Gravitational energy-momentum in small regions according to the tetrad-teleparallel expressions

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Abstract

The gravitational energy-momentum within a small region as determined by two tetrad-teleparallel expressions is evaluated with the aid of an orthonormal frame adapted to Riemann normal coordinates. We find that the gauge current “tensor” does enjoy the highly desired and rare property of being a positive multiple of the Bel-Robinson tensor, whereas Møller’s expression does not.

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1 Introduction: energy-momentum localization

The localization of energy-momentum for gravitating systems is still an outstanding fundamental problem [1]. The classical attempts to identify a gravitational energy-momentum density for Einstein’s covariant theory, general relativity, had all led to various non-covariant expressions which could be written as the partial derivative of some particular non-covariant, coordinate system dependent superpotential, (see, e.g., [2, 3, 4, 5]). As coordinate systems have no physical significance, these energy-momentum density pseudotensors had no clear physical meaning. This led some to argue that there was no physically meaningful gravitational energy-momentum density, and, moreover, that this is just what we should expect from the equivalence principle (see in particular [6], §20.4).

In 1961 Møller constructed an energy-momentum density which, although itself still a pseudotensor, nevertheless has a superpotential which is a tensor under coordinate transformations [7]. Møller achieved this “tensor” form by introducing an orthonormal frame, a tetrad (a.k.a. vierbein). His superpotential depends on the local choice of the orthonormal frame and behaves as a tensor with respect to coordinate transformations. Like many other energy-momentum expressions, the value Moller’s expression assigns to a spatial region is not as ambiguous as one might have first thought, it is quasi-local [1]: it depends on the fields only at the boundary of the region. More precisely the energy-momentum
Møller’s tetrad expression assigns to a spacetime region depends—like other pseudotensors—on the boundary choice of the coordinates, but unlike the other pseudotensors this dependence is tensorial. Moreover, it also depends on an additional object which includes non-physical information, namely the choice of tetrad on the boundary.

Møller noted that his tetrad description could be given an interesting reformulation in terms of teleparallel geometry. The teleparallel reformulation of Einstein’s GR (a.k.a. the teleparallel equivalent of GR (TEGR) and GR||) has attracted interest not only for its presumed advantages for describing energy-momentum but also as a gauge theory of spacetime translations. Within the context of the tetrad-teleparallel theory investigators (see [8, 9, 10, 11, 12, 13, 14, 15, 16] and the works cited therein) have proposed another energy-momentum expression. It can be identified as the teleparallel translational gauge current density.

Nevertheless, largely because of its perceived advantages for energy-momentum localization, Møller’s tetrad expression (even though there is no generally accepted frame gauge condition)—especially in its interesting teleparallel description—has continued to attract interest over the years (see, e.g., [17, 18, 19, 10, 11, 20] and the works cited therein).

In certain special cases, however, there is a natural orthonormal frame; then both Møller’s expression and the gauge current yield an unambiguous energy-momentum. In particular this is so asymptotically—at spatial infinity. In that case Møller’s expression (like most others) works well (see [17] for an explicit verification; moreover Møller’s tetrad expression in fact also works well at future null infinity [18]). This asymptotic success is actually not at all surprising; having the proper asymptotic behavior is a relatively weak requirement, for in this weak field region an expression need only have the proper linear theory limit.

The situation is different in the one other situation where there is a natural frame—a case which has, to our knowledge, not been previously investigated for the tetrad expressions—namely the small region limit. In this limit, to zeroth order, one should get the material energy-momentum density—a quite weak requirement which follows from the equivalence principle. On the other hand the proposed small vacuum region limit is that, to second order, one gets a positive multiple of the Bel-Robinson tensor [21] (that would be sufficient to guarantee that the energy of a small region was positive). Now this latter requirement is especially interesting as a test of proposed energy-momentum densities, since it probes the expression beyond the linear order. It is a strong criterion, capable of excluding many otherwise acceptable expressions, in particular none of the classical pseudotensors satisfy this requirement (although certain artificial combinations of them do [22, 23, 24]).

Here, using Riemann normal coordinates and the associated “normal” tetrad, we examine Møller’s expression and the gauge current in the small region limit. We find that the gauge current naturally satisfies this highly desirable vacuum Bel-Robinson property while Møller’s expression does not.

