On the flat interaction of foundations and punches with complex forms of contacting surfaces

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Abstract. Current work is devoted to solving the problem of the action of an arbitrary system of punches on a coated base in the case of plane strain. It is assumed that all the punches are different and their shapes and coating thickness can be described by complex rapidly changing functions. In this case the classical methods are ineffective and it is necessary to use a special approach, which consists in highlighting the features of the individual terms and factors and further application of the A.V. Manzhirov generalized projection method.

Introduction

The problems of multiple contact for bodies with coatings and regular punch systems were considered in previous papers [1–5]. The present work is devoted to the study of the case when the punches acting on the coating are different and shapes of contacting surfaces can be described by complicated functions.

1. Statement of the plane contact problem

Viscoelastic aging foundation lies on a rigid undeformable basis (figure 1). This foundation consist of two layers: the lower homogeneous layer of constant thickness $h_{\text{lower}}$ made of a viscoelastic aging material at time $\tau_{\text{lower}}$ and the upper layer (coating) of variable thickness $h_{\text{upper}}(x)$ made of another viscoelastic aging material at time $\tau_{\text{upper}}$ ($x$ is longitudinal coordinate).

We assume that the coating rigidity is less than the rigidity of the lower layer or they are of the same order of magnitude, i.e., Young’s modulus $E_{\text{upper}}(t)$ of the upper layer is less than Young’s modulus $E_{\text{lower}}(t)$ of the lower layer. We consider the plain strain problem.

At instant $\tau_0 \geq \tau_{\text{lower}}$, the forces $P_i(t)$ with eccentricities $e_i(t)$ starts to indent rigid punches into the surface of such a foundation ($i = 1, 2, \ldots, n$, $n$ is the number of the punches). The contact area lengths $\bar{a}_i = (b_i - a_i) \ll h(x)$ for all $x$, where $a_i$ and $b_i$ are left and right coordinates of ith punch. The backlash between ith punch and the coating in undeformable state is described by a function $g_i(x)$ ($g_i(x) \geq 0$, $\exists x_{0i} \in [a_i, b_i]: g_i(x_{0i}) = 0$). As a result of the interaction, the punches are immersed into the layered foundation to the depths $\delta_i(t)$ and rotated through the angles $\alpha_i(t)$.

To derive the mathematical model of the problem, we replace the system of punches by some normally distributed load acting on regions $x \in [a_i, b_i]$, $i = 1, 2, \ldots, n$ (under the punches), and...
equal to zero outside these regions. Then the vertical displacement of the upper face of the 
foundation described above on each region (under each punch) can be written as (we use known 
solutions for thick and thin elastic and viscoelastic layers \([6–8]\))

\[
    u_{iz}(x, t) = -(1 - \nu^2_{upper})h(x) \left[ \frac{q_i(x, t)}{E_{upper}(t - \tau_{upper})} \right. \\
    \left. - \int_{\tau_0}^{t} K_{upper}(t - \tau_{upper}, \tau - \tau_{upper}) \frac{q_i(x, \tau)}{E_{upper}(\tau - \tau_{upper})} \, d\tau \right] \\
    - \frac{2(1 - \nu^2_{lower})}{\pi} \sum_{j=1}^{n} \left[ \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi, t)}{E_{lower}(\tau - \tau_{lower})} \, d\xi \\
    \right. \\
    \left. - \int_{\tau_0}^{t} K_{lower}(t - \tau_{lower}, \tau - \tau_{lower}) \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi, \tau)}{E_{lower}(\tau - \tau_{lower})} \, d\xi \, d\tau \right], \quad (1)
\]

\(x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n,\)

where \(\nu_{lower}\) and \(\nu_{upper}\) are Poisson’s ratios of lower and upper layers; \(k_{pl}(s)\) is known kernel of the plane contact problem, which has the form \([9]\)

\[
k_{pl}(s) = \int_0^{\infty} \frac{L(u)}{u} \cos(su) \, du, \quad L(u) = \begin{cases} 
    \frac{\cosh(2u) - 1}{\sinh(2u) + 2u}, & \text{smooth layer-basis contact}, \\
    \frac{2\cosh(2u) + 4u^2}{2x\cosh(2u) + 4u^2 + 1 + x^2}, & \text{perfect layer-basis contact};
\end{cases}
\]

\(K_k(t, \tau)\) are creep kernels which has a form (see \([7,10,11]\))

\[
    K_k(t, \tau) = E_k(\tau) \frac{\partial}{\partial \tau} \left[ \frac{1}{E_k(\tau)} + C_k(t, \tau) \right], \quad k = \text{lower, upper},
\]

\(C_k(t, \tau)\) are tensile creep functions.

