ASYMPTOTICS FOR HECKE EIGENVALUES OF AUTOMORPHIC FORMS ON
COMPACT ARITHMETIC QUOTIENTS WITH NON-TRIVIAL $K$-TYPES

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Abstract. In this paper, we describe the asymptotic distribution of Hecke eigenvalues in the Laplace
eigenvalue aspect for certain families of Hecke-Maass forms on compact arithmetic quotients. In
particular, we treat not only spherical, but also non-spherical Hecke-Maass forms, including remainder
estimates. Our asymptotic formulas are available for arbitrary simple connected algebraic groups over
number fields with cocompact arithmetic subgroups. In preceding studies on distributions of Hecke
eigenvalues the trace formula played an important role. Since the latter is currently not available for
our families, we rely on Fourier integral operator methods instead.

CONTENTS

1. Introduction 1
2. Asymptotics for Hecke eigenvalues and low-lying zeros for $\operatorname{PGL}(n, \mathbb{R})$ 2
3. Spectral asymptotics for kernels of Hecke operators 6
4. Non-equivariant asymptotics for Hecke eigenvalues and Sato-Tate equidistribution 13
5. Equivariant asymptotics for Hecke eigenvalues and Sato-Tate equidistribution 19
6. Examples 27
References 28

1. INTRODUCTION

In this paper, we describe the asymptotic distribution of Hecke eigenvalues for automorphic forms
on compact arithmetic quotients of semisimple Lie groups in the Laplace eigenvalue aspect in both
spherical and non-spherical settings. In the spherical case, such asymptotics were studied previously by
Sarnak [43] for $\operatorname{SL}(2, \mathbb{R})$, Imamoglu-Raulf [21] for $\operatorname{SL}(2, \mathbb{C})$, Matz [33] for $\operatorname{SL}(n, \mathbb{C})$, and Matz-Templier
[34] for $\operatorname{SL}(n, \mathbb{R})$, as well as Finis-Matz [11] and Finis-Lapid [10] for all simple reductive groups. To
our knowledge, non-spherical asymptotics have not been considered so far. Asymptotic formulas for
Hecke eigenvalues are related to the asymptotic distribution of Laplace eigenvalues of Hecke-Maas
forms, which is the content of Weyl’s law [9, 36, 29, 42], and imply Plancherel density theorems and
the Sato-Tate equidistribution theorems for Hecke eigenvalues [43, 7, 47, 49, 50, 3]. From the latter,
statistics of low-lying zeros of automorphic $L$-functions can be inferred [34, 50].

Our results cover basically two settings. Let $H$ denote a semisimple connected linear algebraic
group over the rational number field $\mathbb{Q}$, and write $\mathbb{A}$ for the adele ring of $\mathbb{Q}$. As usual, regard $H(\mathbb{Q})$
as a subgroup of $H(\mathbb{A})$ by the diagonal embedding, and assume that $H(\mathbb{Q}) \backslash H(\mathbb{A})$ is compact. On the
one hand, we derive non-equivariant asymptotics with remainder for Hecke eigenvalues of automorphic
forms in the space $L^2(H(\mathbb{Q}) \backslash H(\mathbb{A}))$ of square integrable functions on $H(\mathbb{Q}) \backslash H(\mathbb{A})$, see Theorem 4.3. On
the other hand, we prove equivariant asymptotics with remainder for Hecke eigenvalues of automorphic
forms belonging to specific $\sigma$-isotypic components $L^2_{\sigma}(H(\mathbb{Q}) \backslash H(\mathbb{A})) = L^2(H(\mathbb{Q}) \backslash H(\mathbb{A}))$
for any simple connected algebraic group $H$ and any $K$-type $\sigma \in K\hat{\sigma}$, where $K$ is a maximal compact subgroup of
$G := H(\mathbb{R})$, see Theorem 5.4. If $G$ is compact, by functoriality our non-equivariant results also cover
certain families of automorphic forms whose corresponding automorphic representations have discrete

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series representations at the archimedean places. Such automorphic forms have been considered before in \cite{1, 17, 19}, and \cite{20}, but the families considered by us are different. Our asymptotic formulas for Hecke eigenvalues do imply corresponding Plancherel density theorems and Sato-Tate equidistribution theorems, see Corollaries \ref{cor1} and \ref{cor2} for the non-equivariant, as well as Corollaries \ref{cor3} and \ref{cor4} for the equivariant case. They can also be applied to the study of statistics of low-lying zeros of automorphic $L$-functions as initiated by Katz-Sarnak in \cite{25}. The heuristics developed by them suggests that $L$-functions can be grouped into families according to the symmetry type exhibited in the distribution of their zeros, and was refined and confirmed for larger classes in \cite{50, 44}. In fact, following the approach of \cite{50, section11} and under some additional assumptions, our results can be used to study low-lying zeros in rather general situations, see Theorem \ref{th2}.

The major novelty of our approach, which was initiated in \cite{41}, consists in applying methods from the modern theory of partial differential equations, more precisely, the theory of Fourier integral operators, to the analysis of Hecke–Maass forms, since the trace formula, which is the main tool of previous approaches, is not available in our situation. In fact, the usual way to proceed in the spherical case is to test the trace formula with a spherical function, and then apply the Fourier inversion formula for the derivation of upper bounds \cite{34, 11}. However, for non-trivial $K$-types the Fourier inversion formula is unknown in general \cite{32}, and so far was established only for one dimensional $K$-types \cite{37, 18}. An alternative way could consist in combining Herb’s explicit formula for the inverse Fourier transform of semisimple orbital integrals \cite{19} with the invariant Paley-Wiener theorem \cite{6}, since in the cocompact case all $\mathbb{Q}$-rational points are semisimple. But since multiplicities of $K$-types are unknown in general, it is difficult to count non-spherical automorphic forms within this approach. Let us also mention that in the spherical situations considered previously, spectral asymptotics for the whole algebra of invariant differential operators were derived, which leads to more refined statements in the higher rank case, compare \cite{9, section8}. Within the FIO approach, one only considers eigenfunctions of a single elliptic differential operator, yielding less refined asymptotics. Nevertheless, they already suffice to derive Plancherel and Sato-Tate theorems, yielding equidistribution results in a simpler way. In a future paper, we intend to generalize our approach to non-compact arithmetic quotients.

The structure of this paper is as follows. In Section 2 we explain our results within a non-adelic framework in the case where $G = \text{PGL}(n, \mathbb{R})$ and $K = \text{PO}(n)$, and indicate how statements about statistics of low-lying zeros of principal $L$-functions can be derived from them in Section 2.4. We commence our analysis in Section 3 by establishing spectral asymptotics for kernels of Hecke operators by means of Fourier integral operators. Based on this, we first describe the asymptotic distribution of Hecke eigenvalues in Section 4 in the non-equivariant setting, and prove corresponding equidistribution theorems. After this, we turn to the study of the equivariant situation in Section 5 which besides the spectral asymptotics for kernels of Hecke operators derived in Section 3 relies on the Fourier inversion formula for orbital integrals. Examples are discussed in the final section. Throughout this paper, we shall use the notation $N := \{0, 1, 2, \ldots\}$ and $N_* := \{1, 2, \ldots\}$.

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2. **Asymptotics for Hecke eigenvalues and low-lying zeros for $\text{PGL}(n, \mathbb{R})$**

In this section, we shall explain our results within a non-adelic framework in the case where $G = \text{PGL}(n, \mathbb{R})$, $K = \text{PO}(n)$, and $n \geq 2$. We then apply them to the study of statistics of low-lying zeros of principal $L$-functions following \cite{50} and \cite{34}.

2.1. **Division algebras and projective linear groups.** Let $D$ be a central division algebra over $\mathbb{Q}$ such that $D \otimes _{\mathbb{Q}} \mathbb{R}$ is isomorphic to the ring $\mathcal{M}(n, \mathbb{R})$ of $(n \times n)$-matrices with real entries. Fix a specific ring isomorphism $D \otimes _{\mathbb{Q}} \mathbb{R} \cong \mathcal{M}(n, \mathbb{R})$, and define a reduced norm $\text{Nrd}$ on $D$ in terms of the determinant on $\mathcal{M}(n, \mathbb{R})$. By construction, $\text{Nrd}$ is a homogeneous polynomial with $\mathbb{Q}$-coefficients. Let $H := \text{PGL}(1, D)$ denote the projective linear group over $\mathbb{Q}$ defined by $D$, by which one understands
the automorphism group $\text{Aut}(D)$ of $D$, a simple connected algebraic group, see [27] Chapter VI, Section 23. Take a maximal order $O$ in $D$, and choose a $\mathbb{Z}$-basis $e_1 = 1, e_2, \ldots, e_n$ of $O$ so that $O = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$. Then, a morphism $f : \hat{H} := \text{GL}(1, D) \to \text{SL}(n^2 - 1)$ over $\mathbb{Q}$ is defined by the adjoint action $g \cdot x \mapsto g x g^{-1}$ of $\hat{H}$ on $D$, and the elements $e_2, e_3, \ldots, e_n$ of the basis. Here $\text{GL}(1, D)$ means the general linear group over $\mathbb{Q}$ defined by $D$, see [27] Chapter VI, Section 20], so that the set $\hat{H}(F)$ of $F$-rational points in $\hat{H}$ is the multiplicative group of $D \otimes \mathbb{Q} F$ for any extension field $F$ of $\mathbb{Q}$. The image of $f$ is identified with $H$, and the group of integral points $\Gamma := H(\mathbb{Z}) = H(\mathbb{Q}) \cap \text{SL}(n^2 - 1, \mathbb{Z})$ is a cocompact lattice of the real Lie group $G := H(\mathbb{R}) \cong \text{PGL}(n, \mathbb{R})$. Note that our main theorems are available for arbitrary cocompact arithmetic congruence subgroups, but we consider only $H(\mathbb{Z})$ here for simplicity.

Let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the $p$-adic number field and the ring of integers, respectively, and define the finite adele ring $\mathbb{A}_\text{fin}$ of $\mathbb{Q}$ as the restricted direct product $\mathbb{A}_\text{fin} := \prod_p \mathbb{Q}_p$, where $p$ moves over all prime numbers. Note that $\mathbb{Q}$ is regarded as a subring of the adele ring $\mathbb{A} := \mathbb{R} \times \mathbb{A}_\text{fin}$ of $\mathbb{Q}$ via the diagonal embedding. Set $K_p := H(\mathbb{Z}_p)$, and consider the open compact subgroup $K_\mathbb{Q} := \prod_p K_p$ of $H(\mathbb{A}_\text{fin})$, which satisfies $H(\mathbb{Q}) \cap K_0 = \Gamma$. Since $H(\mathbb{A}) = H(\mathbb{Q})(GK_0)$, it follows that the arithmetic quotient $\Gamma \backslash G$ is topologically isomorphic to $H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0$, because $\text{SL}(1, D) := \{ g \in \text{SL}(1, D) \mid \text{Nrd}(g) = 1 \}$ satisfies the Strong Approximation Property to $\infty$ and one has $\text{Nrd}(K_0) = \prod_p \mathbb{Z}_p$, compare [38]. It is known that there exists a finite set $S_0$ of primes such that $D \otimes \mathbb{Q}_p$ is isomorphic to $M(n, \mathbb{Q}_p)$ iff $p$ does not belong to $S_0$.

### 2.2. Asymptotics of Hecke eigenvalues

In what follows, and for the convenience of the reader who is not familiar with the adelic language, we shall translate our main theorems, which are stated in that framework, to the non-adelic setting. Let us begin by introducing for each element $\alpha \in H(\mathbb{Q}) \subset \text{SL}(n^2 - 1, \mathbb{Q})$ a Hecke operator $T_{\Gamma \alpha \Gamma}$ on the space $L^2(\Gamma \backslash G)$ of square integrable functions on $\Gamma \backslash G$ by setting

$$(T_{\Gamma \alpha \Gamma} \phi)(x) := \sum_{\Gamma \beta \in \Gamma \backslash \Gamma \alpha \Gamma} \phi(\beta x), \quad \phi \in L^2(\Gamma \backslash G).$$

For details, we refer the reader to Section 3.3. Consider further the right action of the maximal compact subgroup $K := \text{PO}(n) \subset G$ on $\Gamma \backslash G$, and recall the Peter-Weyl decomposition

$$L^2(\Gamma \backslash G) = \bigoplus_{\alpha \in \check{K}} L^2_{\sigma}(\Gamma \backslash G)$$

of $L^2(\Gamma \backslash G)$ into $\sigma$-isotypic components $L^2_{\sigma}(\Gamma \backslash G) := (L^2(\Gamma \backslash G) \otimes \sigma^\vee)^K$, where $\sigma^\vee$ denotes the contragredient representation of $\sigma \in \check{K}$, and $\check{K}$ is the unitary dual of $K$. The operator $T_{\Gamma \alpha \Gamma}$ obviously commutes with the right action of $K$ so that $T_{\Gamma \alpha \Gamma}$ acts on each subspace $L^2_{\sigma}(\Gamma \backslash G)$.

From the viewpoint of automorphic representations, there exists an orthonormal basis $\{ \phi_j \}_{j \geq 0}$ in $L^2(\Gamma \backslash G)$ such that each $\phi_j$ is a simultaneous eigenfunction for the Beltrami-Laplace operator $\Delta$ on $H(\mathbb{Q})$ and the Hecke operators $T_{\Gamma \alpha \Gamma}$ for every $\alpha \in H(\mathbb{Q})$ such that the denominators of the entries of $\alpha$ in $\text{SL}(n^2 - 1, \mathbb{Q})$ are prime to $\prod_{p \in S_0} p$, where $S_0$ was introduced above. Their eigenvalues are denoted by

$$\Delta \phi_j = \lambda_j \phi_j, \quad T_{\Gamma \alpha \Gamma} \phi_j = \lambda_j(\alpha) \phi_j,$$

and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Furthermore, in the present case the strong multiplicity-one theorem implies that $\phi_j$ belongs to a single $K$-type $\sigma$, compare [2]. Set $\mu_j := \sqrt{\lambda_j}$, $d := \dim G = n^2 - 1$, and $d_\sigma := \dim \sigma$. Our first main result deals with the asymptotic distribution of Hecke eigenvalues and is a special case of Theorems 1.3 and 5.4.

**Theorem 2.1.** For any prime $p \not\in S_0$, $\kappa \in \mathbb{N}$, $\alpha \in H(\mathbb{Q}) \cap M(n^2 - 1, p^{-\kappa} \mathbb{Z})$, and small $\varepsilon > 0$,

$$\sum_{\mu_j \leq \mu} \lambda_j(\alpha) = \delta_\alpha \frac{\text{vol}(\Gamma \backslash G) \omega_d}{(2\pi)^d} \mu^d + O(\mu^{d-1} p^{n^2 \kappa}).$$
\[
\sum_{\phi \in L^2(\Gamma, G)} \lambda_j(\phi) = \delta_\alpha \frac{d\sigma \operatorname{vol}(\Gamma \setminus G / K) \varpi^{d-\dim K}}{(2\pi)^{d-\dim K}} \mu^{d-\dim K} + O_\varepsilon \left( \mu^{d-\dim K - \frac{d-\dim K}{d-\dim K - 1} + \varepsilon} p^{-n^2} \right),
\]
where \( \delta_\alpha := 1 \) if \( \alpha \in \Gamma \), \( \delta_\alpha := 0 \) otherwise, \( \operatorname{vol} \) denotes the Riemannian volume, and \( \varpi_m := \pi^m / \Gamma(1 + m) \) stands for the volume of the unit m-sphere.

2.3. Equidistribution theorems for Satake parameters. As a consequence of Theorem 2.1, we obtain certain equidistribution statements which are related to the generalized Ramanujan conjecture. Choose a prime \( p \not\in S_0 \) and define a subring \( \mathbb{Z}(p) \) of \( \mathbb{Q} \) by setting \( \mathbb{Z}(p) := \{ p^{-1}m \mid m \in \mathbb{Z}, l \in \mathbb{Z} \} \). We have \( G_p := H(\mathbb{Q}_p) \cong \text{PGL}(n, \mathbb{Q}_p) \) as well as \( K_p \cong \text{PGL}(n, \mathbb{Z}_p) \), and in view of \( H(\mathbb{A}) = H(\mathbb{Q})(GK_0) \) and the strong approximation property for \( SL(1, D) \) one can prove that the mapping
\[
K_p \setminus G_p / K_p \ni K_p \alpha K_p \mapsto H(\mathbb{Q}) \cap K_0 \alpha K_0 \in H(\mathbb{Q}(p)) / \Gamma
\]
is bijective. As a consequence, the unramified Hecke algebra \( H^u(K_p \setminus H(\mathbb{Q}(p)) / K_p) \) is isomorphic to \( C_c(\Gamma \setminus H(\mathbb{Q}(p)) / \Gamma) \), and it is known that \( H^u(K_p \setminus H(\mathbb{Q}(p)) / K_p) \) is generated by the characteristic functions of \( K_p \gamma K_p, 1 \leq t \leq n-1 \), where
\[
\gamma_t := \text{diag}(p, p, 1, \ldots, 1) \mathbb{Q}_p^\times \in \text{PGL}(n, \mathbb{Q}_p).
\]

Next, define \( \beta_t \) by \( \beta_t \Gamma := H(\mathbb{Q}) \cap K_0 \gamma^{-1} K_0 \), and the Satake parameter of a Hecke-Maass form \( \phi_j \) at \( p \) as the n-tuple \( \alpha^{(j)}(p) = (\alpha_1^{(j)}(p), \alpha_2^{(j)}(p), \ldots, \alpha_n^{(j)}(p)) \in \hat{T} / \mathfrak{S}_n \) consisting of roots of the equation
\[
x^n - \frac{1}{\pi^4} \lambda_j(\beta_1)x^{n-1} + \cdots + (-1)^{p-n} \frac{1}{\pi^2} \lambda_j(\beta_t)x^{n-t} + \cdots + (-1)^n \frac{1}{\pi^4} \lambda_j(\beta_{n-1})x + (-1)^n \]0,\]
where \( \hat{T} := \{(u_1, \ldots, u_n) \in (\mathbb{C}^\times)^n \mid u_1 \cdots u_n = 1\} \), and \( \mathfrak{S}_n \) denotes the symmetric group of degree \( n \). The generalized Ramanujan conjecture predicts that for \( j > 0 \) the Satake parameter \( \alpha^{(j)}(p) \) belongs to \( \hat{T}_c / \mathfrak{S}_n \), where \( \hat{T}_c := \{(u_1, \ldots, u_n) \in \hat{T} \mid |u_t| = 1 \text{ for all } 1 \leq t \leq n\} \). We now introduce the Plancherel measure \( \hat{\mu}^\text{Pl, ur} \) and the Sato-Tate measure \( \hat{\mu}^\text{ST} \) on \( \hat{T} / \mathfrak{S}_n \). Their supports are contained in \( \hat{T}_c \), and with respect to the coordinates \( (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \hat{T}_c \) are defined as
\[
\hat{\mu}^\text{Pl, ur} := c_p \prod_{1 \leq k < j \leq n} \frac{|e^{i\theta_k} - e^{i\theta_j}|^2}{|p^{-1}e^{i\theta_k} - e^{i\theta_j}|^2} d\theta_1 \cdots d\theta_{n-1},
\]
\[
\hat{\mu}^\text{ST} := c_{\infty} \prod_{1 \leq k < j \leq n} \frac{|e^{i\theta_k} - e^{i\theta_j}|^2}{|\pi^2 e^{i\theta_k} - e^{i\theta_j}|^2} d\theta_1 \cdots d\theta_{n-1},
\]
where \( c_p \) and \( c_{\infty} \) are constants determined by requiring \( \hat{\mu}^\text{Pl, ur} / \mathfrak{S}_n = \hat{\mu}^\text{ST} / \mathfrak{S}_n = 1 \). Set
\[
\mathfrak{F}(\mu) := \{ j \in \mathbb{N} \mid \mu_j \leq \mu \}, \quad \mathfrak{F}_\sigma(\mu) := \{ j \in \mathbb{N} \mid \mu_j \leq \mu, \phi_j \in L^2(\hat{T} / \mathfrak{S}_n) \}.
\]
The following two corollaries are direct consequences of Theorem 2.1 and present some evidence towards the generalized Ramanujan conjecture. They are special cases of Corollaries 4.5 and 4.6 and Corollaries 5.3 and 5.4 respectively.

