A COAREA-TYPE INEQUALITY IN CARNOT GROUPS

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ABSTRACT. We prove a coarea-type inequality for continuously Pansu differentiable functions, with everywhere surjective Pansu differential and uniformly lower Ahlfors regular level sets, acting between two Carnot groups endowed with homogeneous distances.

CONTENTS

1. Introduction 1
2. Preliminary definitions and results 3
3. Coarea-type Inequality 9
4. Applications 19
Acknowledgement 22
References 22

1. INTRODUCTION

Geometric measure theory in non-Euclidean metric spaces has been relevantly developed during the last decades. One of the first goals in this line of research is the study of Carnot groups, that are connected, simply connected, nilpotent, stratified Lie groups. These are the simplest models of sub-Riemannian manifolds. One can canonically associate to each Carnot group a family of non-isotropic dilations defined according to the stratification of the Lie algebra of the group. We study Carnot groups endowed with a distance that is homogeneous with respect to these dilations. Within the study of these metric spaces, a long-standing open problem is the validity of the coarea formula for Lipschitz maps acting between two Carnot groups. Up to now, for these mappings only a coarea-type inequality is available [11]. Some stronger results have been proved for specific situations. For instance, one can refer to [3], [13], [20] for Lipschitz real-valued maps acting on a generic Carnot group, to [9], [15], [19], [21] for continuously Pansu differentiable mappings from a Heisenberg group $\mathbb{H}^n$ to $\mathbb{R}^k$ (where, depending on $k$, higher regularity on the Pansu differential may be required) and to [8], [12], [14] for Euclidean regular maps from a Carnot group to $\mathbb{R}^k$. Moreover, a very general result has recently been proved in [7]. The authors consider two Carnot groups $G$ and $M$, endowed with homogeneous distances, an open set $\Omega \subset G$ and a map $f : \Omega \to M$, with Pansu differential $Df(x)$ continuous on $\Omega$. Then, the coarea formula holds for $f$ if, at every point $x \in \Omega$, either $Df(x)$ is surjective and $\ker(Df(x))$ can be complemented with a homogeneous subgroup (Definition 2) or $Df(x)$ is not surjective.

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A key step in the proof of the coarea formula [7, Theorem 1.3] is a suitable implicit function theorem (Theorem 2.6). Fix a value \( m \in \mathbb{M} \) and consider a point \( x \in f^{-1}(m) \) such that \( Df(x) \) is surjective. Assume that there exists a homogeneous subgroup \( V \) complementary to \( \ker(Df(x)) \) and choose any homogeneous subgroup \( W \) complementary to \( V \). Then there exist an open neighbourhood \( \Omega \subset \mathbb{G} \) of \( x \), an open set \( U \subset \mathbb{W} \) and a map \( \phi : U \subset \mathbb{W} \to V \) such that \( \Omega \cap f^{-1}(m) \) is the intrinsic graph of \( \phi \) (Definition 3). It is not clear how to prove the existence of an analogous parametrization if we assume the Pansu differential \( Df(x) \) only to be surjective. In this work we bypass this lack and we prove a weaker coarea-type result, that permits, under a further regularity condition, to deal with more general situations. More precisely we prove the following result.

**Theorem 1.1.** Let \((\mathbb{G},d_1), \ (\mathbb{M},d_2)\) be two Carnot groups endowed with homogeneous distances, of metric dimension \( Q, \ P \) and topological dimension \( q, \ p \), respectively. Let \( f \in C^1_\mathbb{G}(\mathbb{G},\mathbb{M}) \) be a function and assume that \( Df(x) \) is surjective at every point \( x \in \mathbb{G} \). Let \( \Omega \) be a closed bounded subset of \( \mathbb{G} \). Assume that there exist two constants \( \tilde{r}, \tilde{C} > 0 \) such that for any \( m \in \mathbb{M} \), the level set \( f^{-1}(m) \) is \( \tilde{r} \)-locally \( C \)-lower Ahlfors \((Q-P)\)-regular with respect to the measure \( S^{Q-P} \). Then there exists a constant \( L = L(\mathbb{G},\mathbb{P},p) \) such that

\[
\int_{\Omega} C_P(Df(x))dS^Q(x) \leq L \int_{\mathbb{M}} S^{Q-P}(f^{-1}(m) \cap \Omega)dS^P(m).
\]

The factor \( C_P(Df(x)) \) is the coarea factor of the Pansu differential \( Df(x) \) (Definition 9) and \( S^\alpha \) denotes the \( \alpha \)-dimensional spherical Hausdorff measure built with respect to homogeneous distance. Refer to Definition 11 for the notion of locally lower Ahlfors regular set.

It is immediate to extend Theorem 1.1 to the case when \( \Omega \) is a measurable subset of \( \mathbb{G} \) (Theorem 3.3). As an example of its generality, notice that Theorem 1.1 can be applied to any continuously Pansu differentiable functions \( f : \mathbb{H}^1 \to \mathbb{R}^2 \) satisfying the requirements.

The proof of Theorem 1.1 is inspired to an abstract procedure presented in [23], where it is used to prove a coarea-type inequality for functions from a metric space to a measure space, for packing-type measures. Analogous argument involving suitable packing measures is adapted here to prove Claim 1 of Theorem 3.2.

By applying Theorem 1.1, we deduce new results about slicing of measurable functions on the level sets of \( f \) (Corollaries 4.2 and 4.3).

We conclude our introduction with a brief discussion about the hypotheses of Theorem 1.1. In particular we compare them with assumptions of available results in literature.

The hypothesis of Theorem 1.1 about Ahlfors regularity of the level sets of the map \( f \) is not redundant: if we consider \( f \in C^1_\mathbb{G}(\mathbb{G},\mathbb{M}) \), with Pansu differential everywhere surjective on \( \mathbb{G} \), the lower Ahlfors regularity of level sets is not guaranteed, even locally. One can refer to [9, Corollary 6.2.4], where the author presents some explicit examples of this phenomenon. In addition, in [9], a particular situation is considered: a class of higher regular mappings from the Heisenberg group \( \mathbb{H}^n \) to the Euclidean \( \mathbb{R}^{2n} \) is studied. Let us consider a homogeneous distance \( d \) on \( \mathbb{H}^n \). The author considers functions \( f \in C^1_\mathbb{G}(\mathbb{H}^n,\mathbb{R}^{2n}) \) with \( \alpha > 0 \) i.e. continuously Pansu differentiable maps such that, for every \( a,b \in \mathbb{H}^n \)

\[
d_{L(\mathbb{H}^n,\mathbb{R}^{2n})}(Df(a), Df(b)) \lesssim d(a,b)^\alpha.
\]
By [9, Corollary 5.5.6], if we assume that the Pansu differential of \( f \) is everywhere surjective, the level sets of \( f \) are uniformly locally Ahlfors 2-regular with respect to \( S^2 \). Therefore, the validity of the inequality of Theorem 1.1 for this class of regular functions is ensured by our result. This confirms the coarea-type equality proved in [9, Theorem 6.2.5]. To summarize, in this setting, we can weaken the hypothesis adopted in [9, Theorem 6.2.5] about the required regularity of the considered map, passing from \( C^1,\alpha_G \)-regular maps, with \( \alpha > 0 \), to continuously Pansu differentiable functions. We compensate the lower regularity of the map with the more geometrical hypothesis about the Ahlfors regularity of its level sets. We need to remark that these considerations are limited, up to now, to maps from the Heisenberg group \( H^m \) to \( \mathbb{R}^{2n} \), about which more results are available.

We stress that in Theorem 1.1 the assumption about the uniform local lower Ahlfors regularity of the level sets of the map \( f \) can be read also as a substitute of the existence of a suitable splitting of \( G \). In fact, this condition is automatically verified if one assumes the existence of a \( p \)-dimensional homogeneous subgroup \( V \subset G \) complementary to \( \ker(Df(x)) \) for every point \( x \in G \) (Corollary 4.6). The proof of this observation uses tools of the theory of intrinsic Lipschitz graphs (6). We stress that Corollary 4.6 is just an example of application of Theorem 1.1. In fact, as we discussed above, it can be derived also by the coarea formula in [7, Theorem 1.3].

### 2. Preliminary definitions and results

In this section we introduce some definitions and notations.

When we write \( a \lesssim b \), we mean that there exists some positive constant \( C \) such that \( a \leq Cb \). If \( C \) depends on some parameter \( d \), it will be specified with a subscript. For instance, \( a \lesssim_d b \) means that there exists a constant \( C \) depending on \( d \) such that \( a \leq Cb \). Analogous notations are assumed for \( \gtrsim \).

**Definition 1.** A Carnot group \( G \) is a connected, simply connected, nilpotent Lie group such that its Lie algebra \( \text{Lie}(G) \) is stratified i.e. there exist linear subspaces \( V_1, V_2, \ldots, V_k \) such that

\[
\text{Lie}(G) = V_1 \oplus \cdots \oplus V_k
\]

and

\[
[V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \quad \text{if} \quad i > k,
\]

where \( [V_1, V_i] = \text{span}\{[X, Y] : X \in V_1, Y \in V_i\} \).

The number \( k \) is called the step of \( G \).

The topological dimension of \( G \) is \( q = \sum_{i=1}^{k} \dim(V_i) \); the number \( Q = \sum_{i=1}^{k} (i \dim(V_i)) \) is called homogeneous dimension of \( G \).

We denote the left translation associated to an element \( x \in G \) by \( \tau_x : G \to G, \tau_x(y) = xy \).

We can naturally introduce on \( \text{Lie}(G) \) a family of non-isotropic linear dilations

\[
\delta_t(v) = \sum_{i=1}^{k} t^i v_i \quad \text{if} \quad v = \sum_{i=1}^{k} v_i \quad \text{with} \quad v_i \in V_i.
\]

Since \( G \) is simply connected, the exponential map \( \exp : \text{Lie}(G) \to G \) is a global diffeomorphism, then we can identify \( G \) with \( \text{Lie}(G) \) and any dilation \( \delta_t \) can be identified with the function \( \exp \circ \delta_t \circ \exp^{-1} : G \to G \), we denote this map again by \( \delta_t \).
A Lie subgroup $\mathcal{W} \subseteq \mathcal{G}$ is called homogeneous if it is closed with respect to the family of anisotropic dilations, hence if for every $t > 0$, $\delta_t(\mathcal{W}) \subseteq \mathcal{W}$.

Through the exponential map and according to the Baker-Campbell-Hausdorff formula we can move the group product of $\mathcal{G}$ to an isomorphic polynomial group product on Lie($\mathcal{G}$); for $X, Y \in \text{Lie}(\mathcal{G})$, we call it $\text{BCH}(X, Y)$. In particular Lie($\mathcal{G}$) endowed with $\text{BCH}(\cdot, \cdot)$ is isomorphic to $\mathcal{G}$ itself ($[21, \text{Theorem 4.2}]$), so we identify $\mathcal{G}$ and Lie($\mathcal{G}$) as Lie groups.

