GROUND STATE OF A MAGNETIC NONLINEAR CHOQUARD EQUATION

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Abstract. We consider the stationary magnetic nonlinear Choquard equation

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u,$$

where $A: \mathbb{R}^N \to \mathbb{R}^N$ is a vector potential, $V$ is a scalar potential, $f: \mathbb{R} \to \mathbb{R}$ and $F$ is the primitive of $f$. Under mild hypotheses, we prove the existence of a ground state solution for this problem. We also prove a simple multiplicity result by applying Ljusternik-Schnirelmann methods.

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1. Introduction

We consider the problem

$$- (\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u$$

where $\nabla + iA(x)$ is the covariant derivative with respect to the $C^1$ vector potential $A: \mathbb{R}^N \to \mathbb{R}^N$. (After stating our hypotheses, the form of equation (1) will be changed to (3)). The constant $\alpha$ belongs to the intervals $(0, N)$ and $\lim_{|x| \to \infty} A(x) = A_\infty \in \mathbb{R}^N$.

The scalar potential $V: \mathbb{R}^N \to \mathbb{R}$ is a continuous, bounded function satisfying

(V1) $\inf_{\mathbb{R}^N} V > 0$;
(V2) $V_\infty = \lim_{|y| \to \infty} V(y)$;
(V3) $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

We also suppose that

$$(AV) |A(y)|^2 + V(y) < |A_\infty|^2 + V_\infty.$$

The function $F$ is the primitive of the nonlinearity $f: \mathbb{R} \to \mathbb{R}$, which is non-negative in $(0, \infty)$ and satisfies, for any $r \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$,

$$(f1) \lim_{t \to 0} \frac{f(t)}{t} = 0,$$
$$(f2) \lim_{t \to \infty} \frac{f(t)}{t^{r-1}} = 0,$$
$$(f3) \frac{f(t)}{t} \text{ is increasing if } t > 0 \text{ and decreasing if } t < 0.$$
For example, if \( t \in \mathbb{R} \), the functions \( t \ln(1 + |t|) \) and \( |t|^{q_1 - 2}t + |t|^{q_2 - 2}t \) (where \( 2 < q_1, q_2 < r \)) satisfy hypothesis \((f1), (f2)\) and \((f3)\).

We denote

\[
\tilde{f}(t) = \begin{cases} 
\frac{f(t)}{t}, & \text{if } t \neq 0, \\
0, & \text{if } t = 0.
\end{cases}
\]

Our hypotheses imply that \( \tilde{f} \) is continuous. Therefore, problem (1) can be written in the form

\[
-(\nabla + iA(x))^2u + V(x)u = \left( \frac{1}{|x|^\alpha} \ast F(|u|) \right) \tilde{f}(|u|)u.
\]

The composition of \( f \) and \( F \) with \( |u| \) gives a variational structure to the problem, allowing the application of the Mountain Pass Theorem. So, the right-hand side of problem (2) generalizes the term

\[
\left( \frac{1}{|x|^\alpha} \ast |u|^{p} \right) |u|^{p-2}u,
\]

which was studied by Cingolani, Clapp and Secchi in [7]. In some particular cases, similar forms of problem (2) were studied in [5] and [6].

Our aim in this paper is to prove the existence of a ground state solution for problem (2). This is accomplished by showing that the mountain pass geometry is satisfied and then considering the asymptotic form of problem (2) and applying Struwe’s splitting lemma.

The main part of the interesting paper by Cingolani, Clapp and Secchi [7] is devoted to the existence of multiple solutions of equation (2) - with (3) as the right-hand side - under the action of a closed subgroup \( G \) of the orthogonal group \( O(N) \) of linear isometries of \( \mathbb{R}^N \) if \( A(gx) = gA(x) \) and \( V(gx) = V(x) \) for all \( g \in G \) and \( x \in \mathbb{R}^N \). The authors look for solutions satisfying

\[
u(gx) = \tau(g) u(x), \quad \text{for all } g \in G \text{ and } x \in \mathbb{R}^N,
\]

where \( \tau: G \rightarrow S^1 \) is a given continuous group homomorphism into the unit complex numbers \( S^1 \). In this paper we also address the multiplicity of solutions in a particular case of that treated in [7].

