A Babuška-Aziz type proof of the circumradius condition

Kenta Kobayashi · Takuya Tsuchiya

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Abstract In this paper the error of polynomial interpolation of degree 1 on triangles is considered. The circumradius condition, which is more general than the maximum angle condition, is explained and proved by the technique given by Babuška-Aziz.

Keywords interpolation error · finite element methods · the maximum angle condition · the circumradius condition

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1 Introduction — the circumradius condition

Let \( \mathcal{P}_1 \) be the set of polynomials whose degree are at most 1. Let \( K \subset \mathbb{R}^2 \) be any triangle with apexes \( x_i, i = 1, 2, 3 \). Then, for a function \( v \in W^{2,p}(K) \) the \( \mathcal{P}_1 \) interpolation \( I_h v \) on \( K \) is defined by \( (I_h v)(x_i) = v(x_i) \). Note that the interpolation \( I_h v \) is well-defined for \( v \in W^{2,p}(K) \) since \( W^{2,p}(K) \) is imbedded to \( C(\bar{K}) \) for any \( p \in [1, \infty] \). Analyzing the error

\[ \|v - I_h v\|_{1,p,K} \]

is particularly important for the error analysis of finite element methods. There is a long history of research into this error bound. We present some well-known results.

Let \( h_K \) be the diameter (or the length of the longest edge) of \( K \), and \( \rho_K \) be the radius of the inscribed circle of \( K \).

\footnote{For the critical imbedding \( W^{2,1}(K) \subset C(\bar{K}) \), see \cite{7} p.300.}
– The minimum angle condition. Zlámal [9] (1968).
Let \( \theta_0 \), \( 0 < \theta_0 < \pi/3 \) be a constant. If any angle \( \theta \) of \( K \) satisfies \( \theta \geq \theta_0 \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\theta_0) \) independent of \( h_K \) such that
\[
\| v - I_h v \|_{1,2,K} \leq C h_K \| v \|_{2,2,K}, \quad \forall v \in H^2(K).
\]

– The regularity condition. see, for example, Ciarlet [2].
Let \( \sigma > 0 \) be a constant. If \( h_K / \rho_K \leq \sigma \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\sigma) \) independent of \( h_K \) such that
\[
\| v - I_h v \|_{1,2,K} \leq C h_K \| v \|_{2,2,K}, \quad \forall v \in H^2(K).
\]

– The maximum angle condition. Babuška-Aziz [1], Jamet [4] (1976).
Let \( \theta_1 \), \( 2\pi/3 \leq \theta_1 < \pi \) be a constant. If any angle \( \theta \) of \( K \) satisfies \( \theta \leq \theta_1 \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\theta_1) \) independent of \( h_K \) such that
\[
\| v - I_h v \|_{1,2,K} \leq C h_K \| v \|_{2,2,K}, \quad \forall v \in H^2(K).
\]

It is easy to show that the minimum angle condition is equivalent to the regularity condition [2] Exercise 3.1.3, p130. Liu and Kikuchi presented an explicit form of the constant \( C \) in [8].

Inspired by Liu-Kikuchi’s result, Kobayashi obtained the following epoch-making result [5], [6]. Let \( A \), \( B \) and \( C \) be the lengths of the three edges of \( K \) and \( S \) be the area of \( K \).

– Kobayashi’s formula. Kobayashi [5], [6]
Define the constant \( C(K) \) by
\[
C(K) := \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2 + B^2 + C^2}{30} \cdot \frac{S^2}{5} \left( \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} \right)},
\]
then the following holds:
\[
\| v - I_h v \|_{1,2,K} \leq C(K) \| v \|_{2,2,K}, \quad \forall v \in H^2(K).
\]

Let \( R_K \) be the radius of the circumcircle of \( K \). Using the formula \( R_K = ABC/4S \), we can show that \( C(K) < R_K \) and obtain a corollary of Kobayashi’s formula.

– A corollary of Kobayashi’s formula
For any triangle \( K \subset \mathbb{R}^2 \), the following estimate holds:
\[
\| v - I_h v \|_{1,2,K} \leq R_K \| v \|_{2,2,K}, \quad \forall v \in H^2(K).
\]

This corollary demonstrates that even if the minimum angle is very small or the maximum angle is very close to \( \pi \), the error \( \| v - I_h v \|_{1,2,K} \) converges to 0 if \( R_K \) converges to 0. For example, consider the isosceles triangle \( K \) depicted in Figure 1. If \( 0 < h < 1 \) and \( \alpha > 1 \), then \( h^\alpha < h \) and the circumradius of \( K \) is \( h^\alpha/2 + h^{2-\alpha}/8 \). Hence, if \( 1 < \alpha < 2 \) and \( \| v \|_{2,2,K} \) is bounded, the error \( \| v - I_h v \|_{1,2,K} \) converges to 0 even though the maximum angle is tending to \( \pi \) as \( h \to 0 \), although the convergence rate becomes inferior.
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Fig. 1 An example of a triangle which violates the maximum angle condition but satisfies $R_K \to 0$ as $h \to 0$.

Suppose that \( \{ \tau_h \}_{h>0} \) is a series of triangulations of a convex polygonal domain \( \Omega \subset \mathbb{R}^2 \) such that
\[
\lim_{h \to 0} \max_{K \in \tau_h} R_K = 0. \tag{1}
\]

Let \( S_{\tau_h} \) be the set of all piecewise linear functions on \( \tau_h \), defined by
\[
S_{\tau_h} := \{ v_h \in H^1_0(\Omega) \cap C(K) \mid v_h|_K \in \mathcal{P}_1, \forall K \in \tau_h \}.
\]

