On Connected Co-Independent Hop Domination in Graphs

Sandra A. Nanding¹,* , Helen M. Rara²

¹ Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines
² Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra, and Analysis-Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines

Abstract. Let G be a connected graph. A subset S of V(G) is a connected co-independent hop dominating set in G if the subgraph induced by S is connected and V(G) \ S is an independent set where for each v ∈ V(G) \ S, there exists a vertex u ∈ S such that d_G(u, v) = 2. The smallest cardinality of such an S is called the connected co-independent hop domination number of G. This paper presents the characterizations of the connected co-independent hop dominating sets in the join, corona and lexicographic product of two graphs. It also discusses the corresponding connected co-independent hop domination numbers of the aforementioned graphs.

2020 Mathematics Subject Classification: 05C69

Key Words and Phrases: Connected co-independent hop dominating set, connected co-independent hop domination number, strictly co-independent set, strictly co-independent number, join, corona, lexicographic product

1. Introduction

In the late 1950’s and 1960’s, the study on domination in graphs was developed, beginning with C. Berge [1] in 1958. There are now many studies involving domination and its variations. One of its variation is the connected co-independent domination number of graphs introduced by B. Gayathri and S. Kaspar in 2010 [3]. Also, connected co-independent domination number in graphs were studied in [2, 6, 12]. Years later, new domination parameter called hop domination in graph is introduced by Natarajan and Ayyaswamy [8]. Hop domination in graphs were also studied in [7, 9–11, 13].

In this study, the researcher defines and establishes a new concept of hop domination called a connected co-independent hop domination and generates some characterizations

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i4.4069
Email addresses: sandra.nanding@msuiit.edu.ph (S. Nanding), helen.rara@msuiit.edu.ph (H. Rara)
of connected co-independent hop domination in graphs. Connected co-independent hop domination in graphs can have real world applications. For an application, in [5], Desormeaux, Haynes, and Henning inspired their research on these concepts through social networking applications. They considered a factory with a large number of employees and needed to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this. In other words, the manager will select some workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible to minimize costs (extra costs for inspectors) and protect privacy (keep the inspectors’ identity confidential).

To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed in which each worker is represented by a vertex and an edge between two workers represents possible bias, that is, whether the two workers are close friends or enemies. Ideally, an inspector should not be adjacent to any worker who is being inspected.

In connected co-independent hop domination, every worker will be inspected by the nearest non-biased inspector. That is, an inspector who is a close friend (or an enemy) of a close friend (or enemy) of a worker. This is to save time and effort of locating a particular worker. Also, the inspectors should be acquainted with each other and all non-inspector workers are neither friends nor enemies, that is, they are not adjacent or there is no edge between them. The connected co-independent hop domination number will give the minimum number of inspectors needed.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [4] for elementary Graph Theoretic concepts. An independent set $S$ in a graph $G$ is a subset of the vertex-set of $G$ such that no two vertices in $S$ are adjacent in $G$. The cardinality of a maximum independent set is called the independence number $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a $\beta$-set of $G$. A dominating set $D \subseteq V(G)$ is called a connected dominating set of $G$ if $D$ is a connected dominating set of $G$ and $V(G) \setminus D$ is an independent set.

