Complex trajectories in a classical periodic potential

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Abstract
This paper examines the complex trajectories of a classical particle in the potential $V(x) = -\cos(x)$. Almost all the trajectories describe a particle that hops from one well to another in an erratic fashion. However, it is shown analytically that there are two special classes of trajectories $x(t)$ determined only by the energy of the particle and not by the initial position of the particle. The first class consists of periodic trajectories; that is, trajectories that return to their initial position $x(0)$ after some real time $T$. The second class consists of trajectories for which there exists a real time $T$ such that $x(t+T) = x(t) \pm 2\pi$. These two classes of classical trajectories are analogous to valence and conduction bands in quantum mechanics, where the quantum particle either remains localized or else tunnels resonantly (conducts) through a crystal lattice. These two special types of trajectories are associated with sets of energies of measure 0. For other energies, it is shown that for long times the average velocity of the particle becomes a fractal-like function of energy.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, there has been a substantial research effort whose aim is to extend classical mechanics into the complex domain \cite{1-3}. Intriguing analogies with quantum mechanics emerge when conventional classical mechanics is generalized to complex classical mechanics. For instance, complex trajectories of particles under the influence of double-well potentials display tunneling-like behavior when the classical particle has complex energy \cite{4, 5}.

In a previous work complex trajectories in periodic potentials were studied numerically \cite{6}. These numerical investigations identified two types of motion for a complex classical particle in a periodic potential and found a striking analogy with the behavior of a quantum
particle in a periodic potential. Under the influence of the potential \( V(x) = -\cos(x) \), a classical particle having complex energy appears either to remain localized or to move in one direction from site to site. These types of motion are analogous to the valence-band and the conduction-band behavior of a quantum particle in a periodic potential.

A recent paper showed that the complex classical equations of motion for a quartic potential can be solved analytically in terms of doubly periodic elliptic functions [7]. This paper uses a similar procedure to study analytically the classical motion of a particle in the potential \( V(x) = -\cos(x) \). The analytic solution allows us to reexamine the cosine potential, which until now was only examined numerically. In agreement with earlier research, our analysis in this paper leads us to conclude that there exist two special sets of energies that give rise to these two kinds of classical behavior, and for these energies we find that this classical behavior does not depend on the initial position \( x(0) \) of the particle. For energies in the first set the classical trajectories \( x(t) \) are periodic; that is, they satisfy \( x(t + T) = x(t) \) for some real \( T \). Energies from the second set give pseudoperiodic trajectories with the property that there is a real time \( T \) such that \( x(t + T) = x(t) \pm 2\pi \).

However, this work finds a previous claim about the periodic potential to be false. Earlier numerical work suggested that when the trajectory is classified as a function of the energy of the particle, one observes complex energy bands of nonzero thickness, with some bands giving rise to localized motion and others resulting in conducting motion [6]. This paper shows that this observation of continuous bands of nonzero thickness was an artifact of tracking the motion of the particle for a limited amount of time. While accumulating numerical error tends to limit the time that a trajectory can be followed numerically, our new analytic solution enables us to predict the behavior of a particle for times that are arbitrarily large. We find that as we increase the time \( t \) that the particle is observed, the behavior of the particle as a function of energy becomes more complicated. As \( t \to \infty \), the average velocity as a function of the energy of the particle converges to a fractal-like function that is nonzero for a set of measure zero of energies and zero for all other energies.

The paper is arranged as follows. In section 2 we derive an analytic solution to the complex classical equations of motion for the periodic potential \( V(x) = -\cos(x) \). Next, in section 3 we examine the two types of special complex trajectories and describe the sets of complex energies associated with each kind of trajectory. Then, in section 4 we show how a simple function of the energy predicts the hopping of the classical particle from well to well. In section 5 we use our new analytic work to evaluate previous research about the band-like structure of the complex-classical periodic potential. We also examine the average velocity of the particle as a function of the energy as the observation period tends to infinity. We give some concluding remarks in section 6, and in the appendix we explain our methods in greater detail.

