String thresholds and Renormalisation Group Evolution

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Abstract

We consider the calculation of threshold effects due to Kaluza Klein and winding modes in string theory. We show that for a large radius of compactification these effects may be approximated by an effective field theory applicable below the string cut-off scale. Using this formalism we show that the radiative contribution to gauge couplings involving only massive Kaluza Klein and winding modes may be calculated to all orders in perturbation theory and determine the full two loop contribution involving light modes and estimate the magnitude of the higher-order contributions. For the case of the weakly coupled heterotic string we also discuss how an improved calculation can be made incorporating the string theory threshold corrections which avoids the limitations of the effective field theory approach. Using this formalism we determine the implications for gauge coupling unification for one representative model including the effects of two loop corrections above the compactification scale. Finally we discuss the prospects for gauge unification in Type I models with a low string scale and point out potential fine tuning problems in this case.

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1 Introduction

The possibility of large spatial dimensions in addition to the four dimensional Minkowski spacetime has recently received considerable attention, particularly within the context of the unification of the gauge couplings in some string models. In such models, the presence of power-law “running” of the gauge couplings induces a significantly different picture from the familiar logarithmic running normally found in four dimensional field theory. From the four dimensional point of view the different running of the gauge couplings is due to the presence in the spectrum of the additional excitations (Kaluza-Klein states) associated with the extra spatial (compact) dimensions. Alternatively it may be viewed from the higher dimensional point of view as simply due to the different behaviour of propagators in higher dimensions.

Due to the point-like nature of the couplings in field theory, the radiative corrections due to the Kaluza Klein states are intrinsically ill defined because the infinite sum over the Kaluza Klein tower diverges. To make sense of these corrections it is necessary to have a regularization procedure such as is provided by the string. For the closed string the two dimensional structure of the string worldsheet means that, at distance scales at which string excitations are relevant, the couplings are no longer point-like and indeed have a formfactor which falls exponentially fast at short distances above the string scale. For the open string the situation is somewhat different because states far above the string scale can contribute. However, due to supersymmetry only massless ($\mathcal{N} = 1$) states (with cut-off at the string scale) and massive ($\mathcal{N} = 2$) states (with cut-off at the largest compactification scale) contribute. As a result, in both cases only a finite number of Kaluza Klein or winding states (KKW) make significant contributions to the effective low energy theory. For this reason it is possible to determine the effect of the KKW states in an effective field theory approach, applicable at scales below the cut-off scale. In this approach the radiative corrections due to those KKW states with mass less than the cut-off scale are calculated in the effective field theory, while the radiative corrections of the remaining KKW states are included in the boundary condition for the operators of the effective theory. This formalism provides a convenient method for determining perturbative corrections to gauge and other couplings.

While the string gives finite, and in principle calculable, radiative corrections due to the massive states of the theory, it does not guarantee that the perturbative calculation converges even if the couplings, $\alpha_i$, of the effective field theory are small. The reason is that, if the scale of compactification is far below the string scale, there is a very large number, $N$, of light KKW states. Given this the naive condition for convergence of the perturbative series is that $\alpha_i N$ be small. However, as we will discuss, due to supersymmetry, the situation is somewhat better. We show that the one loop contribution of massive KKW modes is perturbatively exact and calculate it in the limit of large compactification scale. We also discuss the perturbative contribution involving massless modes and determine the effect on this contribution of the massive modes at two loop order. This contribution is not perturbatively exact and gives an inherent uncertainty to the calculation. We calculate the correction to the inverse couplings at two loop order and show that the leading contribution occurs at $O(\alpha_i)$ relative to the leading order which is $O(N)$. Higher order contributions occur at order $(\alpha_i N)^m$ relative to this two loop contribution and we discuss the implications of these terms for the calculability of the couplings.

The outline of this paper is as follows. We first discuss the radiative corrections in string theories and develop the effective field theory approach to the calculation of corrections to the gauge couplings of the theory. In Section 2 we discuss the string regularization of the contribution of towers of massive Kaluza Klein and winding states in the context of the weakly coupled heterotic string and also in the context of Type I (Type I) string theories which may admit a low string scale.
We discuss how the contribution of these states may be calculated in the context of an effective field theory applicable below the string cut-off scale and determine the limits of its applicability. In Section 3 we calculate the radiative corrections to gauge couplings using the effective field theory approach and discuss the convergence of the series in the presence of the KKW states. In Section 4 we determine the two loop radiative corrections to gauge couplings using the full string threshold calculation, for the case of the weakly coupled heterotic string. The advantage of the full string calculation is that the boundary conditions for the effective field theory are determined and we use the heterotic string example to illustrate the relevance of these terms for the determination of gauge couplings. In Section 5 we apply the methods developed here to the analysis of one specific model. Finally in Section 6 we discuss potential fine tuning problems in (Type I) models with a low string scale.

2 String threshold corrections

As we have noted the inclusion of the effects of infinite towers of Kaluza Klein and winding modes associated with new compact dimensions is only well defined in the context of string theories. Two classes of string threshold corrections have been explored. The first is for the weakly coupled heterotic string theory. For it threshold corrections due to Kaluza-Klein and winding states have been under extensive study both for orbifolds and smooth manifold compactifications [15, 16, 17, 18, 19]. In this class of string theory the relation of the string scale to the Planck scale is given by

\[ M_s = \frac{2e^{(1-\gamma_E)/2} 3^{-3/4}}{\sqrt{2\pi\alpha'}} \simeq 0.527 g_s \times 10^{18} \text{GeV} \]  

where \( g_s \) is the string coupling at the unification. Given the discrepancy of this scale with the gauge unification scale, \( M_G \approx 3 \times 10^{16} \text{GeV} \), found by continuing the gauge couplings up in energy using the Minimal Supersymmetric Standard Model (MSSM) spectrum, it is clearly of importance to determine the threshold effects to see if they may explain this discrepancy [20].

The second class is the Type I (Type I') string at weak coupling. In this case eq.(1) no longer applies and we have instead the relation [21]

\[ M_s \sim g M_P e^{-\phi} \]  

where \( g \) is the gauge coupling, \( \phi \) is the dilaton and \( M_P \) is the Planck mass. Such a relation allows for a low string scale through the choice of the dilaton v.e.v. \( \langle \phi \rangle \) and this has caused much interest for it may bring [21] the string scale prediction into agreement with the gauge unification scale, \( M_G \). It was further noticed [22] that the mechanism can actually be applied to lower the string scale even further, perhaps even down to the “TeV region”. In such scenarios one may therefore have a low compactification scale [23] and a low string scale as well. At first sight, to preserve the unification of the gauge couplings, the presence of such a low string scale requires a significant change in the running of the gauge couplings, this change being accounted for by the presence of additional thresholds associated with the extra spatial dimensions. The threshold corrections to the gauge couplings have been calculated for various Type I string models in [11, 12] to give either a (linear) power-law running or a logarithmic running. As we shall discuss in Section 6.2 the second case raises the possibility for the gauge couplings to unify at a much larger scale than the string [12], perhaps even at the original unification scale \( \approx 10^{16} \text{GeV} \).
To discuss the threshold effects it is necessary to review briefly some of the details of the string calculation. In string theory the form of the gauge couplings is

\[ \alpha_a^{-1}(Q) = k_a \alpha_{\text{string}}^{-1} + \frac{b_a}{2\pi} \log \frac{M_s}{Q} + \Delta_a \]  

where \( k_a \) is the Kac Moody level, \( \Delta_a \) is the string threshold correction (a function of the moduli fields) and \( b_a \) is the one-loop contribution of the light modes (\( \mathcal{N} = 1 \) sector).

### 2.1 Weakly coupled heterotic strings

We consider first the case of the weakly coupled heterotic string in which six of the dimensions are compactified on an orbifold, \( T^6/G \). In such models the spectrum splits into \( \mathcal{N} = 1, \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) sectors, the latter two associated with a \( T^2 \times T^4 \) split of the \( T^6 \) torus. Due to the supersymmetric non-renormalisation theorem, the \( \mathcal{N} = 4 \) sector does not contribute to the running of the couplings. The \( \mathcal{N} = 1 \) sector gives the usual running associated with light states but does not contain any moduli dependence. The latter comes entirely from the \( \mathcal{N} = 2 \) sector. For the heterotic string all states are closed string states and at one loop the string world sheet has the topology of the torus \( T^2 \). For the case of a six-dimensional supersymmetric string vacuum compactified on a two torus \( T^2 \) the string correction takes the form \([13, 17]\).

\[ \Delta_i = \frac{\overline{b}_i}{4\pi} \int \frac{d\tau_1 d\tau_2}{\tau_2} (Z_{\text{torus}} - 1) \]  

with

\[ Z_{\text{torus}} = \sum_{n_{1,2},m_{1,2} \in \mathbb{Z}} \exp\left[ 2\pi i T (m_1 n_1 + m_2 n_2) \right] \exp\left[ -\frac{\pi T^2}{T_2 U_2} (\tau U_2 + T n_1 - U m_1 + m_2)^2 \right] \]

Here \( T \propto R_1 R_2 \) and \( U \propto R_1 / R_2 \) where \( T, U \) are moduli and \( R_1, R_2 \) are the radii associated with \( T^2 \). For the case of a two torus \( T^2 \) of common internal radius \( T = i T_2 \) (the subscript 2 denotes the imaginary part) and \( U = i \). Making the dimensions explicit \( T_2 \) should be replaced by \( T_2 \rightarrow T_2 / (2\alpha') \equiv 2 R^2 / (2\alpha') \) with \( \alpha' \) as in eq.\([3]\). The sum \( m_1, m_2 \) is over the Kaluza Klein modes associated with \( T^2 \) and the sum \( n_1, n_2 \) is over the winding modes. In eq.\([3]\) \( \Gamma = \{ \tau_2 > 0, |\tau_1| < 1/2, |\tau| > 1 \} \) is the fundamental domain and \( \tau = \tau_1 + i \tau_2 \) is the modulus of the world sheet torus. Performing the sums and doing the integral over the torus world sheet gives the string corrections \( \Delta_i^{\text{string}} \) of the following form \([13, 17]\).