The cardinality of such a minimum set $D$ is called a connected co-independent domination number of $G$ denoted by $\gamma_{c,coi}(G)$. A connected co-independent dominating set $D$ with $|D| = \gamma_{c,coi}(G)$ is called a $\gamma_{c,coi}$-set of $G$. Let $G$ be a connected graph. A set $S \subseteq V(G)$ is a hop dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality of a hop dominating set of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$-set. A vertex $v$ in $G$ is a hop neighbor of vertex $u$ in $G$ if $d_G(u,v) = 2$. The set $N_G(u,2) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the open hop neighborhood of $u$. The closed hop neighborhood of $u$ in $G$ is given by $N_G[u,2] = N_G(u,2) \cup \{u\}$. The open hop neighborhood of $X \subseteq V(G)$ is the set $N_G(X,2) = \bigcup_{u \in X} N_G(u,2)$. The closed hop neighborhood of $X$ in $G$ is the set $N_G[X,2] = N_G(X,2) \cup X$. Let $G$ be a graph. A subset $S$ of $V(G)$ is a strictly co-independent set of $G$ if $V(G) \setminus S$ is an independent set and $N_G(v) \cap S \neq S$ for all $v \in V(G) \setminus S$. The minimum cardinality of a strictly co-independent
set in $G$, denoted by $sci(G)$ is called the strictly co-independent number of $G$. A strictly co-independent set $S$ with $|S| = sci(G)$ is called an $sci$-set of $G$. Let $G$ be a connected graph. A hop dominating set $S \subseteq V(G)$ is a connected co-independent hop dominating set of $G$ if $\langle S \rangle$ is connected and $V(G) \setminus S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of $G$, denoted by $\gamma_{ch, coi}(G)$, is called the connected co-independent hop domination number of $G$. A connected co-independent hop dominating set $S$ with $|S| = \gamma_{ch, coi}(G)$ is called a $\gamma_{ch, coi}$-set of $G$.

2. Preliminary Results

Remark 1. Every connected co-independent hop dominating set in a connected graph $G$ is hop dominating. Hence, $\gamma_h(G) \leq \gamma_{ch, coi}(G)$.

Remark 2. Let $G$ be a connected graph of order $n$. Then $1 \leq \gamma_{ch, coi}(G) \leq |V(G)|$. Moreover, $\gamma_{ch, coi}(G) = 1$ if and only if $G = K_1$.

Example 1. The equations below give the connected co-independent hop domination number of the path $P_n$ and cycle $C_n$.

$$
\gamma_{ch, coi}(P_n) = \begin{cases} 
1 & \text{if } n = 1 \\
2 & \text{if } n = 2, 3 \\
 n - 2 & \text{if } n \geq 4
\end{cases}
$$

$$
\gamma_{ch, coi}(C_n) = \begin{cases} 
3 & \text{if } n = 3 \\
 n - 1 & \text{if } n \geq 4
\end{cases}
$$

Remark 3. If $G$ is a complete graph, then $\gamma_{ch, coi}(G) = n$.

Theorem 1. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{ch, coi}(G) = 2$ if and only if there exist adjacent vertices $x$ and $y$ of $G$ such that for each $z \in V(G) \setminus \{x, y\}$, $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$ and $z \notin N_G(x) \cap N_G(y)$.

Proof: Suppose $\gamma_{ch, coi}(G) = 2$. Let $S = \{x, y\}$ be a $\gamma_{ch, coi}$-set of $G$. Since $S$ is connected, $xy \in E(G)$. Let $z \in V(G) \setminus \{x, y\}$. Then $z \notin S$. Since $S$ is a hop dominating set of $G$, $z \in N_G(x, 2) \cup N_G(y, 2)$. Suppose $z \in N_G(x, 2)$. Then there exist $w \in N_G(z) \cap N_G(x)$. Since $V(G) \setminus S$ is an independent set, $w \in S$. Thus, $w = y$, that is, $N_G(z) = \{y\}$. Similarly, if $z \in N_G(y, 2)$, then $N_G(z) = \{x\}$. Since $z \in N_G(x, 2) \cup N_G(y, 2)$, $z \notin N_G(x) \cap N_G(y)$.

