An Optimality Gap Test for a Semidefinite Relaxation of a Quadratic Program with Two Quadratic Constraints

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Abstract

We propose a necessary and sufficient test to determine whether a solution for a general quadratic program with two quadratic constraints (QC2QP) can be computed from that of a specific convex semidefinite relaxation, in which case we say that there is no optimality gap. Originally intended to solve a nonconvex optimal control problem, we consider the case in which the cost and both constraints of the QC2QP may be non-convex. We obtained our test, which also ascertains when strong duality holds, by generalizing a closely-related method by Ai and Zhang. An extension was necessary because, while the method proposed by Ai and Zhang also allows for two quadratic constraints, it requires that at least one is strictly convex. In order to illustrate the usefulness of our test, we applied it to two examples that do not satisfy the assumptions required by prior methods. Our test guarantees that there is no optimality gap for the first example—a solution is also computed from the relaxation—and we used it to establish that an optimality gap exists in the second. We also verified using the test in a numerical experiment that there is no optimality gap in most instances of a set of randomly generated QC2QP, indicating that our method is likely to be useful in applications other than that of our original motivation.

Keywords. quadratically constrained quadratic program, semidefinite relaxation, strong duality, nonconvex optimization

AMS subject classifications. 90C26, 90C46

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1 Introduction

We consider the following real-valued quadratic program with two quadratic constraints (QC2QP):

minimize \( q_0(z) = z^T Q_0 z + 2b_0^T z \)
subject to \( q_1(z) = z^T Q_1 z + 2b_1^T z + c_1 \leq 0 \), \( q_2(z) = z^T Q_2 z + 2b_2^T z + c_2 \leq 0 \),

where \( Q_0, Q_1, \) and \( Q_2 \) are \( n \times n \)-dimensional real symmetric matrices; \( b_0, b_1 \) and \( b_2 \) are \( n \)-dimensional real vectors; and \( c_1 \) and \( c_2 \) are real constants.

Main problem: We seek to solve (QP0) without positive semidefiniteness restrictions on \( Q_0, Q_1, \) and \( Q_2 \), which, generally, makes the problem nonconvex.

1.1 Brief overview of existing related work

Existing work explored two distinct approaches to obtain a globally optimal solution to (QP0). The first approach, which we adopt to develop our method, uses a semidefinite relaxation of (QP0) whose (Lagrange) dual is convex and identical to that of (QP0). The second approach seeks to exploit the structure of the QC2QP, possibly subject to additional restrictions, to characterize globally optimal solutions in a way that numerically tractable methods can be used. Subsequently, we proceed to describe previous work on both approaches.

Following the first approach, Ai and Zhang [1] introduce a necessary and sufficient condition for strong duality for the Celis-Dennis-Tapia (CDT) subproblem of minimizing a nonconvex quadratic cost over the intersection of an ellipsoid and an elliptical cylinder [4] (corresponding to \( Q_1 \) being positive definite and \( Q_2 \) being positive semidefinite, respectively, in (QP0)), which is a special case of QC2QP used in the extended trust region method [18]. Their result shows that strong duality holds, and a primal optimal solution can be obtained from a semidefinite relaxation, if and only if optimal solutions of the dual and the relaxation violate the so-called Property I comprising three algebraic conditions. Subsequent work by Yuan et al. in [19] shows that adding second-order cone (SOC) constraints to a CDT subproblem for which Property I holds may narrow or even eliminate the duality gap. In the latter case, a globally optimal solution to the original problem can be computed from a solution of the semidefinite relaxation with an SOC reformulation.

Another relaxation technique is to solve the QC2QP in the complex domain. In [2], Beck and Eldar use such a methodology to introduce a necessary and sufficient condition for strong duality, using the classical extended S-Lemma of Fradkov and Yakubovich [7]. If strong duality holds, a globally optimal solution to the original problem can be obtained by solving a quadratic feasibility problem. Using the necessary and sufficient condition and the convexity of a quadratic mapping, they subsequently prove a sufficient condition for strong duality for the real-valued QC2QP. Huang and Zhang [12] propose a sufficient condition for strong duality in the complex-valued problem in which a globally optimal solution to the original problem can be obtained from a semidefinite relaxation if strong duality holds. Their result is derived using a matrix rank-one decomposition for complex Hermitian matrices.

Following the second approach, Peng and Yuan [14] prove a necessary condition for global optimality in QC2QP. Specifically, the number of negative eigenvalues of
the Hessian of the Lagrangian is characterized at a globally optimal solution. For the CDT subproblem, Bomze and Overton [3] prove necessary and sufficient conditions for global and local optimality using copositivity.

1.2 Our main contribution

We seek to use a specific semidefinite relaxation to find a solution for (QP0) for the case in which there are no positive semidefiniteness restrictions on $Q_0$, $Q_1$, and $Q_2$. When a solution for (QP0) can be determined from that of the relaxation we say that there is no optimality gap. The relaxation is cast as a convex semidefinite program (SDP) for which an optimal solution can be determined efficiently using existing software. The dual of the semidefinite relaxation is also convex and is also the dual of (QP0). This motivates the analysis in section 3, where we propose the so-called Property $I^+$ defined by four algebraic conditions that determine based on solutions of the relaxation and its dual when an optimality gap exists. Our main result is Theorem 3.2, which states precisely a necessary and sufficient condition for the existence of an optimality gap based on Property $I^+$. As we discuss in detail in section 5, Theorem 3.2 extends the closely-related result of [1] in the following ways:

• The assumption in [1] that either $Q_1$ or $Q_2$ must be positive definite is replaced in our work by a weaker requirement that the dual of (QP0) satisfies Slater’s condition.

• In the particular case when $Q_1$, or $Q_2$, is positive definite the above-mentioned Property $I^+$ is equivalent to Property I used in [1] to determine when there is no optimality gap. Hence, our work presents no advantage relative to [1] when $Q_1$, or $Q_2$, is positive definite.

A nonconvex optimal control problem studied by Cheng and Martins in [6] motivated the unexampled QC2QP considered here, in which neither $Q_1$ nor $Q_2$ is assumed positive definite. The authors used Theorem 3.2 of this paper to propose conditions for the problem data in [6] with which the problem can be solved using an SDP.

1.3 Structure of the article

We start with reviewing in section 2 the key results of [1]. In doing so, we also present the essential concepts used in [1], which include the semidefinite relaxation used here. We define Property $I^+$ and subsequently state our main result (Theorem 3.2) in section 3, where we also describe a procedure to obtain a solution for (QP0) from that of the semidefinite relaxation for the case in which there is no optimality gap. In section 4, we describe algorithm to implement the test of Theorem 3.2 and we also discuss relevant numerical considerations. In section 4, we apply our algorithms to QC2QP examples that do not satisfy the assumptions required by previous methods. More specifically, we use our algorithms to compute the optimal solution for the first example after we establish that it has no optimality gap. In contrast, we establish that there is an optimality gap for the second. In section 5, we also apply the test to a randomly generated set of QC2QP to conclude that there would be no optimality gap in the vast majority. In section 5, we provide a detailed comparison between our method and that of [1], and in section 6, we present the proof of Theorem 3.2.
1.4 Notation and conventions

Throughout the paper, we adopt the following notation, which is mostly borrowed from [1]: We denote the set of real numbers with $\mathbb{R}$. We use $\mathcal{S}^n$ to denote the set of symmetric matrices in $\mathbb{R}^{n \times n}$. We use $S_n$ to denote the set of symmetric matrices in $\mathbb{R}^{n \times n}$. We use the dot notation to denote the matrix inner product, that is, $A \cdot B := \text{Tr}(AB)$ for $A, B \in \mathbb{R}^{n \times n}$, where $\text{Tr}(AB)$ denotes the trace of $AB$. We use $\det(C)$ to denote the determinant of a square matrix $C$. We use $\text{rank}(D)$ and $\text{rank}(D, \epsilon)$ to denote the rank and the numerical rank with tolerance $\epsilon$, respectively, of a matrix $D$. A positive (semi)definite matrix $M$ is denoted by $M \succ (\succeq) 0$. We use $0_{n \times m}$ to denote a matrix in $\mathbb{R}^{n \times m}$ with all entries being 0 and $I_n$ to denote an $n \times n$-dimensional identity matrix. A diagonal matrix is denoted by $\text{diag}(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are the diagonal entries. The null space of a linear mapping $L : \mathcal{V} \rightarrow \mathcal{W}$ between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ is denoted by $\mathcal{N}(L)$. We use $|a|$ to denote the absolute value of a real-valued constant or variable $a$. We use the term \textit{polynomial time}, which is defined in [18], to indicate the total number of basic operations (for example, addition, subtraction, multiplication, division, and comparison) of a procedure is bounded by a polynomial of the problem data. We use boldface font, such as in $\mathbf{x}$, to represent the scalar or vector-valued variables of a feasibility problem, or, more generally, the optimization variables with respect to which we seek to minimize a cost subject to constraints. We adopt the following format to represent an optimization problem over a subset $\mathcal{X}$ of a real coordinate space, in which we seek to minimize a cost $f : \mathcal{X} \rightarrow \mathbb{R}$ subject to an additional constraint set $\mathcal{C}$.

\[
\begin{align*}
\text{minimize} & \quad f(\mathbf{x}) \\
\text{subject to} & \quad \mathbf{x} \in \mathcal{X}, \quad \mathbf{x} \in \mathcal{C},
\end{align*}
\]

We use $\mathcal{V}_p$ to denote the optimal value of $(P)$.

