The dependence on parameters of the inverse functor to the $K$–finite functor

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Abstract

An interpretation of the Casselman-Wallach (C-W) Theorem is that the $K$–finite functor is an isomorphism of categories from the category of finitely generated, admissible smooth Fréchet modules of moderate growth to the category of Harish-Chandra modules for a real reductive group, $G$ (here $K$ is a maximal compact subgroup of $G$). In this paper we study the dependence of this functor on parameters. Our main result implies that holomorphic dependence implies holomorphic dependence. The work uses results from the excellent thesis of Vincent van der Noort. Also a remarkable family of universal Harish-Chandra modules developed in this paper plays a key role.

Introduction

An interpretation of the Casselman-Wallach (C-W) Theorem is that the $K$–finite functor is an isomorphism of categories from the category of finitely generated, admissible smooth Fréchet modules of moderate growth to the category of Harish-Chandra modules for a real reductive group, $G$ (here $K$ is a maximal compact subgroup of $G$). This variant will be explained in detail in the next section. Also, the inverse functor, $V \to \overline{V}$, to the $K$–finite functor is described therein. In this paper we study the dependence of this functor on parameters. Our main result implies that holomorphic dependence implies holomorphic dependence (see Theorem 8.2). This work rests on the excellent thesis of Vincent van der Noort ([VdN]) which contains several remarkable theorems including his finiteness theorem that is given a slightly simplified proof in Appendix E for the benefit of the reader. In addition to the work of van der Noort our technique involves the study of a
class of standard modules in the Harish-Chandra category with remarkable properties. In particular, they are free modules for the universal enveloping algebra of a maximal unipotent subgroup of $G$. Also, every Harish-Chandra module has a resolution by these modules.

The technical general results not specific to the main results of this paper are the content of the many appendices to this paper. In particular, several of the appendices involve the study of continuous families of Hilbert representations. The first step of the proof of the main theorem is to show that locally an analytic family of Harish-Chandra modules can be globalized to a continuous family of (strongly continuous) Hilbert representations.

One can ask if the work of Bernstein and Krötz [BK] contains our main result? To be blunt, it is not clear what exactly they meant by dependence on parameters. Their proof of the C-W theorem for linear real reductive groups contains a proof of the automatic continuity theorem that takes into account the parameters of the principal series. Their proof of the full C-W theorem is identical in its last stages to the one in [RRG] involving an ingenious argument due to Casselman that doesn’t take into account dependence on parameters. In any event, since only the original C-W theorem is used in this paper their version of the result would serve equally well as a basis for the proof.

The appendices take up more space than the body of the paper. Hopefully this separation will help the reader see the flow of the proof of the main theorem. Among the appendices there are results that are of interest beyond this paper. For example, Appendix A gives a proof that the $C^\infty$ vectors relative to $G$ of a finitely generated, admissible Hilbert representation are the same as the $C^\infty$ vectors relative to $K$ (see Proposition A.2). Also, as mentioned above, Appendix E contains a proof of an important result of van der Noort.

The proof of the main theorem follows the following lines. First a class of Harish-Chandra modules is constructed which we call $J$-modules (for lack of a name) that have the property that every Harish-Chandra module has a resolution by $J$-modules. It is shown that if one has an analytic family of Harish-Chandra modules $D$ and if $U$ is an open set with compact closure in the parameter space there is a family of $J$-modules over $U$ mapping surjectively onto the restriction of the family to $U$. The next step is to locally (in the parameter) globalize a continuous family of $J$-modules to a continuous family of Hilbert representations satisfying a technical condition (smoothable) that implies that the corresponding family of $C^\infty$-vectors defines a
continuous family of smooth Fréchet representations of locally uniform moderate growth (in the parameter). The last stage is to start with a holomorphic family of Harish–Chandra modules use the Hilbert modules corresponding to the resolving $J$–module to construct a local (in the parameter) Hilbert globalizations of the the family satisfying the accessibility condition. The fact that the C-W theorem is an isomorphism of categories shows that the corresponding local families of smooth Fréchet representations “glue” together and complete the proof of the main theorem.

1 The C-W Theorem

Let $G$ be a real reductive group and let $K$ be a maximal compact subgroup of $G$. Throughout the paper, if $H$ is a Lie group over $\mathbb{R}$ then its (real) Lie algebra will be denoted $\mathfrak{h}_o$ (i.e lower case fractur $H$ sub-o) and its complexification denoted $\mathfrak{h}$. Let $\theta$ denote the Cartan involution of $G$ (and of $\mathfrak{g}_o$) corresponding to $K$. Set $t_o = \text{Lie}(K)$, $t = t_o \otimes \mathbb{C}$ and $p_o = \{X \in \mathfrak{g} | \theta X = -X\}$.

Fix a symmetric $\text{Ad}(G)$–invariant bilinear form, $B$, on $\mathfrak{g}_o$ such that $B|_{k_o}$ is negative definite and $B|_{p_o}$ is positive definite. Let $v_1, ..., v_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to $B$ and set $C = \sum v_i^2$, the corresponding Casimir operator. Let $C_K$ be the Casimir operator for $K$ corresponding to $B|_{\mathfrak{k}}$. Let $\mathcal{H}(\mathfrak{g}, K)$ denote the category of Harish-Chandra modules, that is, the finitely generated, admissible, $(\mathfrak{g}, K)$–modules. The other category of representations that appear in the C-W theorem is the category $\mathcal{H}F(G)$ of admissible finitely generated smooth Fréchet representations of moderate growth. An object in $\mathcal{H}F(G)$ is a pair $(\pi, V)$ with $V$ a Fréchet space and $\pi$ a homomorphism of $G$ into the group of continuous bijections of $V$ such the the following 3 conditions are satisfied

1. The map $G \times V \to V$ given by $g, v \mapsto \pi(g)v$ is continuous and is $C^\infty$ in $G$.

2. Let $\|...\|$ be a norm on $G$ (see Appendix C). If $p$ is a continuous seminorm on $V$ then there exists $q$ a continuous seminorm on $V$ and $r$ such that $p(\pi(g)v) \leq \|g\|^r q(g)$.

3. The representation of $\mathfrak{g}$, $d\pi$, defines on the $K$–finite vectors of $V, V_K$, an object in $\mathcal{H}(\mathfrak{g}, K)$.

One version of the C-W Theorem is (see [RG])

**Theorem 1.1** If $(\pi, V), (\mu, W) \in \mathcal{H}F(G)$ and $T : (d\pi, V_K) \to (d\sigma, W_K)$ is a
morphism in $\mathcal{H}(g,K)$ then $T$ extends to a morphism in $\mathcal{HF}(G)$ with closed image that is a topological summand.

Let $(\pi, V) \in \mathcal{H}(g,K)$ then a $K$–invariant Hermitian inner product on $V$, $\langle \ldots, \ldots \rangle$, will be called $G$–integrable if there exists a strongly continuous action, $\sigma$, of $G$ on the Hilbert space completion of $V$, $H(\langle \rangle)$, relative to $\langle \ldots, \ldots \rangle$, such that the $(g, K)$–module of $K$–finite $C^\infty$ vectors, $(d\sigma, (H(\langle \rangle))^\infty_K) = (\pi, V)$.

The subquotient theorem implies that there exists at least one $G$–integrable inner product on $V$. Let $I(\pi, V)$ be the set of integrable $K$–invariant inner products on $V$. If $\langle \ldots, \ldots \rangle \in I(\pi, V)$ then $(H(\langle \ldots, \ldots \rangle))^\infty \in \mathcal{HF}(G)$.

The theorem implies that if $\langle \ldots, \ldots \rangle_i \in I(V), i = 1, 2$ then

$$(H(\langle \ldots, \ldots \rangle)_1)^\infty = (H(\langle \ldots, \ldots \rangle)_2)^\infty.$$ 

In particular this implies that the norm $v \mapsto \|v\|_2$ is continuous on $(H(\langle \ldots, \ldots \rangle)_1)^\infty$. Proposition A.2 implies that there exists $C$ and $l$ such that

$$\|v\|_2 \leq \|d\sigma_1(g)(1 + CK)^l v\|_1.$$ 

Note that if $\langle \ldots, \ldots \rangle \in I(V)$ then the $K$–invariant inner product $\langle v, w \rangle_1 = \langle \pi(1 + C K)^l v, w \rangle$ is also in $I(V)$. This allows us to define an inverse to the $K$–finite functor. Set

$$\nabla = \{\{v_\gamma\} \in \prod_{\gamma \in K} V(\gamma) | \sum_{\gamma \in K} \langle v_\gamma, v_\gamma \rangle^2 < \infty, \forall \langle \ldots, \ldots \rangle \in I(V)\}.$$ 

Noting (as above) that this space is equal to $(H(\langle \rangle))^\infty$ for any $\langle \ldots, \ldots \rangle \in I(V)$ the space $\nabla$ endowed with the topology given by the norms $\{\|\ldots\|_{\langle \ldots, \ldots \rangle}\}_{\langle \ldots, \ldots \rangle \in I(V)}$ is an object in $HF(V)$ with $\nabla_K = V$.

This leads to

**Theorem 1.2** The functor $V \rightarrow V_F$ from $\mathcal{HF}(G)$ to $\mathcal{H}(g,K)$ is an isomorphism of categories with inverse functor $V \rightarrow \nabla$.

The rest of this paper will be devoted to the study of the dependence of this functor on parameters. For this we will use a class of universal modules with remarkable properties related to ones in [RRG], Section 11.3 and in [HOW].
2 The subalgebra $D$ of $Z(g)$

Let $G$ be a real reductive group of inner type. That is, if $g_o = Lie(G)$, $g = g_o \otimes \mathbb{C}$ then $Ad(G)$ is contained in the identity component of $Aut(g)$. Keep the notation of the previous section. Let $p$ be the projection of $g$ onto $p = p_o \otimes \mathbb{C}$ corresponding to $g_o = \mathfrak{t}_o \oplus p_o$. Extend $p$ to a homomorphism of $S(g)$ onto $S(p)$. Then $p$ is the projection corresponding to

$$S(g) = S(p) \oplus S(g)\mathfrak{k}.$$ 

In [HOW] we found homogeneous elements $w_1, ..., w_l$ of $S(g)^G$ with $w_1 = C$ satisfying the following two properties:

1. $p(w_1), ..., p(w_l)$ are algebraically independent.
2. There exists a finite dimensional homogeneous subspace $E$ of $S(p)^K$ such that the map $\mathbb{C}[p(w_1), ..., p(w_l)] \otimes E \to S(p)^K$ given by multiplication is an isomorphism.

If $g$ contains no simple ideals of type $E$ (we will list the real forms of type $E$ that need to be avoided later in this section) one can take $E = \mathbb{C}1$. If $g$ is split over $\mathbb{R}$ then $\mathbb{C}[w_1, ..., w_l] = S(g)^G, E = \mathbb{C}1$.

Let $H$ denote the space of harmonic elements of $S(p)$, that is, the orthogonal complement to the ideal $S(p) (S(p)p)^K$ in $S(p)$ relative to the Hermitian extension of inner product $B_{p_o}$. Then the Kostant-Rallis theorem ([KR]) implies that the map

$$H \otimes S(p)^K \to S(p)$$

given by multiplication is a linear bijection. This and 2. easily imply

**Lemma 2.1** *The map*

$$H \otimes E \otimes \mathbb{C}[w_1, ..., w_l] \otimes S(\mathfrak{t}) \to S(g)$$

given by multiplication is a linear bijection.

Let $a_o$ be a maximal abelian subspace of $p_o$ and let (as usual)

$$W = W(a) = \{s \in GL(a) | s = Ad(k)|a, k \in K\}.$$

Let $h \in a_o$ be such that $a_o = \{X \in p_o|[h, X] = 0\}$. If $\lambda \in \mathbb{R}$ then set $g_o^\lambda = \{X \in g_o|[h, X] = \lambda X\}$. Set $n_o = \oplus_{\lambda>0} g_o^\lambda$ and $n_o = \theta n_o = \oplus_{\lambda>0} g_o^{-\lambda}$. Then

$$p = p(n) \oplus a$$
and \( p(n) \) is the orthogonal complement to \( a \) in \( p \) relative to \( B \). Let \( q \) be the projection of \( p \) onto \( a \) corresponding to this decomposition. Then the Chevalley restriction theorem implies that

\[
q : S(p)^K \rightarrow S(a)^W
\]

is an isomorphism of algebras. Also, as above, if \( H \) is the orthogonal complement to \( (S(a)a)^W \) \( S(a) \) in \( S(a) \). Then the map

\[
S(a)^W \otimes H \rightarrow S(a)
\]

given by multiplication is a linear bijection. Putting these observations together the map

\[
S(n) \otimes S(a)^W \otimes H \otimes S(t) \rightarrow S(g)
\]

given by multiplication is a linear bijection. We also note that the map

\[
\mathbb{C}[w_1, \ldots, w_l] \otimes E \rightarrow S(a)^W
\]

given by

\[
w \otimes e \mapsto q(p(w))q(e)
\]

is a linear bijection. This in turn implies

**Lemma 2.2** The map

\[
S(n) \otimes \mathbb{C}[w_1, \ldots, w_l] \otimes E \otimes H \otimes S(t) \rightarrow S(g)
\]

given by multiplication is a linear bijection.

