Pluricomplex Green functions on Stein manifolds and certain linear topological invariants

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Abstract
In this paper, we explore the existence of pluricomplex Green functions for Stein manifolds from a functional analysis point of view. For a Stein manifold $M$, we will denote by $O(M)$ the Fréchet space of analytic functions on $M$ equipped with the topology of uniform convergence on compact subsets. In the first section, we examine the relationship between existence of pluricomplex Green functions and the diametral dimension of $O(M)$. This led us to consider negative plurisubharmonic functions on $M$ with a nontrivial relatively compact sublevel set (semi-proper). In Section 2, we characterize Stein manifolds possessing a semi-proper negative plurisubharmonic function through a local, controlled approximation type condition, which can be considered as a local version of the linear topological invariant $\tilde{\Omega}$ of Vogt. In Section 3, we look into pluri-Greenian and locally uniformly pluri-Greenian complex manifolds introduced by Poletsky. We show that a complex manifold is locally uniformly pluri-Greenian if and only if it is pluri-Greenian and give a characterization of locally uniformly pluri-Greenian Stein manifolds in terms of the notions introduced in Section 2.

MSC 2020
32A70, 32U35 (primary), 46E10 (secondary)
INTRODUCTION

Spaces of analytic functions form an essential class of nuclear Fréchet spaces. In recent years, there have been significant advances in the structure theory of nuclear Fréchet spaces (see [14]). In attempts to utilize this theory in complex analysis, one is led to analyze the complex analytic properties shared by Stein manifolds whose analytic function spaces possess a common linear topological invariant.

In this note, we look into the connection between the existence of pluricomplex Green functions and certain linear topological properties of the Fréchet space of analytic functions on a Stein manifold; a topic that was touched upon in [4].

Throughout the note, we will denote the space of analytic functions on a Stein manifold $M$ with compact open topology by $\mathcal{O}(M)$.

After compiling some background material, in Section 1 we recall a result of [4] that relates the existence of pluricomplex Green functions on $M$ to the diametral dimension of $\mathcal{O}(M)$ and look at an example.

In Section 2, we examine semi-proper pluricomplex Green functions (see definition below) and give a characterization of their existence in terms of a controlled approximation property for analytic functions on the complex manifold.

We then relate the existence of semi-proper pluricomplex Green functions to the linear topological properties $\Omega$ of Vogt [22], and the diametral dimension for $\mathcal{O}(M)$.

In Section 3, we look into pluri-Greenian and locally uniformly pluri-Greenian complex manifolds introduced by Poletsky [18], [19]. We show that a complex manifold is locally uniformly pluri-Greenian if and only if it is pluri-Greenian and give a characterization of locally uniformly pluri-Greenian Stein manifolds in terms of the notions introduced in Section 2.

In the last section, we recall some research done for the existence of pluricomplex Green functions, with the intention of putting our results in perspective.

The manifolds considered in this note are always assumed to be connected. We will use the standard terminology and results from functional analysis and complex potential theory as presented in [13, 14], respectively.

1 PLURICOMPLEX GREEN FUNCTIONS ON A STEIN MANIFOLD $M$ AND THE DIAMETRAL DIMENSION OF $\mathcal{O}(M)$

We start by defining the terms appearing in the heading.

Let $\mathcal{X}$ be a Fréchet space and $\{|| \cdot ||_k\}_{k=1}^{\infty}$ an increasing system of semi-norms generating the topology of $\mathcal{X}$. Let $U_k := \{x \in \mathcal{X} : ||x||_k \leq 1\}$ denote the unit ball of the corresponding local Banach spaces $\mathcal{X}_k, k = 1, 2, \ldots$. For a subspace $L$ of $\mathcal{X}$ and for $k^+ > k$, let

$$\delta(U_{k^+}; U_k, L) := \inf\{t > 0 : U_{k^+} \subseteq tU_k + L\}.$$ 

The $n$th Kolmogorov diameter of $U_{k^+}$ with respect to $U_k$ is defined as

$$d_n(U_{k^+}, U_k) := \inf\{\delta(U_{k^+}; U_k, L)\},$$
as \( L \) varies over subspaces of \( \mathcal{X} \) of dimension at most \( n \). There is a huge literature on techniques of computing Kolmogorov diameters for specific Banach spaces (see [16]).

The **diametral dimension** of \( \mathcal{X} \) is defined as

\[
\Delta(\mathcal{X}) := \{(t_n)_n \in \mathbb{C}^N; \forall k, \exists k^+ > k; t_n d_n(U_{k^+}, U_k) \to 0\}.
\]

The diametral dimension of \( \mathcal{X} \) does not depend on the generating semi-norm system and is a linear topological invariant of the Fréchet space \( \mathcal{X} \) in the sense that if \( \mathcal{X} \) is linear topologically isomorphic to a Fréchet space \( \mathcal{Y} \), then \( \Delta(\mathcal{X}) = \Delta(\mathcal{Y}) \).

In the case of nuclear Fréchet spaces, one can choose a generating system of Hilbertian semi-norms \( \{\|\cdot\|_k\}_k \) for \( \mathcal{X} \) and represent the compact linking maps \( i^k_{k^+} : \mathcal{X}_{k^+} \to \mathcal{X}_k \) as

\[
i^k_{k^+} := \sum_n \lambda_n \langle \cdot, f_n \rangle_{k^+} e_n,
\]

where \( \{f_n\} \) and \( \{e_n\} \) are orthonormal sequences in \( \mathcal{X}_{k^+} \) and \( \mathcal{X}_k \), respectively and \( (\lambda_n) \) is a sequence of nonnegative numbers converging to 0.

In this framework, the sequence of Kolmogorov diameters \( \{d_n(U_{k^+}, U_k)\} \) coincides with the nonincreasing rearrangement of the sequence \( \{\lambda_n\}_n \) of the eigenvalues.

This information is quite useful in computing diametral dimensions. For example, in view of this observation, one can readily verify that

\[
\Delta(\mathcal{O}(\Delta^d)) = \left\{ (\xi_n) : \sum_n |\xi_n|^2 e^{-\frac{2}{d} n^{1/d}} < \infty, \forall k = 1, 2, \ldots \right\}
\]

\[
\Delta(\mathcal{O}(\mathbb{C}^d)) = \left\{ (\xi_n) : \exists R > 1; |\xi_n| \leq CR^{n^{1/d}} \right\},
\]

where \( \Delta^d \) is the unit polydisc in the \( d \) dimensional complex plane \( \mathbb{C}^d \), \( d = 1, 2, \ldots \).

For more information about diametral dimension, we refer the reader to [15, 20].

We now recall some background material from Complex Potential Theory.

Let \( M \) be a connected complex manifold and \( z_0 \in M \). Let us denote plurisubharmonic functions on an open \( \Omega \subseteq M \) by the symbol \( \text{PSH}(\Omega) \). The **pluricomplex Green function** \( g_M(\cdot; z_0) \) with a pole at \( w_0 \) is defined as

\[
g_M(z; z_0) := \sup\{u \in \text{PSH}(M) : u < 0 \text{ on } M, \exists c : u(z) \leq \ln |z - w_0| + c, z \in U_{z_0}\},
\]

where \( U_{z_0} \) is some local neighborhood near \( z_0 \) (see [13]).

It turns out that either \( g_M \) is a plurisubharmonic function that is maximal on \( M \) or is \( -\infty \). We will say that pluricomplex Green function with a pole at \( w_0 \) exists in case \( g_M(\cdot, z_0) \neq -\infty \).

Pluricomplex Green functions are natural analogues in several variables, of the classical Green functions in the theory of open Riemann surfaces and were studied in diverse settings (see [9, 13, 18]).

Our first result, connecting pluricomplex Green functions and diametral dimension comes basically from [4]. Indeed Corollary 4.3, Theorem 2.9 and Proposition 4.1 of [4] combined yields:

**Theorem 1.1.** Let \( M \) be a Stein manifold of dimension \( d \), and suppose \( \Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\Delta^d)) \). Then pluricomplex Green function exists for each point of \( M \).
However, the statement of the above theorem is not an if and only if statement as the example below reveals.

