Systematic Convergence of Nonlinear Stochastic Estimators on the Special Orthogonal Group $\text{SO}(3)$

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Abstract—This paper introduces two novel nonlinear stochastic attitude estimators developed on the Special Orthogonal Group $\text{SO}(3)$ with the tracking error of the normalized Euclidean distance meeting predefined transient and steady-state characteristics. The tracking error is confined to initially start within a predetermined large set such that the transient performance is guaranteed to obey dynamically reducing boundaries and decrease smoothly and asymptotically to the origin in probability from almost any initial condition. The proposed estimators produce accurate attitude estimates with remarkable convergence properties using measurements obtained from low-cost inertial measurement units. The estimators proposed in continuous form are complemented by their discrete versions for the implementation purposes. The simulation results illustrate the effectiveness and robustness of the proposed estimators against uncertain measurements and large initialization error, whether in continuous or discrete form.

Index Terms—Attitude estimates, transient, steady-state error, nonlinear filter, special orthogonal group, $\text{SO}(3)$, stochastic system, stochastic differential equations, Ito, Stratonovich, asymptotic stability, Wong-Zakai, inertial measurement unit, IMU, prescribed performance function.

I. INTRODUCTION

EXECUTION of successful autonomous maneuvers requires accurate information regarding the orientation of the vehicle. Nonetheless, the orientation, commonly known as attitude, cannot be obtained directly. Instead, attitude has to be acquired, using a set of measurements made on the body-frame. These measurements are subject to unknown bias and noise components, especially if they are supplied by low-cost inertial measurement units (IMU). There are multiple ways to approach the problem of the attitude estimation. For instance, the attitude can be established algebraically using QUEST [1] and singular value decomposition (SVD) [2]. However the static methods of estimation presented in [1] and [2] are characterized by poor results which differ significantly from the true attitude [3–6].

Historically, the attitude estimation problem has been tackled using Gaussian filters, such as Kalman filter (KF) [7], extended KF (EKF) [8], multiplicative EKF [9]. A complete survey of Gaussian attitude filters can be found in [3]. However, rapid development of low-cost IMU and natural non-linearity of the attitude problem led to the proposal of multiple nonlinear deterministic attitude estimators, for instance [10–14]. In fact, nonlinear deterministic attitude estimators received considerable attention and have notable comparative advantages over Gaussian attitude filters, namely, they are characterized by better tracking performance and less computational power in comparison with Gaussian attitude filters [3,5,6,10]. Nonlinear deterministic attitude estimators can be easily constructed knowing a rate gyroscope measurement and two or more vector measurements. A crucial part of the attitude estimator design is the selection of the error function which has significant influence on the transient performance and the steady-state error. The error function proposed in [13] has been slightly modified in [10,12]. However, the overall performance has not changed notably. The main limitation of the error functions proposed in [10,12,13] is the slow convergence, in particular, when faced with large initial attitude error. An alternative error function presented in [11,15] allows for faster error convergence to the origin. Nevertheless, the transient performance and steady-state error of the estimators in [11,15] cannot be predicted, and hence no systematic convergence is observed. In addition, nonlinear deterministic estimators consider only constant bias attached to the angular velocity measurements disregarding irregularity of the noise behavior. Therefore, successful maneuvering applications, for instance [16–18], might not be achieved without robust attitude estimators.

The systematic convergence of the tracking error can be achieved by enclosing the error to initially begin inside a predetermined large set and diminish gradually and smoothly to a predetermined small set [19]. This can be accomplished by transforming the constrained error to its unconstrained shape, known by transformed error. This transformation enables to maintain the error within dynamically reducing sets enabling the attainment of the prescribed performance measures. The systematic convergence with predefined measures has been applied successfully in different applications, for instance 2 DOF planar robot [19], unknown high-order nonlinear networked systems [20], event-triggered control [21].

Taking into consideration all of the above-mentioned challenges, two robust nonlinear stochastic attitude estimators on the Special Orthogonal Group $\text{SO}(3)$ with predetermined transient as well as steady-state measures have been proposed. As part of the proposed approach, the attitude error is defined in terms of normalized Euclidean distance. This error function is bound to start within a known large set and decay in a systematic fashion to a given small set. As such, the error function is constrained and as it converges to the origin, the transformed error approaches zero in probability. The main contributions of this work are as follow:

1) The estimator design takes into consideration the unknown bias and random noise attached to angular ve-
locity measurements.

2) Taking into account the noise components allows to approach the attitude problem in stochastic sense, unlike [10–13,15] which disregarded the noise and tackled the problem in deterministic sense.

3) The transformed error and normalized Euclidean distance of the attitude error are regulated to the origin in probability from almost any initial condition.

4) The convergence of the estimators guarantees prescribed measures of transient and steady-state performance, unlike deterministic estimators in [10–13,15]. The proposed attitude estimators guarantee faster convergence properties owing to the dynamic behavior of the estimator gains.

The remainder of the paper is organized as follows: An overview of SO (3) parameterization and mathematical notation are presented in Section II. In Section III the attitude problem is framed in stochastic sense and the attitude error is formulated in terms of prescribed performance. Nonlinear stochastic attitude estimators with prescribed performance characteristics and the related stability analysis are contained in Section IV. The robustness of the proposed estimators is illustrated in Section V. Finally, Section VI summarizes the work.

II. NOTATIONS AND PRELIMINARIES

In this paper, the set of non-negative real numbers is represented by \( \mathbb{R}_+ \), \( \mathbb{R}^n \) is the real \( n \)-dimensional space, and \( \mathbb{R}^{n \times m} \) is the real \( n \times m \) dimensional space. The set of integer numbers is represented by \( \mathbb{N} \). The Euclidean norm of \( x \in \mathbb{R}^n \) is given by \( ||x|| = \sqrt{x^\top x} \), where superscript \( ^\top \) indicates transpose of a vector or a matrix. \( \lambda (\cdot) \) stands for a set of eigenvalues of a matrix with \( \lambda (\cdot) \) being the minimum eigenvalue within \( \lambda (\cdot) \). \( \mathcal{C}^n \) is a set of functions each of which is characterized by \( n \)-th continuous partial derivative. \( \mathbb{I}_n \) is an identity matrix with \( n \)-by-\( n \) dimensions. \( \mathbb{E} \{ \cdot \} , \mathbb{P} \{ \cdot \} \), and \( \exp (\cdot) \) signify expected value, probability, and exponential of a component, respectively. Define the Special Orthogonal Group as \( \text{SO} (3) \). Bearing in mind the notation above, the attitude of a rigid-body is stated as a rotational matrix \( R \):

\[
\text{SO} (3) := \{ R \in \mathbb{R}^{3 \times 3} | R^\top R = RR^\top = \mathbb{I}_3, \det (R) = 1 \}
\]

where \( \det (\cdot) \) is the determinant of a matrix. \( \text{so} (3) \) is the Lie-algebra associated with \( \text{SO} (3) \) defined by

\[
\text{so} (3) := \{ \lambda' \in \mathbb{R}^{3 \times 3} | \lambda'^\top = -\lambda' \}
\]

with \( \lambda' \) being a skew-symmetric matrix. The map \([ \cdot ]_\times : \mathbb{R}^3 \rightarrow \text{so} (3)\) is defined by

\[
[\lambda']_\times = \begin{bmatrix}
0 & -\lambda_3 & \lambda_2 \\
\lambda_3 & 0 & -\lambda_1 \\
-\lambda_2 & \lambda_1 & 0
\end{bmatrix}
\]

For any \( x, y \in \mathbb{R}^3 \), we have \([ x]_\times y = x \times y \) with \( \times \) being a cross product of the two vectors. The inverse mapping of \([ \cdot ]_\times \) is defined by a vex operator, which in turn can be expressed as \( \text{vex} : \text{so} (3) \rightarrow \mathbb{R}^3 \) such that \( \text{vex} (\lambda') = x, \forall x \in \mathbb{R}^3 \) and \( \lambda' \in \text{so} (3) \). Let \( \mathcal{P}_a \) be the anti-symmetric projection operator on the Lie-algebra of \( \text{so} (3) \) [22], given by \( \mathcal{P}_a : \mathbb{R}^{3 \times 3} \rightarrow \text{so} (3) \) such that

\[
\mathcal{P}_a (Y) = \frac{1}{2} (Y - Y^\top) \in \text{so} (3)
\]

where \( Y \in \mathbb{R}^{3 \times 3} \). Define the composition mapping \( \Upsilon (\cdot) \) as

\[
\Upsilon (Y) = \text{vex} (\mathcal{P}_a (Y)) \in \mathbb{R}^3, \forall Y \in \mathbb{R}^{3 \times 3}
\]

where \( \Upsilon : = \text{vex} \circ \mathcal{P}_a \). For a rotational matrix \( R \in \text{SO} (3) \), the normalized Euclidean distance is defined by

\[
||R||_1 := \frac{1}{4} \text{Tr} \{ I_3 - R \}
\]

with \( \text{Tr} \{ \cdot \} \) being trace of a matrix and \( ||R||_1 \in [0,1] \).

