On The Expected Photon Spectrum in $B \to X_s + \gamma$ and Its Uses

Ikaros Bigi $^{a,\alpha}$
Nikolai Uraltsev $^{a,b,c}$

$^a$Department of Physics, University of Notre Dame du Lac, Notre Dame, IN 46556 USA
$^b$INFN, Sezione di Milano, Milan, Italy
$^c$Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg, 188350, Russia

$\alpha$ e-mail address: bigi.1@nd.edu

Contributed to the Workshop on the CKM Unitarity Triangle
CERN, February 13-16, 2002

Abstract

Measuring the photon energy spectrum in radiative $B$ decays provides essential help for gaining theoretical control over semileptonic $B$ transitions. The hadronic recoil mass distribution in $B \to X_u \ell \nu$ promises the best environment for determining $|V_{ub}|$. The theoretical uncertainties are largest in the domain of low values of the lepton pair mass $q^2$. Universality relations allow to describe this domain reliably in terms of the photon spectrum in $B \to X_s + \gamma$. A method is proposed to incorporate $1/m_b$ corrections into this relation. The low-$E_\gamma$ tail in radiative decays is important in the context of extracting $|V_{ub}|$. We argue that CLEO’s recent fit to the spectrum underestimates the fraction of the photon spectrum below 2 GeV. Potentially significant uncertainties enter in the theoretical evaluation of the integrated end-point lepton spectrum or the $B \to X_u \ell \nu$ width with a too high value of the lower cut on $q^2$ in alternative approaches to $|V_{ub}|$. 
1 Introduction

Since $B \rightarrow X + \gamma$ decays represent a loop effect within the Standard Model (SM), they are viewed as a window onto New Physics. There is a second motivation that has attracted considerable attention: measuring the inclusive photon spectrum allows us to infer the motion of the heavy quark inside the $B$ meson. This yields new insights into the inner workings of QCD which significantly affect, for example, the lepton spectrum in $b \rightarrow u \ell \nu$ decays. The moments of the heavy quark distribution function allow in principle to determine experimentally a number of nonperturbative parameters intrinsic to the heavy quark expansion the knowledge of which is of primary practical importance \[1, 2\].

One should note, however, that we are not in the dark about the values of these quantities: we know the running beauty quark mass $m_b$ and the kinetic $\mu^2_\pi$ and chromomagnetic $\mu^2_G$ expectation values with good accuracy due to intensive theoretical investigations over the last few years (for a review see Ref. [3]). The most accurate value for $m_b$ has been determined from $e^+e^-$ annihilation to beauty hadrons close to threshold. Heavy quark sum rules impose tight constraints; in particular they ensure $\mu^2_\pi > \mu^2_G$, which in practice allows for only a limited range for the former. Extracting the values of $m_b$, $\mu^2_\pi$ etc. from the moments or shape of energy spectra thus serves as a cross check – and a highly valued one at that – of our theoretical control. In lucky cases this can allow to narrow down the theoretical uncertainties by excluding some of the corners in parameter space. This comparison provides us also with information about the scale of higher order effects which will be very important in assessing the numerical accuracy of concrete applications of the heavy quark expansion [4].

It has always been tempting to extract $|V_{ub}|$ from the lepton energy endpoint spectrum in $B \rightarrow X_u \ell \nu$ using information on the $b$ quark distribution function in-
ferred from the photon spectrum in $B \to X_s + \gamma$. We will illustrate that theoretical problems arise in such a procedure which may severely limit its accuracy. It is more profitable to use the information from the photon spectrum to control possible theoretical uncertainties in the hadronic recoil mass spectrum in $B \to X_u \ell \nu$ originating from low-$q^2$ kinematics.

Very recently the CLEO collaboration has presented a new measurement of the branching ratio and photon spectrum for $B \to X_s + \gamma$ [5]. The bulk of their spectrum shown in Fig. 1 is quite similar to the theoretical predictions (Fig. 3 of Ref. [6]) made already in 1995. Since then our knowledge of $m_b$ and $\mu^2_\pi$ has become more precise; the value of the latter, which determines the width of the energy distribution is now expected in the lower half of the interval used in 1995. This brings the main part of the spectrum into even better agreement with the data. One should keep in mind, though, that the experimental photon spectrum is additionally Doppler-smeared by the nonvanishing velocity $\beta \approx 0.063$ of $B$ mesons in decays of $\Upsilon(4S)$.

![Figure 1: CLEO’s photon spectrum and the model fit to the data from Ref. [5].](image)

The model spectrum used by CLEO (hereafter referred to as CLEO’s fit) assigns only a tiny fraction of the photon spectrum to fall below 2 GeV. This would be a welcome feature since the background becomes much more severe there; it means that the branching ratio for radiative decays can therefore be measured with good accuracy. The total radiative branching ratio is little affected by the possible uncertainties in this fraction. Yet the photon spectrum below 2 GeV has considerable intrinsic interest. Since it is under better theoretical control as far as perturbative and nonperturbative effects are concerned, it provides a testbed for studying the
onset of local parton-hadron duality. Its possible violation, although of no practical concern for integrated widths, can significantly affect the decay rates integrated over limited kinematic domains, or interfere with extracting parameters describing strong dynamics from higher moments. Besides, data from this kinematical domain are important for establishing precision control over the low-\( q^2 \) region in \( B \to X_u \ell \nu \), referred to above.

It is therefore important to have accurate data on the photon spectrum below 2 GeV despite the experimental challenge it poses. CLEO has performed a preliminary measurement of that domain. However, we think that the experimental fit in Ref. \[5\] significantly underestimates the spectrum that has to exist there. The theoretical framework itself used to translate the observed spectrum into heavy quark parameters assumes that a more significant fraction of the photon spectrum falls below 2 GeV. We make the case for future data with even higher statistics to clarify this issue, and suggest they will show that more than 10% of the photon spectrum resides below 2 GeV. An intriguing lesson can, however, be drawn if CLEO’s fit were to be confirmed.

The remainder of the paper is organized as follows: after briefly recalling in Sect. 2 how the photon spectrum in radiative \( B \) decays is obtained theoretically, we discuss more specifically the perturbative spectrum mainly responsible for the lower tail in Sect. 2.1. The complete spectrum is considered in Sect. 2.2. In Sect. 3 arguments are given calling for caution when connecting the lepton energy endpoint spectrum in \( B \to X_u \ell \nu \) with the photon spectrum in \( B \to X_s + \gamma \), and a simplified method to account for a part of subleading corrections is suggested. We then describe a way to further reduce the theoretical uncertainties in extracting \( |V_{ub}| \) from the hadronic recoil mass spectrum in \( B \to X_u \ell \nu \) in Sect. 4, before commenting on some issues related to extracting \( |V_{ub}| \) and presenting our conclusions in Sect. 5.

2 Theoretical expectation for the photon spectrum

While the integrated width for the inclusive transition \( B \to X + \gamma \) can be affected by the intervention of New Physics, the shape of the prompt photon spectrum is more robust against it, at least within non-exotic scenarios. Two classes of effects govern the photon spectrum: the lowest order process \( b \to q + \gamma \) would produce a monoenergetic line \( E_\gamma = \frac{m_b^2 - m_q^2}{2m_b} \simeq m_b/2 \) characteristic of two-body kinematics; gluon bremsstrahlung adds a continuous hadronic (partonic) mass spectrum all the way up to \( m_b \) thus producing a radiative tail covering the whole range of smaller photon energies.

Gluon field modes with momenta of order \( \Lambda_{\text{QCD}} \) introduce additional nonperturbative effects: most notably they induce a primordial energy-momentum distribution of the \( b \) quark inside the \( B \) meson – an effect that had been introduced on phenomenological grounds more than 20 years ago \[4, 8\]. The emergence of this
‘Fermi motion’ in the OPE-based QCD heavy quark expansion was pointed out in Ref. [1]. The related Doppler-smearing further broadens the photon spectrum and populates also the gap $\frac{m_b}{2} < E_\gamma < \frac{M_B}{2}$ distinguishing quark-level and hadronic kinematics. To leading order in $1/m_b$ the effect of the primordial distribution is given by the convolution of the short-distance–generated parton spectrum $d\Gamma_{\text{pert}}/dE$ with the heavy quark distribution function $F$:

$$\frac{d\Gamma}{dE_\gamma} = \int dk_+ \frac{d\Gamma_{\text{pert}}(E_\gamma - \frac{k_+}{\Lambda})}{dE} F(k_+).$$

(1)

The natural variable for $F$ is the light-cone momentum $k_+ = k_0 - k_z$ carried by the heavy quark in the parton description. In analogy with the deep inelastic scattering structure functions it is often traded for a dimensionless variable $x = \frac{k_+}{\Lambda}$ with $\Lambda = \lim_{m_b \to \infty} (M_B - m_b)$. The function $F(x)$ has support from $-\infty$ to 1, and its moments are given by the expectation values of local heavy quark operators:

$$\int_{-\infty}^{1} F(x) \, dx = 1, \quad \int_{-\infty}^{1} x F(x) \, dx = 0$$

$$\int_{-\infty}^{1} x^2 F(x) \, dx = \frac{1}{2} \mu_x^2, \quad \int_{-\infty}^{1} x^3 F(x) \, dx = -\frac{1}{3} \rho_D^3,$$

etc. ($\mu_x^2, \rho_D^3$ denote the expectation values of the kinetic energy operator and the Darwin term, respectively). At large negative $x$ this function must rapidly vanish. In actual QCD the moments and likewise the distribution function itself depend on the normalization point, and this dependence is actually far more pronounced than in the usual leading-twist functions of light hadrons. The exact separation between quantum modes that are short-distance and belong to the hard “perturbative” kernel, and those which are considered soft and thus incorporated into the nonperturbative ‘primordial’ effects, is to some extent arbitrary and is set by fixing a normalization scale $\mu$. In this way both elements on the right hand side of Eq. (1) are $\mu$-dependent, while the physical spectrum on the left hand side is not.