We define

\[\nabla_A u = \nabla u + iA(x)u\]

and consider the space

\[H_{A,V}^1(\mathbb{R}^N, \mathbb{C}) = \{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}) \}\]

endowed with scalar product

\[\langle u, v \rangle_{A,V} = \Re \int_{\mathbb{R}^N} (\nabla_A u \cdot \overline{\nabla_A v} + V(x) u \overline{v})\]

and, therefore

\[||u||_{A,V}^2 = \int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2.\]

Observe that the norm generated by this scalar product is equivalent to the norm obtained by considering \( V \equiv 1 \), see [13 Definition 7.20].

If \( u \in H_{A,V}^1(\mathbb{R}^N, \mathbb{C}) \), then \( |u| \in H^1(\mathbb{R}^N) \) and the diamagnetic inequality is valid (see [13 Theorem 7.21], [7])

\[|\nabla |u|(x)| \leq |\nabla u(x) + iA(x)u(x)|, \quad \text{a.e. } x \in \mathbb{R}^N.\]
As a consequence of the diamagnetic inequality, we have the continuous immersion
\[ \mathcal{H}^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C}) \]
for any \( q \in \left[ 2, \frac{2N}{N-2} \right] \). We denote \( 2^* = \frac{2N}{N-2} \).

It is well-known that \( C^\infty_c(\mathbb{R}^N, \mathbb{C}) \) is dense in \( \mathcal{H}^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \), see \cite[Theorem 7.22]{13}.

**Remark 1.1.** It follows from \((f1)-(f2)\) that, for any fixed \( \xi > 0 \), there exists a constant \( C_\xi \) such that
\[ |f(t)| \leq \xi t + C_\xi t^{\tau-1}, \quad \forall \ t \geq 0. \]
Similarly, there exists \( D_\xi > 0 \) such that
\[ |F(t)| \leq \xi t^2 + D_\xi t^{\tau}, \quad \forall \ t \geq 0. \]
Furthermore, \((f3)\) implies that \( f \) satisfies the Ambrosetti-Rabinowitz inequality
\[ 2F(t) < f(t)t, \quad \forall \ t > 0. \]
Observe that the function \( f(t) = t \ln(1 + |t|) \) satisfies the last inequality, but does not satisfy \( \theta F(t) \leq tf(t) \) for any \( \theta > 2 \).

We state our results:

**Theorem 1.** Suppose that \( \alpha \in (0, N) \) and that conditions \((V1)-(V3), (AV)\) and \((f1)-(f3)\) are valid. Then, problem \((1)\) has a ground state solution.

In order to obtain our multiplicity result, we define the space
\[ \mathcal{H}^1_A(\mathbb{R}^N, \mathbb{C})^\tau = \{ u \in \mathcal{H}^1_A(\mathbb{R}^N, \mathbb{C}) : u(gx) = \tau(g)u(x), \ \forall \ g \in G, \ \forall \ x \in \mathbb{R}^N \} \]
and suppose that the closed subgroup \( G \subset O(N) \) satisfies the decomposition
\[ G = O(N_1) \times O(N_2) \times \cdots \times O(N_k), \]
where \( \sum_{j=1}^k N_j = N \), \( N_j \geq 2 \) for all \( j \in \{1, \ldots, k\} \). Then we have

**Theorem 2.** Let \( G \) be a closed subgroup of \( O(N) \) satisfying the decomposition \((7)\). Assume that \( A(gx) = gA(x) \) and \( V(gx) = V(x) \) for all \( g \in G \) and \( x \in \mathbb{R}^N \). Then problem \((2)\) has a sequence \( (u_n) \subset \mathcal{H}^1_A(\mathbb{R}^N, \mathbb{C})^\tau \) such that \( \lim_{n \to \infty} \|u_n\|_{A,V}^2 = \infty \).

The paper is organized as follows: Section 2 shows the mountain pass geometry and some basic results concerning the right-hand side of equation \((2)\). Theorem \((1)\) is proved in Section 3 and our multiplicity result in Section 4.

