BOUNDARY HÖLDER GRADIENT ESTIMATES FOR THE
MONGE-AMPÈRE EQUATION

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Abstract. We investigate global Hölder gradient estimates for solutions to the Monge-Ampère equation
\[ \det D^2 u = f \quad \text{in } \Omega, \]
where the right-hand side \( f \) is bounded away from 0 and \( \infty \). We consider two main situations when a) the domain \( \Omega \) is uniformly convex and b) \( \Omega \) is flat.

1. Introduction

In this paper, we consider boundary Hölder gradient estimate for solutions to the Dirichlet problem
\[ \begin{cases} 
\det D^2 u = f & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega,
\end{cases} \]
where \( \Omega \) is a convex domain in \( \mathbb{R}^n \) and \( 0 < \lambda \leq f \leq \Lambda \) for some constants \( \lambda, \Lambda \).

The regularity of solutions for the Monge-Ampère equation has been extensively studied by many authors, see for instance [6, 14, 21, 1, 2, 3, 10, 4, 5, 11, 13, 15, 16, 17, 18, 20] and references therein.

Concerning gradient Hölder estimates, Caffarelli proved in [1, 3] that solutions \( u \) of (1.1) which are strictly convex satisfy \( u \in C^{1,\delta} \) in the interior of \( \Omega \), for some small \( \delta > 0 \) depending on \( \lambda, \Lambda \) and the dimension \( n \). Moreover, the strict convexity of solutions can be guaranteed if the boundary data is above a critical regularity level \( \varphi \in C^{1,\beta} \) with \( \beta > 1 - \frac{2}{n} \). This exponent is optimal in view of Pogorelov’s famous example of singular solutions in [14]. However, as we will see later, even in this case the \( C^{1,\delta} \) norm of \( u \) may degenerate near the boundary of \( \Omega \).

Here we investigate the \( C^{1,\alpha} \) estimates up to the boundary of \( \Omega \), under minimal conditions on the domain and the boundary data. While there is a rich literature addressing \( C^{2,\alpha} \) boundary estimates for solutions of (1.1), to the authors knowledge there is no work concerning sharp \( C^{1,\alpha} \) boundary estimates which we discuss in this paper.

We consider two main situations when a) the domain \( \Omega \) is uniformly convex and b) \( \Omega \) is flat. In both cases we state two results similar in nature, one of them regarding the pointwise \( C^{1,\alpha} \) estimate at a point on \( \partial \Omega \) and the other one about the global version of this estimate.

For uniformly convex domains, Theorem 1.1 below states that if \( \partial \Omega \) and the boundary data \( \varphi \) are pointwise \( C^{2,\alpha} \) at a boundary point for some \( \alpha \in (0, 1) \), then the solution \( u \) is \( C^{1,\delta_0} \) at this point for some small \( \delta_0 > 0 \).

**Theorem 1.1.** Let \( u : \overline{\Omega} \rightarrow \mathbb{R} \) be a convex, continuous solution to (1.1). Assume \( \Omega \subset \mathbb{R}^n_+ \), \( 0 \in \partial \Omega \), \( \Omega \) is uniformly convex at 0, and \( \partial \Omega, \varphi \in C^{2,\alpha}(0) \); i.e., we assume that on \( \partial \Omega \) we have
\[ x_n = q(x') + O(|x'|^{2+\alpha}), \]
\[ \varphi(x) = p(x') + O(|x'|^{2+\alpha}), \]
where \( p(x'), q(x') \) are quadratic polynomials. Then
\[ u \in C^{1,\delta_0}(0) \]
for some constant \( \delta_0 > 0 \) depending only on \( n \) and \( \alpha \).

For the definition of \( C^{1,\delta}(0), \delta \in (0, 1) \), see Section 2.

The corresponding global Hölder gradient estimate when \( \partial \Omega, \varphi \in C^{2,\alpha} \) in the classical sense is given in the next theorem.
Theorem 1.2. Let \( u : \Omega \to \mathbb{R} \) be a convex, continuous solution to (1.1). Assume \( \Omega \) is uniformly convex, \( \partial \Omega, \varphi \in C^{2,\alpha} \) for some \( \alpha \in (0, 1) \). Then
\[
  u \in C^{1,\beta}(\Omega)
\]
for some constant \( \beta \in (0, 1) \) depending only on \( n, \lambda, \Lambda \) and \( \alpha \).

We will give an example to show that our results are optimal: if \( \varphi \) is only \( C^2 \), the solution may fail to be globally \( C^{1,\delta} \) for any \( \delta \in (0, 1) \).

Next we discuss case b) when the domain \( \Omega \) is flat in a neighborhood of a boundary point. We have the following pointwise \( C^{1,\alpha} \) estimate at a boundary point.

Theorem 1.3. Let \( u : \Omega \to \mathbb{R} \) be a convex, continuous solution to (1.1) with \( \Omega = B_1^+ \). Assume \( \varphi \in C^{1,\alpha}(0) \) with \( \alpha > \frac{1}{3} \) and
\[
  \varphi(0) = 0, \quad \nabla \varphi(0) = 0,
\]
and \( \varphi \) separates quadratically on \( \partial B_1^+ \) in a neighborhood of \( \{ x_n = 0 \} \) from 0. Then
\[
  u \in C^{1,\alpha'}(0)
\]
for some \( \alpha' > 0 \) depending only on \( n \) and \( \alpha \).

The Hölder gradient estimate near the boundary in the flat case is as follows.

We denote by \( B_R^\prime \) the ball in \( \mathbb{R}^{n-1} \) centered at 0 with radius \( R > 0 \).

Theorem 1.4. Let \( u : \Omega \to \mathbb{R} \) be a convex, Lipschitz continuous solution to (1.1) with \( \Omega = B_1^+ \). Assume \( \varphi|_{x_n=0} \in C^{1,\alpha}(B_{3/4}^\prime) \) with \( \alpha > \max\{ \frac{1}{3}, 1 - \frac{2}{n} \} \), and for any \( x_0' \in B_{3/4}^\prime \), \( \varphi|_{x_n=0} \) separates quadratically on \( B_1^\prime \) from its tangent plane at \( x_0' \). Then
\[
  u \in C^{1,\beta}(B_{1/2}^\prime)
\]
for some small constant \( \beta \in (0, 1) \).

In the particular case \( \alpha = 1 \), Theorems 1.3 and 1.4 can be obtained from the work of the first author in [15] and [19, Proposition 2.6]. The novelty here is that they hold when \( \alpha < 1 \).

The paper is organized as follows. In Section 2 we introduce some notation and give the quantitative versions of Theorems 1.1-1.4 (see Theorems 2.1-2.4 respectively). Section 3 is devoted to the proof of Theorem 2.1. In Section 4, we give the proof of Theorem 2.2, and then present an example which shows that the assumptions in Theorem 1.2 are sharp. In Sections 5 and 6, we give the proofs of Theorems 2.3 and 2.4 respectively.

2. Statement of main results

We introduce some notation. We denote points in \( \mathbb{R}^n \) as
\[
x = (x_1, \ldots, x_n) = (x', x_n), \quad x' \in \mathbb{R}^{n-1}.
\]
Let \( u \) be a convex function defined on a convex set \( \Omega \), we denote by
\[
  l_{x_0} := u(x_0) + \nabla u(x_0) \cdot (x - x_0)
\]
a supporting hyperplane for the graph of \( u \) at \( x_0 \) and \( S_h(x_0) \) the section centered at \( x_0 \) and at height \( h > 0 \),
\[
  S_h(x_0) := \{ x \in \Omega | u(x) < l_{x_0}(x) + h \}.
\]
When \( x_0 \in \partial \Omega \), the term \( \nabla u(x_0) \) is understood in the sense that
\[
  x_{n+1} = u(x_0) + \nabla u(x_0) \cdot (x - x_0)
\]
is a supporting hyperplane for the graph of \( u \) at \( x_0 \) but for any \( \epsilon > 0 \),
\[
  x_{n+1} = u(x_0) + (\nabla u(x_0) + \epsilon \nu_{x_0}) \cdot (x - x_0)
\]
is not a supporting hyperplane, where \( \nu_{x_0} \) denotes the unit inner normal to \( \partial \Omega \) at \( x_0 \). We denote for simplicity \( S_h = S_h(0) \), and sometimes when we specify the dependence on the function \( u \) we use the notation \( S_h(u) = S_h \).

