Signed Chord Length Distribution
Part II

Alexander Yu. Vlasov

Abstract
This paper continues description of applications of signed chord length distribution started in [Part I], arXiv:0711.4734 [math-ph]. It is shown simple relation between equation for some transfer integrals with source and target bodies and different geometrical distributions for union of this bodies. The union of disjoint bodies is always nonconvex object and for such a case derivatives of correlation function (used for definition of signed radii and chord lengths distributions) always produce (quasi)densities with negative values. Many equations used in this part are direct consequences of analogue formulas in [Part I].

Contents
1 Introduction to Part II
2 Distances and correlations
   2.1 Distribution of distances
   2.2 Correlation function
3 Signed matrix-valued distributions
   3.1 Integration by parts
   3.2 Some properties of distributions for two bodies
   3.3 Radii (signed matrix) density function
   3.4 Chord length (signed matrix) density function
   3.5 Signed $\lambda$–chords
4 Nonuniform case
5 Arbitrary paths

1 INTRODUCTION TO PART II

In first part [Part I] was considered a basic theory of signed chord length distribution. Here is discussed an extension for specific case. Let us consider two bodies $\mathcal{V}_1$, $\mathcal{V}_2$ (see Fig. 1) with volumes $V_1$, $V_2$ and integral

$$
\delta_{\mathcal{V}_1}^{\mathcal{V}_2}(\Phi) = \int_{\mathcal{V}_1} \int_{\mathcal{V}_2} \Phi(R) \, dr \, dr' = \delta_{\mathcal{V}_2}^{\mathcal{V}_1}(\Phi), \quad R = |r - r'|, \quad \delta_{\mathcal{V}_1}^{\mathcal{V}_2}(1) = V_1 V_2,
$$

where $r \in \mathcal{V}_1$, $r' \in \mathcal{V}_2$, and $dr$, $dr'$ are two three-dimensional volume elements.

\[\text{c} \odot \text{A.Yu.Vlasov, 2009}\]

Signed CLD II
Figure 1: Scheme of integration on “source” and “target” bodies

A particular case is two equivalent bodies \( \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V} \) and integral [Part I, Eq. A1] \( \mathcal{I}_\mathcal{V}(\Phi) = \int_{\mathcal{V}} \Phi(R) \). On the other hand, for two different disjoint bodies it is possible to consider union \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \) as a single compound nonconvex object and due to simple decomposition

\[
\int_{\mathcal{V}_1 \cup \mathcal{V}_2} dr \int_{\mathcal{V}_1 \cup \mathcal{V}_2} dr' \Phi(R) = \int_{\mathcal{V}_1} dr \int_{\mathcal{V}_1} dr' \Phi(R) + \int_{\mathcal{V}_2} dr \int_{\mathcal{V}_2} dr' \Phi(R) + \int_{\mathcal{V}_1} dr \int_{\mathcal{V}_2} dr' \Phi(R) + \int_{\mathcal{V}_2} dr \int_{\mathcal{V}_1} dr' \Phi(R)
\]

(1.2)

it is possible to express Eq. (1.1) using single-body integrals \( \mathcal{I} \) from [Part I]

\[
2 \mathcal{I}_{\mathcal{V}_1}(\Phi) = (V_1 + V_2)^2 \mathcal{I}_{\mathcal{V}_1 \cup \mathcal{V}_2}(\Phi) - V_1^2 \mathcal{I}_{\mathcal{V}_1}(\Phi) - V_2^2 \mathcal{I}_{\mathcal{V}_2}(\Phi).
\]

(1.3)

This consideration justifies application of signed chords and radii distribution for calculation of “transfer integrals” like Eq. (1.1). It is shown below, that (quasi)density functions for signed radii and chord length distributions introduced in [Part I] for Eq. (1.1) with nonoverlapping bodies are always have negative values, even if all three terms in Eq. (1.3) are convex.

Figure 2: Two overlapping bodies

It is also possible to consider two overlapping bodies \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) Fig. 2. Such a case may be described using three nonoverlapping bodies: \( \mathcal{B}_3 = \mathcal{V}_1 \cap \mathcal{V}_2, \mathcal{B}_1 = \mathcal{V}_1 \setminus \mathcal{B}_3, \mathcal{B}_2 = \mathcal{V}_2 \setminus \mathcal{B}_3 \), i.e., \( \mathcal{V}_1 = \mathcal{B}_1 \cup \mathcal{B}_3, \mathcal{V}_2 = \mathcal{B}_2 \cup \mathcal{B}_3 \) and equations for nonoverlapping or equal bodies

\[
\mathcal{J}_{\mathcal{V}_1} = \mathcal{J}_{\mathcal{B}_1 \cup \mathcal{B}_3} = \mathcal{J}_{\mathcal{B}_1} + \mathcal{J}_{\mathcal{B}_3} + \mathcal{J}_{\mathcal{B}_2} + \mathcal{J}_{\mathcal{B}_3}.
\]

(1.4)
Such a scheme lets consider only disjoint bodies without lost of generality. Presentation suggests an acquaintance with first part [Part I] and may be considered as extension of corresponding sections. Sec. 2 develops methods discussed in [Part I] App. A-1, A-2]. Sec. 2 is relevant with [Part I] Sec. 3] and generalizations briefly discussed in Sec. 4, 5 are analogues of [Part I] Sec. 4, 5.

