Estimating Full Lipschitz Constants of Deep Neural Networks

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Abstract

We estimate the Lipschitz constants of the gradient of a deep neural network and the network itself with respect to the full set of parameters. We first develop estimates for a deep feed-forward densely connected network and then, in a more general framework, for all neural networks that can be represented as solutions of controlled ordinary differential equations, where time appears as continuous depth. These estimates can be used to set the step size of stochastic gradient descent methods, which is illustrated for one example method.

1 Introduction

The training of a neural network can be summarized as an optimization problem which consists of making steps towards extrema of a loss function. Variants of the stochastic gradient descent (SGD) are generally used to solve this problem. They give surprisingly good results, even though the objective function is not convex in most cases. The adaptive gradient methods are a state-of-the-art variation of SGD. In particular, AdaGrad [Duchi et al., 2011], RMSProp [Tieleman and Hinton, 2012], and ADAM [Kingma and Ba, 2014] are widely used methods to train neural networks [Melis et al., 2017, Xu et al., 2015]. In most of the SGD methods, the rate of convergence depends on the Lipschitz constant of the gradient of the loss function with respect to the parameters [Reddi et al., 2018, Li and Orabona, 2019]. Therefore, it is essential to have an upper bound estimate on the Lipschitz constant in order to get a better understanding of the convergence and to be able to set an appropriate step-size.

In this paper, we provide a general and efficient estimate for the upper bound on the Lipschitz constant of the gradient of any loss function applied to a feed-forward fully connected DNN with respect to the parameters. Naturally, this estimate depends on the architecture of the DNN (i.e. the activation function, the depth of the NN, the size of the layers) as well as on the norm of the input and on the loss function.

As a concrete application, we show how our estimate can be used to set the (hyper-parameters of the) step size of the AdaGrad [Li and Orabona, 2019] SGD method, such that convergence of this optimization scheme is guaranteed (in expectation). In particular, the convergence rate of AdaGrad with respect to the Lipschitz estimate of the gradient of the loss function can be calculated.

In addition, we provide Lipschitz estimates for any neural network that can be represented as solution of a controlled ordinary differential equation (controlled ODE) [Cuchiero et al., 2019]. This
includes classical DNN as well as continuously deep neural networks, like neural ODE [Chen et al., 2018], ODE-RNN [Rubanova et al., 2019] and neural SDE [Liu et al., 2019, Tzen and Raginsky, 2019, Jia and Benson, 2019]. Therefore, having such a general Lipschitz estimate allows us to cover a wide range of architectures and to study their convergence behaviour.

Moreover, controlled ODE can provide us with neural-network based parametrized families of invertible functions (cf. Cuchiero et al. [2019]), including in particular feed forward neural networks. Not only is it important to have precise formulas for their derivatives, these formulas also appear prominently in financial applications.

In Deep Pricing algorithms the prices of financial derivatives are encoded in feed forward neural networks with market factors and/or market model parameters as inputs. It is well known that sensitivities of prices with respect to those parameters or underlying factors are the crucial hedging ratios for building hedging portfolios. It is therefore important to control some of those sensitivities in the training process, which can be precisely done by our formulas.

controlled ODE also provides a new view on models in mathematical finance, which are typically given by stochastic differential equations to capture the complicated dynamics of financial markets. If the control $u$ is a driving stochastic process, e.g. a semi-martingale, and the state $X$ describes some prices, then the output of controlled ODE is, with respect to continuous depth, a price process. Derivatives and Lipschitz constants describe global properties of such models and can, again, be used to facilitate and shape the training process by providing natural bounds to it. This can for example be applied to the controlled ODE that appears in Cuchiero et al. [2020].

2 Related work

Very recently, several estimates of the Lipschitz constants of neural networks were proposed [Scaman and Virmaux, 2018, Combettes and Pesquet, 2019, Fazlyab et al., 2019, Jin and Lavaei, 2018, Raghunathan et al., 2018, Latorre et al., 2020]. In contrast to our work, those estimates are upper bounds on the Lipschitz constants of neural networks with respect to the inputs and not with respect to the parameters as we provide here. In addition we give upper bounds on the Lipschitz constants of the gradient of a loss function applied to a DNN and not only on the DNN itself. Those other works are mainly concerned with the sensitivity of neural networks to their inputs, while our main goal is to study the convergence properties of neural networks.

In the classical setting, results similar to our work were given in Baes et al. [2019] for one specific loss function. In comparison, we provide in the classical setting a simplified proof. To the best of our knowledge, neither for the classical setting of deep feed-forward fully connected neural networks, nor for the controlled ODE framework, general estimates of the Lipschitz constants with respect to the parameters are available.

3 Ordinary deep neural network setting

3.1 Problem setup

The norm we shall use in the sequel is a natural extension of the standard Frobenius norm to finite lists of matrices of diverse sizes. Specifically, for any $\gamma \in \mathbb{N}$, $m_1, \ldots, m_\gamma, n_1, \ldots, n_\gamma \in \mathbb{N}$, and $(M^1, \ldots, M^\gamma) \in \mathbb{R}^{m_1 \times n_1} \times \cdots \times \mathbb{R}^{m_\gamma \times n_\gamma}$, we let

$$\|(M^1, \ldots, M^\gamma)\| := \left(\sum_{i=1}^{m_1} \sum_{j=1}^{n_1} (M^1_{i,j})^2 + \cdots + \sum_{i=1}^{m_\gamma} \sum_{j=1}^{n_\gamma} (M^\gamma_{i,j})^2\right)^{1/2}. \quad (1)$$

Consider positive integers $\ell_u$ for $u = 0, \ldots, m + 1$. We construct a deep neural network (DNN) with $m$ layers of $\ell_u$, $u \in \{1, \ldots, m\}$ neurons, each with an (activation) function $\sigma_u : \mathbb{R} \to \mathbb{R}$, such that there exist $\sigma_{\max}'$, $\sigma_{\max}''$ > 0, so that for all $u \in \{1, \ldots, m\}$ and all $x \in \mathbb{R}$ we have $|\sigma_u(x)| \leq \sigma_{\max}'$, $|\sigma_u'(x)| \leq \sigma_{\max}''$ and $|\sigma_u''(x)| \leq \sigma_{\max}''$. This assumption is met by the classical sigmoid and tanh functions, but excludes the popular ReLU activation function. However, our main results of this section can easily be extended to allow for ReLU as well, as outlined in Remark 3.4. For each $u \in \{1, \ldots, m + 1\}$, let $A^{(u)} = [A^{(u)}_{i,j}]_{i,j} \in \mathbb{R}^{\ell_u \times \ell_{u-1}}$ be the weights and
Let $b(u) = \left[ b_i(u) \right] \in \mathbb{R}^{\ell_u}$ be the bias. Let $\theta_u = \left( A(u), b(u) \right)$ and define for every $u \in \{1, \ldots, m+1\}$
\begin{align*}
 f_{\theta_u} : \mathbb{R}^{\ell_{u-1}} \to \mathbb{R}^{\ell_u}, \quad x \mapsto A(u)x + b(u), \\
 \sigma_u : \mathbb{R}^{\ell_u} \to \mathbb{R}^{\ell_u}, \quad x \mapsto (\tilde{\sigma}_u(x_1), \ldots, \tilde{\sigma}_u(x_{\ell_u}))^\top.
\end{align*}

We denote for every $u \in \{1, \ldots, m+1\}$ the parameters $\Theta_u := (\theta_1, \ldots, \theta_u)$, and by a slight abuse of notation, considering $\theta_u$ and $\Theta_u$ as flattened vectors, we write $\theta_u \in \mathbb{R}^{d_u}$ and $\Theta_u \in \mathbb{R}^{d_u}$. Moreover, we define $\Omega \subset \mathbb{R}^{d_{m+1}}$ as the set of possible neural network parameters. Then we define the $m$-layered feed-forward neural network as the function
\begin{align*}
 N_{\Theta_{m+1}} : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_{m+1}}, \quad z \mapsto f_{\theta_{m+1}} \circ f_{\theta_m} \circ \cdots \circ f_{\theta_1}(z).
\end{align*}

By $L_{m+1} := d_{m+1} = \sum_{u=1}^{m+1} (\ell_u \ell_{u-1} + \ell_u)$ we denote the number of trainable parameters of $N_{\Theta_{m+1}}$.

We now assume that there exists a (possibly infinite) set of possible training samples $Z \subset \mathbb{R}^{\ell_0} \times \mathbb{R}^k$, for $k \in \mathbb{N}$, equipped with a sigma algebra $\mathcal{A}(Z)$ and a probability measure $\mathbb{P}$, the distribution of the training samples. Let $Z \sim \mathbb{P}$ be a random variable following this distribution. We use the notation $Z = (Z_x, Z_y) = (\text{proj}_x(Z), \text{proj}_y(Z))$ to emphasize the two components of a training sample $Z$. In a standard supervised learning setup we have $k = \ell_{m+1}$, where $Z_x \in \mathbb{R}^{\ell_0}$ is the input and $Z_y \in \mathbb{R}^k$ is the target. However, we also allow any other setup including $k = 0$, corresponding to training samples consisting only of the input, i.e. an unsupervised setting. Let $g : \mathbb{R}^{\ell_{m+1}} \times \mathbb{R}^k \to \mathbb{R}, \quad (x, y) \mapsto g(x, y),$

be a function which is twice differentiable in the first component. We assume there exist $g'_\text{max}, g''_\text{max} > 0$ such that for all $(x, y) \in \mathbb{R}^{\ell_{m+1}} \times \mathbb{R}^k$ we have $\|\frac{\partial}{\partial x} g(x, y)\| \leq g'_\text{max}$ and $\|\frac{\partial^2}{\partial x^2} g(x, y)\| \leq g''_\text{max}$. We use $g$ to define the cost function, given one training sample $\zeta := (\zeta_x, \zeta_y) \in \mathbb{R}^{\ell_0} \times \mathbb{R}^k$, as
\begin{align*}
 \varphi : \mathbb{R}^{\ell_{m+1}} \times (\mathbb{R}^{\ell_0} \times \mathbb{R}^k) \to \mathbb{R}, \quad (\Theta_{m+1}, \zeta) \mapsto g(N_{\Theta_{m+1}}(\zeta_x), \zeta_y).
\end{align*}

Then we define the cost function (interchangeably called objective or loss function) as
\begin{align*}
 \Phi : \mathbb{R}^{\ell_{m+1}} \to \mathbb{R}, \quad \Theta_{m+1} \mapsto \mathbb{E}[\varphi(\Theta_{m+1}, Z)],
\end{align*}

where we denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

**Remark 3.1.** For a finite set of training samples $Z = \{\zeta_1, \ldots, \zeta_N\}$, with equal probabilities (Laplace probability model) we obtain the standard neural network objective function $\Phi(\Theta_{m+1}) = \frac{1}{N} \sum_{i=1}^{N} \varphi(\Theta_{m+1}, \zeta_i)$.

