Sharp gradient estimates for a heat equation in Riemannian manifolds

HA TUAN DUNG, NGUYEN THAC DUNG

Abstract

In this paper, we prove sharp gradient estimates for a positive solution to the heat equation \( u_t = \Delta u + au \log u \) in complete noncompact Riemannian manifolds. As its application, we show that if \( u \) is a positive solution of the equation \( u_t = \Delta u \) and \( \log u \) is of sublinear growth in both spatial and time directions then \( u \) must be constant. This gradient estimate is sharp since it is well-known that \( u(x,t) = e^{x+t} \) satisfying \( u_t = \Delta u \). We also emphasize that our results are better than those given by Jiang ([Jia16]), Souplet-Zhang ([SZ06]), Wu ([Wu15, Wu17]), and others.

1 Introduction

In 1993, Hamilton [Ham93] proved a gradient estimate for a positive solution \( u \) to the heat equation

\[ u_t = \Delta u \tag{1.1} \]

in complete compact Riemannian manifolds with the Ricci curvature bounded from below by \(-K, (K \leq 0)\). If \( u \leq A \) then it was verified by Hamilton that

\[ \frac{\|\nabla u\|}{u} \leq \left( \sqrt{2K} + \frac{1}{\sqrt{t}} \right) \sqrt{\log \left( \frac{A}{u} \right)}. \]

Motivated by Hamilton’s result, in 2006, Souplet and Zhang [SZ06] introduced a new gradient estimate for the heat equation (1.1) in complete non-compact Riemannian manifolds. Assume that the Ricci curvature bounded from below \(-K (K \geq 0)\) and \( u \) is a positive solution to the
heat equation in \( Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times \mathbb{R} \), they showed that if \( u \leq A \) in \( Q_{R,T} \) then there exists a dimensional constant \( c(n) \) such that the following local gradient estimate

\[
\frac{|\nabla u|}{u} \leq c(n) \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \frac{\sqrt{K}}{R} \right) \left( 1 + \log \frac{A}{u} \right)
\]

holds true. As an its consequence, Souplet and Zhang obtained a Liouville theorem for positive ancient solutions \( u \) of the heat equation provided that \( u = e^{o(d(x)+\sqrt{t})} \). Since \( u = e^{x+t} \) satisfies \( u_t = \Delta u \), their growth condition in the spatial direction is sharp. However, it is clear that their gradient estimate is not sharp in the time direction by the same example. Recently, gradient estimate of Hamilton type or Souplet-Zhang type are extendly studied in several works. For example, as pointed out by google scholar, there are more than one hundred papers cited Souplet-Zhang’s gradient estimate. For further details, we also refer the reader to [DKN18, Ham93, SZ06, Wu15, Wu17] and the references there in. On the other hand, we also remark that Souplet-Zhang gave an example in [SZ06] showing that Hamilton’s estimate for the compact case cannot be extended directly, in a localized form, to noncompact manifolds. Moreover, as Souplet-Zhang’s observation, it has been known that the right hand side term in Souplet-Zhang’s gradient estimate is slightly bigger than that in Hamilton’s theorem (in the power of the log term).

In this paper, inspired by Hamilton’s idea and Souplet-Zhang’s work, we want to complete the picture about sharp gradient estimates in time direction for positive solution of the linear heat equation \( u_t = \Delta u \) on complete non-compact Riemannian manifolds and local Hamilton’s gradient for complete noncompact manifolds. For this purpose, our first aim is to point out a gradient estimate for a positive solution to the following equation

\[
(1.2) \quad u_t = \Delta u + au \log u,
\]

where \( a \) is a real constant, on complete non-compact Riemannian manifolds. As showed later, due to Souplet-Zhang’s example, to extend Hamilton’s gradient estimate to the non-compact case, we need to insert necessary correction term in the right hand side. However, as good as Souplet-Zhang’s gradient estimate, our estimate holds for noncompact manifolds and it also has a localized version as the Cheng-Yau estimate and the Hamilton’s gradient estimate. The second aim of this paper is to derive a sharp gradient estimate in both spatial and time directions. Namely, we will prove that the Liouville property holds true if a positive solution \( u \) of the equation \( u_t = \Delta u \) is of growth \( e^{o(d(x)+|t|)} \). This Liouville result confirms that our gradient estimate is sharp, exactly. Now, this is suitable time for us to introduce our main theorem.