The following identities will prove useful in the subsequent derivations:

\[
[\alpha \times \beta]_\times = -\beta \alpha^\top + \alpha \beta^\top, \quad \alpha, \beta \in \mathbb{R}^3
\]

\[
[R\alpha]_\times = R [\alpha]_\times R^\top, \quad R \in \text{SO} (3), \alpha \in \mathbb{R}^3
\]

\[
[\alpha]^2 = -||\alpha||^2 \mathbb{1}_3 + \alpha \alpha^\top, \quad \alpha \in \mathbb{R}^3
\]

\[
[A, B] = AB - BA, \quad A, B \in \mathbb{R}^{3 \times 3}
\]

\[
\text{Tr} \{ [A, B] \} = \text{Tr} \{ AB - BA \} = 0, \quad A, B \in \mathbb{R}^{3 \times 3}
\]

\[
\text{Tr} \{ B [\alpha]_\times \} = 0, \quad B = B^\top \in \mathbb{R}^{3 \times 3}, \alpha \in \mathbb{R}^3
\]

\[
\text{Tr} \{ A [\alpha]_\times \} = \text{Tr} \{ \mathcal{P}_a (A) [\alpha]_\times \} = -2\text{vex} (\mathcal{P}_a (A))^\top \alpha, \quad A \in \mathbb{R}^{3 \times 3}, \alpha \in \mathbb{R}^3
\]

\[
B [\alpha]_\times + [\alpha]_\times B = \text{Tr} \{ B [\alpha]_\times - [B \alpha]_\times \}, \quad B = B^\top \in \mathbb{R}^{3 \times 3}, \alpha \in \mathbb{R}^3
\]

For a unit-axis \( u \in \mathbb{R}^3 \) rotating at a rotational angle \( \theta \in \mathbb{R} \) in a 2-sphere \( \mathbb{S}^2 \), the attitude of a rigid-body can be established through the mapping of angle-axis parameterization to \( \text{SO} (3) \) defined by \( \mathcal{R}_\theta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \text{SO} (3) \) such that [23]

\[
\mathcal{R}_\theta (\theta, u) = \exp (\theta [u]_\times )
\]

\[
= \mathbb{I}_3 + \sin (\theta) [u]_\times + (1 - \cos (\theta)) [u]^2_\times
\]

A more thorough overview of attitude mapping, important properties and helpful notes can be found in [24].

III. PROBLEM FORMULATION WITH PRESCRIBED PERFORMANCE

This section aims to provide an overview of the body-frame and gyroscope measurements in the attitude estimation context, and develop the attitude problem in stochastic sense. Subsequently, the attitude problem is reformulated to follow predefined measures of transient and steady-state performance.

A. Measurements and Attitude Kinematics in Stochastic Sense

Let \( R \in \text{SO} (3) \) represent the relative orientation of a rigid-body in the body-frame \( \{ B \} \) with respect to the inertial-frame \( \{ I \} \) as demonstrated in Fig. 1.

For the sake of clarity, the superscripts \( I \) and \( B \) refer to inertial-frame and body-frame vectors, respectively. Define
Remark 1. The attitude can be reconstructed if at least two unknown bias, and \( \omega \) instantaneously measured inertial-frame vectors as in (12). It is common practice where \( \Omega \) free. In the Simulation Section, however, \( v \) of stability analysis, normalized Euclidean distance as measured relative to the rigid-body fixed coordinate system \( v \) being an unknown noise vector for \( i \) and \( j \) denoting an unknown noise vector for all \( v_i^B \) and \( v_j^B \), respectively.

In accordance with Remark 1, in this work it is assumed that \( n \geq 2 \), and therefore, three non-collinear vectors can be obtained using the expression in (12). It is common practice to consider the normalized values of the inertial-frame and body-frame vectors when calculating the attitude

\[
v_i^T = \frac{v_i^T}{||v_i^T||}, \quad v_i^B = \frac{v_i^B}{||v_i^B||}
\]

(13)

Thereby, the vectors defined in (12) will be normalized according to (13) prior to estimating the attitude. For simplicity of stability analysis, \( v_i^B \) is considered to be noise and bias free. In the Simulation Section, however, \( v_i^B \) is regarded to be contaminated with noise and bias. The true attitude kinematics are described by

\[
\dot{R} = R [\Omega]_x
\]

(14)

where \( \Omega \in \{B\} \) denotes the true angular velocity. Considering the normalized Euclidean distance of \( R \) in (2) and the identity in (9), the kinematics in (14) can be expressed in terms of normalized Euclidean distance as

\[
\frac{d}{dt} ||R||_I = -\frac{1}{4} \text{Tr} \{ \dot{R} \} = -\frac{1}{4} \text{Tr} \{ \mathcal{P}_a (R) [\Omega]_x \} = \frac{1}{2} \Upsilon (R)^\top \Omega
\]

(15)

where \( \Upsilon (R) = \text{vex} (\mathcal{P}_a (R)) \). The measurement of angular velocity is:

\[
\Omega_m = \Omega + b + \omega \in \mathbb{R}^3
\]

(16)

where \( b \) and \( \omega \) represent the unknown constant bias and random noise components attached to angular velocity measurements, respectively, for all \( b, \omega \in \mathbb{R}^3 \). \( \omega \) is a bounded Gaussian random noise vector with zero mean. Derivative of any Gaussian process is a Gaussian process, which allows \( \omega \) to be expressed as a function of Brownian motion process vector \([25,26]\). Assume that \( (\omega, t \geq t_0) \) is a vector process of an independent Brownian motion process given by

\[
\omega = Q \frac{d\beta}{dt}
\]

(17)

with \( \beta \in \mathbb{R}^3 \) and \( Q = \text{diag} (Q_{1,1}, Q_{2,2}, Q_{3,3}) \in \mathbb{R}^{3 \times 3} \) being an unknown time-variant matrix where \( Q_{i,i} \subseteq \mathbb{R} \) is a non-negative bounded component for all \( i = 1, 2, 3 \). Also, \( \text{diag} (\cdot) \) denotes diagonal of a matrix. The covariance of the noise vector \( \omega \) is defined by \( Q^2 = QQ^\top \). The properties of the Brownian motion process are given as \([26–28]\)

\[
P \{ \beta (0) = 0 \} = 1, \quad \mathbb{E} [\beta] = 0, \quad \mathbb{E} [d\beta/dt] = 0
\]

According to (15), (16), and (17), the kinematics of the normalized Euclidean distance in (15) become

\[
d ||R||_I = \frac{1}{2} \Upsilon (R)^\top ((\Omega_m - b) dt - Q d\beta)
\]

(18)

Let us present Lemma 1 which will prove useful in the subsequent estimator derivation.

Lemma 1. Consider \( R \in \mathbb{S}_0 (3) \), \( M^B = (M^B)^\top \in \mathbb{R}^{3 \times 3} \), \( \text{Tr} \{ M^B \} = 3 \), and \( M^B = \text{Tr} \{ M^B \} I_3 - M^B \). Let the minimum singular value of \( M^B \) be given by \( \lambda := \lambda (M^B) \).

Then, the following holds:

\[
\frac{2}{\lambda} ||\text{vex} (\mathcal{P}_a (M^B R))||^2 \geq ||M^B R||_I
\]

(19)

Proof. See Appendix A.

Let \( \hat{R} \) be the estimate of the true attitude \( R \). The objective of any attitude estimator is to drive \( \hat{R} \rightarrow R \) asymptotically with fast convergence properties. Let the error in attitude from body-frame to estimator-frame be given by

\[
\tilde{b} = b - \hat{b}
\]

(22)

B. Attitude Kinematics with Prescribed Performance

In this subsection the normalized Euclidean distance of the attitude error \( ||\hat{R} (t) ||_I \) is reformulated to satisfy predefined measures of transient and steady-state performance set by the user. The objective of the reformulation is to force the error \( ||\hat{R} (t) ||_I \) to initiate within a predefined large set and reduce
smoothly and systematically to a given small set guided by a prescribed performance function (PPF) [19]. Consider defining the PPF as $\xi(t)$ which is a positive and time-decreasing function that satisfies $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\lim_{t \rightarrow \infty} \xi(t) = \xi_\infty > 0$ such that

$$\xi(t) = (\xi_0 - \xi_\infty) \exp(-\ell t) + \xi_\infty \quad (23)$$

with $\xi_0 = \xi(0)$ being the upper bound of the predetermined large set, and $\xi_\infty$ being the maximum value of the predetermined small set, implying that the steady-state error is confined by $\pm \xi_\infty$. Meanwhile, $\ell$ is a positive constant that regulates the convergence rate of $\xi(t)$ from $\xi_0$ to $\xi_\infty$. In order for the error $||\hat{R}(t)||_I$ to follow the dynamically decreasing boundaries of the PPF in (23), the following conditions should be met:

$$-\delta \xi(t) < ||\hat{R}(t)||_I - \xi(t), \text{ if } ||\hat{R}(0)||_I \geq 0, \forall t \geq 0 \quad (24)$$

$$-\xi(t) < ||\hat{R}(0)||_I < -\delta \xi(t), \text{ if } ||\hat{R}(0)||_I < 0, \forall t \geq 0 \quad (25)$$

where $\delta$ satisfies $1 \geq \delta \geq 0$. The systematic convergence of $||\hat{R}(t)||_I$ guided by the dynamically decreasing constraints of PPF in accordance with (24) and (25) is depicted in Fig. 2.

From (27) the transformed error can be expressed by

$$E \left(||\hat{R}(t)||_I, \xi(t)\right) = Z^{-1} \left(\frac{||\hat{R}(t)||_I}{\xi(t)}\right) \quad (28)$$

with $E \in \mathbb{R}$, $Z \in \mathbb{R}$, and $Z^{-1} \in \mathbb{R}$ being smooth functions. For simplicity, define $\xi := \xi(t)$, $||\hat{R}||_I := ||\hat{R}(t)||_I$, and $E := E(t, \cdot)$. Combining (27) and (28), the transformed error can be expressed as follows:

$$E = \frac{1}{2} \ln \left(\frac{\delta + ||\hat{R}||_I/\xi}{\delta - ||\hat{R}||_I/\xi}\right) \quad (29)$$

Remark 3. The prescribed performance is achieved, when the transient and the steady-state performance of the tracking error is bounded by the dynamic boundaries of $\xi(t)$, and the transformed error $E(t)$ is bounded for all $t \geq 0$.

