Quantum fields bounded by one-dimensional crystal plates

J M Munoz-Castaneda and M Bordag

Institut für Theoretische Physik, Universität Leipzig, Germany

E-mail: jose.munoz-castaneda@uni-leipzig.de and bordag@itp.uni-leipzig.de

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Abstract
We obtain a deeper understanding of the role played by boundary conditions in quantum field theory by studying the structure of a scalar massless quantum field theory bounded by two one-dimensional planar crystal plates. The system can also be understood as a massless scalar confined to propagate in the surface of a finite cylinder. We classify the most general type of regular behaved boundary conditions that the quantum field can satisfy, in accordance with the unitarity principle of quantum field theory. Also, we characterize the frequency spectrum for each quantum field theory by computing the holomorphic spectral function. The spectral function is the starting point to compute the Casimir energy as a global function over the space of allowed boundary conditions for the quantum field theory.

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1. Introduction

Boundary conditions play an important role in quantum field theory (QFT). On the one hand, they are generalizations of interactions which are concentrated on a region of co-dimension 1 (or higher). Typical examples are conductor boundary conditions which are a good idealization if the skin depth drops below the wavelength of the relevant electromagnetic field modes [1]. On the other hand, periodic boundary conditions appear naturally when considering topological objects, for example a quantum field on a torus, or QFT at finite temperature. Other examples for the use of boundary conditions can be found in the quantum Hall effect [2, 3], in the physics of graphene [4, 5], in planar plasma systems (see [6, 7]), etc. Boundary phenomena are rather relevant for the structure of the vacuum and the low-energy spectrum than for high-energy phenomena. For field theories without mass scale, i.e. conformal theories, these effects are amplified because the long-distance correlations allow these boundary effects to be observed over all the interior region. Precisely, the conformal field theories related to statistical models with second-order phase transitions and string theories are the ones where the boundary
effects have more remarkable applications. Most of the physical systems appearing in nature or implemented in a laboratory have very large but finite physical dimensions. This makes of great interest the study of quantum fields confined in bounded domains.

A more mathematical question is about the most general boundary conditions compatible with the general principles of self-adjointness of the Hamiltonian and unitarity. These questions are discussed, for example, in [8, 9] in detail. It is worth mentioning that these boundary conditions are in general non-local and can encode essential physical information to a large extent. In other words, many physical problems can be formulated in terms of boundary conditions. From a more mathematical point of view, boundary conditions are used to ensure the discrete spectrum of linear operators in Hilbert spaces [10].

This paper is focused on the application of a systematic formalism for boundary conditions in QFT [8] to the Casimir effect. Since the seminal paper of Casimir [11], where the effect was theoretically proposed, and the first experimental measurements in 1958 by Sparnaay [12], many theoretical and experimental works have been performed (see [13–15]). Since the theoretical and subsequent experimental discovery of the Casimir effect, theoretical research on Casimir effect has been focused on two separate directions. On one hand, the study of the dependence of the Casimir energy on the geometry of the objects is subject to the vacuum force. In this area are particularly remarkable references [16, 17], giving for the first time a general method that allows us to calculate (numerically or analytically) the vacuum energy between compact objects of arbitrary shape-like spheres and cylinders which is free of UV divergences. On the other hand, instead of studying the dependence of vacuum energy on the geometry, one may study the dependence on the boundary conditions that the quantum fields satisfy at the surfaces of interacting objects. Actually, of related physical interest is the influence of corrugations on the Casimir force (see, for instance, [14]). In particular, periodic corrugations can be mapped onto non-trivial boundary conditions keeping periodicity [18]. Much work was done on periodic gratings [19, 20]. More general methods developed in [16] allow us to implement some families of boundary conditions in the path integral approach. Also [13] shows how to introduce local boundary conditions on the quantum fields in the path integral approach. But the path integral approach does not allow us to implement the most general type of boundary conditions that the quantum fields can satisfy (see [21, 22]), and problems when implementing Neumann boundary conditions are discussed in [23]). Similar problems appear in the approach [24] using world-line methods.

In the last few years, the advances in experimental nano-scale physics have allowed us to implement a huge family of exotic boundary behavior, producing results of great importance in the Casimir effect. Asorey and Munoz-Castaneda et al developed in the last few years a theoretical framework that classifies the boundary conditions that quantum fields can satisfy, and allows us to study the dependence of the Casimir energy on the most general type possible of allowed boundary conditions (see [8, 25–27, 21, 28]). The geometry of the bounded system in this case is a simple geometry, equivalent to the original geometry studied by Casimir (parallel infinite plates), but the boundary conditions are the most general ones. In the case of infinite parallel homogeneous isotropic plates, Asorey and Munoz-Castaneda et al classified all the allowed boundary conditions for the quantum fields attending to the type of Casimir force they give rise to: attractive, repulsive or null. Focused on the Casimir energy calculation, the central object in Asorey–Munoz-Castaneda (AMC) formalism is the spectral function. The spectral function is defined as the function whose zeros are the frequencies of the quantum field theoretical Hamiltonian of the system. Calculation of this function allowed them to compute the value of the Casimir energy for the most general type of boundary condition, and hence to study the Casimir energy as a function over the space of allowed boundary conditions.
The purpose of this paper is to use the AMC formalism to calculate the space of boundary conditions in a more complicated case, and then give an explicit expression for the corresponding spectral function, in terms of the boundary condition. This calculation will provide the machinery necessary to compute the corresponding Casimir energy as a function over the space of boundary conditions that the quantum fields can satisfy. The case to be studied is the case of two parallel non-homogeneous plates (one-dimensional crystal plates), isomorphic to a cylindrical geometry. Via this calculation we try to obtain a deeper understanding of the behavior of the vacuum interaction in crystallographic systems taking into account the microscopic lattice structure. In order to take into account the lattice structure, we will impose a periodical behavior over the scalar field that gives rise to the vacuum interaction. This interaction can also be understood as a boundary effect due to the finiteness of two-dimensional crystals in one of its directions.

Throughout this paper, we put \( \hbar = c = 1 \).

