Existence of solutions in the $U(1) \times U(1)$ Abelian Chern-Simons model on finite graphs

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Abstract

In this paper, we consider a system of equations arising from the $U(1) \times U(1)$ Abelian Chern-Simons model

$$
\begin{align*}
\Delta u &= \lambda \left(a(b-a)e^u - b(b-a)e^v + a^2e^{2u} - abe^{2v} + b(b-a)e^{w+v}\right) + 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\Delta v &= \lambda \left(-b(b-a)e^u + a(b-a)e^v - abe^{2u} + a^2e^{2v} + b(b-a)e^{w+v}\right) + 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j},
\end{align*}
$$

(1.1)
on finite graphs. Here $\lambda > 0$, $b > a > 0$, $m_j > 0$ ($j = 1, 2, \cdots, k_1$), $n_j > 0$ ($j = 1, 2, \cdots, k_2$), $\delta_p$ is the Dirac delta mass at vertex $p$. We establish the iteration scheme and prove existence of solutions. The asymptotic behaviors of solutions as $\lambda$ goes to infinity are studied by the new method, which is also applicable to the classical Chern-Simons equation

$$
\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}.
$$

Keywords: finite graph, Chern-Simons model, sub-solution, asymptotic behavior

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1. Introduction

In this paper, we consider an elliptic system arising from the $U(1) \times U(1)$ Abelian Chern-Simons model [17, 23]

$$
\begin{align*}
\Delta u &= \lambda \left(a(b-a)e^u - b(b-a)e^v + a^2e^{2u} - abe^{2v} + b(b-a)e^{w+v}\right) + 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\Delta v &= \lambda \left(-b(b-a)e^u + a(b-a)e^v - abe^{2u} + a^2e^{2v} + b(b-a)e^{w+v}\right) + 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j},
\end{align*}
$$

(1.1)
where $\lambda > 0$, $b > a > 0$, $m_j > 0$ ($j = 1, 2, \ldots, k_1$), $n_j > 0$ ($j = 1, 2, \ldots, k_2$), $\delta_p$ is the Dirac delta mass at vertex $p$.

A lot of physical phenomena can be explained by Chern-Simons models, such as particle physics, condensed matter physics, high-temperature superconductors, quantum Hall effect and so on. There are many studies devoted to the system (1.1) on a domain $\Omega \subset \mathbb{R}^2$. In the case $(a, b) = (0, 1)$, Lin and Prajapat [18] proved the existence of the unique maximal solution and the mountain-pass type solution on a torus. For $a > b > 0$ in (1.1), one may refer to [13, 8]. In the case $b > a > 0$, Huang [11] proved the existence of the topological solutions which is the maximal solution to (1.1). More results include [12, 15, 21] and references therein.

Recently, there is a development of elliptic equations with exponential nonlinearity on graph. Grigor’yan et al. [7] got the existence of the Kazdan-Warner equation

$$\Delta u = c - he^u$$

(1.2)
on graph by the variational calculus and a method of upper and lower solutions. Ge [4] reinvestigated the Kazdan-Warner equation (1.2) on graph in the negative case and completed the result in [7]. Keller and Schwarz [16] studied the Kazdan-Warner equation on canonically compactifiable graphs and got analogous results as in the finite case. The $p$th Kazdan-Warner equation on graphs

$$\Delta_p u = c - he^u$$

(1.3)
was studied in [5, 24]. On an infinite graph, the existence of a solution to the Kazdan-Warner equation

$$\Delta f = g - he^f$$

(1.4)
was proved by a heat flow method [6]. Using a variational method, Liu and Yang [19] obtained multiple solutions of the Kazdan-Warner equation

$$\Delta u + \kappa + Ke^u = 0$$

(1.5)
on graphs in the negative case. Sun and Wang [22] defined the Brouwer degree and gave new proofs of some known existence results for Kazdan-Warner equations on finite graphs. For the Chern-Simons equations, Huang et al. [10] studied the equation

$$\Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}$$

(1.6)
on a finite graph where $\lambda > 0$ is a constant and $\delta_p$ is the Dirac delta mass at vertex $p$. They proved the existence of solutions for the non-critical case. The second author and Sun [9] investigated the critical case for (1.6) and considered a general Chern-Simons equation which is also studied by Liu and Zhong [20]. Huang et al. [14] studied the existence of maximal solutions to a Chern-Simons system and also established the existence of multiple solutions. One may refer to [1, 2, 3] for more results.

