Quantum instability for charged scalar particles on charged Nariai and ultracold black hole manifolds

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Received 15 September 2009, in final form 14 January 2010
Published 16 February 2010
Online at stacks.iop.org/CQG/27/055011

Abstract
We analyze in detail the quantum instability which characterizes charged scalar field on three special de Sitter charged black hole backgrounds. In particular, we compute exactly the imaginary part of the effective action for scalar charged fields on the ultracold I, ultracold II and Nariai charged black hole backgrounds. Both the transmission coefficient approach and the zeta-function approach are exploited. Thermal effects on this quantum instability are also taken into account in the presence of a non-zero black hole temperature (ultracold I and Nariai).

PACS numbers: 04.70.Dy, 04.70.−s, 03.65.Pm

1. Introduction

It is well known that quantum effects lead to the loss of charge by charged black holes [1–4], and that this phenomenon on one hand is independent of the fact that there can be a contextual evaporation phenomenon (i.e. extremal black holes, with zero temperature, are also involved in this spontaneous loss of charge), and on the other hand its being related to the Schwinger instability of vacuum in the presence of a constant electric field has been pointed out. The latter topic can be brought back to the more general class of phenomena of quantum instability in the presence of an external field, which, in the Schwinger calculation, still has the most relevant and important contribution [5, 6]. See also [7, 8].

In previous studies devoted to this topic in the case of black hole backgrounds, we mainly focused our attention on black holes of the Kerr–Newman family [9], also in the presence of a cosmological constant [10, 11]. In the latter case, we were able to perform an exact calculation for charged Dirac fields in four dimensions in three special cases [12]: the ultracold I, ultracold II and Nariai-charged black hole backgrounds. The calculations were performed both in the
so-called transmission coefficient approach and in the zeta-function approach, thus obtaining a double check for our calculations.

Herein, we complete our analysis performed in [12] by taking into account the case of charged scalar fields, and provide an exact calculation for their instability on the given black hole backgrounds. We recall that in similar calculations one is far from being able to reach exact results (e.g. in the Reissner–Nordström case only a WKB approximation is available). We also point out that, contrarily to what one could naively expect, the scalar field analysis presents to some extent more difficulties than the analysis for the Dirac field, because of some mathematical subtleties occurring in the scalar case: we limit ourselves to mentioning the (open) problem of a rigorous mathematical setting for the Klein–Gordon equation minimally coupled with an external electrostatic potential in the presence of event horizon(s), requiring an analysis involving the so-called Krein spaces in place of the more standard Hilbert spaces occurring in the analysis of the Dirac equation. Still, we can perform with some ingenuity zeta-function calculations and show that the imaginary part of the effective action coincides with the one calculated by means of the transmission coefficient approach. Another peculiar behavior emerges in the scalar field case when one takes into account the behavior of the field in the ultracold I case: a bad behavior at infinity occurs for the wavefunction, but an analysis in terms of fluxes allows us to determine the transmission coefficient. Moreover, in the Nariai case, the scalar nature of the particle is at the root of the possibility of obtaining a change of sign in a quantity $\Delta_1$ (cf equation (68)) due to the presence of a term $-\frac{1}{4}$ which is instead missing in the analogous quantity for the Dirac case (see [12]). This may cause a change in the behavior of the imaginary part of the effective action, as we shall see.

In the ultracold I and Nariai cases, which are involved with a non-zero background temperature, thermal effects on the quantum instability are also considered.

The plan of the paper is as follows. In section 2 we sum up some aspects of the transmission coefficient approach which are relevant for our paper, and then extend our analysis [12] concerning instability of the thermal state induced by the pair-creation effect to scalar fields. In sections 3–5 we take into account the cases of ultracold II, ultracold I and Nariai-charged black hole backgrounds respectively. In section 6 conclusions are drawn.

2. Vacuum instability and thermal state instability

We discuss in the following some aspects of the problem of vacuum instability and of thermal state instability induced by it. The Dirac case was discussed in [12].

2.1. Vacuum instability

For completeness, we summarize some aspects of the transmission coefficient approach in the case of scalar fields, following [3, 13, 14]. We are mainly interested in the probability of persistence of the vacuum. Let us introduce, for a diagonal scattering process [3],

$$n_i^{in} = R_i n_i^{out} + T_i p_i^{out},$$

where $n_i$ stands for a negative energy mode and $p_i$ for a positive energy one. $T_i$ is the transmission coefficient and $R_i$ is the reflection one. Moreover, as in [3], we define

$$\eta_i := |T_i|^2.$$  \hspace{1cm} (2)

Then, it is possible to show that for bosons one gets

$$|R_i|^2 = 1 + \eta_i,$$  \hspace{1cm} (3)

which accounts for the well-known super-radiance phenomenon.
The persistence of the vacuum is given by [3]
\[ P_0 = \prod_i p_{i,0} = e^{-2\text{Im}W}, \]  
where \( p_{i,0} \) is the probability of having a zero pair in the channel \( i \), and then
\[ 2\text{Im}W = \sum_i \log(1 + \eta_i) = \sum_{k=1}^{\infty} \frac{1}{k} \eta_i^k. \]

2.2. Thermal state instability

We have to take into account that, in the case of the ultracold I manifold and also in the Nariai-charged case, there exists an intrinsic thermality of the background manifold which is associated with the presence of non-degenerate horizons. As a consequence, the real quantum state to be considered is not the vacuum state in a traditional sense (i.e. absence of particles), to be also called the Boulware-like vacuum, but the thermal state associated with the aforementioned temperature (we recall that we are dealing with special manifolds endowed with a single temperature even if two different non-degenerate event horizons are involved). We stress that, both in the transmission coefficient approach and in the zeta-function one, calculations understand the Boulware-like vacuum, even in the thermal situation we have just mentioned. This kind of evaluation of the electrodynamics instability of charged black holes, which in principle would be correct only in the case of zero-temperature backgrounds, is common (cf e.g. [1, 2]), and can be justified e.g. by recalling that in the usual WKB evaluations very massive holes are involved, and, as a consequence, a negligible Hawking effect is expected to be superimposed to the quantum effect associated with the discharge phenomenon. Moreover, we can show that the aforementioned superimposition can be taken into account, when the consideration of the correct thermal state is restored, and that a major role is still deserved by the transmission coefficient computed in the transmission coefficient approach. The path is as follows. We first construct the thermal state, to be identified as the Hartle–Hawking state for our thermal geometries, and then we take into account the electrodynamic instability.

The thermal state (it could also be called the Gibbons–Hawking thermal state, due to the fundamental work [15] in de Sitter space) is constructed by adopting the same attitude as in [12]. The present choice of the aforementioned thermal state can be considered as the natural generalization of the so-called Bunch–Davies vacuum [16] (it is also known as the Euclidean vacuum [15, 17]; see also the discussion in [18]), which is the correct thermal state at the temperature of the cosmological horizon, and is also suitable for describing quantum fluctuations in an expanding universe and is invariant under a de Sitter group. In our case, which represents a more involved situation, the latter symmetry is broken by the presence of a charge.

We point out that the following construction holds true in general, even if we are interested in it for our specific analysis. We adopt the thermofield dynamics formalism, and define a thermal state \( |0(\beta)\rangle \) characterized by an inverse temperature \( \beta \). This state is annihilated by suitable operators \( a_l(\beta), \tilde{a}_l(\beta), b_l(\beta), \tilde{b}_l(\beta) \) (and conjugated ones) which are labeled by a complete set of quantum numbers \( l \) and are related to ‘standard’ annihilation-creation operators \( a_l, \tilde{a}_l, b_l, \tilde{b}_l \) (and conjugated ones) via a formally unitary transformation:
\[ a_l = c^+_l(\beta)a_l(\beta) + s^+_l(\tilde{a}_l)^\dagger(\beta), \]
\[ b_l = c^-_l(\beta)b_l(\beta) + s^-_l(\tilde{b}_l)^\dagger(\beta), \]  

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and analogs for Hermitian conjugates, with
\[ s_i^+ = \frac{1}{\sqrt{e^{\beta(\omega_i - \phi)} - 1}} \]
\[ s_i^- = \frac{1}{\sqrt{e^{\beta(\omega_i + \phi)} - 1}} \]

where \( \phi^+ \) and \( \phi^- \) stand for chemical potentials for particles and antiparticles respectively.