Let \( u_h \) be the piecewise linear finite element solution on the triangulation \( \tau_h \) of the Poisson problem
\[
-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\]
for a given \( f \in L^2(\Omega) \). Then, the well-known Céa’s lemma \([2, \text{ Theorem 2.4.1}]\) claims that, for the exact solution \( u \),
\[
\| u - u_h \|_{1,2,\Omega} \leq \left(1 + C_P^2\right)^{1/2} \| u - u_h \|_{1,2,\Omega} \leq \left(1 + C_P^2\right)^{1/2} \inf_{v_h \in S_{\tau_h}} | u - v_h |_{1,2,\Omega}
\]
\[
\leq \left(1 + C_P^2\right)^{1/2} \| u - I_h u \|_{1,2,\Omega} \leq \left(1 + C_P^2\right)^{1/2} \left( \max_{K \in \tau_h} R_K \right) | u |_{2,2,\Omega},
\]
where \( C_P \) is the Poincaré constant for \( \Omega \).\(^2\)

Thus, the discretization error \( \| u - u_h \|_{1,2,\Omega} \) is bounded by the interpolation error \( \| u - I_h u \|_{1,2,\Omega} \), and the finite element solutions \( \{ u_h \} \) converge to \( u \) even if the maximum angle condition is violated (see the example of triangulation in Fig. 2). Therefore, to obtain the error estimate of \( I_h v \) or to ensure that the finite element solutions converge to the exact solution, \( \max_{K \in \tau_h} R_K \) is more important than the minimum or maximum angles.

A drawback of Kobayashi’s formula is that its proof is very long and needs the assistance of validated numerical computations. However, in many cases the following estimation is good enough for the error analysis of finite element methods:

- **The circumradius condition**

  For an arbitrary triangle \( K \) with \( R_K \leq 1 \), there exists a constant \( C_P \) independent of \( K \) such that the following estimate holds:
\[
\| v - I_h v \|_{1,p,K} \leq C_P R_K | v |_{2,p,K}, \quad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty.
\]

\(^2\) Poincaré’s inequality claims that there exists a constant \( C_P > 0 \) which is called the Poincaré constant such that \( \| v \|_{0,2,\Omega} \leq C_P | v |_{1,2,\Omega} \) for \( v \in H^1_0(\Omega) \). See \([3]\).
2 Preliminary and basic lemmas

Let $K \subset \mathbb{R}^2$ be any triangle. Partial derivatives of a function $u$ with respect to $x, y$ are denoted by

$$u_x := \frac{\partial u}{\partial x}, \quad u_{xx} := \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} := \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.} \quad (x, y) \in K.$$  

The usual Lebesgue and Sobolev spaces on $K$ are denoted by $L^p(K), W^{k,p}(K), k = 1, 2, p \in [1, \infty].$ As usual, $W^{k,2}(K)$ are denoted by $H^k(K).$ We denote their norms and semi-norms by $|u|_{0,p,K} := \int_K |u|^p dx$ for $p \in [1, \infty), \, |u|_{0,\infty,K} := \text{ess sup}_K |u|$ and

$$|u|_{1,p,K} := |u|_{0,p,K}^{p} + |u|_{0,\infty,K}^{p}, \quad \|u\|_{1,p,K} := |u|_{0,p,K}^{p} + |u|_{1,p,K}^{p},$$

$$|u|_{2,p,K}^{p} := |u|_{0,p,K}^{p} + |u|_{0,\infty,K}^{p} + 2|u|_{1,p,K}^{p},$$

$$|u|_{1,\infty,K} := \max\{ |u|_{0,\infty,K}, |u|_{0,\infty,K} |u|_{1,\infty,K} := \max\{ |u|_{0,\infty,K}, |u|_{1,\infty,K} \},$$

$$|u|_{2,\infty,K} := \max\{ |u|_{0,\infty,K}, |u|_{0,\infty,K}, |u|_{1,\infty,K} \}.$$  

Throughout this paper and \cite{1}, the most important tool is the \textbf{orthogonal expansion-contraction} (OEC) transformation $F_{\alpha, \beta} : K \rightarrow \mathbb{R}^2$ defined, for $\alpha, \beta > 0$, as

$$F_{\alpha, \beta}(x, y) := (\alpha x, \beta y), \quad (x, y) \in K.$$
Define $K_{\alpha,\beta} := F_{\alpha,\beta}(K)$ and take the arbitrary function $v \in W^{2,p}(K_{\alpha,\beta})$. Then, defining $u := v \circ F_{\alpha,\beta} \in W^{2,p}(K)$, we have, for $p \in [1,\infty)$,

\[
\frac{|v|^p}{|v|_{0,p,K}} = \frac{\beta^p|u_x|^p_{0,p,K} + \alpha^p|u_y|^p_{0,p,K}}{\alpha^p \beta^p|u|^p_{0,p,K}},
\]

(2)

\[
\frac{|v|^p}{|v|_{0,p,K}} = \frac{\beta^p|u_x|^p_{0,p,K} + \alpha^p|u_y|^p_{0,p,K} + 2|u|^p_{0,p,K}}{\alpha^p \beta^p|u|^p_{0,p,K}},
\]

(3)

\[
\frac{|v|^p}{|v|_{0,p,K}} = \frac{\beta^p|u_x|^p_{0,p,K} + \alpha^p|u_y|^p_{0,p,K} + 2|u|^p_{0,p,K}}{\beta^p|u|^p_{0,p,K} + \alpha^p|u|^p_{0,p,K}},
\]

(4)

and

\[
\frac{|v|}{|v|_{0,p,K}} = \max \left\{ \beta|u_x|_{0,\infty,K}, \alpha|u_y|_{0,\infty,K} \right\},
\]

(5)

\[
\frac{|v|}{|v|_{0,p,K}} = \max \left\{ \frac{\beta}{\alpha}|u_x|_{0,\infty,K}, \frac{\alpha}{\beta}|u_y|_{0,\infty,K} \right\},
\]

(6)

Let $\hat{K}$ be the reference triangle whose apexes are $(0,0), (1,0), (0,1)$. Take $\alpha$, $\beta$ so that

\[
\alpha^2 + \beta^2 = 2, \quad 0 < \beta \leq 1 \leq \alpha < \sqrt{2},
\]

and define $K_{\alpha,\beta} := F_{\alpha,\beta}(\hat{K})$. Note that $\hat{K} = K_{1,1}$ and the circumradii of $\hat{K}$ and $K_{\alpha,\beta}$ are $1/\sqrt{2}$.

