Computation of systemic risk measures: a mixed-integer linear programming approach

Çağın Ararat∗  Nurtai Meimanjanov†
March 19, 2019

Abstract

Systemic risk is concerned with the instability of a financial system whose members are interdependent in the sense that the failure of a few institutions may trigger a chain of defaults throughout the system. Recently, several systemic risk measures are proposed in the literature that are used to determine capital requirements for the members subject to joint risk considerations. We address the problem of computing systemic risk measures for systems with sophisticated clearing mechanisms. In particular, we consider the Eisenberg-Noe network model and the Rogers-Veraart network model, where the former one is extended to the case where operating cash flows in the system are unrestricted in sign. We propose novel mixed-integer linear programming problems that can be used to compute clearing vectors for these models. Due to the binary variables in these problems, the corresponding (set-valued) systemic risk measures fail to have convex values in general. We associate nonconvex vector optimization problems to these systemic risk measures and solve them by a recent nonconvex variant of Benson’s algorithm which requires solving two types of scalar optimization problems. We provide a detailed analysis of the theoretical features of these problems for the extended Eisenberg-Noe and Rogers-Veraart models. We test the proposed formulations on computational examples and perform sensitivity analyses with respect to some model-specific and structural parameters.

Keywords and phrases: systemic risk measure, aggregation function, set-valued risk measure, systemic risk, capital requirement, Eisenberg-Noe model, Rogers-Veraart model, Benson’s algorithm, nonconvex vector optimization.

Mathematics Subject Classification (2010): 26E25, 90C11, 90C29, 91B30.

1 Introduction

Financial contagion is usually associated with a chain of failures in a financial system triggered by external correlated shocks as well as direct or indirect interdependencies among the members of the system leading to, from an economic point of view, undesirable consequences such as financial crisis, necessity for bailout loans, economic regression, rise in national debt and so on. A good example is a bank run, when a large number of holders withdraw their money from a bank due to panic or decrease in confidence in the bank, causing insolvency of the bank. In turn, the bank may call its claims from the other banks, decreasing confidence in them and causing new bank runs. Being unable to meet their liabilities, some of the banks may become bankrupt and, thus, aggravate the contagion even further. Unlike the usual notion of risk, when it is associated with a

∗Bilkent University, Department of Industrial Engineering, Ankara, Turkey, cararat@bilkent.edu.tr.
†Bilkent University, Department of Industrial Engineering, Ankara, Turkey, nurtai@bilkent.edu.tr.
single entity, systemic risk is related to the strength of an entire financial system against financial contagions.

In this paper, we consider financial systems in which members have direct links to each other through contractual liabilities. When the members realize their operating cash flows, the actual interbank payments are determined through a clearing procedure. As an example of such systems, Eisenberg and Noe (2001) models a financial system as a static directed network of banks where interbank liabilities are attached to the arcs. Assuming a positive operating cash flow for each bank, the paper develops two approaches to calculate a clearing vector, that is, a vector of payments to meet interbank liabilities. The first is a simple algorithm, called the fictitious default algorithm, which gradually calculates a clearing vector by finitely many updates. The second is a laconic mathematical programming problem with linear constraints determined by the liabilities, the operating cash flows, and an arbitrary strictly increasing objective function. In particular, one can choose a linear objective function so that a clearing vector is calculated as an optimal solution of a linear programming problem.

The former algorithmic approach is preferred by most of the scholars that work in network models of systemic risk. Suzuki (2002) introduces a similar approach to evaluate clearing vectors as in Eisenberg and Noe (2001). In addition, Suzuki (2002) considers cross-holdings of stock among members of a financial system. Cifuentes et al. (2005) investigates systemic risk in terms of liquidity of institutions in a financial system and considers unsteadiness of asset prices as well. Unlike earlier works, the network model in this paper differentiates between liquid and illiquid assets. Elsinger (2009) extends the work in Eisenberg and Noe (2001) by introducing a cross-holdings structure similar to the one in Suzuki (2002). Additionally, Elsinger (2009) relaxes the positivity assumption on operating cash flows of the members in a system and studies the model by imposing some seniority assumptions. Rogers and Veraart (2013) introduces default costs to the model in Eisenberg and Noe (2001). In addition, one of the main focuses in Rogers and Veraart (2013) is devoted to the investigation of the necessity of bailing out procedures for the defaulting institutions. It is shown that under strictly positive default costs, it might be beneficial for some of the solvent institutions to take over insolvent institutions. Weber and Weske (2017) integrates many of the factors that contribute to systemic risk into one network model. These factors include cross-holdings introduced in Suzuki (2002) and Elsinger (2009), file sales investigated in Cifuentes et al. (2005), and bankruptcy costs viewed in Elsinger (2009) and Rogers and Veraart (2013). Weber and Weske (2017) takes the model in Eisenberg and Noe (2001) as a base and introduces all the above factors simultaneously, making it more realistic and complex at the same time. For a detailed review of network models of systemic risk, the reader is referred to the survey Kabanov et al. (2017), which focuses on the existence and uniqueness of clearing vectors in the models mentioned above as well as their calculations by certain variations of the fictitious default algorithm in Eisenberg and Noe (2001). However, none of the above works builds on the second mathematical programming approach of Eisenberg and Noe (2001).

On the other hand, the operating cash flows of the members of a network are typically subject to uncertainty due to correlated risk factors. Hence, these cash flows can be modeled as one possible realization of a random vector with possibly correlated components. Then, the resulting clearing vector is a deterministic function of the operating cash flow random vector, where the deterministic function is defined through the underlying clearing mechanism. Based on the random clearing vector, one can define various systemic risk measures to calculate the necessary capital allocations for the members of the network in order to control some (nonlinear) averages over different scenarios. This is the main focus of a recent stream of research started with Chen et al. (2013). Using the clearing mechanism, one defines a random aggregate quantity associated to the clearing vector, such as the total debt paid in the system or the total equity made by all members
as a result of clearing. This aggregate quantity can be seen as a deterministic and scalar function, called the aggregation function, of the operating cash flow vector. In Chen et al. (2013), a systemic risk measure is defined as a scalar functional of the operating cash flow vector that measures the risk of the random aggregate quantity through a convex risk measure (Föllmer and Schied, 2011, Chapter 4) such as negative expected value, average value-at-risk or entropic risk measure.

The value of the systemic risk measure in Chen et al. (2013) can be seen as the total capital requirement for the system to keep the risk of the aggregate quantity at an acceptable level. However, since the total capital is used only after the shock is aggregated, the allocation of this total back into the members of the system remains as a question to be addressed by an additional procedure. To that end, set-valued and scalar systemic risk measures that are considered “sensitive” to capital levels are proposed in Feinstein et al. (2017) and Biagini et al. (2018), respectively. These systemic risk measures look for deterministic capital allocation vectors that are directly used to augment the random operating cash flow vector. Hence, the new augmented cash flow vector is aggregated and the risk of the resulting random aggregate quantity is controlled by a convex risk measure as in Chen et al. (2013). In particular, the value of the set-valued systemic risk measure in Feinstein et al. (2017) is the set of all “feasible” capital allocation vectors, which addresses the measurement and allocation of systemic risk as a joint problem.

The sensitive systemic risk measures studied in Feinstein et al. (2017) and Biagini et al. (2018) have convenient theoretical properties when the underlying aggregation function is simple enough. In Ararat and Rudloff (2016), assuming a monotone and concave aggregation function, it has been shown that the set-valued sensitive systemic risk measure is a convex set-valued risk measure in the sense of Hamel et al. (2011) and dual representations are obtained in terms of the conjugate function of the aggregation function. In particular, the aggregation function for the Eisenberg-Noe model, assuming positive operating cash flows as in the original formulation in Eisenberg and Noe (2001), is monotone and concave, and an explicit dual representation is obtained for the corresponding systemic risk measure of this model.

In this paper, we are concerned with the computation of a sensitive systemic risk measure discussed above. We relate the value of this systemic risk measure to a vector (multiobjective) optimization problem whose “efficient frontier” corresponds to the boundary of the systemic risk measure. The vector optimization problem has a risk constraint written in terms of the aggregation function. The main challenge in solving this problem is that the aggregation function needs to be evaluated for every scenario of the underlying probability space as well as for every choice of the capital allocation vector, which is the decision variable of the optimization problem. For the standard Eisenberg-Noe model, thanks to the linear programming characterization of the clearing vectors, one can formulate the aggregation function in terms of a linear programming problem parametrized by the scenario and the capital allocation vector. Hence, the ultimate vector optimization problem can be seen as a nested optimization problem.

We focus particularly on models beyond the standard Eisenberg-Noe framework with positive operating cash flows. In particular, we consider an extension of the Eisenberg-Noe model by relaxing the positivity assumption as well as the Rogers-Veraart model with default costs. It turns out that both models have a common type of singularity that can be formulated in terms of binary variables, a novel feature studied in this paper. One of our main contributions is to develop mixed-integer linear programming problems that calculate clearing vectors in these models. We fix the objective functions of these optimization problems in such a way that the optimal values give the total debts paid at clearing in the corresponding models. Hence, we calculate the aggregation functions as the optimal values of these optimization problems.

The existence of binary variables in optimization problems results in lack of concavity for the corresponding aggregation functions. Consequently, the sensitive systemic risk measures for the
two models do not possess the nice theoretical features such as convexity and dual representations studied in the earlier papers on systemic risk measures. Indeed, we even have that the values of these systemic risk measures fail to be convex sets, in general. Therefore, one of our fundamental observations is that binary variables and the accompanying lack of concavity/convexity show up naturally at the cost of using more sophisticated aggregation mechanisms beyond the standard Eisenberg-Noe framework.

Going back to the computations of systemic risk measures, the associated vector optimization problems are consequently nonconvex, in general. We use the Benson-type algorithm for such problems developed recently in Nobakhtian and Shafiei (2017). The algorithm (as well as its original version in Benson (1998)) has two “blackbox” subroutines for the following two scalar optimization problems: weighted-sum scalarization problem and the problem of calculating the minimum step-length to hit the efficient frontier from an outside point. It should be noted that the algorithm in Nobakhtian and Shafiei (2017) assumes that these scalar problems are solvable by some unspecified methods and the convergence of the algorithm is guaranteed based on this assumption. In our context, we formulate these problems as mixed-integer linear programming problems for a generic aggregation function that is formulated in terms of a mixed-integer linear programming problem. We address further questions regarding the the finiteness of the optimal values and existence of feasible/optimal solutions separately for each of the extended Eisenberg-Noe model and the Rogers-Veraart model.

We perform a detailed computational study for both models as well as sensitivity analyses with respect to some model parameters such as the default cost parameters in the Rogers-Veraart model, the threshold level used in the risk constraint, and also some parameters determining the interconnectedness of the network.

The rest of this paper is organized as follows. We study the Eisenberg-Noe and Rogers-Veraart network models in detail together with the mathematical programming characterizations of clearing vectors in Section 2. In Section 3 we study the sensitive systemic risk measures and their associated nonconvex vector optimization problems. The proofs of some results in Sections 2, 3 are deferred to Appendices A, B, respectively. We present the computational results in Section 4.

2 Network models of systemic risk

In this section, after reviewing the original Eisenberg-Noe network model in Section 2.1 we propose a seniority-based extension of this model by allowing signed operating cash flows in Section 2.2 and provide a novel mixed-integer linear programming (MILP) formulations of clearing vectors in Theorem 2.7. Then, in Section 2.3 we consider the Rogers-Veraart network model and provide a novel MILP formulation of clearing vectors in Theorem 2.17.

Let us introduce the related notation. Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. Given $a, b \in \mathbb{R}$, we write $a \land b = \min \{a, b\}$, $a \lor b = \max \{a, b\}$, $a^+ = 0 \lor a$, and $a^- = 0 \lor (-a)$. Similarly, given $a = (a_1, \ldots, a_n)^T, b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$, we write

$$a \land b = (a_1 \land b_1, \ldots, a_n \land b_n)^T, \quad a \lor b = (a_1 \lor b_1, \ldots, a_n \lor b_n)^T$$

as well as $a^+ = 0 \lor a$, and $a^- = 0 \lor (-a)$, where $0 = (0, \ldots, 0)^T \in \mathbb{R}^n$. We sometimes use $1 = (1, \ldots, 1)^T \in \mathbb{R}^n$ as well. The vector $a \odot b = (a_1b_1, \ldots, a_nb_n)^T$ denotes the Hadamard product of $a, b$. We write $a \leq b$ if and only if $a_i \leq b_i$ for each $i \in \{1, \ldots, n\}$. In this case, we also define the rectangle $[a, b] = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$. Using $\leq$ on $\mathbb{R}^n$, we define $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid 0 \leq x\}$. 


whose elements are said to be positive. Finally,
\[ \|a\|_\infty = \max_{i \in \{1, \ldots, n\}} |a_i| \]
is the \( \ell_\infty \)-norm of \( a \).

### 2.1 Eisenberg-Noe network model

In this section, the original Eisenberg-Noe network model in [Eisenberg and Noe (2001)] and its corresponding aggregation function are provided for completeness.

**Definition 2.1.** A quadruple \((\mathcal{N}, \pi, \bar{p}, x)\) is called an Eisenberg-Noe network if \(\mathcal{N} = \{1, \ldots, n\}\) for some \( n \in \mathbb{N} \), \( \pi = (\pi_{ij})_{i,j \in \mathcal{N}} \in \mathbb{R}^{n \times n}_+ \) is a stochastic matrix with \( \pi_{ii} = 0 \) and \( \sum_{j=1}^n \pi_{ji} < n \) for each \( i \in \mathcal{N} \), \( \bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)^T \in \mathbb{R}^n_+ \), and \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_+ \).

In Definition 2.1, \( \mathcal{N} \) is the index set of nodes in a network that represents a financial system of \( n \) institutions. For every \( i \in \mathcal{N} \), \( \bar{p}_i > 0 \) denotes the total amount of liabilities of node \( i \). We call \( \bar{p} \) the total obligation vector.

For every \( i, j \in \mathcal{N} \) such that \( i \neq j \), \( \pi_{ij} > 0 \) denotes the fraction of the total liability of node \( i \) owed to node \( j \). We call \( \pi \) the relative liabilities matrix. For every \( i \in \mathcal{N} \), the assumption \( \pi_{ii} = 0 \) means that node \( i \) cannot have liabilities to itself. By \( \sum_{j=1}^n \pi_{ji} < n \) for every \( i \in \mathcal{N} \), we assume that no node owns all the claims in the network. Note that, given \( \bar{p} \) and \( \pi \), for every \( i, j \in \mathcal{N} \), the nominal liability \( l_{ij} \) of node \( i \) to node \( j \) can be calculated as \( l_{ij} = \pi_{ij} \bar{p}_i \).

For each \( i \in \mathcal{N} \), \( x_i \geq 0 \) denotes the operating cash flow of node \( i \). We call \( x \) the operating cash flow vector.

Let \((\mathcal{N}, \pi, \bar{p}, x)\) be an Eisenberg-Noe network. For each \( i \in \mathcal{N} \), let \( p_i \geq 0 \) be the sum of all payments made by node \( i \) to the other nodes in the network. Then, \( p = (p_1, \ldots, p_n)^T \in \mathbb{R}^n_+ \) is called a payment vector.

**Definition 2.2.** A vector \( p \in [0, \bar{p}] \) is called a clearing vector for \((\mathcal{N}, \pi, \bar{p}, x)\) if it satisfies the following properties:

- **Limited liability:** for each \( i \in \mathcal{N} \), \( p_i \leq \sum_{j=1}^n \pi_{ji} \bar{p}_j + x_i \), which implies that node \( i \) cannot pay more than it has.

- **Absolute priority:** for each \( i \in \mathcal{N} \), either \( p_i = \bar{p}_i \) or \( p_i = \sum_{j=1}^n \pi_{ji} \bar{p}_j + x_i \), which implies that node \( i \) either meets its obligations in full or else it defaults by paying as much as it has.

Let \( \Phi_{\text{EN}^+} : [0, \bar{p}] \to [0, \bar{p}] \) be defined by
\[
\Phi_{\text{EN}^+} (p) := \left( \pi^T p + x \right) \wedge \bar{p}.
\]

It is shown in [Eisenberg and Noe (2001)] that a clearing vector \( p \) for \((\mathcal{N}, \pi, \bar{p}, x)\) is a fixed point of \( \Phi_{\text{EN}^+} \), that is, \( \Phi_{\text{EN}^+} (p) = p \).

Next, we recall the programming characterization of clearing vectors shown in [Eisenberg and Noe (2001)], which is the basis of our generalizations to follow. We say that a function \( f : \mathbb{R}^n \to \mathbb{R} \) is strictly increasing if \( a \leq b \) and \( a \neq b \) imply \( f(a) < f(b) \) for every \( a, b \in \mathbb{R}^n \).
Proposition 2.3. (Eisenberg and Noe, 2001, Lemma 4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly increasing function. Consider the following optimization problem with linear constraints:

$$\begin{align*}
\text{max} & \quad f(p) \\
\text{s.t.} & \quad p \leq \pi^T p + x, \\
& \quad p \in [0, \bar{p}].
\end{align*}$$

(2.2)

If $p \in \mathbb{R}_+^n$ is an optimal solution to this optimization problem, then it is a clearing vector for $(\mathcal{N}, \pi, \bar{p}, x)$.

Each member in a network has its impact on economy. As in Chen et al. (2013), Biagini et al. (2018), Feinstein et al. (2017), Ararat and Rudloff (2016), we use aggregation functions to summarize these individual effects and provide a total impact of the network on economy. They play a significant role in evaluating systemic risks and in the computation of systemic risk measures. The aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ for the Eisenberg-Noe network $(\mathcal{N}, \pi, \bar{p}, x)$ is defined as

$$\Lambda(x) := \sup \left\{ f(p) \mid p \leq \pi^T p + x, \ p \in [0, \bar{p}] \right\},$$

(2.3)

where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly increasing function, namely, $\Lambda(x)$ is the optimal value of the problem in (2.2).

2.2 Signed Eisenberg-Noe network model

In the original Eisenberg-Noe network model, it is assumed that the operating cash flow vector is positive. In reality, however, it may happen that an institution has liabilities to external entities not modeled as part of the network resulting in a negative operating cash flow or a positive operating cost.

Definition 2.4. A quadruple $(\mathcal{N}, \pi, \bar{p}, x)$ is called a signed Eisenberg-Noe network if $\mathcal{N}$, $\pi$ and $\bar{p}$ are as in Definition 2.1, and $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

Note that Definition 2.4 removes the positivity assumption on the operating cash flow vector $x$. Our aim is to provide a new definition of clearing vector by extending Definition 2.2 with an additional seniority assumption for negative operating cash flows. Based on this definition, we prove a fixed-point and a mathematical programming characterization of clearing vectors. Finally, we introduce an associated aggregation function through a MILP problem.

Let $(\mathcal{N}, \pi, \bar{p}, x)$ be a signed Eisenberg-Noe network. We assume that the nodes that have obligations outside the network, that is, the nodes with negative operating cash flows have to meet these obligations first, and if they do not default in this “first round,” then they should meet their obligations to the other nodes inside the network. At this “second round”, as in the original Eisenberg-Noe network model, they either meet their obligations to the other nodes in full or pay as much as they have at hand and default. This motivates the following definition.

