Representation of intermediate time-scale motions in stochastic modeling:
Analysis on stochastic description of classical Hamiltonian dynamics in relation with measurement imperfection

Jun Chul Park *

Department of Physics, KAIST, Daejeon 305-701, Korea

Abstract

It is a well established result that, in classical dynamical systems with sufficient time-scale separation, the fast chaotic degrees of freedom are well modeled by (Gaussian) white noise. In this paper, we present the stochastic dynamical description for intermediate time-scale motions with insufficient time-scale separation from the slow dynamical system. First, we analyze how the fast deterministic dynamics can be viewed as stochastic dynamics under experimental observation by intrinsic errors of measurement. Then, we present how the stochastic dynamical description should be modified if intermediate time-scale motions exist: the time correlation of the noise $\xi$ is modified to $\langle \xi(t)\xi(t') \rangle = C(x,p)\delta(t-t')$, where $C(x,p)$ is a smooth function of the slow coordinate $(x,p)$, and generally the cumulants of $\xi$ except its average vary as a smooth function of the slow coordinates $(x,p)$. The analysis given in this work actually shows that, regardless of the sufficiency of time-scale separation, any complex (chaotic and ergodic) dynamical system can be well described using Markov process, if we perfectly construct the deterministic part of (extended) stochastic dynamics.

1 Introduction

In classical many-particle systems with time-scale separation such as Brownian motion, the rapid dynamic fluctuations by numbers of fast and chaotic degrees of freedom are well modeled by Gaussian white noise, and the Langevin equation gives successful descriptions for

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*E-mail address: junchul@kaist.ac.kr
such thermodynamic systems [1, 2]. Moreover, recent theoretical works [3, 4, 5, 6] indicate that, even for few fast chaotic degrees of freedom, the dynamic fluctuations are well approximated by suitable stochastic processes with white noise and the stochastic dynamics still gives reasonable descriptions. Such results certify that the stochastic process is generally a well founded representation for fast chaotic degrees of freedom, if the time-scale separation is sufficient appropriately. Actually the stochastic modeling of fast chaotic dynamics based on time-scale separation has been applied successfully in diverse areas of science such as hydrodynamics [7, 8], climate models [9], and chemical and biological systems [8, 10]. In many cases of real physical systems, however, the corresponding dynamical systems contain diverse intermediate time-scale motions and cannot be well decomposed or approximated simply by two sub-systems with the sufficient time-scale separation. In this paper, we address the problem of how the intermediate time-scale motions should be represented in the point of view of stochastic dynamics. We clarify the role of the intermediate time-scale dynamics in the stochastic description of classical Hamiltonian systems and its qualitative difference from the fast and slow dynamics which are represented as Gaussian white noise and the deterministic part in the Langevin equation, respectively.

Consider the Hamiltonian

$$H = H_{\text{slow}}(x, p) + H_{\text{fast}}(q_i, p_i) + V(x, q_i),$$

in which a slow system $\vec{x}_s \equiv (x, p)$ is interconnected with a fast system $\vec{x}_f \equiv (q_i, p_i) \equiv (q_1, p_1, \ldots, q_N, p_N)$ by a potential $V(x, q_i)$. Assuming that the time-scale separation is sufficient, for the motion of $p$ by $H_{\text{slow}}$, the potential $V(x, q_i)$ introduces additional dynamics, which is composed of slow dynamics, such as damping in Brownian motion [11], and rapid fluctuating dynamics. We represent the rapid fluctuating term in the equation of motion for $p$ as $k(x, p, q_i, p_i)$ and write $\frac{dp}{dt} = -\partial_x H_{\text{slow}} - \partial_y V = h(x, p) + k(x, p, q_i, p_i)$, in which the slow dynamics by $V$ is contained in $h(x, p)$ 1. Then, we consider the following deterministic system:

$$\frac{dx}{dt} = p, \quad \frac{dp}{dt} = h(x, p) + k(\vec{x}),$$

$$\frac{dq_i}{dt} = u_i(\vec{x}), \quad \frac{dp_i}{dt} = v_i(\vec{x}),$$

where $\vec{x} \equiv (x, p, q_1, p_1, \ldots, q_N, p_N) \equiv (\vec{x}_s, \vec{x}_f)$ and $i = 1, \ldots, N$. In the case that we cannot

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1As is well known from the projection operator method [11], in equilibrium thermodynamical system, $h(x, p)$ can be extracted by the projection operation, if one can perfectly construct the relevant subspace in Hilbert space.
know or it is unnecessary to know the exact information for the fast variable part in the equations of motion (2), we may approximate (2) to stochastic differential equations, for example,

\[
\frac{dx}{dt} = p, \quad \frac{dp}{dt} = h(x, p) + \xi, \tag{3}
\]

with a suitable time correlation function \(\langle \xi(t)\xi(t') \rangle\) and a Gaussian probability density function (PDF) for a random variable \(\xi\). In this case, the stochasticity in (3) is just a result of mathematical approximation or simplification of the original complex system. Such approximation can be validated only through the observed empirical data for the system (2).

