New features of collective motion of intrinsic degrees of freedom. Toward a possible way to classify the intrinsic states

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Abstract

Three exactly solvable Hamiltonians of complex structure, are studied in the framework of a semi-classical approach. The quantized trajectories for intrinsic coordinates correspond to energies which may be classified in collective bands. For two of the chosen Hamiltonians the symmetry SU(2)⊗SU(2) is the appropriate one to classify the eigenvalues in the laboratory frame. Connections of results presented here with the molecular spectrum and Moszkowski model are pointed out. The present approach suggests that the intrinsic states, which in standard formalisms are heading rotational bands, are forming themselves "rotational" bands, the rotations being performed in a fictitious boson space.

I. INTRODUCTION

Collective motion in nuclei has constantly been an exciting subject for both experimentalists and theoreticians. Some phenomenological models have been successfully in interpreting the data for energies and transition probabilities for certain ranges of atomic mass. The long series of publi-
cations has been opened by the liquid drop model [1] which was later on improved by including anharmonicities, deformed equilibrium shapes or by extending the symmetries for the intrinsic motion [2–4]. Some phenomenological models interpret the properties of collective states of positive parity in terms of quadrupole degrees of freedom. Interacting boson model (IBA) [5] points out the fact that the monopole degrees of freedom should be included which results in obtaining a conceptually new formalism for the description of nuclear dynamics. All phenomenological models define notions like rotational bands, equilibrium shapes, nuclear phases based on the behavior of the system in the intrinsic frame. The coherent state model (CSM) exploits the classical features of collective states of high angular momentum and provides a realistic description for deformed and transitional region in terms of angular momentum projected states from coherent states [8].

Some microscopic theories were developed by paralleling the aims of the phenomenological models. Indeed, the nuclear properties are described in terms of individual degrees of freedom. In this sense to define the optimal collective coordinates for a many body system, was always a central field of activity. Microscopic formalisms use approximations like Bogoliubov-Hartree-Fok, Random Phase Approximation (RPA), boson expansion technique, higher order RPA, extended shell model, etc. [10], which need to be tested and define the circumstances under which they work or not. For such purposes very often one uses completely solvable models where the results of various approaches are compared with the exact result. Most known schematic and solvable models are those of Moszkowski [22], Lipkin-Meshkov [1] and One Level Pairing Hamiltonian [12,13]. Also phenomenological models proposed by Wilets and Jean [14], Davydov and Filippov [15] are even nowadays used to get a reference framework which allows for an interpretation of the results obtained with more sophisticated
models.

In this paper we address the question whether the motion of intrinsic collective coordinates can be described by the irreducible representation of a symmetry group. The chosen model Hamiltonians have a complex structure being of fourth order in quadrupole bosons. They have however a common feature that all of them are, in a semi-classical picture, completely solvable and therefore analytical expressions for energies are obtained.

The model Hamiltonians are explicitly treated according to the following plan: In Section 2 a fourth order Hamiltonian consisting in a quadratic quadrupole boson term plus a fourth order term which is the square of the second order boson invariant. Two sets of parameters are considered, one yielding, after dequantization, an effective potential with one minimum and an other one which produces a pocket potential. Solutions are presented for both cases. Energies were organized in bands like standard bands are in the laboratory frame. The difference is that here the angular momentum is defined by an $\mathcal{R}_3$ group acting in a fictitious space. In Section 3 the two body boson interaction is of a multipole-multipole type with the multipolarity $2^k, k = 0, 2, 4$. In the intrinsic frame the classical Hamiltonian has energies which after quantization might be classified by $L,M$ and therefore the corresponding states are SU(2) states. This pictures implies that in the laboratory frame the symmetry SU(2)$\otimes$SU(2) might be used. In Section 4, the two body boson interaction is of a quadrupole-quadrupole type. Neglecting the correlations with the excited collective bands the classical Hamiltonian is identical with those proposed by Moszkowski and therefore a simple expression for the corresponding spectrum is possible. Final conclusions are drawn in Section 5.
II. THE FIRST EXAMPLE OF AN EXACTLY SOLVABLE HAMILTONIAN

In this Section we analyze the classical behavior of a fourth order boson Hamiltonian with a particular structure. The third order term is missing and the fourth order one depends exclusively on quadrupole coordinates but on their conjugate momenta:

\[
H_1 = A_1 (b_2^\dagger b_2) + A_2 \left[ (b_2^\dagger b_2)_0 + (b_2 b_2)_0 \right] + A_4 \hat{P}^2. \tag{2.1}
\]

Here \( b_{2\mu}^\dagger \) (\( b_{2\mu} \)) denotes the creation (annihilation) quadrupole boson operator while \( \hat{P} \) is the second order boson invariant:

\[
\hat{P} = \frac{1}{2} \sum_\mu (b_{2\mu}^\dagger + (-)^\mu b_{2-\mu}) (b_{2-\mu}^\dagger + (-)^\mu b_{2\mu})(-)^\mu. \tag{2.2}
\]

We consider the variational equation

\[
\delta \int_0^t \langle \psi | H_1 - i\hbar \frac{\partial}{\partial t'} | \psi \rangle dt' = 0 \tag{2.3}
\]

where \( | \Psi \rangle \) denotes the following coherent state

\[
| \psi \rangle = \exp \left[ z_0 b_{20}^+ - z_0^* b_{20} + z_2 (b_{22}^+ + b_{2-2}^+) - z_2^* (b_{22} + b_{2-2}) \right] | 0 \rangle \tag{2.4}
\]

The vacuum state for the quadrupole boson operators \( b_{2\mu} \) is denoted by \( | 0 \rangle \); \( z_0 \) and \( z_2 \) are complex functions of time and play the role of classical coordinates. To simplify the notations, hereafter the quadrupole boson operators will be denoted by \( b_{\mu}^\dagger \) omitting the index specifying the angular momentum carried by the boson operators.