\[ \Delta_i^{\text{string}} = -\frac{\overline{b}_i}{4\pi} \ln \left\{ \frac{8 \pi e^{1-\gamma_E}}{3 \sqrt{3}} |\eta(i)|^{4} \frac{T_2^2}{2\alpha'} \left| \eta \left( \frac{i T_2}{2\alpha'} \right) \right|^{4} \right\} \]

\[ = -\frac{\overline{b}_i}{4\pi} \ln \left\{ 4 \pi^2 |\eta(i)|^{4} \left( \frac{M_s}{\mu_0} \right)^{2} \left| \eta \left( \frac{3 \sqrt{3} \pi}{2e^{1-\gamma}} \left( \frac{M_s}{\mu_0} \right)^{2} \right) \right|^{4} \right\} \]

where \( \overline{b}_i \) is the beta coefficient associated with the \( \mathcal{N} = 2 \) multiplets, \( M_s \) is the string scale, \( \mu_0 \equiv 1/R, \eta(x) \) is the Dedekind eta function and we have replaced \( \alpha' \) in terms of \( M_s \) (eq.\([3]\)) \([10]\). For large \( T_2 \) the eta function is dominated by the leading exponential \([14]\) and so one finds the

\[ \overline{b}_i \approx 5.5 (M_s R)^2 \]  

in the \( \overline{B} \) scheme, so one can easily have \( T_2 \approx 50 - 100 \) while \( R \) is still close to the string length scale, to preserve the weakly coupled regime of the heterotic string. In this section “large” \( R \) corresponds to values of \( T_2 \) in the above range.

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\[ The \text{Dedekind function is defined by } \eta(T) = e^{\pi T/12} \prod_{k=1}^{\infty} (1 - e^{2\pi ikT}) \]
power law behaviour \( \Delta_i \propto T_2 \propto R^2 \) which has a straightforward interpretation as being due to the decompactification associated with \( T^2 \). Note that, due to the fact that the \( N = 4 \) states associated with the \( T^4 \) compactification of four of the six dimensions do not contribute to the running of the couplings, the power law behaviour only corresponds to the decompactification of two of the six compact dimensions.

It is of interest to determine which contributions in eq.(5) dominate in this limit. Using eq.(5) in eq.(4) one sees the contribution of states more massive than the string scale are exponentially suppressed. This is in contrast to the (regulated) field theory result in which all massive states contribute with a logarithmic dependence on the mass. The difference is due to the absence of point-like coupling in the string theory corresponding to the two dimensional distribution of eq.(4). (The field theory result corresponds to taking the lower limit for the \( \tau_2 \) integration to be 0 rather than \( \sqrt{(1 - \tau_1^2)} \) of the string). As a result, at large \( T_2/\text{radius} \), only the Kaluza Klein modes with mass less than the string scale make significant contributions. It is this fact that allows us to reformulate the calculation of the threshold corrections in terms of an effective field theory calculation valid below the string scale. To make this point more explicitly we note that the momentum modes alone give the contribution

\[
\Delta_i = \frac{b_i}{4\pi} \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} d\tau_2 \frac{1}{T_2} \left\{ \sum_{m_1,m_2 \in \mathbb{Z}} \exp \left[ -\frac{\pi \tau_2}{T_2} (m_1^2 + m_2^2) \right] \right\}^{-1} \tag{8}
\]

It is straightforward to compare the result of directly evaluating eq.(8) with the full string result of eq.(6). For example at \( T_2 = 2 \) the Kaluza Klein states alone give 99.5\% of the full threshold correction\(^5\) (6). This means that one may determine the threshold corrections to an excellent accuracy simply by including the contribution of states at or below the string scale. To determine how well this contribution is approximated by the field theory result we restrict the sums over \( m_1, m_2 \) to a finite number of terms and find after some algebra\(^7\)

\[
\Delta_i = \frac{b_i}{4\pi} \sum_{(m_1, m_2) \neq (0,0)} \int_{-1/2}^{1/2} d\tau_1 \left[ \kappa \sqrt{1 - \tau_1^2} \right] \ln \left[ \frac{M_s}{\mu_0 |\vec{m}|} - \frac{b_i}{4\pi} \sum_{(m_1, m_2) \neq (0,0)} \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} dt \left( e^{-t} - 1 \right) \right] \tag{9}
\]

where

\[
\kappa \equiv \frac{\pi |\vec{m}|^2}{T_2}, \quad |\vec{m}|^2 = m_1^2 + m_2^2 \tag{10}
\]

We denote by \( \tilde{\Delta}_i \) the second term in (10) and we will restrict both sums in (10) to terms with

\[
\kappa \tilde{\Delta}_i \equiv \frac{\pi |\vec{m}|^2}{T_2} \ll 1 \tag{11}
\]

\(^5\)As we will see later \( T_2 \approx O(1) \) is already large enough for our purposes, see also footnote 3.
\(^6\)To see this replace the series under the integral (8) by their sum, elliptic theta function, \( \vartheta_3(0, e^{-\pi \tau_2/T_2}) \).
\(^7\)We use the result

\[
E_1(z) \equiv \int_0^{\infty} dt \frac{e^{-t}}{t} \approx -\gamma_E - \ln z - \int_0^{z} dt \frac{e^{-t}}{t} (e^{-t} - 1) \approx -\gamma_E - \ln z \quad \text{if} \quad z \ll 1
\]
First we have that
\[
\Delta_i(\kappa \vec{m} \leq 1) = \frac{b_i}{2\pi} \sum_{(m_1,m_2) \neq (0,0)}^{\kappa \vec{m} \leq 1} \ln \frac{M_s}{\mu_0|m|} - \tilde{\Delta}_i(\kappa \vec{m} \leq 1)
\] (13)

The term \(\tilde{\Delta}_i\) is negligible in the limit (12) and we are therefore left with the logarithmic terms. These are just those of the effective field theory approach, as we will see in Section 3.1 (eq.25).

How good is this approximation? Keeping states for which \(\kappa \vec{m} \leq 0.5\) the correction terms \(\tilde{\Delta}_i\) are 20\% of the field theory result for \(T_2 = 100\) and this rises to 50\% for \(T_2 = 10\). The full threshold corrections require the inclusion of the contribution above \(\kappa \vec{m} = 0.5\) and in the effective field theory approach these must be input as boundary conditions for the RGE at the scale \(\kappa \vec{m} = 0.5\). Using the exact string result we find that the combination of this contribution and the correction terms \(\tilde{\Delta}_i\) amount to roughly 50\% of the full threshold correction for \(T_2\) in the range 10 to 100. The implication of this is that while the effective field theory provides an excellent description of the contribution of those states far below the string scale it fails to give a very accurate determination of the full threshold corrections because of the contribution of Kaluza Klein states close to the string scale which have significant non-point-like coupling. We shall discuss in Section 3 how to avoid this failing of the effective field theory approach by improving the radiatively corrected calculation using the full string threshold effects to determine the boundary conditions and to take account of the non-pointlike coupling of those states close to the string threshold. However, we note that at one loop order this uncertainty only affects the determination of the string scale \(M_s\). This follows because, c.f. eq.(10), \(\tilde{\Delta}_i\) is proportional to \(b_i\) and thus can be absorbed in a change of scale \(M_s \rightarrow M'_s\) in the first term. If we do this we can extend \(\kappa \vec{m}\) to 1 and the “field-theory” term is exact at one loop. Thus, using the effective field theory approach we lose the accurate determination of the string scale, but otherwise the structure is correct - it correctly reproduces the relative evolution of couplings and the power-law behaviour at large radius (according to our discussion of power-law running in Section 3).

2.2 Type I (I\(^{\prime}\)) string theory

Threshold effects have also been calculated in Type I/I\(^{\prime}\) string theories [11, 12, 13, 14]. These are of much current interest because they allow (eq.(2)) for a very low string scale consistent with the 4D Planck mass. They may also accommodate very large new dimensions in which the closed string states (gravitons, etc.) propagate, giving rise to interesting new phenomena [23]. Threshold effects in such models have quite a different character to those of the weakly coupled heterotic string. The reasons are two-fold. Firstly the states of the MSSM correspond to open string states with their quantum numbers supplied by Chan-Paton charges. As a result the geometry associated with one loop contribution involving the propagation of open string degrees of freedom is now given by the annulus, \(\mathcal{A}\), and the Möbius strip, \(\mathcal{M}\). This affects the way the ultraviolet cut-off appears, it being determined by the lightest state in the crossed (closed-string) channel. The second major difference is that in Type I\(^{\prime}\) the open string has only winding modes (of mass \(nM^2_s R\), \(n\) integer) with respect to “off-the-brane” dimensions (or Kaluza Klein modes in Type I). As a result in the case the compactification scale \(R\) is larger than the string length scale, these (winding) excitations are heavier than the string mass scale in Type I\(^{\prime}\) theories. As we shall discuss, for the \(\mathcal{N} = 2\) contributions to threshold corrections, it is this scale and not the string scale that acts as the ultraviolet cut-off. In this case one might worry that an effective field theory (EFT) will not be able to describe such radiative corrections coming from above the string scale. However, even in
this case, the EFT techniques work because the massive string states fill $\mathcal{N} = 4$ representations and do not contribute to the gauge coupling evolution.

