BASES AND BGG RESOLUTIONS OF SIMPLE MODULES
OF TEMPERLEY–LIEB ALGEBRAS OF TYPE B

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ABSTRACT. We construct explicit bases of simple modules and Bernstein–Gelfand–Gelfand (BGG) resolutions of all simple modules of the (graded) Temperley–Lieb algebra of type B over a field of characteristic zero.

INTRODUCTION

Inspired by the study of certain models in physics, Martin and Saleur introduced the main hero of this paper, the Temperley-Lieb algebra of type B or blob algebra, as the diagrammatic two parameter generalisation of the Temperley-Lieb algebra of type A [20]. The blob algebra deals with boundary conditions of the Potts model which arises in statistical mechanics. Despite having its origins in physics, the applications of the blob algebra are vast throughout pure mathematics. It brings together algebra and geometry and more recently categorical, diagrammatic and knot theoretic ideas. The blob algebra controls a portion of the representation theory of the Kac-Moody quantum algebra $U_q(\hat{sl}_2)$ as Martin and Ryom-Hansen [19] established via Ringel duality. More recently Iohara–Lehrer–Zhang [16] connected the Temperley-Lieb algebra of type B and the quantum algebra $U_q(\hat{sl}_2)$ via Schur–Weyl duality. Vaughn Jones used the Temperley-Lieb of type A to introduce the first knot invariant to distinguish between left and right trefoils, the well known Jones’ polynomials. These polynomials were introduced by studying the Temperley–Lieb algebras.

Apart from abundance of applications, the blob algebra has a very fruitful representation theory and it is of great importance and interest itself. The blob algebra is a quotient of the Hecke algebra of type B, hence it controls portion of the representation theory of the affine Hecke algebras. The decomposition numbers for the ungraded blob algebra were determined by Martin and Woodcock [22] and Ryom-Hansen [26]. However the fact that the blob algebra is graded caused a blast in its study. In their pioneering work, Brundan and Kleshchev [6] have proven that the cyclotomic Hecke algebras are isomorphic to the Khovanov-Lauda-Rouquier (KLR) algebras. The blob algebra $B^\kappa_d$, $\kappa = (\kappa_1, \kappa_2)$ a bicharge, is isomorphic to a quotient of the KLR algebra of level 2. In particular it is isomorphic to the algebra with generators

$$\{e(\bar{i}) | \bar{i} = (i_1, \cdots, i_d) \} \cup \{\psi_1, \cdots, \psi_{d-1}\} \cup \{y_1, \cdots, y_d\}$$

subject to KLR relations and two additional blob relations. In particular the blob relations are the following

$$y_1 e(\bar{i}) = 0, \text{ if } i_1 = \kappa_1, \kappa_2 \text{ and } e(\bar{i}) = 0, \text{ if } i_2 = i_1 + 1.$$

Plaza and Ryom–Hansen [25] proved that the blob algebra admits a $\mathbb{Z}$-grading and also constructed a graded cellular basis. Namely the blob algebra $B^\kappa_d$ is a graded cellular algebra with graded cellular basis indexed by standard tableaux of one-column bipartitions of $d$. The graded decomposition numbers for the blob algebra were computed by Plaza [24]. Moreover the blob algebra is quasi-hereditary and in that setting Hazi, Martin and Parker [13] determined the structure of the indecomposable tilting modules using the graded structure.

Having their origins in Lie theory, alcove geometries play an important role in the understanding of the representation theory of Hecke and KLR algebras. Martin and Woodcock [21] were the pioneers of using such techniques in the study of the blob algebra. Roughly speaking the action of the affine Weyl group of type $A_1$ divides the 2-dimensional Euclidean space into
connected components, the *alcoves*, separated by *hyperplanes* or *walls* which we index by half integers. The alcoves are indexed by integers and the alcove $a_m$ is between the hyperplanes $H_{m-1/2}$, $H_{m+1/2}$. Each bipartition $\lambda$ can be embedded in the Euclidean space and there is a natural bijection between standard $\lambda$-tableaux and paths in the Euclidean space terminating at the point $\lambda$.

**Theorem A.** Let $\lambda$ be a bipartition of $d$.

1. If $\lambda \in a_m$, $m \leq 0$, we have that
   \[ L(\lambda) = \text{span}_F \{ \psi_T \mid T \text{ does not intersect } H_{1/2} \text{ and does not last intersect } H_{m-1/2} \}. \]

2. If $\lambda \in H_{m-1/2}$, $m \leq 0$, we have that
   \[ L(\lambda) = \text{span}_F \{ \psi_T \mid T \text{ does not intersect } H_{1/2} \}. \]

*Figure 1.* Here $\lambda_1 = ((1^3), (1^7))$, $\lambda_2 = ((1^2), (1^8)) \in \text{Bip}(10)$ are bipartitions in an alcove and on a hyperplane respectively. The paths indexing basis elements for the simples $L(\lambda_1)$ and $L(\lambda_2)$ are in the shaded areas. The paths drawn are those corresponding to the initial tableaux $t^{\lambda_1}$ and $t^{\lambda_2}$ respectively (see Definition 1.6).

If $m > 0$ the result is similar up to relabelling hyperplanes and it is given in detail into the paper. In order to approach the problem we use the graded cellular structure of the blob algebra known from [25]. We also develop representation theoretic methods by constructing a presentation for the cell modules $\Delta(\lambda)$. In more detail we identify Garnir relations which are sufficient to describe the cell module.

In 1975 [4] Bernstein–Gelfand–Gelfand constructed resolutions of simple modules by Verma modules in the context of finite-dimensional Lie algebras. Those resolutions, known as BGG resolutions, have applications in many areas of mathematics. In the study of the Laplacian space [8], complex representation theory of Kac-Moody algebras [11], algebraic geometry [9]... In the context of modular representation theory of the symmetric group and Hecke algebras, BGG resolutions were first used by Bowman, Norton and Simental [3]. They utilised resolutions of Specht modules in order to provide homological construction of unitary simple modules of Cherednik and Hecke algebras of type A. They used these results in order to calculate Betti numbers and Castelnuovo–Mumford regularity of symmetric linear subspaces and this is another application of BGG resolution in algebraic geometry.

In this paper we generalise [3] by showing that all simple modules of the blob algebra admit BGG resolutions. In particular we construct resolutions of cell modules for each simple $B^\infty_d$-module indexed by a bipartition which belongs to an alcove. Simple modules indexed by bipartitions which belong to hyperplanes have much easier BGG resolutions and they are used in the proof of the second main theorem of this paper which is the following.
Theorem B. Let \( \lambda \) be a bipartition which belongs to an alcove and
\[
C_\bullet(\lambda) := \bigoplus_{\nu \in \Delta} \Delta(\nu)(|\ell(\nu)| - |\ell(\lambda)|).
\]
The complex
\[
0 \rightarrow C_\bullet(\lambda) \rightarrow L(\lambda) \rightarrow 0
\]
with differentials given by one-column homomorphisms is a BGG resolutions for the simple module \( L(\lambda) \). In other words
\[
H_i(C_\bullet(\lambda)) = \begin{cases} L(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}
\]

The paper is organised as follows. In sections 1 we introduce combinatorics of bipartitions and tableaux that arise in the representation theories of KLR algebras. We then describe the alcove geometry of type \( \tilde{A}_1 \) and we write \( \text{Shape}(\lambda) \) for \( \lambda \in \Delta \). In the next section we also provide a presentation of the cell modules of the blob algebra, in terms of the Garnir relations. In section 3 we establish the basic tool for the construction of the bases of simple modules, namely homomorphisms between cell modules. The construction of those homomorphisms utilises the Garnir-type presentation of the cell modules from section 2. The most important construction of section 3 is the spanning set of the images of the aforementioned homomorphisms. The elements of those spanning sets will certainly belong to the radical of the cell module. Section 4 includes the first of the two main results of this paper. We prove that over a field of characteristic zero the spanning set of the images is a basis for the radical of the cell module, hence we prove Theorem A. In section 5 we provide a homological construction of simple modules via the BGG resolutions and hence prove Theorem B. Again this construction is over a field of characteristic zero.

1. Combinatorics of tableaux and paths

1.1. Partitions and tableaux. We fix two positive integers \( d > 0 \) and \( e \in \{2, 3, \ldots \} \). Let \( \mathfrak{S}_d \) be the symmetric group in \( d \) letters, with length function \( L \) and set \( I := \mathbb{Z}/e\mathbb{Z} \). As Coxeter group \( \mathfrak{S}_d \) is generated by the simple transpositions \( s_1, \ldots, s_{d-1} \) subject to the relations
\[
s_i^2 = 1, \quad \text{for } i = 1, \ldots, d-1
\]
\[
s_is_j = s_js_i, \quad \text{for } i = 1 \leq i < j \leq d-1
\]
\[
s_is_{i+1}s_i = s_{i+1}s_is_i, \quad \text{for } i = 1, \ldots, d-2.
\]
We refer two the last two relations as braid relations. An \( e \)-bicarge is a pair \( \kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}^2 \) such that \( 0 < |\kappa_1 - \kappa_2| < e \). A bipartition of the positive integer \( d \) is a pair of partitions \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) such that
\[
d = |\lambda^{(1)}| + |\lambda^{(2)}|
\]
and we denote by \( \text{Bip}(d) \) the set of bipartitions of \( d \). The diagram of the bipartition \( \lambda \) is the set
\[
[\lambda] := \left\{ (r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, 2\} \mid 1 \leq c \leq \lambda_r^{(m)}, m = 1, 2 \right\}
\]
where \( \lambda_r^{(m)} \) is the \( r \)th part of the partition \( \lambda^{(m)} \), \( m = 1, 2 \). The triples \( (r, c, m) \) are called nodes or boxes and by using the usual convention we can think of the diagram as two arrays of boxes in the plane. A \( \lambda \)-tableau is a bijection \( t: [\lambda] \rightarrow \{1, \cdots, d\} \), we say that the tableau \( t \) has shape \( \lambda \) and we write \( \text{Shape}(t) = \lambda \). We can think a \( \lambda \)-tableau as a diagram of \( \lambda \), where the nodes are occupied by the integers \( \{1, \cdots, d\} \). We denote by \( t^{-1}(k) \) the node occupied by the integer \( k \in \{1, \cdots, d\} \) and by \( t(r, c, m) \) the integer occupying the node \( (r, c, m) \in [\lambda] \). A tableau \( t \) is
called standard if the entries increase along rows and down columns in both components. We
denote by $\text{Std}(\lambda)$ the set of standard $\lambda$-tableau and set
\[
\text{Std}(d) := \bigcup_{\lambda \in \text{Bip}(d)} \text{Std}(\lambda).
\]

Convention. Throughout this paper we shall be exclusively interested in one-column biparti-
tions, that is bipartitions of the form $\lambda = ((1^{\lambda_1}), (1^{\lambda_2}))$. By the notion bipartition we shall
always refer to one-column bipartitions and we shall denote the set of one-column bipartitions
of $d$ by $\text{Bip}(d)$. Moreover the nodes of the diagram of such bipartitions will be of the form
$(r, 1, m)$.

Remark 1.1. Consider the set $\Lambda_d = \{-d, -d + 1, \ldots, d - 1, d\}$. There is an obvious bijection
between $\Lambda_d$ and the set $\text{Bip}(d)$ of bipartitions of $d$, given by
\[
\text{Bip}(d) \rightarrow \Lambda_d, \quad ((1^{\lambda_1}), (1^{\lambda_2})) \mapsto \lambda_1 - \lambda_2.
\]
In other words we can identify each bipartition with an integer in the set $\Lambda_d$. Using the above
bijection we freely identify a bipartition $((1^{\lambda_1}), (1^{\lambda_2}))$ and the integer $\lambda_1 - \lambda_2$.

Let $(r, 1, m) \in [\lambda]$ be a node. We define the content of the node $\alpha$ to be
\[
\text{ct}(r, 1, m) := \kappa_m + 1 - r \in \mathbb{Z}
\]
and the residue of the node $(r, 1, m)$ as
\[
\text{res}(r, 1, m) := \text{ct}(r, 1, m) \pmod{e} \in I.
\]
We refer to a node of residue $i \in I$ as $i$-node.

Definition 1.2. Let $(r, 1, m)$ and $(r', 1, m')$ be two nodes. We define a lexicographic order and
we write $(r, 1, m) \preceq (r', 1, m')$ if either
(i) $\text{ct}(r, 1, m) < \text{ct}(r', 1, m')$ or
(ii) $\text{ct}(r, 1, m) = \text{ct}(r', 1, m')$ and $m > m'$.
If in addition $\text{res}(r, 1, m) = \text{res}(r', 1, m')$, we write $(r, 1, m) \lessdot (r', 1, m')$ and we say that $(r, 1, m)$
is less dominant than $(r', 1, m')$.

Let $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})), \mu = ((1^{\mu_1}), (1^{\mu_2})) \in \text{Bip}(d)$ be two bipartitions of $d$. We say that $\lambda$
less than $\mu$ and we write $\lambda \lessdot \mu$, if $\lambda, \mu$ have the same multiset of residues and
$|\lambda_1 - \lambda_2| > |\mu_1 - \mu_2|$.

Now we shall generalise the ordering of Definition 1.2 to an ordering on the set of standard
tableaux. For a tableau $t$ we write $t_{\downarrow \{1, \ldots, k\}}$ for the subtableau of $t$ containing the entries
$\{1, \ldots, k\}$ for $1 \leq k \leq d$.

Definition 1.3. Let $t, s \in \text{Std}(\lambda)$. We say that $t$ precedes $s$ in the lexicographic order and we
write $t \preceq s$ if and only if
\[
\text{Shape}(t_{\downarrow \{1, \ldots, k\}}) \preceq \text{Shape}(s_{\downarrow \{1, \ldots, k\}})
\]
for $1 \leq k \leq d$. In addition if $\text{res}(t) = \text{res}(s)$, we write $t \lessdot s$ and we say that $t$ is less dominant
than $s$.

Another concept that we shall recall is the degree of a given standard tableau $t$. For that
purpose we give a few more definitions. Let $\lambda \in \text{Bip}(d)$ and $A$ be a node of $\lambda$. The node $A$ is
called addable (resp. removable) if $[\lambda] \cup \{A\}$ (resp. $[\lambda] \setminus \{A\}$) is a diagram of a bipartition. We
denote by $\text{Add}(\lambda)$ and $\text{Rem}(\lambda)$ the set of addable and removable nodes of $\lambda$ respectively. For a
residue $i \in I$ we define the sets
\[
\text{Add}_i(\lambda) := \{A \in \text{Add}(\lambda) \mid \text{res}(A) = i\} \subset \text{Add}(\lambda)
\]
and
\[
\text{Rem}_i(\lambda) := \{A \in \text{Rem}(\lambda) \mid \text{res}(A) = i\} \subset \text{Rem}(\lambda).
\]
Then for a \(\lambda\)-tableau \(t\) we denote by \(\text{Add}_t(k)\) and \(\text{Rem}_t(k)\) the following sets:

\[
\text{Add}_t(k) := \{ A \in \text{Add}_{\text{res}(t^{-1}(k))}(\text{Shape}(t_{1,\ldots,k})) \mid A \triangleq t^{-1}(k) \} \quad (1.1)
\]

and

\[
\text{Rem}_t(k) := \{ A \in \text{Rem}_{\text{res}(t^{-1}(k))}(\text{Shape}(t_{1,\ldots,k})) \mid A \triangleq t^{-1}(k) \}. \quad (1.2)
\]

for all \(1 \leq k \leq d\). By using (1.1), (1.2) we define the degree of the tableau \(t\).