Conversely, suppose that there exist adjacent vertices $x$ and $y$ of $G$ satisfying the given condition. Let $S = \{x, y\}$. Since $xy \in E(G)$, $S$ is connected. Let $z \in V(G) \setminus S$. If $N_G(z) = \{x\}$, then since $xy \in E(G)$, $d_G(y, z) = 2$. While on the other hand, if $N_G(z) = \{y\}$, then $d_G(x, z) = 2$. Thus, $S$ is a hop dominating set of $G$. Since $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$, $V(G) \setminus S$ is an independent set. Therefore, $S$ is a connected co-independent hop dominating set of $G$. So, $\gamma_{ch, coi}(G) \leq |S| = 2$. But $G$ is nontrivial. Hence, $\gamma_{ch, coi}(G) \neq 1$ and so $\gamma_{ch, coi}(G) = 2$. 
\qed
Theorem 2. Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma_{ch, coi}(G) = n$ if and only if $G$ is complete.

Proof: Suppose $\gamma_{ch, coi}(G) = n$. Suppose that $G$ is not complete. Then there exist distinct vertices $u, v \in V(G)$ such that $d_G(u, v) = 2$. Let $S = V(G) \setminus \{u\}$. Then $S$ is a connected co-independent hop dominating set of $G$. Therefore, $\gamma_{ch, coi}(G) \leq |S| = n - 1$, a contradiction. Thus, $G$ is a complete graph.

Conversely, by Remark 3, $\gamma_{ch, coi}(K_n) = n$. \qed

3. On Connected Co-Independent Hop Domination in the Join of Graphs

The join of two graphs $G$ and $H$ is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 3. Let $G$ and $H$ be any two graphs. Then $S \subseteq V(G + H)$ is a connected co-independent hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$ where one of the following holds:

(i) $S_G = V(G)$ and $S_H$ is a strictly co-independent set of $H$.

(ii) $S_H = V(H)$ and $S_G$ is a strictly co-independent set of $G$.

Proof: Suppose $S$ is a connected co-independent hop dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$. Since $S$ is a hop dominating set of $G + H$, $S_G \neq \emptyset$ and $S_H \neq \emptyset$. Since $V(G + H) \setminus S$ is an independent set, $S_G = V(G)$ or $S_H = V(H)$. Suppose $S_G = V(G)$. Then $V(H) \setminus S_H = V(G + H) \setminus S$ is an independent set. Let $v \in V(H) \setminus S_H$. Then $v \in V(G + H) \setminus S$. Since $S$ is a hop dominating set, there exists $w \in S$ such that $d_{G+H}(v, w) = 2$. Hence, $w \in S_H \setminus N_H(v)$. Thus, $N_H(v) \cap S_H \neq S_H$, showing that $S_H$ is a strictly co-independent set of $H$. Thus, (i) holds. Similarly, if $S_H = V(H)$, then $S_G$ is a strictly co-independent set of $G$ and (ii) holds.

For the converse, suppose $S = S_G \cup S_H$ where $S_G$ and $S_H$ satisfy the given conditions. Let $v \in V(G + H) \setminus S$. Consider the following cases.

Case 1. $S_G = V(G)$.

Then $v \in V(H) \setminus S_H$. By (i), there exists $w \in S_H \setminus N_H(v)$. Hence, $w \in S$ and $d_{G+H}(v, w) = 2$.

Case 2. $S_H = V(H)$.

Then $v \in V(G) \setminus S_G$. By (ii), there exists $u \in S_G \setminus N_G(v)$. Thus, $u \in S$ and $d_{G+H}(u, v) = 2$. Therefore, in either case, $S$ is a hop dominating set of $G + H$.

Since $V(G + H) \setminus S = V(H) \setminus S_H$ if (i) holds or $V(G + H) \setminus S = V(G) \setminus S_G$ if (ii) holds, $V(G + H) \setminus S$ is an independent set. It is clear from the definition of the join of $G$ and $H$ that $\langle S \rangle$ is connected. Therefore, $S$ is a connected co-independent hop dominating set of $G + H$. \qed
Corollary 1. Let $G$ and $H$ be any two graphs where $|V(G)| = n$ and $|V(H)| = m$. Then $\gamma_{\text{ch,coi}}(G + H) = \min\{n + \text{sci}(H), m + \text{sci}(G)\}$.

