Sphere Decoding for Spatial Modulation Systems with Arbitrary $N_t$

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Abstract—Recently, three Sphere Decoding (SD) algorithms were proposed for Spatial Modulation (SM) scheme which focus on reducing the transmit-, receive-, and both transmit and receive-search spaces at the receiver and were termed as Receiver-centric SD (Rx-SD), Transmitter-centric SD (Tx-SD), and Combined SD (C-SD) detectors, respectively. The Tx-SD detector was proposed for systems with $N_t \leq N_r$, where $N_t$ and $N_r$ are the number of transmit and receive antennas of the system. In this paper, we show that the existing Tx-SD detector is not limited to systems with $N_t \leq N_r$ but can be used with systems $N_r < N_t \leq 2N_r - 1$ as well. We refer to this detector as the Extended Tx-SD (E-Tx-SD) detector. Further, we propose an E-Tx-SD based detection scheme for SM systems with arbitrary $N_t$ by exploiting the Inter-Channel Interference (ICI) free property of the SM systems. We show with our simulation results that the proposed detectors are ML-optimal and offer significantly reduced complexity.

Index Terms—Sphere decoding, ML decoding, spatial modulation, space-time shift keying, complexity.

I. INTRODUCTION

Spatial Modulation (SM) [1, 2] is a recently developed low-complexity Multiple-Input Multiple-Output (MIMO) scheme that exploits the channel for information transmission in an unprecedented fashion. Specifically, the information bits stream is divided into blocks of length $\log_2(N_tM)$ bits, and in each block, $\log_2(M)$ bits select a symbol $s$ from $M$-ary signal set (such as $M$-QAM or PSK), and $\log_2(N_t)$ bits select an antenna out of $N_t$ transmit antennas for the transmission of the symbol $s$. The throughput achieved by this scheme is $R = \log_2(N_tM)$ bpcu. Thus, the SM scheme achieves an increase in spectral efficiency of $\log_2 N_t$ bits over single-antenna systems with a marginal increase in the complexity since it still needs only one RF chain at the transmitter. However, in order to achieve high throughputs either $N_t$ or $M$, or both need to be increased which renders this scheme suitable for low and moderately high spectral efficiencies.

Recently, three specially tailored Sphere Decoding (SD) detectors were proposed for SM systems in [3] which were termed as Receiver-centric SD (Rx-SD) [4], Transmitter-centric SD (Tx-SD), and Combined SD (C-SD) detectors. It was shown in [3] that the Rx-SD detector is suitable for SM systems with large $N_r$, the number of receive antennas, and C-SD detector is suitable for systems operating at relatively high spectral efficiencies i.e., with large $N_t$ or $M$, or both. But, the applicability of the existing Tx-SD detector and hence, the C-SD detector is limited to systems with $N_t \leq N_r$, which is due to the zero diagonal entries in the $R$ matrix of the QR decomposition associated with the underdetermined channel matrix. This problem is inherent to any MIMO system with underdetermined channel matrix that employs SD detector at the receiver. In [5], this problem was addressed by transforming the underdetermined channel matrix into a full-column rank matrix and applying standard SD on it. But, this SD which was termed as $\lambda$-Generalized SD ($\lambda$-GSD) detector is asymptotically Maximum Likelihood (ML) optimal with SNR for non-constant modulus signal sets such as $M$-QAM. Thus, at low Signal-to-Noise Ratios (SNR) the performance of the $\lambda$-GSD detector cannot be expected to be near-ML. The GSD of [6] is also not ML-optimal for non-constant modulus signal sets under all SNR conditions like the $\lambda$-GSD detector, and also the complexity offered by the GSD of [6] is significantly higher than that of the $\lambda$-GSD [5]. In this paper, we do not take these approaches, instead, we show that the existing Tx-SD detector of [4] is not limited to SM systems with $N_t \leq N_r$ but is applicable to systems with $N_r < N_t \leq 2N_r - 1$ as well, we refer to this detector as the Extended Tx-SD (E-Tx-SD) detector, and further, we propose an $ML$-optimal detection scheme termed as Generalized Tx-SD (G-Tx-SD) detector for SM systems with arbitrary $N_t$.

