The Carleman-contraction method to solve quasi-linear elliptic equations

Loc H. Nguyen

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 9201 University City Blvd, Charlotte, 28223, NC, USA.

Corresponding author(s). E-mail(s): loc.nguyen@uncc.edu;

Abstract

We propose a globally convergent numerical method to compute solutions to a general class of quasi-linear PDEs. Combining the quasi-reversibility method and a suitable Carleman weight function, we define a map of which fixed point is the solution to the PDE under consideration. To find this fixed point, we define a recursive sequence with an arbitrary initial term using the same manner as in the proof of the contraction principle. Applying a Carleman estimate, we show that the sequence above converges to the desired solution. On the other hand, we also show that our method delivers reliable solutions even when the given data are noisy. Numerical examples are presented.

Keywords: Carleman estimate, the contraction principle, quasi-linear PDEs, Cauchy data

MSC Classification: 35N25, 35N10, 65N12

Dedicated to the 70th anniversary of Professor Duong Minh Duc.

1 Introduction

Recently, the author and his collaborators numerically solved several inverse problems by a unified framework, see e.g., [7, 8, 9, 15, 19, 24, 30, 33, 35]. This framework has two steps. In step 1, we introduce a change of variable to reduce the given inverse problem to a system of quasi-linear PDEs with
Cauchy boundary data. In step 2, we solve that system. The obtained solution yields the solution to the inverse problem under consideration. For example, we refer the reader to [25, Section 6] for the reduction of the well-known inverse scattering problem in the frequency domain to a system of PDEs with Cauchy boundary data. We also cite [15, 23] for the use of this framework in solving the inverse scattering problem in the time domain with experimental data. The main aim of this paper is to develop a numerical method for step 2. For simplicity, rather than solving a system of quasi-linear elliptic PDEs with Cauchy boundary data, we solve a single equation. This simplification does not weaken the paper because our analysis and numerical implementation can be directly extended for systems of quasi-linear equations.

Our proposed numerical method to solve quasi-linear elliptic equations with Cauchy data in this paper has two important features: fast and global. By “fast”, we mean that the method converges at the exponential rate with respect to the number of iterations. By “global”, we mean that our method does not require an initial guess of the true solution to the problem under consideration. Both features are the important strengths of this paper since it is well-known that the widely used optimization-based methods to solving nonlinear equations are local and time consuming.

Let $\Omega$ be an open and bounded domain in $\mathbb{R}^d$, $d \geq 2$. Assume that $\partial \Omega$ is smooth. Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a real valued function in the class $C^2$. Let $A$ be a matrix valued function $\Omega \to \mathbb{R}^{d \times d}$ satisfying

1. $A$ is in the class $C^2(\overline{\Omega}, \mathbb{R}^{d \times d})$,
2. $A$ is symmetric; i.e. $A^T = A$,
3. there is a positive constant $\Lambda$ such that

$$\Lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda^{-1} |\xi|^2$$

for all $x \in \overline{\Omega}, \xi \in \mathbb{R}^d$.

Consider the over-determined boundary value problem

$$\begin{cases}
\text{Div}(A(x) \nabla u(x)) + F(x, u(x), \nabla u(x)) = 0 & x \in \Omega, \\
u(x) = f(x) & x \in \partial \Omega, \\
A(x)\nabla u(x) \cdot \nu(x) = g(x) & x \in \partial \Omega
\end{cases}$$

(1)

where $f$ and $g$ are two given functions. Since (1) involves both Dirichlet and Neumann conditions, (1) might not have a solution. For example, in the circumstance that

$$\begin{cases}
\text{Div}(A(x) \nabla u(x)) + F(x, u(x), \nabla u(x)) = 0 & x \in \Omega, \\
u(x) = f(x) & x \in \partial \Omega,
\end{cases}$$

has a unique solution, small noise contained in $g$ makes (1) inconsistent. Therefore, computing the solution to (1) might be impossible. However, assuming that (1) with noiseless data has a unique smooth solution $u^*$ and given noisy data, we approximate $u^*$. In other words, the main goal of this paper to solve the following problem.
Problem 1 Let $f^*$ and $g^*$ be the noiseless versions of $f$ and $g$ respectively. Assume that problem (1), with $f$ and $g$ replaced by $f^*$ and $g^*$ respectively, has a unique solution $u^*$ with $\|u^*\|_{C^1(\Omega)} < M$ for some $M > 0$. Given the noisy data $f$ and $g$, compute an approximation of $u^*$.

Problem 1 is interesting and challenging partly because our target is to compute $u^*$ when the noisy data $f$ and $g$ are given while the corresponding noiseless ones $f^*$ and $g^*$ are unknown. A natural approach to compute solution to (1) is to minimize a least squares functional. A typical example of such a functional is

$$H^p(\Omega) \ni u \mapsto J(u) = \int_{\Omega} \left| \text{Div}(A(x)\nabla u(x)) + F(x,u(x),\nabla u(x)) \right|^2 dx$$

subject to the boundary conditions in (1). One takes the minimizer of the functional $J$ in (2) as a solution to Problem 1. This approach is based on optimization. It has three main drawbacks:

1. $J$ might be nonconvex. It might have multiple local minimizers. Therefore, a good initial guess of the true solution $u^*$ is required.
2. The computational cost is expensive.
3. It is not clear that the minimizer is an approximation of $u^*$.

Recently, we have developed the convexification method to solve Problem 1. The key point of the convexification method is to include suitable Carleman weight functions into the formulation of the mismatch functional $J$. By using Carleman estimates, one can prove that the mismatch functional including the Carleman weight function is strictly convex. One also can prove that the unique minimizer is a good approximation of $u^*$. Hence, drawbacks 1 and 3 can be overcome. The convexification method was first introduced in [13] and then was developed intensively. We refer the reader to [12, 10, 11, 1, 14, 17, 9, 18, 15, 25, 16] for some important works in this area and their applications to solve a variety kinds of inverse problems. However, the computation due to the convexification method is time consuming. We have introduced in [27] another method, also based on Carleman estimates, to solve Problem 1. The method in [27] is inspired by Carleman estimates and linearization that is similar to the Newton method. We have showed in [27] that the combination of Carleman estimates and linearization allows us to compute $u^*$ quickly without requesting a good initial guess.

The contribution of this paper is to introduce another globally convergence method, which is based on a Carleman estimate and the classical contraction principle. More precisely, our approach is to first define a map $\Phi$ such that the desired solution is the fixed point of this map. The construction of $\Phi$ is a combination of the Carleman weight function and the quasi-reversibility method to solve over-determined linear PDEs (see [21] for the original work.
for the quasi-reversibility method). By using a suitable Carleman estimate, we rigorously prove that $\Phi$ is a contraction map. This leads to a numerical method to solve Problem 1. We simply approximate the desired solution by $u_n = \Phi^n(u_0)$ where $\Phi^n = \Phi \circ \Phi \cdots \circ \Phi$ ($n$ times) and $u_0$ is an arbitrary function. The main theorems in this paper confirm that our function $\Phi$ is a contraction map and that the sequence $\{u_n\}_{n \geq 0}$ above converges to the true solution. The stability with respect to the noise contained in the given data is of the Lipschitz type. We also refer to [2, 3, 24] for similar works for the case when the data has no noise and refer to [23, 31] for the proof of a similar result for hyperbolic equations. The strengths of our new approach include the facts that

1. it does not require a good initial guess;
2. it is quite general in the sense that no special structure is imposed on the nonlinearity $F$;
3. the convergence rate is $O(\theta^n)$ where $\theta \in (0, 1)$ and $n$ is the number of iterations.

