ON THE $L^1$-LIOUVILLE PROPERTY OF
STOCHASTICALLY INCOMPLETE MANIFOLDS

G. PACELLI BESSA, STEFANO PIGOLA, AND ALBERTO G. SETTI

Abstract. A classical result by Alexander Grigor’yan states that on
a stochastically complete manifold the non-negative superharmonic $L^1$-
functions are necessarily constant. In this paper we address the question
of whether and to what extent the reverse implication holds.

Contents

Introduction 1
1. An example 4
2. Mean exit time and the $L^1$-Liouville property 6
2.1. Intrinsic curvature restrictions 7
2.2. Minimal submanifolds 8
References 11

Introduction

Let $(M, g)$ be an $m$-dimensional Riemannian manifold. We use the symbol
$\Delta$ to denote the negative-definite Laplace-Beltrami operator of $M$. Thus, if
$M = \mathbb{R}$, $\Delta = +d^2/dx^2$. By a superharmonic function we mean a function
$u \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$ satisfying $\Delta u \leq 0$ in the sense of distributions,
namely,
$$- \int_M \langle \nabla u, \nabla \varphi \rangle \, dv \leq 0,$$
for every $\varphi \in W^{1,2}_{\text{loc}}(M)$. By reversing the inequality we obtain the notion
of subharmonic function and by replacing the inequality with an equality
we get a harmonic function. By elliptic regularity, harmonic functions are
necessarily smooth.

In general, there is no obstruction for a manifold to support many (super)harmonic functions. Indeed, according to a theorem due to R. E. Greene
and H. Wu [3], any $m$-dimensional Riemannian manifold can be embedded

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into $\mathbb{R}^{2m+1}$ by harmonic functions. On the other hand, the presence of superharmonic functions enjoying some special property is intimately related to the geometry of the underlying space. Thus, for instance, if the geodesically complete manifold $(M, g)$ supports a non-constant, positive superharmonic function then $M$ is non-parabolic and, in particular,

$$\int_{\mathbb{R}^+} r \frac{dr}{\text{vol}(B_r(o))} < +\infty,$$

for some origin $o \in M$. Here, $B_r(o)$ denotes the geodesic ball of $M$ centered at $o$ and of radius $r > 0$. In this spirit one gives the following

**Definition 1.** A smooth Riemannian manifold $(M, g)$ is said to satisfy the $L^1$-Liouville property, (shortly, $M$ is $L^1$-Liouville), if every non-negative superharmonic function $0 \leq u \in L^1(M)$ must be constant.

According to a nice result by Alexander Grigor’yan [4], later extended to non-linear operators modeled on the $p$-Laplacian (see [6], [10]) in order to understand whether or not a manifold is $L^1$-Liouville one may simply consider the behavior of its Green kernel $G(x, y)$. We recall that this latter is the minimal, positive, fundamental solution of $-\Delta$.

**Theorem 2.** The Riemannian manifold $(M, g)$ is $L^1$-Liouville if and only if, for some (hence any) $x \in M$,

$$\int_M G(x, y) \, dv(y) = +\infty.$$

Note that, in case that $M$ is parabolic, we have $G \equiv +\infty$ and the integrability condition is trivially satisfied. However, in this case, we already know that positive superharmonic functions (without any further restriction) must be constant.

In [4], A. Grigor’yan makes a clever use of the equivalence established in Theorem 2 to obtain a neat geometric conditions implying the $L^1$-Liouville property. This is achieved by relating the (non-)integrability of the Green function with a further stochastic property of the manifold, namely, its stochastic completeness. Recall that $(M, g)$ is stochastically complete (for the Brownian motion with infinitesimal generator $\Delta$) if for some (hence every) $x \in M$,

$$\int_M p_t(x, y) \, dv(y) = 1,$$

where $p_t(x, y)$ stands for the heat kernel of $M$, i.e., the minimal, positive fundamental solution of the heat operator $\Delta - \partial/\partial t$. From the probabilistic viewpoint, this means that the explosion time of the Brownian motion on $M$ is almost surely infinite. Recall also that $G(x, y)$ and $p_t(x, y)$ are related by

$$G(x, y) = \int_0^{+\infty} p_t(x, y) \, dt.$$
Therefore, applying Tonelli’s Theorem, from Theorem 2 we immediately deduce