2. Analytical solution for the periodic potential

The solution to the equation of motion for complex particle trajectories involves several standard elliptic functions [8–10]. The elliptic integral of the first kind is given by

\[
z = F(\phi, m) = \int_0^\phi \frac{dt}{\sqrt{1 - m \sin^2 t}}.
\]

(Here, \( \phi \) and \( m \) are in general complex numbers.) Then, the Jacobi amplitude \( \phi = \text{am}[z|m] \) is defined as the inverse of the elliptic integral of the first kind:

\[
\text{am}[z|m] = \phi \equiv F^{-1}(z, m).
\]
The Jacobi amplitude is a pseudoperiodic function with respect to its first argument,
\[ \text{am}[z + 2rK(m) + 2iK(1 - m)] = \text{am}[z] \quad (\mod \pi) \]
for integers \( r \) and \( s; \) \( K(m) \equiv F(\pi/2, m) \) is the complete elliptic integral of the first kind.\(^4\)

In terms of these functions, we can solve exactly Hamilton’s equations of motion for a particle in the periodic potential \( V(x) = -\cos(x) \) in which the position and momentum of the particle is complex. Hamilton’s equations have one integral of the motion that expresses the conservation of energy. We scale out unnecessary constants to obtain
\[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 - \cos(x) = E. \]

Using a double-angle trigonometric identity and separating variables, we get
\[ \sqrt{2} t = \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{E + 1 - 2 \sin^2(x/2)}}. \]
We then substitute \( m = 2/(E + 1) \) and \( u = x/2: \)
\[ t\sqrt{2E + 2} + C = \int_{u(0)}^{u(t)} \frac{2 du}{\sqrt{1 - m \sin^2 u}}. \]
From the definition of the Jacobi amplitude, we can then write
\[ x(t) = 2\text{am}[at + C|m], \]
where \( a = \sqrt{(E + 1)/2}, m = 2/(E + 1), \) and the complex integration constant \( C \) is determined by the initial position of the particle. This is the general solution because it can incorporate any initial position and velocity.\(^5\)

We emphasize that the constants \( a \) and \( m \) are just functions of the energy, while \( C \) depends on the initial position. Thus, the analytic solution makes it plausible that the type of motion is dependent primarily on the energy and not on the initial position of the particle.

Using the pseudoperiodicity of \( \text{am}[z|m] \), we can identify special energies that give rise to periodic or pseudoperiodic trajectories. Suppose that the path of a particle is periodic (modulo \( 2\pi \)) with period \( T \). Then for some integers \( r \) and \( s \), we get
\[ aT = 2K(m)r + 2iK(1 - m)s. \]
Since \( T \) is a real number, dividing by \( a \) and taking the imaginary part, we get
\[ \frac{s}{r} = -\frac{\text{Im}[K(m)/a]}{\text{Im}[iK(1 - m)/a]} \equiv R(E), \]
where \( m = 2/(1 + E) \) and \( a = \sqrt{(1 + E)/2} \). Observe that the right side \( R(E) \) of (9) is a function of the energy \( E \). (Note that the function \( R(E) \) lies between \(-1 \) and \( 1 \), as one can verify numerically, so the point \( r = 0 \), which would correspond to \( R(E) = \infty \), is not a problem here because there are no solutions for \( E \).)

The fact that the ratio \( R(E) = s/r \) depends only on \( E \) is fundamental to our discussion of the complex trajectories of classical particles. We will show that the trajectories of particles

\(^4\) Regarding equation (3), if we write \( z + 2rK(m) + 2iK(1 - m) = F(\phi_1, m) \) and \( z = F(\phi_2, m) \), then (3) states that \( \phi_1 - \phi_2 \) is equal to \( kn \) where \( k \) is an integer. The ambiguity in the value of \( k \) is related to the fact that when we do an integral in the complex plane, we must choose a particular path in order to evaluate that integral. When calculating \( F(\phi_1, m) \) and \( F(\phi_2, m) \), we must choose a path in the complex plane to evaluate (1). The integrand in (1) is an analytic function of \( t \) defined on a two-sheeted Riemann surface except for square-root branch points at values of \( t \) where \( m \sin^2 t = 1 \). Paths that encircle the branch points multiple times change the value of \( F(\phi, m) \).