But the vertical displacement of the upper face of the foundation connect with displacement 
of the rigid punches and backlash function by the formula

\[
    u_{iz}(x, t) = -[\delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x)] \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n, \quad (2)
\]
where $\eta_i = \frac{1}{2}(a_i + b_i)$ is midpoint of $i$th punch, $\delta_i(t)$ is its settlement, and $\alpha_i(t)$ is its tilt angle.

By equating right sides of (1) and (2) we obtain the system of integral equations of our problem in the form

$$
(1 - \nu_{upper}^2) h_{upper}(x) \left[ \frac{q_i(x,t)}{E_{upper}(t - \tau_{upper})} - \int_{\tau_0}^{t} K_{upper}(t - \tau_{upper}, \tau - \tau_{upper}) \frac{q_i(x,\tau)}{E(\tau - \tau_{upper})} d\tau \right]
$$

$$
+ \frac{2(1 - \nu_{lower}^2)}{\pi} \left[ \sum_{j=1}^{n} \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi,t)}{E_{lower}(t - \tau_{lower})} d\xi \right]
$$

$$
- \int_{\tau_0}^{t} K_{lower}(t - \tau_{lower}, \tau - \tau_{lower}) \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi,\tau)}{E_{lower}(\tau - \tau_{lower})} d\xi d\tau
$$

$$
= \delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x), \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n.
$$

The resulting system of equations should be supplemented with equilibrium conditions for punches under the action of distributed load $q_i(x, t)$ and a concentrated force $P_i(t)$:

$$
\int_{a_i}^{b_i} q_i(\xi,t) d\xi = P_i(t),
$$

$$
\int_{a_i}^{b_i} q_i(\xi,t)(\xi - \eta_i) d\xi = M_i(t) \equiv P_i(t)\alpha_i(t), \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n.
$$

There are exist four different condition types on each punch: 1) the settlement and tilt angle of the punch are given; 2) the punch settlement and the moment of the load application are given; 3) the tilt angle of the punch and the force of the load application are given; 4) the force and the moment of the load application are given. Thus, there are fifteen different versions of the multiple plane contact problem: four versions if all conditions are equal; 2) six versions if exists two punch groups with different condition types; 3) four versions if exists three punch groups with different conditions; and 4) one version if exists four groups of punches with different condition types.

### 2. Dimensionless form

Make a change of variables in (3) and (4) according to the formulas

$$
x^* = \frac{2(x - \eta_i)}{a_i}, \quad \xi^* = \frac{2(\xi - \eta_i)}{a_j}, \quad t^* = \frac{t}{\tau_0}, \quad \tau_i = \frac{\tau_{upper}}{\tau_0}, \quad \tau_2 = \frac{\tau_{lower}}{\tau_0}, \quad \lambda = \frac{2h_{lower}}{a},
$$

$$
\zeta^{i*} = \frac{\bar{a}_i}{a}, \quad \eta^{i*} = \frac{2\bar{a}_i}{a}, \quad \delta^{i*}(t^*) = \frac{2\delta_i(t)}{a}, \quad \alpha^{i*}(t^*) = \zeta^{i*}\alpha_i(t), \quad g^{i*}(x^*) = \frac{2g_i(x)}{a},
$$

$$
\xi^{i*} = \frac{E_{lower}(t - \tau_{lower})}{E_{upper}(t - \tau_{upper})}, \quad \nu^{i*}(t^*) = \frac{h_{upper}(x)(1 - \nu_{lower}^2)}{\bar{a}_i(1 - \nu_{lower}^2)},
$$