Let symm denote the symmetrization map from \( S(g) \) to \( U(g) \) then symm is a linear bijection and symm \( \circ Ad(g) = Ad(g) \circ \text{symm} \). Let \( Z(g) = U(g)^G \) denote the center of \( U(g) \). Set \( z_i = \text{symm}(w_i) \) and

\[
D = \mathbb{C}[z_1, \ldots, z_l].
\]

Note that if \( S_j(g) = \sum_{k \leq j} S^j_k(g) \) and if \( U^j(g) \subset U^{j+1}(g) \) is the standard filtration of \( U(g) \) then

\[
\text{symm}(S_j(g)) = U^j(g).
\]

The above and standard arguments ([HOW] Theorem 2.5 and Lemma 5.2) imply
Theorem 2.3  Let the notation be as above. Then

1. The map
\[ H \otimes E \otimes D \otimes U(\mathfrak{t}) \to U(\mathfrak{g}) \]
given by
\[ h \otimes e \otimes D \otimes k \mapsto \text{symm}(h)\text{symm}(e)Dk \]
is a linear bijection.

2. The map
\[ U(\mathfrak{n}) \otimes E \otimes H \otimes D \otimes U(\mathfrak{t}) \to U(\mathfrak{g}) \]
given by
\[ n \otimes e \otimes h \otimes D \otimes k \mapsto n\text{symm}(e)\text{symm}(h)Dk \]
is a linear bijection.

3  A class of admissible finitely generated \((\mathfrak{g}, K)\)–modules

Retain the notation in the preceding section. Note that Theorem 2.3 implies that the subalgebra \(DU(\mathfrak{t})\) of \(U(\mathfrak{g})\) is isomorphic with the tensor product algebra \(D \otimes U(\mathfrak{t})\) and that \(U(\mathfrak{g})\) is free as a right \(DU(\mathfrak{t})\)–module under multiplication. If \(R\) is a \(DU(\mathfrak{t})\)–module then set
\[ J(R) = U(\mathfrak{g}) \otimes_{DU(\mathfrak{t})} R. \]

Denote by \(\mathcal{H}(\mathfrak{g}, K)\) the Harish–Chandra category of admissible finitely generated \((\mathfrak{g}, K)\)–modules. Let \(R\) be a finite dimensional continuous \(K\)–module that is also a \(D\)–module and the actions commute then \(K\) acts on \(J(R)\) as follows:
\[ k \cdot (g \otimes r) = \text{Ad}(k)g \otimes kr, k \in K, g \in U(\mathfrak{g}), r \in R. \]

Then as a \(K\)–module
\[ J(R) \cong \mathcal{H} \otimes E \otimes R \]
with \(K\) acting trivially on \(E\). Note that \(J(R) \in \mathcal{H}(\mathfrak{g}, K)\) since the multiplicities of \(K\)–types in \(\mathcal{H}\) are finite and \(J(R)\) is clearly finitely generated as a \(U(\mathfrak{g})\)–module. Let \(W(D, K)\) be the category of finite dimensional \((D, K)\)–modules with \(K\) acting continuously and the action of \(D\) and \(K\) commute.
Lemma 3.1 \( R \to J(R) \) defines an exact faithful functor from the category \( W(K, D) \) to \( \mathcal{H}(g, K) \).

**Proof.** This follows since \( U(g) \) is free as a module for \( DU(\mathfrak{g}) \) under right multiplication. ■

As usual, denote the set of equivalence classes of irreducible, finite dimensional, continuous representations of \( K \) by \( \hat{K} \). If \( V \in \mathcal{H}(g, K) \) set \( V(\gamma) \) equal to the sum of all irreducible \( K \)–subrepresentations of \( V \) in the class of \( \gamma \). Then \( V(\gamma) \) is invariant under the action of \( Z(g) \) hence under the action of \( D \).

By definition if \( V \in \mathcal{H}(g, K) \) there is a finite subset \( F \subset \hat{K} \) such that

\[
U(g_C) \sum_{\gamma \in F} V(\gamma).
\]

Set \( R = \sum_{\gamma \in F} V(\gamma) \in W(D, K) \). One has a canonical \( (g, K) \)–module surjection \( J(R) \to V \) given by \( g \otimes r \mapsto gr \). A submodule of an element of \( \mathcal{H}(g, K) \) is in \( \mathcal{H}(g, K) \) so

**Proposition 3.2** If \( V \in \mathcal{H}(g, K) \) then there exists a sequence of elements \( R_j \in W(g, K) \) and an exact sequence in \( \mathcal{H}(g, K) \)

\[
... \to J(R_k) \to \ldots \to J(R_2) \to J(R_1) \to J(R_0) \to V \to 0.
\]

Notice that this exact sequence us a free resolution of \( V \) as a \( U(n) \)–module.

Let \( \beta : D \to \mathbb{C} \) be an algebra homomorphism. Let \( \mathcal{H}(g, K)_\beta \) be the full subcategory of \( \mathcal{H}(g, K) \) consisting of modules \( V \) such that if \( z \in D \) then it acts by \( \beta(z)I \). The next result is an aside that will not be used in the rest of this paper and is a simple consequence of the definition of projective object.

**Lemma 3.3** Let \( F \) be a finite dimensional \( K \)–module and let \( D \) act on \( F \) by \( \beta(z)I \) yielding an object \( R \in W(K, D) \). Then \( J(R) \) is projective in \( \mathcal{H}(g, K)_\beta \).

4 The objects in \( W(K, D) \)

If \( R \in W(K, D) \) then \( R \) has a \( K \)–isotypic decomposition \( R = \oplus_{\gamma \in \hat{K}} R(\gamma) \). Only a finite number of the \( R(\gamma) \) are non-zero. If \( D \in D \) then \( DR(\gamma) \subset R(\gamma) \) for all \( \gamma \in \hat{K} \). If \( \chi : D \to \mathbb{C} \) is an algebra homomorphism then we set
\[ R_\chi = \{ v \in R \mid (D - \chi(D))^kv = 0, \text{for some } k > 0 \} \]

Then setting \( \text{ch}(D) \) equal to the set of all algebra homomorphisms of \( D \) to \( \mathbb{C} \) we have the decomposition

\[ R = \bigoplus_{\gamma \in \mathcal{K}, \chi \in \text{ch}(D)} R_\chi(\gamma). \]

Fix a \( K \)-module \((\tau_\gamma, F_\gamma) \in \gamma\). Then \( R_\chi(\gamma) \) is isomorphic with

\[ \text{Hom}_K(V_\gamma, R_\chi) \otimes F_\gamma \]

with \( K \) acting on \( F_\gamma \) and \( D \) acting on \( \text{Hom}_K(V_\gamma, R) \).

If \( R \) is an irreducible object in \( W(K, D) \) then Schur’s lemma implies that \( D \) acts by a single homomorphism to \( \mathbb{C} \) and \( R \) is irreducible as a \( K \)-module. Set \( V_\gamma, \chi \), equal to the module with \( D \) acting by \( \chi \) and \( K \) acting by an element of \( \gamma \).

We next analyze the homomorphisms \( \chi \). Let \( \chi \) be such a homomorphism then \( \chi(z_i) = \lambda_i \in \mathbb{C} \). Thus one simple parametrization is by \((\lambda_1, ..., \lambda_l) \in \mathbb{C}^l\).

We use the notation \( \beta_\lambda \) for the homomorphism such that \( \beta_\lambda(z_i) = \lambda_i \).

**Definition 4.1** Let \( X \) be an analytic manifold. An analytic family in \( W(K, D) \) over \( X \) is a pair \((\mu, V)\) of a a finite dimensional continuous \( K \)-module, \( V \), and a \( \mu : X \times D \to \text{End}(V) \) such that \( D \mapsto \mu(x, D) \) is a representation of \( D \) on \( V \) and \( x \mapsto \mu(x, D) \) is analytic for all \( D \in D \).

## 5 Parabolically induced families

Let \( A \) and \( N \) be the connected subgroups of \( G \) with \( \text{Lie}(A) = a_o \) and \( \text{Lie}(N) = n_o \). Let \( M \) be the centralizer of \( a \) in \( K \). Set \( Q = MAN \) then \( Q \) is a minimal parabolic subgroup of \( G \).

**Definition 5.1** An analytic family of finite dimensional \( Q \)-modules over a real analytic manifold \( X \) is a pair \((\sigma, S)\) with \( S \) a finite dimensional continuous \( M \)-module and a real analytic map \( \sigma : X \times Q \to \text{GL}(S) \) such that \( x \mapsto \sigma(x, q) \) is analytic and \( \sigma(x, \cdot) = \sigma_x \) is a representation of \( Q \).

Let \((\sigma, S)\) be a continuous finite dimensional representation of \( Q \). Set \( I^\infty(\sigma |_M) \) equal to the space of all smooth functions \( f : K \to S \) satisfying \( f(mk) = \sigma(m)f(k) \). Define and action \( \pi_\sigma \) of \( G \) on \( I^\infty(\sigma |_M) \) as follows: if \( f \in \)
$I^\infty(\sigma|_M)$ then extend $f$ to $G$ by $f_\sigma(qk) = \sigma(q)f(k)$, then, since $K \cap Q = M$ and $QK = G$, $f_\sigma$ is $C^\infty$ on $G$ set $\pi_\sigma(g)f(k) = f_\sigma(kg)$. Also set

$$\pi_\sigma(Y)f(k) = \frac{d}{dt}f_\sigma(k \exp tY)|_{t=0}$$

for $Y \in g$ and $k \in K, f \in I^\infty(\sigma|_M)$. Let $I(\sigma|_M)$ be the space of all right $K$ finite elements of $I^\infty(\sigma|_M)$

Put and $M$–invariant inner product, $\langle \ldots, \ldots \rangle$ on $S$. If $f, h \in I^\infty(\sigma|_M)$ then set

$$\langle f, h \rangle = \int_K \langle f(k), h(k) \rangle \, dk$$

with $dk$ normalized invariant measure on $K$. The following is standard.

**Proposition 5.2** Let $(\sigma, S)$ be an analytic family of finite dimensional representations of $Q$ over the analytic manifold $X$. Set $\lambda(x,y) = \pi_\sigma(x)(y)$ for $x \in X, y \in U(g_C)$. If $\mu$ is the common value of $\sigma_x|_M$, then $(\lambda, I(\mu))$ is an analytic family (see Appendix D) of objects in $H(g, K)$ over $X$.

**Proof.** It is standard that

$$x, g \mapsto (\pi_\sigma(g)f, h)$$

is real analytic and holomorphic in $x$ for $f, h \in I(\mu)$. ■

**Proposition 5.3** Let $(\sigma, S)$ be an analytic family of $Q$–modules based on $Z$. Set $\sigma(m)$ equal to the common value of $\sigma_x(m)$ for $m \in M$ and $H$ equal to the unitarily induced representation of $\sigma$ from $M$ to $K$. Then $z \mapsto (\pi_{\sigma_z}, H)$ is a continuous family of Hilbert representations over $Z$ (see Definition B.1) that is smoothable in the sense of Definition F.3.