Even leaving aside effects of higher order in $1/m_b$, predicting the photon spectrum would require detailed knowledge of the function $F(x)$, which is not quite realistic. Predictions [2] were obtained assuming a reasonable functional form of $F(x)$ and fitting its parameters to reproduce the known and estimated $B$ expectation values in Eq. (2). Such a choice for $F(x)$ is not unique, and one might be concerned that an essential uncertainty is thus introduced into the spectrum. It turns out, however, that its gross features are relatively stable as long as the first three or four moments are fixed.

This stability applies in particular to the lower part of the spectrum, especially when one considers the fraction of events with photon energies below a value $E$:

$$\Phi_\gamma(E) = \frac{1}{\Gamma(B \to X_s + \gamma)} \int_0^E dE_\gamma \frac{d\Gamma}{dE_\gamma}.$$  

(3)
The fraction of decay events with $E_\gamma < 2$ GeV theoretically comes out about 12%, significantly larger than the 5% CLEO’s fit would yield and above 8% one would obtain literally using the central data points. Yet, it is qualitatively evident that sufficiently low parts of the spectrum are shaped by emission of relatively hard gluons where the perturbative description is adequate and the intervention of nonperturbative dynamics is insignificant. This was manifest in the analysis of Ref. [6].

Of course, the uncertainty in the distribution function $F(x)$ can to some extent affect the fraction $\Phi_\gamma(2 \text{ GeV})$ if, say, $F(x)$ has a long tail towards large negative $x$. It is usually missed, however, that such a scenario would imply a rather large value for the Darwin operator $\rho_D^3$ (and other higher-order ones) by virtue of their relation to the moments in Eq. (2). This would not go unnoticed in other applications of the heavy quark expansion. Yet since our direct knowledge of such expectation values is at present still limited, we will not draw conclusions here based on this observation. Instead we merely want to point out that a broader distribution $F(x)$ yields additional smearing, which tends to increase the low-$E_\gamma$ fraction of decay events. To quantify these intuitive assertions we consider in more detail the purely perturbative component of the spectrum $d\Gamma_{\text{pert}}/dE_\gamma$.

### 2.1 The perturbative spectrum

Throughout this paper we assume the radiatively induced decay to be generated by a single local operator

$$L_{\text{weak}} = \frac{\alpha_s}{8\pi} (1 + \gamma_5) b F^{\mu\nu}.$$  \hspace{1cm} (4)

Normalized at the scale $m_b$, this term indeed yields the dominant contribution to the $b \to s + \gamma$ decay rate in the Standard Model. Other contributions to the transition amplitude generated at scales below $m_b$ give sizeable corrections, in particular the usual four-fermion operator $\bar{s}b \bar{c}c$ with the low-scale annihilation of the $c\bar{c}$ pair into photon and hadrons. However, this mostly modifies the total rate, whereas the decay fraction we are interested in is hardly changed.

Perturbative gluon effects in the $b \to s + \gamma$ spectrum contain some theoretical complexities; for doubly logarithmic Sudakov corrections arise due to the emission probability $\propto \frac{d\omega}{\omega} \frac{dk^2}{k^2}$, where $\omega$ and $k_\perp$ are the gluon energy and transverse momentum, respectively. The gluon coupling grows at small $k_\perp$, which on one hand enhances corrections and on the other mandates isolating soft gluons from the perturbative kernel. At the same time, introducing the cutoff at any reasonable factorization scale eliminates logs as a large parameter in actual $B$ decays: with maximal $E_\gamma = m_b/2$ below 2.5 GeV the parameter $\ln \frac{E_\gamma}{k}$ can barely reach even unity. Therefore, we should not expect significant uncertainties from possible higher-order corrections.

The basic features of the perturbatively generated distribution become manifest already in the model for the leading-log spectrum improved to include running of
where $S$ is the square of the Sudakov formfactor, and $w(E_\gamma)$ is the perturbative probability for a sufficiently hard gluon to be emitted so that the photon is left with energy below $E_\gamma$. Using the fact that soft gluons couple with $\alpha_s(k_\perp)$ and cutting the integration at $k_\perp \leq \mu$ where $\mu$ represents the separation scale, the integrals are easily calculated \cite{6}:

$$w(E_\gamma) = \frac{8}{3\pi} \left( \frac{2\pi}{9} \right)^2 \times$$

$$\left\{ \begin{array}{ll}
\frac{1}{\alpha_s(m_b)} \ln \frac{\alpha_s(\sqrt{s}/m_b)}{\alpha_s(m_b)} - \frac{1}{\alpha_s(\mu)} \ln \frac{\alpha_s(\mu)}{\alpha_s(\sqrt{s}/m_b)} & \varepsilon \geq \mu \\
\frac{1}{\alpha_s(m_b)} \ln \frac{\alpha_s(\sqrt{s}/m_b)}{\alpha_s(m_b)} - \frac{1}{\alpha_s(\mu)} \ln \frac{\alpha_s(\mu)}{\alpha_s(\sqrt{s}/m_b)} - \frac{q}{2\pi} \ln \frac{\sqrt{s}}{\varepsilon} \left[ 1 - \ln \frac{\alpha_s(\mu)}{\alpha_s(\sqrt{s}/m_b)} \right] & \frac{\mu^2}{m_b} \leq \varepsilon < \mu \\
\frac{1}{\alpha_s(m_b)} \ln \frac{\alpha_s(\mu)}{\alpha_s(m_b)} - \frac{q}{2\pi} \ln \frac{m_b}{\mu} & \varepsilon \leq \frac{\mu^2}{m_b}
\end{array} \right.$$  

$$\varepsilon = m_b - 2E_\gamma.$$  

The representation of Eqs. (5)-(6) has the added advantage that the integrated fraction $\Phi_\gamma(E)$ of the decay events with $E_\gamma < E$ is directly given by $1 - S(E) = 1 - e^{-w(E)}$. The effective ‘soft’ coupling in Eq. (8) is known to two loops \cite{6}.

The perturbative photon spectrum following from Eqs. (5)-(6) was discussed in Ref. \cite{6}. The relevant point is that $S(E)$ does not depend on $\mu$ as long as $m_b - 2E > \mu$. This means that this part of the spectrum is shaped by gluons harder than $\mu$; in practice this scale is around 0.9 GeV for $E = 1.9$ GeV. Therefore, the bulk of the perturbative spectrum below 2 GeV is indeed generated by sufficiently hard gluons, and can be trusted. Since the fraction $\Phi_\gamma(E)$ is still small, exponentiation of the soft emissions does not produce a radical numerical change. It is important, however, that this applies only as long as the cutoff in soft emissions is implemented.

The upper part of the spectrum is mostly determined by the primordial distribution function $F(x)$ with perturbative radiation affecting mainly the height of the spectrum and somewhat widening its shape. To a crude approximation, the central part of $F(x)$ can be directly taken from the observed distribution at $E_\gamma > 2.2$ GeV.