### 3.2 Main results

The following theorems show that under standard assumptions for neural network training, the neural network $N$, as well as the cost function $\Phi$, are Lipschitz continuous with Lipschitz continuous gradients with respect to the parameters $\Theta_{m+1}$. We explicitly calculate upper bounds on the Lipschitz constants. Furthermore, we apply these results to infer a bound on the convergence rate to a stationary point of the cost function. The proofs are given in Appendix B.

To simplify the notation, we define the following functions for a given training sample $(\zeta_x, \zeta_y) \in Z$ and $2 \leq u \leq m$:
\begin{align*}
 N_1 : \mathbb{R}^{d_1} &\to \mathbb{R}^{\ell_1}, \quad \Theta_1 \mapsto \sigma_1 \circ f_{\theta_1}(\zeta_x), \\
 N_u : \mathbb{R}^{d_u} &\to \mathbb{R}^{\ell_u}, \quad \Theta_u \mapsto \sigma_u \circ f_{\theta_u}(N_{u-1}(\Theta_{u-1})), \\
 N : \Omega &\to \mathbb{R}^{\ell_{m+1}}, \quad \Theta \mapsto N_{\Theta}(\zeta_x).
\end{align*}

First we derive upper bounds on the Lipschitz constants of the neural network $N$ and its gradient.
Theorem 3.2. We assume that the space of network parameters $\Omega$ is non-empty, open and bounded, that is, there exists some $0 < B_\Omega < \infty$ such that for every $\Theta \in \Omega$ we have $\|\Theta\| < B_\Omega$. For any fixed training sample $(\xi, \xi_y) \in \mathcal{Z}$ we set $S := \|\xi\|$. Then, for $1 \leq u \leq m$, each $N_u$ and its gradient $\nabla N_u$ are Lipschitz continuous with constants $L_{N_u}$ and $L_{\nabla N_u}$ and uniformly bounded with constants $B_{N_u}$, $B_{\nabla N_u} = L_{N_u}$, which can be upper bounded as follows:

$$
L_{N_1} = \sigma'_\max \sqrt{S^2 + 1}, \quad B_{N_1} = \sqrt{\ell_1 \sigma'_{\max}}, \quad L_{\nabla N_1} = \sigma''_{\max} \sqrt{(S^2 + 1)(3S^2 + 2)},
$$

$$
L_{N_u} = \sigma'_\max \sqrt{B_\Omega^2 L_{N_{u-1}}^2 + B_{N_{u-1}}^2 + 1}, \quad B_{N_u} = \sqrt{\ell_u \sigma'_{\max}}, \quad L_{\nabla N_u} = \sqrt{\alpha_u + \beta_u},
$$

where

$$
\alpha_u = \max(3S_{N_{u-1}}^2 (\sigma'_{\max})^2 \ell_u + (\sigma''_{\max})^2 B_\Omega^2 B_{N_{u-1}}^2 + 2 (\sigma''_{\max})^2 B_\Omega^2 L_{N_{u-1}}^2),
$$

$$
\beta_u = (\ell_u \sigma'_{\max}) B_\Omega L_{\nabla N_{u-1}} + \sigma''_{\max} B_\Omega L_{N_{u-1}}^2)^2 + L_{N_{u-1}}^2 (\ell_u \sigma'_{\max} + B_\Omega \sigma''_{\max} (B_{N_{u-1}}^2 + 1)^{1/2})^2.
$$

Furthermore, the function $N$ and its gradient $\nabla N$ are Lipschitz continuous with constant $L_N$ and $L_{\nabla N}$. This also implies that $\nabla N$ is uniformly bounded by $B_{\nabla N} = L_{N}$ and these constants can be estimated by

$$
L_N = \sqrt{B_\Omega^2 L_{N_m}^2 + B_{N_m}^2 + 1}, \quad L_{\nabla N} = \sqrt{3L_{N_m}^2 \ell_{m+1} + \ell_{m+1}^2 B_\Omega^2 L_{N_{m+1}}^2 + L_{N_m}^2 \ell_{m+1}},
$$

In the corollary below, we solve the recursive formulas to get simpler (but less tight) expressions of the upper bounds of the constants.

Corollary 3.3. Let $\ell := \max_{1 \leq u \leq m} \{\ell_u\}$. The iteratively defined constants of Theorem 3.2 can be upper bounded for $1 \leq u \leq m$ by

$$
L_{N_u}^2 \leq B_\Omega^{2(u-1)} (\sigma'_{\max})^{2u} (S^2 + 1) + \sum_{k=1}^{u-1} B_\Omega^{2(k-1)} (\sigma'_{\max})^{2k} (\ell_{\max}^2 + 1),
$$

$$
L_{\nabla N_u}^2 = O\left(u (\sigma'_{\max})^2 \sigma''_{\max} ^2 (u-1)^2 (\sigma''_{\max})^2 (u-1)^{3/2} (\sigma''_{\max})^2 B_\Omega^2 (S^4 + 1)\right).
$$

With Corollary 3.3 we see that our estimates of the Lipschitz constants are linear respectively quadratic in the norm of the neural network input but grow exponentially with the number of layers. Since this gives us only an upper bound for the Lipschitz constants, a natural question arising is, whether these constants are optimal. The answer is no. In particular, the factor $B_\Omega^u$ respectively $B_{N_u}^{2u}$ is too pessimistic as we discuss in Remark A.1. On the other hand, the factor $(\sigma'_{\max})^{2u}$ is needed for an approximation of $L_{N_u}$ as shown in Example A.2. Similar examples can be constructed for $L_{\nabla N_u}$.

Let us finally explain in the remark below, how Theorem 3.2 can be extended to ReLU activation functions.

Remark 3.4. We made the assumption that the activation functions $\sigma_u$ are twice differentiable and bounded. This includes the classical sigmoid and tanh functions, but excludes the often used ReLU function $x \mapsto \max\{0, x\}$. However, Theorem 3.2 can be extended to slight modifications of ReLU, which are made twice differentiable by smoothing the kink at 0. Indeed, if $\sigma''_{\max}$ exists, the only part of the proof that has to be adjusted is the computation of $B_{N_u}$. Since ReLU is either the identity or 0, the norm of its output is bounded by the norm of its input, which yields $B_{N_u} = B_\Omega \sqrt{S^2 + 1}$ and $B_{\nabla N_u} = B_\Omega \sqrt{B_{N_{u-1}}^2 + 1}$. To take account for the smoothing of the kink, a small constant $\varepsilon$ can be added, which equals the maximum difference between the smoothed and the original version of ReLU.

Now we derive upper bounds on the Lipschitz constants of the objective function $\Phi$ and its gradient.

Theorem 3.5. We assume that the space of parameters $\Omega$ is non-empty, open and bounded by 0 < $B_\Omega < \infty$. Let $Z \sim \mathcal{P}$ be a random variable following the distribution of the training samples and assume that $S := \|\text{proj}_Z(Z)\|$ is a random variable in $L^2(\mathcal{P})$, i.e. $\mathbb{E}[S^2] < \infty$. Here $\|\cdot\|$ denotes the norm (1) and $\xi$ of Theorem 3.2 as replaced by $S$. Then, the objective function $\Phi$ and its gradient $\nabla \Phi$ are Lipschitz continuous with constants $L_{\Phi}$ and $L_{\nabla \Phi}$. This also implies that $\nabla \Phi$ is uniformly
bounded by \( B_{\nabla \Phi} = L_{\Phi} \). Using the constants of Theorem 3.2 we define

\[ L_{\Phi} = g_{\max}' \sqrt{B_{\Omega}^2 L_{N_m}^2 + B_{N_m}^2} + 1, \quad L_{\nabla \Phi} = \sqrt{\alpha_{m+1} + 3} + \beta_{m+1}, \]

\[ \alpha_{m+1} = \max\{3L_{N_m}^2 (g_{\max}'^2 + (g_{\max}'^2 B_{\Omega}^2 B_{N_m}^2) + 2(g_{\max}'^2 B_{\Omega}^2 L_{N_m}^2), \]

\[ \beta_{m+1} = (g_{\max}' B_{\Omega} L_{\nabla \Phi} + g_{\max}' B_{\Omega} L_{N_m}^2)^2 + L_{N_m} (g_{\max}' + B_{\Omega} g_{\max}' (B_{N_m}^2 + 1)}^{1/2}. \]

and get the following estimates for the above defined constants:

\[ L_{\Phi} = B_{\nabla \Phi} = E[L_{\phi}], \quad L_{\nabla \Phi} = E[L_{\phi}]. \]

**Remark 3.6.** Note that since \( S \) is now a random variable rather than a constant, \( L_{\Phi}, L_{\nabla \Phi}, B_{\nabla \Phi} \), all depend on \( S \) and therefore are random variables as well.

**Remark 3.7.** The assumption that \( \Omega \) is bounded is not very restrictive. Whenever a neural network numerically converges to some stationary point, which is the case for the majority of problems where neural networks can be applied successfully, the parameters can be chosen to only take values in a bounded region.

Furthermore, regularisation techniques as for example \( L^2 \)-regularisation outside a certain domain, can be used to essentially guarantee that the weights do not leave this domain, hence implying our assumption on \( \Omega \). A similar approach was used for example in Ge et al. [2016].

Assume that the (stochastic) gradient scheme in Algorithm 1 is applied to minimize the objective function \( \Phi \).

