Hilbert Genus Fields of Some Number Fields with High Degrees

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Received: 16 November 2021 / Revised: 23 September 2022 / Accepted: 26 September 2022 / Published online: 2 December 2022
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Abstract
The aim of this paper is to give some properties of Hilbert genus fields and construct the Hilbert genus fields of the fields $L_{m,d} := \mathbb{Q}(\zeta_{2^m}, \sqrt{d})$, where $m \geq 3$ is a positive integer and $d$ is a square-free integer whose prime divisors are congruent to ±3 (mod 8) or 9 (mod 16).

Keywords Unramified extensions · Hilbert genus fields

Mathematics Subject Classification (2020) 11R16 · 11R29 · 11R27 · 11R04 · 11R37

1 Introduction

Let $d$ be a square-free integer and $m \geq 3$ a positive integer. The fields $L_{m,d} := \mathbb{Q}(\zeta_{2^m}, \sqrt{d})$ represent the layers of the cyclotomic $\mathbb{Z}_2$-extension of the special Dirichlet fields $\mathbb{Q}(\sqrt{-1}, \sqrt{d})$ and they were the subject of some recent studies, which gave interest to their 2-class groups (cf. [2, 3]). In the present work, we shall construct the Hilbert genus fields of these fields.
This question has been investigated by many mathematicians for fields of small degrees (e.g., 2 and 4). For example, Bae and Yue studied the Hilbert genus field of the fields \( \mathbb{Q}(\sqrt{p}, \sqrt{d}) \), for a prime number \( p \) such that \( p = 2 \) or \( p \equiv 1 \pmod{4} \), and a positive square-free integer \( d \) (cf. [1]).

Thereafter, Ouyang and Zhang have determined the Hilbert genus field of the imaginary biquadratic fields \( \mathbb{Q}(\sqrt{\delta}, \sqrt{d}) \), where \( \delta = -1, -2 \) or \( -p \) with \( p \equiv 3 \pmod{4} \) is a prime number and \( d \) any square-free integer. Thereafter, they constructed the Hilbert genus field of the real biquadratic fields \( \mathbb{Q}(\sqrt{\delta}, \sqrt{d}) \), for any positive square-free integer \( d \), and \( \delta = p, 2p \) or \( p_1 p_2 \) where \( p, p_1, p_2 \) are prime numbers congruent to 3 (mod 4), such that the class number of \( \mathbb{Q}(\sqrt{\delta}) \) is odd (cf. [9, 10]). For more works on this subject, we refer the reader to the papers [5, 10, 11, 13].

In this paper, we shall construct the Hilbert genus field of the fields \( L_{m,d} \), whenever all the prime divisors of \( d \) are congruent to \( \pm 3 \pmod{8} \) or \( 9 \pmod{16} \). Our results generalize some results of Ouyang and Zhang on the fields \( \mathbb{Q}(\sqrt{-1}, \sqrt{d}) \) (cf. [9]).

The plan of this paper is as follows: In Section 2, we introduce some properties of the Hilbert genus field. In the last section, we give the list of Hilbert genus fields of the fields \( L_{m,d} \).

Let us stick the following notations: Let \( k \) be a number field. \( \mathcal{O}_k \) the ring of integers \( k \), \( E(k) \) the Hilbert genus field of \( k \). \( N_{k/k'} \) denotes the norm map of an extension \( k/k' \). Let \( r_2(G) \) denote the 2-rank of an abelian finite group \( G \). Denote by \( \zeta_n \) an \( n \)th primitive root of unity. For more notations, see the beginning of each section below.

## 2 Some Results on Hilbert Genus Fields

Let us start this section by recalling some definitions and results. Let \( k \) be a number field and denote by \( \text{Cl}(k) \) its class group. It is known by class field theory that we have \( G := \text{Gal}(H(k)/k) \cong \text{Cl}(k) \), where \( H(k) \) is the Hilbert class field. The Hilbert genus field of \( k \) is defined as the invariant field \( E(k) \) of \( G^2 \). Then, we have

\[
\text{Cl}(k)/\text{Cl}(k)^2 \cong G/G^2 \cong \text{Gal}(E(k)/k),
\]

and thus, \( r_2(\text{Cl}(k)) = r_2(\text{Gal}(E(k)/k)) \). On the other hand, \( E(k)/k \) is the maximal unramified Kummer extension of exponent 2. Thus, by Kummer theory (cf. [8, p. 14]), there exists a unique multiplicative group \( \Delta(k) \) such that

\[
E(k) = H(k) \cap k(\sqrt{k^*}) = k(\sqrt{\Delta(k)}) \quad \text{and} \quad k^{*2} \subset \Delta(k) \subset k^*.
\]

Therefore, the construction of the Hilbert genus field of \( k \) is equivalent to the determination of a set of generators for the finite group \( \Delta(k)/k^{*2} \).

