Schubert calculus via Grassmann variables

Ken Kuwata

Division of Mathematics, Graduate School of Science
Hokkaido University
Kita-ku, Sapporo, 060-0810, Japan

e-mail address: kuwata@math.sci.hokudai.ac.jp

November 1, 2021

Abstract

In [5], Imanishi, Jinzenji and Kuwata provided a recipe for computing Euler number of Grassmann manifold \( G(k, N) \). In the process, they showed that the cohomology ring of \( G(k, N) \) is represented by Grassmann variables. The purpose of this paper is to compute some integral of Chern classes of the dual bundle of the tautological bundle on \( G(k, N) \) by using only Grassmann variables. In other words, we compute the intersection number of Schubert cycles by using only Grassmann variables.

1 Introduction

1.1 Background

Our goal is computing some intersection number of Schubert cycles. For this purpose, we use Grassmann variables and its integral in [5]. In this section, we explain the background of this paper. The complex Grassmann manifold \( G(k, N) \) is a space which parameterizes \( k \)-dimensional linear subspaces of \( N \)-dimensional complex vector space. Since its cohomology ring is represented by some Poincaré dual of Schubert cycle or subvariety of \( G(k, N) \), the integral of them represents the intersection number of the Schubert cycles. The research is called the Schurbert calculus, and has been studied combinatorics, representation theory, and other fields [9]. It is known that the integral of these cohomology classes can be computed using the localization theory and the Landau-Ginzburg formulation. In the localization theory, the fixed-point theorem for compact manifold with torus action are used. It computes some cohomological invariants by using the fixed point set of the action. In fact, the formula for the intersection number is given by using localization theory [3, 8]. On the other hand, the Landau-Ginzburg formulation [1, 4] uses a potential function, which is given by the total Chern class of the tautological bundle of \( G(k, N) \), and residue. However, we do not use these theories. We use the theory in [5]. Imanishi, Jinzenji and Kuwata construct the physics toy model for computing Euler number of \( G(k, N) \). Then, they found the cohomology ring is represented by Grassmann variables and Euler number is given by the integral of them. So, intersection number of Schubert cycles are obtained by only using the Grassmann integral. In general, it is not expected to be easy to solve. In some cases, however, it can be obtained by direct calculation. The purpose of this paper is to show this and to demonstrate the effectiveness of method of [5].
1.2 Chern classes and Schubert cycles

In this section, we explain the relation between Chern classes and Schubert cycles and our goal in this paper. Let $S$ be a tautological bundle of $G(k, N)$ whose fiber of $\Lambda \in G(k, N)$ is given by complex $k$-dimensional subspace $\Lambda \subset \mathbb{C}^N$ itself ($\text{rk}(S) = k$). Then universal quotient bundle $Q$ ($\text{rk}Q = N - k$) is defined by the following exact sequence

$$0 \to S \to \mathbb{C}^N \to Q \to 0,$$

where $\mathbb{C}^N$ is trivial bundle $G(k, N) \times \mathbb{C}^N$. We write $c_i(E)$ for the $i$-th Chern class of a vector bundle $E$ and $E^*$ for the dual bundle of $E$. Then, $H^*(G(k, N))$ the cohomology ring of $G(k, N)$ is

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \ldots, c_k(S^*), c_1(Q^*), \ldots, c_{N-k}(Q^*)]}{(c(S^*)c(Q^*) = 1)}.$$ (1.2)

If we decompose $S^*$ formally by the line bundle $L_i$ ($i = 1, 2, \cdots, k$):

$$S^* = \bigoplus_{i=1}^k L_i,$$ (1.3)

$c(S^*)$ and $c_i(S^*)$ are written as

$$c(S^*) = \prod_{i=1}^k (1 + t_i), \quad c_i(S^*) = 1 + \sum_{j=1}^k t_j c_j(S^*), \quad (x_i := c_1(L_i)).$$ (1.4)

$c_i(S^*)$ is written by the $j$-th fundamental symmetric polynomial of $x_1, \cdots, x_k$. Then, the relation $c(S^*)c(Q^*) = 1$ is rewritten by

$$c(Q^*) = \frac{1}{c(S^*)} = \frac{1}{1 + \sum_{j=1}^k t_j c_j(S^*)} = \sum_{i=0}^{\infty} a_i t_i.$$ (1.5)

We can rewrite (1.5) as follows:

$$c_i(Q^*) = a_i \quad (i = 1, 2, \cdots, N - k), \quad a_i = 0 \quad (i > N - k).$$ (1.6)

Note that $a_i$ is degree $i$ homogeneous polynomial of $c_j(S^*)$'s ($j = 1, 2, \cdots, k$). Hence we can eliminate generators $c_j(Q^*)$'s from (1.2) and obtain another representation of $H^*(G(k, N))$.

