A new way to classify 2D higher order quantum superintegrable systems

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Abstract
We revise a method by Kalnins, Kress and Miller (2010) for constructing a canonical form for symmetry operators of arbitrary order for the Schrödinger eigenvalue equation $H\Psi \equiv (\nabla^2 + V)\Psi = E\Psi$ on any 2D Riemannian manifold, real or complex, that admits a separation of variables in some orthogonal coordinate system. We apply the method, as an example, to revisit the Tremblay and Winternitz (2010) derivation of the Painlevé VI potential for a 3rd order superintegrable flat space system that separates in polar coordinates and, as new results, we give a listing of the possible potentials on the two-sphere that separate in spherical coordinates and all two-hyperbolic (two-sheet) potentials separating in horocyclic coordinates. In particular, we show that the Painlevé VI potential also appears for a 3rd order superintegrable system on the two-sphere that separates in spherical coordinates, as well as a 3rd order superintegrable system on the two-hyperboloid that separates in spherical coordinates and one that separates in horocyclic coordinates. Our aim is to develop tools for analysis and classification of higher order superintegrable systems on any 2D Riemannian space, not just Euclidean space.

Keywords: quantum superintegrable systems, Painlevé VI equation, Weierstrass equation, elliptic integrable system

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1. Introduction

In the paper [2] the authors constructed a canonical form for symmetry operators of any order in 2D and used it to give the first proof of the superintegrability of the quantum Tremblay, Turbiner, and Winternitz (TTW) system [3] in polar coordinates, for all rational values of the parameter $k$. In the original method the various potentials were given and the problem was the construction of higher order symmetry operators that would verify superintegrability. The method was highly algebraic and required the solution of systems of difference equations on a lattice. Here, we consider an arbitrary space admitting a separation in some orthogonal coordinate system (hence admitting a 2nd order symmetry operator), and search for all potentials $V$ for which the Schrödinger equation admits an additional independent symmetry operator of order higher than 2. Now the problem reduces to solving a system of partial differential equations.

In section 2 we give a brief introduction to the method and then in section 3 we specialize it to 3rd order superintegrable systems. More details can be found in [2] and [1]. In section 4 we treat a few important examples. We revisit the Tremblay and Winternitz derivation of the Painlevé VI potential for a 3rd order superintegrable flat space system that separates in polar coordinates, [4], and we show among other new results that the Painlevé VI potential also appears for a 3rd order superintegrable system on the two-sphere that separates in spherical coordinates, as well as a 3rd order superintegrable system on the two-hyperboloid that separates in horocyclic coordinates.

In section 5 we classify all systems on the complex two-sphere that admit a 3rd order symmetry operator and separate in spherical coordinates. For some of the cases where the system is 3rd order superintegrable we work out the symmetry algebra generated by the Hamiltonian $H$, the 2nd order symmetry operator $A$ associated with separation in spherical coordinates and the 3rd order symmetry operator $B$. In section 6 we analyze the cases where the operators $A$ and $B$ are algebraically dependent and relate them to the Weierstrass $\wp$-function and its degenerations. In section 7 we study superintegrable systems $\{H, A, B\}$ that are algebraically dependent but with $B$ of order higher than 3. In section 8 we classify 3rd order superintegrable systems on the two-sheet two-hyperboloid that separate in horocyclic coordinates. Section 9 is devoted to discussion. Some of our principal results were announced in the proceedings paper [1], but here we give much more detail. We have used Maple for the original set up and the flat space and two-sphere computations, and Mathematica for the two-hyperboloid computations and to double check all results. Our coordinate computations are local.

2. The method

We consider a Schrödinger equation on a 2D real or complex Riemannian manifold with Laplace–Beltrami operator $\Delta_2$ and potential $V$:

$$H \Psi \equiv \left(-\frac{\hbar^2}{2} \Delta_2 + V\right) \Psi = E \Psi$$  \hspace{1cm} (1)

that also admits an orthogonal separation of variables. If $\{u_1, u_2\}$ is the orthogonal separable coordinate system the corresponding Schrödinger operator can always be put in the form

$$H = -\frac{\hbar^2}{2} \Delta_2 + V(u_1, u_2)$$
$$= \frac{1}{f_1(u_1) + f_2(u_2)} \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial u_1^2} - \frac{\hbar^2}{2} \frac{\partial^2}{\partial u_2^2} + V_1(u_1) + V_2(u_2)\right)$$  \hspace{1cm} (2)
and, due to the separability, there is the second-order symmetry operator

\[
L = \frac{f_2(u_2)}{f_1(u_1) + f_2(u_2)} \left( -\frac{\hbar^2}{2} \partial_{u_1}^2 + V_1(u_1) \right) - \frac{f_1(u_1)}{f_1(u_1) + f_2(u_2)} \left( -\frac{\hbar^2}{2} \partial_{u_2}^2 + V_2(u_2) \right),
\]

i.e., \([H, L] = 0\). We look for a partial differential symmetry operator of arbitrary order \(\tilde{L}(H, L, u_1, u_2)\) that satisfies

\[
[H, \tilde{L}] = 0.
\]

We require that the symmetry operator take the standard form

\[
\tilde{L} = \sum_{jk} (A^{jk}(u_1, u_2)\partial_{u_1} u_2 - B^{jk}(u_1, u_2)\partial_{u_1} - C^{jk}(u_1, u_2)\partial_{u_2}

+ D^{jk}(u_1, u_2)) H^1 L^k.
\]

This can always be done. More details of the derivation can be found in \([2]\) and \([1]\).

In this view we can write

\[
\tilde{L}(H, L, u_1, u_2) = A(u_1, u_2)\partial_{u_1} u_2 - B(u_1, u_2)\partial_{u_1} - C(u_1, u_2)\partial_{u_2} + D(u_1, u_2).
\]

and consider \(\tilde{L}\) as an at most second-order order differential operator in \(u_1, u_2\) that is analytic in the parameters \(H, L\). Then the above system of equations can be written in the more compact form

\[
\partial_{u_1}^2 A + \partial_{u_2}^2 A - 2\partial_{u_2} B - 2\partial_{u_1} C = 0,
\]

\[
\frac{\hbar^2}{2}(\partial_{u_1}^2 B + \partial_{u_2}^2 B) - 2\partial_{u_2} A V_2 - \hbar^2 \partial_{u_1} D - AV_2' + (2\partial_{u_1} A f_2 + A f_2') H - 2\partial_{u_2} A L = 0,
\]

\[
\frac{\hbar^2}{2}(\partial_{u_1}^2 C + \partial_{u_2}^2 C) - 2\partial_{u_1} A V_1 - \hbar^2 \partial_{u_2} D - AV_1' + (2\partial_{u_1} A f_1 + A f_1') H + 2\partial_{u_2} A L = 0,
\]

\[
\frac{\hbar^2}{2}(\partial_{u_1}^2 D + \partial_{u_2}^2 D) + 2\partial_{u_1} B V_1 + 2\partial_{u_2} C V_2 + BV_1' + C V_2' - (2\partial_{u_1} B f_1 + 2\partial_{u_2} C f_2 + B f_1' + C f_2') H + (-2\partial_{u_1} B + 2\partial_{u_2} C) L = 0.
\]

We can view (7) as an equation for \(A, B, C\) and (8) and (9) as the defining equations for \(\partial_{u_1} D, \partial_{u_2} D\). Then by \(L\) we mean that the terms in \(H\) and \(L\) are to be interpreted as (5) and considered as partial differential operators.

