THE TRACIAL ROKHLIN PROPERTY FOR DISCRETE GROUPS ACTING ON SIMPLE $\mathbb{Z}$-STABLE $C^*$-ALGEBRAS

MICHAEL SUN

ABSTRACT. For every countable discrete group $G$, we define an action $\gamma$ of $G$ on the Jiang-Su algebra $\mathcal{Z}$. If the group is elementary amenable, then $\gamma$ has the weak Rokhlin property. We use $\gamma$ to show that, when $G$ is elementary amenable, there always exists a $G$-action $\omega$ with the tracial Rokhlin property on any unital simple $\mathbb{Z}$-stable tracially approximately divisible $C^*$-algebra $A$. For the $\omega$ we construct, we show that if $A$ is unital simple and $\mathbb{Z}$-stable with rational tracial rank at most one, and $G$ belongs to the class of countable discrete groups generated by finite and abelian groups under increasing unions and subgroups, then the crossed product $A \rtimes_\omega G$ is also unital simple and $\mathbb{Z}$-stable with rational tracial rank at most one.

INTRODUCTION

One fundamental way to investigate the structure of a $C^*$-algebra is through the study of its group actions. An indispensable part of this theme is the crossed product construction. Given a group $G$, a $C^*$-algebra $A$ and a group action $\alpha$ of $G$ on $A$, we can construct a $C^*$-algebra called the crossed product and denote it by $A \rtimes_\alpha G$. When $G$ is discrete and $A$ is unital, we have the following presentation for the crossed product:

$$A \rtimes_\alpha G = \langle a, u_g | a \in A, g \in G, \alpha_g(a) = u_gau_g^* \rangle.$$

Another aspect of the study of $C^*$-algebras is the success of the classification of simple $C^*$-algebras using the Elliott invariant. It is remarkable that so basic an invariant can determine so much about the structure of a simple $C^*$-algebra. At the forefront of this success are the large classes of unital simple separable nuclear ($\mathbb{Z}$-stable) $C^*$-algebras satisfying the Universal Coefficient Theorem (UCT) of tracial rank zero, tracial rank at most 1, rational tracial rank zero and rational tracial rank at most 1, which were discovered to be classifiable by Lin [12], [13], [15] and by Lin, Niu and Winter [33], [9]. Classifiable $C^*$-algebras necessarily possess a property called $\mathbb{Z}$-stability which is
attracting a lot of attention in work extending Elliott’s classification program, while those $C^*$-algebras of tracial rank at most one are also known to be *tracially approximately divisible* (Lin [15, Theorem 5.4]).

From the point of view of classification, the crossed product construction gives a way to explicitly construct algebras which belong to a large class of classifiable $C^*$-algebras that were otherwise only identifiable by their Elliott invariant. Conversely, the classification of $C^*$-algebras allows one to distinguish between group actions through examining the crossed products, which also brings clarity to the crossed product construction itself by giving an identity to algebras otherwise only defined by generators and relations.

Of major interest with respect to these two themes is a property of group actions called the *tracial Rokhlin property*. Its appeal comes from the promise that algebras with desirable properties combined with actions with the tracial Rokhlin property can be used to construct crossed products with the same desirable properties and in many cases to even have the crossed product belong to the same class of classifiable algebras as the original algebra. Another advantage of great practical importance is that this property is believed to exist in abundance, especially if contrasted to its stronger counterparts such as the Rokhlin property.

The question of how abundant this property is will be the main focus of this article along with a preliminary investigation into the consequences this has on the classification of the crossed product. Originally, the definition of the tracial Rokhlin property was only stated for $\mathbb{Z}$ (Osaka-Phillips [27]) and finite groups (Phillips [28]), and Phillips also showed the abundance of cyclic group actions with the tracial Rokhlin property in [29], while there is an existence type result for $\mathbb{Z}$ in Lin [11] together with uniqueness. These build on the work of Kishimoto and others. The definition was then extended to include discrete elementary amenable groups when acting on $\mathcal{Z}$-stable algebras in Matui-Sato [25] where it was also shown that the crossed product retained the $\mathcal{Z}$-stability of the original algebra. We will construct in this article, for every countable discrete group $G$ and $\mathcal{Z}$-stable $C^*$-algebra $A$, an action $\omega$ of $G$ on $A$ to prove the following theorem in Section 3.

**Theorem** (Corollary 3.7). *Given a countable discrete elementary amenable group $G$ and a unital simple separable $\mathcal{Z}$-stable tracially approximately divisible $C^*$-algebra $A$, then there exists a group action $\omega$ of $G$ on $A$ such that $\omega$ has the tracial Rokhlin property.*

Built into the construction of $\omega$ is a family of $G$-actions $\gamma$ on the Jiang-Su algebra $\mathcal{Z}$, which we introduce in Section 2. A preliminary
investigation is also undertaken to ascertain the classifiability of $\mathbb{Z} \rtimes_{\gamma} G$ and $A \rtimes_{\omega} G$ in Sections 4, 5 and 6, when $G$ is finite or abelian. That is, we show as part of Theorem 6.4 that:

**Theorem.** Suppose $A$ is a unital simple $\mathbb{Z}$-stable $C^*$-algebra with rational tracial rank at most one and $G$ belongs to the class of groups generated by finite and abelian groups under increasing unions and taking subgroups. Then $A \rtimes_{\omega} G$ is a unital simple $\mathbb{Z}$-stable $C^*$-algebra with rational tracial rank at most one.

This is a natural first step towards the goal of determining the status of the crossed product $A \rtimes_{\omega} G$ for all elementary amenable groups. As a consequence of its construction, one can show for all $G$ and all $A$ in the above theorem satisfying certain UCT requirements that $A \rtimes_{\omega} G$ has tracial rank zero when $A$ has tracial rank zero. So there is essentially always an action present on every unital simple separable nuclear tracial rank zero algebra that has the tracial Rokhlin property and a tracial rank zero crossed product, despite the problem of proving this being open in general. Staying with this line of thinking, we then organise some of our results under the broader context of certain conjectures that we state explicitly as “Conjecture Zero” and “Conjecture One” in Section 7. These conjectures are not original and are based on the open problem just mentioned, which has been unresolved for over a decade.

**Acknowledgements.** I would like to thank Huaxin Lin for his vast wealth of knowledge and everyone at the Research Center for $C^*$-algebras at East China Normal University where most of this work was done. I also appreciated helpful conversations with N. C. Phillips.

**Notation.** We adopt the following conventions throughout this article:

- Denote the cardinality of a set $S$ by $|S|$.
- Write $a \approx_\epsilon b$ to stand for $\|a - b\| < \epsilon$.

1. **The tracial Rokhlin property for discrete groups**

We define the tracial Rokhlin property and strong outerness for discrete group actions on $C^*$-algebras with reference to Matui-Sato [25].

**Definition 1.1** ($(F, \epsilon)$-invariance). Let $G$ be a countable discrete group, let $F \subset G$ be a finite subset and let $\epsilon > 0$. We say a finite subset $K$ of $G$ is $(F, \epsilon)$-invariant if $K \neq \emptyset$ and

$$|K \cap \bigcap_{g \in F} g^{-1}K| \geq (1 - \epsilon)|K|.$$
Definition 1.2 (Amenability). A countable discrete group \( G \) is said to be \textit{amenable} if an \((F,\epsilon)\)-invariant subset exists from any \((F,\epsilon)\). The group \( G \) is said to be \textit{elementary amenable} if it is contained in the smallest class of groups that contains all abelian groups, all finite groups and is closed under taking subgroups, quotients, direct limits and extensions.

Definition 1.3 (Cuntz relation). Let \( A \) be a \( C^* \)-algebra, let \( p \in A \) be a projection and let \( a \in A \) be a positive element. We write

\[ p \preceq a, \]

if \( p \) is Murray-von Neumann equivalent to a projection in \( aAa \).

Definition 1.4 (Tracial Rokhlin property). A group action \( \alpha \) of \( G \) on a \( C^* \)-algebra \( A \) has the \textit{tracial Rokhlin property} if for every finite subset \( F \subset G \), any \( \epsilon > 0 \), every finite subset \( \{x_1, \ldots, x_n\} \subset A \) and all non-zero \( a \in A_+ \), there is a finite \((F,\epsilon)\)-invariant subset \( K \) in \( G \), and mutually orthogonal projections \((p_k)_{k \in K}\) such that for all \( h \in K \) and \( g \in K \cap h^{-1}K \), and writing \( p = \sum_{k \in K} p_k \), we have

- \([p_h, x_i] \approx_\epsilon 0 \) for all \( i \leq n \),
- \( \alpha_h(p_g) \approx_\epsilon p_{hg} \),
- \( 1 - p \preceq a. \)

When \( A \) is not stably finite, another condition is required to prevent the projections from being trivial. When \( A \) is separable and has strict comparison (a condition implied by \( \mathcal{Z} \)-stability), the above definition is equivalent to the following one which appears in Matui-Sato [25] and will be our working definition throughout this article.

Let \( T(A) \) denote the tracial state space of \( A \).

Definition 1.5 (Matui-Sato tracial Rokhlin property). A group action \( \alpha \) of \( G \) on a \( C^* \)-algebra \( A \) has the \textit{tracial Rokhlin property} if for every finite subset \( F \subset G \) and \( \epsilon > 0 \), there is a finite \((F,\epsilon)\)-invariant subset \( K \) in \( G \) and a central sequence \((p_n)_{n \in \mathbb{N}} \) in \( A \) consisting of projections such that for \( g, h \in K \) with \( g \neq h \)

- \( \lim_{n \to \infty} \alpha_g(p_n)\alpha_h(p_n) = 0 \)
- \( \lim_{n \to \infty} \max_{\tau \in T(A)} |\tau(p_n) - |K|^{-1}| = 0. \)

There is also the weaker notion of this for algebras without projections called the \textit{weak Rokhlin property} (Matui-Sato [25], Definition 2.5). Here is a condition which is a priori weaker than the properties above.
Definition 1.6 (Strongly outer). Let $A$ be a $C^*$-algebra and $\pi_\tau$ be the representation obtained from the Gelfand-Naimark-Segal construction with respect to the tracial state $\tau$. An automorphism is said to be weakly inner if it is inner when considered as an automorphism of the weak closure $\pi_\tau(A)''$ for some trace $\tau$ invariant under the automorphism. An action $\alpha$ of $G$ on $A$ is strongly outer if $\alpha_g$ is not weakly inner for all $g \in G \setminus \{1\}$.

The next lemma is taken from Matui-Sato [25, Lemmas 6.12, 6.13]. We would also like to take this opportunity to thank Y. Sato for communicating to us a detailed proof.