**Corollary 2.2 (Plancherel density theorem).** Choose a prime \( p \not\in S_0 \). Then, one obtains
\[
\lim_{\mu \to \infty} \frac{1}{\mathfrak{F}(\mu)} \sum_{j \in \mathfrak{F}(\mu)} \delta_{\alpha^{(j)}(p)} = \hat{\mu}^\text{Pl, ur}, \quad \lim_{\mu \to \infty} \frac{1}{\mathfrak{F}_\sigma(\mu)} \sum_{j \in \mathfrak{F}_\sigma(\mu)} \delta_{\alpha^{(j)}(p)} = \hat{\mu}^\text{Pl, ur},
\]
where \( \delta_{\alpha^{(j)}(p)} \) denotes the Dirac delta measure at \( \alpha^{(j)}(p) \in \hat{T} / \mathfrak{S}_n \).

**Corollary 2.3 (Sato-Tate equidistribution theorem).** Choose a prime \( p \not\in S_0 \), and let \( \{ (p_k, \mu_k) \}_{k \geq 1} \) be a sequence such that \( p_k \to \infty \) and \( p_k / \mu_k \to 0 \) as \( k \to \infty \) for any integer \( l \geq 1 \). Then
\[
\lim_{k \to \infty} \frac{1}{\mathfrak{F}(\mu_k)} \sum_{j \in \mathfrak{F}(\mu_k)} \delta_{\alpha^{(j)}(p_k)} = \hat{\mu}^\text{ST}, \quad \lim_{k \to \infty} \frac{1}{\mathfrak{F}_\sigma(\mu_k)} \sum_{j \in \mathfrak{F}_\sigma(\mu_k)} \delta_{\alpha^{(j)}(p_k)} = \hat{\mu}^\text{ST}.
\]
2.4. Low-lying zeros of principal $L$-functions. As another consequence of Theorem 2.1, we are able to give some results about the statistics of low-lying zeros of $L$-functions. By the strong multiplicity-one theorem, each $\phi_j$ determines a unique automorphic representation $\pi_j = \otimes_v \pi_{j,v}$ of $H$, to which one can associate the principal $L$-function

$$L(s, \pi_j) := \prod_p L_p(s, \pi_{j,p})$$

by Godement-Jacquet theory, see [14, 24, 2] for details. In our context, for each prime $p \notin S_0$, the local $L$-factor $L_p(s, \pi_{j,p})$ is given in terms of the Satake parameters by

$$L_p(s, \pi_{j,p}) := (1 - \alpha_1^{(j)}(p)p^{-s})^{-1}(1 - \alpha_2^{(j)}(p)p^{-s})^{-1}\cdots(1 - \alpha_n^{(j)}(p)p^{-s})^{-1}.$$ 

Furthermore, $L(s, \pi_j)$ can be analytically continued to an entire function on $\mathbb{C}$ if $j > 0$. If $j = 0$, $\phi_0$ is constant on $\Gamma \backslash G$, and consequently on $\Gamma \backslash G/K$, every $\pi_{0,v}$ equals the trivial representation, and $L(s, \pi_0) = \prod_{n=1}^{d} \zeta(s + \frac{n+1}{2} - t)$, where $\zeta(s)$ is the Riemann zeta function. As a consequence, and following [44] and [45], we can regard $\mathfrak{F}(\sigma)$ as a family of $L$-functions. Notice that an automorphic representation may contain Hecke-Maass forms $\phi_j$ belonging to several $K$-types $\sigma$. Later, we will introduce families of automorphic representations in a more general setting following [50], $\mathfrak{F}(\sigma)$ being an instance of such a family in the present context.

In order to study the statistics of low-lying zeros of the $L$-functions $L(s, \pi_j)$ belonging to the family $\mathfrak{F}(\sigma)$ one introduces the average analytic conductor

$$\log C(\mathfrak{F}(\sigma)(\mu)) := \frac{1}{|\mathfrak{F}(\sigma)(\mu)|} \sum_{j \in \mathfrak{F}(\sigma)(\mu)} \log C(\pi_j),$$

where $C(\pi_j)$ denotes the analytic conductor of $\pi_j$, and we refer to [23] and [22, Chapter 5] for its definition. It is obvious that $C(\mathfrak{F}(\sigma)(\mu)) \approx \mu^2$ as $\mu \to \infty$. Choose a Paley-Wiener function $\hat{\Phi}$ on $\mathbb{C}$ whose Fourier transform $\hat{\Phi}$ has sufficiently small support on $\mathbb{R}$, and define the average one-level density of the family $\mathfrak{F}(\sigma)(\mu)$ as

$$D_1(\mathfrak{F}(\sigma)(\mu); \Phi) := \frac{1}{|\mathfrak{F}(\sigma)(\mu)|} \sum_{j \in \mathfrak{F}(\sigma)(\mu)} \sum_{\gamma_j} \hat{\Phi}(\frac{j}{2\pi} \log C(\mathfrak{F}(\sigma)(\mu)),$$

where $\gamma_j$ ranges over all non-trivial zeros of $L(s, \pi_j)$. We then have the following

**Theorem 2.4.** If $n \geq 3$,

$$\lim_{\mu \to \infty} D_1(\mathfrak{F}(\sigma)(\mu); \Phi) = \int_{-\infty}^{\infty} \hat{\Phi}(x) W(U)(x) \, dx = \hat{\Phi}(0),$$

while if $n = 2$,

$$\lim_{\mu \to \infty} D_1(\mathfrak{F}(\sigma)(\mu); \Phi) = \begin{cases} \int_{-\infty}^{\infty} \hat{\Phi}(x) W(SO_{even})(x) \, dx & \text{if } \sigma \text{ is trivial,} \\ \int_{-\infty}^{\infty} \hat{\Phi}(x) W(SO_{odd})(x) \, dx & \text{if } \sigma = \text{det}, \\ \int_{-\infty}^{\infty} \hat{\Phi}(x) W(O)(x) \, dx & \text{if } \sigma \text{ is 2-dimensional.} \end{cases}$$

Here the density functions $W$ are given by

$$W(U) = 1, \quad W(SO_{even}) = 1 + \frac{\sin 2\pi x}{2\pi x}, \quad W(SO_{odd}) = 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x),$$

$$W(O) = \frac{1}{2} W(SO_{even}) + \frac{1}{2} W(SO_{odd}).$$

**Proof.** If $\sigma$ is 1-dimensional, the statement is essentially contained in [34, Theorem 2.1] for general $n$, and for $n = 2$ in [11, Theorem 1.4], because our family can be obtained from the families considered there by restricting finitely many local components of automorphic representations. As for the case $n = 2$, the statement can be reduced to the case $\dim \sigma = 1$ by considering $K$-types of irreducible unitary representations of $G$. Hence, we are left with the task of proving the assertion for $n > 2$ and $\dim \sigma > 1$. Now, [2] implies that for $j > 0$ the automorphic representation of $\text{PGL}(n, \mathbb{Q})$ associated to
a Hecke-Maass form $\phi_j$ is cuspidal, so that $\mathfrak{g}_s(\mu)$ is essentially cuspidal, compare also [45]. In addition, Theorem 2.7 implies that $\mathfrak{g}_s(\mu)$ has rank zero in the sense of [45]. Consequently, the assertion follows with the same arguments that were used in [50], Section 12], see also [44], Section 2].

For the non-equivariant family $\mathfrak{g}(\mu)$, we can also define the average analytic conductor $C(\mathfrak{g}(\mu))$ and the average 1-level density $D_1(\mathfrak{g}(\mu); \Phi)$ as above. The multiplicity of a $K$-type $\sigma$ of a parabolically induced representation of $G$ is bounded by the dimension of $\sigma$, cf. [36] p. 293, because for $G = \text{PGL}(n, \mathbb{R})$, any Levi subgroup of a cuspidal parabolic subgroup is isomorphic to a product of copies of $\text{GL}(1, \mathbb{R})$ and $\text{GL}(2, \mathbb{R})$ modulo the center. Therefore, for a single $\pi_j$, the number of Hecke-Maass forms $\phi \in \pi_j$ with $\mu_\phi \leq \mu$ increases by the order $\mu^n$ for $n > 2$ or $\mu$ for $n = 2$, where $\mu_\phi > 0$ is defined by $D_0\phi = \mu_\phi^2\phi$. Consequently, they can be neglected in the total growth of $\mathfrak{g}(\mu)$, and we obtain $C(\mathfrak{g}(\mu)) \asymp \mu^n$. Thus, by the same argument as in the proof of Theorem 2.3 for $n ≥ 3$, we obtain

$$\lim_{\mu \to \infty} D_1(\mathfrak{g}(\mu); \Phi) = \begin{cases} \int_{-\infty}^{\infty} \Phi(x) W(U)(x) \, dx & \text{if } n ≥ 3, \\ \int_{-\infty}^{\infty} \Phi(x) W(O)(x) \, dx & \text{if } n = 2. \end{cases}$$

3. Spectral asymptotics for kernels of Hecke operators

In this section, we begin our analysis by deriving spectral asymptotics for kernels of Hecke operators by means of Fourier integral operators.

3.1. Non-equivariant spectral asymptotics. Let $M$ be a closed Riemannian manifold $M$ of dimension $d$ and $P_0$ an elliptic classical pseudodifferential operator on $M$ of degree $m$, which is assumed to be positive and symmetric. Denote its unique self-adjoint extension by $P$, and let $\{\phi_j\}_{j ≥ 0}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $P$ with eigenvalues $\{\lambda_j\}_{j ≥ 0}$ repeated according to their multiplicity. Let $p(x, \xi)$ be the principal symbol of $P_0$, which is strictly positive and homogeneous in $\xi$ of degree $m$ as a function on $T^*M \setminus \{0\}$, that is, the cotangent bundle of $M$ without the zero section. Here and in what follows $(x, \xi)$ denotes an element in $T^*Y \simeq Y \times \mathbb{R}^d$ with respect to the canonical trivialization of the cotangent bundle over a chart domain $Y \subset M$. Consider further the $m$-th root $Q := \sqrt[\mu]{P}$ of $P$ given by the spectral theorem. It is well known that $Q$ is a classical pseudodifferential operator of order 1 with principal symbol $q(x, \xi) := \sqrt[p(x, \xi)]{q_0(x, \xi)}$ and the first Sobolev space as domain. Again, $Q$ has discrete spectrum, and its eigenvalues are given by $\mu_j := \sqrt[\mu]{\lambda_j}$. The spectral properties of $Q$ can be described by studying the spectral function of $Q$, which in terms of the basis $\{\phi_j\}$ is given by

$$e(x, y, \mu) := \sum_{\mu_j ≤ \mu} \phi_j(x)\overline{\phi_j(y)},$$

and belongs to $C^\infty(M \times M)$ as a function of $x$ and $y$ for any $\mu \in \mathbb{R}$. Let $s_\mu$ be the spectral projection onto the sum of eigenspaces of $Q$ with eigenvalues in the interval $(\mu, \mu + 1]$, and denote its Schwartz kernel by

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu).$$

To obtain an asymptotic description of the spectral function of $Q$, let $g \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ be such that $\hat{g}(0) = 1$ and supp $\hat{g} \in (-\delta/2, \delta/2)$ for an arbitrarily small $\delta > 0$, and define the approximate spectral projection operator

$$\tilde{s}_\mu u := \sum_{j=0}^{\infty} g(\mu - \mu_j)E_j u, \quad u \in L^2(M),$$

where $E_j$ denotes the orthogonal projection onto the subspace spanned by $\phi_j$. Clearly,

$$K_{\tilde{s}_\mu}(x, y) := \sum_{j=0}^{\infty} g(\mu - \mu_j)\phi_j(x)\overline{\phi_j(y)} \in C^\infty(M \times M)$$

constitutes the Schwartz kernel of $\tilde{s}_\mu$. Describing $\tilde{s}_\mu$ as a Fourier integral operator one obtains the following
Proposition 3.1. [41, Proposition 3.1] Suppose that the cospheres \( S^* \subset M := \{(x, \xi) \in T^*M \mid p(x, \xi) = 1\} \) are strictly convex. Then, as \( \mu \to +\infty \), for any fixed \( x, y \in M \), and \( 0 < N < 1, 2, 3, \ldots \) one has the expansion

\[
K_{\tilde{s}_\mu}(x, y) = \mu^{d-1} \frac{d_{x,y}}{2} \left[ \sum_{r=0}^{N-1} L_r(x, y, \mu) + R_N(x, y, \mu) \right]
\]

up to terms of order \( O(\mu^{-\infty}) \), where

\[
\delta_{x,y} := \begin{cases} 0, & y = x, \\ d - 1, & y \neq x. \end{cases}
\]

The coefficients in the expansion and the remainder \( R_N(x, y, \mu) = O_{x,y}(\mu^{-N}) \) term can be computed explicitly; if \( y = x \), they are uniformly bounded in \( x \) and \( y \), while if \( y \neq x \), they satisfy the bounds

\[
L_r(x, y, \mu) \ll \text{dist} (x, y)^{-(d-1)/2-r} \mu^{-r}, \quad R_N(x, y, \mu) \ll \text{dist} (x, y)^{-(d-1)/2-N} \mu^{-N},
\]

where \( \text{dist} (x, y) \) denotes the geodesic distance between two points belonging to the same connected component, while \( \text{dist} (x, y) := \infty \) for points in different components. On the other hand, \( K_{\tilde{s}_\mu}(x, y) \) is rapidly decreasing as \( \mu \to -\infty \).

To describe the leading term more explicitly, note that by [41, (3.5)]

\[
K_{\tilde{s}_\mu}(x, x) = \frac{\mu^d}{(2\pi)^{d+1}} \int_\mathbb{R} \int_\mathbb{R} e^{i\mu(t-Rt)} I(\mu, R, t, x) dR dt
\]

up to terms of order \( O(\mu^{-\infty}) \), where \( I(\mu, R, t, x) \) is a compactly supported smooth function in \((R, t)\) satisfying \( I(\mu, 1, 0, x) = \text{vol} S^*_x(M) \). If we now apply the stationary phase theorem to the above oscillatory integral with phase function \( t - Rt \) we obtain

\[
K_{\tilde{s}_\mu}(x, x) = \frac{\mu^d}{(2\pi)^{d}} \text{vol} S^*_x(M) + O(\mu^{-d-2})
\]

as \( \mu \to +\infty \), the only critical point being \((R, t) = (1, 0)\). This could also be read off directly from [8, (2.2)].

3.2. Equivariant spectral asymptotics. Keeping the notation as above, assume now that \( M \) carries an isometric action of a compact Lie group \( K \), and consider the right regular representation \( \pi \) of \( K \) on \( L^2(M) \) with corresponding Peter-Weyl decomposition

\[
L^2(M) = \bigoplus_{\sigma \in \widehat{K}} L^2_{\sigma}(M), \quad L^2_{\sigma}(M) := \Pi_\sigma L^2(M),
\]

where \( \Pi_\sigma \) denotes the unitary dual of \( K \), which we identify with the set of characters of \( K \), and

\[
\Pi_\sigma := d_{\sigma} \int_K \sigma(k) \pi(k) dk
\]

the orthogonal projector onto the \( \sigma \)-isotypic component \( L^2_{\sigma}(M) \), \( dk \) being Haar measure and \( d_{\sigma} \) the dimension of an irreducible representation \((V_\sigma, \pi_\sigma)\) of \( K \) in the class \( \sigma \in \widehat{K} \). Note that \( L^2_{\sigma}(M) \simeq (L^2(M) \otimes \sigma^*)^K \), where \((L^2(M) \otimes \sigma^*)_K = (L^2(M) \otimes V_\sigma)^K \) consists of \( L^2 \)-functions \( \phi : M \to V_\sigma \) that are \( K \)-equivariant in the sense that \( \phi(m \cdot k) = \pi_\sigma(k)^{-1} \phi(m) \). The components of \( \phi \) as \( L^2 \)-functions from \( M \) to \( \mathbb{C} \) correspond then to elements in \( L^2(M)_\sigma \). Further, suppose that \( P \) commutes with \( \pi \), and that the orthonormal basis \( \{\phi_j\}_{j \geq 0} \) is compatible with the decomposition \((4.3)\) in the sense that each \( \phi_j \) lies in some \( L^2_{\sigma}(M) \). Then every eigenspace of \( P \) is invariant under \( \pi \), and decomposes into irreducible \( K \)-modules spanned by eigenfunctions. The fine structure of the spectrum of \( P \) is described

\footnote{This condition holds, for example, if \( P_0 = \Delta \) equals the Beltrami–Laplace operator, since then \( p(x, \xi) = \|\xi\|^2_x \).}

\footnote{For the case \( x = y \) see also [31, Proposition 2.1].}
by the spectral function of the operator $Q_x := \Pi_\sigma \circ Q \circ \Pi_\sigma = \Pi_\sigma \circ Q = Q \circ \Pi_\sigma$, which is also called the reduced spectral function, and given by

$$e_\sigma(x, y, \mu) := \sum_{\mu_j \leq \mu, \phi_j \in L^2(\mathcal{M})} \phi_j(x)\phi_j(y).$$

To study it, one considers the composition $s_\mu \circ \Pi_\sigma$, or rather $\tilde{s}_\mu \circ \Pi_\sigma$, whose kernel has the spectral expansion

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \sum_{j \geq 0, \phi_j \in L^2(\mathcal{M})} \rho(\mu - \mu_j)\phi_j(x)\phi_j(y).$$

Write $\mathcal{O}_x := x \cdot K$ for the $K$-orbit through $x$. Similarly to Proposition 3.1, using Fourier integral operator methods one proves the following.

