On the Cauchy problem of a periodic 2-component \( \mu \)-Hunter-Saxton system

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Abstract
In this paper, we study the Cauchy problem of a periodic 2-component \( \mu \)-Hunter-Saxton system. We first establish the local well-posedness for the periodic 2-component \( \mu \)-Hunter-Saxton system by Kato’s semigroup theory. Then, we derive the precise blow-up scenario for strong solutions to the system. Moreover, we present some blow-up results for strong solutions to the system. Finally, we give a global existence result to the system.

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1 Introduction
Recently, a new 2-component system was introduced by Zuo in [23] as follows:

\[
\begin{aligned}
\mu(u)_t - u_{txx} &= 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \rho \rho_x - \gamma_1 u_{xxx}, \\
\rho_t &= (\rho u)_x + 2\gamma_2 \rho_x, \\
u(0, x) &= u_0(x), \\
\rho(0, x) &= \rho_0(x), \\
u(t, x + 1) &= u(t, x), \\
\rho(t, x + 1) &= \rho(t, x),
\end{aligned}
\]

(1.1)

where \( \mu(u) = \int_S u dx \) with \( S = \mathbb{R}/\mathbb{Z} \) and \( \gamma_i \in \mathbb{R} \), \( i = 1, 2 \). By integrating both sides of the first equation in the system (1.1) over the circle \( S = \mathbb{R}/\mathbb{Z} \) and using the periodicity of \( u \), one obtain

\[
\mu(u_t) = \mu(u)_t = 0.
\]

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This yields the following periodic 2-component $\mu$-Hunter-Saxton system:

\[
\begin{cases}
-u_{txx} = 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \rho \rho_x - \gamma_1 u_{xxxx}, \\
\rho_t = (\rho u)_x + 2\gamma_2 \rho_x, \\
\rho(0, x) = \rho_0(x), \\
u(0, x) = u_0(x), \\
u(t, x + 1) = u(t, x), \\
\rho(t, x + 1) = \rho(t, x),
\end{cases} \quad t > 0, \ x \in \mathbb{R}, \quad x \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t \geq 0, \ x \in \mathbb{R}, \quad t \geq 0, \ x \in \mathbb{R},
\] (1.2)

with $\gamma_i \in \mathbb{R}, \ i = 1, 2$. This system is a 2-component generalization of the generalized Hunter-Saxton equation obtained in [16]. The author [23] shows that this system is both a bihamiltonian Euler equation and a bivariational equation.

Obviously, (1.1) is equivalent to (1.2) under the condition $\mu(u_i) = \mu(u)_i = 0$. In this paper, we will study the system (1.2) under the assumption $\mu(u_i) = \mu(u)_i = 0$.

For $\rho \equiv 0$ and $\gamma = 0$, and replacing $t$ by $-t$, the system (1.2) reduces to the generalized Hunter-Saxton equation (named $\mu$-Hunter-Saxton equation or $\mu$-Camassa-Holm equation) as follows:

\[-u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \] (1.3)

which is obtained and studied in [16]. Moreover, the periodic $\mu$-Hunter-Saxton equation and the periodic $\mu$-Degasperis-Procesi equation have also been studied in [10, 17] recently. It is worthy to note that the $\mu$-Hunter-Saxton equation has a very closed relation with the periodic Hunter-Saxton and Camassa-Holm equations. For $\mu(u) = 0$, the equation (1.3) reduces to the Hunter-Saxton equation [11]

\[u_{txx} + 2u_xu_{xx} + uu_{xxx} = 0, \] (1.4)

modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. Here, $u(t, x)$ describes the director field of a nematic liquid crystal, $x$ is the space variable in a reference frame moving with the linearized wave velocity, $t$ is a slow time variable. The orientation of the molecules is described by the field of unit vectors $(\cos u(t, x), \sin u(t, x))$. [22]. The single-component model also arises in a different physical context as the high-frequency limit [7, 12] of the Camassa-Holm equation, which is a model for shallow water waves [2, 13] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [5] with a bi-Hamiltonian structure [9] which is completely integrable [6]. The Hunter-Saxton equation also has a bi-Hamiltonian structure [13, 19] and is completely integrable [11, 12]. The initial value problem for the Hunter-Saxton equation (1.4) on the line (nonperiodic case) and on the unit circle $S = \mathbb{R}/\mathbb{Z}$ were studied by Hunter and Saxton in [11] using the method of characteristics and by Yin in [22] using Kato semigroup method, respectively.

For $\rho \not\equiv 0, \ \gamma_i = 0, \ i = 1, 2$, $\mu(u) = 0$ and replacing $t$ by $-t$, peakon solutions of the Cauchy problem of the system (1.2) have been analysed in [4]. Moreover, the Cauchy problem of 2-component periodic Hunter-Saxton system has been discussed in [18, 20]. However, the Cauchy problem of the system (1.2) has not been studied yet. The aim of this paper is to establish the local well-posedness for the system (1.2), to derive the precise blow-up scenario, to prove that the system (1.2) has global strong solutions and also finite time blow-up solutions.

