Some Further Remarks on the Local Fundamental Group Scheme

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Abstract

We prove that the Local Fundamental Group Scheme satisfies the Lefschetz-Bott theorems in characteristic $p$. The proofs are standard applications of the Enriques-Severi-Zariski-Serre vanishing theorems and known facts about the $p$-curvature.

1 Introduction

The results of this paper were already proved by Indranil Biswas and Yogesh Holla (arXiv.math/0603299 v1 [math AG], 13 March 2006, arXiv.math/0603299v1 [math AG], 1st May 2007, and finally published in the Journal of Algebraic Geometry, Vol. 16, No.3, 2007, pages [547-597]. The present work was done later (around late 2006-early 2007) but independently. A preliminary version of the present work was put on the arXiv in January 2007. This has been withdrawn as soon as we were informed that Biswas and Holla had already put up their paper on the arXiv in March 2006.

However since our proofs are shorter and more suited for future applications ("On the Grothendieck-Lefschetz Theorem for a family of Varieties", with Marco Antei, in preparation), we are putting a shortened version on the arXiv.

The Fundamental Group Scheme was introduced by Madhav Nori in [N1,N2]. In order to prove the conjectures made in [loc.cit], S.Subramanian

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and the present author had introduced the "Local Fundamental Group Scheme \(\pi_{loc}(X)\)\([MS1]\), which is the infinitesimal part of \(\pi(X)\)\([MS2]\).

We had also introduced the notion of a \(F\)-trivial vector bundle on a variety in characteristic \(p\)\([loc.cit.]\). One may ask whether the Lefschetz-Bott theorems hold for \(\pi_{loc}(X)\). In other words, let \(X\) be a smooth projective variety over an algebraically closed field of characteristic \(p\), and let \(H\) be a very ample line bundle on \(X\). The question is if \(\pi_{loc}(Y) \to \pi_{loc}(X)\) is a surjection if \(\dim X \geq 2\) and degree \(Y \geq n_0\), where \(n_0\) is an integer depending only on \(X\). Similarly, if \(\dim X \geq 3\), the question is if \(\pi_{loc}(Y) \to \pi_{loc}(X)\) is an isomorphism if \(\deg Y \geq n_1\), where \(n_1\) is an integer depending only on \(X\).

We give a positive answer to both these questions. The methods only involve applying the lemma of Enriques-Severi-Zariski-Serre (E-S-Z-S) over and over again. We also heavily use the well-known facts about the \(p\)-curvature of integrable connections in characteristic \(p\)\([K1]\). The notation is the same as in \([MS2]\), which is briefly recalled. We thank Vijaylaxmi Trivedi and Vittorio for helpful discussions. This work was completed while the author was visiting the ICTP, Trieste as a Senior Research Associate. We would like to thank the ICTP for its support and hospitality during that period.

Our method of proof is heavily influenced by a paper of Karen Smith \([S]\), especially [Thm. 3.5]. All the facts about Tannaka Categories that we use may be found in \([N1]\).

2 The first theorem

**Theorem 2.1.** Let \(X\) be a non-singular projective variety over an algebraically closed field of characteristic \(p\), of dimension \(\geq 2\). Let \(H\) be a very ample line bundle on \(X\). Then there exists an integer \(n_0\), depending only on \(X\), such that for any smooth \(Y \in |nH|\), of \(\deg n \geq n_0\), the canonical map \(\pi_{loc}(Y) \to \pi_{loc}(X)\) is a surjection.

Before we begin the proof, we recall some facts from \([MS2]\). Let \(F : X \to X\) be the Frobenius map. For any integer \(t \geq 1\), denote by \(C_t\) the category of all \(V \in Vect(X)\) such that \(F^t(V)\) is the trivial vector bundle on \(X\). Let \(FT(X)\) denote the union of \(C_t(X)\) for all \(t \geq 1\). Fix a base point \(x_0 \in X\), Consider the functor \(S : C_t \to Vect(X)\) given by \(V \to V_{x_0}\). It is seen that \(C_t\), with the fibre functor \(S\), is a Tannaka category \([MS1, MS2]\). denote the
corresponding Tannaka group by $G_t$. It is also easily seen that

$$
\pi_{\text{loc}}(X) \simeq \lim_{\leftarrow t} G_t
$$

, where $\pi_{\text{loc}}(X)$ is the Tannaka group associated to the category ($V \in \text{Vect}(X)$) $F^t(V)$ is trivial for some $t$.

