ON REPRESENTABLE GRAPHS, SEMI-TRANSITIVE ORIENTATIONS, AND THE REPRESENTATION NUMBERS

Magnús M. Halldórsson
School of Computer Science, Reykjavik University, Kringlan 1, 103 Reykjavik, Iceland
mmh@ru.is

Sergey Kitaev
The Mathematics Institute, Reykjavik University, Kringlan 1, 103 Reykjavik, Iceland
sergey@ru.is

Artem Pyatkin
Sobolev Institute of Mathematics, pr-t Koptyuga 4, 630090, Novosibirsk, Russia
artem@math.nsc.ru

ABSTRACT

A graph $G = (V, E)$ is representable if there exists a word $W$ over the alphabet $V$ such that letters $x$ and $y$ alternate in $W$ if and only if $(x, y) \in E$ for each $x \neq y$. If $W$ is $k$-uniform (each letter of $W$ occurs exactly $k$ times in it) then $G$ is called $k$-representable. It was shown in [4] that a graph is representable if and only if it is $k$-representable for some $k$. Minimum $k$ for which a representable graph $G$ is $k$-representable is called its representation number.

In this paper we give a characterization of representable graphs in terms of orientations. Namely, we show that a graph is representable if and only if it admits an orientation into a so-called semi-transitive digraph. This allows us to prove a number of results about representable graphs, not the least that 3-colorable graphs are representable. We also prove that the representation number of a graph on $n$ nodes is at most $n$, from which one concludes that the recognition problem for representable graphs is in NP. This bound is tight up to a constant factor, as we present a graph whose representation number is $n/2$.

We also answer several questions posed in [4], in particular, on representability of the Petersen graph and local permutation representability.

Keywords: graph, representation, words, orientations, complexity, circle graph, 3-colorable graph, comparability graph, Petersen graph, representation number, semi-transitive orientation

1. INTRODUCTION

A graph $G = (V, E)$ is representable if there exists a word $W$ over the alphabet $V$ such that letters $x$ and $y$ alternate in $W$ if and only if $(x, y) \in E$ for each $x \neq y$. It is $k$-representable if each letter appears exactly $k$ times in it.
times. The notion of representable (directed) graphs was introduced in [5] to obtain asymptotic bounds on the free spectrum of the widely-studied Perkins semigroup which has played central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. In [4], the only paper solely dedicated to the study of representable graphs, numerous properties of representable graphs are derived and numerous types of representable and non-representable graphs are pinpointed. Still, large gaps of knowledge of these graphs have remained, and the purpose of this paper is to address them.

We address the three most fundamental issues about representable graphs:

- Are there alternative representations of these graphs that aid in reasoning about their properties?
- Which types of graphs are representable and which ones are not? And,
- How large words can be needed to represent representable graphs?

These can be viewed as some of the most basic questions of any graph class. We make progress on each of these.

We characterize representable graphs in terms of orientability. The edges can be directed so as to yield a directed graphs satisfying a property that we call semi-transitivity. It properly generalizes the transitivity property of comparability graphs, constraining the subgraphs induced by certain types of cycles. The definition and the characterization is given in Section 3. This formulation allows us to reason fairly easily about the types of graphs that are representable.

We show that the class of representable graphs captures quite involved properties. In particular, all 3-colorable graphs are representable. This resolves a conjecture of [4] regarding the Petersen graph, showing that it is representable. We actually give an explicit construction to show that it is 3-representable. The result also properly captures all the previously known classes of representable graphs: outerplanar, prisms, and comparability graphs. On the negative side, we answer an open question of [4] by presenting a non-representable graphs all of whose induced neighborhoods are comparability graphs.

Finally, we show that any representable graph is \( n \)-representable, again utilizing the semi-transitive orientability. Previously, no non-trivial upper bound was known on the representation number, which is the smallest value \( k \) such that the given graph is \( k \)-representable. This result implies that problem of deciding whether a given graph is representable is contained in NP. This bound on the representation number is tight up to a constant factor, as we construct graphs with representation number \( n/2 \). We also show that deciding if a representable graph is \( k \)-representable is NP-complete for \( 3 \leq k \leq \lceil n/2 \rceil \), while the class of circle graphs coincides with the class of graphs with representation number at most 2.