As we have already discussed, the corrections to gauge couplings come only from the $\mathcal{N} = 1$ (massless) sector and from the $\mathcal{N} = 2$ massive winding (or Kaluza Klein) sector. Let us consider the $\mathcal{N} = 2$ sector first. For Type I strings at weak coupling, a result similar to that of $\mathcal{N} = 1$ has recently been computed in [11]. The massive $\mathcal{N} = 2$ threshold corrections at $M_s$ have the form (for both $\mathcal{A}$ and $\mathcal{M}$ geometries)

$$
\Delta^\text{TypeI}_a = \frac{1}{4\pi} \sum_i b_{ai}^{N=2} \int_0^\infty \frac{dt}{t} e^{-\pi M^2 t} \sum_{(m_1, m_2)} e^{-\frac{\pi}{\sqrt{G_i} \text{Im} U_i} |m_1 + \text{Im} U_i m_2|^2} \tag{14}
$$

where $M_i$ is the mass of the lightest $\mathcal{N} = 2$ state in the KK tower ($M_i \ll 1/\tilde{R}_1, 1/\tilde{R}_2$). Evaluating this gives the following $\mathcal{N} = 2$ correction at string scale

$$
\Delta^\text{TypeI}_a = -\frac{1}{4\pi} \sum_i b_{ai}^{N=2} \ln \left[ \sqrt{G_i} \text{Im} U_i M_s^2 |\eta(U_i)|^4 \right] \tag{15}
$$

$G_i$ is the metric on the torus $T_i$. For the simple case of a two-torus the behaviour of the thresholds in the limit $\text{Im} U = \tilde{R}_1/\tilde{R}_2$ is of order one ($\sqrt{G} = \tilde{R}_1/\tilde{R}_2$) is logarithmic, $\Delta_a \sim \ln(\tilde{R}_1/\tilde{R}_2)$, while if one radius is much smaller than the other, the thresholds are linearly divergent, $\Delta_a \sim \tilde{R}_2/\tilde{R}_1$ (“power-law” running).

Once again, it is of interest to discuss the origin of these results in the context of an effective field theory. In this case the integral over $t$ ranges from 0 to $\infty$ with a lower cut-off. The upper limit corresponds to the IR region and is regulated by the mass of the lightest $\mathcal{N} = 2$ state. The lower limit probes the UV in the open string channel. In order to determine the ultraviolet cutoff imposed by the string it is instructive to rewrite the threshold correction as an integral over $l$ in the crossed (closed string) channel given by $l = 1/t$ for $\mathcal{A}$ and $l = 1/(4t)$ for $\mathcal{M}$. As we discussed above, in the large radius limit it is necessary to adopt the Type I’ interpretation. Then

$$
\Delta^\text{TypeI’}_a = \frac{1}{4\pi} \sum_i b_{ai}^{N=2} \int_0^\infty \frac{dl}{l} e^{-\pi M^2 l} \sum_{(m_1, m_2)} e^{-\pi l [(m_1/R_1)^2 + (m_2/R_2)^2]} \tag{16}
$$

$$
= \frac{1}{4\pi} \sum_i b_{ai}^{N=2} \tilde{R}_1 \tilde{R}_2 \int_0^\infty \frac{dl}{l} e^{-\pi M^2 l} \sum_{(m_1, m_2)} e^{-\pi l [(m_1/R_1)^2 + (m_2/R_2)^2]} \tag{17}
$$

where in deriving eq.(17) we have performed a Poisson resummation of the first term before changing variables. For inverse radii roughly equal the integral over $l$ is cut-off for $l < 1/R^2$ corresponding to $t \geq 1/\tilde{R}^2$ for $\mathcal{A}$ (or $t \geq 1/(4\tilde{R}^2)$ for $\mathcal{M}$). Using this eq.(17) gives rise to the term $\propto \ln(R^2)$ of eq.(15). For the case the radii are hierarchically different, $R_1 \gg R_2$, states in (17) of mass $n/R_1$ contribute to the integral in (17) which is now cut-off for $l < 1/R_2$ (or for $t > 1/R_2^2$). The theory is effectively five dimensional and the couplings evolve as a linear power ($R_1/R_2$) corresponding to the large Im$U$ behaviour of eq.(15). In both cases only a finite number of states contribute allowing for an effective field theory description of the region up to the cut-off scale. Once again the EFT with a sharp cut-off does not accurately describe the effect of the modes close to the cut-off scale, but as we saw in the heterotic case, this just amounts to an uncertainty in the cut-off scale at one loop order. The important feature that allows this interpretation, even for the case the cut-off scale is much larger than the string scale, is the decoupling of the massive string states.

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$^8$ $R$ and $\tilde{R}$ are related by a T duality transformation.
We turn now to the contribution of the massless modes (before “Higgsing”) in the $\mathcal{N} = 1$ sector. This has recently been calculated \cite{11} in $Z_N$ orbifolds with $N$ a prime integer. These are the simplest 4D models with $\mathcal{N} = 1$ Supersymmetry having no 5-branes in the spectrum. The important point to note is that the $\mathcal{N} = 1$ sector does not include any winding modes and is thus independent of the compactification radius. As a result the regulated contribution is cut-off at the string scale $M_s$, rather than the compactification scale.

The analysis of gauge coupling unification is somewhat complicated by the presence of the twisted NS-NS moduli, $m_k$ the “blowing-up” modes of the orbifold. These have non-universal couplings to the gauge fields and this could affect the unification scale, \cite{24, 25, 26}. For example, in the $Z_3$ model, a linear symmetric combination, $M$, of the 27 twisted moduli couples to the gauge fields. The gauge kinetic term has the form

\[ f_a = S + S_a M \]

where $S$ is the dilaton and $S_a$ are gauge group dependent factors. These have been calculated \cite{11} and found to be proportional to the one loop $\beta$ function coefficients, $b_a$. As a result if the vacuum expectation value of $M$ is non-zero, it will lead to a shift in the one-loop unification scale, $M_s \to M_s e^{\langle M \rangle}$. The $Z_3$ model has a single anomalous $U(1)_X$ under which $M$ transforms in the manner needed to cancel the anomalies via the Green-Schwarz mechanism. Associated with this is the Fayet-Iliopoulos D-term which is sensitive to $\langle M \rangle$. Requiring that the non-Abelian gauge group remains unbroken, the vanishing of the Fayet-Iliopoulos term forces $\langle M \rangle$ to vanish \cite{11}. A similar result is found for the other models studied. As a result\footnote{Provided higher order terms do not shift the minimum of the potential for $M$.} the unification scale of the massless sector remains the string scale, $M_s$. In the case this scale is low, the only possible way to achieve gauge unification is via the $\mathcal{N} = 2$ correction of (15). We shall discuss this possibility in Section 6.

### 3 Kaluza-Klein thresholds and RGE equations in field theory

As we have just discussed, many features of the low energy structure of string theories with large compactification radii can actually be described in terms of a four dimensional effective field theory. In toroidal compactification the existence of new compact spatial dimensions requires $\Phi(x, y) = \Phi(x, y + 2\pi R)$ where $\Phi(x, y)$ is a field, $x$ denotes the four dimensional space-time coordinates and $y$ stands for the additional (compactified) spatial dimensions assumed to be all of equal radius, $R$. The coefficients (operators) $\Phi^{(n)}(x)$ of the Fourier expanded field $\Phi$ with respect to the new spatial coordinates $y$ represent the Kaluza-Klein (KK) modes. As we increase the energy scale the new dimensions will open up and this corresponds to the appearance in the spectrum of new heavy states (excited modes). Their (bare) mass, $\mu_0^2$, is determined in terms of the inverse size of the extra (compact) spatial dimensions, $\mu_0 \equiv 1/R$

\[ \mu_0^2 = \mu_0^2(m_1^2 + m_2^2 + \cdots + m_\delta^2) + m_0^2 \]

where $\delta$ is the number of additional dimensions, $\vec{m} \equiv (m_1, m_2, \cdots, m_\delta)$, the integers $m_i$ are Kaluza-Klein excitation numbers with integer values. The parameter $m_0$ denotes the mass of “zero-modes” which we drop in this discussion, assuming it to be much smaller than $\mu_0$.

Once the quantum numbers of the Kaluza Klein states are specified together with their interactions with the light spectrum, one can apply the renormalisation group evolution (RGE) to
analyse the implications such a spectrum has on the unification of the couplings. The use of the RGE equations is justified in this context by the observation that an effective field theory (below the string scale $M_s$) in $4 + \delta$ dimensions can be described by a renormalisable field theory with a finite number of Kaluza Klein states below this scale\(^{10}\). This is just the structure we found in eq.(13) where a finite number of states reproduced the usual field theory logarithmic contributions. At and above the string scale it is necessary to include string excitations and perform a full string calculation. To match the two theories at the string scale we need to introduce string boundary conditions for the RGE evolution below the string scale. We shall discuss the inclusion of boundary conditions in Section 4.

In this Section we determine the effects of the (finite number of) Kaluza-Klein states on the running of the gauge couplings. Part of the contribution to the running of the gauge couplings will be perturbatively exact, while the contribution corresponding to the MSSM matter wave-function renormalisation will be determined to two-loop order only. The RGE equation, expressed in terms of the wave-function renormalisation coefficients, can be derived from the “NSVZ beta function” \(^{27, 28, 29}\) (generalized in \(^{30}\) to local supersymmetric effective field theories).

We shall calculate the evolution of the gauge couplings in a model with the full MSSM spectrum together with the associated set of Kaluza-Klein excitations. The latter may be associated with any combination of the gauge sector, the Higgs fields and even with the fermionic sector, the choice being determined by the particular model one chooses. In this section we will allow for any of these possibilities, leaving the calculation in one specific model to Section 5.

