This short note is devoted to the unraveling of the hidden interactivity of ordinary differential games which is an artefact of predictions of the behaviour of other players by the fixed player and describes deviations of their real behaviour from such predictions. A method to improve the predictions is proposed. Applications to the strategical analysis of interactive games are also specified.

The mathematical formalism of interactive games, which extends one of ordinary games (see e.g. [1]) and is based on the concept of an interactive control, was recently proposed by the author [2] to take into account the complex composition of controls of a real human person, which are often complicated couplings of his/her cognitive and known controls with the unknown subconscious behavioral reactions. This formalism is applicable also to the description of external unknown influences and, thus, is useful for problems in computer science (e.g. the semi-artificially controlled distribution of resources), mathematical economics (e.g. the financial games with unknown dynamical factors) and sociology (e.g. the collective decision making).

However, the interactivity may be unraveled in all differential games. In some sense any prediction of behaviour of other players by the fixed player allows to consider their controls as interactive. The goal of this article is to explain how it is done.

I. INTERACTIVE SYSTEMS AND GAMES

Definition 1 [2]. An interactive system (with \( n \) interactive controls) is a control system with \( n \) independent controls coupled with unknown or incompletely known feedbacks (the feedbacks as well as their couplings with controls are of a so complicated nature that their can not be described completely). An interactive game is a game with interactive controls of each player.

Below we shall consider only deterministic and differential interactive systems. In this case the general interactive system may be written in the form:

\[
\dot{\varphi} = \Phi(\varphi, u_1, u_2, \ldots, u_n),
\]
where \( \varphi \) characterizes the state of the system and \( u_i \) are the interactive controls:

\[
u_i(t) = u_i(u_i^\circ(t), [\varphi(t)]|_{\tau \leq t}),
\]
i.e. the independent controls \( u_i^\circ(t) \) coupled with the feedbacks on \( [\varphi(t)]|_{\tau \leq t} \). One may suppose that the feedbacks are integrodifferential on \( t \).

However, it is reasonable to consider the \textit{differential interactive games}, whose feedbacks are purely differential. It means that

\[
u_i(t) = u_i(u_i^\circ(t), \varphi(t), \ldots, \varphi^{(k)}(t)).
\]

A reduction of general interactive games to the differential ones via the introducing of the so-called \textit{intention fields} was described in [2]. Below we shall consider the differential interactive games only if the opposite is not specified explicitly.

The interactive games introduced above may be generalized in the following ways.

The first way, which leads to the \textit{indeterminate interactive games}, is based on the idea that the pure controls \( u_i^\circ(t) \) and the interactive controls \( u_i(t) \) should not be obligatory related in the considered way. More generally one should only postulate that there are some time-independent quantities \( F_\alpha(u_i(t), u_i^\circ(t), \varphi(t), \ldots, \varphi^{(k)}(t)) \) for the independent magnitudes \( u_i(t) \) and \( u_i^\circ(t) \). Such claim is evidently weaker than one of Def.1. For instance, one may consider the inverse dependence of the pure and interactive controls: \( u_i^\circ(t) = u_i^\circ(u_i(t), \varphi(t), \ldots, \varphi^{(k)}(t)). \)

The inverse dependence of the pure and interactive controls has a nice psychological interpretation. Instead of thinking of our action consisting of the conscious and unconscious parts and interpreting the least as unknown feedbacks which “dress” the first, one is able to consider our action as a single whole whereas the act of consciousness is in the extraction of a part which it declares as its “property”. So interpreted the function \( u_i^\circ(u_i, \varphi, \ldots, \varphi^{(k)}) \) realizes the “filtering” procedure.

The second way, which leads to the \textit{coalition interactive games}, is based on the idea to consider the games with coalitions of actions and to claim that the interactive controls belong to such coalitions. In this case the evolution equations have the form

\[
\dot{\varphi} = \Phi(\varphi, v_1, \ldots, v_m),
\]

where \( v_i \) is the interactive control of the \( i \)-th coalition. If the \( i \)-th coalition is defined by the subset \( I_i \) of all players then

\[
v_i = v_i(\varphi(t), \ldots, \varphi^{(k)}(t), u_j^\circ|j \in I_i).\]

Certainly, the intersections of different sets \( I_i \) may be non-empty so that any player may belong to several coalitions of actions. Def.1 gives the particular case when \( I_i = \{i\} \).

The coalition interactive games may be an effective tool for an analysis of the collective decision making in the real coalition games that spread the applicability of the elaborating interactive game theory to the diverse problems of sociology.

\textbf{Remark 1.} One is able to consider interactive games of discrete time in the similar manner.

