Reflection $K$-matrices for a nineteen vertex model with $U_q(\text{osp}(2|2)^{(2)})$ symmetry

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Abstract

We derive the solutions of the boundary Yang-Baxter equation associated with a supersymmetric nineteen vertex model constructed from the three-dimensional representation of the twisted quantum affine Lie superalgebra $U_q(\text{osp}(2|2)^{(2)}) \simeq U_q[C(2)^{(2)}]$. We found three classes of solutions. The type I solution is characterized by three boundary free-parameters and all elements of the corresponding reflection $K$-matrix are different from zero. In the type II solution, the reflection $K$-matrix is even (every element of the $K$-matrix with an odd parity is null) and it has only one boundary free-parameter. Finally, the type III solution corresponds to a diagonal reflection $K$-matrix with two boundary free-parameters.

1. The model

This letter concerns with the reflection $K$-matrices of a supersymmetric nineteen vertex model introduced by Yang and Zhang in [1] (see also [2–5]). The Yang-Zhang $R$-matrix is constructed from a three-dimensional representation $\mathcal{V}$ of the twisted quantum affine Lie superalgebra $U_q[\text{osp}(2|2)^{(2)}] \simeq U_q[C(2)^{(2)}]$, and the periodic algebraic Bethe Ansatz of this vertex model was performed in [6].

Vertex models with underlying symmetries corresponding to Lie superalgebras are important in several fields of physics and mathematics. For instance, $\mathbb{Z}_2$-graded Lie superalgebras appear in the study of lattice models of strongly correlated electrons, such as the supersymmetric generalizations of the $t$-$J$ model [7–11] and the Hubbard model [12, 13], among others. The integrability of supersymmetric two-dimensional quantum chains had proved to be important as well in the AdS/CFT correspondence, either in the $\mathcal{N} = 4$ super Yang-Mills side of the duality [14–16], or in the $\text{AdS}_5 \times S^5$ string theory side [17]. Furthermore, these mathematical structures also are important in the construction of supersymmetric Hopf algebras and quantum groups [18–21], representation theory of quantum deformed Virasoro and $W$ algebras [22] and so on.

Nineteen vertex models, by their turn, have been studied for a long time ago. The first studied nineteen vertex models were the Zamolodchikov-Fateev vertex model (ZFvm) [23] and the Izergin-Korepin vertex model (IZvm) [24], which are respectively associated with the $A_2^{(1)}$ and $A_2^{(2)}$ Lie algebras. Several important results were obtained for these models. For instance, considering first periodic boundary conditions, the coordinate Bethe Ansatz of the ZFvm was performed in [25] (although its Hamiltonian had been diagonalized before, through the fusion procedure [26], in [27] and, with inhomogeneities, in [28]) and coordinate Bethe Ansatz of the IZvm was obtained in [29]; the algebraic Bethe Ansatz for the ZFvm and IZvm were derived respectively in [25] and [30]. Considering now non-periodic boundary conditions, the diagonal $K$-matrices for the ZFvm were first derived in [31] and the general $K$-matrices were deduced in [32] and [33]; for the IZvm, the diagonal $K$-matrices were found in [34] and the general $K$-matrices were derived in [33] and [35]; besides, the coordinate Bethe Ansätze for both models were presented in [36], while their algebraic versions were performed in [37] (although the ZFvm had been considered before in [31] with the help of the fusion procedure [26]). Among the ZFvm and IZvm, other nineteen vertex models were discovered since then, for instance, the supersymmetric vertex models associated with the $\text{sl}(2|1)$ and $\text{osp}(2|1)$ Lie superalgebras, whose $R$-matrix was presented in [38] and the corresponding reflection $K$-matrices were obtained in [33]; for the Bethe Ansätze of these models, see [36, 39]. Finally, other more complex nineteen vertex models were also discovered in the last decade – see, for instance, [40–45].