We fix a basis of $\mathcal{G}$, $(v_1, \ldots, v_q)$ and we identify $\mathcal{G}$ with $\mathbb{R}^q$ through the chosen basis as follows

$$\varphi : \mathcal{G} \to \mathbb{R}^q, \quad \varphi(p) = (x_1, \ldots, x_q) \quad \text{if} \quad p = \sum_{i=1}^{q} x_i v_i.$$ 

The product on $\mathcal{G}$ can be moved to a polynomial group product on $\mathbb{R}^q$ (see $[1, \text{Proposition 2.2.22}]$). By the identification of $\mathcal{G}$ with Lie($\mathcal{G}$) and $\mathbb{R}^q$, $\mathcal{G}$ can be seen as $\mathbb{R}^q$ endowed at the same time with the structure of Lie group, with a polynomial group product, and the structure of Lie algebra, and hence of linear space. The inverse of an element with respect to the group product is $(x_1, \ldots, x_q)^{-1} = (-x_1, \ldots, -x_q)$ while the identity element is the null vector of $\mathbb{R}^q$ and we denote it by $0$.

We assume that $\mathcal{G}$ is a Carnot group endowed with a homogeneous distance $d$, that is a distance such that $d(zx, zy) = d(x, y)$ for every $x, y, z \in \mathcal{G}$ and $d(\delta_t(x), \delta_t(y)) = td(x, y)$ for every $t > 0$ and $x, y \in \mathcal{G}$.

We fix on $\mathcal{G}$ a scalar product with respect to which $(\mathcal{V}, \mathcal{W})$ is a homogeneous pair of complementary subgroups and about regularity for maps acting between homogeneous subgroups (for more details, please refer to $[25, \text{Section 4}]$).

**Proposition 2.1.** $[1, \text{Proposition 5.15.1}]$ Let $\mathcal{G}$ be a Carnot group of step $k$ endowed with a homogeneous distance $d$. For every compact subset $K \subset \mathcal{G}$ there exists a constant $C_K$ such that for any $x \in K$

$$\frac{1}{C_K} |x| \leq \|x\| \leq C_K |x|^\frac{k}{k+1}.$$ 

We recall some definitions and results about splitting a Carnot groups into the product of complementary subgroups and about regularity for maps acting between homogeneous subgroups (for more details, please refer to $[25, \text{Section 4}]$).

**Definition 2.** Let $\mathcal{G}$ be a Carnot group. Two homogeneous subgroups $\mathcal{W}, \mathcal{V}$ are said complementary subgroups of $\mathcal{G}$ if $\mathcal{W} \cap \mathcal{V} = \{0\}$ and for every $g \in \mathcal{G}$, there exist $w \in \mathcal{W}$, $v \in \mathcal{V}$ such that: $g = wv$, i.e. $\mathcal{G} = \mathcal{WV}$.

We denote by $\pi_\mathcal{W} : \mathcal{G} \to \mathcal{W}$ and $\pi_\mathcal{V} : \mathcal{G} \to \mathcal{V}$ the group projections on the subgroups: if $g = wv$ with $w \in \mathcal{W}$, $v \in \mathcal{V}$, $\pi_\mathcal{W}(g) = w$ and $\pi_\mathcal{V}(g) = v$.

If $r > 0$ and $w \in \mathcal{W}$, we denote by $B_\mathcal{W}(w, r) := B(w, r) \cap \mathcal{W}$ and if $v \in \mathcal{V}$, $B_\mathcal{V}(v, r) := B(v, r) \cap \mathcal{V}$.
Proposition 2.2. [6, Proposition 2.12] If $G = WV$ is the product of two complementary subgroups, there exists $c_0 = c_0(W, V) > 0$ such that

\[(2) \quad c_0(||w|| + ||v||) \leq ||wv|| \leq ||w|| + ||v||\]

for all $w \in W$, $v \in V$.

Definition 3. Let $G = WV$ be the product of two complementary subgroups and let $U \subset W$ be a set. If we consider a map $\phi : U \to V$, we define its intrinsic graph as the set

\[\text{graph}(\phi) = \{w\phi(w) : w \in U\}.\]

The map $\Phi : U \to \text{graph}(\phi)$, $\Phi(w) := w\phi(w)$ is called the graph map of $\phi$.

Definition 4. Let $G = WV$ be the product of two complementary subgroups and $L$ be a constant. Let $U \subset W$ be open, we say that a function $\phi : U \to V$ is called intrinsic $L$-Lipschitz if

\[(3) \quad ||\pi_V(\Phi(w')^{-1}\Phi(w))|| \leq L||\pi_W(\Phi(w')^{-1}\Phi(w))||\]

for every $w, w' \in U$.

Definition 5 (Carathéodory’s construction). Let $F \subset \mathcal{P}(G)$ be a non-empty family of closed subsets of a Carnot group $G$, equipped with a homogeneous distance $d$. Let $\zeta : F \to \mathbb{R}^+$ be a function such that $0 \leq \zeta(S) < \infty$ for any $S \in F$. If $\delta > 0$, and $A \subset G$, we define

\[(4) \quad \phi_{\delta, \zeta}(A) = \inf \left\{ \sum_{j=0}^{\infty} \zeta(B_j) : A \subset \bigcup_{j=0}^{\infty} B_j, \; \text{diam}(B_j) \leq 2\delta, \; B_j \in F \right\},\]

If $F$ coincides with the family of closed balls, $F_b$, with respect to the distance $d$ and $\zeta(B(x, r)) = r^\alpha$ we call

\[\mathcal{S}^\alpha(A) := \sup_{\delta > 0} \phi_{\delta, \zeta}(A)\]

the $\alpha$-spherical Hausdorff measure of $A$.

If $G$ is a Carnot group of topological dimension $q$ and homogeneous dimension $Q$, then $Q$ is the metric dimension of $G$ with respect to any homogeneous distance $d$. The spherical measures $\mathcal{S}^m$ are invariant by left translation, hence, for any positive $m$, $\mathcal{S}^m(\tau_x(A)) = \mathcal{S}^m(A)$ for every $x \in G$ and $A \subset G$. By uniqueness of the Haar measure, $\mathcal{S}^Q$ coincides up to a constant with the Lebesgue measure $\mathcal{L}^q$ (for more details please refer to [25, Propositions 2.19, 2.32]). Moreover, for any positive $m$, $\mathcal{S}^m(\delta t(x)) = t^m \mathcal{S}^m(A)$ for every $t > 0$ and $A \subset G$.

Definition 6 (Packing). Let $N, \ell$ be two natural numbers, with $\ell \geq 1$. Let $X$ be a metric space. A $\ell$-packing is a countable collection of closed balls $\{B_j\}$ such that concentric balls $\ell B_j$ are pairwise disjoint. An $(N, \ell)$-packing is a collection of balls $\{B_j\}$ which is the union of at most $N \ell$-packings

Remark 1. In a doubling metric space it is not restrictive to assume that once fixed a number $\ell \geq 1$, there exists a natural number $N$, only depending on $\ell$, such that there exists a $(N, \ell)$-packing that covers the whole space. For instance, in [23, Remark 4] it is proved that if a metric space $X$ is doubling at small scales, fine covering of $(N, \ell)$-packings exist with $N$ depending only on $\ell$. 

Definition 7 (Packing premeasure). Let $\ell \geq 1$ and $N$ be natural numbers. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance $d$ and let be $\delta > 0$, $\alpha > 0$; let $E \subset \mathbb{G}$, we introduce

$$\mathcal{P} \Delta_{d,\delta,N,\ell}(E) = \sup \left\{ \sum_i r(B_i)^\alpha : \{B_i\} (N, \ell)\text{-packing of } E, \right. \left. E \subseteq \bigcup_i B_i, B_i \text{ centered on } E, r(B_i) \leq \delta \right\}$$

and define

$$\mathcal{P} \Delta_{d,\delta,N,\ell}(E) := \inf_{\delta > 0} \mathcal{P} \Delta_{d,\delta,N,\ell}(E).$$

We define also a modified packing object. In particular, in this case we do not require the packings to cover the set,

$$\tilde{\mathcal{P}} \Delta_{d,\delta,N,\ell}(E) = \sup \left\{ \sum_i r(B_i)^\alpha : \{B_i\} (N, \ell)\text{-packing of } E, B_i \text{ centered on } E, r(B_i) \leq \delta \right\},$$

and

$$\tilde{\mathcal{P}} \Delta_{d,\delta,N,\ell}(E) := \inf_{\delta > 0} \tilde{\mathcal{P}} \Delta_{d,\delta,N,\ell}(E).$$

We denote by $\mathcal{H}^\alpha_E$ the Hausdorff measure on $\mathbb{G}$ i.e. the measure obtained by Carathéodory’s construction assuming $\mathcal{F}$ is the family of all closed sets and

$$\zeta(B) = \frac{\mathcal{L}^\alpha(\{ y \in \mathbb{R}^\alpha : |y| \leq 1 \})}{2^\alpha} \text{diam}(B)^\alpha.$$

We will denote the closed Euclidean ball of center $x$ and radius $r > 0$ by $B_E(x,r) = \{ x \in \mathbb{G} : |x| \leq r \}$.

From now on, we consider two Carnot groups endowed with homogeneous distances $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$, of metric dimension $Q$ and $P$ and topological dimension $q$ and $p$, respectively. The Lie algebras of $\mathbb{G}$ and $\mathbb{M}$ are stratified and we identify as above Lie$(\mathbb{G})$ with $\mathbb{G}$ and Lie$(\mathbb{M})$ with $\mathbb{M}$ so the groups can be seen as direct sum of linear subspaces

$$\mathbb{G} = V_1 \oplus \cdots \oplus V_k \quad \mathbb{M} = W_1 \oplus \cdots \oplus W_M.$$

We denote by $\delta_1^t$ and $\delta_1^t$ the natural anisotropic dilations of parameter $t > 0$ on $\mathbb{G}$ and $\mathbb{M}$, respectively.

A map $L : \mathbb{G} \to \mathbb{M}$ is a h-homomorphism if it is a group homomorphism such that $L(\delta_1^t(x)) = \delta_1^t(L(x))$ for any $x \in \mathbb{G}$ and $t > 0$. In this case we say $L \in \mathcal{L}(\mathbb{G}, \mathbb{M})$.