We state a variant of John’s lemma [12] (see also [7]), which is a classical result in convex geometry.
Let $\Omega \subset \mathbb{R}^n$ be a bounded convex with nonempty interior and $E$ is the ellipsoid of minimum volume containing $\Omega$ centered at the center of mass of $\Omega$, then
\[ \alpha_n E \subset \Omega \subset E, \]
where $\alpha_n = n^{-3/2}$ and $\alpha E$ denotes the $\alpha$-dilation of $E$ with respect to its center.

The following definition is introduced in [15].

**Definition 2.1.** Let $k \geq 0$ be an integer and $0 < \alpha \leq 1$. We say that a function $u$ is pointwise $C^{k, \alpha}$ at $x_0$ and write
\[ u \in C^{k, \alpha}(x_0) \]
if there exists a polynomial $P_{x_0}$ of degree $k$ such that
\[ u(x) = P_{x_0}(x) + O(|x - x_0|^{k+\alpha}). \]
We say that $u \in C^k(x_0)$ if
\[ u(x) = P_{x_0}(x) + o(|x - x_0|^k). \]

We now state the precise quantitative versions of Theorems 1.1-1.4 as follows.

**Theorem 1.1.** Let $u : \overline{\Omega} \rightarrow \mathbb{R}$ be a convex, continuous solution to (1.1). Assume $\Omega \subset \mathbb{R}^n_+, 0 \in \partial \Omega$, $\Omega$ is uniformly convex at 0, and on $\partial \Omega$ near 0 we have
\[ |x_n - q(x')| \leq M|x'|^{2+\alpha}, \]
\[ |\varphi(x) - p(x')| \leq M|x'|^{2+\alpha}, \]
where $p(x'), q(x')$ are quadratic polynomials and $M \geq \max\{|\nabla p(0)|, |D^2_x p|, |D^2_x q|\}$.

Then
\[ u - u(0) - \nabla u(0) \cdot x \leq C|x|^{1+\delta_0}, \]
where $\delta_0 > 0$ depends only on $n, \lambda, \alpha$, the constant $C > 0$ depends only on $n, \lambda, \Lambda, \alpha, M$, the uniform convexity of $\partial \Omega$ at 0, and $\|\varphi\|_{L^\infty(\partial \Omega)}$.

This pointwise estimate combined with the interior estimates of Caffarelli from [3] implies the global $C^{1, \alpha}$ estimate for solutions to (1.1) in the case that the domain is uniformly convex.

**Theorem 2.2.** Let $u : \overline{\Omega} \rightarrow \mathbb{R}$ be a convex, continuous solution to (1.1). Assume $\Omega$ is uniformly convex, $\partial \Omega$, $\varphi \in C^{2, \alpha}$ for some $\alpha \in (0, 1)$. Then
\[ |\nabla u|_{C^{0, \beta} (\overline{\Omega})} \leq C, \]
where $\beta \in (0, 1)$ depends only on $n, \lambda, \Lambda$ and $\alpha$, the constant $C > 0$ depends only on $n, \lambda, \Lambda, \alpha, \lambda$, $\text{diam}(\Omega), \|\partial \Omega, \varphi\|_{C^{2, \alpha}}$ and the uniform convexity of $\Omega$.

In the case that the domain is flat at a boundary point, the quantitative pointwise $C^{1, \alpha}$ estimate is as follows.

**Theorem 2.3.** Let $u : \overline{\Omega} \rightarrow \mathbb{R}$ be a convex, continuous solution to (1.1) with $\Omega = B^+_1$ and
\[ u(0) = 0, \quad \nabla u(0) = 0. \]
Assume $\alpha > \frac{1}{3}$, and
\[ \varphi|_{x_n = 0} \leq \mu^{-1}|x'|^{1+\alpha} \text{ in } B^1_{1/2} \]
and
\[ \varphi \geq \mu|x|^2 \text{ on } \partial B^+_1 \cap \{x_n \leq \rho\}. \]
for some $\mu, \rho > 0$. Then for any $x \in B^+_1$ with $u(x) \leq c$, we have
\[ u(x) \leq C|x|^{1+\alpha'}, \]
where $\alpha' > 0$ depends only on $n$ and $\alpha$, the constants $c, C$ depend only on $n, \lambda, \mu, \alpha$ and $\rho$.

Using Theorem 2.3 and similar techniques as in the uniformly convex case, we can obtain the $C^{1, \alpha}$ estimate near the flat boundary.
Theorem 2.4. Let \( u : \overline{\Omega} \to \mathbb{R} \) be a convex, Lipschitz continuous solution to (1.1) with \( \Omega = B_1^+ \). Assume \( \varphi_{x_n=0} \in C^{1,\alpha}(B_{3/4}') \) with \( \alpha > \max\{\frac{1}{5}, 1 - \frac{2}{n}\} \), and for any \( x_0 = (x_0', 0) \) with \( x_0' \in B_{3/4}' \) and \( x \in \partial B_1^+ \cap \{ x_n = 0 \} \),

\[
\varphi(x) - \varphi(x_0) - \nabla_{x'} \varphi(x_0) \cdot (x' - x_0') \geq \mu |x' - x_0'|^2.
\]

Then

\[
[\nabla u]_{C^{\beta}(\overline{B_{1/2}^+})} \leq C,
\]

where \( \beta \in (0, 1) \) depends only on \( n, \lambda, \Lambda, \alpha \), and the constant \( C > 0 \) depends on \( n, \lambda, \Lambda, \alpha, \mu \), \( \|\varphi_{x_n=0}\|_{C^{1,\alpha}(B_{3/4}')} \) and \( \|u\|_{C^{0,1}} \).

In the proofs below we denote by \( c, C, c', C', c_i, C_i(i = 0, 1, 2, \ldots) \) constants depending only on the data \( n, \lambda, \Lambda, \alpha, \text{diam}(\Omega), M, \|\varphi\|_{L^\infty(\partial \Omega)} \), the uniform convexity of \( \partial \Omega \) etc. Their values may change from line to line whenever there is no possibility of confusion. For \( A, B \in \mathbb{R} \), we write \( A \sim B \) if

\[
c \leq \frac{A}{B} \leq C
\]

for some universal constants \( c, C \).

3. Proof of Theorem 2.1

Let \( \varphi(x) = \tilde{\varphi}(x') \) and \( \tilde{u} = u - l_0 \), where we recall from Section 2 that

\[
l_0(x) = u(0) + \nabla u(0) \cdot x.
\]

Then (after performing a rotation in the \( x' \) subspace) on \( \partial \Omega \) we have

\[
\tilde{u} = \tilde{\varphi} - \tilde{\varphi}(0) - \nabla_{x'} \tilde{\varphi}(0) \cdot x' - u_n(0)x_n = \sum_{i=1}^{n-1} a_i^2 x_i^2 + O(|x'|^{2+\alpha})
\]

for some constants \( a_i \geq 0, i = 1, \ldots, n - 1 \).

Let \( 0 < \alpha' < \alpha \) be a constant to be chosen below. We will prove that

\[
S_h \supset \overline{\Omega} \cap B_{\frac{1}{\text{ch}^{\alpha'+\delta_0} h}} \quad \forall \ h > 0,
\]

where \( \delta_0 > 0 \) is a constant depending only on \( n \) and \( \alpha \).

First, we use a lower barrier of the type

\[
\tilde{\varphi}(0) + \nabla_{x'} \tilde{\varphi}(0) \cdot x' + \Lambda|x|^2 - Cx_n
\]

and obtain that \( u_n(0) \) is bounded. Hence \( \tilde{u} \) is bounded above and therefore we can assume that \( h \) in (3.1) is sufficiently small.

We only need to consider the following cases: \( \min_i a_i^2 \leq h^{\alpha'/\alpha'} \) and \( \min_i a_i^2 \geq h^{\alpha'/\alpha'} \).

Case 1: \( \min_i a_i^2 \leq h^{\alpha'/\alpha'} \).