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \ldots, c_k(S^*)]}{(a_i = 0 \quad (i > N - k))}. $$ (1.7)

Here, we introduce the Schubert cycle and explain the relation between $c_i(S^*)$ and Schubert cycles. For a more detailed discussion, please refer to [2]. For any flag $V : 0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathbb{C}^N$, the Schubert manifold $\sigma_a(V)$ is defined by

$$\sigma_a(V) := \{ \Lambda \in G(k, N) | \dim(\Lambda \cap V_{N-k+i-a_i}) \geq i | 1 \leq i \leq k \},$$ (1.8)

where $a = (a_1, \cdots, a_k)$ is a sequence of natural number that satisfies $0 \leq a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq N - k$. $\sigma_a(V)$ is an analytic subvariety of $G(k, N)$ of codimension $\sum_{i=1}^k a_i$. Then, the homology class of $\sigma_a(V)$ is independent of the flag chosen. Therefore, let $\sigma_a(V)$ as the homology class be denoted by $\sigma_a$. Let $\sigma_a^*$ be the Poincaré dual of the cycle $\sigma_a$. And we abbreviate 0 in $a$. For example, $\sigma_{(a_1, a_2, \cdots, a_n, 0, \cdots, 0)}$ is denoted by $\sigma_{a_1, a_2, \cdots, a_n}$. Since the relation between $i$-th Chern class of a vector bundle $E$ and the one of its dual bundle $E^*$ is given by $c_i(E^*) = (-1)^i c_i(E)$ and the Gauss-Bonnet theorem,

$$c_i(S^*) = (-1)^i c_i(S) = \sigma_{a_i}^* =: \sigma_{i(l)}^*.$$ (1.9)

Our aim of this paper is to prove the following theorem.
Theorem 1.

\[
\int_{G(k,N)} (\sigma^*_{1(1)})^{kN-k^2} = (kN - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. 
\]

(1.10)

\[
\int_{G(k,N)} (\sigma^*_{1(1)})^{kN-k^2-2} (\sigma^*_{1(2)}) = \frac{(kN - k^2 - 2)!((N - k)(N - k + 1)k(k - 1))}{2} \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. 
\]

(1.11)

\[
\int_{G(k,N)} (\sigma^*_{1(1)})^{kN-k^2-4} (\sigma^*_{1(2)}) = \frac{(kN - k^2 - 4)!((N - k)(N - k + 1)k(k - 1))}{4} \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \times \left[ kN - k - 1 + 2(k - 2)(k - 3)(N - k) + 4(k - 2)(N - k - 1) \right]. 
\]

(1.12)

Here, we assume that \( N \) and \( k \) in (1.11) and (1.12) satisfy \( kN - k^2 - 2 \geq 0 \) and \( kN - k^2 - 4 \geq 0 \), respectively.

We note that these are intersection number of \( \sigma_{1(1)} \) and \( \sigma_{1(2)} \). Results of (1.10) are already known \[3\] \[4\], but we prove these results by using Grassmann variables.