The supersymmetric nineteen vertex model which we consider here is not included in the list above. The $R$-matrix associated with this model was constructed in [1] and it can be written (up to a normalizing factor and employing a different notation) as

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (1)
where the amplitudes \( r_i \), \( 1 \leq i \leq 10 \), functions of the spectral parameter \( x \), are given by

\[
\begin{align*}
    r_1 (x) &= q^2 x - 1, \\
    r_2 (x) &= q (x - 1), \\
    r_3 (x) &= \frac{q (q + x) (x - 1)}{qx + 1}, \\
    r_4 (x) &= q (x - 1) - \frac{(q + 1) (q^2 - 1) x}{qx + 1}, \\
    r_5 (x) &= q^2 - 1, \\
    r_6 (x) &= -q^{1/2} \frac{(q^2 - 1) (x - 1)}{qx + 1}, \\
    r_7 (x) &= \frac{(q - 1) (q + 1)^2}{qx + 1}, \\
    r_8 (x) &= (q^2 - 1) x, \\
    r_9 (x) &= -q^{1/2} \frac{(q^2 - 1) x (x - 1)}{qx + 1}, \\
    r_{10} (x) &= \frac{(q - 1) (q + 1)^2 x^2}{qx + 1}.
\end{align*}
\]

This \( R \)-matrix satisfies the graded YB equation [38, 46–53],

\[
R_{12} (x) R_{13} (xy) R_{23} (y) = R_{23} (y) R_{13} (xy) R_{12} (x),
\]

where \( R \) is a matrix defined in the End \((V \otimes V)\) and \( V \) is a three-dimensional complex vector space; the matrices \( R_{12}, R_{23} \) and \( R_{13} \) are defined in End \((V \otimes V \otimes V)\) respectively by the relations \( R_{12} = R \otimes I, R_{23} = I \otimes R \) and \( R_{13} = P_{12} R_{23} P_{12}^{-1} \), where \( I \) denotes the identity matrix defined on End \((V)\) and \( P_{12} = P^q \otimes I \) with \( P^q \) denoting the graded permutator matrix (see below) defined in End \((V \otimes V)\).

Since we are dealing here with a supersymmetric system, it will be useful to review the basics of Lie superalgebras [54–65]. In terms of the Weyl matrices \( e_{ij} \in \text{End} (V) \) —matrices whose elements are all zero, except that one on the \( i \)th line and \( j \)th column, which equals 1 — and always considering a sum on the repeated indices, we can define, in a \( Z_2 \)-graded Lie algebra, the graded tensor product of two homogeneous even elements \( A \in \text{End} (V) \) and \( B \in \text{End} (V) \) as \( A \otimes^g B = (-1)^{p(a)p(b)} A_{ij} B_{jk} (e_{ij} \otimes e_{kl}) \); the graded permutator matrix as \( P^g = (-1)^{p(a)p(l)} (e_{ij} \otimes e_{kl}) \); the graded transposition of \( A \) as \( A^T = (-1)^{p(k)p(l)} A_{ij} e_{ijl} \); and its inverse graded transposition is given by \( A^{T^g} = (-1)^{p(k)p(l)} A_{ij} e_{lij} \), so that \( A^{T^g} A^T = A \); finally, the formula \( t^g (A) = (-1)^{p(i)} A_{ni} e_{ni} \) defines the graded trace of a matrix \( A \). In the current case, we shall actually consider only a three-dimensional representation \( V \) of the twisted quantum affine \( U_q (\mathfrak{osp} (2|2)^{(2)}) \), which is constructed from a supersymmetric 6-dimensional 14-generator Lie superalgebra \( U_q (\mathfrak{osp} (2|2)^{(2)}) \), which is constructed from a \( Z_2 \)-graded Lie algebra with basis \( E = \{ e_1, e_2, e_3 \} \) and grading \( p (e_1) = 0, p (e_2) = 1, p (e_3) = 0 \) — see [1] for details.

We can verify that the \( R \)-matrix (1) enjoys the following symmetries:

\[
\begin{align*}
    &\text{regularity: } R_{12} (1) = f (f (1)^{1/2} P^q_{12}), \\
    &\text{unitarity: } R_{12} (x) = f (x) R_{21} (\frac{1}{x}), \\
    &\text{super PT: } R_{12} (x) = R_{21}^{T^g} (x), \\
    &\text{crossing: } R_{12} (x) = g (x) \left[ V_1 R_{12}^{T^g} \left( \frac{1}{x} \right) V_1^{-1} \right],
\end{align*}
\]

where \( t_1, t_2, t_3 \) mean the (inverse) graded partial transpositions in the first and second vector spaces, respectively, \( x = -q \) is the crossing parameter, \( M = V^{T^g} V = \text{diag} (q^{-1}, 1, q) \) is the crossing matrix and, finally, \( f(x) = (q^2 x - 1) (q^2 - 1) \) and \( g(x) = -qx (x - 1) / (qx + 1) \).

