on \([0, T]\);
- \( D[0, T] \) is the space of real-valued càdâg functions on \([0, T]\) equipped with Skorokhod topology;
- \( AC_0[0, T] \) is the space of functions \( f : [0, T] \to \mathbb{R} \) that are absolutely continuous with \( f(0) = 0 \);
- \( AC_0^+[0, T] \) is the space of non-decreasing functions \( f : [0, T] \to \mathbb{R} \) that are absolutely continuous with \( f(0) = 0 \).

2.1 Fluctuations

In this section, we are interested to study the fluctuations of \( Z^\varepsilon \) around its limit \( Z^0 \). We will obtain a functional central limit theorem for \( Z^\varepsilon \).

As \( \varepsilon \to 0 \), \( Z^\varepsilon_t \) will converge on \( D[0, T] \) to a deterministic function \( Z^0_t \) that satisfies the equation:

\[
Z^0_t = \int_0^t \phi\left( \int_0^s h(s-u) dZ^0_u \right) \, ds. \tag{2.1}
\]

Indeed, this result will follow from the fluctuation result for \( Z^\varepsilon_t \), that is, we will show that \( \frac{Z^\varepsilon_t - Z^0_t}{\sqrt{\varepsilon}} \) converges in distribution on \( D[0, T] \) to a non-trivial stochastic limit, which turns out to be a continuous Gaussian process. Let us notice that the equation (2.1) has a unique locally bounded and non-negative solution under certain assumptions, see Delattre [12]. It is interesting that the mean of the inhomogeneous Poisson process as the mean-field limit for high dimensional Hawkes processes leads to the same limiting equation as in (2.1).

Let us define:

\[
X^\varepsilon_t = \frac{Z^\varepsilon_t - Z^0_t}{\sqrt{\varepsilon}}. \tag{2.2}
\]

Theorem 6. Suppose Assumption 1, Assumption 2 and Assumption 4 hold. \( X^\varepsilon \) converges in distribution on \( D[0, T] \) to a continuous Gaussian process \( X_t \) defined by

\[
X_t = \int_0^t \phi\left( \int_0^s h(s-u) dZ^0_u \right) \left( h(0)X_s + \int_0^s X_u h'(s-u) du \right) \, ds \\
+ \int_0^t \sqrt{\phi\left( \int_0^s h(s-u) dZ^0_u \right)} \, dW_s, \tag{2.3}
\]

where \( W_t \) is a standard Brownian motion.
Remark 7. The Gaussian process defined by (2.3) is also a semimartingale and
\[ h(0)X_s + \int_0^s X_u h'(s-u)du = \int_0^s h(s-u)dX_u. \]

Thus, the Gaussian process \( X_t \) has the following equivalent characterization:
\[ X_t = \int_0^t \phi'(\int_0^s h(s-u)dZ^0_u) \int_0^s h(s-u)dX_u ds + \int_0^t \sqrt{\phi(\int_0^s h(s-u)dZ^0_u)}dW_s. \]  
(2.4)

A key component of the proof of Theorem 6 is the tightness of the sequence \( X^\epsilon \) on \( D[0,T] \) that we will establish in the following lemma.

Lemma 8. Suppose Assumption 1 and Assumption 2 hold, \( X^\epsilon \) is tight on \( D[0,T] \) and the all limits are in \( C[0,T] \).

The proof of the tightness of \( X^\epsilon \), relies on two auxiliary lemmas. The first lemma, i.e. Lemma 9 gives a uniform bound on the first moment of \( Z^T_T \), uniformly in \( \epsilon \), and the second lemma, i.e. Lemma 10 gives us a uniform bound on the second moment of the running maximum of \( X^\epsilon \) process, uniformly in \( \epsilon \).

Lemma 9. Suppose Assumption 1 holds.
\[
\sup_{\epsilon>0} \mathbb{E}[Z^T_T] \leq \phi(0)e^{\epsilon \|h\|_{L^\infty[0,T]}T}. 
\]  
(2.5)

Lemma 10. Suppose Assumption 1 and Assumption 2 hold,
\[
\sup_{\epsilon>0} \mathbb{E}\left[ \sup_{0\leq t\leq T} (X^\epsilon_t)^2 \right] < \infty. 
\]  
(2.6)

The proofs of Theorem 6, Lemma 8, Lemma 9 and Lemma 10 will all be given in Section 4.

Remark 11. Note that [22] studied the large time fluctuations for stationary nonlinear Hawkes processes and more precisely, as a special case for the linear Hawkes process \( \phi(x) = \nu + x \), we have \( \frac{N_n-\nu}{\sqrt{n}} \rightarrow \sigma B(t) \) in distribution on \( D[0,T] \) as \( n \rightarrow \infty \), where \( \mu = \frac{\nu}{1-\|h\|_{L^1}} \) and \( \sigma^2 = \frac{\nu}{(1-\|h\|_{L^1})^2} \), and \( B(t) \) is a standard Brownian motion. Note that for the large time functional central limit theorem, the limiting variance depends on \( \|h\|_{L^1} \) only, while in our Theorem 6, it depends on the entire exciting function \( h(t) \) for \( t \in [0,T] \). Moreover, in our limit, we obtain a Gaussian process that in general is not a Brownian motion.
2.2 Large deviations

We have already seen that $Z^\epsilon_t$ converges to the limit $Z^0_t$ on $D[0,T]$ and have studied the fluctuations around this limit. It is natural to ask about the probability of the rare events that the process $Z^\epsilon_t$ deviates away from its deterministic limit. That is the question of large deviations in probability theory.

We start by giving a formal definition of the large deviation principle. We refer to Dembo and Zeitouni [14] and Varadhan [40] for general background of large deviations and the applications.

A sequence $(P_\epsilon)_{\epsilon \in \mathbb{R}^+}$ of probability measures on a topological space $X$ satisfies the large deviation principle with rate function $I: X \to \mathbb{R}$ and speed $b(\epsilon)$ if $I$ is non-negative, lower semicontinuous and for any Borel set $A$, we have

$$\inf_{x \in A^o} I(x) \leq \liminf_{\epsilon \to 0} \frac{1}{b(\epsilon)} \log P_\epsilon(A) \leq \limsup_{\epsilon \to 0} \frac{1}{b(\epsilon)} \log P_\epsilon(A) \leq -\inf_{x \in \overline{A}} I(x). \quad (2.7)$$

Here, $A^o$ is the interior of $A$ and $\overline{A}$ is its closure.

Now, we are ready to state the main results of large deviations for $Z^\epsilon_t$ on $D[0,T]$.

**Theorem 12.** Suppose Assumption 4, Assumption 5 and Assumption 7 hold. Then, $\mathbb{P}(Z^\epsilon_t \in \cdot)$ satisfies a large deviation principle on $D[0,T]$ equipped with Skorokhod topology with the speed $\epsilon^{-1}$ and the rate function

$$I(\eta) := \int_0^T \ell \left( \eta'(t); \varphi \left( \int_0^t h(t-s)d\eta(s) \right) \right) dt, \quad (2.8)$$

if $\eta \in \mathcal{AC}_{0+}^+[0,T]$ and $+\infty$ otherwise, where

$$\ell(x;y) := x \log \left( \frac{x}{y} \right) - x + y. \quad (2.9)$$

Instead of establishing a full large deviation principle in Theorem 12 directly, our strategy is to first prove a local large deviation principle in Theorem 13 with the main tool being the change of measure technique for simple point processes. We then establish the exponential tightness in order to obtain a full large deviation principle.

We have the following local large deviation principle.

**Theorem 13.** Suppose Assumption 4 and Assumption 5 hold. For any $\eta \in D[0,T]$,

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta(t)| \leq \delta \right) = -I(\eta), \quad (2.10)$$

where $I(\eta)$ is defined in (2.8).
Next, let us establish the exponential tightness of the sequence $Z^\epsilon_t$ on $D[0,T]$. The following Lemma 14 and Lemma 15, together with the local large deviation principle will provide us the full large deviation principle that is desired.

**Lemma 14.** Suppose Assumption 1, Assumption 3 hold. Then,

$$\lim_{K \to \infty} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(Z^\epsilon_T \geq K) = -\infty.$$  (2.11)

**Lemma 15.** Suppose Assumption 1, Assumption 3 hold. For any $\delta > 0$,

$$\lim_{M \to \infty} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\sup_{0 \leq s \leq t \leq T, |t-s| \leq 1/M} |Z^\epsilon_t - Z^\epsilon_s| \geq \delta\right) = -\infty.$$  (2.12)

**Remark 16.** Theorem 13, Lemma 14 and Lemma 15 provide actually the large deviation principle for $Z^\epsilon_t$ with respect to the uniform topology on $D[0,T]$, see e.g. Lemma A.1 in [16], or Theorem 4.14 [17].

**Remark 17.** In [3], they obtained a sample path large deviation principle for the large time scaling for Poisson cluster processes. More precisely, the linear Hawkes process with $\phi(x) = \nu + x$, as a special case of the Poisson cluster process, has the sample path large deviation principle that $\mathbb{P}(\frac{N_t}{n} \in \cdot)$ satisfies a large deviation principle on $D[0,T]$ equipped with the topology of point-wise convergence with the speed $n$ and the rate function $\int_0^T \mathcal{I}(f'(t))dt$ if $f \in AC_0[0,T]$ and $+\infty$ otherwise, where

$$\mathcal{I}(x) := x \log \left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - x + x\|h\|_{L^1} + \nu,$$  (2.13)

for $x \geq 0$ and $+\infty$ otherwise. Note that since the assumption (37) in [3] is not satisfied for the linear Hawkes process, their large deviations results apply to the topology of pointwise convergence, but not the uniform topology. Our results in Theorem 12 differ in two ways. First, our rate function depends on the entire function $h(t)$, $0 \leq t \leq T$, rather than $\|h\|_{L^1}$ as in (2.13). Second, we allow uniform topology for the sample path large deviation principle.