2 DISTANCES AND CORRELATIONS

2.1 Distribution of distances

Let’s write generalization of formula [Part I] Eq. A2] for distribution of distances \( \eta_{12} \) between points in \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \)

\[
\frac{1}{V_1 V_2} \delta_{\mathcal{W}_1}^2 (\Phi) = \frac{1}{V_1 V_2} \int_{\mathcal{W}_1} \int_{\mathcal{W}_2} \Phi(|r - r'|) \, dr \, dr' = \int_0^\infty \Phi(x) \eta_{12}(x) dx, \tag{2.1}
\]

where \( 1/(V_1 V_2) \) is multiplier used for normalization \( \int_0^\infty \eta_{12}(x) dx = 1 \). Proof of Eq. (2.1) is analogue of Lemma 1 in [Part I] Appendix A-1.

From equations for union like Eq. (2.2) or Eq. (2.3) may be derived similar expression for distributions of distances

\[
(V_1 + V_2)^2 \eta_{12}(l) = V_1^2 \eta_1(l) + 2V_1 V_2 \eta_{12}(l) + V_2^2 \eta_2(l), \tag{2.2}
\]

where \( \eta_1, \eta_2 \) and \( \eta_{12} \) correspond to definition of distances distribution for single (convex or nonconvex) body used in first part, Definition 1 [Part I]. There is also reason to generalize such equation for arbitrary number of bodies and write

\[
\left( \sum_{k=1}^n V_k \right)^2 \eta_{ij}(l) = \sum_{k=1}^n V_k^2 \eta_k(l) + 2 \sum_{j=i+1}^n \sum_{i=1}^n V_i V_j \eta_{ij}(l) = \sum_{i,j=1}^n V_i V_j \eta_{ij}(l), \tag{2.3}
\]

where \( \eta_{ij} \) is distribution of distances for union of \( n \) bodies (considered as a single object) and notation \( \eta_{ii} = \eta_i \) is indirectly used for convenience in last expression with single sum \( \sum_{ij} \). Such notation make possible to talk about \( \eta_{ij} \) as about some matrix-valued density \( \eta(l) \).

2.2 Correlation function

Correlation function \( \gamma(r), r \in \mathbb{R}^3 \) or \( \gamma(l), l \in \mathbb{R} \) may be defined for two densities \( \rho_1(r), \rho_2(r) \) \( r \in \mathbb{R}^3 \) as (cf [Part I] Eq. A5])

\[
\gamma_12(r) = \int_{\mathbb{R}^3} \rho_1(r') \rho_2(r + r') \, dr', \quad \gamma_{12}(l) = \frac{1}{4 \pi l^2} \int_{S_l} \gamma_12(r) d\Omega, \quad d\Omega = \sin \theta d\theta d\phi, \tag{2.4}
\]

i.e., \( \gamma_{12}(l) \) is an average of \( \gamma_12(r) \) on sphere with radius \( l \), \( \{ S_l : |r| = l \} \).

In simplest case of two bodies \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) with constant unit density \( \rho_k(r) = 1 \) for \( r \in \mathcal{W}_k \) and zero otherwise. It is possible to rewrite Eq. (2.4)

\[
\delta_{\mathcal{W}_1}^2 (\Phi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho_1(r) \rho_2(r') \Phi(|r' - r|) \, dr \, dr' \tag{2.5a}
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho_1(r) \rho_2(r + R) \Phi(|R|) \, dr \, dR \quad (R = r' - r) \tag{2.5b}
\]

\[
= \int_{\mathbb{R}^3} \gamma_12(R) \Phi(|R|) dR \tag{2.5c}
\]

\[
= 4\pi \int_0^\infty l^2 \gamma_12(l) \Phi(l) dl, \tag{2.5d}
\]
where Eq. (2.5d) is produced from Eq. (2.5c) by integration over spheres $S_l$. It may be compared with analogue integrals for autocorrelation function [Part I] Eq. A6, A7 up to constant multiplier $1/V$, because here is not used normalization multiplier $V^{-1/2}$ for density introduced in [Part I].

Comparison of Eq. (2.5d) and Eq. (2.1) with arbitrary function $\Phi(x)$ discussed earlier, Eq. (2.6) and usual formula for integration by parts

$$\eta_{12}(l) = \frac{4\pi}{V_1 V_2} r^2 \gamma_{12}(l).$$

Due to Eq. (2.2) and Eq. (2.6)

$$\gamma_{12}(l) = \gamma_{11}(l) + 2 \gamma_{12}(l) + \gamma_{22}(l),$$

where $\gamma_{11}$, $\gamma_{12}$ and $\gamma_{22}$ are autocorrelation function without normalization, i.e., $\gamma_{11}(0) = V_1$, $\gamma_{22}(0) = V_2$ and $\gamma_{12}(0) = V_1 + V_2$ (for nonoverlapping bodies). Here is more convenient to do not use normalization vs [Part I] to make expressions like Eq. (2.7) more clear.

For example, due to Eq. (2.3) and Eq. (2.6) there is quite simple expression with (auto)correlation functions for few objects

$$\gamma_{ij}(l) = \sum_{k=1}^{n} \gamma_{kk}(l) + 2 \sum_{j=i+1}^{n} \sum_{k=1}^{n} \gamma_{ij}(l) = \sum_{i,j=1}^{n} \gamma_{ij}(l),$$

Here again appears some matrix $\gamma(l)$ with correlation functions $\gamma_{ij}$.