Now let $t = 1$, and consider $G_1(Y)$ and $G_1(X)$. Let $V \in C_1(X)$ with $V$ stable. $V$ corresponds to a principal $H$ bundle $E \to X$, where $E$ is reduced in the sense of Nori [N1, page 87, Prop. 3], or just $N$-reduced. To prove that $G_1(Y) \to G_1(X)$ is a surjection, it is enough to prove that $V/Y$ is stable, or better still that $E/Y$ is $N$-reduced.

**Lemma 2.2.** If $X$ and $E$ are as above, then there exists an integer $n_0$, depending only on $X$, such that for all $n \geq n_0$, and for all smooth $Y \in |nH|$, $E/Y$ is $N$-reduced.

**Proof.** Let $f : E \to X$ be given. Then $f_* \mathcal{O}_E$ belongs to $C_1$. Denote it by $W$ for simplicity. It is easy to see that $W \in FT(X)$. Also note that for any $V \in \text{Vect}(X), V \in FT(X)$ if and only if $V^* \in FT(X)$. Now consider

$$
0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B^1 \to 0 \tag{1}
$$

, Tensor (1) with $W^*(-n)$. There is an $n_0$ such that for $n \geq n_0$, we have $\text{Hom}(W(n), B^1) = 0$. In fact, we have that $\text{Hom}(V(n), B^1) = 0$ for all $V \in FT(X)$, for all $n > n_0$, where $n_0$ is independent of $V$ in $FT(X)$. So the canonical map $H^1(W(-n) \to H^1(F^*(W(-n)$ is injective. But the last space is just $H^1(\mathcal{O}_X(-np))^r, r = \text{rank}W$. But this is 0 as soon as $n \geq n_1$, independent of $V \in C_1(X)$, as dim $X \geq 2$. Hence $H^0(Y, W/Y) = 1$, which proves that $E$ restricted to $Y$ is also $N$-reduced.

Now assume $t \geq 2$. We shall assume in fact that $t = 2$, because an identical proof works for $t \geq 3$. Consider the map $F^2 : X \to X$. Let $B^1_2$ be the cokernel. We have the exact sequence

$$
0 \to B^1 \to B^1_2 \to F_* B_1 \to 0 \tag{2}
$$

Let $W \in C_2(X)$ and tensor (2) with $W(-n)$. One sees immediately that if $H^1(W(-n) \otimes B^1) = 0$ for all $n \geq n_0$, then also $H^1((W(-n) \otimes B^1_2) = 0$ for all $n \geq n_0$, for the same integer $n_0$. So if $E$ is $N$-reduced on $X$, then $E$ remains $N$-reduced on $Y$, if $\text{deg} \ Y \geq n_0$, for the same integer $n_0$, which worked for $C_1(X)$. The proof for bigger $t$ goes the same way, by taking the cokernel of $F^t$, where $t \geq 3$. This concludes the proof of Theorem 2.1. \qed
3 The second theorem

Theorem 3.1. Let $X$ be smooth and projective of dim $\geq 3$. Then there exists an integer $n_0$, depending only on $X$, such that for any smooth $Y \in |nH|, n \geq n_0$, the canonical map $\pi^{\text{loc}}(Y) \to \pi^{\text{loc}}(X)$ is an isomorphism.

Remark 3.2. In the sequel we shall use the phrase ” for a uniform $n$” or” there exists a uniform integer $n”$ to denote a positive integer $n$, which may depend on $X$, but not on $V$, for $V \in FT(X)$.

Before we begin the proof we need the following lemma:

Lemma 3.3. With $X$ as above, then there exists a uniform integer $n_0$ such that we have $H^1(\Omega^1_X((-n) \otimes V) = 0$, for all $V \in FT(X)$, for all $n \geq n_0$.