2. The quantum scalar field in bounded domains

Let \( \mathbb{R} \times M \subset \mathbb{R}^{D+1} \) be a spacetime, where the spatial surface \( M \in \mathbb{R}^D \) is a bounded domain with regular boundary \( \partial M \). The dynamics of a free complex scalar field propagating in \( \mathbb{R} \times M \) is given by the action

\[
S(\phi) = \int_{\mathbb{R} \times M} d^{D+1}x (\nabla^\mu \phi^* \nabla_\mu \phi + m^2 |\phi|^2) - \int_{\mathbb{R} \times \partial M} d^Dx \phi^* \partial_n \phi,
\]

where \( \partial_n \) denotes the outgoing normal derivative. The boundary term is added in order to obtain the usual free field equations of motion in the bulk without any extra boundary term potential

\[
-\partial_t^2 \phi = \mathbf{L} \phi, \quad \mathbf{L} = -\Delta + m^2,
\]

where \( \Delta \) is the Laplace operator induced in the bounded domain \( M \) by the Euclidean structure of \( \mathbb{R}^D \).

The usual canonical quantization procedure leads to the quantum Hamiltonian

\[
\mathbf{H} = \int_M d^Dx (\hat{\pi}(x)^* \hat{\pi}(x) + \hat{\phi}(x)^* \mathbf{L} \hat{\phi}(x)),
\]

where the canonical momentum density operator is given by \( \hat{\pi}(x) = \frac{1}{2} \partial_t \phi^*(x) \) and satisfies the canonical commutation rules \([\hat{\pi}(x), \hat{\phi}(x')] = -i\delta(x - x')\).

Decomposing the fields in eigenmodes of \( \mathbf{L} \),

\[
\mathbf{L} \phi_\lambda(x) = \lambda_\lambda \phi_\lambda(x),
\]

allows us to write the system as an infinite collection of non-coupled harmonic oscillators. In order to have a unitary QFT, we should require the eigenvalues \( \{\lambda_\lambda\} \) to be real (self-adjointness of \( \mathbf{L} \)) and non-negative (non-negativity of the operator \( \mathbf{L} \)). When \( M = \mathbb{R}^D \), then automatically \( \mathbf{L} \) is self-adjoint and non-negative because \( \Delta \) is self-adjoint and positive definite in \( \mathbb{R}^D \). But when \( M \subset \mathbb{R}^D \) is a bounded domain with regular boundary, \( \Delta \) may not be essentially self-adjoint\(^1\) and in that case one needs to consider an infinite set of self-adjoint extensions \([9, 8]\), which are in general not definite positive \([9, 8]\).

\(^1\) A symmetric operator \( \mathcal{O} \) defined on a dense domain of a Hilbert space \( \mathcal{H} \) is said to be essentially self-adjoint if it admits a unique self-adjoint extension (for more details on the theory of self-adjoint operators and self-adjoint extensions see \([10]\)).
2.1. AIM formalism for self-adjoint extensions

To ensure self-adjointness of the operator \( L \) we might use the Asorey–Ibort–Marmo (AIM) formalism for self-adjoint extensions\(^2\) in quantum mechanics [9]. We will study quantum mechanics in the domain \( M \).

- Quantum physical states: \( L^2(M, \mathbb{C}^N) \).
- Quantum Hamiltonian: \( L = -\Delta + m^2 \Rightarrow \) symmetric operator in \( C_0^\infty(M, \mathbb{C}^N) \) (smooth functions with compact support in \( M \), but not essentially self-adjoint in \( L^2(M, \mathbb{C}^N) \).

The obstruction for \( L \) to be self-adjoint is given by the boundary term obtained integrating by parts:

\[
\langle \psi_1, L\psi_2 \rangle = \int_M d^Dx \psi_1^+(L\psi_2) = \int_M d^Dx (L\psi_1)^+\psi_2 + i\Sigma(\psi_1, \psi_2) \tag{2.5}
\]

\[
\Sigma(\psi_1, \psi_2) \equiv i\int_{\partial M} [(\psi_1, \psi_2) - (\psi_1, \psi_2)] d\mu_{\partial M} \tag{2.6}
\]

being \( \psi \equiv \psi |_{\partial M} \) and \( \psi \equiv \partial_n \psi |_{\partial M} \). The boundary term \( \Sigma(\psi_1, \psi_2) \) can be used to characterize the domain where \( L \) is self-adjoint [9].

- \( L \) will be self-adjoint if \( \Sigma(\psi_1, \psi_2) = 0 \).

For simplicity reasons, in this first approach we will assume that the boundary data lie in \( L^2(\partial M, \mathbb{C}) \). This simplification also concerns the physical behavior of the boundary, so it must be carefully explained. In the most general approach \( \varphi \in H^{-\frac{1}{2}}(\partial M, \mathbb{C}) \) and \( \psi \in H^{-\frac{1}{2}}(\partial M, \mathbb{C}) \oplus \ker(\Delta^\dagger)|_{\partial M} \), to encode all the self-adjoint extensions of \( \Delta \) (for a detailed technical description see chapter 1 in [8]). This restriction must be taken into account in order to allow non-regular behavior of the boundary data originated by physical situations that might include singular charge distributions on the boundary, or some kind of singular impurity distributions over the crystal plates. In order to simplify a first approach to the study of the vacuum interaction between one-dimensional crystal plates, we will avoid such kinds of behaviors and assume for now on that all the boundary data lie in \( L^2(\partial M, \mathbb{C}^N) \), as was originally proposed in [9]\(^3\). Hence, in this first approach we will only deal with those boundary conditions that do not encode physically singular behaviors of the boundaries. We will call this set of boundary conditions, the set of regular self-adjoint extensions (\( \mathcal{M}_r \)).

The first AIM theorem ensures the following.

- Self-adjoint extensions of \( L \) are in one-to-one correspondence with the unitary group \( \mathcal{U}(L^2(\partial M, \mathbb{C}^N)) = \mathcal{M}_r \) of unitary operators over the square integrable functions over the boundary of the space.
- Given a unitary operator \( U \in \mathcal{M}_r \), the associated self-adjoint extension \( L_U \) is characterized by the domain of functions that satisfy the boundary condition

\[
\varphi - i\psi = U (\varphi + i\psi), \tag{2.7}
\]

where now, \( \varphi, \psi \in L^2(M, \mathbb{C}^N) \), and \( U \in \mathcal{U}(L^2(M, \mathbb{C}^N)) \).