**Proposition 3.2.** [41, Proposition 3.3] Suppose that $K$ acts on $M$ with orbits of the same dimension $\kappa \leq d - 1$ and that the cospheres $S_x^\kappa M := \{(x, \xi) \in T_x^*M \mid p(x, \xi) = 1\}$ are strictly convex. Then, for any fixed $x, y \in M$, $\sigma \in \hat{K}$, and $N = 0, 1, 2, 3, \ldots$ one has the expansion

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \mu^{d - \kappa - 1}d_\sigma \left[ \sum_{r=0}^{N-1} L_r^\sigma(x, y, \mu) + R_N^\sigma(x, y, \mu) \right]$$

up to terms of order $O(\mu^{-\infty})$ as $\mu \to +\infty$, where

$$\varepsilon_{x, y} := \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ d - 1 + \kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients in the expansion and the remainder term can be computed explicitly; if $y \in \mathcal{O}_x$, they satisfy the bounds

$$L_r^\sigma(x, y, \mu) \ll \sup_{u \leq 2r} \|D^u\sigma\|_\infty \mu^{-r}, \quad R_N^\sigma(x, y, \mu) \ll \sup_{u \leq 2N + [\frac{d}{2} + 1]} \|D^u\sigma\|_\infty \mu^{-N},$$

uniformly in $x$ and $y$, where $D^u$ denote differential operators on $K$ of order $u$, and if $y \notin \mathcal{O}_x$, the bounds

$$L_r^\sigma(x, y, \mu) \ll \sup_{u \leq 2r} \|D^u\sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d - \kappa - 1}{2}} \mu^{-r},$$

$$R_N^\sigma(x, y, \mu) \ll \sup_{u \leq 2N + [\frac{d}{2} + 1]} \|D^u\sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d - \kappa - 1}{2}} \mu^{-N},$$

where $\text{dist}(x, \mathcal{O}_y) := \min \{\text{dist}(x, z) \mid z \in \mathcal{O}_y\}$. On the other hand, $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ is rapidly decreasing as $\mu \to -\infty$. \hfill \Box

As far as the leading term is concerned, by [41, Proposition 4.1] one has as $\mu \to +\infty$

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \frac{\mu^{d - \kappa - 1}}{(2\pi)^{d - \kappa}d_\sigma[\pi_{\sigma\mid K_x} : 1]} \text{vol} \left( [\Omega \cap S_x^\kappa(\mathcal{M})]/K \right) + O(\mu^{d - \kappa - 2}),$$

where $[\pi_{\sigma\mid K_x} : 1]$ is a Frobenius factor that denotes the multiplicity of the trivial representation in the restriction of $\pi_\sigma$ to the stabilizer $K_x$ of $x$, and $\Omega$ is the zero level of the momentum map corresponding to the Hamiltonian $K$-action on $T^*M$.

\footnote{Note that the additional assumption made in [41, Section 3.2] and [39] that $K$ acts effectively on $M$ is unnecessary.}
3.3. Spectral asymptotics for Hecke operators. In what follows, we shall apply the previous considerations to derive asymptotics for kernels of Hecke operators in the eigenvalue aspect. To introduce the setting, let $G$ be a $d$-dimensional real semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Denote by $\langle X, Y \rangle := \text{tr}(\text{ad} X \circ \text{ad} Y)$ the Cartan-Killing form on $\mathfrak{g}$ and by $\theta$ a Cartan involution of $\mathfrak{g}$. Let

\begin{equation}
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}
\end{equation}

be the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of $\theta$, corresponding to the eigenvalues $+1$ and $-1$, respectively, and denote the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{t}$ by $K$. Put $\langle X, Y \rangle_\theta := -\langle X, \theta Y \rangle$. Then $\langle \cdot, \cdot \rangle_\theta$ defines a left-invariant Riemannian metric on $G$ with corresponding distance function $\text{dist}_G$.

Now, let $\Gamma_1, \Gamma_2, \ldots, \Gamma_h$ be discrete cocompact subgroups of $G$ which are mutually commensurable. The set $\Gamma := \cap_{l=1}^h \Gamma_l$ is a subgroup of finite index and the disjoint union

\[ M := \Gamma \backslash G \bigsqcup \Gamma_2 \backslash G \bigsqcup \cdots \bigsqcup \Gamma_h \backslash G \simeq \{(g, l) \mid 1 \leq l \leq h, \ g \in \Gamma \backslash G\} \]

is a closed manifold, where each point in $x \in M$ can be expressed as a pair $(g, l) \equiv \Gamma_l g$ of a representative $g \in G$ and the subscript of $\Gamma_l$. The left-invariant metric on $G$ induces a Riemannian metric and a distance function $\text{dist}$ on each of the connected components $\Gamma_l \backslash G$ of $M$ according to

\[ \text{dist}(\Gamma_l g, \Gamma_l h) := \inf_{\gamma \in \Gamma_l} \text{dist}_G(g, \gamma h), \]

while for $l \neq j$ one sets $\text{dist}((g, l), (h, j)) := \infty$. In order to introduce Hecke operators on $M$ we consider the commensurator of $\Gamma$

\[ C(\Gamma) := \{g \in G \mid \Gamma \text{ is commensurable with } g^{-1}\Gamma g\}. \]

Since $\Gamma \backslash \Gamma_\alpha \Gamma \simeq (\Gamma \cap \Gamma_\alpha^{-1}) / \alpha \Gamma \alpha^{-1}$, for each element $\alpha \in C(\Gamma)$ one has

\begin{equation}
\Gamma_\alpha \Gamma = \bigsqcup_{u=1}^{U_{\alpha,j}} \Gamma_j \beta_u, \quad \mathcal{U}_{\alpha,j,l} \in \mathbb{N}_*, \beta_u \in \Gamma_j \alpha \Gamma_l, \end{equation}

so that it is natural to define a linear mapping $T_{\Gamma_\alpha \Gamma_l} : L^2(\Gamma_\alpha \Gamma_l \backslash G) \to L^2(\Gamma_\alpha \Gamma_l \backslash G)$ by the expression

\[ (T_{\Gamma_\alpha \Gamma_l} f)(g, l) := \sum_{u=1}^{U_{\alpha,j}} f(\beta_u g, j) =: \sum_{\beta \in \Gamma_\alpha \Gamma_l} f(\beta g, j). \]

**Remark 3.3 (Notation).** According to general convention, $\beta \equiv \Gamma_j \beta$ (resp. $\beta_u \equiv \Gamma_j \beta_u$) denotes both a right coset as well as a suitable representative in $\Gamma_j \alpha \Gamma_l \subset G$, and the products $\beta g$ and $\beta_u g$ are taken in $G$, compare [35 Section 2.8].

Note that the so-called Hecke points $(\beta_u g, j)$ do depend on the representative $g$, while the sums defining $T_{\Gamma_\alpha \Gamma_l}$ do not depend on the representatives $g$ and $\beta_u$. In fact, for a different representative $g_1$, the Hecke points $(\beta_u g_1, j)$ are given by a permutation of the points $(\beta_u g, j)$. We now generalize this definition, and introduce for each tuple $\alpha = (\alpha_{j,l,m})_{1 \leq j,l \leq h, 1 \leq m \leq c_{j,l}}$ with $\alpha_{j,l,m} \in C(\Gamma)$ and $c_{j,l} \in \mathbb{N}$ a Hecke operator $T_{\alpha} := (\sum_{m=1}^{c_{j,l}} T_{\Gamma_\alpha \Gamma_l} g_{\alpha_{j,l,m}})_{1 \leq j,l \leq h}$ on

\begin{equation}
L^2(M) := \left\{ \varphi : G \times \{1, 2, \ldots, h\} \to \mathbb{C} \mid \begin{array}{c}
\varphi is measurable, \\
\sum_{j,l=1}^{h} f(\gamma g, j) = \varphi(g, j) \text{ holds for any } \gamma \in \Gamma_j, \ g \in G, \\
\end{array} \right\}
\end{equation}

4In this paper, the symbol $\bigsqcup$ will denote the disjoint union of possibly intersecting sets. If $G$ is not compact and one identifies the sets $\Gamma_l \backslash G$ with fundamental domains in $G$, the latter can be chosen such that they have no intersections, since they are bounded.

5In this paper, the symbol $\sqcup$ will denote union of disjoint sets. For disjoint sets, the operations $\bigsqcup$ and $\sqcup$ coincide.
by setting
\[
(T_\alpha f)(g, l) := \sum_{j=1}^{h} \sum_{m=1}^{c_{j,l}} (T_{\alpha j,l,m} f)(g, l) = \sum_{j=1}^{h} \sum_{m=1}^{c_{j,l}} \sum_{\beta \in \Gamma_j \backslash \Gamma_j} f(\beta g, j).
\]

Next, let \( P_0 \) be an elliptic left-invariant differential operator on \( G \) of degree \( m \) which gives rise to a positive and symmetric operator \( P \) on \( L^2(M) \) with strictly convex cospheres \( S^*_x(M) \). With \( \bar{s}_\mu \) as in Section 3.1 we obtain for the Schwartz kernel of \( T_\alpha \circ \bar{s}_\mu \) the expression
\[
(3.11) \quad K_{T_\alpha \circ \bar{s}_\mu}(x, x) = \sum_{j=1}^{h} \sum_{m=1}^{c_{j,l}} \sum_{\beta \in \Gamma_j} K_{\bar{s}_\mu}(\beta g, j), (g, l)), \quad x = (g, l) \in M.
\]

As a consequence of Proposition 3.4 we now deduce

Lemma 3.4. Choose a Hecke operator \( T_\alpha \) on \( L^2(M) \) given by a tuple \( \alpha \equiv (\alpha_{j,l,m})_{1 \leq j, l \leq h, 1 \leq m \leq c_{j,l}} \) as above. Set
\[
\delta_{\alpha,l} := \# \{ m \mid 1 \leq m \leq c_{l,l}, \Gamma_l \subset \Gamma_l \alpha_{l,l,m} \Gamma_l \}.
\]

Then, for each \( x = (g, l) \in M \) one has as \( \mu \to +\infty \)
\[
K_{T_\alpha \circ \bar{s}_\mu}(x, x) = \delta_{\alpha,l} K_{\bar{s}_\mu}(x, x) = O(\mu^{(d-1)/2} \| D(\alpha, x) \| \sum_{m=1}^{c_{j,l}} |\Gamma_l \alpha_{l,l,m} \Gamma_l| + \mu^{-\infty} \sum_{j=1}^{h} \sum_{m=1}^{c_{j,l}} |\Gamma_j \alpha_{j,l,m} \Gamma_l|)
\]

where
\[
D(\alpha, x) := \max_{1 \neq \beta \in \bigcup_{l=1}^{h} \Gamma_l \backslash \Gamma_l \alpha_{l,l,m} \Gamma_l} \text{dist}(\Gamma_l \beta g, \Gamma_l g)^{-(d-1)/2}.
\]

Proof. To begin, note that by definition of the distance in \( \Gamma_l \backslash G \) there exists a constant \( c_{x,\alpha} > 0 \) such that
\[
\min_{1 \neq \beta \in \bigcup_{l=1}^{h} \Gamma_l \backslash \Gamma_l \alpha_{l,l,m} \Gamma_l} \text{dist}(\Gamma_l \beta g, \Gamma_l g) > c_{x,\alpha}.
\]

Consequently, we infer for any \( x = (g, l) \in M \) from Proposition 3.1 and (3.11) that
\[
K_{T_\alpha \circ \bar{s}_\mu}(x, x) = \delta_{\alpha,l} K_{\bar{s}_\mu}(x, x) \leq \mu^{(d-1)/2} \sum_{m=1}^{c_{j,l}} \sum_{1 \neq \beta \in \Gamma_j \backslash \Gamma_j \alpha_{j,l,m} \Gamma_l} \text{dist}((\beta g, j), (g, l))^{-(d-1)/2}
\]

up to terms of order \( O(\mu^{-\infty}) \) times the cardinality of the sum in (3.11), since \( \text{dist}((\beta g, j), (g, l)) = \infty \) if \( j \neq l \). \( \square \)

Next, let \( K \) be any compact subgroup of \( G \), and recall that \( K \) acts on \( G \) and each \( \Gamma_j \backslash G \) from the right in an isometric and effective way, the isotropy group of a point \( \Gamma_j g \in \Gamma_j \backslash G \) being conjugate to the finite group \( g K g^{-1} \cap \Gamma_j \). Hence, all \( K \)-orbits in \( \Gamma_j \backslash G \) are either principal or exceptional, and of dimension \( \dim K \). Since the maximal compact subgroups of \( G \) are precisely the conjugates of \( K \), exceptional \( K \)-orbits arise from elements in \( \Gamma_j \) of finite order. Consider the right regular representation \( \pi \) of \( K \) on \( L^2(M) \) together with the corresponding Peter-Weyl decomposition of \( L^2(\Gamma_j \backslash G) \), and suppose that \( P_0 \) commutes with \( \pi \) and the Hecke operators \( T_\alpha \), which commute with the right regular \( K \)-representation as well. The Schwartz kernel of \( T_\alpha \circ \bar{s}_\mu \circ \Pi_\pi \) is then given by the expression
\[
(3.12) \quad K_{T_\alpha \circ \bar{s}_\mu \circ \Pi_\pi}(x, x) = \sum_{j=1}^{h} \sum_{m=1}^{c_{j,l}} \sum_{\beta \in \Gamma_j \backslash \Gamma_j} K_{\bar{s}_\mu}(\beta g, j), (g, l)), \quad x = (g, l) \in M.
\]

As a consequence of Proposition 3.2 one now deduces the following generalization of Lemma 3.3

\footnote{Note that \( c_{j,l} = 0 \) corresponds to the trivial mapping from \( L^2(\Gamma_j \backslash G) \) to \( L^2(\Gamma_l \backslash G) \). Also, as subsets in \( G \) the \( \Gamma_j \alpha_{j,l,m} \Gamma_l \) are not disjoint in general.}
Remark 3.6. In the statement of the lemma, keep in mind that according to Remark 3.3 the symbol $\beta \equiv \Gamma_l \beta$ denotes both the coset $\Gamma_l \beta$ as well as a suitable representative $\beta$. In this sense, the relations $\beta = g_k g^{-1}$ and $\beta \in K \cap C(G)$ are to be understood that they are valid for a suitable representative. Furthermore, taking $K = \{1\}$ one recovers Lemma 3.4.

Proof. Since $K$-orbits are closed, one has for $g, g_1 \in G$ the equivalences

$$\text{dist} (\Gamma_1 g K, \Gamma_1 g_1 K) = 0 \iff \Gamma_1 g K \cap \Gamma_1 g_1 K \neq \emptyset$$

$$\iff \Gamma_1 g K = \Gamma_1 g_1 K$$

$$\iff \text{there exist } \gamma \in \Gamma_l \text{ and } k \in K \text{ such that } \gamma g k = g_1.$$ 

Consequently, one deduces for any $\beta \in \bigcup_{m=1}^{c_{\beta,1}} \Gamma_1 \setminus \Gamma_1 \alpha_l \cdot m \Gamma_l$, eventually after choosing a suitable representative, the implications

$$\text{dist} (\Gamma_1 g K, \Gamma_1 \beta g K) = 0 \iff \beta = g_k \beta g^{-1} \text{ for some } k \beta \in K.$$ 

From Proposition 3.2 and (3.12) we then infer for any $x = (g, l) \in M$ that

$$K_{T_\alpha \circ \tilde{s}_\mu \circ \Pi_\sigma} (x, x) - \sum_{y \in T(\alpha, x)} K_{\tilde{s}_\mu \circ \Pi_\sigma} (y, x)$$

$$\ll \mu \left( d - \dim K - 1 \right) D_K (\alpha, x) \sum_{m=1}^{c_{\beta,1}} \left| \Gamma_1 \setminus \Gamma_1 \alpha_l \cdot m \Gamma_l \right| + \mu^{-\infty} \sum_{j=1}^{h} \sum_{m=1}^{c_{\beta,1}} \left| \Gamma_j \setminus \Gamma_j \alpha_l \cdot m \Gamma_l \right|$$

up to terms of order $O(\mu^{-\infty})$ times the cardinality of the sum in (3.12), since dist $((\beta g, j), (g, l)) = \infty$ if $j \neq l$. Now, as a consequence of the $K$-equivariance of the kernel of $\tilde{s}_\mu \circ \Pi_\sigma$ one deduces for any $y = (\beta g, l) \in T(\alpha, x)$ the equality

$$K_{\tilde{s}_\mu \circ \Pi_\sigma} (y, x) = \sum_{j \geq 0, \phi_j \in L^2 (M)} g (\mu - \mu_j) \phi_j (\Gamma_1 g k_y) \phi_j (\Gamma_1 g) = \sigma (k_y) K_{\tilde{s}_\mu \circ \Pi_\sigma} (x, x),$$

so that

$$\sum_{y \in T(\alpha, x)} K_{\tilde{s}_\mu \circ \Pi_\sigma} (y, x) = \sum_{y \in T(\alpha, x)} \sigma (k_y) K_{\tilde{s}_\mu \circ \Pi_\sigma} (x, x).$$
Since for $y = (\gamma g, l) \in C(\alpha, x)$ one has $k_y = \beta$, the assertion of the lemma follows.

In the remaining of this section, let us assume that $K$ is the maximal compact subgroup given by the Cartan decomposition (3.8), in which case $C(G) \subset K$.

**Lemma 3.7.** For $\beta \in G$ set $N(\beta, K) := \{ h \in G \mid h^{-1}\beta h \in K \}$, $C'_{\beta} := \{ h^{-1}\beta h \mid h \in G \}$, and assume that $C'_{\beta} \cap K \neq \emptyset$. Then $N(\beta, K)$ is an analytic manifold. Moreover, if $G$ has no compact simple factors,

$$\dim N(\beta, K) = d \implies \beta \in C(G) \implies N(\beta, K) = G.$$

**Proof.** By assumption we have $\beta = g k_0 g^{-1}$ for some $g \in G$, $k_0 \in K$. Then $N(\beta, K) = g N(k_0, K)$, and

$$C'_{k_0} \cap K = \{ h^{-1}k_0 h \in K \mid h \in G \} \simeq G_{k_0} \backslash N(k_0, K),$$

where $G_{k_0} = \{ h \in G \mid h k_0 h^{-1} = k_0 \}$. By [39] Theorem 3.1, $C'_{k_0} \cap K$ is an analytic manifold, and consequently also $N(k_0, K)$ and $N(\beta, K)$, proving the first assertion. Next, $\beta \in C(G)$ implies that $N(\beta, K) = G$ has dimension $d$. Conversely, assume that $N(\beta, K)$ has dimension $d$, and consider the global Cartan decomposition corresponding to (3.8), which is given by the diffeomorphism

$$(3.14) \quad p \times K \ni (X, k) \mapsto \exp X \cdot k = g \in G.$$

Then

$$N(k_0, K) \simeq \{(X, k) \in p \times K \mid \exp(-X) \cdot k_0 \cdot \exp X \in K\}$$

has dimension $d$. Next, note that for arbitrary $Y, Z \in \mathfrak{g}$, and $h \in G$ one has $h \cdot \exp(Y) \cdot h^{-1} = \exp(\text{Ad}(h)Y)$, as well as

$$\text{Ad}(\exp(-sZ))Y = Y - s[Z, Y] + O(s^2),$$

provided that $s \in \mathbb{R}$ has small absolute value, see [16] pp. 127 and 128]. Now, let $h = \exp X \cdot k \in N(k_0, K)$ be arbitrary and $U_h \subset N(k_0, K)$ an open neighborhood of $h$. By assumption, $U_h$ is $d$-dimensional, so that

$$(3.15) \quad \exp X \cdot \exp X_1 \cdot k \in U_h \quad \text{for all} \quad X_1 \in \mathfrak{p} \quad \text{with} \quad \|X_1\| \text{ sufficiently small}.$$

By assumption, $\exp(-X) \cdot k_0 \cdot \exp X = k_1$ for some $k_1 \in K$. If $K$ is connected, the exponential map from $\mathfrak{k}_0$ to $K$ is onto, so that we can write $k_1 = \exp Y_1$ for some $Y_1 \in \mathfrak{t}$. In view of $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$, $[\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}$, and $C(G) \subset K$ we conclude that for almost all $X_1 \in \mathfrak{p}$ with $\|X_1\|$ sufficiently small

$$\exp(-X_1) \cdot \exp(-X) \cdot k_0 \cdot \exp X \cdot \exp X_1 = \exp(-X_1) \cdot \exp Y_1 \cdot \exp X_1$$

$$= \exp(Y_1 - [X_1, Y_1] + O(\|X_1\|^2)) \notin K,$$

unless $k_0 \in C(G)$. Note that here we have used the assumption that $G$ has no compact simple factors. If $K$ is not connected, we may suppose that $G \subset \text{SL}(N, \mathbb{R})$ and $K \subset \text{SO}(N)$ for some sufficiently large $N \in \mathbb{N}$, and repeat the above arguments using the surjectivity of the exponential onto $SO(N)$ and the Cartan decomposition of $\text{SL}(N, \mathbb{R})$. Since by (3.14) we must have $\exp X \cdot \exp X_1 \cdot k \in N(k_0, K)$, we conclude that $\beta = k_0 \subset C(G)$, completing the proof.