We can simplify this system by noting that there are two functions \(F(u_1, u_2, H, L), G(u_1, u_2, H, L)\) such that (7) is satisfied by

\[
A = F, \quad B = \frac{1}{2} \partial_{u_2} F + \partial_{u_1} G, \quad C = \frac{1}{2} \partial_{u_1} F - \partial_{u_2} G.
\]
Then the integrability condition for (8) and (9) is (with the shorthand notation $\partial_u F = F_j$, $\partial_u \partial_u F = F_{jj}$, etc, for $F$ and $G$),

$$
\begin{align*}
\hbar^2 G_{1222} + \frac{1}{4} \hbar^2 F_{2222} - 2 F_{22}(V_2 - f_2 H + L) - 3 F_2(V_2' - f_2'H) - F(V_2'' - f_2''H) \\
= -\hbar^2 G_{1112} + \frac{1}{4} \hbar^2 F_{1111} - 2 F_{11}(V_1 - f_1 H - L) - 3 F_1(V_1' - f_1'H) - F(V_1'' - f_1''H),
\end{align*}
$$

and equation (10) becomes

$$
\begin{align*}
\frac{1}{4} \hbar^2 F_{1112} - 2 F_{12}(V_1 - f_1 H) - F_1(V_1' - f_1'H) + \frac{1}{4} \hbar^2 G_{1111} \\
- 2 G_{11}(V_1 - f_1 H - L) - G_1(V_1' - f_1'H) = -\frac{1}{4} \hbar^2 F_{1222} + 2 F_{12}(V_2 - f_2 H) \\
+ F_2(V_1' - f_1'H) + \frac{1}{4} \hbar^2 G_{2222} - 2 G_{22}(V_2 - f_2 H + L) - G_2(V_2' - f_2'H). 
\end{align*}
$$

We remark that any solution of (12) and (13) with $A, B, C$ not identically 0 corresponds to a symmetry operator that does not commute with $L$, hence is algebraically independent of the symmetries $H, L$.

3. 3rd order superintegrability

To show how equations (12) and (13) can be used to find potentials for superintegrable systems, we provide detailed derivations of the determining equations for 3rd order superintegrability. First we note that the most general 3rd order operator must be of the form (5) with

$$
A = A^0(u_1, u_2),
$$

$$
B = B^0(u_1, u_2) + B^H(u_1, u_2)H + B^L(u_1, u_2)L,
$$

$$
C = C^0(u_1, u_2) + C^H(u_1, u_2)H + C^L(u_1, u_2)L,
$$

$$
D = D^0(u_1, u_2) + D^H(u_1, u_2)H + D^L(u_1, u_2)L,
$$

or, in view of (11),

$$
F(u_1, u_2) = F^0(u_1, u_2), \quad G(u_1, u_2) = G^0(u_1, u_2) + G^H(u_1, u_2)H + G^L(u_1, u_2)L. 
$$

Substituting (14) into (12) and (13) and noting that the coefficients of independent powers of $H$ and $L$ in these expressions must vanish, we obtain 9 equations, (the first 3 from (12) and the next 6 from (13)):

$$
0 = -6 V_1 F_1^0 + 6 V_2 F_2^0 - 4 V_1 F_1^0 + 4 V_2 F_2^0 - 2 \hbar^2 G_{1111} - 2 \hbar^2 G_{1222} + 2 F_1^0 V_2'' - 2 F_2^0 V_1'',
$$

$$
0 = F_{11}^0 + F_{22}^0,
$$

$$
0 = -\hbar^2 G_{1111}^H - \hbar^2 G_{1222}^H + 3 f_1^0 F_1^0 - 3 f_2^0 F_2^0 + 2 f_1^0 F_1^0 - 2 f_2^0 F_2^0 - F_1^0 f_2'' + F_2^0 f_1'',
$$

$$
0 = V_1^2 F_1^0 + V_1^2 F_2^0 + V_1^2 G_1^0 + V_2^2 G_2^0 + 2 F_{12}^0 V_2 + 2 F_{12}^0 V_1 + 2 V_1 G_{11}^0 - 2 V_2 G_{22}^0
$$

$$
- \frac{1}{4} \hbar^2 G_{1111}^0 - \frac{1}{4} \hbar^2 G_{2222}^0.
$$
simplify the expressions. Also we can rescale the answer. 

\[ \frac{V}{G} \]

subject to the conditions

\[ 0 = G_1^{\prime} + G_2^{\prime}, \]

\[ 0 = - f_1^{\prime} + f_2^{\prime} + V_1 G_1^{\prime \prime} - f_1^{\prime} G_1^{\prime} + V_2 G_2^{\prime \prime} + f_2^{\prime} G_2^{\prime} - 2 F_{12} f_2 - 2 F_{12} f_1 + 2 V_1 G_1^{\prime \prime} \]

\[ - 2 f_1^{\prime} G_1^{\prime} - 2 V_2 G_2^{\prime \prime} + 2 F_2^{\prime} f_2^{\prime} - \frac{1}{4} h^2 G_1^{\prime \prime \prime} + \frac{1}{4} h^2 G_2^{\prime \prime \prime}, \]

\[ 0 = - f_1^{\prime} G_1^{\prime} + f_2^{\prime} G_2^{\prime} + 2 f_3 G_2^{\prime \prime} - 2 f_1^{\prime} G_1^{\prime} - 2 G_1^{\prime \prime} - 2 G_2^{\prime \prime}. \]

\[ 0 = - f_1^{\prime} G_1^{\prime} + f_2^{\prime} G_2^{\prime} + 2 f_3 G_2^{\prime \prime} - 2 f_1^{\prime} G_1^{\prime}. \]

4. Some examples (mostly new)

For our first examples we are particularly interested in potentials with nonlinear defining equations. First, we show that we can obtain the result of Tremblay and Winternitz [4] that the quantum system separating in polar coordinates in 2D Euclidean space admits potentials that are expressed in terms of the sixth Painlevé transcendent or in terms of the Weierstrass \( \wp \)-function. To do this we must put the system in the canonical form (2). The separable polar coordinates are \((x, y) = (r \cos(\theta), r \sin(\theta))\). For the canonical form we have \( r = \exp(u_1), \theta = u_2 \). Thus, \( f_1(u_1) = \exp(2u_1) \) and \( f_2(u_2) = 0 \). (Of course, here the scalar curvature is \( c = 0 \).) We know that the Painlevé VI can appear only if the potential depends on the angular variable alone, so we set \( V_1(u_1) = 0 \). Since we want only systems that satisfy nonlinear equations alone, whenever an explicit linear equation for the potential appears, we require that it vanish identically. We have the freedom to replace the angular variable \( u_2 \) by \( u_2 + k \) for some constant \( k \) to simplify the expressions. Also we can rescale the answer.

We obtain a solution of the equations in section 3 as

\[ F^0 = 4 h^2 \exp(-u_1) \sin(u_2), \]

\[ G^0 = - U_1(u_2) \exp(-u_1) + U_2(u_2), \]

\[ G^0 = a_5, \]

subject to the conditions

\[ 0 = a_4 \frac{dV_2}{du_2} + 2 \frac{d^2 U_2}{du_2^2}, \]  

\[ 0 = h^2 \frac{d^2 U_2}{du_2^2} + 4 a_4 \frac{dV_2}{du_2} V_2 - 4 \frac{dV_2}{du_2} \frac{dU_2}{du_2} \frac{dV_2}{du_2}, \]

\[ 0 = 8 V_2 \cos(u_2) + 4 \frac{dV_2}{du_2} \sin(u_2) - \frac{d^2 U_1}{du_2^2} - U_1, \]

\[ 0 = \frac{dV_2}{du_2} \frac{dU_1}{du_2} + \frac{dV_2}{du_2} \sin(u_2) - 4 h^2 \frac{d^2 V_2}{du_2^2} \cos(u_2) \]

\[ + 2 \sin(u_2)(h^2 + 4 V_2) \frac{dV_2}{du_2} + 2 V_2 (6 h^2 \cos(u_2) + 8 V_2 \cos(u_2) - U_1). \]

There are basically two cases to consider:

\[ (18) \]
(a) \( a_4 = 0 \).