The lower part of the spectrum is to some extent affected by the primordial Fermi motion as well, since $\Phi_\gamma^{\text{pert}}(E)$ gets convoluted with $F(x)$:

$$\Phi_\gamma^{\text{pert}}(E) = \int dk_\perp \Phi_\gamma^{\text{pert}}(E-k_\perp) F(k_\perp).$$  

\footnote{This ansatz in fact produces even a somewhat harder spectrum than the one from the exact DL result based on soft factorization, which is evident when double emission is considered. If the soft gluons emitted individually lower $E_\gamma$ by amounts $\delta_1$ and $\delta_2$ with probabilities $dw_1$ and $dw_2$ respectively, double emission would yield $E_\gamma = \frac{m_b}{2} - \delta_1 - \delta_2$ with the probability $\propto dw_1 dw_2$, while the ansatz places this contribution at $E_\gamma$ equal to the lower of the two energies, $E_\gamma = \frac{m_b}{2} - \max \{\delta_1, \delta_2\}$.}
First, a long tail of $F(x)$ to negative $x$ can populate the domain of lower $E_\gamma$ even in the absence of hard bremsstrahlung at all. This clearly would only enhance the estimate of $\Phi_\gamma(2 \text{ GeV})$. Secondly, there is an effect of smearing due to typical $k_\perp \sim \Lambda_{\text{QCD}}$. Since the perturbative spectrum decreases sharply with increase of $m_b^2 - E_\gamma$, this normally enhances $\Phi_\gamma(E)$ as well, at least with $E$ in the deep perturbative domain – the perturbative distribution is much wider than the intrinsic primordial smearing in the $1/m_Q$ expansion. However, in practice typical momenta in $F$ could be of similar scale as $m_b^2 - E_\gamma$; when $k_\perp$ kicks up the effective perturbative energy $E_\gamma + k_\perp/2$ to the domain where $\Phi^\text{pert}_\gamma$ is nearly saturated, the above approximation does not work anymore. Therefore, the primordial smearing can, in principle, decrease the purely perturbative estimate – in particular, when $\Phi^\text{pert}_\gamma(E)$ is already significant and $F(x)$ is a broad function. In practice, the nonperturbative smearing only enhances the tail of the spectrum in the relevant domain.

The double log approximation turns out to be far too crude for $B$ decays. This can be assessed by comparing the exact $\mathcal{O}(\alpha_s)$ spectrum [10] with the one obtained expanding the double log expression in $\alpha_s$: the resulting spectrum exceeds the complete expression by more than a factor of 3. Moreover, keeping double and single logs alone would yield an enhancement – rather than suppression – factor in the Sudakov exponent for $E_\gamma$ as large as $m_b/2 - 75 \text{ MeV}$. In this situation one better relies on the exact one loop spectrum:

$$\frac{1}{\Gamma} \frac{d\Gamma^\text{pert}}{dE_\gamma} = \frac{\alpha_s}{\pi} \left( \frac{2x^2 - 3x - 6}{3(1-x)} \right) x + 2 \left( x^2 - 3 \right) \ln \left( \frac{1-x}{x} \right), \quad x = \frac{2E_\gamma}{m_b}. \quad (8)$$

It is clear, however, that using a fixed coupling $\alpha_s(m_b)$ would significantly underestimate the actual spectrum. The BLM part of the second-order corrections has been computed in Ref. [11] and was indeed found to increase the spectrum by about 60% for low enough $E_\gamma$.

There are subtle physical reasons for this BLM approximation to be of limited validity in the domain of relevant $E_\gamma$. The BLM corrections for a given photon energy are obtained integrating over all gluon configurations consistent with the invariant mass of the strange quark-gluon system. In particular, the gluon can be energetic and collinear, or relatively soft yet emitted at large angles:

$$M_X^2 = 2m_b(m_b - E_\gamma) \approx \frac{k_\perp^2}{\omega}. \quad (9)$$

The scale of the strong coupling, however is driven mainly by $k_\perp$. Hence, the effect of soft gluons is underestimated when averaged with those from short-scale configurations.

This becomes important when double gluon emissions are considered. Their probability is proportional to $\alpha_s^2$, and they are missed by the BLM terms. In contrast to ordinary corrections, this effect has an extra enhancement factor of $\ln (1-x)/x$ for each emission. Thus the common wisdom about the BLM dominance is not
expected to apply here. And there is an additional numerical enhancement: double (or any multiple) emission allows lowering $E_\gamma$ by the same amount with softer gluons (e.g. lower $k_\perp$ for same $\omega$); since $\alpha_s$ grows fast, this would yield a significant increase of the radiative tail overlooked by the naive BLM scale fixing. This can also be understood in a complimentary way applying the OPE to soft perturbative bremsstrahlung.

This problem obviously does not arise if the Wilsonian prescription for separating long and short distance dynamics is implemented when computing the perturbative spectrum to avoid double counting of soft gluon modes. Then the infrared domain is excluded and the coupling never grows too large. Moreover, soft gluons are simply excluded from computations regardless of the strength of the coupling. This eliminates both the possible enhancement in the soft emission amplitudes and strongly suppresses multiple emissions altogether; for the latter would now lower $E_\gamma$ by too large an amount. This approach allows using a single-gluon–generated spectrum and makes the BLM-type accounting for the running of $\alpha_s$ trustworthy.

The estimated fraction $\Phi_{\gamma}\text{pert}(E)$ depends mainly on the exact value of the $b$ quark mass. Actually $\Phi_{\gamma}\text{pert}$ is primarily a function of $\frac{m_b}{\mu} - E$ rather than of $E$ itself. It is evident that one should use the perturbative running low-scale mass $m_b(\mu)$ here, not the pole mass: excluding emissions of soft gluons requires – due to gauge invariance – discarding analogous virtual corrections, including the soft corrections in the $b$ quark self-energy. Then the pole mass never appears in the problem, and the heavy quark is seen in perturbation theory as a particle with the mass near $m_b(\mu)$.

In principle, the Wilsonian prescription can be introduced in different ways. A consistent scheme suitable also for genuinely Minkowskian processes and beyond one loop has been elaborated [12, 13]. In essentially one-loop corrections (including arbitrary order in the BLM resummation) it reduces to a cutoff in the gluon energy $\omega > \mu$ (in the perturbative computations) in the heavy quark restframe. The value of the “kinetic” running quark mass and its running is well known (for a review see Ref. [3]):

$$m_b(1\ \text{GeV}) = (4.57 \pm 0.05)\ \text{GeV}$$

(a somewhat larger value – 4.63 GeV – however with twice the uncertainty was reported by CLEO [5]). A subtlety should be kept in mind though: with such a cutoff scheme there are different short-distance masses perturbatively related to each other. The one entering here is $m_0(\mu)$, the mass defining the rest energy, rather than the kinetic mass in Eq. (10). The former slightly exceeds $m_b(\mu)$:

$$m_0(\mu) = m^{\text{kin}}(\mu) + \frac{4}{9} \frac{\alpha_s}{\pi} \mu \left[ 1 + \frac{\beta_0 \alpha_s}{2\pi} \left[ \ln \frac{m_b}{2\mu} + \frac{13}{6} \right] + \ldots \right] + O\left(\frac{\alpha_s \mu^3}{4m_b^3}\right).$$

The difference constitutes about 60 MeV at $\mu = 1$ GeV. (The straightforward all-order BLM summation yields a larger shift, however for consistency we retain only terms through $\beta_0 \alpha_s^2$.) With such $m_b$ the ratio $\frac{2E_\gamma}{m_b^2}$ is about 0.86 at $E_\gamma = 2$ GeV.

The perturbative expressions existing in the literature [10, 11] for the one-loop spectrum do not include the effect of an infrared cut, even though this computation
is not difficult. For low enough $E_\gamma$ this would not affect the single-gluon spectrum as have been already noted for the double log corrections. Strictly speaking, a gluon with $\omega \leq \mu$ can bring $E_\gamma$ as low as $\frac{m_b^2}{2} - \mu$, so the cut affects the spectrum in a wider domain. This is possible, however, only when the gluon and the s quark fly back-to-back, a rather rare and – being governed by a short-distance coupling – suppressed configuration. The enhanced kinematic sensitivity is absent as soon as the gluon is emitted into the forward hemisphere. Hence, to a good approximation the effect of the cutoff can be neglected at $E_\gamma$ below $\frac{m_b^2}{2}$ as it happened in the double log ansatz (8).

Above this borderline the perturbative spectrum is modified more significantly, its growth slowed down with the spectrum vanishing around $\frac{m_b^2}{2} - \frac{\mu^2}{2m_b}$. It also has the two-body $\delta$-function at $E_\gamma = \frac{m_b(\mu)}{2}$ suppressed by a finite amount by virtual corrections.

Figure 2: Perturbative fraction of decay events $\Phi^\text{pert}_\gamma(E)$ having $E_\gamma < E$.

The estimated fraction of the decay events generated solely by perturbative gluons is shown in Fig. 2. We see that one expects $\Phi^\text{pert}_\gamma(2 \text{ GeV})$ to be about 6%.

