Refined Coding Bounds and Code Constructions for Coherent Network Error Correction

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Abstract

Coherent network error correction is the error-control problem in network coding with the knowledge of the network codes at the source and sink nodes. With respect to a given set of local encoding kernels defining a linear network code, we obtain refined versions of the Hamming bound, the Singleton bound and the Gilbert-Varshamov bound for coherent network error correction. Similar to its classical counterpart, this refined Singleton bound is tight for linear network codes. The tightness of this refined bound is shown by two construction algorithms of linear network codes achieving this bound. These two algorithms illustrate different design methods: one makes use of existing network coding algorithms for error-free transmission and the other makes use of classical error-correcting codes. The implication of the tightness of the refined Singleton bound is that the sink nodes with higher maximum flow values can have higher error correction capabilities.

Index Terms

Network error correction, network coding, Hamming bound, Singleton bound, Gilbert-Varshamov bound, network code construction.

I. INTRODUCTION

Network coding has been extensively studied for multicasting information in a directed communication network when the communication links in the network are error free. It was shown by Ahlswede et al. [1] that the network capacity for multicast satisfies the max-flow min-cut theorem, and this capacity can be achieved by network coding. Li, Yeung, and Cai [2] further showed that it is sufficient to consider linear network codes only. Subsequently, Koetter and Médard [3] developed a matrix framework for network coding. Jaggi et al. [4] proposed a deterministic polynomial-time algorithm to construct linear network codes. Ho et al. [5] showed that optimal linear network codes can be efficiently constructed by a randomized algorithm with an exponentially decreasing probability of failure.
A. Network Error Correction

Researchers also studied how to achieve reliable communication by network coding when the communication links are not perfect. For example, network transmission may suffer from link failures [3], random errors [6] and maliciously injected errors [7]. We refer to these distortions in network transmission collectively as errors, and the network coding techniques for combating errors as network error correction.

Fig. 1 shows one special case of network error correction with two nodes, one source node and one sink node, which are connected by parallel links. This is the model studied in classical algebraic coding theory [8], [9], a very rich research field for the past 50 years.

Cai and Yeung [6], [10], [11] extended the study of algebraic coding from classical error correction to network error correction. They generalized the Hamming bound (sphere-packing bound), the Singleton bound and the Gilbert-Varshamov bound (sphere-covering bound) in classical error correction coding to network coding. Zhang studied network error correction in packet networks [12], where an algebraic definition of the minimum distance for linear network codes was introduced and the decoding problem was studied. The relation between network codes and maximum distance separation (MDS) codes in classical algebraic coding [13] was clarified in [14].

In [6], [10], [11], the common assumption is that the sink nodes know the network topology as well as the network code used in transmission. This kind of network error correction is referred to as coherent network error correction. By contrast, network error correction without this assumption is referred to as noncoherent network error correction. When using the deterministic construction of linear network codes [2], [4], the network transmission is usually regarded as “coherent”. For random network coding, the network transmission is usually regarded as “noncoherent”. It is possible, however, to use noncoherent transmission for deterministically constructed network codes and use coherent transmission for randomly constructed network codes.

In [15], Yang et al. developed a framework for characterizing error correction/detection capabilities of network codes for coherent network error correction. Their findings are summarized as follows. First, the error correction/detection capabilities of a network code are completely characterized by a two-dimensional region of parameters which reduces to the minimum Hamming distance when 1) the network code is linear, and 2) the weight measure on the error vectors is the Hamming weight. For a nonlinear network code, two different minimum distances are

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1Coherent and noncoherent transmissions for network coding are analogous to coherent and noncoherent transmissions for multiple antenna channels in wireless communications.
needed for characterizing the capabilities of the code for error correction and for error detection. This led to the discovery that for a nonlinear network code, the number of correctable errors can be more than half of the number of detectable errors. (For classical algebraic codes, the number of correctable errors is always the largest integer not greater than half of the number of detectable errors.) Further, for the general case, an equivalence relation on weight measures was defined and it was shown that weight measures belonging to the same equivalence class lead to the same minimum weight decoder. In the special case of network coding, four weight measures, including the Hamming weight and others that have been used in various works \cite{12}, \cite{16}, \cite{17}, were proved to be in the same equivalence class for linear network codes.

Network error detection by random network coding has been studied by Ho et al. \cite{18}. Jaggi et al. \cite{16}, \cite{7}, \cite{19} have developed random algorithms for network error correction with various assumptions on the adversaries. A part of the work by Zhang \cite{12} considers packet network error correction when the network code is not known by receivers, where a sufficient condition for correct decoding was given in terms of the minimum distance. The distribution of the minimum distance when applying random network coding was bounded by Balli, Yan and Zhang \cite{20}. They also studied decoding network error-correcting codes beyond the error correction capability \cite{21}.

Koetter and Kschischang \cite{22} introduced a general framework for noncoherent network error correction. In their framework, messages are modulated as subspaces, so a code for noncoherent network error correction is also called a subspace code. They proved a Singleton bound, a sphere-packing bound and a sphere-covering bound for subspace codes. Using rank-metric codes, Silva and Kschischang \cite{23} constructed nearly optimal subspace codes and studied the decoding algorithms.

**B. Paper Outline**

In this paper, we follow the framework provided in \cite{15} to study the design of linear network codes for coherent network error correction.

The coding bounds for coherent network error correction obtained in \cite{6}, \cite{10}, \cite{11} take only one sink node with the smallest maximum flow from the source node into consideration. We observe that each sink node can be considered individually and a sink node with larger maximum flow can potentially have higher error correction/detection capability. These observations lead to the refined versions of the Hamming bound, the Singleton bound and the Gilbert-Varshamov bound for network error correction to be proved in this work. By way of the weight properties of network coding, the proof of these bounds are as transparent as their classical counterparts for linear network codes. By contrast, the proofs of the original versions of these bounds (not necessarily for linear network codes) in \cite{10}, \cite{11} are considerably more complicated. The refined Singleton bound was also implicitly obtained by Zhang \cite{12} independently. When applying to classical error correction, these bounds reduce to the classical Hamming bound, the classical Singleton bound and the classical Gilbert-Varshamov bound, respectively.

Similar to its classical counterpart, this refined Singleton bound is tight for linear network codes. The tightness of this refined bound is shown by two construction algorithms of linear network codes achieving the bound. A linear network code consists of two parts, a codebook and a set of local encoding kernels (defined in Section \ref{sec:linear_network_codes}). Our
first algorithm finds a codebook based on a given set of local encoding kernels. The set of local encoding kernels that meets our requirement can be found by the polynomial-time algorithm in [4]. The second algorithm finds a set of local encoding kernels based on a given classical error-correcting code satisfying a certain minimum distance requirement as the codebook. These two algorithms illustrate different design methods. The set of local encoding kernels determines the transfer matrices of the network. The first algorithm, similar to the classical algebraic coding, designs a codebook for the transfer matrices. The second algorithm, instead, designs transfer matrices to match a codebook.

Various parts of this paper have appeared in [24], [25]. Subsequent to [24], based on the idea of static network codes [3], Matsumoto [26] proposed an algorithm for constructing linear network codes achieving the refined Singleton bound. In contrast to ours, Matsumoto’s algorithm designs the codebook and the local encoding kernels together. The complexity and field size requirements of these three algorithms are compared.

This paper is organized as follows. In Section II, we formulate the network error correction problem and review some previous works. The refined coding bounds for coherent network error correction are proved in Section III. In Section IV, the tightness of the refined Singleton bound is proved, and the first construction algorithm is given. In Section V, we introduce another construction algorithm that can achieve the refined Singleton bound. In the last section, we summarize our work and discuss future work.

II. NETWORK ERROR-CORRECTING PROBLEM

A. Problem Formulation

Let \( \mathbb{F} \) be a finite field with \( q \) elements. Unless otherwise specified, all the algebraic operations in this paper are over this field. A communication network is represented by a directed acyclic graph (DAG). (For a comprehensive discussion of directed acyclic graph, please refer to [27] and the references therein.) A DAG is an ordered pair \( G = (\mathcal{V}, \mathcal{E}) \) where \( \mathcal{V} \) is the set of nodes and \( \mathcal{E} \) is the set of edges. There can be multiple edges between a pair of nodes, each of which represents a communication link that can transmit one symbol in the finite field \( \mathbb{F} \). For an edge \( e \) from node \( a \) to \( b \), we call \( a \) (\( b \)) the tail (head) of the edge, denoted by \( \text{tail}(e) \) (\( \text{head}(e) \)). Let \( I(a) = \{ e \in \mathcal{E} : \text{head}(e) = a \} \) and \( O(a) = \{ e \in \mathcal{E} : \text{tail}(e) = a \} \) be the sets of incoming edges and outgoing edges of node \( a \), respectively.

A directed path in \( G \) is a sequence of edges \( \{ e_i \in \mathcal{E} : i = 1, 2, \cdots, k \} \) such that \( \text{head}(e_i) = \text{tail}(e_{i+1}) \) for \( i = 1, 2, \cdots, k - 1 \). Such a directed path is also called a path from \( \text{tail}(e_1) \) to \( \text{head}(e_k) \). A directed acyclic graph gives rise to a partial order \( \leq \) on its nodes, where \( a \leq b \) when there exists a directed path from \( a \) to \( b \) in the DAG. Similarly, a DAG gives rise to a partial order \( \leq \) on the edges, where \( e \leq e' \) when \( e = e' \) or \( \text{head}(e) \leq \text{tail}(e') \). In other word, \( e \leq e' \) if there exists a directed path from \( \text{tail}(e) \) to \( \text{head}(e') \) that uses both \( e \) and \( e' \). We call this partial order on the edges the associated partial order on the edges. We extend the associated partial order on the edges to a total order on the edges such that for all \( e \) and \( e' \) in \( \mathcal{E} \), either \( e \leq e' \) or \( e' \leq e \). Such an extension is not unique, but we fix one in our discussion and write \( \mathcal{E} = \{ e_i : i = 1, 2, \cdots, |\mathcal{E}| \} \).
A multicast network is an ordered triple \((G, s, T)\) where \(G\) is the network, \(s \in \mathcal{V}\) is the source node and \(T \subseteq \mathcal{V}\) is the set of sink nodes. The source node contains the messages that are demanded by all the sink nodes. Without loss of generality (WLOG), we assume \(I(s) = \emptyset\). Let \(n_s = |O(s)|\). The source node \(s\) encodes its message into a row vector \(x = [x_e, e \in O(s)] \in \mathbb{F}^{n_s}\), called the codeword. The set of all codewords is the codebook, denoted by \(C\). Note that we do not require \(C\) to be a subspace. The source node \(s\) transmits a codeword by mapping its \(n_s\) components onto the edges in \(O(s)\). For any node \(v \neq s\) with \(I(v) = \emptyset\), we assume that this node outputs the zero element of \(\mathbb{F}\) to all its outgoing edges.