Proof: Let $S$ be a $\gamma_{\text{ch,coi}}$-set of $G + H$. Then $S$ is a connected co-independent hop dominating set of $G + H$. Hence, $S = A \cup B$ where $A \subseteq V(G)$ and $B \subseteq V(H)$ satisfying conditions (i) or (ii) of Theorem 3. Thus, $\gamma_{\text{ch,coi}}(G + H) = |S| = |A| + |B|$. By condition (i), $|S| = |V(G)| + |B| \geq n + \text{sci}(H)$. By condition (ii), $|S| = |V(H)| + |A| \geq m + \text{sci}(G)$. Hence, $\gamma_{\text{ch,coi}}(G + H) = |S| \geq \min\{n + \text{sci}(H), m + \text{sci}(G)\}$.

Next, let $X$ and $Y$ be the minimum strictly co-independent sets of $G$ and $H$, respectively. Then by Theorem 3, $S = V(G) \cup Y$ or $S = V(H) \cup X$ is a connected co-independent hop dominating set of $G + H$. Thus,

$$\gamma_{\text{ch,coi}}(G + H) \leq |S| = |V(G)| + |Y| = n + \text{sci}(H)$$

or

$$\gamma_{\text{ch,coi}}(G + H) \leq |S| = |V(H)| + |X| = m + \text{sci}(G)$$

It follows that, $\gamma_{\text{ch,coi}}(G + H) \leq \min\{n + \text{sci}(H), m + \text{sci}(G)\}$. Therefore, $\gamma_{\text{ch,coi}}(G + H) = \min\{n + \text{sci}(H), m + \text{sci}(G)\}$. \hfill \Box

4. On Connected Co-Independent Hop Domination in the Corona of Graphs

The corona of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining every vertex of the $i$th copy of $H$ to the $i$th vertex of $G$. For $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.

Theorem 4. Let $G$ be a nontrivial connected graph and $H$ be any graph. A set $S \subseteq V(G \circ H)$ is a connected co-independent hop dominating set of $G \circ H$ if and only if $S = V(G) \cup ( \bigcup_{v \in V(G)} S_v )$, where $S_v \subseteq V(H^v)$ and $V(H^v) \setminus S_v$ is an independent subset of $V(H^v)$ for each $v \in V(G)$.

Proof: Suppose $S$ is a connected co-independent hop dominating set of $G \circ H$ and let $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S_v \subseteq V(H^v)$. Since $\langle S \rangle$ is connected, $S = V(G) \cup ( \bigcup_{v \in V(G)} S_v )$. Since $V(G \circ H) \setminus S$ is independent and $V(G \circ H) \setminus S = \bigcup_{v \in V(G)} (V(H^v) \setminus S_v)$,
\( V(H^v) \setminus S_v \) is an independent subset of \( V(H^v) \), for each \( v \in V(G) \).

For the converse, suppose that \( S = V(G) \cup \bigcup_{v \in V(G)} S_v \) where \( S_v \subseteq V(H^v) \) and \( V(H^v) \setminus S_v \) is an independent set. Clearly, \( \langle S \rangle \) is connected. Let \( w \in V(G \circ H) \setminus S \). Then \( w \in V(H^v) \setminus S_v \) for some \( v \in V(G) \). Since \( G \) is nontrivial connected graph, there exists \( x \in V(G) \) such that \( vx \in E(G) \). Thus, \( d_{G \circ H}(w, x) = 2 \). This implies that \( S \) is a hop dominating set of \( G \circ H \). Since \( V(G \circ H) \setminus S = \bigcup_{v \in V(G)} (V(H^v) \setminus S_v) \) and \( V(H^v) \setminus S_v \) is an independent set for each \( v \in V(G) \), \( V(G \circ H) \setminus S \) is independent. Therefore, \( S \) is a connected co-independent hop dominating set of \( G \circ H \). \( \square \)

**Corollary 2.** Let \( G \) be a nontrivial connected graph of order \( n \) and \( H \) be any graph of order \( m \). Then \( \gamma_{\text{ch,coi}}(G \circ H) = n(1 + m - \beta(H)) \).

