Approximating Flexible Graph Connectivity via Räcke Tree based Rounding

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Abstract
Flexible graph connectivity is a new network design model introduced by Adjiashvili [1]. It has seen several recent algorithmic advances [2, 3, 4, 7, 5, 10]. Despite these, the approximability even in the setting of a single-pair \((s, t)\) is poorly understood. In [10] we raised the question of whether there is poly-logarithmic approximation for the survivable network design version (Flex-SNDP) when the connectivity requirements are fixed constants. In this paper, we adapt a powerful framework for survivable network design recently developed by Chen, Laekhanukit, Liao, and Zhang [12] to give an affirmative answer to the question. The framework of [12] is based on Räcke trees and group Steiner tree rounding. The algorithm and analysis also establishes an upper bound on the integrality gap of an LP relaxation for Flex-SNDP [10].

1 Introduction
The Survivable Network Design Problem (SNDP) is an important problem in combinatorial optimization that generalizes many well-known problems related to connectivity, and is also motivated by practical problems related to the design of fault-tolerant networks. The input to this problem is an undirected graph \(G = (V, E)\) with non-negative edge costs \(c : E \to \mathbb{R}_+\) and an integer function \(r : V \times V \to \mathbb{Z}_+\) which specifies a connectivity requirement for each node pair \((u, v)\). The goal is to find a minimum-cost subgraph \(H\) of \(G\) such that \(H\) has \(r(u, v)\) connectivity for each pair \((u, v)\).

Our focus in this paper is on edge-connectivity requirements; the resulting problem is referred to as EC-SNDP. VC-SNDP refers to the problem in which each pair \((u, v)\) requires \(r(u, v)\) vertex connectivity. EC-SNDP contains as special cases classical problems such as \(s\)-\(t\) shortest path, minimum spanning tree (MST), minimum \(k\)-edge-connected subgraph (\(k\)-ECSS), Steiner tree, Steiner forest and several others. It is NP-Hard and APX-Hard to approximate. Jain’s seminal 2-approximation for EC-SNDP via iterated rounding [21] is the currently the best known approximation ratio.

In this paper we are interested in a new network design model suggested by Adjiashvili [1] for which there are several recent developments [2, 3, 4, 7, 10, 5]. In this model, the edge set \(E\) is partitioned to safe edges \(S\) and unsafe edges \(U\). Vertices \(s, t \in V\) are \((p, q)\)-flex-connected \(^1\) if \(s\) and \(t\) are \(p\)-edge-connected after deleting any subset of at most \(q\) unsafe edges. The Flex-SNDP problem is the following: the input is a graph \(G = (V, E)\) with edge costs \(c : E \to \mathbb{R}_+\), a partition \(U \cup S\) of the edge set, and functions \(p, q : V \times V \to \mathbb{Z}_+\). The goal is to find a min-cost subgraph \(H\) of \(G\) such that

\(^1\) We follow the terminology from our recent work [10] that is influenced by [3, 7].
each \( u, v \in V \) is \((p(u, v), q(u, v))\)-flex-connected in \( H \). We denote by \((p, q)\)-Flex-SNDP the special case where for each vertex pair \( u, v \), either \( p(u, v) = q(u, v) = 0 \) or \( p(u, v) = p, \ q(u, v) = q \). Note that if all edges are safe, i.e. \( E = S \), then \((p, q)\)-flex-connectivity is the same as \( p \)-connectivity, and if all edges are unsafe, i.e. \( E = U \), then \((p, q)\)-flex-connectivity is the same as \((p + q)\)-connectivity. Flex-SNDP thus generalizes EC-SNDP.

The work so far in flexible connectivity has been on two special cases. The first is the spanning case, which requires \((p, q)\)-flex-connectivity for all pairs of vertices. This is the \((p, q)\)-FGC problem [7]. The other is when the requirement is for a single pair \((s, t)\) \cite{1, 4}. Following \cite{10} we refer to this as \((p, q)\)-Flex-ST. For \((p, q)\)-FGC, \cite{7} obtained an \( O(q \log n) \)-approximation, and constant factor approximations have been developed for small values of \( p, q \) \cite{7, 5, 10}. For \((p, q)\)-Flex-ST, the only non-trivial approximations known are for \((1, q)\)-Flex-ST and \((p, 1)\)-Flex-ST \cite{4} and \((2, 2)\)-Flex-ST \cite{10}. In fact, no non-trivial approximation is known even for \((3, 2)\)-Flex-ST or \((2, 3)\)-Flex-ST. No non-trivial result is known for \((2, 2)\)-Flex-SNDP. We refer the reader to \cite{7, 10, 5} for a more detailed description of existing results.

Adjiashvili et al. \cite{4} show that when \( p \) is part of the input and large, \((p, 1)\)-Flex-ST is \text{NP}-Hard to approximate to almost polynomial factors. Thus \((p, q)\)-Flex-SNDP is a harder problem than EC-SNDP. In our earlier paper \cite{10} we raised the following question: does \((p, q)\)-Flex-SNDP admit an approximation ratio of the form \( f(p, q) \) or \( f(p, q)\text{polylog}(n) \) for all fixed \( p, q \) where \( f \) is some integer valued function? \cite{10} formulated an LP relaxation which can be solved in \( n^{O(\alpha)} \)-time and a corresponding question on its integrality gap was also implicitly raised. The known techniques for EC-SNDP and related problems rely on the requirement function satisfying structural properties such as skew-supermodularity and uncrossability. These properties are crucial in primal-dual and iterated rounding based algorithms \cite{16, 21}. Recent work on flexible connectivity network design \cite{7, 5, 10} extended some of these ideas in non-trivial and interesting ways to the special cases that we mentioned. However, the requirement function for flexible connectivity is not as well-behaved (see \cite{10} for some examples) and it seems challenging to obtain any non-trivial algorithm for say \((2, 2)\)-Flex-SNDP or \((3, 2)\)-Flex-ST.