Proof. Let $T_X$ be the tangent bundle of $X$. For some $s$, depending only on $X$, $T_X(s)$ is generated by global sections. So we have

$$0 \to S^* \to \mathcal{O}^N_X \to T_X(s) \to 0.$$ Dualizing, we get

$$0 \to \Omega^1(-s) \to \mathcal{O}^N_X \to S \to 0$$ (1)

. Tensor (1) with $V(-t)$, to get

$$0 \to V(-t) \otimes \Omega^1(-s) \to V(-t) \otimes \mathcal{O}^N_X \to V(-t) \otimes S \to 0$$ (2)

. Applying $F^r$ to (2) we get

$$0 \to [V(-t) \otimes \Omega^1(-s)]^{p^r} \to [V(-t) \otimes \mathcal{O}^N_X]^{p^r} \to [V(-t) \otimes S]^{p^r} \to 0$$ (3)

. It is clear that $H^0(V(-t) \otimes S) = 0$ for $t >> 0$, independent of $V$. Now look at

$$H^1(V(-t) \otimes \Omega^1(-s)) \to H^1(V(-t) \otimes \mathcal{O}^N_X) \to H^1(V(-t) \otimes \mathcal{O}^N_X)^{p^r} \to 0$$ (4)

where the right arrow is the map induced by $F^r$. But one knows that $H^1(V(-t) \otimes \mathcal{O}^N_X)^{p^r}$ vanishes for $t >> 0$, for a uniform $t$. And $H^1[V(-t) \otimes \Omega^1(-s)] \to H^1[V(-t) \otimes \mathcal{O}^N_X]$ is injective for $t >> 0$, for a uniform $t$.

Also $H^1[V(-t) \otimes \mathcal{O}^N_X] \to H^1[V(-t) \otimes \mathcal{O}^N_X]^{p^r}$ is injective for $t >> 0$, for a uniform $t$. Hence $H^1(V(-t) \otimes \Omega^1)$ vanishes for all $t >> 0$, for a uniform $t$.

This concludes the proof of Lemma 3.3. □
Remark 3.4. In fact, the above proves the following: Let $W$ be an arbitrary vector bundle on $X$. Then there exists a uniform $t_0$, such that for all $t \geq t_0$, and for all $V \in FT(X)$, we have $H^1(V \otimes W(-t)) = 0$. This lemma and remark are the key points in the paper, and will be used over and over again, without explicit mention.

Lemma 3.5. Any $W \in C_t(Y)$ lifts uniquely to an element $V \in C_t(X)$, if $\text{deg}Y = n > n_0$, $n$ is uniform.

Proof. First, we give an idea of the proof. We assume that dimension $X \geq 3$, and we pick an arbitrary smooth $Y \in |nH|$. First assume $t = 1$. Take a $W \in C_1(Y)$ and we show that it lifts uniquely to $V \in C_1(X)$. Such a $W$ is trivialized by the Frobenius, so there exists $M$, an $r \times r$ matrix of 1-forms on $Y$, which gives an integrable connection $\nabla$ on $O_Y$, with $p$-curvature 0.

Consider
\[
0 \to \frac{I}{I^2} \to \Omega^1_X/Y \to \Omega^1_Y \to 0
\]
and
\[
0 \to \Omega^1_X(-n) \to \Omega^1_X \to \Omega^1_Y \to 0
\]

Here $I$ is the idealsheaf of $Y$, $I = O_X(-n)$. We get an integer $n_0$, such that for all $n \geq n_0$, $H^0(X, \Omega^1_X) \to H^0(Y, \Omega^1_Y)$ is an isomorphism. So $M$ lifts to $M_1$, a $r \times r$ matrix of 1-forms on $X$. Now $\nabla$ has $p$-curvature 0 on $Y$. We now show that $\nabla_1$ defined by $M_1$ has $p$-curvature 0 on $X$. The curvature of $\nabla$ is an element of $H^0(\text{End}O_Y \otimes \Omega^2_Y)$. Again by E-S-Z-S, the curvature of $\nabla_1$ is 0 if degree $Y \geq n_1$, for some integer $n_1$, depending only on $X$. The $p$-curvature of $\nabla_1$ is an element of $H^0(F^*\Omega^1_X \otimes \text{End}O_X)$, which again vanishes if $\text{deg} Y \geq n_2$. So any element $W \in C_1(Y)$ lifts uniquely to an element $V \in C_1(X)$.

