QUASI-CYCLIC SUBCODES OF CYCLIC CODES

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Abstract. We completely characterize possible indices of quasi-cyclic subcodes in a cyclic code for a very broad class of cyclic codes. We present enumeration results for quasi-cyclic subcodes of a fixed index and show that the problem of enumeration is equivalent to enumeration of certain vector subspaces in finite fields. In particular, we present enumeration results for quasi-cyclic subcodes of the simplex code and duals of certain BCH codes. Our results are based on the trace representation of cyclic codes.

1. Introduction

Let \( F_q \) denote the finite field with \( q \) elements, where \( q \) is a prime power, and let \( m \) and \( \ell \) be positive integers. A linear code \( C \subseteq F_q^{m\ell} \) is called a quasi-cyclic (QC) code of index \( \ell \) if it is invariant under shift of codewords by \( \ell \) units and \( \ell \) is the minimal number with this property. Clearly QC codes are generalizations of cyclic codes, for which \( \ell = 1 \). QC codes drew much attention in the literature since they yield codes with good parameters (see for instance [3, 4]). The class of QC codes and some of its subclasses also perform well asymptotically and reach the Gilbert-Varshamov bound ([5, 8, 9, 10]).

Studying subcodes in well-known classes of codes is a common theme in coding theory for various purposes. Our motivation to study QC subcodes in cyclic codes stems from [7], where the number of rational points of supersingular curves is related to weight analysis of certain subcodes of cyclic codes. It is shown in [7] that these subcodes are QC codes.

We consider cyclic codes of length \( q^n - 1 \) over \( F_q \) and assume throughout that the dual code’s zeros all have \( q \)-cyclotomic cosets of length \( n \) over the base field \( F_q \) (cf. Section 2). Note that this is true for a broad class of cyclic codes. We have two particular problems addressed: to determine all possible indices of QC subcodes in a given cyclic code and to count the number of QC subcodes for a fixed index. We solve the first problem completely and list all positive integers that are indices of some QC subcode (Theorem 2.6). In particular, we observe that not every divisor of the cyclic code’s length need to be the index of some QC subcode (Remark 2.9). For the second problem, we show that the enumeration of QC subcodes in a cyclic code is related to the count of vector subspaces in finite fields. For the class of cyclic codes we study, we show that these two problems are equivalent (cf. Theorem 3.1). Using this observation, we can count QC subcodes of a given index in certain well-known cyclic codes. Enumeration results require counting subspaces in \( F_q^n \) which are defined over a subfield \( F_{q^d} \) in a maximal way (i.e. these spaces do not have a vector space

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structure over a subfield that contains \( \mathbb{F}_q \)). We give exact count of such vector spaces in Theorem 3.3 using inclusion-exclusion principle. We utilize the trace representation of cyclic codes (III) to obtain our results.

Organization of the paper is as follows: We determine possible indices of QC subcodes in a given cyclic code in Section 2. Enumeration of QC subcodes is addressed in Section 3 where the relation to counting vector subspaces in a finite field is given. In Section 4 we enumerate QC subcodes of the \( q \)-ary simplex code of length \( q^n - 1 \) for any prime power \( q \) and any \( n \). Section 5 contains enumeration results for QC subcodes of duals certain BCH codes. Proofs of our enumeration results yield an algorithm that counts indices and their appearances (multiplicities) for certain cyclic codes. Section 6 consists of some examples produced by the algorithm. Magma code of the algorithm is made available on-line for interested readers (I).

2. Indices of QC Subcodes

Let \( n \) and \( N \) be positive integers with \( N = q^n - 1 \) and let \( \alpha \) a primitive \( N \)th root of unity. Throughout this work, we will concentrate on the cyclic code \( C \) over \( \mathbb{F}_q \) of length \( N \) with basic dual zeros

\[ BZ(C^\perp) = \{ \alpha^{i_1}, \ldots, \alpha^{i_s} \}, \]

where \( i_j \geq 1 \) for all \( j \) and \( i_j \)'s come from pairwise distinct \( q \)-cyclo
tomic cosets mod \( N \). This means that the generating polynomial of \( C^\perp \) is the product of the minimal polynomials of \( \alpha^{i_j} \)'s over \( \mathbb{F}_q \). Since \( N \) and \( q \) are relatively prime these minimal polynomials are distinct. Moreover, we will assume throughout that the \( q \)-cyclo
tomic coset mod \( N \) for each \( i_j \) has size \( n \). Note that this amounts to saying that the minimal polynomials of \( \alpha^{i_j} \)'s over \( \mathbb{F}_q \) are all of degree \( n \); or equivalently \( \mathbb{F}_q^n = \mathbb{F}_q(\alpha^{i_j}) \) for each \( j \).

Trace representation of \( C \) is as follows (III Proposition 2.1):

\[ C = \left\{ \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(\lambda_1 \alpha^{ki_1} + \cdots + \lambda_s \alpha^{ki_s}) \right)_{0 \leq k \leq N-1} ; \lambda_j \in \mathbb{F}_q^n, 1 \leq j \leq s \right\}. \]

Coordinates of length \( N \) codewords of \( C \) are obtained by evaluating the trace expression for each \( 0 \leq k \leq N - 1 \).

Consider a subcode of \( C \),

\[ C' = \left\{ \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(\beta_1 \alpha^{ki_1} + \cdots + \beta_s \alpha^{ki_s}) \right)_{0 \leq k \leq N-1} ; \beta_j \in V_j \subseteq \mathbb{F}_q^n, 1 \leq j \leq s \right\}. \]

The following result will play a crucial role in this article.

**Theorem 2.1.** [6 Theorem 2.5] Let \( i_j \geq 1 \) be positive integers (for \( 1 \leq j \leq s \)) which are in different \( q \)-cyclo
tomic cosets mod \( N = q^n - 1 \). For \( \lambda_1, \ldots, \lambda_s \in \mathbb{F}_q^n \), we have

\[ \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(\lambda_1 x_1^{i_1} + \cdots + \lambda_s x_s^{i_s}) = 0, \quad \text{for all } x \in \mathbb{F}_q^n \]

if and only if each \( i_j \) has \( q \)-cyclo
tomic coset mod \( N \) of length \( |Cyc_q(i_j)| = \delta_j < n \) and \( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(\lambda_j) = 0 \) for all \( j = 1, \ldots, s \). In particular, if each \( i_j \) has \( q \)-cyclo
tomic coset of length \( n \), then (2.3) holds if and only if \( \lambda_j = 0 \) for all \( j \).
Theorem 2.1 justifies our assumption on the sizes of $q$-cyclo-
tomic cosets of $i_j$’s in the trace represen-
tation of the cyclic code $C$, as can be seen in the next results and in Section 3 when we con-
sider enumeration of QC subcodes.

**Lemma 2.2.** $C'$ is an $\mathbb{F}_q$-linear subcode if and only if $V_j$ is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$ for all $j$.

**Proof.** Choose two arbitrary codewords from $C'$:

$$v_\beta = \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s \beta_j \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1}, \quad v_\gamma = \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s \gamma_j \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1}$$

(i.e. $\beta_j, \gamma_j \in V_j$ for all $j$). $C'$ is an $\mathbb{F}_q$-linear subcode of $C$ if and only if for any $a \in \mathbb{F}_q$ we have

$$av_\beta + v_\gamma = \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s (a \beta_j + \gamma_j) \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1} \in C'.$$

That is, there exist $\rho_j \in V_j$ (for all $j$) such that

$$\left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s (a \beta_j + \gamma_j) \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1} = \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s \rho_j \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1},$$

or equivalently

$$\left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \sum_{j=1}^s [(a \beta_j + \gamma_j) - \rho_j] \alpha^{k_{ij}} \right)_{0 \leq k \leq N-1} = 0 \text{ for all } 0 \leq k \leq N - 1.$$

Since we assumed that every $i_j$ has $q$-cyclo-
tomic coset mod $N$ of size $n$, Theorem 2.1 implies that

$$\left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \sum_{j=1}^s [(a \beta_j + \gamma_j) - \rho_j] \alpha^{k_{ij}} \right)_{0 \leq k \leq N-1} = 0 \text{ for all } 0 \leq k \leq N - 1.$$

We will assume from now on that $V_j$’s describing the subcode $C'$ in (2.2) are all $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$. Let $T$ denote the cyclic shift operator on $\mathbb{F}_q^n$, i.e. $T(u_1, u_2, \ldots, u_N) = (u_N, u_1, \ldots, u_{N-1})$. The following is easy to observe.

**Lemma 2.3.**

$$T \left[ \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s \beta_j \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1} \right] = \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \sum_{j=1}^s (\beta_j \alpha^{-i_j}) \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1}.$$

For each $1 \leq j \leq s$, let us now define the following subcode of $C'$:

$$C'_j = \left\{ \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \left( \beta_j \alpha^{k_{ij}} \right) \right)_{0 \leq k \leq N-1} ; \beta_j \in V_j \right\}.$$

Next, we obtain a criterion for quasi-cyclicity of $C'_j$.

**Proposition 2.4.** $C'_j$ is an index $\ell_j$ QC code if and only if $V_j$ is an $\mathbb{F}_q(\alpha^{\ell_j i_j})$-subspace of $\mathbb{F}_q^n$ and $\ell_j$ is the minimal such number.
Proof. According to Lemma 2.3, \( C'_j \) is closed under shift of codewords by \( t \) units if \( V_j \) is closed under multiplication by \( \alpha^{-t_i j} \). Since \( V_j \) is an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_{q^n} \), this is equivalent to saying that \( V_j \) is closed under scalar multiplication by elements in

\[
\mathbb{F}_q(\alpha^{-t_i j}) = \mathbb{F}_q(\alpha^{-t_i j}) = \mathbb{F}_q(\alpha^{t_i j}).
\]

Hence, if \( V_j \) is an \( \mathbb{F}_q(\alpha^{t_i j}) \)-subspace of \( \mathbb{F}_{q^n} \) and \( \ell_j \) is the minimal such number, then \( C'_j \) is an index \( \ell_j \) QC code.

