A TOPOLOGICAL DESCRIPTION OF THE SPACE OF PRIME IDEALS OF A MONOID

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ABSTRACT. We describe which topological spaces can arise as the prime spectrum of a commutative monoid, in the spirit of Hochster’s and Brenner’s theses.

1. Introduction

1.1. In this note, all monoids are commutative and the unit element of a monoid $M$ is written as 1. Associated to a commutative monoid is a space called the prime spectrum. We will refer to it as the Kato spectrum in order to avoid confusion with the prime spectrum of a commutative ring.

Definition. Let $M$ be a monoid. A subset $I \subset M$ is called an ideal if for all $x \in I$ and $m \in M$, we have $xm \in I$. An ideal $p$ is prime if $M \setminus p$ is a submonoid of $M$ (in particular, $1 \in M \setminus p$.)

We define $\text{spec}(M)$ to be the set of all prime ideals of $M$. For $f \in M$, we define

$$D(f) = \{ p \in \text{spec}(M) : f \not\in p \}.$$  

The Kato spectrum of $M$ is the set $\text{spec}(M)$ with the topology with a base given by the $D(f)$.

The Kato spectrum can also be equipped with a sheaf of monoids, but in this paper we are concerned only with the underlying space.

1.2. The notion of prime ideal in a commutative semigroup goes “back to antiquity” according to [Gri01]. The Zariski topology on the set of prime ideals goes back at least to Kist [Kis63]. The Kato spectrum was introduced by Kato [Kat94] in the study of toric singularities. It was later used by Deitmar [Dei05] to construct a theory of “schemes over the field with one element”.

If $R$ is a commutative ring and $M$ denotes the underlying multiplicative monoid $(R, \cdot)$ of $R$ then $\text{spec}(M)$ is the set of unions of prime ideals of $R$. This space has appeared in some constructions of spaces associated functorially to an arbitrary ring. See [Ary10] and references therein.

1.3. The aim of the present paper is to prove an analogue of the following theorem of Hochster. In this paper, we say a topological space $X$ is compact if every open cover of $X$ has a finite subcover (in algebraic geometry, this is often called quasi-compactness.)

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Theorem. [[Hoc69]] A topological space $X$ is homeomorphic to $\text{Spec}(R)$ for some commutative ring $R$ if and only if the following three properties hold.

- $X$ is $T_0$.
- The set of open compact subsets of $X$ is a base of $X$ which contains $X$ and is closed under finite intersections.
- Every irreducible closed subset of $X$ is the closure of a unique point.

Theorem 1.3 is useful in constructing examples of affine schemes whose underlying spaces behave pathologically. It also began the study of spectral spaces.

1.4. Various authors have generalized Hochster’s Theorem to other kinds of spectrum. For example, Hochster [[Hoc71]] proved an analogue of the theorem with $\text{Spec}$ replaced by the set of minimal prime ideals of a ring and Echi [[Ech00]] proved an analogue for the Goldman spectrum of a ring. Also, Kist [[Kis63]] studied topological properties of the space of minimal prime ideals in a commutative semigroup. Brenner [[Bre94]] proved the following analogue of Hochster’s Theorem for monoids.

Definition. Let $X$ be a topological space. A subset $A \subset X$ is called a blob if there exists $a \in X$ such that $A$ is the intersection of all open subsets of $X$ which contain $a$.

Theorem. [[Bre94], Satz 2.3.2] A topological space $X$ is homeomorphic to $\text{spec}(M)$ for some commutative monoid $M$ if and only if the following properties hold

- $X$ is $T_0$.
- The set of open blobs of $X$ is a base of $X$ which contains $X$ and is closed under finite intersections.
- Every intersection of irreducible closed subsets of $X$ is the closure of a unique point.

and $X$ also satisfies $(\ast)$ where

$(\ast)$ If $\{U_\lambda : \lambda \in \Lambda\}$ is a collection of open blobs in $X$ and $U$ is an open set

\[
\bigcap_\lambda U_\lambda \subset U, \text{ then there exist } \lambda_1, \ldots, \lambda_n \in \Lambda \text{ with } \bigcap_{i=1}^{n} U_{\lambda_i} \subset U.
\]

1.5. In this note, we will show that the first three properties in Theorem 1.4 may be expressed by saying that $X$ possesses a certain algebraic structure, essentially coming from the fact that the set of prime ideals of a monoid is closed under union. This yields a nice description of those spaces which can be the prime spectrum of a finitely-generated monoid. See Corollary 1.3.