## 3 SIGNED MATRIX-VALUED DISTRIBUTIONS

### 3.1 Integration by parts

Similarly with [Part I] Eq. 3.1 it is possible to write for Eq. (1.1) with $\Phi(x) = \varphi(x)/(4\pi x^2)$ due to Eq. (2.1), Eq. (2.6) and usual formula for integration by parts

$$\delta_{\mathcal{B}_1}^{\mathcal{B}_2}(\varphi) = \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \frac{\varphi(R)}{4\pi R^2} \, dr \, dr' = \frac{V_1 V_2}{4\pi} \int_0^\infty \frac{\varphi(x)}{x^2} \eta_{12}(x) \, dx$$

$$\gamma_{12}(l) = \int_0^\infty \gamma_{12}(x) \varphi(x) \, dx$$

$$\gamma_{12}(l) = - \int_0^\infty \gamma_{12}(x) \left( \int_0^x \varphi(r) \, dr \right) \, dx$$

$$\gamma_{12}(l) = \int_0^\infty \gamma_{12}(x) \left( \int_0^p \varphi(r) \, dr \right) \, dx,$$

where $R = |r - r'|$. Here $\mathcal{B}_1$ and $\mathcal{B}_2$ may be without lost of generality considered nonoverlapping due to adaptability of decompositions like Fig. 2 and Eq. (1.1) discussed earlier, Eq. (3.1a) and Eq. (3.1b) follows from Eq. (2.3) and Eq. (2.6) respectively, but Eq. (3.1c) and Eq. (3.1d) are produced by formal integrations by parts and need for further explanation.

Comparison with definition of Dirac integral used in [Part I]

$$D_{\mathcal{B}}(\varphi) = \frac{1}{V} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{\varphi(|r - r'|)}{4\pi |r - r'|^2} \, dr \, dr'$$

produces links with considered integrals, if to choose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and use equations like Eq. (1.2) and Eq. (1.4)

$$(V_1 + V_2)D_{\mathcal{B}_1 \cup \mathcal{B}_2}(\varphi) = \delta_{\mathcal{B}_1 \cup \mathcal{B}_2}^{\mathcal{B}_1 \cup \mathcal{B}_2}(\varphi) = 2\delta_{\mathcal{B}_2}^{\mathcal{B}_1}(\varphi) + V_1 D_{\mathcal{B}_1}(\varphi) + V_2 D_{\mathcal{B}_2}(\varphi)$$

© A. Yu. Vlasov, 2009
3.2 Some properties of distributions for two bodies

Signed chords $\mu_\pm(l)$ and radii $\iota_\pm(l)$ distributions was defined in first part [Part I] via formulas

$$
\iota_\pm(l) = -\gamma'(l), \quad \mu_\pm(l) = \langle l \rangle \gamma''(l), \quad \langle l \rangle = \int_0^\infty l \mu_\pm(l) \, dl = \frac{1}{\gamma'(0)} = \frac{4V}{S},
$$

(3.4)

where $\gamma(l)$ is (normalized) autocorrelation function, $S$ and $V$ are surface area and volume of given body utilized due to Cauchy relation for average chord length $\langle l \rangle = 4V/S$.

Distribution of distances between two bodies $\eta_{12}$ in Eq. (2.1) and Eq. (3.1a) has clear geometrical meaning. Integrals Eq. (3.1c) and Eq. (3.1d) makes reasonable to introduce analogues of Eq. (3.4)

$$
C_{12}^{\iota} \iota_{12}(l) = -\gamma'(l), \quad C_{12}^{\mu} \mu_{12}(l) = \gamma''(l),
$$

(3.5)

where $C_{12}^{\iota}$ and $C_{12}^{\mu}$ are some constants. It is useful to apply Eq. (5.1), Eq. (5.3) and Eq. (3.4) in analogues of Eq. (2.2) for disjoint bodies

$$(V_1 + V_2) \iota_\pm^{\mathcal{B}_1 \cup \mathcal{B}_2}(l) = V_1 \iota_\pm^{\mathcal{B}_1}(l) + V_2 \iota_\pm^{\mathcal{B}_2}(l) + 2C_{12}^{\iota} \iota_{12}(l)
$$

(3.6)

and due to Cauchy relation for $\mathcal{B}_1$ and $\mathcal{B}_2$ with surface areas $S_1$ and $S_2$

$$(S_1 + S_2) \mu_\pm^{\mathcal{B}_1 \cup \mathcal{B}_2}(l) = S_1 \mu_\pm^{\mathcal{B}_1}(l) + S_2 \mu_\pm^{\mathcal{B}_2}(l) + 2C_{12}^{\mu} \mu_{12}(l).
$$

(3.7)

Here bodies with common parts of boundaries Fig. 3a and total area $S_{12} < S_1 + S_2$ are excluded for simplicity, but may be considered using infinitesimal displacement of overlapped surfaces Fig. 3b.

![Figure 3: a) Common boundaries. b) Displacement](image)

The distributions of distances $\eta_1$, $\eta_2$, $\eta_{12}$ and even $\eta_{12}$ in Eq. (2.2) are traditional density functions with simple geometrical and statistical interpretation as distribution of distances between points in single object or two different bodies. For convex body radii and chord density functions $\iota$ and $\mu$ also have clear meaning. In first part [Part I] was represented interpretation of signed radii and chord length (quasi)density functions for nonconvex body $\iota_\pm$ and $\mu_\pm$ via composition of some density functions with alternating signs.

Similar decompositions for $\iota_{12}$ and $\mu_{12}$ are represented below, but there is additional problem with definition of $C_{12}^{\iota}$ and $C_{12}^{\mu}$ due to impossibility to use idea of unit normalization for distribution, because integrals over $\iota_{12}$ and $\mu_{12}$ for two nonoverlapping bodies are zeros. It is clear already from integration of Eq. (3.6) and Eq. (3.7).