**Corollary 3.** A stochastically complete manifold is $L^1$-Liouville.

In particular, since a geodesically complete manifold $(M, g)$ is stochastically complete provided, for some origin $o \in M$,

\[ \int_{r_{0}}^{+\infty} \frac{r}{\log(\text{vol}(B_r(o)))} \, dr = +\infty \]

one may conclude the validity of the next

**Corollary 4.** A geodesically complete Riemannian manifold $(M, g)$ is $L^1$-Liouville provided the volume growth condition (1) is satisfied.

So far we have essentially celebrated A. Grigor’yan work on the subject. In this paper we address the following

**Problems** (a) Does the converse of Corollary 3 hold? (b) If not, to what extent and under which conditions the validity of the $L^1$-Liouville property implies that the manifold is stochastically complete?

In the case of a model manifold

\[ M^m_\sigma = \left( [0, +\infty) \times S^{m-1}, dt^2 + \sigma(t)^2 d\theta^2 \right), \]

it is easy to see that stochastic completeness is in fact equivalent to the $L^1$-Liouville property. Indeed, the Green’s kernel with pole at $o$ of $M^m_\sigma$ is given by

\[ G(x, o) = c_m \int_{r_{0}}^{+\infty} \frac{1}{\sigma^{m-1}(t)} \, dt \]

so that, interchanging the order of integration,

\[ \int_{M} G(x, o) \, dx = c_m \int_{0}^{\infty} \sigma^{m-1}(r) dr \int_{r_{0}}^{+\infty} \frac{1}{\sigma^{m-1}(t)} \, dt \]

\[ = c_m \int_{0}^{+\infty} dt \int_{0}^{t} \frac{\sigma^{m-1}(r) dr}{\sigma^{m-1}(t)} \]

which shows that the condition for stochastic completeness and that for the validity of the $L^1$-Liouville property of a model manifold coincide.

The investigation around these very natural questions would benefit of different viewpoints on the notion of stochastic completeness. We will make a constant use of the following equivalent description in the language of maximum principles at infinity (see [8], [9]).

**Theorem 5.** A Riemannian manifold $(M, g)$ is stochastically complete if and only if, for every $u \in C^2(M)$ satisfying $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_k\} \subset M$ along which

\[ (i) \ u(x_k) > u^* - \frac{1}{k}, \quad (ii) \ \Delta u(x_k) < \frac{1}{k}, \]
1. An example

This section is devoted to show that, in general, an \( L^1 \)-Liouville manifold may be stochastically incomplete. This answers in the negative Problem (a) stated in the previous section. To this end, we construct an explicit example in two steps.

**First Step.** Recall that the connected sum \( M_1 \# M_2 \) of equidimensional Riemannian manifolds is stochastically incomplete provided either \( M_1 \) or \( M_2 \) are stochastically incomplete. See [1, Lemma 3.1]. This is a very special case of the following general fact which follows quite easily using the viewpoint of Theorem 5.

**Proposition 6.** Let \((M, g)\) be a complete manifold and let \( E_1, \ldots, E_k \) be the ends of \( M \) with respect to any smooth, compact domain \( \Omega \subset M \). Then \( M \) is stochastically complete if and only if, for every \( j = 1, \ldots, k \), either of the following conditions is verified:

(i) There exists a compact domain \( D_j \) together with a diffeomorphism \( f_j : \partial D_j \to \partial E_j \) such that the gluing \( M_j = D_j \cup f_j E_j \) is a stochastically complete manifold (without boundary).

(ii) The Riemannian double \( D(E_j) \) is a stochastically complete manifold (without boundary).