Proposition 1. Let the normalized Euclidean distance error be $||\hat{R}||_I$ defined according to (2). Also, let the transformed error be expressed as in (29) provided that $\delta = \delta$. Then, the following holds:

(i) $E > 0$ for all $||\hat{R}||_I \neq 0$ and $E = 0$ only at $||\hat{R}||_I = 0$.

(ii) The critical point of $E$ coincides with $||\hat{R}||_I = 0$.

(iii) The only critical point of $E$ is $\hat{R} = I_3$.

Proof. Provided that $\delta = \delta$ and $||\hat{R}||_I \leq \xi$, 

$$\left(\frac{\delta + ||\hat{R}||_I/\xi}{\delta - ||\hat{R}||_I/\xi}\right) \quad (29)$$

is always greater than or equal to 1. Therefore, it becomes evident that $E > 0 \forall ||\hat{R}||_I \neq 0$ and $E = 0$ at $||\hat{R}||_I = 0$ which justifies (i). The definition of the normalized Euclidean distance in (2) states that $||\hat{R}||_I = 0$ if and only if $\hat{R} = I_3$. Hence, the only critical point of $E$ coincides with $\hat{R} = I_3$ and $||\hat{R}||_I = 0$ which proves (ii) and (iii). Define the following variable as

$$\mu(||\hat{R}||_I, E) = \frac{1}{4\xi} \left(\frac{1}{\delta + ||\hat{R}||_I/\xi} + \frac{1}{\delta - ||\hat{R}||_I/\xi}\right) \quad (30)$$

According to (26) one has

$$||\hat{R}||_I = \xi \mathcal{F}(E) = \xi \frac{\delta \exp(E) - \delta \exp(-E)}{\exp(E) + \exp(-E)} \quad (31)$$

Therefore, from (30) and (31) one may find that the expression in (30) is equivalent to

$$\mu(E) = \frac{\exp(2E) + \exp(-2E) + 2}{8\xi \delta} \quad (32)$$

It follows that the transformed error dynamics can be

$$\dot{E} = 2\mu \left(\frac{d}{dt} ||\hat{R}||_I - \frac{\xi}{\xi} ||\hat{R}||_I\right) \quad (33)$$

where $\mu := \mu(E, \xi)$ which could also be expressed by

$$dE = f(E, \hat{\xi})dt + g(E) Q d\beta \quad (34)$$
where both $f(\mathcal{E}, \hat{b})$ and $g(\mathcal{E})$ are to be defined in Section IV with $g: \mathbb{R} \to \mathbb{R}^{\times 3}$ and $f: \mathbb{R} \times \mathbb{R}^{3} \to \mathbb{R}$. $g(\mathcal{E})$ is locally Lipschitz in $\mathcal{E}$, and $f(\mathcal{E}, \hat{b})$ is locally Lipschitz in $\mathcal{E}$ and $\hat{b}$ with $g(0) = 0_{3 \times 3}$ and $f(0, \hat{b}) = 0$ for all $t \geq 0$. Therefore, it can be concluded that there exists a solution in the mean square sense for the dynamic system in (34) for $t \in [t_{0}, T]$ and $\forall t_{0} \leq T < \infty$ [26,28,29].

**Definition 1.** Define $S \subseteq \mathbb{S}(3)$ as a non-attractive, forward invariant unstable set:

$$S = \left\{ \hat{R}(0) \in \mathbb{S}(3) \mid \text{Tr}\{\hat{R}(0)\} = -1 \right\}$$  \hspace{1cm} (35)

where the only three possible scenarios for $\hat{R}(0) \in S$ are:

$\hat{R}(0) = \text{diag}(1,-1,-1)$, $\hat{R}(0) = \text{diag}(-1,-1,-1)$, and $\hat{R}(0) = \text{diag}(-1,-1,1)$.

For any $\mathcal{E}(t) \in \mathbb{R}$ that satisfies $t \neq t_{0}$ and $\hat{R}(0) \notin S$, $\mathcal{E} - \mathcal{E}_{0}$ is independent of $\{\beta(\tau), \tau \geq t\}$, $\forall t \in [t_{0}, T]$ (Theorem 4.5 [26]). With the objective of achieving adaptive stabilization, let us define an unknown time-variant covariance matrix $Q^{\sigma}$. For this purpose, let us also specify the upper bound of $Q^{\sigma}$ as follows:

$$\sigma = \left[ \max \{Q^{2}_{1,1} \}, \max \{Q^{2}_{2,2} \}, \max \{Q^{2}_{3,3} \} \right] \in \mathbb{R}^{3}$$  \hspace{1cm} (36)

where $\max \{ \cdot \}$ denotes the maximum value of a component.

**Definition 2.** [28] For the stochastic dynamics in (34), and for a given potential function $V(\mathcal{E}) \in C^{2}$, the differential operator $LV$ is defined by

$$LV(\mathcal{E}) = V_{\mathcal{E}}^{T} f(\mathcal{E}, \hat{b}) + \frac{1}{2} \text{Tr}\{g(\mathcal{E}) Q^{\sigma} g(\mathcal{E})^{T}(\mathcal{E}) V_{\mathcal{E}} \}$$

where $V_{\mathcal{E}} = \partial V/\partial \mathcal{E}$ and $V_{\mathcal{E}E} = \partial^{2} V/\partial \mathcal{E}^{2}$.

**Lemma 2.** (Stochastic LaSalle Theorem [30]) Consider the stochastic dynamics in (34) and suppose that there exists a positive definite function which is twice continuously differentiable $V(\mathcal{E}) \in C^{2}$. Let $V(\mathcal{E})$ be decrescent and radially unbounded, and suppose that there is a non-negative continuous class $K$ function $\mathcal{H}(\mathcal{E}) \geq 0$, such that $V(\mathcal{E})$ of the stochastic kinematics in (34) satisfies

$$LV(\mathcal{E}) = V_{\mathcal{E}}^{T} f(\mathcal{E}, \hat{b}) + \frac{1}{2} \text{Tr}\{g(\mathcal{E}) Q^{\sigma} g(\mathcal{E})^{T}(\mathcal{E}) V_{\mathcal{E}} \} \leq - \mathcal{H}(\mathcal{E}), \hspace{1cm} \forall \mathcal{E} \in \mathbb{R}, t \geq 0$$  \hspace{1cm} (37)

Thus, for $\mathcal{E}_{0} \in \mathbb{R}$ and the set in (35) $\hat{R}(0) \notin S$, there exists almost a unique strong solution on $[0, \infty)$ for the dynamic system in (34). Additionally, the equilibrium point $\mathcal{E} = 0$ of the dynamic system in (34) is almost globally asymptotically stable in probability such that

$$\mathbb{P}\{ \lim_{t \to \infty} \mathcal{H}(\mathcal{E}(t)) = 0 \} = 1, \hspace{1cm} \forall \hat{R}(0) \notin S, \mathcal{E}(0) \in \mathbb{R}$$  \hspace{1cm} (38)

**IV. NONLINEAR STOCHASTIC ESTIMATORS ON $\mathbb{S}(3)$ WITH PRESCRIBED PERFORMANCE**

This section introduces two nonlinear stochastic complementary attitude estimators on $\mathbb{S}(3)$ which are capable of guiding the normalized Euclidean distance error to follow the predetermined transient and steady-state measures set by the user. $||\hat{R}||_{F}$, initially constrained, is transformed to an unconstrained error denoted by $\mathcal{E}$. The first estimator is termed semi-direct nonlinear stochastic estimator with prescribed performance as it demands attitude reconstruction in addition to angular velocity measurements. The other estimator is called direct nonlinear stochastic estimator with prescribed performance since it uses a set of vector measurements in (13) and (16) avoiding the need for attitude reconstruction. Let us define the error of the upper bound $\sigma$ as follows:

$$\sigma = \sigma - \sigma$$  \hspace{1cm} (39)

with $\sigma$ being the estimate of $\sigma$.

**A. Semi-direct Stochastic Estimator with Prescribed Performance**

Define $R_{\bar{y}}$ as a reconstructed attitude of the rotational matrix $R$. Consider the following estimator

$$\mu = \exp(2\mathcal{E}) + \exp(-2\mathcal{E}) + \frac{8\epsilon \delta}{\mathcal{E}}$$  \hspace{1cm} (40)

$$\dot{\hat{R}} = R_{\bar{y}} \left[ \Omega_{m} - \hat{b} - W \right]\times$$  \hspace{1cm} (41)

$$\dot{\hat{b}} = \gamma_{1}(\mathcal{E} + 1) \exp(\mathcal{E}) \mu \mathcal{Y}(\hat{R})$$  \hspace{1cm} (42)

$$\dot{\sigma} = \gamma_{2}(\mathcal{E} + 2) \exp(\mathcal{E}) \mu^{2} \text{diag} \left( \mathcal{Y}(\hat{R}) \right) \mathcal{Y}(\hat{R})$$  \hspace{1cm} (43)

$$W = \frac{\mathcal{E} + 2}{\mathcal{E} + 1} \mu \text{diag} \left( \mathcal{Y}(\hat{R}) \right) \sigma + 2k_{w} \mathcal{E} \mu - \frac{\xi}{4\mathcal{E}}$$  \hspace{1cm} (44)

with $\hat{R} = R_{\bar{y}}^{T} \hat{R}$, $\mathcal{Y}(\hat{R}) = \text{vec}(\mathcal{P}_{\bar{y}}(\hat{R}))$, $\mathcal{E}$ being defined in (29), $\gamma_{1}$, $\gamma_{2}$, and $k_{w}$ being positive constants, $||\hat{R}||_{F}$ is defined in (23), and $\hat{b}$ and $\hat{\sigma}$ being the estimates of $b$ and $\sigma$, respectively.