So, such functions always must have both positive and negative values and in considered approach analogue of Dirac integrals may never use expression with probability density function satisfying usual definition with nonnegativity and unit normalization conditions. Both $\iota_{12}(l)$ and $\mu_{12}(l)$ are reasonable examples of “Feynman’s negative probabilities” discussed in first part [Part I].

It may be simpler to consider (signed) density functions as elements of some transition matrices. Let us use distributions of distances as simple example. In Sec. 2.1 was introduced matrix $\eta$ with all elements $\eta_{ij}(l)$ are density functions with unit integral $\int_0^\infty \eta_{ij}(l) \, dl = 1$. 

© A.Yu.Vlasov, 2009

Signed CLD II
On the other hand it is possible to consider only $\eta_\cup$ as density function with property
\[ \int_0^\infty \eta_\cup(l)dl = 1 \]
and introduce matrix $\tilde{\eta}$ with components
\[
\tilde{\eta}_{ij}(l) = \frac{V_i V_j}{V_\cup} \eta_\cup(l), \quad V_\cup = \sum_{k=1}^n V_k \tag{3.8}
\]
Instead of Eq. (2.3) it is possible to write
\[
\eta_\cup(l) = \sum_{i,j=1}^n \tilde{\eta}_{ij}(l), \tag{3.9}
\]
i.e., $\eta_\cup$ is density function for distribution of distances and Eq. (3.9) is a sum of contributions $\tilde{\eta}_{ij}$
for $n^2$ possible combinations with different pairs of bodies.
Here due to Eq. (3.8)
\[
p_{ij} = \int_0^\infty \tilde{\eta}_{ij}(l) dl = \frac{V_i V_j}{V_\cup^2} \tag{3.10}
\]
is probability for first and second points to lay in $\mathcal{B}_i$ and $\mathcal{B}_j$ respectively.
Let’s introduce a similar matrix $\tilde{\iota}$ for assembly with few disjoint bodies as
\[
\tilde{\iota}_{ij}(l) = -\frac{V_i V_\cup}{V_j} \gamma'_{ij}(l). \tag{3.11}
\]
For particular case $i = j$
\[
\tilde{\iota}_{ii}(l) = \frac{V_i}{V_\cup} \iota_{\pm}(l) \tag{3.12}
\]
Due to such definition and Eq. (2.8) signed radii distribution for union of such bodies may be expressed as
\[
\iota_{\pm}(l) = \sum_{i,j=1}^n \tilde{\iota}_{ij}(l). \tag{3.13}
\]
It is an analogue of Eq. (3.9).
Now it is possible rewrite Eq. (3.1c) for a pair in such collection
\[
\frac{1}{V_\cup} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{\varphi(|r - r'|)}{4\pi|r - r'|^2} dr dr' = \int_0^\infty \tilde{\iota}_{ij}(x) \left( \int_0^x \varphi(r) dr \right) dx. \tag{3.14}
\]
The $\tilde{\iota}_{ij}$ may be also directly associated with terms used in [Part I] for decomposition of $\iota_{\pm}$ and it is revisited below in Sec. 3.3.
Finally, matrix chord length distribution $\tilde{\mu}$ may be introduced
\[
\tilde{\mu}_{ij}(l) = \frac{4}{S_\cup} \gamma''_{ij}(l). \tag{3.15}
\]
Signed chord length distribution may be expressed for the union due to Eq. (2.3) as
\[
\mu_{\pm}(l) = \sum_{i,j=1}^n \tilde{\mu}_{ij}(l). \tag{3.16}
\]
Analogue of Dirac integral for two bodies may be derived from Eq. (3.1d)
\[
\frac{1}{V_\cup} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{\varphi(|r - r'|)}{4\pi|r - r'|^2} dr dr' = \frac{S_\cup}{4V_\cup} \int_0^\infty \tilde{\mu}_{ij}(x) \left( \int_0^x \varphi(r) dr dp \right) dx. \tag{3.17}
\]
For $\mathfrak{B}_i = \mathfrak{B}_j = \mathfrak{B}$ Eq. (3.17) coincides with integral [Part I] Eq. 2.5] used in Dirac’s method of chords.

Relation of $\tilde{\mu}_{ij}$ with decomposition used in [Part I] for construction of $\mu_{\pm}$ is discussed further in Sec. 3.4.

### 3.3 Radii (signed matrix) density function

Consideration below is very similar with [Part I] Section 3.2]. After transition to spherical coordinates discussed in first part [Part I] Eq. 2.8, A9] it is possible to express Eq. (3.1) via analogue of [Part I] Eq. 3.2]

$$
\int_{\mathfrak{B}_1} \int_{\mathfrak{B}_2} \frac{\varphi(|r - r'|)}{4\pi |r - r'|^2} dr dr' = \frac{1}{4\pi} \int_{\mathfrak{B}_1} dr \int d\Omega \int_{R \cap \mathfrak{B}_2} \varphi(R) dR,
$$

(3.18)

where $R \cap \mathfrak{B}_2$ is intersection of a body $\mathfrak{B}_2$ with a ray from a point inside the body $\mathfrak{B}_2$, e.g., segment $[R_2, R_3]$ on Fig. 4.