**Definition 1.4.** Let \(t \in \text{Std}(d)\) be a standard tableau. We define the degree of the node \(t^{-1}(k)\) to be

\[
\deg(t^{-1}(k)) := |\text{Add}_t(k)| - |\text{Rem}_t(k)|.
\]

The degree of the tableau \(t\) is the sum of the degrees of its nodes, namely

\[
\deg(t) = \sum_{k=1}^{d} \deg(t^{-1}(k)).
\]

**Definition 1.5.** Let \(\lambda \in \text{Bip}(d)\) and \(t\) be a \(\lambda\)-tableau. We define the residue sequence of \(t\) to be the \(d\)-tuple:

\[
\text{res}(t) := (\text{res}(t^{-1}(1)), \ldots, \text{res}(t^{-1}(d))) \in I^d.
\]

**Definition 1.6.** [24, Section 3] Let \(\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \text{Bip}(d)\) and \(m = \min\{\lambda_1, \lambda_2\}\). We define the initial tableau \(t^\lambda \in \text{Std}(\lambda)\) to be the tableau obtained by filling the nodes increasingly down to columns as follows:

1. even numbers less than or equal to \(2m\) in the first component,
2. odd numbers less than \(2m\) in the second component,
3. numbers greater than \(2m\) in the remaining nodes.

For a given bipartition \(\lambda = ((1^{\lambda_1}), (1^{\lambda_2}))\) the standard tableau \(t^\lambda\) is the highest in the lexicographic order among all standard \(\lambda\)-tableaux, in the sense that \(t \leq t^\lambda\) for any \(\lambda\)-tableau \(t \in \text{Std}(\lambda)\). Moreover, in order to simplify the notation, in later sections we shall write \(t^\lambda = (i_1^\lambda, \ldots, i_d^\lambda) \in I^d\) instead of \(\text{res}(t^\lambda)\) for the residue sequence of the most dominant tableau \(t^\lambda\).

**Remark 1.7.** The symmetric group \(\mathfrak{S}_d\) acts in a natural way on the set of tableaux. In particular if \(t\) is a tableau and \(s_i\) is a simple transposition, the tableau \(s_it\) obtained by interchanging the entries \(i, i+1\). For any \(\lambda\)-tableau \(t\) we define the word \(w_i \in \mathfrak{S}_d\) to be the unique element of the symmetric group such that \(w_it^\lambda = t\).

**Definition 1.8.** Let \(d, d' \in \mathbb{Z}\) be two positive integers with \(d' < d\). If \(\lambda \in \text{Bip}(d)\) and \(\nu \in \text{Bip}(d')\) we define the skew bipartition \(\lambda \setminus \nu\) to be the bipartition with diagram the set difference \([\lambda] - [\nu]\).

**Definition 1.9.** Let \(d, d' \in \mathbb{Z}\) with \(d' < d\), \(\lambda \in \text{Bip}(d)\), \(\nu \in \text{Bip}(d')\) and let \(\lambda \setminus \nu\) be the skew bipartition. If \(t \in \text{Std}(\nu)\) and \(s \in \text{Std}(\lambda \setminus \nu)\) then the \(\lambda\)-tableau with entries

\[
(t^{-1}(1), \ldots, t^{-1}(d'), s^{-1}(1), \ldots, s^{-1}(d-d')).
\]

is the composition \(t \circ s \in \text{Std}(\lambda)\) of the tableaux \(t\) and \(s\).

### 1.2 Paths and alcove geometry

Let \(\{\varepsilon_1, \varepsilon_2\}\) be formal symbols. We consider the 2-dimensional Euclidean space

\[
V := \bigoplus_{i=1,2} \mathbb{R}\varepsilon_i
\]

with basis \(\{\varepsilon_1, \varepsilon_2\}\) and let \(V_{\mathbb{Z}_{\geq 0}}\) be the \(\mathbb{Z}_{\geq 0}\)-span of \(\{\varepsilon_1, \varepsilon_2\}\). To any bipartition \(\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \text{Bip}(d)\) we attach a point of the Euclidean space \(V\) via the embedding \(((1^{\lambda_1}), (1^{\lambda_2})) \mapsto \sum_{i=1,2} \lambda_i \varepsilon_i\).

We consider the affine Weyl group \(W_{\text{aff}} \cong \mathfrak{S}_d\) of type \(A_1\) with \(\alpha_1 = \varepsilon_1 - \varepsilon_2\) the corresponding simple root. Let \((\cdot, \cdot)\) be a symmetric bilinear form on \(V\) determined by \((\alpha_1, \alpha_1) = 2\). For a given \(\varepsilon\)-bicharge \(\kappa = (\kappa_1, \kappa_2)\) we set \(\rho := (\kappa_2 - \kappa_1)\varepsilon_1\). For any \(m \in \mathbb{Z}\) we define the hyperplane

\[
H_{\alpha_1, m-\frac{1}{2}} := \{ v \in V \mid (v + \rho, \alpha_1) = me \}. \quad (1.3)
\]
and we sometimes refer such a hyperplane as a wall. For any \( m \in \mathbb{Z} \) there exists a unique reflection \( s_{\alpha, m-1/2} \) such that

\[
s_{\alpha, m-1/2} \cdot v = v - ((v + \rho, \alpha) - m \epsilon) \alpha
\]

for any \( v \in V \). In other words \( s_{\alpha, m-1/2} \) acts on \( V \) by reflection with respect to the hyperplane \( H_{\alpha, m-1/2} \). From now on, since we have only one simple root \( \alpha \), we shall write simply \( H_m \); \( m \) for the wall and the reflection corresponding to the integer \( m \in \mathbb{Z} \), respectively.

For any two integers \( r, s \in \mathbb{Z} \) we denote by \([r, s]\) the set \([r, s] = \{ t \in \mathbb{Z} \mid r \leq t \leq s \}\). For \( d \in \mathbb{Z}_{>0} \) we define \( \text{Path}(d) \) to be the set of maps \( \pi: [0, d] \rightarrow \mathbb{Z}_{\geq 0} \) such that

\[
\pi(0) = 0 \text{ and } \pi(k + 1) - \pi(k) \in \{ \epsilon_1, \epsilon_2 \}
\]

for all \( k \in [0, d - 1] \) and we call its elements \textbf{paths} from 0 to \( d \). Given a standard tableau \( t \in \text{Std}(d) \) we define the point \( \pi_t(k) \) in the space \( V_{\mathbb{Z}_{\geq 0}} \) by the formula

\[
\pi_t(k) := c_{k, 1}(t) \epsilon_1 + c_{k, 2}(t) \epsilon_2 \tag{1.4}
\]

where \( c_{k, i}(t) \) is the number of nodes of the tableau \( t_{i \downarrow (1, \ldots, k)} \) in the \( i \text{th} \) component. Using the aforementioned notation we shall define the path in \( V_{\mathbb{Z}_{\geq 0}} \) attached to a standard tableau \( t \in \text{Std}(\lambda) \).

**Definition 1.10.** Let \( t \in \text{Std}(d) \) be a standard tableau. We define the path \( \pi_t \) corresponding to the tableau \( t \) given by the sequence of points

\[
\pi_t = (\pi_t(0), \ldots, \pi_t(d))
\]

in the sense of relation (1.4). There is a bijection between the set \( \text{Std}(d) \) of standard tableau and the set of paths \( \text{Path}(d) \), given by \( t \mapsto \pi_t \).

Using the notation above we shall define the reflected path through a hyperplane of \( V \).

**Definition 1.11.** Let \( t \in \text{Std}(d) \) and suppose that \( \pi_t(a) \in H_{m-1/2} \) is the \( i \text{th} \) intersection point of \( \pi_t \) with the hyperplane \( H_{m-1/2} \). We define the path \( s_{m-1/2}^i \cdot \pi_t \) as follows

\[
(s_{m-1/2}^i \cdot \pi_t)(k) := \begin{cases} 
\pi_t(k) & \text{if } 0 \leq k \leq a \\
 s_{m-1/2} \cdot \pi_t(k) & \text{if } a < k \leq d 
\end{cases}
\]

We refer to the path \( s_{m-1/2}^i \cdot \pi_t \) as the \textbf{reflected path} through the \( i \text{th} \) intersection point of \( \pi_t \) with the hyperplane \( H_{m-1/2} \).

**Remark 1.12.** Note that if the path \( \pi_t \) intersects the hyperplane \( H_{m-1/2} \) at a unique point, then we shall denote the reflected path simply by \( s_{m-1/2} \cdot \pi_t \).

In Figure 2 we visualise the last definition. We draw a path with endpoint the bipartition \( \lambda \) and we also draw the reflected path through its second intersection point with the hyperplane \( H_{-1/2} \).
Let \( u, v \in V_{\mathbb{Z} \geq 0} \) such that \( u - v = \varepsilon_i, i = 1, 2 \). Then we define the degree of the pair \((u, v)\) as follows

\[
\text{deg}(u, v) := \begin{cases} 
1 & \text{if } u \in H_{m - \frac{1}{2}} \text{ and } |(v + \rho, \alpha_1)| < |mc| \text{ for some } m \in \mathbb{Z}; \\
-1 & \text{if } v \in H_{m - \frac{1}{2}} \text{ and } |(u + \rho, \alpha_1)| > |mc| \text{ for some } m \in \mathbb{Z}; \\
0 & \text{otherwise}. 
\end{cases}
\] (1.5)

By using relation (1.5) we are able to give a reinterpretation of the degree of a tableau in terms of paths. Let \( t \in \text{Std}(d) \) and \( \pi_t \in \text{Path}(d) \) be the path corresponding to \( t \). The integer

\[
\text{deg}(\pi_t) := \sum_{k=0}^{d-1} \text{deg}(\pi_t(k), \pi_t(k+1))
\]

is the degree of the path \( \pi_t \) and it is straightforward that it coincides with the degree of \( t \).

Using the aforementioned notions we are able to describe an alcove geometry on the Euclidean space \( V \). We say that for any \( m \in \mathbb{Z} \), the set of points \( a_m := \left\{ v \in V \mid m - \frac{1}{2} < (v + \rho, \alpha_1) < m + \frac{1}{2} \right\} \)

forms an alcove. By the definition of the hyperplane as presented in (1.3), we can deduce that the origin, namely the point \((0,0)\), will always lie in an alcove and not on a hyperplane. We consider a Pascal triangle with points corresponding to integers and the top of the triangle is the reflected paths is the bipartition \( \mu \) corresponding to 0. We can represent the paths in \( V \) as paths in the Pascal triangle starting from the top and moving downwards. Let \( \lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \text{Bip}(d) \) and \( t \in \text{Std}(\lambda) \) be a standard tableau. The path \( \pi_t \) is a path starting from the top of the Pascal triangle and ending at a point corresponding to the integer \( \lambda_1 - \lambda_2 \) at the level \( d \) of the triangle.

**Notation.** From now on we shall not distinguish between the standard tableau and the corresponding path. Namely, we will denote the path corresponding to the tableau \( t \) by

\[
T = (T(0), \cdots, T(d)) \in \text{Path}(d).
\]

Moreover for a given bipartition \( \lambda \in \text{Bip}(d) \) we denote by \( \text{Path}(\lambda) \) the set of paths with endpoint the bipartition \( \lambda \). By using the above notation we have that

\[
\text{Path}(d) = \bigcup_{\lambda \in \text{Bip}(d)} \text{Path}(\lambda).
\]

In the following example we shall summarise most of the facts we discussed above. Note that if \( t \in \text{Std}(d) \) is a standard tableau, we denote by \( t^t \) the transpose of \( t \). For ease of notation we will often use the transpose tableau.

**Example 1.13.** Suppose \( e = 4 \), \( d = 9 \), \( \kappa = (0, 2) \) and let \( \lambda = ((1^2), (1^7)), \mu = ((1^5), (1^4)) \in \text{Bip}(9) \). We consider the \( \lambda \)-tableau

\[
t^t = \begin{pmatrix} 4 & 7 & 1 & 2 & 3 & 5 & 6 & 8 & 9 \end{pmatrix}.
\]

By following the description above we can construct the path corresponding to the tableau \( t \) as in the left picture in Figure 2. We observe that the path \( T \) intersects the hyperplane \( H_{-\frac{1}{2}} \) at two points which correspond to the steps \( t_{-1/2}^{1/2} \) and \( t_{-1/2}^{2} \) of the path. Then we obtain the reflected paths \( s_{-1/2}^{1} \cdot T, s_{-1/2}^{2} \cdot T \) and the later is also pictured in Figure 2. The endpoint of the reflected paths is the bipartition \( \mu \).

Moreover one can easily calculate the degree of the path \( T \) to be equal to \(-1\). To see this note that \( \text{deg}(T(3), T(4)) = -1 \) and degree is zero otherwise. This is something we expect since \( \text{deg}(t^{-1}(4)) = -1 \) and the rest nodes of the tableau \( t \) are of degree 0.

The residue sequence of the tableau \( t \) is

\[
\text{res}(t) = (2, 1, 0, 0, 3, 2, 3, 1, 0)
\]
and we observe that \( \text{res}(s^{2-1/2} \cdot t) = \text{res}(t) \).

More generally given any two tableaux \( t, s \in \text{Std}(d) \) we have that
\[
\text{res}(t) = \text{res}(s) \iff T = s^{j_1_{i_1-1/2}} \cdots s^{j_a_{i_a-1/2}} \cdot S
\]
for some simple reflections \( s_{i_l-1/2}, 1 \leq l \leq a \). Given two bipartitions \( \lambda, \mu \in \text{Bip}(d) \) and \( T \in \text{Path}(\lambda) \), we define the set of \( \mu \)-paths which can be obtained by \( T \) by a series of reflections as follows:
\[
\text{Path}(\mu, T) := \{ S \in \text{Path}(\mu) \mid S = s^{j_1_{i_1-1/2}} \cdots s^{j_a_{i_a-1/2}} \cdot T, \text{ for some } s_{i_l-1/2} \in \hat{S}_2 \}.
\]
Now we equip our alcove geometry with a \textit{length function}
\[
\ell: \text{Bip}(d) \longrightarrow \frac{1}{2} \mathbb{Z}, \quad \lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \mapsto \begin{cases} m & \text{if } \lambda_1 - \lambda_2 \in a_m \\ m - \frac{1}{2} & \text{if } \lambda_1 - \lambda_2 \in H_m - \frac{1}{2}. \end{cases}
\]
We will also give a useful geometric interpretation of the dominance order on tableaux, mentioned in Definition 1.3 in terms of the alcove geometry. Given two tableau \( t, s \in \text{Std}(d) \) with \( \text{res}(t) = \text{res}(s) \) we say that the node \( t^{-1}(k) \) is less dominant that the node \( s^{-1}(k) \) in the sense of Definition 1.2 if and only if
\[
|\ell(\text{Shape}(t \downarrow_{\{1, \ldots, k\}}))| > |\ell(\text{Shape}(s \downarrow_{\{1, \ldots, k\}}))|.
\]
The tableau \( t \) is less dominant than \( s \) if and only if \( t^{-1}(k) \preceq s^{-1}(k), 1 \leq k \leq d, \) and there is at least one node of \( t \) strictly less dominant than the corresponding node of \( s \).