### 2.2 Combined spectrum

The overall spectrum is obtained by convoluting the perturbative contribution with the light-cone distribution function $F(x)$. Hence it requires the perturbative part in the whole domain including the interval $m_b - 2\mu < 2E_\gamma \leq m_b$ where it is modified by the gluon cutoff $\mu$. However, the exact details in this relatively narrow domain are not too significant due to primordial smearing, as long as the overall normalization, $\Phi^\text{pert}_\gamma(E) = 1$ at $E > \frac{m_b}{2}$ is maintained. Using the double log computation as a guide, we extrapolate $\Phi^\text{pert}_\gamma(E)$ linearly above $E_0 = \frac{m_b - \mu}{2}$; the end-point $\delta$-function in the spectrum is fixed by the normalization constraint. This approximation is expected to slightly underestimate $\Phi^\text{pert}_\gamma$ in the lower end of the window $\frac{m_b}{2} - \mu$, and overestimate it near the maximal $E_\gamma$ except the endpoint itself. The bias in the resulting total fraction should then be insignificant and presumably only underestimates $\Phi_\gamma(E)$ at $E$ near 2 GeV.
As illustrated in Ref. [6], while the purely perturbative spectrum and \( \Phi_{\gamma}^{\text{pert}}(E_{\gamma}; \mu) \) depend strongly on the choice of \( \mu \), the overall \( \Phi_{\gamma}(E) \) comes out fairly insensitive to it. For the \( \mu \) dependence of \( \Phi_{\gamma}^{\text{pert}} \) is strongly offset by the renormalization of \( m_b(\mu) \), i.e. by the shift in the starting point of the parton-level spectrum. Therefore we can infer numerical estimates using a particular choice of the cutoff scale, namely \( \mu = 1 \) GeV for which the nonperturbative parameters are routinely fixed.

The expected fraction \( \Phi_{\gamma}(E) \) is shown in Fig. 3 and the spectrum itself in Fig. 4; we superimposed the experimental points onto our predictions in the latter. We took the ansatz of Ref. [6] for the distribution function \( F(x) \), but simplified it to an essentially a single-parameter one:

\[
F(k_+) = N (k_+ - \Lambda) \alpha e^{-ck_+} \theta(k_+ - \Lambda) .
\] (12)

\( c \) gauges the scale of the light-cone momentum, and \( \alpha \) determines the dimensionless ratio \( \mu_2^2 \frac{\pi}{3\Lambda} = 1 + \alpha \). Numerically \( \alpha = 2 \) yields a value very close to the experimental ones with \( \mu_2^2(1 \text{ GeV}) = 0.43 \text{ GeV}^2 \) and \( \Lambda(1 \text{ GeV}) = 0.65 \text{ GeV} \). It is curious to note that ansatz (12) yields for the Darwin expectation value

\[
\frac{\rho_D^3}{3} = \frac{2}{\Lambda} \left( \frac{\mu_2^2}{3} \right)^2
\] (13)

for arbitrary \( \alpha \). Such a value is favored by theory and would hold exactly if the SV sum rules were saturated by the \( P \)-wave states at a single mass.

To check the sensitivity to the shape of \( F \) we have also employed the alternative ansatz

\[
\tilde{F}(k_+) = \tilde{N} (k_+ - \Lambda)^\beta e^{-(d(k_+ - \Lambda))^2} \theta(k_+ - \Lambda)
\] (14)

with the tail fading out much faster, and \( \beta \) adjusted as to reproduce the same kinetic expectation value for a given \( m_b \). (At \( \alpha = 2 \) this corresponds to \( \beta \approx 0.658 \) and yields a
factor of 2/3 smaller Darwin expectation value). As expected, $\Phi_\gamma(E)$ is practically insensitive to the concrete ansatz; even the difference in the spectrum itself lies below the effects of higher-order nonperturbative corrections. It should be noted that the predictions shown in Fig. 4 do not include Doppler smearing due to the initial velocity of $B$ mesons produced at $\Upsilon(4S)$. Assuming absence of a correlation of experimental measurements with the direction of the decay relative to the $B$ momentum, the additional broadening is easily incorporated. The corresponding plots are given in Fig. 5.

Therefore, we arrive at the conclusion that about 12% of decay events should have $E_\gamma < 2$ GeV. Half of this probability is generated perturbatively, and it already exceeds the 5% of CLEO’s fit. The fact that CLEO’s fit represents too narrow a spectrum can be inferred also from the size of the second moment or dispersion through an alternative perspective: the value quoted by CLEO is essentially oversaturated by perturbative contributions alone.

It should be noted that although the fit was used in the analysis of Ref. 4, the total decay probability was obtained there adopting $\Phi_\gamma(2$ GeV$) = 8.5^{+5.5}_{-2.7}$% from Ref. 14, even though the fit itself yielded about 5%.

At present the part of the spectrum below 2 GeV does not seem very certain. If, however, future measurements establish that significantly less than 10% of the spectrum resides there, this would send the alarming message that the standard treatment of strong interaction effects – in particular the gluon corrections as they...
are routinely applied to decay distributions for beauty hadrons – is unreliable. This
in turn would cast doubts on the numerical accuracy of extracting $m_b$, $\mu_s^2$ etc. from
the photon spectrum and even more on relating the lepton endpoint spectrum in
$b \to u \ell\nu$ to the $b \to s + \gamma$ spectrum.

It can well be that the fully integrated spectrum is still robust against such
effects; however the higher moments and the shape in general would be affected. If
CLEO’s fit were to be vindicated by future data, one could not extract the quantity
$\bar{\Lambda}$ (let alone the kinetic expectation value) using the quoted formulae; for those
expressions were based on the assumption that gluon radiation can be described
erthropatively over the whole range up to $E_\gamma = 2$ GeV.

The simplest resolution of a possible discrepancy between the CLEO spectrum
and the theoretical expectation is that in particular the data point at the 1.9 GeV
bin represents a low fluctuation. Future data will settle this issue.

Of course, the first and, in particular, second moment of the photon energy
decrease if a spectrum closer to QCD expectations is used. Unfortunately, one
cannot re-evaluate them based only on the data points given in Ref. [5]. Extracting
the value of the $b$ quark mass it relied only on the moments evaluated over the range
$E_\gamma > 2$ GeV, hence there was a limited sensitivity of the purely experimental input
to the lower part of the spectrum. However, this simply relegated the sensitivity
to actual physics of gluon bremsstrahlung to theoretical evaluation of such defined
moments.

Figure 5: Photon spectrum in the lab frame where $\Upsilon(4S)$ are at rest, including Doppler
smearing with $\beta_B = 0.067$. The relative normalization is arbitrary.
3 Skeptical remarks

One of the practical applications of measuring the prompt photon spectrum is an attempt to estimate more reliably the fraction of semileptonic $b \to u$ decays expected theoretically with $E_\ell$ in the end-point region beyond the kinematic range for $b \to c$. While appealing as a nontrivial physical statement, in practice the relation between the integrated semileptonic end-point spectrum

$$\frac{1}{\Gamma_{sl}(b \to u)} \int_E^{M_B} dE_\ell \frac{d\Gamma_{sl}(b \to u)}{dE_\ell}$$

(15)

and the corresponding weighted $B \to X_s + \gamma$ spectrum

$$\frac{1}{\Gamma_{bs\gamma}} \int_E^{M_B} dE_\gamma \left( \frac{M_B}{2} - E_\gamma \right) \frac{d\Gamma_{bs\gamma}}{dE_\gamma}$$

(16)

is rather fragile – a fact emphasized already in the first dedicated paper [15]: the actual value of $m_b$ turns out to be insufficiently large to make higher order corrections irrelevant, and the allowed interval of lepton energies $E_\ell > M_B^2 - M_B^2/2M_B$ is too narrow to suppress higher-order corrections violating universality of the Fermi motion. There are two basic nonperturbative effects which are potentially dangerous here; we illustrate them in more detail than it was done in Ref. [15].

The first one is related to a particular class of $1/m_b^3$ corrections associated with four-fermion operators responsible for generalized Weak Annihilation (WA) effects. While typically at a few percent level in fully integrated $B$ widths, they are concentrated in the end-point domain in semileptonic decays and thus are enhanced by an order of magnitude for the narrow slice in $E_\ell$ [16]. Their size is controlled by the poorly known nonfactorizable terms in the expectation values $\langle B|\bar{b}\gamma_\mu(1-\gamma_5)u\bar{u}\gamma_\nu(1-\gamma_5)b|B\rangle$; they are different in charged and neutral $B$ and in general do not vanish even in $B^0$:

$$\delta\Gamma_{sl}(b \to u) \simeq \Gamma_{sl}(b \to u) \frac{32\pi^2}{m_b^3} \frac{2}{2M_B} \frac{\langle B|\bar{b}\gamma_k(1-\gamma_5)u\bar{u}\gamma^k(1-\gamma_5)b|B\rangle}{M_B^2}.$$  

(17)

The nonfactorizable contributions were addressed in Ref. [17]; the non-valence operators, however, are rather uncertain. The reasonable estimate of Ref. [18] suggests that this effect constitutes $10 \div 30\%$ of the semileptonic width with $E_\ell > 2.3$ GeV. A more detailed analysis can be found in the original papers [16, 18].