Definition 2.5. A vector $p \in [0, \bar{p}]$ is called a clearing vector for $(\mathcal{N}, \pi, \bar{p}, x)$ if it satisfies the following properties:

- **Immediate default:** for each $i \in \mathcal{N}$, if $\sum_{j=1}^n \pi_{ji}p_j + x_i \leq 0$, then $p_i = 0$.
- **Limited liability:** for each $i \in \mathcal{N}$, if $\sum_{j=1}^n \pi_{ji}p_j + x_i > 0$, then $p_i \leq \sum_{j=1}^n \pi_{ji}p_j + x_i$, which implies that if node $i$ has a strictly positive operating cash flow, then it cannot pay more than it has.
• Absolute priority: for each \( i \in \mathcal{N} \), if \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0 \), then either \( p_i = \bar{p}_i \) or \( p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \), which implies that if node \( i \) has a strictly positive operating cash flow, then it either meets its obligations in full or else it defaults by paying as much as it has.

Let \( \Phi^{\text{EN}} : [0, \vec{p}] \to [0, \vec{p}] \) be defined by

\[
\Phi^{\text{EN}}(p) := (\vec{p} \land (\pi^T p + x))^+ ,
\]

or more explicitly, for each \( i \in \mathcal{N} \),

\[
\Phi^{\text{EN}}_i(p) = \begin{cases} 0 & \text{if } \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0, \\ \bar{p}_i & \text{if } 0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i, \\ \bar{p}_i & \text{if } \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i. \end{cases}
\]

Observe that, if \( x \in \mathbb{R}_+^n \), then \( \Phi^{\text{EN}} \) coincides with the function \( \Phi^{\text{EN}+} \) in (2.1) defined for the original Eisenberg-Noe network model.

We establish the fixed point characterization of clearing vectors next.

**Proposition 2.6.** A vector \( p \in [0, \vec{p}] \) is a clearing vector for \((\mathcal{N}, \pi, \vec{p}, x)\) if and only if it is a fixed point of \( \Phi^{\text{EN}} \).

**Proof.** To prove the “only if” part, let \( p = (p_1, \ldots, p_n)^T \in [0, \vec{p}] \) be a clearing vector. To show that \( p \) is a fixed point of \( \Phi^{\text{EN}} \), let \( i \in \mathcal{N} \).

If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0 \), then \( p_i = 0 \), by immediate default, and \( \Phi^{\text{EN}}_i(p) = 0 \), by (2.4). Hence, \( \Phi^{\text{EN}}_i(p) = p_i. \)

If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0 \), then, by absolute priority, either \( p_i = \bar{p}_i \) or \( p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \).

If \( p_i = \bar{p}_i \), then, by limited liability, \( \bar{p}_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i \) and, thus, by (2.4), \( \Phi^{\text{EN}}_i(p) = \bar{p}_i \).

Hence, \( \Phi^{\text{EN}}_i(p) = p_i \). On the other hand, if \( p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i < \bar{p}_i \), then, by (2.4), \( \Phi^{\text{EN}}_i(p) = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \). Hence, again \( \Phi^{\text{EN}}_i(p) = p_i \). Thus, \( p \) is a fixed point of \( \Phi^{\text{EN}} \).

To prove the “if” part, let \( p = (p_1, \ldots, p_n)^T \) be a fixed point of \( \Phi^{\text{EN}} \). In other words, for every \( i \in \mathcal{N} \), \( \Phi^{\text{EN}}_i(p) = p_i \). To show that \( p \) is a clearing vector, let \( i \in \mathcal{N} \).

If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0 \), then \( \Phi^{\text{EN}}_i(p) = p_i = 0 \), by (2.4). Hence, immediate default holds.

If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0 \), then \( \Phi^{\text{EN}}_i(p) = p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i \), by (2.4). Hence, limited liability holds.

Now assume \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0 \). If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i \), then \( \Phi^{\text{EN}}_i(p) = p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \).

If \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i \), then \( \Phi^{\text{EN}}_i(p) = p_i = \bar{p}_i \), by (2.4). Hence, absolute priority holds as well. Hence, \( p \) is a clearing vector. \( \square \)

The next theorem is the main result of Section 2.2. It extends Proposition 2.3 for the signed Eisenberg-Noe network model by showing that a clearing vector can be calculated as an optimal solution of a certain MILP. Hence, relaxing the positivity assumption on the operating cash flow vector is at the cost of using binary variables in the mathematical programming characterization of clearing vectors, hence, adding a discrete feature to the originally continuous optimization problem.

**Theorem 2.7.** Let \( \Lambda^{\text{EN}} : \mathbb{R}^n \to \mathbb{R} \) be a MILP aggregation function defined by

\[
\Lambda^{\text{EN}}(y) := \sup \left\{ f(p) \mid p \leq \left[ \pi^T p + y + M(1-s) \right] \land (\vec{p} \circ s), \pi^T p + y \leq Ms, p \in [0, \vec{p}], s \in \{0,1\}^n \right\},
\]

where \( \pi^T p + y \leq Ms, p \in [0, \vec{p}] \).
where \( f : \mathbb{R}^n \to \mathbb{R} \) is a strictly increasing linear function and \( M = n \| \bar{p} \|_{\infty} + \| x \|_{\infty} \). If \((p, s)\) is an optimal solution to MILP for \( \Lambda^{EN}(x) \), then \( p \) is a clearing vector for \((N, \pi, \bar{p}, x)\).

**Remark 2.8.** \( \Lambda^{EN} \) fails to be concave in general.

Observe that \( \Lambda^{EN}(x) \) can be written more explicitly as

\[
\text{maximize} \quad f(p) \quad \text{subject to} \quad p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i + M (1 - s_i), \quad i \in \mathcal{N},
\]

\[
p_i \leq \bar{p}_i s_i, \quad i \in \mathcal{N},
\]

\[
\sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq M s_i, \quad i \in \mathcal{N},
\]

\[
0 \leq p_i \leq \bar{p}_i, \quad i \in \mathcal{N},
\]

\[
s_i \in \{0, 1\}, \quad i \in \mathcal{N}.
\]

Let \( u = (u_1, \ldots, u_n)^T \in \{0, 1\}^n \) be a binary vector, where \( u_i = 0 \) if \( x_i < 0 \), and \( u_i = 1 \) if \( x_i \geq 0 \), for each \( i \in \mathcal{N} \). Then \((p, s) = (0, u) \in \mathbb{R}^n \times \mathbb{Z}^n \) is a feasible solution to the MILP in (2.7). Moreover, since \( f \) is a bounded function on the rectangle \([0, \bar{p}] \subseteq \mathbb{R}^n\), by \cite{Meyer1974}, Theorem 2.1), the MILP has an optimal solution. Observe that, by Theorem 2.7, the existence of an optimal solution to the MILP in (2.7) proves the existence of a clearing vector for the network \((N, \pi, \bar{p}, x)\).

**Remark 2.9.** In Theorem 2.7, \( M = n \| \bar{p} \|_{\infty} + \| x \|_{\infty} \) is taken to ensure the feasibility of the constraint (2.10). In other words, it is enough to choose \( M \) such that \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq M \), for each \( i \in \mathcal{N} \) and for every \( p \in [0, \bar{p}] \). Furthermore, for each \( i \in \mathcal{N} \) and for every \( p \in [0, \bar{p}] \), since \( \sum_{j=1}^{n} \pi_{ji} < n \), it holds \( \sum_{j=1}^{n} \pi_{ji} p_j < n \| \bar{p} \|_{\infty} \). Hence, \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq n \| \bar{p} \|_{\infty} + \| x \|_{\infty} = M \).

**Remark 2.10.** The linearity of \( f \) is not a necessary condition for Theorem 2.7 to hold.

The proof of Theorem 2.7 is based on the following lemma.

**Lemma 2.11.** Let \((p, s)\) be an optimal solution to the MILP for \( \Lambda^{EN}(x) \). Let \( i \in \mathcal{N} \) such that \( 0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \). Then, \( p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} \).

The proof of Lemma 2.11 can be found in Appendix A.1.

**Proof of Theorem 2.7.** Let \((p, s)\) be an optimal solution to the MILP for \( \Lambda^{EN}(x) \). To prove that \( p \) is a clearing vector, by Proposition 2.6, we equivalently show that \( \Phi^{EN}(p) = p \). Let \( i \in \mathcal{N} \). Recalling (2.5), we consider three cases:

1. \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0 \).
2. \( 0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i \).
3. \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i \).

(1) Assume that \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0 \). Then, by (2.4), \( \Phi^{EN}_i(p) = 0 \). By the arguments from the proof of the Lemma 2.11 for this case, \( p_i = 0 \). Hence, \( p_i = 0 = \Phi^{EN}_i(p) \).

(2) Assume that \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i \). Then, by (2.4), \( \Phi^{EN}_i(p) = \bar{p}_i \). By the arguments from the proof of the Lemma 2.11 for this case, \( p_i = \bar{p}_i \). Hence, \( p_i = \bar{p}_i = \Phi^{EN}_i(p) \).

(3) Assume that \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i \). Then, by (2.4), \( \Phi^{EN}_i(p) = \bar{p}_i \). By the arguments from the proof of the Lemma 2.11 for this case, \( p_i = \bar{p}_i \). Hence, \( p_i = \bar{p}_i = \Phi^{EN}_i(p) \).
(2) Assume that \(0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i\). Then, by (2.4), \(\Phi_i^{EN}(p) = \sum_{j=1}^{n} \pi_{ji} p_j + x_i\). Since \(0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i\), by Lemma 2.11

\[
p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} = \sum_{j=1}^{n} \pi_{ji} p_j + x_i.
\]

Hence, \(p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i = \Phi_i^{EN}(p)\).

(3) Assume \(\sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i\). Then, by (2.4), \(\Phi_i^{EN}(p) = \bar{p}_i\). Since \(\sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i > 0\), again by Lemma 2.11

\[
p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} = \bar{p}_i.
\]

Hence, \(p_i = \bar{p}_i = \Phi_i^{EN}(p)\).

Therefore, \(p\) is a clearing vector for \((N, \pi, \bar{p}, x)\). \(\square\)

Remark 2.12. Instead of the seniority-based approach developed above, a naive approach would be to introduce an additional node and consider negative operational cash flows of the nodes as liabilities to this additional node, which itself has neither obligations nor an operating cash flow, as suggested in [Eisenberg and Noe (2001)]. This approach is valid for the fictitious default algorithm described in [Eisenberg and Noe (2001)] and this way a clearing vector for the original network can be found. However, the modified network lacks a solid interpretation in terms of the original network since the relative liabilities matrix of the new network depends on the operational cash flow vector. Hence, we do not follow this route here.

2.3 Rogers-Veraart network model

In [Rogers and Veraart (2013)], the original Eisenberg-Noe network model is extended by including default costs. It is assumed that a defaulting node is not able to use all of its liquid assets to meet its obligations. Unlike the Eisenberg-Noe model, the possibility of a mathematical programming formulation for clearing vectors seems to be an open problem for the Rogers-Veraart. We fill up this gap by proposing a MILP whose optimal solution includes a clearing vector for the Rogers-Veraart network model. Based on this characterization, we define an aggregation function and provide its relationship to the network model. Finally, inspired by Definition 2.2, we propose a weaker definition of a clearing vector for the Rogers-Veraart network model.

Definition 2.13. A sextuple \((N, \pi, \bar{p}, x, \alpha, \beta)\) is called a Rogers-Veraart network if \(N = \{1, \ldots, n\}\) for some \(n \in \mathbb{N}\), \(\pi = (\pi_{ij})_{i,j \in N} \in \mathbb{R}_{+}^{n \times n}\) is a stochastic matrix with \(\pi_{ii} = 0\) and \(\sum_{j=1}^{n} \pi_{ji} < n\) for each \(i \in N\), \(\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)^{T} \in \mathbb{R}_{+}^{n}\), \(x = (x_1, \ldots, x_n)^{T} \in \mathbb{R}_{+}^{n}\) and \(\alpha, \beta \in (0, 1]\).

As in Definition 2.1, \(N\) is the set of nodes in a network with \(n\) institutions, \(\bar{p}\) is the total obligation vector, \(\pi\) is the matrix of relative liabilities and \(x\) is the operating cash flow vector. It is assumed that a defaulting node may not be able to use all of its liquid assets to meet its obligations. For this purpose, we use \(\alpha\) as the fraction of the operating cash flow and \(\beta\) as the fraction of the cash inflow from other nodes that can be used by a defaulting node to meet its obligations.

Let \((N, \pi, \bar{p}, x, \alpha, \beta)\) be a Rogers-Veraart network. For each \(i \in N\), let \(p_i \geq 0\) be the sum of all payments made by node \(i\) to the other nodes in the network. Then, \(p = (p_1, \ldots, p_n)^{T} \in \mathbb{R}_{+}^{n}\) is called a payment vector.
Motivated by Definition 2.2 of a clearing vector for an Eisenberg-Noe network, we suggest the following similar definition of a clearing vector for the Rogers-Veraart network \((N, \pi, \bar{\pi}, x, \alpha, \beta)\).

**Definition 2.14.** A vector \(p \in [0, \bar{\pi}]\) is called a clearing vector for \((N, \pi, \bar{\pi}, x, \alpha, \beta)\) if it satisfies the following properties:

- **Limited liability:** for each \(i \in N\), \(p_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j\), which implies that node \(i\) cannot pay more than it has.

- **Absolute priority:** for each \(i \in N\), either \(p_i = \bar{p}_i\) or \(p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j\), which implies that node \(i\) either has to meet its obligations in full or else it defaults by paying as much as it can.

Let \(\Phi_{RV}^+ : [0, \bar{\pi}] \to [0, \bar{\pi}]\) be defined by

\[
\Phi_{RV}^+ (p) := \begin{cases} 
\bar{p}_i & \text{if } \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j, \\
\alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j & \text{if } \bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j,
\end{cases}
\]

for each \(i \in N\).

Observe that, if \(\alpha = 1\) and \(\beta = 1\), then the function \(\Phi_{RV}^+\) becomes the usual \(\Phi_{EN}^+\) in (2.1) from the original Eisenberg-Noe network model.

**Proposition 2.15.** A fixed point \(p \in [0, \bar{\pi}]\) of \(\Phi_{RV}^+\) is a clearing vector for \((N, \pi, \bar{\pi}, x, \alpha, \beta)\).

*Proof.* Let \(p = (p_1, \ldots, p_n)^T\) be a fixed point of \(\Phi_{RV}^+\). To show that \(p\) is a clearing vector for \((N, \pi, \bar{\pi}, x, \alpha, \beta)\), let \(i \in N\).

If \(\bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j\), then \(\Phi_{RV}^+ (p) = \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j\), and if \(\bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j\), then \(\Phi_{RV}^+ (p) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j = p_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j\), by the definition of \(\Phi_{RV}^+\) in (2.13) and since \(p\) is a fixed point of \(\Phi_{RV}^+\). Hence, both limited liability and absolute priority in Definition 2.14 hold. Hence, \(p\) is a clearing vector for \((N, \pi, \bar{\pi}, x, \alpha, \beta)\).

**Remark 2.16.** The converse of Proposition 2.15 fails to hold in general. Here is a counterexample. Consider a Rogers-Veraart network \((N, \pi, \bar{\pi}, x, \alpha, \beta)\) and payment vector \(p\), where \(N = \{1, 2\}\),

\[
\pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{\pi} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}, \quad x = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad p = \begin{bmatrix} 20 \\ 15 \end{bmatrix},
\]

\(\alpha = 0.5, \beta = 0.5\). According to Definition 2.14, \(p\) is a clearing vector for \((N, \pi, \bar{\pi}, x, \alpha, \beta)\) since it satisfies absolute priority and limited liability. However, by (2.13), \(\Phi_{RV}^+ (p) = 25 > p_2 = 15\). Hence, \(p\) is not a fixed point of \(\Phi_{RV}^+\).

The next theorem is the main result of Section 2.3. In the spirit of Theorem 2.7 for a signed Eisenberg-Noe network, it provides a MILP characterization of clearing vectors for the Rogers-Veraart network \((N, \pi, \bar{\pi}, x, \alpha, \beta)\).

**Theorem 2.17.** Let \(\Lambda_{RV}^+ : \mathbb{R}^n \to \mathbb{R}\) be a MILP aggregation function defined by

\[
\Lambda_{RV}^+ (y) := \begin{cases} 
\sup \left\{ f(p) \mid p \leq \alpha y + \beta \pi^T p + \bar{\pi} \odot s, \quad \bar{\pi} \odot s \leq y + \pi^T p, \quad p \in [0, \bar{\pi}], \quad s \in \{0, 1\}^n \right\}, & \text{if } y \in \mathbb{R}^n_+, \\
-\infty, & \text{if } y \notin \mathbb{R}^n_+.
\end{cases}
\]
where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a strictly increasing linear function. If \((p, s)\) is an optimal solution to the MILP for \(\Lambda^{RV+}(x)\), then \(p\) is a clearing vector for \((N, \pi, \bar{p}, x, \alpha, \beta)\).

**Remark 2.18.** \(\Lambda^{RV+}\) fails to be concave in general.

**Remark 2.19.** Let us comment on the MILP problems in Theorem 2.7 and Theorem 2.17. While both problems have a discrete feature through the binary variables, the natures of this feature are quite different from each other. In Theorem 2.7, the binary variables serve for quantifying the switch from the “first round” to the “second round” in the definition of \(\Phi^{EN}\), which is described above Definition 2.5. In this case, in addition to the binary variables, one also uses a large constant \(M\) in the problem formulation. On the other hand, binary variables are used in Theorem 2.17 to model the *discontinuity* in \(\Phi^{RV+}\) which occurs when \(\alpha < 1\) or \(\beta < 1\). In this case, a formulation without using a large constant \(M\) is possible.

Note that \(\Lambda^{RV+}(x)\) can be written more explicitly as

\[
\begin{align*}
\text{maximize} & \quad f(p) \\
\text{subject to} & \quad p_i \leq \alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j + \bar{p}_i s_i, \quad i \in N, \quad (2.16) \\
& \quad \bar{p}_i s_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j, \quad i \in N, \quad (2.17) \\
& \quad 0 \leq p_i \leq \bar{p}_i, \quad i \in N, \quad (2.18) \\
& \quad s_i \in \{0, 1\} \quad i \in N. \quad (2.19)
\end{align*}
\]

It is easy to check that \((p, s) = (0, 0) \in \mathbb{R}^n \times \mathbb{Z}^n\) is a feasible solution to the MILP in (2.15). Moreover, since \(f\) is a bounded function on the interval \([0, \bar{p}] \subseteq \mathbb{R}^n\), by Meyer (1974, Theorem 2.1), the MILP in (2.15) has an optimal solution. Observe that, by Theorem 2.17, the existence of an optimal solution to the MILP in (2.15) proves the existence of a clearing vector for \((N, \pi, \bar{p}, x, \alpha, \beta)\). Hence, Theorem 2.17 provides an alternative argument for the proof of Rogers and Veraart (2013, Theorem 3.1) on the existence of a clearing vector.

The proof of Theorem 2.17 relies on the following three lemmata.

**Lemma 2.20.** Let \((p, s)\) be an optimal solution to the MILP for \(\Lambda^{RV+}(x)\). Let \(i \in N\) such that

\[
\alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j < \bar{p}_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j.
\]

Then, \(s_i = 1\).

**Lemma 2.21.** Let \((p, s)\) be an optimal solution to the MILP for \(\Lambda^{RV+}(x)\). Let \(i \in N\) with \(\bar{p}_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j\). Then, \(p_i = \bar{p}_i\).

**Lemma 2.22.** Let \((p, s)\) be an optimal solution to the MILP for \(\Lambda^{RV+}(x)\). Let \(i \in N\) with \(\bar{p}_i > x_i + \sum_{j=1}^n \pi_{ji} p_j\). Then, \(p_i = \alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j\).

The proofs of Lemmata 2.20, 2.21, 2.22 can be found in Appendices A.2, A.3, A.4 respectively.