The coordinate transformation:

\[
q_i = 2^{k+2} Re(z_k), \quad p_i = 2^{k+2} Im(z_k), \quad k = 0, 2, \quad i = \frac{k + 2}{4}, \tag{2.5}
\]

brings the classical equation of motion to a canonical form:

\[
\frac{\partial \mathcal{H}_1}{\partial q_k} = -\dot{p}_k, \quad \frac{\partial \mathcal{H}_1}{\partial p_k} = \dot{q}_k. \tag{2.6}
\]
Here $H_1$ stands for the average value of the chosen Hamiltonian on the coherent state $|\Psi\rangle$ while “dot” denotes the time derivative operation.

$$H_1 \equiv \langle \psi | H_1 | \psi \rangle = \frac{A'}{2}(p_1^2 + p_2^2) + \frac{A}{2}(q_1^2 + q_2^2) + D(q_1^2 + q_2^2)^2. \quad (2.7)$$

The coefficients involved in the expression of $H_1$ are related to those defining the model Hamiltonian by the following equations:

$$A = \frac{1}{\sqrt{5}}(A_1 + 2A_2) + 14A_4,$$
$$A' = \frac{1}{\sqrt{5}}(A_1 - 2A_2),$$
$$D = 4A_4. \quad (2.8)$$

For what follow, it is useful to introduce the polar coordinates:

$$q_1 = r \cos \theta, \quad q_2 = r \sin \theta. \quad (2.9)$$

In terms of the new coordinates the classical energy function has the expression:

$$H_1 = \frac{\hbar^2}{2A'}(\dot{\theta}^2 + r^2 \dot{\theta}^2) + \frac{A}{2}r^2 + \frac{D}{4}r^4. \quad (2.10)$$

From the above equations we see that the classical system associated to the boson Hamiltonian is a particle which moves in a plane due to the force determined by a potential invariant with respect to rotations around an axes perpendicular to the motion plane:

$$V(r) = \frac{A}{2}r^2 + \frac{D}{4}r^4. \quad (2.11)$$

The potential energy is plotted in Fig.1 for three sets of parameters $(A, D)$. The parameters were chosen so that three distinct situations are pointed out, namely when the equilibrium shape is spherical, deformed and meta-stable. Using the equations of motion (2.6) one can prove that

$$\dot{\mathcal{L}}_3 = 0, \quad \dot{H}_1 = 0. \quad (2.12)$$

where $\mathcal{L}_3$ is defined by the following expression:
\[ L_3 = \frac{\hbar}{2}(q_1p_2 - q_2p_1) = \frac{\hbar^2}{A'}r^2\dot{\theta}. \] (2.13)

Let us define another two classical functions on the phase space, spanned by the coordinates \((q_1, p_1, q_2, p_2)\)

\[ L_1 = \frac{\hbar}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2) \]
\[ L_2 = \frac{\hbar}{2}(q_1q_2 + p_1p_2). \] (2.14)

Given two function \(f_1\) and \(f_2\), defined on the phase space, their Poisson bracket is defined as:

\[ \{f_1, f_2\} = \sum_{k=1,2} \left[ \frac{\partial f_1}{\partial q_k} \frac{\partial f_2}{\partial p_k} - \frac{\partial f_1}{\partial p_k} \frac{\partial f_2}{\partial q_k} \right]. \] (2.15)

The classical functions \(L_k\) obey the following equations:

\[ \{L_1, L_2\} = \hbar L_3, \]
\[ \{L_2, L_3\} = \hbar L_1, \]
\[ \{L_3, L_1\} = \hbar L_2. \] (2.16)

In virtue of Eq.(2.16) the set of functions \(L_k\) with the Poisson brackets as multiplication operation, form a classical \(SU_c(2)\) algebra. Moreover they could be obtained by averaging on \(|\Psi\rangle\), the generators of a boson \(SU_b(2)\) algebra defined with the boson operators \(b_0^\dagger, b_{\pm 2}^\dagger\):

\[ L_k = \langle \psi \mid \hat{L}_k \mid \psi \rangle; k = 1, 2, 3, \]
\[ \hat{L}_1 = \frac{\hbar}{4} \left[ 2b_0^\dagger b_0 - (b_2^\dagger + b_{-2}^\dagger)(b_2 + b_{-2}) \right], \]
\[ \hat{L}_2 = \frac{\hbar}{2\sqrt{2}} \left[ b_0^\dagger (b_2 + b_{-2}) + (b_2^\dagger + b_{-2}^\dagger)b_0 \right], \]
\[ \hat{L}_3 = \frac{\hbar}{2\sqrt{2}i} \left[ b_0^\dagger (b_2 + b_{-2}) - (b_2^\dagger + b_{-2}^\dagger)b_0 \right]. \] (2.17)

The first equation (2.17) and the correspondence between commutators and Poisson brackets

\[ [\cdot, \cdot] \rightarrow \frac{1}{i}\{\cdot, \cdot\}, \] (2.18)
define a homeomorphism of the boson and classical algebras generated by \( \{ \hat{L}_k \}_{k=1,2,3} \) and \( \{ L_k \}_{k=1,2,3} \) respectively. Note that the boson \( SU_b(2) \) algebra does not describe the rotations in the real configuration space but in a fictitious space. The conservation law expressed by (2.12) is determined by the invariance against rotation around the 3-rd axis in the fictitious space mentioned above:

\[
[H_1, \hat{L}_3] = 0. \tag{2.19}
\]