The paper is organized as follows. In Section 2, we establish the local well-posedness of the initial value problem associated with the system (1.2). In Section 3, we derive the precise
Moreover, the solution depends continuously on the initial data, i.e., the mapping $z_0 \rightarrow z(\cdot, z_0) : H^s \times H^{s-1} \rightarrow C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$ is continuous.
Recall that the periodic 2-component Hunter-Saxton system discussed in [18] only has local existence but not local well-posedness because of the lack of uniqueness. The ambiguity disappears in the case of the periodic 2-component $\mu$-Hunter-Saxton system from the Theorem 2.1. This is a very important difference between the 2-component Hunter-Saxton system and the 2-component $\mu$-Hunter-Saxton system.

Consequently, we will give another equivalent form of (1.2). Integrating both sides of the first equation in (1.2) with respect to $x$, we obtain

$$u_{tx} = \frac{1}{2} u_t^2 + uu_{xx} u - \frac{1}{2} \rho^2 + \gamma_1 u_{xx} + a(t),$$

where

$$a(t) = \mu(u)^2 + \frac{1}{2} \int \rho dx.$$

Using the system (1.2), we have

$$\frac{d}{dt} \int S (u_x^2 + \rho^2) dx = \int (u_t u_{xt} + \rho \rho_t) dx$$

$$= \int (2u_{xt} u_t - \int u_{xx} dx - \int u_{xxxx} dx + \int \rho \rho_t dx$$

$$= \int (2\mu u u_{xt} dx - 2 \int u_{xx} dx - \int u_{xxxx} dx + \int \rho \rho_t dx$$

$$= \int u_{x} dx + \int \rho (u \rho) _{x} dx + 2 \gamma_2 \int \rho \rho_t dx$$

$$= \int u_{x} dx + \int \rho (u \rho) _{x} dx = 0.$$

By $\mu(t) = \mu(t) = 0$, we have

$$\frac{d}{dt} a(t) = 0.$$

For convenience, we let

$$\mu_0 := \mu(u_0) = \mu(u) = \int S u(t, x) dx,$$

$$\mu_1 := \left( \int S (u_x^2 + \rho^2) dx \right)^{\frac{1}{2}} = \left( \int S (u_{0,x}^2 + \rho_0^2) dx \right)^{\frac{1}{2}}$$

and write $a := a(0)$ henceforth. Thus,

$$u_{tx} = \frac{1}{2} u_t^2 + uu_{xx} u - \frac{1}{2} \rho^2 + \gamma_1 u_{xx} + a$$

is a valid reformulation of the first equation in (1.2). Integrating (2.3) with respect to $x$, we get

$$u_t - (u + \gamma_1) u_x = \partial_x^{-1}(-2\mu u - \frac{1}{2} u_x^2 - \frac{1}{2} \rho^2 + a) + h(t),$$

where $\partial_x^{-1} g(x) = \int_0^x g(y) dy$ and $h(t) : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function. For the 2-component Hunter-Saxton system, if we follow the above same procedure, then the arbitrariness
of continuous function $h(t)$ will lead to the non-uniqueness of solution to (1.2). In this paper, the condition $\mu(u_t) = 0$ implies that $h(t)$ is unique. Consequently, the solution to (1.2) will be unique.

Thus we get another equivalent form of (1.2)

\[
\begin{cases}
  u_t - (u + \gamma_1)u_x = \partial_x^{-1}(-2\mu_0 u - \frac{1}{\tau}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t), & t > 0, x \in \mathbb{R}, \\
  \rho_t - (u + 2\gamma_2)\rho_x = u_x\rho, & t > 0, x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
  u(t, x + 1) = u(t, x), & t \geq 0, x \in \mathbb{R}, \\
  \rho(t, x + 1) = \rho(t, x), & t \geq 0, x \in \mathbb{R},
\end{cases}
\]

where $\partial_x^{-1}g(x) = \int_0^x g(y)dy$ and $h(t) : [0, \infty) \to \mathbb{R}$ is a continuous function.

### 3 The precise blow-up scenario

In this section, we present the precise blow-up scenario for strong solutions to the system (1.2).

We first recall the following lemmas.

**Lemma 3.1** [15] If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover

\[\|fg\|_{H^r} \leq c(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}),\]

where $c$ is a constant depending only on $r$.

**Lemma 3.2** [15] If $r > 0$, then

\[\|\Lambda^r f \|_{L^2} \leq c(\|\partial_x f\|_{L^\infty}\|\Lambda^{r-1}g\|_{L^2} + \|\Lambda^r f\|_{L^2}\|g\|_{L^\infty}),\]

where $c$ is a constant depending only on $r$.

Next we prove the following useful result on global existence of solutions to (1.2).

**Theorem 3.1** Let $z_0 = \left(\begin{array}{c}
u_0 \\
\rho_0 \end{array}\right) \in H^s \times H^{s-1}$, $s \geq 2$, be given and assume that $T$ is the maximal existence time of the corresponding solution $z = \left(\begin{array}{c}u \\
\rho \end{array}\right)$ to (2.4) with the initial data $z_0$. If there exists $M > 0$ such that

\[\|u_x(t, \cdot)\|_{L^\infty} + \|\rho(t, \cdot)\|_{L^\infty} + \|\rho_x(t, \cdot)\|_{L^\infty} \leq M, \quad t \in [0, T),\]

then the $H^s \times H^{s-1}$-norm of $z(t, \cdot)$ does not blow up on $[0, T)$.