For notation we follow [6] unless otherwise noted. Here Greek indices are
used to refer to spacetime and, unless otherwise noted, a completely general frame. However, in those sections where it is necessary to make the distinction, we use Greek indices to refer to orthonormal frames, with Latin indices reserved for holonomic (coordinate) frames.

2 Conserved energy-momentum densities from the field equations

A gravitational energy-momentum density is easily derived from Einstein’s equations expressed in terms of differential forms:

\[ R^\alpha_\beta \wedge \eta^\beta_\alpha \mu = -2\kappa T_\mu. \]

Here \( \kappa = 8\pi G/c^4 \) is the gravitational coupling constant (we will use units with \( c = 1 \)), \( R^\alpha_\beta \) is the curvature 2-form, \( T_\mu = T^\nu_\mu \eta_\nu \) is the source energy-momentum 3-form, and we are using Trautman’s convenient dual form basis \( \eta^\alpha := \ast(d^\alpha \wedge \ldots) \), where \( d^\alpha \) is the co-frame. The left hand side of (1) is just \(-2G^\nu_\mu \eta_\nu\), the Einstein tensor expressed as a 3-form. Using the definition of the curvature 2-form in terms of the connection one-form and extracting an exact differential leads to

\[ R^\alpha_\beta \wedge \eta^\beta_\alpha \mu := (d\Gamma^\alpha_\beta \wedge \eta^\beta_\alpha \mu) + \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta \wedge \eta^\beta_\alpha \mu \]

\[ \equiv d(\Gamma^\alpha_\beta \wedge \eta^\beta_\alpha \mu) + \Gamma^\alpha_\beta \wedge \Gamma^\lambda_\mu \wedge \eta^\beta_\alpha \lambda - \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta \wedge \eta^\beta_\alpha \mu, \]

where we have used \( D\eta^\alpha_\beta \mu = 0 \), which follows since the connection is metric compatible and torsion free. Using this expansion one can rewrite the Einstein equation (1) in a neat form (which is remarkably similar to the form used by Einstein when he was still searching for a good gravity theory [25]):

\[ dp_\mu = 2\kappa P_\mu, \]

where the energy-momentum (superpotential) 2-form is

\[ p_\mu := -\Gamma^\alpha_\beta \wedge \eta^\beta_\alpha \mu, \]

and the current is the total energy-momentum density (3-form)

\[ P_\mu := t_\mu + T_\mu, \]

which “automatically” satisfies the current conservation relation \( dP_\mu = 0 \) [5]. This total energy-momentum current complex includes the (non-covariant) gravitational energy-momentum density

\[ t_\mu := (2\kappa)^{-1} \left( \Gamma^\alpha_\beta \wedge \Gamma^\lambda_\mu \wedge \eta^\beta_\alpha \lambda - \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta \wedge \eta^\beta_\alpha \mu \right). \]
According to this prescription the total energy-momentum within a region is given by

$$P_\mu(V) := \int_V P_\mu = (2\kappa)^{-1} \int_V dp_\mu \equiv (2\kappa)^{-1} \oint_{\partial V} p_\mu.$$  \hspace{1cm} (7)

The volume integral form would lead one to expect that the value depends on the quantities and choice of frame throughout the region, but the closed 2-surface integral shows that the value is quasi-local. The value is still non-covariant: it depends on the choice of frame—but, as we have already pointed out, only on the choice at (and, through the connection, near) the boundary.

The 2-surface integrand is

$$p_\mu := -\Gamma^\alpha_{\beta\gamma} \eta^\beta_\mu \equiv -\Gamma^\alpha_{\beta\gamma} g^{\beta\sigma} \delta_{\alpha\sigma\mu} \frac{1}{2} \eta_{\tau\rho}. \hspace{1cm} (8)$$

Expanding the components of this compact 2-form expression gives

$$(\Gamma^\rho_{\beta\gamma} g^{\beta\tau} - \Gamma^\tau_{\beta\gamma} g^{\beta\rho}) \delta_\mu^\tau + (\Gamma^\gamma_{\beta\gamma} g^{\beta\rho} - \Gamma^\rho_{\beta\gamma} g^{\beta\gamma}) \delta_\mu^\rho + (\Gamma^\tau_{\beta\gamma} g^{\beta\gamma} - \Gamma^\gamma_{\beta\gamma} g^{\beta\tau}) \delta_\mu^\rho. \hspace{1cm} (9)$$