**Proof of Theorem 2.17.** Let \((p, s)\) be an optimal solution to the MILP for \(\Lambda^{RV+}(x)\). To prove that \(p\) is a clearing vector, thanks to Proposition 2.15, it suffices to show \(\Phi^{RV+}(p) = p\).
Let us fix $i \in \mathcal{N}$. Recalling (2.13), we consider two cases:

1. $\bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j$.

(1) Assume that $\bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j$. Then, by (2.13), $\Phi^{RV+}_i (p) = \bar{p}_i$. By Lemma 2.21, $p_i = \bar{p}_i = \Phi^{RV+}_i (p)$.

(2) Assume that $\bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j$. Then, by Definition (2.13), $\Phi^{RV+}_i (p) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. By Lemma 2.22, $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. Hence, $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j = \Phi^{RV+}_i (p)$.

Therefore, $p$ is a clearing vector.

The MILP aggregation functions $\Lambda^{EN}$ and $\Lambda^{RV+}$ developed in this section are used in Section 3 to define and calculate systemic risk measures.

### 3 Optimization problems for systemic risk measures

In this section, we consider the computation of the (sensitive) systemic risk measures studied in Feinstein et al. (2017), Biagini et al. (2018), Ararat and Rudloff (2016), which are set-valued functionals of a random operating cash flow vector and defined in terms of the aggregation function of the underlying network model. While the aforementioned articles focus mainly on the case where the aggregation function is concave which results in the convex-valuedness of the corresponding systemic risk measure, the aggregation functions we use are not concave and the corresponding systemic risk measures fail to have convex values, in general.

We are mainly interested in the systemic risk measures for the signed Eisenberg-Noe and Rogers-Veraart network models. We follow a unifying approach by using a general aggregation function defined in terms of a mixed-integer optimization problem. To be able to approximate the non-convex values of the corresponding systemic risk measure, we associate a (generally nonconvex) vector optimization problem to it. We solve this problem by the recent Benson-type algorithm in Nobakhtian and Shafiei (2017) (Section 3.3). For this purpose, we study two families of (scalar) optimization problems: the weighted-sum scalarization problem (Section 3.1) and the problem of calculating the minimum step-length to enter a set with a fixed direction (Section 3.2). We prove that both problems in both models can be formulated as MILP problems. We also prove some results related to the feasibility and boundedness of these MILP problems.

Without specifying a particular network model, we consider a financial network with $n \in \mathbb{N}$ institutions. As in Section 2 we write $\mathcal{N} = \{1, \ldots, n\}$. Similarly, let $\mathcal{K} = \{1, \ldots, K\}$ for some $K \in \mathbb{N}$. We consider a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega^1, \ldots, \omega^K\}$, $\mathcal{F}$ is the power set of $\Omega$, and $\mathbb{P}$ is a probability measure determined by the elementary probabilities $q^k := \mathbb{P} \{\omega^k\} > 0$, $k \in \mathcal{K}$. We denote by $L(\mathbb{R}^n)$ the linear space of all random vectors $X: \Omega \to \mathbb{R}^n$. For every $X \in L(\mathbb{R}^n)$, let

$$
\|X\|_{\infty} := \max_{i \in \mathcal{N}, k \in \mathcal{K}} |X_i(\omega^k)|.
$$

We use the notion of grouping, also discussed in Feinstein et al. (2017), to keep the dimension of the systemic risk measure at a reasonable level for computational purposes. This notion allows one to categorize the members of the network into groups and assign the same capital level for all the members of a group. To that end, let $G \geq 1$ be an integer denoting the number of groups
and \( G = \{1, \ldots, G\} \) the set of groups in the network. For the computations in Section 4 we will use \( G = 2 \) or \( G = 3 \) groups. Let \((N_\ell)_{\ell \in G}\) be a partition of \(N\), where \(N_\ell\) denotes the set of all institutions that belong to group \( \ell \in G \). For each \( \ell \in G \), let \( n_\ell := |N_\ell| \) and \( B_\ell \in \mathbb{R}^{G \times n_\ell} \) the matrix having 1’s in the \( \ell \text{th} \) row and 0’s elsewhere:

\[
B_\ell := \begin{bmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{bmatrix}.
\]

Let \( B \in \mathbb{R}^{G \times n} \) be the grouping matrix defined by

\[
B := [B_1 \ B_2 \ \cdots \ B_G]. \tag{3.1}
\]

We consider the (grouped) sensitive systemic risk measure \( R^{\text{OPT}} : L(\mathbb{R}^n) \to 2^{\mathbb{R}^n} \) defined by

\[
R^{\text{OPT}}(X) := \left\{ z \in \mathbb{R}^n \mid \Lambda^{\text{OPT}}(X + B^Tz) \in \mathcal{A} \right\}, \tag{3.2}
\]

where \( \Lambda^{\text{OPT}} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\} \) is an aggregation function and \( \mathcal{A} \subseteq L(\mathbb{R}^n) \) is an acceptance set, that is, the set of all random aggregate outputs that are at an acceptable level of risk. We assume that \( \Lambda^{\text{OPT}} \) is a general optimization aggregation function of the form

\[
\Lambda^{\text{OPT}}(x) := \sup \left\{ f(p) \mid (p,s) \in \mathcal{Y}(x), p \in \mathbb{R}^n, s \in \mathbb{Z}^n \right\}, \tag{3.3}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a strictly increasing and continuous function, and \( \mathcal{Y} : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{Z}^n} \) is a set-valued constraint function such that \( \mathcal{Y}(x) \) is either the empty set or a nonempty compact set for every \( x \in \mathbb{R}^n \). In particular, this general structure covers the aggregation functions \( \Lambda^{\text{OPT}} = \Lambda^{\text{EN}} \) and \( \Lambda^{\text{OPT}} = \Lambda^{\text{RV+}} \) defined in Section 2. On the other hand, we assume that \( \mathcal{A} \) is a halfspace-type acceptance set defined by

\[
\mathcal{A} = \{ Y \in L(\mathbb{R}^n) \mid \mathbb{E}[Y] \geq \gamma \}, \tag{3.4}
\]

where \( \gamma \in \mathbb{R} \) is some suitable threshold. Hence, the corresponding systemic risk measure \( R^{\text{OPT}} \) becomes

\[
R^{\text{OPT}}(X) = \left\{ z \in \mathbb{R}^G \mid \mathbb{E} \left[ \Lambda^{\text{OPT}}(X + B^Tz) \right] \geq \gamma \right\}. \tag{3.5}
\]

We write \( R^{\text{OPT}} = R^{\text{EN}} \) when \( \Lambda^{\text{OPT}} = \Lambda^{\text{EN}} \) and \( R^{\text{OPT}} = R^{\text{RV+}} \) when \( \Lambda^{\text{OPT}} = \Lambda^{\text{RV+}} \), and refer to them as the Eisenberg-Noe and Rogers-Veraart systemic risk measures, respectively.

**Remark 3.1.** For \( R^{\text{RV+}}(X) \), the definition of \( \Lambda^{\text{RV+}} \) in (2.14) implies the implicit condition \( X + B^Tz \geq 0 \).

Next, we introduce a vector optimization problem associated to each value of \( R^{\text{OPT}} \). Let us fix \( X \in L(\mathbb{R}^n) \) and consider the vector optimization problem

\[
\begin{align*}
\text{minimize} & \quad z \in \mathbb{R}^G \text{ with respect to } \leq \\
\text{subject to} & \quad \mathbb{E} \left[ \Lambda^{\text{OPT}}(X + B^Tz) \right] \geq \gamma,
\end{align*} \tag{3.6}
\]

where \( \leq \) denotes the usual componentwise ordering in \( \mathbb{R}^n \). Note that \( R^{\text{OPT}}(X) \) coincides with the
so-called upper image of this vector optimization problem in the sense that
\[ R^{\text{OPT}}(X) = \left\{ z + \mathbb{R}^G_+ \mid \mathbb{E} \left[ \Lambda^{\text{OPT}}(X + B^T z) \right] \geq \gamma \right\}. \quad (3.7) \]
In general, due to the lack of concavity for \( \Lambda^{\text{OPT}} \), the set \( R^{\text{OPT}}(X) \) may fail to be convex (see Remarks 2.8, 2.18). While the majority of the Benson-type approximation algorithms in the literature (Benson (1998), Hamel et al. (2014), Löhne et al. (2014)) work for linear/convex vector optimization problems and are based on creating supporting halfspaces for the upper image, we use the more recent Benson-type algorithm proposed in Nobakhtian and Shafiei (2017), which works for nonconvex upper images and is based on creating sets of the form “point plus cone.” The algorithm relies on the assumption that the associated weighted-sum scalarization and minimum step-length problems can be solved to optimality. In the next two subsections, we propose methods to solve these problems in our case by exploiting the structure of the optimization aggregation function \( \Lambda^{\text{OPT}} \).

3.1 Weighted-sum scalarizations of systemic risk measures
For each \( w \in \mathbb{R}^G_+ \setminus \{0\} \), we consider the weighted-sum scalarization problem
\[ P_1(w) = \inf_{z \in R^{\text{OPT}}(X)} w^T z = \inf_{z \in \mathbb{R}^G} \left\{ w^T z \mid \mathbb{E} \left[ \Lambda^{\text{OPT}}(X + B^T z) \right] \geq \gamma \right\}. \quad (3.8) \]

The following theorem provides an alternative formulation for \( P_1(w) \).

**Theorem 3.2.** Let \( w \in \mathbb{R}^G_+ \setminus \{0\} \). Consider the problem in (3.8) and let
\[ Z_1(w) := \inf_{z \in \mathbb{R}^G} \left\{ w^T z \mid \sum_{k \in \mathcal{K}} q^k f(p^k) \geq \gamma, \right. \]
\[ (p^k, s^k) \in \mathcal{Y}(X(\omega^k) + B^T z), p^k \in \mathbb{R}^n, s^k \in \mathbb{Z}^n \forall k \in \mathcal{K} \}. \quad (3.9) \]

Then, \( P_1(w) = Z_1(w) \). In particular, if one of the problems in (3.8) and (3.9) has a finite optimal value, then so does the other one and the optimal values coincide.

**Proof.** Let \((z, (p^k, s^k)_{k \in \mathcal{K}})\) be a feasible solution for the problem in (3.9). Then, for each \( k \in \mathcal{K}, (p^k, s^k) \) is a feasible solution to \( \Lambda^{\text{OPT}}(X(\omega^k) + B^T z) \) in (3.3) because the optimization problem in (3.9) includes the constraints of (3.3). Hence, for every \( k \in \mathcal{K}, \)
\[ \Lambda^{\text{OPT}}(X(\omega^k) + B^T z) \geq f(p^k), \]
which implies
\[ \mathbb{E} \left[ \Lambda^{\text{OPT}}(X + B^T z) \right] \geq \sum_{k=1}^{K} q^k f(p^k) \geq \gamma, \]
where the second inequality holds by feasibility of \((z, (p^k, s^k)_{k \in \mathcal{K}})\). Hence, \( z \) is a feasible solution for the problem in (3.8). So \( P_1(w) \leq Z_1(w) \).

Conversely, let \( \tilde{z} \) be a feasible solution for the problem in (3.8). For each \( k \in \mathcal{K}, \) there exists an
optimal solution \((\hat{p}^k, \hat{s}^k)\) to the problem for \(\Lambda^{\text{OPT}}(X(\omega^k) + B^T \hat{z})\). Then,

\[
\sum_{k=1}^{K} q^k f(\hat{p}^k) = \mathbb{E}\left[\Lambda^{\text{OPT}}(X + B^T \hat{z})\right] \geq \gamma,
\]

by the definition of \(P_1(w)\). Hence, \((\hat{z}, (\hat{p}^k, \hat{s}^k)_{k \in K})\) is a feasible solution for the problem in (3.9). So \(P_1(w) \geq Z_1(w)\).

**Remark 3.3.** Let \(\ell \in \mathcal{G}\) and \(e^\ell\) the corresponding standard unit vector in \(\mathbb{R}^G\). Observe that the weighted-sum scalarization problem

\[
P_1(e^\ell) = \inf_{z \in \mathbb{R}^G} \left\{ z_\ell \mid \mathbb{E}\left[\Lambda^{\text{OPT}}(X + B^T z)\right] \geq \gamma \right\}
\]

is a single-objective optimization problem of the vector optimization problem in (3.6). By Theorem 3.2, \(P_1(e^\ell) = Z_1(e^\ell)\).

**Remark 3.4.** Let \(z_{\text{ideal}} \in \mathbb{R}^G\) be the ideal point of the vector optimization problem in (3.6) in the sense that the entries of \(z_{\text{ideal}}\) minimize each of the objective functions of the vector optimization problem. In other words, one can define

\[
z_{\text{ideal}} := (P_1(e^1), \ldots, P_1(e^G))^T \in \mathbb{R}^G
\]

assuming that \(P_1(e^\ell)\) is finite for each \(\ell \in \mathcal{G}\). Theorem 3.2 allows one to solve \(G\) optimization problems with compact feasible sets, namely, the problems \((Z_1(e^\ell))_{\ell \in \mathcal{G}}\), to obtain the ideal point of the vector optimization problem in (3.6).

In the following two subsections, we apply Theorem 3.2 to the special cases \(\Lambda^{\text{OPT}} = \Lambda^{\text{EN}}\) and \(\Lambda^{\text{OPT}} = \Lambda^{\text{RV+}}\), respectively. For this purpose, we fix the function \(f : \mathbb{R}^n \to \mathbb{R}\) in the objective functions of the MILP aggregation functions as

\[
f(p) := 1^T p.
\]

It is clear that \(f\) is a strictly increasing continuous linear function bounded on the interval \([0, \bar{p}] \subseteq \mathbb{R}^n\). Moreover, since the vector optimization algorithm in Section 3.3 requires solving weighted-sum scalarization problems only for the standard unit vectors, we state our results for such direction vectors.

### 3.1.1 Weighted-sum scalarizations of Eisenberg-Noe systemic risk measure

Let \((N, \pi, \bar{p}, X)\) be a signed Eisenberg-Noe network.

**Corollary 3.5.** Let \(\ell \in \mathcal{G}\). Consider the single-objective optimization problem

\[
P_1^{\text{EN}}(e^\ell) := \inf_{z \in \mathbb{R}^G} \left\{ z_\ell \mid \mathbb{E}\left[\Lambda^{\text{EN}}(X + B^T z)\right] \geq \gamma \right\},
\]

(3.12)
Proposition 3.8. Let \( \text{optimal value, that is, } Z \) where \( M \) is a finite optimal value, then so does the other one and the optimal values coincide.

Proof. Let \( \mathcal{Y}_{EN} : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n \times \mathbb{Z}^n} \) be a set-valued function defined by

\[
\mathcal{Y}_{EN} (x) := \left\{ (p, s) \in \mathbb{R}^n \times \mathbb{Z}^n \mid p \leq \left[ \Pi^T p + x + M (1 - s) \right] \land (\bar{p} \odot s), \right. \\
\left. \Pi^T p + x \leq Ms, \ p \in [0, \bar{p}], \ s \in \{0, 1\}^n \right\}.
\]

(3.14)

Then, applying Theorem 3.2 with \( \mathcal{Y} = \mathcal{Y}_{EN} \) gives \( \mathcal{P}_{EN}^1 (e^\ell) = Z_{EN}^1 (e^\ell) \).

The proofs of Propositions 3.6, 3.7, 3.8 can be found in Appendices B.1, B.2, B.3 respectively.

Remark 3.9. Let \( \ell \in \mathcal{G} \). Suppose that there exists an optimal solution \( (z, (p_k^k, s_k^k)_{k \in \mathcal{K}}) \) of the MILP problem in (3.13). By the structure of the matrix \( B \), for each \( i \in \mathcal{N} \), it holds \( (B^T z)_i = z_i \) for some \( t \in \mathcal{G} \). Hence, by Proposition 3.6, \( (B^T z)_i \leq \| X \|_{\infty} + \| \bar{p} \|_{\infty} \) holds for each \( i \in \mathcal{N} \). In addition, for every \( i \in \mathcal{N} \), \( k \in \mathcal{K} \), and \( p^k \in [0, \bar{p}] \), it holds \( \sum_{j=1}^n \pi_{ji} p_j^k < n \| \bar{p} \|_{\infty} \) and \( X_i (\omega^k) \leq \| X \|_{\infty} \). Hence, the choice of \( M = 2 \| X \|_{\infty} + (n + 1) \| \bar{p} \|_{\infty} \) in Corollary 3.3 is justified, since, to ensure the feasibility of the third constraint in (3.13), it is enough to choose \( M \) such that

\[
\sum_{j=1}^n \pi_{ji} p_j^k + (X_i (\omega^k) + (B^T z)_i) \leq M
\]

for every \( i \in \mathcal{N} \), \( k \in \mathcal{K} \) and \( p^k \in [0, \bar{p}] \).
3.1.2 Weighted-sum scalarizations of Rogers-Veraart systemic risk measure

Let \((N, \pi, \bar{p}, X, \alpha, \beta)\) be a Rogers-Veraart network.

**Corollary 3.10.** Let \(\ell \in G\). Consider the single-objective optimization problem

\[
P_{1}^{RV+}(e^{\ell}) := \inf_{z \in \mathbb{R}^G} \left\{ z_{\ell} \mid \mathbb{E} \left[ A^{RV+}(X + B^T z) \right] \geq \gamma \right\},
\]

and let

\[
Z_{1}^{RV+}(e^{\ell}) := \inf_{z \in \mathbb{R}^G} \left\{ z_{\ell} \mid \sum_{k \in K} q^k \mathbb{1}^T p^k \geq \gamma, \right. \\
\left. p^k \leq \alpha (X(\omega^k) + B^T z) + \beta \Pi^T p^k + \bar{p} \odot s^k, \right. \\
\bar{p} \odot s^k \leq (X(\omega^k) + B^T z) + \Pi^T p^k, \\
X(\omega^k) + B^T z \geq 0, \; p^k \in [0, \bar{p}], \; s^k \in \{0, 1\}^n \; \forall k \in K \}. 
\]

Then, \(P_{1}^{RV+}(e^{\ell}) = Z_{1}^{RV+}(e^{\ell})\). In particular, if one of the problems in (3.15) and (3.16) has a finite optimal value, then so does the other one and the optimal values coincide.

**Proof.** Let \(Y_{RV+} : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{Z}^n}\) be a set-valued function defined by

\[
Y_{RV+}(x) := \left\{ (p, s) \in \mathbb{R}^n \times \mathbb{Z}^n \mid p \leq \alpha x + \beta \Pi^T p + \bar{p} \odot s, \right. \\
\left. \bar{p} \odot s \leq x + \Pi^T p, p \in [0, \bar{p}], s \in \{0, 1\}^n \right\}. 
\]

for each \(x \in \mathbb{R}^n_+\) and \(Y_{RV+}(x) = \emptyset\) for each \(x \in \mathbb{R}^n \setminus \mathbb{R}^n_+\). Then, applying Theorem 3.2 with \(Y = Y_{RV+}\) gives \(P_{1}^{RV+}(e^{\ell}) = Z_{1}^{RV+}(e^{\ell})\).

The next three propositions present some boundedness and feasibility results for the MILP problem of computing \(Z_{1}^{RV+}(e^{\ell}), \ell \in G\), in (3.16).

**Proposition 3.11.** Let \(\ell \in G\). If the problem in (3.16) has an optimal solution, then

\[
P_{1}^{RV+}(e^{\ell}) = Z_{1}^{RV+}(e^{\ell}) \leq \|X\|_\infty + \frac{1}{\alpha} \|\bar{p}\|_\infty.
\]

**Proposition 3.12.** Let \(\ell \in G\). If the problem in (3.16) has a feasible solution, then it has a finite optimal value, that is, \(Z_{1}^{RV+}(e^{\ell}) \in \mathbb{R}\).

