Preconditioned Iterative Solves in Model Reduction of Second Order Linear Dynamical Systems

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Abstract

Recently a new algorithm for model reduction of second order linear dynamical systems with proportional damping, the Adaptive Iterative Rational Global Arnoldi (AIRGA) algorithm \cite{1}, has been proposed. The main computational cost of the AIRGA algorithm is in solving a sequence of linear systems. These linear systems do change only slightly from one iteration step to the next. Here we focus on efficiently solving these systems by iterative methods and the choice of an appropriate preconditioner. We propose the use of GMRES and the Sparse Approximate Inverse (SPAI) preconditioner. A technique to cheaply update the SPAI preconditioner in each iteration step of the model order reduction process is given. Moreover, it is shown that under certain conditions the AIRGA algorithm is stable with respect to the error introduced by iterative methods. Our theory is illustrated by an experiment. It is demonstrated that SPAI preconditioned iterative solves work well for model reduction of a one dimensional beam model with AIRGA algorithm. Moreover, the preconditioner with updates takes only one third of the time than the preconditioner without updates.

Keywords: Model Order Reduction, Global Arnoldi Algorithm, Moment Matching, Iterative Methods, Preconditioner and Stability Analysis

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1. Introduction

A continuous time-invariant second order linear dynamical system is of the form

\begin{align}
    M \ddot{x}(t) &= -D \dot{x}(t) - K x(t) + F u(t), \\
    y(t) &= C_p x(t) + C_v \dot{x}(t),
\end{align}

where $M, D, K \in \mathbb{R}^{n \times n}$ are mass, damping and stiffness matrices, respectively, $F \in \mathbb{R}^{n \times m}$, $C_p$, $C_v \in \mathbb{R}^{q \times n}$ are constant matrices. In (1), $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^q$.
is the output. If \( m \) and \( q \) both are one, then we have a Single-Input Single-Output (SISO) system. Otherwise \((m \text{ and } q > 1)\) the system is called Multi-Input Multi-Output (MIMO). We assume the case of proportional damping, i.e., \( D = \alpha M + \beta K \), where the coefficients \( \alpha \) and \( \beta \) are chosen based on experimental results \(^1\) \(^2\). For our derivations, the system matrices \( M, D \) and \( K \) need not be symmetric.

It is assumed that the order \( n \) of the system \(^1\) is extremely high. The simulation of large dynamical systems can be unmanageable due to high demands on computational resources, which is the main motivation for model reduction. The goal of model reduction is to produce a low dimensional system that has, as best as possible, the same characteristics as the original system but whose simulation requires significantly less computational effort. The reduced system of \(^1\) is described by

\[
\begin{align*}
\ddot{x}(t) &= \frac{1}{\alpha} \dot{\hat{x}}(t) - \hat{K} \hat{x}(t) + \hat{F} u(t), \\
\dot{\hat{y}}(t) &= \hat{C}_p \hat{x}(t) + \hat{C}_r \hat{\dot{x}}(t),
\end{align*}
\]

(2)

where \( \hat{M}, \hat{K}, \hat{D} \in \mathbb{R}^{r \times r}, \hat{F} \in \mathbb{R}^{r \times m}, \hat{C}_p, \hat{C}_r \in \mathbb{R}^{q \times r} \) and \( r \ll n \). The damping property of the original system needs to be reflected in the reduced system. That is, \( \hat{D} = \alpha \hat{M} + \beta \hat{K} \) is required, where \( \alpha \) and \( \beta \) remain unchanged from the original system.

Model reduction can be done in many ways, see, e.g., \(^3\). We will focus on a projection based method, specifically Galerkin projection \(^1\). For this a matrix \( V \in \mathbb{R}^{n \times r} \) with orthonormal columns is chosen and the system \(^1\) is projected

\[
\begin{align*}
&V^T (M \hat{x}(t) + D \dot{\hat{x}}(t) + K \hat{x}(t) - Fu(t)) = 0, \\
&\dot{\hat{y}}(t) = C_p V \hat{x}(t) + C_r V \dot{\hat{x}}(t).
\end{align*}
\]

(3)

Comparing \(^3\) with \(^2\) yields

\[
\begin{align*}
\hat{M} &= V^T M V, \\
\hat{D} &= V^T D V, \\
\hat{K} &= V^T K V, \\
\hat{F} &= V^T F, \\
\hat{C}_p &= C_p V \quad \text{and} \\
\hat{C}_r &= C_r V.
\end{align*}
\]

(4)

The matrix \( V \) can be obtained in many ways, see, e.g., \(^3\). The focus in this paper will be on the Adaptive Iterative Rational Global Arnoldi (AIRGA) algorithm \(^1\) in which \( V \) is generated by an Arnoldi based approach.

The main contributions of this paper are as follows: Section 2 summarizes the AIRGA model reduction process which uses a direct solver for solving the linear systems arising in each iteration step. In Section 3 we discuss the use of iterative solvers and preconditioners for these linear systems. Preconditioned iterative solvers are a good choice here since they scale well. They have time complexity \( \mathcal{O}(n \cdot nnz) \), where \( n \) is the size of the system and \( nnz \) is the number of nonzeros in system matrices as compared to \( \mathcal{O}(n^3) \) for direct solvers \(^4\). We show that Sparse Approximate Inverse (SPAI) preconditioners are well suited for solving the linear systems arising in the model reduction process. These linear systems change at each model reduction iteration, but this change is small. Exploiting this fact we propose a cheap preconditioner update. Using an iterative solver introduces additional errors in the computation since the linear systems are not solved exactly. Hence, we discuss the stability of AIRGA in Section 4. In Section 5 an numerical experiment is given to support our preconditioned iterative solver theory. The cheap updates to the SPAI preconditioner reduce the computational time by a third. Finally, we give some conclusions and point out future work in Section 6.

For the rest of this paper, \( || \cdot ||_F \) denotes the Frobenius norm, \( || \cdot || \) the 2-norm, \( || \cdot ||_{H_2} \) the \( H_2 \)-norm, \( || \cdot ||_{H_r} \) the \( H_r \)-norm, and \( || \cdot ||_{H_\infty} \) the \( H_\infty \)-norm \(^5\). Also, \( qr \) denotes the QR factorization \(^5\).
2. Arnoldi Based Projection Method

In this section, we first describe how to obtain $V$ such that the first few moments of the transfer functions of the original and the reduced order transfer function are matched. We then state the AIRGA algorithm \([1]\) based on this approach.