Since the classical system is characterized by two degrees of freedom and, on the other hand, there are two constants of motion

\[
\mathcal{H}_1 = E, \quad \mathcal{L}_3 = \frac{L}{2}, \tag{2.20}
\]

the equations of motion are exactly solvable. Indeed, inserting the constants of motion in Eq. (2.10) the resulting differential equation

\[
\frac{\hbar^2 \dot{r}^2}{2A'} + V_{eff}(L;r) = E, \tag{2.21}
\]

with

\[
V_{eff}(L;r) = \frac{A'L^2}{2r^2} + \frac{A}{2} r^2 + \frac{D}{4} r^4, \tag{2.22}
\]

provides the time variable as function of \( x = r^2 \) by

\[
t = \frac{\hbar}{\sqrt{-2A'D}} \int_{x_0}^{x} \frac{dy}{\sqrt{y^3 + \frac{2A'}{D} y^2 - 4EDy + \frac{2A'L^2}{D} \frac{\hbar^2}{4}}}. \tag{2.23}
\]

Note that \( L \) has the meaning of the third component of the angular momentum in the space spanned by the coordinates \( q_1, q_2, q_3 \) with \( q_3 \) a third coordinate which might be associated in a more complex situation to an additional degree of freedom.

In our numerical application we considered two sets of parameters:

I) \( A' = 0.01 MeV, A = 3 MeV, D = 0.4 MeV \);

II) \( A' = 0.01 MeV, A = 3 MeV, D = -0.04 MeV \). \( \tag{2.24} \)
The two situations are represented pictorially in Figs. 2,3 for several values of $L$. One notes that while the first effective potential has only one extreme value, a minimum, which depends on angular momentum, in the case labeled with I, there are two extremes, one minimum and one maximum. While the minima are depending on angular momenta, maxima are almost independent. Moreover there is a critical value for angular momentum where the two extremes get unified into an inflexion point. Below the critical angular momentum the effective potential has a pocket shape and is similar to that obtained in the study of heavy ion collision with two centers harmonic potential. Equation (2.21) suggests that the motion is allowed only for energies obeying the restriction:

$$E \geq V_{\text{eff}}^{\text{min}}$$  \hspace{1cm} (2.25)

For a given pair of $(E,L)$ the intervals of $r$ were the motion takes place are:

$$r_1 \leq r \leq r_2, \quad \text{and} \quad r \geq r_3 \quad \text{for} \quad E \leq V_{\text{eff}}^{\text{max}},$$

$$r \geq r_1 \quad \text{for} \quad E \geq V_{\text{eff}}^{\text{max}},$$  \hspace{1cm} (2.26)

were $r_k$ denotes the values of $r$ were $E = V_{\text{eff}}$. The minimum and maximum values of effective potential are denoted by $V_{\text{eff}}^{\text{min}}$ and $V_{\text{eff}}^{\text{max}}$, respectively. Classical trajectories may evolve, depending on the initial conditions, either on a finite trajectory in the interval $r_1 \leq r \leq r_2$ or on an unbound one with $r \geq r_3$. As we shall see, in the first interval the classical motion could be quantized while in the second one the system undergoes a fission process. In our semi-classical quantization procedure the system cannot go, through a tunneling effect, to the unstable region. However, in a full quantum mechanical description the wave function describing the system inside the pocket region, is spread also to the region $r \geq r_3$.

Since $t$ is a real quantity the integration interval in (2.23) is chosen so that inside it the polynomial is positive. As shown in Figs. 4 and 5 these intervals are:
\[ e_2 \leq x \leq e_3 \text{ for case I, } e_1 \leq x \leq e_2 \text{ for case II.} \] (2.27)

This suggests that a possible initial condition is:

\[ x_0 \equiv x(0) = \begin{cases} 
  e_2, & \text{case I} \\
  e_1, & \text{case II, bound trajectory} \\
  e_3, & \text{case II, unbound trajectory} 
\end{cases} \] (2.28)

The integral involved in Eq. (2.23) can be analytically performed \[16\] and the final result for time is:

\[ t = \frac{\sqrt{2h}}{\sqrt{|A'|/|D|}} \text{sn}^{-1}(u, k), \]

\[ u = \sqrt{\frac{(e_3 - e_1)(x - e_2)}{(e_3 - e_2)(x - e_1)}}, \quad k = \sqrt{\frac{e_3 - e_2}{e_3 - e_1}}, \text{ for the case I} \]

\[ u = \sqrt{\frac{x - e_1}{e_2 - e_1}}, \quad k = \sqrt{\frac{e_2 - e_1}{e_3 - e_1}}, \text{ for the case II, } e_1 \leq x \leq e_2 \]

\[ u = \sqrt{\frac{e_3 - e_1}{x - e_1}}, \quad k = \sqrt{\frac{e_2 - e_1}{e_3 - e_1}}, \text{ for the case II, } x \geq e_3. \] (2.29)

Here \( \text{sn}(u, k) \) denotes the elliptic sinus function. Its argument "u" depends on 'x', the upper limit of the integral defining the time as well as on the coordinates \( e_1, e_2, e_3 \) where the polynomial lying under the square root symbol vanishes. Actually these are obtained by squaring \( r_1, r_2, r_3 \) respectively, the turning points of the effective potential mentioned above. The roots \( e_k \) are ordered as follows:

\[ e_1 \leq e_2 \leq e_3. \] (2.30)

This function can be inverted

\[ r^2 = g(t), \] (2.31)

and finally the coordinate \( r \) is expressed as a function of time. The explicit expression for the coordinate \( x \) as a function of time is given in Appendix A.