For $p \in [1,\infty]$, define $\Xi^1_p(K_{\alpha,\beta}), \Xi^2_p(K_{\alpha,\beta}), \mathcal{F}_p(K_{\alpha,\beta})$ by

\[
\Xi^1_p(K_{\alpha,\beta}) := \left\{ v \in W^{1,p}(K_{\alpha,\beta}) \left| \int_0^\alpha v(x,0)dx = 0 \right\},
\]

\[
\Xi^2_p(K_{\alpha,\beta}) := \left\{ v \in W^{1,p}(K_{\alpha,\beta}) \left| \int_0^\beta v(0,y)dy = 0 \right\},
\]

\[
\mathcal{F}_p(K_{\alpha,\beta}) := \left\{ v \in W^{2,p}(K_{\alpha,\beta}) \left| v(0,0) = v(0,\alpha) = v(0,\beta) = 0 \right\}.
\]

The following lemma is an extension of [1] Lemma 2.1] to any $p \in [1,\infty]$. Although the proof is very similar, we include it here for the readers’ convenience.

**Lemma 1** For $p \in [1,\infty]$, define the constants $A_{p1}, A_{p2}$ by

\[
A_{p1} := \inf_{w \in \Xi^1_p(\hat{K})} \frac{|w|_{1,p,\hat{K}}}{|w|_{0,p,\hat{K}}}, \quad A_{p2} := \inf_{z \in \Xi^2_p(\hat{K})} \frac{|z|_{1,p,\hat{K}}}{|z|_{0,p,\hat{K}}},
\]

Then, we have $A_p := A_{p1} = A_{p2} > 0$. 
Proof The equality $A_{p1} = A_{p2}$ is clear from the symmetry of $\hat{K}$. The proof of $A_{p1} > 0$ is by contradiction. Suppose that $A_{p1} = 0$. Then, there exists $\{w_k\}_{k=1}^{\infty} \subset \mathcal{X}_{p}^{1}$ such that

$$|w_k|_{0,p,\hat{K}} = 1, \quad \lim_{k \to \infty} |w_k|_{1,p,\hat{K}} = 0.$$ 

Let $\mathcal{P}_0 = \mathbb{R}$ be the set of polynomials of degree 0. By [2] Theorem 3.1.1, there is a constant $C(\hat{K}, p)$ such that

$$\inf_{q \in \mathcal{P}_0} \|v + q\|_{1,p,\hat{K}} \leq C(\hat{K}, p)\|v\|_{1,p,\hat{K}}, \quad \forall v \in W^{1,p}(\hat{K}).$$

Therefore, there exists $\{q_k\} \subset \mathcal{P}_0$ such that

$$\inf_{q \in \mathcal{P}_0} \|w_k + q\|_{1,p,\hat{K}} \leq \inf_{q \in \mathcal{P}_0} \|w_k + q_k\|_{1,p,\hat{K}} \leq \inf_{q \in \mathcal{P}_0} \|w_k + q\|_{1,p,\hat{K}} + \frac{1}{k},$$

$$\lim_{k \to \infty} |w_k + q_k|_{1,p,\hat{K}} \leq \lim_{k \to \infty} \left( C(\hat{K}, p)\|w_k\|_{1,p,\hat{K}} + \frac{1}{k} \right) = 0.$$ 

Since the sequence $\{w_k\} \subset W^{1,p}(\hat{K})$ is bounded, $\{q_k\} \subset \mathcal{P}_0 = \mathbb{R}$ is also bounded. Thus, there exists a subsequence $\{q_{k_i}\}$ such that $q_{k_i}$ converges to $\bar{q} \in \mathcal{P}_0$. In particular, we have

$$\lim_{k_i \to \infty} \|w_{k_i} + \bar{q}\|_{1,p,\hat{K}} = 0.$$ 

Let $\gamma : W^{1,p}(\hat{K}) \to W^{1-1/p,p}(\Gamma)$ be the trace operator, where $\Gamma$ is the edge of $\hat{K}$ connecting $(1,0)$ and $(0,0)$. The boundedness of $\gamma$ and the inclusion $W^{1-1/p,p}(\Gamma) \subset L^{1}(\Gamma)$ yield

$$0 = \lim_{k_i \to \infty} \int_{\Gamma} \gamma(w_{k_i} + \bar{q})ds = \int_{\Gamma} \bar{q}ds,$$

since $w_{k_i} \in \mathcal{X}_{p}^{1}$. Hence, we conclude that $\bar{q} = 0$ and $\lim_{k_i \to \infty} \|w_{k_i}\|_{1,p,\hat{K}} = 0$, which contradicts $\lim_{k \to \infty} \|w_k|_{0,p,\hat{K}} = 1.$ \hfill $\Box$

Remark 1 The constant $1/A_2$ is called the Babuška-Aziz constant. According to [8] pp40-41, the Babuška-Aziz constant $1/A_2$ is the maximum positive solution of the equation $1/x + \tan(1/x) = 0$ and its approximated value is $1/A_2 \approx 0.49291$.

Lemma 2 Define the constants $A_{p1}(K_{\alpha, \beta}), A_{p2}(K_{\alpha, \beta})$ by

$$A_{p1}(K_{\alpha, \beta}) := \inf_{w \in \mathcal{X}_{p}^{1}(K_{\alpha, \beta})} \frac{|w|_{1,p,K_{\alpha, \beta}}}{|w|_{0,p,K_{\alpha, \beta}}}, \quad A_{p2}(K_{\alpha, \beta}) := \inf_{z \in \mathcal{X}_{p}^{1}(K_{\alpha, \beta})} \frac{|z|_{1,p,K_{\alpha, \beta}}}{|z|_{0,p,K_{\alpha, \beta}}}.$$ 

Then $A_{p1}(K_{\alpha, \beta}) \geq A_p/\sqrt{2}, \quad A_{p2}(K_{\alpha, \beta}) \geq A_p/\sqrt{2}.$