\(^2\) The AIM formalism for self-adjoint extensions is equivalent to the classical formalism for self-adjoint extensions of symmetric operators with dense domains in Hilbert spaces developed by Von Neumann (see [10] for modern description) in terms of defect spaces. The advantage of the AIM formalism in physics is that the theory is formulated just in terms of boundary values (and normal derivatives) of the functions that belong to the domain of the self-adjoint extension. The boundary data have a clear physical interpretation, as the boundary data of wavefunctions (see [9]).

\(^3\) To include non-regular behavior of the boundary data in the formalism of quantum fields over bounded domains with regular boundary, and the re-formulation of the two AIM theorems see [8].
Some examples of well-known boundary conditions written in this formalism are as follows:

- **Dirichlet boundary condition:** \( U_d = -I \).
- **Neumann boundary condition:** \( U_n = I \).

When \( \pm 1 \not\in \sigma(U) \) the unitary operator \( U \) admits non-singular Cayley transform, and the boundary condition in this case can be rewritten in a more simple way

\[
\dot{\phi} = A\phi; \quad \phi = A^{-1}\dot{\phi} \tag{2.8}
\]

where \( A \) is a self-adjoint operator in the domain of the corresponding self-adjoint extension. The Cayley transformation takes unitary operators \( (\pm 1 \not\in \sigma(U)) \) into self-adjoint operators.

Now, we should ensure the positivity of the self-adjoint extension \( L_U \). This condition is described by the second AIM theorem that ensures the existence of unitary operators \( U \in \mathcal{M} \) for which \( L_U \) has negative energy eigenstates (also the theorem characterizes these operators).

- Given \( U \in \mathcal{M}_r \) with \( -1 \in \sigma(U) \), there are near \( U \) uni-parametric families of unitary operators which give rise to self-adjoint extensions of \( L \) with arbitrary negative energy states.

Now with these two results, we should go back to the QFT and compute which of those \( U \in \mathcal{M}_r \) give rise to self-adjoint extensions \( L_U \) that are consistent with the principles of QFT.

### 3. Consistency conditions in QFT

As we saw in the first two sections, a massless scalar QFT over the bounded domain \( M \) will be consistent if \( L_U \) is a non-negative self-adjoint extension of \( L \). By the first AIM theorem it is ensured that \( L_U \) is self-adjoint for any \( U \in \mathcal{M}_r \). It is easy to check from the expression for the boundary condition that

\[
\langle \phi, L_U \phi \rangle = \| \nabla \phi \|^2 + m^2 \| \phi \|^2 + i \left( \phi, \frac{I - U}{I + U} \phi \right), \tag{3.1}
\]

By the second AIM theorem there exist some unitary operators \( U \in \mathcal{M} \) for which (3.1) is negative, so not any \( U \in \mathcal{M}_r \) is consistent with the principles of QFT, even when \( L_U \) defines a consistent quantum mechanical system. Now, observe that on the right-hand side of equation (3.1),

- \( \| \nabla \phi \|^2 \geq 0 \)
- \( m^2 \| \phi \|^2 \geq 0 \)
- \( \Rightarrow \| \nabla \phi \|^2 + m^2 \| \phi \|^2 \geq 0 \).

**Strong consistency condition:** we will call strongly consistent boundary conditions or consistent boundary conditions those boundary conditions which give rise to a consistent QFT for any size \( \delta \) of the edge, i.e. those given by unitary operators \( U \in \mathcal{M}_r \) satisfying

\[
i \left( \phi, \frac{I - U}{I + U} \phi \right) \geq 0, \quad \forall \ \phi \in L^2(\partial M, \mathbb{C}^N). \tag{3.2}
\]

That is to say that strongly consistent boundary conditions are only those characterized by unitary operators \( U \in \mathcal{M}_r \) that satisfy

\[
i \frac{I - U}{I + U} \geq 0. \tag{3.3}
\]
The consistency condition can be written more explicitly in terms of the eigenvalues of the operator $U$. Since $U$ is unitary, its spectrum is a set of uni-modular complex numbers $\sigma(U) = \{e^{i\theta}\}$, and taking the diagonal terms in the operator inequality given by (3.3), we get a condition over the eigenvalues:

$$i \left( \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right) \geq 0 \iff \tan \left( \frac{\theta}{2} \right) \geq 0.$$  \hspace{1cm} (3.4)

Finally, with condition (3.4) we can specify the space $\mathcal{M}_{rF}$:

$$\mathcal{M}_{rF} = \left\{ U \in \mathcal{M} | \forall \lambda = e^{i\theta} \in \sigma(U), \tan \left( \frac{\theta}{2} \right) \geq 0 \right\},$$  \hspace{1cm} (3.5)

which is the space of consistent boundary conditions for the QFT.

### 4. 2+1 QFT: one-dimensional crystal plates

The analog of the (1+1)-dimensional example explained above is the case of a (2+1)-dimensional QFT, in a bounded space limited by two infinite homogeneous isotropic parallel wires. This case and its generalizations to general $D+1$ scalar QFTs have been studied in detail in [8]. The simplest extension that can be done for all these plate geometries is to break the isotropy of the plates, by introducing a $(D-1)$-dimensional lattice over each plate. To learn how the boundary conditions behave in this case, the simplest case we can study is the one given by two parallel and identical crystal wires, as shown in figure 1.

We are going to study the system of one scalar field governed by the action

$$S(\phi) = \frac{1}{2} \int_{\mathbb{R} \times M} dx d^{D-1}x (\partial^\mu \phi^* \partial_\mu \phi + m^2 |\phi|^2) + \frac{1}{2} \int_{\mathbb{R} \times M} d^2 x \phi^* \partial_n \phi$$  \hspace{1cm} (4.1)

with the physical space given in figure 1. From the geometrical point of view, the physical space where the quantum field is confined is $M = [0, L] \times \mathbb{R}$, but there are two identical lattices at $x = 0$ and $x = L$, respectively. This fact implies that the boundary data of the quantum field
should preserve this lattice structure, i.e. the boundary data should have the same periodicity of the lattice. Since the two lattices are identical, and no displacement between them appears, in this case boundary data should only have a unique periodicity. This geometry is isomorphic to a cylindrical spacetime where the longitudinal direction corresponds to the direction orthogonal to the plates, and the circled direction is the parallel direction to the plates. In addition, the simple periodic structure imposed implies that the field over the cylinder has winding number 1.