Using this result in connection with the second equation (2.13), the equation for the coordinate \( \theta \) is solved

\[ \theta = \int_0^t \frac{A'L}{h^2g(t')} dt' \equiv h(t) \] (2.32)
and thereby the classical trajectory is fully determined.

An alternative way to derive the implicit equation of the classical trajectory is described in Appendix A.

For the variable $x$, the motion is periodical with the period:

$$T = \frac{\pi}{\sqrt{2|A'D|(e_3 - e_1)}} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

(2.33)

where $_2F_1$ denotes the confluent hyper-geometric function with the argument $k^2$, where $k$ is defined by Eq.(2.29).

Note that the period depends on the energy $E$, by means of the roots $e_1, e_2, e_3$. Energy is discretitized through the quantization equation of Bohr-Sommerfeld type:

$$I(E) \equiv \frac{1}{2\pi\hbar} \int_{V_{\text{min}}^\text{eff}}^{E} T(E')dE' = n.$$  

(2.34)

The function appearing in the left hand side of the above equations is plotted in Figs. 6, 7 as function of E for the two particular potentials considered here. One remarks the fact that this integral depends almost linearly on energy, irrespective of L. However the slope of these straight lines depends on the potential characterizing the motion of the system.

Quantized energies for $n=0,2$ and $L=\text{even}$ and those with $n=1$ and $L \geq 2$ obtained for the pocket like potential are plotted in Fig. 8. One should note that although these states describe the intrinsic degrees of freedom, they might be organized in bands as it happens with the states characterizing the whole set of degrees of freedom. In the case II the bands are finite since trajectories with energy larger than $V_{\text{max}}^\text{eff}$ cannot be quantized, describing unbound systems. Obviously the number of states which could appear inside the pocket depends on angular momentum. The larger angular momentum, the smaller the number of the bound states. We mention again that in a pure quantum mechanical treatment these states are quasi-bound. In contrast with this situation, for the case I, the bands are all infinite.
From Figs. 6-8, it results that for a given value of angular momentum the energy spacings of states with different values for the quantum number \( n \) is almost constant. This feature is determined by the relative values of the coefficients defining the model Hamiltonian. For this particular situation, the harmonic approximation seems to be quite reasonable. Indeed, expanding the effective potential around its minimum and ignoring the terms of order higher than two, one finds for the harmonic frequency the value:

\[
\omega_L = \left[ 3A' \left( \frac{A'L}{r_0^3} + \frac{1}{3}A + Dr_0^2 \right) \right]^{\frac{1}{2}}.
\]  

Thus, the lowest three bands can be defined in an approximative way by:

\[
E^{(\alpha)}_L = V_{\text{eff}}^{\min}(L, r_0), L = 0, 2, 4, 6, ...
\]

\[
E^{(\gamma)}_L = V_{\text{eff}}^{\min}(L, r_0) + \omega_L, L = 2, 3, 4, ...
\]

\[
E^{(\beta)}_L = V_{\text{eff}}^{\min}(L, r_0) + 2\omega_L, L = 0, 2, 4, 6, ...
\]  

(2.36)

Exact energies of the levels in the three bands are given in Fig. 8 as function of angular momentum \( L \) for the case II. From Fig. 7 one may see that the number of the bound states in the effective potential characterizing the case II depends on the value of the angular momentum. This dependence is seen more clearly in Fig. 9. From there one sees that the number of bound states varies from 18, for \( L = 0 \), to 0, for \( L \geq 46 \).

Concluding this section, although the starting Hamiltonian is anharmonic and has a complex structure we derived analytical solutions for the classical trajectories. The corresponding energies where quantized using a Bohr Sommerfeld type quantization condition. Energy levels are labeled by two indices, \( n, L \), one provided by the quantization procedure and the other one by a conservation law. We fixed, conventionally, the constant value of \( L_3 \) to be equal to \( L/2 \). However the classical trajectory is not invariant to the fictitious group \( SU_c(2) \). Therefore it is deformed being a mixture of components of different \( L \). This can be easily seen if we evaluate \( L^2 \) for a
trajectory of energy $E$ and projection of angular momentum on z-axis equal to $M$:

$$\mathcal{L}^2 \equiv \langle \Psi | \hat{L}^2 | \Psi \rangle = \frac{1}{4A'} \left[ E + (A' - A)r^2 - \frac{D}{4}r^4 \right]$$

$$+ \frac{1}{4A'^2} \left[ E + (A' - A)r^2 - \frac{D}{4}r^4 \right]^2 - \frac{1}{2} \frac{M^2}{\hbar^2}. \quad (2.37)$$

From this equation it is manifest that $\mathcal{L}^2$ is not a constant of motion since this quantity depends on $r$ and therefore changes its value when the system moves from one point to another.

The problem treated in this Section is a real challenge for looking for a quadrupole boson Hamiltonian which might yield in the intrinsic frame a motion which admits both $\mathcal{L}_z$ and $\mathcal{L}^2$ as constants of motion. This would allow us to classify the intrinsic motion with a $SU(2)$ symmetry.