1.6. Schemes over $F_1$. Vezzani [[Vez10]] has recently shown that Deitmar’s definition of a scheme over $F_1$ is equivalent to another definition due to Toën and Vaiqué [[TV09]], in which schemes are constructed by glueing together objects of the opposite of the category of monoids via a general construction which mimics (and generalizes) the usual definition of a scheme. See [[JLP09], Section 1.2] for a fuller description of this.
Theorem 1.4 may therefore be useful in understanding the nature of the spaces which play the role of affine schemes over \( F_1 \) in these theories.

2. Exponentiation

2.1. In this section, we characterize those spaces which satisfy the first three conditions of Theorem 1.4 in order to understand how the condition \((\ast)\) fits in.

**Definition.** A poset \((P, \leq)\) is called a join semilattice if every pair of elements \(x, y \in P\) has a least upper bound \(\text{sup}\{x, y\} \in P\). A join semilattice \(P\) is complete if every \(A \subset P\) has a least upper bound \(\text{sup}(A) \in P\).

By definition, a map \(f : (P, \leq_P) \to (Q, \leq_Q)\) of complete join semilattices is an order-preserving function which also satisfies \(f(\text{sup}(A)) = \text{sup}(f(A))\) for all \(A \subset P\).

**Definition.** Let \(X\) be a topological space. We call a base \(B\) of open sets of \(X\) monoidal if \(X \in B\) and \(B\) is closed under finite intersections.

2.2. We define a category \(M\) as follows. Objects of \(M\) are pairs \((X, B)\) where \(X\) is a \(T_0\) space and \(B\) is a monoidal base of \(X\). A morphism \((X, B) \to (Y, C)\) is a function \(f : X \to Y\) such that \(f^{-1}(U) \in B\) for all \(U \in C\).

**Definition.** We call an object \((X, B)\) of \(M\) an \(M\)-complete join semilattice if there is a partial order \(\leq\) on \(X\) such that \((X, \leq)\) is a complete join semilattice, and such that for all \(A \subset X\) and all \(U \in B\), we have

\[
A \subset U \iff \text{sup}(A) \in U.
\]

A morphism of \(M\)-complete join semilattices is a morphism in \(M\) which is also a map of complete join-semilattices (ie. preserves suprema.)

2.3. Recall that the specialization order \(\leq\) on a \(T_0\) space \(X\) is defined by \(x \leq y\) if and only if for all open sets \(U, y \in U \implies x \in U\). Alternatively, \(x \leq y\) if and only if \(y \in \overline{\{x\}}\). One can check that if the partial order \(\leq\) makes \((X, B)\) into an \(M\)-complete join semilattice, then in fact \(\leq\) must be the specialization order.

**Example.** Let \(M\) be a commutative monoid. Let \(X = \text{spec}(M)\) and let \(B = \{D(f) : f \in M\}\), a monoidal base of \(X\). Define a partial order on \(X\) by \(p \leq q\) if and only if \(p \subset q\). Then \((X, B)\) is an \(M\)-complete join semilattice, because if \(A \subset X\), we may take \(\text{sup}(A) = \bigcup_{p \in A} p\), which is a prime ideal of \(M\).

We aim to show that each object \((X, B)\) of \(M\) can be completed to an \(M\)-complete join semilattice.