An error vector \(z\) is an \(|\mathcal{E}|\)-dimensional row vector over \(\mathbb{F}\) with the \(i\)th component representing the error on the \(i\)th edge in \(\mathcal{E}\). An error pattern is a subset of \(\mathcal{E}\). Let \(\rho_z\) be the error pattern corresponding to the non-zero components of error vector \(z\). An error vector \(z\) is said to match an error pattern \(\rho\) if \(\rho_z \subseteq \rho\). The set of all error vectors that match error pattern \(\rho\) is denoted by \(\rho^*\). Let \(\bar{F}_e\) and \(F_e\) be the input and output of edge \(e\), respectively, and let the error on the edge be \(z_e\). The relation between \(F_e\), \(\bar{F}_e\) and \(z_e\) is given by

\[
F_e = \bar{F}_e + z_e. \tag{1}
\]

For any set of edges \(\rho\), form two row vectors

\[
F_\rho = [F_e, e \in \rho],
\]

and

\[
\bar{F}_\rho = [\bar{F}_e, e \in \rho].
\]

A network code on network \(G\) is a codebook \(C \subseteq \mathbb{F}^{n_s}\) and a family of local encoding functions \(\{\beta_e : e \in \mathcal{E} \setminus O(s)\}\), where \(\beta_e : \mathbb{F}^{|I(tail(e))|} \to \mathbb{F}\), such that

\[
\bar{F}_e = \beta_e(F_{I(tail(e))}). \tag{2}
\]

Communication over the network with the network code defined above is in an upstream-to-downstream order: a node applies its local encoding functions only after it receives the outputs from all its incoming edges. Since the network is acyclic, this can be achieved in light of the partial order on the nodes. With \(\bar{F}_{O(s)} = x\) and an error vector \(z\), the symbol \(\bar{F}_e, \forall e \in \mathcal{E}\), can be determined inductively by (1) and (2). When we want to indicate the dependence of \(\bar{F}_e\) and \(F_e\) on \(x\) and \(z\) explicitly, we will write them as \(\bar{F}_e(x, z)\) and \(F_e(x, z)\), respectively.

A network code is linear if \(\beta_e\) is a linear function for all \(e \in \mathcal{E} \setminus O(s)\), i.e.,

\[
\bar{F}_e = \sum_{e' \in \mathcal{E}} \beta_{e', e} F_{e'},
\]

where \(\beta_{e', e}\) is called the local encoding kernel from edge \(e'\) to edge \(e\). The local encoding kernel \(\beta_{e', e}\) can be non-zero only if \(e' \in I(tail(e))\). Define the \(|\mathcal{E}| \times |\mathcal{E}|\) one-step transformation matrix \(K = [K_{i,j}]\) in network \(G\) as \(K_{i,j} = \beta_{e_i, e_j}\). For an acyclic network, \(K^N = 0\) for some positive integer \(N\) (see [3] and [28] for details). Define the transfer matrix of the network by \(F = (I - K)^{-1}\) [3].
For a set of edges \( \rho \), define a \(|\rho| \times |E|\) matrix \( A_\rho = [A_{i,j}] \) by

\[
A_{i,j} = \begin{cases} 
1 & \text{if } e_j \text{ is the } i \text{th edge in } \rho, \\
0 & \text{otherwise.}
\end{cases}
\]  

(3)

By applying the order on \( E \) to \( \rho \), the \(|\rho|\) nonzero columns of \( A_\rho \) form an identity matrix. To simplify notation, we write \( F_\rho,\rho' = A_\rho F A_\rho^\top \). For input \( x \) and error vector \( z \), the output of the edges in \( \rho \) is

\[
F_\rho(x, z) = (xA_{O(s)} + z)FA_\rho^\top
\]  

(4)

\[
= xF_{O(s),\rho} + zFA_\rho^\top.
\]  

(5)

Writing \( F_v(x, z) = F_1(v)(x, z) \) for a node \( v \), the received vector for a sink node \( t \) is

\[
F_t(x, z) = xF_{s,t} + zF_t,
\]  

(6)

where \( F_{s,t} = F_{O(s),1(t)} \) and \( F_t = FA_1^\top \). Here \( F_{s,t} \) and \( F_t \) are the transfer matrices for message transmission and error transmission, respectively.

### B. An Extension of Classical Error Correction

In this paper, we study error correction coding over the channel given in (6), in which \( F_{s,t} \) and \( F_t \) are known by the source node \( s \) and the sink node \( t \). The channel transformation is determined by the transfer matrices. In classical error correction given in Fig.1, the transfer matrices are identity matrices. Thus, linear network error correction is an extension of classical error correction with general transfer matrices. Our work follows this perspective to extend a number of results in classical error correction to network error correction.

Different from classical error correction, network error correction provides a new freedom for coding design—the local encoding kernels can be chosen under the constraint of the network topology. One of our coding algorithm in this paper makes use of this freedom.

### C. Existing Results

In [15], Yang et al. developed a framework for characterizing error correction/detection capabilities of linear network codes for coherent network error correction. They define equivalence classes of weight measures on error vectors. Weight measures in the same equivalence class have the same characterizations of error correction/detection capabilities and induce the same minimum weight decoder. Four weight measures, namely the Hamming weight and the others that have been used in the works [12], [16], [17], are proved to be in the same equivalence class for linear network codes. Henceforth, we only consider the Hamming weight on error vectors in this paper. For sink node \( t \) and nonnegative integer \( c \), define

\[
\Phi_t(c) = \{ zF_t : z \in \mathbb{F}^{|E|}, w_H(z) \leq c \},
\]  

(7)

where \( w_H(z) \) is the Hamming weight of a vector \( z \).
**Definition 1:** Consider a linear network code with codebook $C$. For each sink node $t$, define the distance measure

$$D_t(x_1, x_2) = \min \{ c : (x_1 - x_2)F_{s,t} \in \Phi_t(c) \}$$

and define the minimum distance of the codebook

$$d_{\text{min},t} = \min_{x_1 \neq x_2 \in C} D_t(x_1, x_2).$$

We know that $D_t$ is a translation-invariant metric \textup{[15]}. Consider $x_1, x_2 \in C$. For any $z$ with $zF_t = (x_1 - x_2)F_{s,t}$, we have $(x_1 - x_2)F_{s,t} \in \Phi_t(w_H(z))$. Thus

$$D_t(x_1, x_2) \leq \min_{z : (x_1 - x_2)F_{s,t} = zF_t} w_H(z).$$

On the other hand, we see that $(x_1 - x_2)F_{s,t} \in \Phi_t(D_t(x_1, x_2))$. So, there exists $z \in \mathbb{F}^{n_{s,t}}$ with $w_H(z) = D_t(x_1, x_2)$ and $(x_1 - x_2)F_{s,t} = zF_t$. Thus,

$$D_t(x_1, x_2) \geq \min_{z : (x_1 - x_2)F_{s,t} = zF_t} w_H(z).$$

Therefore, we can equivalently write

$$D_t(x_1, x_2) = \min_{z : (x_1 - x_2)F_{s,t} = zF_t} w_H(z).$$

**Definition 2:** Minimum Weight Decoder I at a sink node $t$, denoted by $\text{MWD}_t^{I}$, decodes a received vector $y$ as follows: First, find all the solutions of the equation

$$F_t(x, z) = y$$

with $x \in C$ and $z \in \mathbb{F}^{n_{s,t}}$ as variables. A pair $(x, z)$, consisting of the message part $x$ and the error part $z$, is said to be a solution if it satisfies (11), and $(x, z)$ is a minimum weight solution if $w_H(z)$ achieves the minimum among all the solutions. If all the minimum weight solutions have the identical message parts, the decoder outputs the common message part as the decoded message. Otherwise, the decoder outputs a warning that errors have occurred.

A code is $c$-error-correcting at sink node $t$ if all error vectors $z$ with $w_H(z) \leq c$ are correctable by $\text{MWD}_t^{I}$.

**Theorem 1 (\textup{[15]})**: A linear network code is $c$-error-correcting at sink node $t$ if and only if $d_{\text{min},t} \geq 2c + 1$.

For two subsets $V_1, V_2 \subset \mathbb{F}^{n_s}$, define

$$V_1 + V_2 = \{ v_1 + v_2 : v_1 \in V_1, v_2 \in V_2 \}.$$

For $v \in \mathbb{F}^{n_s}$ and $V \subset \mathbb{F}^{n_s}$, we also write $\{ v \} + V$ as $v + V$. For sink node $t$ and nonnegative integer $c$, define the decoding sphere of a codeword $x$ as

$$\Phi_t(x, c) = \{ F_t(x, z) : z \in \mathbb{F}^{n_{s,t}}, w_H(z) \leq c \} = xF_{s,t} + \Phi_t(c)$$

**Definition 3:** If $\Phi_t(x, c)$ for all $x \in C$ are nonempty and disjoint, Minimum Weight Decoder II at sink node $t$, denoted by $\text{MWD}_t^{II}(c)$, decodes a received vector $y$ as follows: If $y \in \Phi_t(x, c)$ for some $x \in C$, the decoder
outputs x as the decoded message. If y is not in any of the decoding spheres, the decoder outputs a warning that errors have occurred.

A code is c-error-detecting at sink node t if \( \text{MWD}_t^{\text{IL}}(0) \) exists and all error vector z with \( 0 < w_H(z) \leq c \) are detectable by \( \text{MWD}_t^{\text{IL}}(0) \).

