Modified Variational Iteration Method for the Numerical Solutions of some Non-Linear Fredholm Integro-Differential Equations of the Second Kind

Aloko MD¹, Fenuga OJ* and Okunuga SA²

¹National Agency for Science and Engineering Infrastructure, FMST, Abuja, Nigeria
²Department of Mathematics, University of Lagos, Nigeria

Abstract

This paper provides approximate solutions to some nonlinear Fredholm–Integrodifferential equations of the second kind by using a Modified Variational Iteration Method. Comparison of the approximate solutions of this method with other known methods shows that the modified Variational Iteration scheme is more accurate, reliable and readily implemented.

Keywords: Modified variational iteration method; Nonlinear Fredholm-integro-differential equations of the second kind

Introduction

Many researchers in engineering and physical sciences have used different numerical methods to solve Fredholms Integro-differential equations. Many of these numerical methods gave reliable and accurate solutions [1] applied multi-wavelet direct method for solving integro-differential equations; Ghasemi et al. [2] used Homotopy perturbation method to solve integrodifferential equations [3-6]. Maleknejad et al. [7] use integral mean value theorem II [8-10] adopted Bernstein collocation method find approximate solution in Fredholm integrodifferential equation, and Jianhua et al. [4] used Hybrid Function Operational Matrix techniques in Solving Fredholm Integro-differential Equations. Lakestani et al. used spline wavelets method to solve the integro-differential equations, Rashidinia and Tahmasebi. Used modified Taylor expansion Method in solving Fredholm integro-differential equations and Shahooth et al. use Bernstein Polynomials Method. In this paper, nonlinear Fredholm integro-differential equations of the second kind where solved by modified variational iteration method which uses few numbers of iterations. Numerical examples and graphical results will demonstrate the efficiency of the method and will be shown that the method is accurate and readily implemented compared to some existing methods.

Non-Linear Fredholm Integro-Differential Equations

Consider the general non-linear, second kind Fredholm Integro-Differential equations of the form:

\[ u^{(n)}(x) = f(x) + \sum_{k=1}^{n} \lambda_k \int_a^b K(x,t) u(t) dt \quad u^{0}(0) = c_i \quad 0 \leq k \leq (n-1) \]  

\( u^{(n)}(x) \) indicate the n-th derivatives of \( u(x) \), \( c_i \) are constants that represent the initial conditions and \( F(u(t)) \) is non-linear. \( u(x), f(x) \) assumed to be real and, \( \lambda_k \) is real finite constants \( F, f, \) and \( k \) are continuous functions and is the unknown function to be determined.

Derivation of Modified Variational Iteration Method

To illustrate the basic concepts of Modified Variational Iteration Method, we consider the differential equation:

\[ Lu + Nu = g(x) \]  

L, N are linear, nonlinear operators respectively and are the inhomogeneous term. The variational iteration method presents a correction functional for eqn. (2) in the form:

\[ u_{n+1}(x) = u_n(x) + \int_0^L A(\xi) \{ L u_n(\xi) + Nu_n(\xi) - g(\xi) \} d\xi \]  

\( \lambda \) a general Lagrange’s multiplier, which can be identified optimally via variational theory, that is, integration by parts and by using a restricted variation.

Determining the Lagrange’s multiplier, this can be identified optimally via integration by parts and by using a restricted variation.

Setting \( L u_n(\xi) = u(\xi) \)  

\[ \int_0^L A(\xi) \{ u_n(\xi) \} d\xi = \lambda \int_0^L \{ u_n(\xi) \} d\xi \]  

\[ \int_0^L \{ u_n(\xi) \} d\xi = \lambda \int_0^L \{ u_n(\xi) \} d\xi \]

The generalized integration by parts is

\[ \int_0^L A(\xi) \{ u_n(\xi) \} d\xi = \lambda \int_0^L \{ u_n(\xi) \} d\xi \]

Noting that in this method may be a constant or a function, and is a restricted value that means it behaves as a constant, is considered as restricted variation, i.e., where is the variational derivative. The extremum condition of requires that and this yields the stationary conditions:

\[ 1 + \lambda \int_0^L \{ u_n(\xi) \} d\xi = 0 \]  

\[ \lambda = -1 \]  

The successive approximations \( u_{n+1} \), \( n \geq 0 \) of the solution \( u(x) \) will be readily obtained upon using selective function \( u(x) \)