Lemma 1.7 (Matui-Sato). Let $A$ be a simple $C^*$-algebra with unique trace $\tau$ and $\alpha$ an action of a group $G$ on $A$ such that $\alpha_g \neq \text{id}$ when $g \neq 1$. Then $\alpha^{\otimes N} = \otimes_{n \in \mathbb{N}} \alpha$ is a strongly outer action on $A^{\otimes N} = \otimes_{n \in \mathbb{N}} A$.

Proof. Let $g \in G \setminus \{1\}$. Since the unitary group $U(A)$ spans $A$ and $\alpha_g \neq \text{id}$, there is $v_g \in U(A)$ such that $\alpha_g(v_g) \neq v_g$. Now define a central sequence in $A^{\otimes N}$ as follows:

$$v_g(n) = 1 \otimes \cdots \otimes 1 \otimes v_g \otimes 1 \otimes \cdots \otimes 1 \otimes \ldots,$$

where $v_g$ appears in the $n$-th factor and 1 appears in every other factor. We can define another norm on simple $C^*$-algebras $B$ by $\|a\|_2 = \sup_{\tau \in T(B)} \tau(a^*a)^{1/2}$ for all $a \in B$. We will show that if $\alpha_g^{\otimes N}$ is weakly inner then

$$\|\alpha_g^{\otimes N}(v_g(n)) - v_g(n)\|_2 \to 0,$$

which contradicts the sequence being constant and non-zero as follows:

$$\|\alpha_g^{\otimes N}(v_g(n)) - v_g(n)\|_2^2 = \tau^{\otimes N}(1_{A^{\otimes (N\setminus \{n\})}} \otimes ((\alpha_g(v_g) - v_g)^*(\alpha_g(v_g) - v_g)))$$

$$= \tau((\alpha_g(v_g) - v_g)^*(\alpha_g(v_g) - v_g))$$

$$= \|\alpha_g(v_g) - v_g\|_2^2.$$

Now assume there is a unitary $u \in \pi_\tau(A)''$ such that $\alpha_g^{\otimes N} = \text{Ad} u$ on $\pi_\tau(A)''$. Then there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $\pi_\tau(A)$ such that $x_k \to u$ strongly. Such a $u$ allows us to extend $\pi_\tau$ to a representation of $A \rtimes_{\alpha_g} \mathbb{Z}$ and $\tau$ to a trace on $A \rtimes_{\alpha_g} \mathbb{Z}$ by

$$\tau_{A \rtimes \mathbb{Z}}(x) = \langle \pi_{A \rtimes \mathbb{Z}}(x) \tilde{1}, \tilde{1} \rangle,$$

where $\tilde{1}$ is the cyclic vector that appears in the definition of the GNS-representation. Let $\epsilon > 0$, fix $k$ so that $\|u - x_k\|_{A \rtimes \mathbb{Z}} \approx \epsilon/2$ 0 by way of $x_k$ strongly converging to $u$, and let $n$ be large enough so that
[x_k, v_g(n)] \approx 0, \text{ which is possible because } v_g(n) \text{ is a central sequence.}

We now calculate:

\[ \| \alpha_g \otimes A(v_g(n)) - v_g(n) \|_{2,A} = \| uv_g(n) u^* - v_g(n) \|_{2,A \otimes Z} \]
\[ = \| uv_g(n) - v_g(n) u \|_{2,A \otimes Z} \]
\[ \leq 2 \| u - x_k \|_{2,A \otimes Z} + \| x_k v_g(n) - v_g(n) x_k \| \]
\[ \approx \| x_k v_g(n) - v_g(n) x_k \| \]
\[ \approx 0. \]

\[ \square \]

**Lemma 1.8.** Let \( A \) be a simple \( C^\ast \)-algebra with unique trace and \( \alpha \) an action of a group \( G \) on \( A \) such that \( \alpha_g \neq \text{id} \) when \( g \neq 1 \). Let \( A_0 \) be any simple \( C^\ast \)-algebra and \( \alpha_0 \) any action of \( G \) on \( A_0 \), then \( \alpha_0 \otimes \alpha_\otimes N \) is a strongly outer action on \( A_0 \otimes A_0 \otimes G \).

**Proof.** Since the argument used in the proof of Lemma 1.7 only relies on the properties of the limit of \( \| \alpha_g \otimes A\otimes N (v_g(n)) - v_g(n) \|_2 \), we are free to change the first factor without affecting the conclusion. \( \square \)

**Theorem 1.9** (Matui-Sato). Let \( A \) be a unital simple separable \( C^\ast \)-algebra with tracial rank zero and with finitely many extremal tracial states. Let \( G \) be a countable discrete elementary amenable group. Then an action of \( G \) on \( A \) has the tracial Rokhlin property if and only if it is strongly outer.

**Proof.** This is Matui-Sato [25, Theorem 3.7] specialized to actions and to elementary amenable groups. \( \square \)

2. **A FAMILY OF GROUP ACTIONS ON THE JIANG-SU ALGEBRA**

We will introduce the Jiang-Su algebra \( Z \) as a very important example of a strongly self-absorbing \( C^\ast \)-algebra. Most of what we prove only uses this property of \( Z \) (along with its simplicity and having a unique tracial state, which are implied by the property). Another important example is the universal UHF-algebra \( Q \).

**Definition 2.1** ([32]). A \( C^\ast \)-algebra \( A \) is called strongly self-absorbing if there is an isomorphism \( \Psi : A \to A \otimes A \) and a sequence of unitaries \( (v_n)_{n \in \mathbb{N}} \) in \( A \) such that for any \( a \in A \), we have

\[ \lim_{n \to \infty} \text{Ad } v_n \circ \Psi(a) = a \otimes 1. \]

**Definition 2.2.** A \( C^\ast \)-algebra \( A \) is called \( Z \)-stable if \( A \otimes Z \cong A \).

**Lemma 2.3.** There is an isomorphism

\[ Z \cong \lim_{\rightarrow} (Z^\otimes, \text{id}_Z \otimes 1). \]
Proof. This is well-known and first appeared as [4, Corollary 8.8]. \qed

Definition 2.4. Let $G$ be a countable discrete group and identify $\mathcal{Z}$ with $\mathcal{Z}^{\otimes G} = \bigotimes_{g \in G} \mathcal{Z}$ using Lemma 2.3 and countability of $G$. Now define an action $\beta$ of $G$ on $\mathcal{Z}$ via this identification by

$$
\beta_g : \bigotimes_{h \in G} z_h \mapsto \bigotimes_{h \in G} z_{g^{-1}h}.
$$

Again using Lemma 2.3 to identify $\mathcal{Z}$ with $\mathcal{Z}^{\otimes \mathbb{N}} = \bigotimes_{n \in \mathbb{N}} \mathcal{Z}$, we define the action $\gamma$ of $G$ on $\mathcal{Z}$ by

$$
\gamma_g : \bigotimes_{n \in \mathbb{N}} z_n \mapsto \bigotimes_{n \in \mathbb{N}} \beta_g(z_n).
$$

For any group automorphism $\varphi$, we define $\beta^\varphi$ to be the action of $G$ on $\mathcal{Z}$ given by $g \mapsto (\varphi \circ \beta)_g$. So we have that $\beta^\text{id}_G = \beta$. We can define $\gamma^\varphi$ analogously using $\beta^\varphi$ instead of $\beta$ and also get that $\gamma^\text{id}_G = \gamma$.

We first show that all of the $\beta^\varphi$ are conjugate for different $\varphi$ so we only need to consider $\beta^\text{id}_G = \beta$ from here without loss of generality. A similar thing happens when a different ordering of $G$ is taken to define the infinite tensor product.

Lemma 2.5. Let $\varphi$ be a group automorphism of $G$ and let $\hat{\varphi}$ denote the induced automorphism on $\mathcal{Z}$ given by

$$
\hat{\varphi} : z = \bigotimes_{h \in G} z_h \mapsto \bigotimes_{h \in G} z_{\varphi(h)}.
$$

Then for all $g \in G$, we have

$$
\beta^\varphi_g = \hat{\varphi} \circ \beta_g \circ \hat{\varphi}^{-1}.
$$

Proof. Let $g \in G$ and let $z = \bigotimes_{h \in G} z_h \in \mathcal{Z}^{\otimes G}$. We have

$$
(\hat{\varphi}^{-1} \circ \beta^\varphi_g \circ \hat{\varphi})(z) = (\hat{\varphi}^{-1} \circ \beta^\varphi_g \circ \hat{\varphi}) \left( \bigotimes_{h \in G} z_h \right)
$$

$$
= (\hat{\varphi}^{-1} \circ \beta^\varphi_g) \left( \bigotimes_{h \in G} z_{\varphi(h)} \right)
$$

$$
= \hat{\varphi}^{-1} \left( \bigotimes_{h \in G} z_{\varphi(g^{-1})\varphi(h)} \right)
$$

$$
= \bigotimes_{h \in G} z_{g^{-1}h}
$$

$$
= \beta_g(z).
$$

\qed
Lemma 2.6. The automorphism $\beta_g$ is not inner for any $g \in G \setminus \{1\}$.

