Local supersymmetry without SUSY partners

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Abstract

A gauge theory for a superalgebra that could describe the low energy particle phenomenology is constructed. The system includes an internal gauge connection one-form $A$, a spin-1/2 Dirac fermion $\psi$ in the fundamental representation of the internal symmetry group, and a Lorentz connection $\omega^{ab}$. The important distinctive features between this theory and standard supersymmetries are: i) the number of fermionic and bosonic states are not necessarily equal; ii) no superpartners appear, “bosoninos” or s-leptons commonly found in standard supersymmetry theories are absent; iii) although the supersymmetry is local and gravity is included, there is no gravitino; iv) fermions get mass from their coupling to the background or from a higher order self-coupling, while bosons remain massless. In odd dimensions, the Chern-Simons action is fully invariant under the entire local superalgebra. In even dimensions, the Yang-Mills form $L = \kappa \langle F \otimes F \rangle$ is the only natural option and the symmetry breaks down to (Internal Gauge) $\otimes$ (Lorentz). In four dimensions, following the Townsend-MacDowell-Mansouri construction out of a $osp(4|2) \sim usp(2,2|1)$ connection produces a Lagrangian invariant under the subalgebra $u(1) \oplus so(3,1)$, where the only non-standard additional piece is the Nambu-Jona Lasinio term. In this case, the Lagrangian depends on a single dimensionful parameter that sets the values of Newton’s constant, the cosmological constant and the NJL coupling.

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1 Introduction

Supersymmetry (SUSY), a symmetry that unifies the spacetime invariances and internal gauge symmetries and combines bosons and fermions, presents a curious paradox. On the one hand, there is a large consensus among theoretical—and even experimental—physicists that this unification must exist and be reflected in the particle spectrum of the standard model \[1, 2\]. On the other, there seems to be no trace of SUSY in spite of four decades of extensive search at accelerators \[3\] and on blackboards, making even some original proponents recant their support of SUSY, at least in its simplest form \[4\].

A distinct signal of standard SUSY would be the existence of partners that duplicate the spectrum of observed particles \[5\]. In the minimal supersymmetric scenario \((\mathcal{N} = 1 \text{ SUSY})\), for every lepton, quark and gauge quanta a corresponding particle/field with identical quantum numbers but differing by \(\hbar/2\) in intrinsic angular momentum would exist. In more sophisticated SUSY models, more partners would accompany every observed state.

In an unbroken supersymmetric phase these SUSY partners would have degenerate masses and since no partners have even approximately been observed, it is believed that SUSY must be severely “spontaneously broken” at current experimental energies. This idea could be contrasted with Heisenberg’s proposal of isospin, based on the observed slight difference in mass between the proton and the neutron, \(\Delta m/m \sim 10^{-3}\). In SUSY there is no approximate degeneracy to be explained by the symmetry; instead, there seems to be a need to explain the complete absence of a symmetry for which there are compelling mathematical arguments. The idea of unification has proven a strong guide for progress in physics ever since Newton. Therefore the possibility of a nontrivial combination of Poincaré invariance and internal gauge symmetries cannot be underestimated. The fact that this is the only known mechanism for such combination would be sufficient to justify the interest in SUSY. But there is more.

On the practical side, it was soon observed that supersymmetric models exhibit improved renormalizable features, offering a mechanism to protect some parameters in the action from running under renormalization. This offers a natural possible solution for the so-called hierarchy problem \[1\], and therefore it is an interesting ingredient in all unified models of electroweak and strong interactions. Improved renormalizability has also allowed for attractive ways to unify the standard model with gravity in supergravity \[6\], or perhaps in the ultimate unification scheme like superstring theory \[7\].

In spite of SUSY’s undeniable appeal, the skeptic still has the right to question its logical necessity: does it solve a problem no other scheme can? If no trace of
SUSY is ever experimentally found it can always be argued that SUSY breaking takes place at an energy beyond reach. But, can this be a falsifiable statement [8]?

While its logical necessity may not proven, SUSY does constrain the field contents of a theory. It restricts the interaction patterns, and relates different coupling constants and particle masses, which are all valid falsifiable predictions. Unfortunately, the detailed form of those constraints and parameter relations depend on the particular form in which supersymmetry is presented. In fact, there is a plethora of supersymmetric models resulting from the freedom in the choice of internal gauge group, the number of SUSY generators ($N$), the number of spacetime dimensions and the dimensional reduction schemes invoked.

One critical assumption in standard (global) SUSY is that all fields are in a vector (fundamental) representation of the SUSY generator, and that the supercharge commutes with the Hamiltonian. This is the origin of the mass degeneracy, and although it may be a natural assumption in a globally flat Poincaré-invariant spacetime, it would be unrealistic to expect a symmetry based on the isometries of spacetime to play a fundamental role, since spacetime evolves and need not possess any given symmetry at all times.

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It seems more natural to have supersymmetry built as an extension of the symmetries of the tangent space. In fact, spinors like the supercharge are in a particular representation of the Lorentz group, which, according to the equivalence principle is a feature of any spacetime, regardless of how warped it may be. The idea of extending the tangent space symmetry into a supergroup can be successfully exploited to construct gauge theories with exact, off-shell, local supersymmetry [10, 11, 12].

In [13], a model based on a representation in which the fields are part of a connection for the supersymmetric extension of the tangent space symmetry is shown to give rise to a completely different scenario: no pairs of SUSY fields, different numbers of bosonic and fermionic degrees of freedom, and particles of different spins that need not have the same mass, canonically coupled. Just as in the standard model, fermions describe charged spin-1/2 particles in a vector representation (sections in the gauge bundle) producing matter currents interacting via bosonic gauge fields. The bosonic particles on the other hand, are interaction carriers described by connection fields in the adjoint representation of the gauge group. In other words, there can be SUSY without superpartners of degenerate mass.

1If the spacetime admitted a different set of global Killing vectors –like (anti-) de Sitter space–, the supercharges would not necessarily commute with the Hamiltonian and the mass degeneracy should be lifted, but for the best current estimate of the cosmological constant, the difference in mass would be extremely small, $\Delta m/m \sim G \sqrt{|\Lambda|} \sim M_{Pl}^{-2} \sqrt{10^{-120} M_{Pl}^4} \sim 10^{-60}$ [9].
2 Spin-1/2 fields as part of a connection

The main lesson from [13] is that the representation is crucial not only for the field content of the model, but for the dynamical relations among the different constituents. Although this is well known, it seems to have been systematically ignored by the model building industry, putting all fields by default on the vector (fundamental) representation under supersymmetric rotations. The only widespread exception is in supergravity, where the gravitino is expected to behave as a connection required by local SUSY transformations.