If \( a_i^2 \leq h^{\alpha'/\alpha'} \), then by the uniform convexity of \( \partial \Omega \) at 0, we have on \( \partial \Omega \cap \{ x_i = 0, i = 2, \ldots, n - 1 \} \)

\[
\tilde{u} \leq h^{\frac{\alpha'}{2+\alpha'}} x_1^2 + O(|x_1|^{2+\alpha}) \leq C \left[ h^{\frac{\alpha'}{2+\alpha'}} x_n + x_n^{1+\frac{\alpha'}{2}} \right],
\]

this together with the convexity of \( u \) implies that

\[
\tilde{u}(t e_n) \leq C \left[ h^{\frac{\alpha'}{2+\alpha'}} t + t^{1+\frac{\alpha'}{2}} \right].
\]

It follows that

\[
\{ t e_n : 0 \leq t \leq c_0 h^{\frac{\alpha'}{2+\alpha'}} \} \subset S_h
\]

for some small constant \( c_0 > 0 \).

The domain of definition of \( \partial S_h \cap \partial \Omega \) contains a ball in \( \mathbb{R}^{n-1} \) of radius \( ch^{\frac{1}{2}} \), and by the uniform convexity of \( \partial \Omega \) at 0, we have

\[
x_n \geq c_1 h \quad \forall x \in \partial S_h \cap \partial \Omega \cap \{ |x'| = ch^{\frac{1}{2}} \}.
\]
and therefore 

$$S_h \supset \mathbb{B} \cap \{x_n \leq c_1 h\}.$$ 

Then the convex set generated by \(\Omega \cap \{x_n = c_1 h\}\) and the point \((0, c_0 h^{\frac{n}{2+n}})\) is contained in \(S_h\). Since this convex set contains a half-ball centered at \((0, c_1 h)\) of radius \(ch^{\frac{n}{2+n}}\). We obtain that 

$$S_h \supset \mathbb{B} \cap B_{ch^{\frac{n}{2+n}}}. $$

**Case 2:** \(\min_i a_i^2 \geq h^{\frac{n}{2+n'}}\). Then on \(\partial \Omega\) near 0 we have 

$$\tilde{u} \geq \frac{1}{2n} h^{\frac{n}{2+n'}} |x'|^2 \quad \forall x \in \{|x'| \leq c'h^{\frac{n}{2+n'}}\},$$

for some \(c'\) small.

Let \(x^*_h\) be the center of mass of \(S_h\) and denote \(d_h := x^*_h \cdot e_n\). We claim that 

$$d_h \geq c_2 h^{\frac{n}{n+1}}$$

for some constant \(c_2\) small. Otherwise, by the uniformly convexity of \(\partial \Omega\) at 0 and Lemma 2.1, we have 

$$S_h \subset \{0 \leq x_n \leq C(n)c_2 h^{\frac{n}{n+1}} \leq h^{\frac{n}{n+1}} \} \cap \{|x'| \leq c'h^{\frac{n}{2+n'}}\}. $$

Let \(C_2\) be a large constant to be chosen and define 

$$w := \epsilon x_n + \left[\frac{1}{2} \left(\frac{|x'|}{C_1 h^{\frac{n}{n+1}}}\right)^2 + C_2 \left(\frac{x_n}{h^{\frac{n}{n+1}}}\right)^2\right] \cdot h.$$ 

Since 

$$\partial S_h \cap \partial \Omega \subset \{|x'| \leq C_1 h^{\frac{n}{2(n+1)}}\} \subset \{|x'| \leq c'h^{\frac{n}{2+n'}}\}$$

if we choose \(c'\) small, then on \(\partial S_h \cap \partial \Omega\) we have 

$$w \leq \left[ C\epsilon + \frac{1}{2C_1^2} h^{\frac{1}{n+1}} + C_2 Ch^{\frac{1}{n+1}} \right] |x'|^2 \leq \tilde{u},$$

where we choose \(\epsilon\) and \(c'\) small such that 

$$C\epsilon \leq \frac{h^{\frac{n}{2+n'}}}{6n} \quad \text{and} \quad C_2 Ch^{\frac{1}{n+1}} \leq \frac{h^{\frac{n}{2+n'}}}{6n}.$$

In \(S_h\) we have 

$$w \leq \epsilon + \left[ \frac{1}{2} + C_2 C(n)c_2 \right] h \leq h.$$ 

Moreover, 

$$\det D^2 w > \Lambda$$

by choosing \(C_2\) large.

In conclusion, \(w \leq \tilde{u}\) in \(S_h\), which together with the convexity of \(u\) implies that \(\tilde{u} \geq \epsilon x_n\) in \(\Omega\). This is a contradiction. Thus (3.3) holds.

The uniform convexity of \(\Omega\) at 0 and (3.3) imply that 

$$\{te_n : 0 \leq t \leq c_2 h^{\frac{n}{2(n+1)} + \frac{1}{2}}\} \subset S_h$$

for some small constant \(c_2 > 0\). Similar to **Case 1** we have 

$$S_h \supset \mathbb{B} \cap B_{c_2 h^{\frac{n}{2(n+1)} + \frac{1}{2}}}. $$

Combining **Cases 1-2**, we obtain (3.1).
4. Proof of Theorem 2.2

Using the uniform pointwise estimate of Theorem 2.1 we obtain by standard arguments the Hölder continuity of $\nabla u$ on $\partial \Omega$.

**Lemma 4.1.** *Under the assumptions of Theorem 2.2, we have*

$$[\nabla u]_{C^0(\partial \Omega)} \leq C,$$

*where $\delta_0$ is the constant in Theorem 2.1.*

It remains to show that $\nabla u$ is uniformly Hölder continuous also at interior points of $\Omega$. Assume $0 \in \partial \Omega$ and $\Omega \subset \mathbb{R}^n_+$. We divide the proof of Theorem 2.2 into three steps.

**Step 1.** Let $y \in \Omega$ and consider the maximal interior section $S_{\bar{h}}(y)$ centered at $y$, that is,

$$\bar{h} = \max\{h \mid S_h(y) \subset \Omega\}.$$

Assume $0 \in \partial S_{\bar{h}}(y) \cap \partial \Omega$. We prove that

$$S_{\bar{h}}(y) \subset B_{\bar{h}^\epsilon_0}$$

for any $\bar{h} > 0$ small.

For any $h > 0$ small, let $x_h^*$ be the center of mass of $S_h$ and $d_h := x_h^* \cdot e_n$. We claim that

$$d_h \leq C_0 h^{\frac{1}{2}}$$

for some large constant $C_0 > 0$.

Indeed, if (4.2) does not hold, then as in the proof of Theorem 2.1 we have

$$S_h \supset \Omega \cap \{x_n \leq c_1 h\}$$

for some $c_1$ small, and therefore $S_h$ contains the convex set generated by $\Omega \cap \{x_n = c_1 h\}$ and the point $x_h^*$. We also have

$$|x'| \geq ch^{\frac{1}{2}}, \quad \forall x \in \partial \Omega \cap \{x_n = c_1 h\}.$$

It follows that

$$|S_h| \geq c(n)(ch^{\frac{1}{2}})^{n-1} \frac{C_0 h^{\frac{1}{2}}}{2}.$$

On the other hand, by Lemma 2.1, $S_h$ is equivalent to an ellipsoid $E$ centered at $x_h^*$, i.e.,

$$E \subset S_h \subset C(n)E,$$

where the dilation is with respect to $x_h^*$. Let $P$ be the quadratic polynomial that solves

$$\det D^2 P = \lambda \quad \text{in} E, \quad P = h \quad \text{on} \partial E.$$

