Infinite-Dimensional Fisher Markets: Equilibrium, Duality and Optimization

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Abstract
This paper considers a linear Fisher market with \( n \) buyers and a continuum of items. In order to compute market equilibria, we introduce (infinite-dimensional) convex programs over Banach spaces, thereby generalizing the Eisenberg-Gale convex program and its dual. Regarding the new convex programs, we establish existence of optimal solutions, the existence of KKT-type conditions, as well as strong duality. All these properties are established via non-standard arguments, which circumvent the limitations of duality theory in optimization over infinite-dimensional vector spaces. Furthermore, we show that there exists a pure equilibrium allocation, i.e., a division of the item space. Similar to the finite-dimensional case, a market equilibrium under the infinite-dimensional Fisher market is Pareto optimal, envy-free and proportional. We also show how to obtain the (a.e. unique) equilibrium price vector and a pure equilibrium allocation from the (unique) \( n \)-dimensional equilibrium bang-price vector. When the item space is the unit interval \([0, 1]\) and buyers have piecewise linear utilities, we show that \( \epsilon \)-approximate equilibrium prices can be computed in time polynomial in the market size and \( \log 1/\epsilon \). This is achieved by solving a finite-dimensional convex program using the ellipsoid method. To this end, we give nontrivial and efficient subgradient and separation oracles. For general buyer valuations, we propose computing market equilibrium using stochastic dual averaging, which finds an approximate equilibrium price vector with high probability.

1 Introduction
Market equilibrium is a classical concept from economics, where the goal is to find an allocation of a set of items to a set of buyers, as well as corresponding prices, such that the market clears. One of the simplest equilibrium models is the (finite-dimensional) linear Fisher market. A Fisher market consists of a set of \( n \) buyers and \( m \) divisible items, where the utility for a buyer is linear in their allocation. Each buyer \( i \) has a budget \( B_i \) and valuation \( v_{ij} \) for each item \( j \). A ME consists of an allocation (of items to buyers) and prices (of items) such that (i) each buyer receives a bundle of items that maximizes their utility subject to their budget constraint, and (ii) the market clears (all items such that \( p_j > 0 \) are exactly allocated). In spite of its simplicity, this model has several applications. Perhaps the most well-known application is in the competitive equilibrium from equal incomes (CEEI), where \( m \) items must be fairly divided among \( n \) agents. By giving each agent one unit of faux currency, the allocation from the resulting ME can be used as the fair division. This approach guarantees several fairness desiderata. Linear Fisher markets also have applications in large-scale ad markets (Conitzer et al. 2018; 2019) and fair recommender systems (Kroer et al. 2019; Kroer and Peysakhovich 2019). For the case of finite-dimensional linear Fisher markets, the Eisenberg-Gale convex program computes a market equilibrium (Eisenberg and Gale 1959; Eisenberg 1961; Jain and Vazirani 2010; 2010; Nisan et al. 2007; Cole et al. 2017). However, in settings like Internet ad markets and fair recommender systems, the number of items is often huge (Kroer et al. 2019; Kroer and Peysakhovich 2019; Balseiro, Besbes, and Weintraub 2015), if not infinite or even uncountable. For example, each item can be characterized by a set of features, where features come from a compact set in a Euclidean space. In order to capture this, we study Fisher markets and ME for a continuum of items.

A problem closely related to our infinite-dimensional Fisher-market setting is the cake-cutting or fair division problem. There, the goal is to efficiently partition a “cake” – often modeled as a compact measurable space, or simply the unit interval \([0, 1]\) – among \( n \) agents so that certain fairness and efficiency properties are satisfied (Weller 1985; Brams and Taylor 1996; Cohler et al. 2011; Procaccia 2013; Cohler et al. 2011; Brams et al. 2012; Chen et al. 2013; Aziz and Ye 2014; Aziz and Mackenzie 2016; Legut 2017; 2020). See (Procaccia 2014) for a survey for the various problem setups, algorithms and complexity results. Weller (1985) shows the existence of a fair allocation, that is, a measurable division of a measurable space satisfying weak Pareto optimality and envy freeness. As will be seen shortly, when all buyers have the same budget, our definition of a pure ME, i.e., where the allocation consist of indicator functions of a.e.-disjoint measurable sets, is in fact equivalent to this notion of fair division, that is, a Pareto optimal, envy-free division (also see, e.g., (Cohler et al. 2011; Chen et al. 2013)). Additionally, we also give an explicit
characterization of the unique equilibrium prices based on a pure equilibrium allocation under arbitrary budgets, generalizing the result of Weller (1985) for uniform budgets.

Under piecewise constant valuations over the cake \([0, 1]\), the equivalence of fair division and market equilibrium in certain setups has been discovered and utilized in the design of cake-cutting algorithms (Brams et al. 2012; Aziz and Ye 2014). Here, we extend this connection to arbitrary valuations in the \(L_1\) function space: we propose Eisenberg-Gale-type convex programs that characterize all ME (and hence all fair divisions). As a concrete example, we show that we can efficiently compute approximate equilibrium bang-per-buck variables and prices under piecewise linear valuations, partially addressing an open problem posed by Aziz and Ye (2014). A recent related work is that of Legut (2017), which proposes optimization formulations for “optimal” fair division under linear utilities. This work considers optimality and fairness notions that are different from the more commonly used one we adopt. Furthermore, the optimization formulation is non-convex, and does not lead to market equilibrium properties.

**Summary of contributions.** First, we propose a natural generalization of market equilibrium (ME) for a Fisher market with \(n\) buyers and a continuum of items \(\Theta\). We then give two infinite-dimensional convex programs over Banach spaces of measurable functions on \(\Theta\), generalizing the EG convex program and its dual for a finite-dimensional Fisher market. For the new convex programs, we first establish existence of optimal solutions (Lemma 1, 2 and 3). Due to the lack of a compatible constraint qualification, general duality theory does not apply to these convex programs. Instead, we establish various duality properties directly through nonstandard arguments (Lemma 4). Based on these duality properties and the existence of a minimizer in the “primal” convex program (1), we show that an allocation and prices pair is a ME if and only if they are optimal solutions to our convex programs (Theorem 2). Furthermore, since (1) has a pure optimal solution (i.e., buyers get disjoint measurable item subsets), there exists a pure equilibrium allocation, i.e., a division (module zero-value item) of the item space. Finally, we show that a ME under the infinite-dimensional Fisher market satisfies (budget-weighted) proportionality, Pareto optimality and envy-freeness. Our results on the existence of ME and its fairness properties can be viewed as generalizations of those in (Weller 1985), in which every buyer (agent) has the same budget. All results, characterizations of the unique equilibrium prices based on a pure equilibrium allocation under arbitrary budgets, generalize the result of Weller (1985) for uniform budgets.

2 **Infinite-dimensional Fisher Markets**

First we introduce the measure-theoretic concepts that we will need. This paragraph can be skimmed and referred back to later. The items will be represented by \(\Theta\), a compact subset of \(\mathbb{R}^d\). Denote the Lebesgue measure on \(\mathbb{R}^d\) as \(\mu\). Let \(\mathcal{M}\) be the set of real-valued (Borel) measurable functions on \(\Theta\). Since \(\Theta\) is compact, it is (Borel) measurable and \(\mu(\Theta) < \infty\). Note that functions that are equal a.e. on \(\Theta\) form an equivalence class, which we treat as the same function. In fact, any function \(f\) in this equivalence class give the same functional \(g \mapsto \int f g d\mu\). Meanwhile, for any Lebesgue measurable \(f\) on \(\Theta\), there exists \(f \in \mathcal{M}\) such that \(f = f\). The suffix a.e. will be omitted unless the emphasis is necessary. For any \(A \subseteq \mathcal{M}\), denote \(S_+ = \{ f \in S : f \geq 0 \}\). For \(g, f \in \mathcal{M}_+\), denote \(\langle f, g \rangle = \int f g d\mu\). This notation aligns with the usual notation for bilinear form, i.e., applying a bounded linear functional \(g \in \mathcal{X}^*\) to \(f \in \mathcal{X}\). Let \(I\) be the constant function taking value 1 on \(\Theta\) and \(1_A\) be the indicator function of measurable \(A \subseteq \Theta\). For a cone \(\mathcal{P} \subseteq \mathcal{X}\), denote its dual cone as \(\mathcal{P}^* = \{ g \in \mathcal{X}^* : \langle f, g \rangle \geq 0, \forall f \in \mathcal{X} \}\). For \(g \in [1, \infty]\), let \(L_q(\Theta)\) be the Banach space of \(L_q\) (integrable) functions on \(\Theta\) with the usual \(L_q\) norm. When \(q = \infty\), the norm is \(\| f \| = \inf \{ M > 0 : \| f \| \leq M \text{ a.e.} \}\). Any \(\tau \in L_1(\Theta)\) can also be viewed as a measure on \(\Theta\) via \(\mu_\tau(A) := \int \tau d\mu\) for any measurable \(A \subseteq \Theta\). Here, \(\tau\) is the Radon-Nikodym derivative of \(\mu_\tau\) w.r.t. \(\mu\). We will denote \(\mu_\tau(A)\) simply as \(\tau(A)\) for a measurable \(A \subseteq \Theta\) whenever there is no confusion. In this work, any measure \(\mu\) used or constructed is absolutely continuous w.r.t. the Lebesgue measure \(\mu\) and hence atomless: for any measurable set \(A \subseteq \Theta\) such that \(\mu(A) > 0\) and any \(c \in (0, \mu(A))\), there exists measurable \(B \subseteq A\) such that \(\mu(B) = \Theta\). Two measurable sets \(A, B \subseteq \Theta\) are a.e.-disjoint if \(\mu(A \cap B) = 0\). We say that \(\{ \Theta_i \}\) is a measurable partition of \(\Theta\) if all \(\Theta_i \subseteq \Theta\) are measurable and pairwise a.e.-disjoint. As is customary, we often use equations and inequalities of measurable functions in \(\mathcal{M}\) to denote the corresponding (measurable) preimages in \(\Theta\). For example, \(\{ f \leq 0 \} := \{ \theta \in \Theta : f(\theta) \leq 0 \}\) and \(\{ f = 0 \} := \{ \theta \in \Theta : f(\theta) = 0 \}\).

**Fisher market.** There are \(n\) buyers and a compact continuum of items \(\Theta \subseteq \mathbb{R}^d\), as described above. Let valuations and allocations \(\mathcal{V}, \mathcal{X} \subseteq \mathcal{M}\) be Banach spaces. Unless otherwise stated, we assume \(\mathcal{V} = L_1(\Theta)^*\) and \(\mathcal{X} = L_\infty(\Theta)\). Each buyer has some valuation \(v_i \in \mathcal{V}\). The prices \(p \in \mathcal{V}_+\) live in the same space as valuations. An allocation of items to a buyer \(i\) is denoted by \(x_i \in \mathcal{X}_+\). We use \(x = (x_1, \ldots, x_n) \in \mathcal{X}_+^n\) to denote the overall allocation.

An allocation \(x\) is said to be a pure allocation if for all \(i\), \(x_i = 1_{\Theta_i}\) for a.e.-disjoint measurable \(\Theta_i \subseteq \Theta\) (where left-over is possible, i.e., \(\Theta \setminus \bigcup_i \Theta_i \neq \emptyset\)). In this case, we also denote \(x\) as \(\{ \Theta_i \}\). An allocation is mixed if it is not pure, or equivalently, \(\{ 0 < x_i < 1 \}\) has positive measure for some \(i\). Each buyer has a budget \(B_i > 0\) and all items have unit supply, i.e., \(x\) is supply-feasible if \(\sum_i x_i \leq 1\). Without loss of generality, we also assume that \(v_i(\Theta) = \| v_i \| > 0\) for all \(i\) (otherwise buyer \(i\) can be removed).
Given prices $p \in \mathcal{X}_+$, the demand set of buyer $i$ is
\[ D_i(p) = \arg \max \{ \langle u_i, x_i \rangle : x \in \mathcal{X}_+, \langle p, x \rangle \leq B_i \}. \]

Generalizing the finite-dimensional counterpart (Eisenberg and Gale 1959; Eisenberg 1961; Jain and Vazirani 2007; 2010; Nisan et al. 2007), a market equilibrium is defined as a pair $(x^*, p^*) \in \mathcal{X}^* \times \mathcal{V}$ satisfying the following:

- Buyer optimality: for every $i \in [n]$, $x_i^* \in D_i(p^*)$.
- Market clearance: $\sum_i x_i^* \leq 1$ and $(p^*, 1 - \sum_i x_i^*) = 0$.