No such effect arises for the $B \to X_s + \gamma$ width at tree level. It is generated to order $\alpha_s$ [15] and is given by a different four-fermion operator. The correction to $\Gamma(B \to X + \gamma)$ is expected smaller here; a more careful consideration reveals that the end-point fraction can be affected, yet less significantly than in $B \to X_u \ell \nu$.

The second problem is of a general origin and related to the fact that the interval in the electron (or photon) energy $E_\ell > 2.3$ GeV is too narrow in practice [15]. To
quantify this observation, let us recall the $1/m_Q^2$ corrections in the $b \to u$ lepton spectrum [13, 1]:

$$\frac{1}{\Gamma_0} \frac{d\Gamma_{sl}}{dE_\ell} = 2 \left( \frac{2E_\ell}{m_b} \right)^2 \left\{ 3 - \frac{4E_\ell}{m_b} + \frac{10E_\ell}{m_b^2} \delta\left( \frac{m_b}{2} - E_\ell \right) - \frac{E_\ell^2}{3m_b^2} (m_b - E_\ell) \delta\left( \frac{m_b}{2} - E_\ell \right) \right\} \mu_\pi^2 +$$

$$\left[ \frac{2}{m_b^2} + \frac{10E_\ell}{3m_b^2} - \frac{11}{12m_b} \delta\left( \frac{m_b}{2} - E_\ell \right) \right] \mu_\pi^2 \right\}, \quad (18)$$

and examine the effect of the chromomagnetic operator. It has a negative $\delta$-function contribution in the end point:

$$\left. \frac{1}{\Gamma_0} \frac{d\Gamma_{sl}}{dE_\ell} \right|_{E_\ell \simeq \frac{m_b}{2}} \simeq 2 \left\{ 1 - \frac{11}{12} \frac{\mu_\pi^2}{m_b} \delta\left( \frac{m_b}{2} - E_\ell \right) \right\} + \text{term } \propto \mu_\pi^2. \quad (19)$$

The spectrum can be neither singular nor negative, of course. What this means is that one has to average the lepton energy at the very least over an interval $\Delta_{\min} = \frac{11}{12} \frac{\mu_\pi^2}{m_b} \simeq 70 \text{ MeV};$ a similar shift and a need for smearing applies a priori to $b \to s + \gamma$ events as well. On the other hand it is quite evident that this effect is completely missing from the Fermi motion and from the distribution function $F(x)$ – in contrast to the latter it is a non-universal process-dependent nonperturbative correction and would even have the opposite sign in decays of $B^*.$ Thus it is absent from the naive leading in $1/m_b$ relations between the observables in Eqs. (15) and (18).

The shift by 70 MeV does not affect much the total decay rate – yet it is very significant for the narrow end-point slice: for the parton spectrum it would decrease the rate by a factor $1 - \frac{11}{12} \frac{\mu_\pi^2}{m_b (m_b - 2E_\ell)}.$ This is more than a 50 percent reduction for $E = 2.3 \text{ GeV}$ even with conservative assumptions about $m_b.$

We can suggest a simplified way to improve the end-point relation between the photon and lepton spectrum close in spirit to the proposal to be made below in Sect. 5.2 for the invariant mass $M^2_{\chi_1}$ distribution. To account for the chromomagnetic interaction effect in Eq. (19) one can make a shift in the lepton vs. photon energy by a formally $O(1/m_b)$ amount $\delta E_\ell = \frac{2}{3} \frac{\mu_\pi^2}{m_b}$:

$$\frac{1}{\Gamma_{sl}(b \to u)} \int_{E_\ell}^{M_B} dE_\ell \frac{d\Gamma_{sl}(b \to u)}{dE_\ell} \propto \frac{1}{\Gamma_{bs\gamma}} \int_{E_\ell + \delta E_\ell}^{M_B} dE_\gamma \left( \frac{M_B}{2} - E_\gamma \right) \frac{d\Gamma_{bs\gamma}}{dE_\gamma}. \quad (20)$$

Physically this is motivated by the different interaction of fast light-like light quarks with the chromomagnetic field in the two processes leading to a relative energy shift. However this is not a rigorous prescription and its accuracy is difficult to quantify.

\textsuperscript{2} Chromomagnetic field $\vec{B}(0)$ emerges as a commutator of different spacelike momentum operators of $b$ quark, while Fermi motion depends only on a single space momentum and energy operator. The term $\propto \mu_\pi^2$ is the first term in the effect of the primordial motion.
Since the effect is significant, it may introduce an essential uncertainty even with this improvement.

It is important to keep in mind that the above constraint on the minimal resolution is of a rather fundamental nature which goes beyond a conceivable improvement of the Fermi motion description [9]. Knowledge of the subleading distribution functions would not enable one to shrink this interval. Of course, the literal expression for higher-order terms like Eqs. (17), (19) or (20) going beyond the leading-twist expansion in this end-point domain cannot be used literally to improve the naive leading-order expressions, since they are superimposed onto the nonperturbative leading-twist Fermi motion. In principle, it cannot be excluded that the latter somewhat softens the numerical instability of the end-point expansion – yet one cannot count on a radical improvement of the accuracy: the fact that allowing only a 70 MeV interval of the end point energy yields a 100% error in the spectrum in the best scenario, is a benchmark which is difficult to get around.

From the general perspective the problem is related to presence of another parameter in the problem – the ratio of the scale $\frac{m^2}{m_b}$ to the typical hadronic mass. With this scale amounting to only a few times $\frac{\mu^2}{m_b}$ the relation for the end-point spectrum is violated by a fixed amount even for asymptotic values of $m_b$. The accuracy of the relation is not governed solely by $1/m_b$ as is usually stated, and the actual numbers favor a possibility for significant corrections. This point is often missed in the literature.

4 Improving the $M_X^2$ distribution in $B \to X_u \ell \nu$

The integrated width $\Gamma(B \to X_u \ell \nu)$ is under good theoretical control [20]. Yet to measure it one has to disentangle it from the dominant $B \to X_c \ell \nu$ width. Vertex detection is not so highly efficient that one could achieve this goal solely or even mainly by rejecting the $b \to c$ background through finding the secondary charm decays. One has to apply cuts of a mainly kinematic nature. The most direct way is to consider the hadronic recoil mass spectrum $\frac{d\Gamma}{dM_X}$ and impose a cut in the invariant hadronic mass $M_X < M_D$, which has the advantage of retaining almost all $b \to u$ events. For experimental reasons one will have to lower the cut to confidently reject the $b \to c$ background. The situation is quite favorable since even a cut as low as 1.5 GeV leaves more than half of the signal events. Yet one still wants to place it as high as possible, preferably above 1.65 or even 1.7 GeV. Considering the huge statistics that can be accumulated at modern $B$ factories the central issue becomes how well one can theoretically evaluate the fraction of the $b \to u$ width falling into a restricted domain of phase space. Raising the cut reduces the rejected fraction and thus suppresses the theoretical uncertainty in the measured rate even without precise control over the fraction.

While the $M_X$-spectrum is sensitive to the values of the nonperturbative parameters as well as the heavy quark distribution function $F(x)$ describing Fermi
motion, the dependence on the latter is moderate if the current knowledge and the constraints on \( F(x) \) are incorporated in full. This particularly applies to the partially integrated spectrum. Following Refs. \[21\] we consider the fraction of events with the hadronic recoil mass below \( M_{\text{max}} \):

\[
\Phi_{\text{sl}}(M) = \frac{1}{\Gamma_{\text{sl}}(b \to u)} \int_{0}^{M} dM_X \frac{d\Gamma_{\text{sl}}(b \to u)}{dM_X} 
\]

(21)

Even without a dedicated analysis of all the uncertainties it is evident that \( \Phi_{\text{sl}}(1.7 \text{ GeV}) \) must exceed 0.55 and be below 0.9. This observable is thus known with at least \( \pm 24\% \) accuracy translating into a \( \pm 12\% \) error bar in \( |V_{ub}| \). Even allowing for additional stretching of uncertainties this would represent at the moment a good benchmark for quantities relevant for \( |V_{ub}| \). We found no substance to claims of unrecoverably large uncertainties in this observable.