Let $V$ be the vertex set and $E$ be the edge set. A finite graph is written as $G = (V, E)$. We assume that $G$ is connected which means that any two vertices can be connected by finite edges. The weight on the edge $xy \in E$ is defined by $\omega_{xy}$ which is assumed symmetric i.e. $\omega_{xy} = \omega_{yx}$. Denote by $\mu : V \to \mathbb{R}^+$ a finite measure. We define the $\mu$-Laplace operator for any function $u : V \to \mathbb{R}$ by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)),$$
where \( y \sim x \) means that \( xy \in E \). For any two functions \( u \) and \( v \), the gradient form is defined by

\[
\Gamma(u, v) = \frac{1}{2\mu(x)} \sum_{j=1}^k \omega_{ij} (u(y) - u(x))(v(y) - v(x)).
\]

If \( u = v \), we write \( \Gamma(u) = \Gamma(u, u) \). Define

\[
|\nabla u(x) = \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{j=1}^k \omega_{ij} (u(y) - u(x))^2 \right)^{1/2}.
\]

The integral over \( V \) is defined by

\[
\int_V ud\mu = \sum_{x \in V} \mu(x)u(x),
\]

for any function \( u : V \to \mathbb{R} \). Analogous to the Euclidean case, we also define the Sobolev space

\[
W^{1,2}(V) = \left\{ u : V \to \mathbb{R}, \int_V (|\nabla u|^2 + u^2) \, d\mu < +\infty \right\}
\]

under the norm

\[
||u||_{W^{1,2}(V)} = \left( \int_V (|\nabla u|^2 + u^2) \, d\mu \right)^{1/2}.
\]

We first consider the system (1.1). Set

\[
\begin{align*}
f_1(u, v) &= a(b - a)e^u - b(b - a)e^v + a^2e^{2u} - abe^{2v} + b(b - a)e^{u+v}, \\
f_2(u, v) &= -b(b - a)e^u + a(b - a)e^v - abe^{2u} + a^2e^{2v} + b(b - a)e^{u+v}.
\end{align*}
\]

Then we rewrite (1.1) as

\[
\begin{align*}
\Delta u = \lambda f_1(u, v) + 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\Delta v = \lambda f_2(u, v) + 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j}.
\end{align*}
\]

Let \((u_0, v_0)\) be the solution to the system

\[
\begin{align*}
\Delta u = -\frac{4\pi N_1}{|V|} + 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\Delta v = -\frac{4\pi N_2}{|V|} + 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j},
\end{align*}
\]

where \( N_1 = \sum_{j=1}^{k_1} m_j, N_2 = \sum_{j=1}^{k_2} n_j \), and \(|V|\) is the volume of \( V \).

If \((\tilde{u}, \tilde{v})\) is a solution to (1.9), setting \((u, v) = (\tilde{u}, \tilde{v}) - (u_0, v_0)\), we see that

\[
\begin{align*}
\Delta u = \lambda f_1(u + u_0, v + v_0) + \frac{4\pi N_1}{|V|}, \\
\Delta v = \lambda f_2(u + u_0, v + v_0) + \frac{4\pi N_2}{|V|}.
\end{align*}
\]

We get the following theorem.
Theorem 1.1. Assume that \( b > a > 0 \). There is \( \lambda_0 > 0 \) such that if \( \lambda > \lambda_0 \),

1. The system (1.11) has a solution \((u, \nu)\) which is a maximal solution.
2. As \( \lambda \to \infty \), \((u_0, \nu_0) \to (u_0, -\nu_0)\).
3. As \( \lambda \to \infty \), \( \lambda f_1(u_0 + u_0, \nu_0 + \nu_0) \to -4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}\), \( \lambda f_2(u_0 + u_0, \nu_0 + \nu_0) \to -4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j}\).