The paper is organized as follows. In Section 2 we give definitions of objects of interest and review some of the known results. In Section 3 we give a characterization of representable graphs in terms of orientations and discuss some important corollaries of this fact. In Section 4 we consider the problems concerning the representation numbers, and show that it is always at most \( n \) but can be as much as \( n/2 \). We explore in Section 5 which classes of graphs are representable, showing, in particular, 3-colorable graphs to be representable, but numerous others to be orthogonal to representability. Finally, we conclude with a discussion of algorithmic complexity and some open problems in Section 6.

2. Definitions, notation, and known results

In this section we follow [4] to define the objects of interest.

Let \( W \) be a finite word over an alphabet \( \{x_1, x_2, \ldots\} \). If \( W \) involves the letters \( x_1, x_2, \ldots, x_n \) then we write \( \text{Var}(W) = \{x_1, \ldots, x_n\} \). Let \( X \) be a subset of \( \text{Var}(W) \). Then \( W \setminus X \) is the word obtained by eliminating all
letters in \( X \) from \( W \). A word is \( k \)-uniform if each letter appears in it exactly \( k \) times. A 1-uniform word is also called a permutation. Denote by \( W_1W_2 \) the concatenation of words \( W_1 \) and \( W_2 \). We say that the letters \( x_i \) and \( x_j \) alternate in \( W \) if the word induced by these two letters contains neither \( x_ix_i \) nor \( x_jx_j \) as a factor. If a word \( W \) contains \( k \) copies of a letter \( x \) then we denote these \( k \) appearances of \( x \) by \( x^1, x^2, \ldots, x^k \). We write \( x^i_j < x^l_k \) if \( x^i_j \) stays in \( W \) before \( x^l_k \), i.e., \( x^i_j \) is to the left of \( x^l_k \) in \( W \).

Let \( G = (V, E) \) be a graph with the vertex set \( V \) and the edge set \( E \). We say that a word \( W \) represents the graph \( G \) if there is a bijection \( \phi : \text{Var}(W) \rightarrow V \) such that \((\phi(x_i), \phi(x_j)) \in E \) if and only if \( x_i \) and \( x_j \) alternate in \( W \). It is convenient to identify the vertices of a representable graph and the corresponding letters of a word representing it. We call a graph \( G \) representable if there exists a word \( W \) that represents \( G \). If \( G \) can be represented by a \( k \)-uniform word, then we say that \( G \) is \( k \)-representable. Clearly, the complete graphs are the only examples of 1-representable graphs. So, in what follows we assume that \( k \geq 2 \). Let the representation number of a graph \( G \) be the minimum \( k \) such that \( G \) is \( k \)-representable.

We call a graph permutationally representable if it can be represented by a word of the form \( P_1P_2 \ldots P_k \) where all \( P_i \) are permutations. In particular, all permutationally representable graphs are \( k \)-representable.

A digraph (directed graph) is transitive if the adjacency relation is transitive, i.e., for every vertices \( x, y, z \in V \) the existence of the arcs \( xy, yz \in E \) yields that \( xz \in E \). A comparability graph is an undirected graph having an orientation of the edges that yields a transitive digraph.

The following results on representable graphs are known. They were proved in [4] except for the Lemma 3 that was proved in [3].

**Proposition 1.** Let \( W = AB \) be a \( k \)-uniform word representing a graph \( G \). Then the word \( W' = BA \) also \( k \)-represents \( G \).

**Proposition 2.** Let the graphs \( G_1 \) and \( G_2 \) be \( k \)-representable and \( x \in V(G_1), y \in V(G_2) \). Assume that the graph \( G \) is obtained from \( G_1 \) and \( G_2 \) by identifying the vertices \( x \) and \( y \) into a new vertex \( z \). Then \( G \) is also \( k \)-representable.

**Lemma 3.** A graph is permutationally representable if and only if it is a comparability graph. In particular, all bipartite graphs are permutationally representable.

For a vertex \( x \in V(G) \) denote by \( N(x) \) the set of all its neighbors.

**Theorem 4.** If \( G \) is representable, then for every \( x \in V(G) \) the graph induced by \( N(x) \) is permutationally representable.

**Theorem 5.** Outerplanar graphs are 2-representable.

**Theorem 6.** For every graph \( G \) there exists a 3-representable graph \( H \) that contains \( G \) as a minor. In particular, a 3-subdivision of every graph \( G \) is 3-representable.