**Theorem 1.1.** Let \((M^n, g)\) be an \( n \)-dimensional complete Riemannian manifold with

\[
\text{Ric} \geq -(n-1)K
\]
for some constant $K \geq 0$ in $B(x_0, R)$ some fixed point $x_0$ in $M$, and some fixed radius $R \geq 2$. Assume that $0 < u(x, t) < A$ for some constant $A$, is a smooth solution to the general heat equation (1.2) in the cylinder $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$, where $t_0 \in R$ and $T > 0$, then there exists a constant $c(n)$ such that

\begin{equation}
\frac{\|\nabla u\|}{u} \leq c(n) \left( \sqrt{\log A - \log \left( \inf_{Q_{R,T}} u \right)} + \frac{1}{\sqrt{T}} + 4 \sqrt{\frac{K}{R^2}} + \sqrt{H} \right) \sqrt{\log \frac{A}{u}} \tag{1.3}
\end{equation}

in $Q_{\frac{R}{2}, \frac{T}{2}}$, where

$$H = \max \{ 2(n - 1)K + a, 0 \} + \max \{ 2a \log A, 0 \}.$$ 

Note that if $T \geq 2t_0$, then (1.3) holds true at $(x_0, -\frac{T}{2})$. From this observation, we obtain local a Hamilton’s gradient estimate by using (1.3). As applications of Theorem 1.1, we have the following Liouville theorems.

**Corollary 1.2.** Let $M$ be a complete, noncompact manifold with nonnegative Ricci curvature. If $u$ is a positive ancient solution to the linear heat equation (1.1) (that is, a solution defined in all space and negative time) such that $u(x, t) = e^{o(d(x) + |t|)}$ near infinity. Then $u$ is a constant.

Note that $u(x, t) = e^{x+t}$ is a positive solution to $u_t = \Delta u$. Corollary (1.2) implies that our gradient estimate is sharp in both spatial and time directions. Therefore, this results is better than those in [DKN18, SZ06, Wu15, Wu17]. It is also worth to mention that such Liouville theorem still holds true for eternal solution of the heat equation (1.1) (that is, a solution defined in all space and positive time).

Now, we emphasize that our gradient estimate is not only better than Souplet-Zhang’s gradient estimate for positive solution $u$ of (1.1) but also better than recent gradient estimates for positive solutions of the general heat equation $u_t = \Delta u + au \log u$, where $a \neq 0$. In fact, we obtain the following Liouville type result.

**Corollary 1.3.** Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric} \geq -(n - 1)K$ for some constant $K \geq 0$. Suppose $u$ is a positive and bounded solution to the equation (1.2) with $a \leq 0$ and $u$ is independent of time. Assume $1 \leq u \leq A$. If $a \leq -2(n - 1)K$ then $u \equiv 1$.

**Remark 1.4.** Note that if $a \leq 0$ and $1 \leq u \leq A$, using the inequality $\log x \leq x$ for all $x \geq 1$, then (1.3) can be rewritten as following

$$\frac{\|\nabla u\|}{u^{1/2}} \leq c(n) \sqrt{A} \left( \frac{\sqrt{\log A}}{R} + \frac{1}{\sqrt{T}} + 4 \sqrt{\frac{K}{R^2}} + \sqrt{\max \{ 2(n - 1)K + a, 0 \}} \right).$$
Therefore, our estimate is better than those in [Jia16, Wu17] for the case $a \leq 0$. Moreover, the Liouville type results we obtained in the case $a \leq 0$ for the equation (1.2) are also better than those in [DKN18, HM15, Li15, Jia16, Wu17].

Finally, by using the same argument in the proof of Theorem 1.1 and a Laplacian comparison theorem given by Wei and Wylie [WW09] (see also [Bri13]), it is worth to notice that our gradient estimates are also valid for a positive solution to

$$u_t = \Delta_f u + au \log u$$

on smooth metric measure spaces $(M, g, e^{-f}du)$, where $f$ is the weighted function (see Remark 2.3). Hence, we can improve a Liouville property in Theorem 3.2 by Wu ([Wu15]). Moreover, due to Example 1.2 in [Wu15], our gradient estimate is sharp. The paper has two sections. Beside this section, we use section 2 to prove Theorem 1.1 and its corollaries.