Moreover, it holds
\[ (c_i^+)^2 - (s_i^+)^2 = 1 \]
\[ (c_i^-)^2 - (s_i^-)^2 = 1 \]

Thermofield dynamics, as discussed in [12], allows us to find out a straightforward generalization of quantum instability to the case where the initial (in) and the final (out) states are thermal states (at the same temperature) instead of vacuum ones. In order to check how thermal effects affect instability of quantum fields, we consider, in place of the usual \( \langle 0_{\text{in}} | (a_{\text{out}}^i)^\dagger a_{\text{out}}^i | 0_{\text{in}} \rangle \), which gives the number of out-particles in the in-vacuum, the following quantity:
\[ N_{\text{out}}^+ := \langle 0_{\beta_{\text{in}}}^\beta | (a_{\text{out}}^i)^\dagger a_{\text{out}}^i | 0_{\beta_{\text{in}}}^\beta \rangle \]

and check if deviations from pure thermality appears in the distribution. Equivalently, as in [19], we can define
\[ \bar{N}_{\text{out}}^+ := \langle 0_{\beta_{\text{in}}}^\beta | (a_{\text{out}}^i)^\dagger a_{\text{out}}^i - (a_{\text{in}}^i)^\dagger a_{\text{in}}^i | 0_{\beta_{\text{in}}}^\beta \rangle \]

which just shows us the deviation part (or it is zero). We also introduce standard Bogoliubov relations between ‘in’ and ‘out’ operators as follows:
\[ a_{\text{out}}^i = \rho_\ell a_{\text{in}}^i + T_{\ell\ell}^\ast (b_{\text{in}}^\dagger) \]
\[ b_{\text{out}}^i = \rho_\ell b_{\text{in}}^i + T_{\ell\ell}^\ast (a_{\text{in}}^\dagger) \]

where \( |\rho_\ell|^2 - |T_{\ell\ell}|^2 = 1 \). Compare also [19]. Note that we limit ourselves to considering diagonal transformations, as it is the interesting case for our considerations.

It is easily shown that
\[ \bar{N}_{\text{out}}^+ = |T_{\ell\ell}|^2 \left[ (s_i^+)^2 + (s_i^-)^2 + 1 \right] \]

where \( |T_{\ell\ell}|^2 \) is the transmission coefficient. When \( \phi^+ = \phi^- = \phi \), as in the case of our black hole background, we obtain
\[ \bar{N}_{\text{out}}^+ = |T_{\ell\ell}|^2 \frac{1}{2} \left[ \coth \left( \frac{\beta(|\omega| - \phi)}{2} \right) + \coth \left( \frac{\beta(|\omega| + \phi)}{2} \right) \right] \]

which is easily realized to coincide with the result displayed for the boson case in [19] when \( \phi = 0 \), and matches the results in [12] for the Dirac case. Note that (17) can also be used for the Reissner–Nordström case, where the coefficient \( |T_{\ell\ell}|^2 \) is known only in the WKB approximation [1, 2].

3. Ultracold II case

The ultracold II metric is obtained from the Reissner–Nordström–de Sitter one in the limit of coincidence of the Cauchy horizon, of the black hole event horizon and of the cosmological
event horizon. See [20, 21]. In particular, the metric we are interested in is
\[ ds^2 = -dt^2 + dy^2 + \frac{1}{2\Lambda} (d\theta^2 + \sin^2(\theta) \, d\phi^2), \]
(18)
with \( y \in \mathbb{R} \) and \( t \in \mathbb{R} \). The electromagnetic field strength is \( F = -\sqrt{\Lambda} dt \wedge dy \), and we can choose \( A_0 = \sqrt{\Lambda} y \) and \( A_j = 0 \), \( j = 1, 2, 3 \). It is also useful to define \( E := \sqrt{\Lambda} \), which represents the modulus of the electrostatic field on the given manifold. We note that it is uniform, and then one expects naively to retrieve at least some features of Schwinger’s result, as in the Dirac case [12].

3.1. The transmission coefficient approach

Let us consider the Klein–Gordon equation in the given manifold
\[ \left[ -\left( -i\partial_t + eEy \right)^2 - \partial_y^2 - 2\Lambda \nabla_r^2 + \mu^2 \right] \phi = 0, \]
(19)
where \( \mu \) and \( e \) are the mass and the charge of the scalar particle. We assume \( eE > 0 \) for definiteness. In agreement with the possibility of performing variable separation, let us set
\[ \phi(t, y, \Omega) = e^{-i\omega t} Y_{lm}(\Omega) \psi(y); \]
(20)
then one obtains the following equation for \( \psi \):
\[ \frac{d^2 \psi}{dy^2}(y) = \left( \mu_l^2 - (\omega + eEy)^2 \right) \psi(y), \]
(21)
where \( \mu_l^2 = 2\Lambda(l + 1) + \mu^2 \). By defining (cf [3])
\[ \xi = \frac{1}{\sqrt{eE}}(\omega + eEy), \quad \lambda = \frac{eE \mu_l^2}{\mu}, \]
\[ k = \frac{1}{2} - i\frac{\lambda}{2}, \quad u = \sqrt{2} e^{-i\xi}, \]
(22)
one obtains
\[ \frac{d^2 \psi}{d\xi^2}(\xi) = (\lambda - \xi^2) \psi(\xi), \]
(23)
whose solution is
\[ \psi = D_u(u), \]
(24)
which is a parabolic cylinder function. The calculation is completely analogous to the one performed in [3], and as in [3] one can easily show that the transmission coefficient satisfies
\[ |T_l|^2 = e^{-\pi \lambda} = e^{-\frac{\mu_l^2}{2\pi}}. \]
(25)
The latter expression coincides with the WKB approximation for the same coefficient [11] (that calculation is for Dirac particles, but it is easy to realize that for scalar particles the result is the same, apart for the obvious replacement \( k^2 \mapsto l(l + 1) \)). This means that the WKB approximation is actually exact for the given case. We have the exact transmission coefficient. As in [2, 3] we can determine the degeneracy factor, and one obtains
\[ W = \frac{eES}{2\pi} \sum_{l=0}^{\infty} (2l + 1) \log \left( 1 + e^{-\frac{\mu_l^2}{2\pi}} \right), \]
(26)
where \( S \) is the spacetime volume of the \((t, y)\) part of the manifold.
3.2. The zeta-function approach

We can use the zeta-function regularization to compute the effective action. The spectral zeta function for the Euclidean Klein–Gordon equation is given by

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} K(z) \, dz, \]  

(27)

with kernel

\[ K(z) = \text{Tr} e^{-\left(-i\tau + iE y^2 - \nabla^2 + \mu^2\right) \frac{1}{2}} z, \]  

(28)

where \( \tau \) stands for the Euclidean time and \( \gamma \) turns out to be a renormalization scale [22].

To compute the trace, note that the operator \(-2\Lambda \nabla^2 + \mu^2\) commutes with the Klein–Gordon operator so that it contributes with the eigenvalues \( \mu^2_l = 2\Lambda(l + 1) + \mu^2 \) with degeneration \((2l + 1)\). Next, noting that the operator \( \hat{p} = -i\partial_\tau \) commutes with \( \hat{A} = -(\partial_t + iE y)^2 - \partial_y^2 \) which describes a harmonic oscillator with eigenvalues \( eE(2n + 1) \). Independence on \( \omega \) shows that such eigenvalues are degenerate so that if \( D \) is the degeneration we can write

\[ K(z) = \sum_{l=0}^\infty \sum_{n=0}^\infty l(l+1)D e^{-\left(eE(2n+1) + \mu^2_l\right) \frac{1}{2}} z \]  

(29)

We can determine the degeneration factor as done in [22]. We then obtain \( D = \frac{e^E}{\pi} S \), where \( S \) is the spacetime volume of the \((t, y)\) part of the manifold.