Proof. Let $\epsilon = 1/7$, let $g \in G \setminus \{1\}$ and suppose $\beta_g = \text{Ad} u$ for some $u \in Z^{\otimes G}$. We annotate with subscripts our copies of $Z$ in the tensor product decomposition of $Z^{\otimes G}$ to emphasise their position. That is,

$$Z^{\otimes G} = \bigotimes_{h \in G} Z_h.$$ 

We note that for each $h$, the factor $Z_h$ can be further decomposed using Lemma 2.3 into an infinite tensor product of copies of $Z$, where we denote each copy by $Z_h^{(l)}$ to emphasise its placement in the decomposition of $Z_h$. That is,

$$Z_h \cong \bigotimes_{l \in \mathbb{N}} Z_h^{(l)}.$$ 

For any subset $H$ of $G$, we set

$$Z^{\otimes H} = \bigotimes_{h \in H} Z_h,$$ 

where we understand this to be identified with the obvious subalgebra of $Z^{\otimes G}$. There is a finite subset $K$ of $G$, $m > 0$ and $z_{k,i} \in Z_k$ for $k \in K$ and $i \leq m$ such that

$$u \approx \sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes \bigotimes_{k \in K} z_{k,i} \right).$$ 

Without loss of generality, we can assume that $1, g \in K$. Now let

$$\delta_1 = \frac{\epsilon}{m} \left( \max_{h \neq 1, i \leq m} \|z_{h,i}\| \right)^{-|K|},$$

and use the tensor product decomposition of $Z_1$ to get, for each $j \leq m$, some $l_j \in \mathbb{N}$ and $z_{1,j}^{(l_j)} \in \bigotimes_{l \leq l_j} Z_1^{(l)}$ such that $z_{1,j} \approx \delta_1 z_{1,j}^{(l_j)}$ and thus

$$\sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes \bigotimes_{h \in K} z_{h,i} \right) \approx \sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes z_{1,i}^{(l_1)} \otimes \bigotimes_{h \in K \setminus \{1\}} z_{h,i} \right).$$

Iterating this process once more, we get $z_{g,i}^{(l_g)} \in \bigotimes_{l \leq l_g} Z_g^{(l)}$ such that

$$\sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes \bigotimes_{h \in K} z_{h,i} \right) \approx 2 \sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes z_{1,i}^{(l_1)} \otimes z_{g,i}^{(l_g)} \otimes \bigotimes_{k \in K \setminus \{1,g\}} z_{k,i} \right).$$

Thus if we write

$$u^{(l_1,l_g)} = \sum_{i=1}^{m} \left( 1_{Z^{\otimes (G \setminus K)}} \otimes z_{1,i}^{(l_1)} \otimes z_{g,i}^{(l_g)} \otimes \bigotimes_{k \in K \setminus \{1,g\}} z_{k,i} \right),$$
we see that \( u \approx_{3\epsilon} u^{(l_1,l_g)} \) and \( u^{(l_1,l_g)} \in \bigotimes_{l \leq l_1} Z_1^{(l)} \otimes \bigotimes_{l \leq l_g} Z_1^{(l)} \otimes \mathbb{Z}^{\otimes K} \).

We now define an element in \( Z^G \), which we denote by \( a \otimes b \), to commute with \( u^{(l_1,l_g)} \) but be orthogonal to \( \beta_g(a \otimes b) \).

Let \( L = \max(l_1, l_g) + 1 \) and choose positive, orthogonal elements \( a, b \in Z \) of norm 1. Writing

\[
a \otimes b = a \otimes b \otimes 1_{Z^G(K^{\setminus\{1, g\}})} \in Z_1^{(L)} \otimes Z_g^{(L)} \otimes Z^{\otimes (K^{\setminus\{1, g\}})},
\]

we have that

\[
\beta_g(a \otimes b) = 1_{Z^G(K^{\setminus\{1, g\}})} \otimes a \otimes b \in Z_1^{(L)} \otimes Z_g^{(L)} \otimes Z_{g^2}^{(L)},
\]

\[
(a \otimes b) \beta_g(a \otimes b) = 0 \quad \text{and} \quad \|a \otimes b\| = \|\beta_g(a \otimes b)\| = 1.
\]

We get immediately that

\[
\|\beta_g(a \otimes b) - a \otimes b\| = \max(\|\beta_g(a \otimes b)\|, \|a \otimes b\|) = 1.
\]

On the other hand,

\[
\|\beta_g(a \otimes b) - a \otimes b\| = \|u(a \otimes b)u^* - a \otimes b\|
\]

\[
= \|u(a \otimes b) - (a \otimes b)u\|
\]

\[
\approx 6\epsilon \|u^{(l_1,l_g)}(a \otimes b) - (a \otimes b)u^{(l_1,l_g)}\| = 0.
\]

With our choice of \( \epsilon \), this is a contradiction. \( \square \)

**Theorem 2.7.** If \( G \) is a countable discrete elementary amenable group, then \( \gamma \) has the weak Rokhlin property in the sense of Matui-Sato \[25\] Definition 2.5.

**Proof.** Since \( \beta_g \neq \text{id} \) for all \( g \in G \setminus \{1\} \) and \( Z \) has unique trace, we apply Lemma 1.7 with \( \alpha = \beta \) and \( A = Z \). We see that in this case \( \alpha^{\otimes N} = \gamma \) is strongly outer. We now apply Matui-Sato \[25\] Theorem 3.6 specialized to actions and to elementary amenable groups to conclude that \( \gamma \) has the weak Rokhlin property. \( \square \)

### 3. Group actions on tracially approximately divisible \( C^* \)-algebras

**Definition 3.1.** Let \( \gamma \) be as in Definition 2.4. For any \( C^* \)-algebra \( A \), define the action \( \omega^A \) on \( A \otimes Z \) by

\[
\omega^A = \text{id}_A \otimes \gamma.
\]

**Remark 3.2.** If \( A \) is unital, the action \( \omega^A \) is pointwise approximately inner because all automorphisms on \( Z \) are approximately inner.

Let \( M_k \) denote the full \( k \times k \) matrix algebra with identity written \( 1_k \).
Lemma 3.3. For any finite subset $F \subset G$ and $\epsilon > 0$, there is a finite $(F, \epsilon)$-invariant subset $K$ of $G$ such that for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ and a projection $q_n \in M_{m(n)} \otimes Z$ satisfying the following

$$\omega_{m(n)}^i(q_n) \omega_{m(n)}^j(q_n) \approx_{1/n} 0 \text{ for all } g, h \in K \text{ with } g \neq h,$$

$$\tau(q_n) \approx_{1/n} |K|^{-1}.$$

Proof. Let $F$ be a finite subset of $G$ and let $\epsilon > 0$. By Lemma 1.8, $\text{id} \otimes \gamma$ is a strongly outer action on $Q \otimes Z$. Therefore by Theorem 1.9 it also has the tracial Rokhlin property. So we have from Definition 1.5 a finite subset $K$ in $G$ and a central sequence $(q'_n)$ consisting of projections in $Q \otimes Z$ such that for all $g, h \in K$ with $g \neq h$, we have

- $\lim_{n \to \infty} \text{id}_Q \otimes \gamma(g)(q'_n) \approx_{1/3n} 0$ for all $g, h \in K$ with $g \neq h,$

- $\lim_{n \to \infty} \max_{r \in T(Q \otimes Z)} |r(q'_n) - |K|^{-1}| = 0.$

By passing to a subsequence if necessary and noting that $Q \otimes Z$ has unique trace, we have

$$(\text{id}_Q \otimes \gamma)(q'_n)(\text{id}_Q \otimes \gamma)(q'_n) \approx_{1/3n} 0 \text{ for all } g, h \in K \text{ with } g \neq h,$$

$$\tau(q'_n) \approx_{1/n} |K|^{-1}.$$

Since $Q$ is a UHF-algebra, there are $m(n)$ and $q''_n \in M_{m(n)}$ self-adjoint such that $q'_n \approx_{1/15n} q''_n$. Hence by functional calculus there is a projection $q_n \in M_{m(n)}$ such that $q_n \approx_{5/15n} q'_n$ (see for example [17 Lemma 2.5.5]). We see now, since automorphisms are isometric and $\text{id}_Q \otimes \gamma$ restricts to $\text{id}_{m(n)} \otimes \gamma$ on $M_{m(n)} \otimes Z$, that

$$(\text{id}_{m(n)} \otimes \gamma)(q_n)(\text{id}_{m(n)} \otimes \gamma)(q_n) \approx_{1/3m} (\text{id}_Q \otimes \gamma)(q'_n)(\text{id}_Q \otimes \gamma)(q'_n)$$

$$\approx_{1/3m} (\text{id}_Q \otimes \gamma)(q'_n)(\text{id}_Q \otimes \gamma)(q'_n)$$

$$\approx_{1/3m} 0 \text{ for all } g, h \in K \text{ with } g \neq h.$$

We also have

$$\tau(q_n) = \tau(q'_n) \approx_{1/n} |K|^{-1}.$$

□

Definition 3.4 (Tracially approximately divisible). Let $A$ be a unital simple separable $C^*$-algebra. We say that $A$ is tracially approximately divisible if for every $\epsilon > 0$, every $l \in \mathbb{N}$, every finite subset $\{a_1, a_2, \ldots, a_k\} \subset A$ and any non-zero $y \in A_+$, there exists a finite dimensional algebra $B$ with each simple summand’s rank exceeding $l$, and a $*$-homomorphism $\varphi : B \to A$, such that for all $i \leq k$ and $e \in B$ with $\|e\| \leq 1$,

- $[a_i, \varphi(e)] \approx_{\epsilon} 0$, 

Remark. Simple tracially approximately divisible algebras automatically satisfy strict comparison because they are tracially $\mathcal{Z}$-stable (see [3, Definition 2.1, Theorem 3.3]). If we assume strict comparison, then the definition above is equivalent to the following definition, which will serve as our working definition.

**Definition 3.5 (Tracially approximately divisible with strict comparison).** Let $A$ be a unital simple separable $C^*$-algebra with strict comparison. We say that $A$ is tracially approximately divisible if for every $\epsilon > 0$, every $n \in \mathbb{N}$, every finite subset $\{a_1, a_2, \ldots, a_k\} \subset A$, there exists $N > n$ and a $*\$-homomorphism $\varphi : M_N \to A$ such that for all $i \leq k$, $e \in M_N$ with $\|e\| \leq 1$ and $\tau \in T(A)$,

- $[a_i, \varphi(e)] \approx_\epsilon 0$,
- $\sup_{\tau \in T(A)} |1 - \tau(\varphi(1_N))| \approx_\epsilon 0$.

**Theorem 3.6.** If $A$ is a unital simple separable tracially approximately divisible $C^*$-algebra, $G$ is a countable discrete elementary amenable group and $\omega^A$ is an action of $G$ on $A \otimes \mathcal{Z}$ as in Definition 3.4, then $\omega^A$ has the tracial Rokhlin property.

**Proof.** Let $F$ be a finite subset of $G$ and let $\epsilon > 0$. We aim to show that there is a finite $(F, \epsilon)$-invariant subset $K$ in $G$ and a central sequence $(p_n)_{n \in \mathbb{N}}$ consisting of projections in $A \otimes \mathcal{Z}$ such that

- $\lim_{n \to \infty} \omega^A_g(p_n) \omega^A_h(p_n) = 0$ for all $g, h \in K$,
- $\lim_{n \to \infty} \max_{\tau \in T(A \otimes \mathcal{Z})} |\tau(p_n) - |K|^{-1}| = 0$.