Then condition (15) says that \( U_2 \) is linear in \( u_2 \). Thus condition (16) is a linear equation for \( V_2(u_2) \) which must vanish. Then condition (17) can be solved for \( U_1(u_1) \) and the result substituted into condition (18) to obtain an equation for \( V_2(u_2) \). After some manipulation (using the fact that \( V_2 \) is unchanged by transformations \( W \rightarrow W + k \), where \( k \) is a constant), we obtain an equation characterizing Painlevé VI, in agreement with [4], equation (4.27):

\[
h^2 \left( \sin(u_2) \frac{d^4 W}{du_2^4} + 4 \cos(u_2) \frac{d^3 W}{du_2^3} - 6 \sin(u_2) \frac{d^2 W}{du_2^2} - 4 \cos(u_2) \frac{dW}{du_2} \right) - 12 \sin(u_2) \frac{d^2 W}{du_2^2} - 4 \cos(u_2) W \frac{d^2 W}{du_2^2} - 4(\beta_1 \sin(u_2) - \beta_2 \cos(u_2)) \frac{d^2 W}{du_2^2} - 16 \cos(u_2) \left( \frac{dW}{du_2} \right)^2 + 8 \sin(u_2) W \frac{dW}{du_2} - 8(\beta_1 \cos(u_2) + \beta_2 \sin(u_2)) \frac{dW}{du_2} = 0
\]  

Here \( V_2(u_2) = \frac{dW(u_2)}{du_2} \).

(b) \( a_4 \neq 0 \).

Solving condition (15) for \( V_2(u_2) \) and substituting the result and (15) into (16) we obtain the equation that characterizes the Weierstrass \( \wp \)-function (in fact it is a translated and rescaled version):

\[
h^2 \frac{d^3 V_2}{du_2^3} - 12 \frac{dV_2}{du_2} V_2 + 12 a_1 \frac{dV_2}{du_2} = 0,
\]  

where \( a_1 \) is an arbitrary constant. Thus \( V_2(u_2) = N \wp(u_2 - u_{2,0}; g_2, g_3) + a_1 \), where \( u_{2,0}, g_2, \) and \( g_3 \) are arbitrary constants. As shown in [4] this solution is subject to the compatibility condition (17) and (18), which leads to a complicated nonlinear differential equation for \( V_2(u_2) \) that we will treat later.

Now we consider the analogous system on the two-sphere, separable in spherical coordinates. Here \( s_1 = \sin(\theta) \cos(\phi), s_2 = \sin(\theta) \sin(\phi), s_3 = \cos(\theta) \) with \( s_1^2 + s_2^2 + s_3^2 = 1 \). This system is in canonical form with coordinates \( \{u_1, u_2\} \) where

\[
\sin(\theta) = \cosh(u_1)^{-1}, \quad \phi = u_2, \quad f_1(u_1) = \cosh(u_1)^{-2}, \quad f_2(u_2) = 0.
\]

The scalar curvature is \( c = 2 \). As before we look for solutions such that \( V_1(u_1) = 0 \) and \( V_2(u_2) \) satisfies a nonlinear equation only.

The computation is very similar to that for the Euclidean space example. We obtain the solution

\[
F^0 = 4h^2 \cosh(u_1) \sin(u_2), \quad G^L = 8 \sinh(u_1) \cos(u_2) + a_1 y + a_3, \quad G^H = a_5,
\]

subject to the conditions (15)–(18), exactly the same as for Euclidean space. Thus the system on the two-sphere also admits Painlevé VI and special Weierstrass \( \wp \)-function potentials for 3rd order superintegrability. It is clear from these results that these systems in Euclidean space can be obtained as Bôcher contractions, [9, chapter 15], of the corresponding systems on the two-sphere.
Next we consider spherical coordinates on the hyperboloid \( s_1^2 - s_2^2 - s_3^2 = 1 \),
\[
s_1 = \cosh(x), \quad s_2 = \sinh(x) \cos(\phi), \quad s_3 = \sinh(x) \sin(\phi).
\]
For the canonical form we find
\[
\tanh \left( \frac{u_1}{2} \right) = \exp(x), \quad u_2 = \phi, \quad f_1(u_1) = \sinh(u_1)^{-2}, \quad f_2(u_2) = 0,
\]
(the scalar curvature is \( c = -2 \)), and we look for solutions such that \( V_1(u_1) = 0 \) and \( V_2(u_2) \) satisfies only a nonlinear equation. We obtain the solution
\[
F^0 = 4\hbar^2 \sin(u_1) \sinh(u_1), \quad G^L = 8 \cos(u_2) \cosh(u_1) + a_4u_2 + a_5,
\]
\[
G^0 = \cosh(u_1) U_1(u_2) + U_2(u_2), \quad G^H = a_5,
\]
subject to the conditions (15)–(18), again exactly the same as for flat space. Thus the system on the two-hyperboloid admits Painlevé VI and special Weierstrass \( \wp \)-function potentials for 3rd order superintegrability.

For our next example we consider horocyclic coordinates \( \{u_1, u_2\} \) on the hyperboloid \( s_1^2 - s_2^2 - s_3^2 = 1 \), e.g. [8, section 7.7]:
\[
s_1 = \frac{1}{2} \left( u_1 + \frac{u_2^2 + 1}{u_1} \right), \quad s_2 = \frac{1}{2} \left( u_1 + \frac{u_2^2 - 1}{u_1} \right), \quad s_3 = \frac{u_2}{u_1},
\]

These coordinates are separable and the canonical system is defined by \( f_1(u_1) = 1/u_1^2 \), \( f_1(u_2) = 0 \). We look for systems such that \( V_1(u_1) = 0 \), in analogy with our first three examples.

We obtain the solution
\[
F^0 = -\frac{1}{2} a_8 \hbar^2 u_1, \quad G^L = \frac{u_1^2(a_8u_2 + a_9)}{2} - \frac{a_8u_2^3}{6} - \frac{a_9u_2^2}{2} + a_1u_2,
\]
\[
G^0 = \frac{u_1^2}{2} U_1(u_2) + U_2(u_2), \quad G^H = a_7,
\]
subject to the conditions
\[
0 = a_8 \frac{dV_2}{du_2} + 2 \frac{d^2U_1}{du_2^2},
\]
\[
0 = \frac{1}{2} \hbar^2 a_8 \frac{d^3V_2}{du_2^3} - 4a_8 \frac{dV_2}{du_2} V_2 + 4 \frac{dV_2}{du_2} \frac{dU_1}{du_2} \quad (25)
\]
\[
0 = (2a_{10} - 2a_9u_2 - a_9u_2^2) \frac{dV_2}{du_2} - 4(a_9 + a_8u_2)V_2 + 4U_1 + 4 \frac{d^2U_2}{du_2^2} \quad (26)
\]
\[
0 = -2\hbar^2 a_8u_2^2 \frac{d^3V_2}{du_2^3} + 16(a_9 + a_8u_2)V_2^2 - 4(2a_{10} - 2a_9u_2 + a_9u_2^2) \frac{dV_2}{du_2} V_2
\]
\[
+ \frac{\hbar^2}{2} (2a_{10} - 2a_9u_2 - a_9u_2^2) \frac{d^3V_2}{du_2^3} - 4\hbar^2(a_9 + a_8u_2)
\]
\[
- 16V_2 U_1 + 8 \frac{dV_2}{du_2} \frac{dU_2}{du_2} \frac{d^2U_2}{du_2^2} \quad (27)
\]
There are again two basic cases here:

(a) \( \alpha_8 = 0 \).