In this letter, we shall derive the reflection \( K \)-matrices, solutions of the boundary graded YB equation [66–70]

\[
R_{12} (x/y) K_1 (x) R_{21} (xy) K_2 (y) = K_2 (y) R_{12} (xy) K_1 (x) R_{21} (x/y),
\]

(also known as the graded reflection equation) for the Yang-Zhang model [1]. In (17), \( R_{12} \) denotes just the \( R \)-matrix defined on End \((V \otimes V)\) and \( R_{21} = P^q R_{12} P^q \); besides, the reflection \( K \)-matrix is defined in End \((V)\), while \( K_{1} = K \otimes I \) and \( K_{2} = I \otimes K \). The properties (13-16) ensure the existence of the standard isomorphism \( K_{1,2} (x) \rightarrow K^{T^g} (x^{-1} \eta^{-1}) M \), so that, if \( K (x) \) is a solution of the graded boundary YB equation (17), then \( K_{1,2} (x) \) will be the corresponding solution of the dual reflection equation, as described in [70].

We highlight that vertex-models associated with Lie superalgebras, in particular those associated with twisted Lie superalgebras, are generally the most complex ones [1–5, 53–65]. In fact, the classification of the reflection \( K \)-matrices for models associated with Lie superalgebras is not yet complete. A great advance in this direction was obtained in the last years by Lima-Santos and collaborators, where a classification of the reflection \( K \)-matrices associated with the \( U_q [sl (r|2m)^{(2)}] \), \( U_q [osp (r|2m)^{(1)}] \), \( U_q [spo (2n|2m)] \) and \( U_q [sl (m|n)]^{(1)} \) vertex models were given [71–74]. More recently, Vieira and Lima-Santos derived the reflection \( K \)-matrices associated with \( (2m + 2) \)-dimensional representations of the \( U_q [osp (2|2m)^{(2)}] \approx U_q [C (m + 1)^{(2)}] \) quantum affine Lie superalgebra [75]. It is necessary to remark, however, that the \( K \)-matrices considered in [75] for \( m = 1 \) (see also the appendix in [75]), are not related to the \( K \)-matrices considered here, although the symmetry behind both models is the same — namely, \( U_q [osp (2|2)^{(2)}] \approx U_q [C (2)^{(2)}] \) quantum affine Lie superalgebra. This is because the \( R \)-matrix considered in [75], for \( m = 1 \), is constructed from a four-dimensional representation of the \( U_q [osp (2|2)^{(2)}] \) twisted quantum affine Lie superalgebra, which leads to a thirty-six vertex model instead of a nineteen vertex-model as we have here. Moreover, the \( R \)-matrix considered in [75] is obtained from a reduction \((n = 0)\) of a most general \( R \)-matrix derived by Galleas and Martins in [76], which is associated with the
\[ U_q[\text{osp}(2n + 2|2m)^{(2)}] \simeq U_q[D(n + 1, m)^{(2)}] \] Lie superalgebra (we remark that this \( R \)-matrix corresponds to a graded generalization of Jimbo’s \( R \)-matrix associated with the \( U_q[\text{osp}(2n + 2|2)^{(2)}] \simeq U_q[D^{(2)}_{n+1}] \) vertex model [75–82]).

2. The reflection \( K \)-matrices

In order to solve the boundary YB equation (17), we shall employ the standard derivative method [23, 33]. This method consists in taking the derivative of (17) with respect to one of the variables, say \( y \), and evaluate the result at \( y = 1 \). With this, all \( y \) dependence vanishes, but, on the other hand, the following set of boundary parameters,

\[
\beta_{i,j} = \frac{d k_{i,j}(y)}{dy} \bigg|_{y=1}, \quad 1 \leq i, j \leq 3, \tag{18}
\]

is introduced. The result is a linear system of univariate functional equations for the \( K \)-matrix elements, \( k_{i,j}(x) \). The system is actually over-determined, since we have 81 equations against 9 unknowns \( k_{i,j}(x) \). Notwithstanding, the system is consistent thanks to the existence of others boundary parameters \( \beta_{i,j} \). In fact, all the unknowns \( k_{i,j}(x) \) can be eliminated from a subset of the functional equations and the remaining equations can be solved imposing certain constraints between the boundary parameters \( \beta_{i,j} \).