For the converse, suppose \( C'_j \) is an index \( \ell_j \) QC code. So, given \( \beta_j \in V_j \), there exists \( \gamma_j \in V_j \) such that

\[
\left( \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \left( (\beta_j \alpha^{-\ell_j} \alpha^{\gamma_j}) \right) \right)_{0 \leq k \leq N-1} = \left( \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \left( \gamma_j \alpha^{\gamma_j} \right) \right)_{0 \leq k \leq N-1},
\]

or equivalently

\[
\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \left( (\beta_j \alpha^{-\ell_j} - \gamma_j) \alpha^{\gamma_j} \right) = 0 \text{ for all } 0 \leq k \leq N-1.
\]

By assumption \( i_j \) has size \( n \) \( q \)-cyclotomic coset mod \( N \). Therefore by Theorem 2.1, the above equality holds for all \( k \) if and only if \( \beta_j \alpha^{-\ell_j} - \gamma_j = 0 \). In other words \( V_j \) is closed under multiplication by \( \alpha^{-\ell_j} \) and \( \ell_j \) is the minimal such positive integer. Hence the result follows. \( \square \)

We need two more facts on finite fields. Proofs are clear, hence omitted.

**Proposition 2.5.** (i) Consider \( \mathbb{F}_q \subseteq \mathbb{F}_{q^d} \subseteq \mathbb{F}_{q^n} \) and let \( \alpha \) be a primitive element in \( \mathbb{F}_{q^n} \). Then

\[
L_d = \frac{q^n - 1}{q^d - 1}
\]

is the least positive integer such that \( \alpha^{L_d} \in \mathbb{F}_{q^d} \). Moreover, \( \alpha^{L_d} \) is a primitive element of \( \mathbb{F}_{q^d} \).

(ii) For \( \alpha^i \in \mathbb{F}_{q^n} \), the least positive integer \( \ell_d \) satisfying \( \alpha^{\ell_d} \in \mathbb{F}_{q^d} \) is

\[
\ell_d = \frac{\text{lcm}(i, L_d)}{i}.
\]

The following result describes the set of possible indices for QC subcodes in \( C \).

**Theorem 2.6.** (i) Consider the \( \mathbb{F}_q \)-linear subcode \( C' \) of the cyclic code \( C \) (cf. (2.1) and (2.2)). For \( 1 \leq j \leq s \), let \( \mathbb{F}_{q^{d_j}} \) be the largest intermediate field in \( \mathbb{F}_{q^n}/\mathbb{F}_q \) such that \( V_j \) is an \( \mathbb{F}_{q^{d_j}} \)-subspace of \( \mathbb{F}_{q^n} \). Let

\[
L_j = \frac{q^n - 1}{q^{d_j} - 1} \text{ and } \ell_j = \frac{\text{lcm}(i_j, L_j)}{i_j}, \quad \forall j
\]

and let \( \ell = \text{lcm}(\ell_1, \ldots, \ell_s) \). If \( \ell \neq q^n - 1 = N \), then \( C' \) is an index \( \ell \) QC subcode of \( C \).

(ii) Let \( \mathbb{F}_{q^{i_1}}, \ldots, \mathbb{F}_{q^{i_s}} \) be the intermediate fields of the extension \( \mathbb{F}_{q^n}/\mathbb{F}_q \), where \( d_1 = 1 \) and \( d_u = n \). Let \( L_j = \frac{q^n - 1}{q^{d_j} - 1} \) for each \( 1 \leq j \leq u \) and define the integers

\[
\ell_1^1 = \frac{\text{lcm}(i_1, L_1)}{i_1}, \quad \ell_1^s = \frac{\text{lcm}(i_s, L_1)}{i_s}, \quad \ell_s^1 = \frac{\text{lcm}(i_1, L_s)}{i_1}, \quad \ell_s^s = \frac{\text{lcm}(i_s, L_s)}{i_s}.
\]
For
\[ I = \{ \text{lcm}(\ell_{t_1}^a, \ldots, \ell_{t_m}^a) \mid 1 \leq m \leq s, 1 \leq a_1, \ldots, a_m \leq u, 1 \leq t_1, \ldots, t_m \leq s, t_i \neq t_j \text{ for any } i \neq j \} \setminus \{ q^n - 1 \}, \]

C has QC subcodes of index ℓ for every ℓ ∈ I and has no QC subcode of different index.

Proof. (i) Note that if some \( V_j = \{0\} \), then it can be considered “maximally” as a vector space over \( \mathbb{F}_{q^n} \). Then the index of \( C'_j \) should be 1 since \( \{0\} \) is a vector space over any subfield of \( \mathbb{F}_{q^n} \), and in particular over \( \mathbb{F}_q(\alpha^j) \) (cf. Proposition 2.4). In this case, indeed, \( L_j = 1 \) and hence \( \ell_j = 1 \).

Now suppose \( V_j \) is not zero and let \( \mathbb{F}_{q^d_j} \) be the largest field over which it is a vector space. By Proposition 2.6 the least power of \( \alpha^j \) that lies in \( \mathbb{F}_{q^d_j} \) is \( \ell_j \). So, \( \mathbb{F}_q(\alpha^{\ell_j}) \subseteq \mathbb{F}_{q^d_j} \) and hence \( V_j \) is an \( \mathbb{F}_q(\alpha^{\ell_j}) \)-space. By maximality of \( d_j \), any field over which \( V_j \) has a vector space structure must be contained in \( \mathbb{F}_{q^d_j} \). Therefore, \( \ell_j \) is indeed the index of \( C'_j \) by Proposition 2.4. It is clear that the index of \( C' \) is the least common multiple of indices of all \( C'_j \)’s.

(ii) Part (i) shows that the index of a QC subcode of \( C \) lies in \( I \). Take an element \( \ell = \text{lcm}(\ell_{t_1}^a, \ldots, \ell_{t_m}^a) \) of \( I \). Consider the subcode \( C' \) where \( V_{t_j} \) (for \( j = 1, \ldots, m \)) is “maximally” defined over the intermediate field \( \mathbb{F}_{q_{s_j}} \) (e.g. \( V_{t_j} = \mathbb{F}_{q_{s_j}} \)) and all other \( V_j \)’s equal \( \{0\} \). Then the index of \( C'_{t_j} \) is \( \ell_{t_j}^a \) for all \( j = 1, \ldots, m \). Hence the index of \( C' \) is \( \ell \). Therefore for any element of \( I \), there is a QC subcode of \( C \) of that index.

Remark 2.7. Let us note that the exponents \( i_j \)’s in the trace representation of the cyclic code \( C \) can be any representative of a \( q \)-cyclic coset mod \( N \). In other words, replacing \( i_j \) by \( qi_j \) in (2.1) still yields the same code. Let us observe that the choice of cyclotomic coset representatives in \( C \)’s trace representation does not affect indices of QC subcodes either. For this, let \( j \in \{1, \ldots, s\} \) and note that \( L_a \) is relatively prime to \( q \) for any \( 1 \leq a \leq u \) (with the notation of Theorem 2.6). Then,
\[ \ell_{qi_j}^a = \frac{\text{lcm}(qi_j, L_a)}{qi_j} = \frac{\text{lcm}(i_j, L_a)}{i_j} = \ell_{i_j}^a. \]

Remark 2.8. Let us now show that a different primitive element choice for \( \mathbb{F}_{q^n} \) does not change the set \( I \) of possible indices for QC subcodes. Let \( \eta \) be another primitive element in \( \mathbb{F}_{q^n} \). Then \( \eta = \alpha^r \) for some \( 1 < r < q^n - 1 \) and \( \gcd(r, q^n - 1) = 1 \). Then,
\[ BZ(C^\perp) = \{ \alpha^{i_1}, \ldots, \alpha^{i_s} \} = \{ \eta^{j_1}, \ldots, \eta^{j_s} \}, \]
for some \( j_1, \ldots, j_s \). Note that for each \( t \in \{1, \ldots, s\} \), this means \( \alpha^{rj_t} = \alpha^{i_t} \). In other words,
\[ rj_t = i_t + k_t(q^n - 1), \quad 1 \leq t \leq s, \]
for some integers \( k_1, \ldots, k_s \). For \( a \in \{1, \ldots, u\} \), let \( \nu \) be such that
\[ (i_t + k_t(q^n - 1)) \nu = \text{lcm}(i_t + k_t(q^n - 1), L_a). \]
Since \( L_a \) divides \( k_t(q^n - 1) \), we conclude that \( \nu \) is the smallest positive integer so that \( i_t \nu \) is a multiple of \( L_a \) (i.e. \( i_t \nu = \text{lcm}(i_t, L_a) \)). Hence,
\[ \frac{\text{lcm}(i_t + k_t(q^n - 1), L_a)}{i_t + k_t(q^n - 1)} = \frac{(i_t + k_t(q^n - 1)) \nu}{i_t + k_t(q^n - 1)} = \nu = \frac{\text{lcm}(i_t, L_a)}{i_t}. \]
Hence, contributions of $i_t$ and $j_t$ to the index relative to the intermediate field $F_{q^{d_t}}$ are the same.

