BILINEAR GENERATING FUNCTIONS
FOR ORTHOGONAL POLYNOMIALS

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Abstract

Using realisations of the positive discrete series representations of the Lie algebra $\mathfrak{su}(1,1)$ in terms of Meixner-Pollaczek polynomials, the action of $\mathfrak{su}(1,1)$ on Poisson kernels of these polynomials is considered. In the tensor product of two such representations, two sets of eigenfunctions of a certain operator can be considered and they are shown to be related through continuous Hahn polynomials. As a result, a bilinear generating function for continuous Hahn polynomials is obtained involving the Poisson kernel of Meixner-Pollaczek polynomials. For the positive discrete series representations of the quantised universal enveloping algebra $U_q(\mathfrak{su}(1,1))$ a similar analysis is performed and leads to a bilinear generating function for Askey-Wilson polynomials involving the Poisson kernel of Al-Salam and Chihara polynomials.

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1 Introduction

Representation theory of Lie algebras and quantum algebras (or quantised universal enveloping algebras [8]) is closely related to special functions of (basic) hypergeometric type [20, Ch. 14], [8, Ch. 13]. In this paper we are dealing with positive

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discrete series representations of the Lie algebra $\mathfrak{su}(1,1)$ and of its quantum analogue $\mathcal{U}_q(\mathfrak{su}(1,1))$. Using realisations of these representations in terms of orthogonal polynomials $p_n(x)$, the action of the Lie or quantum algebra on Poisson kernels of these polynomials is considered. In the realisation of the tensor product of two representations, one can either use the product $p_{n_1}(x_1)p_{n_2}(x_2)$ as basis vectors, or a new basis in which different orthogonal polynomials $q_n(x)$ appear [14]. Here it is shown how bilinear generating functions for the $q_n(x)$ appear naturally in this framework.

For the Lie algebra $\mathfrak{su}(1,1)$ it was shown in [18, 15], using ideas of Granovskii and Zhedanov [11], that Laguerre, Meixner-Pollaczek and Meixner polynomials appear as overlap coefficients in the positive discrete series representations. As a consequence, basis vectors of these representations have a realisation in terms of these polynomials. In the realisation with Laguerre polynomials, the tensor product of two representations has a so-called uncoupled basis with basis vectors the product of two Laguerre polynomials, or a coupled basis with basis vectors the product of a Jacobi and a Laguerre polynomial. The Clebsch-Gordan coefficients of $\mathfrak{su}(1,1)$ relate one basis to another, and as a result a new convolution formula for Laguerre polynomials involving a Jacobi polynomial follows [18]. This result was extended yielding a new convolution formula for Meixner-Pollaczek (resp. Meixner) polynomials involving a continuous Hahn (resp. Hahn) polynomial [13].

In the present paper we use the action of the Lie algebra on Poisson kernels. Let us briefly describe the situation and the technique when the realisation is in terms of Meixner-Pollaczek polynomials. Then a positive discrete series representation is identified with $L^2(\mathbb{R}, d\mu(x))$, $d\mu(x)$ being the orthogonality measure of Meixner-Pollaczek polynomials, and its basis vectors are realised in terms of Meixner-Pollaczek polynomials. The Lie algebra $\mathfrak{su}(1,1)$ has a realisation in terms of operators acting in $L^2(\mathbb{R}, d\mu(x))$. The Poisson kernel can be seen as an expansion of some function in terms of Meixner-Pollaczek polynomials, thus $\mathfrak{su}(1,1)$ has a natural action on it. In this case the Poisson kernel is proportional to a $_2\!F_1$-series. The Poisson kernel is shown to be an eigenfunction of some element $X_t$ (in this realisation) of $\mathfrak{su}(1,1)$. This element $X_t$ is closely related to a recurrence operator that led to the interpretation of Meixner-Pollaczek polynomials as overlap coefficients in the positive discrete series representations. Then we go on considering eigenfunctions of $X_t$ in the tensor product of two representations. There are again so-called uncoupled eigenfunctions, being the product of two Poisson kernels, and coupled eigenfunctions. The Clebsch-Gordan coefficients relating these two are the polynomials that appear in the convolution formula, i.e. continuous Hahn polynomials. As a consequence of this relation, one obtains a bilinear generating function for continuous Hahn polynomials (Theorem 2.3), involving the Poisson kernel of Meixner-Pollaczek polynomials. This appears to be a new formula, which also has an interpretation as a multiplication formula for hypergeometric series (Theorem 2.6). Theorem 2.3 generalises Bateman’s 1904 bilinear generating function for the Jacobi polynomials. The same technique can be applied to the realisation in terms of Meixner polynomials, for
which we give only the final result.

In section 3 we show how this approach can be applied to the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(1,1))$. The basis vectors of the positive discrete series representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$ can be realised in terms of Al-Salam and Chihara polynomials \cite{15}. The general convolution identity for Al-Salam and Chihara polynomials, see \cite{15}, Theorem 4.5, involves an Askey-Wilson polynomial. For Al-Salam and Chihara polynomials the Poisson kernel needed here was given recently in a paper by Askey, Rahman and Suslov \cite{3}; another method by which this Poisson kernel can be obtained is given in \cite{19}. This Poisson kernel is expressed by means of a very-well-poised basic hypergeometric series $\mathcal{8}\phi_7$, often denoted by $\mathcal{8}\mathcal{W}_7$, where we follow the notation of Gasper and Rahman \cite{10} for basic hypergeometric series. In the present case, our technique leads to a bilinear generating function for Askey-Wilson polynomials; the only other functions appearing in this formula being $\mathcal{8}\mathcal{W}_7$ series (Theorem 3.3). Limiting cases yield a bilinear generating function for continuous dual $q$-Hahn polynomials and for Al-Salam and Chihara polynomials. The bilinear generating function for Askey-Wilson polynomials (Theorem 3.3) is a $q$-analogue of Bateman’s bilinear generating function for Jacobi polynomials.

2 Polynomials related to $\mathfrak{su}(1,1)$

The Lie algebra $\mathfrak{su}(1,1)$ is generated by $H, B, C$ subject to

$$[H, B] = 2B, \quad [H, C] = -2C, \quad [B, C] = H.$$ 

There is a $*$-structure by $H^* = H$ and $B^* = -C$. The positive discrete series representations $\pi_k$ of $\mathfrak{su}(1,1)$ are unitary representations labelled by $k > 0$. The representation space is $\ell^2(\mathbb{Z}_+)$ equipped with an orthonormal basis $\{e_n^k\}_{n \in \mathbb{Z}_+}$, and the action is given by

$$\pi_k(H) e_n^k = 2(k + n) e_n^k,$$
$$\pi_k(B) e_n^k = \sqrt{(n + 1)(2k + n)} e_{n+1}^k,$$
$$\pi_k(C) e_n^k = -\sqrt{n(2k + n - 1)} e_{n-1}^k.$$ 

The tensor product of two positive discrete series representations decomposes as

$$\pi_{k_1} \otimes \pi_{k_2} = \bigoplus_{j=0}^\infty \pi_{k_1+k_2+j},$$

and the corresponding intertwining operator can be expressed by means of the Clebsch-Gordan coefficients

$$e_n^{(k_1 k_2)} = \sum_{n_1, n_2} C_{n_1, n_2, n}^{k_1, k_2} e_{n_1}^{k_1} \otimes e_{n_2}^{k_2}.$$ 

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The Clebsch-Gordan coefficients are non-zero only if \( n_1 + n_2 = n + j, k = k_1 + k_2 + j \) for \( j, n_1, n_2, n \in \mathbb{Z}_+ \), and normalised by \( \langle e^k_0, e^{k_1}_0 \otimes e^{k_2}_j \rangle > 0 \). For the above results see Vilenkin and Klimyk [20, §8.7], or [15].