\footnote{For the trace operator, see, for example, [8]. If $p = \infty$, the boundedness of $\gamma$ is obvious since $W^{1,\infty}(\Omega) = C^{0,1}()$ and $W^{1,\infty}(\Gamma) = C^{0,1}(\Gamma)$.}
Proof Suppose that \( p \in [1, \infty) \). Then, (2) yields, for any \( v \in \mathcal{Z}_p(K_{\alpha, \beta}) \), \( i = 1, 2, \)

\[
\frac{|v|_{2,p,K_{\alpha, \beta}}}{|v|_{1,p,K_{\alpha, \beta}}} = \frac{|u_{i,0,p,K}| + \frac{\beta}{p} |u_{y,0,p,K}|}{\alpha |u|_{0,p,K}} \geq \frac{|u_{i,0,p,K}| + |u_{y,0,p,K}|}{2^{p/2} |u|_{0,p,K}} \geq A_p^p 2^{p/2}.
\]

Taking the infimum of the left-hand side with respect to \( v \in \mathcal{Z}_p(K_{\alpha, \beta}) \), we obtain \( A_p(K_{\alpha, \beta}) \geq A_p/\sqrt{2} \). The case \( p = \infty \) is similarly proved.

The proof of the following lemma is very similar to that of Babuška-Aziz’s [1] Lemma 2.2. We present it here for the readers’ convenience.

Lemma 3 Define the constant \( B_p(K_{\alpha, \beta}) \) by

\[
B_p(K_{\alpha, \beta}) := \inf_{v \in \mathcal{Z}_p(K_{\alpha, \beta})} \frac{|v|_{2,p,K_{\alpha, \beta}}}{|v|_{1,p,K_{\alpha, \beta}}}
\]

Then \( B_p(K_{\alpha, \beta}) \geq A_p/\sqrt{2} \).

Proof Let \( 1 \leq p < \infty \). Take any \( v \in \mathcal{Z}_p(K_{\alpha, \beta}) \) and define \( u := v \circ F_{\alpha, \beta} \in \mathcal{Z}_p(\hat{K}) \). It follows from (4) that

\[
\frac{|v|_{2,p,K_{\alpha, \beta}}}{|v|_{1,p,K_{\alpha, \beta}}} = \frac{|u_{x,0,p,K}| + \frac{\beta}{p} |u_{y,0,p,K}|}{\alpha |u|_{0,p,K}} \geq \frac{|u_{x,0,p,K}| + |u_{y,0,p,K}|}{2^{p/2} |u|_{0,p,K}} \geq A_p^p 2^{p/2}.
\]

Setting \( w := u_{x,0} \), we notice \( w \in \mathcal{Z}_p(\hat{K}) \), and

\[
|u_{x,0,p,K}| + |u_{y,0,p,K}| \geq A_p^p |w|_{1,p,K} = A_p^p |u_{x,0,p,K}|
\]

by Lemma 3 Similarly, setting \( z := u_{y,0} \), we have \( z \in \mathcal{Z}_p(\hat{K}) \) and hence

\[
|u_{y,0,p,K}| + |u_{x,0,p,K}| \geq A_p^p |z|_{1,p,K} = A_p^p |u_{y,0,p,K}|
\]

Therefore,

\[
\frac{|v|_{2,p,K_{\alpha, \beta}}}{|v|_{1,p,K_{\alpha, \beta}}} \geq \frac{A_p^p (|u_{x,0,p,K}| + \frac{\beta}{p} |u_{y,0,p,K}|)}{2^{p/2} (|u_{x,0,p,K}| + \frac{\beta}{p} |u_{y,0,p,K}|)} = A_p^p 2^{p/2}.
\]

Taking the infimum with respect to \( v \), we conclude \( B_p(K_{\alpha, \beta}) \geq A_p/\sqrt{2} \). The proof of the case \( p = \infty \) is similar.

The following lemma is an extension of [1] Lemma 2.3] to any \( p \in [1, \infty] \). As the proof is relatively simple, we omit the details.
Lemma 4 Define the constant $D_p$ by

$$D_p := \inf_{u \in \mathcal{F}(\mathcal{K})} |u|_{2,p,\mathcal{K}}^p / |u|_{0,p,\mathcal{K}}$$

Then $D_p > 0$.

Lemma 5 Define the constant $D_p(K_{\alpha,\beta})$ by

$$D_p(K_{\alpha,\beta}) := \inf_{v \in \mathcal{F}(K_{\alpha,\beta})} |v|_{2,p,K_{\alpha,\beta}}^p / |v|_{0,p,K_{\alpha,\beta}}$$

Then $D_p(K_{\alpha,\beta}) \geq D_p/2$.

Proof Suppose that $p \in [1,\infty)$. Then, \cite{8} yields, for any $v \in \mathcal{F}(K_{\alpha,\beta})$,

$$|v|_{2,p,K_{\alpha,\beta}}^p \geq |u_{xx}|_{0,p,\mathcal{K}}^p + |u_{yy}|_{0,p,\mathcal{K}}^p + |\alpha^2 v|_{0,p,\mathcal{K}}^p + 2|u_{xy}|_{0,p,\mathcal{K}}^p / 2p|u|^p_{0,p,\mathcal{K}}$$

$$\geq |u_{xx}|_{0,p,\mathcal{K}}^p + |u_{yy}|_{0,p,\mathcal{K}}^p + 2|u_{xy}|_{0,p,\mathcal{K}}^p / 2p |u|^p_{0,p,\mathcal{K}} \geq D_p(\mathcal{K})^p / 2p.$$

Taking the infimum of the left-hand side with respect to $v \in \mathcal{F}(K_{\alpha,\beta})$, we obtain $D_p(K_{\alpha,\beta}) \geq D_p(K)/2$. The case $p = \infty$ can be proved in the same manner. \hfill \Box

Remark 2 According to \cite{8} pp40-41, the approximated value of $D_2$ is $1/0.167$.