### 2.3 Moderate Deviations

In this section, we are interested in the moderate deviations for $Z^\epsilon_t$. The moderate deviation principle fills in the gap between the central limit theorem and the large deviation principle. For a brief introduction to moderate deviations, we refer to Chap. 3.7. in Dembo and Zeitouni [14].

Our approach to the proof of the moderate deviations is similar to that of the large deviations. That is, we first establish a local moderate deviation principle by using the change of measure technique, i.e. Theorem 19 and then establish the appropriate exponential tightness estimates, i.e. Lemma 20 and Lemma 21.

Our main result is the following:
Theorem 18. Suppose Assumption 1, Assumption 2, Assumption 3, Assumption 4, and Assumption 5 hold. Let $a(\epsilon)$ be a positive sequence such that $a(\epsilon) \frac{\epsilon}{a(\epsilon)^2} \to 0$ as $\epsilon \to 0$. Then, $\mathbb{P}(\frac{Z_t - Z_0}{a(\epsilon)} \in \cdot)$ satisfies a large deviation principle on $D[0, T]$ equipped with Skorokhod topology with speed $a(\epsilon)^2$ and the rate function

$$J(\eta) := \frac{1}{2} \int_0^T \left( \eta'(t) - \phi' \left( \int_0^t h(t-u)dZ_u^0 \right) \int_0^t h(t-u)d\eta_u \right)^2 dt,$$

(2.14)

if $\eta \in \mathcal{AC}_0[0, T]$ and $+\infty$ otherwise.

We first establish a local moderate deviation principle:

Theorem 19. Suppose Assumption 1, Assumption 4, and Assumption 5 hold. For any $\delta \in D[0, T]$,

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \frac{Z_t^\epsilon - Z_0^\epsilon}{a(\epsilon)} - \eta_t \right| \leq \delta \right) = -J(\eta),$$

where $J(\eta)$ is given in (2.14).

Next, we establish the exponential tightness of the sequence $\frac{Z_t^\epsilon - Z_0^\epsilon}{a(\epsilon)}$ on $D[0, T]$ in the following lemmas.

Lemma 20. Suppose Assumption 1, Assumption 2, and Assumption 3 hold.

$$\limsup_{K \to \infty} \limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| Z_t^\epsilon - Z_0^\epsilon \right| \geq Ka(\epsilon) \right) = -\infty.$$  (2.15)

Lemma 21. Suppose Assumption 1, Assumption 2, and Assumption 3 hold. For any $\delta > 0$,

$$\limsup_{M \to \infty} \limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \sup_{0 \leq s \leq t \leq T, \frac{t-s}{M} \leq 1} \left| Z_s^\epsilon - Z_t^\epsilon - Z_s^0 + Z_t^0 \right| \geq \delta a(\epsilon) \right) = -\infty.$$  (2.16)

Remark 22. (i). Theorem 19, Lemma 20, and Lemma 21 provide actually the moderate deviation principle for $Z_t^\epsilon$ with respect to the uniform topology on $D[0, T]$, see e.g. Lemma A.1 in [16], or Theorem 4.14 [17]. (ii). For stochastic dynamics driven by Brownian motion, the limit of its standardization is an Ornstein-Uhlenbeck process driven by the same Brownian motion, and so, the fluctuations and the moderate deviations can be established by estimating deviation inequality of the standardization with the Ornstein-Uhlenbeck process (see, e.g. [18]). That approach cannot be applied to stochastic dynamics with jumps in our paper.
3 Asymptotics for the mean process for high-dimensional Hawkes processes

All the previous results that we derived in Theorem 6, Theorem 12 and Theorem 18 for the univariate Hawkes process can be transferred to the mean process of a multivariate Hawkes process. Consider the \( N \)-dimensional multivariate Hawkes process using the Poisson embeddings representation: \((Z_{t_{1}}^{N,1}, \ldots, Z_{t_{N}}^{N,N})_{t \geq 0}\):

\[
Z_{t}^{N,i} = \int_{0}^{t} \int_{0}^{\infty} 1_{\{z \leq \phi(N^{-1} \sum_{j=1}^{N} \int_{0}^{\infty} h(s-u)dz_{u}^{N,j})\}} \pi^{i}(dsdz),
\]

and its mean process \( \bar{Z}_{t}^{N} = \frac{1}{N} \sum_{i=1}^{N} Z_{t}^{N,i} \), which satisfies

\[
\bar{Z}_{t}^{N} = \int_{0}^{t} \int_{0}^{\infty} 1_{\{z \leq \phi(\int_{0}^{\infty} h(s-u)dz_{u}^{N})\}} \frac{1}{N} \sum_{i=1}^{N} \pi^{i}(dsdz).
\]

Theorem 6, Theorem 12 and Theorem 18 for the univariate Hawkes process can be transferred to the following Theorem 23, Theorem 24 and Theorem 25 respectively for the mean process of a multivariate Hawkes process.

**Theorem 23.** Suppose Assumption 1, Assumption 2 and Assumption 4 hold. Set \( X_{t}^{N} := \sqrt{N}(Z_{t}^{N} - m_{t}) \). Then \( X_{t}^{N} \) converges in distribution on \( D[0,T] \) to a continuous Gaussian process \( X_{t} \) defined in Theorem 6.

**Theorem 24.** Suppose Assumption 1, Assumption 3 and Assumption 5 hold. \( \mathbb{P}(\bar{Z}_{t}^{N} \in \cdot) \) satisfies a large deviation principle on \( D[0,T] \) equipped with Skorokhod topology with the speed \( N \) and the rate function given in Theorem 12.

**Theorem 25.** Suppose Assumption 1, Assumption 2, Assumption 3, Assumption 4 and Assumption 5 hold. Let \( a(N) \) be a positive sequence such that \( a(N) = \frac{1}{N} \rightarrow \infty \) as \( N \rightarrow \infty \). Then, \( \mathbb{P}\left( \sqrt{N}(Z_{t}^{N} - m_{t}) a(N)^{-1} \in \cdot \right) \) satisfies a large deviation principle on \( D[0,T] \) equipped with Skorokhod topology with speed \( a(N)^{2} \) and the rate function given in Theorem 18.

4 Proofs

4.1 Proofs for Section 2.1

Before we prove Theorem 6, let us first give the proofs of Lemma 9, Lemma 10 and Lemma 8.
Firstly, for any \( \theta \in \mathbb{R} \), \( e^{\theta N_T^\alpha} - \int_0^T (e^{\theta - 1})^{\frac{1}{2}} \phi(f_0^v e^{h(t-s)}dN_s^v)dt \) is a positive local martingale, hence a supermartingale, and thus for any \( \theta > 0 \), we have
\[
1 \geq \mathbb{E} \left[ e^{\theta N_T^\alpha} - \int_0^T (e^{\theta - 1})^{\frac{1}{2}} \phi(f_0^v e^{h(t-s)}dN_s^v)dt \right] \quad (4.1)
\]
\[
\geq \mathbb{E} \left[ e^{\theta N_T^\alpha} - (e^{\theta} - 1) \phi(0)T - (e^{\theta} - 1)\alpha \int_0^T \|h(t-s)\|dN_s \right] \\
= \mathbb{E} \left[ e^{\theta N_T^\alpha} - (e^{\theta} - 1) \phi(0)T(\alpha \|h\|_{L^1[0,T]} N_T) \right] \\
\geq \mathbb{E} \left[ e^{\theta N_T^\alpha} - (e^{\theta} - 1) \phi(0)T(\alpha \|h\|_{L^1[0,T]} N_T) \right].
\]

Since we assumed that \( \alpha \|h\|_{L^1[0,T]} < 1 \), for sufficiently small \( \theta > 0 \), we have \( \theta - (e^{\theta} - 1)\alpha \|h\|_{L^1[0,T]} > 0 \). It follows that
\[
\mathbb{E} \left[ e^{(\theta - (e^{\theta} - 1)\alpha \|h\|_{L^1[0,T]}) N_T^\alpha} \right] \leq e^{(e^{\theta} - 1)\phi(0)T \frac{1}{\alpha}}. \quad (4.2)
\]

In particular, for any \( k \geq 1 \),
\[
\mathbb{E}[|Z_T^\alpha|^k] < \infty. \quad (4.3)
\]

**Proof of Lemma 7.** Notice that for any \( 0 \leq t \leq T \),
\[
\mathbb{E}[Z_t^\alpha] = \mathbb{E} \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\alpha \right) ds \quad (4.4)
\]
\[
\leq \phi(0)t + \alpha \mathbb{E} \int_0^T \int_0^s \|h(s-u)\| dZ_u^\alpha ds \leq \phi(0)t + \alpha \|h\|_{L^\infty[0,T]} \int_0^T \mathbb{E}[Z_s^\alpha] ds.
\]

The result follows from the Gronwall’s inequality. \( \square \)

**Proof of Lemma 10.** Notice that
\[
X_t^\epsilon = \frac{1}{\sqrt{\epsilon}} M_t^\epsilon + \frac{1}{\sqrt{\epsilon}} \left[ \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) ds - \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^0 \right) ds \right], \quad (4.5)
\]
where
\[
M_t^\epsilon := \epsilon Z_t^\epsilon - \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) ds \quad (4.6)
\]
is a martingale. For any \( 0 \leq t \leq T \),
\[
|X_t^\epsilon| \leq \frac{1}{\sqrt{\epsilon}} \sup_{0 \leq s \leq T} |M_t^\epsilon| + \frac{\alpha}{\sqrt{\epsilon}} \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) \quad (4.7)
\]
\[
\leq \frac{1}{\sqrt{\epsilon}} \sup_{0 \leq s \leq T} |M_t^\epsilon| + \alpha \left[ \sup_{0 \leq s \leq T} \left| h(0) \right| + \sup_{0 \leq s \leq T} \left| h'(s) \right| \right] \int_0^t \mathbb{E}[|Z_s^\epsilon|^2] ds.
\]

12
By Gronwall's inequality,
\[
\sup_{0 \leq t \leq T} |X_t^\epsilon| \leq \frac{1}{\sqrt{\epsilon}} \sup_{0 \leq t \leq T} |M_t^\epsilon| e^{\alpha [\int_0^T |h(t)| + \int_0^T |h'(t)|] T}.
\]

(4.7)

Finally, by Doob's inequality
\[
\frac{1}{\epsilon} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} |M_t^\epsilon| \right)^2 \right] \leq \frac{4}{\epsilon} \mathbb{E} \left[ (M_T^\epsilon)^2 \right]
\]
\[
= 4 \mathbb{E} \int_0^T \phi \left( \int_0^s h(s - u) dZ_u^\epsilon \right) ds
\]
\[
= 4 \mathbb{E} [Z_T^\epsilon],
\]
where we have proved in Lemma 9 that \( \mathbb{E} [Z_T^\epsilon] \) is uniformly bounded in \( \epsilon \).