In particular, consider the 2-dimensional model manifolds

\[ M_{\sigma_j}^2 = \left( [0, +\infty) \times S^1, dt^2 + \sigma_j(t)^2 d\theta^2 \right), \]

\( j = 1, 2 \), where we require

\[ \int^{+\infty} \sigma_1(t) \, dt = +\infty \]

and

\[ \int^{+\infty} \int_{0}^{r} \frac{\sigma_2(t) \, dt}{\sigma_2(r)} \, dr < +\infty. \]

The first condition means that \( M_{\sigma_1}^2 \) has infinite volume. On the other hand, by a well known characterization, [5, Prop. 3.2], the second condition is equivalent to requiring that \( M_{\sigma_2}^2 \) be stochastically incomplete. Let

\[ M = M_{\sigma_1}^2 \# M_{\sigma_2}^2 \]

where the connected sum is performed using embedded disks \( D \) centered at the poles of the manifolds. By the above considerations, \((M, g)\) is stochastically incomplete. In particular, \( M \) is non-parabolic, therefore, it possesses a Green function \( G < +\infty \).

**Second Step.** We now perform a conformal change of the metric \( g \). We define

\[ \tilde{g} = \lambda^2 g, \]

where \( \lambda > 0 \) is any smooth function with the following properties:
(a) Outside a neighborhood of \( M_{\sigma_1}^2 \setminus D \subset M \), \( \lambda \equiv 1 \).

(b) Outside a neighborhood of \( M_{\sigma_2}^2 \setminus D \subset M \), \( \lambda \) satisfies

\[
\lambda(t, \theta) \geq \frac{1}{\sqrt{\min_{[t_1, t]} G_{x_0}(x)}}.
\]

where \( x_0 \) is any point in \( M_{\sigma_2}^2 \setminus D \subset M \) and, without loss of generality, \([t_1, +\infty) \times S^1 \subset M_{\sigma_1}^2\) has the original metric \( dt^2 + \sigma_1^2(t) d\theta^2 \).

**Conclusion.** We claim that \( \tilde{M} = (M, \tilde{g}) \) is stochastically incomplete and possesses the \( L^1 \)-Liouville property. Indeed, according to (a), and using Proposition 6, we see that \( \tilde{M} \) is stochastically incomplete. In particular, \( \tilde{M} \) is non-parabolic. Actually, since

\[
\Delta \tilde{g} = \frac{1}{\lambda^2} \Delta g,
\]

it follows that the Green function \( \tilde{G} \) of \( \tilde{M} \) satisfies

\[
\tilde{G} = G.
\]

Therefore,

\[
\int_{\tilde{M}} \tilde{G}(x_0, y) d\tilde{v}(y) = \int_{\tilde{M}} G(x_0, y) \lambda^2(y) dv(y)
\]

\[
\geq \lim_{t \to +\infty} \int_{[t_1, t] \times \partial M_{\sigma_1}^2} G(x_0, y) \lambda^2(y) dv(y)
\]

\[
\geq C \int_{t_1}^{+\infty} \sigma_1(t) dt
\]

\[
= +\infty,
\]

and by Theorem 2 we conclude that the Riemannian manifold \( \tilde{M} \) is \( L^1 \)-Liouville.

Actually, a variation of the above construction allows us to produce an example of a stochastically incomplete \( L^1 \)-Liouville manifold with only one end. As above, we start with a 2-dimensional stochastically incomplete model

\[
M_{\sigma}^2 = \left( [0, +\infty) \times S^1, g = dr^2 + \sigma(r)^2 d\theta^2 \right),
\]

with \( \sigma(r) \) increasing and diverging to infinity at infinity, and (radial) Green’s function with pole at \( o \), \( G(o, x) = G(o, r(x)) \), and perform the conformal change of metric

\[
\tilde{g} = \lambda g
\]