**Proposition 7.** All prisms are 3-representable. Moreover, the triangular prism has the representation number 3.

Theorem [4] provides an easy way to construct non-representable graphs: just take a graph that is not a comparability graph and add an all-adjacent vertex to it. The wheel \( W_5 \) is the smallest non-representable graph. Some other small non-representative graphs are given in Fig. 1 (this figure appears in [4]).

Paper [4] contains several open problems. In this paper we solve some of them.
The word representation of representable graphs is simple and natural. Yet it does not lend to easy arguments for the characteristic of representable graphs. Non-representability is even harder to argue in terms of the many possible corresponding words. The main result of this section is a new characterization of representable graphs that leads easily to various results about representability.

We give a characterization in terms of orientability, which implies that representability corresponds to a property of a digraph obtained by directing the edges in a certain way. Recall that Lemma 3 states that a graph is permutationally representable if and only if it has a transitive orientation. We prove a similar fact on representable graphs, namely, that a graph is representable if and only if it has a so-called semi-transitive orientation. Our definition, in fact, generalizes that of a transitive orientation.

Other orientations have been defined in order to capture generalizations of comparability graphs. As transitive orientations form constraints on the orderings of induced $P_3$, these generalizations form constraints on the orderings of induced $P_4$. These include perfectly orderable graphs (and its subclasses) and opposition graphs [1]. Classes such as chordal graphs are defined in terms of vertex-orderings, and imply therefore indirectly acyclic orientations. None of these properties captures our definition below, nor does our characterization subsume any of them.

We turn to the characterization and start with definitions of certain directed graphs. A semi-cycle is the directed acyclic graph obtained by reversing the direction of one arc of a directed cycle. An acyclic digraph is a shortcut if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_0, v_1, \ldots, v_t\}$, is a shortcut if it contains a directed path $v_0v_1 \ldots v_t$, the arc $v_0v_t$ and it is missing the arcs $v_iv_j$ for $0 \leq i < j \leq t$ (in particular, $t \geq 3$).

**Definition 8.** A digraph is semi-transitive if it is acyclic and contains no shortcuts.

A graph is semi-transitively orientable if there exists an orientation of the edges that results in a semi-transitive graph.

Our main result in this paper is the following.

**Theorem 9.** A graph is representable if and only if it is semi-transitively orientable.
We say that a node $D$. We claim that this word and the second occurrences are as early as possible. The ordering of other nodes is arbitrary, within these symmetry of the two copies of $D$ appear before $v$. Let $S$ be a topological ordering of $a D^i$ for some $t$. We first observe that any topological ordering of $D^t$ preserves arcs.

**Lemma 10.** Let $D$ be a digraph with distinct node-labels. Let $S$ be a topological ordering of a $D^i$. Then $G_D$ is a subgraph of $G_S$.

**Proof.** Consider an edge $uv$ in $G_D$, and suppose without loss of generality that it is directed as $uv$ in $D$. Then, in $D^i$, there is a directed path $u^1v^1u^2v^2\ldots u^tv^t$. Thus, occurrences of $u$ and $v$ in a topological ordering of $D^t$ are alternating. Hence, $uv \in G_S$. 

To prove equivalence, we now give a method to produce a topological ordering that generates all non-arcs. We say that a subgraph $H$ covers a set $A$ of non-arcs if each non-arc in $A$ is also found in $H$. A word covers the non-arcs if the graph it represents covers them.

**Lemma 11.** The non-arcs incident with a path in a semi-transitive digraph can be covered with a 2-uniform word.

**Proof.** Let $P$ be a path in a semi-transitive digraph $D$. We shall form a topsort $S$ of the 2-string digraph $D^2$ and show that it covers all non-arcs having at least one endpoint on $P$.

We say that a node $x$ of $D^2$ depends on node $y$ if there is a directed path from $y$ to $x$ in $D^2$, i.e. $y$ must appear before $x$ in a topological ordering of $D^2$. We use the notation $y \rightarrow x$ if $x$ depends on $y$. A node is listed earliest possible if it is listed as soon as all nodes that it depends on have been listed. A node is listed latest possible if it is listed after all nodes that do not depend on it.

Let $S$ be any topological ordering of $D^2$ where the first occurrences of nodes in $P$ are as late as possible and the second occurrences are as early as possible. The ordering of other nodes is arbitrary, within these constraints.