For $t > 1$, we use induction on $t$. Assume that for lower values of $t$, any element $W \in C_t(Y)$ has been lifted uniquely to an element $V \in C_t(X)$, where degree $Y = n$, where $n$ is uniform. We show in fact that there exists a uniform $n$ with the property: if $W \in C_{t+1}(Y)$, then $W$ lifts to $X$. This proceeds by showing that $F^*W$ lifts uniquely to $X$, and this lifted bundle, say $V_t$, on $X$ has an integrable $p$-flat connection. So on $X$, $V_t$ will descend under $F$ to $V_{t+1}$. And $V_{t+1}$ restricts to $W_{t+1}$ on $Y$. We begin the proof by establishing a couple of claims:

Claim (1): There exists a uniform $n$ such that for $V \in FT(X)$ and for a $Y \in |nH|$ , we have $H^0(\text{End}V \otimes \Omega^1_Y(-n)) = 0.$
Proof of Claim 1): Look at
\[ 0 \to \mathcal{O}_Y(-n) \to \Omega^1_X/Y \to \Omega^1_Y \to 0 \] (1)
and
\[ 0 \to \Omega^1_X(-n) \to \Omega^1_Y \to \Omega^1_X/Y \to 0 \] (2)

Tensor (2) with \( \mathcal{E}ndV(-n) \), we get \( H^0(\mathcal{E}ndV(-n) \otimes \Omega^1_X/Y) = 0 \), for a uniform \( n \). From (1) after tensoring with \( \mathcal{E}ndV \), one sees that \( H^1(\mathcal{E}ndV \otimes \mathcal{O}_Y(-2n)) = 0 \) implies that \( H^0(\mathcal{E}ndV(-n) \otimes \Omega^1_Y) = 0 \). All this is for a uniform \( n \). Now we prove that \( H^1(\mathcal{E}ndV \otimes \mathcal{O}_Y(-2n)) = 0 \), for a uniform \( n \), in

Claim (2): One has \( H^1(\mathcal{E}ndV \otimes \mathcal{O}_Y(-2n)) = 0 \), for any \( V \in FT(X) \) and smooth \( Y \in |nH| \), for a uniform \( n \).

Proof of Claim (2): Look at
\[ 0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B^1 \to 0 \] (1)
\[ 0 \to B^1 \to Z^1 \to \Omega^1 \to 0 \] (2)
\[ 0 \to Z^1 \to F_* \Omega^1 \to B^2 \to 0 \] (3)

This are sequences obtained from the Cartier operator applied to the DeRham complex \( F_*(\Omega^1_X) \). Tensor all the 3 sequences by \( \mathcal{E}ndV(-n) \), and take \( H^0 \). From (2), we get \( H^1(B^1 \otimes \mathcal{E}ndV(-n)) \) injects into \( H^1(Z^1 \otimes \mathcal{E}ndV(-n)) \) for a uniform \( n \). But by 3, we see that \( H^1(Z^1 \otimes \mathcal{E}ndV(-n)) \) injects into \( H^1(F_* \Omega^1 \otimes \mathcal{E}ndV(-n)) \) for a uniform \( n \). But the last cohomology group vanishes for \( n \gg 0 \), with \( n \) uniform (Lemma 2.2 and [S, Th. 3.5]). Therefore \( H^1(B^1 \otimes \mathcal{E}ndV(-n)) \) vanishes for \( n \gg 0 \), with \( n \) uniform. Now look at
\[ 0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B^1 \to 0 \] (4)

Tensor with \( \mathcal{E}ndV(-n) \) and take cohomology. As \( H^1(B^1 \otimes \mathcal{E}ndV(-n)) \) vanishes for \( n \gg 0 \), one gets that \( H^2(\mathcal{E}ndV(-n)) \) injects into \( H^2(\mathcal{E}ndV(-n))^{pr} \) for all \( r \gg 0 \), and for \( n \) uniform. But this last group \( \text{vanishes for} \ n \gg 0 \), and uniform, and for all \( r \) (Lemma 2.2 again). (This is where the hypothesis \( \dim X \geq 3 \) is used). Now finally look at
\[ 0 \to \mathcal{O}_X(-3n) \to \mathcal{O}_X(-2n) \to \mathcal{O}_Y(-2n) \to 0 \] (5)
and tensor with $\mathcal{E}ndV$. We get $H^1(\mathcal{O}_V(-2n) \otimes \mathcal{E}ndV)$ vanishes for $n \gg 0$, and $n$ uniform. Hence, we get $H^0(\mathcal{E}ndV \otimes \Omega^1_Y(-n)) = 0$, with $n$ uniform, which completes the proof of Claim 1.