**Contribution:** In this paper we take a substantially different approach for \((p, q)\)-Flex-SNDP, motivated by a very recent work of Chen, Laekhanukit, Liao, and Zhang \cite{12}. They developed a new algorithmic approach for survivable network design to tackle a generalization of EC-SNDP to the group/set connectivity setting. We use their framework to obtain the following theorem.

**Theorem 1.1.** There is a randomized algorithm that yields an \( O((p+q)^3 \log^7 n) \)-approximation for \((p, q)\)-Flex-SNDP and runs in expected \( n^{O(q)} \)-time. The approximation is based on an LP relaxation for the problem.

**Remark 1.2.** The algorithm for \((p, q)\)-Flex-SNDP easily extends to the setting where the maximum connectivity requirement is dominated by \((p, q)\).

The preceding theorem sheds light on the approximability of the problem — as discussed, this has been challenging via past techniques. It suggests that there may be an \( f(p, q) \)-approximation for \((p, q)\)-Flex-SNDP via the LP relaxation. It also showcases the generality of the approach in \cite{12} which is likely to have further impact in network design. For instance, the algorithm and analysis extend to the Set Connectivity version of flexible connectivity problem.

### 1.1 Technical Overview

We give a brief technical overview of the algorithm. We follow an augmentation approach for \((p, q)\)-Flex-SNDP following recent work \cite{7, 5, 10}. The idea is to start with a subgraph \( H_0 \) that satisfies
analyses the correctness and approximation ratio of the 
sets up the relevant background on the LP relaxation, Räcke tree em-
test for
properties of $G$ is to consider a probabilistic approximation of
separating a terminal pair must have at least $p$ $\frac{1}{k}$
subgraph such that each terminal pair
Recall the definition of the $2.1$ LP Relaxation for Augmentation
verify the feasibility of a solution in
$n$ shrinking 
$x$ subsets. We say that $x$ supports a flow of value $f$ between $A$ and $B$ if, in the graph $G'$ obtained by shrinking $S$ to $s$ and $T$ to $t$, the max $s$-$t$ flow is at least $f$.

2 Preliminaries, Augmentation LP, and Background

Let $G = (V, E)$ be a graph with edge capacities $x : E \to \mathbb{R}_+$, and let $S, T$ be two disjoint vertex subsets. We say that $x$ supports a flow of value $f$ between $A$ and $B$ if, in the graph $G'$ obtained by shrinking $S$ to $s$ and $T$ to $t$, the max $s$-$t$ flow is at least $f$.

2.1 LP Relaxation for Augmentation

Recall the definition of the $(p, q)$-Flex-SNDP problem: given a graph $G = (V, E)$ with edge costs $c(e)$, a partition $H \cup S$ on the edge set, and terminal pairs $(s_i, t_i) \in V \times V$, $i \in [k]$, find the cheapest subgraph such that each terminal pair $(s_i, t_i)$ is $(p, q)$-flex-connected. Equivalently, any cut $\delta(S)$ separating a terminal pair must have at least $p$ safe edges or at least $p + q$ total edges. One can verify the feasibility of a solution in $n^{O(q)}$-time: for each subset of $q$ unsafe edges, remove them and test for $p$-connectivity between terminal pairs. We employ the augmentation methodology. Suppose we are given a partial solution $H \subseteq E$ that satisfies $(p, q - 1)$-flex-connectivity for the given terminal
pairs. We call a cut \( S \) violated with respect to \( H \) if \( |S \cap \{s_i, t_i\}| = 1 \) for some \( i \), and \( |\delta_H(S) \cap \mathcal{U}| < p \) and \( |\delta_H(S)| = p + q - 1 \). Note that any cut separating a terminal pair in \( H \) has at least \( p \) safe edges or at least \( p + q - 1 \) total edges. Let \( \mathcal{C} = \{ S \subseteq V : S \text{ is violated} \} \). We can augment \( H \) to obtain a feasible solution to \((p, q)\)-Flex-SNDP instance by covering all cuts in \( \mathcal{C} \). This naturally leads to a cut-based LP relaxation for the augmentation problem with variables \( x_e \in [0, 1] \) for \( e \in E \setminus H \):

\[
\min_{e \in E \setminus H} \sum_{e \in E \setminus H} c(e)x_e \quad \text{Augment-LP}
\]

subject to \( \sum_{e \in \delta_{E \setminus H}(S)} x_e \geq 1 \) \( \forall S \in \mathcal{C} \)

\( x_e \in [0, 1] \) \( e \in E \setminus H \)

**Claim 2.1.** Augment-LP admits an \( n^{O(q)} \)-time separation oracle and hence can be solved in polynomial time for each fixed \( q \).

