INFINITE SUMS OF ADDITIVE UNSTABLE ADAMS OPERATIONS AND COBORDISM

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ABSTRACT. The elements of the ring of bidegree $(0, 0)$ additive unstable operations in complex $K$-theory can be described explicitly as certain infinite sums of Adams operations. Here we show how to make sense of the same expressions for complex cobordism $MU$, thus identifying the “Adams subring” of the corresponding ring of cobordism operations. We prove that the Adams subring is the centre of the ring of bidegree $(0, 0)$ additive unstable cobordism operations.

For an odd prime $p$, the analogous result in the $p$-local split setting is also proved.

1. INTRODUCTION

In [6] an injective ring map was defined from the ring of stable degree zero operations in $p$-local $K$-theory to the corresponding ring of cobordism operations. The main theorem identified the image of this map as the centre of the target ring.

Here we show that that same thing happens in the additive unstable setting. For a cohomology theory $E$, we work with additive unstable bidegree $(0, 0)$ operations, that is natural transformations $E^0(-) \to E^0(-)$, where the functor $E^0(-)$ is viewed as taking values in abelian groups.

For $E = KU$, the complex $K$-theory spectrum, all such operations can be described in terms of Adams operations, where certain specified infinite sums of Adams operations are allowed. This goes back to work of Adams [2].

Since unstable Adams operations also exist for cobordism [10, 7], we can consider the corresponding expressions for $MU$. The same infinite sums converge and this allows us to define a ring map from the additive unstable bidegree $(0, 0)$ $K$-theory operations to the corresponding $MU$ operations.

The main work of this paper is devoted to showing that the image of this map is precisely the centre of the target. The methods are close to those used in the stable case, but suitably adapted to incorporate the Hopf ring techniques necessary in the unstable case. They exploit duality between operations and cooperations for $K$-theory and for cobordism and they rely on the fact that the operations under consideration are determined by their actions on homotopy groups. In one respect the additive unstable case is
simpler than the stable situation: we are able to produce integral results directly rather than by piecing together \( p \)-local results for each prime.

In the final section we prove the analogous result in the \( p \)-local split setting.

This paper is based on work in the Ph.D. thesis of the first author \[9\], produced under the supervision of the second author.

2. Action on homotopy groups

In this section we note the important fact that the operations we will be considering act faithfully on homotopy groups.

First we introduce some notation. Our main reference for background on cohomology operations is \[4\] and we adopt their notation and grading conventions. The cohomology theories we will be concerned with are complex \( K \)-theory \( KU \) and complex cobordism \( MU \). For an odd prime \( p \), we will also consider the Adams summand of \( p \)-local complex \( K \)-theory, which we denote by \( G \), and the Brown-Peterson theory \( BP \).

For a cohomology theory \( E \), we denote by \( E_0 \) the infinite loop spaces in an \( \Omega \)-spectrum representing \( E \). The unstable bidegree \((0,0)\) operations of \( E \)-theory are given by \( E^0(E_0) \cong [E_0, E_0] \). This is given the profinite topology and, as noted in \[4\], it is complete with respect to this topology for all of our examples. Inside here are the additive unstable bidegree \((0,0)\) operations \( P E^0(E_0) \), which we will denote simply by \( \mathcal{A}(E) \). Again, in all the theories we consider, \( \mathcal{A}(E) \) is complete with respect to the profinite filtration.

All our theories have good duality properties (see \[4\]). In particular, operations are dual to cooperations: we have an isomorphism of \( E^* \)-modules

\[
E^*(E_0) \cong \text{hom}_{E^*}(E_*(E_0), E^*).
\]

The right-hand side is given the dual-finite topology: we filter by

\[
\ker \left( \text{hom}_{E^*}(E_*(E_0), E^*) \to \text{hom}_{E^*}(L, E^*) \right),
\]

where \( L \) runs through finitely generated \( E^* \)-submodules of \( E_*(E_0) \). Then the above isomorphism is a homeomorphism with respect to the profinite topology on the left-hand side and the dual-finite topology on the right-hand side.

The additive operations \( \mathcal{A}(E) \) are dual to \( QE_*(E_0) \), the indecomposable quotient of the cooperations for the \( \ast \)-product:

\[
\mathcal{A}(E) = PE^*(E_0) \cong \text{hom}_{E^*}(QE_*(E_0), E^*).
\]

Let \( \text{Ab}_s \) denote the category of \( \mathbb{N} \)-graded abelian groups and degree zero morphisms of abelian groups. So \( \text{Ab}_s(M, N) \) denotes the degree zero homomorphisms between two graded abelian groups \( M \) and \( N \).