\( \square \)

**Proof of Lemma 8**. Let us recall that
\[
X_t^\epsilon = \frac{1}{\sqrt{\epsilon}} M_t^\epsilon + \frac{1}{\sqrt{\epsilon}} \left[ \int_0^t \phi \left( \int_0^s h(s - u) dZ_u^\epsilon \right) ds - \int_0^t \phi \left( \int_0^s h(s - u) dZ_u^0 \right) ds \right],
\]
where \( M_t^\epsilon \) is a martingale and we can show that for any \( \eta > 0 \),
\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{0 \leq s, t \leq T, |s - t| \leq \delta} \left| \frac{1}{\sqrt{\epsilon}} M_t^\epsilon - \frac{1}{\sqrt{\epsilon}} M_s^\epsilon \right| \geq \eta \right) = 0.
\]

(4.10)

To show (4.10), w.l.o.g., assume \( T/\delta \in \mathbb{N} \) and by using Doob’s martingale inequality, Chebychev’s inequality, and Burkholder-Davis-Gundy inequality, we have
\[
\mathbb{P} \left( \sup_{0 \leq s, t \leq T, |s - t| \leq \delta} \left| \frac{1}{\sqrt{\epsilon}} M_t^\epsilon - \frac{1}{\sqrt{\epsilon}} M_s^\epsilon \right| \geq \eta \right)
\]
\[
\leq \sum_{n=1}^{T/\delta} \mathbb{P} \left( \left| \frac{1}{\sqrt{\epsilon}} M_n^\epsilon - \frac{1}{\sqrt{\epsilon}} M_{n-1}^\epsilon \right| \geq \frac{\eta}{2} \right)
\]
\[
\leq \frac{C}{\epsilon^2 \eta^4} \sum_{n=1}^{T/\delta} \mathbb{E} \left[ (M_n^\epsilon - M_{n-1}^\epsilon)^4 \right]
\]
\[
= \frac{C'}{\eta^4} \sum_{n=1}^{T/\delta} \mathbb{E} \left[ \left( \int_{(n-1)\delta}^{n\delta} \phi \left( \int_0^{s'} h(s - u) dZ_u^\epsilon \right) ds' \right)^2 \right]
\]
\[
\leq \frac{C'}{\eta^4} \delta T \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \phi \left( \int_0^t h(t - u) dZ_u^\epsilon \right) \right)^2 \right]
\]
\[
\leq \frac{C'}{\eta^4} \delta T \mathbb{E} \left[ (\phi(0) + \alpha ||h||_{L^\infty[0,T]} Z_T^\epsilon)^2 \right].
\]

(4.11)
Note that for every $0 \leq t \leq T$,

$$E[(Z_t^\varepsilon)^2] \leq 2E[(M_t^\varepsilon)^2] + 2E \left[ \left( \int_0^t \phi \left( \int_0^{s-} h(s-u)dZ_u \right) \, ds \right)^2 \right]$$

$$\leq 2E[Z_t^\varepsilon] + 2T E \int_0^t \phi^2 \left( \int_0^{s-} h(s-u)dZ_u \right) \, ds$$

$$\leq 2E[Z_t^\varepsilon] + 2T E \int_0^t (\phi(0) + \alpha \|h\|_{L^\infty[0,T]} Z_s^\varepsilon)^2 \, ds$$

$$\leq 2E[Z_T^\varepsilon] + 2T \phi(0)^2 + 4T^2 \phi(0) \alpha \|h\|_{L^\infty[0,T]} E[Z_T^\varepsilon] + 2T (\alpha \|h\|_{L^\infty[0,T]})^2 \int_0^t E[(Z_s^\varepsilon)^2] \, ds.$$

Recall that we have proved in Lemma 9 that $E[Z_T^\varepsilon]$ is uniformly bounded in $\varepsilon$. By Gronwall’s inequality, $E[(Z_T^\varepsilon)^2]$ is uniformly bounded in $\varepsilon$. Hence, we conclude that (4.10) follows from (4.11) since it goes to zero as $\delta \to 0$ uniformly in $\varepsilon$.

Moreover, for any $0 \leq t \leq t + \delta \leq T$,

$$\frac{1}{\sqrt{\varepsilon}} \int_t^{t+\delta} \left[ \phi \left( \int_0^s h(s-u)dZ_u \right) - \phi \left( \int_0^{s'} h(s-u)dZ_u \right) \right] \, ds$$

$$\leq \frac{1}{\sqrt{\varepsilon}} \int_t^{t+\delta} \left| \phi \left( \int_0^s h(s-u)dZ_u \right) - \phi \left( \int_0^{s'} h(s-u)dZ_u \right) \right| \, ds$$

$$\leq \delta \alpha \left[ |h(0)| + \int_0^T |h'(t)| \, dt \right] \sup_{0 \leq t \leq T} |X_t^\varepsilon|.$$ (4.12)

It follows from Lemma 10 that the sequence

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ \phi \left( \int_0^s h(s-u)dZ_u \right) - \phi \left( \int_0^{s'} h(s-u)dZ_u \right) \right] \, ds$$

is tight on $C[0,T]$. Hence, for any $\eta > 0$,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{0 \leq s,t \leq T, |s-t| \leq \delta} |X_t^\varepsilon - X_s^\varepsilon| \geq \eta \right) = 0,$$ (4.14)

which implies that the sequence $X_t^\varepsilon$ is tight on $D[0,T]$ and the all limits are in $C[0,T]$ by Theorem 15.5 [2].

We are now finally ready to give the proof of the fluctuations results in Theorem 6.
Proof of Theorem 6. We can write

\[ X_t^\epsilon \sim \int_0^t \phi' \left( \int_0^s h(s-u) dZ_u^0 \right) \left( h(0) X_s^\epsilon + \int_0^s X_u^\epsilon h'(s-u) du \right) ds \]

\[ = X_t^\epsilon - \int_0^t \phi' \left( \int_0^s h(s-u) dZ_u^0 \right) \int_0^s h(s-u) dX_u^\epsilon ds \]

\[ = \frac{M_t^\epsilon}{\sqrt{\epsilon}} + \mathcal{E}_t^{(2)}, \]

and

\[ \left( X_t^\epsilon - \int_0^t \phi' \left( \int_0^s h(s-u) dZ_u^0 \right) \int_0^s X_u^\epsilon h'(s-u) duds \right)^2 - \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) ds \]

\[ = \frac{1}{\epsilon} \left( (M_t^\epsilon)^2 - \langle M^\epsilon \rangle_t \right) + \left( \mathcal{E}_t^{(2)} \right)^2 + 2 \left( \frac{M_t^\epsilon}{\sqrt{\epsilon}} \right) \mathcal{E}_t^{(2)}, \]

where \( M_t^\epsilon \) is defined in (4.6) and

\[ \mathcal{E}_t^{(2)} := \frac{1}{\epsilon} \left[ \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^0 \right) ds - \int_0^t \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) ds \right] \]

\[ - \frac{1}{\epsilon} \int_0^t \phi' \left( \int_0^s h(s-u) dZ_u^0 \right) \left[ \int_0^s h(s-u) dZ_u^\epsilon - \int_0^s h(s-u) dZ_u^0 \right] ds. \quad (4.15) \]

Then, by Doob’s martingale inequality, and Burkholder-Davis-Gundy inequality, there exists a constant \( 0 < C_T < \infty \)

\[ \frac{1}{\epsilon^2} \mathbb{E} \left( \left( \sup_{0 \leq t \leq T} |M_t^\epsilon| \right)^4 \right) \leq C_T \mathbb{E} \int_0^T \left( \phi \left( \int_0^s h(s-u) dZ_u^\epsilon \right) \right)^2 \] \[ \leq C_T T \mathbb{E} \left[ (\phi(0) + \alpha \|h\|_{L^\infty[0,T]} Z_T^\epsilon)^2 \right], \]

where we have proved in Lemma 10 that \( \mathbb{E}[(Z_T^\epsilon)^2] \) is uniformly bounded in \( \epsilon \). Thus, \( \frac{1}{\epsilon^2} \mathbb{E} \left( \sup_{0 \leq t \leq T} |M_t^\epsilon|^4 \right) \) is uniformly bounded in \( \epsilon \), and by (4.17), \( \frac{1}{\epsilon^2} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\epsilon|^4 \right) \) is also uniformly bounded in \( \epsilon \).

