Abstract  We investigate the metric behavior of the Kähler-Ricci flow on the Hirzebruch surfaces, assuming the initial metric is invariant under a maximal compact subgroup of the automorphism group. We show that, in the sense of Gromov-Hausdorff, the flow either shrinks to a point, collapses to \( \mathbb{P}^1 \) or contracts an exceptional divisor, confirming a conjecture of Feldman-Ilmanen-Knopf. We also show that similar behavior holds on higher-dimensional analogues of the Hirzebruch surfaces.

1 Introduction

The behavior of the Kähler-Ricci flow on a compact manifold \( M \) is expected to reveal the metric and algebraic structures on \( M \). If \( M \) is a Kähler manifold with \( c_1(M) = 0 \) then the Kähler-Ricci flow \( \frac{\partial \omega}{\partial t} = -\text{Ric} (\omega) \) starting at a metric \( \omega_0 \) in any Kähler class \( \alpha \), converges to the unique Ricci-flat metric in \( \alpha \) \([\text{Cao1}, \text{Y1}]\). If \( c_1(M) < 0 \), the normalized Kähler-Ricci flow

\[
\frac{\partial}{\partial t} \omega = \lambda \omega - \text{Ric}(\omega), \quad \omega(0) = \omega_0,
\]

with \( \lambda = -1 \) and \( \omega_0 \in \lambda c_1(M) \) converges to the unique Kähler-Einstein metric \([\text{Cao1}, \text{Y1}, \text{A}]\).

If \( c_1(M) > 0 \) then Kähler-Einstein metrics do not exist in general. If one assumes the existence of a Kähler-Einstein metric then, according to unpublished work of Perelman [P2] (see [TZhu]), the normalized Kähler-Ricci flow (1.1) with \( \lambda = 1 \) and \( \omega_0 \in \lambda c_1(M) \) converges to a Kähler-Einstein metric (this is due to [H], [Ch] in the case of one complex dimension). By a conjecture of Yau [Y2], a necessary and sufficient condition for \( M \) to admit a Kähler-Einstein metric is that \( M \) be ‘stable in the sense of geometric invariant theory’. Tian [T] later proposed the condition of \( K\)-stability and this concept has been refined and extended by Donaldson [D]. One might expect that the sufficiency part of the Yau-Tian-Donaldson conjecture can be proved via the flow (1.1). Indeed, the problem of using stability conditions to prove convergence properties of the Kähler-Ricci flow is an area of considerable current interest and we refer the reader to [PS2], [PSS], [PSSW1], [PSSW2], [R], [Sz2], [PSSW3], [To] and [CW] for some recent advances (however, this list of references is far from complete). We also remark that if \( M \) is toric, it turns out that the stability condition can be replaced by a simpler criterion involving the Futaki invariant, and the behavior of (1.1) is then well-understood [WZ], [Zhu].

There has also been much interest in understanding the behavior of the flow (1.1) with \( \lambda = -1 \) on manifolds with \( c_1(M) \leq 0 \) (and not strictly definite). In this case, smooth Kähler-Einstein metrics cannot exist and the Kähler class of \( \omega(t) \) must degenerate in the limit. If \( M \) is a minimal

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model of general type, it is shown in [Ts] and later generalized in [TZha] that the Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric. If $M$ is not of general type, the Kähler-Ricci flow collapses and converges weakly to a generalized Kähler-Einstein metric if the canonical line bundle $K_M$ is semi-ample [SoT1, SoT2].

In contrast, the case when $c_1(M)$ is nonnegative or indefinite has been little studied. In complex dimension two, it is natural to consider the rational ruled surfaces, known as the Hirzebruch surfaces and denoted $M_0, M_1, M_2, \ldots$. Indeed, all rational surfaces can be obtained from $\mathbb{P}^2$ and Hirzebruch surfaces via consecutive blow-ups. In this paper we describe several distinct behaviors of the Kähler-Ricci flow on the manifolds $M_k$. We consider the unnormalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \omega(0) = \omega_0. \quad (1.2)$$

for $\omega_0$ in any given Kähler class. Write $\omega(t) = \sqrt{-1} \sum_{i} g_i dz^i \wedge d\bar{z}^i$, for $g = g(t)$ the Kähler metric associated to $\omega(t)$. We find that, in the Gromov-Hausdorff sense, the flow $g(t)$ may: shrink the manifold to a point, collapse to a lower dimensional manifold or contract a divisor on $M_k$. As we will see, the particular outcome depends on $k$ and the initial Kähler class of $\omega_0$. Much of this behavior was conjectured by Feldman, Ilmanen and Knopf in their detailed analysis [FIK] of self-similar solutions of the Ricci flow. We confirm the Feldman-Ilmanen-Knopf conjectures, under the assumption that the initial metric is invariant under a maximal compact subgroup of the automorphism group.

Our results in this paper give some evidence that the Kähler-Ricci flow may indeed provide an analytic approach to the classification theory of algebraic varieties as suggested in [SoT2]. In general, if the canonical line bundle $K_M$ is not nef, the unnormalized Kähler-Ricci flow (1.2) must become singular at some finite time, say $T$. If the limiting Kähler class is big as $t \to T$, a number of conjectures about the behavior of the flow have been made in [SoTT, SoT2]. It is conjectured that the limiting Kähler metric has a metric completion $(M', d_{T})$, where $M'$ is an algebraic variety obtained from $M$ by an algebraic procedure such as a divisorial contraction or flip. It is further proposed in [SoT3] that $(M, \omega(t))$ should converge to $(M', d_{T})$ in the sense of Gromov-Hausdorff. Our main result in this paper confirms this speculation in the case of $\mathbb{P}^2$ blown up at one point (with any initial Kähler class) and a family of higher-dimensional analogues of Hirzebruch surfaces if the initial Kähler metric is invariant under a maximal compact subgroup of the automorphism group.

The Hirzebruch surfaces $M_0, M_1, \ldots$ are projective bundles over $\mathbb{P}^1$ which can be described as follows. Write $H$ and $\mathbb{C}_{\mathbb{P}^1}$ for the hyperplane line bundle and trivial line bundle respectively over $\mathbb{P}^1$. Then we define the Hirzebruch surface $M_k$ to be

$$M_k = \mathbb{P}(H^k \oplus \mathbb{C}_{\mathbb{P}^1}). \quad (1.3)$$

One can check that $M_0$ and $M_1$ are the only Hirzebruch surfaces with positive first Chern class. $M_0$ is the manifold $\mathbb{P}^1 \times \mathbb{P}^1$ and will not be dealt with in this paper (see instead [PS1], [P2], [TZhu], [PSSW2], [Zhu]). $M_1$ can be identified with $\mathbb{P}^2$ blown up at one point. It is already known that the normalized Kähler-Ricci flow (1.1) with $\lambda = 1$ on $M_1$ starting at a toric metric $\omega_0$ in $c_1(M_1)$ converges to a Kähler-Ricci soliton after modification by automorphisms (see [Zhu] and also [PSSW3]). However, the manifold $M_1$ is still of interest to us since we are considering the more general case of the initial metric $\omega_0$ lying in any Kähler class.
Assume now that \( k \geq 1 \) and denote by \( D_\infty \) the divisor in \( M_k \) given by the image of the section \((0,1)\) of \( H^k \oplus \mathbb{C} \). Since the complex manifold \( M_k \) can also be described by \( \mathbb{P}(\mathbb{C} \oplus H^{-k}) \) we can define another divisor \( D_0 \) on \( M_k \) to be that given by the image of the section \((1,0)\) of \( \mathbb{C} \oplus H^{-k} \).

All of the Hirzebruch surfaces \( M_k \) admit Kähler metrics. Indeed, the cohomology classes of the line bundles \([D_0]\) and \([D_\infty]\) span \( H^{1,1}(M;\mathbb{R}) \) and every Kähler class \( \alpha \) can be written uniquely as

\[
\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0] \tag{1.4}
\]

for constants \( a, b \) with \( 0 < a < b \). If \( \alpha_t \) denotes the Kähler class of a solution \( \omega(t) \) of the flow (1.2) then a short calculation shows that the associated constants \( a_t, b_t \) satisfy

\[
b_t = b_0 - t(k + 2) \quad \text{and} \quad a_t = a_0 + t(k - 2). \tag{1.5}
\]

Our goal is to understand the behavior of the Kähler-Ricci flow with initial metric \( \omega_0 \) in any given Kähler class \( \alpha \). We focus on the case when \( \omega_0 \) is invariant under the action of a maximal compact subgroup \( G_k \cong U(2)/\mathbb{Z}_k \) of the automorphism group of \( M_k \). We will say that \( \omega_0 \) satisfies the Calabi symmetry condition. This symmetry is explained in detail in Section 2.2 and was used by Calabi \([\text{Cal}]\) to construct extremal Kähler metrics on \( M_k \) (see also \([\text{Sz}]\)). Our first result shows that under this symmetry condition we can describe the convergence of the flow \( (M, g(t)) \) in the sense of Gromov-Hausdorff.

**Theorem 1.1** On the Hirzebruch surface \( M_k \), let \( \omega = \omega(t) \) be a solution of the Kähler-Ricci flow (1.2) with initial Kähler metric \( \omega_0 \) satisfying the Calabi symmetry condition. Assume that \( \omega_0 \) lies in the Kähler class \( \alpha_0 \) given by \( a_0, b_0 \) satisfying \( 0 < a_0 < b_0 \). Then we have the following:

(a) If \( k \geq 2 \) then the flow (1.2) exists on \([0, T)\) with \( T = (b_0 - a_0)/2k \) and \( (M_k, g(t)) \) converges to \((\mathbb{P}^1, a_T g_{FS})\) in the Gromov-Hausdorff sense as \( t \to T \), where \( g_{FS} \) is the Fubini-Study metric and \( a_T \) is the constant given by (1.5).

(b) If \( k = 1 \) there are three subcases.

(i) If \( b_0 = 3a_0 \) then the flow (1.2) exists on \([0, T)\) with \( T = a_0 \) and \((M_1, g(t))\) converges to a point in the Gromov-Hausdorff sense as \( t \to T \).

(ii) If \( b_0 < 3a_0 \) then the flow (1.2) exists on \([0, T)\) with \( T = (b_0 - a_0)/2k \) and, as in (a) above, \((M_1, g(t))\) converges to \((\mathbb{P}^1, a_T g_{FS})\) in the Gromov-Hausdorff sense as \( t \to T \).