**Remark 2.9.** A natural question is whether a cyclic code has an index $\ell$ QC subcode for every divisor $\ell$ of its length. This is not necessarily the case as the following example shows. Consider the binary cyclic code $C$ of length 15 with the trace representation

$$C = \left\{ \left( \text{Tr}_{F_{2^4}/F_2} \left( \lambda \alpha^k \right) \right)_{0 \leq k \leq 14} : \lambda \in F_{2^4} \right\}.$$  

With the notation of Theorem 2.6 we have $i_1 = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 4$. Moreover,

$$L_1 = 15, \quad L_2 = 5, \quad L_3 = 1,$$

and

$$\ell_1^1 = \frac{\text{lcm}(1, 15)}{1} = 15, \quad \ell_1^2 = \frac{\text{lcm}(1, 5)}{1} = 5, \quad \ell_1^3 = \frac{\text{lcm}(1, 1)}{1} = 1.$$  

Hence, $C$ has only cyclic and index 5 QC subcodes but no index 3 QC subcode, which is the other divisor of its length. An index 5 QC subcode can be easily obtained from Theorem 2.6 and here is an example:

$$C' = \left\{ \left( \text{Tr}_{F_{2^4}/F_2} \left( \beta \alpha^k \right) \right)_{0 \leq k \leq 14} : \beta \in F_{2^2} \right\}.$$  

Note that $C'$ is a rather “small” code, let us list its nonzero codewords:

$$\begin{align*}
c_1 &= (0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0), \\
c_2 &= (0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1), \\
c_3 &= (0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1).
\end{align*}$$  

Clearly, the codewords of $C'$ are invariant relative to 5-shift.

Observe that $C$ is the binary simplex code of length 15. We will have a complete investigation of QC subcodes of the simplex code in Section 4.

### 3. Enumeration of QC Subcodes

We saw in Section 2 that QC subcodes of a cyclic code $C$ and their indices are determined by the vector subspaces of $F_{q^n}$ that determine the coefficients of terms in the trace representation and the maximal intermediate fields over which they have vector space structure. Hence, enumeration of QC subcodes of a fixed index in $C$ is clearly related to the number of vector subspaces in $F_{q^n}$. Due to trace, however, various choices of coefficient vector spaces may yield the same subcode. The following result shows that this is not the case in our setting.

We will continue considering the cyclic code $C$ with the trace representation in (2.1). For vector spaces $V_1, \ldots, V_s \subseteq F_{q^n}$, let us denote the subcode determined by them as in (2.2) by $C_V$.  

**Theorem 3.1.** Consider the cyclic code $C$ in (2.1) and let $V = (V_1, \ldots, V_s)$ and $W = (W_1, \ldots, W_s)$ be collection of vector subspaces in $F_{q^n}$ that determine subcodes $C_V$ and $C_W$ as in (2.2). Then, $C_V = C_W$ if and only if $V_j = W_j$ for all $j$. In particular, the number of QC subcodes of index $\ell$ in $C$ is the same as the number of vector space choices $V_1, \ldots, V_s \subseteq F_{q^n}$ that yield this index.
Proof. Let \( \lambda_j \in V_j \) for all \( j \) and consider the codeword

\[
c_\lambda = \left( \text{Tr}_{F_q^n/F_q} \left( \sum_{j=1}^s \lambda_j \alpha^k \right) \right)_k \in C_V.
\]

This codeword also belongs to \( C_W \) if and only if there exists \( \beta_j \in W_j \) such that

\[
c_\lambda = c_\beta = \left( \text{Tr}_{F_q^n/F_q} \left( \sum_{j=1}^s \beta_j \alpha^k \right) \right)_k.
\]

This holds exactly when

\[
(3.1) \quad \text{Tr}_{F_q^n/F_q} \left( (\lambda_1 - \beta_1)x^{i_1} + \cdots + (\lambda_n - \beta_n)x^{i_n} \right) = 0 \text{ for all } x \in F_q^n.
\]

Since each \( i_j \) has full size \( q \)-cyclotomic coset by assumption, Theorem 2.1 implies that \((3.1)\) holds if and only if \( \lambda_j = \beta_j \) for all \( j \).

\[
\square
\]

Remark 3.2. We can loosen the assumption on the cyclotomic cosets of \( i_j \)'s and still write a criterion for equality of subcodes \( C_V \) and \( C_W \). Namely, suppose that

\[
|Cyc_q(i_j)| = n \text{ for } 1 \leq j \leq r, \quad |Cyc_q(i_j)| = \delta_j < n \text{ for } r + 1 \leq j \leq s.
\]

Theorem 2.1 implies that \( C_V = C_W \) if and only if

\[
V_j = W_j \text{ for } 1 \leq j \leq r \text{ and } \text{Tr}_{F_q^n/F_q^{|\delta_j|}}(V_j) = \text{Tr}_{F_q^n/F_q^{|\delta_j|}}(W_j) \text{ for } r + 1 \leq j \leq s \text{ (cf. } 3.1).\]

Let us recall that the number of nonzero \( F_q \)-subspaces in \( F_q^n \) is determined by \( q \)-binomial coefficients as

\[
(3.2) \quad \binom{n}{k}_q = \frac{n!_q}{k!_q \cdot (n-k)!_q} = \frac{\sum_{k=1}^n \binom{n}{k}_q \sum_{k=1}^n \binom{n}{k}_q \cdot \binom{n}{k}_q}{\binom{n}{k}_q \cdot \binom{n}{k}_q \cdot \binom{n}{k}_q}.
\]

Note that each \( k \in \{1, \ldots, n\} \) counts \( F_q \)-subspaces of dimension \( k \). In the rest of the manuscript, the number of subspaces will refer to this number, which excludes the zero subspace.

The following result provides the number of nonzero vector subspaces in \( F_q^n \) maximally defined over an intermediate field of the extension \( F_q^n/F_q \). It will be used in the following sections to obtain enumeration results for QC subcodes of certain cyclic codes. We first introduce some notation.

Let \( n = u_1^{a_1} u_2^{a_2} \cdots u_t^{a_t} \), where \( u_i \)'s are pairwise distinct prime numbers and \( a_i \)'s are nonnegative integers. For \( \vec{i} = (i_1, \ldots, i_t) \in \prod_{1 \leq j \leq t} \{0,1, \ldots, a_j\} \), we denote the intermediate field \( F_{q_i^{a_1} \cdots q_i^{a_t}} \) by \( F_{i_1 \cdots i_t} \). For any \( 1 \leq j_1 < j_2 < \cdots < j_v \leq t \), we let

\[
(3.3) \quad N_{\vec{i}}(j_1, \ldots, j_v) := \binom{n}{j_1 \ldots j_v} \binom{n}{j_1 \ldots j_v} \cdots \binom{n}{j_1 \ldots j_v}.
\]

In \((3.3)\), only the terms corresponding to \( j_v \)'s are written. For any \( \mu \in \{1, \ldots, t\} \setminus \{j_1, \ldots, j_v\} \), the exponent of \( u_\mu \) in the dimension part of the expression is \( a_\mu - i_\mu \) and the exponent of \( u_\mu \) in the
field size part is \( i_\mu \). Finally, we will assume that

\[
N_i(j_1, \ldots, j_v) = 0, \quad \text{if } i_{j_\nu} + 1 > a_{j_\nu} \text{ for some } \nu \in \{1, \ldots, v\}.
\]

**Theorem 3.3.** Let \( n = u_1^{a_1} u_2^{a_2} \cdots u_t^{a_t} \), where \( u_i \)'s are pairwise distinct prime numbers and \( a_i \)'s are nonnegative integers, and consider the extension \( \mathbb{F}_{q^n}/\mathbb{F}_q \). For any \( \vec{i} = (i_1, \ldots, i_t) \in \prod_{1 \leq j \leq t} \{0, 1, \ldots, a_j\} \), the number of nonzero subspaces of \( \mathbb{F}_{q^n} \) that are maximally defined over \( \mathbb{F}_{i_1, \ldots, i_t} \) is given by

\[
N(\vec{a}; \vec{i}) = \sum_{1 \leq j \leq t} N_i(j) + \sum_{1 \leq j_1 < j_2 \leq t} N_i(j_1, j_2) - \cdots - (-1)^{t-1} N_i(1, 2, \ldots, t).
\]

**Proof.** The number of nonzero vector spaces in \( \mathbb{F}_{q^n} \) defined over \( \mathbb{F}_{i_1, \ldots, i_t} \) is \( N(u_1^{a_1-i_1} \cdots u_t^{a_t-i_t}, q^{u_1^{i_1} \cdots u_t^{i_t}}) \). Note that if a vector space \( V \subseteq \mathbb{F}_{q^n} \) is defined over an intermediate field properly containing \( \mathbb{F}_{i_1, \ldots, i_t} \), then it also has a vector space structure over \( \mathbb{F}_{i_1, \ldots, i_t} \). Therefore we need to subtract the number of all such vector spaces from \( N(u_1^{a_1-i_1} \cdots u_t^{a_t-i_t}, q^{u_1^{i_1} \cdots u_t^{i_t}}) \). An intermediate field properly containing \( \mathbb{F}_{i_1, \ldots, i_t} \) has to contain at least one of the following fields:

\[
\mathbb{F}_{i_1+1, i_2, \ldots, i_t}, \quad \mathbb{F}_{i_1, i_2+1, i_3, \ldots, i_t}, \quad \mathbb{F}_{i_1, \ldots, i_{t-1}, i_t+1}.
\]

For each \( 1 \leq j \leq t \), set

\[
S^j_i := \{ V \subseteq \mathbb{F}_{q^n} : V \neq 0, \ V: \text{vector space over } \mathbb{F}_{i_1, \ldots, i_{j-1}, i_j+1, i_{j+1}, \ldots, i_t} \}.
\]