In the following examples we use Theorem 3.5 to set the step sizes of gradient descent (GD) (Example 3.8) in the case of a finite training set \( \mathcal{Z} = \{\zeta_1, \ldots, \zeta_N\} \) (in particular using \( M = N \)) and of stochastic gradient descent (SGD) (Example 3.9) with adaptive step-sizes (a state-of-the-art neural network training method first introduced in Duchi et al. [2011]) in the case of a general training set \( \mathcal{Z} \). In particular, the step sizes respectively hyper-parameters for the step sizes can be chosen depending on the computed estimates for \( L_{\nabla \Phi} \), such that the GD respectively SGD method are guaranteed to converge (in expectation). At the same time, these examples give bounds on the convergence rates.

**Example 3.8 (Gradient descent).** Assume that \( \mathcal{Z} = \{\zeta_1, \ldots, \zeta_N\} \) with equal probabilities and that in each step of the gradient method the true gradient of \( \Phi \) is computed, i.e. gradient descent and not a stochastic version of it is applied. Furthermore, assume that there exists

\[ 0 < B_{\Omega} < \infty \]

such that \( \sup_{j \geq 0} \|\Theta(j)\| < B_{\Omega} \). Choosing the step sizes \( h_j := \frac{L_{\nabla \Phi}}{L_{\nabla \Phi}}, \) the following inequality

\[ \Phi(\Theta(j)) - \Phi(\Theta(j+1)) \geq \frac{1}{2L_{\nabla \Phi}} \|\nabla \Phi(\Theta(j))\|^2, \]

is always satisfied as shown in Section 1.2.3 of Nesterov [2013]. Furthermore, it follows that for every \( n \in \mathbb{N} \) we have

\[ \min_{0 \leq j \leq n} \|\nabla \Phi(\Theta(j))\| \leq \frac{1}{\sqrt{n+1}} \left[ 2L_{\nabla \Phi} (\Phi(\Theta(0)) - \Phi^*) \right]^{1/2}, \]

where \( \Phi^* := \min_{\|\Phi\| \leq B_{\Omega}} \Phi(\Theta) \). In particular, for every tolerance level \( \varepsilon > 0 \) we have

\[ n + 1 \geq \frac{L_{\nabla \Phi}}{\|\Phi^*\|} (\Phi(\Theta(0)) - \varphi^*) \implies \min_{0 \leq j \leq n} \|\nabla \Phi(\Theta(j))\| \leq \varepsilon. \]

**Example 3.9 (Stochastic gradient descent).** Assume that the random variable \( S := \|\text{proj}_{\overline{1}}(Z)\| \) lies in \( L^2(\mathbb{P}) \), i.e. \( E[S^2] < \infty \). Furthermore, assume that there exists \( 0 < B_{\Omega} < \infty \) such that \( \sup_{j \geq 0} \|\Theta(j)\| < B_{\Omega} \). Choose the adaptive step-sizes \( h_j \) of the stochastic gradient method in Algorithm 1 as

\[ h_j := \alpha \frac{\beta + \sum_{i=1}^{j-1} \|G_j\|^2}{\beta + \sum_{i=1}^{j-1} \|G_j\|^2} + \varepsilon. \]
for constants $\alpha, \beta > 0, \epsilon \in [0, 1/2)$ that satisfy $2\alpha L_{\nabla \psi} < \beta^{2+\epsilon}$. One possibility is to use $\alpha = \frac{1}{2}, \epsilon = 0, \beta = L_{\nabla \psi} + \epsilon$ for some $\epsilon > 0$. Then there exists a constant $C$ depending on $L_{\nabla \psi}, B_{\nabla \psi}, \alpha, \beta, \epsilon, \Phi(\Theta^{(0)}) - \Phi^*$, where $\Phi^* := \min_{\|\theta\| \leq L} \Phi(\theta)$, such that for every $n \in \mathbb{N}$,

$$
\mathbb{E}\left[ \min_{0 \leq j \leq n} \| \nabla \Phi(\Theta^{(j)}) \|^{1-2\epsilon} \right] \leq \frac{C}{(n+1)^{\frac{1}{2}-\epsilon}}.
$$

In particular, for every tolerance level $\eta > 0$ we have

$$
n + 1 \geq \left( \frac{C}{\eta} \right)^{\frac{1}{2-2\epsilon}} \quad \implies \quad \mathbb{E}\left[ \min_{0 \leq j \leq n} \| \nabla \Phi(\Theta^{(j)}) \|^{1-2\epsilon} \right] \leq \eta.
$$

**Remark 3.10.** The exact value of the constant $C$ can be looked up in Theorem 4 and its proof in Li and Orabona [2019].

The statements of Example 3.8 and 3.9 are proven in the Appendix. They are just one possibility how the estimates of Theorem 3.5 can be used in practice. Similarly, the step sizes of other gradient descent methods can be chosen and convergence rates can be computed using Theorem 3.5. In Baes et al. [2019], the same result as presented in Example 3.8 was already used to provide a bound on the convergence rate to a stationary point of their algorithm.

### 4 Deep neural networks as controlled ODEs

#### 4.1 Framework & definitions

We introduce a slightly different notation than in the previous section. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$. Let $\ell \in \mathbb{N}$ be the fixed dimension of the problem. In particular, if we want to define a neural network $\mathcal{N}$ mapping some input $x$ of dimension $\ell_0 \in \mathbb{N}$ to an output $\mathcal{N}(x)$ of dimension $\ell_2 \in \mathbb{N}$ with $\ell_1 \in \mathbb{N}$ the maximal dimension of some hidden layer $\tilde{\mathcal{N}}(x)$, then we set $\ell := \max\{\ell_0, \ell_1, \ell_2\}$. We use “zero-embeddings” to write (by abuse of notation) $x, \mathcal{N}(x), \tilde{\mathcal{N}}(x) \in \mathbb{R}^\ell$, i.e. we identify $x \in \mathbb{R}^{\ell_0}$ with $(x^\top, 0)^\top \in \mathbb{R}^\ell$. This is important, since we want to describe the evolution of an input through a neural network to an output by an ODE, which means that the dimension has to be fixed and cannot change. To do so, we fix $d \in \mathbb{N}$ and define for $1 \leq i \leq d$ vector fields

$$V_i : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \to \mathbb{R}^\ell, (\theta, t, x) \mapsto V_i^\theta(t, x),$$

which are càdlàg in the second variable and Lipschitz continuous in $x$. Furthermore, we define scalar càdlàg functions for $1 \leq i \leq d$, which we refer to as controls

$$u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}, t \mapsto u_i(t),$$

which are assumed to have finite variation (also called bounded variation) and start at 0, i.e. $u_i(0) = 0$. With these ingredients, we can define the following controlled ordinary differential equation (controlled ODE)

$$dX^\theta_t = \sum_{i=1}^d V_i^\theta(t, X^\theta_{t-})du_i(t), \quad X^\theta_0 = x,$$

where $x \in \mathbb{R}^\ell$ is the starting point, respectively input to the “neural network”. We fix some $T > 0$. $X^\theta, x$ is called a solution of (3), if it satisfies for all $0 \leq t \leq T$,

$$X^\theta_t = x + \sum_{i=1}^d \int_0^t V_i^\theta(s, X^\theta_{s-})du_i(s).$$

(4)

Then (4) describes the evolution of the input $x$ through a “neural network” to the output $X^\theta, x$. Here, the “neural network” is defined by $V_i^\theta$ and $u_i$ for $1 \leq i \leq d$.

**Remark 4.1.** The assumption on $u_i$ to have finite variation is needed for the integral (4) to be well defined. Indeed, a deterministic càdlàg function of finite variation is a special case of a semimartingale, whence we could also take $u_i$ to be semimartingales.
Before we discuss this framework, we define the loss functions similarly to Section 3.1. Let $Z \subset \mathbb{R}^k \times \mathbb{R}^k$ be the set of (0-embedded) training samples, again equipped with a sigma algebra and a probability measure $P$. Let $Z \sim P$ be a random variable. For a fixed function $g : \mathbb{R}^l \times \mathbb{R}^k \rightarrow \mathbb{R}$, $(x, y) \rightarrow g(x, y)$, we define the loss (or objective or cost) function by

$$
\varphi : \Omega \times Z \rightarrow \mathbb{R}, \ (\theta, (x, y)) \mapsto g(X^\theta_{T-}, y), \quad \text{and} \quad \Phi : \Omega \rightarrow \mathbb{R}, \ \theta \mapsto \mathbb{E}[\varphi(\theta, Z)].
$$

The framework (4) is much more general and powerful than the standard neural network definition. In Example C.1 we show that the neural network $N_{\Theta_{m+1}}$ defined in (2) is a special case of the controlled ODE solution (4). This example clarifies why we speak of a solution $X^\theta_{x}$ of (4) as a “neural network”, respectively the evolution of the input $x$ through a neural network. If $u_i$ respectively $u_i$ are not pure step functions, (4) defines a neural network of “infinite depth”, which we refer to as continuously deep neural networks. Their output can be approximated using a stepwise scheme to solve ODEs. Doing this, the continuously deep neural network is approximated by a deep neural network of finite depth. Using modern ODE solvers with adaptive step sizes as proposed in Chen et al. [2018], the depth of the approximation and the step sizes change depending on the wanted accuracy and the input.

Continuously deep neural networks are already used in practice. The neural ODE introduced in Chen et al. [2018], is an example of such a continuously deep neural network that can be described in our framework by (4), when choosing $d = 1$ and $u_1(t) = t$, i.e. $d u_1(t) = dt$. However, our framework allows to describe more general architectures, which combine jumps (as occurring in Example C.1 with continuous evolutions as in Chen et al. [2018]). One example of such an architecture is the ODE-RNN introduced in Rubanova et al. [2019]. Furthermore, allowing $u_i$ to be semimartingales instead of deterministic processes of finite variation, neural SDE models as described e.g. in Liu et al. [2019], Jia and Benson [2019], Peluchetti and Favaro [2019], Tzen and Raginsky [2019] are covered by our framework (4).

4.2 Gradient and existence of solutions

Although we are in a deterministic setting, it is reasonable to make use of Itô calculus (also called stochastic calculus) in the above framework, since integrands are predictable. See for instance Protter [1992] for an extensive introduction. We make use of the typical differential notation that is common in stochastic calculus (as for example in (3)) and we treat our ODEs with methods for stochastic differential equations (SDEs). Again we emphasize that all $u_i$ could be general semimartingales.