3 The circumradius condition for right triangular elements

Take $R > 0$ and let the linear map $G_R$ be defined by

$$G_R : \mathbb{R}^2 \to \mathbb{R}^2, \quad G_R(x) := Rx, \quad x \in \mathbb{R}^2.$$

Two bounded domains $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are called similar if there exists a map $\varphi$ which consists of a rotation and a parallel translation, such that $\Omega_2 = \varphi \circ G_R(\Omega_1)$. If $\varphi$ is a parallel translation, $\varphi$ preserves the Sobolev norms of functions in $W^{m,p}(\Omega_1)$, $(m = 0, 1, 2, p \in [1,\infty])$. Hence, we may ignore $\varphi$ in the following discussion without loss of generality. Set $K_{\alpha,\beta}^R := G_R(K_{\alpha,\beta})$. The circumradius of $K_{\alpha,\beta}^R$ is $R$. Take $v \in W^{m,p}(K_{\alpha,\beta}^R)$ and define $\bar{v} := v \circ G_R \in W^{m,p}(K_{\alpha,\beta})$. Then, we have

$$|v|_{m,p,K_{\alpha,\beta}^R} = (\sqrt{2}R)^{2/p-m} |v|_{m,p,K_{\alpha,\beta}}, \quad m = 0, 1, 2.$$ (7)
For the domain $K_{\alpha,\beta}^R$, we define $\mathcal{F}_p(K_{\alpha,\beta}^R)$ as before and find that

$$B_p(K_{\alpha,\beta}^R) := \inf_{v \in \mathcal{F}_p(K_{\alpha,\beta}^R)} |v|_{2,p,K_{\alpha,\beta}^R} = B_p(K_{\alpha,\beta}) = \frac{A_p}{\sqrt{2R}} \geq \frac{A_p}{2R},$$

$$D_p(K_{\alpha,\beta}^R) := \inf_{v \in \mathcal{F}_p(K_{\alpha,\beta}^R)} |v|_{2,p,K_{\alpha,\beta}^R} = D_p(K_{\alpha,\beta}) = \frac{D_p}{2R^2} \geq \frac{D_p}{4R^2}.$$  

Combining these estimates we obtain the following lemma.

**Lemma 6** Let $K \subset \mathbb{R}^2$ be a right triangle whose circumradius is $R$. Suppose that the two edges which contain the right angle are parallel to $x$- and $y$-axis. Then, we have the following estimates:

$$B_p(K) \geq \frac{A_p}{2R}, \quad D_p(K) \geq \frac{D_p}{4R^2}, \quad 1 \leq p \leq \infty. \quad (8)$$

As is stated in the introduction, the linear interpolation operator $I_h v \in \mathcal{P}_1$ for $v \in W^{2,p}(K_R)$ is defined by $(I_h v)(x_i) = v(x_i)$, where $x_i, i = 1, 2, 3$ are apexes of $K_R$.

**Theorem 1** Let $K \subset \mathbb{R}^2$ be a right triangle whose circumradius is $R$. Suppose that the two edges that contain the right angle are parallel to the $x$- and $y$-axes. Then, the error of $I_h$ on $K$ is estimated as

$$\|v - I_h v\|_{1,p,K} \leq C_p R |v|_{2,p,K}, \quad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty,$$

$$C_p := \begin{cases} 2A_p^p + 4R^p D_p^p & 1 \leq p < \infty, \\ \max \{ 2/A_p, 4R/D_p \} & p = \infty. \end{cases}$$

**Proof** Since $v - P v \in \mathcal{F}_p(K)$ for $v \in W^{2,p}(K)$, we may apply Lemma 6 and obtain

$$\|v - I_h v\|_{1,p,K} \leq C_p R |v - I_h v|_{2,p,K} = C_p R |v|_{2,p,K}.$$  

\[\square\]

### 4 The circumradius condition for general triangular elements

In this section, we prove the circumradius condition for general triangular elements. Let $K_{s,t}$ be the right triangle with apexes $N_1(-1,0), N_2(1,0), N_3(s,t), s > 0, s^2 + t^2 = 1$. The circumradius of $K_{s,t}$ is 1. Define $K_{s,t}^\eta := F_1 \eta(K_{s,t})$ using the OEC transformation $F_1 \eta$. Note that any triangle $K$ is similar to $K_{s,t}^\eta$ with appropriate $(s,t)$ and $\eta > 0$ (see Fig. 3). We then try to write a lower bound of

$$\inf_{v \in \mathcal{F}_p(K_{s,t}^\eta)} \frac{|v|_{2,p,K_{s,t}^\eta}}{|v|_{1,p,K_{s,t}^\eta}}$$

using $A_p$ and the circumradius of $K_{s,t}^\eta$. We may assume without loss of generality that the baseline $N_1N_2$ is the longest edge of $K_{s,t}^\eta$. Under this assumption, $\eta$ is in the interval $(0, \sqrt{3}].$
4.1 Preliminary

Define the constants \(a, b, X, Y\) by

\[
(a, b) := \frac{1}{\sqrt{2}} \left( \sqrt{1 + s}, \sqrt{1 - s} \right), \quad X := \sqrt{a^2 \eta^2 + b^2}, \quad Y := \sqrt{a^2 + b^2 \eta^2}.
\]

Note that \(a^2 + b^2 = 1\), \(2ab = t\) and the vector \((a, b)\) is parallel to the edge \(N_1N_3\). We also have

\[
X \leq \sqrt{3a^2 + 3b^2} \leq \sqrt{3}, \quad Y \leq \sqrt{3},
\]

\[
\frac{X}{\eta} = \sqrt{\frac{a^2 + b^2}{\eta^2}} \geq \frac{1}{\sqrt{3}}, \quad \frac{Y}{\eta} \geq \frac{1}{\sqrt{3}}.
\]

Note that the circumradius \(R\) of \(K^\eta_{s,t}\) is

\[
R = \frac{2 \sqrt{(1 - s)^2 + \eta^2 t^2} \sqrt{(1 + s)^2 + \eta^2 t^2}}{4 \eta t} = \frac{\sqrt{a^2 + b^2 \eta^2} \sqrt{b^2 + a^2 \eta^2}}{\eta} = \frac{XY}{\eta}.
\]