## 2. Variational Formulation

The energy functional associated to problem \((1)\) is given by
\[ J_{A,V}(u) = \frac{1}{2} \|u\|^2_{A,V} - D(u), \]
where
\[ D(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} + F(|u|) \right) F(|u|). \]
The energy functional is well-defined as a consequence of the Hardy-Littlewood-Sobolev (see [13, Theorem 4.3], since

\begin{equation}
\left| \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * F(|u|) \right) F(|u|) \right| \leq C (\|u\|^4 + \|u\|^{2r}).
\end{equation}

**Remark 2.1.** Let us consider the case \( F(t) = |t|^r \). By applying the Hardy-Littlewood-Sobolev inequality we have that

\[ \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * F(|u|) \right) F(|u|) \]

is well-defined if \( F(|u|) \in L^p(\mathbb{R}^N) \) for \( p > 1 \) defined by

\[ \frac{2}{p} + \frac{\alpha}{N} = 2 \Rightarrow \frac{1}{p} = \frac{1}{2} \left( 2 - \frac{\alpha}{N} \right). \]

Consequently, in order to apply the immersion \([4]\), we must have

\[ pr \in [2, 2^{*}] \Rightarrow \frac{2N - \alpha}{N} \leq r \leq \frac{N}{N - 2} \left( 2 - \frac{\alpha}{N} \right) = \frac{2N - \alpha}{N - 2}. \]

This condition (taking the open interval satisfied by \( r \)) justifies hypothesis \((f2)\).

Since the derivative of the energy functional \( J_{A,V}(u) \) is given by

\[ J_{A,V}'(u) \cdot \psi = \langle u, \psi \rangle_{A,V} - D'(u) \cdot \psi = \langle u, \psi \rangle_{A,V} - 2 \Re \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * F(|u|) \right) \bar{\psi} f(|u|) u \psi, \]

we see that critical points of \( J_{A,V}'(u) \) are weak solutions of \([2]\). Note that, if \( \psi = u \) we obtain

\begin{equation}
J_{A,V}'(u) \cdot u := \|u\|^2_{A,V} - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * F(|u|) \right) f(|u|) |u|. \tag{10}
\end{equation}

**Lemma 2.1.** The functional \( J_{A,V} \) satisfies the Mountain Pass geometry. Precisely,

(i) there exist \( \rho, \delta > 0 \) such that \( J_{A,V}|_S \geq \delta > 0 \) for any \( u \in S \), where

\[ S = \{ u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) : \|u\|_{A,V} = \rho \}; \]

(ii) for any \( u_0 \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \) there exists \( \tau \in (0, \infty) \) such that \( \|\tau u_0\| > \rho \) and \( J_{A,V}(\tau u_0) < 0 \).

**Proof.** Inequality \([9]\) yields

\[ J_{A,V}(u) \geq \frac{1}{2} \|u\|^2_{A,V} - C (\|u\|^4_{A,V} + \|u\|^{2r}_{A,V}), \]

thus implying (i) when we take \( \|u\|_{A,V} = \rho > 0 \) small enough.

In order to prove (ii), fix \( u_0 \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \) and consider the function \( g_{u_0} : (0, \infty) \to \mathbb{R} \) given by

\[ g_{u_0}(t) = D \left( \frac{tu_0}{\|u_0\|_{A,V}} \right) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * F \left( \frac{t|u_0|}{\|u_0\|_{A,V}} \right) \right) F \left( \frac{t|u_0|}{\|u_0\|_{A,V}} \right). \]
We have
\[
g'_u(t) = \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} * F \left( \frac{|t|u_0|}{\|u_0\|_{A,V}} \right) \right] - f \left( \frac{|t|u_0|}{\|u_0\|_{A,V}} \right) \frac{|u_0|}{\|u_0\|_{A,V}} \\
= \frac{4}{t} \int_{\mathbb{R}^N} \frac{1}{2} \left[ \frac{1}{|x|^\alpha} * F \left( \frac{|t|u_0|}{\|u_0\|_{A,V}} \right) \right] \frac{1}{2} f \left( \frac{|t|u_0|}{\|u_0\|_{A,V}} \right) \frac{|t|u_0|}{\|u_0\|_{A,V}} \\
\geq \frac{4}{t} g_u(t)
\]
as a consequence of the Ambrosetti-Rabinowitz condition \([3]\). Observe that \(g'_u(t) > 0\) for \(t > 0\).