**Proof** Let $z = \left(\begin{array}{c}u \\
\rho \end{array}\right)$ be the solution to (2.4) with the initial data $z_0 \in H^s \times H^{s-1}$, $s \geq 2$, and let $T$ be the maximal existence time of the corresponding solution $z$, which is guaranteed
by Theorem 2.1. Throughout this proof, $c > 0$ stands for a generic constant depending only on $s$.

Applying the operator $\Lambda^s$ to the first equation in (2.4), multiplying by $\Lambda^s u$, and integrating over $\mathcal{S}$, we obtain

$$\frac{d}{dt} \|u\|_{H^s}^2 = 2(u u_x, u) + 2(u, \partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u^2 - \frac{1}{2}\rho^2 + a) + h(t)), \quad (3.1)$$

Let us estimate the first term of the right-hand side of (3.1).

$$|(u u_x, u)| = |(\Lambda^s (u \partial_x u), \Lambda^s u)|$$

$$= |(\Lambda^s u \partial_x u, \Lambda^s u) + (u \Lambda^s \partial_x u, \Lambda^s u)|$$

$$\leq \|[(\Lambda^s u \partial_x u)_{L}, \Lambda^s u]_{L^2} + \frac{1}{2}(u \Lambda^s \partial_x u, \Lambda^s u)|$$

$$\leq (c \|u_x\|_{L^\infty} + \frac{1}{2} \|u_x\|_{L^\infty}) \|u\|_{H^s}^2$$

$$\leq c \|u_x\|_{L^\infty} \|u\|_{H^s}^2,$$

where we used Lemma 3.2 with $r = s$. Let $f \in H^{s-1}, s \geq 2$. We have

$$|\partial_x^{-1} f| = |f |_{L^2} \leq \int_{\mathcal{S}} |f| dx \leq \|f\|_{L^2}$$

and

$$\|\partial_x^{-1} f\|_{L^2} = \left(\int_0^1 (\partial_x^{-1} f)^2 dx\right)^{1/2} \leq \left(\int_0^1 \|f\|_{L^2}^2 dx\right)^{1/2} = \|f\|_{L^2}.$$

Thus

$$\|\partial_x^{-1} f\|_{H^{s}} \leq \|\partial_x^{-1} f\|_{L^2} + \|f\|_{H^{s-1}} \leq 2\|f\|_{H^{s-1}}.$$}

Then, we estimate the second term of the right-hand side of (3.1) in the following way:

$$|(\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u^2 - \frac{1}{2}\rho^2 + a) + h(t), u)|$$

$$\leq \|\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u^2 - \frac{1}{2}\rho^2 + a) + h(t)||u||_{H^s}$$

$$\leq (\|\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u^2 - \frac{1}{2}\rho^2 + a)||u||_{H^s} + \|h(t)||H^s\|)||u||_{H^s}$$

$$\leq (2\| - 2\mu_0 u - \frac{1}{2}u^2 - \frac{1}{2}\rho^2 + a)||u||_{H^{s-1}} + \|h(t)||H^s\|)||u||_{H^s}$$

$$\leq (4\mu_0||u||_{H^s} + ||u_x||_{H^{s-1}} + \|\rho||_{H^{s-1}} + 2\|a||_{H^{s-1}} + \|h(t)||H^s\|)||u||_{H^s}$$

$$\leq c(||u||_{H^s} + ||u_x||_{L^\infty} + ||u||_{H^{s-1}} + ||\rho||_{H^{s-1}} + ||a||_{H^{s-1}} + \max_{t\in[0,T]} |h(t)||u||_{H^s}$$

$$\leq c(||u_x||_{L^\infty} + ||\rho||_{H^{s-1}} + 1)(||u||_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1),$$

where we used Lemma 3.1 with $r = s - 1$. Combining (3.2) and (3.3) with (3.1), we get

$$\frac{d}{dt} \|u\|_{H^s}^2 \leq c(||\rho||_{L^\infty} + ||u_x||_{L^\infty} + 1)(||u||_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1). \quad (3.4)$$

In order to derive a similar estimate for the second component $\rho$, we apply the operator $\Lambda^{s-1}$ to the second equation in (2.4), multiply by $\Lambda^{s-1} \rho$, and integrate over $\mathcal{S}$, to obtain

$$\frac{d}{dt} \|\rho\|_{H^{s-1}}^2 = 2(u \rho_x, \rho)_{s-1} + 2(u_x \rho, \rho)_{s-1}. \quad (3.5)$$
Let us estimate the first term of the right hand side of (3.5)