### 1.3 Grassmann variables and Cohomology ring of \( G(k, N) \) (Review of \[5\])

We summarize the representation of the cohomology ring of \( G(k, N) \) by Grassmann variables \[5\]. We introduce Grassmann variables \( \psi_s^i, \psi_s^j \) \((s = 1, \ldots, N - k, \ j = 1, \ldots, k)\) and \((k \times k)\) matrix

\[
\Phi := \begin{pmatrix}
\psi_1^1 \psi_1^1 & \cdots & \psi_1^1 \psi_1^k \\
\vdots & \ddots & \vdots \\
\psi_k^1 \psi_1^1 & \cdots & \psi_k^1 \psi_1^k
\end{pmatrix}
\]

(1.13)

These satisfy the following conditions.

\[
\psi_s^i \psi_s^j = \psi_s^j \psi_s^i = 0, \quad \psi_s^i \psi_s^j = -\psi_s^j \psi_s^i, \quad \psi_s^i \psi_s^j = -\psi_s^j \psi_s^i, \quad \psi_s^i \psi_s^j = -\psi_s^j \psi_s^i
\]

(1.14)

\((s, l = 1, 2, \ldots, N - k, \ i, j = 1, 2, \ldots, k)\). Let us define the Grassmann integral as follows.

\[
\int D\psi \prod_{s=1}^{N-k} \psi_s^1 \psi_s^1 \cdots \psi_s^1 \psi_s^k = 1,
\]

(1.15)

where \( D\psi := \prod_{s=1}^{N-k} d\psi_s^1 d\psi_s^1 \cdots d\psi_s^k d\psi_s^k \). Let us define \( \tau_j \) \((j = 1, 2, \ldots, k)\) by

\[
1 + \tau_1 t + \cdots + \tau_k t^k := \det(J_k + t\Phi) = \prod_{j=1}^{k} (1 + \lambda_j t).
\]

(1.16)

\( \lambda_j \((j = 1, \ldots, k)\) are eigenvalues of \( \Phi \). In other words, \( \tau_j \) is the \( j \)-th fundamental symmetric polynomial of \( \lambda_1, \ldots, \lambda_k \). Note that \( \tau_k \) is identified with \( \det(\Phi) \) and \( \tau_1 \) is identified with \( \text{tr}(\Phi) \). In \[5\], following theorems are proved.

**Theorem 2.** \[5\]

\[
\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\det(\Phi))^{N-k} = 1.
\]

(1.17)
Theorem 3. \[5\]

\[H^*(G(k, N)) = \mathbb{R}[c_1(S^*), \ldots, c_k(S^*)] \subset \mathbb{R}[\tau_1, \ldots, \tau_k]. \quad (1.18)\]

Theorem 3 was denoted by a ring homomorphism \(f : \mathbb{R}[c_1(S^*), \ldots, c_k(S^*)] \to \mathbb{R}[\tau_1, \ldots, \tau_k]\), which is defined by

\[f(c_j(S^*)) = \tau_j \quad (j = 1, 2, \ldots, k). \quad (1.19)\]

From the isomorphism \(H^*(G(k, N)) \cong \mathbb{R}[\tau_1, \ldots, \tau_k]\), \(x_j\) is identified with \(\lambda_j\). The normalization condition of integration on \(G(k, N)\) is given by

\[\int_{G(k, N)} (\sigma^*_{1(\nu)})^{N-k} = 1. \quad (1.20)\]

Therefore, Theorem 2 also leads us to the following equality:

\[\int_{G(k, N)} g(x_1, \ldots, x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=0}^{N-k-j} j!} \int D\psi g(\lambda_1, \ldots, \lambda_k), \quad (1.21)\]

where \(g(x_1, \ldots, x_k)\) is a symmetric polynomial of \(x_1, \ldots, x_k\) that represents an element of \(H^*(G(k, N))\).

2 Proof of theorems

2.1 Proof of Theorem 1

Proof. From (1.10), (1.16) and (1.21),

\[
\int_{G(k, N, 2)} (\sigma^*_{1(\nu)})^{kN-k^2} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-j} j!} \int D\psi (\operatorname{tr} (\Phi))^{kN-k^2} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-j} j!} \int D\psi \left( \sum_{s=1}^{N-k} \sum_{j=1}^{k} \psi_s \bar{\psi}_s \right)^{kN-k^2}.
\]

(2.22)

From the multinomial theorem and conditions of Grassmann variables \(\psi_s \bar{\psi}_s = \bar{\psi}_s \psi_s = 0\), we get the following result.

\[
\int_{G(k, N, 2)} (\sigma^*_{1(\nu)})^{kN-k^2} = (Nk - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-j} j!} \int D\psi \prod_{s=1}^{N-k} \prod_{j=1}^{k} \psi_s \bar{\psi}_s = (Nk - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-j} j!}.
\]

