Spectral characteristics of a fully-superconducting SQUIPT

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We consider properties of a fully superconducting variant of the superconducting quantum interference proximity transistor, a magnetic flux sensor. We study the density of states in a finite-size superconducting metal wire in the diffusive limit, and how it depends on the phase gradient of the order parameter. We describe the dependence on the junction length and interface transparency, and discuss properties relevant for using the structure in magnetic flux detection applications.

I. INTRODUCTION

Superconductivity on the scale of the coherence length is sensitive to its surroundings. This can be used to modulate the density of states in a mesoscopic metal wire via magnetic flux, by imposing a phase gradient via embedding the wire as a weak link in a superconducting ring. Detecting the modulation with a tunnel junction probe attached to the weak link is the basis of the superconducting quantum interference proximity transistor (SQUIPT), which can be used as a magnetic field sensor. [1]

Previous experiments mostly employed normal-state metal wires. [1] For practical purposes, it however can be advantageous if also the weak link is made of a superconducting rather normal-state material: sample fabrication is simpler, quality of the contacts can be better, and the quality of the energy gap can improve, resulting to better sensitivity. The intrinsic superconductivity however modifies the current-phase relation, which determines what phase gradient can be imposed, and affects the detailed form of the density of states.

The current-phase relation in SS'S junctions has been extensively studied in the past, and is largely understood. [9–13] The density of states (DOS) is moreover well-studied in the normal-state case. [14–19] The DOS in a phase-biased superconducting wire, which is the case relevant for the fully superconducting SQUIPT, appears to have received somewhat less comment. [20–22]

In this work, we investigate the density of states and other characteristics of a phase-biased superconducting wire, such as those embedded as a part of a SQUIPT structure, as depicted in Fig. 1. We approach the problem of finding the current-phase relation and spectral characteristics from a functional minimization perspective. We describe the evolution of the current-phase relation and the density of states between short and long junction limits. Finite-size effects in the DOS turn out to decay as a power law with increasing system length, rather than exponentially as they do for the current-phase relation. We discuss the effect of differing materials in the ring and wire parts, the impact of interface transparency, and some practical consequences for the S-SQUIPT application.

II. MODEL

We consider the SQUIPT setup depicted in Fig. 1, where a mesoscopic conventional superconductor (S) metal wire of length $L$ and cross section $A$ is embedded as a part of a superconducting ring. The density of states in the wire is probed by the current-voltage characteristics of a normal (N) or superconducting tunnel junction. The critical temperature of the ring material is $T_{c,R}$, and that of the wire material is $T_{c,w}$, corresponding to the superconducting coupling constants $\lambda_R$, $\lambda_c$ and the zero-temperature values $\Delta_{0,R}$, $\Delta_{0,w}$ of the energy gap via the BCS relations. Possible dependence of the superconducting coupling on the film thickness is included in these parameters. We assume the dimensions of the wire are small compared to the ring, so that the presence of the wire has small effect on the superconductivity of the ring.

The superconducting properties in the diffusive limit at equilibrium are conveniently described by the non-linear $\sigma$-model. [23–26] The approach encompasses the well-known quasiclassical Green function theory of diffusive superconducting systems as a special case, but can also be used to study eg. fluctuation effects. Here, we consider only equilibrium properties of quasi-1D systems in the semiclassical approximation — in this case
the advantage is in directly specifying the problem in variational form. At the classical saddle point the free energy contribution is \( \delta F = \mathcal{A} \nu_F [F_b + \int_0^L dxF_0] \), where the density reads (cf. Ref. [28])

\[
F_0 = \frac{|\Delta|^2}{\lambda} + 2\pi T \sum_{\omega_n > 0} \left\{ \frac{D(\partial_x \chi - 2A)^2}{2} \sin^2 \vartheta + \frac{D(\partial_x \vartheta)^2}{2} \right\} \tag{1}
\]

\[
+ 2\omega_n (1 - \cos \vartheta) - 2|\Delta| \cos(\chi - \phi) \sin \vartheta \cdot \delta F\]

Here, \( \omega_n = 2\pi T (n + \frac{1}{2}) \) are Matsubara frequencies, \( \nu_F \) the DOS per spin at the Fermi level, and \( \lambda \) the vector potential. \( D \) is the diffusion constant and \( \Delta = |\Delta| e^{i\phi} \) the order parameter. Moreover, \( \lambda \) is the superconducting interaction constant. The parameters \( \chi \) and \( \vartheta \) are real-valued, and at saddle points related to the quasiclassical anomalous Green function \( f(x, \omega_n) = e^{i\chi(x, \omega_n)} \sin \theta(x, \omega_n) \). [27]