We claim that this word $S$ covers all non-arcs involving nodes in $P$. Consider a pair $u,v$, where $uv \notin G_D$ and $u \in P$. Note that $v$ may also belong to $P$, in which case we may assume that the path goes from $u$ to $v$. Consider the listings of $u^1, v^1, u^2, v^2$, where the subscript refers to the occurrence number of the node. Observe that $u$ may depend on $v$, or vice versa, but not both. There are three cases to consider.

Case (i): There is a path from $u$ to $v$ in $D$. We claim that $u^2$ does not depend on $v^1$. Suppose it does, i.e. $v^1 \rightarrow u^2$. Then, there is an arc $x^1y^2 \in D^2$ such that $v^1 \rightarrow x^1$ and $y^2 \rightarrow u^2$. By the assumptions and the symmetry of the two copies of $D^2$, we follow that $y^1 \rightarrow u^1 \rightarrow v^1 \rightarrow x^1$. By the definition of 2-string graphs, $y^1x^1 \in E(D^2)$. Then, by semi-transitivity, $u^1v^1 \in E(D^2)$, or $uv \notin E(G_D)$, which is a contradiction. It now follows that the nodes will occur as $u^1u^2v^1v^2$ in $S$, i.e. $uv \notin E(G_S)$.

Case (ii): There is a path from $v$ to $u$ in $D$. This is symmetric to case (i), with $u$ replaced by $v$. Thus, the nodes will occur as $v^1v^2u^1u^2$ in $S$.

Case (iii): The nodes $u$ and $v$ are incomparable in $D$. In particular, $v$ is not in $P$. Then, $u^1$ and $v^1$ do not depend on each other, nor do $u^2$ and $v^2$. If $v^2$ depends on $u^1$ then the nodes occur as $v^1u^1u^2v^2$ in $S$. Otherwise, their order is $v^1v^2u^1u^2$. 

□
We now return to the proof of Theorem 9 starting with the forward direction. Given a word-representant $S$, we direct an edge from $x$ to $y$ if the first occurrence of $x$ is before that of $y$ in the word. Let us show that such orientation $D$ of $G_S$ is semi-transitive. Indeed, assume that $x_0x_i \in E(D)$ and there is a directed path $x_0x_1 \ldots x_i$ in $D$. Then in the word $S$ we have $x_0 \prec x_1 \prec \ldots \prec x_i$ for every $i$. Since $x_0x_i \in E(D)$ we have $x_i \prec x_{i+1}$. But then for every $j < k$ and $i$ there must be $x_j \prec x_k \prec x_{j+1}$, i.e. $x_ix_j \in E(D)$. So, $D$ is semi-transitive.

For the other direction, denote by $G$ the graph and by $D$ its semi-transitive orientation. Let $P_1, P_2, \ldots, P_\tau$ be the set of directed paths covering all vertices of $D$. For every $i = 1, 2, \ldots, \tau$ denote by $S_i$ the topsort of the digraph $D^2$ satisfying the conditions of Lemma 11 for the path $P_i$. Put $S = S_1S_2 \ldots S_\tau$. Clearly, $S$ is a $2\tau$-uniform word; it can be treated as a topsort of a $2\tau$-string $D^{2\tau}$. Then $G = G_S$. Indeed, by Lemma 10 we have $E(G) \subseteq E(G_S)$. On the other hand, if $uv \notin E(G)$ then $u \in P_i$ for some $i$, and thus by Lemma 11 the letters $u$ and $v$ are not alternating in the subword $S_i$. Therefore, $uv \notin E(S)$. Theorem 9 is proved.

Theorem 9 makes clear the relationship to comparability graphs, which are those that have transitive orientations. Since transitive digraphs are also semi-transitive, this immediately implies that comparability graphs are representable.

4. The Representation Number of Graphs

We focus now on the following question: Given a representable graph, how large is its representation number? In [4], certain classes of graphs were proved to be 2- or 3-representable, and an example was given of a graph (the triangular prism) with the representation number of 3. On the other hand, no examples were known of graphs with representation numbers larger than 3, nor were there any non-trivial upper bounds known. We show here that the maximum representation number of representable graphs is linear in the number of vertices.

For the upper bound, we use the results of the preceding section. We have the following directly from the proof of Theorem 9.