Specializing to the case where the frame is holonomic this expression is exactly the superpotential found by Freud [26]; in that case the associated gravitational energy-momentum density is the Einstein pseudotensor 3-form. On the other hand one can choose the frame to be orthonormal, then these same formal expressions become the those of the tetrad-teleparallel translational gauge current [9, 10, 12, 13, 14], which—as we will elaborate on in the next section—are closely related (see [19]) to those proposed by Møller [7] in 1961 (by the way, a differential form construction of these expressions virtually the same as ours was presented some time ago by Wallner [27], and similar arguments appear in [11] and [15]); the noteworthy thing is that these tetrad expression are tensors—under coordinate transformations. Although they are completely independent of the choice of coordinates (i.e., they are covariant under coordinate transformations), they do depend on the choice of tetrad (in this important sense they are still non-covariant). More specifically the energy-momentum values they determine are quasi-local: they depend on the choice of tetrad, but only on the choice at and near the boundary.

### 3 Møller’s expression and the gauge current

The traditional metric approach to gravitational energy-momentum had led to various pseudotensors (see, e.g., [2, 3, 11, 14]), none really satisfactory. Then Møller [7] replaced the metric by an orthonormal frame (a.k.a. tetrad, vierbein). The resulting formulation admits an interesting alternate geometric interpretation in terms of teleparallel geometry [7, 9, 10, 12, 13, 14, 20, 16]. It has been argued that this framework is more suitable for identifying a good gravitational energy-momentum density. Indeed, using this approach Møller put forward his well-known energy-momentum expression.

Recall that the Einstein pseudotensor can be obtained as the (Noether) canonical energy-momentum density from the Hilbert scalar curvature Lagrangian
after a certain (non-covariant) divergence has been removed (which removes all the second derivatives of the metric):

\[ L_E(g, \partial g) := \sqrt{-g}R - \partial_l(\sqrt{-g}g^{im}\Gamma^i_{jmk}\delta^l_{km}), \]  

\[ t^i_{E,j} := \frac{\partial L_E}{\partial g_{kl}}\partial_j g_{kl} - \delta^i_j L_E; \]  

(10)  

(11)

this is related to the aforementioned Einstein pseudotensor 3-form by \( t^i_{E,j} \eta^j \). Similarly, one can obtain Møller’s expression by using the tetrad \( e^{\alpha_i} \) (related to the metric by \( g_{ij} = \bar{g}_{\alpha\beta}e^{\alpha_i}e^{\beta_j} \), where \( \bar{g} = \text{diag}(-1,+1,+1,+1) \) is the Minkowski metric) as a variable and removing an appropriate divergence which contains all the second derivatives of the tetrad:

\[ L_M(e, \partial e) := eR - \partial_l(ee^{\alpha_i}\Gamma^{\alpha\beta}_k\delta^l_{ij}), \]  

\[ t^i_{M,j} := \frac{\partial L_M}{\partial e^{\alpha_k}}\partial_j e^{\alpha_k} - \delta^i_j L_M. \]  

(12)  

(13)

(Here \( \epsilon := \det e^{\alpha_i} \), the dual frame satisfies \( e^{\alpha_i}e^{\alpha_j} = \delta^i_j \) and \( e^{\alpha_i}e^{\beta_j} = \delta^\alpha_\beta \), \( \Gamma^\alpha_{\beta k} = \Gamma^\alpha_{\beta k}(\partial_k) \), and Greek and Latin indices are transveected using respectively \( \bar{g}_{\alpha\beta} \) and \( g_{ij} \).) The associated Møller 3-form is \( t_{Mj} = t^i_{M,j}\eta^i \). From this perspective Møller’s expression is quite natural, namely it is the (Noether) canonical energy-momentum density associated with the tetrad variable. It should be noted that exactly this same density can also be obtained from our considerations in the previous section—simply by formally replacing \( \mu \) by \( j \), while keeping all the other indices referring to the orthonormal frame. For more on these two closely related expressions see [10, 14].