The alternative approach discussed in [13] puts instead all the fields (bosons and fermions) into a connection one-form,

\[ A \sim A^a B_a + \bar{Q} \psi, \]  

(1)

where \( A^a = A^a_\mu dx^\mu \) is a connection one-form and \( \psi \) is a Dirac or Majorana spinor. The bosonic generators \( B_a \) and and the fermionic ones \( Q \), span a superalgebra of the form

\[ \{ \bar{Q}, Q \} \sim B, \quad [B, Q] \sim Q, \quad [B, \bar{Q}] \sim -\bar{Q}, \quad [B, B] \sim B. \]  

(2)

The crucial ingredient necessary to make this scheme viable is the introduction of a “soldering” one-form in the right representation to turn the zero-form \( \bar{Q}_\alpha \psi^\alpha \) into a one form. This can be done if, instead of \( \bar{Q} \psi \), one writes

\[ dx^\mu (M_\mu)^\alpha_\beta \bar{Q}_\alpha \psi^\beta. \]  

(3)

The coefficients \( (M_\mu)^\alpha_\beta \) represent a mapping between the spinorial representation defined on the tangent space and the coordinate basis of the manifold where the fields are defined. A natural –and possibly the simplest– option is to take

\[ (M_\mu)^\alpha_\beta = e^a_\mu (\Gamma_a)^\alpha_\beta \equiv (\Gamma_\mu)^\alpha_\beta, \]  

(4)

where \( e^a_\mu \) are the components of the vielbein and \( \Gamma_a \) are the Dirac matrices [13]. This ansatz does the job of connecting the Lorentz symmetry belonging to the tangent, to the coordinates of the manifold. In this way, the Lorentz transformations are locally realized in a fibre bundle structure. The direct consequence of this construction is that the resulting theory will be a gauge theory of the Lorentz group by construction. In this framework the presence of spinors has two effects: the need to introduce a metric structure (the vielbein \( e^a_\mu \)), and the addition of the Lorentz symmetry to the SUSY algebra. These two ingredients make the incorporation of gravity practically unavoidable.
2.1 Supersymmetry acting on the connection

The idea is to define a connection for a superalgebra $G$ that extends the Lorentz algebra $so(2n,1)$ by certain number of supercharges and possibly some internal generators. As explained above, if the fermionic field has spin $1/2$ requires the introduction of a soldering form $\xi = e^a \Gamma_a = \Gamma_\mu dx^\mu$,

$$A \sim A^r B_r + \overline{Q} \xi \psi + \overline{\psi} \xi Q .$$

This construction was carried out in [13] as a demonstrative example for one of the simplest superalgebras in $2 + 1$ dimensions. The resulting Lagrangian turns out to be a good description of graphene [14] (see also [15, 16]).

Under gauge transformations, the connection transforms as

$$A_\mu \to A'_\mu = g^{-1} A g + g^{-1} d g ,$$

where $g(x) = \exp \Lambda(x)$ is an element of the gauge group, and $\Lambda$ is in the algebra $G$, and for an infinitesimal transformation, $\delta A = d \Lambda + [A, \Lambda]$. In particular, under an infinitesimal local supersymmetry transformation

$$\Lambda(x) = \overline{Q} \epsilon - \bar{\epsilon} Q ,$$

the connection changes by

$$\delta A = (\delta_e A^r) B_r + \overline{Q} \delta_s (\Gamma \psi) - \delta_s (\overline{\psi} \Gamma) Q ,$$

which translates to the component fields as

$$\delta_e A^r = -i \left[ \overline{\epsilon} \{ \Gamma \}^r \psi + \overline{\psi} \{ \Gamma \}^r \epsilon \right]$$

$$\delta_e (\Gamma_\mu \psi) = -\nabla_\mu \epsilon$$

$$\delta_e (\overline{\psi} \Gamma_\mu) = -\bar{\epsilon} \nabla_\mu ,$$

where $\{ \Gamma \}^r$ denotes a properly (anti-) symmetrized product of Dirac matrices, and $\nabla$ is the covariant derivative for the connection in the spin-1/2 representation of the bosonic subalgebra. Finally, it can also be checked that successive gauge transformations of $A$ form a closed off-shell algebra, $[\delta_\Lambda, \delta_\Delta] A = \delta_{[\Lambda,\Delta]} A$. We can see at this point that there is no need of extra fields in order to close the algebra. This is a general feature of supersymmetric theories based on a super-connection field (see, e.g., [12]), instead of a vector super multiplet in which bosons and fermions are rotated into each other in a vector (fundamental) representation of SUSY [17].
2.2 Absence of gravitini

A sufficient condition to solve Equation (8) is to assume $\delta \Gamma_\mu = 0$, and therefore

$$\delta \psi = \frac{1}{D} \nabla \epsilon \quad \delta \bar{\psi} = -\frac{1}{D} \bar{\epsilon} \bar{\nabla}, \quad (12)$$

where $D$ is the spacetime dimension. The condition $\delta \Gamma_\mu = 0$ is equivalent to requiring that the vielbein remain invariant under SUSY. This is because $\delta \Gamma_\mu = \delta (\Gamma_\alpha e^\alpha_\mu) = \Gamma_\alpha \delta e^\alpha_\mu = 0$, and therefore $\delta e^a_\mu = 0$. This in turn implies that the metric is invariant under supersymmetry,

$$\delta_{\text{susy}} (g_{\mu\nu}) = 0, \quad (13)$$

which explains why there is no need for gravitini in spite of it being a locally supersymmetric theory.

Plugging $\delta \psi$ from (12) back in (10) yields the constraint

$$\left( \delta_{\nu}^{\mu} - \frac{1}{D} \Gamma_\nu \Gamma^\mu \right) \nabla_\mu \epsilon = 0. \quad (14)$$

It is easy to see that $P_\nu^{\mu} \equiv \delta_{\nu}^{\mu} - \frac{1}{D} \Gamma_\nu \Gamma^\mu$ is a projector, $P_\nu^{\mu} P_\mu^{\lambda} = P_\nu^{\lambda}$ that acts on spinorial vectors $\chi^a_\mu \in 1/2 \otimes 1$, projecting onto the spin-3/2 subspace. Its orthogonal complement, $\delta_{\nu}^{\mu} - P_\nu^{\mu} = \frac{1}{D} \Gamma_\nu \Gamma^\mu$ projects on the spin 1/2 component. Consequently, the meaning of (14) is to eliminate from $\nabla_\mu \epsilon$ the spin-3/2 component. Hence, there are no spin 3/2 components generated on the right hand side of (10) — again, no gravitini.

This condition is consistent with the starting assumption that no spin-3/2 particles are included in the action. Moreover, the projection operator itself is invariant under this form of supersymmetry. Writing $P_\nu^{\mu}$ more explicitly,

$$P_\nu^{\mu} = e^a_\nu E_b^\mu \left( \delta_a^b - \frac{1}{D} \Gamma_a \Gamma^b \right). \quad (15)$$

Since $e^a_\mu$ is invariant under SUSY, so is its inverse $E_a^\mu$, and therefore the projector commutes with supersymmetry. Therefore, eq. (14) represents a consistent truncation of the representation.

