Backreaction to wormhole by classical scalar field: 
Will classical scalar field destroy wormhole?

Sung-Won Kim*
Department of Science Education
Ewha Womans University
Seoul 120-750, Korea

Abstract

There are two effects of extra matter fields on the Lorentzian traversable wormhole. The “primary effect” says that the extra matter can afford to be a part of source or whole source of the wormhole when the wormhole is being formed. Thus the matter does not affect the stability of wormhole and the wormhole is still safe. If the extra matter is exotic, it can be the whole part of the source of the wormhole.

The “auxiliary effect” is that the extra matter plays the role of the additional matter to the stably-existed wormhole by the other exotic matter. This additional matter will change the geometry of wormhole enough to prevent from forming the wormhole by backreaction. In the minimally coupled massless scalar field case, the self-consistent solution was found. The backreaction of the scalar field can dominate the exotic matter part so that it will hinder the formation of the wormhole.
I. INTRODUCTION

One of the most important issues in making a practically usable Lorentzian wormhole is just the traversability [1, 2]. If it is traversable, there is a good usability, such as the short-cut in space [1], the time machine [2, 3], and the inspector of the interior of black hole [4]. For the realization problem, it is believed that the wormhole is created in Planckian era as the quantum foam [5]. As the other approach about wormhole creation, it is suggested that the wormhole can be made from a black hole by adding some special exotic matter [6]. The wormhole model as the final state of black hole evaporation was very recently established in two dimensions [7].

To make a Lorentzian wormhole traversable, one has usually used an exotic matter which violates the well-known energy conditions [1, 2]. For instance, a wormhole in the inflating cosmological model still requires the exotic matter to be traversable and to maintain its shape [8]. It is known that the vacuum energy of the inflating wormhole does not change the sign of the exoticity function. A traversable wormhole in the Friedmann-Robertson-Walker(FRW) cosmological model, however, does not necessarily require the exotic matter at the very early times [9]. The result means that there were an exotic period in the early universe.

The problem about the maintaining wormhole by other fields relating with the exotic property has also been interesting to us. There are two ways to generalize or modify the Lorentzian traversable wormhole spacetime. (From now on ‘wormhole’ will be simply used as the meaning of the ‘Lorentzian traversable wormhole’ unless there is a confusion.) One way is the generalization of the wormhole by alternative theory, for example, Brans-Dicke theory, Einstein-Cartan theory, etc. The other is the generalization by adding the extra matter.

In the case of the latter generalization, the added matter will play two kinds of roles in affecting the wormhole spacetime. The first role of the other physical matters (for example, scalar field, charge, spin, etc.) is the “primary effect”, which says that the added matter is the partial or total source of the wormhole. The matter gets involved in the constructing stage of the wormhole. The wormhole is safe under the addition of the matter, since the matter is a part of the sources for constructing wormhole. This means that if this added extra matter field has the exotic properties, it shares the exoticity with other matter which is exotic. When the other fields are not exotic, the added matter will monopoly the exotic property. In this case, wormhole cannot be performed without this extra matter. Its example is the case of the wormhole solution with scalar field [10].

The second role is the “auxiliary effect”. In this effect, the added matter is not a source, but an extra effect to the existed wormhole. Therefore, it is not involved with the constructing stage of the wormhole, but relates with the wormhole after the construction. Since the matter makes the extra geometry, the wormhole is not safe when this effect dominate the exotic matter for constructing wormhole. It is the backreaction to the wormhole by the additional field.

In this paper, the self-consistent solution of the wormhole with classical, minimally-coupled, massless scalar field is found. The backreaction of the scalar field on wormhole spacetime is also found to see the stabilities of the wormhole. The result is that the modified wormhole can be broken by the large variation of the scalar field.
The similar works about the scalar field effect on black hole and wormhole have been done by several authors. For the example on black hole, Fonarev [11] generated new exact solutions of Einstein-scalar field equations from static vacuum solutions of Einstein equations. In that paper, the cosmological black hole solution was obtained.