We need the following proposition which is a particular case of the result in [7, p. 239].

**Proposition 2.1** ([7, Theorem 5.20]) Let \( k/k' \) be a quadratic extension of number fields and \( \alpha \) an element of \( k' \), coprime with 2, such that \( k = k'(\sqrt{\alpha}) \). The extension \( k/k' \) is unramified at all finite primes of \( k' \) if and only if the following two items hold:

1. The ideal generated by \( \alpha \) is the square of the fractional ideal of \( k' \),
2. There exists a nonzero number \( \xi \) of \( k' \) verifying \( \alpha \equiv \xi^2 \pmod{4} \).

**Corollary 2.2** Let \( k/k' \) be an extension of number fields and let \( \alpha \in k' \) be a square-free in \( k \). If \( k'(\sqrt{\alpha})/k' \) is unramified, then \( k(\sqrt{\alpha})/k \) is also unramified.
Proof As \( k'(\sqrt{\alpha})/k' \) is unramified, then there exist an ideal \( b \) of \( \mathcal{O}_{K'} \) such that \( \alpha \mathcal{O}_{K'} = b^2 \) and \( \xi \in k' \) verifying \( \alpha \equiv \xi^2 \pmod{4} \). Therefore, \( \alpha \mathcal{O}_K = (b \mathcal{O}_K)^2 \) and \( \xi \in k' \) verifies \( \alpha \equiv \xi^2 \pmod{4} \). So, the result follows by Proposition 2.1.

Let us put the following definition.

**Definition 2.3** A number field \( k \) is said \( \mathbb{QO} \)-field if it is a quadratic extension of certain number field \( k' \) whose class number is odd. We shall call \( k' \) a base field of the \( \mathbb{QO} \)-field \( k \) and the \( k/k' \) is a \( \mathbb{QO} \)-extension.

**Lemma 2.4** Let \( k/k' \) be a \( \mathbb{QO} \)-extension. Let \( \Delta(k) \) denote the multiplicative group such that \( k^* \subset \Delta(k) \subset k^* \) and \( k(\sqrt{\Delta(k)}) \) the Hilbert genus field of \( k \) (see the beginning of this section). Then,

\[
r_2(\Delta(k)/k^*) = t - e - 1,
\]

with \( t \) is the number of ramified primes (finite or infinite) in the extension \( k/k' \) and \( e \) is defined by \( 2^e = [E_{k'} : E_{k'} \cap N_{k'/k}(k^*)] \), where \( E_{k'} \) is the unit group of \( k' \).

**Proof** Put \( G = \text{Gal}(H(k)/k) \). By the definition of \( E(k) \) (see the beginning of this section) we have

\[
\text{Cl}(k)/\text{Cl}(k)^2 \simeq G/G^2 \simeq \text{Gal}(E(k)/k).
\]

Since \( E(k) = k(\sqrt{\Delta(k)}) \), then by class field theory

\[
r_2(\Delta(k)/k^*) = \log_2[k(\sqrt{\Delta(k)}) : k] = \log_2[E(k) : k] = r_2(\text{Cl}(k)).
\]

So, the result follows by the well-known ambiguous class number formula (cf. [12, Lemma 2.4]).

**Theorem 2.5** Let \( k = k'(\sqrt{\mu}) \) be a ramified quadratic extension of \( \mathbb{QO} \)-fields that have the same base field. Assume furthermore that \( r_2(\text{Cl}(k)) = r_2(\text{Cl}(k')) \). Then, we have

\[
E(k) = k(E(k')).
\]

**Proof** Let \( \gamma_1, \ldots, \gamma_r \) be \( r = r_2(\text{Cl}(k')) \) elements of \( k' \) such that,

\[
E(k') = k'(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_r}).
\]

By Corollary 2.2, \( k(\sqrt{\gamma_i})/k \) is unramified for all \( i \). Therefore, \( \{\gamma_1, \ldots, \gamma_r\} \) is a representative set of \( \Delta(k)/k^2 \). In fact, it suffices to prove that it is linearly independent modulo \( k^* \). Put \( \beta = \gamma_1 \gamma_2 a_2 \cdots \gamma_r a_r \), for some \( a_i \in \{0, 1\} \), and assume that \( \beta \in k^2 \). One can easily check that we have \( \beta \in k^2 \) or \( \mu \beta \in k^2 \).