The Euclidean action is

\[ S_E = -\zeta'(0), \]

and one finds

\[ W = \text{Im} S_L, \]

where \( S_L \) is the Lorentzian action, in our case obtained by \( E \to iE \). Explicitly

\[ \zeta(s) = \sum_{l=0}^\infty \frac{eE}{2\pi} S(2l + 1) \left( \frac{\gamma^2}{2eE} \right)^s \zeta_H \left( s; \frac{\mu^2_l}{2eE} + \frac{1}{2} \right) \]  

(30)

where \( \zeta(s; a) \) is the Hurwitz zeta function. We also put

\[ W = \sum_{l=0}^\infty W_l. \]  

(31)

After Lorentzian continuation we find

\[ W_l = \frac{eE S}{2\pi} (2l + 1) \text{Re} \left[ \left( \frac{\mu^2_l}{2eE} \right) \log \left( 2ieE \frac{1}{\gamma^2} \right) + \frac{1}{2} \log 2\pi - \zeta_H \left( 0; -i \frac{\mu^2_l}{2eE} + \frac{1}{2} \right) \right]. \]

We point out that the renormalization scale \( \gamma \) does not affect \( W \). One can note that

\[ \text{Re} \left[ \left( \frac{\mu^2_l}{2eE} \right) \frac{\pi}{2} + \frac{1}{2} \log 2\pi - \log \Gamma \left( \frac{1}{2} - i \frac{\mu^2_l}{2eE} \right) \right] = \frac{1}{2} \log (1 + e^{-\pi^2 \frac{\mu^2_l}{2eE}}). \]

Thus, the final expression for the imaginary part of the Lorentzian action is

\[ W_l = \frac{eE S}{2\pi} (2l + 1) \frac{1}{2} \log \left( 1 + e^{-\pi^2 \frac{\mu^2_l}{2eE}} \right), \]  

(32)

which coincides with (26).
4. The ultracold I case

A second extremal limit of the Nariai background is given by the type I ultracold solution when \( r_- = r_+ = r_c \). The metric is \([21]\)

\[
d s^2 = - \chi^2 d\psi^2 + d\chi^2 + \frac{1}{2\Lambda}(d\theta^2 + \sin^2(\theta) d\phi^2),
\]
with \( \chi \in (0, \infty) \) and \( \psi \in \mathbb{R} \), and the electromagnetic field strength is \( F = \sqrt{\Lambda} \chi d\chi \wedge d\psi \). The spacetime presents the structure of a 2D Rindler manifold times a two-dimensional sphere (with a constant warping factor). One gets \( \Gamma_{01}^0 = \frac{1}{\chi}, \Gamma_{10}^1 = \chi, \Gamma_{13}^2 = - \sin(\theta) \cos(\theta), \Gamma_{23}^3 = \cot(\theta) \).

We can choose \( A_0 = \sqrt{\Lambda/\chi^2} \) and \( A_j = 0, j = 1, 2, 3 \), as potential. This case is a little bit more tricky than the ultracold II and we will adopt a different strategy to define the transmission and reflection coefficients. However, we will again be able to compare this approach with the zeta-function method and the two results are the same. The expression of the effective action for a scalar field in this background fits our previous result for the Dirac case \([12]\).

4.1. The transmission coefficient approach

In order to compute the wavefunctions for a scalar field in this background we search the solution of the Klein–Gordon equation, for the variable \( \chi \), that for this metric is

\[
- \frac{1}{\chi^2} \left( \omega + eE \frac{\chi^2}{2} \right)^2 - \frac{1}{\chi} \partial_\chi (\chi \partial_\chi) + \mu^2 F(t) = 0,
\]
where to perform variable separation we pose \( \phi(\tau, \chi, \Omega) = e^{-i\omega \tau} Y_{lm}(\Omega)\Phi(\chi) \). Making the change of variable \( t = \chi^2 \):

\[
(t \partial_t)^2 + \frac{1}{4} \left( \omega + eEt \right)^2 - \frac{1}{2} \mu^2 F(t) = 0.
\]

Next, we set, as usual, the wavefunction in the factorized form:

\[
\Psi(\chi(t)) = e^{i\omega t} t^{1/2} F(t),
\]
so we obtain the following confluent hypergeometric equation or Kummer’s equation:

\[
t^2 \partial_t^2 F + (1 + i\omega + i eEt) \partial_t F - \frac{1}{2} (\mu^2 - i eE) F = 0,
\]
whose general solution is

\[
F(t) = \alpha \Phi \left( \frac{1}{2} - \frac{\mu^2}{2ieE}; 1 + i\omega; -ieEt \right) + \beta t^{-i\omega} \Phi \left( \frac{1}{2} - \frac{\mu^2}{2ieE} - i\omega; 1 - i\omega; -ieEt \right),
\]
where \( \Phi(a; c; z) \) is the usual Kummer function (or first kind confluent hypergeometric function). Then, the general solution is

\[
\Psi(\chi(t)) = e^{i\omega t} \alpha t^{1/2} \Phi \left( \frac{1}{2} - \frac{\mu^2}{2ieE}; 1 + i\omega; -ieEt \right) + \beta t^{-i\omega} \Phi \left( \frac{1}{2} - \frac{\mu^2}{2ieE} - i\omega; 1 - i\omega; -ieEt \right).
\]

The asymptotic behavior of the wavefunction can be determined by using for \(|z| \approx \infty \)

\[
\Phi(a; c; z) \approx \frac{\Gamma(c)}{\Gamma(c - a)} (-z)^{-a}(1 + O(1/z)) + \frac{\Gamma(c)}{\Gamma(a)} e^{z^{-a}}(1 + O(1/z)).
\]

In the ultracold I background the electric field vanishes in \( t = 0 \) and it grows indefinitely for \( t \approx \infty \). Thus, for large \( t \) a charged particle is subjected to an increasing force and
it is accelerated toward infinity. For this reason the particle, for \( t \approx \infty \), does not behave as a free particle. The asymptotic behavior of the Kummer function reflects these physical considerations: the presence in the asymptotic expansion of the wavefunctions of terms proportional to \( r^{-1/2} \) shows that the behavior of the particle is far from that of a free particle. Thus, one cannot define the transmission or reflection coefficients in the usual way. However, we can define them using a slightly different strategy. First we compute the Klein–Gordon conserved current. Then, in the region \( t \approx 0 \) one expects to find a flux of matter coming from \( t \approx 0 \) and also a reflected one from large \( t \). Instead, in the region \( t \approx \infty \), due to the electric field, one expects just the presence of a transmitted flux, the one started from a small \( t \) region, and no reflected one. These considerations allow us, in a quite straightforward way, to define the transmission coefficient as the ratio between the transmitted flux at \( t \approx \infty \) and the incoming flux at \( t \approx 0 \) and the reflection coefficient as the ratio between the reflected flux and the incoming flux at \( t \approx 0 \). Let \( v = \frac{\dot{x}}{\dot{t}} \) be the 4-velocity of the static coordinate observer, \( j \) be the conserved current and \( d\Sigma^{\mu\nu} \) be the surface element through which we would compute the flux. Then, the associated infinitesimal flux is \( \Phi_{\Sigma}(j) = \frac{2\pi}{\sqrt{\gamma}} e^{\iota \Theta} \int \gamma^{\nu} d\Sigma^{\mu\nu} \). As we are interested in the flux in the \( x \) direction (\( \chi = e^{2x} \)), \( d\Sigma^{\mu\nu} = \left[(\delta_\mu^\nu \delta_\rho^\sigma - \delta_\rho^\nu \delta_\mu^\sigma)/2\right] d^2b \), where \( d^2b \) is an infinitesimal surface element, and then

\[
\Phi_{\Sigma}(j) = \sqrt{\gamma} j^x d^2b = \frac{1}{2\Lambda} j_x d^2b.
\]

Thus, dropping the unessential factor \( 1/2\Lambda \), we can define the transmission and reflection coefficients by looking at the covariant current only.