We say that $x^* \in \mathcal{X}^*$ is an equilibrium allocation if $(x^*, p^*)$ is a ME for some $p^* \in \mathcal{V}$. A pair $(x^*, p^*)$ is called a pure ME if it is a ME and $x^*$ is a pure allocation. From the definition of ME, we can assume $v_i(\Theta) = 1$ for all $i$ w.l.o.g.

We can also assume $\|B\|_1 = \sum_i B_i = 1$ w.l.o.g., since $(x^*, p^*)$ is a ME under $B = (B_i)$, then $(x^*, p^*/\|B\|_1)$ is a ME under $(B_i/\|B\|_1)$.

## 3 Equilibrium and duality

Since infinite-dimensional convex duality does not allow us to conclude strong duality, we cannot start with a convex program and then derive its dual. Instead, we directly propose three convex programs, and then proceed to show from first principles that a strong duality-like relationship holds. First, we give a direct generalization of the Eisenberg-Gale first principles that a strong duality-like relationship holds. We show the existence of an optimal (pure) solution of (1) and (2) satisfies Slater’s condition does not hold. If we choose $\mathcal{V}_+ = L_\infty(\Theta)$ instead of $L_1(\Theta)$, then (2) satisfies Slater’s condition (Luenberger 1997, §8.8 Problem 2).

### Lemma 1. The supremum $z^*$ in (1) is attained at some $x^*$ such that $x_i^* = 1_{\Theta_i}$, where $\{\Theta_i\}$ are disjoint measurable subsets of $\Theta$ (can also require partition, i.e., $\cup_i \Theta_i = \Theta$).

Motivated by the dual of the finite-dimensional EG convex program (Cole et al. 2017, Lemma 3), we also consider the following convex program:

\[ w^* = \inf_{p \in \mathcal{V}_+, \beta \in \mathbb{R}^n_+} \left[ \langle p, 1 \rangle - \sum_i B_i \log \beta_i \right] \]
\[ \text{s.t. } p \geq \beta_i v_i \text{ a.e.}, \forall i. \]

For a fixed $\beta > 0$, intuitively, we should set $p = \max_i \beta_i v_i$. The following lemma ensures that this choice is well-defined.

### Lemma 2. When $\mathcal{V} = L_q(\Theta)$ for $q \in [1, \infty]$, for any $v_i \in \mathcal{V}_+$ and $\beta \in \mathbb{R}^n_+$, we have $\max_i \beta_i v_i \in \mathcal{V}_+$.

Thus we can consider the following finite-dimensional convex program:

\[ \inf_{\beta \in \mathbb{R}^n_+} \left[ \max_i \beta_i v_i, 1 \right] - \sum_i B_i \log \beta_i \right]. \]

\[ \text{(3)} \]

### Lemma 3. The minimum of problem (3) is attained at a unique minimizer $\beta^* > 0$. The optimal solution $(p^*, \beta^*)$ of (2) is also (a.e.) unique and satisfies $p^* = \max_i \beta_i^* v_i$.

**Remark.** Here, our approach is to first define a pair of convex programs (1) and (2), and then show that they exhibit various duality properties. Specifically, when (2) is viewed as the “primal” and (1) the associated “dual”, conditions (4) and (5) can be interpreted as first-order stationarity conditions and (6) is the complementary slackness condition. However, these properties cannot be concluded directly from the general convex optimization duality theory in Banach spaces (see, e.g., (Ponstein 2004, §3) and (Luenberger 1997, §8.6)). For example, if we view (2) as the primal, then its dual is (1) and weak duality follows. However, we cannot conclude strong duality, or even primal or dual optimal attainment, since $\mathcal{V}_+$ has an empty interior (Luenberger 1997, §8.8 Problem 1) and hence Slater’s condition does not hold. If we choose $\mathcal{V}_+ = L_\infty(\Theta)$ instead of $L_1(\Theta)$, then (2) satisfies Slater’s condition (Luenberger 1997, §8.8 Problem 2).

Due to the above remark, we now establish weak duality and necessary and sufficient conditions for strong duality directly. Our proofs of optimum attainment (Lemmas 1 and 3) were necessary for the same reason.

### Lemma 4. Let $C = \|B\|_1 - \sum_i B_i \log B_i$. We have

- Weak duality: $C + z^* \leq w^*$.
- When $C + z^* = w^*$ and $z^*$, $w^*$ are attained by $x^*$ feasible to (1) and $(p^*, \beta^*)$ feasible to (2), respectively, we have

\[ \langle p^*, 1 - \sum_i x_i^* \rangle = 0, \]
\[ \langle v_i, x_i^* \rangle = u_i^* := \frac{B_i}{\beta_i^*}, \forall i, \]
\[ \langle p^* - \beta^*_i v_i, x_i^* \rangle = 0, \forall i. \]

- Suppose $x^*$ is feasible to (1), $(p^*, \beta^*)$ is feasible to (2) and (4), (5) hold. Then, $x^*$ and $(p^*, \beta^*)$ are optimal to (1) and (2), respectively. Meanwhile, $C + z^* = w^*$.

At the moment, the above lemma is “vacuous”: we do not know whether $C + z^* = w^*$ holds or, equivalently, whether there exists feasible solutions satisfying (4) and (5). This will be answered affirmatively by Theorem 1, where we show that, from a a pure optimal solution of (1), we can construct the (a.e. unique) optimal solution of (2).

### Theorem 1. Let $x^*$ solves (1) and be that $x_i^* = 1_{\Theta_i}$, where $\Theta_i$ are a.e.-disjoint subsets of $\Theta$ (existence of such $x^*$ given by Lemma 1). Let $u_i^* = v_i(\Theta_i)$ and $\beta_i^* = B_i/u_i^*$. Then, for each $i$, we have $\beta_i^* v_i \geq \beta_j^* v_j$ a.e. for all $j \neq i$ on $\Theta_i$. 

• Define \( p^* = \max_i \beta_i^* v_i \). Then, for any measurable \( A \subseteq \Theta \), it holds that \( p^*(A) = \sum_i \beta_i^* v_i(A \cap \Theta) \).
• The constructed \((p^*, \beta^*)\) solves (2) and satisfies (4)-(6).

The above theorem ensures that there exists optimal solutions of (1) and (2) satisfying all predicates in Lemma 4. Thus, we see that in spite of the general difficulties with duality theory in infinite dimensions, we have shown that (1) and (2) behave like duals of each other: strong duality theory in infinite dimensions, we have shown that (1) and (2) behave like duals of each other: strong duality holds, and we have KKT-like conditions (see, e.g., [Nisan et al. 2007, §5.2]) for the finite-dimensional counterparts. In fact, we can even strengthen the above to show that optimal solutions of (1) and (2) are ME, and vice versa.

**Theorem 2.** Assume \( x^* \) and \((p^*, \beta^*)\) solve (1) and (2), respectively. Then \( (x^*, p^*) \) is a ME such that \( p^*(x_i^*) = B_i \) and the equilibrium utility of buyer \( i \) is \( u_i^* = \langle v_i, x_i^* \rangle = B_i / \beta_i^* \). Conversely, if \( (x^*, p^*) \) is a ME, then \( x^* \) solves (1) and \((p^*, \beta^*)\), where \( \beta_i^* := \frac{B_i}{\langle v_i, x_i^* \rangle} \), solves (2).

We list some useful direct consequences of the results we have so far. Theorems 1 and 2 give the following.

**Corollary 1.** Let \( x^* \) be as in Theorem 1, that is, \( x^* \) solves (1) and is pure. Then, \( x^* \) is a pure equilibrium allocation.

**Lemma 4 and Theorem 2** give the following.

**Corollary 2.** Let \((x^*, p^*)\) be a ME. Then, \( x^* \) and \((p^*, \beta^*)\), where \( \beta_i^* := \frac{B_i}{\langle v_i, x_i^* \rangle} \), satisfy (6). In other words, for each \( i \), \( x_i^* = 0 \) a.e. on \( \{p^* > \beta_i^* v_i\} \).

**Properties of ME.** Let \( x \in \mathcal{X}^n \), \( \sum_i x_i \leq 1 \) be an allocation. It is (strongly) Pareto optimal if there does not exist \( \tilde{x} \in \mathcal{X}^n \), \( \sum_i \tilde{x}_i \leq 1 \) such that \( \langle v_i, \tilde{x}_i \rangle \geq \langle v_i, x_i \rangle \) for all \( i \) and the inequality is strict for at least one \( i \) (Cohler et al. 2011). It is envy-free (w.r.t. the budgets) if \( \frac{B_i}{\beta_i} \langle v_i, x_i \rangle \geq \langle v_i, x_j \rangle \) for any \( j \neq i \). When all \( B_i = 1 \), this is sometimes referred to as being “equitable” (Weller 1985). It is proportional if \( \langle v_i, x_i \rangle \geq \frac{B_i}{\beta_i} v_i(\Theta) \) for all \( i \). Similar to the finite-dimensional case (Jain and Vazirani 2010; Nisan and Ronen 2001), market equilibria in infinite-dimensional Fisher markets also have these properties.

**Theorem 3.** Let \((x^*, p^*)\) be a ME. Then, \( x^* \) is Pareto optimal, envy-free and proportional.

**Bounds on equilibrium quantities.** We can establish upper and lower bounds on equilibrium quantities (recall that \( v_i(\Theta) = 1 \) and \( \|\beta_i\| = 1 \) W.L.O.G.). These bounds will be useful in efficient optimization of (3). Similar bounds hold in the finite-dimensional case (Gao and Kroer 2020).

**Lemma 5.** For any ME \((x^*, p^*)\), we have \( p^*(\Theta) = 1 \). Furthermore, \( B_i \leq u_i^* = \langle v_i, x_i^* \rangle \leq 1 \) and hence \( \beta_i \geq 1 \leq \beta_i^* := \frac{B_i}{u_i^*} \leq B_i^* := \frac{1}{B_i} \) for all \( i \).

**Discrete item space \( \Theta \).** All of the results so far, except the existence of a pure solution of (1) (hence a pure equilibrium allocation), hold even when \( \Theta \) is discrete (finite or countably infinite). In this case, \( \mu(A) = |A| \), the cardinality of \( A \subseteq \Theta \), is the counting measure on \( \Theta \) and \( v_i \in E(\Theta) \) are summable sequences, i.e., \( v_i(\Theta) = \langle v_i, 1 \rangle = \sum_{i \in \Theta} v_i(\theta) < \infty \).

\[ \langle f, g \rangle := \sum_{\theta \in \Theta} f(\theta)g(\theta) \] for any \( f, g : \Theta \rightarrow \mathbb{R} \). If \( v_i \) is summable, its induced measure is clearly finite and absolutely continuous w.r.t. \( \mu \). In this case, Lemma 1 partially holds. The supremum can still be attained by some \( x^* \) (c.f. proof of Lemma 10 and its use of [Dvoretzky et al. 1951, Theorem 1]). However, it may not be attained by a pure solution. This is because the proof of existence of a pure optimal solution relies on [Dvoretzky et al. 1951, Theorem 5], which requires \( v_i \) to be atomless. When \( \Theta \) is discrete, any nonzero \( v_i \), as a measure, is necessarily atomic: \( v_i(\{\theta\}) > 0 \) for some singleton \( \{\theta\} \), which clearly does not contain any subset of a smaller nonzero measure.

### 4 Efficient optimization of (3)

In the rest of the paper, unless otherwise stated, we always use \( x^* \) or \( \{\Theta_i\} \) to denote a pure equilibrium allocation. We also use \( \beta^* \) to denote the unique optimal solution of (3) and \( p^* \) the a.e. unique equilibrium prices, which satisfy \( p^* = \max_i \beta_i^* v_i \) and (4)-(6) together with \( x^* \) (Lemma 4 and Theorem 2). The convex program (3) is finite-dimensional and has a real-valued, convex and continuous objective function (Lemma 3). By Lemma 5, we can also add the constraint \( \beta \in [\beta, \bar{\beta}] \) without affecting the optimal solution. This makes the “dual” (3) more computationally tractable than its “primal” (1).