Recall that \((p, 0)\)-Flex-SNDP is equivalent to EC-SNDP with all terminal pairs having requirement \( p \). We can obtain a \( 2 \)-approximate feasible solution. For \( \ell = 0 \) to \( q - 1 \) we solve an augmentation problem in each stage to go from \((p, \ell)\) to \((p, \ell + 1)\)-flex-connectivity. Thus an \( \alpha \)-approximation for the augmentation problem implies an overall \((2 + qa)\)-approximation for \((p, q)\)-Flex-SNDP.

**Remark 2.2.** There is an LP relaxation for \((p, q)\)-Flex-SNDP with the property that a feasible fractional solution to it is also feasible for Augment-LP for each stage [10]. Thus, proving an integrality gap bound for Augment-LP gives upper bounds on the integrality gap of the LP for \((p, q)\)-Flex-SNDP.

### 2.2 Räcke Tree Embeddings

The results in this paper use Räcke’s capacity-based probabilistic tree embeddings. We borrow the notation from [12]. Given \( G = (V, E) \) with capacity \( x : E \to \mathbb{R}^+ \) on the edges, a capacitated tree embedding of \( G \) is a tree \( T \), along with two mapping functions \( M_1 : V(T) \to V(G) \) and \( M_2 : E(T) \to 2^{E(G)} \) that satisfy some conditions. \( M_1 \) maps each vertex in \( T \) to a vertex in \( G \), and has the additional property that it gives a one-to-one mapping between the leaves of \( T \) and the vertices of \( G \). \( M_2 \) maps each edge \( (a, b) \in E(T) \) to a path in \( G \) between \( M_1(a) \) and \( M_1(b) \). For notational convenience we view the two mappings as a combined mapping \( M \). For a vertex \( u \in V(G) \) we use \( M^{-1}(u) \) to denote the leaf in \( T \) that is mapped to \( u \) by \( M_1 \). For an edge \( e \in E(G) \) we use \( M^{-1}(e) = \{ f \in E(T) \mid e \in M_2(f) \} \). It is sometimes convenient to view a subset \( S \subseteq V(G) \) both as vertices in \( G \) and also corresponding leaves of \( T \).

The mapping \( M \) induces a capacity function \( y : E(T) \to \mathbb{R}_+ \) as follows. Consider \( f = (a, b) \in E(T) \). \( T - f \) induces a partition \((A, B)\) of \( V(T) \) which in turn induces a partition/cut \((A', B')\) of \( V(G) \) via the mapping \( M \): \( A' \) is the set of vertices in \( G \) that correspond to the leaves in \( A \) and similarly \( B' \). We then set \( y(f) = \sum_{e \in \delta(A')} x(e) \), in other words \( y(f) \) is the capacity of cut \((A', B')\) in \( G \). The mapping also induces loads on the edges of \( G \). For each edge \( e \in G \), we let \( \text{load}(e) = \sum_{f \in E(T), e \in M(f)} y(f) \). The relative load or congestion of \( e \) is \( \text{rload}(e) = \text{load}(e)/x(e) \). The congestion of \( G \) with respect to a tree embedding \((T, M)\) is defined as \( \max_{e \in E(G)} \text{rload}(e) \). Given a probabilistic distribution \( \mathcal{D} \) on trees embeddings of \((G, x)\) we let

\[
\beta_{\mathcal{D}} = \max_{e \in E(G)(T, M) \sim \mathcal{D}} \mathbf{E}[\text{rload}(e)]
\]
denote the maximum expected congestion. Räcke showed the following fundamental result on probabilistic embeddings of a capacitated graph into trees.

**Theorem 2.3** ([22]). Given a graph $G$ and $x : E(G) \to \mathbb{R}_+$, there exists a probability distribution $\mathcal{D}$ on tree embeddings such that $\beta_{\mathcal{D}} = O(\log |V(G)|)$. All trees in the support of $\mathcal{D}$ have height at most $O(\log (nC))$, where $C$ is the ratio of the largest to smallest capacity in $x$. Moreover, there is a randomized polynomial-time algorithm that can sample a tree from the distribution $\mathcal{D}$.

In the rest of the paper we use $\beta$ to denote the guarantee provided by the preceding theorem where $\beta = O(\log n)$ for a graph on $n$ nodes.

**Implication for flows:** The original motivation for capacitated tree embeddings is oblivious routing of multicommodity flows. A multicommodity flow instance in a graph $G = (V, E)$ is specified by a demand matrix $D : V \times V \to \mathbb{R}_+$ and the goal is to simultaneously route $D(u, v)$ amount of flow between $u$ and $v$ for each vertex pair $(u, v)$. We say that $D$ is routable in $G$ with congestion $\alpha$ if there is a feasible multicommodity flow in $G$ that satisfies all demands such that the total flow on each edge $e$ is at most $\alpha \cdot x(e)$ (it is routable if $\alpha \leq 1$). It can be seen that given a tree $(T, \mathcal{M})$ in $\mathcal{D}$, any multicommodity flow that can be routed in $G$ with capacities $x$ can also be routed in $T$ with capacities given by $y$ with congestion $1$ — this is because cut-condition is necessary for routing and in trees it is also sufficient. Moreover, any routable multicommodity flow in $T$ with demands only between leaves can be routed in $G$ with congestion $\max_{e \in E(G)} \text{load}(e)$. The mapping of the routing in $T$ to $G$ is simple and follows the paths given by $\mathcal{M}$. The implication of this connection is the following corollary where we use $\text{maxflow}_T^y(A, B)$ to denote the maxflow between two disjoint vertex subsets $A, B$ in a capacitated graph $H$ with capacities given by $y : E(H) \to \mathbb{R}_+$.