Thus,
\[
\ln g_u(t) = 4 \ln t + \ln g_u(1) \geq (\tau\|u_0\|_{A,V})^4,
\]
proving that
\[
D(\tau u_0) = g_u(\tau u_0) \geq M(\tau u_0)^4
\]
for a constant \(M > 0\). So,
\[
J_{A,V}(\tau u_0) = \frac{\tau^2}{2}\|u_0\|_{A,V}^2 - D(\tau u_0) \leq C_1\tau^2 - C_2\tau^4
\]
yields that \(J_{A,V}(\tau u_0) < 0\) when \(\tau\) is large enough. \(\square\)

The mountain pass theorem without the PS condition (see \([21\text{, Teorema. 1.15}])\) yields a Palais-Smale sequence \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) such that
\[
J'_{A,V}(u_n) \to 0 \quad \text{and} \quad J_{A,V}(u_n) \to c,
\]
where
\[
c = \inf_{\alpha \in \Gamma} \max_{t \in [0,1]} J_{A,V}(\alpha(t)),
\]
and \(\Gamma = \{ \alpha \in C^1([0,1], H^1_{A,V}(\mathbb{R}^N, \mathbb{C})) : \alpha(0) = 0, \alpha(1) < 0 \} \).

We now consider the Nehari manifold
\[
\mathcal{N}_{A,V} = \left\{ u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} : J'_{A,V}(u) \cdot u = 0 \right\}
\]
\[
= \left\{ u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} : \|u\|^2_{A,V} = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * F(|u|) \right) f(|u|)|u| \right\}.
\]
It is not difficult to see that \(\mathcal{N}_{A,V}\) is a manifold in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}\). The next result shows that \(\mathcal{N}_{A,V}\) is a closed manifold in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\).

**Lemma 2.2.** There exists \(\beta > 0\) such that \(\|u\|_{A,V} \geq \beta\) for all \(u \in \mathcal{N}_{A,V}\).

Another characterization of \(c\) in terms of the Nehari manifold is now standard: for \(u \neq 0\), consider the function \(\Phi(t) = (1/2)\|tu\|_{A,V}^2 - D(tu)\), preserving the notation of Lemma [22]. The proof of Lemma [22] assures that \(\Phi(tu) > 0\) for \(t\) small enough, \(\Phi(tu) < 0\) for \(t\) large enough and \(g'_u(t) > 0\) if \(t > 0\). Therefore, \(\max_{t \geq 0} \Phi(t)\) is achieved at a unique \(t_u = t(u) > 0\) and \(\Phi'(t_u) > 0\) for \(t < t_u\) and \(\Phi'(t_u) < 0\) for \(t > t_u\). Furthermore, \(\Phi'(t_u) = 0\) implies that \(t_u u \in \mathcal{N}_{A,V}\).

The map \(u \mapsto t_u (u \neq 0)\) is continuous and \(c = c^*\), where
\[
c^* = \inf_{u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V}(tu).
\]
Lemma 2.4. Let \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) be a sequence such that \(J_{A,V}(u_n) \to c\) and \(J'_{A,V}(u_n) \to 0\), where
\[
c = \inf_{u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V}(tu).
\]
Then \((u_n)\) is bounded and (for a subsequence) \(u_n \to u_0\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\).

Lemma 2.5. Let \(U \subseteq \mathbb{R}^N\) be any open set. For \(1 < p < \infty\), let \((f_n)\) be a bounded sequence in \(L^p(U, \mathbb{C})\) such that \(f_n(x) \to f(x)\) a.e. Then \(f_n \to f\).

The proof of Lemma 2.4 follows by adapting the arguments given for the real case, as in [11, Lemme 4.8, Chapitre 1].