- Since \( \{\gamma_1, \ldots, \gamma_r\} \) is a representative set of \( \Delta(k')/k^2 \), the case \( \beta \in k^2 \) is clearly impossible.
- Assume that \( \mu \beta \in k^2 \). Thus, \( \mu \gamma_1 \gamma_2 a_2 \cdots \gamma_r a_r = \alpha^2 \), for some \( \alpha \in k' \). Since \( k/k' \) is ramified, this equality implies that there is a square-free element \( \mu' \) of \( k' \) such that \( \mu' = \alpha^2 \), for some \( \alpha' \in k' \), which is impossible.

Since \( r_2(\text{Cl}(k)) = r \), we have the result.

Thus, we have the following corollary.
Corollary 2.6  Let $k$ be a QO-number field, $k'$ its base field, and $k_n$ the $n$th layer of its cyclotomic $\mathbb{Z}_2$-extension. Let $n_0$ be an integer such that every prime which ramifies in $k_\infty/k_{n_0}$ is totally ramified and $r_2(\text{Cl}(k_{n_0})) = r_2(\text{Cl}(k_{n_0+1})) = r$. Assume furthermore that $k_{n_0+1}$ is a QO-field whose base field is the $(n_0 + 1)$th layer of the cyclotomic $\mathbb{Z}_2$-extension of $k'$. If $E(k) = k_{n_0}(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_r})$ and $k_n'$ has odd class number, then

$$E(k_n) = k_n(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_r})$$

for all $n \geq n_0$. Furthermore, $E(k_\infty) = k_\infty(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_r})$.

Proof  Under the hypothesis of the corollary, one can use the well-known result of Fukuda (cf. [4, Theorem 1]) to check what follows:

- The class number of $k_n'$, the $n$th layer of cyclotomic $\mathbb{Z}_2$-extension of $k'$, is odd for all $n$.
- $r_2(\text{Cl}(k_n)) = r$ for all $n \geq n_0$.

Therefore, a recurrent application of Theorem 2.5 gives the corollary.

3  The Hilbert Genus Field of $L_{m,d} = \mathbb{Q}(\sqrt[2^m]{d})$

Keep notations fixed in the previous sections. Now, we state the main result of the paper. For the existence of solutions of the diophantine equations mentioned on the theorem below, one can see [9] and [6, p. 324].

Theorem 3.1  Let $d$ be a square-free integer whose prime divisors are congruent to $\pm 3$ (mod 8) or 9 (mod 16). Let $n$ be the number of prime divisors of $d$, $s$ the number of prime divisors of $d$ which are congruent 5 (mod 8), $t$ the number of prime divisors of $d$ which are congruent 3 (mod 8), and $r$ the number of those which are congruent 9 (mod 16). We have $n = r + t + s$. Let $d = \ell_1 \ldots \ell_r p_1 \ldots p_s q_1 \ldots q_t$ be the factorization of $d$ such that $\ell_i \equiv 9 \pmod{16}$ and $p_i \equiv 5 \equiv -q_i \pmod{8}$. Put $L_{m,d} = \mathbb{Q}(\sqrt[2^m]{d})$, $m \geq 3$.

1. Assume that $r = 0$ and $s \geq 2$. Let $(x_i, y_i)$ be a primitive solution of $p_1 p_i = x^2 + y^2$ ($i \geq 2$) such that $x_i \equiv 1 \pmod{2}$ and $y_i \equiv 0 \pmod{4}$. Then for all $m \geq 3$,

$$E(L_{m,d}) = L_{m,d}(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_{n-1}}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_n})$$

where $\gamma_i = x_i + y_j \sqrt{-1}$.

2. Assume that $r = s = 0$ and $t \geq 2$. Let $(x'_i, y'_i)$ be a primitive solution of $q_1 q_i = x^2 + 2 y^2$ ($i \geq 2$) such that $x'_i \equiv 1 \pmod{2}$ and $y'_i \equiv 0 \pmod{2}$. Then for all $m \geq 1$,

$$E(L_{m,d}) = L_{m,d}(\sqrt{q_1}, \ldots, \sqrt{q_{n-1}}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n})$$

where $\alpha_i = x'_i + y'_i \sqrt{-2}$.

3. Assume that $r = 0$ and $t, s \geq 2$. Then for all $m \geq 3$,

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{s} \mathbb{Q}(\sqrt{p_i}) \right) \left( \prod_{i=1}^{t} \mathbb{Q}(\sqrt{q_i}) \right) \left( \prod_{i=2}^{s} \mathbb{Q}(\sqrt{\gamma_i}) \right) \left( \prod_{i=2}^{t} \mathbb{Q}(\sqrt{\alpha_i}) \right),$$

where $\alpha_i$ and $\gamma_i$ are defined in the first two items.
(4) Assume that $r \geq 1$ and $t, s \geq 2$. Let $(a_i, b_i), (e_i, f_i)$, and $(u_i, v_i)$ such that $\ell_i = a_i^2 + 16b_i^2 = e_i^2 - 32f_i^2 = u_i^2 + 8v_i^2$ and $a_i \equiv e_i \equiv u_i \equiv \pm 1 \pmod{4}$. Then for all $m \geq 3$,