The covariant components of the Klein–Gordon conserved current are \( j_\mu = -\frac{i}{2} [\Psi^* D_\mu \Psi - (D_\mu \Psi^*) \Psi] \). As we will compute it for the two asymptotic regions \( t \approx 0 \) and \( t \approx \infty \), we need the expansion of the wavefunctions for small and large \( t \). For \( t \approx 0 \) we obtain

\[
\Psi(\chi(t)) \approx \alpha e^{\iota \epsilon E t} t^{\frac{1}{2} \omega} + \beta e^{\iota \epsilon E t} t^{-\frac{1}{2} \omega},
\]

making the change of variable \( t = \frac{1}{2} e^{2x} \),

\[
\Psi(\chi) \approx \alpha e^{\iota \epsilon E} e^{\iota \omega t_{\text{aux}}} + b e^{\iota \epsilon E} e^{\iota \omega t_{\text{aux}}},
\]

and restoring the time dependence \( \psi \):

\[
\Psi(\chi) \approx \alpha e^{\iota \epsilon E} e^{\iota \omega t_{\text{aux}} - \iota \omega \psi} + b e^{\iota \epsilon E} e^{\iota \omega t_{\text{aux}} - \iota \omega \psi},
\]

with \( a = \alpha (\frac{1}{2})^{\frac{1}{2} \omega} \) and \( b = \beta (\frac{1}{2})^{-\frac{1}{2} \omega} \). Finally we obtain the following expression for the \( x \)-component of the conserved current:

\[
j_x \approx \omega (|a|^2 - |b|^2).
\]

For \( t \approx \infty \), using the expansion of the Kummer function and making the change of variable as before, the asymptotic behavior of \( \Psi(\chi(t)) \) is

\[
\Psi(\chi(t)) \approx c_1 e^{\iota (2k_1 x + \frac{\omega}{2} e^{2x} - x)} + c_2 e^{-\iota (2k_1 x + \frac{\omega}{2} e^{2x} - x)}
\approx c_1 e^{\iota \frac{\omega}{2} e^{2x} - x} + c_2 e^{-\iota \frac{\omega}{2} e^{2x} - x},
\]

with \( k_1 = \frac{\omega}{2} - \frac{\mu_1^2}{2E} \), \( k_2 = \frac{\epsilon E}{2} \) and

\[
c_1 = \alpha \frac{\Gamma(1 + \iota \omega)}{\Gamma(\frac{1}{2} + \iota (\omega - \frac{\mu_1^2}{2E}))} (\iota \epsilon E)^{-\frac{1}{2} - \frac{\mu_1^2}{2E}} 2^{-\frac{1}{2} - i \frac{\omega}{2E} + \frac{1}{2}}
+ \beta \frac{\Gamma(1 - \iota \omega)}{\Gamma(\frac{1}{2} - \iota (\omega + \frac{\mu_1^2}{2E}))} (\iota \epsilon E)^{-\frac{1}{2} + \frac{\mu_1^2}{2E}} 2^{\frac{1}{2} - i \frac{\omega}{2E} - \frac{1}{2}}
\]
Finally, for the coefficients $|T_l|^2$ and $|R_l|^2$ we obtain

$$|T_l|^2 = -\frac{eE}{2} \left| c_1 \right|^2 \quad |R_l|^2 = \frac{|b|^2}{|a|^2}.$$  

As explained before, to avoid particles coming from $t \approx \infty$, we impose the condition $c_2 = 0$ and we obtain

$$\beta = -\frac{\alpha}{\Gamma(1 - i\omega)} \frac{\Gamma(\frac{1}{2} + i(\omega - \frac{\mu_i}{2eE})) (1 + i\omega)^{-i\omega}}{(1 - i\omega)^{-i\omega}}.$$  

To obtain the transmission coefficient we have to compute $|c_1|^2$:

$$|c_1|^2 = \alpha^2 \frac{2eE}{\pi^2} e^{\frac{\mu_i}{2eE}} \left[ \frac{\Gamma(1 + i\omega)}{\Gamma(\frac{1}{2} + i(\omega - \frac{\mu_i}{2eE}))} - \frac{\Gamma(1 + i\omega)}{\Gamma(\frac{1}{2} + i(\omega - \frac{\mu_i}{2eE}))} e^{-\pi\omega} \right].$$

Finally, for the coefficients $|T_l|^2$ and $|R_l|^2$ we obtain

$$|T_l|^2 = -\frac{eE}{2} \sqrt{\left| c_1 \right|^2} \quad |R_l|^2 = e^{-\pi\omega} \frac{\cos(\pi(\omega - \frac{\mu_i}{2eE}))}{\sin(\pi\omega) \cos(\pi(\omega - \frac{\mu_i}{2eE}))}.$$  

Observe that $|R_l|^2 - |T_l|^2 = 1$, as expected for bosons.

As in the Dirac case, the level-crossing region, assuming $eE > 0$, is determined by $\omega < 0$.  

Pair production is expected to happen only in this region; thus, for $eE > 0$, we must calculate (cf 31)

$$W_i = \frac{1}{2} \sum_\omega \log(1 + |T(\omega)|^2),$$

for $\omega < 0$. We have $\sum_\omega \rightarrow \frac{T}{2\pi} \int d\omega$ (cf [23, 24]), where $T$ stands for a finite-time interval. An easy computation shows that

$$\log(1 + |T|^2) = \log(|R|^2) = \log(1 + e^{-\pi\frac{\mu_i}{2eE}}) - \log(1 + e^{2\pi\omega - \frac{\mu_i}{2eE}}),$$

and we have to evaluate the integral

$$\int_{-\infty}^{0} d\omega \log \left( 1 + e^{2\pi\omega - \frac{\mu_i}{2eE}} \right) = \frac{1}{\pi} \text{Li}_2 \left( -e^{-\frac{\mu_i}{2eE}} \right).$$

In strict analogy with [12] we obtain

$$W_i = \frac{1}{2} \frac{T}{2\pi} \left[ \left( \int_{-\infty}^{0} d\omega \right) \log \left( 1 + e^{-\frac{\mu_i}{2eE}} \right) + \frac{1}{2\pi} \text{Li}_2 \left( -e^{-\frac{\mu_i}{2eE}} \right) \right].$$  

(42)
The factor \( T \frac{2}{\pi} \int_{-\infty}^{0} d\omega \) amounts to a degeneracy factor and the same geometric considerations done in [12] allow us to evaluate it following [23]. The degeneracy factor for the scalar case is the same as in the Dirac case and its value is \( T \frac{2}{\pi} \int_{-\infty}^{0} d\omega = e E S / 2\pi \), with \( S = T L \) where \( T \) and \( L \) are the sizes of the space-time box over which \( E \) is non-vanishing. This value is exactly the same as the one obtained in the zeta-function approach. The final result (42) for the imaginary part of the effective action coincides with result (62) that we will find using the zeta-function approach. It is worth mentioning that the above background implements the physical model analyzed in [24], apart from the fact that in [24] one deals with a 2D model and a further parameter \( a \) appears (which in our case is equal to 1). The fact that all our geometries allow a Kaluza–Klein reduction (compare the discussion in [12]) explains why a correspondence with a 2D model is found: the only substantial difference is represented in our case by the presence of an effective mass which is given by \( \mu^2_1 = \mu^2 + 2\Lambda l(l + 1) \) replacing the mass \( \mu^2 \) of the aforementioned 2D model.