**Theorem 1.** Consider the attitude dynamics in (14), vector measurements in (12), and angular velocity measurements in (16) coupled with the estimator in (41), (42), (43), and (44). Assume that at least two body-frame non-collinear vectors are available for measurements. Recall the set defined in Definition 1. Thus, for attitude error $\hat{R}(0) \notin S$, and $\mathcal{E}(0) \in \mathbb{R}$, all the error signals are bounded, $\mathcal{E}$ asymptotically approaches the origin in probability, and $\hat{R}$ asymptotically approaches $I_{3}$ in probability.

Theorem 1 guarantees that the estimator kinematics in (41), (42), (43), and (44) are stable with $\mathcal{E}(t)$ being almost globally asymptotically stable in probability. Since $\mathcal{E}(t)$ is bounded, $||\hat{R}(t)||_{F}$ follows the dynamically decrescent boundaries of the PPF introduced in (23).

**Proof.** Consider $\check{R} = R_{\bar{y}}^{T} \check{R}$, $\check{b} = b - \check{b}$, and $\check{\sigma} = \sigma - \check{\sigma}$ as in (21), (22), and (39), respectively. From (14) and (41), the error kinematics can be translated into

$$d\check{R} = R_{\bar{y}}^{T} \check{R} \left[ \Omega_{m} - \check{b} - W \right]\times dt + \left[ \mathcal{P}_{\bar{y}} \right]^{T} R_{\bar{y}} \check{R} dt$$

$$= \left( [\check{R}, \mathcal{P}_{\bar{y}}] + \check{R} [\check{b} - W] \right) dt + \check{R} \mathcal{P}_{\bar{y}} \check{b} - \check{b} \mathcal{P}_{\bar{y}} \check{R} dt$$  \hspace{1cm} (45)
where $[\tilde{R}, [\Omega_x]] = \tilde{R} [\Omega_x] + [\Omega_x] \tilde{R}$ is the Lie bracket as defined in (6). In view of (14) and (15), the error dynamics in (45) can be expressed in the sense of normalized Euclidean distance as
\[
d||\tilde{R}||_I = -\frac{1}{4} \text{Tr} \left\{ \tilde{R} (\dot{b} - W)_x + \left[ \tilde{R}, [\Omega_x] \right] \right\} dt \\
- \frac{1}{4} \text{Tr} \left\{ \tilde{R} [Q d\beta] \right\} \\
= \frac{1}{2} \Upsilon (\tilde{R})^\top \left( (\dot{b} - W)_x + Q d\beta \right)
\] (46)

where $\text{Tr} \{[\tilde{R}, [\Omega_x]]\} = -2 \Upsilon (\tilde{R})^\top \dot{b}$ as presented in identity (9), and $\text{Tr} \{[\tilde{R}, [\Omega_x]]\} = 0$ as defined in identity (7). From (46), the transformed error dynamics in incremental form are as follows:
\[
d\mathcal{E} = f(\mathcal{E}, \dot{b}) dt + g(\mathcal{E}) Q d\beta
\] (47)

where $f(\mathcal{E}, \dot{b}) = 2\mu \left( \frac{1}{2} \Upsilon (\tilde{R})^\top (\dot{b} - W) - \xi ||\tilde{R}||_I \right)$ and $g(\mathcal{E}) = \mu \Upsilon (\tilde{R})^\top$. Ito [25,27] and Stratonovich [31] are the two most commonly used methods of stochastic integral representation. These two approaches have the following two advantages: $\beta (t)$ and $d\beta / dt$ are continuous and Lipschitz, and their mean square exists. However, the Ito integral does not obey the chain rule and can be a good fit for measurements corrupted only by white noise signals. Stratonovich, on the other hand, is a well-defined Riemann integral, and as opposed to Ito, has a continuous partial derivative with respect to $\beta$ which means that it follows the chain rule. Additionally, Stratonovich is suitable for both white and colored noise signals [3,26,31]. Suppose that the dynamics in (47) are defined in the sense of Stratonovich [31]. Thereby, the transformation of (47) from Stratonovich to Ito is given by
\[
d\mathcal{E} = \mathcal{F}(\mathcal{E}, \dot{b}) dt + g(\mathcal{E}) Q d\beta
\] (48)

such that
\[
\mathcal{F}(\mathcal{E}, \dot{b}) = f(\mathcal{E}, \dot{b}) + \mathcal{W}(\mathcal{E})
\]

where $\mathcal{W}(\mathcal{E})$ is the Wong-Zakai factor introduced to allow for the transition from Stratonovich to Ito and is given by [26,31]
\[
\mathcal{W}(\mathcal{E}) = g(\mathcal{E}) \frac{Q \mathcal{Q}^\top}{2} \frac{\partial g(\mathcal{E})}{\partial \mathcal{E}} \\
= \frac{\Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R}) \partial \mu}{2} \\
= \frac{\exp (2\mathcal{E}) - \exp (-2\mathcal{E}) \mu \Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R})}{8\xi \delta}
\] (49)

Remark 4. [26,31] The Wong-Zakai factor introduced in (48) has a prominent role in attaining $\mathbb{E} [\mathcal{E}] = \mathbb{E} \int_0^t f(\mathcal{E} (\tau), \dot{b} (\tau)) d\tau$ whether $d\beta / dt$ has a zero mean or not.

Consider the following candidate Lyapunov function
\[
V(\mathcal{E}, \dot{b}, \dot{\sigma}) = \mathcal{E} \exp (\mathcal{E}) + \frac{1}{2\gamma_1} ||\dot{b}||^2 + \frac{1}{\gamma_2} ||\dot{\sigma}||^2
\] (50)

For $V : = V(\mathcal{E}, \dot{b}, \dot{\sigma})$ the first and second partial derivatives of (50) with respect to $\mathcal{E}$ are as follows
\[
\frac{\partial V}{\partial \mathcal{E}} = (\mathcal{E} + 1) \exp (\mathcal{E})
\] (51)
\[
\frac{\partial^2 V}{\partial \mathcal{E}^2} = (\mathcal{E} + 2) \exp (\mathcal{E})
\] (52)

Consider the differential operator $\mathcal{L} \mathcal{V}$ in Definition 2. From (50), (48), (51), and (52), $\mathcal{L} \mathcal{V}$ is equivalent to
\[
\mathcal{L} \mathcal{V} = V_{\mathcal{E}} \mathcal{F}(\mathcal{E}, \dot{b}) + \frac{1}{2} \text{Tr} \{ g(\mathcal{E}) \mathcal{Q}^2 g^\top (\mathcal{E}) V_{\mathcal{E} \mathcal{E}} \} \\
- \frac{1}{\gamma_1} \dot{b}^\top \dot{b} - \frac{2}{\gamma_2} \dot{\sigma}^\top \dot{\sigma} \\
= (\mathcal{E} + 1) \exp (\mathcal{E}) \mu \left( \Upsilon (\tilde{R})^\top (\dot{b} - W) - 2\frac{\xi}{\mathcal{E}} ||\tilde{R}||_I \right) \\
+ \exp (2\mathcal{E}) - \exp (-2\mathcal{E}) \Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R}) \\
+ \frac{1}{2} (\mathcal{E} + 2) \exp (\mathcal{E}) \mu^2 \Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R}) \\
- \frac{1}{\gamma_1} \dot{b}^\top \dot{b} - \frac{2}{\gamma_2} \dot{\sigma}^\top \dot{\sigma}
\] (53)

From property (ii) and (iii) of Proposition 1, $\mathcal{E} > 0 \forall ||\tilde{R}||_I \neq 0$ and $\mathcal{E} = 0$ only at $||\tilde{R}||_I = 0$. From (32) and (40), one has $\mu > 0 \forall t \geq 0$. Thus, one has $\mathcal{E} \exp (\mathcal{E}) \geq 0, \mu > 0$ and $\Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R}) \geq 0$ for all $t \geq 0$. Also, it becomes apparent that $\mu > \exp (2\mathcal{E}) - \exp (-2\mathcal{E})$. As such, the differential operator in (53) can be written in inequality form as
\[
\mathcal{L} \mathcal{V} \leq 2 (\mathcal{E} + 1) \exp (\mathcal{E}) \mu \left( \frac{1}{2} \Upsilon (\tilde{R})^\top (\dot{b} - W) - \frac{\xi}{\mathcal{E}} ||\tilde{R}||_I \right) \\
+ 2 (\mathcal{E} + 2) \exp (\mathcal{E}) \mu^2 \Upsilon (\tilde{R})^\top \mathcal{Q}^2 \Upsilon (\tilde{R}) \\
- \frac{1}{\gamma_1} \dot{b}^\top \dot{b} - \frac{2}{\gamma_2} \dot{\sigma}^\top \dot{\sigma}
\] (54)

where $\sigma$ is defined in (36). Directly substituting $\dot{b}, \dot{\sigma}$, and $W$ with their definitions in (42), (43), and (44), respectively, and considering $||\Upsilon (\tilde{R})||^2 = 4 (1 - ||\tilde{R}||_I)||\tilde{R}||_I$ as defined in (19), the inequality in (54) becomes
\[
\mathcal{L} \mathcal{V} \leq -4k_w ||\tilde{R}||_I \mu^2 (\mathcal{E} + 1) \exp (\mathcal{E})
\] (55)