**Proposition 3.13.** Let \(\ell \in G\). The problem in (3.16) has a feasible solution if and only if \(\gamma \leq \mathbb{1}^T \bar{p}\).

The proofs of Propositions 3.11, 3.12 and 3.13 can be found in Appendices B.4, B.5 and B.6, respectively.
3.2 Minimum step-length function

Weighted-sum scalarizations are used to calculate supporting halfspaces for the value of a systemic risk measure and they can be sufficient to characterize the entire risk set when the set is convex. In our nonconvex case, we make use of additional scalarizations that are used to calculate the minimum step-lengths to the enter the risk set from possibly outside points. Such scalarizations are well-known in vector optimization; see Gerstewitz and Iwanow (1985), Göpfert et al. (2003), for instance.

For each \( \mathbf{v} \in \mathbb{R}^G \), we consider

\[
\mathcal{P}_2(\mathbf{v}) := \inf \left\{ \mu \in \mathbb{R} \mid B^T \mathbf{v} + \mu 1 \in R^\text{OPT}(X) \right\} 
= \inf \left\{ \mu \in \mathbb{R} \mid \mathbb{E} \left[ \Lambda^\text{OPT}(X + B^T \mathbf{v} + \mu 1) \right] \geq \gamma \right\},
\]

which can be interpreted as the minimum step-length in the direction \( 1 \) from the point \( \mathbf{v} \) to the boundary of the set \( R^\text{OPT}(X) \).

The following theorem provides an alternative formulation for \( \mathcal{P}_2(\mathbf{v}) \).

**Theorem 3.14.** Let \( \mathbf{v} \in \mathbb{R}^n \). Consider the problem in (3.18) and let

\[
Z_2(\mathbf{v}) := \inf \left\{ \mu \in \mathbb{R} \mid \sum_{k \in \mathcal{K}} q^k f(p^k) \geq \gamma, \right. \\
\left. \quad (p^k, s^k) \in \mathcal{Y}(X(\omega^k) + B^T \mathbf{v} + \mu 1), \ p^k \in \mathbb{R}^n, \ s^k \in \mathbb{Z}^n \ \forall k \in \mathcal{K} \right\}.
\]

Then, \( \mathcal{P}_2(\mathbf{v}) = Z_2(\mathbf{v}) \). In particular, if one of the problems in (3.18) and (3.19) has a finite optimal value, then so does the other one and the optimal values coincide.

**Proof.** Let \( (\mu, (p^k, s^k)_{k \in \mathcal{K}}) \) be a feasible solution of the problem in (3.19). For each \( k \in \mathcal{K} \), \( (p^k, s^k) \) is a feasible solution to \( \Lambda^\text{OPT}(X(\omega^k) + B^T \mathbf{v} + \mu 1) \) in (3.3) because the problem in (3.19) includes the constraints of (3.3). Hence, for each \( k \in \mathcal{K} \),

\[
\Lambda^\text{OPT}(X(\omega^k) + B^T \mathbf{v} + \mu 1) \geq f(p^k),
\]

which implies

\[
\mathbb{E} \left[ \Lambda^\text{OPT}(X + B^T \mathbf{v} + \mu 1) \right] \geq \sum_{k=1}^K q^k f(p^k) \geq \gamma,
\]

where the second inequality holds by feasibility of \( (\mu, (p^k, s^k)_{k \in \mathcal{K}}) \). Then, \( \mu \) is a feasible solution for the problem in (3.18). Hence, \( \mathcal{P}_2(\mathbf{v}) \leq Z_2(\mathbf{v}) \).

Conversely, let \( \bar{\mu} \in \mathbb{R} \) be a feasible solution for the problem in (3.18). Then, for each \( k \in \mathcal{K} \), \( \Lambda^\text{OPT}(X(\omega^k) + B^T \mathbf{v} + \bar{\mu} 1) \in \mathbb{R} \) and, by the compactness of \( \mathcal{Y}(X(\omega^k) + B^T \mathbf{v} + \bar{\mu} 1) \), there exists an optimal solution \( (\bar{p}^k, \bar{s}^k) \) for the problem \( \Lambda^\text{OPT}(X(\omega^k) + B^T \mathbf{v} + \bar{\mu} 1) \) in (3.3). Then,

\[
\sum_{k=1}^K q^k f(\bar{p}^k) = \mathbb{E} \left[ \Lambda^\text{OPT}(X + B^T \mathbf{v} + \bar{\mu} 1) \right] \geq \gamma
\]

by the definition of \( \mathcal{P}_2(\mathbf{v}) \). Hence, \( (\bar{\mu}, (\bar{p}^k, \bar{s}^k)_{k \in \mathcal{K}}) \) is a feasible solution for the problem in (3.19). Hence, \( \mathcal{P}_2(\mathbf{v}) \geq Z_2(\mathbf{v}) \).
The following two sections apply Theorem 3.14 to the special cases $\Lambda^{\text{OPT}} = \Lambda^{\text{EN}}$ and $\Lambda^{\text{OPT}} = \Lambda^{\text{RV}}$, respectively.

### 3.2.1 Minimum step-length function for Eisenberg-Noe systemic risk measure

Let $(\mathcal{N}, \pi, \bar{p}, X)$ be an Eisenberg-Noe network.

**Corollary 3.15.** Let $v \in \mathbb{R}^G$. Consider the problem

$$\mathcal{P}^\text{EN}_2(v) := \inf \left\{ \mu \in \mathbb{R} \mid \mathbb{E} \left[ \Lambda^{\text{EN}}(X + B^T v + \mu 1) \right] \geq \gamma \right\},$$

and let

$$\mathcal{Z}^\text{EN}_2(v) := \inf \left\{ \mu \in \mathbb{R} \mid \sum_{k \in K} q^k 1^T p^k \geq \gamma, \right\}$$

where $M = 2 \|X\|_{\infty} + 2 \|v\|_{\infty} + (n + 1) \|\bar{p}\|_{\infty}$. Then, $\mathcal{P}^\text{EN}_2(v) = \mathcal{Z}^\text{EN}_2(v)$. In particular, if one of the problems in (3.20) and (3.21) has a finite optimal value, then so does the other one and the optimal values coincide.

**Proof.** Let $\mathcal{Y} = \mathcal{Y}^\text{EN}$ as in the proof of Corollary 3.5. Then, applying Theorem 3.14 gives $\mathcal{P}^\text{EN}_2(v) = \mathcal{Z}^\text{EN}_2(v)$. \qed

The next three propositions present some boundedness and feasibility results for the MILP problem in (3.21).

**Proposition 3.16.** Let $v \in \mathbb{R}^G$. If the problem in (3.21) has an optimal solution, then

$$\mathcal{P}^\text{EN}_2(v) = \mathcal{Z}^\text{EN}_2(v) \leq \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}.$$

**Proposition 3.17.** Let $v \in \mathbb{R}^G$. If the problem in (3.21) has a feasible solution, then it has a finite optimal value, that is, $\mathcal{Z}^\text{EN}_2(v) \in \mathbb{R}$.

**Proposition 3.18.** Let $v \in \mathbb{R}^G$. The problem in (3.21) has a feasible solution if and only if $\gamma \leq 1^T \bar{p}$.

The proofs of Propositions 3.16, 3.17, 3.18 are presented in Appendices B.7, B.8, B.9 respectively.

**Remark 3.19.** Let $v \in \mathbb{R}^G$ and $(\mu, (p^k, s^k)_{k \in K})$ an optimal solution of the MILP problem in (3.21). By Proposition 3.16, $\mu \leq \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}$. By the structure of the matrix $B$, for each $i \in \mathcal{N}$, it holds $(B^T v)_i = v_i$ for some $t \in \mathcal{G}$. Hence, for every $v \in \mathbb{R}^G$, $(B^T v)_i \leq \|v\|_{\infty}$. In addition, for every $i \in \mathcal{N}$, $k \in \mathcal{K}$, and $p^k \in [0, \bar{p}]$, it holds $\sum_{j=1}^{n} \pi_{ij} p^k < n \|\bar{p}\|_{\infty}$ and $X_i(\omega^k) \leq \|X\|_{\infty}$.
Hence, the choice of $M$ as $M = 2\|X\|_\infty + 2\|v\|_\infty + (n + 1)\|\bar{p}\|_\infty$ in Corollary 3.15 is justified, since, to ensure the feasibility of the third constraint in (3.21), it is enough to choose $M$ such that

$$\sum_{j=1}^{n} \pi_j p_j^k + \left( X_i(\omega^k) + (B^T v)_i + \mu \right) \leq M$$

for every $i \in \mathcal{N}$, $k \in \mathcal{K}$, $v \in \mathbb{R}^G$ and $p^k \in [0, \bar{p}]$.

**Remark 3.20.** Proposition 3.7 shows that if the MILP problem $Z_{EN}^1(e^\ell)$ in (3.13) is feasible for every $\ell \in \mathcal{G}$, then the ideal point $z_{\text{ideal}} \in \mathbb{R}^n$ exists for the vector optimization problem in (3.6) with $\Lambda_{\text{OPT}} = \Lambda_{EN}^1$. Proposition 3.12 provides the same result for the vector optimization problem in (3.6) with $\Lambda_{\text{OPT}} = \Lambda_{PV}^1$. In addition, the results of Propositions 3.6, 3.7, 3.16 and 3.17 allow one to choose the exact value for the upper bound $M$ in the corresponding MILP problems instead of assuming some heuristic values.

### 3.2.2 Minimum step-length function for Rogers-Veraart systemic risk measure

Let $(\mathcal{N}, \pi, \bar{p}, X, \alpha, \beta)$ be a Rogers-Veraart network.

**Corollary 3.21.** Let $v \in \mathbb{R}^G$. Consider the problem

$$P_{RV^+}^2(v) := \inf \left\{ \mu \in \mathbb{R} : E \left[ \Lambda_{RV^+}(X + B^T v + \mu \mathbbm{1}) \right] \geq \gamma \right\}, \quad (3.22)$$

and let

$$Z_{RV^+}^2(v) := \inf \left\{ \mu \in \mathbb{R} : \sum_{k \in \mathcal{K}} q^k \mathbbm{1}^T p^k \geq \gamma, \right. \left. p^k \leq \alpha \left( X(\omega^k) + B^T v + \mu \mathbbm{1} \right) + \beta \Pi^T p^k + \bar{p} \odot s^k, \bar{p} \odot s^k \leq \left( X(\omega^k) + B^T v + \mu \mathbbm{1} \right) + \Pi^T p^k, \right. \left. X(\omega^k) + B^T v + \mu \mathbbm{1} \geq 0, \right. \left. p^k \in [0, \bar{p}], s^k \in \{0, 1\}^n \forall k \in \mathcal{K} \right\}. \quad (3.23)$$

Then, $P_{RV^+}^2(v) = Z_{RV^+}^2(v)$. In particular, if one of the problems in (3.22) and (3.23) has a finite optimal value, then so does the other one and the optimal values coincide.

**Proof.** Let $\mathcal{Y} = \mathcal{Y}_{RV^+}$ as in the proof of Corollary 3.10. By Theorem 3.14, the result follows.

The next three propositions present some boundedness and feasibility results for the problem in (3.23).

**Proposition 3.22.** Let $v \in \mathbb{R}^G$. If the problem in (3.23) has an optimal solution, then

$$P_{RV^+}^2(v) = Z_{RV^+}^2(v) \leq \|X\|_\infty + \|v\|_\infty + \frac{1}{\alpha} \|\bar{p}\|_\infty.$$

**Proposition 3.23.** Let $v \in \mathbb{R}^G$. If the problem in (3.23) has a feasible solution, then it has a finite optimal value, that is $Z_{RV^+}^2(v) \in \mathbb{R}$. 20
Proposition 3.24. Let $v \in \mathbb{R}^G$. The problem in (3.23) has a feasible solution if and only if $\gamma \leq 1^T \bar{p}$.

The proofs of Propositions 3.22, 3.23, 3.24 can be found in Appendices B.10, B.11, B.12, respectively.

Remark 3.25. For $\ell \in \mathcal{G}$ and $v \in \mathbb{R}^G$, the threshold $\gamma$ appearing in $R^{EN}$, $R^{RV+}$ can be taken as some percentage of $1^T \bar{p}$, the sum of the debts of all nodes in the network. Then this threshold ensures that the expected total amount of payments exceeds this fraction of the total debt in the system. Indeed, Corollaries 3.8, 3.13, 3.18, 3.24 show that the MILP problems for calculating $Z_1^{EN}(e^t)$, $Z_1^{RV+}(e^t)$, $Z_2^{EN}(v)$, $Z_2^{RV+}(v)$ are feasible if and only if $\gamma \leq 1^T \bar{p}$. Hence, this choice of $\gamma$ is justified.

3.3 The nonconvex Benson-type algorithm

In this section, we present an algorithm that approximates the Eisenberg-Noe and Rogers-Veraart systemic risk measures. The risk measures are approximated with respect to a user-defined approximation error $\epsilon > 0$ and an upper bound point $z^{UB} \in \mathbb{R}^G$. The algorithm is based on the Benson-type algorithm for nonconvex multi-objective programming problems described in Nobakhtian and Shafiei (2017). The following definitions are borrowed from Nobakhtian and Shafiei (2017).

Let $L \subseteq \mathbb{R}^G$. A point $v \in L$ is called a vertex of $L$ if there exists a neighborhood $N$ of $v$ for which $v$ cannot be expressed as a strict convex combination of two distinct points in $L \cap N$. The set of all vertices of $L$ is denoted by $\operatorname{vert} L$. The notation $\operatorname{int} L$ denotes the interior of $L$. Given a point $z \in \mathbb{R}^G$ and $L \subseteq \mathbb{R}^G$, we define $L|_z := \{v \in L \mid v \leq z\}$.

Let $R, L, U \subseteq \mathbb{R}^G$, $z \in \mathbb{R}^G$ and $\epsilon > 0$ be given. The set $L$ is called an outer approximation for $R$ with respect to $\epsilon$ and $z$, if $R \subseteq L$ and $L|_z \subseteq R + B(0, \epsilon)$, where $B(0, \epsilon)$ is the closed ball in $\mathbb{R}^G$ centered at 0 with radius $\epsilon$. The set $U$ is called an inner approximation for $R$ with respect to $\epsilon$ and $z$ if $R$ is an outer approximation for $U$ with respect to $\epsilon$ and $z$.

The algorithm that calculates inner and outer approximations of a systemic risk measure works as follows. It is provided in detail only for the Eisenberg-Noe systemic risk measures, since it works similarly for the Rogers-Veraart systemic risk measures. Let $(\mathcal{N}, \pi, \bar{p}, \bar{X})$ be a signed Eisenberg-Noe network. Let $G$ be the number of groups in the network and $\mathcal{G} = \{1, \ldots, G\}$. Consider the corresponding Eisenberg-Noe systemic risk measure $R^{EN}(X)$. Let $z_{\text{ideal}} \in \mathbb{R}^G$ be the ideal point of the vector optimization problem in (3.6) with $A^{OPT} = A^{EN}$, see Remark 3.4 for its definition. One can calculate $z_{\text{ideal}} = (Z_1^{EN}(e^1), \ldots, Z_1^{EN}(e^G))^T$ by Corollary 3.5. In addition, for $v \in \mathbb{R}^G$, the minimum step-length $p_{2v}^{EN}(v)$ can be obtained by solving the MILP problem $Z_2^{EN}(v)$ in (3.21), by Corollary 3.15.

The algorithm starts with the initial inner approximation $U^0 := z^{UB} + \mathbb{R}^G_+$ and the initial outer approximation $L^0 := z_{\text{ideal}} + \mathbb{R}^G_+$, which satisfy $U^0 \subseteq R^{EN}(X) \subseteq L^0$. Let $\epsilon = \epsilon_1$ and initially set $t \leftarrow 0$. At the $t$th iteration, for a vertex $v^t \in (\operatorname{vert} L^t|_{z^{UB}})$ such that $v^t + \epsilon \notin \operatorname{int} U^t$, the algorithm solves $Z_2^{EN}(v^t)$ to obtain the minimum step-length $\mu^t$ from the point $v^t$ to the boundary of $R^{EN}(X)$ in the direction $1 \in \mathbb{R}^G$. In other words, $y^t = v^t + \mu^t 1$ is a boundary point of the set $R^{EN}(X)$. Then the algorithm excludes the cone $y^t - \mathbb{R}^G_+$ from $L^t$ to obtain $L^{t+1}$ by $L^{t+1} : = L^t \setminus (y^t - \mathbb{R}^G_+)$, and adds the cone $y^t + \mathbb{R}^G_+$ to $U^t$ to obtain $U^{t+1}$ as follows: $U^{t+1} := U^t \cup (y^t + \mathbb{R}^G_+)$. Therefore, at each step of the algorithm, we have $U^t \subseteq U^{t+1} \subseteq R^{EN}(X) \subseteq L^{t+1} \subseteq L^t$. At the end of the $t$th iteration, $\operatorname{vert} L^{t+1}$ is computed. The computation of $\operatorname{vert} L^{t+1}$ is described in detail in Gourion and Luc (2010). The above process repeats for $t \leftarrow t + 1$. The algorithm stops at $T$th iteration, when $(\operatorname{vert} L^T|_{z^{UB}}) + \epsilon \subseteq \operatorname{int} U^T$. The sets $U^T$ and $L^T$ are the inner and outer approximations for $R^{EN}(X)$.
with respect to $\epsilon > 0$ and $z^{UB} \in \mathbb{R}^G$. Note that $z^{UB}$ has to be chosen such that $z^{UB} \in R^{EN}(X)$ to get nonempty approximations. The pseudocode of the algorithm for the Eisenberg-Noe systemic risk measures is provided in Algorithm 1.

Algorithm 1. Inner and outer approximation algorithm for $R^{EN}(X)$

Initialization.
(1) Let $z^{UB} \in R^{EN}(X)$, $L^0 = z^{ideal} + \mathbb{R}^G_+, U^0 = z^{UB} + \mathbb{R}^G_+$ and $\epsilon > 0$.
(2) Put $\epsilon = \epsilon \mathbb{1}$ and set $t \leftarrow 0, S \leftarrow \emptyset$.

Iteration steps.
(k1) If $(\text{vert} L^t|_{z^{UB}}) \subseteq S$, then set $T = t$ and stop. Otherwise, choose $v^t \in (\text{vert} L^t|_{z^{UB}}) \setminus S$.
(k2) If $v^t + \epsilon \in \text{int} U^t$, then set $S \leftarrow S \cup \{v^t\}$ and go to (k1).
(k3) Suppose that $\mu^t = \mathcal{P}^{EN}_2(v^t)$. Define $y^t = v^t + \mu^t \mathbb{1}$.
(k4) Define $L^{t+1} := L^t \setminus (y^t - \mathbb{R}^G_+)$ and $U^{t+1} := U^t \cup (y^t + \mathbb{R}^G_+)$. 
(k5) Determine $\text{vert} L^{t+1}$ and set $t \leftarrow t + 1$. Go to (k1).

Results.
(1) $L^T$ is an outer approximation and $U^T$ is an inner approximation for $R^{EN}(X)$.

4 Computational results and analysis

In this section, we present some computational results to illustrate the approximation of the Eisenberg-Noe and Rogers-Veraart systemic risk measures by the Benson-type algorithm described in Section 3.3. We implement the algorithm on Java Photon (Release 4.8.0) calling Gurobi Interactive Shell (Version 7.5.2) and run it on an Intel(R) Core(TM) i7-4790 processor with 3.60 GHz and 4 GB RAM. We approximate the Eisenberg-Noe and Rogers-Veraart systemic risk measures within a two-group framework and perform a detailed sensitivity analysis. In the last part, we present several computational results for three-group networks.