The paper is organized as follows. In Section 2, we introduce the map $\Phi$ mentioned in the previous paragraph and show that $\Phi$ is a contraction map. In Section 3, we show that the fixed point of $\Phi$ is an approximation of solution to Problem 1. We also study the convergence of this approximation as the noise in the boundary data tends to zero. Section 4 is for the numerical study. Section 5 is for some concluding remarks.

2 The Carleman-contraction principle

In this section, we combine Carleman estimates and the classical contraction principle to develop a numerical method to solve (1).

2.1 A Carleman estimate

In this section, we present a simple form of Carleman estimates. Carleman estimates are great tools in the study of PDEs. They were first used to prove the unique continuation principle, see e.g., [6, 34]. The use of Carleman estimates quickly became a powerful tool in many areas of PDEs, especially in both theoretical and numerical methods for inverse problems, see e.g., [5, 4, 16, 7, 9, 19, 26, 29]. Carleman estimates were used in cloaking [28] and in the area of computing solution to Hamilton-Jacobi equations [20, 27]. We recall a useful Carleman estimate which is important for us in the proof of the main theorem in this paper. Let $x_0$ be a point in $\mathbb{R}^d \setminus \Omega$ such that $r(x) = |x - x_0| > 1$ for all $x \in \Omega$. For each $\beta > 0$, define

$$\mu_\beta(x) = r^{-\beta}(x) = |x - x_0|^{-\beta} \quad \text{for all } x \in \Omega. \quad (3)$$

We have the following lemma.
Lemma 1 (Carleman estimate) There exist positive constants $\beta_0, \lambda_0$ depending only on $x_0$, $\Omega$, $\Lambda$, and $d$ such that for all function $v \in C^2(\overline{\Omega})$ satisfying
\[ v(x) = \partial_n v(x) = 0 \quad \text{for all } x \in \partial \Omega, \tag{4} \]
the following estimate holds true
\[ \int_{\Omega} e^{2\lambda_0(x)} |\text{Div}(A \nabla v)|^2 \, dx \geq C \int_{\Omega} e^{2\lambda_0(x)} \left| \nabla v(x) \right|^2 \, dx + C \lambda_0 \int_{\Omega} e^{2\lambda_0(x)} \left| v(x) \right|^2 \, dx \tag{5} \]
for all $\beta \geq \beta_0$ and $\lambda \geq \lambda_0$. Here, $C = C(x_0, \Omega, \Lambda, d, \beta)$ depends only on the listed parameters.

Lemma 1 is a direct consequence of [28, Lemma 5]. We refer the reader to [27, Lemma 2.1] for details of the proof. An alternative way to obtain (5), with another Carleman weight function, is to apply the Carleman estimate in [22, Chapter 4, Section 1, Lemma 3] for general parabolic operators. The arguments to obtain (5) using [22, Chapter 4, Section 1, Lemma 3] are similar to that in [26, Section 3] with the Laplacian replaced by the operator $\text{Div}(A \nabla \cdot)$.

We specially draw the reader’s attention to different forms of Carleman estimates for all three main kinds of differential operators (elliptic, parabolic and hyperbolic) and their applications in inverse problems and computational mathematics [4, 5, 16, 29]. It is worth mentioning that some Carleman estimates hold true for all functions $v$ satisfying $v|_{\partial \Omega} = 0$ and $\partial_n v|_{\Gamma} = 0$ where $\Gamma$ is a part of $\partial \Omega$, see e.g., [20, 32], which can be used to solve quasilinear elliptic PDEs partly given boundary data.

2.2 A fixed point theorem

Let $p > [d/2] + 2$ be a number such that $H^p(\Omega)$ can be continuously embedded into $C^2(\overline{\Omega})$. Recall that the function $u^*$ satisfies $\|u^*\|_{C^1(\overline{\Omega})} < M$ for a known number $M$. Hence, $u^*$ satisfies
\[
\begin{aligned}
\text{Div}(A(x) \nabla u(x)) + F_M(x, u(x), \nabla u(x)) &= 0 \quad x \in \Omega, \\
\ u(x) &= f^*(x) \quad x \in \partial \Omega, \\
\ A(x) \nabla u(x) \cdot \nu(x) &= g^*(x) \quad x \in \partial \Omega,
\end{aligned}
\tag{6}
\]

where $F_M = \chi_M F$ with $\chi_M \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{d \times d}, [0, \infty))$ being defined as
\[
\chi_M(x, s, p) = \begin{cases}
1 & (|s|^2 + |p|^2)^{1/2} \leq M, \\
(0, 1) & M < (|s|^2 + |p|^2)^{1/2} < 2M, \\
0 & (|s|^2 + |p|^2)^{1/2} \geq 2M.
\end{cases}
\tag{7}
\]

Since $F_M$ is smooth and has compact support, it is Lipschitz. That means, there is a number $C_{F,M} > 0$ depending only on $F$ and $M$ such that
\[
|F_M(x, s_1, p_1) - F_M(x, s_2, p_2)| \leq C_{F,M}(|s_1 - s_2| + |p_1 - p_2|) \tag{8}
\]
A numerical method for PDEs

for all \( x \in \Omega, s_1, s_2 \in \mathbb{R} \) and \( p_1, p_2 \in \mathbb{R}^{d \times d} \).

We assume that the true solution \( u^* \) of (1), with \( f \) and \( g \) replaced by \( f^* \) and \( g^* \) respectively, belongs to \( H^p(\Omega) \). Fix \( \lambda \) and \( \beta \) such that Lemma 1 and the Carleman estimate (5) hold true. Let \( H \) be the set of admissible solutions

\[
H = \{ \varphi \in H^p(\Omega) : \varphi|_{\partial \Omega} = f, A\nabla \varphi \cdot \nu|_{\partial \Omega} = g \}. \tag{9}
\]

Assume that \( H \neq \emptyset \). Define \( \Phi : H \to H \) as

\[
\Phi(u) = \arg\min_{\varphi \in H} J_u(\varphi)
\]

where

\[
J_u(\varphi) = \int_{\Omega} e^{2\lambda \mu(x)} |\text{Div}(A(x)\nabla \varphi(x)) + F_M(x, u(x), \nabla u(x))|^2 \, dx \\
+ \epsilon \| \varphi \|^2_{H^p(\Omega)} \tag{10}
\]

for all \( u \in H^p(\Omega) \) where \( \epsilon > 0 \) is the regularization parameter.

**Remark 1** (The well-definedness of \( \Phi \)) It is not hard to verify that the functional \( J_u \) has a unique minimizer \( \Phi(u) \in H \) for all function \( u \in H \). In fact, using the compact embedding theorem from \( H^p(\Omega) \) to \( H^2(\Omega) \), together with the trace theory, one can verify that \( H \) is weakly closed in \( H^p(\Omega) \) and \( J_u \) is weakly lower semicontinuous on \( H \). The presence of the regularization term implies that \( J_\varphi \) is coercive in the sense that \( \lim_{\| \varphi \| \to \infty} J_u(\varphi) = \infty \). Therefore, by a standard argument in analysis, we can conclude that \( J_u \) has a minimizer. The uniqueness of the minimizer is due to the strict convexity of \( J_u \).

