Deciding Word Problems of Semigroups using Finite State Automata

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Abstract

We explore natural classes of finitely generated semigroups that have word problem decidable by synchronous or asynchronous two-tape finite state automata. Synchronous two-tape automata decide regular word problems and asynchronous automata decide rational word problems. We argue that asynchronous two-tape automata are a more suitable choice than synchronous two-tape automata and show that the word problem being rational is invariant under change of generating set and some basic algebraic constructions. We also examine some algebraic properties of semigroups with rational word problem.

Keywords: finite state automata, rational relations, semigroups, word problems

1. Introduction

Finite state automata are a simple concept that is well-established in the theory of computation. For finite state automata there is an established set of algorithms that solve relevant problems such as minimisation, comparison, and others. For a reference see for example [1], or any other standard monograph on the subject.

The word problem of finitely generated semigroups, monoids and groups is the decision problem whether two strings over the generators represent the same element.

We will define two distinct natural classes of semigroups: The class of semigroups with regular word problem and the class of semigroups with rational word problem.
problem. Regular word problems can be decided by deterministic, synchronous two tape finite state automata, which in every step read exactly one symbol from each tape. Rational word problems can be decided by asynchronous two tape automata that in every step read at most one symbol from each tape. The class of semigroups with regular word problem is a subclass of the class of semigroups with rational word problem by Proposition 2.8 and we give an example of a semigroup with rational word problem that do not have regular word problem in 4.4.

The notions of regular and rational languages are well-established in the literature. The notion of a regular language is commonly used for rational subsets of free monoids. The notion of a rational subset of a monoid or semigroup is more general. See for example [2] for reference.

The topic of group word problems and formal language theory has been extensively researched, and yielded strong structural results. Two relevant results to be referenced here are Anisimov’s characterisation of groups with regular word problem [3] and Muller and Schupp’s [4] characterisation of groups with context-free word problem.

Previous research in the area of word problems of semigroups and formal language theory includes work by Sakarovitch and Pelletier [5, 6] on what they call rational monoids and semigroups. Note that it is currently open whether our notion of semigroups with rational word problem coincides with the notion of rational semigroups in Sakarovitch’s sense. It is easy to show that every semigroup that is rational in Sakarovitch’s sense has rational word problem in our sense. Therefore both notions have to be treated as not being equal. We make a clear mention of which notion is meant where necessary, and for most of the paper we will be talking about semigroups with rational word problem in our sense.

Kambites shows in [7] and [8] that semigroups that have a presentation which fulfills small overlap conditions are rational in Sakarovitch’s sense. From results in this paper we can conclude that these semigroups have rational word problem in our sense.

Holt et al. in [9] investigate properties of semigroups with one-counter word problem. In particular they show that the free semigroup on one generator has one-counter word problem, and that the free semigroup on more than one generator does not have one-counter word problem. They also show that the free group on one generator has one-counter word problem. As we will show in Example 4.3, free semigroups on arbitrary finite generating sets have regular word problem and rational word problem in our sense, but we also show in Theorem 7.4 that infinite groups cannot have rational word problem.

This paper is structured as follows. Section 2 will give the necessary definitions and results from automata theory to define our notions of regular and rational word problem. In Section 3 we will then define precisely what we understand a word problem to be, and when a word problem is regular or rational.
respectively. We then continue to show some initial results in Section 4 such as regularity of the word problem of finite semigroups and free semigroups over a finite generating set. In Section 5 we prove the first main result: invariance of rational word problem under change of generating sets, and Section 6 establishes the related result that a finitely generated monoid has rational word problem when generated as a semigroup if and only if it has rational word problem when generated as a monoid. Since we established having rational word problem as a property of the semigroup we examine structural properties of semigroups with rational word problem in Section 7. Sections 8 and 9 then treat constructions involving semigroups with rational word problem and whether the property of having rational word problem is closed under these constructions. In Section 10 we give an outlook on further research to be undertaken.

2. Automata and Rational Relations

We introduce the basic notions of alphabet, strings and automata accepting subsets of the sets of all strings and the sets of all pairs of strings over an alphabet. This will enable us to define the notions of regular and rational word problem in the following section.

Let in the following $A$ be a finite set. A string over $A$ is a finite sequence of elements of $A$. We denote the special case of the empty string by $\varepsilon_A$, or simply $\varepsilon$ if there is no ambiguity. We denote by $A^*$ the set of all strings over $A$ and by $A^+$ the set of all nonempty strings over $A$.

Literal strings will be typeset in bold, thus for $a$, $b$ and $c$ in $A$ the literal $abc$ denotes a string of length three in $A^*$.

Let $s$ and $t$ be elements of $A^*$. We denote by $|s|$ the length of the sequence $s$ and by $|s|_a$ the number of occurrences of the letter $a \in A$ in $s$.

One of the most immediate operations on strings is concatenation. We denote the concatenation of $s$ and $t$ by $st$.

For any natural number $i$, we denote by $s^i$ the $i$–fold concatenation of copies of $s$. The special case $s^0$ is defined to be $\varepsilon$.

We call $s$ a prefix of $t$, if there is a string $u$ such that $t = su$, and we call $s$ a suffix of $t$, if there is a string $u$ such that $t = us$.

**Definition 2.1** (finite state automaton). A finite state automaton $A$ is a tuple $A = \langle Q, A, q_0, F, \Delta \rangle$

consisting of a finite set $Q$ of states, an alphabet $A$, an initial state $q_0$ in $Q$, a set $F \subseteq Q$ of final states and a transition relation $\Delta \subseteq Q \times (A \cup \{\varepsilon\}) \times Q$.

We also denote elements $(q, a, r)$ from $\Delta$ by $q \xrightarrow{a} r$. 

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A computation of \( A \) from \( q_1 \) to \( q_{n+1} \) with label \( a_1 a_2 \cdots a_n \) is a finite sequence of transitions

\[
\gamma : q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} q_n \xrightarrow{a_n} q_{n+1}.
\]

The computation \( \gamma \) is said to be accepting if \( q_1 \) is the initial state and \( q_{n+1} \) is an element of \( F \). Note that the label of a computation is an element of \( (A \cup \{\varepsilon\})^* \).

Consider the map

\[
p : (A \cup \{\varepsilon\}) \to A^*, a \mapsto \begin{cases} a & \text{for } a \in A \\ \varepsilon_A & \text{for } a = \varepsilon, \end{cases}
\]

which extends to a surjective monoid homomorphism \( \pi : (A \cup \{\varepsilon\})^* \to A^* \).

We say that \( A \) accepts a string \( s \) in \( A^* \) if there is an accepting computation labelled by a string \( t \) in \( (A \cup \{\varepsilon\})^* \) such that \( \pi(t) = s \). The set of all strings in \( A^* \) that are accepted by \( A \) is called the language of \( A \), denoted \( L(A) \).

Conversely, subsets \( L \) of \( A^* \) with \( L = L(A) \) for some finite state automaton \( A \) are called regular.

A slight generalisation of the concept of a finite state automaton is the notion of a synchronous two tape finite state automaton. For this we take an alphabet \( A \) and add a padding character \( \Box \) forming \( A^\Box = A \cup \{\Box\} \). As alphabet for a two tape synchronous finite state automaton we take \( A^\Box \times A^\Box \). To be able to feed pairs \( (s, t) \) from \( A^* \times A^* \) of strings of differing length to such an automaton we pad the shorter of the two strings by using the padding symbol, more formally

\[
(s, t)^\Box := (s'_1, t'_1)(s'_2, t'_2) \cdots (s'_n, t'_n),
\]

where \( n = \max\{|s|, |t|\} \) and

\[
z'_i = \begin{cases} z_i & i \leq |z| \\ \Box & \text{otherwise} \end{cases}
\]

for \( 1 \leq i \leq n \) and \( z \in \{s, t\} \).

We call a subset \( R \) of \( A^* \times A^* \) regular if there is a synchronous two tape finite state automaton that accepts a padded pair \( (s, t)^\Box \) if and only if \( (s, t) \) is in \( R \).

Note that \( (A \times B)^* \) is isomorphic to the submonoid of pairs of strings of equal lengths in \( A^* \times B^* \) and we will use this isomorphism implicitly.

An asynchronous two tape finite state automaton has the ability to read its two tapes at different speeds.