Given an unstable operation \( \theta \in E^0(E_0) \cong [E_0, E_0] \), we may consider the induced homomorphism of graded abelian groups \( \theta_* : \pi_*(E_0) \to \)
\[ \pi_*(E_0) \] given by the action of \( \theta \) on homotopy groups. Sending an operation to its action on homotopy groups in this way gives a homomorphism of rings
\[
E^0(E_0) \to \text{Ab}_*(\pi_*(E_0), \pi_*(E_0)) \\
\theta \mapsto \theta_*. 
\]

We will consider the restriction of this map to the additive \( E \)-operations \( \mathcal{A}(E) \) and denote this by \( \beta_E \):
\[
\beta_E : \mathcal{A}(E) \to \text{Ab}_*(\pi_*(E_0), \pi_*(E_0)) \\
\theta \mapsto \theta_*. 
\]

**Proposition 1.** For \( E = MU, BP, KU \) or \( G \), the map \( \beta_E : \mathcal{A}(E) \to \text{Ab}_*(\pi_*(E_0), \pi_*(E_0)) \) is injective.

**Proof.** As noted above, each of these theories has good duality, so any \( \theta \in \mathcal{A}(E) \) is uniquely determined by the corresponding \( E^* \)-linear functional \( \bar{\theta} : QE_*(E_0) \to E^* \).

As \( QE_*(E_0) \) \( \text{ and } E^* \) have no torsion, it is enough to show that \( \theta_* \) determines
\[
\bar{\theta} \otimes 1_Q : QE_*(E_0)_Q \to E^*_Q, 
\]
where we are writing \( M_Q \) for \( M \otimes \mathbb{Q} \).

By [4, 12.4], the action of an operation on homotopy is given in terms of the corresponding functional by
\[
\theta_*(t) = \bar{\theta}(e^{2h}\eta_R(t)) 
\]
for \( t \in E^{-2h} \). (Note that each of our theories \( E \) has coefficients \( E^* \) concentrated in even degrees.) Here \( \eta_R : E^* \to QE_*(E_0) \) is the right unit map and \( e \in QE_1(E_0) \) is the suspension element.

Now for each of our theories \( E \), every element of \( QE_*(E_0)_Q \) is an \( E^*_Q \)-linear combination of elements of the form \( e^{2h}\eta_R(t) \), where \( t \in E^{-2h} \). (See [4]; this may be proved by an inductive argument using the relations in \( QE_*(E_0) \).) It follows that \( \bar{\theta} \otimes 1_Q \) is completely determined by \( \theta_* \) as required. \( \square \)

### 3. Adams Operations

We begin by discussing the definition and properties of unstable Adams operations in \( K \)-theory and cobordism. We will denote our ground ring by \( R \). Thus \( R = \mathbb{Z} \) for \( E = MU \) or \( E = KU \) and \( R = \mathbb{Z}((p)) \) for \( E = BP \) or \( E = G \).

The unstable Adams operations in \( K \)-theory \( \psi_{KU}^k \), for \( k \in \mathbb{Z} \), were constructed in [11]. Unstable Adams operations for complex cobordism \( MU \) and for the Brown-Peterson theory \( BP \) were defined by Wilson [10] and...
also discussed by Kashiwabara [7]. These sources give us the following proposition.

**Proposition 2.** Let $E = MU$, $BP$, $KU$ or $G$. For $k \in R$, there is an unstable Adams operation $\Psi^k_E \in A(E)$ such that $(\Psi^k_E)_* : \pi_{2n}(E_0) \to \pi_{2n}(E_1)$ is multiplication by $k^n$. These operations satisfy $\Psi^k_E \Psi^l_E = \Psi^{kl}_E$. Furthermore $\Psi^k_E$ is multiplicative. □

For $KU$, all additive unstable bidegree $(0, 0)$ operations can be described in terms of Adams operations.

**Theorem 3.** [2] The topological ring $A(KU)$ may be identified with the collection of infinite sums $\{\sum_{n=0}^{\infty} a_n \sigma^k_{KU} \mid a_n \in \mathbb{Z}\}$, where

$$
\sigma^k_{KU} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \Psi^k_{KU}.
$$

These expressions are added termwise and multiplied using $\Psi^k_{KU} \Psi^l_{KU} = \Psi^{kl}_{KU}$. □

Explicit multiplication and comultiplication formulas for this topological basis, as well as further results, can be found in [9].

We now show that the corresponding infinite sums of Adams operations are also defined for $MU$. First we note some information about the Adams operations viewed as functionals on the cooperations.

**Lemma 4.** The Adams operation $\Psi^k_{MU}$ considered as a functional

$$
\Psi^k_{MU} : QMU_*(MU_0) \to MU^*
$$

is determined by

$$
e^{2h} \eta_R(x) \mapsto k^h x \quad \text{for} \; x \in MU^{-2h}.
$$

**Proof.** As noted in the proof of Proposition [1], $\Psi^k_{MU} \otimes 1_Q$ determines $\Psi^k_{MU}$ and it is enough to specify this $MU^*$-linear map on elements of the form $e^{2h} \eta_R(x)$ for $x \in MU^{-2h}$ since these generate $QMU_*(MU_0)_Q$ as a module over $MU^*_Q$.