**Remark 2** (The Carleman quasi-reversibility method) Fix a function \( u \in H \). Let \( \varphi = \Phi(u) \). Since \( \varphi \) is in \( H \) and it minimizes \( J_u \), roughly speaking, the function \( \varphi \) “almost” solves

\[
\begin{align*}
\text{Div}(A(x)\nabla \varphi(x)) + F_M(x, u(x), \nabla u(x)) &= 0 \quad x \in \Omega, \\
\varphi(x) &= f(x) \quad x \in \partial \Omega, \\
A(x)\nabla \varphi(x) \cdot \nu(x) &= g(x) \quad x \in \partial \Omega.
\end{align*} \tag{11}
\]

One might also say that \( \varphi \) is the least squares solution to (11). Due to the presence of the regularization term \( \epsilon \| \varphi \|^2_{H^p(\Omega)} \), we call \( \varphi \) the regularized solution to (11). The method to compute the regularized solution to the linear equation (11) by minimizing \( J_u \) is named as the Carleman quasi-reversibility method. This name is suggested by the presence of the Carleman weight function in the formula of \( J_u \) and by the quasi-reversibility method to solve linear PDEs with Cauchy data. See [21] for the original work on the quasi-reversibility method.

The contraction behavior of \( \Phi \) is confirmed by the following theorem and its consequence mentioned in Corollary 1.
Theorem 1 There is a number $C$ depending only on $x_0$, $\Omega$, $\Lambda$, $\beta$ and $d$ such that
\[
\|\Phi(u) - \Phi(v)\|_{H_{\lambda,\beta}(\Omega)}^2 + \frac{\epsilon}{\lambda} \|\Phi(u) - \Phi(v)\|_{H^p(\Omega)}^2 \leq \frac{C_{F,M}}{C_{\lambda}} \|u - v\|_{H_{\lambda,\beta}(\Omega)} \tag{12}
\]
for all $u, v \in H^p(\Omega)$. Here, $C_{F,M}$ is the constant in (8) and the Carleman weighted norm $H_{\lambda,\beta}$ is defined as
\[
\|\varphi\|_{H_{\lambda,\beta}} = \left( \int_{\Omega} e^{2\lambda\mu_\beta(x)} (|\varphi(x)|^2 + |\nabla\varphi(x)|^2) dx \right)^{1/2} \tag{13}
\]
for all $\varphi \in H^1(\Omega)$.

Corollary 1 Choose $\lambda \gg 1$ such that $\theta = \sqrt{\frac{C_{F,M}}{C_{\lambda}}} \in (0, 1)$. It follows from (12) that
\[
\|\Phi(u) - \Phi(v)\|_{H_{\lambda,\beta}(\Omega)} \leq \theta \|u - v\|_{H_{\lambda,\beta}(\Omega)}, \tag{14}
\]
which means $\Phi$ is a contraction map with respect to the norm $H_{\lambda,\beta}$. Although the norm $H_{\lambda,\beta}$ is not equivalent to the norm in $H^p(\Omega)$, we will prove that $\Phi$ has a fixed point in $H^p(\Omega)$. The convergence of the fixed point method due to the contraction behavior of $\Phi$ will be discussed in Section 3.

Remark 3 For each fixed $\lambda, \beta$, the norm $\|\cdot\|_{H_{\lambda,\beta}}$ is equivalent to the classical norm of $H^1(\Omega)$. Hence, any convergence with respect to the $H_{\lambda,\beta}$ norm implies that in $H^1(\Omega)$.

Proof of Theorem 1 Define the set of test functions
\[
H_0 = \{ \varphi \in H^p(\Omega) : \varphi|_{\partial \Omega} = 0, A\nabla \varphi \cdot v|_{\partial \Omega} = 0 \}. \tag{15}
\]
Recall the admissible set of solutions $H$ defined in (9). Take two arbitrary functions $u$ and $v$ in $H$. Let $u_1 = \Phi(u)$ and $v_1 = \Phi(v)$. Since $u_1$ is the minimizer of $J_u$ in $H$, by the variational principle, we have for all $h \in H_0$
\[
\left\langle e^{2\lambda\mu_\beta(x)} \left[ \text{Div}(A(x)\nabla u_1(x)) + F_M(x, u(x), \nabla u(x)) \right], \text{Div}(A(x)\nabla h(x)) \right\rangle_{L^2(\Omega)} + \epsilon \left\langle u_1(x), h(x) \right\rangle_{H^p(\Omega)} = 0. \tag{16}
\]
Similarly, for all $h \in H_0$,
\[
\left\langle e^{2\lambda\mu_\beta(x)} \left[ \text{Div}(A(x)\nabla v_1(x)) + F_M(x, v(x), \nabla v(x)) \right], \text{Div}(A(x)\nabla h(x)) \right\rangle_{L^2(\Omega)} + \epsilon \left\langle v_1(x), h(x) \right\rangle_{H^p(\Omega)} = 0. \tag{17}
\]
Combining (16) and (17), using the inequality $2ab \leq a^2 + b^2$ and taking the test function
\[
h = u_1 - v_1 \in H_0,
\]
we have
\[
\frac{1}{2} \int_{\Omega} e^{2\lambda\mu_\beta} |\text{Div}(A(x)\nabla h(x))|^2 dx + \epsilon \|h\|_{H^p(\Omega)}^2
\]
\[ \leq \frac{1}{2} \int_{\Omega} e^{2\lambda \beta} |F_M(x, u(x), \nabla u(x)) - F_M(x, v(x), \nabla v(x))|^2 \, dx. \]  

(18)

Using (8) and (18), we have

\[ \frac{1}{2} \int_{\Omega} e^{2\lambda \beta} |\text{Div}(A(x) \nabla h(x))|^2 \, dx + \epsilon \|h\|_{H^p(\Omega)}^2 \]

\[ \leq \frac{1}{2} C_{F,M} \int_{\Omega} e^{2\lambda \beta} (|u(x) - v(x)| + |\nabla u(x) - \nabla v(x)|)^2 \, dx. \]

(19)

Note that \( h|_{\partial \Omega} = 0 \) and \( A \nabla h \cdot \nu|_{\partial \Omega} = 0 \). We apply the Carleman estimate (5) for \( h \) to get

\[ \int_{\Omega} e^{2\lambda \beta(x)} |\text{Div}(A \nabla h)|^2 \, dx \]

\[ \geq C \lambda \int_{\Omega} e^{2\lambda \beta(x)} |\nabla h(x)|^2 \, dx + C \lambda^3 \int_{\Omega} e^{2\lambda \beta(x)} |h(x)|^2 \, dx \]

(20)

where \( C = C(x_0, \Omega, \Lambda, d, \beta) > 0 \) depends only on the listed parameters. Combining (19) and (20), we have

\[ C \lambda \int_{\Omega} e^{2\lambda \beta(x)} |\nabla h(x)|^2 \, dx + C \lambda^3 \int_{\Omega} e^{2\lambda \beta(x)} |h(x)|^2 \, dx + \epsilon \|h\|_{H^p(\Omega)}^2 \]

\[ \leq C_{F,M} \int_{\Omega} e^{2\lambda \beta} (|u(x) - v(x)| + |\nabla u(x) - \nabla v(x)|)^2 \, dx. \]

(21)