$$
q^{i*}(x^*, t^*) = \frac{2s^{i*}q_i(x,t)(1 - \nu_{lower}^2)}{E_{lower}(t - \tau_{lower})}, \quad P^{i*}(t^*) = \frac{4P_i(t)(1 - \nu_{lower}^2)}{\bar{a}E_{lower}(t - \tau_{lower})},
$$

$$
M^{i*}(t^*) = \frac{8M_i(t)(1 - \nu_{lower}^2)}{\bar{a}E_{lower}(t - \tau_{lower})}, \quad k^{i*}(x^*, \xi^*) = \frac{1}{\pi} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right),
$$

$$
K^{i*}(t^*, \tau^*) = \frac{E(t - \tau_{upper})E_{lower}(\tau - \tau_{lower})}{E(\tau - \tau_{upper})E_{lower}(t - \tau_{lower})} K_{upper}(t - \tau_{upper}, \tau - \tau_{upper})\tau_0,
$$

$$
K^{2*}(t^*, \tau^*) = K_{lower}(t - \tau_{lower}, \tau - \tau_{lower})\tau_0,
$$

$$
F^{ij*}f(x^*) = \int_{-1}^{1} k^{ij*}(x^*, \xi^*) f(\xi^*) d\xi^*, \quad V^{k*}f(t^*) = \int_{0}^{t^*} K^{k*}(t^*, \tau^*) f(\tau^*) d\tau^*, \quad k = 1, 2.
$$
Here \( \bar{a} = \min_{i=1,2,...,n} a_i \).

Then we obtain following system of mixed integral equations and additional conditions:

\[
c^*(t^*)m^{i*}(I - V^{1*})(x^*)q^{i*}(x^*, t^*) + (I - V^{2*}) \sum_{j=1}^{n} F^{ij*} q^{j*}(x^*, t^*) = \delta^{i*}(t^*) + \alpha^{i*}(t^*)x^* - g^*(x^*),
\]

\[
\int_{-1}^{1} q^{i*}(\xi, t^*) d\xi = P^{i*}(t^*), \quad \int_{-1}^{1} q^{i*}(\xi, t^*)\xi d\xi = M^{i*}(t^*),
\]

\( i = 1, 2, \ldots, n, \quad x^* \in [-1, 1], \quad t^* \geq 1. \) \hspace{1cm} (6)

Here \( I \) is identity operator. These equations can be represented as

\[
c^*(t^*)D(x^*) \cdot (I - V^{1*})q^*(x^*, t^*) + (I - V^{2*})F^*q^*(x^*, t^*) = \delta^*(t^*) + \alpha^*(t^*)x^* - g^*(x^*),
\]

\[
\int_{-1}^{1} q^*(\xi, t^*) d\xi = P^*(t^*), \quad \int_{-1}^{1} q^*(\xi, t^*)\xi d\xi = M^*(t^*),
\]

\( x^* \in [-1, 1], \quad t^* \geq 1, \) \hspace{1cm} (7)

where

\[
q^*(x^*, t^*) = \sum_{i=1}^{n} q^{i*}(x^*, t^*)i^i, \quad P^*(t^*) = \sum_{i=1}^{n} M^{i*}(t^*)i^i, \quad M^*(t^*) = \sum_{i=1}^{n} M^{i*}(t^*)i^i,
\]

\[
\delta^*(t^*) = \sum_{i=1}^{n} \delta^{i*}(t^*)i^i, \quad \alpha^*(t^*) = \sum_{i=1}^{n} \alpha^{i*}(t^*)i^i, \quad k^*(x, \xi) = \sum_{i,j=1}^{n} k^{ij*}(x, \xi)i^i\xi^j,
\]

\[
F^*f(x) = \int_{-1}^{1} k^*(x, \xi) \cdot f(\xi) d\xi, \quad D(x) = \text{diag}\{m^{1*}(x), m^{2*}(x), \ldots, m^{n*}(x)\}.
\]

Further, we will omit asterisks and it will be the summation over repeated upper indices \( i \) and \( j \) from 1 to \( n \) if the left side of the formula is independent of the index.

Note that vector-functions \( m(x) \) and \( g(x) \) relate with coating width and punch base forms. These parameters can be described by a rapidly changing function.