\(^5\) We must take care when using (7) because the Jacobi amplitude is a multivalued function. This function must be evaluated in such a way that \( x(t) \) is continuous.
with energies $E$ such that $R(E)$ is rational give rise to a stable time evolution. We refer to the energies that give rise to this special behavior as classical eigenenergies. We will show that the set of eigenenergies has measure zero in the set of all complex numbers.

It is worth noting that if the energy $E_0$ satisfies (9), then we can find $T$ by reducing the fraction in (9) to lowest terms to obtain the numbers $s$ and $r$. We then substitute these numbers into (8) to solve for $T$. Choosing $E$ so that (9) holds guarantees that $T$ will be a real number.

This analysis reveals a numerical subtlety. If we use (3), we get $x(t + T) - x(t) = k\pi$ where $k$ is an integer. When we solve the equations numerically, we find that solutions come in two types, one for which $x(t + T) = x(t)$ and $k = 0$, and another for which $x(t + T) = 2\pi k + x(t)$, where $k = \pm 1$. The appearance of these two different kinds of solutions is related to taking the functional inverse of the integral in (6).

3. Description of classical trajectories

This paper focuses on the special periodic and pseudoperiodic trajectories of the cosine potential, but it is useful first to examine a typical nonperiodic trajectory. Figure 1 displays such a trajectory beginning at $x(0) = 0.1i$. The particle in this figure hops randomly from site to site without exhibiting any periodic or pseudoperiodic behavior.

To gain a heuristic understanding of the behavior of trajectories with energies satisfying (9), we use a numerical approach. We find that when $s$ is odd, the trajectories are periodic (see figures 2 and 3). When $s$ is even, the trajectories satisfy $x(t + T) = x(t) \pm 2\pi$ (see figures 4 and 5). Thus, our numerical work shows that we must take great care in interpreting the inversion of the integral in (6).

The classical eigenenergies form a special subset of complex energies (of all possible complex energies) and the behavior of the trajectories of particles having these energies is independent of the initial position of the particle at $t = 0$. Had we not determined the motion of the particle analytically from (7), it would have been quite difficult to find energies such that the trajectory obeys $x(t + T) - x(t) \in \{0, \pm 2\pi\}$.

Let us first suppose that $E$ is real. In this case the trajectory is periodic and it is confined to one well if $E < -1$. Even if $E < -1$, where the energy of the particle lies below the bottom of the potential well, the particle loops around a pair of turning points (see [6]). If $E > 1$, as $x$ ranges from 0 to $2\pi$, the function $\cos(x)$ dips from its maximum value of 1 down to its minimum value of $-1$ at $x = \pi$ and then goes back up to 1. Thus, for a $-\cos x$ potential, if a classical particle on the real axis has energy less than $E = 1$ and has an initial position in $(-\pi, \pi)$, the particle will remain confined to the region on the real-$x$ axis inside $(-\pi, \pi)$ where its kinetic energy is positive. We call this region a well. Similarly, the regions $\pi, 3\pi, 5\pi, -\pi$, and so on, are also wells. If the particle has real energy is allowed to move in the complex-$x$ plane, we generalize the idea of a well to mean that the real part of $x$ lies in $(-\pi, \pi)$ while the imaginary part of $x$ is arbitrary.

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Figure 1. A typical trajectory of a particle governed by the cosine potential. Here, $E = 1.99 + i$ and $x(0) = 0.1i$. This particle does not exhibit periodic or pseudoperiodic behavior; it just appears to hop at random from site to site. The dots in this and in the next four figures represent the turning points, which satisfy $V(x) = E$. 