For any \( 1 \leq j_1 < j_2 < \cdots < j_v \leq t \), note that \( S^j_{j_1} \cap S^j_{j_2} \cap \cdots \cap S^j_{j_v} \) consists of nonzero subspaces \( V \subseteq \mathbb{F}_{q^n} \) which are defined over the composite of the intermediate fields

\[
\mathbb{F}_{i_1, \ldots, i_{j_1-1}, i_{j_1}+1, i_{j_1+1}, \ldots, i_t}, \ldots, \mathbb{F}_{i_1, \ldots, i_{j_v-1}, i_{j_v}+1, i_{j_v+1}, \ldots, i_t}.
\]

Hence,

\[
|S^j_{j_1} \cap S^j_{j_2} \cap \cdots \cap S^j_{j_v}| = N_i(j_1, \ldots, j_v).
\]

Observe that \( S^j_{j_1} \cap S^j_{j_2} \cap \cdots \cap S^j_{j_v} = \emptyset \) if \( i_{j_\nu} + 1 > a_{j_\nu} \) for some \( \nu \in \{1, \ldots, v\} \) (cf. (3.4)). Since the number of nonzero subspaces in \( \mathbb{F}_{q^n} \) that are defined maximally over the subfield \( \mathbb{F}_{i_1, \ldots, i_t} \) is

\[
N(u_1^{a_1-i_1} \cdots u_t^{a_t-i_t}, q^{u_1^{i_1} \cdots u_t^{i_t}}) - |S^j_1 \cup S^j_2 \cup \cdots \cup S^j_t|,
\]

and the inclusion-exclusion principle states that

\[
|S^j_1 \cup S^j_2 \cup \cdots \cup S^j_t| = \sum_{1 \leq j \leq t} |S_j| - \sum_{1 \leq j_1 < j_2 \leq t} |S_{j_1} \cap S_{j_2}| + \cdots + (-1)^{t-1}|S^j_1 \cap S^j_2 \cap \cdots \cap S^j_t|,
\]

the result follows. \( \square \)
4. QC Subcodes of the Simplex Code

The \( q \)-ary simplex code of length \( N = q^n - 1 \) is defined as

\[
C = \left\{ \left( \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \left( \lambda \alpha^k \right) \right)_{0 \leq k \leq N-1} : \lambda \in \mathbb{F}_{q^n} \right\}.
\]

We will let \( n \) be as general as possible: \( n = u_1^{a_1} u_2^{a_2} \cdots u_t^{a_t} \), where \( u_i \)'s are pairwise distinct prime numbers and \( a_i \)'s are nonnegative integers. For \( \mathbf{i} = (i_1, \ldots, i_t) \in \prod_{1 \leq j \leq t} \{0, 1, \ldots, a_j\} \), let

\[
L_{\mathbf{i}} = L_{i_1, \ldots, i_t} := \frac{q^n - 1}{q^{u_{i_1} \cdots u_{i_t}} - 1}.
\]

By Theorem 2.6, the set of indices of QC subcodes in \( C \) is

\[
I = \{ L_{i_1, \ldots, i_t} : 0 \leq i_j \leq a_j \text{ for all } j \},
\]

when \( q \neq 2 \). For \( q = 2 \), one has to exclude \( L_0, \ldots, 0 = 2^n - 1 \) from the set above.

Let us denote the intermediate field \( \mathbb{F}_{q^{u_{i_1} \cdots u_{i_t}}} \) of \( \mathbb{F}_{q^n}/\mathbb{F}_q \) by \( \mathbb{F}_{i_1, \ldots, i_t} \). By Theorem 3.1, the number of QC subcodes of index \( L_{i_1, \ldots, i_t} \) in the simplex code is equal to the number of nonzero subspaces in \( \mathbb{F}_q^{n} \) that are defined maximally over the subfield \( \mathbb{F}_{i_1, \ldots, i_t} \). Using Theorem 3.3, we obtain the following.

**Theorem 4.1.** Let \( C \) be the \( q \)-ary simplex code of length \( N = q^n - 1 \), where \( n = u_1^{a_1} u_2^{a_2} \cdots u_t^{a_t} \) for pairwise distinct primes \( u_i \) and nonnegative integers \( a_i \). With the notations in (4.1), (3.3) and (3.4), for any \( \mathbf{i} = (i_1, \ldots, i_t) \in \prod_{1 \leq j \leq t} \{0, 1, \ldots, a_j\} \), \( C \) has

\[
N(u_1^{a_1-i_1} \cdots u_t^{a_t-i_t}, q^{u_{i_1} \cdots u_{i_t}}) - \sum_{1 \leq j \leq t} N_t(j) + \sum_{1 \leq j_1 < j_2 \leq t} N_t(j_1, j_2) - \cdots - (-1)^{t-1} N_t(1, 2, \ldots, t)
\]

QC subcodes of index \( L_{\mathbf{i}} \). For \( q = 2 \), exclude \( \mathbf{i} = (0, \ldots, 0) \) from this conclusion.

5. QC Subcodes of Duals of BCH Codes

5.1. Dual of the binary double-error-correcting BCH code. Dual of the binary double-error-correcting BCH code of length \( N = 2^n - 1 \) is defined as

\[
C = \left\{ \left( \text{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2} \left( \lambda \alpha^k + \beta \alpha^{3k} \right) \right)_{0 \leq k \leq N-1} : \lambda, \beta \in \mathbb{F}_{2^n} \right\}.
\]

With the notation of Theorem 2.6, \( i_1 = 1 \) and \( i_2 = 3 \). Note that \( |\text{Cyc}(3)| = n \) for any \( q \) and \( n \) except for \( q = 2 \) and \( n = 2 \). We will investigate this code for two families of \( n \) values.

5.1.1. \( n = u^{a-1} \): power of a prime. In order to have full cyclotomic coset for 3, we will exclude the case \( u = 2 = a \), in which case \( C \) is a rather short and uninteresting code.

Intermediate fields of the extension \( \mathbb{F}_{2^{u^{a-1}}}/\mathbb{F}_2 \) are

\[
\mathbb{F}_2 \subset \mathbb{F}_{2^u} \subset \mathbb{F}_{2^{u^2}} \subset \mathbb{F}_{2^{u^3}} \subset \cdots \subset \mathbb{F}_{2^{u^{a-1}}}.
\]
We have
\[ L_1 = 2^{u^a-1} - 1 = N, \quad L_2 = \frac{2^{u^a-1} - 1}{2^u - 1}, \quad L_3 = \frac{2^{u^a-1} - 1}{2^{u^2} - 1}, \ldots, L_a = 1 \] (cf. Theorem 2.6).

It is clear that \( \ell_i^1 = L_i \) for all \( i = 1, \ldots, a \). Now we need to compute \( \ell_i^2 \)'s. Since \( 2^2 \equiv 1 \) (mod 3), we have
\[ 2^{u^a-1} - 1 \equiv \begin{cases} 1 \text{ mod } 3, & \text{if } u \text{ is odd}, \\ 0 \text{ mod } 3, & \text{if } u = 2. \end{cases} \]

Note that \( L_i \mid L_1 \) for all \( i \). Hence for \( u \) an odd prime, we have \( 3 \nmid L_i \) for all \( i \) (since \( 3 \nmid L_1 \) in this case) and
\[ \ell_i^2 = \frac{\text{lcm}(3, L_i)}{3} = L_i. \]

For \( u = 2 \) however, \( 3 \mid L_1 \). Therefore \( \ell_2^2 = L_1/3 = L_2 \) in this case. For \( i > 1 \), we have
\[ L_i = \frac{L_1}{2^{2i-1} - 1} = 1 + 2^{2i-1} + 2^{2^2 \cdot 2i-1} + \cdots + 2^{(2^{a-i-1}) \cdot 2^i-1} \equiv 2^{a-i} \text{ mod 3} \]
\[ \equiv 0 \text{ mod 3} \]
whether \( a - i \) is odd or even. Hence, \( 3 \nmid L_i \) for \( i > 1 \) when \( u = 2 \). Therefore for \( u = 2 \),
\[ \ell_i^2 = L_i \text{ for } i = 2, \ldots, a. \]

Our conclusions are summarized in Table 1.

| \( u > 2 \) | \( u = 2 \) |
|---|---|
| \( \ell_1^1 = L_1 \) | \( \ell_1^1 = L_1 \) |
| \( \ell_2^1 = L_2 \) | \( \ell_2^1 = L_2 \) |
| \( \ell_1^2 = L_2 \) | \( \ell_1^2 = L_2 \) |
| \( \ell_2^2 = L_2 \) | \( \ell_2^2 = L_2 \) |
| \( \ell_3^1 = L_3 \) | \( \ell_3^1 = L_3 \) |
| \( \ell_3^2 = L_3 \) | \( \ell_3^2 = L_3 \) |
| \( \vdots \) | \( \vdots \) |
| \( \ell_a^1 = L_a \) | \( \ell_a^1 = L_a \) |
| \( \ell_a^2 = L_a \) | \( \ell_a^2 = L_a \) |

**Table 1.** \( \ell_i^1 \) and \( \ell_i^2 \) values for the dual of the binary BCH code for \( n = u^{a-1} \)

The following result describes all possible indices for QC subcodes of the dual of the binary double-error-correcting BCH code when \( n = u^{a-1} \).

**Proposition 5.1.** For the dual of the binary double-error-correcting BCH code \( C \) of length \( N = 2^{u^{a-1}} - 1 \), indices of QC subcodes are
\[ I = \{ L_a = 1, L_{a-1}, \ldots, L_2 \}. \]
Proof. By Theorem 2.6, \( I \) consists of \( \ell_1 \)'s, \( \ell_2 \)'s and other values that \( \text{lcm}(\ell_1, \ell_2) \)'s can produce. Of course, \( L_1 = N \) is excluded from \( I \). By Table 1, \( L_2, L_3, \ldots, L_a \) are all contained in \( I \). We need to check the outcomes of \( \text{lcm}(\ell_1, \ell_2) \)'s.