Since $\mu > 0 \forall t \geq 0$, from (31) the inequality in (55) is equivalent to
\[
\mathcal{L} \mathcal{V} \leq -4k_w \xi \mu^2 (\mathcal{E} + 1) \exp (\mathcal{E}) \exp (\mathcal{E}) - \exp (\mathcal{E}) \exp (\mathcal{E}) \exp (\mathcal{E})
\] (56)

with $\bar{\delta} > 0, k_w > 0$, and $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. As stated by property (ii) and (iii) of Proposition 1, $\mathcal{E} > 0 \forall ||\tilde{R}||_I \neq 0$ and $\mathcal{E} = 0$ only at $||\tilde{R}||_I = 0$ for $R (0) \notin S$ and $\mathcal{E} (0) \in \mathbb{R}$. According to the fact that $\mathcal{L} \mathcal{V}$ is bounded and $V$ is radially unbounded for any $R (0) \notin S$ and $\mathcal{E} (0) \in \mathbb{R}$, there exists a unique strong solution for the stochastic system in (48) with probability of one [25]. From property (ii) and (iii) of Proposition 1, $\mathcal{E} = 0$ implies that $\tilde{R} = \mathbb{I}_3$. Thus, based on the result presented in (56), $\mathcal{E} = 0$ and $\tilde{R} = \mathbb{I}_3$ are independent of the unknown
noise and bias attached to the angular velocity measurements. Therefore, on the basis of the stochastic LaSalle Theorem [25], it can be concluded that $\mathcal{E} = 0$ is almost globally stable in probability with $E(t)$ being regulated asymptotically to the origin in probability for all $\hat{R}(0) \notin \mathcal{S}$ and $\mathcal{E}(0) \in \mathbb{R}$ which implies that $\mathbb{P}\{\lim_{t \to \infty} \hat{R} = I_3\} = 1$ for all $\hat{R}(0) \notin \mathcal{S}$ and $\mathcal{E}(0) \in \mathbb{R}$. Moreover, it can be noticed that the estimation of $b$ and $\sigma$ is achievable and has a finite limit in probability ( [28], Theorem 3.1).

**B. Direct Stochastic Estimator with Prescribed Performance**

Let us define $\tilde{R}_g$ as the reconstructed attitude of the true $\tilde{R}$. The stochastic estimator introduced in the above subsection requires attitude reconstruction. Several methods have been proposed to obtain static estimation of the attitude, such as QUEST [1] and SVD [2]. Nonetheless, the aforementioned methods add complexity to the estimator design process [10,32]. In an effort to simplify the estimator design, this section presents a novel nonlinear stochastic attitude estimator with prescribed performance. The merit of the proposed estimator consists in its ability to receive direct input from the measurement units bypassing attitude reconstruction. Consider the normalized inertial and body-frame vectors in (13) for $i = 1, \ldots, n$ and define

$$M^L = (M^L)^\top = \sum_{i=1}^{n} s_i \hat{v}_i^L \times (v_i^B)^\top$$

$$M^B = (M^B)^\top = \sum_{i=1}^{n} s_i \hat{v}_i^B \times (v_i^B)^\top = R^\top M^2 R$$

(57)

with $s_i > 0$ representing the confidence level of the $i$th sensor measurement. In this work, $s_i$ has been selected to satisfy $\sum_{i=1}^{n} s_i = 3$. It becomes apparent that $M^L$ and $M^B$ are symmetric matrices, and therefore suppose that at least two non-collinear inertial-frame and body-frame vectors are available at every time instant. In case when $n = 2$, the third vector is defined using the cross product of the two known vectors as stated in Subsection III-A. As such, $M^B$ is nonsingular with rank($M^B$) = 3 at every time instant. In addition, the three eigenvalues of the symmetric matrix $M^B$ are greater than zero [33]. Let us introduce $M^B = \text{Tr}(M^B)I_3 - M^B \in \mathbb{R}^{3 \times 3}$, given that $n \geq 2$. Accordingly, the following statements are met ([33] page. 553):

1) The matrix $M^B$ is symmetric and positive-definite.
2) The eigenvectors of $M^B$ have values similar to those of the eigenvectors of $M^B$.
3) Consider the three eigenvalues of $M^B$ stated as $\lambda(M^B) = \{\lambda_1, \lambda_2, \lambda_3\}$. It follows that $\lambda(M^B) = \{\lambda_3 + \lambda_2, \lambda_3 + \lambda_1, \lambda_2 + \lambda_1\}$ with $\lambda(M^B) > 0$ being the minimum eigenvalue of $M^B$.

In the remainder of this subsection it is supposed that $n \geq 2$ such that rank($M^B$) = 3. Define the estimate of $\hat{v}_i^B$ by

$$\hat{\hat{v}}_i^B = \hat{\hat{R}}^\top v_i^B$$

(58)

In order to avoid attitude reconstruction, it is necessary to introduce a set of expressions in terms of vector measurements. With the aid of identity (3), one obtains

$$\left[\sum_{i=1}^{n} \frac{s_i}{2} \hat{\hat{v}}_i^B \times v_i^B\right] = \sum_{i=1}^{n} \frac{s_i}{2} \left( (v_i^B)^\top (v_i^B)^\top - \hat{\hat{v}}_i^B (\hat{\hat{v}}_i^B)^\top\right)$$

$$\begin{align*}
&= \frac{1}{2} R^\top M^2 R \tilde{R} - \frac{1}{2} \tilde{R}^\top R^\top M^2 R \\
&= \mathcal{P}_\sigma(M^B \tilde{R})
\end{align*}$$

such that

$$\text{vex} \left( \mathcal{P}_\sigma(M^B \tilde{R}) \right) = \sum_{i=1}^{n} \frac{s_i}{2} \hat{\hat{v}}_i^B \times v_i^B$$

(59)

It is straightforward to find

$$||M^B \tilde{R}||_1 = \frac{1}{4} \text{Tr}(I_3 - M^B \tilde{R})$$

$$= \frac{1}{4} \text{Tr} \left\{ I_3 - \sum_{i=1}^{n} s_i v_i^B (\hat{\hat{v}}_i^B)^\top \right\}$$

$$= \frac{1}{4} \sum_{i=1}^{n} s_i \left( 1 - (\hat{\hat{v}}_i^B)^\top v_i^B \right)$$

(60)

Consider the following variable

$$\mathcal{J}(M^B, \tilde{R}) = \text{Tr} \left\{ (M^B)^{-1} M^B \tilde{R} \right\}$$

$$= \text{Tr} \left\{ \left( \sum_{i=1}^{n} s_i v_i^B (\hat{\hat{v}}_i^B)^\top \right)^{-1} \sum_{i=1}^{n} s_i v_i^B (\hat{\hat{v}}_i^B)^\top \right\}$$

(61)

In the subsequent estimator derivations the variables $\text{vex}(\mathcal{P}_\sigma(M^B \tilde{R})), ||M^B \tilde{R}||_1, \text{ and } \mathcal{J}(M^B, \tilde{R})$ are going to be obtained through a set of vector measurements as defined in (59), (60), and (61), respectively. Recall the discussion presented in Subsection III-B: every $||\hat{R}||_1$ is replaced by $||M^B \hat{R}||_1$ such that $\mathcal{E} := \mathcal{E}(||M^B \hat{R}||_1, \hat{\xi}, \xi)$, and $\mu := \mu(\hat{\xi}, \xi)$. Consider the following estimator design

$$\mu = \exp(2\mathcal{E} + \exp(-2\mathcal{E} + 2) \frac{8\mathcal{E}}{\mathcal{E} + 1})$$

$$\dot{\hat{R}} = \hat{\hat{R}} \left[ \Omega_m - \hat{b} - \hat{W} \right]$$

$$\hat{b} = \sigma_1 (\mathcal{E} + 1) \exp(\mathcal{E}) \mathcal{Y}(M^B \tilde{R})$$

$$\dot{\hat{\xi}} = \sigma_2 (\mathcal{E} + 2) \exp(\mathcal{E}) \mu^2 \text{diag}(\mathcal{Y}(M^B \tilde{R})) \mathcal{Y}(M^B \tilde{R})$$

$$\dot{\hat{\xi}} = \sigma_2 (\mathcal{E} + 2) \exp(\mathcal{E}) \mu^2 \text{diag}(\mathcal{Y}(M^B \tilde{R})) \mathcal{Y}(M^B \tilde{R})$$

(65)

$$W = k_w \mathcal{E} + \frac{2}{\mathcal{E} + 1} \mu \text{diag}(\mathcal{Y}(M^B \tilde{R})) \mathcal{Y}(M^B \tilde{R})$$

(66)

with $\mathcal{J}(M^B, \tilde{R})$ and $\mathcal{Y}(M^B \tilde{R}) = \text{vex}(\mathcal{P}_\sigma(M^B \tilde{R}))$ being defined in (61) and (59), respectively, $\mathcal{E}$ being PPF given in (23), and $\Delta := \Delta(M^B)$ being the minimum eigenvalue of $M^B$. $\mathcal{E} := \mathcal{E}(||M^B \hat{R}||_1, \hat{\xi}, \xi)$ is defined in (29), while $\gamma_1$, $\gamma_2$, and $k_w$ are positive constants. $\hat{b}$ and $\hat{\xi}$ are the estimates of $b$ and $\sigma$, respectively.

**Theorem 2.** Let the estimator in (63), (64), (65), and (66) be combined with the measurements in (13) and the angular
Recalling the identities in (9) and (7), it becomes apparent that such that $f$ where $g$ verifying the result in (69), the transformed error can be expressed for $\tilde{M}$ velocity measurements in (16). Suppose that two or more instantaneous non-collinear body-frame vectors are available for measurements such that $M^B$ has the rank of 3. Therefore, for $R(0) \notin S$, as defined in Definition 1, and $E(0) \in \mathbb{R}$, all error signals are bounded, while $E(t)$ asymptotically approaches 0 and $\dot{R}$ asymptotically approaches $I_3$ in probability from almost any initial condition.