**Example 1.14.** We continue on the Example 1.13 and we have that \( \ell(\lambda) = -1 \) while \( \ell(\mu) = 0 \). The paths \( S_1, S_2 \) drawn in the following figure are the elements of \( \text{Path}(\mu, T) \).

![Diagram of paths](image)

**Figure 3.** The paths \( S_1 \) and \( S_2 \) are solid. The path \( T \) is dotted.

In particular we have that \( S_1 = s_{-1/2} \cdot T \) and \( S_2 = s_{-1/2} \cdot T \). Moreover we observe that \( S_1 \triangleright S_2 \triangleright T \).

We say that two bipartitions \( \lambda, \mu \) are \textit{linked} with respect to the alcove geometry of type \( \hat{A}_1 \) and we write \( \lambda \sim \mu \) if they belong to the same \( W_{\text{aff}} \)-orbit, i.e \( \lambda \in W_{\text{aff}} \cdot \mu \). If \( T \in \text{Path}(\lambda) \) then the paths linked with \( T \) are the paths of \( \text{Path}(\mu, T) \), defined above, for \( \mu \sim \lambda \). The paths linked with the path \( T^\lambda \) will be of particular interest when we construct homomorphisms of the blob algebra. We define
\[
\text{Path}_\sim(\lambda) := \bigcup_{\mu \triangleright \lambda} \text{Path}(\mu, T^\lambda).
\]

**Remark 1.15.** If \( \lambda, \mu \in \text{Bip}(d) \) are two bipartitions, we note that \( \lambda \) is less dominant than \( \mu \) if and only if \( \lambda \sim \mu \) and \( |\ell(\lambda)| > |\ell(\mu)| \), i.e. \( \lambda \) is further away from the origin of the Pascal triangle than \( \mu \).

Let us see an example regarding the notions we discussed above.
Example 1.16. Let $d = 9, e = 4, \kappa = (0, 2)$ as in Example 1.13 and $\lambda = ((1), (1^8))$. Then
\[(t^\lambda)^t = \begin{pmatrix} 2 & 1 & 3 & 4 & 5 & 6 & 8 & 9 \end{pmatrix}\]
and the paths linked with $T^\lambda$ are drawn in the following diagram.

![Diagram](image)

Figure 4. The red path is the path $T^\lambda$ and the black ones are those linked with $T^\lambda$.

The new linked paths correspond to the tableaux
\[s_1^t = \begin{pmatrix} 2 & 9, & 1 & 3 & 4 & 5 & 6 \end{pmatrix}\]
\[s_2^t = \begin{pmatrix} 2 & 5 & 6 & 7 & 8, & 1 & 3 & 4 \end{pmatrix}\]
and
\[s_3^t = \begin{pmatrix} 2 & 5 & 6 & 7 & 8 & 9, & 1 & 3 & 4 \end{pmatrix}\]

Thus $\text{Path}_\sim(\lambda) = \{T^\lambda, S_1, S_2, S_3\}$ and one can easily see that the bipartitions linked with $\lambda$ are $\lambda$ itself and $((1^2), (1^7)), ((1^5), (1^4)), ((1^6), (1^3))$.

2. The Blob Algebra

2.1. Definition and basic properties. In this section we shall introduce the main object of our study, namely the blob algebra. The blob algebra was first introduced by Martin and Saleur [20], but we shall present the equivalent definition given in [25]. We will not give many details regarding the structure of the blob algebra. The interested reader may look for further details in the literature, for example [24] and [25]. As in the last section we fix two positive integers $d > 0$ and $e \in \{2, 3, \cdots\}$ with $I := (\mathbb{Z}/e\mathbb{Z})^d$ and let $F$ be a field of characteristic $p \geq 0$.

Definition 2.1. Let $\kappa = (\kappa_1, \kappa_2)$ be an $e$-bicharge. The blob algebra $B_d^e$ is the $F$-algebra on the generators
\[
\{e(i) | i \in I^d\} \cup \{y_1, \cdots, y_d\} \cup \{\psi_1 \cdots, \psi_{d-1}\}
\]
subject to the usual KLR relations
\[e(i) = 0, \quad \text{if } i_1 \neq \kappa_1, \kappa_2 \quad (2.1)
\]
\[y_1 e(i) = 0, \quad \text{if } i_1 = \kappa_1, \kappa_2 \quad (2.2)
\]
\[e(i)e(j) = \delta_{i,j} e(i), \quad (2.3)
\]
\[\sum_{i \in I^d} e(i) = 1, \quad (2.4)
\]
\[y_r e(i) = e(i)y_r, \quad (2.5)
\]
\[\psi_r e(i) = e(s_r i) \psi_r, \quad (2.6)
\]
\[y_r y_s = y_s y_r, \quad (2.7)
\]
\[\psi_r y_s = y_s \psi_r, \quad \text{if } |r - s| > 1 \quad (2.8)
\]
\[\psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r - s| > 1 \quad (2.9)
\]
\[\psi_r y_{r+1} e(i) = (y_r \psi_r + \delta_{i, i_{r+1}}) e(i). \quad (2.10)
\]
We define the and we shall refer to the word \( w \) along with the additional blob relation \( e(z) = 0 \), if \( i_2 = i_1 + 1 \). (2.14)

The function \( \text{deg}: B^s_0 \rightarrow \mathbb{Z} \) determined by

\[
\text{deg}(e(z)) = 0, \quad \text{deg}(y_r e(z)) = 2, \quad \text{deg}(\psi_s e(z)) = \begin{cases} -2 & \text{if } i_s = i_{s+1} \\ 0 & \text{if } i_s \neq i_{s+1} \pm 1 \\ -1 & \text{if } i_s = i_{s+1} + 1 \end{cases}
\]

for \( 1 \leq r \leq d \) and \( 1 \leq s \leq d - 1 \) is a degree function of \( B^s_0 \). Thus the blob algebra is a \( \mathbb{Z} \)-graded algebra with \text{deg} being the degree function. We also let \(*: B^s_0 \rightarrow B^s_0\) be the anti-involution defined by fixing the KLR generators.

Note that there is a diagrammatic presentation of the blob algebra in terms of KLR diagrams. For a more detailed description about the KLR diagrams we refer to [18], [13]. Each KLR diagram of \( B^s_0 \) consists of \( d \) strings and each string carries a residue \( i \in \mathbb{Z}/c\mathbb{Z} \). The bottom and the top of the KLR diagram are sequences of residues. The product of two KLR diagrams is given by concatenation. If \( i = (i_1, i_2, \ldots, i_d) \in I^d \) we have that that

\[
e(z) = \begin{array}{cccccc}
& \cdots & & \cdots & & \\
i_1 & i_2 & i_d & i_1 & i_{s+1} & i_d \\
\end{array}
\]

\[
\psi_s e(z) = \begin{array}{cccccc}
& \cdots & & \cdots & & \\
i_1 & i_s & & & & \\
\end{array}
\]

\[
y_r e(z) = \begin{array}{cccccc}
& \cdots & & \cdots & & \\
i_1 & i_r & i_d & i_1 & i_d \\
\end{array}
\]

The diagrammatic interpretation of (2.1)-(2.14) is given in [13].

If \( w = s_{i_1} \cdots s_{i_t} \in \mathcal{S}_d \) is a reduced expression of an element of the symmetric group, we set \( \psi_w = \psi_{i_1} \cdots \psi_{i_t} \in B^s_0 \).

Recall that for any \( \lambda \)-tableau \( t, \lambda \in \text{Bip}(d) \), we have defined the reduced expression \( w_t = s_{i_1} \cdots s_{i_t} \in \mathcal{S}_d \) such that \( t = w_t \cdot \lambda \). Later we shall prove that this reduced expression is unique up to the commuting relations of the symmetric group and the KLR algebra. We define the element \( \psi_t := \psi_{i_1} \cdots \psi_{i_t} e((z)^\lambda) \).

and we shall refer to the word \( w_t \) as the reduced expression of \( t \).

**Definition 2.2.** Suppose that \( \lambda \in \text{Bip}(d) \) and \( t, s \in \text{Std}(\lambda) \). We define the element

\[
\psi_{st} := \psi_s e((z)^\lambda) \psi_t^* \in B^s_0.
\]

**Theorem 2.3** ([25, Theorem 6.10]). The blob algebra \( B^s_0 \) is a graded \( F \)-algebra with basis \n
\[
\{ \psi_{st} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \text{Bip}(d) \}.
\]

We let \( B^s_{d,\lambda} \) be the \( F \)-submodule of \( B^s_0 \) with basis

\[
\{ \psi_{uv} \mid u, v \in \text{Std}(\mu) \text{ for } \mu \in \text{Bip}(d), \mu \trianglerighteq \lambda \}.
\]
Under the anti-involution \(*\) : \(B_d^\kappa \rightarrow B_d^\kappa\), we have \(\psi_{st}^* = \psi_{ts}\). For any \(\lambda \in \text{Bip}(d)\), \(t \in \text{Std}(\lambda)\) and \(a \in B_d^\kappa\) there exists \(\alpha_u \in F\) such that for all \(s \in \text{Std}(\lambda)\)

\[
a\psi_{st} = \sum_{u \in \text{Std}(\lambda)} \alpha_u \psi_{su} \mod B_d^{\kappa, \lambda}.
\]

In particular the blob algebra \(B_d^\kappa\) is a graded cellular algebra.

Remark 2.4. Let \(\lambda \in \text{Bip}(d)\) be a bipartition of \(d\) and \(r, r + 1, r + 2, 1 \leq r \leq d - 2\), be three successive positive integers. Then the four different cases for a standard \(\lambda\)-tableau are those in Figure 5.

![Tableau Diagram](image)

**Figure 5.** We refer to standard tableaux of the above form as tableaux of form (T1)-(T4), respectively.

The subword \(s_rs_{r+1}s_r\) cannot appear in the reduced expression of any of the above tableaux, as if we apply it to any standard tableau we get a non-standard tableau. In particular if \(t \in \text{Std}(\lambda)\) is the initial tableau, the non-standard tableau would be the one coming from the interchange of the nodes occupied by the entries \(r, r + 2\), which can be denoted by \(t_{r+r+2}\). Hence the reduced expression for each tableau is unique up to the commuting relations of the symmetric group. It follows that for any tableau \(t\) with \(w_t = s_{i_1} \cdots s_{i_k}\), the coset \(\psi_{i_1} \cdots \psi_{i_k}\) is unique up to the KLR relation (2.9).

By the classical theory of cellular algebras as presented for example in [23, Chapter 2] we know that there exists a family of modules \(\{\Delta(\lambda) \mid \lambda \in \text{Std}(\lambda)\}\) with \(F\)-basis

\[
\{\psi_t \mid t \in \text{Std}(\lambda)\}
\]

called cell or standard modules and there is a unique bilinear form \(\langle \cdot, \cdot \rangle : \Delta(\lambda) \times \Delta(\lambda) \rightarrow F\) such that \(\langle \psi_s, \psi_t \rangle\) for \(s, t \in \text{Std}(\lambda)\), is given by

\[
\langle \psi_s, \psi_t \rangle \psi_{uv} = \psi_{us} \psi_{tu} \mod B_d^{\kappa, \lambda}.
\]

The radical of a cell module \(\Delta(\lambda)\) is given by

\[
\text{rad} \Delta(\lambda) := \{x \in \Delta(\lambda) \mid \langle x, y \rangle \text{ for all } y \in \Delta(\lambda)\}.
\]

We let \(L(\lambda) := \Delta(\lambda)/\text{rad} \Delta(\lambda)\). Using the above notation and [23, Theorem 2.16] we have that \(\{L(\lambda) \mid \lambda \in \text{Std}(d)\}\) is a complete set of pairwise inequivalent irreducible \(B_d^\kappa\)-modules. Therefore by [17] the blob algebra is quasi-hereditary.
2.2. Graded decomposition numbers of the blob algebra. As we discussed in the previous subsection the blob algebra $B^a_d$ is a $\mathbb{Z}$-graded algebra. Let $M$ be a finite dimensional graded $B^a_d$-module and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be its decomposition into direct sum of homogeneous components. We define the graded dimension of $M$ to be the polynomial
\[ \dim_t(M) := \sum_{i \in \mathbb{Z}} (\dim M_i) t^i \in \mathbb{Z}[t, t^{-1}]. \]
Moreover if $L(\lambda)$ is a simple graded $B^a_d$-module then we denote by $L(\lambda)(k)$ the graded $B^a_d$-module obtained by shifting the grading on $\hat{L}(\lambda)$ up by $k$, namely
\[ L(\lambda)(k) = \bigoplus_{i \in \mathbb{Z}} L(\lambda)(k)_i = \bigoplus_{i \in \mathbb{Z}} L(\lambda)_{i-k}. \]

The simple modules for the ungraded blob algebra were given by Martin and Woodcock [22]. The following theorem summarises the work of Plaza and Ryom Hansen using the Hu and Mathas’ work on graded cellular structure of KLR algebras, [25, 14].

**Theorem 2.5.** Let $L(\lambda)$, $\lambda \in \text{Bip}(d)$ be a simple module. Then
\[ \{ L(\lambda)(k) \mid \lambda \in \text{Bip}(d) \text{ and } k \in \mathbb{Z} \} \]

is a complete set of pairwise non-isomorphic simple graded $B^a_d$-modules.

We also have the following useful proposition from [14].

**Proposition 2.6.** For any $\lambda, \mu \in \text{Bip}(d)$ the simple module $L(\lambda)$ is bar-invariant (i.e., fixed under interchanging $t$ and $t^{-1}$).

**Proof.** See [14, Proposition 1.8].

It is important to know the block structure of the blob algebra. In our case of study the block structure is controlled by a linkage property with respect to the affine Weyl group $W_{\text{aff}}$ of type $\tilde{A}_1$.

**Proposition 2.7.** Let $\lambda, \mu \in \text{Bip}(d)$. Two simple modules $L(\lambda)$, $L(\mu)$ are in the same block of $B^a_d$ if and only if $\lambda$ and $\mu$ are in the same orbit, i.e $\lambda \in W_{\text{aff}} \cdot \mu$.

For a graded $B^a_d$-module $M$ we denote by $[M : L(\lambda)(k)]$ the graded multiplicity of the simple module $L(\lambda)(k)$ as a graded composition factor of $M$. Then the graded decomposition number is
\[ [M : L(\lambda)]_t := \sum_{k \in \mathbb{Z}} [M : L(\lambda)(k)] t^k \in \mathbb{Z}[t, t^{-1}]. \]
In particular we are interested in the decomposition matrix $D = (d_{\mu,\lambda})_{\mu,\lambda \in \text{Bip}(d)},$ that is the decomposition numbers
\[ d_{\mu,\lambda} = [\Delta(\mu) : L(\lambda)]_t \]
which were computed, over a field $F$ of characteristic zero, by Plaza [24]. The closed formula for the graded decomposition number $[\Delta(\mu) : L(\lambda)]_t$ depends on whether the bipartition $\lambda$ lies in an alcove or on a hyperplane. However using the length function we defined before we can amalgamate the two distinct formulas into one. In what follows we assume that $\mu \geq \lambda$, since this is the only case we can have a non-zero decomposition number, by Theorem 2.3. The following theorem gives the graded decomposition numbers of the blob algebra.