**Proof.**

$$f(mk) = \sigma(m)f(k), m \in M, k \in K.$$  

Recall that $f_{\sigma_x}(g) = f_{\sigma_x}(namk) = \sigma_x(nam)f(k)$ for $g = namk, n \in N, a \in A, m \in M, k \in K$. Let $\{n_1, n_2, \ldots\}, \{a_1, a_2, \ldots\}$ be respectively bases of $U(n)$ and $U(a)$. Let $Y_1, \ldots, Y_n$ be a basis of $\mathfrak{t}$, such that $B(Y_i, Y_j) = -\delta_{ij}$. The monomials $Y^I = Y_1^{i_1} \cdots Y_n^{i_n}$ form a basis of $U(\mathfrak{t})$. If $u \in U(g)$ and if $f \in H^\infty$ then

$$d\pi_{\sigma_x}(u)f(k) = L(Ad(k^{-1})u^T)f_{\sigma_x}(k)$$
with $L$ the left action of $U(g)$ on $C^\infty(G, S)$. Also

$$Ad(k^{-1})u^T = \sum_{i,j,I} a_{i,j,I}(k)n_i a_j Y^I$$

finite sum. Thus

$$d\pi_{\sigma_z}(u)f(k) = \sum a_{i,j,I}(k) d\sigma_z(n_i a_j) \left( L(Y^I)f \right)(k)$$

$$= \sum a_{i,j,I}(k) d\sigma_z(n_i a_j) \left( Ad(k)^{-1} \left( Y^I \right)^T f \right)(k).$$

Writing

$$Ad(k)^{-1} \left( Y^I \right)^T = \sum_{|J| \leq |I|} b_{j,I}(k) Y^J$$

we have

$$d\pi_{\sigma_z}(u)f(k) = \sum_{i,j,I,J} b_{j,I}(k)a_{i,j,I}(k) d\sigma_z(n_i a_j) Y^J f(k).$$

Since the sum is finite all of the indices are bounded. Let $\omega$ be a compact subset of $Z$ then for each fixed $J$

$$\sum_{i,j,I} |b_{j,I}(k)a_{i,j,I}(k)| \| d\sigma_z(n_i a_j) \| \leq C^1_{u,\omega,J}, k \in K, z \in \omega.$$

Thus

$$\| d\pi_{\sigma_z}(u)f \|^2 \leq \sum C^1_{u,\omega,J} C^1_{u,\omega,J} \langle Y^I f, Y^J f \rangle$$

$$\leq C_{u,\omega} \| (1 + C_K)^l f \|^2$$

with $l$ the maximum of the $|J|$ for the multi-indices that appear in the formulas above. ■

### 6 Analytic families of $J$–modules

Notation as in the previous section. Throughout this section analytic will mean complex analytic in the context of a complex analytic manifold and real analytic in the contest of a real analytic manifold.
Thus \( k \) be a basis of \( V \) be given by \( \lambda \) be a family of objects in \( W(K,D) \) over \( X \) and define \( R_x \in W(K,D) \) to be the module with action \( \lambda(x,\cdot) \). Let
\[
V = \mathcal{H} \otimes E \otimes R
\]

\((K \text{ act by the tensor product action with its action on } E \text{ trivial})\) and let \( T_x : V \to J(R_x) \) be given by \( T_x(h \otimes e \otimes r) = \alpha_x(\text{symm}(h)e)(1 \otimes r) \) with \( \alpha_x \) the action of \( U(g_C) \) on \( J(R_x) \). If \( \lambda(x,y) = T_x^{-1}\alpha_x(y)T_x \) then \( (\lambda,V) \) is an analytic family of objects in \( H(g,K) \) based on \( X \).

**Proof.** Let \( \{h_i\} \) be a basis of \( \mathcal{H} \) such that for each \( i \) there exists \( \gamma \in \hat{K} \) such that \( h_i \in \mathcal{H}(\gamma) \), let \( e_j \) be a basis of \( E \), let \( r_m \) be a basis of \( R \) and let \( Y_1, ..., Y_n \) be a basis of \( \mathfrak{t} \). Then if \( y \in U(g_C) \)
\[
ysymm(h_i)e_jz^{L_1}y^{J_1} = \sum_{i_1,j_1,l_1} b_{i_1,j_1,l_1,i_1,j_1,l_1} \text{symm}(h_{i_1})e_{j_1}z^{L_1}y^{J_1}.
\]

Thus
\[
T_x^{-1}\alpha_x(y)T_x(h_i \otimes e_j \otimes r_k) = \sum b_{i_1,j_1,l_1,i_1,j_0,k_0}(y) h_{i_1} \otimes e_{j_1} \otimes (\lambda_x(z^{L_1})y^{J_1}r_k).
\]

The proposition follows. \( \blacksquare \)

**Theorem 2.3** implies

**Lemma 6.2** Let \( R \in W(K,D) \) then
\[
J(R)/n^{k+1}J(R) \cong (U(n)/n^{k+1}U(n)) \otimes E \otimes H \otimes R_{|M}
\]
as an \((n,M)\)-module with \( n \) and \( M \) acting trivially on \( E \otimes H \) and \( n \) acting trivially on \( R \).

Let \((\mu,R)\) be an analytic family of objects in \( W(K,D) \) over \( X \). Let \( R_x, x \in X \) be the object in \( W(K,D) \) with \( K \) acting by its action on \( R \) and \( D \) acting by \( \mu_x = \mu(x,\cdot) \).

**Proposition 6.3** Let \( p_{x,k} : J(R_x) \to (U(n)/n^{k+1}U(n)) \otimes E \otimes H \otimes R_{|M} \) be given by the projection of \( J(R_x) \) onto \( J(R)/n^{k+1}J(R) \) composed with the isomorphism of \( J(R)/n^{k+1}J(R) \) with \( (U(n)/n^{k+1}U(n)) \otimes E \otimes H \otimes R \). If \( v \in J(R_x) \) and \( u \in U(g_C) \) then the map
\[
x \mapsto p_{x,k}(uv)
\]
is analytic from \( X \) to \( (U(n)/n^{k+1}U(n)) \otimes E \otimes H \otimes R|_M \). In particular, if \( p_k \) is the canonical projection of projection of \( U(n) \otimes E \otimes H \otimes R|_M \) onto \( (U(n)/n^{k+1}U(n)) \otimes E \otimes H \otimes R|_M \) then define \( \sigma_{k,x}(q)p_k(v) = p_{x,k}(qv) \) for \( q \in U(L(Q)) \).

**Proof.** Let \( x_1, x_2, \ldots, x_r \) be a linearly independent set in \( U(n) \) that projects to a basis in \( U(n)/n^{k+1}U(n) \) and let \( x_{r+1}, \ldots \) be a basis of \( n^{k+1}U(n) \). Theorem 2.3 implies that if \( Y_1, \ldots, Y_n \) is a basis of \( \mathfrak{f} \) and \( h_1, \ldots, h_r \) is a basis for \( \text{symm}(E)\text{symm}(H) \) then if \( J \) is a multi-index of size \( n \) and \( I \) is a multi-index of size then the set of elements

\[
x_{i}z^{I}h_{m}Y^{J}
\]

is a basis of \( U(\mathfrak{g}_C)(z^I = z^{i_1}_1 \cdots z^{i_r}_r \) and the \( z_i \) are the generators of \( D \)). This implies that if \( u \in U(\mathfrak{g}_C) \) then

\[
ux_{i}z^{I}h_{m}Y^{J} = \sum_{t_1, t_1, t_1} a_{s_1, s_1, s_1, s_1} h_{i_1} x_{j_1} h_{i_1} Y^{J_1}.
\]

This implies that if we take a basis \( v_1, \ldots, v_d \) of \( R \) then the elements \( X^J h_i \otimes v_j \) form a basis of \( J(V^*_x) \). Thus if \( u \in U(\mathfrak{g}_C) \) then

\[
ux_{i}h_{t} \otimes v_j = \sum_{j_1, j_1, j_1} a_{s_1, s_1, s_1, s_1} h_{i_1} x_{j_1} h_{i_1} Y^{J_1} \otimes v_j = \sum_{j_1, j_1, j_1} a_{s_1, s_1, s_1, s_1} h_{i_1} x_{j_1} h_{i_1} Y^{J_1} \otimes v_j
\]

Now apply \( p_{k,x} \) getting the image of

\[
\sum_{s_1 \leq r} a_{s_1, s_1, s_1, s_1} h_{i_1} x_{j_1} h_{i_1} Y^{J_1}(z^{l_1}) Y^{J_1}(z^{l_1})
\]

The proposition follows from this formula. \( \blacksquare \)

If \( R \in W(K, D) \) the space \( J(R)/n^{s+1}J(R) \) has a natural structure of an \( M \) module and an \( n + a \) module. Since \( \text{dim } J(R)/n^{s+1}J(R) = \infty \) and \( AN \) is a simply connected Lie group \( J(R)/n^{s+1}J(R) \) has a natural structure of a finite dimensional continuous \( Q \)-module with action \( \sigma_{s,R} \). Let (as above) \( p_s \) denote the natural surjection

\[
p_s : J(R) \rightarrow J(R)/n^{s+1}J(R).
\]

If \( k \in K, v \in J(R) \), define \( S_{s,R}(v)(k) = p_{s,R}(kv) \), then \( S_{s,R}(v) \in I(\sigma_{s,R}|M) \) and it is easily seen that \( S_{s,R} \in \text{Hom}_{H(\mathfrak{g}, K)}(J(R), (\pi_{s,R}, I(\sigma_{s,R}|M))) \). Combining the above results we have
**Theorem 6.4** Let \((\mu, R)\) be an analytic (resp. continuous) family in \(W(K, D)\) based on the manifold \(X\). Let \((\lambda, V)\) be the analytic family (as in Theorem 6.1) corresponding to \(x \to J((\mu_x, R))\). Then recalling that \(V = H \otimes E \otimes R\) define \(T_s(x)(h \otimes e \otimes r) = S_{s-R_x}(\text{symm}(h)e \otimes r)\). Then \(T_s\) defines a homomorphism of the analytic family \((\lambda, V)\) to \((\xi, I(\sigma_{s,R_x}|M))\) (in the sense of Definition D.3) with \(\xi(x, y) = \pi_{\sigma_{s,R_x}}(y)\) and \(\sigma_{s,R_x}\) is defined as above.

We will use the notation \(J(R)\) for the analytic family associated with \(x \to J((\mu_x, R))\).

**7 Imbeddings of \(J\)–modules and their Hilbert family completions**

Let \(X\) be a connected real or complex analytic manifold and let \((\mu, R)\) be an analytic family of objects in \(W(K, D)\) based on \(X\). The purpose of this section is to prove

**Theorem 7.1** Let the representation of \(Q, \sigma_{k,x}\), on

\[ W_k = \left( U(n)/n^{k+1}U(n) \right) \otimes E \otimes H \otimes R_{|M} \]

be as in Proposition 6.3 and let \(T_k(x)\) be the analytic family as in Theorem 6.4. If \(\omega\) is a compact subset of \(X\) then there exists \(k_\omega\) such that if \(x \in \omega\) then \(T_k(x)\) is injective for \(k \geq k_\omega\).

**Proof.** This is a slight extension of a result in [HOW]. Given \(k\) then \((\sigma_{k,x}, W_k)\) as a composition series \(W_{k,x} = W_{k,x}^1 \supset W_{k,x}^2 \supset \ldots \supset W_{k,x}^r \supset W_{k,x}^{r+1} = \{0\}\) and each \(W_{k,x}^i/W_{k,x}^{i+1}\) is isomorphic with the representation \((\lambda_j, M)\) with \(\lambda_j \in \mathfrak{a}_C^{\ast}\) and \(\lambda_j(m) = a^{\mu+\nu_j(m)}\) with \(m \in M, a \in A\) and \(n \in N\). Also note that there is a natural \(Q\)–module exact sequence

\[ 0 \to \left( n^{k+1}U(n)/n^{k+2}U(n) \right) \otimes E \otimes H \otimes R_{|M} \to W_{k+1,x} \to W_{k,x} \to 0. \]

We may assume that the composition series is consistent with this exact sequence. This implies that the \(\nu_j\) that appear in \(W_k/W_{k+1}\) are of the form \(\mu + \alpha_1 + \ldots + \alpha_{k+1}\) with \(\alpha_i\) a restricted positive root (i.e. a weight of \(\mathfrak{a}\) on \(n\)).
Now consider the corresponding exact sequence of \((g, K)\)-modules.

\[(*) 0 \to I(\eta_{k,x}) \to I(\sigma_{k+1,x}) \to I(\sigma_{k,x}) \to 0.\]

The \((g, K)\)-modules \(I(\sigma_\nu)\) with \(\sigma\) an irreducible representation of \(M\) with Harish-Chandra parameter \(\Lambda_\sigma\) (for \(\text{Lie}(M)_C\)) and \(\nu \in a_C^*\) have infinitesimal character with Harish-Chandra parameter \(\Lambda_\sigma + \nu\). We are finally ready to prove the theorem.

Let \(C_\omega\) be the compact set \(\bigcup_{x \in \omega} \text{ch}(J(R_x))\). Let \(C_\omega = \bigcup_{i=1}^{k_\omega} (\Lambda_i + D_i)\) with \(D_i\) compact in \(a_C^*\) and \(k_\omega < \infty\). Assume that the result is false for \(\omega\). Then for each \(j\) there exists \(k \geq j\) and \(x\) such that \(\ker T_k(x) \neq 0\) but \(\ker T_{k+1}(x) = 0\). Label the Harish-Chandra parameters that appear in \(I(\sigma_\nu,x)\), \(\Lambda_1 + \nu_1, \ldots, \Lambda_s + \nu_s\) with \(\Lambda_i \in \text{Lie}(T)^*\) and \(\nu_i \in a_C^*\) (recall that we have fixed a maximal torus of \(M\)). The above observations imply that \(\text{ch}(J(R_x))\) contains an element of the form \(\Lambda + \nu_i + \beta_k\) with \(\beta_k\) a sum of \(k\) positive roots, \(\Lambda \in \text{Lie}(T)^*\) and \(1 \leq i \leq s\). We now have our contradiction \(\nu_i + \beta_k \in \bigcup D_i\) which is compact. But the set of \(\nu_i + \beta_k\) is unbounded.