By Taylor expansion,

\[ \sup_{0 \leq t \leq T} |\mathcal{E}_t^{(2)}| \leq \frac{1}{\sqrt{\epsilon}} \|\phi''\| \int_0^T \left[ \int_0^s h(s-u) dZ_u^\epsilon - \int_0^s h(s-u) dZ_u^0 \right]^2 ds \]

\[ \leq \sqrt{\epsilon} T \|\phi''\| \alpha^2 \left[ |h(0)| + \int_0^T |h'(t)| dt \right]^2 \sup_{0 \leq t \leq T} (X_t^\epsilon)^2. \quad (4.16) \]
Therefore, we have that
\[
\frac{M_t}{\sqrt{\epsilon}}, \quad \frac{1}{\epsilon} \left( (M_t^\epsilon)^2 - \langle M^\epsilon_t \rangle_t \right), \quad 0 < \epsilon \leq \epsilon_0
\]
are uniformly integrable martingales, and
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\mathcal{E}_t^{(2)}| \right] = O(\epsilon).
\] (4.17)
These yield that as \( \epsilon \to 0 \), in probability,
\[
\sup_{0 \leq t \leq T} |\mathcal{E}_t^{(2)}| \to 0,
\]
and there exists a square integrable martingale \( \tilde{M}_t, t \in [0, T] \) such that
\[
\sup_{0 \leq t \leq T} \left| \frac{M_t}{\sqrt{\epsilon}} - \tilde{M}_t \right| \to 0 \quad \text{and} \quad \sup_{0 \leq t \leq T} \left| \frac{1}{\epsilon} \left( (M_t^\epsilon)^2 - \langle M^\epsilon_t \rangle_t \right) - \left( \tilde{M}_t^2 - \langle \tilde{M}_t \rangle_t \right) \right| \to 0.
\]
In Lemma 8, we showed that the sequence \( X_t^\epsilon \) is tight on \( D[0, T] \). Let \( X \) be a limit point of \( X^\epsilon \), and \( X \) is continuous in \( t \). We conclude that,
\[
M_t := X_t - \int_0^t \phi' \left( \int_0^s h(s-u)dZ_u^0 \right) \int_0^s X_u h'(s-u)du ds,
\]
and
\[
N_t := \left( X_t - \int_0^t \phi' \left( \int_0^s h(s-u)dZ_u^0 \right) \right) \left( h(0)X_s + \int_0^s X_u h'(s-u)du \right) ds
\]
\[
- \int_0^t \phi \left( \int_0^s h(s-u)dZ_u^0 \right) ds,
\]
are martingales. Since \( X_t \) is continuous in time \( t \), by Lévy’s characterization of Brownian motion and martingale representation theorem, see e.g. Chapter IV Theorem 3.6. and Chapter V Proposition 3.8. [34], there exists a standard Brownian motion \( W_t \), such that (2.3) holds.

By Gronwall’s inequality, the stochastic differential equation (2.3) only has a unique solution which implies that as \( \epsilon \to 0 \), the set of limit points of \( \{X^\epsilon\} \) is a singleton. Thus, \( X^\epsilon \) converges in distribution on \( D[0, T] \) to the solution of the equation (2.3).

Finally, let us show that the limit \( X_t \) is a Gaussian process. Set \( X_t^{(b)} := 0 \) and
\[
X_t^{(1)} := \int_0^t \sqrt{\phi \left( \int_0^s h(s-u)dZ_u^0 \right)} dW_s,
\] (4.18)
and for every \( n \geq 1 \),
\[
X_t^{(n+1)} := \int_0^t \phi' \left( \int_0^s h(s - u) dZ^0_u \right) \left( h(0) X_s^{(n)} + \int_0^s X_u^{(n)} h'(s - u) du \right) ds
\]
\[+ \int_0^t \sqrt{\phi \left( \int_0^s h(s - u) dZ^0_u \right)} dW_s, \tag{4.19}
\]

Then \( \{X_t^{(n)}, t \in [0, T]\}_{n \geq 1} \) is a sequence of Gaussian processes. Moreover, we can compute that
\[
X_t^{(n+1)} - X_t^{(n)}
\]
\[= \int_0^t \phi' \left( \int_0^s h(s - u) dZ^0_u \right) \left[ h(0)(X_s^{(n)} - X_s^{(n-1)}) + \int_0^s (X_u^{(n)} - X_u^{(n-1)}) h'(s - u) du \right] ds,
\]
where we used the integration by parts and \( X_0^{(n)} = X_0^{(n-1)} = 0 \). Set \( \Phi^{(n)}(t) := \sup_{0 \leq s \leq t} |X_s^{(n)} - X_s^{(n-1)}| \). Then for every \( t \in [0, T] \),
\[
\Phi^{(n+1)}(t) \leq \alpha \int_0^t \left( |h(0)| + \int_0^s |h'(s - u)| du \right) \Phi^{(n)}(s) ds
\]
\[\leq \alpha (|h(0)| + \|h'\|_{L^1[0,T]}) \int_0^t \Phi^{(n)}(s) ds,
\]
which implies that
\[
\Phi^{(n+1)}(T) \leq \frac{\alpha (|h(0)| + \|h'\|_{L^1[0,T]}) T^n}{n!} \sup_{0 \leq t \leq T} \Phi^{(1)}(t), \tag{4.20}
\]
which yields that
\[
\mathbb{E} \left[ \sum_{n=1}^{\infty} (\Phi^{(n)}(T)) \right] < \infty. \tag{4.21}
\]
Thus, almost surely, \( \sum_{n=1}^{\infty} \Phi^{(n)}(T) < \infty \). Thus, by Proposition 6.1 (Chapter 0) in [34],
\[
\tilde{X}_t = \sum_{n=0}^{\infty} (X_t^{(n+1)} - X_t^{(n)})
\]
is a continuous Gaussian process such that
\[
\sup_{0 \leq t \leq T} |X_t^{(n)} - \tilde{X}_t| \to 0 \quad \text{almost surely.}
\]

Therefore,
\[
\tilde{X}_t = \int_0^t \phi' \left( \int_0^s h(s - u) dZ^0_u \right) \left( h(0) \tilde{X}_s + \int_0^s \tilde{X}_u h'(s - u) du \right) ds
\]
\[+ \int_0^t \sqrt{\phi \left( \int_0^s h(s - u) dZ^0_u \right)} dW_s. \tag{4.22}
\]
By the uniqueness of the solution of the equation \( (2.3) \), we have \( \tilde{X} = X \). Therefore, \( \{X_t, t \in [0, T]\} \) is a Gaussian process. \qed
4.2 Proofs for Section 2.2

Proof of Theorem 12. Theorem 12 follows from the local large deviation principle in Theorem 13 and the super-exponential estimates in Lemma 14 and Lemma 15.

Proof of Theorem 13. Set

\[ M_0[0, T] = \{ \eta \in D[0, T]; \eta(0) = 0, \eta(t) \text{ is non-decreasing in } t \in [0, T] \}. \]

Then \( M_0[0, T] \) is a closed subset in \( D[0, T] \) and \( P(Z^\varepsilon \in M_0[0, T] \text{ for all } \varepsilon \in (0, 1)) = 1 \).

Thus, for any \( \eta \notin M_0[0, T] \),

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon \log P \left( \sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta(t)| \leq \delta \right) = -\infty.
\]

Next, we assume that \( \eta \in M_0[0, T] \).

Let \( \tilde{P} \) be the probability measure under which \( N^\varepsilon \) is a standard Poisson process with intensity \( \frac{1}{\varepsilon} \). Since \( \phi \) is \( \alpha \)-Lipschitz, we have

\[
\frac{1}{\varepsilon} \phi \left( \int_0^t e \phi(t-s) dN^\varepsilon_s \right) \leq \frac{1}{\varepsilon} \phi(0) + \alpha \int_0^t e |h(t-s)| dN^\varepsilon_s \leq \frac{1}{\varepsilon} \phi(0) + \alpha \|h\|_{L^\infty[0, T]} N^\varepsilon_t.
\]

That is, the intensity has at most the linear growth in \( N^\varepsilon_t \). Moreover, under our assumption, we have \( \inf_{x \geq 0} \phi(x) > 0 \). Thus, \( P \) and \( \tilde{P} \) are equivalent, and the Radon-Nikodym is given by, see e.g. [36],

\[
\frac{dP}{d\tilde{P}} \bigg|_{\mathcal{F}_T} = e \int_0^T \log \left( \frac{\frac{1}{\varepsilon} \phi \left( \int_0^t \phi(t-s) dZ^\varepsilon_s \right) - \frac{1}{\varepsilon} }{1} \right) dN^\varepsilon_t - \int_0^T \left[ \frac{1}{\varepsilon} \phi \left( \int_0^t \phi(t-s) dN^\varepsilon_s \right) - \frac{1}{\varepsilon} \right] ds.
\]

By changing of the probability measure \( P \) to \( \tilde{P} \),

\[
P \left( \sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta(t)| \leq \delta \right) \quad \text{(4.25)}
\]

\[
= \tilde{E} \left[ e \int_0^T \log \left( \frac{\frac{1}{\varepsilon} \phi \left( \int_0^t h(t-s) dZ^\varepsilon_s \right) - \frac{1}{\varepsilon} }{1} \right) dZ^\varepsilon_t - \int_0^T \left[ \frac{1}{\varepsilon} \phi \left( \int_0^t h(t-s) dZ^\varepsilon_s \right) - \frac{1}{\varepsilon} \right] ds \right] \quad \text{1}_{\sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta(t)| \leq \delta}
\]

\[
= \tilde{E} \left[ e \int_0^T \log \phi \left( \int_0^t e h(t-s) dZ^\varepsilon_s \right) dZ^\varepsilon_t - \int_0^T \left[ \frac{1}{\varepsilon} \phi \left( \int_0^t h(t-s) dZ^\varepsilon_s \right) - 1 \right] ds \right] \quad \text{1}_{\sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta(t)| \leq \delta}.
\]