**Theorem 2.8** ([24, Theorem 5.11, 5.15]). Let $\lambda \in \text{Bip}(d)$ be a bipartition and $\mu \supseteq \lambda$ be a bipartition linked with $\lambda$, i.e $\mu \sim \lambda$. Then
\[ [\Delta(\mu) : L(\lambda)]_t = t^{|\ell(\mu)| - |\ell(\lambda)|}. \]

**Example 2.9.** We continue with the example 1.16 and let $\lambda_1 = ((1^2), (1^7)), \lambda_2 = ((1^5), (1^4)),$ $\lambda_3 = ((1^6), (1^3))$. According to Theorem 2.8, the (non-zero) graded decomposition numbers are the following:
\[ [\Delta(\lambda_1) : L(\lambda)] = 1, \ [\Delta(\lambda_2) : L(\lambda)] = t, \ [\Delta(\lambda_3) : L(\lambda)] = t^2 \text{ and } [\Delta(\lambda_3) : L(\lambda)] = t. \]
2.3. Garnir relations for the blob algebra. In this subsection we provide a presentation for the cell modules of $B_d^\lambda$. In particular we derive Garnir relations for the blob algebra. The above, apart from its importance on the structure of the cell modules, will be important in the next section where we wish to construct homomorphisms between cell modules.

Recall from [7] the Bruhat order $\leq$ on $\mathfrak{S}_d$: for $u, w \in \mathfrak{S}_d$ we say that $u \leq w$ if and only if the reduced expression of $u$ is a subexpression of the reduced expression of $w$ (see [15, Theorem 5.10]). We can connect the Bruhat order on permutations with the dominance order on tableaux.

**Lemma 2.10.** Let $\lambda \in \text{Bip}(d)$ and $t, s$ be two $\lambda$-tableaux with $w_{t}, w_{s} \in \mathfrak{S}_d$ be the unique permutations such that $w_{t}t^{\lambda} = t$ and $w_{s}t^{\lambda} = s$. Then

$$s \preceq t \iff w_{t} \leq w_{s}.$$

**Proof.** The result is straightforward from the fact that $t = w_{t}t^{\lambda}$ and $s = w_{s}t^{\lambda}$. $\square$

**Definition 2.11.** Let $\lambda \in \text{Bip}(d)$ and $A = (r, 1, m) \in [\lambda]$ be a node of the diagram of $\lambda$. The node $A$ is called Garnir node if it is not removable.

Suppose that $A = (r, 1, m) \in [\lambda]$ is a Garnir node and let $u := t^{\lambda}(r, 1, m)$ and $v := t^{\lambda}(r + 1, 1, m)$. It is clear from the definition of $t^{\lambda}$ that -for $(r, 1, m)$ be Garnir node- there are two distinct cases for $u$ and $v$. In particular it will either be $v = u + 1$ or $v = u + 2$. The Garnir belt $B^{A}$ is a set of nodes which is defined as follows.

1. If $v = u + 1$,

$$B^{A} := \{(r, 1, m), (r + 1, 1, m)\}.$$

2. If $v = u + 2$ we have two subcases.

   $\circ$ If $m = 1$,

   $$B^{A} := \{(r, 1, 1), (r + 1, 1, 1), (r + 1, 1, 2)\}.$$

   $\circ$ If $m = 2$,

   $$B^{A} := \{(r, 1, 2), (r + 1, 1, 2), (r, 1, 1)\}.$$

The Garnir tableau $G^{A}$ of the Garnir node $A$ is the $\lambda$-tableau which coincides with $t^{\lambda}$ outside the Garnir belt $B^{A}$. Inside the Garnir belt:

1. If $v = u + 1$, then $G^{A}$ has the entries $u, u + 1$ from the bottom to the top in the $m^{th}$ column, $m = 1, 2$.

2. If $v = u + 2$, then $G^{A}$ has the entries $u, u + 1, u + 2$ from the bottom to the top in both components.

**Remark 2.12.** Let $A = (r, 1, m)$ be a Garnir node. When $v = u + 1$ there is an easy form for the Garnir tableau. In particular we have

$$G_{1}^{A} = s_{u}t^{\lambda}.$$  (2.15)

When $v = u + 2$ there are two Garnir tableaux. We denote by $G_{1}^{A}$ the Garnir tableau derived by first filling the first component and by $G_{2}^{A}$ the Garnir tableau derived by first filling the second component. In particular we have that

$$G_{1}^{A} = s_{u}s_{u+1}t^{\lambda} \text{ and } G_{2}^{A} = s_{u+1}s_{u}t^{\lambda}, \text{ if } m = 1$$  (2.16)

and

$$G_{1}^{A} = s_{u+1}s_{u}t^{\lambda} \text{ and } G_{2}^{A} = s_{u}s_{u+1}t^{\lambda}, \text{ if } m = 2. \quad (2.17)$$

When we are not interested in distinguishing between (2.15), (2.16) and (2.17) we shall write $G^{A}$ for the Garnir tableau of the Garnir node $A$.

**Lemma 2.13.** Let $\lambda \in \text{Bip}(d)$ and $t \not\in \text{Std}(d)$. Suppose that $A = (r, 1, m) \in [\lambda]$ is a node such that $t(r, 1, m) > t(r + 1, 1, m)$. Then there exists $w \in \mathfrak{S}_d$ such that $t = wG_{2}^{A'}$ for some Garnir node $A' \in [\lambda]$ and $L(w_{t}) = L(w) + L(w_{G_{2}^{A'}}).$ Conversely, if $t = wG_{2}^{A'}$ with $L(w_{t}) = L(w) + L(w_{G_{2}^{A'}})$ then $t \in \text{Std}(\lambda).$
Proof. Let $u := t^\lambda(r, 1, m)$, $v := t^\lambda(r + 1, 1, m)$, $a := t(r, 1, m)$ and $b := t(r, 1, m)$. First consider the case that $v = u + 1$, i.e $G^A_t = s_u t^\lambda$ and without loss of generality we assume that $(r, 1, m)$ is the unique node with $t(r, 1, m) > t(r + 1, 1, m)$. If $t = G^A_t$ we have nothing to prove, so let $t \neq G^A_t$. If $a = v$ and $b = u$ the result is straightforward. Assume that $a \neq v$, $b = a - 1$. Let $s := t_{a+a-1} \in \text{Std}(\lambda)$, hence $t = s_{a-1}s$. Then the word $s_{a-1}s_as_{a-1}$ appears as subword of $w_t$ and by successively applying the braid Coxeter relations we end up with a subword of the form $s_us_{u-1}s_u$ with $s_u$ being right exposed. Note that if $b \neq a - 1$ then $t$ will be of the form

for $2 \leq k \leq a - 1$ and we simply have the subword $s_{a-k} \cdots s_{a-2}$ on the left of $s_us_{u-1}s_u$. Similarly if $t$ is of the form

for $1 \leq k \leq d - a$, we have the subword of the form $s_{a+k-1} \cdots s_a$ on the left of $s_us_{u-1}s_u$. In any case we have that

$$t = w's_us_{u-1}s_ut^\lambda = w's_us_{u-1}G^A_1$$

(2.18)

for some $w' \in S_d$.

Now consider the case that $v = u + 2$ and we may assume that $m = 1$, i.e $G^A_t = s_u s_{u+1} t^\lambda$, $G^A_2 = s_{u+1} s_u t^\lambda$. If the entries $u, u + 1, u + 2$ occupy the nodes in $B^A$ in $t$ then the result is straightforward, that is $t = s_uG^A_2$. Now suppose that the numbers $u, u + 1, u + 2$ do not occupy the nodes of $B^A$, but those nodes contain consecutive numbers $a, a + 1, a + 2$. Then if $a < u$ we have that on of the subwords $s_{a+3}s_{a+2}$ or $s_{a+2}s_{a+3}$ will appear in $w_t$ and it will be right exposed, hence

$$t = wG^B_t$$

(2.19)

for some $w \in S_d$. If $a > u$ then either $s_{a-2}s_{a-1}$ or $s_{a-1}s_{a-2}$ will appear as subword of $w_t$ and it will be right exposed, hence

$$t = wG^C_t$$

(2.20)

for some $w \in S_d$. Note that everything works similarly for $m = 2$. From (2.18), (2.19) and (2.20) we have the desired result.

Conversely, suppose that $t \in \text{Std}(\lambda)$. Since $G^A_w$ is non-standard, we should have $L(w_t) < L(w) + L(w G^A_d)$ which is a contradiction. □
Theorem 2.14 (Garnir relations for the blob algebra). Let $\lambda \in \text{Bip}(d)$. Then
\[
e(\bar{i})\psi_{t^\lambda} = \delta_{\bar{i}^\lambda}^{i^\lambda} \psi_{t^\lambda}, \quad \delta_{\bar{i}^\lambda}^{i^\lambda} \text{ the Kronecker delta},
\]
\[
\psi_{t^\lambda} = \begin{cases} 
\psi_{t^\lambda_{r+1}} & \text{if } r, r+1 \text{ are in different components} \\
0 & \text{otherwise}
\end{cases}
\]
\[
y_y \psi_{t^\lambda} = 0
\]
\[
\psi_{t+1} \psi_{t} \psi_{t^\lambda} = 0
\]
\[
\psi_{t} \psi_{t+1} \psi_{t^\lambda} = 0
\]
for all $1 \leq r \leq d - 1$, $1 \leq s \leq d$ and $1 \leq t \leq d - 2$.

Proof. Let $e(\bar{i})$, $\bar{i} \in I^d$ be a KLR idempotent of $B_{d}^{\sigma}$. By the orthogonality relation we have that
\[
e(\bar{i})\psi_{t^\lambda} = e(\bar{i}) e(\bar{i}^\lambda) = \begin{cases} 
\psi_{t^\lambda} & \text{if } \bar{i} = \bar{i}^\lambda \\
0 & \text{otherwise}.
\end{cases}
\]
The element $\psi_{r} e(\bar{i}^\lambda)$ corresponds to a tableau with residue sequence $(i_1^\lambda, \cdots, i_{r+1}^\lambda, i_r^\lambda, \cdots, i_d^\lambda)$. We use the fact that for any standard tableau $t$ its coset is unique up to KLR relation (2.9). If the nodes of $t^\lambda$ occupied by the entries $r$, $r+1$ are in the same component, then there does not exist any standard tableau with such residue sequence and coset consisting of one generator, hence $\psi_{r} e(\bar{i}^\lambda) = 0$. If they are in different components then the only choice for a tableau with the above residue sequence and coset consisting of the generator $\psi_{r}$ is the tableau $t^\lambda_{r+1}$, hence $\psi_{r} e(\bar{i}^\lambda) = \psi_{t^\lambda_{r+1}}$. The element $y_y e(\bar{i}^\lambda)$ corresponds to a tableau with residue sequence $i^\lambda_{r+1} \in I^d$. The unique tableau with that residue sequence is $t^\lambda$. However
\[
\text{deg}(y_y e(\bar{i}^\lambda)) = 2 \neq 0 = \text{deg}(e(\bar{i}^\lambda))
\]
thus $y_y e(\bar{i}^\lambda)$. Regarding relation (2.24), if $t, t+1$ are in the same component then the result follows from (2.22). If $t, t+1$ are in different components then the element $\psi_{t+1} \psi_{t} e(\bar{i}^\lambda)$ corresponds to a tableau with residue sequence $(i_1^\lambda, \cdots, i_{r+2}^\lambda, i_r^\lambda, i_{r+1}^\lambda, \cdots, i_d^\lambda)$. But such standard tableau does not exist hence $\psi_{t+1} \psi_{t} e(\bar{i}^\lambda) = 0$. Similarly we prove relation (2.25).

Now we are using the theory we developed in this section in order to get a presentation for the cell modules of $B_{d}^{\sigma}$.

Proposition 2.15. Let $\lambda \in \text{Bip}(d)$. The generator $\psi_{t^\lambda}$ and relations of Theorem 2.14 form a presentation for the cell module $\Delta(\lambda)$.

Proof. By Theorem 2.14 the desired relations are satisfied. By Lemma 2.13 we know that every non-standard tableau factorises through a Garnir tableau. This is enough for proving that the relations, together with $\psi_{t^\lambda}$, form a presentation for $\Delta(\lambda)$.

The connection we described in Lemma 2.10 between the Bruhat order on words and the dominance order on tableaux will be useful throughout this paper. In particular it can be used for proving technical results such as the following proposition which deals with the action of the KLR generators on specific elements of the cell module.

In order to make the notation simpler we introduce the notion of the left and right exposed transposition. Let $t \in \text{Std}(d)$ with reduced expression $w_t = s_{i_1} \cdots s_{i_k}$. A simple transposition $s_r$ is called left exposed (resp. right exposed) if $s_r = s_j$ for some $j \in \{1, \cdots, k\}$ and $s_r$ commutes with $s_l$ for all $l < j$ (resp. $l > j$).

Proposition 2.16. Let $\lambda \in \text{Bip}(d)$ be a bipartition and $t \in \text{Std}(\lambda)$ be a standard $\lambda$-tableau.
Proof. Suppose that \( w = s_1 \cdots s_k \in \mathfrak{S}_d \) is the reduced expression of the tableau \( t \), i.e \( \psi_t = \psi_{i_1} \cdots \psi_{i_k} \epsilon(\lambda^t) \).

(1) We have

\[
y_r \psi_t = \sum_{s \in \text{Std}(\lambda)} \alpha_s \psi_s
\]

with \( \alpha_s \in F \).

(2) If \( s, t \notin \text{Std}(\lambda) \), then

\[
\psi_r \psi_t = \sum_{s \in \text{Std}(\lambda), \text{res}(s) = \text{res}(s, t)} \alpha_s \psi_s
\]

with \( \alpha_s \in F \).

Proof. Suppose that \( w = s_1 \cdots s_k \in \mathfrak{S}_d \) is the reduced expression of the tableau \( t \), i.e \( \psi_t = \psi_{i_1} \cdots \psi_{i_k} \epsilon(\lambda^t) \).

(1) We have

\[
y_r \psi_t = y_r \psi_{i_1} \cdots \psi_{i_k} \epsilon(\lambda^t)
\]

By taking into account relations (2.8), (2.10) and (2.11) we have that \( y_r \) commutes with the generators \( \psi_{i_1}, \cdots, \psi_{i_k} \) apart from the case that residues \( i_r, i_{r+1} \) coincide, i.e \( i_r = i_{r+1} \). In that case there exists an error term. The error term does not contain the generator \( \psi_r \) in its coset. Then

\[
s_{i_1} \cdots \hat{s}_{i_r} \cdots s_{i_k} \leq s_{i_1} \cdots s_{i_k}
\]

under the Bruhat order, where \( \hat{s}_{i_r} \) means that the simple transposition is missing from the reduced expression. By Lemma 2.10 we know that all the error terms will correspond to tableaux strictly more dominant than \( t \). Note that in the case we get elements of the form \( \psi_{s_k}^2 \) for some \( 1 \leq s \leq d - 1 \), by relation (2.12) we will either have the generators \( y_s, y_{s+1} \) or the term \( \psi_{s_k}^2 \) acts as identity (if \( i_s = i_{s+1} \), \( \psi_{s_k}^2 \) kills the element). We proceed inductively by applying relations (2.8), (2.10) and (2.11). Then the desired result follows.