**Example 1.2.** Let $M = \Delta \times \mathbb{C} \subseteq \mathbb{C}^2$. Clearly, pluricomplex Green function exists for each point $\zeta_0 = (z_0, w_0)$ of $M$, indeed the function:

$$g((z, w); \zeta_0) = \ln \left| \frac{z - z_0}{1 - z \overline{z_0}} \right|$$

is the pluricomplex Green function for $M$ with a pole at $\zeta_0$.

We now examine the diametral dimension of $\mathcal{O}(M)$.

The nuclear Fréchet space $\mathcal{O}(M)$ admits a Schauder basis $\{z^n w^m\}$, where

$$z^n(z, w) = z^n, \quad w^m(z, w) = w^m, \quad n, m = 0, 1, \ldots .$$

We choose the exhaustion $D_k = \{(z, w) \in M : |z| \leq e^{-1/k}, |w| \leq e^k\}$ of $M$, $k = 1, 2, \ldots$, and set $h_k(\theta) := (1 - \theta)k - \frac{\theta}{k}$, for $0 \leq \theta \leq 1, k = 1, 2, \ldots$.

Using the $H^2(D_k)$ norms, $k = 1, 2, \ldots$, to generate the topology of $\mathcal{O}(M)$, we obtain a sequence space representation of $\mathcal{O}(M)$ of the form:

$$\mathcal{O}(M) \cong \left\{ (\xi_{n,m})_{n,m=0}^\infty : |(\xi_{n,m})|_k \right\}$$

$$= \left\{ \sum_{\tilde{\eta}=(n,m)} |\xi_{n,m}|^2 e^{2|\tilde{\eta}|_k h_k(\frac{n}{n+m})} \right\}^{1/2} < \infty, \forall k = 1, 2, \ldots,$$

where, as usual, we used the notation $|\tilde{\eta}| := n + m$ for $\tilde{\eta} = (n, m) \in \mathbb{N} \times \mathbb{N}$.

The corresponding fundamental system of neighborhoods are:

$$\mathcal{U}_k = \left\{ f = \sum_{n,m} \xi_{n,m} z^n w^m : |(\xi_{n,m})|_k \leq 1, k = 1, 2, \ldots \right\} .$$

Fix a $k \in \mathbb{N}, k^+ \gg 2k$ and $\varepsilon, 0 < \varepsilon < 1$, whose values we will specify later. We will use the usual lexicographical order for $\mathbb{N} \times \mathbb{N}$, and set $\theta(\tilde{\eta}) := \frac{n}{n+m}$ for $\tilde{\eta} = (n, m), n, m = 0, 1, \ldots$.

As was mentioned in the introduction, the sequence of Kolmogorov diameters $\{d_0(\mathcal{U}_{k^+}, \mathcal{U}_k), \ldots, d_n(\mathcal{U}_{k^+}, \mathcal{U}_k), \ldots\}$ coincides with the nonincreasing rearrangement of

$$\{e^{2|\tilde{\eta}|(h_k(\theta(\tilde{\eta}))-h_{k^+}(\theta(\tilde{\eta}))\} \in \mathbb{N} \times \mathbb{N}.$$

Evidently, for a fixed $j$

$$j < j + 1 \leq \#\{\tilde{\eta} : |\tilde{\eta}|(h_k(\theta(\tilde{\eta}))-h_{k^+}(\theta(\tilde{\eta}))) \leq \varepsilon_j(k^+; k)\}, \quad (1.1)$$
where \( \#(A) \) denotes the number of elements of the set \( A \), and \( \varepsilon_j(k^+, k) = -\ln d_j(U_{k^+}; U_k) \), \( j = 0, 1, ... \).

On the lattice points \( \tilde{\eta}; \vartheta(\tilde{\eta}) \leq 1 - \varepsilon \), since \( h_{k^+}(\vartheta(\tilde{\eta})) - h_k(\vartheta(\tilde{\eta})) \geq \varepsilon(k^+ - k) \), we have

\[
\{ \tilde{\eta} : \vartheta(\tilde{\eta}) \leq 1 - \varepsilon, |\tilde{\eta}|(h_{k^+}(\vartheta(\tilde{\eta})) - h_k(\vartheta(\tilde{\eta}))) \leq \varepsilon_j(k^+; k) \}
\leq \{ \tilde{\eta} : |\tilde{\eta}| \leq \frac{1}{\varepsilon(k^+ - k)} \varepsilon_j(k^+; k) \}
\leq \frac{1}{2} \left\{ \left( \frac{1}{\varepsilon(k^+ - k)} \varepsilon_j(k^+; k) \right)^2 + \frac{1}{\varepsilon(k^+ - k)} \varepsilon_j(k^+; k) \right\}
\leq K \varepsilon_j(k^+; k)^2,
\]
where \( K = \frac{1}{2} \left( \frac{1}{\varepsilon^2(k^+ - k)} + \frac{1}{\varepsilon(k^+ - k)} \right) \).

On the remaining lattice points \( \tilde{\eta}; \vartheta(\tilde{\eta}) > 1 - \varepsilon \), since \( k^+ \gg 2k \), we estimate

\[
\{ \tilde{\eta} : \vartheta(\tilde{\eta}) > 1 - \varepsilon, |\tilde{\eta}|(h_{k^+}(\vartheta(\tilde{\eta})) - h_k(\vartheta(\tilde{\eta}))) \leq \varepsilon_j(k^+; k) \}
\leq \{ \tilde{\eta} : \vartheta(\tilde{\eta}) > 1 - \varepsilon, |\tilde{\eta}| \leq 2k \varepsilon_j(k^+; k) \}.
\]

A crude estimate on the number of lattice points in the wedge with base an interval in the unit simplex starting from \((0,1)\) with length \( \sqrt{2\varepsilon} \) and with norm less than or equal to \( 2k \varepsilon_j(k^+; k) \) yields

\[
\{ \tilde{\eta} : \vartheta(\tilde{\eta}) > 1 - \varepsilon, |\tilde{\eta}| \leq 2k \varepsilon_j(k^+; k) \} \leq 1 + \frac{\varepsilon}{2}(4k^2 + 2k) = K'.
\]

Now choose \( \varepsilon \) so that \( K' \leq 2 \) and \( k^+ \) so that \( K < 1 \). It follows from these choices and inequality (1.1) that

\[
j \leq 3\varepsilon_j(k^+; k)^2 = 3 \ln \left( \frac{1}{d_j(k^+, k)} \right)^2, \text{ or}
\]

\[
d_j(U_{k^+}, U_k) \left( e^{\frac{1}{\sqrt{3}} \sqrt{j}} \right)^j \leq 1.
\]

Since \( j \) was arbitrary, the nuclearity of \( \mathcal{O}(M) \) implies that \( (e^{\frac{1}{\sqrt{3}} \sqrt{j}})_j \in \Delta(\mathcal{O}(M)) \) (see [10]). On the other hand, since \( \Delta(\mathcal{O}(\Delta^2)) = \{ (\xi_j) : \sum_j |\xi_j|^2 e^{-2\sqrt{j}/k} < \infty, \forall k = 1, 2, ... \} \), \( (e^{\frac{1}{\sqrt{3}} \sqrt{j}})_j \notin \Delta(\mathcal{O}(\Delta^2)) \).

For more information about the diametral dimension of spaces of analytic functions on complete Reinhardt domains in \( \mathbb{C}^n \), we refer the reader to [5, 7].

2 | SEMI-PROPER PLURICOMPLEX GREEN FUNCTIONS

The sublevel sets of the pluricomplex Green functions of the above example are far from being relatively compact. Moreover, the poles of these pluricomplex Green functions are not strict in the sense of Poletsky. Recall that a function \( g \) on a complex manifold is said to have a strict logarithmic
pole at a point $w$, in case there are a coordinate neighborhood $U$ and constants $c_1$ and $c_2$ such that

$$\ln \|z - w\| + c_1 \leq g(w) \leq \ln \|z - w\| + c_2$$

on $U$ (see [18]). One can ask as to whether these features account for the difference; $\Delta(\mathcal{O}(M)) \neq \Delta(\mathcal{O}(\Delta^2))$ in the example above.