In sharp contrast to the metric formulation, within the tetrad/teleparallel formulation investigators [7, 9, 10, 12, 13, 14, 20, 15, 16] have been led to only these two (closely related) quasi-local boundary term expressions for the energy-momentum within a volume \( V \):

\[ P^G_\mu(V) := \oint_{\partial V} p_\mu, \quad P^M_j(V) := \oint_{\partial V} p_j \equiv \oint_{\partial V} e^{\mu j}p_\mu, \]  

(14)

respectively, the translational gauge current and the Møller expression [7]. Møller had pointed out that his superpotential (which appears here as a 2-form integrand) is tensorial (i.e., it transforms homogeneously under a change of coordinates); however its differential,

\[ t_{Mj} = dp_j = d(e^{\mu j}p_\mu) = de^{\mu j} \wedge p_\mu + e^{\mu j}dp_\mu, \]  

(15)

the Møller tetrad-teleparallel energy-momentum 3-form, is not a tensor with respect to coordinate transformations (as Møller himself noted)—because of the factor \( de^{\mu j} \). In contrast, it should be emphasized that both the translation gauge current superpotential 2-form \( p_\mu \) and its differential, the gauge current 3-form [10], are true tensors—under changes of coordinates.

The tetrad theory, however, does have local Lorentz gauge freedom. The gauge current expressions do depend on the choice of orthonormal frame, and
thus still contain some observer dependent information mixed in with the physical information in the energy-momentum expression. Nevertheless one can regard the gauge current expression as preferable to any of the pseudotensors or Møller’s tetrad expression, since an orthonormal frame is more physical than an arbitrary choice of coordinates.

Concerning the ambiguity re the choice of frame, it is important to note that the quasi-local values depend only on the choice of frame on the boundary, and not on the choice within the interior of the region.

It should also be mentioned that, unfortunately, in some earlier investigations by our group \[28, 23, 24, 29\] we misidentified the gauge current as the expression of Møller. (From our perspective the gauge current is the natural choice, and we just assumed that was what Møller had used—without actually carefully reading his work. While we can appreciate that his expression is—from the Noether approach \[12, 13\]—also a natural choice, the coordinate non-covariance of his energy-momentum density is certainly a liability.)

4 Riemann normal coordinates and normal tetrads

To find the energy-momentum within a small region surrounding a particular point, we look to the 3-forms $\mathcal{P}_\mu, \mathcal{P}_i$ expanding them in a power series. For this purpose we choose Riemann normal coordinates $x^i$ centered at the selected point. The Maclaurin-Taylor expansion of the holonomic components of the metric and connection are well known (see, e.g. [6], §11.6):

\[
\left. g_{ij} \right|_0 = \bar{g}_{ij}, \quad \partial_k g_{ij} \right|_0 = 0, \quad 3 \partial_{kl} g_{ij} \right|_0 = -R_{ikjl} - R_{djk}, \quad (16)
\]

\[
\left. \Gamma^i_{jk} \right|_0 = 0, \quad 3 \partial_l \Gamma^i_{jk} \right|_0 = -R^i_{jkl} - R^i_{kjl}. \quad (17)
\]

Here $\bar{g}_{ij} = \text{diag}(- + + +)$ is the Minkowski metric. In the associated “normal” orthonormal frame, the coframe $\theta^a = e^a_i dx^i$ and connection one-form $\Gamma^a_{\beta k} dx^k$ components take closely related analogous values:

\[
\left. e^a_j \right|_0 = \delta^a_j, \quad \partial_k e^a_j \right|_0 = 0, \quad 6 \partial_{kl} e^a_j \right|_0 = -R^a_{kjl} - R^a_{ljk}, \quad (18)
\]

\[
\left. \Gamma^a_{\beta j} \right|_0 = 0, \quad 2 \partial_k \Gamma^a_{\beta j} \right|_0 = R^a_{\beta kl}. \quad (19)
\]

It is readily verified that these values satisfy, to the appropriate order, the two relations which transform the metric and connection coefficients between the holonomic and orthonormal frames:

\[
\left. g_{ij} \right|_0 = \bar{g}_{\alpha \beta} e^\alpha_i e^\beta_j, \quad \left. e^\beta_j \Gamma^\alpha_{\beta i} \right|_0 = \Gamma^k_{ji} e^\alpha_k - \partial_i e^\alpha_j. \quad (20)
\]