(2) Let \( s, t \notin \text{Std}(\lambda) \). We can distinguish two cases depending on the form of the reduced expression \( w_t \):

(i) \( r = i_a \), for some \( 1 \leq a \leq k \),
(ii) \( r \neq i_a \), for all \( 1 \leq a \leq k \).

(i) Since \( t \in \text{Std}(\lambda) \), \( s_{i_k} = s_{i_k+1} = \cdots = s_{i_b} \), for some \( b \), is left exposed in the reduced expression \( w_t \).

We assume that \( s_{i_b} = s_{r+1} \) since everything works similarly when \( s_{i_b} = s_{r-1} \). We shall proceed by induction on the dominance order. The most dominant tableau such that the transposition \( s_r \) exists in its reduced expression and \( s_{r+1} \) is left exposed, is the tableau \( t = s_{r+1} t^\lambda \). Then by (2.24) we get that

\[
\psi_r \psi_t \psi_r \psi_{r+1} \epsilon(\lambda^t) = 0.
\]

With the base case done, suppose that the proposition holds for any tableau \( s \triangleright t \) and \( s, t \notin \text{Std}(\lambda) \). Note that since \( s, t \notin \text{Std}(\lambda) \) we have that the tableau \( t \) will be of the form (T1) or (T3). We write

\[
t = s_{r+1} s_r (s_{i_1} \cdots s_{i_{b-1}} s_{i_{b+1}} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_k} t^\lambda)
\]

and let us denote the bracketed term by

\[
u := s_{i_1} \cdots s_{i_{b-1}} s_{i_{b+1}} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_k} t^\lambda.
\]

Since the tableau \( u \) is standard, we have that \( t \) will be of the form (T1). By applying the braid Coxeter relation we have that

\[
s_r t = s_r s_{r+1} s_r u = s_{r+1} s_r s_{r+1} u
\]

with \( s_{r+1} u \) being non-standard and more dominant than \( t \), since

\[
s_{r+1} s_{i_1} \cdots s_{i_{b-1}} s_{i_{b+1}} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_k} \leq s_{i_1} \cdots s_{i_k}
\]

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under the Bruhat order, i.e $w_d \leq w_t$. The result follows by induction.

(ii) There exists $i_b$, $1 \leq b \leq k$, such that $s_{i_b} := s_{r+1}$ and consider the case that $s_{i_b} = s_{r+1}$. Note that the transposition $s_{r-1}$ cannot exist in the reduced expression, as in that case the tableau $s,t$ would be standard. Moreover the transposition $s_{r+1}$ is right exposed. Hence we can rearrange the product $\psi_r \psi_t$ as follows

$$\psi_r \psi_t = \psi_{i_1} \cdots \psi_{i_{b-1}} \psi_{i_b+1} \cdots \psi_{i_k} \psi_r \psi_{r+1} e^{(\lambda^\dagger)}$$

By (2.25) we have that $\psi_r \psi_t = 0$. Similar arguments apply in the case that $s_{i_b} = s_{r-1}$. □

3. Homomorphisms between cell modules

3.1. Construction of homomorphisms. In this section we shall construct homomorphisms between certain cell modules of the blob algebra. By using the fact that $\mathcal{B}_d$ is quasi-hereditary, we know that for a given bipartition $\nu \in \text{Bip}(d)$ we have $\text{Hom}_{\mathcal{B}_d}(\Delta(\nu'), \Delta(\nu)) \neq 0$ only if $\nu' \succeq \nu$. For the purposes of this paper we need to construct homomorphisms between cell modules indexed by linked bipartitions which also have lengths with absolute value differing by one.

Let $\mu \in \text{Bip}(d)$ be a bipartition with $\ell(\mu) = m$ or $\ell(\mu) = m - 1/2$ for some integer $m \in \mathbb{Z}$. Equivalently $\mu$ lies in the alcove $a_m$ or in the hyperplane $H_{m-1/2}$.

(1) Suppose that $\ell(\mu) = m$, $m \in \mathbb{Z}$. We fix two bipartitions $\lambda, \lambda' \sim \mu$ and $|\ell(\lambda)| = |\ell(\lambda')| = |\ell(\mu)| + 1$.

- If $m \leq 0$ is a non-positive integer then we let $\ell(\lambda) < 0$ and $\ell(\lambda') > 0$.
- If $m > 0$ is a positive integer then we let $\ell(\lambda) > 0$ and $\ell(\lambda') < 0$.

In this case we shall construct maps in the sets $\text{Hom}_{\mathcal{B}_d}(\Delta(\lambda'), \Delta(\mu))$ and $\text{Hom}_{\mathcal{B}_d}(\Delta(\lambda), \Delta(\mu))$.

(2) Suppose that $\ell(\mu) = m - 1/2$, $m \in \mathbb{Z}$. In this case we fix the unique bipartition $\lambda'$ with $\lambda' \sim \mu$ and $|\ell(\lambda')| = |\ell(\mu)| + 1$.

- If $m \leq 0$ is a non-positive integer then $\ell(\lambda') > 0$.
- If $m > 0$ is a positive integer then $\ell(\lambda') < 0$.

In this case we shall construct map in the set $\text{Hom}_{\mathcal{B}_d}(\Delta(\lambda'), \Delta(\mu))$.

Notation. From now on we will make the following abuse of notation. We shall not distinguish between the tableau $t$ and the attached path $T$ and both will be denoted by $T$. Moreover we shall denote by $T_{m-1/2}$ the $i$th intersection point of the path $T$ with the hyperplane $H_{m-1/2}$.

In what follows we shall restrict ourselves in the case that $m \leq 0$ and we shall construct the maps we discussed above. Note that the results are not affected by whether $m \leq 0$ or $m > 0$. This is just a convention in order to save space since everything is analogous for $m > 0$.

Definition 3.1. Let $\mu \in \text{Bip}(d)$ be a bipartition.

(1) If $\ell(\mu) = m$, $m \leq 0$, we define the maps $\varphi^\mu_{\lambda} : \Delta(\lambda) \longrightarrow \Delta(\mu)$ and $\varphi^\mu_{\lambda'} : \Delta(\lambda') \longrightarrow \Delta(\mu)$ as follows:

$$\varphi^\mu_{\lambda}(\psi_{T_\lambda}) := \psi_{s_{m-1/2} T_\lambda}$$

and

$$\varphi^\mu_{\lambda'}(\psi_{T_{\lambda'}}) := \psi_{s_{1/2} T_{\lambda'}}$$

where the paths $s_{m-1/2} T_\lambda$ and $s_{1/2} T_{\lambda'}$ are the reflections of the paths $T_\lambda$, $T_{\lambda'}$ through the hyperplanes $H_{m-1/2}$, $H_{1/2}$ respectively.

(2) If $\ell(\mu) = m - 1/2$, $m \leq 0$, then we define the map $\varphi^\mu_{\lambda'} : \Delta(\lambda') \longrightarrow \Delta(\mu)$ on the same way as in equation (3.2).

Remark 3.2. Note that each of the paths $T_\lambda$ and $T_{\lambda'}$ intersects the hyperplanes $H_{m-1/2}$ and $H_{1/2}$ at precisely one point and we have dropped the superscripts.
In order to prove that relation (2.23) holds we need to consider the case of homomorphisms. We cover the Proposition 3.4.

Since there does not exist element in $\text{Std}(\mu)$ with residue sequence $\res(S)$ and degree equal to $\deg(S) + 2$. Hence the Garnir relation (2.22) holds.

Consider the element $\psi_r\varphi^\mu_\lambda(\psi_{T\lambda})$, for some $1 \leq r \neq n < d$. Then

$$
\psi_r\varphi^\mu_\lambda(\psi_{T\lambda}) = \begin{cases} 
\psi_{S_{r+r+1}} & \text{if } r, r+1 \text{ are in different components} \\
0 & \text{otherwise}
\end{cases}
$$

In order to prove that relation (2.23) holds we need to consider the case $r = n$. By construction the simple transposition $s_n$ exists in $w_S$ and it is left exposed. Hence $\psi_{S} = \psi_n\psi_{i_1} \cdots \psi_{i_r} e(\ell^\mu)$. Since $\res(S^{-1}(n)) = \res(S^{-1}(n+1)) + 1$ (since we reflected through a hyperplane at this point) we have that

$$
\psi_n\psi_{S} = \psi_{n}^2\psi_{s_n} S = (y_{n+1} - y_n)\psi_{s_n} S.
$$

But both summands are zero since there does not exist standard $\mu$-tableau with residue sequence $\res(s_n S)$ and degree $\deg(s_n S) + 2$. Thus $\psi_n\psi_{S} = 0$ and so the Garnir relation (2.23) holds.

Consider $\psi_{r+1}\psi_r\varphi^\mu_\lambda(\psi_{T\lambda})$, $1 \leq r \leq d - 2$. If $r, r+1$ are in the same component or $r = n$ the Garnir relation 2.24 holds by using the fact that relation (2.23) is satisfied. If $r, r+1$ are in different components then

$$
\psi_{r+1}\psi_r\varphi^\mu_\lambda(\psi_{T\lambda}) = \begin{cases} 
\psi_{r+1}\psi_{r} S & \text{if } r, r+1 \text{ are in different components} \\
0 & \text{otherwise}
\end{cases}
$$
since there does not exist standard \( \mu \)-tableau with residue sequence \( \text{res}(s_{r+1}s,S) \). Hence the Garnir relation (2.24) is satisfied. Similarly we prove that the Garnir relation (2.25) is also satisfied. 

In the following definition we define a type of paths which shall be useful in the next subsection when we will prove some of the main results of the paper.

**Definition 3.5.** Let \( T \in \text{Std}(d) \) be a standard tableau. The path \( T \) is called **length increasing** if

\[
|\ell(\text{Shape}(T|_{1,\ldots,k}))| \leq |\ell(\text{Shape}(T|_{1,\ldots,k+1}))|
\]

for all \( 1 \leq k < d \).

For a given bipartition \( \mu \in \text{Bip}(d) \), an example of a length increasing path is the following:

An alternative criterion to be a length increasing path of shape \( \mu \) is that for a given bipartition \( \mu \in \text{Bip}(d) \) with \( \ell(\mu) = m < 0 \) (resp. \( \ell(\mu) = m > 0 \)) every length increasing path in \( \text{Path}(\mu) \) intersects with the hyperplanes

\[
H_{m+3/2}, H_{m+5/2}, \cdots \quad \text{(resp.} \quad H_{m-3/2}, H_{m-5/2}, \cdots \). \]

In the exceptional case that \( \ell(\mu) = 0 \) we have that a length increasing path is a path that does not leave the fundamental alcove at any point.

The following lemma will be useful in the sequel.

**Lemma 3.6.** Let \( \nu \in \text{Bip}(d) \) and \( T \in \text{Path}(\nu) \) be a length increasing path. Then the element \( \psi_T \) belongs to the simple module \( L(\nu) \).

**Proof.** Let \( \text{res}(T) \in I^d \) be the residue sequence of \( T \). Since \( T \) is length increasing, the set \( \text{Path}(\nu', T') \) is non-empty only if \( \nu' \trianglerighteq \nu \). Hence we have that \( e(\text{res}(T))\Delta(\nu') = 0 \), for any bipartition \( \nu' \prec \nu \). Thus \( e(\text{res}(T))L(\nu') = 0 \), for any bipartition \( \nu' \prec \nu \). This shows that the element \( \psi_T \) belongs to a composition factor of \( \Delta(\mu) \) not of the form \( L(\nu') \), \( \nu' \prec \nu \), so it belongs to the simple head \( L(\nu) \). \( \square \)

3.2. **Image of the homomorphisms.** In this subsection we shall construct the image of the homomorphisms \( \varphi^\mu_\lambda \) and \( \varphi^\mu_\lambda \) of Definition 3.1. Same as in last subsection we cover the case that \( m \leq 0 \), since all the arguments work equally in the case \( m > 0 \) up to relabelling hyperplanes.

In the alcove case we compute the image of both \( \varphi^\mu_\lambda \), \( \varphi^\mu_\lambda \), whereas in the hyperplane case it is only necessary to consider the homomorphism \( \varphi^\mu_\lambda \).

Suppose that \( T_1 \in \text{Path}(\lambda') \) is a length increasing path. The image of the element \( \psi_{T_1} \) under the homomorphism \( \varphi^\mu_\lambda \) is

\[
\varphi^\mu_\lambda(\psi_{T_1}) = \psi_{s_{1/2}T_1}
\]

since the path \( s_{1/2} \cdot T_1 \) is the unique path with residue sequence equal to \( \text{res}(T) \) terminating at the bipartition \( \mu \). For the same reason, if \( \ell(\mu) = m, m \leq 0 \) and \( T_2 \in \text{Path}(\lambda) \) is a length
increasing path then the image of the element $\psi_{T_2}$ under the homomorphism $\varphi^\mu_\lambda$ is

$$\varphi^\mu_\lambda(\psi_{T_2}) = \psi_{s_{m+1/2}T_2}.$$  

The following proposition is one of the main results of the section and describes a spanning set for the image of the homomorphism $\varphi^\mu_\lambda$. Note that the result holds for both $\ell(\mu) = m$ and $\ell(\mu) = m - 1/2$, $m \leq 0$.

**Proposition 3.7.** The homomorphism $\varphi^\mu_\lambda : \Delta(\lambda') \to \Delta(\mu)$ of Definition 3.1 is an injective homomorphism. Moreover

1. if $m \leq 0$
   $$\text{Im} \varphi^\mu_\lambda = \text{span}_F \{ \psi_U \mid U \in \text{Path}(\mu), U \text{ intersects } H_{1/2} \},$$
2. if $m > 0$
   $$\text{Im} \varphi^\mu_\lambda = \text{span}_F \{ \psi_U \mid U \in \text{Path}(\mu), U \text{ intersects } H_{-1/2} \}.$$

**Proof.** We cover the case $m \leq 0$ as the other one works similarly. Take any path $U \in \text{Path}(\mu)$ and suppose that it intersects the hyperplane $H_{1/2}$ at $n$-many points and let $U_{n/2}^m$ be the final point. Then we notice that the reflection $s_{n/2}^m \cdot U$ through the final point that $U$ intersects the hyperplane $H_{1/2}$ gives a path terminating at $\lambda'$. This shows that there is a bijection between the paths in $\text{Path}(\mu)$ intersecting $H_{1/2}$ and the paths in $\text{Path}(\lambda')$. We will prove that any path intersecting the hyperplane $H_{1/2}$ belongs indeed to the image of $\varphi^\mu_\lambda$, and thus the result will follow from the fact that the dimension of $\Delta(\lambda')$ is equal to the dimension of the image of $\varphi^\mu_\lambda$.

We consider the path $T^\mu$ and we fix integer $a \in \mathbb{Z}$, $1 \leq a \leq d - 1$. Let $T \in \text{Path}(\mu)$ be the minimal path, under the lexicographic order, with the property $T(a) \in H_{1/2}$ (see Figures 6, 7). We proceed by considering each value, $a$, one at a time. Since $T$ must intersect $H_{1/2}$ at some point by assumption, this allows us to consider all such paths.