The role of the soldering from $e^a_\mu$ is that of projecting the Clifford algebra from the tangent space down to the base manifold. At the same time, $\Gamma_a e^a_\mu$ turns a spinorial zero-form into a one-form. Therefore the vielbein does not play an active dynamical role, which is consistent with assuming that it does not transform under supersymmetry.
2.3 Historical Note

In their seminal 1974 paper [17], Wess and Zumino presented the blueprint of a local conformal supersymmetric system, where the spinorial parameter \( \alpha(x) \) that defines a local SUSY transformation is expected to obey the constraint

\[
\left( \frac{1}{4} \Gamma_\mu \Gamma^\mu - \delta^\mu_\nu \right) \partial_\mu \alpha = 0. \tag{16}
\]

After some manipulations, they conclude that the spinor \( \alpha \) should satisfy the condition \( \partial_\mu \partial_\nu \alpha = 0 \) which implies that \( \alpha(x) \) must be a linear function of the form

\[
\alpha(x) = \alpha^{(0)} + x^\mu \Gamma_\mu \alpha^{(1)}. \]

This has two important consequences: i) these SUSY transformations are parametrized by two constant spinors \( \alpha^{(0)} \) and \( \alpha^{(1)} \), it is not by a gauge symmetry described by arbitrary local functions at every point; and ii) The spacetime manifold must be flat (four-dimensional Minkowski space).

It is intriguing that an alternative branch of solutions of (16) was ignored in [17] without comments, \( \partial_\mu \alpha = \Gamma_\mu \beta(x) \), \( \tag{17} \) where \( \beta(x) \) is an arbitrary spinor field. This second solution is a genuine gauge transformation defined by an arbitrary local function, and requires no assumptions about the underlying spacetime geometry; in fact equation (17) in generalized coordinates on a curved background should be exactly (14), and (17) should be \( \nabla_\mu \alpha = \Gamma_\mu \beta(x) \). The question of how many independent solutions this latter equation might have is an interesting one as there might be obstructions to the global existence of \( \alpha \) in a manifold of nontrivial topology. This question, however, lies beyond the scope of this work.

3 Constructing Lagrangian \( D \)-forms

The dynamical features of a system described in terms of these fields should be obtained from a Lagrangian \( L(A) \) that is expected to be either invariant or quasi-invariant polynomials in \( A \) and \( dA \). The associated curvature \( F = dA + A \wedge A \) (field strength), is a tensor under gauge transformations in the adjoint representation, \( F \rightarrow F' = g^{-1}Fg \). The obvious invariant choice in even dimensions

\[
P_{2n} = \langle F \cdots F \rangle, \tag{18}
\]

Aquasi-invariant function \( f(A) \) is one that, under a gauge transformation changes by a locally exact form, \( \delta f = d\phi \).
where \( \langle \cdots \rangle \) is an invariant (super) trace in the Lie algebra, does not yield a suitable Lagrangian. In fact, the Chern-Weil theorem asserts that any invariant polynomial of this form is necessarily also locally exact: \( P_{2n} = dC_{2n-1} \), and therefore its variations (under appropriate boundary conditions) vanish identically. This means, in particular, that there are no Lagrangians \( L(F) \) constructed using only exterior products, invariant under the entire gauge group; the Euler-Lagrange equations for such “invariant Lagrangians” would have the trivial form \( 0 = 0 \). In other words, one must give up gauge invariance under the entire gauge group. This leaves essentially two interesting possibilities for the Lagrangian to be considered: i) instead of being plainly invariant, it can be quasi-invariant—that is, changing by a total derivative—under gauge transformations; or ii) it can be invariant under a subgroup of the entire gauge group.

The first case corresponds to Lagrangians defined by Chern-Simons forms\(^3\) that define viable dynamical theories in odd dimensions. The second case occurs when the form \( (18) \), is constructed with a symmetric trace \( \langle \cdots \rangle \) invariant only under a subgroup of the entire gauge symmetry group. The latter case is the only alternative in even dimensions and is the approach taken by MacDowell and Mansouri \[23\], and by Townsend \[24\] in the construction of four-dimensional (super)gravity out of a super algebra containing the (A)dS symmetry (MM-T approach).

For example, given this Lie algebra-valued connection in \( 2n + 1 \) dimensions, the CS form is naturally defined,

\[
L_{CS} \equiv C(A) = \langle AdA^n + ... \rangle ,
\]

where \( \langle \cdots \rangle \) stands for a supertrace, invariant under the entire gauge group. Under gauge transformations continuously connected to the identity, the CS form changes by a boundary term by construction, \( \delta C = d\Omega \). Thus, the problem in odd dimensions is to find an invariant bracket \( \langle \cdots \rangle \).

In even dimensions, Yang-Mills Lagrangians can be constructed in any dimension provided the spacetime is equipped with a metric structure with which the Hodge dual of \( F \) is defined. Thus, we tentatively define

\[
L_{YM} = -\frac{1}{4}Str [F \wedge \& F] ,
\]

where \( \& F \) is the dual of \( F \). This operation requires a metric structure, which in this scheme is provided by the soldering form \( e^a \).

\(^3\)Chern-Simons theories have been extensively discussed in the physics literature, starting with the pioneering works of Cremmer, Julia and Sherk \[19\], Schoenfeld \[20\], and Deser, Jackiw and Templeton \[21\]. For a recent review, see \[22\].
3.1 Three-dimensional example

In three dimensions, the construction outlined above leads to the model discussed in [13]. We summarize the results here to illustrate the idea, further details can be found in that reference. The connection (1) takes the form

$$A = AK + \overline{Q}_\beta \gamma^\beta \psi^\alpha + \overline{\psi}_\alpha \gamma^\alpha \psi^\beta + \omega^\alpha \mathbb{J}_a,$$  \hspace{1cm} (21)

where $K$, $Q$, $\overline{Q}$, and $\mathbb{J}$ are the generators of $U(1)$, supersymmetry and Lorentz transformations in 2+1 dimensions, respectively. Here $\omega^\alpha = \frac{1}{2} \epsilon^a_{bc} \omega_{\mu}^{bc}$ is the Lorentz connection. The Chern-Simons 3-form provides a Lagrangian for the connection $A$ without additional ingredients,

$$L = \langle A dA + \frac{2}{3} \Lambda^3 \rangle,$$

where $\langle \cdots \rangle$ stands for the supertrace. Using the standard conventions, the Lagrangian reads

$$L = 2 AdA + \frac{1}{4} [\omega^a_d \omega^b_a + \frac{2}{3} \omega^a_b \omega^b_c \omega^c_a] - 2 \overline{\psi} \psi e^a T_a$$
$$+ 2 \overline{\psi} (\partial - i/A + \frac{1}{2} \kappa - \frac{1}{4} \Gamma_a \gamma^a \gamma^b) \psi \overline{e} d^3x,$$

where $|e| = \text{det}[e^a_\mu] = \sqrt{-g}$, and $T^a = de^a + \omega^a_b e^b$ is the torsion 2-form. This is a standard Lagrangian for a Dirac field minimally coupled to CS electrodynamics in a gravitational background. This system is invariant under local $U(1)$ and $SO(2,1)$ transformations. It may be surprising that this rather ordinary-looking system is also invariant under local supersymmetry because it is a genuine gauge theory for the $osp(2|2)$ superalgebra. Although this supersymmetry is local and contains 2+1 gravity, there is no gauging of local translations and hence, no gravitino is required.