That these values are as claimed follows from the relation

$$
\Psi^k_{MU_*(x)} = \Psi^k_{MU}(e^{2h} \eta_R(x))
$$

for $x \in MU^{-2h} \cong \pi_{2n}(MU_0)$ and the action of $\Psi^k_{MU}$ on homotopy groups. □

The following combinatorial lemma will be useful.

**Lemma 5.** For $m, n \geq 0$,

$$
\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^m = n! \left\{ \frac{m}{n} \right\},
$$

where $\{ \frac{m}{n} \}$ denotes the $n$th Eulerian number of the first kind.
where \( \left\{ \begin{array}{c} \frac{m}{n} \\
\end{array} \right\} \) denotes a Stirling number of the second kind. In particular,
\[
\sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} k^m = m!,
\]
\[
\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^m = 0 \quad \text{if } n > m.
\]

**Definition 6.** For \( n \in \mathbb{N} \), define \( \sigma^{MU}_{n} \in \mathcal{A}(MU) \) by
\[
\sigma^{MU}_{n} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \Psi^{k}_{MU}.
\]

**Proposition 7.** The infinite sums \( \sum_{n=0}^{\infty} a_n \sigma^{MU}_{n} \), where \( a_n \in \mathbb{Z} \), are well-defined operations in \( \mathcal{A}(MU) \).

**Proof.** By Proposition 2, we have the Adams operations \( \Psi^{k}_{MU} \in \mathcal{A}(MU) \). So clearly finite sums of the \( \sigma^{MU}_{n} \) are well-defined operations in \( \mathcal{A}(MU) \).

To see that the same is true for the infinite sums, it suffices by completeness to show that \( \sigma^{MU}_{n} \to 0 \) as \( n \to \infty \) in the profinite topology on \( \mathcal{A}(MU) \).

Now \( \mathcal{A}(MU) \) is homeomorphic to \( \text{hom}_{MU^*}(QMU_{*}(MU_{0}), MU^*) \) with the dual-finite topology. Since \( QMU_{*}(MU_{0}) \) and \( MU^* \) are torsion-free, we have an injective map
\[
\text{hom}_{MU^*}(QMU_{*}(MU_{0}), MU^*) \hookrightarrow \text{hom}_{MU_{Q}^*}(QMU_{*}(MU_{0})_{Q}, MU_{Q}^*),
\]
given by
\[
f \mapsto f \otimes 1_{Q}.
\]
This is a homeomorphism to its image, where the target is also endowed with the dual-finite topology. Thus it is enough to show that \( \sigma^{MU}_{n} \otimes 1_{Q} \to 0 \) as \( n \to \infty \).

Using Lemmas 4 and 5 for \( x \in MU^{-2h} \), we have
\[
\sigma^{MU}_{n} \left( e^{2h} \eta_{R}(x) \right) = \left( \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^{h} \right) x = n! \left\{ \frac{h}{n} \right\} x.
\]
Thus, \( \sigma^{MU}_{n} \left( e^{2h} \eta_{R}(x) \right) = 0 \) if \( h < n \) and \( \sigma^{MU}_{n} \otimes 1_{Q} \) is zero on the \( MU_{Q}^* \)-submodule of \( QMU_{*}(MU_{0})_{Q} \) generated by the finite collection of elements of the form \( e^{2h} \eta_{R}(x) \) where \( x \) runs through a \( \mathbb{Z} \)-basis of \( MU^{-2h} \) and \( h < n \).

Since \( QMU_{*}(MU_{0})_{Q} \) is generated as an \( MU_{Q}^* \)-module by \( e^{2h} \eta_{R}(x) \) where \( x \) runs through a \( \mathbb{Z} \)-basis of \( MU^{-2h} \) and \( h \geq 0 \), it follows that \( \sigma^{MU}_{n} \otimes 1_{Q} \to 0 \) as \( n \to \infty \) in the dual-finite topology.

**Proposition 8.** The map
\[
\iota: \mathcal{A}(KU) \to \mathcal{A}(MU)
\]
given by
\[ \sum_{n=0}^{\infty} a_n \sigma_n^{KU} \mapsto \sum_{n=0}^{\infty} a_n \sigma_n^{MU} \]
is an injective ring homomorphism.

**Proof.** Consider \( \sum_{n=0}^{\infty} a_n \sigma_n^{MU} = \iota \left( \sum_{n=0}^{\infty} a_n \sigma_n^{KU} \right) \) in \( \mathcal{A}(MU) \) and suppose \( a_m \neq 0 \) with \( m \) minimal. Since
\[ \sum_{n=0}^{\infty} a_n \sigma_n^{MU} = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \Psi_k^{MU}, \]
this operation acts on \( \pi_{2m}(MU_0) \neq 0 \) as multiplication by
\[ \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^m. \]

Since we have assumed that \( a_n = 0 \) for \( n < m \), it follows from Lemma 5 that \( \sum_{n=0}^{\infty} a_n \sigma_n^{MU} \) acts on \( \pi_{2m}(MU_0) \) as multiplication by \( a_m m! \neq 0 \). So \( \sum a_n \sigma_n^{MU} \) is a non-trivial operation in \( \mathcal{A}(MU) \) and therefore \( \iota \) is injective.

