Efficient Convex Optimization with Membership Oracles

Yin Tat Lee∗ Aaron Sidford† Santosh S. Vempala ‡

June 23, 2017

Abstract

We consider the problem of minimizing a convex function over a convex set given access only to an evaluation oracle for the function and a membership oracle for the set. We give a simple algorithm which solves this problem with \(O(n^3)\) oracle calls and \(O(n^3)\) additional arithmetic operations. Using this result, we obtain more efficient reductions among the five basic oracles for convex sets and functions defined by Grötschel, Lovasz and Schrijver [5].
1 Introduction

Minimizing a convex function over a convex set is a fundamental problem with many applications. The problem stands at the forefront of polynomial-time tractability and its study has lead to the development of numerous general algorithmic techniques. In recent years, improvements to important special cases (e.g., maxflow) have been closely related to ideas and improvements for the general problem [4, 15, 13, 7, 8, 9, 10, 11, 16].

Here we consider the very general setting where the objective function and feasible region are both presented only as oracles that can be queried, specifically an evaluation oracle for the function and a membership oracle for the set. We study the problem of minimizing a convex function over a convex set provided only these oracles as well as bounds $0 < r < R$ and a point $x_0 \in K$ s.t. $B(x_0, r) \subseteq K \subseteq B(x_0, R)$ where $B(x_0, r)$ is the ball of radius $r$ centered at $x_0 \in \mathbb{R}^n$.

It is well-known that with a stronger separation oracle for the set (and subgradient oracle for the function), this problem can be solved with $\tilde{O}(n)$ oracle queries using any of [18, 2, 11] or with $\tilde{O}(n^2)$ queries by the classic ellipsoid algorithm [5]. Moreover, it is known that the problem can be solved with only evaluation and membership oracles through reductions shown by Grötschel, Lovasz and Schrijver in their classic book [5]. However, the reduction in [5] appears to take at least $n^{10}$ calls to the membership oracle. This has been improved using the random walk method and simulated annealing to $n^{4.5}$ [6, 12] and [1] provides further improvements of up to a factor of $\sqrt{n}$ for more structured convex sets.

Our main result in this paper is an algorithm that minimizes a convex function over a convex set using only $\tilde{O}(n^2)$ membership and evaluation queries. Interestingly, we obtain this result by first showing that we can implement a separation oracle for a convex set and a subgradient oracle for a function using only $\tilde{O}(n)$ membership queries (Section 3) and then using the known reduction from optimization to separation (Section 4). We state the result informally below. The formal statements, which allow an approximate membership oracle, are Theorem 14 and Theorem 15.

**Theorem 1.** Let $K$ be a convex set specified by a membership oracle, a point $x_0 \in \mathbb{R}^n$, and numbers $0 < r < R$ such that $B(x_0, r) \subseteq K \subseteq B(x_0, R)$. For any convex function $f$ given by an evaluation oracle and any $\epsilon > 0$, there is a randomized algorithm that computes a point $z \in B(K, \epsilon)$ such that

$$f(z) \leq \min_{x \in K} f(x) + \epsilon \left( \max_{x \in K} f(x) - \min_{x \in K} f(x) \right)$$

with constant probability using $O \left( n^2 \log^{O(1)} \left( \frac{nR}{\epsilon r} \right) \right)$ calls to the membership oracle and evaluation oracle and $O(n^3 \log^{O(1)} \left( \frac{nR}{\epsilon r} \right))$ total arithmetic operations.

Protasov [14] gives an algorithm for approximately minimizing a convex function defined over an explicit convex body in $\mathbb{R}^n$, using $O(n^2 \log(n) \log(1/\epsilon))$ function evaluations, a logarithmic factor higher. Unfortunately, each iteration of his algorithm requires computing the convex hull, John ellipsoid and centroid of a set maintained by the algorithm, thereby making a very large number of calls to the membership oracle (in [14] the focus is on the number of function calls and it is assumed that the set is known to the algorithm). We remark that using the main idea from our algorithm, Protasov’s method can be made more efficient, resulting in oracle complexity that is only a logarithmic factor higher, although still with a much higher arithmetic complexity than the results of this paper.

In Section 5 we consider to consequences of our main result. In [3], the authors describe five basic problems over convex sets as oracles (OPTimization, SEparation, MEMbership, VIOLation and
VALidity) and give polynomial-time reductions between them. With our new algorithm, several of these reductions become significantly more efficient, as summarized in Theorem 21. In discussing these reductions, it is natural to introduce oracles for convex functions. The relationships between set oracles and function oracles are described in Lemma 19 and those between function oracles in Lemma 20. Figure 1.1 illustrates these relationships and is an updated version of Figure 4.1 from [5]. We suspect that the resulting complexities of reductions are all asymptotically optimal in terms of the dimension, up to logarithmic factors.

2 Preliminaries

Here we introduce notation and terminology. Our conventions are chosen for simplicity and consistency with Grötschel, Lovasz and Schrijver [5]. We use \([n] \overset{\text{def}}{=} \{1, \ldots, n\}\). For a convex function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) and \(x \in \mathbb{R}^n\) we use \(\partial f(x)\) to denote the set of subgradients of \(f\) at \(x\). For \(p > 1, \delta \geq 0, \) and \(K \subseteq \mathbb{R}^n\) we let

\[
B_p(K, \delta) \overset{\text{def}}{=} \{x \in \mathbb{R}^n : \exists y \in K \text{ such that } \|x - y\|_p \leq \delta\}
\]

denote the set of points at distance at most \(\delta\) from \(K\) in \(\ell_p\) norm. For convenience we overload notation and for \(x \in \mathbb{R}^n\) let \(B_p(x, \delta) \overset{\text{def}}{=} B_p(\{x\}, \delta)\) denote the ball of radius \(\delta\) around \(x\). We also let

\[
B_p(K, -\delta) \overset{\text{def}}{=} \{x \in \mathbb{R}^n : B_p(x, \delta) \subseteq K\}
\]

denote the set of points such that the \(\delta\) radius balls centered on them are contained in \(K\). In this notation, whenever \(p\) is omitted it is assumed that \(p = 2\). Furthermore, for any set \(K \subseteq \mathbb{R}^n\) we let \(1_K\) denote a function from \(\mathbb{R}^n\) to \(\mathbb{R} \cup \{+\infty\}\) such that \(1_K(x) = 0\) if \(x \in K\) and \(1_K(x) = +\infty\) otherwise.