We also observe that the inequality

\[
X^2Y^2 = \eta^2 + a^2 b^2 (1 - \eta^2) \geq a^2 b^2 (1 - \eta^2)^2
\]
yields

\[
XY - ab|1 - \eta^2| = \frac{X^2Y^2 - a^2 b^2 (1 - \eta^2)^2}{XY + ab|1 - \eta^2|} \geq \frac{\eta^2}{2XY},
\]

\[
XY - \eta^2 \geq ab|1 - \eta^2|.
\]

We notice that the following inequalities hold for any positive numbers \(U, V\) and \(p \geq 1:\)

\[
U^p + V^p \leq 2^{(p)} (U^2 + V^2)^{p/2}, \quad \tau(p) := \begin{cases} 1 - p/2, & 1 \leq p \leq 2, \\ 0, & 2 \leq p < \infty, \end{cases}
\]

\[
(U^2 + V^2)^{p/2} \leq 2^{(p)} (U^p + V^p), \quad \gamma(p) := \begin{cases} 0, & 1 \leq p \leq 2, \\ p/2 - 1, & 2 \leq p < \infty. \end{cases}
\]
4.2 A congruent transformation 

Define the congruent transformation $F : (x, y) \mapsto (z, w)$ by

$$
(z, w) = \left( -a - b \begin{pmatrix} x - s \\ y - t \end{pmatrix} \right).
$$

This transformation defines the $(z, w)$-coordinate on $K_{s, t}$ and maps the three apexes $(-1, 0), (1, 0), (s, t)$ of $K_{s, t}$ to $(2a, 0), (0, 2b), (0, 0)$. See Figure 4.

For a sufficiently smooth function $f$ defined on $K_{s, t}$, we can write upper and lower bounds of $\eta^p|f_x|_{0, p, K_{s, t}} + |f_y|_{0, p, K_{s, t}}$ using $|f_z|_{0, p, K_{s, t}}, |f_w|_{0, p, K_{s, t}}$. Let us suppose $1 \leq p < \infty$ at first. It follows from (11), (12), (13) and $f_x = -af_z + bf_w, \quad f_y = -bf_z - af_w$, that

$$
\eta^p|f_x|_{0, p, K_{s, t}} + |f_y|_{0, p, K_{s, t}} = \int_{K_{s, t}} (\eta^p| - af_z + b f_w| + |bf_z + af_w|^p) \, dx
$$

(by (12))

$$
\leq 2^{\varepsilon(p)} \int_{K_{s, t}} (\eta^2(-af_z + b f_w)^2 + (bf_z + af_w)^2)^{p/2} \, dx
$$

$$
= 2^{\varepsilon(p)} \int_{K_{s, t}} (X^2|f_z|^2 + Y^2|f_w|^2 + 2ab(1 - \eta^2)f_z f_w)^{p/2} \, dx
$$

(by (11))

$$
\leq 2^{\varepsilon(p)} \int_{K_{s, t}} (X|f_z| + Y|f_w|)^p \, dx
$$

$$
\leq 2^{\varepsilon(p) + p-1} \left( X^p|f_z|_{0, p, K_{s, t}} + Y^p|f_w|_{0, p, K_{s, t}} \right), \quad (14)
$$

and

$$
\eta^p|f_x|_{0, p, K_{s, t}} + |f_y|_{0, p, K_{s, t}}
$$

(by (13))

$$
\geq 2^{-p\eta} \int_{K_{s, t}} (X^2|f_z|^2 + Y^2|f_w|^2 + 2ab(1 - \eta^2)f_z f_w)^{p/2} \, dx
$$
\[
\geq 2^{-\gamma(p)} \int_{K_{\epsilon,t}} \left( X^2 |f_\epsilon|^2 + Y^2 |f_w|^2 \right. \\
\left. - ab |1 - \eta^2| \left( \frac{X}{Y} |f_\epsilon|^2 + \frac{Y}{X} |f_w|^2 \right) \right)^{p/2} dx \\
\left( \text{by (11)} \right) \geq 2^{-\gamma(p)} \int_{K_{\epsilon,t}} \left( X^2 |f_\epsilon|^2 + Y^2 |f_w|^2 \right. \\
\left. - \left( XY - \frac{\eta^2}{2XY} \right) \left( \frac{X}{Y} |f_\epsilon|^2 + \frac{Y}{X} |f_w|^2 \right) \right)^{p/2} dx \\
= 2^{-\gamma(p) - p/2} \eta^p \int_{K_{\epsilon,t}} \left( \frac{1}{Y^2} |f_\epsilon|^2 + \frac{1}{X^2} |f_w|^2 \right)^{p/2} dx \\
\left( \text{by (12)} \right) \geq 2^{-\gamma(p) - \tau(p) - p/2} \eta^p \left( \frac{|f_\epsilon|_{L^p(K_{\epsilon,t})}^p}{Y^p} + \frac{|f_w|_{L^p(K_{\epsilon,t})}^p}{X^p} \right). \tag{15}
\]

If \( p = \infty \), we obtain
\[
\max \{ \eta |f_\epsilon|_{L^\infty(K_{\epsilon,t})}, |f_w|_{L^\infty(K_{\epsilon,t})} \} \leq 2 \max \{ X |f_\epsilon|_{L^\infty(K_{\epsilon,t})}, Y |f_w|_{L^\infty(K_{\epsilon,t})} \},
\]
in exactly the same manner. (In the following, we have denoted the \( L^p(K_{\epsilon,t}) \)-norm by \( | \cdot |_p \).) To obtain the opposite inequality, let \( x \in K_{\epsilon,t} \) be any point. Then, using the previous technique we have
\[
\eta^2 |f_\epsilon(x)|^2 + |f_w(x)|^2 = X^2 |f_\epsilon(x)|^2 + Y^2 |f_w(x)|^2 + 2ab (1 - \eta^2) f_\epsilon(x) f_w(x) \\
\geq X^2 |f_\epsilon(x)|^2 + Y^2 |f_w(x)|^2 \\
- ab |1 - \eta^2| \left( \frac{X}{Y} |f_w(x)|^2 + \frac{Y}{X} |f_\epsilon(x)|^2 \right) \\
\geq \frac{\eta^2}{2} \left( \frac{|f_\epsilon(x)|^2}{Y^2} + \frac{|f_w(x)|^2}{X^2} \right).
\]
This inequality yields
\[
\max \{ \eta |f_\epsilon|_{L^\infty}, |f_w|_{L^\infty} \} \geq \max \{ \eta |f_\epsilon(x)|, |f_w(x)| \} \\
\geq \frac{1}{\sqrt{2}} \left( \eta^2 |f_\epsilon(x)|^2 + |f_w(x)|^2 \right)^{1/2} \\
\geq \frac{\eta}{2} \left( \frac{|f_\epsilon(x)|^2}{Y^2} + \frac{|f_w(x)|^2}{X^2} \right)^{1/2} \geq \frac{\eta |f_\epsilon(x)|}{2Y},
\]
\[
\max \{ \eta |f_\epsilon|_{L^\infty}, |f_w|_{L^\infty} \} \geq \frac{\eta |f_w(x)|}{2X}.
\]
Since \( x \in K_{\epsilon,t} \) is arbitrary, we obtain
\[
\max \{ \eta |f_\epsilon|_{L^\infty}, |f_w|_{L^\infty} \} \geq \frac{\eta}{2} \max \left\{ \frac{|f_\epsilon|_{L^\infty}}{Y}, \frac{|f_w|_{L^\infty}}{X} \right\}.
\]
4.3 A proof of the circumradius condition