Therefore,

\[ \int_{\Omega} e^{2\lambda \beta(x)} |\nabla u_1(x) - \nabla v_1(x)|^2 \, dx + |u_1(x) - v_1(x)|^2 \, dx + \frac{\epsilon}{\lambda} \|u_1 - v_1\|_{H^p(\Omega)}^2 \]

\[ \leq \frac{C_{F,M}}{C \lambda} \int_{\Omega} e^{2\lambda \beta} (|u(x) - v(x)|^2 + |\nabla u(x) - \nabla v(x)|^2) \, dx. \]

(22)

We have proved (12).

\[ \square \]

Corollary 1 and estimate (14) suggest that when \( \lambda \) is sufficiently large, the “fixed-point” like sequence \( \{u_n\}_{n \geq 0} \subset H \) defined as

\[
\begin{align*}
\begin{cases}
&u_0 \in H, \\
&u_n = \Phi(u_{n-1}) \quad n \geq 1,
\end{cases}
\end{align*}
\]

(23)

converges to a function \( \bar{u} \in H^1(\Omega) \) with respect to the norm \( \| \cdot \|_{H^1_{\lambda, \beta}} \). Two questions arise immediately if \( \bar{u} \) belongs to \( H^p(\Omega) \) and if \( \bar{u} \) is an approximation of the solution \( u^* \) to (1). The affirmative answers are given in the next section.

3 The convergence of the fixed point method

Recall that \( f \) and \( g \) are the noisy versions of the boundary data \( f^* \) and \( g^* \) respectively. Let \( \delta > 0 \) be the noise level. By noise, we mean that we assume

\[ \inf\{\|e\|_{H^p(\Omega)} : e \in E\} < \delta \]

(24)
where \( E = \{ e \in H^p(\Omega) : e|_{\partial\Omega} = f - f^*, A\nabla e \cdot \nu|_{\partial\Omega} = g - g^* \} \). Note that \( E \) is nonempty because \( u_0 - u^* \in E \). Due to (24), there exists a function \( e \in E \) such that
\[
\| e \|_{H^p(\Omega)} < 2\delta. \tag{25}
\]
By the continuous embedding from \( H^p(\Omega) \) to \( C^2(\Omega) \), we have
\[
\| e \|_{C^2(\Omega)} \leq C\delta \tag{26}
\]

**Remark 4** The existence of the “error” function \( e \) satisfying (25) and (26) implies that the differences \( f - f^* \) and \( g - g^* \) are smooth. That means the noise must be smooth. This smoothness condition is significant for the proof of the convergence theorem, see Theorem 2. In practice, one can smooth out the data by many existing methods; for e.g., by using the well-known spline curves or Tikhonov regularization. However, we can relax this step in the numerical study. That means our method is stronger than what we can prove. In our numerical study, we do not have to smooth out the noisy data. We directly compute the desired numerical solutions to (1) from the given raw noisy data
\[
f = f^*(1 + \delta \text{rand}) \quad \text{and} \quad g = g^*(1 + \delta \text{rand}) \tag{27}
\]
where \( \text{rand} \) is a function taking uniformly distributed random numbers in the range \([-1, 1]\).

We have the theorem.

**Theorem 2** Let \( \lambda \) and \( \beta \) be such that (5) holds true and the number \( \theta \) in Corollary 1 is in \((0, 1)\). Let \( \{ u_n \}_{n \geq 1} \subset H \) be the sequence defined in (23). The following statements hold true.

1. The sequence \( \{ u_n \}_{n \geq 1} \) converges in to a function \( u \in H \) with respect to the norm \( \| \cdot \|_{H_{\lambda, \beta}} \).
2. Let \( u^* \) be the solution to (1). Assume that \( \| u^* \|_{C^1(\Omega)} < M \). Then,
\[
\| u - u^* \|_{H_{\lambda, \beta}}^2 \leq C \left[ \int_\Omega e^{2\lambda\mu_\beta(x)} \left( |\text{Div} A(x)\nabla e(x)|^2 + |e(x)|^2 + |\nabla e(x)|^2 \right) dx + \epsilon \| e \|_{H^p(\Omega)}^2 + \epsilon \| u^* \|_{H^p(\Omega)}^2 \right] \tag{28}
\]
where \( C \) is a positive constant depending only on \( M, F, x_0, \Omega, \Lambda, \beta \) and \( d \).

Estimate (28) is interesting in the sense that it guarantees that \( u^* \) is an approximation of \( u^* \) as the noise level \( \delta \) and the regularization parameter \( \epsilon \) tends to 0. See the discussion in the Corollary below.

**Corollary 2** Estimate (28) affirmatively answers the question how the fixed point of \( \Phi \) approximate the true solution \( u^* \). In fact, when \( \lambda \) and \( \beta \) are fixed, using (25) and
(26) and Remark 3, we obtain the following estimate with respect to the noise level
\[ \| \overline{u} - u^* \|_{H^1(\Omega)}^2 \leq C \left[ \delta^2 + \epsilon \| u^* \|_{H^p(\Omega)}^2 \right]. \] (29)

Here, \( C \) is a constant depending on \( M, F, x_0, \Omega, d, \Lambda, \lambda \) and \( \beta \). If we choose \( \epsilon = O(\delta^2) \) as \( \delta \to 0 \) then, due to (29), our method is Lipschitz stable.

**Proof of Theorem 2** The convergence of \( \{ u_n \}_{n \geq 1} \) to a function \( \overline{u} \) with respect to the norm \( \| \cdot \|_{H^\lambda, \beta} \) is proved by the same argument as in the proof of the contraction principle. In fact, we can rewrite the definition of \( \{ u_n \}_{n \geq 1} \) in (23) as \( u_n = \Phi^n(u_0) \) where \( \Phi^n = \Phi \circ \cdots \circ \Phi \) (n times), \( n \geq 1 \). For \( n \geq m \), we have
\[
\| u_n - u_m \|_{H^\lambda, \beta} \leq\sum_{j=m+1}^{n} \| u_j - u_{j-1} \|_{H^\lambda, \beta} = \sum_{j=m+1}^{n} \| \Phi^{j-1} u_1 - \Phi^{j-1} u_0 \|_{H^\lambda, \beta}
\]
which goes to 0 as \( m \to \infty \) because \( \theta \in (0,1) \). We have used (14) in the estimate above. Consequently, \( \{ u_n \}_{n \geq 0} \) is a Cauchy sequence in \( H^1(\Omega) \) with respect to the norm \( \| \cdot \|_{H^\lambda, \beta} \). As a result, the sequence \( \{ u_n \}_{n \geq 0} \) converges to an element \( \overline{u} \) of \( H^1(\Omega) \).

We next prove that \( \overline{u} \) belongs to \( H \). We first show that \( \{ u_n \}_{n \geq 0} \) is bounded in \( H^p(\Omega) \). In fact, for all \( n \geq 1 \), since \( u_n \) is the minimizer of \( J_{u_{n-1}} \), we have
\[
\epsilon \| u_n \|_{H^p(\Omega)}^2 \leq J_{u_{n-1}}(u_n) \leq J_{u_{n-1}}(u_0)
\]
\[
= \int_\Omega e^{2\lambda \beta(x)} \left[ \| \text{Div}(A(x) \nabla u_0(x)) + F_M(x, u_{n-1}(x), \nabla u_{n-1}(x)) \|_{L^2(\Omega)}^2 + \epsilon \| u_0 \|_{H^p(\Omega)}^2 \right] dx
\]
\[
\leq 2 \int_\Omega e^{2\lambda \beta(x)} \left[ \| \text{Div}(A(x) \nabla u_0(x)) \|_{L^2(\Omega)}^2 + \| F_M \|_{L^\infty(\Omega)}^2 \right] dx + \epsilon \| u_0 \|_{H^p(\Omega)}^2,
\]
which is finite. Since \( H^p(\Omega) \) is a Hilbert space, \( \{ u_n \}_{n \geq 0} \) has a subsequence, still named as \( \{ u_n \}_{n \geq 0} \), converges weakly to \( \tilde{u} \) with respect to the norm \( H^p(\Omega) \) and strongly to \( \tilde{u} \) with respect to the norm \( H^1(\Omega) \). Since \( H \) is weakly closed in \( H^p(\Omega) \) (see Remark 1), \( \tilde{u} \in H \). Recall that \( \{ u_n \}_{n \geq 0} \) converges strongly to \( \overline{u} \) in \( H^1(\Omega) \) (see Remark 3). We conclude that \( \overline{u} = \tilde{u} \). As a result, \( \overline{u} \in H \).