**Definition 2.2** (asynchronous finite state automaton). An asynchronous finite state automaton \( A \) is a tuple

\[
A := (Q, A, B, q_0, F, \Delta)
\]
consisting of a finite set $Q$ of states, two alphabets $A$ and $B$, an initial state $q_0$ in $Q$, a set $F \subseteq Q$ of final states and a transition relation

$$\Delta \subseteq Q \times (A \cup \{\varepsilon\}) \times (B \cup \{\varepsilon\}) \times Q$$

As in case of finite state automata, we denote elements $(p, a, b, q)$ of the transition relation by

$$p \xrightarrow{(a,b)} q,$$

and a computation $\gamma$ of $\mathfrak{A}$ from $q_1$ to $q_{n+1}$ with label $(a_1, b_1) \cdots (a_n, b_n)$ is a finite sequence of transitions, denoted

$$\gamma : q_1 \xrightarrow{(a_1,b_1)} q_2 \xrightarrow{(a_2,b_2)} q_3 \cdots q_n \xrightarrow{(a_n,b_n)} q_{n+1}.$$  

We shorten this to $\gamma : q_1 \rightarrow^* q_{n+1}$ to say that there exists a computation of finite length from $q_1$ to $q_{n+1}$. A computation $\gamma$ is said to be accepting if $q_1 = q_0$ and $q_{n+1}$ is in $F$.

In the case of an asynchronous automaton the label of a computation is an element of $((A \cup \{\varepsilon\}) \times (B \cup \{\varepsilon\}))^\ast$.

To get a pair of strings from the label of a computation we apply maps $\pi_A$ and $\pi_B$ to the components of the pair of strings that arises from the label of the computation.

We also say that a pair $(v, w)$ of strings in $A^* \times B^*$ has a computation $\gamma : q_1 \rightarrow^* q_n$ if $\gamma$ has label $(s, t)$ such that $(\pi_A(s), \pi_B(t)) = (v, w)$.

An asynchronous automaton $\mathfrak{A}$ accepts a pair $(s, t) \in A^* \times B^*$ if there is an accepting computation of $\mathfrak{A}$ with label $(v, w)$ such that $(\pi_A(v), \pi_B(w)) = (s, t)$.

The set of all pairs $(v, w)$ in $A^* \times B^*$ that are accepted by a finite state automaton $\mathfrak{A}$ is called the language of $\mathfrak{A}$ and is denoted $L(\mathfrak{A})$.

Subsets $R$ of $A^* \times B^*$ for which there is an asynchronous finite state automaton $\mathfrak{A}$ with $L(\mathfrak{A}) = R$ are called rational relations.

For any of the above automaton models, we call a state $q$ in $\mathfrak{A}$ accessible if there is a computation in $\mathfrak{A}$ from the initial state $q_0$ to $q$ and co-accessible if there is computation from $q$ to a final state. An automaton is unambiguous if any string $s$ has at most one computation from any given state $q$. Furthermore, an automaton $\mathfrak{A}$ is deterministic, if it is unambiguous and any $s$ has at least one computation from the initial state.

The above automaton models have a natural interpretation as finite, directed, labelled graphs where the set of vertices is the set of states and there is a labelled edge between two states if and only if there is a transition between them.

We now recall the well known Pumping Lemmas which enable us to prove that a given set is not regular or rational respectively. For proofs of the two lemmas we refer the reader to [2].
Proposition 2.3 (Pumping Lemma for finite state automata). Let \( \mathcal{A} \) be a finite state automaton. There is a natural number \( n_0 \) such that for every string \( s \) accepted by \( \mathcal{A} \) with \( |s| > n_0 \) there is a decomposition \( s = xuy \) into strings \( x, u \) and \( y \) such that

- \( |u| \geq 1 \)
- \( |xu| \leq n_0 \)
- For all \( i \in \mathbb{N} \) the string \( xu^iy \) is also accepted by \( \mathcal{A} \).

Proposition 2.4 (Pumping Lemma for asynchronous finite state automata). Let \( \mathcal{A} \) be an asynchronous finite state automaton. Then there is a natural number \( n_0 \) such that for every pair \( (s_1, s_2) \) of strings accepted by \( \mathcal{A} \) with \( |s_1| + |s_2| > n_0 \) there is a decomposition \( (s_1, s_2) = (x_1u_1y_1, x_2u_2y_2) \) into pairs \( (x_1, x_2), (u_1, u_2) \) and \( (y_1, y_2) \) such that

- \( |u_1| + |u_2| \geq 1 \)
- \( |x_1| + |x_2| + |u_1| + |u_2| \leq n_0 \)
- For all \( i \in \mathbb{N} \) the pair \( (x_1u_1^iy_1, x_2u_2^iy_2) \) is also accepted by \( \mathcal{A} \).

The following proposition states that the composition of rational relations is again a rational relation.

Proposition 2.5. Let \( A, B \) and \( C \) be alphabets and let \( R \subseteq A^* \times B^* \) and \( S \subseteq B^* \times C^* \) be rational relations. Then \( R \circ S \) is also a rational relation, where

\[
R \circ S = \{(r, s) \in A^* \times C^* \mid \text{there is } x \in B^* \text{ such that } (r, x) \in R \text{ and } (x, s) \in S\}
\]

Proof. See [2].

J.H. Johnson in his PhD thesis [10] examined rational equivalence relations over strings, that is rational relations that are equivalence relations. He proved the following theorem which we will use in a later section to show that infinite semigroups with rational word problem cannot be periodic. The proof can be found in the referenced paper.

Proposition 2.6. Let \( A \) be an alphabet and \( R \subseteq A^* \times A^* \) be a rational equivalence relation. Then there is a regular language \( D \subseteq A^* \) such that

- for any equivalence class \( \gamma \) of \( R \) it holds that \( 0 < |\gamma \cap D| < \infty \), and
- \( R \cap (D \times D) \) is a rational equivalence relation on \( D \).

Proof. The idea of the proof is to remove loops from an automaton that decides \( R \) that are labelled by \( (s, \varepsilon) \) for some \( s \in (A \cup \{\varepsilon\})^* \). This puts a bound on the length of representatives accepted for each equivalence class. See [11].
The following three propositions will help to simplify the proofs of a few theorems. The proofs are straightforward and can be found in [2].

**Proposition 2.7.** Let $A$ and $B$ be two alphabets. If $L_1$ is a regular language over $A$ and $L_2$ is a regular language over $B$, then $L_1 \times L_2$ is a rational relation.

**Proposition 2.8.** Let $A$ be an alphabet. If $R \subseteq A^* \times B^*$ is regular, then $R$ is rational.

**Proposition 2.9.** Let $A$ and $B$ be alphabets and $R \subseteq A^* \times B^*$ be a rational relation. Then the languages \(\{w \in B^* \mid (v, w) \in R\}\) and \(\{v \in A^* \mid (v, w) \in R\}\) are regular for all $v \in A^*$ and $w \in B^*$.

### 3. Semigroup Word Problems

This section will introduce the central objects of study for this paper: semigroups, monoids, groups and their word problems. We will also define our notion of regular and rational word problem using the definitions in Section 5.

A **semigroup** is a set $S$ together with a binary associative operation, usually denoted $s \cdot t$ or simply $st$ for $s$ and $t$ in $S$. A **monoid** is a semigroup which contains an element $e$ such that for all $a$ it holds that $ea = ae = a$. We call $e$ an identity element. A **group** is a monoid with the additional condition that for each element $g$ there is an element $h$ such that $gh = hg = e$. We call $h$ an inverse of $g$.

To any semigroup $S$ we can adjoin an **identity element** $e$, that is an element $e$ not previously contained in $S$, and extend the binary operation such that $e$ is an identity element. We denote the semigroup with an adjoined identity element by $S^1$. Note that we allow to add an identity to $S$ even if $S$ already contains one.

An element $z$ of a semigroup is a **zero** if $zs = sz = z$ for all $s$ in $S$. As in the case of identity elements, a zero can be adjoined to any semigroup $S$. We denote the resulting semigroup by $S^0$.

The set $A^+$ together with the concatenation operation is isomorphic to the free semigroup on $A$, and the set $A^*$ together with concatenation is isomorphic to the free monoid on $A$. We denote the free group on a finite generating set by $F(A)$.

A semigroup $S$ is **finitely generated** if there is a finite subset $A$ of $S$ such that the inclusion map from $A$ into $S$ extends to a surjective semigroup homomorphism $\overline{\cdot} : A^+ \to S$. We write $S = Sg(A)$. Note that although $A$ is a subset of $S$, the set $A^+$ is a set of strings and is not a subset of $S$, and for any $v \in A^+$ we denote by $\overline{v}$ the image of the string $v$ in $S$.

Similarly to the above, a monoid $M$ is **finitely generated** if there is a finite subset $A$ of $M$ such that the inclusion map from $A$ into $M$ extends to a surjective monoid homomorphism $\overline{\cdot} : A^+ \to M$, and a group $G$ is finitely generated if there
is a finite subset $A$ of $G$ such that the inclusion map from $A$ into $G$ extends to a group homomorphism $\varphi : F(A) \to G$.

Note that for monoids we defined two notions of finite generation, and for groups we defined three notions of finite generation. More specifically a monoid can be finitely generated as a semigroup or as a monoid, and a group can be finitely generated as a semigroup, as a monoid or as a group.