The transfer function of \((1)\) is given by
\[
H(s) = (C_p + sC_v)(s^2M + sD + K)^{-1}F = (C_p + sC_v)X(s),
\]
where $X(s) = (s^2M + sD + K)^{-1}F$ is the state variable in frequency domain. The power series expansion of state variable $X(s)$ around expansion point $s_0 \in \mathbb{C}$ is given as \([6]\)
\[
X(s) = \sum_{j=0}^{\infty} X^{(j)}(s_0)(s - s_0)^j,
\]
where,
\[
\begin{align*}
X^{(0)}(s_0) &= (s_0^2M + s_0D + K)^{-1}F, \\
X^{(1)}(s_0) &= (s_0^2M + s_0D + K)^{-1}[-(2s_0M + D)X^{(0)}(s_0)], \\
X^{(2)}(s_0) &= (s_0^2M + s_0D + K)^{-1}[-(2s_0M + D)X^{(1)}(s_0) - MX^{(0)}(s_0)], \\
&\vdots \\
X^{(j)}(s_0) &= (s_0^2M + s_0D + K)^{-1}[-(2s_0M + D)X^{(j-1)}(s_0) - MX^{(j-2)}(s_0)].
\end{align*}
\]
Here, $X^{(j)}(s_0)$ is called the $j^{th}$-order system moment of $X(s)$ at $s_0$.

Similarly, the transfer function of the reduced system \((2)\) is given by
\[
\hat{H}(s) = (\hat{C}_p + s\hat{C}_v)\hat{X}(s),
\]
where $\hat{X}(s) = (s^2\hat{M} + s\hat{D} + \hat{K})^{-1}\hat{F}$. The power series expansion of the reduced state space $\hat{X}(s)$ around expansion point $s_0 \in \mathbb{C}$ is
\[
\hat{X}(s) = \sum_{j=0}^{\infty} \hat{X}^{(j)}(s_0)(s - s_0)^j.
\]
Here, $\hat{X}^{(j)}(s_0)$ is defined analogously to the $X^{(j)}(s_0)$. It is called the $j^{th}$-order system moment of $\hat{X}(s)$ at $s_0$.

The goal of moment-matching approaches is to find a reduced order model such that the first few moments of \((5)\) and \((7)\) are matched, that is, $X^{(j)}(s_0) = \hat{X}^{(j)}(s_0)$ for $j = 0, 1, 2, \ldots, t$ for some $t$.

Define
\[
\begin{align*}
\mathcal{P}_1 &= -(s_0^2M + s_0D + K)^{-1}(2s_0M + D), \\
\mathcal{P}_2 &= -(s_0^2M + s_0D + K)^{-1}M, \\
\mathcal{Q} &= (s_0^2M + s_0D + K)^{-1}F,
\end{align*}
\]
then from (6) we have

\[ X^{(0)}(s_0) = \mathcal{D}, \]
\[ X^{(1)}(s_0) = \mathcal{P}_1 X^{(0)}(s_0), \quad \text{and} \]
\[ X^{(j)}(s_0) = \mathcal{P}_1 X^{(j-1)}(s_0) + \mathcal{P}_2 X^{(j-2)}(s_0) \]

for \( j \geq 2 \).

The second order Krylov subspace \( \mathcal{K} \) is defined as

\[ \mathcal{G}_j(\mathcal{P}_1, \mathcal{P}_2, \mathcal{D}) = \text{span}\{ \mathcal{D}, \mathcal{P}_1 \mathcal{D}, (\mathcal{P}_1^2 + \mathcal{P}_2) \mathcal{D}, \ldots, \mathcal{G}_j(\mathcal{P}_1, \mathcal{P}_2) \mathcal{D} \}, \]

where \( \mathcal{G}_j(\mathcal{P}_1, \mathcal{P}_2) = \mathcal{P}_1 \cdot \mathcal{G}_{j-1}(\mathcal{P}_1, \mathcal{P}_2) + \mathcal{P}_2 \cdot \mathcal{G}_{j-2}(\mathcal{P}_1, \mathcal{P}_2) \) for \( j \geq 2 \).

Let \( \tilde{K} = (\tilde{s}_0 M + s_0 D + \tilde{K}) \). For the special case of proportionally damped second-order systems, it has been observed in [2]

\[ \mathcal{G}_j(\mathcal{P}_1, \mathcal{P}_2, \mathcal{D}) = \mathcal{G}_j(-\tilde{K}^{-1}(2s_0 M + D), -\tilde{K}^{-1} M, -\tilde{K}^{-1} F), \]
\[ = \mathcal{G}_j(-\tilde{K}^{-1}(2s_0 + \alpha) M + \beta K, -\tilde{K}^{-1} M, -\tilde{K}^{-1} F), \]
\[ = \mathcal{K}_j(\mathcal{P}_1, \mathcal{D}), \]

where \( \mathcal{K}_j(\mathcal{P}_1, \mathcal{D}) \) is the standard block Krylov subspace

\[ \mathcal{K}_j(\mathcal{P}_1, \mathcal{D}) = \text{span}\{ \mathcal{D}, \mathcal{P}_1 \mathcal{D}, \mathcal{P}_1^2 \mathcal{D}, \ldots, \mathcal{P}_1^{j-1} \mathcal{D} \}. \]

Thus, we need a good basis of \( \mathcal{K}_j(\mathcal{P}_1, \mathcal{D}) \). This can be obtained efficiently by, e.g., the block or the global Arnoldi algorithm [4, 8, 9].

The AIRGA algorithm, as proposed in [1], is one of the latest methods based on the global Arnoldi method. It is given in Algorithm 1. In this method, moment matching is done at multiple expansion points \( s_i \), \( i = \{1, \ldots, l\} \), rather than just at \( s_0 \) as earlier. This ensures a better reduced model in the entire frequency domain of interest.

The initial selection and further the computation of expansion points has been discussed in, e.g., [1] and [10]. The initial expansion points could be either real or imaginary, both of which have their merits. Here we adopt the choice of the expansion points as described in Section 5.0.1 of [1]. After the first AIRGA iteration, the expansion points are chosen from the eigenvalues of the quadratic eigenvalue problem \( \lambda^2 \tilde{M} + \lambda \tilde{D} + \tilde{K} \) (at line 32).

The method is adaptive, i.e., it automatically chooses the number of moments to be matched at each expansion point \( s_i \). This is controlled by the while loop at line 8. The variable \( j \) stores the number of moments matched. The upper bound on \( j \) is \( \lfloor r_{\text{max}} / m \rfloor \), where \( r_{\text{max}} \) is the maximum dimension to which we want to reduce the state variable (input from the user), and \( m \) is the dimension of the input; see [1] for a detailed discussion. At exit of this while loop, \( J = j \).

At line 2, no convergence implies that the \( H_2 \) norm of the difference between two consecutive reduced systems, computed at line 33, is greater than a certain tolerance. Similarly, at line 8, no convergence implies that the \( H_2 \) norm of the difference between two consecutive intermediate reduced systems, computed at line 25, is greater than a certain tolerance.