6 As $x$ ranges from 0 to $2\pi$, the function $\cos(x)$ dips from its maximum value of 1 down to its minimum value of $-1$ at $x = \pi$ and then goes back up to 1. Thus, for a $-\cos x$ potential, if a classical particle on the real axis has energy less than $E = 1$ and has an initial position in $(-\pi, \pi)$, the particle will remain confined to the region on the real-$x$ axis inside $(-\pi, \pi)$ where its kinetic energy is positive. We call this region a well. Similarly, the regions $\pi, 3\pi, 5\pi, -\pi$, and so on, are also wells. If the particle having real energy is allowed to move in the complex-$x$ plane, we generalize the idea of a well to mean that the real part of $x$ lies in $(-\pi, \pi)$ while the imaginary part of $x$ is arbitrary.
Figure 2. Periodic trajectory of a particle of complex energy $E = -0.30905 + i$. The particle starts at $x = 0.2i$ and the value of the ratio $R(E)$ is $17/20$. Because the $R(E)$ is reduced to lowest terms and the integer 17 in the numerator is odd, the trajectory $x(t)$ satisfies the phenomenological rule that $x(t + T) = x(t)$. If the values of the numerator and denominator are increased, the trajectories get progressively more complicated.

Figure 3. Periodic trajectories of a particle of energy $E = 1.43696 + i$. Again, the trajectories are periodic because the numerator in the ratio $R(E) = 5/12$ is odd. Three trajectories are shown, one starting at $-2\pi + 0.2i$, a second starting at $0 + 0.2i$, and a third starting at $2\pi + 0.2i$ (respectively purple, blue and magenta in the electronic version). Note that if a trajectory is translated by $2\pi$ to the left or to the right, it still does not intersect another trajectory.

Figure 4. Pseudoperiodic trajectory of a particle with energy $E = -0.779757 + i$ starting at $x = 0$. For this particle the numerator 8 in the ratio $R(E) = 8/9$ is even, so the trajectory $x(t)$ is pseudoperiodic; that is, it satisfies $x(t + T) = 2\pi + x(t)$.

then the particle moves with an average velocity along the positive- or negative-real $x$ axis. If $E = 1$, the particle approaches a turning point as $t \to \infty$.

If we solve (9) for complex $E$ for different rational numbers $s/r$, we can plot the energies in the complex plane (see figure 6). Since $R(E)$ is a continuous function almost everywhere, the relationship between the set of special energies and the set of all complex energies is the
Figure 5. Pseudoperiodic trajectory $x(t)$ of a particle of energy $E = 2.428\,91 + i$ starting at $x = 0$. As in figure 4, the even numerator in the ratio $R(E) = 2/9$ indicates that the trajectory is pseudoperiodic.

Figure 6. A plot of the complex $E$ plane. Going counterclockwise, the radial curves correspond to energies with $R(E) = 1/10, 2/10, \ldots, 9/10, 1$ in the upper-half complex plane. In the lower half complex plane, $R(E)$ has the opposite sign. Numerical work shows that the ratio $R(E)$ falls between $-1$ and $1$. Because the potential is real, if $x(t)$ is a solution with energy $E$, the complex conjugate of $x(t)$ with complex-conjugate energy $E^*$ is also a solution. Thus, there is a symmetrical plot of eigenenergies in the lower-half complex plane.

same as the relationship between the set of rational numbers and the set of all real numbers; that is, the set of all special energies is dense in the complex plane, but it is also of measure zero in the complex plane. Observe that there is a singular point in the complex plane at $E = 1$. It is not surprising that $E = 1$ corresponds to a singular point because this corresponds to a particle being at the top of the potential wells.