The field equations for this system are

$$F_{\mu \nu} = \epsilon_{\mu \nu \lambda} j^\lambda$$  \hspace{1cm} (22)
$$R^{ab} = 2 \overline{\psi} \psi e^a e^b$$  \hspace{1cm} (23)

$$[\partial - i A + \frac{1}{2} \kappa - \frac{1}{4} \Gamma_a \phi_{ab} \Gamma^b + \frac{1}{2} |e| \partial_\mu (|e| E^a_\mu \Gamma^a)] \psi = 0,$$  \hspace{1cm} (24)

where $j^\lambda = -i \overline{\psi} \Gamma^\lambda \psi |e|$, is the electric current density of a charged spin 1/2 field, and $|e|k d^3x = e^a T_a$. Equation (24) implies that the torsion is covariantly constant, $DT^a = 0$ and therefore, $\kappa$ is a constant that can be identified with the fermion mass.
As shown in [13], the matter-free configurations $\psi = 0 = F$, describe manifold whose local geometry has constant torsion and constant negative curvature. These anti-de Sitter spaces include magnetically charged BTZ black holes which for some values of mass, angular momentum and magnetic charge, are BPS states. Additionally, the dynamics of fermions around these vacua describes the propagation of carriers of electric charge in graphene [14, 15, 16].

The above construction can be naturally extended to other gauge groups. For example, instead of the $U(1)$ piece one could have included a $SU(2)$ group, replacing the connection (21) by

$$A = A^i K_i + \bar{\psi}\phi\psi + \bar{\psi}\phi Q + \omega^a J_a, \quad (25)$$

where $K_i$ are the $SU(2)$ generators. The resulting action is the natural extension of the three-dimensional one for the $osp(2|2)$ algebra [25].

### 3.2 Four-dimensional action

The fact that spinors form a representation of the Lorentz group implies that the superalgebra should contain at least the Lorentz generators along with other internal symmetry generators in the bosonic sector.

In Appendix B the simplest SUSY in four dimensional space containing one complex supercharge $Q$ in the Dirac representations is reviewed.

#### 3.2.1 SUSY algebra, connection and curvature

Starting with a $U(1)$-charged supercharge generator $Q$ in four dimensions, the closure of the superalgebra also requires, besides the $U(1)$ generator $K$ and the Lorentz generators $J_{ab}$ for the symmetry of the tangent space, four additional (A)dS generators $J_a$ (see Appendix B). The resulting superalgebras can be found in standard texts such as [6, 12],

$$\{Q^\alpha, \overline{Q}_\beta\} = -\frac{i}{s}(\Gamma^a)_{\alpha}^\beta J_a + \frac{i}{2}(\Gamma^{ab})_{\alpha}^\beta J_{ab} - \delta_\beta^\alpha K, \quad (26)$$

where $s = i$ corresponds to de Sitter, and $s = 1$ to anti-de Sitter. The connection can be written as

$$A = AK + \bar{Q}\phi\psi + \bar{\psi}\phi Q + f^a J_a + \frac{1}{2} \omega^{ab} J_{ab}, \quad (27)$$

where $A = A_\mu dx^\mu$, $e^a = e_\mu^a dx^\mu$, $f^a = f^a_\mu dx^\mu$ and $\omega^{ab} = \omega^{a\mu} dx^\mu$ are 1-form fields (spinorial indices omitted), and we define $\phi \equiv \Gamma_a e^a$. The curvature

F = dA + AA

is antisymmetric, i.e., $F_{\alpha\beta} = -F_{\beta\alpha}$, $F_{\alpha\beta\gamma} = F_{\beta\alpha\gamma}$. The field strength tensor $F = dA + AA$.

\footnote{Exterior (wedge) products of differential forms will be implicitly assumed throughout.}
takes the form
\[ F = F_0 K + \overline{Q}_a F^a + F^a Q^a + F^a J_a + \frac{1}{2} F^{ab} J_{ab}, \]
where
\[ F_0 = F - \bar{\psi} \psi \Rightarrow (28) \]
\[ F = \left[ d - iA + \frac{s}{2} f + \frac{1}{2} \phi \right] (\phi \psi) \Rightarrow (29) \]
\[ \overline{F} = (\bar{\psi} \phi) \left[ -\overline{d} - iA + \frac{s}{2} f + \frac{1}{2} \phi \right] \Rightarrow (30) \]
\[ F^a = D f^a - \frac{i}{s} \bar{\psi} \Gamma^a \psi \Rightarrow (31) \]
\[ F^{ab} = R^{ab} + s^2 f^a f^b + i \bar{\psi} \phi \Gamma^{ab} \phi \psi, \Rightarrow (32) \]
Here \( F = dA, D f^a = df^a + \omega^a_b f^b, \) and \( R^{a b} = d \omega^a_b + \omega^a_c \omega^c_b. \) We have also used the notation \( f = \Gamma_a f^a \) and \( \phi = \frac{1}{2} \Gamma_{ab} \omega_{ab} \) (see Appendix A).

### 3.2.2 Supersymmetry transformations

The variation of \( \mathcal{A} \) under supersymmetry generated by \( \Lambda = \overline{Q} \epsilon - \tau Q \) is
\[ \delta \mathcal{A} = d\Lambda + [\mathcal{A}, \Lambda]. \]
Using the (anti-) commuting relations of the superalgebra \([56, 61]\), one finds
\[ \delta A_\mu = - (\tau \Gamma_\mu \psi + \bar{\psi} \Gamma_\mu \epsilon) \] \( \Rightarrow (33) \)
\[ \delta f^a = - i \frac{s}{2} (s \Gamma^a \phi \psi + \bar{\psi} \Gamma^a \epsilon) \] \( \Rightarrow (34) \)
\[ \delta \omega^{ab} = i (s \Gamma^{ab} \phi \psi + \bar{\psi} \Gamma^{ab} \epsilon) \] \( \Rightarrow (35) \)
\[ \delta [\Gamma_c \epsilon^c \psi] = \left[ d - iA + \frac{1}{2} s f^a \Gamma_a + \frac{1}{4} \omega^{ab} \Gamma_{ab} \right] \epsilon. \] \( \Rightarrow (36) \)
As discussed in Sect. 2.2, we postulate \( \delta \Gamma_\mu = \delta (\Gamma_a e^a_\mu) = \Gamma_a \delta e^a_\mu = 0, \) and therefore
\[ \delta \psi = \frac{1}{4} \Gamma^\mu \nabla_\mu \epsilon, \] \( \Rightarrow (37) \)
and we impose the consistency condition
\[ \left( \delta^\mu_\nu - \frac{1}{4} \Gamma^\mu_\nu \Gamma^\nu \right) \nabla_\mu \epsilon = 0. \] \( \Rightarrow (38) \)
The combination \( \chi_\mu \equiv \nabla_\mu \epsilon \) is in the \( 1/2 \otimes 1 = 1/2 \oplus 3/2 \) representation of the Lorentz group, but the projection \([58]\) eliminates the spin-3/2 piece.
3.2.3 Invariant supertraces

Here we proceed along the lines of the MM-T constructions, except that for our choice of connection we follow closely the construction in \cite{13} in which the metric structure does not transform under the action of the supercharges. This has the added effect that there are no gravitini associated to the SUSY generators, and the residual symmetry is simply $SO(3,1) \times U(1)$, as we have seen in section 2.2.