(iii) If \( b_0 > 3a_0 \) then the flow (1.2) exists on \([0, T)\) with \( T = a_0 \). On compact subsets of \( M_1 \setminus D_0 \), \( g(t) \) converges smoothly to a Kähler metric \( g_T \). If \((\overline{M}, d_T)\) denotes the metric completion of \((M_1 \setminus D_0, g_T)\), then \((M_1, g(t))\) converges to \((\overline{M}, d_T)\) in the Gromov-Hausdorff sense as \( t \to T \). \((\overline{M}, d_T)\) has finite diameter and is homeomorphic to the manifold \( \mathbb{P}^2 \).

We now make some remarks about this theorem. As mentioned above, the manifold \( M_1 \) can be identified with \( \mathbb{P}^2 \) blown up at one point. The case (b).(i) occurs precisely when the initial Kähler form \( \omega_0 \) lies in the first Chern class \( c_1(M_1) \). This situation has been well-studied and the
convergence result of (b).(i) is an immediate consequence of the diameter bound of Perelman for the normalized Kähler-Ricci flow \cite{P2, SeT}. In the case (b).(iii), we use the work of \cite{TS, TZha, Zha} to obtain the smooth convergence of the metric outside \( D_0 \). Our result shows that the Kähler-Ricci flow ‘blows down’ the exceptional curve on \( M_1 \). For more details about this see Section 5.

We see from the above that, assuming the Calabi symmetry, there are three distinct behaviors of the Kähler-Ricci flow on a Hirzebruch surface, depending on \( k \) and the initial Kähler class:

- The \( \mathbb{P}^1 \) fiber collapses (cases (a) and (b).(ii))
- The manifold shrinks to a point (case (b).(i))
- The exceptional divisor is contracted (case (b).(iii)).

We can say some more about the cases (a) and (b).(ii) when the fiber collapses. If \( D_H \) denotes any fiber of the map \( \pi : M_k \to \mathbb{P}^1 \) then the line bundle associated to \( D_H \) is given by \([D_H] = \pi^*H\). The cohomology class of \( D_H \) is represented by the smooth (1,1) form \( \chi = \pi^*\omega_{FS} \) where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{P}^1 \). We can show that in the case \( k \geq 2 \) or \( k = 1 \) with \( b_0 < 3a_0 \), the Kähler form \( \omega(t) \) along the flow converges to \( a_T \chi \) in a certain weak sense which we now explain. Define for \( 0 \leq t < T = (b_0 - a_0)/2k \) a reference Kähler metric

\[
\hat{\omega}_t = a_t \chi + \frac{(b_t - a_t)}{2k} \theta,
\]

in \( \alpha_t \), where \( \theta \) is a certain closed nonnegative (1,1) form in \( 2[D_\infty] \) (see Lemma 2.1 below). Observe that \( \hat{\omega}_t \) converges to \( a_T \chi \) as \( t \to T \). Now define a potential function \( \tilde{\varphi} = \tilde{\varphi}(t) \) by

\[
\omega(t) = \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\varphi}(t),
\]

where \( \tilde{\varphi} \) is subject to a normalization condition \( \tilde{\varphi}|_{\mu=0} = 0 \) (see Section 4). Then we have:

**Theorem 1.2** Assume that \( k \geq 2 \) or \( k = 1 \) and \( b_0 < 3a_0 \). Let \( \omega(t) \) be a solution of the flow \( (1.2) \) on \( M_k \) with \( \omega_0 \) satisfying the Calabi symmetry condition. Then for all \( \beta \) with \( 0 < \beta < 1 \),

(i) \( \tilde{\varphi}(t) \) tends to zero in \( C^{1,\beta}_{a_0}(M_k) \) as \( t \to T \).

(ii) For any compact set \( K \subset M_k \setminus (D_\infty \cup D_0) \), \( \tilde{\varphi}(t) \) tends to zero in \( C^{2,\beta}_{a_0}(K) \) as \( t \to T \). In particular, on such a compact set \( K \), \( \omega(t) \) converges to \( a_T \chi \) on \( C^{2,\beta}_{a_0}(K) \) as \( t \to T \).

In addition, we can extend our results to higher dimensions by considering \( \mathbb{P}^1 \) bundles over \( \mathbb{P}^{n-1} \) for \( n \geq 2 \). Write \( H \) and \( \mathbb{C}_{\mathbb{P}^{n-1}} \) for the hyperplane line bundle and trivial line bundle respectively over \( \mathbb{P}^{n-1} \). Then we define the \( n \)-dimensional complex manifold \( M_{n,k} \) by

\[
M_{n,k} = \mathbb{P}(H^k \oplus \mathbb{C}_{\mathbb{P}^{n-1}}).
\]

We can define divisors \( D_0 \) and \( D_\infty \) in the same manner as for \( M_k \) above. The cohomology classes \([D_0]\) and \([D_\infty]\) again span \( H^{1,1}(M; \mathbb{R}) \) (see for example \cite{GH} or \cite{IS}) and the Kähler classes are described as in \( (1.4) \). Similarly, we have a Calabi symmetry condition, where the maximal compact subgroup of the automorphism group is now \( G_k \cong U(n)/\mathbb{Z}_k \) (see Section 2.2). We have the following generalization of Theorem 1.1.
Theorem 1.3 On $M_{n,k}$, let $\omega = \omega(t)$ be a solution of the Kähler-Ricci flow (1.2) with initial Kähler metric $\omega_0$ satisfying the Calabi symmetry condition. Assume that $\omega_0$ lies in the Kähler class $\alpha_0$ given by $a_0, b_0$ satisfying $0 < a_0 < b_0$. Then we have the following:

(a) If $k \geq n$ then the flow (1.2) exists on $[0,T)$ with $T = (b_0 - a_0)/2k$ and $(M_{n,k}, g(t))$ converges to $(\mathbb{P}^{n-1}, a_T g_{FS})$ in the Gromov-Hausdorff sense as $t \to T$.

(b) If $1 \leq k \leq n - 1$ there are three subcases.

(i) If $a_0(n+k) = b_0(n-k)$ then the flow (1.2) exists on $[0,T)$ with $T = a_0$ and $(M_{n,k}, g(t))$ converges to a point in the Gromov-Hausdorff sense as $t \to T$.

(ii) If $a_0(n+k) > b_0(n-k)$ then the flow (1.2) exists on $[0,T)$ with $T = (b_0 - a_0)/2k$ and, as in (a) above, $(M_{n,k}, g(t))$ converges to $(\mathbb{P}^{n-1}, a_T g_{FS})$ in the Gromov-Hausdorff sense as $t \to T$.

(iii) If $a_0(n+k) < b_0(n-k)$ then the flow (1.2) exists on $[0,T)$ with $T = a_0/(n-k)$. On compact subsets of $M_{n,k} \setminus D_0$, $g(t)$ converges smoothly to a Kähler metric $g_T$. If $(\bar{M}, d_T)$ denotes the metric completion of $(M_{n,k} \setminus D_0, g_T)$, then $(M_{n,k}, g(t))$ converges to $(\bar{M}, d_T)$ in the Gromov-Hausdorff sense as $t \to T$. $(\bar{M}, d_T)$ has finite diameter and is homeomorphic to the orbifold $\mathbb{P}^n/\mathbb{Z}_k$ (see Section 2.3).

We also prove an analog of Theorem 1.2 in higher dimensions (see Theorem 4.2 below). Since Theorem 1.3 includes Theorem 1.1 as a special case, we will prove all of our results in this paper in the general setting of complex dimension $n$. We will often write $M$ for $M_{n,k}$.

Finally, we mention some known results about Kähler-Ricci solitons on these manifolds. For $1 \leq k \leq n - 1$, the manifold $M_{n,k}$ has positive first Chern class and admits a Kähler-Ricci soliton [Koi, Cao2]. In addition, Kähler-Ricci solitons have been constructed on the orbifolds $\mathbb{P}^n/\mathbb{Z}_k$ for $2 \leq k \leq n - 1$ [FIK] and, in the noncompact case, on line bundles over $\mathbb{C}\mathbb{P}^{n-1}$ [Cao2, FIK]. The limiting behavior of such solitons is studied and used in [FIK] to construct examples of extending the Ricci flow through singularities. Theorem 1.3 shows in particular that if the initial Kähler is not proportional to $c_1(M_{n,k})$, the Kähler-Ricci flow on $M_{n,k}$ ($1 \leq k \leq n$) will not converge to a Kähler-Ricci soliton on the same manifold after normalization.

The outline of the paper is as follows. In Section 2, we describe some background material including the details of the Calabi ansatz. In Section 3, we prove some estimates for the Kähler-Ricci flow on the manifolds $M_k$ which hold without any symmetry condition. We then impose the Calabi symmetry assumption in Section 4 to give stronger estimates, and in particular, we prove Theorem 1.2 (cf. Theorem 4.2). In Section 5, we give proofs of the Gromov-Hausdorff convergence of the flow, thus establishing Theorems 1.1 and 1.3.