### 3.1 Case 1: Kaluza Klein states with mass below string scale

For generality, consider first the effect of a tower of Kaluza-Klein states each state having $\Delta b_i = T(R_i)$, $(i = 1, 2, 3)$, as its contribution to the one-loop $\beta$ function coefficients\(^{11}\). This tower could be associated with any low-energy state, function of the value of $T(R_i)$. The number of the Kaluza-Klein states (including the zero-modes) associated with any such low-energy state in representation $R_i$ is set by the number of solutions (including the trivial one) \(^2\) to the equation which sets the upper limit to the mass of these excited modes, corresponding to the cut-off in the effective field theory. This is given by

\[
0 \leq (m_1^2 + m_2^2 + \cdots + m_\delta^2) \leq \left[ \frac{\Lambda}{\mu_0} \right]^2
\]

where $\Lambda$ is the high scale cut-off of the 4-dimensional effective field theory and $\mu_0$ is the compactification scale, equal to the inverse of the radius of the extra spatial dimensions (we assume that all additional spatial dimensions have equal radius, thus the mass splitting of KK states is proportional to $\mu_0$). The number of states is approximated by the volume of a $\delta$-dimensional sphere \(^4\) of radius $\Lambda/\mu_0$, given by

\[
N(\Lambda, \mu_0; \delta) \approx \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} \left[ \frac{\Lambda}{\mu_0} \right]^{\delta}
\]

The “radial” degeneracy $\sigma$ of a level of fixed energy (i.e. a fixed value for $\mu_m^2 = \mu_0^2 (m_1^2 + m_2^2 + \cdots + m_\delta^2)$) is therefore given by

\[
\sigma(\mu_m^2, \mu_0; \delta) \approx \frac{dN}{d\mu_m^2} \bigg|_{\Lambda = \mu_m^2} = \frac{\delta \pi^{\delta/2}}{\Gamma(1 + \delta/2)} \left[ \frac{\mu_m^2}{\mu_0^2} \right]^{\delta-1}
\]
which increases with the scale. This means that the Kaluza-Klein states will be “turned on” in an increasing number at a higher scale and with a strong effect on the RGE equations within a small range of energy. Eqs \([21], [22]\) will prove useful in evaluating the contribution of the tower states to the RGE equations.

The RGE equations for the gauge couplings \([27, 28, 29]\) include contributions from the gauge wave-function renormalisation and matter wave-function renormalisation, which can be easily obtained by formally integrating the exact NSVZ beta function \([27, 28, 29]\)

\[
\beta(\alpha_i)^{NSVZ} \equiv \frac{d\alpha_i}{d(\ln Q)} = -\frac{\alpha_i^2}{2\pi} \left[ 3T_i(G) - \sum_{\psi} T(R^i_\psi)(1 - 2\gamma_\psi) \right] \left( 1 - T_i(G)\frac{\alpha_i}{2\pi} \right)^{-1}
\]

with the definition \((Q\) is the scale\)

\[
\gamma_\psi = -\frac{1}{4\pi} \frac{d\ln Z_\psi}{dt}, \quad t = \frac{1}{2\pi} \ln Q
\]

and where \(T_i(G)\) and \(T(R^i_\psi)\) represent the Dynkin index for the adjoint representation and for \(R^i_\psi\) representation respectively (not necessarily the fundamental one) which contribute to the running of \(\alpha_i, i = \{1, 2, 3\}\). The above sum runs over all matter fields \(\psi\) in the representation \(R^i_\psi\) and this includes the high energy (Kaluza Klein) and the low energy (MSSM) spectrum. Eq.(23) can be re-written as follows

\[
-\frac{d\alpha_i^{-1}}{d\ln Q} = \frac{1}{2\pi} \left[ T_i(G)\frac{d\ln \alpha_i}{d\ln Q} - 3T_i(G) + \sum_{\psi} T(R^i_\psi)(1 - 2\gamma_\psi) \right]
\]

One can integrate this equation \([27, 28]\) to give the exact contributions to all orders in perturbation theory to the running of the gauge couplings for the particular spectrum considered. The results depend however on the wave-function renormalisation coefficients which are not known to all orders and this makes the results less useful. Still, for the effects of the heavy states to \(\alpha_i^{-1}\) one finds exact results \([27, 29]\) which depend on the bare mass of the states and not on the factors \(Z\) and this is useful for our phenomenological predictions.

More explicitly, consider the Kaluza-Klein tower of states of bare mass \(\mu_0 \vec{m}\) (eq.(19)) associated with a particular low-energy state in representation \(R^i, i = 1, 2, 3\), with Dynkin index \(T(R^i) = \Delta b_i\). In the RGE they decouple at a scale equal to their physical mass, \(\mu_0 \vec{m}\) \([29]\). After performing an integration of eq.(24), we obtain the generic contribution of this tower of states to the RGE equations for \(\alpha_i^{-1}(\mu_0)\)

\[
A_i = \frac{\Delta b_i}{2\pi} \sum_{\psi} \int_{\mu_0}^{\Lambda} (1 - 2\gamma_\psi) d\ln \mu = \frac{\Delta b_i}{2\pi} \sum_{\vec{m}} |\vec{m}| \leq \Lambda/\mu_0 \left( \frac{\Lambda Z(\Lambda)}{\mu_0 Z_\vec{m}(\mu_0)} \right) = \frac{\Delta b_i}{2\pi} \sum_{\vec{m}} \frac{|\vec{m}| \leq \Lambda/\mu_0}{\mu_0^{\vec{m}}} \ln \Lambda \mu_0^{\vec{m}} \sigma_{\vec{m}} \ln \frac{\Lambda}{\mu_0^{\vec{m}}}
\]

\[\text{[25]}\]

\[\text{[26]}\]

\[\text{[27]}\]

\[\text{[28]}\]

\[\text{[29]}\]

\[\text{[30]}\]

\[\text{[31]}\]

\[\text{[32]}\]
\[ m_\sigma m \] is the “radial” degeneracy which can be approximated by eq. (22). The degeneracy accounts for all possible configurations \[ m \equiv (m_1, m_2, \ldots, m_\delta) \] with fixed value for \[ m_0^2 \equiv m_0^2 (m_1^2 + m_2^2 + \cdots + m_\delta^2) \]. In eq. (25) we have also used the mass renormalisation equation

\[ \mu m Z m (\mu m) = \mu m^0 Z (\Lambda) \] (26)

The mass renormalisation equation is exact to all orders in perturbation theory [27, 28] and gives a result (eq. (25)) depending on the bare mass of these states only. This mechanism is general, as long as Supersymmetry is present [27, 28] and shows how to sum up the heavy modes’ contribution to the RGE equations. Strictly speaking the mechanism applies only to \( \mathcal{N} = 1 \) Kaluza Klein states (i.e. zero modes) while for \( \mathcal{N} = 2 \) states we have \[ Z = 1 \].

Note that the result of eq. (25) is exactly the string theory term of eq. (13) (with \( \delta = 2 \)) provided that the Kaluza Klein spectrum is the same in both cases. The interpretation of the scale \( \Lambda \) is obvious from eq. (13). It shows that the string and field theory results for the Kaluza Klein thresholds are equal for the sum over those states whose mass satisfies

\[ \mu m \equiv \mu_0 |m| \ll M_s \] (27)

In terms of the RGE evaluation of these terms what is important is the logarithmic \( \mu m^0 \) dependence of the result and one may readily check that this is given by eq. (13) to an accuracy of 10% for \( \mu m^0 \leq M_s / 10 \). This is equivalent to using the RGE up to the scale \( \Lambda = M_s / 10 \). Above this scale the field theory estimate of the Kaluza Klein contribution deviates from the string result because of the pointlike coupling assumed in the effective field theory result. The effect of the states in this region must be included as boundary conditions for the RGE evolution using the full string theory calculation. This is the approach we take in Section 4 to show how to determine the radiative corrections using the string theory threshold corrections directly.

The result of eq. (25) can also be written as

\[ A_i = \frac{\Delta b_i}{2\pi} \left[ N(\Lambda, \mu_0; \delta) - 1 \right] \ln \frac{\Lambda}{\mu_0} - \frac{\Delta b_i}{2\pi} \sum_{|m| \leq \Lambda/\mu_0} \sigma m \ln \frac{\mu m^0}{\mu_0} \] (28)

The sum \( \sum_{|m|} \sigma m \) was replaced by \( N(\Lambda, \mu_0, \delta) - 1 \) which accounts for the number of excited modes only, and this excludes the configuration of “zero modes” \( \bar{m} = (0, 0, \ldots, 0) \). To evaluate the last sum of eq. (28) we can approximate it by an integral with upper mass limit set by eq. (20) while the lower limit is the mass of the lowest excited mode, \( |\bar{m}| = 1 \). The result obtained for the sum of eq. (28) is then given by

\[ \sum_{|m|} \sigma m \ln \frac{\mu m^0}{\mu_0} \approx \frac{\delta}{\Gamma(1 + \delta/2)} \int_1^{\Lambda/\mu_0} z^{\delta-1} \ln z \, dz \]

\[ = N(\Lambda, \mu_0; \delta) \ln \frac{\Lambda}{\mu_0} - \frac{\pi^{\delta/2}}{\delta \Gamma(1 + \delta/2)} \left[ \left( \frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] \] (29)

15 One could simply use [28] for the RGE evolution, which is just the integral of NSVZ function. The \( \mathcal{N} = 2 \) Kaluza Klein states (in addition to the MSSM) which in \( \mathcal{N} = 2 \) language do not have wavefunction renormalisation, give only an overall contribution to \( 1/\alpha_i \), equal to \( b_i/(2\pi) \sum \ln \Lambda/\mu m_0 \) with \( b_i \), accounting for all \( \mathcal{N} = 2 \) Kaluza Klein (gauge bosons and/or matter fields) contributions. For a self-contained approach and explicit presentation of the mass renormalisation of the \( \mathcal{N} = 1 \) components we used here the “NSVZ” \( \beta \) function and \( \mathcal{N} = 1 \) language, with the same result for the RGE.
Using this, one finds
\[ \mathcal{A}_i \equiv \frac{\Delta b_i}{2\pi} \left\{ \frac{\pi^{\bar{\delta}/2}}{\bar{\delta} \Gamma(1 + \bar{\delta}/2)} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\bar{\delta}} - 1 \right] - \ln \frac{\Lambda}{\mu_0} \right\} \]  
(30)
where \( \mathcal{A}_i \) represents the exact contribution, to all orders in perturbation theory of a tower of Kaluza-Klein states to the running of the gauge couplings. This is the result for a tower of states associated with a low-energy state in representation \( R^i \), with the one-loop beta function contribution \( \Delta b_i = T(R^i) \). Equation (30) also takes account of the heavy threshold effects. The power-law behaviour of eq.(30) is a just a consequence of the (large) number of states we are summing over, each of them giving an individual contribution to the RGE equations of logarithmic type. The result is similar to the “one-loop” result of [2] obtained in the “standard” way. Note that the computation of the last sum in equation (28) is an approximation which works well for large number of states and also uses the approximation (21) which for \( \delta = 1, 2, 3 \) is indeed reliable.\(^{16}\) One should however ensure that this is true for all phenomenological cases, particularly when the number of Kaluza-Klein states is relatively low when eq.(21) may not be reliable. This approximation will be eliminated in Section 4 (for the case of weakly coupled heterotic string) were we match the field theory results with those from string theory. This concludes the calculation of the generic contribution of a tower of Kaluza-Klein states associated with a low-energy state of Dynkin index \( T(R^i) \).