4.2. Zeta-function approach

Also, for this background we analyze pair-production with the zeta-function method. This technique confirms the results obtained with the transmission coefficients approach. The Euclidean Klein–Gordon (KG) operator on ultracold I is

\[
KG = -\frac{1}{\chi^2} \partial^2_t + \frac{1}{\chi} \partial_{\chi}(\chi \partial_{\chi}) - 2\Lambda \nabla^2_{\Omega} + \mu^2 + 2i e E \frac{1}{2} \partial_{t} + (e E)^2 \frac{1}{4} \chi^2.
\]  (43)

In the eigenvalue equation \( KG \phi = \lambda \phi \) we put

\[
\phi = e^{-i \omega t} Y_{lm}(\Omega) \psi(\chi),
\]  (44)

which leads to variable separation, where \( Y_{lm}(\Omega) \) are the usual spherical harmonics appearing in every problem with spherical symmetry. Then we obtain

\[
-\frac{1}{\chi} \partial_{\chi}(\chi \partial_{\chi} \psi) + \left[ \mu^2_1 + \frac{1}{\chi^2} \left( \omega + e E \frac{1}{2} \chi^2 \right)^2 \right] \psi = \lambda \psi.
\]  (45)

We also introduce \( t = \frac{1}{2} \chi^2 \), and then we obtain

\[
(t \partial_t)^2 \psi + \left[ \frac{1}{2} (\lambda - \mu^2_1) t - \frac{1}{4} (\omega + e Et)^2 \right] \psi = 0.
\]  (46)

By choosing

\[
\psi = e^{-\frac{i}{4} e Et} t^{-\frac{i}{4} \omega} g(t),
\]  (47)

and introducing \( z = e Et \), we obtain the confluent hypergeometric equation

\[
z \partial_z^2 g + (1 - \omega - z) \partial_z g - \frac{1}{2} \left( 1 - \frac{\lambda - \mu^2_1}{e E} \right) g = 0.
\]  (48)

We require that solutions \( \psi \) belong to \( L^2(0, \infty), \frac{d}{z} \) (the measure is inherited from that of the usual scalar product for scalar particles). It is easy to realize that this requires the consideration of different solutions for \( \omega < 0 \) and for \( \omega > 0 \). Let us first consider

\[
g(t) = \Phi \left( \frac{1}{2} \left( 1 - \frac{\lambda - \mu^2_1}{e E} \right) , 1 - \omega, e Et \right).
\]  (49)

We need the quantization condition

\[
\frac{1}{2} \left( 1 - \frac{\lambda - \mu^2_1}{e E} \right) = -n.
\]  (50)
with \( n \in \mathbb{N} \), and then
\[
\lambda_{n,l} = (2n + 1)eE + \mu_l^2. \tag{51}
\]
We have \( \psi \in L^2((0, \infty), \frac{dz}{z}) \) iff \( \omega < 0 \).

The solution
\[
g(t) = t^\omega \Phi \left( \frac{1}{2} \left( 1 - \frac{\lambda - \mu_l^2}{eE} \right) + \omega, 1 + \omega, eEt \right) \tag{52}
\]
requires a further quantization condition
\[
\frac{1}{2} \left( 1 - \frac{\lambda - \mu_l^2}{eE} \right) + \omega = -n, \tag{53}
\]
and then
\[
\lambda_{n,l,\omega} = (2n + 1)eE + 2eE\omega + \mu_l^2, \tag{54}
\]
and we can conclude that \( \psi \in L^2((0, \infty), dz) \) iff \( \omega > 0 \).

We obtain that the heat kernel
\[
K(s) = \sum_l (2l + 1)k_l(s) \tag{55}
\]
receives different contributions from different ranges for \( \omega \). In particular, for \( \omega < 0 \) there is, as in the ultracold II case, a degeneracy in \( \omega \) to be determined, being \( \lambda_{n,l} \) independent of \( \omega \) in that region. We get
\[
k_l(s) = De^{-\mu_l^2\gamma^2}e^{-eE\gamma^2} \frac{1}{1 - e^{-2eE\gamma^2}}, \tag{56}
\]
where \( \gamma \) is a renormalization scale (which will not affect the imaginary part of the effective action), and where formally
\[
D = \int_{-\infty}^0 d\omega. \tag{57}
\]
We determine \( D \) as in \([22]\), by comparing the expansion of \( k_l(s) \) as \( s \to 0^+ \) with the heat kernel expansion. We obtain
\[
D = \frac{eES}{2\pi}, \tag{58}
\]
where \( S \) is the volume of the first 2D factor of the metric.

We obtain
\[
\zeta_l(s) = \frac{eES}{2\pi} \left( \frac{2eE}{\gamma^2} \right)^{-s} \zeta_H \left( \frac{1}{2} \left( 1 + \frac{\mu_l^2}{eE} \right); s \right) + \frac{T}{2\pi} \left( \frac{2eE}{\gamma^2} \right)^{-s} \frac{1}{s - 1} \zeta_H \left( \frac{1}{2} \left( 1 + \frac{\mu_l^2}{eE} \right); s - 1 \right). \tag{59}
\]
By rotating \( eE \mapsto ieE \) and looking for \( \text{Im}\zeta'(0) \), we obtain a first contribution from the \( \omega < 0 \) region which is easily realized to be the same as in the ultracold II case and a further contribution from the \( \omega > 0 \) region which is given by
\[
\frac{\pi}{2} \text{Re}\zeta_H \left( \frac{1}{2} \left( 1 + \frac{\mu_l^2}{ieE} \right); -1 \right) + \left( \log \left( \frac{2eE}{\gamma^2} \right) - 1 \right) \text{Im}\zeta_H \times \left( \frac{1}{2} \left( 1 + \frac{\mu_l^2}{ieE} \right); -1 \right) - \text{Im}\zeta_H' \left( \frac{1}{2} \left( 1 + \frac{\mu_l^2}{ieE} \right); -1 \right); \tag{59a}
\]
as to the first term we get
\[
\frac{\pi}{2} \text{Re} \zeta_H \left( \frac{1}{2} \left( 1 + \frac{\mu^2}{ieE} \right), -1 \right) = \frac{1}{8\pi} \left[ -\text{Li}_2(-e^{-\pi \mu l e E}) - \text{Li}_2(-e^{\pi \mu l e E}) \right];
\]  
(60)
the second one is zero, whereas the third one is
\[
\text{Im} \zeta'_H \left( \frac{1}{2} \left( 1 + \frac{\mu^2}{ieE} \right), -1 \right) = \frac{1}{8\pi} \left[ \text{Li}_2(-e^{-\pi \mu l e E}) - \text{Li}_2(-e^{\pi \mu l e E}) \right].
\]  
(61)
As a consequence, we obtain
\[
\text{Im} \zeta'_l (0) = -\frac{eES}{2\pi} \frac{1}{2} \log \left( 1 + e^{-\pi \mu l e E} \right) - \frac{T}{2\pi} \frac{1}{4\pi} \text{Li}_2(-e^{-\pi \mu l e E})
\]  
(62)
which leads to a full accord with (42).

As to thermal effects, in this case we limit ourselves to pointing out that equation (17) holds, but with a pathological behavior associated with the fact that the chemical potential \( \phi \) is ill-defined unless a spatial cut-off is introduced at \( \chi = \chi_0 < \infty \). The same phenomenon affects Dirac particles [12].

