Diffeomorphism invariant measure for finite dimensional geometries

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Abstract

We consider families of geometries of $D$–dimensional space, described by a finite number of parameters. Starting from the De Witt metric we extract a unique integration measure which turns out to be a geometric invariant, i.e. independent of the gauge fixed metric used for describing the geometries. The measure is also invariant in form under an arbitrary change of parameters describing the geometries. We prove the existence of geometries for which there are no related gauge fixing surfaces orthogonal to the gauge fibers. The additional functional integration on the conformal factor makes the measure independent of the free parameter intervening in the De Witt metric. The determinants appearing in the measure are mathematically well defined even though technically difficult to compute.

1 Introduction

The aim of discrete formulations of quantum gravity is to regularize the theory by reducing the degrees of freedom to a finite number. The underlying idea is to obtain the continuum theory by letting go to infinity the number of degrees of freedom.

Quantum gravity in the functional approach is specified by an action invariant under diffeomorphisms and an integration measure. If one follows the analogy with gauge theories the analogous of the field $A_\mu$ is the metric field $g_{\mu\nu}$ and the analogous of the gauge invariant metric is the De Witt supermetric, which is the unique ultra–local distance in the space of the metrics. The requirement of ultra–locality i.e. the absence of derivatives in the metric is dictated by the fact that the integration measure should not play a dynamical role but only a kinematical one. While in dynamical triangulations one replaces the functional integral by a discrete sum, a typical example of the reduction to a finite number of degrees of freedom is provided by Regge gravity [1]. We shall consider here a general situation in which the class of geometries described by a finite number of parameters is not necessarily the Regge model.

In previous papers [2], concerned with the $D = 2$ case, the breakdown to a finite number of degrees of freedom was achieved by restricting the functional integral in the conformal gauge to those conformal factors describing Regge surfaces.
This was possible due to the simplifying feature occurring in $D = 2$ where all geometries can be described by the conformal factor and a finite number of Teichmüller parameters. In $D > 2$, to which we shall address here, the scheme has to be enlarged.

Diffeomorphisms play a key role in the formulation of gravity and the viewpoint we shall adopt is to treat them exactly at every stage. We shall consider a class of geometries parameterized by a finite number of invariants $l_i$ and described by a gauge fixed metric $\tilde{g}_{\mu\nu}(x, l)$. The functional integration will be performed on the entire class $[f*\tilde{g}_{\mu\nu}(l)](x)$ with $f$ denoting the diffeomorphisms. In other words the reduction to a finite number of degrees of freedom will involve the geometries only, not the diffeomorphisms. Since the integration on the latter is infinite dimensional the related contribution will be a true functional integral (the Faddeev–Popov determinant).

We recall that the differential structure of a manifold, i.e. the charts and the transition functions, are to be given before imposing on the differential manifold a metric structure. In other words if we consider families of metrics on the same differential manifolds the transition functions have to be independent of the metric themselves. Such a feature is essential if we want that the variations of the metric tensor appearing in the De Witt distance are to be tensors under diffeomorphisms, or equivalently if the De Witt distance has to be an invariant under diffeomorphisms.

In sect. 2 after setting up the general framework we shall show that it is possible to obtain from the De Witt supermetric a unique integration measure given by a functional of $\tilde{g}_{\mu\nu}(x, l)$. This will be a geometric invariant that remains unchanged also under arbitrary $l$–dependent diffeomorphisms. Thus while the De Witt metric is invariant only under $l$–independent diffeomorphisms the final expression of the associated measure can be computed on charts with $l$–dependent transition functions. In addition the approach turns out invariant in form under an arbitrary change of the parameters which describe our geometries.

Great simplifications occur if it is possible to choose a gauge fixing surface in such a way that the variations of the metric under a variation of the $l_i$ result orthogonal to the gauge fibers. On the other hand we shall show that this can be realized only for special classes of geometries. In different words only for selected minisuperspaces this simplifying feature can
be achieved.

Despite a gauge fixing procedure is necessary in order to factorize the infinite volume of the diffeomorphisms we shall see that, provided the chosen parameters $l_i$ are geometric invariants, no Gribov phenomenon occurs. It is well known that in the De Witt supermetric an arbitrary parameter $C$ appears. Keeping the number of parameters $l_i$ finite we shall show that in general such a dependence does not disappear. In sect.3 we enlarge the integration to the inclusion of the conformal factor in addition to a finite number of parameters $\tau_i$ which describe deformations transverse (i.e. non collinear) to the orbits generated by the conformal and the diffeomorphism groups. As a result of the functional integration on the conformal factor the dependence on $C$ disappears. In $D = 2$ the relevant functional determinant is given by the exponential of the Liouville action \( [4] \). The analogous functional determinant in $D > 2$ is also perfectly defined, being the Lichnerowicz operator elliptic in any dimension. On the other hand the usual technique which works in $D = 2$, based on the local variation of the conformal factor \( [4] \), fails to work in $D > 2$ due to the lack of ellipticity of one of the operators entering in the conformal variation.

## 2 Geometric invariant measure

In this paper we shall confine ourselves to Euclidean gravity, which allows a positive definite De Witt supermetric. We shall consider a class of geometries parameterized by a finite number $N$ of parameters which we shall call $l = \{l_i\}$. In the case of Regge geometry one can think of the $l_i$ as the link lengths, but any other parameterization is equally possible, as our treatment will be invariant under the change of parameterization. For a given $l$ the geometry is described by an infinite family of metrics, related by diffeomorphism transformations. With $\bar{g}_{\mu\nu}(x, l)$ we shall denote a special choice (gauge fixing) of the metric describing the given geometries. The choice is widely arbitrary; we shall see that the result will be independent of such a choice. The diffeomorphism transformations act on $\bar{g}_{\mu\nu}(x, l)$ as follows

$$g_{\mu\nu}(x, l, f) = [f^* \bar{g}_{\mu\nu}(l)](x) = \bar{g}_{\mu'\nu'}(x'(x, l), l) \frac{\delta x'^\mu}{\partial x^\mu} \frac{\delta x'^\nu}{\partial x^\nu}. \quad (1)$$
As explained in the introduction we shall consider the metric \( g_{\mu\nu}(x) \) as the basic integration variable and as functional integration measure we adopt the one induced by the DeWitt supermetric

\[
(\delta g, \delta g) = \int \sqrt{g(x)} \, d^D x \, \delta g_{\mu\nu}(x) G^{\mu\nu'\rho'\sigma'}(x) \delta g_{\rho'\sigma'}(x)
\]  