Let \( \bar{u}_0 \) be the solution to

\[
\Delta u = -\frac{4\pi N}{|V|} + 4\pi \sum_{j=1}^{N} \delta_{p_j}.
\]

(1.12)

If \( \tilde{u} \) is a solution to (1.6), writing \( \tilde{u} = \bar{u}_0 + u \), then we get

\[
\Delta u = \lambda e^{\bar{u}_0 + u}(e^{\bar{u}_0 + u} - 1) + \frac{4\pi N}{|V|}.
\]

(1.13)

The results in [10, 9] show that there exists a critical value \( \lambda_c \geq \frac{16\pi N}{|V|} \) such that (1.6) admits a maximal solution if and only if \( \lambda \geq \lambda_c \), where \( |V| \) is the volume of \( V \).

Analogous to Theorem 1.1, we get the result for (1.6).

Theorem 1.2. Denote by \( u_\lambda \) the maximal solution to (1.13) for \( \lambda > \lambda_c \). There holds that

1. \( u_\lambda \to -\bar{u}_0 \) as \( \lambda \to \infty \);
2. \( \lambda e^{\bar{u}_0 + u}(e^{\bar{u}_0 + u} - 1) \to -4\pi \sum_{j=1}^{N} \delta_{p_j} \), as \( \lambda \to \infty \).

We arrange the rest of the paper as follows. In Section One, we establish the iteration scheme for any sub-solution and get the monotone sequence. That implies that if (1.1) has a sub-solution, then it admits a solution. We prove the existence of the solution by constructing the sub-solution. Furthermore, the asymptotic behavior is obtained by the different method from [11]. For Eq. (1.6), the behavior of the solution as \( \lambda \) goes to the critical value has been investigated in [9]. But no one considers the behavior of the solution to (1.6) as \( \lambda \) goes to infinity. Such behavior is studied and the proof of Theorem 1.2 is completed in Section Three by the same method as in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

First, we introduce the following maximum principle which is Lemma 4.1 in [10].

Lemma 2.1. Let \( G = (V, E) \) be a finite graph. If there is a positive constant \( K \) such that \( \Delta u(x) - Ku(x) \geq 0 \) for all \( x \in V \), there holds \( u \leq 0 \) on \( V \).

We call \((u_-, \nu_-)\) a sub-solution to (1.11) if it satisfies

\[
\begin{align*}
\Delta u_- & \geq \lambda f_1(u_- + u_0, \nu_- + \nu_0) + \frac{4\pi N_1}{|V|}, \\
\Delta \nu_- & \geq \lambda f_2(u_- + u_0, \nu_- + \nu_0) + \frac{4\pi N_2}{|V|}.
\end{align*}
\]

(2.1)
2.1. Monotone sequence

In order to get the existence of solutions to (1.11), letting \((u_1, v_1) = (\neg u_0, \neg v_0)\), we first perform the following iteration scheme

\[
\begin{align*}
\left(\Delta - K\right) u_{n+1} &= \lambda f_1(u_n + u_0, v_n + v_0) - Ku_n + \frac{4\pi N_1}{|V|}, \\
\left(\Delta - K\right) v_{n+1} &= \lambda f_2(u_n + u_0, v_n + v_0) - Kv_n + \frac{4\pi N_2}{|V|},
\end{align*}
\]

(2.2)

where \(K > 2\lambda (a + b)(b - a) + 2\alpha^2\).