First we note that by Theorem 7 of Chapter V in Protter [1992], a solution of (4) exists and is unique, given that all $V^\theta_i(t, x)$ are Lipschitz continuous in $x$. Starting from (4), we derive the ODE describing the first derivative of $X^\theta_i$ with respect to $\theta$. For this, let us define $\partial X^\theta_i := \frac{\partial X^\theta_i}{\partial \theta}$, $\partial V^\theta_i := \frac{\partial V^\theta_i}{\partial \theta}$ and for $a, b \in \{x, \theta\}$ we use the standard notation $\partial_a V^\theta_i := \frac{\partial V^\theta_i}{\partial a}(t, x)$ and $\partial_{ab} V^\theta_i := \frac{\partial}{\partial a} \frac{\partial}{\partial b} V^\theta_i(t, x)$. Assuming that all required derivatives of $V^\theta_i$ exist, we have

$$
\partial X^\theta_i = \frac{\partial X^\theta_i}{\partial \theta} = \sum_{i=1}^d \int_0^t \frac{\partial}{\partial \theta} \left( V^\theta_i(s, X^\theta_{s-}) \right) du_i(s)
$$

$$
= \int_0^t \sum_{i=1}^d \left( \partial V^\theta_i(s, X^\theta_{s-}) + \partial_x V^\theta_i(s, X^\theta_{s-}) \partial X^\theta_{s-} \right) du_i(s).
$$

Therefore, we obtain the following controlled ODE (with differential notation)

$$
d \partial X^\theta_i = \sum_{i=1}^d \left( \partial V^\theta_i(t, X^\theta_{t-}) + \partial_x V^\theta_i(t, X^\theta_{t-}) \partial X^\theta_{t-} \right) du_i(t),
$$

$$
\partial X^\theta_0 = 0 \in \mathbb{R}^{l \times n}.
$$

We remark that (5) is a linear ODE, and therefore, by Theorem 7 of Chapter V in Protter [1992], a unique solution exists, given that all $\partial V_i$ and $\partial_x V_i$ are uniformly bounded.
4.3 Lipschitz regularity in the setting of controlled ODE

In the following, we provide similar results for the controlled ODE setting as for the standard DNN setting in Section 3. The proofs are again given in Appendix D.

Let us denote the total variation process (cf. Chapter I.7 Protter [1992]) of \( u_i \) as \( |u_i| \). We then define \( v := \sum_{i=1}^d |u_i| \), which is an increasing function of finite variation with \( v(0) = 0 \). Furthermore, we define \( B_v := v(T) \) and note that \( \sum_{i=1}^d \int_0^T |u_i| = B_v \).

With this we are ready to state our main results of this section. We start with the equivalent result to Theorem 3.2, giving bounds on the Lipschitz constants of the neural network and its gradient.

**Theorem 4.2.** Let \( \Omega \) be non-empty and open. We assume that there exist constants \( B_V, B_{\partial_0 V}, B_{\partial_0 e V}, B_{\partial_0 x V}, B_{\partial_x e V} \geq 0 \) and \( p_0, p_\theta, p_{x\theta}, p_{0x}, p_{xx} \in \mathbb{R} \) such that for all \( 1 \leq i \leq d \), \( \theta \in \Omega \), \( 0 \leq t \leq T \) and \( x \in \mathbb{R}^\ell \) we have

\[
\|V_i^\theta(t, x)\| \leq B_V (1 + \|x\|), \quad \text{and} \quad \|\partial_\theta V_i^\theta(t, x)\| \leq B_{\partial_0 V} (1 + \|x\|^{p_\theta}),
\]

and similarly for \( \partial_0 V_i, \partial_x V_i, \partial_{x\theta} V_i \) and \( \partial_{xx} V_i \). We also assume that for any \( 0 \leq i \leq d \), \( 0 \leq t \leq T \) and \( \theta \in \Omega \) the map

\[
\mathbb{R}^\ell \rightarrow \mathbb{R}^\ell, \quad x \mapsto V_i^\theta(t, x)
\]

is Lipschitz continuous with constants \( L_{V_i} \) independent of \( i, t \) and \( \theta \). Then, for any fixed training sample \( (x, y) \in \mathcal{Z} \), the neural network output \( X_T^{\theta, x} \) is uniformly bounded in \( \Omega \) by a constant \( B_X \) and the map and its gradient

\[
\Omega \rightarrow \mathbb{R}^\ell, \quad \theta \mapsto X_T^{\theta, x}, \quad \text{and} \quad \Omega \rightarrow \mathbb{R}^{\ell \times n}, \quad \theta \mapsto \partial_\theta X_T^{\theta, x},
\]

are Lipschitz continuous on \( \Omega \) with constants \( L_X \) and \( L_{\partial X} \). This also implies that \( \partial_\theta X_T^{\theta, x} \) is uniformly bounded by \( B_{\partial X} = B_X \). Upper bounds for these constants can be computed as

\[
B_X = (\|x\| + B_V B_v) \exp(B_V B_v), \\
L_X = B_{\partial_0 V} (1 + B_X^{p_\theta} B_v) \exp(L_{V_i} B_v), \\
C_{\theta \theta} = B_v \left[ B_{\partial_0 e V}(1 + B_X^{p_x}) + B_{\partial_0 x V}(1 + B_X^{p_{x\theta}}) L_X \right] + B_{\partial_0 e V}(1 + B_X^{p_x}) L_X + B_{\partial_0 x V}(1 + B_X^{p_{x\theta}}) L_X^2, \\
L_{\partial X} = C_{\theta \theta} \exp(L_{V_i} B_v).
\]

A remark about the bounding constants is given in Remark C.2. Next, we present the equivalent result to Theorem 3.5, giving bounds on the Lipschitz constant of the objective function and its gradient.

**Theorem 4.3.** We make the same assumptions as in Theorem 4.2. Furthermore, we assume that for any fixed \( y \in \text{proj}_y(\mathcal{Z}) \), the functions

\[
\mathbb{R}^\ell \rightarrow \mathbb{R}, \quad x \mapsto g(x, y), \quad \text{and} \quad \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell, \quad x \mapsto \frac{\partial}{\partial x} g(x, y)
\]

are Lipschitz continuous on \( \text{proj}_y(\mathcal{Z}) \) with constants \( L_g, L_{\partial_y g} \) independent of \( y \). Let \( Z \sim \mathbb{P} \) be a random variable following the distribution of the training samples and assume that the random variable \( S := ||\text{proj}_y(\mathcal{Z})|| \) lies in \( L^p(\mathbb{Z}, \mathcal{A}(\mathbb{Z}), \mathbb{P}) \), i.e. \( \mathbb{E}[S^p] < \infty \), where \( p := \max\{1, p_\theta, p_{x\theta} + p_\theta, p_{0x} + p_0, p_{xx} + 2p_0 \} \). Then, the objective function and its gradient

\[
\Omega \rightarrow \mathbb{R}, \quad \theta \mapsto \Phi(\theta), \quad \text{and} \quad \Omega \rightarrow \mathbb{R}^n, \quad \theta \mapsto \nabla \Phi(\theta),
\]

are Lipschitz continuous with Lipschitz constants \( L_\Phi \) and \( L_{\nabla \Phi} \). This also implies that \( \nabla \Phi \) is uniformly bounded by \( B_{\nabla \Phi} = L_\Phi \). Upper bounds for these constants can be computed as

\[
L_\Phi = \mathbb{E}[L_g L_X], \quad L_{\nabla \Phi} = \mathbb{E}[L_{\partial_y g} L_X^2 + L_g L_{\partial X}].
\]

A comparison of the Theorems of this Section with those of Section 3 is given in Remark C.3.

Theorem 4.3 can be used exactly like Theorem 3.5 to set the step sizes of gradient descent methods. In particular, if a (stochastic) gradient descent scheme as in Algorithm 1 is used, we get the same results as in Example 3.8 and 3.9.
5 Conclusion

We used two different frameworks to estimate the Lipschitz constants of the gradient of a neural network with respect to the full set of parameters. First the standard DNN definition, and secondly a novel modelization introduced in Cuchiero et al. [2019], where neural networks are expressed as solutions of controlled ODEs. Here, time appears as continuous depth. The first modelization has the main advantage of being practical and easily understandable. The second modelization gives the opportunity to study a wider range of neural networks, while having a simple mathematical expression. Methods borrowed from Stochastic Calculus were used for this purpose. Moreover, we discussed several applications and implications of our main results.

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Appendix

A Auxiliary results in the ordinary DNN setting

Remark A.1. Revisiting the proof of Theorem 3.2 we see that in each step applying Lemma B.1, we set $D = B_3$. However, in the $u$-th layer, $D$ needs only to be a bound for $|\theta_u|$, while the norm of the entire parameter vector has to satisfy $\|\Theta_m+1\| < B_3$. Therefore, we can replace $B_3$ by $D_u \geq 0$ in the constants for the $u$-th layer in Theorem 3.2. Doing this for all layers, including the last one, $L_N$ and $L_{\nabla N}$ become functions of $(D_1, \ldots, D_{m+1})$, where the constraint $\sum_{u=1}^{m+1} D_u^2 < B_3^2$ has to be satisfied. Therefore, computing upper bounds of the Lipschitz constants now amounts to solving the optimization problems

$$\max \left\{ L_N(D_1, \ldots, D_{m+1}) \left| \sum_{u=1}^{m+1} D_u^2 < B_3^2 \right. \right\} \quad \text{and} \quad \max \left\{ L_{\nabla N}(D_1, \ldots, D_{m+1}) \left| \sum_{u=1}^{m+1} D_u^2 < B_3^2 \right. \right\}.$$

Due to the iterative definition of $L_N$ and $L_{\nabla N}$, both objective functions are complex polynomials in a high dimensional constraint space where the maximum is achieved at some boundary point, i.e. where $\sum_{u=1}^{m+1} D_u^2 = B_3^2$. In particular, numerical methods have to be used to solve these optimization problems.