Given two h-homomorphisms $L, T \in \mathcal{L}(\mathbb{G}, \mathbb{M})$, we define the distance $d_{\mathcal{L}(\mathbb{G}, \mathbb{M})}(L, T) := \sup_{q \in B(0,1)} d_2(L(q), T(q))$ and we denote by $\|L\|_{\mathcal{L}(\mathbb{G}, \mathbb{M})} := d_{\mathcal{L}(\mathbb{G}, \mathbb{M})}(L, I)$, where $I : \mathbb{G} \to \mathbb{M}$ denotes the map that associates to any point of $\mathbb{G}$ the unit element of $\mathbb{M}$.

If we identify $\mathbb{G}$ with $\mathbb{R}^q$ and $\mathbb{M}$ with $\mathbb{R}^p$ through two fixed bases as in (1), $L$ is a linear map from $\mathbb{R}^q$ to $\mathbb{R}^p$. Then we can consider its Jacobian $|L| = \sqrt{\det(LL^*)}$. Observe that $|L|$ is the Euclidean algebraic Jacobian of $L$ from $\mathbb{R}^q$ to $\mathbb{R}^p$, or, equivalently, it is the Jacobian of $L$ between the two Lie algebras $\mathbb{G}$ and $\mathbb{M}$ with respect to the fixed scalar products. For more details about h-homomorphisms, please refer to [10] Section 3.1.

An invertible h-homomorphism is called a h-isomorphism.

If $L : \mathbb{G} \to \mathbb{M}$ is a surjective h-homomorphism and $\mathbb{N}$ is its kernel, we call $L$ a h-epimorphism if there exists a homogeneous subgroup of $\mathbb{G}$, $\mathbb{H}$, complementary to $\mathbb{N}$. In this
case the restriction $L|_{\mathbb{H}}$ is a $h$-isomorphism (for more details please refer to [14] Definition 2.2, Proposition 7.14).

We denote by $\|x\|_1 := d_1(x, 0)$ for every $x \in \mathbb{G}$ and by $\|x\|_2 := d_2(x, 0)$ for every $x \in \mathbb{M}$.

Let $\Omega$ be an open set in $\mathbb{G}$ and $f : \Omega \to \mathbb{M}$ be a continuous function. Fix a point $x \in \Omega$. If there exists a $h$-homomorphism $L : \mathbb{G} \to \mathbb{M}$ that satisfies

$$\|L(x^{-1}y)^{-1}f(x)^{-1}f(y)\|_2 = o(\|x^{-1}y\|_1) \quad \text{as} \quad \|x^{-1}y\|_1 \to 0,$$

then $f$ is said to be Pansu differentiable at $x$. If such a map $L$ exists, it is unique and it is called the Pansu differential of $f$ at $x$; we denote it by $Df(x)$. This definition has been introduced in [22]. We say that $f \in C^1_0(\Omega, \mathbb{M})$ or that $f$ is continuously Pansu differentiable on $\Omega$ if the function $Df : \Omega \to L(\mathbb{G}, \mathbb{M})$ is continuous.

**Proposition 2.3.** Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups and let $\Omega \subset \mathbb{G}$ be an open set. If $f \in C^1_0(\Omega, \mathbb{M})$, the function $\|Df\|_{L(\mathbb{G}, \mathbb{M})} : \Omega \to \mathbb{R}, x \to \|Df(x)\|_{L(\mathbb{G}, \mathbb{M})}$ is continuous.

**Definition 8.** [16] Definition 4.5 Let $f : K \to Y$ be a vector-valued continuous function from a compact metric space $(K, d_1)$ to a vector-valued metric space $(Y, d_2)$. Then we define the modulus of continuity of $f$ on $K$ as

$$\omega_{K,f}(t) = \max_{x,y \in K \atop d_1(x,y) \leq t} d_2(f(x), f(y)).$$

If we consider an open set $\Omega \subset \mathbb{G}$ and a map $f : \Omega \subset \mathbb{G} \to \mathbb{M}$; for $j = 1, \ldots, M$, we call $F_j := \pi_j \circ f$, where $\pi_j : \mathbb{M} \to W_j$ is the orthogonal projections onto the $j$-th layers of $\mathbb{M}$. If $f$ is Pansu differentiable at $x \in \Omega$, by [16] Theorem 4.12, $F_1$ is Pansu differentiable at $x$ and $DF_1(x) = \pi_1 \circ Df(x)$ (notice that $DF_1 : \Omega \to W_1$).

**Theorem 2.4.** [16] Theorem 1.2 Let $(\mathbb{G}, d_1)$ and $(\mathbb{M}, d_2)$ be two Carnot groups endowed with homogeneous distances. Let $k$ be the step of $\mathbb{G}$ and $\Omega \subset \mathbb{G}$ be an open subset. Let us consider a map $f \in C^1_0(\Omega, \mathbb{M})$. Let $\Omega_1, \Omega_2 \subset \mathbb{G}$ be two open subsets of $\mathbb{G}$ such that

$$\{x \in \mathbb{G} : \text{d}(x, \Omega_1) \leq c\text{diam}(\Omega_1)\} \subset \Omega_2$$

with $\Omega_2$ compactly contained in $\Omega$, and $c = c(\mathbb{G}, d_1)$ and $H = H(\mathbb{G}, d_1)$ geometric constants, only depending on $(\mathbb{G}, d_1)$. Then there exists a constant $C$, only depending on $\mathbb{G}$, $\max_{x \in \Omega_2} \|DF_1(x)\|_{L(\mathbb{G}, W_1)}$, and on the modulus of continuity $\omega_{\Omega_1, DF_1}$ such that

$$\frac{d_2(f(x)^{-1}f(y), Df(x)(x^{-1}y))}{d_1(x, y)} \leq C[\omega_{\Omega_1, DF_1}(cHd_1(x, y))]^{1/2}.$$ 

for every $x, y \in \Omega_1$ with $x \neq y$.

**Remark 2.** By [16] Theorem 4.12, if $f$ is continuously Pansu differentiable, then $x \to DF_1(x)$ is a continuous map from $\Omega$ to $L(\mathbb{G}, W_1)$, and so by Proposition 2.3 the modulus of continuity $\omega_{\Omega_1, DF_1}$ goes to zero as $s$ goes to zero.

**Definition 9** (Coarea factor). Let $L : \mathbb{G} \to \mathbb{M}$ be a $h$-homomorphism and let be $Q \geq P$. We call coarea factor of $L$, $C_P(L)$, the unique constant such that

$$S^Q(B(0, 1))C_P(L) = \int_{\mathbb{M}} S^{Q-P}(L^{-1}(\xi) \cap B(0, 1))dS^P(\xi).$$

By [11] Proposition 1.12, and so in particular by the left invariance of the involved spherical Hausdorff measures, by uniqueness of the Haar measure on Carnot groups and
by the Euclidean coarea formula, $C_P(L)$ is well defined, and it is not zero if and only if $L$ is surjective and in this case it can be computed as follows. By $B_G(0,1)$ and $B_M(0,1)$ here we denote the closed unitary balls in $G$ and $M$, respectively, then

$$C_P(L) = \frac{S^{Q-P}(\ker(L) \cap B_G(0,1))}{\mathcal{H}^{Q-P}(\ker(L) \cap B_G(0,1))} \frac{S^P(B_M(0,1))}{\mathcal{L}^P(B_M(0,1))} \frac{\mathcal{L}^q(B_G(0,1))}{S^q(B_M(0,1))} |L|$$

(5)

$$= Z \frac{S^{Q-P}(\ker(L) \cap B_G(0,1))}{\mathcal{H}^{Q-P}(\ker(L) \cap B_G(0,1))} |L|$$

$$= Z \frac{S^{Q-P}(\ker(L) \cap B(0,1))}{\mathcal{H}^{Q-P}(\ker(L) \cap B(0,1))} |L|,$$

where $Z = \frac{S^q(B_M(0,1))}{\mathcal{L}^q(B_G(0,1))} \frac{\mathcal{L}^q(B_M(0,1))}{S^q(B_M(0,1))}$. Observe that $Z$ is a geometrical constant not depending on $L$. Notice that in the last line of (5) we have dropped the subscript $G$ to denote the ball $B_G(0,1)$. This will be done also later in the paper.

We introduce now the definition of Federer density.

**Definition 10.** Let $G$ be a Carnot group endowed with a homogeneous distance $d$. Let $\mathcal{F}_b$ be the family of closed balls with positive radius in $G$. Let $\alpha > 0$, $x \in G$ and $\mu$ be a Borel regular measure on $G$. We call spherical $\alpha$-Federer density of $\mu$ at $x$ the real number

$$\theta^\alpha(\mu, x) := \inf_{\epsilon > 0} \sup \left\{ \mu(B) : x \in B \in \mathcal{F}_b, \text{ diam}(B) < \epsilon \right\}.$$ 

This density is the right object to represent the abstract way to differentiate any Borel regular measure absolutely continuous with respect to the $\alpha$-spherical one in a metric space satisfying general hypothesis.

**Theorem 2.5 ([13 Theorem 7.2]).** Let $\alpha > 0$ and let $\mu$ be a Borel regular measure on $G$ such that there exists an open numerable covering of $G$ whose elements have $\mu$-finite measure. Let $d$ be a homogeneous distance. If $B \subset A \subset G$ are Borel sets, then $\theta^\alpha(\mu, \cdot)$ is a Borel function on $A$. In addition, if $S^\alpha(A) < \infty$ and $\mu \ll A$ is absolutely continuous with respect to $S^\alpha \ll A$, then we have

$$\mu(B) = \int_B \theta^\alpha(\mu, x) \ dS^\alpha(x).$$

**Definition 11.** Let $(X, d, \mu)$ be a metric measure space, consider a subset $E \subset X$ and two positive numbers $\alpha, C > 0$. We say that $E$ is locally $C$-lower Ahlfors $\alpha$-regular with respect to $\mu$ if there is $\tilde{r} > 0$ such that for every $x \in E$, $0 < r < \tilde{r}$,

$$\mu(B(x, r) \cap E) \geq Cr^\alpha.$$ 

If we need to stress the value of $\tilde{r}$, we say that $E$ is $\tilde{r}$-locally $C$-lower Ahlfors $\alpha$-regular with respect to $\mu$. If $\tilde{r} = \infty$, we say that $E$ is $C$-lower Ahlfors $\alpha$-regular with respect to $\mu$.

Now we report a result contained in [7, Lemma 2.10], that is, up to now, the most general available implicit function theorem into this setting. Similar statements are proved in [16, Theorem 1.4] and [4, Theorem 3.27].