Here, it should be noted that experimental data always contains unpredictable intrinsic errors due to the imperfection of measurement process, which destroy the information of fine dynamical structures. The ‘unpredictable’ nature of error makes such fine dynamics change into stochastic dynamics in the resultant experimental data. If we assume all obtainable deterministic information for (1) in our measurement systems is described by the term \(h(x, p)\), actually the mathematically introduced stochasticity by \(\xi\) should correspond to the inevitable stochasticity in the experimental data originated by the intrinsic measurement errors: the stochasticity contained in the experimental data gives the physical identification for the mathematically introduced. In what follows, we investigate how the stochasticity originated by time measurement errors is related with the fine dynamical structures.

## 2 Stochasticity induced by time measurement error

Let us assume that there are inherent errors in our measurement system for coordinates and time. We obtain an erroneous value \((\vec{x}, t)\) in measurement at an arbitrary instant, while the ideal coordinates and ideal time, which is measured by a perfect measurement system without error at the same instant, is \((\vec{x}^*, t^*) = (\vec{x}, t) + (\Delta \vec{x}, \Delta t)\); that is, \((\Delta \vec{x}, \Delta t)\) is the error in our measurement. Under our measurement system, the dynamics of (2) is observed as the time series \(\vec{x}(t)\), which is deformed from the original dynamics of (2), as follows:

\[
\vec{x}(t) = \vec{x}^*(t^*) + \Delta \vec{x}, \tag{4}
\]

where \(\vec{x}^*(t^*)\) satisfies the equation of motion (2) exactly. In this work, we analyze how the dynamics of \(\vec{x}(t)\) changes as the accuracy of our measurement system changes; if we assume that, for an arbitrary measurement system, the statistical properties of the error \((\Delta \vec{x}, \Delta t)\) is always independent of time and the error occurrences at different times are probabilistically
independent of each other, the time-dependence of $\vec{x}(t)$ only comes from $\vec{x}^*(t^*)$, and the coordinate measurement error $\Delta \vec{x}$ does not have any role in our analysis—only the time measurement error $\Delta t$ is important in analyzing the variation of the dynamical behavior of $\vec{x}(t)$ in relation with the change of the accuracy of measurement system.\footnote{In this respect, time measurement error plays a special role in dynamics.} As can be easily checked, all results given in this work essentially hold, regardless of the presence of the effect by coordinates measurement error $\Delta \vec{x}$. Thus, to avoid notational complexes and to simplify the arguments, we only consider the effect by time measurement error $\Delta t$, which does not make any loss of the generality of the analysis. We assume that, except time, we are perfectly informed about all values of the observables $x, p, q_i$, and $p_i$ in (2) at each instant from a perfect observer. The situation under our consideration is that there is a missed time-parametrization for the perfectly observed $\vec{x}$ at each instant, i.e.,

$$\vec{x}(t) = \vec{x}^*(t^*),$$

but generally

$$t \neq t^*. \quad (6)$$

In the following, we use the notation for $\vec{x}^*(t^*)$ as

$$\vec{x}^*(t^*) \equiv \vec{x}(t^*) \quad (7)$$

in order to emphasize that $\vec{x} = \vec{x}^*$ at an arbitrary instant; we obtain the time series $\vec{x}(t)$, while the perfect observer obtains $\vec{x}(t^*)$, which satisfies the equations of motion (2) exactly.

We can characterize the time measurement errors by giving the PDF $f_i(\Delta t)$, for which the probability to obtain an error in the range $(\Delta t, \Delta t + d(\Delta t))$ is given as $f_i(\Delta t)d(\Delta t)$. Then, we can be confident about the measured data $t$ within an error range as

$$t^* - \frac{\varepsilon}{2} < t < t^* + \frac{\varepsilon}{2}, \quad (8)$$

where we can define the suitable $\frac{\varepsilon}{2}$ based on the PDF $f_i(\Delta t)$. Thus, for two measured times $t_1$ and $t_2$ such that $0 < t_2 - t_1 < \varepsilon$, we cannot be certain $t_2^* > t_1^*$ because their error-ranges are overlapped, i.e., to ensure that the measured times give the correct time order of $t_1^*$ and $t_2^*$, the measured times should be separated as $|t_2 - t_1| \geq \varepsilon$. In other words, approximately there is a minimum length of time elapse that can be identified by a given time measurement
system (one cannot chase correctly the causality of the events occurring within a time interval shorter than the minimum), and this minimum length of time elapse is determined by the inherent errors in the measurement system. In translating the deterministic system (2) into a stochastic system like (3) to explain experimental data, an important physical meaning is added to the differential $dt$. The physical meaning of $dt$ in (3) is the minimum length of time elapse identifiable in our measurement system. We can determine the minimum length of time elapse $dt$ to be $\varepsilon$ providing that the probability to obtain a time value outside of the range $t^* - \frac{\varepsilon}{2} < t < t^* + \frac{\varepsilon}{2}$ is very small:

$$dt = \varepsilon. \quad (9)$$

Let us assume that the dynamical system (2) starts from an initial value of $\vec{x}$. Then, if we measure an observable $A(\vec{x})$ at $t$ for system (2), the measured value of $A$ at $t$, i.e., $A(t)$ is exactly given by the equations of motion (2) as $A(t^*) \equiv A(\vec{x}(t^*))$, where $t^*$ is a value in $t - \frac{\varepsilon}{2} < t^* < t + \frac{\varepsilon}{2}$: $t^*$ randomly has a value in the range with the PDF $f_t(\Delta t)$. Thus, $A(t)$ is given by one of the elements in the following set:

$$\mathcal{S}_A(t, \varepsilon) = \left\{ A(t^*) \mid t - \frac{\varepsilon}{2} < t^* < t + \frac{\varepsilon}{2} \right\}, \quad (10)$$

which is the set of all values obtainable in the measurement of $A$ at $t$. Exactly which value in $\mathcal{S}_A(t, \varepsilon)$ is given for $A$ in our measurement at $t$ is totally a probabilistic problem determined by the following two components: (i) PDF $f_t(\Delta t)$ and (ii) the set $\mathcal{S}_A(t, \varepsilon)$, where we assume that $f_t(\Delta t)$ is invariant for time translation and the error occurrences at different times are probabilistically independent of each other. In what follows, we assume $f_t(\Delta t)$ is well approximated as $f_t(\Delta t) \approx \frac{1}{\varepsilon}$ for $-\frac{\varepsilon}{2} < \Delta t < \frac{\varepsilon}{2}$, otherwise $f_t(\Delta t) \approx 0$.

3 Conventional Langevin equation system as a special case

Let $A$ in (10) be the fast coordinates $\vec{x}_f$. Then, the set $\mathcal{S}_{\vec{x}_f}(t, \varepsilon) \equiv \mathcal{S}(t, \varepsilon)$ is the trajectory of $\vec{x}_f(t^*)$ during $t - \frac{\varepsilon}{2} < t^* < t + \frac{\varepsilon}{2}$ in the fast phase space $\Gamma$ defined by $\vec{x}_f$. As the value of $\varepsilon$ increases, i.e., as our measurement system is less fine, the set $\mathcal{S}(t, \varepsilon)$ occupies larger part of $\Omega$, where $\Omega$ is the set of all solutions $\vec{x}_f$ of (2) in $\Gamma$. Let us consider the extremal case where $\varepsilon$ is sufficiently large for $\vec{x}_f$ to wander over almost all areas of $\Omega$ during $t - \frac{\varepsilon}{2} < t^* < t + \frac{\varepsilon}{2}$,
providing that $\vec{x}_f$ is ergodic and chaotic \cite{12}. Then, the set $\mathcal{S}(t,\varepsilon)$ is almost the same as $\Omega$ and independent of time, which also guarantees the time independence of the set $\mathcal{S}_F(t,\varepsilon)$ for any fast observable $F(\vec{x}_f)$. Thus, if $\vec{x}_f$ is fast enough to satisfy the extremal case for a given $\varepsilon$, the statistical properties of the set $\mathcal{S}(t,\varepsilon)$ are almost stationary for time: that is, the statistical properties of the measured values of $F(\vec{x}_f)$ at $t$ are stationary, and $F(\vec{x}_f)$ behaves as a random variable with stationary statistical properties. On the other hand, it should be noted that the set $\mathcal{S}(t,\varepsilon)$ is to be slowly dependent on time, because $\mathcal{S}(t,\varepsilon)$ is determined by the fast equations of motion from the Hamiltonian

$$\mathcal{H}' \equiv \mathcal{H}_{fast}(q_i, p_i) + V(x, q_i)$$

and the value of $\mathcal{H}'$ has a slow time-dependence through the slow motion of $x$ originated from $h(x, p)$ in \cite{2} \cite{13}. Thus, the extremal case is actually the case where the potential $V(x, q_i)$ is regarded as an extremely slow varying function for the deterministic slow motion of $\vec{x}_s$, i.e., $V(x, q_i)$ is effectively invariant for the slow motion of $\vec{x}_s$ and the trajectory of $\vec{x}_f$ is almost confined near the hypersurface $\mathcal{H}' = const$ (oscillating around the hypersurface) \cite{3}.

Also, in the extremal case, by the ergodic property, the time correlation function of any fast observable $F(t^*) \equiv F(\vec{x}_f(t^*))$ approximately satisfies

$$\langle F(t_1^*) F(t_2^*) \rangle \approx 0 \text{ for } |t_1^* - t_2^*| \geq \varepsilon,$$

where we assume $\langle F(t^*) \rangle = 0$. In our measurement system, (concerning the time measurement errors) the experimentally ascertained quantity as the correlation function of $F(t)$ is

$$\langle F(t_1) F(t_2) \rangle_{exp} \equiv \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \langle F(t_1 + \Delta t_1) F(t_2 + \Delta t_2) \rangle$$

$$\times f_1(\Delta t_1) f_1(\Delta t_2) d(\Delta t_1) d(\Delta t_2),$$

where, in the integrand, we use $F(t_{1,2}^*) = F(t_{1,2} + \Delta t_{1,2})$. If $|t_1 - t_2| \geq 2\varepsilon$, we obtain $\langle F(t_1 + \Delta t_1) F(t_2 + \Delta t_2) \rangle \approx 0$ from \cite{12} for any $\Delta t_1$ and $\Delta t_2$ in the integration ranges, and

$$\langle F(t_1) F(t_2) \rangle_{exp} \approx 0 \text{ for } |t_1 - t_2| \geq 2\varepsilon.$$
Thus, the correlation actually behaves as a $\delta$-function in our measurement system, i.e.,
$$\langle \mathcal{F}(t_1)\mathcal{F}(t_2) \rangle_{\text{exp}} \sim \delta(t_1 - t_2).$$