Ultimately we would like to be able to decrease the theoretical uncertainties in \( \Phi_{\text{sl}}(M) \) further still – and achieve it without having to push the cutoff \( M \) too close to \( M_D \) or degrading the confidence in the assessed error bars. One obvious possibility is to introduce an additional cut rejecting small invariant mass \( q^2 \) (\( q \) denotes the momentum of the lepton pair) where the strong interaction effects on \( M_X^2 \) are maximal due to the significant recoil.\[3\] Choosing, for instance \( q^2 \geq 0.35 m_b^2 \) practically eliminates the impact of the primordial Fermi motion on \( M_X < 1.7 \text{ GeV} \) events. However, imposing additional cuts has the serious drawback of introducing a significant dependence on higher-order effects which are difficult to control and which are all too often missed in estimates. The overall energy scale governing the intrinsic ‘hardness’ of the decay rate deteriorates being driven at large \( q^2 \) by \( m_b - \sqrt{q^2} \) instead of \( m_b \) itself.\[13\] Fortunately, there is an alternative way not employing additional cuts.

The idea is based on the realization that the dangerous small-\( q^2 \) domain in \( B \to X_u \ell \nu \) decays is nearly identical to the case of \( B \to X_s + \gamma \) in respect to the hadronic mass distribution, at least in the limit of large \( m_b \). Moreover, its accuracy is governed by a genuinely \( 1/m_b \) expansion. Therefore, we can directly utilize experimental information about the \( B \to X_s + \gamma \) spectrum to control this kinematics, in a far more reliable way than the charged lepton end-point spectrum.

### 4.1 Scaling behavior and universality relations

To illustrate the idea, let us consider first \( B \to X_u \ell \nu \) at \( q^2 = 0 \). The concrete Lorentz structure of the decay vertex (\( V - A \) in \( b \to u \ell \nu \) and tensor in \( b \to s + \gamma \)) does not matter in the heavy quark limit; the effects of strong interactions rather depend on kinematics which are equal for vanishing invariant mass of the particles recoiling against the final state hadronic system. The decay distributions are governed by

\[3\] A cut, say on the lepton pair space momentum \(|\vec{q}|\) achieves a similar purpose since it is kinematically related to \( M_X^2 \) and \( q^2 \).
transitions when refer to semileptonic decays. Thus, at least for \( q^2 \) directly obtain \( 1 - q^2/M^2 \). However, at \( q^2 \) away from the point \( q^2 = 0 \) in the semileptonic decays. However, there is a scaling over the variable \( M^2_{\text{had}}/(1 - q^2/m_q^2) \) reflecting the universality of the Fermi motion \[15\]. It simply states that the hadronic mass squared is proportional to the decay light quark momentum \( |k_u| \):

\[
\frac{1}{\Gamma_{s\!l}(q^2)} \frac{d\Gamma_{s\!l}(q^2=0)}{dM_X^2}(M_X^2) = \frac{1}{2M_B} \frac{m_b^2}{m_b^2 - q^2} \frac{1}{\Gamma_{b\!s\gamma}} \frac{d\Gamma_{b\!s\gamma}}{dE_\gamma} \left( \frac{M_B^2 - M_X^2 m_b^2}{2M_B m_b^2 - q^2} \right).
\] (23)

This relation deteriorates for larger \( q^2 \) and breaks down when \( m_b^2 - q^2 \) becomes too low. However, at \( q^2 \) constituting a significant fraction of \( M_B^2 \) the effect of Fermi motion on the integrated fraction \( \Phi(M) \) disappears which simply becomes unity in this approximation.

Having this in mind, we can implement the scaling relation in practice to evaluate \( \Phi_{s\!l}(M) \). For the required rate with \( M_X \) exceeding \( M \) we have

\[
\Phi(M) = \int dq^2 \frac{1}{\Gamma_{s\!l}} \frac{d\Gamma_{s\!l}}{dq^2} \int M^2 \frac{1}{\Gamma_{s\!l}(q^2)} \frac{d^2\Gamma_{s\!l}(M_X^2;q^2)}{dM_X^2}.
\] (24)

with the last differential width in the scaling approximation given by Eq. (23). \( (\Gamma_{s\!l}(q^2) \) and \( d\Gamma_{s\!l}/dq^2 \) denote the same total fixed-\( q^2 \) width.) Then we have

\[
1 - \Phi_{s\!l}(M;\chi) = \int_0^{M_B^2 - M^2} dq^2 \chi(q^2) \frac{1}{\Gamma_{s\!l}} \frac{d\Gamma_{s\!l}}{dq^2} \int_0^{M_B^2 - M^2 - m_b^2} dE_\gamma \frac{1}{\Gamma_{b\!s\gamma}} \frac{d\Gamma_{b\!s\gamma}}{dE_\gamma},
\] (25)

where \( \chi(q^2) \) is a weight function one may (but does not have to) introduce to suppress contributions from small \( q^2 \) kinematics. Straightforward arithmetics yield

\[
1 - \Phi_{s\!l}(M;\chi) = \int_0^{M_B^2 - M^2 - m_b^2} dE_\gamma \phi(E_\gamma, M;\chi) \frac{1}{\Gamma_{b\!s\gamma}} \frac{d\Gamma_{b\!s\gamma}}{dE_\gamma},
\] (26)

with

\[
\phi(E_\gamma, M;\chi) = \int_0^{m_b^2 - M^2 - m_b^2} dq^2 \chi(q^2) \frac{1}{\Gamma_{s\!l}} \frac{d\Gamma_{s\!l}}{dq^2}.
\]
For not too large $q^2$ relevant in Eq. (25) the parton approximation

$$\frac{1}{\Gamma_{\text{sl}} m_b^2} \frac{d\Gamma_{\text{sl}}}{dq^2} = 6 \left(1 - \frac{q^2}{m_b^2}\right)^2 - 4 \left(1 - \frac{q^2}{m_b^2}\right)^3$$

(27)

is quite accurate (see Ref. [21]). In what follows we assume $\chi(q^2) = 1$ (no additional discrimination at all); then we arrive at the following universality relation

$$1 - \Phi_{\text{sl}}(M) = \int_0^M \frac{dE_{\gamma} \phi(E_{\gamma}, M)}{1 - \Phi_{\text{sl}}(M)} \frac{1}{\Gamma_{b\gamma}} \frac{d\Gamma_{b\gamma}}{dE_{\gamma}}.$$  

(28)

$$\phi(E_{\gamma}, M) = 1 - \frac{2r^3}{(1-y)^3} + \frac{r^4}{(1-y)^4}, \quad y = \frac{2E_{\gamma}}{M_B}, \quad r = \frac{M^2}{M_B^2}.$$  

(29)

It allows one – as promised – to express $\Phi_{\text{sl}}(M)$ directly via the photon spectrum.

The weight functions $\phi(E_{\gamma})$ depend on the chosen $M_X$-cutoff $M$ and vary from 1 at $E_{\gamma} \ll M_B/2$ down to zero at the maximal $E_{\gamma} = \frac{M_B^2 - M^2}{2M_B}$ where the single point $q^2 = 0$ contributes. They are shown in Fig. 6 for $M = 1.6$ GeV, $M = 1.7$ GeV and $M = 1.8$ GeV. As anticipated, at $E_{\gamma} < 2$ GeV the weight $\phi(E_{\gamma})$ is practically unity for all reasonable cut masses $M$. This means that the low part of the photon spectrum contributes only as a total integral, in other words, enters as the overall fraction of events discussed in the previous sections. The spectrum at $E_{\gamma} \gg 2$ GeV enters with decreasing weight:

$$1 - \Phi_{\text{sl}}(M) \simeq \Phi_{\gamma}(2 \text{ GeV}) + \int_{2 \text{ GeV}}^M \frac{dE_{\gamma} \phi(E_{\gamma}, M)}{1 - \Phi_{\text{sl}}(M)} \frac{1}{\Gamma_{b\gamma}} \frac{d\Gamma_{b\gamma}}{dE_{\gamma}}.$$  

(29)

The practical importance of clarifying the actual value of $\Phi_{\gamma}(2 \text{ GeV})$ becomes then manifest.

The consequences of the Fermi motion description of the large-recoil light-cone kinematics similar to explicit relations (24) have been discussed in the literature more than once (see, e.g. the recent paper [22]). Taking a different perspective, though, it is usually claimed that the outcome for $\Phi_{\text{sl}}$ suffers from significant uncertainties. We do not share this pessimistic viewpoint for actual $B$ decays. For example, the obvious constraint that $\Phi_{\text{sl}}(M)$ is exactly unity regardless of dynamics at sufficiently high $M$ approaching $M_B$ is usually neglected. As illustrated earlier in this section, in reality we need to evaluate only the $ab$ initio suppressed fraction $1 - \Phi_{\text{sl}}(M)$, and therefore even its crude estimate does not introduce a significant uncertainty in $|V_{ub}|$.