(2.23)

Second, we show (1.11) and (1.12). In the same way as for (1.10),

\[
\int_{G(k, N, 2)} (\sigma^*_{1(\nu)})^{kN-k^2-2l} (\sigma^*_{1(\nu)})^l = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-j} j!} \int D\psi (\tau_1)^{kN-k^2-2l} (\tau_2)^l \quad (l = 1, 2).
\]

(2.24)

(2.25)

Since \(\tau_2 = \frac{1}{2} \{ (\operatorname{tr}(\Phi))^2 - \operatorname{tr}(\Phi^2) \} \),

\[
\int D\psi (\tau_1)^{kN-k^2-2l} (\tau_2)^l = \frac{1}{2^l} \int D\psi \{ \operatorname{tr}(\Phi) \}^{kN-k^2-2l} \{ (\operatorname{tr}(\Phi))^2 - \operatorname{tr}(\Phi^2) \}^l
\]

(2.26)

\[
= \frac{1}{2^l} \sum_{m=0}^{l} \binom{l}{m} (-1)^m \int D\psi (\operatorname{tr}(\Phi))^{kN-k^2-2m} (\operatorname{tr}(\Phi^2))^m.
\]

(2.27)
Let us define
\[ P_m := \int D\psi (\text{tr } (\Phi))^k N - k^2 - 2m (\text{tr } (\Phi^2))^m \quad (m = 0, 1, 2). \] (2.28)

\[ P_0 = (kN - k^2)! \] from the calculation of (1.10). We obtain the following result for \( P_1 \) and \( P_2 \).

**Proposition 1.**
\[ P_1 = (kN - k^2 - 2)! k(N - k)(N - 2k). \] (2.29)
\[ P_2 = (kN - k^2 - 4)! k(N - k) \left[ k(N - k)^3 - 2(N - k)^2 (k^2 + 2) + (N - k)(k^3 + 10k) - 4k^2 - 2 \right]. \] (2.30)

We prove these results later. Since
\[ \int D\psi (\tau_1)^{kN - k^2 - 2} (\tau_2) = \frac{1}{2} (P_0 - P_1) = \frac{1}{2} (kN - k^2 - 2)! k(N - k) \{ (kN - k^2 - 1) - (N - 2k) \} \]
\[ = \frac{1}{2} (kN - k^2 - 2)! (N - k)(N - k + 1)k(k - 1), \] (2.31)

We obtain (1.10). \( \int D\psi (\tau_1)^{kN - k^2 - 4} (\tau_2)^2 = \frac{1}{4} (P_0 - 2P_1 + P_2) \). We put together an equation with \((kN - k^2 - 4)! k(N - k) / 4\) as a common factor. If we organize it for \((N - k)\), we obtain (1.12).

**Proof. (Proposition 1)**
Let \( \omega^i \) be \( \sum_{n=1}^{N-k} \psi^i \psi^j \). By definition,
\[ \text{tr } (\Phi^2) = \sum_{i,j=1}^{k} \omega^i \omega^j = \sum_{i=1}^{k} (\omega^i)^2 + \sum_{i \neq j} \omega^i \omega^j, \] (2.32)
\[ P_1 = \int D\psi (\text{tr } (\Phi))^{kN - k^2 - 2} (\text{tr } (\Phi^2))^1, \] (2.33)
\[ = \sum_{i=1}^{k} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN - k^2 - 2} (\omega^i)^2 + \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN - k^2 - 2} \omega^i \omega^j \] (2.34)
\[ = \sum_{i=1}^{k} \sum_{p_n} \frac{(kN - k^2 - 2)!}{\Pi_{n=1}^{k} p_n!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) (\omega^i)^2 \]
\[ + \sum_{i \neq j} \sum_{p_n} \frac{(kN - k^2 - 2)!}{\Pi_{n=1}^{k} p_n!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) \omega^i \omega^j. \] (2.35)

\( \sum_{p_n} \) means summing so that \( kN - k^2 - 2 \geq p_1, \ldots, p_k \geq 0 \) satisfy condition \( \sum_{n=1}^{k} p_n = kN - k^2 - 2 \).