(2)

with

\[
G^{\mu\nu'\rho'\sigma'} = g^{\mu\rho'} g^{\nu'\sigma'} + g^{\mu\sigma'} g^{\nu'\rho'} - \frac{2}{D} g^{\mu\nu} g^{\rho'\sigma'} + C g^{\mu\nu} g^{\rho'\sigma'}. 
\]  

(3)

Eq. (2) is the most general ultra–local distance, invariant under diffeomorphisms. In fact it must be a bilinear in \( \delta g_{\mu\nu} \); the metric tensor \( G^{\mu\nu'\rho'\sigma'}(x, y) \) must have support in \( x = y \) and should be formed only by the \( g_{\mu\nu} \) excluding its derivatives. Introducing derivatives in \( G^{\mu\nu'\rho'\sigma'} \) would give a dynamical role to the measure for the field \( g_{\mu\nu} \). The analogous metric for Euclidean Yang-Mills theory is

\[
(\delta A, \delta A) = \int d^D x \, \text{Tr}(\delta A_\mu(x) \delta A_\mu(x)).
\]  

(4)

Metric (2) will be requested to be positive definite, and this requirement puts a restriction on \( C \).

In fact after writing \( \delta g_{\mu\nu} = \delta g^T_{\mu\nu} + \frac{g_{\mu\nu}}{D} \delta g^\lambda_{\lambda} \), being \( \delta g^T_{\mu\nu} \) the traceless part, we have

\[
(\delta g, \delta g) = 2 \int \sqrt{g} \, d^D x \, \delta g^T_{\mu\nu}(x) g^{\mu'\nu'}(x) \delta g^T_{\mu'\nu'}(x) + C \int \sqrt{g} \, d^D x \, \delta g^\lambda_{\lambda}(x) \delta g^\rho_{\rho}(x)
\]  

(5)

from which we see that \( C > 0 \) if we want a positive definite metric. The next problem is to factor out from \( D[g] \) the infinite volume of the diffeomorphisms and leave an integral on the \( dl_i \) multiplied by a proper Jacobian; the calculation of such a Jacobian is the most relevant part in the process of the reduction of the integral on the metrics to the integral over the geometries.

We have to generalize to an infinite dimensional space the usual procedure which relates a distance (metric) to a volume element (measure). We stress that even in the case when our parameters \( l_i \) are finite in number, the integration space on the metric is always infinite dimensional due to the presence of the diffeomorphisms \( f \). In a finite dimensional space
$t_1, t_2, \ldots, t_n$ with distance $(\delta t, \delta t) = \delta t_i M^{ij}(t) \delta t_j$ the integration measure is given by

$$J(t) \prod_i dt_i = \sqrt{\det M(t)} \prod_i dt_i$$

and such $J(t)$ can be computed by means of an integration on the tangent space at the point $t$ i.e.

$$1 = \frac{J(t)}{(2\pi)^{N/2}} \int \prod_i d\delta t_i e^{-\frac{1}{2}(\delta t,\delta t)}.$$  \tag{6}$$

Similarly one proceeds on the infinite dimensional space generated by the diffeomorphisms i.e. $g_{\mu\nu}(x, l, f) = [f^\star \bar{g}_{\mu\nu}(l)](x)$ by writing, apart from an irrelevant multiplicative constant

$$1 = \int \mathcal{D}[\delta g] e^{-\frac{1}{2}(\delta g,\delta g)} = J(l, f) \int \prod_i d\delta l_i \mathcal{D}[\xi] e^{-\frac{1}{2}(\delta g,\delta g)}$$

where

$$\delta g_{\mu\nu}(x) = f^\star [\nabla_\mu \bar{\xi}_\nu + \nabla_\nu \bar{\xi}_\mu](x) + \left[ f^\star \frac{\partial \bar{g}_{\mu\nu}(l)}{\partial l_i} \delta l_i \right](x) = \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) + \frac{\partial g_{\mu\nu}(x, l, f)}{\partial l_i} \delta l_i.$$  \tag{8}$$

The first term in the variation can be understood as $g_{\mu\nu}(x, l, f_1 \cdot f) - g_{\mu\nu}(x, l, f)$, where $f_1$ is the infinitesimal diffeomorphism $x^\mu \to x^\mu + \bar{\xi}^\mu(x)$; $\nabla_\mu$ is the covariant derivative in the metric $[f^\star \bar{g}_{\mu\nu}(l)](x) = g_{\mu\nu}(x, l, f)$. In eq.(8) $\mathcal{D}[\xi]$ is defined by adopting, analogously to eq.(6) the diffeomorphism invariant distance

$$\langle \xi, \xi \rangle = \int \sqrt{g(x)} d^D x \xi_\mu(x) g^{\mu\nu}(x) \xi_\nu(x)$$

i.e.

$$1 = \int \mathcal{D}[\xi] e^{-\frac{1}{2}(\xi,\xi)}. \tag{10}$$

Eq.(10) defines, analogously to what happens in Yang–Mills theory for the gauge transformations, the ultra–local distance between two diffeomorphisms. As a result $J$ appearing in eq.(8) is independent of $f$. To compute $J$ we shall need to decompose $\delta g_{\mu\nu}$ in a part orthogonal to the gauge orbits generated by the diffeomorphisms and a remainder. In order to do this we have to discuss in more detail the operator $F$ defined by $(F \xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. $F^\dagger$ is the adjoint of $F$ according the positive definite metric (4). $F$ acts on the vector
fields $\xi_\mu$ whose Hilbert space, equipped with the norm provided by the Lebesgue measure $L^2(\sqrt{g} d^D x g^{\mu\nu})$, will be denoted by $\Xi$. The result of $F$ acting on $\Xi$ are symmetric tensor fields. We shall denote by $\mathcal{H}$ the Hilbert space of the symmetric tensor fields $h$ equipped with the norm provided by the Lebesgue measure $L^2(\sqrt{g} d^D x G^{\mu\nu\rho\sigma})$. It is well known that $\mathcal{H}$ can be decomposed as $\mathcal{H} = \text{Im}(F) \oplus \text{Ker}(F^\dagger)$ and $\Xi$ as $\Xi = \text{Im}(F^\dagger) \oplus \text{Ker}(F)$. In physical terms $\text{Ker}(F)$ represents the Killing vector fields of the metric (if they exist) while $\text{Ker}(F^\dagger)$ corresponds to the true variations of the geometry. On a finite dimensional space the operator $F^\dagger F$ acting on $\text{Im}(F^\dagger)$ onto $\text{Im}(F^\dagger)$ has a well defined inverse. With infinite dimensions, as it is our case, some assumption is needed. Let us consider the equation