Universality relations and the scaling relations in general are clearly not exact. In the given form they are robust against perturbative effects; in particular the Sudakov double log corrections would not violate them. Yet they hold only to leading order in $1/m_b$. For example, terms of order $\Lambda^2$ were neglected in $M_X^2$ compared to the leading ones $\sim m_b \Lambda$. Since $1 - \Phi_{\text{sl}}(M)$ itself is a small fraction and thus introduces only a correction in extracting $|V_{ub}|$, its approximate knowledge already suffices for our purpose. Nevertheless further improvement can be achieved by evaluating $1/m_b$ corrections to the universality relations Eqs. (28), (29).
4.2 Leading $1/m_b$ corrections

Accounting for the $\Lambda^2$ terms referred to above is quite simple, and in general they can easily be incorporated in both semileptonic and radiative decays. However, this would not give the full answer. The universality relations make use of two basic facts, namely the process-independence of the leading-twist Fermi motion in the light-cone kinematics and the scaling of $M_X^2$ (or a certain function of it) with $1 - q^2/m_b^2$.

While it is easy to account for the $1/m_b$ corrections to the former, $1/m_b$ scaling violation is governed by independent distribution functions.

There are two circumstances which help to overcome these obstacles. In practice, only a limited range of $q^2$ yields the unwanted background due to Fermi motion; therefore the scaling violation in passing to $q^2 > 0$ is not dramatic and can be dealt with. Besides, we do not aim at detailed distributions, but rather need an integrated probability. In addition, in the small-$q^2$ domain where the Fermi motion is potentially most important, its universality is ensured by the full $b$ quark mass rather than a lower momentum scale. Below a way is suggested to implement these ideas in practice.

Instead of assuming the leading twist scaling in $M_X^2/(1 - q^2/m_b^2)$ for decay distributions, we can employ its $1/m_b$ improved ansatz:

$$\frac{1}{\Gamma_{sl}(q^2)} \frac{d\Gamma_{sl}(\mu^2; q^2)}{d\mu^2} = \frac{1}{2M_B} \frac{m_b^2 - q^2}{m_b^2} \frac{1}{\Gamma_{bs\gamma}} \frac{d\Gamma_{bs\gamma}}{dE_{\gamma}} \left( \frac{M_B}{2 - m_b^2/q^2} - \mu^2 \right)$$

(30)

with $\mu^2$ given by

$$\mu^2 = M_X^2 + A\left(\frac{q^2}{m_b^2}\right) + B\left(\frac{q^2}{m_b^2}\right)M_X^2 + C\left(\frac{q^2}{m_b^2}\right)\frac{M_X^4}{M_B^2}$$

(31)

where $A \sim \Lambda^2_{QCD}$, $B \sim \Lambda_{QCD}/m_b$, and $C \sim \mathcal{O}(1)$ are effective parameters accounting
for $1/m_b$ corrections. In other words, we assume $\mu^2$ instead of $M_X^2$ itself to scale proportional to $1 - \frac{q^2}{m_b}$.

In the simplest parton model one has, as an example,

$$A = \vec{X}^2, \quad B = \frac{\vec{X}^2}{M_B} \frac{1 + 2x}{1 - x}, \quad C = \frac{x}{(1-x)^2}; \quad x = \frac{q^2}{M_B^2},$$

which would incorporate kinematic effects. This is analogous to deep inelastic lepton-nucleon scattering, where the introduction of Nachtmann variable leads to an extension of the scaling regime. Simultaneously, this can also account for corrections to the heavy quark symmetry for different weak decay currents ($V - A$ in $b \to u \ell \nu$ vs. tensor in $b \to s + \gamma$). We do not rely on a model here, but rather can fix the parameters theoretically minimizing the strong interaction corrections computed at given $q^2$ in the standard $1/m_b$ expansion \[19, 1\]. This determines the ansatz parameters in a model-independent way. After that, the straightforward integration (24) would yield the $1/m_b$ and perturbatively improved version of the universality relation (28).

A clarifying note is in order here. The standard $1/m_b$ expansion yields non-perturbative effects only as an expansion around the free quark kinematics, e.g. $M_X^2 = (M_B - m_b)^2 + (M_B - m_b)m_b(1 - \frac{q^2}{m_b^2})$ for $b \to u \ell \nu$ consisting of the $\delta$-function and its higher derivatives. Their direct integration in the observables of interest makes little sense in the present context (likewise, it cannot be used to quantify the accuracy of various constrained inclusive probabilities in the case of limited energy release, as can be sometimes seen in the literature). Instead, here we can compute at given $q^2$ the hadronic moments ($\langle M_X^2 \rangle$, $\langle M_X^4 \rangle$, ...) in terms of heavy quark parameters $m_b$, $\mu_\pi^2$, $\mu_G^2$, $\rho_D^3$, ... directly and in the ansatz (30)-(31), and require them to match. In this way $1/m_b$ corrections are properly accounted for.

5 Conclusions and outlook

By the turn of the third Millennium $|V_{ub}|$ had been determined through a measurement of the inclusive rate $\Gamma(B \to X_u \ell \nu)$. Most recently – after this paper had largely been completed – a new result from CLEO has appeared \[24\] with an extraction of $|V_{ub}|$ from the end point lepton spectrum. Yet the theoretical reliability level of these methods varies considerably. The inclusive rate $\Gamma(B \to X_u \ell \nu)$ suffers from little model dependence. The main problem arises in distinguishing it against the dominant $b \to c$ background and how to model the latter \[23\].

In our considered judgement measuring the hadronic recoil mass spectrum in $B \to X_u \ell \nu$ will provide one with the most reliable extraction. Placing a high premium on model-independence we think that a measurement of the lepton energy endpoint spectrum per se cannot provide a competitive extraction of $|V_{ub}|$ as long as a cut around $E_\ell \simeq \frac{M_B^2 - M_X^2}{2M_B}$ remains, whether or not the $B \to X_s + \gamma$ spectrum is known.
Nevertheless studying the end point lepton spectrum is of high theoretical interest. The fact that the analysis of Ref. [24] yielded a value for \( |V_{ub}| \) very close to other determinations places bounds on the expectation values of four-quark operators. Naive estimates result in a typical limit around

\[
\left| \frac{\langle B| \bar{b}\gamma_i(1-\gamma_5)u \bar{u}\gamma_i(1-\gamma_5)b|B \rangle}{2M_B} \right| < 0.012 \text{ GeV}^3
\]  

(33)

(an average over charged and neutral \( B \) is implied) which lies in the range of previous estimates [16, 23, 17, 18]. A more careful analysis is required, though, to obtain reliable numbers. In particular, an improvement in treating the end-point relation between the semileptonic and radiative spectra may turn the bound into a nontrivial measurement of these expectation values. The interpretation would be much better supported if measurements were performed separately for charged and neutral \( B \) mesons.

High statistics studies of the hadronic recoil mass \( M_X \) in \( B \to X \ell \nu \) provide good means of obtaining an accurate value for \( |V_{ub}| \). The photon spectrum in \( B \to X_s + \gamma \) decay gives a more or less direct measurement of the heavy quark distribution function, far beyond indirect constraints on its gross features inferred from estimates of a few moments available a few years ago. The universality relations allow to convert the spectrum into quantitative estimates of the required fraction \( \Phi_{sl} \) of \( B \to X_u \ell \nu \) decays with high \( M_X \) rejected to suppress the \( b \to c \) background. For example, using the simplest version – Eqs. (28), (29) – we obtain \( \Phi_{sl}(1.7 \text{ GeV}) \simeq 0.65 \) assuming \( \Phi_{\gamma}(2 \text{ GeV}) = 12\% \). We have argued that already the present uncertainty in this \( b \to u \) fraction is not significant, and suggested the way to improve it for further studies by incorporating \( 1/m_b \) effects.

There are statements in the literature that extracting \( |V_{ub}| \) from the \( M_X^2 \) distribution is seriously plagued by uncertainties in the heavy quark distribution function \( F(x) \), while decay rates with a cut \( q^2 > (M_B - M_D)^2 \) are accurately calculable and stable against higher order effects. Such arguments miss, however, some basics of the heavy quark expansion and in particular how the Fermi motion itself emerges in the OPE. The claim that some fraction of \( \Gamma_{sl}(b \to u) \) constrained by a cut on \( q^2 \) does not depend on \( F(x) \) is actually justified only in a negative way: this width is governed by a whole series of unknown expectation values of higher-order heavy quark operators which cannot be resummed or otherwise related to \( F(x) \). The latter is known reasonably well since it is strongly constrained by sum rules and can directly be bounded by data. The corrections to widths with a cut in \( q^2 \) are obscure in this respect; there is actually indirect evidence that they are quite significant [18]. A similar reservation was expressed recently in Ref. [26].