Here and below, we use units \( k_B = e = \hbar = 1 \). For taking the weak coupling limit \( \lambda \to 0 \), \( \Delta_{0,0} = \) const., it is useful to add and subtract terms to eliminate the implicit cutoffs,

\[
F_0 \approx |\Delta|^2 \ln \frac{T}{T_c} + 2\pi T \sum_{\omega_n > 0} \frac{|\Delta|^2}{\omega_n} + \frac{D(\partial_x \chi - 2A)^2}{2} \sin^2 \vartheta \tag{2}
\]

\[
+ \frac{D(\partial_x \vartheta)^2}{2} + 2\omega_n (1 - \cos \vartheta) - 2|\Delta| \cos(\chi - \phi) \sin \vartheta \cdot \delta F\]

The dependence on \( \lambda \) is now contained in \( T_c \), assumed to be of the BDS weak-coupling form.

The connection of the wire to the superconducting ring is described via a tunneling boundary term. [29] [30]

\[
F_b = \frac{2\pi TD}{r} \sum_{\omega_n > 0, j = \pm} \left[ 1 - \cos \vartheta(x_j) \cos \vartheta_S \right] \tag{3}
\]

\[
- \cos(\chi(x_j) - \chi_S) \sin \vartheta(x_j) \sin \vartheta_S \]

\[
= \cos(\chi(x_j) - \chi_S) \sin \vartheta(x_j) \sin \vartheta_S \]

where \( r = 2R_t A \nu_F \) is the ratio of interface resistance \( R_t \) to the resistance per length of the wire, and \( \theta_S = \arctan(\Delta_{0,0}) \), \( \chi_s = 0 \) and \( \chi_{s+} = \phi \) are the values inside the ring at the left \( (x = 0) \) and right \( (x = L) \) interfaces.

Requiring variation of Eq. (2) vs. \( \vartheta \) and \( \chi \) to vanish, and analytical continuation to real axis \( i\omega \to E + i0^+ \) and defining \( \theta \equiv -i\vartheta \) produces the standard quasiclassical real-time description of the system via the Usadel equation, [27] [31] which can be written in the form

\[
D \partial_x^2 \theta = -2ie\sin \theta + \frac{D(\partial_x \chi)^2}{2} \sinh 2\theta \tag{4}
\]

\[
+ 2|\Delta| \cos(\phi - \chi) \cosh \theta \cdot \delta F\]

\[
D \partial_x \cdot (\partial_x \chi \sin^2 \theta) = -2|\Delta| \sin(\chi - \phi) \sinh \theta \cdot \delta F\]

The self-consistency equation for the order parameter \( \Delta \) is obtained similarly,

\[
|\Delta| \ln \frac{T}{T_c} = 2\pi T \sum_{\omega_n > 0} \left[ e^{i(\chi - \phi) - \frac{|\Delta|}{E}} \right]_{E = i\omega_n} \tag{5}
\]

The boundary term generates the boundary conditions, [29]

\[
\mp r \partial_x \chi = \sin(\chi - \chi_S) \sinh \theta \cdot \delta F\]

\[
\mp r \partial_x \theta = \sin \theta \cosh \theta_S - \cos(\chi - \chi_S) \cosh \theta \sinh \theta \cdot \delta F\]

at the left (-) and right (+) interfaces. In the clean-interface limit \( r \to 0 \) these reduce to continuity conditions \( \chi(x_{\pm}) = \chi_{S_{\pm}}, \theta(x_{\pm}) = \theta_{S_{\pm}} \). Note also that in this case \( F_b \to 0 \) as \( r \to 0 \).

The reduced density of states \( N(E, x) = \nu(E, x) / \nu_F \) is \( \nu_F = \text{Re} \cosh \theta(E, x) \) and current \( I(x) = -\frac{\partial F}{\partial x} |_{A=0} = 4\pi T A \nu_F \sum_{\omega_n > 0} \partial_x \chi \sin^2 \vartheta \cdot \delta F \) follow directly from the solutions of the equation set.

It is well known that there are multiple classical solutions, corresponding to different windings of the superconducting phase along the superconducting wire. In the numerical solution of the equation set, to handle this and to obtain also the solutions along the unstable branches, we use the pseudo-arclength continuation method applied on the self-consistency equation. This method is useful for tracing a continuous solution branch \( (\varphi, \Delta) \) without requiring the existence of a single-valued function \( \Delta(\varphi) \) (see Appendix A for details).