For the general case the coefficient \( \Delta b_i = T(R^i) \) in eq.(31) should be replaced by the total contribution to the one loop beta function of the massive KK states considered, which we call \( \bar{b}_i \).

Then the gauge couplings at the compactification scale have the generic form
\[ \alpha^{-1}_i(\mu_0) = \alpha^{-1}_i(\Lambda) + \frac{\bar{b}_i}{2\pi} \sum_{\vec{m}} |\vec{m}| \leq \Lambda/\mu_0 \sigma_{\vec{m}} \ln \frac{\Lambda}{\mu_0} + l.s. \text{ contribution} \]  
(31)
\[ \cong \alpha^{-1}_i(\Lambda) + \frac{\bar{b}_i}{2\pi} \left\{ \frac{\pi^{\bar{\delta}/2}}{\bar{\delta} \Gamma(1 + \bar{\delta}/2)} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\bar{\delta}} - 1 \right] - \ln \frac{\Lambda}{\mu_0} \right\} + l.s. \text{ contribution} \]  
(32)
where the power-law term is perturbatively exact. On the other hand, the last term due to the light states (l.s.) (such as those of the MSSM) can only be computed perturbatively up to some given order. This concludes the calculation of the explicit contribution of the heavy Kaluza-Klein states to the RGE equations.

We now evaluate the contribution of the matter fields of the MSSM to the running of the gauge couplings, in two-loop order. For this we need to evaluate the anomalous dimensions of the MSSM matter fields (which we call \( \gamma_\phi \)) in one-loop order only \([27, 29]\). These quantities evaluated above the decompactification scale \( \mu_0 \) are changed from their usual MSSM values due to the presence of the tower of Kaluza-Klein states. In this way Kaluza Klein states give a second, indirect contribution, of two-loop (and higher) order to the RGE equations. To obtain a definite expression for the value of the anomalous dimensions of the MSSM matter fields one needs to determine further model-dependent properties i.e. the interaction Lagrangian between the low-energy fields and the excited Kaluza-Klein states which can contribute to the wavefunction renormalisation of the MSSM states. To keep a model independent approach, we separate the contributions to \( \gamma_\phi \) in two terms, one due to the MSSM states alone (denoted by \( \gamma_\phi^o \)) and one including Kaluza Klein states, which we call \( \gamma_\phi^i \).

\(^{16}\)For this compare the number of Kaluza Klein states of Table 3 of [7] with that given by eq.(21) for the ratio \( M_s/\mu_0 \) given in Table 3 of [7].

\(^{17}\)Kaluza Klein states are \( N = 2 \) multiplets hence \( \bar{b}_i \) is the \( N = 2 \) \( \beta \) function, \( \bar{b}_i = 2 \sum_\psi T_i(\psi) - 2T_i(G) \).
\[ \Delta \gamma_\phi. \text{ Therefore} \]

\[ \gamma_\phi = \gamma_\phi^0 + \Delta \gamma_\phi \]

If we ignore any Yukawa contribution\(^{18}\) the value of \(\gamma_\phi^0\) has the following expression, valid between the scale \(M_z\) and \(\Lambda\)

\[
4\pi \gamma_\phi^0(Q) = -\sum_{k=1}^{3} 2\alpha_k C_k(\phi) + \mathcal{O}(\alpha^2) \quad M_z < Q < \Lambda
\]

where \(\phi\) can stand for the chiral superfields of the MSSM and \(C_k(\phi)\) represents the quadratic Casimir operator. We therefore find the following contribution to perturbation theory [27] and is equal to

\[
T_{\text{function renormalisation of the MSSM gauge bosons. The latter is actually exact to all orders in result (using (24))}
\]

\[
\text{MSSM beta function, and T}_\gamma
\]

\[
\text{if we ignore any Yukawa contribution the scale } \mu \text{ and } \Lambda
\]

\[
\text{two-loop order only.}
\]

\[
\text{We now add together all the contributions to the RGE equations for the gauge couplings which we evaluate just below the scale } \mu_0, \text{ in two loop order. This includes the contributions of eq.(33) modified by the factor } \tilde{b}_i \text{ of [32] and the contributions of eq.(35) as well as those due to the wave-function renormalisation of the MSSM gauge bosons. The latter is actually exact to all orders in perturbation theory [27] and is equal to } T_i(G)/(2\pi) \ln(\alpha_\Lambda/\alpha_i(\mu_0)). \text{ We thus obtain the following result (using (24))}
\]

\[
\alpha_i^{-1}(\mu_0) \cong \alpha_i^{-1}(\Lambda) + \frac{\tilde{b}_i}{2\pi} \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} T_\phi \left(1 - 2\gamma_\phi\right) + \sum_{a=u,d}^{3} \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \left(1 - 2\gamma_H\right) + \frac{b_i}{2\pi} \ln \frac{\Lambda}{\mu_0} + \frac{T_i(G)}{2\pi} \ln \frac{\alpha_i(\Lambda)}{\alpha_i(\mu_0)} \]

\[
- \sum_{k=1}^{3} \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \frac{1}{4\pi} \left(2_b T_k(G)\delta_{ik}\right) \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \alpha_k(\mu) - \sum_{a=u,d}^{3} \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \frac{2\Delta \gamma_H}{2\pi} \Delta \gamma_H
\]

where \(\tilde{b}_i\) is the one loop contribution to beta function due to those (light) states having Kaluza-Klein towers. Beyond the scale \(\Lambda\) a more fundamental theory (of strings) is supposed to be valid. From the scale \(\mu_0\) down to \(M_z\) scale, only the usual MSSM running is supposed to take place and it affects the running of the gauge couplings in a manner presented in [3].
Equation (36) provides the general result for the running of the gauge couplings in the presence of Kaluza-Klein states, derived on purely field theoretical grounds. The symbol \( \cong \) accounts for the approximation (29) of the discrete sum over the KK thresholds by an integral which gives the power-law term in (36). Although this is a good approximation, it may be avoided by doing the sum explicitly as we show it for the case of weakly coupled heterotic string in Section 4. In the limit \( \delta \to 0 \) one recovers the standard RGE of the MSSM.

To go further we need to know something about \( \Delta \gamma \)'s. Let us keep only gauge contributions to \( \Delta \gamma \) and work with the weakly coupled heterotic string. For the orbifold compactification procedure we adopt, we assume that the Higgs and MSSM fields are situated at the fixed points of the orbifold and thus do not have Kaluza Klein excitations. In orbifold compactification the gauge coupling between two massless twisted modes (i.e. a low-energy state without KK modes) situated at the same fixed point of the orbifold and the (untwisted) gauge excitations of level \( \vec{m} \) is modified \([33,34]\) from the ordinary gauge coupling \( \alpha_j \), giving

\[
4\pi \Delta \gamma_\phi(Q) = -f_\phi(Q, \mu_0; \delta) \sum_{k=1}^3 2\alpha_k(Q) C_k(\phi) \quad \mu_0 \leq Q \leq \Lambda
\]

with

\[
f(Q, \mu_0, \delta) = \sum_{\vec{m}} \sigma(\mu, \vec{m}, \mu_0, \delta) \rho \frac{|\vec{m}|^2_{\mu_0}^2}{M^2}
\]

where \( \rho \) is a number which depends on the orbifold twist \([33]\), of order 10 or larger. Here \( \sigma(\mu, \vec{m}, \mu_0, \delta) \) is the degeneracy of Kaluza-Klein states coupling to the two massless twisted modes and accounts for the sum of all effects due to excited KK modes of fixed \( |\vec{m}| \). We see that the field theory result is again obtained in the limit \( \mu_0^2 \equiv |\vec{m}|^2 \mu_0^2 \ll M^2 \) when the orbifold information (\( \rho \) dependence in (38)) is lost and \( f \) becomes equal to a finite number of excited Kaluza Klein gauge bosons, equal to \( N(Q, \mu_0, \delta) - 1 \) with \( Q \) restricted to values below the string scale.

Since we are keeping only gauge interactions, the three generations of the MSSM have all the same function \( f_\phi(Q, \mu_0; \delta) \) and with the corresponding function for the Higgs fields which we denote by \( g(\mu, \mu_0; \delta) \), we obtain the final form for the running of the gauge couplings (with \( \alpha_i(\Lambda) \rightarrow \alpha_\Lambda \))

\[
\alpha_i^{-1}(\mu_0) \approx \alpha_\Lambda^{-1} + \frac{b_i}{2\pi} \left( \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} \left[ \left( \frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] - \ln \frac{\Lambda}{\mu_0} \right) + \frac{b_i}{2\pi} \ln \frac{\Lambda}{\mu_0} + \frac{T_i(G)}{2\pi} \ln \frac{\alpha_\Lambda}{\alpha_i(\mu_0)}
\]

\[
\quad + \frac{1}{4\pi} \sum_{k=1}^3 \left( b_{ik} - 2 b_k T_k(G) \delta_{ik} \right) \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \alpha_k(\mu)
\]

\[
\quad + \frac{1}{4\pi} \sum_{k=1}^3 \left( b_{ik} - 2 b_k T_k(G) \delta_{ik} - b_k^H \right) \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \alpha_k(\mu) f(\mu, \mu_0; \delta)
\]

\[
\quad + \frac{1}{4\pi} \sum_{k=1}^3 b_k^H \int_{\mu_0}^\Lambda \frac{d\ln \mu}{2\pi} \alpha_k(\mu) g(\mu, \mu_0; \delta)
\]

(39)

For the present case when there are only Kaluza Klein modes for the gauge bosons, \( f \) and \( g \) are equal (and given by the number of Kaluza Klein excitations of the gauge bosons) in (24). For the general case when the Higgs fields and (or) MSSM fermions have KK excitations as well, then \( g \) and (or) \( f \) are equal to zero as the states in the loop which generates \( \Delta \gamma_{\phi,H} \) belong to complete \( \mathcal{N} = 2 \) multiplets.