## 2 Sharp gradient estimates

In this section, we will give a proof of Theorem 1.1. By now a standard routine, we need two basic lemmas. To begin with, let us introduce some notations. Consider the nonlinear heat equation

(2.1) \[ u_t = \Delta u + au \log u, \]

where $a$ is a real constant, on an an $n$-dimensional complete Riemannian manifold $(M^n, g)$. Suppose that $u(x,t)$ is a solution of (2.1) and $0 < u \leq A$ for some constant $A$ in the cylinder

$$Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$$

where $t_0 \in \mathbb{R}$ and $T > 0$. We introduce a new smooth function

$$h = \sqrt{\log \frac{A}{u}} \geq 0$$

in $Q_{R,T}$. Observe that

$$u = Ae^{-h^2} \quad \text{and} \quad \log u = \log A - h^2.$$

This implies,

$$u_t = -2Ah\nabla e^{-h^2}, \quad \nabla u = -2Ah\nabla e^{-h^2},$$
and
\[
\Delta u = \nabla (\nabla u) = \nabla \left( -2Ah \nabla h e^{-h^2} \right) \\
= -2A \left[ \nabla (h \nabla h) e^{-h^2} + h \nabla h \nabla \left( e^{-h^2} \right) \right] \\
= -2A \left[ |\nabla h|^2 e^{-h^2} + h \Delta h e^{-h^2} - 2h^2 |\nabla h|^2 e^{-h^2} \right] \\
= -2Ahe^{-h^2} \left[ \Delta h + |\nabla h|^2 \left( \frac{1}{h} - 2h \right) \right].
\]

From the equation (2.1), we obtain
\[
-2Ah_i e^{-h^2} = -2Ahe^{-h^2} \left[ \Delta h + |\nabla h|^2 \left( \frac{1}{h} - 2h \right) \right] + 2Ahe^{-h^2} \langle \nabla f, \nabla h \rangle \\
+ aAe^{-h^2} \left( \log A - h^2 \right),
\]
or equivalently
\[
(2.2) \quad h_t = \Delta h + |\nabla h|^2 \left( \frac{1}{h} - 2h \right) - \frac{a}{2} \left( \log A - h \right).
\]

Using the above equality, we establish the first computational lemma.

**Lemma 2.1.** Let \( w = |\nabla h|^2 \). For any \((x, t) \in Q_{R, T}\),
\[
\Delta w - w_t \geq - \left[ \max \left\{ 2(n - 1) K + a, 0 \right\} + \frac{1}{h^2} \max \left\{ a \log A, 0 \right\} \right] w \\
+ 2 \left( 2h - \frac{1}{h} \right) \langle \nabla w, \nabla h \rangle + 2 \left( 2 + \frac{1}{h^2} \right) w^2.
\]

**Proof.** By the Bochner-Weitzenböck formula, for any function \( \psi \), we have
\[
\frac{1}{2} \Delta |\nabla \psi|^2 = |\nabla^2 \psi|^2 + \text{Ric}(\nabla \psi, \nabla \psi) + \langle \nabla \Delta \psi, \nabla \psi \rangle.
\]
Therefore, under the assumption \( \text{Ric} \geq -(n - 1) K \), after choosing \( \psi = |\nabla h|^2 \) we deduce that
\[
\Delta w - w_t \geq 2|\nabla^2 h|^2 - 2(n - 1) K |\nabla h|^2 + 2 \langle \nabla \Delta h, \nabla h \rangle - w_t \\
\geq -2(n - 1) Kw + 2 \langle \nabla \Delta h, \nabla h \rangle - w_t.
\]