**Theorem 7.2** Let \(U \subset Z\) be open with compact closure. There exists a continuous family \((\pi, H)\) of Hilbert representations of \(G\) (see Definition \[B.1\]) based on \(U\) such that the continuous family of \((g, K)\)–modules \((d\pi, H_\infty)\) is isomorphic with the analytic family \(z \mapsto J(L_z)\) of objects in \(\mathcal{H}(g, K)\) based on on \(U\) (thought of as a continuous family). Furthermore, the family \((\pi, H)\) is smoothable (see Definition \[??\]).

**Proof.** Let \(\gamma \in \hat{K}\) then Theorem \[2.3\] implies

\[\dim J(L_z)(\gamma) = \dim \gamma \dim \text{Hom}_K(V_\gamma, \mathcal{H} \otimes L).\]

for every \(z \in Z\). In particular it is independent of \(z\). Theorem \[7.1\] implies that there exists \(k\) and for each \(u \in U\) the map

\[T_{k,L_u} : J(L_u) \to I(\sigma_{k,L_u})\]

is injective. Note that the space of \(K\)-finite vectors in \(I(\sigma_{k,L_u})\) is the \(K\)-finite induced representation \(Ind^K_M(\sigma_{k,L_u})\) and hence independent of \(u\). Let \((H_1, \langle \ldots, \ldots \rangle)\) be the Hilbert space completion of \(Ind^K_M(\sigma_{k,L_u})\) corresponding to unitary induction from \(M\) to \(K\). This gives an smoothable analytic family of Hilbert representations of \(G\) (Proposition \[5.3\], \(\mu_z\). Proposition \[G.3\] now implies the result. ■
8 The main theorem

Theorem 8.1 Let \((\pi,V)\) be an analytic family of objects in \(\mathcal{H}(\mathfrak{g},K)\) based on the analytic manifold \(X\). Let \(x_\alpha \in X\) then there exists, \(U\), an open neighborhood of \(x_\alpha\) in \(X\), and a smoothable, continuous family of Hilbert representations \((\mu_U,H_U)\) such that the family \((d\mu_U,(H_U)_K^\infty)\) is isomorphic with \((\pi|_U,V)\) (as a continuous family).

Proof. Let \(U_1\) be an open neighborhood of \(x_\alpha\) in \(X\) with compact closure. Then Theorem 8.6 implies that there exists \(F_0 \subset \hat{K}\) a finite subset such that \(\pi_x(U(\mathfrak{g}_C))\sum_{\gamma \in F_0} V(\gamma) = V\). Let \(R^0 = \sum_{\gamma \in F_0} V(\gamma)\). \(R^0\) is invariant under the action \(\pi_x(D)\) for all \(x \in X\). This implies that \(((\pi|_U)|_D,R^0)\) defines an analytic family of objects in \(W(K,D)\) based on \(U_1\). Let \(J(R^0)\) be the corresponding \(J\)-family. Then we have the surjective analytic homomorphism of families

\[
T_0 : J(R^0) \rightarrow V_{|U} \rightarrow 0
\]

with \(T_0(x)\) mapping \(J(R^0)\) onto \(V\) for all \(x \in U_1\). Let \((\sigma,(H^0,(...,...)))\) be the smoothable, continuous family of Hilbert representations based on \(U_1\) corresponding to \(J(R^0)\) as in Theorem 7.2. Let \(U\) be an open neighborhood of \(x_\alpha\) contained in \(U_1\) such that \(\overline{U}\) is contractible. The theorem now follows from Proposition G.4. ■

The main result is

Theorem 8.2 Let \((\pi,V)\) be a holomorphic family of objects in \(\mathcal{H}(\mathfrak{g},K)\) based on the connected complex manifold \(X\). Then there exists a holomorphic family of smooth Fréchet representations, \((\lambda,W)\) based on \(X\) that globalizes \((\pi,V)\).

Proof. The above theorem implies that there is an open covering, \(\{U_\alpha\}\), of \(X\) and for each \(\alpha\) a continuous family of smoothable, admissible Hilbert representations based on \(U_\alpha\), \((\sigma_\alpha,H_\alpha)\), such that \(((d\sigma_\alpha)_x,(H_\alpha)_K) = (\pi_x,V), x \in U_\alpha\). Proposition G.2 combined with Theorem G.3 implies that \((\sigma_\alpha,H_\alpha^\infty)\) (here \(H_\alpha^\infty\) is the space of \(C^\infty\) vectors with respect to \(K\)) is a holomorphic family of smooth Fréchet representations of \(G\) based on \(U_\alpha\). The isomorphism of categories implies that if \(x \in U_\alpha \cap U_\beta\) then \(H_\alpha^\infty = H_\beta^\infty\) and \((\sigma_\alpha)_x(g)|_{H_\alpha^\infty} = (\sigma_\beta(g))_x|_{H_\beta^\infty}\) for all \(g \in G\). Thus we can define \(W = H_\alpha^\infty\)
the common value for all $\alpha$ (since $X$ is connected) and if $x \in X$ then
\[ \sigma_x(g) = (\sigma_{\alpha})_x(g) \] for $\alpha$ such that $x \in U_\alpha$. ■

This can be interpreted in the following way:

**Corollary 8.3** Let $T$ be the inverse functor to the $K$–finite functor $\mathcal{H}F(G) \to \mathcal{H}(\mathfrak{g},K)$ and let $(\pi,V)$ is an holomorphic family of objects in $\mathcal{H}(\mathfrak{g},K)$ over the connected complex manifold $X$. If $T((\pi_x,V)) = (\lambda_x,\nabla_x)$ then

1. For all $x,y \in X, V_x = V_y$ as subspaces of $\prod_{\gamma \in \hat{K}} V(\gamma)$ and as Fréchet spaces. Set $V$ equal to the common value.

2. The map $x,g,v \mapsto \lambda_x(g)v$ is continuous from $X \times G \times V$ to $\overline{V}$, linear in $v$ and $C^\infty$ in $g$ and holomorphic in $x$.

**Appendices**

**A  $G$–$C^\infty$ vectors and $K$–$C^\infty$ vectors**

Let $G$ be a real reductive group with a fixed maximal compact subgroup $K$ and let $\theta$ be the corresponding Cartan involution. Fix a symmetric bilinear form $B$ on $\text{Lie}(G)$ such that $\langle X,Y \rangle = -B(\theta X,Y)$ is positive definite. Let $C$ and $C_K$ be the Casimir operators of $G$ and $K$ respectively corresponding to $B$ and we set $\Delta = C - 2C_K$. We observe that $\Delta = \sum X_i^2$ for $X_1, \ldots, X_m$ an orthonormal basis of $\text{Lie}(G)$ relative to $\langle \ldots, \ldots \rangle$. As a left invariant operator on $G$, $\Delta$ is an elliptic and invariant under $K$. Let $(\pi,H)$ be a Hilbert representation of $G$ and set $V = (H^\infty)_K$. Let $Z$ be the completion of $V$ relative to the seminorms $q_l(v) = \|\Delta^l v\|$, $l = 0, 1, 2, \ldots$ Then since $q_0 = \|\ldots\|$, $Z$ can be looked upon as a subspace of $H$. Also $H^\infty$ is the completion of $V$ using the seminorms $s_x(v) = \|xv\|$ with $x \in U(\mathfrak{g})$. Thus $Z \supset H^\infty$.

**Lemma A.1** $Z = H^\infty$. Furthermore, the topology on $H^\infty$ is given by the semi-norms $q_l$. That is $H^\infty = H^{\infty_K}$.

**Proof.** We note that the second assertion is a direct consequence of the closed graph theorem (c.f. [1]) and the first assertion. We will now prove the first assertion. Let $v \in Z \subset H$. We must prove that $v \in H^\infty$. Let $v_j \in V$ be a sequence converging to $v$ in the topology of $Z$. Let $w \in H$ then for all $j$

\[ \Delta^k(\pi(g)v_j,w) = (\pi(g)\Delta^k v_j,w). \]
Set $p_k(v) = \sum_{j=0}^{k} q_k(v)$. Noting that $p_l(\Delta^k v) + p_{k-1}(v) = p_{l+k}(v)$ and $p_l(v) \leq p_{l+1}(v)$, we see that for fixed $k$ the sequence $\{\Delta^k v\}_j$ converges to $u_k$ in $Z$.

We assert that the function $g \mapsto (\pi(g)v, w)$ is $C^\infty$. Since $w \in H$ is arbitrary, this would imply that the map $g \mapsto \pi(g)v$ is weakly $C^\infty$. But a weakly $C^\infty$ map of a finite dimensional manifold into a Hilbert space is strongly $C^\infty$ (c.f. [G2]). This is exactly the statement that $v$ is a $C^\infty$ vector. We now prove the assertion. We first show that if we look upon the continuous function $\Delta^k f(g) = (\pi(g)v, w)$ as a distribution on $G$ (using the Haar measure on $G$) then in the distribution sense

$$\Delta^k h(g) = (\pi(g)u_k, w).$$

Indeed, let $f \in C^\infty_c(G)$ then

$$\int_G h(g)\Delta^k f(g)dg = \lim_{j \to \infty} \int_G (\pi(g)v_j, w)\Delta^k f(g)dg =$$

$$\lim_{j \to \infty} \int_G \Delta^k(\pi(g)v_j, w)f(g)dg = \lim_{j \to \infty} \int_G (\pi(g)\Delta^k v_j, w)f(g)dg =$$

$$\int_G (\pi(g)u_k, w)f(g)dg$$

as asserted. Since $\Delta$ is elliptic, local Sobolev theory (c.f. [F, Chapter 6]) implies that $h \in C^\infty(G)$. ■

**Proposition A.2** If $(\pi, H)$ is an admissible Hilbert representation of $G$ such that there exists a polynomial

$$f(x) = x^m - \sum_{j=0}^{m-1} c_j x^j$$

such that $f(C) = 0$ on $H^\infty$ then the topology of $H^\infty$ is given by the seminorms $p_l(v) = \|(I + C_K)^l v\|$, $l = 0, 1, 2, \ldots$ That is, the $K - C^\infty$ vectors, $H^\infty_K$ are the same as the $G - C^\infty$ vectors, $H^\infty$.

**Proof.** This result will be proved by induction on $m$. If $m = 1$ then $C$ acts by $c = c_0$ on $H^\infty$. Note that $\Delta = C - 2C_K$. So if $v \in H^\infty$

$$\|\Delta^k v\| = \left\| \sum_{j=0}^{k} (-2)^j \binom{k}{j} C^{k-j} C^j_K v \right\| \leq$$
\[
\sum_{j=0}^{k} (2)^j \binom{k}{j} |c|^{k-j} \|C_K^j v\| \leq \sum_{j=0}^{k} (2)^j \binom{k}{j} |c|^{k-j} \|(1 + C_K)^j v\|.
\]

Now assume the result if the degree is \(m - 1 \geq 1\). Let \(H^\omega\) denote the space of analytic vectors in \(H^\infty\). Then the \(K\)-finite vectors \(V = H_K\) are contained in \(H^\omega\). If \(c \in \mathbb{C}\) and \(V_c = \{v \in V | Cv = cv\} \neq 0\) then let \(H_1\) be the Hilbert space completion of \(V_c\) then \(H_1\) is \(G\)-invariant and \((H_1)_K = V_c\) and \((H/H_1)_K = H_K/V_c\). The first part of the proof implies that the seminorms \(p_l\) define the topology on \(H_1^\infty\). The correspondence \(U \to U^\infty\) is an exact functor from the category of strongly continuous representations of \(G\) to the category of smooth Fréchet modules ([Wa], Proposition 4.4.1.11 p. 260). Setting \(g(x) = \frac{f(x)}{x - c}\) we have \(g(C)\) is zero on \((H/H_1)^\infty\) (since it is 0 on \(V/V_c\)). Thus we have the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & H_1^\infty & \to & H^\infty & \to & (H/H_1)^\infty & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_1^\infty K & \to & H^\infty K & \to & (H/H_1)^\infty K & \to & 0
\end{array}
\]

with the right-most and left-most vertical arrows isomorphisms. This implies that the middle vertical arrow is also an isomorphism completing the induction. ■

**Corollary A.3** If \((\pi, H)\) is an admissible finitely generated Hilbert representation of \(G\) then the \(K\)-\(C^\infty\) vectors are the same as the \(G\)-\(C^\infty\) vectors.