**Theorem 2 ([15]):** A code is c-error-detecting at sink node t if and only if \( d_{\min, t} \geq c + 1 \).

Furthermore, we can use \( \text{MWD}_t^{\text{IL}}(c) \), \( c > 0 \), for joint error correction and detection. Erasure correction is error correction with the potential positions of the errors in the network known by the decoder. We can similarly characterize the erasure correction capability of linear network codes by \( d_{\min, t} \). Readers are referred to [15] for the details.

There exist coding bounds on network codes that corresponding to the classical Hamming bound, Singleton bound and Gilbert-Varshamov bound. We review some of the results in [10], [11]. The maximum flow from node \( a \) to node \( b \) is the maximum number of edge-disjoint paths from \( a \) to \( b \), denoted by maxflow\((a, b)\). Let

\[
d_{\min} = \min_{t \in T} d_{\min, t},
\]

and

\[
n = \min_{t \in T} \text{maxflow}(s, t).
\]

In terms of the notion of minimum distance, the Hamming bound and the Singleton bound for network codes obtained in [10] can be restated as

\[
|C| \leq \frac{q^n}{\sum_{i=0}^{\tau_t} \binom{r_t}{i} (q - 1)^i},
\]

where \( \tau_t = \lfloor \frac{d_{\min, t} - 1}{2} \rfloor \), and

\[
|C| \leq q^{n-d_{\min}+1},
\]

respectively, where \( q \) is the field size. The tightness of (14) has been proved in [11].

**III. REFINED CODING BOUNDS**

In this section, we present refined versions of the coding bounds in [10], [11] for linear network codes. In terms of the distance measures developed in [15], the proofs of these bounds are as transparent as the their classical counterparts.

**A. Hamming Bound and Singleton Bound**

**Theorem 3:** Consider a linear network code with codebook \( C \), \( \text{rank}(\text{F}_{s,t}) = r_t \) and \( d_{\min, t} > 0 \). Then \( |C| \) satisfies

1) the refined Hamming bound

\[
|C| \leq \min_{t \in T} \frac{q^{r_t}}{\sum_{i=0}^{\tau_t} \binom{r_t}{i} (q - 1)^i},
\]

where \( \tau_t = \lfloor \frac{d_{\min, t} - 1}{2} \rfloor \), and

2) the refined Singleton bound

\[
|C| \leq q^{r_t-d_{\min, t}+1},
\]
for all sink nodes $t$.

Remark: The refined Singleton bound can be rewritten as
\[ d_{\min, t} \leq r_t - \log_q |C| \leq \max \text{flow}(s, t) - \log_q |C| + 1, \]
for all sink nodes $t$, which suggests that the sink nodes with larger maximum flow values can potentially have higher error correction capabilities. We present network codes that achieve this bound in Section IV and V.

Proof: Fix a sink node $t$. Since $\text{rank}(F_{s, t}) = r_t$, we can find $r_t$ linearly independent rows of $F_{s, t}$. Let $\rho_t \subset O(s)$ such that $|\rho_t| = r_t$ and $F_{\rho_t, l(t)}$ is a full rank submatrix of $F_{s, t}$. Note that $\rho_t$ can be regarded as an error pattern. Define a mapping $\phi_t : C \rightarrow F^{r_t}$ by $\phi_t(x) = x'$ if $x'F_{\rho_t, l(t)} = xF_{s, t}$. Since the rows of $F_{\rho_t, l(t)}$ form a basis for the row space of $F_{s, t}$, $\phi_t$ is well defined. The mapping $\phi_t$ is one-to-one because otherwise there exists $x' \in F^{r_t}$ such that $x'F_{\rho_t, l(t)} = x_1F_{s, t} = x_2F_{s, t}$ for distinct $x_1, x_2 \in C$, a contradiction to the assumption that $d_{\min, t} > 0$. Define
\[ C_t = \{ \phi_t(x) : x \in C \}. \]
Since $\phi_t$ is a one-to-one mapping, $|C_t| = |C|$.

We claim that, as a classical error-correcting code of length $r_t$, $C_t$ has minimum distance $d_{\min}(C_t) \geq d_{\min, t}$. We prove this claim by contradiction. If $d_{\min}(C_t) < d_{\min, t}$, it means there exist $x'_1, x'_2 \in C_t$ such that $w_H(x'_1 - x'_2) < d_{\min, t}$. Let $x_1 = \phi_t^{-1}(x'_1)$ and $x_2 = \phi_t^{-1}(x'_2)$. We know that $x_1, x_2 \in C$, and
\[ (x_1 - x_2)F_{s, t} = (x'_1 - x'_2)F_{\rho_t, l(t)} = z' F_t, \]
where $z' = (x'_1 - x'_2)A_{\rho_t}$. Thus,
\[ D_t(x_1, x_2) \leq w_H(z') = w_H(x'_1 - x'_2) < d_{\min, t}, \]
where the first inequality follows from (10). So we have a contradiction to $d_{\min, t} \leq D_t(x_1, x_2)$ and hence $d_{\min}(C_t) \geq d_{\min, t}$ as claimed. Applying the Hamming bound and the Singleton bound for classical error-correcting codes to $C_t$, we have
\[ |C_t| \leq q^{r_t} \leq \frac{q^{r_t}}{\sum_{i=0}^{\tau_t} \binom{r_t}{i}(q - 1)^i} \leq \frac{q^{r_t}}{\sum_{i=0}^{\tau_t'} \binom{r_t'}{i}(q - 1)^i}, \]
where $\tau_t' = \lfloor \frac{d_{\min}(C_t) - 1}{2} \rfloor \geq \tau_t$, and
\[ |C_t| \leq q^{r_t - d_{\min}(C_t) + 1} \leq q^{r_t - d_{\min, t} + 1}. \]
The proof is completed by noting that $|C| = |C_t|$.

Remark: Let $f$ be an upper bound on the size of a classical block code in terms of its minimum distance such that $f$ is monotonically decreasing. Examples of $f$ are the Hamming bound and the Singleton bound. Applying this bound to $C_t$, we have
\[ |C_t| \leq f(d_{\min}(C_t)). \]
Since $f$ is monotonically decreasing, together with $d_{\min}(C_t) \geq d_{\min,t}$ as shown in the above proof, we have

$$|C| = |C_t| \leq f(d_{\min}(C_t)) \leq f(d_{\min,t}).$$  \hspace{1cm} (17)

In other words, the bounds in (17) is simply the upper bound $f$ applied to $C$ as if $C$ is a classical block code with minimum distance $d_{\min,t}$.

**Lemma 4:**

$$\sum_{i=0}^{\tau} \binom{m}{i} (q-1)^i < \sum_{i=0}^{\tau} \binom{m+1}{i} (q-1)^i$$

for $\tau \leq m/2$.

**Proof:** This inequality can be established by considering

$$\sum_{i=0}^{\tau} \binom{m}{i} (q-1)^i = \frac{q^m}{\sum_{i=0}^{\tau} \binom{m}{i} (q-1)^i} < \frac{q^m}{\sum_{i=0}^{\tau} \binom{m+1}{i} (q-1)^i},$$

where (18) holds because $\frac{q(m-i+1)}{m+1} > 1$ given that $q \geq 2$ and $i \leq \tau \leq m/2$.

The refined Hamming bound and the refined Singleton bound, as we will show, imply the bounds shown in (13) and (14) but not vice versa. The refined Hamming bound implies

$$|C| \leq \sum_{i=0}^{\tau_t} \binom{r_t}{i} (q-1)^i \leq \sum_{i=0}^{\tau_t} \binom{r_t}{i} (q-1)^i \leq \sum_{i=0}^{\tau_t} \binom{\text{maxflow}(s,t)}{i} (q-1)^i$$

(19)

for all sink nodes $t$, where (19) follows from $\tau = \lceil \frac{d_{\min,t} - 1}{2} \rceil \leq \lceil \frac{d_{\min,t+1} - 1}{2} \rceil = \tau_t$, and (20) follows from $r_t \leq \text{maxflow}(s,t)$ and the inequality proved in Lemma 4. By the same inequality, upon minimizing over all sink nodes $t \in \mathcal{T}$, we obtain (13). Toward verifying the condition for applying the inequality in Lemma 4 in the above, we see $r_t \geq d_{\min,t} - 1$ since $1 \leq |C| \leq q^{r_t - d_{\min,t} + 1}$. Then

$$\tau \leq \tau_t \leq \frac{d_{\min,t} - 1}{2} \leq \frac{r_t}{2}$$

for all $t \in \mathcal{T}$.

The refined Singleton bound is maximized when $r_t = \text{maxflow}(s,t)$ for all $t \in \mathcal{T}$. This can be achieved by a linear broadcast code whose existence was proved in [2], [14]. To show that the refined Singleton bound implies (14), consider

$$|C| \leq q^{r_t - d_{\min,t} + 1} \leq q^{r_t - d_{\min} + 1} \leq q^{\text{maxflow}(s,t) - d_{\min} + 1}$$

for all sink nodes $t$. Then (14) is obtained upon minimizing over all $t \in \mathcal{T}$.
By the definition of $q$, together with (24), we obtain the first inequality in (22).

We have $F$ has dimension $\Phi$, where $\alpha$ and $r$ are defined in (7). Since the rank of $F|_{s,t}$ is defined in (8), we have $\Delta_t(0,d) = \Phi t$, and all such $x$ are in $\Delta_i(0,d)$. Thus, 

$$|\Delta_i(0,d)| = q^{n_s-r_t}|\Phi_t(d)|.$$ 

(24)

By the definition of $\Phi_t$, we have 

$$|\Phi_t(d)| \leq \left\{|z\in F^{[\mathcal{E}]_{d}}: w_H(z) \leq d]\right\| < \left(\frac{|\mathcal{E}|}{d}\right)q^d.$$ 

(25)

Together with (24), we obtain the first inequality in (22).

Since $\text{rank}(F|_{s,t}) = r_t$, we can find $r_t$ linearly independent rows of $F|_{s,t}$. Let $\rho_t \subset O(s)$ such that $|\rho_t| = r_t$ and $F|_{\rho_t,t}$ is a full row rank submatrix of $F|_{s,t}$. Note that $F|_{\rho_t,t}$ is also a submatrix of $F_t$. Since, 

$$\Phi_t(d) = \{zF_t : w_H(z) \leq d\} \supset \{z'F|_{\rho_t,t} : w_H(z') \leq d\},$$ 

we have 

$$|\Phi_t(d)| \geq |\left\{z'F|_{\rho_t,t} : w_H(z') \leq d\right\}|$$

$$= \left|\left\{z' \in F^{r_t} : w_H(z') \leq d\right\}\right|$$

$$= \sum_{i=0}^{d} \binom{r_t}{i}(q-1)^i.$$
The proof is complete.