By the equality (2.2), we obtain
\[
\Delta w - w_t \geq -2(n - 1) Kw + 2 \left\langle \nabla \left( h_t + |\nabla h|^2 \left( 2h - \frac{1}{h} \right) \right) + \frac{a}{2} \left( \log A - h \right), \nabla h \right\rangle \]
\[
\geq -2(n - 1) Kw + 2 \langle \nabla (h_t), \nabla h \rangle + 2 \left( 2h - \frac{1}{h} \right) \langle \nabla \left( |\nabla h|^2 \right), \nabla h \rangle \\
+ 2|\nabla h|^2 \left\langle \nabla \left( 2h - \frac{1}{h} \right), \nabla h \right\rangle + a \left\langle \nabla \left( \frac{\log A}{h} - h \right), \nabla h \right\rangle - w_t.
\]
Observe that
\[ 2 \langle \nabla (h_t), \nabla h \rangle = (|\nabla h|^2)_t = w_t, \]
\[ \nabla \left( 2h - \frac{1}{h} \right) = 2 \nabla h + \frac{\nabla h}{h^2} = \left( 2 + \frac{1}{h^2} \right) \nabla h, \]
and
\[ \nabla \left( \frac{\log A}{h} - h \right) = - \log A \frac{\nabla h}{h^2} - \nabla h = - \left( \frac{\log A}{h^2} + 1 \right) \nabla h. \]
Hence, the inequality (2.4) implies
\[ \Delta w - w_t \geq - [2 (n - 1) K + a] w - \frac{a \log A}{h^2} w + 2 \left( 2h - \frac{1}{h} \right) \langle \nabla w, \nabla h \rangle + 2 \left( 2 + \frac{1}{h^2} \right) w^2. \]
Notice that
\[ \frac{a \log A}{h^2} \leq \frac{1}{h^2} \max \{ a \log A, 0 \}, \]
\[ [2 (n - 1) K + a] \leq \max \{ 2 (n - 1) K + a, 0 \}. \]
Therefore, we have
\[ \Delta w - w_t \geq - \left[ \max \{ 2 (n - 1) K + a, 0 \} + \frac{1}{h^2} \max \{ a \log A, 0 \} \right] w \]
\[ + 2 \left( 2h - \frac{1}{h} \right) \langle \nabla w, \nabla h \rangle + 2 \left( 2 + \frac{1}{h^2} \right) w^2. \]
The proof is complete. \( \square \)

Next, we introduce a smooth cut-off function originated in [LY86] (see also [SZ06, Jia16]).

**Lemma 2.2.** There exists a smooth cut-off function \( \psi = \psi(x, t) \) supported in \( Q_{R,T} \), satisfying following propositions

(i) \( \psi = \psi(d(x, x_0), t) \equiv \psi(r, t); \psi(r, t) = 1 \) in \( Q_{\frac{R}{2}, \frac{T}{2}} \), \( 0 \leq \psi \leq 1 \).

(ii) \( \psi \) is decreasing as a radial function in the spatial variables, and \( \frac{\partial \psi}{\partial r} = 0 \) in \( Q_{R/2, T} \).

(iii) \( \left| \frac{\partial \psi}{\partial r} \right| \frac{1}{\psi^{1/2}} \leq \frac{C}{T} \).

(iv) \( \left| \frac{\partial \psi}{\partial r} \right| \leq \frac{C_\varepsilon \psi^\varepsilon}{R^2} \) and \( \left| \frac{\partial^2 \psi}{\partial r^2} \right| \leq \frac{C_\varepsilon \psi^\varepsilon}{R^2} \), when \( 0 < \varepsilon < 1 \).

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.1.** Due to the standard argument of Calabi (see [Cal57]), we may assume that \( (\psi w) \) obtains its maximal value at \( (x_1, t_1) \) and we also may assume that that \( x_1 \) is not on the cut-locus of \( M \). At \( (x_1, t_1) \), we have
\[ \nabla (\psi w) = 0, \quad \nabla (\psi w) \leq 0 \quad \text{and} \quad (\psi w)_t \geq 0. \]
Hence, still being at \((x, t)\), we obtain
\[
0 \geq \Delta (\psi w) - (\psi w)_t = \psi (\Delta w - w_t) + w (\Delta \psi - \psi_t) + 2 \langle \nabla w, \nabla \psi \rangle.
\]

Using the fact that
\[
0 = \nabla (\psi w) = w \nabla \psi + \psi \nabla w,
\]
we get
\[
0 \geq \psi (\Delta w - w_t) + w \Delta \psi - w \psi_t - 2 \frac{|\nabla \psi|^2}{\psi} w.
\]

This inequality combining with \((2.3)\) implies
\[
0 \geq - \left[ \max \{2 (n - 1) K + a, 0\} + \frac{1}{h^2} \max \{a \log A, 0\} \right] \psi w - 2 \left( 2h - \frac{1}{h} \right) \langle \nabla h, \nabla \psi \rangle w + \frac{2}{1 + 2h^2} \psi w^2 + w \Delta \psi - w \psi_t - 2 \frac{|\nabla \psi|^2}{\psi} w.
\]
at \((x, t)\). In other words, we have just proved that
\[
4 \psi^2 \leq \left[ \frac{2h^2}{1 + 2h^2} \max \{2 (n - 1) K + a, 0\} + \frac{2}{1 + 2h^2} \max \{a \log A, 0\} \right] \psi w - \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w - \frac{2h^2}{1 + 2h^2} w \Delta \psi - \frac{2h^2}{1 + 2h^2} w \psi_t - \frac{4h^2}{1 + 2h^2} \frac{|\nabla \psi|^2}{\psi} w.
\]