Up to a scalar multiple, the most general self-adjoint element in \( \mathfrak{su}(1, 1) \) is of the form

\[
X_\alpha = B - C - \alpha H, \quad \alpha \in \mathbb{R},
\]

thus \( X_\alpha^* = X_\alpha \). The action of \( X_\alpha \) on a basis vector \( e^k_n \) in the representation \( \pi_k \) is then given by

\[
\pi_k(X_\alpha)e^k_n = a_ne^k_{n+1} + b_ne^k_n + a_{n-1}e^k_{n-1}
\]

where \( a_n = \sqrt{(n+1)(2k+n)} \) and \( b_n = -2\alpha(k+n) \).

Thus \( \pi_k(X_\alpha) \) acts as a three-term recurrence operator. Since the coefficients \( a_n \) and \( b_n \) in (2.5) satisfy \( a_n > 0 \) and \( b_n \in \mathbb{R} \), there exist orthonormal polynomials associated with \( \pi_k(X_\alpha) \) defined by [6]

\[
xp_n(x) = a_np_{n+1}(x) + b_np_n(x) + a_{n-1}p_{n-1}(x)
\]

with \( p_1(x) = 0 \) and \( p_0(x) = 1 \).

In [14] it was shown that these polynomials are Meixner-Pollaczek polynomials if \( |\alpha| < 1 \), Meixner polynomials if \( |\alpha| > 1 \), and Laguerre polynomials if \( |\alpha| = 1 \) (this last case was also considered in [18]). A consequence of this is that the basis vectors of the positive discrete series representations can be realised as these polynomials, and the elements of \( \mathfrak{su}(1, 1) \) are realised as operators in the corresponding \( L^2 \) space such that \( X_\alpha \) is realised as a multiplication operator.

Here the realisation in terms of Meixner-Pollaczek polynomials is considered. It is shown how the \( \mathfrak{su}(1, 1) \) operators act on Poisson kernels of these polynomials, and that they are eigenfunctions of some operator \( X_t \). Then two different eigenfunctions of this operator in the tensor product are introduced, and their relationship is shown to lead to a bilinear generating function for continuous Hahn polynomials. The case of Laguerre polynomials (and related Jacobi polynomials) is obtained as a limit. The case of Meixner polynomials (and related Hahn polynomials) is completely analogous, and we just mention the final result at the end of this paragraph.

The Meixner-Pollaczek polynomials are defined by

\[
P_n^{(k)}(x; \phi) = \frac{(2k)^n}{n!} e^{in\phi} 2F_1\left(-n, k + ix; \frac{1 - e^{-2i\phi}}{2k}\right),
\]

with the usual notation for Pochhammer symbols and hypergeometric series [10]. For \( k > 0 \) and \( 0 < \phi < \pi \) these are orthogonal polynomials with respect to a positive measure on \( \mathbb{R} \), see [14], [17] App.[. The orthonormal Meixner-Pollaczek polynomials

\[
p_n^{(k)}(x; \phi) = \sqrt{\frac{n!}{\Gamma(n + 2k)}} P_n^{(k)}(x; \phi)
\]
satisfy the orthogonality relation
\[ \int_{\mathbb{R}} p_n^{(k)}(x; \phi) p_m^{(k)}(x; \phi) \frac{(2 \sin \phi)^{2k}}{2\pi} e^{(2\phi-\pi)x} |\Gamma(k + ix)|^2 dx = \delta_{n,m}. \]

We shall denote this absolutely continuous measure by \( d\mu_{k,\phi}(x) \). There exists \( ^{[15, \text{Proposition 3.1}]} \) a unitary operator \( \Lambda : l^2(\mathbb{Z}_+) \to L^2(\mathbb{R}, d\mu_{k,\phi}(x)) \) mapping each \( e_n^k \) onto \( p_n^{(k)}(x; \phi) \), and every element \( X \) of \( \mathfrak{s}\mathfrak{u}(1,1) \) has a realisation \( \rho_{k,\phi}(X) \) in \( L^2(\mathbb{R}, d\mu_{k,\phi}(x)) \).

**Proposition 2.1** For \( t \in \mathbb{C} \) with \( |t| < 1 \) and \( 0 < \phi < \pi \) denote
\[ r = -4t \sin^2 \phi/(1 - t)^2. \]

Let \( y \in \mathbb{R} \) and define
\[ v_t^k(x, y; \phi) = \sum_{n=0}^{\infty} p_n^{(k)}(x; \phi) p_n^{(k)}(y; \phi) t^n. \tag{2.8} \]

Then
(i) \( v_t^k(x, y; \phi) \in L^2(\mathbb{R}, d\mu_{k,\phi}(x)) \).
(ii) Explicitly,
\[ v_t^k(x, y; \phi) = \frac{1}{1/(2k)}(1 - te^{2i\phi})^{ix+y}(1 - t)^{-2k-ix-iy} \text{F}_1 \left( \frac{k + ix, k + iy}{2k}; r \right). \tag{2.9} \]
(iii) Let
\[ X_t = -\cos \phi H + tB - t^{-1}C \in \mathfrak{s}\mathfrak{u}(1,1). \]
Then \( v_t^k(x, y; \phi) \) is an eigenfunction of \( \rho_{k,\phi}(X_t) \) for the eigenvalue \( 2y \sin \phi \).