![Figure 4: Scheme of intervals for radii in two disjoint bodies](image)

Let’s denote for a ray from $\mathfrak{B}_1$ as $R_{ij}^{(l)}$ the distance between origin and $k$-th intersection with $\mathfrak{B}_j$, e.g., for two convex bodies on Fig. 4 $R_1^{(12)} = R_2$ and $R_2^{(12)} = R_3$. Origin is considered further as first intersection. i.e., $R_1^{(11)} = 0$ and $R_2^{(11)} = R_1$ on Fig. 4 In more general case with nonconvex source body $R_1^{(12)} = R_{2m}$ with $m \geq 1$.

With such notation

$$
\int_{R \cap \mathfrak{B}_2} \varphi(R) dR \equiv \sum_{k=1}^{n_{12}} \int_{R_{2k-1}^{(12)}}^{R_{2k}^{(12)}} \varphi(R) dR = \sum_{k=1}^{2n_{12}} (-1)^k \int_0^{R_{2k}^{(12)}} \varphi(R) dR,
$$

(3.19)

where $n_{12}$ is amount of intervals of ray from $\mathfrak{B}_1$ inside body $\mathfrak{B}_2$. A similar equation is appropriate for any number of bodies and any pair $\mathfrak{B}_i, \mathfrak{B}_j$. It may be written formally due to Eq. (3.19)

$$
\tilde{\iota}_{ij}(l) = \sum_{k=1}^{2n_{ij}^{(l)}} (-1)^k \iota_k^{(ij)}(l),
$$

(3.20)

where $\iota_k^{(ij)}(l)$ is density function for length of $k$-th intersection with $\mathfrak{B}_j$ of ray originated in $\mathfrak{B}_i$. It is an analogue of [Part I] Eq. 3.5] up to sign $(-1)^{k+1}$ due to formally zero-based indexes in initial numeration (e.g., $R_0 = 0$) used in [Part I].

It is analogy of [Part I] Section 3.2] with representation of $\iota_k$ as an alternating sum of $\iota_k$ [Part I] Eq. 3.2]. The main difference of representation $\tilde{\iota}_{ij}$ as a sum $\iota_k$ is requirement for origin of ray to be inside $\mathfrak{B}_i$ and inclusion in sum only intervals inside $\mathfrak{B}_j$. For a case depicted on Fig. 4 $\iota_1^{(11)}(l)$,
\( \iota_1^{(12)}(l) \) and \( \iota_2^{(12)}(l) \) are distributions of radii \( R_1, R_2, \) and \( R_3 \) respectively and would correspond in initial notation \([\text{Part I}]\) to \( \iota_1, \iota_2 \) and \( \iota_3 \).

For nonconvex target \( R_{ji}^{(ij)} = R_{2m} \) with different \( m \geq 1 \) (if \( i \neq j \)) there is no direct relation between \( \iota_k^{(ij)} \) in Eq. (3.20) and \( \iota_{k'} \) with some fixed \( k' \) in notation used in first part \([\text{Part I}]\) Eq. 3.2]. It is rather formal rearrangement, because due to Eq. (3.13) together with Eq. (3.20) there is an analogue of \([\text{Part I}]\) Eq. 3.5

\[
\iota_{\pm}(l) = \sum_{i,j,k=1} (-1)^k \iota_k^{(ij)}(l),
\]

there \( \iota_{\pm}(l) \) is signed radii distribution for collection \( \mathcal{B}_i \). Such correspondence produce normalization for all \( \iota_{ij} \). It is not always convenient for analysis of single integral like Eq. (1.1), Eq. (3.14), Eq. (3.18) etc.

Say, for case with two bodies \( V_{ij} = V_1 + V_2 \) in Eq. (3.14) it corresponds to already noted in Eq. (3.12) scaling \( \iota_{ij}^{(1)} = \iota_{11}(l) V_2 / V_1 \) and \( \iota_{ij}^{(2)} = \iota_{22}(l) V_2 / V_2 \). Asymmetric normalization on source body produces yet another notation

\[
\iota_{ij}(l) = \frac{V_{ij}}{V_{i}} \iota_{ij}^{(i)}(l) \implies \iota_{ij}^{(i)}(l) = \frac{V_{ij}}{V_{i}} \iota_{ij}^{(i)}(l)
\]

with last equation due to symmetry of initial definition \( \iota_{ij} = \iota_{ji} \).

Here is also convenient to use stochastic model similar with introduced in \([\text{Part I}]\). Each ray is “primary event” and for any given body \( \mathcal{B}_i \) there are \( n \) kinds of “secondary events”: intersection of the ray with boundary of \( \mathcal{B}_i \) and intersections with other \( (n-1) \) bodies. Each such event has “negative sign” if the ray enters into the body and positive one otherwise.

For all bodies nonoverlapping with \( \mathcal{B}_i \) number of “negative” and “positive” intersection are equal and so formal “balance” \( N_{ij}^{(ij)} = N_{ij}^{(ij)} - N_{ij}^{(ij)} \) for such events is zero. It is yet another demonstration of zero integral \( \int_0^\infty \iota_{ij}(l) = 0 \) for \( i \neq j \) and disjoint bodies. More direct explanation is rather obvious equality of such integrals over corresponding “negative” and “positive” terms \( \iota_k^{(ij)}(l) \) and \( \iota_{k+1}^{(ij)}(l) \) in Eq. (3.20).

### 3.4 Chord length (signed matrix) density function

This section is based on \([\text{Part I}]\) Section 3.3. For calculation of \( \mu_{12}(l) \) it is necessary to save in \([\text{Part I}]\) Eq. 3.14, 3.15 only terms with \( x \in \mathcal{B}_1, x' \in \mathcal{B}_2 \). Each such term corresponds to an integration on some rectangle \( x \times x' \in [L_{2k},L_{2k+1}] \times [L_{2m},L_{2m+1}] \) represented on Fig. 5 derived from analogous scheme in \([\text{Part I}]\) Figure 4.