2 Preliminary results and concepts

We start with introducing assumptions and reviewing the key results in [1]. For the reader’s convenience, we follow the notation in [1] and rewrite $(QP_0)$ in a homogeneous quadratic form:

\[
\begin{align*}
\text{minimize} & \quad M(q_0) \cdot \begin{bmatrix} t \\ z \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix}^T = \begin{bmatrix} t^T Q_0 z + 2tb_0^T z \\ c_i^T Q_i z + 2tb_i^T z + t^2 c_i \end{bmatrix} \leq 0, & \quad i \in \{1, 2\}, \\
\text{subject to} & \quad t^2 = 1,
\end{align*}
\]

where

\[
M(q_0) = \begin{bmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{bmatrix}, \quad M(q_i) = \begin{bmatrix} c_i & b_i^T \\ b_i & Q_i \end{bmatrix}, \quad i \in \{1, 2\}.
\]

In the rest of the paper, we use $\mathbf{x}$ to represent an $(n+1)$-dimensional vector concatenating a scalar $t$ and an $n$-dimensional vector $z$ as follows

\[
\mathbf{x} = \begin{bmatrix} t \\ z \end{bmatrix}.
\]
The semidefinite relaxation of \((QP)\) is the following:

\[
\begin{align*}
    \text{minimize} & \quad M(q_0) \cdot X \\
    \text{subject to} & \quad M(q_i) \cdot X \leq 0, \quad i \in \{1, 2\}, \\
    & \quad I_{00} \cdot X = 1, \\
    & \quad X \succeq 0,
\end{align*}
\]

\((SP)\)

where \(I_{00} = [\begin{smallmatrix} 1 & 0_{n \times 1} \\ 0_{n \times 1} & 0_{n \times n} \end{smallmatrix}]\).

The dual problem of \((SP)\) is the following:

\[
\begin{align*}
    \text{maximize} & \quad y_0 \\
    \text{subject to} & \quad y_0 I_{00} - y_1 M(q_1) - y_2 M(q_2) + Z = M(q_0), \\
    & \quad y_i \geq 0, \quad i \in \{1, 2\}, \\
    & \quad Z \succeq 0,
\end{align*}
\]

\((SD)\)

Note that \((SD)\) is also the dual of \((QP)\).

**Assumption 2.1.** Problem \((SP)\) satisfies Slater’s condition, that is, there exists a symmetric positive definite \((n+1) \times (n+1)\)-dimensional real matrix \(X\) such that

\[
\begin{align*}
    M(q_i) \cdot X < 0, & \quad i \in \{1, 2\}, \\
    I_{00} \cdot X = 1.
\end{align*}
\]

\((2.3a)\)

\((2.3b)\)

**Remark 2.2.** Assumption 2.1 holds when Slater’s condition holds for \((QP_0)\) [1] (and hence for \((QP)\)), that is, there exists an \(n\)-dimensional vector \(z\) such that

\[
\begin{align*}
    z^T Q_i z + 2 b_i^T z + c_i < 0, & \quad i \in \{1, 2\}
\end{align*}
\]

\((2.4)\)

**Assumption 2.3.** Slater’s condition holds for \((SD)\), that is, there exist a scalar \(y_0\) and positive scalars \(y_1\) and \(y_2\) such that

\[
M(q_0) - y_0 I_{00} + y_1 M(q_1) + y_2 M(q_2) \succ 0.
\]

\((2.5)\)

**Remark 2.4.** For problem data \(M(q_0), M(q_1),\) and \(M(q_2)\), one can numerically check whether Assumption 2.1 and Assumption 2.3 are met by solving the feasibility problem of \((SP)\) and \((SD)\), respectively, using an SDP solver.

**Remark 2.5.** The inequality \((2.5)\) holds, by Schur complement, if and only if there exist a scalar \(y_0\) and positive scalars \(y_1\) and \(y_2\) such that

\[
Q_0 + y_1 Q_1 + y_2 Q_2 \succ 0,
\]

\((2.6a)\)

\[
(\xi(y_1, y_2))^T (Q_0 + y_1 Q_1 + y_2 Q_2)^{-1} (\xi(y_1, y_2)) < -y_0 + y_1 c_1 + y_2 c_2,
\]

\((2.6b)\)

where \(\xi(y_1, y_2) := b_0 + y_1 b_1 + y_2 b_2\).

**Remark 2.6.** By Proposition 2.1 of [18], Assumption 2.3 holds if the objective function of \((QP_0)\) is strictly convex (that is, \(Q_0\) is positive definite), or at least one of the constraints of \((QP_0)\) is elliptical (that is, \(Q_i \succ 0\) and \(b_i^T Q_i^{-1} b_i - c_i > 0\) for \(i\) being 1 or 2, or both). The latter condition, as a special case of Assumption 2.3 implied by Remark 2.6, is required by Ai and Zhang in [1] which ensures that Slater’s condition holds for \((SD)\).
Remark 2.7. Assumption 2.1 and Assumption 2.3 together imply that both (SP) and (SD) have attainable optimal solutions which yield an identical optimal value, which further implies that (SP) is a tight relaxation of (QP) (that is, the optimal value of (SP) is identical to that of (QP)) if and only if strong duality holds for (QP). This observation is going to be relevant to our optimality gap test later.

We denote optimal solutions of (QP), (SP), and (SD), respectively, by \( x^*, \hat{X}, \hat{x}_1 \) and \( (Z, \hat{y}_0, \hat{y}_1, \hat{y}_2) \), and their optimal values, respectively, by \( \hat{V}^*(\text{QP}), \hat{V}^*(\text{SP}), \text{ and } \hat{V}^*(\text{SD}) \). Note that a primal-dual feasible pair, \( (\hat{X}, (\hat{y}_0, \hat{y}_1, \hat{y}_2)) \), is optimal if and only if it satisfies the complementary conditions:

\[
\begin{align*}
X Z &= 0_{(n+1) \times (n+1)}, \quad (2.7a) \\
y_1 M(q_1) \cdot X &= 0, \quad (2.7b) \\
y_2 M(q_2) \cdot X &= 0. \quad (2.7c)
\end{align*}
\]

Property I, which we shall state in Definition 2.8, is the key to the necessary and sufficient condition for an optimality gap (or, equivalently, for a duality gap between (QP) and (SD)) when \( Q_1 \) is positive definite.

Definition 2.8 (Definition 4.1 of [1]). For \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \), a given pair of optimal solutions for (SP) and (SD), respectively, we say that this pair has Property I if:

1. \( \hat{y}_1 \hat{y}_2 \neq 0 \);
2. \( \text{rank}(\hat{Z}) = n - 1 \);
3. \( \text{rank}(\hat{X}) = 2 \) and there is a rank-one decomposition of \( \hat{X} \), \( \hat{X} = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T \) such that

\[
(M(q_1) \cdot \hat{x}_i \hat{x}_i^T = 0, \quad i \in \{1, 2\}, \quad (2.8)
\]

\[
(M(q_2) \cdot \hat{x}_1 \hat{x}_1^T)(M(q_2) \cdot \hat{x}_2 \hat{x}_2^T) < 0. \quad (2.9)
\]

Theorem 2.9 (Theorem 4.2 of [1]). Consider (QP) where Slater’s condition is satisfied and \( Q_1 \) is positive definite. Suppose that \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \) is a pair of optimal solutions for the semidefinite relaxation (SP) and dual problem (SD), respectively. Then, \( \hat{V}^*(\text{SP}) \leq \hat{V}^*(\text{SD}) \) holds if and only if the pair \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \) has Property I.