**Proof.** The first statement is true when \( \sum_n |p_n^{(k)}(y; \phi)t^n|^2 < \infty \), and this holds since \( p_n^{(k)}(y; \phi) \) are polynomials of the first kind associated with a three-term recurrence operator \( ^{[3, \text{Ch. VII, (1.24)}]} \). Statement (i) also follows for \( y \in \mathbb{C} \) from the asymptotic behaviour of the Meixner-Pollaczek polynomials. For \( k = 1/2 \) see Szegő \( ^{[17, \text{App. \S 6}]} \), and the general case follows similarly using Darboux’s method \( ^{[17, \text{\S 8.4}]} \) on \( ^{[17, \text{App. (4.1)}]} \). The explicit formula (2.9) is the Poisson kernel for the Meixner-Pollaczek polynomials and follows from \( ^{[3, 2.5.2 (12)]} \). The last statement follows from (2.1) and the three-term recurrence relation
\[ 2y \sin \phi \ p_n^{(k)}(y; \phi) = a_n p_{n+1}^{(k)}(y; \phi) - 2(n+k) \cos \phi \ p_n^{(k)}(y; \phi) + a_{n-1} p_{n-1}^{(k)}(y; \phi), \]
where \( a_n = \sqrt{(n+1)(n+2k)} \). \( \square \)

Note that the nonsymmetric Poisson kernel \( \sum_{n=0}^{\infty} p_n^{(k)}(x; \phi) p_n^{(k)}(y; \psi)t^n \), i.e. (2.8) with \( \phi \neq \psi \), can also be determined explicitly using \( ^{[3, 2.5.2 (12)]} \), but it does not lead to a more general result.
In the tensor product $\pi_{k_1} \otimes \pi_{k_2}$, there is a unitary operator $\Upsilon : \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}^2, d\mu_{k_1,\phi_1}(x_1) d\mu_{k_2,\phi_2}(x_2))$ mapping $e_{n_1}^{(k_1)} \otimes e_{n_2}^{(k_2)}$ onto $p_{n_1}^{(k_1)}(x_1; \phi_1) p_{n_2}^{(k_2)}(x_2; \phi_2)$. We shall only consider the case $\phi_1 = \phi_2 = \phi$, because then we can simplify the basis functions in the tensor product decomposition. The realisation of $\Delta(X) = X \otimes 1 + 1 \otimes X$ with $X \in su(1, 1)$ in $L^2(\mathbb{R}^2, d\mu_{k_1,\phi}(x_1) d\mu_{k_2,\phi}(x_2))$ is denoted by $\sigma_{k_1,k_2,\phi}(X)$. For $k = k_1 + k_2 + j$, with $j \in \mathbb{Z}_+$, the above-mentioned simplification reads [13, (3.7)]

$$e_{n}^{(k_1,k_2)}(x_1, x_2; \phi) = \sum_{n_1,n_2} C_{n_1,n_2}^{k_1,k_2} p_{n_1}^{(k_1)}(x_1; \phi) p_{n_2}^{(k_2)}(x_2; \phi) = p_n^{(k)}(x_1 + x_2; \phi) S_j^{(k_1,k_2)}(x_1, x_2; \phi),$$

where

$$S_j^{(k_1,k_2)}(x_1, x_2; \phi) = (-2 \sin \phi)^j \sqrt{\frac{j! (2j + 2k_1 + 2k_2 - 1) \Gamma(j + 2k_1 + 2k_2 - 1)}{\Gamma(2k_1 + j) \Gamma(2k_2 + j)}} \times p_j(x_1; k_1, k_2 - i(x_1 + x_2), k_1, k_2 + i(x_1 + x_2)).$$

Herein, $p_j$ is a continuous Hahn polynomial introduced by Atakishiyev and Suslov [4], see also [3, 4],

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)n(a+d)n}{n!} {}_3F_2 \left( -n, n + a + b + c + d - 1, a + i\gamma ; 1 \right).$$

Next, we have

**Proposition 2.2** Let $y, t$ and $\phi$ be as in Proposition [2.1], and

$$w_t^{(k_1,k_2)}(x_1, x_2, y; \phi) = \sum_{n=0}^{\infty} p_n^{(k)}(y; \phi) e_{n}^{(k_1,k_2)}(x_1, x_2; \phi) t^n$$

for $k = k_1 + k_2 + j$ with $j \in \mathbb{Z}_+$. Let again $X_t = -\cos \phi H + tB - t^{-1}C \in su(1, 1)$, then

(i) $w_t^{(k_1,k_2)}(x_1, x_2, y; \phi) \in L^2(\mathbb{R}^2, d\mu_{k_1,\phi}(x_1) d\mu_{k_2,\phi}(x_2))$ and moreover it is an eigenfunction of $\sigma_{k_1,k_2,\phi}(X_t)$ for the eigenvalue $2y \sin \phi$.

(ii) Explicitly, one has

$$w_t^{(k_1,k_2)}(x_1, x_2, y; \phi) = S_j^{(k_1,k_2)}(x_1, x_2; \phi) v_t^k(x_1 + x_2, y; \phi).$$

(iii) For $y_1, y_2 \in \mathbb{R}$,

$$v_t^{k_1}(x_1, y_1; \phi) v_t^{k_2}(x_2, y_2; \phi) = \sum_{j=0}^{\infty} t^j S_j^{(k_1,k_2)}(y_1, y_2; \phi) w_t^{(k_1,k_2)}(x_1, x_2, y_1 + y_2; \phi).$$
The proposition states that in $L^2(\mathbb{R}^2, d\mu_{k_1,\phi}(x_1) d\mu_{k_2,\phi}(x_2))$ the uncoupled eigenfunctions of $\sigma_{k_1,k_2,\phi}(X_t)$ for the eigenvalue $2(y_1 + y_2)\sin \phi$ are $u^{k_1}_t(x_1, y_1; \phi)v_{t}^{k_2}(x_2, y_2; \phi)$; the coupled eigenfunctions for the same eigenvalue are $w^{(k_1,k_2)}_t(x_1, x_2, y_1 + y_2; \phi)$; and the Clebsch-Gordan coefficients relating these two are $t^j S_j^{(k_1,k_2)}(y_1, y_2; \phi)$.

Proof. Since the action of $\sigma_{k_1,k_2,\phi}(H)$, $\sigma_{k_1,k_2,\phi}(B)$ and $\sigma_{k_1,k_2,\phi}(C)$ on the basis functions $e_n^{(k_1,k_2)}k(x_1, x_2; \phi) \in L^2(\mathbb{R}^2, d\mu_{k_1,\phi}(x_1) d\mu_{k_2,\phi}(x_2))$ is the standard one given by (2.1), one can use again the three-term recurrence relation for the orthonormal Meixner-Pollaczek polynomials to see that $\sigma_{k_1,k_2,\phi}(X_t)w^{(k_1,k_2)}_t(x_1, x_2, y; \phi) = 2y \sin \phi w^{(k_1,k_2)}_t(x_1, x_2, y; \phi)$. Using (2.11) in (2.13), and then definition (2.8), (2.14) follows directly. Finally, from (2.8).

$$u^{k_1}_t(x_1, y_1; \phi)v^{k_2}_t(x_2, y_2; \phi) = \sum_{n_1=0}^{\infty} p^{(k_1)}_{n_1}(x_1; \phi)p^{(k_1)}_{n_1}(y_1; \phi) t^{n_1} \sum_{n_2=0}^{\infty} p^{(k_2)}_{n_2}(x_2; \phi)p^{(k_2)}_{n_2}(y_2; \phi) t^{n_2}. \quad (2.16)$$

Herein, use

$$p^{(k_1)}_{n_1}(x_1; \phi)p^{(k_2)}_{n_2}(x_2; \phi) = \sum_{k,n} C_{n_1,n_2,n}^{k_1,k_2,k_1} e_n^{(k_1,k_2)}k(x_1, x_2; \phi).$$