II. SYSTEM MODEL

We consider a MIMO system having $N_t$ transmit as well as $N_r$ receive antennas and a quasi-static, frequency-flat fading channel, yielding:

$$ y = Hx + n, \quad (1) $$

where $x \in \mathbb{C}^{N_t \times 1}$ is the transmitted vector, $y \in \mathbb{C}^{N_r \times 1}$ is the received vector, $H \in \mathbb{C}^{N_r \times N_t}$ is the channel matrix, and $n \in \mathbb{C}^{N_r \times 1}$ is the noise vector. The entries of the channel matrix and the noise vector are from circularly symmetric complex-valued Gaussian distributions $\mathcal{CN}(0,1)$ and $\mathcal{CN}(0, \sigma^2)$, respectively, where $\sigma^2$ is the noise variance per dimension.
A. Spatial Modulation

In SM scheme [1], we have
\[
x = [0, \ldots, 0, s, 0, \ldots, 0]^T \in \mathbb{C}^{N_t \times 1},
\]
where \( s \) is a complex symbol from the signal set \( S \) with \(|S| = M\). Throughout this paper we assume \( S \) to be a lattice constellation such as QAM. Thus, for an SM system, eq.(1) becomes
\[
y = \text{H}x_{l,s} + n,
\]
where \( l \in L = \{1\}^{N_t} \) and the subscript \( s \) captures the dependence of \( x \) on the signal set \( S \). Assuming perfect channel state information and ML decoding at the receiver, we have
\[
(i, s)_{ML} = \arg \min_{i \in L, s \in S} \|y - \text{H}x_{l,s}\|_2^2, \tag{4}
\]
\[
= \arg \min_{i \in L, s \in S} \left\{ \sum_{i=1}^{N_t} |y_i - h_{i,l,s}|^2 \right\}, \tag{5}
\]
where \( y_i \) and \( h_{i,l,s} \) are the \( i \)th and the \( (i, l) \)th entry of the received vector \( y \) and the channel matrix \( \text{H} \), respectively. The computational complexity in terms of number of real multiplications involved in computing ML solution of eq.(4) is given by
\[
C_{ML} = 8NM_tN_r, \tag{6}
\]
since, 8 real multiplications are required in computing \(|y_i - h_{i,l,s}|^2\) for any legitimate \((s, l, i)\). If \( N_t \geq N_r \), then from the well known division algorithm we have unique non-negative integers \( q \) and \( r \) \((0 \leq r < N_r)\) such that \( N_t = qN_r + r \). Thus, from eq.(6) we can write
\[
C_{ML}^{N_t \times N_r} = C_{ML}^{(qN_r+r) \times N_r} = qC_{ML}^{N_r \times N_r} + rC_{ML}^{r \times N_r}. \tag{7}
\]

B. Review of Tx-SD detector [7]

The complex-valued system in eq.(3) can be expressed in terms of real variables as
\[
\begin{bmatrix}
\Re(y) \\
\Im(y) \\
\text{H} \\
x_{l,s} \\
n
\end{bmatrix} =
\begin{bmatrix}
\Re(\text{H}) & -\Im(\text{H}) \\
\Im(\text{H}) & \Re(\text{H}) \\
\Re(x_{l,s}) & -\Im(x_{l,s}) \\
\Re(n) & -\Im(n)
\end{bmatrix} \begin{bmatrix}
\Re(y) \\
\Im(y) \\
\text{H} \\
x_{l,s} \\
n
\end{bmatrix} +
\begin{bmatrix}
\Re(n) \\
\Im(n)
\end{bmatrix}, \tag{8}
\]
where, \( \Re(\cdot) \) and \( \Im(\cdot) \) represent the real and imaginary parts, \( \bar{y} \) and \( \bar{n} \in \mathbb{R}^{2N_t \times 1} \), \( \bar{\text{H}} \in \mathbb{R}^{2N_r \times 2N_t} \), and \( x_{l,s} \in \mathbb{R}^{2N_t \times 1} \).