Then

$$P \geq \tilde{u} := u - l_0 \geq 0 \quad \text{in} E.$$

It follows

$$h^n \geq \min_{E} P^n \geq c(n, \lambda) |E|^2 \geq c |S_h|^2.$$

We reach a contradiction if we choose $C_0$ sufficiently large. Hence (4.2) holds, which gives

$$S_h \subset \{0 \leq x_n \leq C'h^{\frac{1}{2}}\} \quad \forall h > 0.$$

This together with the uniform convexity of $\Omega$ gives

$$S_h \subset B_{C'h^{\frac{1}{2}}} \quad \forall h > 0.$$

Since $\partial \Omega, \varphi \in C^2$, we find that

$$S_h(y) = \{x \in \Omega \mid v(x) < 0\},$$

where

$$v(x) = (u - l_0)(x) - [u_n(y) - u_n(0)]x_n.$$

Choose $h > 0$ such that

$$u_n(y) - u_n(0) = (2C')^{-1} h^{\frac{1}{2}},$$

and we are done.
where $C'$ is the constant in (4.3). Then we have by (4.3)

$$S_h(y) \cap \{x_n \leq 2C'h^{\frac{1}{2}}\} \subset S_h \subset \{x_n \leq C'h^{\frac{1}{2}}\},$$

which implies that

$$S_h(y) \subset \{x_n \leq 2C'h^{\frac{1}{2}}\}.$$  

Using (4.4) we obtain that

(4.5) $$S_h(y) \subset S_h \subset B_{C'h^{\frac{1}{2}}}.$$  

On the other hand, let $\theta > 0$ be a small constant to be chosen below and denote

$$\Lambda_0 := \frac{1 + \delta_0}{2\delta_0},$$

where $\delta_0$ is the constant in Theorem 2.1. By Theorem 2.1, we can choose a point $z = t\epsilon_n \in \partial S_{\theta h^{\Lambda_0}}$ with $t \geq c(\theta h^{\Lambda_0})^{\frac{1}{1+\delta_0}}$. It follows that

$$v(z) \leq \theta h^{\Lambda_0} - (2C')^{-1}h^{\frac{1}{2}}c(\theta h^{\Lambda_0}) \frac{1}{1+\delta_0}$$

$$= \theta^{\frac{1}{1+\delta_0}}h^{\Lambda_0}[\theta^{\frac{1}{1+\delta_0}} - (2C')^{-1}c]$$

$$\leq -\theta^{\frac{1}{1+\delta_0}}h^{\Lambda_0}(4C')^{-1}c < 0$$

if $\theta > 0$ is sufficiently small. This implies that

(4.6) $$\tilde{h} \geq c|S_h(y)|^{\frac{1}{2}} \geq c \min v \geq ch^{\Lambda_0}.$$  

This together with (4.5) gives (4.1) with $0 < \epsilon_0 < \frac{1}{4\delta_0}$.

**Step 2.** Let $S_{\tilde{h}}(y)$ be a maximal interior section tangent to $\partial \Omega$ at 0 as in **Step 1**. We prove that

(4.7) $$B_{\tilde{h}^{1-\epsilon_1}}(y) \subset S_{\tilde{h}}(y)$$

for any $\tilde{h} > 0$ small, where $\epsilon_1 > 0$ is a small constant.

By Lemma 2.1, $S_{\tilde{h}}(y)$ is equivalent to an ellipsoid $E$ centered at $y$, i.e.,

$$E \subset S_{\tilde{h}}(y) \subset C_1 E,$$

where $C_1$ is a constant depending only on $n, \lambda, \Lambda$, and

$$E = y + U^t \text{diag}(\mu_1, \ldots, \mu_n)B_1,$$

where $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $U$ is an orthogonal matrix. We only need to prove that

(4.8) $$\mu_1 \geq \tilde{h}^{1-\epsilon_1}.$$  

Assume by contradiction that

$$\mu_1 < \tilde{h}^{1-\epsilon_1}.$$  

Let $\nu = U^t e_1$ be a unit vector which is parallel to the shortest axis of $E$. Then for any $x \in C_1 E$, we have

$$|(x - y) \cdot \nu| = |e_1^t \text{diag}(\mu_1, \ldots, \mu_n) \text{diag}(\mu_1^{-1}, \ldots, \mu_n^{-1})U(x - y)|$$

$$\leq C_1 |e_1^t \text{diag}(\mu_1, \ldots, \mu_n)|$$

$$= C_1 \mu_1 \leq C_1 \tilde{h}^{1-\epsilon_1}.$$  

Define $w^+ = v(x) - \frac{\tilde{h}^{\epsilon_1}}{2C_1}(x - y) \cdot \nu$ and $a^+ = \min_{S_{\tilde{h}}(y)} w^+ = w^+(x_0)$. Since

$$w^+(y) = v(y) = -\tilde{h}, \quad a^+ = -\frac{\tilde{h}^{\epsilon_1}}{2C_1}(x - y) \cdot \nu \geq -\frac{\tilde{h}}{2} \text{ on } \partial S_{\tilde{h}}(y),$$

we find that $x_0 \in S_{\tilde{h}}(y)$ and

$$v \geq \frac{\tilde{h}^{\epsilon_1}}{2C_1}(x - y) \cdot \nu + w^+(x_0) \text{ in } S_{\tilde{h}}(y).$$
It follows from the convexity of $v$ that
\[ v \geq \frac{\tilde{h}^{\epsilon_1}}{2C_1} (x - y) \cdot \nu + w^+(x_0) \geq \frac{\tilde{h}^{\epsilon_1}}{2C_1} (x - y) \cdot \nu - 2\tilde{h} \quad \text{in} \ \Omega. \]
Similarly we have
\[ v \geq \frac{\tilde{h}^{\epsilon_1}}{2C_1} (x - y) \cdot (-\nu) - 2\tilde{h} \quad \text{in} \ \Omega. \]
The last two estimates imply
\[ \text{(4.9)} \quad v \geq \tilde{c} h^{\epsilon_1} |(x - y) \cdot \nu| - 2\tilde{h} \quad \text{in} \ \Omega. \]
Recall that
\[ v = u - l_y - \bar{h} = u - l_0 - [u_n(y) - u_n(0)]x_n \]
satisfies
\[ \text{(4.10)} \quad v < 0 \quad \text{in} \ S_h(y), \quad v \geq 0 \quad \text{in} \ \Omega \setminus S_h(y). \]
Since $v \geq 0$ on $\partial \Omega$ and $\partial \Omega, \varphi \in C^{2, \alpha}$, we have (after performing a rotation in the $x'$ subspace)
\[ \text{(4.11)} \quad v|_{\partial \Omega} = \sum_{i=1}^{n-1} \lambda_i^2 x_i^2 + O(|x'|^{2+\alpha}) \]
for some bounded constants $\lambda_i, i = 0, \ldots, n - 1$.

We only need to consider the cases: $|\nu'| \geq \tilde{h}^{\frac{\alpha}{2}}$ and $|\nu'| \leq \tilde{h}^{\frac{\alpha}{2}}$, where $\epsilon_0$ is the constant in (4.1).

**Case 1.** $|\nu'| \geq \tilde{h}^{\frac{\alpha}{2}}$. We choose $x \in \partial \Omega$ with $x' = \tilde{h}^{\frac{\alpha}{2}} |\nu'|$. Then we have
\[ x_n \leq C|x'|^2 \leq C\tilde{h}^{\frac{\alpha}{2}}|x'|, \]
and
\[ |y| \leq \tilde{h}^{\epsilon_0} \leq \tilde{h}^{\frac{\alpha}{2}}|x'|. \]
It follows that
\[ \tilde{h}^{\epsilon_1} (x - y) \cdot \nu \geq \tilde{h}^{\epsilon_1} (x' \cdot \nu' - x_n - |y|) \]
\[ \geq \tilde{h}^{\epsilon_1} \left( |\nu'| |x'| - C\tilde{h}^{\frac{\alpha}{2}}|x'| \right) \]
\[ \geq \tilde{h}^{\epsilon_1 + \frac{\alpha}{2}} (1 - C\tilde{h}^{\frac{\alpha}{2}})|x'| \]
\[ \geq \frac{\tilde{h}^{\epsilon_1 + \frac{\alpha}{2}}}{2} |x'| \]
if $\epsilon_1, \epsilon_0 > 0$ are sufficiently small and $\tilde{h}$ is small. It follows from (4.9) and (4.11) that
\[ c\tilde{h}^{\epsilon_1} \tilde{h}^{\frac{\alpha}{2}} |x'| \leq v(x) \leq C|x'|^2 = C\tilde{h}^{\frac{\alpha}{2}}|x'|. \]
Choose $0 < \epsilon_1 < \frac{\alpha}{4}$ and then we reach a contradiction. Thus (4.8) holds.

**Case 2.** $|\nu'| \leq \tilde{h}^{\frac{\alpha}{2}}$. Then we have
\[ |\nu \cdot e_n| \geq 1 - \tilde{h}^{\frac{\alpha}{2}} > \frac{1}{2} \]
if $\tilde{h}$ is small.