4. Determining the periodic motion from the ratio $s/r$

In [7] the ratio $s/r$ was used to predict the shape of a complex trajectory. Thus, it is natural to ask the same question here: using just the ratio $s/r$, what can we say about the behavior of the trajectories? For example, the particle might move right two wells, left one well, right two wells, left one well, and so on. We will show that the pattern of wells the particle visits is an easily calculated function of the energy.

We consider a specific example in order to demonstrate how to determine the pattern of motion from the ratio. In figure 7 we record the motion of a particle with ratio $9/13$ and $E = 0.999\,931 + 0.0001i$ as follows: We write $+$ if the particle continues moving in the same direction at a red dot and write $-$ if the particle changes directions at a red dot. (We begin by assuming that $\text{Im} E$ is small because the pattern of hopping from well to well is relatively easy to understand.) Then the pattern is

$$- + - - + - - + - - - + - - - - + - + - - - - - - - - - - - - -$$

(10)
Figure 7. Plot of Re $x$ as a function of time for $E = 0.999931 + 0.0001i$ for the ratio $9/13$ and the initial position $x(0) = 1 + i$. The dots (red in the electronic version) indicate when the velocity of the particle is zero in the Re $x$ direction. The dashed lines correspond to Re $x = −\pi, \pi, 3\pi$, or the $x$ values that correspond to the tops of the potential $V(x) = −\cos(x)$. The region bounded by the lowest pair of dashed lines corresponds to the particle being in the well centered at $x = 0$ with boundaries at $x = ±\pi$. There is a definite pattern; at each red point the particle decides whether to reverse direction and stay in the well in which it started or to move to an adjacent well.

Note that there is a pair of minuses, a plus, a pair of minuses, a $+$, three minuses, a $+$, and then the pattern repeats. Our goal is to predict this pattern of plus and minus signs using just the ratio $9/13$.

We define the function

$$f_\alpha(n) = \lfloor (n + 1)\alpha \rfloor - \lfloor n\alpha \rfloor,$$

where $\alpha = 9/13$ and the floor function $\lfloor \rfloor$ takes the lower integer part. Then consider the sequence $\{f_\alpha(n)\}_{n=1}^\infty$. The first few terms are as follows:

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1.$$

If we associate $f = 1$ with $-$ and $f = 0$ with $+$, then comparing (10) and (12), we find that these two sequences match exactly. Numerical work shows that this method of determining the order in which a particle with a particular energy visits the wells works for all energies (see appendix).

5. Long-time behavior

Having established that we can use the ratio $s/r$ to predict the pattern of particles hopping from well to well, we can then predict the long-time behavior of the classical particle. Whenever a particle closely approaches a boundary between wells, it can either conduct into the next well or it can turn around and stay in the same well. Therefore, we introduce the following measure. Let $g_n(E)$ be the net number of wells to which the particle has moved after $n$ close approaches to boundaries between wells divided by $n$ for a particular energy $E$. For example, consider a case in which a particle conducts twice, turns around twice, and the pattern repeats. If the particle is initially in the well $[−\pi, \pi]$ and moving in the $+$Re $x$ direction, the state of the particle, from start to finish, would progress as $([−\pi, \pi]$, $+$), $([\pi, 3\pi]$, $+$), $([3\pi, 5\pi]$, $+$), $([3\pi, 5\pi]$, $−$), $([3\pi, 5\pi]$, $+$), and so on. This
Figure 8. This plot shows the behavior of a particle under the influence of the periodic potential as a function of a complex energy. The Xs refer to the particle conducting (being delocalized) and the -s refer to the particle staying localized. (The terms conduction and (de)localization are adopted from the quantum-mechanical picture of a particle in a crystal lattice. By localized motion we mean that the particle hops (or tunnels) from lattice site to lattice site in a random fashion. Thus, on the average the expected position of the particle remains fixed at the initial position of the particle. On the other hand, if the particle is in a conduction band, resonant tunneling occurs and the expected position of the particle drifts with constant velocity through the lattice.) On the basis of this figure one might conclude that there are sharply defined boundaries between the energies that give rise to conducting behavior and the energies that give rise to localized behavior. However, as explained in this paper, this conclusion is not correct; the apparent sharpness of the band is an artifact of observing the particle for a fixed amount of time. Reproduced with permission from [6].

will result in a net displacement of two wells in a total of four close approaches to the boundaries between wells. So \( g_n(E) \) would be \( 2/4 = 1/2 \) in this example.