2 Background

2.1 The anti-canonical bundle

Let $M = M_{n,k}$ be the manifold given by (1.8). The anti-canonical bundle $K_M^{-1}$ of $M$ can be described as follows. If $\pi : M \to \mathbb{P}^{n-1}$ is the bundle map then write $D_H = \pi^{-1}(H_{n-1})$ where
$H_{n-1}$ is a fixed hyperplane in $\mathbb{P}^{n-1}$. Then the anti-canonical line bundle $K_M^{-1}$ is given by

$$K_M^{-1} = 2[D_\infty] - (k - n)[D_H] = \frac{(k + n)}{k}[D_\infty] + \frac{(k - n)}{k}[D_0].$$

(2.1)

and we have

$$k[D_H] = [D_\infty] - [D_0].$$

(2.2)

2.2 The Calabi ansatz

We now briefly describe the ansatz of [Cal] following, for the most part, Calabi’s exposition. We use coordinates $(x_1, \ldots, x_n)$ on $\mathbb{C}^n \setminus \{0\}$. Then the manifold $\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ can be described by $n$ coordinate charts $U_1, \ldots, U_n$, where $U_i$ is characterized by $x_i \neq 0$. For a fixed $i$, the holomorphic coordinates $z_i^j$ on $U_i$, for $1 \leq j \leq n$, $j \neq i$ are given by $z_i^j = x_j/x_i$. Then $M = M_{n,k}$ can be defined as the $\mathbb{P}^1$ bundle over $\mathbb{P}^{n-1}$ with a projective fiber coordinate $y(i)$ on $\pi^{-1}(U_i)$ which transforms by

$$y(i) = \left(\frac{x_i}{x_\ell}\right)^k y(\ell), \quad \text{on } \pi^{-1}(U_i \cap U_\ell),$$

(2.3)

where $\pi : M \to \mathbb{P}^{n-1}$ denotes the bundle map. Then the divisors $D_\infty$ and $D_0$ are given by $y(i) = 0$ and $y(i) = \infty$ respectively. We parametrize $M \setminus (D_0 \cup D_\infty)$ by a $k$-to-one map $\mathbb{C}^n \setminus \{0\} \to M \setminus (D_0 \cup D_\infty)$ described as follows. The point $(x_1, \ldots, x_n)$, with $x_i \neq 0$ say, maps to the point in $M \setminus (D_0 \cup D_\infty) \cap \pi^{-1}(U_i)$ with coordinates $z_i^j = x_j/x_i$, $y(i) = x_k$, for $j \neq i$.

It is shown in [Cal] that the group $G_k \cong U(n)/\mathbb{Z}_k$ is a maximal compact subgroup of the automorphisms of $M$ via the natural action on $\mathbb{C}^n \setminus \{0\}$. Moreover, any Kähler metric $g_{ij}$ on $M$ which is invariant under $G_k$ is described on $\mathbb{C}^n \setminus \{0\}$ as $g_{ij} = \partial_i \partial_j u$ for a potential function $u = u(\rho)$, where

$$\rho = \log \left(\sum_{i=1}^n |x_i|^2\right).$$

(2.4)

The potential function $u$ has to satisfy certain properties in order to define a Kähler metric. Namely, a Kähler metric $g_{ij}$ with Kähler form $\omega = \sqrt{2\pi} g_{ij} dz^i \wedge d\bar{z}^j$ on $M$ in the class (see (1.4))

$$\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$$

(2.5)

is given by the potential function $u : \mathbb{R} \to \mathbb{R}$ with $u' > 0$, $u'' > 0$ together with the following asymptotic condition. There exist smooth functions $u_0, u_\infty : [0, \infty) \to \mathbb{R}$ with $u_0'(0) > 0$, $u_\infty'(0) > 0$ such that

$$u_0(e^{k\rho}) = u(\rho) - a\rho, \quad u_\infty(e^{-k\rho}) = u(\rho) - b\rho,$$

(2.6)

for all $\rho \in \mathbb{R}$. It follows that

$$\lim_{\rho \to -\infty} u'(\rho) = a < b = \lim_{\rho \to -\infty} u'(\rho).$$
Note that the divisor $D_0$ corresponds to $\rho = -\infty$ while $D_\infty$ corresponds to $\rho = \infty$. The Kähler metric $g_{ij}$ associated to $u$ is given in the $x_i$ coordinates by

$$g_{ij} = \partial_i \partial_j u = e^{-\rho} u'(\rho) \delta_{ij} + e^{-2\rho} x_i x_j (u''(\rho) - u'(\rho)).$$

(2.7)

Conversely, a Kähler metric $g$ determines the function $u$ up to the addition of a constant. The metric $g$ has determinant

$$\det g = e^{-n \rho} (u'(\rho))^{n-1} u''(\rho).$$

(2.8)

Thus, if we define

$$v = -\log \det g = n\rho - (n-1)\log u'(\rho) - \log u''(\rho)$$

(2.9)

then the Ricci curvature tensor $R_{ij} = \partial_i \partial_j v$ is given by

$$R_{ij} = e^{-\rho} v'(\rho) \delta_{ij} + e^{-2\rho} x_i x_j (v''(\rho) - v'(\rho)).$$

(2.10)

Finally, we construct a reference metric $\hat{\omega}$ in the class $\alpha$ given by (2.5). Define a potential function $\hat{u}$ by

$$\hat{u}(\rho) = a\rho + \frac{(b-a)}{2k} \log(e^{k\rho} + 1).$$

(2.11)

Then one can check by the above definition that the associated Kähler form $\hat{\omega}$ lies in the class $\alpha$. We observe in addition that $\hat{\omega}$ can be decomposed into a sum of nonnegative $(1,1)$ forms. The smooth $(1,1)$ form $\chi = \pi^* \omega_{FS} \in [D_H]$ is represented by the straight line function $u_\chi$:

$$u_\chi(\rho) = \rho, \quad \chi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_\chi.$$  

(2.12)

In addition, let $u_\theta$ and $\theta$ be respectively the potential and associated closed $(1,1)$-form defined by

$$u_\theta = 2 \log(e^{k\rho} + 1), \quad \theta = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_\theta.$$  

(2.13)

The form $\theta$ lies in the cohomology class $2[D_\infty]$. Moreover,

$$\hat{u} = au_\chi + \frac{(b-a)}{2k} u_\theta, \quad \text{and} \quad \hat{\omega} = a\chi + \frac{(b-a)}{2k} \theta \in \alpha.$$  

(2.14)

The following lemma will be useful later.

**Lemma 2.1** The smooth nonnegative closed $(1,1)$ form $\theta$ in $2[D_\infty]$ given by (2.13) satisfies

$$\chi^{n-1} \wedge \theta > 0 \quad \text{and} \quad \int_M \theta^n > 0.$$  

**Proof** Note that

$$\chi^{n-1} \wedge \theta = \frac{2k}{b-a} \chi^{n-1} \wedge \hat{\omega} > 0.$$  

Also, from the construction of $\theta$ and the formula (2.8), we have $\int_M \theta^n > 0$. □
2.3 The orbifold $\mathbb{P}^{n}/\mathbb{Z}_{k}$

Using homogeneous coordinates $Z_1, \ldots, Z_{n+1}$, we let the group $\mathbb{Z}_{k}$ act on $\mathbb{P}^{n}$ by

$$j \cdot [Z_1, \ldots, Z_{n+1}] = [Z_0, \ldots, Z_n, e^{2\pi j \sqrt{-1}/k} Z_{n+1}],$$

for $j = 0, 1, 2, \ldots, k - 1$. The quotient space $\mathbb{P}^{n}/\mathbb{Z}_{k}$ has a natural orbifold structure, branched over the point $[0, \ldots, 0, 1]$ and the hyperplane $\{Z_{n+1} = 0\}$. In addition, there is a holomorphic map $f : M_{n,k} \to \mathbb{P}^{n}/\mathbb{Z}_{k}$ given as follows. A point in $\pi^{-1}(U_i)$ with coordinates $z_{(i)}^j$ (for $j \neq i$) and fiber coordinate $y_{(i)}$ maps to the point in $\mathbb{P}^{n}/\mathbb{Z}_{k}$ with homogeneous coordinates

$$[z_{(i)}^1, \ldots, z_{(i)}^{i-1}, 1, z_{(i)}^{i+1}, \ldots, z_{(i)}^n, y_{(i)}^{-1/k}],$$

or $[0, \ldots, 0, 1]$ if $y_{(i)} = 0$. The map $f$ is well-defined because of the group action. Note that the inverse image $f^{-1}([0, \ldots, 0, 1])$ is the divisor $D_0$ and

$$f|_{M_{n,k}\setminus D_0} : M_{n,k} \setminus D_0 \to (\mathbb{P}^{n}/\mathbb{Z}_{k}) \setminus \{[0, \ldots, 0, 1]\}$$  \hspace{1cm} (2.15)

is an isomorphism. In the case $n = 2$, $k = 1$, the divisor $D_0$ is the exceptional curve on $\mathbb{P}^{2}$ blown up at one point and $f : M_{2,1} \to \mathbb{P}^{2}$ is the blow-down map.

Finally we note that there is a map $\mathbb{C}^{n} \to (\mathbb{P}^{n}/\mathbb{Z}_{k}) \setminus \{Z_{n+1} = 0\}$ given by

$$(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n, 1].$$

This map is $k$-to-one on $\mathbb{C}^{n} \setminus \{0\}$. With respect to these coordinates (and the corresponding coordinates $(x_1, \ldots, x_n)$ on $M_{n,k}$ as described above) one can check that $f$ is the identity map on $\mathbb{C}^{n} \setminus \{0\}$.

3 Estimates for the Kähler-Ricci flow

In this section we prove some estimates for a solution $\omega(t)$ to the Kähler-Ricci flow (1.2) on $M = M_{n,k}$ that hold without the assumption of Calabi symmetry. If the initial metric $\omega_0$ lies in the class $\alpha_0$ given by constants $0 < a_0 < b_0$ then the Kähler class $\alpha_t$ of $\omega(t)$ evolves by

$$\alpha_t = \frac{b_t}{k} [D_{\infty}] - \frac{a_t}{k} [D_0] = \frac{(b_t - a_t)}{k} [D_{\infty}] + a_t [D_H],$$  \hspace{1cm} (3.1)

where

$$b_t = b_0 - (k + n)t \quad \text{and} \quad a_t = a_0 + (k - n)t.$$  \hspace{1cm} (3.2)

Define

$$T = \sup\{t \geq 0 \mid \alpha_t \text{ is Kähler}\}.$$  \hspace{1cm} (3.3)

Note that the Kähler metric $\hat{\omega}_t$ given by

$$\hat{\omega}_t = a_t \chi + \frac{(b_t - a_t)}{2k} \theta$$  \hspace{1cm} (3.4)

for $t \in [0, T)$ and $\theta$ from Lemma 2.1 lies in $\alpha_t$.

We observe that the Kähler-Ricci flow exists on $[0, T)$. 
Theorem 3.1 There exists a unique smooth solution of the Kähler-Ricci flow (1.2) on $M$ starting with $\omega_0 \in \alpha_0$ for $t$ in $[0, T)$.

Proof This follows from a general and well-known result in the Kähler-Ricci flow. Indeed, let $X$ be any Kähler manifold and $\alpha_0$ be a Kähler class on $X$ with $\omega_0 \in \alpha_0$. If

$$T = \sup\{t \geq 0 \mid \alpha_0 + t[K_X] > 0\},$$

then it is shown in [Cao1], [Ts], [TZha] that there is a unique smooth solution $\omega = \omega(t)$ of the Kähler-Ricci flow (1.2) on $X$ starting at $\omega_0$, for $t$ in $[0, T)$.