Typically, $g_{r_1\ldots r_n}$ is a symmetrized (super-)trace of all possible products of $n$ generators in the Lie (super-)algebra, which we will denote as $\langle \cdots \rangle$. The Lagrangian is a quadratic invariant of the Lorentz and internal gauge groups of the form

$$L = \kappa \langle F \otimes F \rangle,$$

where $\otimes F$ stands for the dual of $F$. Here we take duality as the Hodge dual in the spacetime, the $\Gamma_5$-conjugate in spinor indices, and the dual in the AdS algebra, to wit,

$$\otimes F = *F_0 K + (\overline{Q})_a (\Gamma_5 F)^a + (\overline{F})_a (\Gamma_5 Q)^a + \hat{\Gamma} \left[ F^a J_a + \frac{1}{2} F^{ab} J_{ab} \right].$$

Here $*F_0$ is the Hodge-dual of $F_0$, and the matrix $\hat{\Gamma}$ is both Lorentz and $U(1)$-invariant. In the $6 \times 6$ representation of Appendix B,

$$\hat{\Gamma} = \begin{bmatrix}
\Gamma_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \tag{41}$$

or $(\hat{\Gamma})^A_B = (\Gamma_5)^\alpha_\beta \delta^A_B$. The three dualities square to minus the identity in their respective subspaces:

$$(*)^2 = (\Gamma_5)^2 = (\hat{\Gamma})^2 = -1,$$

and therefore, $(\otimes)^2 = -1$. Since $\hat{\Gamma}$ commutes with $K$ and $J_{ab}$, but not with $Q^a_i$ or $J_a$, the resulting quadratic form \eqref{39} will be invariant under $SO(3,1) \times U(1)$, the only remaining symmetry of the action out of the full AdS supersymmetry \eqref{26}.

3.2.4 Lagrangian

The nonvanishing supertraces, bilinear in the generators that appear in $L$, are

$$\langle KK \rangle = 2, \quad \langle Q^a Q_\beta \rangle = 2i \delta^\beta_\alpha, \quad \langle Q_\alpha Q^\beta \rangle = -\langle Q_\alpha Q^\beta \rangle, \quad \langle J_{ab} \Gamma J_{cd} \rangle = \epsilon_{abcd}.$$
Therefore,
\[
\langle F \otimes F \rangle = 2F_0 \ast F_0 + 4i\mathcal{F}_a (\Gamma_5)_{\beta}^{\alpha} \mathcal{F}^\beta + \frac{1}{4} \epsilon_{abcd} F^{ab} F^{cd} .
\] (43)

Using the conventions in Appendix A, this expression can be written as
\[
L = -\frac{1}{4} \langle F \wedge \bar{F} \rangle = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} |e| d^4 x - \frac{1}{16} \epsilon_{abcd} (R^{ab} + s^2 f^a f^b) (R^{cd} + s^2 f^c f^d)
\]
\[
+ \frac{i}{2} \bar{\psi} \left[ \bar{D} \Gamma_5 \slashed{f} \psi + \Gamma_5 \slashed{f} \psi \bar{D} \right] \psi + \frac{i}{2} \bar{\psi} \left[ \Gamma_5 (\slashed{T} f \psi - \phi \slashed{T} f) \right] \psi
\]
\[
- \frac{i}{2} \epsilon^{abc} \Gamma_5 \slashed{f} \psi \Gamma_5 \psi + 12 \left[ (\bar{\psi} \Gamma_5 \psi)^2 - (\bar{\psi} \psi)^2 \right] |e| d^4 x \, ,
\] (44)

where \( \bar{D} \psi \equiv (d - i A + \frac{1}{2} \phi) \psi \), and \( \psi \bar{D} \equiv \bar{\psi} (\slashed{d} + i A - \frac{1}{2} \phi) \). The quartic fermionic expression is the Nambu–Jona-Lasinio term,
\[ g[(\bar{\psi} \psi)^2 - (\bar{\psi} \psi)^2] . \]

The field \( f^a \) appears undifferentiated and can in principle be solved from its own field equation; it is an auxiliary, not dynamically independent field, its form being determined by the other dynamical fields in the theory. Since \( f \) is a connection component, this means that the invariance of the theory under local AdS boosts is not a dynamical gauge freedom, and setting \( f^a = f^a(A, e, \omega, \psi) \) corresponds to a gauge choice that fixes it.

Using the identities (I, II) in the Appendix, the terms involving \( f \) can be written as
\[
\Gamma_5 \slashed{f} \psi = \epsilon_{abcd} e^a e^b f^c \Gamma^d
\]
\[
\Gamma_5 \slashed{f} \slashed{f} \slashed{f} \psi = \epsilon_{abcd} e^a e^b e^c f^d - 2 \Gamma_5 (f \cdot e)(f \cdot e)
\]
\[
\Gamma_5 (\slashed{T} f \psi - \phi \slashed{T} f) = 2 \Gamma_5 \Gamma_a [(f \cdot e) T^a + (T \cdot f) e^a + (T \cdot e) f^a] ,
\] (45)

where \( v \cdot w \equiv v^a w_a \). The last two expressions contain parity-odd terms. Thus, a natural way to avoid parity violation is to impose \( f^a \wedge e_a = 0 \). This condition is automatically satisfied if \( f^a \) is identified with the vielbein \[24\] or, more generally if one assumes \( f^a = \mu e^a \), as we do here. Then, the last identity in (45) gives the coupling between the fermion and torsion, \( t^\mu j_5^\mu \), where \( t^\mu \sim \epsilon^{\mu\nu\lambda\rho} T_{\tau\lambda\rho} \), and \( j_5^\mu \equiv \bar{\psi} \Gamma_5 \Gamma_\mu \psi |e| \frac{1}{2} \).

\[5\] This is the standard coupling between the chiral current and the axial vector part of torsion \[26, 27\]. Note that both \( j_5^\mu \) and \( t^\mu \) are parity-odd and their product is even, so this coupling does not violate parity. This coupling between a spinor field and torsion has been shown to lead to a chiral anomaly given by the Nieh-Yan topological invariant of torsional origin \[28\] that can have observational effects \[29\].
Since \( e^a \) and \( \psi \) are defined modulo local scale transformations (the connection is invariant under \( e^a \to \lambda(x)e^a, \psi \to [\lambda(x)]^{-1}\psi \)), it is sufficient to take \( \mu \) constant. Hence, the second term in the first line of (44) is a linear combination of the Euler-Gauss-Bonnet invariant, and the Einstein-Hilbert plus cosmological terms, so the Lagrangian becomes

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} |e| d^4x - \frac{1}{16} \epsilon_{abcd} (R^{ab} + s^2 \mu^2 e^a e^b)(R^{cd} + s^2 \mu^2 e^c e^d)
\]

\[
-\frac{i}{2} s \mu \left[ (\nabla^\mu D^a \psi - \bar{\psi} \Gamma^a (D \psi) \right] \epsilon_{abcd} e^b e^c e^d + 2i s \mu \bar{\psi} \Gamma_5 \epsilon_a (T \cdot e) e^a
\]

(46)