\[ |(u \rho_x, \rho)_{s-1}| \]
\[ = |(\Lambda^{s-1}(u \partial_x \rho), \Lambda^{s-1} \rho)| \]
\[ = |(u \Lambda^{s-1} \partial_x \rho, \Lambda^{s-1} \rho)| \]
\[ \leq \frac{1}{2} |(u \Lambda^{s-1} \rho, \Lambda^{s-1} \rho)| \]
\[ \leq c(u_{xL} \| \rho \|_{H^{s-1}} + \| \rho_x \|_{L^\infty} \| u \|_{H^{s-1}}) \| \rho \|_{H^{s-1}} + \frac{1}{2} \| u_x \|_{L^\infty} \| \rho \|_{H^{s-1}}^2 \]

where we applied Lemma 3.2 with \( r = s - 1 \). Then we estimate the second term of the right hand side of (3.5). Based on Lemma 3.1 with \( r = s - 1 \), we get

\[ \|(u_x \rho, \rho)_{s-1}\| \leq \|u_x \rho\|_{H^{s-1}} \| \rho \|_{H^{s-1}} \]
\[ \leq c(u_{xL} \| \rho \|_{H^{s-1}} + \| \rho_x \|_{L^\infty} \| u \|_{H^{s-1}}) \| \rho \|_{H^{s-1}} \]
\[ \leq c(u_{xL} \| \rho \|_{L^\infty} + \| \rho \|_{L^\infty})(\| \rho \|_{H^{s-1}}^2 + \| u \|_{H^s}^2). \]

Combining the above two inequalities with (3.5), we get

\[ \frac{d}{dt} \| \rho \|_{H^{s-1}}^2 \leq c(u_{xL} \| \rho \|_{L^\infty} + \| \rho \|_{L^\infty} + \| \rho_x \|_{L^\infty})(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-1}}^2 + 1). \] (3.6)

By (3.4) and (3.6), we have

\[ \frac{d}{dt}(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-1}}^2 + 1) \]
\[ \leq c(u_{xL} \| \rho \|_{L^\infty} + \| \rho \|_{L^\infty} + \| \rho_x \|_{L^\infty} + 1)(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-1}}^2 + 1). \]

An application of Gronwall’s inequality and the assumption of the theorem yield

\[ (\| u \|_{H^s}^2 + \| \rho \|_{H^{s-1}}^2 + 1) \leq \exp(c(M + 1)t)(\| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-1}}^2 + 1). \]

This completes the proof of the theorem.

Given \( z_0 \in H^s \times H^{s-1} \) with \( s \geq 2 \), Theorem 2.1 ensures the existence of a maximal \( T > 0 \) and a solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (2.4) such that

\[ z = z(\cdot, z_0) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2}). \]

Consider now the following initial value problem

\[ \begin{cases} q_t = u(t, -q) + 22, & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \] (3.7)

where \( u \) denotes the first component of the solution \( z \) to (2.4). Then we have the following two useful lemmas.

Similar to the proof of Lemma 4.1 in [21], applying classical results in the theory of ordinary differential equations, one can obtain the following result on \( q \) which is crucial in the proof of blow-up scenarios.
Theorem 3.2
This proves (3.8). By Lemma 3.3, in view of (3.8) and the assumption of the lemma, we obtain existence time of the corresponding solution

Lemma 3.3 Let \( u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}), s \geq 2 \). Then Eq.(3.7) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_x(t, x) = \exp \left( - \int_0^t u_x(s, -q(s, x)) \, ds \right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

Lemma 3.4 Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, s \geq 2 \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (1.2). Then we have

\[
\rho(t, -q(t, x)) q_x(t, x) = \rho_0(-x), \quad \forall (t, x) \in [0, T) \times \mathbb{S}.
\]

Moreover, if there exists \( M > 0 \) such that \( u_x \leq M \) for all \( (t, x) \in [0, T) \times \mathbb{S} \), then

\[
\|\rho(t, \cdot)\|_{L^\infty} \leq e^{MT} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0, T).
\]

Proof Differentiating the left-hand side of the equation (3.8) with respect to \( t \), and applying the relations (2.4) and (3.7), we obtain

\[
\frac{d}{dt} \rho(t, -q(t, x)) q_x(t, x)
\]

\[
= (\rho_t(t, -q) - \rho_x(t, -q) q_t(t, x)) q_x(t, x) + \rho(t, -q(t, x)) q_{xx}(t, x)
\]

\[
= (\rho_t - (u(t, -q) + 2\gamma_2) \rho_x) q_x(t, x) - u_x \rho q_x(t, x)
\]

\[
= (\rho_t - (u + 2\gamma_2) \rho_x - u_x \rho) q_x(t, x) = 0
\]

This proves (3.8). By Lemma 3.3, in view of (3.8) and the assumption of the lemma, we obtain

\[
\|\rho(t, \cdot)\|_{L^\infty(\mathbb{S})} = \|\rho(t, \cdot)\|_{L^\infty(\mathbb{R})}
\]

\[
= \|\rho(t, -q(t, \cdot))\|_{L^\infty(\mathbb{R})}
\]

\[
= \|\exp \left( \int_0^t u_x(s, -q(s, x)) \, ds \right) \rho_0(-x)\|_{L^\infty(\mathbb{R})}
\]

\[
\leq e^{MT} \|\rho_0(\cdot)\|_{L^\infty(\mathbb{R})} = e^{MT} \|\rho_0(\cdot)\|_{L^\infty(\mathbb{S})}, \quad \forall t \in [0, T).
\]

Our next result describes the precise blow-up scenario for sufficiently regular solutions to (1.2).