In order to determine the set $\mathcal{M}_{rF}$ of regular self-adjoint extensions, we need to construct the most general wavefunction for the quantum mechanical system associated with the QFT between crystal plates. Since there is no potential between plates, and in the plates, the only requirement to be imposed is the periodicity of the wavefunction the direction parallel to the crystal (the $y$-direction):

$$\psi_{n,k}(x, y) = A e^{i(kx+q_y)} + B e^{i(kx-q_y)} + C e^{i(-kx+q_y)} + D e^{-i(kx+q_y)}$$

being $q_n = 2\pi n/a$ with $n \in \mathbb{N}$. Since the momenta in the $y$-direction are discrete (due to the requirement of periodicity imposed in order to take into account the crystal structure and Bloch’s theorem), and the $x$-direction is compact and bounded ($x \in [0, L]$), the allowed momenta in the $x$-direction are also a discrete set. The allowed values of momentum $k$ should arise from the boundary condition.

5. The set of regular boundary conditions: $\mathcal{M}_{rF}$

In order to exactly determine the set $\mathcal{M}_r$ of regular boundary conditions (reduced self-adjoint extensions), we need to explicitly compute the equations that arise from the general boundary condition given by equation (2.7). Using the same notation used in [8] the boundary data can be written as

$$\varphi \pm i\psi = \Psi_{\pm} \equiv \begin{pmatrix} \psi_{L,n}(0, y) \mp i\partial_y \psi_{L,n}(0, y) \\ \psi_{L,n}(L, y) \mp i\partial_y \psi_{L,n}(L, y) \end{pmatrix}$$

and the boundary condition takes the form

$$\Psi_{-} = U(\Psi_{+}) .$$

From equation (4.2), we can obtain an explicit expression for $\Psi_{\pm}$:

$$\psi_{k,n}(0, y) \mp i\partial_y \psi_{k,n}(0, y) = (A(1 \pm k) + C(1 \mp k)) e^{i\varphi_{y}} + (B(1 \pm k) + D(1 \mp k)) e^{-i\varphi_{y}}$$

(5.3)

$$\psi_{k,n}(L, y) \mp i\partial_y \psi_{k,n}(L, y) = (A e^{ikL}(1 \mp k) + C e^{-ikL}(1 \pm k)) e^{i\varphi_{y}} + (B e^{ikL}(1 \mp k) + D e^{-ikL}(1 \pm k)) e^{-i\varphi_{y}}$$

(5.4)

In view of these last expressions, and taking into account that the functions $e^{i\varphi_{y}}$ and $e^{-i\varphi_{y}}$ are linearly independent, we can rewrite them all in terms of four-dimensional column vectors and $4 \times 4$ matrices. The two-dimensional column vector $\Psi_{\pm}$ now becomes

$$\tilde{\Psi}_{\pm} = \begin{pmatrix} A(1 \pm k) + C(1 \mp k) \\ B(1 \pm k) + D(1 \mp k) \\ A e^{ikL}(1 \mp k) + C e^{-ikL}(1 \pm k) \\ B e^{ikL}(1 \mp k) + D e^{-ikL}(1 \pm k) \end{pmatrix}$$

(5.5)

$^4$ ( $\tilde{\Psi}_{\pm}^{(1)}$ ) the $e^{i\varphi_{y}}$ components, and ( $\tilde{\Psi}_{\pm}^{(2)}$ ) the $e^{-i\varphi_{y}}$ components. Also we could have imposed the convention of choosing ( $\tilde{\Psi}_{\pm}^{(1)}$ ) to be the $e^{i\varphi_{y}}$ components, and ( $\tilde{\Psi}_{\pm}^{(2)}$ ) the $e^{-i\varphi_{y}}$ components, but the spectral function does not change under this change, due to the properties of the determinant ($\det(AB) = \det(A) \det(B)$).
and the boundary condition is written in terms of a $4 \times 4$ unitary matrix:
\[
\tilde{\Psi}_-(k; L) = U \left( \tilde{\Psi}_+(k; L) \right) ; \quad U \in U(4).
\] (5.6)

**Observation.** Not all matrices $U \in U(4)$ are allowed boundary conditions. This means that for all $U \in U(4)$ the theory of quantum fields obtained is not unitary and well defined.

In order to classify which $U \in U(4)$ give rise to unitary QFTs we must make use of the consistency lemma (see [8]). From the consistency lemma, it is rapidly deduced that the space of regular boundary conditions is given by the subset
\[
\mathcal{M}_r = \left\{ U \in U(4) \mid \forall e^{i\theta} \in \text{spec}(U), \tan \left( \frac{\theta}{2} \right) \geq 0 \right\}.
\] (5.7)

$\mathcal{M}_r$ is a space with boundary, given by
\[
\partial \mathcal{M}_r = \left\{ U \in U(4) \mid \exists e^{i\theta} \in \text{spec}(U), \tan \left( \frac{\theta}{2} \right) = 0 \right\}.
\] (5.8)

Hence, $\mathcal{M}_r$ is a 15-dimensional manifold, with 14-dimensional boundary $\partial \mathcal{M}_r$. Since the sequence
\[
1 \leftrightarrow U(1) \leftrightarrow U(4) \rightarrow U(1) \rightarrow 1
\] (5.9)
is exact via the determinant projection of $U(4)$ into $U(1)$, we can write $U(4) = U(1) \times SU(4)$ to get a simplified coordinate system in $U(4)$, and hence also for $\mathcal{M}_r$:
\[
U(4) = \left\{ e^{i\theta} U \mid U \in SU(4), \ e^{i\theta} \in U(1) \right\}.
\] (5.10)

An explicit parametrization for $SU(4)$ will be given later.

### 6. Explicit computation of the spectral function

In this section, we will compute explicitly the spectral function whose zeros give the allowed momenta for the transverse direction, in terms of the boundary condition. This computation has never been done before and is very useful from the computational point of view.