Let us analyze the behavior of the rapid fluctuating time series $k(\vec{x})$ in (2) under our measurement system. Expanding $k(\vec{x})$ for the slow coordinates $\vec{x}_s$, we obtain $k(\vec{x}) = \mathcal{F}_0(\vec{x}_f) + \sum_{l,m=1}^{\infty} \mathcal{F}_{lm}(\vec{x}_f) x^l p^m$. As previously argued, the first term $\mathcal{F}_0(\vec{x}_f)$ behaves as a white noise with stationary statistical properties in the extremal case. The remaining part $\Delta k(\vec{x}) \equiv k(\vec{x}) - \mathcal{F}_0(\vec{x}_f)$ also should behave as a white noise in the extremal case, as shown in the following argument. Let us denote by $\langle \cdots \rangle_{\mathcal{G}(t,dt)}$ the average for all elements of $\mathcal{G}(t,dt)$. Assuming that all linear increment of $p(t)$ during $dt$ is given by the deterministic part as $h(x,p)dt$, we request $\langle k(\vec{x}) \rangle_{\mathcal{G}(x,t,dt)} = 0$, which means that $\langle \Delta k(\vec{x}) \rangle_{\mathcal{G}(x,t,dt)} = 0$, because $\langle \mathcal{F}_0(\vec{x}_f) \rangle_{\mathcal{G}(x,t,dt)} = \text{const} = 0$. Thus, we can conclude that the time series $\Delta k(\vec{x})$ is composed of the fast fluctuating motion of $\vec{x}_s$ and the motion of $\vec{x}_f$. If we assume that the fast fluctuating motion of $\vec{x}_s$ has the same time-scale as $\vec{x}_f$, then the motion satisfies a $\delta$-function correlation as (14), and $\Delta k(\vec{x})$ also behaves as a white noise. Thus, we can replace $k(\vec{x})$ by a single white noise $\xi$, and the equation of motion of $p$ is observed as

$$\frac{dp}{dt} = h(x,p) + \xi,$$

$$\langle \xi(t)\xi(t') \rangle = 2C\delta(t-t'),$$

in the measurement system with $dt(=\varepsilon)$, where $C$ is a constant. Also, it is reasonable to assume that the PDF for $\xi$, which is determined basically by $f_t(\Delta t)$ and $\mathcal{G}(t,\varepsilon) \approx \Omega$, is Gaussian $[14]$: $\xi$ can be regarded as the fluctuation from an ideal thermal reservoir in equilibrium, because the behavior of $\xi$ is independent from $\vec{x}_s$ and the energy flux between the fast and slow system during $dt$ is averagely 0.

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4The time $t = 0$ and $t = \varepsilon$ respectively corresponds to $t = 0$ and $t \to 0$ in the case where $t$ is continuously variable. Since $\lim_{t\to0}\delta(t)$ needs not to be 0, (14) is sufficient for $\delta$-function correlation.

5The term $h(x,p)$ gives the averaged dynamics for all possible measurement errors; the errors cannot deform this deterministic dynamical structure. Basically, we assume that the times series $h(x,p)$ is sufficiently slow so that $h(t^* + \varepsilon) - h(t^*) \approx h(t^*)\varepsilon$, and the error-induced variation in the measured value of $h(x,p)$ is comparatively negligible to that of $k(\vec{x})$. Thus, we analyze the fluctuating term $k(\vec{x})$ only.
4 Stochastic dynamical description for insufficiently time-scale separated dynamical system

Now, we generalize the above argument. *Let us denote by* \( \tau \) *the minimum of the time length* \( \delta \) *for which we can approximate* \( \mathcal{S}(t, \delta) \approx \Omega \) *for the fast dynamical system. Then, if the relative accuracy of our measurement system, denoted by* \( M_{\text{old}} \), *to the fast dynamics is not fine so that* \( dt(= \varepsilon) \geq \tau \), *it corresponds to the extremal case.*

*Let us observe the dynamical system (2) with a finer new measurement system denoted by* \( M_{\text{new}} \) *such that* \( dt < \tau \). *Then, there is some non-ignorable area in* \( \Omega \) *which can not be reached by* \( \vec{x}_f \) *during* \( dt \), *and we cannot simply approximate* \( \mathcal{S}(t, dt) \approx \Omega \), *i.e., we have* \( \mathcal{S}(t, dt) \nsubseteq \Omega \). *The set* \( \mathcal{S}(t, dt) \) *becomes dependent on* \( t \), *and the statistical properties of the set* \( \mathcal{S}(t, \varepsilon) \) *are generally non-stationary:* \( \mathcal{F}(\vec{x}_f) \) *generally behaves as a random variable with non-stationary statistical properties. *Considering* \( \mathcal{S}(t, dt) \) *is the trajectory of* \( \vec{x}_f(t^*) \) *during* \( t - \frac{dt}{2} < t^* < t + \frac{dt}{2} \), *its time-dependence indicates that* \( \mathcal{S}(t, dt) \) *is dependent on the value of* \( \vec{x}(t^*) = (\vec{x}_s, \vec{x}_f)(t^*) \) *at* \( t^* = t \). *In here, each* \( \vec{x}_s \)-dependence *and* \( \vec{x}_f \)-dependence *has some different dynamical meaning. In the following analysis, we will show that the* \( \vec{x}_f \)-dependence *is actually related with the deterministic slow motions which are newly emerged in* \( M_{\text{new}} \) *and the* \( \vec{x}_s \)-dependence *is related with intermediate time-scale motions.*