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{l_i}, \sqrt{\pi_{1,i}, \sqrt{\pi_{2,i}, \sqrt{\pi_{3,i}}}}) \right) \left( \prod_{i=1}^{s} \mathbb{Q}(\sqrt{p_i}) \right) \left( \prod_{i=1}^{t} \mathbb{Q}(\sqrt{q_i}) \right),$$

where $\alpha_i$ and $\gamma_i$ are defined in the first two items, and $\pi_{1,i} = a_i + 4ib_i$, $\pi_{2,i} = e_i + 4fi\sqrt{2}$, and $\pi_{3,i} = u_i + 2v_i\sqrt{-2}$.

(5) Assume that $t = 0$, $r \geq 1$, and $s \geq 2$. For all $m \geq 3$, we have

- If $\exists i \in \{1, \ldots, r\}$ such that $\left(\frac{2}{\ell_i}\right)_4 = 1$, then

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{l_i}, \sqrt{\pi_{1,i}, \sqrt{\pi_{2,i}, \sqrt{\pi_{3,i}}}}) \right) \left( \prod_{i=1}^{s} \mathbb{Q}(\sqrt{p_i}) \right) \left( \prod_{i=2}^{s} \mathbb{Q}(\sqrt{\gamma_i}) \right),$$

- Else,

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{l_i}, \sqrt{\pi_{1,i}, \sqrt{\pi_{2,i}, \sqrt{\pi_{3,i}}}}) \right) \left( \prod_{i=1}^{s-1} \mathbb{Q}(\sqrt{p_i}) \right) \left( \prod_{i=2}^{s} \mathbb{Q}(\sqrt{\gamma_i}) \right),$$

where $\gamma_i$, $\pi_{1,i}$, $\pi_{2,i}$, and $\pi_{3,i}$ are defined as above.

(6) Assume that $s = 0$, $r \geq 1$, and $t \geq 2$. For all $m \geq 3$, we have

- If $\exists i \in \{1, \ldots, r\}$ such that $\left(\frac{2}{\ell_i}\right)_4 = -1$, then

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{l_i}, \sqrt{\pi_{1,i}, \sqrt{\pi_{2,i}, \sqrt{\pi_{3,i}}}}) \right) \left( \prod_{i=1}^{r-1} \mathbb{Q}(\sqrt{l_i}) \right) \left( \prod_{i=1}^{t} \mathbb{Q}(\sqrt{q_i}) \right) \left( \prod_{i=2}^{t} \mathbb{Q}(\sqrt{\alpha_i}) \right),$$

- Else,

$$E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{l_i}, \sqrt{\pi_{1,i}, \sqrt{\pi_{2,i}, \sqrt{\pi_{3,i}}}}) \right) \left( \prod_{i=1}^{t-1} \mathbb{Q}(\sqrt{q_i}) \right) \left( \prod_{i=2}^{t} \mathbb{Q}(\sqrt{\alpha_i}) \right),$$

where $\alpha_i$, $\pi_{1,i}$, $\pi_{2,i}$, and $\pi_{3,i}$ are defined as above.
(7) Assume that \( t = s = 0 \) and \( r \geq 2 \). For all \( m \geq 3 \), we have
- If there exist \( i, j \in \{1, \ldots, r\} \) such that \( \left( \frac{2}{r} \right)_4 \neq \left( \frac{2}{r} \right)_4 \), then
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}) \right) \left( \prod_{i=1}^{r-1} \mathcal{Q}(\sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right).
\]
- Else,
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}) \right) \left( \prod_{i=1}^{r-1} \mathcal{Q}(\sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right),
\]
where \( \pi_{1,i}, \pi_{2,i}, \) and \( \pi_{3,i} \) are defined as above.

(8) Assume that \( s = 1, r \geq 1, \) and \( t \geq 2 \). For all \( m \geq 3 \), we have
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right) \left( \prod_{i=1}^{t} \mathcal{Q}(\sqrt[q_i]{}) \right) \left( \prod_{i=2}^{t} \mathcal{Q}(\sqrt[q_i]{}) \right).
\]

(9) Assume that \( t = 1, r \geq 1, \) and \( s \geq 2 \). For all \( m \geq 3 \), we have
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right) \left( \prod_{i=1}^{s} \mathcal{Q}(\sqrt[p_i]{}) \right) \left( \prod_{i=2}^{s} \mathcal{Q}(\sqrt[y_i]{}) \right).
\]

(10) Assume that \( s = t = 1 \) and \( r \geq 1 \). For all \( m \geq 3 \), we have
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right) \mathcal{Q}(\sqrt[p_1]{}).
\]