This equation follows from (2.10) and the orthogonality of the Clebsch-Gordan coefficients; the finite sum is over all $k = k_1 + k_2 + j$ ($j \in \mathbb{Z}_+$) and $n$ such that $j + n = n_1 + n_2$. Then (2.10) reduces to

$$u^{k_1}_t(x_1, y_1; \phi)v^{k_2}_t(x_2, y_2; \phi) \quad \sum_{n_1,n_2} \sum_{k,n} C_{n_1,n_2,n}^{k_1,k_2,k_1} e_n^{(k_1,k_2)}k(x_1, x_2; \phi) \sum_{n_1,n_2} \sum_{k,n} C_{n_1,n_2,n}^{k_1,k_2,k_1} e_n^{(k_1,k_2)}k(x_1, x_2; \phi)$$

$$= \sum_{k,n} \sum_{j=0}^{\infty} t^{n+j} e_n^{(k_1,k_2)}k(x_1, x_2; \phi) S_j^{(k_1,k_2)}(y_1, y_2; \phi)$$

$$= \sum_{k,n} \sum_{j=0}^{\infty} t^j S_j^{(k_1,k_2)}(y_1, y_2; \phi) w^{(k_1,k_2)}_t(x_1, x_2, y_1 + y_2; \phi),$$

proving the final statement. All series manipulations above are allowed because of the absolute convergence of the power series in $t$ involved.

The basic type of identity in this paper follows from (2.14) and (2.17):

$$u^{k_1}_t(x_1, y_1; \phi)v^{k_2}_t(x_2, y_2; \phi) \quad \sum_{j=0}^{\infty} t^{j} u^{k_1+k_2+j}_t(x_1 + x_2, y_1 + y_2; \phi) S_j^{(k_1,k_2)}(x_1, x_2; \phi) S_j^{(k_1,k_2)}(y_1, y_2; \phi).$$

Using (2.3) and (2.11) some simplifications take place, and (2.17) becomes ($k = k_1 + k_2 + j)$

$$\begin{align*}
\sum_{j=0}^{\infty} t^{j} u^{k_1+k_2+j}_t(x_1 + x_2, y_1 + y_2; \phi) S_j^{(k_1,k_2)}(x_1, x_2; \phi) S_j^{(k_1,k_2)}(y_1, y_2; \phi) &= 2F1 \left( \begin{array}{c} k_1 + ix_1, k_1 + iy_1 \\ 2k_1 \end{array} ; r \right) 2F1 \left( \begin{array}{c} k_2 + ix_2, k_2 + iy_2 \\ 2k_2 \end{array} ; r \right)
\end{align*}$$
This bilinear generating function for the Jacobi polynomials $P_n^{\alpha,\beta}$ both sides of (2.19) as power series in which can be rewritten in terms of Bessel functions. Jacobi polynomials, cf. [14, § from Theorem 2.3 using a standard limit of the continuous Hahn polynomials to where the $c$ and $|x|$ for the right hand side and equating coefficients.

One can interpret (2.18) as a bilinear sum formula for continuous Hahn polynomials. Using the relabelling $(x, y) = (x_1, y_1)$, $a = k_1$, $b = k_2 - i(x_1 + x_2)$, $c = k_2 - i(x_1 + x_2)$, $b' = k_2 - i(y_1 + y_2)$, and $c' = k_2 + i(y_1 + y_2)$, we have the following

**Theorem 2.3** Let $a > 0$, $\Re(b, c, b', d') > 0$ with $b + d = b' + d'$, $\bar{d} = b$ and $\bar{d}' = b'$. Then the continuous Hahn polynomials satisfy a bilinear sum formula given by:

$$\begin{align*}
\sum_{j=0}^{\infty} h_j p_j(x; a, b, a, d) p_j(y, a, b', a, d') r^j &= \\
&= \frac{(-1)^j j!}{(2a, b + d, 2a + b + d + j - 1)_j} \binom{2a + d + j, a + d' + j}{2a + b + d + 2j} \binom{a + d + j, a + d' + j}{2a + b + d + 2j},
\end{align*}$$

where

$$h_j = \frac{(-1)^j j!}{(2a, b + d, 2a + b + d + j - 1)_j} \binom{a + d + j, a + d' + j}{2a + b + d + 2j},$$

and $|r| < 1$, $x, y \in \mathbb{R}$.

**Remark 2.4** The bilinear generating function of Theorem 2.3 is an extension of

$$\begin{align*}
\sum_{j=0}^{\infty} \frac{(-1)^j j! (\alpha + \beta + 2j + 1)\Gamma(\alpha + \beta + j + 1)}{\Gamma(\alpha + j + 1)\Gamma(\beta + j + 1)} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y) J_{\alpha+\beta+2j+1}(z) &= 2^{\alpha+\beta-1}((1-x)(1-y))^{-\alpha/2}((1+x)(1+y))^{-\beta/2} \\
&\times z J_\alpha \left(\frac{z}{2} \sqrt{(1-x)(1-y)}\right) J_\beta \left(\frac{z}{2} \sqrt{(1+x)(1+y)}\right).
\end{align*}$$

This bilinear generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}$ can be obtained from Theorem 2.3 using a standard limit of the continuous Hahn polynomials to Jacobi polynomials, cf. [14, §2.8], and in this rescaling the $2F1$’s go over into $\alpha F1$’s, which can be rewritten in terms of Bessel functions $J_\nu$ of the first kind [21]. This formula is due to Bateman (1904), see [21, §11.6]. It should be noted that Bateman’s formula

$$\begin{align*}
\sum_{k=0}^{n} c_{k,n} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) &= (x + y)^n P_n^{(\alpha,\beta)} \left(\frac{1 + xy}{x + y}\right),
\end{align*}$$

where the $c_{k,n}$ follow by specialising $y = 1$, can be obtained from (2.19) by writing both sides of (2.19) as power series in $z$ after division by $z^{\alpha+\beta+1}$ using [1, 4.3(12)] for the right hand side and equating coefficients.

In the same way we can obtain from Theorem 2.3 an extension of (2.20).
Corollary 2.5 Assume \( b + d = b' + d' \), \( \bar{d} = b \) and \( \bar{d}' = b' \), then

\[
\sum_{j=0}^{k} \frac{(-k)^j(2a + b + d)_{2j}j!}{(2a, b + d, 2a + b + d + j - 1, a + d, a + d', 2a + b + d + k)_{j}} \times p_j(x; a, b, a, d) p_j(y, a, b', a, d') \\
= \frac{(d - ix, d' - iy, 2a + b + d)^k}{(a + d, a + d', b + d)_{k}} 4F_3 \left( \begin{array}{c} -k, 1 - k - b - d, a + ix, a + iy \\ 2a, 1 - k - d + ix, 1 - k - d' + iy \end{array} ; 1 \right).
\]

Note that for \( b = d' \), and thus \( d = b' \) and \( \bar{b} = d \), the \( 4F_3 \)-series is balanced. In this case we can use Whipple’s transformation, see e.g. [10, (2.10.5)], to see that the \( 4F_3 \)-series in Corollary 2.3 equals

\[
\frac{(a + d)_{k}(a + b + i(x - y))_{k}}{(d - ix)_{k}(b - iy)_{k}} 4F_3 \left( \begin{array}{c} -k, 2a + b + d + k - 1, a + ix, a - iy \\ 2a, a + d, a + b + i(x - y) \end{array} ; 1 \right),
\]

which is a Wilson polynomial for \( x = y \).