For \( N_t \leq N_r \), the Tx-SD detector \([4]\) is given by
\[
(i, s)_{T \times S \text{D}} = \arg \min_{(i, s) \in \Theta_R} \|\bar{y} - \bar{\text{H}}x_{l,s}\|_2^2, \tag{9}
\]
where, \( \Theta_R = \{(l, s) \mid l \in L, s \in S, \text{ and } \|\bar{y} - \bar{\text{H}}x_{l,s}\|_2^2 \leq R^2 \} \) with \( R \) as the initial search radius of the sphere decoder. From [4] and [7], we have \( R = \alpha N_r \sigma^2 \), where \( \alpha \) is a parameter chosen to maximize the probability of detection. By expressing \( \bar{\text{H}} \) in terms of its QR decomposition we have \( \|\bar{y} - \bar{\text{H}}x_{l,s}\|_2^2 \leq R^2 \) equivalent to \( \|\bar{z} - \bar{R}x_{l,s}\|_2^2 \leq R^2 \) where, \( \bar{R} \) is an upper triangular matrix given by
\[
\begin{bmatrix}
\bar{R} & (2N_r \times 2N_t) \\
0 & (2N_t - 2N_r \times 2N_t)
\end{bmatrix}, \tag{10}
\]
and
\[
\bar{z} = \bar{Q}^T \bar{y} \quad \text{with} \quad \bar{Q} = \begin{bmatrix} \bar{Q}_1 & (2N_r \times 2N_t) \bar{R}_2 \end{bmatrix} \begin{bmatrix} \bar{Q}_1 & (2N_r \times 2N_t) \end{bmatrix} \tag{11}
\]
where \( \bar{Q}_1 \) is an upper triangular matrix.

III. PROPOSED DETECTOR FOR SM SYSTEMS WITH ARBITRARY \( N_t \)

The Tx-SD detector discussed in the previous section was proposed for SM systems with \( N_t \leq N_r \). In this section, we show that the existing Tx-SD detector is applicable to systems with \( N_r < N_t \leq 2N_r - 1 \) as well, and further extend it to systems with arbitrary \( N_t \).

A. Tx-SD detector in SM systems with \( N_r < N_t \leq 2N_r - 1 \)

Proposition 1: The Tx-SD detector \([4]\) originally proposed for SM systems with \( N_t \leq N_r \) is applicable to systems with \( N_r < N_t \leq 2N_r - 1 \) as well.

Proof: Consider an SM system with \( N_t > N_r \). The QR decomposition of \( \bar{H} \) is given by \( \bar{Q} \bar{R} \) where \( \bar{Q} = \begin{bmatrix} \bar{Q}_1 & (2N_r \times 2N_t) \end{bmatrix} \) and \( \bar{R} = \begin{bmatrix} \bar{R}_1 & (2N_r \times 2N_t) \bar{R}_2 \end{bmatrix} \) where \( \bar{R}_1 \) is an upper triangular matrix. Recall that only two elements are...
non-zero in $\bar{x}_{l,s}$ and are apart by $N_t - 1$ zero elements as shown below:

$$\bar{x}_{l,s} = \left[0, \ldots, 0, \Re(s), 0, \ldots, 0, \Im(s), 0, \ldots, 0\right]^T \in \mathbb{R}^{2N_t \times 1}.$$  

(13)

Considering $l = N_t$ and some $s \in S$ we have

$$\bar{x}_{N_t,s} = \left[0, \ldots, 0, \Re(s), 0, \ldots, 0, \Im(s)\right]^T \in \mathbb{R}^{2N_t \times 1},$$  

(14)

and the last element of $\bar{p} = R\bar{x}_{N_t,s}$ is given by $\bar{p}_2N_r = \Re(2N_r,;)\bar{x}_{N_t,s}$. It is easy to see that $R(2N_r,;)\bar{x}_{N_t,s}$ is upper triangular, and if the number of non-zero elements $2N_t - 2N_r + 1$ in $R(2N_r,;)\bar{x}_{N_t,s}$ is less than or equal to $N_t = (N_t - 1) + 1$, the number of zeros between $\Re(s)$ and $\Im(s)$ plus one non-zero element $\Im(s)$ in $\bar{x}_{N_t,s}$, we see that $\bar{p}_{2N_r}$ depends only on $\Im(s)$. Since this is true for all $\{\bar{p}_i\}^{2N_r}_{i=2}$ and any legitimate pair $(l, s)$, we have the condition $2N_t - 2N_r + 1 \leq N_t = N_t \leq 2N_r - 1$ for independent detection of imaginary components in $\bar{x}_{l,s}$. This results in the following intervals analogous to those in eq. (17):