We use presentations as means to specify semigroups and monoids. We denote by $\text{Sg} (A \mid R)$ the semigroup generated by $A$ with relations $R$ and by $\text{Mon} (A \mid R)$ the monoid generated by $A$ with relations $R$. For a reference on presentations the reader is referred to [12, Ch. 1].

We now define the central notions for this paper, the word problems of finitely generated semigroups monoids and groups. For a semigroup $S$ with finite generating set $A$ we define the semigroup word problem to be the set 

$$\text{SgWP}(S, A) := \{(v, w) \in A^+ \times A^+ \mid v = w\} \subseteq A^+ \times A^+,$$

for a monoid $M$ finitely generated by the set $A$ we define the monoid word problem to be the set 

$$\text{MonWP}(M, A) := \{(v, w) \in A^* \times A^* \mid v = w\} \subseteq A^* \times A^*.$$

Commonly, for a group $G$ with finite monoid generating set $A$, the group word problem is defined as 

$$\text{GrpWP}(G, A) := \{v \in (A \cup A^{-1})^* \mid v = e\} \subseteq (A \cup A^{-1})^*.$$

Note that there are three notions of word problem for finitely generated groups, and two for finitely generated monoids.

If $G$ is a group, finitely generated as a group, then $G$ is also finitely generated as a semigroup and $\text{SgWP}(G, A)$ consists of pairs $(v, w)$ such that $v \cdot w^{-1} = e$.

The question that we want to address is the decidability of the word problems defined above by finite state automata and properties of semigroups with word problem decidable by finite state automata.

Some of the results for groups are already known, results concerning semigroups and monoids are original work of the authors.

4. Initial Results

This section will give a foundation for the presented work starting from a well known result about groups: Anisimov showed in [3] that the class of groups with regular one tape word problem is the class of finite groups.

We will show that finite semigroups and finite monoids have regular word problem, and give an example of an infinite semigroup with regular word problem. We then show that the notion of regular word problem is not invariant under change of generating sets. We argue that the notion of rational word problem is therefore more suitable.
Theorem 4.1 (Anisimov). Let $G$ be a group and let $A$ be a finite monoid generating set for $G$. Then $\text{GrpWP}(G, A) \subseteq A^*$ is regular if and only if $G$ is finite.

Proof. Suppose $G$ is finite and consider the automaton

$$\mathfrak{A} = \langle G, A, 1, \{1\}, \Delta \rangle,$$

where $(g, a, h)$ is in $\Delta$ if and only if $ga = h$. This automaton is the Cayley graph of $G$ with respect to the generating set $A$, extended by predicates for the initial state and final states.

A string $s$ in $A^*$ is accepted by $\mathfrak{A}$ if and only if there is a computation from 1 to 1 labelled by $s$. This also means that $s = 1$ by the definition of $\mathfrak{A}$.

Conversely, assume that there is a finite state automaton $\mathfrak{A} = \langle Q, A, q_0, F, \Delta \rangle$ that has as its language all strings $s$ with $s = 1$. Without loss of generality we can assume $\mathfrak{A}$ to be deterministic. If it was not, we can construct an equivalent deterministic automaton by applying the powerset construction, a standard tool in the theory of finite state automata, which can for example be found in [1].

Let $s$ and $t$ be two strings that label paths in $\mathfrak{A}$ from $q_0$ to some state $q$ in $Q$. Since $G$ is a group, and $\mathfrak{A}$ is deterministic, there has to be a path from $q$ labelled $u$ to an accept state. Thus

$$su = s \cdot u = 1 = tu = t \cdot u,$$

which implies $s = t$. Therefore $G$ is finite.

One direction of the above theorem stays true for semigroup and monoid word problems.

Theorem 4.2. Let $S$ be a finite semigroup or monoid. Then $S$ has regular word problem with respect to all generating sets.

Proof. Let $S$ be a finite semigroup and let $A$ be any generating set for $S$. Consider the following automaton.

$$\mathfrak{A} = \langle Q, A^\square \times A^\square, q_0, F, \Delta \rangle$$

consisting of

- $Q = \{q_0\} \cup (S \times S \times \{L, N, R\})$
- $F = \{(s, s, i) \mid s \in S, i \in \{L, N, R\}\}$
- $\Delta = \{[q_0, (x, y), (x, y, N)]\}$
  \hspace{1em} \cup \{[(s, t), (x, y), (sx, ty, i)] \mid x, y \neq \square, i \in \{L, N, R\}\}$
  \hspace{1em} \cup \{[(s, t, N), (x, \square), (sx, t, R)]\}$
  \hspace{1em} \cup \{[(s, t, N), (\square, y), (s, ty, L)]\}$
  \hspace{1em} \cup \{[(s, t, R), (x, \square), (sx, t, R)]\}$
  \hspace{1em} \cup \{[(s, t, L), (\square, y), (s, ty, L)]\}$. 


This automaton consists of three copies of the direct product of two copies of the Cayley graph of $S$ together with an initial state. Reading a pair of symbols it keeps track of right multiplication by a generator with the $\square$ symbol acting as identity. The automaton determines the elements represented by the input strings, and accepts if and only if these are the same.

The copies indexed by $L$, $N$ and $R$ are needed to take care of padding symbols: if a padding symbol is read on one tape for the first time, then the automaton is only allowed to read padding symbols from that tape, and non padding symbols from the other tape.

This automaton accepts a pair $(v, w)$ of padded strings if and only if $v = w$. The proof for monoids is similar.

In contrast with the group case there are examples of infinite semigroups and monoids that do have word problem that is decidable by a finite state automaton. The first examples are the free semigroup and the free monoid on any finite set. Additionally, in Sections 8 and 9 we will show that we can construct infinite semigroups with rational word problem from semigroups which are known to have rational word problem.

**Example 4.3.** Let $A$ be a finite, non-empty set. The free semigroup $A^+$ and the free monoid $A^*$ are infinite and $\text{SgWP}(A^+, A)$ and $\text{MonWP}(A^*, A)$ are regular.

**Proof.** The automaton depicted in Figure 1.3 accepts pairs of equal strings. In a free semigroup on a finite set $A$ two strings $v$ and $w$ over $A$ represent the same element if and only if they are equal. Therefore $\text{SgWP}(A^+, A)$ is regular. For an automaton that decides $\text{MonWP}(A^*, A)$ we turn $q_0$ into an accept state. 

One important aspect of our definitions of the word problem is that they depend on the choice of a generating set. If the set $\text{SgWP}(S, A)$ is regular for some choice of $A$, then in general it need not be regular for other finite generating sets of $S$.

We will show in Section 5 that using asynchronous finite state automata is an appropriate choice to achieve independence of choice of generators. Additionally Theorem 7.2 states that semigroups with regular word problem with respect to all finite generating set are precisely the finite semigroups.

The following example shows that there are semigroups that are finitely generated, not finitely presentable, and have rational word problem.
Example 4.4. Let $S = \langle a, b \mid (ab^n a = aba)_{n \geq 2} \rangle$. This semigroup is infinite, not finitely presentable, and $\text{SgWP}(S, \{a, b\})$ is rational. Furthermore, there is no generating set $A'$ for $S$ such that $\text{SgWP}(S, A')$ is regular.

Proof. The monoid $S$ is infinite, because the submonoid generated by $a$ is infinite.

A result first proven for groups by Baumslag [13, Theorem 12], and subsequently extended to semigroups by Ruškuc [14, Chapter 1, Proposition 3.1] states for any two finite generating sets $A$ and $B$ for a semigroup $S$, the semigroup is finitely presented with respect to $A$ if and only if it is finitely presented with respect to $B$. Hence if $S$ had a finite presentation, then it would have a finite presentation with respect to the given generating set, and there would be a finite set $X \subseteq \{ab^n a = aba \mid n \geq 2\}$ such that $S \cong \langle a, b \mid X \rangle$. This would mean that there is an $N \in \mathbb{N}$ such that $ab^N a = aba$ is a consequence of $ab^k a = aba$ for $k$ less than $N$, which is impossible.

To show that $S$ has rational word problem, we give an asynchronous finite state automaton in Figure [4.3] that decides the word problem of $S$. To prove correctness of this automaton first let $v$ and $w$ be strings over $A$. Write $v$ and $w$ as

$$v = \prod_{1 \leq i \leq k} a^{\alpha_i} b^{\beta_i}, \quad w = \prod_{1 \leq i \leq k'} a^{\alpha'_i} b^{\beta'_i},$$

for $k > 0$ and $\alpha_i, \beta_i > 0$ for $1 \leq i \leq k$ with the exception of $\alpha_1$ and $\beta_k$, which can be zero, and likewise for $k' > 0$ and $\alpha'_i, \beta'_i > 0$ for $1 \leq i \leq k'$ with the exception of $\alpha'_1$ and $\beta'_k$, which can be zero.