2.1 Oracles for Convex Sets

Here we provide the five basic oracles for a convex set, \(K \subseteq \mathbb{R}^n\), defined by Grötschel, Lovasz and Schrijver [5]. We simplify notation slightly by using the same parameter, \(\delta > 0\), to bound both the approximation error and the probability of failure.

**Definition 2** (Optimization Oracle (OPT)). Queried with a unit vector \(c \in \mathbb{R}^n\) and a real number \(\delta > 0\), with probability \(1 - \delta\), the oracle either

- finds a vector \(y \in \mathbb{R}^n\) such that \(y \in B(K, \delta)\) and \(c^T x \leq c^T y + \delta\) for all \(x \in B(K, -\delta)\), or
• asserts that $B(K, -\delta)$ is empty.

We let $\text{OPT}_\delta(K)$ be the time complexity of this oracle.

**Definition 3** (Violation Oracle (VIOL)). Queried with a unit vector $c \in \mathbb{R}^n$, a real number $\gamma$ and a real number $\delta > 0$, with probability $1 - \delta$, the oracle either

• asserts that $c^T x \leq \gamma + \delta$ for all $x \in B(K, -\delta)$, or
• finds a vector $y \in B(K, \delta)$ with $c^T y \geq \gamma - \delta$.

We let $\text{VIOL}_\delta(K)$ be the time complexity of this oracle.

**Definition 4** (Validity Oracle (VAL)). Queried with a unit vector $c \in \mathbb{R}^n$, a real number $\gamma$, and a real number $\delta > 0$, with probability $1 - \delta$, the oracle either

• asserts that $c^T x \leq \gamma + \delta$ for all $x \in B(K, -\delta)$, or
• asserts that $c^T x \geq \gamma - \delta$ for some $x \in B(K, \delta)$.

We let $\text{VAL}_\delta(K)$ be the time complexity of this oracle.

**Definition 5** (Separation Oracle (SEP)). Queried with a vector $y \in \mathbb{R}^n$ and a real number $\delta > 0$, with probability $1 - \delta$, the oracle either

• assert that $y \in B(K, \delta)$, or
• find a unit vector $c \in \mathbb{R}^n$ such that $c^T x \leq c^T y + \delta$ for all $x \in B(K, -\delta)$.

We let $\text{SEP}_\delta(K)$ be the time complexity of this oracle.

**Definition 6** (Membership Oracle (MEM)). Queried with a vector $y \in \mathbb{R}^n$ and a real number $\delta > 0$, with probability $1 - \delta$, either

• assert that $y \in B(K, \delta)$, or
• assert that $y \notin B(K, -\delta)$.

We let $\text{MEM}_\delta(K)$ be the time complexity of this oracle.

### 2.2 Oracles for Convex Functions

Let $f$ be a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$. Recall that the dual function $f^*$ is the convex (Fenchel) conjugate of $f$, defined as

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \langle y, x \rangle - f(x).$$

In particular $f^*(0) = \inf f$. We will use the following two oracles for functions.

**Definition 7** (Evaluation Oracle (EVAL)). Queried with a vector $y$ with $\|y\|_2 \leq 1$ and real number $\delta > 0$ the oracle finds an extended real number $\alpha$ such that

$$\min_{x \in B(y, \delta)} f(x) - \delta \leq \alpha \leq \max_{x \in B(y, \delta)} f(x) + \delta. \quad (2.1)$$

We let $\text{EVAL}_\delta(f)$ be the time complexity of this oracle.
Algorithm 1: $\text{Separate}_{\varepsilon, \rho}(K, x)$

Require: $B_2(0, r) \subset K \subset B_2(0, R)$.

if $\text{MEM}_\varepsilon(K)$ asserts that $x \in B(K, \varepsilon)$ then

Output: $x \in B(K, \varepsilon)$.

else if $x \notin B_2(0, R)$ then

Output: the half space \{ $y : 0 \geq \langle y - x, x \rangle$ \}.

end

Let $\kappa = R/r$, $\alpha_x(d) = \max_{d + \alpha x \in K} \alpha$ and $h_x(d) = -\alpha_x(d) \|x\|_2$.

The evaluation oracle of $\alpha_x(d)$ can be implemented via binary search and $\text{MEM}_\varepsilon(K)$.

Compute $\bar{g} = \text{SeparateConvexFunc}(h_x, 0, r_1, 4\varepsilon)$ with $r_1 = n^{1/6}\varepsilon^{1/3}R^{2/3}\kappa^{-1}$ and the evaluation oracle of $\alpha_x(d)$.

Output: the half space \{ $y : 50\rho n^{7/6}R^{2/3}\kappa^{1/3} \geq \langle \bar{g}, y - x \rangle$ \}.

Definition 8 (Subgradient Oracle (GRAD)). Queried with a vector $y$ with $\|y\|_2 \leq 1$ and real numbers $\delta > 0$, the oracle outputs an extended real number $\alpha$ satisfying (2.1) and a vector $c \in \mathbb{R}^n$ such that

\[
\alpha + c^T(x - y) < \max_{z \in B(x, \delta)} f(z) + \delta \quad \text{for all} \quad x \in \mathbb{R}^n 
\]

(2.2)

We let $\text{GRAD}_\delta(f)$ be the time complexity of this oracle.

3 From Membership to Separation

In this section, we show that how to implement a separation oracle for a convex set using only a nearly linear number of queries to a membership oracle. We divide the construction into two steps. In Section 3.1, we show how to compute an approximate subgradient of a Lipschitz convex function via finite differences. Using this, in Section 3.2 we compute an approximate separating hyperplane for a convex set using a membership oracle for the set. The algorithms are stated in Algorithm 1 and Algorithm 2.

The output of the algorithm for separation is a halfspace that approximately contains $K$ and the input point $x$ is close to its bounding hyperplane. It uses a call to a an subgradient function given below.

3.1 Separation for Lipschitz Convex Function

Here we show how to construct a separation oracle for Lipschitz convex function given an evaluation oracle. Our construction is motivated by the following property of convex functions proved by Bubeck and Eldan [3, Lem 6]: for any Lipschitz convex function $f$, there exists a small ball $B$ such that $f$ restricted on $B$ is close to a linear function. By a small modification of their proof, we show this property in fact holds for almost every small ball (Lemma 9). This can be viewed as a quantitative version of the Alexandrov theorem for Lipschitz convex functions.

Leveraging this powerful fact, our algorithm is simple: we compute a random partial difference in each coordinate to get a subgradient (Algorithm 2). We prove that as long as the box we compute
over sufficiently small and the additive error in our evaluation oracle is sufficiently small, this yields an accurate separation oracle in expectation (Lemma 10). We then obtain high probability bounds using Markov’s inequality.