Recall that $n$ is the number of institutions in a financial system, referred to as banks here, $n_\ell$ is the number of nodes in a group $\ell \in \mathcal{G}$, $K$ is the number of scenarios, $\epsilon$ is a user-defined approximation error and $z^{UB}$ is a user-defined upper-bound vector that limits the approximated region of a systemic risk measure. Throughout the computation of systemic risk measures, except for the Rogers-Veraart case in a three-group framework (Section 4.6), $z^{UB}$ is taken as $z^{UB} = z^{ideal} + 2 \| \bar{p} \|_\infty$, where $z^{ideal}$ is the ideal point of the corresponding systemic risk measure (Remark 3.4) for the case $\gamma = 1^T \bar{p}$, that is, when it is required that the expected total value of payments is at least as much as the total amount of liabilities in the network.

For convenience, let us write $\gamma = \gamma^p (1^T \bar{p})$, where $\gamma^p \in [0, 1]$.

4.1 Data generation

We consider a network with $n$ banks forming $G = 2$ or $G = 3$ groups. Recall that $\mathcal{G} = \{1, \ldots, G\}$, $\mathcal{N} = \bigcup_{\ell \in \mathcal{G}} \mathcal{N}_\ell = \{1, \ldots, n\}$, and $n_\ell = |\mathcal{N}_\ell|$. When $G = 2$, the groups $\ell = 1$ and $\ell = 2$ correspond to big and small banks, respectively. When $G = 3$, the groups $\ell = 1$, $\ell = 2$ and $\ell = 3$ correspond to big, medium and small banks, respectively.
In order to construct a signed Eisenberg-Noe network \( (N, \pi, \bar{p}, X) \) and a Rogers-Veraart network \( (N, \pi, \bar{p}, X, \alpha, \beta) \), the corresponding interbank liabilities matrix \( l := (l_{ij})_{i,j \in N} \in \mathbb{R}^{n \times n} \) and the random operating cash flow vector \( X \) are generated in the following fashion. For \( l \), we use an Erdős-Rényi random graph model \( \text{(Erdős and Rényi, 1959), (Gilbert, 1953)} \). First, we fix a connectivity probabilities matrix \( q^{\text{con}} : = (q^{\text{con}}_{i,j})_{i,j \in G} \in \mathbb{R}^{G \times G} \) and an intergroup liabilities matrix \( l^{gr} : = (l^{gr}_{i,j})_{i,j \in G} \in \mathbb{R}^{G \times G} \). For any two banks \( i,j \in N \) with \( i \in N_t, j \in N_l \) and \( l, \hat{l} \in G \), \( q^{\text{con}}_{l,\hat{l}} \) is interpreted as a probability that bank \( i \) owes \( l^{gr}_{l,\hat{l}} \) amount to bank \( j \). Then, the liability \( l_{ij} \) is generated by the Bernoulli trial

\[
 l_{ij} = \begin{cases} 
 l^{gr}_{l,\hat{l}}, & \text{if } U_{ij} < q^{\text{con}}_{l,\hat{l}}, \\
 0, & \text{otherwise},
\end{cases}
\]

where \( U_{ij} \) is the realization of a continuous random variable with a standard uniform distribution on a separate probability space. Then, the relative liabilities matrix \( \pi \) and the total obligation vector \( \bar{p} \) are calculated accordingly.

Recall that the operating cash flow vector \( X = (X_1, \ldots, X_n) \in L(\mathbb{R}^n) \) is a multivariate random vector and \( \Omega \) is a finite set of \( K \) scenarios. It is assumed that all scenarios are equally likely to happen, the operating cash flows have a common standard deviation \( \sigma \), and there is a common correlation \( \varrho \) between any two operating cash flows. Then, each entry \( X_i, i \in N \), is generated as a random sample of size \( K \) as described below.

For the Eisenberg-Noe network, the mean values of operating cash flows in each group, \( \nu := (\nu_l)_{l \in G} \), are fixed and the random vector \( X \) is generated from \( K \) instances of a Gaussian random vector. For the Rogers-Veraart network, first, shape parameters \( \kappa := (\kappa_l)_{l \in G} \) and scale parameters \( \theta := (\theta_l)_{l \in G} \) are fixed in accordance with the choices of \( \sigma, \varrho \) and then, \( X \) is generated from \( K \) instances of a random vector whose cumulative distribution function is stated in terms of a Gaussian copula with gamma marginal distributions with the chosen parameters. In particular, \( \nu_l = \kappa_l \theta_l \) and \( \sigma = \sqrt{\kappa_l \theta_l} \) for each \( l \in G \).

### 4.2 A two-group signed Eisenberg-Noe network with 50 nodes

We consider a two-group Eisenberg-Noe network with \( n = 50 \) banks that consists of \( n_1 = 15 \) big banks, \( n_2 = 35 \) small banks. We take \( K = 100 \), \( \sigma = 100 \), \( \varrho = 0.05 \),

\[
 q^{\text{con}} = \begin{bmatrix} 0.9 & 0.3 \\ 0.7 & 0.5 \end{bmatrix}, \quad l^{gr} = \begin{bmatrix} 10 & 5 \\ 8 & 5 \end{bmatrix}, \quad \nu = \begin{bmatrix} -50 \\ -100 \end{bmatrix}.
\]

In the corresponding Eisenberg-Noe systemic risk measure, we take \( \gamma^p = 0.7 \).

| \( \epsilon \) | Inner approx. vertices | Outer approx. vertices | \( P_2 \) problems | Avg. time per \( P_2 \) prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|---|---|---|---|---|---|---|
| 20 | 18 | 19 | 18 | 663.546 | 11944 | 3.318 |
| 10 | 35 | 36 | 35 | 541.419 | 18950 | 5.264 |
| 5 | 73 | 74 | 73 | 512.998 | 37449 | 10.403 |
| 1 | 394 | 395 | 394 | 492.597 | 194083 | 53.912 |

**Table 1:** Computational performance of the algorithm for a network of 15 big and 35 small banks, 100 scenarios and approximation errors \( \epsilon \in \{1, 5, 10, 20\} \).
Figure 1: Zoomed inner approximations of the Eisenberg-Noe systemic risk measure for $\epsilon \in \{1, 5, 10, 20\}$.

The Benson-type algorithm is run with four different approximation errors $\epsilon$ to demonstrate different inner approximation levels. Table I presents the computational performance of the algorithm for $\epsilon \in \{1, 5, 10, 20\}$. Figure II consists of the zoomed inner approximations.

One can easily observe from Figure I that as $\epsilon$ decreases the algorithm gives a more precise inner approximation of the systemic risk measure. In addition, as the number of $P_2$ problems increases, the average computation time per $P_2$ problem decreases. This may be attributed to the warm start feature of the Gurobi solver. When a sequence of mixed-integer programming problems are solved, the solver constructs an initial solution out of the previously obtained optimal solution. This feature is explained in detail in Gurobi Optimizer Reference Manual (2018, Chapter 10.2, pp. 594-595).

In the rest of this section, we perform sensitivity analyses on this network with respect to the connectivity probabilities between big and small banks and on the number of scenarios.

4.2.1 Connectivity probabilities

Connectivity probabilities play a major role in determining the topology of the network because they define the existence of liabilities between the banks. We would like to identify the sensitivity of the systemic risk measure with respect to the changes in the connectivity probability $q_{1,2}^{\text{con}}$ corresponding


| $q_{1,2}^{\text{con}}$ | Inner approx. vertices | Outer approx. vertices | $P_2$ problems | Avg. time per $P_2$ prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|------------------------|------------------------|------------------------|----------------|---------------------------------|-------------------------------|-----------------------------|
| 0.1                    | 279                    | 280                    | 358            | 294.07                          | 105 277                       | 29.244                      |
| 0.3                    | 394                    | 395                    | 394            | 492.597                         | 194 083                       | 53.912                      |
| 0.5                    | 360                    | 361                    | 360            | 556.795                         | 200 447                       | 55.680                      |
| 0.7                    | 364                    | 365                    | 364            | 633.644                         | 230 647                       | 64.069                      |
| 0.9                    | 377                    | 378                    | 377            | 772.76                          | 291 331                       | 80.925                      |

Table 2: Computational performance of the algorithm for $q_{1,2}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

to the liabilities of big banks to small banks, and the probability $q_{2,1}^{\text{con}}$ corresponding to the liabilities of small banks to big banks.

For the sensitivity analysis with respect to $q_{1,2}^{\text{con}}$, originally taken as $q_{1,2}^{\text{con}} = 0.3$, we present in Table 2 the computational performance of the algorithm for $q_{1,2}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Figure 2 consists of the corresponding inner approximations.

Observe from Table 2 that the average time per $P_2$ problem increases with $q_{1,2}^{\text{con}}$. This is the case because as $q_{1,2}^{\text{con}}$ increases, big and small banks in the network become more connected in terms of liabilities. Hence, the corresponding MILP formulations of $P_2$ problems need more time to be solved. This seems to be the only factor behind the increase because most of the algorithm runtime is devoted to solving $P_2$ problems and the number of $P_2$ problems in each case does not change much.

It can be observed that, as $q_{1,2}^{\text{con}}$ increases, the corresponding inner approximations of systemic risk measures in Figure 2 shift from the top left corner towards the bottom right corner. It can be interpreted as follows: as $q_{1,2}^{\text{con}}$ increases, the first group, the group of big banks, loses capital allocation options, while the second group, the group of small banks, gains a wider range of capital allocation options. It can also be observed from Figure 2 that generating a network with $q_{1,2}^{\text{con}} = 0.1$ results in a nonconvex Eisenberg-Noe systemic risk measure. However, for the
Table 3: Computational performance of the algorithm for \( q_{2,1}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\} \).

| \( q_{2,1}^{\text{con}} \) | Inner approx. vertices | Outer approx. vertices | \( P_2 \) problems | Avg. time per \( P_2 \) prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|----------------|----------------------|----------------------|----------------------|--------------------------------|-----------------------------|-----------------------------|
| 0.1            | 257                  | 258                  | 257                  | 233.243                       | 59943                       | 16.651                      |
| 0.3            | 294                  | 295                  | 294                  | 319.511                       | 93936                       | 26.093                      |
| 0.5            | 328                  | 329                  | 328                  | 377.398                       | 123787                      | 34.385                      |
| 0.7            | 394                  | 395                  | 394                  | 492.597                       | 194083                      | 53.912                      |
| 0.9            | 435                  | 436                  | 512                  | 487.547                       | 249624                      | 69.340                      |

For these cases, there might be some breakpoint between 0.1 and 0.3 that switches these Eisenberg-Noe systemic risk measures from a nonconvex shape to a convex one, meaning that, whenever the probability \( q_{1,2}^{\text{con}} \) is less than this breakpoint, big banks are less likely to be liable to small banks and have even more capital allocation options than they have in the other cases.

Next, for the sensitivity analysis with respect to \( q_{2,1}^{\text{con}} \), we present in Table 3 the computational performance of the algorithm for \( q_{2,1}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\} \). Figure 3 consists of the corresponding inner approximations.

As in the previous sensitivity analysis, observe from Table 3 that the average time per \( P_2 \) problem increases with \( q_{2,1}^{\text{con}} \). Hence, it is another justification of the presumption that this happens because with higher connectivity probabilities the network becomes more connected in terms of liabilities and the corresponding MILP formulations of \( P_2 \) problems need more time to be solved.

Note that as \( q_{2,1}^{\text{con}} \) increases, the inner approximations of the corresponding Eisenberg-Noe systemic risk measures in Figure 3 shift from the bottom right corner towards the top left corner. Conversely to the previous sensitivity analysis, it can be interpreted as follows: as \( q_{2,1}^{\text{con}} \) increases, the first group gains a wider range of capital allocation options, while the second group loses capital allocation options. It can also be observed from Figure 3 that generating a network with

![Figure 3: Inner approximations of the Eisenberg-Noe systemic risk measure for \( q_{2,1}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\} \).](image-url)

for various values \( q_{1,2}^{\text{con}} \in \{0.3, 0.5, 0.7, 0.9\} \), the corresponding Eisenberg-Noe systemic risk measures seem to be convex sets.
$q_{2,1}^\text{con} = 0.9$ results in a nonconvex Eisenberg-Noe systemic risk measure. However, for the values $q_{2,1}^\text{con} \in \{0.1, 0.3, 0.5, 0.7\}$, the corresponding Eisenberg-Noe systemic risk measures seem to be convex sets. As in the previous sensitivity analysis, it can be presumed that for these cases there is some breakpoint between 0.7 and 0.9 that switches these Eisenberg-Noe systemic risk measures from a convex shape to a nonconvex one, meaning that, whenever the probability $q_{2,1}^\text{con}$ is higher than this breakpoint, small banks are more likely to be liable to big banks and the latter have even more capital allocation options than they have in the other cases.

### 4.2.2 Number of scenarios

![Figure 4: Inner approximations of the Eisenberg-Noe systemic risk measure for $K \in \{10, 20, \ldots, 100\}$.](image)

Next, we analyze how computation times and the corresponding systemic risk measures change with the number $K$ of scenarios. Since the network structure remains the same all the time, it is expected that there will be no major changes in Eisenberg-Noe systemic risk measures. However, since each scenario adds $n$ continuous and $n$ binary variables to the corresponding $P_2$ problem and its MILP formulation $Z_2^\text{EN}$, given in (3.21), one would expect major changes in computation times.
Table 4 shows the computational performance of the algorithm for $K \in \{10, 20, \ldots, 100\}$ and Figure 4 provides the inner approximations of the corresponding Eisenberg-Noe systemic risk measures. Finally, the plots in Figure 5 suggest that the average time per $P_2$ problem and the total algorithm time increase faster than linearly with $K$. At the same time, it can be observed from Figure 5 that the corresponding inner approximations of the Eisenberg-Noe systemic risk measures do not change much. Hence, the results obtained justify the expectations.

### 4.3 A two-group signed Eisenberg-Noe network with 70 nodes

In this section, we consider an Eisenberg-Noe network $(N, \pi, \bar{p}, X)$ with $n = 70$, $n_1 = 10$, $n_2 = 60$, $K = 50$, $\sigma = 100$, $\varrho = 0.05$ and

$$q^\text{con} = \begin{bmatrix} 0.7 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}, \quad l^\text{gr} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}, \quad \nu = [-50, -100].$$

In the corresponding Eisenberg-Noe systemic risk measure, we take $\gamma^p = 0.9$. The approximation error in the algorithm is taken as $\epsilon = 1$.

On this network, we perform sensitivity analyses with respect to the threshold $\gamma^p$, the distribution of nodes among groups, and the number of scenarios.

#### 4.3.1 Threshold level

We investigate how the Eisenberg-Noe systemic risk measures and their computation times change when the requirement that some fraction of the total amount of liabilities in the network should be met on average gets more strict. Table 5 illustrates the computational performance of the algorithm for $\gamma^p \in \{0.01, 0.1, 0.2, \ldots, 0.9, 0.95, 0.99, 1\}$ and Figure 6 represents the corresponding inner approximations of the Eisenberg-Noe systemic risk measures.

It can be noted from Table 5 that the average times per $P_2$ problem are high for the values of $\gamma^p$ around 0.3, and the number of $P_2$ problems are high for the values of $\gamma^p$ around 0.5. These two factors result in high total algorithm times for the values of $\gamma^p$ around 0.4. In addition, it can be observed that the difference between the number of inner and outer approximation vertices and the number of $P_2$ problems increase drastically for the values of $\gamma^p$ around 0.5. This happens because the boundaries of the corresponding Eisenberg-Noe systemic risk measures in Figure 6 contain “flat” regions, which makes the algorithm solve more $P_2$ problems without actually improving the approximation. Observe from Figure 6 that as $\gamma^p$ increases, each subsequent Eisenberg-Noe systemic risk measure is contained in the previous one. This result is fully consistent with the
Table 5: Computational performance of the algorithm for $\gamma^p \in \{0.01, 0.1, \ldots, 0.9, 0.95, 0.99, 1\}$.

4.3.2 Distribution of nodes among groups

In this part, we perform a sensitivity analysis with respect to the distribution of nodes among the groups for a fixed total number of nodes $n = 70$. We take the number of big banks $n_1$ in the set \{5, 10, 20, \ldots, 60, 65\}. Then, the number of small banks is $n_2 = n - n_1$. The generated random operating cash flows remain the same all the time, while the network structure changes at each run.

Figure 6: Inner approximations of the Eisenberg-Noe systemic risk measure for $\gamma^p \in \{0.01, 0.1, \ldots, 0.9, 0.95, 0.99, 1\}$.
Hence, the corresponding Eisenberg-Noe systemic risk measures are expected to vary significantly.

Table 6: Computational performance of the algorithm for $n_1 \in \{5, 10, 20, \ldots, 60, 65\}$.

| $n_1$ | Inner approx. vertices | Outer approx. vertices | $\mathcal{P}_2$ problems | Avg. time per $\mathcal{P}_2$ prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|-------|-------------------------|------------------------|--------------------------|--------------------------------------------|-----------------------------|-----------------------------|
| 5     | 93                      | 94                     | 1096                     | 16.88                                      | 18 501                      | 5.139                       |
| 10    | 234                     | 235                    | 461                      | 15.285                                     | 7 047                       | 1.957                       |
| 20    | 209                     | 210                    | 209                      | 38.512                                     | 8 049                       | 2.236                       |
| 30    | 201                     | 202                    | 201                      | 45.225                                     | 9 090                       | 2.525                       |
| 40    | 213                     | 214                    | 213                      | 55.444                                     | 11 809                      | 3.280                       |
| 50    | 250                     | 251                    | 250                      | 61.329                                     | 15 332                      | 4.259                       |
| 60    | 403                     | 404                    | 639                      | 79.577                                     | 50 850                      | 14.125                      |
| 65    | 205                     | 206                    | 1092                     | 131.431                                    | 143 523                     | 39.867                      |

Table 6 shows the computational performance of the algorithm for $n_1 \in \{5, 10, 20, \ldots, 60, 65\}$ and Figure 7 represents the corresponding inner approximations of the Eisenberg-Noe systemic risk measures.

Note that the average time per $\mathcal{P}_2$ problem in Table 6 tends to increase as the number of big banks increases. This happens because the highest connectivity probability, $q_{1,1}^\text{con} = 0.7$, is the probability that one big bank is liable to another big bank. Hence, as the number of big banks increases, the nodes in the network become more connected with liabilities and it takes more time to solve a $\mathcal{P}_2$ problem because the MILP formulations of $\mathcal{P}_2$ problems get more complex in terms of constraints. In addition, it can be observed that the difference between the numbers of inner and outer approximation vertices and the number of $\mathcal{P}_2$ problems increases as the distribution of nodes changes toward the two extreme cases: 5 big banks and 65 big banks. As in the previous sensitivity analysis, this happens because the boundaries of the Eisenberg-Noe systemic risk measures around these extreme cases in Figure 7 contain “flat” regions, which makes the algorithm solve more $\mathcal{P}_2$ problems.

Figure 7: Inner approximations of the Eisenberg-Noe systemic risk measure for $n_1 \in \{5, 10, 20, \ldots, 60, 65\}$.
problems without actually improving the approximation.

We observe from Figure 7 that as the number of big banks increases and the number of small banks decreases, the small banks get a wider range of capital allocation options, as opposed to the big banks. This happens because the total number of banks is fixed and the group with less number of banks has a wider range of capital allocation options since it has more claims to the other group’s banks. When the number of banks in each group is evenly distributed, the group of big banks has a wider range of capital allocation options. The reason lies behind connectivity probabilities. Recall that for this set-up it is assumed that the connectivity probability from big banks to small banks is $q_{12}^{\text{con}} = 0.1$, while the connectivity probability from small banks to big banks is $q_{21}^{\text{con}} = 0.5$. It means that small banks are more likely to be liable to big banks and, since big banks have more claims compared to small banks, they have a wider range of capital allocation options.