![Figure 5](image-url)
It may be written instead of [Part I] Eq. 3.13
\[
\int_{B_1} \int_{B_2} \frac{\varphi(R)}{4\pi R^2} dr \, dr' = \frac{1}{4\pi} \int_{L \cap B_1} \int_{L \cap B_2} \varphi(x' - x) dx \, dx' dT,
\]
(3.23)
there \(L \cap B_k\) is intersection of line \(L\) with body \(B_k\) and \(dT\) is measure of integration on space of lines used earlier in [Part I]. Here area of integration \((L \cap B_1) \times (L \cap B_2)\) is union of rectangles mentioned above and denoted as \(\square_{k,m}\) in [Part I] Eq. 3.12c.

If to apply method developed in [Part I] Section 3.3 to Dirac integral for body \(\Omega = B_1 \cup B_2\), then area of integration considered here becomes subset of \((L \cap (B_1 \cup B_2))_2\) used in [Part I] Eq. 3.13.

More generally, decomposition Eq. (3.10) corresponds to expressions of Dirac integral [Part I], Eq. 3.20 for \(\bigcup_k B_k\) via sum of Dirac integrals for \(B_k\) and integrals Eq. (3.23) for pairs \(B_i, B_j\).

Necessary expressions may be found in first part [Part I], Eq. 3.14, 3.15.

Let’s denote intersections of line with boundary of body \(B_j\) as \(L^{(j)}_{k} = L^{(j)}_{2k} = x \leq L^{(j)}_{2k+1} = L^{(j)}_{2m} = x' \leq L^{(j)}_{2m+1}\).

Such terms corresponds to contributions of \(\bar{\mu}_{ij}\) in two last sums in [Part I] Eq. 3.15 via quadruplets expressed by [Part I], Eq. 3.14c.

Let us use notation [Part I], Eq. 3.11
\[
\Delta^L(\varphi) = \int_{0}^{L} \int_{0}^{L} \varphi(r) dr \, dp
\]
(3.25)
From [Part I], Eq. 3.14c rewritten with new notation Eq. (3.24) follows
\[
\square_{k,m}^{(ij)}(\varphi) \equiv - \int_{L^{(i)}_{2k+1}}^{L^{(i)}_{2m}} \int_{L^{(j)}_{2k+1}}^{L^{(j)}_{2m}} \varphi(x' - x) dx \, dx' = - \int_{L^{(i)}_{2k+1}}^{L^{(i)}_{2m+1}} \int_{L^{(j)}_{2k+1}}^{L^{(j)}_{2m+1}} \varphi(x' - x) dx \, dx
\]
(3.26)
\[
\Delta^L_{m} = \sum_{k,m}^{(ij)}(\varphi) = \Delta^{L^{(i)}_{2m}-L^{(i)}_{2k+1}}(\varphi) = \Delta^{L^{(j)}_{2m+1}-L^{(j)}_{2k+1}}(\varphi) = \Delta^{L^{(j)}_{2m+1}-L^{(j)}_{2k+1}}(\varphi).
\]
For example, Eq. (3.24) may be rewritten using such notation as
\[
\int_{B_1} \int_{B_2} \frac{\varphi(R)}{4\pi R^2} dr \, dr' = \frac{1}{4\pi} \int dT \sum_{k,m}^{(12)}(\varphi),
\]
(3.27)
where \(k\) and \(m\) are zero-based indexes of intervals of a chord inside of first and second body respectively, e.g., for two convex bodies it would be only one term and for scheme on Fig. 5 there are up to two terms.

Due to Eq. (3.23) each such a term formally corresponds to four segments of a chord Fig. 5.

So, for two different bodies each pair of intervals of intersection with the same line [\(A, B\)] in \([C, D] \in B_i\), generates quadruplets of chord segments with “positive” pair [\(AD\)], [\(BC\)] and “negative” pair [\(AC\)], [\(BD\)].

Let us denote \(\mu_{mn}^{(ij)}\) distribution of length \(L^{(j)}_{m} - L^{(i)}_{k}\). It is clear from Eq. (3.20) that sign is always equal to \((-1)^{m-k+1}\) and so it is possible to define
\[
\mu_{ij}(l) = \sum_{k,m}^{(ij)}(-1)^{m-k+1} \mu_{mn}^{(ij)}
\]
(3.28)
Let’s also rewrite Eq. (3.27) using mentioned rule for signs of intervals
\[
\int_{B_1} \int_{B_2} \frac{\varphi(R)}{4\pi R^2} dr \, dr' = \frac{1}{4\pi} \int dT \sum_{k,m}^{(ij)}(\varphi) \Delta^L(\varphi),
\]
(3.29)
Application of methods represented in [Part I] Sec. 2.2, Sec. 3.3, App. A-5] to Eq. (3.28) and Eq. (3.29) produces yet another version of Eq. (3.17)

\[ \int_{\mathbf{B}_1} \int_{\mathbf{B}_2} \frac{\phi(|r-r'|)}{4\pi |r-r'|^2} dr \, dr' = S_{\mathbf{B}_2} \int_0^\infty \mu_{ij}(x) \left( \int_0^\infty \frac{\phi(r)}{r} dr \right) dx, \]  

(3.30)

where \( S_{\mathbf{B}_2} \) is constant. Due to principles discussed in [Part I] App. A-5] it is equivalent with average overlap of projection of \( \mathbf{B}_1 \) and \( \mathbf{B}_2 \) on the same plane.