Defining the four-vector
\[
\Phi = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}
\] (6.1)
and having in mind equation (5.5), rapidly allows us to write $\Psi_\pm$ as
\[
\Psi_\pm = M_\pm(k) \cdot \Phi.
\] (6.2)

The matrices $M_\pm(k)$ are defined as
\[
M_\pm(k) = \begin{pmatrix} 1 \pm k & 0 & 1 \mp k & 0 \\ 0 & 1 \pm k & 0 & 1 \mp k \\ e^{ikL} (1 \mp k) & 0 & e^{-ikL} (1 \pm k) & 0 \\ 0 & e^{ikL} (1 \mp k) & 0 & e^{-ikL} (1 \pm k) \end{pmatrix}.
\] (6.3)

Now, the boundary condition is written in terms of $M_\pm(k)$ and $\Phi$ as
\[
[M_- - U \cdot M_+(k)] \cdot \Phi = 0.
\] (6.4)

**Observation.** The matrices $M_\pm(k)$ have determinant
\[
D_\pm(k) = -4[2ik \cos(kL) \pm (k^2 + 1) \sin(kL)]^2
\] (6.5)
and therefore they are singular for a discrete but infinite number of values of \( k \), i.e., the equation \( D_\pm(k) = 0 \) has an infinite number of solutions. Nevertheless, \( D_\pm(k) = 0 \) does not determine the allowed values of \( k \) because roots of \( D_+ (k) \) do not coincide with roots of \( D_- (k) \).

In order to have non-trivial solutions satisfying (6.4), and in view of the observation made above, we must require the condition
\[
\det[M - U \cdot M_\pm] = 0.
\]  
(6.6)

This condition, as is known from [8], gives rise to the spectral function, which is now defined as
\[
h_U(k; L) \equiv \det[M - U \cdot M_\pm].
\]  
(6.7)

**Lemma 1.** The infinite discrete set of allowed values for the transverse momentum \( k \) is given by the set of real positive zeros of the spectral function \( h_U(k; L) \). Note that the AIM theorem ensures the self-adjointness; meanwhile the consistency lemma ensures the unitarity of the QFT. This means that after imposing both conditions, the boundary condition cannot give rise to bound states, i.e., these two conditions ensure that the spectral function does not have imaginary zeros.

Explicit computation of the spectral function can be very useful from a purely computational point of view. Even when the spectral function has a very large expression, it is fundamentally composed of trigonometric functions with argument \( kL \) and polynomials in \( k \) whose coefficients depend on the boundary condition. This means that calculating its zeros numerically is not a difficult task with most recent methods. We now proceed to give the explicit algebraic structure of the spectral function. Later on some particular cases for matrix \( U \) will be studied.

For studying the algebraic structure of \( h_U(k; L) \), we introduce an auxiliary real positive parameter \( \epsilon \) and define
\[
\tilde{h}_U(k; L; \epsilon) = \det[M - U \cdot M_\pm].
\]  
(6.8)

Trivially, we can restore the spectral function by making \( \epsilon = 1 \). The key point is that \( \tilde{h}_U(k; L; \epsilon) \) is a four-order polynomial in \( \epsilon \), and the order in \( \epsilon \) coincides with the number of \( U \)-matrix elements appearing. Hence, we can rapidly identify where the algebraic invariants of matrix \( U \) appear.

- Terms of order 4 in \( \epsilon \) are the only ones that include \( \det(U) \).
- Terms of order 1 in \( \epsilon \) are the only ones where \( \text{tr}(U) \) can appear.
- Terms of order 0 in \( \epsilon \) do not depend on the matrix \( U \) and hence are independent of the boundary condition.

We introduce at this point the notation for the coefficients in \( \epsilon \) of \( \tilde{h}_U(k; L; \epsilon) \):
\[
\tilde{c}_0(k, L) = c_0(k, L) + c_1(k, L, U)\epsilon + c_2(k, L, U)\epsilon^2 + c_3(k, L, U)\epsilon^3 + c_4(k, L, U)\epsilon^4.
\]  
(6.9)

Coefficients itemized before as the ones that include algebraic invariants of the \( U \) matrix will be named from now on algebraic terms of \( \tilde{h}_U(k; L; \epsilon) \).

6.1. The algebraic terms of \( \tilde{h}_U(k; L; \epsilon) \)

In this sub-section, we show explicitly how the algebraic invariants of the matrix \( U \) enter in the spectral function via the \( h_U \) function.

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The coefficient \( c_0(k, L) \) intuitively can only depend on the matrix \( M_\epsilon \) since it is the only part of \( \tilde{h}_\epsilon(k, L; \epsilon) \) that is not multiplied by \( U \). The explicit calculation ensures that intuition points are in the right direction, and the 0-order coefficient is given by

\[
c_0(k, L) = \det (M_\epsilon(k, L)) = -4|2ik \cos(kL) - (k^2 + 1) \sin(kL)|^2.
\] (6.10)

The coefficient \( c_1(k, L, U) \) is the one where \( \text{tr}(U) \) can appear. In general, non-diagonal terms can also appear. After calculation, the final expression for the coefficient \( c_1(k, L, U) \) is

\[
c_1(k, L, U) = (2ik \cos(kL) - (k^2 + 1) \sin(kL)) \times (4(k^2 - 1) \text{tr}(U) \sin(kL) + 8ik(U_{13} + U_{31} + U_{24} + U_{42})).
\] (6.11)

As can be rapidly seen, \( c_1(k, L, U) \) depends on \( \text{tr}(U) \) but also has a non-algebraic part depending on non-diagonal matrix elements.

The coefficient \( c_4(k, L, U) \) is the one where the determinant appears. Since it comes from the fourth order in the \( \epsilon \) expansion, in this case no non-algebraic terms can appear, as all combinations of four matrix elements of \( U \) appear in \( \det(U) \). After calculation and simplification, the corresponding coefficient reads

\[
c_4(k, L, U) = \det(U) \det(M_\epsilon) = -4 \det(U)[2ik \cos(kL) + (k^2 + 1) \sin(kL)]^2.
\] (6.12)

6.2. The non-algebraic terms of \( \tilde{h}_\epsilon(k, L; \epsilon) \)

Calculation of the non-algebraic terms is more tedious than the algebraic ones. Nevertheless, they can also be computed using standard simplification techniques.