Using the previous lemma one deduces

**Lemma 3.8.** In the situation of Lemma 3.7, suppose that $G$ has no compact simple factors. Then the function defined by

$$F(x) := \sum_{y \in T(\alpha, x) - C(\alpha, x)} \sigma(k_y)$$

if $T(\alpha, x) - C(\alpha, x) \neq \emptyset$, $F(x) := 0$ else, is supported on a set of measure zero in $M$. 

[16]
Proof. To begin, let $U \subset G$ be a sufficiently small open neighbourhood of the identity and $h \in U$. For $x = (g, l)$ and $x_1 = (gh, l) \in M$, one has the one-to-one correspondence of Hecke points

$$(\beta, g, l) \in H(\alpha, x) \leftrightarrow (\beta h, g, l) \in H(\alpha, x_1);$$

furthermore, points in $C(\alpha, x)$ correspond to points in $C(\alpha, x_1)$. Now, take any $y = (\beta g, l) \in H(\alpha, x) - C(\alpha, x)$. Then, by Lemma 3.7, $\dim N(\gamma \beta, K) < d$ for all $\gamma \in \Gamma_l$. In order that $y_1 = (\beta g, l) \in T(\alpha, x_1) - C(\alpha, x_1)$ we must have $\gamma \beta = g h k_1 h^{-1} g^{-1}$ for some $\gamma \in \Gamma_l$ and $k_1 \in K$, which is equivalent to

$$h \in N(g^{-1} \gamma \beta g, K) = g^{-1} N(\gamma \beta, K).$$

That is, $h$ must belong to a lower dimensional set in $U$. In other words, Hecke points in $T(\alpha, x_1) - C(\alpha, x_1)$ can only arise from Hecke points in $H(\alpha, x) - C(\alpha, x)$ by deformation along a measure zero set. Consequently, if $x \in \text{supp } F$ we can only have $x_1 \in \text{supp } F$ if $h$ belongs to a measure zero set in $U$, and the assertion follows. \qed

As a consequence of the previous lemma,

$$\int_{\Gamma \backslash G} \left[ \sum_{k \in \Gamma(\alpha, l)} \sigma(k) + F(x) \right] K_{\alpha, \infty}(x, x) dx = \sum_{k \in \Gamma(\alpha, l)} \sigma(k) \int_{\Gamma \backslash G} K_{\alpha, \infty}(x, x) dx,$$

so that non-central torsion elements do not contribute to the leading term in Lemma 3.5 after integration over $\Gamma \backslash G$.

4. Non-equivariant asymptotics for Hecke eigenvalues and Sato-Tate equidistribution

We commence our study of the asymptotic distribution of Hecke eigenvalues by considering first the non-equivariant setting.

4.1. Preliminaries. Let $H$ denote a semisimple connected linear algebraic group over the rational number field $\mathbb{Q}$. We may suppose that $H$ is a closed subgroup of $\text{SL}(N)$ over $\mathbb{Q}$ for a fixed $N \in \mathbb{N}$. We write $\mathbb{Q}_p$ for the $p$-adic number field, and $\mathbb{A}$ (resp. $\mathbb{A}_{\text{fin}}$) for the adele (resp. finite adele) ring of $\mathbb{Q}$. We choose an open compact subgroup $K_0$ of $H(\mathbb{A}_{\text{fin}})$. As usual, we regard $H(\mathbb{Q})$ as a subgroup of $H(\mathbb{A})$ by the diagonal embedding. By the finiteness of class numbers [38, Theorems 5.1 and 8.1], there exist elements $x_1, x_2, \ldots, x_{c_H}$ in $H(\mathbb{A}_{\text{fin}})$ such that

$$(4.1) \quad H(\mathbb{A}) = \bigcup_{l=1}^{c_H} H(\mathbb{Q}) x_l H(\mathbb{R}) K_0.$$

There exists also a finite set $S_0$ of primes such that

(C1) $H$ is unramified over $\mathbb{Q}_p$ for every prime $p \notin S_0$;
(C2) one has $K_0 = K_0 \prod_{p \notin S_0} K_p$, where $K_0$ is an open compact subgroup of $\prod_{p \in S_0} G(\mathbb{Q}_p)$ and $K_p = H(\mathbb{Q}_p) \cap \text{SL}(N, \mathbb{Z}_p)$ is a hyperspecial compact subgroup of $H(\mathbb{Q}_p)$ for every $p \notin S_0$;
(C3) for $x_1 = (x_{1p})_p$, we have $x_{1p} \in K_p$ for any $p \notin S_0$. In other words, we may suppose that $x_{1p} = 1$ for all $p \notin S_0$ without loss of generality.

For details, we refer to [52], and normalize the Haar measures on $H(\mathbb{A}_{\text{fin}})$ and $H(\mathbb{Q}_p)$ by setting

$$\text{vol}(K_0) = 1 \quad \text{and} \quad \text{vol}(K_p) = 1 \quad \text{for all} \quad p \notin S_0.$$

Next, fix a prime $p \notin S_0$. The group $G_p := H(\mathbb{Q}_p)$ has a Borel subgroup that contains a maximal $\mathbb{Q}_p$-torus $T_p$, and its Cartan decomposition reads $G_p = K_p T_p K_p$. Let $A_p$ denote the maximal $\mathbb{Q}_p$-split subtorus in $T_p$. Since the inclusion mapping $A_p \subset T_p$ induces an isomorphism $A_p / A_p \cap K_p \cong T_p / T_p \cap K_p$, one gets $G_p = K_p A_p K_p$. Let $X_A(A_p)$ denote the abelian group of co-characters of $A_p$. A light function $\| \cdot \|_p$ on $G_p$ is defined by $\| g \|_p := \max_{i,j} \{|g_{i,j}|_p, |g'_{i,j}|_p\}$ for $g = (g_{i,j})_{1 \leq i,j \leq N} \in G_p \subset \text{SL}(N, \mathbb{Q}_p)$ and
Since the Plancherel measure \( \hat{f} \) holds for any \( k_1, k_2 \in K_p \) and \( g \in G_p \). Further, we define a right function \( \| \cdot \|_p \) on \( X_*(A_p) \) by

\[
\| \omega \|_p := \frac{\log \| \omega(p) \|_p}{\log p} \in \mathbb{N}, \quad \omega \in X_*(A_p),
\]

\( \text{as well as the unramified Hecke algebra } \mathcal{H}^{ur}(G_p) := C_c^\infty(K_p \backslash G_p/K_p) \), which is generated by the family of characteristic functions \( \tau_\omega \) of the double cosets \( K_p \omega(p) K_p \) with \( \omega \in X_*(A_p) \). Also, for each \( \kappa \in \mathbb{N} \), a truncated unramified Hecke algebra \( \mathcal{H}_\kappa^{ur}(G_p) \) is defined by

\[
\mathcal{H}_\kappa^{ur}(G_p) := \langle \tau_\omega \mid \omega \in X_*(A_p), \| \omega \|_p \leq \kappa \rangle.
\]

For other, essentially equivalent definitions of \( \mathcal{H}^{ur}(G_p) \) see [50] Section 2.3, [31] Section 3.4, and [31] Lemma 3.5.

In what follows, we write \( G_p^{\t^\text{ur}, \text{temp}} \) (resp. \( G_p^{\text{ur, temp}} \)) for the unramified (resp. unramified and tempered) part of the unitary dual of \( G_p \). Let \( T_p \) denote the \( \mathbb{Q}_p \)-rational Weyl group for \( (G_p,A_p) \). By the canonical map given in [50] pp. 33–34, we have the topological injective mapping \( G_p^{\t^\text{ur}, \text{temp}} \to T_p / \Omega_p \) and the topological isomorphism

\[
G_p^{\text{ur, temp}} \cong \hat{A}_p / \Omega_p
\]

where \( \hat{A}_p \) denotes the dual torus of \( A_p \) and \( \hat{A}_p \) denotes the compact subtorus of \( \hat{A}_p \). For \( f \in \mathcal{H}^{ur}(G_p) \), a continuous function \( \hat{f} \) on \( \hat{A}_p / \Omega_p \) is defined by

\[
\hat{f}(\pi) := \text{Tr} \left( f(\pi) \right), \quad \pi \in G_p^{\text{ur, temp}},
\]

and it is well-known that the Plancherel measure \( \hat{m}_p^{\text{Pl, ur}} \) on \( G_p^{\text{ur, temp}} \) satisfies

\[
\hat{m}_p^{\text{Pl, ur}}(\hat{f}) = f(1), \quad f \in \mathcal{H}^{ur}(G_p).
\]

Notice that the support of \( \hat{m}_p^{\text{Pl, ur}} \) is included in \( G_p^{\text{ur, temp}} \cong \hat{A}_p / \Omega_p \). For explicit descriptions of the Plancherel measure \( \hat{m}_p^{\text{Pl, ur}} \), we refer to [30] and [50] Proposition 3.3.

Next, let \( S \) be a finite set of prime numbers. Suppose that \( S \) has no intersection with \( S_0 \), and set

\[
\mathbb{Q}_S := \prod_{p \in S} \mathbb{Q}_p \quad \text{and} \quad \mathcal{H}^{ur}(\mathbb{Q}(S)) := \bigotimes_{p \in S} \mathcal{H}^{ur}(\mathbb{Q}(p)).
\]

Each element \( (q_p)_{p \in S} \in \mathbb{Q}(S) \) is identified with the element \( (y_v)_{v < \infty} \in \mathbb{H}_{\text{et}}^e \) such that \( y_p = q_p \) for all \( p \in S \) and \( y_v = 1 \) for all \( v \not\in S \). Write \( H(\mathbb{Q}(S))^{\text{ur, temp}} \) (resp. \( H(\mathbb{Q}(S))^{\text{ur}} \)) for the unramified (resp. unramified and tempered) part of the unitary dual of \( H(\mathbb{Q}(S)) \). Clearly, there is an injective mapping \( H(\mathbb{Q}(S))^{\text{ur, temp}} \to \prod_{p \in S} \hat{A}_p / \Omega_p \), and one has an isomorphism

\[
H(\mathbb{Q}(S))^{\text{ur, temp}} \cong \prod_{p \in S} \hat{A}_p / \Omega_p.
\]

Further, for each \( f_S \in \mathcal{H}^{ur}(H(\mathbb{Q}(S))) \) define the continuous function

\[
\hat{f}_S : H(\mathbb{Q}(S))^{\text{ur}} \ni \pi_S \mapsto \hat{f}_S(\pi_S) := \text{Tr} \pi_S(f_S).
\]

Since the Plancherel measure \( \hat{m}_S^{\text{Pl, ur}} \) on \( H(\mathbb{Q}(S))^{\text{ur}} \) satisfies \( \hat{m}_S^{\text{Pl, ur}} = \prod_{p \in S} \hat{m}_p^{\text{Pl, ur}} \), one has \( \hat{m}_S^{\text{Pl, ur}}(\hat{f}_S) = f_S(1) \) by (4.2) and the support of \( \hat{m}_S^{\text{Pl, ur}} \) is contained in \( H(\mathbb{Q}(S))^{\text{ur, temp}} \).

We shall now recall briefly some fundamental facts about Sato-Tate measures for \( H \), and refer the reader to [50] Section 5 for details. Denote by \( \hat{H} \) the dual group of \( H \) and by \( \hat{T} \) the maximal torus in \( \hat{H} \), which is a constituent of the root datum, compare [3]. The Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) acts on \( \hat{H} \) via its natural action on the root datum, and there exists a finite extension \( F_1 \) of \( \mathbb{Q} \) such that \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) acts on \( \hat{H} \) through the faithful action of \( \Gamma_1 := \text{Gal}(F_1/\mathbb{Q}) \). Let \( \hat{K}_1 \) be a \( \Gamma_1 \)-invariant maximal compact subgroup of \( \hat{H} \), set \( \hat{T}_c := \hat{T} \cap \hat{K}_1 \), and let \( \Omega_c \) denote the Weyl group for \( (\hat{K}_1,\hat{T}_c) \). For each \( \theta \in \Gamma_1 \), set

\[
\hat{T}_{c,\theta} := \hat{T}_c / (\theta - \text{id}) \hat{T}_c, \quad \Omega_{c,\theta} := \{ w \in \hat{T}_c \mid \theta(w) = w \}.
\]
and denote by $\hat{m}^{ST}_p$ the $\theta$-Sato-Tate measure on $\hat{T}_{\theta}/\Omega_{c,\theta}$ introduced in [50] Definition 5.1. It can be characterized by a limit of Plancherel measures as follows. Write $\mathcal{C}(\Gamma_1)$ for a set of representatives of conjugacy classes in $\Gamma_1$, and consider the corresponding partition of the set of primes outside $S_0$

$$\{\mathcal{V}(\theta)\}_{\theta \in \mathcal{C}(\Gamma_1)}.$$

Fix $\theta \in \mathcal{C}(\Gamma_1)$, and for each $p \in \mathcal{V}(\theta)$ choose an inclusion $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ such that the Frobenius $\text{Fr}_p$ in $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ has image $\theta$ in $\Gamma_1$, yielding the identification

$$\hat{T}_{c,\theta}/\Omega_{c,\theta} = \hat{T}_{c,\text{Fr}_p}/\Omega_{c,\text{Fr}_p} \cong G_{c,\text{ur},\text{temp}},$$

see [50] (5.2)]. By [50] Proposition 5.3 one then has the weak convergence

$$\hat{m}^{\text{pl},\text{ur}}_p \rightarrow \hat{m}^{ST}_p \text{ as } p \rightarrow \infty \text{ in } \mathcal{V}(\theta).$$

Consequently, there is a unique Sato-Tate measure $\hat{m}^{ST}$, which coincides with the limit $\lim_{p \rightarrow \infty} \hat{m}^{\text{pl},\text{ur}}_p$ in the weak topology.

4.2. Non-equivariant asymptotics and equidistribution results. In this paper we will mainly deal with the case where $H(\mathbb{Q})\backslash H(\mathbb{A})$ is compact, which we assume from now on. In this situation, $L^2(H(\mathbb{Q})\backslash H(\mathbb{A}))$ decomposes into a countable orthogonal direct sum of irreducible unitary representations $\pi$ of $H(\mathbb{A})$, so that

$$L^2(H(\mathbb{Q})\backslash H(\mathbb{A})) \cong \bigoplus_{\pi \in \hat{H}(\mathbb{A})} V^{\otimes m_\pi}_\pi,$$

where $\hat{H}(\mathbb{A})$ denotes the unitary dual of $H(\mathbb{A})$, $m_\pi \in \mathbb{N}$ the multiplicity of $\pi$, and $V_\pi$ a representative space of $\pi$, see [13]. For each double coset $K_0\alpha K_0$ with $\alpha \in H(\mathbb{A}_0)$, a Hecke operator $T_{K_0\alpha K_0}$ on $L^2(H(\mathbb{Q})\backslash H(\mathbb{A})/K_0)$ is defined by

$$(T_{K_0\alpha K_0}\phi)(x) := \sum_{\beta \in K_0\alpha K_0/K_0} \phi(x\beta), \quad \phi \in L^2(H(\mathbb{Q})\backslash H(\mathbb{A})/K_0).$$

Write $G := H(\mathbb{R})$, and let $K$ be a maximal compact subgroup of $G$, and $\Delta$ the Beltrami-Laplace operator on $G$. It is known that $G$ is a $d$-dimensional semisimple real Lie group with finite center [38] and that $\Delta = -\mathcal{C} + 2\mathcal{C}_K$, where $\mathcal{C}$ (resp. $\mathcal{C}_K$) denotes the Casimir operator of $G$ (resp. $K$), compare [41]. We choose an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ in $L^2(H(\mathbb{Q})\backslash H(\mathbb{A})/K_0)$ such that each $\phi_j$ is a $\Delta$-eigenfunction included in a single space $V_\pi$. Since any automorphic representation $\pi$ factors as a tensor product $\pi = \otimes_v \pi_v$ of irreducible unitary representations $\pi_v$ of $H(\mathbb{Q}_v)$ for all places $v$ of $\mathbb{Q}$ [12], any $\phi_j$ is a simultaneous eigenfunction for $\Delta$ and $T_{K_0\alpha K_0}$ for any $\alpha \in H(\mathbb{A}_0)$, where

$$\mathbb{A}_0^{S_0} := \{(\alpha_v)_{v < \infty} \in \mathbb{A}_0 \mid \alpha_p = 1 \quad \forall \, p \in S_0\}.$$

Let $\lambda_j$ and $\lambda_j(\alpha)$ denote the eigenvalue of $\phi_j$ for $\Delta$ and $T_{K_0\alpha K_0}$ respectively, so that

$$\Delta \phi_j = \lambda_j \phi_j \quad \text{and} \quad T_{K_0\alpha K_0} \phi_j = \lambda_j(\alpha) \phi_j, \quad \alpha \in H(\mathbb{A}_0^{S_0}).$$