In what follows, we will call an extended real valued function $f$ on a topological space $X$ semi-proper in case there exists a $c \in \mathbb{R}$ such that the sublevel set $\emptyset \neq \{x \in X : f(x) < c\}$ is relatively compact in $X$.

To answer the question posed above, we will enquire into the existence of semi-proper pluricomplex Green functions.

**Definition 2.1.** Let $M$ be a Stein manifold and $\{\| \cdot \|_k\}_k$ be a generating norm system for $\mathcal{O}(M)$.

For $z \in M$, we will say that $M$ has the property $\tilde{\Omega}_z$ in case there exists a ball with radius $\varepsilon > 0$ in some coordinate system around $z$, $B(z; \varepsilon)$, $B(z; \varepsilon) \subset M$ such that

$$\exists k_1, \lambda > 0; \forall k > k_1, \exists C > 0; \quad U_{k_1} \subseteq C + r^\lambda U_k, \forall r > 0,$$

where $U_0 := \{f \in \mathcal{O}(M) : \sup_{w \in B(z; \varepsilon)} |f(w)| \leq 1\}$.

Clearly, this condition does not depend on the generating norm system of $\mathcal{O}(M)$.

Now we can state the main theorem of this section.

**Theorem 2.2.** Let $M$ be a Stein manifold and $\zeta_0 \in M$. The following statements are equivalent.

1. There exists a negative plurisubharmonic function on $M$ that has a relatively compact sublevel set containing $\zeta_0$.

2. $M$ possesses a semi-proper pluricomplex Green function with a strict logarithmic pole at $\zeta_0$.

3. $M$ has the property $\tilde{\Omega}_{\zeta_0}$.

**Proof.** (1) $\Rightarrow$ (2) The proof of this implication is a gluing argument (see, for example [3]). We give it in detail since we will utilize the explicit form of some of the functions appearing in the proof in a later discussion.

Let $\sigma_1$ be a negative plurisubharmonic function given in the assumption and assume that $\{z : \sigma_1(z) < -c\}$ is relatively compact for some $c > 0$, and contains $\zeta_0$. Choose a ball $B(\zeta_0, \varepsilon)$ in a coordinate neighborhood of $\zeta_0$ such that

$$\overline{B(\zeta_0, \varepsilon)} \subset \{z : \sigma_1(z) < -c\}$$

and set $\sigma_2(z) := \sigma_1 + \bar{c}$, where $-\bar{c} := \max_{\xi \in \overline{B(\zeta_0, \varepsilon)}} \sigma_1(\xi)$.

Fix a relatively compact, smooth strictly pseudo-convex domain $D \subset M$ containing

$$\{z : \sigma_2(z) < \beta\}, \beta := -\bar{c} - c,$

and let $g_D$ denote the pluricomplex Green function on $D$ with a pole at $\zeta_0$. The function $g_D$ is continuous on $D$ and is a proper exhaustion on $D$ (see Theorem 4.3 of [9], and Theorem 4.3 of [18]).

Choose $c_1 > 0, c_2 > 0$ so that

$$\{z : g_D(z) < -c_1\} \subset B(\zeta_0, \varepsilon) \subset \{z : \sigma_2(z) < \beta\} \subset \{z \in D : g_D(z) < -c_2\} \subset D \subset M.$$
Set $\psi(z) := \left(\frac{c_1 - c_2}{\beta}\right)z^2 - c_1$, and let

$$U := \{z \in D : g_D(z) < -c_2\}, \quad V := \{z \in D : g_D < -c_1\} \cap \{z \in D : g_D < -c_2\}.$$ 

For any $\xi \in \partial V \cap U$, $\lim_{z \to \xi} \psi(z) \leq g_D(\xi)$ by construction. Hence the function $u$ defined by

$$u := \begin{cases} 
\max(g_D, \psi) & \text{on } V, \\
g_D & \text{on } U - V 
\end{cases}$$ 

is a plurisubharmonic function on $U$.

Since on $\{z \in D : \sigma(z) < \beta\} \cap \{z \in D : g_D < -c_2\}$, $\max\{g_D, \psi\} = \psi$, we can extend $u$ to be a bounded plurisubharmonic function on $M$ by setting it equal to $\psi$ off $\{z \in D : g_D(z) < -c_2\}$.

This modification makes $u - \sup_{z \in M} u(z)$ a semi-proper, negative plurisubharmonic function on $M$. Note that $u$ is equal to $g_M$ in a neighborhood of $\xi_0$. The domain $D$ is hyperconvex in the sense of Poletsky and so $g_D$, and hence $u$ has a strict logarithmic pole at $\xi_0$ according to Theorem 4.3 of [18]. Since $g_M(\cdot, \xi_0)$ dominates $u - \sup_{z \in M} u(z)$ on $M$, $g_M(\cdot, \xi_0)$ also has a strict logarithmic pole at $\xi_0$.

(2) $\Rightarrow$ (3) To simplify the notation, we will denote the pluricomplex Green function with a strict logarithmic pole at $\xi_0$ by $p_M$. Pick a coordinate chart $U$ around $\xi_0$, such that $g_M(\cdot, \xi_0) \geq \ln || \cdot - \xi_0 || + c$ for some $c$, on $U$. Choose an $\varepsilon > 0$ such that the ball around $\xi_0$ with radius $\varepsilon > 0$, $B(\xi_0, \varepsilon)$, in this coordinate system, that satisfies $B(\xi_0, \varepsilon) \subset U$. Now for $\beta \ll \ln \varepsilon - c$ small enough, we can find an $0 < \varepsilon^- < \varepsilon$ such that $\Omega_{\beta} \cap B(\xi_0, \varepsilon^-) \subset B(\xi_0, \varepsilon)$, where $\Omega_{\beta} := \{z \in M : g_M(z, \xi_0) < \beta\}$. Since $\Omega_{\beta}$ is connected, ([17]), we conclude that $\Omega_{\beta} \subset B(\xi_0, \varepsilon)$ for this choice of $\beta$. Hence $g_M(\cdot, \xi_0)$ is a semi-proper negative plurisubharmonic function on $M$.

Choose $\varepsilon > 0, c > 0, 0 < c < c^+$ such that

$$\overline{B(\xi_0, \varepsilon)} \subset \{z : p_M(z) < -c^+\} \subset \{z : p_M(z) < -c\} \subset M.$$ 

Pick a $C^\infty$ strictly plurisubharmonic, proper function $\sigma : M \to [0, \infty)$ and set $D_k = \{z : \sigma(z) < k\}, k = 1, 2, \ldots$. For some $k_1$ large, we have

$$\overline{B(\xi_0, \varepsilon)} \subset \{z : p_M(z) < -c^+\} \subset \{z : p_M(z) < -c\} \subset D_{k_1}.$$ 

Let $\Omega_- := D_{k_1}$, $\Omega_+ := \{z \in M : p_M(z) < -c\}$.

Define a plurisubharmonic function $\rho_t$ on $M$ via $\rho_t(z) := \frac{t}{c^+} p_M(z) + t, t > 0$.

Clearly:

1. $\rho_t(z) \leq 0$ for $z \in B(\xi_0, \varepsilon), t > 0$;
2. $\rho_t(z) \geq (1 - \frac{\varepsilon}{c^+})t$ for $z \in \Omega_+, t > 0$.

We choose a Hermitian metric on $M$ and denote by $dV$ the volume form coming from this metric. Using the notation of Lemma 1 of [1], we set $d\varepsilon = cd\mu$, where $\mu$ is a measure on $M$ that is equivalent to the volume form, and $c$ is a strictly positive real valued function on $M$.