In our analysis we use * to denote the convolution operator, i.e. \((f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dy\).

**Lemma 9.** For any \(0 < r_2 \leq r_1\) and twice differentiable convex function \(f\) defined on \(B_\infty(x, r_1 + r_2)\) with \(\|\nabla f(z)\|_\infty \leq L\) for any \(z \in B_\infty(x, r_1 + r_2)\) we have

\[
\mathbb{E}_{y \in B_\infty(x_1, r_1)} \mathbb{E}_{z \in B_\infty(y, r_2)} \|\nabla f(z) - g(y)\|_1 \leq n^{3/2} r_2 L
\]

where \(g(y)\) is the average of \(\nabla f\) over \(B_\infty(y, r_2)\).

**Proof.** Let \(h = \frac{1}{(2r_1)^n} f * 1_{B_{\infty}(0, r_2)}\). Integrating by parts, we have that

\[
\int_{B_\infty(x, r_1)} \Delta h(y) dy = \int_{\partial B_\infty(x, r_1)} \langle \nabla h(y), n(y) \rangle dy
\]

where \(\Delta h(y) = \sum_i \frac{\partial^2 h}{\partial x_i^2}(y)\) and \(n(y)\) is the normal vector on \(\partial B_\infty(x, r_1)\) the boundary of the box \(B_\infty(x, r_1)\), i.e. standard basis vectors. Since \(f\) is \(L\)-Lipschitz with respect to \(\|\cdot\|_\infty\) so is \(h\), i.e. \(\|\nabla h(z)\|_\infty \leq L\). Hence, we have that

\[
\mathbb{E}_{y \in B_\infty(x, r_1)} \Delta h(y) \leq \frac{1}{(2r_1)^n} \int_{\partial B_\infty(x, r_1)} \|\nabla h(y)\|_\infty \|n(y)\|_1 dy \leq \frac{1}{(2r_1)^n} \cdot 2n(2r_1)^{-n} \cdot L = \frac{nL}{r_1}.
\]

By the definition of \(h\), we have that

\[
\mathbb{E}_{y \in B_\infty(x, r_1)} \mathbb{E}_{z \in B_\infty(y, r_2)} \Delta f(z) = \mathbb{E}_{y \in B_\infty(x, r_1)} \Delta h(y) \leq \frac{nL}{r_1}. \tag{3.1}
\]

Let \(\omega_i(z) = \langle \nabla f(z) - g(y), e_i \rangle\) for all \(i \in [n]\). Since \(\int_{B_\infty(y, r_2)} \omega_i(z) dz = 0\), the Poincare inequality for a box (see e.g. [17]) shows that

\[
\int_{B_\infty(y, r_2)} |\omega_i(z)| dz \leq r_2 \int_{B_\infty(y, r_2)} \|\nabla \omega_i(z)\|_2^2 dz.
\]

**Algorithm 2: SeparateConvexFunc\((f, x, r_1, \varepsilon)\)**

```plaintext```
Require: \(r_1 > 0\), \(\|\partial f(z)\|_\infty \leq L\) for any \(z \in B_\infty(x, 2r_1)\).

Set \(r_2 = \sqrt{\frac{r_1}{2r_1}}\).

Sample \(y \in B_\infty(x, r_1)\) and \(z \in B_\infty(y, r_2)\) independently and uniformly at random.

for \(i = 1, 2, \cdots, n\) do

  Let \(\alpha_i\) and \(\beta_i\) denote the end points of the interval \(B_\infty(y, r_2) \cap \{z + se_i : s \in \mathbb{R}\}\).

  Set \(\hat{g}_i = \frac{f(\beta_i) - f(\alpha_i)}{2r_2}\) where we compute \(f\) with \(\varepsilon\) additive error.

end

Output \(\hat{g}\) as the approximate subgradient of \(f\) at \(x\).
```
Since \( f \) is convex, we have that \( \|\nabla^2 f(z)\|_F \leq \text{Tr}\nabla^2 f(z) = \Delta f(z) \) and hence
\[
\sum_{i \in [n]} \|\nabla \omega_i(z)\|_2 = \sum_{i \in [n]} \|\nabla^2 f(z)e_i\|_2 \leq \sqrt{n}\|\nabla^2 f(z)\|_F \leq \sqrt{n}\Delta f(z).
\]
Using this with \( \|\nabla f(z) - g(y)\|_1 = \sum_i |\omega_i(z)| \), we have that
\[
\int_{B_{\infty}(y,r_2)} \|\nabla f(z) - g(y)\|_1 \, dz \leq \sqrt{nr_2} \int_{B_{\infty}(y,r_2)} \Delta f(z) \, dz.
\]
Combining with the inequality (3.1) yields the result. \( \square \)

**Lemma 10.** Given \( r_1 > 0 \). Let \( f \) be a convex function on \( B_{\infty}(x, 2r_1) \). Suppose that \( \|\partial f(z)\|_\infty \leq L \) for any \( z \in B_{\infty}(x, 2r_1) \). Also, assume that we can compute function \( f \) with \( \varepsilon \) additive error with \( \varepsilon \leq r_1\sqrt{nL} \). Let \( \tilde{g} = \text{SeparateConvexFunc}(f, x, r_1, \varepsilon) \). Then, there is random variable \( \zeta \geq 0 \) with
\[
\mathbb{E}\zeta \leq 3\sqrt{\frac{L}{r_1}}n^{5/4}
\]
such that
\[
f(q) \geq f(x) + \langle \tilde{g}, q - x \rangle - \zeta \|q - x\|_\infty - 4nr_1L \text{ for all } q \in \Omega.
\]