Since \(0 < \frac{2h^2}{1 + 2h^2} \leq 1\) and \(0 < \frac{2}{1 + 2h^2} \leq 2\), we get
\[
4 \psi^2 \leq H \psi w - \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w - \frac{2h^2}{1 + 2h^2} w \Delta \psi + w |\psi_t| + \frac{2}{\psi} \frac{|\nabla \psi|^2}{\psi} w
\]
at \((x, t)\). Since \(\text{Ric} \geq -(n - 1) K\), we can apply the Laplacian comparison theorem to get
\[
\Delta r \leq \frac{n - 1}{r} \left( 1 + \sqrt{K} r \right).
\]

Using the Laplacian comparison theorem again and Lemma \((2.2)\) we first have
\[
- \frac{2h^2}{1 + 2h^2} w \Delta \psi = - \frac{2h^2}{1 + 2h^2} w \left[ \psi_r \Delta r + \psi_{rr} |\nabla r|^2 \right]
\]
\[
\leq \frac{2h^2}{1 + 2h^2} w \left| \psi_r \right| \left( \frac{n - 1}{r} \left( 1 + \sqrt{K} r \right) \right) + |\psi_{rr}| \left| \psi_r \right|
\]
\[
\leq \psi^{1/2} w \frac{\psi_{rr}}{\psi^{1/2}} + \psi^{1/2} \left( \frac{n - 1}{r} \left( 1 + \sqrt{K} r \right) \right) \left| \psi_r \right| \psi^{1/2} \left( \frac{n - 1}{r} \left( 1 + \sqrt{K} r \right) \right) \left| \psi_r \right| \psi^{1/2}
\]
\[
\leq \frac{3}{5} \psi w^2 + c \left[ \left( \frac{|\psi_{rr}|}{\psi^{1/2}} \right)^2 + \left( \frac{n - 1}{r} \left( 1 + \sqrt{K} r \right) \right)^2 \left| \psi_r \right| \psi^{1/2} \right]^2
\]
\[
\leq \frac{3}{5} \psi w^2 + c \frac{K}{R^2}.
\]
On the other hand, by the Young inequality, we have
\[
- \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w \leq 4h \frac{|1 - 2h^2|}{1 + 2h^2} |\nabla \psi| |\nabla h| w
\leq 4h |\nabla \psi| w^{3/2} = 4h |\nabla \psi| \psi^{-3/4} (\psi w^2)^{3/4}
\leq \frac{3}{5} \psi w^2 + ch^4 \frac{|\nabla \psi|^4}{\psi^3}
\]
\[
\leq \frac{3}{5} \psi w^2 + cD^2 \frac{R^4}{R^4},
\]
(2.7)
where
\[
D = \log A - \log \left( \inf_{Q_{R,T}} u \right).
\]
By using the Cauchy-Schwarz inequality several times, it is not hard for us to see that the following estimates hold true: first for \(\psi w\)
\[
H \psi w \leq \frac{3}{5} \psi w^2 + cH^2,
\]
(2.8)
then for \(w \psi_t\) as follows
\[
w \psi_t = \psi^{1/2} w \psi_t \psi^{1/2} \leq \frac{3}{5} \left( \psi^{1/2} w \right)^2 + c \left( \frac{|\psi_t|}{\psi^{1/2}} \right)^2
\leq \frac{3}{5} \psi w^2 + \frac{c}{T^2}
\]
(2.9)
and finally for as the following
\[
\frac{2|\nabla \psi|^2}{\psi} w = 2 \left( |\nabla \psi|^2 \psi^{-3/2} \right) \left( \psi^{1/2} w \right)
\leq \frac{3}{5} \psi w^2 + c \frac{|\nabla \psi|^4}{\psi^3}
\leq \frac{3}{5} \psi w^2 + \frac{c}{R^4}.
\]
(2.10)
We now substitute (2.6)-(2.10) into the right hand side of (2.5), and get that
\[
\psi w^2 \leq c \left( \frac{D^2 + 1}{R^4} + \frac{K}{R^2} + \frac{1}{T^2} + H^2 \right)
\]
(2.11)
at \((x_1, t_1)\). Then, for all \((x, t) \in Q_{R,T}\), using (2.11) we obtain
\[
\psi^2 w^2 (x, t) \leq \psi^2 w^2 (x_1, t_1)
\leq \psi w^2 (x_1, t_1)
\leq c \left( \frac{D^2 + 1}{R^4} + \frac{K}{R^2} + \frac{1}{T^2} + H^2 \right).
\]
(2.12)
Sharp gradient estimates