In accordance with Theorem 2 the stability of the estimator kinematics in (63), (64), (65), and (66) is guaranteed, since $E(t)$ approaches the origin asymptotically. Consequently, $E(t)$ is bounded and well-defined, which in turn implies that $||\dot{R}||_I$ obeys the dynamic boundaries of transient and steady-state PPF as defined in (23) in consistence with Remark 3.

Proof. Let $\ddot{R} = R^T \dot{R}$, $\ddot{b} = b - \dot{b}$, and $\ddot{\sigma} = \sigma - \dot{\sigma}$ as defined in (21), (22), and (39), respectively. From (14) and (63), the attitude error kinematics are analogous to (45). From (57), it can be found that

$$M^B = \ddot{R}^T M^T R + R^T \dot{M}^T \dot{R} = \begin{bmatrix} [\Omega]_x R^T M^T R + R^T [\Omega]_x \dot{R} \\ [\Omega]_x M^B + [\Omega]_x M^B [\Omega]_x \end{bmatrix}$$

Hence, from (45) and (67), one has

$$\frac{d}{dt}||M^B \ddot{R}||_I = -\frac{1}{4} \text{Tr} \{M^B \ddot{R} + M^B \dot{\ddot{R}}\}$$

such that

$$\frac{d}{dt}||M^B \ddot{R}||_I = -\frac{1}{4} \text{Tr} \left\{ \begin{bmatrix} [\Omega]_x M^B + [\Omega]_x M^B [\Omega]_x \end{bmatrix} \ddot{R} \right\} dt$$

Recall that $E > 0$ and $E = 0$ only at $||M^B \ddot{R}||_I = 0$, according to property (ii) and (iii) of Proposition 1. In this regard, from (32) and (62) one has $\mu > 0 \forall t \geq 0$. This implies that $E \exp{E(t)} \geq 0$, $\mu > 0$ and $\mathbf{Y}(M^B \ddot{R})^T Q^2 \mathbf{Y}(M^B \ddot{R}) \geq 0$ for all $t \geq 0$. In addition, it can be deduced that $\mu > \frac{\exp{2E(t)}}{\exp{2E(t)}}$. Accordingly, the differential operator in (74) could be expressed in a form of inequality as follows

$$LV \leq (E + 1) \exp{E(t)} \mu \mathbf{Y}(M^B \ddot{R})^T (\ddot{b} - W)$$

$$- 2 (E + 1) \exp{E(t)} \mu \ddot{\sigma} - 2 (E + 2) \exp{E(t)} \mu \ddot{\sigma} - 1 \frac{\ddot{b}^T \ddot{b}}{\gamma_1}$$

with $\mathbf{W}(E)$ being the Wong-Zakai factor. In view of (49), $\mathbf{W}(E)$ is defined in the following manner

$$\mathbf{W}(E) = \frac{\exp{(2E)} - \exp{(-2E)}}{8 \xi \delta} \mu \mathbf{Y}(M^B \ddot{R})^T Q^2 \mathbf{Y}(M^B \ddot{R})$$

Consider the following candidate Lyapunov function

$$V_E = V(E, \ddot{b}, \ddot{\sigma}) = E \exp{E(t)} + \frac{1}{2 \gamma_1} ||\ddot{b}||^2 + \frac{1}{2 \gamma_2} ||\ddot{\sigma}||^2$$

For $V := V(E, \ddot{b}, \ddot{\sigma})$, the first and second partial derivatives of (72) are equivalent to (51) and (52), respectively. Thus, from (72), (71), (51), and (52), the differential operator $LV$ becomes

$$LV = V_E \mathbf{F}(E, \ddot{b}) + \frac{1}{2} \text{Tr} \{g \mathbf{E} \mathbf{Q} \mathbf{Y} \mathbf{Q}^2 \mathbf{Y} \mathbf{E} E \}$$

$$- \frac{1}{\gamma_1} \ddot{b}^T \ddot{b} - \frac{2}{\gamma_2} \ddot{\sigma}^T \ddot{\sigma}$$

(73)
It can be easily found that

\[
(\mathcal{E} + 1) \exp(\mathcal{E}) \mu_\mathcal{E} \frac{\delta}{\Xi} \left( 2 \frac{\|\mathcal{Y}(M^B R)\|^2}{\lambda_1 + \mathcal{J}(M^B, R)} - \|M^B R\|^2 \right) \leq 0
\]

(77)

where \( \delta \) is a negative strictly increasing component which satisfies \( \delta \rightarrow 0 \) as \( t \rightarrow \infty \), while \( \mathcal{E} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( \mathcal{E} \rightarrow \mathcal{E}_\infty \) as \( t \rightarrow \infty \). Thus, \( \mathcal{E} / \delta \leq 0 \). According to the equation (20) of Lemma 1, the expression in (77) is negative semi-definite. Thus, the inequality in (76) becomes

\[
\mathcal{L} V \leq - k_w \exp(\mathcal{E}) (\mathcal{E} + 1) \mathcal{E} \mu_\mathcal{E}^2 \|M^B R\|_I \leq 0
\]

(78)

Due to the fact that

\[
\|M^B R\|_I = \frac{\mathcal{E}}{\exp(\mathcal{E}) + \exp(-\mathcal{E})} \leq \mathcal{E}
\]

(79)

The inequality in (78) can be expressed as

\[
\mathcal{L} V \leq - \delta k_w \mathcal{E} \mu_\mathcal{E}^2 (\mathcal{E} + 1) \mathcal{E} \exp(\mathcal{E}) - \exp(-\mathcal{E}) \exp(\mathcal{E}) \leq 0
\]

(80)

where \( \delta > 0, k_w > 0, \) and \( \mathcal{E} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). On the basis of property (ii) and (iii) of Proposition 1, \( \mathcal{E} > 0 \) only at \( \|M^B R\|_I \neq 0 \) and \( \mathcal{E} = 0 \) only at \( \|M^B R\|_I = 0 \) for \( \mathcal{R}(0) \notin \mathcal{S} \) and \( \mathcal{E}(0) \in \mathbb{R} \), \( \mathcal{E} = 0 \) implies that \( \mathcal{R} = I_3 \). Since \( \mathcal{L} V \) is bounded and \( V \) is radially unbounded, it can be stated that for \( \mathcal{R}(0) \notin \mathcal{S} \) and \( \mathcal{E}(0) \in \mathbb{R} \) there exists a unique strong solution to the stochastic system in (71) with the probability of one [25]. Thus, from the result in (80), \( \mathcal{E} = 0 \) as well as \( \mathcal{R} = I_3 \) are independent of the unknown noise and bias attached to angular velocity measurements. Therefore, based on the stochastic LaSalle Theorem [25], one has \( \mathbb{P}(\lim_{t \rightarrow \infty} \mathcal{E}(t) = 0) = 1, \forall \mathcal{R}(0) \notin \mathcal{S}, \mathcal{E}(0) \in \mathbb{R} \) which in turn indicates that \( \mathbb{P}(\lim_{t \rightarrow \infty} \mathcal{R} = I_3) = 1 \) for all \( \mathcal{R}(0) \notin \mathcal{S} \) and \( \mathcal{E}(0) \in \mathbb{R} \). Furthermore, the estimation of \( b \) and \( \sigma \) has a finite limit in probability ([28], Theorem 3.1).

C. Remarks on Stability Analysis and Implementation Steps

It becomes apparent that the gains associated with the vex operator \( \mathcal{Y}(\cdot) = \text{vex}(\mathcal{P}_\mathcal{A}(\cdot)) \) of \( \hat{b}, \hat{\sigma}, \) and \( \hat{W} \) in (42), (43), and (44), or in (64), (65), and (66), respectively, are dynamic. Their dynamic behavior forces the attitude error to comply with the prescribed performance constraints. Consequently, both proposed estimators are characterized by highly favorable features which cause the dynamic gains to become increasingly aggressive as \( \|\mathcal{R}\| \) approaches the unstable equilibria +1. On the other side, these gains decrease significantly as \( \|\mathcal{R}\| \rightarrow 0 \). This dynamic behavior allows the proposed nonlinear stochastic estimators to force the attitude error to follow the predefined PPF imposed by the user and thereby to achieve the predetermined measures of transient as well as steady-state performance. Let us summarize the design process of the estimator proposed in Subsection IV-A:

1. Set the following parameters: \( \delta = 2 > \|\hat{\mathcal{R}}(0)\|, \xi_0 = \delta, \) a small set \( \xi_\infty \) and a convergence rate \( \ell \).
2. Evaluate \( \mathcal{Y}(\hat{\mathcal{R}}) = \text{vex}(\mathcal{P}_\mathcal{A}(\hat{\mathcal{R}})) \) and \( \|\hat{\mathcal{R}}\|_I = \frac{1}{2} \text{Tr}(I_3 - \hat{\mathcal{R}}) \),
3. Evaluate the PPF \( \mathcal{E} \) from (23).
4. Evaluate \( \mu(\|\hat{\mathcal{R}}\|_I, \mathcal{E}) \) and \( \mathcal{E}(\|\hat{\mathcal{R}}\|_I, \mathcal{E}) \) from (30) and (29), respectively.
5. Evaluate the estimator design \( \hat{b}, \hat{\sigma} \) and \( \hat{W} \) from (41), (42), (43), and (44), respectively.
6. Go to Step 2.

In the same manner, the steps stated above are applicable for the estimator proposed in Subsection IV-B.

D. Stochastic Estimators with Prescribed Performance: Discrete-Form

For the purpose of implementation, this subsection presents a discrete-form of the stochastic estimators proposed in Subsection IV-A and Subsection IV-B. The exact integration of (14) is equivalent to

\[
R[k+1] = R[k] \exp(\mathcal{O}[k]_x \Delta t)
\]

(81)

where \( \Delta t \) is a small time sample and \( [k] \) associated with a variable refers to its value at the \( k \)th sample for \( k \in \mathbb{N} \). In view of (81), the discrete analogue of the semi-direct nonlinear stochastic estimator with prescribed performance outlined in (63), (64), (65), and (66), is given in Algorithm 1.