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The term with the coefficient $b_{ik}^H$ accounts for two loop effects to $\alpha_i^{-1}$ due to the MSSM Higgs fields through their one-loop wavefunction renormalisation (due to $\text{excited}$ KK modes in one line of the loop)$^{19}$ while the term $(b_{ik} - 2b_k T_k(G) \delta_{ik} - b_{ik}^H)$ is the equivalent contribution of the one-loop wavefunction renormalisation of the three MSSM families (due to $\text{excited}$ KK modes). The term with the coefficient $(b_{ik} - 2b_k T_k \delta_{ik})$ accounts for the “standard” two-loop MSSM effects to $\alpha_i^{-1}$ due to the wavefunction renormalisation of the three MSSM generations and Higgs fields. The one-loop gauge wavefunction renormalisation effects are accounted for by the term $T^i(G)/(2\pi) \ln(\alpha_\Lambda/\alpha_i(\mu_0))$. Since the Kaluza-Klein states belong to $N = 2$ supermultiplets this term is perturbatively exact.

Note that the departure from point-like coupling observed at one-loop level occurs at two loops as well due to the presence of the functions $f,g$. For $\Lambda$ well below the string scale two loop terms may be reliably calculated using point-like couplings. The modification due to the non-pointlike coupling of states close to the string scale is a relatively small effect. Finally, due to the power-law behaviour of the couplings, the familiar two-loop running $\sim \log(\alpha(\Lambda)/\alpha(Q))$ of the MSSM is replaced by a leading dependence of the type $\log(\Lambda/m)$ characteristic of the one-loop terms in the MSSM. This can be seen by inserting the one-loop power-like running for the gauge coupling in the integrals of eq.(39).

3.1.1 Convergence of the perturbative expansion

We are now in a position to determine whether the perturbative expansion converges in the presence of the massive tower of Kaluza Klein states. As we have noted the contribution involving only Kaluza Klein modes is perturbatively exact. However the term corresponding to the wave function renormalisation of the light states has contributions to all orders in perturbation theory. From eq.(39) we see that the two loop term is of $O(\alpha N/4\pi)$ where $N$ counts the number of Kaluza Klein states. This is to be compared with the one loop contribution which occurs at $O(N)$ so the two loop term is suppressed at $O(\alpha)$ with respect to the one loop term, just as in the case without Kaluza Klein towers. However at higher order we expect terms of $O((\alpha N/4\pi)^m)$ due to the contribution of Kaluza Klein modes to the wavefunction renormalisation of the MSSM spectrum, see eq.(37) where higher order contributions have been neglected. This means that these terms are suppressed with respect to the two loop terms only if $N\alpha \ll O(4\pi)$. As we will see in a specific example (Section 5), this condition may be satisfied; if not corrections to $\alpha$ at $O(\alpha^2)$ will have an undetermined coefficient$^{20}$.

3.2 Case 2: Type I$'$ strings with $M_s < M_{\text{Winding}}$

For the case of Type I$'$ strings with the winding mode scale above the string scale the situation is simpler. Below the string scale we have standard MSSM running. Above the string scale only $N = 2$ massive multiplets contribute at one loop order and the contribution is cut-off near the first winding mode excitation $^{12}$. As discussed above, this contribution is perturbatively exact. Its implications are discussed in Section 5.

---

$^{19}$The corresponding Feynman diagram of this contribution is similar to that leading to eq.(34).

$^{20}$If we reach the limit $N\alpha \approx O(4\pi)$, a resummed $O(1/N)$ perturbation series for anomalous dimensions of the light fields, eq.(33) should be used $^3$ although its small radius of convergence ($N\alpha < O(6\pi)$) brings little improvement. Beyond this limit we reach the non-perturbative regime.
4 RGE equations with full (heterotic) string thresholds

As we discussed in Section 2 the sum of Kaluza Klein modes with mass less than or equal to the string scale accurately gives the string threshold corrections even for moderate compactification radii, as small as only a factor of 2 larger than the string scale. However the field theory approximation fails to generate the correct contribution of the Kaluza Klein modes with mass close to the string scale because it assumes a pointlike coupling for all states. It was for this reason we argued in Section 3 that the cutoff for the effective field theory had to be set somewhat lower than the string scale but in this case significant corrections arise from the need to impose boundary conditions at this scale. These boundary conditions are determined by the effect of the states lying between the cutoff scale and the string scale. It is possible to improve on this situation in a given string theory by including the full string theory threshold effects. We shall do this here for the case of the orbifold compactification of the weakly coupled heterotic string but the method immediately generalizes to the case of Type I strings or any other case in which one can calculate the one-loop threshold effects. The full one-loop string threshold effects are simply included in the general form for the RGE equations by substituting the full string term \( \Lambda \rightarrow M_s \), \( \alpha \Lambda \rightarrow \alpha_s \equiv \alpha(M_s) \)

\[
\alpha_i^{-1}(\mu_0) = \alpha_s^{-1} - \frac{b_i}{4\pi} \ln \left\{ 4\pi^2 |\eta(i)|^4 \left( \frac{M_s}{\mu_0} \right)^2 \left| \eta \left[ \frac{i3\sqrt{3\pi}}{2e^{1-\gamma}} \left( \frac{M_s}{\mu_0} \right)^2 \right] \right|^4 \right\} \\
+ \frac{b_i}{2\pi} \ln \left( \frac{M_s}{\mu_0} \right) + T_i(G) \ln \left( \frac{\alpha_s}{\alpha_i(\mu_0)} \right) \\
+ \frac{1}{4\pi} \sum_{k=1}^{3} \left( b_{ik} - 2 b_k T_k(G) \delta_{ik} \right) \int_{\mu_0}^{M_s} d\ln \mu \frac{\ln \mu}{2\pi} \alpha_k(\mu) \\
+ \frac{1}{4\pi} \sum_{k=1}^{3} \left( b_{ik} - 2 b_k T_k(G) \delta_{ik} - b_{ik}^H \right) \int_{\mu_0}^{M_s} d\ln \mu \frac{\ln \mu}{2\pi} \alpha_k(\mu) f(\mu, \mu_0; \delta) \\
+ \frac{1}{4\pi} \sum_{k=1}^{3} b_{ik}^H \int_{\mu_0}^{M_s} d\ln \mu \frac{\ln \mu}{2\pi} \alpha_k(\mu) g(\mu, \mu_0; \delta) \\
\]

This is an integral equation for \( \alpha_i(\mu_0) \) and can be solved numerically with \( \mu_0 \) kept relatively close to \( M_s \) to preserve the weakly coupled regime of the theory. Below the compactification scale \( \mu_0 \) only the MSSM spectrum and RGE “running” applies (see for example [3] for its explicit form). The appropriate matching field theory-string theory results should therefore be done at the compactification scale, \( \mu_0 \).

The use of the string boundary conditions eliminates the discrepancy \( \approx \) between the one loop terms evaluated on pure field theory grounds and the exact string result, which originated in the point-like nature of the couplings in field theory. Regarding the two loop terms which we computed using a mixed field theory-string theory formalism, their contribution relative to one loop terms is in general small since they are suppressed by an \( O(\alpha) \) (see next section for an example). This contribution is about 4% for the prediction for the strong coupling at \( M_z \) and thus the residual effect of the discrepancy is suppressed at two loop level below \( 4\% \times 50\% = 2\% \) (for a conservative

\[21\] See next section for an example.
\[22\] Of up to 50%, see discussion at the end of Section 2.1
estimate of the discrepancy between the field theory and string result of 50%). This “hybrid” construction for two loop terms and their small effect enable us to argue that eq.(40) is close to a full two-loop string result for the effects of string (Kaluza Klein and winding) thresholds on the gauge couplings.

5 A numerical example

We present for illustration of the methods developed here a simple model and investigate its predictions following from the RGE equations, the unification of the gauge couplings and some additional assumptions. We use heterotic boundary conditions (10) for an illustrative purpose, but the analysis is similar if one uses Type I conditions, (13). Below the scale $\mu_0$ only the MSSM spectrum and “running” apply. The model is built by assuming that above the scale $\mu_0$ only Kaluza Klein towers for the MSSM gauge bosons exist, and they come in as $\mathcal{N} = 2$ vector multiplets. The “zero-level” modes of these Kaluza Klein states are identified with the corresponding MSSM bosons. To ensure that “zero-level” modes are indeed only $\mathcal{N} = 1$ supersymmetric multiplets, additional symmetry conditions must be imposed and these depend on the type of manifold. This example corresponds to an orbifold compactification in which the $\mathcal{N} = 1$ chiral adjoint component of the $\mathcal{N} = 2$ vector multiplet is odd under the discrete group of the orbifold so that it does not have zero modes23. Furthermore the three generations of the MSSM are assumed to lie all at the (same) fixed points of the orbifold considered (to avoid the presence of KK states for these states). In this simple model the coefficient $\mathcal{b}_i$ of (10) is given by

$$\mathcal{b}_i = T_i(G) - 3T_i(G) = \{0; -4; -6\}_i, \quad i = \{1, 2, 3\}$$

(41)

where the first and second term account for the chiral adjoint and massless vector component of the $\mathcal{N} = 2$ vector multiplet, respectively.