**Theorem 2.6.** Let $G$ and $M$ be two Carnot groups and let $\Omega \subset G$ be open. Let $f \in C^1_0(\Omega, M)$ be a function and fix $x_0 \in \Omega$. Assume that $Df(x_0)$ is a surjective $h$-epimorphism
and consider $V$ a subgroup complementary to $\ker(Df(x_0))$. Fix a homogeneous subgroup $W$ complementary to $V$. Write $x_0 = w_0 a_0$ with respect to the splitting $WV$.

Then there exist an open set $A \subset W$, with $w_0 \in A$, and a continuous map $\phi : A \to V$ such that $f(a\phi(a)) = f(x_0)$ for every $a \in A$.

**Remark 3.** By [7, Corollary 2.16], the parametrization $\phi$ given by Theorem 2.6 is $L$-Lipschitz for some positive constant $L$.

**Proposition 2.7.** [6, Theorem 3.9] Let $G = WV$ be the product of two complementary subgroups endowed with a homogeneous distance $d$. Let be $N$ be the metric dimension of $W$. If $\phi : W \to V$ is intrinsic $L$-Lipschitz in $W$, then
\[
\left( \frac{c_0(W, V)}{1 + L} \right)^N r^N \leq S^N(\text{graph}(\phi) \cap B(x, r))
\]
for all $x \in \text{graph}(\phi)$ and $r > 0$.

**Remark 4.** The proof of Proposition 2.7 is based on local arguments. Let us now assume that $\phi$ is defined from an open set $A \subset W$ to $V$ and $w \in A$. Assume that, for a positive constant $\tilde{r} > 0$, $\pi_W(B(\Phi(w), \tilde{r})) \subset A$, then for any $0 < r < \tilde{r}$ we have that $S^N(\text{graph}(\phi) \cap B(\Phi(w), r)) \geq \left( \frac{c_0(W, V)}{1 + L} \right)^N r^N$.

In [11], the author proved a coarea-type inequality. We recall it here, adapting it to our context.

**Theorem 2.8.** [11, Theorem 2.6] Let $A \subseteq G$ be a measurable set and $f : A \to M$ be a Lipschitz map, then
\[
\int_M S^{Q-P}(f^{-1}(m) \cap A) dS^P(m) \leq \int_A C_P(Df(x))dS^Q(x).
\]

### 3. Coarea-type Inequality

We will need the following simple proposition in order to prove the main theorem.

**Proposition 3.1.** Let $G$ be a Carnot group endowed with a homogeneous distance $d$. Let $W \subset G$ be a homogeneous subgroup of topological dimension $n$ and metric dimension $N$. Then for every Borel set $B \subset W$ we have that
\[
\mathcal{L}^n(B) = \sup_{w \in B(0,1)} \mathcal{H}^n_E(B(w, 1) \cap W) S^N(B)
\]

**Proof.** Let us consider $\mu_W(A) := \mathcal{H}^n_E \cap W(A) = \mathcal{L}^n(W \cap A)$ for any set $A \subset G$. Since both $\mu_W$ and $S^N \cap W$ are left invariant on $W$, they coincide up to a constant, so we can apply Theorem 2.5, hence we know that $\mu_W(A) = \int_A \theta^N(\mu_W, x) S^N(x)$. Let us then compute for any $x \in W$
\[
\theta^N(\mu_W, x) = \inf_{r>0} \sup_{\{z : x \in B(z, t)\}} \frac{\mu_W(B(z, t))}{t^N}
\]
notice that for every \( t > 0 \), \( z \in B(x,t) \),
\[
\frac{\mu_{\mathcal{W}}(B(z,t))}{t^n} = \mathcal{L}^n(z\delta_t(B(0,1)) \cap \mathbb{W})
= \mathcal{L}^n(x^{-1}z\delta_t(B(0,1)) \cap x^{-1}\mathbb{W})
= \mathcal{L}^n(\delta_{1/t}(x^{-1}z)B(0,1) \cap \mathbb{W})
\]
hence
\[
\theta^N(\mu_{\mathcal{W}}, x) = \inf_{r>0} \sup_{\{w: d(w,0) \leq 1\}} \mathcal{H}^n_E(B(w, 1) \cap \mathbb{W}) = \sup_{w \in B(0,1)} \mathcal{H}^n_E(B(w, 1) \cap \mathbb{W}).
\]

Before proving our main result, Theorem 1.1, we make a preliminary observation.

**Remark 5.** It is immediate to observe that any continuously Pansu differentiable function is locally metric Lipschitz, hence by [15, Theorem 2.1] we have that for every measurable set \( A \subset \mathbb{G} \) the function \( m \to \mathcal{S}^Q_{\mathcal{P}}(A \cap f^{-1}(m)) \) is \( \mathcal{S}_{\mathcal{P}} \)-measurable. In particular this result follows by [15, Theorem 2.1] combined with Monotone Convergence Theorem applied, for every \( m \in \mathbb{M} \), to the sequence of characteristic functions \( 1_{\Omega_m \cap f^{-1}(m)} \to 1_{A \cap f^{-1}(m)} \), where we have considered an exhaustion of compact sets for \( A, \Omega_m \to A \). Notice that the measurability of the map \( x \to \mathcal{C}_\mathcal{P}(Df(x)) \) follows by [11, Theorem 2.6].

Theorem 1.1 is a direct consequence of the following result. In fact, it is analogous to Theorem 3.2 where hypotheses have been slightly weakened even if they are definitely more technical.

**Theorem 3.2.** Let \((\mathbb{G}, d_1), (\mathbb{M}, d_2)\) be two Carnot groups, endowed with homogeneous distances, of metric dimension \( Q \), \( \mathcal{P} \) and topological dimension \( q, p \), respectively. Let \( \Omega' \) be an open subset of \( \mathbb{G} \). Let \( f \in C_0^1(\Omega', \mathbb{M}) \) be a function and assume \( Df(x) \) is surjective at every \( x \in \Omega' \).

Let \( \Omega \subset \Omega' \) be a closed bounded set such that there exists an open set \( \Omega'' \) and a positive number \( s > 0 \) such that \( \Omega'' \) is compactly contained in \( \Omega \) and if we call \( \Omega_s := \{ x \in \mathbb{G} : d(x, \Omega) < s \} \) and we call \( R := cHdiam(\Omega_s) \),
\[
(7) \quad \Omega^*_R := \{ x \in \mathbb{G} : d(x, \Omega_s) \leq R \} \subset \Omega''
\]
where \( H = H(\mathbb{G}, d_1), c = c(\mathbb{G}, d_1) \) are geometric constants that depend on \( \mathbb{G} \) and \( d_1 \) as in Theorem 2.2. Assume that there exist two constants \( \tilde{r}, C > 0 \) such that for any \( m \in \mathbb{M} \), the level set \( f^{-1}(m) \) is \( \tilde{r} \)-locally \( C \)-lower Ahlfors \((Q-P)\)-regular with respect to the measure \( \mathcal{S}^Q_{\mathcal{P}} \). Then there exists a constant \( L = L(C, \mathbb{G}, p) \) such that
\[
\int_\Omega C_P(Df(x)) d\mathcal{S}^Q(x) \leq L \int_M \mathcal{S}^Q_{\mathcal{P}}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m).
\]

**Proof.** In the proof we will denote by \( B(x,r) \) the closed metric ball of \( \mathbb{G} \) with respect of \( d_1 \) with center \( x \) and radius \( r \). Let us preliminarily introduce for \( \delta > 0 \), and \( E \subset \mathbb{G} \)
\[
\mathcal{S}_{d_1, \delta, \zeta}(E) = \inf \left\{ \sum \zeta_1(B_i) : B_i \text{ closed balls, } E \subset \bigcup_i B_i, \ r(B_i) \leq \delta \right\}
\]
with \( \zeta_{1}(B(x, r)) = r^{Q-P} S^P(f(B(x, r))) \). Define the Carathéodory's measure
\[
\mathcal{S}_{d_1, \zeta_1}(E) := \sup_{\delta > 0} \mathcal{S}_{d_1, \delta, \zeta_1}(E).
\]

Let be \( \ell = 1 \) and fix \( N = N(2, \mathbb{G}) \) the minimum natural number such that there exists a \((N, 2)\)-packing of \( \mathbb{G} \) that covers \( \mathbb{G} \) itself. We define
\[
\mathcal{P}_{d_1, \delta, \zeta_1}(E) = \sup \{ \sum \zeta_1(B_i) : \{ B_i \} (N, \ell)\)-packing of \( E \),
\]
\[
\text{B}_i \text{ centered on } E, \ E \subseteq \bigcup \text{B}_i, \ r(B_i) \leq \delta \}.
\]
and
\[
\mathcal{P}_{d_1, \delta, \zeta_1}(E) = \inf_{\delta > 0} \mathcal{P}_{d_1, \delta, \zeta_1}(E).
\]

Claim 1
\[
\mathcal{S}_{d_1, \zeta_1}(\Omega) \leq \mathcal{P}_{d_1, \zeta_1, N, 2}(\Omega) \leq \int_{\mathcal{M}} \mathcal{P}_{\Delta d_1, \zeta_1, N, 1}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m)
\]

Proof. We follow the scheme of [23, Proposition 15]. Let \( \{ B_i \} \) be a \((N, 2)\)-packing of \( \Omega \) that covers \( \Omega \) with \( r(B_i) \leq \delta \) and with \( B_i \) centered on \( \Omega \). Consider any ball \( B_i \) and choose \( m \in f(B_i) \). Pick \( x_i \in B_i \cap f^{-1}(m) \cap \Omega \) and call \( B_i, m \) the smallest ball centered at \( x_i \) that contains \( B_i \). Observe that \( B_i, m \subset 2B_i \), so that, for each \( m \in \mathcal{M} \), the collection \( \{ B_i, m \} \) is a \((N, 1)\)-packing of \( f^{-1}(m) \cap \Omega \) consisting of balls with radius less or equal than \( 2\delta \) centered on \( f^{-1}(m) \cap \Omega \). Then
\[
\sum_i r(B_i)^{Q-P} S^P(f(B_i)) = \sum_i (\int_{M} 1_{f(B_i)}(m) d\mathcal{S}^P(m))(r(B_i))^{Q-P}
\]
\[
= \int_{M} \sum_i 1_{f(B_i)}(m)(r(B_i))^{Q-P} d\mathcal{S}^P(m)
\]
\[
= \int_{M} \sum_{i : m \in f(B_i)} (r(B_i))^{Q-P} d\mathcal{S}^P(m)
\]
\[
\leq \int_{M} \sum_{i : m \in 2B_i, m} (r(B_i, m))^{Q-P} d\mathcal{S}^P(m)
\]
\[
\leq \int_{M} \mathcal{P}_{\Delta d_1, \delta, N, 1}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m).
\]
Hence, by Monotone Convergence Theorem
\[
\mathcal{P}_{d_1, \zeta_1, N, 2}(\Omega) \leq \int_{M} \mathcal{P}_{\Delta d_1, N, 1}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m).
\]