**Proof:** Let \( C \) be a \( \gamma_{\text{ch,coi}} \)-set of \( G \circ H \). Then \( C \) is a connected co-independent hop dominating set of \( G \circ H \). By Theorem 4, \( C = V(G) \cup \bigcup_{v \in V(G)} S_v \) where \( V(H^v) \setminus S_v \) is an independent set of \( H^v \) for every \( v \in V(G) \). Then

\[
\gamma_{\text{ch,coi}}(G \circ H) = |C| = |V(G)| + \left| \bigcup_{v \in V(G)} S_v \right|
= |V(G)| + \sum_{v \in V(G)} |S_v|
= |V(G)| + \sum_{v \in V(G)} (|V(H^v)| - |V(H^v) \setminus S_v|)
\geq |V(G)| + |V(G)|(|V(H^v)| - \beta(H))
= n + n(m - \beta(H))
= n(1 + m - \beta(H)).
\]

Therefore, \( \gamma_{\text{ch,coi}}(G \circ H) \geq n(1 + m - \beta(H)) \).

Let \( D \) be a maximum independent set of \( H \). For each \( v \), let \( D_v \subseteq V(H^v) \) such that \( \langle D_v \rangle \cong \langle D \rangle \). Let \( S_v = V(H^v) \setminus D_v \). Then \( C = V(G) \cup \bigcup_{v \in V(G)} S_v \) is a connected co-independent hop dominating set of \( G \circ H \) by Theorem 4. Thus,

\[
\gamma_{\text{ch,coi}}(G \circ H) \leq |C| = |V(G)| \cup \left( \bigcup_{v \in V(G)} S_v \right|
= |V(G)| + \sum_{v \in V(G)} |S_v|
= |V(G)| + |V(G)|(|V(H^v)| - |D_v|)
= |V(G)| + |V(G)|(|V(H^v)| - \beta(H))
= n + n(m - \beta(H))
= n(1 + m - \beta(H)).
\]
Therefore, $\gamma_{ch,co}(G \circ H) \leq n(1 + m - \beta(H))$.
Consequently, $\gamma_{ch,co}(G \circ H) = n(1 + m - \beta(H))$. \hfill \Box

5. On Connected Co-Independent Hop Domination in the Lexicographic Product of Graphs

The lexicographic product of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

**Theorem 5.** Let $G$ and $H$ be nontrivial connected graphs with $|V(G)| = n$. A subset $C = \bigcup \{(x) \times T_x\}$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ of $V(G[H])$ is a connected co-independent hop dominating set if and only if

(i) $S = V(G)$.
(ii) For every $x \in V(G)$ such that $T_x \neq V(H)$, $V(H) \setminus T_x$ is an independent set and $T_y = V(H)$ for every $y \in N_G(x)$ where $T_x$ is a hop dominating set of $H$ if $deg_G(x) = n - 1$.