*If we expand* \( k(\vec{x}) \) *in (2) for the fast coordinates* \( \vec{x}_f \), *we obtain* \( k(\vec{x}) = \sum_{|\alpha| \geq 0} S_{\alpha}(\vec{x}_s)\vec{x}_f^\alpha \), *where* \( \alpha = (\alpha_1, \ldots, \alpha_{2N}) \) *is a multi-index and we can set* \( S_0(\vec{x}_s) = 0 \). *Then, the linear term* \( \tilde{k}(\vec{x}_f) \equiv \sum_{|\alpha| = 1} S_{\alpha}(0)\vec{x}_f^\alpha \) *generates the most slow dynamics in* \( k(\vec{x}) \) *—while all* \( \vec{x}_f^\alpha \) *for* \( |\alpha| \geq 1 \) *are detected as white noises in* \( M_{\text{old}} \), *\( q_i \) *and* \( p_i \) *generate the most slow time series among* \( \vec{x}_f^\alpha \). *As our measurement system changes from* \( M_{\text{old}} \) *to* \( M_{\text{new}} \), *the variation of the statistical properties (the moments or cumulants) of the measured values of* \( k(\vec{x}) \) *at* \( t \) *mainly comes from the most slow time series* \( \tilde{k}(\vec{x}_f) \). *In the following, we analyze how the most slow time series* \( \tilde{k}(\vec{x}_f) \) *is viewed in* \( M_{\text{new}} \) *(*\( dt < \tau \)).*

4.1 \( \vec{x}_f \)-dependence of the set \( \mathcal{S}(t, dt) \)

*Let us assume that* \( \mathcal{S}(t, dt) \) *is only dependent on* \( \vec{x}_f \) *and almost independent of* \( \vec{x}_s \). *As previously argued in the extremal case, the* \( \vec{x}_s \)-independence *of* \( \mathcal{S}(t, dt) \) *actually means that the motion of* \( \vec{x}_f \) *is nearly confined on a hypersurface* \( \mathcal{H}' = \text{const} \); *\( \Omega \) *is approximately given as the* \( \mathcal{H}' = \text{const} \) *surface. Thus, in this case, the* \( \vec{x}_f \)-dependence *means that* \( dt \) *is not long enough for* \( \vec{x}_f \) *to wander over all areas of the* \( \mathcal{H}' = \text{const} \) *surface. Consider the*
averaged dynamics $\langle \vec{x}_f \rangle_{\Theta(t,dt)}$. Because $\vec{x}_f$ cannot wander over all areas of the $\mathcal{H}' = \text{const}$ surface during $dt < \tau$, generally $\langle \vec{x}_f \rangle_{\Theta(t,dt)} \neq \text{const}$ for time in $\mathbb{M}_{\text{new}}$; on the contrary, in $\mathbb{M}_{\text{old}}$ with $dt \geq \tau$, $\langle \vec{x}_f \rangle_{\Theta(t,dt)} = \text{const} = 0$ (with a suitable setting of the origin of $\Gamma$), because $\mathcal{S}(t,dt) \approx \mathcal{S}(t,\tau) \approx \Omega$ from the definition of $\tau$. In $\mathbb{M}_{\text{old}}$ with $dt \geq \tau$, $\langle \vec{x}_f \rangle_{\Theta(t,dt)}$ is stationary as it is fixed at the origin of $\Gamma$ but, as our measurement system is finer so that $dt < \tau$, $\langle \vec{x}_f \rangle_{\Theta(t,dt)}$ is not fixed any more and starts to move slowly with time in $\Gamma$. That is, the ensemble average of the measured values of $\vec{x}_f$ at each instant varies slowly over time in $\mathbb{M}_{\text{new}}$, which is an undetectable deterministic motion in $\mathbb{M}_{\text{old}}$ and viewed as a stochastic process in $\mathbb{M}_{\text{old}}$. Considering $\langle \vec{k}(\vec{x}_f) \rangle_{\Theta(t,dt)} = \sum_{|\alpha|=1} S_{\alpha}(0) \langle \vec{x}'_f \rangle_{\Theta(t,dt)}$ also slowly varies by the motion of $\langle \vec{x}_f \rangle_{\Theta(t,dt)}$, eventually, this newly emerged slow dynamics induces the observer using $\mathbb{M}_{\text{new}}$ to change the deterministic term $h(x,p)$ in (3) to $h(x,p) + \delta h(x,p)$ by adding the more detailed information of the dynamics.