We now deal with the behavior of the flow as $t \to T$ in various different cases.

3.1 The case $k \geq n$

In this case, the class $\alpha_t$ remains Kähler for $0 < t < T$ where

$$T = \frac{b_0 - a_0}{2k}.$$

As $t \to T$, the difference $b_t - a_t$ tends to zero, while the constant $a_t$ remains bounded below away from zero.

The reference metric $\hat{\omega}_t$ in $\alpha_t$ is given by

$$\hat{\omega}_t = a_t \chi + \frac{(b_t - a_t)}{2k} \theta = a_t \chi + (T - t) \theta \in \alpha_t. \quad (3.5)$$

From (2.1) we see that the closed (1,1) form $\theta - (k - n)\chi$ lies in the first Chern class $c_1(M)$. Hence there is a smooth volume form $\Omega$ on $M$ such that

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \Omega = -\theta + (k - n)\chi.$$

We consider the parabolic Monge-Ampère equation:

$$\frac{\partial \phi}{\partial t} = \log \left( \frac{\hat{\omega}_t + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi}{(T - t)\Omega} \right), \quad \phi|_{t=0} = \phi_0, \quad (3.6)$$

where $\hat{\omega}_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_0 = \omega_0 \in \alpha_0$. If $\phi = \varphi(t)$ solves (3.6) then

$$\omega(t) = \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$$

solves the Kähler-Ricci flow (1.2).

Note that in the following two lemmas, we only use the assumption $k \geq n$ to obtain a uniform lower bound of the constant $a_t$ away from zero, for $t \in [0, T)$.

Lemma 3.1 There exists a constant $C$ depending only on the initial data such that

$$|\varphi(t)| \leq C, \quad \omega^n(t) \leq C\Omega.$$
By Lemma 2.1, since
\[ \hat{\omega}_t^n = (a_t \chi + (T - t) \theta)^n \geq n (T - t) a_t^{n-1} \chi^{n-1} \wedge \theta, \]
there exist constants \( C_1, C_2 > 0 \) independent of \( t \) such that
\[ C_1 (T - t) \Omega \leq \hat{\omega}_t^n \leq C_2 (T - t) \Omega. \tag{3.7} \]
Note that we are making use of the fact that \( a_t \) is bounded from below away from zero. To obtain the upper bound of \( \varphi \), consider the evolution of \( \psi = \varphi - (1 + \log C_2) t \). We claim that
\[ \sup_{M \times [0,T)} \psi = \sup_M \psi |_{t=0}. \]
Otherwise there exists a point \((x, t) \in M \times (0, T)\) at which \( \partial \psi / \partial t \geq 0 \) and \( \frac{\partial \psi}{\partial t} \leq 0 \). Thus, at that point,
\[ 0 \leq \frac{\partial \psi}{\partial t} \leq \log \hat{\omega}_t^n (T - t) \Omega - 1 - \log C_2 \leq -1, \]
a contradiction. Hence \( \sup_{M \times [0,T)} \psi = \sup_M \psi |_{t=0} \) and thus \( \varphi \) is uniformly bounded from above. A lower bound on \( \varphi \) is obtained similarly.

For the upper bound of \( \omega^n \), we will bound \( H = \log \frac{\omega^n}{T} - A \varphi \), where \( A \) is a constant to be determined later. Writing \( \text{tr} \omega' \omega' = n \frac{\omega^{n-1} \wedge \omega'}{\omega^n} \) where \( \omega' \) is any \((1, 1)\)-form, we compute
\[ \frac{\partial}{\partial t} H = \Delta \varphi + \text{tr}_\omega \left( \frac{\partial}{\partial t} \hat{\omega}_t \right) - A \hat{\varphi}, \]
where \( \Delta \) denotes the Laplace operator associated to \( g(t) \). Since \( \varphi = H + A \varphi - \log(T - t) \) and \( \theta \geq 0 \), we have
\[ \frac{\partial}{\partial t} H = \Delta H + A \Delta \varphi + (k - n) \text{tr}_\omega \chi - \text{tr}_\omega \theta - A (H + A \varphi - \log(T - t)) \]
\[ \leq \Delta H + An - Aa_t \text{tr}_\omega \chi + (k - n) \text{tr}_\omega \chi - AH - A^2 \varphi + A \log T. \]
Choosing \( A \) sufficiently large so that \( Aa_t \geq (k - n) \) and using the fact that \( \varphi \) is uniformly bounded, we see that \( H \) is bounded from above by the maximum principle. \( \square \)

We have, in addition, the following estimate.

**Lemma 3.2** There exists a uniform constant \( C > 0 \) such that
\[ \text{tr}_\omega \chi = n \frac{\omega^{n-1} \wedge \chi}{\omega^n} \leq C. \]

**Proof** This is a ‘parabolic Schwarz lemma’ similar to the one given in [SoT1]. We use the maximum principle. Let \( \omega_{\text{FS}} = \sqrt{\frac{1}{2\pi}} h_{\alpha \beta} d_\alpha \wedge d_\beta \) be the Fubini-Study metric on \( \mathbb{P}^{n-1} \) and let \( \pi : M \to \mathbb{P}^{n-1} \) be the bundle map. We will calculate the evolution of
\[ w = \text{tr}_g (\pi^* h) = g^T \pi_\alpha \pi_\beta h_{\alpha \beta} = n \frac{\omega^{n-1} \wedge \chi}{\omega^n}. \tag{3.8} \]
A standard computation shows that

\[
\Delta w = g^{kl} \partial_k \partial_l \left( g^{ij} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} \right) \\
= g^{ij} g^{kl} R_{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} + g^{ij} g^{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} - g^{ij} g^{kl} S_{\alpha \beta \gamma \delta} \pi_i^\alpha \pi_j^\beta \pi_k^\gamma \pi_l^\delta, 
\]

(3.9)

where \( S_{\alpha \beta \gamma \delta} \) is the curvature tensor of \( h_{\alpha \beta} \). By the definition of \( w \) we have

\[
\Delta w \geq g^{ij} g^{kl} R_{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} + g^{ij} g^{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} - 2w^2. 
\]

(3.10)

Now

\[
\frac{\partial w}{\partial t} = -g^{ij} g^{kl} \frac{\partial g_{kl}}{\partial t} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} \\
= g^{ij} g^{kl} R_{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta}, 
\]

(3.11)

Combining (3.10) and (3.11), we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) w \leq -g^{ij} g^{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} + 2w^2. 
\]

(3.12)

On the other hand, by a standard argument (see [Y1] for example)

\[
\frac{|\nabla w|^2}{w} \leq g^{ij} g^{kl} \pi_i^\alpha \pi_j^\beta h_{\alpha \beta} 
\]

(3.13)

and thus

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log w \leq 2w. 
\]

(3.14)

We now compute the evolution of the quantity \( L = \log w - A\varphi \), where \( A \) is a constant to be determined later. From the arithmetic-geometric means inequality

\[
\frac{\lambda_1 + \cdots + \lambda_n}{n} \geq (\lambda_1 \cdots \lambda_n)^{1/n}, \quad \text{for } \lambda_1, \ldots, \lambda_n \geq 0, 
\]

and (3.7) we have,

\[
\text{tr}_w \omega_t \geq n \left( \frac{\omega^n}{\omega_t} \right)^{1/n} \geq c \left( \frac{(T-t)\Omega}{\omega^n} \right)^{1/n}, 
\]

(3.15)

for a uniform constant \( c > 0 \). Then, using the inequality \( \text{tr}_w \omega_t \geq a_t w \) together with (3.14) and (3.15),

\[
\left( \frac{\partial}{\partial t} - \Delta \right) L \leq 2w - A \log \left( \frac{\omega^n}{(T-t)\Omega} \right) + An - A \text{tr}_w \omega_t \\
\leq -w + A \log \left( \frac{(T-t)\Omega}{\omega^n} \right) + An - c \left( \frac{(T-t)\Omega}{\omega^n} \right)^{1/n}, 
\]

where we have chosen \( A \) sufficiently large so that \( (A - 1)a_t \geq 3 \). Note that the function \( \mu \mapsto A \log \mu - c\mu^{1/n} \) for \( \mu > 0 \) is uniformly bounded from above. Hence if the maximum of \( L \) is achieved at a point \((x_0, t_0) \in M \times (0, T)\) then, at that point, \( w \) is uniformly bounded from above. Since we have already shown in Lemma 3.1 that \( \varphi \) is uniformly bounded along the flow, the required upper bound of \( w \) follows by the maximum principle.  \( \Box \)
3.2 The case $1 \leq k \leq n - 1$

There are three distinct types of behavior here which depend on the choice of initial Kähler class $\alpha_0$. We deal with each in turn.

3.2.1 The subcase $a_0 (n + k) = b_0 (n - k)$.

In this case $\alpha_0 = (a_0 / (n - k)) c_1 (M)$ and the class $\alpha_t = (a_0 / (n - k) - t) c_1 (M)$ is proportional to the first Chern class. After renormalizing, this is the Kähler-Ricci flow (1.1) with $\lambda = 1$ on a manifold with $c_1 (M) > 0$. It is shown in [FIK] that $M$ admits a Kähler-Ricci soliton and we refer the reader to the results of [TZhu] and also [PSSW3]. For our purposes, we only need the fact that the diameter of the metric is bounded along the normalized Kähler-Ricci flow [P1], [SeT].

3.2.2 The subcase $a_0 (n + k) > b_0 (n - k)$.

In this case the Kähler class $\alpha_t$ remains Kähler until time $T = (b_0 - a_0) / 2k$. We have $\lim_{t \to T} b_t = \lim_{t \to T} a_t > 0$ as in the case $k \geq n$. Lemmas 3.1 and 3.2 hold with the same proofs in this case since, as pointed out in Section 3.1, we only used there the fact that $a_t$ is uniformly bounded from below away from zero.