3. Analytical Solution

We should see the solution for all versions of the problem in the form

\[
q(x, t) = \bar{q}(x, t) - (I - V^{1})^{-1} \frac{1}{c(t)} D^{-1}(x) \cdot g(x), \hspace{1cm} (8)
\]

where \( \bar{q}(x, t) \) is new function to be determined. In this case main equation and additional conditions (7) take a form

\[
c(t)D(x) \cdot (I - V^{1})\bar{q}(x, t) + (I - V^{2})F\bar{q}(x, t) = \delta(t) + \alpha(t)x - \bar{c}(t)g(x),
\]

\[
\int_{-1}^{1} \bar{q}(\xi, t) d\xi = \bar{P}(t), \quad \int_{-1}^{1} \bar{q}(\xi, t)\xi d\xi = \bar{M}(t), \quad x \in [-1, 1], \quad t \geq 1, \hspace{1cm} (9)
\]

where

\[
\bar{g}(x) = \int_{-1}^{1} k(x, \xi) \cdot [D(x) \cdot g(\xi)] d\xi, \quad \bar{c}(t) = -(I - V^{2})\cdot(I - V^{1})^{-1} \frac{1}{c(t)},
\]

\[
\bar{P}(t) = P(t) + (I - V^{1})^{-1} \frac{1}{c(t)} \int_{-1}^{1} D(x) \cdot g(\xi) d\xi,
\]

\[
\bar{M}(t) = M(t) + (I - V^{1})^{-1} \frac{1}{c(t)} \int_{-1}^{1} [\xi D(x) \cdot g(\xi)] d\xi.
\]
We obtain operator equation with different integral operators and one group of rapidly changing functions $m^i(x)$ supplemented by two vector conditions. Last term in right-hand side of operator equation (9) is “good”: its smoothness defined by kernel $k(x,\xi)$. Obtained operator equation with additional conditions has the same form as the main equation and additional conditions in [1]. (Only the last known term in the right-hand side contain $t$-dependent factor.) Hence the solution method will be similar.

In this paper we will show how to construct the solution if we have one punch group with known force conditions. So we have following version of problem: $P_i(t)$ and $M_i(t)$ are known, $\alpha(t)$ and $\delta(t)$ are unknown for all $i = 1, 2, \ldots, n$. Solution constructing will be similar for other versions of the problem.

3.1. Solution form and special basis

By introducing notations in (9)

$$Q(x,t) = \sqrt{m(x)}q(x,t), \quad K(x,\xi) = D^{-1/2}(x) \cdot k(x,\xi) \cdot D^{-1/2}(\xi),$$

$$Gf(x) = \int_{-1}^{1} K(x,\xi) \cdot f(\xi) \, d\xi,$$

we obtain new operator equation and additional conditions

$$c(t)(I - V^1)Q(x,t) + (I - V^2)GQ(x,t) = D^{-1/2}(x) \cdot [\delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x)],$$

$$\int_{-1}^{1} D^{-1/2}(\xi) \cdot Q(\xi,t) \, d\xi = \tilde{P}(t), \quad \int_{-1}^{1} D^{-1/2}(\xi) \cdot Q(\xi,t) \, d\xi = \tilde{M}(t), \quad x \in [-1,1], \quad t \geq 1. \quad (12)$$

We will find solution of operator equation with additional conditions (12) in the class of vector–functions continuous in time $t$ in Hilbert space $L_2([-1,1],V)$ (see [12]). First of all we should construct special basis. To this end we will construct special orthonormal basis in $L_2([-1,1],V)$ by the formulas [13]:

$$p_i^0(x) = D^{-1/2}(x) \cdot p_i^0(x), \quad p_i^0(x) = p_i^0(x)t^i, \quad d_{-i,1} = 1, \quad J_{k,i} = \int_{-1}^{1} \frac{\xi^k \, d\xi}{m^i(\xi)},$$

$$d_{k,i} = \begin{vmatrix} J_{0,i} & \cdots & J_{k,i} \\ \vdots & \ddots & \vdots \\ J_{k,i} & \cdots & J_{2k,i} \end{vmatrix}, \quad p_i^0(x) = \frac{1}{\sqrt{d_{-i-1,i}d_{ki}}} \begin{vmatrix} J_{0,i} & J_{1,i} & \cdots & J_{k,i} \\ \vdots & \ddots & \vdots \\ J_{k-1,i} & J_{k,i} & \cdots & J_{2k-1,i} \\ 1 & x & \cdots & x^k \end{vmatrix}, \quad (13)$$