On the other hand, the motion of the fast fluctuating part

$$\vec{y}_f \equiv \vec{x}_f - \langle \vec{x}_f \rangle_{\Theta(t,dt)}$$

is confined on the energy surface $\mathcal{H}' = \text{const}$ in the $\vec{y}_f$ phase space (note that $\langle \vec{x}_f \rangle_{\Theta(t,dt)}$ is a function of $\vec{x}_f$, and the set $\mathcal{S}_{\vec{y}_f}(t,dt)$ contains all points of the $\mathcal{H}' = \text{const}$ surface, under the assumption that the motion of $\vec{y}_f$ is still ergodic and chaotic: if $\vec{y}_f$ cannot wander over all areas of the energy surface during $dt$, generally $\langle \vec{y}_f \rangle_{\Theta(t,dt)} \neq \text{const}$, which contradicts with the definition of $\vec{y}_f$. Thus, the motion of $\vec{y}_f$ becomes the extremal case again in $\mathbb{M}_{\text{new}}$ with $dt < \tau$.

There is just a replacement of the deterministic part $h(x,p)$ by $h(x,p) + \delta h(x,p)$ and the fluctuating noise part is identical to the case of the Langevin equation or the extremal case, i.e., white noise with stationary statistical properties; the $\vec{x}_f$-dependence of the set $\mathcal{S}(t,dt)$ changes the time series $\vec{k}(\vec{x}_f)$ from white noises in $\mathbb{M}_{\text{old}}$ to colored noises in $\mathbb{M}_{\text{new}}$ as

6$\langle \vec{x}_f \rangle_{\Theta(t,dt)}$ is a function $\vec{x}_f$ from the $\vec{x}_f$-dependence of $\mathcal{S}(t,dt)$.

7Strictly speaking, $\delta h(x,p)$ is an over-simplified expression for the newly emerged slow dynamics. If we define a new slow variable $x' \equiv \langle \vec{k}(\vec{x}_f) \rangle_{\Theta(t,dt)}$, then $\frac{dx'}{dt} = h(x,p) + x' + \sum_{|\alpha|=1} S_{\alpha}(0) \vec{y}'_f$. Generally, $x'$ may be independent of $x$ and $p$ in the sense that, for some $t_1 \neq t_2$, $(x,p)(t_1) = (x,p)(t_2)$ but $x'(t_1) \neq x'(t_2)$. We have to contain the new additional equation of motion in the deterministic part as (in the most general expression) $\frac{dx'}{dt} = h_{\text{new}}(x,p,x',t)$. But, $x'(t)$ is very slow varying time series, and there can be one-to-one correspondence between $(x,p,x')$ and $t$ for very long time interval. Thus, we can write the deterministic part of the equations of motion as $\frac{dx}{dt} = p$, $\frac{dx'}{dt} = h_{\text{new}}(x,p,x')$, and $\frac{dp}{dt} = h(x,p) + x'$ for very long time. Under $\mathbb{M}_{\text{new}}$, generally the number of the slow variables may be increased, e.g., $\vec{x}_s = (x,p,x')$; thus, in our notation, we can regard $(x,p)$ under $\mathbb{M}_{\text{new}}$ as a more expanded slow variable set than $(x,p)$ under $\mathbb{M}_{\text{old}}$. If we intend to write the equations of motion only using $x$ and $p$ except $x'$, generally the equations of motion have to contain the memory term for $(x,p)$ instead of $x'$. 

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the combination of the deterministic motion $\delta h(x, p)$ and white noises.

### 4.2 $\vec{x}_s$-dependence of the set $\mathcal{S}(t, dt)$

Next, we analyze the $\vec{x}_s$-dependence of the set $\mathcal{S}(t, dt)$. Let us assume that $\mathcal{S}(t, dt)$ is only dependent on $\vec{x}_s$ and almost independent of $\vec{x}_f$; the $V(x, q_i)$ is not effectively invariant for the slow motion of $\vec{x}_s$ any more. Actually, the $\vec{x}_f$-independence means $\langle \vec{x}_f \rangle_{\mathcal{S}(t, dt)} = \text{const} = 0$ for time, as shown in the following argument. If we fix the value of $\vec{x}_s$ arbitrarily, the set $\mathcal{S}(t, dt)$ is invariant for any value of $\vec{x}_f$ on the hypersurface $\mathcal{H}' = \text{const}$ corresponding to the fixed value of $\vec{x}_s$ (note that $\mathcal{H}$ is conserved), which means that the set $\mathcal{S}(t, dt)$ contains all points $\vec{x}_f$ on the hypersurface. Also, since $\mathcal{H}'$ varies slowly with the slow motion of $\vec{x}_s$, the elements of $\mathcal{S}(t, dt)$ are confined near the hypersurface. Thus, actually the average for the set $\mathcal{S}(t, dt)$ can be approximated as the average for all points on the $\mathcal{H}' = \text{const}$ surface and $\langle \vec{x}_f \rangle_{\mathcal{S}(t, dt)} = 0$, which holds for arbitrary values of $\vec{x}_s$.

The $\vec{x}_s$-dependence induces the change of the set $\mathcal{S}(t, dt)$ while keeping $\langle \vec{x}_f \rangle_{\mathcal{S}(t, dt)} = 0$ invariant for time, which means the following two results:

(i) the variation of $\vec{x}_s$ induces the variation of the other statistical properties of $\mathcal{S}(t, dt)$, such as the averages of $q_i^2$, $q_i p_j$, $q_i q_j p_k$, . . . , which are stationary in $\mathcal{M}_{old}$, i.e., generally the statistical properties $\langle \vec{x}_f^\alpha \rangle_{\mathcal{S}(t, dt)}$ for $|\alpha| \geq 2$ are non-stationary and vary as a smooth function of $\vec{x}_s$ in $\mathcal{M}_{new}$

(ii) the $\vec{x}_s$-dependence does not make any change in the deterministic part $h(x, p)$ in (3).