Example A.2. We assume to have a 1-dimensional input $\zeta_x = 0$ and use $m \in \mathbb{N}$ layers each with only 1 hidden unit. Furthermore we use as activation function a smoothed version of $\sigma(x) := -cR1_{[x \leq -R]} + cx1_{[-R < x < R]} + cR1_{[x \geq R]}$, for some $c > 0$ and $R > e^m B_3^m$. Then $\sigma'_{\max} = c$. We define the 1-dimensional weights $\alpha_u := A(u) \in \mathbb{R}$ and $\beta_u := b(u) \in \mathbb{R}$. Then we have $\Theta := \Theta_m = (\alpha, \beta)$ for $\alpha = (\alpha_u)_{u=1}^m, \beta = (\beta_u)_{u=1}^m$. We choose $\Theta = (\alpha, \beta)$ with $\alpha = (0, \alpha_2, \ldots, \alpha_m), \beta = (\beta_1, 0, \ldots, 0)$ and $\Theta = (\tilde{\alpha}, \tilde{\beta})$ with $\tilde{\alpha} = \alpha, \tilde{\beta} = -\beta$, such that $\|\Theta\|, |\Theta| < B_3$. Then we have

$$\|\Theta - \tilde{\Theta}\| = 2\beta_1,$$

$$\|N_m(\Theta) - N_m(\tilde{\Theta})\| = 2\beta_1 e^m \prod_{u=2}^m \alpha_u = 2e^m \prod_{u=2}^m \alpha_u\|\Theta - \tilde{\Theta}\|.$$

Therefore,

$$L_{N_m} \geq (\sigma'_{\max})^m (\prod_{u=2}^m \alpha_u (\sum_{u=2}^m (\alpha_u)^2 < B_3^2)),$$

which is in line with Remark A.1.

B Proofs in the ordinary DNN setting

Before we start to prove the theorems, we establish some helpful results.

Lemma B.1. Let $n_1, n_2, n_3 \in \mathbb{N}_{> 0}$, let $C \in \mathbb{R}^{n_3 \times n_2}$, $d \in \mathbb{R}^{n_2}$, $l_2 = n_3(n_2 + 1)$ and let $\lambda \in \mathbb{R}^2$ be a flattened version of $(C, d)$. We restrict $\lambda$ to be an element of $\mathcal{L} := \{ \lambda \in \mathbb{R}^{l_2} \mid \|\lambda\| < D \}$, for some $D > 0$. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a function with bounded first and second derivatives, i.e. there exist $c_1, c_2 \geq 0$ such that for all $x \in \mathbb{R}$ we have $|\psi'(x)| \leq c_1$ and $|\psi''(x)| \leq c_2$. Let $\psi : \mathbb{R}^{n_3} \rightarrow \mathbb{R}^{n_3}, x \mapsto (\psi(x_1), \ldots, \psi(x_{n_3}))$ and define $\Psi_{\lambda} : \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3}, x \mapsto \psi(Cx + d)$. Furthermore, let $l_1 \in \mathbb{N}_{\geq 0}$ and $\kappa \in \mathbb{K} \subset \mathbb{R}^{l_1}$. Let $\rho_\kappa : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}, x \mapsto \rho_\kappa(x)$ be a function depending on the parameters $\kappa$. Let $\mu = (\kappa, \lambda) \in \mathcal{M} := \mathbb{K} \times \mathcal{L} \subset \mathbb{R}^{l_1}$, where $l = l_1 + l_2$. For a fixed $\zeta \in \mathbb{R}^{n_3}$ we define the function

$$\chi : \mathcal{M} \rightarrow \mathbb{R}^{n_3}, \mu = (\kappa, \lambda) \mapsto \chi(\mu) := \Psi_{\lambda}(\rho_\kappa(\zeta)).$$

We use the notation $\Psi'_{\lambda}(x) := \frac{\partial}{\partial x} \Psi_{\lambda}(x), \nabla \Psi_{\lambda}(x) := \left[ \frac{\partial}{\partial x_j} (\Psi_{\lambda}(x)) \right]_{i,j}$ and similar for $\nabla \rho_\kappa(\zeta)$. If $\kappa \mapsto \rho_\kappa(\zeta)$ is Lipschitz continuous with constant $L_1$ and $\kappa \mapsto \nabla \rho_\kappa(\zeta)$ with constant $L_2$ and if $\|\rho_\kappa(\zeta)\| \leq B_3$ and $\|\nabla \rho_\kappa(\zeta)\| \leq B_4$, where $0 \leq L_1, L_2, B_1, B_2 < \infty$, then we have that
i) \( \chi \) is Lipschitz continuous with constant \( L_\chi = c_1 \sqrt{D^2 L^2_1 + B^2_1 + 1} \).

ii) \( \nabla \chi \) is Lipschitz continuous with constant \( L_{\nabla \chi} = \sqrt{m_1 + m_2} \), where

\[
m_1 := \max \{ 3L_1^2 (c_1^2 n_3 + c_1^2 D^2 B_1^2) + 2c_1^2 D^2 L_1^2, c_1^2 (B_1^2 + 1)(3B_1^2 + 2) \},
\]

\[
m_2 := (n_3c_1 DL_2 + B_2c_2 D^2 L_1^2) + B_2^2 (n_3c_1 + Dc_2(B_1^2 + 1)^{1/2})^2,
\]

iii) the gradient \( \nabla \chi(\mu) \) of \( \chi \) is bounded by \( B_{\nabla \chi} = L_\chi \). If we also assume that \( \tilde{\psi} \) is bounded by \( 0 < B_3 < \infty \), i.e. for all \( x \in \mathbb{R} : |\tilde{\psi}(x)| \leq B_3 \), then \( \chi(\mu) \) is bounded by \( B_\chi = \sqrt{n_3 B_3} \).

**Proof of Lemma B.1.** Let \( \mu = (\kappa, \lambda), \bar{\mu} = (\bar{\kappa}, \bar{\lambda}) \in \mathcal{M} \) with \( \lambda = (C, d) \). For \( i \), we use that \( x \mapsto \psi(x) \) is Lipschitz with constant \( c_1 \) and compute

\[
\| \chi(\mu) - \chi(\bar{\mu}) \|^2 = \| \psi(C \rho_\kappa(C) + d) - \psi(\tilde{C} \rho_{\bar{\kappa}}(\zeta) + \bar{d}) \|^2
\]

\[
\leq c_1^2 (\| C \rho_\kappa(C) + d - C \rho_{\bar{\kappa}}(\zeta) - d \| + \| C \rho_{\bar{\kappa}}(\zeta) + d - \tilde{C} \rho_{\bar{\kappa}}(\zeta) - \bar{d} \|)^2
\]

\[
\leq c_1^2 (\| C \| \| \rho_\kappa(C) - \rho_{\bar{\kappa}}(\zeta) \| + \| \rho_{\bar{\kappa}}(\zeta) \| C - \tilde{C} \| + \| d - \bar{d} \|)^2
\]

\[
\leq c_1^2 (DL_1 \| \kappa - \bar{\kappa} \| + B_1 \| C - \tilde{C} \| + \| d - \bar{d} \|)^2
\]

\[
\leq c_1^2 (D^2 L_1^2 + B_1^2 + 1) \| \mu - \bar{\mu} \|^2,
\]

where we used the Cauchy–Schwarz inequality in the last step.

For \( ii) \) we first compute some partial derivatives of the functions under consideration with respect to \( C_{i,j}, 1 \leq i \leq n_3, 1 \leq j \leq n_2 \) and \( d_i, 1 \leq i \leq n_3 \) and \( \kappa \). Denoting by \( e_i \) the canonical basis vectors in \( \mathbb{R}^{n_3} \), one has

\[
\frac{\partial}{\partial C_{i,j}} \Psi_\lambda(x) = \tilde{\psi}'(C_i, x + d_i) e_j e_i,
\]

\[
\frac{\partial}{\partial d_i} \Psi_\lambda(x) = \tilde{\psi}'(C_i, x + d_i) e_i,
\]

\[
\frac{\partial}{\partial \kappa} \chi(\mu) = \Psi_\lambda'(\rho_\kappa(C)) \nabla \rho_\kappa(C),
\]

\[

\Psi_\lambda'(x) = \text{diag}(\tilde{\psi}'(C \kappa + d) C).
\]

We then compute the Lipschitz constants of the different partial derivatives of \( \chi \). As above, we use the triangle inequality extensively to get:

\[
\| \frac{\partial}{\partial C_{i,j}} \chi(\mu) - \frac{\partial}{\partial C_{i,j}} \chi(\bar{\mu}) \|^2 = \| \frac{\partial}{\partial C_{i,j}} \Psi_\lambda(\rho_\kappa(C)) - \frac{\partial}{\partial C_{i,j}} \Psi_\lambda(\rho_{\bar{\kappa}}(\zeta)) \|^2
\]

\[
\leq \left( \| \tilde{\psi}'(C_i, \rho_\kappa(C) + d_i) \| (\rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \right)
\]

\[
+ \left( \| \tilde{\psi}'(C_i, \rho_{\bar{\kappa}}(\zeta) + d_i) - \tilde{\psi}'(C_i, \rho_{\bar{\kappa}}(\zeta) + d_i) \| \| \rho_{\bar{\kappa}}(\zeta) \| \right)^2
\]

\[
\leq \left( c_1 \| (\rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \| + c_2 \| C_i \| \| \rho_\kappa(C) - \rho_{\bar{\kappa}}(\zeta) \| \| (\rho_{\bar{\kappa}}(\zeta))^j \| \right)
\]

\[
+ c_2 \| C_i - \tilde{C}_i \| \| \rho_\kappa(C) \| + \| d_i - \bar{d}_i \| \| (\rho_{\bar{\kappa}}(\zeta))^j \| \|^2
\]

\[
\leq 3c_2^2 \| (\rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \|^2 + 3c_2^2 \| C_i \|^2 \| \rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \|^2
\]

\[
+ 3c_2^2 \| C_i - \tilde{C}_i \| \| \rho_\kappa(C) \| + \| d_i - \bar{d}_i \| ^2 \| (\rho_{\bar{\kappa}}(\zeta))^j \|^2,
\]

where in the last equation we used the Cauchy–Schwarz inequality. Summing over \( j \), yields

\[
\sum_{j=1}^{n_2} \left\| \frac{\partial}{\partial C_{i,j}} \chi(\mu) - \frac{\partial}{\partial C_{i,j}} \chi(\bar{\mu}) \right\|^2 \leq 3c_1^2 \| (\rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \|^2 + 3c_2^2 \| C_i \|^2 \| (\rho_\kappa(C) - (\rho_{\bar{\kappa}}(\zeta)))_j \|^2
\]