Set $\mu_j := \sqrt{\lambda_j}$. Our goal is to study the asymptotics of the sum

$$\sum_{\mu_j \leq \mu} \lambda_j(\alpha), \quad \alpha \in H(\mathbb{A}_0^{S_0}),$$

of Hecke eigenvalues with respect to the spectral parameter $\mu \in \mathbb{R}_{>0}$. For this, let $1 \leq l \leq c_H$ and set

$$\Gamma_l := H(\mathbb{Q}) \cap x_l K_0 x_l^{-1},$$

where $c_H$ and $x_l$ are as in [41], and we regard $H(\mathbb{Q})$ as a subgroup of $H(\mathbb{A}_0)$ via the diagonal embedding. Then, one gets a diffeomorphism

$$M := \bigsqcup_{l=1}^{c_H} \Gamma_l \backslash G \cong \bigsqcup_{l=1}^{c_H} \Gamma_l \backslash G \cdot x_l \cong H(\mathbb{Q}) \backslash H(\mathbb{A})/K_0,$$
which defines an isomorphism from $L^2(M)$ to $L^2(H(\mathbb{Q}) \setminus H(\mathbb{A})/K_0)$ given by the mapping

$$L^2(M) \ni \varphi_M \mapsto \varphi \in L^2(H(\mathbb{Q}) \setminus H(\mathbb{A})/K_0),$$

$$\varphi(\gamma x g_{k_0}) := \varphi_M(g_{k_0}, 1), \quad \gamma \in H(\mathbb{Q}), \quad g_{k_0} \in G, \quad k_0 \in K_0.$$

**Lemma 4.1.** For any element $\alpha$ in $H(\mathbb{A}_f)$, there exist elements $\alpha_{j,k,m} \in H(\mathbb{Q})$ with $1 \leq m \leq c_{j,k}$ such that

$$H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1} = \bigcup_{m=1}^{c_{j,k}} \Gamma_j \alpha_{j,k,m} \Gamma_k.$$

**Proof.** By Lemma 4.1, there obviously exist $\gamma_{jm} \in H(\mathbb{Q})$, $R \in \mathbb{N}$, $1 \leq n_m \leq c_H$ such that $x_j K_0 \alpha^{-1} K_0 = \bigcup_{m=1}^{R} \gamma_{jm} x_{n_m} K_0$. For this reason, it is sufficient to show that

$$H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1} = \bigcup_{1 \leq m \leq R, \ x_{n_m} = x_k} \Gamma_j \gamma_{jm} \Gamma_k.$$

If $x_{n_m} = x_k$, one gets $\gamma_{jm} \in x_j K_0 \alpha^{-1} K_0 x_j^{-1}$, hence $\Gamma_j \gamma_{jm} \Gamma_k \subset x_j K_0 \alpha^{-1} K_0 x_j^{-1}$. It follows that $H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1} \supset \bigcup_{1 \leq j \leq R, \ x_{n_m} = x_k} \Gamma_j \gamma_{jm} \Gamma_k$. Next, we suppose that $\gamma \in H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1}$. For some $1 \leq j \leq R$ and $k_0 \in K_0$, one has $\gamma x_k = \gamma_{jm} x_{n_m} k_0$. This implies $x_{n_m} = x_k$ by (4.1), and therefore $\gamma = \gamma_{jm} x_k \gamma x_k^{-1} \in \gamma_{jm} \Gamma_k$. Thus, we obtain $H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1} \subset \bigcup_{1 \leq m \leq R, \ x_{n_m} = x_k} \Gamma_j \gamma_{jm} \Gamma_k$, and the assertion follows. \qed

**Lemma 4.2.** Let $H(\mathbb{A}_f) \ni \alpha \equiv (\alpha_{j,k,m})_{1 \leq j,k \leq c_H, 1 \leq m \leq c_{j,k}}$ be as in Lemma 4.1. In terms of the isomorphism $L^2(M) \simeq L^2(H(\mathbb{Q}) \setminus H(\mathbb{A})/K_0)$, $T_{\alpha} K_0 \alpha K_0$ coincides with the Hecke operator $T_\alpha$ defined in Section 3.3 and

$$T_{K_0 \alpha K_0} \varphi = T_\alpha \varphi_M, \quad \text{where } (T_\alpha \varphi_M)(g,k) := \sum_{j=1}^{c_H} \sum_{m=1}^{c_{j,k}} (T_{\Gamma_j \alpha_{j,k,m}} \varphi_M)(g,j).$$

**Proof.** For $x_j$ and $\beta \in K_0 \alpha K_0 / K_0$, there exists an element $x_j$ such that $x_j \beta K_0 \cap H(\mathbb{Q}) x_k K_0 \neq \emptyset$ by (4.1). Hence, one has

$$\varphi(g \infty x_j \beta) = \varphi((\gamma_{\text{fin}} g \infty x_j) = \varphi((\gamma_{\infty}^{-1}) g \infty x_j)$$

for some $\gamma \in H(\mathbb{Q})$, where $(\gamma_{\text{fin}})$ (resp. $(\gamma_{\infty})$) denotes the embedding of $\gamma$ into $H(\mathbb{A}_f)$ (resp. $H(\mathbb{R})$). For this reason, we have only to prove the one-to-one correspondence between the left cosets of $K_0 / K_0 = K_0$ and the left cosets of $\bigcup_{j=1}^{c_H} \bigcup_{m=1}^{c_{j,k}} \Gamma_j \gamma_{jm} \Gamma_k$, but this is obvious because the left $x_j K_0 x_j^{-1}$-equivalence coincides with the left $\Gamma_j$-equivalence in $H(\mathbb{Q}) \cap x_j K_0 \alpha^{-1} K_0 x_j^{-1}$. \qed

We can now state the first main result of this paper. Set

$$p_S := \prod_{p \in S} p \quad \text{and} \quad K_S := \prod_{p \in S} K_p.$$

For each $\alpha \in H(\mathbb{Q}_S)$, write $\|\alpha\|_S \leq \kappa$ if the characteristic function of $K_S \alpha K_S$ belongs to $H^\infty_{\kappa}(H(\mathbb{Q}_S))$.

**Theorem 4.3 (Asymptotic distribution of Hecke eigenvalues).** Let $\{\phi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(H(\mathbb{Q}) \setminus H(\mathbb{A})/K_0)$ as above and $d := \dim H(\mathbb{R})$. Then there exists a constant $0 < c < d + N(N + 1)$ such that for any finite set $S$ of primes in the complement of $S_0$ and any $\alpha \in H(\mathbb{Q}_S)$ with $\|\alpha\|_S \leq \kappa$ one obtains

$$\sum_{p < \mu} \lambda_j(\alpha) = \delta_\alpha \frac{\text{vol}(M)}{(2\pi)^d} \mu^d + O(\mu^{d-1} p_S^c),$$

where $\delta_\alpha := 1$ if $\alpha \in K_S$, $\delta_\alpha := 0$ otherwise, $\text{vol}(M)$ denotes the Riemannian volume of $M$, and $\infty := \pi^2 / (\Gamma(1 + \frac{d}{2})$ means the volume of the unit $d$-sphere.
Proof. The assertion is essentially a consequence of Lemma 4.4. As a consequence of the two previous lemmata, any \( \alpha \in H(\beta_{0m}) \) can be identified with a tuple \((\alpha_{j,k,m}^i)_{1 \leq j,k \leq c_H, 1 \leq m \leq c_{j,k}}\) up to \((\Gamma_j, \Gamma_k)\)-equivalence via the decomposition \((4.6)\), and be associated to a Hecke operator \( T_\alpha \) on \( L^2(M) \) as in \((4.7)\). By Lemma 4.1 we can assume for each pair \((j, k)\), that \( \Gamma_j \alpha_{j,k,m_1} \Gamma_k \neq \Gamma_j \alpha_{j,k,m_2} \Gamma_k \) if \( m_1 \neq m_2 \), where \( 1 \leq m_1, m_2 \leq c_{j,k} \). Also, assume that either of the conditions

(i) \( \Gamma_l \) lies in \( \sqcup_{m=1}^{c_{l,j}} \Gamma \alpha_{l,m} \Gamma_l \) for every \( l \),

(ii) \( \Gamma_l \) does not lie in \( \sqcup_{m=1}^{c_{l,j}} \Gamma \alpha_{l,m} \Gamma_l \) for any \( l \),

holds, and set \( \delta'_\alpha := 1 \) if (i) and \( \delta'_\alpha := 0 \) if (ii) is fulfilled. Then, since the double cosets \( \Gamma \alpha_{l,m} \Gamma_l \) are disjoint in the present case, Lemma 4.4 implies for each \( x = (g, l) \in M \) that

\[
K_{T_\alpha \circ \delta'_\alpha}(x, x) - \delta'_\alpha K_{\delta'_\alpha}(x, x) = O\left( \mu^{(d-1)/2} D(\alpha, x) \sum_{j=1}^{c_{j,i}} |\Gamma_j \setminus \Gamma_l \alpha_{l,m} \Gamma_l| + \mu^{-\infty} \sum_{j=1}^{c_{j,i}} |\Gamma_j \setminus \Gamma_l \alpha_{l,m} \Gamma_l| \right).
\]

Denote by \( \text{lcm}(\gamma) \) the least common multiple of denominators of components of a matrix \( \gamma \in H(\mathbb{Q}) \subset \text{SL}(N, \mathbb{Q}) \), and consider an element \( \alpha \in H(\mathbb{Q}) \) with \( ||\alpha||_S \leq \kappa \). By Lemma 4.1 there is a constant \( c_1 \in \mathbb{N} \) such that \( \text{lcm}(\gamma) < c_1 \mathcal{P}_S^N \) holds for any \( \gamma \in \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_k \), \( 1 \leq j, k \leq c_H \). Hence, for some constant \( c_2 \) and \( c_3 = N(N + 1) \) one has

\[
\sum_{j=1}^{c_{j,i}} \sum_{m=1}^{c_{j,i}} \zeta(\Gamma_j \setminus \Gamma_l \alpha_{j,k,m} \Gamma_k) < c_2 \mathcal{P}_S^N,
\]

because \( \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_k \) is contained in \( \text{SL}(N, \mathbb{Q}) \cap \mathcal{M}(N, \mathcal{C}_{-1} \mathcal{P}_S^N \mathbb{Z}) \). Furthermore, for any \( x = (g, k) \in M = G \times \{1, \ldots, c_H\} \) and any \( \gamma \in \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_k \), \( 1 \leq j, k \leq c_H \), one has dist \((\Gamma_j \gamma, \Gamma_k g) > c_{\kappa} \mathcal{P}_S^N \) for some constant \( c_{\kappa} \) whenever \( \gamma = 1 \), because \( M \) is compact and the distance dist on \( M \) is locally equivalent to the distance induced by the Euclidean distance on \( \mathcal{M}(N, \mathbb{R}) \), see [11] Section 2. Therefore, setting \( c_5 = c_3 + (d - 1)/2 \) Equation (4.8) would imply

\[
K_{T_\alpha \circ \delta'_\alpha}(x, x) - \delta'_\alpha K_{\delta'_\alpha}(x, x) = O(\mu^{(d-1)/2} \mathcal{P}_S^N)
\]

provided that we prove the necessary conditions

(I) If \( \alpha \in K_\kappa \), 1 belongs to \( \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_j \) for every \( j \),

(II) If \( \alpha \not\in K_\kappa \), 1 does not belong to \( \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_j \) for any \( j \).

The condition (I) is obvious by Lemma 4.1 so suppose that \( 1 \in \sqcup_{m=1}^{c_{j,i}} \Gamma_j \alpha_{j,k,m} \Gamma_j \) for some \( j \). This means that \( 1 \in H(\mathbb{Q}) \cap x_j K_0 \alpha \kappa x_j^{-1} \) and in particular \( 1 \in K_0 \alpha K_0 \) together with \( \alpha \in K_\kappa \). Hence (II) holds by contraposition, and (4.11) is proved. Integrating this equality over \( x \) and \( \mu \) we arrive at

\[
\int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \varrho(t - \mu_j) \lambda_j(\alpha) dt - \delta_\alpha \varrho_d \frac{\text{vol}(M)}{(2\pi)^d} \mu^d = O(\mu^{(d+1)/2} \mathcal{P}_S^N + \mu^{-d-1}),
\]

where we took into account (3.2), (3.3), and (III), together with the fact that \( K_\kappa(x, y) \) is rapidly decreasing as \( \mu \to -\infty \). Besides, in the present case we have \( S_x^r M = \{(x, \xi) \in T_x^r(M) \mid |\xi|_x = 1\} \), so that

\[
\int_M \text{vol}(S_x^r M) dx = d \int_M \text{vol}(B_x^r(M)) dx = d \varrho_d \text{vol}(M),
\]

where \( B_x^r M := \{(x, \xi) \in T_x^r(M) \mid |\xi|_x \leq 1\} \). Now, for each eigenfunction \( \phi_j \) we can choose a point \( y_j \in H(\mathbb{Q}) \setminus H(\mathbb{R}) / K_0 \) such that \( |\phi_j(y_j)| = \max_{x \in H(\mathbb{Q}) \setminus H(\mathbb{R}) / K_0} |\phi_j(x)| \), yielding the trivial bound

\[
|\lambda_j(\alpha)| = \sum_{\beta \in K_0 \alpha K_0 / K_0} |\phi_j(\gamma \beta)|/|\phi_j(y_j)| \leq \varrho_N(K_0 \alpha K_0 / K_0)
\]

uniformly in \( j \). Furthermore, by 50, Lemma 2.13 one has \( \varrho_N(K_0 \alpha K_0 / K_0) \ll \mathcal{P}_S^N \) for \( c_6 = d + N - 1 + \frac{1}{2} N(N + 1) \). With the arguments in [5] Proof of Corollary 2.5 one therefore deduces for any \( K > 0 \)
the estimate
\[
\int_{-\infty}^{\mu} \sum_{j=0}^{\infty} \phi(t - \mu j) \lambda_j(\alpha) \, dt = \sum_{\mu_j \leq \mu - \kappa} \lambda_j(\alpha) \int_{-\infty}^{\infty} \phi(t - \mu_j) \, dt + O(\mu^{d-1} p_{S}^{r\kappa}).
\]
Since \( \hat{\phi}(0) = \int \phi(t) \, dt = 1 \), the assertion of the theorem follows from (4.11), since \( d \geq 3 \). \( \square \)

Following Shin and Templier \[50\], we now define a certain family of automorphic representations of \( H \) depending on \( \mu \). Fix an automorphic representation \( \pi \) of \( H \). In view of \( H(\mathbb{A}) = H(\mathbb{R}) \times H(\mathbb{A}_{\mathbb{F}}) \), one has the decompositions \( \pi = \pi_{\infty} \otimes \pi_{\mathbb{F}} \) and \( V_{\pi} = V_{\pi_{\infty}} \otimes V_{\pi_{\mathbb{F}}} \), where \( \pi_{\infty} \in \widehat{H}(\mathbb{R}) \) and \( \pi_{\mathbb{F}} \in \widehat{H}(\mathbb{A}_{\mathbb{F}}) \), and for each eigenfunction \( \phi_{\infty} \) in \( V_{\pi_{\infty}} \) of \( \pi_{\infty}(\Delta) \) we write
\[
\pi_{\infty}(\Delta) \phi_{\infty} = \lambda_{\phi_{\infty}} \phi_{\infty}.
\]
We can then define the finite dimensional subspace
\[
V_{\pi_{\infty}}^{\leq \mu} := \{ \phi_{\infty} \in V_{\pi_{\infty}} \mid \text{\( \phi_{\infty} \) is an eigenfunction of \( \pi_{\infty}(\Delta) \) and \( \sqrt{\lambda_{\phi_{\infty}}} \leq \mu \)} \}.
\]
Note that \( \dim V_{\pi_{\infty}}^{\leq \mu} > 0 \) means that the Casimir eigenvalue of \( \pi_{\infty} \) is less than or equal to \( \mu^2 \). In addition, we denote the subspace of \( \mathcal{K}_{\mathbb{F}} \)-fixed vectors in \( V_{\pi_{\mathbb{F}}} \) by
\[
V_{\pi_{\mathbb{F}}}^{K_{0}} := \{ u \in V_{\pi_{\mathbb{F}}} \mid \pi_{\mathbb{F}}(k_0)u = u \ \forall k_0 \in K_{0} \}.
\]
Now, define \( \mathcal{F} = \mathcal{F}(\mu) \) as the finite multi-set consisting of those automorphic representations \( \pi \in \widehat{H}(\mathbb{A}) \) with \( m_{\pi} > 0 \) for which the positive integer
\[
a_{\pi}(\mu) := m_{\pi} \dim V_{\pi_{\infty}}^{\leq \mu} \dim V_{\pi_{\mathbb{F}}}^{K_{0}}
\]
is strictly positive, where for each such \( \pi \) appears in \( \mathcal{F} \) with the multiplicity \( a_{\pi}(\mu) \). As an immediate consequence of Theorem 4.3, we now obtain the following

**Corollary 4.4 (Asymptotic trace formula).** In the setting of Theorem 4.3, there exists a constant \( c' > 0 \) such that for each finite set \( S \) of primes outside \( S_{0} \) and each \( f_{S} \in \mathcal{H}^{ur}(H(\mathbb{Q}_{S})) \),
\[
\sum_{\pi \in \mathcal{F}(\mu)} \text{Tr} \pi_{S}(f_{S}) = f_{S}(1) \cdot \frac{\text{vol}(M) \prod_{p \in S} j_{\mathbb{F}}}{(2\pi)^{d}} \mu^{d} + O(\mu^{d-1} p_{S}^{r\kappa} \|f_{S}\|_{\infty}),
\]
where \( \|f_{S}\|_{\infty} := \max_{x \in H(\mathbb{Q}_{S})} f_{S}(x) \).