Fix $f \in \mathcal{O}(\Omega_-)$ with $(\int_{D_{k_1}} |f|^2 d\varepsilon)^{1/2} \leq 1$. 
In view of Lemma 1 of [1] we can, for each \( t > 0 \), compose \( f \) as, \( f = f_+ + f_-; f_+ \in \mathcal{O}(\Omega_+), f_- \in \mathcal{O}(\Omega_-) \) such that

\[
\int_{\Omega_+} |f_+|^2 e^{-\rho t} \, d\varepsilon \leq C_1 e^{-(1 - \frac{c}{c+})t}
\]

\[
\int_{\Omega_-} |f_-|^2 e^{-\rho t} \, d\varepsilon \leq C_1 e^{-(1 - \frac{c}{c+})t}
\]

and for some \( C_1 > 0 \) which is independent of \( f \) and \( t \).

Since \( \exists C_2 > 0 \) such that

\[
\int_{\Omega_+ \cap \Omega_-} |f|^2 e^{-\rho t} \, d\varepsilon \leq C_2 e^{-(1 - \frac{c}{c+})t},
\]

we have

\[
\int_{\Omega_{\pm}} |f_{\pm}|^2 e^{-\rho t} \, d\varepsilon \leq C_1 e^{\frac{c}{c+}t}
\]

and

\[
\int_{B(\zeta_0, \varepsilon)} |f_\pm|^2 \, d\varepsilon \leq \int_{B(\zeta_0, \varepsilon)} |f_-|^2 e^{-\rho t} \, d\varepsilon \leq C_1 e^{-(1 - \frac{c}{c+})t}.
\]

Set

\[
F := \begin{cases} f_+ & \text{on } \Omega_+, \\ f_+ - f_- & \text{on } \Omega_-.
\end{cases}
\]

The analytic function \( F \in \mathcal{O}(M) \) satisfies

\[
\exists K > 0; \int_M |F|^2 \, d\varepsilon \leq K e^{(\frac{c}{c+})t}, \tag{2.1}
\]

\[
\int_{B(\zeta_0, \varepsilon)} |F - f|^2 \, d\varepsilon = \int_{B(\zeta_0, \varepsilon)} |f_-|^2 \leq c_1 e^{-(1 - \frac{c}{c+})t}.
\]

We set for \( f \in \mathcal{O}(M) \)

\[
\| f \|_k := \left( \int_{D_k} |f|^2 \, d\varepsilon \right)^{1/2}, \quad k = 1, 2, \ldots
\]

Plainly, \( \{\| \cdot \|_k\}_{k=1}^{\infty} \) forms a fundamental system of norms generating the topology of \( \mathcal{O}(M) \), and as usual we denote the unit ball corresponding to \( \| \cdot \|_k \) by \( U'_k, k = 1, 2, \ldots \).

Let \( B := \{ g \in \mathcal{O}(M) : \int_M |g|^2 \, d\varepsilon \leq 1 \} \). Observe that \( B \subseteq U'_k \) for each \( k \in \mathbb{N} \).

The above analysis can be summarized as

\[
\exists k_1, \lambda > 0; \forall k > k_1, \exists \varepsilon > 0; U_{k_1} \subset \frac{1}{r^k} U_{B(\zeta_0, \varepsilon)} + cr U_k, \forall r \geq 1, \tag{2.2}
\]

where \( 0 < r^\varepsilon < \varepsilon \) and \( U_{B(\zeta_0, \varepsilon)} = \{ f \in \mathcal{O}(M) : \sup_{\xi \in B(\zeta_0, \varepsilon)} |f(\xi)| \leq 1 \} \).

Since for \( 0 < r \leq 1 \), (2.2) is trivial, we conclude that \( M \) has the property \( \tilde{\Omega}_{\zeta_0} \).
Choose a volume form $dV$ on $M$ and an open strictly pseudoconvex exhaustion $\{D_k\}_{k=1}^\infty$, $\overline{D}_k \subset \overline{D}_{k+1}$, $k = 1, 2, ..., \bigcup D_k = M$, of $M$ with $\zeta_0 \in D_1$. For $f \in \mathcal{O}(M)$, set $\| \cdot \|_k := (\int_{D_k} |\cdot|^2 dV)^{1/2}$, $k = 1, 2, ...$. In view of our assumption $\exists \epsilon > 0$ such that $B(\zeta_0, \epsilon) \subset \subset D_1$ and

$$\exists k, \beta > 0; \forall k > k_1 \exists C_k > 0: U_{k_1} \subset rU_k + \frac{C_k}{r^\beta} U_0, \forall r > 0,$$

where the set $U_k$ is the closed unit ball of $\| \cdot \|_k$, $k = 1, 2, ...$, and

$$U_0 = \{ f \in \mathcal{O}(M) : \sup_{\xi \in B(\zeta_0, \epsilon)} |f(\xi)| \leq 1 \}.$$

In the first part of the proof, we employ the argument given in 29.16 Lemma of [14] to construct a Hilbert space densely imbedded in $\mathcal{O}(M)$ and satisfies a certain interpolation condition. We will include this line of reasoning here for completeness.

In view of nuclearity of $\mathcal{O}(M)$, for each $k = 1, 2, ..., and

$$0 < \epsilon_k := \min \left\{ \frac{1}{2^k}, \frac{1}{C_{k+1}^\beta} \right\},$$

we can find a finite set $M_k \subset U_{k+1}$ such that

$$U_{k+1} \subset \epsilon_k U_k + M_k.$$

Since for each $k = 1, 2, ..., M_s \subset U_{s+1}$ for $s > k$, the set $\Lambda = \bigcup_{s=1}^\infty M_s$ is bounded in $\mathcal{O}(M)$. For each $g \in \mathcal{O}(M)^*$, we have

$$\| g \|_n^* \leq \epsilon_k \| g \|_k^* + \sup_{x \in \Lambda} |g(x)|,$$

where $\| h \|_s^* := \sup_{x \in U_s} |h(x)|$ is the dual norm of $\| \cdot \|_s$, $s = 1, 2, ...$. Iterating inequality (2.3) we get

$$\forall g \in \mathcal{O}(M)^*, \forall n > k_1$$

$$\| g \|_n^* \leq (1 + \epsilon_{n-1} + \epsilon_{n-2} + \cdots + \epsilon_0) \sup_{x \in \Lambda} |g(x)| + \epsilon_0 \cdots \epsilon_{n-1} \| g \|_0^*$$

$$\leq 2 \sup_{x \in \Lambda} |g(x)| + \epsilon_0 \cdots \epsilon_{n-1} \| g \|_0^*,$$

where $\| g \|_n^* = \sup_{x \in U_n} |g(x)|$, and $\epsilon_0 = 1$.

From (2.2), it follows that $\exists k_1, \beta > 0; \forall n > k_1 \exists C_n$ such that $\forall g \in \mathcal{O}(M)^*$

$$\| g \|_{k_1}^* \leq r \| g \|_n^* + \frac{C_n}{r^\beta} \| g \|_0^*, \quad \forall r > 0.$$

Inserting (2.4) into this expression, one gets

$$\| g \|_{k_1}^* \leq 2r \sup_{x \in \Lambda} |g(x)| + (r \epsilon_0 \cdots \epsilon_{n-1} + \frac{C_n}{r^\beta}) \| g \|_0^*, \quad \forall r, \forall g \in \mathcal{O}(M)^*.$$
Now for $C_n \leq r^{\beta/2} \leq C_{n+1}$,

$$r \varepsilon_0 \cdots \varepsilon_{n-1} \leq C_{n+1}^{2/\beta} \varepsilon_0 \cdots \varepsilon_{n-1} \leq \frac{C_{n+1}^{2/\beta}}{C_{n+1}^{1/\beta+1}} = \frac{1}{C_{n+1}^{1/\beta+1}} \leq \frac{1}{r^{\beta/2}}.$$ 

Hence $\exists R_0$,

$$\| g \|_{k_1}^* \leq 2r \sup_{x \in \Lambda} |g(x)| + \frac{2}{r^{\beta/2}} \| g \|_0^* \quad \forall r \geq R_0.$$ 

To get a Hilbert ball from $\Lambda$, we first consider the closed absolutely convex hull of $\Lambda$; call it $\hat{\Lambda}$, and consider the Hilbert norm $| \cdot |^2 := \sum_{k=1}^{\infty} \| \cdot \|_k^2 \frac{1}{2^k D_k^2}$, where $D_k = \sup_{\xi \in \Lambda} \| \xi \|_k$, $k = 1, 2, \ldots$, and let $B := \{ f \in \mathcal{O}(M) : |f| \leq 1 \}$. We denote by $H_B$ the Hilbert space imbedded in $\mathcal{O}(M)$, whose unit ball is $B$.