2.4. The exponential. We define a functor \(E : M \to M\) as follows. Let \((X, B)\) be an object of \(M\). Let \(\mathcal{P}(X)\) be the power set of \(X\). Let \(B_0 = \{\mathcal{P}(U) : U \in B\}\). Then \(B_0\) is a base for a topology on \(\mathcal{P}(X)\). The resulting topological space \(\mathcal{P}(X)\) may not be \(T_0\), but we define an equivalence relation on \(\mathcal{P}(X)\) by \(A \sim B\) if and only if \(A\) and \(B\) belong to the same open sets of \(\mathcal{P}(X)\).
**Definition.** The exponential $E(X,B)$ is the quotient space $\mathcal{P}(X)/\sim$ equipped with the monoidal base $\bar{B} = \{\bar{U} : U \in B\}$ where for $U \in B$, we define

$$\bar{U} = \{[A] \in \mathcal{P}(X)/\sim \text{ such that } A \subseteq U\}.$$ 

Here, we write $[A]$ for the equivalence class in $\mathcal{P}(X)/\sim$ of $A \in \mathcal{P}(X)$.

**Example.** Let $X = \{x\}$ be a one-point space. Let $B = \{\emptyset, X\}$, a monoidal base of $X$. Then $\mathcal{P}(X) = \{\emptyset, X\}$ with base $\{\{\emptyset\}, \mathcal{P}(X)\}$. This space is $T_0$, and so $E(X,B) = \mathcal{P}(X)$ is the two-point Sierpinski space. If we had taken the base $B' = \{X\}$ instead, we would get $E(X,B') \cong (X,B')$. Thus, $E(X,B)$ depends on the choice of the base $B$.

2.5. The following example was the original motivation for defining $E$.

**Example.** Let $R$ be a commutative ring and let $X = \text{Spec}(R)$. For $f \in R$, let $D(f) = \{p \in \text{Spec}(R) : f \notin p\}$. Let $B$ be the base of $X$ consisting of all the $D(f)$. Let $M = (R, \cdot)$ be the underlying multiplicative monoid of $R$. Then $E(X,B) \cong \text{spec}(M)$.

2.6. It is easy to see that $E : \mathcal{M} \to \mathcal{M}$ is a functor and that if $(X,B)$ is an object of $\mathcal{M}$ then there is a map $i = i_{(X,B)} : (X,B) \to E(X,B)$ defined by $i(x) = \{\{x\}\}$, the equivalence class of the singleton $\{x\}$. The map $i_{(X,B)}$ is injective and its image is dense in $E(X,B) \setminus \{\emptyset\}$. The $i_{(X,B)}$ define a natural transformation $\text{id} \to E$ of functors $\mathcal{M} \to \mathcal{M}$.

2.7. If $(X,B)$ is an object of $\mathcal{M}$ then $E(X,B)$ is an $\mathcal{M}$–complete join semilattice under the specialization order. Indeed, the reader can check that if $A_\lambda$, $\lambda \in \Lambda$ are subsets of $X$, then the supremum of $\{[A_\lambda] : \lambda \in \Lambda\}$ is $\bigcup_{\lambda \in \Lambda} A_\lambda$.

2.8. We now show that $E(X,B)$ is the smallest $\mathcal{M}$–complete join semilattice which contains $(X,B)$.

**Proposition.** Let $(X,B)$ be an object of $\mathcal{M}$.

The natural map $i : (X,B) \to E(X,B)$ is the universal map from $(X,B)$ to an $\mathcal{M}$–complete join-semilattice in the following sense:

If $\theta : (X,B) \to (Y,C)$ is a map in $\mathcal{M}$ and $(Y,C)$ is an $\mathcal{M}$–complete join-semilattice, then there exists a unique map of $\mathcal{M}$–complete join semilattices $\hat{\theta} : E(X,B) \to (Y,C)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
(X,B) & \xrightarrow{i} & (Y,C) \\
\downarrow{\theta} & & \circlearrowright{\hat{\theta}} \\
E(X,B) & \xrightarrow{i} & \\
\end{array}
$$

*Proof.* The map $\hat{\theta}$ is defined by $\hat{\theta}([A]) = \sup(\theta(A))$ for all $A \subseteq X$. It is routine to check that $\hat{\theta}$ is the unique morphism of $\mathcal{M}$–complete join semilattices which makes the diagram commute. 

\[\Box\]
2.9. We have the following corollary of Proposition 2.8.