It is easy to see that we have an algebra map: the product of two infinite sums is determined in both the source and the target by the products of Adams operations. \( \square \)

We note that the injective map \( \iota \) above also respects the coalgebra structure that we have on each side, since the comultiplication on a general infinite sum is determined by the fact that the Adams operations are group-like.

We think of the image of \( \iota \) as the “Adams subring” of \( \mathcal{A}(MU) \). Our main result (Theorem 17) is that this is the centre of \( \mathcal{A}(MU) \). We can prove one inclusion immediately.

**Lemma 9.** The image Im(\( \iota \)) is contained in the centre \( Z(\mathcal{A}(MU)) \).

**Proof.** It is enough to show that the operations \( \sigma_n^{MU} \) commute with all elements of \( \mathcal{A}(MU) \). It is clear from the action of \( \Psi_k^{MU} \) on homotopy that \( \beta_{MU}(\Psi_k^{MU}) \) commutes with all elements of \( \text{Ab}_* (\pi_*(MU_0), \pi_*(MU_0)) \). So the same holds for \( \beta_{MU}(\sigma_n^{MU}) \). But by Proposition 11 \( \beta_{MU} \) is injective, so \( \sigma_n^{MU} \) commutes with all elements of \( \mathcal{A}(MU) \). \( \square \)

4. DIAGONAL OPERATIONS AND CONGRUENCES

**Definition 10.** Let \( E = MU \) or \( BP \). Write \( \mathcal{D}(E) \) for the subring of \( \mathcal{A}(E) \) consisting of operations whose action on each homotopy group \( \pi_{2n}(E_n) \) is multiplication by an element \( \lambda_n \) of the ground ring \( R \). We call elements of \( \mathcal{D}(E) \) unstable diagonal operations.

**Lemma 11.** Let \( E = MU \) or \( BP \). There is an inclusion \( Z(\mathcal{A}(E)) \subseteq \mathcal{D}(E) \).
Proof. We note that there is an injection $E^0(E) \hookrightarrow \mathcal{A}(E)$ from the stable degree zero operations to the additive unstable bidegree $(0,0)$ operations, given by sending a stable operation to its zero component. Indeed, this map fits into a commutative diagram

$$
\begin{array}{ccc}
E^0(E) & \longrightarrow & \mathcal{A}(E) \\
\alpha_E & & \beta_E \\
\text{Ab}_*(\pi_*(E_0), \pi_*(E_0)) & \downarrow & \\
\end{array}
$$

where $\alpha_E$ sends a stable operation to its action on $\pi_*(E_0) = \pi_*(E_0)$. But, as noted in [6], $\alpha_E$ is injective, so the map $E^0(E) \rightarrow \mathcal{A}(E)$ is injective.

In [6, Proposition 14] particular Landweber-Novikov operations were exploited in order to show that, for $E = MU$ and $E = BP$, a central stable operation has to act diagonally on homotopy. Using the above inclusion, we consider the images of these Landweber-Novikov operations and then exactly the same argument shows that commuting with these elements forces a central element of $\mathcal{A}(MU)$ or $\mathcal{A}(BP)$ to act diagonally on homotopy. \hfill \Box

We continue to study the injective ring homomorphism $\iota : \mathcal{A}(KU) \rightarrow \mathcal{A}(MU)$ of Proposition 8. By Lemmas 9 and 11, we have

$\text{Im}(\iota) \subseteq Z(\mathcal{A}(MU)) \subseteq \mathcal{D}(MU)$.

Our aim is to show that $\text{Im}(\iota) = \mathcal{D}(MU)$ and thus $\text{Im}(\iota) = Z(\mathcal{A}(MU))$. The strategy is to characterize $\mathcal{D}(MU)$ by a system of congruences and to compare this with a system of congruences governing the $K$-theory operations $\mathcal{A}(KU)$. For future use, we will also set up the corresponding congruences for $BP$ and the Adams summand $G$.

For $E = MU, BP, KU$ or $G$, the relevant congruences arise as follows. Let $\theta \in \mathcal{A}(E)$ be a diagonal operation, so that $\theta$ acts on $\pi_{2n}(E_0)$ as multiplication by an element $\lambda_n$ of the ground ring $R$. Of course, for $KU$ and $G$ all operations are diagonal. Consider the corresponding $E^*$-linear functional $\bar{\theta} : QE_*(E_0) \rightarrow E^*$. We get a set of congruences which must be satisfied by the $\lambda_n$, characterizing diagonal operations, arising from $\bar{\theta}(x) \in E^*$ for all $x \in QE_*(E_0)$. For all these theories, $QE_*(E_0)$ is free as an $E^*$-module and of course, we can let $x$ run through a basis.