\[
+ 3c_2^2 \| C_i - \tilde{C}_i \| \| (\rho_{\bar{\kappa}}(\zeta))^j \| ^2 \| (\rho_{\bar{\kappa}}(\zeta))^j \|^2.
\]
Summing this expression over $i$, again using the norm (1), and using Cauchy–Schwarz for the last term, we get
\[
\| \frac{∂}{∂ν} \chi(μ) - \frac{∂}{∂ν} \chi(μ) \|^2 = \sum_{i=1}^{n_3} \sum_{j=1}^{n_2} \| \frac{∂}{∂c_{i,j}} \chi(μ) - \frac{∂}{∂c_{i,j}} \chi(μ) \|^2 \\
\leq 3c_1^2L_1^2n_3\|\kappa - \bar{κ}\|^2 + 3c_2^2\|C\|^2L_1^2\|\kappa - \bar{κ}\|^2B_1^2 \\
+ 3c_2^2(B_1^2 + 1)\|C_{i,\cdot} - \bar{C}_{i,\cdot}\|^2 + \|d_i - \bar{d}_i\|^2B_1^2 \\
\leq 3L_1^2(c_1^2n_3 + c_2^2D^2B_1^2)\|\kappa - \bar{κ}\|^2 + 3c_2^2(B_1^2 + 1)B_1^2\|\lambda - \bar{λ}\|^2.
\]
With a very similar (but slightly easier) computation we get
\[
\| \frac{∂}{∂ν} \chi(μ) - \frac{∂}{∂ν} \chi(μ) \|^2 \leq 2c_2^2(D^2L_1^2\|\kappa - \bar{κ}\|^2 + (B_1^2 + 1)\|\lambda - \bar{λ}\|^2).
\]
Combining these two results we have
\[
\| \frac{∂}{∂ν} \chi(μ) - \frac{∂}{∂ν} \chi(μ) \|^2 \leq (3L_1^2(c_1^2n_3 + c_2^2D^2B_1^2) + 2c_2^2D^2L_1^2)\|\kappa - \bar{κ}\|^2 \\
+ c_2^2(B_1^2 + 1)(3B_1^2 + 2)\|\lambda - \bar{λ}\|^2 \tag{8}
\]
Now we compute the Lipschitz constant of the last part of the gradient of $\chi$. We proceed similarly to before, using the triangle inequality, yielding
\[
\| \frac{∂}{∂ν} \chi(μ) - \frac{∂}{∂ν} \chi(μ) \|^2 = \| \Psi'_\lambda(ρ_κ(ζ)) \nabla ρ_κ(ζ) - \Psi'_\lambda(ρ_κ(ζ)) \nabla ρ_κ(ζ) \|^2 \\
\leq (\| \Psi'_\lambda(ρ_κ(ζ)) \|\| \nabla ρ_κ(ζ) - \nabla ρ_κ(ζ) \| + \| \Psi'_\lambda(ρ_κ(ζ)) - \Psi'_\lambda(ρ_κ(ζ)) \|\| \nabla ρ_κ(ζ) \|)^2 \\
\leq (n_3C_1DL_2\|\kappa - \bar{κ}\| + B_2\| \Psi'_\lambda(ρ_κ(ζ)) - \Psi'_\lambda(ρ_κ(ζ)) \|)^2.
\]
We compute the second term as
\[
\| \Psi'_\lambda(ρ_κ(ζ)) - \Psi'_\lambda(ρ_κ(ζ)) \| = \| \text{diag}(ψ'(Cρ_κ(ζ) + d))C - \text{diag}(ψ'(Cρ_κ(ζ) + d))C \| \\
\leq \| \text{diag}(ψ'(Cρ_κ(ζ) + d)) \|\| C - \bar{C} \| \\
+ \| \text{diag}(ψ'(Cρ_κ(ζ) + d) - ψ'(Cρ_κ(ζ) + d)) \|\| C \| \\
+ \| \text{diag}(ψ'(Cρ_κ(ζ) + d) - ψ'(Cρ_κ(ζ) + d)) \|\| C \| \\
\leq n_3C_1\| C - \bar{C} \| + c_2\| C \|\| ρ_κ(ζ) - \bar{ρ}_κ(ζ) \|\| D \| \\
+ D\| \sum_{i=1}^{n_3} c_2^2(\|C_{i,\cdot} - \bar{C}_{i,\cdot}\|\| ρ_κ(ζ) \| + \|d_i - \bar{d}_i\|)^2 \\
\leq n_3C_1\| C - \bar{C} \| + c_2D^2L_1\|\kappa - \bar{κ}\| + Dc_2\sqrt{B_1^2 + 1}\|\lambda - \bar{λ}\| \\
\leq (n_3C_1 + Dc_2(B_1^2 + 1)^{1/2})\|\lambda - \bar{λ}\| + c_2D^2L_1\|\kappa - \bar{κ}\|,
\]
where we used Cauchy–Schwarz in the second last step and $\| C - \bar{C} \| \leq \|\lambda - \bar{λ}\|$ in the last step. Inserting this in the previous inequality yields
\[
\| \frac{∂}{∂ν} \chi(μ) - \frac{∂}{∂ν} \chi(μ) \|^2 \leq \left( (n_3C_1DL_2 + B_2c_2D^2L_1)\|\kappa - \bar{κ}\| \\
+ B_2(n_3C_1 + Dc_2(B_1^2 + 1)^{1/2})\|\lambda - \bar{λ}\| \right)^2 \\
\leq \left( (n_3C_1DL_2 + B_2c_2D^2L_1)^2 \\
+ B_2^2(n_3C_1 + Dc_2(B_1^2 + 1)^{1/2})^2 \right)\|\mu - \bar{μ}\|^2 \tag{9}
\]
Combining (8) and (9) we arrive at
\[
\| \nabla χ(μ) - \nabla χ(μ) \|^2 = \| \frac{∂}{∂ν} χ(μ) - \frac{∂}{∂ν} χ(μ) \|^2 + \| \frac{∂}{∂ν} χ(μ) - \frac{∂}{∂ν} χ(μ) \|^2 \\
\leq (m_1 + m_2)\|\mu - \bar{μ}\|^2,
\]
which proves ii).
The second bound in iii) is immediate using the fact that \( \chi \) maps to \( \mathbb{R}^{n_3} \) and that each component of the resulting vector is bounded by \( B_3 \). For the first bound we remark that the Lipschitz constant is always an upper bound for the gradient, which completes the proof. □

**Remark B.2.** Under the same setting as in Lemma B.1, but using a function \( \psi : \mathbb{R}^{n_3} \to \mathbb{R} \) with constants \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{R}^{n_3} \) we have \( \| \frac{\partial^2}{\partial x^2} \psi(x) \| \leq c_1 \) and \( \| \frac{\partial^2}{\partial x^2} \psi(x) \| \leq c_2 \), we get exactly the same constants with \( n_3 = 1 \). Indeed, going through the proof again and replacing \( \psi \) wherever necessary, we first get the partial derivatives with \( \psi'(x) := \frac{\partial}{\partial x} \psi(x) \).

\[
\begin{align*}
\frac{\partial}{\partial x^i} \psi_A(x) &= \psi'(Cx + d)x_i e_i, \\
\frac{\partial}{\partial d} \psi_A(x) &= \psi'(Cx + d)e_i, \\
\frac{\partial}{\partial \xi} \chi &= \Psi_A' \rho_c(\xi) \nabla \rho_c(\xi), \\
\Psi_A'(x) &= \psi'(Cx + d)C.
\end{align*}
\]

Using them in the subsequent steps, we see that we get exactly the same constants with \( n_3 = 1 \).

With Lemma B.1 we can now prove Theorems 3.2 and 3.5 iteratively.

**Proof of Theorem 3.2.** First, we apply Lemma B.1 with \( u := 1, n_1 := \ell_0, n_2 := \ell_{u-1}, n_3 := \ell_u, (C, d) := b_u, \psi := \bar{\sigma}_u, \ell_1 := 0, \rho_u := id \) and \( \xi := \xi_x \). Hence, we have \( D := B_3, c_1 := \sigma_{\text{max}}' \) and \( c_2 := \sigma_{\text{max}}'' \). Therefore, \( N_1 \) and \( \nabla N_1 \) are Lipschitz continuous and bounded with the Lipschitz constants \( L_{N_1}, L_{\nabla N_1} \), and the bounding constants \( B_{N_1} \) and \( B_{\nabla N_1} \), as given in Theorem 3.2. Next, we apply Lemma B.1 iteratively, where for \( 2 \leq u \leq m \) we use the same variables as above except for \( \ell_1 := \ell_{u-1}, \rho_u := N_{u-1} \) and \( \ell := \Theta_{u-1}, \rho_u := N_{u-1} \) and \( \kappa := \Theta_{u-1}, \) yielding \( L_1 := L_{N_{u-1}}, L_2 := L_{\nabla N_{u-1}}, B_1 := B_{N_{u-1}}, B_2 := B_{\nabla N_{u-1}} \). It follows that \( N_u \) and \( \nabla N_u \) are Lipschitz and bounded with constants \( L_{N_u}, L_{\nabla N_u}, B_{N_u} \) and \( B_{\nabla N_u} \) as in Theorem 3.2. To get the Lipschitz constants for \( N_1 \) and \( \nabla N \), we apply Lemma B.1 another time with the same variables for \( u := m + 1 \), except for \( \psi := id \), yielding \( c_1 := 1, c_2 := 0 \) and \( B_3 := \infty \). We conclude that \( N \) and \( \nabla N \) are Lipschitz continuous and that \( \nabla N \) is also bounded with the constants given in Theorem 3.2. □

**Proof of Corollary 3.3.** The first inequality can easily be proven by induction. Furthermore, the following inequality can be shown by induction as well.

\[
\begin{align*}
\gamma_u &:= 2B_{\nabla N_{u-1}}^2 (\sigma_{\text{max}}'' B_{\Omega} \Theta_{u-1})^2 B_{\nabla N_{u-1}}^2 + B_{\nabla N_{u-1}}^2 (\ell \sigma_{\text{max}}' + B_{\Omega} \sigma_{\text{max}}'' \sqrt{2 B_{\nabla N_{u-1}}^2 + 1})^2, \\
L_{N_u}^2 &\leq (2^2 \sigma_{\text{max}}'' B_{\Omega}^{u-1})^2 (\sigma_{\text{max}}'')^2 (S^2 + 1) (3S^2 + 2) \\
&\quad + \sum_{k=1}^{u-1} (2^2 \sigma_{\text{max}}'' B_{\Omega}^2)^{k-1} (\sigma_{\text{max}}'')^2 (S^2 + 1) (3S^2 + 2).
\end{align*}
\]