This algorithm requires solving a linear system at line 4 and 13. As the \( s_i \) change in each iteration step, the linear systems to be solved change in each iteration step. As discussed in Section 1 since solving such systems by direct methods is quite expensive, we propose to use iterative methods. As the change in the \( s_i \) will be small (at least after the first iteration step),
we can develop a cheap update of the necessary preconditioner. Please note, that even if \(M, D\) and \(K\) are symmetric, the linear systems to be solved might not have a symmetric (or Hermitian) system matrix as the expansion points can be complex. In such a case, \(s^2 M + s D + K\) is complex-symmetric.
3. Preconditioned Iterative Method

There are two types of methods for solving linear systems of equations: a) direct methods and b) iterative methods. For large systems, direct methods are not preferred because they are too expensive in terms of storage and operation. On the other hand, iterative methods require less storage and operations than direct methods. For a large linear system $Ax = b$, with $A \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^n$, an iterative method finds a sequence of solution vectors $x_0, x_1, \ldots, x_k$ which (hopefully) converges to the desired solution. Krylov subspace based methods are an important and popular class of iterative methods. If $x_0$ is the initial solution and $r_0 = b - Ax_0$ is the initial residual, then Krylov subspace methods find the approximate solution by projecting onto the Krylov subspace

$$K_k(A, r_0) = \text{span}\{r_0, Ar_0, A^2r_0, \ldots, A^{k-1}r_0\}.$$  

There are many types of Krylov subspace algorithms [4]. Some popular ones include Conjugate Gradient (CG), Generalized Minimal Residual (GMRES), Minimum Residual (MINRES), and BiConjugate Gradient (BiCG). Block versions of these algorithms do exist. The coefficient matrices of the linear systems to be solved by AIRGA are non-Hermitian. For such linear systems GMRES is the method of choice, and hence we use this here.

In Krylov subspace methods, the conditioning of the system is very important. “Conditioning pertains to the perturbation behavior of a mathematical problem [5]”. For example, in a well-conditioned problem, a small perturbation of the input leads to a small change in the output. This is not guaranteed for an ill-conditioned problem, where a small perturbation in the input may change the output drastically [5]. Preconditioning is a technique to well-condition an ill-conditioned problem. We discuss that next.

Preconditioning is a technique for improving the performance of iterative methods. It transforms a difficult system (ill-conditioned system) to another system with more favorable properties for iterative methods. For example, a preconditioned matrix may have eigenvalues clustered around one. This means that the preconditioned matrix is close to the identity matrix, and hence, the iterative method will converge faster. For a Hermitian positive definite (HPD) system, the convergence rate of iterative methods depends on the distribution of the eigenvalues of the coefficient matrix. However, for a non-Hermitian system, the convergence rate may depend on pseudo-spectra as well [11, 12].

If $M$ is a nonsingular matrix which approximates $A$; that is, $M \approx A^{-1}$, then the system

$$MAx = Mb$$

may be faster to solve than the original one. The above system represents preconditioning from left. Similarly, right and split preconditioning is given by two equations below, respectively.

$$AM\tilde{x} = b, \ x = M\tilde{x} \quad \text{and} \quad M_1AM_2\tilde{x} = M_1b, \ x = M_2\tilde{x}.$$  

The type of preconditioning technique to be used depends on the problem properties as well as on the choice of the iterative solver. For example, for HPD systems, $MA$, $AM$, $M_1AM_2$ all have same eigenvalue spectrum, and hence, left, right and split preconditioners behave the same way, respectively. For a general system, this need not be true [13].

Besides making the system easier to solve by an iterative method, a preconditioner should be cheap to construct and apply. Some existing preconditioning techniques include Successive Over Relaxation, Polynomial, Incomplete Factorizations, Sparse Approximate Inverse, and Algebraic Multi-Grid [13, 14, 15, 16].
Successive Over Relaxation and Polynomial preconditioners are very basic, and work when the coefficient matrix of the linear system to be solved has nice properties, e.g. it may be diagonally dominance or Hermitian. Our coefficient matrices do not have such properties. Incomplete Factorization based preconditioners are mature, and are commonly used. However, they also work well when the coefficient matrix has some of the above discussed properties, which we do not have here. Algebraic Multi-Grid preconditioners are extremely useful for solving linear systems arising from full discretization of a Partial Differential Equation (PDE). The linear systems here do not satisfy this.

Fortunately, SPAI preconditioners are useful here. These preconditioners are known to work well in the most general setting. In Section 3.1 we summarize the SPAI preconditioner from [14] and we discuss the use of SPAI in the AIRGA algorithm. Since the change in the coefficient matrix of the linear system to be solved is small from one step of AIRGA to the next, we update the preconditioner from one step to the next. This aspect is covered in Section 3.2.

3.1. Sparse Approximate Inverse (SPAI)

In constructing a preconditioner $P_i$ for a coefficient matrix $K_i$ (i.e., $K_i = s_i^2 M + s_i D + K$), we would like $P_i K_i \approx I$ (for left preconditioning) and $K_i P_i \approx I$ (for right preconditioning). SPAI preconditioners find $P_i$ by minimizing the associated error norm $||I - P_i K_i||$ or $||I - K_i P_i||$ for a given sparsity pattern. If the norm used is Frobenius norm, then the minimization function will be

$$\min_{P \in S} ||I - K_i P||_F,$$

where $S$ is a set of certain sparse matrices. The above approach produces a right approximate inverse. Similarly, a left approximate inverse can be computed by solving the minimization problem $||I - P_i K||_F$. For non-Hermitian matrices, the distinction between left and right approximate inverses is important. There are some situations where it can be difficult to find a right approximate inverse but finding a left approximate inverse can be easy. Whether left or right preconditioning should be used is problem dependent [4]. Since the SPAI preconditioner was originally proposed for right preconditioning [14], we focus on the same here. Similar derivation can be done for the left preconditioning as well.

The minimization problem can be rewritten as

$$\min_{P \in S} ||I - K_i P||_F = \min_{P \in S} \sum_j ||e^{(j)} - K_i P_i^{(j)}||^2_2,$$

where $P_i^{(j)}$ and $e^{(j)}$ are $j^{th}$ columns of the $P_i$ matrix and $I$ (identity matrix), respectively. The minimization problem (8) is essentially just one least squares problem, to be solved for $n$ different right-hand sides. Here it is solved iteratively.

The algorithm for computing a SPAI preconditioner for right preconditioning is given in Algorithm 2. The inputs to this algorithm are $K_i$ (coefficient matrix), $n_0$ (outer loop iteration count) and $n_l$ (inner loop iteration count); $n_0$ and $n_l$ are picked based upon experience. The initial guess for approximate inverse $P_i$ is usually taken as $\alpha I$ where $\alpha = \text{trace}(K_i)/\text{trace}(K_i K_i^T)$ (see line 2). This initial scaling factor $\alpha$ is minimizes the spectral radius of $(I - \alpha K_i)$ [15, 14, 17].