3.2.3 The subcase $a_0 (n + k) < b_0 (n - k)$.

In this case the Kähler class $\alpha_t$ remains Kähler until time $T = a_0 / (n - k)$. As $t \to T$, $a_t$ tends to zero while $b_t$ remains bounded below away from zero. The metrics $\hat{\omega}_t$ and classes $\alpha_t$ satisfy the following properties for all $t \in [0, T)$:

1. The limit $\hat{\omega}_T = \lim_{t \to T} \hat{\omega}_t$ is a smooth closed nonnegative (1,1) form satisfying $\int_M \hat{\omega}_T^n > 0$.

2. For all $\varepsilon > 0$ sufficiently small (independent of $t$), the class $\alpha_t - \varepsilon [D_0]$ is Kähler.

Indeed, (1) follows from Lemma 2.1 and (2) is immediate from the definition of $\alpha_t$. We will use these properties to prove the following (cf. [TS], [TZha], [Zha]).

**Theorem 3.2** Let $\omega (t)$ be a solution of the Kähler-Ricci flow on $M$ starting at $\omega_0$ in $\alpha_0$ with $a_0 (n + k) < b_0 (n - k)$. Then:

(i) If $\Omega$ is a fixed volume form on $M$ then there exists a uniform constant $C$ such that

$$\omega^n (t) \leq C \Omega, \quad \text{for all } t \in [0, T).$$

(ii) There exists a closed semi-positive (1,1) current $\omega_T$ on $M$ which is smooth outside the exceptional curve $D_0$ and has an $L^\infty$-bounded local potential such that following holds. The metric $\omega (t)$ along the Kähler-Ricci flow converges in $C^\infty$ on compact subsets of $M \setminus D_0$ to $\omega_T$ as $t \to T$. 
Proof This result is essentially contained in [TZha], [Zha], but we will include an outline of the proof for the reader’s convenience. First, define a closed (1,1) form \( \eta \) by

\[
\eta = \frac{\partial}{\partial t} \hat{\omega} = (k - n) \chi - \theta,
\]

so that the reference metric \( \hat{\omega} \) is given by \( \hat{\omega} = \hat{\omega}_0 + t\eta \). Since \(-\eta\) is in \( c_1(M) \) there exists a volume form \( \Omega \) on \( M \) with \( \sqrt{\frac{-1}{2\pi}} \partial \bar{\partial} \log \Omega = \eta \). Let \( \varphi = \varphi(t) \) be a solution of the parabolic Monge-Ampère equation

\[
\frac{\partial \varphi}{\partial t} = \log \left( \hat{\omega}_t + \sqrt{\frac{-1}{2\pi}} \partial \bar{\partial} \varphi \right)^n \Omega,
\]

with \( \varphi|_{t=0} = 0 \), for \( t \in [0, T) \). Then \( \omega(t) = \hat{\omega} + \sqrt{\frac{-1}{2\pi}} \partial \bar{\partial} \varphi \) solves the Kähler-Ricci flow (1.2). We will first bound \( \varphi \). Notice that \( \varphi \) is uniformly bounded from above by a simple maximum principle argument. To obtain a uniform \( L^\infty \) bound on \( \varphi \) we will show that \( \dot{\varphi} \) is uniformly bounded from above for \( t \in [T/2, T) \).

Compute

\[
(\frac{\partial}{\partial t} - \Delta) \dot{\varphi} = \text{tr}_\omega \eta.
\]

Then

\[
(\frac{\partial}{\partial t} - \Delta)(t\dot{\varphi} - \varphi - nt) = -\text{tr}_\omega \dot{\omega}_0 \leq 0,
\]

using the fact that \( \Delta \varphi = n - \text{tr}_{\omega}(\dot{\omega}_0 + t\eta) \). It follows from the maximum principle that \( t\dot{\varphi} \) is uniformly bounded from above. Hence \( \dot{\varphi} \) is uniformly bounded from above for \( t \) in \([T/2, T)\).

Rewrite (3.16) as

\[
(\dot{\omega}_t + \sqrt{\frac{-1}{2\pi}} \partial \bar{\partial} \varphi)^n = e^{\dot{\varphi} \Omega}.
\]

By property (1) above, the Kähler metric \( \dot{\omega}_t \) satisfies \( \int_M \dot{\omega}_t^n > c > 0 \) for a uniform constant \( c \) for all \( t \in [T/2, T) \) and the limit \( \dot{\omega}_T \) is a smooth nonnegative (1,1) form. Hence we can apply the results of [Kol], [Zha], [EGZ] on the complex Monge-Ampère equation to obtain an \( L^\infty \) bound on \( \varphi \) which is uniform in \( t \). Note that part (i) of the Theorem follows from the upper bound of \( \dot{\varphi} \).

For (ii), we use property (2) of \( \alpha_t \) as listed above. The argument of Tsuji [Ts] (cf. [Y1]) gives second order estimates for \( \varphi \), depending on the \( L^\infty \) estimate, on all compact sets of \( M \setminus D_0 \). The rest of the theorem follows by standard theory. \( \square \)

4 Estimates under the Calabi symmetry condition

In this section we assume that the initial metric \( \omega_0 \) satisfies the symmetry condition of Calabi. We prove estimates for the solution of the Kähler-Ricci flow and in particular we give a proof of Theorem 1.2 (see Theorem 4.1 below.)

Let \( \omega(t) \) be a solution of the Kähler-Ricci flow (1.2) on a time interval \([0, T)\). The Kähler-Ricci flow can be described in terms of the potential \( u = u(\rho, t) \). Noting that \( \omega \) determines \( u \) only up to the addition of a constant, we consider \( u = u(\rho, t) \) solving

\[
\frac{\partial}{\partial t} u(\rho, t) = \log u''(\rho, t) + (n - 1) \log u'(\rho, t) - n\rho + c_t,
\]

(4.1)
where
\[ c_t = -\log u''(0, t) - (n - 1)u'(0, t). \quad (4.2) \]

The notation \( u'(\rho, t) \) denotes the partial derivative \( (\partial u/\partial \rho)(\rho, t) \) and similarly for \( u''(\rho, t) \). We assume that \( \rho \mapsto u(\rho, 0) \) represents the initial Kähler metric \( g_0 \) and we impose the further normalization condition that
\[ u(0, 0) = 0. \quad (4.3) \]

Then by (2.9), the solution \( g = g(t) \) of (1.2) can be written as \( g_{ij} = \partial_i \partial_j u \). The constant \( c_t \) is chosen so that \( \partial_t u(0, t) = 0 \) and thus \( u(0, t) = 0 \) on \( [0, T) \). The existence of a unique solution of the Kähler-Ricci flow on \( [0, T) \) and the parabolicity of (4.1) ensures the existence of a smooth unique \( u(\rho, t) \) solving (4.1).

The Kähler metric at time \( t \) lies in the cohomology class
\[ \alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0] \]
along the flow, where \( a_t \) and \( b_t \) are given by (3.2). We have
\[ \lim_{\rho \to -\infty} u'(\rho, t) = a_t, \quad \lim_{\rho \to \infty} u'(\rho, t) = b_t \]
and by convexity
\[ a_t < u'(\rho, t) < b_t, \quad \text{for all } \rho \in \mathbb{R}. \]

Next, the evolution equations for \( u' \), \( u'' \) and \( u''' \) are given by
\[
\begin{align*}
\frac{\partial}{\partial t} u' &= \frac{uu''}{u''} + \frac{(n - 1)u''}{u'} - n & \quad (4.4) \\
\frac{\partial}{\partial t} u'' &= \frac{u^{(4)}}{u''} \left( \frac{u''}{u'} \right)^2 + \frac{(n - 1)u''}{u'} - \frac{(n - 1)(u'')^2}{(u')^2} & \quad (4.5) \\
\frac{\partial}{\partial t} u''' &= \frac{u^{(5)}}{u''} \left( \frac{u''}{u'} \right)^2 + 2\left( \frac{u''}{u'} \right)^3 + \frac{(n - 1)u^{(4)}}{u'} - \frac{3(n - 1)u''u'''}{(u')^2} + \frac{2(n - 1)(u'')^3}{(u')^3}, & \quad (4.6)
\end{align*}
\]
as can be seen from differentiating (4.1).

For the rest of this section we assume that \( u = u(\rho, t) \) solves the Kähler-Ricci flow (4.1) with (4.2) and (4.3).

### 4.1 The case \( k \geq n \)

The following elementary lemma shows that as \( t \to T = (b_0 - a_0)/2k \), the potential \( u \) converges pointwise to the function \( a_T u_\chi \).

**Lemma 4.1** The function \( u = u(\rho, t) \) satisfies, for all \( \rho \) in \( \mathbb{R} \),
\[ (i) \quad 0 < u'(\rho, t) - a_t < 2k(T - t), \quad \text{for all } t \in [0, T); \]

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\[(ii) \lim_{t \to T} (u(\rho, t) - a_T \rho) = 0.\]

**Proof** (i) follows immediately from the convexity of \(u\) and the definition of \(a_t\) and \(b_t\). For (ii), recall that \(u(0, t) = 0\) for all \(t \in [0, T)\) and so

\[
u(\rho, t) - a_t \rho = \int_0^\rho (u'(s, t) - a_t) ds\]

Applying (i),

\[
|u(\rho, t) - a_t \rho| \leq 2k(T - t)|\rho| \longrightarrow 0,
\]
as \(t \to T\), while \(a_t \to a_T\).

As a simple application of this we prove a lower bound for the Kähler metric along the flow.

**Lemma 4.2** Along the Kähler-Ricci flow, we have

\[
\omega(t) \geq a_t \chi
\]
for all \(t \in [0, T)\).

**Proof** With a slight abuse of notation, we write \(\chi = \sqrt{-1} \chi_{ij} dz^i \wedge d\overline{z}^j\). Then by (2.7),

\[
g_{ij}(t) = e^{-\rho} u' \delta_{ij} + e^{-2\rho \overline{x}_i x_j} (u'' - u'), \quad \chi_{ij} = e^{-\rho} \delta_{ij} - e^{-2\rho \overline{x}_i x_j}.
\]

(4.7)

Since \(u'' > 0\) and \(u' > a_t\),

\[
g_{ij}(t) \geq u'e^{-\rho} \left( \delta_{ij} - \frac{x_i x_j}{\sum_k |x_k|^2} \right) \geq a_t e^{-\rho} \left( \delta_{ij} - \frac{x_i x_j}{\sum_k |x_k|^2} \right) = a_t \chi_{ij},
\]
as required. \(\square\)

We have the following further estimates for \(u\) which will give an upper bound for the metric \(\omega(t)\).