For the equations of \( L_{N_u}^2 \), the geometric sum equality for \( q \neq 1 \), \( \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \), can be used to rewrite the sum. Using this together with quite rough approximations, the asymptotic approximation of \( L_{N_u}^2 \) can be shown. □

**Proof of Theorem 3.5.** We first prove that for a random variable \( Z = (Z_x, Z_y) \sim \mathcal{P} \), the function \( \phi : \mathbb{R}^{d_{m+1}} \to \mathbb{R}, \quad \Theta \mapsto \varphi(\Theta, Z) \), and its gradient \( \nabla \phi := \nabla \varphi \phi \) are Lipschitz continuous with integrable constants. To see this, we proceed as in the proof of Theorem 3.2 for \( 1 \leq u \leq m \). Then, to get the Lipschitz constants of \( \phi \) and \( \nabla \phi \), we apply Lemma B.1 as in the final step of the proof of Theorem 3.2, but using \( \psi := g(\cdot, Z_y) \), \( c_1 := g_{\text{max}}' \) and \( c_2 := g_{\text{max}}'' \). With Remark B.2 we get the Lipschitz and bounding constants \( L_{\phi}, L_{\nabla \phi} \) and \( B_{\nabla \phi} \) as defined in Theorem 3.5. From Corollary 3.3, we deduce that there exist constants \( a_S, b_S \in \mathbb{R} \) such that

\[ 0 \leq L_{\phi}, L_{\nabla \phi}, B_{\nabla \phi} \leq a_S S^2 + b_S. \]
Since \( \mathbb{E}[S^2] < \infty \), it follows that \( L_\Phi, L_{\nabla \Phi}, B_{\nabla \Phi} \in L^1(\mathbb{P}) \). In the remaining part of the proof we show that we get the constants for \( \Phi \) and \( \nabla \Phi \) as in Theorem 3.5. Let \( \Theta, \bar{\Theta} \in \Omega \). Then we have by the Lipschitz continuity of \( \phi \) that
\[
\|\Phi(\Theta) - \Phi(\bar{\Theta})\| = \|\mathbb{E}[\varphi(\Theta, Z)] - \mathbb{E}[\varphi(\bar{\Theta}, Z)]\|
\leq \mathbb{E} \|\varphi(\Theta, Z) - \varphi(\bar{\Theta}, Z)\|
= \mathbb{E}[L_\phi] \|\Theta - \bar{\Theta}\| = L_\Phi \|\Theta - \bar{\Theta}\|
\]
Next we remark that
\[
\nabla \Phi(\Theta) = \nabla_\Theta \mathbb{E}[\varphi(\Theta, X)] = \mathbb{E}[\nabla_\Theta \varphi(\Theta, X)],
\]
where we used the dominated convergence theorem in the second equality. Indeed, dominated convergence can be used, since \( \nabla_\Theta \varphi(\Theta, X) \) exists and since all directional derivatives (and sequences converging to them) can be bounded by the following integrable random variable
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\varphi(\Theta + \varepsilon \Theta, Z) - \varphi(\Theta, Z)) \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} L_\phi \|\Theta\| \leq L_\phi B_\Omega.
\]
This also implies that a (vector-valued) sequence converging to the gradient \( \nabla_\Theta \varphi(\Theta, X) \) can be bounded by an integrable random variable, yielding that the assumptions for dominated convergence are satisfied. Hence, we have that
\[
\|\nabla \Phi(\Theta)\| = \|\mathbb{E}[\nabla_\Theta \varphi(\Theta, Z)]\| \leq \mathbb{E} \|\nabla_\Theta \varphi(\Theta, Z)\| \leq \mathbb{E}[B_{\nabla \Phi}] = B_{\nabla \Phi}
\]
and
\[
\|\nabla \Phi(\Theta) - \nabla \Phi(\bar{\Theta})\| = \|\mathbb{E}[\nabla_\Theta \varphi(\Theta, Z) - \nabla_\Theta \varphi(\bar{\Theta}, Z)]\|
\leq \mathbb{E} \|\nabla_\Theta \varphi(\Theta, Z) - \nabla_\Theta \varphi(\bar{\Theta}, Z)\|
\leq \mathbb{E}[L_{\nabla \Phi}] \|\Theta - \bar{\Theta}\| = L_{\nabla \Phi} \|\Theta - \bar{\Theta}\|,
\]
which completes the proof.

**Remark B.3.** If \( \text{proj}_x(Z) \) is bounded by \( B_S > 0 \), then in Theorem 3.5, \( S \) can be chosen to be this bound and we get exactly the same constants, but in this case \( L_\Phi, L_{\nabla \Phi}, B_{\nabla \Phi} \) are also constants rather than random variables.

We can now use Theorem 3.5 to prove the two examples.

**Proof of Example 3.8.** In this setting of a finite training set with equal probabilities we have (cf. Remark 3.1) for \( \Theta \in \Omega \),
\[
\nabla \Phi(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \nabla \phi(\Theta, \zeta_i).
\]
In particular, we can compute the true gradient of \( \Phi \). By the assumption \( \sup_{j \geq 0} \|\Theta(j)\| < B_\Omega \), we can use \( \Omega = \{\Theta \in \mathbb{R}^{d_m+1} \mid \|\Theta\| < B_\Omega\} \). Furthermore, since the training set is finite (and hence bounded), we can set \( S := \max_{1 \leq i \leq N} \|\zeta_i\| \leq \infty \), and get by Theorem 3.5 that \( \Phi \) and \( \nabla \Phi \) are Lipschitz continuous on \( \Omega \) with constants \( L_\Phi \) and \( L_{\nabla \Phi} \). The result then follows as outlined in Section 1.2.3 of Nesterov [2013].

**Proof of Example 3.9.** By the assumption \( \sup_{j \geq 0} \|\Theta(j)\| < B_\Omega \), we can use \( \Omega = \{\Theta \in \mathbb{R}^{d_m+1} \mid \|\Theta\| < B_\Omega\} \). Furthermore, since \( S = \|\text{proj}_x(Z)\| \in L^2 \), Theorem 3.5 yields, that \( \Phi \) and \( \nabla \Phi \) are Lipschitz continuous on \( \Omega \) with constants \( L_\Phi \) and \( L_{\nabla \Phi} \). We establish the assumptions of Theorem 4 in Li and Orabona [2019], which in turn establishes our result. We set \( f := \Phi \) and remark first that their results still hold when restricting \( \Phi \) and \( \nabla \Phi \) to be Lipschitz only on the subset \( \Omega \). Indeed, by the assumption \( \sup_{j \geq 0} \|\Theta(j)\| < B_\Omega \) we know that \( \Theta \) stays within \( \Omega \) for the entire training process. In the remainder of the proof we show that all needed assumptions H1, H3 and H4’ (as defined in Li and Orabona [2019]) are satisfied. H1, the Lipschitz continuity of \( \nabla \Phi \), holds as outlined above. Let \( Z_1, \ldots, Z_M \sim \mathbb{P} \) be independent and identically distributed random variables with the distribution of the training set. By the stochastic gradient method outlined in (1), in each step the approximation of the gradient \( \nabla \Phi(\Theta(j)) \) is given by the random variable
\[
G_j := G(\Theta(j); Z_1, \ldots, Z_M) := \frac{1}{M} \sum_{i=1}^{M} \nabla \phi(\Theta(j), Z_i).
\]
By (14) we have \( E[G_i] = \nabla \Phi \), yielding H3.

In the proof of Theorem 4 of Li and Orabona [2019], assumption H4' is only used for the proof of their Lemma 8. In particular, it is only used to show

\[
E \left[ \max_{1 \leq i \leq T} \| \nabla \Phi(\Theta_i) - G_i \|^2 \right] \leq \sigma^2 (1 + \log(T)),
\]

for a constant \( \sigma \in \mathbb{R} \). Instead of showing H4', we directly show that (15) is satisfied. We have

\[
E \left[ \max_{1 \leq i \leq T} \| \nabla \Phi(\Theta_i) - G_i \|^2 \right] \leq E \left[ \max_{1 \leq i \leq T} \left( 2 \| \nabla \Phi(\Theta_i) \|^2 + 2 \| G_i \|^2 \right) \right]
\]

\[
\leq 2H_v^2 + 2E \left[ \max_{1 \leq j \leq T} \frac{1}{M} \sum_{i=1}^{M} \| \nabla \phi_i(\Theta(\hat{\theta}_i), Z_i) \|^2 \right]
\]

\[
\leq 2H_v^2 + 2E[|B_v|^2] = \sigma^2,
\]

where in the second inequality we used Cauchy–Schwarz and in the last equality we used that \( E[|B_v|^2] < \infty \), since \( \hat{S} \in L^2 \). In particular, this implies that (15) is satisfied. For completeness we also remark that H2 holds as well, since \( \Phi \) is Lipschitz. Applying Theorem 4 of Li and Orabona [2019] concludes the proof. \( \square \)

C Auxiliary results in the controlled ODE setting

Example C.1. We define \( u \) as a step function and \( V^\theta \) as a stepwise (with respect to its second parameter) vector field

\[
u(t) := \sum_{i=1}^{m} \mathbb{I}_{[i,i+1)}(t) + (m + 1) \mathbb{I}_{[m+1,\infty)}(t)
\]

\[
V^\theta(t, x) := \sum_{i=1}^{m} \mathbb{I}_{(i-1,i)}(t) (\sigma_i(f_{\theta_i}(x)) - x) + \mathbb{I}_{(m,\infty)}(t) (f_{\theta_{m+1}}(x) - x)
\]

Here, \( \theta = (\theta_1, \ldots, \theta_{m+1}) \) is the concatenation of all the weights needed to define the affine neural network layers, and \( \sigma_i \) and \( f_{\theta_i} \) are defined as in Section 3.1. However, by abuse of notation, we assume that each \( f_{\theta_i} : \mathbb{R}^l \to \mathbb{R}^l \), using “0-embeddings” wherever needed and similar for \( \sigma_i \). Evaluating (4), which amounts to computing the (stochastic) integral with respect to a step function, we get

\[
X^\theta_{i} = x + \sum_{i=1}^{m} \mathbb{I}_{[i,i+1)}(t) V^\theta(i, X^\theta_{i-}) (u(i) - u(i-))
\]

\[
= x + \sum_{i=1}^{m+1} \mathbb{I}_{[i,i+1]}(t) \left( \sigma_i(f_{\theta_i}(X^\theta_{i-})) - X^\theta_{i-} \right),
\]

where we use \( \sigma_{m+1} := id \).