Let us now observe that
\[
\mathcal{S}_{d_1, \zeta_1}(\Omega) \leq \mathcal{P}_{d_1, \zeta_1, N, 2}(\Omega).
\]
Any \((N, 2)\)-packing of \( \Omega \) that covers \( \Omega \) with balls centered on \( \Omega \) of radius smaller that \( \delta \) is a covering of \( \Omega \) of balls of radius smaller than \( \delta \) so, surely, for any \( \delta \)
\[
\mathcal{S}_{d_1, \delta, \zeta_1}(\Omega) \leq \mathcal{P}_{d_1, \delta, \zeta_1, N, 2}(\Omega)
\]
so letting \( \delta \) go to zero, we get the thesis.
Claim 2 There exists a constant $T = T(C, \mathcal{G})$ such that for every $m \in \mathcal{M}$
\[ \mathcal{P} \Delta_{d_1,N,1}^{Q-P}(f^{-1}(m) \cap \Omega) \leq T \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega). \]

Proof. Of course, for any $\delta > 0$ small enough (i.e. $\delta < \hat{r}$) and $m \in \mathcal{M}$ we have that
\[ \mathcal{P} \Delta_{d_1,\hat{r},N,1}^{Q-P}(f^{-1}(m) \cap \Omega) \leq N \hat{\mathcal{P}} \Delta_{d_1,\hat{r},1,1}^{Q-P}(f^{-1}(m) \cap \Omega). \]
Let us consider a $(1,1)$-packing $\{B_i\}$ of balls of radius smaller than $\delta$, centered on $f^{-1}(m) \cap \Omega$ such that
\[ \sum_i r(B_i)^{Q-P} \geq \frac{1}{2} \hat{\mathcal{P}} \Delta_{d_1,\hat{r},1,1}^{Q-P}(f^{-1}(m) \cap \Omega). \]
Then
\[ \mathcal{P} \Delta_{d_1,\hat{r},N,1}^{Q-P}(f^{-1}(m) \cap \Omega) \leq N \hat{\mathcal{P}} \Delta_{d_1,\hat{r},1,1}^{Q-P}(f^{-1}(m) \cap \Omega) \]
\[ \leq 2N \sum_i r(B_i)^{Q-P} \]
\[ \leq 2 \frac{N}{C} \mathcal{S}^{Q-P}(f^{-1}(m) \cap B_i) \]
\[ = 2 \frac{N}{C} \mathcal{S}^{Q-P}(f^{-1}(m) \cap B_i \cap \overline{\Omega_\delta}) \]
\[ \leq 2 \frac{N}{C} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{\Omega_\delta}), \]
where $\Omega_\delta := \{x \in \mathcal{G} : d(x, \Omega) < \delta\}$.

Now we let $\delta$ go to zero. Since $\Omega$ is closed, for every $m \in \mathcal{M}$, $f^{-1}(m) \cap \overline{\Omega_\delta} \setminus f^{-1}(m) \cap \Omega$ as $\delta \to 0$, hence by Monotone convergence theorem, for every $m \in \mathcal{M}$ we get
\[ \mathcal{P} \Delta_{d_1,N,1}^{Q-P}(f^{-1}(m) \cap \Omega) \leq T \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega), \]
where $T = \frac{2N}{C} = \frac{2N(2,\mathcal{G})}{C}$, hence $T = T(C, \mathcal{G})$.

By combining Claim 1. and Claim 2, we obtain the existence of a constant $T = T(C, \mathcal{G})$ such that
\[ \mathcal{S} f_{d_1,\hat{r},1}(\Omega) \leq \mathcal{P} f_{d_1,\hat{r},N,2}(\Omega) \leq T \int_{\mathcal{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega)d\mathcal{S}^P(m). \]

From now on, we denote by $\delta_i^t$ the intrinsic dilations by $t > 0$, on $\mathcal{G}$ for $i = 1$ and on $\mathcal{M}$ for $i = 2$, respectively.

Claim 3 If $k$ is the step of $\mathcal{G}$,
\[ \mathcal{S} f_{d_1,\hat{r},1}(\Omega) \gtrsim_{k,p} \int_{\Omega} |Df(x)|d\mathcal{S}^Q(x). \]

Proof. The proof is composed of two main steps.

First, we can observe that $Df(x)$ is a continuous function on $\Omega'$, so we can consider the following measure on $\Omega'$: for any $A \subset \Omega'$, $\mu(A) := \int_A |Df(x)|d\mathcal{S}^Q(x)$. We want to compare $\mu$ with the Carathéodory’s measure built with coverings of closed balls measured by the function
\[ \zeta(B(x,r)) := |Df(x)|r^Q. \]
We denote this measure by $Sf_{d_1, \zeta_2} = \sup_{\delta > 0} \delta \delta \zeta_2$.

We want to prove that there exists $\bar{r} > 0$ such that for every $0 < r \leq \bar{r}$ and for every $x \in \Omega$, $\mu(B(x, r)) \lesssim \zeta_2(B(x, r))$. By Federer 2.10.17 (1), this implies that $\mu(A) \lesssim Sf_{d_1, \zeta_2}(A)$ for any $A \subseteq \Omega$ (so also for $A = \Omega$).

Since $\Omega$ is closed and bounded, it is compact. The function $|Df(\cdot)| : \Omega \to \mathbb{R}$, $x \to |Df(x)|$ is a continuous function on a compact set. Let us fix $\epsilon = \min_{x \in \Omega} |Df(x)| > 0$; it is positive since $Df(x)$ is everywhere surjective by hypothesis. Moreover, the map $|Df(\cdot)|$ is uniformly continuous, then there exists $r' > 0$ such that $|Df(x) - Df(y)| \leq \epsilon$ if $|x - y| \leq r'$, $x, y \in \Omega$.

Let us fix $\bar{r} < \min\{r', s\}$. Let us fix any $x \in \Omega$ and $0 < r \leq \bar{r}$ and let us study

$$
\frac{\mu(B(x, r))}{|Df(x)| r^Q} = \frac{1}{r^Q} \int_{B(x, r)} \frac{|Df(y)|}{|Df(x)|} dS^Q(y)
\leq \frac{1}{r^Q} \int_{B(x, r)} \frac{|Df(y) - Df(x)|}{|Df(x)|} dS^Q(y) + \frac{1}{r^Q} \int_{B(x, r)} \frac{|Df(x)|}{|Df(x)|} dS^Q(y)
\leq \frac{1}{r^Q} \int_{B(x, r)} \frac{\epsilon}{\min_{z \in \Omega} |Df(z)|} S^Q(y) + S^Q(B(0, 1)) = 2S^Q(B(0, 1)) = 2b;
$$

where $0 < b = S^Q(B(0, 1)) < \infty$, hence for any $x \in \Omega$ and $0 < r \leq \bar{r}$, we have that

$$
\frac{\mu(B(x, r))}{|Df(x)| r^Q} \leq 2b
$$

and then, as we said above, $\mu(\Omega) \leq 2b Sf_{d_1, \zeta_2}(\Omega)$, so $\mu(\Omega) \lesssim Sf_{d_1, \zeta_2}(\Omega)$.

Second part of the proof, we want to compare $Sf_{d_1, \zeta_1}(\Omega)$ with $Sf_{d_1, \zeta_2}(\Omega)$, and, in particular, we want to prove that

$$
Sf_{d_1, \zeta_2}(\Omega) \lesssim_{k, p} Sf_{d_1, \zeta_1}(\Omega).
$$

As we observed, the measure $Sf_{d_1, \zeta_1}$ is defined as a Carathéodory’s measure defined with coverings of closed balls measured by the function $\zeta_1(B(x, r)) = r^{Q - p} S^p(f(B(x, r)))$. The strategy will rely on the comparison between $\zeta_1$ and $\zeta_2$. In particular we fix $h > 0$ such that $h < s$. We want to prove that there exists $\bar{r} > 0$ such that for every $0 < r \leq \bar{r}$ for every $x \in \Omega_h := \{y \in G : \text{dist}(y, \Omega) < h\}$,

$$
\zeta_1(B(x, r)) \gtrsim_{k, p} \zeta_2(B(x, r)), \tag{8}
$$

this would give the desired thesis.

Let us first define for $x \in \Omega_h$ and $r > 0$,

$$
A_{x,r} := \frac{\delta_1^2}{\bar{r}^2} (f(x)^{-1} f(B(x, r))), \quad A_x := Df(x)(B(0, 1)).
$$

The proof will be composed of various steps, and it will be useful to give name to the following conditions:

(a) $\lim_{r \to 0} \sup_{x \in \Omega_h} \left| 1_{A_{x,r}}(m) - 1_{A_x}(m) \right| = 0$ for any $m \in \mathbb{M}$;

(b) $\lim_{r \to 0} \sup_{x \in \Omega_h} |S^p(A_{x,r}) - S^p(A_x)|$.