**Lemma 4.3** There exists a constant \(C\) depending only on the initial data such that

\[
0 < u''(\rho, t) \leq C \min \left( \frac{e^{k\rho}}{(1 + e^{k\rho})^2}, (T - t) \right)
\]
and

\[
|u'''(\rho, t)| \leq C u''(\rho, t)
\]
for all \((\rho, t) \in \mathbb{R} \times [0, T)\).
Proof We begin by establishing the bound
\[
u''(\rho, t) \leq C e^{k\rho} \frac{e^{kp}}{(1 + e^{kp})^2}.
\] (4.10)

Let \( \hat{g}_0 \) be the reference metric with associated potential \( \hat{u}_0 \) (see (3.4)). Then from (2.3) we see that
\[
det \hat{g}_0 = e^{-np}(\hat{u}_0'le(\rho))^{n-1} \hat{u}_0''(\rho) = k(b_0 - a_0)e^{-np} \left(a_0 + (b_0 - a_0) \frac{e^{kp}}{1 + e^{kp}}\right)^{n-1} e^{kp} \frac{e^{kp}}{(1 + e^{kp})^2}
\]
\[
\leq Ce^{-np} \frac{e^{kp}}{(1 + e^{kp})^2}.
\] (4.11)

By Lemma 3.1 the volume form \( \omega^n \) is uniformly bounded from above by a fixed volume form along the flow, and hence \( \det g(t) \leq C \det \hat{g}_0 \) for some uniform constant \( C \). Combining this fact with (2.3) and (4.11), we have
\[
(u'(\rho, t))^{n-1}u''(\rho, t) \leq C e^{kp} \frac{e^{kp}}{(1 + e^{kp})^2},
\] (4.12)

Then (4.10) follows, since \( u'(\rho) \) is uniformly bounded from below away from zero.

Next, we give a proof of the bound (4.9). We will apply the maximum principle to the quantity \( u''' / u'' \). Before we do this, observe that for each fixed \( t \in [0, T) \),
\[
\lim_{\rho \to -\infty} \frac{u'''(\rho, t)}{u''(\rho, t)} = k, \quad \lim_{\rho \to \infty} \frac{u'''(\rho, t)}{u''(\rho, t)} = -k.
\] (4.13)

Indeed, for the first limit, we can compute \( u''' / u'' \) in terms of the function \( s \mapsto u_0(s, t) \) associated to \( u \). From (2.6),
\[
u(\rho, t) = a_1\rho + u_0(e^{kp}, t)
\]
\[
u'(\rho, t) = a_2 + ke^{kp}u'_0(e^{kp}, t)
\]
\[
u''(\rho, t) = k^2e^{kp}u'_0(e^{kp}, t) + k^2e^{2kp}u''_0(e^{kp}, t)
\]
\[
u'''(\rho, t) = k^3e^{3kp}u''_0(e^{kp}, t) + 3k^3e^{2kp}u'''_0(e^{kp}, t) + k^3e^{3kp}u'''_0(e^{kp}, t).
\] (4.14)

Hence
\[
u'''(\rho, t) \frac{u''(\rho, t)}{u'(\rho, t)} = k u'_0(e^{kp}, t) + 3k e^{kp}u''_0(e^{kp}, t) + k e^{2kp}u'''_0(e^{kp}, t) \to k, \quad \text{as} \quad \rho \to -\infty,
\]
since \( u'_0(0, t) > 0 \). The second limit can be proved in a similar way using the function \( u_\infty \).

Using (4.5) and (4.6) we find that \( u''' / u'' \) evolves by
\[
\frac{\partial}{\partial t} \left( \frac{u'''}{u''} \right) = \frac{1}{u''} \left( \frac{u^{(5)}}{u''} - \frac{3u'''u''}{(u'')^2} + \frac{2(u''')^3}{(u'')^3} + \frac{(n-1)u''}{u'} - \frac{3(n-1)u''u'''}{(u')^2} + \frac{2(n-1)(u''')^3}{(u')^3} \right)
\]
\[
- \frac{u''}{(u'')^2} \left( \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{(n-1)u''}{u'} - \frac{(n-1)(u''')^2}{(u'')^2} \right).
\]
To obtain an upper bound of $u'''/u''$ we argue as follows. From (4.13) we may assume without loss of generality that the maximum of $u'''/u''$ occurs at an interior point $(\rho_1, t_1) \in \mathbb{R} \times (0, T)$. At that point,

$$
\left( \frac{u'''}{u''} \right)' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} = 0
$$

and

$$
\left( \frac{u'''}{u''} \right)'' = \frac{u^5}{u''} - \frac{3u'''u^{(4)}}{(u'')^2} + \frac{2(u''')^3}{(u'')^3} \leq 0.
$$

Then at $(\rho_1, t_1)$,

$$
\frac{\partial}{\partial t} \left( \frac{u'''}{u''} \right) \leq -2(n-1)\frac{u''}{(u')^2} + \frac{2(n-1)(u'')^2}{(u')^3}
$$

and hence

$$
\left( \frac{u'''}{u''} \right)(\rho_1, t_1) \leq \left( \frac{u''}{u'} \right)(\rho_1, t_1) \leq C,
$$

using (4.10). This gives the upper bound for $u''/u''$. The argument for the lower bound is similar. In fact, since $u'' > 0$ we obtain $u'''/u'' \geq -k$. This establishes (4.9).

It remains to prove the estimate $u'' \leq C(T - t)$. Fix $t$ in $[0, T)$. Since the bound (4.10) implies that $u''(\rho)$ tends to zero as $\rho$ tends to $\pm \infty$, there exists $\hat{\rho} \in \mathbb{R}$ such that

$$
u''(\hat{\rho}) = \sup_{\rho \in \mathbb{R}} u''(\rho).
$$

By the Mean Value Theorem and (4.9) we have, for all $\rho \in \mathbb{R}$,

$$
u''(\hat{\rho}) - u''(\rho) \leq C u''(\hat{\rho}) |\rho - \hat{\rho}|.
$$

Then for $|\rho - \hat{\rho}| \leq 1/2C$,

$$
u''(\rho) \geq \frac{u''(\hat{\rho})}{2},
$$

and hence

$$
\frac{1}{2C} u''(\hat{\rho}) = \int_{|\rho - \hat{\rho}| \leq 1/2C} \frac{u''(\hat{\rho})}{2} d\rho \leq \int_{-\infty}^{\infty} u''(\rho) d\rho = b_t - a_t = 2k(T - t).
$$

The bound $u'' \leq C(T - t)$ then follows. \(\square\)

We can now use these estimates on $u$ to obtain bounds on the Kähler metric $\omega(t)$ along the Kähler-Ricci flow. In the following, $\omega(t)$ will always denote a solution of the Kähler-Ricci flow (1.2) with initial metric $\omega_0$ satisfying the Calabi symmetry.

**Theorem 4.1** We have

(i) $\sup_M \text{tr}_{\hat{g}_0} g \leq C$.

(ii) For any compact set $K \subset M \setminus (D_\infty \cup D_0)$,

$$
\sup_K |\nabla_{\hat{g}_0} g|_{\hat{g}_0} \leq C_K.
$$

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Proof An elementary computation shows that
\[ \text{tr}_{g_0}g = \frac{u''}{u_0'} + (n - 1) \frac{u'}{u_0'}. \]
But we have \( u'/u_0' \leq b_0/a_0 \) and, making use of Lemma 4.3,
\[ \frac{u''}{u_0''} = \frac{(1 + e^{k\rho})^2}{k(b_0 - a_0)e^{k\rho}} u'' \leq C, \]
and this gives (i).

We now prove (ii). By (i) it suffices to bound
\[ \frac{\partial}{\partial x_k} g_{ij} = e^{-2\rho}(u'' - u')(\pi_k \delta_{ij} + \pi_i \delta_{jk}) + e^{-3\rho} \pi_i x_j \pi_k (u''' - 3u'' + 2u'), \]
in a given compact set \( K \subset M \setminus (D_0 \cup D_\infty) \). But \( u'' \) and \( u''' \) and uniformly bounded from above by Lemma 4.3 and in \( K \), the functions \( \rho \) and \( x_i \) are uniformly bounded. This gives (ii).

From Lemma 4.2 and Theorem 4.1 we obtain the following immediate corollary.

Corollary 4.1 We have
\[ \frac{1}{C} \leq \text{diam}_{g(t)} M \leq C. \]

Moreover we prove:

Theorem 4.2 Define \( \tilde{\varphi} = \tilde{\varphi}(t) \) by
\[ \omega(t) = \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\varphi}, \quad \tilde{\varphi}|_{\rho=0} = 0. \quad (4.15) \]
Then for all \( \beta \) with \( 0 < \beta < 1 \),

(i) \( \tilde{\varphi} \) tends to zero in \( C^{1,\beta}_{g_0}(M) \) as \( t \to T \).

(ii) For any compact set \( K \subset M \setminus (D_0 \cup D_\infty) \), \( \tilde{\varphi} \) tends to zero in \( C^{2,\beta}_{g_0}(K) \) as \( t \to T \). In particular, on \( K \), \( \omega(t) \) converges to \( a_T \chi \) on \( C^{\beta}_{g_0}(K) \) as \( t \to T \).

Proof By the normalization of \( \tilde{\varphi} \), we have \( \tilde{\varphi}(t) = u(t) - \hat{u}_t \). As \( t \to T \), \( \hat{u}_t \) tends to \( a_T u_\chi \). Then by Lemma 4.1, \( \tilde{\varphi} \) converges pointwise to zero. Taking the trace of (4.13) with respect to \( \hat{g}_0 \), applying the first part of Theorem 4.1 and using the fact that \( \text{tr}_{\hat{g}_0}g_t \) is bounded, we see that \( \Delta_{\hat{g}_0} \hat{\varphi} \) is uniformly bounded on \( M \), giving (i). The second part of Theorem 4.1 gives (ii).

Finally, we use the estimates of Lemma 4.3 together with the bound of Lemma 3.2 to show that, away from the divisors \( D_0 \) and \( D_\infty \), the fibers are collapsing.