5. Nariai case
We now consider the more general case, that is the electrically charged Nariai solution. The manifold is described by the metric [20, 21, 25]
\[
ds^2 = \frac{1}{A} (-\sin^2(\chi) d\psi^2 + d\chi^2) + \frac{1}{B} (d\theta^2 + \sin^2(\theta) d\phi^2),
\]  
(63)
with \( \psi \in \mathbb{R}, \chi \in (0, \pi) \), and the constants \( B = \frac{1}{2Q} \left( 1 - \sqrt{1 - 12 \frac{Q^2}{L^2}} \right), A = \frac{6}{L^2} - B \) are such that \( \frac{A}{B} < 1 \), and \( L^2 := \frac{3}{4} \). The black hole horizon occurs at \( \chi = \pi \). This manifold differs from the ultracold cases because it has finite spatial section. In the Euclidean version, it corresponds to two spheres characterized by different radii. One finds the following non-vanishing Christoffel symbols \( \Gamma^0_{01} = \cot(\chi), \Gamma^1_{00} = \sin(\chi) \cos(\chi), \Gamma^2_{33} = -\sin(\theta) \cos(\theta), \Gamma^3_{23} = \cot(\theta) \). For the gauge potential we can choose \( A_i = -Q B \frac{B}{A} \cos(\chi) \delta^0_i \). Also for this more complex case we study pair-production making use of the transmission coefficients and zeta-function approach. Again these two different methods give the same results. For the Nariai case the zeta-function approach requires some mathematical techniques recently developed in [26] and their application is strictly analogous to the Dirac case, exhaustively analyzed in [12].

5.1. Transmission coefficient approach
We perform variable separation and set \( \phi(\psi, \chi, \Omega) = e^{-i\omega \psi} Y_{lm}(\Omega) \Psi(\chi) \); moreover, we define \( \mu^2 l = \frac{\mu^2}{A} + \frac{B}{A} (l + 1) \). We need to find the solution of the Klein–Gordon equation for the variable \( \chi \):
\[
\left[ -\frac{\sin^2(\chi)}{A} \left( \omega - eQ \frac{B}{A} \cos(\chi) \right)^2 - \frac{1}{\sin(\chi)} \frac{\partial}{\partial \chi}(\sin(\chi) \partial(\chi) + \mu^2 l) \right] \Psi(\chi) = 0.
\]  
(64)
Let us first change variable, \( t = -\cos(\chi) \). Then
\[
(1 - t^2) \Psi'' - 2t \Psi' + \left[ \frac{1}{1 - t^2} \left( \omega + eQ \frac{B}{A} l \right)^2 - \mu^2 l \right] \Psi = 0,
\]  
(65)
where the prime is the derivation w.r.t. $t$. Note that this equation is invariant under $[t \rightarrow -t, \, Q \rightarrow -Q]$ so that we can look at the singularity in $t = 1$ only and obtain the properties of the singularity in $t = -1$ by $Q \rightarrow -Q$. Now, near $t = 1$.

$$0 \approx 2(1-t)\Psi'' - 2\Psi' + \frac{1}{2(1-t)} \left( \omega + eQ \frac{B}{A} \right)^2,$$

which has solution $\Psi = (1-t)^{\pm i} (\omega + eQ B)^{\frac{1}{2}}$. This suggests to set

$$\Psi(t) = (1-t)^{l_+} (1+ l) l - \Phi_1(t), \quad (66)$$

so that the equation for the function $\Phi_1$ is

$$(1 - t^2)\Phi'' - 2(t - l_+ (1 - t) + l_+ (1 + t)) \Phi' - \left[ \mu^2_0 - \omega^2 + l_+ - (l_+ - l_-)^2 \right] \Phi = 0.$$

Let us introduce

$$E := Q \frac{B}{A}.$$

We are interested in the level-crossing region, which is, for $eE > 0$,

$$-eE \leq \omega \leq eE.$$

In this region one obtains

$$l_+ = \frac{i}{2} (eE \pm \omega).$$

We also define

$$\Delta = \mu^2_0 + (eE)^2 - \frac{1}{4}. \quad (68)$$

Note that the sign of $\Delta$ is not ensured to be positive. To be precise, we should also indicate the dependence of $\Delta$ on $l$, by writing e.g. $\Delta_l$, but, in order to simplify the notation, we leave implicit this dependence. Note also that, if $\mu^2_0 + (eE)^2 - \frac{1}{4} < 0$, then for sufficiently high values of $l$ the quantity $\Delta$ passes from negative to positive values. The sign of $\Delta$ is associated with a different behavior of the transmission coefficients and then of the imaginary part of the effective action. A little consideration allows us to draw the conclusion that the behaviors in the two different regions (positive and negative) are linked to each other by analytic continuation.

We first consider the case $\Delta > 0$. The general solution of this equation in the level crossing region is easily found to be

$$\Phi(t) = C_\pm F \left( i eE + \frac{1}{2} \pm i \sqrt{\Delta}, i eE + \frac{1}{2} - i \sqrt{\Delta}; i (eE + \omega) + 1; \frac{1 - t}{2} \right)$$

$$+ C_- F \left( i eE + \frac{1}{2} + i \sqrt{\Delta}, i eE + \frac{1}{2} - i \sqrt{\Delta}; i (eE - \omega) + 1; \frac{1 + t}{2} \right), \quad (69)$$

where $F(a, b; c; z)$ is the usual hypergeometric function.

We can use the well-known relation

$$F(a, b; c; z) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} F(a, b; 1 + a + b - c; 1 - z)$$

$$+ \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} (1 - z)^{c - a - b} F(c - a, c - b; 1 - a - b + c; 1 - z) \quad (70)$$

to look at the asymptotic behavior of $\Psi$ near the boundaries.
Let us now introduce the coordinates \( x = \log \tan \frac{\chi}{2} \). In this coordinate \( \chi \approx 0, \pi \) become \( x \approx -\infty, +\infty \) and setting
\[
\eta(x) := \Psi(\chi(x)),
\]
and
\[
\alpha := \frac{\Gamma(1 + i(eE - \omega))}{\Gamma(1/2 - i\omega - i\sqrt{\Delta})} \Gamma(1/2 - i\omega + i\sqrt{\Delta}),
\]
\[
\beta := \frac{\Gamma(1 + i(eE - \omega))}{\Gamma(1/2 + i\omega - i\sqrt{\Delta})} \Gamma(1/2 + i\omega + i\sqrt{\Delta}),
\]
we can write
\[
\eta(x) \approx e^{i\omega x E} [C_+ + C_\alpha \beta] + e^{-i\omega x E} C_+ \beta, \quad x \approx +\infty,
\]
and
\[
\eta(x) \approx e^{i(eE - \omega) x} [C_+ + C_\alpha \beta] + e^{-i(eE - \omega) x} C_- \beta, \quad x \approx -\infty.
\]
If we are searching for the transmission of a particle coming from \( x \approx -\infty \), then at \( t \to +\infty \) we must find only the transmitted particle at \( x \approx +\infty \), with positive momentum. With the chosen condition, the positive momentum is \( (\omega + eE) \) so that we must set \( C_+ = 0 \) and \( c_{\text{out}} = C_- \) is the coefficient of the outgoing particle. At \( x \approx -\infty \), the coefficient of the ingoing particle is then \( c_{\text{in}} = C_- \alpha' \), so that the transmission coefficient is
\[
\tilde{T}_r = \frac{c_{\text{out}}}{{c_{\text{in}}}} = \frac{1}{\alpha'} = \frac{\Gamma(1/2 + i\omega - i\sqrt{\Delta}) \Gamma(1/2 + i\omega + i\sqrt{\Delta})}{\Gamma(1 + i(\omega + eE)) \Gamma(i(\omega - eE))}.
\]
Then, by using known formulas for the Gamma function, one obtains
\[
|\tilde{T}_r|^2 = \frac{eE - \omega \sinh(\pi(eE - \omega)) \sinh(\pi(eE + \omega))}{eE + \omega \cosh(\pi(eE - \sqrt{\Delta})) \cosh(\pi(eE + \sqrt{\Delta}))}.
\]
We used a different notation for \( \tilde{T}_r \) because it is not yet the transmission coefficient such that
\[
|R|^2 = 1 + |\tilde{T}_r|^2.
\]
The latter is obtained by noting that
\[
|\tilde{T}_r|^2 = -\frac{r}{q} |\tilde{T}_r|^2,
\]
where \( r := \omega + eE \) and \( q := \omega - eE \). Compare also [27], where a fine discussion on the topic of the Klein paradox is given.