Notice that \( \psi(r,t) = 1 \) in \( Q_{\frac{1}{2}, \frac{1}{2}} \) and \( w = |\nabla h|^2 \), we have

\[
|\nabla u|_u \leq c(n) \left( \sqrt{\log A - \log \left( \inf_{Q_{R,T}} u \right)} + \frac{1}{R} + \sqrt{\frac{K}{R^2} + \sqrt{H}} \right) \sqrt{\log \frac{A}{u}}
\]

in \( Q_{\frac{1}{2}, \frac{1}{2}} \), where \( c = c(n) \). The proof is complete.

We would like to mention that if we let \( Q_{R,T} = B(x_0, R) \times [t_0, t_0 + T] \) and construct a similarly test function as in Lemma 2.2 then we can obtain a gradient estimate for positive eternal solution to (1.3). Now, we give a verification of Corollary 1.2.

**Proof of Corollary 1.2.** By assumption, we have \( K = H = 0 \). Since \( u_t = \Delta u \), let \( v = u + 1 \), then \( v \) satisfies \( v_t = \Delta v \). Moreover, \( u, v \) has the same growth at infinity. Hence, without loss of generality, we may assume that \( u \geq 1 \). Fixing \( (x_0, t_0) \) in space-time, Theorem 1.1 applying to the cube \( Q_{R,R} = B(x_0, R) \times [t_0 - R, t_0] \) implies that

\[
|\nabla u|_{(x_0, t_0)} \leq c(n) \left( \frac{\sqrt{\log A}}{R} + \frac{1}{\sqrt{R}} \right) \sqrt{\log \frac{A}{u(x_0, t_0)}}
\]

\[
\leq c(n) \left( \frac{\sqrt{o(R + |R|)}}{R} + \frac{1}{\sqrt{R}} \right) \sqrt{o(R + |R|) - \log(u(x_0, t_0))}.
\]

Let \( R \) goes to infinity, it turns out that \( |\nabla u(x_0, t_0)| = 0 \). Since \( (x_0, t_0) \) is arbitrary, we conclude that \( u \) is constant.

**Theorem 1.1** in fact implies Corollary 1.3.

**Proof of Corollary 1.3.** Note that if \( a \leq 0 \) and \( 1 \leq u \leq A \) then the inequality (1.3) can be rewritten as following

\[
|\nabla u|_u \leq c(n) \left( \frac{\sqrt{\log A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{\frac{K}{R^2} + \max\{2(n-1)K + a, 0\}} \right) \sqrt{\log \frac{A}{u}}
\]

Since \( a \leq -2(n-1)K \), we get

\[
\max\{2(n-1)K + a, 0\} = 0.
\]

Letting \( R \to +\infty, T \to +\infty \) in (2.14), we obtain \( u \) is a constant. Using \( \Delta u + au \log u = 0 \), we get \( u \equiv 1 \).
Remark 2.3. Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete smooth metric measure space with
\[
\text{Ric}_f \geq -(n-1)K
\]
for some constant \(K \geq 0\) in \(B(x_0, R)\), for some fixed point \(x_0\) in \(M\), and some fixed radius \(R \geq 2\). Assume that \(0 < u(x, t) < A\) for some constant \(A\), is a smooth solution to the general \(f\)-heat equation.