Remark 2.10. By Lemma 2.2 in [18], one can always obtain a rank-one decomposition \( \hat{X} = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T \) such that (2.8) holds, provided \( M(q_1) \cdot \hat{X} = 0 \). We will briefly introduce how to obtain such \( \hat{x}_1 \) and \( \hat{x}_2 \) by referring to the proof of Lemma 2.2 in [18].

Assume \( \text{rank}(\hat{X}) = 2 \) and \( M(q_1) \cdot \hat{X} = 0 \). Without loss of generality, assume a candidate rank-one decomposition \( \hat{X} = x_1^T(x_1^T)^T + x_2^T(x_2^T)^T \) is such that

\[
M(q_1) \cdot x_1^T(x_1^T)^T = -M(q_1) \cdot x_2^T(x_2^T)^T \neq 0. \quad (2.10)
\]

Consider a quadratic equation of a scalar \( \beta \),

\[
0 = (\beta x_1^T + x_2^T)^T M(q_1) (\beta x_1^T + x_2^T) = \beta^2 (x_1^T)^T M(q_1) x_1^T + 2\beta (x_1^T) M(q_1) x_2^T + (x_2^T)^T M(q_1) x_2^T. \quad (2.11)
\]

Since \( \hat{X} \) is symmetric and positive semidefinite, we can always perform an eigendecomposition to \( \hat{X} \) to obtain the eigenvalue \( \lambda_i \) associated with column eigenvector \( v_i \) for \( i \) in \( \{1, 2, \ldots, n+1\} \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} \geq 0 \). Therefore, a trivial rank-one decomposition is \( \hat{X} = \sum_{i=1}^r \lambda_i v_i v_i^T \) where \( r := \text{rank}(\hat{X}) \).
This equation must have two distinctive real roots with opposite signs because
\[
((x^c_1)^T M(q_1)x^c_1)((x^c_2)^T M(q_1)x^c_2) < 0. \tag{2.12}
\]
Let \( \bar{\beta} \) be one of the roots and
\[
\bar{x}_1 := \frac{\bar{\beta}}{\sqrt{\beta^2 + 1}} x^c_1 + \frac{1}{\sqrt{\beta^2 + 1}} x^c_2, \tag{2.13a}
\]
\[
\bar{x}_2 := -\frac{1}{\sqrt{\beta^2 + 1}} x^c_1 + \frac{\bar{\beta}}{\sqrt{\beta^2 + 1}} x^c_2. \tag{2.13b}
\]
Now we have \( \hat{X} = \bar{x}_1 \bar{x}_1^T + \bar{x}_2 \bar{x}_2^T \) such that
\[
M(q_1) \cdot \hat{x}_i \hat{x}_i^T = 0, \quad i \in \{1, 2\}. \tag{2.14}
\]

3 Main results

We start by modifying Property I by adding an extra condition and naming the combined conditions Property I$^+$ as follows:

**Definition 3.1.** For \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \), a given pair of optimal solutions for (SP) and (SD), respectively, we say that this pair has Property I$^+$ if:

1. Property I holds;
2. \( M(q_1) \cdot \hat{x}_1 \hat{x}_1^T \neq 0 \).

As we shall see in the following theorem, Property I$^+$ is the key to the necessary and sufficient condition for the optimality gap (or, equivalently, for the duality gap between (QP) and (SD)), when the positive definiteness of \( Q_1 \) is not assumed.

**Theorem 3.2.** Consider (QP) and let Assumption 2.1 and Assumption 2.3 hold. Suppose that \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \) is a pair of optimal solutions for the semidefinite relaxation (SP) and dual problem (SD), respectively. Then, \( V^+_\text{SP} < V^\ast \text{QP} \) holds if and only if the pair \( \hat{X} \) and \( (\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \) has Property I$^+$.

**Proof.** See section 6.

The significance of Theorem 3.2 is that whether (QP) has an optimality gap can be efficiently determined by examining whether solutions of (SP) and (SD) violate Property I$^+$. It is worth noting that the examination only takes polynomial time since it only requires to solve one SDP with its dual and conduct a rank-one decomposition, both of which run in polynomial time [13]. Furthermore, if there is no optimality gap, then a solution to the original problem (QP) can be obtained by conducting a rank-one decomposition to a solution of (SP). Otherwise, if an optimality gap exists (or, equivalently, a duality gap exists), then Property I$^+$, which holds in this case, could be applied to explore methods that can tighten, or even eliminate, the gap.
3.1 A procedure to compute an optimal solution for \((QP_0)\)

The following procedure explains how to obtain a solution to \((QP_0)\) when there is no optimality gap. Let \(\hat{X}\) denote an optimal solution of \((SP)\). Since \(\text{rank}(\hat{X})\) is either one or two, we focus only on these two cases.

Case 1. \(\text{rank}(\hat{X}) = 1\).

The unique rank-one decomposition \(\hat{X} = \hat{x}\hat{x}^T\) provides a solution \(z^*\) to \((QP_0)\) by \(z^* = \hat{z}/\hat{t}\) following the partition of \(\hat{x}\) in the form of \((2.2)\).

Case 2. \(\text{rank}(\hat{X}) = 2\).

We show how to obtain a solution to \((QP_0)\) in the following exhaustive and mutually exclusive subcases. As Theorem 3.2 indicates, since there is no optimality gap, each of these three subcases violates at least one of the conditions in Property \(I^+\) except for \(\text{rank}(\hat{X}) = 2\).

Case 2a. \(\hat{y} \neq 0\) and \(\text{rank}(\hat{X}) = 2\).

Two situations are considered in this subcase:

(i) \(\hat{y}_1 = \hat{y}_2 = 0\) and \(\text{rank}(\hat{X}) = 2\). It implies that both inequality constraints of \((SP)\) are inactive at an optimal solution \(\hat{X}\), namely,

\[
M(q_j) \cdot \hat{X} < 0 \quad \forall j \in \{1, 2\}. \tag{3.1}
\]

Consequently, an arbitrary rank-one decomposition \(\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T\) satisfies

\[
M(q_j) \cdot \hat{x}_i\hat{x}_i^T < 0 \quad \forall j \in \{1, 2\} \tag{3.2}
\]

when \(i\) is 1 or 2, or both. This inequality holds because otherwise

\[
M(q_j) \cdot \hat{x}_i\hat{x}_i^T \geq 0 \quad \forall i \in \{1, 2\} \tag{3.3}
\]

holds for \(j\) being 1 or 2, or both, which implies

\[
M(q_i) \cdot \hat{X} = M(q_i) \cdot (\hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T) \geq 0. \tag{3.4}
\]

This is a contradiction to \((3.1)\). Without loss of generality, we assume \(3.2\) holds for \(i = 1\). Consequently, a solution to \((QP_0)\), \(z^*\), can be computed by \(z^* = \hat{z}_1/\hat{t}_1\) following the partition of \(\hat{x}_1\) in the form of \((2.2)\). Note that \(\hat{t}_1 \neq 0\) because otherwise the set of optimal solutions of \((SP)\) is unbounded, which contradicts Assumption \(2.3\) (For detailed relation between the boundedness of the set of optimal solutions of an SDP and the feasibility of its dual problem, one is referred to \[15\].)

Since a solution to \((QP_0)\) can be obtained from an arbitrary rank-one decomposition of \(\hat{X}\), we can proceed to compute such a decomposition by

\[
\hat{x}_i = \sqrt{\lambda_i}v_i, \quad i \in \{1, 2\}, \tag{3.5}
\]

where \(\lambda_1\) and \(\lambda_2\) are the only positive eigenvalues of \(\hat{X}\); and \(v_1\) and \(v_2\) are the associated column eigenvectors.

(ii) Either \(\hat{y}_1\) or \(\hat{y}_2\) is zero and \(\text{rank}(\hat{X}) = 2\). Without loss of generality, we can assume \(\hat{y}_1 \neq 0\) and \(\hat{y}_2 = 0\) which imply

\[
M(q_1) \cdot \hat{X} = 0, \tag{3.6a}
\]

\[
M(q_2) \cdot \hat{X} < 0. \tag{3.6b}
\]
Thus, we can obtain a rank-one decomposition $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ such that
\[
M(q_1) \cdot \hat{x}_i\hat{x}_i^T = 0 \quad \forall i \in \{1, 2\}
\] (3.7)
according to Remark 2.10. Without loss of generality, we assume $\hat{x}_1$ satisfies $M(q_2) \cdot \hat{x}_1\hat{x}_1^T < 0$. Consequently, a solution to \((QP0)\) can be computed from $\hat{x}_1$ in the same way as we showed in part (i), with the same argument on $\hat{t}_1$ being nonzero.