We begin by introducing some notation for this proof. Define

$$
\mathcal{Z}^{\otimes n} = \bigotimes_{j \leq n} \mathcal{Z} \quad \text{with action } \beta^{\otimes n} = \bigotimes_{j \leq n} \beta,
$$

and

$$
\mathcal{Z}^{\otimes (N\setminus n)} = \bigotimes_{j > n} \mathcal{Z} \quad \text{with action } \beta^{\otimes (N\setminus n)} = \bigotimes_{j > n} \beta.
$$

There are obvious action preserving isomorphisms

$$
\rho_n : (\mathcal{Z}, \gamma) \to (\mathcal{Z}^{\otimes n} \otimes \mathcal{Z}^{\otimes (N\setminus n)}, \beta^{\otimes n} \otimes \beta^{\otimes (N\setminus n)})
$$

and

$$
\sigma_n : (\mathcal{Z}, \gamma) \to (\mathcal{Z}^{\otimes (N\setminus n)}, \beta^{\otimes (N\setminus n)}).
$$
Fix a dense sequence $x_1, x_2, \ldots$ in $A \otimes \mathcal{Z}$. We will proceed to define for each $n \in \mathbb{N}$ a projection $p_n$ to satisfy our initial requirements. To do this, it will be helpful to also establish for $j \leq n$,

$$[p_n, x_j] \approx \frac{1}{n} 0.$$  

Let $n \in \mathbb{N}$. Find $a_{i,j} \in A$ and $z_{i,j} \in \mathcal{Z}$ such that for $j \leq n$, we have

$$x_j \approx \frac{1}{n} \sum_{i=1}^{l(j)} a_{i,j} \otimes z_{i,j}.$$  

Write

$$L_n = \sum_{j \leq n} \sum_{i \leq l(j)} \|a_{i,j}\|.$$  

There exists $n' \in \mathbb{N}$ such that for all $j \leq n$ and $i \leq l(j)$, there are $z'_{i,j} \in \mathcal{Z}^{\otimes n'}$ satisfying

$$\rho_{n'}(z_{i,j}) \approx \frac{1}{n L_n} z'_{i,j} \otimes 1_{\mathcal{Z}^{\otimes (n \setminus n')}}.$$  

Define an action preserving isomorphism

$$\chi_n : A \otimes \mathcal{Z} \to A \otimes \mathcal{Z}^{\otimes n'} \otimes \mathcal{Z}$$  

by

$$\chi_n = (\text{id}_{A \otimes \mathcal{Z}^{\otimes n'}} \otimes \sigma_{n'-1}^{-1}) \circ (\text{id}_{A} \otimes \rho_{n'}).$$  

For $j \leq n$ and $i \leq l(j)$, we get

$$\chi_n(x_j) \approx \frac{1}{n} \sum_{i=1}^{l(j)} a_{i,j} \otimes z'_{i,j} \otimes 1_{\mathcal{Z}}.$$  

(1)
(The calculation for (1) is included here for convenience)

\[
\| \chi_n(x_j) - \sum_{i=1}^{l(j)} a_{i,j} \otimes z_{i,j}' \otimes 1_Z \| \approx \frac{1}{n}
\]

\[
= \left\| \sum_{i=1}^{l(j)} a_{i,j} \otimes ((\text{id} \otimes \sigma_{n'})(\rho_{n'}(z_{i,j}) - z_{i,j}' \otimes 1_Z)) \right\|
\]

\[
\leq \sum_{i=1}^{l(j)} \| a_{i,j} \| \|((\text{id} \otimes \sigma_{n'})(\rho_{n'}(z_{i,j}) - z_{i,j}' \otimes 1))\|
\]

\[
= \sum_{i=1}^{l(j)} \| a_{i,j} \| \| \rho_{n'}(z_{i,j}) - z_{i,j}' \otimes 1)\|
\]

\[
\leq \sum_{i=1}^{l(j)} \| a_{i,j} \| \frac{1}{nL_n}
\]

\[
= \frac{1}{nL_n} \sum_{i=1}^{l(j)} \| a_{i,j} \|
\]

\[
\leq \frac{L_n}{nL_n}
\]

\[
= \frac{1}{n}
\]

We now apply Lemma 3.3 to get an \( m \in \mathbb{N} \), a finite \((F, \epsilon)\)-invariant subset \( K \) of \( G \) and a projection \( q_n \in M_m \otimes Z \) satisfying:

\[
((\text{id}_m \otimes \gamma_g)(q_n))((\text{id}_m \otimes \gamma_h)(q_n)) \approx_{1/n} 0 \quad \text{for all} \ g, h \in K \text{ with } g \neq h,
\]

\[
(\tau_m \otimes \tau_Z)(q_n) \approx_{1/n} |K|^{-1}.
\]

Thinking of \( q_n \) as a matrix with entries \( y_{k,l} \in Z \), we have

\[
q_n = \sum_{k,l=1}^{m} e_{k,l} \otimes y_{k,l},
\]

where the \( e_{k,l} \) are standard matrix units. Also define for convenience,

\[
L_n' = \sum_{j=1}^{n} \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} \| y_{k,l} \| \| z_{i,j}' \|.
\]

Since \( A \) is tracially approximately divisible, there exists by Definition 3.5 an \( m' \in \mathbb{N} \), a \(*\)-homomorphism \( \varphi : M_{m'} \to A \), satisfying for all
$j \leq n$, $i \leq l(j)$, and $e \in M_{m'}$ with $\|e\| \leq 1$, 

\begin{align*}
[\alpha_{i,j}, \varphi(e)] & \approx \frac{1}{n_l n} \\ \tau(\varphi(1_{m'})) & \approx \frac{1}{n} \end{align*}

(4) \qquad m' > mn.

Now write $m' = Nm + r$ with $0 \leq r < m$, and $N \in \mathbb{N}$ and define an embedding

$$\psi_n : M_m \otimes \mathbb{Z} \hookrightarrow A \otimes \mathbb{Z}^{n'} \otimes \mathbb{Z}$$

on generators for $e \in M_m$ and $z \in \mathbb{Z}$ by

$$e \otimes z \mapsto \varphi(\text{diag}(e \otimes 1_{N}, 0_{r})) \otimes 1 \otimes z,$$

where $\text{diag}(e \otimes 1_{N}, 0_{r})$ denotes a block diagonal matrix with the first $N$ blocks $e$ and zeros for the remaining $r \times r$ block. We see that this embedding respects the group action and the image of $q_n$ is

$$\psi_n(q_n) = \sum_{k,l=1}^{m} \varphi(\text{diag}(e_{k,l} \otimes 1_{N}, 0_{r})) \otimes 1 \otimes y_{k,l}.$$

We now define the promised projections $p_n \in A \otimes \mathbb{Z}$ by

$$p_n = (\chi_n^{-1} \circ \psi_n)(q_n).$$

We first check for all $j \leq n$ that

$$[p_n, x_j] \approx 5/n 0.$$

Let $j \leq n$. Then

$$\chi_n([p_n, x_j]) = [\chi_n(p_n), \chi_n(x_j)]$$

$$= [\psi_n(q_n), \chi_n(x_j)]$$

(use (1))

$$\approx \frac{1}{n} \left[ \psi_n(q_n), \sum_{i=1}^{l(j)} a_{i,j} \otimes z'_{i,j} \otimes 1_{Z} \right]$$

( use (2) )

$$= \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} [\varphi(\text{diag}(e_{k,l} \otimes 1_{N}, 0_{r})) \otimes 1 \otimes y_{k,l}, a_{i,j} \otimes z'_{i,j} \otimes 1_{Z}]$$

$$= \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} [\varphi(\text{diag}(e_{k,l} \otimes 1_{N}, 0_{r})), a_{i,j}] \otimes z'_{i,j} \otimes y_{k,l}$$

( use (2) )

$$\approx 1/n 0.$$

We now show that $(p_n)_{n \in \mathbb{N}}$ satisfies the conditions in the definition of the tracial Rokhlin property.
• It is clear that $K$ is a finite $(F, \epsilon)$-invariant subset of $G$.

• The sequence $(p_n)_{n \in \mathbb{N}}$ is central: Let $x \in A \otimes \mathcal{Z}$ and $\epsilon > 0$. We have from density that for some $j \in \mathbb{N},$

$$x \approx_\epsilon x_j.$$ 

Now let $n \geq j$ be such that $1/n < \epsilon$, then

$$\chi_n([p_n, x]) = [\chi_n(p_n), \chi_n(x)] 
\approx_{2\epsilon} [\chi_n(p_n), \chi_n(x_j)]
\approx_{5/n} 0.$$ 

Hence for our choice of $n$ we have

$$[p_n, x] \approx_{7\epsilon} 0.$$ 

• Orthogonality: Let $g, h \in K$ with $g \neq h$. Then

$$\chi_n(\omega^A_g(p_n)) = \omega^A_g(\chi_n(p_n))
= (\id \otimes \beta^A_g \otimes \gamma_g)(\psi_n(q_n))
= (\id \otimes \id \otimes \gamma_g)(\psi_n(q_n))
= \psi_n((\id \otimes \gamma_g)(q_n)).$$ 

So

$$\chi_n(\omega^A_g(p_n)\omega^A_h(p_n)) = \chi_n(\omega^A_g(p_n))\chi_n(\omega^A_h(p_n))
= \psi_n((\id \otimes \gamma_g)(q_n))\psi_n((\id \otimes \gamma_h)(q_n))
= \psi_n((\id \otimes \gamma_g)(q_n)(\id \otimes \gamma_h)(q_n))
\approx_{1/n} 0.$$
Trace condition: let $\tau \in T(A)$, and let $\tau_Z$ and $\tau_M$ be the unique tracial states on $\mathcal{Z}$ and $M_k$ respectively. Then

$$(\tau \otimes \tau_Z)(p_n) = (\tau \otimes \tau_Z \otimes \tau_Z)(\chi_n(p_n))$$

$$= (\tau \otimes \tau_Z \otimes \tau_Z)(\psi_n(q_n))$$

$$= (\tau \otimes \tau_Z)(q_n)$$

$$= (\tau \otimes \tau_Z)(\varphi(\text{diag}(1_m \otimes 1_N, 0_r))(\tau_m \otimes \tau_Z)(q_n))$$

$$= \tau(\varphi(\text{diag}(1_m \otimes 1_N, 0_r)))(\tau_m \otimes \tau_Z)(q_n)$$

$$= \tau(\varphi(1_m))(\tau_m \otimes \tau_Z)(q_n)$$

$$= \frac{m' - r}{m'} \tau(\varphi(1_m))(\tau_m \otimes \tau_Z)(q_n)$$

(\text{use (3)})

$$\approx \frac{1}{n} \tau(\varphi(1_m))(\tau_m \otimes \tau_Z)(q_n)$$

(\text{use (4)})

$$\approx \frac{1}{n} \frac{1}{|K|}$$

Therefore we have

$$\max_{\tau \in T(A \otimes \mathcal{Z})} |\tau(p_n) - |K|^{-1}| \leq \frac{3}{n}$$

from which it follows the limit is 0.