For any $x \in \Omega$ near 0 with $x_n^\frac{1}{2} \geq \tilde{h}^{\frac{\alpha}{4}}$, we have
\[ \tilde{h}^{\epsilon_1} |(x - y) \cdot \nu| \geq \tilde{h}^{\epsilon_1} \left( x_n |\nu \cdot e_n| - |x'| |\nu'| - |y| \right) \]
\[ \geq \tilde{h}^{\epsilon_1} \left( \frac{1}{2} x_n - C\tilde{h}^{\frac{\alpha}{2}} x_n^\frac{3}{4} - \tilde{h}^{\epsilon_0} \right) \]
\[ \geq \frac{\tilde{h}^{\epsilon_1}}{4} x_n \]
if $\epsilon_1, \epsilon_0 > 0$ are sufficiently small and $\tilde{h}$ is small. Hence by (4.9),
\[ \text{(4.12)} \quad v \geq c\tilde{h}^{\epsilon_1} x_n, \quad \text{in} \ \Omega \cap \{\tilde{h}^{\frac{\alpha}{2}} \leq x_n^\frac{1}{2} \leq c\}. \]
Let $0 < \delta < \frac{\Lambda}{8}$ be a small constant to be chosen below.

**Case 2.1.** If one of $\lambda_i^2, i = 1, \ldots, n - 1$, say $\lambda_1^2$, satisfies $\lambda_1^2 \leq \tilde{h}^{\delta} \alpha$, then we choose $x = (x_1, 0, \ldots, 0, x_n) \in \partial \Omega$ with $\frac{x_1}{h} = h^\delta$. We have by (4.11) and (4.12)

$$c\tilde{h}^{\delta} x_n \leq v(x) \leq \lambda_1^2 x_n^2 + C|x|^2 + \alpha \leq C\tilde{h}^{\delta} x_n.$$  

Choose $\epsilon_1 < \delta \alpha$ and we reach a contradiction.

**Case 2.2.** $\min_{1 \leq i \leq n-1} \lambda_i^2 \geq \tilde{h}^{\delta} \alpha$. Then we have

$$(4.13) \quad v \geq \frac{1}{2n} \tilde{h}^{\delta} \alpha |x|^2, \quad \text{on } \partial \Omega \cap \{x_n \leq \tilde{c}h\delta\}.$$  

Define

$$w = \sigma x_n + \frac{1}{4n} \tilde{h}^{\delta} \alpha |x|^2 + \frac{C_\sigma}{h^{\delta\alpha(n-1)}} x_n^2.$$  

Then $w$ is a lower barrier for $v$ in $\Omega \cap \{x_n \leq \tilde{h}^{\delta} \alpha\}$ if $\sigma, \delta > 0$ are sufficiently small and $C_\sigma$ is large.

Indeed, on $\partial \Omega \cap \{x_n \leq \tilde{h}^{\delta} \alpha\}$ we have

$$w \leq C\sigma |x|^2 + \frac{1}{4n} \tilde{h}^{\delta} \alpha |x|^2 + C\tilde{h}^{\delta\alpha(n-1)} |x|^2 \leq \frac{1}{2n} \tilde{h}^{\delta} \alpha |x|^2$$  

if $\sigma$ is small and $\delta \alpha \sigma < \frac{\Lambda}{8}$.

On $\Omega \cap \{x_n = \tilde{h}^{\delta} \alpha\}$ we have

$$w \leq \sigma x_n + \tilde{C}h^{\delta} \alpha x_n + C\tilde{h}^{\delta\alpha(n-1)} x_n \leq \tilde{c} \epsilon x_n$$  

if we choose $\epsilon_1 < \min \{\delta \alpha, \frac{\Lambda}{8} - \delta \alpha(n-1)\}$ and $\sigma$ small.

Hence by (4.13) and (4.12) we obtain that $v \geq w \geq \sigma x_n$ in $\Omega \cap \{x_n = \tilde{h}^{\delta} \alpha\}$. Since $y \in \Omega \cap B_{\tilde{h}^{\delta} \alpha} \cap \{x_n \leq \tilde{h}^{\delta} \alpha\}$, we reach a contradiction since $v(y) = -\tilde{h} < 0$.

Combining **Case 2.1** and **Case 2.2**, we prove (4.8) in **Case 2**. Hence (4.7) holds.

**Step 3.** We show that

$$(4.14) \quad ||\nabla u(x) - \nabla u(y)|| \leq C|x - y|^\beta \quad \forall x, y \in \overline{\Omega},$$  

where $\beta \in (0, 1)$ is a constant depending only on $n, \lambda, \Lambda$ and $\alpha$.

This follows from **Steps 1-2** and similar arguments as in [19]. For completeness, we include the proof.

We first note that in **Steps 1-2**, if $\tilde{h} \geq c$ for some small constant $c$, the estimates (4.1) and (4.7) obviously hold since $\tilde{h} \sim |S_h(y)|^{\frac{1}{\tilde{h}}}$ is bounded above. (We only need to replace $\tilde{h}^{1-\epsilon_1}$ by $C^{-1}\tilde{h}^{1-\epsilon_1}$ and $\tilde{h}^{\delta \alpha}$ by $C\tilde{h}^{\delta \alpha}$ for some large constant $C$.)

Let $y \in \Omega$ and assume the maximal interior section $S_h(y)$ is tangent to $\partial \Omega$ at $0 \in \partial \Omega$. Let $Tx = Ax + b$ be an affine transformation that normalizes $S_h(y)$, i.e.,

$$B_{\tilde{h}^{\delta}} \subset TS_h(y) \subset B_1.$$  

By (4.7) we have

$$(4.15) \quad \|T\| \leq C\tilde{h}^{-1-\epsilon_1}, \quad |\det T|^{\frac{1}{\tilde{h}}} \sim |S_h(y)|^{-\frac{1}{\tilde{h}}} \sim \tilde{h}^{-1}.$$  

For $\tilde{x} \in TS_h(y)$, define

$$\tilde{u}(\tilde{x}) = |\det T|^{\frac{1}{\tilde{h}}} |u - \tilde{l}_y - \tilde{h}|(T^{-1}\tilde{x}),$$  

where we recall that

$$\tilde{l}_y(x) = u(y) + \nabla u(y) \cdot (x - y).$$  

The interior $C^{1,\gamma}$ estimate for solutions of the Monge-Ampère equation (see [8]) gives

$$|\nabla \tilde{u}(\tilde{x}_1) - \nabla \tilde{u}(\tilde{x}_2)| \leq C|\tilde{x}_1 - \tilde{x}_2|^\gamma \quad \forall \tilde{x}_1, \tilde{x}_2 \in TS_h(y)$$  

for some $\gamma \in (0, 1)$ depending only on $n, \lambda, \Lambda$. Rescaling back and using

$$\nabla \tilde{u}(\tilde{x}_1) - \nabla \tilde{u}(\tilde{x}_2) = |\det T|^{\frac{1}{\tilde{h}}} [\nabla u(T^{-1}\tilde{x}_1) - \nabla u(T^{-1}\tilde{x}_2)]A^{-1},$$  

where $A \in GL_n$.
we find from (4.15)
\[
|\nabla u(x_1) - \nabla u(x_2)| \leq C|\det T|^{-\frac{2}{n}}|A|^{1+\gamma}|x_1 - x_2|^\gamma
\]
\[
\leq C\bar{h}^{1-(1-\epsilon)\gamma} |x_1 - x_2|
\]
\[
(4.16)
\]
if we choose \(\gamma > 0\) sufficiently small.

The convexity of \(u\) implies that \(S_{\bar{h}}(y) \supset \frac{1}{2} S_{\bar{h}}(y)\), where the rescaling is with respect to \(y\).

Hence by (4.7),
\[
S_{\bar{h}}(y) \supset B_{\delta h^{1-\epsilon}}(y).
\]

Also, by the proof of (4.1) (see (4.6)) we have
\[
|\nabla u(y) - \nabla u(0)| \leq C\bar{h}^{2\epsilon_0}.
\]

For any \(x, y \in \Omega\), assume the maximal interior sections \(S_{\bar{h}_x}(x), S_{\bar{h}_y}(y)\) are tangent to \(\partial \Omega\) at some points \(\bar{x}, \bar{y} \in \partial \Omega\) respectively. Assume without loss of generality that \(\bar{h}_y \geq \bar{h}_x\).