Lemma 2.5. Suppose that \(u_n \rightharpoonup u_0\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) and \(u_n(x) \to u_0(x)\) a.e. in \(\mathbb{R}^N\). Then
\[
(12) \quad \frac{1}{|x|^{\alpha}} \* F(|u_n(x)|) - \frac{1}{|x|^{\alpha}} \* F(|u_0(x)|) \quad \text{in} \quad L^{2N/\alpha}(\mathbb{R}^N).
\]

Proof. In this proof we adapt some ideas of [2].

The growth condition implies that \(F(|u_n|)\) is bounded in \(L_{2N-\alpha}^{\frac{2N-\alpha}{N}}(\mathbb{R}^N)\). Since we can suppose that \(u_n(x) \to u_0(x)\) a.e. in \(\mathbb{R}^N\), it follows from the continuity of \(F\) that \(F(|u_n(x)|) \to F(|u_0(x)|)\). From Lemma 2.4 follows

\[
F(|u_n(x)|) \to F(|u_0(x)|).
\]

As a consequence of the Hardy-Littlewood-Sobolev inequality, we have that
\[
\frac{1}{|x|^\alpha} \* w(x) \in L^{2N/\alpha}(\mathbb{R}^N)
\]
for all \(w \in L_{2N-\alpha}^{\frac{2N-\alpha}{N}}(\mathbb{R}^N)\); this is a bounded linear operator from \(L_{2N-\alpha}^{\frac{2N-\alpha}{N}}(\mathbb{R}^N)\) to \(L^{2N/\alpha}(\mathbb{R}^N)\). A new application of Lemma 2.4 yields (12).}

Corollary 2.1. Consider
\[
D(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} \* F(|u|) \right) F(|u|).
\]

If \(u_n \rightharpoonup u_0\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) and \(u_n(x) \to u_0(x)\) a.e. in \(\mathbb{R}^N\), then \(D(u_n) \to D(u_0)\) and \(D(u_n - u_0) \to 0\).

Proof.
\[
D(u_n) - D(u_0) = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} \* F(|u_n|) \right) F(|u_n|) - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} \* F(|u_0|) \right) F(|u_0|)
\]
\[
= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} \* F(|u_n|) \right) [F(|u_n|) - F(u_0)]
\]
\[
+ \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} \* [F(|u_n|) - F(|u_0|)] \right) F(|u_0|).
\]

(13)

It follows from Lemma 2.4 that
\[
\left( \frac{1}{|x|^{\alpha}} \* F(|u_n|) \right)
\]
Thus, (14) is bounded. Since \( F \) is continuous, we have \( F(|u_n(x)|) - F(|u_0(x)|) = 0 \) a.e. in \( \mathbb{R}^N \).

So, both integrals in (13) go to zero when \( n \to \infty \) and we are done. \( \square \)

**Corollary 2.2.** Suppose that \( u_n \to u_0 \) and consider

\[
D'(u_n) \cdot \psi = \Re \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_n|) \right] \tilde{f}(|u_n|)(u_n)\overline{\psi},
\]

for \( \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{C}) \). Then \( D'(u_n) \cdot \psi \to D'(u_0) \cdot \psi \).

**Proof.** It follows from the growth condition on \( f \) that \( \tilde{f}(|u_n|) \) is bounded in \( L^p(\mathbb{R}^N) \). Since \( u_n(x) \to u_0(x) \) a.e. in \( \mathbb{R}^N \) and \( \tilde{f} \) is continuous, by applying Lemma 2.5 we conclude that

\[
\tilde{f}(|u_n|)u_n \to \tilde{f}(|u_0|)u_0 \quad \text{in} \quad L^q(\mathbb{R}^N, \mathbb{C}).
\]

Thus,

\[
\left| \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_n|) \right] \tilde{f}(|u_n|)(u_n)\overline{\psi} - \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_0|) \right] \tilde{f}(|u_0|)(u_0)\overline{\psi} \right|
\]

\[
\leq \left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \ast F(|u_n|) \left( \tilde{f}(|u_n|)u_n - \tilde{f}(|u_0|)u_0 \right)\overline{\psi} \right|
\]

\[+
\left| \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \ast [F(|u_n|) - F(|u_0|)] \tilde{f}(|u_0|)u_0\overline{\psi} \right|.
\]

The claim follows from Lemma 2.5 and (14). \( \square \)