**III. ANOTHER SOLVABLE BOSON HAMILTONIAN**

Here we study the fourth order boson Hamiltonian:

$$H_2 = \epsilon \sum_{\mu} b_{\mu}^\dagger b_{\mu} + \sum_{J=0,2,4} C_J \left[ (b^\dagger b_J^J) f_J(b_J) \right]_0. \quad (3.1)$$

Along the time, this Hamiltonian has been used by several authors to describe the rotational ground band. First authors were Das, Dreizler and A. Klein [17] who treated this Hamiltonian in a particular basis and obtained an analytical formula for the angular momentum dependence of the yrast states energies. Later on this boson number conserving Hamiltonian was used by Iachello Ref. [18,19] to formulate the first version of the interacting boson approximation. In [20], this Hamiltonian was averaged on a angular momentum projected state obtained from an axially deformed coherent boson state to approximate the energies from a rotational ground band. In the vibrational limit the empirical formula of Ejiri [25] was rigorously derived.
As shown in Appendix B, the Hamiltonian \( H_2 \) can be written in an equivalent otherwise more convenient form:

\[
H_2 = (\gamma + \epsilon)\hat{N}_2 + \beta\hat{N}_2^2 + \delta\hat{J}_2^2 + \alpha(b^\dagger b^\dagger)_0(bb)_0
\]

(3.2)

with the coefficients defined there. The notation \( \hat{N}_2 \) is used for the quadrupole boson number operator while \( \hat{J}_2^2 \) stands for the total angular momentum squared carried by the quadrupole bosons. The classical energy function associated to \( H_2 \) has the expression:

\[
H_2 = A(u_0^2 + v_0^2 + 2u_2^2 + v_2^2) + B(u_0^2 + v_0^2 + 2u_2^2 + v_2^2)^2 + C(u_0v_2 - u_2v_0)^2,
\]

(3.3)

with the coefficients \( A, B, C \) given in Appendix B.

The classical equations of motion are obtained from the time dependent variational equation (2.3) by replacing \( H_1 \) with \( H_2 \). They have a canonical form with respect to the coordinates \( \{q_k, p_k\}_{k=1,2} \) defined by Eq. (2.5). In terms of canonical coordinates the energy function has the expression:

\[
\mathcal{H}_2 = \frac{A}{2}(q_1^2 + q_2^2 + p_1^2 + p_2^2) + \frac{B}{4}(q_1^2 + q_2^2 + p_1^2 + p_2^2)^2 + \frac{C}{8}(q_1p_2 - q_2p_1)^2.
\]

(3.4)

Since the boson Hamiltonian commutes with \( \hat{L}_3 \) and \( \hat{N}_2 \) there are two independent constants of motion:

\[
\mathcal{N}_2 = \langle \Psi | \hat{N}_2 | \Psi \rangle, \quad \mathcal{L}_3 = \langle \Psi | \hat{L}_3 | \Psi \rangle
\]

(3.5)

This results from the equations:

\[
\frac{d}{dt}(q_1^2 + q_2^2 + p_1^2 + p_2^2) = 0,
\]

\[
\frac{d}{dt}(q_1p_2 - q_2p_1) = 0,
\]

(3.6)

implied by the equations of motion. Using these results one finds out that \( \mathcal{H}_2 \) is a constant of motion. On the other hand, considering the expressions of the \( SU_c(2) \) generators one finds that:
\[ \mathcal{L}^2 = \left[ \frac{\hbar}{4} (q_1^2 + q_2^2 + p_1^2 + p_2^2) \right]^2. \]  

(3.7)

and according to the previous equation this is a constant of motion. The classical energy can be quantized in two equivalent ways, namely either one quantizes the angular momentum by fixing the constants of motion such that:

\[ \mathcal{L}^2 = \hbar^2 L(L+1), \quad \mathcal{L}_3 = \hbar M, \quad \text{with} \quad -L \leq M \leq L, \quad L, M \quad \text{integers.} \]  

(3.8)

or by quantizing the classical action

\[ \int (q_1 p_1 + q_2 p_2) dq dp = 2\pi \hbar n, n - \text{positive integer} \]  

(3.9)

and the third component of angular momentum. For the first option the result for the quantized energy is:

\[ E_{LM} = 2A \sqrt{L(L+1)} + 4BL(L+1) + \frac{C}{2} M^2, \]  

(3.10)

while in the second case the result is:

\[ E_{n,M} = A(n+1) + B(n+1)^2 + \frac{C}{2} M^2 \]  

(3.11)

In this equation we introduced the zero point motion for the plane oscillator quanta although the semi-classical quantization is not able to account for it. To compare the two alternative expressions for the quantized energy it is convenient to make the following approximations, which work quite well for large quantum numbers:

\[ (n + 1)^2 \approx n(n + 2), \quad \sqrt{L(L+1)} \approx L + \frac{1}{2} \]  

(3.12)

In this way the two expressions for energy are identical provided \( L = \frac{n}{2} \). It is worth mentioning that the Hamiltonian considered in this section was used in Ref. 17 in the boson basis \( \{|N\nu\alphaJM\} \), where the quantum numbers involved are the number of bosons, seniority, missing quantum number, angular momentum and its projection on z-axis, with the restriction \( N = \)
\( v = \frac{J}{2} \) for the ground band states. By contrast, the connection between the new angular momentum quantum number \( L \) and the number of quanta in plane is \( L = \frac{n}{\sqrt{2}} \). This symmetry obeyed by the plane oscillator was exploited by Moszkowski in Ref. [22] where a schematic solvable many body Hamiltonian was found which describes fairly well the main features of the competition between individual and collective degrees of freedom.