*Corresponding author: Fenuga OJ, Department of Mathematics, University of Lagos, Nigeria, Tel: +234 1 280 2439; E-mail: ofenuga@unilag.edu.ng

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The Non-linear term is expressed in a unique way that gives a better approximation than the Adomian polynomial, Bell polynomial, Orthogonal polynomial just to mention a few. Considering a special case of eqn. (1) as:

\[ u^{(k)}(x) + f(x) + \lambda \int_{\pi}^{(k)} k(x,t)u^{(n+1)}(t)dt = g(x) \quad (8) \]

Subject to the initial conditions \( u^{(r)}(0) = c_r \) where \( c_r, r=0,1,...(n-1) \) are real constant and \( k, m \) are integers with \( k \leq m \leq n \).

In solving the general nth-order nonlinear integro-differential equations, we consider the following general functional equation of the form:

\[ Lu = f + N(u) \]

Where \( N \) is the Non-linear differential operator, \( f \) is a known analytical function, \( N(u_n) \) is the nonlinear operator which is decomposed as:

\[ N = \sum_{j=0}^{n} N(u_j) = \sum_{j=0}^{n} \left( N(u_j + u_{j+1} + \ldots + u_n) - N(u_j) \right) \quad (9) \]

\( u_j \) are polynomials of \( x \).

\[ N(u_0 + u_1 + u_2 + \ldots + u_n) = N(u_0) + \sum N(u_{j+1} + u_{j+2} + \ldots + u_{n}) \]

The recurrence relations are defined as

\[ u_0 = f \]
\[ u_1 = N(u_0) \]
\[ u_2 = N(u_1 + u_2) - N(u_1) \]
\[ \vdots \]
\[ u_n = N(u_{n-1} + u_{n} + \ldots + u_n) - N(u_{n-1}) \quad (10) \]

Assume a series solution of the form:

\[ u = \sum_{j=0}^{n} u_j(x) \quad (11) \]

The non-linear term in eqn. (3) can be written as \( N\tilde{u}(\xi) = N_\xi(\xi) \)

The n-th term approximate solution in eqn. (10) is

\[ u_n + u_{n+1} + \ldots + u_{n+j} = N(u_0 + u_1 + \ldots + u_j) \]

Apply \( L^{-1} \) to the recurrence relation for the determination of the components, the \((n+1)\) th approximation of the exact solutions for the unknown functions \( u(x) \) is obtained as

\[ u_{n+1}(x) = N(u_0 + u_1 + \ldots + u_n) - N(u_0) \quad (12) \]

\[ u_n \]

\[ u_{n+1}(x) = u_n(x) + \int L\tilde{u}(\xi) \left( Lu_0(\xi) + Nu_1(\xi) - g(\xi) \right) d\xi \quad (13) \]

The modified algorithms is formulated as

\[ u_{n+1}(x) = u_n(x) + \int L\tilde{u}(\xi) \left( Lu_0(\xi) + L\sum_{j=0}^{n} u_j(\xi) - g(\xi) \right) d\xi \quad (14) \]

### Numerical Examples

In this section, some numerical examples are given to illustrate the accuracy and effectiveness properties of the method and MAPLE 17 package is used to carry-out the calculation. The absolute errors used is defined as \( |u(x) - u_n|, a \leq x \leq b \) for \( n = 0, 1, 2, \ldots \), \( u(x) \) is the exact solution and \( u_n \) is the approximate solution. The numerical solutions of this method will be compared with the numerical solutions of other known methods.

#### Example 1

Consider the first-order Fredholm integro-differential eqns. (2) and (15)

\[ u^{(3)}(x) = \frac{-x^2}{4} \int_{0}^{1} (x^2 - t)u^2(t)dt \]

With the initial conditions \( u(0) = 0 \) and exact solution \( u(x) = x \), the correction functional for eqn. (15) is constructed as

\[ u_{n+1}(x) = u_n(x) + \int L\tilde{u}(\xi) \left( Lu_0(\xi) - \frac{5}{3} \int_{0}^{1} (\xi^2 - r)u^2_r(r)dr \right) d\xi \]

And making the functional stationary and noting that, \( \tilde{u}_n = 0 \). To find the optimal \( \lambda(\xi) \) and calculate variation with respect to \( u_n \), we have the stationary Conditions by applying eqns. (4) and (5):

\[ \delta u_n : 1 + \lambda_0 = 0 \quad \text{and} \quad \delta u_n : \lambda = 0 \]