$$F^\dagger F \eta = F^\dagger h$$

with $h_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial l_i}$. We can decompose $h$ as $h = h_0 + h_1$ with $h_0 \in \text{Ker}(F^\dagger)$ and $h_1 \in \text{Im}(F^\dagger)$. The regularity assumption will be that the family $\bar{g}_{\mu\nu}(x, l)$ has been chosen so that $h \in D(F^\dagger)$ and $h_1 \in \text{Im}(F)$. Physically it means that the “gauge part” of $h$ can be written as $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = (F \xi)_{\mu\nu}$ and not as a singular limit of gauge transformations. Then it is immediate that the solution of eq.(12) in $\text{Im}(F^\dagger)$ is given by $\eta = \xi_1$ where $\xi = \xi_0 + \xi_1$ with $\xi_0 \in \text{Ker}(F)$ and $\xi_1 \in \text{Im}(F^\dagger)$. Due to the positive definite metric (10) such solution is unique.

We can now decompose the variation of the metric into two orthogonal parts as follows

$$\delta g_{\mu\nu} = [(F \xi)_{\mu\nu} + F(F^\dagger F)^{-1} F^\dagger \frac{\partial g_{\mu\nu}}{\partial l_i} \delta l_i] + [1 - F(F^\dagger F)^{-1} F^\dagger] \frac{\partial g_{\mu\nu}}{\partial l_i} \delta l_i$$

and $(F^\dagger F)^{-1} F^\dagger \frac{\partial g_{\mu\nu}}{\partial l_i} \delta l_i$ can be absorbed in a shift of $\xi_\mu$. Obviously the optimal choice for $\bar{g}_{\mu\nu}(x, l)$ would be such that

$$\frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} \in \text{Ker}(\bar{F}^\dagger)$$

in which case the two terms $(F \xi)_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial l_i}$ would be already orthogonal, saving the effort to compute the inverse of $F^\dagger F$ on $\text{Im}(F^\dagger)$.

However in general this choice cannot be accomplished. In fact we show in Appendix A that for a generic choice of geometries described by the parameters $l$ there is no related gauge fixing surface which is orthogonal to the gauge fibers.
On the other hand if the class of geometries described by the $l$ are such that eq.(14) is satisfied, then such a property holds all along the gauge fiber $f^*\bar{g}_{\mu\nu}(l)$.

Substituting eq.(13) into eq.(8) we have

$$J(l) = \det(t^i, t^j)^\frac{1}{2} \text{Det}(F^\dagger F)^\frac{1}{2}$$

with

$$t^i_{\mu\nu} = [1 - F(F^\dagger F)^{-1} F^\dagger] \frac{\partial g_{\mu\nu}}{\partial l_i}.$$  

$\text{Det}(F^\dagger F)$ is a true functional determinant and it is the Faddeev–Popov corresponding to the gauge fixing $\bar{g}_{\mu\nu}(l)$. We notice in this connection that, provided the parameters $l$ are geometric invariants, no Gribov problem arises, as a diffeomorphism cannot connect two different geometries. As $F$ is a covariant operator the value of $\text{Det}(F^\dagger F)$ does not depend on the diffeomorphism $f$ i.e. $\text{Det}(F^\dagger F) = \text{Det}(\bar{F}^\dagger \bar{F})$, being $\bar{F}$ the operator computed in the metric $\bar{g}_{\mu\nu}(l)$. The same invariance property holds for $\det(t^i, t^j)$ with the result that $J$ does not depend on $f$. We remark that both determinants in eq.(13) depend on the parameter $C$ appearing in the De Witt supermetric. Such a dependence in general does not cancel out (see Appendix B). Actually the dependence on $C$ could also be taken as an index of the approach to the continuum theory when the number of the parameters $l$ becomes large. In the next section we shall see how the integration over the conformal factor makes the result $C$-independent.

As we pointed out at the beginning of this section, the De Witt metric eq.(2) is a diffeomorphism invariant provided the transition functions are independent of the metric. In fact only in this case $\delta g_{\mu\nu}$ transforms like a tensor.

Having reached the two expressions $\text{Det}(F^\dagger F)$ and $\det(t^i, t^j)$ which are invariant under rigid diffeomorphisms, it is of interest to consider $l$–dependent diffeomorphisms, which modifies the gauge fixing surface in an $l$–dependent way. As the Faddeev–Popov term $\text{Det}(F^\dagger F)$ does not depend on any derivative of $g_{\mu\nu}$ with respect to $l$, it is left invariant under $l$–dependent diffeomorphisms. With regard to $\det(t^i, t^j)$ we first examine the behavior of $\frac{\partial g_{\mu\nu}(x, l)}{\partial l_i}$ under such diffeomorphisms

$$\bar{g}_{\mu\nu}(x, l) = A^X_\mu A^\rho_\nu \bar{g}_{\lambda\rho'}(x', (x, l), l)$$  

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with \( A'_\mu \equiv \frac{\partial x'^\lambda (x, l)}{\partial x^\mu} \). Consequently

\[
\frac{\partial \bar{g}_{\mu\nu}(x, l)}{\partial l_i} = A'_\mu A'_\nu \frac{\partial \bar{g}_{\lambda\rho}'(x'(x, l), l)}{\partial l_i} + A'_\mu A'_\nu \frac{\partial x'^\alpha(x, l)}{\partial l_i} \frac{\partial \bar{g}_{\lambda\rho}'(x'(x, l), l)}{\partial x'^\alpha} + 2 \frac{\partial A'_\mu}{\partial l_i} A'_\nu \bar{g}_{\lambda\rho}'(x'(x, l), l),
\]

which shows that \( \frac{\partial \bar{g}_{\mu\nu}(x, l)}{\partial l_i} \) is not a tensor under this class of transformations.