The equation (3.10) shows that for the model Hamiltonian considered in this section the energy of the intrinsic degrees of freedom can be classified by the quantum numbers \((L,M)\) and correspondingly the quantized states by the irreducible representations of the fictitious \(SU_b(2)\) group. Since energies depend on \( M \), the states are deformed despite the fact \( L \) is a good quantum number. From the classical energy function it is clear that one deals with a plane oscillator. For even values of \( n \) one may define a one dimensional oscillator whose number of quanta is determined as \( 2\nu = n \).

Also we fix the projection of angular momentum on z-axis by the condition \( L_3 = \sqrt{L(L+1)} \). Under these circumstances the quantized energy gets the expression:

\[
E_{\nu,L} = 2A(\nu + \frac{1}{2}) + 4B(\nu + \frac{1}{2})^2 + \frac{C}{2}L(L + 1). \quad (3.13)
\]

Apart from an additive constant term this equation coincides with those used by Erb and Bromley to fit the spectrum of \(^{12}\mathrm{C} + ^{12}\mathrm{C}\) system [26,21].

Using a more complex structure for the starting model Hamiltonian, a coupling of rotational and vibrational degrees of freedom is possible. Such a coupling has been described by Iachello [27,29] in the framework of an algebraic model.

Equations derived in this Section for energy refer to the intrinsic degrees of freedom. Passing to the laboratory frame, the total energy accounts also for the rotational degrees of freedom. Assuming that the intrinsic degrees of freedom and Eulerian angles, defining the position of the intrinsic
frame, are only weakly coupled, the total energy can be approximated by:

\[ E_{JLM} = 2A \sqrt{L(L + 1)} + 4BL(L + 1) + \frac{C}{2}M^2 + \delta J(J + 1). \] (3.14)

Concluding this Section one may say that the energies of the exactly solvable Hamiltonian can be classified by a \( SU(2) \otimes SU(2) \) symmetry, one factor describing the motion of intrinsic degrees of freedom while the other one taking care for the motion of Eulerian angles.

In the next section we consider a boson Hamiltonian to which it corresponds a classical energy function which is similar to the mean field underlying the model proposed by Moszkowski [22].

IV. A QUADRUPOLE-QUADRUPOLE BOSON HAMILTONIAN.

In this section we study a quadrupole boson Hamiltonian which, in the intrinsic frame is closely related to the schematic Hamiltonian introduced by Moszkowski [22], long time ago. This Hamiltonian has been used by several authors to test various many-body approaches [23,24]. For the sake of completeness we present first the main ideas underlying the Moszkowski model (MM).

A. Brief review of Moszkowski model

The MM model considers a system of nucleons moving in a mean field, consisting in a two dimensional oscillator potential and a spin-orbit term, and interacting among themselves through a quadrupole-quadrupole interaction. Let us consider first the one body Hamiltonian:

\[ H_{sp} \equiv H_{ho} + H_{so} = \frac{1}{2M} (p_x^2 + p_y^2) + \frac{M \omega_0^2}{2} (x^2 + y^2) - C \vec{l} \cdot \vec{s}. \] (4.1)

It can be checked that \( H_{ho} \) commutes with the quasi-spin operators:
\[ t_x = \frac{1}{4} \left[ M\omega_0 (x^2 - y^2) + \frac{1}{M\omega_0} (p_x^2 - p_y^2) \right], \]
\[ t_y = \frac{1}{2} \left[ M\omega_0 xy + \frac{1}{M\omega_0} p_x p_y \right], \]
\[ t_z = \frac{1}{2} l_z = \frac{1}{2} (x p_y - y p_x). \tag{4.2} \]

and therefore its eigenstates can be classified by the irreducible representation of the \( SU(2) \) group generated by \( \{t_k\}_{k=x,y,z} \):

\[ \hat{t}^2 |Nm\sigma\rangle = \frac{N}{2} \left( \frac{N}{2} + 1 \right) |Nm\sigma\rangle \equiv t(t + 1) |Nm\sigma\rangle, \]
\[ t_z |Nm\sigma\rangle = \frac{1}{2} m |Nm\sigma\rangle. \tag{4.3} \]

Here \( N \) denotes the total number of quanta and \( m \) the azimuthal quantum number. The spin is perpendicular on the \((x,y)\) plane, the two possible orientations being specified by \( \sigma(= \pm) \). It is worthwhile to notice that in terms of stretched coordinates:

\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = \sqrt{\frac{M\omega_0}{\hbar}} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \tag{4.4}
\]

the components of quasi-spin operator are formally identical with those of angular momentum \( L_k \), defined by Eqs. (2.13), (2.14). However the two sets of operators act on different spaces.

Let us consider a many body system moving in the mean field described by \( H_{sp} \) and interacting by the QQ force, which in plane acquires a very simple form:

\[ H_{QQ} = -\frac{1}{4}(T_+ T_- + T_- T_+) = -\frac{X}{2}(T^2 - T_z^2). \tag{4.5} \]

where \( T \) denotes the operator acting on the many body states:

\[ T_\mu = T_\mu(+) + T_\mu(-), \]
\[ T_\mu(\sigma) = \sum_{m,m'} \langle Nm\sigma | t_\mu | Nm'\sigma \rangle c_{Nm\sigma}^\dagger c_{Nm'\sigma}. \tag{4.6} \]

One distinguishes two limiting cases. When \( C=0 \) one obtains the two dimensional version of the Elliott model [30] which is suitable for describing...
the collective rotations and quadrupole vibrations of a many body system.