For the Grassmann integral to be non-zero, in the first term \( p_n = N - k (n \neq i) \) and \( p_i = N - k - 2 \), since each \( \omega^i \) \((i = 1, \ldots, k)\) needs to be \( N - k \). And \( p_n = N - k (n \neq i, j) \) and \( p_i = p_j = N - k - 1 \) in the second term.
\[ P_1 = \sum_{i=1}^{k} \frac{(kN - k^2 - 2)!}{((N-k)!)^{k-i}((N-k-2)!)^i} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{N-k} \right) \]
\[ + \sum_{i \neq j} \frac{(kN - k^2 - 2)!}{((N-k)!)^{k-2}((N-k-1)!)^2} \int D\psi \left( \prod_{n \neq i,j} (\omega^{nn})^{N-k} \right) (\omega^i \omega^j)^{N-k-1} \omega^i \omega^j. \] (2.36)
From $\omega^{ii} = \sum_{s=1}^{N-k} \psi_s^i \bar{\psi}_s^i$, multinominal theorem and conditions of Grassmann variables $\psi_s^i \bar{\psi}_s^i = \bar{\psi}_s^i \psi_s^i = 0$,
\[
P_1 = \sum_{i=1}^{N-k} \frac{(kN - k^2 - 2)!}{(N - k - 2)!} (N - k) \int D\psi \left( \prod_{n \neq i,j}^{N-k} \psi_s^n \psi_s^n \right) \bar{\omega}^{ii} \omega^{jj} (\bar{\omega}^{ii} \omega^{jj})^{N-k-1} \left( \sum_{s,t=1}^{N-k} \psi_s^i \bar{\psi}_s^j \psi_t^j \bar{\psi}_t^j \right).
\]
In the second term, $\omega^{ii} \omega^{jj} (\bar{\omega}^{ii} \omega^{jj})^{N-k-1}$ contains $N - k - 1 \psi_s^i \bar{\psi}_s^i$'s and $\psi_t^j \bar{\psi}_t^j$'s, it must be $s = t$ from conditions of Grassmann variables.
\[
P_1 = (kN - k^2 - 2)! k(N - k)(N - k - 1)
- \sum_{i \neq j}^{N-k} \frac{(kN - k^2 - 2)!}{(N - k - 1)!} \int D\psi \left( \prod_{n \neq i,j}^{N-k} \psi_s^n \psi_s^n \right) \bar{\omega}^{ii} \omega^{jj} \left( \omega^{ii} \omega^{jj} \right)^{N-k-1} \left( \sum_{q=1}^{N-k} \psi_s^i \bar{\psi}_s^j \psi_t^j \bar{\psi}_t^j \right) \left( \sum_{q=1}^{N-k} \psi_s^i \bar{\psi}_s^j \psi_t^j \bar{\psi}_t^j \right)
= (kN - k^2 - 2)! k(N - k)(N - k - 1) - \sum_{i \neq j}^{N-k} \frac{(kN - k^2 - 2)!}{(N - k - 1)!} (N - k) k(k-1)
= (kN - k^2 - 2)! \{ (N-k)(N-k-1) - (N-k)k(k-1) \} = (kN - k^2 - 2)! k(N - k)(N - 2k).
\]
We compute $P_2$.
\[
P_2 = \int D\psi \left( \sum_{n=1}^{N-k} \omega^{nn} \right)^{kN-k^2-4} \left( \sum_{i \neq j}^{k} (\omega^{ii})^2 + \sum_{i \neq j} \omega^{ij} \omega^{ji} \right)^2
= \int D\psi \left( \sum_{n=1}^{N-k} \omega^{nn} \right)^{kN-k^2-4} \left[ \sum_{i,j} (\omega^{ii} \omega^{jj})^2 + 2 \sum_{m=1}^{k} \omega^{rr} \omega^{rr} \omega^{mm} \omega^{mm} \right] .
\]
Let us define
\[
Q_1 := \sum_{i,j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii} \omega^{jj})^2, Q_2 := \sum_{m=1}^{k} \sum_{i,j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{mm} \omega^{mm})^2 \omega^{ii} \omega^{jj},
\]
\[
Q_3 := \sum_{a \neq b} \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} \omega^{ab} \omega^{ba} \omega^{ij} \omega^{ij} .
\]
First, we consider $Q_1$,
\[
Q_1 = \sum_{i=1}^{k} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^4 + \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii} \omega^{jj})^2.
\]
It can be computed in the same way as $P_1$, with the result as follows.