Setting

\[
\bar{F}(\bar{F}^t \bar{F})^{-1} \bar{F}^t \frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} = \bar{B}_{\mu\nu} \frac{\partial \bar{g}_{\lambda\rho}}{\partial l_i}
\]

the second term in eq.(13) transforms as

\[
(\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}) \frac{\partial \bar{g}_{\lambda\rho}}{\partial l_i} = A'_\mu A'_\nu (\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}'(A^{-1})^\alpha_\lambda (A^{-1})^\beta_\rho).
\]

\[
\cdot \left[ A'_\alpha A'_\beta \frac{\partial \bar{g}_{\alpha\beta}'(x, l)}{\partial l_i} + A'_\alpha A'_\beta \frac{\partial x'^\gamma(\bar{B}_{\mu\nu}^t)(A^{-1})^\alpha_\lambda (A^{-1})^\beta_\rho)}{\partial l_i} + 2 \frac{\partial A'_\alpha}{\partial l_i} A'_\beta \bar{g}_{\alpha\beta}' \right] =
\]

\[
A'_\mu A'_\nu (\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}') \left[ \frac{\partial \bar{g}_{\lambda\rho}'}{\partial l_i} + \frac{\partial x'^\gamma}{\partial l_i} \frac{\partial \bar{g}_{\lambda\rho}'}{\partial x'^\gamma} + 2(A^{-1})^\alpha_\lambda \frac{\partial A'_\alpha}{\partial l_i} \bar{g}_{\alpha\beta}' \right] =
\]

\[
A'_\mu A'_\nu (\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}') \left[ \frac{\partial \bar{g}_{\lambda\rho}'}{\partial l_i} + \nabla^\lambda_{\mu} \xi_{\nu} + \nabla^\nu_{\mu} \xi_{\lambda} \right]
\]

where \( \xi_{\nu} = \frac{\partial x'^\nu(x, l)}{\partial l_i} \). But then the projector \( (\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}') \) annihilates the \( \bar{F}^t \xi \) part and we are left with

\[
A'_\mu A'_\nu (\delta^\lambda_{\mu} \delta^\rho_{\nu} - \bar{B}_{\mu\nu}') \frac{\partial \bar{g}_{\lambda\rho}'}{\partial l_i}
\]

i.e. the same covariant expression as under a rigid diffeomorphism. We stress that this invariance is due to the appearance of the projector \( (I - \bar{B}) \). The larger freedom on the diffeomorphism transformations may be useful in the difficult job of computing the functional determinant of the Lichnerowicz operator on the manifold. Physically we found, starting from the De Witt supermetric, a geometric invariant measure which depends only on the geometries given by the \( l_i \) and not on the particular metric used in describing them. We notice finally that a change of the parameters, i.e. \( l_i \rightarrow l'_i(l) \) leaves the result invariant in form.
3 Measure for the conformal factor

We pointed out in the previous section that the integration on the parameters \( l \) leaves a dependence of the result on the constant \( C \) appearing in the De Witt supermetric. In this section we want to enlarge the treatment replacing the integration variables \( l_i \) by a conformal factor \( \sigma(x) \) and a finite number of other parameters \( \tau_i \) describing geometric deformations transverse (i.e. non collinear) both to the diffeomorphism and to the Weyl group.

Thus the set of metrics we shall integrate on is given by

\[
g_{\mu\nu}(x, \tau, \sigma, f) = [f^* e^{2\sigma} \bar{g}_{\mu\nu}(\tau)](x). \tag{22}
\]

In the following we denote by \( \bar{g}_{\mu\nu}(x, \tau, \sigma) \) the combination

\[
\bar{g}_{\mu\nu}(x, \tau, \sigma) = e^{2\sigma} \bar{g}_{\mu\nu}(x, \tau). \tag{23}
\]

We have to evaluate the Jacobian \( J(\sigma, \tau) \) such that

\[
D[g] = J(\sigma, \tau)D[f]D[\sigma] \prod_i d\tau_i. \tag{24}
\]

We proceed as in sect.2. The general variation of the metric can be written as

\[
\delta g_{\mu\nu}(x, \tau, \sigma, f) = (P\xi)_{\mu\nu}(x) + 2[f^* \delta \sigma \bar{g}_{\mu\nu}](x) + [f^* \frac{\partial \bar{g}_{\mu\nu}}{\partial \tau_i} \delta \tau_i](x). \tag{25}
\]

Defining the operator \( P \) by

\[
(P\xi)_{\mu\nu} = (F\xi)_{\mu\nu} - \frac{g_{\mu\nu}}{D} g^{\alpha\beta} (F\xi)_{\alpha\beta} = \\
= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2}{D} g_{\mu\nu} \nabla \cdot \xi
\]

and the traceless tensor

\[
k^i_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \tau_i} - \frac{g_{\mu\nu}}{D} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau_i}
\]

we can rewrite

\[
\delta g_{\mu\nu}(x, \tau, \sigma, f) = (P\xi)_{\mu\nu}(x) + f^* \left[ \left( 2\delta \sigma + \frac{\bar{g}^{\alpha\beta}}{D} \frac{\partial \bar{g}_{\alpha\beta}}{\partial \tau_i} \delta \tau_i + \frac{\bar{g}^{\alpha\beta}}{D} (F\bar{\xi})_{\alpha\beta} \right) \bar{g}_{\mu\nu}(\sigma, \tau) \right](x) + k^i_{\mu\nu}(x) \delta \tau_i. \tag{28}
\]
Setting now
\[ \xi' = \xi + \left( \frac{1}{P^\dagger P} P^\dagger k^i \delta \tau_i \right)_\mu \]
and
\[ \delta \sigma' = \delta \sigma + \frac{\bar{g}^{\alpha \beta}}{2D} \frac{\partial g_{\alpha \beta}}{\partial \tau_i} \delta \tau_i + \frac{\bar{g}^{\alpha \beta}}{2D} (\bar{F} \bar{\xi})_{\alpha \beta} \]
we obtain
\[ \delta g_{\mu \nu}(x, \tau, \sigma, f) = (P \xi')_{\mu \nu} + f^* 2 \delta \sigma' \bar{g}_{\mu \nu}(\tau, \sigma) + \left( 1 - \frac{1}{P^\dagger P} P^\dagger \right) k^i_{\mu \nu} \delta \tau_i. \]
We remark that the three terms are mutually orthogonal and thus
\[ 1 = \int \mathcal{D}[\delta g] e^{-\frac{1}{2}(\delta g, \delta g)} = \]
\[ J(\sigma, \tau) \int \mathcal{D}[\delta \sigma] \mathcal{D}[\xi] \prod_i d\tau_i e^{-\frac{1}{2}(P \xi', P \xi')} \cdot e^{-2CD^2(\delta \sigma', \delta \sigma')} \cdot e^{-\frac{1}{D}(1 - \frac{1}{P^\dagger P} P^\dagger \bar{k}^i \delta \tau_i, (1 - \frac{1}{P^\dagger P} P^\dagger \bar{k}^j \delta \tau_j))}. \]