\begin{corollary}
For any unital simple separable $\mathcal{Z}$-stable tracially approximately divisible $C^*$-algebra $A$ and any discrete countable elementary amenable group $G$, there exists a pointwise approximately inner action $\omega$ of $G$ on $A$ with the tracial Rokhlin property. Furthermore, $\omega$ can be taken to be isomorphic to $\omega^A$ from Theorem 3.6.
\end{corollary}

\begin{proof}
If $A$ is $\mathcal{Z}$-stable, $\omega^A$ in Theorem 3.6 is such an action on $A$.
\end{proof}

\begin{corollary}
For any unital simple separable nuclear tracially approximately divisible $C^*$-algebra $A$ and any discrete countable elementary amenable group $G$, there exists a pointwise approximately inner action $\omega$ of $G$ on $A$ with the tracial Rokhlin property. Furthermore, $\omega$ can be taken to be isomorphic to $\omega^A$ from Theorem 3.6.
\end{corollary}

\begin{proof}
Since simple tracially approximately divisible algebras are tracially $\mathcal{Z}$-stable (\cite[Definition 2.1]{[3]}), then $A$ being nuclear implies it is in fact $\mathcal{Z}$-stable (see \cite[Theorem 4.1]{[3]}). Now use Corollary 3.7.
\end{proof}

\begin{corollary}
If $A$ is a unital simple separable nuclear infinite-dimensional $C^*$-algebra of tracial rank at most one and $G$ is any discrete countable elementary amenable group, then there exists a pointwise approximately inner action $\omega$ of $G$ on $A$ with the tracial Rokhlin property. Furthermore, $\omega$ can be taken to be isomorphic to $\omega^A$ from Theorem 3.6.
\end{corollary}

\begin{proof}
\end{proof}
Proof. Lin [13, Theorem 5.4] shows that $A$ is tracially approximately divisible. Since $A$ is also nuclear, we can apply Corollary 3.8.

4. Relationships between $\gamma$ and $\omega^Z$

We make use of Matui-Sato [25] to show that $\gamma$ and $\omega^Z$ are equivalent in some sense as actions on $\mathbb{Z}$. The following lemma unpackages the definition of cocycle conjugacy as applied to actions on $\mathbb{Z}$.

Lemma 4.1. There exists $\theta \in \text{Aut}(\mathbb{Z})$, and collections of unitaries $(v'_g)_{g \in G}$ and $(v_n)_{n \in \mathbb{N}}$ such that for each $g \in G$,

$$\theta \gamma_g \theta^{-1} = \text{Ad} v'_g \omega^Z_g$$

$$\lim_{n \to \infty} v_n \omega^Z_g (v^*_n) = v'_g.$$ 

Proof. We know that $\gamma$ has the weak Rokhlin property by Theorem 2.7. Therefore [25, Theorem 4.9] applies (specialized to actions on $\mathbb{Z}$).

Definition 4.2. We say that two actions are stably outer conjugate with respect to $\theta$, $v$ and $v'$ if they satisfy the conclusion of Lemma 4.1.

Proposition 4.3. The actions $\gamma$ and $\omega^Z$ are stably outer conjugate. If they are stably outer conjugate with respect to $\theta$, $v$ and $v'$, then there is an isomorphism

$$\Psi : \mathbb{Z} \rtimes_\gamma G \to \mathbb{Z} \rtimes_{\omega^Z} G$$

$$zu_g \mapsto \theta(z)v'_g u'_g,$$

where $u_g$ and $u'_g$ are the standard unitaries implementing $\gamma$ and $\omega^Z$ respectively in the crossed product.

Proof. The first part is a restatement of Lemma 4.1 from which showing the crossed products are isomorphic is standard.

5. The crossed products $\mathbb{Z} \rtimes_\gamma G$

Let $\gamma$ be as in Definition 2.4. Simple $C^*$-algebras with rational tracial rank zero are important because they help define a large class of $C^*$-algebras which can be classified by their Elliott invariants. We investigate the classifiability of $\mathbb{Z} \rtimes_\gamma G$ by examining its rational tracial rank, that is, the tracial rank of $\mathbb{Q} \otimes (\mathbb{Z} \rtimes_\gamma G)$.

We first summarize what can be deduced in a straightforward manner from the literature about $\mathbb{Z} \rtimes_\gamma G$ in the following proposition.

Proposition 5.1. If $G$ is a countable discrete elementary amenable, then $\mathbb{Z} \rtimes_\gamma G$

- is unital and separable,
Proof. It is clear that $\mathcal{Z} \rtimes \gamma G$ is unital and separable. We first give an argument here that $\mathcal{Z} \rtimes \gamma G$ has unique trace with reference to Kishimoto [6]. Now it suffices to show that $\mathcal{Q} \otimes (\mathcal{Z} \rtimes \gamma G)$ has unique trace. But we have $\mathcal{Q} \otimes (\mathcal{Z} \rtimes \gamma G) \cong (\mathcal{Q} \otimes \mathcal{Z}) \rtimes \text{id} \otimes \gamma = \omega \mathcal{Q}$ is a strongly outer action (by Lemma 1.8) on $\mathcal{Q} \otimes \mathcal{Z}$, which is isomorphic to $\mathcal{Q}$. So [6, Theorem 4.5] says that $\omega \mathcal{Q}$ is pointwise uniformly outer. Now the proof of [6, Lemma 4.3] applied to each automorphism gives the result. For simplicity use Kishimoto [5, Theorem 3.1]. For nuclearity use Rosenberg [22, Theorem 1]. For $\mathcal{Z}$-stablity use Matui-Sato [25, Corollary 4.11]. □

Remark. We could also let $A = \mathcal{Z}$ in Proposition 6.3 and apply Proposition 4.3 to obtain the above proposition minus the claim about unique trace.

Definition 5.2. If $A \subset B(H)$ is separable then $A$ is a quasidiagonal set of operators if there exists an increasing sequence of finite rank projections, $q_1 \leq q_2 \leq q_3 \ldots$, such that for all $a \in A$, $[a, q_n] \to 0$ and $q_n \to 1_H$ strongly. A separable $C^*$-algebra is quasidiagonal if it has a faithful representation whose image is a quasidiagonal set of operators.

Remark 5.3. It is clear from this definition that a subalgebra of a quasidiagonal $C^*$-algebra is quasidiagonal.

Corollary 5.4. The following are equivalent:

- $\mathcal{Z} \rtimes \gamma G$ is quasidiagonal
- $\mathcal{Z} \rtimes \gamma G$ has rational tracial rank zero
- $\mathcal{Z} \rtimes \gamma G$ has rational tracial rank at most one.

Proof. If $\mathcal{Z} \rtimes \gamma G$ is quasidiagonal, then Proposition 5.1 combined with Matui-Sato [26, Theorem 6.1] allows us to conclude that $\mathcal{Z} \rtimes \gamma G$ has rational tracial rank zero. The next implication is obvious. Finally, if $\mathcal{Z} \rtimes \gamma G$ has rational tracial rank at most one, then it is isomorphic to a subalgebra of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes \gamma G)$. Since $\mathcal{Q} \otimes (\mathcal{Z} \rtimes \gamma G)$ has tracial rank at
most one, it is quasidiagonal by Lin [18, Corollary 6.7]. Hence Remark 5.3 tells us that $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal. □

Quasidiagonality of $\mathcal{Z} \rtimes_\gamma G$ and $\mathcal{Z} \rtimes_\beta G$.

Lemma 5.5. If $H$ is a subgroup of $G$ and $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal, then $\mathcal{Z} \rtimes_\gamma H$ is quasidiagonal.

Proof. It suffices to show that the obvious map $\mathcal{Z} \rtimes_\gamma H \to \mathcal{Z} \rtimes_\gamma G$ is injective. One way to see this is to recall that $\mathcal{Z} \rtimes_\gamma H$ is simple. □

Lemma 5.6. Suppose $G$ is countable discrete and amenable, then the $C^*$-algebra $\mathcal{Z} \rtimes_\beta G$ is simple.

Proof. By Lemma 2.6 $\beta$ is pointwise outer, so Kishimoto [5, Theorem 3.1] applies, which says the reduced crossed product simple. □

Proposition 5.7. The $C^*$-algebra $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal if and only if $\mathcal{Z} \rtimes_\beta G$ is quasidiagonal.

Proof. We show that $\mathcal{Z} \rtimes_\gamma G$ can be embedded into $\mathcal{Z} \rtimes_\beta G$ and vice versa. We have

$$\mathcal{Z} \rtimes_\gamma G \cong \lim_{\to}(\mathcal{Z}^{\otimes n} \rtimes_{\beta \otimes n} G)$$

$$\cong \lim_{\to}(\mathcal{Z}^{\otimes n} \rtimes_{\beta \otimes n} G)$$

$$\hookrightarrow \lim_{\to}(\mathcal{Z}^{\otimes n} \rtimes_\beta G^n)$$

$$\cong \lim_{\to}(\mathcal{Z} \rtimes_\beta G)^{\otimes n}.$$

From this, we see that if $\mathcal{Z} \rtimes_\beta G$ is quasidiagonal then so is $\lim_{\to}(\mathcal{Z} \rtimes_\beta G)^{\otimes n}$ (see [11]) and hence $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal. Conversely, from the second line we see that $\mathcal{Z}^{\otimes 1} \rtimes_{\beta \otimes 1} G$ is contained in $\mathcal{Z} \rtimes_\gamma G$. □

This proposition allows us to look to $\mathcal{Z} \rtimes_\beta G$ for assistance.

Lemma 5.8. If $G = \lim_{\to}(G_i, \varphi_i)$ with $\varphi_i$ injective and $\mathcal{Z} \rtimes_{\beta_i} G_i$ is quasidiagonal for all $i$, then $\mathcal{Z} \rtimes_\beta G$ is quasidiagonal.