The second-order coefficient \( c_2(k, L, U) \) contains only combinations of two matrix elements of the matrix \( U \). We could in general expect all possible combinations to appear. After calculation and simplification the coefficient is given by

\[
c_2(k, L, U) = -16k^2(U_{14}U_{23} - U_{13}U_{24} - U_{21}U_{34} + U_{32}U_{41} + U_{13}U_{42} - U_{31}U_{42} + 4(k^2 - 1)^2 \sin^2(kL)(U_{12}U_{21} - U_{11}U_{22} + U_{23}U_{32} - U_{22}U_{33} + U_{14}U_{41} + U_{34}U_{43} + U_{11}U_{44} - U_{33}U_{44}) - (16k^2 + 4(k^2 - 1)^2 \sin^2(kL))(U_{11}U_{23} - U_{13}U_{31} + U_{22}U_{44} - U_{42}U_{24}) + 8ik(k^2 - 1) \sin(kL)(U_{12}U_{23} + U_{34}U_{23} - U_{11}U_{24} - U_{22}U_{31} + U_{23}U_{32} - U_{24}U_{31} + U_{34}U_{41} - U_{31}U_{44} - U_{11}U_{42} - U_{33}U_{42} + U_{32}U_{43} + U_{14}U_{41} + U_{14}U_{43} - U_{31}U_{44} - U_{13}U_{22} + U_{13}U_{44}).
\] (6.13)

The coefficient \( c_3(k, L, U) \) is obtained after a similar calculation of the one made for \( c_2 \):

\[
c_3(k, L, U) = 4(k^2 - 1) \sin(kL)(2ik \cos(kL) - (k^2 + 1) \sin(kL))[U_{11}U_{22}U_{33} + U_{12}U_{23}U_{31} + U_{13}U_{23}U_{31} - U_{11}U_{32}U_{23} - U_{13}U_{22}U_{31} - U_{12}U_{21}U_{33} + U_{11}U_{22}U_{44} + U_{12}U_{23}U_{41} + U_{21}U_{34}U_{41} - U_{11}U_{24}U_{42} - U_{12}U_{24}U_{41} - U_{13}U_{34}U_{41} + U_{13}U_{43}U_{41} + U_{31}U_{43}U_{41} + U_{31}U_{43}U_{41} - U_{13}U_{34}U_{41} + U_{22}U_{34}U_{42} + U_{32}U_{42}U_{42} - U_{22}U_{32}U_{43} - U_{23}U_{33}U_{44} - U_{23}U_{33}U_{41}].
\]
If we define the two-variable polynomial $U$ notation of the matrix $U$, it can be rewritten in terms of algebraic invariants of a matrix associated with the matrix $U$: the matrix of minors. Also, this way of writing $c_3$ provides a way to simplify a lot the expression for $c_3$. First of all, recall the definition of the matrix of minors. In the most general case, if $U$ is a square $n \times n$ matrix, the $n^2$ submatrices of order $(n-1) \times (n-1)$ are constructed deleting one row and one column, and we use the notation

$$U^{(i, j)} = \{\text{delete row } n+1-i, \text{ and column } n+1-j\}.$$  

With this notation, we define the $n \times n$ matrix of minors $A^{(U)}$ associated with $U$, as the matrix whose elements $a^{(U)}_{ij}$ are given by

$$a^{(U)}_{ij} = \det(U^{(i, j)}).$$

Using the matrix of minors associated with $U$, the coefficient $c_3$ can be written in a very compact and symmetric way

$$c_3(k, L, U) = -(2ik \cos(kL) + (k^2 + 1) \sin(kL)\left[4(k^2 - 1) \sin(kL) \text{tr}(A^{(U)}) - 8i(k^{(U)}_{11} + a^{(U)}_{41} + a^{(U)}_{42} + a^{(U)}_{24})\right].$$

After this simplification, we have explicitly computed the expression for the spectral function whose zeros are the allowed values for the transverse momentum

$$h_U(k; L) = c_0(k, L) + c_1(k, L, U) + c_4(k, L, \det(U)) + c_3(k, L, A^{(U)}) + c_2(k, L, U).$$

7. Particular cases

7.1. Diagonal and anti-diagonal elements of $M_r$

Diagonal elements of $M_r$ are given by four parameters that are their eigenvalues:

$$U_d = \text{diag} (\lambda_1, \ldots, \lambda_4).$$

Each eigenvalue is a modulus 1 complex number $\lambda_j = e^{\theta_j}$, satisfying the consistency condition $\tan(\theta_j/2) \geq 0$. In this case, the corresponding minors matrix $A^{(U_d)}$ is also diagonal, and given by

$$A^{(U_d)} = \text{diag} (\lambda_1^2 \lambda_2 \lambda_3, \lambda_1 \lambda_2^2 \lambda_4, \lambda_1 \lambda_3 \lambda_4, \lambda_2 \lambda_3 \lambda_4).$$

If we define the two-variable polynomial

$$p(x, y; k, L) = -2i(k^2 - 1) \sin(kL)(x + y) - (D_+)^{1/2}xy + (D_-)^{1/2},$$

the corresponding spectral function can be written in a compact form as

$$h_{U_d}(k, L) = p(\lambda_1, \lambda_2; k, L)p(\lambda_3, \lambda_4; k, L).$$
There are some particular cases of diagonal boundary conditions that should be explicitly computed. The two most important cases are Dirichlet and Neumann boundary conditions, given by

$$U_D = -1; \quad U_N = 1. \quad (7.5)$$

The spectral function for both cases looks very simple, and its roots can be calculated analytically:

$$h_D(k, L) = -64 \sin^2(kL), \quad h_N(k, L) = 64k^4 \sin^2(kL). \quad (7.6)$$

Also, we can mix Neumann and Dirichlet boundary conditions by imposing one boundary condition for the upward modes ($e^{+iy}$ components), and the other one for the downward modes ($e^{-iy}$ components), which is given by a diagonal matrix

$$U_{ND} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7.7)$$

In this case, the spectral function resulting is also very simple, and given by

$$h_{ND}(k, L) = 64k^2 \cos^2(ak). \quad (7.8)$$

Anti-diagonal elements of $\mathcal{M}_r$ can be handled with analytical techniques. Let us write an arbitrary anti-diagonal element of $\mathcal{M}_r$ as

$$U_{ad} = \begin{pmatrix} 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ \lambda_4 & 0 & 0 & 0 \end{pmatrix}. \quad (7.9)$$