$$i = 1, 2, \ldots, n, \quad k = 0, 1, 2, \ldots, \quad x \in [-1,1].$$

3.2. Orthoprojectors, spectral problem, and solution

Divide the space $L_2([-1,1],V)$ into two parts (two subspaces): $L_2^{(1)}([-1,1],V)$ and $L_2^{(2)}([-1,1],V)$ with following bases:

$$\{p_i^0(x)\}_{i=1,2,\ldots,n} \text{ in } L_2^{(1)}([-1,1],V),$$

$$\{p_i^k(x)\}_{i=1,2,\ldots,n; k=2,3,4,\ldots} \text{ in } L_2^{(2)}([-1,1],V).$$

Following [12], we introduce the operators of orthogonal projection $P_1: L_2([-1,1],V) \to L_2^{(1)}([-1,1],V)$ and $P_2: L_2([-1,1],V) \to L_2^{(2)}([-1,1],V)$:

$$P_1f(x) = (p_i^0(x),f(x))p_i^0(x) + (p_i^1(x),f(x))p_i^1(x),$$

$$P_2f(x) = \sum_{k=2}^{\infty} (p_i^k(x),f(x))p_i^k(x) = (I - P_1)f(x). \quad (14)$$
Hence integrand and right-hand side of operator equation (12) can be represented as algebraic sum of terms from $L^2_1([-1,1],V)$ and $L^2_2([-1,1],V)$, i.e.

$$Q(x,t) = Q_1(x,t) + Q_2(x,t), \quad \frac{\delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x)}{\sqrt{m(x)}} = \Delta_1(x,t) + \Delta_2(x,t), \quad (15)$$

where $Q_1(x,t), \Delta_1(x,t) \in L^2_1([-1,1],V), Q_2(x,t), \Delta_2(x,t) \in L^2_2([-1,1],V)$, and

$$\Delta_1(x,t) = \mathbf{P}_1 \left\{ \mathbf{D}(x) \cdot [\delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x)] \right\} = \Delta_0^i(t)p_0^i(x) + \Delta_1^i(t)p_1^i(x),$$

$$\Delta_2(x,t) = \mathbf{P}_2 \left\{ \mathbf{D}(x) \cdot [\delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x)] \right\} = \sum_{m=2}^{\infty} \Delta_m^i(t)p_m^i(x),$$

$$\Delta_0^i(t) = \sqrt{J_{0,i}}\delta^i(t) + \frac{1}{\sqrt{J_{0,i}}}\alpha^i(t) - \tilde{g}_0^i\tilde{c}(t), \quad \Delta_1^i(t) = \sqrt{J_{0,i}J_{2,i} - J_{1,i}^2}\alpha^i(t) - \tilde{g}_1^i\tilde{c}(t),$$

$$\Delta_m^i(t) = -\tilde{g}_m^i\tilde{c}(t), \quad i = 1,2,\ldots,n, \quad m = 2,3,4,\ldots$$

To calculate coefficients $\tilde{g}_m^i$ we should represent kernel $K(x,\xi)$ as

$$K(x,\xi) = \sum_{m=0}^{\infty} K_{m,m}^i p_m^i(x)p_i^i(\xi),$$

$$K_{m,m}^i = \left(\left(K(x,\xi), p_i^i(\xi)\right), p_m^i(x)\right) = \int_{-1}^{1} \int_{-1}^{1} \frac{k_{ij}(x,\xi)p_m^i(x)p_i^0(\xi)}{m^i(x)m^j(\xi)} \, dx \, d\xi.$$

and use relations (10), (11), and (13). Then:

$$\tilde{g}_m^i = (\mathbf{D}^{-1}(x)\tilde{g}(x), p_m^i(x)) = \sum_{l=0}^{\infty} K_{m,l}^{ij} \int_{-1}^{1} \frac{p_l^0(x)g^i(x)}{m^i(x)} \, dx, \quad m = 0,1,2,\ldots, \quad i = 1,2,\ldots,n.$$