Using the idea of sphere packing, we have the following stronger version of the refined Hamming bound in Theorem 3.

**Theorem 6 (Sphere-packing bound):** A linear network code with codebook \( C \) and positive minimum distance \( d_{\min,t} \) for all sink nodes \( t \) satisfies

\[
|C| \leq \frac{q^{\tau_t}}{|\Phi_t(\tau_t)|},
\]

where \( \tau_t = \lfloor \frac{d_{\min,t} - 1}{2} \rfloor \).

**Proof:** For different codewords \( x_1 \) and \( x_2 \), we show that \( \Delta_t(x_1, \tau_t) \) and \( \Delta_t(x_2, \tau_t) \) are disjoint by contradiction. Let

\[
x \in \Delta_t(x_1, \tau_t) \cap \Delta_t(x_2, \tau_t).
\]

By the definition of \( \Delta_t \) in (21), we have \( D_t(x_1, x) \leq \tau_t \) and \( D_t(x_2, x) \leq \tau_t \). Applying the triangle inequality of \( D_t \), we have

\[
D_t(x_1, x_2) \leq D_t(x_1, x) + D_t(x_2, x) \\
\leq 2\tau_t \\
\leq d_{\min,t} - 1,
\]

which is a contradiction to the definition of \( d_{\min,t} \). Therefore, \( q^{n_s} \geq \sum_{x \in C} |\Delta_t(x, \tau_t)| = |C||\Delta_t(0, \tau_t)| \). The proof is complete by considering the equality in Lemma 5.

Applying the second inequality in Lemma 5, Theorem 6 implies the refined Hamming bound in Theorem 3. Thus Theorem 6 gives a potentially tighter upper bound on \( |C| \) than the refined Hamming bound, although the former is less explicit than the latter.

### C. Gilbert Bound and Varshamov Bound

We have the following sphere-covering type bounds for linear network codes.

**Theorem 7 (Gilbert bound):** Given a set of local encoding kernels, let \( |C|_{\max} \) be the maximum possible size of codebooks such that the network code has positive minimum distance \( d_{\min,t} \) for each sink node \( t \). Then,

\[
|C|_{\max} \geq \frac{q^{n_s}}{|\Delta(0)|},
\]

where

\[
\Delta(0) = \bigcup_{t \in T} \Delta_t(0, d_{\min,t} - 1).
\]

**Proof:** Let \( C \) be a codebook with the maximum possible size, and let

\[
\Delta(c) = \bigcup_{t \in T} \Delta_t(c, d_{\min,t} - 1).
\]
For any \( x \in \mathbb{F}^n_s \), there exists a codeword \( c \in \mathcal{C} \) and a sink node \( t \) such that
\[
D_t(x, c) \leq d_{\min,t} - 1,
\]
since otherwise we could add \( x \) to the codebook while keeping the minimum distance. By definition, we know
\[
\Delta(c) = \bigcup_{t \in \mathcal{T}} \{ x \in \mathbb{F}^n_s : D_t(x, c) \leq d_{\min,t} - 1 \}.
\]
Hence, the whole space \( \mathbb{F}^n_s \) is contained in the union of \( \Delta(c) \) over all codewords \( c \in \mathcal{C} \), i.e.,
\[
\mathbb{F}^n_s = \bigcup_{c \in \mathcal{C}} \Delta(c).
\]
Since \( \Delta(c) = c + \Delta(0) \), we have \( |\Delta(c)| = |\Delta(0)| \). So we deduce that \( q^n_s \leq |\mathcal{C}||\Delta(0)| \).

We say a codebook is linear if it is a vector space.

**Lemma 8:** Consider a linear network code with linear codebook \( \mathcal{C} \). The minimum distance \( d_{\min,t} \geq d \) if and only if
\[
\mathcal{C} \cap \Delta_t(0, d-1) = \{0\}.
\]
**Proof:** If there exists \( x \in \mathcal{C} \cap \Delta_t(0, d-1) \) and \( x \neq 0 \), then \( D_t(0, x) < d \). Since \( 0 \in \mathcal{C} \), we have \( d_{\min,t} < d \). This proves the sufficient condition.

Now we prove the necessary condition. For \( x_1, x_2 \in \mathcal{C}, x_1 - x_2 \in \mathcal{C} \). Since
\[
D_t(x_1, x_2) = D_t(x_1 - x_2, 0),
\]
we have
\[
d_{\min,t} = \min_{x \in \mathcal{C}, x \neq 0} D_t(x, 0).
\]
Thus,
\[
\mathcal{C} \cap \Delta_t(0, d_{\min,t} - 1) = \{0\}.
\]
The proof is completed noting that \( \Delta_t(0, d_{\min,t} - 1) \supset \Delta_t(0, d-1) \).

**Theorem 9 (Varshamov bound):** Given a set of local encoding kernels, let \( \omega_{\max} \) be the maximum possible dimension of linear codebooks such that the network code has positive minimum distance \( d_{\min,t} \) for each sink node \( t \). Then,
\[
\omega_{\max} \geq n_s - \log_q |\Delta(0)|,
\]
where \( \Delta(0) \) is defined in (27).

**Proof:** Let \( \mathcal{C} \) be a linear codebook with the maximum possible dimension. By Lemma 8 \( \mathcal{C} \cap \Delta(0) = \{0\} \). We claim that
\[
\mathbb{F}^n_s = \Delta(0) + \mathcal{C}.
\]
If the claim is true, then
\[
q^n_s = |\Delta(0) + \mathcal{C}| \leq |\Delta(0)|||\mathcal{C}| = |\Delta(0)||\omega_{\max},
\]
where \( \omega_{\max} \) is defined in (27).
proving (28).

Since $F_{ns} \supset \Delta(0) + C$, so we only need to show $F_{ns} \subset \Delta(0) + C$. Assume there exists

$$g \in F_{ns} \setminus (\Delta(0) + C).$$  \hfill (30)

Let $C' = C + \langle g \rangle$. Then $C'$ is a subspace with dimension $\omega_{\text{max}} + 1$. If $C' \cap \Delta(0) \neq \{0\}$, then there exists a non-zero vector

$$c + \alpha g \in \Delta(0),$$  \hfill (31)

where $c \in C$ and $\alpha \in F$. Here, $\alpha \neq 0$, otherwise we have $c = 0$ because $C \cap \Delta(0) = \{0\}$. Since $\Delta_t(0, d_{\text{min}, t} - 1)$ is closed under scalar multiplication for all $t \in T$, see from (27) that the same holds for $\Delta(0)$. Thus from (31),

$$g \in \Delta(0) - \alpha^{-1} c \subset \Delta(0) + C,$$

which is a contradiction to (30). Therefore, $C' \cap \Delta(0) = \{0\}$. By Lemma 8, $C'$ is a codebook such that the network code has unicast minimum distance larger than or equal to $d_{\text{min}, t}$, which is a contradiction on the maximality of $C$. The proof is completed.

IV. Tightness of the Singleton Bound and Code Construction

For an $(\omega, (r_t : t \in T), (d_t : t \in T))$ linear network code, we refer to one for which the codebook $C$ is an $\omega$-dimensional subspace of $F_{ns}$, the rank of the transfer matrix $F_{s,t}$ is $r_t$, and the minimum distance for sink node $t$ is at least $d_t$, $t \in T$. In this section, we propose an algorithm to construct $(\omega, (r_t : t \in T), (d_t : t \in T))$ linear network codes that can achieve the refined Singleton bound.

A. Tightness of the Singleton Bound

**Theorem 10:** Given a set of local encoding kernels with $r_t = \text{rank}(F_{s,t})$ over a finite field with size $q$, for every

$$0 < \omega \leq \min_{t \in T} r_t,$$

there exists a codebook $C$ with $|C| = q^\omega$ such that

$$d_{\text{min}, t} = r_t - \omega + 1$$  \hfill (33)

for all sink nodes $t$, provided that $q$ is sufficiently large.

**Proof:** Fix an $\omega$ which satisfies (32). We will construct an $\omega$-dimensional linear codebook which together with the given set of local encoding kernels constitutes a linear network code that satisfies (33) for all $t$. Note that (32) and (33) imply

$$d_{\text{min}, t} \geq 1.$$

We construct the codebook $C$ by finding a basis. Let $g_1, \cdots, g_\omega \in F_{ns}$ be a sequence of vectors obtained as follows. For each $i$, $1 \leq i \leq \omega$, choose $g_i$ such that

$$g_i \notin \Delta_t(0, r_t - \omega) + \langle g_1, \cdots, g_{i-1} \rangle$$  \hfill (34)
for each sink node $t$. As we will show, this implies

$$\Delta_t(0, r_t - \omega) \cap (g_1, \ldots, g_i) = \{0\}$$

(35)

for each sink node $t$. If such $g_1, \ldots, g_{\omega}$ exist, then we claim that $\mathcal{C} = \langle g_1, \ldots, g_{\omega} \rangle$ is a codebook with the desired properties. To verify this claim, first, we see that $i = 1$

$$d_{\min, t} \geq r_t - \omega + 1$$

since (35) holds for $i = \omega$ (ref Lemma 8). Note that by (16), the refined Singleton bound, we indeed have $d_{\min, t} \geq r_t - \omega + 1$, namely (33) for any sink node $t$.