Within the approximations here, the configuration minimizing \( \delta F \) should be considered as the stable solution. Along a continuous solution branch, this can also be evaluated via a standard relation,

\[
\delta F[X_{\ast}(\varphi); \varphi] - \delta F[X_{\ast}(0); 0] = \frac{1}{2} \int_0^\varphi d\varphi' I[X_{\ast}(\varphi'); \varphi'] \tag{9}
\]

based on the current evaluated at a stationary point \( X_{\ast} = (\theta_{\ast}, \chi_{\ast, i}, \Delta_{\ast, i}, \phi_w) \). This follows from Eqs. [2], [3] by noting the gauge transform \( F[\hat{\theta}, \chi, \Delta, \phi; \varphi, A] = F[\hat{\theta}, \chi - \xi - \Delta, \phi - \xi(L), \varphi - \xi(L), 0] \) for \( \xi(x) = 2 \int_0^x dx' A(x') \) and that \( \delta F / \delta X |_{X = X_{\ast}} = 0 \). Energy differences between disconnected branches however need to be determined from Eqs. [2], [3].

III. DENSITY OF STATES AND CURRENT

The density of states in S-SQUIPT is sensitive to the phase point where the system is biased. To specify this, we need to comment on the supercurrent in the junction. The form of the current-phase relation (CPR) in superconducting strips is well studied under the approximations outlined above. [9] [13] CPRs computed from the Usadel equation are illustrated in Fig. 2 for reference; see also Ref. [11] [33]. For junctions short compared to the coherence length, \( L < \xi_w \equiv \sqrt{D / (2\pi T_{c,w})} \), the CPR is a (deformed) sinusoid, described by a known analytical solution. [8] [32] In long junctions, the CPR becomes multivalued, corresponding to multiple winding of the order
FIG. 2. (a–c) Current-phase relation vs. junction length and temperature \((T/T_{c,w} = 0.1, 0.5, 0.9)\), for \(T_{c,R} / T_{c,w} = 1.5\). Here, \(\xi_{w} = \sqrt{D/(2\pi T_{c,w})}\). Analytical short-junction result 33 (dashed) and the GL result discussed in the text (dotted) are also shown. (d) Free energy change corresponding to curves in (c), as obtained from Eq. (2) or (9). Backward branches are shown with dashed lines.

FIG. 3. Normalized inductance of the superconducting wire at \(\varphi = \pi\), for different \(T_{c,R}\) and \(L\), at \(T/T_{c,w} = 0.01, 0.1, 0.5\) (from bottom to top). Solid lines denote the short-junction limit 31.

For \(L \gg \xi_{GL} = \sqrt{D}\pi/[8(T_{c} - T)]\) one expects a CPR \(I = (A\sigma_{N}\pi\Delta^{2}/(4TL))\varphi[1 - (\varphi\xi_{GL}/L)^{2}]\), up to the point \(\varphi < \varphi_{\text{max}} \approx L/(\sqrt{3}\xi_{GL})\). After this, the solution transitions to a backward branch reaching to \(\varphi = \pi, I = 0\),

\[
I = \frac{A\sigma_{N}\pi\Delta^{2}}{2T\xi_{GL}}a\sqrt{a - k}, \quad \varphi = \frac{L}{\xi_{GL}}\sqrt{a - k} + \frac{1}{a}\arcsin\sqrt{\frac{k}{a}},
\]

for \(k \in [0, 1/2]\), \(a = (k + 1)/3\). The phase gradient becomes more non-uniform, corresponding to the formation of a phase slip center in the middle of the junction.

In long junctions the CPR allows several saddle point solutions for the current \(I\) when \(\varphi\) is fixed. The backward solution branches are unstable and not directly accessible. Which of the remaining multiple solutions are accessible depends on the rate \(\Gamma\) of phase slips by which the system can transition to lower-energy states.

The free energy barrier \(U\) in the thermal activation rate \(\Gamma \propto e^{-U/T}\) can be estimated from the difference between the unstable and metastable branch energies in Fig. 2 and is of order \(U \sim \hbar\Delta_{o,w}/(e^{2}R_{c})\), \(R_{c} \equiv \xi_{w}/2L\). Quantum phase slip rate in nanowires is \(\Gamma \propto e^{-\Delta_{R}/(e^{2}R_{c})}\) with \(a \sim 1\). For typical S-SQUlPT cross-section 100 nm \(\times 30\) nm and high-conductivity material, \(R_{c} \sim 1\) Ω, so that both rates are effectively zero except close to \(\varphi \approx \varphi_{\text{max}}\) where the CPR bends backward, or at high temperatures. For low relaxation rates, the experimental CPR has magnetic hysteresis, and the different branches can be accessed by sweeping the magnetic flux.