The other regime, when the long-range interaction is missing (X=0), sim-
ulates the shell model description in realistic situations. The intermediate
situations can be covered by a smooth variation of the two strength param-
eters C and X. Renormalizing the mean field due to the two body interac-
tion one obtains a single particle Hamiltonian which in terms of stretched
coordinates looks as:

\[ H'_{sp} = \frac{\omega}{2} (x'^2 + y'^2 + p_x'^2 + p_y'^2) - \frac{X}{32} (x'^2 + y'^2 + p_x'^2 + p_y'^2)^2 + \frac{X}{8} (x'p_{y'} - y'p_{x'})^2. \]  (4.7)

As we shall see, this expression for the renormalized mean field is very
useful.

B. Two body interaction for quadrupole bosons.

In this subsection we shall treat semi-classically the following
quadrupole boson Hamiltonian:

\[ H_3 = \Omega \hat{N}_2 - \frac{F}{4} \sum_{\mu} Q_{2\mu} Q_{2-\mu}(-)^\mu, \]  (4.8)

where \( Q_{2\mu} \) denotes the quadrupole operators defined as:

\[ Q_{2\mu} = \sqrt{6} \left( b^\dagger b \right)_{2\mu}. \]  (4.9)

Averaging this boson Hamiltonian on the coherent state (2.4) and approx-
imating the average for the two body term by \( \sum_{\mu} \langle Q_{2\mu} \rangle \langle Q_{2-\mu}(-)^\mu \rangle \), one
obtains the following expression for the classical energy function:

\[ H_3 = \Omega(q_1^2 + q_2^2 + p_1^2 + p_2^2) - \frac{3}{7} F[(q_1^2 + q_2^2 + p_1^2 + p_2^2)^2 - 4(q_1 p_2 - q_2 p_1)^2]. \]  (4.10)

Comparing the energy function with the mean field corresponding to
Moszkowski model we realize that they are identical if we accept the follow-
ing relationships between the strengths involved in the two Hamiltonians:

\[ \Omega = \omega, \quad \frac{3}{7} F = \frac{1}{32} X. \]  (4.11)
Therefore the spectrum of the quantized intrinsic Hamiltonian is:

\[ E_{LM} = 2\Omega \sqrt{L(L + 1)} - \frac{48}{7} F[L(L + 1) - M^2]. \] (4.12)

The competition between the individual and collective feature in Moszkowski model corresponds to the interplay of harmonic and anharmonic terms in the boson interacting model.

Again we pointed out the possibility to classify the states describing the intrinsic degrees of freedom by the irreducible representation of an SU(2) group.

V. CONCLUSIONS.

Three boson Hamiltonians of complex structure otherwise exactly solvable, were semi-classically treated. The first is a boson number non-conserving Hamiltonian. Its classical trajectories are analytically expressed in terms of elliptic functions. Closed orbits are quantized and explicit expressions for discrete energies are obtained. They are organized in rotational bands following the traditional scheme of the liquid drop model. The difference is that here the states describe the motion of intrinsic degrees of freedom and moreover the angular momentum components generate a rotation group acting in a fictitious space.

The second Hamiltonian commutes with the quadrupole boson number operator and yields in a classical framework a spectrum which is classified by angular momentum and its projection on z-axes. In the laboratory frame the spectrum is classified by the symmetry SU(2)⊗SU(2). The similarity with the molecular spectrum used for studying the quasi-bound states in \(^{12}C + ^{12}C\) system was pointed out.

The third Hamiltonian includes, as a two-body boson interaction, the separable quadrupole-quadrupole interaction. The classical image of this Hamiltonian is similar to the schematic Hamiltonian introduced by
Moszkowski to study the interplay of individual and collective degrees of freedom. Its spectrum is also classified according to the symmetry SU(2)⊗SU(2).

Concluding, the present paper points out a certain SU(2) symmetry which allows for a complete description of the intrinsic dynamic variables, β and γ. In a subsequent publication we shall investigate the question whether this symmetry is observed experimentally or not. Our optimism is supported by the fact that the considered Hamiltonians were already used in connection with different approaches to describe some of available data.

VI. APPENDIX A

From Eq (2.29) one can obtain analytical expressions for the coordinate x as function of time.

When the effective potential has only one extremum, the case I, the result is:

\[
x(t) = e_1 + \frac{(e_3 - e_1)(e_2 - e_1)}{(e_3 - e_1) - (e_3 - e_2)sn^2 \left[ \frac{|A'D|}{2\hbar^2} (e_3 - e_1)t; \frac{e_1 - e_1}{e_3 - e_1} \right]},
\]

(A.1)

As for the potential exhibiting a pocket structure, one obtains:

\[
x(t) = e_1 + (e_2 - e_1)\frac{sn^2 \left[ \frac{|A'D|}{2\hbar^2} (e_3 - e_1)t; \frac{e_1 - e_1}{e_3 - e_1} \right]}{cn^2 \left[ \frac{|A'D|}{2\hbar^2} (e_3 - e_1)t; \frac{e_1 - e_1}{e_3 - e_1} \right]}, \quad e_1 \leq x \leq e_2, \\
x(t) = e_3 + (e_3 - e_2)\frac{sn^2 \left[ \frac{|A'D|}{2\hbar^2} (e_3 - e_1)t; \frac{e_1 - e_1}{e_3 - e_1} \right]}{cn^2 \left[ \frac{|A'D|}{2\hbar^2} (e_3 - e_1)t; \frac{e_1 - e_1}{e_3 - e_1} \right]}, \quad x \leq e_3.
\]