We can now prove the circumradius condition. Let \( 1 \leq p < \infty \). For \( v \in \mathcal{P}(\mathcal{K}_p) \), define \( u := v \circ F_1, \eta \in \mathcal{P}(\mathcal{K}_p) \). Then, it follows from (4) that

\[
\frac{|v|^p_{2,p;\mathcal{K}_p}}{|v|^p_{1,p;\mathcal{K}_p}} = \frac{\eta^p|u_{x}|_p^p + \frac{1}{n_p}|u_{y}|_p^p + 2|u_{xy}|_p^p}{\eta^p|u_{x}|_p + |u_{y}|_p^p}. 
\]

By (4) the denominator \( \Pi_D^p \) is estimated as

\[
\Pi_D^p \leq 2^{(r(p)+p-1)}(X^p|u_{x}|_p^p + Y^p|u_{w}|_p^p).
\]

By (5) we notice

\[
\eta^p|u_{x}|_p^p + |u_{y}|_p^p \geq c_1 \eta^p \left( \frac{|u_{x}|_p^p}{Y^p} + \frac{|u_{y}|_p^p}{X^p} \right),
\]

\[
\eta^p|u_{y}|_p^p + |u_{x}|_p^p \geq c_1 \eta^p \left( \frac{|u_{y}|_p^p}{Y^p} + \frac{|u_{x}|_p^p}{X^p} \right),
\]

where \( c_1 := 2^{-r(p)-r(p)-p/2} \). Hence, the numerator \( \Pi_N^p \) is estimated as

\[
\Pi_N^p = \eta^p|u_{x}|_p^p + |u_{y}|_p^p + \eta^{-p} \left( \eta^p|u_{y}|_p^p + |u_{x}|_p^p \right)
\]

\[
\geq c_1 \left\{ \eta^p \left( \frac{|u_{x}|_p^p}{Y^p} + \frac{|u_{y}|_p^p}{X^p} \right) + \frac{|u_{y}|_p^p}{Y^p} + \frac{|u_{x}|_p^p}{X^p} \right\}
\]

\[
= c_1 \left\{ \frac{1}{Y^p} \left( \eta^p|u_{x}|_p^p + |u_{x}|_p^p \right) + \frac{1}{X^p} \left( \eta^p|u_{y}|_p^p + |u_{y}|_p^p \right) \right\}
\]

\[
\geq c_1^2 \eta^p \left( \frac{|u_{x}|_p^p}{Y^2p} + \frac{2|u_{x}|_p^p}{X^2p} + \frac{|u_{y}|_p^p}{X^2p} \right)
\]

\[
(by (2)) \geq c_1^2 3^{-p/2} \eta^p \left( \frac{1}{Y^p} \left( |u_{x}|_p^p + |u_{y}|_p^p \right) + \frac{1}{X^p} \left( |u_{x}|_p^p + |u_{y}|_p^p \right) \right)
\]

\[
\geq c_1^2 2^{-p} 3^{-p/2} \eta^p A_p^p \left( \frac{1}{Y^p} |u_{x}|_p^p + \frac{1}{X^p} |u_{y}|_p^p \right)
\]

\[
\geq c_1^2 2^{-p} 3^{-p/2} \eta^p A_p^p \left( X^p|u_{x}|_p^p + Y^p|u_{w}|_p^p \right).
\]

Here, we have used the estimates

\[
|u_{x}|_p^p + |u_{y}|_p^p \geq A_p^p \frac{2}{|u_{x}|_p^p}, \quad |u_{x}|_p^p + |u_{y}|_p^p \geq A_p^p \frac{2}{|u_{y}|_p^p},
\]

since we may apply Lemma 2 to \( u_{x} \) and \( u_{w} \). Combining these estimates, we obtain

\[
\frac{\Pi_N^p}{\Pi_D^p} \geq \frac{A_p^p}{2^{3(r(p)+2p-1)p/2} X^p Y^p} \eta^p = \frac{A_p^p}{2^{3(r(p)+2p-1)p/2} X^p Y^p}. 
\]
Lemma 7. Let $K$ be an arbitrary triangle whose circumradius is $R$. Suppose that the longest edge of $K$ is parallel to the $x$-axis (or to the $y$-axis) and of length $2$. Then

$$B_p(K) := \inf_{v \in \mathcal{B}_p(K)} \frac{|v|_{2,p,K}}{|v|_{1,p,K}} \geq \frac{A_p}{2 \phi(p)3^{1/2}R}, \quad \phi(p) := \begin{cases} 3/2 + 2/p & 1 \leq p \leq 2 \\ 4 - 3/p & 2 \leq p < \infty \end{cases}.$$ 