The AIRGA algorithm with the SPAI preconditioner is given in Algorithm 3. Here, we only show those parts of AIRGA algorithm that require changes.
Algorithm 2: Sparse Approximate Inverse (SPAI) [14]

1: Input: \{\mathcal{K}, n_0, n_t\}
2: \(P_i = \alpha I\) where \(\alpha = \frac{\text{trace}(\mathcal{K}_i)}{\text{trace}(\mathcal{K}_i \mathcal{K}_i^T)}\) and \(n\) is the dimension of \(\mathcal{K}_i\)
3: for outer = 1, ..., \(n_0\) do
4: for \(j = 1, \ldots, n_t\) do
5: Define \(p_{i}^{(j)} = P_i e_j\)
6: for inner = 1, ..., \(n_t\) do
7: \(r = e^{(j)} - \mathcal{K}_i p_{i}^{(j)}\)
8: \(z = P_i r\)
9: \(q = \mathcal{K}_i z\)
10: \(\alpha = (\langle s, q \rangle, \langle q, q \rangle)\)
11: \(p_{i}^{(j)} = p_{i}^{(j)} + \alpha z\)
12: end for
13: Update \(j^{th}\) column of \(P_i\) with \(p_{i}^{(j)}\)
14: end for
15: end for

Algorithm 3: AIRGA Algorithm with SPAI Preconditioner

1: \(j = 1\)
2: while no convergence do
3: for \(i = 1, \ldots, l\) do
4: Let \(\mathcal{K}_i = (s_i^2 M + s_i D + K)\)
5: Compute preconditioner \(P_i\) by solving \(\|I - \mathcal{K}_i P_i\|_F^2\)
6: Solve \(\mathcal{K}_i P_i X^{(0)}(s_i) = F\)
7: end for
8: while no convergence and \(j \leq \lceil r_{\text{max}} / m \rceil\) do
9: for \(i = 1, \ldots, l\) do
10: Only right hand sides are changing, so above preconditioner \(P_i\) can be applied as it is, i.e., Solve \(\mathcal{K}_i P_i X^{(j)}(s_i) = MV_j\)
11: end for
12: end while
13: \(j = j + 1\)
14: end while

3.2. SPAI Update Preconditioner

Usually, the difference between \(\mathcal{K}_i = s_i^2 M + s_i D + K\) and \(\mathcal{K}_{i+1} = s_{i+1}^2 M + s_{i+1} D + K\) is small. One can exploit this while building preconditioners for this sequence of matrices. This has been considered in the quantum Monte Carlo setting [18] and for model reduction of first order linear dynamical systems [19] and [20].

Let \(P_i\) be a good initial preconditioner for \(\mathcal{K}_i\). As will be seen, a cheap preconditioner update can be obtained by asking for \(K_1 P_i \approx K_i P_i\), where \(i = 2, \ldots, l\) and, as earlier, \(l\) denotes
the number of expansion points. Expressing $\mathcal{K}_i$ in terms of $\mathcal{K}_1$, we get

$$\mathcal{K}_i = \mathcal{K}_1 (I + (s_i^2 - s_i^2)\mathcal{K}_1^{-1}M + (s_i - s_i)\mathcal{K}_1^{-1}D).$$

Now we enforce $\mathcal{K}_i P_i = \mathcal{K}_i P_i$ or

$$\mathcal{K}_i P_i = \mathcal{K}_1 (I + (s_i^2 - s_i^2)\mathcal{K}_1^{-1}M + (s_i - s_i)\mathcal{K}_1^{-1}D)^{-1} P_i \approx (I + (s_i^2 - s_i^2)\mathcal{K}_1^{-1}M + (s_i - s_i)\mathcal{K}_1^{-1}D) P_i = \mathcal{K}_i P_i,$$

where $P_i = (I + (s_i^2 - s_i^2)\mathcal{K}_1^{-1}M + (s_i - s_i)\mathcal{K}_1^{-1}D)^{-1} P_i$. Let $Q_i \approx (I + (s_i^2 - s_i^2)\mathcal{K}_1^{-1}M + (s_i - s_i)\mathcal{K}_1^{-1}D)^{-1}$, then the above implies $\mathcal{K}_i P_i \approx \mathcal{K}_i Q_i P_i$ or $\mathcal{K}_i \approx \mathcal{K}_i Q_i$. This leads us to the following idea: instead of solving for $P_i$ from $\mathcal{K}_i P_i = \mathcal{K}_i P_i$, we solve a simpler problem

$$\min ||\mathcal{K}_i - \mathcal{K}_i Q_i||^2_F = \min \sum_{j=1}^n ||k_{i}^{(j)} - \mathcal{K}_i q_{i}^{(j)}||^2_2,$$

where $k_{i}^{(j)}$ and $q_{i}^{(j)}$ denote the $j$th columns of $\mathcal{K}_i$ and $Q_i$, respectively. Compare this minimization problem with the one in SPAI (Equation 8 in Section 3.1). Earlier, we were finding the preconditioner $P_i$ by solving $\min ||I - \mathcal{K}_i P_i||^2_F$. Now we are finding the preconditioner $P_i$ (i.e., $P_i = Q_i P_i$) by solving $\min ||\mathcal{K}_i - \mathcal{K}_i Q_i||^2_F$. The second formulation is much easier to solve since in the first $\mathcal{K}_i$ could be very different from $I$, while in the second $\mathcal{K}_i$ and $\mathcal{K}_1$ are similar (change only in the expansion points $s_i$).

The SPAI algorithm (Algorithm 2) adapted for finding the preconditioner by minimizing this new expression is given in Algorithm 4. The inputs to this algorithm include $\mathcal{K}_1$ (initial coefficient matrix), $\mathcal{K}_i$ (next coefficient matrix), $n_0$ (outer iteration count) and $n_i$ (inner iteration count); $n_0$ and $n_i$ are picked based upon experience. The initial guess for the approximate inverse $Q_i$ is usually taken as $\alpha I$. Similar to before, $\alpha$ is chosen to minimize the spectral radius of $(\mathcal{K}_1 - \alpha \mathcal{K}_i)$:

$$\frac{\partial}{\partial \alpha} ||\mathcal{K}_1 - \alpha \mathcal{K}_i||^2_F = 0 \quad \text{or} \quad \frac{\partial}{\partial \alpha} \text{trace}((\mathcal{K}_1 - \alpha \mathcal{K}_i)^T (\mathcal{K}_1 - \alpha \mathcal{K}_i)) = 0 \quad \text{or} \quad \frac{\partial}{\partial \alpha} \text{trace}([\mathcal{K}_1^T \mathcal{K}_1 - (\mathcal{K}_1 + \alpha \mathcal{K}_i^T \mathcal{K}_1 + \alpha^2 \mathcal{K}_i^T \mathcal{K}_i] = 0 \quad \text{or} \quad 2\alpha \cdot \text{trace}((\mathcal{K}_i^T \mathcal{K}_i) = \text{trace}((\mathcal{K}_1^T \mathcal{K}_1 + \mathcal{K}_i^T \mathcal{K}_i) \quad \text{or} \quad \alpha = \frac{1}{2} \cdot \frac{\text{trace}((\mathcal{K}_i^T \mathcal{K}_i) - \mathcal{K}_1^T \mathcal{K}_i)}{\text{trace}(\mathcal{K}_i^T \mathcal{K}_i)}.