In short junctions where the CPR is single-valued, the kinetic inductance of the wire decreases around \(\varphi = \pi\), and can become small compared to that of the ring, inducing behavior similar to hysteretic rf-SQUIDs. To quantify this, we show in Fig. 3 the Josephson inductance \(\mathcal{L} = \hbar/(2e\partial_{\varphi}I)\) at \(\varphi = \pi\) for different temperatures and wire lengths. In the short-junction limit \(L \ll \xi_{w}\), from the known expression for the CPR, 32 after some rewriting we have

\[
\frac{1}{\mathcal{L}(\pi)} = -\frac{\pi\Delta_{R}}{\hbar R_{N}}\int_{0}^{\Delta_{R}/(2T)} dz \tanh \frac{z}{T}.
\]

For \(T \lesssim \Delta_{R}/2\), we have \(\mathcal{L}(\pi)^{-1} \approx \frac{\pi\Delta_{R}}{\hbar R_{N}} \ln \frac{T}{T_{c,R}}\). As visible in Fig. 3 for \(L > \xi_{w}\) the normalized inductance in general further decreases from the short-junction value, regardless of the ratio of \(T_{c,R}\) and \(T_{c,w}\). However, the value is tunable by material choices.

Let us now consider the density of states (DOS) in the junction. Analytical solutions to Eq. 4 providing access to the DOS are known in the long-junction \((L \gg \xi)\) and short-junction \((L \ll \xi)\) limits. In the short-junction limit, we have a well-known result (see eg. Ref. 39),

\[
N(x, E) = \text{Re} \sqrt{\frac{E^{2} - \Delta_{R}^{2}\cos^{2}\frac{x}{2}}{E^{2} - \Delta_{R}^{2}\cos^{2}\frac{x}{2}}} \cosh \left(\frac{2x - L}{L}\right) \times \text{arccosh} \left(\sqrt{\frac{E^{2} - \Delta_{R}^{2}\cos^{2}\frac{x}{2}}{E^{2} - \Delta_{R}^{2}}}\right),
\]

independent of the superconductivity of the wire itself. For long wires \((L \gg \xi_{w})\), on the other hand, the result converges to the density of states of a bulk superconductor, affected by depairing from the supercurrent flow.
The cross-over is not very rapid, verges towards that of a bulk superconductor affected by except in narrow energy regions around the gap edges short-junction solution is fairly accurate up to $L/\xi_w$. The approximation in Eq. (13) is however not $\ell \ell \ell_0 (\ell_0) = \sqrt{\hbar D/E}$, which can be understood to originate from energy dependent phase shifts between electrons and the almost retroreflected holes (cf. e.g. Ref. [22] for review). The possibility of Andreev reflections also inside the superconducting wire itself however alters $\ell (E)$, e.g., inhibiting low-energy electrons from reaching the ring boundary. Moreover, in a superconductor the decay length diverges around the gap edge $E_g$, instead of around $E = 0$. This can be seen by considering small perturbations $\theta = \theta_0 + \eta, \chi = \phi + \alpha, |\eta| \ll 1$ around the uniform solution $(\theta_0, \phi)$ described by Eq. (13). Substituting such ansatz in Eq. (14) and linearizing around $(\theta_0, \chi_0)$, produces solutions of the form $\eta, \chi \propto e^{\pm x/\ell}$, where $\ell$ are the decay lengths. Consequently, the factor $\sim e^{-L/(2\ell)}$ indicates how much the boundary conditions affect the solution at the center of the wire. The linearized equation can be written here as $\partial_x (\partial_x \eta, \eta, \partial_x \alpha) = M(\partial_x \eta, \eta, \partial_x \alpha)\ell$ where the matrix $M$ is

$$M = \begin{pmatrix}
0 & \ell_0^{-2} g \sinh 2\theta_0 & 0 \\
1 & 0 & 0 \\
-2g \coth \theta_0 & 0 & -2i|\Delta| \text{csch} \theta_0
\end{pmatrix},$$

where $\ell_0^{-2} = -2i|\Delta| \text{csch} \theta_0 + g^2 \sinh^2 \theta_0$. The longest decay length is given by the eigenvalue with the smallest real part, $\ell^{-1} = \text{min} |\text{Re} \lambda|$. They are solutions to

$$\lambda^2 + (5g^2 \cosh^2 \theta_0 - g^2 - 2\ell_0^{-2}) \lambda^2 = \frac{2i|\Delta| \ell_0^{-2}}{\text{sinh} \theta_0}.$$