Lemma 2.2. The sequence \((u_n, v_n)\) is a monotone, i.e.,

\[
\begin{align*}
u_1 > u_2 > \cdots > u_n > \cdots > u_-,
\end{align*}
\]

(2.3)

where \((u_-, v_-)\) is any sub-solution to (1.11). Hence if (1.11) have a sub-solution, it admits a solution \((u_n, v_n) = \lim_{n \to \infty} (u_n, v_n)\).

Proof. When \(n = 1\), we see that

\[
\begin{align*}
\left(\Delta - K\right) (u_2 - u_1) &= 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\left(\Delta - K\right) (v_2 - v_1) &= 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j}.
\end{align*}
\]

(2.4)

Applying Lemma 2.1 to the first equation in (2.4), we conclude that \(u_2 \leq u_1\). Observing that \(u_2 - u_1\) has only a finite number of values, we have that there is some \(x_0\) in \(V\) such that \((u_2 - u_1)(x_0)\) achieves the maximum. If \((u_2 - u_1)(x_0) = 0\), then (2.4) implies

\[
\Delta(u_2 - u_1)(x_0) = 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j} \geq 0.
\]

(2.5)

In view of the definition of the \(\mu\)-Laplace, we obtain \(\Delta(u_2 - u_1)(x_0) = 0\), which means that \((u_2 - u_1)(x) = 0\) if \(x \neq x_0\). Furthermore, since \(G\) is connected, for any \(x \in V\), \((u_2 - u_1)(x) = 0\), which contradicts (2.4). Thus, we get \(u_1 > u_2\). Similarly, \(v_1 > v_2\).

If \(u_1 > u_2 > \cdots > u_n\) and \(v_1 > v_2 > \cdots > v_n\) for some \(n\), it follows from the first equation in (2.2) that

\[
\begin{align*}
\left(\Delta - K\right)(u_{n+1} - u_n) &= \lambda f_1(u_n + u_0, v_n + v_0) - \lambda f_1(u_{n-1} + u_0, v_{n-1} + v_0) - K(u_n - u_{n-1}) \\
&= \left(\lambda \frac{\partial f_1}{\partial u}(\xi, \eta) - K\right)(u_n - u_{n-1}) + \lambda \frac{\partial f_1}{\partial u}(\xi, \eta)(v_n - v_{n-1}).
\end{align*}
\]

where \(\xi\) lies between \(u_0 + u_n\) and \(u_0 + u_{n-1}\), \(\eta\) lies between \(v_0 + v_n\) and \(v_0 + v_{n-1}\). Noting that

\[
\lambda \frac{\partial f_1}{\partial u}(\xi, \eta) = \lambda \left(a(b - a)e^\xi + 2a^2e^{2\xi} + b(b - a)e^{\xi + \eta}\right)
\]

\[
\leq \lambda \left(a(b - a) + 2a^2 + b(b - a)\right) = \lambda \left((a + b)(b - a) + 2a^2\right),
\]

5
and
\[
\frac{\partial f_1}{\partial u}(\xi, \eta) = \left(-b(b-a)e^{0} - 2abe^{2} + b(b-a)e^{\xi+\eta}\right)
\]
\[
= \lambda \left(-2abe^{2} - b(b-a)e^{0}(1 - e^{\xi})\right) < 0,
\]
we deduce that
\[
(\Delta - K)(u_{n+1} - u_{n}) > 0. \tag{2.6}
\]
Using Lemma 2.1 again, we get \(u_{n+1} \leq u_{n}\). By taking the similar process as in the case when \(n = 1\), we get \(u_{n+1} < u_{n}\). Hence \(u_{1} > u_{2} > \cdots > u_{n} > \cdots\). Similarly, \(v_{1} > v_{2} > \cdots > v_{n} > \cdots\).