Proof. Write the \( 2F_1 \) in \( h_j \) in the left hand side of Theorem 2.3 as

\[
\sum_{k=0}^{\infty} \frac{(a + d + j)_{k-j}(a + d' + j)_{k-j}}{(k - j)! (2a + b + d + 2j)_{k-j}}
\]

to see that the coefficient of \( r^k \) in the left hand side equals

\[
\sum_{j=0}^{k} \frac{(a + d + j)_{k-j}(a + d' + j)_{k-j}}{(k - j)! (2a + b + d + 2j)_{k-j}} \frac{(-1)^j j!}{(2a, b + d + 2a + b + d + j - 1)_j} \times p_j(x; a, b, a, d) p_j(y, a, b', a, d').
\]

The coefficient of \( r^k \) in the right hand side of Theorem 2.3 follows from [9, 4.3(14)]. Equating and rewriting gives the result.

Formula (2.18) can be extended to more general values. Relabelling the parameters of the first (resp. second) \( 2F_1 \) in (2.18) by \( a, b, c \) (resp. \( a', b', c' \)), and expressing the continuous Hahn polynomials in terms of a \( 3F_2 \), we obtain:

Theorem 2.6 For \( a, b, c, a', b', c' \in \mathbb{C} \) with \( c \) and \( c' \) no negative integers, and \( |z| < 1 \), the following multiplication formula holds:

\[
2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) 2F_1 \left( \begin{array}{c} a', b' \\ c' \end{array} ; z \right) = \sum_{j=0}^{\infty} C_j z^j 2F_1 \left( \begin{array}{c} a + a' + j, b + b' + j \\ c + c' + 2j \end{array} ; z \right),
\]

where

\[
C_j = \frac{(c, a + a', b + b')_{j}}{j!(c', c + c' + j - 1)_j} 3F_2 \left( \begin{array}{c} -j, a, c + c' + j - 1 \\ a + a', c \end{array} ; 1 \right) \times 3F_2 \left( \begin{array}{c} -j, b, c + c' + j - 1 \\ b + b', c \end{array} ; 1 \right).
\]

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The theorem holds because as a power series in $z$ the coefficient of $z^k$ in both sides of the equation is the same (this follows from (2.18)). Moreover, for $|z| < 1$ and $c$ and $c'$ no negative integers the series are well defined and absolutely convergent. Note that for $a + a'$ (or $b + b'$) a negative integer, $C_j$ has to be interpreted with care and in fact it is not zero for $j > -a - a'$ unless both $a$ and $a'$ are negative integers. In that case $C_j = 0$ for $j > -a - a'$ and we obtain a polynomial identity. The polynomial identity is equivalent to the dual statement of the convolution identity [15, Thm. 3.4, Rem. 3.5(ii)] as is easily seen by specialising the parameters such that each $2F_1$ in the left hand side of Theorem 2.6 corresponds to a Meixner-Pollaczek polynomial (2.7).

A special case of Theorem 2.6 occurs when $(a', b', c') = (a, b, c)$. Then the $3F_2$'s in $C_j$ can be summed by Watson’s theorem [9, 4.4(7)] and we obtain the Burchnall-Chaundy (1948) formula for the square of a $2F_1$-series, cf. [3, 2.5.2(7)].

An interesting limit of Theorem 2.6 is found by putting $z = x/b$ and $b' = yb/x$, and taking the limit for $b \to \infty$.

**Corollary 2.7**

$$1F_1 \left( a; c, x \right) \times 2F_1 \left( -j, c + c' + j - 1; \frac{x}{x+y} \right) 1F_1 \left( a + a' + j; c + c' + 2j; x + y \right) = \sum_{j=0}^{\infty} D_j (x+y)^j,$$

where

$$D_j = \frac{(c, a + a')_j}{j!(c', c + c' + j - 1)_j} 3F_2 \left( -j, a, c + c' + j - 1; a + a', c; 1 \right).$$

Again for $a$ and $a'$ negative integers we obtain the dual statement of the general convolution identity [18, (1.1)], [15, Cor. 3.6(i)] for Laguerre polynomials.

As mentioned in the beginning of this section, the other case to study is the realisation in terms of Meixner polynomials. However, the analysis is completely similar to that of Meixner-Pollaczek polynomials (the continuous Hahn polynomials being replaced by Hahn polynomials), and the results are essentially the same. We mention one formula here, namely a bilinear generating function for Hahn polynomials. The definition of the Hahn polynomial reads [14]

$$Q_n(x; \alpha, \beta, N) = 3F_2 \left( -n, n + \alpha + \beta + 1, -x; \alpha + 1, -N; 1 \right), \quad n = 0, 1, 2, \ldots, N. \quad (2.21)$$

They satisfy a discrete orthogonality relation, with support $\{0, 1, \ldots, N\}$. We obtain:
Theorem 2.8 Let $M$ and $N$ be positive integers, $\alpha, \beta, z \in \mathbb{C}$, $x \in \{0, 1, \ldots, M\}$, and $y \in \{0, 1, \ldots, N\}$. Then
\[
\sum_{j=0}^{\min(M,N)} Q_j(x; \alpha, \beta, M)Q_j(y; \alpha, \beta, N) \frac{(\alpha + 1, -M, -N)_j}{(\beta + 1, \alpha + \beta + j + 1)_j} \\
\times {}_2F_1\left( \begin{array}{c} j - M, j - N \\ \alpha + \beta + 2j + 2 \end{array} ; z \right) z^j \\
= {}_2F_1\left( \begin{array}{c} -x, -y \\ \alpha + 1 \end{array} ; z \right) {}_2F_1\left( \begin{array}{c} x - M, y - N \\ \beta + 1 \end{array} ; z \right).
\]

It is easy to see how to obtain this result from Theorem 2.6.