Observe that the pair $(v, w)$ is an element of $\text{SgWP}(S, A)$ if and only if $k = k'$, $\alpha_i = \alpha'_i$, and $\beta_k = \beta'_k$, and if $\alpha_1 = 0$, then $\beta_1 = \beta'_1$.

Now $(v, w)$ is accepted by the automaton, if and only if $v$ and $w$ are exactly of the form described as above.

To show that $\text{SgWP}(S, B)$ is not regular for any finite $B$, we first show that $\text{SgWP}(S, A)$ is not regular. For this assume $\text{SgWP}(S, A)$ to be regular and to be accepted by a finite state automaton with $n_0$ states. Choose $n > n_0$ and consider the pair

$$(ab)^n ab^2n a, ab^{2n} (ab)^n a).$$

Both strings represent the same element $(ab)^{n+1} a$ of $S$, and therefore the pair is an element of $\text{SgWP}(S, A)$.

Since $n > n_0$, there are two natural numbers $i$ and $j$ with $i < j$ such that, after reading $((ab)^i, ab^{2i-1})$ and $((ab)^j, ab^{2j-1})$, the automaton is in some state $q$. From $q$ the automaton can reach an accept state by reading the pair $(ab^{2j-1} a, (ab)^j a)$. Hence the automaton also accepts

$$(ab)^j ab^{2i-1} a, ab^{2j-1} (ab)^i a)$$

which would mean that $(ab)^{j+1} a$ is equal to $(ab)^{i+1} a$ in contradiction to $j > i$. 


Now suppose for a contradiction that $B$ is a finite generating set for $S$ such that $SgWP(S, B)$ is regular. There are strings $v$ and $w$ in $B^*$ with $\overline{v} = a$ and $\overline{w} = b$. Let $l$ be the least common multiple of $|v| + |w|$ and $|w|$ and let $k_1$ and $k_2$ be such that $(|v| + |w|) k_1 + |w| k_2 = l$. Let $n > n_0$ where $n_0$ is the the number of states in a finite state automaton accepting $SgWP(S, B)$. Now the pair

$$(vw)^{nk_1+1} v w^{nk_2+1} v, v w^{nk_2+1} (vw)^{nk_1+1} v$$

is a pair of representatives of $(ab)^{nk_1+2} a$. Therefore there is an accepting computation, and natural numbers $i < j$ such that the automaton reaches some state $q$ after having read $((vw)^{ik_1+1}, v w^{ik_2+1})$ and $((vw)^{jk_1+1}, v w^{jk_2+1})$. The rest of the argument is analogous to the above.

Therefore there is no finite generating set $B$ for $S$ such that $SgWP(S, B)$ is regular.

The following lemma shows that a free commutative semigroup of rank at least two does not have rational word problem. Together with results in Section 5 this will give a method of showing that a semigroup does not have rational word problem.

**Lemma 4.5.** Let $A = \{a, b\}$ and $S = Sg \langle A \mid ab = ba \rangle$. Then $SgWP(S, A)$ is not rational.

**Proof.** For completeness we demonstrate how to apply the Pumping Lemma.
Two strings $s$ and $t$ over $A$ represent the same element of $M$ if and only if $|s|_a = |t|_a$ and $|s|_b = |t|_b$.

For a contradiction assume that $\text{SgWP}(M, A)$ is rational. According to Proposition 2.3, there exists a natural number $n_0$ such that for all pairs $(s, t) \in \text{SgWP}(S, A)$ with $|s| + |t| > n_0$ there is a factorisation

$$(s, t) = (x_1, x_2)(u_1, u_2)(y_1, y_2)$$

with $|x_1| + |x_2| + |u_1| + |u_2| < n_0$ and $|u_1| + |u_2| \geq 1$ such that

$$(x_1, x_2)(u_1, u_2)^i(y_1, y_2)$$

is also in $\text{SgWP}(S, A)$ for all $i \in \mathbb{N}$.

Consider the two representatives $s_1 = a^{n_0}b^{n_0}$ and $s_2 = b^{n_0}a^{n_0}$ of the same element of $S$. The pair $(s_1, s_2)$ is an element of $\text{SgWP}(S, A)$, and since $|s_1| + |s_2| = 2n_0 > n_0$, there is a factorisation as above of $(s_1, s_2)$ with $|x_1| + |x_2| + |u_1| + |u_2| < n_0$ and $|u_1| + |u_2| \geq 1$ such that $(x_1, x_2)(u_1, u_2)^i(y_1, y_2)$ is also in $\text{SgWP}(S, A)$. Assume the factors are as follows.

- $(x_1, x_2) = (a^{k_1}, b^{k_2})$
- $(u_1, u_2) = (a^{l_1}, b^{l_2})$
- $(y_1, y_2) = (a^{n_0-k_1-l_1}b^{n_0-k_2-l_2}a^{n_0})$ for $k, l \in \mathbb{N}$ with $k_1 + k_2 + l_1 + l_2 < n_0$ and $l_1 + l_2 > 0$.

Then it follows by Proposition 2.3 that $(a^{k_1}a^{l_1}b^{n_0-k_1-l_1}, b^{k_2}b^{l_2}a^{n_0-k_2-l_2})$ is an element of $\text{SgWP}(S, A)$, which is a contradiction. Therefore $\text{SgWP}(S, A)$ is not rational.

Note that from the previous lemma it also follows that the word problem of $\langle a, b \mid ab = ba \rangle$ is not regular by Proposition 2.3.

As a closing example for this section, we state that the bicyclic monoid does not have rational word problem.

**Lemma 4.6.** The bicyclic monoid $B = \text{Mon}\langle b, c \mid bc = 1 \rangle$ does not have rational word problem.

**Proof.** This can be proven by applying the Pumping Lemma 2.4 to $(b^{n_0}c^{n_0}, \varepsilon)$ where $n_0$ is the constant guaranteed to exist in the statement of Proposition 2.4.

5. Change of Generators and Subsemigroups

We show that if word problem of a finitely generated semigroup is rational with respect to one finite generating set, then the word problem is rational with
Figure 3: Automaton deciding the replacement relation in Lemma 5.1 respect to all finite generating sets. The proof relies on the closure of rational relations under composition, which is itself not a trivial result.

To prove the main result of this section we first give a few technical lemmas. We observe that the graph of a map that replaces every occurrence of some symbol in a string by a string is a rational relation, after that we use closure of rational relations under compositions to prove the main theorem.

**Lemma 5.1.** Let $A$ be an alphabet and $B = A \cup \{b\}$, where $b$ is not an element of $A$. For some string $w$ over $A$ consider the following map:

$$\varphi : B \rightarrow A^*, x \mapsto \begin{cases} w & \text{if } x = b \\ x & \text{otherwise} \end{cases}.$$  

This map extends to a surjective morphism $\Phi : B^* \rightarrow A^*$ that replaces all occurrences of $b$ in a string over $B$ with $w$. The sets

$$R := \{(v, \Phi(v)) \in B^* \times A^* \mid v \in B^*\}$$

and

$$R' := \{(\Phi(v), v) \in A^* \times B^* \mid v \in B^*\}$$

are rational relations.

**Proof.** Let $w = t_1 \ldots t_n$ and consider $\mathcal{R} = \langle Q, B, A, q_0, F, \Delta \rangle$, where

- $Q = \{q_0, \ldots, q_n\}$
- $F = \{q_0\}$
- $\Delta = \{(q_0, a, a, q_0) \mid a \in A\}$
- $\cup \{(q_{i-1}, \varepsilon, t_i, q_i) \mid 1 \leq i \leq n\}$
- $\cup \{(q_n, b, \varepsilon, q_0)\}$

Note that the states of $\mathcal{R}$ correspond to prefixes of $w$. Figure 5.1 makes the situation much easier to understand. \qed

This next lemma uses composition of rational relations and Proposition 2.5 and is the key lemma for the proof of Theorem 5.3.
Lemma 5.2. Let $S$ be a semigroup generated by the finite set $A$ and let $B = A \cup \{b\}$, where $b$ is an element of $S$ not in $A$. Choosing $w$ in $A^+$ such that $w = b$, define $R$ and $R'$ as in Lemma 5.1. Then the word problem $\mathrm{SgWP}(S, B)$ can be written in terms of $R$, $R'$ and $\mathrm{SgWP}(S, A)$ as follows:

$$\mathrm{SgWP}(S, B) = R \circ \mathrm{SgWP}(S, A) \circ R'.$$

If $\mathrm{SgWP}(S, A)$ is rational, then so is $\mathrm{SgWP}(S, B)$.