with a conformality factor \( \lambda(x) \geq 1 \) such that \( \lambda(x) = 1 \) if \( x = re^{i\theta} \) with \(-\pi/2 \leq \theta \leq \pi/2\) and \( \lambda(x) \geq G(o, r)^{-1/2} \) if \( x = re^{i\theta} \) with \( r > 1 \) and \( 3\pi/4 \leq \theta \leq 5\pi/4 \). Denoting as above with a tilde the quantities relative to
the conformal metric \( \tilde{g} \) and, using again the fact that \( \tilde{G}(o, x) = G(o, x) \) and that \( d\tilde{v} = \lambda dv = \lambda \sigma drd\theta \) we see that

\[
\int_{\tilde{M}} \tilde{G}(o, x)d\tilde{v} \geq \int_{[1, \infty) \times [3\pi/4, 5\pi/4]} G\left(o, re^{i\theta}\right) \lambda\left(re^{i\theta}\right) \sigma(r)drd\theta
\]

\[
\geq \pi/2 \int_{1}^{\infty} \sigma(r)dr = +\infty,
\]

and \( \tilde{M} \) is \( L^1 \)-Liouville. On the other hand, let

\[
v_o(r) = \int_{0}^{r} \sigma(t)^{-1} \int_{0}^{t} \sigma(s)ds dt,
\]

and let \( v(re^{i\theta}) = v_o(r)\cos(\theta) \). Then \( v \) tends to its supremum along the ray \( re^{i\theta} \) and, using \( \tilde{\Delta} = \lambda^{-2}\Delta \) and \( \Delta v_o = 1 \) we deduce that in the region where \(-\pi/4 \leq \theta \leq \pi/4 \) and \( \sigma(r)^2 > 2 \sup v_o \) we have

\[
\tilde{\Delta}v\left(re^{i\theta}\right) = \frac{1}{\lambda(re^{i\theta})^2}\left(\Delta v_o(r)\cos(\theta) - \frac{1}{\sigma(r)^2}v_o(r)\cos(\theta)\right) \geq \frac{\sqrt{2}}{4}.
\]

Thus \( v \) does not satisfy the maximum principle at infinity and \( \tilde{M} \) is not stochastically complete.

The examples above stress the fact that equivalence between stochastic completeness and the validity of the \( L^1 \)-Liouville property depends very much on the rotational invariance of the models. In the presence of a strong anisotropy, it is possible that Brownian motion may explode in finite time in certain directions and yet the \( L^1 \)-Liouville property holds, due to the fact that the Green’s kernel is big enough in other regions (or ends of the manifold).

The first example constructed in Section \ref{sec:examples} fits very well in this order of ideas. In fact, inspection of that example shows that the stochastically incomplete end remains essentially untouched whereas the background metric is conformally modified only on the end responsible for the validity of the \( L^1 \)-Liouville property.

While, the second example shows that the \( L^1 \)-Liouville property does not imply even a weak form of stochastic completeness where it is required that at least one of the ends of the manifold is stochastically complete.

2. MEAN EXIT TIME AND THE \( L^1 \)-LIOUVILLE PROPERTY

As remarked above, stochastic completeness and the \( L^1 \)-Liouville property are equivalent on models, but, in general, \( L^1 \)-Liouville manifolds may be stochastically incomplete. We are thus naturally led to investigate general geometric conditions that guarantee that a a stochastically incomplete manifold is not \( L^1 \)-Liouville. In this section we will focus our attention on curvature conditions both of intrinsic and of extrinsic nature. In both cases we shall use the notion of “global mean exit time” that we are going to introduce.
Let \((M, g)\) be a complete Riemannian manifold and let \(o \in M\) be a fixed reference point. The mean exit time of the Brownian motion from the ball \(B_R(o)\) is defined as the (positive) solution of the Dirichlet problem

\[
\begin{cases}
\Delta E_R = -1 & \text{on } B_R(o) \\
E_R = 0 & \text{on } \partial B_R(o).
\end{cases}
\]