$$-R_Q + \bar{x}_{2N_t - i + 1} \leq R_Q + \bar{x}_{2N_t - i + 1} \leq -R_Q + \bar{x}_{2N_t - i + 1} \leq R_Q + \bar{x}_{2N_t - i + 1} \leq -R_Q + \bar{x}_{2N_t - i + 1} \leq R_Q + \bar{x}_{2N_t - i + 1},$$  

(15)

for $1 \leq i \leq N_t$. Proceeding in the lines similar to that of [4] for real components in $\bar{x}_{l,s}$ we get the intervals of eq. (16) given in the next page. From eq. (16) we observe that $\bar{x}$. With $N_t = N_t$, it can be checked that the intervals of eq. (15) and eq. (16) reduce to those of eq. (17) and eq. (18), respectively.

We refer to this Tx-SD detector as the Extended Tx-SD (E-Tx-SD) detector in the rest of the paper.

B. Tx-SD detector for SM systems with arbitrary $N_t$

We have shown in the previous subsection that the Tx-SD detector of [4] can be used in SM systems with $N_r \leq N_t \leq 2N_r - 1$. In this subsection we propose a SD detection scheme for arbitrary $N_t$ by partitioning the antenna search space into disjoint subsets each of size $2N_r - 1$ and running E-Tx-SD decoders sequentially.

Consider an SM system with $N_t > 2N_r - 1 = N'$. From the division algorithm we have unique non-negative integers $q$ and $r$ such that $N_t = qN' + r$, where $0 \leq r < N'$. For the ease of presentation we assume $r = 0$ and hence $N_t = qN'$ for now, and later generalize our results for non-zero $r$. Let $L$, the set of antenna indices, be partitioned into disjoint subsets $L_k = \{i\}^{kN'}_{i=(k-1)N'+1}$ so that $L = \bigcup_{k=1}^{q} L_k$ and $|L_k| = N'$ for all $1 \leq k \leq q$.

Let $J(l, s)$ represent the ML metric $\sum_{i=1}^{N_r} |y_i - h_{l,i}s|^2$ of eq. (5). Then, the ML solution in terms of $J(l, s)$ is $\hat{(l, s)}_{ML} = \arg\min_{l \in L, s \in S} J(l, s)$ and we have

$$\min_{l \in L, s \in S} J(l, s) = \min_{l \in L} \left\{ \min_{s \in S} J(l, s) \right\},$$  

(17)

$$= \min \left[ \bigcup_{k=1}^{q} \left\{ \min_{l \in L_k} \left\{ \min_{s \in S} J(l, s) \right\} \right\} \right],$$  

(18)

where $\Theta(k) = \min_{l \in L_k} \left\{ \min_{s \in S} J(l, s) \right\}$. In the above, (a) follows from the assumption that the antenna index and the transmitted symbol are encoded by independent sets of bits, and (b) follows directly from the fact that $L$ partitions into $L_k$'s. It is straightforward that each of the $\Theta(k)$'s can be obtained by running E-Tx-SD detector discussed in the previous subsection. The sphere radius $R_k$ for all $1 \leq k \leq q$ is taken as $R = \alpha N_r \sigma^2$. Thus, we have

$$\Theta(k) = \{l(s) \mid l \in L_k, s \in S, \|\bar{z}_k - R_k\bar{x}_{l,s}\|^2 \leq R^2 \},$$  

(20)

where, $Q_k R_k = H(:, [L_k])$, $R^2 = 2^2$ and $\bar{z}_k = Q_k^T \bar{y}$ for all $1 \leq k \leq q$. Equations (15) and (16) directly give the elements of $\Theta(k)$ for all $1 \leq k \leq q$.