3. Ground state

In order to consider the general case of the potential \( V(y) \), we adapt a well-known result due to M. Struwe:

Let \((u_n)\) be the minimizing sequence given as consequence of Lemma 2.1 that is, \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) such that

\[
J'_{A,V}(u_n) \to 0 \quad \text{and} \quad J_{A,V}(u_n) \to c,
\]

where

\[
c = \inf_{u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V}(tu).
\]

We assume that \( u_n \rightharpoonup u_0 \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \). We define \( u^1_n = u_n - u_0 \) and consider the limit problem

\[
-(\nabla + iA_\infty)^2 u + V_\infty u = \left( \frac{1}{|x|^\alpha} \ast F(|u|) \right) \frac{f(|u|)}{|u|} u,
\]

where \( A_\infty = \lim_{|x| \to \infty} A(x) \) and \( V_\infty = \lim_{|x| \to \infty} V(y) \). The energy functional attached to this problem is, of course,

\[
J_\infty(u) = \frac{1}{2} \| u \|_{A_\infty,V_\infty}^2 - D(u).
\]

**Lemma 3.1** (Splitting Lemma). Let \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) be such that

\[
J_{A,V}(u_n) \to c, \quad J'_{A,V}(u_n) \to 0
\]

and \( u_n \rightharpoonup u_0 \) weakly on \( H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \). Then \( J'_{A,V}(u_0) = 0 \) and we have either

(i) \( u_n \to u_0 \) strongly on \( H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \);
\((ii)\) or there exist \(k \in \mathbb{N}\), \((y^j) \in \mathbb{R}^N\) such that \(|y^j_n| \to \infty\) for \(j \in \{1, \ldots, k\}\) and nontrivial solutions \(u^1, \ldots, u^k\) of problem \((15)\) so that

\[ J_{A,V}(u_n) \to J_{A,V}(u_0) + \sum_{j=1}^{k} J_{\infty}(u_j) \]

and

\[ \left\| u_n - u_0 - \sum_{j=1}^{k} u^j (\cdot - y^j_n) \right\| \to 0. \]

Proof. (Sketch) We simply adapt the arguments presented in \([10, \text{Lemma 2.3}]\) and \([21, \text{Theorem 8.4}]\). Since \(D'(u_n) \phi \to D'(u_0) \phi\), it follows that \(J'_{A,V}(u_0) \phi = 0\).

By setting \(u^1_n = u_n - u_0\), we have

1. \(\|u^1_n\|_{A,V}^2 = \|u_n\|_{A,V}^2 - \|u_0\|_{A,V}^2 + o_n(1);\)
2. \(J_{\infty}(u^1_n) \to c - J_{A,V}(u_0);\)
3. \(J'_{\infty}(u^1_n) \to 0.\)

Let us define

\[ \delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B^1(y)} |u^1_n|^2 \, dx. \]

If \(\delta = 0\), it follows that \(u^1_n \to 0\) in \(L'(\mathbb{R}^N)\) for all \(t \in (2, 2^*)\). It follows that \(u^1_n \to 0\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\), since \(J'_{\infty}(u^1_n) \to 0\). In this case, the proof of Lemma \(3.1\) is complete.

So, let us suppose that \(\delta > 0\). Then, we obtain a sequence \((y^1_n) \subset \mathbb{R}^N\) such that

\[ \int_{B^1(y_n)} |u^1_n|^2 \, dx \geq \frac{\delta}{2}. \]

By setting \(v^1_n = u^1_n (\cdot + y^1_n)\), we obtain a new bounded sequence \((v^1_n)\). Therefore, we assume that \(v^1_n \to v_1\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) and \(v^1_n \to v\) a.e. in \(\mathbb{R}^N\).

Since

\[ \int_{B^1(0)} |v^1_n|^2 \, dx \geq \frac{\delta}{2}, \]

we conclude that \(u^1 \neq 0\) as consequence of Sobolev’s immersion. We also conclude that \((y_n)\) is unbounded, since \(u^1_n \to 0\) in \(H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\). Therefore, we may assume that \(|y^1_n| \to \infty\). Then, it is easy to see that \(J'_{\infty}(u^1) = 0\).