Taking into account that $\mathcal{O}(M)$ is separable, by enlarging $\Lambda$ if necessary, we can, without loss of generality, assume that $H_B$ is dense in $\mathcal{O}(M)$.

In view of (2.6)

$$\exists D > 0; \forall g \in \mathcal{O}(M)^*; \quad \| g \|_{k_1}^* \leq r \| g \|_{B}^* + \frac{D}{r^{\beta/2}} \| g \|_{0}^* \quad \forall r > 0,$$

where $\| g \|_{B}^* := \sup_{x \in B} |g(x)|$.

Taking minimum (over $r$) of the expression (2.7), we obtain an equivalent condition to $\tilde{\Omega}_{\xi_0}$, namely, there exists a Hilbert space $H_\infty$ densely imbedded in $\mathcal{O}(M)$ and $\exists k_1, 0 < \gamma < 1, c > 0$:

$$\| g \|_{k_1}^* \leq c (\| g \|_0^*)^\gamma (\| g \|_{B}^*)^{1-\gamma}, \forall g \in \mathcal{O}(M)^*,$$

where $\| \cdot \|_\infty^* = \sup_{\nu_0} | \cdot |$.

Let $H$ denote the closure of $\mathcal{O}(M)$ with respect to the Hilbertian norm

$$\| \cdot \| = \left( \int_{B(\xi, \varepsilon)} |\cdot|^2 dV \right)^{1/2}.$$

Since for some constant $c > 0$, $\| g \|_0^* \leq c \| g \|_\infty^*$, the above statement holds for the dual norm $\| \cdot \|_\infty^*$ as well.

The imbedding $\iota : H_\infty \hookrightarrow H$ is compact since $\mathcal{O}(M)$ is nuclear, and consequently has a representation of the form:

$$\iota(x) = \sum_{n=0}^{\infty} \lambda_n \langle x, f_n \rangle \_\infty e_n,$$

for an orthonormal basis $\{f_n\}$, an orthonormal system $\{e_n\}$ for $H_\infty$, and $H$, respectively, for some null sequence $\{\lambda_n\}$ consisting of strictly positive numbers.
Note that \( \| f_n \| = \lambda_n \) and \( \| f_n^* \| = \frac{1}{\lambda_n} \), where \( f_n^*(\cdot) = \langle \cdot, f_n \rangle_\infty \), \( n = 0, 1, \ldots \).

Set \( \varepsilon_n := \ln\left(\frac{1}{\lambda_n}\right) \). Since \( \{f_n\} \) is a bounded sequence in \( \mathcal{O}(M) \), and \( \varepsilon_n \to \infty \), the function

\[
\psi(z) := \lim_{\xi \to z} \lim_{n} \frac{\ln |f_n(\xi)|}{\varepsilon_n}
\]

defines a plurisubharmonic function on \( M \). Plainly, \( \psi(z) \leq 0 \) for \( z \in M \), and since \( \| f_n \| = \lambda_n \) for \( z \in B(\zeta_0; \varepsilon^-) \), \( \varepsilon^- < \varepsilon \), \( \psi(z) \leq -1 \). Choose a \( 0 < \gamma^- < 1 \) with \( \gamma^+ > \gamma \).

We will look at the sublevel set

\[
\Omega_{\gamma^+} = \{ z : \psi(z) < -\gamma^+ \}.
\]

This set is nonempty since \( B(\zeta_0; \varepsilon) \subseteq \Omega_{\gamma^+} \).

Choose a point \( z \in \Omega_{\gamma^+} \). In view of Hartog's Lemma (p.50 in [13]), \( \exists c = c(z) > 0 \) such that

\[
|f_k(z)| \leq ce^{-\gamma^+ \varepsilon_k}, k = 1, 2, \ldots \tag{2.9}
\]

For an \( f \in H_\infty \), \( f(z) = \sum_n f_n^*(f) f_n(z) \), in view of (2.8) and (2.9), we have

\[
|f(z)| \leq \sum_n |f_n^*(f)||f_n(z)| \leq \left( \sum_n \| f_n^* \|_{k_1} \| f_n(z) \| \right) \| f \|_{k_1}
\]

\[
\leq \tilde{c} \left( \sum_n \frac{1}{\lambda_n} e^{-\gamma^+ \varepsilon_n} \right) \| f \|_{k_1}
\]

\[
= \tilde{c} \left( \sum_n e^{(\gamma-\gamma^+)\varepsilon_n} \right) \| f \|_{k_1} \leq \kappa \| f \|_{k_1},
\]

for some \( \kappa = \kappa(z) \). In the above estimate we have used the fact that increasing permutation of the sequence \( (\varepsilon_n)_n \) dominates by a constant times the sequence \( (\frac{1}{d_d(U, U_k)})_n \), \( U \) being the unit ball of \( H \), which in turn dominates a constant times \( (n^{\frac{3}{d}})_n \), where \( d = \text{dim} M \) (See Proposition 1.1 of [5]). Since \( H_\infty \) is dense in \( \mathcal{O}(M) \), the above inequality is true for each \( f \in \mathcal{O}(M) \). Choose a relatively compact holomorphically convex \( \bar{D}_{k_1} \subset M \) that contains \( \bar{D}_{k_1} \). The analysis above yields

\[
\exists C > 0 : \forall f \in \mathcal{O}(M) \quad |f(z)| \leq C \sup_{\xi \in \bar{D}_{k_1}} |f(\xi)|, \quad j = 1, 2, \ldots
\]

Employing a well-known trick of applying this inequality to \( f_n^* \) for a given \( f \in \mathcal{O}(M) \), taking \( n \)th root of both sides as \( n \to \infty \), we conclude that \( z \in \bar{D}_{k_1} \).

This finishes the proof of the theorem.

\[\square\]

We now look into the question posed in the beginning of the section.

**Theorem 2.3.** Let \( M \) be a Stein manifold of dimension \( d \), and suppose it satisfies one of the equivalent conditions of Theorem 2.2 for some \( \zeta_0 \in M \). Then \( \Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\Delta^d)) \).
Proof. Choose a $C^\infty$-strictly plurisubharmonic exhaustion function $p : M \to (-\infty, \infty)$ and let $\Omega_k = \{ z : p < k \}, k \in \mathbb{Z}^+$ with $p$ chosen such that $\zeta_0 \in \Omega_0$. We assume that $M$ has the property $\hat{\Omega}_n$. Choose $\varepsilon < \varepsilon^+, B(\zeta_0, \varepsilon^+) \subset \Omega_{k_1}$ and a sequence of integers $\{k_n\}$ with $k_1 < k_2 < \ldots$ tending to infinity, where $\varepsilon$ and $k_1$ are as in the definition of $\widehat{\Omega}_n$. Using the notation of Theorem 2.2, ‘(3) $\Rightarrow$ (1)’, we set

$$
\mathcal{V}_0 = \left\{ f \in \mathcal{O}(M) : \left( \int_{B(\zeta_0, \varepsilon^+)} |f|^2 d\varepsilon \right)^{1/2} \leq 1 \right\}
$$

and

$$
\mathcal{V}_n = \left\{ f \in \mathcal{O}(M) : \left( \int_{D_{k_n}} |f|^2 d\varepsilon \right)^{1/2} \leq 1 \right\}, \quad n = 1, 2, \ldots
$$

In view of Proposition 3.8 of [6] and the condition $\hat{\Omega}_n$, we have

$$
\forall n = 0, 1, \ldots, \forall k > n, \exists j \text{ and } C > 0 \text{ with } \mathcal{V}_{k+1} \subset \subset r^j \mathcal{V}_n + \frac{C}{r} \mathcal{V}_k, \quad \forall r > 0. \tag{2.10}
$$