Note that \(E_R\) is a smooth function on \(B_R(o)\). Moreover, if \(G_R(x, y)\) denotes the Dirichlet Green function of \(B_R(o)\), then, we have the representation formula

\[E_R(x) = \int_{B_R(o)} G_R(x, y) \, dv(y).\]

Since \(G_R(x, y) \nearrow G(x, y)\) as \(R \nearrow +\infty\), by monotone convergence we deduce that

\[E_R(x) \nearrow E(x) = \int_M G(x, y) \, dv(y).\]

We call \(E(x)\) the global mean exit time of \(M\). With this terminology and notation, \(M\) is not \(L^1\)-Liouville if and only if \(E\) is a genuine (say, finite) function. In particular, on a stochastically complete manifold, the global mean exit time must be infinite. On the other hand, we point out that, according to Section 1, there exist stochastically incomplete manifolds with infinite global mean exit time, thus showing that, in general, the global mean exit time does not carry enough information on the explosion of the Brownian motion because of the possible presence of direction along which explosion can occur in finite time.

### 2.1. Intrinsic curvature restrictions

We shall prove the following

**Theorem 7.** Let \((M, g)\) be a complete Riemannian manifold of dimension \(m\) with a pole \(o\). Assume that the distance function \(r(x) = d_M(x, o)\) satisfies

\[\Delta r \geq (m - 1) \frac{\sigma'}{\sigma}, \text{ on } M\]

where \(\sigma : [0, +\infty) \rightarrow [0, +\infty)\) is the warping function of the \(m\)-dimensional model manifold \(M^m_\sigma\). If \(M^m_\sigma\) is not \(L^1\)-Liouville (equivalently, stochastically incomplete) then \(M\) is not \(L^1\)-Liouville.

**Remark 8.** By standard comparison arguments, condition (2) follows from the radial sectional curvature condition

\[\text{Sec}_{\text{rad}}(x) \leq -\frac{\sigma''}{\sigma}(r(x)).\]

Note that the above result allows us to recover the already noted equivalence of stochastic completeness and \(L^1\)-Liouville property of model manifolds.

**Proof.** Define

\[F_R(r) = \int_r^R \int_0^t \frac{\sigma^{m-1}(s) \, ds}{\sigma^{m-1}(t)} \, dt\]
and

\[ F(r) = \int_{r}^{+\infty} \frac{\int_{t}^{s} \sigma^{-1}(s)\,ds}{\sigma^{-1}(t)}\,dt. \]

Since, by assumption, \( M^m_\sigma \) is stochastically incomplete, we have

\[ F(r) < +\infty, \forall r. \]

Direct computations show that the transplanted function \( F_R(r(x)) \) satisfies

\[
\begin{aligned}
\Delta F_R &\leq -1 \quad \text{on } B_R(o) \\
F_R & = 0 \quad \text{on } \partial B_R(o).
\end{aligned}
\]

Therefore, by comparison on bounded domains,

\[ E_R(x) \leq F_R(r(x)) \quad \text{on } B_R, \]

and letting \( R \to +\infty \) we conclude

\[ E(x) \leq F(r(x)). \]

This proves that \( M \) is not \( L^1 \)-Liouville. \( \square \)

2.2. Minimal submanifolds. This subsection aims to showing that the \( L^1 \)-Liouville property of a proper minimal submanifold \( \Sigma \) of manifold with a pole \( N \) depends on the curvature of the ambient space. In particular, in the case where \( N \) is a model with warping function \( \sigma \), if the \( m \)-dimensional model manifold \( M^m_\sigma \) is not \( L^1 \)-Liouville, and \( \sigma \) satisfies a technical condition, then \( \Sigma \) is not \( L^1 \)-Liouville. As alluded to above, we shall use a global mean exit time comparison argument which extends a previous result by S. Markovsen, \[7\] (see also \[2\]).