Then conditions (24) and (25) say that \( U_1 \) is a constant: \( U_1(u_2) = d_1 \). Then condition (26) can be solved for \( U_2(u_2) \) and the result substituted into condition (27) to obtain an equation for \( V_2(u_2) \):

\[
-4a_8(\frac{dW}{du_2})^2 + \left( (-3a_8u_2 + 3a_{10})\frac{d^2W}{du_2^2} + 4d_1 \right) \frac{dW}{du_2}
+ (-a_8W + 2d_1u_2 - 2d_3)\frac{d^2W}{du_2^2} + h^2a_9\frac{d^3W}{du_2^3} - \frac{1}{4}h^2(a_10 - a_9u_2)\frac{d^4W}{du_2^4} = 0,
\]

(28)

where \( V_2(u_2) = \frac{dW(u_2)}{du_2} \).

With the integrating factor \( \mu(u_2) = u_2a_9 - a_{10} \), the fourth order nonlinear differential equation admits the following first integral

\[
J = \frac{1}{4}h^2(a_{10} - a_9u_2)^2\frac{dW}{du_2} - \frac{h^2}{2}(a_{10}a_9h^2 - a_9^2h^2)\frac{d^2W}{du_2^2}
- \frac{3}{2}(a_{10}^2 - 2a_{10}a_9u_2 - a_9^2u_2^2)\left( \frac{dW}{du_2} \right)^2 + (a_{10}a_9 - a_9^2u_2)W\frac{dW}{du_2}
+ (2a_{10}d_1 - a_9^2h^2 - 2a_{10}d_1u_2 - 2a_9d_3u_2 + 2a_9d_1u_2^2)\frac{dW}{du}
+ \frac{1}{2}a_9^2W^2 + 2a_9d_3W - 2a_{10}d_1.
\]

(29)

We now consider two subcases.

(a) \( a_0 \neq 0 \). Using the following transformation of the independent and dependent variables

\[
W(u_2) = u_2 \tilde{W}(u_2) + b + cu_2^2 + du_2^2, \quad z = \frac{a_{10}}{a_9} + u
\]

(30)

and further using

\[
y = z^2
\]

(31)

and using the constraint \( d_3 = \frac{a_{10}d_1}{a_9} \) (for \( a_9 \neq 0 \)), we obtain a third order differential equation can be related to the Chazy I equation. This equation appears in the classification, \([14, \text{equation (A.3)}]\). It has the form

\[
\tilde{W}'''' = -\frac{2}{f'(y)} \left( 3c_1y(\tilde{W}' - \tilde{W})^2 + c_2 \left( y\tilde{W}' - \tilde{W} \right) (3y\tilde{W}' - \tilde{W})
+ c_3 \tilde{W}' (3y\tilde{W}' - 2\tilde{W}) + 3c_4 \left( \tilde{W}' \right)^3 + 2c_5y (y\tilde{W}' - \tilde{W})
+ c_6 (2y\tilde{W}' - \tilde{W}) + 2c_7\tilde{W}' + c_8y + c_9 \right) - \frac{f''}{f} \tilde{W}'',
\]
where

\[ f(y) = c_1y^3 + c_2y^2 + c_3y + c_4 \]

Our case corresponds to the following choice of parameters:

\[
\begin{align*}
   c_1 &= c_2 = 0, & c_3 &= 2a_9, & c_4 &= c_5 = 0, & c_6 &= -2(2a_9c + d_1), \\
   c_7 &= \frac{1}{4}a_9(8b - a_9), & c_8 &= 2(a_9c^2 + cd_1), \\
   c_9 &= \frac{1}{2}(-8a_9bc - 4bd_1 - a_9c), & d &= 0, & a_9 &= -h^2.
\end{align*}
\]

This equation can be further integrated, [15, equation (A.21)] and the resulting equation takes the following form

\[
(\tilde{W}''') = -4f^2( c_1(x\tilde{W}' - \tilde{W})^2 + c_2\tilde{W}'(y\tilde{W}' - \tilde{W})^2 + c_3(\tilde{W}')^2(y\tilde{W}' - \tilde{W})
+ c_4(\tilde{W}')^3 + c_5(y\tilde{W}' - \tilde{W})^2 + c_6\tilde{W}'(y\tilde{W}' - \tilde{W}) + c_7(\tilde{W}')^2
+ c_8(y\tilde{W}' - \tilde{W}) + c_9\tilde{W}' + c_{10})
\]

(32)

where \( c_{10} \) is an integration constant. This equation is known as SD-I and was discussed in [15, equation (4.9)]. It has 6 subcases: SD-Ia, SD-Ib, SD-Ic, SD-Id, SD-Ie, SD-If. Here the non zero parameters are \( c_1, c_6, c_8, c_9 \), according to [14]. As described in [15, equation (5.5)] the equation is related to Painlevé III and Painlevé V equations. Explicit formulas are given in [15, equations (5.26)–(5.40)].

(b) \( a_9 = 0 \). We obtain the third order equation

\[-2d_1W + (2d_3 - 2d_1u_2)W' - \frac{3}{2}a_{10}W'^2 + \frac{1}{4}a_{10}h^2W'W'' = 0,\]

which is a special case of [7, equation (83)]:

\[
\alpha h^2W''' - 6\alpha W'^2 - 4(c_1u^3 - \alpha c_2u^2 + b_1u + b_0)W'
- 4(3c_1u^2 - 2c_2\alpha u + b_1)W - \frac{2}{3}c_2\alpha u^4 + 4(\frac{1}{3}c_2b_1 - c_1a_2)u^3
- 2(3c_1a_1 - 2c_2b_0)u^2 + k_2u + k_4 = 0.
\]

This equation admits the first integral

\[
J = \alpha h^2W' - \alpha W^2 - \left( \alpha a_1 - 2(b_0 - \alpha a_2)u - 2b_1u^2 - \frac{2}{3}\alpha c_2u^3 - 2c_1u^4 \right)W
- \frac{1}{9}\alpha c^2u^6 + \frac{1}{6}(3a_2c_1 - b_1c_2)u^5 + \left( \frac{2}{3}\alpha a_2c_2 - b_0c_2 + a_1c_1 \right)u^4
+ \left( \frac{4}{9}\alpha a_1c_2 - \frac{k_2}{4} \right)u^3 - \frac{1}{8}(\alpha k_1 + 4k_4)u^2 + k_3u - \frac{k_3}{2} = 0.
\]

(34)

Then, a Cole-Hopf transformation \( W = -h^2\frac{U'}{U} \) gives a second order linear equation for \( U \).
Here we can solve (24) for $V_2(u_2)$ and substitute the result into (25) to obtain the equation
\[
\hbar^2 \frac{d^3 V_2}{du_2^3} - 12 V_2 \frac{dV_2}{du_2} + 12 a_1 \frac{dV_2}{du_2} = 0, \tag{35}
\]
where $a_1$ is an arbitrary constant. Solutions of (35) are further subject to the requirement that a solution $U_2(u_2)$ of equations (26) and (27) exists. The general solution of (35) is
\[
V_2(u_2) = \hbar^2 \varphi(u_2 - u_2, g_2, g_3) + a_1,
\]
where $u_2, g_2$, and $g_3$ are arbitrary constants.