Next, we try to estimate $\inf_{v \in \mathcal{B}_p(K)} |v|_{2,p,K}/|v|_{1,p,K}$. Let $1 \leq p < \infty$. Take any function $v \in \mathcal{B}_p(K)$, and define $u := v \circ F_1, \eta \in \mathcal{B}_p(K, \eta)$. From [3], we need to estimate

$$\frac{|v|_{2,p,K}^p}{|v|_{1,p,K}^p} = \frac{\eta^p |u_{x1}|_p^p + \frac{1}{\eta^p} |u_{y1}|_p^p + 2 |u_{x3}|_p^p}{\eta^p |u|^p_p} = \frac{\Pi_N^p}{\eta^p |u|^p_p}.$$ 

Using [10], we see that

$$\Pi_N^p \geq c^2 \eta^p \left( \frac{|u_{x1}|_p^p + 2 |u_{x3}|_p^p}{X^{2p}Y^{2p}} + \frac{|u_{y1}|_p^p}{X^{2p}} \right) \geq c^2 \frac{\eta^3 p}{X^{2p}Y^{2p}} \left( \frac{X^{2p}}{\eta^p} |u_{x1}|_p^p + 2 \frac{X^{2p}Y^{2p}}{\eta^p} |u_{x3}|_p^p + \frac{Y^{2p}}{\eta^p} |u_{y1}|_p^p \right) \geq c^2 \frac{\eta^p}{3^{2p}R^{2p}} \left( |u_{x1}|_p^p + 2 |u_{x3}|_p^p + |u_{y1}|_p^p \right).$$
Therefore, applying Lemma 5 we obtain
\[
\frac{|v|^{p}{_{2,p,K}}}{|v|^{p}{_{0,p,K}}} = \frac{\Pi^{p}{_0}}{\eta^{p}} \frac{|u|^{p}}{|u|^{p}} \geq \frac{c^{2}}{3^{p}R^{2p}} \left(|u_{zz}|^{p} + 2|u_{zn}|^{p} + |u_{wn}|^{p}\right)
\]
\[
\geq \frac{c^{2}}{3^{p}R^{2p}} \frac{D^{p}}{2^{2p}} \geq \frac{D^{p}}{2^{2p} + 2\tau(p) + 3\gamma(p)R^{2p}}.
\]

The proof of the case \( p = \infty \) is very similar.

**Lemma 8** Let \( K \) be an arbitrary triangle whose circumradius is \( R \). Suppose that the longest edge of \( K \) is parallel to the \( x \)-axis (or to the \( y \)-axis) and of length \( 2 \). Then
\[
D_{p}(K) := \inf_{v \in \mathcal{J}_{p}(K)} \frac{|v|^{p}{_{2,p,K}}}{|v|^{p}{_{0,p,K}}} \geq \frac{D_{p}}{2^{2p} + 2\tau(p) + 3\gamma(p)R^{2p}}, \quad \mu(p) := \begin{cases} 2 + 2/p & 1 \leq p \leq 2, \\ 4 - 2/p & 2 \leq p \leq \infty. \end{cases}
\]

When we apply Lemmas 7 and 8 to an arbitrary triangle \( K \), we have to notice an orthogonal matrix (or a rotation) \( \varphi \) may change the Sobolev norms. More precisely, the Sobolev norms \( |v \circ \varphi|_{m,p,K} \), \( m = 0, 1, 2 \) of \( v \in W^{m,p}(\varphi(K)) \) are different from \( |v|_{m,p,\varphi(K)} \) in general, and are estimated as
\[
2^{-m(\tau(p) + \gamma(p))} |v|_{m,p,\varphi(K)} \leq |v \circ \varphi|_{m,p,K} \leq 2^{m(\tau(p) + \gamma(p))} |v|_{m,p,\varphi(K)}, \quad p \in [1, \infty),
\]
and
\[
2^{-m/2} |v|_{m,\infty,\varphi(K)} \leq |v \circ \varphi|_{m,\infty,K} \leq 2^{m/2} |v|_{m,\infty,\varphi(K)}.
\]

Gathering Lemma 7 and 8, we have obtained the following theorems:

**Theorem 2** Let \( K \) be an arbitrary triangle whose circumradius is \( R \). Then for any \( w \in \mathcal{J}_{p}(K) \), \( 1 \leq p \leq \infty \), there exist constants \( E_{1}(p) \) and \( E_{2}(p) \) that depend only on \( p \) such that the following is true:
\[
|w|_{1,p,K} \leq M_{p}R|w|_{2,p,K}, \quad M_{p} := (E_{1}(p))^{p} + E_{2}(p)R^{p})^{1/p}.
\]

**Theorem 3** Let \( K \) be an arbitrary triangle whose circumradius is \( R \). Then the following estimate holds:
\[
|w - I_{p}w|_{1,p,K} \leq M_{p}R|w|_{2,p,K}, \quad \forall w \in W^{2,p}(K), \quad 1 \leq p \leq \infty.
\]

### 5 Concluding remarks

In this paper, we have proved the circumradius condition for triangular elements in \( \mathbb{R}^{2} \) using a Babuška-Aziz type technique, without validated numerical computation. Since the error analysis of \( I_{p} \) is very important, generalizations of Kobayashi’s formula and/or the circumradius condition are required. Some of the unanswered questions on which to focus subsequent research are:

- In nonlinear finite element error analysis, the inverse inequality plays an important role [2] pp139-143. It is interesting to consider whether or not we are able to obtain a condition similar to the circumradius condition for the inverse inequality.
– Does the circumradius condition hold on three dimensional tetrahedrons? Unfortunately, one of the authors, Kobayashi, has already found a counter example which shows that the circumradius condition does not hold on tetrahedrons. A very interesting problem is to find an essential condition, similar to the circumradius condition, which can be used for error estimate on tetrahedrons.

– In [3], Hannukainen-Korotov-Křížek show that the maximum angle condition is not necessary for convergence of the finite element method by showing simple examples. In their examples, the circumradii of triangles are very close to constants while $h \to 0$. Thus, the circumradius condition cannot explain the convergence of the finite element solutions in [3]. Therefore, the question remains “what is the essential triangulation condition for the convergence of finite element solutions?”. This is a very important question, which we wish to answer.

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