**Proof.** By limiting argument, we assume that \( f \) is twice differentiable.

First, we assume that we can compute \( f \) exactly, namely \( \varepsilon = 0 \). Fix \( i \in [n] \). Let \( g(y) \) is the average of \( \nabla f \) over \( B_{\infty}(y, r_2) \). Then, we have that
\[
\mathbb{E}_z |\tilde{g}_i - g(y)_i| = \mathbb{E}_z \left| \frac{f(\beta_i) - f(\alpha_i)}{2r_2} - g(y)_i \right|
\]
\[
\leq \mathbb{E}_z \frac{1}{2r_2} \int \left| \frac{df}{dx_i}(z + se_i) - g(y)_i \right| \, ds
\]
\[
= \mathbb{E}_z \left| \frac{df}{dx_i}(z) - g(y)_i \right|
\]
where we used that both \( z + se_i \) and \( z \) are uniform distribution on \( B_{\infty}(y, r_2) \) in the last line. Hence, we have
\[
\mathbb{E}_z \|\tilde{g} - \nabla f(z)\|_1 \leq \mathbb{E}_z \|\nabla f(z) - g(y)\|_1 + \mathbb{E}_z \|\tilde{g} - g(y)\|_1 \leq 2\mathbb{E}_z \|\nabla f(z) - g(y)\|_1.
\]
Now, applying the convexity of \( f \) yields that
\[
f(q) \geq f(z) + \langle \nabla f(z), q - z \rangle
\]
\[
= f(z) + \langle \tilde{g}, q - x \rangle + \langle \nabla f(z) - \tilde{g}, q - x \rangle + \langle \nabla f(z), x - z \rangle
\]
\[
\geq f(z) + \langle \tilde{g}, q - x \rangle - \|\nabla f(z) - \tilde{g}\|_1 \|q - x\|_\infty - \|\nabla f(z)\|_\infty \|x - z\|_1.
\]
Now, \( \|\nabla f(z)\|_\infty \leq L \) and \( \|x - z\|_1 \leq n \cdot \|x - z\|_\infty \leq 2n(r_1 + r_2) \) by assumption. Furthermore, we can apply Lemma 9 to bound \( \|\nabla f(z) - \tilde{g}\|_1 \) and use that \( r_2 = \sqrt{\frac{\varepsilon}{\sqrt{nl}}} \leq r_1 \) to get
\[
f(q) \geq f(z) + \langle \tilde{g}, q - x \rangle - \zeta \|q - x\|_\infty - 4nr_1L
\]
with \( \mathbb{E}\zeta \leq 2n^{3/2}r_2L \).

Since we only compute \( f \) up to \( \varepsilon \) additive error, it introduces \( \frac{\varepsilon}{r_2} \) additive error into \( \tilde{g}_i \). Hence, we instead have that
\[
\mathbb{E}\zeta \leq 2n^{3/2}r_2L + \frac{\varepsilon n}{r_2}.
\]
Putting \( r_2 = \sqrt{\frac{\varepsilon}{\sqrt{nl}}} \), we get the bound. \( \square \)
3.2 Separation for Convex Set

Throughout this subsection, let $K \subseteq \mathbb{R}^n$ be a convex set that contains $B_2(0,r)$ and is contained in $B_2(0,R)$. Given some point $x \notin K$, we wish to separate $x$ from $K$ using a membership oracle. To do this, we reduce this problem to computing an approximate subgradient of a Lipschitz convex function, called $h_x(d)$, the “height” of a point $d$ in the direction of $x$. We let $\alpha_x(d) = \max_{d + x \in K} \alpha$ and define $h_x(d) = -\alpha_x(d) \|x\|_2$. Note that $d + \alpha_x(d)x$ is the last point on the line passing through $d$ and $d + x$ that is in $K$ and that $-h_x(d)$ is the $\ell_2$ distance from this point to $d$.

**Lemma 11.** $h_x(d)$ is convex on $K$.

*Proof.* Let $d_1, d_2 \in K$ and $\alpha \in [0, 1]$ be arbitrary. Now $d_1 + \alpha_x(d_1)x \in K$ and $d_2 + \alpha_x(d_2)x \in K$ and consequently,

$$[\lambda d_1 + (1 - \lambda)d_2] + [\alpha \cdot \alpha_x(d_1) + (1 - \lambda) \cdot \alpha_x(d_2)] \in K.$$ 

Therefore, if we let $d \overset{\text{def}}{=} \lambda d_1 + (1 - \lambda)d_2$ we see that $\alpha_x(d) \geq \lambda \cdot \alpha_x(d_1) + (1 - \lambda) \cdot \alpha_x(d_2)$ and $h_x(\lambda d_1 + (1 - \lambda)d_2) \leq \lambda h_x(d_1) + \lambda h_x(d_2)$.

**Lemma 12.** $h_x$ is $\frac{R + \delta}{r - \delta}$ Lipschitz over points in $B_2(0, \delta)$ for $\delta < r$.

*Proof.* Let $d_1, d_2$ be arbitrary points in $B(0, \delta)$. We wish to upper bound $|h_x(d_1) - h_x(d_2)|$ in terms of $\|d_1 - d_2\|_2$. We assume without loss of generality that $\alpha_x(d_1) \geq \alpha_x(d_2)$ and therefore

$$|h_x(d_1) - h_x(d_2)| = |\alpha_x(d_1) \|x\|_2 - \alpha_x(d_2) \|x\|_2| = (\alpha_x(d_1) - \alpha_x(d_2)) \|x\|_2.$$ 

Consequently, it suffices to lower bound $\alpha_x(d_2)$. We split the analysis into two cases.

Case 1: $\|d_2 - d_1\|_2 \leq r - \delta$. We consider the point $d_3 = d_1 + \frac{d_2 - d_1}{\lambda}$ with $\lambda = \|d_2 - d_1\|_2/(r - \delta)$. Note that

$$\|d_3\|_2 \leq \|d_1\|_2 + \frac{1}{\lambda} \|d_2 - d_1\|_2 \leq \delta + \frac{1}{\lambda} \|d_2 - d_1\|_2 \leq r.$$ 

Hence, $d_3 \in K$. Since $\lambda \in [0, 1]$ and $K$ is convex, we have that $\lambda \cdot d_3 + (1 - \lambda) \cdot [d_1 + \alpha_x(d_1) x] \in K$. Now, we note that

$$\lambda \cdot d_3 + (1 - \lambda) \cdot [d_1 + \alpha_x(d_1)x] = d_2 + (1 - \lambda) \cdot \alpha_x(d_1)x$$

and this shows that

$$\alpha_x(d_2) \geq (1 - \lambda) \cdot \alpha_x(d_1) = \left(1 - \frac{\|d_2 - d_1\|_2}{r - \delta}\right) \cdot \alpha_x(d_1).$$

Since $d_1 + \alpha_x(d_1)x \in K \subset B_2(0,R)$, we have that $\alpha_x(d_1) \cdot \|x\|_2 \leq R + \delta$ and hence

$$|h_x(d_1) - h_x(d_2)| = (\alpha_x(d_1) - \alpha_x(d_2)) \cdot \|x\|_2 \leq \alpha_x(d_1) \cdot \|x\|_2 \cdot \frac{\|d_2 - d_1\|_2}{r - \delta} \leq \frac{R + \delta}{r - \delta} \|d_2 - d_1\|_2.$$ 

Case 2: $\|d_2 - d_1\|_2 \geq r - \delta$. Since $0 \geq h_x(d_1), h_x(d_2) \geq -R - \delta$, we have that

$$|h_x(d_1) - h_x(d_2)| \leq R + \delta \leq \frac{R + \delta}{r - \delta} \|d_2 - d_1\|_2.$$ 

In either case we have that

$$|h_x(d_1) - h_x(d_2)| \leq \frac{R + \delta}{r - \delta} \|d_2 - d_1\|_2$$

yielding the desired result. \qed

\[7\]
Lemma 13. Let $K$ be a convex set satisfying $B_2(0,r) \subset K \subset B_2(0,R)$. Given any $0 < \rho < 1$ and $0 \leq \varepsilon \leq r$. With probability $1 - \rho$, $\text{Separate}_{\varepsilon,\rho}(K, x)$ outputs a half space that contains $K$.