3 Polynomials related to $U_q(\mathfrak{su}(1, 1))$

Assume that $0 < q < 1$. Let $U_q(\mathfrak{sl}(2, \mathbb{C}))$ be the complex unital associative algebra generated by $A, B, C, D$ subject to the relations
\[
AD = 1 = DA, \quad AB = q^{1/2} BA, \quad AC = q^{-1/2} CA, \quad BC - CB = \frac{A^2 - D^2}{q^{1/2} - q^{-1/2}}.
\]

It is a Hopf algebra with comultiplication
\[
\Delta(A) = A \otimes A, \quad \Delta(B) = A \otimes B + B \otimes D, \\
\Delta(C) = A \otimes C + C \otimes D, \quad \Delta(D) = D \otimes D
\]
on the level of generators and extended as an algebra homomorphism. The $\ast$-structure corresponding to $U_q(\mathfrak{su}(1, 1))$ is
\[
A^\ast = A, \quad B^\ast = -C, \quad C^\ast = -B, \quad D^\ast = D.
\]

The positive discrete series representations $\pi_k$ of $U_q(\mathfrak{su}(1, 1))$ are unitary representations labelled by $k > 0$. They act in $\ell^2(\mathbb{Z}_+)$ and the action of the generators is given by
\[
\pi_k(A) e_n^k = q^{(k+n)/2} e_n^k, \\
\pi_k(C) e_n^k = q^{1/(4-(k+n))} \frac{\sqrt{(1-q^n)(1-q^{2k+n-1})}}{q^{1/2}-q^{-1/2}} e_{n-1}^k, \\
\pi_k(B) e_n^k = q^{1/(4-(k+n))} \frac{\sqrt{(1-q^{n+1})(1-q^{2k+n})}}{q^{1/2}-q^{-1/2}} e_{n+1}^k.
\]

Recall that the tensor product of two representations is defined by use of the comultiplication. The tensor product of two positive discrete series representations decomposes as for the Lie algebra $\mathfrak{su}(1, 1)$, see eq. (2.2). The Clebsch-Gordan coefficients for $U_q(\mathfrak{su}(1, 1))$ appearing in
\[
e_{n}^{(k_1,k_2)k} = \sum_{n_1,n_2} C_{n_1,n_2,n}^{k_1,k_2,k}(q) e_{n_1}^{k_1} \otimes e_{n_2}^{k_2}
\]
are non-zero only if \( n_1 + n_2 = n + j, k = k_1 + k_2 + j \) for \( j, n_1, n_2, n \in \mathbb{Z}_+ \), and are normalised by \( \langle e_j^{(k_1 k_2) j}, e_j^{(k_1 k_2) j} \rangle \geq 0 \). In [13, Lemma 4.4] they have been computed in terms of \( q \)-Hahn polynomials.

The above results can be found in Burban and Klimyk [7] and K. Chihara, Manocha and Miller [13]. See Chari and Pressley [8] for general information on quantised universal enveloping algebras.

Next, we consider the polynomials related to these representations. Our notation for \( q \)-special functions and \( q \)-shifted factorials is as in [10]. The Askey-Wilson polynomials are, see Askey and Wilson [10] or e.g. [10, §7.5],

\[
p_n(x; a, b, c, d|q) = a^{-n}(ab, ac, ad; q)_n \ {}_4\phi_3 \left( \begin{array}{c} q^{-n}, abcdq^{-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} ; q, q \right),
\]

with \( x = \cos \theta \). The Al-Salam and Chihara polynomials are obtained by taking \( c = d = 0 \) in the Askey-Wilson polynomials;

\[
R_n(x; a, b|q) = p_n(x; a, b, 0, 0|q) = a^{-n}(ab; q)_n \ {}_3\phi_2 \left( \begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{array} ; q, q \right).
\]

The corresponding normalised polynomials are denoted by

\[
r_n(x; a, b|q) = R_n(x; a, b|q)/\sqrt{(q, ab; q)_n}.
\]

For \( a \) and \( b \) real, or complex conjugates, with \( \max(|a|, |b|) < 1 \) the orthogonality measure \( d\mu(x; a, b|q) \) is absolutely continuous on \([-1, 1]\) and given by

\[
\frac{1}{2\pi} \int_{-1}^{1} (q, ab; q)_{\infty} \frac{w(x)}{\sqrt{1 - x^2}} r_m(x; a, b|q)r_n(x; a, b|q)dx = \delta_{m,n},
\]

where

\[
w(x) = \frac{h(x, 1)h(x, -1)h(x, q^{1/2})h(x, -q^{1/2})}{h(x, a)h(x, b)},
\]

with

\[
h(x, \alpha) = (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_{\infty}, \quad x = \cos \theta.
\]

If \( a > 1, |b| < 1 \) and \( |ab| < 1 \), the orthogonality measure contains a discrete part. Here we shall assume that the parameters are such that we have an absolute continuous measure; in the final result this condition can be weakened.

Let \( k > 0 \) and assume that \( s \in \mathbb{R} \) satisfies \( q^k < |s| < q^{-k} \), or that \( s \) is complex with \( |s| = 1 \). There is a unitary operator \( \Lambda : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}, d\mu(x; q^k s, q^k s^{-1}|q)) \) mapping each \( e_j^k \) onto \( r_n(x; q^k s, q^k s^{-1}|q) \), and for elements \( X \in \mathcal{U}_q(\mathfrak{su}(1, 1)) \) their realisation in \( L^2(\mathbb{R}, d\mu(x; q^k s, q^k s^{-1}|q)) \) will be denoted by \( \rho_{k,s}(X) \).
Proposition 3.1 Let $t \in \mathbb{C}$, $|t| < 1$, $x, y \in [-1, +1]$, $|s|, |\sigma| \in (q^k, q^{-k})$, and

$$v^k_t(x, s; y, \sigma) = \sum_{n=0}^{\infty} r_n(x; q^k s, q^k s^{-1}|q)r_n(y; q^k \sigma, q^k \sigma^{-1}|q)t^n. \quad (3.9)$$

Then

(i) $v^k_t(x, s; y, \sigma) \in L^2(\mathbb{R}, d\mu(x; q^k s, q^k s^{-1}|q))$.

(ii) Explicitly, with $x = \cos \theta$ and $y = \cos \phi$

$$v^k_t(x, s; y, \sigma) = \frac{(q^k te^{-i\phi} s, q^k te^{-i\phi} s^{-1}, q^k te^{-i\theta} \sigma, q^k te^{-i\theta} \sigma^{-1}; q)_\infty}{(te^{i(\theta-\phi)}, te^{-i(\theta-\phi)}, te^{-i(\theta+\phi)}, q^{2k} te^{-i(\theta+\phi)}; q)_\infty}$$

$$\times_s W_t(q^{2k-1} te^{-i(\theta+\phi)}; q^k e^{-i\phi} s, q^k e^{-i\phi} s^{-1}, q^k e^{-i\phi} \sigma, q^k e^{-i\phi} \sigma^{-1}, te^{-i(\theta+\phi)}; q, te^{i(\theta+\phi)}). \quad (3.10)$$

Herein, $r_{r+1} W_r$ is a very-well-poised $r+1\varphi_r$ series, see [10, (2.1.11)].

(iii) Let

$$X_t = q^{1/4} t B - q^{-1/4} t^{-1} C + \frac{\sigma^{-1} + \sigma}{q^{-1/2} - q^{1/2}}(A - D).$$

Then $v^k_t(x, s; y, \sigma)$ is an eigenfunction of $\rho_{k, n}(X_t A)$ for the eigenvalue $(2y - \sigma - \sigma^{-1})/(q^{-1/2} - q^{1/2})$.