**Theorem 9.** Let \( f : \Sigma \hookrightarrow N \) be an \( m \)-dimensional properly immersed minimal submanifold into an \( n \)-dimensional Riemannian manifold \( N \). Assume that the sectional curvature of \( N \) satisfies

\[ K^N \leq -G(\rho(y)) \]

where \( G \) is a smooth even function on \( \mathbb{R} \) and \( \rho(y) = d_N(y,p) \) denotes the Riemannian distance function from a fixed point \( p \in N \). Let \( \sigma \) be the solution of the initial value problem

\[
\begin{aligned}
\sigma'' & = G\sigma \\
\sigma(0) & = 0, \sigma'(0) = 1
\end{aligned}
\]

and assume that

\[ \sigma' \geq 0 \]

and

\[ \frac{\int_{0}^{t} \sigma^{-1}(s)\,ds}{\sigma^{-1}(t)} \leq m \frac{\sigma'}{\sigma}, \]
on \((0, +\infty)\). Then, for every extrinsic ball \(B^N_R\) centered at \(p\) in \(N^n\) with radius \(R < \text{inj}_N(p)\), the mean exit time \(E_{f^{-1}(B^N_R)}(x)\) satisfies

\[
E_{f^{-1}(B^N_R)}(x) \leq F_R \circ \rho \circ f(x),
\]

where \(F_R\) is defined in (3). In particular, if \(\text{inj}_N(p) = +\infty\), and

\[
t \to \int_0^t \frac{\sigma^m}{\sigma^m - 1} \in L^1(\infty)
\]

then \(\Sigma\) is not \(L^1\)-Liouville.

Proof. S. Markvorsen [7], considered the case of constant curvature reference spaces, which correspond to the choices

\[
\sigma(t) = t, \quad \sigma(t) = k^{-1}\sin(kt), \quad \sigma(t) = k^{-1}\sinh(kt),
\]

with \(k > 0\). Actually, it is possible to extend Markvorsen arguments to the more general setting of Theorem 9 by using the function

\[
F_R(t) = F_R \circ i^{-1}(t),
\]

with

\[
i(t) = \int_0^t \sigma(s) ds.
\]

We are going to exhibit a more straightforward argument which avoids the use of the auxiliary function \(F_R\).

Recall that if \(f : M \to N\) is an isometric immersion and \(\varphi : N \to \mathbb{R}\) and \(F : \mathbb{R} \to \mathbb{R}\) are smooth, then for every \(X \in T_xM\) we have

\[
\text{Hess}(F \circ \varphi \circ f)(X, X) = F''(\varphi(f(x)))\langle \nabla^N\varphi, dfX \rangle^2
\]

\[
+ F'(\varphi(f(x))) \left[ \text{Hess}^N \varphi(dfX, dfX) + \langle \nabla^N\varphi, I_t(X, X) \rangle \right].
\]

If \(\varphi = \rho\) is the distance function, then the assumption on the sectional curvature of \(N\) implies

\[
\text{Hess}^N \rho(Y, Y) \geq \frac{\sigma'}{\sigma} \left[ \langle Y, Y \rangle - \langle \nabla^N\rho, Y \rangle^2 \right].
\]

Thus, assuming that \(f\) is minimal and that \(F' \leq 0\) and setting for ease of notation \(\rho_x = \rho(f(x))\), we obtain

\[
\Delta(F \circ \rho \circ f)(x) \leq m \left( F' \frac{\sigma'}{\sigma}(\rho_x) + (F'' - F' \frac{\sigma'}{\sigma})(\rho_x) \sum_{i=1}^m \langle \nabla^N\rho, dfX_i \rangle^2 \right),
\]

where \(\{X_i\}\) is an orthonormal basis on \(T_xM\).