Except $\langle \vec{k}(\vec{x}_f) \rangle_{\mathcal{S}(t, dt)}$, generally $\langle \vec{k}^\alpha(\vec{x}_f) \rangle_{\mathcal{S}(t, dt)}$ for $|\alpha| \geq 2$ vary as a function of $\vec{x}_s$ in $\mathcal{M}_{new}$, i.e., the statistical properties of the measured values of $\vec{k}(\vec{x}_f)$ at $t$, except the ensemble average of $\vec{k}(\vec{x}_f)$, vary slowly as a function of $\vec{x}_s$ in $\mathcal{M}_{new}$.

Also, the time correlation function of $\vec{k}(t^*)$ is given as $\langle \vec{k}(t_1^*) \vec{k}(t_2^*) \rangle \approx 0$ for $|t_1^* - t_2^*| \geq dt$, because $dt$ is sufficient for $\vec{x}_f$ to wander over all areas of an energy surface—the $\mathcal{S}(t, dt)$ contains all $\vec{x}_f$ on the energy surface. Thus, considering $\langle \vec{k}^2(\vec{x}_f) \rangle_{\mathcal{S}(t, dt)}$ should be a smooth function of $\vec{x}_s$ and using (13) and (14), the motions inducing the $\vec{x}_s$-dependence give the following time correlation for $k(t)$ in $\mathcal{M}_{new}$ with $dt < \tau$:

$$\langle \xi(t_1) \xi(t_2) \rangle_{\text{exp}} = C(x, p) \delta(t_1 - t_2),$$

(17)

The smoothness of $\langle \vec{x}_f \rangle_{\mathcal{S}(t, dt)}$ as a function of $\vec{x}_s$ comes from the fact that $\lim_{t_1 \to t_2} \mathcal{S}(t_1, dt) = \mathcal{S}(t_2, dt)$ and the trajectory of $\vec{x}_f$ in $\Gamma$ is smooth.
where the notation $\tilde{k}(\vec{x}_f)$ is replaced by $\xi$ and $C(x, p)$ is a smooth function of $\vec{x}_s$. Therefore, under the $\vec{x}_s$-dependence, $\tilde{k}(\vec{x}_f)$ still behaves as a white noise as satisfies (17), but, differently from the conventional white noise with stationary statistical properties, its moments or cumulants except its average are non-stationary for time as a smooth function of the slow coordinates $\vec{x}_s$.

4.3 Extension in Stochastic dynamical formalism

As is clear from the above analysis, the motions of $\vec{x}_f$ generating the $\vec{x}_s$-dependence have a faster time-scale than the motions generating the $\vec{x}_f$-dependence which are observed as the deterministic slow dynamics $\delta h(x, p)$: under the $\vec{x}_f$-dependence, $\vec{x}_f$ is not fast enough to wander over all areas of an energy surface during $dt$ but, under the $\vec{x}_s$-dependence, $\vec{x}_f$ is fast enough to do so. Also, the motions of $\vec{x}_f$ generating the $\vec{x}_s$-dependence have a slower time-scale than the motions of $\vec{y}_f \equiv \vec{x}_f - \langle \vec{x}_f \rangle_S(t, dt)$ in the case of $\vec{x}_f$-dependence, i.e., the fast system in the extremal case (the fast system described as white noise in the Langevin equation): under the $\vec{x}_s$-dependence, the motions of $\vec{x}_f$ cannot be approximated as confined near an energy surface, and, since the elements of $\mathcal{S}(t, dt)$ are confined near an energy surface, $\vec{x}_f$ cannot wander over the entire areas of $\Omega$ during $dt$. Thus, actually the motions related with the $\vec{x}_s$-dependence are intermediate time-scale motions.

Consequently, the previously obtained results for the case of $\vec{x}_s$-dependence show that, if there exist intermediate time-scale motions, generally the fast dynamics, which cannot be written in the deterministic part of stochastic dynamics, should be described as the white noise $\xi$ of which the cumulant

$$c_n \equiv \left[ (-i)^n \frac{d^n}{dk^n} \ln \sum_{m=0}^{\infty} \frac{1}{m!} \langle \xi^m \rangle (ik)^m \right]_{k=0}$$

for $n \geq 2$ is a smooth function of the slow coordinates $\vec{x}_s$. Thus, we have to treat the cumulants $c_n(x, p)$ as dynamical variables in stochastic dynamics and have the following stochastic dynamics for the coordinates $(x, p, c_2, c_3, \ldots)$ with the setting $c_1 = 0$:

$$\frac{dx}{dt} = p, \quad \frac{dp}{dt} = h(x, p) + \xi,$$  \hspace{1cm} (19a)