We will first prove that (b) implies the thesis [3].
Let \( x \in \Omega_h \) and denote by \( V(x) := (\ker(Df(x))^\perp \). For every \( r \) small enough
\[
\zeta_1(B(x, r)) = r^{Q-P} S^P(B(x, r)) = r^Q \frac{S^P(f(B(x, r)))}{r^P} = r^Q \frac{S^P(f^{-1}(B(x, r)))}{r^P}
\]
hence
\[
\zeta_1(B(x, r)) = \frac{S^P(\delta^2_{1/r}(f^{-1}f(B(x, r))))}{|Df(x)|}.
\]
Observe that the map \( Df(x)|_{V(x)} : V(x) \rightarrow \mathbb{M} \) is injective and surjective and that
\( |Df(x)| = |Df(x)|_{V(x)} | \), by the choice of \( V(x) \). If \( \pi_{V(x)} \) is the orthogonal projection on
\( V(x) \), for some geometric constant \( G \) we have that
\[
S^P(Df(x)(B(0, 1))) = G \mathcal{L}^P(Df(x)(B(0, 1))) = G|Df(x)| \mathcal{L}^p(\pi_{V(x)}(B(0, 1))
\]
by the classical Euclidean area formula; (see also \[17, Lemma 9.2\]), that ensures that one can see any element \( y \in \mathbb{G} \) as \( y = m_y(\pi_{V(x)}(y)) \), with \( m_y \in \ker(Df(x)) \). Observe that for every \( x \in \Omega_h, V(x) = (\ker(Df(x))^\perp \) is a linear subspace of constant topological dimension \( p > 0 \), since \( Df(x) \) is surjective at any point \( x \in \Omega_h \). We notice that the factor \( \mathcal{L}^p(V(x) \cap B_{E}(0, 1)) \) does not depend on \( x \). Remember now that by Proposition \[2.1\] applied to \( K = B(0, 1) \) there exists a constant \( C_{B(0,1)} \) such that
\[
\frac{1}{C_{B(0,1)}} |x| \leq \|x\|_1 \leq C_{B(0,1)} |x|_1^\frac{1}{p},
\]
where \( k \) is the step of \( \mathcal{G} \). Hence
\[
\mathcal{L}^p(\pi_{V(x)}(B(0, 1))) \geq \mathcal{L}^p(V(x) \cap B(0, 1)) \]
\[
\geq \mathcal{L}^p \left( V(x) \cap B_E \left( 0, \frac{1}{(C_{B(0,1)})^k} \right) \right) \]
\[
= \frac{1}{(C_{B(0,1)})^k} \mathcal{L}^p(V(x) \cap B_E(0, 1)) := D(k, p) > 0.
\]
Hence for every \( x \in \Omega_h \), we have that
\[
\frac{S^P(A_x)}{|Df(x)|} \geq GD(k, p) := D'(k, p) = D' > 0.
\]
If we now assume \((b)\) to be true, and we fix \( \epsilon = \frac{D' \min_{x \in \Omega_h} |Df(x)|}{2} > 0 \), there exists
\( 0 < \bar{r} \leq s - h \) such that for every \( 0 < r \leq \bar{r} \) and for every \( x \in \Omega_h \),
\[
|S^P(A_{x,r}) - S^P(A_x)| \leq \sup_{x \in \Omega_h} |S^P(A_{x,r}) - S^P(A_x)| \leq \epsilon,
\]
so that for every \( 0 < r \leq \bar{r} \) and for every \( x \in \Omega_h \),
\[
S^P(A_{x,r}) \geq S^P(A_x) - \epsilon
\]
and so for every \( x \in \Omega_h \) and \( 0 < r \leq \bar{r} \)
\[
\frac{\zeta_1(B(x, r))}{\zeta_2(B(x, r))} = \frac{S^P(A_{x,r})}{|Df(x)|} \geq \frac{S^P(A_x)}{|Df(x)|} - \frac{\epsilon}{|Df(x)|} \geq D' - \frac{\epsilon}{|Df(x)|} \geq D' - \frac{\epsilon}{\min_{x \in \Omega_h} |Df(x)|} = \frac{D'}{2} > 0
\]
by the choice of $\epsilon$. This gives the thesis.

Second point, we prove that (a) implies (b). Surely, we have that
\[
\lim_{r \to 0} \sup_{x \in \Omega_h} \left| \mathcal{S}^P(A_{x,r}) - \mathcal{S}^P(A_x) \right|
\]
\[
\leq \lim_{r \to 0} \sup_{x \in \Omega_h} \left| \int_{M} \mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m) d\mathcal{S}^P(m) \right|
\]
\[
\leq \lim_{r \to 0} \int_{M} \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m) \right| d\mathcal{S}^P(m).
\]
(9)

We want now to apply the dominated Lebesgue convergence theorem using (a). In order to do this, we prove that for $r \leq s - h$, for any $m \in \mathbb{M}$
\[
\sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{s,r}}(m) - \mathbf{1}_{A_x}(m) \right| \leq 21 B_{0(W)}(m)
\]
for some constant $W > 0$. Notice that $21 B_{0(W)} \in L^1 G(M)$.

First, consider that $\sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{s,r}}(m) - \mathbf{1}_{A_x}(m) \right| \leq \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{s,r}}(m) \right| + \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_x}(m) \right|$. For any $x \in \Omega_h$ and $m \in \mathbb{M}$, if we assume $\mathbf{1}_{A_x}(m) = 1$, it implies that $m = Df(x)(\eta)$ for some $\eta \in B(0, 1)$, then $||m||_2 = ||DF(x)\eta||_2 \leq ||DF(x)||_{\mathcal{L}(G,M)} \leq \max_{x \in \Omega_h} ||DF(x)||_{\mathcal{L}(G,M)} =: ||DF||_{\mathcal{L}(\Omega_h)}$.

For any $x \in \Omega_h$, $r \leq s - h$ and $m \in \mathbb{M}$, if $\mathbf{1}_{A_{s,r}}(m) = 1$, $m = \delta_{1/r}(f(x)^{-1}f(q_r))$ for some $q_r \in B(x, r) \subseteq \Omega_s$, hence
\[
||m||_2 = ||\delta_{1/r}(f(x)^{-1}f(q_r))||_2 = ||DF(x)\delta_{1/r}(x^{-1}q_r)\delta_{1/r}(DF(x)(x^{-1}q_r)^{-1}f(x)^{-1}f(q_r))||_2
\]
\[
\leq ||DF(x)\delta_{1/r}(x^{-1}q_r)||_2 + ||\delta_{1/r}(DF(x)(x^{-1}q_r)^{-1}f(x)^{-1}f(q_r)||_2
\]
\[
\leq ||DF(x)||_{\mathcal{L}(G,M)} + K(\omega_{\mathcal{L}(DF)}(Hc(s - h)))^{1/2}
\]
\[
\leq ||DF||_{\mathcal{L}(\Omega_h)} + K(\omega_{\mathcal{L}(DF)}(Hc(s - h)))^{1/2},
\]
where $\omega_{\mathcal{L}(DF)}$ is the modulus of continuity of $x \to DF_1(x)$ defined in Definition 8 and $K$ is a constant that plays the role of $C$ of Theorem 2.4.

Hence
\[
\sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{s,r}}(m) - \mathbf{1}_{A_x}(m) \right| \leq \sup_{x \in \Omega_h} \mathbf{1}_{A_{s,r}}(m) + \sup_{x \in \Omega_h} \mathbf{1}_{A_x}(m)
\]
\[
\leq 1_{B(0(DF||_{\mathcal{L}(\Omega_h)})(Hc(s - h)))^{1/2}}(m) + 1_{B(0(DF||_{\mathcal{L}(\Omega_h)})(m)}
\]
\[
\leq 21 B_{0(DF||_{\mathcal{L}(\Omega_h)})(Hc(s - h)))^{1/2}}(m),
\]
and this implies that (10) is true, with $W = ||DF||_{\mathcal{L}(\Omega_h)} + (\omega_{\mathcal{L}(DF)}(Hc(s - h)))^{1/2})$.

We can then apply the dominated Lebesgue convergence Theorem to (9), and since we have assumed (a) to be true, we obtain (b).

It remains to prove (a).
By contradiction, we assume (a) to be false. Then, there exists at least one element \( m \in \mathbb{M} \) such that the limit

\[
\lim_{r \to 0} \sup_{x \in \Omega_h} |1_{A_{x,r}}(m) - 1_{A_x}(m)|
\]

does not exist or

\[
\lim_{r \to 0} \sup_{x \in \Omega_h} |1_{A_{x,r}}(m) - 1_{A_x}(m)| > 0.
\]

In both cases, since all the considered elements are positive, there exists at least a positive infinitesimal sequence \( r_n \) such that

\[
\lim_{n \to \infty} \sup_{x \in \Omega_h} |1_{A_{x,r_n}}(m) - 1_{A_x}(m)| > 0.
\]

This implies that there exists \( \tilde{n} > 0 \) such that for every \( n \geq \tilde{n} \),

\[
\sup_{x \in \Omega_h} |1_{A_{x,r_n}}(m) - 1_{A_x}(m)| > 0 \quad \text{and} \quad r_n \leq s - h.
\]

Hence for every \( n \geq \tilde{n} \) there exists at least an element \( x_n \in \Omega_h \subseteq \overline{\Omega_h} \) such that

\[
|1_{A_{x_n,r_n}}(m) - 1_{A_{x_n}}(m)| > 0
\]

and then

\[(11) \quad |1_{A_{x_n,r_n}}(m) - 1_{A_{x_n}}(m)| = 1.\]

Since \( \overline{\Omega_h} \) is a compact set, the sequence \( x_n \) converges up to a subsequence to some \( \tilde{x} \in \overline{\Omega_h} \).

Let us first prove that there exists some \( \tilde{n} \) such that for every \( n \geq \tilde{n} \)

\[(12) \quad |1_{A_{x_n,r_n}}(m) - 1_{A_{x_n}}(m)| = 1.\]

Let us then assume by contradiction that there exists a subsequence \( x_{n_k} \) such that

\[(13) \quad \lim_{k \to \infty} \left|1_{A_{x_{n_k},r_{n_k}}}(m) - 1_{A_{x_k}}(m)\right| = 0\]

then, on this subsequence, we have that

\[(14) \quad \left|1_{A_{x_{n_k},r_{n_k}}}(m) - 1_{A_{x_{n_k}}}(m)\right| \leq \left|1_{A_{x_{n_k},r_{n_k}}}(m) - 1_{A_{x_k}}(m)\right| + \left|1_{A_{x_{n_k}}}(m) - 1_{A_{x_k}}(m)\right|.
\]

Let us prove that \(14\) goes to zero as \( k \to \infty \) and this would give a contradiction with \(11\). The fact that \(14\) goes to zero follows by the assumption \(13\) and by the fact that

\[(15) \quad \left|1_{A_{x_{n_k}}}(m) - 1_{A_{x_k}}(m)\right| \to 0 \quad \text{and} \quad k \to \infty.
\]

Let us prove \(15\): assume by contradiction that on a subsequence of \( x_{n_k}, x_{n_{kp}} \), for \( p \) sufficiently large,

\[(16) \quad \left|1_{A_{x_{n_{kp}}}}(m) - 1_{A_{\tilde{x}}}(m)\right| = 1.
\]

Let us assume \( m \notin A_{\tilde{x}} \), then \( m \in A_{x_{n_{kp}}} \) so that \( m = Df(x_{n_{kp}})(\eta_{n_{kp}}) \) for \( \eta_{n_{kp}} \in B(0,1) \), hence we can extract a converging subsequence such that \( \eta_{n_{kp}} \to \tilde{\eta} \in B(0,1) \) and letting \( q \) go to infinite, by the continuity of the differential, we know that \( m = Df(\bar{x})(\bar{\eta}) \in A_{\bar{x}} \), but this is not possible.
So it must be true that \( m \in A_\bar{x} \). Hence, by the definition of \( A_\bar{x} \) and the continuity of the Pansu differential, it is true that for some \( \eta \in B(0,1) \), \( m = Df(\bar{x})(\eta) = \lim_{p \to \infty} Df(x_{n_p})(\eta) = \lim_{p \to \infty} X_p \) where for every \( p \), \( X_p := Df(x_{n_p})(\eta) \in A_{x_{n_p}} \); hence \( m \in \limsup_{p \to \infty} A_{x_{n_p}} \), and then \( \limsup_{p \to \infty} A_{x_{n_p}} (m) = \limsup_{p \to \infty} 1_{A_{x_{n_p}}} (m) = 1 \) but at the same time, by (16), \( m \notin A_{x_{n_p}} \) for \( p \) sufficiently large and this implies that \( \lim_{p \to \infty} 1_{A_{x_{n_p}}} (m) = 0 = \limsup_{p \to \infty} 1_{A_{x_{n_p}}} (m) \) and this gives a contradiction.