**Ellipsoid method for piecewise linear \( v_i \).** We show that, for piecewise linear (p.w.l.) \( v_i \) over \( \Theta = [0, 1] \), we can compute a solution \( \beta \) such that \( \|\beta - \beta^*\| \leq \epsilon \) in time polynomial in \( \log_2 \frac{1}{\epsilon}, n \) and \( K = \sum_i K_i \). This is achieved via solving (3) using the ellipsoid method. Consider the following generic convex program (Ben-Tal and Nemirovski 2019, §4.1.4):

\[ f^* := \min \ f(x) \text{ s.t. } x \in X \] (7)

where \( f \) is convex and continuous (and hence subdifferentiable) on a convex compact \( X \subseteq \mathbb{R}^n \). Assume we have access to the following oracles:

- The separation oracle \( S \): given any \( x \in \mathbb{R}^n \), either report \( x \in int X \) or return a \( g \neq 0 \) (representing a separating hyperplane) such that \( \langle g, x \rangle \geq \langle g, y \rangle \) for any \( y \in X \).
- The first-order or subgradient oracle \( G \): given \( x \in int X \) (the interior of \( X \)), return a subgradient \( f'(x) \) of \( f \) at \( x \), that is, \( f(y) \geq f(x) + \langle f'(x), y - x \rangle \) for any \( y \).

The time complexity of the ellipsoid method is as follows.

**Theorem 4.** (Ben-Tal and Nemirovski 2019, Theorem 4.1.2)

Let \( V = \max_{x \in X} f(x) - f^* \), \( R = \sup_{x \in X} \|x\| \), and \( r > 0 \) be the radius of a Euclidean ball contained in \( X \). For any \( \epsilon > 0 \), it is possible to find an \( \epsilon \)-solution \( x_\epsilon \), (i.e., \( f(x_\epsilon) \leq f^* + \epsilon \)) with no more than \( N(\epsilon) \) calls to \( S \) and \( G \), followed by no more than \( O(1)n^2 N(\epsilon) \) arithmetic operations to process the answer of the oracles, where \( N(\epsilon) = O(1)n^2 \log \left( 2 + \frac{VR}{\epsilon^2} \right) \).

In order to make use of the ellipsoid method for (2) for p.w.l. \( v_i \), we need to derive efficient oracles \( S \) and \( G \). To this end, we need some elementary lemmas regarding p.w.l. linear functions.
Lemma 6. For any \( \beta \in \mathbb{R}_{+}^{d} \), the function \( \theta \mapsto \max_i \beta_i v_i(\theta) \) is piecewise linear with at most \( n(K + n + 1) \) pieces.

Lemma 7. Suppose \( f_i(\theta) = c_i \theta + d_i \geq 0 \), for all \( \theta \in [l, u] \subseteq [0, 1], i \in [n] \). Then, \( h_n(\theta) = \max_i f_i(\theta) \) is piecewise linear on \([l, u]\) with at most \( n \) pieces. Furthermore, the breakpoints of \( h_n, l = a_0 < a_1 < \cdots < a_n = u \) \((n' \leq n) \) can be found in \( O(n^2) \) time.

Assume that each \( v_i \) is \( K_i \)-piecewise linear (possibly discontinuous). There are in total \( K = \sum_i K_i \) pieces. Denote \( \phi(\beta) = (\max_i \beta_i v_i, 1) \). From the proof of Lemma 3, we know that \( \phi \) is finite, convex and continuous on \( \mathbb{R}_{+}^{d} \). Hence, it is subdifferentiable on \( \mathbb{R}_{+}^{d} \) (Ben-Tal and Nemirovski 2019, Proposition C.6.5). First, we show that, if all \( v_i \) are linear on a common interval and zero otherwise, a subgradient of \( \phi(\beta) \) can be constructed in \( O(n^2) \) time. This utilizes the additivity (in terms of integration or expectation) property of subgradients, as formalized in the following lemma. Here, \( \Theta \subseteq \mathbb{R}^d \) can be a generic compact set and \( e(i) \) is the \( i \)th unit vector in \( \mathbb{R}^d \).

Lemma 8. Let \( f(\beta, \theta) = \max_i \beta_i v_i(\theta) \). For any \( \theta \in \Theta \), a subgradient of \( f(\cdot, \theta) \) at \( \beta = g(\beta, \theta) \) is \( v_i(\theta) \theta^{i'} \), where \( i' \in \{\max_i \beta_i v_i(\theta) \} \) (taking the smallest index if there is a tie). Hence, a subgradient of \( \phi(\beta) = \int_{\Theta} g(\beta, \theta) d\theta = \mu(\Theta) \cdot E_{\theta} g(\beta, \theta) \) where the expectation is over \( \Theta \sim \text{Unif}(\Theta) \).

Using Lemma 8 and the p.w.l. structure of \( v_i \), we have the following for computing a subgradient of \( \phi \).

Lemma 9. For each \( i \), assume that \( v_i(\theta) = c_i \theta + d_i \geq 0 \) on an interval \([l, u] \subseteq [0, 1]\).

- The function \( \theta \mapsto \max_i \beta_i v_i(\theta) \) has at most \( n \) linear pieces on \([l, u]\), with breakpoints \( l = a_0 < a_1 < \cdots < a_n = u \) (depending on \( \beta \)).
- We can construct \( \phi(\beta) \in \partial \phi(\beta) \) for any \( \beta > 0 \) as follows: the \( i \)th component of \( \phi(\beta) \) is

\[
\sum_{k \in [i], i_k = i} \left( \frac{c_{i_k}}{2} (a_{i_k}^2 - a_{i_k-1}^2) + d_{i_k} (a_{i_k} - a_{i_k-1}) \right),
\]

where \( i_k \) is the (unique) winner (with the smallest index among ties) on \([a_{i_k-1}, a_{i_k}]\).

- The above construction of \( \phi(\beta) \) takes \( O(n^2) \) time.

When \( v_i \) are \( K_i \)-piecewise linear on \([0, 1]\), using Lemma 9, we can compute a subgradient \( \phi(\beta) \) by summing up the above construction over the intervals given by the breakpoints of all \( v_i \), and there are at most \( K \) such intervals.

The ellipsoid method can be applied to (3) more generally than for the case of p.w.l. \( v_i \). It finds a solution \( \beta \) that is \( \epsilon \)-close to \( \beta^* \) in time \( \log \frac{1}{\epsilon} \) as long as we can compute \( \phi'(\beta) \) efficiently. By Lemma 8, since a “pointwise” subgradient \( g(\beta, \theta) \) of \( f(\beta, \theta) \) is much easier to compute, we can compute a “full” subgradient \( \phi'(\beta) \) efficiently as long as the integral (expectation) can be evaluated efficiently.

Stochastic optimization for general \( \Theta \) and \( v_i \). When a full subgradient \( \phi'(\beta) \) is difficult to compute, we can still utilize the expectation characterization in Lemma 8 to use a stochastic optimization algorithm to solve (3). The problem structure is particularly suitable for the stochastic dual averaging (SDA) algorithm (Xiao 2010; Nesterov 2009). It solves problems of the following form:

\[
\min_{\beta} E_{\theta} f(\beta, \theta) + \Psi(\beta),
\]

where \( \Psi \) is a strongly convex regularization function such that dom \( \Psi = \{ \beta : \Psi(\beta) < \infty \} \) is closed. Let \( \theta \sim \mathcal{D} \) be a random variable with distribution \( \mathcal{D} \) and \( f(\cdot, \theta) \) is convex and subdifferentiable on dom \( \Psi \) for all \( \theta \in \Theta \). The algorithm works as follows (Xiao 2010, Algorithm 1). Again, assume \( \mu(\Theta) = 1 \).

Theorem 7. Assume \( v_i \in L_2(\Theta) \), that is, \( v_i^2, 1 \) is \( E_{\theta} |v_i(\theta)|^2 < \infty \) for all \( i \). Let \( G^2 := E_{\theta} |v_i(\theta)|^2 < \infty \) and \( \sigma = \min_i B_i^4 \). Let \( \beta^t := \frac{1}{t} \sum_{i=1}^{t} \beta_i^t \) and \( \beta_i^t := \frac{1}{t} \sum_{i=1}^{t} \beta_i^t \). Then,

\[
\mathbb{E}[\|\beta^t - \beta^*\|^2] < \frac{6(1+\log t)}{t^2} \cdot \frac{M_4}{\sigma^2},
\]

Next, further assume that \( v_i \leq G \) a.e. for all \( i \). Then, for any \( \delta > 0 \), with probability at least \( 1 - 4\delta \log t \), we have \( \|\hat{\beta}^t - \beta^*\|^2 \leq 2M_4 \), where

\[
M_4 = \frac{4G^2}{t} \cdot \frac{\Delta_t^2}{(6+ \log t)} \quad \text{and} \quad V = \frac{2n}{\min_i B_i^4}.
\]

Remark. In the above theorem, the bound on \( \mathbb{E}[\|\beta^t - \beta^*\|^2] \) (MSE) is of order \( O \left( \frac{\log(t)^2}{t} \right) \), where the constant degrades...
upon buyer heterogeneity, i.e., a smaller $\min_i B_i$ leads to a larger bound (recall that $|B_i| = 1$ and therefore $\min_i B_i \leq \frac{1}{n}$). For the second half regarding $\|\beta^t - \beta^*\|^2$, substituting $\delta = \frac{1}{n} (\alpha \geq 1)$ yields a bound of order $O(\frac{\log t}{n})$ (also depending inversely on $\min_i B_i$), with probability at least $1 - \frac{4\log t}{n}$. In addition, the added assumptions $\mathbb{E}_n[\max_i v_i^2] < \infty$ and $v_i \leq G$ a.e. for all $i$ are always satisfied as long as they are (a.e.) bounded (e.g., p.w.l. functions).

**Deterministic optimization using $\phi(\beta)$.** When $\phi(\beta)$ can be computed, such as when $v_i$ are piecewise linear on $\Theta = [0, 1]$ (Theorem 5), in Algorithm 1, we can replace $g'$ with a full subgradient $\phi'(\beta^t)$. Then, $\|\beta^t - \beta^*\|^2$ is deterministic and bounded by the same right hand side as the first half of Theorem 7. In this case, if $v_i \leq G$ a.e., then it can be easily verified that $\|\phi(\beta)\|^2 \leq nG^2 < \infty$ for all $\beta > 0$. Then, we can also use a projected subgradient descent method that achieves $\|\beta^t - \beta^*\|^2 = O(n/t)$, where $\beta^t$ is a weighted average of $\beta_1^*, \ldots, \beta_n^*$ (see, e.g., (Lacoste-Julien, Schmidt, and Bach 2012) and (Bubeck 2015, Theorem 3.9)).

**Approximate equilibrium prices.** Suppose we have obtained an approximate solution $\beta$ such that $\|\beta - \beta^*\| < \epsilon$. Then, define $\bar{p} = \max_i \bar{p}_i$, which is in $L_1(\Theta)$, by Lemma 2. We have $\|\bar{p} - p^*\| = \int_{\Theta} \max_i \bar{p}_i - \max_i \beta^*_i v_i \, \mu \leq \int_{\Theta} \|\beta - \beta^*\| \, \mu \leq n\epsilon$. Recall that for any $\beta$, the prices $\bar{p}$ defined above is a p.w.l. function with at most $n(K - 1)$ pieces (Lemma 6). By Lemma 7, finding its p.w.l. representation (breakpoints and linear coefficients on each piece) takes $O(nK)$ time. Therefore, under the same complexity as in Theorem 6 (where the additional factor $n$ is inside log and is absorbed into the constant), we can compute an approximate equilibrium prices $\bar{p}$ such that $\|\bar{p} - p^*\| < \epsilon$. Furthermore, under prices $\bar{p}$, the true $x^*$ may violate buyers’ budget constraints: $(\bar{p}, x^*) = (p^*, x^*_i) + (\bar{p} - p^*, x^*_i) \leq B_i + (\bar{p} - p^*) = B_i + n\epsilon$, where the inequality uses Theorem 2 (budget of buyer $i$ depleted, i.e., $p^*, x^*_i = 1$ and $x^*_i \leq 1$. Hence, consider the mixed allocation $\bar{x}_i = \frac{B_i + n\epsilon}{B_i + n\epsilon}$ for all $i$. This allocation satisfies $\sum_i \bar{x}_i \leq 1$, and, for each $i$, its budget constraint is satisfied: $\langle \bar{p}, \bar{x}_i \rangle = \frac{B_i}{B_i + n\epsilon} \langle \bar{p}, x^*_i \rangle \leq B_i$. Meanwhile, the utility of buyer $i$ from $\bar{x}_i$ is $u_i = \langle v_i, \bar{x}_i \rangle = \frac{B_i}{B_i + n\epsilon} u_i^*$, which is close to the equilibrium utility $u_i^*$ as long as $n\epsilon \ll B_i$.