Now we show that $g_i$ satisfying (34) exists if the field size $q$ is sufficiently large. Observe that

$$|\Delta_t(0, r_t - \omega) + (g_1, \ldots, g_i)|$$

$$\leq |\Delta_t(0, r_t - \omega)|q^{i-1}$$

$$\leq \left(\frac{|\mathcal{E}|}{r_t - \omega}\right)q^{r_t - \omega - |\Delta_t(0, r_t - \omega)|}q^{i-1}$$

$$= \left(\frac{|\mathcal{E}|}{r_t - \omega}\right)q^{n_s - \omega + i - 1},$$

(36)

where (36) follows from Lemma 5. If

$$q^{n_s} > \sum_{i \in T} \left(\frac{|\mathcal{E}|}{r_t - \omega}\right)q^{n_s + i - 1},$$

(37)

we have

$$\mathbb{F}^{n_s} \setminus \bigcup_{\mathcal{E}} (\Delta_t(0, r_t - \omega) + (g_1, \ldots, g_i)) \neq \emptyset,$$

i.e., there exists a $g_i$ satisfying (34). Therefore, if $q$ satisfies (37) for all $i = 1, \ldots, \omega$, or equivalently

$$q > \sum_{i \in T} \left(\frac{|\mathcal{E}|}{r_t - \omega}\right),$$

(38)

then there exists a vector that can be chosen as $g_i$ for $i = 1, \ldots, \omega$.

Fix $g_1, \ldots, g_i$ that satisfy (34). We now prove by induction that (35) holds for $g_1, \ldots, g_i$. If (35) does not hold for $i = 1$, then there exists a non-zero vector $\alpha g_1 \in \Delta_t(0, r_t - \omega)$, where $\alpha \in \mathbb{F}$. Since $\Delta_t(0, r_t - \omega)$ is closed under scalar multiplication and $\alpha \neq 0$, we have $g_1 \in \Delta_t(0, r_t - \omega)$, a contradiction to (34) for $i = 1$. Assume (35) holds for $i \leq k - 1$. If (35) does not hold for $i = k$, then there exists a non-zero vector

$$\sum_{i=1}^{k} \alpha_i g_i \in \Delta_t(0, r_t - \omega),$$

where $\alpha_i \in \mathbb{F}$. If $\alpha_k = 0$,

$$\sum_{i=1}^{k-1} \alpha_i g_i \in \Delta_t(0, r_t - \omega),$$

a contradiction to the assumption that (35) holds for $i = k - 1$. Thus $\alpha_k \neq 0$. Again, by $\Delta_t(0, r_t - \omega)$ being closed under scalar multiplication, we have

$$g_k \in \Delta_t(0, r_t - \omega) - \alpha_k^{-1} \sum_{i=1}^{k-1} \alpha_i g_i \in \Delta_t(0, r_t - \omega) + (g_1, \ldots, g_{k-1}),$$
a contradiction to $g_k$ satisfying (34). The proof is completed.

B. The First Construction Algorithm

The proof of Theorem 10 gives a construction algorithm for an $(\omega, (r_t : t \in T), (d_t : t \in T))$ linear network code and it also verifies the correctness of the algorithm when the field size is sufficiently large. This algorithm, called Algorithm 1, makes use of existing algorithms (e.g., the Jaggi-Sanders algorithm [4]) to construct the local encoding kernels. The pseudo code of Algorithm 1 is shown below.

```
Algorithm 1: Construct network codes that achieve the refined Singleton bound.

input: $(G, s, T), (r_t : t \in T), \omega, (d_t : t \in T)$ with $r_t \leq \text{maxflow}(s, t) \forall t \in T$

output: local encoding kernels and $C$

1 begin
2 Construct a set of local encoding kernels such that $\text{rank}(F_{s,t}) = r_t$;
3 for $i \leftarrow 1, \omega$ do
4 find $g_i$ such that $g_i \notin \bigcup t \Delta t(0, d_t - 1) + \langle g_1, \cdots, g_{i-1} \rangle$;
5 end
6 end
```

The analysis of the complexity of the algorithm requires the following lemma implied by Lemma 5 and 8 in [4].

**Lemma 11:** Suppose $m \leq q$, the field size, and $B_k \subset \mathbb{F}^n, k = 1, \cdots, m$, are subspaces with $\text{dim}(B_k) < n$. A vector $u \in \mathbb{F}^n \setminus \bigcup_{k=1}^m B_k$ can be found in time $O(n^3m + nm^2)$.

**Proof:** For each $B_k$ find a vector $a_k \in \mathbb{F}^n$ such that $a_k b^\top = 0, \forall b \in B_k$. This vector $a_k$ can be obtained by solving the system of linear equations

$$B_k a_k^\top = 0,$$

where $B_k$ is formed by juxtaposing a set of vectors that form a basis of $B_k$. The complexity of solving this system of linear equations is $O(n^3)$. We inductively construct $u_1, u_2, \cdots, u_m$ such that $u_i a_k^\top \neq 0$ for all $1 \leq k \leq i \leq m$. If such a construction is feasible, then $u_m \notin B_k, \forall k \leq m$. Thus, $u = u_m \notin \bigcup_{k=1}^m B_k$ is the desired vector.

Let $u_1$ be any vector such that $u_1 a_i^\top \neq 0$. For $1 \leq i \leq m - 1$, if $u_i a_{i+1}^\top \neq 0$, we set $u_{i+1} = u_i$. Otherwise, find $b_{i+1}$ such that $b_{i+1} a_i^\top \neq 0$. We choose

$$\alpha \in \mathbb{F} \setminus \{-(b_{i+1} a_{j}^\top)/(u_j a_{j}^\top) : 1 \leq j \leq i\},$$

and define

$$u_{i+1} = \alpha u_i + b_{i+1}.$$
The existence of such an $\alpha$ follows from $q \geq m > i$.

By construction, we know that

$$u_{i+1}a_{i+1}^T = \alpha u_i a_i^T + b_i a_{i+1}^T,$$

$$= b_{i+1} a_{i+1}^T \neq 0.$$ 

If $u_{i+1}a_j^T = \alpha u_i a_j^T + b_i a_{i+1}^T = 0$ for some $1 \leq j \leq i$, we have $\alpha = -(b_{i+1} a_j^T)/(u_i a_j^T)$, a contradiction to (39). So, $u_{i+1}a_j^T \neq 0$ for all $j$ such that $1 \leq j \leq i+1$.

Similar to the analysis in [4, Lemma 8], the construction of $u$ takes time $O(nm^2)$. Therefore, the overall time complexity is $O(n^3 m + nm^2)$.

We analyze the time complexity of Algorithm 1 for the representative special case that $r_t = r$ and $d_t = d$ for all $t \in T$, where $r \leq \min_{t \in T} \text{maxflow}(s, t)$ and $d \leq r - \omega + 1$. In the pseudo code, Line 2 can be realized using the Jaggi-Sanders algorithm with complexity $O(|E| |T| n (n + |T|))$, where $n = \min_{t \in T} \text{maxflow}(s, t)$ [4]. Line 3-5 is a loop that runs Line 4 $\omega$ times. Considering $\Delta_t(0, d - 1)$ as the union of $(|E| - 1)$ subspaces of $\mathbb{F}^r$, Line 4 can be realized in time $O(n_s^2 |T| (|E| - 1)^2) + n_s (|T| (|E| - 1)^2)$ as proved in Lemma [11]. Repeating $\omega$ times, the complexity of Line 3-5 is

$$O(\omega n_s^3 |T| \xi + \omega n_s |T| ^2 \xi^2),$$

where $\xi = (|E| - 1)$. The overall complexity is

$$O(\omega n_s |T| \xi (n_s^2 + |T| \xi) + |E| |T| (n + |T|)).$$

Comparing the complexities of constructing the local encoding kernels (Line 2) and finding the codebook (Line 3-5), the latter term in the above dominates when $d > 1$.

To guarantee the existence of the code, we require the field size to be sufficiently large. From (38) in the proof of Theorem [10] all finite fields with size larger than $|T| (|E| - 1)$ are sufficient. It is straightforward to show that this algorithm can also be realized randomly with high success probability if the field size is much larger than necessary.

V. THE SECOND CONSTRUCTION ALGORITHM

Algorithm 1 can be regarded as finding a codebook for the given transfer matrices. In this section, we study network error correction from a different perspective by showing that we can also shape the transfer matrices by designing proper local encoding kernels. Following this idea, we give another algorithm that constructs an $(\omega, (r_t : t \in T), (d_t : t \in T))$ linear network code.

A. Outline of Algorithm 2

We first give an informal description of this algorithm. The second algorithm, called Algorithm 2, starts with a classical error-correcting code as the codebook. The main task of the algorithm is to design a set of local encoding
kernels such that the minimum distances of the network code, roughly speaking, are the same as the classical error-correcting code.

It is complicated to design all the local encoding kernels altogether. Instead, we use an inductive method: we begin with the simplest network that the source and the sink nodes are directed connected with parallel edges; we then extend the network by one edge in each iteration until the network becomes the one we want. For each iteration, we only need to choose the local encoding kernels associated with the new edge.

We have two major issues to solve in the above method: the first is how to extend the network; the second is how to choose the local encoding kernels. In Section \[V-B\] we define a sequence of networks \( G^i \) for a given network \( G \). The first network is the simplest one as we described, the last one is the network \( G \), and \( G^{i+1} \) has one more edge than \( G^i \). In Section \[V-C\] we give an algorithm that designs the local encoding kernels inductively. Initially, we choose a classical error-correcting code that satisfies certain minimum distance constraint. The local encoding kernels of \( G^{i+1} \) is determined as follows: Except for the new edge, all the local encoding kernels in \( G^{i+1} \) are inherited from \( G^i \). The new local encoding kernels (associated with the new edge) is chosen to guarantee 1) the preservation of the minimum distance of the network code, and 2) the existence of the local encoding kernels to be chosen in the next iteration. We find a feasible condition on the new local encoding kernels to be chosen such that these criteria are satisfied.

When \( d_t = 1 \) for all sink nodes \( t \), this algorithm degenerates to the Jaggi-Sanders algorithm for designing linear network codes for the error-free case.

### B. Iterative Formulation of Network Coding

In this and the next subsections, we describe the algorithm formally. At the beginning, the algorithm finds \( r_t \) edge-disjoint paths from the source node \( s \) to each sink node \( t \) using a maximum flow algorithm (for example, finding the augmenting paths). We assume that every edge in the network is on at least one of the \( \sum_{t \in T} r_t \) paths we have found. Otherwise, we delete the edges and the nodes that are not on any such path, and consider the coding problem for the new network. Note that a network code for the new network can be extended to the original network without changing the minimum distances by assigning zero to all the local encoding kernels associated with the deleted edges.