The Lagrange multiplier can be identified as \( \lambda = -1 \)

\[ u_{n+1}(x) = u_n(x) - \int L\tilde{u}(\xi) \left( Lu_0(\xi) - \frac{5}{3} \int_{0}^{1} (\xi^2 - r)u^2_r(r)dr \right) d\xi \]

Consequently, we have the following approximations (Table 1)

| \( u_n(x) \) | 1.25000000000 x - 0.11111111111 \times \n | 1.0319576576 \times x - 0.01481615250 \times \n | 0.9888349561 + 0.00518701277 \times \n | 1.0033829240 - 0.001782108797 \times \n | 0.9986730892 x + 0.000616215870 \times \n |

#### Example 2

Consider the following nonlinear system of third-order Fredholm integrodifferential equation (Table 2)

\[ u^{(3)}(x) = \sin(x) - x \int_{0}^{\frac{\pi}{2}} \sin(u') dt \]

With the initial conditions \( u(0) = 1, u'(0) = 1, u''(0) = -1 \) for \( x \in [0, \frac{\pi}{2} ] \). The exact solution \( u(x) = \cos(x) \) the correction functional for

(16) is constructed as

\[ u_{n+1}(x) = u_n(x) + \int L\tilde{u}(\xi) \left( Lu_0(\xi) - \sin(\xi) + \int_{\xi}^{\pi} \sum_{j=0}^{n} u'(r)dr \right) d\xi \]

And making the functional stationary and noting that, \( \tilde{u}_n \) is a restriction variation, \( \delta u_n = 0 \). To find the optimal \( \lambda(\xi) \) and calculate variation with respect to \( u_n \), we have the stationary Conditions by applying eqns. (4) and (5):

\[ \delta u_n : \lambda = 0 \quad \text{and} \quad \delta u_n : \lambda = 0 \]

The Lagrange multiplier can be identified as \( \lambda = \frac{1}{2!} (\xi - x)^2 \)
we have the stationary conditions by applying in . To find the optimal

Consequently, we have the following approximations and errors

The correction functional for eqn. (17) is constructed as

and making the functional stationary and noting that, , a restriction

For n=3

Using the natural conditions, we have

Applying as a natural condition, the Lagrange multiplier can be

Table 1: Computations showing comparison of results for example 1.

| X | Exact | Bernstein Polynomials Method (BPM) | MVIM |
|---|---|---|---|
| 0.0 | 0.0 | 0.0000 | 0.00E+00 |
| 0.1 | 0.1 | 0.08810 | 1.19E-02 |
| 0.2 | 0.2 | 0.17602 | 2.20E-02 |
| 0.3 | 0.3 | 0.26784 | 3.22E-02 |
| 0.4 | 0.4 | 0.35861 | 4.14E-02 |
| 0.5 | 0.5 | 0.45067 | 4.93E-02 |
| 0.6 | 0.6 | 0.54433 | 5.75E-02 |
| 0.7 | 0.7 | 0.63991 | 6.01E-02 |
| 0.8 | 0.8 | 0.73773 | 6.23E-02 |
| 0.9 | 0.9 | 0.83811 | 6.19E-02 |
| 1.0 | 1.0 | 0.93898 | 5.9934 |

Table 2: Error for example 2.

| X | Direct Redial Basis Function Method (DRBFM) n=15 | MVIM n=3 |
|---|---|---|
| 5.0 | 6.82E-01 | 2.06E+01 |
| 10.0 | 1.19E+04 | 4.17E+02 |
| 15.0 | 1.33E+03 | 2.11E+03 |

Conclusion

In this paper, numerical methods for approximating solution of
Non-linear Fredholm-integral differential equation of the second kind are considered by using the modified vibrational iteration (MVIM) and this:

- Help to reduce some inherited problems and weakness associated with other method as outlined in literatures.

- Comparison of the approximate, exact solutions shows that MVIM is more an efficient tool and more practical for solving non-linear systems of integral-differential equations and plot confirm.

- It will now be possible to investigate the approximate solution of nonlinear applied problems, particularly of the nonlinear problems in dynamic model of a chemical reactor and the present method reduces the computational difficulty of other traditional methods and all the calculation are made simple.

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