5 Small region values

Here we present the energy-momentum values for small regions obtained from the expressions mentioned above. For non-vacuum regions all of the expressions reduce in zeroth order to the material energy-momentum density, in accord with
the equivalence principle. The value obtained for vacuum regions using the holonomic Einstein pseudotensor has long been known \[6, 21, 22\]. To second order in RNC it is

\[
2\kappa \mathcal{P}_j^E = 2\kappa t_{Ej} \simeq x^k x^l \frac{1}{3} \cdot \frac{4}{6} (4B - S)^j_{\ kli} \eta_i,
\]

(21)

where

\[
S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\ \lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}^{\ \lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\sigma\kappa\rho} R_{\lambda\sigma\kappa\rho}^{\ ^{\lambda\sigma\kappa\rho}},
\]

(22)

and

\[
B_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\ \lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}^{\ \lambda\sigma} - \frac{1}{2} g_{\alpha\beta} R_{\gamma\delta}^{\gamma\delta} R_{\mu\nu}^{\ ^{\mu\nu}},
\]

(23)

is the celebrated Bel-Robinson tensor. (This tensor has many interesting properties, in particular in vacuum—where the Riemannian curvature reduces to the Weyl curvature—it is totally symmetric and traceless.)

5.1 The tetrad-teleparallel gauge current

For the gauge current, expanding \(\mathcal{P}_\mu\) using Riemann normal coordinates and the associated normal tetrad gives, to zeroth order (unsurprisingly) only the source energy momentum density—just as it should according to the equivalence principle. In vacuum regions \(\mathcal{P}_\mu\) reduces to \(t_\mu\ [9]\), and the leading non-vanishing value—using (19)—appears at the second order:

\[
2\kappa \mathcal{P}_\mu = \Gamma^{\alpha\beta} \wedge \Gamma^{\gamma} \wedge \eta_{\alpha\beta\gamma} - \Gamma^{\alpha} \wedge \Gamma^{\beta} \wedge \eta_{\alpha\beta}\mu
\]

(24)

\[
\simeq \frac{x^l x^m}{4} \left( R_{\alpha\beta\lambda\mu} R_{\mu\lambda}^{\ \gamma} \delta_{\gamma}^{\ ^{\gamma}} \delta_{\alpha}^{\ ^{\alpha}} \right) \eta_{\gamma}
\]

\[
\simeq \frac{x^l x^m}{4} \left( R_{\alpha\beta\lambda\mu} R_{\mu\lambda}^{\ \gamma} \delta_{\gamma}^{\ ^{\gamma}} \delta_{\alpha}^{\ ^{\alpha}} \right) \eta_{\gamma}
\]

\[
= \frac{x^l x^m}{4} \left( 2 R_{\mu\lambda\delta\nu} R_{\lambda\delta}^{\ \mu} R_{\gamma\delta}^{\ ^{\gamma}} \delta_{\alpha}^{\ ^{\alpha}} \right) \eta_{\gamma}
\]

\[
= \frac{x^l x^m}{4} B_{\nu}^{\mu} \eta_{\nu},
\]

(25)

proportional to the Bel-Robinson tensor. In this calculation we have used the vanishing of the Ricci tensor in vacuum and some well known curvature tensor symmetry properties.

To see why it is so desirable to get just the Bel-Robinson tensor one can integrate (26) over a small coordinate sphere in the surface \(x^0 = 0\), using (with \(a, b, \ldots = 1, 2, 3\))

\[
\int x^a x^b d^3 x = \frac{1}{3} \delta^{ab} \int r^2 d^3 x = \frac{4 \pi}{3} \cdot \frac{2}{5} r^5,
\]

(27)

and the traceless property of the Bel-Robinson tensor to get for the gauge current energy-momentum

\[
P^G_{\mu} \simeq (2\kappa)^{-1} B^0_{\mu} \delta_{\mu}^{ab} \cdot \frac{4 \pi}{3} \cdot \frac{2}{5} r^5 = B^0_{\mu} \frac{4 \pi}{5!} \kappa r^5 = \frac{1}{2G} B^0_{\mu} \frac{4 \pi}{5!} \kappa r^5.
\]