**Corollary.** $E^2 = E$ as functors $\mathcal{M} \to \mathcal{M}$.

**Proof.** This follows directly from Proposition 2.8 if we note that if $(X, B)$ and $(Y, C)$ are $\mathcal{M}$–complete join semilattices and $f : (X, B) \to (Y, C)$ is a morphism in $\mathcal{M}$, then $f$ necessarily preserves suprema.

2.10. Now we give a topological characterization of which spaces can arise as $E(X, B)$ for some $(X, B)$. First, we need the following Lemma.

**Lemma.** Let $(X, B)$ be an $\mathcal{M}$–complete join semilattice. Let $U \subset X$ be open. Then $U$ is a blob (see Definition 7.4) if and only if $U \in B$.

**Proof.** Suppose $U \subset X$ is an open blob. Then there exist $U_\lambda \in B$ with $U = \bigcup U_\lambda$ because $B$ is a base for the topology on $X$. Since $U$ is a blob, there exists $a \in X$ such that $U$ is the intersection of all the open sets which contain $a$. Therefore, $a \in U_\lambda$ for some $\lambda$, and so $U \subset U_\lambda$. So $U = U_\lambda \in B$. Conversely, suppose $U \in B$. Then $\text{sup}(U) \in U$ by Definition 2.2. Since the order $\leq$ on $X$ coincides with the specialization order (see Section 2.9), for all $x \in X$ we have $x \in U$ if and only if $x \leq \text{sup}(U)$. Therefore, $U$ is the intersection of all the open sets which contain $\text{sup}(U)$, so $U$ is a blob.

**Theorem.** Let $X$ be a $T_0$ space. The following are equivalent.

1. The open blobs form a monoidal base of $X$, and for all $A \subset X$, $\bigcap_{x \in A} \overline{\{x\}}$ is the closure of a point.
2. If $B$ denotes the set of open blobs of $X$, then $(X, B)$ is an $\mathcal{M}$–complete join semilattice.
3. There exists a monoidal base $B$ of $X$ such that $(X, B)$ is an $\mathcal{M}$–complete join semilattice.
4. There exists a monoidal base $B$ of $X$ such that $(X, B) \cong E(X, B)$ in $\mathcal{M}$.
5. There exists a monoidal base $B$ of $X$ and an object $(Y, C)$ of $\mathcal{M}$ such that $(X, B) \cong E(Y, C)$ in $\mathcal{M}$.

**Proof.** The equivalence of (3), (4) and (5) follows directly from Proposition 2.8 and Corollary 2.10. Also, (2) trivially implies (3). It remains to show that (1) and (3) are equivalent.

(1) $\implies$ (2): Let $B$ be the set of all open blobs of $X$. We claim that $(X, B)$ is an $\mathcal{M}$–complete join semilattice. Define $\leq$ to be the specialization order on $X$, so that $x \leq y$ if and only if $y \in \overline{\{x\}}$. For $A \subset X$, there is a point $y$ such that $\bigcap_{a \in A} \overline{\{a\}} = \overline{\{y\}}$, and we see that $y$ is the supremum of $A$ in the ordering $\leq$. So $(X, \leq)$ is a complete join-semilattice. Now let $U$ be an open blob and $A \subset X$. We must show that $A \subset U$ if and only if $\text{sup}(A) \in U$. There is some $a \in X$ such that $U$ is the intersection of all open sets containing $a$. In other words, $U = \{x \in X : x \leq a\}$. So if $A \subset U$ then $\text{sup}(A) \in U$, while if $\text{sup}(A) \in U$ then $A \subset U$ because $x \leq \text{sup}(A) \leq a$ for all $x \in A$. Therefore, $(X, B)$ is an $\mathcal{M}$–complete join semilattice.