Now, if \(F = F_R\), then we have

\[
F'_R(r) = -\frac{\int_0^r \sigma^m - 1(s) ds}{\sigma^m - 1(t)} < 0
\]

and

\[
F''_R(r) = -1 - (m - 1) \frac{\sigma'}{\sigma} F'_R,
\]
so substituting,
\[
\Delta(F \circ \rho \circ f)(x) \leq m\left(F_{\sigma}^{\rho} \sigma'(\rho_x) - \left[(1 + m\frac{\sigma'}{\sigma} F_{\rho}^{\sigma}(\rho_x)\right] \sum_{i=1}^{m} \langle \nabla^N \rho, df X_i \rangle^2. \]

Complete \(\{df X_i\}_{i=1}^{m}\) to an orthonormal basis \(\{df X_i\}_{i=1}^{m} \cup \{X_j\}_{j=m+1}^{n}\) on \(T_{f(x)}N\), and note that
\[
\sum_{i} \langle \nabla^N \rho, df X_i \rangle^2 + \sum_{j} \langle \nabla^N \rho, Y_j \rangle^2 = 1.
\]
Inserting this in the above inequality and using the assumption
\[
m\frac{\sigma'}{\sigma} F_{\rho}^{\sigma} = -m\frac{\sigma'}{\sigma} \int_{0}^{t} \sigma^{m-1}(s) ds \leq -1
\]
we finally obtain
\[
\Delta(F \circ \rho \circ f)(x) \leq -\sum_{i} \langle \nabla^N \rho, df X_i \rangle^2 + m\frac{\sigma'}{\sigma} F_{\rho}^{\sigma} \sum_{j} \langle \nabla^N \rho, Y_j \rangle^2
\]
\[
\leq -\sum_{i} \langle \nabla^N \rho, df X_i \rangle^2 - \sum_{j} \langle \nabla^N \rho, Y_j \rangle^2 = -1.
\]
Thus,
\[
\Delta(F_R \circ \rho \circ f) \leq -1 \text{ on } f^{-1}(B_{R}^{N}),
\]
and since both \(E_{f^{-1}(B_{R}^{N})}\) and \(F_{R} \circ \rho \circ f\) vanish on \(\partial f^{-1}(B_{R}^{N})\), the first assertion in the statement follows from the comparison principle.

The second assertion follows letting \(R \to +\infty\), so that \(f^{-1}(B_{R}^{N}) \nrightarrow M\) and \(E_{R}(x) \nrightarrow E(x)\), while
\[
F_{R}(r) \nrightarrow \int_{r}^{R} \frac{\int_{0}^{t} \sigma^{m-1}(s) ds}{\sigma^{m-1}(t)} dt,
\]
and, as seen above, the assumption that \(M_{\sigma}^{m}\) is not \(L^{1}\)-Liouville amounts to the fact that the integral on the right hand side is finite.

Note that (6) amounts to the fact that the \(m\)-dimensional model manifold \(M_{\sigma}^{m}\) is not \(L^{1}\)-Liouville.

We also remark that, since \(\sigma\) is non-decreasing, for every \(n \geq m\)
\[
\int_{r}^{R} \frac{\int_{0}^{t} \sigma^{n-1}(s) ds}{\sigma^{n-1}(t)} dt \leq \int_{r}^{R} \frac{\int_{0}^{t} \sigma^{m-1}(s) ds}{\sigma^{m-1}(t)} dt,
\]
so condition (6) also implies that the \(n\)-dimensional model \(N_{\sigma}^{n}\) is not \(L^{1}\)-Liouville, and therefore, by Theorem (4), the same holds for the manifold \(N\).
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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO CEARÁ-UF, 60455-760 FORTALEZA, CE, BRAZIL

E-mail address: bessa@mat.ufc.br

SEZIONE DI MATEMATICA - DI SAT, UNIVERSITÀ DELL’INSUBRIA - COMO, VIA VALLEGGIO 11, I-22100 COMO, ITALY

E-mail address: stefano.pigola@uninsubria.it

SEZIONE DI MATEMATICA - DI SAT, UNIVERSITÀ DELL’INSUBRIA - COMO, VIA VALLEGGIO 11, I-22100 COMO, ITALY

E-mail address: alberto.setti@uninsubria.it