Proof. Statement (i) follows from the asymptotic behaviour of the Al-Salam and Chihara polynomials [2, §3.1], see also [12, 22]. Expression (3.9) is the nonsymmetric Poisson kernel for the Al-Salam and Chihara polynomials, and has been deduced by Askey, Rahman and Suslov [3]. Using [3, (14.5)] gives

$$v^k_t(x, s; y, \sigma) = \frac{(t^2, q^k e^{-i\theta} s^{-1}, q^k e^{-i\theta} \sigma, q^k e^{-i\theta} \sigma^{-1}, q^k e^{-i\theta} e^{-i\phi}, q^k e^{-i\theta} e^{-i\phi}; q)_\infty}{(q^{2k} te^{i\theta} s, te^{i(\theta+\phi)}, te^{-i(\theta+\phi)}, te^{-i(\theta+\phi)}, q^{2k} te^{-i(\theta+\phi)}; q)_\infty}$$

$$\times_s W_t(q^{k-1} st e^{i\theta} ; te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q^k e^{i\theta} s, \sigma^{-1} st \sigma; q, q^k s^{-1} e^{-i\theta}). \quad (3.11)$$

Using (III.24) of [10] on this equation yields (3.10). The rest of the statements are proved in a similar way as Proposition 2.1, using in this case the three-term recurrence relation for Al-Salam and Chihara polynomials.

In the tensor product $\pi_{k_1} \otimes \pi_{k_2}$, there is a unitary operator $\Upsilon : \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \rightarrow L^2(\mathbb{R}^2, d\mu(x_1, q^k s, q^k s^{-1}|q)d\mu(x_2, q^k s, q^k s^{-1}|q))$ mapping $e^{k_1}_{n_1} \otimes e^{k_2}_{n_2}$ onto $r_{n_1}(x_1; q^k s, q^k s^{-1}|q)r_{n_2}(x_2; q^k s, q^k s^{-1}|q)$. Here we shall only consider the case $s_2 = s$ and $s_1 = e^{i\theta_2}$ ($x_2 = \cos \theta_2$) and denote the corresponding measure by

$$d\mu(x_1, x_2) = d\mu(x_1; q^k s, q^k s^{-1}|q)d\mu(x_2; q^k s, q^k s^{-1}|q).$$

Then we can simplify the basis functions in the tensor product decomposition. The realisation of $\Delta(X)$ with $X \in \mathcal{U}_q(\mathfrak{su}(1, 1))$ in $L^2(\mathbb{R}^2, d\mu(x_1, x_2))$ is denoted by
\( \sigma_{k_1,k_2,s}(X) \). For \( k = k_1 + k_2 + j \), with \( j \in \mathbb{Z}_+ \), and \( x_i = \cos \theta_i \), the above-mentioned simplification reads \([3.41]\)

\[
\begin{align*}
\sigma^{(k_1,k_2)}_n(x_1, x_2; s) &= \sum_{n_1,n_2} C_{n_1,n_2,n}^{k_1,k_2,k}(q) r_{n_1}(x_1; q^{k_1} e^{i\theta_1}, q^{k_2} e^{-i\theta_2} | q) r_{n_2}(x_2; q^{k_2} s, q^{k_2} s^{-1} | q) \\
&= r_n(x_1; q^k s, q^k s^{-1} | q) S_{j}^{(k_1,k_2)}(x_1, x_2; s),
\end{align*}
\]

where in this case \( S_{j} \) is given in terms of an Askey-Wilson polynomial:

\[
S_{j}^{(k_1,k_2)}(x_1, x_2; s) = \frac{p_j(x_2; q^{k_1} e^{i\theta_1}, q^{k_1} e^{-i\theta_1}, q^{k_2} s, q^{k_2} s^{-1} | q)}{\sqrt{(q, q^{2k_1}, q^{2k_2}, q^{2k_1+2k_2+j-1}, q)}_j}. \tag{3.13}
\]

Now we have the following

**Proposition 3.2** Let \( t \in \mathbb{C} \), \( \ |t| < 1 \), \( x_1, x_2, y \in [-1, +1] \), \( |s|, |\sigma| \in (q^k, q^{-k}) \), and

\[
\begin{align*}
\sigma^{(k_1,k_2)}_n(x_1, x_2; s; y, \sigma) &= \sum_{n=0}^{\infty} r_n(y; q^k \sigma, q^k \sigma^{-1} | q) e^{(k_1,k_2)_n}(x_1, x_2; s) t^n \tag{3.14}
\end{align*}
\]

for \( k = k_1 + k_2 + j \) with \( j \in \mathbb{Z}_+ \). Let again \( X_t \) be as in Proposition \([3.4]\)(iii), then

(i) \( \sigma^{(k_1,k_2)}(x_1, x_2, s; y, \sigma) \in L^2(\mathbb{R}^2, d\mu(x_1, x_2)) \) is an eigenfunction of \( \sigma_{k_1,k_2,s}(X_t A) \) for the eigenvalue \((2y - \sigma - \sigma^{-1})/(q^{-1/2} - q^{1/2})\).

(ii) Explicitly, one has

\[
\sigma^{(k_1,k_2)}(x_1, x_2, s; y, \sigma) = S_{j}^{(k_1,k_2)}(x_1, x_2; s) v_t^k(x_1, s; y, \sigma). \tag{3.15}
\]

(iii) For \(-1 \leq y_1, y_2 \leq 1\),

\[
v_t^{k_1}(x_1, e^{i\theta_2}; y_1, e^{i\theta_2}) v_t^{k_2}(x_2, s; y_2, \sigma) = \sum_{j=0}^{\infty} b^j S_{j}^{(k_1,k_2)}(y_1, y_2; \sigma) \sigma^{(k_1,k_2)}(x_1, x_2, s; y_1, \sigma), \tag{3.16}
\]

where \( x_i = \cos \theta_i \) and \( y_i = \cos \phi_i \).

In this case, the basic identity follows from \([3.15]\) and \([3.16]\):

\[
v_t^{k_1}(x_1, e^{i\theta_2}; y_1, e^{i\theta_2}) v_t^{k_2}(x_2, s; y_2, \sigma) = \\
\sum_{j=0}^{\infty} b^j v_t^{k_1+k_2+j}(x_1, s; y_1, \sigma) S_{j}^{(k_1,k_2)}(x_1, x_2; s) S_{j}^{(k_1,k_2)}(y_1, y_2; \sigma). \tag{3.17}
\]