It was already shown that the minimally coupled massless scalar field taken as a source of Einstein gravity admitted only the Schwarzschild black hole as a solution [12]. The result for the conformally invariant system was same [13]. Chamber et al [14] tried to find the evolution of a Kerr black hole emitting purely scalar radiation via Hawking process.

There were also some works for the effect on wormholes. Taylor and Hiscock [15] examined whether the stress-energy of quantized fields in fact will have the appropriate form to support a wormhole geometry. They do not attempt to solve the self-consistent semiclassical Einstein equations. They found that the stress-energy tensor of the quantized scalar field is not even qualitatively of the correct form to support the wormhole. To maintain a wormhole classically, Vollick [16] found the effect of the coupling a scalar field to matter which satisfies the weak energy condition.

In Sec. II, these “primary” and “auxiliary” effects are described more concretely in analytic form. Breakdown of the wormhole by the backreaction of classical scalar field is calculated in Sec. III. Finally the summary and the further problems are discussed in Sec. IV.

II. GENERALIZATION OF WORMHOLE BY EXTRA MATTER

The Einstein equation for the simplest normal (usual) wormhole spacetime is given as

\[ G_{\mu\nu}^{(0)} = 8\pi T_{\mu\nu}^{(0)} \]  

The left hand side, \( G_{\mu\nu}^{(0)} \), is the wormhole geometry and the right hand side, \( T_{\mu\nu}^{(0)} \), is the exotic matter violating the known energy conditions, which is needed to construct the wormhole.

The Einstein equation with the “primary effect” is

\[ G_{\mu\nu}^{(0)} = 8\pi [T_{\mu\nu} + T_{\mu\nu}^{(1)}] = 8\pi T_{\mu\nu}^{(0)} \]  

While the left hand side is usual wormhole geometry, the right hand side is divided into two terms: the original matter \( T_{\mu\nu} \) and the added matter \( T_{\mu\nu}^{(1)} \). Since the sum of the right hand side is still \( T_{\mu\nu}^{(0)} \), the added matter does not affect on the geometry of the wormhole any more. The original matter part, \( T_{\mu\nu} = T_{\mu\nu}^{(0)} - T_{\mu\nu}^{(1)} \), will be determined according to the magnitude of the additional matter \( T_{\mu\nu}^{(1)} \). When \( T_{\mu\nu}^{(1)} \) is not exotic, the original matter part \( T_{\mu\nu} \) should be exotic. If the additional matter \( T_{\mu\nu}^{(1)} \) is exotic, then \( T_{\mu\nu} \) is not necessarily exotic. In this case, we can construct wormhole with usual matter, while the exotic part will be replaced by the additional \( T_{\mu\nu}^{(1)} \), which makes total matter \( T_{\mu\nu}^{(0)} \) exotic. The special model of this case will be \( G_{\mu\nu}^{(0)} = 8\pi T_{\mu\nu}^{(1)} \), i.e., the additional matter takes over the whole \( T_{\mu\nu}^{(0)} \) without any other matter. It was already tried in scalar field case (if only \( T_{\mu\nu}^{(1)} \) is exotic), for example, the conformally-coupled case [10].

For the “auxiliary effect”, the additional matter \( T_{\mu\nu}^{(1)} \) is added to the right hand side of Eq. (1) and its backreaction \( G_{\mu\nu}^{(1)} \) is added to the geometry, the left hand side, so that the Einstein equation becomes
The sum of the matters of the right hand side already satisfies the conservation law. However, there is no guarantee on the exoticity of the right hand side. When the effect on the geometry of wormhole, \( G^{(1)}_{\mu\nu} \), is large enough to dominate any structure, the additional matter, for example, scalar field has the preventing mechanism from the formation of the wormhole.