**Corollary 2.4.** Let $\mathcal{D}$ be the distribution guaranteed in Theorem 2.3. Let $A, B \in V(G)$ be two disjoint sets. Then (i) for any tree $(T, \mathcal{M})$ in $\mathcal{D}$, $\text{maxflow}_T^y(A, B) \leq \text{maxflow}_G^y(\mathcal{M}^-(A), \mathcal{M}^-(B))$ and (ii) $\frac{1}{\beta} E_{(T, \mathcal{M}) \sim \mathcal{D}}[\text{maxflow}_T^y(\mathcal{M}^-(A), \mathcal{M}^-(B))] \leq \text{maxflow}_G^y(A, B)$.

### 2.3 Group Steiner Tree, Set Connectivity and Tree Rounding

The group Steiner tree problem was introduced in [23] and studied in approximation by Garg, Konjevod and Ravi [14]. The input is an edge-weighted graph $G = (V, E)$, a root vertex $r \in V$, and $k$ groups $S_1, S_2, \ldots, S_k$ where each $S_i \subseteq V$. The goal is to find a min-weight subgraph $H$ of $G$ such that there is a path in $H$ from $r$ to each group $S_i$ (that is, to some vertex in $S_i$). The approximability of this problem has attracted substantial attention. Garg et al. [14] described a randomized algorithm to round a fractional solution to a cut-based LP relaxation when $G$ is a tree — it achieves an $O(\log n \log k)$-approximation. This has been shown to be essentially tight from both an integrality gap and a hardness point of view [19, 20]. Their algorithm also yields an $O(\log^2 n \log k)$-approximation in general graphs via embeddings into tree metrics [6, 13]. Better approximation in quasi-polynomial time are known [11, 17, 15].

Set Connectivity is a generalization of group Steiner tree problem. Here we are given pairs of sets $(S_1, T_1), (S_2, T_2), \ldots, (S_k, T_k)$ and the goal is to find a min-cost subgraph $H$ such that there is an $(S_i, T_i)$ path in $H$ for each $i$. Chekuri et al. [9] obtained a poly-logarithmic approximation and integrality gap by generalizing the ideas from group Steiner tree. Chalermsook, Grandoni and Laekhanukit [8] studied Survivable Set Connectivity problem, motivated by earlier work in [18]. Here each pair $(S_i, T_i)$ has a connectivity requirement $r_i$ which implies that one seeks $r_i$ edge-disjoint paths between $S_i$ and $T_i$ in the chosen subgraph $H$; [8] obtained a bicriteria-approximation via Räcke tree and group Steiner tree rounding. The recent work of Chen et al [12] uses related
but more sophisticated ideas to obtain the first true approximation for this problem. They refer to the problem as Group Connectivity problem and obtain an $O(r^3 \log r \log^7 n)$-approximation where $r = \max_i r_i$ connectivity requirement (see [12] for more precise bounds).

**Oblivious tree rounding:** The rounding algorithm for Set Connectivity in trees given in [9] establishes a poly-logarithmic integrality gap, however, the rounding is not oblivious to the pairs. In [8] a randomized oblivious algorithm based on the group Steiner tree rounding from [14] is described. This is useful since the sets to be connected during the course of their algorithm are implicitly generated. We encapsulate their result in the following lemma. The tree rounding algorithm in [8, 12] is phrased slightly differently since they combine aspects of group Steiner rounding and the congestion mapping that comes from Räcke trees. We separate these two explicitly to make the idea more transparent. We refer to the algorithm from the lemma below as TreeRounding.

**Lemma 2.5 ([8, 12]).** Consider an instance of Set Connectivity on an $n$-node tree $T = (V, E)$ with height $h$ and let $x : E \to [0, 1]$. Suppose $A, B \subseteq V$ are disjoint sets and suppose $K \subseteq E$ such that $x$ restricted to $K$ supports a flow of $f \leq 1$ between $A$ and $B$. There is a randomized algorithm that is oblivious to $A, B, K$ (hence depends only on $x$ and value $f$) that outputs a subset $E' \subseteq E$ such that (i) The probability that $E' \cap K$ connects $A$ to $B$ is at least a fixed constant $\phi$ and (ii) For any edge $e \in E$, the probability that $e \in E'$ is $\min\{1, O(\frac{1}{h} \log^2 n) x(e)\}$.

### 3 Rounding Algorithm for the Augmentation Problem

We adapt the algorithm and analysis in [12] to Flex-SNDP. Let $\beta$ be the expected congestion given by Theorem 2.3. Consider an instance of $(p, q + 1)$-Flex-SNDP specified by a graph $G$ and a set of pairs $(s_i, t_i), i \in [k]$. Assume that we have a partial solution $H$ in which each $(s_i, t_i)$ is $(p, q)$-flex-connected. We augment $H$ to ensure that each $(s_i, t_i)$ is $(p, q + 1)$-flex-connected.