It is well-known that the polynomial \( x^{u^{t-1}} - 1 \) (for any \( t \)) factors over \( \mathbb{Q} \) into cyclotomic polynomials:

\[
x^{u^{t-1}} - 1 = (x - 1)\phi_u(x)\phi_{u^2}(x)\phi_{u^3}(x) \cdots \phi_{u^{t-1}}(x).
\]

Therefore for any \( 1 \leq i \leq a \), we have

\[
L_i = \frac{2^{u^{a-1}} - 1}{2^{u^{i-1}} - 1} = \frac{\phi_u(2)\phi_{u^2}(2) \cdots \phi_{u^{a-1}}(2)}{\phi_u(2)\phi_{u^2}(2) \cdots \phi_{u^{i-1}}(2)} = \phi_u(2)\phi_{u+1}(2) \cdots \phi_{u^{a-1}}(2).
\]

Hence for \( i < j \), we have \( L_j \mid L_i \) and \( \text{lcm}(L_i, L_j) = L_i \). Using Table 1, \( \text{lcm}(\ell_1, \ell_2) = \text{lcm}(L_i, L_j) \)'s do not bring any new value to \( I \) for both odd and even prime values of \( u \). \( \square \)

We are ready to count QC subcodes.

**Theorem 5.2.** Consider the dual of the binary double-error-correcting BCH code \( C \) of length \( N = 2^{u^{a-1}} - 1 \) for a prime number \( u \) and an integer \( a \geq 2 \) except for \( u = 2 = a \). Let

\[
A_j = N(u^{a-j-1}, 2^j) - N(u^{a-j-2}, 2^{j+1}) \quad \text{for} \quad 0 \leq j \leq a - 1.
\]

(i) If \( u \) is odd, then \( C \) has 3 cyclic subcodes (including itself),

\[
3A_{a-2} + A_{a-2}N(u, 2^{u^{a-2}})
\]

QC subcodes of index \( L_{a-1} \), and

\[
2A_{j-1} + A_{j-1}(N(u^{a-j}, 2^{j-1}) + N(u^{a-j-1}, 2^j)) \quad \text{for} \quad 2 \leq j \leq a - 2.
\]

(ii) If \( u = 2 \), then \( C \) has 3 cyclic subcodes (including itself) for any \( a \). For \( a \geq 4 \), there are

\[
3A_{a-2} + A_{a-2}N(u, 2^{u^{a-2}})
\]

QC subcodes of index \( L_{a-1} \),

\[
2A_{j-1} + A_{j-1}(N(u^{a-j}, 2^{j-1}) + N(u^{a-j-1}, 2^j)) \quad \text{for} \quad 3 \leq j \leq a - 2, \text{ and}
\]

\[
2A_1 + A_0 + A_1N(u^{a-1}, 2) + (A_0 + A_1)N(u^{a-3}, 2^{u^2}) \quad \text{for} \quad 2 \leq j \leq a - 2.
\]

QC subcodes of index \( L_2 \). If \( a = 3 \), there are

\[
3A_1 + 2A_0 + A_1N(u^2, 2)
\]

QC subcodes of index \( L_2 \).
Proof. A QC subcode of $C$ is of the form

$$C' = \left\{ \left( \text{Tr}_{F_{2^u}/F_{2}} (\lambda \alpha^k + \beta \alpha^{3^k}) \right)_{0 \leq k \leq N-1} : \lambda \in V, \beta \in W \right\},$$

where $V, W \subseteq F_{2^u}$ are subspaces defined over some intermediate field of the extension $F_{2^u}/F_2$. Note that this extension’s subfield structure is rather simple (cf. (5.1)). Hence it is easy to see that $A_j$ describes the number of nonzero subspaces in $F_{2^u}$ that yield QC subcodes of index $1$, $2$, $\ldots$, $j$ for $0 \leq j \leq a - 1$ (cf. (5.5) and note that $A_{a-1} = 1$ by the convention in (3.1)).

The following observation in the proof of Proposition 5.1 will be the main tool for our analysis below:

If $i < j$, then $\text{lcm}(L_i, L_j) = L_i$.

Note that in any case, subcodes of index $L_a = 1$ (i.e., cyclic subcodes) are obtained by letting $(V = F_{2^u}, W = 0)$, $(V = 0, W = F_{2^u})$ or $(V = F_{2^u}, W = F_{2^u})$ (the code $C$ itself).

Moreover when $a = 2$, we have an extension $F_2/F_2$ and $C$ only has cyclic subcodes and there are 3 cyclic subcodes as noted above. Hence, we will assume that $a \geq 3$ below.

(i) Choices of $V$ and $W$ that yield QC subcodes of index $L_{a-1}$ can be systematically listed as follows:

$V$: maximally defined over $F_{2^u}$, $W = 0$

$V = 0, W$: maximally defined over $F_{2^u}$

$V$: maximally defined over $F_{2^u}$, $W$: defined over $F_{2^u}$

$V$: maximally defined over $F_{2^u}$, $W$: maximally defined over $F_{2^u}$

The number of such choices are, respectively,

$$A_{a-2}, A_{a-2}, A_{a-2}N(u, 2^{u-2}), A_{a-1}A_{a-2},$$

whose sum is $3A_{a-2} + A_{a-2}N(u, 2^{u-2})$. Note that for $a = 3$, there are only QC subcodes of index $L_{a-1} = L_2$ (other than cyclic subcodes) whose count is given above and (5.2) does not apply to this situation. For $a \geq 4$ and $2 \leq j \leq a - 2$, choices of $V$ and $W$ that yield QC subcodes of index $L_j$ are

$V$: maximally defined over $F_{2^u}$, $W = 0$

$V = 0, W$: maximally defined over $F_{2^u}$

$V$: maximally defined over $F_{2^u}$, $W$: defined over $F_{2^u}$

$V$: maximally defined over $F_{2^u}$, $W$: maximally defined over $F_{2^u}$

$: $V$: maximally defined over $F_{2^u}$, $W$: maximally defined over $F_{2^u}$

The number of choices for the first three combinations are, respectively,

$$A_{j-1}, A_{j-1}, A_{j-1}N(u^{a-j}, 2^{u-j-1}).$$

The remaining choices add up to $N(u^{a-j-1}, 2^{u-j-1})A_{j-1}$. The total is

$$2A_{j-1} + A_{j-1}(N(u^{a-j}, 2^{u-j-1}) + N(u^{a-j-1}, 2^{u-j-1})).$$
(ii) For \( a \geq 5 \), the count of \( L_j \) for \( 3 \leq j \leq a \) follows as in part (i) and the results for index \( L_{a-1} \) subcodes, (5.3) and (5.4) are identical. For \( a = 4 \), counting goes similarly again but note that (5.3) does not apply to this situation. So, what needs special attention here is the number of index \( L_2 \) QC subcodes for \( a \geq 4 \) and \( a = 3 \). For \( a \geq 4 \), choices of \( V \) and \( W \) that yield QC subcode of index \( L_2 \) are as follows:

\[
\begin{align*}
V & : \text{maximally defined over } \mathbb{F}_{2^u}, W = 0 \\
V = 0, W : \text{maximally defined over } \mathbb{F}_2 \\
V = 0, W : \text{maximally defined over } \mathbb{F}_{2^u} \\
V : \text{maximally defined over } \mathbb{F}_{2^u}, W : \text{defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^2}, W : \text{maximally defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^3}, W : \text{maximally defined over } \mathbb{F}_2 \\
& \vdots \\
V : \text{maximally defined over } \mathbb{F}_{2^u^{a-1}}, W : \text{maximally defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^{a-1}}, W : \text{maximally defined over } \mathbb{F}_{2^u}
\end{align*}
\]

The number of choices for the first four combinations are, respectively,

\[ A_1, A_0, A_1, A_1 N(u^{a-1}, 2). \]

The remaining choices add up to \((A_0 + A_1) N(u^{a-3}, 2u^2)\). Total is the desired value.

When \( a = 3 \), choices of \( V \) and \( W \) that yield QC subcode of index \( L_2 \) are as follows:

\[
\begin{align*}
V & : \text{maximally defined over } \mathbb{F}_{2^u}, W = 0 \\
V = 0, W : \text{maximally defined over } \mathbb{F}_2 \\
V = 0, W : \text{maximally defined over } \mathbb{F}_{2^u} \\
V : \text{maximally defined over } \mathbb{F}_{2^u}, W : \text{defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^2}, W : \text{maximally defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^3}, W : \text{maximally defined over } \mathbb{F}_2 \\
& \vdots \\
V : \text{maximally defined over } \mathbb{F}_{2^u^{a-1}}, W : \text{maximally defined over } \mathbb{F}_2 \\
V : \text{maximally defined over } \mathbb{F}_{2^u^{a-1}}, W : \text{maximally defined over } \mathbb{F}_{2^u}
\end{align*}
\]

The number of choices for the first four combinations are, respectively,

\[ A_1, A_0, A_1, A_1 N(u^2, 2). \]

The remaining two choices add up to \((A_0 + A_1)\). Hence the total is \(3A_1 + 2A_0 + A_1 N(u^2, 2)\) and the proof is finished. \( \square \)

5.1.2. \( n = uv \): \textit{product of two distinct primes}. We consider the dual of the binary double-error-correcting BCH code of length \( N = 2^uv - 1 \), where \( u \) and \( v \) are distinct primes.