**5 Construction of an equilibrium allocation**

Throughout this section (as in §4), $p^*$ is the optimal solution of (3), $p^* = \max_i \beta_i^* v_i$ is the equilibrium prices and $v_i^* = \frac{B_i}{p^*}$ is the equilibrium utility. Given $\beta^*$, we can construct a pure equilibrium allocation $x^*$ explicitly if we allow arbitrary division of measurable subsets. This is possible only if $v_i$, and hence $p^*$, are atomless.

**Theorem 8.** For any $S \subseteq [n], \Theta_S = \{p^* = \beta_i^* v_i > \beta_j^* v_j, \forall i \in S, \ell \notin S\}$ if $S \neq \emptyset$, \{p^* = \beta_i^* v_i = 0, \forall i \in [n]\} if $S = \emptyset$, $\Theta_S \subseteq [n]$ is a measurable partition of $\Theta$.

**5.1 Existence of an equilibrium allocation.** There exists $\Theta_S$ such that $\bigcup_{i \in S} \Theta_S = \Theta$, $\bigcap_{i \notin S} \Theta_S = \emptyset$, and $\Theta_S \subseteq \Theta_S$ for all $S \subseteq [n]$. $\Theta$ is an optimal solution of (1) and, by Theorem 2, a pure equilibrium allocation.

The above construction of $x^*$ requires constructing $\Theta_S$, $S \subseteq [n]$ via partitioning measurable sets and solving a large linear system of size $2^n$. Nonetheless, often an equilibrium allocation, or even a pure one, can be constructed more easily. We give some concrete examples below.

**Finite item space $\Theta$.** In this case, simply solve the linear feasibility problem $\sum_{\Theta \subseteq \Theta} (\Theta) = u_i^*$, $\forall i$ and $\sum_{i \in S}(\Theta) \leq 1$, $\forall \Theta$ to obtain an equilibrium allocation. This can be cast as a linear program and solved efficiently. It has a solution since an equilibrium allocation exists.

**P.w.l. valuations.** For the case of p.w.l. $v_i$ in §4, $p^*$ can have at most $n(K + n) - 1$ linear pieces with breakpoints $0 = a_0 < a_1 < \ldots < a_N = 1, N \leq n(K - 1)$. These breakpoints consist of (i) the “static” breakpoints of all $v_i$ and (ii) the intersections of the linear pieces of $\beta_i^* v_i$ between these breakpoints. Each interval between two breakpoints, on which every $v_i$ is linear, is further divided into at most $n$ small subintervals (Lemma 7), leading to the subintervals $[a_k, a_{k+1}]$, $k \in [N]$ above. On each $[a_k, a_{k+1}]$, for $i \neq j$, the line segments $\beta_i^* v_i$ and $\beta_j^* v_j$ either separate completely or overlapping completely, except possibly at endpoints. For $k \in [N]$, let $I_k \neq \emptyset$ be the set of “winners” on $[a_k, a_{k+1}]$, that is, for $i \in I_k$ and $j \notin I_k$, $p^* = \beta_i^* v_i > \beta_j^* v_j$ on $[a_k, a_{k+1}]$. In the language of Theorem 8, we only need to consider $\Theta_{i,k}, i \in [n], k \in [N]$ such that $i \in I_k$. This is because, by Corollary 2, $x^*$ must not allocate anything on $[a_k, a_{k+1}]$ such that $p^* > \beta_i^* v_i$ (i.e., $x^*_i = 0$ on $[a_k, a_{k+1}]$ if $i \notin I_k$). Therefore, we only need to solve for $b = (b_{i,k}) \in \mathbb{R}^n$.

Meanwhile, $p^*(x^*([a_k, a_{k+1}]) = \beta_i^* v_i, \forall i \in I_k$) for some $i \in I_k$ can be easily computed, since $v_i$ is linear on $[a_k, a_{k+1}]$. Splitting each $\Theta_k := [a_k, a_{k+1}]$ into $\Theta_{i,k}$, $i \in I_k$ according to each winner’s $b_{i,k}$ is also trivial: since $p^* = \beta_i^* v_i$ ($i \in I_k$) is linear on $[a_k, a_{k+1}]$, we can partition $[a_k, a_{k+1}]$ into consecutive intervals $\Theta_{i,k}$, each having $p^*(\Theta_{i,k}) = b_{i,k}$, $i \in I_k$.

**Multidimensional linear valuations.** Let $\Theta = \{\Theta, \Theta_{i,j} \subseteq \mathbb{R}^d \text{ be a rectangle where } \theta \leq \bar{\theta}\}$, $\mu(\Theta) = \prod_{i}(\bar{\theta}_i - \theta_i) \in (0, \infty)$, $\alpha_i^0 \in \mathbb{R}$ and $\alpha_i(\theta) = \alpha_i^0 + \alpha_i(\theta) \geq 0$ for all $\theta \in \Theta$. For simplicity, further assume that all $v_i$ are distinct. Define $\Theta_i := \{p^* = \beta_i^* v_i \geq \beta_j^* v_j, \forall i \neq j\}$. It is prescribed by linear inequalities and hence is a polytope too. For any $i \neq j$, we have $\Theta_i \varcap \Theta_j \subseteq \{p^* = \beta_i^* v_i = \beta_j^* v_j \} = \emptyset \in \Theta$ (where $\beta_i^* \neq \beta_j^* \neq 0$. If $\beta_i^* = \beta_j^* = 0$, the right-hand side is a hyperplane in $\mathbb{R}^d$, whose intersection with $\Theta$ has measure zero. Hence, $\mu(\Theta \cap \Theta_j) = 0$. If $\beta_j^* \alpha_j^0 = \beta_i^* \alpha_i^0$, then we have $\beta_j^* \alpha_j^0 = \beta_i^* \alpha_i^0$, which implies $\mu(\Theta_i \cap \Theta_j) = 0$. Therefore, $\Theta_i$ defined
this way is already a.e.-disjoint. By Theorem 8, \(x_i^* = \mathbf{1}_\Theta\) gives a pure equilibrium allocation (in this case, \(\mu(\Theta_S) = 0\) for any \(S \subset [n] \), \(|S| \geq 2\)). More generally, suppose that \(v_i\) are distinct and, for any \(i \neq j\), if \(v_i = \lambda v_j\) for some \(\lambda > 0\), then \(v_i = v_j\). Then, analogously, \(\Theta_i = \{\beta_i^* v_j \geq \beta_j^* v_i, \forall j \neq i\}\) are also pairwise a.e.-disjoint and constitute a pure equilibrium allocation.

Next, we discuss how to handle buyers with identical distributions. Let \(x^* (\{\Theta_i\})\) be a pure equilibrium allocation. For any \(i\) and \(0 < \lambda < 1\), we can split \(v_i\) into two buyers with valuations \(\lambda v_i, (1 - \lambda) v_i\) and budgets \(\lambda B_i, (1 - \lambda) B_i\), respectively. Then, it can be easily verified that \(\lambda x_i^*, (1 - \lambda) x_i^*\) are in the new demand sets of the two split buyers (i.e., buyer-wise optimal under price \(p^*\)) and achieve utilities \(\lambda u_i^*, (1 - \lambda) u_i^*\), respectively. This “buyer divisibility” property of ME is well-known in the finite-dimensional case. Since \(v_i\) is atomless, we can partition \(\Theta_i\) into \(\Theta_i'\) and \(\Theta_i''\) such that \(v_i(\Theta_i') = \lambda u_i^*, v_i(\Theta_i'') = (1 - \lambda) u_i^*.\) Hence, when \(v_i, i \in [n]\) are distinct, we can first merge the buyers with identical valuations, compute the equilibrium allocation in the reduced market and then allocate proportionally (w.r.t. the budgets) among buyers having identical valuations. Let the reduced market consist of \(s\) buyers \(I_1, \ldots, I_s, s \leq n (v_i, i \in I_s\) are identical and \(v_i \neq v_j\) for \(i \in I_j, j \in I_s, i \neq j\)). Given a pure equilibrium allocation \(x_i^* = 1_{\Theta_i}, i \in [s]\) for the reduced market, for each \(\ell \in [s]\), let \(x_i^* = \sum_{j \in I_s} B_j^{-1} x_{i,\ell}^*\) for each \(i \in I_s\) (i.e., proportional to its budget). It gives an equilibrium allocation \(x^*\), possibly mixed, for the original market. Since all \(v_i\) are atomless, we can also partition \(\Theta_i\) into \(\Theta_i', i \in I_s\) such that \(v_i(\Theta_i') = \sum_{j \in I_s} B_j^{-1} v_i(\Theta_{i,\ell})\). However, depending on the \(v_i\), finding these partitions may be computationally intractable. For example, under linear \(v_i\), dividing a given polytope into two smaller ones with given volumes is hard, since computing the volume of a polytope is \#P-hard in general (Khachiyan 1989; 1993).

One future work is to construct an approximate pure equilibrium allocation \(\tilde{\Theta}_i\), with any provable guarantee, from an approximate solution \(\tilde{\beta}\) of (3). For example, in the previous p.w.l. example, the construction of \(x^*\) given \(\tilde{\beta}\) relies critically on the medley of properties of the equilibrium quantities \(x^*, p^*\) and \(\beta^*\), especially (6), i.e., buyer \(i\) only allocates on where it “wins” (in the p.w.l. example, \(i \in I_k\)). In contrast, for piecewise constant \(v_i\), we can divide \([0, 1]\) into static intervals independent of \(\beta\), regarded as discrete items, and compute a ME of the resulting finite-dimensional Fisher market (Aziz and Ye 2014, Theorem 2 & Algorithm 2). As noted in (Cohler et al. 2011), this observation does not apply when \(v_i\) are p.w.l. Indeed, for piecewise linear \(v_i\), the “correct” discretization of \([0, 1]\) cannot be determined unless we have computed \(\beta^*\) (and hence \(p^*\)) exactly. Another necessary ingredient in this regard is the construction of a pure allocation from a known mixed one, preserving the utilities of all buyers. In other words, for an arbitrary (possibly mixed) \(x \in X^n, \sum_i x_i \leq 1, \) we must find a.e. disjoint \(\{\Theta_i\},\)
\(\Theta_i \subseteq \Theta\) such that \(v_i(\Theta_i) \geq u_i := (v_i, x_i)\) for all \(i\) and each \(\Theta_i\) is the union of polynomially many intervals. Such \(\{\Theta_i\}\) must exist, since \(v_i\) as measure are atomless (c.f. Lemma 1

\textbf{Figure 1:} A division of \([0, 1]\) given by \(\tilde{\beta}\), where each buyer \(i\) gets the interval shown in the legend, on which \(\tilde{\mu} = \tilde{\beta}_i v_i\)

and (Dvoretzky et al. 1951, Theorem 5)). For general \(v_i\), in the utility-preserving pure allocation \(\{\Theta_i\}\) some \(\Theta_i\) may be the union of an exponential number of intervals (Dvoretzky et al. 1951, Theorem 3). When each \(v_i\) is \(K\)-p.w.l., each \(\Theta_i\) can be the union of at most \(K_i\) intervals. In this case, however, a tractable construction still remains a challenge.

An illustrative example. Consider \(n\) buyers with distinct linear valuations \(v_i\) on the item space \(\Theta = [0, 1]\) such that \(v_i(\Theta) = 1\) for all \(i\). In this case, \(p^* = \max_i \beta_i^* v_i\) is piece-wise linear with exactly \(n\) pieces, since (i) it has at most \(n\) pieces by Lemma 7 and (ii) each buyer \(i\) “wins” at least one piece, i.e., \(p^* = \beta_i^* v_i\) on a nonempty closed interval (otherwise, by Corollary 2, buyer \(i\) has \(x_i^* = 0\) and gets \(u_i^* = 0\), contradicting to Lemma 5). We construct a small instance with \(n = 4\) and solve its convex program (3) using Algorithm 1 (SDA). When \(\beta^*\) converges numerically, we construct a final division, which is given by the \(n\) intervals as well as the (unique) winners on each interval, as shown in Figure 1. More details can be found in Appendix B.