Proof. When $x \notin B_2(0,R)$, the algorithm outputs a valid separation for $B_2(0,R)$. For the rest of the proof, we assume $x \notin B(K,-\varepsilon)$ (due to the membership oracle) and $x \in B_2(0,R)$.

By Lemma 11 and Lemma 12, $h_x$ is convex with Lipschitz constant $3\kappa$ on $B_2(0, \frac{\varepsilon}{2})$. By our assumption on $\varepsilon$ and our choice of $r_1$, we have that $B_{\infty}(0,2r_1) \subset B_2(0, \frac{\varepsilon}{2})$. Hence, we can apply Lemma 10 to get that

$$h_x(y) \geq h_x(0) + \langle \hat{g}, y \rangle - \zeta \|y\|_{\infty} - 12nr_1\kappa$$

(3.2)

for any $y \in K$. Note that $-\frac{\varepsilon}{\kappa} \in K$ and $h_x(-\frac{\varepsilon}{\kappa}) = h_x(0) - \frac{1}{\kappa} \|x\|_2$. Hence, we have

$$h_x(0) - \frac{1}{\kappa} \|x\|_2 = h_x(-\frac{1}{\kappa} x) \geq h_x(0) + \left\langle \hat{g}, -\frac{1}{\kappa} x \right\rangle - \frac{1}{\kappa} \zeta \|x\|_{\infty} - 12nr_1\kappa.$$

Therefore, we have

$$\langle \hat{g}, x \rangle \geq \|x\|_2 - \zeta \|x\|_{\infty} - 12nr_1\kappa^2. \quad (3.3)$$

Now, we note that $x \notin B(K,-\varepsilon)$. Using that $B(0,r) \subset K$, we have $(1 - \frac{\varepsilon}{\kappa})K \subset B(K,-\varepsilon)$. Hence,

$$h_x(0) \geq -\left(1 - \frac{\varepsilon}{r}\right) \|x\|_2 \geq -\|x\|_2 + \varepsilon \kappa.$$

Therefore, we have

$$h_x(0) + \langle \hat{g}, x \rangle \geq -\zeta \|x\|_{\infty} - 12nr_1\kappa^2 - \varepsilon \kappa$$

Combining this with (3.2), we have that

$$h_x(y) \geq \langle \hat{g}, y - x \rangle - \zeta \|y\|_{\infty} - \zeta \|x\|_{\infty} - 12nr_1\kappa - 12nr_1\kappa^2 - \varepsilon \kappa$$

$$\geq \langle \hat{g}, y - x \rangle - 2\zeta R - 24nr_1\kappa^2 - \varepsilon \kappa.$$

for any $y \in K$. Recall from Lemma 10 that $\zeta$ is a positive random scalar independent of $y$ satisfying $\mathbb{E} \zeta \leq 3\sqrt{\frac{12\kappa}{r_1}}n^{5/4}$. For any $y \in K$, we have that $h_x(y) \leq 0$ and hence $\hat{\zeta} \geq \langle \hat{g}, y - x \rangle$ where $\hat{\zeta}$ is a random scalar independent of $y$ satisfying

$$\mathbb{E} \hat{\zeta} \leq 6\sqrt{\frac{12\kappa}{r_1}}n^{5/4}R + 24nr_1\kappa^2 + \varepsilon \kappa$$

$$\leq 45n^{7/6}R^{2/3}\varepsilon^{1/3} + \varepsilon \kappa$$

$$\leq 50n^{7/6}R^{2/3}\varepsilon^{1/3} + \varepsilon \kappa$$

where we used $0 \leq \varepsilon \leq r$ at the end. The result then follows from this and Markov inequality.

Theorem 14. Let $K$ be a convex set satisfying $B_2(0,1/\kappa) \subset K \subset B_2(0,1)$. For any $0 \leq \eta < \frac{1}{2}$, we have that

$$\text{SEP}_\eta(K) \leq O\left(n \log \left(\frac{nk}{\eta}\right)\right) \mathbb{M}w_{(1)}(K).$$
Proof. First, we bound the running time. Note that the bottleneck is to compute $h_x$ with $\varepsilon$ additive error. Since $-O(1) \leq h_x(y) \leq 0$ for all $y \in B_2(0, O(1))$, one can compute $h_x(y)$ by binary search with $O(\log(1/\delta))$ calls to the membership oracle.

Next, we check that $\text{Separate}_{\delta, \rho}(K, x)$ is indeed a separation oracle. Note that $\tilde{g}$ may not be an unit vector and we need to re-normalize the $\tilde{g}$ by $1/\|\tilde{g}\|_2$. So, we need to a lower bound $\|\tilde{g}\|_2$.

From (3.3) and our choice of $r_1$, if $\delta \leq \rho$, then we have that

$$\langle \tilde{g}, x \rangle \geq \|x\|_2 - \zeta \|x\|_\infty - 12nr_1\kappa^2 \geq \frac{r}{4}.$$ 

Hence, we have that $\|\tilde{g}\|_2 \geq \frac{1}{4\kappa}$. Therefore, this algorithm is a separation oracle with error $\frac{2n\eta^7/6\kappa^2\delta^{1/3}}{\rho}$ and failure probability $O(\rho + \log(1/\delta)\delta)$.

$$\text{SEP}_{\Omega(\max(n^{7/6}\kappa^2\delta^{1/3}/\rho + \rho + \log(1/\delta))\delta)}(K) \leq O(\log(1/\delta))\text{MEM}_{\delta}(K).$$

Setting $\rho = \sqrt{n^{7/6}\kappa^2\delta^{1/3}}$ and $\delta = \Theta\left(\frac{n^6}{n^{7/6}\kappa^2}\right)$, we have that

$$\text{SEP}_{\eta}(K) \leq O(\log(n^{7/6}\kappa^2\delta))\text{MEM}_{\eta^6/(n^{7/6}\kappa^2)}(K).$$

\[ \Box \]

4 From Separation to Optimization

Once we have a separation oracle, our running times follow by applying a recent convex optimization algorithm by Lee, Sidford and Wong [11]. Previous algorithms also achieved $\tilde{O}(n)$ oracle complexity, but needed a higher polynomial number of arithmetic operations. We remark that the theorem stated in [11] is slightly more general then the one we give below, but since we only need to minimize linear functions over convex sets, we state a simplified version here.