We have

\[
\begin{align*}
L_1 &= 2^uv - 1 \\
L_2 &= \frac{2^uv - 1}{2^u - 1} = 1 + 2^u + 2^{2u} + \cdots + 2^{(v-1)u} \\
L_3 &= \frac{2^uv - 1}{2^v - 1} = 1 + 2^v + 2^{2v} + \cdots + 2^{(u-1)v} \\
L_4 &= 1.
\end{align*}
\]
Depending on whether the product $uv$ is odd or even, we can compute the values of $\ell_1^i$ and $\ell_2^j$ as in Section 5.1.1. The results are presented in Table 2.

$$
\begin{array}{|c|c|}
\hline
u, v: \text{ odd} & u = 2, v: \text{ odd} \\
\hline
\ell_1^1 = L_1 & \ell_1^1 = L_1 \\
\ell_2^1 = L_2 & \ell_1^2 = \frac{L_1}{3} \\
\ell_1^2 = L_3 & \ell_2^2 = \frac{L_2}{3} (v = 3) \\
\ell_1^3 = L_4 & \ell_2^2 = L_2 (v \neq 3) \\
\ell_2^3 = L_3 & \ell_3^2 = \frac{L_3}{3} \\
\ell_2^4 = L_4 & \ell_4^2 = L_4 \\
\hline
\end{array}
$$

Table 2. $\ell_1^i$ and $\ell_2^j$ values for the dual of the BCH code for $n = uv$

The following result describes all possible indices for QC subcodes in the case $n = uv$.

**Proposition 5.3.** For the dual of the binary double-error-correcting BCH code $C$ of length $N = 2^u - 1$, indices of QC subcodes are follows:

$$I = \begin{cases} 
\{L_2, L_3, L_4\} & \text{if } u, v \text{ are both odd} \\
\{L_2, L_3, L_3/3, L_4\} & \text{if } u = 2, v \neq 3: \text{ odd} \\
\{1, 3, 7, 9, 21\} & \text{if } u = 2, v = 3
\end{cases}$$

**Proof.** We use the $\ell_1^i$ and $\ell_2^j$ values in Table 2. Let us note that

$$2^{uv} - 1 = \phi_u(2)\phi_v(2)\phi_{uv}(2),$$

$$2^u - 1 = \phi_u(2),$$

$$2^v - 1 = \phi_v(2).$$

Hence, $L_2 = \phi_v(2)\phi_{uv}(2)$ and $L_3 = \phi_u(2)\phi_{uv}(2)$.

For the case $u$ and $v$ are both odd, other than $L_2, L_3$ and $L_4$, the only possible index for a QC subcode is $\text{lcm}(L_2, L_3)$. However,

$$\text{lcm}(L_2, L_3) = \phi_{uv}(2)\text{lcm}(2^u - 1, 2^v - 1) = \phi_{uv}(2)(2^u - 1)(2^v - 1) = L_1.$$ 

Hence $I = \{L_2, L_3, L_4\}$ in this case.

For $u = 2$, $v$ odd and $v \neq 3$, note that $L_1/3 = L_2$. Therefore, other than $L_2, L_3, L_3/3$ and $L_4$, the only other possible index for a QC subcode is $\text{lcm}(L_2, L_3/3)$. Since $\phi_u(2) = 2^u - 1 = 3$ in this case, we have $L_3/3 = \phi_{uv}(2)$ and this is a divisor of $L_2$. Hence $\text{lcm}(L_2, L_3/3) = L_2$ and there is no new contribution to $I$ in this case.

When $u = 2$ and $v = 3$, it is easy to verify the set $I$ is as stated in the proposition.

We are ready to count QC subcodes of the dual of the BCH code in the case $n = uv$.

**Theorem 5.4.** Consider the dual of the binary double-error-correcting BCH code $C$ of length $N = 2^u - 1$, where $u$ and $v$ are distinct prime numbers. Then $C$ has 3 cyclic subcodes (including itself) for any $u$ and $v$. Moreover:
(i) If $u$ and $v$ are odd, then $C$ has $(2N(v, 2^u) + N(v, 2^u)^2 - 3)$ QC subcodes of index $L_2$ and $(2N(u, 2^v) + N(u, 2^v)^2 - 3)$ QC subcodes of index $L_3$.

(ii) If $u = 2$ and $v \neq 3$ and odd, then $C$ has $(N(u, 2^v) + N(v, 2^u) + N(u, 2^v) - 2N(u, 2^v) - 1)$ QC subcodes of index $L_2$, $(N(u, 2^v)^2 - 1)$ QC subcodes of index $L_3$ and $(2N(u, 2^v) - 2)$ QC subcodes of index $L_3/3$.

(iii) If $u = 2$ and $v = 3$, then $C$ has 124194 QC subcodes of index 21, 99 QC subcodes of index 9, 84 QC subcodes of index 7 and 18 QC subcodes of index 3.

Proof. Note that the number of subspaces in $F_2^{v, u}$ that are maximally defined over $F_2$, $F_{2^u}$ and $F_{2^v}$ are respectively given by $(N(u, 2^v) - N(u, 2^v) - N(v, 2^u) + 1)$, $(N(v, 2^u) - 1)$ and $(N(u, 2^v) - 1)$. Consider a subcode $C'$ as in (5.6) of $C$. We will use Table 2 and the combination of lcm$(\ell_1, \ell_2)$’s that lead to the corresponding index. For all possibilities of $u$ and $v$, it is clear that there are 3 cyclic subcodes of $C$ obtained by the choices $(V = F_{2^u}, W = 0)$, $(V = 0, W = F_{2^v})$ and $(V = F_{2^{uv}}, W = F_{2^{uv}})$.

If $u$ and $v$ are both odd, then the choices of subspaces $V, W$ that yield the indices $L_2, L_3$ are as follows:

$L_2$: $V$: maximally defined over $F_{2^v}$, $W = 0$

$V = 0, W$: maximally defined over $F_{2^u}$

$V$: maximally defined over $F_{2^v}$, $W$: maximally defined over $F_{2^v}$

$V$: maximally defined over $F_{2^v}$, $W$: maximally defined over $F_{2^{uv}}$

$V$: maximally defined over $F_{2^{uv}}, W$: maximally defined over $F_{2^u}$

Total number of such subspaces is

$$(N(v, 2^u) - 1) + (N(v, 2^u) - 1)N(v, 2^u) + (N(v, 2^u) - 1) = 2N(v, 2^u) + N(v, 2^u)^2 - 3.$$

$L_3$: $V$: maximally defined over $F_{2^v}$, $W = 0$

$V = 0, W$: maximally defined over $F_{2^v}$

$V$: maximally defined over $F_{2^v}$, $W$: maximally defined over $F_{2^v}$

$V$: maximally defined over $F_{2^v}$, $W$: maximally defined over $F_{2^{uv}}$

$V$: maximally defined over $F_{2^{uv}}, W$: maximally defined over $F_{2^v}$

Total number of such subspaces is

$$2N(u, 2^v) + N(u, 2^v)^2 - 3.$$
Total number of such subspaces is
\[ N(uv, 2) + N(v, 2^u) + N(uv, 2^u)N(v, 2^u) - 2N(u, 2^v) - 1. \]

\[ L_3 : \quad V: \text{maximally defined over } \mathbb{F}_{2v}, \quad W = 0 \]
\[ V: \text{maximally defined over } \mathbb{F}_{2v}, W: \text{maximally defined over } \mathbb{F}_{2v} \]
\[ V: \text{maximally defined over } \mathbb{F}_{2v}, W: \text{maximally defined over } \mathbb{F}_{2v} \]

Total number of such subspaces is
\[ N(uv, 2^u)^2 - 1. \]

\[ L_3/3 : \quad V = 0, W: \text{maximally defined over } \mathbb{F}_{2v} \]
\[ V: \text{maximally defined over } \mathbb{F}_{2v}, W: \text{maximally defined over } \mathbb{F}_{2v} \]

Total number of such subspaces is
\[ 2(N(u, 2^v) - 1). \]

Finally, if \( u = 2 \) and \( v = 3 \), we have
\[ \ell_1^2 = 21, \quad \ell_1^3 = 9, \quad \ell_1^4 = 1, \quad \ell_1^2 = 21, \quad \ell_2^2 = 7, \quad \ell_2^3 = 3, \quad \ell_2^4 = 1. \]

Moreover, \( N(3, 4) = 43 \), \( N(2, 8) = 10 \) and \( N(6, 2) = 2824 \). Then,

\[ 21 : \quad V: \text{maximally defined over } \mathbb{F}_{22}, W = 0 \]
\[ V = 0, W: \text{maximally defined over } \mathbb{F}_2 \]
\[ V: \text{maximally defined over } \mathbb{F}_{22}, W: \text{defined over } \mathbb{F}_2 \]
\[ V: \text{maximally defined over } \mathbb{F}_{2v}, W: \text{maximally defined over } \mathbb{F}_2 \]

Total number of such subspaces is
\[
\left( (N(3, 4) - 1) + (N(6, 2) - N(3, 4) - N(2, 8) + 1) + (N(3, 4) - 1)N(6, 2) + \right.
\]
\[
\left. (N(6, 2) - N(3, 4) - N(2, 8) + 1) \right) = 124194.
\]

\[ 9 : \quad V: \text{maximally defined over } \mathbb{F}_{23}, W = 0 \]
\[ V: \text{maximally defined over } \mathbb{F}_{23}, W: \text{maximally defined over } \mathbb{F}_{23} \]
\[ V: \text{maximally defined over } \mathbb{F}_{23}, W: \text{maximally defined over } \mathbb{F}_{26} \]

Total number of such subspaces is
\[ (N(2, 8) - 1) + (N(2, 8) - 1)N(2, 8) = 99. \]

\[ 7 : \quad V = 0, W: \text{maximally defined over } \mathbb{F}_{22} \]
\[ V: \text{maximally defined over } \mathbb{F}_{26}, W: \text{maximally defined over } \mathbb{F}_{22} \]