**Case 1.** \(x \in S_{\bar{h}_y}(y)\). Then by (4.16) we have
\[
|\nabla u(x) - \nabla u(y)| \leq C|x - y|^\gamma.
\]

**Case 2.** \(x \notin S_{\bar{h}_y}(y)\). Then we have
\[
|x - y| \geq c\bar{h}_y^{1-\epsilon_1} \geq c\bar{h}_x^{1-\epsilon_1}.
\]

and therefore by (4.1)
\[
|\bar{x} - \bar{y}| \leq |\bar{x} - x| + |x - y| + |y - \bar{y}| \leq C[\bar{h}_x^{\epsilon_0} + |x - y| + \bar{h}_y^{\epsilon_0}] \leq C|x - y|^{\epsilon_0}.
\]

Hence by (4.17) and Lemma 4.1,
\[
|\nabla u(x) - \nabla u(y)| \leq |\nabla u(x) - \nabla u(\bar{x})| + |\nabla u(\bar{x}) - \nabla u(\bar{y})| + |\nabla u(\bar{y}) - \nabla u(y)|
\]
\[
\leq C[\bar{h}_x^{2\epsilon_0} + |x - y|^{\epsilon_0} + \bar{h}_y^{2\epsilon_0}]
\]
\[
\leq C|x - y|^{\epsilon_0}.\]

Hence we obtain (4.14). The proof of Theorem 2.2 is complete.

We conclude this section with an example which shows that if the boundary data \(\varphi\) is only \(C^2\) in Theorem 2.2, then \(u\) may fail to be of class \(C^{1,\delta}(\Omega)\) for any \(\delta \in (0,1)\).

**Example.** Let \(\Omega = B_\rho(\rho e_n)\), where \(\rho\) is small depending only on \(n\) to be chosen below. Let \(u\) solves
\[
\begin{align*}
\text{det} \, D^2 u &= 1 \\
u &= \frac{x_n}{-\log x_n} \quad \text{in} \, \Omega, \\
u &= 0 \quad \text{on} \, \partial \Omega,
\end{align*}
\]
where we define \(u(0) = \lim_{|x| \to 0} \frac{x_n}{-\log x_n} = 0\).

In a neighborhood of 0, the boundary data \(\varphi = u|_{\partial \Omega}\) can be written as
\[
\varphi(x') = \frac{\rho - \sqrt{\rho^2 - |x'|^2}}{-\log(\rho - \sqrt{\rho^2 - |x'|^2})}.
\]

By straightforward computation we find that \(\varphi \in C^2(\partial \Omega)\).

Next we show that \(u_n(0) \leq 0\).

Indeed, for any \(0 < t < \rho\), we choose \(y = (y', t) \in \partial \Omega\). Then the convexity of \(u\) gives
\[
\frac{u(te_n)}{t} \leq \frac{1}{2} \left[ \frac{u(y', t)}{t} + \frac{u(-y', t)}{t} \right] = \frac{1}{-\log t},
\]
which implies \(u_n(0) \leq 0\).

Now we construct a lower barrier for \(u\) in \(\Omega\).
Let
\[ w := \frac{1}{2} \frac{x_n}{\log x_n} + |x'|^2 x_n^{\frac{1}{n}}. \]
Then we can compute
\[ \det D^2 w = 2^{n-2} \left[ \frac{1}{x_n^2 (\log x_n)^2} \left( 1 + \frac{2}{-\log x_n} \right) - \frac{2(n+1)}{n^2} |x'|^2 x_n^{-1} \right] \text{ in } \Omega. \]
Since \(|x'|^2 x_n^{-1}\) and \(x_n\) are bounded by \(2\rho\), we can choose \(\rho > 0\) small depending only on \(n\) such that
\[ \det D^2 w > 1 \text{ in } \Omega \]
and
\[ w \leq \frac{1}{2 - \log x_n} + 2\rho x_n^{1+\frac{1}{n}} \leq \frac{x_n}{-\log x_n} \text{ on } \partial \Omega. \]
Therefore \(u \geq w \geq 0\) in \(\Omega\), which implies \(u_n(0) \geq 0\). Hence,
\[ u_n(0) = 0. \]
It follows that
\[ u(0, x_n) - u_n(0)x_n = u(0, x_n) \geq w(0, x_n) = \frac{1}{2} \frac{x_n}{-\log x_n}. \]
This implies that \(u \notin C^{1, \delta}(0)\) for any \(\delta \in (0, 1)\).

5. PROOF OF THEOREM 2.3

We divide the proof into two steps.

**Step 1**: We construct an explicit barrier for \(u\). Let
\[ \bar{w}(r, y) := \frac{r}{2 - t^\sigma} + \frac{|x'|}{C'x_n}, \]
for some \(\sigma > 0, \epsilon \in (0, 1)\) to be chosen below. Then the function
\[ w_1(x', x_n) := \epsilon \bar{w}(|x'|, C'x_n), \]
is a lower barrier for \(u\) provided that \(\epsilon\) (small), \(C'\) (large) are appropriate constants.

Since
\[ \frac{dt}{dr} = -\sigma r^{-1} t, \quad \frac{dt}{dy} = r^{-\sigma}, \]
we compute in the set \(B_1^+ \cap \{ w > 0 \}\) (i.e. \(t \in (0, 1)\)):
\[
\begin{align*}
\bar{w}_r &= 2r(1-t^\sigma) + \sigma \epsilon t^\epsilon, \\
\bar{w}_{rr} &= 2(1-t^\sigma) + (3-\sigma \epsilon) \sigma \epsilon t^\epsilon, \\
\bar{w}_{ry} &= (\sigma \epsilon - 2) \epsilon t^{\epsilon - 1} r^{1-\sigma}, \\
\bar{w}_{yy} &= \epsilon(1-\epsilon) t^{\epsilon - 2} r^{2-2\sigma}. 
\end{align*}
\]
We have
\[ \det D^2_{r,y} \bar{w} \geq \epsilon^2 r^{2-2\sigma} t^{2\epsilon - 2} \left[ \sigma (1-\epsilon)(3-\sigma \epsilon) - (\sigma \epsilon - 2)^2 \right] \]
and
\[ \frac{\bar{w}_r}{r} \geq \sigma \epsilon t^\epsilon. \]
If we choose
\[ \sigma > \frac{4}{3} \]
and \(\epsilon > 0\) sufficiently small, then
\[ \det D^2 w_1(x', x_n) \geq c' n^{2} C_2^2 c(n, \sigma, \epsilon) r^{2-2\sigma} t^{n \epsilon - 2}, \quad r := |x'|, \quad t := C'x_n |x'|^{-\sigma}. \]
The right hand side of the last inequality is sufficiently large if we choose \(C' > 0\) sufficiently large.
Now we choose $c'$ small such that 
\[ c'|x'|^2 \leq \mu|x'|^2 \leq u \quad \text{on } \partial B_1^+ \cap \{x_n \leq \rho\} \]
and then $C'$ large such that 
\[ C'|x_n|x'|^{-\sigma} \geq 1 \quad \text{on } B_1^+ \cap \{x_n = \rho\}, \]
and 
\[ \det D^2w_1 > \Lambda \quad \text{in } B_1^+ \cap \{w_1 > 0\}. \]
Then we have $w_1 \leq u$ on $\partial(B_1^+ \cap \{x_n \leq \rho\})$ and $\det D^2w_1 > \det D^2u$ on the set where $w_1 > 0$. Hence $u \geq w_1$ in $B_1^+ \cap \{x_n \leq \rho\}$.

It follows that 
\[ S_h \cap \{x_n \leq \rho\} \subset \left\{c'|x'|^2 \leq 2h\right\} \cup \left\{[1 - (C'|x_n|x'|^{-\sigma})^c] \leq \frac{1}{2}\right\} \]
or
\[ (5.2) \quad S_h \cap \{x_n \leq \rho\} \subset \{|x'| \leq Ch^\frac{2}{n}\} \cup \{x_n \geq c|x'|^\sigma\}. \]

**Step 2:** Let $x_h^*$ be the center of mass of $S_h$ and denote $d_h = x_h^* \cdot e_n$, then we claim that for all small $h > 0$ we have
\[ d_h \geq \tilde{c}h^{\frac{2}{n}} \]
for some small $\tilde{c} > 0$.

Otherwise from (5.2) and Lemma 2.1 we obtain 
\[ S_h \subset \{x_n \leq C(n)\tilde{c}h^{\frac{2}{n}} \leq h^\frac{2}{n}\} \cap \{|x'| \leq C_1h^\frac{2}{n}\} \]
for some large constant $C_1$. Then the function 
\[ w = cx_n + \left[ c_0 \frac{x'}{C_1h^{\frac{2}{n}}} \right]^2 + C_0 \frac{x_n}{h^{\frac{2}{n}}} \cdot h \]
is a lower barrier for $u$ in $S_h$ if $c_0$ is sufficiently small and $C_0$ is large.