6 Discussion and conclusions

Motivated by applications in ad auctions and fair recommender systems, we considered a Fisher market with a continuum of items and the concept of a market equilibrium in this setting. Based on the finite-dimensional Eisenberg–Gale convex program and its dual, we proposed convex programs whose optimal solutions are ME, and vice versa. Duality properties of the convex programs parallel various structural properties of ME. Due to the limitation of the general duality theory for optimization over infinite-dimensional vector spaces, we established these properties via directly exploiting the problem structure. In particular, we showed that a ME must exist, and it satisfies various duality and fairness criteria. We also gave efficient algorithms for solving the finite-dimensional convex program (3) in the bang-per-buck variables. To this end, we showed that, for one-dimensional piecewise linear valuations, we can derive efficient first-order and separation oracles, which allowed us to apply the ellipsoid method. Lastly, we discussed the construction of an equilibrium allocation from the optimal solution and gave an illustrative example.
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A Proofs

A.1 Proof of Lemma 1

We first show that the “utility feasible region” is compact.

**Lemma 10.** Define

\[ U = \left\{ (u_1, \ldots, u_n) : u_i = \langle v_i, x_i \rangle, x \in X_+^n, \sum_i x_i \leq 1 \right\} \subseteq \mathbb{R}_+^n \]

and

\[ U' = \left\{ (v_1(\Theta_1), \ldots, v_n(\Theta_n)) : \Theta_i \subseteq \Theta \text{ measurable and disjoint} \right\} \subseteq \mathbb{R}_+^n. \]

We have \( U = U' \). Furthermore, it is convex and compact.

**Proof.** Note that \( v_i \) as measures are atomless. Consider the set \( \tilde{U} = \{ (x, x_{n+1}) \in X_+^{n+1} : \sum_i x_i + x_{n+1} = 1 \} \). Define

\[ \tilde{U} = \left\{ u \in \mathbb{R}^{n \times (n+1)} : u_{ij} = \langle v_i, x_j \rangle, (i, j) \in [n] \times [n+1], x_i \in X_+^n, \sum_i x_i = 1 \right\} \]

and

\[ \tilde{U}' = \left\{ u \in \mathbb{R}^{n \times (n+1)} : u_{ij} = v_i(\Theta_j), (i, j) \in [n] \times [n+1], \{\Theta_1, \ldots, \Theta_{n+1}\} \text{ is a measurable partition of } \Theta \right\}. \]

By (Dvoretzky et al. 1951, Theorem 1), \( U \) is a convex and compact set in \( \mathbb{R}^{(n+1)} \). By (Dvoretzky et al. 1951, Theorem 4) and \( v_i \) being atomless, we have \( U = U' \). Therefore, their (Euclidean) projections on to the \( n \) dimensions corresponding to \( u_{ii}, i \in [n] \) are also equal, that is, \( U = U' \). It is convex compact (in \( \mathbb{R}^n \)) since projection preserves compactness in a finite-dimensional Euclidean space.

Next, we show Lemma 1. By Lemma 10, the set \( \tilde{U} = U' \subseteq \mathbb{R}_+^n \) is convex and compact. Let \( \rho(u) = -\sum_i B_i \log u_i \). Taking \( u_i^0 = \langle v_i, 1/n \rangle = \frac{1}{n} v_i(\Theta) \) for all \( i \) ensures \( u^0 \in \tilde{U} \) and \( \rho(u^0) \in \mathbb{R} \) (finite). Since

\[ u_i \leq (v_i, 1) = u_i(\Theta) < \infty \]

for all \( u \in \tilde{U} \), we have

\[ \sum_i B_i \log u_i \leq M := \sum_i B_i \log v_i(\Theta) < \infty. \]

Hence, there exists \( \epsilon > 0 \) sufficiently small such that, for \( u \in \tilde{U} \), if some \( u_i \leq \epsilon \), then

\[ \rho(u) \geq -B_i \log \epsilon - M > \rho(u^0). \]

In fact, it suffices to have \(-B_i \log \epsilon > \rho(u^*) + M \) for all \( i \), or simply

\[ \epsilon < e^{-\rho(u^*) - M}. \]

Hence, removing all \( u \in \tilde{U} \) such that \( \min_i u_i < \epsilon \) does not affect the infimum of \( \rho \) over \( \tilde{U} \), that is,

\[ \inf_{u \in \tilde{U}} \rho(u) = \inf_{u \in \tilde{U}, u \geq \epsilon} \rho(u). \]

Since \( \{ u \in \tilde{U} : u \geq \epsilon \} \) is compact (as a closed subset of the compact set \( \tilde{U} \)) and \( \rho \) is continuous on it, by the extreme value theorem, there exists a minimizer \( u^* \in \tilde{U} \) (such that \( u \geq \epsilon \)). By the definition of \( \tilde{U}' = \tilde{U} \) (c.f. Lemma 10), there exists \( x^* \) such that \( x_i^* = 1_{\Theta_i} \) (where \( \Theta_i \) are disjoint measurable subsets of \( \Theta \))

\[ u_i^* = \langle v_i, x_i^* \rangle = v_i(\Theta_i). \]

Finally, if \( \sup_i \Theta_i \subseteq \Theta \), then assign \( \Theta \setminus (\cup_i \Theta_i) \) to buyer 1, that is, augment \( \Theta_1 \) so that \( \sup \Theta_i = \Theta \). This does not affect \( \sum_i x_i \leq 1 \), nor does it affect optimality (since \( v_i(S) \leq v_i(T) \) if \( S \subseteq T \subseteq \Theta \), all measurable). In fact, \( \Theta_0 := \Theta \setminus (\cup_i \Theta_i) \) corresponds to the subset on which all buyers have valuation 0 a.e.

\[ \square \]

A.2 Proof of Lemma 2

Suppose \( \mathcal{V} = L_q(\Theta), q \in [1, \infty] \). Then the Banach space norm is

\[ \|f\| = \begin{cases} (\int_\Theta |f| q \, d\mu)^{1/q} & \text{if } q < \infty, \\ \inf\{M \geq 0 : |f| \leq M \ \text{a.e.} \} & \text{if } q = \infty. \end{cases} \]

Note that they are also well-defined even if \( \|v_i\| = \infty \). Since \( 0 \leq p := \max_i \beta_i v_i \leq \sum_i \beta_i v_i \), we have

\[ 0 \leq p \leq \sum_i \beta_i \|v_i\| < \infty. \]

Therefore, \( p \in \mathcal{V}_+ \).
A.3 Proof of Lemma 3

Denote the objective function of (3) as \( \psi(\beta) \). Notice that

\[
\max_i \beta_i v_i \leq \sum_i \beta_i v_i
\]

and therefore

\[
0 \leq \left\langle \max_i \beta_i v_i, 1 \right\rangle \leq \sum_i \beta_i \langle v_i, 1 \rangle, \forall \beta > 0.
\]

Meanwhile, for any \( \lambda \in [0, 1], \beta, \beta' \in \mathbb{R}^n_+, \theta \in \Theta, \)

\[
\max_i (\lambda \beta_i + (1 - \lambda) \beta'_i) v_i(\theta) \leq \lambda \max_i \beta_i v_i(\theta) + (1 - \lambda) \max_i \beta'_i v_i(\theta).
\]

Therefore,

\[
\left\langle \max_i (\lambda \beta_i + (1 - \lambda) \beta'_i) v_i, 1 \right\rangle \leq \lambda \max_i \beta_i v_i + (1 - \lambda) \max_i \beta'_i v_i, 1 \right\rangle.
\]

In other words, the function

\[
\beta \mapsto \left\langle \max_i \beta_i v_i, 1 \right\rangle
\]

is convex. Since \( \beta \mapsto - \sum_i B_i \log \beta_i \) is strongly convex on \( \mathbb{R}^n_+ \), we know that \( \psi \) is real-valued, strongly convex and hence continuous on \( \mathbb{R}^n_+ \). Furthermore, for any \( i \), when \( \beta_i \to 0 \) or \( \beta_i \to \infty \), we have \( \psi(\beta) \to \infty \). Hence, for \( \beta^0 = (1, \ldots, 1) > 0 \), there exists \( 0 < \beta < \beta < \infty \) (cf. the proof of Lemma 1; tighter bounds are given in Lemma 5) such that

\[
\beta \notin [\ddot{\beta}, \dddot{\beta}] \Rightarrow \psi(\beta) > \psi(\beta^0).
\]

Therefore, we can restrict \( \beta \) inside a closed interval without affecting the infimum:

\[
\inf_{\beta \in \mathbb{R}^n_+} \psi(\beta) = \inf_{\beta \in [\ddot{\beta}, \dddot{\beta}]} \psi(\beta).
\]

The right-hand side is the infimum of a continuous function on a compact set. Therefore, the infimum is attained at some \( \beta^* \in [\ddot{\beta}, \dddot{\beta}] \). Clearly, \( \beta^* > 0 \). It is unique since \( \psi \) is strongly convex.

Finally, when solving (2), for any fixed \( \beta \), the objective is clearly minimized at \( p = \max_i \beta_i v_i \in \mathcal{V}_+ \) (cf. Lemma 2). Therefore, we can eliminate \( p \) in this way and obtain (3). In other words, for the optimal solution \( \beta^* \) of (3), setting \( p^* := \max_i \beta^*_i v_i \) gives an optimal solution \((p^*, \beta^*)\) of (2), which is (a.e.) unique.

A.4 Proof of Lemma 4

View (2) as the “primal” convex program. Associate Lagrange multipliers \( x_i \in (\mathcal{V}_+)^* \) to constraints

\[
p \geq \beta_i v_i \iff p - \beta_i v_i \in \mathcal{V}_+.
\]

The Lagrangian is

\[
\mathcal{L}(p, \beta; x) = \langle p, 1 \rangle - \sum_i B_i \log \beta_i - \sum_i \langle p - \beta_i v_i, x_i \rangle = \sum_i (\beta_i \langle v_i, x_i \rangle - B_i \log \beta_i) + \left\langle p, 1 - \sum_i x_i \right\rangle.
\]

If not \( 1 \geq \sum_i x_i \) a.e., then \( \langle p, 1 - \sum_i x_i \rangle \) can be arbitrarily negative. If \( 1 \geq \sum_i x_i \), then \( \langle p, 1 - \sum_i x_i \rangle \) is minimized at any \( p \in \mathcal{V}_+ \) such that \( \langle p, 1 - \sum_i x_i \rangle = 0 \), e.g., \( p = 0 \). Meanwhile,

\[
\beta_i \langle v_i, x_i \rangle - B_i \log \beta_i
\]

is clearly minimized at \( \beta_i = \frac{B_i}{\langle v_i, x_i \rangle} \). Therefore, the dual objective is, for \( x \in (\mathcal{V}_+)^n \),

\[
g(x) = \inf_{(p, \beta) \in \mathcal{V}_+ \times \mathbb{R}^n_n} \mathcal{L}(p, \beta; x) = \begin{cases} \|B\| - \sum_i B_i \log B_i + \sum_i B_i \log \langle v_i, x_i \rangle & \text{if } \sum_i x_i \leq 1, \\ -\infty & \text{o.w.} \end{cases}
\]
For any $x \in (\mathcal{X}_+)^n$ such that $\sum_i x_i \leq 1$, we have the following chain of inequalities:

$$C + \sum_i \log(v_i, x_i) = \min \left\{ \mathcal{L}(p, \beta; x) : (p, \beta) \in \mathcal{V}_+ \times \mathbb{R}_+^n \right\}$$  [by the above derivation of $g(x)$]

$$\leq \inf \left\{ \mathcal{L}(p, \beta; x) : (p, \beta) \in \mathcal{V}_+ \times \mathbb{R}_+^n, p \geq \beta v_i, \forall i \right\}$$  [due to the additional constraints]

$$\leq \inf \left\{ (p, 1) - \sum_i \log(\beta_i) : (p, \beta) \in \mathcal{V}_+ \times \mathbb{R}_+^n, p \geq \beta v_i, \forall i \right\}$$

$$= w^*.$$  \hspace{1cm} (9)

Taking supremum over $x \in \mathcal{X}_+^n, \sum_i x \leq 1$ on both sides yields weak duality:\footnote{Omitting the constant, the Lagrange dual $\sup_{x \in (\mathcal{X}_+^n)^*} g(x)$ is precisely (1) but with the cone $\mathcal{X}_+$ replaced by the much larger $(\mathcal{V}_+^n)^*$ (recall that $L_\infty(\Theta)^*$ is a much richer space than $L_1(\Theta)$). This is the Lagrange dual of (2) (see, e.g., (Luenberger 1997, §8.6) and (Ponstein 2004, §3)). We do not need to work with it. Again, we first define the convex programs (1), (2) instead and then show that they exhibit various duality properties.}

$$C + z^* \leq w^*.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm}

Meanwhile, when $z^*$ is attained by some $x^*$ feasible to (1), we have strong duality:

$$n + \sum_i \log(v_i, x_i^*) = n + z^* = w^*.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm}

Hence, when $x = x^*$, all inequalities in (9) become equalities. In particular,

$$\mathcal{L}(p^*, \beta^*; x^*) = \min \left\{ \mathcal{L}(p, \beta; x^*) : (p, \beta) \in \mathcal{V}_+ \times \mathbb{R}_+^n \right\},$$  \hspace{1cm} (10)

$$\mathcal{L}(p^*, \beta^*; x^*) = \langle p^*, 1 \rangle - \sum_i B_i \log \beta_i^*.$$  \hspace{1cm} (11)

By (10), similar to the above derivation of $g(x)$, we know that $p^*$ minimizes

$$p \mapsto \left\langle p, 1 - \sum_i x_i^* \right\rangle$$

over $p \in \mathcal{V}_+$, which implies (4). Meanwhile, $\beta^*$ minimizes

$$\beta \mapsto \sum_i (\beta_i \langle v_i, x_i^* \rangle - B_i \log \beta_i)$$

over $\beta \in \mathbb{R}_+^n$, which implies (5). Furthermore, complementary slackness follows immediately from (11).