Proof. Note that for all $u \in B^*$ the equality $u = \overline{\Phi(u)}$ holds. For all $(v, w) \in A^+ \times A^+$

$$(v, w) \in \mathrm{SgWP}(S, B) \iff (\Phi(v), \Phi(w)) \in \mathrm{SgWP}(S, A).$$

Also observe that for all $(u, w) \in B^+ \times A^+$

$$(u, w) \in R \iff w = \Phi(u) \iff (w, u) \in R'.$$

Therefore

$$(v, w) \in \mathrm{SgWP}(S, B) \iff (v, \Phi(v)) \in R, (\Phi(w), w) \in R' \text{ and } (\Phi(v), \Phi(w)) \in \mathrm{SgWP}(S, A) \iff \exists v', w' \in A^+ \text{ with } (v, v') \in R, (w', w) \in R' \text{ and } (v', w') \in \mathrm{SgWP}(S, A) \iff (v, w) \in R \circ \mathrm{SgWP}(S, A) \circ R'.$$

It follows from Proposition 2.5 and Lemma 5.1 that if $\mathrm{SgWP}(S, A)$ is rational, then $\mathrm{SgWP}(S, B)$ is rational.

The preceding lemmas are tied together to form the following theorem.

Theorem 5.3. Let $S$ be a semigroup and let $A$ be a finite generating set for $S$ such that $\mathrm{SgWP}(S, A)$ is rational.

1. If $B := A \cup \{b\}$ where $b$ is an element of $S$ not in $A$, then $\mathrm{SgWP}(S, B)$ is rational.

2. For the subsemigroup $S'$ generated by $C := A \setminus \{c\}$ for any $c \in A$ the word problem $\mathrm{SgWP}(S', C)$ is rational.

Proof. To prove statement 1 the relation $\mathrm{SgWP}(S, B)$ can be written in terms of $\mathrm{SgWP}(S, A)$ as shown in Lemma 5.2 and is rational. For 2 assume $\mathfrak{A} = (Q, A, A, q_0, F, \Delta)$ to be the asynchronous finite state automaton that decides $\mathrm{SgWP}(S, A)$, then removing all transitions involving $a$ results in a new automaton that decides $\mathrm{SgWP}(S', C)$.

The remaining main results for this section are now corollaries of Theorem 5.3.

Corollary 5.4. Let $S$ be a semigroup. If there exists a finite generating set $A$ for $S$ such that $\mathrm{SgWP}(S, A)$ is rational, then for all finite generating sets $B$ of $S$ the set $\mathrm{SgWP}(S, B)$ is rational.
Proof. If there is a generating set $A$ such that $\text{SgWP}(S, A)$ is rational, and given any other generating set $B$ for $S$, we add generators from $B \setminus A$ to $A$ using Theorem 5.3 (1). Then we remove everything in $A \setminus B$ by application of Theorem 5.3 (2).

Corollary 5.5. Let $S$ be a semigroup and let $A$ be a finite generating set for $S$ such that $\text{SgWP}(S, A)$ is rational. Then for every subsemigroup $T$ of $S$ the word problem $\text{SgWP}(T, A')$ is rational.

Proof. Given any finitely generated subsemigroup $T$ of $S$, this follows from 5.3 by first adding a generating set for $T$ to $A$ and then removing superflous generators from the resulting set.

The preceding corollaries give means of proving non-rationality of the word problem of semigroups. For example, if a semigroup contains a finitely generated free commutative semigroup of rank greater than one, it cannot have rational word problem. This is discussed further in Section 7 in which we discuss structural properties of semigroups with rational word problem.

6. Change of Type

In this section we will show that groups with rational word problem are finite, and that a monoid has rational word problem generated as a monoid if and only if it has rational word problem generated as a semigroup.

A group $G$ with rational monoid word problem is finite. We will extend this result further by showing that the group of units of a monoid with rational word problem is finite in Theorem 7.3 and by showing that in fact any group contained in a semigroup with rational word problem is finite in Theorem 7.4.

Theorem 6.1. Let $G$ be a group finitely generated by $A$ as a monoid. Then $\text{MonWP}(G, A)$ is rational if and only if $G$ is finite.

Proof. If $G$ is finite, it follows from Theorem 4.2 that $\text{MonWP}(G, A)$ is regular and thus rational.

Suppose that $G$ is an infinite group, finitely generated by $A$. Furthermore let $\mathfrak{A}$ be an asynchronous finite state automaton that decides $\text{MonWP}(G, A)$. Without loss of generality assume $\mathfrak{A}$ to be accessible and co-accessible. If a state is not reachable from the initial state, or if no final state is reachable from it, it cannot occur in an accepting computation, and can be removed without changing the accepted relation.

Let $q$ be a state of $\mathfrak{A}$ and let $(v_1, w_1)$ and $(v_2, w_2)$ be two pairs of strings that have computations $\gamma_1 : q_0 \to^* q$ and $\gamma_2 : q_0 \to^* q$. The quotients $w_1^{-1}v_1$ and $w_2^{-1}v_2$ coincide, because $q$ is co-accessible and there is a pair $(s, t)$ that
has a computation \( \delta : q \rightarrow^* q_f \) to some accept state \( q_f \), because \( G \) is a group. Therefore \( \overline{w_1}s = w_1f \) and \( \overline{w_2}s = w_2f \) which after rearrangement yields
\[
\overline{w_1}^{-1}w_1 = \overline{w_2}^{-1}w_2.
\]
In particular, if there are computations \( \gamma_1 : q_0 \rightarrow^* q \) and \( \gamma_2 : q_0 \rightarrow^* q \) labelled by \((v_1, \varepsilon|v_1|)\) and \((v_2, \varepsilon|v_2|)\) respectively, then \( v_1 = v_2 \).

Since \( G \) is infinite, there have to be two strings \( w_1 \) and \( w_2 \) such that \( w_1 \neq w_2 \), and such that \((w_1, \varepsilon|w_1|)\) and \((w_2, \varepsilon|w_2|)\) have computations \( \gamma_1 : q_0 \rightarrow^* q \) and \( \gamma_2 : q_0 \rightarrow^* q \) for some state \( q \). This contradicts the choice of \( w_1 \) and \( w_2 \). \( \square \)

Moving from semigroup generation to monoid generation and vice versa is possible for monoids with rational word problem.

**Theorem 6.2.** Let \( M \) be a monoid finitely generated as a monoid by \( A \). Let \( S = \text{Sg}(A) \) be the subsemigroup of \( M \) generated by \( A \). Then \( \text{SgWP}(S, A) \) is rational if and only if \( \text{MonWP}(M, A) \) is rational.

**Proof.** Let \( M \) be a monoid finitely generated by \( A \) as a monoid. If \( A \) contains the identity then \( \text{Sg}(A) \) is isomorphic to \( \text{Mon}(A) \). Assume now that \( A \) does not contain the identity element of \( M \) and let \( S = \text{Sg}(A) \). Suppose that \( \text{SgWP}(S, A) \) is rational. The set
\[
E = \{ v \in A^+ \mid \overline{v} = e \},
\]
where \( e \) is the identity element of \( M \), is regular, because if \( e \in \text{Sg}(A) \), then there is a string \( w \) over \( A \) with \( \overline{w} = e \) and thus \( E \) is regular by Proposition 2.9, and if \( e \notin \text{Sg}(A) \) then \( E \) is empty. Therefore the set
\[
W = \text{SgWP}(S, A) \cup (E \times \{ \varepsilon \}) \cup (\{ \varepsilon \} \times E) \cup \{(\varepsilon, \varepsilon)\}
\]
is rational and in fact \( W = \text{MonWP}(M, A) \).

Conversely, assume \( \text{MonWP}(M, A) \) is rational. We observe that
\[
\text{SgWP}(S, A) = \text{MonWP}(M, A) \cap (A^+ \times A^+).
\]
It remains to be shown that the intersection on the left hand side is rational. For this we use that the intersection of a rational and a recognisable subset of a monoid is rational. This result can be found in [2]. Since \( \text{MonWP}(M, A \cup \{ \varepsilon \}) \) is rational and \( A^+ \times A^+ \) as a subset of \( A^* \times A^* \) is recognisable, the result follows. \( \square \)

### 7. Structural Properties

We have shown in Section 5 that \( \text{SgWP}(S, A) \) being rational is independent of the choice of \( A \), and thus a property of \( S \). The natural way to proceed is now to establish structural results about such semigroups. We prove that semigroups with rational word problem cannot be periodic, monoids with rational

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word problem have a finite group of units, and that all groups contained in a semigroup with rational word problem are finite.

We first show that an infinite semigroup with rational word problem cannot be periodic.

**Theorem 7.1.** Let $S$ be an infinite semigroup with rational word problem. Then there is an element $y \in S$ such that the subsemigroup $\langle y \rangle$ of $S$ is infinite.

**Proof.** Let $A$ be a finite generating set for $S$. Proposition 2.6 ensures existence of a regular language $D$ over $A$ that contains at least one, and only finitely many representatives of each element of $S$.

Since we assume $S$ to be infinite, $D$ has to be infinite.