We start by obtaining a solution $\{x_e\}_{e \in E \setminus H}$ for the LP relaxation to the augmentation problem, Augment-LP, described in Section 2. Let $E' = E \setminus H$. We define LARGE = $\{e \in E' : x_e \geq \frac{1}{4(p+q)\beta}\}$, and SMALL = $\{e \in E : x_e < \frac{1}{4(p+q)\beta}\}$. The LP has paid for each $e \in$ LARGE a cost of $(\frac{c(e)}{4(p+q)\beta})$, hence adding all of them to $H$ will cost $O((p+q)\beta \cdot$ OPT$_{LP}$). If LARGE $\cup$ H is a feasible solution to the augmentation problem, then we are done since we obtain a solution of cost $O((p+q) \log n \cdot$ OPT$_{LP}$). Thus, the interesting case is when LARGE $\cup$ H is not a feasible solution. In effect, we can assume that LARGE = $\emptyset$. One can assume this situation without loss of generality by creating sufficiently many parallel edges for each $e$ and splitting the $x(e)$ amongst them.

Following [12] we employ a Räcke tree based rounding. A crucial step is to set up a capacitated graph appropriately. We can assume, with a negligible increase in the fractional cost, that for each edge $e \in E \setminus H$, $x(e) = 0$ or $x(e) \geq \frac{1}{n}$; this can be ensured by rounding down to 0 the fractional value of any edge with very small value, and compensating for this loss by scaling up the fractional value of the other edges by a factor of $(1 + 1/n)$. It is easy to check that the new solution satisfies the cut covering constraints, and we have only increased the cost of the fractional solution by a $(1 + 1/n)$-factor. In the subsequent steps we can ignore edges with $x_e = 0$ and assume that there are no such edges.

Consider the original graph $G = (V, E)$ where we set a capacity for each $e \in E$ as follows. If $e \in$ LARGE $\cup$ H we set $\hat{x}_e = \frac{1}{4(p+q)\beta}$. Otherwise we set $\hat{x}_e = x_e$. Since the ratio of the largest to smallest capacity is $O(n^3)$, the height of any Räcke tree for $G$ with capacities $\hat{x}$ is at most $O(\log n)$. Then, we repeatedly sample Räcke trees. For each tree, we sample edges by the rounding algorithm given by Chalermsook et al in [8] (see Section 2 for details). A formal description of the algorithm
is provided below where \( t' \) and \( t \) are two parameters that control the number of trees sampled and the number of times we run the tree rounding algorithm in each sampled tree. We will analyze the algorithm by setting both \( t \) and \( t' \) to \( \Theta((p + q) \log n) \).

Algorithm 1 Augmentation from \((p, q)\) to \((p, q + 1)\)-flex-connectivity

\[
\begin{align*}
H &\leftarrow \text{partial solution satisfying } (p, q)\text{-flex-connectivity for given instance} \\
\{x\}_{e \in E} &\leftarrow \text{fractional solution to Augment-LP} \\
\text{LARGE} &\leftarrow \{e \in E : x_e \geq \frac{1}{4(p+q)3}\} \\
\text{SMALL} &\leftarrow \{e \in E : x_e < \frac{1}{4(p+q)3}\} \\
H &\leftarrow H \cup \text{LARGE} \\
\text{if } &H \text{ is a feasible solution to } (p, q + 1)\text{-Flex-SNDP then return } H \\
\text{else} & \\
\tilde{x}_e &\left\{ \begin{array}{ll}
\frac{1}{4(p+q)3} & e \in H \\
0 & x_e < \frac{1}{n^2} \\
x_e & \text{otherwise}
\end{array} \right.
\end{align*}
\]

\(D \leftarrow \text{Räcke tree distribution for } (G, \tilde{x})\)

for \( i = 1, \ldots, t' \) do

- Sample a tree \((T, M, y) \sim D\)

  for \( j = 1, \ldots, t \) do

    - \(K \leftarrow \text{output of oblivious TreeRounding algorithm on } (G, T)\)

    - \(H \leftarrow H \cup M(K)\)

  end for

end for

return \( H \)

\section{Analysis}

We will assume, following earlier discussion, that LARGE = \( \emptyset \) and focus on the case when the algorithm proceeds to the TreeRounding step. Let \( H \) denote the set of edges that satisfies \((p, q)\)-flex-connectivity for the given pairs. Augment-LP is a cut covering LP. Consider any violated cut \( S \) with respect to \( H \); \( S \) is violated because \( S \) separates a pair \((s_i, t_i)\) and \( \delta_H(S) \) has exactly \((p + q)\) edges, of which at most \( p - 1 \) are safe. Let \( F = \delta_H(S) \). We call \( F \) a violating edge set. There are at most \(|H|^{(p+q)}\) violating edge sets, and since \(|H| \leq n^2\), this is upper bounded by \( O(n^{2(p+q)}) \). We say that a set of edges \( H' \subseteq E \setminus H \) is a feasible augmentation for violating edge set \( F \) if for each pair \((s_i, t_i)\), there is a path from \( s_i \) to \( t_i \) in the graph \((H \cup H') \setminus F\). The following is a simple observation.

\textbf{Claim 4.1.} \( H' \subseteq E \setminus H \) is a feasible solution to the augmentation problem iff for each violating edge set \( F \), \( H' \) is a feasible augmentation for \( F \).