Exploiting invariance under translations of the integrals on the tangent space and the definition of \( \mathcal{D}[\delta \sigma] \)
\[ \int \mathcal{D}[\delta \sigma] e^{-\frac{1}{2}(\delta \sigma, \delta \sigma)} = 1 \]
with \( (\delta \sigma, \delta \sigma) = \int \sqrt{g(x)} d^D x \delta \sigma(x) \delta \sigma(x) \) we have apart for a multiplicative constant
\[ J(\sigma, \tau) = \text{Det}(P^\dagger P)^{\frac{1}{2}} \left[ \text{det} \left( \bar{k}^i_{\mu}, (1 - \frac{1}{P^\dagger P} P^\dagger) \bar{k}^j_{\mu} \right) \right]^{\frac{1}{2}}. \]

The dependence on \( f \) has disappeared due to the invariance of the De Witt metric under diffeomorphisms and thus in eq.(24) the infinite volume of the diffeomorphisms can be factorized away.

We notice that in eq.(34) the dependence on \( C \) has been absorbed in an irrelevant multiplicative constant, as it happens in two dimensions. This is the result of having integrated over all the conformal deformations. On the other hand the \( \bar{k}^i_{\mu \nu} \), depend both on \( \tau \) and \( \sigma \) and also the operators \( \bar{P}, \bar{P}^\dagger \) depend on \( \tau \) and \( \sigma \) through the metric \( \bar{g}_{\mu \nu} \)
\[ \bar{P} = e^{2\sigma} \hat{P} e^{-2\sigma}, \quad \bar{P}^\dagger = e^{-D\sigma} \hat{P}^\dagger e^{(D-2)\sigma}, \quad \bar{P}^\dagger \bar{P} = e^{-D\sigma} \hat{P}^\dagger e^{D\sigma} \hat{P} e^{-2\sigma}. \]
For the subsequent discussion, it will be useful to examine $\text{Ker}(\bar{P})$ and $\text{Ker}(\bar{P}^\dagger)$. As already mentioned, $\text{Ker}(\bar{P})$ is given by the conformal Killing vectors of the geometry $\bar{g}_{\mu\nu}(x, \tau, \sigma)$. In fact under the infinitesimal diffeomorphism generated by $\bar{\xi}$

$$
\bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + (\nabla_\mu \bar{\xi}_\nu + \nabla_\nu \bar{\xi}_\mu - \frac{2}{D} \bar{g}_{\mu\nu} \nabla \cdot \bar{\xi}) + \frac{2}{D} \bar{g}_{\mu\nu} \nabla \cdot \bar{\xi} = 
\bar{g}_{\mu\nu}(1 + \frac{2}{D} \nabla \cdot \bar{\xi}) + (\bar{P}\bar{\xi})_{\mu\nu} = \bar{g}_{\mu\nu}(1 + \frac{2}{D} \nabla \cdot \bar{\xi})
$$

if $(\bar{P}\bar{\xi})_{\mu\nu} = 0$. Contrary to what happens in two dimensions, where every topology carries its own conformal Killing vectors (6 for the sphere, 2 for the torus and 0 for higher genus), here we shall have no conformal Killing vectors for a generic $\hat{g}_{\mu\nu}(\tau)$, and thus the geometries with $\text{Ker} \bar{P} \neq \emptyset$ have zero relative measure. The null eigenvectors $\bar{\xi}$ of $\bar{P}$ are related to those of $\hat{P}$ by $\bar{\xi} = e^{2\sigma} \hat{\xi}$ where $\hat{\xi}$ are independent of $\sigma$. Similarly the null eigenvectors $\bar{h}$ of $\bar{P}^\dagger$ are related to those of $\hat{P}^\dagger$ by $\bar{h} = e^{(2-D)\sigma} \hat{h}$. The $\text{Ker}(\bar{P}^\dagger)$ is the analogous of the pure Teichmüller deformations in two dimensions. On the other hand we have

$$
\bar{k}^i = e^{2\sigma} \hat{k}^i .
$$

We notice that given an orthonormal basis $\hat{h}^n$ of $\text{Ker}(\hat{P}^\dagger)$, the vectors $\bar{h}^n = e^{(2-D)\sigma} \hat{h}^n$, even though complete in $\text{Ker}(\bar{P}^\dagger)$ do not remain orthogonal as

$$
(\bar{h}^m, \bar{h}^n) = 2 \int \sqrt{\hat{g}} d^D x e^{-D\sigma} \hat{h}^m_{\mu\nu} \hat{g}^{\mu\nu'} \hat{g}^{\mu\nu'} \hat{h}^n_{\mu'\nu'}.
$$

On the other hand from eq.$(37)$

$$
(\bar{h}^m, \bar{k}^i) = (\hat{h}^m, \hat{k}^i).
$$

The operator $1 - \bar{P} \bar{P}^\dagger$ appearing in eq.$(34)$ projects on $\text{Ker}(\bar{P}^\dagger)$ and thus can be written in terms of the $\bar{h}^n_{\mu\nu}$ as

$$
\left(1 - \bar{P} \bar{P}^\dagger\right)_{\mu\nu, \lambda\rho} (x, x') = \bar{h}^m_{\mu\nu}(x) M^{-1}_{mn} \bar{h}^n_{\lambda\rho}(x')
$$

where $M_{mn}$ is the infinite matrix

$$
M_{mn} = (h^m, h^n).
$$
The \((1 - \bar{P} \frac{1}{P^t} P^t) \bar{k}^i\) span an \(N\)-dimensional subspace of \(\text{Ker}(P^t)\). In fact if \(a_i\) exist such that \((1 - \bar{P} \frac{1}{P^t} P^t) \sum_{i=1}^{N} a_i \bar{k}^i = 0\) then it would mean that \( \sum_{i=1}^{N} a_i \bar{k}^i = \bar{P} \xi \). Thus \( \sum_{i=1}^{N} a_i \frac{\partial \bar{\omega}}{\partial \tau_i} \) would be the sum of a Weyl and a diffeomorphism transformation.