Using the explicit expressions \([3.10]\) and \([3.13]\), relabelling \((x_2, y_2)\) by \((x, y)\) and putting

\[
\begin{align*}
(a, b, c, d) &= (q^{k_1} e^{i\theta_1}, q^{k_1} e^{-i\theta_1}, q^{k_2} s, q^{k_2} s^{-1}), \\
(a', b', c', d') &= (q^{k_1} e^{i\phi_1}, q^{k_1} e^{-i\phi_1}, q^{k_2} \sigma, q^{k_2} \sigma^{-1}),
\end{align*}
\]

we have the following
Theorem 3.3 For $|t| < 1$, $(a, b, c, d)$ with $\max(|a|, |b|, |c|, |d|) < 1$, $(a', b', c', d')$ with $\max(|a'|, |b'|, |c'|, |d'|) < 1$, and $|a't/b| < 1$, and

$$ab = a'b', \quad cd = c'd', \quad (3.18)$$

we have, with $x = \cos \theta$ and $y = \cos \phi$,

$$
\sum_{j=0}^{\infty} H_j p_j(x; a, b, c, d|q)p_j(y; a', b', c', d'|q)t^j = \\
\frac{(bt^{i\theta}, bte^{-i\phi}, cte^{-i\phi}, dte^{-i\phi}, b'te^{i\theta}, b'te^{-i\theta}, c'te^{-i\theta}, d'te^{-i\theta}; q)_{\infty}}{(b't, te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)}, cdte^{-i(\theta+\phi)}; q)_{\infty}} \\
\times_8 W_7(bb't/q; be^{-i\theta}, be^{-i\phi}, b'e^{i\phi}, b'e^{-i\phi}, bt/a'; q, a't/b) \\
\times_8 W_7(cdte^{-i(\theta+\phi)}/q; ce^{-i\theta}, de^{-i\theta}, c'e^{-i\phi}, d'e^{-i\phi}, te^{-i(\theta+\phi)}; q, te^{i(\theta+\phi)}), \quad (3.19)
$$

where

$$H_j = \frac{(bc'q^j t, b'c'q^j t, bd'q^j t, b'dq^j t; q)_{\infty}}{(bb'cdq^{2j} t; q)_{\infty}(q, ab, cd, abcdq^{2j-1}; q)_{j}} \\
\times_8 W_7(bb' cdq^{2j-1} t; bcq^j, bdq^j, b'c'q^j, b'dq^j; bt/a'; q, a't/b).$$

The proof follows directly from (3.17). The result (3.19) holds for arbitrary parameters satisfying (3.18) as long as all the series in (3.19) converge absolutely. The absolute convergence follows from the fact that for $j \to \infty$ the coefficients $H_j$ go to $(t^2; q)_{\infty}/(q, ab, cd, a't/b; q)_{\infty}$, together with the asymptotic behaviour of Askey-Wilson polynomials [12, 24]. The condition $|a't/b| < 1$ is not essential as one can use Bailey’s transform [10] (III.23-24) to rewrite the $8W_7$-series.

Rahman [16, (4.4)] derives a formula remotely similar to (3.19) in the sense that the summand also consists of the product of a $8W_7$-series and the product of two Askey-Wilson polynomials. However, the right hand side in (3.19) is much less complicated than the right hand side of [16, (4.4)], which consists of four infinite sums each involving $10W_9$-series. Rahman’s formula is a $q$-analogue of Feldheim’s bilinear generating function for the Jacobi polynomials, whereas (3.19) is a $q$-analogue of Bateman’s bilinear generating function for the Jacobi polynomials, see (2.19). It should be interesting to investigate whether (3.19) could lead to a $q$-analogue of (2.20) or Corollary 2.5.

The previous formulas can be specialised by putting $d = d' = 0$; then the Askey-Wilson polynomials become continuous dual $q$-Hahn polynomials denoted by $p_n(x; a, b, c|q)$. One finds :

Corollary 3.4 For given $|t| < 1$, $(a, b, c)$ and $(a', b', c')$ with $\max(|a|, |b|, |c|) < 1$ and $\max(|a'|, |b'|, |c'|) < 1$, $|a't/b| < 1$, and

$$ab = a'b', \quad (3.20)$$
we have, with $x = \cos \theta$ and $y = \cos \phi$,

$$\sum_{j=0}^{\infty} G_j p_j(x; a, b, c|q)p_j(y; a', b', c'|q)t^j =$$

$$\frac{(bte^{-i\phi}, bte^{i\phi}, cte^{-i\phi}, bte^{i\phi}, bte^{-i\theta}, cte^{-i\theta}; q)_{\infty}}{(bt, te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)}; q)_{\infty}}$$

$$\times_8 W_7(bbt/q; be^{i\phi}, be^{-i\phi}, b'e^{-i\phi}, b'e^{i\phi}, bt/a'; q, a't/b)$$

$$\times_{3\varphi_2} \left( ce^{-i\phi}, c' e^{-i\phi}, te^{-i(\theta+\phi)} ; q, te^{i(\theta+\phi)} \right),$$

where

$$G_j = \frac{(bcq^j t, b'c't^j q;q)_{\infty}}{(q, ab;q)_j} 3\varphi_2 \left( bcq^j, b'c'q^j, bt/a' ; b'c't^jt, b'c't^j ; q, a't/b \right).$$

By making the further specialisation $c = c' = 0$ the identity reduces to the non-symmetric Poisson kernel for the Al-Salam and Chihara polynomials [3, (14.8)], i.e. essentially the formula that we started from and used in (3.10).

Another specialisation of Corollary 3.4 can be performed as follows. In $G_j$, use transformation [10, (III.9)] for the $3\varphi_2$-series; on the right hand side of (3.21) use transformation [10, (III.23)] for the $8W_7$-series with $(b, c, d)$ corresponding to $(be^{-i\phi}, b'e^{-i\phi}, bt/a)$; finally multiply both sides of (3.21) by $(a't/b; q)_{\infty}$ and take $b = b' = 0$. The result is a bilinear generating function for Al-Salam and Chihara polynomials without the extra assumption that the product of the parameters should be equal.

**Corollary 3.5** For $|t| < 1$ and max$(|a|, |c|) < 1$, max$(|a'|, |c'|) < 1$ and with $x = \cos \theta$, $y = \cos \phi$ we have

$$\sum_{j=0}^{\infty} \frac{t^j}{(q, a'ct^j q)_{\infty}} 2\varphi_1 \left( ct/c', aqc^j ; q, t^j/c \right) R_j(x; a, c | q)R_j(y; a', c'|q) =$$

$$\frac{(cte^{-i\phi}, c'te^{i\phi}, ate^{i\phi}, a' te^{i\phi} ; q)_{\infty}}{(te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, ct/c, a'ct ; q)_{\infty}} 3\varphi_2 \left( a'e^{i\phi}, ae^{i\phi}, te^{i(\theta+\phi)} ; q, te^{-i(\phi+\theta)} \right)$$

$$\times_{3\varphi_2} \left( ce^{-i\phi}, c' e^{-i\phi}, te^{-i(\theta+\phi)} ; q, te^{i(\theta+\phi)} \right).$$

**Remark 3.6** Take $a = a' = 0$ in Corollary 3.3, so that the $2\varphi_1$-series and the first $3\varphi_2$-series reduce to $1\varphi_0$-series which can be summed by the $q$-binomial theorem [10, (II.3)]. The resulting formula is the non-symmetric Poisson kernel for the continuous big $q$-Hermite polynomials, i.e. Askey-Wilson polynomials with three parameters set to zero. So for $a = a' = 0$ Corollary 3.3 overlaps with Askey, Rahman and Suslov [3, (14.8)].
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