By Proposition 2.3 there exists a natural number $n_0$ such that for every $v$ in $D$ with $|v| > n_0$ there exists a factorisation of $v$ into three substrings $x$, $y$, and $z$, such that $|y| \geq 1$ and $xy^iz \in D$ for all $i \in \mathbb{N}$.

Let $v$ be an element of $D$ with $|v| > n_0$. The set $R = \{ xy^iz \mid i \in \mathbb{N}\}$ is an infinite subset of $D$, and since $D$ only contains finitely many representatives for each element of $S$, the set $\overline{R} = \{w \mid w \in R\}$ is an infinite subset of $S$.

This means that the set $\{y^i \mid i > 0\}$ is an infinite subset of $S$, since if there was $i, j \in \mathbb{N}$ with $y^i = y^j$ then $\overline{R}$ would be a finite subset of $S$. Hence $\langle y \rangle$ is infinite.

We briefly return to semigroups with regular word problem. The following theorem characterises the semigroups that have regular word problem with respect to every generating set.

**Theorem 7.2.** Let $S$ be a finitely generated semigroup. Then $\text{SgWP}(S, A)$ is regular for all finite generating sets $A$ if and only if $S$ is finite.

**Proof.** The if part is precisely Theorem 1.2.

Suppose $S$ is infinite and has regular word problem. Then $S$ also has rational word problem by Proposition 2.8 and by Theorem 7.1 there exists some $s$ in $S$ that has infinite order. Let $A$ be a generating set for $S$. The set $B = A \cup \{s, t\}$ where $t = s^2$ also generates $S$. Applying the Pumping Lemma 2.3 to the pair $(t^{n_0}, s^{2n_0})$ shows that the set $\text{SgWP}(S, B)$ is not regular.

This shows that for infinite semigroups regularity of $\text{SgWP}(S, A)$ depends on the choice of the generating set. We have also shown in Example 4.4 that there exist semigroups that have rational word problem but do not possess any finite generating set such that their word problem is regular.

We show in the following that a monoid with rational word problem can only have a finite group of units.

**Theorem 7.3.** Let $M$ be a finitely generated monoid and let $U(M)$ denote the group of units of $M$. If $M$ has rational word problem then $U(M)$ is finite.
Proof. Let $M$ be a monoid finitely generated by $A$ with MonWP($M$, $A$) rational and let $C = M \setminus U(M)$.

Note that $C$ is an ideal if and only if every right-invertible element is also left-invertible, if and only if every left-invertible element is right-invertible.

If $C$ is an ideal, then $U(M)$ is finitely generated by $U(M) \cap A$ and has rational word problem by Corollary 5.5. This means that by Theorem 6.1, the group $U(M)$ is finite.

If $C$ is not an ideal, we can pick $a$ from $C$ and $b$ in $U(M)$ with the property that $ab = 1$ and $ba \neq 1$. It follows from [13, Corollary 1.32], that the submonoid of $M$ that is generated by $a$ and $b$ is a bicyclic monoid.

Since by Lemma 4.6 the bicyclic monoid does not have rational word problem, it follows from Corollary 5.5 and Lemma 4.6 that this is impossible for a monoid with rational word problem.

More generally than Theorem 7.3 for semigroups, every group that is contained in a semigroup with rational word problem is finite. This is straightforward for groups that are finitely generated subsemigroups by Theorem 5.3.

**Theorem 7.4.** Let $S$ be a semigroup with rational word problem. Then all subsemigroups of $S$ that are groups are finite.

Proof. Let $S$ be a semigroup finitely generated by $A$ with rational word problem and assume there exists an infinite subsemigroup $G$ of $S$ that is a group. Let $\mathfrak{A}$ be an asynchronous finite state automaton that decides SgWP($S$, $A$) and let $N$ be the number of states of $\mathfrak{A}$. Let $e$ be the identity of $G$, let $f$ be a string in $A^+$ with $\overline{f} = e$, and let $n$ be the length of $f$.

Since $G$ is infinite, there exist $g = \overline{w}$ in $G$ with the property that a shortest string $w'$ such that $wfw' = e$ has length greater than $(n + 1)N + n$.

The automaton accepts $(wf, f)$, therefore it has to go into a loop while reading a subword of $w'$ on the first and reading nothing on the second tape. This means that there are strings $a$, $b$ and $c$ with $|b| \geq 1$ such that $w' = abc$ and $(w'abc, f)$ is accepted by the automaton for all $i \in \mathbb{N}$, in particular $(wfabc, f)$ is accepted by $\mathfrak{A}$. Therefore

$$e = \overline{w}f\overline{w} = ge\overline{w}$$

which implies

$$g^{-1} = g^{-1}ge\overline{w} = e\overline{w} = \overline{f}\overline{w} = \overline{ac}$$

in contradiction to the choice of $w'$ as a shortest string such that $fu'$ represents of $g^{-1}$ of this form.

8. Constructions

In this section we examine natural algebraic constructions or decompositions involving semigroups and show which of them preserve rational word problem.
In particular we show that rational word problem is preserved under adding a
zero element or an identity and that a semigroup that is a disjoint union of an
infinite semigroup and a finite ideal has rational word problem if and only if the
infinite semigroup has rational word problem.

**Theorem 8.1.** Let $S$ be a finitely generated semigroup. Then the following
statements are equivalent.

1. $S$ has rational word problem.
2. $S^0$ has rational word problem.
3. $S^1$ has rational word problem.

**Proof.** We only prove the equivalence of (1) and (3), the equivalence of (1) and
(2) is a special case of Theorem 8.2.

Let $\mathfrak{A} = (Q, A, A, q_0, F, \Delta)$ be an asynchronous finite state automaton that
decides $\text{SgWP}(S, A)$.

For $S^1$ we add 1 to the set of generators. To form an automaton that decides
$\text{SgWP}(S^1, A \cup \{1\})$ we add transitions $(q, \varepsilon, 1, q)$ and $(q, 1, \varepsilon, q)$ for all $q \in Q$.

If $S^1$ has rational word problem, we remove 1 from the generating set. By
Theorem 5.3 $S$ has rational word problem.

We show that an infinite semigroup that consists of a finite ideal and an
infinite semigroup has rational word problem if and only if the infinite semigroup
has rational word problem. In particular the equivalence of (1) and (2) in
Theorem 8.1 is a special case of Theorem 8.2.

**Theorem 8.2.** Let $T = S \cup I$ be a finitely generated semigroup where $I$ is a
finite ideal of $T$ and $S$ is an infinite subsemigroup of $T$. Then $S$ has rational
word problem if and only if $T$ has rational word problem.

**Proof.** To show that $S$ has rational word problem if $T$ has rational word problem,
let $A$ be a finite generating set for $T$. The set $B = A \cap S$ generates $S$ and
therefore $S$ has rational word problem by Theorem 5.3.

Conversely, let $S$ be finitely generated by $B$ and let $\text{SgWP}(S, B)$ be rational.
Denote by $l_b$ for $b \in B$ the map that maps every element $i$ of $I$ to $bi$ and let

$$\varphi_l : B \to T_I, b \mapsto l_b,$$

where $T_I$ is the full transformation monoid of the set $I$. We denote concatenation
for $T_I$ by $\circ$ for better readability, and $\alpha \circ \beta$ for $\alpha$ and $\beta$ in $T_I$ means that
we first apply $\beta$ and then $\alpha$. The map $\varphi_l$ uniquely extends to a homomorphism
$\varphi$ from $B^*$ to $T_I$. Also note that since $T_I$ is finite, we may use it as a subset of
the set of states in a finite state automaton.

Let $\mathfrak{B} = (Q, B, B, q_0, F, \Delta)$ be an asynchronous finite state automaton de-
ciding the word problem for $S$ with respect to the finite generating set $B$. The
idea of the constructed automaton is as follows.

Given two strings $v$ and $w$ over the generating set $A$ of $T$, to decide whether
$v = w$ we can distinguish the following cases.
1. None of the two strings contain an element of $I$ and both elements lie in $S$, or
2. precisely one string contains an element of $I$, or
3. both strings contain an element of $I$ and both elements lie in $I$.

To construct an automaton that decides the word problem of $T$ we need three components that provide accepting runs for the cases (1) and (3), and for (2) we have to make sure that there is no run that accepts. For (1), we include the automaton $B$, for (3) we use a direct product of two copies of $T_I$ that memorises left-transformations of $I$ by $S$ that are read on both tapes and a direct product of two copies of $I$ to compare elements of $I$.