For \(\sigma = 0\) we can choose the \(h^n\) with the properties \((h^n, k^i) = 0\) for \(n > N\) and a non zero determinant of the \(N \times N\) matrix \((h^n, k^i)\) with \(i, n \leq N\). Such properties are maintained for \(\sigma \neq 0\), due to the independence on \(\sigma\) in eq. (39), but all matrix elements of \(M_{mn}\) are needed to compute the top left \(N \times N\) sub-matrix of \(M_{mn}^{-1}\) whose determinant we shall denote by \(\det M_{N \times N}^{-1}\). The computation of such a sub-matrix and of the Faddeev–Popov determinant \(\mathcal{D} \det(\bar{P}^t \bar{P})\) are the two technically difficult points.

We now examine the functional dependence on \(\sigma\) of the two determinants. We notice that

\[
\det \left( (1 - \bar{P} \frac{1}{P^t} P^t) \bar{k}^i, (1 - \bar{P} \frac{1}{P^t} P^t) \bar{k}^j \right) = [\det(\bar{k}^i, \bar{h}^n)]^2 \det \bar{M}_{N \times N}^{-1}
\]

\[(42)\]

due to eq. (39). Thus the dependence on \(\sigma\) is restricted to \(\mathcal{D} \det(\bar{P}^t \bar{P})\) and to \(\det \bar{M}_{N \times N}^{-1}\). We want to stress at this point the main differences between \(D = 2\) and \(D > 2\). In \(D = 2\), \(\text{Ker}(P^t)\) is finite dimensional (quadratic differentials), and thus the number of the \(k^i\) is finite and \((1 - P(P^t P)^{-1} P^t) k^i\) span completely \(\text{Ker}(P^t)\). As for \(\mathcal{D} \det(\bar{P}^t \bar{P})\) its dependence on \(\sigma\) in \(D = 2\) is obtained by computing the variation under \(\delta \sigma\) and then integrating back as a result one obtains the Liouville action. In \(D > 2\) (referring to the generic case in which there are no conformal Killing vectors) from

\[
\log \mathcal{D} \det(\bar{P}^t \bar{P}) = -\frac{d}{ds} Z(0) = -\frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{Tr}(e^{-t \bar{P}^t \bar{P}}) \right]
\]

\[(43)\]

we have

\[-\delta \log \mathcal{D} \det(\bar{P}^t \bar{P}) = \gamma_E \delta Z_{\bar{P}^t \bar{P}}(0) + \text{Finite}_{\epsilon \rightarrow 0} \int_\epsilon^\infty dt \{ (2 + D) \text{Tr}(e^{-t \bar{P}^t \bar{P} \delta \sigma}) - D \text{Tr}'(e^{-t \bar{P}^t \bar{P} \delta \sigma}) \}
\]

\[(44)\]

where \(\text{Tr}'\) excludes the 0-modes of \(\bar{P}^t\), which now \((D > 2)\) are infinite in number. We notice that \(\mathcal{D} \det(\bar{P}^t \bar{P})\) is well defined because \(\bar{P}^t \bar{P}\) is an elliptic operator \([7]\).
In fact
\[ P^i P \xi_\nu = -4[\nabla^2 \xi_\nu + \nabla^\mu \nabla_\nu \xi_\mu - \frac{2}{D} \nabla_\nu \nabla \cdot \xi] \quad (45) \]
and the determinant of the leading symbol [7]
\[ 4[k^2 \delta_{\mu}' + (1 - \frac{2}{D})k_\mu k^{\mu'}] \quad (46) \]
vanas only for \( k = 0 \). On the other hand
\[ PP^ih_{\mu\nu} = -4(\nabla_\nu \nabla^\lambda h_{\lambda\mu} + \nabla_\mu \nabla^\lambda h_{\lambda\nu} - \frac{2}{D} g_{\mu\nu} \nabla^\rho \nabla^\lambda h_{\rho\lambda}) \quad (47) \]
whose leading symbol
\[ 2[k_\nu k^{\nu'} \delta_{\mu'} + k_\mu k^{\mu'} \delta_{\nu'} + k_\nu k^{\nu'} \delta_{\mu'} + k_\mu k^{\mu'} \delta_{\nu'}] - \frac{8}{D} \delta_{\mu\nu} k^{\mu'} k^{\nu'} \quad (48) \]
has zero determinant for \( k \neq 0 \) as it is immediately seen by applying it to a tensor of the form \( h_{\mu\nu} = v_\mu w_\nu + v_\nu w_\mu \) with \( w \cdot v = w \cdot k = v \cdot k = 0 \). Thus the variation with respect to \( \sigma \) cannot be computed in terms of local quantities as the usual heat kernel technique is not available.

An exception is the variation with \( \delta \sigma = \text{const.} \) under which due to \( \text{Tr}(e^{-tP^i P}) = \text{Tr}'(e^{-tP^i P}) \) the calculation can be reduced to the heat kernel of the elliptic operator \( \hat{P}^i \hat{P} \). For the expression of such variation see [8].

4 Conclusions

Summarizing, for a class of metrics of the type \( f^* \tilde{g}_{\mu\nu}(l) \) the De Witt supermetric induces unambiguously the \( C \)-dependent measure eq.(15)
\[ \prod_k dl_k [\det(t^i, t^j) \det(F^i F)]^{\frac{1}{2}} \quad (49) \]
Similarly for the class \( f^* e^{2\sigma} \tilde{g}_{\mu\nu}(\tau) \) we have the \( C \)-independent measure eq.(34)
\[ \prod_k d\tau_k \mathcal{D}[\sigma] \left[ \det(P^i P) \det \left( k^i, (1 - \frac{1}{P^i P} k^i) \right) \right]^{\frac{1}{2}} \quad (50) \]
with $\mathcal{D}[\sigma]$ the measure induced by the distance

$$
(\delta \sigma, \delta \sigma) = \int d^Dx \sqrt{\hat{g}(x)} \ e^{D\sigma} \delta \sigma(x) \delta \sigma(x) .
$$

(51)

In the first case we have a finite dimensional integral and as such more suitable to numerical calculations. A finite dimensional approximation to eq.(50) is obtained by restricting to a family of conformal factors parameterized by a finite numbers of parameters $s = \{ s_i \}$. Thus to the family $f^* e^{2\sigma(s)} \hat{g}_{\mu\nu}(\tau)$ it is associated the measure

$$
\prod_k d\tau_k \prod_i ds_i \left[ \det (J_{ij}^s) \det \left( k^i, (1 - P \frac{1}{P^\dagger P} P^\dagger) k^j \right) \mathcal{D} \det (P^\dagger P) \right]^{\frac{1}{2}} ,
$$

(52)

where $J_{ij}^s = \int d^Dx \sqrt{\hat{g}} \ e^{D\sigma} \frac{\partial \sigma}{\partial s_i} \frac{\partial \sigma}{\partial s_j}$. If now we denote by $l = \{ \tau_i, s_j \}$ and use the first scheme eq.(49), we get a different result (e.g. in the first scheme the result depends on $C$ while in the second it does not). The reason is that in deriving the measure (50) $\sigma$ is a generic function, for which the shift (30) is allowed. With $\sigma$ depending on a finite number of parameters $s_i$ such shift will be the more accurate the higher the numbers of the parameters. In this limit one expects complete equivalence of the measure (50) and (52) for metrics of the type $f^* e^{2\sigma(s)} \hat{g}_{\mu\nu}(\tau)$. As remarked in the sect.4 the dependence on $C$ of the measure (49) is expected to drop out for a large number of $l_i$ describing the conformal deformations.