In standard units, \( \hbar = c = 1 \), \( \mu \) should have units of (length\(^{-1}\) or mass. The spin-1/2 field with the right “physical” dimensions is obtained if \( \psi_{\text{physical}} = \sqrt{\frac{\pi}{6}} \mu \psi \), where we have included a factor \( \sqrt{6} \) for later convenience. Rewriting the Lagrangian in this convention, one obtains

\[
L = \left[ L_F - \frac{1}{4} F_{\mu\nu} G^{\mu\nu} \right] |e| d^4x - \frac{1}{16} \epsilon_{abcd} [R^{ab} + s^2 \mu^2 e^a e^b] [R^{cd} + s^2 \mu^2 e^c e^d],
\]

where the fermionic Lagrangian is

\[
L_F = -\frac{i}{2} \left[ \bar{\psi} (\nabla - D) \psi + 4 \mu s \bar{\psi} \psi \right] - i s t^\mu \bar{\psi} \Gamma_5 \epsilon_a (T \cdot e) e^a - \frac{1}{3 \mu^2} [\bar{\psi} \psi]^2 - \frac{1}{\mu^2} [\bar{\psi} \Gamma_5 \psi]^2.
\]

(47)

Here, following [27], we used \( t^\mu \equiv -\frac{1}{3!} \epsilon^{\mu\rho\sigma} T_{\nu\rho\sigma} |e| \), and

\[
\overrightarrow{\nabla} \psi = \Gamma^\alpha E^\nu_a (\partial_\nu - i A_\nu + \frac{1}{4} \omega^b_{\nu c} \Gamma_{bc}) \psi = \Gamma^\nu \nabla_\nu \psi
\]

\[
= (\bar{\phi} - i A + \frac{1}{2} \psi) \psi
\]

\[
\overleftarrow{\nabla} \psi \equiv \bar{\psi} (\overleftarrow{\partial}_\nu + i A_\nu - \frac{1}{4} \omega^b_{\nu c} \Gamma_{bc}) E^\nu_a \Gamma^a = \bar{\psi} \nabla_\nu \Gamma^\nu
\]

\[
= \psi (\bar{\phi} + i A - \frac{1}{2} \psi)
\]

(49)

are the covariant derivatives for the connection of the [(anti-)de Sitter] \( \times U(1) \) gauge group in the spinorial representation, and \( E^\nu_a \) is the inverse vielbein. The correct sign of Newton’s constant in (47) is obtained for \( s^2 = -1 \), that is, for the de Sitter group only.

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3.2.5 Field equations

Varying the action (46) with respect to the dynamical fields yields the following:

\[ \delta A_\nu : \quad \partial_\mu F_{\mu \nu} + s \bar{\psi} \Gamma_{\nu} \psi = 0 \]

\[ \delta \omega^{ab}_\mu : \quad i \bar{\psi} \Gamma_{ab} \psi E^\mu_c + 3s \mu^2 [E^\nu_a E^\lambda_b E^\mu_c + 2E^\mu_a E^\nu_b E^\lambda_c] T^c_{\nu \lambda} = 0 . \] (50)

From the last equation it follows that \( T^c_{\nu \mu} E^\nu_c = 0 \), which means that torsion is determined by the local presence of fermions.

\[ T^a_{\mu \nu} = -\frac{i}{3 \mu^2} \bar{\psi} \Gamma^a_{bc} \psi e^b_\mu e^c_\nu . \] (51)

The remaining field equations are

\[ \delta \bar{\psi}_\alpha : \quad +is \bar{\psi} \psi - 2is^2 \bar{\psi} \psi - is \Gamma_5 \Gamma_\mu \psi t^\mu + \frac{2}{3 \mu^2} [(\bar{\psi} \Gamma_5 \psi) \Gamma_5 - (\bar{\psi} \psi)] \psi = 0 \]

\[ \delta e^a : \quad \epsilon_{abcd} (R^{bc} + s^2 \mu^2 e^b e^c) e^d = \tau_a . \] (52)

where \( \tau_a \) is the stress-energy 3-form defined by \( \delta (|e|[L_F + L_{EM}]) = \delta e^a \wedge \tau_a \). Contracting these equations with \( \bar{\psi}_\alpha \), and the conjugate with \( \psi^\alpha \), gives

\[ \bar{\psi}_\alpha \frac{\delta L}{\delta \bar{\psi}_\alpha} - \frac{\delta L}{\delta \psi^\alpha} \psi^\alpha = \partial_\mu (is |e| \bar{\psi} \Gamma^\mu \psi) d^4 x = 0 , \] (53)

which expresses the conservation of electric charge.

4 Discussion

The three- and four-dimensional models outlined here describe the interactions we experience in our daily life: QED and gravitation, the low energy/classical limit of the standard model. The relation between these systems and supersymmetry is indirect and can be traced to the presence of the fields (\( \psi, A_\mu, \omega^{ab}_\mu, e^a_\mu \)) required by the symmetry and the particular couplings among those fields.

Another vestige of supersymmetry in the model is the fact that the actions have no dimensionful constants. However, if one wants to fit the vielbein \( e^a_\mu \) with dimensions of length into a dimensionless connection, it is necessary to bring in an arbitrary dimensionful constant (\( \mu \)). In three dimensions, this appears in the integration constant for \( DT^a = 0 \) (\( T^a = \mu e^{abc} e^b_\mu e^c_\rho \)); in for dimensions appears in the identification between the vielbein and the the part of the connection related to
AdS boosts \( f^a = \mu e^a \). It is the dimensionful constant \( \mu \) which fixes all remaining parameters of the theory.

In three dimensions,

| Electric charge | \( e = 1 \) |
|-----------------|---------------|
| Newton’s constant | \( G = 1 \) |
| Cosmological constant | \( \Lambda = -\mu^2 \) |
| Fermion mass | \( m = \mu \) |

And in four dimensions,

| Electric charge | \( e = 1 \) |
|-----------------|---------------|
| Newton’s constant | \( G = -s^2(4\pi\mu^2)^{-1} \) |
| Cosmological constant | \( \Lambda = -s^2\mu^2 \) |
| Nambu–Jona-Lasinio coupling | \( g = (3\mu)^{-2} \) |

The fermionic Lagragians \( L_F \) describe an electrically charged spin-1/2 field minimally coupled to the spacetime background, plus non-minimal couplings that depend on the spacetime dimension. The coupling to torsion is not a new feature of this model but, as noted long ago by H. Weyl \([26]\), it is present whenever the Dirac equation is written in a curved spacetime with torsion. The only new fermionic piece in the Lagrangian is the Nambu–Jona-Lasinio (NJL) term in the four dimensional theory, which can be viewed as the main modification predicted by this model.\(^6\)

Both the four-fermion NJL coupling and the Einstein-Hilbert gravitational action are perturbatively non-renormalizable. This strongly suggests that the whole system should be considered as a low energy, effective model and not a fully consistent quantum theory. However, the fact that the parameters of the theory are so tightly constrained that it is conceivable that the two evils may cancel each other. The exploration of this problem, however, lies well beyond the scope of this work.