**Proof:** Suppose $C$ is a connected co-independent hop dominating set of $G[H]$ and $S \neq V(G)$. Then, a vertex $v \in V(G) \setminus S$ exists. Thus, $(v, z) \in V(G[H]) \setminus C$ for all $z \in V(H)$. Since $H$ is a nontrivial connected graph, an edge $pq \in E(H)$ exists. Hence, $(v, p), (v, q) \in V(G[H]) \setminus C$ and $(v, p)(v, q) \in E(G[H])$. This contradicts the independence of $V(G[H]) \setminus C$. It follows that $S = V(G)$ and (i) holds. Now, let $x \in V(G)$ such that $T_x \neq V(H)$. We claim that $V(H) \setminus T_x$ is an independent set. Let $u, w \in V(H) \setminus T_x$ where $u \neq w$. Then $(x, u), (x, w) \in V(G[H]) \setminus C$. Since $V(G[H]) \setminus C$ is independent, $(x, u)(x, w) \notin E(G[H])$. Thus, $uw \notin E(H)$. Hence, $V(H) \setminus T_x$ is an independent set. Now, we show that $T_y = V(H)$ for every $y \in N_G(x)$. Suppose $T_y \neq V(H)$. Then there exists $p \in V(H) \setminus T_y$. Thus, $(y, p) \in V(G[H]) \setminus C$. Since $y \in N_G(x), (y, p)(x, q) \in E(G[H])$ for all $q \in V(H)$. This contradicts the independence of $V(G[H]) \setminus C$. Hence, $T_y = V(H)$. Lastly, suppose $deg_G(x) = n - 1$. Then $xa \in E(G)$ for all $a \in V(G) \setminus \{x\}$. Since $V(G) \neq T_x$, a vertex $b \in V(H) \setminus T_x$ exists. Thus, $(x, b) \in V(G[H]) \setminus C$. Since $C$ is a hop dominating set and $(x, b)(a, d) \in E(G[H])$ for all $a \in V(G) \setminus \{x\}$ and $d \in T_a$, there exists $z \in T_x$ such that $d_G[H]((x, b), (x, z)) = 2$. Hence, $d_H(b, z) = 2$, showing that $z \in T_x \setminus N_H(b)$. Hence, $N_H(b) \cap T_x \neq T_x$ implying that $T_x$ is strictly co-independent set of $H$. Thus, (ii) holds.

Conversely, suppose $C = \bigcup \{(x) \times T_x\}$ satisfies conditions (i) and (ii). First, we claim that $C$ is connected in $G[H]$. Let $(x, a)$ and $(y, b)$ be two distinct vertices in $C$, $(x, a)(y, b) \notin E(G[H])$. Consider the following cases.

**Case 1.** $x = y$

Since $(x, a) \neq (y, b)$ and $(x, a)(y, b) \notin E(G[H])$, $a \neq b$ and $ab \notin E(H)$. Since $G$ is a nontrivial connected graph and $S = V(G)$ by (i), there exists $z \in S \cap N_G(x)$. Thus, $(z, w) \in C$ for some $w \in V(H)$. It follows that $[(x, a), (z, w), (y, b)]$ is a path in $C$.

**Case 2.** $x \neq y$

Since $G$ is a nontrivial connected graph, there exists an $x$-$y$ path $[v_1, v_2, ..., v_n]$ where $x = v_1$ and $y = v_n$, $n > 2$. By (i) $v_i \in S$ for all $i \in \{1, 2, ..., n\}$. Let $u_i \in T_{v_i}$, $u_1 = a$ and
$u_n = b$. Then $[(x, a), (v_2, u_2), (v_3, u_3), \ldots, (y, b)]$ is an $(x, a)$-$(y, b)$ path in $C$.

Therefore in either case, $C$ is connected.

Now, let $(u, v), (w, p) \in V(G[H]) \setminus C$ where $(u, v) \neq (w, p)$. Consider the following cases.

**Case 1.** $u = w$

Since $(u, v), (w, p) \notin C$, $v, p \notin T_w$ and $v \neq p$. Hence, $v, p \in V(H) \setminus T_u$. Since

$V(H) \setminus T_u$ is independent by $(ii)$, $vp \notin E(H)$. Thus,

$(u, v)(w, p) \notin E(G[H])$.

**Case 2.** $u \neq w$

Since $v \notin T_u$ and $p \notin T_w$, $T_u \neq V(H)$ and $T_w \neq V(H)$. By $(ii)$, $u \notin N_G(w)$. Thus,

$(u, v)(w, p) \notin E(G[H])$.

Therefore, in any case $V(G[H]) \setminus C$ is an independent set.