The measure (52), with the modifications due to the presence of the conformal Killing vectors, has been adopted in [2] for the $D = 2$ Regge case. The $\tau_i$ are the Teichmüller parameters of the Riemann surfaces and the $s_i$ parameterize the positions and the angular defects of the conical singularities of the Regge geometries. An exact calculation of $\mathcal{D} \det (P^\dagger P)$ and an explicit form of $J_{ij}^s$ were given.

In higher dimension $D \geq 3$ the route leading to the measure (49) appears more proficient as up to now we do not know how to extract analytically the dependence of $\mathcal{D} \det (P^\dagger P)$ and $\det \left( \bar{k}^i, (1 - \bar{P} \frac{1}{P^\dagger \bar{P}} P^\dagger) \bar{k}^j \right) \mathcal{D} \sigma(x)$ on $\sigma(x)$.
Appendix A. A geometric property of gauge fixing surfaces

In the text we remarked that whenever the gauge fixing surface $g_{\mu \nu}(x, l)$ satisfies the orthogonality condition, i.e.

$$F^\dagger \left( \frac{\partial g_{\mu \nu}(x, l)}{\partial l_i} \right) \equiv -4\nabla^\mu \frac{\partial g_{\mu \nu}(x, l)}{\partial l_i} - 2 \left( C - \frac{2}{D} \right) \partial_\nu \left[ g^{\alpha \beta}(x, l) \frac{\partial g_{\alpha \beta}(x, l)}{\partial l_i} \right] = 0$$

(53)
great simplifications occur, because the inverse of $F^\dagger F$ is not to be computed. If

$$F^\dagger \left( \frac{\partial g_{\mu \nu}(x, l)}{\partial l_i} \right) \neq 0$$

(54)
it is natural to ask for the existence of a family of diffeomorphisms $f(l)$ such that

$$g'_{\mu \nu}(x, l) = [f(l)^* g_{\mu \nu}(l)](x) = \frac{\partial x'^\alpha(x, l)}{\partial x^\mu} g_{\alpha \beta}(x'(x, l), l) \frac{\partial x'^\beta(x, l)}{\partial x^\nu}$$

(55)
satisfies

$$F'^\dagger \left( \frac{\partial g'_{\mu \nu}(x, l)}{\partial l_i} \right) \equiv -4\nabla'^\mu \frac{\partial g'_{\mu \nu}(x, l)}{\partial l_i} - 2 \left( C - \frac{2}{D} \right) \partial_\nu \left[ g'^{\alpha \beta}(x, l) \frac{\partial g'_{\alpha \beta}(x, l)}{\partial l_i} \right] = 0 .$$

(56)

We shall prove that there exist $g_{\mu \nu}(x, l)$ transverse to the gauge fibers, violating eq.(53) for which no $f$ satisfying eq.(56) can be found. In particular we shall show that if an $f(l) : x \to x'(x, l)$ satisfying eq.(56) is supposed to exist then the integrability condition

$$\frac{\partial^2 x'(x, l)}{\partial l_i \partial l_j} = \frac{\partial^2 x'(x, l)}{\partial l_j \partial l_i}$$

leads to a contradiction.

Let us consider a metric $g_{\mu \nu}(x, l)$ on the $D$–dimensional torus represented by an hypercube $0 \leq x^\mu < 1$ with opposite faces identified, with the properties

$$g_{\mu \nu}(x, 0) = \delta_{\mu \nu}, \quad \delta^{\alpha \beta} \frac{\partial g_{\alpha \beta}(x, l)}{\partial l_i} \bigg|_{l=0} = 0, \quad \delta^{\alpha \beta} \partial_\alpha \frac{\partial g_{\beta \nu}(x, l)}{\partial l_i} \bigg|_{l=0} = 0,$$

(57)

for a pair of indexes $i$ and $j$. Suppose there exists $x'^\mu = x'^\mu(x, l)$ such that

$$g'_{\mu \nu}(x, l) = \frac{\partial x'^\alpha(x, l)}{\partial x^\mu} g_{\alpha \beta}(x'(x, l), l) \frac{\partial x'^\beta(x, l)}{\partial x^\nu}$$

(58)
satisfies eq. (56). As any finite $l$–dependent diffeomorphism can be written as $f(l) = f_1(l) \cdot f(0)$ with $f_1(0) = \text{identity}$ and under an $l$–independent diffeomorphism (like $f(0)$) eq. (53) is covariant, we can restrict ourselves to transformations with $x'^\mu(x, 0) = x^\mu$. For $l = 0$ eq. (56) implies

$$2 \partial_\mu \left|_{l=0} \frac{\partial g'_{\mu\nu}(x, l)}{\partial l_i} \right| + \left( C - \frac{2}{D} \right) \partial_\nu \left|_{l=0} \left( \delta^{\alpha\beta} \frac{\partial g'_{\alpha\beta}(x, l)}{\partial l_i} \right) \right| = 0$$

(59)
i.e. taking into account eq. (57)

$$\partial_\mu \left|_{l=0} \frac{\partial g_{\mu\nu}(x, l)}{\partial l_i} \right| + \partial^2 \xi^i + (C + 1 - \frac{2}{D}) \partial_\nu \partial \cdot \xi^i = 0$$

(60)
where we defined

$$\xi^i = \left. \frac{\partial x'^i(x, l)}{\partial l_i} \right|_{l=0}$$

(61)
Thus because of eq. (57) we have $\partial^2 \xi^i + (C + 1 - \frac{2}{D}) \partial_\nu \partial \cdot \xi^i = 0$, which implies $\partial^2 \partial \cdot \xi^i = 0$, i.e. on the torus $\partial \cdot \xi^i = \text{const}$. Thus $\partial^2 \xi^i = 0$ and then $\xi^i = \text{const}$.