We next prove the second part of the theorem. We employ the notation \( H_0 \) defined in (15). Fix \( n \geq 1 \), since \( u_n \) is the minimizer of \( J_{u_{n-1}} \) in \( H \), for all \( h \in H_0 \),
\[
\left\langle e^{2\lambda \beta(x)} \left[ \text{Div}(A(x) \nabla u_0(x)) + F_M(x, u_{n-1}(x), \nabla u_{n-1}(x)) \right], \right. 
\left. \text{Div}(A(x) \nabla h(x)) \right\rangle_{L^2(\Omega)} + \epsilon \left\langle u_n(x), h(x) \right\rangle_{H^p(\Omega)} = 0.
\] (30)

Since \( \| u^* \|_{C^1(\overline{\Omega})} < M \), the function \( u^* \) satisfies (6). Hence,
\[
\left\langle e^{2\lambda \beta(x)} \left[ \text{Div}(A(x) \nabla u^*(x)) + F_M(x, u^*(x), \nabla u^*(x)) \right], \text{Div}(A(x) \nabla h(x)) \right\rangle_{L^2(\Omega)} + \epsilon \left\langle u^*(x), h(x) \right\rangle_{H^p(\Omega)} = \epsilon \left\langle u^*(x), h(x) \right\rangle_{H^p(\Omega)}.
\] (31)

Combining (30) and (31), we have
\[
\left\langle e^{2\lambda(x)} [\text{Div}(A(x)\nabla(u_n(x) - u^*(x))) + F_M(x, u_{n-1}(x), \nabla u_{n-1}(x))
- F_M(x, u^*(x), \nabla u^*(x))] \right\rangle_{L^2(\Omega)} + \epsilon \left\langle u_n(x) - u^*(x), h(x) \right\rangle_{H^p(\Omega)}
= -\epsilon \left\langle u^*(x), h(x) \right\rangle_{H^p(\Omega)}. \tag{32}
\]

Recall \( e \) the function satisfying (25) and (26). Using the test function \( h_n = u_n - u^* - e \in H_0 \), or \( u_n - u^* = h_n + e \) in (32), we have
\[
\left\langle e^{2\lambda(x)} [\text{Div}(A(x)\nabla(h_n(x) + e(x))) + F_M(x, u_{n-1}(x), \nabla u_{n-1}(x))
- F_M(x, u^*(x), \nabla u^*(x))] \right\rangle_{L^2(\Omega)} + \epsilon \left\langle h_n(x) + e(x), h_n(x) \right\rangle_{H^p(\Omega)}
= -\epsilon \left\langle u^*(x), h_n(x) \right\rangle_{H^p(\Omega)}. \tag{34}
\]

It follows from (34) and the inequality \( 2|ab| \leq 4a^2 + b^2/4 \) that
\[
\int_{\Omega} e^{2\lambda(x)} |\text{Div}(A(x)\nabla h_n(x))|^2 \, dx + \epsilon \|h_n\|^2_{H^p(\Omega)}
\leq C \int_{\Omega} e^{2\lambda(x)} |\text{Div}(A(x)\nabla e(x))|^2 \, dx
+ C \int_{\Omega} e^{2\lambda(x)} |F_M(x, u_{n-1}(x), \nabla u_{n-1}(x)) - F_M(x, u^*(x), \nabla u^*(x))|^2 \, dx
+ C\epsilon\|e\|^2_{H^p(\Omega)} + C\epsilon\|u^*\|^2_{H^p(\Omega)}. \tag{35}
\]

Using (8), we can estimate the second integral in the right hand side of (37) as
\[
\int_{\Omega} e^{2\lambda(x)} \left[ |u_{n-1}(x) - u^*(x)|^2 + |\nabla(u_{n-1}(x) - u^*(x))|^2 \right] \, dx \tag{36}
\]

Combining (35) and (36), we get
\[
\int_{\Omega} e^{2\lambda(x)} |\text{Div}(A(x)\nabla h_n(x))|^2 \, dx + \epsilon \|h_n\|^2_{H^p(\Omega)}
\leq C \int_{\Omega} e^{2\lambda(x)} |\text{Div}(A(x)\nabla e(x))|^2 \, dx
+ C \int_{\Omega} e^{2\lambda(x)} \left[ |u_{n-1}(x) - u^*(x)|^2 + |\nabla(u_{n-1}(x) - u^*(x))|^2 \right] \, dx
+ C\epsilon\|e\|^2_{H^p(\Omega)} + C\epsilon\|u^*\|^2_{H^p(\Omega)}. \tag{37}
\]

Applying the Carleman estimate (5) for the function \( h_n \), we obtain
\[
\int_{\Omega} e^{2\lambda(x)} |\text{Div}(A\nabla h_n)|^2 \, dx
\geq C\lambda \int_{\Omega} e^{2\lambda(x)} |\nabla h_n(x)|^2 \, dx + C\lambda^3 \int_{\Omega} e^{2\lambda(x)} |h_n(x)|^2 \, dx. \tag{38}
\]

Combining (37) and (38) and recalling \( \lambda \gg 1 \), we have
A numerical method for PDEs

\[
\lambda \left[ \int_{\Omega} e^{2\lambda \mu(x)} |\nabla h_{n}(x)|^{2} \, dx + \int_{\Omega} e^{2\lambda \mu(x)} |h_{n}(x)|^{2} \, dx \right] \\
\leq C \int_{\Omega} e^{2\lambda \mu(x)} |\text{Div}(A(x)\nabla e(x))|^{2} \, dx \\
+ C \int_{\Omega} e^{2\lambda \mu(x)} \left[ |u_{n-1}(x) - u^{*}(x)|^{2} + |\nabla (u_{n-1}(x) - u^{*}(x))|^{2} \right] \, dx \\
+ C \epsilon \|e\|_{H^{p}(\Omega)}^{2} + C \epsilon \|u^{*}\|_{H^{p}(\Omega)}^{2}.
\] (39)

Let \( n \to \infty \) and recall that \( \{u_{n}\}_{n \geq 0} \) strongly converges to \( \overline{u} \) in \( H_{\lambda, \beta} \). Due to (33), we have

\[
\lambda \left[ \int_{\Omega} e^{2\lambda \mu(x)} \left( |\nabla (\overline{u}(x) - u^{*}(x) - e(x))|^{2} + |\overline{u}(x) - u^{*}(x) - e(x)|^{2} \right) \, dx \right] \\
\leq C \int_{\Omega} e^{2\lambda \mu(x)} |\text{Div}(A(x)\nabla e(x))|^{2} \, dx \\
+ C \int_{\Omega} e^{2\lambda \mu(x)} \left[ |\overline{u}(x) - u^{*}(x)|^{2} + |\nabla (\overline{u}(x) - u^{*}(x))|^{2} \right] \, dx \\
+ C \epsilon \|e\|_{H^{p}(\Omega)}^{2} + C \epsilon \|u^{*}\|_{H^{p}(\Omega)}^{2}.
\] (40)