(A.2)

Now we describe briefly how to obtain the implicit equation for the classical trajectory. Inserting the expression of the time derivatives given by Eqs. (2.21) and (2.20) in the identity:

\[
\frac{d\theta}{dt} = \frac{d\theta}{dr} \frac{dr}{dt} \tag{A.3}
\]

one obtains:
\[ \theta = \sqrt{\frac{A'L^2}{2Dh^2}} \int_{x_0}^{x} dy \frac{dy}{y^3 + \frac{2A'}{2D}y^2 - 4EDy + \frac{2A'}{2D}L^2}, \quad x = r^2. \quad (A.4) \]

The integral involved in the above equation can be analytically performed and the final result is:

\[ \theta = L \frac{2|A'|}{\hbar (e_3 - e_1)|D|} \begin{cases} 
-\frac{1}{e_1} \Pi \left( \arcsin(u), k^2 \frac{e_1}{e_2}, k \right) + \frac{e_3}{e_2 - e_1} \text{sn}^{-1}(u, k), \text{ case I, } e_2 \leq x \leq e_3 \\
\Pi \left( \arcsin(u), \frac{e_1 - e_2}{e_2}, k \right), \text{ case II, } e_1 \leq x \leq e_2 \\
\Pi \left( \arcsin(u), \frac{e_2}{e_3 - e_2}, k \right) + \frac{e_3}{e_3 - e_1} \text{sn}^{-1}(u, k), \text{ case II, } x \geq e_3 
\end{cases} \]  

(A.5)

The arguments \( u \) and \( k \) depend on the interval and are those defined by Eq. (2.29). We denoted by \( \Pi \) the elliptic function of third rank defined as:

\[ \Pi(\phi, n, k) = \int_{\phi}^{0} \frac{d\phi}{(1 + n \sin^2 \alpha)\sqrt{1 - k^2 \sin^2 \alpha}}. \quad (A.6) \]

### VII. APPENDIX B

The anharmonic terms involved in Eq.(3.1) defining the Hamiltonian \( H_2 \) can be easily expressed in terms of boson number and total angular momentum operators:

\[ \left[ (b^\dagger b^\dagger)(bb) \right]_0 = -\frac{7}{5} \sqrt{5} b^\dagger b^\dagger)_0(bb)_0 + \frac{4}{35} \sqrt{5} \hat{N}_2(\hat{N}_2 - 1) - \frac{1}{35} \sqrt{5}(\hat{J}_2 - 6\hat{N}_2), \]

\[ \left[ (b^\dagger b^\dagger)_{4}(bb)_{4} \right]_0 = \frac{1}{7} (b^\dagger b^\dagger)_0(bb)_0 + \frac{1}{7} \hat{N}_2(\hat{N}_2 - 1) + \frac{1}{21}(\hat{J}_2 - 6\hat{N}_2). \quad (B.1) \]

Using these expressions in connection with Eq. (3.1) one obtains \( H_2 \) in the form given by (3.2) with:

\[ \alpha = C_0 - \frac{2}{7} \sqrt{5}C_2 + \frac{1}{7} C_4, \]

\[ \beta = \frac{4}{35} \sqrt{5}C_2 + \frac{1}{7} C_4, \]

\[ \gamma = \frac{2}{35} \sqrt{5}C_2 - \frac{3}{7} C_4, \]

\[ \delta = -\frac{1}{6}(\beta + \gamma). \quad (B.2) \]

Averaging \( H_2 \) on the coherent state (2.4) one obtains the classical energy from Eq. (3.4) where the following notations have been used:
\[ A = \gamma + \epsilon + \beta + 6\delta, \]
\[ B = \beta + \frac{\alpha}{5}, \]
\[ C = -\frac{8\alpha}{5}. \]  

(B.3)
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FIG. 1. The potential energy involved in Eq. (2.11) is plotted as function of $r$ for three sets of parameters $(A,D)$. 
FIG. 2. The effective potential $V_{\text{eff}}(L, r)$ is plotted as a function of $r$ for the parameters specified by Eq. (2.24) for the case I.
FIG. 3. The same as in Fig. 2, but for the case II
FIG. 4. The third order polynomial standing under the square root symbol in Eq. (2.23), multiplied with $\text{sign}(-A'D)$ is plotted as function of $x = r^2$, in the interval $x \geq e_2$ where $e_1 < e_2 < e_3$ denotes its roots. The parameters correspond to the case I, listed in Eq. (2.24). The plotted function is positive in the interval $e_2 \leq x \leq e_3$. 
FIG. 5. The same as in Fig. 4 but for the case II. The plotted function is positive in the interval $e_1 \leq x \leq e_1$. 
FIG. 6. The function $I(E)$, defined by Eq. (2.34) is plotted as function of $E$ for several angular momenta. The parameters involved are those specified in Eq. (2.24) by the label I.
FIG. 7. The same as in Fig. 6, but for the case II
FIG. 8. The energies associated to the motion of intrinsic degrees of freedom are classified in rotational bands in the following way. The lowest band is characterized by \( n=0 \) and even values for \( L \). The second band is similar to the gamma band and corresponds to \( n=1 \) and \( L \geq 2 \). The third band is similar to the band \( \beta \) and has \( n=2 \) and even angular momenta.
FIG. 9. The number of bound states in the pocket like potential is plotted as function of angular momentum.