(3) $\implies$ (1): Suppose $(X, B)$ is an $\mathcal{M}$–complete join semilattice. By Lemma 2.10, $B$ is precisely the set of open blobs, and so these form a monoidal base. Now let $A \subset X$. It is easy to check that $\bigcap_{a \in A} \overline{\{a\}} = \overline{\{\text{sup}(A)\}}$, and so we are done. □
3. Monoids

3.1. In this section, we give a proof of Theorem 1.4 following [Bre94].

**Proposition.** Let $X$ be a topological space. The following are equivalent.

1. $X$ is homeomorphic to $\text{spec}(M)$ for some monoid $M$.
2. $X$ is $T_0$ and there is a monoidal base $B$ of $X$ such that $(X, B)$ is an $M$-complete join semilattice and the map

$$
\varphi : X \to \text{spec}(B, \cap) \\
x \mapsto \{U \in B : x \notin U\}
$$

is a bijection.

**Proof.** Suppose $X = \text{spec}(M)$. Then take $B = \{D(f) : f \in M\}$. Then $B$ is a monoidal base of $X$ and $(X, B)$ is an $M$-complete join semilattice. The map $\varphi : X \to \text{spec}(B, \cap)$ is given by $p \mapsto \{D(g) : p \notin D(g)\} = \{D(g) : g \in p\}$. This has an inverse given by sending a prime ideal $Q$ of the monoid $(B, \cap)$ to the prime ideal \{f \in M : D(f) \in Q\} of $M$.

Conversely, suppose $X$ has a monoidal base $B$ and the map $x \mapsto \{U \in B : x \notin U\}$ is bijective. We need only show that this map is in fact a homeomorphism. This follows from [Kis63, Lemma 4.3]. Alternatively, we may define a function $\theta : \text{spec}(B, \cap) \to X$ by $\theta(p) = \sup(\bigcap_{U \in B \setminus p} U)$. Then for all $V \in B$ and $x \in X$, $\theta(\varphi(x)) = \theta(\{U \in B : x \notin U\}) \in V$ if and only if $x \in V$. Therefore, $\theta \varphi(x) = x$ since $X$ is $T_0$. So $\theta$ is a one-sided inverse to $\varphi$, and therefore $\theta$ is inverse to $\varphi$ since $\varphi$ is a bijection. To show that $\theta$ is continuous, observe that $\varphi$ is an open map because for $V \in B$, $\varphi(V) = \{p \in \text{spec}(B, \cap) : V \notin p\}$. \qed

3.2. **Proposition.** Let $X$ be a topological space. The following are equivalent.

1. $X$ is homeomorphic to $\text{spec}(M)$ for some monoid $M$.
2. $X$ is $T_0$ and there is a monoidal base $B$ of $X$ such that $(X, B)$ is an $M$-complete join semilattice and the condition (⋆) of Theorem 1.4 holds for $X$.

**Proof.** Suppose $X$ is homeomorphic to $\text{spec}(M)$. Then $X$ is $T_0$. Let $B = \{D(f) : f \in M\}$, a monoidal base of $X$. By Example 2.3, $(X, B)$ is an $M$-complete join semilattice. By Lemma 2.10, the open blobs of $X$ are precisely the elements of $B$. Suppose $U_{\lambda}$, $\lambda \in \Lambda$, are open blobs. Then

$$
p = \{U \in B : U \text{ does not contain any finite intersection of the } U_{\lambda}\}
$$

is a prime ideal of the monoid $(B, \cap)$. By the proof of Proposition 3.1, there exists $x \in M$ such that $p = \{U \in B : x \notin U\}$. In other words, for any $U \in B$, $x \in U$ if and only if $U$ contains some $\bigcap_{i=1}^{n} U_{\lambda_i}$. In particular, $x \in U_{\lambda}$ for all $\lambda$ and so $x \in \bigcap_{\lambda} U_{\lambda}$. Suppose $U$ is an open set and $U \supseteq \bigcap_{\lambda} U_{\lambda}$. Then $x \in U$ and
so \(x\) is contained in some open blob \(U' \subset U\), since the open blobs are a base. So \(U' \subset U\) contains some finite intersection of the \(U_s\). Thus, \((*)\) holds.