Estimate (28) is a direct consequence of (40) and the inequality \((a-b)^{2} \geq \frac{1}{2} a^{2} - b^{2}\). □

4 Numerical study

For simplicity, we consider the case \( d = 2 \) and \( A \) is the identity matrix. The computational domain \( \Omega \) is chosen to be \((-1, 1)^{2}\). We solve (1) by the finite difference method. That means we compute the values of the solution \( u^{*} \) on the grid

\[
\mathcal{G} = \{(x_{i} = -1 + (i - 1)d_{x}, y_{j} = -1 + (j - 1)d_{x}) : 1 \leq i, j \leq N\}
\]

where \( d_{x} = \frac{2}{N-1} \) and \( N \) is a large integer. In our numerical study, \( N = 150 \).

Theorem 1 and Theorem 2 guarantee that \( u_{n} \), see (23) with \( n \) sufficiently large, is an approximation of \( u^{*} \). This suggests a procedure to compute \( u^{*} \). This procedure is written in Algorithm 1.

### Algorithm 1

The procedure to compute the numerical solution to (1)

1: Choose a regularization parameter \( \epsilon \) and a threshold number \( \kappa_{0} > 0 \).
2: Set \( n = 0 \). Choose an arbitrary initial solution \( u_{0} \in H \).
3: Compute \( u_{n+1} = \Phi(u_{n}) \) by minimizing \( J_{u_{n}} \) in \( H \).
4: if \( \|u_{n+1} - u_{n}\|_{L^{2}(\Omega)} > \kappa_{0} \) then
5: Replace \( n \) by \( n + 1 \).
6: Go back to Step 3.
7: else
8: Set the computed solution \( u_{\text{comp}} = u_{n+1} \).
9: end if
We choose $\epsilon$ and $\kappa_0$ in Step 1 manually by a trial and error process. That means, we take a reference test in which we know the true solution. Then, we choose $\epsilon$ and $\kappa_0$ such that Algorithm 1 delivers acceptable numerical solution with noiseless data; i.e. $\delta = 0$. Then, we use these parameters for all other tests and all other noise levels $\delta$. The reference test is test 1 below. In all of our numerical results, $\epsilon = 10^{-6}$ and $\kappa = 10^{-3}$. The Carleman weight function used in this section is $e^{\lambda|x-x_0|^{-\beta}}$ with $\lambda = 3$, $x_0 = (0, 9)$ and $\beta = 10$.

In Step 2 of Algorithm 1, we need to choose a function $u_0$ in $H$. A natural way to compute such a function is to solve the linear problem, obtained by removing from (1) the nonlinearity $F$, by the Carleman quasi-reversibility method, see Remark 2. We do not present the numerical implementation to solve linear PDEs using the Carleman quasi-reversibility method in this paper. The reader can find the details about this in [24, 30, 33].

In Step 3, we minimize $J_{u_n}$ in $H$. Similarly to the discussion in Remark 2, the obtained minimizer $u_{n+1}$ is actually the regularized solution to

\[
\begin{cases}
\Delta u_{n+1}(x) + F(x, u_n(x), \nabla u_n(x)) = 0 & x \in \Omega, \\
u_n + 1(x) = f(x) & x \in \partial \Omega, \\
\partial_\nu u_{n+1}(x) = g(x) & x \in \partial \Omega.
\end{cases}
\] (41)

The details in implementation to compute the regularized solution $u_{n+1}$ were presented in [24, 30, 33]. We do not repeat here. In our implementation of Step 3, we do not have to truncate the function $F$ using the cut off function $\chi_M$ as in (7). That means, we compute $u_{n+1}$ by directly solving (41) using the quasi-reversibility method mentioned in Remark 2.

We next display our numerical examples.

4.1 Test 1

In this test, we compute the solution to

\[
\Delta u(x) + u(x) + \sqrt{|\nabla u|^2 + 1} - \left[ -2\pi^2 \sin(\pi(x + y)) + \sin(\pi(x + y)) + \sqrt{\pi^2 \cos(\pi(x + y)) + 1} \right] = 0
\] (42)

for all $x = (x, y) \in \Omega$. The boundary data are given by

\[
u_n + 1(x) = \sin(\pi(x + y))(1 + \delta \text{rand}_1),
\] (43)

\[
\partial_\nu u(x) = \pi(\cos(\pi(x + y)), \cos(\pi(x + y))) \cdot \nu(1 + \delta \text{rand}_2)
\] (44)

for all $x = (x, y) \in \partial \Omega$, where $\delta > 0$ is the noise level and $\text{rand}_i$, $i = 1, 2$, is the function taking uniformly distributed random numbers in the rank $[-1, 1]$. The true solution to (42), (43) and (44) when $\delta = 0$ is $u^*(x) = \sin(\pi(x + y))$ for all $x = (x, y) \in \Omega$. The graphs of the true and computed solution as well as the relative $L^\infty$ error in computation are displayed in Figure 1.
Fig. 1: Test 1. The graphs of the true and computed solution to (42), (43) and (44) with noiseless and noisy boundary data.

It is evident from Figure 1 that the numerical solutions to (42), (43) and (44) are computed with a good accuracy. The relative errors in computation are in Table 1. On the other hand, one can observe from Figure 1f that our method converges fast after only 4 iterations.

Table 1: Test 1. The relative errors in computation.

4.2 Test 2

We consider a more complicated test with the nonlinearity $F(x, s, p)$ grows as $O(|p|^2)$ as $p \to \infty$ and is not convex with respect to $p$. We solve the equation

$$\Delta u(x) + u_x - u_y^2 - \left[ -2 + 2x - 16y^2 \right] = 0 \quad (45)$$

for all $x = (x, y) \in \Omega$. The boundary data are given by

$$u(x) = (x^2 - 2y^2)(1 + \delta \text{rand}_1), \quad (46)$$
\[
\partial_\nu u(x) = (2x, -4y) \cdot \nu(1 + \delta \text{rand}_2) \tag{47}
\]
for all \(x = (x, y) \in \partial \Omega\), where \(\delta > 0\) is the noise level and \(\text{rand}_i, i = 1, 2\), is the function taking uniformly distributed random numbers in the rank \([-1, 1]\). The true solution to (45), (46) and (47) when \(\delta = 0\) is \(u^*(x) = x^2 - 2y^2\) for all \(x = (x, y) \in \Omega\). The graphs of the true and computed solution as well as the relative \(L^\infty\) error in computation are displayed in Figure 2.

![Graphs of true and computed solutions](image)

**Fig. 2:** Test 2. The graphs of the true and computed solution to (45), (46) and (47) with noiseless and noisy boundary data.