4.4 A two-group Rogers-Veraart networks with 45 nodes

In this section, we consider a Rogers-Veraart network $(\mathcal{N}, \pi, \bar{p}, X, \alpha, \beta)$ generated with the following parameters: $n = 45$, $n_1 = 15$, $n_2 = 30$, $K = 50$, $\rho = 0.05$ and

$$q^{\text{con}} = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & 0.5 \end{bmatrix}, \quad l^{\text{gr}} = \begin{bmatrix} 200 & 100 \\ 50 & 50 \end{bmatrix}.$$ 

In addition, the liquid fraction of the random operating cash flows available to a defaulting node is fixed as $\alpha = 0.7$, and the liquid fraction of the realized claims available to a defaulting node is fixed as $\beta = 0.9$. The shape and scale parameters of gamma distributions of the random operating cash flows $X_i, i \in \mathcal{N}_\ell, \ell \in \mathcal{G}$, are chosen as

$$\kappa = [100 \ 64], \quad \theta = [1 \ 2.25].$$

Then the mean values of the random operating cash flows in the corresponding groups are

$$\nu = [100 \ 80]$$

and the common standard deviation is $\sigma = 10$. In the corresponding Rogers-Veraart systemic risk measure, we take $\gamma^p = 0.9$. The approximation error in the algorithm is taken as $\epsilon = 1$.

4.4.1 Rogers-Veraart $\alpha$ parameter

In this part, we perform a sensitivity analysis with respect to $\alpha$, the liquid fraction of the operating cash flow that can be used by a defaulting node to meet its obligations. The generated network

| $\alpha$ | Inner approx. vertices | Outer approx. vertices | $P_2$ problems | Avg. time per $P_2$ prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|----------|------------------------|------------------------|---------------|-----------------------------------|-------------------------------|-------------------------------|
| 0.1      | 273                    | 274                    | 333           | 12.165                            | 4,051                         | 1.125                         |
| 0.3      | 461                    | 462                    | 484           | 10.572                            | 5,117                         | 1.421                         |
| 0.5      | 592                    | 593                    | 602           | 5.231                             | 3,149                         | 0.875                         |
| 0.7      | 583                    | 584                    | 584           | 3.876                             | 2,264                         | 0.629                         |
| 0.9      | 589                    | 590                    | 589           | 3.395                             | 2,000                         | 0.555                         |

Table 7: Computational performance of the algorithm for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.
$\langle \mathcal{N}, \pi, \bar{p}, X, \alpha, \beta \rangle$ remains the same in all cases. Table 7 illustrates the computational performance of the algorithm for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and Figure 8 consists of the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

Figure 8: Inner approximations of the Rogers-Veraart systemic risk measures for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Note from Table 7 that the average time per $\mathcal{P}_2$ problem decreases with $\alpha$. It can be presumed that this happens because of the following observation: as $\alpha$ parameter increases, the discontinuity in the fixed-point characterization of clearing vectors in the Rogers-Veraart model in (2.13) decreases and it gets easier to solve the corresponding MILP formulation of a $\mathcal{P}_2$ problem because it contains the constraints of (2.14), the MILP characterization of clearing vectors in the Rogers-Veraart model.

Observe from Figure 8 that the Rogers-Veraart systemic risk measures expand significantly as $\alpha$ increases. It means that both big and small banks get less strict capital requirements as default costs decrease. One can also observe that in each case allocating zero capital requirement to the groups is not an available option. In addition, in each case big banks can be allocated a negative amount of capital requirement given that the capital requirements for small banks are high enough. On the other hand, small banks do not have this privilege.

| $\beta$ | Inner approx. vertices | Outer approx. vertices | $\mathcal{P}_2$ problems | $\mathcal{P}_2$ Avg. time per prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|---------|------------------------|------------------------|--------------------------|------------------------------------------|-------------------------------|-----------------------------|
| 0.1     | 187                    | 189                    | 214                      | 5.014                                    | 1 073                         | 0.298                       |
| 0.3     | 223                    | 225                    | 270                      | 5.561                                    | 1 502                         | 0.417                       |
| 0.5     | 323                    | 324                    | 350                      | 3.733                                    | 1 307                         | 0.363                       |
| 0.7     | 394                    | 395                    | 401                      | 3.710                                    | 1 488                         | 0.413                       |
| 0.9     | 583                    | 584                    | 584                      | 3.876                                    | 2 264                         | 0.629                       |

Table 8: Computational performance of the algorithm for $\beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.
4.4.2 Rogers-Veraart $\beta$ parameter

In this part, we perform a sensitivity analysis with respect to $\beta$, the liquid fraction of the realized claims from the other nodes that can be used by a defaulting node to meet its obligations. The generated network $(\mathcal{N}, \pi, \bar{p}, X, \alpha, \beta)$ remains the same in all cases. Table 8 shows the computational performance of the algorithm for $\beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and Figure 9 provides the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

Note from Table 8 that the total number of $\mathcal{P}_2$ problems increases with $\beta$. We can observe smaller average times per $\mathcal{P}_2$ problem for higher values of $\beta$. As in the case of the $\alpha$ parameter, it can be presumed that this happens because of the following observation: as $\beta$ parameter increases, the discontinuity in the fixed-point characterization of clearing vectors in the Rogers-Veraart model in (2.13) decreases, which makes it easier to solve the MILP formulation of a $\mathcal{P}_2$ problem.

Observe from Figure 9 that the Rogers-Veraart systemic risk measures expand significantly as $\beta$ increases. It means that both big and small banks get less strict capital requirements if defaulting banks are able to use larger fractions of realized claims. It can also be observed that in each case allocating zero capital requirement to the groups is not an available option. In addition, if $\beta = 0.9$ then big banks can be allocated a negative amount of capital requirement given that the capital requirements for small banks are high enough. On the other hand, small banks do not have this privilege.

4.4.3 Threshold level

In this part, different $\gamma^p$ levels are compared. Table 9 shows the computational performance of the algorithm for $\gamma^p \in \{0.1, 0.2, \ldots, 0.9, 0.95, 0.99, 1\}$ and Figure 10 consists of the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

It can be noted from Table 9 that the average time per $\mathcal{P}_2$ problem and the total algorithm time are high for $\gamma^p$ values around 0.7. In addition, the number of $\mathcal{P}_2$ problems increases up to $\gamma^p = 0.9$ and then decreases. Similar to the structure in Figure 6, we observe in Figure 10 that the Rogers-Veraart systemic risk measures with smaller $\gamma^p$ values contain the ones that have higher $\gamma^p$
values, which is consistent with the definition of these risk measures.

### 4.4.4 Distribution of nodes among groups

In this part, we perform a sensitivity analysis by changing the distribution of nodes among the groups for a fixed total number of nodes \( n = 45 \) where the number of big banks \( n_1 \) takes values in \( \{5, 10, 15, 20, 25, 30, 35, 40\} \). Then, the number of small banks is \( n_2 = n - n_1 \). Table 10 shows the computational performance of the algorithm and Figure 11 provides the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

Note that the average time per \( P_2 \) problem in Table 10 is relatively high for the values \( n_1 \in \{0.1, 0.2, \ldots, 0.9, 0.95, 0.99, 1\} \).

| \( \gamma^p \) | Inner approx. vertices | Outer approx. vertices | \( P_2 \) problems | Avg. time per \( P_2 \) prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|----------------|------------------------|------------------------|-------------------|---------------------------------|-------------------------------|-----------------------------|
| 0.1            | 1                      | 1                      | 1                 | 0.384                           | 0.384                         | 0                           |
| 0.2            | 13                     | 14                     | 13                | 13.809                          | 180                           | 0.050                       |
| 0.3            | 51                     | 52                     | 51                | 30.273                          | 1544                          | 0.429                       |
| 0.4            | 94                     | 95                     | 94                | 36.645                          | 3445                          | 0.957                       |
| 0.5            | 165                    | 166                    | 165               | 98.625                          | 16273                         | 4.520                       |
| 0.6            | 223                    | 224                    | 223               | 138.532                         | 30089                         | 8.581                       |
| 0.7            | 389                    | 390                    | 389               | 204.288                         | 79468                         | 22.075                      |
| 0.8            | 395                    | 396                    | 395               | 91.600                          | 36182                         | 10.051                      |
| 0.9            | 583                    | 584                    | 584               | 3.876                           | 2264                          | 0.629                       |
| 0.95           | 418                    | 419                    | 431               | 2.946                           | 1270                          | 0.353                       |
| 0.99           | 66                     | 67                     | 74                | 1.639                           | 121                           | 0.034                       |
| 1.00           | 1                      | 1                      | 1                 | 0.132                           | 0.132                         | 0                           |

Table 9: Computational performance of the algorithm for \( \gamma^p \in \{0.1, 0.2, \ldots, 0.9, 0.95, 0.99, 1\} \).
Figure 11: Inner approximations of the Rogers-Veraart systemic risk measure for \( n_1 \in \{5, 10, 15, 20, 25, 30, 35, 40\} \). In addition, the number of \( P_2 \) problems is greater for the values around \( n_1 = 20 \). Observe from Figure 11 that as the number of big banks increases and the number of small banks decreases, the small banks get a wider range of capital allocation options, as opposed to the big banks. This happens because the total number of banks is fixed and the group with less number of banks has a wider range of capital allocation options since it has more claims to the other group’s banks in the scope of this set-up.

| \( n_1 \) | Inner approx. vertices | Outer approx. vertices | \( P_2 \) problems | Avg. time per \( P_2 \) prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|--------|------------------------|------------------------|----------------|-----------------------------|-----------------------------|-----------------------------|
| 5      | 6                      | 7                      | 6              | 1.006                       | 6                           | 0.002                       |
| 10     | 436                    | 437                    | 436            | 3.994                       | 1742                        | 0.484                       |
| 15     | 583                    | 584                    | 584            | 3.876                       | 2264                        | 0.629                       |
| 20     | 516                    | 517                    | 517            | 7.887                       | 4078                        | 1.133                       |
| 25     | 557                    | 558                    | 557            | 6.118                       | 3408                        | 0.947                       |
| 30     | 371                    | 372                    | 371            | 5.786                       | 2147                        | 0.596                       |
| 35     | 187                    | 188                    | 187            | 6.100                       | 1141                        | 0.317                       |
| 40     | 106                    | 107                    | 108            | 5.196                       | 561                         | 0.156                       |

Table 10: Computational performance of the algorithm for \( n_1 \in \{5, 10, 15, 20, 25, 30, 35, 40\} \).

| Inner approx. vertices | Outer approx. vertices | \( P_2 \) problems | Avg. time per \( P_2 \) prob. (seconds) | Total algorithm time (seconds) | Total algorithm time (hours) |
|------------------------|------------------------|----------------|-----------------------------|-----------------------------|-----------------------------|
| 413                    | 516                    | 1250           | 2.904                       | 3631                        | 1.009                       |

Table 11: Computational performance of the algorithm for a signed Eisenberg-Noe network with 10 big, 20 medium and 30 small banks, 50 scenarios and approximation error \( \epsilon = 20 \).
4.5 A three-group signed Eisenberg-Noe network with 60 nodes

In this section, we consider a three-group signed Eisenberg-Noe network \((N, \pi, \bar{p}, X)\) generated with \(n = 60, n_1 = 10, n_2 = 20, n_3 = 30, K = 50, \sigma = 100, \varrho = 0.05\) and
\[
q^\text{con} = \begin{bmatrix}
0.4 & 0.2 & 0.1 \\
0.3 & 0.4 & 0.1 \\
0.2 & 0.3 & 0.4
\end{bmatrix}, \quad l^\text{gr} = \begin{bmatrix}
20 & 15 & 8 \\
15 & 10 & 6 \\
8 & 6 & 5
\end{bmatrix}, \quad \nu = [-50, -100, -150].
\]

In the corresponding Eisenberg-Noe systemic risk measure, we take \(\gamma^p = 0.95\).

Table 11 shows the computational performance of the algorithm for \(\epsilon = 20\). Figure 12 represents the inner approximation of the corresponding three-group Eisenberg-Noe systemic risk measure. It can be presumed that the value of this Eisenberg-Noe systemic risk measure is convex.

![Figure 12: Inner approximation of the three-group Eisenberg-Noe systemic risk measure with 60 nodes, 50 scenarios and approximation error \(\epsilon = 20\).](image)

4.6 A three-group Rogers-Veraart network with 60 nodes

In this section, we consider a Rogers-Veraart network \((N, \pi, \bar{p}, X, \alpha, \beta)\) generated with \(n = 60, n_1 = 10, n_2 = 20, n_3 = 30, K = 50, \varrho = 0.05\), and
\[
q^\text{con} = \begin{bmatrix}
0.4 & 0.2 & 0.1 \\
0.2 & 0.3 & 0.2 \\
0.1 & 0.2 & 0.2
\end{bmatrix}, \quad l^\text{gr} = \begin{bmatrix}
200 & 190 & 180 \\
190 & 190 & 180 \\
180 & 180 & 170
\end{bmatrix}.
\]

In addition, the liquid fraction of the random operating cash flows and the liquid fraction of the realized claims available to defaulting banks are fixed as \(\alpha = \beta = 0.9\). The shape and scale parameters of gamma distributions of \(X_i, i \in N_\ell, \ell \in G\), are chosen as
\[
\kappa = [100, 81, 64], \quad \theta = [1, \frac{10}{9}, 1.25].
\]
Table 12: Computational performance of the algorithm for a Rogers-Veraart network with 10 big, 20 medium and 30 small banks, 50 scenarios and approximation error $\epsilon = 40$.

In the corresponding Rogers-Veraart systemic risk measure, we take $\gamma^p = 0.99$. The upper bound point in the approximation is taken as $z^{UB} = z^{ideal} + \frac{1}{5} \|\bar{p}\|_{\infty}$.

Table 12 shows the computational performance of the algorithm for $\epsilon = 40$. Figure 13 provides the inner approximation of the corresponding three-group Rogers-Veraart systemic risk measure. It can be observed that the value of this Rogers-Veraart systemic risk measure is not convex.

A Proofs of some results in Section 2

A.1 Proof of Lemma 2.11

Proof. If $s_i = 0$, then constraint (2.10) is infeasible by assumption. Hence, $s_i = 1$, and this yields $p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i$ and $p_i \leq \bar{p}_i$, by constraints (2.8) and (2.9), respectively. Hence,

$$p_i \leq \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\}.$$
To get a contradiction to the claim of the lemma, suppose that \( p_i < \min \{ \sum_{j=1}^{n} \pi_j p_j + x_i, \bar{p}_i \} \).

Now let \( p^\epsilon \in \mathbb{R}_+^n \) be equal to \( p \) in all components except the \( i \)th one, and let \( p_i' = p_i + \epsilon \), where

\[
\epsilon := \min \left\{ \min \left\{ \sum_{j=1}^{n} \pi_j p_j + x_i, \bar{p}_i \right\} - p_i, M - \max_{l \in \mathcal{N}} \left( \sum_{j=1}^{n} \pi_j p_j + x_l \right) \epsilon' \right\} > 0,
\]

and

\[
\epsilon' := \min \left\{ \left| \sum_{j=1}^{n} \pi_j p_j + x_l \right| \left| \sum_{j=1}^{n} \pi_j p_j + x_l < 0, l \in \mathcal{N} \right\}. \tag{2.7}
\]

(Here, we assume that \( \epsilon' = +\infty \) if there is no qualifying \( l \in \mathcal{N} \) in the above definition.) This choice of \( \epsilon \) ensures

\[
p_i' \leq \bar{p}_i \quad \text{and} \quad p_i' \leq \sum_{j=1}^{n} \pi_j p_{j}^\epsilon + x_i,
\]

and will also be justified by other technical details later in this proof.

Let \( s^\epsilon \in \{0,1\}^n \) be a vector of binaries, where \( s_i^\epsilon = 0 \) if \( \sum_{j=1}^{n} \pi_j p_{j}^\epsilon + x_l < 0 \) and \( s_i^\epsilon = 1 \) if \( \sum_{j=1}^{n} \pi_j p_{j}^\epsilon + x_l \geq 0 \), for each \( l \in \mathcal{N} \). We show that \( (p^\epsilon, s^\epsilon) \) is a feasible solution to \( \Lambda^{\mathcal{EN}}(x) \) by showing that all constraints in \((2.7)\) are satisfied. First, for fixed \( k \in \mathcal{N} \setminus \{i\} \), we verify the \( k \)th constraints in \((2.7)\) for \((p^\epsilon, s^\epsilon)\). We consider three cases:

1. Assume that \( \sum_{j=1}^{n} \pi_j p_{j} + x_k < 0 \). If \( s_k = 1 \), then, by constraint \((2.8)\),

\[
p_k \leq \sum_{j=1}^{n} \pi_j p_{j} + x_k + M (1 - 1) = \sum_{j=1}^{n} \pi_j p_{j} + x_k < 0,
\]

which is a contradiction to the feasibility of \((p, s)\) in constraint \((2.8)\). Hence, \( s_k = 0 \), which in its turn implies \( p_k = 0 \) by \((2.9)\) and \((2.11)\).

By the definitions of \( p^\epsilon \) and \( s^\epsilon \), it holds that \( p_k^\epsilon = p_k = 0 \) since \( k \neq i \), and \( s_k^\epsilon = 0 \). Constraint \((2.8)\) holds as

\[
p_k^\epsilon = p_k = 0 \leq \sum_{j=1}^{n} \pi_j p_{j}^\epsilon + x_k + M (1 - s_k^\epsilon) = \sum_{j=1}^{n} \pi_j p_{j} + x_k + M \epsilon \pi_{ik}
\]

by the feasibility of \( p_k = 0 \) and \( s_k = 0 \), and since \( \epsilon > 0 \) and \( \pi_{ik} \geq 0 \). Constraint \((2.10)\) holds as

\[
\sum_{j=1}^{n} \pi_j p_{j}^\epsilon + x_k = \sum_{j=1}^{n} \pi_j p_{j} + x_k + \epsilon \pi_{ik} \leq \sum_{j=1}^{n} \pi_j p_{j} + x_k + \epsilon \leq M s_k^\epsilon
\]

since \( \sum_{j=1}^{n} \pi_j p_{j} + x_k < 0 \), \( \pi_{ik} \leq 1 \) and since a small enough \( \epsilon > 0 \) is taken to ensure \( \sum_{j=1}^{n} \pi_j p_{j} + x_k + \epsilon \leq 0 \). Constraints \((2.9)\), \((2.11)\), and \((2.12)\) for node \( k \) hold trivially by the feasibility of \( p_k = 0 \) and \( s_k = 0 \). Hence, \( p_k^\epsilon = 0 \) and \( s_k^\epsilon = 0 \) satisfy the corresponding constraints in \((2.7)\).
(2) Assume that $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$. Now, either $s_k = 0$ or $s_k = 1$. If $s_k = 0$, then $p_k = 0$ by constraints (2.9) and (2.11). If $s_k = 1$, then, by the assumption of this case and (2.8), $p_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + M (1 - 1) = 0$, which, together with (2.11), implies $p_k = 0$.