Also we have

$$
\forall k = 0, \ldots, \exists 0 < \gamma < 1, C > 0 : \| f \|_{k+1} \leq C \| f \|_k^{\gamma} \| f \|_{k+2}^{1-\gamma}. \tag{2.11}
$$

This ‘two-constants theorem’ type condition can be perceived in many ways, one of which is to employ $p$-measures in its proof as in [2, p. 125]. Now, from the discussion leading to Proposition 1.1 of [5], we conclude that

$$
0 < \lim \frac{-\ln d_n(\mathcal{V}_1; \mathcal{V}_0)}{\alpha_n} \leq \lim \frac{-\ln d_n(\mathcal{V}_1; \mathcal{V}_0)}{\alpha_n} < \infty,
$$

where $\alpha_n := n^{1/d}, d = \dim M$, is the associated exponent sequence of $\mathcal{O}(M)$ ([5]). Choose $R_0$ large, $1 \ll R_0$ such that

$$
\exists n_0 : n \geq n_0 \Rightarrow \frac{1}{R_0^\alpha} \leq d_n(\mathcal{V}_1; \mathcal{V}_0). \tag{2.12}
$$

We estimate $d_n(\mathcal{V}_1; \mathcal{V}_0), n \in \mathbb{N}$ using $\Omega_{\zeta_0}$ as in [21]. In view of 29.13 Lemma of [14] and (2.6) of the proof ‘(2) $\Rightarrow$ (3)’ of Theorem 2.2, an equivalent form of $\Omega_{\zeta_0}$ in our notation reads as; $\exists$ bounded set $B$ and $\lambda > 0$ such that

$$
\mathcal{V}_1 \subset \frac{1}{r^\lambda} \mathcal{V}_0 + CrB; \quad \forall r > 0. \tag{2.13}
$$

Now suppose that $\rho > 0$ satisfies $B \subset \rho \mathcal{V}_0 + \mathcal{L}$, for some finite-dimensional subspace $\mathcal{L}$ with $\dim(\mathcal{L}) \leq n$.

In view of (2.13), this leads to

$$
\mathcal{V}_1 \subset \left( \frac{1}{r^\lambda} + Cr\rho \right) \mathcal{V}_0 + \mathcal{L}, \quad \forall r > 0,
$$
and hence to
\[ d_n(V_1; V_0) \leq \left( \frac{1}{r^\lambda} + Cr^\rho \right) V_0 + \mathcal{L}, \quad \forall r > 0. \]

Taking infimum of \( \left( \frac{1}{r^\lambda} + Cr^\rho \right) \) over \( r \) first, then taking infimum over \( \rho \), we get
\[ \exists \kappa > 0; \quad d_n(V_1; V_0) \leq \kappa d_n(B, V_0)^{\lambda/1+\lambda}. \] (2.14)

We claim that
\[ \Delta(\mathcal{O}(M)) \neq \Delta(\mathcal{O}(\mathbb{C}^d)) = \{(\xi_n) : \exists R \geq 1 \text{ and } C > 0; |\xi_n| \leq CR^\alpha_n\}. \]

Anticipating a contradiction, we assume \( \Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\mathbb{C}^d)) \). Choose an \( R > R_0^{1+1/\lambda} \). Since \( \{R^\alpha_n\}_n \in \Delta(\mathcal{O}(\mathbb{C}^d)) \), \( \exists m \in \mathbb{N} \) such that
\[ \lim_{n} R^\alpha_n d_n(V_m; V_0) = 0. \] (2.15)

So by (2.12) and (2.14),
\[ \left( \frac{R}{R_0^{1+1/\lambda}} \right)^{\alpha_n} \leq R^\alpha_n d_n(V_1; V_0)^{1+1/\lambda} \leq \left( \kappa^{1+1/\lambda} \right) R^\alpha_n d_n(B; V_0) \leq CR^\alpha_n d_n(V_m; V_0). \]

Hence, (2.15) gives us the desired contradiction.

In view of Theorem 4.4 of [4], we conclude that \( \Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\Delta^d)) \). This finishes the proof of the theorem.

We now look at some immediate corollaries of Theorem 2.3.

**Corollary 2.4.** Let \( M \) be an open Riemann surface. Then the following are equivalent.

1. \( M \) is hyperbolic.
2. \( M \) possesses a semi-proper negative subharmonic function.
3. \( \Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(B(0, 1))) = \{(\xi_n) : \sup_n |\xi_n|^r < +\infty, \forall r < 1\}. \)

**Proof.** If \( M \) is hyperbolic, it has a nontrivial Green’s function \( g_M(\cdot, \cdot) \) at each point of \( M \), which is harmonic except at its singularity. Fix a point \( \xi \in M \) and a small ball \( B(\xi, \varepsilon) \) in some chart around \( \xi \).

Consider the sublevel set \( \{z : g_M(z; \xi) < -c\} \) where \( -c < \min_{z \in B(\xi, \varepsilon)} g_M(\xi, \varepsilon) \). Such a \( c > 0 \) exists since \( g_M(\cdot, \cdot) \) is continuous. Taking into account that the sublevel sets of the Green functions are connected ([17]), we conclude that
\[ \{z : g_M(z, \xi) < -c\} \subset B(\xi, \varepsilon). \]

So \( M \) has a semi-proper negative subharmonic function. The other implications are consequences of Theorem 2.3 and Theorem 1.1, respectively. \[ \square \]
Another consequence of Theorem 2.3 and Theorem 1.1 is:

**Corollary 2.5.** Let $M$ be a Stein manifold. If $M$ possesses a semi-proper complex Green function with a pole at a point $\zeta_0 \in M$, then for each $\zeta \in M$ pluricomplex Green function with a pole at $\zeta$ exists.

We would like to close this section by recalling a linear topological invariant introduced by Vogt [22] from the structure theory of nuclear Fréchet spaces.

**Definition 2.6.** Let $X$ be a Fréchet space and $\{\| \cdot \|_k\}$ a fundamental system of seminorms generating the topology. $X$ is said to have the property $\Omega$ in case

\[ \forall k_0 \exists k_1, \lambda > 0; \forall k > k_1, \exists C > 0, \quad U_{k_1} \subseteq C U_{k_0} + r^\lambda U_k, \quad \forall r > 0, \]

where as usual, $U_s$ denotes the closed unit ball of the local Banach space $X_s$, $s = 1, 2, \ldots$

This property does not depend on the generating system $\{\| \cdot \|_k\}$ and is in fact a linear topological invariant; that is, if $X$ has $\Omega$ then any Fréchet space isomorphic to $X$ as a linear topological space, also enjoys this property.

Like all $\Omega$-type conditions, $\Omega$ passes to quotients spaces. For more information about the property $\Omega$, we refer the reader to [11, 22]. Stein manifolds $M$, for which $\mathcal{O}(M)$ possesses the property $\Omega$, were characterized in [3].

Plainly the property $\Omega$ for $\mathcal{O}(M)$, $M$ a Stein manifold, implies that $M$ possesses the property $\Omega_z$ for every point $z \in M$. Consequently, $\Omega$ appears as a linear topological invariant of $\mathcal{O}(M)$ that ensures for every $\zeta \in M$ the existence of a semi-proper negative plurisubharmonic on $M$ with a relatively compact sublevel set containing $\zeta$.

On the other hand, possessing $\Omega_z$ at every point for a Stein manifold $M$ need not imply that $\mathcal{O}(M)$ has the property $\Omega$. To see this, consider the punctured disc $B(0, 1) \setminus \{0\} \subseteq \mathbb{C}$. This domain obviously enjoys the property $\Omega_z$ for each of its points. Employing Laurent expansion for analytic functions on the punctured disc, it is not difficult to show that $\mathcal{O}(B(0, 1) \setminus \{0\}) \approx \mathcal{O}(\mathbb{C}) \times \mathcal{O}(B(0, 1))$ as Fréchet spaces (see [20], p. 376). If the space of analytic functions on this punctured disc had the property $\Omega$, $\mathcal{O}(\mathbb{C})$, being a quotient of $\mathcal{O}(B(0, 1) \setminus \{0\})$, would also enjoy this property. But this cannot be, for example, in view of Theorem 2.3 and the above remarks.