**Corollary 12.** A representative graph $G$ is $2\tau(G)$-representable, where $\tau(G)$ is the minimum number of paths covering all nodes in some semi-transitive orientation of $G$.

This immediately gives an upper bound of $2n$ on the representation number. We can improve this somewhat with an effective procedure.

**Theorem 13.** Given a semi-transitive digraph $D$ on $n$ vertices, there is a polynomial time algorithm that generates an $n$-uniform word representing $G_D$. Thus, each representable graph is $n$-representable.

**Proof.** The algorithm works as follows.

Step 0. Start with $A = \emptyset$ and $i = 1$.

Step $i$. If $D$ contains a path $P_i$ covering at least two vertices from $V \setminus A$ then let $A := A \cup V(P_i)$ and $i := i + 1$. Otherwise, let $B = V \setminus A$ and go to the Final Step.

Final Step. Let $S_i$ be the topsort of the digraph $D^2$ satisfying the conditions of Lemma 11 for the path $P_i$ and put $S' = S_1S_2 \ldots S_t$ where $t$ is the number of paths found at previous steps. If $|B| \leq 1$ then let $S = S'$. Otherwise, consider a topsort $S_0$ of $D$ where the vertices of $B$ are listed in a row (since the vertices of $B$ do not depend on each other, such a topsort must exist) and in particular in the reverse order of their appearance in $S_1$. Let $S = S'S_0$. 


Clearly, $G_D = G_S$ (the proof is the same as in Theorem 9). It is easy to verify that each letter appears in $S$ at most $n$ times.

Theorem 13 implies that the graph representability is polynomially verifiable, answering an open question in [4]. Indeed, having a representable graph $G$, we may ask for a word $W$ $k$-representing it and verify this fact in time bounded by the polynomial on $k$ and $n$. Since $k \leq n$, this is a polynomial on $n$. So, we have proved

**Corollary 14.** The recognition problem for representable graphs is in NP.

We now show that there are graphs with representation number of $n/2$, matching the upper bound within a factor of 2.

The **cocktail party graph** $H_{k,k}$ is the graph obtained from the complete bipartite graph $K_{k,k}$ by removing a perfect matching. Denote by $G_k$ the graph obtained from a cocktail party graph $H_{k,k}$ by adding an all-adjacent vertex.

**Theorem 15.** The graph $G_k$ has representation number $k = \lfloor n/2 \rfloor$.

The proof is based on three statements.

**Lemma 16.** Let $H$ be a graph and $G$ be the graph obtained from $H$ by adding an all-adjacent vertex. Then $G$ is $k$-representable if and only if $H$ is permutationally $k$-representable.

**Proof.** Let 0 be the letter corresponding to the all-adjacent vertex. Then every other letter of the word $W$ representing $G$ must appear exactly once between two consecutive zeroes. We may assume also that $W$ starts with 0. Then the word $W \setminus \{0\}$ is a permutational $k$-representation of $H$. Conversely, if $W'$ is a word permutationally $k$-representing $H$, then we insert 0 in front of each permutation to get a $k$-representation (in fact permutational) of $G$. □

Recall that the **dimension** of a poset is the minimum number of linear orders such that their intersection induces this poset.

**Lemma 17.** A comparability graph is permutationally $k$-representable if and only if the poset induced by this graph has dimension at most $k$.

**Proof.** Let $H$ be a comparability graph and $W$ be a word permutationally $k$-representing it. Each permutation in $W$ can be considered as a linear order where $a < b$ if $a$ meets before $b$ in the permutation (and vice versa). We want to show that the comparability graph of the poset induced by the intersection of these linear orders coincides with $H$.

Two vertices $a$ and $b$ are adjacent in $H$ if and only if their letters alternate in the word. So, they must be in the same order in each permutation, i.e. either $a < b$ in every linear order or $b < a$ in every linear order. But this means that $a$ and $b$ are comparable in the poset induced by the intersection of the linear orders, i.e. $a$ and $b$ are adjacent in its comparability graph. □

The next statement most probably is known but we give its proof here for the sake of completeness.