Next we compare \((u_{n}, v_{n})\) with \((u_{n}, v_{n-1})\). Observing that \(G\) has only finite vertices, we conclude that there exist \(x_{a}\) and \(x_{b}\) such that \(u_{n} + u_{0}\) and \(v_{n} + v_{0}\) attain the maximum at \(x_{a}\) and \(x_{b}\), respectively. In view of (1.10) and (2.1) and the definition of the \(\mu\)-Laplace, we have
\[
0 \geq \Delta (u_{n} - u_{1})(x_{a})
\]
\[
\geq A_{1}(u_{n} + u_{0}, v_{n} + v_{0})(x_{a}) + 4\pi \sum_{j=1}^{k_{1}} m_{j} \delta_{p_{j}}
\]
\[
= \lambda \left(a(b-a)e^{(u_{n}, u_{0})(x_{a})} - b(b-a)e^{(u_{n}, v_{1})(x_{a})} + a^{2}e^{2(u_{n}, u_{0})(x_{a})} - abe^{2(u_{n}, v_{1})(x_{a})}\right)
\]
\[
+ b(b-a)e^{(u_{n}, v_{0})(x_{a})}(1 - e^{u_{n}, v_{1})(x_{a})}) + 4\pi \sum_{j=1}^{k_{1}} m_{j} \delta_{p_{j}} > 0, \tag{2.7}
\]
Assume that \((u_{n} - u_{1})(x_{a}) > 0\) and \((u_{n} - u_{1})(x_{a}) \geq (v_{n} - v_{1})(x_{a})\). Then the right hand side of (2.7) can be rewritten as
\[
\lambda \left(a(b-a) + abe^{(u_{n}, u_{0})(x_{a})} + a^{2}e^{(u_{n}, u_{0})(x_{a})} - e^{(u_{n}, v_{1})(x_{a})}\right)
\]
\[
+ (b-a)^{2}e^{(u_{n}, v_{0})(x_{a})}(1 - e^{(u_{n}, v_{1})(x_{a})}) + 4\pi \sum_{j=1}^{k_{1}} m_{j} \delta_{p_{j}} > 0, \tag{2.8}
\]
which leads to a contradiction.

Assume that \((u_{n} - u_{1})(x_{a}) > 0\) and \((u_{n} - v_{1})(x_{a}) > (u_{n} - u_{1})(x_{a})\). Obviously, \((u_{n} - v_{1})(x_{a}) \geq (u_{n} - u_{1})(x_{a}) > 0\). Analogous to (2.7), we obtain
\[
0 \geq \Delta (v_{n} - v_{1})(x_{a})
\]
\[
\geq A_{1}(u_{n} + u_{0}, v_{n} + v_{0})(x_{a}) + 4\pi \sum_{j=1}^{k_{1}} n_{j} \delta_{q_{j}}
\]
\[
= \lambda \left(-b(b-a)e^{(u_{n}, u_{0})(x_{a})} + a(b-a)e^{(u_{n}, v_{1})(x_{a})} - abe^{2(u_{n}, u_{0})(x_{a})} + a^{2}e^{2(u_{n}, v_{1})(x_{a})}\right)
\]
\[
+ b(b-a)e^{(u_{n}, v_{0})(x_{a})}(1 - e^{(u_{n}, v_{1})(x_{a})}) + 4\pi \sum_{j=1}^{k_{1}} n_{j} \delta_{q_{j}} > 0, \tag{2.9}
\]
Again, rewrite the right hand side of (2.9) as
\[
\lambda \left(a(b-a) + abe^{(u_{n}, u_{0})(x_{a})} + a^{2}e^{(u_{n}, u_{0})(x_{a})} - e^{(u_{n}, v_{1})(x_{a})}\right)
\]
\[
+ (b-a)^{2}e^{(u_{n}, v_{0})(x_{a})}(1 - e^{(u_{n}, v_{1})(x_{a})}) + 4\pi \sum_{j=1}^{k_{1}} n_{j} \delta_{q_{j}} > 0, \tag{2.10}
\]
which is a contradiction. Hence \((u_\cdash u_\bar{1})(x_n) \leq 0\), and similarly, \((u_\cdash v_\bar{1})(x_n) \leq 0\).