$$Q_1 = (kN - k^2 - 4)!k(N - k)\{N - k - 1)(N - k - 2)(N - k - 3) + (k - 1)(N - k)(N - k - 1)^2\}. \tag{2.47}$$

Next, we calculate $Q_2$.

$$Q_2 = 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn}^{\alpha} \right)^{kN-k^2-4} (\omega^{ij})^2 \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn}^{\alpha} \right)^{kN-k^2-4} (\omega^{ij})^2 \omega^{ij} \omega^{ji}$$

$$+ 2 \sum_{i \neq j} \sum_{m \neq i, j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn}^{\alpha} \right)^{kN-k^2-4} (\omega^{mn})^2 \omega^{ij} \omega^{ji}. \tag{2.48}$$

From $\omega^{ij} \omega^{ji} = \omega^{ji} \omega^{ij}$, if we replace $i$ with $j$ and $j$ with $i$ in the second term, it is the same as the first term.

$$Q_2 = 4 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn}^{\alpha} \right)^{kN-k^2-4} (\omega^{ii})^2 \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \sum_{m \neq i, j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn}^{\alpha} \right)^{kN-k^2-4} (\omega^{mn})^2 \omega^{ij} \omega^{ji}$$

$$= 4 \sum_{i \neq j} \sum_{p_n} \frac{(kN-k^2-4)!}{\prod_{q=1}^{k} P_q!} \int D\psi \left( \prod_{n=1}^{k} (\omega_{nn}^{\alpha})^{p_n} \right) (\omega^{ii})^2 \omega^{ij} \omega^{ji}$$

$$+ 2 \sum_{i \neq j} \sum_{m \neq i, j} \sum_{p_n} \frac{(kN-k^2-4)!}{\prod_{q=1}^{k} P_q!} \int D\psi \left( \prod_{n=1}^{k} (\omega_{nn}^{\alpha})^{p_n} \right) (\omega^{mn})^2 \omega^{ij} \omega^{ji}. \tag{2.49}$$

$\sum_{p_n}$ means summing so that $kN - k^2 - 4 \geq p_1, \ldots, p_k, 0 \geq \sum_{n=1}^{k} p_n = kN - k^2 - 4$. From the condition of integration and the condition of Grassmann variables $\psi^i_s \psi^i_s = 0$, in the first term, $p_n = N - k(n \neq i, j)$ and $p_i = N - k - 3, p_j = N - k - 1$. In the second term, $p_n = N - k(n \neq i, j, m)$ and $p_i = p_j = N - k - 1, p_m = N - k - 2$.

$$Q_2 = 4 \sum_{i \neq j} \frac{(kN-k^2-4)!}{(N-k-3)!(N-k-1)!} \int D\psi \left( \prod_{n \neq i, j, l=1}^{N-k} \prod_{i} \psi_{ii}^{n} \bar{\psi}_{ii}^{n} \right) (\omega^{ii})^2 \omega^{ij} \omega^{ji}$$

$$+ 2 \sum_{i \neq j} \sum_{m \neq i, j} \frac{(kN-k^2-4)!}{(N-k-2)!(N-k-1)!} \int D\psi \left( \prod_{n \neq i, j, l=1}^{N-k} \prod_{i} \psi_{ii}^{n} \bar{\psi}_{ii}^{n} \right) (\omega^{ij})^2 \omega^{ij} \omega^{ji}. \tag{2.50}$$