18
For any \( \{Z_t^i, 0 \leq t \leq T\} \) with \( \sup_{0 \leq t \leq T} |Z_t^i - \eta(t)| \leq \delta \), we have
\[
\left| \int_0^T \log \left( \phi \left( \int_0^t h(t-s) dZ_s^i \right) \right) dZ_t^i - \int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) d\eta(t) \right| \leq \alpha \inf_{x \geq 0} \phi(x) \left| h \right| L_{\infty} [0, T] \eta(T),
\]
and
\[
\int_0^T |d\nu_\eta(t)| \leq \alpha (|h(0)| + \left| h' \right| L_{\infty} [0, T]) (\eta(T) + T \eta(T)).
\]

It follows from integration by parts that
\[
\left| \int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) Z_t^i - \int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) d\eta(t) \right| \leq \sup_{0 \leq t \leq T} |Z_t^i - \eta(t)| \frac{1}{\inf_{x \geq 0} \phi(x)} \int_0^T |d\nu_\eta(t)| + \sup_{0 \leq t \leq T} |Z_t^i - \eta(t)| \left| \log (\nu_\eta(T)) \right| \eta(T)
\]
\[
\leq \sup_{0 \leq t \leq T} |Z_t^i - \eta(t)| \frac{\alpha}{\inf_{x \geq 0} \phi(x)} \left( |h(0)| + \left| h' \right| L_{\infty} [0, T] (\eta(T) + T \eta(T)) \right)
\]
\[
\quad + \sup_{0 \leq t \leq T} |Z_t^i - \eta(t)| \left[ \left| \log (\phi(0)) \right| + \frac{\alpha}{\inf_{x \geq 0} \phi(x)} \left| h \right| L_{\infty} [0, T] \eta(T) \right].
\]

On the other hand, we can estimate that
\[
\left| \int_0^T \log \left( \phi \left( \int_0^t h(t-s) dZ_s^i \right) \right) dZ_t^i - \int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) dZ_t^i \right| \leq \sup_{0 \leq t \leq T} \left| \log \left( \phi \left( \int_0^t h(t-s) dZ_s^i \right) \right) - \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) \right| \left| Z_t^i \right|
\]
\[
\leq \sup_{0 \leq t \leq T} \left| \log \left( \phi \left( \int_0^t h(t-s) dZ_s^i \right) \right) - \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) \right| \left| Z_t^i \right|
\]
\[
\leq \sup_{0 \leq t \leq T} \left| h(t-s) dZ_s^i - \int_0^t h(t-s) d\eta(s) \right| \left| \eta(T) + \delta \right|
\]
\[
\leq \sup_{0 \leq t \leq T} \left| h(t-s) dZ_s^i \right| \left| \eta(T) + \delta \right|
\]
Finally, we can estimate that
\[
\left| \int_0^T \phi \left( \int_0^t h(t-s) dZ^s_t \right) ds - \int_0^T \phi \left( \int_0^t h(t-s) d\eta(s) \right) ds \right| \leq T \alpha \left[ |h(0)| + \int_0^T |h'(t)| dt \right] \sup_{0 \leq t \leq T} |Z^t_t - \eta(t)|.
\]

Since \( N^t \) is a standard Poisson process with intensity \( \frac{1}{\tau} \) under the probability measure \( \tilde{P} \), it is well known that, see, e.g. [29, 30], \( \tilde{P}(Z^t_t \in \cdot) \) satisfies a large deviation principle on \( D[0, T] \) with the rate function
\[
I_{\text{Pos}}(\eta) = \begin{cases} 
\int_0^T \ell(\eta(t); 1) dt & \text{if } \eta \in \mathcal{AC}^+_0[0, T], \\
+\infty & \text{otherwise}
\end{cases}
\]
(4.29)
where \( \ell(\cdot; 1) \) is defined in (2.9).

Hence, we conclude that
\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon \log \tilde{E} \left[ e^{\int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) d\eta(t) - \int_0^T \phi \left( \int_0^t h(t-s) d\eta(s) \right) ds} \right] \leq e^{\int_0^T \log \left( \phi \left( \int_0^t h(t-s) d\eta(s) \right) \right) d\eta(t) - \int_0^T \phi \left( \int_0^t h(t-s) d\eta(s) \right) ds - I_{\text{Pos}}(\eta)}
\]
(4.30)

which yields the desired result.

**Proof of Lemma 14** By (4.2), and Chebychev’s inequality, for sufficiently small \( \theta > 0 \),
\[
\mathbb{P}(Z^T_T \geq K) \leq \mathbb{E} \left[ e^{(\theta - (e^\theta - 1)\alpha\|h\|_{L^1[0, T]} \mathbb{N}^T_T)} e^{-(\theta - (e^\theta - 1)\alpha\|h\|_{L^1[0, T]}) K} \right] \leq e^{(e^\theta - 1)\phi(0) T \mathbb{N}^T_T} e^{-(\theta - (e^\theta - 1)\alpha\|h\|_{L^1[0, T]}) K},
\]
(4.31)
which yields the desired result.
Proof of Lemma 15. Without loss of generality, let us assume that $MT \in \mathbb{N}$. For any $\delta > 0$,

$$
P \left( \sup_{0 \leq s \leq t \leq T, |t-s| \leq \frac{s}{MT}} |Z^c_t - Z^e_s| \geq \delta \right) \leq P \left( \exists j, 1 \leq j \leq MT : \frac{Z^c_t - Z^e_s}{\mathcal{M}} \geq \frac{\delta}{2} \right) \quad (4.32)
$$

$$
\leq \sum_{j=1}^{MT} P \left( N^c_{t-j} - N^e_{t-j} \geq \frac{\delta}{2\epsilon} \right).
$$

For any $\theta > 0$,

$$
1 = E \left[ e^{\theta(N^c_{j/M} - N^e_{(j-1)/M}) - (e^{\theta-1}) \int_{(j-1)/M}^{j/M} \frac{1}{2} \phi(\int_0^1 \epsilon h(t-s) dN^c_t) dt} \right]
$$

$$
= e^{-(e^{\theta-1})\phi(0) \frac{1}{2} \mathcal{M}} E \left[ e^{\theta(N^c_{j/M} - N^e_{(j-1)/M}) - (e^{\theta-1}) \alpha \int_{(j-1)/M}^{j/M} \int_0^1 \epsilon h(t-s) dN^c_t dt} \right]
$$

$$
\geq e^{-(e^{\theta-1})\phi(0) \frac{1}{2} \mathcal{M}} E \left[ e^{\theta(N^c_{j/M} - N^e_{(j-1)/M}) - (e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^c_{j/M} \mathcal{M}} \right].
$$

Therefore, by Cauchy-Schwarz inequality,

$$
E \left[ e^{\frac{\theta}{2}(N^c_{j/M} - N^e_{(j-1)/M})} \right]
$$

$$
= E \left[ e^{\frac{\theta}{2}(N^c_{j/M} - N^e_{(j-1)/M}) - \frac{1}{2}(e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^c_{j/M} \mathcal{M}} e^{\frac{1}{2}(e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^e_{(j-1)/M} \mathcal{M}} \right]
$$

$$
\leq \left( E \left[ e^{\frac{\theta}{2}(N^c_{j/M} - N^e_{(j-1)/M}) - (e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^c_{j/M} \mathcal{M}} \right] \right) \frac{1}{2} \left( E \left[ e^{(e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^e_{(j-1)/M} \mathcal{M}} \right] \right) \frac{1}{2}
$$

$$
\leq e^{\frac{1}{2}(e^{\theta-1})\phi(0) \frac{1}{2} \mathcal{M}} \left( E \left[ e^{(e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^e_{(j-1)/M} \mathcal{M}} \right] \right) \frac{1}{2},
$$

which is uniform in $1 \leq j \leq TM$. By Chebychev's inequality,

$$
P \left( \sup_{0 \leq s \leq t \leq T, |t-s| \leq \frac{s}{MT}} |Z^c_t - Z^e_s| \geq \delta \right) \geq \frac{\delta}{2\epsilon}
$$

$$
\leq \sum_{j=1}^{MT} E \left[ e^{\frac{\theta}{2}(N^c_{j/M} - N^e_{(j-1)/M})} \right] e^{-\frac{\theta}{2} \mathcal{M}}
$$

$$
\leq MT e^{\frac{1}{2}(e^{\theta-1})\phi(0) \frac{1}{2} \mathcal{M}} \left( E \left[ e^{(e^{\theta-1}) \alpha \|h\|_{L^\infty[0,T]} N^e_{(j-1)/M} \mathcal{M}} \right] \right) \frac{1}{2} e^{-\frac{\theta}{2} \mathcal{M}}.
$$

It follows from (4.32) that for any sufficiently small $\epsilon > 0$,

$$
E[e^{tN^c_T}] \leq e^{C(t)^{1/2}},
$$

(4.36)
where $C(\iota)$ is a positive constant that depends only on $\iota$, $\alpha$, $\|h\|_{L^1[0,T]}$, $\phi(0)$ and $T$.

Let $\gamma$ be a sufficiently small fixed constant, independent of all the other parameters. We define $\theta := \log(1 + \gamma M)$, and thus

$$(e^{\theta} - 1)\alpha\|h\|_{L^\infty[0,T]} \frac{1}{M} = \gamma \alpha\|h\|_{L^\infty[0,T]}$$

is sufficiently small since $\gamma$ is sufficiently small and we can apply (4.36) and the Chebychev’s inequality and get

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P} \left( \sup_{0 \leq s \leq t \leq T, |t-s| \leq \frac{1}{M}} |Z^\epsilon_s - Z^\epsilon_t| \geq \delta \right) \leq \frac{1}{2} (e^{\theta} - 1) \phi(0) \frac{1}{M} + \frac{1}{2} C \left( (e^{\theta} - 1)\alpha\|h\|_{L^\infty[0,T]} \frac{1}{M} \right) - \frac{1}{4} \theta \delta.$$

Since $\theta = \log(1 + \gamma M)$, we get

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P} \left( \sup_{0 \leq s \leq t \leq T, |t-s| \leq \frac{1}{M}} |Z^\epsilon_s - Z^\epsilon_t| \geq \delta \right) \leq \frac{1}{2} \phi(0) \gamma + \frac{1}{2} C \left( \gamma \alpha\|h\|_{L^\infty[0,T]} \right) - \frac{1}{4} \log(1 + \gamma M) \delta,$$

which yields the desired result by letting $M \to \infty$.