**Case 2b.** $y_1y_2 \neq 0$, rank($\hat{Z}$) $\neq n - 1$, and rank($\hat{X}$) = 2.

By Remark 2.10 we can start with an initial rank-one decomposition $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ such that
\[
M(q_1) \cdot \hat{x}_i\hat{x}_i^T = 0, \quad i \in \{1, 2\}.
\] (3.8)
If this decomposition also satisfies
\[
M(q_2) \cdot \hat{x}_i\hat{x}_i^T = 0, \quad i \in \{1, 2\},
\] (3.9)
than without loss of generality, we can compute a solution to \((QP0)\) from $\hat{x}_1$ using the same way showed in the previous subcases. Otherwise, there exists a nontrivial vector $y$ in $\mathbb{R}^{n+1}$ such that $y$ is in the intersection of $\mathcal{N}(\hat{X})$ and $\mathcal{N}(\hat{Z})$ (for details, see Case 4 of the sufficiency proof in section 6). Furthermore, the matrix $\hat{X} + yy^T$ is rank-one decomposable at a vector $x$ in $\mathbb{R}^{n+1}$ such that
\[
M(q_j) \cdot xx^T = 0, \quad j \in \{1, 2\}.
\] (3.10)
The proof of Lemma 3.3 of [1] provides the procedure to find $x$. Therefore, a solution of \((QP0)\) is $z^* = z/t$ following the partition of $x$ in (2.2). Note that $t \neq 0$ follows from the argument in Case 4 of the sufficiency proof in section 6.

**Case 2c.** $y_1y_2 \neq 0$, rank($\hat{Z}$) = $n - 1$, and rank($\hat{X}$) = 2.

Similar to Case 2b, if an initial rank-one decomposition $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ is such that
\[
M(q_j) \cdot \hat{x}_i\hat{x}_i^T = 0, \quad i \in \{1, 2\},
\] (3.11)
then without loss of generality, we can compute a solution to \((QP0)\) from $\hat{x}_1$ using the same approach showed in the previous subcases. Otherwise, it must hold that $M(q_1) \cdot \hat{x}_1\hat{x}_2^T = 0$. Thus, there exists another rank-one decomposition $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ such that
\[
M(q_j) \cdot \hat{x}_i\hat{x}_i^T = 0, \quad i \in \{1, 2\},
\] (3.12)
where we show the procedure to obtain $\hat{x}_1$ and $\hat{x}_2$ in Case 5 of the sufficiency proof in section 6. Therefore, without loss of generality, a solution to \((QP0)\) can be computed from $\hat{x}_1$ and the first element of $\hat{x}_1$ is nonzero, both shown in Case 2b(i).

**Remark 3.3.** Besides the numerical results that are going to be presented in the next section, Theorem 3.2 was used in [6] to propose conditions for the QC2QP problem data with which the optimality gap is guaranteed to not exist, for example, Theorem 2 of [6].

## 4 Testing Property $I^+$ Numerically

Theorem 3.2 enables a simply verifiable optimality gap test for a semidefinite relaxation of a QC2QP. This test only requires to solve one SDP with its dual and conducting a rank-one decomposition, both of which run in polynomial time. However, in general, SDP solvers (for example, SDPT3 [17] and SeDuMi [16]) give approximate,
rather than exact, solutions within certain tolerance. Hence, it is useful to establish an optimality gap test utilizing Property I\(^*\) in an approximation sense. The following procedures refer to the purified \((\epsilon_1, \epsilon_2)\)-approximation in \(\Pi\):

Let \(\hat{X}\) and \((\hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2)\) denote a pair of numerical solutions of \(\text{SP}\) and its dual \(\text{SD}\), respectively, solved by an SDP solver whose tolerance is \(\epsilon_1 > 0\). Let \(\epsilon_2 > 0\) be the tolerance for purification. First, conduct an eigendecomposition of \(\hat{X}\) and \(\hat{Z}\), that is,

\[
\hat{X} = P_1^T \Lambda_1 P_1, \quad \hat{Z} = P_2^T \Lambda_2 P_2,
\]

where \(P_1\) and \(P_2\) are \((n + 1) \times (n + 1)\)-dimensional orthogonal matrices and \(\Lambda_1\) and \(\Lambda_2\) are \((n + 1) \times (n + 1)\)-dimensional diagonal matrices. Let \(\Lambda_i := \text{diag}(\lambda_{i1}, \ldots, \lambda_{i(n+1)})\), and let

\[
\lambda_{ij} := \begin{cases} 
\lambda_{ij} \geq \epsilon_2, & i \in \{1, 2\}, \ j \in \{1, 2, \ldots, n + 1\}, \\
0, & \lambda_{ij} < \epsilon_2.
\end{cases}
\]

Define the purified solutions by

\[
X^* := P_1^T \text{diag}(\lambda_{11}^*, \ldots, \lambda_{(n+1)}^*) P_1, \\
Z^* := P_2^T \text{diag}(\lambda_{21}^*, \ldots, \lambda_{2(n+1)}^*) P_2, \\
y_i^* := \hat{y}_i, \quad i \in \{0, 1, 2\}.
\]

The above step essentially purifies the rank of \(\hat{X}\) and \(\hat{Z}\) so that

\[
\text{rank}(\hat{X}, \epsilon_2) = \text{rank}(X^*, \epsilon_2), \\
\text{rank}(\hat{Z}, \epsilon_2) = \text{rank}(Z^*, \epsilon_2).
\]

**Remark 4.1.** We use a criteria that satisfies the definition of the numerical \(\epsilon\)-rank defined in \(\Omega\) to determine the numerical rank, which is stated as the following: The numerical rank of a matrix \(A\) in \(\mathbb{R}^{m \times n}\) with tolerance \(\epsilon\), denoted by \(\text{rank}(A, \epsilon)\), is \(r\), if the singular values, \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0\), of matrix \(A\) satisfy \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq \epsilon > \sigma_{r+1} \geq \cdots \geq \sigma_{\min(m,n)}\). For more discussion, see \(\Sigma\) and \(\Pi\).

We call \((X^*, y_0^*, y_1^*, y_2^*)\) a pair of purified \((\epsilon_1, \epsilon_2)\)-approximate optimal solutions of \(\text{SP}\) and \(\text{SD}\), respectively.

**Definition 4.2.** For \(X^* \) and \((Z^*, y_0^*, y_1^*, y_2^*)\), a given pair of purified \((\epsilon_1, \epsilon_2)\)-approximate optimal solutions of \(\text{SP}\) and \(\text{SD}\), respectively, we say that this pair has Property I\(^*\)\((\epsilon_2)\) if:

1. \(y_1^* > \epsilon_2\) and \(y_2^* > \epsilon_2\);
2. \(\text{rank}(Z^*, \epsilon_2) = n - 1\);
3. \(\text{rank}(X^*, \epsilon_2) = 2\) and there is a rank-one decomposition \(X^* = x_1^*(x_1^*)^T + x_2^*(x_2^*)^T\) such that
   \[
   |M(q_1) \cdot x_i^*(x_i^*)^T| < \epsilon_2, \quad i \in \{1, 2\},
   \]
   \[
   M(q_2) \cdot x_1^*(x_1^*)^T > \epsilon_2, \\
   M(q_2) \cdot x_2^*(x_2^*)^T < -\epsilon_2;
   \]
4. \(\epsilon_2 > 0\).
Now, we introduce a polynomial-time algorithm to test the optimality gap for the relaxation (SP), with \( \epsilon_1 \)-precision SDP solutions \( \hat{X} \) and \(( \hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \) as well as Property I\(^+\)(\( \epsilon_2 \)).

**Algorithm 4.1** Optimality gap test

**Input:** \( \epsilon_1, \epsilon_2, Q_0, Q_1, Q_2, y_0, q_1, q_2, c_1, c_2 \)

1. Solve (SP) and (SD) for \( \hat{X} \) and \(( \hat{Z}, \hat{y}_0, \hat{y}_1, \hat{y}_2) \), respectively, using an SDP solver with \( \epsilon_1 \)-precision
2. Compute the purified \(( \epsilon_1, \epsilon_2)\)-approximate optimal pair of solutions \( X^* \) and \(( Z^*, y_0^*, y_1^*, y_2^*) \)
3. Conduct a rank-one decomposition on \( X^* \) and check if Property I\(^+\)(\( \epsilon_2 \)) is satisfied
4. If Property I\(^+\)(\( \epsilon_2 \)) is satisfied then
5. an optimality gap exists
6. else
7. there is no optimality gap
8. end if

By Remark 2.7, the test described by Algorithm 4.1 can also be applied to check whether a duality gap exists for (QP). Note that it could be difficult to check the existence of the duality gap using other methods, for example, solving for the global minimum of the primal problem. The primal problem is nonconvex, though smooth. Hence, to find the global minimum, a good initial point is necessary for the convergence of a gradient-based algorithm. However, to the best of the authors’ knowledge, there is no efficient method to pick an initial point for convergence to the global minimum.