To give a brief illustration of missing pieces, let us consider a constrained fraction of the \( B \to X_s + \gamma \) events

\[
1 - \Phi_\gamma(E) = \frac{1}{\Gamma_{bs\gamma}} \int_E^{M_B} \frac{dE_\gamma}{dE_\gamma} \frac{d\Gamma_{bs\gamma}}{dE_\gamma}.
\]  

(34)
Ignoring the perturbative corrections, the usual $1/m_b$ expansion always yields the spectrum in the form of expanding around the free-quark kinematics:

$$\int_{E_\gamma} \frac{d\Gamma_{bb\gamma}}{dE_\gamma} = a \delta(E_\gamma - \frac{m_b}{2}) + b \delta'(E_\gamma - \frac{m_b}{2}) + c \delta''(E_\gamma - \frac{m_b}{2}) + \ldots ,$$

(35)

where $a$, $b$, ... are given by the expectation values of local $b$-quark operators over $B$. Naively computing $1-\Phi_\gamma(E)$ in this way would yield unity for any $E > \frac{m_b}{2}$ – the result clearly unjustified until ‘hardness’ $M_B - 2E$ is high enough in hadronic scale. Moreover, if $\Phi_{sl}(M)$ were computed in the same naive $1/m_b$ expansion as was used to quantify the corrections to the $q^2$-constrained semileptonic widths, one would have $\Phi_{sl}(M) = 1$ for any $M \approx 1.8 \text{ GeV}$, with the decrease only due to perturbative effects.

Similar reservations apply to the computation of ‘incomplete’ moments evaluated over the restricted domain $E_\gamma > 2 \text{ GeV}$ which have been used by CLEO to extract $m_b$. The offered expressions depend on the value of the lower cutoff (2 GeV) only in the perturbative piece. Switching off the perturbative corrections, there would be no dependence on the cutoff, which cannot be true in general and thus introduces an additional uncertainty. It is likewise obvious that the true dependence is more complicated and involves both perturbative and nonperturbative effects in a more intricate way than a simple sum. This complication would be reduced if the immediate part of the spectrum below 2 GeV can be incorporated into the analysis.

In radiative $B$ decays one indeed expects only a small fraction of the spectrum to reside below 2 GeV, yet not an insignificant amount: we anticipate about 12%. It contributes a barely significant amount to the overall $\text{BR}(B \to X_s + \gamma)$. Nevertheless its accurate measurement is important for various reasons. We expect – and hope – that new data sets with even higher statistics will reveal a low energy spectrum at the predicted level. Detailed study of this part of the spectrum are instrumental for a comprehensive program of exploring local quark-hadron duality; here it is probed in the somewhat special case of hard heavy-to-light flavor transitions which has its own advantages as well as complications. Should it be confirmed, we expect the (properly defined) heavy quark parameters $\bar{\Lambda}(1 \text{ GeV}), \mu^2(1 \text{ GeV})$ – and possibly $\rho^3_D$, $\rho^3_{LS}$ – to be measured with the comparable precision and to add confidence in our present knowledge of these parameters. Moreover, combining the whole set of emerging new data with tight theoretical constraints and precisely extracted $m_b(1 \text{ GeV})$ will allow one to address in a quantitative fashion higher-order corrections – a possibility considered so far less than remote in conventional approaches.

One of the important quantities in this quest is just the integrated low-$E_\gamma$ fraction of radiative decays $\Phi_\gamma(2 \text{ GeV})$ – even leaving aside the aspect that it is directly related to the determination of $\Phi_{sl}(M)$ at experimentally relevant cuts $M \approx 1.7 \text{ GeV}$. Finding in the data a fraction of only, say, about 5% would tell us that some basic elements of the description had to be revisited. A possible interpretation would be that processes involving the emission (or excitation) of “real” gluonic degrees
of freedom start to exhibit perturbative local duality at higher scales than when only quark degrees of freedom are involved. This actually represents a conceivable scenario [20]. The higher onset of duality here would mean that a perturbative treatment of gluon radiation is not reliable till one reaches a higher scale. From the specific case under study here with the separation scale around \( E_\gamma \simeq 2 \text{ GeV} \) one would be sensitive to the duality onset scale near 1.5 GeV; the lower part of the spectrum would allow to probe duality for even harder gluons. The total integrated decay rate still can be fairly insensitive to such effects. The photon decay spectrum from this perspective is a complementary tool to study details of local parton-hadron duality compared to hadronic mass distribution in semileptonic \( b \to c \) decays [20]. It is more ambiguous as a probe of lower scales due to more significant interference with the nonperturbative Fermi motion, yet may improve at lower \( E_\gamma \) if prompt photons and their production mechanism can be reliably identified.

Although physics of gluon bremsstrahlung is quite different in semileptonic and radiative \( B \) decays, finding a significant violation of gluon-hadron duality in the latter at \( E_\gamma \lesssim 2 \text{ GeV} \) would raise concerns in treating, say the hadronic moments in the former. This especially applies to higher moments in the environment of present CLEO measurements: the cut on the lepton energy \( E_\ell > 1.5 \text{ GeV} \) effectively decreases the expansion mass parameter down from the usual energy release \( m_b - m_c \).

Acknowledgements: N.U. thanks U. Aglietti, Yu. Dokshitzer and M. Misiak for useful clarifications. Discussions with M. Artuso and P. Roudeau are gratefully acknowledged. This work has been supported in part by the National Science Foundation under grant number PHY00-87419.

References

[1] I.I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, \textit{Phys. Rev. Lett.} \textbf{71} (1993) 496.

[2] I.I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, \textit{Phys. Rev.} \textbf{D 52} (1995) 196.

[3] N. Uraltsev, in \textit{Boris Ioffe Festschrift “At the Frontier of Particle Physics – Handbook of QCD”}, Ed. M. Shifman (World Scientific, Singapore, 2001), Vol. 3, p. 1577; [hep-ph/0010328].

[4] D. Cronin-Hennessy \textit{et al.} (CLEO Collaboration) \textit{Phys. Rev. Lett.} \textbf{87} (2001) 251808.

[5] S. Chen \textit{et al.}, (CLEO Collab.), \textit{Phys. Rev. Lett.} \textbf{87} (2001) 1807.

[6] R.D. Dikeman, M. Shifman and N.G. Uraltsev, \textit{Int. Journ. Mod. Phys.} \textbf{A11} (1996) 571.
[7] A. Ali, E. Pietarinen, *Nucl. Phys. B154* (1979) 519.

[8] G. Altarelli *et al.*, *Nucl. Phys. B208* (1982) 365.

[9] S. Catani and L. Trentadue, *Nucl. Phys. B327* (1989) 323.

[10] A. Ali and C. Greub, *Phys. Lett. B361* (1995) 146.

[11] Z. Ligeti, M. Luke, A. Manohar and M. Wise, *Phys. Rev. D60* (1999) 034019.

[12] N.G. Uraltsev, *Nucl. Phys. B491* (1997) 303.

[13] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev. D56* (1997) 4017.

[14] A. Kagan and M. Neubert, *Eur. Phys. J. C7* (1999) 5.

[15] I.I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Int. Journ. Mod. Phys. A9* (1994) 2467.

[16] I. Bigi and N. Uraltsev, *Nucl. Phys. B423* (1994) 33.

[17] D. Pirjol and N. Uraltsev, *Phys. Rev. D59* (1999) 034012.

[18] N. Uraltsev, *Int. Journ. Mod. Phys. A14* (1999) 4641.

[19] I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, *The Fermilab Meeting*, Proc. of the 1992 DPF meeting of APS, eds. C.H. Albright *et al.* (World Scientific, Singapore 1993), vol. 1, p. 610; [hep-ph/9212227](http://arxiv.org/abs/hep-ph/9212227).

[20] I. Bigi and N. Uraltsev, *Int. J. Mod. Phys. A16* (2001) 5201.

[21] R.D. Dikeman and N.G. Uraltsev, *Nucl. Phys. B509* (1998) 378
I. Bigi, R.D. Dikeman and N. Uraltsev, *Eur. Phys. J. C4* (1998) 453.

[22] C.W. Bauer, Z. Ligeti and M. Luke, *Phys. Rev. D 64* (2001) 113004.

[23] P. Abreu *et al.* (DELPHI Collab.), *Phys. Lett. B478* (2000) 14.

[24] A. Bornheim *et al.* (CLEO Collaboration), *Preprint CLNS 01/1767*, CLEO 01-23; [hep-ex/0202019](http://arxiv.org/abs/hep-ex/0202019).

[25] I. Bigi and N. Uraltsev, *Z. Phys. C62* (1994) 623.

[26] M.B. Voloshin, *Phys. Lett. B515* (2001) 74.