For a formal construction consider the automaton

$\mathfrak{A} = \langle R, A, A, r_0, G, \Gamma \rangle$,

over the alphabet $A = B \cup I$, with the set

$R = \{r_0\} \cup Q \cup T_I \times T_I \cup I \times I$,

of states and the following transition relation in which we denote by $\alpha$ and $\beta$ elements of $T_I$, by $i$ and $j$ elements of $I$, by $x$ and $y$ elements of $B$ and by $a$ and $b$ elements of $A$,

$\Gamma = \{(r_0, \varepsilon, \varepsilon, q_0)\} \cup \{(r_0, \varepsilon, \varepsilon, (id, id))\} \\
\cup \Delta \\
\cup \{((\alpha, \beta), x, \varepsilon, (\alpha \circ (\varphi x), \beta)) \mid x \in B\} \\
\cup \{((\alpha, \beta), \varepsilon, y, (\alpha, \beta \circ (\varphi y))) \mid y \in B\} \\
\cup \{((\alpha, \beta), a, b, (\alpha a, \beta b) \mid a, b \in I\} \\
\cup \{((i, j), a, \varepsilon, (ia, j)) \mid a \in A\} \\
\cup \{((i, j), \varepsilon, b, (i, jb)) \mid b \in A\}.$

The set $G$ of accept states is $F \cup \{(i, i) \mid i \in I\}$.

To prove correctness we show that $(v, w)$ is accepted by $\mathfrak{A}$ if and only if $\overline{v} = \overline{w}$.

Assume that a pair $(v, w)$ is accepted by $\mathfrak{A}$. This means there is an accepting computation $\gamma$ of $\mathfrak{A}$ on $(v, w)$. If the computation has the form

$\gamma : r_0 \xrightarrow{(x, x)} q_0 \xrightarrow{(v, w)} q \in F$,

if we are in case (1). By assumption $\mathfrak{B}$ decides $\text{SgWP}(S, B)$, and therefore $\overline{v} = \overline{w}$.

If $\gamma$ has the form

$\gamma : r_0 \xrightarrow{(x, x)} (id, id) \xrightarrow{(v, w)} (i, i)$

for some $i$ in $I$, we are in case (3) and by construction $\overline{v} = \overline{w}$ because they represent equal elements of $I$. 

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Conversely assume $v = w$ in $T$. In case (1) there is an accepting computation on $\mathcal{B}$ by assumption, and thus an accepting computation on $\mathcal{A}$ exists by construction. In case (3) we can decompose $v$ and $w$ as $v = v_1 a v_2$ and $w = w_1 b w_2$, where $v_1$ and $w_1$ are elements of $B^*$, $a$ and $b$ are elements of $I$ and $v_2$ and $w_2$ are elements of $A^*$. By construction of $\mathcal{A}$ the following computation of $\mathcal{A}$ on $(v, w)$ exists:

$$
\gamma : r_0 \xrightarrow{(e,e)} (id, id) \xrightarrow{(v_1, w_1)} ((\varphi_l v_1, \varphi_l w_1)) \xrightarrow{(a,b)} ((\varphi_l v_1) a, (\varphi_l w_1) b) \\
\xrightarrow{(v_2, w_2)} ((\varphi_l v_1) a) v_2, ((\varphi_l w_1) b) w_2.
$$

What is left to show is that $(\varphi_r v_2) ((\varphi_l v_1)) a = (\varphi_r w_2) ((\varphi_l w_1)) b$.

$$(\varphi_l v_1) a v_2 = v_1 a v_2 = v_1 w_2 = v = w_1 b v_2 = w_1 b w_2 = ((\varphi_l w_1) b) w_2$$

\hfill \Box

9. Products

In this section we examine products of semigroups with rational word problem. The direct product of two semigroups with rational word problem does not have rational word problem in general, even if we assume the direct product to be finitely generated. This can most easily be seen by considering $\mathbb{N}_0 \times \mathbb{N}_0$ which does not have rational word problem by Lemma 4.5.

It follows from results in [16, Section 2] that for two finitely generated semigroups $S$ and $T$ the direct product $S \times T$ of $S$ and $T$ is finitely generated if and only if one of the following conditions is true.

1. $S$ and $T$ are finite,
2. $S$ is finite and $S^2 = S$,
3. $T$ is finite and $T^2 = T$, or
4. $S^2 = S$ and $T^2 = T$.

Following an example from [16, Remark 7.5] we consider a finitely generated infinite semigroup $S$ with rational word problem that has the property $S^2 = S$, effectively enabling us to form the finitely generated infinite semigroup $S \times S$.

Example 9.1. Let $S$ be given by the presentation

$$S = \langle a, b \mid a^2 = a, ba = b \rangle.$$

The semigroup $S$ is infinite, finitely generated and has rational word problem. The direct product $S \times S$ is finitely generated but does not have rational word problem.
Figure 4: Asynchronous finite state automaton $\mathfrak{A}$ that decides $\text{SgWP}(S, \{a, b\})$ for $S = S\langle \{a, b\} \mid a^2 = a, ba = b \rangle$.

There is an easily described set of representatives of elements of $S$ consisting of non-empty strings of the form $a^\alpha b^\beta$ for $\alpha \in \{0, 1\}$ and $\beta \in \mathbb{N}$.

Consider the automaton $\mathfrak{A}$ depicted in Figure 4. We prove that two non-empty strings $v$ and $w$ over $\{a, b\}$ are accepted by $\mathfrak{A}$ if and only if $v = w$.

Let $v$ and $w$ be two non-empty strings such that $v = w$. Then either both begin with $a$ or they both begin with $b$. In either case the automaton ends up in a final state after reading the first character of both strings. After that, both strings can contain any number of $a$s as long as there is an equal number of $b$s in both strings. The automaton can just skip occurrences of $a$ until it reaches a $b$ on each tape which it can read then.

Conversely, assume that the pair $(v, w)$ of strings are accepted by the automaton. Then they begin with the same letter and contain an equal number of $b$s, since otherwise they could not be read at all by the automaton. Thus $v = w$.

Now consider $T = S \times S$. Following [16, Corollary 2.11], the resulting semigroup $T$ is finitely generated and finitely presented. A generating set is for example

$$B = \{(a, a), (a, b), (b, a), (b, b)\}.$$  

The elements $(b^2, b)$ and $(b, b^2)$ generate a free commutative semigroup of rank 2 in $T$ and therefore $T$ does not have rational word problem by Theorem 5.3.

The following theorems characterise direct products of semigroups that have rational word problem. Given a direct product of two semigroups with rational word problem, it follows that the factors have rational word problem. Conversely, the direct product of two semigroups with rational word problem gives a semigroup with rational word problem only if the direct product is finitely generated and one of the factors is finite.

**Theorem 9.2.** Let $S$ and $T$ be semigroups. If $S \times T$ is finitely generated, and has rational word problem, then $S$ and $T$ are finitely generated and have rational word problem.

**Proof.** It is sufficient to prove the statement for $S$. Assume $S \times T$ to be generated by the finite set $C$. Applying the projection

$$\pi_S : S \times T \rightarrow S, (s, t) \mapsto s,$$

to $C$ gives the finite generating set $\pi_S(C)$ for $S$. 

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Assume that \( S \times T \) has rational word problem and that
\[
\mathfrak{A} = \langle Q, A, A, q_0, F, \Delta \rangle
\]
is an asynchronous finite state automaton that decides \( \text{SgWP}(S \times T, A) \). The following automaton then decides \( \text{SgWP}(S, \pi_S(A)) \).
\[
\mathfrak{A}' = \langle Q, \pi_S(A), \pi_S(A), q_0, F, \Delta' \rangle,
\]
where
\[
\Delta' = \{(p, \pi_S a, \pi_S b, q) \mid (p, a, b, q) \in \Delta \}.
\]

Lemma 9.3. Let \( S \) be a finite semigroup and \( T \) be a finitely generated semigroup with rational word problem. If \( S \times T \) is finitely generated, then \( S \times T \) has rational word problem.

Proof. Let \( C \) be a finite generating set for \( S \times T \). We denote by \( \pi_S \) and \( \pi_T \) the projections from \( S \times T \) onto \( S \) and \( T \) respectively.

Since \( T \) has rational word problem there is an asynchronous finite state automaton
\[
\mathfrak{B} = \langle R, \pi_T(C), \pi_T(C), r_0, G, \Gamma \rangle
\]
that decides \( \text{SgWP}(T, \pi_T(C)) \).