5. Classification of systems on the two-sphere separating in spherical coordinates

We use the coordinates (21) with $x = \sin(\theta), y = \phi$ and list the systems that admit a 3rd order symmetry operator. They fall into 4 classes:

(a) Systems that are 2nd order superintegrable.
These systems are all known and they are classical (all parameters in the potential are arbitrary).

(b) Systems that are neither 2nd or 3rd order superintegrable.
These are special cases of 2nd order superintegrable systems, except that they depend on $\hbar$, and admit symmetry generators distinct from the classical ones.

(c) Systems with algebraically dependent generators $A$ and $B$.

(d) Systems that are truly 3rd order superintegrable.

5.1. Systems that are 2nd order superintegrable

\[
V_1(x) = \frac{c_1 \sqrt{1 - x^2}}{x} + c_2, \quad V_2(y) = \frac{c_3}{\cos^2(y)} + \frac{c_4 \sin(y)}{\cos^2(y)}. \tag{36}
\]
This is the 2nd order superintegrable system $S_7$, [6].

\[
V_1(x) = \frac{c_1}{1 - x^2} + c_2, \quad V_2(y) = \frac{c_3}{\cos^2(y)} + \frac{c_4}{\sin^2(y)}. \tag{37}
\]
This is the 2nd order superintegrable system $S_9$, [6]. It is characterized by the fact that all of the parameters $\alpha_j$ and $\beta_{jk}$ are zero. These are the only 2nd order four-parameter superintegrable systems.

\[
V_1(x) = 0, \quad V_2(y) = c_1 \exp(-2iy). \tag{38}
\]
This complex potential system admits a 3rd order symmetry and also a 1st order symmetry, so it is 2nd order superintegrable, the 2nd order system $S_5$, [6]. It is PT-symmetric so the energy eigenvalues are real.

All other 2nd order systems are special cases of these. We will omit all classical special cases of these systems and include only purely quantum special cases.

(a) Quantum special cases.
\[ V_1(x) = \frac{\hbar^2}{c_1(1-x^2)}, \quad V_2(y) = 0. \]  
\hspace{1cm} (39)

This is a special case of \( S_9 \), but only quantum. In these cases the symmetry operators differ from those for which \( \hbar \) is an arbitrary constant \( c_1 \). The system admits a 1st order symmetry, so it is 2nd order superintegrable.

\[ V_1(x) = 0, \quad V_2(y) = \frac{27\hbar^2}{\cos^2(y)}. \]  
\hspace{1cm} (40)

This system is 2nd order superintegrable and only quantum, a special case of \( S_9 \). It admits a 1st order symmetry.

5.2. Systems that are 3rd order superintegrable but do not generate a cubic algebra

\[ V_1(x) = 0, \quad V_2(y) = -3\hbar^2(4 \cos^4(y) - 3) \frac{\cos^2(y) - \cos^2(y) - 3)^2}{\cos^2(y) - \cos^2(y) - 3)^2}. \]  
\hspace{1cm} (41)

This system is truly 3rd order superintegrable but only quantum. The generators \( \{H, A, B, C\} \) where \( C = [A, B] \) do not close under commutation to form a cubic algebra. This is a special case of an isospectral deformation of the trigonometric Scarf potential, [12]. The bound state spectrum of \( A \) is

\[ \lambda = 2\hbar^2 \left( \frac{1}{2} + \nu \right)^2, \quad \nu = 0, 1, 2, \ldots, \]

and the \( A \) eigenfunctions are proportional to Jacobi exceptional orthogonal polynomials of degree \( \nu + 2 \) in \( \sin(y) \).

5.3. Systems with algebraically dependent generators \( A \) and \( B \)

These systems are not 2nd order superintegrable and \( C = [A, B] = 0 \). Thus the set \( \{H, A, B\} \) is commutative so (by Burchnall–Chaundy theory), [7],[11] there must be an algebraic relation between \( A \) and \( B \) that we can compute. This permits us to exhibit an explicit first integral for the eigenfunctions of \( A \). Thus these systems are special. We give the details for our first example. The other cases are similar.

\[ V_1(x) = 0, \quad V_2(y) = \frac{4\hbar^2}{\cos^2(2y)}. \]  
\hspace{1cm} (42)

Here \( B \) is the formally self-adjoint operator

\[ B = i\hbar^3 \frac{\partial}{\partial y} + 8i\hbar \frac{\cos^3(y) - 2 \cos^2(y) - 1}{(2 \cos(y)^2 - 1)^2} \frac{\partial}{\partial y} - \frac{48i\hbar^3 \sin(y) \cos(y)}{(2 \cos(y)^2 - 1)^3}. \]  
\hspace{1cm} (43)
The relationship is
\[ A^3 - \frac{1}{8} B^3 - 4\hbar^2 A^2 + 4\hbar^4 A = 0. \]

However, since \( C = 0 \), the formally self-adjoint operators \( B \) and \( A \) admit common eigenfunctions \( g(y) \):
\[ B g = \mu g, \quad A g = \lambda g, \quad \lambda^3 - \frac{1}{8} \mu^2 - 4\hbar^2 \lambda^2 + 4\hbar^4 \lambda = 0. \]

Solving for \( \mu \) we find
\[ \mu = \pm 2\sqrt{2}(\lambda^2 - 2\hbar^2) \sqrt{\lambda}. \]

Now consider the equation
\[ (B g - \mu g) + a \frac{d}{dy}(A g - \lambda g) = 0 \]
for any constant \( a \). Choosing \( a = 2i\hbar \) we eliminate the 3rd derivative term in \( y \) and obtain the 1st order differential equation
\[ - \frac{2i\hbar}{g(y)} \frac{dg(y)}{dy} = \frac{-8 \cos^6(y)\mu + 16i \cos(y) \sin(y) \hbar^3 + 12 \cos^4(y)\mu - 6 \cos^2(y)\mu + \mu}{(2 \cos^2(y) - 1)(8 \cos^4(y)\hbar^2 - 4 \cos^4(y)\lambda - 8 \cos^2(y)\hbar^2 + 4 \cos^2(y)\lambda - \lambda^3)}, \tag{44} \]
into which we substitute the two possibilities for \( \mu \). These are explicit first integrals for the 2nd order differential equations satisfied by the eigenfunctions of \( A \).

**Another example:**

\[ V_1(x) = 0, \quad V_2(y) = \frac{N(y)}{D(y)}, \tag{45} \]
where
\[ N(y) = -4\hbar^2 \left( 2\sqrt{3} \sin(y) \cos^3(y) + 2 \cos^4(y) + 18\sqrt{3} \sin(y) \cos(y) \sin(y) - 21 \cos^2(y) - 81 \right), \]
\[ D(y) = 16 \cos^6(y) + 72\sqrt{3} \sin(y) \cos^3(y) + 48 \cos^4(y) + 108\sqrt{3} \cos(y) \sin(y) - 207 \cos^2(y) + 243. \]

This system admits a 3rd order symmetry \( B \), but then \( C = 0 \). Thus \( A \) and \( B \) are algebraically dependent and the system is not 3rd order superintegrable. The algebraic relationship between \( A \) and \( B \) is
\[ A^3 - \frac{1}{8} B^3 - \hbar^2 A^2 + \frac{i\sqrt{3}\hbar^3}{3} B + \frac{h^4}{4} A + \frac{2}{3} \hbar^6 = 0. \]

The general \( C = 0 \) case:
This differential equation admits the Weierstrass \( \wp \)-function as a solution. The system admits a 3rd order symmetry provided the additional condition

\[
h^3 \frac{d^3 V_2}{dy^3} - 12h V_2 \frac{dV_2}{dy} - 4Z_1 \frac{dV_2}{dy} = 0
\]

is satisfied for some constant \( Z_1 \). However, this condition is always satisfied since (47) is a first integral of (46). Equation (47) implies \( C = 0 \), so that \( A \) and \( B \) are algebraically dependent. Indeed, they obey the relation

\[-B^2 + A^3 + \alpha A^2 = 0,
\]

where

\[-iB = \frac{3h}{2} \frac{dV_2}{dy} - h^3 \partial_y^3 + (3hV_2 + \alpha h) \partial_y.
\]

Thus this general system is not 3rd order superintegrable.