Note that the reflection $s_{1/2} \cdot T$ of the path $T$ through the hyperplane $H_{1/2}$ is a length increasing path in $\text{Path}(\lambda')$. Hence the element $\psi_T$ belongs to the image of the homomorphism $\varphi^\mu_\lambda$. Let $U \in \text{Path}(\mu)$ be any path which intersects the hyperplane $H_{1/2}$ at the point $U(a) = T(a)$, with $w_U = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_d$ its reduced expression. Since $T$ is the minimal path, under the lexicographic order, with $T(a) \in H_{1/2}$ we have that $w_T$ is a subword of $w_U$.

Note that the subword $w_T$ will be right exposed, as otherwise the condition $U(a) \in H_{1/2}$ would not hold. We can rewrite the reduced expression $w_U$ as

$$w_U = s_{i_1} \cdots s_{i_l} w_T, \ 1 \leq l \leq k.$$  

hence $U = s_{i_1} \cdots s_{i_l} T$, $1 \leq l \leq k$.

Then the basis element $\psi_U$ corresponding to the path $U$ can be written as

$$\psi_U = \psi_{i_1} \cdots \psi_{i_l} \psi_T$$

and it belongs to the image of $\varphi^\mu_\lambda$, since $\psi_T$ does. By repeating the same procedure for all admissible integers $a \in \mathbb{Z}$, $1 \leq a \leq d$, we prove that all paths in $\text{Path}(\mu)$ which intersect the hyperplane $H_{1/2}$ correspond to elements in the image of the homomorphism $\varphi^\mu_\lambda$. \hfill $\Box$

**Example 3.8.** Let $d = 20$, $e = 4$, $\kappa = (0, 2)$ and $\mu = ((1^7), (1^{13}))$. If $T = s_4 s_3 s_5 T^\mu$ then we observe that the path

$$S = s_5 s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} T^\mu$$

can be written as

$$S = s_5 s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_4 s_3 s_5 T^\mu$$

$$= s_5 s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_4 s_3 s_5 T^\mu$$

by using the Coxeter relations of the symmetric group (see Figure 8).
Figure 6. For $d = 20$, $\mu = ((1^{13}), (1^7))$, $\ell(\mu) = -3/2$, $e = 4$, $\kappa = (0, 2)$ and $a = 4$ the red path is the path $T$. The shaded area corresponds to all the paths that can be obtained from $T$ and the elements corresponding to them belong to the image of $\varphi^\mu_\lambda$. Any path within the shaded region has $s_4 s_3 s_5$ as a subword.

Figure 7. For $d = 20$, $\mu = ((1^{14}), (1^5))$, $\ell(\mu) = -2$, $e = 4$, $\kappa = (0, 2)$ and $a = 4$ the blue path is the path $T$. The shaded area corresponds to all the paths that can be obtained from $T$ and the elements corresponding to them belong to the image of $\varphi^\mu_\lambda$. Any path within the shaded region has $s_4 s_3 s_5$ as a subword.

Hence the basis element $\psi_S$ can be written as

$$\psi_S = \psi_{15}\psi_{14}\psi_{16}\psi_{18}\psi_{13}\psi_{15}\psi_{17}\psi_3\psi_5\psi_{12}\psi_{14}\psi_{16}\psi_6\psi_{11}\psi_{13}\psi_{15}\psi_1\psi_7\psi_{10}\psi_{12}\psi_{14}\psi_T \in \Delta(\mu)$$

and since the element $\psi_T$ belongs to the image of $\varphi^\mu_\lambda$, we have that $\psi_S$ also belongs to the image of $\varphi^\mu_\lambda$.

Recall that the homomorphism $\varphi^\mu_\lambda : \Delta(\lambda) \rightarrow \Delta(\mu)$ only exists when $\ell(\mu) = m$, that is the bipartition $\mu$ lies in the alcove $a_m$. The construction of the spanning set for the image of the homomorphism $\varphi^\mu_\lambda$ is the next important result of our paper towards our aim to construct bases.
for the irreducible representations of $B_d^\kappa$. For completeness we give the spanning sets for both $m \leq 0$ and $m > 0$.

**Proposition 3.9.** The homomorphism $\varphi_\mu^\kappa \colon \Delta(\lambda) \to \Delta(\mu)$ of Definition 3.1 is an injective homomorphism. Moreover

1. if $m \leq 0$

   $$\text{Im} \varphi_\mu^\kappa = \text{span}_F \left\{ \psi_U \mid U \in \text{Path}(\mu), U \text{ last intersects } H_{m-1/2} \text{ or intersects } H_{1/2} \text{ after intersecting } H_{-1/2} \right\}.$$

2. if $m > 0$

   $$\text{Im} \varphi_\mu^\kappa = \text{span}_F \left\{ \psi_U \mid U \in \text{Path}(\mu), U \text{ last intersects } H_{m+1/2} \text{ or intersects } H_{-1/2} \text{ after intersecting } H_{1/2} \right\}.$$

Before presenting the proof, we shall give an example which illustrates which paths we are referring to in the statement of Proposition 3.9.

**Example 3.10.** Let $d = 20, e = 4, \kappa = (0,2)$ and consider the bipartition $\mu = ((1^6), (1^{14}))$ with $\ell(\mu) = m = -2$. Then $\lambda = ((1^4), (1^{16}))$ is the bipartition linked with $\mu$ with $\ell(\lambda) = -3$ (see Figure 9). The hyperplanes that we shall be interested in are $H_{-1/2}, H_{1/2}$ which are the hyperplanes of the fundamental alcove and the hyperplane $H_{m-1/2} = H_{-5/2}$ which is the left hyperplane of the alcove $a_-1$.

The black path is a path which intersects the hyperplane $H_{1/2}$ after intersecting the hyperplane $H_{-1/2}$. The blue path is a path last intersecting the hyperplane $H_{-5/2}$. Both paths belong to the image of the homomorphism $\varphi_\mu^\kappa$. On the other hand the red path does intersect the hyperplane $H_{-5/2}$, but it last intersects $H_{3/2}$ and it does not belong to the image of $\varphi_\mu^\kappa$.
the integer with the property $T(a) \in H_{m-1/2}$ and $c_{a,1}(T) = c_{j,1}(T)$, $c_{a,2}(T) = c_{n,2}(T) + |\ell(\mu)|e$. Finally, let $T(b) \in H_{m+1/2}$ be the second intersection point of $T$ with $H_{m+1/2}$. Note that the integers $j, n, a, b$ determine the path $T$. The diagram corresponding to the basis element $\psi_T$ is presented in Figure 10.

Notation. Let $\nu \in \text{Bip}(d)$ and $T \in \text{Path}(\nu)$ be a path. We denote by $T_{n-1/2}^{\text{last}}$ the last intersection point of the path $T$ with the hyperplane $H_{n-1/2}$, for some $n \in \mathbb{Z}$. Also we denote by $s_{n-1/2}^{\text{last}}$ the reflection through that point with respect to the hyperplane $H_{n-1/2}$.

Proof of Proposition 3.9. Same as in the proof of Proposition 3.7 we cover the case $m \leq 0$. Let $a \in \mathbb{Z}$, $1 \leq a < d$ be a fixed integer such that if $\alpha = ((1^{\alpha_1}), (1^{\alpha_2}))$ is a bipartition of $a$, then $\alpha_1 - \alpha_2 \in H_{m-1/2}$. Also let $T^\alpha \in \text{Std}(\alpha)$ be the $\alpha$-tableau highest in the lexicographic order. Consider the skew bipartition $\lambda \setminus \alpha$ and let $S, S' \in \text{Path}(\lambda \setminus \alpha)$ length increasing paths which are highest and lowest in the lexicographic order respectively (see Figure 11). Note that all length increasing $\lambda \setminus \alpha$-paths placed between $S$ and $S'$ can be obtained by multiplying with a product of transpositions on the tableau $S$ and we focus on $S, S'$ for the ease of notation. We define the standard $\lambda$-tableaux $T := T^\alpha \circ S$ and $T' := T^\alpha \circ S'$ and let $\tilde{T} := s_{m-1/2} \cdot T$ and $\tilde{T'} := s_{m-1/2} \cdot T'$ be the reflection of those paths through the unique point they intersect

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{The blue and the black path label elements which belong in the image of $\varphi^n_{\lambda}$ whereas the red path labels an element not in the image of $\varphi^n_{\lambda}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{The general form of the diagram corresponding to the element $\psi_T$.}
\end{figure}
the hyperplane \( H_{m-1/2} \). Note that since the paths \( \mathbf{T}, \mathbf{T}' \) are length increasing paths, the basis elements \( \psi_{\mathbf{T}}, \psi_{\mathbf{T}'} \) corresponding to the paths \( \mathbf{T}, \mathbf{T}' \) belong to the image of the homomorphism \( \varphi_{\lambda}^\mu \).

We shall prove that if the generators \( \psi_{s}, a < r < d \) act on \( \psi_{\mathbf{T}} \) then \( \psi_{s} \psi_{\mathbf{T}} \) is a non-zero element and it corresponds to a path which either last intersects \( H_{m-1/2} \) or intersects \( H_{1/2} \) after intersecting \( H_{-1/2} \). Since \( \psi_{r} \psi_{\mathbf{T}} \) belongs to the image of \( \varphi_{\lambda}^\mu \), the new element will also belong to the image of \( \varphi_{\lambda}^\mu \). For any \( a < r < d \) such that \( s_{r} \mathbf{T} \) does not intersect \( H_{m-1/2}, H_{m+1/2} \), it is straightforward that \( \psi_{r} \psi_{\mathbf{T}} = \psi_{s_{r} \mathbf{T}} \) because \( s_{r} \mathbf{T} \) is the unique tableau with the desired residue sequence. Let \( b \in \mathbb{Z}, a < b < d \), such that \( (s_{b}\mathbf{T})(b) \in H_{m+1/2} \). Since \( s_{b}\mathbf{T} \leq \mathbf{T} \) we also have that

\[
\psi_{b}\psi_{\mathbf{T}} = \psi_{s_{b}\mathbf{T}}
\]

and the element \( \psi_{b}\psi_{\mathbf{T}} \) is a non zero element which belongs to the image of the homomorphism \( \varphi_{\lambda}^\mu \). We also need to prove that \( \psi_{r}\psi_{\mathbf{T}}, 1 < r < d \) is a non zero element which belongs to the radical. Consider the element \( \psi_{\mathbf{T}} \), and let \( b \in \mathbb{Z} \) be such that \( (s_{b}\mathbf{T})(b) \in H_{m-3/2} \). This is the only interesting case as for the rest cases the result is straightforward. The transposition \( s_{b} \) will appear in the reduced expression of \( \mathbf{T} \) and it will be left exposed. Hence

\[
\psi_{b}\psi_{\mathbf{T}} = \psi_{b}^{2}\psi_{1}\cdots\psi_{b}\psi_{1}e(\lambda^{\mu})
\]

with \( \psi_{s_{b}\mathbf{T}} = \psi_{1}\cdots\psi_{b}\psi_{1}e(\lambda^{\mu}) \), where by \( \psi_{b} \) we mean that the generator \( \psi_{b} \) does not appear in the coset. Since \( \text{res}((s_{b}\mathbf{T})^{-1}(b)) = \text{res}((s_{b}\mathbf{T})^{-1}(b+1)) + 1 \), by applying the KLR relation (2.12) we have that

\[
\psi_{b}\psi_{\mathbf{T}} = (y_{b+1} - y_{b})\psi_{s_{b}\mathbf{T}}.
\]

**Step 1:** We shall prove that \( y_{b+1}\psi_{s_{b}\mathbf{T}} = 0 \). Let \( (s_{b}\mathbf{T}')(n) \in H_{-1/2} \), for some \( n \in \mathbb{Z} \), be the unique intersection point of the path \( s_{b}\mathbf{T}' \) with the hyperplane \( H_{-1/2} \).

![Figure 11](image)

**Figure 11.** Let \( d = 20, e = 4 \) and \( \kappa = (0,2) \). For \( a = 14 \) and \( b = 18 \) the paths \( S \) (red), \( S' \) (blue) and \( \mathbf{T} \) (black) are depicted above. In this case \( n = 6 \).

In order to compute the product \( y_{b+1}\psi_{s_{b}\mathbf{T}} \), it is easier to consider the diagrammatic presentation of our algebra. In particular the diagram of the element \( s_{b}\psi_{\mathbf{T}} \) is of the form of Figure 10.

Note that the diagram consists of strands moving towards up to the right (UR-strands) and strands moving towards up to the left (UL-strands). If the \( i^{th} \) UR-strand (resp. UL-strand) carries the residue \( i \in \mathbb{Z}/e\mathbb{Z} \) then the \((i + xe)^{th}\), \( x \in \mathbb{Z}_{>0} \), UR-strand (resp. UL-strand) also carries the residue \( i \in \mathbb{Z}/e\mathbb{Z} \). We colour strands carrying the same residue with the same colour.
We apply the generator \( y_{b+1} \) on the element \( \psi_{s_b \hat{T}} \), and we obtain the element corresponding to the following diagram.

By applying the KLR-relation (2.10) in the case that the residues coincide we get the following combination of diagrams:

We take the first summand of (3.3) and by reapplying the KLR relation (2.10) we obtain two more summands.

Those new summands are both equal to zero. The first one by the Garnir relation (2.22) and the second one because it corresponds to a non standard tableau. We now consider the second summand of (3.3) and we apply the KLR relation (2.13). We obtain the element

which is equal to zero because of the Garnir relations. Finally we get that \( y_{b+1} \psi_{s_b \hat{T}} = 0 \).

**Step 2:** Now we shall consider the product \( y_b \psi_{s_b \hat{T}} \) and we shall distinguish between two cases according to the length of \( \mu \). If \( |\ell(\mu)| > 1 \), then the unique element of \( \text{Path}(\mu, s_b \hat{T}) \) with degree equal to \( \deg(s_b \hat{T}) + 2 \) is the path \( V_1 := s_{m+1/2} s_{m+3/2} s_{1/2} s_{-1/2} \cdot (s_b \hat{T}) \), hence

\[
\psi_b \psi_{s_b \hat{T}} = \alpha_{V_1} \psi_{V_1}
\]

with \( \alpha_{V_1} \in F \). If \( |\ell(\mu)| = 1 \), then the unique path in \( \text{Path}(\mu, s_b \hat{T}) \) with the desired property is the path \( V_2 := s_{2/3}^{-1} s_{1/2} \cdot (s_b \hat{T}) \), hence

\[
\psi_b \psi_{s_b \hat{T}} = \alpha_{V_2} \psi_{V_2}
\]

with \( \alpha_{V_2} \in F \). In order to prove that the homomorphism \( \varphi^\mu_{s_b} \) is injective it suffices to prove that the scalars \( \alpha_{V_1} \), \( \alpha_{V_2} \in F \) are non-zero. We prove it for the scalar \( \alpha_{V_1} \in F \) since the proof for \( \alpha_{V_2} \in F \) will be a subcase. The element \( y_b \psi_{s_b \hat{T}} \) corresponds to the diagram

By the KLR relation (2.11) we have that the above element is equal to the following combination of diagrams:
By using similar arguments as above, the first summand is zero. We now consider the second summand and we apply the KLR-relation (2.13), hence we obtain the diagram

\[\text{diagram image}\]

We apply the KLR-relation (2.12) for the case that the residues are not equal and do not differ by one. The we obtain the diagram

\[\text{diagram image}\]

in which we can apply the KLR relation (2.12) for the case that the residues differ by one. Then we get the following sum of diagrams.

\[\text{diagram image}\]

where strands with different colours carry different residues which differ by one. We apply the KLR relations (2.10), (2.11) and (2.13) appropriately until we obtain reduced diagrams. Then the only non-zero summand is of the form

\[\text{diagram image}\]

Hence we have proven that the scalar $\alpha_{V_1} \in F$ is equal to $\pm 1$ and we shall not be interested in keeping track of its value. As a result the homomorphism $\varphi^\mu_\lambda$ is injective homomorphism.