**Definition 12.** We write $S_E$ for the subring of $\prod_{n=0}^{\infty} R$ consisting of sequences $(\lambda_n)_{n \geq 0}$ satisfying this system of congruences.

To be more explicit about these congruences we recall some further information about the Hopf rings of these theories. Let $x^E \in E^2(\mathbb{C}P^\infty)$ be a choice of complex orientation class, so that $E^*(\mathbb{C}P^\infty) = E^*[[x]]$. (Later it will be convenient to choose $x^{KU} = \varphi_*(x^{MU})$ and $x^G = \hat{\varphi}_*(x^{BP})$ where $\varphi : MU \rightarrow KU$ and $\hat{\varphi} : BP \rightarrow G$ are the standard maps of ring spectra.) For any space $X$, there is a coaction map

$$
\rho : E^k(X) \rightarrow E^*(X) \hat{\otimes} QE_*(E_0);
$$

where $\rho(E^k(X)) \subseteq E_*(X) \hat{\otimes} \pi_*(E_0)$ and $\pi_*(E_0)$ is the homotopy group of $E_0$. This coaction respects the congruences arising from $\bar{\theta}(x)$ and restricts to the usual coaction on $\mathcal{A}(KU)$. By the previous discussion, $\text{Im}(\iota) \subseteq Z(\mathcal{A}(MU))$. The proof of Theorem 12 will show that $\text{Im}(\iota) = Z(\mathcal{A}(MU))$.
see \cite{4} 6.26]. The standard elements $b_i^E \in QE_2(\underline{E}_2)$ are characterized by the property that

$$\rho(x^E) = \sum_{i=0}^{\infty} b_i^E (x^E)^i \in E^*(\mathbb{C}P^\infty) \otimes Q E_*(\underline{E}_2) = Q E_*(\underline{E}_2)[[x^E]].$$

Note that $b_0^E = 0$ and $b_1^E = e^2$, where $e$ is the suspension element.

For $E = KU$, a completely explicit formulation of the congruences, which can be found in \cite{5} Theorem 4], is

$$\sum_k (-1)^{n-k} \binom{n}{k} \lambda_k \in \mathbb{Z} \quad \text{for all } n \geq 0,$$

where the $\binom{n}{k}$ are Stirling numbers of the first kind. We will abbreviate this system of congruences to

$$C_n \cdot \lambda \in \mathbb{Z} \quad \text{for all } n \geq 0,$$

where we are adopting vector notation and

$$C_n = \left( \sum_k (-1)^{n-k} \binom{n}{k} \lambda_k \right),$$

$$\lambda = (\lambda_k)_{k \geq 0}.$$

The coefficient ring $KU^*$ is given by $KU^* = \mathbb{Z}[u, u^{-1}]$ where $|u| = -2$. As is usual, we write $v = \eta_R(u) \in Q KU_0(KU_{-2})$. One finds the congruences above by letting $x$ run through the $KU^*$-basis for $Q KU_*(KU_n)$ given by the elements 1 and $b_n^{KU} v$ for $n > 0$; see \cite{4} Theorem 16.15. In more detail, the periodicity of $K$-theory gives an isomorphism of $KU^*$-modules

$$Q KU_*(KU_n) \cong KU^* \otimes_{\mathbb{Z}} Q KU_0(KU_0)$$

and $Q KU_0(KU_n)$ may be identified with the ring of integer-valued polynomials

$$A = \{ f(w) \in \mathbb{Q}[w] \mid f(\mathbb{Z}) \subseteq \mathbb{Z} \};$$

see \cite{5} 8. The explicit congruences above arise from the binomial polynomial basis for the ring of integer-valued polynomials and the expansion of the binomial polynomials in terms of Stirling numbers. Another way of expressing the same thing is that $\pi_\lambda (b_n^{KU} v) = C_n \cdot \lambda$, where $\pi_\lambda : Q KU_*(KU_n) \to \mathbb{Z}$ is the map determined rationally by

$$u^a e^{2b} v^b \mapsto \lambda_b.$$

Examples 13. The first non-trivial congruences in this family are

$$\frac{\lambda_2 - \lambda_1}{2} \in \mathbb{Z},$$

$$\frac{\lambda_3 - 3\lambda_2 + 2\lambda_1}{6} \in \mathbb{Z},$$

$$\frac{\lambda_4 - 6\lambda_3 + 11\lambda_2 - 6\lambda_1}{24} \in \mathbb{Z}.$$
Since all operations in $A(KU)$ are diagonal and diagonal operations are precisely characterized by the congruences, it is immediate that we have an isomorphism of rings $A(KU) \cong S_{KU}$, given by sending an operation to its action on homotopy.