The preceding observations allows us to focus on a fixed violating edge set \( F \), and ensuring that the algorithm outputs a set \( H' \) that is a feasible augmentation for \( F \) with high probability. We observe that the algorithm is oblivious to \( F \). Thus, if we obtain a high probability bound for a fixed \( F \), since there are \( O(n^{2(p+q)}) \) violating edge sets, we can use the union bound to argue that \( H' \) is feasible solution for \emph{all} violating edge sets. For the remainder of this section, until we do the final cost analysis, we work with a fixed violating edge set \( F \).
We call $\mathcal{M}^{-1}(F)$ denote the set of all tree edges corresponding to edges in $F$, i.e. $\mathcal{M}^{-1}(F) = \cup_{e \in F} \mathcal{M}^{-1}(e)$. We call $(T, \mathcal{M}, y)$ good with respect to $F$ if $y(\mathcal{M}^{-1}(F)) \leq \frac{1}{2}$; equivalently, $F$ blocks a flow of at most $\frac{1}{2}$ in $T$.

**Lemma 4.2.** For a violating edge set $F$, a randomly sampled Räcke tree $(T, \mathcal{M}, y)$ is good with respect to $F$ with probability at least $\frac{1}{2}$.

Proof. For each $e \in F$, $\tilde{x}_e = \frac{1}{4(p+q)\beta}$. Since the expected congestion of each edge is at most $\beta$, $E[\text{load}(e)] \leq \beta \tilde{x}_e \leq \frac{1}{4(p+q)}$ for each $e \in F$. Note that $y(\mathcal{M}^{-1}(F)) = \sum_{e \in F} \text{load}(e)$, hence by linearity of expectation, $E[y(\mathcal{M}^{-1}(F))] = \sum_{e \in F} E[\text{load}(e)] \leq |F|\frac{1}{4(p+q)} = \frac{1}{4}$. Applying Markov’s inequality to $y(\mathcal{M}^{-1}(F))$ proves the lemma.

Given the preceding lemma, a natural approach is to sample a good tree $T$ and hope that $T \setminus \mathcal{M}^{-1}(F)$ still has good flow between each terminal pair. However, since we rounded down all edges in LARGE, it is possible that $\mathcal{M}^{-1}(F)$ contains an edge whose removal would disconnect a terminal pair in $T$, even if $T$ is good. See [12] for a more detailed discussion and example.

We note that our goal is to find a set of edges $Q \subseteq E$ such that each shattered component is fully contained in $A \cup B$. Let $Q$ be the component containing $s_i$, and $Q_{t_i}$ be the component containing $t_i$. Note that $Q_{s_i}$ may be the same as $Q_{t_i}$ for some $i$, but if $F$ is a violating edge set then there is at least one $i$ such that $Q_{s_i} \neq Q_{t_i}$. Now, we define a Set Connectivity instance that is induced by $F$ and $T$. Consider two disjoint vertex subsets $A, B \subseteq V$. We say that $(A, B)$ partitions the set of shattered components if each shattered component $Q$ is fully contained in $A$ or fully contained in $B$. Formally let

$$Z_F = \{(A \cup Q_{s_i}, B \cup Q_{t_i}) : (A, B) \text{ partitions the shattered components, } i \in [k]\}.$$ 

In other words, $Z_F$ is set of all partitions of shattered components that separate some pair $(s_i, t_i)$. Since the leaves of $T$ are in one to one correspondence with $V$ we can view $Z_F$ as inducing a Set Connectivity instance in $T$; technically we need to consider the pairs $\{(\mathcal{M}^{-1}(A), \mathcal{M}^{-1}(B)) \mid (A, B) \in Z_F\}$; however, for simplicity we conflate the leaves of $T$ with $V$. We claim that it suffices to find a feasible solution that connects the pairs defined by $Z_F$ in the tree $T$.

**Claim 4.3.** Let $E' \subseteq \mathcal{T} \setminus \mathcal{M}^{-1}(F)$. Suppose there exists a path in $E' \subseteq T \setminus \mathcal{M}^{-1}(F)$ connecting $A$ to $B$ for all $(A, B) \in Z_F$. Then, there is an $s_i$-$t_i$ path for each $i \in [k]$ in $(\mathcal{M}(E') \cup H) \setminus F$.

Proof. Let $E' \subseteq T \setminus \mathcal{M}^{-1}(F)$ such that there is a path from $A$ to $B$ in $E'$ for each $(A, B) \in Z_F$. Assume for the sake of contradiction that $\exists i \in [k]$ such that $(s_i, t_i)$ are disconnected in $(\mathcal{M}(E') \cup H) \setminus F$. Then, there must be some cut $S$ such that $\delta_{(\mathcal{M}(E') \cup H) \setminus F}(S) = \emptyset$ and $|S \cap \{s_i, t_i\}| = 1$.  

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We observe that no component $Q \in Q_F$ can cross $S$ since each $Q$ is connected in $H \setminus F$. Assume without loss of generality that $s_i \in S$. Then, let $A = Q_{s_i} \cup \{Q \in Q_F : Q$ is shattered, $Q \subseteq S\}$, and $B = Q_{t_i} \cup \{Q \in Q_F : Q$ is shattered, $Q \subseteq S\}$. Clearly, $A \subseteq S$, $B \subseteq S$. Furthermore, $(A, B) \in Z_F$. By assumption, there is a path $P$ in $E'$ between $A$ and $B$. Since $E' \cap \mathcal{M}^{-1}(F) = \emptyset$, $\mathcal{M}(E')$ cannot contain any edges in $F$. Therefore, $\mathcal{M}(P)$ contains a path that crosses $S$ which implies that $|\delta_{\mathcal{M}(E')}(S)| = |\delta_{\mathcal{M}(E')} \cap F(S)| \geq 1$, contradicting the assumption on $S$. \hfill $\square$

We now argue that $(T, \mathcal{M}, y)$ routes sufficient flow for each pair in $Z_F$ without using the edges in $\mathcal{M}^{-1}(F)$; in other words $y$ is fractional solution (modulo a scaling factor) to the Set Connectivity instance $Z_F$ in the graph/forest $T \setminus \mathcal{M}^{-1}(F)$. We can then appeal to TreeRounding lemma to argue that it will connect the pairs in $Z_F$ without using any edges in $F$.