In any case the element $\psi_{\lambda_0^\mu}^\lambda$, corresponds to the path $V_1 := s_{m+1/2}s_{m+3/2}s_{1/2}s_{-1/2} \cdot (s_b^\hat{T})$ which intersects the hyperplane $H_{1/2}$ after intersecting the hyperplane $H_{-1/2}$. By repeating the same procedure for all admissible integers $a \in \mathbb{Z}$ we prove that the paths which either last intersect $H_{m-1/2}$ or intersect $H_{1/2}$ after intersecting $H_{-1/2}$, correspond to elements in the image of $\varphi^\mu_\lambda$.

**A bijection**: In order to complete the proof we need to prove that any element in the image of $\varphi^\mu_\lambda$ either last intersects $H_{m-1/2}$ or intersects $H_{1/2}$ after intersecting $H_{-1/2}$. For that purpose it suffices to show that the map

$$\Phi : \text{Path}(\lambda) \rightarrow \text{Path}(\mu)$$

defined by

$$\Phi(U) := \begin{cases} s_{m-1/2} \cdot U, & \text{if } U \text{ last intersects } H_{m-1/2} \\ s_{m-1/2}s_{1/2}s_{-1/2} \cdot U, & \text{otherwise} \end{cases}$$

is a bijection.
is an injective map of degree one, with image containing the paths in Path(µ) which either last intersect \( H_{m-1/2} \) or intersect \( H_{1/2} \) after intersecting \( H_{-1/2} \). Let \( U \in \text{Path}(µ) \) be a path which last intersects the hyperplane \( H_{m-1/2} \) at the point \( u_{\text{last}}^{m-1/2} \). Then we have that

\[ \Phi(s_{m-1/2}^{\text{last}} \cdot U) = U \]

hence \( U \) belongs to the image of the map \( \Phi \). Consider an arbitrary path \( V \in \text{Path}(µ) \) which intersects both hyperplanes \( H_{-1/2} \) and \( H_{1/2} \). Suppose that if \( v_{\text{last}}^{j_{1/2}} = V(n_{2}) \) is the last intersection point with the hyperplane \( H_{1/2} \), then there exists an intersection point \( v_{-1/2}^{j} = V(n_{1}) \) with \( n_{1} < n_{2} \) and assume that \( n_{1} \) is the greatest integer with that property. Moreover let \( v_{\text{last}}^{j_{m+1/2}} = V(n_{3}) \) be the last intersection point of \( V \) with \( H_{m+1/2} \). Then

\[ \Phi(s_{-1/2}^{\text{last}}s_{1/2}^{\text{last}}s_{m+1/2}^{\text{last}} \cdot V) = V \]

hence \( V \) belongs to the image of \( \Phi \). Since both those types of paths belong to the image of \( \varphi_{\lambda}^{\mu} \), we have proven that any element in the image corresponds to a path of that form. The fact that \( \Phi \) is of degree 1 is straightforward by its construction. \( \square \)

4. Bases of simple modules

In this section we assume that \( F \) is a field of characteristic 0 and we shall construct the bases of simple modules for the algebra \( B^{\mathbb{Z}_{2}}_{\mathcal{T}} \). Recall from Section 3 that for a given bipartition \( \mu \in \text{Bip}(d) \) with \( \ell(\mu) \leq 0 \) we fix two bipartitions \( \lambda, \lambda' \) and consider the homomorphisms \( \varphi_{\lambda}^{\mu}, \varphi_{\lambda'}^{\mu} \) of Definition 3.1. Note that everything works on the same way if \( \ell(\mu) > 0 \), so we restrict ourselves to the previous case. Let us denote by \( \text{Im} \varphi_{\lambda}^{\mu} \) and \( \text{Im} \varphi_{\lambda'}^{\mu} \) the images of the above homomorphisms, constructed in Propositions 3.7 and 3.9 respectively. We denote by \( E(\mu) \) the quotient module

\[ E(\mu) := \Delta(\mu)/(\text{Im} \varphi_{\lambda}^{\mu} + \text{Im} \varphi_{\lambda'}^{\mu}) \]

of \( \Delta(\mu) \) modulo the sum of the images of the homomorphisms. From the results of the previous section we have that when \( \mu \) belongs to an alcove, \( E(\mu) \) is spanned by elements corresponding to paths which do not intersect the hyperplane \( H_{1/2} \) and they do not last intersect the hyperplane \( H_{m-1/2} \). In the hyperplane case we have that the module \( E(\mu) \) is spanned by elements \( \psi_{\mathcal{T}} \) where \( \mathcal{T} \) is a path which does not intersect the hyperplane \( H_{1/2} \).

We consider a path \( \mathcal{T} \in \text{Path}(\mu) \) with \( \psi_{\mathcal{T}} \in E(\mu) \) and let us denote by \( t_{n-1/2}^{1}, t_{n-1/2}^{2}, \ldots \) the intersection points of \( \mathcal{T} \) with the hyperplane \( H_{n-1/2} \) for some \( n \leq 0 \). We shall construct a new path \( \mathcal{T} \in \text{Path}(\mu) \) as follows: between two consecutive intersection points \( t_{n-1/2}^{i}, t_{n-1/2}^{i+1} \) we have that \( \mathcal{T}(a) = (s_{n-1/2}^{i} \cdot \mathcal{T})(a) \) if and only if all the points between \( t_{n-1/2}^{i} \) and \( t_{n-1/2}^{i+1} \) are either in the alcove \( a_{n} \) or in the alcove \( a_{n-1} \). Otherwise we have that \( \mathcal{T}(a) = (t_{n-1/2}^{i} \cdot \mathcal{T})(a) \). If we apply the aforementioned rule to any hyperplane we obtain a path \( \mathcal{T} \in \text{Path}(\mu) \) with \( \psi_{\mathcal{T}} \in E(\mu) \) and \( \deg(\mathcal{T}) = -\deg(\mathcal{T}) \).

Note that the above construction does not depend on whether the bipartition \( \mu \) lies in an alcove or on a hyperplane.

Example 4.1. Suppose that \( d = 24, e = 4 \) and \( \kappa = (0, 2) \). We consider the bipartition \( \mu = ((1^{8}), (1^{16})) \in \text{Bip}(24) \) and let \( \mathcal{T} \in \text{Path}(\mu) \) be the black path in Figure 12 which corresponds to the basis element \( \psi_{\mathcal{T}} \in E(((1^{8}), (1^{16}))) \). The path \( \mathcal{T} \) has degree \( \deg(\mathcal{T}) = -2 \).

The path \( \mathcal{T} \in \text{Path}(\mu) \) obtained by the procedure we described before, is the red path in Figure 12. One can readily check that \( \deg(\mathcal{T}) = 2 = -\deg(\mathcal{T}) \).

Remark 4.2. Suppose that \( \mu \in \text{Bip}(d) \) with \( \ell(\mu) = m > 0 \). Then the elements spanning the module \( E(\mu) \) are of the form \( \psi_{\mathcal{T}} \) where \( \mathcal{T} \) is a path which does not intersect \( H_{-1/2} \) and does not last intersect the hyperplane \( H_{m+1/2} \).
Using the notions we defined above we can state and prove the following theorems. Those theorems are two of the main results of our paper and gives a precise description of the basis of an irreducible representation of the blob algebra over a field of characteristic 0, in the alcove and hyperplane cases.

**Theorem 4.3.** Let $\mu \in \text{Bip}(d)$ with $\ell(\mu) = m$. The module $E(\mu)$ is equal to the simple head $L(\mu)$, hence

1. if $m \leq 0$
   
   $$L(\mu) = \text{span}_F \left\{ \psi_T \mid T \in \text{Path}(\mu), \ T \text{ does not intersect } H_{1/2} \right\},$$

2. if $m > 0$
   
   $$L(\mu) = \text{span}_F \left\{ \psi_T \mid T \in \text{Path}(\mu), \ T \text{ does not intersect } H_{-1/2} \right\}.$$

*Proof.* As we discussed above, for any path $T \in \text{Path}(\mu)$ with $\psi_T \in E(\mu)$ there exists a path $\bar{T}$ with $\deg(\bar{T}) = -\deg(T)$. Using this we have that the polynomial $\dim_t(E(\mu))$ is bar-invariant, i.e. fixed under interchanging $t$ and $t^{-1}$. Moreover by using Proposition 2.6 and Theorem 2.8 we get that $E(\mu) = L(\mu)$. □

The following theorem is the analogous of Theorem 4.3 in the hyperplane case.

**Theorem 4.4.** Let $\mu \in \text{Bip}(d)$ with $\ell(\mu) = m - 1/2$. The module $E(\mu)$ is equal to the simple head $L(\mu)$, hence

1. if $m \leq 0$
   
   $$L(\mu) = \text{span}_F \{ \psi_T \mid T \in \text{Path}(\mu), \ T \text{ does not intersect } H_{1/2} \},$$

2. if $m > 0$
   
   $$L(\mu) = \text{span}_F \{ \psi_T \mid T \in \text{Path}(\mu), \ T \text{ does not intersect } H_{-1/2} \}.$$

*Proof.* The proof is identical to the proof of Theorem 4.3. □
5. BGG resolutions of simple representations

5.1. Composition of one-column homomorphisms. In this section we shall compute the composition of certain one-column homomorphisms. We consider two bipartitions \( \alpha, \gamma \in \text{Bip}(d) \) such that \( |\ell(\alpha)| = |\ell(\gamma)| + 2 \) and without loss of generality we may assume that \( \ell(\gamma) < 0 \). Then we can either have \( \ell(\alpha) < 0 \) or \( \ell(\alpha) > 0 \) and let \( \beta, \beta' \in \text{Bip}(d) \) be the bipartitions with \( |\ell(\beta)| = |\ell(\beta')| = |\ell(\gamma)| + 1 \) for which we have constructed the homomorphisms \( \varphi^\gamma_\beta, \varphi^\gamma_{\beta'} \) of Section 3. In a case as above we can consider the following “diamond” diagram:

\[
\begin{array}{ccc}
\Delta(\gamma) & \rightarrow & \Delta(\beta) \\
\varphi^\gamma_\beta & & \varphi^\gamma_{\beta'} \\
\downarrow & & \downarrow \\
\Delta(\alpha) & \rightarrow & \Delta(\beta') \\
\varphi^\alpha_\beta & & \varphi^\alpha_{\beta'}
\end{array}
\]

The aim of this section is to compute the compositions of the homomorphisms in such diamonds and prove that those are commutative or anti-commutative.

In the next proposition we shall assume that \( \ell(\alpha) < 0 \), as everything works similarly when \( \ell(\alpha) > 0 \).

**Proposition 5.1.** Let \( \alpha, \gamma \in \text{Bip}(d) \) with \( |\ell(\alpha)| = |\ell(\gamma)| + 2 \). Then

\[
(\varphi^\gamma_\beta \circ \varphi^\alpha_\beta)(\psi_T) = (-1)^{|\ell(\gamma)|} \psi_{s_{\ell(\gamma)+1/2}(s_{-1/2}s_{1/2})T}
\]

and

\[
(\varphi^\gamma_{\beta'} \circ \varphi^\alpha_{\beta'})(\psi_T) = \psi_{s_{1/2}s_{-1/2}T}.
\]

In particular the diamond will either be commutative or anti-commutative, depending on the number \( |\ell(\gamma)| \).

**Proof.** Let \( \alpha = ((1^{a_1}), (1^{a_2})) \), \( \beta = ((1^{b_1}), (1^{b_2})) \) and \( \gamma = ((1^{c_1}), (1^{c_2})) \). The composition \( \Delta(\alpha) \rightarrow \Delta(\beta) \rightarrow \Delta(\gamma) \) is harder to compute than the composition \( \Delta(\alpha) \rightarrow \Delta(\beta') \rightarrow \Delta(\gamma) \) and we shall start by computing it. Consider the generator \( \psi_T \) of the cell module \( \Delta(\alpha) \). Then

\[
(\varphi^\gamma_\beta \circ \varphi^\alpha_\beta)(\psi_T) = \varphi^\gamma_\beta(\varphi^\alpha_\beta(\psi_T)) = \varphi^\gamma_\beta(\psi_{s_{\ell(\beta)-1/2}T}).
\]

Let \( T := s_{\ell(\beta)-1/2} \cdot T \in \text{Path}(\beta) \) and \( T = w_T T^\beta \). Then we have that

\[
\varphi^\gamma_\beta(\psi_{s_{\ell(\beta)-1/2}T}) = \varphi^\gamma_\beta(\psi_{w_T} T^\beta) = \psi_{w_T} \varphi^\gamma_\beta(\psi_T) = \psi_{w_T} \psi_{s_{\ell(\beta)-1/2}T^\beta}.
\]

The element \( \psi_T \in \Delta(\beta) \) corresponds to a diagram of the form

```
  2\alpha_1 + 2
 / | | \
/  | | \
/   |   \\
2\beta_1
```
while the element $\psi_{s_{\ell(\beta)} - 1/2} \cdot T^\beta \in \Delta(\gamma)$ corresponds to a diagram of the form

```
...  
```

For the multiplication $\psi_{w_T} \cdot \psi_{s_{\ell(\beta)} - 1/2} \cdot T^\beta$ we concatenate the above diagrams. Hence we obtain the diagram

```
...  
```

and by applying the KLR-relation (2.13) in the case the middle residue is less by one that the adjacent residues we get the diagram

```
...  
```

Then we apply the KLR relation (2.12) in the case that the residues are not equal and they do not differ by one and we get the diagram

```
...  
```

Since the strands we have marked in red carry the same residue, we apply the KLR-relation (2.12) and by using similar computations as in the proof of Proposition 3.9 we have that

\[
(\phi^\gamma_{\beta'} \circ \phi^\beta_{\alpha})(\psi_{T^\alpha}) = \psi_{w_T} \cdot \psi_{s_{\ell(\beta)} - 1/2} \cdot T^\beta = (-1)^{l(\gamma)} \cdot \psi_{s(\gamma) + 1/2} \cdot T^\alpha.
\]

Now we shall compute the composition $\Delta(\alpha) \rightarrow \Delta(\beta') \rightarrow \Delta(\gamma)$ on the generator $\psi_{T^\alpha}$ of the cell module $\Delta(\alpha)$. We have that

\[
(\phi^\gamma_{\beta'} \circ \phi^\beta_{\alpha})(\psi_{T^\alpha}) = \phi^\gamma_{\beta'}(\phi^\beta_{\alpha}(\psi_{T^\alpha})) = \phi^\gamma_{\beta'}(\psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha).
\]

But the element $\psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha \in \Delta(\beta')$ is simply of the form

\[
\psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha = \psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha
\]

hence

\[
\phi^\gamma_{\beta'}(\psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha) = \psi_{s_{\ell(\beta)} - 1/2} \cdot T^\alpha.
\]

□
5.2. BGG resolutions for the alcove and hyperplane case. Let $F$ be a field of characteristic zero. In this section we attach to any bipartition $\lambda \in \text{Bip}(d)$ a complex $C_\bullet(\lambda)$ called the BGG resolution for the irreducible representation $L(\lambda)$.

