On nonimbeddability of topologically trivial domains and Thin Hartogs figures of $P_2(\mathbb{C})$ into Stein spaces

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abstract. A question of Poletsky was to know if there exists a thin Hartogs figure such that any of its neighborhoods cannot be imbedded in Stein spaces. In [2], Chirka and Ivashkovitch gave such an example arising in an open complex manifold. In this paper, we answer to the question of the existence of such a figure in compact surfaces by giving an example arising in $P_2(\mathbb{C})$. By smoothing it, we obtain a smooth (non-analytic) disc with boundary $\overline{D} \subset P_2(\mathbb{C})$ having the same property. Consequently, this disc intersects all algebraic curves of $P_2(\mathbb{C})$. Moreover, as $\overline{D}$ is topologically trivial, it has a neighborhood diffeomorphic to the unit ball of $\mathbb{C}^2$. This gives a negative answer to the following question of S. Ivashkovitch: Is the property for a domain $B$ of $P_2(\mathbb{C})$ to be diffeomorphic to the unit ball of $\mathbb{C}^2$ a sufficient condition for the existence of non-constant holomorphic functions on it?

1 Introduction

The understanding of the relationship between the theory of holomorphic functions on two-dimensional complex manifolds and their differential topology has been a subject of main interest, specially in the case of domains of $P_2(\mathbb{C})$. In [8] (see also [9] for a review on related questions), Nemirovski proved that if an embedded two-sphere in $P_2(\mathbb{C})$ is not homologous to zero, then every holomorphic function in a neighborhood of this sphere is constant.

2000 Mathematics Subject Classification: Primary:32Q55,32d10 Secondary: 32Q40, 32v10, 32v30

Key words : Thin Hartogs figure, projective space, imbeddability, Stein, topology
In [10] [11], we proved that if $M$ is a real hypersurface of $P_2(\mathbb{C})$ dividing it into two domains $\Omega_1$ and $\Omega_2$, then any holomorphic function defined in a neighborhood of at least one of this two domains is constant. Thus topology can be an obstruction to the existence of non-constant holomorphic functions. In this paper, we prove that topology is not the only obstruction. Indeed, answering a question of S. Ivashkovitch, we prove that there exists a domain of $P_2(\mathbb{C})$ diffeomorphic to the unit ball of $\mathbb{C}^2$ which admits no non-constant holomorphic function.

A related question is the study of the embedding of thin Hartogs figures in Stein manifolds. Let $\Delta$ be the unit disc of $\mathbb{C}$, $S^1$ be its boundary, $[0,1] \subset \mathbb{C}$ be a segment in the real line and $X$ be a complex surface. We call thin Hartogs figure the embedding of the the set $W = \Delta \times \{0\} \cup S^1 \times [0,1] \subset \mathbb{C}^2$ by any continuous imbedding $f : W \to X$ which is holomorphic on $\Delta \times \{0\}$. A question of Poletsky was to know if a thin Hartogs figure has always a neighborhood imbeddable in a Stein space. In [2], Chirka and Ivashkovitch gave a counter-example arising in an open complex manifold. We begin by answering the question of the existence of such figures in compact manifolds by giving a counter-example arising in $P_2(\mathbb{C})$.

This two problems are related because a thin Hartogs figure being topologically equivalent to the unit disc, it has a neighborhood homeomorphic to the unit ball of $\mathbb{C}^2$. More, by smoothing the thin Hartogs figure we constructed in $P_2(\mathbb{C})$, we obtain a smooth and closed disc with boundary $\overline{D} \subset P_2(\mathbb{C})$ such that any holomorphic function defined on any of its open neighborhoods is constant. This disc admitting neighborhoods diffeomorphic to the unit ball of $\mathbb{C}^2$, we obtain a negative answer to Ivashkovitch’s question.

For any algebraic curve $C$, $P_2(\mathbb{C}) \setminus C$ is Stein and cannot contain $\overline{D}$. Thus, $\overline{D}$ intersects all algebraic curves of $P_2(\mathbb{C})$. Real surfaces of $P_2(\mathbb{C})$ having this last property have been constructed by B. Fabre [3] then by Nemirovski [7]. Those examples are in some sense contrary to our’s because they admit Stein neighborhoods.