**Lemma 4.4.** Let $(A, B) \in Z_F$. Let $S \subset V_T$ such that $A \subseteq S$ and $B \subseteq V_T \setminus S$. Then $y(\delta_{T \setminus \mathcal{M}^{-1}(F)}(S)) \geq \frac{1}{4(p+q)\beta}$.

**Proof.** Let $S$ be a vertex set of $T$ that separates $A$ from $B$. First, suppose there exists a component $Q \in Q_F$ such that $Q$ crosses $S$, i.e. $S \cap Q \neq \emptyset$ and $\overline{S} \cap Q \neq \emptyset$. Since $(A, B)$ partitions the set of shattered components, $Q$ must be intact in $T$. Let $u$ be a leaf in $Q \cap S$ and $v$ be a leaf in $Q \cap \overline{S}$. Since $Q$ is intact in $T$ the unique path connecting $u$ to $v$ in $T$ crosses $S$ and let $e$ be an edge on this path that crosses $S$. It suffices to show that $y(e) \geq \frac{1}{4(p+q)\beta}$. This follow from properties of the Räcke tree. Since $u$ and $v$ are connected in $G'$ with a path using only edges in $\text{LARGE} \cup H$ each of which has a capacity of $\frac{1}{4(p+q)\beta}$, $u-v$ maxflow in $G'$ is at least $\frac{1}{4(p+q)\beta}$. From Corollary 2.4, for any tree $T$, the $u-v$ maxflow in $T$ with capacities $y$ must be at least $\frac{1}{4(p+q)\beta}$. This in particular implies that $y(e) \geq \frac{1}{4(p+q)\beta}$ for every edge $e$ on the unique path from $u$ to $v$ in $T$.

We can now restrict attention to the case that no connected component of $Q_F$ crosses $S$. Consider $S'$ be the set of leaves in $S$ and consider the cut $(S', V \setminus S')$ in $G$. It follows that $(S', V - S')$ partitions the connected components in $Q_F$ and $\delta_{H \setminus F}(S') = \emptyset$. Since $(A, B) \in Z_F$ there is a pair $(s_i, t_i)$ such that $Q_{s_i} \in S'$ and $Q_{t_i} \in V \setminus S'$. Thus $(S', V \setminus S')$ is a violated cut with $F$ as its witness. Since $x$ is a feasible solution to **Augment-LP** it follows that $x(\delta_{E \setminus H}(S')) \geq 1$. Recall that we assumed that LARGE $= \emptyset$, and hence all edges in $\delta_{E \setminus H}(S')$ are in SMALL. Therefore, $x(\delta_{E \setminus H}(S')) = \tilde{x}(\delta_{E \setminus H}(S')) \geq 1$.

The Räcke tree property guarantees that $y(\delta_T(S)) \geq \tilde{x}(\delta_{G'}(S')) \geq 1$ (via Corollary 2.4). We note that

$$y(\delta_{T \setminus \mathcal{M}^{-1}(F)}(S)) \geq y(\delta_T(S)) - y(\mathcal{M}^{-1}(F)) \geq 1 - 1/2 \geq 1/2.$$ 

where we used the fact that $y(\mathcal{M}^{-1}(F)) \leq 1/2$ since $T$ is good for $F$. Thus in both cases we verify the desired bound. \hfill $\square$

**Bounding $Z_F$:** A second crucial property is a bound on $|Z_F|$, the number of pairs in the Set Connectivity instance induced by $F$ and a good tree $T$ for $F$.

**Lemma 4.5.** For a good tree $T$, $|Z_F| \leq 2^{2(p+q)\beta}k$.

**Proof.** Let $\ell$ be the number of shattered components and let them be $Q_1, \ldots, Q_\ell$. For each $Q_i$ pick a pair of vertices $u_i, v_i$ that are in separate components of $T - \mathcal{M}^{-1}(F)$. Let $A = \{u_1, \ldots, u_\ell\}$ and $B = \{v_1, v_2, \ldots, v_\ell\}$. Since the paths connecting $u_i, v_i$ are in different connected components of $H \setminus F$, it follows that the $(A, B)$-maxflow in $H \setminus F$ is at least $\ell$. In the graph $G'$ obtained by scaling down the capacity of edges of $H$, the maxflow is at least $\frac{\ell}{4(p+q)\beta}$ which implies that it is at least this quantity in $T$. Since $T$ is good, the total decrease of flow can be at most $y(\mathcal{M}^{-1}(F)) \leq 1/2$. By construction
there is no flow between \( A \) and \( B \) in \( \mathcal{T} - \mathcal{M}^{-1}(F) \) which implies that \( \frac{\ell}{4(p+q)\beta} \leq 1/2 \Rightarrow \ell \leq 2(p+q)\beta \). Each pair in \( Z_F \) corresponds to a subset of shattered components and a demand pair \((s_i, t_i)\), and hence \( |Z_F| \leq 2^{\ell} k \leq 2^{2(p+q)\beta} k \).