In the case that the simple representation is indexed by a bipartition $\lambda \in \text{Bip}(d)$ with $\lambda \in H_{n-1/2}$, $n \in \mathbb{Z}$, the BGG resolution has an easy form. In the next proposition we construct a BGG resolution for the irreducible representation $L(\lambda)$, $\lambda \in H_{n-1/2}$, for some $n \in \mathbb{Z}$.

**Proposition 5.2.** Let $\lambda \in \text{Bip}(d)$ with $\lambda \in H_{n-1/2}$, for some $n \in \mathbb{Z}$. We have the short exact sequence

$$
C_\bullet(\lambda): \quad 0 \to \Delta(\mu)(\ell(\mu) + |\ell(\lambda)|) \to \Delta(\lambda) \to L(\lambda) \to 0
$$

where $\mu \in \text{Bip}(d)$ with $|\ell(\mu)| = |\ell(\lambda)| + 1$ and $\ell(\mu) = -|\ell(\lambda)| + 1$, with

$$
H_i(C_\bullet(\lambda)) = \begin{cases} L(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}
$$

**Proof.** The result is straightforward by using the fact that $\text{Coker}(\varphi^\lambda_\mu) = L(\lambda)$. \hfill \Box

Now we shall construct BGG resolutions for the simple modules indexed by bipartitions which belong to an alcove. Let $\lambda \in \text{Bip}(d)$ be a bipartition such that $\lambda \in \mathfrak{a}_n$, $n \in \mathbb{Z}$ and let us denote by $\nu, \nu'$ the bipartitions in the same linkage class as $\lambda$ such that $|\ell(\nu)| = |\ell(\nu')| = |\ell(\lambda)| + i$. We set

$$
C_\bullet(\lambda) := (C_i(\lambda))_{i \geq 0}
$$

where

$$
C_0(\lambda) := \Delta(\lambda)
$$

and

$$
C_i(\lambda) := \bigoplus_{\nu = \nu, \nu'} \Delta(\mu)(|\ell(\nu)| - |\ell(\lambda)|)
$$

for $i > 0$. We define the maps

$$
\delta_i: C_{i+1}(\lambda) \to C_i(\lambda)
$$

between those components. For $i = 0$ we have that

$$
\delta_0 := \begin{pmatrix} \varphi^\lambda_{\nu_1} & \varphi^\lambda_{\nu'_1} \\ -\varphi^\lambda_{\nu_1+1} & \varphi^\lambda_{\nu'_1+1} \\ \varphi^\lambda_{\nu'_1+1} & -\varphi^\lambda_{\nu_1+1} \end{pmatrix} \tag{5.1}
$$

For $i > 0$ we shall distinguish between two cases on the number $|\ell(\lambda)| + i$. In particular if $|\ell(\lambda)| + i$ is even, we set

$$
\delta_i := \begin{pmatrix} \varphi^\lambda_{\nu_1+1} & \varphi^\lambda_{\nu'_1+1} \\ -\varphi^\lambda_{\nu_1+1} & \varphi^\lambda_{\nu'_1+1} \\ \varphi^\lambda_{\nu'_1+1} & -\varphi^\lambda_{\nu_1+1} \end{pmatrix} \tag{5.2}
$$

whereas if it is odd, we set

$$
\delta_i := \begin{pmatrix} \varphi^\lambda_{\nu_1+1} & -\varphi^\lambda_{\nu'_1+1} \\ \varphi^\lambda_{\nu'_1+1} & \varphi^\lambda_{\nu_1+1} \end{pmatrix} \tag{5.3}
$$

Note that there is the possibility that not both the rightmost and leftmost alcove contain bipartitions linked with $\lambda$. In that case let $\nu_1, \nu'_1, \ldots, \nu_k \in \text{Bip}(d)$ be the bipartitions linked with $\lambda$. Then we define the maps $\delta_i: C_{i+1}(\lambda) \to C_i(\lambda)$ are defined exactly like the maps (5.1), (5.2) and (5.3) for $0 \leq i \leq k - 1$. For $i = k$ we define

$$
\delta_k := \begin{pmatrix} \varphi^\lambda_{\nu_k-1} \\ \nu_k \end{pmatrix} \tag{5.4}
$$
Proposition 5.3. Let \( \lambda \in \text{Bip}(d) \) be a bipartition such that \( \lambda \in a_n, n \in \mathbb{Z} \). For the pair \((C^\bullet(\lambda), (\delta_i)_{i \geq 0})\) we have that
\[
\text{Im}(\delta_{i+1}) \subset \text{Ker}(\delta_i).
\]
for any \( i \geq 0 \), in other words the pair \((C^\bullet(\lambda), (\delta_i)_{i \in \mathbb{Z}})\) is a (chain) complex.

**Proof.** The result is straightforward from Proposition 5.1. \( \square \)

Recall that \( I = \mathbb{Z}/e\mathbb{Z} \) and let \( r \in I \) be a given residue. The \( r \)-restriction functor \( r - \text{res}^d_{d-1} : \text{mod} - B^\kappa_d \rightarrow \text{mod} - B^\kappa_{d-1} \)
is defined by
\[
M \mapsto \sum_{i=(i_1, i_2, \ldots, i_{d-1}, r) \in I^{d-1} \cup \{r\}} e(i) M
\]
and we have that
\[
\text{res}^d_{d-1} = \sum_{r \in I} r - \text{res}^d_{d-1}.
\]

**Remark 5.4.** Suppose that \( \lambda \in \text{Bip}(d) \) and \( \lambda \in a_n, n \in \mathbb{Z} \). If \( r \in I \) then we have that \( \lambda \) has either 0 or 1 removable \( r \)-nodes. We shall denote by \( E_r(\lambda) \) the unique bipartition which differs from \( \lambda \) by removing an \( r \)-node. Consider the cell module \( \Delta_d(\lambda) \in \text{mod} - B^\kappa_d \). We have that
\[
\text{res}^d_{d-1}(\Delta_d(\lambda)) = \begin{cases} 
\Delta_{d-1}(E_r(\lambda)), & \text{if } \text{Rem}_r(\lambda) \neq \emptyset \\
0, & \text{otherwise}
\end{cases}
\]
where \( \Delta_{d-1}(E_r(\lambda)) \) is a cell module in \( \text{mod} - B^\kappa_{d-1} \).

**Definition 5.5.** Let \( \lambda \in \text{Bip}(d) \). The complex
\[
0 \longrightarrow C^\bullet(\lambda) \longrightarrow L(\lambda) \longrightarrow 0
\]
is called BGG resolution of \( L(\lambda) \) if
\[
H_i(C^\bullet(\lambda)) = \begin{cases} 
L(\lambda), & \text{if } i = 0 \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 5.6.** Let \( \lambda \in \text{Bip}(d) \) be a bipartition such that \( \lambda \in a_n, n \in \mathbb{Z} \) and
\[
C^\bullet(\lambda) := \bigoplus_{\nu \in \Lambda} \Delta(\nu) |[\ell(\nu)] - |\ell(\lambda)|\).
\]
The \( B^\kappa_d \)-complex
\[
0 \longrightarrow C^\bullet(\lambda) \longrightarrow L(\lambda) \longrightarrow 0
\]
with differentials \( \delta_i : C^i+1(\lambda) \longrightarrow C^i(\lambda) \) the maps defined above is a BGG resolution for the simple representation \( L(\lambda) \). Moreover we have that
\[
\text{res}^d_{d-1}(L_d(\lambda)) = \bigoplus_{\square \in \text{Rem}(\lambda)} L_{d-1}(\lambda - \square).
\]

**Proof.** Let \( \lambda \in \text{Bip}(d) \) with \( \lambda \in a_n, \) for some \( n \leq 0 \). Note that everything works analogously when \( n > 0 \). In order to prove that our \( B^\kappa_d \)-complex is a BGG resolution for the simple representation \( L(\lambda) \) we need to show that
\[
H_i(C^\bullet(\lambda)) = \begin{cases} 
L_d(\lambda), & \text{if } i = 0 \\
0, & \text{otherwise}
\end{cases}
\]
Recall that BGG resolutions and bases for the hyperplane case are already done. We assume, by induction, that the theorem holds for any bipartition \( \lambda \in \text{Bip}(d-1) \) where \( \lambda \) belongs to an alcove. We also have that
\[
\text{res}^d_{d-1}(C^\bullet(\lambda)) = \bigoplus_{r \in I} E_r(C^\bullet(\lambda)).
\]
We shall consider one residue at a time. The bipartition $\lambda$ belongs to an alcove, hence as we mentioned in Remark 5.4 there will be either 0 or 1 removable $r$-nodes. For each residue we have 3 different cases.

- Suppose that the bipartition $E_r(\lambda)$ belongs to the hyperplane $H_{n-1/2}$. In terms of the alcove geometry, one can think of it as the hyperplane of the alcove $a_n$ which is further away from the origin than $\lambda$. Note that since the $r$-restriction functor is left exact, we have that $r - \operatorname{res}^d_{d-1}(C_n(\lambda))$ is a complex. Consider a bipartition $\mu \in \text{Bip}(d)$ such that $\mu$ is less dominant than $\lambda$ and $E_r(\mu)$ is a bipartition. In the case we examine, all the bipartitions of $d$ with the aforementioned property come into pairs $(\nu^+, \nu^-)$ with $\nu^+ \prec \nu^-$ and $|\ell(\nu^+)| = |\ell(\nu^-)| + 1$. They also have the additional property

$$E_r(\nu^+) = E_r(\nu^-) = \nu$$

where $\nu$ is linked with $E_r(\lambda)$. Then we have that

$$r - \operatorname{res}^d_{d-1}(\Delta_d(\nu^+)) = r - \operatorname{res}^d_{d-1}(\Delta_d(\nu^-)) = \Delta_{d-1}(\nu).$$

Now consider the homomorphism $\varphi^\nu_{\nu^+} \in \operatorname{Hom}_{B_n^d}(\Delta_d(\nu^+), \Delta_d(\nu^-))$ for some bipartition $\nu \prec E_r(\lambda)$. Under the $r$-restriction functor we have that

$$r - \operatorname{res}^d_{d-1}(\varphi^\nu_{\nu^+}) = 1_\nu \in \operatorname{End}_{B_n^d}(\Delta_{d-1}(\nu)).$$

In other words the identity morphism appears into all the differentials of the $r$-restricted complex $r - \operatorname{res}^d_{d-1}(C_\bullet(\lambda))$, hence the complex is exact. In particular the homology

$$H_i(r - \operatorname{res}^d_{d-1}(C_\bullet(\lambda))) = 0$$

for any $i \geq 0$.

- Suppose that the bipartition $E_r(\lambda)$ belongs to the hyperplane $H_{n+1/2}$, that is the hyperplane closer to the origin. Recall that we denote by $\nu'_i \in \text{Bip}(d)$ the bipartition such that $|\ell(\nu'_i)| = |\ell(\lambda)| + 1$ with $\nu'_i$ belonging to the positive alcoves. The above pair of bipartitions are of the form of Proposition 5.2. Apart from those bipartitions, all the rest bipartitions $\mu \in \text{Bip}(d)$ which are strictly less dominant than $\lambda$ and $E_r(\mu) \in \text{Bip}(d)$, pair up in the exact same way as in the previous case when restricted under the $r$-restriction functor. Hence

$$H_i(r - \operatorname{res}^d_{d-1}(C_\bullet(\lambda))) = 0$$

for $i > 0$. From Proposition 5.2 we have that

$$H_0(r - \operatorname{res}^d_{d-1}(C_\bullet(\lambda))) = L(\lambda).$$

- Suppose that the bipartition $E_r(\lambda)$ remains to the alcove $a_n$. Then the complex $r - \operatorname{res}^d_{d-1}(C_\bullet(\lambda))$ is given by

$$r - \operatorname{res}^d_{d-1}\left(\bigoplus_{\nu \in \lambda} \Delta_d(\nu)(\ell(\nu))\right)$$

with differentials given by

$$r - \operatorname{res}^d_{d-1}(\delta_i): r - \operatorname{res}^d_{d-1}(C_{i+1}(\lambda)) \longrightarrow r - \operatorname{res}^d_{d-1}(C_i(\lambda)).$$

Note that if $\text{Rem}_r(\nu) \neq \emptyset$, we have that

$$r - \operatorname{res}^d_{d-1}(\Delta_d(\nu)(\ell(\nu))) = \Delta_{d-1}(E_r(\nu))(\ell(\nu))$$

since $\ell(\nu) = \ell(E_r(\nu))$, otherwise we have that

$$r - \operatorname{res}^d_{d-1}(\Delta_d(\nu)) = 0.$$

Let $\nu, \nu' \in \text{Bip}(d)$ be bipartitions such that $\text{Rem}_r(\nu), \text{Rem}_r(\nu') \neq \emptyset$. Then

$$r - \operatorname{res}^d_{d-1}(\varphi_{\nu'}) = \varphi_{E_r(\nu')}$$
Hence we get
\[ r - \text{res}_{d-1}^d(C_\bullet(\lambda)) = C_\bullet(E_r(\lambda)) \]
and by the induction hypothesis we have that \( H_0(C_\bullet(E_r(\lambda))) = L_{d-1}(E_r(\lambda)) \), while \( H_i(C_\bullet(E_r(\lambda))) = 0 \), for all \( i > 0 \). Thus \( r - \text{res}_{d-1}^d(H_0(C_\bullet(\lambda))) = H_0(C_\bullet(E_r(\lambda))) = L_{d-1}(E_r(\lambda)) \) and \( r - \text{res}_{d-1}^d(H_i(C_\bullet(\lambda))) = 0 \), for all \( i > 0 \).

Using the work we have done above we have proven that
\[ \text{res}_{d-1}^d(H_i(C_\bullet(\lambda))) = \begin{cases} \bigoplus_{r \in I} L_{d-1}(E_r(\lambda)), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases} \]

Moreover we have that the cokernel of the differential \( \delta_1 \) projects onto the simple representation \( L(\lambda) \). The above argument gives us
\[ \text{res}_{d-1}^d(L_d(\lambda)) \subseteq \bigoplus_{r \in I} L_{d-1}(E_r(\lambda)). \]

In addition, by Theorem 4.3, we have that the cardinality of the basis of the simple representation \( L_d(\lambda) \) is equal to the sum of the cardinalities of the bases for the simple representations \( L_{d-1}(E_r(\lambda)) \), for all \( r \in I \). Thus
\[ \text{res}_{d-1}^d(L_d(\lambda)) = \bigoplus_{r \in I} L_{d-1}(E_r(\lambda)). \]

and we conclude that
\[ \text{res}_{d-1}^d(H_i(C_\bullet(\lambda))) = \begin{cases} \text{res}_{d-1}^d(L_d(\lambda)), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases} \]

Since \( \text{res}_{d-1}^d(L_{d-1}(\nu)) \neq 0 \), for any \( \nu \leq \lambda \), despite the fact that \( r - \text{res}_{d-1}^d(L_{d-1}(\nu)) = 0 \), for some \( r \in I \), we have that
\[ H_i(C_\bullet(\lambda)) = \begin{cases} L_d(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases} \]

and the proof is complete.

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