Claim (3): There is a uniform $n$ with the property: let $Y \in |nH|$ and assume that $W \in FT(Y)$ has been lifted to $V \in FT(X)$. Then if $W$ has a connection on $Y$, then $V$ on $X$ has a connection.

Proof of Claim (3): Look at

$$0 \to \mathcal{O}_Y(-n) \to \Omega^1_X/Y \to \Omega^1_Y \to 0$$

(1)

and

$$0 \to \Omega^1_X(-n) \to \Omega^1_X \to \Omega^1_X/Y \to 0$$

(2)

Tensor with $\mathcal{E}ndV$. From (2) we get $H^1(\mathcal{E}ndV \otimes \Omega^1_X) \to H^1(\mathcal{E}ndV \otimes \Omega^1_X/Y)$. And from (1) and the proof of claim (2), we get $H^1(\mathcal{E}ndV \otimes \Omega^1_X/Y)$ injects into $H^1(\mathcal{E}ndV \otimes \Omega^1_Y)$. But $V$ restricted to $Y$ is $W$. Hence if $W$ has a connection, so does $V$.

Claim (4): If this connection on $W$ is integrable, then the connection on $V$ is also integrable.

Proof of Claim (4): Look at

$$0 \to \Omega^2_X(-n) \to \Omega^2_X/Y \to \Omega^2_Y \to 0$$

(1)

and

$$0 \to \Omega^2_X(-n) \to \Omega^2_X \to \Omega^2_X/Y \to 0$$

(2)

Tensor with $\mathcal{E}ndV$ and take $H^0$. One knows that $H^0(\mathcal{E}ndV \otimes \Omega^2_Y(-n))$ vanishes for $n \gg 0$ with $n$ uniform, by the proof of Claim 1. Similarly, $H^0(\mathcal{E}ndV \otimes \Omega^2_X(-n))$ also vanishes for $n \gg 0$, and $n$ uniform. So $H^0(\mathcal{E}ndV \otimes \Omega^2_X) \to H^0(\mathcal{E}ndV \otimes \Omega^2_Y)$ is injective for $n \gg 0$, $n$ uniform. So if $W$ has an integrable connection, so does $V$. Finally, the $p$-curvature of $V$ is an element of $H^0(\mathcal{E}ndV \otimes F^*\Omega^2_Y)$ which injects into $H^0(\mathcal{E}ndV \otimes F^*\Omega^2_Y)$. So if the $p$-curvature on $W$ is 0, so is the $p$-curvature on $V$.

Claims 1-4 imply that on $X$, if $V$ restricts to $W$ on $Y$, with degree $Y = n$ with $n$ uniform and $W$ has a $p$-flat connection, then $V$ also has a $p$-flat connection. So if $W$ on $Y$ descends to $W_1$, then $V$ also descends under $F$ to $V_1$, and that $V_1$ restricts to $W_1$ on $Y$. This continues to hold for $V \in C_t(X)$, restricting to $W \in C_t(Y)$, any $t$, for $Y$ degree a uniform $n$, depending only on $X$. Hence the canonical map of Tannaka Categories: $C_t(X) \to C_t(Y)$, given by restriction from $X$ to $Y$, induces an isomorphism:

$$G_t(Y) \to G_t(X),$$

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for all $t$. But

$$\pi^{loc}(X) \simeq \lim_{\leftarrow t} G_t(X)$$

and similarly for $Y$. Since there are only finitely many choices of $n$, depending only on $X$, this completes the proof of Lemma 3.5 and hence of Theorem 3.1.

**Remark 3.6.** It is also interesting to determine if the category of $F$-trivial vector bundles, on a smooth $X$, is $m_0$-regular. This is the case if $\dim X \leq 3$.

**Remark 3.7.** The discerning reader will notice that the only property of $F$-trivial bundles which is used is the following: the set of isomorphism classes of stable bundles, which occur in a stable filtration of $F^{n*}(V), n = 0, 1, \ldots$ is only finite in number.

If $V$ is only assumed to be essentially finite [N1,p.82], then $\exists$ a Galois etale covering $\pi : Z \to X$ such that $F^{m*}V$ is trivial on $Z$, for some $m$. Now one sees using [D1, Thm 2.3.2.4], that the set of stable components of $F^{n*}(V), n = 0, 1, \ldots$ is again finite. So all the proofs and propositions carry over with $\pi^{loc}X$ replaced by $\pi(X)$, making use of Lemma 3.3. We leave the details as an exercise.

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