Such normalization produces some difficulty, due to absence of simple generalization of Cauchy formula for average area of surface of single body, e.g., \( S_{\mathbf{B}} = S/4 \). Average chord length [Part I] also may not be used, because it is zero

\[ \int_0^\infty l \mu_{ij}(l) dl - \int_0^\infty l \nu_{ij}(l) dl = \int_0^\infty \nu_{ij} dl = 0 \]  

(3.31)

Let’s discuss that more explicitly. In [Part I] App. A-5] for each line \( L \) intersecting single convex body is obviously defined a chord length \( L(L) \). So the chord length \( L(L) \) may be considered as some function on space of lines. This space \( T \) may be considered as some abstract space and Dirac “chord integral” may be expressed [Part I] A20 via integration on this space.

For nonconvex body with \( n_1 \) intervals of intersection in [Part I] Sec. 3.3 similar integral was defined via sum of \( 2n_1^2 - n_1 \) terms with different signs. Formally instead of one function \( L(L) \) on the space of lines \( T \) it may be considered \( 2n_1^2 - n_\max \) different functions \( L_{k,m}(L) \), \( k, m = 1, \ldots, 2n_\max \), \( k < m \), where \( n_\max \) is maximal number of intervals, \( k \) and \( m \) are indexes of intersections.

For each function \( L_{k,m}(L) \) may be defined “density function of interval \( [k, m] \)” \( \mu_{k,m}(l) \) and it is very similar with definition of \( L(L) \) for intersection with single convex body. It justifies application of methods developed in [Part I] App. A-5] for nonconvex case. In [Part I] Sec. 3.3 notation \( L_{k,m}(L) \), \( \mu_{k,m}(l) \) was not used, because these distributions was combined for brevity in three groups: \( \mu_1(l) \), \( \mu_+(l) \), \( \mu_-(l) \).

The same consideration may be used here, if to consider few bodies as one compound nonconvex object with two additional indexes like in Eq. (3.24) to mark “source” and “target” bodies. So, there are distributions of length \( L^{(ij)}(L) \) used in equations like Eq. (3.28). For two bodies only combination of indexes 12 (or 21) is useful for calculation of integrals like Eq. (1.1) and all other terms are used for connection with theory developed in [Part I].

It is possible again to use common normalization on union of bodies similar with Sec. 3.3 to produce Eq. (3.17), coinciding with Eq. (3.30) up to constant and term \( \tilde{u}_{ij} \). In such a case sum Eq. (3.29) is considered as part of sum [Part I] Eq. 3.15] in expressions for Dirac integral with single nonconvex object constructed as the union of all \( \mathfrak{N} = \bigcup_k \mathbf{B}_k \).

Area of integration denoted in [Part I] Eq. 3.13] as \( (L \cap \mathfrak{N})_x \) for each line may be represented as union of \( (L \cap \mathbf{B}_j)_x \) for each body with already considered sets \( (L \cap \mathbf{B}_j) \times (L \cap \mathbf{B}_i) \) for each pair of bodies. It may be clarified by comparison of Fig. 5a and [Part I] Fig. 4a]. So signed chord distribution for union of bodies \( \mu^{(ij)}_L \) is constructed as sum of distributions \( \tilde{u}^{(ij)} \) for all pairs of bodies like Eq. (3.16).

The only difference between \( \mu^{(ij)} \) and \( \tilde{u}^{(ij)} \) is normalization. First one is normalized on set of lines intersecting both bodies \( \mathbf{B}_i \) and \( \mathbf{B}_j \). Second one is normalized on intersection with union, i.e., at least with one body \( \mathbf{B}_k \). Second method produce normalization using simply defined values \( V_j = \sum_k V_k \) and \( S_j = \sum_k S_k \).

Here is again may be used stochastic model similar with discussed at end of Section 3.3 of [Part I]. It is used uniform isotropic set of lines, and for each chord are generated \( n^2 \) signed distributions instead of only one, because each segment started in \( \mathbf{B}_i \) and finished in \( \mathbf{B}_j \) is marked by two additional indexes \( i \) and \( j \).
Different kinds of segments due to intersection with such lines produce distributions of lengths \( \vec{\mu}_{k,m}(l) \) with signed sums equivalent to Eq. (3.32) up to multiplier

\[
\vec{\mu}_{ij}(l) = \sum_{m,k} (-1)^{m-k+1} L_{k,m}(l)
\]

and so generate matrix \( \vec{\mu}(l) \) representing decomposition of \( \mu_{\pm}(l) \) for \( \bigcup_k \mathcal{B}_k \) already presented earlier Eq. (3.10).

There are two “positive” and two “negative” segments in each quadruplet Eq. (3.26) and “event balance” \( N^{(ij)} \equiv N^{(ij)}_+ - N^{(ij)}_- \) is zero. It is confirmation of property \( \int_0^\infty \mu_{ij}(l) = 0 \) for \( i \neq j \) and disjoint bodies. Sums of lengths for two positive and two negative segments are equivalent \( (L^{(i)}_{2m} - L^{(j)}_{2k+1}) + (L^{(i)}_{2m+1} - L^{(j)}_{2k}) = (L^{(i)}_{2m} - L^{(j)}_{2k}) + (L^{(i)}_{2m+1} - L^{(j)}_{2k+1}) \), so contribution to average length is zero, i.e., \( \int_0^\infty l \mu_{ij}(l) = 0 \), cf. Eq. (3.31).