### 4.3 Proofs for Section 2.3

**Proof of Theorem 18**. Theorem 18 follows from the local large deviation principle in Theorem 19 and the super-exponential estimates in Lemma 20 and Lemma 21.

**Proof of Theorem 19**. Set

$$\mathcal{V}_0[0,T] = \{ \eta \in D[0,T]; \eta(0) = 0, \eta(t) \text{ has finite variation in } t \in [0,T] \}.$$

Then $\mathcal{V}_0[0,T]$ is a closed subset in $D[0,T]$ and $\mathbb{P}(Z^\epsilon - Z^0 \in \mathcal{V}_0[0,T] \text{ for all } \epsilon \in (0,1]) = 1$. Thus, for any $\eta \notin \mathcal{V}_0[0,T]$,

$$\lim\lim_{\delta \to 0} \epsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \frac{Z^\epsilon_t - Z^0_t}{a(\epsilon)} - \eta(t) \right| \leq \delta \right) = -\infty.$$

Next, we assume that $\eta \in \mathcal{V}_0[0,T]$.

Let $\mathbb{P}$ be the probability measure under which $N^\epsilon$ is an inhomogeneous Poisson process with intensity $\frac{1}{\epsilon} \phi \left( \int_0^t h(t-s) dZ^0_s \right)$. Denote by

$$\eta^\epsilon(t) := Z^0_t + a(\epsilon) \eta(t), \quad \delta(\epsilon) := \delta a(\epsilon).$$
By changing the probability measure \( \mathbb{P} \) to \( \hat{\mathbb{P}} \) (see the discussions about the change of measure and the Radon-Nikodym derivative in the proof of Theorem 13 and 36),

\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \frac{Z_t^\varepsilon - Z_t^0}{a(\varepsilon)} - \eta(t) \right| \leq \delta \right) = \mathbb{E} \left[ 1_{\sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta(t) \right| \leq \delta(\varepsilon) \right] .
\]

Replacing \( \eta \) in the proof of Theorem 13 by \( \eta' \), we have firstly,

\[
\left| \int_0^T \log \left( \phi \left( \int_0^t h(t-s)d\eta(s) \right) \right) dZ_t^\varepsilon - \int_0^T \log \phi \left( \int_0^t h(t-s)d\eta(s) \right) d\eta(t) \right| \leq \sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta'(t) \right| \left[ \log \phi(0) + \frac{\alpha}{\inf_{x \geq 0} \phi(x)} |h(0)| + ||h'||_{L^\infty[0,T]} (\eta'(T) + T\eta'(T)) \right] ,
\]

and secondly,

\[
\left| \int_0^T \log \phi \left( \int_0^t h(t-s)d\eta(s) \right) dZ_t^\varepsilon - \int_0^T \log \phi \left( \int_0^t h(t-s)d\eta(s) \right) d\eta(t) \right| \leq \frac{\alpha}{\inf_{x \geq 0} \phi(x)} \left[ |h(0)| + \int_0^T |h'(t)|dt \right] \sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta'(t) \right| |\eta'(T) + \delta| ,
\]

and thirdly,

\[
\left| \int_0^T \phi \int_0^t h(t-s) dZ_s^\varepsilon \right| ds - \int_0^T \phi \int_0^t h(t-s) d\eta(s) \right| ds \leq T\alpha \left[ |h(0)| + \int_0^T |h'(t)| dt \right] \sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta'(t) \right| ,
\]

and finally,

\[
\left| \int_0^T \log \phi \left( \int_0^t h(t-s)d\eta^0(s) \right) dZ_t^\varepsilon - \int_0^T \log \phi \left( \int_0^t h(t-s)d\eta^0(s) \right) d\eta(t) \right| \leq \sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta'(t) \right| \frac{\alpha}{\inf_{x \geq 0} \phi(x)} \left[ |h(0)| + ||h'||_{L^\infty[0,T]} (Z^0(T) + TZ^0(T)) \right] + \sup_{0 \leq t \leq T} \left| Z_t^\varepsilon - \eta'(t) \right| \left[ |\log \phi(0)| + \frac{\alpha}{\inf_{x \geq 0} \phi(x)} ||h||_{L^\infty[0,T]} Z^0(T) \right] .
\]
Thus, by (4.40), (4.41), (4.42), and (4.43), there exists a constant $C$ such that for any $\sup_{0 \leq t \leq T} |Z^\epsilon_t - \eta^\epsilon(t)| \leq \delta(\epsilon)$,

$$
\left| \frac{1}{\epsilon} \int_0^T \log \frac{\phi \left( \int_0^t h(t-s) dZ^\epsilon_s \right)}{\phi \left( \int_0^t h(t-s) dZ^0_s \right)} dZ^\epsilon_t \\
- \frac{1}{\epsilon} \int_0^T \left( \phi \left( \int_0^t h(t-s) dZ^\epsilon_s \right) - \phi \left( \int_0^t h(t-s) dZ^0_s \right) \right) ds \right|
$$

$$
\leq \frac{\delta \alpha^2(\epsilon) C}{\epsilon}.
$$

Moreover, by a deterministic time change, we have

$$
Z^\epsilon_t - Z^0_t = Y^\epsilon(0^T_t) = Y^\epsilon \left( \int_0^t \phi \left( \int_0^s h(s-u) dZ^0_u \right) ds \right), \quad (4.44)
$$

where the first equality above holds in distribution, where $Y^\epsilon(t) := \epsilon \bar{N}^\epsilon_t - t$, and $\bar{N}^\epsilon_t$ is a standard Poisson process with constant intensity $\frac{1}{\epsilon}$ under the probability measure $\bar{P}$.

It is well known that, see e.g. [30], that $\bar{P}(\cdot) \in \mathcal{D}[0, Z^0_T]$, with the speed $\frac{a^2(\epsilon)}{\epsilon}$ and the rate function

$$
J_{\text{Pos}}(\xi) = \begin{cases} 
\frac{1}{2} \int_0^T |\xi(t)|^2 dt & \text{if } \xi \in \mathcal{AC}_0[0, Z^0_T], \\
+\infty & \text{otherwise},
\end{cases} \quad (4.45)
$$
Therefore,

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\delta^2} \log \hat{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |Z_t - \eta'(t)| \leq \delta(\epsilon) \right)
\]

\[
= \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\delta^2} \log \hat{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |Z_t - Z_t^0 - a(\epsilon) \eta(t)| \leq \delta a(\epsilon) \right)
\]

\[
= \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\delta^2} \log \hat{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |Y'(Z_t^0) - a(\epsilon) \eta(t)| \leq \delta a(\epsilon) \right)
\]

\[
= \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\delta^2} \log \hat{\mathbb{P}} \left( \sup_{0 \leq t \leq Z_T^0} |Y'(t) - a(\epsilon) \xi(t)| \leq \delta a(\epsilon) \right)
\]

\[= J_{Pos}(\xi), \]

where \(\xi(.)\) is defined via \(\xi(Z_t^0) = \eta(t)\), for every \(0 \leq t \leq T\), so that if \(\eta \notin \mathcal{AC}_0[0, T]\), then \(J_{Pos}(\xi) = +\infty\), and if \(\eta \in \mathcal{AC}_0[0, T]\), then \(\eta'(t) = (Z_t^0)'\xi'(Z_t^0)\) and

\[
\int_0^T |\xi'(t)|^2 dt = \int_0^T |\xi'(Z_t^0)|^2 dZ_t^0 = \int_0^T |\eta'(t)|^2 (Z_t^0)^2 dt = \int_0^T \frac{|\eta'(t)|^2}{\varphi \left( \int_0^t h(t-s)dZ_s^0 \right)} dt.
\]

Thus, we have

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\delta^2} \log \hat{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |Z_t - \eta'(t)| \leq \delta(\epsilon) \right)
\]

\[
= \begin{cases}-\frac{1}{2} \int_0^T \frac{|\eta'(t)|^2}{\varphi \left( \int_0^t h(t-s)dZ_s^0 \right)} dt & \text{if } \xi \in \mathcal{AC}_0[0, Z_T^0], \\ -\infty & \text{otherwise.} \end{cases}
\]