The NJL term provides a mechanism for spontaneous symmetry breaking that gives mass to the fermionic excitations in superconductivity and was originally proposed as a way to describe massive excitations in strong interactions \([30]\). The value of the fermion mass \( m \) is produced through the gap equation for a cut-off \( \mathcal{M} \),

\[
\frac{m^2}{\mathcal{M}^2} \log \left[ 1 + \frac{\mathcal{M}^2}{m^2} \right] = 1 - \frac{2\pi^2}{g\mathcal{M}^2}.
\]

For \( m = m_e \approx 0.5\,\text{MeV} \) and \( \mathcal{M} = M_{\text{Planck}} = G^{-1/2} \approx 2.5 \times 10^{22}m_e \), so that \( m^2/\mathcal{M}^2 \approx 10^{-45} \), the relation between the NJL coupling \( g \) and the UV cut-off \( \mathcal{M} \)

\(^6\)If instead of the \( U(1) \) gauge group, one had considered \( SU(2) \) or \( SU(3) \), NJL term would have been of the form \( C_{abcd}[(\psi^a \psi^b)(\psi^c \psi^d) - (\psi^a \Gamma_5 \psi^b)(\psi^c \Gamma_5 \psi^d)] \), where \( C_{abcd} \) is an invariant tensor in the algebra.
must be extremely fine-tuned in the range $1 < gM^2/2\pi^2 < 1 + 10^{-43}$.

In four dimensions, the spacetime geometry is described by the Einstein-Hilbert action with cosmological constant $\Lambda = -3s^2\mu^2$, where $s^2 = -1$(resp. + 1) for dS (resp. AdS) algebra [see, e.g., [56]]. The sign of the kinetic term in the gravitational action (46) is $-s^2$, which in the standard convention should be positive. Hence, both $\Lambda$ and $G$ should be positive. However, depending on the vacuum structure of the theory it might be worth considering the alternative where both $G$ and $\Lambda$ are negative, as in the case of the so-called topologically massive gravity in three dimensions [31, 32]. At any rate, the effective cosmological constant in the nontrivial vacuum should be given by $\Lambda_{\text{Eff}} = \Lambda - 2i\mu s^2\langle \bar{\psi}\psi \rangle + g^2[\langle (\bar{\psi}\Gamma_5\psi)^2 \rangle - \langle (\bar{\psi}\psi)^2 \rangle]$. It would be premature to claim something about the sign of $\Lambda_{\text{Eff}}$, specially in view of the fine tuning between $g$, $G$, $\Lambda$ and the cut-off $M$.

The gravitational Lagrangian is a particular combination of the three Lovelock terms that occur in 4D that has the form of the Pfaffian of the (A)dS curvature. This combination can also be viewed as the gravitational analogue of Born-Infeld electrodynamics [33], and although the Gauss-Bonnet term has no affect on the field equations and hence is usually ignored, it can give a significant contribution to the global charges of the theory, and acts as a regulator that renders the charges well defined and finite in the case of nontrivial asymptotics [34, 35]. It is therefore an interesting bonus of the model that the gravitational action is regularized by construction and no ad hoc counterterms are necessary to correctly define its thermodynamics.

Even as an effective low energy model, a healthy theory should have a well defined (stable) ground state, a vacuum around which it would make sense to expand perturbatively to study the quantum features of the theory (Killing spinors, BPS vacua). A vacuum without fermions (trivial vacuum, $\psi = 0$) would be invariant under supersymmetry provided $\delta\bar{\psi} = \nabla\lambda = 0$, which means that $\lambda$ must be a covariantly constant (Killing) spinor. The number of linearly independent, globally defined solutions of this equation characterizes the residual supersymmetries of a particular background configuration. Such background have been studied and a number of nontrivial BPS backgrounds are known [36, 37].

5 Summary

• Supersymmetry as an extension of the Lorentz group that mixes bosons and fermions is captured by the three-dimensional model discussed in [13] and extended here to four dimensions. This supersymmetry is moreover local and contains the
Lorentz group. Consequently, the theory includes gravity in a natural manner.

- The representation for this symmetry is chosen so that the fields are packaged into a connection one-form. Some features of standard supersymmetry are recovered – reduced number of free parameter in the action; the need to include gravity in order to close the algebra of local supersymmetry –, while others are not – equal number of bosonic and fermionic degrees of freedom, a susy partner for each particle with equal quantum number (except for the spin). Unlike standard local susy (supergravity) this theory requires no spin-3/2 fields; in fact, no new fields are needed at all. SUSY partners are a consequence of the representation, not a necessary feature of supersymmetry.

- The introduction of gravity is ultimately forced by supersymmetry because fermions are spinors, which form an irreducible representation of the Lorentz group and this brings in the spin connection $\omega^{ab}$. Additionally, the fact that the spinors are zero-forms (scalars under general coordinate transformations) makes it necessary to introduce the soldering form $e^{\mu}_a$, and consequently, the metric structure. Thus, both the affine and the metric structures are necessary to accommodate fermions in the scheme: gravitation can be viewed a necessary consequence of having fermionic matter in nature.

- In four dimensions (A)dS symmetry is broken by the fact that there is no $SO(4, 1)$- or $SO(3, 2)$-invariant tensor in four dimensions. Since the (A-)dS symmetry is broken, supersymmetry is also necessarily broken. This is not a spontaneous breaking in which the Hamiltonian selects the vacuum from an equivalence class of degenerate minima. This breaking comes from some sort of frustration: there are no invariant tensors of the $OSp(4|2)$ in the right representation in four dimensions. It would be very interesting to see this symmetry breaking emerging from the dimensional reduction to four dimensions from a fully gauge invariant CS theory, based on transgression forms \[38] [39].

## Appendix A. Conventions

### Lorentz Group invariant tensors

The signature we choose is such that the tangent space metric is $\eta_{ab} = diag(-1, 1, 1, 1)$;
the tangent space Levi-Civita invariant tensor of the Lorentz group is defined as

\[ \epsilon_{abcd} = \begin{cases} 
0 & \text{if any two indices repeat} \\
+1 & \text{even permutation of 0123} \\
-1 & \text{odd permutation of 0123} 
\end{cases} \]

so that, in particular \( \epsilon_{0123} = +1 = -\epsilon_{0123} \).