3 | **PLURI-GREENIAN AND LOCALLY UNIFORMLY PLURI-GREENIAN COMPLEX MANIFOLDS**

Poletsky in [18] introduced the concepts of pluri-Greenian and locally uniformly pluri-Greenian complex manifolds. In this section, we look into these notions from our point of view.

A complex manifold $M$ is called pluri-Greenian if for each $\omega \in M$, the pluricomplex Green function $g_M(\cdot, \omega)$ has a strict logarithmic pole at $\omega$.

On pluri-Green manifolds the pluricomplex Green function $g_M(\cdot, \omega)$ satisfies $(dd^c g_M(\cdot, \omega))^d = (2\pi)^d \delta_\omega$ if $d$ is the dimension of the manifold.

Consequently, in view of Theorem 2.2, a Stein manifold $M$ is pluri-Greenian if and only if it has the property $\Omega_z$ for every $z \in M$.

Recall that a complex manifold $M$ is called locally uniformly pluri-Greenian in case every point $w_0 \in M$ has a coordinate neighborhood $U$ with the following property: there is an open set $W \subseteq U$...
containing $w_0$ and a constant $c$ such that $g_M(z, w) \geq \ln \|z - w\| + c$ on $U$ whenever $w \in W$ (see [18], [19]).

Balls in $\mathbb{C}^N$ plainly satisfy the required property; hence bounded domains in $\mathbb{C}^N$, are \textit{locally uniformly pluri-Greenian}. It turns out that existence of a pluricomplex Green function with a strict logarithmic pole at each point of a complex manifold ensures that it is locally uniformly pluri-Greenian.

\textbf{Theorem 3.1.}

(1) A complex manifold is locally uniformly pluri-Greenian if and only if it is pluri-Greenian.

(2) A Stein manifold $M$ is locally uniformly pluri-Greenian if and only if it has the property $\Omega_\beta$ for each $z \in M$.

\textbf{Proof.} Let $M$ be a complex manifold and $w_0 \in M$. Suppose there exists a pluricomplex Green function $g_M(., w_0)$ with a strict logarithmic pole at $w_0$. That is there exists numbers $A$ and $B$ such that

$$\ln \|z - w_0\| + A \leq g_M(z, w_0) \leq \ln \|z - w_0\| + B$$

on a coordinate neighborhood $U_0$ around $w_0$. Choose $R > 0$ and $\beta < 0$ for the time being, so that

$$\{z : z \in U_0, g_M(z, w_0) < \beta\} \subset B\left(w_0, e^{\beta - A}\right)$$

and

$$B\left(w_0, e^{\beta - A}\right) \subset \subset B(w_0, R) \subset \subset U_0$$

where $B(w, r)$ denotes the ball around $w \in U_0$ with radius $r > 0$ in $U_0$. Since the sub-level sets of $g_M(., w_0)$ are connected,

$$\Omega_\beta = \{z : z \in M, g_M(z, w_0) < \beta\} \subset \subset B(w_0, R) \subset \subset U_0$$

Choose $R^- << R$ such that $B(w_0, R^-) \subset \subset \Omega_\beta$, and set $\beta_0 = \sup_{\zeta \in B(w_0, R^-)} g_M(\zeta, w_0)$. Set

$$\sigma(z) = g_M(z, w_0) - \beta_0, \ z \in M.$$ 

The plurisubharmonic function $\sigma$ is bounded and semi-proper, since

$$\{z : \sigma(z) < \beta - \beta_0\} \subset \subset B(w_0, R).$$

Let $g$ denote the pluricomplex Green function of $B(w_0, R)$ with a pole at $w_0$. Choose constants $r_1$ and $r_2$, $r_1 < r_2$, such that

$$\{z \in B(w_0, R) : g(z) < r_1\} \subset \subset B(w_0, R^-)$$

$$B\left(w_0, e^{\beta - A}\right) \subset \subset \{z \in B(w_0, R) : g(z) < r_2\} \subset \subset B(w_0, R).$$

Set

$$\Phi(z) = \frac{r_2 - r_1}{\beta - \beta_0}\sigma(z) + r_1, \ z \in M \text{ and } \Omega = \{z \in B(w_0, R) : g(z) < r_2\}.$$
Define a function $u$ on $\Omega$, by the formula:

$$
u(z) = \begin{cases}  
\max \{g, \Phi\} & \text{on } \{z \in B(w_0, R) : g(z) < r_1\} \cap \Omega \\
g(z) & \text{on } \{z \in B(w_0, R) : g(z) \leq r_1\}.
\end{cases}$$

By our construction, $u$ becomes a plurisubharmonic function. Note that outside $B(w_0, e^{\beta - A})$, $g_M(z, w_0) \geq \beta$, so $\sigma \geq \beta - \beta_0$ outside $B(w_0, e^{\beta - A}) \cap \Omega$. Hence by setting $u = \Phi$ outside $\Omega$, we can extend $u$ to $M$ to a bounded plurisubharmonic function on $M$. Note that on $M$,

$$
u(z) \leq \sup_{\zeta \in M} \Phi(\zeta) \leq \frac{r_2 - r_1}{\beta - \beta_0}(\beta_0) + r_1.
$$

Setting $u_M = \frac{r_2 - r_1}{\beta - \beta_0}(\beta_0) + r_1$, the function $v = u - u_M$ becomes a negative plurisubharmonic function on $M$ that is equal to $g - u_M$ on a ball around $w_0$ where $g$ is the pluricomplex Green function of a ball of radius $R$ with center $w_0$ in a coordinate chart around $w_0$.

Note that the dependence of $\Phi$ to the function $g$ is through the constants $r_1$ and $r_2$ that satisfy the conditions above. If one could select $r_1$ and $r_2$ such that the conditions required by these numbers are fulfilled by all pluricomplex Green functions of $B(w_0, R)$ with poles that lie in a fixed small ball around $w_0$, one can then use the function $\Phi$ defined above and carry out the above argument simultaneously for all pluricomplex Green functions with poles in this neighborhood and thereby obtain a negative plurisubharmonic function on $M$, for each $x$ in this neighborhood, which equals to $g_x + C$, for some $C$ that does not depend upon $x$, on a fixed neighborhood $W$ of $w_0$, where $g_x$ denotes the pluricomplex Green function of $B(w_0, R)$ with a pole at $x$. We claim that such a choice of constants is possible.

In other words, we wish to show that

$$\exists s_1 \text{ and } s_2, \quad s_1 < s_2 < 0 \text{ and } \delta > 0 :$$

$$B(w_0, \delta) \subset \{z \in U_0 : g_x(z) < s_1\} \subset B(w_0, R^-) \text{ and}$$

$$B\left(w_0, e^{\beta - A}\right) \subset \{z \in B(w_0, R) : g_x(z) < s_2\} \subset B(w_0, R),$$

for every $x$ in a fixed neighborhood of $w_0$.

To this end, choose $r_1$ and $r_2$ such that the condition is satisfied for $x = w_0$. In view of Lemma 6.2.4 of [13], choose balls around $w_0$, $B$ and $B_0$ such that $w_0 \in B \subset B_0 \subset \{z \in B(w_0, R) : g_{w_0}(z) < r_1\}$,

$$(1 + \varepsilon)g_{w_0}(z) \leq g_x(z) \leq \frac{1}{1 + \varepsilon}g_{w_0}(z), \quad x \in B, \ z \in U_0 - B_0,$$

and $\varepsilon > 0$ to be chosen later.