The automaton \( \mathfrak{C} \) that decides \( \text{SgWP}(S \times T, C) \) can then be given as follows.
\[
\mathfrak{C} = \langle S^1 \times S^1 \times R, C, C, (1, 1, r_0), H, \Pi \rangle,
\]
where
\[
H = \{(s, s, g) \mid s \in S, g \in G\},
\]
and the transition relation \( \Pi \) is given as
\[
\Pi = \{(s, t, q), c, d, (s \cdot \pi_S(c), t \cdot \pi_S(d), r) \mid (q, \pi_T(c), \pi_T(d), r) \in \Gamma\}
\]
\[
\cup \{(s, t, q), \varepsilon, d, (s, t \cdot \pi_S(d), r) \mid (q, \pi_T(c), \pi_T(d), r) \in \Gamma\}
\]
\[
\cup \{(s, t, q), c, \varepsilon, (s \cdot \pi_S(c), t, r) \mid (q, \pi_T(c), \pi_T(d), r) \in \Gamma\}
\]
\[
\cup \{(s, t, q), \varepsilon, \varepsilon, (s, t, r) \mid (q, \pi_T(c), \pi_T(d), r) \in \Gamma\}.
\]

We show that \((v, w)\) is accepted by \( \mathfrak{C} \) if and only if \( \overline{v} = \overline{w} \). Let \( \mathfrak{C} \) accept the pair \((v, w)\) in \( C^+ \). Then, by construction, there exists an accepting computation on \( \mathfrak{B} \), thus \( \pi_T(\overline{v}) = \pi_T(\overline{w}) \). Also by construction \( \pi_S(\overline{v}) = \pi_S(\overline{w}) \).

Now let \( \overline{v} = \overline{w} \). In particular \( \pi_T(\overline{v}) = \pi_T(\overline{w}) \), and hence there exists an accepting run of \( \mathfrak{B} \). One can immediately find a run on \( \mathfrak{C} \) by lifting this run from \( \mathfrak{B} \) to \( \mathfrak{C} \). Since also \( \pi_S(\overline{v}) = \pi_S(\overline{w}) \) the lifted run is accepting.

Lemma 9.4. Let \( S \) and \( T \) be finitely generated infinite semigroups with rational word problem. Then \( S \times T \) contains a free commutative semigroup of rank 2.
Proof. By Theorem 7.1 there are elements \( s \) in \( S \) and \( t \) in \( T \) that generate infinite monogenic subsemigroups in \( S \) and \( T \) respectively. The elements \((s^2, t)\) and \((s, t^2)\) generate a free commutative semigroup of rank 2 in \( S \times T \). Theorem 5.3 now implies that \( S \times T \) cannot have rational word problem. \( \square \)

We summarise the above in the following theorem.

**Theorem 9.5.** Let \( S \) and \( T \) be two semigroups such that \( S \times T \) is finitely generated. Then \( S \times T \) has rational word problem if and only if \( S \) and \( T \) have rational word problem and at least one of \( S \) or \( T \) is finite.

Proof. If \( S \times T \) has rational word problem, then Theorem 9.2 implies that \( S \) and \( T \) have rational word problem. If both \( S \) and \( T \) are infinite, then by Lemma 9.4 it follows that \( S \times T \) does not have rational word problem.

Conversely, if both \( S \) and \( T \) are finite, then the direct product \( S \times T \) is finite and therefore has rational word problem. If \( S \) is finite and \( T \) is infinite or vice versa, we use Lemma 9.3 \( \square \)

Inductively it follows that any finite direct product \( S_1 \times \cdots \times S_n \) of semigroups has rational word problem if and only if it is finitely generated and there is at most one \( S_i \) that is infinite and has rational word problem.

In general the monoid free product of two monoids with rational word problem does not have rational word problem. For consider the cyclic group \( C_2 \). The monoid free product \( C_2 \ast C_2 \) is an infinite group, but infinite groups do not have rational word problem by Theorem 6.1. The situation is different for semigroup free products.

**Theorem 9.6.** Let \( S \) and \( T \) be two semigroups generated by finite sets \( A \) and \( B \) respectively. The semigroup free product \( S \ast T \) has rational word problem if and only if \( S \) and \( T \) have rational word problem.

Proof. Let \( S \) and \( T \) be semigroups with rational word problem and let \( \mathfrak{A} \) be an asynchronous automaton that decides \( S_{\text{gWP}}(S, A) \), and \( \mathfrak{B} \) be an asynchronous automaton that decides \( S_{\text{gWP}}(T, B) \). An automaton that decides \( S_{\text{gWP}}(S \ast T, A \cup B) \) can be constructed by using both \( \mathfrak{A} \) and \( \mathfrak{B} \) and adding a new initial state \( q_0 \) and \((\varepsilon, \varepsilon)\) transitions from \( q_0 \) to the initial states of \( \mathfrak{A} \) and \( \mathfrak{B} \) as well as from the accept states of both automata to \( q_0 \).

The converse follows directly from Theorem 5.3 \( \square \)

Another product construction that is possible for semigroups is the zero union of two semigroups. We define the zero union as follows.

**Definition 9.7.** Let \( U \) be a semigroup with zero element 0. If there exist subsemigroups \( S \) and \( T \) of \( U \) such that \( S \cap T = \{0\} \) and \( U = S \cup T \cup \{0\} \) and \( st = 0 = ts \) for all \( s \in S \) and \( t \in T \), then \( U \) is a zero union of \( S \) and \( T \), denoted by \( S \cup_0 T \).
Note that \( S \cup_0 T \) is finitely generated if and only if \( S \) and \( T \) are finitely generated. A generating set for \( S \) can be obtained from a generating set \( C \) for \( S \cup_0 T \) by intersecting \( C \) with \( S \), a generating set for \( T \) can be obtained by intersecting \( C \) with \( T \). Given generating sets for \( S \) and \( T \) the union of those generating sets together with the zero element gives a generating set for \( S \cup_0 T \). Rational word problem is preserved under zero union.

**Theorem 9.8.** Let \( U \) be a finitely generated semigroup that is a zero union of two subsemigroups \( S \) and \( T \). Then \( U \) has rational word problem if and only if \( S \) and \( T \) have rational word problem.

**Proof.** If \( U = S \cup_0 T \) has rational word problem, then \( S \) and \( T \) are finitely generated subsemigroups of \( U \) and therefore have rational word problem by Theorem 5.3

Conversely, let \( C \) be a generating set for \( U \). Let \( A = C \cap S \) and let \( B = C \cap T \) be generating sets for \( S \) and \( T \) respectively and assume that \( \text{SgWP}(S, A) \) and \( \text{SgWP}(T, B) \) are rational.

Additionally we observe that the set
\[
Z = \{ v \in C^+ \mid v = 0 \},
\]
is regular by Proposition 2.9 and hence \( Z \times Z \) is rational by Proposition 2.7. We show that
\[
\text{SgWP}(S \cup_0 T, C) = \text{SgWP}(S, A) \cup \text{SgWP}(T, B) \cup (Z \times Z).
\]

Let \((v, w)\) be in \( \text{SgWP}(S \cup_0 T, C) \), which is the case if and only if \( v = w \) and we distinguish three cases

1. \( v \) is a non-zero element of \( S \),
2. \( v \) is a non-zero element of \( T \), or
3. \( v \) is zero.

In the first two cases \((v, w)\) is contained in the right hand side, because it is either contained in \( \text{SgWP}(S, A) \) or in \( \text{SgWP}(T, B) \) respectively. A string \( v \) over \( C \) represents the zero element of \( S \cup_0 T \) if and only if it is contained in \( Z \), thus if \( v = 0 \) then \((v, w)\) is contained in \( Z \times Z \).

\[ \square \]

**10. Conclusion and Outlook**

We have introduced a natural class of finitely generated semigroups with the property that the word problem is decidable by an asynchronous finite state automaton. We examined the behaviour of this property under a few constructions and gave some basic structural properties of semigroups with rational word problem. We were not yet able to achieve a full characterisation of all semigroups with rational word problem.
Further research will be aimed at finding such a characterisation. A first step is to consider a notion that is ubiquitous in semigroup theory: Green’s relations. Do semigroups with rational word problem have rational Green’s relations? How many $\mathcal{R}$- or $\mathcal{L}$-classes can a semigroup with rational word problem have? Are all $\mathcal{H}$-classes of such semigroups finite?

Once one has characterised all semigroups with rational word problem, one also has classified all rational congruences. This is because the word problem of a semigroup $S$ finitely generated by a subset $A$ is the kernel of the canonical map $\varepsilon: A^+ \to S$, and every rational congruence is the kernel of such a map. An open question that is tied to this is whether rational equivalence relations have regular cross sections. A regular cross section of a rational equivalence relation is regular language of unique representatives for the equivalence classes. This problem was investigated in [11], and to this day has not been solved. If the answer is positive, semigroups with rational word problem in our sense are rational in the sense of Sakarovitch [5, 6], and therefore a semigroup has rational word problem if and only if it is rational.

We remark that it is undecidable whether a given finitely generated semigroup has rational word problem. For if it was decidable, then it would be decidable whether a given finitely generated semigroup was trivial. One question to pursue is to find an algorithm that, given a finite presentation of a monoid, finds an automaton that decides the word problem if it exists, and proves its correctness, as it is done for automatic groups in [17].

Extending the notion of rational word problem to intersections of rational relations is another natural direction of research, and additionally more automaton models should be considered. This will provide a more complete picture of the complexity of word problems that arise.

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