Conversely, if \(X\) satisfies (2), then by Proposition 3.1 we just need to show that the function \(\varphi : X \to \text{spec}(B, \cap)\) defined by \(\varphi(x) = \{U : x \notin U\}\) is a bijection. We define \(\theta : \text{spec}(B, \cap) \to X\) by \(\theta(p) = \sup(\bigcap_{W \in B \setminus \{p\}} W)\). We have seen in the proof of Proposition 3.1 that \(\theta\varphi\) is the identity, so we need to show that if \(p\) is a prime ideal of \((B, \cap)\), then \(\varphi\theta(p) = p\). We have \(\varphi\theta(p) = \{U \in B : U \not\supseteq \bigcap_{W \in B \setminus \{p\}} W\}\). Thus, for \(V \in B\), we see that \(V \notin \varphi\theta(p)\) if and only if \(V \supseteq \bigcap_{W \in B \setminus \{p\}} W\). By \((*)\) combined with Lemma 2.10 this holds if and only if there are \(W_1, \ldots, W_n \notin p\) with \(V \supseteq \bigcap_{i=1}^n W_i\). Since \(p\) is a prime ideal, this holds if and only if \(V \notin p\). Thus, \(\varphi\theta(p) = p\) as required.

3.3. We can also prove directly from the definition of \(\text{spec}(M)\) that \(\text{spec}(M)\) satisfies the condition \((*)\) of Theorem 1.4. This can be done by first reducing to the case of an inclusion of basic open sets \(\bigcap_{\lambda \in \Lambda} D(f_\lambda) \subset D(g)\) and then considering the prime ideal \(p = M \setminus q\) where

\[
q = \{h \in M : \exists t \in M, \lambda_1, \ldots, \lambda_n \in \Lambda, k_1, \ldots, k_n \geq 0 \text{ with } th = f_{\lambda_1}^{k_1} \cdots f_{\lambda_n}^{k_n}\}.
\]

The details are left as an exercise.

3.4. We are now ready to prove Theorem 1.4. We use the following lemma.

**Lemma.** Let \(M\) be a monoid. Then every irreducible closed subset of \(X = \text{spec}(M)\) is the closure of a point.

**Proof.** This is like the analogous fact from algebraic geometry. By [Vez10 Section 2], every closed subset of \(X\) has the form \(V(I) = \{p : p \supseteq I\}\) where \(I\) is an intersection of prime ideals of \(M\). If \(V(I)\) is irreducible and \(x, y \notin I\), then \(V(I) \cap D(x) \neq \emptyset\) and \(V(I) \cap D(y) \neq \emptyset\), and so \(V(I) \cap D(x) \cap D(y) = V(I) \cap D(xy) \neq \emptyset\) by irreducibility. Therefore, \(xy \notin I\) and so \(I\) is prime. Similarly, if \(I\) is prime then \(V(I)\) is irreducible.

Thus, the irreducible closed sets of \(X\) are precisely the sets \(V(p)\) for \(p \in X\). But \(V(p) = \overline{\{p\}}\), as required. \(\square\)

3.5. **Proof of Theorem 1.4** Let \(X = \text{spec}(M)\) for a monoid \(M\). By Proposition 3.2 there is a monoidal base of \(X\) such that \((X, B)\) is an \(M\)-complete join semilattice. By Theorem 2.10 the open blobs of \(X\) form a monoidal base of \(X\). By Lemma 3.3 every closed irreducible subset of \(X\) is the closure of a point. By Theorem 2.10 again, every intersection of closed irreducible subsets of \(X\) is the closure of a point. This point is unique because \(X\) is a \(T_0\) space. By Theorem 3.2 the condition \((*)\) of Theorem 1.4 holds. Thus, all the conditions of Theorem 1.4 are satisfied by \(X\).