Theorem 3.2 Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, s > \frac{5}{2} \) be given and let \( T \) be the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (2.4) with the initial data \( z_0 \). Then the corresponding solution blows up in finite time if and only if

\[
\limsup_{t \to T} \sup_{x \in \mathbb{S}} \{u_x(t, x)\} = +\infty \quad \text{or} \quad \limsup_{t \to T} \{\|\rho_x(t, \cdot)\|_{L^\infty}\} = +\infty.
\]
Proof By Theorem 2.1 and Sobolev’s imbedding theorem it is clear that if
\[
\limsup_{t \to T} \sup_{x \in S} \{u_x(t, x)\} = +\infty \quad \text{or} \quad \limsup_{t \to T} \{\|\rho_x(t, \cdot)\|_{L^\infty}\} = +\infty,
\]
then \( T < \infty \).

Let \( T < \infty \). Assume that there exists \( M_1 > 0 \) and \( M_2 > 0 \) such that
\[
u_x(t, x) \leq M_1, \quad \forall (t, x) \in [0, T) \times S,
\]
and
\[
\|\rho_x(t, \cdot)\|_{L^\infty} \leq M_2, \quad \forall t \in [0, T).
\]
By Lemma 3.4, we have
\[
\|\rho(t, \cdot)\|_{L^\infty} \leq e^{M_1 T} \|\rho_0\|_{L^\infty}, \quad \forall t \in [0, T).
\]

By (2.2) and the first equation in (2.4), a direct computation implies
\[
\frac{d}{dt} \int_S u^2(t, x)dx = 2 \int_S u \left( (u + \gamma_1)u_x + \partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t) \right) dx \\
\leq \int_S u^2 dx + \int_S \left( \int_0^x (-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) dy \right)^2 dx + 2|h(t)| \int_S |u(t, x)| dx \\
\leq \int_S u^2 dx + 8\mu_0^2 \left( \int_S |u| dx \right)^2 + 2 \left( \int_S (\frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + a) dx \right)^2 \\
+ \max_{t \in [0, T]} |h(t)| + \max_{t \in [0, T]} |h(t)| \int_S u^2(t, x) dx \\
= (1 + 8\mu_0^2 + \max_{t \in [0, T]} |h(t)|) \int_S u^2 dx + \frac{1}{2} \left[ \int_0^1 (u_{0,x}^2 + \rho_0^2 + 2a) dx \right]^2 + \max_{t \in [0, T]} |h(t)|
\]
for \( t \in (0, T) \).

Multiplying the first equation in (1.2) by \( m = u_{xx} \) and integrating by parts, we find
\[
\frac{d}{dt} \int_S m^2 dx = -4\mu \int_S m u_x dx + 4 \int_S u_x m^2 dx + 2 \int_S umm_x dx = 3 \mu m^2 dx - 2 \int_S m \rho \rho_x dx \\
\leq 3M_1 \int_S m^2 dx + \|\rho\|_{L^\infty} \int_S m^2 + \rho_x^2 dx \\
\leq (3M_1 + \|\rho\|_{L^\infty}) \int_S m^2 dx + \|\rho\|_{L^\infty} \int_S \rho_x^2 dx.
\]
Differentiating the first equation in (1.2) with respect to $x$, multiplying the obtained equation by $m_x = u_{xxx}$, integrating by parts and using Lemma 3.4, we obtain
\[
\frac{d}{dt} \int_S m_x^2 dx = 3 \int_S u_x m_x dx + 2 \int_S m_x^2 dx + \int_S m_x u_{xxx} dx + \int_S m_x^2 m_x dx\]
\[
= -4 \mu \int_S m m_x + 4 \int_S m^2 m_x dx + 6 \int_S u_x m_x^2 + 2 \int_S u m_x m_x dx + 2 \int_S \rho_x m_x m_x dx + 2 \gamma_1 \int_S m_x m_x dx\]
\[
= 5 \int_S u_x m_x^2 dx - 2 \int_S \rho_x m_x dx - 2 \int_S \rho_x m_x m_x dx\]
\[
\leq 5M_1 \int_S m_x^2 dx + 2 \int_S m_x |dx| + \int_S |\rho_x| L^2 \int_S (\rho_x^2 + m_x^2) dx\]
\[
\leq 5M_1 \int_S m_x^2 dx + \int_S (\rho_x^2 + m_x^2) dx + 2 \int_S |\rho_x| L^2 \int_S m_x^2 dx\]
\[
\leq 5M_1 + \int_S |\rho_x| L^2 + 2M_2^2 \int_S m_x^2 dx + \int_S m_x^2 dx\].

Differentiating the second equation in (1.2) with respect to $x$, multiplying the obtained equation by $\rho_x$ and integrating by parts, we obtain
\[
\frac{d}{dt} \int_S \rho_x^2 dx = 3 \int_S u_x \rho_x^2 dx + 2 \int_S m \rho_x dx\]
\[
= 3M_1 \int_S \rho_x^2 dx + \int_S (m^2 + \rho_x^2) dx\]
\[
\leq 3M_1 + \int_S \rho_x^2 dx + \int_S m^2 dx\].