**Levi-Civita tensor on a coordinate basis of the base manifold**

The alternating symbols in the base manifold \( (\varepsilon) \) are related to those on the tangent space \( (\epsilon) \) through

\[ \varepsilon_{0123} = \epsilon_{0123} = +1 \]

\[ \varepsilon_{0123} = |e|^{-2} \epsilon_{0123} = -|e|^{-2} = |g|^{-1} , \]

\[ c^a \epsilon^b \epsilon^c \epsilon^d \epsilon_{abcd} = |e| |\epsilon_{\mu\nu\lambda\rho}| , \quad E^\mu_a E^\nu_b E^\lambda_c E^\rho_d \epsilon_{abcd} = |e| |\epsilon_{\mu\nu\lambda\rho}| = |E|^{-1} |\epsilon_{\mu\nu\lambda\rho}| , \]

With these definitions, the volume form is

\[ dx^\mu dx^\nu dx^\lambda dx^\rho = -|e|^2 \epsilon^{\mu\nu\lambda\rho} d^4 x = -|e| E^\mu_a E^\nu_b E^\lambda_c E^\rho_d \epsilon_{abcd} d^4 x , \]

and hence the oriented volume form is \( e^0 e^1 e^2 e^3 = |e| d^4 x \). Also, if \( \sigma = \frac{1}{2} \sigma_{\mu\nu} dx^\mu dx^\nu \) is a two-form, its Hodge-dual is

\[ *\sigma = \frac{1}{4} |e| |\epsilon_{\mu\nu\alpha\beta}| \sigma^{\alpha\beta} dx^\mu dx^\nu . \]

**Dirac matrices**

The \( \Gamma \)-matrices are in a 4 \( \times \) 4 spinorial-representation of the Clifford algebra \( \{ \Gamma^a, \Gamma^b \} = 2\eta^{ab} \), and \( \Gamma^{ab} = \frac{1}{2} [\Gamma^a, \Gamma^b] \). The indices of the tangent space \( a, b \) take the values 0,1,2 and 3. Consistently with the choice of signature, we take

\[ \Gamma^0 = -\Gamma_0, \quad (\Gamma^0)^2 = (\Gamma_5)^2 = -1 , \quad \Gamma_5 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 . \tag{55} \]

From this, a number of useful identities follow:

| I | \[ \Gamma_5 \Gamma_a \Gamma_b \Gamma_c = \Gamma_5 [\eta_{ab} \Gamma_c - \eta_{ac} \Gamma_b + \eta_{bc} \Gamma_a] + \epsilon_{abcd} \Gamma^d \] |
| II | \[ \Gamma_5 \Gamma_a \Gamma_b \Gamma_c \Gamma_d = \Gamma_5 \eta_{abcd} + \Gamma_5 [\eta_{ab} \eta_{cd} - \eta_{ac} \eta_{bd} + \eta_{ad} \eta_{bc}] + \epsilon_{abcd} \Gamma^d \] |
| III | \[ \Gamma_a \Gamma_b \Gamma_c = \eta_{ab} \Gamma_c - \eta_{ac} \Gamma_b + \eta_{bc} \Gamma_a - \epsilon_{abcd} \Gamma_5 \Gamma^d \] |

**Slashed notation**

Gamma matrices can be used to write Lorentz tensors in a spinorial basis, which
is convenient sometimes when working with spin-1/2 fields. In our case, we have defined \( \varphi \equiv e^a \Gamma_a = \Gamma_\mu dx^\mu \) and \( \varphi \equiv \frac{1}{2} \omega^{ab} \Gamma_{ab} \). The covariant derivative of a spinor \( \xi \) in the Lorentz connection becomes

\[
D\xi = d\xi + \frac{1}{2} \omega \xi, \quad DD\xi = \hat{R}\xi,
\]

where

\[
\hat{R} = \frac{1}{2} R_{ab} \Gamma_{ab} = \left[ d\varphi + \frac{1}{2} \varphi^2 \right].
\]

If \( M = \frac{1}{p!} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} M_{\mu_1 \ldots \mu_p}^{a_1 \ldots a_r} \Gamma_{a_1 \ldots a_r} \) is a p-form spinorial tensor, then its covariant derivative reads

\[
D M = d M + \frac{1}{2} (\varphi M - (-1)^p M \varphi),
\]

which verifies Leibnitz’s rule, \( D(M\xi) = (DM)\xi + (-1)^p M D\xi \).

Appendix B
Supersymmetric extension of \( SO(3,1) \times U(1) \)

The generators \( J_a \) and \( J_{ab} \) form the 4D algebra

\[
[J_a, J_b] = s^2 J_{ab}, \quad [J_a, J_{bc}] = \eta_{ab} J_c - \eta_{ac} J_b, \quad (56)
\]

\[
[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} - \eta_{ac} J_{bd} + \eta_{bc} J_{ad} - \eta_{bd} J_{ac}, \quad (57)
\]

which corresponds to anti-de Sitter \( (so(3,2)) \) for \( s = 1 \) and to de Sitter \( (so(4,1)) \) for \( s = i \). The supercharge \( Q \) belongs to a spin 1/2 representation, that is

\[
[J_a, Q^\alpha] = -\frac{s}{2} (\Gamma_a)^\alpha_\beta Q^\beta, \quad [J_a, \bar{Q}_\alpha] = \frac{s}{2} \bar{Q}_\beta (\Gamma_a)^\beta_\alpha, \quad (58)
\]

\[
[J_{ab}, Q^\alpha] = -\frac{1}{2} (\Gamma_{ab})^\alpha_\beta Q^\beta, \quad [J_{ab}, \bar{Q}_\alpha] = \frac{1}{2} \bar{Q}_\beta (\Gamma_{ab})^\beta_\alpha. \quad (59)
\]

Since \( Q \) is complex, it has the following commutators with the \( U(1) \) generator

\[
[K, Q^\alpha] = i Q^\alpha, \quad [K, \bar{Q}_\alpha] = -i \bar{Q}_\alpha. \quad (60)
\]

The algebra is completed by the anticommutators of supercharges,

\[
\{Q^\alpha, \bar{Q}_\beta\} = -\frac{i}{s} (\Gamma^\alpha)^\alpha_\beta J_a + \frac{i}{2} (\Gamma_{ab})^\alpha_\beta J_{ab} - \delta^\alpha_\beta K, \quad (61)
\]

\[20\]
together with the trivial anticommutators \( \{\overline{Q}^\alpha, \overline{Q}^\beta\} = 0 = \{Q^\alpha, Q^\beta\} \). An explicit 6 × 6 representation for the supercharges is the following

\[
(Q^\alpha)_{AB} = -i \begin{bmatrix}
0_{4 \times 4} & C^\alpha A \\
\delta^\alpha_5 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} = -i(\delta^A_5 \delta^\alpha_5 + C^\alpha A \delta^\alpha_5) \tag{62}
\]

\[
(\overline{Q}_\alpha)_{AB} = \begin{bmatrix}
0_{4 \times 4} & \delta^A_\alpha \\
\delta^A_\alpha & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
C_\alpha B & 0
\end{bmatrix} = \delta^A_\alpha \delta^5_5 + \delta^A_6 C_{\alpha B}, \tag{63}
\]

where \( C_{\alpha\beta} = -C_{\beta\alpha} \) is the conjugation matrix, and \( C^{\alpha\beta} \) is its inverse. In this representation, the \( U(1) \) and \( (A)dS \) generators are

\[
(K)^A_B = \begin{bmatrix}
0_{4 \times 4} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
i & 0 \\
0 & 0
\end{bmatrix} = i(\delta^A_5 \delta^\alpha_5 - \delta^A_6 \delta^\alpha_6), \tag{64}
\]

\[
(J_a)^A_B = \begin{bmatrix}
\frac{s}{2} \Gamma_a & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} = \frac{s}{2} (\Gamma_a)^\alpha_\beta \delta^A_5 \delta^\beta_5 \tag{65}
\]

\[
(J_{ab})^A_B = \begin{bmatrix}
\frac{1}{2} \Gamma_{ab} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} = \frac{1}{2} (\Gamma_{ab})^\alpha_\beta \delta^A_5 \delta^\beta_5 \tag{66}
\]
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