There are also elementary function solutions of the nonlinear defining equation, none of which lead to 3rd order superintegrability. In particular the following systems are solutions:

*\[V_2(y) = h^2 \left( \frac{1}{\sin^2(y)} + \frac{1}{\cos^2(y)} \right), \quad (48)\]*

This can be considered as a zero-parameter potential within \( S_6 \). It has a two-parameter 3rd order symmetry, \( \beta_3, \alpha_3 \), but if \( \beta_3 \neq 0 \) the system does not close to a cubic algebra. If \( \beta_3 = 0, \alpha_3 \neq 0 \) we find \( C = 0 \). Then \( A \) and \( B \) are related by

\[A^3 - \frac{1}{8} B^2 - 4h^2 A^2 + 4h^4 A = 0.
\]

*\[V_2(y) = \frac{4h^2}{(2 \sin^2(y) - 1)^2}. \quad (49)\]*

This system admits a 3rd order symmetry but \( C = 0 \) so \( A \) and \( B \) are algebraically dependent:

\[A^3 - \frac{1}{8} B^2 - 4h^2 A^2 + 4h^4 A = 0.
\]

*\[V_2(y) = \frac{h^2(4b^2 + c^2)}{(2b \cos^2(y) \pm c \sin(y) \cos(y) - b)^2}.
\]

\[ (50)\]
This can be obtained from the Weierstrass \( \wp \)-function solution. The system admits a 3rd order symmetry but \( C = 0 \) so \( A \) and \( B \) are algebraically dependent. The relationship between \( A \) and \( B \) is

\[
A^3 - \frac{1}{8} B^2 - 4h^2 A^2 + i \frac{48b^3 h^3 + 12bc^2 h^3}{c^3} B + 4h^2 A + 288 \frac{(4b^2 + c^2)^2 h^6}{c^6} = 0.
\]

### 5.4. Systems that are truly 3rd order superintegrable

In addition to the nonlinear solution (22) we have:

- \( V_1(x) = 0 \), \( V_2(y) = \frac{c}{(4 \cos^2(y) - 3)^2 \cos(y)} \).

This system is truly 3rd order superintegrable. It satisfies a cubic algebra of the form.

\[
[A, C] = \alpha A^2 + \beta \{A, B\} + \gamma A + \delta B + \epsilon,
\]

\[
[B, C] = \mu A^3 + \nu A^2 - \beta B^2 - \alpha \{A, B\} + \xi A - \gamma B + \zeta,
\]

where \( \{A, B\} = AB + BA \) and

\[
\gamma = \gamma_0 + \gamma_1 H, \quad \delta = \delta_0 + \delta_1 H, \quad \epsilon = \epsilon_0 + \epsilon_1 H + \epsilon_2 H^2,
\]

\[
\nu = \nu_0 + \nu_1 H, \quad \xi = \xi_0 + \xi_1 H + \xi_2 H^2, \quad \zeta = \zeta_0 + \zeta_1 H + \zeta_2 H^2 + \zeta_3 H^3.
\]

The final closure relation is

\[
C^2 = \frac{9h^2}{2} [A, B^2] + 576h^2 A^4 + 9h^2 B A^2 - 576h^2 H^3 A
\]

\[
+ 1728h^2 H^2 A^2 - 1728h^2 H A^3 - \frac{243h^2}{4} B^2 + (16704h^4 - 576c h^3) A^3
\]

\[
+ (10080h^4 - 1728c h^5) H^2 A + (-26784h^4 + 1728c h^3) H A^2
\]

\[
+ 576c h^2 H^3 + (27792h^6 - 8928c h^4) A^2 - 2304c h^6 H^2
\]

\[
+ (-17280h^6 + 11232c h^5) H A + 1728c h^3 H + (2592c h^8 - 5760c h^6) A.
\]

The possible spectra for \( H \) can be computed from these algebraic relations.

### 6. Systems with algebraically dependent generators as degenerations of \( \wp \)-potentials

The systems with algebraically dependent generators \( A \) and \( B \) can all be obtained as degenerations of \( \wp \)-potentials. The Weierstrass \( \wp \)-function with invariants \( g_2, g_3 \) satisfies the 1st order, nonlinear differential equation

\[
(\wp')^2 = 4\wp^3 - g_2 \wp - g_3.
\]

We can parameterize the invariants \( g_2 \) and \( g_3 \) as

\[
g_2 = 2(e_1^2 + e_2^2 + e_3^2), \quad g_3 = 4e_1 e_2 e_3.
\]
so that
\[(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_2), \quad e_1 + e_2 + e_3 = 0. \tag{56}\]

When two of the roots \(e_i\) coincide, (56) can be integrated in terms of elementary functions. We take \(e_1 = e_2\) so that \(e_3 = -2e_1\). The general solution of

\[(\wp')^2 = 4(\wp - e_1)^2(\wp + 2e_1) \tag{57}\]

is

\[\wp(z - z_0; 12e_1^2, -8e_1^2) = 3e_1 \csc^2\left(\sqrt{3e_1(z - z_0)}\right) + e_1 \tag{58}\]

or, making the replacement \(e_1 \rightarrow -e_1\),

\[\wp(z - z_0; 12e_1^2, -8e_1^2) = 3e_1 \csc^2\left(\sqrt{3e_1(z - z_0)}\right) - e_1. \tag{59}\]

Either of these expressions degenerates to a rational solution as \(e_1 \rightarrow 0:\)

\[\wp(z - z_0; 0, 0) = \frac{1}{(z - z_0)^2}. \tag{60}\]

6.1. Potentials from third-order superintegrability

On the two-sphere we have the potential

\[V_2(u_2) = \hbar^2 \wp(u_2 - u_{2,0}; g_2, g_3) + a_1 \tag{61}\]

Only the trigonometric degeneration is relevant here:

\[V_2(u_2) = \hbar^2 \kappa^2 \csc^2(\kappa(u_2 - u_{2,0})) + \frac{\kappa^2}{3} + a_1, \tag{62}\]

for an arbitrary real parameter \(\kappa\).

On the two-hyperboloid we have the potential

\[V_2(u_2) = \hbar^2 \wp(u_2 - u_{2,0}; g_2, g_3) + a_1. \tag{63}\]

There are three degenerate potentials:

(a) \(V_2(u_2) = \hbar^2 \kappa^2 \csc^2(\kappa(u_2 - u_{2,0})) + \frac{\kappa^2}{3} + a_1\)

(b) \(V_2(u_2) = \hbar^2 \kappa^2 \csc^2(\kappa(u_2 - u_{2,0})) + \frac{\kappa^2}{3} + a_1\)

(c) \(V_2(u_2) = \hbar^2(u_2 - u_{2,0})^{-2} + a_1,\)

where \(\kappa\) is an arbitrary complex parameter.