Proof. Use the injectivity of $\varphi_i$ to define a $*$-homomorphism

$$\Psi_i : \mathcal{Z}^{\otimes G_i} \to \mathcal{Z}^{\otimes G_{i+1}},$$

on generators by

$$z = \bigotimes_{g \in G_i} z_g \mapsto 1_{\mathcal{Z}^{\otimes (G_{i+1} \setminus \varphi_i(G_i))}} \otimes \bigotimes_{h \in \varphi_i(G_i)} z_{\varphi_i^{-1}(h)}.$$

This proposition allows us to look to $\mathcal{Z} \rtimes_\beta G$ for assistance.
We check this is covariant with respect to $\beta^i$. For $g \in G_i$, we have

$$\varphi_i(g)\Psi_i(z)\varphi_i(g)^* = \beta^{i+1}_i 1_{\mathcal{Z} \otimes (G_{i+1} \setminus \varphi_i(G_i))} \otimes \bigotimes_{h \in \varphi_i(G_i)} z_{\varphi_i^{-1}(h)}$$

$$= 1_{\mathcal{Z} \otimes (G_{i+1} \setminus \varphi_i(G_i))} \otimes \bigotimes_{h \in \varphi_i(G_i)} z_{\varphi_i^{-1}(\varphi_i(g)^{-1}h)}$$

$$= 1_{\mathcal{Z} \otimes (G_{i+1} \setminus \varphi_i(G_i))} \otimes \bigotimes_{h \in \varphi_i(G_i)} z_{\varphi_i^{-1}(h)}$$

$$= \Psi_i\left(\bigotimes_{h \in G_i} z_{\varphi_i^{-1}(h)}\right)$$

$$= \Psi_i(\beta^i_g(z)).$$

Hence we have a sequence of injective maps $\Psi_i \rtimes \varphi_i : \mathcal{Z} \rtimes G_i \rightarrow \mathcal{Z} \rtimes G_{i+1}$ of quasidiagonal algebras, hence the limit is quasidiagonal (see [1, Section 9]). Now we show that this limit is isomorphic to $\mathcal{Z} \otimes G$. First notice that $\lim (\mathcal{Z} \otimes G_i, \Psi_i) \cong \mathcal{Z} \otimes G$ and get the obvious maps

$$\mathcal{Z} \otimes G_i \hookrightarrow \mathcal{Z} \otimes G \hookrightarrow \mathcal{Z} \otimes G \rtimes G$$

and

$$G_i \hookrightarrow G \hookrightarrow \mathcal{Z} \otimes G \rtimes G.$$

We can show covariance in much the same way as before to get

$$\mathcal{Z} \otimes G_i \rtimes G_i \hookrightarrow \mathcal{Z} \otimes G \rtimes G.$$

We check that these maps are compatible with the increasing $i$ and conclude there is an injective map

$$\lim (\mathcal{Z} \otimes G_i \rtimes G_i, \Psi_i \rtimes \varphi_i) \hookrightarrow \mathcal{Z} \otimes G \rtimes G.$$

This map is also surjective because its image contains $\mathcal{Z} \otimes G$ and $G$, which generate the target algebra. \hfill $\square$

**Remark.** For the following discussion we assume for convenience that $G$ is finite to avoid difficulties associated with infinite tensor products of Hilbert spaces. Though we do not anticipate that this will be our main obstacle when $G$ is infinite.

The algebra $\mathcal{Z} \rtimes G$ also has the following natural faithful representation. Let $\pi : \mathcal{Z} \hookrightarrow B(H)$ be a faithful quasidiagonal representation. Then we have a representation of

$$\mathcal{Z} \otimes G = \bigotimes_{g \in G} \mathcal{Z}.$$
on
\[ H^\otimes G = \bigotimes_{g \in G} H \]
given by
\[ \pi^\otimes G = \bigotimes_{g \in G} \pi, \]
as well as a unitary representation \( \rho \) of \( G \) on \( H^\otimes G \) via \( \rho_g : \bigotimes_{h \in G} x_h \mapsto \bigotimes_{h \in G} \xi_{g^{-1}h} \). These representations are a covariant pair by Lemma 5.9 below and hence give a representation of \( \mathcal{Z} \rtimes_\beta G \) on \( H^\otimes G \), which we will denote by \( \pi^\otimes G \rtimes_\beta \rho \).

**Lemma 5.9.** \( \pi^\otimes G \) and \( \rho \) are covariant with respect to \( \beta \).

**Proof.** It suffices to check the relation on generators. Let \( \xi = \bigotimes_{h \in G} \xi_h \in H^\otimes G \) and \( z = \bigotimes_{k \in G} z_k \in \mathcal{Z}^\otimes G \) with \( \xi_h \in H \) and \( z_k \in \mathcal{Z} \) of norm 1. Then
\[
(\rho_g \pi^\otimes G(z) \rho_g^*) (\xi) = \rho_g \bigotimes_{h \in G} \pi(z_h) \bigotimes_{h \in G} \xi_{gh}
\]
\[
= \rho_g \bigotimes_{h \in G} \pi(z_h) (\xi_{gh})
\]
\[
= \bigotimes_{h \in G} \pi(z_{g^{-1}h})(\xi_h)
\]
\[
= \pi^\otimes G \left( \bigotimes_{h \in G} z_{g^{-1}h} \right) \left( \bigotimes_{h \in G} \xi_h \right)
\]
\[
= \pi^\otimes G (\beta_g(z))(\xi).
\]
\[ \square \]

**Lemma 5.10.** \( \pi^\otimes G \rtimes_\beta \rho \) is a faithful representation.

**Proof.** One way to see this is to recall \( \mathcal{Z} \rtimes_\beta G \) is simple. \[ \square \]

**Lemma 5.11.** If \( G \) is a finite group, then \( \pi^\otimes G \rtimes_\beta \rho \) is a quasidiagonal representation.

**Proof.** Suppose \( (q_n)_{n \in \mathbb{N}} \) is a sequence of projections strongly converging to 1 such that for all \( z \in \mathcal{Z}, [\pi(z), q_n] \to 0 \). Then define
\[
q_n^\otimes G = \bigotimes_{g \in G} q_n.
\]
Number the elements of \( G \) so that \( G = (g_j)_{j=1}^{|G|} \) and fix a collection \( (z_g)_{g \in G} \) of elements of \( \mathcal{Z} \). Let \( \epsilon > 0 \) and let \( n \) be large enough so that
\[
\max_{g \in G} \|[\pi(z_g), q_n]\| < \frac{\epsilon}{\sum_{i=1}^{|G|} (\max_{g \in G} \|[\pi(z_g)]\|)^{|G|-1}}.
\]
We first check the commutator condition for \( q^G_n \). Since it is obvious from the definition of \( \rho \) that \([\rho_g, q^G_n] = 0\) for all \( n \in \mathbb{N} \) and \( g \in G \), so we will only check the image of \( \pi \), which we do using a telescoping sum. We have

\[
\left\| \bigotimes_{g \in G} \pi(g) q_n \right\| = \left\| \bigotimes_{g \in G} \pi(g) \bigotimes_{g \in G} q_n \right\|
\]

\[
= \left\| \bigotimes_{g \in G} \pi(g) q_n - \sum_{i=1}^{G-1} \bigotimes_{j=i+1}^{|G|} \pi(g_j) q_n \bigotimes_{j=1}^i q_n \pi(g_j) \right\|
\]

\[
+ \sum_{i=1}^{|G|} \bigotimes_{j=i+1}^{|G|} \pi(g_j) q_n \bigotimes_{j=1}^i q_n \pi(g_j) - \bigotimes_{g \in G} q_n \pi(g) \right\|
\]

\[
\leq \sum_{i=1}^{|G|} \| \pi(g_i) q_n \| \prod_{j=i+1}^{|G|} \| \pi(g_j) \| \| q_n \| \prod_{j=1}^i \| \pi(g_j) \|
\]

\[
\leq \sum_{i=1}^{|G|} \| \pi(g_i) q_n \| (\max_j \| \pi(g_j) \|)^{|G|-1}
\]

\[
\leq \max_{g \in G} \| \pi(g) q_n \| \sum_{i=1}^{|G|} (\max_k \| \pi(g_k) \|)^{|G|-1}
\]

\[
< \epsilon.
\]

Now we verify the strong convergence of \( q^G_n \) to 1 with a telescoping sum similar to that of the above calculation. Let \( \xi \in H^G \) and \( \epsilon > 0 \). Find \( m \) and \( \xi_g(i) \) for \( g \in G \) and \( 1 \leq i \leq m \) so that

\[
\xi \approx \epsilon \sum_{i=1}^m \bigotimes_{g \in G} \xi_g(i).
\]

Let \( n \) be large enough so that

\[
\max_{i,j} \| (q_n(\xi_{h_j}(i)) - \xi_{h_j}(i)) \| < \frac{\epsilon}{|G| \sum_{i=1}^m (\max_k \| \xi_{g_k}(i) \|)^{|G|-1}}.
\]
Then
\[ \| q_n \otimes G^G \xi - \xi \| \approx 2 \epsilon \| q_n \otimes G^G \sum_{g \in G} \otimes_{i=1}^{m} \xi_g(i) - \sum_{g \in G} \otimes_{i=1}^{m} \xi_g(i) \| \]
\[ = \| \sum_{i=1}^{m} \otimes_{g \in G} q_n(\xi_g(i)) - \sum_{i=1}^{m} \otimes_{g \in G} \xi_g(i) \| \]
\[ \leq \sum_{i=1}^{m} \sum_{j=1}^{G} \sum_{k=1}^{j-1} |G| \| q_n(\xi_{g_k}(i)) \otimes (q_n(\xi_{h_j}(i)) - \xi_{h_j}(i)) \otimes \otimes_{k=j+1}^{G} \xi_{g_k}(i) \| \]
\[ \leq |G| \sum_{i=1}^{m} (\max_k |\xi_{g_k}(i)|)^{|G|-1} (\max_j |q_n(\xi_{h_j}(i)) - \xi_{h_j}(i)|) \]
\[ < \epsilon. \]

**Proposition 5.12.** If $G$ is a finite group, then $\mathbb{Z} \rtimes_{\gamma} G$ is quasidiagonal.

**Proof.** Combine Lemmas 5.10 and 5.11 with Proposition 5.7. □

**Remark.** We note that this is also a consequence of N. C. Phillips [28, Theorem 2.6] applied to $\mathbb{Q} \rtimes_{\omega} \mathbb{Q}$, which is isomorphic to $\mathbb{Q} \otimes (\mathbb{Z} \rtimes_{\gamma} G)$.

**Proposition 5.13.** If $G$ is a countable discrete abelian group, then $\mathbb{Z} \rtimes_{\gamma} G$ is quasidiagonal.

**Proof.** Lin [19, Theorem 9.11] shows that if the crossed product of any AH-algebra and any finitely generated abelian group has an invariant tracial state, then the crossed product is quasidiagonal. We apply this to $(\mathbb{Q} \otimes \mathbb{Z}) \rtimes_{\text{id} \otimes \gamma} G \cong \mathbb{Q} \otimes (\mathbb{Z} \rtimes_{\gamma} G)$ when $G$ is a finitely generated abelian group to show the crossed product is quasidiagonal. Now $\mathbb{Z} \rtimes_{\gamma} G$ is a subalgebra of $\mathbb{Q} \otimes (\mathbb{Z} \rtimes_{\gamma} G)$ and hence quasidiagonal. By Lemma 5.7, $\mathbb{Z} \rtimes_{\beta} G$ is quasidiagonal for finitely generated groups $G$. The condition on $G$ being finitely generated can be removed by Lemma 5.8. Finally, one last application of Lemma 5.7 gives the result. □

**Remark.** The quasidiagonality of $\mathbb{Z} \rtimes_{\gamma} \mathbb{Z}$ can be shown directly using earlier results of Brown [2] and even earlier that of Voiculescu [31], applied again to $\mathbb{Q} \rtimes_{\omega} \mathbb{Z}$.