Next we give further details of the congruences characterizing $D(MU)$. Consider the restriction of the map

$$\beta_{MU} : A(MU) \to \text{Ab}_*(\pi_*(MU_0), \pi_*(MU_0))$$

to diagonal operations $D(MU)$. By the definition of $D(MU)$, this restriction may be viewed as a map

$$\beta_{MU|} : D(MU) \to \prod_{n=0}^{\infty} \mathbb{Z},$$

where we implicitly identify the homomorphism given by multiplication by an integer $\lambda$ on an abelian group with the integer $\lambda$.

Let $\theta \in D(MU)$ with $\theta$ acting on $\pi_{2n}(MU_0)$ as multiplication by $\lambda_n \in \mathbb{Z}$; that is, $\beta_{MU|}(\theta) = (\lambda_n)_{n \geq 0}$. We give some simple examples of the congruences satisfied by the $\lambda_n$ to illustrate how these arise.

The coefficient ring of $MU$ is given by $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, where $|x_i| = -2i$. Using [4, Theorem 16.9], the free $MU^*$-module $QM_{U_*}(MU_0)$ has generators $\langle b_{MU}^{\alpha} \eta_R(x) \rangle$, where $b_{MU}^{\alpha} = (b_{MU}^{\alpha_1})^{\alpha_2} \cdots$, for any finite sequence of non-negative integers $\langle \alpha_1, \alpha_2, \ldots \rangle$, and $x \in MU^{-2|\alpha|}$ where $|\alpha| = \sum_i \alpha_i$.

As noted in the proof of Proposition 11, the main relation in the Hopf ring shows that rationally $QM_{U_*}(MU_0)$ has $MU^*$-module generators $e^{2h \eta_R(x)}$.

**Examples 14.** Consider $b_{2}^{MU} \eta_R(x_1) \in QM_{U_*}(MU_0)$. We rewrite $b_{2}^{MU}$ rationally as $\frac{1}{2}(e^{4 \eta_R(x_1)} - x_1 e^{2})$. Then

$$\bar{\theta}(b_{2}^{MU} \eta_R(x_1)) = \frac{(\lambda_2 - \lambda_1)}{2} x_1^2 \in MU^*.$$ 

This gives the congruence $\frac{\lambda_2 - \lambda_1}{2} \in \mathbb{Z}$.

Now consider $b_{3}^{MU} \eta_R(x_1) \in QM_{U_*}(MU_0)$. The same procedure as above shows that

$$\bar{\theta}(b_{3}^{MU} \eta_R(x_1)) = \frac{(\lambda_3 - 3\lambda_2 + 2\lambda_1)}{6} x_1^3 + \frac{(\lambda_3 - \lambda_1)}{3} a_{2,1} x_1 \in MU^*,$$

where $a_{2,1} \in MU^{-4}$.

So this gives us the two congruences

$$\frac{\lambda_3 - 3\lambda_2 + 2\lambda_1}{6} \in \mathbb{Z} \quad \text{and} \quad \frac{\lambda_3 - \lambda_1}{3} \in \mathbb{Z}.$$ 

Notice that the three $MU$ congruences we have produced here are equivalent to the first two $K$-theory ones: the first two are the same and the third is redundant.
It follows from the definitions that we have an equality $\beta_{MU}(D(MU)) = S_{MU}$ and since $\beta_{MU}$ is an injective ring homomorphism, this gives an isomorphism of rings $D(MU) \cong S_{MU}$.

5. The Centre

Our goal is now to compare the solution sets $S_{KU}$ and $S_{MU}$ for the congruences coming from $K$-theory and from complex cobordism. We will show that they are equal and this will allow us to prove the main result about the centre of $A(MU)$.

One inclusion follows directly from the existence of the map $\iota: A(KU) \hookrightarrow A(MU)$.

**Proposition 15.** We have the inclusion $S_K \subseteq S_{MU}$.

**Proof.** Let $\lambda = (\lambda_n)_{n \geq 0} \in S_{KU} \subseteq \prod_{n=0}^{\infty} \mathbb{Z}$. Then $\lambda = \theta$ for some $\theta \in A(KU)$ and $\iota(\theta) \in A(MU)$, with $(\iota(\theta))_* = \theta_* = \lambda$. Hence, $\lambda \in S_{MU}$. □

To prove the reverse inclusion we consider the relationship between the Hopf rings for $MU$ and $KU$. The standard map of ring spectra $\varphi: MU \to K$ induces a map of Hopf rings $MU_* = KU_*$. Hence there is an induced ring map on indecomposables $QM U_* \to QKU_*$, which we denote by $\tilde{\phi}$. We now choose the orientation class $x_{KU}$ for $K$-theory to be $\varphi(x_{MU})$. With this choice it is routine to check that $\varphi(b^n_{KU}) = b^n_{MU}$.

Fixing $\theta \in D(MU)$, we consider the $MU$ congruences satisfied by $\theta_* = (\lambda_n)_{n \geq 0}$. It turns out, as the proof of the next proposition shows, that we obtain the $K$-theory congruences among these by considering the coefficient of $x_1$ in $\tilde{\theta}(b^n_{MU} \eta_R(x_1))$.