If \((u_\cdash u_\bar{1})(x_n) = 0\), using the similar technique as in (2.7) and (2.8), we see that

\[
0 \geq \Delta (u_\cdash u_\bar{1})(x_n) \geq 4\pi \sum_{j=1}^{k_n} m_j \delta_{p_j} \geq 0.
\] (2.11)

Hence \((u_\cdash u_\bar{1})(x) \equiv 0\) since \(G\) is connected, which is contradiction. Therefore, \(u_\cdash < u_\bar{1}\), and similarly, \(v_\cdash < v_\bar{1}\).

Now we assume that \(u_n > u_\cdash\) and \(v_n > v_\cdash\) for some \(n\). By (2.1) and (2.2), we have

\[
(\Delta - K)(u_n - u_{n+1}) \geq \lambda f_1(u_\cdash + u_0, v_\cdash + v_0) - \lambda f_1(u_\cdash + u_0, v_\cdash + v_0) - K(u_n - u_0)
\]

\[
= \left(\lambda \frac{\partial f_1}{\partial u}(\xi, \eta) - K\right)(u_n - u_0) + \lambda \frac{\partial f_1}{\partial u}(\xi, \eta)(v_\cdash - v_0).
\] (2.12)

Utilizing the similar technique as in proving (2.6), we deduce that

\[
(\Delta - K)(u_n - u_{n+1}) > 0.
\] (2.13)

By the same arguments as in proving \(u_{n+1} < u_n\), we show that \(u_{n+1} > u_\cdash\) and \(v_{n+1} > v_\cdash\). We finish the proof of Lemma 2.2.

\[\square\]

2.2. The sub-solution

Let \((u_\bar{i}, v_\bar{i})\) be a solution to (1.11) determined by Lemma 2.2. Noting that a solution to (1.11) is also a sub-solution, we always have \(u_\bar{i} \geq u\) and \(v_\bar{i} \geq v\) for any solution \((u, v)\) to (1.11). In this sense, we say that \((u_\bar{i}, v_\bar{i})\) is a maximal solution. Next we give the existence of the sub-solution.

**Lemma 2.3.** There exists \(\lambda_0 > 0\) such that the system (1.11) has a sub-solution \((u_\cdash, v_\cdash)\) for all \(\lambda > \lambda_0\).

**Proof.** If \(u = v\), then \(f_1(u, v)\) and \(f_2(u, v)\) are reduced to \((a - b)^2(e^{2u} - e^0)\). Let \(c > 0\) be a fixed constant. Set

\[
(u_\cdash, v_\cdash) = (-u_0 - c, -v_0 - c).
\]

We see that

\[
\begin{align*}
\Delta u_- &= \frac{4\pi N_1}{|V|} - 4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \\
\Delta v_- &= \frac{4\pi N_2}{|V|} - 4\pi \sum_{j=1}^{k_1} n_j \delta_{q_j}.
\end{align*}
\] (2.14)

Observing that

\[
f_1(u_\cdash + u_0, v_\cdash + v_0) = (a - b)^2(e^{-c} - e^{-c})
\]

and

\[
f_2(u_\cdash + u_0, v_\cdash + v_0) = (a - b)^2(e^{-c} - e^{-c}),
\]

we conclude that

\[
\begin{align*}
\Delta u_- &\geq \lambda f_1(u_\cdash + u_0, v_\cdash + v_0) + \frac{4\pi N_1}{|V|}, \\
\Delta v_- &\geq \lambda f_2(u_\cdash + u_0, v_\cdash + v_0) + \frac{4\pi N_2}{|V|}.
\end{align*}
\] (2.15)

if \(\lambda\) is large enough. Hence, there exists \(\lambda_0 > 0\) such that if \(\lambda > \lambda_0\), \((u_\cdash, v_\cdash)\) is a sub-solution to (1.11).

\[\square\]
2.3. Asymptotic behavior

Lemma 2.4. Let \((u_0, v_0)\) be the maximal solution obtained by Lemma 2.2. Then there holds that

1. \((u_0, v_0) \to (-u_0, -v_0)\) as \(\lambda \to \infty\);

2. \(f_1(u_0 + u_0, v_0 + v_0) \to -4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}, \quad f_2(u_0 + u_0, v_0 + v_0) \to -4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j}\) as \(\lambda \to \infty\).