Finally, we show that $C$ is a hop dominating set. Let $(x, y) \in V(G[H]) \setminus C$. Then $y \notin T_x$, that is, $V(H) \neq T_x$. Suppose $deg_G(x) = n - 1$. By $(ii)$, $T_x$ is a strictly co-independent set of $H$. Since $y \notin T_x$, $N_H(y) \cap T_x \neq T_x$. This implies that there exists $a \in T_x$ such that $d_H(a, y) = 2$. Hence, $(x, a) \in C$ and $d_{G[H]}((x, y), (x, a)) = 2$. Suppose $deg_G(x) < n - 1$. Then a vertex $z \in V(G) \setminus N_G(x)$ exists. Choose $z$ such that $d_G(x, z) = 2$. Since $S = V(G)$ by $(i)$, there exists $b \in T_z$, that is, $(z, b) \in C$. It follows that $d_{G[H]}((x, y), (z, b)) = 2$.

Therefore, $C$ is a hop dominating set of $G[H]$.

Accordingly, $C$ is a connected co-independent hop dominating set of $G[H]$. \hfill \Box

**Corollary 3.** Let $G$ be any connected noncomplete graph of order $m$ and $H$ be any nontrivial connected graph of order $n$. Then

$$
\gamma_{ch, coi}(G[H]) = m(n - \beta(H)) + r(G)\beta(H),
$$

where $r(G) = \min\{|D| : V(G) \setminus D \text{ is an independent set}\}$ and $\beta(H)$ is an independence number of $H$.

**Proof:** Let $r(G) = \min\{|D| : V(G) \setminus D \text{ is an independent set}\}$. Let $D_0 \subseteq V(G)$ such that $V(G) \setminus D_0$ is an independent set and $|D_0| = r(G)$. Let $T$ be a $\beta$-set of $H$. Let $T_x = V(H) \setminus T$ for each $x \in V(G) \setminus D_0$ and let $T_y = V(H)$ for each $y \in D_0$. Then

$$
C = \bigcup_{x \in V(G)} \{x \times T_x\} \cup \bigcup_{y \in D_0} \{y \times T_y\} \cup \bigcup_{x \in V(G) \setminus D_0} \{x \times T_x\}
$$

is a connected co-independent hop dominating set of $G[H]$, by Theorem 5. Hence,

$$
\gamma_{ch, coi}(G[H]) \leq |C| = nr(G) + (m - r(G))(n - \beta(H)) = nr(G) + mn - m\beta(H) - nr(G) + r(G)\beta(H) = mn - m\beta(H) + r(G)\beta(H) = m(n - \beta(H)) + r(G)\beta(H)
$$

$$
\gamma_{c, coi}(G[H]) \leq m(n - \beta(H)) + r(G)\beta(H)
$$
Let $C_0 = \bigcup_{x \in V(G)} \{ \{x \} \times R_x \}$ be a $\gamma_{ch,coi}$-set of $G[H]$. Let $D = \{ x \in V(G) : R_x = V(H) \}$. Then $V(G) \setminus D$ is an independent set of $G$. Then

$$C_0 = \left( \bigcup_{x \in D} R_x \right) \cup \left( \bigcup_{x \in V(G) \setminus D} T_x \right).$$

Moreover,

$$\gamma_{ch,coi}(G[H]) = |C_0| = n|D| + \sum_{x \in V(G) \setminus D} |T_x|.\]

Since $V(H) \setminus T_x$ is an independent set of $H$ for each $x \in V(G) \setminus D$, it follows that $|V(H) \setminus T_x| \leq \beta(H)$ for each $x \in V(G) \setminus D$. Thus, $|V(H) \setminus T_x| = |V(H)| - |T_x| \leq \beta(H).$ Hence $|T_x| \geq n - \beta(H)$.