The energy dependence of $\ell (E)$ is shown in Fig. 5, with a divergence clearly visible. From Eq. (13) it follows that the gap edge is located at $E_g = \Delta^2 / (g^2/2\Delta)^{2/3}/3/2$, and $\theta_0 (E_g) = -\pi/2 + \text{arccosh}(2\Delta g^{-2})^{1/3}$. Consequently, $\ell_0^{-2} (E_g) = 0$, so that also $\ell^{-2} (E_g) = 0$. Around the gap wire, this effect decoheres on a certain decay length scale $\ell (E) \propto \sqrt{\hbar D/E}$, which can be understood to originate from energy dependent phase shifts between electrons and the almost retroreflected holes (cf. e.g. Ref. [22] for review). The possibility of Andreev reflections also inside the superconducting wire itself however alters $\ell (E)$, e.g., inhibiting low-energy electrons from reaching the ring boundary. Moreover, in a superconductor the decay length diverges around the gap edge $E_g$, instead of around $E = 0$. This can be seen by considering small perturbations $\theta = \theta_0 + \eta, \chi = \phi + \alpha, |\eta| \ll 1$ around the uniform solution $(\theta_0, \phi)$ described by Eq. (13). Substituting such ansatz in Eq. (14) and linearizing around $(\theta_0, \chi_0)$, produces solutions of the form $\eta, \chi \propto e^{\pm x/\ell}$, where $\ell$ are the decay lengths. Consequently, the factor $\sim e^{-L/(2\ell)}$ indicates how much the boundary conditions affect the solution at the center of the wire. The linearized equation can be written here as $\partial_x (\partial_x \eta, \eta, \partial_x \alpha) = M(\partial_x \eta, \eta, \partial_x \alpha)\ell$ where the matrix $M$ is

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FIG. 6. (a),(b) CPR in SIS’IS system for \( L/\xi_w = 2.5 \), \( T/T_{c,w} = 0.1 \), and \( T_{c,R}/T_{c,w} = 1.5 \). The dimensionless interface resistances are \( r/\xi_w = 5 \) (black), 7.5 (cyan) on the left panel, and 13 on the right. Higher energy solutions may also exist, but are not shown. (c),(d) Corresponding free energy contribution from Eqs. (2–3).

edge, the length scale diverges as \( \ell \propto (E - E_g)^{-1/4} \). As the decay length is larger than \( L \) in an energy range of width \( \delta E \propto L^{-4} \), the deviations from Eq. (13) at the middle of a long wire do not decay exponentially with increasing \( L \), and the scaling of Fig. 5 can occur.

### A. Interface resistance

Imperfect interface transparency in SQUID influences both the CPR and the DOS. In particular, it enables additional stationary solutions, where phase drops across the barriers at the interfaces. With increasing interface resistance \( r \), the current-phase relation crosses over to \( I(\varphi) \propto \pm I_c \sin(\varphi/2) \), that of two Josephson junctions in series. \[22, 29 \] For two identical Josephson junctions and the superconducting wire in series, with phase drop \( \varphi_1 \) at the interfaces, the free energy is

\[
\delta F(\varphi, \varphi_1) \sim -2E_J(r)\cos(\varphi_1) - E_{J,\text{wire}}\cos(\varphi - 2\varphi_1).
\]

For \( r \to 0 \) (\( E_J(r) \gg E_{J,\text{wire}} \)), lowest-energy solutions have \( \varphi_1 \approx 2\pi n \) and \( I(\varphi) \approx E_{J,\text{wire}}(\varphi) \), whereas for \( r \to \infty \) (\( E_J(r) \ll E_{J,\text{wire}} \)), \( \varphi_1 \approx \varphi/2 + \pi n \) and consequently \( I(\varphi) \approx \mp I_J \sin(\varphi/2) \).

The cross-over is illustrated in Fig. 6(a),(b). A nonzero but small \( r \) only effectively adds to the length of the junction, as shown in panel (a) compared to Fig. 2. A larger \( r \) splits the solution to disconnected branches as seen in panel (b) where there are three different solutions at \( \varphi = 0 \) yielding \( I = 0 \). These correspond to: (i) no phase drop across the wire or interfaces, (ii) \( \pi \) phase drop at both interfaces, and (iii) similar to the second solution but smaller \( |\Delta| \). As \( \varphi \) is varied continuously from 0 to \( 2\pi \), the solution (i) transforms to (ii) and vice versa. The solution (iii) is not connected to the two; instead, when \( \varphi \) varies from 0 to \( \pi \), a phase slip center forms in the center of the wire; cf. also Ref. [22]. The evolution is somewhat more clear in the free energy shown in panels (c),(d). The energy differences between the solutions in (d) must be obtained from Eqs. (2–3) instead of Eq. (9) as not all are continuously connected. Note that little qualitative change occurs in the lowest-energy solution.

The proximity effect from the ring diminishes with increasing barrier height \( r \), and controlling the properties of the weak link via the superconducting ring becomes less effective. This is illustrated in Fig. 7 which shows the modulation of the density of states for a long junction with two different interface resistances. As \( r \) increases, and the critical current of the interfaces becomes small compared to that of the wire, the DOS approaches that of the bulk superconductor.

FIG. 7. DOS in SIS’IS system, for \( L/\xi_w = 13 \); other parameters are as in Fig. 4. Position along the solution branch is shown in the inset.