Define \( x \in \mathbb{R}^3 \) as an axis that is rotating by an angle \( \beta \in \mathbb{R} \) in a 2-sphere \( \mathbb{S}^2 \). The rigid-body’s attitude is given by \( \mathcal{R}_\beta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{S}^2(3) \) visit (11)

\[
\mathcal{R}_\beta(\beta, x) = I_3 + \sin(\beta) [x]_x + (1 - \cos(\beta)) [x]^2_x
\]

The expression \( \mathcal{R}[k+1] = \mathcal{R}[k] \exp(\mathcal{O}[k]_x \Delta t) \) uses exact integration. Thereby, to guarantee that \( \exp(\mathcal{O}[k]_x \Delta t) \in \mathbb{S}^2(3) \), define \( u = \mathcal{O}[k]_x \Delta t, \) where \( x = u/\|u\| \) and \( \beta = \|u\| \). Hence, \( \exp(\mathcal{O}[k]_x \Delta t) = \mathcal{R}_\beta(\beta, x) \) can be obtained as follows

\[
\exp(\mathcal{O}[k]_x \Delta t) = I_3 + \sin(\beta) [x]_x + (1 - \cos(\beta)) [x]^2_x
\]

(82)

Likewise, in the light of Algorithm, the discrete filter of (40), (41), (42), (43), and (44) is given by

\[
\begin{align*}
\mu[k] &= \exp(2\mathcal{E}[k] + \exp(-2\mathcal{E}[k] + 2) \mathcal{E}[k]) \\
\hat{R}[k+1] &= \mathcal{R}[k] \exp(\left[ \mathcal{O}_m[k] - \hat{b}[k] - W[k] \right] \times \mathcal{E}[k] \Delta t) \\
W[k] &= \left( \frac{\Delta \mathcal{E}[k+1]}{2} + \mathcal{E}[k] \mathcal{Y}([\mathcal{E}[k]]) \hat{\sigma}[k] + 2 \mathcal{E}[k] \mathcal{Y}([\mathcal{E}[k]]) \right) \mathcal{Y}([\mathcal{E}[k]]) \mathcal{Y}([\mathcal{E}[k]]) \\
\hat{b}[k+1] &= \hat{b}[k] + \gamma_1 (\mathcal{E}[k] + 1) \exp(\mathcal{E}[k]) \mu[k] \mathcal{Y}(\hat{R}[k]) \Delta t \\
\hat{\sigma}[k+1] &= \hat{\sigma}[k] + \gamma_2 \mathcal{E}[k] \mathcal{Y}(\hat{R}[k]) \Delta t \\
\end{align*}
\]

(83)

with

\[
\begin{align*}
\mathcal{E}[k] &= \frac{\mathcal{Y}([\mathcal{E}[k]])}{\mathcal{Y}(\hat{R}[k])} \\
\mathcal{Y}([\mathcal{E}[k]]) &= \frac{\mathcal{Y}([\mathcal{E}[k]])}{\mathcal{Y}(\hat{R}[k])} \\
\end{align*}
\]

(84)
Algorithm 1 Complete implementation steps of the proposed direct stochastic estimator

Initialization:
1. Set $\hat{R}[0] \in SO(3)$. Alternative solution, construct $\hat{R}[0] \in SO(3)$ using one of the methods of attitude determination, visit [6]
2. Set $\hat{b}[0] = 0_{3 \times 1}$ and $\hat{\sigma}[0] = 0_{3 \times 1}$
3. Select $\xi_0 = \delta = \xi_infty$, $\ell$, $k_{sw}$, $\gamma_1$, and $\gamma_2$ as positive constants. Also, $s_i \geq 0$ with $\sum_{i=1}^{n} s_i = 3$
4. While $\forall i = 1, 2, \ldots, n$
5. Consider the measurements and observations in (12) and compute the normalization $v_i^T = \frac{v_i^T}{||v_i^T||}$ and $\tilde{v}_i = \frac{\hat{v}_i^T}{||\hat{v}_i^T||}$ as in (13)
6. $\hat{v}_i^B = \hat{R}^T v_i^T$ as in (58)
7. $M^B = \sum_{i=1}^{n} s_i \hat{v}_i^B (\hat{v}_i^B)^T$ as in (57) with $M^B = \text{Tr}(M^B)I_3 - M^B$
8. $\Upsilon = \sum_{i=1}^{n} s_i \hat{v}_i^B \times \hat{v}_i^B$ as in (59)
9. $||M^B||_f = \sum_{i=1}^{n} s_i \left(1 - (\hat{v}_i^B)^T \hat{v}_i^B\right)$ as in (61)
10. $\xi = (\xi_0 - \xi_{infty}) \exp(-k \Delta t) + \xi_{infty}$ as in (23) and $\xi_d = (\xi[k] - \xi[k-1]) / \Delta t$
11. $E[k] = \frac{1}{2} \ln \frac{||M^B||_f + ||M^B||_f / ||M^B||_f / ||M^B||_f}{\sin \delta}$ as in (29)
12. $\mu = \frac{\exp(2\xi[k]) + \exp(-2\xi[k]) + 2}{8 \xi \delta}$ as in (32)
13. $W[k] = \frac{\xi[k]}{\xi[k]} \text{diag}(\Upsilon) \hat{\sigma}[k] \frac{4}{\Delta t} k_{sw} \mu E[k] - \xi_{infty / 4} \Upsilon$ $\text{diag}(\Upsilon)$
14. $\Omega[k] = \Omega_m[k] - \hat{b}[k] - W[k]$
15. $\hat{R}[k+1] = \hat{R}[k] \exp \left(\Omega[k] \Delta t\right)$, visit (82)
16. $\hat{b}[k+1] = b_0[k] + \Delta t \gamma_1 \mu (E[k] + 1) \text{exp}(E[k]) \Upsilon$
17. $\hat{\sigma}[k+1] = \hat{\sigma}[k] + \Delta t \gamma_2 (E[k] + 2) \text{exp}(E[k]) \mu^2 \text{diag}(\Upsilon) \Upsilon$
18. $k + 1 \rightarrow k$

V. Simulations

In this section the performance of the proposed nonlinear stochastic attitude estimators on $SO(3)$ with prescribed performance is examined and tested against large initialization error and high level of noise and bias components associated to the measurements. Let the true attitude dynamics be as in (14) with the following angular velocity

$$\Omega = \left[ \sin(0.4 t), \sin(0.7 t + \frac{\pi}{4}), 0.4 \cos(0.3 t) \right]^T \text{rad/sec}$$

where $R(0) = I_3$. Let the measurement of angular velocity be $\hat{\Omega}_m = \Omega + b + \omega$ with $b = 0.1 [1, -1, 1]^T$ and $\omega$ being a wide-band of random white noise process with a standard deviation (STD) 0.3 (rad/sec). Consider two non-collinear inertial-frame vectors given by $v_1^T = \frac{1}{\sqrt{2}} [1, -1, 1]^T$ and $v_2^T = [0, 0, 1]^T$, while their measured values in the body-frame are $v_1^B = R^T v_1^T + b_1 + \omega_1^B$ for $i = 1, 2$. Let $b_1 = 0.1 [1, -1, 0]^T$ and $b_2 = 0.1 [0, 0, 1]^T$ and suppose that $\omega_i^B$ is a white noise vector with zero mean and a STD =0.12 for $i = 1, 2$. Let $v_i^T = \frac{\hat{v}_i^B}{||\hat{v}_i^B||}$ and $v_i^B = \frac{\hat{v}_i^B}{||\hat{v}_i^B||}$ for $i = 1, 2$ with $v_1^T = v_1^T \times v_2^T$ and $v_3^T = v_1^T \times v_2^T$. Also, $s_1 = 1.4, s_2 = 1.4$, and $s_3 = 0.2$. The given measurements are uncertain and signify gyro and two vector measurements which is characteristic of a low-cost IMU module. For the semi-direct estimator, the reconstructed attitude $R_y$ is evaluated by SVD [2] such that $\hat{R} = R_y \hat{R}$. The total simulation time is 30 seconds.

A very large initial attitude error has been considered with the initial attitude rotation of $\hat{R}$ defined according to angle-axis parameterization in (11). $\hat{R}(0) = R_{\theta}(\theta, u/||u||)$ where $\theta = 178$ (deg) and $u = [4, 1, 5]^T$, and $||R||_f$ is very near to the unstable equilibria (+1) with $||R||_f \approx 0.9999$

$$R(0) = I_3, \quad \hat{R}(0) = \begin{bmatrix} -0.2377 & 0.1635 & 0.9575 \\ 0.2173 & -0.9518 & 0.2165 \\ 0.9467 & 0.2596 & 0.1907 \end{bmatrix}$$

Initial estimates are $\hat{b}(0) = [0, 0, 0]^T$ and $\hat{\sigma}(0) = [0, 0, 0]^T$. The design parameters are $\gamma_1 = 1, \gamma_2 = 0.1, k_{sw} = 3, \delta = 1.2, \xi_0 = 1.2, \xi_{infty} = 0.04$, and $\ell = 4$.

The color notation used in the simulation graphs below is as follows: green color indicates true value, red refers to the performance of the semi-direct nonlinear stochastic estimator on $SO(3)$ discussed in Subsection IV-A, and blue demonstrates the performance of the direct nonlinear stochastic estimator on $SO(3)$ presented in Subsection IV-B. Magenta shows measured values while orange and black refer to the prescribed performance response.

The high levels of noise and bias components associated with the angular velocity and body-frame measurements are depicted against the true values in Fig. 3 and Fig. 4.