**Proof.** To begin, note that \( \mathcal{H}^{ur}(H(\mathbb{Q}_{S})) \) is spanned by the elements \( \tau_{\omega} := \otimes_{p \in S} \tau_{\omega_{p}} \), where \( \omega = (\omega_{p})_{p \in S} \in B_{S,\mathbb{R}} \) and \( B_{S,\mathbb{R}} := \prod_{p \neq \infty} X_{s}(A_{p}) \mid \|\omega_{p}\|_{p} \leq \kappa \). Next, define \( a_{\omega} := (\omega_{p}(p))_{p \in S} \in \prod_{p \in S} A_{p} \). Then \( \|a_{\omega}\|_{S} \leq \kappa \), and \( \tau_{\omega} \) can be interpreted as the characteristic function of \( K_{S}a_{\omega}K_{S} \). As a consequence, \( f_{S} \in \mathcal{H}^{ur}(H(\mathbb{Q}_{S})) \) can be written as \( f_{S} = \sum_{\omega \in B_{S,\mathbb{R}}} f_{S}(a_{\omega}) \tau_{\omega} \), and if \( K_{S}aK_{S} = K_{S}a_{\omega}K_{S} \), the sum
\[(4.13) \sum_{\pi \in \mathcal{F}} \text{Tr} \pi_{S}(\tau_{\omega})
\]
coincides with (4.13), where for \( \pi = \otimes_{v} \pi_{v} \) we set \( \pi_{S} := \otimes_{p \in S} \pi_{p} \). Thus, the assertion follows from Theorem 4.3 in view of the bound \( |B_{S,\mathbb{R}}| \leq (c'_{1} \kappa)^{\text{rank}_{\mathbb{Z}}X_{s}(A_{p})} \) for some suitable \( c'_{1} \in \mathbb{N} \). \( \square \)

With the preceding asymptotic trace formula, we are able to prove a Plancherel density theorem and a Sato-Tate equidistribution theorem. For this, let us first introduce the relevant measures. Define a counting measure on \( H(\mathbb{Q}_{S})^{\wedge,ur} \) for the \( S \)-component of \( \mathcal{F} \) by setting
\[
\hat{m}_{\mu,S}^{\text{count}} := \frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{F}} \delta_{\pi_{S}},
\]
where \( \delta_{\pi_{S}} \) denotes the Dirac delta measure at \( \pi_{S} \in H(\mathbb{Q}_{S})^{\wedge,ur} \). We then have the following

**Corollary 4.5 (Plancherel density theorem).** For any \( f_{S} \in \mathcal{H}^{ur}(H(\mathbb{Q}_{S})) \) we have
\[
\lim_{\mu \to \infty} \hat{m}_{\mu,S}^{\text{count}}(f_{S}) = \hat{m}_{S}^{\text{Pl,ur}}(f_{S}).
\]
Proof. For any \( f_S \in \mathcal{H}^u(H(\mathbb{Q}_S)) \), we can choose a constant \( \kappa > 0 \) such that \( f_S \) is in \( \mathcal{H}_\kappa^u(H(\mathbb{Q}_S)) \). Corollary 4.4 implies that

\[
\frac{(2\pi)^d}{\mu^d \text{vol}(M)} \sum_{\pi \in \mathcal{F}(\mu)} \hat{f}_S(\pi) = f_S(1) + O(\mu^{-1} p_S^\epsilon \|f_S\|_\infty).
\]

Since integration of \( \mathcal{F}(\mu) \) over \( x \) and \( \mu \) yields Weyl’s law \( |\mathcal{F}(\mu)| = \# \{ j \mid \mu_j \leq \mu \} = \frac{\mu^d \text{vol}(M) \pi d}{(2\pi)^d} \mu^d + O(\mu^{d-1}) \), the assertion follows by taking the limit \( \mu \to \infty \) in the last equality for each \( \kappa \) separately. \( \square \)

Corollary 4.6 (Sato-Tate equidistribution theorem). Fix \( \theta \in \mathscr{C}(G) \), and let \( \hat{f} \) be a continuous function on \( T_{\mu_0}/\Omega_{\mu_0} \). By (4.3), \( \hat{f} \) can be extended to a continuous function \( \hat{f}_p \) on \( G_{\mu,\text{ur,temp}} \) for any \( p \in \mathcal{V}(\theta) \). Let \( \{(p_k, \mu_k)\}_{k \geq 1} \) be a sequence in \( \mathcal{V}(\theta) \times \mathbb{R}_{>0} \) such that \( p_k \to \infty \) and \( p_k^l / \mu_k \to 0 \) as \( k \to \infty \) for any integer \( l \geq 1 \). Then

\[
\lim_{k \to \infty} \hat{m}_{\mu_k,p_k}^\text{count}(\hat{f}_p) = \hat{m}_{\mu_k,p_k}^{\text{ST}}(\hat{f})
\]

where we wrote \( \hat{m}_{\mu_k,p_k}^\text{count} \) for \( \hat{m}_{\mu_k,p_k} \).

Proof. To begin, notice that there exists a constant \( \kappa > 0 \) such that \( f_{p_k} \) belongs to \( \mathcal{H}_\kappa^u(H(\mathbb{Q}_S)) \) for any \( k \), where \( f_{p_k} \) denotes the inverse image of \( \hat{f}_{p_k} \). Now, by Corollary 4.4 we have

\[
\frac{(2\pi)^d}{\mu^d \text{vol}(M)} \sum_{\pi \in \mathcal{F}(\mu_k)} \hat{f}_{p_k}(\pi) \hat{f}_{p_k}^{\text{Pl}}(\pi) = \hat{m}_{p_k}^{\text{Pl,ur}}(\hat{f}_{p_k}) + O(\mu_k^{-1} p_k^\epsilon \|f_{p_k}\|_\infty).
\]

Since \( \|f_{p_k}\|_\infty \) does not depend on \( p_k \), and by assumption we have \( \mu_k^{-1} p_k^\epsilon \to 0 \) as \( k \to \infty \), the assertion is proved by using the same argument than in the proof of Corollary 4.5. \( \square \)

5. Equivariant asymptotics for Hecke eigenvalues and Sato-Tate equidistribution

Let us now turn to the equivariant situation. To begin, we collect some basic facts about orbital integrals needed in the sequel.

5.1. Orbital integrals. Choose \( \Theta : g \to g^{-1} \) as Cartan involution on \( G \), and suppose as we may that \( K = G \cap \mathrm{SO}(N) \), where \( K \) denotes a maximal compact subgroup of \( G \). Let \( T \) denote a Cartan subgroup in \( G \), and suppose that \( T \) is \( \Theta \)-stable. Notice that any semisimple element is conjugate to an element of a \( \Theta \)-stable Cartan subgroup in \( G \), see [26, Theorem 5.22]. For each \( \gamma \in T \), let \( G_\gamma \) denote the centralizer of \( \gamma \) in \( G \), and \( \mathfrak{g}_\gamma := \{ X \in \mathfrak{g} \mid \text{Ad}(\gamma)X = X \} \) its Lie algebra. One then introduces the orbital integral

\[
J(\gamma, f) := J^{G/T}(\gamma, f) := |D(\gamma)|^{1/2} \int_{G_\gamma \backslash G} f(g^{-1} \gamma g) \, dg, \quad \gamma \in T, \ f \in C_c^\infty(G),
\]

where \( D(\gamma) := D^{G}(\gamma) := \text{det}((1 - \text{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{p}}) \) denotes the Weyl discriminant. Let \( t \) denote the Lie algebra of \( T \), and write \( \mathfrak{g}_C \) and \( \mathfrak{t}_C \) for the respective complexifications. It is well-known that \( J(\gamma, f) \) defines a compactly supported smooth function on the subset \( T' \subset T \) of regular elements of \( T \), see [26, Propositions 11.7]. Since the structure of a single \( J(\gamma, f) \) is rather involved, it is convenient to consider superpositions of orbital integrals of the following form. Let \( W_T \) denote the Weyl group generated by reflections corresponding to the imaginary roots in \( \{ \mathfrak{g}_C, \mathfrak{t}_C \} \). One then defines the stable orbital integral

\[
\tilde{J}(\gamma, f) := \tilde{J}^{G/T}(\gamma, f) := \sum_{w \in W_T} J(w \gamma, f).
\]

To describe the structure of the stable orbital integrals more explicitly, let \( \mathfrak{a} \) be a maximal Abelian subspace in \( \mathfrak{p} \) with respect to the Cartan decomposition \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \), and put \( A := \exp(\mathfrak{a}) \). Consider the corresponding Iwasawa decomposition

\[
G = AU \mathcal{K}
\]
of $G$, $U$ being a unipotent subgroup $U$ in $G$. There is an algebra isomorphism $C_c^\infty(K\backslash G/K) \ni f \mapsto \mathcal{A}f \in C_c^\infty(a/W)$ called the *Abel transform* given by

\begin{equation}
(\mathcal{A}f)(X) := e^{\theta(X)} \int_U f(\exp(X)u) \ du,
\end{equation}

where $W$ is the Weyl group of $(\mathfrak{g}_C, \mathfrak{a}_C)$ and $\theta$ denotes the half sum of positive roots of $(A, U)$. We may suppose that $T = T_AT_K$, where $T_K := T \cap K$ and $T_A := T \cap A$. Further, for any integrable function $h \in L^1(a)$ let

\[
\widehat{h}(\lambda) := \int_a h(x)e^{\lambda(x)} \ dx, \quad \lambda \in i\mathfrak{a}^*,
\]

denote its Fourier transform. Now, let $f \in C_c^\infty(K\backslash G/K)$ be arbitrary.

In what follows, we state a Fourier inversion formula for $J^{G/T}(\gamma, f)$, which expresses the latter in terms of the Fourier transform $\widehat{\mathcal{A}f}$ of $\mathcal{A}f$. First, we consider the case where $\gamma \in T'$. Let $M$ denote the centralizer of $T_A$ in $G$. Clearly, $A \subset M$. Since $T$ is commutative and $T_A \subset T$, we have $T \subset M$ as well, and by [9, Proposition 4.7] or [26, (11.42)] one has

\[
J^{G/T}(\gamma, f) = J^{M/T}(\gamma, \hat{f}_U),
\]

where $U$ is a subgroup of $M$ such that $G = M \cup K$, and for $m \in M$ we set $f_U(m) := \eta(m) \int_U f(mu) \ du$, $\eta$ being a non-negative real-valued quasi-character on $M$. Notice that $\eta$ is trivial on $T_K$, and that $W_I$ does not act on $T_A$. Now, since $G = H(\mathbb{R})$ and $H$ is connected, there exists an algebraic torus $T$ over $\mathbb{R}$ such that $\gamma \in T(\mathbb{R})$ and $T(\mathbb{R}) \subset T$, compare [31 Corollary 13.3.8 (i)]. It is known that $T(\mathbb{R})$ is isomorphic to $\left(\mathbb{R}^+\right)^{n_1} \times (\mathbb{R}_{>0})^{n_2} \times (\mathbb{C}^1)^{n_3}$ for some $n_1, n_2, n_3 \in \mathbb{N}$, with $\mathbb{C}^1 := \{z \in \mathbb{C} | |z| = 1\}$. The part $(\mathbb{R}^+)^{n_1} \times (\mathbb{R}_{>0})^{n_2}$ is included in the center $C(M)$ of $M$. Since the Fourier transform on $C(M)$ is obvious, the problem is reduced to the case $J^{M/T}(\gamma', \hat{f}_U)$ where $\gamma'$ denotes the $(\mathbb{C}^1)^{n_3}$-part of $\gamma$. Note that rank $M := \text{rank} K$, $K := M \cap K$. Since $(\mathbb{C}^1)^{n_3}$ is connected, $\gamma'$ belongs to the connected component $M^0$ of the identity in $M$. Therefore, we may assume that $M$ is connected in view of the equality $J^{M/T}(\gamma', f) = \int_{[M^0]^{\gamma', \hat{f}_U}} J^{M^0/(T \cap M^0)}(\gamma', f)$. By the above mentioned conditions on $M$ and $K$, we can now apply Herb’s Fourier inversion formula [18 Theorem 1], [17 Theorem 2] to $J^{M/T}(\gamma', f)$, and consequently obtain an explicit formula for $J^{G/T}(\gamma, f)$. Without explaining the details, we obtain as a result for any $f \in C_c^\infty(K\backslash G/K)$ and regular $\gamma \in T'$ the expression

\[
J^{G/T}(\gamma, f) = \int_{i\mathfrak{a}^*} \hat{\mathcal{A}f}(\lambda) \Phi(\gamma, \lambda) \ d\lambda,
\]

where $\Phi(\gamma, \lambda)$ is an explicitly given smooth function on $T' \times i\mathfrak{a}^*$. Next, let us consider the case where $\gamma \in T \setminus T'$ is a singular element. For each $y \in G$, set

\[
f_\gamma(y) := \frac{|D_G^\gamma(\gamma y)|^{1/2}}{|D_G^\gamma(y)|^{1/2}} \int_{G \setminus G} f(x^{-\gamma}yx) \ dx,
\]

and fix a chamber $c := \{H \in t | \alpha(H) > 0 \text{ for all } \alpha \in \Delta_+\}$ with respect to a positive root system $\Delta_+$ in $(\mathfrak{g}_c, \mathfrak{t}_c)$. Let $\Delta_\gamma$ denote a positive root system in $(\mathfrak{g}_\gamma, \mathfrak{t}_\gamma)$ and for $\alpha \in \Delta_\gamma$, write $D_\alpha$ for the invariant differential operator on $T$ corresponding to $H_\alpha \in t$, where $H_\alpha$ is defined via the relation $\alpha(H) = \langle H, H_\alpha \rangle$ for all $H \in t$. Then, by Harish-Chandra’s limit formula [15 Theorem 4],

\[
J^{G/T}(\gamma, f) = f_\gamma(1) = \lim_{\delta \to 1, \delta \in \exp(c)} D_\gamma J^{G/T}(\delta, f_\gamma) = \lim_{\delta \to 1, \delta \in \exp(c)} D_\gamma J^{G/T}(\gamma \delta, f),
\]

where $D_\gamma := c_\gamma \prod_{\alpha \in \Delta_\gamma} D_\alpha$ acts on the variable $\delta \in \exp(c)$ and $c_\gamma$ is some constant. Since $D_\gamma$ remains unchanged under the action of $W_I$ on $\delta$, we have

\[
J(\gamma, f) \leq J(\gamma, f) = \sum_{w \in W_I} \delta \to 1, \delta \in \exp(c)} D_\gamma (w(\gamma) \delta, f) = \lim_{\delta \to 1, \delta \in \exp(c)} D_\gamma J(\gamma \delta, f).
\]

Now, let $\gamma \in T_K$ be arbitrary, and without loss of generality suppose that $\mathfrak{a} \cap \mathfrak{g}_\gamma$ is a maximal Abelian subspace in $\mathfrak{p} \cap \mathfrak{g}_\gamma$. Let $B$ denote a $\Theta$-stable Cartan subgroup of $G$ containing $A$, and $\mathfrak{b}$ the Lie algebra of $B$. Set $\mathfrak{b}_\gamma := \mathfrak{b} \cap \mathfrak{g}_\gamma$. Assume that the root system of $(\mathfrak{g}_\gamma, \mathfrak{c}_\gamma, \mathfrak{b}_\gamma, \mathfrak{c}_\gamma)$ does not contain all
non-compact roots in \((g_\mathbb{C}, b_\mathbb{C})\). Then \(\Phi(\gamma, \lambda)\) is uniformly bounded for regular \(\gamma\). Moreover, if \(\gamma\) is singular, Herb’s explicit formula \([17, 18]\) implies that \(D_\gamma \Phi(\gamma \delta, \lambda)\) is uniformly bounded by \((1 + |\lambda|)^{r/2}\) for any \(\delta \in \text{exp}(\mathbb{C})\), where \(r\) is the number of non-compact roots in \((g_\mathbb{C}, b_\mathbb{C}, \mathbb{C})\). Consequently, by \((5.3)\) and \((5.4)\) we arrive at the uniform bound

\[
J(\gamma, f) \ll \int_{\mathfrak{s}_a} |\hat{A}(\lambda)| \left(1 + |\lambda|\right)^{r/2} d\lambda, \quad \gamma \in T_K, \ f \in C^\infty_c(K\backslash G/K),
\]

where \(l = 2 \dim U\) denotes the number of non-compact roots in \((g_\mathbb{C}, b_\mathbb{C})\).

### 5.2. Equivariant asymptotics and equidistribution results

We are now ready to derive asymptotics for Hecke eigenvalues in the equivariant setting. With the notation as in Section 4, let \(K\) be a maximal compact subgroup of \(G = H(\mathbb{R})\), so that \(C(G) \subset K\). Further, we may suppose that \(G\) is not compact. Denote by \(Z_H\) the center of \(H\), and set

\[
Z := Z_H(\mathbb{Q}) \cap K_0.
\]

Clearly, \(Z \subset C(G) = Z_H(\mathbb{R})\). Choose an irreducible representation \(\sigma\) in \(\tilde{K}\). It is obvious that \(L^2_k(\mathbb{R}) \backslash H(\mathbb{A}) / K_0 = 0\) if \(\sigma\) is not trivial on \(Z\). Hence, we may suppose that \(\sigma\) is trivial on \(Z\), so that

\[
Z_{\sigma} := \text{Ker}(\sigma) \supset Z.
\]

Notice that \(C(\Gamma_j) = \Gamma_j \cap Z_H(\mathbb{Q}) = Z\) for any \(j\), where \(C(\Gamma_j)\) denotes the center of \(\Gamma_j\), since \(\Gamma_j\) is Zariski dense in \(H\), see \([38]\) Theorem 4.10. To begin, we need the following variant of Lemma 5.5.

**Lemma 5.1.** Let \(T_{\alpha}\) be a Hecke operator as in Lemma 4.2. For \(\varepsilon \geq 0\), denote by \(f_\varepsilon : G \to \{0, 1\}\) the characteristic function of \(K_\varepsilon := \{g \in G \mid \text{dist}_G(K, gK) \leq \varepsilon\}\). Then, one has for each \(x = (g, l) \in M\) and \(\varepsilon > 0\) the asymptotic formula

\[
K_{T_{\alpha} \circ \tilde{T}_{\beta} \circ \Pi_{\alpha}}(x, x) - \left[ \sum_{k \in \Gamma(\alpha, l)} \sigma(k) + \sum_{y \in T(\alpha, x) - C(\alpha, x)} \sigma(ky) \right] K_{\tilde{T}_{\beta} \circ \Pi_{\alpha}}(x, x)
\]

\[
= O\left(\mu^{d - \dim K - 1} \sum_{\beta \in \Gamma(\beta \gamma, \Gamma_\varepsilon)} \left( f_\varepsilon(g^{-1} \beta g) - f_0(g^{-1} \beta g) \right) \right)
\]

\[
+ O\left( (\mu / \varepsilon)^{(d - \dim K) / 2} \sum_{m = 1}^{c_\varepsilon,l} |\Gamma_\varepsilon| / \Gamma_\varepsilon \alpha_{l, m} |\Gamma_\varepsilon| + \mu^{-\infty} \sum_{j = 1}^{c_\varepsilon,l} \sum_{m = 1}^{c_\varepsilon,l} |\Gamma_\varepsilon| / \Gamma_\varepsilon \alpha_{j, l, m} |\Gamma_\varepsilon| \right).
\]

**Proof.** Let \(x = (g, l)\) be fixed and \(y = (\beta g, l) \in H(\alpha, x)\). Then

\[
\text{dist}(xK, yK) = \text{dist}(\Gamma gK, \Gamma \beta gK) = \inf_{\gamma \in \Gamma_\beta} \text{dist}_G(K, g^{-1} \gamma g K).
\]

Assuming as we may that \(\beta \equiv \Gamma_\beta \) has been chosen such that \(\inf_{\gamma \in \Gamma_\beta} \text{dist}_G(K, g^{-1} \gamma g K)\) is attained by \(\text{dist}_G(K, g^{-1} \beta g K)\) we obtain

\[
\text{dist}(xK, yK) \leq \varepsilon \iff f_\varepsilon(g^{-1} \beta g) = 1.
\]

Furthermore, \(f_0(g^{-1} \beta g) = 1\) iff \(y \in T(\alpha, x)\). Consequently, for \(\text{dist}(xK, yK) \leq \varepsilon\) we have

\[
f_\varepsilon(g^{-1} \beta g) - f_0(g^{-1} \beta g) = \begin{cases} 1 & \text{iff } y \in H(\alpha, x) - T(\alpha, x), \\ 0 & \text{iff } y \in T(\alpha, x). \end{cases}
\]

Thus,

\[
K_{T_{\alpha} \circ \tilde{T}_{\beta} \circ \Pi_{\alpha}}(x, x) - \sum_{y \in T(\alpha, x)} K_{\tilde{T}_{\beta} \circ \Pi_{\alpha}}(y, x) = \sum_{y \in H(\alpha, x) - T(\alpha, x)} K_{\tilde{T}_{\beta} \circ \Pi_{\alpha}}(y, x) + \sum_{y \in H(\alpha, x) - T(\alpha, x), \text{dist}(xK, yK) > \varepsilon} K_{\tilde{T}_{\beta} \circ \Pi_{\alpha}}(y, x)
\]
up to terms of order $O(\mu^{-\infty})$ times the cardinality of the sum in (3.12). The assertion now follows from Proposition 3.2 along the lines of the proof of Lemma 3.3 by taking into account [41] Remark 3.4.