As a consequence, we find
\[
|\tilde{T}_r|^2 = \frac{\sinh(\pi(eE - \omega)) \sinh(\pi(eE + \omega))}{\cosh(\pi(eE - \sqrt{\Delta})) \cosh(\pi(eE + \sqrt{\Delta}))}
\]
In the limit \( eE \gg \omega \) one finds
\[
|\tilde{T}_r|^2 \approx e^{-2\pi eE \sqrt{1 + \frac{1}{\pi x^2} - 1}},
\]
which, apart for the term \(-\frac{1}{4}\), is the result which can be obtained in the WKB approximation.
It is easy to show, in the case $\Delta < 0$, that the only change consists in the replacement $i\sqrt{\Delta} \mapsto -i\sqrt{|\Delta|}$ in the above formulas for the solution and also for $\alpha, \alpha', \beta, \beta'$. As a consequence, we find the following result:

$$|T_l|^2 = \frac{|\Gamma\left(\frac{1}{2} + i\omega - \sqrt{|\Delta|}\right)|^2 |\Gamma\left(\frac{1}{2} + i\omega + \sqrt{|\Delta|}\right)|^2}{|\Gamma(1 + i(\omega + eE))|^2 |\Gamma(i(\omega - eE))|^2};$$

the denominator is the same as in the case $\Delta > 0$. As to the numerator one finds

$$\left|\Gamma\left(\frac{1}{2} + i\omega - \sqrt{|\Delta|}\right)\right|^2 \left|\Gamma\left(\frac{1}{2} + i\omega + \sqrt{|\Delta|}\right)\right|^2 = \frac{\pi}{\cos(\pi z_1)} \frac{\pi}{\cos(\pi z_2)}$$

where $z_1 := \sqrt{|\Delta|} + i\omega$ and $z_2 := \sqrt{|\Delta|} - i\omega$ and standard relations for the Gamma function are used. As a consequence we get

$$|\tilde{T}_l|^2 = eE - \omega eE + \omega \frac{\sinh(\pi(\omega - eE))}{\cosh(2\pi \omega) + \cos(2\pi \sqrt{|\Delta|})}.$$

We need to calculate

$$W_l = \sum_{i=1}^{2l+1} \log(1 + |T_l|^2);$$

$(85)$

we do not perform the sum over $l$, and then we calculate

$$W_l = (2l + 1) \frac{1}{2} \sum_{\omega} \log(1 + |T_l(\omega)|^2).$$

$(86)$

Let us start from the case $\Delta > 0$. We have to perform the following integral:

$$I := \int_{-e\epsilon}^{e\epsilon} \frac{e\epsilon - e\epsilon}{e\epsilon + e\epsilon} \log\left(1 + \frac{\cosh[2\pi e\epsilon] - \cosh[2\pi \omega]}{\cosh[2\pi \sqrt{|\Delta|}] + \cosh[2\pi \omega]}\right),$$

where the dependence on $l$ is implicit in $\Delta$; the integral can be rewritten as follows:

$$I = 2e\epsilon \log(\cosh[2\pi \sqrt{|\Delta|}] + \cosh[2\pi e\epsilon]) - II,$$

$(88)$

where

$$II := \int_{-e\epsilon}^{e\epsilon} \frac{1}{e\epsilon} \int_{-2\pi e\epsilon}^{2\pi e\epsilon} \text{dy} \log(p + \cosh[\text{y}]),$$

which is formally the same integral as in the Dirac case. Then we find

$$W_l = (2l + 1) \frac{T}{2\pi} \left(e\epsilon \log[2(\cosh[2\pi \sqrt{|\Delta|}] + \cosh[2\pi e\epsilon])]ight) + \frac{1}{4\pi} \left[-\text{Li}_2(-e^{-2\pi(\sqrt{\Delta} + e\epsilon)}) + \text{Li}_2(-e^{2\pi(\sqrt{\Delta} - e\epsilon)})
- \text{Li}_2(-e^{2\pi(\sqrt{\Delta} - e\epsilon)}) + \text{Li}_2(-e^{-2\pi(\sqrt{\Delta} + e\epsilon)})\right].$$

$(90)$

As to the case $\Delta < 0$, it can be obtained by the replacement $\sqrt{\Delta} \mapsto i\sqrt{|\Delta|}$. In particular, if $\mu^2 + (eE)^2 - \frac{1}{4} < 0$, one finds that there exists $l_c$ such that

$$W = \sum_{l \leq l_c} W_l + \sum_{l > l_c} W_l,$$

$(91)$
where
\[
\hat{W}_l = \frac{T}{2\pi} (2l + 1) \left( eE \log[2(\cos[2\pi \sqrt{|\Delta|}] + \cosh[2\pi eE])] + \frac{1}{4\pi} \left[ -L_l^2 (e^{-2\pi i\sqrt{\Omega} t e E}) - L_l^2 (e^{2\pi i\sqrt{\Omega} t e E}) + L_l^2 (e^{2\pi i\sqrt{\Omega} t e E}) + L_l^2 (e^{-2\pi i\sqrt{\Omega} t e E}) \right] \right).
\]
By taking into account that \(L_l^2 (\hat{Z}) = \overline{L_l^2 (\hat{Z})}\), it is evident that the latter expression is real.

### 5.2. Nariai in the zeta-function approach

The Euclidean Klein–Gordon operator for the Nariai solution is given by
\[
-\frac{A}{\sin^2 \chi} (\partial_t - ie E \cos \chi)^2 - \frac{A}{\sin \chi} \partial_x (\sin \chi \partial_x) - B \nabla^2 \Omega + \mu^2 \equiv KG(E),
\]
where \(\tau = i\psi\). Let us search for the eigenfunctions of this differential operator. We can perform variable separation, as usual. Note that \(B \nabla^2 \Omega\) and \(-i\partial_t\) commute with \(KG(E)\) and then one can restrict the study of its eigenvalue equation to the eigenspaces of the aforementioned operators, i.e. we can write for its eigenfunctions in these eigenspaces

\[
\psi(\tau, \chi, \Omega) = e^{-(1+i)t} Y_{l,m}(\Omega) g(\chi),
\]
where \(Y_{l,m}(\Omega)\) are the spherical harmonics. Moreover, note that

\[
(-B \nabla^2 \Omega + \mu^2) Y_{l,m} = \mu^2 Y_{l,m}, \quad \mu^2 = \frac{\mu^2}{A} + \frac{B}{A} (l + 1).
\]
The operator \(KG(E)|_{a, l, m}\) restricted to the above eigenspaces takes the following form:

\[
KG(E)|_{a, l, m} = \left[ -\frac{A}{\sin^2 \chi} (i\omega - ie E \cos \chi)^2 - \frac{A}{\sin \chi} \partial_x (\sin \chi \partial_x) + A \mu^2 \right].
\]
Let us introduce the new variable \(t = -\cos \chi\); then the eigenvalue equation for \(KG(E)|_{a, l, m}\) becomes

\[
\left[ -\frac{1}{1 - t^2} (\omega + eEt)^2 - \mu^2 + \frac{\lambda}{A} \right] g + (1 - t^2) g'' - 2tg' = 0.
\]
To transform this equation into a hypergeometric, we set \(g(t) = (1 + t)^l (1 - t)^l \psi(t)\), with
\[
l = \frac{1}{2} (\omega \pm eE).
\]
This choice ensures that the solutions belong in the Hilbert space \(L^2((1, 1))\) for all values of \(\omega \in \mathbb{R}\). As a consequence, equation (96) becomes

\[
(1 - t^2)\psi''(t) + [2t + 2(1 - t) - 2l_+ (1 + t)] \psi'(t) - \omega^2 - \mu^2 + \frac{\lambda}{A} - 2l_+ l_- + l_+^2 - l_+ l_- - l_-^2 \psi(t) = 0.
\]
The general solution of this equation is
\[
\psi(t) = 2F_1 \left( a_+, a_-; 2l_+ + 1; \frac{1 + t}{2} \right) + 2F_1 \left( a, b; 2l_+ + 1; \frac{1 - t}{2} \right),
\]
with
\[
a_+ = \frac{1}{2} + l_+ + l_- \pm \sqrt{\frac{1}{4} - \omega^2 - \mu^2 + \frac{\lambda}{A} + (l_+ + l_-)^2 (l_+ - l_-)^2}.
\]
Note that this solution has a bad behavior in \( t = \pm 1 \). The only possibility for it to lie in \( L^2((-1, 1)) \) is that \( a_\pm \in -\mathbb{N} \), that is

\[
\frac{1}{2} I_+ + I_+ - \sqrt{\frac{1}{4} - \frac{\mu^2}{A} + \frac{\lambda}{A} + (eE)^2} = -n, \quad n \in \mathbb{N}.
\]  