Theorem 4.3 Let \( \pi^{-1}(z) \) be the fiber of \( \pi : M \to \mathbb{P}^{n-1} \) over the point \( z \in \mathbb{P}^{n-1} \). Define \( \omega_z(t) = \omega(t)|_{\pi^{-1}(z)} \). Then for any compact set \( K \subset M \setminus (D_0 \cup D_\infty) \), there exists a constant \( C_K \) such that
\[ \sup_{z \in \mathbb{P}^{n-1}} \|\omega_z(t)\|_{C^0(\pi^{-1}(z) \cap K)} \leq C_K(T - t). \quad (4.16) \]
Proof Fix a compact set \( K \subset M \setminus (D_0 \cup D_\infty) \). From (2.8) we see that on \( K \), the quantity \( \omega^n/\Omega \) is uniformly equivalent to \( u'' \). Then by Lemma 4.3, there exists a constant \( C_K \) such that
\[
\omega^n(x) \leq C_K(T - t)\Omega(x), \quad \text{for } x \in K.
\] (4.17)

Now at a point \( x \in K \), choose complex coordinates \( z^1, \ldots, z^n \) so that
\[
\chi = \sqrt{-\frac{1}{2\pi}} \sum_{i=1}^{n-1} dz^i \wedge d\overline{z}^i \quad \text{and} \quad \omega = \sqrt{-\frac{1}{2\pi}} \sum_{i=1}^{n} \lambda_i dz^i \wedge d\overline{z}^i,
\]
for \( \lambda_1, \ldots, \lambda_n > 0 \). The coordinate \( z^n \) is in the fiber direction and we wish to obtain an upper bound for \( \lambda_n \). From Lemma 3.2 we have
\[
\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \leq C.
\]
Hence \( \lambda_1, \ldots, \lambda_{n-1} \) are uniformly bounded from below away from zero and we have, for uniform constants \( C, C' \),
\[
\lambda_n \leq C \frac{1}{\lambda_1 \cdots \lambda_{n-1}} \frac{\omega^n(x)}{\Omega} \leq C' \cdot C_K(T - t)
\]
from (4.17). This proves (4.16).

As a corollary of this and Theorem 4.1, the diameter of the fibers goes to zero as \( t \) tends to \( T \).

Corollary 4.2 As above, let \( \pi^{-1}(z) \) be the fiber of \( \pi : M \to \mathbb{P}^{n-1} \) over the point \( z \in \mathbb{P}^{n-1} \). Then
\[
\lim_{t \to T} \left( \sup_{z \in \mathbb{P}^{n-1}} \text{diam}_{g(t)} \pi^{-1}(z) \right) = 0.
\]

Proof Fix \( \varepsilon > 0 \). By Theorem 4.1, (i) there exists a tubular neighborhood \( N_\varepsilon \) of \( D_0 \cup D_\infty \) such that for all \( z \in \mathbb{P}^{n-1} \) and all \( t \in [0, T) \),
\[
\text{diam}_{g(t)}(\pi^{-1}(z) \cap N_\varepsilon) < \frac{\varepsilon}{2}.
\] (4.18)

On the other hand, applying Theorem 4.3 with \( K = M \setminus N_\varepsilon \) we see that for \( t \) sufficiently close to \( T \),
\[
\text{diam}_{g(t)}(\pi^{-1}(z) \cap K) < \frac{\varepsilon}{2},
\] (4.19)
for all \( z \in \mathbb{P}^{n-1} \). Combining (4.18) and (4.19) completes the proof.

4.2 The case \( 1 \leq k \leq n - 1 \)

As in section 3.2, there are three subcases. We do not require any further estimates when \( a_0(n + k) = b_0(n - k) \) and so we move on to the other two cases.
4.2.1 The subcase \( a_0(n + k) > b_0(n - k) \).

We obtain all the results of subsection 4.1 by identical proofs. The key point is that, in this case, \( a_t \) is uniformly bounded from below away from zero.

4.2.2 The subcase \( a_0(n + k) < b_0(n - k) \).

Recall that in this case, \( a_t = a_0 + (k - n)t \) tends to zero as \( t \) tends to the blow-up time \( T \). Note that by part (ii) of Theorem 3.2 we already have \( C^\infty \) estimates for the metric \( g(t) \) on \( M \setminus D_0 \). In this subsection we will obtain estimates for the metric in a neighborhood of \( D_0 \). First we have the following estimate on \( u' \).

**Lemma 4.4** There exists a uniform constant \( C \) such that for all \( t \in [0, T) \) and all \( \rho \in \mathbb{R} \),

\[
0 < u'(\rho, t) - a_t \leq Ce^{k\rho/n}.
\]

**Proof** The first inequality follows from the definition of \( a_t \) and the convexity of \( u \). For the upper bound of \( u'(\rho, t) - a_t \) we argue as follows. By part (i) of Theorem 3.2 the volume form of \( \omega(t) \) is uniformly bounded along the flow. Then by the same argument as in the proof of (4.12), we have

\[
(u'(\rho, t))^{n-1}u''(\rho, t) \leq C \frac{e^{kp}}{(1 + e^{kp})^2} \leq Ce^{kp}.
\]

Hence

\[
((u'(\rho, t))^n)' \leq Ce^{kp}.
\]

Integrating in \( \rho \) we obtain

\[
(u'(\rho, t))^n - a_t^n \leq Ce^{kp}
\]

and thus

\[
u'(\rho, t) \leq a_t + Ce^{\frac{k}{n}p},
\]

as required. \( \square \)

Note that the conclusion of Lemma 4.4 could be strengthened for \( \rho > 0 \). However, in this subsection we need only concern ourselves with the case of negative \( \rho \) since the metric is bounded away from \( D_0 \).

**Lemma 4.5** There exists a uniform constant \( C \) such that

\[
u'' \leq C(u' - a_t)(b_t - u').
\]

In particular,

\[
u'' \leq Ce^{kp/n}.
\]
Proof We evolve the quantity \( H = \log u'' - \log(u' - a_t) - \log(b_t - u') \). Using (4.4) and (4.5) we compute
\[
\frac{\partial H}{\partial t} = \frac{1}{u''} \left( \frac{u'}{u''} - \frac{(u'')^2}{(u'')^2} + \frac{(n - 1)u''}{u'} - \frac{(n - 1)(u'')^2}{(u')^2} \right) - \frac{1}{u' - a_t} \left( \frac{u''}{u'} + \frac{(n - 1)u''}{u'} - k \right) - \frac{1}{b_t - u'} \left( \frac{u''}{u'} - \frac{(n - 1)u''}{u'} - k \right).
\]
\[ (4.20) \]

Before applying the maximum principle we check that \( H \) remains bounded from above as \( \rho \) tends to \( \pm \infty \). For the case of \( \rho \) negative we use (4.14) to obtain
\[
\frac{u''(\rho, t)}{(u'(\rho, t) - a_t)(b_t - u'(\rho, t))} = \frac{ke^{k_\rho}u'_0(e^{k_\rho}, t) + ke^{2k_\rho}u''_0(e^{k_\rho}, t)}{e^{k_\rho}u'_0(e^{k_\rho}, t)(b_t - a_t - ke^{k_\rho}u'_0(e^{k_\rho}, t))} \leq \frac{k}{b_t - a_t - ke^{k_\rho}u'_0(e^{k_\rho}, t)} + \frac{ke^{k_\rho}u''_0(e^{k_\rho}, t)}{u'_0(e^{k_\rho}, t)(b_t - a_t)} \leq \frac{k}{b_t - a_t} + 1,
\]
as \( \rho \) tends to \( -\infty \), where we are using the fact that \( u'_0(0, t) > 0 \). Note that \( b_t - a_t \) remains bounded from below away from zero. Similarly, we can show that \( H \) is bounded from above as \( \rho \) tends to positive infinity.

Suppose then that \( H \) has a maximum at a point \((\rho_0, t_0) \in \mathbb{R} \times (0, T)\). Then at this point, we have
\[
\frac{u''}{u'} - \frac{u''}{u' - a_t} + \frac{u''}{b_t - u'} = 0 \quad (4.21)
\]
and
\[
\frac{u^{(4)}}{u''} - \frac{(u'')^2}{(u')^2} - \frac{u''}{u' - a_t} + \frac{(u'')^2}{(u' - a_t)^2} + \frac{u''}{b_t - u'} + \frac{(u'')^2}{(b_t - u')^2} \leq 0. \quad (4.22)
\]
Combining (4.20), (4.21) and (4.22) we see that at \((\rho_0, t_0)\),
\[
0 \leq -u'' \left( \frac{1}{(u' - a_t)^2} + \frac{1}{(b_t - u')^2} \right) - \frac{(n - 1)u''}{(u')^2} + \frac{k}{u' - a_t} + \frac{k}{b_t - u'}
\]
and hence
\[
\frac{u''}{(u' - a_t)(b_t - u')} \leq \frac{k(b_t - a_t)}{(u' - a_t)^2 + (b_t - u')^2} \leq C,
\]
and the result follows by the maximum principle. \( \square \)

5 Gromov-Hausdorff convergence

In this section we prove Theorem 1.3 (and hence also Theorem 1.1) on the Gromov-Hausdorff convergence of the Kähler-Ricci flow. We assume in this section that \( g(t) \) is a solution of the Kähler-Ricci flow (1.2) on \([0, T)\) with initial metric \( g_0 \) satisfying the Calabi symmetry condition.
We begin by recalling the definition of Gromov-Hausdorff convergence. It will be convenient to use the characterization given in, for example, [F] or [GW]. Let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces. We define the Gromov-Hausdorff distance $d_{GH}(X, Y)$ to be the infimum of all $\varepsilon > 0$ such that the following holds. There exist maps $F : X \to Y$ and $G : Y \to X$ such that
\[
|d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \leq \varepsilon, \quad \text{for all } x_1, x_2 \in X
\] (5.1) and
\[
d_X(x, G \circ F(x)) < \varepsilon, \quad \text{for all } x \in X
\] (5.2) and the two symmetric properties for $Y$ also hold. Note that we do not require the maps $F$ and $G$ to be continuous.

We say that a sequence of compact metric spaces $(X_i, d_{X_i})$ converges to $(Y, d_Y)$ in the sense of Gromov-Hausdorff if $d_{GH}(X_i, Y)$ tends to zero as $i$ tends to infinity.

### 5.1 The case $k \geq n$

Again we consider first the case when $k \geq n$. We prove the following.