7. Higher order FD-superintegrable systems

A standard superintegrable system on an \(n\)-dimensional Riemannian manifold (real or complex) is a system that possesses \(2n - 1\) functionally independent constants of the motion in the classical case and \(2n - 1\) algebraically independent symmetry operators in the operator case. If a system possess \(2n - 1\) linearly independent symmetries but they are not functionally (or algebraically) independent, we will call it functionally dependent—superintegrable,
(or FD-superintegrable for short). (These are to be distinguished from Functionally Linearly Independent systems, [10], of which there is only one example in two dimensions).

In this paper we have found a number of FD-superintegrable systems of a rather simple type: \( C = 0 \). For these systems the operators \( A \) and \( B \) commute. Further, they are ordinary differential operators in the variable \( y \) alone. The Abelian algebra generated by \( \{ H, A, B \} \) does have structure because \( A \) and \( B \) satisfy an algebraic equation. It is clear, moreover that the machinery constructed here will work almost unchanged on any manifold with metric of the form \( ds^2 = dx^2 + F(x)dy^2 \), not just those of constant curvature.

Consider the 1D Hamiltonian

\[
A = -\frac{\hbar^2}{2} \frac{d^2}{dy^2} f(y) + V_0(y)f(y)
\]

and the symmetry operator

\[
B = W_0(y) + W_1(y) \frac{d}{dy} + W_2(y) \frac{d^2}{dy^2} + W_3(y) \frac{d^3}{dy^3}.
\]

We require that [\( A, B \)] = 0 and that \( B \) is linearly independent of \( A \). The solution, unique up to a constant factor, is

\[
B = \frac{d^3}{dy^3} - \frac{b_1}{\hbar^2} \frac{d}{dy} \frac{dV_0}{dy} - \frac{3}{2\hbar^2} \frac{dV_0}{dy},
\]

subject to the condition

\[
-4b_1 \frac{dV_0}{dy} \frac{1}{\hbar^2} - \frac{d^3V_0}{dy^3} \frac{1}{\hbar^2} + 12V_0 \frac{dV_0}{dy} = 0.
\]

This can be identified with equation (47), so the treatment of (46) carries over immediately to this 1D system.

To summarize, the differential equation

\[
\hbar^2 \left( -\frac{dV_2}{dy} \frac{d^4V_2}{dy^4} + \frac{d^2V_2}{dy^2} \frac{d^2V_2}{dy^2} \right) + 12 \left( \frac{dV_2}{dy} \right)^3 = 0.
\]

can be integrated in terms of the Weierstrass \( \wp \)-function. This equation passes the Painlevé test with movable double poles as the only singularities. We can find the general solution.

This construction can be extended to general \( n \). We present an example that is pertinent to 5th order FD-superintegrability for the two-sphere. The 5th order operator that commutes with \( A \) is

\[
B = \frac{d^3}{dy^3} + \frac{d^4}{dy^4} - \frac{5V_0}{2\hbar^2} \frac{d^3}{dy^3} - \frac{(-2c_2h^2 + 8c_4V_0 + 15\frac{dV_0}{dy})}{2\hbar^2} \frac{d^2}{dy^2} - \frac{4c_1h^4 + 16c_4\frac{dV_0}{dy}h^2 + 25\frac{d^2V_0}{dy^2}h^2 - 30V_0}{4\hbar^4} \frac{d}{dy} - \frac{-8c_0h^4 + 16c_4\frac{dV_0}{dy}h^2 + 16c_2h^2V_0 + 15\frac{dV_0}{dy}h^2 - 32V_0}{8\hbar^4} \frac{dV_0}{dy},
\]

(67)
subject to the condition

$$16c_1 \frac{dV_0}{dy} h^4 + \frac{d^5 V_0}{dy^5} h^4 - 20V_0 \frac{d^3 V_0}{dy^3} h^2 - 40 \frac{dV_0}{dy} \frac{d^2 V_0}{dy^2} h^2 + 120V_0 \frac{dV_0}{dy} = 0.$$  \hspace{1cm} (68)

Here, $A$ and $B$ are related by the expression

$$B^2 + a_1 A^5 + a_2 B A^2 + a_3 A^4 + a_4 B A + a_5 A^3 + a_6 B + a_7 A^2 + a_8 A = 0,$$  \hspace{1cm} (69)

where

$$a_1 = \frac{32}{h^{10}}, \quad a_2 = \frac{8c_4}{h^4}, \quad a_3 = \frac{16c_4^2}{h^8}, \quad a_4 = \frac{4c_2}{h^2},$$

$$a_5 = \frac{c_1 - c_2 c_4}{h^6}, \quad a_6 = -2c_0, \quad a_7 = -\frac{k_1}{2h^{10}},$$

and $V_0(y)$ satisfies the equation

$$48V_0^5 + \left( 64c_1 h^4 + 80h^2 \frac{d^2 V_0}{dy^2} V_0^3 + (-192c_4 c_0 h^6 - 96c_2^2 h^6 \right.$$

$$- 12h^2 \left( -20 \frac{d^2 V_0}{dy^2} \right)^2 h^4 + 32h^6 c_1 \frac{d^2 V_0}{dy^2} h^2 + 96h^8 (a_0 h^2 + 4c_0 c_2 - 2c_1^2) V_0 + \left( -16h^6 c_1 + 4h^4 \frac{d^2 V_0}{dy^2} \right) \frac{dV_0}{dy}$$

$$- 2h^2 (16c_0 c_4 h^6 + 8c_2^2 h^6 + k_1) \frac{d^2 V_0}{dy^2} + 128c_0 c_1 c_4 h^{10} + 64c_1 c_2 h^{10}$$

$$- 64c_2^2 h^{10} + \left( \frac{d^3 V_0}{dy^3} \right)^2 h^6 + 8c_1 h^2 k_1 - a_{10} = 0. \quad \hspace{1cm} (70)$$

Here (70) is an integrated form of (68). See [11] for the theory behind these results.

However, since $[A, B] = 0$, the formal operators $B$ and $A$ admit common eigenfunctions $g(y)$:

$$Bg = \mu g, \quad Ag = \lambda g,$$

and

$$\mu^2 + a_1 \lambda^5 + a_2 \lambda^4 + a_3 \lambda^3 + a_4 \mu \lambda + a_5 \lambda^2 + a_6 \mu + a_7 \lambda^2 + a_8 \lambda = 0, \quad \hspace{1cm} (71)$$

a quadratic equation for $\mu$ as a function of $\lambda$. Differentiating the equation $Ag = \lambda g = 0$ repeatedly we can derive linear equations for $\frac{d^2 g}{dx^2}, \frac{d^4 g}{dx^4}, \frac{d^6 g}{dx^6}, \frac{d^8 g}{dx^8}$, as functions of $\frac{d^2 g}{dx^2}, g$. Substituting these results into the equation $Bg - \mu g = 0$ we obtain a first order differential equation for $g(y)$. Thus this construction enables one to express the eigenfunctions of $A$ in terms of a single integral.
The equation (68) passes the Painlevé test. It appears in the Cosgrove list, [13] and is equivalent to

\[ u^{(5)} = 20u u^{(3)} + 40u' u'' - 120u^2 u' + \alpha u' \]

via the change of variables

\[ V_0(y) = u(x), \quad y = \hbar x, \quad c_1 = -\frac{\hbar^5}{16} \alpha. \]

This is Cosgrove’s equation Fif-III [13, equation (2.71)] with \( \lambda = \kappa = 0 \). The equation can be integrated once to

\[ u^{(4)} = 20u u'' + 10(u')^2 - 40u^3 + \alpha y + \beta. \]

This is Cosgrove’s equation F-V [2, equation (1.7)] with \( \kappa = 0 \). The general solution is in terms of genus two hyperelliptic functions, but there are particular solutions in terms of elliptic functions and degenerations. This Burchnall–Chaundy construction occurs in many fields, see for example the connection with the KdV hierarchy, equation [16, equation (2.11)]. Here we are pointing out its significance in superintegrability theory.