Notice, however, that in general some of the boundary parameters do not need to be fixed — they are the boundary free-parameters of the solution. Although, theoretically, the order in which the equations are solved should not change the final results (up to a different choice of the free-parameters, of course), solving the equations in an inappropriate order generally makes the system complexity to increase very fast — it is not uncommon for the system to become practically incomputable if such an unfortunate order is chosen. In the following, we shall describe an appropriate order to solve the functional equations on which the complexity of the system can be kept under control so that the solution can be achieved.

We found three classes of regular \( K \)-matrices for the Yang-Zhang vertex model [1]. (A regular \( K \)-matrix is a matrix \( K(x) \) that satisfies the property \( K(1) = I \).) The first type of solutions is the most general one, on which no element of the reflection \( K \)-matrix is null; this solution contains three boundary free-parameters. The second type of solutions corresponds to the most general even reflection \( K \)-matrix, which means that all non-null elements of the \( K \)-matrix have zero parity (the parity of the \( K \)-matrix elements being defined by \( p(k_{i,j}(x)) = p(i) + p(j) \mod 2 \)); this solution contains only one boundary free-parameter. Finally, the third type of solutions consists in a diagonal \( K \)-matrix with two boundary free-parameters.

2.1. The type I solution

In the type I solution all elements of the reflection \( K \) matrix are different from zero. The boundary YB equation (17) represents a system of 81 functional equations which can be labeled as \( E_{i,j} \) with \( 1 \leq i, j \leq 9 \). The simplest equations are those not containing any diagonal elements of the \( K \)-matrix. Indeed, from the group of equations \( \{E_{1,8}, E_{8,1}, E_{2,9}, E_{9,2}\} \) we can eliminate the elements \( k_{1,2}(x), k_{2,3}(x), k_{3,1}(x) \) and \( k_{3,2}(x) \), from which we get the expressions,

\[
k_{1,2}(x) = \frac{(q + x) \beta_{1,2} - q^{1/2} (x - 1) \beta_{2,3} k_{3,1}(x)}{(q^{2} + q) \beta_{1,3}}, \tag{19}
\]

\[
k_{2,1}(x) = \frac{(q + x) \beta_{2,1} + q^{1/2} (x - 1) \beta_{3,2} k_{3,1}(x)}{(q^{2} + q) \beta_{1,3}}, \tag{20}
\]

\[
k_{2,3}(x) = \frac{(q + x) \beta_{2,3} + q^{1/2} (x - 1) \beta_{1,2} x k_{1,3}(x)}{(q^{2} + q) \beta_{1,3}}, \tag{21}
\]

\[
k_{3,2}(x) = \frac{(q + x) \beta_{3,2} - q^{1/2} (x - 1) \beta_{2,1} x k_{1,3}(x)}{(q^{2} + q) \beta_{1,3}}. \tag{22}
\]

Then we can use \( E_{4,6} \) and \( E_{5,8} \) simultaneously to get the expressions of \( k_{2,2}(x) \) and \( k_{3,3}(x) \) in terms of \( k_{1,1}(x) \) and \( k_{1,3}(x) \):

\[
k_{2,2}(x) = -x k_{1,1}(x) + \left\{ \frac{q^{1/2} (q - x) (x - 1) \beta_{1,2}^{2} + \beta_{1,2}^{1}}{(q - 1) (x^{2} + q) \beta_{1,3}^{2}} + \frac{(\beta_{2,2} - \beta_{3,3} + 2) x + \beta_{3,3} - \beta_{2,2}}{(x - 1) \beta_{1,3}} \right\} k_{1,3}(x), \tag{23}
\]

\[
k_{3,3}(x) = x^{2} k_{1,1}(x) + \left\{ \frac{q^{1/2} (q - x) (x - 1) (\beta_{1,2}^{2} + \beta_{2,3}^{2})}{(q - 1) (x^{2} + q) \beta_{1,3}^{2}} + \frac{\beta_{3,3} - \beta_{1,1} - 2}{\beta_{1,3}} \right\} x k_{1,3}(x). \tag{24}
\]