Function $Q_1(x,t)$ can be represented as

$$Q_1(x,t) = \tilde{z}_0^i(t)p_0^i(x) + \tilde{z}_1^i(t)p_1^i(x),$$

(19)

where $\tilde{z}_0^i(t)$ and $\tilde{z}_1^i(t)$ are unknown functional coefficients. They can be determined from the additional conditions (12). To this end substitute the representation (15) for $Q(x,t)$ into left-hand side of the additional conditions (12) in components with known forces and moments and use (13) and (19):

$$\int_{-1}^{1} \frac{Q^i(\xi,t)}{m^i(\xi)} \, d\xi = \tilde{z}_0^i(t)\sqrt{J_{0,i}},$$

$$\int_{-1}^{1} \frac{Q^i(\xi,t)}{m^i(\xi)} \, d\xi = \tilde{z}_0^i(t)\frac{J_{1,i}}{J_{0,i}} + \tilde{z}_1^i(t)\sqrt{\frac{J_{0,i}J_{2,i} - J_{1,i}^2}{J_{0,i}}}.$$

Equating derived expressions to the right-hand sides of additional conditions we obtain the expressions for $\tilde{z}_0^i(t)$ and $\tilde{z}_1^i(t)$:

$$\tilde{z}_0^i(t) = \frac{\tilde{P}_i(t)}{\sqrt{J_{0,i}}}, \quad \tilde{z}_1^i(t) = \frac{\tilde{Q}_i(t) + J_{1,i}\tilde{M}_i(t)}{\sqrt{J_{0,i}(J_{0,i}J_{2,i} - J_{1,i}^2)}}.$$
The formula (12) for $Q(x,t)$ contains known term $Q_1(x,t) \in L_2^1([-1,1],V)$ and the term $Q_2(x,t) \in L_2^2([-1,1],V)$ must be found. For the right-hand side, one should find $\Delta_1(x,t) \in L_2^1([-1,1],V)$, while $\Delta_2(x,t) \in L_2^2([-1,1],V)$ is known. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [12].

We apply the orthogonal projection operator $P_2$ to operator equation (12). As a result, we obtain the equation for determining $Q_2(x,t)$ with a known right-hand side

$$c(t)(I - V^1)Q_2(x,t) + (I - V^2)P_2GQ_2(x,t) = -(I - V^2)P_2GQ_1(x,t) + \Delta_2(x,t). \quad (21)$$

It is necessary to construct its solution in the form of a series in the eigenfunctions of the operator $P_2G$, which, as one can show, is a compact strongly positive self-adjoint operator $L_2^2([-1,1],V) \rightarrow L_2^2([-1,1],V)$. The system of eigenfunctions of such an operator is a basis in the space $L_2^2([-1,1],V)$. The spectral problem for the operator $P_2G$ can be written in the form

$$P_2G\varphi_k(x) = \gamma_k \varphi_k(x),$$

where $\varphi_k(x)$ is linearly combination of basis functions (13) in $L_2([-1,1],V)$, i.e.

$$\varphi_k(x) = \sum_{m=2}^{\infty} \psi_{km}^i P_m(x), \quad k = 2, 3, 4, \ldots \quad (22)$$

Using representations (11) and (14) for operators $G$ and $P_2$ and (17) for kernel decomposition we can obtain spectral problem about coefficients $\gamma_k$ and $\psi_{km}^i$:

$$\sum_{l=2}^{\infty} K_{ml}^{ij} \psi_{kl}^i = \gamma_k \psi_{km}^i, \quad i = 1, 2, \ldots, n, \quad k, m = 2, 3, 4, \ldots$$