Let us then continue from (12). We need to prove that (11) is not possible. Since \( m \) is fixed, there are only two possibilities:

(a1) \( m \in A_\bar{x} \);

(a2) \( m \notin A_\bar{x} \).

We show that neither (a1) nor (a2) can be true. Assume that (a1) is true, then \( m = Df(\bar{x})(\eta) \) for some \( \eta \in B(0,1) \). Then by (12) \( m \notin A_{x_n, r_n} \) for \( n \geq \bar{n} \) and so clearly there exists the following limit

\[
\lim_{n \to \infty} 1_{A_{x_n, r_n}} (m) = 0.
\]

Let us define for any \( n, q_n := x_n \delta_{\bar{x},n}^1(\eta) \in B(x_n, r_n) \subseteq \Omega_s \). Consider for any \( n \geq \bar{n} \)

\[
\delta_{1/r_n}^2(f(x_n)^{-1}f(q_n)) = Df(\bar{x})(\eta)\delta_{1/r_n}^2(Df(\bar{x})(x_n^{-1}q_n)^{-1}Df(x_n)(x_n^{-1}q_n))
\]

and observe that by Theorem 2.4 since \( x_n, q_n \in \Omega_s \) and (7) holds

\[
||\delta_{1/r_n}^2(Df(x_n)^{-1}f(x_n)^{-1}f(q_n))||_2 \leq K(\omega_x, Df_{\bar{x}})C_H \to 0
\]
as \( n \to \infty \) and

\[
||Df(\bar{x})(\eta)||_2 \leq d_L(\bar{G}, M)(Df(x_n), Df(\bar{x}) \to 0
\]
as \( n \to \infty \) by the continuity of \( Df(x) \). Hence \( \lim_{n \to \infty} \delta_{1/r_n}^2(f(x_n)^{-1}f(q_n)) = m \). This permits to conclude that \( m \in \limsup_{n \to \infty} A_{x_n, r_n} \) and so that

\[
\limsup_{n \to \infty} 1_{A_{x_n, r_n}} (m) = \limsup_{n \to \infty} 1_{A_{x_n, r_n}} (m) = 1.
\]

At the same time, (17) implies that there exists the limit

\[
\limsup_{n \to \infty} 1_{A_{x_n, r_n}} (m) = \lim_{n \to \infty} 1_{A_{x_n, r_n}} (m) = 0,
\]

so we reach a contradiction.

Assume now (a2), then \( m \in A_{x_n, r_n} \) for every \( n \geq \bar{n} \) and then by (12), \( m \notin A_\bar{x} \). For every \( n \geq \bar{n} \) there exists \( q_n \in B(x_n, r_n) \subseteq \Omega_s \) such that

\[
m = \delta_{1/r_n}^2(f(x_n)^{-1}f(q_n)) = Df(\bar{x})(\delta_{1/r_n}^1(x_n^{-1}q_n))Df(\bar{x})(\delta_{1/r_n}^1(x_n^{-1}q_n))^{-1}
\]

\[
Df(x_n)(\delta_{1/r_n}^1(x_n^{-1}q_n))\delta_{1/r_n}^2(Df(x_n)^{-1}f(x_n)^{-1}f(q_n))
\]

and again by Theorem 2.4 and by continuity of \( Df \) we obtain that

\[
m = \lim_{n \to \infty} \delta_{1/r_n}^2(f(x_n)^{-1}f(q_n)) = \lim_{n \to \infty} Df(\bar{x})(\delta_{1/r_n}^1(x_n^{-1}q_n))
\]
that up to a subsequence is equal to $Df(x)(\eta)$ for some $\eta \in B(0,1)$, so that $m \in A_x$ that is a contradiction with (11). Hence, finally (a) is proved.

**Claim 4** For every $x \in \Omega'$, 
\[ C_P(Df(x)) \lesssim_{q,p,k} |Df(x)|. \]

**Proof.** Since we have assumed that $|Df(x)|$ is surjective at every $x$, by (5), we have that
\[ C_P(Df(x)) = Z \frac{S^{Q-P}(\ker(Df(x)) \cap B(0,1))}{\mathcal{H}^{q-p}_E(\ker(Df(x)) \cap B(0,1))} |Df(x)|. \]

By Proposition 3.1 for any $x \in \Omega'$ and any Borel set $B \subset \ker(Df(x))$
\[ S^{Q-P}(B) = \frac{1}{\sup_{w \in B(0,1)} \mathcal{H}^{q-p}_E(\ker(Df(x)) \cap B(w,1))} \mathcal{H}^{q-p}_E(B), \]

hence, by taking into account Proposition 2.1, we have that
\[ \frac{S^{Q-P}(\ker(Df(x)) \cap B(0,1))}{\mathcal{H}^{q-p}_E(\ker(Df(x)) \cap B(0,1))} = \frac{1}{\sup_{w \in B(0,1)} \mathcal{H}^{q-p}_E(B(w,1) \cap \ker(Df(x)))} \leq \frac{1}{\mathcal{H}^{q-p}_E(B(0,1) \cap \ker(Df(x)))} \leq \frac{1}{\mathcal{L}^{q-p}(B_E(0, \frac{1}{(C_B(0,1))} \cap \ker(Df(x)))} =: D^{q-p}(q,p,k) > 0. \]

In the last passage we considered that $\ker(Df(x))$ is a linear subspace of constant topological dimension $q-p$. By combining all claims the proof is achieved.

It is easy to extend Theorem 1.1 to the case in which $\Omega$ is not necessarily compact but it is any measurable set.

**Theorem 3.3.** Let $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$ be two Carnot groups, endowed with homogeneous distances, of metric dimension $Q$, $P$ and topological dimension $q$, $p$, respectively. Let $f \in C^1_b(\mathbb{G}, \mathbb{M})$ be a function and assume $Df(x)$ to be surjective at any point $x \in \mathbb{G}$. Let $A \subset \mathbb{G}$ be a measurable set. Assume that there exist two constants $\tilde{r}, C > 0$ such that for any $m \in \mathbb{M}$, the level set $f^{-1}(m)$ is $\tilde{r}$-locally $C$-lower Ahlfors $(Q-P)$-regular with respect to the measure $S^{Q-P}$. Then there exists a constant $L = L(C, \mathbb{G}, p)$ such that
\[ \int_A C_P(Df(x))dS^Q(x) \leq L \int_M S^{Q-P}(f^{-1}(m) \cap A) dS^P(m). \]

**Proof.** Let us consider an increasing sequence of compact sets in $\Omega_n \subseteq A$ such that $\Omega_n \nearrow A$. Hence by Theorem 1.1 there exists $L = L(C, \mathbb{G}, p)$ such that for every $n \in \mathbb{N}$
\[ \int_{\Omega_n} C_P(Df(x))dS^Q(x) \leq L \int_M S^{Q-P}(f^{-1}(m) \cap \Omega_n) dS^P(m) \leq L \int_M S^{Q-P}(f^{-1}(m) \cap A) dS^P(m), \]
so if we let \( n \) go to \( \infty \), by Monotone Convergence Theorem we get the thesis. \( \Box \)

4. Applications

**Corollary 4.1.** In the hypothesis of Theorem 3.3, let \( u : A \to \mathbb{R} \) be a non-negative measurable function, then there exists a constant \( L = L(C, \mathcal{G}, p) \) such that

\[
\int_A u(x)C_P(Df(x))dS^Q(x) \leq L \int_M \int_{f^{-1}(m) \cap A} u(x)dS^{Q-P}(x)dS^P(m).
\]

**Proof.** We can write \( u = \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{1}_{A_k} \) with \( A_k \) measurable sets (see [2] Theorem 7]). By Monotone Convergence Theorem we have that

\[
\int_A u(x)C_P(Df(x))dS^Q(x) = \sum_{k=1}^{\infty} \frac{1}{k} \int_{A \cap A_k} C_P(Df(x))dS^Q(x)
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{k} L \int_M S^{Q-P}(f^{-1}(m) \cap A \cap A_k)dS^P(m)
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{k} L \int_M \int_{f^{-1}(m) \cap A} \mathbf{1}_{A_k}(x)dS^{Q-P}(x)dS^P(m)
\]

\[
= L \int_M \int_{f^{-1}(m) \cap A} \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{1}_{A_k}(x)dS^{Q-P}(x)dS^P(m)
\]

\[
= L \int_M \int_{f^{-1}(m) \cap A} u(x)dS^{Q-P}(x)dS^P(m).
\]

**Corollary 4.2.** In the hypothesis of Theorem 3.3, let \( u : A \to \mathbb{R} \) be a measurable function. If we assume

- \( u \) is \( S^{Q-P} \)-summable on \( f^{-1}(m) \cap A \) for \( S^P \)-a.e. \( m \in M \)
- \( \int_M \int_{f^{-1}(m) \cap A} |u(x)|dS^{Q-P}(x)dS^P(m) < \infty \)

then \( u \) is summable on \( A \).

**Proof.** We can write \( u = u^+ - u^- \). By applying passages analogous to (18), considering Theorem 2.8 instead of Theorem 1.1, we then obtain that

\[
- \int_A u^-(x)C_P(Df(x))dS^Q(x) \leq - \int_M \int_{f^{-1}(m) \cap A} u^-(x)dS^{Q-P}(x)dS^P(m).
\]

Hence by (18) applied to \( u^+ \), (19), and our hypothesis, we have that

\[
\int_A u(x)C_P(Df(x))dS^Q(x) = \int_A u^+(x)C_P(Df(x))dS^Q(x) - \int_A u^-(x)C_P(Df(x))dS^Q(x)
\]

\[
\leq L \int_M \int_{f^{-1}(m) \cap A} u^+(x)dS^{Q-P}(x)dS^P(m) - \int_M \int_{f^{-1}(m) \cap A} u^-(x)dS^{Q-P}(x)dS^P(m)
\]

\[
\leq L \int_M \int_{f^{-1}(m) \cap A} |u(x)|dS^{Q-P}(x)dS^P(m) < \infty.
\]
Now, it is enough to prove that $C_P(Df(x)) > 0$, for every $x \in A$. This follows from the facts that $|Df(x)| > 0$ for every $x \in A$ and that, taking into consideration Proposition 4.4, for every $x$ we have the following

\[
\frac{\mathcal{S}^Q-P(\ker(Df(x)) \cap B(0,1))}{\mathcal{H}^p_{E}(\ker(Df(x)) \cap B(0,1))} \\
= \sup_{w \in B(0,1)} \frac{1}{\mathcal{H}^p_{E}(B(w,1) \cap \ker(Df(x)))} \\
\geq \frac{1}{\mathcal{H}^p_{E}(B(0,2) \cap \ker(Df(x)))} \\
= \frac{1}{\mathcal{L}^p(B(0,2) \cap \ker(Df(x)))} \\
= \frac{1}{(2C_{B(0,2)})^q-p} > 0.
\]

(20)

**Corollary 4.3.** In the hypothesis of Theorem 3.3, if $1_A(x) = 0$ for $\mathcal{S}^Q-P$-a.e. $x \in f^{-1}(m)$, for $\mathcal{S}^P$-a.e $m \in \mathbb{M}$, then $1_A(x) = 0$ for $\mathcal{S}^Q$-a.e $x \in \mathbb{G}$.