Differentiating the second equation in (1.2) with respect to $x$ twice, multiplying the obtained equation by $\rho_{xx}$, integrating by parts and using Lemma 3.4, we obtain
\[
\frac{d}{dt} \int_S \rho_{xx}^2 dx = 5 \int_S u_x \rho_{xx}^2 dx + \int_S u_{xxx} (2 \rho_{xx} - 3 \rho_x^2) dx\]
\[
\leq 5M_1 \int_S \rho_{xx}^2 dx + \int_S m_x (2 \rho_{xx} - 3 \rho_x^2) dx\]
\[
\leq 5M_1 \int_S \rho_{xx}^2 dx + 3 \int_S \rho_x^2 dx + \int_S m_x |dx| + \int_S m_x^2 m_x dx\]
\[
\leq 5M_1 + \int_S \rho_x^2 + (3M_2^2 + \int_S m_x^2 dx + 3M_2^2\].

Summing (2.2) and (3.9)-(3.13), we have
\[
\frac{d}{dt} \int_S (u^2 + u_x^2 + m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx\]
\[
\leq K_1 \int_S (u^2 + u_x^2 + m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx + K_2,
\]
where
\[ K_1 = 1 + 8\mu_0^2 + \max_{t \in [0,T]} |h(t)| + 8e^{M_1T}\|\rho_0\|_{L^\infty} + 16M_1 + 5M_2^2, \]
\[ K_2 = \frac{1}{2} \left[ \int_S (u_0^2 x + \rho_0^2 + 2a) dx \right]^2 + \max_{t \in [0,T]} |h(t)| + 5M_2^2. \]

By means of Gronwall’s inequality and the above inequality, we deduce that
\[ \|u(t, \cdot)\|_{H^3}^2 + \|\rho(t, \cdot)\|_{H^2}^2 \leq e^{K_1 t} \left( \|u_0\|_{H^3}^2 + \|\rho_0\|_{H^2}^2 + \frac{K_2}{K_1} \right), \quad \forall \ t \in [0, T]. \]

The above inequality, Sobolev’s imbedding theorem and Theorem 3.1 ensure that the solution \( z \) does not blow-up in finite time. This completes the proof of the theorem.

For initial data \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), we have the following precise blow-up scenario.

**Theorem 3.3** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), and let \( T \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (2.4) with the initial data \( z_0 \). Then the corresponding solution blows up in finite time if and only if
\[ \limsup_{t \to T} \sup_{x \in S} \{|u_x(t, x)|\} = +\infty. \]

**Proof** Let \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) be the solution to (2.4) with the initial data \( z_0 \in H^2 \times H^1 \), and let \( T \) be the maximal existence time of the solution \( z \), which is guaranteed by Theorem 2.1.

Let \( T < \infty \). Assume that there exists \( M_1 > 0 \) such that
\[ u_x(t, x) \leq M_1, \quad \forall \ (t, x) \in [0, T] \times S. \]

By Lemma 3.4, we have
\[ \|\rho(t, \cdot)\|_{L^\infty} \leq e^{M_1 T} \|\rho_0\|_{L^\infty}, \quad \forall \ t \in [0, T]. \]

Combining (2.2), (3.9)-(3.10) and (3.12), we obtain
\[ \frac{d}{dt} \int_S (u^2 + u_x^2 + m^2 + \rho^2 + \rho_x^2) dx \leq K_3 \int_S (u^2 + u_x^2 + m^2 + \rho^2 + \rho_x^2) dx + K_4, \]
where
\[ K_3 = 1 + 8\rho_0^2 + \max_{t \in [0,T]} |h(t)| + 6M_1 + 4e^{M_1 T} \|\rho_0\|_{L^\infty}, \]
\[ K_4 = \frac{1}{2} \left[ \int_0^1 (u_{0,x}^2 + \rho_0^2 + 2a) dx \right]^2 + \max_{t \in [a,T]} |h(t)|. \]
By means of Gronwall’s inequality and the above inequality, we get
\[ \|u(t, \cdot)\|^2_{H^2} + \|\rho(t, \cdot)\|^2_{H^1} \leq e^{K_3 t}(\|u_0\|^2_{H^2} + \|\rho_0\|^2_{H^1} + \frac{K_4}{K_3}). \]

The above inequality ensures that the solution \(z\) does not blow-up in finite time.

On the other hand, by Sobolev’s imbedding theorem, we see that
\[ \limsup_{t \to T} \sup_{x \in S} \{|u_x(t, x)|\} = +\infty, \]
then the solution will blow up in finite time. This completes the proof of the theorem.

**Remark 3.1** Note that Theorem 3.2 shows that
\[ T(\|z_0\|_{H^s \times H^{s-1}}) = T(\|z_0\|_{H^{s'} \times H^{s'-1}}), \quad \forall s, s' > \frac{5}{2}, \]
while Theorem 3.3 implies that
\[ T(\|z_0\|_{H^s \times H^{s-1}}) \leq T(\|z_0\|_{H^2 \times H^1}), \quad \forall s, s' \geq 2. \]

### 4 Blow-up

In this section, we discuss the blow-up phenomena of the system (1.2) and prove that there exist strong solutions to (1.2) which do not exist globally in time.