![Angular Velocities](image)

Fig. 3. Angular velocities: true and measured values.

A. Results of Stochastic Estimators in Continuous Form

The systematic and smooth convergence of the normalized Euclidean distance error $||\hat{R}||_f = \frac{1}{2} \text{Tr}(I_3 - R^T \hat{R})$ from a predetermined large set to a given small residual set is presented in Fig. 5. For an explicit illustration of the systematic convergence, the transient response has been presented over a period of (0-10) seconds, while the steady-state behavior has been plotted in a sub-figure of Fig. 5 over a period of
The output performance of the three Euler angle estimates of the proposed estimators plotted against the true Euler angles are given in Fig. 6. It can be noticed that a smooth and fast tracking performance is achieved in a short period of time. As Fig. 5 shows, the tracking error started very near to the unstable equilibria within a given large set and obeyed the dynamic decreasing boundaries of the PPF. (4-30) seconds. As such, the prescribed performance has been successfully achieved by utilizing the proposed robust stochastic estimators. The output performance of the three Euler angle estimates of the proposed estimators examined in Fig. 6. In comparison with the direct nonlinear stochastic estimator with prescribed performance, there is a remarkable smaller mean and STD of $||\tilde{R}||_I$ in comparison with the proposed estimators in a short period of time. Fig. 7 illustrates the output performance of the Euler angle estimates ($\phi[k], \theta[k], \psi[k]$) and the robustness of the proposed estimators against large error initialization and uncertainties in sensor measurements as demonstrated in Fig. 3, 4, 5, and 6.

Table I presents a summary of statistical details of the error ($||\tilde{R}||_I$), namely the mean and STD. This comparison allows to assess the steady-state error performance of the proposed estimators examining their oscillatory behavior. It can be noticed that both estimators have extremely small mean and STD of $||\tilde{R}||_I$. Nonetheless, the semi-direct nonlinear stochastic attitude estimator requires attitude reconstruction, and in our case SVD [2] was employed to obtain $\hat{R} = R_0^T \tilde{R}$.

### Table I

| Estimator  | Semi-direct | Direct |
|------------|-------------|--------|
| Mean       | $3.8 \times 10^{-3}$ | $5.2 \times 10^{-3}$ |
| STD        | $2.1 \times 10^{-3}$ | $2.6 \times 10^{-3}$ |

#### B. Results of Stochastic Estimators in Discrete Form

This subsection demonstrates the output performance of the proposed estimators in their discrete form in Subsection IV-D at small sampling interval. Set the sampling time to $\Delta t = 0.01$ seconds. The angular velocity and body-frame measurements be analogous to Fig. 3 and 4, respectively. Consider the following definition of the initial attitude (true and estimate):

$$R(0) = I_3, \ \hat{R}(0) = \begin{bmatrix} -0.8959 & -0.1209 & 0.4275 \\ 0.3824 & -0.6998 & 0.6034 \\ 0.2262 & 0.7041 & 0.6731 \end{bmatrix}$$

Let the initial estimates be $\hat{b}(0) = [0, 0, 0]^T$, $\hat{\sigma}(0) = [0, 0, 0]^T$, and set the design parameters to $\gamma_1 = 1$, $\gamma_2 = 0.1$, $k_w = 3$, $\delta = \hat{\delta} = 1.2$, $\xi_0 = 1.2$, $\xi_\infty = 0.04$, and $\ell = 4$. Fig. 7 reveals that the tracking error of $||\tilde{R}||_I$ started very near to the unstable equilibria and reduced virtually close to the origin. Additionally, Fig. 7 illustrates the output performance of the Euler angle estimates ($\phi[k], \theta[k], \psi[k]$) of the proposed estimators against the true Euler angles. Overall, Fig. 7 demonstrates smooth and fast tracking performance of the proposed estimators in a short period of time.

The simulation results reveal the effectiveness and confirm the robustness of the two proposed nonlinear stochastic estimators against uncertain measurements and large initialization error. In addition, both estimators are able to guide the tracking error to comply with the dynamically decreasing constraints set by the user, and thereby the proposed estimator design allows to achieve guaranteed measures of transient and steady-state performance. These advantageous features are not offered by the earlier proposed attitude estimators such as [10–13,15,32]. It should be noted that the semi-direct nonlinear stochastic attitude estimator requires attitude reconstruction, and in our case SVD [2] was employed to obtain $\hat{R} = R_0^T \tilde{R}$.  

Fig. 4. Body-frame vectors: true and measured values.

Fig. 5. Tracking performance of normalized Euclidean distance within PPF.

Fig. 6. Euler angles: true vs estimate.
I. Display remarkable convergence properties as detailed in Table Nonethelss, despite the higher computational power requirement of the semi-direct estimator, both proposed estimators display remarkable convergence properties as detailed in Table I.

VI. Conclusion

Successful robotic applications rely heavily on the convergence properties and steady-state behavior or the attitude estimators. Hence, it is crucial to account for the fact that attitude estimators are subject to large error in initialization and uncertainties in measurements, in particular when low-cost inertial measurement units are utilized for data collection. This paper addresses the above-mentioned challenge by proposing two nonlinear stochastic attitude estimators on $S\mathbb{O}(3)$ with guaranteed performance of transient and steady-state error. Both estimators reformulate the attitude error in terms of normalized Euclidean distance and then relax it from its constrained to unconstrained form, termed transformed error. This allows the transformed error and the normalized Euclidean distance of the attitude error to be regulated to the origin in probability from almost any initial condition. In addition, the equilibrium point has been proven to be independent of the unknown bias and noise components attached to the angular velocity measurements. Simulation results demonstrate the ability of the proposed estimators to obey the predefined characteristics of transient and steady-state performance set by the user and show their robustness against high level of uncertainties in the measurements and large initial attitude error.

APPENDIX A

Proof of Lemma 1

Define the attitude as $R \in S\mathbb{O}(3)$. Rodriguez parameters vector $\rho \in \mathbb{R}^3$ is employed for attitude representation such that the related map from vector form to $S\mathbb{O}(3)$ is given by $\mathcal{R}_\rho : \mathbb{R}^3 \rightarrow S\mathbb{O}(3)$ \cite{23,24} such that

$$\mathcal{R}_\rho (\rho) = \frac{1}{1 + ||\rho||^2} \left( (1 - ||\rho||^2) \mathbf{I}_3 + 2 \rho \rho^\top + 2 [\rho]_\times \right)$$  \hspace{1cm} (85)

with direct substitution of (85) in (3) it is straight forward to find

$$||R||_I = \frac{||\rho||^2}{1 + ||\rho||^2}$$  \hspace{1cm} (86)

Similarly, for $\mathcal{R}_\rho = \mathcal{R}_\rho (\rho)$ one has

$$\mathcal{P}_a (R) = \frac{1}{2} \left( \mathcal{R}_\rho - \mathcal{R}_\rho^\top \right) = 2 \frac{1}{1 + ||\rho||^2} [\rho]_\times$$

such that the vex of the above anti-symmetric operator is governed by

$$\text{vex} (\mathcal{P}_a (R)) = 2 \frac{\rho}{1 + ||\rho||^2}$$  \hspace{1cm} (87)

Hence, from (86) one obtains

$$1 - ||R||_I \cdot ||R||_I = \frac{||\rho||^2}{(1 + ||\rho||^2)^2}$$  \hspace{1cm} (88)

and from (87) it can be shown that

$$||\text{vex} (\mathcal{P}_a (R))||^2 = 4 \frac{||\rho||^2}{(1 + ||\rho||^2)^2}$$  \hspace{1cm} (89)

Therefore, (88) and (89) justify (19) in Lemma 1. In Subsection IV-B it is assumed that $\sum_{i=1}^{n} s_i = 3$ which means that $\text{Tr} \{M^B\} = 3$. Since $||M^B R||_I = \frac{1}{4} \text{Tr} \{M^B (I_3 - R)\}$, from angle-axis parameterization in (11), one has

$$||M^B R||_I = \frac{1}{4} \text{Tr} \left\{ -M^B \left( \sin(\theta) \ [u]_\times + (1 - \cos(\theta)) \ [u]_\times^2 \right) \right\}$$

$$= -\frac{1}{4} \text{Tr} \left\{ M^B (1 - \cos(\theta)) \ [u]_\times^2 \right\}$$  \hspace{1cm} (90)

with $\text{Tr} \{M^B [u]_\times \} = 0$ as defined in identity (8). The following relation holds \cite{22}

$$||R||_I = \frac{1}{4} \text{Tr} \{I_3 - R\} = \frac{1}{2} (1 - \cos(\theta)) = \sin^2 \left( \frac{\theta}{2} \right)$$  \hspace{1cm} (91)

such that the relation between Rodriguez parameters vector and angle-axis parameterization is given by \cite{23}

$$u = \cot \left( \frac{\theta}{2} \right) \rho$$

From identity (5), $||u||^2 = -||u||^2 I_3 + uu^\top$. Therefore, the result in (90) becomes

$$||M^B R||_I = \frac{1}{2} ||R||_I u^\top M^B u$$

$$= \frac{1}{2} ||R||_I \cot^2 \left( \frac{\theta}{2} \right) \rho^\top M^B \rho$$

Also, from (91), one has $\cos^2 (\theta/2) = 1 - ||R||_I$ which shows that

$$\tan^2 (\theta/2) = \frac{||R||_I}{1 - ||R||_I}$$
Thus, from (94) and (95) the following inequality holds
\[ \|\text{vex} (\mathcal{P}_a(M^B R))\|_F \geq \frac{\lambda}{2} \left( 1 + \text{Tr} \left( (M^B)^{-1} M^B R \right) \right) \|M^B R\|_F \]
this proves (20) in Lemma 1.

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