(11) Assume that \( t = 0, s = 1, \) and \( r \geq 1 \). For all \( m \geq 3 \), we have
- If \( \exists i \in \{1, \ldots, r\} \) such that \( \left( \frac{2}{r} \right)_4 = 1 \), then
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right) \left( \prod_{i=1}^{r-1} \mathcal{Q}(\sqrt[l_i]{}) \right).
\]
- Else,
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}) \right) \left( \prod_{i=1}^{r-1} \mathcal{Q}(\sqrt[l_i]{}) \right).
\]

(12) Assume that \( s = 0, t = 1 \) and \( r \geq 1 \). For all \( m \geq 3 \), we have
- If \( \exists i \in \{1, \ldots, r\} \) such that \( \left( \frac{2}{r} \right)_4 = -1 \), then
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right) \left( \prod_{i=1}^{r-1} \mathcal{Q}(\sqrt[l_i]{}) \right).
\]
- Else,
\[
E(L_{m,d}) = L_{m,d} \left( \prod_{i=1}^{r} \mathcal{Q}(\sqrt[l_i]{}, \sqrt[\pi_{1,i}]{}, \sqrt[\pi_{2,i}]{}, \sqrt[\pi_{3,i}]{}) \right).
\]

(13) Assume that \( r = 0 \) and \( s = t = 1 \). For all \( m \geq 3 \), we have
\[
E(L_{m,d}) = L_{m,d}(\sqrt[p_1]{}) = L_{m,d}(\sqrt[q_1]{}).
\]
Thus, β

Proof. To simplify notations, let us put the following set

\[ \pi_{1,1} = a_1 + 4i b_1 \text{ and } \pi_{2,1} = e_1 + 4 f_1 \sqrt{2} \text{ such that } l_1 = a_1^2 + 16b_1^2 = e_1^2 - 32f_1^2. \]

Note that by [2, Theorem 1], we have

\[ \eta = \{p_1, \ldots, p_{n-1}, \gamma_2, \ldots, \gamma_n\}. \]

Let us show that the elements of η are linearly independent modulo \( L^{*2} \). Put

\[ \beta = \left( \prod_{i=1}^{n-1} p_i^{a_i} \right) \left( \prod_{j=2}^{n} \gamma_j^{b_j} \right), \]

where \( a_i, b_j \in \{0, 1\} \) are not all zero. Assume that \( \beta \in L^{*2} \).

Thus, \( \beta \in K^{*2} \) or \( d\beta \in K^{*2} \).

Note that \( \exists j \in \{2, 3, \ldots, n\} \) with \( b_j \neq 0 \) (else, we get \( \beta = \prod_{i=1}^{n-1} p_i^{a_i} \in K^{*2} \), then \( \sqrt{\beta} \in K^{*} \). So \( \mathbb{Q}(\sqrt{\beta}) \) is a quadratic subfield of \( K = \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) \). But this is impossible, since the only quadratic subfields of \( K = \mathbb{Q}(\zeta) \) are \( k = \mathbb{Q}(\sqrt{-1}), k' = \mathbb{Q}(\sqrt{2}) \) and \( k'' = \mathbb{Q}(\sqrt{-2}) \). We similarly show that the case \( d\beta \in K^{*2} \) is also impossible. We therefore distinguish two cases

- If \( \beta \in K^{*2} \). Then, \( N_{K/k}(\gamma_j) = x_j^2 + \gamma_j^2 = p_1 p_j \).

Thus,

\[ N_{K/k}(\beta) = \left( \prod_{i=1}^{n-1} p_i^{a_i} \right)^2 \left( \prod_{j=2}^{n} p_j^{b_j} \right) = p_1^\ell \left( \prod_{j=2}^{n} p_j^{b_j} \right) Z^2, \]

such that \( \ell \in \{0, 1\} \) and \( Z \in \mathbb{Q} \). This clearly implies that \( p_1^\ell \left( \prod_{j=2}^{n} p_j^{b_j} \right) \) is a square in \( k \), which is impossible. Thus, \( \beta \notin K^{*2} \).