There might be an interaction term \( T^{(\text{int})}_{\mu\nu} \) between \( T^{(0)}_{\mu\nu} \) and \( T'^{\mu\nu} \), such as Ref. [16]. The model is the maintaining wormhole by the coupling a scalar field to matter that is not exotic, but the coupling is not. Then there also might be the interaction term \( G^{(\text{int})}_{\mu\nu} \) in geometry. In some case, \( G^{(\text{int})}_{\mu\nu} \) may be joined in Eq. (3), even though \( T^{(\text{int})}_{\mu\nu} \) does not appear, so that the equation can have the form as \( G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(\text{int})}_{\mu\nu} = 8\pi(T^{(0)}_{\mu\nu} + T'^{(1)}_{\mu\nu}) \). This example is the charged wormhole case which will be discussed in separate paper [17].

III. BACKREACTION TO WORMHOLE

A. Static wormhole

Before we study the wormhole model with scalar field, the metric of the static wormhole is given by

\[
ds^2 = -e^{2\Lambda(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}\]

The arbitrary functions \( \Lambda(r) \) and \( b(r) \) are lapse and wormhole shape functions, respectively. The shape of the wormhole is determined by \( b(r) \). Beside the spherically symmetric and static spacetime, we further assume a zero-tidal-force as seen by stationary observer, \( \Lambda(r) = 0 \), to make the problem simpler.

The Einstein equation Eq. (1) is given as

\[
\begin{align*}
\frac{b'}{8\pi r^2} &= \rho^{(0)}, \\
\frac{b}{8\pi r^3} &= \tau^{(0)}, \\
\frac{b - b'r}{8\pi r^3} &= P^{(0)}. \tag{7}\end{align*}
\]

Assuming a spherically symmetric spacetime, one finds the components of \( T^{(0)}_{\mu\nu} \) in orthonormal coordinates

\[
T^{(0)}_{tt} = \rho^{(0)}(r), \quad T^{(0)}_{\phi\phi} = -\tau^{(0)}(r), \quad T^{(0)}_{\theta\theta} = P^{(0)}(r), \tag{8}\]

where \( \rho^{(0)}(r) \), \( \tau^{(0)}(r) \) and \( P^{(0)}(r) \) are the mass energy density, radial tension per unit area, and lateral pressure, respectively, as measured by an observer at a fixed \( r, \theta, \phi \).
B. Primary effect by scalar field

Firstly, we study the simplest case of a static Lorentzian wormhole with a minimally-coupled massless scalar field. The additional matter Lagrangian due to the scalar field is given by

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \quad (9)$$

and the equation of motion for \( \varphi \) by

$$\Box \varphi = 0. \quad (10)$$

The stress-energy tensor for \( \varphi \) is obtained from Eq. (9) as

$$T^{(1)}_{\mu\nu} = \varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \varphi_{,\rho} \varphi_{,\sigma}. \quad (11)$$

Now the Einstein equation Eq. (2) has an additional stress-energy tensor (11)

$$G^{(0)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{(0)}_{\mu\nu} = 8\pi (T_{\mu\nu} + T^{(1)}_{\mu\nu}), \quad (12)$$

where \( T_{\mu\nu} \) is the stress-energy tensor of the background matter. Here \( T^{(1)}_{\mu\nu} \) will be a portion of \( T^{(0)}_{\mu\nu} \) as the “primary effect” to the wormhole. One also finds the components of \( T^{(1)}_{\mu\nu} \) in orthonormal coordinates

$$T^{(1)}_{tt} = \rho(r), \quad T^{(1)}_{rr} = -\tau(r), \quad T^{(1)}_{\theta\theta} = P(r), \quad (13)$$

as the similar way of Eq. (8). Thus not only the scalar field \( \varphi \) but also the matter \( T_{\mu\nu} \) are assumed to depend only on \( r \). The components of \( T^{(1)}_{\mu\nu} \) in the static wormhole metric (4) with \( \Lambda = 0 \) have the form

$$T^{(1)}_{tt} = \frac{1}{2} \left( 1 - \frac{b}{r} \right) \varphi'^2, \quad (14)$$