We consider a special order on the set of edges such that 1) it is consistent with the partial order on the set of edges; 2) the first \( n_s \) edges are in \( O(s) \). The order on the paths to a particular sink node is determined by the first edges on the paths.

Given a DAG \( G \), we construct a sequence of graphs \( G^i = (V^i, E^i), i = 0, 1, \cdots, |E| - n_s \) as follows. First, \( G^0 \) consists of a subgraph of \( G \) containing only the edges in \( O(s) \) (and the associated nodes) and all the sink nodes. Following the order on \( E \), in the \( i \)th iteration \( G^{i-1} \) is expanded into \( G^i \) by appending the next edge (and the associated node) in \( E \). This procedure is repeated until \( G^i \) eventually becomes \( G \). Note that \( G^i \) contains \( n_s + i \) edges and \( G^{i+n_s} = G \). A sink node \( t \) has \( r_t \) incoming edges in \( G^i \), where the \( j \)th edge is the most downstream edge in the truncation in \( G^i \) of the \( j \)th edge-disjoint path from the source node \( s \) to sink node \( t \) in \( G \). With a slight
abuse of notation, we denote the set of incoming edges of a sink node \( t \) in \( G^i \) as \( I(t) \), when \( G^i \) is implied by the context. Fig. 2 illustrates \( G^0 \) and \( G^1 \) when \( G \) is the butterfly network.

The network \( G^i \) is a multicast network with the source node \( s \) and the set of sinks \( T \). The algorithm chooses a proper codebook, and then constructs local encoding kernels starting with \( G^0 \). Except for the new edge, all the local encoding kernels in \( G^{i+1} \) are inherited from \( G^i \). We define \( K_i, F_i, F_i^ρ, z_i \) and \( A_i^ρ \) for \( G^i \) in view of \( K, F, F^ρ, z \) and \( A^ρ \) defined for \( G \) in Section II, respectively. Writing \( F_i^t(x, z_i) = (xA_i^{O(s)} + z_i)F_i(A_i^{I(t)})^\top \),

\[
\text{(40)}
\]

in view of (4). Further, we can define the minimum distance \( d_{\text{min}}^{i,t} \) corresponding to the sink node \( t \) at the \( i \)th iteration as in (9).

Consider a matrix \( M \). Let \((M)_L \) be the submatrix of \( M \) containing the columns with indices in \( L \), and \( M^\perp_L \) be the submatrix obtained by deleting the columns of \( M \) with indices in \( L \). If \( L = \{j\} \), we also write \( M^\perp_j \) and \((M)_j \) for \( M^\perp\{j\} \) and \((M)_{\{j\}} \), respectively.

In the following, we give an iterative formulation of \( F_i^t \) for \( i > 0 \). Let \( e \) be the edge added to \( G^{i-1} \) to form \( G^i \), and let \( k_e = [β_{e',e} : e' \in E_{i-1}] \) be an \((n_s + i - 1)\)-dimensional column vector. In the \( i \)th iteration, we need to determine the component \( β_{e',e} \) of \( k_e \) with \( e' \in I(\text{tail}(e)) \). All other components of \( k_e \) are zero. Using \( k_e \), we have

\[
F^t = (I - K^i)^{-1}
= \left( I - \begin{bmatrix} K^{i-1} & k_e \\ 0 & 0 \end{bmatrix} \right)^{-1}
= \begin{bmatrix} I - K^{i-1} & -k_e \\ 0 & 1 \end{bmatrix}^{-1}
= \begin{bmatrix} (I - K^{i-1})^{-1} & (I - K^{i-1})^{-1}k_e \\ 0 & 1 \end{bmatrix}
\]
The matrix $A_{O(s)}^i$ has one more column with zero components than $A_{O(s)}^{i-1}$, i.e.,

$$A_{O(s)}^i = \begin{bmatrix} A_{O(s)}^{i-1} & 0 \end{bmatrix}. \quad (42)$$

If the edge $e$ is not on any path from the source node $s$ to sink node $t$, we only need to append a column with zero components to $A_{I(t)}^{i-1}$ to form $A_{I(t)}^i$, i.e.,

$$A_{I(t)}^i = \begin{bmatrix} A_{I(t)}^{i-1} & 0 \end{bmatrix}. \quad (43)$$

For this case, we can readily obtain from (40), (41), (42) and (43) that

$$F_t^i(x, z^i) = (x A_{O(s)}^i + z^i) F_{I(t)}^i (A_{I(t)}^i)^\top$$

$$= (x A_{O(s)}^i + z^i) \begin{bmatrix} F_{I(t)}^{i-1} \\ 0 \end{bmatrix} (A_{I(t)}^{i-1})^\top$$

$$= (x A_{O(s)}^i + (z^i)^{\perp i}) F_{I(t)}^{i-1} (A_{I(t)}^{i-1})^\top$$

$$= F_{I(t)}^{i-1} (x, (z^i)^{\perp i}). \quad (44)$$

Note that $(z^i)^{\perp i}$ is an $(n_s + i - 1)$-dimensional error vector obtained by deleting the $i$th component of $z^i$, which corresponds to $e$.

If edge $e$ is on the $j$th edge-disjoint path from the source node $s$ to sink node $t$, to form $A_{I(t)}^i$, we need to first append a column with zero components to $A_{I(t)}^{i-1}$, and then move the ‘1’ in the $j$th row to the last component of that row. That is, if

$$A_{I(t)}^{i-1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{r_1} \end{bmatrix},$$

then

$$A_{I(t)}^i = \begin{bmatrix} b_1 & 0 \\ \vdots & \vdots \\ b_{j-1} & 0 \\ 0 & 1 \\ b_{j+1} & 0 \\ \vdots & \vdots \\ b_{r_t} & 0 \end{bmatrix}. \quad (45)$$
We can then obtain $F_i(t, z_i)$ from (40), (41), (42) and (45) as
\[
(F_i(t, z_i))_j = (x A_{O(s)}^i + z_i^t) F^i ((A_i(t)_{l(t)})^\top)_j \\
= (x A_{O(s)}^i + z_i^t) \begin{bmatrix} F^{i-1} k_e \\ 1 \end{bmatrix} \\
= (x A_{O(s)}^i + z_i^t)^i F^{i-1} k_e + (z_i^t)_i, \tag{46}
\]
and
\[
(F_i(t, x, z_i^t))_j \\
= (x A_{O(s)}^i + z_i^t)^i F^i ((A_i(t)_{l(t)})^\top)_j \\
= (x A_{O(s)}^i + z_i^t)^i \begin{bmatrix} F^{i-1} \\ 0 \end{bmatrix} ((A_i(t)_{l(t)})^\top)_j \\
= (F_i(t, (z_i^t)_i))_j. \tag{47}
\]

C. Algorithm 2

Let $e$ be the edge appended to the graph in the $i$th iteration for $i > 0$. We choose $k_e$ such that the following feasible condition is satisfied:
\[
(F_i(t, x, -z_i^t))_{\mathcal{L}} \neq 0 \tag{48}
\]
for all combinations of

C1) $t \in \mathcal{T}$,
C2) $\mathcal{L} \subset \{1, 2, \ldots, r_t\}$ with $0 \leq |\mathcal{L}| \leq d_t - 1$,
C3) non-zero $x \in \mathcal{C}$, and
C4) error vector $z_i$ with $w_H(z_i) \leq d_t - 1 - |\mathcal{L}|$.

If the feasible condition is satisfied for sink node $t$ and $\mathcal{L} = \emptyset$, we have
\[
x A_{O(s)}^i F^i (A_i(t)_{l(t)})^\top \neq z_i^{i-1} F_i,
\]
for all $z_i^t$ and $x$ satisfying C3 and C4. If $\mathcal{C}$ is a subspace, we have $d_{\text{min},t}^i \geq d_t$. Since the feasible condition is required for each iteration, when the algorithm terminates, the code constructed for $\mathcal{G}$ satisfies $d_{\text{min},t}^i \geq d_t$. Algorithm 2 is also called the distance preserving algorithm since the algorithm keeps the minimum distance larger than or equal to $d_t$ in each iteration. Even though the feasible condition is stronger than necessary for $d_{\text{min},t}^i \geq d_t$, $t \in \mathcal{T}$, as we will see, it is required for the existence of the local encoding kernels for $k > i$ such that the feasible condition is satisfied.

**Theorem 12:** Given a linear codebook with $d_{\text{min},t}^i \geq d_t$ for all $t \in \mathcal{T}$, there exist local encoding kernels such that the feasible condition is satisfied for $i = 1, \ldots, |\mathcal{E}| - n_s$ when the field size is larger than $\sum_{t \in \mathcal{T}} (r_t + |\mathcal{E}| - 1)^2$. 

January 22, 2010
Algorithm 2: Construct \((\omega, (r_t : t \in \mathcal{T}), (d_t : t \in \mathcal{T}))\) linear network code

\textbf{input}: \((G, s, \mathcal{T}), (r_t : t \in \mathcal{T}), \omega, (d_t : t \in \mathcal{T})\)

\textbf{output}: local encoding kernels and codebook \(C\)

\begin{verbatim}
1 begin
2     \textbf{for each sink node } t \textbf{ do}
3         choose \(r_t\) edge disjoint paths from \(s\) to \(t\);
4         initialize \(A_{l(t)}\);
5     \textbf{end}
6     Find a linear codebook \(C\) with \(d_{\text{min},t}^0 \geq d_t, \forall t \in \mathcal{T}\);
7     \(F \leftarrow I, A_{O(s)} \leftarrow I\);
8     \textbf{for each } e \in \mathcal{E} \setminus O(s) \textbf{ in an upstream to downstream order do}
9         \(\Gamma \leftarrow \emptyset\);
10        \textbf{for each sink node } t \textbf{ do}
11            if no chosen path from \(s\) to \(t\) crosses \(e\) then
12                \(A_{l(t)} \leftarrow [A_{l(t)} \ 0] ;\)
13            else \(e\) is on the \(j\)th path from \(s\) to \(t\)
14                \textbf{for each } L \textbf{ with } |L| \leq d_t - 1 \textbf{ and } j \notin L \textbf{ do}
15                    \textbf{for each } \rho \textbf{ with } |\rho| = d_t - 1 - |L| \textbf{ do}
16                        find \(x_0 \neq 0\) and \(z_0\) matching \(\rho\) such that \((F_t(x_0, -z_0))^{\backslash (L \cup \{j\})} = 0\);
17                        if exist \(x_0\) and \(z_0\) then
18                            \(\Gamma \leftarrow \Gamma \cup \{k : (x_0A - z_0)Fk = 0\}\);  
19                        end
20                    end
21            end
22        end
23        update \(A_{l(t)}\) using \([45]\);
24     \textbf{end}
25     choose a vector \(k_e\) in \(F_q^{[O(tail(e))]} \setminus \Gamma\);
26     \(F \leftarrow [F \ Fk_e] ;\)
27 \end{verbatim}
Proof Outline: (See the complete proof in Section V-E) The linear codebook satisfies the feasible condition for \( i = 0 \). Assume we can find local encoding kernels such that the feasible condition is satisfied for \( i < k \), where \( 0 \leq k - 1 < |\mathcal{E}| - n_s \). In the \( k \)th iteration, let \( e \) be the edge appended to \( \mathcal{G}^{k-1} \) to form \( \mathcal{G}^k \). We find that \( k_e \) only affects (43) for the case such that

1) \( e \) is on \( j \)th path from \( s \) to \( t \),
2) \( j \notin \mathcal{L} \), and
3) \( (F^{k-1}_i(x, -z))^{\mathcal{L} \cup \{ j \}} = 0 \), where \( x \neq 0 \in \mathcal{C} \), \( z \in \mathbb{F}^{n_s+k-1} \), \( w_H(z) = d_t - 1 - |\mathcal{L}| \).