Indeed, we have on $\partial S_h \cap \partial B_1^+ \subset \{x_n = 0\}$, 
\[ w = \frac{c_0}{C_1h^{\frac{2}{n}}} \cdot h \leq \mu|x'|^2. \]
On $\partial S_h \cap B_1^+$, 
\[ w \leq c + [c_0^2 + C_0C(n)\tilde{c}] \cdot h \leq h \]
if $c_0, \tilde{c}$ are sufficiently small. Moreover, since $\sigma > \frac{4}{3} > 1$ (see (5.1)),
\[ \det D^2w \geq 2^n h^n \left( \frac{c_0}{C_1h^{\frac{2}{n}}} \right)^2 \left( \frac{C_0}{h^{\frac{2}{n}}} \right)^2 \geq \Lambda \]
if $C_0$ is sufficiently large.

Hence, we obtain $w \leq u$ in $S_h$. This is a contradiction with $\nabla u(0) = 0$. Thus (5.3) is proved.

Since $S_h \cap \{x_n = 0\}$ contains a ball in $\mathbb{R}^{n-1}$ of radius $(\mu h)^{\frac{1}{1+\alpha}}$, we obtain from (5.2) and (5.3) that 
\[ \{te_n : 0 \leq t \leq c_0h^{\frac{1}{1+\alpha}} \} \subset S_h \]
for some small constant $c_0 > 0$. Hence we only need to choose $\sigma$ satisfying
\[ (5.4) \quad \frac{\sigma - 1}{2} + \frac{1}{1+\alpha} < 1. \]
By the assumption $\alpha > \frac{1}{5}$, we can choose $\sigma$ satisfying (5.1) and (5.4). The conclusion of the theorem follows.
6. Proof of Theorem 2.4

The proof of Theorem 2.4 is similar to that of Theorem 2.2.

**Step 1.** Let \( y \in B_1^+ \) and assume the maximal interior section \( S_h(y) \) is tangent to \( \partial B_1^+ \) at 0. We prove that

\[
S_h(y) \subset B_{\bar{h}} 
\]

for any \( \bar{h} > 0 \) small, where \( \epsilon_0 \) is a small constant.

For any \( h > 0 \) small, let \( x_h^* \) be the center of mass of \( S_h \) and \( d_h := x_h^* \cdot e_n \). Since \( S_h \) contains the convex set generated by \( \{ |x'| \leq ch^{-1} \cdot e_n, x_n = 0 \} \) and the point \( x_h^* \), we can use the estimate of the upper bound of \( |S_h| \) (see the proof of Theorem 2.2) and obtain that

\[
d_h \leq C_0 h^\gamma, \quad \tau := \frac{n}{2} - \frac{n-1}{1+\alpha} \in (0, \frac{1}{2}]
\]

for some large constant \( C_0 > 0 \).

By Lemma 2.1,

\[
S_h \subset \{ x_n \leq C(n)C_0 h^\gamma \} \subset \{ x_n \leq C'h^\gamma \} \quad \forall h > 0.
\]

Choose \( \sigma = 2 > \frac{4}{3} \) in (5.2) and we obtain that

\[
S_h \subset B_{Ch^{2/3}} \quad \forall h > 0.
\]

We have

\[
v(x) = (u - u_0)(x) - [u_n(y) - u_n(0)] x_n.
\]

Choose \( h > 0 \) such that

\[
u_n(y) - u_n(0) = (2C')^{-1}h^{1-\tau},
\]

where \( C' \) is the constant in (6.3). Then we have by (6.3) and (6.4) that

\[
S_h(y) \subset S_h \subset B_{Ch^{2/3}}.
\]

On the other hand, let \( \theta > 0 \) be a small constant to be chosen below and denote

\[
\Lambda_0 := \frac{(1 + \alpha')(1 - \tau)}{\alpha'},
\]

where \( \alpha' \in (0,1) \) is the constant obtained by Theorem 2.3. If \( h \leq 1 \), by Theorem 2.3, we can choose a point \( z = te_n \in \partial S_{h\Lambda_0} \) with \( t \geq c(\theta h^{\Lambda_0})^{\frac{1}{1+\alpha'}} \). It follows that

\[
v(z) \leq \theta h^{\Lambda_0} - (2C')^{-1}h^{1-\tau}c(\theta h^{\Lambda_0})^{\frac{1}{1+\alpha'}}
\]

\[
= \frac{1}{1+\alpha'} h^{\Lambda_0} [\theta^{\frac{\alpha'}{1+\alpha'}} - (2C')^{-1}c]
\]

\[
< \frac{1}{1+\alpha'} h^{\Lambda_0} (4C')^{-1}c < 0
\]

if \( \theta > 0 \) is sufficiently small. On the other hand, if \( h \geq 1 \), then we can choose a point \( z = te_n \in \partial S_{c'} \) with \( t \geq cc'^{\frac{1}{1+\alpha'}} \) and therefore

\[
v(z) \leq c' - (2C')^{-1}h^{1-\tau}cc'^{\frac{1}{1+\alpha'}}
\]

\[
= cc'^{\frac{1}{1+\alpha'}} [c'^{\frac{\alpha'}{1+\alpha'}} - (2C')^{-1}ch^{1-\tau}]
\]

\[
< cc'^{\frac{1}{1+\alpha'}} h^{1-\tau} (4C')^{-1}c < 0
\]

if \( c' > 0 \) is sufficiently small. Combining the last two estimates we obtain

\[
\bar{h} \geq c|S_h(y)|^{\frac{1}{h}} \geq c \min_{S_h(y)} |v| \geq c \min \{ h^{\Lambda_0}, h^{1-\tau} \}.
\]

If \( \bar{h} \) is sufficiently small, then (6.6) together with (6.5) gives (6.1) with \( 0 < \epsilon_0 < \frac{1}{2\Lambda_0} \).
Step 2. Assume as in Step 1 that $S_h(y)$ is a maximal interior section tangent to $\partial B^+_1$ at 0. We prove that

\begin{equation}
B_{\tilde{h}_{1-\epsilon_1}}(y) \subset S_h(y)
\end{equation}

for any $\tilde{h} > 0$ small, where $\epsilon_1 > 0$ is a small constant.

By Lemma 2.1, $S_h(y)$ is equivalent to an ellipsoid $E$ centered at $y$ of axes $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. We only need to prove that

\begin{equation}
\mu_1 \geq \tilde{h}^{1-\epsilon_1}.
\end{equation}

Assume by contradiction that

\begin{equation}
\mu_1 < \tilde{h}^{1-\epsilon_1}.
\end{equation}

Then as in the proof of Theorem 2.2 we have

\begin{equation}
v \geq ch^{\epsilon_1} |x - y| \cdot \nu - 2\tilde{h} \quad \text{in} \quad B^+_1,
\end{equation}

where $\nu$ is a unit vector.

We have

\begin{equation}
v \big|_{\partial B^+_1 \cap \{x_n=0\}} = \varphi - \varphi(0) - \nabla_x \varphi(0) \cdot x' \leq C|x'|^{1+\alpha}.
\end{equation}

We only need to consider the cases: $|\nu'| \geq \tilde{h}^{\alpha_0}$ and $|\nu'| \leq \tilde{h}^{\alpha_0}$, where $\epsilon_0$ is the constant in (6.1).

**Case 1.** $|\nu'| \geq \tilde{h}^{\alpha_0}$. We choose $x = (x',0)$ with $x' = \tilde{h}^{\alpha_0} \frac{\nu'}{||\nu||}$. Then we have

\begin{equation}
|y| \leq \tilde{h}^{\epsilon_0} \leq \tilde{h}^{\epsilon_0} |x'|,
\end{equation}

which implies

\begin{equation}
\tilde{h}^{\epsilon_1} (x - y) \cdot \nu \geq \tilde{h}^{\epsilon_1} (x' \cdot \nu' - |y|) \geq \tilde{h}^{\epsilon_1 + \frac{\alpha_0}{2}} (1 - C\tilde{h}^{\frac{\alpha_0}{2} - \frac{\alpha_0}{2}}) |x'| \geq \frac{\tilde{h}^{\epsilon_1 + \frac{\alpha_0}{2}}}{2} |x'|
\end{equation}

if $\epsilon_1, \epsilon_0 > 0$ are sufficiently small and $\tilde{h}$ is small. Choose $0 < \epsilon_1 < \frac{\alpha_0}{4}$, then by (6.9) and (6.10), we reach a contradiction. Thus (6.8) holds.