$$T^{(1)}_{rr} = \frac{1}{2} \varphi'^2, \quad (15)$$

$$T^{(1)}_{\theta\theta} = -\frac{1}{2} r^2 \left( 1 - \frac{b}{r} \right) \varphi'^2, \quad (16)$$

$$T^{(1)}_{\phi\phi} = -\frac{1}{2} r^2 \left( 1 - \frac{b}{r} \right) \varphi'^2 \sin^2 \theta, \quad (17)$$

where and hereafter a prime denoted the differentiation with respect to \( r \). In the spacetime with the metric (4), the field equation Eq. (10) of \( \varphi \) becomes

$$\frac{\varphi''}{\varphi'} + \frac{1}{2} \frac{(1 - b/r)'}{1 - b/r} + \frac{2}{r} = 0 \quad \text{or} \quad r^4 \varphi'^2 \left( 1 - \frac{b}{r} \right) = \text{const}, \quad (18)$$

and the Einstein equations are given explicitly by
\[
\frac{b'}{8\pi r^2} = \rho^{(0)} = \rho + \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right), \tag{19}
\]
\[
\frac{b}{8\pi r^3} = \tau^{(0)} = \tau - \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right), \tag{20}
\]
\[
\frac{b - b'}{16\pi r^3} = P^{(0)} = P - \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right). \tag{21}
\]

Thus one sees that the conservation law of the effective stress-energy tensor \( T^{\mu\nu} + T^{(1)}_{\mu\nu} \) still obeys the same equation
\[
\tau^{(0)}' + \frac{2}{r} (\tau^{(0)} + P^{(0)}) = 0. \tag{22}
\]

With the equation of state, \( P^{(0)} = \beta \rho^{(0)} \) and the appropriate asymptotic flatness, we find the matter as functions of \( r \)
\[
\rho^{(0)}(r) \propto r^{-2(1+3\beta)/(1+2\beta)}, \tag{23}
\]
\[
\tau^{(0)}(r) \propto r^{-2(1+3\beta)/(1+2\beta)}, \tag{24}
\]
\[
b(r) \propto r^{1/(1+2\beta)}, \quad \beta < -\frac{1}{2}. \tag{25}
\]

Here the wormhole is stable, because the added scalar field does not affect any spacetime geometry.

**C. Auxiliary effect by scalar field**

Next we think the “auxiliary effect” by the scalar fields, the additions of the scalar field \( T^{(1)}_{\mu\nu} \) to the existing wormhole matter \( T^{(0)}_{\mu\nu} \). If we add \( G^{(1)}_{\mu\nu} \) as an additional geometry to \( G^{(0)}_{\mu\nu} \) by scalar field, the Einstein equation Eq. (4) is changed from Eq. (5)-(7) into
\[
\frac{b'}{8\pi r^2} + \frac{1}{8\pi} \frac{\alpha}{r^4} = \rho^{(0)} + \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right), \tag{26}
\]
\[
\frac{b}{8\pi r^3} - \frac{1}{8\pi} \frac{\alpha}{r^4} = \tau^{(0)} - \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right), \tag{27}
\]
\[
\frac{b - b'}{16\pi r^3} - \frac{1}{8\pi} \frac{\alpha}{r^4} = P^{(0)} - \frac{1}{2} \varphi^2 \left( 1 - \frac{b}{r} \right). \tag{28}
\]

when the interaction between the matter and additional scalar fields is neglected. Here \( \alpha \) is defined as the positive value. The term \( \alpha/r^4 \) is added to the left hand side, because the field equation Eq. (18) of field \( \varphi \) shows that \( \varphi^2 \left( 1 - \frac{b}{r} \right) \propto r^{-4} \).