The AIRGA algorithm with the SPAI update preconditioner is given in Algorithm 5. Here, we only show those parts of the AIRGA algorithm that require changes.
Algorithm 4: SPAI Update Preconditioner

1: Input: $\{\mathcal{K}_1, \mathcal{K}_i, n_0, n_i\}$
2: $Q_i = \alpha I$ where $\alpha = \frac{1}{2} \cdot \frac{\text{trace}(\mathcal{K}_1^T \mathcal{K}_i + \mathcal{K}_i^T \mathcal{K}_1)}{\text{trace}(\mathcal{K}_i^T \mathcal{K}_i)}$ and $n$ is the dimension of $\mathcal{K}_i$
3: for outer = 1, ..., $n_0$ do
4: for $j = 1, ..., n_i$ do
5: Define $q_{(j)}^i = Q_i e_{(j)}$
6: for inner = 1, ..., $n_i$ do
7: $r = k_{(j)}^i - \mathcal{K}_i q_{(j)}^i$
8: $z = Q r$
9: $q = \mathcal{K}_i z$
10: $\alpha = \frac{(r, q)}{(q, q)}$
11: $q_{(j)}^i = q_{(j)}^i + \alpha z$
12: end for
13: Update $j^{th}$ column of $Q_i$ with $q_{(j)}^i$
14: end for
15: end for

Algorithm 5: AIRGA with SPAI Update Preconditioner

1: $j = 1$
2: while no convergence do
3: Let $\mathcal{K}_1 = (s_1^2 M + s_1 D + K)$
4: Compute initial preconditioner $P_1$ by solving $\min ||I - \mathcal{K}_1 P_1||_F^2$
5: Solve $\mathcal{K}_1 P_1 X^{(0)}(s_1) = F$
6: for $i = 2, ..., l$ do
7: Let $\mathcal{K}_i = (s_i^2 M + s_i D + K)$
8: Compute $Q_i$ by solving $\min ||\mathcal{K}_1 - \mathcal{K}_i Q_i||_F^2$
9: Compute next preconditioners $P_i = Q_i P_1$
10: Solve $\mathcal{K}_i P_i X^{(0)}(s_i) = F$
11: end for
12: while no convergence and $j \leq \lceil r_{\text{max}} / m \rceil$ do
13: for $i = 1, ..., l$ do
14: Only right hand sides are changing, so above preconditioner $P_i$ can be applied as it is, i.e., solve $\mathcal{K}_i P_i X^{(j)}(s_i) = MV_j$
15: end for
16: end while
17: $j = j + 1$
18: end while
4. Stability Analysis of AIRGA

An algorithm \( \tilde{f} \) for computing the solution of a continuous problem \( f \) on a digital computer is said to be stable [5] if

\[
\tilde{f}(x) = f(\tilde{x}) \text{ for some } \tilde{x} \text{ with } \frac{||\tilde{x} - x||}{||x||} = O(\epsilon_{\text{machine}}),
\]

where \( \epsilon_{\text{machine}} \) is the machine precision. Here, we study the stability of AIRGA algorithm with respect to the errors introduced by iterative methods.

Suppose \( X^{(j)}(s_i) \) at line 4 and 13 in AIRGA algorithm (Algorithm 1) are computed using a direct method of solving linear system. This gives us the matrix \( V \) at line 30 in Algorithm 1. Let \( f \) be the functional representation of the moment matching process that uses \( V \) in AIRGA (i.e., exact AIRGA). Similarly, suppose \( X^{(j)}(s_i) \) at line 4 and 13 in Algorithm 1 are computed using an iterative method of solving linear systems. Since iterative methods are inexact, i.e., they solve the linear systems up to a certain tolerance, we denote the resulting matrix \( \tilde{V} \) as \( \tilde{V} \).

Let \( \tilde{f} \) be the functional representation of the moment matching process that uses \( \tilde{V} \) in AIRGA (i.e., inexact AIRGA). Then we will say that AIRGA is stable with respect to iterative solvers if

\[
\tilde{f}(H(s)) = f(\tilde{H}(s)) \text{ for some } \tilde{H}(s) \text{ with } \frac{||H(s) - \tilde{H}(s)||_{H_2 \text{ or } H_\infty}}{||H(s)||_{H_2 \text{ or } H_\infty}} = O(||Z||),
\]

(10)

where \( \tilde{H}(s) \) is a perturbed original full model corresponding to the error in the linear solves for computing \( \tilde{V} \) in inexact AIRGA. This perturbation is denoted by \( Z \). Further, we denote \( f(H(s)) = \hat{H}(s) \) and \( \tilde{f}(H(s)) = \hat{H}(s) \).

In Algorithm 1 the linear systems at line 4 are computed for different expansion points as

\[
(s_i^2 M + s_i D + K)X^{(0)}(s_i) = F,
\]

where \( s_i \in \{s_1, s_2, \ldots, s_l\} \). As discussed earlier, we solve these linear systems inexactly (i.e., by an iterative method). Let the residual associated with inexact linear solves for computing \( X^{(0)}(s_i) \) be \( \eta_{hi} \) for \( i = 1, \ldots, l \)

\[
(s_i^2 M + s_i D + K)X^{(0)}(s_i) = F + \eta_{hi}.
\]

(11)

Further, in Algorithm 1 at line number 10, \( \tilde{V}_1 \) is computed as

\[
\tilde{V}_1 = [X^{(0)}(s_{h_0})/||X^{(0)}(s_{h_0})||],
\]

(12)

where \( s_{h_0} \) is the expansion point corresponding to the maximum moment error of the reduced system.

Solving the linear systems for \( X^{(j)}, j = 1, \ldots, J - 1 \) at line number 13 in Algorithm 1 inexactly yields

\[
(s_i^2 M + s_i D + K)X^{(j)}(s_i) = M\tilde{V}_j + \eta_{ji} \text{ for } i = 1, \ldots, l.
\]

(13)

Next, \( \tilde{V}_{j+1} \) is computed as

\[
\tilde{V}_{j+1} = [X^{(j)}(s_{ij})/||X^{(j)}(s_{ij})||],
\]

(14)
where \( s_j \) is the expansion point corresponding to the maximum moment error of the reduced system.