For \( t, \mathcal{L}, x \) and \( z \) satisfying the above condition, we need to choose \( k_e \) such that

\[
(xA^{k-1}_{O(s)} - z)F^{k-1}k_e \neq 0.
\]

We verify that if \( q > \sum_{t \in \mathcal{T}} \left( \frac{r_t}{d_t} - \frac{1}{2} \right) \), we can always find such a \( k_e \).

Refer to the pseudo code of Algorithm 2 above. At the beginning, the algorithm finds \( r_t \) edge-disjoint paths from the source node to each sink node \( t \), and initializes \( F \), \( A_{O(s)} \), and \( A_{I(t)} \), \( t \in \mathcal{T} \) by \( F^0 \), \( A^0_{O(s)} \), and \( A^0_{I(t)} \), respectively. The algorithm takes as the input a linear codebook \( \mathcal{C} \) such that \( d_{\text{min}, t} \geq d_t \) for all sink nodes \( t \). Such a codebook can be efficiently constructed by using Reed-Solomon codes. The main part of this algorithm is a loop starting at Line 7 for updating the local encoding kernels for the edges in \( \mathcal{E} \setminus O(s) \) in an upstream-to-downstream order. The choosing of \( k_e \) is realized by the pseudo codes between Line 8 and Line 25.

We analyze the time complexity of the algorithm for the representative special case that \( r_t = r \) and \( d_t = d \) for all \( t \in \mathcal{T} \), where \( r \leq \min_{t \in \mathcal{T}} \text{maxflow}(s, t) \) and \( d \leq r - \omega + 1 \). For Line 3, the augmenting paths for all the sinks can be found in time \( O(|\mathcal{T}|\|\mathcal{E}|r) \) [4]. Line 16 and 18 can be realized by solving a system of linear equations which take time \( O(r^3) \) and \( O(1) \), respectively, and each of these two lines is repeated \( O(d\|\mathcal{E}||\mathcal{T}|\left( \frac{|\mathcal{E}|}{d-1} \right) \) times. Line 26 can be solved by the method in Lemma [11] in time \( O(d|\mathcal{T}|\left( \frac{r+|\mathcal{E}|-2}{d-1} \right) + |\mathcal{T}| \left( \frac{r+|\mathcal{E}|-2}{d-1} \right) \delta^2 \) times. Under the assumption that each edge is on some chosen path from the source to the sinks, \( \delta \leq r|\mathcal{T}| \). Summing up all the parts, we obtain the complexity

\[
O(d|\mathcal{T}|\|\mathcal{E}||\mathcal{T}|\xi' \left( \delta^2 + |\mathcal{T}|\xi' \right) + r^3d|\mathcal{E}||\mathcal{T}|\xi),
\]

where \( \xi' = \left( \frac{r+|\mathcal{E}|-2}{d-1} \right) \).

Subsequent to a conference paper of this work [24], Matsumoto [26] proposed an algorithm to construct network codes that achieve the refined Singleton bound. In Table [1] we compare the performances of Algorithm 1, Algorithm 2 and Matsumoto’s algorithm. When \( n_s, \omega, \delta, d \) and \( r \) are fixed (i.e., we regard \( |\mathcal{T}| \) and \( |\mathcal{E}| \) as variables) and \( d > 1 \), the complexities of these algorithms are \( O(|\mathcal{T}|^2|\mathcal{E}|^{2d-2}) \), \( O(|\mathcal{T}|^2|\mathcal{E}|^{2d-1}) \) and \( O(|\mathcal{T}|^2|\mathcal{E}|^{2d-1}) \), respectively.

D. An Example of Algorithm 2

We give an example of applying Algorithm 2 to the network \((\mathcal{G}, s, \{ t, u \})\) shown in Fig. 3. In this network the maximum flow to each sink node is three. We show how Algorithm 2 outputs a network code with \( \omega = 1 \), \( r_t = r_u = 3 \) and \( d_{\text{min}, t} = d_{\text{min}, u} = 3 \). Here the finite field \( \mathbb{F} = GF(2^2) = \{ 0, 1, \alpha, \alpha^2 \} \), where \( \alpha^2 + \alpha + 1 = 0 \).
We first consider node \( k \) kernels satisfying the minimum distance constraints. Together with the local encoding kernels associated with nodes \( e \), we need to choose \( k \) so that (49) is unchanged by multiplying a nonzero elements in \( F \) (see also Section V-E).

We choose the codebook \( C = \{(1, \alpha, \alpha^2)\} \), which is a Reed-Solomon code. Let \( x = (1, \alpha, \alpha^2) \). Note that we only need to check \( x \) with the feasible condition. The reason is that the constraint to choose \( k \) in (49) is unchanged by multiplying a nonzero elements in \( F \) (see also Section V-E).

Notice that nodes \( b, c, d, e \) only copy and forward their received symbols. We refer the reader to [29, Section 17.2] for an explanation that this assumption does not change the optimality of our coding design.

In the following, we show that Algorithm 2 can give \( \beta = \beta_3 = \beta_4 = \beta_5 = 1 \) and \( \beta_6 = \beta_7 = 0 \). Together with the local encoding kernels associated with nodes \( b, c, d, e \), we have a set of local encoding kernels satisfying the minimum distance constraints.

We skip the first two iterations, in which we assign \( \beta_1 = 1 \) and \( \beta_2 = 1 \). In the third iteration, edge 6 is added to the graph and we need to determine

\[
k_6 = \begin{bmatrix} 0 & 0 & \beta_{3,6} & \beta_{4,6} & \beta_{5,6} \end{bmatrix}^T.
\]

We have

\[
F^2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

We first consider node \( t \). We see that edge 6 is on the third path to \( t \). In this iteration, \( I(t) = \{1, 5, 6\} \). We consider the following four cases of \( L \) such that \( 3 \notin L \):

1) \( L = \emptyset \): Since \( z_1 = (1, 0, 0, 0, \alpha) \) satisfies \( (F^2)(x, -z_1))^{13} = 0 \), we need to choose \( k_6 \) such that \( (xA^2_{O(s)} - \ldots}
Fig. 3. This network is used to demonstrate Algorithm 2, in which $s$ is the source node, $t$ and $u$ are the sink nodes. The edges in the network is labelled by the integers beside. We design a code with $\omega = 1$, $r_t = r_u = 3$ and $d_{\min,t} = d_{\min,u} = 3$ over $\text{GF}(2^2)$.

$z_1)F^2k_6 \neq 0$. This gives

$$\beta_{3,6} \neq 0.$$  \hfill (51)

We also have $z_2 = (1, \alpha, 0, 0, 0)$ satisfies $(F^2_t(x, -z_2))^3 = 0$. This error vector imposes the same constraint that $\beta_{3,6} \neq 0$.

2) $L = \{1\}$: Since $z_3 = (0, 0, 0, 0, \alpha)$ satisfies $(F^2_t(x, -z_3)^{\{1,3\}} = 0$, we need to choose $k_6$ such that $(xA_{O(s)}^2 - z_3)F^2k_6 \neq 0$. This gives

$$\beta_{3,6}\alpha^2 + \beta_{4,6} \neq 0.$$  \hfill (52)

3) $L = \{2\}$: Similar to the above case, we have

$$\beta_{3,6}\alpha^2 + \beta_{5,6}\alpha \neq 0.$$  \hfill (53)

4) $L = \{1, 2\}$: We need $xA_{O(s)}^2 F^2k_6 \neq 0$, i.e.,

$$\beta_{3,6}\alpha^2 + \beta_{4,6} + \beta_{5,6}\alpha \neq 0.$$  \hfill (54)

Similarly, we can analyze sink node $u$ and obtain the following constraints on $k_6$:

$$\beta_{4,6} \neq 0$$  \hfill (55)

$$\beta_{3,6}\alpha^2 + \beta_{4,6} \neq 0$$  \hfill (56)

$$\beta_{4,6} + \beta_{5,6}\alpha \neq 0$$  \hfill (57)

$$\beta_{3,6}\alpha^2 + \beta_{4,6} + \beta_{5,6}\alpha \neq 0.$$  \hfill (58)

Form (51) to (58), we have six distinct constraints, which are satisfied by $\beta_{3,6} = \beta_{4,6} = 1$ and $\beta_{5,6} = 0$.