We summarize the findings of this section into the next theorem.
**Theorem 5.14.** Let $\gamma$ be as in Definition 2.4 and let $\mathcal{C}$ be the class of countable discrete groups generated by abelian groups and finite groups under increasing unions and taking subgroups. Then $\mathcal{Z} \rtimes_\gamma G$ is a unital separable simple nuclear $\mathcal{Z}$-stable $C^*$-algebra with rational tracial rank zero for any $G \in \mathcal{C}$.

**Proof.** We combine Lemmas 5.5 and 5.8 with Propositions 5.12 and 5.13 to get $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal for all $G \in \mathcal{C}$ and hence has rational tracial rank zero by Corollary 5.4. The remaining properties are those listed in Proposition 5.1. $\square$

**Corollary 5.15.** Let $\gamma$ be as in Theorem 5.14. Then $\mathcal{Z} \rtimes_\gamma \mathcal{Z}$ is unital separable simple nuclear $\mathcal{Z}$-stable with rational tracial rank zero and has a unique tracial state as well as and satisfying the UCT. We also have for $i = 0$ or $1$ that

$$K_i(\mathcal{Z} \rtimes_\gamma \mathcal{Z}) = \mathcal{Z}.$$ 

Moreover, if $\alpha$ is any other strongly outer $\mathcal{Z}$-action on $\mathcal{Z}$, then there exists an automorphism $\sigma$ of $\mathcal{Z}$ and a unitary $u \in \mathcal{Z}$ such that

$$\alpha = \text{Ad } u \circ \sigma \circ \gamma \circ \sigma^{-1}.$$ 

In particular, $\mathcal{Z} \rtimes_\alpha \mathcal{Z} \cong \mathcal{Z} \rtimes_\gamma \mathcal{Z}$.

**Proof.** Since $\mathcal{Z} \in \mathcal{C}$, putting $G = \mathcal{Z}$ in Theorem 5.14 shows that $\mathcal{Z} \rtimes_\gamma \mathcal{Z}$ is unital separable simple nuclear $\mathcal{Z}$-stable with unique tracial state and rational tracial rank zero. Crossed products by $\mathcal{Z}$ always satisfy the UCT. The $K$-groups are obtained using the Pimsner-Voiculescu six-term exact sequence. The uniqueness statement is due to Sato [30, Theorem 1.3]. $\square$

### 6. The Crossed Products $A \rtimes_\omega G$

Let $A$ be a unital $C^*$-algebra and $G$ a discrete group. Let $\omega$ be as in Theorem 3.6 and $\gamma$ as in Definition 2.4. If $A$ is $\mathcal{Z}$-stable, then $\omega^A$ is isomorphic to an action of $G$ on $A$ that we will also call $\omega$.

**Proposition 6.1.** For $g \in G$ let $u_g$ and $u'_g$ be the implementing unitaries for $\omega_g$ and $\gamma_g$ respectively. There is an isomorphism

$$i : (A \otimes \mathcal{Z}) \rtimes_\omega G \to A \otimes (\mathcal{Z} \rtimes_\gamma G),$$

such that

$$i : (a \otimes z)u_g \mapsto a \otimes (zu'_g).$$

**Proof.** This is standard. $\square$
Lemma 6.2. If $A$ is $\mathcal{Z}$-stable, then there is a $*$-isomorphism
\[ Q \otimes (A \rtimes_\omega G) \cong (Q \otimes A) \otimes ((\mathcal{Z} \rtimes_\gamma G) \otimes Q). \]

Proof. We use Proposition 6.1 and $Q \cong Q \otimes Q$ to write
\[ Q \otimes ((A \otimes \mathcal{Z}) \rtimes_\omega G) \cong Q \otimes (A \otimes (\mathcal{Z} \rtimes_\gamma G)) \]
\[ \cong (Q \otimes A) \otimes ((\mathcal{Z} \rtimes_\gamma G) \otimes Q). \]

Now since $A$ is $\mathcal{Z}$-stable, we are done. \hfill \Box

Proposition 6.3. Suppose $A$ is a unital separable $\mathcal{Z}$-stable $C^*$-algebra and $G$ is any countable discrete amenable group. Then $A \rtimes_\omega G$ is a unital separable $\mathcal{Z}$-stable $C^*$-algebra and we also have:

- If $A$ is simple, then $A \rtimes_\omega G$ is simple.
- If $A$ is nuclear, then $A \rtimes_\omega G$ is nuclear.
- $T(A \rtimes_\omega G) = T(A)$.
- If $A$ has real rank zero, then $A \rtimes_\omega G$ has real rank zero.

Proof. It is clear that the crossed product is unital and separable since $A$ is unital and separable, and $G$ is countable. If $A$ is simple, then Lemma 1.8 shows that $\omega$ is pointwise outer. Hence Kishimoto [5, Theorem 3.1] shows the (reduced) crossed product is simple. Nuclearity follows from Rosenberg [22, Theorem 1]. For $\mathcal{Z}$-stablity we use Proposition 6.1 to get
\[ \mathcal{Z} \otimes (A \rtimes_\omega G) \cong \mathcal{Z} \otimes (A \otimes (\mathcal{Z} \rtimes_\gamma G)) \]
\[ \cong (\mathcal{Z} \otimes A) \otimes (\mathcal{Z} \rtimes_\gamma G). \]
\[ \cong A \otimes (\mathcal{Z} \rtimes_\gamma G) \]
\[ \cong A \rtimes_\omega G. \]

For the claim about the tracial state spaces, let $\tau_\gamma$ be the unique tracial state on $\mathcal{Z} \rtimes_\gamma G$ (Proposition 5.1) and define a map $T(A) \to T(A \otimes (\mathcal{Z} \rtimes_\gamma G))$ given by $\tau \mapsto \tau \otimes \tau_\gamma$. This map is obviously affine and injective, while for surjectivity we make use of a brief argument which can be found as [8, Lemma 5.15]. Now by Proposition 6.1 we have that $T(A \otimes (\mathcal{Z} \rtimes_\gamma G)) \cong T(A \rtimes_\omega G)$. Since our algebras are $\mathcal{Z}$-stable, we use the characterization of real rank zero by Rørdam [21, Theorem 7.2] that $K_0(A)$ is uniformly dense in the space of affine functions on $T(A)$ under the standard mapping $\rho_A$ gotten by evaluation. Since $K_0(A) \subset K_0(A \rtimes_\omega G)$ via $p \mapsto p \otimes 1$, under the identification $T(A) = T(A \rtimes_\omega G)$, $\rho_{A \rtimes_\omega G}(K_0(A)) = \rho_A(K_0(A))$, which is already uniformly dense. Hence the image of $\rho_{A \rtimes_\omega G}$ is uniformly dense and we are done. \hfill \Box
Theorem 6.4. Suppose $A$ is a unital separable simple nuclear $\mathcal{Z}$-stable $C^*$-algebra. Let $\mathcal{C}$ be as in Theorem 5.14, let $G \in \mathcal{C}$, let $\omega$ be as in Theorem 3.6 and let $\gamma$ be as in Definition 2.4. Then $\omega$ is isomorphic to an action of $G$ on $A$ and $A \rtimes_\omega G$ is a unital separable simple nuclear $\mathcal{Z}$-stable $C^*$-algebra. Furthermore:

- If $A$ has rational tracial rank at most one, then $A \rtimes_\omega G$ has rational tracial rank at most one.

- If $A$ has rational tracial rank zero, then $A \rtimes_\omega G$ has rational tracial rank zero.

- If $A$ has tracial rank at most one, satisfies the UCT and $\mathcal{Z} \rtimes_\gamma G$ satisfies the UCT, then $A \rtimes_\omega G$ has tracial rank at most one and satisfies the UCT.

- If $A$ has tracial rank zero, satisfies the UCT and $\mathcal{Z} \rtimes_\gamma G$ satisfies the UCT, then $A \rtimes_\omega G$ has tracial rank zero and satisfies the UCT.

Proof. Since $A$ is $\mathcal{Z}$-stable, $\omega$ is isomorphic to an action on $A$. Since $G \in \mathcal{C}$ and $A$ satisfy the hypotheses of Proposition 6.3, the conditions of being unital, separable, simple, nuclear and $\mathcal{Z}$-stable, are retained by the crossed product. To determine the rational tracial rank of $A \rtimes_\omega G$ we use Lemma 6.2 and apply Hu-Lin-Xue [17, Theorem 4.8], which says that the tracial rank of a tensor product is bounded by the sum of the tracial ranks of the factors, to the algebra on the right hand side of the lemma. Since $G \in \mathcal{C}$, the tracial rank of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ is zero by Theorem 5.14, which means the tracial rank is bounded by the rational tracial rank of $A$. This gives us both claims about rational tracial rank. Now we address the claim for $A$ being of tracial rank at most one. We will use [16, Theorem 4.7] with our $A$ as their $B$ and $\mathcal{Z} \rtimes_\gamma G$ as their $A$ to show that $A \otimes (\mathcal{Z} \rtimes_\gamma G)$ has tracial rank at most one and satisfies the UCT. Hence by Proposition 6.3, $A \rtimes_\omega G$ has tracial rank at most one and satisfies the UCT. Now if $A$ was also tracial rank zero, then it is real rank zero and Proposition 6.3 tells us that $A \rtimes_\omega G$ is real rank zero. But we know that if an algebra is unital simple of tracial rank at most one and real rank zero, then it has tracial rank zero (see for example [14, Lemma 3.2]). \qed
Here is a curious result in the converse direction.

**Theorem 6.5.** Suppose $A$ is a unital separable simple nuclear $\mathbb{Z}$-stable $C^*$-algebra satisfying the UCT. Let $C$ be as in Theorem 5.14, let $\omega$ be as in Theorem 3.6, and let $\gamma$ be as in Definition 2.4. If there exists $G \in \mathcal{C}$ such that $\mathbb{Z} \rtimes_\gamma G$ satisfies the UCT and $A \rtimes_\omega G$ has rational tracial rank at most one, then $A$ has rational tracial rank at most one.