- If \( d\beta \in K^{*2} \), then \( N_{K/k}(d\beta) = d^2 N_{K/k}(\beta) \). As above, we show that this implies that

\[ p_1^\ell \left( \prod_{j=2}^{n} p_j^{b_j} \right) \in k^2, \]

which is also impossible. So \( \beta \) cannot be a square in \( L \). Therefore, the elements of \( \eta \) are linearly independent modulo \( L^{*2} \). It follows that \( \eta \) is a representative set of \( \Delta(L)/L \). Hence,

\[ E(L_{3,d}) = \mathbb{Q}(\zeta, \sqrt{d}, \sqrt{\beta_1}, \ldots, \sqrt{\beta_{n-1}}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_n}). \]

Note that by [2, Theorem 1], we have \( r_2(\text{Cl}(L_{m,d})) = 2n - 2 \) for all \( m \geq 3 \). Therefore, by Corollary 2.6, we get

\[ E(L_{m,d}) = L_{m,d}(\sqrt{\beta_1}, \ldots, \sqrt{\beta_{n-1}}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_n}). \]

(2) By [2, Theorem 1], we have \( r_2(\text{Cl}(L)) = 2n - 2 \). Consider the set

\[ \eta = \{q_1, \ldots, q_{n-1}, \alpha_2, \ldots, \alpha_n\}. \]
Since $\forall i \in \{2, \ldots, n\}: \alpha_i = x_i' + y_i'\sqrt{-2}$, where the couple $(x_i', y_i')$ is a primitive solution of $q_1 q_i = x_i'^2 + 2y_i'^2$ ($i \geq 2$) such that $x_i' \equiv 1$ (mod 2) and $y_i' \equiv 0$ (mod 2), then $\alpha_i \equiv x_i' + y_i' + \frac{1+\sqrt{-2}}{2}2y_i'$ (mod 4) $\equiv x_i' + y_i'$ (mod 4) $\equiv \pm 1$ (mod 4). Proceeding as above, we show that the extensions $L(\sqrt{\alpha_i})/L$ and $L(\sqrt{\alpha_i})/L$ are unramified and the elements of $\eta$ are linearly independent modulo $L^2$. Thus, $\eta$ is the set of generators of $\Delta(L^*)/L^2$. Therefore,

$$E(L) = L_{3,d}(\sqrt{q_1}, \ldots, \sqrt{q_{n-1}}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n}),$$

and so by [2, Theorem 1] and Corollary 2.6, we get

$$E(L_{m,d}) = L_{m,d}(\sqrt{q_1}, \ldots, \sqrt{q_{n-1}}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n}).$$

(3) By [2, Theorem 1], we have $r_2(CL(L)) = 2n - 3$. Put

$$\eta = \{p_1, \ldots, p_s, q_1, \ldots, q_{t-1}, \gamma_2, \ldots, \gamma_s, \alpha_2, \ldots, \alpha_t\}.$$

Let us show that the elements of $\eta$ are linearly independent modulo $L^2$. Set $\beta = (\prod_{i=1}^s p_i^{a_i}) (\prod_{j=2}^s q_i^{a_j}) (\prod_{j=2}^s \gamma_j^{b_j}),$ where $a_i, b_j, a_j, b_j \in \{0, 1\}$ are not all zero. Assume that $\beta \in L^2$. Then, $\beta \in K^2$ or $d\beta \in K^2$. Note that $\exists j \in \{2, \ldots, s\}$, $b_j \neq 0$ or $\exists j \in \{2, \ldots, t\}$, $b_j' \neq 0$.

- Assume that $\beta \in K^2$. Since $N_{K/k}p(\alpha_j) = x_j'^2 + 2y_j'^2 = q_1 q_j$ and $N_{K/k}\gamma_j) = x_j^2 + y_j^2 = p_1 p_j$, we have

$$N_{K/k}p(\beta) = \left(\prod_{i=1}^{s} p_i^{a_i}\prod_{j=2}^{s} q_j^{a_j}\prod_{j=2}^{s} \gamma_j^{b_j}\right)^2 \left(\prod_{j=2}^{t} (q_1 q_j)^{b_j}\right) = q_1^{2}\left(\prod_{j=2}^{t} q_j^{b_j}\right) Z_1^2,$$

$$N_{K/k}(\beta) = \left(\prod_{i=1}^{s-1} p_i^{a_i}\prod_{j=2}^{s} q_j^{a_j}\prod_{j=2}^{s} \alpha_j^{b_j}\right)^2 \left(\prod_{j=2}^{t} (p_1 p_j)^{b_j}\right) = p_1^{2}\left(\prod_{j=2}^{t} p_j^{b_j}\right) Z_2^2,$$

for some $\ell, \ell' \in \{0, 1\}$ and $Z_i \in \mathbb{Q}$. This implies that $q_1^{2}\left(\prod_{j=2}^{t} q_j^{b_j}\right)$ is a square in $k''$ and $p_1^{2}\left(\prod_{j=2}^{t} p_j^{b_j}\right)$ is a square in $k$, which is impossible. Thus, $\beta \notin K^2$.

- Analogously, we check that $d\beta \in K^2$ is impossible.