Theorem 15 (Theorem 42 of [11] Rephrased). Let $K$ be a convex set satisfying $B_2(0, r) \subset K \subset B_2(0, 1)$ and let $\kappa = 1/r$. For any $0 < \varepsilon < 1$, with probability $1 - \varepsilon$, we can compute $x \in B(K, \varepsilon)$ such that

$$c^T x \leq \min_{x \in K} c^T x + \varepsilon \|c\|_2$$

with an expected running time of

$$O\left(n\text{SEP}_{\delta}(K) \log\left(\frac{n^6}{\varepsilon}\right) + n^3 \log^{O(1)}\left(\frac{n^6}{\varepsilon}\right)\right),$$

where $\delta = \left(\frac{n^6}{n^{7/6}\kappa^2}\right)^{\Theta(1)}$. In other words, we have that

$$\text{OPT}_{\varepsilon}(K) = O\left(n\text{SEP}_{\Theta(1)}(K) \log\left(\frac{n^6}{\varepsilon}\right) + n^3 \log^{O(1)}\left(\frac{n^6}{\varepsilon}\right)\right).$$

5 Reductions Between Oracles

In this section, we provide all other reductions among oracles defined in Section 2.1. To simplify notation we assume the convex set is contained in the unit ball and convex function is defined on the unit ball. This can be done without loss of generality by scaling and shifting.

We remark that it is known that OPT and VIOL are equivalent up to the cost of a binary search.
Lemma 16 (Equivalence between OPT and VIOL). Given a convex set $K$ contained in the unit ball, we have that $\text{VIOL}_\delta(K) \leq \text{OPT}_\delta(K)$ and $\text{OPT}_\delta(K) \leq O(\log(1 + 1/\delta)) \cdot \text{VIOL}_\delta(K)$ for any $\delta > 0$.

Hence, we ignore VIOL for the remainder of this section.

5.1 Relationships between Set oracles and Function Oracles

Next, to handle all these relationships efficiently, we find it convenient to instead look at oracles on convex functions and connect them to set oracles. For this purpose we note the following simple relationship between MEM($K$) and EVAL($1_K$) and between SEP($K$) and GRAD($1_K$).

Lemma 17 (MEM($K$) and SEP($K$) are membership and subgradient oracle of $1_K$). For any convex set $K \subseteq \mathbb{R}^n$, we have that $\text{MEM}_\delta(K) = \text{EVAL}_\delta(1_K)$ and $\text{SEP}_\delta(K) = \text{GRAD}_\delta(1_K)$ for any $\delta > 0$.

Next, we note that the relationship between VAL($K$) and EVAL($1_K^*$) and between OPT($K$) and GRAD($1_K^*$).

Lemma 18 (VAL($K$) and OPT($K$) are membership and subgradient oracle of $1_K^*$). Given a convex set $K$. Suppose that $B(\bar{0}, r) \subseteq K \subseteq B(\bar{0}, 1)$ and let $\kappa = 1/r$. For any $\delta > 0$, we have that

- $\text{VAL}_\delta(K) \leq \text{EVAL}_\delta(1_K^*)$ and $\text{EVAL}_\delta(1_K^*) \leq O(\log(\kappa/\delta)) \cdot \text{VAL}_\delta(\delta/(\kappa \cdot \log(1/\delta)))(K)$.
- $\text{OPT}_\delta(K) \leq \text{GRAD}_\delta(1_K^*)$ and $\text{GRAD}_\delta(1_K^*) \leq \text{OPT}_\delta/(3+\kappa)(K)$.

where the oracle for $1_K^*$ is only defined on the unit ball.

Proof. For the first inequality, to implement the validity oracle, we need to compute $\beta$ such that

$$\max_{x \in B(K, -\delta)} c^T x - \beta \leq \min_{x \in B(K, \delta)} c^T x + \delta$$

for any unit vector $c$ and $\delta > 0$. We note that

$$\min_{x \in B(c, \frac{\delta}{\kappa})} 1_K^*(x) \geq 1_K^*(c) - \delta = \max_{x \in K} c^T x - \delta \geq \max_{x \in B(K, -\delta)} c^T x.$$ 

Therefore, (5.1) shows that the output $\alpha$ by $\text{EVAL}_\delta(1_K^*)$ with input $-c$ satisfies $\max_{x \in B(K, -\delta)} c^T x \leq \alpha + \delta$. Similarly, we have that $\min_{x \in B(K, \delta)} c^T x \geq \alpha - \delta$. Thus, the output of $\text{EVAL}_\delta(1_K^*)$ satisfies the condition (5.1). Hence, we have that $\text{VAL}_\delta(K) \leq \text{EVAL}_\delta(1_K^*)$.

For the second inequality, to implement the evaluation oracle of $1_K^*$, we need to compute $1_K^*(c) = \max_{x \in K} c^T x$ for any vector $c$ with $\|c\|_2 \leq 1$. Using that $B(0, r) \subseteq K$, we have $(1 - \frac{\delta}{r}) K \subseteq B(K, -\delta)$. Hence, we have that

$$\max_{x \in B(K, -\delta)} c^T x \geq (1 - \frac{\delta}{r}) \max_{x \in K} c^T x \geq \max_{x \in K} c^T x - \kappa \delta.$$ 

On the other hand, we have that

$$\max_{x \in B(K, \delta)} c^T x \leq \max_{x \in K} c^T x + \delta.$$ 

Hence, by binary search on $\gamma$, $\text{VAL}_\delta(K)$ allows us to estimate $\max_{x \in K} c^T x$ up to $2(2 + \kappa)\delta$ additive error.
For the third inequality, to implement the optimization oracle, we let \( c \) be the vector we want to optimize. Let \( x \) be the output of \( \text{GRAD}_\eta(1_K^*) \) on input \( c \). Using (2.2) and (2.1), we have that

\[
\min_{z \in B(c, \eta)} 1_K^*(z) + x^T(d - c) < \max_{z \in B(c, \eta)} 1_K^*(z) + 2\eta
\]

for any vector \( d \). Since \( 1_K^* \) is \( R \)-Lipschitz, we have that

\[
1_K^*(c) + x^T(d - c) < 1_K^*(d) + 4\eta.
\]

Putting \( d = 0 \), we have

\[
\max_{x \in K} c^T x = 1_K^*(c) \leq c^T x + 4\eta.
\]

Setting \( \eta = \delta/4 \), we see that \( x \) is a maximizer of \( \max_{x \in K} c^T x \) up to \( \delta \) additive error.