To proceed, we need the following

**Proposition 5.2.** Let $K$ denote the maximal compact normal subgroup of $G$, that is, the product of the center $C(G)$ and all compact simple factors of $G$. Fix $m \in \mathbb{N}$, and let $\beta \in H(\mathbb{Q}) \cap \mathcal{M}(N, \frac{1}{m} \mathbb{Z})$ be such that $\beta \notin K$. Choose a bounded domain $D$ in $G$. Then, for any $0 < \varepsilon < \log(1 + 1/Nm^2)$ and any $0 < s < 1$ we have

$$
\int_D f_\varepsilon(g^{-1} \beta g) \, dg \ll_D m^N s^{-1} \varepsilon^{1-s}.
$$

**Proof.** Recall the notations and the setting in Section 6.1. To begin, we define an inner product on $\mathcal{M}(N, \mathbb{R})$ by setting $(X, Y) := \text{Tr}(X Y)$ and a norm $\|X\| := (X, X)^{1/2}$. The corresponding distance is locally equivalent to the distance dist$_G$ on $G \subset \mathcal{M}(N, \mathbb{R})$. Denote by $\Delta$ the root system of $(\mathfrak{g}, \mathfrak{a})$, by $\Delta_0$ the set of positive simple roots, and by $W$ the corresponding Weyl group. Further, recall the polar decomposition $G = KA_K$, by which every $g \in G$ can be written as $g = k_1 \cdot \exp(X(g)) \cdot k_2$ where $k_i \in K$, and $X(g) \in \mathfrak{a}$ uniquely determined up to conjugation by $W$. Introducing the positive Weyl chamber $\mathfrak{a}^+ := \{X \in \mathfrak{a} \mid \alpha(X) > 0 \forall \alpha \in \Delta_0\}$, this decomposition induces a mapping $X : G \to \mathfrak{a}^+$ such that

$$
\|X(g)\| \ll \text{dist}_G(K, gK) \ll \|X(g)\|
$$

uniformly in $g \in D$. Further, by [34] Lemma 4.2,

$$
(5.6) \quad \|X(g)\|^2 \ll_D 2 \log\left(\frac{\|g\|^2}{N}\right) \leq 2 \|X(g)\|, \quad g \in D,
$$

with $\mathcal{L}(g) = 0$ iff $g \in K$. Now, let $\beta \in G$ be arbitrary. By the $K$-bi-invariance of $f_\varepsilon$ one computes with respect to the global Cartan decomposition (3.14)

$$
\int_D f_\varepsilon(g^{-1} \beta g) \, dg = \int_K \int_{D_\mathfrak{p}} f_\varepsilon(\exp(-X) \cdot \beta \cdot \exp X) \, dX \, dk \ll \int_{D_\mathfrak{p}} f_\varepsilon(\exp(-X) \cdot \beta \cdot \exp X) \, dX,
$$

where $D_\mathfrak{p} \subset \mathfrak{p}$ is a bounded domain and $dX$ a suitable measure on $\mathfrak{p}$. Let us examin the last integral more closely by introducing the $\beta$-displacement function

$$
\delta_\beta(X) := \text{dist}_G(K, \exp(-X) \cdot \beta \cdot \exp X \cdot K), \quad X \in \mathfrak{p},
$$

which can also be regarded as a function on the Riemannian symmetric space $G/K$ in view of the diffeomorphism $G/K \simeq \mathfrak{p}$. If $\beta$ is semisimple, the infimum of $\delta_\beta$ is reached, and the points where it is reached constitute a submanifold $S_\beta \subset \mathfrak{p}$, see [16] p. 279 and [9] Proposition 5.7. Furthermore, the minimum of $\delta_\beta$ is given by $\|X_\beta\|$ if one writes $\beta = \exp(X_\beta) \cdot k_\beta$ with respect to the decomposition (3.14). Notice that since $H(\mathbb{Q}) \cap H(\mathbb{A})$ is compact, all elements in $H(\mathbb{Q})$ are semisimple.

Now, assume that $\beta \in H(\mathbb{Q}) \cap \mathcal{M}(N, \frac{1}{m} \mathbb{Z})$ for some $m \in \mathbb{N}$, but $\beta \notin K$, so that $X_\beta \neq 0$. By the above,

$$
f_\varepsilon(\exp(-X) \cdot \beta \cdot \exp X) = 0 \quad \text{for all } X \in \mathfrak{p} \text{ if } \varepsilon < \|X_\beta\|,
$$

and by (5.6) we have $\mathcal{L}(\beta) \leq 2 \|\beta\| \leq 2 \|X_\beta\|$, while

$$
\frac{\text{Tr}(\beta^2 \beta)}{N} = \frac{L}{Nm^2} > 1, \quad L \in \{Nm^2 + 1, Nm^2 + 2, \ldots\}.
$$

Consequently, we conclude for all $X \in \mathfrak{p}$ that

$$
f_\varepsilon(\exp(-X) \cdot \beta \cdot \exp X) = 0 \quad \text{if } \varepsilon \ll \log\left(1 + \frac{1}{Nm^2}\right),
$$

yielding the assertion for $\beta \notin K$.

Next, let us suppose that $\beta \in H(\mathbb{Q}) \cap \mathcal{M}(N, \frac{1}{m} \mathbb{Z}) \cap K$, so that inf $\delta_\beta = 0$. Our intention is to make use of the upper bound (5.3) for orbital integrals to show the desired estimate in this case. For this

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5This fact is crucial for the following.
sake notice that since $H$ is a closed subgroup of $\text{SL}(N)$ over $\mathbb{Q}$, the Lie algebra $\mathfrak{h} \subset \mathcal{M}(N, \mathbb{Q})$ of $H$ is a $\mathbb{Q}$-vector space, so that $\mathfrak{h} \otimes \mathbb{R} \simeq \mathfrak{g}$ and $\mathfrak{h} = \mathfrak{g} \cap \mathcal{M}(N, \mathbb{Q})$. Further, $\beta \in H(\mathbb{Q})$ implies that the $\mathbb{R}$-subspace $\mathfrak{g}_\beta := \{ X \in \mathfrak{g} \mid \beta X = X\beta \}$ has a basis $\{ Y_j \}_{1 \leq j \leq \dim \mathfrak{g}_\beta}$ consisting of matrices $Y_j \in \mathcal{M}(N, \mathbb{Z})$. Consequently, $L := \mathfrak{g}_\beta^1 \cap \mathcal{M}(N, \mathbb{Z})$ must be a $\mathbb{Z}$-lattice in the orthogonal complement $\mathfrak{g}_\beta^2 := \{ X \in \mathfrak{g} \mid (X, Y_j) = 0, \ 1 \leq j \leq \dim \mathfrak{g}_\beta \}$. In addition, $\mathfrak{g}_\beta^2$ is $\text{Ad}(\beta)$-stable since $(\beta X \beta^{-1}, Y) = (X, \beta Y \beta^{-1})$ for any $\beta \in K$. Thus, $m^2 \beta L|\beta \subset L$, and we conclude that
\[
|D(\beta)| := \left| \det \left( (1 - \text{Ad}(\beta))|_{\mathfrak{g}_\beta^2} \right) \right| \in \frac{1}{m^2 \mathbb{N}}.
\]
In view of $\beta \notin C(G)$, this implies that $1 \leq m^N|D(\beta)|^{1/2}$. Now, choose a $\Theta$-stable Cartan subgroup $T$ in $G$ such that $T_K := T \cap K$ is a maximal torus in $K$. There exists an element $k_0 \in K$ such that $k_0^{-1} \beta k_0 = \beta_0 \in T_K$, and without loss of generality we may suppose that $D$ is left $K$-invariant. Since $|D(\beta)| = |D(\beta_0)|$, we obtain
\[
\int_D f_\epsilon(g^{-1} \beta g) \ dg = \int_D f_\epsilon(g^{-1} \beta_0 g) \ dg \leq m^N|D(\beta_0)|^{1/2} \int_D f_\epsilon(g^{-1} \beta_0 g) \ dg.
\]
Further, there are only finitely many possibilities for centralizers of elements in $T_K$, so that normalizing their Haar measures we arrive at
\[
\int_D f_\epsilon(g^{-1} \beta g) \ dg \leq m^N|D(\beta_0)|^{1/2} \int_{D(\beta_0) \times D} f_\epsilon(g^{-1} \beta_0 g) \ dg \leq m^N J^G/T(\beta_0, f_\epsilon),
\]
where we wrote $D_{\beta_0} \subset G_{\beta_0}$ and $D' \subset G_{\beta_0} \backslash G$ for the projections of $D$ with respect to the decomposition $G \simeq G_{\beta_0} \times G_{\beta_0} \backslash G$.

In order to use the upper bound \[\text{(5.5)}\], we have to replace $f_\epsilon$ by a test function $\tilde{f}_\epsilon \in C^\infty_c(K \backslash G/K)$ in a suitable way. Recall that $f_\epsilon$ is the characteristic function of the $K$-bi-invariant compact set $K_\epsilon := \{ g \in G \mid \text{dist}_G(K, gK) \leq \epsilon \}$. Using standard techniques one can construct a function $\tilde{f}_\epsilon \in C^\infty_c(K \backslash G/K)$ which, say, equals 1 on $K_\epsilon$ and is supported inside $K_3 \epsilon$, compare \[\text{[20, Theorem 1.4.1]}\]. Furthermore, one can achieve that the restriction of $\tilde{f}_\epsilon$ to $A U \simeq a \times u \simeq p$ with respect to the Iwasawa decomposition \[\text{(5.1)}\] is essentially of the form
\[
\tilde{f}_\epsilon \equiv f_\epsilon \ast \chi_\epsilon,
\]
where $\chi \in C^\infty_c(a \times u)$ denotes a non-negative function with support in the unit ball, $\int_{a \times u} \chi = 1$, and $\chi_\epsilon := \epsilon^{-\dim (A \times U)} \chi(\cdot / \epsilon)$. Furthermore, writing the integral over $U$ in \[\text{(5.2)}\] as an integral over its Lie algebra $a$ one computes
\[
\hat{A} f_\epsilon(\lambda) = \int_a \left( \int_u e^{\phi(\lambda)} \tilde{f}_\epsilon \chi \right) \frac{\chi^{2X}}{e^{2X}}(X, Y) \ dY \ e^{\lambda(X)} \ dX
\]
\[
= \int_{a \times u} \left( \int_{a \times u} f_\epsilon(X', Y') \chi_\epsilon(X - X', Y - Y') \ dY' \ dX' \right) \ e^{\lambda+\phi(X)} \ dY \ dX
\]
\[
= \int_{a \times u} \left( \int_{a \times u} f_\epsilon(X', Y') \chi(X - X', Y - Y') \ dY' \ dX' \right) \ e^{\lambda+\phi(X)} \ dY \ dX
\]
\[
= \epsilon^{\dim (A \times U)} \int_a \ e^{\lambda(X)} \left( \int_u \int_{a \times u} f_\epsilon(X', Y') \chi(X - X', Y - Y') \ dY' \ dX' \right) \ dX
\]
\[
=: \hat{A}_\epsilon(\lambda) \in C^\infty_c(a),
\]
where $\hat{B}_\epsilon(\lambda) \in \mathcal{S}(ia^*)$ is rapidly decreasing in $\lambda$ uniformly in $\epsilon$. Now, by assumption, $\beta_0 \notin \tilde{K}$, which implies that
\[
(5.9) \quad \text{the root system of } (\mathfrak{g}_{\beta_0, C}, \mathfrak{b}_{\beta_0, C}) \text{ does not contain all non-compact roots in } (\mathfrak{g}_C, \mathfrak{b}_C).
\]
In fact, let $G_1$ be a non-compact simple linear algebraic $\mathbb{R}$-group, let $A_1$ denote the $\mathbb{R}$-connected component of the identity in a maximal split algebraic $\mathbb{R}$-torus in $G_1$, and $B_1$ a Cartan subgroup of $G_1$ containing $A_1$. Set $g_1 := \text{Lie}(G_1)$, $a_1 := \text{Lie}(A_1)$, and $b_1 := \text{Lie}(B_1)$. Suppose that a semisimple element $\gamma_1$ in $G_1$ commutes with all root spaces of $(g_1, a_1)$. Then $\gamma_1$ commutes with $A_1$. It is clear that any non-compact root space in $(g_1, c_1, b_1, c)$ is a subspace of a root space of $(g_1, c, a_1, c)$. Since by $\mathfrak{sl}_2$-triple theory every non-trivial nilpotent element has a non-trivial factor in a root space of $A_1$, $\gamma_1$ commutes with all unipotent elements. The Bruhat decomposition then implies that $\gamma_1 \in C(G_1)$ because $G_1$ is simple, yielding (5.1).

In view of (5.9) we can now apply the bound (5.7) to estimate $J^{G/T}(\beta_0, \tilde{f}_\varepsilon)$, and with (5.8) we obtain for sufficiently large $s' > 0$ the estimate

$$
J^{G/T}(\beta_0, f_\varepsilon) \leq J^{G/T}(\beta_0, \tilde{f}_\varepsilon) \leq \int_{\text{ia}^*} |\tilde{A}f(\lambda)(1 + \|\lambda\|^s)\text{dim} U^{-1} d\lambda
$$

$$
= \varepsilon^{\text{dim} U + \text{dim} A} \int_{\text{ia}^*} |\tilde{B}(\varepsilon\lambda)| (1 + \|\lambda\|^{s'})^{\text{dim} U^{-1} s'} d\lambda
$$

$$
= \varepsilon^{\text{dim} U + \text{dim} A - s'} \sup_{\lambda \in \text{ia}^*} |\tilde{B}(\varepsilon\lambda)(1 + \|\lambda\|^{s'})^{\text{dim} U^{-1} s'} d\lambda.
$$

Taking $s' = \text{dim} U + \text{dim} A - 1 + s$, the assertion of the proposition follows with (5.7).

Remark 5.3. Note that in the situation of the proposition above, one can actually show by stationary phase analysis that

$$
\int_D f_\varepsilon(g^{-1}\beta g) dg \leq C_{D,m} \varepsilon
$$

for any $0 < \varepsilon \leq \varepsilon(D, m)$, yielding a better power in $\varepsilon$. Nevertheless, both $\varepsilon(D, m)$ and the constant $0 < C_{D,m}$ cannot be specified in their dependence of $m$ with this method, which is essential for the obtention of Sato-Tate equidistribution results. But since the proof of (5.10) does not require the theory of orbital integrals and has an interest in its own, we include it below. In the case $\beta \notin K$, the proof of (5.10) is identical to the one in Proposition 5.2. Let us therefore suppose that $\beta \in K \cap H(\mathbb{Q}) \cap \mathcal{M}(N, \frac{1}{m}Z)$ for some $m \in \mathbb{N}$ and $\beta \notin K$. Then $\text{inf} \delta_\beta = 0$, and

$$
S_\beta = \{X \in p \mid \delta_\beta(X) = 0\} = \{X \in p \mid \exp(-X) \cdot \beta \cdot \exp X \in K\}
$$

is a submanifold of lower dimension by Lemma 3.7. In fact, $S_\beta \subset p$ coincides with the critical set of $\Delta_\beta(X) := \delta_\beta(X)^2/2$, see [10, p. 279] and [9, Proposition 5.7]. Furthermore, $S_\beta$ is clean as critical set of $\Delta_\beta$, compare [9, p. 64], and setting

$$
U_\beta(\varepsilon) := \{X \in D_p \mid \delta_\beta(X) < \varepsilon\}
$$

we conclude by definition of $f_\varepsilon$ that

$$
\int_{D_p} f_\varepsilon(\exp(-X) \cdot \beta \cdot \exp X) dX = \text{vol} (D_p \cap U_\beta(\varepsilon)).
$$

The fact that $S_\beta$ is clean as critical set of $\Delta_\beta$ means that the transversal Hessian $\text{Hess}_\beta^\perp \Delta_\beta$ of $\Delta_\beta$ is non-degenerate, which together with the fact that $\Delta_\beta$ takes its minimum at $S_\beta$ implies that $\text{Hess}_\beta^\perp \Delta_\beta$ has strictly positive eigenvalues. By compactness we can therefore choose an $\varepsilon(D_p, m) > 0$ independent of $\beta$ such that

- there exist finitely many charts $\{\langle \kappa_i, \mathcal{O}_i \rangle \}_{i \in I}$ which cover $D_p \cap U_\beta(\varepsilon(N, D_p, m))$ such that for each $i \in I$

$$
\kappa_i^{-1}(x, y) \in S_\beta \iff y = 0,
$$

where $\kappa_i : p \supset \mathcal{O}_i \subset X \mapsto (x, y) \in \mathbb{R}^{\text{dim} S_\beta} \times \mathbb{R}^{\text{dim} p - \text{dim} S_\beta}$ are the corresponding local coordinates;
• for each \( i \in I \) and \( x \), the function \( y \mapsto \Delta_\beta \circ \kappa^{-1}_i(x, y) \) has a non-degenerate critical point \( y = 0 \);
• for each \( i \in I \) and \((x, y) \in \kappa_i(O_i)\), the real symmetric matrix

\[
 \mathcal{H}_i(x, y) := \left( \frac{\partial^2}{\partial y_i \partial y_j} (\Delta_\beta \circ \kappa^{-1}_i)(x, y) \right)_{i,j}
\]

has only strictly positive eigenvalues, \( \mathcal{H}_i(x, 0) \) being equal to \( \text{Hess}^\perp \Delta_\beta(\kappa^{-1}_i(x), 0) \).

Now, let \( X = \kappa^{-1}_i(x, y) \) for some \( i \) and \((x, y) \in \kappa_i(O_i)\). Since \( \Delta_\beta \circ \kappa^{-1}_i \) vanishes in second order at \( y = 0 \), Taylor expansion in transversal direction at \( y = 0 \) yields

\[
 \Delta_\beta(X) = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} (\Delta_\beta \circ \kappa^{-1}_i)(x, y_0) y_i y_j = \frac{1}{2} \langle y, \mathcal{H}_i(x, y_0)y \rangle
\]

for some \( y_0 \) lying on the line segment \([0, y]\) joining 0 and \( y \). By the theorem of Courant-Fischer, for \( y \neq 0 \) one has

\[
 \lambda_{\min} \leq \frac{\langle y, \mathcal{H}_i(x, y_0)y \rangle}{\langle y, y \rangle} \leq \lambda_{\max},
\]

\( \lambda_{\min} \) and \( \lambda_{\max} \) denoting the minimal and maximal eigenvalue of \( \mathcal{H}_i(x, y_0) \), so with (5.11) we infer for any \( 0 < \varepsilon < \varepsilon(D_p, m) \) that

\[
 \delta_\beta(X) < \varepsilon \implies \|y\| \leq \lambda_{\min}^{-1} \varepsilon.
\]

Thus, by compactness there is a constant \( C_{D_p, m} > 0 \) such that

\[
 D_p \cap U_\beta(\varepsilon) \subset \bigcup_{i \in I} \{ X = \kappa^{-1}_i(x, y) \in O_i \mid \|y\| \leq C_{D_p, m} \varepsilon \}.
\]

Since \( S_\beta \) has at least codimension 1 we conclude that

\[
 \text{vol}(D_p \cap U_\beta(\varepsilon)) = O_{D_p, m}(\varepsilon)
\]

for any \( 0 < \varepsilon < \varepsilon(D_p, m) \), finishing the proof of (5.10).