Proof. Without loss of generality, assume that \(N_2 \geq N_1\). Denote by \(\{\lambda_m\}\) a sequence with \(\lambda_m \to \infty\) as \(m \to \infty\). Let \(\eta = \max_{x \in V} \frac{1}{\lambda_m}, \lambda_m \geq \frac{16\pi N_1}{a b}, \) and \((u_-, v_-) = (-u_0 - c, -v_0 - c)\) where

\[
\eta = -\ln \left(1 + \frac{\sqrt{1 - \frac{4\pi N_1}{\lambda_m(a-b)}}}{2}\right).
\]

It is easy to check that

\[
\begin{align*}
\Delta u_- & \geq \lambda_m f_1(u_-, v_- + v_0) + \frac{4\pi N_1}{|V|}, \\
\Delta v_- & \geq \lambda_m f_2(u_-, v_- + v_0) + \frac{4\pi N_2}{|V|}.
\end{align*}
\]

Hence \((u_-, v_-)\) is a sub-solution to the system \((1.11)\) with \(\lambda = \lambda_m\). Lemma 2.2 yields that

\[
-u_0 + \ln \left(1 + \frac{\sqrt{1 - \frac{4\pi N_1}{\lambda_m(a-b)}}}{2}\right) \leq u_{\lambda_m} < -u_0
\]

and

\[
-v_0 + \ln \left(1 + \frac{\sqrt{1 - \frac{4\pi N_2}{\lambda_m(a-b)}}}{2}\right) \leq v_{\lambda_m} < -v_0.
\]

We get \((u_{\lambda_m}, v_{\lambda_m}) \to (-u_0, -v_0)\) as \(\lambda_m \to \infty\) immediately. As a conclusion, \((u_0, v_0) \to (-u_0, -v_0)\) as \(\lambda \to \infty\) since \(\{\lambda_m\}\) may be any sequence with \(\lambda_m \to \infty\).

Let \(V^n = \{u \mid u \text{ is a real function : } V \to \mathbb{R}\}\). For any \(\phi \in V^n\), it follows from \((1.11)\) that

\[
\int_V\lambda_m f_1(u_0 + u_{\lambda_m}, v_0 + v_{\lambda_m}) \phi d\mu = \int_V u_{\lambda_m} \Delta \phi d\mu = \frac{4\pi N_1}{|V|} \int_V \phi d\mu
\]

\[
= -\int_V u_0 \Delta \phi d\mu = \left[-4\pi \sum_{j=1}^{k_1} m_j \phi(p_j)\right] - 4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j} \int_V \phi d\mu
\]

as \(\lambda_m\) goes to infinity. We deduce that

\[
\lambda_m f_1(u_0 + u_{\lambda_m}, v_0 + v_{\lambda_m}) \to -4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j},
\]
as $\lambda_m$ goes to infinity. Similarly,
\[
\lambda_m f_2(u_{0} + u_{\lambda_m}, v_{0} + v_{\lambda_m}) \to -4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j},
\]
as $\lambda_m$ goes to infinity. Furthermore,
\[
\lambda f_1(u_{\lambda} + u_{0}, v_{\lambda} + v_{0}) \to -4\pi \sum_{j=1}^{k_1} m_j \delta_{p_j}
\]
and
\[
\lambda f_2(u_{\lambda} + u_{0}, v_{\lambda} + v_{0}) \to -4\pi \sum_{j=1}^{k_2} n_j \delta_{q_j},
\]
as $\lambda \to \infty$.