Also, $p_k' = p_k = 0$ and $s_k' = 1$, by the definitions of $p^\epsilon$ and $s^\epsilon$. Constraint (2.8) holds as

$$p_k' = p_k = 0 \leq \sum_{j=1}^{n} \pi_{jk} p_j^\epsilon + x_k + M (1 - s_k^\epsilon)$$

$$= \sum_{j=1}^{n} \pi_{jk} p_j + x_k + M (1 - 1) + \epsilon \pi_{ik} = \epsilon \pi_{ik},$$

since $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$, $\epsilon > 0$ and $\pi_{ik} \geq 0$. Constraint (2.10) holds as

$$\sum_{j=1}^{n} \pi_{jk} p_j^\epsilon + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} = \epsilon \pi_{ik} \leq M s_k^\epsilon = M$$

since $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$, $\epsilon \leq \min_{l \in \mathcal{N}} \left\{ M - \left( \sum_{j=1}^{n} \pi_{jl} p_j + x_l \right) \right\} \leq M$ by the definition of $\epsilon$, and $0 \leq \pi_{ik} \leq 1$. It is easy to observe that all other constraints in (2.7) for node $k$ are satisfied trivially by $p_k' = 0$ and $s_k' = 1$.

(3) Assume that $0 < \sum_{j=1}^{n} \pi_{jk} p_j + x_k$. If $s_k = 0$, then, by constraint (2.10),

$$\sum_{j=1}^{n} \pi_{jk} p_j + x_k \leq M s_k = 0,$$

which is a contradiction to the assumption. Hence, $s_k = 1$. Also, $s_k' = 1$, by the definition of $s^\epsilon$.

Since $s_k = 1$, (2.9) and (2.11) hold by the feasibility of $p_k$ since $p_k' = p_k$ for $k \neq i$. Also, (2.10) holds since $\epsilon > 0$ is taken small enough to ensure

$$\sum_{j=1}^{n} \pi_{jk} p_j^\epsilon + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} \leq M. \quad \text{(A.1)}$$

Indeed, recall the assumption $\sum_{j=1}^{n} \pi_{jl} < n$, for each $l \in \mathcal{N}$. Hence, for each $l \in \mathcal{N}$ and for every $p \in [0, \bar{p}]$, $\sum_{j=1}^{n} \pi_{jl} p_j + x_l < M$, where $M = n \|\bar{p}\|_{\infty} + \|x\|_{\infty}$. So, (A.1) is guaranteed by the choice of $\epsilon$. (This is the reason behind including the term $M - \max_{l \in \mathcal{N}} \left( \sum_{j=1}^{n} \pi_{jl} p_j + x_l \right)$ in the definition of $\epsilon$.)

Note that, since $s_k = 1$, $p_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k$ holds. Then constraint (2.8) is satisfied since

$$p_k' = p_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik}$$

$$= \sum_{j \in \mathcal{N}} \pi_{jk} p_j + \pi_{ik} (p_i + \epsilon) + x_k = \sum_{j=1}^{n} \pi_{jk} p_j^\epsilon + x_k.$$
Constraint (2.12) is satisfied trivially. Hence, \( p'_k \) and \( s'_k \) satisfy the corresponding constraints in (2.7).

Next, we show that \( p'_i \) and \( s'_i \) satisfy the constraints in (2.7) for \( i \). It holds \( s'_i = 1 \), since \( \sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0 \) by the assumption of Lemma 2.11. Then, constraints (2.9) and (2.11) hold since \( p'_i = p_i + \epsilon > 0 \) and \( p'_i = p_i + \epsilon \leq p_i + \bar{p}_i - p_i \leq \bar{p}_i \) holds since \( \epsilon \leq \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} - \bar{p}_i \). Constraint (2.8) holds as

\[
p'_i = p_i + \epsilon \leq p_i + \sum_{j=1}^{n} \pi_{ji} p_j + x_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i
\]

\[
\leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} = \sum_{j \in N \setminus \{i\}} \pi_{jk} p_j + \pi_{ik} (p_i + \epsilon) + x_k = \sum_{j=1}^{n} \pi_{jk} p'_j + x_k,
\]

where \( \epsilon \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i - p_i \) holds since \( \epsilon \leq \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} - \bar{p}_i \). Constraint (2.10) holds as

\[
\sum_{j=1}^{n} \pi_{ji} p'_j + x_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i + \epsilon \pi_{ii} = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq M
\]

by the feasibility of \( p \) and since \( \pi_{ii} = 0 \), for each \( l \in N \). Constraint (2.12) is satisfied trivially. Hence, \( p'_i \) and \( s'_i \) satisfy the corresponding constraints in (2.7).

Hence, \((p^\epsilon, s^\epsilon)\) is a feasible solution to \( \Lambda^{\text{EN}}(x) \). However, since \( p^\epsilon \geq p \) with \( p^\epsilon \neq p \) and \( f \) is a strictly increasing function, it holds that \( f(p^\epsilon) > f(p) \), which is a contradiction to the optimality of \( p \). Hence, \( p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} \).

**A.2 Proof of Lemma 2.20**

**Proof.** To get a contradiction, suppose that \( s_i = 0 \). Then \( p_i \leq x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i \) by constraint (2.16) and the assumption. Let \( p' \in \mathbb{R}_{+}^{n} \) be equal to \( p \) in all components except the \( i \)-th one, and let \( p'_i = \bar{p}_i \). Also, let \( s' \in \mathbb{R}^{n} \) be equal to \( s \) in all components except the \( i \)-th one, and let \( s'_i = 1 \).

We show that \((p', s')\) is a feasible solution to \( \Lambda^{\text{RV+}}(x) \) by checking that all constraints in (2.15) are satisfied. First, for fixed \( k \in N \setminus \{i\} \), we verify the \( k \)-th constraints in (2.15) for \((p', s')\). Constraints (2.16) and (2.17) hold as

\[
p'_k = p_k \leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k
\]

\[
\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k + \pi_{ik} (\bar{p}_i - p_i) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p'_j + \bar{p}_k s'_k,
\]

and

\[
\bar{p}_k s'_k = \bar{p}_k s_k \leq x_k + \sum_{j=1}^{n} \pi_{jk} p'_j \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j + \pi_{ik} (\bar{p}_i - p_i) = x_k + \sum_{j=1}^{n} \pi_{jk} p'_j,
\]

since \( p'_k = p_k, s'_k = s_k \) for every \( k \in K \) such that \( k \neq i, \bar{p}_i - p_i > 0, \pi_{ik} \geq 0 \), and by the feasibility of \((p, s)\). Constraints (2.18), (2.19) hold trivially by the feasibility of \((p, s)\).
Next, we verify the $i^{th}$ constraints in (2.15) for $p'_i = \bar{p}_i$, $s'_i = 1$. Constraints (2.16) and (2.17) hold as

$$p'_i = \bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p_j + \bar{p}_is'_i$$

$$= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p_j + \bar{p}_i + \pi_{ii}(\bar{p}_i - p_i) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p'_j + \bar{p}_i,$$

and

$$\bar{p}_is'_i = \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji}p_j = x_i + \sum_{j=1}^{n} \pi_{ji}p_j + \pi_{ii}(\bar{p}_i - p_i) = x_i + \sum_{j=1}^{n} \pi_{ji}p'_j,$$

since $\alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p'_j \geq 0$, $\pi_{ii} = 0$ and by the assumption of Lemma 2.20. Constraints (2.18), (2.19) are satisfied trivially.

Hence, $(p', s')$ is a feasible solution to $\Lambda^{RV^+}(x)$. However, since $p' \geq p$ with $p' \neq p$ and $f$ is a strictly increasing function, it holds that $f(p') > f(p)$, which is a contradiction to the optimality of $p$. Hence, $s_i = 1$. 

**A.3 Proof of Lemma 2.21**

**Proof.** To get a contradiction, suppose that $p_i < \bar{p}_i$. Let $p' \in \mathbb{R}^n_+$ be equal to $p$ in all components except the $i^{th}$ one, and let $p'_i = \bar{p}_i$.

We show that $(p', s)$ is a feasible solution to $\Lambda^{RV^+}(x)$ by showing that all constraints in (2.15) are satisfied. First, for fixed $k \in \mathcal{K}\setminus\{i\}$, we verify the $k^{th}$ constraint in (2.15) for $(p', s)$. Constraints (2.16) and (2.17) hold as

$$p'_k = p_k \leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk}p_j + \bar{p}_ks_k$$

$$\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk}p_j + \bar{p}_ks_k + \pi_{ik}(\bar{p}_i - p_i) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk}p'_j + \bar{p}_ks_k,$$

and

$$\bar{p}_ks_k \leq x_k + \sum_{j=1}^{n} \pi_{jk}p_j \leq x_k + \sum_{j=1}^{n} \pi_{jk}p_j + \pi_{ik}(\bar{p}_i - p_i) = x_k + \sum_{j=1}^{n} \pi_{jk}p'_j,$$

since $p'_k = p_k$ for every $k \in \mathcal{K}$ such that $k \neq i$, $\bar{p}_i - p_i > 0$, $\pi_{ik} \geq 0$ and by the feasibility of $(p, s)$. Constraints (2.18), (2.19) hold trivially by the feasibility of $(p, s)$.

Next, we verify the $i^{th}$ constraints in (2.15) for $p'_i = \bar{p}_i$, $s_i$. We consider two cases:

1. $\bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p_j$,  
2. $\alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji}p_j < \bar{p}_i$. 

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(1) If the first case holds, then constraints (2.16) and (2.17) hold for both \( s_i = 0 \) and \( s_i = 1 \) as

\[
p'_i = \bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i
\]

\[
= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i + \pi_{ii} (\bar{p}_i - p_i) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j + \bar{p}_i s_i,
\]

and

\[
\bar{p}_i s_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} (\bar{p}_i - p_i) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,
\]

since \( \pi_{ii} = 0 \) and by the assumption of Lemma 2.21. Constraints (2.18), (2.19) hold trivially.

(2) If the second case holds, then, by Lemma 2.20, constraints (2.16) and (2.17) hold for both \( s_i = 0 \) and \( s_i = 1 \). Then constraints (2.16) and (2.17) hold as

\[
p'_i = \bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i
\]

\[
= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i + \pi_{ii} (\bar{p}_i - p_i) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j + \bar{p}_i,
\]

and

\[
\bar{p}_i s_i = \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} (\bar{p}_i - p_i) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,
\]

since \( \pi_{ii} = 0 \) and by the assumption of Lemma 2.21. Constraints (2.18), (2.19) are satisfied trivially.

Hence, \((p', s)\) is a feasible solution to \( \Lambda^{RV+} (x) \). However, since \( p' \geq p \) with \( p' \neq p \) and \( f \) is a strictly increasing function, it holds that \( f(p') > f(p) \), which is a contradiction to the optimality of \( p \). Hence, \( p_i = \bar{p}_i \).

A.4 Proof of Lemma 2.22

Proof. To get a contradiction, suppose that \( p_i \neq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j \). If \( s_i = 1 \), then constraint (2.17) is not satisfied by assumption. Hence, \( s_i = 0 \) and \( p_i < \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j \) by constraint (2.16). Let \( p' \in \mathbb{R}^n_+ \) be equal to \( p \) in all components except the \( i \)th one, and let \( p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j \).

We show that \((p', s)\) is a feasible solution to \( \Lambda^{RV+} (x) \) by checking that all constraints in (2.15) are satisfied. First, for fixed \( k \in \mathcal{N} \setminus \{i\} \), we verify the \( k \)th constraints in (2.15) for \((p', s)\). Constraints (2.16) and (2.17) hold as

\[
p'_k = p_k \leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k
\]

\[
\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k + \pi_{ik} (\bar{p}_i - p_i) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p'_j + \bar{p}_k s_k,
\]

42
and
\[
\bar{p}_k s_k \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j + \pi_{ik} (\bar{p}_i - p_i) = x_k + \sum_{j=1}^{n} \pi_{jk} p_j,
\]
since \(p'_k = p_k\) for every \(k \in K\) such that \(k \neq i\), \(\bar{p}_i - p_i > 0\), \(\pi_{ik} \geq 0\) and by the feasibility of \((p, s)\). Constraints (2.18), (2.19) hold trivially by the feasibility of \((p, s)\).

Next, we verify the \(i\)th constraints in (2.15) for \(p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j, s_i = 0\). Constraints (2.16) and (2.17) hold as
\[
p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i
\]
\[= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} (\bar{p}_i - p_i) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j,
\]
and
\[
\bar{p}_i s_i = 0 \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} (\bar{p}_i - p_i) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,
\]
since \(\pi_{ii} = 0\) and \(x_i + \sum_{j=1}^{n} \pi_{ji} p_j \geq 0\). Constraints (2.18), (2.19) are satisfied trivially.

Hence, \((p', s)\) is a feasible solution to \(\Lambda_{RV+}(x)\). However, since \(p' \geq p\) with \(p' \neq p\) and \(f\) is a strictly increasing function, it holds that \(f(p') > f(p)\), which is a contradiction to the optimality of \(p\). Hence, \(p_i = \bar{p}_i\).

**B  Proofs of some results in Section 3**

For convenience, let us rewrite the mixed-integer linear programming problem in (3.13) more explicitly as

\[
\text{minimize} \quad z_{\ell}, \quad (B.1)
\]

\[
\text{subject to} \quad \sum_{k \in K} q^k 1^T p^k \geq \gamma, \quad (B.2)
\]

\[
p^k_i = \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T z)_i) + M(1 - s^k_i), \quad \forall i \in N, k \in K, \quad (B.3)
\]

\[
p^k_i \leq \bar{p}_i s^k_i, \quad \forall i \in N, k \in K, \quad (B.4)
\]

\[
\sum_{j=1}^{n} \pi_{ji} p^k_j + (X_i(\omega^k) + (B^T z)_i) \leq M s^k_i, \quad \forall i \in N, k \in K, \quad (B.5)
\]

\[
0 \leq p^k_i \leq \bar{p}_i, \quad \forall i \in N, k \in K, \quad (B.6)
\]

\[
s^k_i \in \{0, 1\}, \quad \forall i \in N, k \in K, \quad (B.7)
\]

\[
z \in \mathbb{R}^G. \quad (B.8)
\]
B.1 Proof of Proposition 3.6

Proof. Let \((z, (p^k, s^k))_{k \in K}\) be an optimal solution of the problem in (3.13). To get a contradiction, suppose that \(z_\ell > \|X\|_\infty + \|\bar{p}\|_\infty\). Let \(z' \in \mathbb{R}^G\) be the vector such that \(z'_\ell = \|X\|_\infty + \|\bar{p}\|_\infty\) and \(z'_\ell = z_\ell\) for each \(\ell \in G \setminus \{\ell\}\). We claim that \((z', (p^k, s^k))_{k \in K}\) is a feasible solution of the problem in (3.13). Indeed, for each \(i \in \mathcal{N}\), \(k \in K\) such that \((B^T z')_i = \|X\|_\infty + \|\bar{p}\|_\infty\), constraint (B.3) holds as

\[
p^k_i \leq \sum_{j=1}^n \pi_{ji}p_j^k + \left(X_i(\omega^k) + (B^T z')_i\right) + M(1 - s^k_i)
\]

\[
= \sum_{j=1}^n \pi_{ji}p_j^k + X_i(\omega^k) + \|X\|_\infty + \|\bar{p}\|_\infty + M(1 - s^k_i)
\]

since

\[
\sum_{j=1}^n \pi_{ji}p_j^k \geq 0, \quad X_i(\omega^k) + \|X\|_\infty \geq 0, \quad p^k_i \leq \bar{p}_i \leq \|\bar{p}\|_\infty, \quad M(1 - s^k_i) \geq 0.
\]

Also, for each \(i \in \mathcal{N}\), \(k \in K\) such that \((B^T z')_i = \|X\|_\infty + \|\bar{p}\|_\infty\), constraint (B.5) holds as

\[
\sum_{j=1}^n \pi_{ji}p_j^k + \left(X_i(\omega^k) + (B^T z')_i\right) = \sum_{j=1}^n \pi_{ji}p_j^k + X_i(\omega^k) + \|X\|_\infty + \|\bar{p}\|_\infty
\]

\[
< \sum_{j=1}^n \pi_{ji}p_j^k + X_i(\omega^k) + z_\ell \leq M s^k_i,
\]

which holds by the supposition \(\|X\|_\infty + \|\bar{p}\|_\infty < z_\ell\) and the feasibility of \((z, (p^k, s^k))_{k \in K}\). All the other constraints in (B.1) hold by the feasibility of \((z, (p^k, s^k))_{k \in K}\), since they are free of \(\|X\|_\infty + \|\bar{p}\|_\infty\). Hence, the claim follows, which yields \(z_\ell = Z^{\mathrm{EN}}_1(e^\ell) \leq z'_\ell = \|X\|_\infty + \|\bar{p}\|_\infty\). As this is a contradiction, the result follows.

B.2 Proof of Proposition 3.7

Proof. To get a contradiction, suppose that the problem in (3.13) has a feasible solution but \(Z^{\mathrm{EN}}_1(e^\ell) = -\infty\). Since \(Z^{\mathrm{EN}}_1(e^\ell) = -\infty\), there exist \(\epsilon > 0\) and \((z, (p^k, s^k))_{k \in K}\), where \(z \in \mathbb{R}^G\) and \((p^k, s^k) \in \mathbb{R}^n \times \mathbb{Z}^n\) for each \(k \in K\), such that \(e^T z = z_\ell = -2M\) and \((z - \epsilon e^\ell, (p^k, s^k))_{k \in K}\) is a feasible solution for the problem in (3.13). Fix \(i \in \mathcal{N}\), \(k \in K\) such that \((B^T z)_i = z_\ell = -2M\). Then, constraint (B.3) contradicts constraint (B.0) as

\[
p^k_i \leq \sum_{j=1}^n \pi_{ji}p_j^k + \left(X_i(\omega^k) + (B^T (z - \epsilon e^\ell))_i\right) + M(1 - s^k_i)
\]

\[
\leq \sum_{j=1}^n \pi_{ji}p_j^k + X_i(\omega^k) - 2M - \epsilon + M
\]

\[
= \sum_{j=1}^n \pi_{ji}p_j^k + X_i(\omega^k) - \epsilon - 2 \|X\|_\infty - (n + 1) \|\bar{p}\|_\infty
\]
Hence, \( \mathbf{Z} \) holds as
\[
\sum_{j=1}^{n} \pi_{ji} p^k_j - n \| \mathbf{p} \|_\infty + \left( X_i (\omega^k) - 2 \| X \|_\infty \right) - \| \mathbf{p} \|_\infty - \epsilon < 0
\]
since
\[
\sum_{j=1}^{n} \pi_{ji} p^k_j < n \| \mathbf{p} \|_\infty , \quad X_i (\omega^k) \leq 2 \| X \|_\infty , \quad - \| \mathbf{p} \|_\infty < 0, \quad -\epsilon < 0.
\]

Hence, \((z - \epsilon \mathbf{e}, (\mathbf{p}^k, s^k)_{k \in \mathcal{K}})\) is infeasible, which is a contradiction to the assumption. Hence, \(Z_1^{EN}(\mathbf{e}_\ell) > -\infty\). In addition, the existence of a feasible solution implies that \(Z_1^{EN}(\mathbf{e}_\ell) < +\infty\). Hence, \(Z_1^{EN}(\mathbf{e}_\ell) \in \mathbb{R}\). \hfill \square

### B.3 Proof of Proposition 3.8

**Proof.** Assume that \(\gamma \leq \mathbf{1}^T \mathbf{p}\). Let \(z = (\| X \|_\infty + \| \mathbf{p} \|_\infty) \mathbf{1}, \mathbf{p}^k = \mathbf{p}, s^k = \mathbf{1}\) for each \(k \in \mathcal{K}\). We show that \((z, (\mathbf{p}^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution for the problem in (B.3). Since \(\mathbf{p}^k = \mathbf{p}\) for each \(k \in \mathcal{K}\), it is clear that \(\sum_{k \in \mathcal{K}} q^k \mathbf{1}^T \mathbf{p}^k = \mathbf{1}^T \mathbf{p} \geq \gamma\). Hence, constraint (B.2) holds. Let \(i \in \mathcal{N}, k \in \mathcal{K}\). Constraint (B.3) holds as
\[
\sum_{j=1}^{n} \pi_{ji} p^k_j + \left( X_i (\omega^k) + (B^T z)_i \right) + M(1 - s^k_i)
\]
\[
= \sum_{j=1}^{n} \pi_{ji} p^k_j + X_i (\omega^k) + (\| X \|_\infty + \| \mathbf{p} \|_\infty)(B^T \mathbf{1})_i + M(1 - 1)
\]
\[
= \sum_{j=1}^{n} \pi_{ji} p^k_j + X_i (\omega^k) + \| X \|_\infty + \| \mathbf{p} \|_\infty \geq \bar{p}_i = p^k_i
\]
since
\[
\sum_{j=1}^{n} \pi_{ji} p^k_j \geq 0, \quad X_i (\omega^k) + \| X \|_\infty \geq 0, \quad (B^T \mathbf{1})_i = 1, \quad s^k_i = 1.
\]

Constraint (B.5) holds as
\[
\sum_{j=1}^{n} \pi_{ji} p^k_j + \left( X_i (\omega^k) + (B^T z)_i \right) = \sum_{j=1}^{n} \pi_{ji} p^k_j + X_i (\omega^k) + \| X \|_\infty + \| \mathbf{p} \|_\infty
\]
\[
\leq 2 \| X \|_\infty + (n + 1) \| \mathbf{p} \|_\infty = M = Ms^k_i,
\]
since \(\sum_{j=1}^{n} \pi_{ji} p^k_j \leq n \| \mathbf{p} \|_\infty\). All the other constraints in (B.1) hold trivially by the choice of \(z, \mathbf{p}^k\) and \(s^k\), for each \(k \in \mathcal{K}\). Hence, \((z, (\mathbf{p}^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution of the problem in (B.3).