Finally, let $B$ be an open neighborhood of $\overline{D}$ diffeomorphic to the unit ball and with smooth boundary $\partial B$. Then, by combining our construction, the result of [8] and the Plemedj decomposition in $P_2(\mathbb{C})$, we prove that $\partial B$ is an example of a smooth CR-hypersurface of $P_2(\mathbb{C})$, diffeomorphic to the unit sphere of $\mathbb{C}^2$ such that all continuous CR functions defined on $\partial B$ and all holomorphic functions defined on any connected component of $P_2(\mathbb{C}) \setminus \partial B$ are constant.
2 Envelopes of holomorphy of open sets in projective space

The study of the envelopes of holomorphy has been treated by authors as Fujita \cite{4,5}, Takeuchi \cite{12}, Kiselman \cite{6} or Ueda \cite{13}. Let us recall some well known results.

**Definition 1** Let $X$ be a complex manifold. A domain over $X$ is a connected complex manifold $W$ equipped with a locally biholomorphic map $\pi : U \to X$. We say that a domain $(U, \pi)$ contains another domain $(V, \Pi)$ if there is a map $j : V \to U$ respecting the projections, $\pi \circ j = \Pi \circ j$. The envelope of holomorphy $(\tilde{U}, \tilde{\pi})$ of a domain $(U, \pi)$ over $X$ is the maximal domain over $X$ containing $(U, \pi)$ such that every holomorphic function in $U$ extends holomorphically to $\tilde{U}$.

Let us consider the complex projective space $P_n(\mathbb{C})$ as the quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ by the action of $\mathbb{C}^*$. Holomorphic functions in domains over $P_n(\mathbb{C})$ correspond to holomorphic functions over $\mathbb{C}^{n+1}$ constant on the lines passing through the origin. The analytic continuation in $\mathbb{C}^{n+1}$ preserves this property because it can be represented by the differential equation

$$\sum_{j=1}^{n+1} z_j \frac{\partial f}{\partial z_j} = 0.$$ 

Hence, envelopes over $P_n(\mathbb{C})$ correspond to envelopes over $\mathbb{C}^{n+1}$.

**Proposition 1** Let $U$ be a domain over $P_n(\mathbb{C})$, we have two cases, if the envelope over $\mathbb{C}^{n+1}$ contains the origin, then all holomorphic functions on $U$ are constant and the envelope over $P_n(\mathbb{C})$ is the entire space. Otherwise, the envelope over $\mathbb{C}^{n+1}$ is a Stein manifold.

**Proposition 2** The envelope of holomorphy of a domain over $P_n(\mathbb{C})$ is either a Stein manifold or coincides with the entire $P_n(\mathbb{C})$. Equivalently, the envelope is Stein if and only if there exists non-constant holomorphic functions on the domain.
3 Continuity principle

Let $X$ be a complex manifold, an *analytic disc of $X$* is a continuous map $A : \Delta \to X$ which is holomorphic on $\Delta$. The *boundary* $\partial A$ of the analytic disc $A$ is by definition the restriction of $A$ to the unit circle $S^1 = \partial \Delta$. A family of discs $\{A_t\}_{t \in [0,1]}$ is called *continuous* if the map $\tilde{A} : [0,1] \times \Delta \to X$ defined by $\tilde{A}(t,w) = A_t(w)$ is continuous. Let us recall the following well known continuity principle (see [1]):

**Proposition 3 (Behnke-Sommer)** Let $\{A_t\}_{t \in [0,1]}$ be a continuous family of analytic discs of a complex manifold $X$. Let $U \subset X$ be an open set and $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that $U$ verifies the following:

1. $A_0 \subset U$.
2. For any $t \in [0,1]$, the boundary $\partial A_t \subset U$.

Then for any $t \in [0,1]$, there exists a neighborhood $U_t$ of the disc $A_t$ such that $f$ extends holomorphically to $U_t$.

4 Construction of the thin Hartogs figure

For any point $z \in \mathbb{C}^3$, let $L_z$ be the complex line passing through $z$ and the origin, this line defines a point $\tilde{L}_z$ in $P_2(\mathbb{C})$. Let $\Phi : \mathbb{C}^3 \setminus \{0\} \to P_2(\mathbb{C})$ be the map defined by $\Phi(z) = \tilde{L}_z$. If $\{A_t\}_{t \in [0,1]}$ is a smooth family of closed analytic discs properly imbedded in $\mathbb{C}^3$, such that $A_1(0) = 0$, then the smooth family of analytic discs $\{\Phi \circ A_t\}_{t \in [0,1]}$ is well defined.