\[
(2.15) \quad u_t = \Delta_f u + au \log u,
\]
in the cylinder \(Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)\), where \(a \in \mathbb{R}\) is fixed, \(t_0 \in \mathbb{R}\) and \(T > 0\). Following the proof of Theorem 1.1 and using Laplacian comparison theorem as in [Bri13] instead of classical Laplacian comparison theorem, we can show that there exists a constant \(c(n)\) such that

\[
(2.16) \quad \frac{\left|\nabla u\right|}{u} \leq c(n)\left(\sqrt{\log A - \log \left(\inf_{Q_{R,T}} u\right)} + \sqrt{|\alpha R|} + \frac{1}{\sqrt{T}} + \sqrt{K + \sqrt{H}}\right) \sqrt{\log \frac{A}{u}}.
\]
in \(Q_{\frac{R}{2}, \frac{T}{2}}\), where
\[
H = \max \{a, 0\} + \max \{2a \log A, 0\}.
\]
Here, \(\alpha := \max_{x \in \partial B(x_0, 1)} \Delta_f \rho(x)\), where \(\rho(x)\) is the distance from \(x\) to \(x_0\). Since the proof of this gradient estimate is similar to the proof of Theorem 1.1, we omit the detail.

From this remark, we observe that if \(\text{Ric}_f \geq 0\), let \(u\) be a positive solution to \((2.15)\), where \(a \leq 0\) and \(u = e^{\alpha(d(x) + \sqrt{t})}\) then \(u\) must be constant, namely \(u \equiv 1\). The Example 1.2 in [Wu15] implies that the estimate \((2.16)\) is sharp. Indeed, let us recall the example.

Example 2.4. For any \(a, b > 0\) let
\[
u = e^{ax + (a^2 + ab)t}, \quad f = -bx
\]
them \(u\) satisfies the heat equation \(u_t = \Delta_f u\).

This example confirms that our gradient estimate is sharp in both spatial and time directions. In [Wu15], to obtain a Liouville property, the author had to require that the potential function \(f\) is bounded and \(u = e^{\alpha(d(x) + \sqrt{t})}\).
Acknowledgment

This work was initiated when second author visited ICTP to participate in the international school on Extrinsic Geometric Flows. He would like to thank the staff there for hospitality and financial support. This work was partially supported by NAFOSTED under grant number 101.02-2017.313.

References

[Bri13] K. Brighton, *A Liouville-type theorem for smooth metric measure spaces*, Jour. Geom. Anal., **23** (2013), 562–570.

[Cal57] E. Calabi, *An extension of E. Hopf’s maximum principle with an application to Riemannian geometry*, Duke Math. Jour. **25** (1957), 45-56.

[DKN18] N. T. Dung, N. N. Khanh, and Q. A. Ngo, *Gradient estimates for some f-heat equations driven by Lichnerowicz’s equation on complete smooth metric measure spaces*, Manuscripta Math., **155** (2018), 471 - 501.

[Ham93] R. S. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom **1** (1993), 113-126.

[HM15] G. Huang, B. Ma, *Gradient estimates and Liouville type theorems for a nonlinear elliptic equation*, Arch. Math., **105** (2015), 491 - 499.

[LY86] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math, **156** (1986) (3-4) 153 - 201.

[Li15] Y. Li, *Li-Yau-Hamilton estimates and Bakry-Émery Ricci curvature*, Nonlinear Analysis: TMA, **113** (2015), 1-32.

[Jia16] X. R. Jiang, *Gradient estimate for a nonlinear heat equation on Riemannian manifolds*, Proc. Amer. Math. Soc., **144** (2016) 3635-3642.

[SZ06] P. Souplet, Q.S. Zhang, *Sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc., **38** (2006), 1045-1053.

[WW09] G. F. Wei and W. Wylie, *Comparison geometry for the Bakry-Émery Ricci tensor*, Jour. Differ. Geom., **83** (2009), 377-405.
[Wu15] J. Y. Wu, *Elliptic gradient estimates for a weighted heat equation and applications*, Math. Zeits., **280** (2015), 451-468. 1, 2, 3, 4, 10

[Wu17] J. Y. Wu, *Elliptic gradient estimates for a nonlinear heat equation and applications*, Nonlinear Analysis: TMA, **151** (2017), 1-17. 1, 2, 3, 4

Ha Tuan Dung
FACULTY OF MATHEMATICS
HANOI PEDAGOGICAL UNIVERSITY NO. 2
XUAN HOA, VINH PHUC, VIETNAM
E-mail address: hatuandung.hpu2@gmail.com

Nguyen Thac Dung
FACULTY OF MATHEMATICS - MECHANICS - INFORMATICS
HANOI UNIVERSITY OF SCIENCE (VNU)
HA NOI, VIET NAM
E-mail address: dungmath@gmail.com