The PDF for $\xi$ is a function of $\vec{x}_s$, and generally the functional form of the PDF varies as $\vec{x}_s$ varies.
\[
\frac{dc_n}{dt} = \frac{\partial c_n}{\partial x} \frac{dx}{dt} + \frac{\partial c_n}{\partial p} \frac{dp}{dt} \equiv h_n(x, p, \xi),
\]  

where \(n = 2, 3, \ldots\), and

\[
\langle \xi(t)\xi(t') \rangle = c_2(x, p)\delta(t-t')
\]

by the result (17); the cumulants \(c_n\) and (20) completely determine any time correlations \(\langle \xi^{n_1}(t_1) \cdots \xi^{n_l}(t_l) \rangle\), where \(n_1, \ldots, n_l\) are positive integers, and thus the formulation (19) together with (20) gives a complete stochastic dynamical description. In deriving (19), similarly as in the extremal case, \(k(\vec{x})\) can be replaced by a single noise \(\xi\): assuming that all linear increment of \(p(t)\) during \(dt\) is given by the deterministic term as \(h(x, p)dt\), we have \(\langle \tilde{\Delta}k(\vec{x}) \rangle_{x(t, dt)} = 0\), where \(\tilde{\Delta}k(\vec{x}) \equiv k(\vec{x}) - k(\vec{x}_f)\), and, also if we assume that the fast fluctuation of \(\vec{x}_s\) has the same time-scale as \(\vec{x}_f\), \(\tilde{\Delta}k(\vec{x})\) should satisfy at least (17). However, generally the time-scale of \(\vec{x}_f\) becomes faster as \(|\alpha|\) increases, and, for some \(|\alpha| = M\), the term \(\sum_{|\alpha| \geq M} \mathcal{S}_\alpha(\vec{x}_s)\vec{p}_f\) behaves as a white noise. Thus, more precisely, we can write as \(\frac{dp}{dt} = h(x, p) + \xi + \eta\), where \(\eta\) is the conventional Gaussian white noise and \(\langle \xi(t)\eta(t') \rangle \propto \delta(t-t')\).

While the trivial case \((c_2 = \text{const} \neq 0\) and \(c_n = 0\) for \(n \geq 3\)) of formulation (19) gives the conventional Langevin equation, in the most simple nontrivial case as all \(c_n = 0\) except \(c_2 \neq \text{const}\), (as shown in the subsequent argument) the formulation can be reduced to the conventional multiplicative noise method.

The results obtained so far can be summarized as follows. Intermediate time-scale motions cannot be described by deterministic trajectories in phase space, because they satisfy a \(\delta\)-function time correlation as (17), and also they cannot be described simply by random forces acting on the trajectories, because they have deterministic properties represented by the cumulants \(c_n(x, p)\) for \(n \geq 2\). Intermediate time-scale motions should be described by the trajectories in the cumulant space defined by the coordinates \((c_2, c_3, \ldots)\). This indicates that the simple trajectory-based description in phase space, such as the Langevin or generalized Langevin equation, cannot describe systematically the dynamical effect from diverse intermediate time-scale motions. Also, since intermediate time-scale motions satisfy a \(\delta\)-function time correlation, it means that, regardless of the sufficiency of time-scale separation, any complex (chaotic and ergodic) dynamical system can be well described using Markov process, if we perfectly construct the deterministic part in stochastic dynamics.

In formulation (19), \(\xi\) and \((x, p)\) exchange their influence with each other. Thus, the random noise \(\xi\) cannot be considered as the thermal fluctuation from an ideal thermal reservoir in that the statistical properties of \(\xi\) are not independent from the system described by \((x, p)\)—the thermal reservoir by the fast chaotic dynamics of \(\vec{x}_f\) is treated as a finite
thermal system not having infinite capacity.

In the most simple case of formulation \((19)\) as all \(c_n = 0\) except \(c_2 = 2C(x,p)\), we have \(\langle \xi(t)\xi(t') \rangle = 2C(x,p)\delta(t-t')\) from \((20)\), and the PDF \(w(\xi)\) for \(\xi\) is given as a Gaussian: \(w(\xi) \propto e^{-\xi^2/4C(x,p)}\). If we define \(\zeta \equiv \xi/C(x,p)^{1/2}\), the variance of the random variable \(\zeta\) is constant, and, using \(\zeta\), we can reformulate the most simple case of \((19)\) in a conventional multiplicative noise system, as follows:

\[
\frac{dp}{dt} = h(x,p) - \frac{1}{2} \frac{\partial C(x,p)}{\partial p} + C(x,p)^{1/2} \zeta,
\]

\[
\langle \zeta(t)\zeta(t') \rangle = 2\delta(t-t'),
\]

where the equation of motion for \(x\) is the same as in \((19)\), and the PDF for \(\zeta\) is Gaussian: \(w(\zeta) \propto e^{-\zeta^2/4}\). As can be easily checked, these two systems give a same Fokker-Planck equation; in this respect, the multiplicative noise method is equivalent to the most simple case of formulation \((19)\).

5 Additional remarks

Finally, we point out that the arguments for intrinsic time measurement errors symmetrically hold for space measurement errors in static problems, e.g., \(\partial_x\psi(x,t) = h_s(x,t) + h_f(x,t)\) where \(h_s\) and \(h_f\) are slow and fast varying parts for \(x\), respectively. There is the experimentally identifiable minimum distance \(dx\) corresponding to \(dt\) (in dynamics), and we obtain the stochastic statics, \(\partial_x\psi(x,t) = h_s(x,t) + \xi(x)\) for a random variable \(\xi(x)\). The stochasticity originated by space measurement errors reflects the fine structure of statics on the statistical properties of \(\xi(x)\).

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