Above the scale $\mu_0$ the values of $\Delta \gamma_i$ due to the excited KK gauge effects are fixed by the following value for $f_{\delta i}(Q, \mu_0; \delta)$ 24:

$$f_{\delta i}(Q, \mu_0; \delta) = N(Q, \mu_0; \delta) - 1, \quad Q > \mu_0$$

(42)

and a similar expression exists for $\delta_{H_u,d}(Q, \mu_0; \delta)$. This follows from the fact that above the scale $\mu_0$ only excited $\mathcal{N} = 1$ massless vector KK states contribute (we ignore possible string form factor effects, eq. (18)). Using (10) the RGE equations of the model take the following form (for $\delta = 2$, 24)

$$\alpha_i^{-1}(\mu_0) = \alpha_s^{-1} - \frac{b_i}{4\pi} \ln \left\{ \frac{4\pi^2 |\eta(i)|^4}{\eta \left[ i \frac{3\sqrt{3} \pi}{2e^{1-\gamma}} \left( \frac{M_s}{\mu_0} \right)^2 \right]^4} \right\} + \frac{b_i}{2\pi} \ln \frac{M_s}{\mu_0} + \frac{T_i(G)}{2\pi} \ln \frac{\alpha_s}{\alpha_i(\mu_0)} + \frac{1}{4\pi} \sum_{k=1}^3 (b_{ik} - 2b_k T_k(G)\delta_{ik}) \int_{\mu_0}^{M_s} \frac{d\ln \mu}{2\pi} N(\mu, \mu_0, 2) \alpha_k(\mu)$$

(43)

with $\mathcal{b}_i$ defined in eq.(11). If we set $\mu_0 = M_s$ only the MSSM spectrum and “running” would apply below the unification scale. However, there would be a difference from the familiar MSSM case because of the string theory boundary condition (eq.(10) with $\mu_0 = M_s$), with effects on $\alpha_3(M_z)$ to be discussed later. From the scale $\mu_0$ down to the scale $M_z$ only the usual MSSM spectrum and RGE running apply (see 3 for explicit form of the RGE below $\mu_0$). We use the values of

23 This ensures that the chiral adjoint field is not present in the low energy spectrum.

24 See discussion following eq.(10) of Section 2.1 which motivates this choice for $\delta$. 

17
\[ \frac{M_s}{\mu_0} \quad M_s \text{ (GeV)} \quad \mu_0 \text{ (GeV)} \quad \alpha_3(M_Z) \quad \alpha_s \]

| \( \frac{M_s}{\mu_0} \) | \( M_s \) (GeV) | \( \mu_0 \) (GeV) | \( \alpha_3(M_Z) \) | \( \alpha_s \) |
|----------------|-------------|-------------|----------------|----------|
| 1.00           | \( 10^{16} \) | \( 10^{16} \) | 0.1224         | 0.0411   |
| 1.11           | \( 6.82 \times 10^{15} \) | \( 6.17 \times 10^{15} \) | 0.1212 | 0.0405 |
| 1.22           | \( 4.22 \times 10^{15} \) | \( 3.45 \times 10^{15} \) | 0.1197 | 0.0396 |
| 1.35           | \( 2.31 \times 10^{15} \) | \( 1.71 \times 10^{15} \) | 0.1180 | 0.0387 |
| 1.50           | \( 1.08 \times 10^{15} \) | \( 7.27 \times 10^{14} \) | 0.1158 | 0.0375 |
| 1.65           | \( 4.25 \times 10^{14} \) | \( 2.58 \times 10^{14} \) | 0.1133 | 0.0361 |
| 1.82           | \( 1.33 \times 10^{14} \) | \( 7.32 \times 10^{13} \) | 0.1103 | 0.0346 |
| 2.01           | \( 3.19 \times 10^{14} \) | \( 1.58 \times 10^{14} \) | 0.1068 | 0.0328 |
| 2.23           | \( 5.47 \times 10^{14} \) | \( 2.46 \times 10^{14} \) | 0.1028 | 0.0309 |

Table 1: The values of the unification scale \( M_s \), decoupling scale \( \mu_0 \), the strong coupling at the electroweak scale and the bare coupling \( \alpha_s \); in terms of the ratio \( M_s/\mu_0 \) which is related to the number of additional Kaluza Klein states approximately by eq. (21). Our two-loop results are based on \( [\text{X}] \) and correspond to the case when gauge bosons have Kaluza-Klein towers. The model fails to increase the unification scale closer to the heterotic scale, eq. (1). However, using string boundary conditions for the RGE and a compactification scale equal to the string scale (first line in the table) allows one to reduce the strong coupling from the MSSM value (0.125) to 0.122 which is closer to the experimental value. This is achieved at the expense of a very small decrease (by a factor of 2) of the unification scale from the MSSM scale (of \( \approx 2 \times 10^{16} \) GeV).

\( \alpha_1(M_z) \) and \( \alpha_2(M_z) \) as a numerical input and keep \( M_s/\mu_0 \) as a parameter of the model. Using \( [\text{X}] \) and the RGE equations below the scale \( \mu_0 \) we can compute the values of \( M_s, \alpha_3(M_z) \) and \( \alpha_s \). The low-energy supersymmetric thresholds are taken into account as an effective threshold \( [\text{XX}] \) 300 GeV which in the MSSM case gives \( [\text{YY}] \) a value for the strong coupling of \( \alpha_3(M_z) \approx 0.125 \), above the experimental limit \( [\text{ZZ}] \), \( \alpha_3(M_z) = 0.119 \pm 0.002 \).

The numerical results are presented in Table 1. We examined the importance of the two loop effects above \( \mu_0 \) on \( \alpha_3(M_z) \), relative to the case when they are neglected and found they are about 3 – 4%. The values of the unification scale \( M_s \) and of the decoupling scale \( \mu_0 \) are more sensitive to these effects, their relative variation being \( \approx 10\% \). As the results of Table 1 show, the model fails to give a string scale close to the weakly coupled heterotic string prediction eq. (1). There is however one important observation regarding the strong coupling at \( M_z \). Assuming the value of \( \mu_0 \) to be equal to the string scale, one finds a value for the strong coupling \( \approx 0.122 \), closer to the experimental value than that predicted \( [\text{ZZ}] \) by the two loop MSSM of \( \approx 0.125 \). This effect is due entirely to the string boundary condition (eq. (41)) with \( \mu_0 = M_s \) and cannot be computed on pure field theory grounds. The boundary conditions used in this model considered Kaluza Klein towers for the gauge bosons only but one could investigate other possibilities which may increase the unification scale, to bring it closer to the string prediction, eq. (1). One can also include Kaluza-Klein towers for the fermions as well (embedded in \( N = 2 \) hypermultiplets) which will enhance the two-loop contribution above the scale \( \mu_0 \) due to their positive contribution to \( \partial_1 \) (which increases the coupling and the higher order effects). Other constructions are also possible \( [\text{X}] \), where supersymmetry is broken by a Scherk-Schwarz mechanism to reproduce the standard

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25 A larger, less favoured value for this threshold is 1000 GeV which further reduces \( \alpha_3(M_z) \).
26 The MSSM value for the strong coupling is computed using two loop running \textbf{without} any string boundary condition of the type given by eq. (40) with \( \mu_0 = M_s \).
27 with identical low energy Supersymmetric (effective) threshold of 300 GeV
We have scale is very high (see Table 1), as the boundary conditions (40) are those of the weakly coupled heterotic string.

we assume the MSSM $N$ requires $10^{19}$, that there are a relatively small number of Kaluza Klein modes below the string scale.

threshold effects fixes the values of the one loop coefficients, $b_i$. One finds at one-loop, up to Supersymmetric threshold effects

$$\alpha_3^{-1}(M_z) = -\frac{b_2 - b_3}{b_1 - b_2} \alpha_1^{-1}(M_z) - \frac{b_3 - b_1}{b_1 - b_2} \alpha_2^{-1}(M_z)$$ (44)

How does this equation change in models with KKW states? This depends on the $(N = 2)$ multiplet content which causes the couplings to run.

### 6.1 Power-law running

We assume the asymmetric case in eq.(15) leading to the linear power-law running in Type I theories [11]. Ignoring a small term $\ln(M_s/\mu_0)$, at the one-loop level the equations have the form

$$\alpha_i^{-1}(M_z) \approx \alpha_s^{-1} + \frac{b_i}{2\pi} \ln \frac{M_s}{M_z} + \frac{\bar{b}_i}{12} \frac{M_s}{\mu_0}$$ (45)

which follows from eqs.(3), (15) and where $\bar{b}_i$ is the $N = 2$ one loop $\beta$ function coefficient. In this we assume the MSSM $N = 1$ light spectrum (but see below for a discussion of this point).

The relative evolution of the gauge couplings is what determines whether the couplings unify.

We have

$$\alpha_i^{-1}(M_z) - \alpha_j^{-1}(M_z) \approx \frac{b_i - b_j}{2\pi} \ln \frac{M_s}{M_z} + \frac{\bar{b}_i - \bar{b}_j}{12} \frac{M_s}{\mu_0}$$ (46)

Clearly if

$$(\bar{b}_i - \bar{b}_j) = K(b_i - b_j)$$ (47)

where $K$ is a positive constant, then the MSSM unification will persist with the replacement

$\ln(M_G/M_z) \rightarrow \ln(M_s/M_z) + 2\pi K(M_s/\mu_0)/12$. The fact that the MSSM is consistent with unification with $M_G \approx 3 \times 10^{16}$ GeV gives a value of unification scale for our example of $M_s/\mu_0 \approx 63/K$.

One can also determine the string coupling in terms of the value $\alpha_g$ of the unified coupling in the MSSM ($\alpha_g \approx 1/24$) to find $\alpha_s^{-1} = \alpha_g^{-1} - \sigma M_s/(12 \mu_0)$ where $\sigma = \bar{b}_i - K b_i$.

Given $\alpha_s g$ and $M_s/\mu_0$ we may determine whether higher order radiative corrections are under control. Following the discussion of Section 3.1.1 we see that the corrections to the running coupling at $O(\alpha^2)$ are calculable provided $\alpha_s N_{KK}/(4\pi) < 1$. Using $N_{KK} \approx 2M_s/\mu_0 \approx 126/K$ this requires $10\alpha_s g < K < 1$. Whether or not this is true is model dependent but it will be the case for a small unified coupling $\alpha_s g \approx \alpha_g$. The reason higher order corrections may remain under control is that there are a relatively small number of Kaluza Klein modes below the string scale.