We now define \(u^2_n = u_n - u^1 (\cdot - y_n)\). We then have

1. \(\|u^2_n\|_{A,V}^2 = \|u_n\|_{A,V}^2 - \|u_0\|_{A,V}^2 - \|u^1\|_{A,V}^2 + o_n(1);\)
2. \(J_{\infty}(u^2_n) \to c - J_{A,V}(u_0) - J_{\infty}(u^1);\)
3. \(J'_{\infty}(u^2_n) \to 0.\)

Proceeding by iteration, we observe that, if \(u\) is a nontrivial critical point of \(J_{\infty}\) and \(\tilde{u}\) a ground state of problem \((15)\), then the Ambrosetti-Rabinowitz condition implies that

\[ J_{\infty}(u) \geq J_{\infty}(\tilde{u}) = \int_{\mathbb{R}^N} \left( \frac{1}{2} f(|\tilde{u}|) |\tilde{u}|^2 - F(|\tilde{u}|) \right) =: \beta > 0. \]

Therefore, it follows from \((b_2)\) that the iteration process must end at some index \(k \in \mathbb{N}\). \(\Box\)
Remark 3.1. Observe that, in particular, the proof shows that the sequence \( u_n \) converges to \( \bar{u} \) and we have a solution of problem (15).

The next result also follows [10] Corollary 2.3, see also [4]. We present the proof for the convenience of the reader.

**Lemma 3.2.** The functional \( J_{A,V} \) satisfies \((PS)_c\) for any \( 0 \leq c < c_{\infty} \).

**Proof.** Let us suppose that \((u_n)\) satisfies \( J_{A,V}(u_n) \to c < c_{\infty} \) and \( J'_{A,V}(u_n) \to 0 \).

According to Lemma 2.3, we can suppose that the sequence \((u_n)\) is bounded. Therefore, for a subsequence, we have \( u_n \rightharpoonup u_0 \) in \( H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \). It follows from the Splitting Lemma 3.1 that \( J'_{A,V}(u_0) = 0 \). Since

\[
J'_{A,V}(u_0) \cdot u_0 = \|u_0\|_{A,V}^2 - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} \ast F(|u_0|) \right) f(|u_0|)|u_0|
\]

we conclude that

\[
J_{A,V}(u_0) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_0|) \right] (f(|u_0|)|u_0| - 2F(|u_0|))
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_0|) \right] F(|u_0|)
\]

\[
> \frac{1}{2} \int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|u_0|) \right] (f(|u_0|)|u_0| - 2F(|u_0|)) > 0
\]

(16)

as a consequence of the Ambrosetti-Rabinowitz condition.

If \( u_n \not\rightharpoonup u_0 \) in \( H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \), by applying again the Splitting Lemma we guarantee the existence of \( k \in \mathbb{N} \) and nontrivial solutions \( u^1, \ldots, u^k \) of problem (15) satisfying

\[
\lim_{n \to \infty} J_{A,V}(u_n) = c = J_{A,V}(u_0) + \sum_{j=1}^{k} J_{\infty}(u^j) \geq kc_{\infty} \geq c_{\infty}
\]

contradicting our hypothesis. We are done.

We prove the next result by adapting the proof given in Furtado, Maia e Medeiros [10] Proposition 3.1, see also [4]:

**Lemma 3.3.** Suppose that \( V(y) \) satisfies \((V_3)\). Then

\[
0 < c < c_{\infty},
\]

where \( c \) is characterized in Lemma 2.3.

**Proof.** Let \( \bar{u} \) be the weak solution of (15) obtained in the proof of the Splitting Lemma (see Remark 3.1) and \( t_\bar{u} > 0 \) the unique number such that \( t_\bar{u} \bar{u} \in \mathcal{N}_{A,V} \). We claim that \( t_\bar{u} < 1 \). Indeed, it follows from the condition \((AV)\) that