For all systems with \( C = 0 \) we can use this method to reduce the solution of the \( A \) eigenvalue equation to a single quadrature. The method also extends to systems where \( B \) is of arbitrarily high order, \( A \) is 2nd order and \( C = 0 \), and always yields structure equations relating \( A \) and \( B \) and a solution for the eigenvalues of \( A \) up to a single quadrature.

8. The two-hyperboloid in horocyclic coordinates

We classify 3rd order superintegrable systems on the two-sheet hyperboloid \( s_1^2 - s_2^2 - s_3^2 = 1 \) with horocyclic coordinates. In this section, we set \( \hbar = \sqrt{2}i \) for simplicity. Then, with horocyclic coordinates \( u_1, u_2, (23) \), a separable Hamiltonian takes the form

\[ H = u_2^2 \left( \partial_{u_1}^2 + \partial_{u_2}^2 \right) + u_3^2 (V_1(u_1) + V_2(u_2)). \]

and

\[ A = \partial_{u_1}^2 + V_1(u_1) \]

We obtain the following systems.

8.1. Systems that are 2nd order superintegrable

- \( V_1(u) = c_1 + c_2 u_1 + 4c_3 u_1^2, \quad V_2(v) = c_3 u_2^2 \)

This is the 2nd order superintegrable system S1, [6].

- \( V_1(u) = \frac{c_1}{u_1^2} + c_2 + c_3 u_1^2, \quad V_2(v) = c_3 u_2^2 \)

This is the 2nd order superintegrable system S2, [6].
8.2. Systems with algebraically dependent generators \( A \) and \( B \).

**A truly 3rd order system, but no cubic algebra.**

Here, \( V_2(v) = 0 \) and \( V_1(u) \) satisfies the nonlinear equation

\[
8V_1 + 4u_1V'_1 + 6cV_1V'_1 + cV''_1 = 0
\]

It admits the first integral

\[
V''_1 = -\frac{4V'_1e^2 + 16V_1^2eu_1 - c^2(V'_1)^2 + 16u_1^2V_1 - 4V'_1c + 2c_1c}{2c(cV_1 + 2u_1)}.
\]

This equation does not satisfy the Painlevé property. The structure equation is

\[
[A, B] = 2A - C
\] (74)

which gives no information about the eigenvalues of the Hamiltonian, but shows that \( A \) is a raising operator for the eigenvalues of \( B \). Here,

\[
B = \frac{3}{4}cV_1 + \left( u_1 + \frac{3}{2}cV_1 \right) \partial_{u_1} + w_2 \partial_{u_2} + c\partial^3_{u_1}.
\]

**Systems with \( C = 0. \)**

There are formally the same functionally superintegrable solutions for the hyperboloid as for the sphere, except that now \( V_2(u_2) \) is arbitrary, and the ranges of the variables \( u_1, u_2 \) differ from those of \( x, y \).

8.3. Systems that are truly 3rd order superintegrable.

Here, we have not classified the quantum special cases, i.e. the quantum \( h \)-systems that are special cases of classical superintegrable systems.

**The TTW solution:**

The TTW method applied to the system 57 with horocyclic coordinates leads to an infinite family of higher order superintegrable systems, and exactly one system is 3rd order superintegrable:

\[
U(u) = U_0 + U_1u_1 + U_2u_1^2, \quad V(v) = U_2u_2^2.
\] (75)

Thus \( A = \partial^2_{u_1u_1} + U_0 + U_1u_1 + U_2u_1^2 \). We can verify that the Hamiltonian with this potential is third order superintegrable with

\[
B = \frac{2u_1U_2}{U_1} \partial^2_{u_1u_1} \partial_{u_2} + \left( \frac{2U_2u_1 + U_1}{U_1} \right) \partial^3_{u_1} - \left( \frac{2U_2u_1^2}{U_1} + 2U_2u_1 + \frac{1}{2}U_1 \right) u_2 \partial_{u_2}
\]  

\[
+ \frac{3U_2^2u_1 + (4U_2u_2^2 + 6u_2^2u_2^2 - U_1u_1 + 2U_0U_1 + 8u_1U_2u_2^2 + \frac{1}{2}U_2u_1^2 + \frac{1}{2}U_0U_2)u_2}{2U_1} \partial_{u_1}
\]

\[
+ \left( \frac{2u_2U_2^2}{U_1} + \frac{2U_2^2u_2^2}{U_1} + 2U_2u_1 \right).
\] (76)

**The structure equations:**
With $C = [A, B]$ we have

$$[A, C] = -16 U_2 B - \frac{32 U_2^2}{U_1} A + \frac{4 U_2(4U_0 U_2 - 3U_1^2)}{U_1},$$

$$[B, C] = a_1 A^3 + a_2 A^2 + a_3 H A + a_4 B + a_5 A + a_6 H + a_7,$$

with

$$a_1 = -\frac{32 U_2^2}{U_1^2}, \quad a_2 = \frac{12 U_2(4U_0 U_2 - U_1^2)}{U_1^3}, \quad a_3 = -\frac{64 U_2^3}{U_1^3}, \quad a_4 = \frac{32 U_2^2}{U_1^2},$$

$$a_5 = \frac{(16U_0^2 U_2^2 - 8U_0 U_1^2 U_2 + U_1^4 - 160U_1^2)}{U_1^3}, \quad a_6 = \frac{16 U_2^3(4U_0 U_2 - U_1^2)}{U_1^2},$$

$$a_7 = \frac{-4 U_2^2(20U_0 U_2 - 9U_1^2)}{U_1^2}.$$

where we have omitted the constants $b_j$.

The eigenvalues of $A$ are of the form $\lambda = -4\sqrt{U_2} n + \text{constant, linear in } n$. Note that this system is Stäckel equivalent to a 3rd order Euclidean space superintegrable system in Cartesian coordinates.

The remaining 3rd order nonlinear superintegrable systems.

These are the systems treated in (28).

9. Discussion and conclusions

We have developed a new approach to the classification of higher order superintegrable systems on any 2D manifold that admit a separation of variables, not just Euclidean space. As a check we have reproduced the striking Tremblay and Winternitz result that for Euclidean space and polar coordinates one of the 3rd order superintegrable potentials can be expressed in terms of Painlevé VI. As new results we show that Painlevé VI also appears for 3rd order superintegrable systems on the two-sphere and the two-hyperboloid; this occurrence is common for constant curvature spaces. We derived the possible separable systems on the two-sphere in spherical coordinates and the two sheet two-hyperboloid in horocyclic coordinates that admit a third order symmetry operator and provide information about their symmetry algebras. We showed that the Weierstrass function appears for a multiplicity of spaces. Our aim is to develop a convenient tool that can be used study higher order superintegrability on spheres, hyperboloids and more general Riemannian spaces, and to compare results between spaces. In [16], and references contained therein ‘exotic potentials’ are studied, potentials that satisfy nonlinear equations. The authors of that paper advanced a ‘Painlevé conjecture’ for 2D Euclidean space to the effect that all higher order (> 2) superintegrable systems separating in Cartesian coordinates with potentials satisfying nonlinear equations must be of Painlevé type. The computations in our paper provide some evidence that if the conjecture is true an analogous result may also hold for systems on the two-sphere and two-hyperboloid.
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