**Proof.** It follows by Theorem 3.3 and (20).

Now we see how to apply Theorem 1.1 to the particular geometrical case in which there exists a $p$-dimensional homogeneous subgroup $\mathcal{V}$ complementary to $\ker(Df(x))$ for any point $x$ of a neighbourhood of a fixed compact set $\Omega$. We fix then again $(\mathbb{G},d_1)$, $(\mathbb{M},d_2)$ two Carnot groups, endowed with homogeneous distances, of metric dimension $Q$, $P$ and topological dimension $q$, $p$, respectively. For any set $\Omega \subset \mathbb{G}$, and any real number $D > 0$ we set $\Omega_D = \{ y \in \mathbb{G} : d(y,\Omega) < D \}$.

By modifying the proof of [7, Lemma 2.9], combining it with an easy compactness argument and Theorem 2.4 the following immediately follows.

**Proposition 4.4.** Let us consider a map $f \in C^1_0(\mathbb{G},\mathbb{M})$ and a compact set $\Omega \subset \mathbb{G}$. Assume that there exists a $p$-dimensional homogeneous subgroup $\mathcal{V}$ such that $Df(x)|_\mathcal{V} : \mathcal{V} \to \mathbb{M}$ is an $h$-isomorphism for every $x \in \Omega$. Then there exists a constant $R > 0$ such that for every $x \in \Omega$, for every $y \in B(x,R)$ and $v \in \mathcal{V}$ such that $yv \in B(x,R)$

\[d_2(f(y), f(yv)) \geq R\|v\|_1.\]

Notice that our hypothesis implies that $\mathcal{V}$ is complementary to $\ker(Df(x))$ for every $x \in \Omega$.

Indeed, any continuously Pansu differentiable map is locally metric Lipschitz. Hence, combining Proposition 1.1 with the proof of [7, Corollary 2.16] we get the following.

**Proposition 4.5.** Let us consider a map $f \in C^1_0(\mathbb{G},\mathbb{M})$ and a compact set $\Omega \subset \mathbb{G}$. Let us assume that there exists a $p$-dimensional homogeneous subgroup $\mathcal{V}$ such that $Df(x)|_\mathcal{V} : \mathcal{V} \to \mathbb{M}$ is an $h$-isomorphism for every $x \in \Omega_D$ for some $D > 0$. Then there exists a constant $L$ such that for every $m \in \mathbb{M}$, $x \in f^{-1}(m) \cap \Omega$, the set $f^{-1}(m) \cap B(x,R)$ is an intrinsic Lipschitz graph with constant $L$, where $R$ is the constant of Proposition 4.4 applied to $\Omega$.  

\[\]
Proof. By hypothesis, at any point \( x \in \Omega_D \), \( \ker Df(x) \) is a normal homogeneous subgroup complementary to \( \mathbb{V} \), hence \( Df(x) \) is a surjective h-epimorphism. Assume \( R \) is smaller than \( D \). Let us fix a homogeneous subgroup \( \mathbb{W} \) complementary to \( \mathbb{V} \). For every \( m \in \mathbb{M} \) and \( x \in f^{-1}(m) \cap \Omega \), the set \( f^{-1}(m) \cap B(x,R) \) is contained into the intrinsic graph of a function \( \phi_{m,x} : U_{m,x} \subset \mathbb{W} \rightarrow \mathbb{V} \), for some open set \( U_{m,x} \subset \mathbb{W} \). The map \( \phi_{m,x} \) is given by Theorem 2.6 repeatedly applied to different points of \( f^{-1}(m) \cap B(x,R) \), if necessary.

Now we need to observe that the notion of intrinsic Lipschitz function introduced in [7] is equivalent to our notion (it is immediate to compare Definition in [7] with [6, Definition 9, Definition 10, Proposition 3.1]). Then by [7, Corollary 2.16], \( f^{-1}(m) \cap B(x,R) \) is the intrinsic Lipschitz graph of an intrinsic L-Lipschitz function \( \phi_{m,x} \) for some constant \( L \) depending on \( R \), and on the Lipschitz constant of \( f|_{B(x,R)} \), that can be uniformly bounded by the \( \sup_{x \in \Omega} \text{Lip}(f|_{B(x,R)}) \leq \text{Lip}(f|_{B(x,R)}) < \infty \). As a consequence, the sets \( f^{-1}(m) \cap B(x,R) \) are intrinsic L-Lipschitz for some positive \( L \) independent of \( x \in \Omega \) and \( m \in \mathbb{M} \).

Corollary 4.6. Let \( f \in C^1_G(\mathbb{G},\mathbb{M}) \) be a function with \( Df(x) \) surjective at every \( x \in \mathbb{G} \) and let \( \Omega \subset \mathbb{G} \) be a compact set. Assume that there exists a \( p \)-dimensional subgroup \( \mathbb{V} \) of \( \mathbb{G} \) such that \( Df(x)|_{\mathbb{V}} \) is an h-isomorphism for every \( x \in \Omega_D \) for some \( D > 0 \). Set \( \lambda = \sup_{x \in \Omega} \text{Lip}(f|_{B(x,R)}) \), where \( R \) is the constant given by Proposition 4.4 applied to \( \Omega \). Then there exists a constant \( 1 \leq T(\mathbb{G},\lambda, R, p) < \infty \), such that
\[
\int_{\Omega} C_P(Df(x))d\mathcal{S}^{Q-p}(x) \leq T \int_{\mathbb{M}} \mathcal{S}^{Q-p}(f^{-1}(m) \cap \Omega)d\mathcal{S}^{P}(m).
\]

Proof. We can assume \( R < D \). Set \( \mathbb{W} \) any homogeneous subgroup complementary to \( \mathbb{V} \). By Propositions 2.3 and 2.7 there exists a constant \( K > 0 \) such that for every \( m \in \mathbb{M} \) and \( x \in f^{-1}(m) \cap \Omega \), for every \( 0 < r < R \), \( \mathcal{S}^{Q-p}(f^{-1}(m) \cap B(x,r)) \geq Kr^{Q-p} \), where \( K \) is a constant depending on \( c_0(\mathbb{W},\mathbb{V}) \) > 0 and on the intrinsic Lipschitz constant of the parametrizing map \( \phi_{m,x} : U_{m,x} \subset \mathbb{W} \rightarrow \mathbb{V} \) of \( \{ f^{-1}(m) \cap B(x,r) \} \{ m \in \mathbb{M}, x \in f^{-1}(m) \cap \Omega \} \). Moreover observe that by Proposition 4.3 \( \phi_{m,x} \) are intrinsic L-Lipschitz, for some constant \( L \) independent of \( m \) and \( x \). Now notice that our observation can take the place of the hypothesis that level sets \( f^{-1}(m) \) are uniformly locally lower Ahlfors \( (Q-P) \)-regular with respect to \( \mathcal{S}^{Q-P} \) in Theorem 1.1 (more precisely in Claim 2 on Theorem 3.2), hence we can apply our result to this situation, and we directly get the thesis.

Remark 6. We have seen, in the proof of Corollary 4.6, that the existence of a \( p \)-dimensional homogeneous subgroup \( \mathbb{V} \) complementary to \( \ker(Df(x)) \) for every point \( x \in \mathbb{G} \), implies that the level sets of \( f \) are \( R \)-locally \( C \)-lower Ahlfors \( (Q-P) \)-regular with respect to \( \mathcal{S}^{Q-P} \), for some positive constants \( C \) and \( R \), locally independent of the choice of the level set. We want to highlight that the opposite may be false. In fact, there exist continuously Pansu differentiable maps between Carnot groups, with everywhere surjective differential, such that their level sets are lower Ahlfors regular, but at the same time \( \ker(Df(x)) \) does not admit any complementary subgroup.

We present a simple example related to the first Heisenberg group \( \mathbb{H}^1 \), that is the simplest non-commutative Carnot group. It can be represented as a direct sum of two linear subspaces \( \mathbb{H}^1 = H_1 \oplus H_2 \), where \( H_1 = \text{span}(e_1,e_2) \), \( H_2 = \text{span}(e_3) \) with unique non trivial relation \( [e_1,e_2] = e_3 \). For every \( p, q \in \mathbb{H}^1 \), \( pq = p + q + \frac{1}{2}[p,q] \).
Let us consider the map
\[ f : \mathbb{H}^1 \to \mathbb{R}^2, \quad f(x, y, z) = (ax + by, cx + dy), \]
with \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \).

Observe that \( f \in \mathcal{L}(\mathbb{H}^1, \mathbb{R}^2) \), then the Pansu differential of \( f \) is constant on \( \mathbb{H}^1 \): for every \( \bar{x} \in \mathbb{H}^1 \),
\[ Df(\bar{x})(x, y, z) = f(x, y, z), \]
hence, \( \ker(Df(\bar{x})) = \text{span}(e_3) \) for every \( \bar{x} \in \mathbb{H}^1 \). Notice that \( \text{span}(e_3) \) is a normal homogeneous subgroup of metric dimension 2 that does not admit any complementary subgroup (see for instance [5, Proposition 4.1]). Let us now focus on the level sets of \( f \). If we fix \( v \in \mathbb{R}^2 \), we have that \( f^{-1}(v) = w\text{span}(e_3) \) for some \( w = w(v) \in H_1 \), hence any level set is a coset of \( \text{span}(e_3) \). Then, by left invariance and homogeneity of the distance, the level sets \( f^{-1}(v) \) are \( C \)-lower Ahlfors 2-regular with respect to \( S^2 \), for some positive constant \( C \), independent of the choice of \( v \).

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