Finally, we notice that

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_0^T \log \left( \frac{\phi \left( \int_0^t h(t-s)d\eta'(s) \right)}{\phi \left( \int_0^t h(t-s)dZ_s^0 \right)} \right) d\eta'(t)
\]

\[
= \int_0^T \phi' \left( \int_0^t h(t-s)dZ_s^0 \right) \int_0^t h(t-s) d\eta(s) dt
\]

\[
- \frac{1}{2} \int_0^T \frac{\left| \phi' \left( \int_0^t h(t-s)dZ_s^0 \right) \right|^2}{\phi \left( \int_0^t h(t-s)dZ_s^0 \right)} dt.
\]
Hence, we conclude that

\[
\lim\lim_{\delta \to 0} \epsilon \to 0 \frac{\epsilon}{a^2(\epsilon)} \log \mathbb{E} \left[ \frac{1}{\epsilon} \int_0^T \log \left( \frac{\phi(f_0^{t_0} h(t-s) d\tilde{Z}_s^t)}{\phi(f_0^{t_0} h(t-s) d\tilde{Z}_s^0)} \right) d\tilde{Z}_s^t - \frac{1}{\epsilon} \int_0^T \phi(f_0^{t_0} h(t-s) d\tilde{Z}_s^0) - \phi(f_0^{t_0} h(t-s) d\tilde{Z}_s^t) dt \right]
\]

\[= \lim_{\epsilon \to 0} \frac{1}{a^2(\epsilon)} \int_0^T \log \left( \frac{\phi(f_0^{t_0} h(t-s) d\eta^s)}{\phi(f_0^{t_0} h(t-s) d\tilde{Z}_s^0)} \right) d\eta^s(t)\]

\[- \lim_{\epsilon \to 0} \frac{1}{a^2(\epsilon)} \left( \epsilon \int_0^T \phi \left( \int_0^t h(t-s) d\eta^s \right) dt - \phi \left( \int_0^t h(t-s) d\tilde{Z}_s^0 \right) dt \right)\]

\[+ \lim_{\delta \to 0} \epsilon \to 0 \frac{\epsilon}{a^2(\epsilon)} \log \mathbb{E} \left( \sup_{0 \leq t \leq T} |Z_\epsilon^t - \eta^t(t)| \leq \delta(\epsilon) \right)\]

\[= -J(\eta).\]

Hence, the conclusion of the Theorem 19 holds.

\[\square\]

**Proof of Lemma 20.** Let us recall that $Z_\epsilon^t$ satisfies the dynamics:

\[
Z_\epsilon^t = \int_0^t \int_0^\infty 1_{[0,\phi(f_0^{t_0} h(s-u) d\tilde{Z}_u^t)]}(z) e^{s-1} (dz ds)
\]

\[= M_\epsilon^t + \int_0^t \phi \left( \int_0^s h(s-u) d\tilde{Z}_u^t \right) ds,
\]

where $M_\epsilon^t$ is a martingale. Therefore, for any $0 \leq t \leq T,$

\[
|Z_\epsilon^t - Z_0^t| \leq \sup_{0 \leq t \leq T} |M_\epsilon^t| + \int_0^t \left| \phi \left( \int_0^s h(s-u) d\tilde{Z}_u^t \right) \right| \left. - \phi \left( \int_0^s h(s-u) d\tilde{Z}_u^0 \right) \right| ds
\]

\[\leq \sup_{0 \leq t \leq T} |M_\epsilon^t| + \alpha \left[ |h(0)| + \int_0^T |h'(t)| dt \right] \int_0^t \sup_{0 \leq u \leq s} |Z_\epsilon^u - Z_u^t| ds.
\]

It follows from Gronwall’s inequality that

\[
\sup_{0 \leq t \leq T} |Z_\epsilon^t - Z_0^t| \leq \sup_{0 \leq t \leq T} |M_\epsilon^t| e^{\alpha \left[ |h(0)| + \int_0^T |h'(t)| dt \right]} T.
\]

(4.48)
For any $\theta > 0$, by Doob’s martingale inequality,

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t^\varepsilon - Z_0^\varepsilon| \geq K a(\varepsilon) \right) \leq \mathbb{P} \left( \sup_{0 \leq t \leq T} |M_t^\varepsilon e^\alpha [h(0) + \int_0^T |h'(t)| dt] T | \geq K a(\varepsilon) \right)
$$

$$
\leq \mathbb{P} \left( \sup_{0 \leq t \leq T} M_t^\varepsilon e^\alpha [h(0) + \int_0^T |h'(t)| dt] T \geq K a(\varepsilon) \right) + \mathbb{P} \left( \sup_{0 \leq t \leq T} (-M_t^\varepsilon) e^\alpha [h(0) + \int_0^T |h'(t)| dt] T \geq K a(\varepsilon) \right)
$$

$$
\leq \mathbb{E} \left[ \frac{a(\varepsilon) e^\alpha [h(0) + \int_0^T |h'(t)| dt] T M_T^\varepsilon}{\varepsilon} \right] e^{-\theta K a(\varepsilon)^2 / \varepsilon} + \mathbb{E} \left[ e^{-\frac{a(\varepsilon) e^\alpha [h(0) + \int_0^T |h'(t)| dt] T M_T^\varepsilon}{\varepsilon}} \right] e^{-\theta K a(\varepsilon)^2 / \varepsilon}.
$$

Let us define

$$
R_t^\varepsilon = \frac{a(\varepsilon)}{\varepsilon} \theta e^\alpha [h(0) + \int_0^T |h'(t)| dt] T M_T^\varepsilon.
$$

Then $R_t^\varepsilon$ is a martingale and $R_0^\varepsilon = 0$. Moreover, $|\Delta M^\varepsilon| \leq \varepsilon$ and

$$
|\Delta R_t^\varepsilon| \leq c := a(\varepsilon) \theta e^\alpha [h(0) + \int_0^T |h'(t)| dt] T.
$$

By Lemma 26.19 in Kallenberg [28], if $M$ is a local martingale starting at 0 with $|\Delta M| \leq c$ then $e^{M - b(M)}$ is a supermartingale where $b = g(c) := (e^c - 1 - c)c^{-2}$. Hence,

$$
\mathbb{E}[e^{R_T^\varepsilon}] = \mathbb{E}[e^{R_T^\varepsilon - \frac{1}{2} g(2c)(R^\varepsilon)^2 T} e^{\frac{1}{2} g(2c)(R^\varepsilon)^2 T}]
$$

$$
\leq \mathbb{E} \left[ e^{2R_T^\varepsilon - g(2c)(R^\varepsilon)^2 T} \right]^{1/2} \mathbb{E} \left[ e^{g(2c)(R^\varepsilon)^2 T} \right]^{1/2}
$$

$$
\leq \mathbb{E} \left[ e^{4g(2c)(R^\varepsilon)^2 T} \right]^{1/2}.
$$

Similarly,

$$
\mathbb{E}[e^{-R_T^\varepsilon}] \leq \mathbb{E} \left[ e^{4g(2c)(R^\varepsilon)^2 T} \right]^{1/2}.
$$
As $\epsilon \to 0$, it is easy to see that $c \to 0$ and $g(2c) \to \frac{1}{2}$. Therefore, for sufficiently small $\epsilon$,

$$
E[e^{R_T}] \leq E\left[e^{4(R_T^\epsilon)_T}\right]^{1/2} \leq E\left[\left.\frac{a(\epsilon)^2}{\alpha}e^{2\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T}\left.\right|\frac{\theta}{2}\right|^{\frac{1}{2}}\right]^{1/2} \leq \frac{2\theta^2e^{2\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T}||h||_{L^\infty[0,T]}T\bar{C}-\theta K}.
$$

where $C(\theta)$ for small $\theta > 0$ is defined in the proof of the exponential tightness for the large deviation principle under the assumption that $\alpha||h||_{L^1[0,T]} < 1$ and the fact that $a(\epsilon)^2$ is sufficiently small as $\epsilon \to 0$. It is easy to check that for some $\bar{C} > 0$

$$
\bar{C} := \limsup_{\theta \to 0} \frac{C(\theta)}{\theta} < \infty.
$$

Therefore, we conclude that

$$
\limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log P\left(\sup_{0 \leq t \leq T} |Z_t^\epsilon - Z_t^0| \geq K a(\epsilon)\right) \leq 2\theta^2e^{2\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T+\alpha\phi(0)T}||h||_{L^\infty[0,T]}T\bar{C}-\theta K.
$$

The desired result follows by letting $K \to \infty$. \qed

Proof of Lemma 21 Without loss of generality, let us assume that $MT \in \mathbb{N}$. For any $\delta > 0$,

$$
P\left(\sup_{0 \leq t \leq T, |t-s| \leq \frac{1}{M}} |Z_t^\epsilon - Z_t^0 - Z_s^0 + Z_s^0| \geq \delta a(\epsilon)\right) \leq \sum_{j=1}^{MT} P\left(\left|Z_t^\epsilon - Z_t^{j-\frac{1}{M}} - Z_t^{0} + Z_t^{0}\right| \geq \delta a(\epsilon)\right).
$$
We can estimate that
\[
\left| Z^e_{1/\Delta t} - Z^e_{1-1/\Delta t} - Z^0_{1/\Delta t} + Z^0_{1-1/\Delta t} \right| \leq |M^e_{j/M} - M^e_{(j-1)/M} + \frac{1}{M} \alpha \left[ h(0) + \int_0^T |h'(t)| dt \right] \sup_{0 \leq t \leq T} |Z^e_t - Z^0_t|.
\]

Thus,
\[
P \left( \left| Z^e_{1/\Delta t} - Z^e_{1-1/\Delta t} - Z^0_{1/\Delta t} + Z^0_{1-1/\Delta t} \right| \geq \frac{\delta a(\epsilon)}{2} \right) \leq P \left( \left| M^e_{j/M} - M^e_{(j-1)/M} \right| \geq \frac{\delta a(\epsilon)}{2} \right) + P \left( \frac{1}{M} \alpha \left[ h(0) + \int_0^T |h'(t)| dt \right] \sup_{0 \leq t \leq T} |Z^e_t - Z^0_t| \geq \frac{\delta a(\epsilon)}{2} \right).
\]