Solving the sum iteratively, we get for \( 1 \leq i \leq m + 1 =: T \),

\[
X^\theta_0 = x, \quad X^\theta_{i} = \sigma_i \circ f_{\theta_i} \circ \cdots \circ \sigma_1 \circ f_{\theta_1}(x),
\]

in particular, \( X^\theta_{i} \) is the output of the \( i \)-th layer of the neural network \( N_{\theta_{m+1}} \) defined in (2).

Remark C.2. If \( \partial_x V^\theta_i(t, x) \) is continuously differentiable with respect to \( \theta \), then Schwarz’s theorem, as for example outlined in Chapter 2.3 of Königsberger [2013], implies that \( \partial_\theta \partial_x V^\theta(t, x) = \partial_x \partial_\theta V^\theta(t, x) \).

In particular, the bounding constants \( B_{\partial_\theta \partial_x V} \) and \( B_{\partial_\theta V} \) are equal.

Remark C.3. Comparing the theorems of Section 4 to the theorems of Section 3, we see that here we did not make assumptions on the boundedness of \( \Omega \). As we discussed before, the controlled ODE setting (4) is a generalization of the setting in Section 3, hence, Theorem 4.2 and 4.3 can be applied to a classical DNN. Does this mean that the assumption of \( \Omega \) being bounded is in fact unnecessary? The answer is no, because for the assumptions (6) on the vector fields \( V_i \) to be satisfied in the case of DNN, it is necessary to assume that \( \Omega \) is bounded. In that sense, this assumption is now just hidden inside another assumption.

Furthermore, it is easy to see that the constants estimated in Theorem 3.2 and 3.5 are smaller than the respective constants that we get from Theorem 4.2 and 4.3.
D Proofs in the controlled ODE setting

For the proofs of the Lipschitz results we extensively use the following stochastic version of Grönwall’s Inequality, which is presented as Lemma 15.1.6 in Cohen and Elliott [2015].

**Lemma D.1.** Let \( Y \) be a (1-dimensional) càdlàg process, \( U \) an increasing real process and \( \alpha > 0 \) a constant. If for all \( 0 \leq t \leq T \),

\[
Y_t \leq \alpha + \int_0^t Y_s \, dU_s,
\]

then \( Y_t \leq \alpha \mathcal{E}(U)_t \) for all \( 0 \leq t \leq T \).

Here \( \mathcal{E}(U) \) is the stochastic exponential as defined in Definition 15.1.1 and Lemma 15.1.2 of Cohen and Elliott [2015]. Note also that \( 0 \leq \mathcal{E}(U)_t \leq \exp(U_t) \) holds, if no jump of \( U \) is smaller than \(-1\), i.e. \( \Delta U_s \geq -1 \) for all \( 0 \leq s \leq t \).

**Remark D.2.** If \( \partial_x V^\theta_i(t, x) \) is continuously differentiable with respect to \( \theta \), then Schwarz’s theorem, as for example outlined in Chapter 2.3 of Königsberger [2013], implies that \( \partial_\theta \partial_x V^\theta_i(t, x) = \partial_x \partial_\theta V^\theta_i(t, x) \). In particular, the bounding constants \( B_{\partial_\theta V} \) and \( B_{\partial_x V} \) are equal.

**Proof of Theorem 4.2.** Starting from (4) we get

\[
\|X^\theta_t\| \leq \|x\| + \sum_{i=1}^d \int_0^t V^\theta_i(s, X^\theta_s) \, du_i(s)
\]

\[
\leq \|x\| + \int_0^t \max_{1 \leq i \leq d} \|V^\theta_i(s, X^\theta_s)\| \, dv(s)
\]

\[
\leq \|x\| + \int_0^t B_V (1 + \|X^\theta_s\|) \, dv(s)
\]

\[
= \|x\| + B_V \|x\| + \int_0^t \|X^\theta_s\| \, d\tilde{v}(s),
\]

where \( \tilde{v} = B_V v \). Hence, Lemma D.1 implies that for \( 0 \leq t \leq T \),

\[
\|X^\theta_t\| \leq \left( \|x\| + B_V \|x\| \right) \mathcal{E}(B_V v)_t
\]

\[
\leq \left( \|x\| + B_V \|x\| \right) \exp(B_V v)_t = B_X,
\]

using the fact that all the jumps of \( u \) are positive, since \( u \) is increasing. In the following we do the same for (5) and (16), showing that the first and second derivatives of \( X^\theta_t \) with respect to \( \theta \) are bounded, which implies that \( \theta \mapsto X^\theta_T \) and \( \theta \mapsto \partial X^\theta_T \) are Lipschitz continuous on \( \Omega \) with these constants. Using all the given bounds and using that \( \|\partial_x V\| \) is bounded by the Lipschitz constant \( L_{V_x} \), we obtain from (5) the following inequality:

\[
\|\partial X^\theta_t\| \leq B_{\partial_x V} (1 + B_{V_x}^\theta) B_v + \int_0^t L_{V_x} \|\partial X^\theta_s\| \, dv(s).
\]

Hence, by Lemma D.1, we have for \( 0 \leq t \leq T \),

\[
\|\partial X^\theta_t\| \leq B_{\partial_x V} (1 + B_{V_x}^\theta) B_v \exp(L_{V_x} v)_t = L_X,
\]

which therefore is a Lipschitz constant of the map \( \theta \mapsto X^\theta_T \). Similarly, we need the corresponding ODE for the second derivative of \( X^\theta_t \) with respect to \( \theta \) in order to obtain the Lipschitz constant of the map \( \theta \mapsto \partial X^\theta_T \). Assuming that all needed derivatives of \( V^\theta_i \) exist, similarly to (5), we obtain the ODE for the second derivative

\[
d_{\theta \theta} X^\theta_t = \sum_{i=1}^d \left[ \partial_{\theta \theta} V^\theta_i \frac{\partial X^\theta}{\partial x} + \partial_{x \theta} V^\theta_i \frac{\partial X^\theta}{\partial x} \partial X^\theta_T + \partial_{x x} V^\theta_i \frac{\partial X^\theta}{\partial x} \partial X^\theta_T + \partial_{\theta x} V^\theta_i \frac{\partial X^\theta}{\partial x} \partial X^\theta_T + \partial_{x \theta} V^\theta_i \frac{\partial X^\theta}{\partial x} \partial X^\theta_T + \partial_{x x} V^\theta_i \frac{\partial X^\theta}{\partial x} \partial X^\theta_T \right] \, du_i(t),
\]

\[
\partial_{\theta \theta} X^\theta_0 = 0 \in \mathbb{R}^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}.
\]
Here, we have an equation for tensors of third order. We implicitly assume that for each term the correct tensor product is used, such that the term has the required dimension. Writing down the equation component wise clarifies which tensor products are needed. Observe that (16) is also a linear ODE, and therefore, by Theorem 7 of Chapter V in Protter [1992], a unique solution exists. Finally, for (16) we get

\[ \|\partial_{\theta\theta} X^\theta_t\| \leq C_{\theta\theta} + \int_0^t L_{V^\theta} \|\partial_{\theta\theta} X^\theta_s\| \, dv(s). \]

Hence, by Lemma D.1, we have for \( 0 \leq t \leq T \),

\[ \|\partial_{\theta\theta} X^\theta_t\| \leq C_{\theta\theta} \exp(L_{V^\theta} B_{\theta}) = L_{\partial X^\theta}, \]

which is therefore a Lipschitz constant of \( \theta \mapsto \partial X^\theta_T \).

**Proof of Theorem 4.3.** We first prove that for a random variable \( Z = (Z_x, Z_y) \sim P \), the function \( \phi: \Omega \to \mathbb{R}, \theta \mapsto \phi(\theta) := \varphi(\theta, Z) \),

and its gradient \( \nabla \phi := \nabla \varphi \) are Lipschitz continuous with integrable constants. In the following, \( L_X \) and \( L_{\partial X} \) are as defined in the proof of Theorem 4.2, except that \( \|x\| \) is now exchanged with \( S \) (in the definition of \( B_X \)). Let \( \theta, \bar{\theta} \in \Omega \). Then

\[ \|\phi(\theta) - \phi(\bar{\theta})\| = \|g(\theta, Z_x, Z_y) - g(\bar{\theta}, Z_x, Z_y)\| \]

\[ \leq L_g L_X \|\theta - \bar{\theta}\|, \]

which shows the first part of the claim. We define \( L_\phi := L_g L_X \) and also notice that, by Lipschitz continuity of \( \phi \), it follows that the gradient \( \nabla \phi \) is bounded by \( L_\phi \). Furthermore, we have

\[ \|\nabla \phi(\theta) - \nabla \phi(\bar{\theta})\| = \|\partial_z g(\theta, Z_x, Z_y) \partial X^\theta_T Z_x - \partial_z g(\bar{\theta}, Z_x, Z_y) \partial X^\bar{\theta}_T Z_x\| \]

\[ \leq (L_{\partial_z g} L_X^2 + L_g L_{\partial X}) \|\theta - \bar{\theta}\|, \]

where we used again the technique to introduce intermediate terms and split up the norm using the triangle inequality. Defining \( L_{\nabla \phi} := (L_{\partial z g} L_X^2 + L_g L_{\partial X}) \), this shows the second part of the claim. Since \( \mathbb{E}[S^p] < \infty \), it follows that \( L_\phi, L_{\nabla \phi} \in L^1(\mathbb{P}) \).

Using this and the same steps as in the second part of the proof of Theorem 3.5, we now get that \( \Phi \) and \( \nabla \Phi \) are Lipschitz continuous with constants \( L_\Phi = \mathbb{E}[L_\phi] \) and \( L_{\nabla \Phi} = \mathbb{E}[L_{\nabla \phi}] \) and that \( \nabla \Phi \) is bounded by \( B_{\nabla \phi} := L_{\Phi} \).