We have implemented Algorithm 4.1 in a MATLAB script which is available online [5]. We invite the interested readers to use this script with their own QC2QP problem data.

We provide numerical examples which contain two nonconvex QC2QPs, in order to illustrate the test described in Algorithm 4.1. The optimality gap does not exist in the first example but does in the second one. Note that Theorem 2.9 cannot be applied to the optimality gap test because the constraints in both examples are nonconvex, which violates the assumption in [1] which requires at least one of the constraints to be strictly convex.

To solve SDPs, we use CVX [9][10] with solver SDPT3 [17] and the default tolerance \( \epsilon_1 = 1.49 \times 10^{-8} \). We set the purification tolerance \( \epsilon_2 \) to \( 1 \times 10^{-5} \). The same solver and tolerances have been applied to the numerical experiment which displays the proportion of the randomly generated feasible nonconvex QC2QP instances of which there is no optimality gap.

### 4.1 Example: (there is no optimality gap)

Consider the following data in (QP0):

\[
Q_0 = \begin{bmatrix} 2 & -4 \\ -4 & -2 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4 & -5 \\ -5 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix},
\]

\[
b_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad c_1 = -1, \quad c_2 = -4.
\]
Both the objective function and constraints are hyperbolic and nonconvex. Assumption 2.1 and Assumption 2.3 are verified to hold in this example. Solving (SP) and (SD), we obtain the purified $(\epsilon_1, \epsilon_2)$-approximate optimal solutions:

$$X^* \approx \begin{bmatrix} 1.000000 & -0.7547192 & -3.9916123 \\ -0.7547192 & 0.5696011 & 3.0125464 \\ -3.9916123 & 3.0125464 & 15.9329684 \end{bmatrix},$$  \hspace{1cm} (4.14)

$$Z^* \approx \begin{bmatrix} 45.5612496 & 0.3855596 & 11.3413460 \\ 0.3855596 & 2.7711193 & -0.4273607 \\ 11.3413460 & -0.4273607 & 2.9220980 \end{bmatrix},$$  \hspace{1cm} (4.15)

$$y_0^* \approx -54.8271062, \hspace{0.5cm} y_1^* \approx 0.1927798, \hspace{0.5cm} y_2^* \approx 2.2682692,$$  \hspace{1cm} (4.16)

where $\text{rank}(X^*, \epsilon_2) = 1$, $\text{rank}(Z^*, \epsilon_2) = 2$, and a rank-one decomposition yields $X^* = x^*(x^*)^T$ such that

$$x^* \approx \begin{bmatrix} -1.000000 \\ -0.7547192 \\ -3.9916123 \end{bmatrix}^T.$$

It can be easily verified that

$$|M(q_1) \cdot x^*(x^*)^T| < \epsilon_2,$$  \hspace{1cm} (4.18)

$$|M(q_2) \cdot x^*(x^*)^T| < \epsilon_2.$$  \hspace{1cm} (4.19)

Denote the normalized solutions by $\hat{z}$, where $\hat{z} = z^*/t^*$ follows the partition of $x^*$ in the form of (2.2). Thus,

$$\hat{z} \approx \begin{bmatrix} -0.7547192 \\ -3.9916123 \end{bmatrix}^T$$  \hspace{1cm} (4.20)

is marked in Figure 1.

Since Property I$^\epsilon(\epsilon_2)$ is violated, we claim that there is no optimality gap. This can be verified since the primal optimal solution

$$z^+ \approx \begin{bmatrix} -0.7547192 \\ -3.9916123 \end{bmatrix}^T$$  \hspace{1cm} (4.21)

coincides with $\hat{z}$. The globally optimal value is $-54.8271061$, which is identical to that of (SP).

4.2 Example: (there is an optimality gap)

Consider the following data in (QP0):

$$Q_0 = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}, \hspace{0.5cm} Q_1 = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}, \hspace{0.5cm} Q_2 = \begin{bmatrix} 4 & 5 \\ 5 & 1 \end{bmatrix},$$  \hspace{1cm} (4.22)

$$b_0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \hspace{0.5cm} b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \hspace{0.5cm} b_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \hspace{0.5cm} c_1 = -2, \hspace{0.5cm} c_2 = 4.$$  \hspace{1cm} (4.23)

Both the objective function and constraints are hyperbolic and nonconvex. Assumption 2.1 and Assumption 2.3 are verified to hold in this example. Solving (SP) and
Figure 1: Illustration of the numerical example in which there is no optimality gap. The contour plot represents the objective function, while the filled region represents the feasible set. The primal optimum $z^+$ coincides with the normalized solution $\hat{z}$ from (SP).

where rank($X^*$, $\epsilon_2$) = 2, rank($Z^*$, $\epsilon_2$) = 1, and a rank-one decomposition yields $X^* = x_1^*(x_1^*)^T + x_2^*(x_2^*)^T$ such that

$$x_1^* \approx \begin{bmatrix} 0.1712233 & -0.9403767 & -1.2494816 \end{bmatrix}^T,$$

$$x_2^* \approx \begin{bmatrix} 0.9852322 & 1.1766611 & -1.0835160 \end{bmatrix}^T.$$

It can be easily verified that

$$|M(q_1) \cdot x_i^*(x_i^*)^T| < \epsilon_2, \quad i \in \{1, 2\},$$

$$M(q_2) \cdot x_1^*(x_1^*)^T > \epsilon_2,$$

$$M(q_2) \cdot x_2^*(x_2^*)^T < -\epsilon_2,$$

$$|M(q_1) \cdot x_1^*(x_2^*)^T| > \epsilon_2.$$
Denote the normalized solutions by \( \hat{z}_1 \) and \( \hat{z}_2 \), where \( \hat{z}_1 = z_1^* / t_1^* \) and \( \hat{z}_2 = z_2^* / t_2^* \) follow the partition of \( x_1^* \) and \( x_2^* \), respectively, in the form of (2.2). Thus,

\[
\hat{z}_1 \approx \begin{bmatrix} -5.4921056 \\ -7.2973787 \end{bmatrix}^T, \tag{4.33}
\]

\[
\hat{z}_2 \approx \begin{bmatrix} 1.1942982 \\ -1.0997569 \end{bmatrix}^T, \tag{4.34}
\]

are marked in Figure 2.

Since Property I\((\epsilon_2)\) is met, we claim that there is an optimality gap. This can be verified since the primal optimal solution

\[
z^+ \approx \begin{bmatrix} 0.5251114 \\ -0.3446140 \end{bmatrix}^T, \tag{4.35}
\]

which is marked in Figure 2 yields the globally optimal value \(-1.5335857\), whereas the optimal value of \( \text{SP} \) is \(-3.1269177\).

Figure 2: Illustration of the numerical example in which there is an optimality gap. The contour plot represents the objective function, while the filled region represents the feasible set.

### 4.3 Numerical experiment

We conduct a numerical experiment to determine the proportion of randomly generated feasible nonconvex QCQPs that has no optimal gap (and hence can be solved by the semidefinite relaxation \(\text{SP}\) using the test described by Algorithm 4.1). For a specific positive integer \( n \), we generate the matrix \( M_i \) in \( \mathbb{R}^{(n+1)\times(n+1)} \) whose entries are uniformly distributed in \([0, 1]\) and compute the problem data \( M(q_i) \) by

\[
M(q_i) = (M_i + M_i^T) / 2, \quad i \in \{0, 1, 2\}. \tag{4.36}
\]

However, we only keep the problem data that satisfy the following qualifications while discarding the rest:
1. Each of the matrices $Q_1$ and $Q_2$ (corresponding to $M(q_1)$ and $M(q_2)$, respectively, in (2.1)) must have at least one negative eigenvalue.