Since the extensions $L(\sqrt{p_i})/L, L(\sqrt{q_i})/L, L(\sqrt{y_j})/L,$ and $L(\sqrt{\alpha_j})/L$ are unramified, then the elements of $\eta$ are generators of $\Delta(L^*)/L^2$. Hence,

$$E(L_{m,d}) = L_{m,d}(\sqrt{p_1}, \ldots, \sqrt{p_s}, \sqrt{q_1}, \ldots, \sqrt{q_{t-1}}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_s}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_t})$$

$$= L_{m,d}(\prod_{i=1}^{s} \mathbb{Q}(\sqrt{p_i})\prod_{j=2}^{t-1} \mathbb{Q}(\sqrt{q_j})\prod_{j=2}^{s} \mathbb{Q}(\sqrt{y_j})\prod_{j=2}^{t} \mathbb{Q}(\sqrt{\alpha_j})).$$

(4) In this case, we have $r_2(CL(L)) = n - 3 = 4r + 2s + 2t - 3$. Consider the set

$$\eta = \{l_1, \ldots, l_r, p_1, \ldots, p_s, q_1, \ldots, q_{t-1}, \pi_1, \ldots, \pi_1, \pi_2, \ldots, \pi_2, \pi_3, \ldots, \pi_3, \pi_4, \ldots, \pi_4, \gamma_2, \ldots, \gamma_s, \alpha_2, \ldots, \alpha_t\}.$$
Put
\[ \beta = \left( \prod_{i=1}^{r} l_i^{\theta_{1,i} \pi_{1,i} \theta_{2,i} \pi_{2,i} \theta_{3,i} \pi_{3,i} \theta_{4,i}} \right) \left( \prod_{i=1}^{s} p_i^{\theta_{1,i}} \right) \left( \prod_{j=2}^{t-1} q_i^{\theta_{2,i}} \right) \left( \prod_{j=2}^{s} \gamma_j^{\theta_{3,j}} \right) \left( \prod_{j=2}^{t} \alpha_j^{\theta_{4,j}} \right), \]
where \( \theta_{1,i}, \theta_{2,i}, \theta_{3,i}, \theta_{4,i}, \theta_{1,i}', \theta_{2,i}', \theta_{3,i}', \theta_{4,i}' \in \{0, 1\} \) are not all zero.

Assume that \( \beta \in L^{*2} \), then \( \beta \in K^{*2} \) or \( d\beta \in K^{*2} \) (note that, as in the first item, the exponents \( \theta_{2,i}, \theta_{3,i}, \theta_{4,i}, \theta_{1,i}', \theta_{2,i}', \theta_{3,i}', \theta_{4,i}' \) cannot all vanish, since elsewhere we get that \( \beta = (\prod_{i=1}^{s} q_i^{\theta_{2,i}'}) \left( \prod_{i=1}^{s} (p_i p_j)^{\theta_{3,i}'} \right) \left( \prod_{j=2}^{t} (q_i q_j)^{\theta_{4,j}'} \right) \) is a square in \( K^{*} \), which implies that \( Q(\sqrt{\beta}) \) is a quadratic subfield of \( K \) which is impossible).

- Let \( \beta \in K^{*2} \). Since \( N_{K/k}(\pi_{1,i}) = a_i^2 + 16b_i^2 = l_i, N_{K/k}(\gamma_j) = x_j^2 + y_j^2 = p_j p_{j'}, N_{K/k'}(\pi_{5,i}) = e_i^2 - 32b_i^2 = l_i, N_{K/k''}(\alpha_j) = x_j^2 + 2y_j^2 = q_j q_{j'} \), then