Finally, Galerkin projection is used to generate the reduced model (obtained by inexact AIRGA)

\[
\hat{M} = \hat{V}^T M \hat{V}, \quad \hat{D} = \hat{V}^T D \hat{V}, \quad \hat{K} = \hat{V}^T K \hat{V},
\]

\[
\hat{F} = \hat{V}^T F, \quad \hat{C}_p = C_p \hat{V}, \quad \text{and} \quad \hat{C}_v = C_v \hat{V},
\]

where \( \hat{V} = [\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_J] \). \( \text{(15)} \) states \( f(\hat{H}(s)) \). Now we have to find a perturbed original model whose exact solution, \( f(\hat{H}(s)) \), will give the reduced model as obtained by the inexact solution of the original full model, \( f(H(s)) \). That is, find \( \hat{H}(s) \) such that \( f(\hat{H}(s)) = f(H(s)) \). This would satisfy the first condition of stability \( \text{(9)} \).

Assume that \( \hat{H} \) is given by the original matrices \( M \) and \( D \) and a perturbed matrix \( \hat{K} = K + Z \). Then, for \( \hat{H} \) we have

\[
(s_j^2 M + s_j D + (K + Z) X(0)(s_j) = F \quad \text{for} \quad i = 1, \ldots, l.
\]

Further, assume that the linear systems can be solved exactly as

\[
(s_j^2 M + s_j D + (K + Z) X^{(j)}(s_i) = M \hat{V}_j \quad \text{for} \quad j = 1, \ldots, J - 1 \text{ and } i = 1, \ldots, l.
\]

(17)

Again, \( \hat{V} = [\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_J] \) where \( \hat{V}_1 \) and \( \hat{V}_{J+1} \) for \( j = 1, \ldots, J - 1 \) are given by \( \text{(12)} \) and \( \text{(14)} \) since \( X^{(0)}(s_i) \) and \( X^{(j)}(s_i) \) for \( j = 1, \ldots, J - 1 \) and \( i = 1, \ldots, l \) are kept same as in \( \text{(11)} \) and \( \text{(13)} \), respectively.

As earlier, applying Galerkin projection to the perturbed original system gives

\[
\hat{M} = \hat{V}^T M \hat{V}, \quad \hat{D} = \hat{V}^T D \hat{V}, \quad \hat{K} = \hat{V}^T (K + Z) \hat{V} = \hat{K} + \hat{V}^T Z \hat{V},
\]

\[
\hat{F} = \hat{V}^T F, \quad \hat{C}_p = C_p \hat{V}, \quad \text{and} \quad \hat{C}_v = C_v \hat{V}.
\]

(18)

Our goal now is to find \( Z \) such that \( \hat{K} = K \) (recall, that we assumed that the error can be attributed solely to \( K; M \) and \( D \) do not change). Comparing \( \text{(11)} \) with \( \text{(16)} \), and \( \text{(13)} \) with \( \text{(17)} \), we get

\[
Z X^{(0)}(s_0) = -\eta \eta_0 \quad \text{and} \quad Z X^{(j)}(s_j) = -\eta \eta_j \quad \text{for} \quad j = 1, \ldots, J - 1,
\]

where \( \eta_0 \) and \( \eta_j \) are residuals corresponding to the maximum moment error of the reduced system in inexact AIRGA. We can rewrite \( Z \) as

\[
Z X = -\eta,
\]

where \( Z \in \mathbb{R}^{n \times n}, \ X = [X^{(0)}(s_0), \ldots, X^{(J-1)}(s_{l_{J-1}})] \in \mathbb{R}^{n \times mJ}, \) and \( \eta = [\eta_0, \ldots, \eta_{l_{J-1}}] \in \mathbb{R}^{mJ} \). As discussed in Section 2 the upper bound for \( J \) is \( [r/m] \), and hence, \( mJ < r \). Using the fact that \( r < n \), we have \( mJ < n \). Thus, we have an under-determined system of equations.

One solution of this is

\[
Z = -\eta X^T (XX^T)^{-1}.
\]
Multiplying both sides of (19) with $\tilde{V}$, we get

$$\tilde{V}^T Z \tilde{V} = -\tilde{V}^T \eta X^T (XX^T)^{-1} \tilde{V}. \tag{20}$$

For Ritz-Galerkin based iterative solvers, the solution space of linear systems is orthogonal to the residuals, i.e., $\tilde{V}_1 \perp \eta_{\theta_0}$, $\tilde{V}_2 \perp \eta_{\theta_1}$, ..., and $\tilde{V}_j \perp \eta_{(j-1)(j-1)}$ [21]. Hence,

$$\tilde{V}^T \eta = \begin{bmatrix} \tilde{V}_1^T \\ \vdots \\ \tilde{V}_j^T \end{bmatrix} \begin{bmatrix} \eta_{\theta_0} \\ \eta_{\theta_1} \\ \vdots \\ \eta_{(j-1)(j-1)} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{V}_1^T \eta_{\theta_1} & \cdots & \tilde{V}_1^T \eta_{(j-2)(j-2)} & \tilde{V}_1^T \eta_{(j-1)(j-1)} \\ \tilde{V}_2^T \eta_{\theta_0} & 0 & \cdots & \tilde{V}_2^T \eta_{(j-2)(j-2)} & \tilde{V}_2^T \eta_{(j-1)(j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{V}_{j-1}^T \eta_{\theta_0} & \tilde{V}_{j-1}^T \eta_{\theta_1} & \cdots & 0 & \tilde{V}_{j-1}^T \eta_{(j-1)(j-1)} \\ \tilde{V}_j^T \eta_{\theta_0} & \tilde{V}_j^T \eta_{\theta_1} & \cdots & \tilde{V}_j^T \eta_{(j-2)(j-2)} & 0 \end{bmatrix}.$$

Further, $\tilde{V}^T \eta X^T$

$$= \begin{bmatrix} 0 & \tilde{V}_1^T \eta_{\theta_1} & \cdots & \tilde{V}_1^T \eta_{(j-2)(j-2)} & \tilde{V}_1^T \eta_{(j-1)(j-1)} \\ \tilde{V}_2^T \eta_{\theta_0} & 0 & \cdots & \tilde{V}_2^T \eta_{(j-2)(j-2)} & \tilde{V}_2^T \eta_{(j-1)(j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{V}_{j-1}^T \eta_{\theta_0} & \tilde{V}_{j-1}^T \eta_{\theta_1} & \cdots & 0 & \tilde{V}_{j-1}^T \eta_{(j-1)(j-1)} \\ \tilde{V}_j^T \eta_{\theta_0} & \tilde{V}_j^T \eta_{\theta_1} & \cdots & \tilde{V}_j^T \eta_{(j-2)(j-2)} & 0 \end{bmatrix} \begin{bmatrix} X^{(0)}(\theta_0)^T \\ X^{(1)}(\theta_1)^T \\ \vdots \\ X^{(j-2)}(\theta_{j-2})^T \\ X^{(j-1)}(\theta_{j-1})^T \end{bmatrix}.$$