\textbf{Remark 2.} In the most cases one may exclude the first derivative of \( \varphi \) from feedbacks. First, one should transform the feedbacks to the inverse form, i.e. to express
pure controls via interactive ones. Second, one should substitute the first derivative of \( \varphi \) by its value specified by the evolution equations. Third, one should return to the direct dependence of controls. To exclude the higher derivatives it is necessary to perform such procedure several times. However, the time derivatives of controls will appear. The highest derivative should be considered as a new control variable as it is often done in the control theory whereas other derivatives and the control itself will be defined as states to make the procedure consistent. Nevertheless, sometimes there some difficulties to manipulate with feedbacks with derivatives of \( \varphi \). One may either practically postulate that the feedbacks depend on \( \varphi \) only or to consider the discrete time approximation and to use the left difference of \( \varphi \) in feedbacks and the right difference of \( \varphi \) in the evolution equations.

II. The \( \varepsilon \)-representations

Interactive games are games with incomplete information by their nature. However, this incompleteness is in the unknown feedbacks, not in the unknown states. The least situation is quite familiar to specialists in game theory and there is a lot of methods to have deal with it. For instance, the unknown states are interpreted as independent controls of the virtual players and some muppositions on their strategies are done. To transform interactive games into the games with an incomplete information on the states one can use the following trick, which is called the \( \varepsilon \)-representation of the interactive game.

**Definition 2.** The \( \varepsilon \)-representation of the differential interactive game is a representation of the interactive controls \( u_i(t) \) in the form

\[
u_i(t) = u_i(u_i^0(t), \varphi(t), \ldots, \varphi^{(k)}(t); \varepsilon_i(t))\]

with the known function \( u_i \) of its arguments \( u_i^0, \varphi, \ldots, \varphi^{(k)} \) and \( \varepsilon_i \), whereas

\[
\varepsilon_i(t) = \varepsilon_i(u_i^0(t), \varphi(t), \ldots, \varphi^{(k)}(t))
\]

is the unknown function of \( u_i^0 \) and \( \varphi, \ldots, \varphi^{(k)} \).

**Remark 3.** The derivatives of \( \varphi \) may be excluded from the feedbacks in the way described above.

**Remark 4.** One is able to consider the \( \varepsilon \)-representations of the indeterminate and coalition interactive games.

\( \varepsilon_i \) are interpreted as parameters of feedbacks and, thus, characterize the internal existential states of players. It motivates the notation \( \varepsilon \). Certainly, \( \varepsilon \)-parameters are not really states being the unknown functions of the states and pure controls, however, one may sometimes to apply the standard procedures of the theory of games with incomplete information on the states. For instance, it is possible to regard \( \varepsilon_i \) as controls of the virtual players. The naively introduced virtual players only double the ensemble of the real ones in the interactive games but in the coalition interactive games the collective virtual players are observed. More sophisticated procedures generate ensembles of virtual players of diverse structure.

Precisely, if the derivatives of \( \varphi \) are excluded from the feedbacks (at least, from the interactive controls \( u_i \) as functions of the pure controls, states and the \( \varepsilon \)-parameters) the evolution equation will have the form

\[
\dot{\varphi}(t) = \Phi(\varphi, u_1(u_1^0(t), \varphi(t); \varepsilon_1(t)), \ldots, u_n(u_n^0(t), \varphi(t); \varepsilon_n(t))),
\]
so it is consistent to regard the equations as ones of the controlled system with the ordinary controls $u_1, \ldots, u_n, \varepsilon_1, \ldots, \varepsilon_n$. One may consider a new game postulating that these controls are independent. Such game will be called the ordinary differential game associated with the $\varepsilon$-representation of the interactive game.

III. Unraveling the interactivity of ordinary differential games

Let us consider an arbitrary ordinary differential game with the evolution equations

$$\dot{\varphi} = \Phi(\varphi, u_1, u_2, \ldots, u_n),$$

where $\varphi$ characterizes the state of the system and $u_i$ are the ordinary controls.

Let us fix a player. For simplicity of notations we shall suppose that it is the first one. As a rule the players have their algorithms of predictions of the behaviour of other players. For a fixed moment $t_0$ of time let us consider the prediction of the first player for the game. It consists of the predicted controls $u^0_{[t_0];i}(t)$ ($t > t_0$; $i \geq 2$) of all players and the predicted evolution of the system $\varphi^0_{[t_0]}(t)$. Let us fix $\Delta t$ and consider the real and predicted controls for the moment $t_0 + \Delta t$. Of course, they may be different because other players use another algorithms for the game prediction. One may interpret the real controls $u_i(t)$ ($t = t_0 + \Delta t$; $i \geq 2$) of other players as interactive ones whereas the predicted controls $u^0_{[t_0];i}(t)$ as pure ones, i.e. to postulate their relation in the form:

$$u_i(t) = u_i(u^0_{[t_0]-\Delta t};i(t); \varphi_{[t_0]}(\tau)|\tau \leq t).$$

In particular, the feedbacks may be either reduced to differential form via the introducing of the intention fields or simply postulated to be differential. Thus, we constructed an interactive game from the initial ordinary game. One may use $\varphi(\tau)$ as well as $\varphi^0_{[t_0]}(\tau)$ in the feedbacks.