Conversely, suppose \(X\) satisfies the conditions of Theorem 1.4. Let \(B\) be the set of open blobs of \(X\). Since \(\overline{\{x\}}\) is an irreducible closed set for any \(x \in X\), condition (1) of Theorem 2.10 holds for \(X\). By Theorem 2.10 \((X, B)\) is an \(M\)-complete join semilattice. Since \(X\) satisfies \((*)\), we get from Proposition 3.2 that \(X\) is homeomorphic to \(\text{spec}(M)\) for some \(M\).

This completes the proof of Theorem 1.4.
4. Remarks and applications

4.1. The proof of Theorem 1.4 implies that if \( M \) is a monoid then \( \text{spec}(M) \) is homeomorphic to \( \text{spec}(B, \cap) \) where \( B = \{ D(f) : f \in M \} \). The monoid \( (B, \cap) \) is known as the \textit{universal semilattice} of \( M \) ([Gri01, III.1.4]). Thus, every monoid spectrum is the spectrum of a semilattice. From this, we get a second characterization of monoid spectra as those spaces which can be obtained as the soberification (space of closed irreducible subsets) of some meet-semilattice \( (P, \leq) \) with greatest element, equipped with the topology whose open sets are the lower order ideals.

4.2. The necessity of \((*)\). The condition \((*)\) of Theorem 1.4 is ugly-looking. However, it is not possible to remove it (or some equivalent condition). To see this, we must exhibit a space which satisfies the first three conditions of Theorem 1.4 but is not homeomorphic to \( \text{spec}(M) \) for any monoid \( M \). In view of Theorem 2.10, we just need to exhibit a space \((X, B)\) such that \( E(X, B) \) does not satisfy \((*)\).

Example. Let \( X \) be a space with a monoidal base \( B \) such that there are \( U_n \in B \) with \( \bigcap_{n \in \mathbb{N}} U_n = \emptyset \), but no finite intersection of the \( U_n \) is empty. For example, take \( X = \mathbb{R} \) with \( B \) the collection of all open subsets of \( \mathbb{R} \), and let \( U_n = (0, \frac{1}{n}) \) for \( n \in \mathbb{N} \). Then from Definition 2.4, we see that \( \bigcap_n \widetilde{U}_n = \{ \{A\} \in \mathcal{P}(X) / \sim \text{ such that } A \subset \bigcap U_n \} = \{\emptyset\} \). Thus, \( \bigcap_n \widetilde{U}_n \subset \emptyset \). But no finite intersection of the \( \widetilde{U}_n \) is contained in \( \emptyset \), and thus \((*)\) does not hold for \( E(X, B) \).

4.3. A topological space \( X \) is \textit{artinian} if every descending chain \( U_1 \supset U_2 \supset \cdots \) of open sets terminates. Any artinian space (or even a space whose open blobs satisfy the descending chain condition) automatically satisfies \((*)\). Combining this observation with [Kat94, Proposition 5.5], we have the following corollary of Theorem 1.4.

\textbf{Corollary.} Let \( X \) be a topological space. The following are equivalent.

1. \( X \) is homeomorphic to \( \text{spec}(M) \) for some finite monoid \( M \).
2. \( X \) is homeomorphic to \( \text{spec}(M) \) for some finitely-generated monoid \( M \).
3. \( X \) is the underlying space of \( E(Y, B) \) for some finite space \( Y \) with a monoidal base \( B \).

4.4. Comparing Theorem 1.4 to Theorem 1.3, and noting that the compact open subsets of \( \text{spec}(M) \) are just the finite unions of sets of the form \( D(f) \), we observe that the underlying space of the Kato spectrum is always the underlying space of the spectrum of some commutative ring.

4.5. Questions. Theorem 1.4 gives rise to some natural questions.

- Is there a better way to express the condition \((*)\) of Theorem 1.4?
- Is there a characterization of spaces of the form \( \text{spec}(M) \) analogous to Hochster’s characterization ([Hoc69, Proposition 10]) of the underlying spaces of ring spectra as projective limits of finite \( T_0 \) spaces?
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