**Lemma 4.1** ([13 [14] If \( f \in H^1(S) \) is such that \( \int_S f(x) dx = 0 \), then we have
\[ \max_{x \in S} f^2(x) \leq \frac{1}{12} \int_S f_x^2(x) dx. \]

Note that \( \int_S (u(t, x) - \mu_0) dx = \mu_0 - \mu_0 = 0 \). By Lemma 4.1, we find that
\[ \max_{x \in S} [u(t, x) - \mu_0]^2 \leq \frac{1}{12} \int_S u_x^2(t, x) dx \leq \frac{1}{12} \mu_1^2. \]

So we have
\[ \|u(t, \cdot)\|_{L^\infty(S)} \leq |\mu_0| + \frac{\sqrt{3}}{6} \mu_1. \quad (4.1) \]

**Theorem 4.1** Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \neq 0 \in H^s \times H^{s-1}, s \geq 2 \), and \( T \) be the maximal time of the solution \(z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (1.2) with the initial data \(z_0\). If \( \gamma_1 = 2\gamma_2 \), \( \mu_0 = 0 \) and there exists a point \( x_0 \in S \), such that \( \rho_0(-x_0) = 0 \), then the corresponding solution to (1.2) blows up in finite time.
Proof Let \( m(t) = u_x(t, -q(t, x_0)) \), \( \gamma(t) = \rho(t, -q(t, x_0)) \), where \( q(t, x) \) is the solution of Eq.(3.7). By Eq.(3.7) we can obtain

\[
\frac{dm}{dt} = (u_t - (u + \gamma_1)u_{xx})(t, -q(t, x_0)).
\]

Evaluating the integrated representation (2.3) at \((t, -q(t, x_0))\) with the assumption \( \mu_0 = 0 \), we get

\[
\frac{d}{dt}m(t) = \frac{1}{2}m(t)^2 - \frac{1}{2}\gamma(t)^2 + a.
\]

Since \( \gamma(0) = 0 \), we infer from Lemmas 3.3-3.4 that \( \gamma(t) = 0 \) for all \( t \in [0, T) \). Note that \( a = 2\mu(u)^2 + \frac{1}{2}\int_S(u_x^2 + \rho^2)dx > 0 \). (Indeed, if \( a(t) = 0 \), then \( (u, \rho) = (0, 0) \). This contradicts the assumption of the theorem.) Then we have \( \frac{d}{dt}m(t) \geq a > 0 \). Thus, it follows that \( m(t_0) > 0 \) for some \( t_0 \in (0, T) \). Solving the following inequality yields

\[
\frac{d}{dt}m(t) \geq \frac{1}{2}m(t)^2.
\]

Therefore

\[
0 < \frac{1}{m(t)} \leq \frac{1}{m(t_0)} - \frac{1}{2}(t - t_0), \quad t \in [t_0, T).
\]

The above inequality implies that \( T < t_0 + \frac{2}{m(t_0)} \) and \( \lim_{t \to T} m(t) = +\infty \). In view of Theorem 3.2, this completes the proof of the theorem.

Theorem 4.2 Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2 \), and \( T \) be the maximal time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (1.2) with the initial data \( z_0 \). If \( \gamma_1 = 2\gamma_2, \mu_0 \neq 0, |\mu_0| + \frac{\sqrt{a}}{6}\mu_1 < \frac{a}{2|\mu_0|} \) and there exists a point \( x_0 \in S \), such that \( \rho_0(-x_0) = 0 \), then the corresponding solution to (1.2) blows up in finite time.

Proof Let \( m(t) = u_x(t, -q(t, x_0)) \), \( \gamma(t) = \rho(t, -q(t, x_0)) \), where \( q(t, x) \) is the solution of Eq.(3.7). By Eq.(3.7) we can obtain

\[
\frac{dm}{dt} = (u_t - (u + \gamma_1)u_{xx})(t, -q(t, x_0)).
\]

Evaluating the integrated representation (2.3) at \((t, -q(t, x_0))\) we have

\[
\frac{d}{dt}m(t) = \frac{1}{2}m(t)^2 - \frac{1}{2}\gamma(t)^2 + a - 2\mu_0 u.
\]

Since \( \gamma(0) = 0 \), we infer from Lemmas 3.3-3.4 that \( \gamma(t) = 0 \) for all \( t \in [0, T) \). In view of (4.1) and the condition \( |\mu_0| + \frac{\sqrt{a}}{6}\mu_1 < \frac{a}{2|\mu_0|} \), we have \( a - 2\mu_0 u \geq a - 2|\mu_0|u > 0 \). Then we have \( \frac{d}{dt}m(t) \geq a - 2\mu_0 u > 0 \). The left proof is the same as that of Theorem 4.1, so we omit it here.
5 Global Existence

In this section, we will present a global existence result. Firstly, we give two useful lemmas.