Total number of such subspaces is
\[ 2(N(3, 4) - 1) = 84. \]

\[ 3 : \quad V = 0, W: \text{maximally defined over } \mathbb{F}_{23} \]
\[ V: \text{maximally defined over } \mathbb{F}_{26}, W: \text{maximally defined over } \mathbb{F}_{23} \]

Total number of such subspaces is
\[ 2(N(2, 8) - 1) = 18. \]
5.2. Dual of the $p$-ary BCH Code of Designed Distance 3. Let $p$ be an odd prime. Dual of the $p$-ary BCH code of length $N = p^n - 1$ and designed distance 3 has the following trace representation:

$$C = \left\{(\text{Tr}_{F_{p^n}/F_{p}}(\lambda \alpha^k + \beta \alpha^{2k}))_{0 \leq k \leq N-1} : \lambda, \beta \in F_{p^n}\right\}.$$ 

As in Section 5.1.2 we will consider the case $n = uv$ where $u$ and $v$ are distinct prime numbers. In this case $|Cyc_p(2)| = n$ for any odd prime $p$ and hence Theorem 3.1 applies. We have

$$L_1 = \frac{p^{uv} - 1}{p - 1} = \phi_u(p)\phi_v(p)$$
$$L_2 = \frac{p^{uv} - 1}{p^v - 1} = 1 + p^u + p^{2u} + \cdots + p^{(v-1)u} = \phi_v(p)\phi_{uv}(p)$$
$$L_3 = \frac{p^{uv} - 1}{p^v - 1} = 1 + p^v + p^{2v} + \cdots + p^{(u-1)v} = \phi_u(p)\phi_{uv}(p)$$
$$L_4 = 1.$$ 

Note that any positive power of $p$ is congruent to 1 mod 2. Therefore $L_2 \equiv v$ and $L_3 \equiv u$ mod 2. This implies that when $u$ and $v$ are both odd primes, no $L_i$ is divisible by 2. If $u = 2$ and $v$ is an odd prime, then $L_1$ and $L_3$ are divisible by 2, $L_2$ is not. Combining these observations, values of $\ell_1$ and $\ell_2$ are presented in Table 3. Note that when $u = 2$, $L_1/2$ and $L_3/2$ are not equal to $L_i$ for any $1 \leq i \leq 4$ regardless of choice of the odd primes $p$ and $v$.

| $u, v$: odd | $u = 2, v$: odd |
|-------------|-----------------|
| $\ell_1 = L_1$ | $\ell_1 = L_1$ |
| $\ell_2 = L_2$ | $\ell_2 = L_1/2$ |
| $\ell_3 = L_3$ | $\ell_3 = L_3/2$ |
| $\ell_4 = L_4$ | $\ell_4 = L_4$ |

Table 3. $\ell_1$ and $\ell_2$ values for the dual of the $p$-ary BCH code for $n = uv$

The following result describes all possible indices for QC subcodes in this case.

**Proposition 5.5.** For the dual of the $p$-ary BCH code $C$ of length $N = p^{uv} - 1$ and designed distance 3, indices of QC subcodes are follows:

$$I = \begin{cases} \{L_1, L_2, L_3, L_4\} & \text{if } u, v \text{ are both odd} \\
\{L_1, L_1/2, L_2, L_3, L_3/2, L_4\} & \text{if } u = 2, v: \text{ odd} \end{cases}$$

**Proof.** By Table 3, all index values in the statement belong to $I$. Again, we need to check that $\text{lcm}(\ell_1, \ell_2)$ values do not bring any different index to $I$. One of the key observations for this purpose is the following:

$$\text{lcm}(\phi_u(p), \phi_v(p)) = \phi_u(p)\phi_v(p) = \frac{(p^u - 1)(p^v - 1)}{(p - 1)^2}.$$
Therefore \( \text{lcm}(L_2, L_3) = L_1 \) in any case. Moreover, it is clear that \( L_i \mid L_1 \) for \( i = 2, 3, 4 \). Therefore the result follows immediately for the case \( u \) and \( v \) are both odd.

For \( u = 2, v \): odd case, the following extra least common multiple values, compared to previous case, can be easily verified:

\[
\begin{align*}
\text{lcm}(L_1, L_1/2) &= L_1 \\
\text{lcm}(L_1, L_3/2) &= L_1 \\
\text{lcm}(L_2, L_1/2) &= L_1/2 \\
\text{lcm}(L_2, L_3/2) &= L_1/2 \\
\text{lcm}(L_3, L_1/2) &= L_1 \\
\text{lcm}(L_3, L_3/2) &= L_3
\end{align*}
\]

Therefore the result also follows in the second case. \( \square \)

We count the QC subcodes in the following result.

**Theorem 5.6.** Consider the dual of the \( p \)-ary BCH code \( C \) of length \( N = p^{uv} - 1 \), where \( u \) and \( v \) are distinct prime numbers. Let \( A = N(uv, p) - N(u, pv) - N(v, pu) + 1 \). Then \( C \) has 3 cyclic subcodes (including itself) for any \( u \) and \( v \). Moreover:

(i) If \( u \) and \( v \) are odd, then \( C \) has

\[
\begin{align*}
A + A(N(uv, p) + N(u, pv) + N(v, pu)) &+ 2(N(u, pv)N(v, pu) - N(u, pv) - N(v, pu) + 1)
\end{align*}
\]

QC subcodes of index \( L_1 \),

\[
2N(v, pu) + N(v, pu)^2 - 3
\]

QC subcodes of index \( L_2 \), and

\[
2N(u, pv) + N(u, pv)^2 - 3
\]

QC subcodes of index \( L_3 \).

(ii) If \( u = 2 \) and \( v \) an odd prime, \( C \) has

\[
A + AN(uv, p) + (N(u, pv) - 1)(A + N(v, pu) - 1)
\]

QC subcodes of index \( L_1 \),

\[
2A + (N(v, pu) - 1)(A + N(u, pv) - 1)
\]

QC subcodes of index \( L_1/2 \),

\[
2N(v, pu) + N(v, pu)^2 - 3
\]

QC subcodes of index \( L_2 \),

\[
N(u, pv)^2 - 1
\]

QC subcodes of index \( L_3 \) and

\[
2(N(u, pv) - 1)
\]

QC subcodes of index \( L_3/2 \).

**Proof.** Let us note that the number of subspaces in \( \mathbb{F}_{p^{uv}} \) that are maximally defined over \( \mathbb{F}_p, \mathbb{F}_{p^u} \) and \( \mathbb{F}_{p^v} \) are given by \( A, (N(v, pu) - 1) \) and \( (N(u, pv) - 1) \), respectively. Consider a subcode \( C' \) of \( C \).

\[
C' = \left\{ \left( \text{Tr}_{\mathbb{F}_{p^u}/\mathbb{F}_p}(\lambda \alpha^k + \beta \alpha^{2k}) \right)_{0 \leq k \leq N-1} : \lambda \in V, \beta \in W \right\}.
\]
We will proceed as in the proof of Theorem 5.4 and use Table 3 and the combination of lcm($\ell_i^1, \ell_j^2$)'s that lead to the corresponding index (cf. proof of Proposition 5.5). For all possibilities of $u$ and $v$, it is clear that there are 3 cyclic subcodes of $C$.

If $u$ and $v$ are both odd, then the choices of subspaces $V, W$ that yield index $L_1$ are as follows:

1. $L_1$: $V$: maximally defined over $\mathbb{F}_p$, $W = 0$
   - $V = 0$, $W$: maximally defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: maximally defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_{p^u}$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_{p^u}$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_p$

Total number of such subspaces is

$$2A + AN(uv, p) + (N(v, p^u) - 1)(A + N(u, p^v) - 1) + (N(u, p^v) - 1)(A + N(v, p^u) - 1) + A,$$

which yields the desired result. For index $L_2$, we have

1. $L_2$: $V$: maximally defined over $\mathbb{F}_{p^u}$, $W = 0$
   - $V = 0$, $W$: maximally defined over $\mathbb{F}_{p^u}$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: maximally defined over $\mathbb{F}_{p^u}$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: maximally defined over $\mathbb{F}_{p^v}$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: maximally defined over $\mathbb{F}_{p^v}$

Total number of such subspaces is

$$2(N(v, p^u) - 1) + (N(v, p^u) - 1)N(v, p^u) + (N(v, p^u) - 1) = 2N(v, p^u) + N(v, p^u)^2 - 3.$$

The result for index $L_3$ follows identically.