Now, for a system with $N_t = qN' + r$ and $r \neq 0$, it is straightforward that there is an additional set $\Theta(q + 1)$ in eq. (20) which is same as eq. (20) with $L_{q+1} = L \setminus \bigcup_{k=1}^{q} L_k$. Thus, the E-Tx-SD based solution for systems with arbitrary $N_t$ is given by

$$\hat{(l, s)}_{E-Tx-SD} = \arg\min_{(l,s) \in \bigcup_{k=1}^{q+1} \Theta(k)} \|\bar{y} - \bar{H}\bar{x}_{l,s}\|^2.$$  

(21)

This solution is referred to as the Generalized Tx-SD (G-Tx-SD) detector. We note here that the partitions $L_k$ considered here are of size $N' = 2N_r - 1$, and it is straightforward that the detector in eq. (21) can be run over partitions of any size, for example $N' = N_r$ as well.

From eq. (7) it is clear that the ML complexity scales linearly with the number of transmit antennas. As the proposed G-Tx-SD detector runs over partitioned channel blocks, it is obvious that the complexity of the proposed detector can be expected to be less than that of the ML detector, since the complexity of the individual Tx-SD/E-Tx-SD detector is much lesser than that of the ML detector [4]. We note here that the proposed E-Tx-SD and G-Tx-SD detectors are not limited to SM scheme alone, but are applicable to any system with the ICI-free property, for example, the Space-Time Shift Keying (S-TSK). The C-SD detector proposed in [4] uses both the Tx-SD and the Rx-SD detectors in order to reduce the overall search complexity. It is straightforward that the proposed detectors can be further extended by incorporating Rx-SD detector for reduction in the receive search space complexity as well.

IV. Simulation results

Consider an SM system with $N_r = 4$ and $N_t = 7$, employing 16- and 64-QAM signal sets. Let $N_{ML} = MN_t$ denote the number of search points in the ML detection, and $E[N_E-Tx-SD]$ denote the expected number of points that lie inside the hypersphere of radius $R$ under E-Tx-SD detection. In this paper, we use the metric $E[N_E-Tx-SD]/N_{ML}$ to measure the reduction in search space with respect to the ML detection. From Fig. (4), it is clear that the Symbol Error Rate (SER) performance of the ML and the E-Tx-SD detectors overlap for both the modulation schemes considered, and from
\[ -R_Q + \bar{z}_{N_t-i+1} - \bar{r}_{N_t-i+1,2N_t-i+1} \bar{x}_{N_t-i+1} \leq \bar{x}_{N_t-i+1} \leq R_Q + \bar{z}_{N_t-i+1} - \bar{r}_{N_t-i+1,2N_t-i+1} \bar{x}_{2N_t-i+1} \text{ for } 1 \leq i \leq N_t \]

Fig. 2. SER curves of ML and E-Tx-SD detectors in an SM system with \( N_r = 4, N_t = 7 \), employing 16- and 64-QAM signal sets.

Specifically, at an SNR of 20 dB, we see a reduction of about 90% and 70% for 16- and 64-QAM signal sets, respectively.

Fig. 3 shows the SER performance and the reduction in complexity due to the G-Tx-SD detector in an SM system with \( N_r = 4, N_t = 8 \), and 16-QAM signal set. We observe from Fig. 4(a) that the SER curves of the proposed and the ML detector overlap, and from Fig. 4(b) that there is about 85% reduction in complexity with respect to ML detection at an SNR of about 18 dB. The partitions considered here are of size \( |L_1| = |L_2| = 4 \). The reduction in complexity involved in precomputations such as QR decomposition, etc., by using the E-Tx-SD detector with different \( |L_i|'s \) instead of the Tx-SD detector with equal \( |L_i|'s \) is left for our future study.

V. CONCLUSION

We have shown that the existing Tx-SD detector is not limited to SM systems with \( N_t \leq N_r \) but is applicable to systems with \( N_r < N_t \leq 2N_r - 1 \) as well. Further, we have proposed a generalized Tx-SD detector for SM systems with arbitrary \( N_t \) by exploiting the ICI-free nature of the system, and have shown with our simulation results that the proposed detectors give ML performance with significantly reduced complexity.

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