We expand the functions $Q_2(x,t)$, $P_2GQ_1(x,t)$, and $\Delta_2(x,t)$ with respect to the new basis functions $\varphi_k(x)$ ($k = 2, 3, 4, \ldots$) in $L_2^2([-1,1],V)$, i.e.,

$$Q_2(x,t) = \sum_{k=2}^{\infty} z_k(t) \varphi_k(x), \quad P_2GQ_1(x,t) = \sum_{k=2}^{\infty} \sigma_k(t) \varphi_k(x), \quad \Delta_2(x,t) = \sum_{k=2}^{\infty} \Delta_k(t) \varphi_k(x), \quad (23)$$

where coefficients $\sigma_k(t)$ and $\Delta_k(t)$ defined by (we use formulas (11), (14), (16), (17), (19), and (22))

$$\sigma_k(t) = (P_2GQ_1(x,t), \varphi_k(x)) = \sum_{m=2}^{\infty} K_{m0}^{ij} \psi_{km}^i z_0(t) + \sum_{m=2}^{\infty} K_{m1}^{ij} \psi_{km}^i z_1(t),$$

$$\Delta_k(t) = (\Delta_2(x,t), \varphi_k(x)) = \sum_{m=2}^{\infty} \Delta_{km}^i(t) \psi_{km}^i.$$ 

Considering the above, the equation (21) is represented as

$$\sum_{k=2}^{\infty} [c(t)(I - V^1)z_k(t) + (I - V^2)\gamma_kz_k(t)] \varphi_k(x) = \sum_{k=2}^{\infty} [- (I - V^2)\sigma_k(t) + \Delta_k(t)] \varphi_k(x).$$
We can find functions $z_k(t)$ from this equation:

$$z_k(t) = (I + W_k)\frac{-(I - V^2)\sigma_k(t) + \Delta_k(t)}{c(t) + \gamma_k}, \quad W_k f(t) = \int_1^t R_k(t, \tau) f(\tau) d\tau,$$

where $R_k(t, \tau)$ is is the resolvent of the kernel

$$K_k^*(t, \tau) = \frac{c(t)K_1(t, \tau) + \gamma_k K_2(t, \tau)}{c(t) + \gamma_k}.$$

Note that the final expression (according to formulas (8), (11), (15), (19), and (23)) for contact pressure vector-function has the following structure

$$q(x, t) = D^{-1}(x)\left[z_0^0(t)P_0^0(x) + z_1^1(t)P_1^0(x) + \sum_{k=2}^{\infty} z_k(t)\varphi_k^k(x)\right] - (I - V^1)^{-1}\frac{1}{c(t)}D^{-1}(x)\cdot g(x),$$

where $\varphi_k^k(x) = D^{1/2}, \varphi_k(x)$ are polynomials. It means that there is a “good” function in square brackets and one can explicitly write out the weight functions $D(x)$ and $g(x)$ in the solution. Note that the coating thickness is related to $D(x)$ and backlash function is related to $g(x)$ (see (5)). This formula permit obtaining efficient analytic solutions for the cases when forms of contact surfaces described by complicated and rapidly changing functions. Such a result can hardly be done by other known methods.

Knowing the stresses we can find settlements and tilt angles under the punches from equation (12). To this end we apply the orthogonal projection operator $P_1$ to operator equation (12). As a result, we obtain the equation for determining $\delta^i(t)$ and $\alpha^i(t)$:

$$c(t)(I - V^1)Q_1(x, t) + (I - V^2)P_1 GQ(x, t) = \Delta_1(t).$$

Using representations (11), (14)–(17), (19), (22), (23) we can obtain following formulas:

$$\alpha^i(t) = \sqrt{\frac{J_0}{J_{0,i}}} - \frac{J_{0,i}}{J_{0,i}} - \frac{\tilde{g}_0^1 c(t) + (I - V^1)c(t)z_1^1(t)}{J_{0,i}},$$

$$\delta^i(t) = \frac{1}{\sqrt{J_{0,i}}} - \frac{J_{0,i}}{J_{0,i}} - \frac{\tilde{g}_0^1 c(t) + (I - V^1)c(t)z_0^1(t)}{J_{0,i}},$$

Main Results and Conclusions

- Plane contact problem for base with coating and arbitrary system of punches is posed. The corresponding mathematical model is given and analyzed. Possible variants of the problem statement are formulated.

- The analytical solution for one version of statement of this problem is obtained. In relation of contact stresses forms of contact surfaces are represented by separate terms and factors. It is allows one to perform effective computations for actual shapes.

- Analytical representation of the solution allow one to analyze carefully the behavior of the punches on the layer, taking into account the complex shapes of the contacting surfaces and the mutual influence of the punches.

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