If we examine \( g_n(E) \) for progressively longer times (or equivalently, for more and more times where the particle remains in the same well or conducts into an adjacent well), the graph of \( g_n(E) \) becomes more complicated. This procedure produces a fractal-like graph where \( g_\infty(E) = 0 \) except when the energy \( E \) is such that \( R(E) \) is a rational number with an even numerator.

To produce the following graphs we use \( E \) to calculate the ratio \( R(E) \) from (9). The details of extending from rational to irrational ratios are discussed in the appendix. Then from \( f \) we determine whether or not the particle is going to change direction. Finally, we use this information to calculate \( g \), our measure of the motion of the particle.

5.1. The measure \( g_n(E) \) compared with previous research

Using \( R(E) \) to study the motion of the particles involves detailed numerical observations, so it is helpful to show that the method described in section 4 is consistent with previous research on the potential \( V(x) = -\cos x \). Consider figure 8 (reproduced with permission from [6]). This plot indicates that there are bands of energies that give rise to delocalized behavior.

If our method is to agree with the research reported in [6], then the energies that have Xs in figure 8 will have a nonzero value of \( g_n(E) \). For energies that have minuses in figure 8, \( g_n(E) \) should be near zero. Indeed, figure 9 verifies the correspondence between the two methods. We get an almost identical band of energies such that \( g_{100}(E) = 0.2 \) where there were plus signs in figure 8 [11]. In the regions of minuses of figure 8, \( g_{100}(E) \) is near zero.

However, as discussed earlier, the dynamics are not as simple as figure 8 suggests. When we examine the long-time behavior (that is, when we measure the average displacement for an even longer time), we find that the band fractures into multiple bands as shown in figure 10. Yet,
Figure 9. Average displacement of the complex classical particle after 100 close approaches to the boundaries between wells. The horizontal and vertical axes give the real and imaginary parts of the energy. The intensity of each region indicates the value of $g_{100}(E)$. The energy range is the same as in figure 8. The plots are in good agreement; the boundaries between the so-called conducting and localized regions match very closely.

Figure 10. Same as figure 9 except that $n = 300$ instead of 100. Observe that the center of the band still corresponds to an average displacement of one well per five close approaches to the turning points between wells, but that the apparent band in figure 9 has fractured. If we look still further into the future of the particle, we see that the energy band edge is not sharp.

even though the band divides into multiple bands, the center of both bands still corresponds to an average displacement of one well for every five close approaches to well boundaries.

As we let $n$ approach infinity, the bands continue to divide and we get a fractal-like structure. In figure 11 we remove the unnecessary dimension of the imaginary part of the energy and just focus on a plot of $g$ as a function of $\text{Re} E$ where $\text{Im} E$ is fixed to be 1. This figure demonstrates how the fractal-like nature of $g_n(E)$ develops when $n$ becomes large.

5.2. Energy dependence of long-time behavior

We have demonstrated numerically that if $R(E) = \alpha = p/q$ [in the sense of (9)] where $p$ and $q$ are relatively prime, then $g_\infty(E) = 1/q$ if $p$ is even and $g_\infty(E) = 0$ if $p$ is odd. If $\alpha$
is irrational, then $g_\infty = 0$. We examine $g_n$ as $n$ goes to infinity for $E$ such that $R(E) = \alpha$ is irrational. To study $g_n(E)$ we approximate $\alpha$ by a suitably close rational number. Our numerical experience shows that if we examine only up to a fixed number $n$ of close approaches to boundaries between wells, we can always find such a rational number $p/q$ and corresponding energy $E'$ that reproduces the behavior demonstrated by $\alpha$ and $E$. As we choose better rational-number approximations for $\alpha$, we note that the denominators of such approximations approach infinity. Since $g_n(E')$ is either 0 or $1/q$ with $q$ becoming arbitrarily large as we choose better rational approximations for $\alpha$, $g_n(E) \rightarrow 0$ as $n \rightarrow \infty$. Thus, the particle motion is localized in the sense that the average velocity is zero for all but a set of measure zero of energies.