FIG. 8. Energy gap \( E_g \) achieved at \( \varphi = \varphi_{\text{max}} \) where CPR starts to bend back, and at \( \varphi = \pi \) in the lowest-energy state. Wire lengths and interface resistances are varied, other parameters are as in Fig. 4.
FIG. 9. Predicted IV characteristics of devices with $L = 2.5\xi_w$ and $L = 13\xi_w$ junctions, at $T/T_{c,w} = 0.1$ and $T_{c,R} = 1.5T_{c,w}$. For $L = 13\xi_w$, the IV characteristics has flux hysteresis for $|\Phi - n\Phi_0| \gtrsim 0.25\Phi_0$.

For the S-SQUIPT device application, a relevant metric is how much the modulation of the superconducting gap is suppressed. In Fig. 8 we show the smallest magnitude of the energy gap achievable with phase biasing. Based on the above discussion, the minimum is achieved at the point $\varphi = \varphi_{\text{max}}$ where the CPR transitions to the backward branch. In the lowest-energy state, the minimum gap is achieved at $\varphi = \pi$, which is also shown. We can note that in Fig. 8 the curves for different wire lengths $L$ tend to collapse onto a single curve for $L \gtrsim 2.5\xi_w$. The interface resistance in this case acts similarly as an extension of the junction length by $2r$, consistent with the increase in the total resistance, although at $\varphi = \varphi_{\text{max}}$ the behavior is complicated by the crossover illustrated in Fig. 5.

IV. DEVICE PERFORMANCE

We can now discuss the performance of the SS's devices in a magnetometric measurement. In this mode, the tunnel contact [(T) in Fig. 1] connected to the S' wire is current biased to a working point $I = I_{\text{bias}}$. The tunneling current depends on the DOS,

$$I(V, \Phi) = \frac{1}{R_T} \int_{-\infty}^{\infty} dE N(E, \Phi) N_{\text{probe}}(E - V) \times$$

$$\times \left[ f(E) - f(E - V) \right],$$

and the variation of the observed voltage $V(I_{\text{bias}}, \Phi)$ as a function of the flux $\Phi$ can be used to infer $\Phi$. The sensitivity can be characterized by the flux-voltage transfer function

$$F(\Phi) = \frac{dV}{d\Phi}.$$  (18)

The resolution is intrinsically limited by the voltage noise in the probe junction, which can be described by an equivalent flux noise

$$S_{\Phi,T} = \frac{S_V}{F(\Phi)^2},$$  (19)

where $S_V = \langle \frac{dV}{d\Phi} \rangle^2 S_I$ and $S_I = 2eI \coth \frac{eV}{2kT}$ is the tunneling shot noise. Preamplifier voltage noise will also give a similar contribution.

The current-voltage characteristic $I(V, \Phi)$ is shown in Fig. 9 for the short $L = 2.5\xi_w$ and long $L = 13\xi_w$ junction cases. Here, we assume the tunnel probe is superconducting, with $T_{c,\text{probe}} = T_{c,R}$. For the short device fixed at $I = I_{\text{bias}}$, the external flux corresponds to a single measured voltage value in a wide bias range, shown in Fig. 10a. The long-junction device on the other hand has magnetic hysteresis in the range $|\Phi - n\Phi_0| \gtrsim 0.25\Phi_0$, see Fig. 3 where each flux value is associated with two possible $V$. Which one is realized depends on the initialization of the device. Note that this assumes the relaxation rates $\Gamma(\Phi)$ of metastable states are very low, as was estimated in Sec. III. If $\Gamma(\Phi)$ is not low compared to measurement time scales, transitions can contribute additional voltage noise, which reduces the usefulness of the device. We will not consider this case here.

The voltage response at different bias currents and the corresponding transfer function are shown in Fig. 10b for a short-junction device. The overall behavior and magnitude of $F$ is relatively similar to that of an N-SQUIPT in this case. [3] For long-junction devices, as also visible in Fig. 9, we expect decreasing device sensitivity with increasing system length, as the gap suppression depends on the phase gradient which scales $\propto 1/L$.

Calculated equivalent flux noise from the tunnel junction is displayed in Fig. 10b, where we chose representative values for $R_T$ and $\Delta_{0,w}$. The tunnel junction noise is of the order of $S_{\Phi,T}^{1/2} \approx 10n\Phi_0/\sqrt{\text{Hz}}$ at bias currents of order $1\text{nA}$. Preamplifier voltage noise for a typical value of $S_V^{1/2} \approx 1n\sqrt{\text{VHz}}$ on the other
hand yields $s_{\phi,\text{pre}}^{1/2} > 500 n \Phi_0/\sqrt{\text{Hz}}$ for the parameters of Fig. 10b. The results depend on the energy gap $\Delta_{0,w}$ via $s_{\phi,\text{pre}}^{1/2} \propto \sqrt{\text{RT}/\Delta_{0,w}}$ and $s_{\phi,\text{pre}}^{1/2} \propto 1/\Delta_{0,w}$, so that the performance is expected to improve with larger-gap superconductors. Based on these estimates, the performance of the device is in practice expected to be mostly limited by the external amplifier noise.