Note that the controls of the first player may be also considered as interactive if the corrections to the predictions are taken into account when controls are chosen.

The obtained construction may be used in practice to make more adequate predictions. Namely, a posteriori analysis of the differential interactive games allows to make the short-term predictions in such games [3]. One should use such predictions instead of the initial ones. Note that at the moment $t_0$ the first player knows the pure controls of other players at the interval $[t_0, t_0 + \Delta t]$ whereas their real freedom is interpreted as an interactivity of their controls $u_i(t)$. So it is reasonable to choose $\Delta t$ not greater than the admissible time depth of the short-term predictions. Estimations for this depth were proposed in [3].

Naïvely, the proposed idea to improve the predictions is to consider deviations of the real behaviour of players from the predicted ones as a result of the interactivity, then to make the short-term predictions taking the interactivity into account and, thus, to receive the corrections to the initial predictions. Such corrections may be regarded as “psychological” though really they are a result of different methods of predictions used by players.

Such procedure can be performed also for the discrete time games and, thus, is applicable to a wide class of the model entertainment games such as domino or chess. On the other hand, the procedure is useful for almost all practically important games.

Constructions above motivates the following definition.
**Definition 3.** The *scenario* of an interactive game is the set of pure controls $u_i^c(t)$ as functions of time. The *performance* is an interactive game with a *priori* fixed scenario.

Thus, one may said that at any moment $t_0$ an ordinary differential game coincides with certain performance in the nearest future $t \in [t_0, t_0 + \Delta t]$.

**Remark 5.** Performances have some specific features. The real evolution of the game does not *strategically* differ from the scenario. Their divergence is only *tactical* that strengthens the role of short-term predictions in the analysis of game. It explains why ordinary games can be represented as performances only in the nearest future.

**Remark 6.** The interpretation of the ordinary differential game as an interactive game also allows to perform the strategical analysis of interactive games. Indeed, let us consider an arbitrary differential interactive game $A$. Specifying its $\varepsilon$-representation one is able to construct the associated ordinary differential game $B$ with the doubled number of players. Making some predictions in the game $B$ one transform it back into an interactive game $C$. Combination of the strategical long-term predictions in the game $B$ with the short-term predictions in $C$ is often sufficient to obtain the adequate strategical prognosis for $A$.

**Remark 7.** Some procedures of predictions are applicable only to the proper classes $U$ of scenarios and specific initial positions. For instance, one may consider such class of scenarios and initial positions for which certain quantities $Z(t) = Z(u_i^c(t), \varphi(t), \varepsilon_i(t))$ are time-independent or their time derivatives are expressed via $Z$ themselves. Besides local-in-time expressions the functional expressions of the form $Z(t) = Z(u_i^c(t-\tau), \varphi(t-\tau), \varepsilon_i(t-\tau) | \tau \in [0, \Delta t])$ are also admitted. Also one may use the generalized functions (distributions) of the same arguments. The knowledge of such quantities allows to make some specific predictions and their detection in the concrete game may be rather natural. For instance, in the kaleidoscope-roulettes [4] the appearing of similar quantities (but of discrete time) indicates the presence of resonances for the class of scenarios that allows to make predictions, on the other hand, they also describe the interpretational figures detected visually by the player, which should be regarded therefore as “omens”. The unraveling of such omens and their interpretation provide a way to success in the kaleidoscope-roulettes. Because the controls of various players are independent it is important to know whether the situation is stable, it means that if one consider a scenario $\{u_i^c(t)\}$ in the neigbourhood of the class $U$ (in certain topology on the space of all scenario) then slightly improving $Z$ it is possible to provide its invariance for this scenario. Bifurcations and their controlling are also interesting. For instance, the controlling of bifurcations of $Z$ may be interpreted as the “omen-manipulation” in the kaleidoscope-roulettes that is perhaps the main intrigue in these games.

**IV. Conclusions**

Thus, the hidden interactivity of the ordinary differential games is unraveled. It is shown that such games may be regarded as interactive games of special kind. In these interactive games the pure controls of players coincide with the predicted ones and, hence, are known for the nearest future $t \in [t_0, t_0 + \Delta t]$, whereas the real freedom of players is interpreted as an interactivity of their really observed controls. Such observation leads to a method which allows to improve the initial predictions.
for the game. Applications to the strategical analysis of interactive games are also briefly specified.

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