4.2 Correctness and Cost

Now we analyze the correctness and cost of the algorithms output.

**Lemma 4.6.** Suppose \( \mathcal{T} \) is good for a violating edge set \( F \). Then after \( t \) rounds of TreeRounding with flow parameter \( \frac{1}{4(p+q)\beta} \), the probability that \( H' \) is not a feasible augmentation for \( F \) is at most \((1 - \phi)^t |Z_F| \leq 1/4\).

**Proof.** Suppose \( \mathcal{T} \) is good for \( F \). Let \((A, B) \in Z_F\). From Lemma 4.4 the flow for \((A, B)\) in \( \mathcal{T} - \mathcal{M}^{-1}(F) \) is at least \( \frac{1}{4(p+q)\beta} \). From Lemma 2.5, with probability at least \( \phi \), the pair \((A, B)\) is connected via a path in \( \mathcal{T} - \mathcal{M}^{-1}(F) \). If all pairs are connected, then via Claim 4.3, \( H' \) is a feasible augmentation for \( F \). Thus, \( H' \) is not a feasible augmentation if for some \((A, B) \in Z_F\) the TreeRounding does not succeed after \( t \) rounds. The probability of this, via the union bound over the pairs in \( Z_F \), is at most \((1 - \phi)^t |Z_F| \). From Lemma 4.5, \( |Z_F| \leq 2^{2(p+q)\beta} k \). Consider \( t = \frac{1}{\beta} \log(4k \cdot 2^{2\beta(p+q)}) = O((p+q) \log n) \), since \( \beta = O(\log n) \). Then, \((1 - \phi)^t |Z_F| = 2^{2(p+q)\beta} k (1 - \phi)^t \leq 2^{2(p+q)\beta} k e^{-\phi t} \leq \frac{1}{4} \).

**Lemma 4.7.** The algorithm outputs a solution \( H' \) such that \( H \cup H' \) is a feasible augmentation to the given instance with probability at least \( \frac{1}{2} \).

**Proof.** For a fixed \( F \) the probability that a sampled tree is good is at least \( 1/2 \). By Claim 4.6, conditioned on the sampled tree being good for \( F \), \( t \) iterations of TreeRounding fail to augment \( F \) with probability at most \( 1/4 \). Thus the probability that all \( t' \) iterations of sampling trees fail is \((1 - 3/8)^t'\). There are at most \( n^{2(p+q)} \) violating edge sets \( F \). Consider \( t' = \frac{8}{\beta} \log(2n^{2(p+q)}) = O((p+q) \log n) \). By applying the union bound over all violating edge sets \( F \), the probability of the algorithm failing is at most \( n^{2(p+q)}(1 - 3/8)^{t'} \leq n^{2(p+q)} e^{-3t'/8} \leq \frac{1}{2} \). Therefore, the output of the algorithm is a feasible augmentation for all violating edge sets with probability at least \( \frac{1}{2} \).

Now we analyze the expected cost of the edges output by the algorithm for augmentation with respect to \( \text{OPT}_{LP} \), the cost of the fractional solution.

**Lemma 4.8.** The total expected cost of the algorithm is \( O((p+q)^3 \log^7 n) \cdot \text{OPT}_{LP} \).

**Proof.** Fix an edge \( e \in \text{SMALL} \) with fractional value \( x_e \). Consider one outer iteration of the algorithm in which it picks a random tree \( \mathcal{T} \) from the Räcke tree distribution and then runs \( t \) iterations of TreeRounding with flow parameter \( \alpha = \frac{1}{4(p+q)\beta} \). Via Lemma 2.5, the probability of an edge \( f \in \mathcal{T} \) being chosen is at most \( O(\frac{1}{\alpha} h \log^2 n) g(f) \). Thus the expected cost for \( e \) for one round of TreeRounding is \( O(\frac{1}{\alpha} h \log^2 n) \sum_{f \in \mathcal{M}^{-1}(e)} g(f) = O(\frac{1}{\alpha} h \log^2 n) \text{load}(e) \). By the Räcke distribution property, \( E_{\mathcal{T}}[\text{load}(e)] \leq \beta x_e \). By linearity of expectation, since there are a total of \( t \cdot t' \) iterations of TreeRounding, the total expected cost is at most \( (t \cdot t') \cdot O(\frac{1}{\alpha} h \log^2 n) \beta \sum_{e \in E} c(e) x_e \). By the analysis in Section 3, \( h = O(\log n) \), and \( \beta = O(\log n) \). Substituting in the values of \( t \) and \( t' \) stated in Lemmas 4.6 and 4.7, the total expected cost is at most \( O((p+q)^3 \log^7 n) \cdot \text{OPT}_{LP} \).

Combining the correctness and cost analysis we obtain the following.

**Theorem 4.9.** There is a randomized \( O((p+q)^3 \log^7 n) \) approximation for the augmentation problem via Augment-LP.
Starting with a solution for \((p,0)\)-flex-connectivity, and using \(q\) augmentation iterations, we obtain an \(O(q(p+q)^3 \log^7 n)\)-approximate solution for the given instance of \((p,q)\)-Flex-SNDP, proving Theorem 1.1.

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