\[
N_{K/k}(\beta) = \left( \prod_{i=1}^{r} l_i^{\theta_{1,i} \pi_{1,i} \theta_{2,i} \pi_{2,i} \theta_{3,i} \pi_{3,i} \theta_{4,i}} \right) \left( \prod_{i=1}^{s} p_i^{\theta_{1,i}} \right) \left( \prod_{j=2}^{t-1} q_i^{\theta_{2,i}} \right) \left( \prod_{j=2}^{s} \gamma_j^{\theta_{3,j}} \right) \left( \prod_{j=2}^{t} \alpha_j^{\theta_{4,j}} \right)^2 = p_1^\ell \left( \prod_{i=1}^{r} l_i^{\theta_{1,i}} \right) \left( \prod_{j=2}^{t} p_j^{\theta_{4,j}} \right) Z_1^2,
\]
\[
N_{K/k'}(\beta) = \left( \prod_{i=1}^{r} l_i^{\theta_{1,i} \pi_{1,i} \theta_{2,i} \pi_{2,i} \theta_{3,i} \pi_{3,i} \theta_{4,i}} \right) \left( \prod_{i=1}^{s} p_i^{\theta_{1,i}} \right) \left( \prod_{j=2}^{t-1} q_i^{\theta_{2,i}} \right) \left( \prod_{j=2}^{s} \gamma_j^{\theta_{3,j}} \right) \left( \prod_{j=2}^{t} \alpha_j^{\theta_{4,j}} \right)^2 = \left( \prod_{i=1}^{r} l_i^{\theta_{3,i}} \right) Z_2^2,
\]
\[
N_{K/k''}(\beta) = \left( \prod_{i=1}^{r} l_i^{\theta_{1,i} \pi_{1,i} \theta_{2,i} \pi_{2,i} \theta_{3,i} \pi_{3,i} \theta_{4,i}} \right) \left( \prod_{i=1}^{s} p_i^{\theta_{1,i}} \right) \left( \prod_{j=2}^{t-1} q_i^{\theta_{2,i}} \right) \left( \prod_{j=2}^{s} \gamma_j^{\theta_{3,j}} \right) \left( \prod_{j=2}^{t} (q_i q_j)^{\theta_{4,j}} \right) = q_1^\ell \left( \prod_{i=1}^{r} l_i^{\theta_{3,i}} \right) \left( \prod_{j=2}^{t} q_j^{\theta_{4,j}} \right) Z_3^2.
\]
such that \( \ell, \ell' \in \{0, 1\} \) and \( Z_i \in \mathbb{Q} \). Thus, \( p^\ell_i \left( \prod_{i=1}^{r} l_i^{\theta_{i, j}} \right) \left( \prod_{j=2}^{r} p^\ell_i \theta_{j, i} \right) \) is a square in \( k \), \( \left( \prod_{i=1}^{r} l_i^{\theta_{i, j}} \right) \) is a square in \( k' \), and \( q^\ell_i \left( \prod_{i=1}^{r} l_i^{\theta_{i, j}} \right) \left( \prod_{i=2}^{r} q_i^{\ell_{j, i}} \right) \) is a square in \( k'' \), then the above three equations are impossible.

- We similarly show that \( d \beta \in K_{*2} \) is impossible. Thus, the elements of \( \eta \) are linearly independent modulo \( L_{*2} \).

On the other hand, as above, we check that the primes of \( L \) generated respectively by \( \pi_{1, i} \), \( \pi_{2, i} \), and \( \pi_{3, i} \) are the squares of certain fractional ideals of \( L \) and as \( a_i \equiv e_i \equiv \pm 1 \mod 4 \), since they are odd, the equations \( \pi_{1, i} \equiv \xi^2 \mod 4 \) and \( \pi_{2, i} \equiv \xi^2 \mod 4 \) have solutions in \( L \) (since \( i \in L \)). Furthermore, we have \( \pi_{3, i} = u_i + 2v_i \sqrt{-2} = u_i + 2v_i + 4v_i \sqrt{\frac{1 + \sqrt{-2}}{2}} \equiv u_i + 2v_i \) (mod 4) \( \equiv \pm 1 \) (mod 4) \( \) (\( u_i \) is odd), then the equation \( \pi_{3, i} \equiv \xi^2 \mod 4 \) also has solutions in \( L \). Thus, by Proposition 2.1, the extensions \( L(\sqrt{\pi_{1, i}})/L \), \( L(\sqrt{\pi_{2, i}})/L \), and \( L(\sqrt{\pi_{3, i}})/L \) are unramified for \( i \in \{1, \ldots, k\} \). It follows that \( \eta \) is a set of generators of \( \Delta(L)/L_{*2} \). Hence, we have the fourth item.

(5) By [2, Theorem 1], if \( \exists i \in \{1, \ldots, r\} \) such that \( \left( \frac{2}{l_i} \right)_4 = -1 \), then we have \( r_2(\text{Cl}(L)) = 4r + 2s - 3 \). Put
\[
\eta = \{l_1, \ldots, l_r, p_1, \ldots, p_s, \pi_{1, 1}, \ldots, \pi_{1, r}, \pi_{2, 1}, \ldots, \pi_{2, r}, \pi_{3, 1}, \ldots, \pi_{3, r}, \gamma_2, \ldots, \gamma_s\}.
\]
As above, we show that \( \eta \) is a set of generators of \( \Delta(L)/L_{*2} \). Therefore,
\[
E(L_{m, d}) = L_{m, d} \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{\pi_{1, i}}, \sqrt{\pi_{2, i}}, \sqrt{\pi_{3, i}}) \right) \left( \prod_{i=1}^{r} \mathbb{Q}(\sqrt{\pi_i}) \right) \left( \prod_{i=2}^{s} \mathbb{Q}(\sqrt{\pi_i}) \right).
\]
With an analogous reasoning, we prove the rest of our theorem. \( \square \)

Acknowledgements The authors are so grateful to the unknown referee for his/her several helpful suggestions that helped to improve our paper, and for calling our attention to the missing details.

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