Here, the integral value is calculated in the same way with $P_1$. In $i \neq j$,

$$\int D\psi \left( \prod_{n \neq i, j, l=1}^{N-k} \prod_{i} \psi_{ii}^{n} \bar{\psi}_{ii}^{n} \right) (\omega^{ii})^2 \omega^{ij} \omega^{ji} = -(N-k)((N-k-1))!^2. \tag{2.51}$$

$$Q_2 = -4 \sum_{i \neq j} \frac{(kN-k^2-4)!}{(N-k-3)!} (N-k) - 2 \sum_{i \neq j} \sum_{m \neq i, j} \frac{(kN-k^2-4)!}{(N-k-2)!} (N-k)$$

$$= -4 \frac{(kN-k^2-4)!}{(N-k-3)!} k(k-1) - 2 \frac{(kN-k^2-4)!}{(N-k-2)!} k(k-1)(k-2)$$

$$= (kN-k^2-4)!k(N-k)(k-1)[-4(N-k-1)(N-k-2) - 2(N-k)(N-k-1)(k-2)]. \tag{2.52}$$
Finally, we compute $Q_3$.

$$Q_3 = \sum_{a \neq b} \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn} \right)^{kN-k^2-4} \omega^{ab} \omega^{i_j} \omega^{i_j}.$$  \hspace{1cm} (2.55)

The sum $\sum_{a \neq b} \sum_{i \neq j}$ can be divided into the following seven cases.

| Sum patterns of $(i, j)$ and $(a, b)$ |
|---------------------------------------|
| (1) $i = a$, $j = b$. (2) $i = b$, $j = a$. (3) $i = a$, $j \neq b$. (4) $i = b$, $j \neq a$. (5) $i \neq a$, $j = b$. (6) $i \neq b$, $j = a$. (7) $i \neq a$, $b \neq j = a$. |

From the symmetry of $a, b$ and $i, j$, (1) and (2) have the same form. Similarly, (3), (4), (5) and (6) have the same form. Therefore,

$$Q_3 = 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_{nn} \right)^{kN-k^2-4} (\omega^{i_i} \omega^{i_j})^{N-k-2} (\omega^{i_i} \omega^{i_j})^{2}$$

$$+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^{2}((N-k-1)!)^{2}} \int D\psi \left( \sum_{n \neq i, j} \omega_{nn} \right)^{N-k} (\omega^{i_i} \omega^{i_j})^{N-k-2} (\omega^{i_i} \omega^{i_j})^{N-k-1}$$

$$+ \frac{(kN-k^2-4)!}{((N-k)!)^{k-4}((N-k-1)!)^{4}} \int D\psi \left( \sum_{n \neq a, b, i, j} \omega_{nn} \right)^{N-k} (\omega^{i_i} \omega^{i_j})^{N-k-1}$$

$$\times \omega^{ab} \omega^{i_i} \omega^{i_j}.$$  \hspace{1cm} (2.56)

Here, $\sum'_{i, j, a, b}$ implies that $i, j, a$, and $b$ sum so that they are different each other.

$$Q_3 = 2 \sum_{i \neq j} \frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^{2}} \int D\psi \left( \prod_{n \neq i, j} \omega_{nn} \right)^{N-k} (\omega^{i_i} \omega^{i_j})^{N-k-2}$$

$$+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^{2}((N-k-1)!)^{2}} \int D\psi \left( \prod_{n \neq i, j} \omega_{nn} \right)^{N-k} (\omega^{i_i} \omega^{i_j})^{N-k-2}$$

$$+ \frac{(kN-k^2-4)!}{((N-k)!)^{k-4}((N-k-1)!)^{4}} \int D\psi \left( \prod_{n \neq a, b, i, j} \omega_{nn} \right)^{N-k} (\omega^{i_i} \omega^{i_j})^{N-k-1}$$

$$\times \omega^{ab} \omega^{i_i} \omega^{i_j}.$$  \hspace{1cm} (2.57)
We consider sum of $s_1, s_2, t_1, t_2$. In the first term, the summation can be divided into two ways, $(s_1 = t_1, s_2 = t_2, s_1 \neq s_2)$ and $(s_1 = t_2, s_2 = t_1, s_1 \neq s_2)$. In the second term, it must be $(s_1 = t_1, s_2 = t_2, s_1 \neq s_2)$. In the third term, it must be $(s_1 = t_1, s_2 = t_2)$. Since the first term is symmetric for $s_1$ and $s_2$, and $t_1$ and $t_2$,