Conversely, for $x^*$ feasible to (1) and $(p^*, \beta^*)$ feasible to (2) such that (4), (5) hold, we have

$$C + z^* \geq C + \sum_i \log(v_i, x_i^*) \ [x^* \text{ is feasible to (1)}]$$

$$= \min \left\{ \mathcal{L}(p, \beta, x^*) : (p, \beta) \in \mathcal{V}_+ \times \mathbb{R}_+^n \right\} \ [\text{same as the first equality of (9)}]$$

$$= \mathcal{L}(p^*, \beta^*; x^*) \ [\text{by (4) and (5)}]$$

$$\geq \langle p^*, 1 \rangle - \sum_i B_i \log \beta_i^* \ [\text{since } \langle p^* - \beta^*_i v_i, x_i^* \rangle \geq 0]$$

$$\geq w^*. \ [\langle p^*, \beta^* \rangle \text{ is feasible to (2)}]$$

Since $C + z^* \leq w^*$, the above chain of inequalities must all be equalities. Therefore, strong duality holds with both optima attained:

$$C + z^* = C + \sum_i B_i \log(v_i, x_i^*) = \langle p^*, 1 \rangle - \sum_i B_i \log \beta_i^* = w^*.$$
A.5 Proof of Theorem 1

Suppose that, for some \( i \), there exists a measurable \( A \subseteq \Theta_i \) \((\mu (A) \neq 0)\) such that
\[
\beta_i^* v_i(A) < \beta_j^* v_j(A). \tag{12}
\]

Then, for \( \alpha \in (0, 1) \), consider
\[
\hat{x}_i = \mathbf{1}\Theta_i - \alpha \mathbf{1}_A, \quad \hat{x}_j = \mathbf{1}\Theta_j + \alpha \mathbf{1}_A.
\]
Clearly, replacing \( x_i^* \) by \( \hat{x}_i \), \( \hat{x}_j \) does not affect feasibility to (1). We analyze the chance in the objective value. Note that
\[
B_i \log \langle v_i, \hat{x}_i \rangle + B_j \log \langle v_j, \hat{x}_j \rangle \tag{13}
\]
\[
= B_i \left[ \log u_i^* - v_i(A) \right] + B_j \log (u_j^* + \alpha v_j(A))
\]
\[
= B_i \left[ \log u_i^* - v_i(A) \right] + B_j \left[ \log u_j^* + v_j(A) \right] + \alpha \left[ \frac{v_j(A)^2}{2B_j} \right] - \left( \frac{1}{B_j} \right) \alpha^2
\]
\[
= (B_i \log u_i^* + B_j \log u_j^*) + (\beta_i^* v_i(A) - \beta_j^* v_j(A)) \alpha - \frac{v_j(A)^2}{2B_j} + \left( \frac{1}{B_j} \right) \alpha^2 \tag{14}
\]
for some \( \xi_i \in (u_i^* - \alpha v_i(A), u_i^*) \) and \( \xi_j \in (u_j^*, u_j^* + \alpha v_j(A)) \), both depending on \( \alpha \) (Taylor expansion of log). Let \( \alpha_0 \) be such that
\[
u_i^* - \alpha_0 v_i(A) \geq \frac{u_i^*}{2} > 0.
\]
Then, for all \( \alpha \in (0, \alpha_0) \), we have
\[
\frac{v_i(A)^2 + v_j(A)^2}{2} \cdot \left( \frac{1}{\xi_j} + \frac{1}{\xi_i} \right) \leq M := \left( \max_{i} v_i(\Theta) \right) \left( \frac{4}{(u_i^*)^2} + \frac{1}{(u_j^*)^2} \right) < \infty. \tag{15}
\]
By (12) and (15), we know that when
\[
\alpha \in \left( 0, \frac{\beta_i^* v_i(A) - \beta_j^* v_j(A)}{M} \right),
\]
we have
\[
\beta_j^* v_j(A) - \beta_i^* v_i(A) \alpha - \frac{v_i(A)^2 + v_j(A)^2}{2} \cdot \left( \frac{1}{\xi_j} + \frac{1}{\xi_i} \right) \alpha^2 \geq (\beta_j^* v_j(A) - \beta_i^* v_i(A)) \alpha - M \alpha^2 > 0.
\]
Substituting the above into (14) yields
\[
B_i \log \langle v_i, \hat{x}_i \rangle + B_j \log \langle v_j, \hat{x}_j \rangle > B_i \log u_i^* + B_j \log u_i^*.
\]
In other words, replacing \( x_i^* \), \( x_j^* \) by \( \hat{x}_i \), \( \hat{x}_j \) strictly increases the objective of (1), contradicting to the optimality of \( x^* \). Therefore, on each \( \Theta_i \), we have \( \beta_i^* v_i \geq \beta_j^* v_j \) a.e. for all \( j \neq i \).

Let \( p^* = \max_i \beta_i^* v_i \). Then, on each \( \Theta_i \), we have \( p^* = \beta_i^* v_i \geq \beta_j^* v_j \) a.e. for all \( j \neq i \). Therefore, for any measurable \( A \subseteq \Theta \), we have \( p^*(A \cap \Theta_i) = \beta_i^* v_i(A \cap \Theta_i) \) and hence
\[
p^*(A) = \sum_i p^*(A \cap \Theta_i) = \sum_i \beta_i^* v_i(A \cup \Theta_i).
\]
Finally, we verify that \( x^* \) and \((p^*, \beta^*)\) satisfy the sufficient conditions for optimality in Lemma 4. Note that \((p^*, \beta^*)\) is feasible to (2) by the definition of \( p^* \). Since \( \{ \Theta_i \} \) is a partition of \( \Theta \), we have \( \sum_i x_i = 1 \). Therefore, (4) trivially holds. Meanwhile, our construction of \( \beta^* \) ensures (5), i.e., \( \langle v_i, x_i^* \rangle = \frac{B_i}{\beta_i^*} \). Hence, by Lemma 4, \((p^*, \beta^*)\) solves (2) and (6) holds (in addition to (4) and (5)).
A.6 Proof of Theorem 2
We first show the forward direction (optimal solutions ⇒ ME). By Theorem 2, the equalities (4)-(6) in Lemma 4 hold for \( x^* \) and \( (p^*, \beta^*) \). First, (4) gives market clearance. Next, we show buyer optimality. For each \( i \), by (5) and (6),
\[
\langle p^*, x_i^* \rangle = \beta_i^* \langle v_i, x_i^* \rangle = B_i.
\]
In words, \( x_i^* \) depletes buyer \( i \)'s budget \( B_i \) and the utility buyer \( i \) receives is
\[
\langle v_i, x_i^* \rangle = \frac{B_i}{\beta_i^*}.
\]
Meanwhile, for any \( x_i \in \mathcal{X}_+ \) such that \( \langle p^*, x_i \rangle \leq B_i \), since \( p^* \geq \beta_i^* v_i \) (by the choice of \( p^* \)), it holds that
\[
\langle v_i, x_i \rangle \leq \frac{1}{\beta_i^*} \langle p^*, x_i \rangle \leq \frac{B_i}{\beta_i^*} = \langle v_i, x_i^* \rangle.
\]
Therefore,
\[
x_i^* \in D_i(p^*).
\]
Hence, \( (x^*, p^*) \) is a ME, where buyer \( i \)'s equilibrium utility is clearly \( u_i^* := \langle v_i, x_i^* \rangle = \frac{B_i}{\beta_i^*} \).

Conversely, let \( (x^*, p^*) \) be a ME and \( \beta_i^* := \frac{B_i}{u_i^*} \), where \( u_i^* := \langle v_i, x_i^* \rangle \). We first check that \( (p^*, \beta^*) \) is feasible to (2). For any \( i \), suppose there exists a measurable \( A \subseteq \Theta \) such that \( p^*(A) < \beta_i v_i(A) \). Then, consider \( x_i = \frac{B_i}{p^*(A)} \cdot 1_A \). We have
\[
\langle p^*, x_i \rangle \leq B_i
\]
and
\[
\langle v_i, x_i \rangle = B_i \cdot \frac{v_i(A)}{p^*(A)} > \frac{B_i}{\beta_i^*} = \langle v_i, x_i^* \rangle,
\]
contradicting to buyer optimality \( x_i^* \in D_i(p^*) \). Therefore, \( p^* \geq \beta_i^* v_i \) a.e., \( \forall i \)
and \( (p^*, \beta^*) \) is feasible to (2). Meanwhile, \( x^* \) is feasible to (1), since ME requires \( \sum_i x_i^* \leq 1 \). Furthermore, \( x^* \) and \( (p^*, \beta^*) \) satisfy (5) (by definition of \( u^* \) and \( \beta^* \)) and (4) (since ME requires market clearance). Therefore, by Lemma 4, they must be optimal to (1) and (2), respectively.

\( \square \)

A.7 Proof of Theorem 3
Since \( x^* \) is an equilibrium allocation, by Theorem 2, it is also an optimal solution of (1). If there exists \( \bar{x} \in \mathcal{X}_n \), \( \sum_i \bar{x}_i \leq 1 \) such that \( \langle v_i, \bar{x}_i \rangle \geq \langle v_i, x_i^* \rangle \) for all \( i \) and at least one inequality is strict, then
\[
\sum_i B_i \log \langle v_i, \bar{x}_i \rangle > \sum_i B_i \log \langle v_i, x_i^* \rangle,
\]
i.e., \( x^* \) is not an optimal solution of (1), a contradiction. Therefore, \( x^* \) is Pareto optimal.

For any \( j \neq i \), since \( \langle p^*, x_i^* \rangle = B_i \) (Theorem 2) and \( \langle p^*, x_j^* \rangle = B_j \), we have
\[
\langle p^*, \frac{B_i}{B_j} x_j^* \rangle = B_i.
\]
Since \( x_i^* \in D_i(p^*) \) and \( \frac{B_i}{B_j} x_j^* \geq 0 \), we have
\[
\langle v_i, x_i^* \rangle \geq \langle v_i, \frac{B_i}{B_j} x_j^* \rangle \Rightarrow \frac{\langle v_i, x_i^* \rangle}{B_i} \geq \frac{\langle v_i, x_j^* \rangle}{B_j}.
\]
Therefore, \( x^* \) is envy-free.