(28)
This result is best appreciated when expressed in terms of the (traceless, symmetric) electric and magnetic parts of the Weyl tensor, \( E_{ab} := R_{a0b0} \), \( H_{ab} := \frac{1}{2} \epsilon_{acde} R^c_{\ e0b0} \). We then have a value similar to that in electrodynamics:

\[
P^\mu_G = (P^0_G, P^c_G) \simeq \frac{\eta^b}{3G} \left( \frac{1}{2} (E_{ab} E^{ab} + H_{ab} H^{ab}), \epsilon^{acb} E_{ad} H^d_b \right);
\] (29)

hence \( P^\mu_G \) satisfies an important energy condition: it is future pointing and non-spacelike since \( P^0_G \geq |P^c_G| \geq 0 \).

### 5.2 Møller’s expression

Turning now to Møller’s expression \( P^c_j \):

\[
2\kappa(T^c_j + t^c_j) = dp_j = d(e^\mu_j p_\mu) = e^\mu_j dp_\mu + de^\mu_j \wedge p_\mu = e^\mu_j (2\kappa(T^c_\mu + t^c_\mu) + \partial_\ell e^\mu_j dx^\ell \wedge (-\Gamma^{\alpha\beta}_{\ m} dx^m) \wedge \eta_{\alpha\beta\mu} \ (30)
\]

To zeroth order this is again the material result one expects in accord with the equivalence principle. For small vacuum regions we find to lowest non-vanishing order

\[
2\kappa t^c_j = 2\kappa t^G_j + \partial_\ell e^\mu_j (-\Gamma^{\alpha\beta}_{\ m} \delta_{\alpha\beta\mu} \eta_i \ (31)
\]

\[
\simeq \frac{x^l x^m}{4} B^j_{ilm} \eta_i - \frac{1}{6} (R^\mu_{ljm} + R^\mu_{nlj}) x^n (-\frac{1}{2} R^{\alpha\beta}_{\ km} x^k) \delta_{\alpha\beta\mu} \eta_i \ (32)
\]

\[
= \frac{x^l x^m}{4} B^j_{ilm} \eta_i - \frac{x^l x^m}{24} (2B + S)^i j_{ilm} \eta_i \ (33)
\]

\[
= x^l x^m \frac{1}{4 \cdot 6} (4B - S)^i j_{ilm} \eta_i \ (34)
\]

Remarkably it turns out to be proportional to the Einstein value (21). (As far as we can see this is just an accidental coincidence.)

According to this measure the energy within a small sphere of radius \( r \) is

\[
P^0 = \frac{1}{12\kappa} (B - \frac{1}{4} S)^{00} \int x^a x^b d^3 x = \frac{4\pi}{12\kappa} (B - \frac{1}{4} S)^{00} \frac{r^5}{3 \cdot 5} (35)
\]

\[
= \frac{1}{3G} \frac{r^5}{5!} (7E_{ab} E^{ab} - 3H_{ab} H^{ab}), (36)
\]

which can be negative.

### 6 Conclusion

One reason that the tetrad-teleparallel formulation of GR has been favored is because it has been believed to have some advantage with respect to the long-standing problem of how to localize gravitational energy. Within this framework two energy-momentum expressions have been advocated. Here we have
shown that the tetrad-teleparallel gauge current (which had already been recognized as one of the best descriptions of the gravitational energy-momentum for GR) satisfies another important criterion. Whereas the desired small region Bel-Robinson property is not satisfied by Møller’s expression, it is naturally satisfied for the tetrad-teleparallel gauge current energy-momentum density. An important consequence is that the gravitational energy according to the latter measure is positive, at least to this order. (We expected this positivity result since in fact there is a positivity proof for the energy associated with the tetrad gauge current expression [9].)

We stress that the vacuum small region Bel-Robinson property is, as exemplified by the two cases considered here, a strong test capable of excluding many otherwise acceptable expressions; indeed none of the classical pseudotensors (and in this category one can include Møller’s tetrad-teleparallel expression) satisfies this requirement (although certain quite artificial combinations of them do [22, 23, 24]).

Compared to these the tetrad-teleparallel gauge current energy-momentum density stands out. It is certainly a better description for gravitational energy-momentum. In addition to being a tensor (under coordinate transformations) it also enjoys the highly desired and rare property of having its small region value be a positive multiple of the Bel-Robinson tensor.

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