Conversely, if \(\gamma > \mathbf{1}^T \mathbf{p}\), then constraint (B.2) is infeasible, since \(\sum_{k \in \mathcal{K}} q^k \mathbf{1}^T \mathbf{p}^k \leq \mathbf{1}^T \mathbf{p} < \gamma\) by constraint (B.6). Hence, the problem in (3.13) is infeasible, which concludes the proof. \hfill \square

The mixed-integer linear programming problem of calculating \(Z_1^{RV+}(\mathbf{e}_\ell)\) in (3.16) can be written more explicitly as
\[
\text{minimize} \quad z_\ell \tag{B.9}
\]
subject to \[ q^k \mathbf{1}^T p^k \geq \gamma, \quad (B.10) \]
\[ p^k_i \leq \alpha \left( X_i (\omega^k) + (B^T z)_i \right) + \beta \sum_{j=1}^{n} \pi_{ji} p^k_j + \tilde{p}_i s^k_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad (B.11) \]
\[ \tilde{p}_i s^k_i \leq \left( X_i (\omega^k) + (B^T z)_i \right) + \sum_{j=1}^{n} \pi_{ji} p^k_j, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad (B.12) \]
\[ X_i (\omega^k) + (B^T z)_i \geq 0, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad (B.13) \]
\[ 0 \leq p^k_i \leq \tilde{p}_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad (B.14) \]
\[ s^k_i \in \{0, 1\}, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad (B.15) \]
\[ z \in \mathbb{R}^G. \quad (B.16) \]

Here, constraint \[(B.13)\] ensures \(X + B^T z \geq 0\) so that \(\Lambda^\text{RV} \left( X (\omega^k) + B^T z \right) \neq -\infty\) for every \(k \in \mathcal{K}\).

### B.4 Proof of Proposition 3.11

**Proof.** Let \((z, (p^k, s^k)_{k \in \mathcal{K}})\) be an optimal solution of the problem in \((3.16)\). To get a contradiction, suppose that \(x^b > \|X\|_{\infty} + \frac{1}{\alpha} \|\bar{p}\|_{\infty}\). Let \(z' \in \mathbb{R}^n\) be the vector such that \(z' = \|X\|_{\infty} + \frac{1}{\alpha} \|\bar{p}\|_{\infty}\) and \(z'_i = z_i\) for each \(i \in \mathcal{N}\). Similar to the argument in the proof of Proposition 3.6, it can be checked that \((z', (p^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution of the problem in \((3.16)\). Hence, \(x^b = 2^\text{RV} \left( e^f \right) \leq z'_i = \|X\|_{\infty} + \frac{1}{\alpha} \|\bar{p}\|_{\infty}\). As this is a contradiction, the result follows. \(\square\)

### B.5 Proof of Proposition 3.12

**Proof.** To get a contradiction, suppose that the problem in \((3.16)\) has a feasible solution but \(Z_1^\text{RV} (e^f) = -\infty\). Let \(M = \|X\|_{\infty} + \frac{1}{\alpha} (n + 1) \|\bar{p}\|_{\infty}\). Since \(Z_1^\text{RV} (e^f) = -\infty\), there exist \(\epsilon > 0\) and \((z, (p^k, s^k)_{k \in \mathcal{K}})\), where \(z \in \mathbb{R}^n\) and \((p^k, s^k) \in \mathbb{R}^n \times \mathbb{Z}^n\) for each \(k \in \mathcal{K}\), such that \(e^f z = z^f = -M\) and \((z - \epsilon e^f, (p^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution for the problem in \((3.16)\). Fix \(i \in \mathcal{N}\), \(k \in \mathcal{K}\) such that \((B^T z)_i = z^f = -M\). Similar to the argument in the proof of Proposition 3.7, it can be checked that constraint \[(B.11)\] contradicts constraint \[(B.14)\]. Hence, \((z - \epsilon e^f, (p^k, s^k)_{k \in \mathcal{K}})\) is infeasible, which is a contradiction to the assumption. Hence, \(Z_1^\text{RV} (e^f) > -\infty\). In addition, the existence of a feasible solution implies that \(Z_1^\text{RV} (e^f) < +\infty\). Hence, \(Z_1^\text{RV} (e^f) \in \mathbb{R}\). \(\square\)

### B.6 Proof of Proposition 3.13

**Proof.** Assume that \(\gamma \leq 1^T \tilde{p}\). Let \(z = (\|X\|_{\infty} + \frac{1}{\alpha} \|\bar{p}\|_{\infty}) 1, \quad p^k = \tilde{p}, \quad s^k = 1 \) for each \(k \in \mathcal{K}\). As in the proof of Proposition 3.8, it can be checked that \((z, (p^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution for the problem in \((3.16)\). Conversely, if \(\gamma > 1^T \tilde{p}\), then constraint \[(B.10)\] is infeasible, since \(\sum_{k \in \mathcal{K}} q^k 1^T p^k \leq 1^T \tilde{p} < \gamma\) by constraint \[(B.14)\]. Hence, the problem in \((3.16)\) is infeasible, which concludes the proof. \(\square\)

The mixed-integer linear programming problem of computing \(Z_2^\text{EN} (v)\) in \((3.21)\) can be written more explicitly as

\[
\text{minimize} \quad \mu \quad \text{subject to} \quad \sum_{k \in \mathcal{K}} q^k 1^T p^k \geq \gamma \quad \text{(B.17)}
\]
subject to \[ \sum_{k \in \mathcal{K}} q^k \mathbf{1}^T p^k \geq \gamma, \quad \text{(B.18)} \]

\[ p_i^k \leq \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T v)_i + \mu) + M(1 - s^k_i), \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad \text{(B.19)} \]

\[ p_i^k \leq \bar{p}_i s^k_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad \text{(B.20)} \]

\[ \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T v)_i + \mu) \leq M s^k_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad \text{(B.21)} \]

\[ 0 \leq p_i^k \leq \bar{p}_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \quad \text{(B.22)} \]

\[ s^k_i \in \{0, 1\}, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}. \quad \text{(B.23)} \]

B.7 Proof of Proposition 3.16

Proof. Let \((\mu, (p^k, s^k)_{k \in \mathcal{K}})\) be an optimal solution of the problem in (3.21). To get a contradiction, suppose that \(\mu > \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}\). We claim that \((\mu_{\text{max}}, (p^k, s^k)_{k \in \mathcal{K}})\) is a feasible solution of the problem in (3.21). Let \(i \in \mathcal{N}, k \in \mathcal{K}\). Note that constraint (B.19) holds as

\[ p_i^k \leq \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T v)_i + \mu_{\text{max}}) + M(1 - s^k_i) \]

\[ = \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i(\omega^k) + (B^T v)_i + \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty} + M(1 - s^k_i) \]

\[ = \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + \|X\|_{\infty}) + ((B^T v)_i + \|v\|_{\infty}) + \|\bar{p}\|_{\infty} + M(1 - s^k_i), \]

since

\[ \sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad X_i(\omega^k) + \|X\|_{\infty} \geq 0, \quad (B^T v)_i + \|v\|_{\infty} \geq 0, \]

\[ p_i^k \leq \|\bar{p}\|_{\infty}, \quad M(1 - s^k_i) \geq 0. \]

Constraint (B.21) holds as

\[ \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T v)_i + \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}) \]

\[ < \sum_{j=1}^{n} \pi_{ji} p_j^k + (X_i(\omega^k) + (B^T v)_i + \mu) \leq M s^k_i = M \]

by the assumption \(\|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty} < \mu\) and the feasibility of \((\mu, (p^k, s^k)_{k \in \mathcal{K}})\). All the other constraints in (B.17) hold by the feasibility of \((\mu, (p^k, s^k)_{k \in \mathcal{K}})\), since they are free of \(\|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}\). Hence, the claim follows, which yields \(\mu = 2_{\text{EN}}^2(v) \leq \|X\|_{\infty} + \|v\|_{\infty} + \|\bar{p}\|_{\infty}\). As this is a contradiction, we obtain the desired result. \(\square\)
B.8 Proof of Proposition 3.17

Proof. To get a contradiction, suppose that the problem in \((3.21)\) has a feasible solution but \(\mathcal{Z}_{2}^{\text{EN}}(\nu) = -\infty\). Then, there exist \(\epsilon > 0\) and \((p^{k}, s^{k})_{k \in \mathcal{K}}\), where \((p^{k}, s^{k}) \in \mathbb{R}^{n} \times \mathbb{Z}^{n}\) for each \(k \in \mathcal{K}\), such that \((-2M - \epsilon, (p^{k}, s^{k})_{k \in \mathcal{K}})\) is a feasible solution of the problem in \((3.21)\). Fix \(i \in \mathcal{N}\), \(k \in \mathcal{K}\). Then constraint \((B.19)\) violates constraint \((B.22)\) as

\[
p^{k}_{i} \leq \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + \left( X_{i}(\omega^{k}) + (B^{T} \nu)_{i} - 2M - \epsilon \right) + M(1 - s^{k}_{i})
\]

\[
\leq \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + X_{i}(\omega^{k}) + (B^{T} \nu)_{i} - \epsilon - M
\]

\[
= \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + X_{i}(\omega^{k}) + (B^{T} \nu)_{i} - \epsilon - 2 \|X\|_{\infty} - 2 \|\nu\|_{\infty} - (n + 1) \|\bar{p}\|_{\infty}
\]

\[
= \left( \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} - (n + 1) \|\bar{p}\|_{\infty} \right) + \left( X_{i}(\omega^{k}) - 2 \|X\|_{\infty} \right) + \left( (B^{T} \nu)_{i} - 2 \|\nu\|_{\infty} \right) - \epsilon < 0,
\]

since

\[
\sum_{j=1}^{n} \pi_{ji} p^{k}_{j} < (n + 1) \|\bar{p}\|_{\infty}, \quad X_{i}(\omega^{k}) \leq 2 \|X\|_{\infty}, \quad (B^{T} \nu)_{i} \leq 2 \|\nu\|_{\infty}, \quad -\epsilon < 0.
\]

Hence, \((-2M - \epsilon, (p^{k}, s^{k})_{k \in \mathcal{K}})\) is infeasible, which is a contradiction to the assumption. Hence, \(\mathcal{Z}_{2}^{\text{EN}}(\nu) > -\infty\). On the other hand, \(\mathcal{Z}_{2}^{\text{EN}}(\nu) < +\infty\) by the existence of a feasible solution. So \(\mathcal{Z}_{2}^{\text{EN}}(\nu) \in \mathbb{R}\).

B.9 Proof of Proposition 3.18

Proof. Assume that \(\gamma \leq \mathbf{1}^{T} \bar{p}\). Let \(\mu = \|X\|_{\infty} + \|\nu\|_{\infty} + \|\bar{p}\|_{\infty}\), \(p^{k} = \bar{p}\), \(s^{k} = \mathbf{1}\) for each \(k \in \mathcal{K}\). We show that \((\mu, (p^{k}, s^{k})_{k \in \mathcal{K}})\) is a feasible solution for the problem in \((3.21)\). Since \(p^{k} = \bar{p}\) for each \(k \in \mathcal{K}\), it holds that \(\sum_{k \in \mathcal{K}} q^{k} \mathbf{1}^{T} p^{k} = \mathbf{1}^{T} \bar{p} \geq \gamma\). Hence, constraint \((B.18)\) holds. Now fix \(i \in \mathcal{N}\), \(k \in \mathcal{K}\). Constraint \((B.19)\) holds as

\[
\sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + \left( X_{i}(\omega^{k}) + (B^{T} \nu)_{i} + \mu \right) + M(1 - s^{k}_{i})
\]

\[
= \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + X_{i}(\omega^{k}) + (B^{T} \nu)_{i} + \mu + M(1 - 1)
\]

\[
= \sum_{j=1}^{n} \pi_{ji} p^{k}_{j} + X_{i}(\omega^{k}) + (B^{T} \nu)_{i} + \|X\|_{\infty} + \|\nu\|_{\infty} + \|\bar{p}\|_{\infty} \geq \bar{p}_{i} = p^{k}_{i},
\]

since

\[
\sum_{j=1}^{n} \pi_{ji} p^{k}_{j} \geq 0, \quad X_{i}(\omega^{k}) + \|X\|_{\infty} \geq 0, \quad (B^{T} \nu)_{i} + \|\nu\|_{\infty} \geq 0,
\]
and $s_i^k = 1$, by the choice of $s^k$. Constraint (B.21) holds as
\[\sum_{j=1}^n \pi_{ji} p_j^k + \left( X_i(\omega^k) + (B^T v)_i + \mu \right) \]
\[= \sum_{j=1}^n \pi_{ji} p_j^k + X_i(\omega^k) + (B^T v)_i + \|X\|_\infty + \|v\|_\infty + \|\bar{p}\|_\infty \]
\[\leq 2 \|X\|_\infty + 2 \|v\|_\infty + (n + 1) \|\bar{p}\|_\infty = M = Ms_i^k,\]
since $\sum_{j=1}^n \pi_{ji} p_j^k \leq n \|\bar{p}\|_\infty$. All the other constraints hold trivially by the choice of $\mu$, $p^k$ and $s^k$, for each $k \in K$. Hence, $(\mu, (p^k, s^k)_{k \in K})$ is a feasible solution for the problem in (3.21).

Conversely, if $\gamma > 1^T \bar{p}$, then constraint (B.18) is infeasible, since $\sum_{k \in K} q^k 1^T p^k \leq 1^T \bar{p} < \gamma$, by constraint (B.22). Hence, the problem in (3.21) is infeasible, which finishes the proof. □

The mixed-integer linear programming problem for calculating $Z_{2,\lambda}^R (v)$ in (3.23) can be written more explicitly as

\[\text{minimize} \quad \mu \quad \text{subject to} \quad \sum_{k \in K} q^k 1^T p^k \geq \gamma, \quad \text{(B.24)}\]
\[p_i^k \leq \alpha \left( X_i(\omega^k) + (B^T v)_i + \mu \right) + \beta \sum_{j=1}^n \pi_{ji} p_j^k + \bar{p}_i s_i^k, \quad \forall i \in \mathcal{N}, k \in K, \quad \text{(B.25)}\]
\[\bar{p}_i s_i^k \leq \left( X_i(\omega^k) + (B^T v)_i + \mu \right) + \sum_{j=1}^n \pi_{ji} p_j^k, \quad \forall i \in \mathcal{N}, k \in K, \quad \text{(B.26)}\]
\[X_i(\omega^k) + (B^T v)_i + \mu \geq 0, \quad \forall i \in \mathcal{N}, k \in K, \quad \text{(B.27)}\]
\[0 \leq p_i^k \leq \bar{p}_i, \quad \forall i \in \mathcal{N}, k \in K, \quad \text{(B.28)}\]
\[s_i^k \in \{0, 1\}, \quad \forall i \in \mathcal{N}, k \in K. \quad \text{(B.29)}\]

Here, constraint (B.28) ensures $X + B^T v + \mu 1 \geq 0$ so that $\Lambda_{2,\lambda}^R (X(\omega^k) + B^T v + \mu 1) \neq +\infty$ for every $k \in K$.

**B.10 Proof of Proposition 3.22**

*Proof.* Let $(\mu, (p^k, s^k)_{k \in K})$ be an optimal solution for the problem in (3.23). To get a contradiction, suppose that $\mu > \|X\|_\infty + \|v\|_\infty + 1 T \bar{p}_\infty$. Following similar arguments as in the proof of Proposition 3.16, it can be shown that $(\|X\|_\infty + \|v\|_\infty + 1 T \bar{p}_\infty, (p^k, s^k)_{k \in K})$ is a feasible solution for the problem in (3.23). Hence, $\mu = Z_{2,\lambda}^R (v) \leq \|X\|_\infty + \|v\|_\infty + 1 T \bar{p}_\infty$, which is a contradiction. Hence, the result follows. □

**B.11 Proof of Proposition 3.23**

*Proof.* To get a contradiction, suppose that the problem in (3.23) has a feasible solution but $Z_{2,\lambda}^R (v) = -\infty$. Let $M = \|X\|_\infty + \|v\|_\infty + 1 T \bar{p}_\infty$. Then, there exist $\epsilon > 0$ and $(p^k, s^k)_{k \in K}$, where $(p^k, s^k) \in \mathbb{R}^n \times \mathbb{Z}^n$ for each $k \in K$, such that $(-M - \epsilon, (p^k, s^k)_{k \in K})$ is a feasible solution for the problem in (3.23). Fix $i \in \mathcal{N}, k \in K$. As in the proof of Corollary 3.17, it can be checked
that constraint (B.26) violates constraint (B.29). Hence, \((-M - \epsilon, (p^k, s^k)_{k \in K})\) is infeasible, which is a contradiction to the assumption. Hence, \(Z^{RV+}_2(\nu) > -\infty\). Together with the feasibility of the problem, it follows that \(Z^{RV+}_2(\nu) \in \mathbb{R}\). □

### B.12 Proof of Proposition 3.24

**Proof.** Assume that \(\gamma \leq 1^T\bar{p}\). Let \(\mu = \|X\|_\infty + \|v\|_\infty + \frac{1}{\sigma} \|\bar{p}\|_\infty\), \(p^k = \bar{p}, s^k = 1\) for each \(k \in K\). As in the proof of Corollary 3.24, it can be shown that \((\mu, (p^k, s^k)_{k \in K})\) is a feasible solution for the problem in (3.23). Conversely, if \(\gamma > 1^T\bar{p}\), then constraint (B.25) is infeasible, since \(\sum_{k \in K} q^k 1^T p^k \leq 1^T \bar{p} < \gamma\), by constraint (B.29). Hence, the problem in (3.23) is infeasible, which concludes the proof. □

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