The corresponding matrix of minors is easily computed and has the form

$$A^{(ad)} = \begin{pmatrix} 0 & 0 & 0 & -\lambda_2 \lambda_3 \lambda_4 \\ 0 & -\lambda_1 \lambda_3 \lambda_4 & 0 & 0 \\ -\lambda_1 \lambda_2 \lambda_4 & 0 & 0 & 0 \end{pmatrix}. \quad (7.10)$$

Simplifying the general expression for the spectral function in the case of anti-diagonal boundary conditions, we obtain the spectral function

$$h_{ad}(k, L) = D_+ + D_+ \lambda_1 \lambda_2 \lambda_3 \lambda_4 + 4(k^2 - 1)^2 \sin^2(kL)(\lambda_2 \lambda_3 + \lambda_1 \lambda_4) - 16k^2(\lambda_1 \lambda_2 + \lambda_3 \lambda_4). \quad (7.11)$$

A representative example of this case is given by the boundary condition whose matrix is given by

$$U_{lad} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (7.12)$$

The spectral function obtained has the form

$$h_{lad}(k, L) = 64k^2 \sin^2(kL), \quad (7.13)$$

and hence has the same spectrum as Dirichlet boundary conditions. The case $U_{1lad} = -U_{lad}$ leads to the same spectral function. We can also mix $U_{-1lad}$ and $U_{1lad}$ boundary conditions.
making the system distinguish between upward propagating modes and downward propagating modes:

\[
U_{\pm \text{ad}} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

(7.14)

The spectral function generated in this case is given by

\[
h_{\pm \text{ad}}(k, L) = 8(k^2 + 1)^2 \cos(2kL) - 8(k^4 - 6k^2 + 1).
\]

7.2. Box-anti-diagonal elements of \( M_r \)

Other well-known important boundary conditions, studied for the case of homogeneous isotropic plates, arise in the case of one-dimensional perfect lattices, as box-anti-diagonal boundary conditions, i.e. boundary conditions given by matrices of the form

\[
U = \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix},
\]

(7.15)

where \( A, B \in U(2) \). This subset of boundary conditions is contained in \((U(2) \times U(2)) \cap \mathcal{M}_F\). The most well-known boundary conditions belonging to this subset are the periodic, anti-periodic and pseudo-periodic boundary conditions. In order to have a complete compilation of most well-known boundary conditions, we will include the calculation of the spectral function for quasi-periodic boundary conditions. Both examples correspond to a one-parameter set of boundary conditions that interpolate between periodic and anti-periodic boundary conditions.

For the periodic boundary conditions the corresponding matrix is

\[
U_p = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

(7.16)

and the spectral function obtained has the form

\[
h_p(k, L) = 256k^2 \sin^4 \left( \frac{kL}{2} \right).
\]

(7.17)

For the case of anti-periodic boundary conditions, the corresponding matrix is \( U_{ap} = -U_p \), which leads to the same spectral function as for periodic boundary conditions: \( h_{ap}(k, L) = h_p(k, L) \). We can also mix periodic and anti-periodic boundary conditions making the system distinguish between upward propagating modes and downward propagating modes:

\[
U_{pap} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

(7.18)

The spectral function generated in this case is given by

\[
h_{pap}(k, L) = -64k^2 \sin^2(kL).
\]

(7.19)

For this case, it is immediately observed from the spectral function that the spectrum contains the double of states compared with periodic or anti-periodic, but now the degeneration of each state is lower.
The pseudo-periodic boundary conditions physically correspond to a situation in which both plates are identified (cylinder geometry), with a magnetic flux passing along the cylinder ($\phi(L) = e^{i\alpha}\phi(0)$; $\dot{\phi}(L) = -e^{-i\alpha}\dot{\phi}(0)$). This situation can be developed with a boundary condition given by the box-anti-diagonal matrix

$$U_{pp} = \begin{pmatrix}
0 & 0 & e^{-i\alpha} & 0 \\
0 & 0 & 0 & e^{i\alpha} \\
e^{i\alpha} & 0 & 0 & 0 \\
e^{-i\alpha} & 0 & 0 & 0
\end{pmatrix}. \quad (7.20)$$

It must be pointed out that if the upward modes are suffering a flux $\alpha$, then the downward modes coupled to the same flux are suffering a flux $-\alpha$, as is perfectly shown in the corresponding unitary matrix. The resulting spectral function is given by

$$h_{pp}(k, L) = 64k^2(\cos(kL) - \cos(\alpha))^2. \quad (7.21)$$

If we now couple downward and upward modes to different fluxes, the corresponding boundary condition is given by

$$U_{mp} = \begin{pmatrix}
0 & 0 & e^{-i\alpha} & 0 \\
0 & 0 & 0 & e^{i\beta} \\
e^{i\alpha} & 0 & 0 & 0 \\
e^{-i\beta} & 0 & 0 & 0
\end{pmatrix}, \quad (7.22)$$

which leads to the spectral function

$$h_{mp}(k, L)64k^2(\cos(kL) - \cos(\alpha))(\cos(kL) - \cos(\beta)). \quad (7.23)$$

As expected, when $\beta = \alpha$ the periodic boundary conditions are restored, because the flux suffered by upward modes is compensated for by the effect of the flux suffered by the downward modes.

Quasi-periodic boundary conditions are mathematically very similar to pseudo-periodic boundary conditions, but the corresponding physical situation is very different. Quasi-periodic boundary conditions correspond to an identification of both plates via a delta function, i.e. a step discontinuity in the first derivatives at the junction point ($\phi(L) = \tan\left(\frac{\alpha}{2}\right)\phi(0)$; $\dot{\phi}(L) = -\tan\left(\frac{\beta}{2}\right)\dot{\phi}(0)$). These boundary conditions are given by the matrix

$$U_{qp} = \begin{pmatrix}
\cos(\alpha) & 0 & \sin(\alpha) & 0 \\
0 & \cos(\alpha) & 0 & \sin(\alpha) \\
\sin(\alpha) & 0 & -\cos(\alpha) & 0 \\
0 & \sin(\alpha) & 0 & -\cos(\alpha)
\end{pmatrix}. \quad (7.24)$$

and the corresponding spectral function reads

$$h_{qp}(k, L) = 64k^2(\cos(kL) - \sin(\alpha))^2. \quad (7.25)$$

8. Conclusions

In the first part of the paper, we classified the set of boundary conditions that gives rise to a unitary massless scalar quantum field theory with regular behavior of the field in the boundary (given by the two one-dimensional crystal plates, or equivalently by two $S^1$ circles). In the second part, we determined uniquely the spectrum of $L_u$ by calculating the spectral function for each self-adjoint extension $L_U \in \mathcal{M}_F$. The importance of the spectral function in relation to the Casimir effect is that having the spectral function explicitly computed the Casimir
energy can be computed as a global function over the 16-dimensional space $M_{rF}$. Taking into account that the vacuum energy is basically given by

$$E_c(U) = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sqrt{k^2 + \frac{4\pi^2 n^2}{a^2}}$$