For the fourth inequality, to implement the subgradient oracle, we let \( c \) be the point we want to compute the subgradient such that \( \|c\|_2 \leq 1 \). Let \( y \) be the output of \( \text{OPT}_\delta(K) \) with input \( c \). Since \( (1 - \frac{\delta}{4}) K \subset B(K, -\delta) \), we have that

\[
\max_{x \in K} c^T x \leq c^T y + \delta + \kappa\delta.
\]

Therefore,

\[
c^T y + (d - c)^T y \leq \max_{x \in K} c^T x + \delta + (d - c)^T y \leq d^T y + (2 + \kappa)\delta \leq \max_{x \in K} d^T x + (3 + \kappa)\delta.
\]

Let \( \alpha = c^T y \). Since \( y \in B(K, \delta) \) and satisfies the guarantee of optimization oracle, \( \alpha \) satisfies (2.1) with additive error \( \delta \). Furthermore, we note that

\[
\alpha + y^T(d - c) \leq 1_K^*(d) + (3 + \kappa)\delta.
\]

Hence, it satisfies (2.2) with additive error \( (3 + \kappa)\delta \). \( \square \)

**Lemma 19.** Given a convex function \( f : B_n \to [0, 1] \), let \( K_f = \{(\frac{2}{3}, \frac{1}{4})\} \) such that \( x \in B(0, 1) \) and \( f(x) \leq t \leq 2 \). Then,

- \( \text{MEM}_\delta(K_f) \leq \text{EVAL}_{\delta/10}(f) \) and \( \text{EVAL}_{\delta}(f) \leq O(\log(1/\delta)) \text{MEM}_{\delta/(\log(1/\delta))}(K_f) \).
- \( \text{SEP}_{\delta}(K_f) \leq \text{GRAD}_{\delta/10}(f) \) and \( \text{GRAD}_{\delta}(f) \leq O(\log(1/\delta)) \text{SEP}_{\delta/(\log(1/\delta))}(K_f) \).
- \( \text{GRAD}_{\delta}(f^*) \leq \text{OPT}_{\delta/6}(K_f) \).

**Proof.** The first two sets of reductions are clear.

For the last one, to implement the subgradient oracle, we let \( c \) be the point we want to compute the subgradient such that \( \|c\|_2 \leq 1 \). Let \( (y, t') \) be the output of \( \text{OPT}_\delta(K_f) \) with input \((c, -1)\). Since \( (1 - 4\delta) K_f \subset B(K_f, -\delta) \), we have that

\[
\max_{(z, t) \in K_f} (c^T x - t) \leq c^T y - t' + 5\delta.
\]

Since \( (y, t') \in B(K_f, \delta) \), for any vector \( d \), we have that

\[
(c^T y - t') + (d - c)^T y \leq \max_{(z, t) \in K_f} (c^T x - t) + (d - c)^T y \leq d^T y - t' + 5\delta \leq \max_{(z, t) \in K_f} (d^T x - t) + 6\delta.
\]
Let $\alpha = c^T y - t'$. Since $y \in B(K, \delta)$ and satisfies the guarantee of optimization oracle, $\alpha$ is a good enough approximation of $f^*(c)$. Furthermore, we note that

$$\alpha + y^T (d - c) \leq \max_{(x, t) \in K_f} d^T x + 5\delta = f^*(d) + 6\delta$$

Hence, it satisfies with additive error $6\delta$.

### 5.2 Relationships Between Convex Function Oracles

Due to the equivalences above, we can focus on the more general problem: the relationships between

- $EVAL_{\delta}(f)$, $GRAD_{\delta}(f)$, $EVAL_{\delta}(f^*)$, $GRAD_{\delta}(f^*)$.

**Lemma 20.** Given a convex function $f$ defined on unit ball with value between 0 and 1. For any $0 \leq \delta \leq \frac{1}{2}$, we have that

- $EVAL_{\delta}(f) \leq GRAD_{\delta}(f) \leq O(n \log^2 (\frac{n}{\delta})) MEM_{(\delta/n)O(1)}(K_f) \leq O(n \log^2 (\frac{n}{\delta})) EVAL_{(\delta/n)O(1)}(f)$
- $GRAD_{\delta}(f^*) \leq OPT_{\delta/(6\delta)}(K_f)$ and
  $$OPT_{\delta/(6\delta)}(K_f) \leq O\left(n \text{SEP}_{(\delta/(n)O(1)}(K_f) \log \left(\frac{n}{\delta}\right) + n^3 \log^{O(1)} \left(\frac{n}{\delta}\right)\right)$$
  $$\leq O\left(n \log \left(\frac{n}{\delta}\right) \cdot GRAD_{(\delta/n)O(1)}(f) + n^3 \log^{O(1)} \left(\frac{n}{\delta}\right)\right).$$

**Proof.** The bound $EVAL_{\delta,\eta}(f) \leq GRAD_{\delta,\eta}(f)$ is immediate from definition.

To bound $GRAD(f)$ by $EVAL(f)$, we use Lemma [19] and get that

$$GRAD_{\delta}(f) \leq O(\log(\delta^{-1})) \text{SEP}_{\Omega(\delta/\log(\delta^{-1}))}(K_f).$$

Next, we note that $B(0, 0.1) \subset K_f \subset B(0, 1)$. Hence, Theorem [14] shows that

$$\text{SEP}_{\delta}(K_f) \leq O(n \log(n/\delta)) MEM_{(\delta/n)O(1)}(K_f).$$

Hence, we have that

$$GRAD_{\delta}(f) \leq O(n \log^2 (n/\delta)) MEM_{(\delta/n)O(1)}(K_f).$$

Applying Lemma [19] again, we have the result.