\[
\int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\alpha} \ast F(|t_\bar{u} \bar{u}|) \right] f(|t_\bar{u} \bar{u}|)|t_\bar{u} \bar{u}| = t_\bar{u}^2 \| \bar{u} \|_{A,V}
\]
\[
< t_\alpha^2 \| \bar{u} \|_{L^\infty} \\
= t_\alpha^2 \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(|\bar{u}|) f(|\bar{u}|)|\bar{u}| \\
= t_\alpha^2 \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(|\bar{u}|) f(|\bar{u}|)|\bar{u}| + \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(t_\alpha \bar{u}) f(|\bar{u}|)|\bar{u}| \\
- \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(t_\alpha \bar{u}) f(|\bar{u}|)|\bar{u}| \right) 
\]

thus yielding
\[
0 > \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(t_\alpha \bar{u}) \left( \tilde{f}(t_\alpha \bar{u}) - \tilde{f}(|\bar{u}|) \right) \\
+ t_\alpha^2 \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} \left( F(t_\alpha \bar{u}) - F(|\bar{u}|) \right) f(|\bar{u}|)|\bar{u}|. 
\]

If \( t_\alpha \geq 1 \), since \( \tilde{f} \) is increasing, the first integral is non-negative and the second as well, since \( F \) is also increasing. We conclude that \( t_\alpha < 1 \).

Lemma 2.3 and its previous comments show that
\[
c \leq \max_{t \geq 0} J_{A,V}(t \bar{u}) = J_{A,V}(t_\alpha \bar{u}) \\
= \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(t_\alpha \bar{u}) \left( \frac{1}{2} f(|t_\alpha \bar{u}|)|t_\alpha \bar{u}| - F(|t_\alpha \bar{u}|) \right). 
\]

Since
\[
g(t) = \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(|t \bar{u}|) \left( \frac{1}{2} f(|t \bar{u}|)|t \bar{u}| - F(|t \bar{u}|) \right) 
\]
is a strictly increasing function, we conclude that
\[
c = g(t_\alpha) < g(1) = \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} F(|\bar{u}|) \left( \frac{1}{2} f(|\bar{u}|)|\bar{u}| - F(|\bar{u}|) \right) = c_{\infty}, 
\]
proving our result. \( \square \)

**Proof of Theorem 7** Let \((u_n)\) be the minimizing sequence given by Lemma 2.1. It follows from Lemmas 3.2 and 3.3 that \(u_n\) converges to \(u \in H_{1,A,V}(\mathbb{R}^N, \mathbb{C})\) satisfying \(J_{A,V}(u) = c\) and \(J_{A,V}(u) = 0\). \( \square \)

4. **ON THE MULTIPOLICY OF SOLUTIONS**

In order to obtain multiplicity of solutions, we consider in this section a particular case of that considered by Cingolani, Clapp and Secchi in [7]. We think that the direct proof we present is interesting.

So, let \( G \) be a closed subgroup of \( O(n) \), the group of orthogonal transformations in \( \mathbb{R}^N \). As in [7], we suppose that \( A(gx) = gA(x) \) and \( V(gx) = V(x) \) for all \( g \in G \) and \( x \in \mathbb{R}^N \) and take a continuous group homomorphism \( \tau : G \to S^1 \) into the unit complex numbers \( S^1 \).

We consider the space
\[
H_{1}^1(\mathbb{R}^N, \mathbb{C})^\tau = \{ u \in H_{A}^1(\mathbb{R}^N, \mathbb{C}) : u(gx) = \tau(g)u(x), \forall \ g \in G, \forall \ x \in \mathbb{R}^N \}.
\]

We apply the following compactness result due to P.L. Lions:
Lemma 4.1 (Lions). Let $G$ be a closed subgroup of $O(N)$ and denote

$$H^1_G = \{ u \in H^1(\mathbb{R}^N) : gu = u, \forall g \in G \}.$$ 

Suppose that $\sum_{j=1}^k N_j = N$, $N_j \geq 2$ for all $j \in \{1, \ldots, k\}$, and $G = O(N_1) \times O(N_2) \times \cdots \times O(N_k)$. Then, the immersion $H^1_G(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is compact for $2 < p < 2^*$.

Observe that, if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1_G(\mathbb{R}^N)$.

Proof of Theorem 3. It follows from applying Theorem 10.10 from Ambrosetti e Malchiodi to the Nehari manifold $M = \mathcal{N}_{A,V}$. \qed

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