V. DISCUSSION AND CONCLUSIONS

For weak links short compared to the coherence length, there is little difference between the normal and superconducting cases, on the level of the present description. As the junction length increases, the change in the current-phase relation starts to limit the maximum achievable modulation of the density of states. This is reflected in the decrease of the voltage modulation observed by the tunnel probe. Moreover, with increasing length of the junction, the gap edge singularities of the BCS DOS transform to smaller peaks, in a way that is sensitive to the finite size of the weak link.

The long perturbation decay length at gap edges has an impact on how generic the results discussed here are in practice, even within the mean field approximations. In reality, even if the superconducting wire is well described by the quasi-1D equations used above, the boundary conditions at $x = 0, L$ may not be as accurate. Namely, the contact region often has nontrivial 3D structure, and the phase gradient also extends to the terminals. In such cases, we expect that in the interior of the wire the results will follow Eq. 13, but deviations appear at the gap edges, which in practice are likely to be sensitive to the details not necessarily described with a single parameter $r$.

In summary, we discussed the current-phase relation, density of states, and free energy of superconducting wires, focusing on points relevant for the S-SQUIPT application. The results point out that superconducting material is a viable choice, provided the junction length does not significantly exceed the coherence length, in order to retain sensitivity and avoid magnetic hysteresis.

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[22] A. Levy Yeyati, Á. Martín-Rodero, and F. J. García-Vidal, “Self-consistent theory of superconducting mesoscopic weak links,” Phys. Rev. B 51, 3743–3753 (1995).

[23] D. Belitz and T. R. Kirkpatrick, “The Anderson-Mott transition,” Rev. Mod. Phys. 66, 261–380 (1994).

[24] A. Altland, B. D. Simons, and D. Taras-Semchuk, “Field theory of mesoscopic fluctuations in superconductor/normal-metal systems,” JETP Lett. 67, 22 (1998).

[25] D. Taras-Semchuk and A. Altland, “Quantum interference and the formation of the proximity effect in chaotic normal-metal/superconducting structures,” Phys. Rev. B 64, 014512 (2001).

[26] I. V. Yurkevich and I. V. Lerner, “Nonlinear σ model for disordered superconductors,” Phys. Rev. B 63, 064522 (2001).

[27] W. Belzig, F. K. Wilhelm, C. Bruder, G. Schönen, and A. D. Zaikin, “Quasiclassical Green’s function approach to mesoscopic superconductivity,” Superlatt. Microstruct. 25, 1251 (1999).

[28] G. Eilenberger, “Transformation of Gorkov’s equation for type ii superconductors into transport-like equations,” Z. Phys. B 19, 195 (1975).

[29] M. Yu. Kupriyanov and V. F. Lukichev, “Influence of boundary transparency on the current critical of dirty SSS structures,” Sov. Phys. JETP 9, 164 (1975).

[30] A. Altland, B. D. Simons, and D. Taras Semchuk, “Field theory of mesoscopic fluctuations in superconductor-normal-metal systems,” Adv. Phys. 49, 321 (2000).

[31] K. D. Usadel, “Generalized diffusion equation for superconducting alloys,” Phys. Rev. Lett. 25, 507 (1970).

[32] I. O. Kulik and A. N. Omelyanchuk, “Contribution to the microscopic theory of the Josephson effect in superconducting bridges,” JETP Lett. 21, 96 (1975).

[33] For the parameters of Ref. [11] we obtain coinciding results for the CRPs.

[34] J. S. Langer and V. Ambegaokar, “Intrinsic resistive transition in narrow superconducting channels,” Phys. Rev. 164, 498–510 (1967).

[35] B. I. Ivlev and N. B. Kopnin, “Theory of current states in narrow superconducting channels,” Sov. Phys. Usp. 27, 206 (1984).

[36] K. Yu. Arutyunov, D. S. Golubev, and A. D. Zaikin, “Superconductivity in one dimension,” Phys. Rep. 464, 1–70 (2008).

[37] A. Zharov, A. Lopatin, A. E. Koshelev, and V. M. Vinokur, “Microscopic theory of thermal phase slips in clean narrow superconducting wires,” Phys. Rev. Lett. 98, 197005 (2007).

[38] A. V. Semenov, P. A. Krutitskii, and I. A. Devyatov, “Microscopic theory of phase slip in a narrow dirty superconducting strip,” JETP Lett. 92, 762–766 (2010).