Even this test is challenging, Algorithm 1 delivers out of expectation numerical solutions. The relative errors in computation are in Table 2. On the other hand, one can observe from Figure 2f that our method converges fast after only 7 iterations. The number of iterations in this test is greater than that in test 1 probably because the nonlinearity \(F\) in this test grows faster at \(|p| \to \infty\).

| Noise level | \(\|u^* - u^{\text{comp}}\|_{L^\infty(\Omega)}\) | \(\|u^* - u^{\text{comp}}\|_{L^2(\Omega)}\) |
|-------------|---------------------------------|---------------------------------|
| \(\delta = 0\%\) | 1.3727 \times 10^{-4} | 1.3134 \times 10^{-4} |
| \(\delta = 2\%\) | 0.0198 | 0.0130 |
| \(\delta = 5\%\) | 0.0702 | 0.0694 |
| \(\delta = 10\%\) | 0.1760 | 0.1721 |

**Table 2:** Test 2. The relative errors in computation
4.3 Test 3

In this test, we try the efficiency of Algorithm when the nonlinearity $F$ is not smooth. We solve the equation

$$
\Delta u(x) + |u_x(x)| - |u_y(x)| + 4\pi(\pi(x^2 + y^2) \sin(\pi(x^2 + y^2)) - \cos(\pi(x^2 + y^2)))
- 2\pi(|x \cos(\pi(x^2 + y^2))| - |y \cos(\pi(x^2 + y^2))|) = 0 \tag{48}
$$

for all $x = (x, y) \in \Omega$. The boundary data are given by

$$
u(x) = \sin(\pi(x^2 + y^2))(1 + \delta \text{rand}_1),$$

$$\partial_n \nu(x) = 2\pi(x \cos(\pi(x^2 + y^2)), y \cos(\pi(x^2 + y^2))) \cdot \nu(1 + \delta \text{rand}_2) \tag{50}
$$

for all $x = (x, y) \in \partial \Omega$, where $\delta > 0$ is the noise level and $\text{rand}_i$, $i = 1, 2$, is the function taking uniformly distributed random numbers in the rank $[-1, 1]$. The true solution to (48), (49) and (50) when $\delta = 0$ is $u^*(x) = \sin(\pi(x^2 + y^2))$ for all $x = (x, y) \in \Omega$. The graphs of the true and computed solution as well as the relative $L^\infty$ error in computation are displayed in Figure 3.

![Graphs of the true and computed solution](image)

**Fig. 3**: Test 3. The graphs of the true and computed solution to (48), (49) and (50) with noiseless and noisy boundary data.

Even this test is challenging, Algorithm 1 delivers out of expectation numerical solutions. The relative errors are compatible with the noise, which can be
found in Table 3. On the other hand, one can observe from Figure 3f that our method converges fast. The stopping criterion meets after only 4 iterations.

| Noise level | $\|u^* - u^{\text{comp}}\|_{L^\infty(\Omega)}$ | $\|u^* - u^{\text{comp}}\|_{L^2(\Omega)}$ |
|-------------|---------------------------------|---------------------------------|
| $\delta = 0\%$ | 0.0026                          | 0.0018                          |
| $\delta = 2\%$ | 0.0200                          | 0.0062                          |
| $\delta = 5\%$ | 0.0509                          | 0.0168                          |
| $\delta = 10\%$ | 0.0983                          | 0.0332                          |

Table 3: Test 3. The relative errors in computation

4.4 Test 4

We now test a more interesting problem when the nonlinearity $F(x, s, p)$ grows at the quadratic rate in $s$ and discontinuous with respect to $p$. Let

$$G(x, s, p) = \begin{cases} 
  s^2 - e^{p_2} & \text{if } e^{p_2} < 30, \\
  0 & \text{otherwise}
\end{cases}$$

for all $x \in \Omega$, $s \in \mathbb{R}$, $p = (p_1, p_2) \in \mathbb{R}^2$. We numerically solve the equation

$$\Delta u(x) + G(x, u(x), \nabla u(x)) - \left[ (\sin(4\pi x - 2\pi y^2) + y)^2 - e^{-4\pi y \cos(4\pi x - 2\pi y^2) + 1} \right] = 0 \quad (51)$$

for all $x = (x, y) \in \Omega$. The boundary data are given by

$$u(x) = (\sin(4\pi x - 2\pi y^2) + y)(1 + \delta \text{rand}_1), \quad (52)$$
$$\partial_\nu u(x) = 4\pi(\cos(4\pi x - 2\pi y^2), -y \cos(4\pi x - 2\pi y^2) + 1) \cdot \nu(1 + \delta \text{rand}_2) \quad (53)$$

for all $x = (x, y) \in \partial \Omega$, where $\delta > 0$ is the noise level and $\text{rand}_i$, $i = 1, 2$, is the function taking uniformly distributed random numbers in the rank $[-1, 1]$. The true solution to (51), (52) and (53) when $\delta = 0$ is $u^*(x) = \sin(4\pi x - 2\pi y^2) + y$ for all $x = (x, y) \in \Omega$. The graphs of the true and computed solution as well as the absolute error in computation are displayed in Figure 4.

Even when (51) involves a term that is not continuous with respect to $u_y$, Algorithm 1 delivers acceptable numerical solutions. The relative errors in computation are in Table 4. On the other hand, one can observe from Figure 4f that our method converges fast. The stopping criterion meets after only 10 iterations.

Remark 5 We use a Macbook Pro 6-Core Intel Core i7 (2.6 GHz) to compute the numerical solutions above. The computational time for tests 1, 2, 3, and 4 are about 5 seconds, 7 seconds, 5 seconds and 10 seconds respectively. The computational cost is not expensive.
(a) The true solution $u^*$

(b) The computed solution $u$ when $\delta = 0\%$

(c) The relative error $\frac{|u^* - u|}{\|u^*\|_{L^\infty(\Omega)}}$ when $\delta = 0\%$

(d) The computed solution $u$ when $\delta = 10\%$

(e) The relative error $\frac{|u^* - u|}{\|u^*\|_{L^\infty(\Omega)}}$ when $\delta = 10\%$

(f) The difference $|u_{n+1} - u_n|$  

**Fig. 4**: Test 4. The graphs of the true and computed solution to (51), (52) and (53) with noiseless and noisy boundary data.

| Noise level $\delta$ | $\|u^* - u_{\text{comp}}\|_{L^\infty(\Omega)}$ | $\|u^* - u_{\text{comp}}\|_{L^2(\Omega)}$ |
|----------------------|-----------------------------------------------|-----------------------------------------------|
| $\delta = 0\%$      | 0.0066                                        | 0.0055                                        |
| $\delta = 2\%$      | 0.0184                                        | 0.0104                                        |
| $\delta = 5\%$      | 0.0496                                        | 0.0179                                        |
| $\delta = 10\%$     | 0.0904                                        | 0.0325                                        |

**Table 4**: Test 4. The relative errors in computation

**Remark 6** The numerical method in Algorithm 1 is stronger than what we can prove in Theorem 1 and Theorem 2. It can deliver reliable solutions even when the nonlinearity is not smooth (see test 3) and even is not continuous (see test 4).

**5 Concluding remarks**

We have solved the problem of computing solutions to quasi-linear PDEs. Although this problem is nonlinear, we do not require a good initial guess of the true solution. We first define an operator $\Phi$ such that the true solution to the given quasilinear PDE is the fixed point of $\Phi$. We construct a recursive sequence $\{u_n\}_{n \geq 0}$ whose initial term $u_0$ can be taken arbitrary and the $n^{th}$ term $u_n = \Phi(u_{n-1})$. We next apply a Carleman estimate to prove the convergence of this sequence. Moreover, we have proved that the stability of our method with respect to noise is of the Lipschitz type. Some interesting numerical examples are presented.
Acknowledgement

This work was supported by US Army Research Laboratory and US Army Research Office grant W911NF-19-1-0044 and was supported, in part, by funds provided by the Faculty Research Grant program at UNC Charlotte, Fund No. 111272.