When $u = 2$ and $v$ an odd prime, the choices of $V, W$ yielding index $L_1$ are as follows:

1. $L_1$: $V$: maximally defined over $\mathbb{F}_p$, $W = 0$
   - $V$: maximally defined over $\mathbb{F}_p$, $W$: defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_{p^u}$

Total number of such subspaces is

$$A + AN(uv, p) + (N(u, p^v) - 1)(A + N(v, p^u) - 1).$$

For the other indices, we have the following:

1. $L_1/2$: $V = 0$, $W$: maximally defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^u}$, $W$: maximally defined over $\mathbb{F}_p$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_{p^u}$
   - $V$: maximally defined over $\mathbb{F}_{p^v}$, $W$: maximally defined over $\mathbb{F}_p$
The total number of such subspaces is

\[ A + (N(v, p^u) - 1)(A + N(u, p^v) - 1) + A. \]

\[ L_2 : \quad V: \text{maximally defined over } \mathbb{F}_{p^v}, \ W = 0 \]
\[ V = 0, \ W: \text{maximally defined over } \mathbb{F}_{p^v} \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^u}, \ W: \text{maximally defined over } \mathbb{F}_{p^u} \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^u}, \ W: \text{maximally defined over } \mathbb{F}_{p^u} \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^u}, \ W: \text{maximally defined over } \mathbb{F}_{p^u} \]

Total number of such subspaces is

\[ 2(N(v, p^u) - 1) + (N(v, p^u) - 1)N(v, p^u) + (N(v, p^u) - 1). \]

\[ L_3 : \quad V: \text{maximally defined over } \mathbb{F}_{p^v}, \ W = 0 \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^v}, \ W: \text{maximally defined over } \mathbb{F}_{p^v} \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^v}, \ W: \text{maximally defined over } \mathbb{F}_{p^v} \]

Total number of such subspaces is

\[ 2(N(u, p^v) - 1) + (N(u, p^v) - 1)^2. \]

\[ L_3/2 : \quad V = 0, \ W: \text{maximally defined over } \mathbb{F}_{p^v} \]
\[ V: \text{maximally defined over } \mathbb{F}_{p^v}, \ W: \text{maximally defined over } \mathbb{F}_{p^v} \]

Total number of such subspaces is \( 2(N(u, p^v) - 1). \)

6. Examples

Our results in Sections 2 and 3 yield an algorithm, which can compute possible indices of QC subcodes of a given cyclic code based on Theorem 2.6, together with the number of these subcodes by using Theorems 3.1 and 3.3. In [1], we provide a Magma code ([2]), which illustrates the algorithm for length \( q^n - 1 \) cyclic codes of the form

\[ C = \left\{ \left( \text{Tr}_{q^n/q} \left( \lambda_1 \alpha^{k_1} + \lambda_2 \alpha^{k_2} + \lambda_3 \alpha^{k_3} \right) \right)_{0 \leq k \leq q^n - 2} ; \lambda_j \in \mathbb{F}_{q^n}, \ 1 \leq j \leq 3 \right\}, \]

where

\[ n = u_1^{a_1} u_2^{a_2} u_3^{a_3}, \text{ such that } a_i \geq 1, u_i : \text{distinct primes for } 1 \leq i \leq 3. \]

In the following examples, we used the code in [1] for some binary \( (q = 2) \) and ternary \( (q = 3) \) cyclic codes satisfying the conditions above. Namely, Table 1 presents the indices and the appearances of QC subcodes of binary simplex codes \((i_1 = 1, i_2 = i_3 = 0)\), dual of the double-error-correcting BCH codes \((i_1 = 1, i_2 = 3, i_3 = 0)\) and dual of the triple-error-correcting BCH codes \((i_1 = 1, i_2 = 3, i_3 = 5)\), for various \( n \) values. The results are listed in the form \([i, M_i]\), where \(M_i\) is the number of proper nonzero QC subcodes of index \( i \). In particular, for \( i = 1 \) the corresponding count \( M_1 \) is the number of proper nonzero cyclic subcodes. Table 5 presents the similar results for ternary simplex codes \((i_1 = 1, i_2 = i_3 = 0)\), dual of the BCH codes of designed distance 3 \((i_1 = 1, i_2 = 2, i_3 = 0)\) and dual of the BCH codes of designed distance 5 \((i_1 = 1, i_2 = 2, i_3 = 4)\). Note that we do not consider prime \( n \) values, since no proper QC subcode occurs in this case. We do not consider cases where \( q \)-cyclotomic coset mod \( q^n - 1 \) for some \( i_j \) has size less than \( n \).
| $n \backslash C$ | Simplex | Dual of Double-E.-C. BCH | Dual of Triple-E.-C. BCH |
|-------|--------|--------------------------|-------------------------|
| 6     | [1,0], [9,9], [21,42] | [1,2], [3,18], [7,84], [9,99], [21,124194] | [1,6], [3,36], [7,168], [9,1287], [21,5468988] |
| 8     | [1,0], [17,17], [85,510] | [1,2], [17,357], [85,220697910] | [1,6], [17,190961], [51,150417870], [85,11674914390] |
| 9     | [1,0], [73,146] | [1,2], [73,21900] | [1,6], [73,3241784] |
| 10    | [1,0], [33,33], [341,12276] | [1,2], [11,66], [33,1155], [341,2820939318120] | [1,6], [11,132], [33,42735], [341,3463549248736680] |
| 12    | [1,0], [65,65], [273,546], [585,5910], [1365,565721] | [1,2], [65,4485], [91,1092], [195,391950], [273,299208], [455,37897860], [585,34614450], [1365,27617278737667730] | [1,6], [13,260], [65,300495], [91,73164], [117,23400], [195,26260560], [273,169888806360], [455,2539156620], [585,207132845400], [819,146601246105077400], [1365,15623729801897799851310] |
| 14    | [1,0], [129,129], [5461,51409854] | [1,2], [43,258], [129,16899], [5461,20943210052503796112058] | [1,6], [43,516], [129,2247567], [5461,107668741349346608587731025396] |
| 15    | [1,0], [1057,2114], [4681,617892] | [1,2], [1057,4477452], [4681,381792995232] | [1,6], [1057,9474296888], [4681,235907600998352976] |
| 16    | [1,0], [257,257], [4369,78642], [21845,9370980720] | [1,2], [257,67077], [4369,6225300720], [21845,2\cdot3\cdot5\cdot17\cdot257], [9632900474097094857135899] | [1,6], [257,17373971], [4369,5838292582507289442], [13107,83875802178495526312038290], [21845,2\cdot3\cdot5\cdot7\cdot11\cdot17\cdot59\cdot257], [20627476\cdot9632900474097094857135899] |
| 18    | [1,0], [513,513], [4161,8322], [37449,195118125], [87381,2^3\cdot3\cdot5^2], [11\cdot19\cdot73\cdot370091] | [1,2], [171,1026], [513,264195], [1387,16644], [4161,69272328], [12483,162409399914], [29127,2820077158670472], [37449,38069459320434786], [87381,2^3\cdot3^2\cdot5^3\cdot7\cdot19\cdot23\cdot73\cdot911\cdot10067265549\cdot1237940881586443] | [1,6], [171,2052], [513,136588815], [1387,33288], [4161,576761402928], [12483,13518958457429676], [29127,234743222688194168928], [37449,74283548873680265257616], [87381,2^3\cdot7^2\cdot5^3\cdot7\cdot19\cdot23\cdot2382323\cdot2528261\cdot25131697\cdot143372569\cdot5369043671728307] |
| 20    | [1,0], [1025,1025], [33825,1151070], [69905,36070980], [349525,5\cdot7\cdot41\cdot43\cdot3922132101] | [1,2], [1025,1054725], [1127,11811101350], [33825,1323796103850], [69905,1301115742444320], [349525,2\cdot3^2\cdot5^2\cdot11\cdot17\cdot19^2\cdot31\cdot41\cdot5113\cdot4182209\cdot18840858933\cdot14764156982759] | [1,6], [205,4100], [1025,1083202575], [6765,4600200], [1127,121299086450], [13981,144283920], [33825,1525150786424500200], [69905,320508193887157765425602895040], [209715,2^3\cdot3^3\cdot5^2\cdot11\cdot31\cdot37\cdot41\cdot43\cdot9089307795660878604236175349273961], [349525,2^3\cdot3^2\cdot5^3\cdot11\cdot31\cdot41\cdot89\cdot126963961\cdot795792305160258988205007754270563107343898999] |

**Table 4.** Duals of binary BCH codes with designed distances 1, 3 and 5.
| n \ C | Simplex | Dual of BCH with \(d = 3\) | Dual of BCH with \(d = 5\) |
|------|---------|----------------|----------------|
| 4    | [1,0], [10,10], [40,200] | [1,2], [5,20], [10,120], [20,2400], [40,4240] | [1,6], [5,280], [10,30240], [20,508800], [40,8988800] |
| 6    | [1,0], [28,28], [91,182], [364,56630] | [1,2], [14,56], [28,840], [91,33852], [182,10386376], [364,3196762296] | [1,6], [7,112], [14,1680], [28,25200], [91,1917332872], [182,588204415344], [364,181039089892752] |
| 8    | [1,0], [82,82], [820,9020], [3280,127893760] | [1,2], [41,164], [82,6888], [410,757680], [820,82118080], [3280,1635797819728640] | [1,6], [41,14104], [82,578592], [205,1515360], [410,6960048480], [820,209223259971646824960] |
| 9    | [1,0], [757,1514], [9841,13721227572] | [1,2], [757,2298252], [9841,188272127685375013488] | [1,6], [757,34834156088], [9841,25832449845624928215332653376] |
| 10   | [1,0], [147,147], [29524,2 · 3² · 17 · 61 · 136334867] | [1,2], [122,488], [29524,60024], [14762,1501232661500], [29524,120048], [14762,148923033457497538560] | [1,6], [61,976], [122,120048], [244,14765904], [410,6960048480], [820,209223259971646824960] |
| 12   | [1,0], [195,195], [6643,13286], [20440,593480], [551881,806850022], [7174453,2 · 179 · 4561 · 357509 · 3559979741071921] | [1,2], [365,1460], [6643,176570940], [20440,35179817920], [33215,717565536140], [66430,291618191728640] | [1,6], [61,976], [122,120048], [244,14765904], [410,6960048480], [820,209223259971646824960] |
| 15   | [1,0], [59293,118586], [551881,806850022], [7174453,2 · 179 · 4561 · 357509 · 3559979741071921] | [1,2], [59293,14063113740], [551881,651006961228800572], [7174453,2 · 5 · 11² · 13 · 267 · 4561 · 101209 · 822407 · 100842919 · 9770548580137061374107091] | [1,6], [59293,166771652673464], [551881,525264982291624814236813816], [7174453,2 · 5 · 11² · 13 · 267 · 4561 · 9993125731 · 152673840083 · 4006805689324561 · 13019832459914677 · 377833767094685409] |

Table 5. Duals of ternary BCH codes with designed distances 1, 3 and 5
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