**Proof.** There exists a finite subset \(F \subset \hat{K}\) such that \(U(g) \sum_{\gamma \in F} H(\gamma)\). Clearly, \(C \sum_{\gamma \in F} H(\gamma) \subset \sum_{\gamma \in F} H(\gamma)\). Let \(p(x)\) be the minimal polynomial of \(C \sum_{\gamma \in F} H(\gamma)\). Then \(p(C) = 0\) on \(H_K\) and hence on \(H^\infty\). ■

**B Hilbert families**

Let \(G\) be a locally compact topological group and let \(X\) be a locally compact metric space. All Hilbert spaces in this appendix (indeed, in this paper) will be separable.
Definition B.1 A continuous family of Hilbert representations of $G$ over $X$ is a pair $(\sigma, H)$ with $H$ a Hilbert space and $\sigma : X \times G \to GL(H)$ (the continuous invertible operators on $H$ with the strong operator topology) a continuous map such that $\sigma_x(g) = \sigma(x, g)$ defines a representation of $G$ for every $x \in G$.

The following lemma is Lemma 1.1.3 in [RRG] taking into account dependence on parameters. The proof is essentially the same using the local compactness of $X$.

Lemma B.2 Let $X$ be a locally compact metric space and let $H$ be a Hilbert space. Assume that for each $x \in X$, $\pi_x : G \to GL(H)$ (bounded invertible operators such that

1) If $\omega \subset X$ and $\Omega \subset G$ are compact subsets of $X$ and of $G$ respectively then there exists a constant $C_{\omega,\Omega}$ such that $\|\pi_x(g)\| \leq C_{\omega,\Omega}$ for $x \in \omega, g \in \Omega$.

2) The map $x, g \to \langle \pi_x(g)v, w \rangle$ is continuous for all $v, w \in H$.

Then $(\pi, H)$ is a continuous family of representations of $G$ based on $X$ and conversely if $(\pi, H)$ is a continuous family of Hilbert representations then 1) and 2) are satisfied.

An immediate corollary is

Corollary B.3 Let $(\pi, H)$ be an admissible, continuous family of Hilbert representations of $G$ based on the locally compact metric space $X$. Set for each $x \in X$, $\hat{\pi}_x(g) = \pi_x(g^{-1})^*$ then $(\hat{\pi}, H)$ is a continuous, admissible family of Hilbert representations of $G$ based on $X$.

C Norms

Let $\|g\|$ be a norm on $G$, that is a continuous function from $G$ to $\mathbb{R}_{>0}$ (the positive real numbers) such that

1. $\|k_1gk_2\| = \|g\|, k_1, k_2 \in K, g \in G$,
2. $\|xy\| \leq \|x\| \|y\|, x, y \in G$,
3. The sets $\|g\| \leq r < \infty$ are compact.
4. If $X \in p$ then if $t \geq 0$ then $\log \|\exp tX\| = t \log \|\exp X\|$.

If $(\sigma, V)$ is a finite dimensional representation of $G$ with compact kernel. Assume also that $(..., ...)$ is an inner product on $V$ that is $K$–invariant and is such that the elements $\sigma(\exp(X))$ are self adjoint for $X \in p_o$. If $\|\sigma(g)\|$ is
the operator norm of \( \sigma(g) \) then \( \|g\| = \|\sigma(g)\| \) is a norm on \( G \). Taking the representation on \( V \oplus V ^* \) given by

\[
\begin{bmatrix}
\sigma(g) \\
\sigma(g^{-1})^*
\end{bmatrix}
\]

then we may (and will) assume in addition

5. \( \|g\| = \|g^{-1}\|. \)

Note that 5. implies that \( \|g\| \geq 1 \).

Using the same proof as Lemma 2.A.2.1 in [RRG] (which we give for the sake of completeness) one can prove

**Lemma C.1** If \( (\pi, H) \) is a continuous family of Hilbert representations over \( X \) and if \( \omega \) is a compact subset of \( X \) then there exist constants \( C_\omega, r_\omega \) such that

\[\|\pi_\omega(g)\| \leq C_\omega \|g\|^{r_\omega}.\]

**Proof.** Let

\[B_1 = \{g \in G | \|g\| \leq 1\}.\]

If \( v \in H \) and \( (x, g) \in \omega \times B_1 \) then \( \sup \|\pi_\omega(x)g\| < \infty \) by strong continuity. The principle of uniform boundedness (c.f. [RS], III.9, p.81) implies that there exists a constant, \( R \), such that \( \|\pi_\omega(x)g\| \leq R \) for \( (x, g) \in \omega \times B_1 \). Let \( r = \log R \). In particular if \( k \in K \) then

\[\|\pi_\omega(kg)\| \leq \|\pi_\omega(k)\| \|\pi_\omega(g)\| \leq R \|\pi_\omega(g)\|.\]

Also,

\[\|\pi_\omega(g)\| = \|\pi_\omega(k^{-1})\pi_\omega(kg)\| \leq R \|\pi_\omega(kg)\|.\]

Thus for all \( k \in K, g \in G \)

\[R^{-1} \|\pi_\omega(g)\| \leq \|\pi_\omega(kg)\| \leq R \|\pi_\omega(g)\|.\]

Let \( X \in \mathfrak{p}, X \neq 0 \) and let \( j \) be such that

\[j < \log \|\exp X\| \leq j + 1\]

then

\[\log \|\pi_\omega(\exp X)\| \leq \log \|\pi_\omega(\exp (X/j + 1))^{j+1}\| \leq r(j + 1) \leq r + r \log \|\exp X\|.\]
Thus
\[ \| \pi_x (\exp X) \| \leq R \| \exp X \|^r, \quad X \in \mathfrak{p}. \]
If \( g \in G \) then \( g = k \exp X \) with \( k \in K \) and \( X \in \mathfrak{p} \) so
\[ \| \pi_x (g) \| = \| \pi_x (k \exp X) \| \leq R^2 \| \exp X \|^r = R^2 \| g \|^r. \]
Take \( C_\omega = R^2 \) and \( r_\omega = r. \]

D Continuous and analytic families of \((\mathfrak{g}, K)\) modules.

Let \( G \) be a reductive group with fixed maximal compact subgroup, \( K \). In this section \( X \) will denote a connected, paracompact real analytic or complex manifold.

**Definition D.1** If \( V \) is a vector space over \( \mathbb{C} \) then a continuous, real analytic or holomorphic function from \( X \) to \( V \) is a map \( f : X \to V \) such that for each \( x \in X \) there exists, \( U \), an open neighborhood of \( x \) in \( X \) such that the following two conditions are satisfied:
1. \( \dim \text{span}_\mathbb{C} \{ f(x) \mid x \in U \} < \infty. \)
2. \( f : U \to \text{span}_\mathbb{C} \{ f(x) \mid x \in U \} \) is respectively continuous, real analytic or continuous.

**Definition D.2** A holomorphic, analytic or continuous family of admissible \((\mathfrak{g}, K)\)-modules over \( X \) is a pair, \( (\mu, V) \), of an admissible \((\mathfrak{t}, K)\)-module, \( V \), and
\[ \mu : X \times U(\mathfrak{g}) \to \text{End}(V) \]
such that \( x \mapsto \mu(x, y)v \) is respectively holomorphic, analytic or continuous for all \( y \in U(\mathfrak{g}) \), \( v \in V \) and if we set \( \mu_x(y) = \mu(x, y) \) for \( y \in U(\mathfrak{g}) \) then \( (\mu_x, V) \) is an admissible \((\mathfrak{g}, K)\)-module. It will be called a family of objects in \( \mathcal{H}(\mathfrak{g}, K) \) if each \( (\mu_x, V) \) is finitely generated.

**Definition D.3** If \( (\lambda, V) \) and \( (\mu, W) \) are analytic or continuous families of objects in \( \mathcal{H}(\mathfrak{g}, K) \) over \( X \) then a homomorphism of the family \( (\lambda, V) \) to \( (\mu, W) \) is a map
\[ T : X \to \text{Hom}_\mathbb{C}(V, W) \]
such that
1. \( x \mapsto T(x)v \) is an analytic or continuous map of \( X \) to \( W \) for all \( v \in V \).
2. \( T(x) \in \text{Hom}_{H(\mathfrak{g},K)}(V_x, W_x) \) with \( V_x = (\lambda_x, V), W_x = (\mu_x, W) \).

**Lemma D.4** Let \((\pi, H)\) be a continuous family of admissible Hilbert representations of \( G \) over \( X \) and denote by \( d\pi_x \) the action of \( \mathfrak{g} \) on \( H_K^\infty \) (the \( K \)-finite \( C^\infty \)-vectors). Then \((d\pi, H_K)\) is a continuous family of admissible \((\mathfrak{g}, K)\)-modules based on \( X \).

**Proof.** If \( \gamma \in \hat{K} \) then \( C_c^\infty(\gamma; G) \) denotes the space of all \( f \in C_c^\infty(G) \) such that
\[
\int_K \chi_\gamma(k)f(k^{-1}g)dk = f(g), g \in G
\]
with \( \chi_\gamma \) the character of \( \gamma \). Then
\[
H(\gamma) = \pi_x(C_c^\infty(\gamma; G))H.
\]
We also note that if \( Y \in \mathfrak{g}, f \in C_c^\infty(\gamma; G) \) and \( v \in H \) then
\[
d\pi_x(Y)\pi_x(f)v = \pi_x(Yf)v
\]
with \( Yf \) the usual action of \( Y \in \mathfrak{g} \) on \( C^\infty(G) \) as a left invariant vector field. Thus, if \( v \in H_K \) and \( y \in U(\mathfrak{g}_C) \) then the map
\[
x \mapsto d\pi_x(y)v
\]
is continuous. \( \blacksquare \)

**E Some results of Vincent van der Noort**

Throughout this section \( Z \) will denote a connected real or complex analytic manifold. We will use the terminology analytic to mean complex analytic or real analytic depending on the context.

We continue the notation of the previous sections. In particular \( G \) is a real reductive group of inner type.

We denote (as is usual) the standard filtration of \( U(\mathfrak{g}) \), by
\[
... \subset U^j(\mathfrak{g}) \subset U^{j+1}(\mathfrak{g}) \subset ...
\]
Let $V$ be an admissible $(\text{Lie}(K), K)$ module. We note that if $E \subset V$ is a finite dimensional $K$–invariant subspace of $V$ then there exists a finite subset $F_{j,E} \subset \hat{K}$ such that

$$U^j(g) \otimes E \cong \sum_{\gamma \in F_{j,E}} m_{\gamma,j} V_{\gamma}.$$ 

If $v \in V$ we denote by $E_v$ the span of $Kv$ in $V$.

The purpose of this section is to prove a theorem of van der Noort which first appeared in his thesis [VdN]. We include the details only because he is not expected to publish it. In his thesis he studied the holomorphic case. Our exposition follows his original line.

Fix a maximal torus, $T$, of $M$ then $\mathfrak{h}_o = \text{Lie}(T) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. As usual, set $\mathfrak{h}$ equal to the complexification of $\mathfrak{h}_o$. We parametrize the homomorphisms of $Z(\mathfrak{g})$ to $\mathbb{C}$ by $\chi_\Lambda$ for $\Lambda \in \mathfrak{h}^*$ using the Harish–Chandra parametrization. Since $M$ is compact we endow $\hat{M}$ with the discrete topology. Note that if $C$ is a compact subset (sorry of the over use of $C$ the Casimir operator will not appear in this section) of $\hat{M} \times \mathfrak{a}^*$ then there exist a finite number of elements $\xi_1, \ldots, \xi_r \in \hat{M}$ and compact subsets, $D_j$, of $\mathfrak{a}^*$ such that

$$C = \bigcup_{j=1}^r \xi_j \times D_j.$$ 

If $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*_C$ then set $\sigma_{\xi,\nu}(\text{man}) = \xi(m)a^{\nu+\rho} (\rho(H) = \frac{1}{2} tr(adH|_{\text{Lie}(\mathfrak{N})}))$, $H \in \mathfrak{a}$, $a^\nu = \exp(\nu(H)) a = \exp(H)$, $\xi$ is taken to be a representative of the class $\xi$. $H^{\xi,\nu}$ is $I(\sigma_{\xi,\nu})$ which equals as a $K$–module $H^\xi = \text{Ind}_M^K(\xi)$. If $f \in H^\xi$ set $f_\nu(nak) = a^{\nu+\rho} f(k), n \in N, a \in A, k \in K$. $A_\rho(\nu)$ is the corresponding Kunze-Stein intertwining operator (c.f. [W1], 8.10.18. p.241).

**Proposition E.1** Let $\xi \in \hat{M}$ and let $\Omega \subset \mathfrak{a}^*_C$ be compact. There exists $F \subset \hat{K}$ such that $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( \sum_{\gamma \in F} H^\xi(\gamma) \right) = H^\xi$ for all $\nu \in \Omega$.