To bound $GRAD(f^*)$ by $GRAD(f)$, we again use Lemma [19] and Theorem [15] to get

$$GRAD_{\delta}(f^*) \leq OPT_{\delta/(6\delta)}(K_f) \leq O\left(n \text{SEP}_{(\delta/(n)O(1)}(K_f) \log \left(\frac{n}{\delta}\right) + n^3 \log^{O(1)} \left(\frac{n}{\delta}\right)\right)$$

$$\leq O\left(n \log \left(\frac{n}{\delta}\right) \cdot GRAD_{\delta}(f) + n^3 \log^{O(1)} \left(\frac{n}{\delta}\right)\right).$$

\[\square\]
5.3 Relationships Between Convex Set Oracles

Theorem 21. For any convex set $K$ such that $B(0, 1/\kappa) \subset K \subset B(0, 1)$, for any $0 < \delta < \frac{1}{2}$, we have that

1. $\text{VIOL}_\delta(K) \leq \text{OPT}_\delta(K)$ and $\text{OPT}_\delta(K) \leq O(\log(1 + \frac{1}{\delta})) \cdot \text{VIOL}_{\Theta(\delta / \log(1/\delta))}(K)$.

2. $\text{MEM}_\delta(K) \leq \text{SEP}_\delta(K)$ and $\text{SEP}_\delta(K) \leq O(n \log(\frac{n\delta}{\kappa})) \cdot \text{MEM}_{\Theta(\delta/n\kappa)}(K)$.

3. $\text{VAL}_\delta(K) \leq \text{OPT}_\delta(K)$ and $\text{OPT}_\delta(K) \leq O(n \log(3) \cdot \text{VAL}_{\Theta(\delta/n\kappa)}(K))$.

4. $\text{OPT}_\delta(K) = O\left(n \log\left(\frac{n\delta}{\kappa}\right) \cdot \text{SEP}_{\Theta(\delta/n\kappa)}(K) + n^3 \log O(1)\left(\frac{n\delta}{\kappa}\right)\right)$.

5. $\text{SEP}_\delta(K) = O\left(n \log\left(\frac{n\delta}{\kappa}\right) \cdot \text{OPT}_{\Theta(\delta/n\kappa)}(K) + n^3 \log O(1)\left(\frac{n\delta}{\kappa}\right)\right)$.

Proof. (1) follows from Lemma 11. (2) follows from Theorem 12. (4) follows from Theorem 15. For (3), we use Lemma 18, 20 and 18 to get

$\text{OPT}_\delta(K) \leq \text{GRAD}_{\delta/4}(1^*_K) \leq O(n \log(\frac{n}{\delta})) \cdot \text{EVAL}_{\Theta(\delta/n)}(1^*_K)$

$\leq O(n \log(\frac{n\delta}{\kappa}) \cdot \text{VAL}_{\Theta(\delta/n\kappa)}(K))$

where we used that $1^*_K$ is a function between 0 and 1.

For (5), we use Lemma 13 and 18 to get

$\text{SEP}_\delta(K) = \text{GRAD}_{\delta/4}(1^*_K) \leq O\left(n \log\left(\frac{n\delta}{\kappa}\right) \cdot \text{GRAD}_{\Theta(\delta/n\kappa)}(1^*_K) + n^3 \log O(1)\left(\frac{n\delta}{\kappa}\right)\right)$

$\leq O\left(n \log\left(\frac{n\delta}{\kappa}\right) \cdot \text{OPT}_{\Theta(\delta/n\kappa)}(K) + n^3 \log O(1)\left(\frac{n\delta}{\kappa}\right)\right)$

where we used that $1^*_K$ is a function between 0 and 1. \hfill $\Box$

Acknowledgments

The authors thank Sébastien Bubeck, Ben Cousins, Sham M. Kakade and Ravi Kannan for helpful discussions, and Yan Kit Chim for making the illustrations.

References

[1] Jacob D. Abernethy and Elad Hazan. Faster convex optimization: Simulated annealing with an efficient universal barrier. In Proceedings of the 33nd International Conference on Machine Learning, ICML 2016, New York City, NY, USA, June 19-24, 2016, pages 2520–2528, 2016.

[2] Dimitris Bertsimas and Santosh Vempala. Solving convex programs by random walks. Journal of the ACM (JACM), 51(4):540–556, 2004.

[3] Sébastien Bubeck and Ronen Eldan. Multi-scale exploration of convex functions and bandit convex optimization. arXiv preprint arXiv:1507.06580, 2015.
[4] Paul Christiano, Jonathan A Kelner, Aleksander Madry, Daniel A Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 273–282. ACM, 2011.

[5] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Algorithms and Combinatorics, 1988.

[6] A. T. Kalai and S. Vempala. Simulated annealing for convex optimization. *Math. Oper. Res.*, 31(2):253–266, 2006.

[7] Jonathan A Kelner, Yin Tat Lee, Lorenzo Orecchia, and Aaron Sidford. An almost-linear-time algorithm for approximate max flow in undirected graphs, and its multicommodity generalizations. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 217–226. SIAM, 2014.

[8] Yin Tat Lee, Satish Rao, and Nikhil Srivastava. A new approach to computing maximum flows using electrical flows. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 755–764. ACM, 2013.

[9] Yin Tat Lee and Aaron Sidford. Path finding methods for linear programming: Solving linear programs in \( \alpha(\sqrt{\text{rank}}) \) iterations and faster algorithms for maximum flow. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pages 424–433. IEEE, 2014.

[10] Yin Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 230–249. IEEE, 2015.

[11] Yin Tat Lee, Aaron Sidford, and Sam Chiu-wai Wong. A faster cutting plane method and its implications for combinatorial and convex optimization. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 1049–1065. IEEE, 2015.

[12] L. Lovász and S. Vempala. Fast algorithms for logconcave functions: sampling, rounding, integration and optimization. In *FOCS*, pages 57–68, 2006.

[13] Aleksander Madry. Navigating central path with electrical flows: From flows to matchings, and back. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 253–262. IEEE, 2013.

[14] V. Yu. Protasov. Algorithms for approximate calculation of the minimum of a convex function from its values. *Mathematical Notes*, 59(1):69–74, 1996.

[15] Jonah Sherman. Nearly maximum flows in nearly linear time. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 263–269. IEEE, 2013.

[16] Jonah Sherman. Area-convexity, \( l_\infty \) regularization, and undirected multicommodity flow. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 452–460, 2017.

[17] Stefan Steinerberger. Sharp \( l_1 \)-poincaré inequalities correspond to optimal hypersurface cuts. *Archiv der Mathematik*, 105(2):179–188, 2015.
[18] P. M. Vaidya. A new algorithm for minimizing convex functions over convex sets. *Math. Prog.*, 73:291–341, 1996.