2. The data $M(q_0)$, $M(q_1)$, and $M(q_2)$ must hold Slater’s condition for (SP) and (SD).

Qualification 1 yields two consequences. First, the problem data characterize non-convex QC2QP instances. Second, neither $Q_1$ or $Q_2$ is positive definite so that Theorem 2.9 is excluded for testing the optimality gap (or, equivalently, for testing the duality gap). Qualification 2 makes sure that Assumption 2.1 and Assumption 2.3 are met.

For each dimension $n$, we generate 1000 instances of problem data (all satisfying the above qualifications and available online [5]), test them using Algorithm 4.1, and count the number of instances which do not have an optimality gap. The result is summarized in Table 1. It is interesting to note that most of the randomly generated feasible instances do not have an optimality gap and hence can be solved using a solution of the relaxation, which indicates the usefulness of the test in applications other than that of our original motivation [6].

| $n$ | 2   | 3   | 4     | 5     | 6     | 7     |
|-----|-----|-----|-------|-------|-------|-------|
| $m$ | 855 | 780 | 775   | 760   | 744   | 747   |

Table 1: Result of the numerical experiment. The integer $n$ represents the problem dimension while the integer $m$ represents the number of instances that have no optimality gap out of a total of 1000.

5 On why Theorem 3.2 is an extension of Theorem 2.9

Theorem 3.2 is an extension of Theorem 2.9 in the following aspects: First, Theorem 3.2 characterizes the necessary and sufficient condition for an optimality gap (or, equivalently, for a duality gap) under weaker assumptions than Theorem 2.9 (see the discussion in Remark 2.6). Second, the necessary and sufficient condition in Theorem 3.2 involves an extra condition in Property I$^+$ compared with Property I which is required by Theorem 2.9.

On the other hand, when $Q_1 \succ 0$, Theorem 3.2 coincides with Theorem 2.9 because the extra condition in Property I$^+$, $M(q_1) \cdot \hat{x}_1\hat{x}_2^T \neq 0$, is redundant when Property I holds. We show the redundancy in the following proposition:

**Proposition 5.1.** Consider (QP) where Slater’s condition holds and $Q_1 \succ 0$. Let $X = (\tilde{Z}, y_0, y_1, y_2)$ denote a pair of optimal solutions for (SP) and (SD), respectively. Suppose rank($X$) = 2, $\tilde{y}_1\tilde{y}_2 \neq 0$, and there exists a rank-one decomposition $X = \tilde{x}_1\tilde{x}_1^T + \tilde{x}_2\tilde{x}_2^T$ such that $M(q_1) \cdot \tilde{x}_1\tilde{x}_2^T = M(q_1) \cdot \tilde{x}_2\tilde{x}_1^T = 0$. Then $M(q_1) \cdot \tilde{x}_1\tilde{x}_2^T \neq 0$.

**Proof.** First, we conduct a change of coordinates to make

$$M(q_1) = \begin{bmatrix} -1 & 0_{1 \times n} \\ 0_{n \times 1} & I_n \end{bmatrix}.$$
Adopting the partition of $x$ in (2.2), we apply the following linear transformation in the change of coordinates:

\[
\begin{bmatrix}
\hat{t} \\
\hat{z}
\end{bmatrix} = R \begin{bmatrix}
t \\
z
\end{bmatrix},
\]

where $\hat{t}$ and $t$ are real numbers, $\hat{z}$ and $z$ are $n$-dimensional vectors, and

\[
R := \begin{bmatrix}
(b_1^T Q_1^{-1} b_1 - c_1)^{1/2} & 0_{1 \times n} \\
Q_1^{-1/2} b_1 & Q_1^{1/2}
\end{bmatrix}.
\]

The vector $\begin{bmatrix} \hat{t} \\ \hat{z} \end{bmatrix}$ is a partition of the new variable $\hat{x}$ after the change of coordinates. Hence,

\[
\hat{q}_1(\hat{x}) := \hat{z}^T M(q_1) \hat{x}
\]

\[
= \begin{bmatrix} \hat{t} \\ \hat{z} \end{bmatrix}^T \begin{bmatrix} -I_n & 0_{n \times 1} \\ 0_{n \times 1} & I_n \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{z} \end{bmatrix}
\]

\[
= \begin{bmatrix} \hat{t} \\ \hat{z} \end{bmatrix}^T \begin{bmatrix} Q_1^{-1/2} b_1 \\ Q_1^{1/2} \end{bmatrix} R \begin{bmatrix} \hat{t} \\ \hat{z} \end{bmatrix}
\]

\[
= \hat{z}^T Q_1 z + 2 b_1^T Q_1 z + t^2 c_1 = q_1(x).
\]

By contradiction, assume $M(q_1) \cdot \hat{x}_1 \hat{x}_2^T = 0$. Hence, adopting the partition of $\hat{x}_1$ and $\hat{x}_2$ in the form of (2.2) and using the $M(q_1)$ in (5.4), we have

\[
\begin{cases}
M(q_1) \cdot \hat{x}_1 \hat{x}_1^T = 0, \\
M(q_1) \cdot \hat{x}_2 \hat{x}_2^T = 0, \\
M(q_1) \cdot \hat{x}_1 \hat{x}_2^T = 0,
\end{cases}
\Rightarrow \begin{cases}
\hat{z}_1^T \hat{z}_1 = \hat{t}_1^2, \\
\hat{z}_2^T \hat{z}_2 = \hat{t}_2^2, \\
\hat{z}_1^T \hat{z}_2 = \hat{t}_1 \hat{t}_2,
\end{cases}
\Rightarrow (\hat{z}_1^T \hat{z}_1)(\hat{z}_2^T \hat{z}_2) = (\hat{z}_1^T \hat{z}_2)^2. \tag{5.5}
\]

By Cauchy–Schwarz inequality, the last equality implies $\hat{z}_1$ and $\hat{z}_2$ are linearly dependent, and so are $\hat{x}_1$ and $\hat{x}_2$, which contradicts the fact that rank($\hat{X}$) = 2 and $\hat{X} = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T$.

\section{Appendix}

In order to prove Theorem 3.2, we shall use the following result:

\textbf{Lemma 6.1} (Theorem 2.4 of [13]). Let $X = VV^T$ be a solution of (SP), where $V$ is a matrix in $\mathbb{R}^{n \times r}$ and $r = \text{rank}(X)$. Define a linear mapping $A_V : S^r \to \mathbb{R}^r$ as

\[
A_V(\Delta) = \begin{bmatrix}
V^T M(q_1) V \cdot \Delta \\
V^T M(q_2) V \cdot \Delta \\
(V^T I_0 V) \cdot \Delta
\end{bmatrix}.
\]

Then $X$ is the unique solution of (SP) if and only if

1. $X$ has the maximum rank among all solutions;
2. $\mathcal{N}(A_V) = \{0_{r \times r}\}$.