**Proposition 4** Let $W$ be the thin Hartogs figure $\Delta \times \{0\} \cup S^1 \times [0,1] \subset \mathbb{C}^2$. Their exists two complex lines $L_1$ and $L_2$ of $\mathbb{C}^3$ and a continuous (even smooth) family $\{A_t\}_{t \in [0,1]}$ of closed analytic discs of $\mathbb{C}^3$ such that the family of analytic discs $\{\Phi \circ A_t\}_{t \in [0,1]}$ is

1. continuous and properly embedded in $P_2(\mathbb{C})$.
2. For $0 \leq t_1 < t_2 < 1$ the discs $\Phi \circ A_{t_1}$ and $\Phi \circ A_{t_2}$ intersects only at the points $\tilde{L}_1$ and $\tilde{L}_2$.
3. For any $t \in [0,1]$ the disc $A_t$ is transversal to $L_1$ and to $L_2$. 

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4. The restriction of the map \( \Phi \circ A \) defined by \( \Phi \circ A(w, t) = \Phi \circ A_t(w) \) to \( \mathcal{W} \) is a continuous (even smooth) proper imbedding of \( \mathcal{W} \) into \( P_2(\mathbb{C}) \).

\( (\Phi \circ A(\mathcal{W}) \) is a thin Hartogs figure of \( P_2(\mathbb{C}) \).

**Proof.** Let \( P(z_1, z_2, z_3) : \mathbb{C}^3 \to \mathbb{C} \) be a generic polynomial of degree 2 such that the complex hypersurface \( H = \{ P(z) = 0 \} \) is a smooth and generic quadric which contains the origin. Thus, \( H \) contains only two complex lines \( L_1 \) and \( L_2 \) passing through the origin. According to the Bezout theorem, for any point \( z \in (H \setminus (L_1 \cup L_2)) \), the line \( L_z \) intersects \( H \) only at the point \( z \) and at the origin. Then, the restriction of the map \( \Phi \) on the Zarisky open set \( H \setminus (L_1 \cup L_2) \) is open, one to one and holomorphic (it defines a biholomorphism on its image). Let \( F : \mathbb{C}^3 \to \mathbb{C} \) be a holomorphic submersion and note \( F_c \) the smooth hypersurface \( F_c = \{ F(z) = c \} \). Suppose \( F \) is chosen such that \( F_0 \) is transversal at the origin to \( L_1, L_2 \) and \( H \). Then \( F_0 \) intersects \( L_1 \) and \( L_2 \) only at the origin and intersects \( H \) on a smooth curve \( S_0 = H \cap F_0 \). Let us note, for any \( c \in \mathbb{C}, S_c = H \cap F_c \). Then their exists a small neighborhood \( V \) of the origin in \( \mathbb{C} \) such that \( \{ S_c \}_{c \in V} \) is a smooth family of complex curves of \( H \) transversal to the lines \( L_1 \) and \( L_2 \). If \( V \) is taken small enough, their exists \( \epsilon > 0 \) such that, for any \( c \in V \) the ball \( B(0, \epsilon) \subset \mathbb{C}^3 \) intersects \( S_c \) on an analytic disc \( B_c \). One can always choose the parametrization of the discs \( B_c \) and an imbedding \( \phi \) of the set \( [0, 1] \) in \( V \) with \( \phi(1) = 0 \) such that the family of disc \( \{ A_t \}_{t \in [0, 1]} = \{ B_{\phi^{-1}(t)} \}_{t \in [0, 1]} \) is a smooth and properly imbedded family of analytic discs of \( \mathbb{C}^3 \) with \( A_1(0) = 0 \). By the transversality assumption (for \( V \) chosen small enough), this family of analytic discs verify the Lemma.

**Remark.** By exploding \( P_2(\mathbb{C}) \) at the points \( \tilde{L}_1 \) and \( \tilde{L}_2 \) (let us denote \( \tilde{P}_2(\mathbb{C}) \) this manifold) our construction gives an imbedding of the family \( \{ A_t \}_{t \in [0, 1]} \) in \( \tilde{P}_2(\mathbb{C}) \).