6. Concluding remarks

In this paper we have tried to characterize the motion of a complex classical particle in a periodic potential. We have identified interesting trajectories and have described the pattern of jumping from one well to another.

The results presented here are largely consistent with earlier work in which it was observed that in the complex domain classical and quantum mechanics exhibit many similar features. However, we have discovered new and interesting behaviors. Not surprisingly, when we look at the long-time behavior of a dynamical system, we observe fractal-like behavior. We have shown that for irrational numbers, the so-called average velocity is zero. However, this does not eliminate the possibility that the displacement grows as a function such as $\log t$ that is much smaller than $t$. It may be that the motion of the particle is chaotic for energies having an irrational ratio. That is, the particle jumps from well to well in a pattern that does not have a well-defined asymptotic behavior. However, because the particle does not coordinate its hops in a particular direction, the average displacement is much less than the number of jumps of the particle.
Figure A1. A trajectory of a particle with $E = 1.49555 + i$ starting at $x = 0$ and $r(E) = 2/5$ plotted for precisely one period. The circles correspond to points on the trajectory where $\text{Re} x = k\pi$ where $k$ is an odd integer. The squares correspond to parts of the trajectory where $\text{Re} x(t)$ changes sign.

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Appendix. Details concerning the pattern

A.1. Using the analytic solution to explain the pattern

To explain how to use $g_n$ to predict the motion of a particle, we consider an explicit example of the motion of a particle and the corresponding elliptic function. Figure A1 shows the motion of a particle with a particular initial position and energy. As time increases, the particle moves to the right and eventually crosses the line at $\text{Re} x = \pi$ at the circular dot. In figure A2 the particle starts at the bottom left of each plot. As time increases, the particle reaches the first dot at $t = t_1$. While the real part of $f(u, v)$ is continuous and reaches a maximum of $\pi$, the imaginary part of $f(u, v)$ is discontinuous and changes sign at the first circle. The discontinuity occurs because we are plotting the principle value for the Jacobi amplitude function. To make the function continuous $x(t_1 - \epsilon) = f(u, v)$, but $x(t_1 + \epsilon) = 2\pi - f(u, v)$.

Again, as time increases, the particle moves along the blue line in figure A1 until it hits the square at $t = t_2$. This part of the trajectory corresponds to the straight line from the bottom left circle to the closest square in A2. Here, we see that at $t = t_2$, the value of $\text{Re} x$ increases and then decreases.