The proof of this result will use the following

**Lemma E.2** If $\nu_o \in \mathfrak{a}^*_C$ then there exists an open neighborhood of $\nu_o$, $U_{\nu_o}$, and a finite subset $F = F_{\nu_o}$ of $\hat{K}$ such that $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( \sum_{\gamma \in F} H^\xi(\gamma) \right) = H^\xi$ for all $\nu \in U_{\nu_o}$.

**Proof.** If $\gamma \in \hat{K}$ fix $W_\gamma \in \gamma$. If $\text{Re}(\nu, \alpha) > 0$ for all $\alpha \in \Phi^+$ and if $\gamma \in \hat{K}$ and $A_{\overline{\nu}}(\nu) H^\xi(\gamma) \neq 0$ then $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( H^\xi(\gamma) \right) = H^\xi$ (c.f. [RRG], Theorem 5.4.1
(1)). Fix such a $\gamma_\nu$ (which always exists since the operator $A_\nu(\nu) \neq 0$), take $F_\nu = \{\gamma_\nu\}$ and $U_\nu$ an open neighborhood of $\nu$ such that $A_\nu(\mu)H^\xi(\gamma_\nu) \neq 0$ for $\mu \in U$. Let $\nu \in \mathfrak{a}_C^*$ be arbitrary. There exists a positive integer, $k$, such that $\Re(\nu + kp, \alpha) > 0$ for all $\alpha \in \Phi^+$ and such that $kp$ is the highest weight of a finite dimensional spherical representation, $V^{kp}$, of $G$ relative to $\mathfrak{a}$. The lowest weight of $V^{kp}$ relative to $\mathfrak{a}$ is $-kp$ and $M$ acts trivially on that weight space thus $H^\xi_{K}(\mu) \otimes V^{kp}$ has $H^\xi_{K}$ as a quotient representation (see [W1],8.5.14,15). Take $F_\nu$ to be the set of $K$–types that occur in both $W_{\nu+k\rho} \otimes V^{kp}$ and $H^\xi_{K}$ and $U_\nu = U_{\nu+k\rho} - k\rho$. 

We now prove the proposition. By the lemma above for each $\nu \in \Omega$ there exists $F_\nu$ and $U_\nu$ as in the statement of the lemma. The $U_\nu$ form an open covering of $\Omega$ which is assumed to be compact. Thus there exist a finite number $\nu_1, ..., \nu_r \in \Omega$ such that

$$\Omega \subset \bigcup_{i=1}^{r} U_{\nu_i}.$$ 

Take $F = \bigcup_{i=1}^{r} F_{\nu_i}$. This proves the proposition.

**Lemma E.3** Let $\chi_{\xi,\nu}$ denote the infinitesimal character of $\pi_{\xi,\nu}$. If $C$ is a compact subset of $\mathfrak{h}^*$ then

$$\{(\xi, \nu) \in \hat{M} \times \mathfrak{a}_C^* | \chi_{\xi,\nu} = \chi_\Lambda, \Lambda \in C\}$$ 

is compact.

**Proof.** Fix a system of positive roots for $(M^0, T)$ ($M^0$ the identity component of $M$). If $\lambda_\xi$ is the highest weight of $\xi$ relative to this system of positive roots and if $\rho_M$ is the half sum of these positive roots then $\chi_{\xi,\nu} = \chi_\Lambda$ with $\Lambda = \lambda_\xi + \rho_M + \nu$. This implies the lemma. 

The following result is the reason for the assumption of analyticity.

**Lemma E.4** Let $(\pi, V)$ be an analytic family of admissible $(\mathfrak{g}, K)$ modules over $\mathbb{Z}$. Assume that $z_0 \in \mathbb{Z}$ is such that $(\pi_{z_0}, V)$ is finitely generated. If $T$ is an element of $Z(\mathfrak{g})$ there exist analytic functions $a_0, ..., a_{n-1}$ on $\mathbb{Z}$ such that if

$$f(z, x) = x^n + \sum_{j=0}^{n-1} a_j(z)x^j.$$ 

for $z$ then $f(z, \pi_z(T)) = 0$. 

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Proof. Let $F$ be a finite number of elements of $\hat{K}$ such that $\pi_{z_0}(U(\mathfrak{g})) \sum_{\gamma \in F} V(\gamma) = V$. Let $L = \sum_{\gamma \in F} V(\gamma)$. Then we define the functions $a_j$ the by the formula

$$f(z, x) = \det(x I - \pi_z(T)|_L) = x^n + \sum_{j=0}^{n-1} a_j(z)x^j.$$ 

The Cayley-Hamilton theorem implies that $h(z) = T^n + \sum_{j=0}^{n-1} a_j(z)T^j \in Z(\mathfrak{g})$ vanishes on $L$. Let $\gamma \in \hat{K}$ then there exist $x_1, ..., x_r \in U(\mathfrak{g})$ and $v_1, ..., v_r \in L$ such that $\{\pi_{z_0}(x_i)v_i\}_{i=1}^r$ is a basis of $V(\gamma)$. Let $P_\gamma$ be the projection onto the $\gamma$–isotypic component of $V$. Thus

$$(P_\gamma \pi_z(x_1)v_1) \wedge (P_\gamma \pi_z(x_2)v_2) \wedge \cdots \wedge (P_\gamma \pi_z(x_r)v_r) \in \wedge^r V(\gamma)$$

(a one dimensional space) is non–zero for $z = z_0$. This implies that there exists an open neighborhood, $U$, of $z_0$ in $\Omega$ such that

$$P_\gamma \pi_z(x_i)v_i, P_\gamma \pi_z(x_2)v_2, ..., P_\gamma \pi_z(x_r)v_r$$

is a basis of $V(\gamma)$ for $z \in U$. That

$$h(z)P_\gamma \pi_z(x_i)v_i = P_\gamma \pi_z(x_i)h(z)v_i = 0$$

implies that $h(z)V(\gamma) = 0$ for $z \in U$. The connectedness of $Z$ and the analyticity imply that $h(z)V(\gamma) = 0$ for $z \in Z$. Thus $h(z) = 0$ for all $z \in Z$. This proves the Lemma. 

If $V$ is a $(\mathfrak{g}, K)$–module then set $ch(V)$ equal to the set of $\Lambda \in \mathfrak{h}^*$ such that there exists $v \in V$ with $Tv = \chi_\Lambda(T)v$ for all $T \in Z(\mathfrak{g})$.

Corollary E.5 Keep the notation and assumptions of the previous lemma, If $\omega \subset Z$ is compact then there exists a compact subset $C_\omega$ of $\mathfrak{h}^*$ such that $ch(\pi_z, V) \subset C_\omega$ for all $z \in \omega$.

Proof. Let $T_1, ..., T_m$ be a generating set for $Z(\mathfrak{g})$ and let $f_j(z, x)$ be the function in the previous lemma corresponding to $T_j$. Then

$$f_j(z, x) = x^{n_j} + \sum_{i=0}^{n_j-1} a_{j,i}(z)x^i.$$
with $a_{j,i}$ analytic in on $Z$. If $\chi_{\Lambda} \in \text{ch}(\pi_z,V)$ then

$$|\chi_{\Lambda}(T_j)| \leq \max_{0 \leq i < n_j} |a_{j,i}(z)| + 1$$

(c.f. [RRG], 7.A.1.3). If $L \subset Z$ is compact then there exists a constant $r < \infty$ such that $|a_{j,i}(z)| \leq r$ for all $i, j$ and $z \in L$. This implies the corollary.

**Theorem E.6** Let $(\pi,V)$ be an analytic family of admissible $(g,K)$ modules over $Z$. Assume that there exists $z_0 \in Z$ such that $(\pi_{z_0},V)$ is finitely generated. If $\omega$ is a compact subset of $Z$ then there exists $S_{\omega} \subset \hat{K}$ a finite subset such that if $y \in \omega$ then

$$\pi_y(U(g)) \left( \sum_{\gamma \in S_{\omega}} V(\gamma) \right) = V.$$

**Proof.** Let $C_{\omega}$ as in the above corollary for $\omega$. Let

$$X = \{ (\xi, \nu) \in \hat{M} \times \mathfrak{a}_C^* | \chi_{\xi,\nu} = \chi_{\Lambda}, \Lambda \in C_{\omega} \}.$$

$X$ is compact so there exist $\xi_1, ..., \xi_r \in \hat{M}$ and $D_1, ..., D_r$, compact subsets of $\mathfrak{a}_C^*$, such that $X = \bigcup_j \xi_j \times D_j$. Let $S_j \subset \hat{K}$ be the finite set corresponding to $\xi_j \times D_j$ in Proposition E.1. Set $S_{\omega} = \cup S_j$. Let $L_1 \subset L_2 \subset ... \subset L_j \subset ...$ be an exhaustion of the $K$–types of $V$ with each $L_j$ finite.

We will use the notation $V_y$ for the $(g,K)$–module $(\pi_y,V)$. Let $y \in C$. Set $W_j = \pi_y(U(g)) \left( \sum_{\gamma \in L_j} V(\gamma) \right) \text{then } W_j \subset W_{j+1}$ and $\cup W_j = V$. Each $W_j$ is finitely generated and admissible, hence of finite length. Therefore $V_y$ has a finite composition series

$$0 = V_y^0 \subset V_y^1 \subset ... \subset V_y^N$$

or a countably infinite composition series

$$0 = V_y^0 \subset V_y^1 \subset ... \subset V_y^n \subset V_y^{n+1} \subset ...$$

with $V_y^i/V_y^{i-1}$ irreducible. Thus by the dual form of the subrepresentation theorem there exists for each $i, \xi_i \in \hat{M}$ and $\nu_i \in \mathfrak{a}_C^*$ so that $V_{y_i}/V_y^{i-1}$ is a quotient of $(\pi_{\xi_i,\nu_i},H^{\xi_i,\nu_i})$. Observe that $(\xi_i,\nu_i) \in X$. Thus $V_y^i/V_y^{i-1}(\gamma_i) \neq 0$ for some $\gamma_i \in S_{\omega}$. Let $L$ be a quotient module of $V_y$. Then $L = V_y/U$ with $U$ a submodule of $V_y$. There must be an $i$ such that $V_y^i/(V_y^{i-1} \cap U) \neq 0$. Let
i be minimal subject to this condition. Then $V^i_y \subset U$. Thus $V^i_y/V^{i-1}_y$ is a submodule of $L$. Hence $M(\gamma) \neq 0$ for some $\gamma \in S_\omega$. This implies that

$$\pi_y(U(\mathfrak{g}))(\sum_{\gamma \in S_\omega} V(\gamma)) = V.$$ 

Indeed, we have shown that

$$\left(V_y/\pi_y(U(\mathfrak{g}))\left(\sum_{\gamma \in S_\omega} V(\gamma)\right)\right)(\gamma) = 0, \gamma \in S_\omega.$$ 

\[\blacksquare\]

**Corollary E.7** *(To the proof)* Let $(\pi, V)$ be an analytic family of finitely generated admissible $(\mathfrak{g}, K)$ modules over $Z$. Let $W \subset Z$ be compact. Let for each $z \in W$, $U_z$ be a $(\mathfrak{g}, K)$–submodule of $V_z$. Then there exists a finite subset $F_W \subset \hat{K}$ such that

$$\pi_z(U(\mathfrak{g}))\left(\sum_{\gamma \in F_W} U_z(\gamma)\right) = U_z.$$ 

**Proof.** In the proof of the theorem above all that was used was that the set of possible infinitesimal characters is compact. \[\blacksquare\]

**F Continuous and Holomorphic families of smooth Fréchet representations.**

Let $G$ be a reductive group with fixed maximal compact subgroup, $K$. Let $\mathcal{F}(G)$ denote the category of smooth Fréchet representations (here smooth means that the map $g \mapsto \pi(g)v$ is $C^\infty$).