References

[1] A. B. Bakushinskii, M. V. Klibanov, and N. A. Koshev. Carleman weight functions for a globally convergent numerical method for ill-posed Cauchy problems for some quasilinear PDEs. *Nonlinear Anal. Real World Appl.*, 34:201–224, 2017.

[2] L. Baudouin, M. de Buhan, and S. Ervedoza. Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation. *SIAM J. Numer. Anal.*, 55:1578–1613, 2017.

[3] L. Baudouin, M. de Buhan, S. Ervedoza, and A. Osses. Carleman-based reconstruction algorithm for the waves. *SIAM Journal on Numerical Analysis*, 59(2):998–1039, 2021.

[4] L. Beilina and M. V. Klibanov. *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*. Springer, New York, 2012.

[5] A. L. Bukhgeim and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Soviet Math. Doklady*, 17:244–247, 1981.

[6] T. Carleman. Sur les systèmes linéaires aux dérivées partielles du premier ordre a deux variables. *C. R. Acad. Sci. Paris*, 197:471–474, 1933.

[7] V. A. Khoa, G. W. Bidney, M. V. Klibanov, L. H. Nguyen, L. Nguyen, A. Sullivan, and V. N. Astratov. Convexification and experimental data for a 3D inverse scattering problem with the moving point source. *Inverse Problems*, 36:085007, 2020.

[8] V. A. Khoa, G. W. Bidney, M. V. Klibanov, L. H. Nguyen, L. Nguyen, A. Sullivan, and V. N. Astratov. An inverse problem of a simultaneous reconstruction of the dielectric constant and conductivity from experimental backscattering data. *Inverse Problems in Science and Engineering*, 29(5):712–735, 2021.

[9] V. A. Khoa, M. V. Klibanov, and L. H. Nguyen. Convexification for a 3D inverse scattering problem with the moving point source. *SIAM J. Imaging Sci.*, 13(2):871–904, 2020.

[10] M. V. Klibanov. Global convexity in a three-dimensional inverse acoustic problem. *SIAM J. Math. Anal.*, 28:1371–1388, 1997.

[11] M. V. Klibanov. Global convexity in diffusion tomography. *Nonlinear World*, 4:247–265, 1997.

[12] M. V. Klibanov. Carleman weight functions for solving ill-posed Cauchy problems for quasilinear PDEs. *Inverse Problems*, 31:125007, 2015.

[13] M. V. Klibanov and O. V. Ioussoupova. Uniform strict convexity of a cost functional for three-dimensional inverse scattering problem. *SIAM
[14] M. V. Klibanov and A. E. Kolesov. Convexification of a 3-D coefficient inverse scattering problem. *Computers and Mathematics with Applications*, 77:1681–1702, 2019.

[15] M. V. Klibanov, T. T. Le, L. H. Nguyen, A. Sullivan, and L. Nguyen. Convexification-based globally convergent numerical method for a 1D coefficient inverse problem with experimental data. *to appear on Inverse Problems and Imaging, DOI: https://www.aimsciences.org/article/doi/10.3934/ipi.2021068*, 2021.

[16] M. V. Klibanov and J. Li. *Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data*. De Gruyter, 2021.

[17] M. V. Klibanov, J. Li, and W. Zhang. Convexification of electrical impedance tomography with restricted Dirichlet-to-Neumann map data. *Inverse Problems*, 35:035005, 2019.

[18] M. V. Klibanov, Z. Li, and W. Zhang. Convexification for the inversion of a time dependent wave front in a heterogeneous medium. *SIAM J. Appl. Math.*, 79:1722–1747, 2019.

[19] M. V. Klibanov and L. H. Nguyen. PDE-based numerical method for a limited angle X-ray tomography. *Inverse Problems*, 35:045009, 2019.

[20] M. V. Klibanov, L. H. Nguyen, and H. V. Tran. Numerical viscosity solutions to Hamilton-Jacobi equations via a Carleman estimate and the convexification method. *Journal of Computational Physics*, 451:110828, 2022.

[21] R. Lattès and J. L. Lions. *The Method of Quasireversibility: Applications to Partial Differential Equations*. Elsevier, New York, 1969.

[22] M. M. Lavrent’ev, V. G. Romanov, and S. P. Shishat-skii. *Ill-Posed Problems of Mathematical Physics and Analysis*. Translations of Mathematical Monographs. AMS, Providence: RI, 1986.

[23] T. T. Le, M. V. Klibanov, L. H. Nguyen, A. Sullivan, and L. Nguyen. Carleman contraction mapping for a 1D inverse scattering problem with experimental time-dependent data. *to appear in Inverse Problems, https://doi.org/10.1088/1361-6420/ac50b8, see also arXiv:2109.11098*, 2022.

[24] T. T. Le and L. H. Nguyen. A convergent numerical method to recover the initial condition of nonlinear parabolic equations from lateral Cauchy data. *Journal of Inverse and Ill-posed Problems, DOI: https://doi.org/10.1515/jiip-2020-0028*, 2020.

[25] T. T. Le and L. H. Nguyen. The gradient descent method for the convexification to solve boundary value problems of quasi-linear PDEs and a coefficient inverse problem. *preprint Arxiv:2103.04159*, 2021.

[26] T. T. Le, L. H. Nguyen, T-P. Nguyen, and W. Powell. The quasi-reversibility method to numerically solve an inverse source problem for hyperbolic equations. *Journal of Scientific Computing*, 87:90, 2021.

[27] T. T. Le, L. H. Nguyen, and H. V. Tran. A Carleman-based numerical
method for quasilinear elliptic equations with over-determined boundary data and applications. preprint arXiv:2108.07914, 2021.

[28] H. M. Nguyen and L. H. Nguyen. Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations. Transaction of the American Mathematical Society, 2:93–112, 2015.

[29] L. H. Nguyen. An inverse space-dependent source problem for hyperbolic equations and the Lipschitz-like convergence of the quasi-reversibility method. Inverse Problems, 35:035007, 2019.

[30] L. H. Nguyen. A new algorithm to determine the creation or depletion term of parabolic equations from boundary measurements. Computers and Mathematics with Applications, 80:2135–2149, 2020.

[31] L. H. Nguyen and M. V. Klibanov. Carleman estimates and the contraction principle for an inverse source problem for nonlinear hyperbolic equations. Inverse Problems, 38:035009, 2022.

[32] L. H. Nguyen, Q. Li, and M. V. Klibanov. A convergent numerical method for a multi-frequency inverse source problem in inhomogenous media. Inverse Problems and Imaging, 13:1067–1094, 2019.

[33] P. M. Nguyen and L. H. Nguyen. A numerical method for an inverse source problem for parabolic equations and its application to a coefficient inverse problem. Journal of Inverse and Ill-posed Problems, 38:232–339, 2020.

[34] M. H. Protter. Unique continuation for elliptic equations. Trans. Amer. Math. Soc., 95(1):81–91, 1960.

[35] A. V. Smirnov, M. V. Klibanov, and L. H. Nguyen. On an inverse source problem for the full radiative transfer equation with incomplete data. SIAM Journal on Scientific Computing, 41:B929–B952, 2019.