(8.1)

and the fact that the spectral function $h_U(k)$ is holomorphic in $k$, we can replace the summation in $k$ by an integration in the complex $k$-plane over a contour $C$ that encloses all the zeros of $h_u(k)$ (see [29]):

$$\sum_{k \in \mathbb{Z}(h_u)} f(k) \rightarrow \oint_C \frac{dk}{2\pi i} f(k) \frac{d}{dk} \ln (h_U(k)).$$

(8.2)

Hence, using the heat kernel regularization

$$E_c(U; \epsilon) = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}(h_u)} \sqrt{k^2 + \frac{4\pi^2 n^2}{a^2}} e^{-\epsilon \sqrt{k^2 + \frac{4\pi^2 n^2}{a^2}}}$$

(8.3)

in the same way as done in [8], the spectral function allows us to obtain a computable (numerically in general and analytically in particular simple cases) expression for the vacuum energy and the corresponding force.

Finally in the third part of the paper, we have computed explicitly the spectral function for the most well-known types of boundary conditions, as well as some generalizations to one-parameter and two-parameter families of boundary conditions.

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References

[1] Jackson J D 1975 Classical Electrodynamics 2nd edn (New York: Wiley)
[2] Halperin B I 1982 Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential Phys. Rev. B 25 2185–90
[3] Michael S and Fisher P A M 1994 Laughlin states at the edge Int. J. Mod. Phys. B 8 2539–53
[4] Berry M V and Mondragon R J 1987 Neutrino billiards: time reversal symmetry breaking without magnetic fields Proc. R. Soc. A 412 53
[5] Peres N M R, Guinea F and Neto A H Castro 2006 Electronic properties of disordered two-dimensional carbon Phys. Rev. B 73 125411
[6] Fetter A L 1973 Electrodynamics of a layered electron-gas: 1. Single layer Ann. Phys. 81 367–93
[7] Bordag M 2007 On the interaction of a charge with a thin plasma sheet Phys. Rev. D 76 065011
[8] Muñoz-Castañeda J M 2009 Boundary Effects in Quantum Field Theory (Zaragoza, Spain: PUZ) (in Spanish, English version in preparation)
[9] Asorey M, Ibort A and Marmo G 2005 Global theory of quantum boundary conditions and topology change Int. J. Mod. Phys. A 20 1001–26
[10] Dunford N and Schwartz J T 1988 Linear Operators: Part II. Spectral Theory. Self-adjoint Operators in Hilbert Space (Wiley Classics Library) (New York: Wiley) (With the assistance of William G Bade and Robert G Bartle, reprint of the 1963 original)
[11] Casimir H B G 1948 On the attraction between two perfectly conducting plates Proc. Kon. Ned. Akad. Wetensch. 51 793–5
[12] Casimir H B G 1948 Indag. Math. 10 261–3
[13] Sparnaay M J 1958 Measurements of attractive forces between flat plates Physica 24 751–64
[14] Bordag M, Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 Advances in the Casimir Effect (Oxford: Oxford University Press)
[14] Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 The Casimir force between real materials: experiment and theory Rev. Mod. Phys. 81 1827–85
[15] Milton K A 2001 The Casimir Effect: Physical Manifestations of Zero-Point Energy (River Edge, NJ: World Scientific)
[16] Emig T, Graham N, Jaffe R L and Kardar M 2007 Casimir forces between arbitrary compact objects Phys. Rev. Lett. 99 170403
[17] Bulgac A, Magierski P and Wirzba A 2006 Scalar Casimir effect between Dirichlet spheres or a plate and a sphere Phys. Rev. D 73 025007
[18] Golestanian R and Kardar M 1998 Path integral approach to the dynamic Casimir effect with fluctuating boundaries Phys. Rev. A 58 1713–22
[19] Lambrecht A and Marachevsky V N 2008 Casimir interaction of dielectric gratings Phys. Rev. Lett. 101 160403
[20] Chiu H C, Klimchitskaya G L, Marachevsky V N, Mostepanenko V M and Mohideen U 2010 Lateral Casimir force between sinusoidally corrugated surfaces: asymmetric profiles, deviations from the proximity force approximation, and comparison with exact theory Phys. Rev. B 81 115417
[21] Asorey M, Munoz-Castaneda J M and Clemente-Gallardo J 2007 Boundary conditions: the path integral approach J. Phys.: Conf. Ser. 87 012004
[22] Asorey M, Ibort A and Marmo G 2006 Path integrals and boundary conditions arXiv:quant-ph/0609023
[23] Grosche C 1995 $\Delta$'-function perturbations and Neumann boundary conditions by path integration J. Phys. A: Math. Gen. 28 L99–105
[24] Gies H and Klingmuller K 2006 Worldline algorithms for Casimir configurations Phys. Rev. D 74 045002
[25] Asorey M, Garcia-Alvarez D and Munoz-Castaneda J M 2006 Casimir effect and global theory of boundary conditions J. Phys. A: Math. Gen. 39 6127–36
[26] Asorey M, Garcia-Alvarez D and Munoz-Castaneda J M 2007 Vacuum energy and renormalization on the edge J. Phys. A: Math. Theor. 40 6767–76
[27] Asorey M and Munoz-Castaneda J M 2008 Vacuum structure and boundary renormalization group J. Phys. A: Math. Theor. 41 164043
[28] Asorey M and Munoz-Castaneda J M 2008 Vacuum boundary effects J. Phys. A: Math. Theor. 41 304004
[29] Cartan H 1995 Elementary Theory of Analytic Functions of One or Several Complex Variables (New York: Dover)