We can now state the second main result of this paper. As before, let \( \{\phi_j\}_{j \in \mathbb{N}} \) be an orthonormal basis of \( L^2(H(\mathbb{Q}) \backslash H(\mathbb{A})/K_0) \) such that each \( \phi_j \) is an eigenfunction of \( \Delta \) included in a single space \( V_\sigma \). In particular, each \( \phi_j \) is a simultaneous eigenfunction of \( \Delta \) and the Hecke operators \( T_{K_0^\sigma K_0^\sigma} \), \( \alpha \in H(\mathbb{A}_\mathbb{Q}) \).

**Theorem 5.4 (Equivariant distribution of Hecke eigenvalues).** Let \( H_1 \) be a simply connected algebraic group over a number field \( F \), and set \( H := \text{Res}_{F/\mathbb{Q}}(H_1) \). With the notation of the beginning of Section 5.2, write \( d := \text{dim } H(\mathbb{R}) \) and let \( N \geq d - \text{dim } K + 1 \) be a sufficiently large integer such that one has an embedding \( H(\mathbb{R}) \subset \text{SL}(N, \mathbb{R}) \). Suppose that \( \sigma \in \widehat{K} \) is trivial on \( Z \). Then, there exists a constant \( 0 < c < N^2 + 2N \) such that for any finite set \( S \) of primes in the complement of \( S_0 \), any \( \alpha \in H(\mathbb{Q}_S) \) with \( \|\alpha\|_S \leq \kappa \), and any \( 0 < s < 1 \)

\[
 \sum_{\phi_j \in L^2(H(\mathbb{Q}) \backslash H(\mathbb{A})/K_0)} \lambda_j(\alpha) \frac{n_Z(\alpha) d_\sigma \text{ vol}(M/K) \sigma_{d - \text{dim } K} \mu^{d - \text{dim } K}}{(2\pi)^{d - \text{dim } K} \mu^{d - \text{dim } K}} + O(\mu^{d - \text{dim } K} \cdot \frac{d - \text{dim } K - 1}{s - 1} \cdot \frac{1}{s - 1} \cdot \frac{N}{s - 1} \cdot \frac{1}{s - 1}), \quad \mu \gg N^{(d - \text{dim } K + 1)/(d - \text{dim } K - 1)},
\]

where \( n_Z(\alpha) := |Z_H(\mathbb{Q}) \cap (K \cdot K_0^\sigma K_0^\sigma)| \) and \( \text{vol}(M/K) \) denotes the orbifold volume of \( M/K \).

**Proof.** To begin, note that \( G \) might have a compact simple factor. Nevertheless, the conclusions of Lemmata 5.7 and 5.8 are still true if \( \beta \in H(\mathbb{Q}) \). Indeed, let \( F_v \) denote the completion of \( F \) at a place \( v \) of \( F \). Since \( G \) is isomorphic to \( \prod_v H_1(F_v) \), where \( v \) moves over infinite places of \( F \), there exists an infinite place \( w \) of \( F \) such that \( H_1(F_w) \) is not compact. In addition, \( H_1(F_w) \) is simple by assumption. Therefore, if \( \beta \in H(\mathbb{Q}) \), it is sufficient to apply Lemma 5.7 to \( H_1(F_w) \), \( H(\mathbb{Q}) = H_1(F) \) being directly

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8This means that \( H_1 \) has no nontrivial connected normal subgroups.
9Here \( \text{Res}_{F/\mathbb{Q}} \) means the restriction of scalars from \( F \) to \( \mathbb{Q} \), see [25, Section 2.1.2].
embedded into $\prod_{r|\infty} H_1(F_r)$. In view of Lemma \ref{lem:bundle_reduction} we are therefore left with the task of deriving suitable upper bounds for the sums of orbital integrals

$$\sum_{\beta \in \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1} \int_{\Gamma_1 \setminus G} (f_\varepsilon(g^{-1}\beta g) - f_0(g^{-1}\beta g)) \, dg,$$

where $l$ ranges from 1 to $c_H$, $m$ from 1 to $c_H$. For this, choose a connected compact domain $D$ in $G$ including a fundamental domain of $\Gamma_1 \setminus G$, and let $\beta_1, \ldots, \beta_r \in \Gamma_1 \setminus \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1$ be a set of representative elements, that is, $\Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 = \Gamma_1 \beta_1 \cup \cdots \cup \Gamma_1 \beta_r$ where $r = |\Gamma_1 \setminus \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1|$. Now, observe that

$$f_\varepsilon(g^{-1}\beta g) - f_0(g^{-1}\beta g) = 0 \iff \begin{cases} \text{dist}_G(K, g^{-1}\beta gK) > \varepsilon & \text{or} \\ \text{dist}_G(K, g^{-1}\beta gK) = 0. \end{cases}$$

The condition $\text{dist}_G(K, g^{-1}\beta gK) = 0$ is equivalent to $N(\beta, K)$ not being empty, in which case Lemma \ref{lem:neumann} asserts that $N(\beta, K)$ has full measure iff $\beta \in C(G)$. In addition, by the argument above, $\beta \in \tilde{K} \cap H_1(F)$ implies $\beta \in C(G)$. Therefore, taking into account Proposition \ref{prop:zeta_integration} the integrals in question can be estimated according to

$$\sum_{\beta \in \Gamma_1 \setminus \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1 \cap \Gamma_1} \int_{\Gamma_1 \setminus G} (f_\varepsilon(g^{-1}\beta g) - f_0(g^{-1}\beta g)) \, dg \ll \sum_{\beta \in \tilde{K}} \int_D (f_\varepsilon(g^{-1}\beta g) - f_0(g^{-1}\beta g)) \, dg \leq \varepsilon^{1-s} p_S^{(N+c_3)\varepsilon s^{-1}}$$

uniformly in $l$ and $m$, provided that

$$0 < \varepsilon \ll \log(1 + N^{-1} p_S^{-2\kappa} c_1^{-2}),$$

where $c_j$ denote the same constants as in the proof of Theorem \ref{thm:main}. Here we put $m = c_1 p_S^2$ in Proposition \ref{prop:zeta_integration} and took into account that $m$ is bounded by $p_S^{2\kappa}$ up to a constant. Integrating over $x$ and $\mu$ we now infer from Lemmata \ref{lem:neumann} and \ref{lem:bundle_reduction} that

$$\int_{-\infty}^{\mu} \int_{\Gamma_1 \setminus G} \left[ K_{t_0, o_{t_0}} \Pi_t(x, x) - n_Z(\alpha) K_{t_0, o_{t_0}}(x, x) \right] \, dx \, dt = O\left( \mu^{d-\dim K} \varepsilon^{-1-s} p_S^{(N+c_3)\varepsilon s^{-1}} + \mu^{(d-\dim K+1)/2} \varepsilon^{-(d-\dim K-1)/2} p_S^{2\kappa} \right),$$

where we took into account \ref{eq:dimension}. Putting

$$\varepsilon = \mu^{-(d-\dim K-1)/(d-\dim K+1)} p_S^{-2N\kappa/(d-\dim K+1)}$$

the assertion of the theorem now follows from \ref{lem:neumann} by the same arguments than those in the proof of Theorem \ref{thm:main}. Notice hereby that for $N \geq d - \dim K + 1$ the choice \ref{eq:dimension} in particular fulfills the requirement \ref{eq:mu} for $\mu \gg N(d-\dim K+1)/(d-\dim K-1)$. Furthermore,

$$\int_M [\pi_K : 1] \text{vol} \left[ (\Omega \cap S^*_x(M))/K \right] \, dx = [\pi_K : 1] \int_{M(K_{\text{prin}})} \text{vol} \left[ (\Omega \cap S^*_x(M))/K \right] \, dx,$$

where $(K_{\text{prin}})$ denotes the principal isotropy type of the $K$-action on $M$, $K_{\text{prin}} \subset K$ being a closed subgroup, and $M(K_{\text{prin}})$ the corresponding stratum of $M$, the latter being dense. But since there are only finitely many torsion points on a fundamental domain of an arithmetic quotient \ref{thm:arithmetic_quotient} we have $K_{\text{prin}} = \{1\}$, and consequently $[\pi_K : 1] = 1$. In addition, by singular cotangent bundle reduction \ref{rem:bundle_reduction} we have $\Omega/K \simeq T^*(M/K)$ as orbifolds, so that as in \ref{eq:dimension} one deduces

$$\int_M \text{vol} \left[ (\Omega \cap S^*_x(M))/K \right] \, dx = (d-\dim K) \int_{M/K} \text{vol} \left[ B^*_x(M/K) \right] \, d(x \cdot K)$$

$$= (d-\dim K) \omega_{d-\dim K} \text{vol} (M/K).$$
Next, we shall introduce for each $\sigma \in \hat{K}$ a family of $\sigma$-isotypic automorphic representations of $H$, and recall for this purpose the notations $\pi = \pi_\infty \otimes \pi_\text{fin}$, $V_\pi = V_\pi^\infty \otimes V_\pi^\text{fin}$, $V_{\pi_\infty}^{\leq \mu}$, and $V_{\pi_\text{fin}}^{K_0}$ introduced in Section 4.2 for each $\pi \in \hat{H}(k)$. The Peter-Weyl theorem implies the decompositions

$$V_{\pi_\infty}^{\leq \mu} = \bigoplus_{\sigma \in \hat{K}} V_{\pi_\infty, \sigma}^{\leq \mu},$$

where $V_{\pi_\infty, \sigma}^{\leq \mu}$ denotes the $\sigma$-isotypic component in $V_{\pi_\infty}^{\leq \mu}$. Let now $\mathcal{F}_\sigma := \mathcal{F}_\sigma(\mu)$ be the finite multi-set consisting of those automorphic representations $\pi \in \hat{H}(k)$ that satisfy

$$a_{\mathcal{F}_\sigma}(\pi) := m_\pi \dim V_{\pi_\infty, \sigma}^{\leq \mu} \dim V_{\pi_\text{fin}}^{K_0} > 0,$$

where each such $\pi$ appears in $\mathcal{F}_\sigma$ with multiplicity $a_{\mathcal{F}_\sigma}(\pi)$. Notice that $\dim V_{\pi_\infty, \sigma}^{\leq \mu}$ means the multiplicity of $\sigma$ in $\pi_\infty|_K$ if $V_{\pi_\infty, \sigma}^{\leq \mu}$ is not empty. We also define a counting measure $\widehat{m}_{\mu, \sigma, S}^{\text{count}}$ on $H(Q_S)^{\wedge, \text{ur}}$ for the $S$-component of $\mathcal{F}_\sigma$ by setting

$$\widehat{m}_{\mu, \sigma, S}^{\text{count}} := \frac{1}{|\mathcal{F}_\sigma|} \sum_{\pi \in \mathcal{F}_\sigma} \delta_{\pi_S}.$$

The following results are direct consequences of Theorem 5.4 and can be proved by the same arguments used in Section 4.2 to prove their non-equivariant versions.

**Corollary 5.5 (Equivariant asymptotic trace formula).** Choose a $K$-type $\sigma \in \hat{K}$, and for simplicity suppose that $Z_H(Q) \cap K$ is contained in $K_0$. Then, there exists a constant $c' > 0$ such that for each finite set $S$ of primes outside $S_0$, each $f_S \in H^\text{ur}(H(Q_S))$ with $|f_S| \leq 1$, and each $0 < s < 1$

$$\sum_{\pi \in \mathcal{F}_\sigma(\mu)} \mathrm{Tr} \pi_S(f_S) = f_S(1) \cdot \frac{|Z| d_\sigma \text{vol}(M/K) \omega_{d-\dim K}}{(2\pi)^{d-\dim K}} \mu^{d-\dim K} + O(\mu^{d-\dim K - \frac{\dim K - 1}{2}} (1-s) \rho_{\mu, \sigma, S}^{c'} s^{-1})$$

for any $\mu \gg N^{(d-\dim K+1)/(d-\dim K-1)}$.

□

**Corollary 5.6 (Equivariant Plancherel theorem).** For any $f_S \in H^\text{ur}(H(Q_S))$,

$$\lim_{\mu \to \infty} \widehat{m}_{\mu, \sigma, S}^{\text{count}}(\hat{f}_S) = \widehat{m}_{S}^{\text{Pl, ur}}(\hat{f}_S).$$

□

**Corollary 5.7 (Equivariant Sato-Tate equidistribution theorem).** Fix $\theta \in \mathcal{C}(\Gamma_1)$, and let $\hat{f}$ be a continuous function on $\hat{T}_{\epsilon, \theta}/\Omega_{\epsilon, \theta}$, which can be extended to a continuous function $\hat{f}_p$ on $\mathbf{G}^\text{ur, temp}$ for any $p \in \mathcal{V}(\theta)$ by (4.3). If one now chooses a sequence $\{(p_k, \mu_k)\}_{k \geq 1}$ in $\mathcal{V}(\theta) \times \mathbb{R}_{>0}$ such that $p_k \to \infty$ and $p_k^l/\mu_k \to 0$ as $k \to \infty$ for any integer $l \geq 1$, then

$$\lim_{k \to \infty} \widehat{m}_{\mu_k, \sigma, p_k'}(\hat{f}_{p_k}) = \hat{m}^{\text{ST}}(\hat{f}).$$

□

6. EXAMPLES

To conclude, we shall specify some concrete situations to which our results apply.

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\footnote{This ensures that $Z = Z_H(Q) \cap K$. Otherwise, a central character has to be fixed.}
6.1. Algebraic groups arising from division algebras. Let $D$ denote a central division algebra of index $n$ over a number field $F$. The algebraic group

$$H := \text{Res}_{F/Q} \text{SL}(1, D)$$

over $\mathbb{Q}$ is semisimple and simply connected, and a cocompact lattice $\Gamma$ of $G := H(\mathbb{R})$ is given by $\Gamma := H(\mathbb{Q}) \cap K_0$ for each open compact subgroup $K_0$ in $H(\mathbb{A}^\infty_F)$. In this case, $H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0 \cong \Gamma \backslash G$, and our results apply. The real Lie group $G$ can be expressed as $G = \prod_v H(F_v)$, where $v$ moves over infinite places of $F$ and $F_v$ denotes the completion of $F$ at an arbitrary place $v$ of $F$, and for each infinite place $v$, the real group $H(F_v) = \text{SL}(1, D \otimes F_v)$ is isomorphic to $\text{SL}(n, \mathbb{C})$, $\text{SL}(n, \mathbb{R})$, or $\text{SL}(n/2, \mathbb{H})$, where $\mathbb{H}$ denotes Hamilton's quaternion field and $n$ is even in $\text{SL}(n/2, \mathbb{H})$. One could also consider a quadratic extension $E$ of a number field $F$, together with a central division algebra $D$ over $E$ with $E/F$-involution $\iota$. Such division algebras have been classified in [40, Chapter 10]. For a fixed $\iota$ one can then take

$$H := \text{Res}_{F/Q} \text{SU}(1, D)$$

as algebraic group, which is semisimple, simply connected, and connected, and $\Gamma := H(\mathbb{Q}) \cap K_0$ as cocompact discrete subgroup of $G := H(\mathbb{R})$.

6.2. Special orthogonal and quaternion special unitary groups. Our next class of examples consists of special orthogonal and quaternion unitary groups constructed with Borel’s method [4]. Choose a positive rational number $d \in \mathbb{Q}^\times \setminus (\mathbb{Q}^\times)^2$ and consider the real quadratic field $F := \mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$. Set

$$J_{p,q} := \text{diag}(1, \ldots, 1, -\sqrt{d}, \ldots, -\sqrt{d}).$$

Then $\text{SO}(J_{p,q}) := \{g \in \text{SL}(n) \mid g J_{p,q} g^t = J_{p,q}\}$ constitutes a semisimple algebraic group over $F$, and we set $H := \text{Res}_{F/Q} \text{SO}(J_{p,q})$. In this case,

$$G := H(\mathbb{R}) \cong G_1 \times G_2, \quad G_1 := \text{SO}(p, q), \quad G_2 := \text{SO}(p + q),$$

where $\text{SO}(p, q)$ denotes the special orthogonal group of signature $(p, q)$ over $\mathbb{R}$, and it is well known that $\Gamma := H(\mathbb{Q}) \cap K_0$ is cocompact. Let $\Gamma_1$ denote an arithmetic lattice in $G$ defined as in (4.5). If the $K$-type $\sigma \in \hat{K}$ is trivial on $G_2$, then $L_2^2(K_0) \backslash H(\mathbb{A}) / K$ can be identified with a sum $\bigoplus_{l=1}^m L_2^2(\Gamma_{l,1}) \backslash G_1$, where $\Gamma_{l,1}$ denotes the projection of $\Gamma_l$ into $G_1$. In this case, our results imply asymptotics for the single orthogonal group $G_1$. Next, let $\sigma$ denote the non-trivial element of the Galois group $\text{Gal}(F/\mathbb{Q})$, and recall that there exist quaternion division algebras $D_1$ and $D_2$ over $F$ such that $D_1 \otimes_F \mathbb{R} \cong \mathcal{M}(2, \mathbb{R})$, $D_1 \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{H}$ and $D_2 \otimes_{\mathbb{R}} \mathbb{R} \cong D_2 \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{H}$. Introducing a conjugation map $x \mapsto \overline{x}$ on $D_1$, we can define over $F$ the semisimple algebraic groups $\text{SU}(n, D_1) := \{g \in \text{SU}(n, D_1) \mid g \overline{\sigma} = I_n\}$ and $\text{SU}(J_{p,q}, D_2) := \{g \in \text{SU}(n, D_2) \mid g J_{p,q} \overline{\sigma} = J_{p,q}\}$, and we set

$$H := \text{Res}_{F/Q} \text{SU}(n, D_1) \text{ or } \text{Res}_{F/Q} \text{SU}(J_{p,q}, D_2).$$

In these cases, $\Gamma = K_0 \cap H(\mathbb{Q})$ is cocompact. In the first case, the real group $G := H(\mathbb{R})$ is isomorphic to $\text{Sp}(2n) \times \text{SU}(n, \mathbb{H})$ where $\text{Sp}(2n)$ denotes the split symplectic group of rank $n$ over $\mathbb{R}$ and $\text{SU}(n, \mathbb{H}) := \{g \in \text{SU}(n, \mathbb{H}) \mid g \overline{\sigma} = I_n\}$, while in the second case the real group is isomorphic to $\text{SU}(p, q, \mathbb{H}) \times \text{SU}(n, \mathbb{H})$, where $\text{SU}(p, q, \mathbb{H})$ denotes the quaternion special unitary group of signature $(p, q)$.

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