**Proof.** Let $B = \mathbb{Z} \rtimes_\gamma G$ and note $A \otimes B = A \rtimes_\omega (G)$ by Proposition 6.1. We now apply Lin-Sun [16, Theorem 4.8 (1,2,13)] with reference to Propositions 5.1 and 6.3 and Theorem 5.14 to verify the hypotheses there. Hence we get a conclusion that is equivalent (by [8, Theorem 3.6]) to our claim. 

We specialise Theorems 6.4 and 6.5 to the case of the integers.

**Corollary 6.6.** Suppose $A$ is a unital separable simple nuclear $\mathbb{Z}$-stable $C^*$-algebra satisfying the UCT and $\omega$ is as in Theorem 3.6. Then $A \rtimes_\omega \mathbb{Z}$ is a unital separable simple nuclear $\mathbb{Z}$-stable $C^*$-algebra satisfying the UCT. We also have

- $A$ has rational tracial rank at most one if and only if $A \rtimes_\omega \mathbb{Z}$ has rational tracial rank at most one.
- If $A$ has rational tracial rank zero, then $A \rtimes_\omega \mathbb{Z}$ has rational tracial rank zero.
- If $A$ has tracial rank at most one, then $A \rtimes_\omega \mathbb{Z}$ has tracial rank at most one.
- If $A$ has tracial rank zero, then $A \rtimes_\omega \mathbb{Z}$ has tracial rank zero.
- $K_i(A \rtimes_\omega \mathbb{Z}) = K_0(A) \oplus K_1(A)$ for $i = 0$ or 1.

**Proof.** The group $\mathbb{Z}$ is amenable and so Proposition 6.3 tells us that $A \rtimes_\omega \mathbb{Z}$ is unital separable simple nuclear and $\mathbb{Z}$-stable. The UCT will always be preserved by crossed products by $\mathbb{Z}$. For the claim about rational tracial rank, we note that $\mathbb{Z} \in \mathcal{C}$ and that the forward implication is given by Theorem 6.4 with $G = \mathbb{Z}$ and the converse by Theorem 5.1 with $G = \mathbb{Z}$. Since $\mathbb{Z} \rtimes_\gamma \mathbb{Z}$ always satisfies the UCT, the next three claims all follow from Theorem 6.4 with $G = \mathbb{Z}$. For the calculation of $K$-groups, we use the Kunneth Theorem for tensor products combined with knowing the $K$-groups of $\mathbb{Z} \rtimes_\gamma \mathbb{Z}$ from Corollary 5.15. 

7. Questions and conjectures

One interesting open problem, proposed implicitly by Kishimoto and others, is that of determining whether automorphisms (\(\mathbb{Z}\)-actions) with the tracial Rokhlin property preserve tracial rank zero. Our action \(\omega\) from Theorem 6.4 provides an example of an instance where the conditions of the problem and the conclusion are satisfied for all unital separable simple nuclear (\(\mathbb{Z}\)-stable) \(C^*\)-algebras of tracial rank zero and hence affirms that the conditions themselves are compatible with the conclusion for all of these algebras. It also suggests this compatibility for certain generalisations of the problem that are meaningful given recent developments in the classification program. We will give a formal statement of this conjecture in a more general contemporary form in an attempt to make the problem more explicit. We take the liberty to name the conjecture concerning tracial rank zero “Conjecture Zero” and that concerning tracial rank at most one “Conjecture One”.

**Conjecture Zero** \((A, G, \alpha)\). Suppose \(A\) is a unital simple separable nuclear \(\mathbb{Z}\)-stable \(C^*\)-algebra satisfying the UCT, \(G\) a countable discrete elementary amenable group, and \(\alpha\) an action of \(G\) on \(A\) with the weak Rokhlin property. Then \(A \rtimes_{\alpha} G\) is a unital simple separable nuclear \(\mathbb{Z}\)-stable \(C^*\)-algebra satisfying the UCT and:

(i) If \(A\) has rational tracial rank zero, then \(A \rtimes_{\alpha} G\) has rational tracial rank zero.

(ii) If \(A\) has tracial rank zero and \(\alpha\) has the tracial Rokhlin property, then \(A \rtimes_{\alpha} G\) has tracial rank zero.

**Conjecture One** \((A, G, \alpha)\). Suppose \(A\) is a unital simple separable nuclear \(\mathbb{Z}\)-stable \(C^*\)-algebra satisfying the UCT, \(G\) a countable discrete elementary amenable group, and \(\alpha\) an action of \(G\) on \(A\) with the weak Rokhlin property. Then \(A \rtimes_{\alpha} G\) is a unital separable simple nuclear \(\mathbb{Z}\)-stable \(C^*\)-algebra satisfying the UCT and:

(i) If \(A\) has rational tracial rank at most one, then \(A \rtimes_{\alpha} G\) has rational tracial rank at most one.

(ii) If \(A\) has tracial rank at most one and \(\alpha\) has the tracial Rokhlin property, then \(A \rtimes_{\alpha} G\) has tracial rank at most one.

We have included the parameters \((A, G, \alpha)\) to emphasise the dependence of the conjectures on the algebra \(A\), the group \(G\) and the action \(\alpha\). Part of the problem statement implicit for these conjectures is to determine the largest class \(\{(A, G, \alpha)\}\) for which the conjectures are true. There have been many partial results on these conjectures, see
We conclude this article with a restatement of some of our results in terms of the conjectures just stated. Let $\omega$ be any action isomorphic to the one from Theorem 6.4, let $C$ be from Theorem 5.14 and let $\gamma$ be as in Definition 2.4. Then:

(Theorem 6.3) Conjecture Zero $(A, G, \omega)$ and Conjecture One $(A, G, \omega)$ are true for all $A$ in the statement of the conjectures and all $G \in C$ such that $Z \rtimes_\gamma G$ satisfies the UCT.

(Corollary 6.6) Conjecture Zero $(A, Z, \omega)$ and Conjecture One $(A, Z, \omega)$ are true for all $A$ in the statement of the conjectures.

References

[1] N. P. Brown, *On quasidiagonal C*-algebras*, arxiv:0008181v1.
[2] N. P. Brown, *AF embeddability of crossed products of AF algebras by the integers*, J. Funct. Anal. 160, 150-175 (1998).
[3] I. Hirshberg, J. Orozit, *Tracially $\mathcal{Z}$-absorbing C*-algebras*, arXiv:1208.2444v2.
[4] X. Jiang, H. Su, *On a simple unital projectionless C*-algebra*, American Journal of Mathematics, Volume 121, Number 2, April 1999, pp. 359–413.
[5] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C*-algebras*, Comm. Math. Phys. 81, 429-435 (1981).
[6] A. Kishimoto, *The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras*, J. Funct. Anal. 140 (1996).
[7] H. Lin, *An introduction to the classification of amenable C*-algebras*, World Scientific 2001.
[8] H. Lin, N. Zhuang, *The range of a class of classifiable separable simple amenable C*-algebras*, J. Funct. Anal. 260 (2011), no. 1, 1-29.
[9] H. Lin, N. Zhuang, *Lifting KK-elements, asymptotic unitary equivalence and classification of simple C*-algebras*, Adv. Math. 219 (2008), no. 5, 1729-1769.
[10] H. Lin, *On Local AH-algebras*, [arXiv:1104.0445v1].
[11] H. Lin, *Kishimoto’s Conjugacy Theorems in simple C*-algebras of tracial rank one*, [arXiv:1305.4994v1].
[12] H. Lin, *Classification of simple C*-algebras of tracial topological rank zero*, Duke Mathematical Journal, Vol. 125, No. 1 (2004).
[13] H. Lin, *Classification of simple tracially AF C*-algebras*, Canad. J. Math. Vol. 53, No. 1, pp 161-194 (2001).
[14] H. Lin, *The Rohlin property for automorphisms on simple C*-algebras*, Operator theory, operator algebras, and applications, 189–215, Contemp. Math., 414, Amer. Math. Soc., Providence, RI, 2006.
[15] H. Lin, *Simple nuclear C*-algebras of tracial topological rank one*, J. Funct. Anal. 160 (2007).
[16] H. Lin, W. Sun, *Tensor products of classifiable C*-algebras*, 2012 arXiv:1203.3737v1.
[17] S. Hu, H. Lin, Y. Xue, *The tracial topological rank of C*-algebras (II)*, Indiana Univ. Math. J. 53, No. 6 (2004), 1579-1606.
[18] H. Lin, *The tracial topological rank of C*-algebras*, Proc. London Math. Soc. (3) 83 (2001) 199–234.
[19] H. Lin, *AF-embeddings of the crossed products of AH-algebras by finitely generated abelian groups*, Int. Math. Res. Pap. (2008), no. 3.

[20] H. Lin *The Rokhlin property for automorphisms on simple C*-algebras*, arxiv:0602513v2.

[21] M. Rørdam, *The stable and real rank of Z-absorbing C*-algebras*, International Journal of Mathematics Vol. 15, No. 10 (2004) 10657-1084.

[22] J. Rosenberg, *Amenability of crossed products of C*-algebras*, Comm. Math. Phys. 57, 187–191 (1977).

[23] H. Matui, Y. Sato, *Strict comparison and Z-absorption of nuclear C*-algebras*, Acta Math. 209 (2012), no. 1, 179-196.

[24] H. Matui, Y. Sato, *Z-stability of crossed products by strongly outer actions I*, Commun. Math. Phys. 314, 193–228 (2012).

[25] H. Matui, Y. Sato, *Z-stability of crossed products by strongly outer actions II*, arXiv:1205.1590v2.

[26] H. Matui, Y. Sato, *Decomposition rank of UHF-absorbing C*-algebras*, arXiv:1303.4371v2.

[27] H. Osaka, N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergodic Theory Dynam. Systems 26 (2006), no.5, 1579–1621.

[28] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C*-algebras*, Amer. J. Math. (2011), no. 3, 581–636.

[29] N. C. Phillips, *The tracial Rokhlin property is generic*, arXiv:1209.3859v1.

[30] Y. Sato, *The Rohlin property for automorphisms of the Jiang–Su algebra*, J. Funct. Anal. 259 (2010), 453–476.

[31] D. Voiculescu, *Almost inductive limit automorphisms and embeddings into AF-algebras*, Ergodic Theory Dynam. Systems 6 (1986), no. 3, 475–484.

[32] A. Toms, W. Winter, *Strongly self-absorbing C*-algebras*, Trans. AMS Vol 359, No. 8 (2007), 3999–4029.

[33] W. Winter *Localizing the Elliott conjecture at strongly self-absorbing C*-algebras*, arXiv:0708.0283v3.

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA, Research Center for Operator Algebras, East China Normal University, China
E-mail address: msun@uoregon.edu