On $g_{w_0} = r_1$, $g_x \leq \frac{r_1}{1 + \varepsilon}$ hence $g_x(z) < \frac{r_1}{1 + \varepsilon}$, on $(z : g_{w_0} < r_1)$ for all $x \in B$. Consequently, we can find a $\delta > 0$, such that

$$B(w_0, \delta) \subset \{z \in U_0 : g_{w_0} < r_1\} \subset \{z \in U_0 : g_x(z) < \frac{r_1}{1 + \varepsilon}\}$$
for all \( x \in B \). On the other hand for \( z \notin B_0 \), and \( x \in B \), one has
\[
g_x(z) < \frac{r_1}{1 + \varepsilon} \implies g_{w_0}(z) < \frac{r_1}{(1 + \varepsilon)^2}.
\]

Choose \( \varepsilon > 0 \) so small that \( \{ z : g_{w_0}(z) < \frac{r_1}{(1 + \varepsilon)^2} \} \subset \subset B(w_0, R^-) \). It follows that \( \{ z \in B(w_0, R) : g_x(z) < \frac{r_1}{1 + \varepsilon} \} \subset \subset B(w_0, R^-) \), \( \forall x \in B \).

Choose a ball \( \Delta \) around \( w_0 \) that is contained compactly in \( \{ z \in B(w_0, R) : g_{w_0}(z) < r_2 \} \) and contains \( \overline{B(w_0, e^{\delta - A})} \).

On the boundary of \( \Delta \), \( g_x \leq \frac{r_2}{1 + \varepsilon} \) for \( x \in B \). So,
\[
B(w_0, e^{\delta - A}) \subset \subset \left\{ z \in B(w_0, R) : g_x(z) < \frac{r_2}{1 + \varepsilon} \right\}.
\]

We put another condition on \( \varepsilon \), by requiring that
\[
\left\{ z \in B(w_0, R) : g_{w_0}(z) < \frac{r_2}{(1 + \varepsilon)^2} \right\} \subset \subset B(w_0, R).
\]

It now follows that
\[
\left\{ z \in B(w_0, R) : g_x(z) < \frac{r_2}{1 + \varepsilon} \right\} \subset \subset B(w_0, R).
\]

By choosing \( \varepsilon > 0 \) satisfying the requirements stated above and setting \( s_1 = \frac{r_1}{1 + \varepsilon}, s_2 = \frac{r_1}{1 + \varepsilon} \), one proves the claim.

Now since \( B(w_0, R) \) is locally uniformly pluri-Greenian, there are open neighborhoods \( W \) and \( U \) of \( w_0, W \subseteq U \), contained in \( B \) and a constant \( c \) such that \( g_x(z) \geq \ln ||z - x|| + c \) on \( U \) whenever \( x \in W \). Combining this with the analysis above finishes the proof of the first part of the theorem.

The second part of the theorem follows directly from the first part and Theorem 2.2. \( \square \)

## 4 CONCLUDING REMARKS

We would like to point out that the results about locally uniformly pluri-Greenian complex manifolds obtained by Poletsky in his papers [18], [19] are valid, in view of Theorem 3.1, for pluri-Greenian complex manifolds. This, in particular, applies to the construction in [19] of pluri-potential compactifications for locally uniformly pluri-Greenian complex manifolds. Poletsky introduced biholomorphically invariant pluri-potential compactifications for complex manifolds in [19] as a counterpart to the Martin compactification in the classical potential theory. Consequently, Theorems 3.1 and 2.1 implies

**Corollary 4.1.** Let \( M \) be a Stein manifold and suppose the Fréchet space \( \mathcal{O}(M) \) has the property \( \tilde{\Omega} \). Then \( M \) admits a pluri-potential compactification in the sense of Poletsky.
II

In [12], Harz, Shcherbina and Tomassini introduced the notion of a core of a complex manifold \( M \), \( c(M) \), as the set of all points \( w \in M \) where every smooth global plurisubharmonic function that is bounded from above fails to be strictly plurisubharmonic near \( w \).

In the example of Section 1, the core of the manifold is plainly itself.

Poletsky and Shcherbina in [17, Corollary 3.3] showed that complex manifolds possess pluricomplex Green functions with strict logarithmic singularities at points outside their core. Combining this result with Theorem 2.2 of Section 2, we get

**Corollary 4.2.** A Stein manifold \( M \) satisfies \( \tilde{\Omega}_\zeta \), for all points \( \zeta \in M \) that lie outside its core.

III

Chen and Zhang, in their paper [8], considered Stein manifolds whose pluricomplex Green functions are semi-proper and specified them as manifolds that satisfy the property \((B1)\). They showed, among other things, that Stein manifolds satisfying property \((B1)\) possesses a Bergman metric ([8] Theorem 1). This result and Theorem 2.2 of Section 2 yields:

**Corollary 4.3.** A Stein manifold \( M \) satisfying \( \tilde{\Omega}_\zeta \) for all its points, possesses a Bergman metric.

ACKNOWLEDGEMENT

The author sincerely thanks Professor E. Poletsky for his valuable comments.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. A. Aytuna, *On stein manifolds M for which \( \mathcal{O}(M) \) is isomorphic to \( \mathcal{O}(\Delta^n) \) as Fréchet spaces*, Manuscripta Math. **62** (1988), no. 3, 297–315.
2. A. Aytuna, *Stein Spaces M for which \( \mathcal{O}(M) \) is isomorphic to a power series space*, in T. Terzioğlu (ed.), Advances in the theory of Fréchet spaces, NATO ASI Series (Series C: Mathematical and Physical Sciences), vol. 287, Springer, Dordrecht, 1989, pp. 115–154.
3. A. Aytuna, *Stein manifolds M for which \( \mathcal{O}(M) \) has the property \( \tilde{\Omega} \)*, Math. Forum **7** (2013), 45–57.
4. A. Aytuna, *Tameness in Fréchet spaces of analytic functions*, Studia Math. **232** (2016).
5. A. Aytuna, J. Krone, and T. Terzioğlu, *Imbedding of power series spaces and spaces of analytic functions*, Manuscripta Math. **67** (1990), 125–142.
6. A. Aytuna and A. Sadullaev, *Parabolic Stein manifolds*, Math. Scand. **114** (2014), 86–109.
7. A. Aytuna and T. Terzioğlu, *Some applications of a decomposition method*, in K. D. Bierstedt, J. Bonet, and J. Horváth (eds.), Progress in functional analysis, Noth-Holland Math. Stud., vol. 170, Elsevier, Amsterdam, 1992, pp. 85–95.
8. B. Y. Chen and J. H Zhang, *The Bergman metric on a Stein manifold with a bounded plurisubharmonic function*, Trans. Amer. Math. Soc. **354** (2002), 2997–3009.
9. J. P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. **194** (1987), 519–564.

10. L. Demeulenaere, L. Frerick, and J. Wengenroth, *Diametral dimensions of Fréchet spaces*, Studia Math. **234** (2016), no. 3, 271–280.

11. S. Dineen, R. Meise, and D. Vogt, *Characterization of nuclear Fréchet spaces in which every bounded set is polar*, Bull. Soc. Math. France **112** (1984), 41–68.

12. T. Harz, N. Shcherbina, and G. Tomassini, *On defining functions and cores for unbounded domains I*, Math. Z. **286** (2017), 987–1002.

13. M. Klimek, *Pluripotential theory*, Clarendon Press Publications, 1991.

14. R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, Clarendon Press Publications, 1997.

15. A. Pietsch, *Nuclear locally convex spaces*, Springer, Berlin, 1972.

16. A. Pinkus, *n-Widths in approximation theory*, Springer, Berlin–Heidelberg, 1985.

17. E. A. Poletsky and N. Shcherbina, *Plurisubharmonically seperable complex manifolds*, Proc. Amer. Math. Soc. **147** (2019), 2413–2424.

18. E. A. Poletsky, *Pluricomplex green functions on manifolds*, J. Geom. Anal. **30** (2020), no. 2, 1396–1410.

19. E. Poletsky, *Pluripotential Compactifications*, Potential Anal. **53** (2020), no. 1, 231–245.

20. S. Rolewicz, *Metric linear spaces*, PWN-Polish Sci. Publ. Warszawa and Reidel Dordrecht, 1985.

21. T. Terzioglu, *On diametral dimension of some classes of F-spaces*, J. Karadeniz Univ. **8** (1985), 1–13.

22. D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. reine angew. Math. **345** (1983), 182–200.