We can compute that for any \( \theta > 0 \), for sufficiently small \( \epsilon > 0 \),
\[
P \left( \left| M^e_{j/M} - M^e_{(j-1)/M} \right| \geq \frac{\delta a(\epsilon)}{2} \right) \leq P \left( \sup_{(j-1)/M \leq t \leq j/M} (M^e_t - M^e_{(j-1)/M}) \geq \frac{\delta a(\epsilon)}{2} \right) + P \left( \sup_{(j-1)/M \leq t \leq j/M} (-M^e_t + M^e_{(j-1)/M}) \geq \frac{\delta a(\epsilon)}{2} \right) \leq \left( \mathbb{E} \left[ e^{\frac{\theta a(\epsilon)}{\epsilon} (M^e_{j/M} - M^e_{(j-1)/M})} \right] + \mathbb{E} \left[ e^{\frac{\theta a(\epsilon)}{\epsilon} (M^e_{(j-1)/M} - M^e_{j/M})} \right] \right) e^{-\theta \frac{\delta a(\epsilon)^2}{\epsilon}} \leq 2 \mathbb{E} \left[ e^{4\theta^2 \frac{a(\epsilon)^2}{\epsilon^2} \frac{j}{M} \phi(\int_0^s h(s-u)dZ^e_u) ds} \right]^{1/2} e^{-\theta \frac{\delta a(\epsilon)^2}{\epsilon}},
\]

where the last line uses (4.54). From here, we can further estimate that
\[
P \left( \left| M^e_{j/M} - M^e_{(j-1)/M} \right| \geq \frac{\delta a(\epsilon)}{2} \right) \leq 2 e^{4\theta^2 \frac{a(\epsilon)^2}{\epsilon^2}} \mathbb{E} \left[ e^{2\theta^2 a(\epsilon)^2 \frac{j}{M} \phi(\int_0^s h(s-u)dZ^e_u) ds} \right]^{1/2} e^{-\theta \frac{\delta a(\epsilon)^2}{\epsilon}},
\]

29
which is uniform in $j$. Moreover,
\[
\limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \left| M_{j/M}^\epsilon - M_{(j-1)/M}^\epsilon \right| \geq \frac{\delta a(\epsilon)}{2} \right) \leq 2\theta^2 \frac{1}{M} \phi(0) + 2\theta^2 \tilde{C} \frac{1}{M} \alpha \|h\|_{L^\infty[0,T]} - \frac{\delta}{2}.
\] (4.63)

The choice of $\theta > 0$ is arbitrary. Let us choose $\theta = \sqrt{M}$, then
\[
\limsup_{M \to \infty} \limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \left| M_{j/M}^\epsilon - M_{(j-1)/M}^\epsilon \right| \geq \frac{\delta a(\epsilon)}{2} \right) = -\infty. \tag{4.64}
\]

Finally, by Lemma 20,
\[
\limsup_{M \to \infty} \limsup_{\epsilon \to 0} \frac{\epsilon}{a(\epsilon)^2} \log \mathbb{P} \left( \left| M_{j/M}^\epsilon - M_{(j-1)/M}^\epsilon \right| \geq \frac{\delta a(\epsilon)}{2} \right) = -\infty. \tag{4.65}
\]

Hence, we have proved the desired result. □

**Acknowledgements**

We are very grateful to the Associate Editor and two anonymous referees for their helpful comments and suggestions. Fuqing Gao acknowledges support from NSFC Grant 11571262 and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20130141110076). Lingjiong Zhu is grateful to the support from NSF Grant DMS-1613164.

**References**

[1] Bacry, E., Delattre, S., Hoffmann, M. and Muzy, J. F. (2013). Scaling limits for Hawkes processes and application to financial statistics. *Stochastic Processes and their Applications* 123, 2475-2499.

[2] Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd edition. Wiley-Interscience, New York.

[3] Bordenave, C. and Torrisi, G. L. (2007). Large deviations of Poisson cluster processes. *Stochastic Models*, 23, 593-625.

[4] Brémaud, P. and Massoulié, L. (1996). Stability of nonlinear Hawkes processes. *Ann. Probab.*, 24, 1563-1588.

[5] Budhiraja, A., Chen, J. and Dupuis, P. (2013). Large deviations for stochastic partial differential equations driven by a Poisson random measure. *Stochastic Processes and their Applications*. 123, 523-560.
[6] Budhiraja, A., Dupuis, P. and Maroulas, V. (2011). Variational representations for continuous time processes. *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*. **47**, 725-747.

[7] Budhiraja, A., Dupuis, P. and Ganguly, A. (2016). Moderate deviation principle for stochastic differential equations with jumps. *Annals of Probability*. **44**, 1723-1775.

[8] Chevallier, J. (2017). Mean-field limit of generalized Hawkes processes. *to appear in Stochastic Processes and their Applications*.

[9] Chevallier, J., Duarte, A., Löcherbach, E. and G. Ost. (2017). Mean field limits for nonlinear spatially extended Hawkes processes with exponential memory kernels. *arXiv:1703.05031*.

[10] Dai Pra, P., Runggaldier, W., Sartori, E. and M. Tolotti. (2009). Large portfolio losses: a dynamic contagion model. *Ann. Appl. Probab*. **19**, 347-394.

[11] Delattre, S. and Fournier, N. (2016). Statistical inference versus mean field limit for Hawkes processes. *Electronic Journal of Statistics*. **10**, 1223-1295.

[12] Delattre, S., Fournier, N. and Hoffmann, M. (2016) Hawkes processes on large networks. *Annals of Applied Probability*. **26**, 216-261.

[13] Dembo, A., J.-D. Deuschel, and D. Duffie. (2004). Large portfolio losses. *Finance and Stochastics*. **8**, 3-16.

[14] Dembo, A. and Zeitouni, O. *Large Deviations Techniques and Applications*. 2nd Edition, Springer, New York, 1998.

[15] Ditlevsen, S. and E. Löcherbach. (2017). Multi-class oscillating systems of interacting neurons. *Stochastic Processes and their Applications*. **127**, 1840-1869.

[16] Djellout, H., Guillin, A. and Wu L. M. (1999). Large and moderate deviations for estimators of quadratic variational processes of diffusion. *Statistical Inference for Stochastic Processes*. **2**, 195-225.

[17] Feng, J. and Kurtz, T. G. *Large Deviations for Stochastic Processes*. American Mathematical Society, 2006.

[18] Gao, F. Q. and Wang, S. C. (2016). Asymptotic behaviors for functionals of random dynamical systems. *Stochastic Analysis and Applications*. **34**, 258-277.

[19] Gao, F. Q. and Zhu, L. (2017). Precise deviations for Hawkes processes. *arXiv:1702.02962*.

[20] Gao, X. and Zhu, L. (2015). Limit theorems for linear Markovian Hawkes processes with large initial intensity. *arXiv:1512.02155*.

[21] Gao, X. and Zhu, L. (2016). Large deviations and applications for Markovian Hawkes processes with a large initial intensity. *to appear in Bernoulli*.
[22] Gao, X. and Zhu, L. (2016). A functional central limit theorem for stationary Hawkes processes and its application to infinite-server queues. arXiv:1607.06624.

[23] Giesecke, K., Spiliopoulos, K., Sowers, R. B. and J. A. Sirignano. (2015). Large portfolio asymptotics for loss from default. Mathematical Finance. 25, 77-114.

[24] Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. Biometrika 58, 83-90.

[25] Hawkes, A. G. and Oakes, D. (1974). A cluster process representation of a self-exciting process. J. Appl. Probab. 11, 493-503.

[26] Jaisson, T. and Rosenbaum, M. (2015). Limit theorems for nearly unstable Hawkes processes. Annals of Applied Probability. 25, 600-631.

[27] Jaisson, T. and Rosenbaum, M. (2016). Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. Annals of Applied Probability. 26, 2860-2882.

[28] Kallenberg, O. Foundations of Modern Probability. 2nd Edition. Springer, 2002.

[29] Lynch, J. and Sethuraman, J. (1987). Large deviations for processes with independent increments. Ann. Probab. 15, 610-627.

[30] Mogulskii, A. A. (1993). Large deviations for processes with independent increments. Ann. Probab. 21, 202-215.

[31] Pernice, V., Staude B., Carndanobile, S. and S. Rotter. (2012). How structure determines correlations in neuronal networks. PLoS Computational Biology. 85:031916.

[32] Pernice, V., Staude B., Carndanobile, S. and S. Rotter. (2011). Recurrent interactions in spiking networks with arbitrary topology. Physical Review E. 7:e1002059.

[33] Reynaud-Bouret, P., Rivoirard, V. and C. Tuleau-Malot. (2013). Inference of functional connectivity in Neurosciences via Hawkes processes. 1st IEEE Global Conference on Signal and Information Processing.

[34] Revuz, D. and Yor, M. Continuous Martingales and Brownian Motion. Springer, 3rd Edition, 1998.

[35] Reynaud-Bouret, P. and S. Schbath. (2010). Adaptive estimation for Hawkes processes; application to genome analysis. Ann. Statist. 38, 2781-2822.

[36] Sokol, A. and Hansen, N. R. (2015). Exponential martingales and changes of measure for counting processes. Stochastic Analysis and Applications. 33, 823-843.

[37] Takeuchi, A. (2017). Malliavin calculus for marked Hawkes processes. Preprint.

[38] Torrisi, G. L. (2016). Gaussian approximation of nonlinear Hawkes processes. Annals of Applied Probability. 26, 2106-2140.

[39] Torrisi, G. L. Poisson approximation of point processes with stochastic intensity, and application to nonlinear Hawkes processes. to appear in AIHP.
Varadhan, S. R. S. *Large Deviations and Applications*, SIAM, Philadelphia, 1984.

Zhu, L. (2013). *Nonlinear Hawkes Processes*. PhD thesis, New York University.

Zhu, L. (2013). Central limit theorem for nonlinear Hawkes processes. *Journal of Applied Probability*, 50, 760-771.

Zhu, L. (2013). Moderate deviations for Hawkes processes. *Statistics & Probability Letters*, 83, 885-890.

Zhu, L. (2014). Process-level large deviations for nonlinear Hawkes point processes. *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*, 50, 845-871.

Zhu, L. (2015). Large deviations for Markovian nonlinear Hawkes Processes. *Annals of Applied Probability*, 25, 548-581.