**Lemma 18.** The poset $P$ over $2k$ elements $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ such that $a_i < b_j$ for every $i \neq j$ and all other elements are not comparable has dimension $k$. 
Proof. Assume that this poset is the intersection of \( t \) linear orders. Since \( a_i \) and \( b_i \) are not comparable for each \( i \), their must be a linear order where \( b_i < a_i \). If we have in some linear order both \( b_i < a_i \) and \( b_j < a_j \) for \( i \neq j \), then either \( a_i < a_j \) or \( a_j < a_i \) in it. In the first case we have that \( b_i < a_j \), in the second that \( b_j < a_i \). But each of these inequalities contradicts the definition of the poset. Therefore, \( t \geq k \).

In order to show that \( t = k \) we can consider a linear order \( a_1 < a_2 < \ldots < a_{k-1} < b_k < a_k < b_{k-1} < \ldots < b_2 < b_1 \) together with all linear orders obtained from this order by the simultaneous exchange of \( a_k \) and \( b_k \) with \( a_m \) and \( b_m \) respectively (\( m = 1, 2, \ldots, k - 1 \)). It can be verified that the intersection of these \( k \) linear orders coincides with our poset.

Now we can prove Theorem 15. Since the cocktail party graph \( H_{k,k} \) is a comparability graph of the poset \( P \), we deduce from Lemmas 18 and 17 that \( H_{k,k} \) is permutationally \( k \)-representable but not permutationally \((k - 1)\)-representable. Then by Lemma 16 we have that \( G_k \) is \( k \)-representable but not \((k - 1)\)-representable. Theorem 15 is proved.

The above arguments help us also in deciding the complexity of determining the representation number. From Lemmas 16 and 17 we see that it is as hard as determining the dimension \( k \) of a poset. Yannakakis \[8\] showed that the latter is NP-hard, for any \( 3 \leq k \leq \lceil n/2 \rceil \). We therefore obtain the following.

**Proposition 19.** Deciding whether a given graph is \( k \)-representable, for any given \( 3 \leq k \leq \lceil n/2 \rceil \), is NP-complete.

It was further shown by Hegde and Jain \[3\] that it is NP-hard to approximate the dimension of a poset within almost a square root factor. We therefore obtain the following hardness for the representation number.

**Proposition 20.** Approximating the representation number within \( n^{1/2 - \epsilon} \)-factor is NP-hard, for any \( \epsilon > 0 \).

In contrast with these hardness results, the case \( k = 2 \) turns out to be easier and admits a succinct characterization. The following fact essentially appears in \[2\]. Recall that a graph is called a circle graph if we can arrange its vertices as chords on a circle in such a way that two nodes in the graph are adjacent if and only if the corresponding chords overlap.

**Observation 21.** A graph is 2-representable if and only if it is a circle graph.

Indeed, given a circle graph \( G \), consider the ends of the chords on a circle as letters and read the obtained word in a clockwise order starting from an arbitrary point. It is easy to see that two chords intersect if and only if the corresponding letters alternate in the word. For the opposite direction, place \( 2n \) nodes at a circle in the same order as they meet in the word and connect the same letters by chords.

It follows from Theorems 21 and 5 that outerplanar graphs are circle graphs. Theorem 21 can also be useful as a tool in proving that a graph is not a circle graph. For example, non-representable graphs (for instance, all odd wheels \( W_{2t+1} \) for \( t \geq 2 \)) are not circle graphs.

5. **Characteristics of Representable Graphs**

When faced with a new graph class, the most basic questions involve the kind of properties it satisfies: which known classes are properly contained (and which not), which graphs are otherwise contained (and which not), what operations preserve representability (or non-representability), and which properties hold for these graphs.
Previously, it was known that the class of representable graphs includes comparability graphs, outerplanar graphs, subdivision graphs, and prisms. The purpose of this section is to clarify this situation significantly, including resolving some conjectures. We start with exploring the impact of colorability on representability.

**Chromatic number and representability.**

**Theorem 22.** 3-colorable graphs are semi-transitive, and thus representable.

**Proof.** Given a 3-coloring of a graph, direct its edges from the first color class through the second to the third class. It is easy to see that we obtain a semi-transitive digraph. □

This implies a number of earlier results on representability, including that of outerplanar graphs, subdivision graphs, and prisms (see Theorems 5, 6 and Proposition 7). The theorem also shows that 2-degenerate are representable, as well as graphs of maximum degree 3 (via Brooks theorem).