Since $\tilde{V}_1 \perp \eta_{\theta_0}$, $\tilde{V}_2 \perp \eta_{\theta_1}$, ..., $\tilde{V}_j \perp \eta_{(j-1)(j-1)}$, and, due to (14), $\tilde{V}_{j+1}$ is just the normalized $X^{(j)}(\theta_j)$, we have $X^{(0)}(\theta_0) \perp \eta_{\theta_0}$, $X^{(1)}(\theta_1) \perp \eta_{\theta_1}$, ..., and $X^{(j-1)}(\theta_{j-1}) \perp \eta_{(j-1)(j-1)}$. Therefore from (20), we get $\tilde{V}^T Z \tilde{V} = 0$. Thus, $\tilde{K} = \tilde{K}$ or

$$\tilde{f}(H(s)) = f(H(s)) = \tilde{H}(s),$$

where $H(s) = (C_p + sC_v)(s^2\hat{M} + s\hat{D} + K)^{-1} F$, $\tilde{H}(s) = (C_p + sC_v)(s^2\hat{M} + s\hat{D} + (K + Z))^{-1} F$, and $\tilde{f}(s) = (\tilde{C}_p + s\tilde{C}_v)(s^2\tilde{M} + s\tilde{D} + \tilde{K})^{-1} \tilde{F} = (\tilde{C}_p + s\tilde{C}_v)(s^2\tilde{M} + s\tilde{D} + \tilde{K})^{-1} \tilde{F}$. Thus, we have satisfied the first condition of stability.
According to the second condition of stability, given in (10), the difference between the original full model and the perturbed full model should be of the order of the perturbation \[5\]. This can be easily shown (Theorem 4.3 from [22]).

**Theorem 1.** If \[ ||Z|| < \frac{1}{||\mathcal{X}(s)^{-1}||_{H_\infty}} \] then

\[
||H(s) - \tilde{H}(s)||_{H_2} \leq \frac{||C(s)\mathcal{X}(s)^{-1}||_{H_2}||\mathcal{X}(s)^{-1}F||_{H_\infty}||Z||}{1 - ||\mathcal{X}(s)^{-1}||_{H_\infty}||Z||},
\]

where \( \mathcal{X}(s) = (s^2M + sD + K) \) and \( C(s) = (C_p + sC_v) \).

Hence,

\[
||H(s) - \tilde{H}(s)||_{H_2} = O(||Z||).
\]

The above result holds in a relative sense too. This proves the stability of AIRGA. The next theorem summarizes this.

**Theorem 2.** If the linear systems arising in AIRGA are solved by a Ritz-Galerkin based solver (i.e., the residual is orthogonal to the generated Krylov subspace) and

\[
||(s^2M + sD + K)^{-1}||_{H_\infty} \cdot ||Z|| < 1,
\]

where \( Z \) given by (19), then AIRGA is stable.

5. Numerical results

Consider a one dimensional beam model [2], which is of the form (1)

\[
M \ddot{x}(t) + D \dot{x}(t) + Kx(t) = Fu(t),
\]

\[
y(t) = C_p x(t),
\]

where \( m = q = 1 \), \( F \in \mathbb{R}^{n \times 1} \) and \( C_p \in \mathbb{R}^{1 \times n} \). The model has proportional damping, i.e., \( D = \alpha M + \beta K \), where the damping coefficients \( \alpha \) and \( \beta \) belong to \((0, 1)\) [2]. We consider the model with two different sizes, \( n = 500 \) and \( n = 1000 \).

We compute a reduced order model by the AIRGA algorithm given in Algorithm 1. We implemented AIRGA in MATLAB (2014a). We take \( r_{\text{max}} \), i.e. the maximum dimension to which we want to reduce the system, as 25 based on similar values selected in [2]. We take three expansion points that are linearly spaced between 1 and 1000 based on initial data used in [2]. As discussed in Section 3, we use iterative methods to solve the linear systems at lines 4 and 13 of Algorithm 1.

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1 Since the system sizes we work with are 500 and 1000 (that are small), we do not compare with direct methods for solving linear systems, e.g., Gaussian elimination. Iterative methods are competitive when working with much larger system sizes. This is future work since the goal of this paper is to show applicability of SPAI preconditioners and SPAI preconditioner updates to AIRGA as well as stability analysis of AIRGA with respect to iterative solves. All of these three aspects are independent of the size of the input problem.
\[ s^2M + sD + K \] are also symmetric initially. As discussed in Section 2 after the first AIRGA iteration, the expansion points are chosen from the eigenvalues of the quadratic eigenvalue problem
\[ \lambda^2M + \lambda D + K. \]
For our example, after the first AIRGA iteration, the eigenvalues turn out to be complex. Thus, we get complex expansion points. This implies that the coefficient matrices of the linear systems to solved after the first AIRGA iteration become non-Hermitian, specifically complex-symmetric.

For ensuring stable iterative solves in AIRGA, from Theorem 2 we know that we need to use a Ritz-Galerkin based solver. Conjugate Gradient (CG) is the most popular solver based on this theory. However, CG works for Hermitian positive definite (HPD) systems. There are ways of making CG work for non-Hermitian systems as well but achieving \( ||\mathcal{K}(s)^{-1}||_{H_\infty} ||Z|| < 1 \) is non-trivial. Hence, we use the Generalized Minimum Residual method (GMRES) \(^3\), which is the most popular method for solving non-Hermitian systems. It is observed that unpreconditioned GMRES stagnates for all model sizes.

As discussed in Section 3 preconditioning has to be employed when iterative methods fail or have very slow convergence. As discussed above, the coefficient matrices of the linear systems to be solved become non-Hermitian after the first AIRGA iteration. We use SPAI and SPAI update as discussed in Section 3.1 and 3.2 respectively. That is, we use Algorithm 3 with Algorithm 2 and Algorithm 5 with Algorithm 4. In SPAI (Algorithm 2) we set \( n_i = 4 \) and \( n_0 = 6 \). In using SPAI with update (Algorithm 4), we set \( n_i = 4 \) and \( n_0 = 6 \) for the first preconditioner in the sequence, and \( n_i = 1 \) and \( n_0 = 1 \) for subsequent preconditioners. This is because for the first preconditioner \( P_1 \), \( \mathcal{K}_1 \) could be very different from \( I \), while for subsequent preconditioners \( P_i \), \( \mathcal{K}_i \) and \( \mathcal{K}_1 \) are similar (change in expansion points \( s_i \) only).