Now we can understand the pattern. The right plot in figure A2 shows that there are curves of discontinuity where the color of the plot jumps between red and green [$\text{Im} f(u, v)$ changes sign]. These curves correspond to when $f(u, v) = \pm \pi$ in the left plot. These curved lines have an average slope of 1, whereas the black line has a slope of $2/5$. When the particle spends time without crossing a curve of discontinuity, the particle turns around. In each period the particle goes through three circles and two squares. This is in exact correspondence to the sequence $\{f_n(2/5)\} = \{0, 1, 0, 1, 0, 0, 1, 0, 1, 0, \ldots\}$. Evidently, if the line in figure A2 had a different slope, the pattern of circles and squares would be different. The fact that we can predict the pattern of circles and squares using the $f_n$ sequence can be seen to work for all rational numbers, even though we do not offer a detailed proof.
Figure A2. The analytic solution for the trajectory in figure A1 is \( x(t) = 2am[|m|] \). In order to see the usefulness of the analytic solution, we consider \( 2am[|m|] \), where \( z \) is an arbitrary complex number and the black line corresponds to \( z = at \). We emphasize that we are plotting the principal value of the Jacobi amplitude (that is, \( 2am[|m|] \in [-\pi, \pi] \)). We write \( z = up + vq \), where \( p = 2K(m) \) and \( q = 2iK(1 - m) \) are the two complex periods of this elliptic function and \( u \) and \( v \) are real numbers. Let \( f(u, v) = 2am[|m|] \). On the left we plot \( \text{Re} f(u, v) \) and on the right we plot \( \text{Im} f(u, v) \). The color of a particular pixel corresponds to the value of one of those functions for a particular ordered pair of real numbers \((u, v)\). The circles and squares correspond to the same points as in figure A1. Because \( \text{Im} f(u, v) \) diverges at certain points, the scale on the right plot is cut off at an arbitrary value of 5.

Figure A3. Plots of \( \text{Re} x \) as a function of time for \( x(0) = 0.1i \), \( R(E) = 4/5 \), and varying \( \text{Im} E \). The dotted lines correspond to the maxima of \( V(x) \) on the real axis. From left to right, \( E = 0.986436 + 0.01i, 0.874181 + 0.1i, 0.0627904 + i \) and the time is taken out to \( t = 215, 152, 94 \). Notice that as \( \text{Im} E \) increases, the trajectories deform and the time axis deforms. These plots demonstrate qualitatively that as \( \text{Im} E \) decreases, the time required to ‘tunnel’ from one well to another increases. Nevertheless, we can see that even though the trajectories deform, the general behavior of moving from one well to another remains the same even when \( \text{Im} E \) is not small. Furthermore, these plots demonstrate that the function \( g_{\alpha}(E) \) is a relatively good approximation to the average velocity as a function of energy.

A.2. Numerical justification of the use of the pattern

Extending from small \( \text{Im} E \) to any \( \text{Im} E \). To identify the hopping pattern we have assumed that the imaginary part of the energy is small so that the boundaries of the wells are relatively well defined (as in figure 7). However, extensive numerical work indicates that even if we let the imaginary part of the energy be large, we still obtain the same behavior.

As the imaginary part of the energy increases, two things happen. First, the locations of the wells become less defined. However, the pattern of moving from well to well remains the same, where we define the wells by the intervals \([−\pi, \pi], [\pi, 3\pi], \) etc. This can be verified, for example, by plotting \( \text{Re} x \) as a function of \( t \) for energies that have a ratio of 9/13 with \( \text{Im} E = 0.01, 0.1, 1 \) (see figure A3).
Second, the time between close approaches to the boundaries of the wells changes. However, this change happens slowly. Using the trajectories in figure A3, one can also see that the time needed to move from one well to another varies, but even for energies having small imaginary parts, this time does not approach $\infty$ like $\frac{1}{\text{Im} E}$. For instance, for a ratio of $9/13$, and $\text{Im} E = 0.1$, this time is about 5 and for $\text{Im} E = 10^{-5}$, this time is about 10. This is contrary to previous research that had suggested that the tunneling time is inversely proportional to the imaginary part of the energy.

Extending from $E$ with a rational ratio to $E$ with an irrational ratio. We have assumed that the energy is such that $R(E)$ is rational. However, whether or not $R(E)$ is rational is unimportant if we are tracking the particle for a fixed amount of time because the difference between the motion for energies that are close does not show up until a large time $t$. Thus, if we examine the motion of a particle where $R(E)$ is irrational up to some fixed time $t$, we can choose an energy $E'$ such that $R(E')$ is rational and such that the two trajectories are practically identical in the time interval $[0, t]$ for any $t$.

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[11] The density plots in figures 9 and 10 were generated by using Colorbarplot v0.6 (available at http://www.walkingrandomly.com/?p=2960)