\textbf{Proof of Theorem 3.2}. \textbf{Necessity}: We first show $\hat{X}$ is the unique solution of (SP) using Lemma 6.1. Thus, $\mathcal{V}_{\hat{X}}^{\text{SP}} < \mathcal{V}_{\hat{X}}^{\text{SP}}$ is a trivial consequence.
Let $\hat{X}$ denote an optimal solution of $\text{SP}$. The Sylvester’s Inequality and the complementary condition $2.4$ imply that
\[
\operatorname{rank}(\hat{X}) + \operatorname{rank}(\hat{Z}) - (n + 1) \leq \operatorname{rank}(\hat{X}\hat{Z}) = 0 \Rightarrow \operatorname{rank}(\hat{X}) \leq 2,
\] (6.2)
that is, the maximum rank of an optimal solution of $\text{SP}$ is two. Since the rank of $\hat{X}$ is two already, in order to show that $\hat{X}$ is the unique solution of $\text{SP}$, we only need to show $\mathcal{N}(A_V) = \{0_{n, 2}\}$, where $V$ is a matrix in $\mathbb{R}^{(n+1) \times 2}$ and is defined as $\hat{X} = VV^T$. Equivalently, we need to show that the solution $\Delta$, which is a $2 \times 2$-dimensional symmetric matrix, of the following equation
\[
\begin{bmatrix}
(V^T M(q_1) V) \cdot \Delta \\
(V^T M(q_2) V) \cdot \Delta \\
(V^T I_{00} V) \cdot \Delta
\end{bmatrix} = 0_{3 \times 1}
\] (6.3)
is $\Delta = 0_{2 \times 2}$ only. Consider the following partitions:
\[
V = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix}, \quad \hat{x}_1 = \begin{bmatrix} \hat{t}_1 \\
\hat{z}_1 \end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix} \hat{t}_2 \\
\hat{z}_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\
\Delta_2 & \Delta_3 \end{bmatrix},
\] (6.4)
where $\hat{z}_1$ and $\hat{z}_2$ are $n$-dimensional real vectors and $\hat{t}_1, \hat{t}_2, \Delta_1, \Delta_2, \text{ and } \Delta_3$ are real numbers. Since $\hat{y}_1\hat{y}_2 \neq 0$, the inequality constraints in $\text{SP}$ are all active at $\hat{X}$. Especially,
\[
M(q_2) \cdot \hat{X} = M(q_2) \cdot \hat{x}_1\hat{x}_1^T + M(q_2) \cdot \hat{x}_2\hat{x}_2^T = 0.
\] (6.5)
Let $\alpha \neq 0$ be such that
\[
\alpha = M(q_2) \cdot \hat{x}_1\hat{x}_1^T = -M(q_2) \cdot \hat{x}_2\hat{x}_2^T
\] (6.6)
and
\[
\Gamma := \begin{bmatrix} 0 & 2M(q_1) \cdot \hat{x}_1\hat{x}_1^T & 0 \\
\alpha & 2M(q_2) \cdot \hat{x}_1\hat{x}_1^T & -\alpha \\
\hat{t}_1 & 2\hat{t}_1 \hat{t}_2 & \hat{t}_2 \end{bmatrix}.
\] (6.7)
Therefore, (6.3) is a linear equation of $\Delta_1, \Delta_2$ and $\Delta_3$ as
\[
\begin{bmatrix}
(V^T M(q_1) V) \cdot \Delta \\
(V^T M(q_2) V) \cdot \Delta \\
(V^T I_{00} V) \cdot \Delta
\end{bmatrix} = 0_{3 \times 1} \Rightarrow \Gamma \begin{bmatrix} \Delta_1 \\
\Delta_2 \\
\Delta_3 \end{bmatrix} = 0_{3 \times 1}.
\] (6.8)
Notice that $I_{00} \cdot \hat{X} = 1$ implies $\hat{t}_1^2 + \hat{t}_2^2 = 1$. Now, the matrix $\Gamma$ is full rank because
\[
det(\Gamma) = -(2M(q_1) \cdot \hat{x}_1\hat{x}_1^T)(\alpha(\hat{t}_1^2 + \hat{t}_2^2)) \neq 0.
\] (6.9)
So $\Delta_1, \Delta_2$, and $\Delta_3$ are all zero and the only solution of $\text{(6.3)}$ is $\Delta = 0_{2 \times 2}$. By Lemma 6.1, $\hat{X}$ is the unique solution of $\text{SP}$.

Let $z^*$ denote an optimal solution of $\text{QP}_0$. Hence, $\left[\begin{bmatrix} 1 \\
\hat{z} \end{bmatrix}\right]^T$ is an optimal solution of $\text{QP}$ and $\begin{bmatrix} 1 \\
\hat{z} \end{bmatrix}\begin{bmatrix} 1 \\
\hat{z} \end{bmatrix}^T$ is feasible to $\text{SP}$. The corresponding optimal value of $\text{QP}$ is
\[
M(q_0) \cdot \begin{bmatrix} 1 \\
\hat{z} \end{bmatrix}\begin{bmatrix} 1 \\
\hat{z} \end{bmatrix}^T = (z^*)^T Q_0 z^* + 2b^T_0 z^* = \mathcal{V}_{\text{QP}}.
\] (6.10)
Since $\hat{X}$ is the unique solution to $\text{SP}$, we conclude that $\mathcal{V}_{\text{QP}} > \mathcal{V}_{\text{SP}}$.

**Sufficiency:** We prove by contraposition. We will enumerate five exhaustive (but not mutually exclusive) possibilities, denoted by Case $i$, with $i \in \{1, 2, 3, 4, 5\}$. 


Case 1. \( \hat{y}_1 \hat{y}_2 = 0 \).

The proof showing that there is no optimality gap in this case can be found in \[13\].

Case 2. \( \hat{y}_1 \hat{y}_2 \neq 0 \) and \( \text{rank}(\hat{X}) \neq 2 \).

The condition \( \hat{y}_1 \hat{y}_2 \neq 0 \) implies that, by the complementary condition (2.1),

\[
(M(q_1)) \cdot \hat{X} = M(q_2) \cdot \hat{X} = 0.
\]

(6.11)

Let \( r = \text{rank}(\hat{X}) \). Thus, \( r > 0 \) because \( I_{00} \cdot \hat{X} = 1 \). If \( r = 1 \), then the rank-one decomposition of \( \hat{X} = \hat{x} \hat{x}^T \) provides an optimal solution to (QP). And we do not consider the case \( r \geq 3 \) because (SP) satisfies Slater’s condition and hence (SP) is solvable, and hence \( r \leq 2 \) by Theorem 2.1 of [13].

Case 3. \( \hat{y}_1 \hat{y}_2 \neq 0 \), \( \text{rank}(\hat{X}) = 2 \), and \( M(q_1) \cdot \hat{x}_1 \hat{x}_1^T = M(q_2) \cdot \hat{x}_1 \hat{x}_1^T = 0 \) for \( i \) being 1 and 2.

We adopt the partition of \( \hat{x}_1 \) and \( \hat{x}_2 \) in (6.4). Consequently, \( I_{00} \cdot \hat{X} = 1 \) implies that \( \hat{t}_1^2 + \hat{t}_2^2 = 1 \), that is, at least one of \( \hat{t}_1 \) and \( \hat{t}_2 \) is nonzero. Without loss of generality, assume \( \hat{t}_1 \neq 0 \). Hence, \( \hat{x}_1 \hat{x}_1^T / \hat{t}_1^2 \) is a solution of (SP) due to (2.7). Therefore, \( \hat{x}_1 / \hat{t}_1 \) is a homogenized solution of (QP).

Case 4. \( \hat{y}_1 \hat{y}_2 \neq 0 \), \( \text{rank}(\hat{X}) = 2 \), and \( M(q_1) \cdot \hat{x}_2 \hat{x}_2^T = M(q_2) \cdot \hat{x}_2 \hat{x}_2^T = 0 \) for \( i \) being 1 and 2.

Since

\[
\text{rank}(\hat{Z}) + \text{rank}(\hat{X}) \leq n + 1,
\]

(6.12)

\[
\text{rank}(\hat{X}) = 2,
\]

(6.13)

\[
\text{rank}(\hat{Z}) \neq n - 1,
\]

(6.14)

it follows that

\[
\text{rank}(\hat{Z}) < n - 1,
\]

(6.15)

and hence

\[
\text{rank}(\hat{Z}) + \text{rank}(\hat{X}) < n + 1.
\]

(6.16)

Now \( \hat{X} + \hat{Z} \) is singular because \( \text{rank}(\hat{X} + \hat{Z}) \leq \text{rank}(\hat{X}) + \text{rank}(\hat{Z}) \). Also, both \( \hat{X} \) and \( \hat{Z} \) are positive semidefinite. So there must be a nontrivial \((n + 1)\)-dimensional real vector \( y \) in the intersection of the null space of \( \hat{X} \) and the null space of \( \hat{Z} \). Let

\[
X := \hat{X} + yy^T = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T + yy^T.
\]

(6.17)

Obviously, \( \text{rank}(X) = 3 \) and \( \hat{Z}X = 0_{(n+1) \times (n+1)} \) because \( \hat{Z} \hat{X} = 0_{(n+1) \times (n+1)} \). By Lemma 3.3 of [11], we know there exists an \((n + 1)\)-dimensional real vector \( x \) such that \( X \) is rank-one decomposable at \( x \) and that

\[
M(q_1) \cdot xx^T = M(q_2) \cdot xx^T = 0.
\]

(6.18)

Since \( x \) is in the range space of \( X \), it must be in the null space of \( \hat{Z} \). Therefore, \( \hat{Z} \cdot xx^T = 0 \) and the complementary condition (2.7) implies that \( xx^T / t^2 \) is an optimal solution of (QP), where \( t \) comes from the partition of \( x \) in (2.2). Hence, \( x/t \) is an optimal solution of (QP). Note that \( t \neq 0 \) for the following reason: By contradiction, assume \( t = 0 \). Thus, \( \hat{Z} \cdot xx^T = 0 \) implies

\[
z^T(Q_0 + \hat{y}_1Q_1 + \hat{y}_2Q_2)z = 0,
\]

(6.19)