In Algorithms 3 and 5 at line 2 the overall iteration (while-loop) terminates when the change in the reduced model (computed as \( H_2 \)-error between the reduced models at two consecutive AIRGA iterations) is less than a certain tolerance. We take this tolerance to be \( 10^{-04} \) based on values in \(^1\). There is one more stopping criteria in these algorithms, at lines 8 and 12, respectively. This checks the \( H_2 \)-error between two temporary reduced models. We take this tolerance to be \( 10^{-06} \) based on values in \(^1\). Since this is an adaptive algorithm, the optimal size of the reduced model is determined by the algorithm itself, and is denoted by \( r \).

Table 1 gives the GMRES iteration count and time when using basic SPAI (that is without SPAI update; labeled as GMRES in the table) and SPAI with update (labeled as GMRES with Update in the table) for model size 500. Table 2 gives the same data for model size 1000. From Tables 1 and 2 it is observed that the GMRES computation time for both variants of SPAI is fairly small. Although the time is higher for the case of with update, it fades when we look at savings in computing a preconditioner (see below).

Table 3 gives the time for computing the basic SPAI preconditioner (labeled as Preconditioner) and SPAI with updated preconditioner (labeled as Preconditioner with Update) for model size 500. Table 4 gives the same data for model size 1000. From Tables 3 and 4 it is observed that considerable amount of time is saved by using SPAI with updates.

Table 5 shows the total computation time for iterative solves (i.e., GMRES time plus the preconditioner time) when using basic SPAI preconditioner (labeled as Iterative Solves) and

\(^2\)The coefficient matrices of the linear systems to solved are non-Hermitian but they have a nicer structure than a general non-Hermitian system. That is, they are complex-symmetric. For such systems, although specialized linear solvers have been developed, e.g. Minimum Residual methods for complex-symmetric systems \(^2\), GMRES does remain the most popular method. Moreover, here we are showing advantages of using SPAI preconditioners and their updates, and hence, the choice of underlying linear solver is flexible to a great extent.
Table 1: GMRES iterations and computation time for model size 500

| AIRGA Iteration# | GMRES             | GMRES with Update |
|------------------|------------------|-------------------|
|                  | Iter | Time (secs) | Iter | Time (secs) |
| 1                | 1    | 0.017      | 1    | 0.017       |
|                  | 3    | 0.018      | 11   | 0.039       |
|                  | 3    | 0.018      | 11   | 0.039       |
| 2                | 2    | 0.018      | 2    | 0.018       |
|                  | 2    | 0.018      | 12   | 0.041       |
|                  | 2    | 0.018      | 7    | 0.026       |

Table 2: GMRES iterations and computation time for model size 1000

| AIRGA Iteration# | GMRES             | GMRES with Update |
|------------------|------------------|-------------------|
|                  | Iter | Time (secs) | Iter | Time (secs) |
| 1                | 1    | 0.057      | 1    | 0.057       |
|                  | 3    | 0.061      | 11   | 0.159       |
|                  | 3    | 0.061      | 11   | 0.159       |
| 2                | 2    | 0.059      | 2    | 0.059       |
|                  | 2    | 0.059      | 12   | 0.165       |
|                  | 2    | 0.059      | 6    | 0.095       |

when using SPAI with update preconditioner (labeled as Iterative Solves with Update) for model sizes 500 and 1000. We can notice from this table that iterative solves with update take about one third of time as needed for basic iterative solves (or without SPAI update).

Table 6 lists the relative error between the original model and the reduced model as well as the size to which the model is reduced ($r$) when using a direct method (LU factorization), GMRES with SPAI, and GMRES with SPAI updates. Since the error and $r$ values for all the above three cases are almost the same, we can conclude that by using iterative solves in AIRGA, the quality of reduced system is not compromised.

6. Conclusion and Future work

We discussed the application of preconditioned iterative methods for solving large linear systems arising in AIRGA. The SPAI preconditioner works well here and SPAI update (where we reuse the preconditioner) leads to substantial savings. This is demonstrated by experiments on two different size of one model. We also presented conditions under which the AIRGA algorithm is stable with respect to the errors introduced by iterative methods.

Future work includes applying preconditioned iterative methods in other model reduction algorithms for second order dynamical systems (besides AIRGA). For example, Alternate Direction Implicit (ADI) methods for model reduction of second order linear dynamical systems [24]. Based upon our studies on AIRGA and ADI based methods, we also plan to propose a class of preconditioners that would work for most model reduction algorithms for second order linear dynamical systems.
Table 3: SPAI and SPAI update computation time for model size 500

| AIRGA Iteration# | Preconditioner (secs) | Preconditioner with Update (secs) |
|------------------|-----------------------|----------------------------------|
| 1                |                       |                                  |
|                  | 0.033                 | 0.033                            |
|                  | 1.971                 | 1.962                            |
| 2                | 3.113                 | 3.112                            |
|                  | 3.642                 | 0.982                            |
|                  | 3.123                 | 0.991                            |

Table 4: SPAI and SPAI update computation time for model size 1000

| AIRGA Iteration# | Preconditioner (secs) | Preconditioner with Update (secs) |
|------------------|-----------------------|----------------------------------|
| 1                |                       |                                  |
|                  | 0.087                 | 0.078                            |
|                  | 20.105                | 3.378                            |
| 2                | 34.201                | 34.412                           |
|                  | 34.201                | 9.372                            |
|                  | 34.412                | 6.240                            |

Table 5: Total computation time of GMRES with Preconditioner

| Size    | Iterative Solve (secs) | Iterative Solve with Update (secs) |
|---------|------------------------|-----------------------------------|
| 500     | 13.898                 | 5.952                             |
| 1000    | 135.78                 | 50.00                             |

Table 6: Accuracy of reduced system

| Problem        | n  | Method                | Error     | r  |
|----------------|----|-----------------------|-----------|----|
| 1-D Beam Model | 500| Direct                | $4.5 \cdot 10^{-07}$ | 17 |
|                |    | GMRES with SPAI       | $2.5 \cdot 10^{-07}$ | 17 |
|                |    | GMRES with SPAI-Update| $4.3 \cdot 10^{-07}$ | 16 |
|                | 1000| Direct                | $1.2 \cdot 10^{-06}$ | 17 |
|                |    | GMRES with SPAI       | $1.2 \cdot 10^{-06}$ | 17 |
|                |    | GMRES with SPAI-Update| $2.8 \cdot 10^{-07}$ | 17 |
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