**Theorem 5.1**  $(M, g(t))$ converges to $(\mathbb{P}^{n-1}, a_T g_{FS})$ in the Gromov-Hausdorff sense as $t \to T$.

**Proof** We write $d_{g(t)}$ and $d_T$ for the distance functions on $M$ and $\mathbb{P}^{n-1}$ associated to $g(t)$ and $a_T g_{FS}$ respectively. Let $\varepsilon > 0$ be given. Let $\sigma : \mathbb{P}^{n-1} \to M$ be any smooth map satisfying $\pi \circ \sigma = \text{id}_{\mathbb{P}^{n-1}}$ such that $\sigma(\mathbb{P}^{n-1})$ does not intersect $D_0 \cup D_\infty$. We first verify that (5.2) (and its analog for $Y$) hold whenever $t$ is sufficiently close to $T$. We are taking here $(X, d_X) = (M, d_{g(t)})$, $(Y, d_Y) = (\mathbb{P}^{n-1}, d_T)$, $F = \pi$ and $G = \sigma$ in the definition of Gromov-Hausdorff convergence.

For $x \in M$, by Corollary 4.2
\[
d_{g(t)}(x, \sigma \circ \pi(x)) \leq \text{diam}_{g(t)}(\pi^{-1}(\pi(x))) \to 0
\]
uniformly in $x$ as $t \to T$. Moreover, for any $y$ in $\mathbb{P}^{n-1}$, $d_{FS}(y, \pi \circ \sigma(y)) = 0$ holds trivially.

Now we verify (5.1). First, again by Corollary 4.2 we choose $t$ close enough to $T$ so that for all $z$ in $\mathbb{P}^{n-1}$,
\[
\text{diam}_{g(t)}(\pi^{-1}(z)) < \varepsilon/4.
\] (5.3)

For any $x_1, x_2 \in M$, let $y_i = \pi(x_i) \in \mathbb{P}^{n-1}$. Let $\gamma$ be a geodesic in $\mathbb{P}^{n-1}$ such that $d_T(y_1, y_2) = L_{a_T g_{FS}}(\gamma)$, where $L_{a_T g_{FS}}(\gamma)$ is the arc length of $\gamma$ with respect to $a_T g_{FS}$. Choose a small tubular neighborhood $N$ of $D_0 \cup D_\infty$ so that $\sigma(\mathbb{P}^{n-1})$ does not intersect $N$. Then $\tilde{\gamma} = \sigma \circ \gamma$ is a smooth path in $M \setminus N$ joining the points $x_1' = \sigma(y_1)$ and $x_2' = \sigma(y_2)$. By Theorem 4.2 $g(t)$ converges to $a_T \pi^* g_{FS}$ uniformly on $M \setminus N$, and hence for $t$ sufficiently close to $T$,
\[
L_{g(t)}(\tilde{\gamma}) < L_{a_T g_{FS}}(\gamma) + \varepsilon/2.
\] (5.4)

Then from (5.3) and (5.4),
\[
d_{g(t)}(x_1, x_2) \leq d_{g(t)}(x_1', x_2') + \frac{\varepsilon}{2} \leq L_{g(t)}(\tilde{\gamma}) + \frac{\varepsilon}{2} \leq L_{a_T g_{FS}}(\gamma) + \varepsilon = d_T(y_1, y_2) + \varepsilon.
\] (5.5)
On the other hand, by Lemma 4.2,

\[ d_{g(t)}(x_1, x_2) \geq \left( \frac{a_t}{a_T} \right)^{1/2} d_T(y_1, y_2), \]  

(5.6)

and \( a_t \to a_T \) as \( t \to T \). Combining (5.5) and (5.6) gives

\[ |d_{g(t)}(x_1, x_2) - d_T(\pi(x_1), \pi(x_2))| \leq \varepsilon, \]

and hence the first case of (5.1).

The second case of (5.1) is simpler. Let \( y_1, y_2 \) be in \( \mathbb{P}^{n-1} \) and write \( x_i = \sigma(y_i) \). Since \( \sigma(\mathbb{P}^{n-1}) \) does not intersect \( D_0 \cup D_\infty \), we can apply Theorem 4.2 to obtain the following convergence uniformly in \( t \) and the choice of \( y_1, y_2, x_1, x_2 \):

\[ \lim_{t \to T} d_{g(t)}(x_1, x_2) = d_T(y_1, y_2), \]

as required. \( \square \)

5.2 The case \( 1 \leq k \leq n - 1 \).

If \( a_0(n + k) > b_0(n - k) \) then one can apply verbatim the argument as in the case \( k \geq n \) and the Kähler-Ricci flow collapses to \( \mathbb{P}^{n-1} \).

If \( a_0(n + k) = b_0(n - k) \), then \( \alpha_0 \) is proportional to the first Chern class \( c_1(M) \). As discussed in subsection 3.2.1, the diameter is uniformly bounded along the normalized Kähler-Ricci flow with initial Kähler metric in \( c_1(M) \). Then after scaling, the diameter tends to 0 along the unnormalized Kähler-Ricci flow.

We consider then just the subcase \( a_0(n + k) < b_0(n - k) \). We will show that as \( t \to T \) the divisor \( D_0 \) in \( M \) contracts. First, we have the following lemma.

Lemma 5.1 There exists a uniform constant \( C \) such that the metric \( g_\overline{\sigma} = g_{\overline{j}}(t) \) on \( \mathbb{C}^n \setminus \{0\} \) satisfies the estimate

\[ g_{\overline{j}}(t) \leq a_t \chi_{\overline{j}} + C e^{(k-n)\rho/n} \delta_{ij}, \] 

(5.7)

where \( \chi_{\overline{j}} = e^{-\rho} \delta_{ij} - e^{-2\rho} x_i x_j \).

Proof This follows from Lemmas 4.4 and 4.5. Indeed,

\[
\begin{align*}
g_{\overline{j}} & = e^{-\rho} u^t \delta_{ij} + e^{-2\rho} x_i x_j (u'' - u') \\
& \leq C e^{(k-n)\rho/n} \delta_{ij} + e^{-\rho} a_t \delta_{ij} + C e^{(k-n)\rho/n} e^{-2\rho} x_i x_j a_t \\
& \leq a_t \chi_{\overline{j}} + C e^{(k-n)\rho/n} \delta_{ij},
\end{align*}
\]

since \( u'(\rho, t) > a_t \). \( \square \)

Recall that by Theorem 3.2, the metric \( g_t \) along the Kähler-Ricci flow converges in \( C^\infty \) on compact subsets of \( M \setminus D_0 \) to a singular metric \( g_T \) which is smooth on \( M \setminus D_0 \). We will apply this and Lemma 5.1 to prove the following result on the metric completion of the manifold \( (M \setminus D_0, g_T) \).
**Theorem 5.2** Let \( g_T \) be the smooth metric on \( M \setminus D_0 \) obtained by
\[
g_T = \lim_{t \to T} g(t),
\]
and let \((\overline{M}, d)\) be the completion of the Riemannian manifold \((M \setminus D_0, g_T)\) as a metric space. Then \((\overline{M}, d)\) is a metric space with finite diameter and is homeomorphic to the orbifold \( \mathbb{P}^n / \mathbb{Z}_k \) (see Section 2.3).

**Proof** Let \( f : M \to \mathbb{P}^n / \mathbb{Z}_k \) be the holomorphic map described in Section 2.3. Recall that \( f \) restricted to \( M \setminus D_0 \) is represented in the \((x_1, \ldots, x_n)\) coordinates by the identity map \( \text{id} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\} \). Now \( f \) is an isomorphism on \( M \setminus D_0 \) and Theorem 3.2 implies that \( g(t) \) converges locally in \( C^\infty(M \setminus D_0) \). Thus it only remains to check the limiting behavior of \( g(t) \) near \( D_0 \) as \( t \to T \) (that is, in a neighborhood of the origin in \( \mathbb{C}^n \)).

We make a simple observation. Suppose \( g_{ij} \) is a continuous Riemannian metric on \( B \setminus \{0\} \), where \( B = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid |x_1|^2 + \cdots + |x_n|^2 \leq 1\} \). Suppose in addition that \( g_{ij} \) satisfies the inequality
\[
g_{ij} \leq \frac{C}{r^\beta \delta_{ij}},
\]
for some \( \beta < 2 \), where \( r = (|x_1|^2 + \cdots + |x_n|^2)^{1/2} \). Then the completion of \((B \setminus \{0\}, g)\) as a metric space has finite diameter and is homeomorphic to \( B \) with topology induced from \( \mathbb{C}^n \).

Now from Lemma 5.1 since \( a_t \to 0 \) as \( t \to T \), we see that on \( \mathbb{C}^n \setminus \{0\} \),
\[
(g_T)_{ij} \leq \frac{C}{r^{2(n-k)/n} \delta_{ij}}
\]
and hence the theorem follows from the observation above with \( \beta = 2(n-k)/n \). \( \square \)

In addition, the proof of Theorem 5.2 gives:

**Lemma 5.2** Let \( N_\varepsilon \) be an \( \varepsilon \)-tubular neighborhood of \( D_0 \) in \( M \) with respect to the fixed metric \( \hat{g}_0 \). Then
\[
\lim_{\varepsilon \to 0} \limsup_{t \to T} \text{diam}_{g(t)} N_\varepsilon = 0.
\]

We can then prove:

**Theorem 5.3** \((M, g(t))\) converges to \((\overline{M}, d)\) in the Gromov-Hausdorff sense as \( t \to T \), where \((\overline{M}, d)\) is the metric space as described in Theorem 5.2.

**Proof** This is a simple consequence of the results described above. Identifying \( \overline{M} \) with \( \mathbb{P}^n / \mathbb{Z}_k \) as in the proof of Theorem 5.2, let \( F : M \to \overline{M} \) be the map corresponding to \( f : M \to \mathbb{P}^n / \mathbb{Z}_k \). Let \( G : \overline{M} \to M \) be any map satisfying \( F \circ G = \text{id}_{\overline{M}} \). Then it is left to the reader to check that, using these functions \( F \) and \( G \), the Gromov-Hausdorff distance between \((M, g(t))\) and \((\overline{M}, d)\) tends to zero as \( t \to T \). \( \square \)

This completes the proof of Theorem 1.3.

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