Next, \( k_{1,1}(x) \) can be determined from \( E_{3,9} \):

\[
k_{1,1}(x) = \left\{ \frac{(q - 1) (x - 1) \beta_{1,2} + (x + q) \beta_{1,1}}{(q + 1) x (x + 1) \beta_{1,3}} \right\} \left\{ \frac{(q x + 1) \beta_{1,3}}{(q + 1) x (x + 1) \beta_{1,3}} + \frac{q^{1/2} (q - x) (x - 1) (\frac{x q + 1) \beta_{1,2}^{2} + (x + q) \beta_{2,3}^{2})}{(q - 1) (x^{2} + q) x (x + 1) \beta_{1,3}^{2}} \right. \tag{25}\]

\[
\left. - \frac{q^{2} (2 x - 1) + q (x + 1) x - 2 x^{2} (x - 2)}{(q + 1) (x^{2} + q) x (x + 1) \beta_{1,3}^{2}} \right. \tag{25}\]

\[
+ \frac{2 [q x^{2} + (q^{2} + 1) (2 x - 1)]}{(q + 1) x (x^{2} - 1) \beta_{1,3}} k_{1,3}(x). \tag{25}\]

Now, after the diagonal elements are eliminated from the system, several equations will contain only \( k_{1,3}(x) \). Supposing \( k_{1,3}(x) \neq 0 \), these equations will be satisfied only if some constraints between the parameters \( \beta_{i,j} \) are imposed. For instance, from the equations \( E_{9,2}, E_{1,5}, E_{2,7} \) and \( E_{2,5} \), we can fix the parameters \( \beta_{3,3}, \beta_{2,2}, \beta_{2,3} \) and
\( \beta_{3,2} \), respectively. After simplifying the expressions, we shall get, thus,

\[
\beta_{2,3} = -\frac{(q - 1) \beta_{1,3}}{q^{1/2}} \left[ \frac{2q^{1/2}}{(q + 1)} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{1,2}} \right], \quad (26)
\]

\[
\beta_{3,2} = \frac{(q - 1) \beta_{1,3} \beta_{2,1}}{q^{1/2}} \left[ \frac{2q^{1/2}}{(q + 1)} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{1,2}} \right], \quad (27)
\]

and

\[
\beta_{2,2} = \beta_{1,1} + \frac{2}{(q + 1)} - \left[ \frac{q^{1/2} \beta_{1,2}^2}{(q - 1) \beta_{1,3}^2} - \frac{(q - 1) \beta_{2,1}}{q^{1/2}} \frac{\beta_{1,3}}{\beta_{1,2}} \right] \beta_{1,3}, \quad (28)
\]

\[
\beta_{3,3} = \beta_{1,1} + \frac{4}{(q + 1)} - \left[ \frac{q^{1/2} \beta_{1,2}^2}{(q - 1) \beta_{1,3}^2} - \frac{(q - 1) \beta_{2,1}}{q^{1/2}} \frac{\beta_{1,3}}{\beta_{1,2}} \right] \beta_{1,3}
\]

\[
+ \frac{q - 1}{q^{1/2}} \left[ 1 - \frac{\beta_{1,3}}{\beta_{1,2}^2} \left( \frac{2q^{1/2}}{q + 1} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{1,2}} \right) \right] \times \left( \frac{2q^{1/2}}{q + 1} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{1,2}} \right). \quad (29)
\]

The use of these constraints makes the system simpler. In fact, the equation \( E_{1,4} \) provides now a nice expression of \( k_{3,1}(x) \) in terms of \( k_{1,3}(x) \), namely,

\[
k_{3,1}(x) = (\beta_{3,1}/\beta_{1,3}) k_{1,3}(x), \quad (30)
\]

and, finally, equation \( E_{8,6} \) enable us to fix \( \beta_{3,1} \),

\[
\beta_{3,1} = -(\beta_{2,1} \beta_{1,2}^2) / \beta_{1,3}. \quad (31)
\]