**Definition F.1** A continuous family of objects in $\mathcal{F}(G)$ over a metric space $X$ is a pair $(\pi, V)$ of a Fréchet space $V$ and a continuous map

$$\pi : X \times G \to \text{End}(V)$$ 

(here $\text{End}(V)$ is the algebra of continuous operators on $V$ with the strong topology) such that such that for each $x \in X$, if $\pi_x(g) = \pi(x, g)$ then
\((\pi_x, V) \in \mathcal{F}(G)\). We will say that the family has local uniform moderate growth if for each \(\omega\) a compact subset of \(X\) and each continuous seminorm on \(V, p\), there exists a continuous seminorm \(q_\omega\) on \(V\) and \(r_\omega\) such that if \(v \in V\) then
\[
p(\pi_x(g)v) \leq q_\omega(v) \|g\|^{r_\omega}.
\]

**Proposition F.2** If \((\pi, H)\) is a continuous family of Hilbert representations over the analytic manifold \(X\) such that the representations \(\pi_x|_K\) are the same for all \(x \in X\) (we denote this common value by \(\pi(k)\)) and the representations \((d\pi_x, H^\infty)_K\) form an analytic family of objects in \(H(\mathfrak{g}, K)\), then

1. The space of \(C^\infty\) vectors in \(H\) with respect to \(\pi_x\) is equal to the space of \(C^\infty\) vectors of the representation \((\pi, H)\) of \(K\).
2. Assume that for each, \(\omega \subset X\), compact, and \(u \in U(\mathfrak{g})\) there exist constants \(C_\omega, \nu, n_\omega, u\) such that
\[
\|d\pi_y(u)v\| \leq C_\omega, u \|d\pi(1 + C_K)^{n_\omega, u}v\|
\]
for \(v \in H^\infty\). Then \(x \mapsto (\pi_x, H^\infty)\) is a continuous family of smooth Fréchet representations of local uniform moderate growth.

**Proof.** 1. follows from Lemma [E.4] and Proposition [A.2].

We now prove 2. To prove the continuity assertion we need to show that if \(l > 0\) and \(x_o \in X\) then
\[
\lim_{x \to x_o} \|d\pi(1 + C_K)^l(\pi_x(g) - \pi_xo(g))v\| = 0.
\]
Let \(\lambda_\gamma\) be the eigen-value of \(C_K\) on \(V(\gamma)\). Recall that if \(v \in H^\infty = \sum_\gamma v_\gamma\) with \(v_\gamma \in H(\gamma)\) and for each \(r\) there exists a constant \(C_{r,v}\) such that
\[
\|v_\gamma\| \leq C_{v, r}(1 + \lambda_\gamma)^{-r}.
\]
As is well known
\[
\sum_{\gamma \in K} (1 + \lambda_\gamma)^{-r} < \infty
\]
if \(r > \frac{\dim T}{2}\) with \(T\) is a maximal torus of \(K\). Fix \(l > 0\) and \(x_o\) in \(X\). Let \(F \subset \hat{K}\) if \(u \in H^\infty\) set \(u(F) = \sum_{\gamma \in F} u_\gamma\). If \(F^c = \hat{K} - F\), then \(u = u(F) + u(F^c)\). If \(u \in H^\infty\) then
\[
d\pi(1 + C_K)^l \pi_x(g)u = \pi_x(g)d\pi_x(Ad(g)^{-1}(1 + C_K)^l)u.
\]
Let $z_1, \ldots, z_d$ be a basis of $U^l(g)$. Then

$$Ad(g)^{-1}(1 + C_K)^l = \sum a_i(g)z_i$$

with $a_i$ real analytic on $G$. Thus

$$d\pi(1 + C_K)^l \pi_x(g)v = \pi_x(g)\sum a_i(g)d\pi_x(z_i)u.$$  

Note that there exists $C_1, m$ such that $|a_i(g)| \leq C_1 \|g\|^m$ for all $i$. Now fix $x_0 \in X$ and fix $U$ a neighborhood of $x_0$ with compact closure. Then

$$\|\pi_x(g)u\| \leq C_2 \|g\|^{m_1} \|u\|, \ x \in U, \ u \in H.$$  

Let $v \in H^\infty$. Let $F_N = \{\gamma \in \hat{K} | \lambda_{\gamma} \leq N\}$ then $F_N$ is a finite set. Let $r = \frac{\dim T}{2} + 1$. Set $n = \max_i n_{u_i, \omega}$ with $\omega$ the closure of $U$ and $C_3 = \max C_{u_i, \omega}$.

If $x \in U$

$$\|d\pi(1 + C_K)^l(\pi_x(g) - \pi_{x_0}(g))v(F_N^c)\|^2 \leq 2d_l^2 C_2^2 C_1^2 C_3^2 \|g\|^{m+m_1} \sum_{\gamma \notin F_N} (1 + \lambda_{\gamma})^{2l+2n} \|\nu_{\gamma}\|^2.$$  

Also

$$\|\nu_{\gamma}\| \leq C_{v,m}(1 + \lambda_{\gamma})^{-m},$$

so

$$\sum_{\gamma \notin F_N} (1 + \lambda_{\gamma})^{2l+2n} \|\nu_{\gamma}\|^2 \leq C_{v,m} \sum_{\gamma \notin F_N} (1 + \lambda_{\gamma})^{2l-m}.$$  

Choose $m = 2l + r + 2n + s$ with $s \geq 1$, Then

$$\sum_{\gamma \notin F_N} (1 + \lambda_{\gamma})^{2l+2n} \|\nu_{\gamma}\|^2 \leq N^{-s} C_{v,m} \sum_{\gamma \notin F_N} (1 + \lambda_{\gamma})^{-r} \leq N^{-s} C_{v,m} \sum_{\gamma \notin \hat{K}} (1 + \lambda_{\gamma})^{-r}.$$  

We therefore have ($C_4 = C_{v,m} \sum_{\gamma \notin \hat{K}} (1 + \lambda_{\gamma})^{-r}$)

$$\|d\pi(1 + C_K)^l(\pi_x(g) - \pi_{x_0}(g))v(F_N^c)\|^2 \leq 2d_l^2 C_2^2 C_1^2 C_3^2 \|g\|^{m+m_1} C_4 N^{-s}.$$  

Let $\varepsilon > 0$ be and let $\omega_1$ be a compact subset of $G$. Choose $N$ so that

$$2d_l^2 C_2^2 C_1^2 C_3^2 \|g\|^{m+m_1} C_4 N^{-s} < \frac{\varepsilon^2}{4}$$

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for \( g \in \omega_1 \). Now if \( x \in U \) then
\[
\| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N) \| \leq \| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N) \|
\]
\[
+ \| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N^c) \| < \| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N) \| + \frac{\varepsilon}{2}.
\]
The function of \( x \), \( \| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N) \|^2 \) is, by our assumption real analytic in \( x, g \) and equal to 0 at \( x = x_0 \). Hence there exists a neighborhood, \( W \), of \( x_0 \) in \( U \) such that if \( g \in \omega_1 \) and \( x \in W \) then
\[
\| d\pi (1 + C_K l) (\pi_x (g) - \pi_{x_0} (g)) v(F_N) \|^2 < \frac{\varepsilon^2}{4}.
\]
This completes the proof of continuity. We leave the condition of uniform moderated growth to the reader (what is needed is in the above argument).

Definition F.3 A continuous family of Hilbert representations of \( G \), \( (\pi, H) \), over \( X \) will be called smoothable if for each compact subset \( \omega \subset X, u \in U(g) \) there exists \( C_{\omega, u}, n_{\omega, u} \) such that
\[
\| d\pi_y (u) v \| \leq C_{\omega, u} \| d\pi (1 + C_K n_{\omega, u}) v \|
\]
for \( y \in \omega, v \in H^\infty \).

Definition F.4 A holomorphic family of objects in \( F(G) \) over the complex manifold \( X \) is a continuous family \( (\pi, V) \) such that the map \( x \mapsto \pi_x (g) v \) is holomorphic from \( X \) to \( V \) for all \( g \in G, v \in V \).

Theorem F.5 If \( (\pi, V) \) is a continuous family of smooth Fréchet representations over the complex manifold \( X \) such that \((d\pi, V_K)\) is a holomorphic family of objects in \( H(g, K) \) then \((\pi, V)\) is a holomorphic family of objects in \( F(G) \) over \( X \).

We will use the next Lemma in the proof.

Lemma F.6 Let \( X \) a complex \( n \)-manifold, \( G \) be connected and let
\[
f : X \times G \rightarrow \mathbb{C}
\]
be continuous and real analytic in \( G \). If \( z f(x, e) \) is holomorphic in \( x \in X \) for all \( z \in U(g) \) (here \( z \) is acting as left invariant differential operators on the \( G \) the second factor) then \( f \) is holomorphic in \( X \).
Proof. Let $x \in X$ and let $z_1, ..., z_n$ be local coordinates on an open neighborhood, $U$, of $x$ in $X$ such that if $\psi = (z_1, ..., z_n)$ then $\psi(x) = 0$ and $\psi(U) \supset D^n$ with $D$ (resp. $\overline{D}$) the (resp. closed) unit disk in $\mathbb{C}$. For simplicity we may assume that $X = \psi(U)$. Define for $z \in D$

$$h(z, g) = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{f(u, g)}{\prod (u_i - z_i)} du_1 \cdots du_n.$$ 

Then $h(z, g)$ is holomorphic in $z$ on $D$. By our assumption $uh(z, e) = uf(z, e)$ for $u \in U(g), z \in D$. Since $f$ is analytic in $G$ and $G$ is connected $h = f$ on $D$. 

We will now prove the theorem. Let $\lambda \in V'$. If $v \in V_K$ then the function

$$f(x, g) = \lambda(\pi_x(g)v)$$

on $X \times G$ is continuous and real analytic in $G$. Now

$$uf(x, e) = \lambda(d\pi_x(u)v)$$

which is holomorphic in $x$. Thus if $v \in V_K$ then

$$x \mapsto \lambda(\pi_x(g)v)$$

is holomorphic in $x$. Let $x \in X$ and $U, \psi$, etc. be as in the previous lemma. Set

$$h(z, v) = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{\lambda(\pi_u(g)v)}{\prod (u_i - z_i)} du_1 \cdots du_n.$$ 

for $v \in V, z \in D^n$. Then $h(z, v)$ is holomorphic in $z$ and continuous in $v$. Furthermore, the first part of this proof showed that $h(z, v) = \lambda(\pi_z(g)v)$ if $v \in V_K$. Since, $V_K$ is dense in $V$ this implies that

$$z \mapsto \lambda(\pi_z(g)v)$$

is holomorphic in $z$. Grothendieck [?] has shown that a weakly holomorphic map of a complex manifold to a Fréchet space is strongly holomorphic thus completing the proof.
G Functorial properties of Hilbert families

In this section we will analyze Hilbert globalizations of subfamilies and quotient families of Harish-Chandra modules.

Lemma G.1 Let $(\tau, V)$ be a finite dimensional continuous representation of $K$ and let $X$ be a locally compact metric space (resp. an analytic manifold). If $u \in X$ let $(\ldots, \ldots)_u$ be an inner product on $V$ such that $\tau(k)$ acts unitarily with respect to $(\ldots, \ldots)_u$ for $k \in K$ and such that the map $u \mapsto \langle v, w \rangle_u$ is continuous (resp. real analytic) for all $v, w \in V$. Then there exists, for each $u$ an ordered orthonormal basis of $V, e_1(u), \ldots, e_n(u)$ such that the map $u \mapsto e_i(u)$ is continuous (resp. real analytic) and the matrix of $\tau(k)$ with respect to $e_1(u), \ldots, e_n(u)$ is independent of $u$. Furthermore, if $X$ is compact and contractible and $(\sigma, W)$ is a finite dimensional continuous representation of $K$ and $u \mapsto B(u) \in \text{Hom}_K(V, W)$ is continuous and surjective for $u \in X$ then $e_1(u), \ldots, e_r(u)$ with $r = \dim V - \dim W$ can be taken in $\ker B(u)$.

Proof. Fix an inner product, $(\ldots, \ldots)$, on $V$ such that $\tau$ is unitary. Then there exists a positive definite Hermitian operator (with respect to $(\ldots, \ldots))$, $A(u)$ such that $\langle v, w \rangle_u = (A(u)v, w), v, w \in V$ and $A(u)$ is continuous (resp. real analytic) in $u$. Note that

$$\tau(k)^{-1}A(u)\tau(k) = A(u), u \in X, k \in K.$$ 

Set $S(u) = A(u)^{\frac{1}{2}}$ then $\langle v, w \rangle_u = (S(u)v, S(u)w)$. Thus if $T(u) = S(u)^{-\frac{1}{2}}$ then $\tau(k)T(u) = T(u)\tau(k), k \in K, u \mapsto T(u)$ is continuous (resp. real analytic) and

$$\langle T(u)v, T(u)w \rangle_u = (v, w), v, w \in V.$$ 

Let $e_1, \ldots, e_n$ be an (ordered) orthonormal basis of $V$ with respect to $(\ldots, \ldots)$ then $e_1(u) = T(u)e_1, \ldots, e_n(u) = T(u)e_n$ is an orthonormal basis of $V$ with respect to $(\ldots, \ldots)_u$. If $\tau(k)e_i = \sum k_{ji}e_j$ then

$$\tau(k)e_i(u) = \tau(k)T(u)e_i = T(u)\tau(k)e_i = \sum k_{ji}T(u)e_j.$$ 

To prove the second assertion note that $u \mapsto \ker B(u)$ is a $K$–vector bundle over $X$ (see the lemma below). Since $X$ compact and contractible the bundle is a trivial $K$–vector bundle ([A] Lemma 1.6.4). Thus there is a representation $(\mu, Z)$ of $K$ and $u \mapsto L(u) \in \text{Hom}_K(Z, V)$ continuous such that $L(u)Z = \ldots$
ker $B(u)$ and $L(u)$ is injective. Notice that $B(u) : \ker B(u)^\perp \to W$ is a $K$–module isomorphism. Now pull back the inner product $\langle \ldots, \ldots \rangle_u$ to $Z$ using $L(u)$ getting a $K$–invariant inner product, $(\ldots, \ldots)_u$, on $Z$ and push the inner product to $W$ getting a $K$–invariant inner product $(\ldots, \ldots)_u^1$ on $W$. Now apply the first part of the lemma to get an orthonormal basis $f_1(u), \ldots, f_r(u)$ of $Z$ with respect to $(\ldots, \ldots)_u$ and an orthonormal basis $f_{r+1}(u), \ldots, f_n(u)$ ($n = \dim V$) with respect to $(\ldots, \ldots)_u^1$ such that the matrices of the action of $K$ with respect to each of these bases is constant. Take $e_i(u) = L(u)f_i(u)$ for $i = 1, \ldots, r$ and $e_i(u) = \left( B(u)_{\ker B(u)^\perp} \right)^{-1} f_i(u)$ for $i = r+1, \ldots, n$.