(101)

Indeed, the spectrum is discrete with eigenvalues

\[
\frac{\lambda_{n,l,\omega}}{A} = \left[ n + \frac{1}{2} + (I_+ + I_-) \right]^2 - \frac{1}{4} + \frac{\mu^2}{A} - (eE)^2,
\]  

(102)

which are degenerate in the azimuthal quantum number \( m \).

It follows that the heat kernel for the operator \( KG(E) \) is

\[
k(s) = \sum_l k_l(s),
\]  

where

\[
k_l(s) = \mathrm{Tr} e^{-sKG(E)} = \sum_{\omega} \sum_n (2l + 1) e^{-sA^{\frac{1}{2}}\left[ (n + \frac{1}{2} + \omega)^2 + \frac{\mu^2}{A} - (eE)^2 - \frac{1}{4} \right]},
\]  

(103)

As in the ultracold cases, \( \gamma \) is a renormalization constant. Actually the sum over \( \omega \) is an integral due to the continuity of the \(-i\partial_t\) spectrum. It is convenient to split such integration into two parts that are the interval \(-eE < \omega < eE\) and its complement in \( \mathbb{R} \). This is because inside the interval the eigenvalues \( \lambda_{n,l,\omega} \) do not depend on \( \omega \). In this way, we get

\[
k_l(s) = \frac{T}{2\pi} \left( 2(2l + 1) \int_{-eE}^{\infty} e^{-sA^{\frac{1}{2}}\left[ (n + \frac{1}{2} + \omega)^2 + \frac{\mu^2}{A} - (eE)^2 - \frac{1}{4} \right]} d\omega + (2l + 1)(eE) \sum_n e^{-sA^{\frac{1}{2}}\left[ (n + \frac{1}{2} + eE)^2 + \frac{\mu^2}{A} - (eE)^2 - \frac{1}{4} \right]} \right),
\]

The spectral Riemann zeta function associated with the Klein–Gordon operator with kernel \( k_l(t) \) is then

\[
\zeta_l(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} k_l(t) dt
\]

\[
= \frac{T}{2\pi} \left( 2(2l + 1) \int_{-eE}^{\infty} \sum_n \frac{d\omega}{A^{\frac{1}{2}}\left[ (n + \frac{1}{2} + \omega)^2 + \frac{\mu^2}{A} - (eE)^2 - \frac{1}{4} \right]^\frac{3}{2}} + (2l + 1)(eE) \sum_n \frac{1}{A^{\frac{1}{2}}\left[ (n + \frac{1}{2} + eE)^2 + \frac{\mu^2}{A} - (eE)^2 - \frac{1}{4} \right]^\frac{3}{2}} \right).
\]

An analogous computation of one performed in [12, 26] leads to the following expression for the imaginary part of the effective action:

\[
W_l = \frac{T}{2\pi} (2l + 1) \left[ eE \log[\cosh[\pi(\sqrt{\Delta} - eE)] \cosh[\pi(\sqrt{\Delta} + eE)]]
\right.
\]

\[
+ 2eE \log 2 + \frac{1}{4\pi} \left[ -\mathrm{Li}_2(-e^{-2\pi(\sqrt{\Delta} + eE)}) + \mathrm{Li}_2(-e^{2\pi(\sqrt{\Delta} - eE)})
\right.
\]

\[
- \mathrm{Li}_2(-e^{-2\pi(\sqrt{\Delta} - eE)}) + \mathrm{Li}_2(-e^{2\pi(\sqrt{\Delta} + eE)}) \right),
\]

(104)

which coincides with the one obtained using the transmission coefficient approach, and does not depend on the renormalization constant \( \gamma \). The same considerations as for the aforementioned approach in the case \( \Delta < 0 \) apply in the zeta-function approach.
5.3. Thermal effects

We find (for definiteness we choose $\Delta > 0$)

$$\langle \bar{N}^{\text{out}} \rangle = \sinh (\pi (eE - \omega)) \sinh (\pi (eE + \omega))$$

$$\times \frac{1}{2} \left( \coth[\pi (\omega - \varphi^+)] + \coth[\pi (|\omega| + \varphi^-)] \right),$$

with $\varphi^+ = e(A_0|\pi - A_0|0) = 2eE = \varphi^-$. We recall that in terms of physical (dimensionful) variables, by taking into account that $Th = \bar{h}c \sqrt{\Delta^2 \pi}$, and that $\omega_{\text{phys}} = \sqrt{\Delta \omega}$, in such a way that $\beta_{\text{phys}} \omega_{\text{phys}} = 2\pi \omega$.

6. Conclusions

Our analysis for the scalar case has confirmed the main features we obtained in [12]: exact calculations have been performed, both in the transmission coefficient approach and in the zeta-function approach. The latter is more involved but it also provides us with much more information with respect to the former, indeed the complete one loop effective action (and not only its imaginary part) can be obtained by using the zeta function, as known. We also stress that, in the case of the ultracold I and Nariai backgrounds, the imaginary part of the effective action is not able to capture the instability associated with Hawking radiation. In order to leave room for the latter instability, a different scheme should be adopted: one should e.g. use the Wilczek–Parikh method [28], which understands a dynamical situation and, moreover, requires that quantum field modes beyond the horizon(s) are also considered (there is a very rich literature on this topic; we limit ourselves to referring to the seminal paper [28], to [29], which provides a fine discussion of the method, and to [30], which deals with de Sitter space).

As in the evaluation of the effective action, as discussed in section 2, we are in principle considering the Boulware-like vacuum in a static situation, and excluding the region beyond the horizons, our effective action is not enabled to also see the Hawking effect; still, its marrying with the given instability is taken into account in the paper. In particular, thermal effects, as in the Dirac case, have been shown to affect the discharge phenomenon with a key role in the pair-creation phenomenon still to be assigned to the transmission coefficient. It is also worth mentioning that, in the ultracold I and Nariai case, a naive approach to the thermal effective action in the imaginary time formalism would not give rise to meaningful results. We mean to come back to this topic in a future publication.

It is also important to remark differences between the analysis in the scalar field case and in the Dirac field one. Differences with the Dirac case are both of general character (indefinite scalar product spaces versus Hilbert spaces) and in particular characteristics of the cases we analyzed: in the ultracold I case a bad behavior at $x = +\infty$, which does not occur for the Dirac case, forced us to refer to fluxes in order to compute the transmission coefficient; moreover, in the Nariai case, the coefficient $\Delta$ is not ensured to be positive definite (whereas it is positive definite in the Dirac case), and then one is forced to consider both cases. It is worth mentioning that this aspect is not new, because an analogous problem occurs in the well-known case of the so-called Sauter potential; nevertheless, a discussion of that problem for the Sauter potential is often missing (cf e.g. [31], where the case associated with our $\Delta < 0$ is considered, and e.g. the results in [27] and in [8] (first paper), for Sauter-like potentials, where the opposite case is given).
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