By the market clearance condition of ME, we have
\[
p^*(\Theta) = \langle p^*, 1 \rangle = \sum_i \langle p^*, x_i^* \rangle = \|B\|_1.
\]
Therefore, for each buyer \( i \), it holds that
\[
\langle p^*, \frac{B_i}{\|B\|_1} 1 \rangle = \frac{B_i}{\|B\|_1} p^*(\Theta) = \frac{B_i}{\|B\|_1} = B_i.
\]
In other words, buyer \( i \) can afford the bundle \( \frac{B_i}{\|B\|_1} 1 \). Hence, its equilibrium utility must be at least
\[
\langle v_i, \frac{B_i}{\|B\|_1} 1 \rangle = \frac{B_i}{\|B\|_1} v_i(\Theta).
\]
\( \square \)
A.8 Proof of Lemma 5

By the characterization of \(p^*\) in Theorem 1, clearly,
\[
p^*(\Theta) = \sum_i p^*(\Theta_i) = \sum_i B_i = \|B\|_1,
\]
where \(\{\Theta_i\}\) is a pure equilibrium allocation. Since \(x_i^* \leq 1\) \((\Theta_i \subseteq \Theta)\), we have
\[
u_i = \langle v_i, x_i^* \rangle \leq v_i(\Theta) = 1.
\]
Meanwhile, since \(\{\Theta_i\}\) is an equilibrium allocation, it is proportional (Theorem 3), that is, \(B_i / \|B\|_1\) is a feasible bundle for buyer \(i\). Hence,
\[
u_i^* \geq \langle v_i, B_i / \|B\|_1 \rangle = B_i / \|B\|_1 v_i(\Theta).
\]
The bounds on \(\beta_i^* = \frac{B_i}{v_i}\) follow immediately. \(\square\)

A.9 Proof of Lemma 6

Since each \(v_i\) has \(K_i\) pieces, i.e., \(K_i - 1\) breakpoints in \((0,1)\), there are at most \((K - n)\) breakpoints in total, denoted as (including the endpoints) \(0 = s_0 < s_1 < \cdots < s_{K'} = 1\) \((K' \leq K - n + 1)\). For \(k \in [K']\), on the interval \([s_k-1, s_k]\), each \(v_i\) (and hence \(\beta_i v_i\)) is linear. Therefore, on \([s_k-1, s_k]\), the function \(\max_i \beta_i v_i\) is the maximum of \(n\) linear functions. By Lemma 7, it has at most \(n\) pieces on \([s_k-1, s_k]\). Hence, on \([0,1]\), \(\max_i \beta_i v_i\) has at most \(nK' \leq n(K - n + 1)\) pieces. \(\square\)

A.10 Proof of Lemma 7

We prove it by induction. Clearly, when \(n = 1\), there is only one piece \(f_1\). It is trivially convex and continuous. Suppose it is true for some \(n \geq 1\), that is,
\[
h_n(\theta) = \max_{i \in [n]} f_i(\theta)
\]
is convex, continuous and \(n\)-piecewise linear. For the case of \((n + 1)\), note that
\[
h_{n+1}(\theta) := \max_{i \in [n+1]} f_i(\theta) = \max \{h_n(\theta), f_{n+1}(\theta)\}
\]
Since \(f_{n+1}\) is linear, \(h_n\) is convex and continuous, the set
\[
\{\theta \in [l,u] : f_{n+1}(\theta) \geq h_n(\theta)\}
\]
must be a closed convex subset (i.e., closed interval) of \([l,u]\). Therefore, \(h_{n+1}\) has at most one more piece than \(h_n\), that is, at most \((n + 1)\) pieces in total. It is also convex and continuous since it is the maximum of two such functions.

To find the breakpoints, consider the following incremental procedure. Suppose we already have the sorted breakpoints of \(h_n\):
\[
l = a_0 < a_1 < \cdots < a_{n'} = u, \ n' \leq n.
\]
We also keep track of \(i_k\), i.e., a “winner” on \([a_{k-1}, a_k]\), that is, \(f_{i_k} \geq f_i\) on \([a_{k-1}, a_k]\) for all \(i \neq i_k\). Clearly, \(h_n\) is a line segment on each \([a_{k-1}, a_k]\). Suppose we add a new linear function \(f_{n+1}(\theta) = c_{n+1} \theta + d_{n+1}\). Then, for each \(k \in [n']\), if \(c \neq c_{n+1}\), find the intersection point (of the two lines of \(f_{n+1}\) and \(f_k\), while \(\theta_{k,n+1} = \infty\) if they are parallel):
\[
\theta_{k,n+1} = \frac{d_{n+1} - d_k}{c_k - c_{n+1}}.
\]
Since \(h_n\) is convex, continuous and piecewise linear, \(f_{n+1}\) can intersect with at most two of its \(n'\) pieces. In other words, there can be at most two \(k, \ell \in [n']\) such that \(\theta_{k,n+1}, \theta_{\ell,n+1} \in [l,u]\). If so \((k < \ell)\), use bisection to find all \(a_k \in [\theta_{k,n+1}, \theta_{\ell,n+1}]\) and replace them with \(\theta_{k,n+1}, \theta_{\ell,n+1}\). The new set of sorted breakpoints are
\[
l = a_0 < \cdots < \theta_{k,n+1} < \theta_{\ell,n+1} < \cdots < a_{n'} = u.
\]
Clearly, there are at most \(n' + 1 \leq n + 1\) breakpoints. A winner on \([\theta_{k,n+1}, \theta_{\ell,n+1}]\) is \((n + 1)\). Winners on other line segments remain unchanged. It can be easily seen that the above procedure takes \(O(n)\) times. To find the breakpoints of \(h_n\), incrementally add \(f_1, \ldots, f_n\) and augment the array of sorted breakpoints. Hence, it takes \(O(n^2)\) time to compute the breakpoints \(a_k, k \in [n']\), \(n' \leq n\) of \(h_n\). \(\square\)
A.11 Proof of Lemma 8
The first half well-known, see, e.g., (Beck 2017, Theorem 3.50). In fact, by this theorem, the subdifferential is
\[ \partial_\beta f(\beta, \theta) = \text{conv} \left\{ v_i(\theta)e^{(i)} : i \in \arg \max \beta_i v_i(\theta) \right\}. \]
For the second half, note that for any \( \beta > 0 \), since \( g(\beta, \theta) \in \partial_\theta f(\beta, \theta) \) we have
\[ f(\beta', \theta) - f(\beta, \theta) \geq (g(\beta, \theta), \beta' - \beta), \forall \beta' > 0. \]
Integrate w.r.t. \( \theta \) over \( \Theta \) on both sides yield
\[ \phi(\beta') - \phi(\beta) \geq \int_\Theta (g(\beta, \theta), \beta' - \beta)d\theta = \left\langle \int_\Theta g(\beta, \theta)d\theta, \beta' - \beta \right\rangle, \]
where the (component-wise) integral \( \int_\Theta g(\beta, \theta)d\theta \) is well-defined and finite, since each component of \( g(\beta, \theta) \) is uniformly bounded by the pointwise maximum \( \max_i v_i \), which is integrable:
\[ 0 \leq \max_i v_i \leq \sum_i v_i \in L_1(\Theta). \]
Therefore, by the definition of subgradient,
\[ \int_\Theta g(\beta, \theta)d\theta \in \partial \phi(\beta). \]
Clearly, the integral can also be written as an expectation over \( \theta \sim \text{Unif}(\Theta) \), since the probability density of \( \text{Unif}(\Theta) \) is \( \frac{1}{\mu(\Theta)} \) for all \( \theta \in \Theta \).

A.12 Proof of Lemma 9
The first statement is a direct consequence of Lemma 7.
For the second statement, By Lemma 8 and using the notation there, since \( g(\beta, \theta) \) is a subgradient of \( f(\beta, \theta) \) (both as functions of \( \beta \)), taking expectation over \( \theta \) gives (in this case, \( \mu([0,1]) = 1 \) and the probability density is 1 for all \( \theta \in [0,1] \))
\[ \phi'(\beta) := E_\theta g(\beta, \theta) \in \partial \phi(\beta), \forall \beta > 0. \]
In the above, the \( i \)-th component of \( \phi'(\beta) \) is the expectation of the \( i \)-th component of \( g(\beta, \theta) \):
\[ \phi'_i(\beta) = E_\theta \left[ v_{i,\beta,\theta}(\theta)1_{i^{*}_{i,\beta,\theta} = i} \right]. \]
For \( \theta \in [a_{k-1}, a_k] \), the buyer \( i^*_{\beta,\theta} \) is a “winner” on the \( k \)-th line segment. Therefore, for \( \theta \in [a_{k-1}, a_k] \), we can denote it as the same buyer \( i^*_{k} \) instead (depending on \( \beta \) and \( k \)). In this way, the above expectation can be decomposed into sum of integrals over intervals \( [a_{k-1}, a_k] \) (if \( n' < n \), then for \( k > n' \) the interval is empty, on which we can assign an arbitrary winner):
\[ \phi'_i(\beta) = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} v_{i,\beta,\theta}(\theta)1_{i^{*}_{i,\beta,\theta} = i}d\theta = \sum_{k=1}^{n} \left( C_i^2 \frac{a_k^2 - a_{k-1}^2}{2} + d_i^* (a_k - a_{k-1}) \right). \]
By Lemma 7, it takes \( O(n^2) \) time to find all breakpoints \( a_k, k \in [n'] \) and winners \( i^*_{k} \). Given all \( a_k \) and \( i^*_{k} \), the above expression of \( \phi'_i(\beta) \) clearly takes \( O(n) \) time to evaluate. Therefore, computing \( \phi'(\beta) \) takes \( O(n^2) + n \cdot O(n) = O(n^2) \) time in total.

A.13 Proof of Theorem 5
By Lemma 6, \( [0,1] \) can be partitioned into \( K' \leq (K - n + 1) \) consecutive intervals \( [s_k, s_{k}], k \in [K'] \). Here, \( s_k \) only depends on the input \( v_i \). On each interval, all \( v_i \) are linear. Using the same argument as in the proof of Lemma 9, we have
\[ \phi'(\beta) = \sum_{k=1}^{K'} \int_{s_k}^{s_{k+1}} g(\beta, \theta)d\theta, \]
where \( g(\beta, \theta) \) is the subgradient of \( \beta \mapsto \max_i \beta_i v_i(\theta) \). By Lemma 9 (where we take \( l = s_{k-1} \) and \( u = s_k \) for each \( k \)), each summand \( \int_{s_k}^{s_{k+1}} g(\beta, \theta)d\theta \) can be computed in \( O(n^2) \) times. Therefore, \( \phi'(\beta) \) can be computed in \( O(n^2 K) \) time.
A.14 Proof of Theorem 6

By Lemma 5, adding the simple "box" constraint $\beta \in [\beta, \bar{\beta}]$, where $\beta_i = 1$, $\bar{\beta}_i = \frac{1}{B_i}$ to (3) does not affect the optimal solution $\beta^\ast$. We will derive all relevant constants under both general $B_i$ and uniform budgets $B_i = \frac{1}{n}$, with the latter in brackets.

When applying the ellipsoid method to (3) with the added constraint $\beta \in [\beta, \bar{\beta}]$, we need to specify the two oracles. By Theorem 5, the first-order oracle $G$ takes $O(n^2K)$ to compute a subgradient for any input $\beta > 0$:

$$ G(\beta) := \phi'(\beta) - \sum_i \frac{B_i}{\beta_i} $$

that is, the sum of $\phi'(\beta)$ and the gradient of the second (smooth) part $-\sum_i B_i \log \beta_i$.