**Proposition 16.** $S_{MU} \subseteq S_K$.

**Proof.** Let $\lambda \in S_{MU}$. Then there is a $\theta \in D(MU)$ such that $\theta_* = \lambda = (\lambda_n)_{n \geq 0} \in \prod_{n=0}^{\infty} \mathbb{Z}$. We define $V_\lambda : QMU_* \to \mathbb{Z}$ by the composite $\pi \tilde{\theta}$ where $\pi : MU^* \to \mathbb{Z}$ is defined to be the ring map determined by

- $x_1 \mapsto 1$,
- $x_i \mapsto 0$, for $i > 1$.

Thus we have a commutative diagram

$$
\begin{array}{ccc}
QM U_* & \xrightarrow{\overline{\theta}} & MU^* \\
\downarrow{V_\lambda} \quad & & \quad \downarrow{\pi} \\
\mathbb{Z} \\
\end{array}
$$

and the diagonal map takes $x \in QMU_* = \prod_{n=0}^{\infty} \mathbb{Z}$ to some rational linear combination of the $\lambda_i$, which the $MU$ congruences tell us is in $\mathbb{Z}$.

It is easy to check that we can factorize $V_\lambda$ as $\pi \tilde{\phi}$ where

$$
\tilde{\phi} : QMU_* \to \text{Im}(\phi)
$$
is the map given by restricting the range of 
\[ \phi : QMU_*(MU_0) \to QKU_*(KU_0), \]
and
\[ \pi_\lambda : \text{Im}(\phi) \to \mathbb{Z} \]
is the \( \mathbb{Q} \)-linear map determined by
\[ \lambda \hookrightarrow \lambda. \]

Now
\[ \tilde{\phi}(b_n^M \eta_R(x_1)) = b_n^K \eta_R(u) = b_n^K v. \]
So
\[ V_\lambda(b_n^M \eta_R(x_1)) = \pi_\lambda \tilde{\phi}(b_n^M \eta_R(x_1)) \]
\[ = \pi_\lambda(b_n^K v) \]
\[ = C_n \cdot \lambda. \]

But \( V_\lambda(b_n^M \eta_R(x_1)) = \pi_\lambda(b_n^M \eta_R(x_1)) \in \mathbb{Z} \). So \( C_n \cdot \lambda \in \mathbb{Z} \) for all \( n \geq 0 \) and thus \( \lambda \in S_K \).
Hence \( S_{MU} \subseteq S_K \). □

**Theorem 17.** The image of the injective ring homomorphism \( \iota : A(KU) \hookrightarrow A(MU) \) is the centre \( Z(A(MU)) \).

**Proof.** We now have the following commutative diagram, where both vertical arrows are given by sending operations to their actions on homotopy.

\[ \begin{array}{ccc}
A(KU) & \cong & \text{Im}(\iota) \hookrightarrow Z(A(MU)) \\
\downarrow \cong & & \downarrow \cong \\
S_{KU} & \xrightarrow{=} & S_{MU}
\end{array} \]

It follows that the two inclusions on the top line of the diagram must be equalities and hence \( \text{Im}(\iota) = Z(A(MU)) = D(MU). \) □

6. **THE SPLIT CASE**

Let \( p \) be an odd prime. In this section we give the analogue of Theorem 17 in the split \( p \)-local setting, that is with the Adams summand \( G \) and Brown-Peterson theory \( BP \) in place of \( KU \) and \( MU \).

Most of the steps in the proof follow those given earlier in the non-split setting. To get started we need to know that we can express all operations in \( A(G) \) in terms of Adams operations.

**Proposition 18.** The topological ring \( A(G) \) may be identified with the collection of infinite sums \( \{ \sum_{n=0}^{\infty} a_n \hat{\sigma}_n^G \mid a_n \in \mathbb{Z}_{(p)} \} \), where each \( \hat{\sigma}_n^G \) is a finite \( \mathbb{Z}_{(p)} \)-linear combination of the Adams operations \( \Psi_k^G \).
Proof. We can obtain \( A(G) \) from \( A(KU) \) by applying the Adams idempotent \( e_0 \). This idempotent operation acts on homotopy as the identity on \( \pi_n(KU_0) \) if \( n \) is a multiple of \( 2(p - 1) \) and as zero otherwise. Thus we see that \( e_0 \Psi^n_{KU} = \Psi^n_G \). We obtain the topological spanning set \( \{ e_0 \sigma^n_{KU} \mid n \geq 0 \} \) of \( A(G) \) from the topological basis of \( A(KU) \) given in Theorem 3. Each element is a finite linear combination of \( G \) Adams operations. Within this spanning set we can find a topological basis \( \{ \sigma_n \mid n \geq 0 \} \) of \( A(G) \). \( \square \)

Explicit formulas for a choice of such topological basis elements for \( A(G) \) are given in [9] Chapter 4, but we do not need these here.