3. Proof of Theorem 1.2

The method in Lemma 2.4 can be applied to the following equation
\[
\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}, \tag{3.1}
\]
which was studied in \cite{10, 9}.

By Lemma 4.2 in \cite{10}, we know that if Eq. (1.13) has a sub-solution, then it admits a solution. Let $u_{\lambda}$ be the maximal solution determined by Lemma 4.2 in \cite{10} with $u_{\cdot} = -\bar{u}_0 - c$. Analogously, we obtain the following lemma.

**Lemma 3.1.** There holds that

1. $u_{\lambda} \to -\bar{u}_0$ as $\lambda \to \infty$;
2. $\lambda e^{\bar{u}_0 + u} (e^{\bar{u}_0 + u} - 1) \to -4\pi \sum_{j=1}^{N} \delta_{p_j}$, as $\lambda \to \infty$.

**Proof.** Define by $\{\lambda_m\}$ a sequence with $\lambda_m \to \infty$ as $m \to \infty$. Let $\lambda_m \geq \frac{16\pi N}{|V|} \eta$ and $u_{\cdot} = -\bar{u}_0 - c$ where
\[
c = -\ln \left( \frac{1 + \sqrt{1 - \frac{16\pi N}{|V|} \eta}}{2} \right)
\]
and $\eta = \max_{x \in V} \frac{1}{\mu(x)}$. We see that
\[
\Delta u_{\cdot} \geq \lambda_m e^{\bar{u}_0 + u} (e^{\bar{u}_0 + u} - 1) + \frac{4\pi N}{|V|}.
\tag{3.2}
\]
Then Lemma 4.2 in \cite{10} implies that
\[
-\bar{u}_0 + \ln \left( \frac{1 + \sqrt{1 - \frac{16\pi N}{|V|} \eta}}{2} \right) \leq u_{\lambda_m} \leq -\bar{u}_0.
\]
Thus, we get $u_{\lambda m} \to -\bar{u}_0$ as $m \to \infty$. Furthermore, $u_\lambda \to -\bar{u}_0$ as $\lambda \to \infty$.

For any $\phi \in V^R$, it is seen from (1.13) that

$$
\int_V \lambda m e^{\bar{u}_0 + u_{\lambda m}} (e^{\bar{u}_0 + u_{\lambda m}} - 1) \phi d\mu = \int_V u_{\lambda m} \Delta \phi d\mu - \frac{4\pi N}{|V|} \int_V \phi d\mu
$$

$$
\to - \int_V \bar{u}_0 \Delta \phi d\mu - \frac{4\pi N}{|V|} \int_V \phi d\mu
$$

$$
e - \int_V \Delta \bar{u}_0 \phi d\mu - \frac{4\pi N}{|V|} \int_V \phi d\mu
$$

$$
= -4\pi \sum_{j=1}^N \phi(p_j),
$$

as $\lambda m$ goes to infinity. We conclude that

$$
\lambda e^{\bar{u}_0 + u_{\lambda m}} (e^{\bar{u}_0 + u_{\lambda m}} - 1) \to -4\pi \sum_{j=1}^N \delta(p_j),
$$

as $\lambda \to \infty$.

**Remark 1.** The method in Lemma 2.4 can be also applied to the system

$$
\begin{cases}
\Delta u_1 = \lambda e^{\alpha_1} (e^{\alpha_1} - 1) + 4\pi \sum_{j=1}^{k_1} \alpha_j \delta(p_j), \\
\Delta u_2 = \lambda e^{\alpha_2} (e^{\alpha_2} - 1) + 4\pi \sum_{j=1}^{k_2} \beta_j \delta(q_j),
\end{cases}
$$

(3.4)

to get the asymptotic behavior of the maximal solutions. The system (3.4) on graphs has been studied in [14].

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