[39] T. T. Heikkinä, J. Särkkä, and F. K. Wilhelm, “Supercurrent-carrying density of states in diffusive mesoscopic Josephson weak links,” Phys. Rev. B 66, 184513 (2002).

[40] J. Romijn, T. M. Klapwijk, M. J. Renne, and J. E. Mooij, “Critical pair-breaking in current superconducting aluminum strips far below Tc,” Phys. Rev. B 26, 3648–3655 (1982).

[41] A. A. Abrikosov and L. P. Gor’kov, “Theory of superconducting alloys with paramagnetic impurities,” Zh. Eksp. Teor. Fiz. 39, 1781 (1960).

[42] B. Pannetier and H. Courtois, “Andreev reflection and proximity effect,” J. Low Temp. Phys. 118, 599 (2000).

[43] A. I. Larkin and Yu. N. Ovchinnikov, “Density of states in inhomogeneous superconductors,” Sov. Phys. JETP 34, 1144 (1972).

[44] L. P. Gor’kov, “Microscopic derivation of the Ginzburg–Landau equations in the theory of superconductivity,” Sov. Phys. JETP 9, 1364 (1959).

**Appendix A: Pseudo-arclength continuation**

For completeness, we describe here the application of pseudo-arclength continuation on the self-consistency equation. The method generates a set of values $(\Delta_k, \varphi_k)$, tracing a curve of solutions. The next point $(\Delta_{k+1}, \varphi_{k+1})$ is generated from the previous by solving

\[ F(\Delta_{k+1}, \varphi_{k+1}) = 0, \quad (A1a) \]

\[ s = (2 - \theta) \varphi_k(\varphi_{k+1} - \varphi_k) + + \theta \text{Re} \int_0^L dx \dot{\Delta}(x)[\Delta_{k+1}(x) - \Delta(x)]. \quad (A1b) \]

Here, $F(\Delta, \varphi) = 0$ denotes the set of equations \ref{eq:varphi} taking $\Delta(x)$ and the phase difference $\varphi$ as unknowns. The value of $\varphi_{k+1}$ is fixed by the pseudo-arclength constraint \ref{eq:varphi}, where the parameter $s > 0$ is an arclength constant and $\theta \in [0,2]$ a weight factor. The tangent approximants can be taken as $\varphi_k = (\varphi_k - \varphi_{k-1})/s$, $\Delta_{k} = (\Delta_{k} - \Delta_{k-1})/s$, with $\delta^2 = \theta(\Delta_k - \Delta_{k-1})^2 + (2 - \theta)(\varphi_k - \varphi_{k-1})^2$. Equations \ref{eq:varphi} are of similar complexity as the self-consistency equation $F'(\Delta) = 0$, and can be solved for $(\Delta_{k+1}, \varphi_{k+1})$ using standard nonlinear solvers, given a spatial discretization of $\Delta$.

**Appendix B: Riccati parameterization**

Equations \ref{eq:varphi} can be written in a Riccati parameterization, $e^{i\theta} \sin(x) \equiv 2\gamma/(1 + |\gamma|^2)$:

\[ \mathcal{F}_0 = |\Delta|^2 T \frac{\varphi}{T} + 2\pi T \sum_{\omega_n > 0} \left| \frac{\Delta}{\omega_n} \right|^2 \frac{\partial_x \gamma}{(1 + |\gamma|^2)^2} \left(1 + |\gamma|^2 \right) \left(1 + |\gamma|^2 \right) + \left. 4 \, \omega_n |\gamma|^2 - \text{Re}[\Delta^* \gamma] \right| \frac{1}{1 + |\gamma|^2}, \quad (B1) \]

and

\[ \mathcal{F}_b = \frac{2\pi T D}{r} \sum_{\omega_n > 0} \sum_{j=1}^M \frac{2 |\gamma(x_j) - \gamma S j|}{(1 + |\gamma|^2)^2 (1 + |\gamma|^2)^2}. \quad (B2) \]

This form has some advantages for numerical implementation. Moreover, the well-known connection to the Ginzburg-Landau functional \cite{26, 44} is straightforward in this form. The minimum of $\mathcal{F}_0$ vs. $\gamma$ is $\gamma(x, \omega_n) = \Delta(x)/(2\omega_n)$, up to corrections $\propto |\partial_x \gamma|^3, |\Delta|^3$. Neglecting
the corrections, substituting the leading term back in, and expanding in small $|\Delta|$ yields

$$\mathcal{F}_0 \approx \frac{\pi}{8T} |\partial_x \Delta|^2 - |\Delta|^2 \ln \frac{T_c}{T} + \frac{7\zeta(3)}{16\pi^2 T^2} |\Delta|^4, \quad (B3)$$

the GL functional.