$$\left. \frac{\partial g'_{\mu\nu}(x, l)}{\partial l_i} \right|_{l=0} = \left. \frac{\partial g_{\mu\nu}(x, l)}{\partial l_i} \right|_{l=0}$$

(62)
Eq. (53) becomes

$$0 = 2g^{\mu\nu'}(x, l) \left[ \partial_\mu \left|_{l=0} \frac{\partial g_{\mu\nu'}(x, l)}{\partial l_i} \right| - \Gamma^{\rho\nu'}_{\mu\nu'}(x, l) \frac{\partial g'_{\rho\nu'}(x, l)}{\partial l_i} - \Gamma^{\nu\nu'}_{\mu\nu'}(x, l) \frac{\partial g'_{\rho\nu'}(x, l)}{\partial l_i} \right] +$$

$$\left( C - \frac{2}{D} \right) \partial_\nu \left[ g^{\alpha\beta}(x, l) \frac{\partial g'_{\alpha\beta}(x, l)}{\partial l_i} \right].$$

(63)
We consider now the antisymmetric part in $(i, j)$ of the derivative of eq. (53) with respect to $l_j$, for $l = 0$ i.e. using eq. (52)

$$0 = 2 \left. \frac{\partial g^{\mu\nu'}(x, l)}{\partial l_j} \right|_{l=0} \left. \frac{\partial g_{\mu\nu'}(x, l)}{\partial l_i} \right|_{l=0} + \partial_\nu \left|_{l=0} \left( \delta^{\mu\nu'} \frac{\partial g_{\mu\nu'}(x, l)}{\partial l_j} \right) \right| \left. \frac{\partial g_{\nu\nu'}(x, l)}{\partial l_i} \right|_{l=0}$$

$$- \partial_\nu \left|_{l=0} \left( \delta^{\nu\nu'} \frac{\partial g_{\nu\nu'}(x, l)}{\partial l_j} \right) \right| \left. \frac{\partial g_{\mu\nu}(x, l)}{\partial l_i} \right|_{l=0} - \left( i \leftrightarrow j \right).$$

(64)
Taking into account that the square bracket is symmetric in $(i, j)$ and condition (57) we have at last

$$0 = \left. \frac{\partial g^{\mu\nu'}(x, l)}{\partial l_j} \right|_{l=0} \left. \frac{\partial g_{\mu\nu'}(x, l)}{\partial l_i} \right|_{l=0} - \frac{1}{2} \left. \frac{\partial g_{\mu\nu}(x, l)}{\partial l_i} \right|_{l=0} \left. \frac{\partial g_{\mu\nu}(x, l)}{\partial l_j} \right|_{l=0} - \left( i \leftrightarrow j \right).$$

(65)
Let us now choose
\[ \frac{\partial g_{\mu \nu}(x, l)}{\partial l_i} \bigg|_{l=0} = k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu = \text{const.} \tag{66} \]
and
\[ \frac{\partial g_{\mu \nu}(x, l)}{\partial l_j} \bigg|_{l=0} = (\eta_\mu \varepsilon_\nu + \eta_\nu \varepsilon_\mu) \sin(k \cdot x) \tag{67} \]
with \( k \cdot \varepsilon = \eta \cdot \varepsilon = k \cdot \eta = 0 \) with \( k_\mu = 2\pi n_\mu \) with \( n_\mu \) integers, to satisfy the torus boundary conditions. Eqs.\((66)\) and \((67)\) satisfy the conditions \((57)\) but if we substitute in eq.\\((65)\) we find instead of zero the value \(-k^2 \varepsilon^2 \eta_\nu \cos(k \cdot x) \neq 0.\\)

### Appendix B. Dependence on the \( C \) parameter

We give here a simple example to show that if one does not integrate on the conformal factor the dependence on \( C \) in \( J(l) = \det(t^i_{\mu \nu}, t^j_{\mu \nu})^{1/2} \text{Det}(F^\dagger F)^{1/2} \) (see eq.\\((15)\)) does not cancel out.

Let us consider a flat torus in \( D \) dimensions with metric \( \bar{g}_{\mu \nu} = \text{diag}(l_1, l_2, \ldots, l_D) \). Being the metric constant we have \( \nabla^\mu \frac{\partial \bar{g}_{\mu \nu}}{\partial l_i} = 0 \) and the integration measure becomes

\[ J = [\det(\frac{\partial \bar{g}_{\mu \nu}}{\partial l_i}, \frac{\partial \bar{g}_{\mu \nu}}{\partial l_j}) \text{Det}'(\bar{F}^\dagger \bar{F})]^{1/2} / V, \tag{68} \]

where the volume \( V = (l_1 \cdots l_D)^{1/2} \) is due to the presence of the \( \text{D} \) Killing vectors. We have

\[ (\frac{\partial \bar{g}_{\mu \nu}}{\partial l_i}, \frac{\partial \bar{g}_{\mu \nu}}{\partial l_j}) = (l_1 l_2 \cdots l_D)^{1/2}[2 \delta_{ij} l_i l_j + (C - \frac{2}{D}) \frac{1}{l_i l_j}] \tag{69} \]

whose determinant is linear in \( C \) and given by

\[ 2^{D-1} \left( \prod_{i=1}^{D} l_i \right)^{D/2 - 2} C D \tag{70} \]

On the other hand, as \( R_{\mu \nu} = 0 \)

\[ \bar{F}^\dagger \bar{F} = 4[\delta d + (C + 2 - \frac{2}{D}) d\delta] . \tag{71} \]

Due to \( \delta d \delta = d\delta d = 0 \) we have that the non zero eigenvalues of \( \bar{F}^\dagger \bar{F} \) are either eigenvalues of \( 4\delta d \) or of \( 4(C + 2 - \frac{2}{D}) d\delta \) and

\[ Z(\bar{F}^\dagger \bar{F})(s) = Z(4\delta d)(s) + Z(4(C+2-\frac{2}{D})d\delta)(s) . \tag{72} \]
Thus
\[
\text{Det}'(\bar{F}^\dagger \bar{F}) = \left[ 16(C + 2 - \frac{2}{D}) \right] Z_{(d\bar{s}j)}^{(0)} e^{-2Z_{(d\bar{s}j)}^{(0)}}
\] (73)
and thus such a power behavior cannot be canceled by the polynomial of eq.(70).
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