### 3.5 Signed \( \lambda \)-chords

Yet another way to set normalizing multiplier is to use an analogue of “\( \lambda \)-randomness” (cf. Part I and references therein), i.e., some function with extra multiplier \( \lambda_{ij}(l) \propto l^4 \mu_{ij}(l) \).

For such a case it is always possible to normalize \( \lambda(l) \) with condition \( \int_0^\infty \lambda_{ij}(l)dl = 1 \) and to write instead of Eq. (3.30)

\[
\int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{\varphi(|r - r'|)}{4\pi|r - r'|^2} d\lambda dr' = C_{ij}^\lambda \int_0^\infty \lambda_{ij}(x) \frac{\varphi(x)}{3} dx \int_0^\infty \varphi(x) dx
\]

where \( C_{ij}^\lambda \) may be simply calculated using \( \varphi(r) = 4\pi r^2 \)

\[
\int_{\mathcal{B}_i} \int_{\mathcal{B}_j} dx dr = V_i V_j = C_{ij}^\lambda \int_0^\infty \frac{\lambda_{ij}(x)}{3} dx \quad \Rightarrow \quad C_{ij}^\lambda = \frac{3}{\pi} V_i V_j
\]

Unlike single convex body here \( \lambda \)-chords formally may not be defined by lines through pair of points with independent uniform distributions inside body. Here \( \lambda_{ij}(l) \) rather should be considered as some formal function produced from \( l^4 \mu_{ij}(l) \) or \( l^4 \gamma_{ij}''(l) \) after normalization on unit. Yet, for \( i = j \) and convex \( \mathcal{B}_i \) it may be derived from initial definition.

Another equation may be produced directly from comparison of Eq. (3.10) and Eq. (3.33)

\[
C_{ij}^\lambda \frac{\lambda_{ij}(x)}{x^4} = \gamma''_{ij}(x) \quad \Rightarrow \quad \lambda_{ij}(x) = \frac{\pi x^4 \gamma''_{ij}(x)}{3 V_i V_j}. \quad (3.34)
\]

Let’s express \( \bar{\mu}_{12}(l) \) from \( \lambda_{12}(l) \) for two bodies using Eq. (3.13) and Eq. (3.34)

\[
\bar{\mu}_{12}(l) = \frac{4}{S_1 + S_2} \gamma''_{12}(l) = \frac{12 V_1 V_2}{\pi(S_1 + S_2)} \frac{\lambda_{12}(l)}{l^4}. \quad (3.35)
\]

### 4 NONUNIFORM CASE

For nonuniform case for two bodies with densities \( \rho_1 \) and \( \rho_2 \) instead of Eq. (1.1) or Eq. (3.1) may be written an analogue of Part I, Eq. B1

\[
\int \int \rho_1(r) \rho_2(r') \frac{\varphi(|r' - r|)}{4\pi|r' - r|^2} d\lambda dr' = \int_0^\infty \gamma_{12}(x) \varphi(x) dx
\]

\[
= \int_0^\infty \frac{\gamma''_{12}(x)}{3} \left( \int_0^\infty \varphi(r) dr dp \right) dx. \quad (4.1)
\]
It follows from definition of correlation function Eq. (2.4) together with Eq. (2.5) and integration by parts, cf Eq. (3.1). For nonoverlapping \( \rho_1 \) and \( \rho_2 \): \( \gamma_{12}(0) = \gamma'_{12}(0) = 0 \) and there is problem with normalization, cf [Part I] Eq. B2. It is useful to define some \( \lambda(l) \propto l^4 \gamma''(l) \) and to write analogue of Eq. (3.33)

\[
\int \int \rho_1(r)\rho_2(r') \frac{\varphi(|r - r'|)}{4\pi |r - r'|^2} dr dr' = C_{12}^\lambda \int_0^\infty \frac{2}{x^4} \left( \int_0^x \varphi(r) dr dp \right) dx,
\]

(4.2a)

where \( C_{12}^\lambda \) may be again calculated using \( \varphi(r) = 4\pi r^2 \)

\[
\int \rho_1(r)\rho_2(r') dr dr' = M_1 M_2 = C_{12}^\lambda \int_0^\infty \frac{2}{3} \lambda_{12}(x) dx \quad \Rightarrow \quad C_{12}^\lambda = \frac{3}{\pi} M_1 M_2,
\]

(4.2b)

where \( M_1 = \int \rho_1(r) dr \) and \( M_2 = \int \rho_2(r') dr' \) are masses of the bodies. So, \( \lambda_{ij}(x) \) may be formally defined by equation similar with Eq. (3.34)

\[
C_{ij}^\lambda \frac{\dot{\lambda}_{ij}(x)}{x^4} = \gamma''_{ij}(x) \quad \Rightarrow \quad \lambda_{ij}(x) = \frac{\pi}{3} x^4 \gamma''_{ij}(x).
\]

(4.3)

5 ARBITRARY PATHS

Discussion in [Part I] Sec. 5] about possibility of applications to arbitrary paths with uniform and isotropic distribution of initial points and directions is certainly true for presented consideration with few bodies, see Fig. 6. The “ray-tracing” schemes like Fig. 1, Fig. 4, Fig. 5, etc. have rather illustrative purposes and the only necessary condition — is spherical and translational symmetry due to argument \( R = |r - r'| \) of function \( \Phi(R) \) in Eq. (1.1) and other expressions.

Figure 6: Different paths with isotropic distribution

References

[Part I] A. Yu. Vlasov, “Signed chord length distribution. I,” Preprint arXiv:0711.4734 [math-ph] (2007) and references therein.