**Theorem 5.1** Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2 \times H^1 \), and \( T \) be the maximal time of the solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (1.2) with the initial data \( z_0 \). If \( \gamma_1 = 2 \gamma_2 \), \( \rho_0(x) \neq 0 \) for all \( x \in S \), then the corresponding solution \( z \) exists globally in time.

**Proof** By Lemma 3.3, we know that \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_x(t, x) = \exp \left( -\int_0^t u_x(s, -q(s, x))ds \right) > 0 \), \( \forall (t, x) \in [0, T) \times \mathbb{R} \).

Moreover,
\[
\sup_{y \in S} u_y(t, y) = \sup_{x \in \mathbb{R}} u_x(t, -q(t, x)), \quad \forall t \in [0, T). \tag{5.1}
\]

Set \( M(t, x) = u_x(t, -q(t, x)) \) and \( \alpha(t, x) = \rho(t, -q(t, x)) \) for \( t \in [0, T) \) and \( x \in \mathbb{R} \). By \( \gamma_1 = 2 \gamma_2 \), (1.2) and Eq.(3.7), we have
\[
\frac{\partial M}{\partial t} = (u_t - (u + \gamma_1)u_{xx})(t, -q(t, x)) \quad \text{and} \quad \frac{\partial \alpha}{\partial t} = \alpha M. \tag{5.2}
\]

Evaluating (2.3) at \( (t, -q(t, x)) \) we get
\[
\partial_t M(t, x) = \frac{1}{2} M(t, x)^2 - \frac{1}{2} \alpha(t, x)^2 + a - 2 \mu_0 u(t, -q(t, x)). \tag{5.3}
\]

Write \( f(t, x) = a - 2 \mu_0 u(t, -q(t, x)) \). By (4.1) we have
\[
|f(t, x)| \leq a + 2|\mu_0||u||_{L^\infty} \leq a + 2|\mu_0|(|\mu_0| + \sqrt{\frac{3}{6}} \mu_1) = 4 \mu_0^2 + \frac{1}{2} \mu_1^2 + \sqrt{\frac{3}{3}} |\mu_0| \mu_1
\]
and
\[
\partial_t M(t, x) = \frac{1}{2} M(t, x)^2 - \frac{1}{2} \alpha(t, x)^2 + f(t, x). \tag{5.4}
\]

By Lemmas 3.3-3.4, we know that \( \alpha(t, x) \) has the same sign with \( \alpha(0, x) = \rho_0(-x) \) for every \( x \in \mathbb{R} \). Moreover, there is a constant \( \beta > 0 \) such that \( \inf_{x \in \mathbb{R}} |\alpha(0, x)| = \inf_{x \in S} |\rho_0(-x)| \geq \beta > 0 \) since \( \rho_0(x) \neq 0 \) for all \( x \in S \) and \( S \) is a compact set. Thus,
\[
\alpha(t, x)\alpha(0, x) > 0, \quad \forall x \in \mathbb{R}.
\]

Next, we consider the following Lyapunov function first introduced in [4].
\[
w(t, x) = \alpha(t, x)\alpha(0, x) + \frac{\alpha(0, x)}{\alpha(t, x)}(1 + M^2), \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{5.4}
\]
By Sobolev’s imbedding theorem, we have
\[
0 < w(0, x) = \alpha(0, x)^2 + 1 + M(0, x)^2
\]
\[
= \rho_0(x)^2 + 1 + u_{0,x}(x)^2
\]
\[
\leq 1 + \max_{x \in \mathbb{S}}(\rho_0(x)^2 + u_{0,x}(x)^2) := C_1.
\]
Differentiating (5.4) with respect to \(t\) and using (5.2)-(5.3), we obtain
\[
\frac{\partial w}{\partial t}(t, x) = \frac{\alpha(0, x)}{\alpha(t, x)} M(t, x)(2f - 1)
\]
\[
\leq |f - \frac{1}{2}\frac{\alpha(0, x)}{\alpha(t, x)}(1 + M^2)
\]
\[
\leq (4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})w(t, x).
\]

By Gronwall’s inequality, the above inequality and (5.5), we have
\[
w(t, x) \leq w(0, x)e^{(4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})t} \leq C_1 e^{(4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})t}
\]
for all \((t, x) \in [0, T) \times \mathbb{R}\). On the other hand,
\[
w(t, x) \geq 2\sqrt{\alpha^2(0, x)(1 + M^2)} \geq 2\beta |M(t, x)|, \quad \forall \ (t, x) \in [0, T) \times \mathbb{R}.
\]

Thus,
\[
|M(t, x)| \leq \frac{1}{2\beta} w(t, x) \leq \frac{1}{2\beta} C_1 e^{(4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})t}
\]
for all \((t, x) \in [0, T) \times \mathbb{R}\). Then by (5.1) and the above inequality, we have
\[
\limsup_{t \to T} \sup_{y \in \mathbb{S}} u_y(t, y) = \limsup_{t \to T} \sup_{x \in \mathbb{R}} u_x(t, -q(t, x)) \leq \frac{1}{2\beta} C_1 e^{(4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})t}.
\]

This completes the proof by using Theorem 3.3.

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