Therefore,

$$\gamma_{ch,coi}(G[H]) = |C_0| = n|D| + |V(G) \setminus D||T_x| \leq n|D| + (m - |D|)(n - \beta(H)) = n|D| + mn - m\beta(H) - n|D| + |D|\beta(H) = m(n - \beta(H)) + |D|\beta(H) = m(n - \beta(H)) + r(G)\beta(H)$$

Therefore, $\gamma_{ch,coi}(G[H]) \geq m(n - \beta(H)) + r(G)\beta(H).$ 

**Corollary 4.** Let $H$ be any nontrivial connected graph of order $m$. Then $\gamma_{ch,coi}(K_n[H]) = m(n - 1) + sci(H)$.

**Proof:** Let $T$ be an $sci$-set of $H$. Let $v \in V(K_n)$ and $T_v = T$. By Theorem 5, $C = \bigcup_{y \in V(K_n) \setminus \{v\}} (\{y\} \times T_y) \cup (\{v\} \times T_v)$ is a connected co-independent hop dominating set of $K_n[H]$. Since $y \in N_{K_n}(v)$ for each $y \in V(K_n) \setminus \{v\}$, $T_y = V(H)$. Thus,

$$\gamma_{ch,coi}(K_n[H]) \leq |C| = (n - 1)|T_y| + |T_v| = (n - 1)m + |T| = (n - 1)m + sci(H).$$

Let $C_o = \bigcup_{x \in V(K_n)} (\{x\} \times R_x)$ be a $\gamma_{ch,coi}$-set of $K_n[H]$. Since $deg_{K_n}(x) = n - 1$ for each $x \in V(K_n)$, by Theorem 5, $R_y = T$ where $T$ is a strictly co-independent set of $H$ for a unique $y \in V(K_n)$ and $R_x = V(H)$ for all $x \in V(K_n) \setminus \{y\}$. Hence, $C_o = (\{y\} \times R_y) \cup \bigcup_{x \in V(K_n) \setminus \{y\}} (\{x\} \times R_x)$ and

$$\gamma_{ch,coi}(K_n[H]) = |C_o|.$$
Therefore, $\gamma_{ch,co} (K_n[H]) = m(n-1) + \text{sci}(H)$. \hfill $\square$

Acknowledgements

This research is funded by the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines.

References

[1] C. Berge. *Theorie des graphes et ses applications*. Metheun and Wiley, London and New York, 1962.

[2] R. Detalla and H. Rara. On Connected Co-independent Domination in the Join and Corona of Graphs. *Undergraduate Thesis*, 2019.

[3] B. Gayathri and S. Kaspar. Connected Co-Independent Domination of a Graph. *International Journal Contemp. Mathematics and Sciences*, 6:423–429, 2011.

[4] F. Harary. *Graph Theory*. Addison-Wesley Publishing Company, USA, 1969.

[5] W. Desormeaux, T. Haynes and M.A. Henning. A Note on Non-Dominating Set Partitions in Graphs. *Networks*, pages 1–8, 2016.

[6] I. Aniversario M. Bonsocan. On Connected Co-independent Domination of Some Graphs. *Undergraduate Thesis*, 2018.

[7] S. Canoy, R. Mollejon and J. Canoy. Hop Dominating Sets in Graphs Under Binary Operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[8] C. Natarajan and S. Ayyaswamy. Hop Domination in Graphs-II. *Versita*, 23(2):187–199, 2015.

[9] S. Ayyaswamy, C. Natarajan and G. Sathiamoorthy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics*, 119(12):11465–14171, 2018.

[10] Y. Pabilona and H. Rara. Some Variants of Hop Domination in Graphs. *Dissertation*, 2017.

[11] Y. Pabilona and H. Rara. Connected Hop Domination in Graphs under Some Binary Operations. *Asian-European Journal of Mathematics*, 2018.
[12] M. Perocho and H. Rara. On Connected Co-Independent Domination in Lexicographic Product of Graphs. *Undergraduate Thesis*, 2020.

[13] G. Salasalan and S. Canoy Jr. Some related concepts of hop domination in a graph. *Dissertation*, 2021.