**Theorem 1** Let \( \{ A_t \}_{t \in [0, 1]} \) be the smooth family of analytic disc constructed in the previous proposition and \( \mathcal{H} = \Phi \circ A(\mathcal{W}) \) the corresponding thin Hartogs figure of \( P_2(\mathbb{C}) \). Then any holomorphic function defined in a connected neighborhood of \( \mathcal{H} \) is constant. Thus no neighborhood of \( \mathcal{H} \) can be embedded in a Stein space.

**Proof.** Let \( U \) be an open and connected neighborhood of \( \mathcal{H} \) in \( P_2(\mathbb{C}) \) and \( f \) be a holomorphic function defined on \( U \). Let us note \( \widehat{U} \) and \( \widehat{f} \) the corresponding open set and holomorphic function of \( \mathbb{C}^3 \setminus \{ 0 \} \). Then, by construction, \( \widehat{U} \)
contains an open neighborhood of $A(W)$. According to the continuity principle, $\hat{f}$ extends holomorphically in a neighborhood of the disc $A_1(\Delta)$. So the envelop of holomorphy of $\hat{U}$ over $\mathbb{C}^3$ contains the origin and according to proposition 1, $f$ has to be constant.

Let us note $\Delta(r) = \{ z \in \mathbb{C}; |z| \leq r \}$ with $r \in [0, 1]$, $S^1(r)$ its boundary and let us define

$$\overline{\mathcal{D}} = A_1(\Delta(1/2)) \cup_{t \in [0,1]} A_t(S^1(1/2 + 2^{1/t-1})).$$

Then $\overline{\mathcal{D}}$ is a smooth disc with boundary and as for the precedent proposition, any holomorphic function defined in any of its neighborhoods has to extend to a domain over $\mathbb{C}^3$ which contains the origin and so, has to be constant. Moreover, for any compact complex curve $C \subset P_2(\mathbb{C})$, the open set $P_2(\mathbb{C}) \setminus C$ is pseudoconvex and Stein. As there exists non-constant holomorphic functions in Stein manifolds, the disc $\overline{\mathcal{D}}$ is not included in $P_2(\mathbb{C}) \setminus C$. Thus $\mathcal{D}$ intersects $C$. We have obtained:

**Corollary 1**. There exists a (non analytic) closed and smooth disc with boundary $\overline{\mathcal{D}} \subset P_2(\mathbb{C})$ such that any holomorphic function defined on its neighborhood is constant. Consequently, $\overline{\mathcal{D}}$ intersects any algebraic curves of $P_2(\mathbb{C})$.

The disc with boundary $\overline{\mathcal{D}}$ being smooth, it has an open neighborhood $B$ diffeomorphic to the unit ball of $\mathbb{C}^2$.

**Corollary 2**. There exists a domain $B \subset P_2(\mathbb{C})$, diffeomorphic to the unit ball of $\mathbb{C}^2$ such that any holomorphic function defined on it is constant.

Moreover, $\partial B$ is a smooth hypersurface dividing $P_2(\mathbb{C})$ into two domains $B$ and $C_B = P_2(\mathbb{C}) \setminus \overline{\mathcal{D}}$. The domain $B$ being topologically trivial, its complementary $C_B$ has the same second homology group than $P_2(\mathbb{C})$, in particular, $C_B$ contains a non contractible real 2-sphere. According to [8], all holomorphic functions defined on $C_B$ are constant. It is well known (see [10] for an example of a proof) that the Plemedj decomposition of CR functions is available in $P_2(\mathbb{C})$. So any continuous CR function $f$ defined on $\partial B$, can be decomposed as $f = f^+ - f^-$ with $f^+$ and $f^-$ the boundary values (in the current sense) of holomorphic functions defined respectively on $B$ and $C_B$. In our case, this two holomorphic functions have to be constant, so $f$ is constant.
Corollary 3 The boundary $\partial B$ of the previously constructed domain $B \subset P_2(\mathbb{C})$ is a smooth hypersurface dividing $P_2(\mathbb{C}) \setminus \partial B$ into two domains $B$ and $C_B$ which verify the following properties:

1. All holomorphic functions defined on $B$ are constant.
2. All holomorphic functions defined on $C_B$ are constant.
3. The CR hypersurface $\partial B \subset P_2(\mathbb{C})$ is diffeomorphic to the unit sphere $S^3$ of $\mathbb{C}^2$ and all continuous CR functions defined on $\partial B$ are constant.

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