This result does not extend to higher chromatic numbers. The examples in Fig. 1 show that 4-colorable graphs can be non-representable. We can, however, obtain a results in terms of the girth of the graph, which is the length of its shortest cycle.

**Proposition 23.** Let $G$ be a graph whose girth is greater than its chromatic number. Then, $G$ is representable.

**Proof.** Suppose the graph is colored with $\chi(G)$ natural numbers. Orient the edges of the graph from small to large colors. There is no directed path with more than $\chi(G) - 1$ arcs, but since $G$ contains no cycle of $\chi(G)$ or fewer arcs, there can be no shortcut. Hence, the digraph is semi-transitive. □

Theorem 22 also implies that the Petersen graph is representable, turning down a conjecture in [4]. We can show that the graph is actually 3-representable. We give here two of its 3-representations, related to the numbering in Fig. 2, that were found in [6]:

- 1, 3, 8, 7, 2, 9, 6, 10, 7, 4, 9, 3, 5, 4, 1, 2, 8, 3, 10, 7, 6, 8, 5, 10, 1, 9, 4, 5, 6, 2
- 1, 3, 4, 10, 5, 8, 6, 7, 9, 10, 2, 7, 3, 4, 1, 2, 8, 3, 5, 10, 6, 8, 1, 9, 7, 2, 6, 4, 9, 5

![Figure 2. Petersen’s graph](image)

The following argument shows that Petersen’s graph is not 2-representable. Suppose that the graph is 2-representable and $W$ is a word 2-representing it. Let $x$ be a letter in $W$ such that there are minimum number of letters between the two appearances of $x$. Clearly, there are exactly three different letters between them. By symmetry, we can assume that $x = 1$ and by Proposition 1 we can assume that $W$ starts with 1. So, letters 2, 5, and 6 are between the two 1’s and because of symmetry, the fact that Petersen’s graph is edge-transitive (that is, each of its edges can be made “internal”), and taking into account that nodes 2,
5, and 6 are pairwise not adjacent, we can assume that \( W = 12561W_16W_25W_32W_4 \) where \( W_i \)'s are some factors for \( i = 1, 2, 3, 4 \). To alternate with 6 but not to alternate with 5, we must have \( 8 \in W_1 \) and \( 8 \in W_2 \). Also, to alternate with 2 but not to alternate with 5, we must have \( 3 \in W_3 \) and \( 3 \in W_4 \). But then 8833 is a subsequence in \( W \) and thus 8 and 3 are not adjacent in the graph, a contradiction.

We explore now further graph properties that are orthogonal to representability.

**Non-representable graphs.** One of the open problems posed in [4] was the following.

**Problem 1.** Are there any non-representable graphs that do not satisfy the conditions of Theorem 4?

We give here the positive answer. A counterexample to the converse of Theorem 4 is given by the graph in Fig. 3 called co-(\( T_2 \)) in [7]. It is easy to check that the induced neighborhood of any node of the graph co-(\( T_2 \)) is a comparability graph.

![Figure 3. Co-(\( T_2 \)) graph](image)

**Theorem 24.** The graph co-(\( T_2 \)) is non-representable.

*Proof.* Assume that the graph in Fig. 3 is \( k \)-representable for some \( k \) and \( W \) is a word-representant for it. The vertices 1,2,3,4 form a clique; so, their appearances \( 1^i, 2^i, 3^i, 4^i \) in \( W \) must be in the same order for each \( i = 1, 2, \ldots, k \). By symmetry and Proposition 1 we may assume that the order is 1234. Now let \( I_1, I_2, \ldots, I_k \) be the set of all \([2^i, 4^i]\)-intervals in \( W \). Two cases are possible.

1. There is an interval \( I_j \) such that 7 belongs to it. Then since 2,4,7 form a clique, 7 must be inside each of the intervals \( I_1, I_2, \ldots, I_k \). But then 7 is adjacent to 1, a contradiction.
2. 7 does not belong to any of the intervals \( I_1, I_2, \ldots, I_k \). Again, since 7 is adjacent to 2 and 4, each pair of consecutive intervals \( I_j, I_{j+1} \) must be separated by a single 7. But then 7 is adjacent to 3, a contradiction.

\( \square \)

**Remark 25.** Note that the existence of edges between vertices 5,6,7 was not used in the proof of Theorem 24.

So, we actually have four counter-examples to the converse of Theorem 4.