Lemma G.2 Let $V$ and $W$ be finite dimensional, continuous $K$–modules and assume that for $x \in X$, $B(x) \in \text{Hom}_K(V,W)$ is surjective and the map $x \mapsto B(x)$ is continuous. Then $x \mapsto \ker B(x)$ is a $K$–vector bundle over $X$.

Proof. Let $x_o \in X$ and let $M \subset V$ be a $K$–invariant subspace of $V$ such that $B(x_o)$ is a $K$–isomorphism of $M$ onto $W$. Then there exists $U \subset X$ an open neighborhood of $x_o$ such that $B(u)|_M$ is invertible for $u \in U$. Set $S(u) = (B(u)|_Z)^{-1}$ on $B(u)V$, for $u \in U$. If $v \in \ker B(x_o)$ and if $u \in U$ then

$$B(u)v = B(u)S(u)B(u)v$$

so

$$B(u)(I - S(u)B(u))v = 0.$$  

Thus $I - S(u)B(u)$ maps $\ker B(x_o)$ to $\ker B(u)$ for $u \in U$. This map is the identity for $u = x_o$, so it is a $K$–isomorphism for $u \in U_1 \subset U$ with $U_1$ open in $U$.

Proposition G.3 If $(\sigma, V)$ is a continuous family of admissible $(\mathfrak{g}, K)$–modules over a metric space $X$, if $(\mu, H)$ is a smoothable (see Definition ??) continuous family of admissible Hilbert representations of $G$ based on $X$ and if

$$T : (\sigma, V) \to (d\mu, H_K)$$

is a continuous family of injective $(\mathfrak{g}, K)$–module homomorphisms then there exists $(\lambda, W)$ a smoothable continuous family of Hilbert representations of $G$ based on $X$ such that $(d\lambda, W_K^\infty)$ and $(\sigma, V)$ are isomorphic as continuous families and a continuous family of injections of $(\lambda, W)$ into $(\mu, H)$.
Proof. If \( x \in X \) then set \( \langle...,... \rangle_x = T^*_x \langle...,... \rangle \). If \( \gamma \in \hat{K}, x \in X \) let \( e_1^\gamma(x),...,e_n^\gamma(x) \) be the orthonormal basis as in Lemma 4.1 corresponding to \( \langle...,... \rangle_x \). Then \( \{e_i^\gamma(x)\} \) is an orthonormal basis of \( V \). Set

\[
f_i^\gamma(x) = T_x(e_i^\gamma(x))
\]

then \( \{f_i^\gamma(x)\} \) is an orthonormal basis of \( T_x V \) for \( x \in X \). If \( v \in H \) then set

\[
P_\gamma(x)v = \sum_{i=1}^{\nu_\gamma} \langle v, f_i^\gamma(x) \rangle f_i^\gamma(x).
\]

The map

\[
x \mapsto P_\gamma(x)
\]

is Strongly continuous from \( X \) to \( \text{Hom}_K(H,H(\gamma)) \) (the continuous \( K \) homomorphisms). Define

\[
P(x)v = \sum P_\gamma(x)v.
\]

Then \( P(x) \) is the orthogonal projection of \( H \) onto the closure of \( T_x V \) in \( H \). Thus in particular \( \|P(x)\| = 1 \). We assert that

\[
x \mapsto P(x)
\]

is strongly continuous from \( X \) to the bounded operators on \( H \). To this end, let \( v \in H \) be a unit vector and \( x_o \in X \). Let \( \varepsilon > 0 \) be given and let \( F \subset \hat{K} \) be such that

\[
\left\| \sum_{\gamma \notin F} v(\gamma) \right\| < \frac{\varepsilon}{4}
\]

then since

\[
P(x) \sum_{\gamma \in F} v(\gamma) = \sum_{\gamma \in F} P_\gamma(x)v(\gamma)
\]

there exists an open neighborhood, \( U \), of \( x_o \) in \( X \) such that

\[
\left\| (P(x) - P(x_o)) \sum_{\gamma \in F} v(\gamma) \right\| < \frac{\varepsilon}{2}.
\]

Thus

\[
\|(P(x) - P(x_o))v\| \leq \left\| (P(x) - P(x_o)) \sum_{\gamma \in F} v(\gamma) \right\| + \left\| (P(x) - P(x_o)) \sum_{\gamma \notin F} v(\gamma) \right\|
\]

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\[ \left\| (P(x) - P(x_0)) \sum_{\gamma \in F} v(\gamma) \right\| + 2 \left\| \sum_{\gamma \notin F} v(\gamma) \right\| < \varepsilon. \]

Let \( \nu_x(g) \) be the action of \( G \) on \( P(x)H \). Define \( L(x, y) : P(y)H \rightarrow P(x)H \)
\[ L(x, y)f_i^\gamma(y) = f_i^\gamma(x). \]

Then \( L(x, y) \) is a unitary operator and a \( K \)-module equivalence. Furthermore,
\[ x, y \mapsto L(x, y)P(y) \]
is strongly continuous (use a slight modification of the argument for the strong continuity of \( P(x) \)). Fix \( x_0 \in X \) and set \( W = L(x_0)H \). Set \( \lambda_x(g) = L(x_0, x)\nu_x(g)L(x, x_0) \).

To complete the proof we need to show that \( (\mu, W) \) is smoothable. Let \( \omega \) be a compact subset of \( X \) and \( u \in U(\mathfrak{g}) \) then there exist \( C_\omega \) and \( n_\omega \) such that if \( v \in H^\infty \) then
\[ \|d\mu_x(u)v\| \leq C_\omega \|d\mu_x(1 + C_K)^{n_\omega}v\|. \]

Now, if \( v \in W^\infty \) then \( v = P_{x_0}w \) with \( w \in H^\infty \). So
\[ \|d\lambda_x(u)v\| = \|L(x_0, x)d\nu_x(u)L(x, x_0)P_{x_0}w\| \]
\[ = \|d\nu_x(u)L(x, x_0)P_{x_0}w\| \leq C_\omega \|d\nu_x(1 + C_K)^{n_\omega}L(x, x_0)P_{x_0}w\| \]
\[ = C_\omega \|L(x_0, x)d\nu_x(1 + C_K)^{n_\omega}L(x, x_0)P_{x_0}w\| \]
\[ = C_\omega \|d\lambda_x(1 + C_K)^{n_\omega}P_{x_0}w\|. \]

Similarly we can prove

**Proposition G.4** If \((\sigma, V)\) is a continuous family of admissible \((\mathfrak{g}, K)\)-modules over a compact contractible metric space \( X \), \((\mu, H)\) is a continuous family of admissible Hilbert representations of \( G \) based on \( X \) and if
\[ T : (d\mu, H_K) \rightarrow (\sigma, V) \]
is a continuous family of surjective \((\mathfrak{g}, K)\)-module homomorphisms then there exists \((\lambda, W)\) an acceptable, continuous family of Hilbert representations of \( G \) based on \( X \) such that \((d\lambda, W^\infty_K)\) and \((\sigma, V)\) are isomorphic as continuous families and a continuous family of surjections of \((\mu, H)\) into \((\lambda, W)\). Furthermore, if \((\mu, H)\) is smoothable the so is \((\lambda, W)\).
Proof. The proof follows the same lines as the previous theorem. Let for each \( x \in X \),
\[
B_{\gamma}(x) = T_x|_{H_{\gamma}}.
\]
Let \( r_\gamma = \dim V(\gamma), m_\gamma = \dim H(\gamma) \). Then Lemma \[G.1\] implies that for each \( x \in X \) there exists an orthonormal basis of \( H(\gamma) \), \( \{e_i^\gamma(x)\} \) with respect to the inner product, \( \langle \cdot, \cdot \rangle \) on \( H \) such that \( \{e_i^\gamma(x)\}_{i \geq r_\gamma} \) is a basis of \( \ker B_{\gamma}(x) \) and \( x \mapsto e_i^\gamma(x) \) is continuous. Let \( f_i^\gamma(x) = B_{\gamma}(x)e_i^\gamma(x) \) and for each \( x \in X \) define an inner product, \( \langle \cdot, \cdot \rangle_x \) on \( V \) by declaring that \( \{f_i^\gamma(x)\} \) is an orthonormal basis. Set
\[
P_\gamma(v) = \sum_{i=1}^{r_\gamma} \langle v, e_i^\gamma(x) \rangle e_i^\gamma(x)
\]
and \( P(x) = \sum P_\gamma(x) \). Essentially the same argument as in the preceding theorem shows that map \( x \mapsto P(x) \) is strongly continuous. Also, \( T_x : P(x)H_K \to V \) is unitary relative to \( \langle \cdot, \cdot \rangle_x \) and an equivalence of representations of \( K \). Let \( H_{1,x} \) be the closure in \( H \) of \( \ker T_x \) for \( x \in X \). Then, as a Hilbert space, \( H/H_{1,x} = P(x)H \). Since \( \ker T_x \) consists of analytic vectors \( H_{1,x} \) is \( G \)-invariant. This defines a Hilbert representation, \( \gamma_x \), on \( P(x)H \). Which in turn defines a Hilbert representation, \( \nu_x \), of \( G \) on the Hilbert space completion of \( V, Z_x \). Let \( L(x,y) : Z_y \to Z_x \) be defined by \( L(x,y)f_i^\gamma(y) = f_i^\gamma(x) \). Then \( L(x,y) \) defines a unitary \( K \)-isomorphism of \( (\nu_y|_K, Z_y) \) with \( (\nu_x|_K, Z_x) \). Fix \( x_0 \) in \( X \) and let \( W = Z_{x_0} \) and \( \lambda_x(g) = L(x_0, x)\nu_x(g)L(x, x_0) \). Then, as in the above theorem, we have defined a Hilbert family globalizing \((\sigma, V)\).

We now assume that \((\mu, H)\) is smoothable. If \( v \in P_xH^\infty \) and then
\[
d\mu_x(u)v = d\gamma_x(u)v + (I - P_x)d\mu_x(u)v.
\]
Thus if \( \omega \) is a compact subset of \( X \) then
\[
\|d\gamma_x(u)v\| \leq \|d\mu_x(u)v\|
\]
\[
\leq C_{u,\omega} \|d\mu_x(1 + C_K)\| = C_{u,\omega} \|d\gamma_x(1 + C_K)\|.
\]
We reinterpret the family \((\lambda, W)\). Let \( M(x,y) : P_yH \to P_xH \) be given by
\[
M(x,y)e_i^\gamma(y) = e_i^\gamma(x).
\]
Fix \( x_o \in X \). Then the family can be defined as \( \delta_x(g) = M(x_o, x)\gamma_x(g)M(x, x_o)v \) for \( v \in P_{x_o}H \). Setting \( U = P_{x_o}H \) then \((\delta, U)\) is an isomorphic continuous family to \((\lambda, W)\). We show that this family is smoothable let \( u \in U(g) \). Then if
$v \in U$ and $\omega$ is a compact subset of $X$ then
\[
\|d\delta_x(u)v\| = \|M(x_o, x)d\gamma_x(u)M(x, x_o)v\| = \|d\gamma_x(u)M(x, x_o)v\| \\
\leq C_{v,\omega}\|d\gamma_x(1 + C_K)^l M(x, x_o)v\| = C_{u,\omega}\|M(x_o, x)d\gamma_x((1 + C_K)^l v)\| \\
= C_{u,\omega}\|d\delta_x((1 + C_K)^l v)\|.
\]

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