$$Q_3 = 4 \sum_{i \neq j} \sum_{s_1 \neq s_2} \frac{(kN - k^2 - 4)!}{((N - k - 2)!)^2} \int D\psi \left( \prod_{n \neq i, j} \prod_{l=1}^{N-k} \psi_i^n \psi_j^b \right) \left( \omega^{i_1 j_1} \right)^{N-k-2} \left( \psi_{s_1}^i \psi_{s_1}^j \psi_{s_2}^i \psi_{s_2}^j \right)$$

$$+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \sum_{s_1 \neq s_2} \frac{(kN - k^2 - 4)!}{((N - k - 2)!)^2} \int D\psi \left( \prod_{n \neq i, j} \prod_{l=1}^{N-k} \psi_i^n \psi_j^b \right) \left( \omega^{b_1 b_2} \omega^{i_1 j_1} \right)^{N-k-2} \left( \psi_{s_1}^i \psi_{s_1}^j \psi_{s_2}^i \psi_{s_2}^j \right)$$

$$+ \sum_{(i,j,a,b)} \sum_{s_1 \neq s_2} \frac{(kN - k^2 - 4)!}{((N - k - 2)!)^2} \int D\psi \left( \prod_{n \neq i, j} \prod_{l=1}^{N-k} \psi_i^n \psi_j^b \right) \left( \omega^{a_1 b_1} \omega^{i_1 j_1} \omega^{i_2 j_2} \right)^{N-k-1}$$

$$\left( \psi_{s_1}^i \psi_{s_1}^j \psi_{s_2}^i \psi_{s_2}^j \right)$$

$$= 4 \sum_{i \neq j} \sum_{s_1 \neq s_2} (kN - k^2 - 4)! + 4 \sum_{i \neq j} \sum_{b \neq i, j} \sum_{s_1 \neq s_2} (kN - k^2 - 4)! + \sum_{(i,j,a,b)} \sum_{s_1 \neq s_2} (kN - k^2 - 4)!$$

$$= (kN - k^2 - 4)! [4(k - 1)(N - k)(N - k - 1) + 4k(k - 1)(k - 2)(N - k)(N - k - 1)]$$

$$+ k(k - 1)(k - 2)(k - 3)(N - k)^2$$

$$= (kN - k^2 - 4)! [4(k - 1)(N - k)(N - k - 1) + (k - 1)(k - 2)(k - 3)(N - k)]$$  

By rearranging $P_3 = Q_1 + Q_2 + Q_3$ for $(N - k)$, we obtain \[Q_3\].

\[\]

**Acknowledgement**

We would like to thank Prof. M. Jinzenji for useful discussions.

**References**

[1] E. Witten, The Verlinde algebra and the cohomology of the Grassmannian, [arXiv:hep-th/9312104](https://arxiv.org/abs/hep-th/9312104).

[2] P. Griffiths; J. Harris, Principles of algebraic geometry, Pure and Applied Mathematics. Wiley–Interscience, New York (1978).

[3] W.Fulton, Intersection Theory, Springer-Verlag Berlin Hidelberg(1998).

[4] N.Chair, Explicit Computations for the Intersection Numbers on Grassmannians, and on the Space of Holomorphic Maps from $CP^1$ into $Gr(C^n)$. [arXiv:hep-th/9808170](https://arxiv.org/abs/hep-th/9808170).

[5] S. Imanishi, M. Jinzenji, K.Kuwata, Evaluation of Euler Number of Complex Grassmann Manifold $G(k, N)$ via Mathai-Quillen Formalism, arXiv:hep-th/2108.13623

[6] D.T.Hiep, IDENTITIES INVOLVING (DOUBLY) SYMMETRIC POLYNOMIALS AND INTEGRALS OVER GRASSMANNIANS, [arXiv:1607.04850](https://arxiv.org/abs/1607.04850).

[7] A.Weber, Equivariant Chern classes and localization theorem, [arXiv:1110.5515](https://arxiv.org/abs/1110.5515).
[8] M. Zielenkiewicz, Integration over homogenous spaces for classical Lie groups using iterated residues at infinity, arXiv:1212.6623.

[9] T. Ikeda, H. Naruse, Modern Schubert calculus, from the special polynomial theory’s point of view (in Japanese, printed in Japan), Sugaku, 63(3), 313-337 (2011).