**Case 2.** $|\nu'| \leq \tilde{h}^{\alpha_0}$. By (6.6) we find that $h$ is bounded above, hence we have $v + \tilde{h} \geq 0$ in $B^+_1$ and

\begin{equation}
v \geq c|x'|^2 \quad \text{on} \quad \partial B^+_1 \cap \{x_n \leq c_s\}
\end{equation}

if $c_s > 0$ is sufficiently small, then similar to the proof of Theorem 2.3, the function

\begin{equation}
\tilde{u}_1 := c'|x'|^2 [1 - (\tilde{C}' x_n |x'|^{-2})^{\epsilon}]^+
\end{equation}

is a lower barrier for both $u - l_0$ and $v + \tilde{h}$ in $B^+_1 \cap \{x_n \leq c_s\}$, where $\epsilon, \tilde{C}'$ are some small constants and $\tilde{C}'$ is a large constant.

On $\{x_n = \frac{1}{2\tilde{C}'} |x'|^2 \} \cap \{|x'| \leq \frac{1}{2}\}$ we have

\begin{equation}
u - l_0 \geq \tilde{u}_1 \geq (1 - 2^{-\epsilon}) \tilde{c}' |x'|^2,
\end{equation}

it follows that

\begin{equation}v \geq (1 - 2^\epsilon) \tilde{c}' |x'|^2 - (2\tilde{C}')^{-1} \tilde{h}^{1-\tau} \frac{1}{2\tilde{C}'} |x'|^2
\end{equation}

\begin{equation}\geq \frac{1}{2}(1 - 2^\epsilon) \tilde{c}' |x'|^2
\end{equation}

if $\tilde{C}'$ is sufficiently large.
On $\{\tilde{h}^{\alpha n} = x_n > \frac{1}{2C^\alpha}|x'|^2\}$, we have by (6.9)

$$v \geq \tilde{c}\epsilon_1 \left( x_n |\nu \cdot e_n | - \frac{\tilde{h}^{\alpha n}}{4} |x'| - |y| \right) - 2\tilde{h}$$

$$\geq \tilde{c}\epsilon_1 \left( \frac{1}{2}x_n - \frac{\tilde{h}^{\alpha n}}{4} (2\bar{C}'x_n)^{\frac{1}{2}} - \tilde{h}\epsilon_0 \right) - 2\tilde{h}$$

(6.12)

Define

$$w = \delta x_n + c_2\tilde{h}\epsilon_1 |x'|^2 + \frac{C_2}{\tilde{h}^{\epsilon_1(n-1)}} x_n^2,$$

then $w$ is a lower barrier for $v$ in $\{ \frac{1}{2C^\alpha}|x'|^2 \leq x_n \leq \tilde{h}^{\alpha n} \}$ if $\delta, c_2 > 0$ are sufficiently small and $C_2$ is large. Indeed, on $\{\tilde{h}^{\alpha n} \geq x_n = \frac{1}{2C^\alpha}|x'|^2\}$, we have by (6.11)

$$w \leq 2\delta\bar{C}'|x'|^2 + c_2\tilde{h}\epsilon_1 |x'|^2 + C\tilde{h}^{\alpha n} - \epsilon_1(n-1)|x'|^2 \leq v$$

if $\delta, c_2$ are small and $\epsilon(n-1) < \frac{\epsilon_1}{4}$. On $\{\tilde{h}^{\alpha n} = x_n > \frac{1}{2C^\alpha}|x'|^2\}$, we have by (6.12)

$$w \leq \delta x_n + c_2\tilde{h}\epsilon_1 2\bar{C}'x_n + C_2\tilde{h}^{\alpha n} - \epsilon_1(n-1)x_n \leq v$$

if $\delta, c_2$ are small and $\epsilon_1 n < \frac{\epsilon_1}{4}$.

Therefore, $v \geq w \geq \delta x_n$ in $\{ \frac{1}{2C^\alpha}|x'|^2 \leq x_n \leq \tilde{h}^{\alpha n} \}$. Since

$$0 = v(y) + \tilde{h} \geq \bar{w}_1(y),$$

we have

$$\tilde{h}\epsilon_0 \geq y_n \geq \frac{1}{C^\alpha}|y'|^2.$$

Hence we reach a contradiction since $v(y) < 0$. Thus (6.8) holds. Combining these two cases, we obtain (6.7).

**Step 3.** Let $c_0 > 0$ be a small constant to be chosen below and $y \in B^+_{c_0}$. Assume the maximal interior section $S_h(y)$ is tangent to $\partial B^+_1$ at $y_0 \in \partial B^+_1$. We prove that

(6.13)

$$B_{\tilde{h}\epsilon_1}(y) \subset S_{\tilde{h}}(y) \subset B_{\tilde{h}n_0}(y_0),$$

where $\epsilon_1, \epsilon_0$ are the constants in (6.1) and (6.7).

We only need to prove that

(6.14)

$$y_0 \in \{|x'| \leq \frac{1}{2}, x_n = 0\}.$$

Then it follows from **Step 1-2** (with 0 replaced by $y_0$) that (6.13) holds.

Now we prove (6.14).

Since $S_{\tilde{h}}(y)$ is equivalent to an ellipsoid centered at $y$, we obtain

$$y_0 \cdot e_n \leq C y_n \leq C c_0.$$

Assume in contradiction that (6.14) does not hold, then we have

$$y_0 \in \left\{ \frac{1}{2} \leq |x'| \leq 1, x_n = 0 \right\} \cup \left\{ |x| = 1, 0 < x_n \leq C c_0 \right\}.$$

Recall that $u$ separates quadratically from $l_0$ on $\partial B^+_1$ in a neighborhood of 0. Hence if $c_0$ is sufficiently small, then

(6.15)

$$(u - l_0)(y_0) \geq c|y_0|^2 \geq c_1$$

for some $c_1$ small. Since

$$0 \leq (u - l_0)(x) = (u - l_0)(x) + (u - l_y)(0) + (\nabla u(0) - \nabla u(y)) \cdot x$$

and $u$ is Lipschitz continuous, we have on $\partial B^+_1 \cap \{ x_n = 0 \}$,

$$[\nabla_x u(y) - \nabla_x \varphi(0)] \cdot x' \leq C[|x'|^{1+\alpha} + |y|].$$
Choose $x' = \frac{\nabla x' u(y) - \nabla x' \varphi(0)}{\nabla x' u(y) - \nabla x' \varphi(0)}$, we obtain that

\begin{equation}
|\nabla x' u(y) - \nabla x' \varphi(0)| \leq C |y|^{\frac{1}{2}}.
\end{equation}

The convexity of $u$ implies that $u_n(0)$ is bounded above. Since $u$ is Lipschitz continuous, $|\nabla u(y)|$ is bounded and moreover, for any $x \in B^+_1$, 

$$u(x) = u(x) - \varphi(x') + \varphi(x') \geq -C x_n + \varphi(0) + \nabla x' \varphi(0) \cdot x',$n which implies that $u_n(0)$ is bounded below.

We have 

$$v = u - l_y - (u - l_y)(y_0) = u - l_0 - (u - l_0)(y_0) - (\nabla u(y) - \nabla u(0)) \cdot (x - y_0).$$

On $\partial B^+_1 \cap \{x_n = 0\}$, since $v \geq 0$, we have by (6.15) and (6.16)

\begin{align*}
C|x'|^{1+\alpha} & \geq \varphi - \varphi(0) - \nabla x' \varphi(0) \cdot x' \\
& \geq (u - l_0)(y_0) + (\nabla x' u(y) - \nabla x' \varphi(0)) \cdot (x' - y_0') \\
& + (u_n(y) - u_n(0))(y_0 - 0) \cdot c_n
\end{align*}
\begin{equation}
(6.17) \geq c_1 - C c_0^{\frac{1}{2}} - C c_0 \geq c_1 - 2
\end{equation}

if $c_0$ is sufficiently small. We reach a contradiction. Hence (6.14) holds.

Using (6.13) and similar arguments as in Step 3 in the proof of Theorem 2.2 we can prove

$$|\nabla u|_{C^{\beta}([B^+_1]_0)} \leq C$$

for some $\beta \in (0, 1)$ depending only on $n, \lambda, \Lambda$ and $\alpha$.

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