Of the important question is whether condition (47) can be satisfied? In the MSSM the one loop coefficient $b_i$ has the following form $b_i = \sum_\psi T_i(\psi) - 3T_i(G)$ where the sum runs

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28In this section we kept MSSM as the low energy model since we investigated the case where the compactification scale is very high (see Table 1), as the boundary conditions (4) are those of the weakly coupled heterotic string.
over all matter fields of the MSSM and the second term accounts for the gauge sector. If all the MSSM states have KK towers, the coefficients $\tilde{b}_i$ (i=1,2,3) have such values that condition (14) is not satisfied. Indeed, in this case $\tilde{b}_i$ is given by $\tilde{b}_i = 2 \sum_\ell T_\ell (\psi) - 2T_i (G)$ where the sum is over the MSSM matter fields which have all KK towers and the factor 2 in front accounts for their embedding in $N = 2$ hypermultiplets. Similarly, the second term in the expression of $\tilde{b}_i$ accounts for the gauge sector with KK states being $N = 2$ vector supermultiplets (which contain a $N = 1$ massless vector and a chiral adjoint multiplet), hence the factor 2 which multiplies it. Thus $\tilde{b}_i = 2b_i + 4T_i (G)$ and (14) is not satisfied.

One way out of this difficulty is to include further KK states which together with the KK excitations of the MSSM spectrum will have one loop coefficients which satisfy (14). An example of this class was found by Kakushadze [39] who constructed a simple $Z_2$ orbifold model which breaks $N = 2$ to $N = 1$. The original $N = 2$ spectrum is that of MSSM KK excitations together with those of additional superfields $F_{\pm}$. These are $SU(3) \times SU(2)$ singlets with $U(1)_Y$ charge $\pm 2$ respectively. For this model condition (14) is indeed satisfied with $K = 1$. However the model is not quite what is desired because it contains additional light $N = 1$ fields with the quantum numbers of $F_{\pm}$ and as a result we find

$$\alpha^{-1}_3 (M_z) = - \frac{b_2 - b_0}{b_1 - b_2} \alpha^{-1}_1 (M_z) - \frac{b_3 - b_1}{b_1 - b_2} \alpha^{-1}_2 (M_z) + \frac{1}{2 \pi} \ln \left( \frac{M_s}{M_F} \right)$$

where the last term is the one-loop contribution of $F_{\pm}$ states to the running of the gauge couplings. This term depends on the (unknown) ratio $M_s/M_F$ and thus eq.(18) makes no prediction for $\alpha_3 (M_z)$ following from the unification condition. A detailed mechanism or further input is needed to fix the value $M_s/M_F$. Thus, although the model may allow the presence of a low (TeV) string scale, it does not predict $\alpha_3 (M_z)$.

For the purpose of exploring the most realistic possibility for unification with a low string scale we will assume the absence of the $N = 1$ states $F_{\pm}$ (this could perhaps be arranged by a different orbifold construction). With this minimal spectrum we can achieve power-law unification following from eq.(19). However there is a potential problem due to the sensitivity of such power-law unification to the thresholds. To see this, note that in eq.(19) we have assumed the same thresholds $T \equiv M_s/\mu_0$ for all three gauge group factors. $T$ is determined by the high scale cut-off $M_s$ and the low scale cut-off $\mu_0$. As discussed in Section 3 the latter is the “bare” mass and is not sensitive to radiative corrections in the supersymmetric limit. However it is sensitive to contributions from SUSY breaking and other spontaneous symmetry breaking sectors which are likely to be flavour dependent effects giving non-universal $T$. Further, as discussed above, the high scale cut-off $M_s$ is determined by the cross-channel exchanges. We have no reason to think that the coupling of the direct channel open string states to the closed channel exchange processes is not sensitive to higher order radiative corrections and SUSY breaking. Such effects are likely to give a non-universal threshold $\bar{T}$ in eq.(19). Given this it is important to determine how sensitive the predictions are to the assumption of universal thresholds in eq.(19). Since $\alpha_3 (M_z)$ is the most poorly determined of the three gauge couplings, we concentrate on it. Requiring unification should reproduce the observed value of $\alpha_3 (M_z)$ to an accuracy of $\delta \alpha_3$, we find

$$\frac{\delta T}{T} \approx \frac{12}{b_3 T} \delta \alpha^{-1}_3 \approx \frac{1}{15} \delta \alpha^{-1}_3$$

Given that $\delta \alpha^{-1}_3 \approx 0.5$ we see that $\delta T/T \approx 1/30$. This limit suggests we should choose the scales at least a factor of 30 above the SUSY breaking scale to avoid having an unacceptable

\[29\text{We take the accuracy } \delta \alpha_3 (M_z) \approx 0.006 \text{ roughly equal to the two loop contribution in the MSSM.}\]
sensitivity to SUSY breaking. This corresponds to $\mu_0 > 10^4 - 10^5$ GeV and an associated string scale $M_s > 10^6 - 10^7$ GeV. Thus we see that power-law running avoids the threshold fine tuning problem only if the string scale is relatively large.

### 6.2 Logarithmic running

An interesting possibility \cite{11, 12} for unification in Type I$'$ theories occurs if the compactification radii are equal. Then from eqs.\eqref{3},\eqref{15} we find

$$\alpha_1^{-1}(M_z) = \alpha_s^{-1} + \frac{b_i}{2\pi} \ln \frac{M_s}{M_z} + \frac{\bar{b}_i}{2\pi} \ln \frac{\mu_0}{M_s}$$

where $\bar{b}_i$ is the $\mathcal{N} = 2$ one-loop beta function coefficient. The presence of the (mild) logarithmic running would suggest that unification at a high scale ($\mu_0$) above $M_s$ is possible while keeping the string scale low (TeV region). Here we investigate this possibility. If eq.\eqref{47} is true then the usual MSSM unification will occur but with $\mu_0/M_s \approx 10^{16}$ GeV. For the case $K = 1$ the unification occurs at $10^{16}$ GeV.

What is the threshold sensitivity in this case? From eq.\eqref{50} we find $\delta T / T \approx 2\pi \delta \alpha_3^{-1}/\mathcal{O}(1)$. However this apparent relaxation of the fine tuning requirement is misleading because here the string cut-off scale $\mu_0 \approx 10^{16}$ GeV corresponds to a cross channel exchange state with mass $M_s^2/\mu_0$. For a 1 TeV string scale this is $10^{-10}$ GeV and corresponds to the mass of the KK modes in the closed string channel. However the exchanged state controlling the UV behaviour of the open string channel has vacuum quantum numbers. Such a state has no symmetry (local gauge symmetry etc) to protect it from receiving contributions to its mass at scales below the composite scale. As we have seen the KK modes have pointlike couplings up to the string scale and so we expect that such a state with vacuum quantum numbers will acquire mass from supersymmetry breaking and other symmetry breaking sectors. Thus we consider it more likely that for the case under discussion the UV divergence in the open string channel will be controlled by an exchange particle with a mass close to the supersymmetry breaking scale i.e $1 \text{TeV} \approx M_s^2/\mu_0$. If this is the case logarithmic running will not be a viable mechanism for gauge unification (with $M_s \approx 1 \text{TeV}$) because $(\mu_0/M_s) \approx 1$ and so there is no large log in eq.\eqref{50}.

### 7 Conclusions

In this paper we have considered whether string threshold corrections can be approximately described by a effective field theory valid below an UV cutoff determined by the underlying string structure. In the weakly coupled heterotic string case the UV cutoff is at the string scale. Below this scale there are only a finite number of degrees of freedom and so one can approximate the string calculation using an effective field theory in which one included all modes with mass less than the string scale and assuming pointlike coupling for these modes. In the case of type I/I$'$ theories the UV cutoff is much larger than the string scale for the contributions of Kaluza Klein (winding ) modes. However in this case too the effective field theory approach may be used because, due to supersymmetry, the string states do not contribute to the radiative corrections of gauge couplings. As a result only a finite number of massless states (before “Higgsing”) contribute together with a finite number of Kaluza Klein (winding) modes. Just as for the heterotic string case, except for the states close to the cut-off scale, these states have pointlike coupling and so an effective field theory approach is again possible.
We found that the most significant error to the calculation of gauge coupling evolution in the effective field theory approach came from the assumption of pointlike couplings for the Kaluza Klein states close to the string scale. However the effect of this approximation can be absorbed in a change of the cut-off scale. As a result the interpretation of the cut-off scale as the string scale has a significant error but the other features of the string threshold effects on gauge couplings are well described by the effective field theory approach. Moreover the effective field theory offers a convenient way to determine the structure of higher order radiative corrections in these theories. If detailed information is needed about the string scale the effective field theory calculation can be modified to take account of the non-pointlike couplings of the states close to the string scale and we presented a simple way to do this.

The string threshold effects show that the gauge couplings can have power law running in the case of large compactification scale although the effective number of “decompactified” dimensions is model dependent. This raises the interesting possibility of achieving unification of gauge couplings at a low scale, something that is desirable in theories in which the string scale is itself small. However the power law running occurs only in the $\mathcal{N} = 2$ sector and hence is governed by the gauge quantum numbers in this sector. As a result in such models the successful prediction of the usual MSSM running for gauge couplings (which comes from the $\mathcal{N} = 1$ sector) must be viewed as an accident. Moreover, while it is possible to choose the $\mathcal{N} = 2$ sector to get a low scale of unification via power law running, in order to avoid a fine tuning problem associated with the threshold of the $\mathcal{N} = 2$ states, it is necessary to have a relatively high string and compactification scale, $> \mathcal{O}(10^6 \text{GeV})$. We also examined the possibility of a theory with a low string scale but with a high scale of unification through logarithmic running. However this scheme seems to suffer from a severe fine tuning problem associated with the need for extremely light closed string states carrying vacuum quantum numbers. In the absence of a symmetry protecting these states we expect them to be driven to the supersymmetry breaking scale spoiling the mechanism needed to have logarithmic running to a very high scale. Finally, while unification through power law running is possible in theories with a low compactification and string scale, it seems much more contrived than the original proposal of logarithmic running and unification at a high scale.

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