Meanwhile, the separation oracle $S$ is trivial due to the simple constraint set. First, normalizing $B$ and computing $\bar{\beta}_i = \frac{1}{B_i}$ and $\bar{\beta} = \frac{1}{n}$ take $O(n)$ time. Checking whether $\beta \in (\bar{\beta}, \bar{\beta})$ takes $O(n)$ time. If $\beta \notin (\bar{\beta}, \bar{\beta})$, then along the way we find $i$ such that $\beta_i \leq \beta_i$ or $\beta_i \geq \bar{\beta}_i$, in the former case, $\langle -\epsilon^{(i)}, \beta \rangle \geq \langle -\epsilon^{(i)}, \beta' \rangle$ for any $\beta' \in [\beta, \bar{\beta}]$; in the latter case, use $\epsilon^{(i)}$ instead of $-\epsilon^{(i)}$ for the separating hyperplane. The constants $R$ and $r$ in Theorem 4 can be specified as follows.

$$ R = \sup_{\beta \in [\bar{\beta}, \bar{\beta}]} \|\beta\| = \|\bar{\beta}\| = \sqrt{\sum_i \frac{1}{B_i^2}} \quad (\text{if } B_i = \frac{1}{n}) $$

$$ r = \min_i (\bar{\beta}_i - \beta_i) = \min_i \frac{1}{B_i} - 1 = \frac{1}{\|B\|_\infty} - 1 \quad (= n - 1 \text{ if } B_i = \frac{1}{n}) $$

For the constant $V$, note that, for any $\beta \in [\bar{\beta}, \bar{\beta}]$, we have

$$ \langle \max_i \beta_i v_i, 1 \rangle \leq \|\beta\|_\infty \sum_i v_i(\Theta) \leq \frac{n}{\min_i B_i} \quad (= n^2) $$

Therefore,

$$ \phi(\beta) \leq \frac{n}{\min_i B_i} - \sum_i B_i \log \beta_i = \frac{n}{\min_i B_i} \quad (= n^2) $$

Similarly, since $\beta_i \geq \beta_i = 1$, we have

$$ \langle \max_i \beta_i v_i, 1 \rangle \geq \langle \min_i v_i(\Theta) = 1 $$

Therefore,

$$ \phi(\beta) \geq 1 - \sum_i B_i \log \bar{\beta}_i = 1 + \sum_i B_i \log B_i \quad (1 - \log n) $$

Hence, the constant $V$ can be bounded by

$$ V \leq \frac{n}{\min_i B_i} - \sum_i B_i \log B_i - 1 \quad (= n^2 + \log n - 1 = O(n^2)) $$

Since $\|B\|_1 = 1$ and $\log$ is concave on $\mathbb{R}_{++}$, we have

$$ -\sum_i B_i \log B_i = \sum_i B_i \log \frac{1}{B_i} \leq \log n. $$

Meanwhile, $\frac{n}{\min_i B_i} \geq n^2$, since $\min_i B_i \leq \frac{1}{n}$. Therefore, the bound above for $V$ has growth at least $\Omega(n^2)$. Substituting the constants $R$, $r$, $V$ and using the bound

$$ \sqrt{\sum_i \frac{1}{B_i^2}} \leq \frac{\sqrt{n}}{\min_i B_i} $$

we have

$$ N(\epsilon) = O(1)n^2 \log \left(2 + \frac{VR}{\epsilon r} \right) = O\left(n^2 \log \frac{n \cdot \max_i B_i}{\epsilon \cdot \min_i B_i} \right) \quad (= O\left(n^2 \log \frac{n}{\epsilon} \right)) $$

By Theorem 4, in order to obtain an $\epsilon'$-solution (in terms of objective error) the total time for oracle calls and additional arithmetic operations is

$$ N(\epsilon') \cdot O(n^2K) + O(n^2N(\epsilon')) = N(\epsilon') \cdot O(n^2K) = O\left(n^4K \log \frac{n \cdot \max_i B_i}{\epsilon' \cdot \min_i B_i} \right) \quad (= O\left(n^4K \log \frac{n}{\epsilon'} \right)) $$
Since \( \beta \mapsto - \sum_i B_i \log \beta_i \) is strongly convex with modulus
\[
\min_i \frac{B_i}{\beta_i^2} = \min_i B_i^3
\]
on \([\beta, \bar{\beta}]\), so is the overall objective function of (3). Therefore, to ensure \( \|\beta - \beta^*\| \leq \epsilon \), it suffices to take \( \epsilon' = \frac{\zeta}{2} \cdot \min_i B_i^3 \). Substituting it into the above time complexity yields the total time needed to achieve \( \|\beta - \beta^*\| \leq \epsilon \):
\[
O \left( n^4 K \log \frac{n \cdot \max_i B_i}{\epsilon \cdot \min_i B_i} \right) = O \left( n^4 K \log \frac{n}{\epsilon} \right).
\]

### A.15 Proof of Theorem 7
The bound on \( \mathbb{E}[\|\beta^t - \beta^*\|^2] \) is derived directly from the proof of (Xiao 2010, Corollary 4). Here, the strong convexity modulus is \( \sigma = \min_i B_i^3 \) due to the term \( \beta \mapsto - \sum_i B_i \log \beta_i \) and the domain \([\beta, \bar{\beta}]\). Since \( \bar{\beta}_i = \frac{1}{n} \) for all \( i \). For the subgradient, we choose \( g^t = g(\beta^t, \theta_t) = v_i(\theta_t) \cdot e(i_t) \), where \( i_t \in \arg \max \beta_i^tv_i(\theta_t) \) (choosing the smallest index in a tie) based on Lemma 8. Clearly, \( i_t \) depends on both \( \beta^t \) and \( \theta_t \). Hence, we have
\[
\|g^t\|^2 = v_i(\theta_t)^2 \leq \max_i v_i(\theta_t).
\]
Therefore,
\[
\mathbb{E}[\|g_t\|^2] \leq \mathbb{E}[\|v_i(\theta_t)^2\|] = G^2.
\]
Here, \( \max_i v_i \in L^2(\Theta) \) since
\[
0 \leq \max_i v_i \leq \sum_i v_i \in L^2(\Theta).
\]
The second half is derived in a straightforward manner from the discussion in (Xiao 2010, pp. 2559). Note that \( \Delta_t = \frac{C^3}{2\sigma} (6 + \log t) \) is an upper bound on the regret in iteration \( t \) in an online optimization setting (Xiao 2010, §3.2, Eq. (20)). Meanwhile, the constant \( V \) here is an upper bound on the difference between maximum and minimum attainable objective values of (8) (across all \( \theta \in \Theta \)). By (17) and (18) in the proof of Theorem 6, it is bounded by
\[
\frac{n}{\min_i B_i} - \sum_i B_i \log B_i \leq \frac{n}{\min_i B_i} + \log n \leq \frac{2n}{\min_i B_i}.
\]

### A.16 Proof of Theorem 8
First, each \( \Theta_S \) is (Borel) measurable since it is the preimage of a (Borel) measurable set
\[
\{ y \in \mathbb{R}^n : y_i = y_j > y_t, \ \forall i, j \in S, \ \ell \notin S \} \subseteq \mathbb{R}^n
\]
under a continuous function
\[
\theta \mapsto (\beta^*_1 v_1(\theta), \ldots, \beta^*_n v_n(\theta)).
\]
Suppose we have solved for \( b \) that satisfy the said conditions (we will show that it does exist) and construct \( x^*_i = 1_{\Theta_i}, \Theta_i = \cup_i \Theta_i, S \) accordingly. By the choice of \( S \), we know that \( \beta^*_i v_i = p^* \) a.e. on \( \Theta_S \) if \( i \in S \) and \( \beta^*_i v_i < p^* \) a.e. on \( \Theta_S \) if \( i \notin S \).
\[
\langle v_i, x^*_i \rangle = \sum_S v_i(\Theta_i,S) = \sum_{S : i \in S} \frac{1}{\beta^*_i} p^*(\Theta_i,S) = \frac{1}{\beta^*_i} \sum_{S : i \in S} b_i,S = \frac{1}{\beta^*_i} \sum_{S} b_i,S = B_i \beta^*_i,
\]
that is, (5) holds. Meanwhile, note that
\[
\cup_{S \neq \emptyset} \Theta_S = \{ \theta \in \Theta : p^*(\theta) > 0 \}.
\]
Therefore,
\[
\sum_i x^*_i = \sum_i \sum_{S : i \notin S} x^*_i = \sum_{S \neq \emptyset} \sum_{i \notin S} 1_{\Theta_i,S} = \sum_{S \neq \emptyset} 1_{\Theta_S} = 1_{\{p^* > 0\}}.
\]
Hence, (4) clearly holds. By Lemma (4), \( x^* \) solves (1). By Theorem 2, \( x^* \) is an equilibrium allocation. It is a pure allocation by construction.

Next, we show that such \( b \) exists. By Theorem 1 and 2, there exists a pure equilibrium allocation \( x^*, x^*_i = 1_{\Theta_i}, \) where \( \Theta_i, \ i \in [n] \) is a partition of \( \Theta \), that solves (1) and satisfies (4)-(6) together with the optimal solution \( (p^*, \beta^*) \) of (2). Note that these \( \Theta_i \) are potentially different from those constructed in the theorem: for the latter,
\[
\cup_i \Theta_i = \cup_{S \neq \emptyset} \Theta_S = \Theta \setminus \Theta_{\emptyset}.
\]
where $\Theta_\emptyset$ is the (possibly nonempty) zero-value subset, on which $v_i = 0$ for all $i$ (and hence $p^*(\Theta_\emptyset) = 0$). By Corollary 2, when $i \notin S$, we must have $x_i^* = 0$ a.e. on $\{p^* > \beta_i v_i\}$, i.e.,

$$\mu(\Theta_i \cap \Theta_S) = 0.$$ 

In other words, in the pure equilibrium allocation $x^*$, buyer $i$ only chooses from where it wins ($i \in S$). Define

$$b_{i,S} = \langle p^*, x^*_{i,S} \rangle,$$

where $x^*_{i,S} = 1_{\Theta_i \cap \Theta_S}$ is the restriction of $x_i^*$ on $\Theta_S$. We can also write it as $b_{i,S} = p^*(\Theta_i \cap \Theta_S)$. Since $p^* \in L_1(\Theta)_+$ (c.f. Lemma 2) and, as a measure, is absolutely continuous w.r.t. $\mu$, we know that $b_{i,S} = 0$ if $i \notin S$. By (4), we also know that on each $\Theta_S$,

$$\left\langle p^*, 1_{\Theta_S} - \sum_i x^*_{i,S} \right\rangle = 0.$$ 

Therefore, for each $S \neq \emptyset$, we have

$$\sum_i b_{i,S} = \sum_{i \in S} b_{i,S} = \left\langle p^*, \sum_i x^*_{i,S} \right\rangle = \langle p^*, 1_{\Theta_S} \rangle = p^*(\Theta_S).$$ 

For $S = \emptyset$, we have $b_{i,S} = 0$ for all $i$ and the above equality also holds (this can also been seen from the choice of $\Theta_S$, which ensures $p^* = 0$ on $\Theta_S$). Meanwhile, since $\Theta_i = \cup S (\Theta_i \cap \Theta_S)$, we have $\sum_S x^*_{i,S} = x_i^*$ and hence

$$\sum_S b_{i,S} = \sum_S \langle p^*, x^*_{i,S} \rangle = \langle p^*, x_i^* \rangle = p^*(\Theta_i) = B_i,$$

where the last equality uses the characterization of $p^*$ in Theorem 1:

$$p^*(A) = \sum_i \beta_i v_i(A \cap \Theta_i).$$ 

The leftover $\Theta_\emptyset$ can be assigned arbitrarily since, by its construction, $p^* = v_i = 0$ a.e. on $\Theta_\emptyset$ for all $i$. \qed

**B More details on the illustrative example**

Here, $n = 4$ and the buyers’ budgets are $B = (B_1, \ldots, B_4) = [0.1, 0.2, 0.3, 0.4]$. The valuations are $v_i(\theta) = c_i(\theta) + d_i$, where

$$c = (c_1, \ldots, c_4) = [0.0298, -0.0194, -0.0269, 0.011],$$

$$d = (d_1, \ldots, d_4) = [0.9851, 1.0097, 1.0135, 0.9945].$$

The numbers above are randomly chosen and normalized so that $v_i([0, 1]) = \int_0^1 v_i(\theta) d\theta = 1$ for all $i$. Meanwhile, we set $|c_i| < |d_i|$ so that no two different $\beta_i v_i$ are nearly parallel. The construction of the breakpoints is straightforward, similar to the description in the proof of Lemma 9. Let $\Theta_i = (\tilde{p} = \tilde{\beta}_i) = [l_i, u_i]$ (i.e., the intervals in the legend of Figure 1, which are determined by $\tilde{\beta}$). Use this as a primal solution, an upper bound on the duality gap is

$$\left(\sum_i \tilde{\beta}_i v_i(\tilde{\Theta}_i) - \sum_i B_i \log \tilde{\beta}_i\right) - C - \sum_i \log v_i(\tilde{\Theta}_i),$$

where $C = \|B\|_1 - \sum_i B_i \log B_i$ and

$$v_i(\tilde{\Theta}_i) = \int_{l_i}^{u_i} v_i(\theta) d\theta = \frac{c_i}{2} (u_i^2 - l_i^2) + d_i (u_i - l_i).$$

After $t = 5 \times 10^4$ SDA iterations, the numerical value of the duality gap is 0.000466. Utility of buyer $i$ is $\tilde{u}_i = v_i(\tilde{\Theta}_i)$, which is

$$\tilde{u}_i = (\tilde{u}_1, \ldots, \tilde{u}_4) = [0.1031, 0.1816, 0.312, 0.4088].$$

The (weighted) envy of buyer $i$ is defined as

$$\gamma_i = \max_{j \neq i} \left(\frac{B_i}{B_j} v_i(\tilde{\Theta}_j) - \tilde{u}_i \right).$$

The final $\{\tilde{\Theta}_i\}$ leads to envy

$$\gamma = (\gamma_1, \ldots, \gamma_4) = [0.0, 0.0259, 0.0, 0.0017].$$