At this point we can verify that all functional equations are satisfied. Besides, by setting

\[
k_{1,3}(x) = \frac{1}{2} (x^2 - 1) \beta_{1,3}, \quad (32)
\]

we can verify that the solution is regular. The derivatives of the \( K \)-matrix elements, however, are not yet consistent with the definition of the boundary parameters given in (18). In order to make the solution consistent with (18), we must further to fix the value of \( \beta_{1,1} \). This can be made by evaluating the derivative of \( k_{1,3}(x) \) at \( x = 1 \), equaling this to \( \beta_{1,1} \) and solving the resulting equation in favor of \( \beta_{1,1} \). This provides us the desired value,

\[
\beta_{1,1} = \frac{q - 1}{q + 1} + \frac{1}{2q - 1} \left[ \frac{\beta_{1,2}^2}{\beta_{1,3}} + 4 \left( \frac{q - 1}{q + 1} \right)^2 \frac{\beta_{1,3}}{\beta_{1,2}^2} \right]
\]

\[
- \frac{1}{2} \frac{q - 1}{q^{1/2}} \left[ 1 + \frac{\beta_{1,3}}{\beta_{1,2}^2} \left( \frac{4q^{1/2}}{q + 1} - \frac{\beta_{1,3} \beta_{2,1}}{\beta_{1,2}} \right) \right] \frac{\beta_{1,3} \beta_{2,1}}{\beta_{1,2}}. \quad (33)
\]

Now the solution is regular and consistent with the derivative method employed. We can therefore simplify all the expressions above and write the final form of the reflection \( K \)-matrix. In this way, we shall get the following expressions for the non-diagonal elements of the \( K \)-matrix:

\[
k_{1,3}(x) = \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,3}, \quad (34)
\]

\[
k_{3,1}(x) = -\frac{1}{2} \left( x^2 - 1 \right) \beta_{1,3} \left( \beta_{2,1}/\beta_{1,2} \right), \quad (35)
\]

\[
k_{1,2}(x) = \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ \frac{\beta_{1,2}}{\beta_{1,3}} \right\}, \quad (36)
\]

\[
k_{2,1}(x) = \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ \frac{\beta_{1,2}}{\beta_{1,3}} \right\}, \quad (37)
\]

\[
k_{2,3}(x) = \frac{1}{2} x \left( x^2 - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ q^{1/2} \left( x - 1 \right) \beta_{1,2} / \beta_{1,3} \right\}, \quad (38)
\]

\[
k_{3,2}(x) = -\frac{1}{2} x \left( x^2 - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ q^{1/2} \left( x - 1 \right) \beta_{1,2} / \beta_{1,3} \right\} - \frac{\beta_{2,1}}{\beta_{1,2}}, \quad (39)
\]

and, for its diagonal elements, we get,

\[
k_{1,1}(x) = \frac{1}{2} \left( x - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ 4q^{1/2} \left( q - 1 \right) \left( \frac{x + 1}{x + q} \right) \frac{\beta_{1,2}}{\beta_{1,3}} \right\}, \quad (40)
\]

\[
k_{2,2}(x) = \frac{1}{2} x \left( x - 1 \right) \beta_{1,3} \left( \frac{x + q}{x^2 + q} \right) \left\{ q^{1/2} \left( q - 1 \right) \left( \frac{x + 1}{x + q} \right) \frac{\beta_{1,3} \beta_{2,1}}{\beta_{1,2}} \right\} - \frac{\beta_{2,1}}{\beta_{1,2}}, \quad (41)
\]
\[ k_{3,3}(x) = -\frac{1}{2} x^2 (x-1) \beta_{1,1} \left( \frac{x + q}{x^2 + q} \right) \left\{ \frac{4q^{1/2}(q-1)}{(q+1)^2 \beta_{1,2}^2} \right. \]
\[ - 2 \frac{(q-1)^2 x + 2q (x^2 - 1)}{(q+1) (x+q) (x^2 + 1)} (x+1) \beta_{1,3} \]
\[ + \frac{q^{1/2} (qz + 1)}{(q-1)(x+q)} \beta_{1,2}^2 \frac{q^{1/2}}{(q-1)^2 \beta_{2,1}} \beta_{1,2} \]
\[ - \frac{(q-1)^2 \beta_{1,3} \beta_{2,1}}{q^{1/2} \beta_{1,2}^4} \left[ \frac{4q^{1/2}}{(q+1)} - \frac{\beta_{1,3} \beta_{2,1}}{\beta_{1,2}} \right]. \] (42)

Notice that the solution depends only on the boundary parameters \( \beta_{1,2}, \beta_{1,3} \) and \( \beta_{2,1} \). These are the boundary free-parameters of the solution, which can assume any complex value.