\footnote{As a convention, a matrix \( X \) is rank-one decomposable at \( x_1 \) if there exist other \( r - 1 \) vectors \( x_2, \ldots, x_r \) such that \( X = x_1x_1^T + x_2x_2^T + \cdots + x_rx_r^T \), where \( r := \text{rank}(X) \) [11].}
where \( z \) comes from the partition of \( x \) in (2.2). On the other hand, by (6.18), we have

\[
z^T Q_1 z = z^T Q_2 z = 0.
\]  

(6.20)

Combining (6.19) and (6.20), we know \( z^T Q_0 z = 0 \). The only \( z \) that satisfies \( z^T Q_0 z, z^T Q_1 z, \) and \( z^T Q_2 z \) all being zero is \( z = 0_{n \times 1} \) for the following reason: By contradiction, assume there exists a nonzero \( n \)-dimensional real vector \( z \) such that \( z^T Q_0 z, z^T Q_1 z, \) and \( z^T Q_2 z \) are all zero. Therefore, for all \( \tilde{y}_1, \tilde{y}_2 > 0 \), we have

\[
z^T (Q_0 + \tilde{y}_1 Q_1 + \tilde{y}_2 Q_2) z = 0.
\]  

(6.21)

This equality contradicts Slater’s condition (2.6a) of (SD). Hence, we have \( z = 0_{n \times 1} \) which implies \( x = 0_{(n+1) \times 1} \). However, the vector \( x \) being zero is a contradiction because, by the definition of rank-one decomposability [1], \( x \neq 0_{(n+1) \times 1} \) must hold. So we have proved \( t \neq 0 \).

**Case 5.** \( \tilde{y}_1 \tilde{y}_2 \neq 0 \), \( \text{rank}(\tilde{Z}) = n - 1 \), \( \text{rank}(\tilde{X}) = 2 \), \( M(q_1) \cdot \hat{x}_i \hat{x}^T_i = 0 \) for \( i \) being 1 and 2, \( M(q_2) \cdot \hat{x}_1 \hat{x}^T_1 (M(q_2) \cdot \hat{x}_2 \hat{x}^T_2) < 0 \), and \( M(q_1) \cdot \hat{x}_1 \hat{x}^T_1 = 0 \).

We show that we can obtain another rank-one decomposition of \( \tilde{X} = \tilde{x}_1 \tilde{x}^T_1 + \tilde{x}_2 \tilde{x}^T_2 \) such that

\[
M(q_1) \cdot \hat{x}_i \hat{x}^T_i = M(q_2) \cdot \hat{x}_i \hat{x}^T_i = 0, \quad i \in \{1, 2\}.
\]  

(6.22)

This is achievable because any rank-one decomposition of \( \tilde{X} \) must be a linear combination of \( \tilde{x}_1 \) and \( \tilde{x}_2 \). And \( M(q_1) \cdot \hat{x}_1 \hat{x}^T_1 = 0 \) together with \( M(q_1) \cdot \hat{x}_1 \hat{x}^T_1 = M(q_1) \cdot \hat{x}_2 \hat{x}^T_2 = 0 \) indicate that an arbitrary linear combination \( \tilde{x} \) of \( \tilde{x}_1 \) and \( \tilde{x}_2 \), that is, \( \tilde{x} = \alpha_1 \tilde{x}_1 + \alpha_2 \tilde{x}_2 \)

(6.23)

for real numbers \( \alpha_1 \) and \( \alpha_2 \), satisfies \( M(q_1) \cdot \tilde{x} \tilde{x}^T = 0 \). Hence, what remains to be done is to obtain \( \hat{x}_1 \) and \( \hat{x}_2 \) which are both linear combinations of \( \tilde{x}_1 \) and \( \tilde{x}_2 \), where we only require

\[
M(q_2) \cdot \hat{x}_i \hat{x}^T_i = 0, \quad i \in \{1, 2\}.
\]  

(6.24)

To compute such \( \hat{x}_1 \) and \( \hat{x}_2 \), we follow the procedure in Remark 2.10 where \( x_i^r \), \( M(q_1) \), and \( \hat{x}_i \) are replaced by \( \tilde{x}_i, M(q_2), \) and \( \hat{x}_i \), respectively, for \( i \) being 1 and 2. Thus, the new rank-one decomposition \( \tilde{X} = \tilde{x}_1 \tilde{x}^T_1 + \tilde{x}_2 \tilde{x}^T_2 \) satisfies

\[
M(q_1) \cdot \hat{x}_i \hat{x}^T_i = M(q_2) \cdot \hat{x}_i \hat{x}^T_i = 0, \quad i \in \{1, 2\}.
\]  

(6.25)

The rest of this case continues in Case 3.

**Remark 6.2.** The proof of Theorem 2.6 of [13] also shows the uniqueness of the solution of (SD) in the CDT subproblem when Property I holds. The proof uses a property of the boundary points of an SOC whereas we use a result on the uniqueness of the solution of a semidefinite program.

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References

[1] Wenbao Ai and Shuzhong Zhang. Strong duality for the CDT subproblem: a necessary and sufficient condition. *SIAM Journal on Optimization*, 19(4):1735–1756, 2009.

[2] Amir Beck and Yonina C. Eldar. Strong duality in nonconvex quadratic optimization with two quadratic constraints. *SIAM Journal on Optimization*, 17(3):844–860, 2006.

[3] Immanuel M. Bomze and Michael L. Overton. Narrowing the difficulty gap for the Celis–Dennis–Tapia problem. *Mathematical Programming*, 151(2):459–476, 2015.

[4] Rosa M. Celis, John E. Dennis, and Richard A. Tapia. A trust region strategy for nonlinear equality constrained optimization. *Numerical Optimization*, 1984:71–82, 1985.

[5] Sheng Cheng. QC2QP-SDR-Optimality-Gap-Test. [https://github.com/Sheng-Cheng/QC2QP-SDR-Optimality-Gap-Test](https://github.com/Sheng-Cheng/QC2QP-SDR-Optimality-Gap-Test), 2019.

[6] Sheng Cheng and Nuno C. Martins. Reaching a target in a time-costly area using a two-stage optimal control method. In *Proceedings of American Control Conference (ACC)*, 2019.

[7] Alexander L. Fradkov and Vladimir A. Yakubovich. The S-procedure and the duality relation in convex quadratic programming problems. *Vestnik Leningrad. Univ.*, 1(1):81–87, 1973.

[8] Gene H. Golub and Charles F. Van Loan. *Matrix computations*, volume 3. Johns Hopkins University Press, Baltimore, MD, 2012.

[9] Michael Grant and Stephen Boyd. Graph implementations for nonsmooth convex programs. In V. Blondel, S. Boyd, and H. Kimura, editors, *Recent Advances in Learning and Control*, Lecture Notes in Control and Information Sciences, pages 95–110. Springer-Verlag Limited, 2008. [http://stanford.edu/~boyd/graph_dcp.html](http://stanford.edu/~boyd/graph_dcp.html).

[10] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. [http://cvxr.com/cvx](http://cvxr.com/cvx) March 2014.

[11] Per C. Hansen. *Rank-deficient and discrete ill-posed problems*. SIAM, Philadelphia, PA, 1998.

[12] Yongwei Huang and Shuzhong Zhang. Complex matrix decomposition and quadratic programming. *Mathematics of Operations Research*, 32(3):758–768, 2007.

[13] Alex Lemon, Anthony Man-Cho So, and Yinyu Ye. Low-rank semidefinite programming: Theory and applications. *Foundations and Trends® in Optimization*, 2(1-2):1–156, 2016.

[14] Ji-ming Peng and Ya-xiang Yuan. Optimality conditions for the minimization of a quadratic with two quadratic constraints. *SIAM Journal on Optimization*, 7(3):579–594, 1997.

[15] Jos F. Sturm. *Primal-dual interior point approach to semidefinite programming*. Ph.D. Thesis, Erasmus University Rotterdam, Rotterdam, Nethernlands, 1997.

[16] Jos F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653, 1999.
[17] Kim C. Toh, Michael J. Todd, and Reha H. Tütüncü. SDPT3—a Matlab software package for semidefinite programming, Version 1.3. *Optimization Methods and Software*, 11(1-4):545–581, 1999.

[18] Yinyu Ye and Shuzhong Zhang. New results on quadratic minimization. *SIAM Journal on Optimization*, 14(1):245–267, 2003.

[19] Jianhua Yuan, Meiling Wang, Wenbao Ai, and Tianping Shuai. New results on narrowing the duality gap of the extended Celis–Dennis–Tapia problem. *SIAM Journal on Optimization*, 27(2):890–909, 2017.