**Theorem 19.** There is an injective ring homomorphism \( \hat{i} : A(G) \to A(BP) \) such that \( \operatorname{Im}(\hat{i}) = Z(A(BP)) \).

**Outline proof of Theorem 19.** By Proposition 18 we can identify \( A(G) \) with

\[
\left\{ \sum_{n=0}^{\infty} a_n \sigma_n^G \mid a_n \in \mathbb{Z}/(p) \right\}.
\]

We define \( \hat{\sigma}_n^{BP} \in A(BP) \) in the obvious way, by replacing \( G \)-Adams operations by the corresponding \( BP \) ones, given by Proposition 2. The method of the proof of Proposition 7 shows that \( \hat{\sigma}_n^{BP} \to 0 \) as \( n \to \infty \) in the filtration topology of \( A(BP) \). Since \( A(BP) \) is complete in this topology, we can define \( \hat{i} : A(G) \to A(BP) \) by \( \sum_{n=0}^{\infty} a_n \hat{\sigma}_n^G \mapsto \sum_{n=0}^{\infty} a_n \hat{\sigma}_n^{BP} \). This is an injective ring homomorphism, just as in the non-split case and the same proof as for Lemma 9 shows that \( \operatorname{Im}(\hat{i}) \subseteq Z(A(BP)) \). By Lemma 11 there is an inclusion \( Z(A(BP)) \subseteq D(BP) \).

Recall that \( \beta_{BP} \) is the injective ring homomorphism

\[
\beta_{BP} : A(BP) \to \text{Ab}_*(\pi_*(BP_0), \pi_*(BP_0))
\]

\[\theta \mapsto \theta^* \]

Restricting to \( D(BP) \), we have a map

\[
\beta_{BP} : D(BP) \to \prod_{n=0}^{\infty} \mathbb{Z}/(p).
\]

Consider the congruences satisfied by \( \beta_{BP}(\theta) = (\mu_n)_{n \geq 0} \), where the operation \( \theta \) acts on \( \pi_2(p-1)_{n}(BP_0) \) as multiplication by \( \mu_n \). We have \( \beta_{BP}(D(BP)) = S_{BP} \), and thus an isomorphism of rings \( D(BP) \cong S_{BP} \).

For \( E = BP \) or \( G \), we write \( b_{(i)}^E = b_{(i)}^G \) and we recall that all the other Hopf ring elements \( b_{(i)}^E \) are redundant. Let \( (b_E)^{\alpha} = (b_{(0)}^E)^{\alpha_0} (b_{(1)}^E)^{\alpha_1} \ldots \) for a finite integer sequence \( \alpha = (\alpha_0, \alpha_1, \ldots) \).

Then, using [4 Theorem 16.11(a)], we find that \( QBP_*(BP_0) \) is free as a \( BP^* \)-module and it is generated by elements of the form \( (b_{BP})^{\alpha} \eta_R(v) \) with \( v \in BP^{2|\alpha|} \) where \( |\alpha| = \sum \alpha_i \).

So the \( BP \) congruences come from \( \overline{D((b_{BP})^{\alpha} \eta_R(v))} \subseteq BP^* \), for \( v \) and \( \alpha \) as above and we now compare the solution sets for the \( BP \) and \( G \) congruences.
The inclusion $S_G \subseteq S_{BP}$ follows directly from the existence of $\hat{i}$.

For the reverse inclusion, we consider the ring map $QBP_n(BP_0) \to QG_n(\mathbb{G})$ coming from the map of ring spectra $\hat{\phi}: BP \to G$. This takes $b_n^{BP}$ to $b_n^G$. Write $G^* = \mathbb{Z}(p)\hat{\eta} \hat{u}^{-1}$ where $|\hat{u}| = 2(p - 1)$. The elements $(b^G)^*\hat{v}^i$ span $QG_n(\mathbb{G})$ as a $G^*$-module, where $\hat{v} = \eta_R(\hat{u})$ and $i \in \mathbb{Z}$ satisfies $\sum \alpha_j = i(p - 1)$. Let $\{\hat{f}_n | n \geq 0\}$ be a $\mathbb{Z}(p)$-basis of $QG_0(\mathbb{G})$. Then we can express each $\hat{f}_n$ as a $G^*$-linear combination of the $(b^G)^*\hat{v}^i$. This means that, up to some shift by a power of $\hat{u}$, each $\hat{f}_n$ is in the image of the map from $QBP_n(BP_0)$. The analogous proof to that in Lemma 16 now shows that $S_{BP} \subseteq S_G$.

Therefore we have the following commutative diagram, where both vertical maps send operations to their actions on homotopy.

$$
\begin{array}{cccc}
A(G) & \xrightarrow{\cong} & \text{Im}(\hat{i}) & \xrightarrow{\cong} Z(A(BP)) & \xrightarrow{\cong} D(BP) \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
S_G & \xrightarrow{\cong} & S_{BP} & & \end{array}
$$

It follows that $\text{Im}(\hat{i}) = Z(A(BP))$. \qed

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