If we set \( b_{\text{eff}} = b - \alpha/r \) instead of \( b \), then the Einstein equations Eq. (26)-(28) satisfy self-consistently. Thus the effect by the scalar field on the wormhole is only by changing the wormhole function \( b \) into \( (b - \alpha/r) \) without any interaction term in the left hand side. Since \( b \) is proportional to \( r^{1/(1+2\beta)} \), with the proper parameter \( \beta \) of equation of state, the wormhole will vary seriously by the additional factor \( -\alpha/r \), according to the values of the parameters
β and α. While β is given as the equation of state by the choice of the appropriate matter, α depends on the changing rate of the scalar field ϕ, if only b is fixed.

For the wormhole function, we can set as

\[ b = b_0^{\frac{2\beta}{2\beta+1}} r^{\frac{1}{2\beta+1}}, \]

where the value of β should be less than \(-\frac{1}{2}\) so that the exponents of \(b_0\) and \(r\) can be negative.

When \(-1 < \beta < -\frac{1}{2}\), the power of \(b(r)\) is less than \(-1\), the power of the second term, so \(b(r)\) vanishes more quickly in the far region. Thus it gives the negative region for \(b_{\text{eff}}\) at large \(r\), even though it has the positive regions near throat when \(b_0 > \alpha\). The region of the positive \(b_{\text{eff}}\) is \(b_0 < r < r_0\), where \(r_0 = \alpha^{(1+2\beta)/(2+2\beta)}/b_0^{\beta/(1+2\beta)}\). If \(b_0 < \alpha\), \(b_{\text{eff}}\) is negative at all \(r\), which is not suitable for wormhole formation.

If only \(\beta \leq -1\) and \(b_0 > \alpha\), the wormhole is safe, because \(b_{\text{eff}}\) is positive at all \(r\). Otherwise, the scalar field effect will change the wormhole structure into others, since the scalar field dominates the exotic matter. Thus the addition of the minimally-coupled, massless scalar field does not guarantee the structure of wormhole.

Now we shall examine the special case of this backreaction problem, for instance, \(\beta = -1\) which is \(b = b_0^2/r\). In this case, the solution of the scalar field is given as

\[ \varphi = \varphi_0 \left[ 1 - \cos^{-1} \left( \frac{b_0}{r} \right) \right]. \]

Thus the proportional constant α becomes

\[ \alpha = \varphi_0^2 b_0^2, \]

where \(b_0\) is the minimum size of the wormhole and \(\varphi_0\) is the value of \(\varphi(r)\) at \(r = b_0\). Therefore, \(\varphi_0 < b_0^{-1/2}\) is the condition that is required for maintaining the wormhole under the addition of the scalar field. In this choice of \(\beta = -1\), there is no \(r_0\) at which \(b_{\text{eff}}\) changes from positive value to negative one.

We can also apply the result to the other form of \(b(r)\), which means the exotic matter distribution in the restricted region only, “absurdly benign” wormhole,

\[ b(r) = \begin{cases} 
  b_0[1 - (r - b_0)/a_0]^2, & \text{if } \Phi(r) = 0, \quad b_0 \leq r \leq b_0 + a_0, \\
  b = \Phi = 0, & \text{if } r \geq b_0 + a_0.
\end{cases} \]

In this case, since the second term \(-\alpha/r\) in the effective shape function by the scalar field extends to over the region \(r \geq b_0 + a_0\), there will be a negative \(b_{\text{eff}}\) within this range of values of \(r\), which means that wormhole will be broken.

IV. DISCUSSION

Here we studied the backreaction to wormhole by the scalar field and found the self-consistent solution. The scalar field effect may break the wormhole structure when the field and the variation of the field is large. Similar consequences are obtained in charged wormhole.
case [17], in which there is the interaction term in geometry, even though no interaction term in matter. It is natural that the addition of the nonexotic matter will break wormhole if the “auxiliary effect” is large.

In this paper, the interaction between the extra field and the original matter is neglected. If the interaction exists and it is large, it can change the whole geometry drastically. If it is very small, it does not change the main structure of the wormhole. The detailed discussion on these interactions will be in separate paper.

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