What about other non-representable graphs? Or, rather, which important classes of graphs are not contained in the class of representable graphs? We establish the following classes to be not necessarily representable:

- chordal (see the rightmost graph in Fig. 1), and thus perfect,
- line (second graph in Fig. 1),
- co-trees (the graph co-\( T_2 \) earlier), and thus co-bipartite and co-comparability,
• 2-outerplanar (the first and last graphs in Fig. 1), and thus planar,
• split (the last graph in Fig. 1),
• 3-trees (the last graph in Fig. 1), and thus partial 3-trees.

On the other hand, the 4-clique, $K_4$, is representable and it belongs to all these classes.

**The effect of graph operations.** One may want to explore which operations on graphs preserve representability (or non-representability). We pinpoint one such operation, and list others that are orthogonal.

1. The following operation on a representable graph yields a representable graph: Replace any node with a comparability graph, connecting all the new nodes to the neighbors of the original node. I. e., replacing a node with a *module* that is a comparability graph.

2. A generalization of Proposition 2 on identifying cliques of size more than 1 from two representable graphs is false. Indeed, consider the rightmost graph in Fig. 1 without a node of degree 2 connected to the end points of edge $e$ (denote this graph by $G$), and identify $e$ with an edge in a triangle $T$ resulting in obtaining the rightmost graph in Fig. 1. Both $G$ and $T$ are representable, but gluing them through an edge (a clique of size 2) results in a non-representable graph.

3. The complement to a non-representable graph can be permutationally representable: see, for example, the second graph in Fig. 1.

4. Not much can be said in general on the taking line graph operation. For example, the second non-representable graph in Fig. 1 is obtained from $K_{2,3}$ together with an edge between the nodes of degree 3, which is representable. On the other hand, there are many easy constructible examples when representable graphs go to representable graphs by taking line graph operation.

6. **Concluding Remarks and Open Questions**

It is natural to ask about optimization problems on representable graphs. Theorem 22 implies that many classical optimization problems are NP-hard on representable graphs:

**Observation 26.** The optimization problems Independent Set, Dominating Set, Graph Coloring, Clique Partition, Clique Covering are NP-hard on representable graphs.

Note that it may be relevant whether the representation of the graph as a semi-transitive digraph is given; solvability under these conditions is open.

However, some problems remain polynomially solvable:

**Observation 27.** The Clique problem is polynomially solvable on representation graphs.

Indeed, we can simply use the fact that the neighborhood of any node is a comparability graph. The clique problem is easily solvable on comparability graphs. Thus, it suffices to search for the largest clique within all induced neighborhoods.

There are still many questions one can ask on representable graphs, some of which are stated below.

1. Is it NP-hard to decide whether a graph is representable?
2. What is a tighter upper bound on the representation number of a graph, in terms of $n$? We know that it lies between $n/2$ and $n$.
3. Can one characterize the forbidden subgraphs of representable graphs? This problem seems to be hard since even for 2-representable (i. e. circle) graphs such a characterization is unknown.
ON REPRESENTABLE GRAPHS, SEMI-TRANSITIVE ORIENTATIONS, AND THE REPRESENTATION NUMBERS

REFERENCES

[1] A. Brandstädt, V. Bang Lee, J. P. Spinrad. Graph Classes: A Survey. Monographs on Discrete Mathematics and Applications. SIAM, 1987.

[2] B. Courcelle: Circle graphs and Monadic Second-order logic, Journal of Applied Logic, to appear.

[3] R. Hegde and K. Jain: The hardness of approximating poset dimension, Electronic Notes in Discrete Mathematics 29 (2007), 435–443.

[4] S. Kitaev, A. Pyatkin: On representable graphs, Automata, Languages and Combinatorics 13 (2008) 1, 45–54.

[5] S. Kitaev and S. Seif: Word problem of the Perkins semigroup via directed acyclic graphs, Order, DOI 10.1007/s11083-008-9083-7 (2008).

[6] A. Konovalov, S. Linton: Search of representable graphs with constraint solvers, University of St Andrews, CIRCA technical report 2008/7.

[7] http://wwteo.informatik.uni-rostock.de/sgci/smallgraphs.html

[8] M. Yannakakis: The complexity of the partial order dimension problem, SIAM J. Algebraic Discrete Methods 3(3) (1982), pp. 351–358.