2.2. The type II solution

The second class of regular solutions consists in the most general even \( K \)-matrix. This means that the only non-null elements of the \( K \)-matrix will be those lying on the main and secondary diagonals (i.e., \( k_{1,2}(x) = k_{2,1}(x) = k_{3,3}(x) = k_{3,1}(x) = 0 \)). We highlight that this solution is not a simple reduction of the previous one.

The method employed to find this solution is the same as that explained in the previous section, thus we shall only report here the final result, which is actually very simple:

\[ K(x) = \begin{pmatrix} 1 & 0 & \frac{1}{2} (x^2 - 1) \beta_{1,3} \\ 0 & \frac{qz + 1}{q^{1/2}} & 0 \\ -2q (x^2 - 1) & 0 & x^2 \end{pmatrix}. \] (43)

Notice that \( \beta_{1,3} \) is the only boundary free-parameter of this solution. The other parameters \( \beta_{1,1} \) must be fixed through (18) in order to the functional equations be satisfied and the solution becomes consistent with the derivative method.

2.3. The type III solution

Finally, there is also a diagonal regular solution, \( K(x) = \text{diag}(k_{1,1}(x), k_{2,2}(x), k_{3,3}(x)) \) in which

\[ k_{1,1}(x) = 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1}, \] (44)

\[ k_{2,2}(x) = x \left[ 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1} \right] \times \left( \frac{\beta_{1,1} - \beta_{1,2} + 2}{\beta_{1,1} - \beta_{2,2} + 2} \right), \] (45)

and

\[ k_{3,3}(x) = x^2 \left[ 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1} \right] \times \left( \frac{\beta_{1,1} - \beta_{1,2} + 2 - q x (\beta_{2,2} - \beta_{1,1})}{\beta_{1,1} - \beta_{2,2} + 2} \right) \times \left( \frac{\beta_{1,1} - \beta_{1,2} + 2 - (\beta_{2,2} - \beta_{1,1})}{\beta_{1,1} - \beta_{2,2} + 2} \right). \] (46)

Here, \( \beta_{1,1} \) and \( \beta_{2,2} \) are the boundary free-parameters of the solution. The boundary parameter \( \beta_{3,3} \) must be fixed through (18) and it is given by

\[ \beta_{3,3} = \beta_{2,2} + \frac{2q}{q+1} \left[ \frac{\beta_{1,1} - \beta_{2,2}}{\beta_{1,1} - \beta_{2,2} + 2} \right]. \] (47)

Particular solutions can be obtained if we fix the boundary parameters further. In particular, setting \( \beta_{1,1} = \beta_{2,2} = 0 \), we obtain the identity solution \( K(x) = I \) which corresponds to the quantum group invariant solution for this model.

3. Conclusion

In this letter we derived and classified the reflection \( K \)-matrices – solutions of the boundary YB equation – for a supersymmetric nineteen vertex model presented by Yang and Zhang in [1], whose \( R \)-matrix is constructed from a three-dimensional representation \( V \) of the quantum affine twisted Lie superalgebra \( U_q[\mathfrak{osp}(2|2)] \). We found three classes of solutions: the type I solution is the most general one, where all elements of the \( K \)-matrix are different from zero and it contains three boundary free-parameters. The type II solution is the most general even solution, on which all elements of the \( K \)-matrix have parity zero; this solution has only one boundary free-parameter. Finally, the type III solution consists in a diagonal \( K \)-matrix with two boundary free-parameters. Particular solutions (including the quantum group invariant solution) can be obtained by given particular values to the boundary free-parameters.

The boundary Bethe Ansatz of this model will be communicated elsewhere [83].

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