Then we go to the fourth iteration, for which edge 7 is added to the graph and we need to determine

$$k_7 = \begin{bmatrix} 0 & 0 & \beta_{3,7} & \beta_{4,7} & \beta_{5,7} & 0 \end{bmatrix}^\top.$$
We have

\[
F^3 = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Edge 7 is on the second path to \( t \). Considering all \( L \) such that \( 2 \notin L \), we obtain the following constraints on \( k_7 \):

\[
\beta_{3,7} \neq 0 \tag{59}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{5,7} \alpha \neq 0 \tag{60}
\]

\[
\beta_{3,7} + \beta_{4,7} + \beta_{5,7} \alpha \neq 0 \tag{61}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{4,7} \alpha^2 + \beta_{5,7} \alpha \neq 0 \tag{62}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{5,7} \alpha \neq 0 \tag{63}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{4,7} + \beta_{5,7} \alpha \neq 0 \tag{64}
\]

Similarly, we can analyze sink node \( u \) and obtain the following constraints on \( k_7 \):

\[
\beta_{3,7} + \beta_{4,7} \neq 0 \tag{65}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{4,7} \neq 0 \tag{66}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{4,7} \alpha^2 + \beta_{5,7} \alpha \neq 0 \tag{67}
\]

\[
\beta_{3,7} \alpha^2 + \beta_{4,7} + \beta_{5,7} \alpha \neq 0 \tag{68}
\]

From (59) to (68), we have seven distinct constraints on \( k_7 \), which are satisfied by \( \beta_{3,7} = \beta_{5,7} = 1 \) and \( \beta_{4,7} = 0 \).

Let us see what would happen if we only consider \( L = \emptyset \). For this case, in iteration 3, we have only two constraints given by (51) and (55), which are satisfied by \( \beta_{3,6} = \beta_{4,6} = 1 \) and \( \beta_{5,6} = \alpha \). We see that these values do not satisfy (53). We now show that it is impossible to find a network code with \( d_{\text{min},t} = 3 \) with these values.

Construct an error vector \( z \) as follows: \( z_1 = 1 \), \( z_7 = -\left( \beta_{3,7} \alpha^2 + \beta_{5,7} \alpha \right) \) and \( z_i = 0 \) for \( i \neq 1, 7 \). We check that

\[
F_t(x, z) = xF_s,t + zF_t
\]

\[
= (1, \alpha, \alpha^2) \begin{bmatrix}
1 & \beta_{4,7} & 1 \\
0 & \beta_{5,7} & \alpha & 0 & \beta_{3,7} & 1
\end{bmatrix} + (1, z_7) \begin{bmatrix}
1 & \beta_{4,7} & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
= 0.
\]

Thus, \( d_{\text{min},t} \leq w_H(z) = 2 \).
E. Proof of Theorem 12

Theorem 12 is proved by induction on $i$. The codebook with $d_{\text{min},i}^0 \geq d_i$ for all $t \in T$ satisfies the feasible condition for $i = 0$. Assume that up to the $(k - 1)$th iteration, where $0 \leq k - 1 < |T| - n_s$, we can find local encoding kernels such that the feasible condition is satisfied for all $i \leq k$. In the $k$th iteration, let $e$ be the edge appended to $G^{k-1}$ to form $G^k$. We will show that there exists $k_e$ such that the feasible condition continues to hold for $i = k$.

We first consider a sink node $t$ for which edge $e$ is not on any path from the source node $s$ to $t$. (Such a sink node does not necessarily exist). For all $L$, $x$ and $z^k$ satisfying C2)-C4) with $i$ in place of $i$, we have

\[ (F^k_t(x, -z^k))^L = (F^{k-1}_t(x, -(z^k)^k))^L \neq 0, \]

(69)

where (69) follows from (44), and (70) follows from the induction hypothesis, i.e., the feasible condition is satisfied for $i = k - 1$ by noting $w_H((z^k)^k) \leq w_H(z^k) \leq d_i - 1 - |L|$. Therefore, (48) holds for $i = k$ regardless of the choice of $k_e$.

For a sink node $t$ such that edge $e$ is on the $j$th edge-disjoint path from the source node $s$ to $t$, we consider two scenarios for $L$, namely $j \in L$ and $j \notin L$. For all $L$ satisfying C2) and $j \in L$, and all $x$ and $z^k$ satisfying C3) and C4) for $i = k$,

\[ (F^k_t(x, -z^k))^L = (F^{k-1}_t(x, -(z^k)^k))^L \neq 0, \]

(71)

where (71) follows from (47) and (72) follows from the induction hypothesis using the same argument as the previous case. Therefore, (48) again holds for $i = k$ regardless of the choice of $k_e$.

For all $L$ satisfying C2) and $j \notin L$, all $x$ satisfying C3) and all $z^k$ satisfying C4) with $i = k$, (48) holds for $i = k$ if and only if either

\[ (F^k_t(x, -z^k))^L \neq 0 \]

(73)

or

\[ (F^k_t(x, -z^k))_j \neq 0. \]

(74)

By (47) and (46), (73) and (74) are equivalent to

\[ (F^{k-1}_t(x, -(z^k)^k))^L \neq 0, \]

(75)

and

\[ (x^A_{O(s)} - (z^k)^k)F^{k-1}_e - (z^k)_k \neq 0, \]

(76)

respectively. Note that $k_e$ is involved in (76) but not in (75).
For an index set \( \mathcal{L} \) satisfying (2) and \( j \notin \mathcal{L} \), let \( \Sigma^k_{\mathcal{L}} \) be the set of all \((x, z^k)\) that do not satisfy (75), where \( x \) satisfies (3) and \( z^k \) satisfies (4) for \( i = k \). We need to find a proper \( k \), such that for any \((x, z^k) \in \Sigma^k_{\mathcal{L}}, (x, z^k) \) satisfies (76). In the following technical lemmas, we first prove some properties of \( \Sigma^k_{\mathcal{L}} \).

**Lemma 13:** If the feasible condition holds for \( i = k - 1 \), then for any \((x, z^k) \in \Sigma^k_{\mathcal{L}}, w_H(z^k) = d_t - 1 - |\mathcal{L}| \) and \((z^k)_k = 0 \).

**Proof:** Fix \((x, z^k) \in \Sigma^k_{\mathcal{L}} \) and \(|\mathcal{L}| = d_t - 1\). Since \( w_H(z^k) \leq d_t - 1 - |\mathcal{L}| = 0 \), the lemma is true. If \( 0 \leq |\mathcal{L}| < d_t - 1 \), we now prove that \( w_H((z^k)^k) > d_t - 2 - |\mathcal{L}| \). If \( w_H((z^k)^k) \leq d_t - 2 - |\mathcal{L}| \), by the assumption that the feasible condition holds for \( i = k - 1 \),

\[
(F_{t-1}^k(x, -(z^k)^k))_{\mathcal{L} \cup \{j\}} \neq 0,
\]

i.e., \((x, z^k)\) satisfies (75), a contradiction to \((x, z^k) \in \Sigma^k_{\mathcal{L}} \). Therefore

\[
d_t - 1 - |\mathcal{L}| \leq w_H((z^k)^k) \tag{78}
\]

\[
\leq w_H(z^k) \tag{79}
\]

\[
\leq d_t - 1 - |\mathcal{L}|. \tag{80}
\]

Hence, \( w_H((z^k)^k) = w_H(z^k) = d_t - 1 - |\mathcal{L}| \). This also implies that \((z^k)_k = 0 \).

**Lemma 14:** Let \( M \) be a matrix, and let \( j \) be a column index of \( M \). If a system of linear equations \( xM = 0 \) with \( x \) as the variable has only the zero solution, then \( xM^\perp = 0 \) has at most a one-dimensional solution space.

**Proof:** The number of columns of \( M \) is at least the number of rows of \( M \), otherwise the system of linear equations \( xM = 0 \) cannot have a unique solution. Let \( m \) be the number of rows in \( M \). We have \( \text{rank}(M) = m \). Let \( \text{Null}(M^\perp) \) be the null space of \( M^\perp \) defined as

\[
\text{Null}(M^\perp) = \{x : xM^\perp = 0\}.
\]

By the rank-nullity theorem of linear algebra, we have

\[
\text{rank}(M^\perp) + \dim(\text{Null}(M^\perp)) = m. \]

Hence,

\[
\dim(\text{Null}(M^\perp)) = \text{rank}(M) - \text{rank}(M^\perp) \leq 1.
\]

The proof is completed by noting that \( \text{Null}(M^\perp) \) is the solution space of \( xM^\perp = 0 \) with \( x \) as the variable.

**Lemma 15:** Let \( \rho \) be an error pattern with \( |ho| = d_t - 1 - |\mathcal{L}| \), where \( 0 \leq |\mathcal{L}| \leq d_t - 1 \). If the feasible condition holds for \( i = k - 1 \), the span of all \((x, z^k) \in \Sigma^k_{\mathcal{L}} \) with \( z^k \in \rho^* \) is either empty or a one-dimensional linear space.

**Proof:** Consider the equation

\[
(F_{t-1}^k(x, -(z^k)^k))_{\mathcal{L}} \neq 0 \tag{81}
\]
with $x \in C$ and $z^k \in \rho^*$ as variables. Since $C$ and $\rho^*$ are both vector spaces, (81) is a system of linear equations. By the assumption that the feasible condition holds for $i = k - 1$, (81) has only the zero solution. By Lemma 14, the system of linear equations

$$ (F_i^{k-1}(x, -(z^k)^k))\{C \cup \{j\} = 0, $$

with $x \in C$ and $z^k \in \rho^*$ as variables, has at most a one-dimensional solution space.

**Lemma 16:** If the feasible condition holds for $i = k - 1$, there exist at most $\binom{n_s + k - 1}{d_t - 1 - |C|}q^{d_t - 1 - |C|} - 1$ values of $k_e$ such that (76) does not hold for some $(x, z^k) \in \Sigma^k_{L}$. 

**Proof:** For $(x_0, z^k_0) \in \Sigma^k_{L}$, by Lemma 13, $(z^k_0)_{k} = 0$. Thus, all $k_e$ satisfying

$$ (x_0 A^k_{O(s)} - (z^k_0)^k) F^k k_e = 0 \tag{82} $$

do not satisfy (76) for $(x_0, z^k_0) \in \Sigma^k_{L}$. To count the number of solutions of (82), we notice that

$$ (F_i^{k-1}(x_0, -(z^k_0)^k))\{C \neq 0, $$

by the feasible condition holding for $i = k - 1$, and

$$ (F_i^{k-1}(x_0, -(z^k_0)^k))\{C \cup \{j\} = 0, \tag{84} $$

since $(x_0, z^k_0) \in \Sigma^k_{L}$. Thus,

$$ (F_i^{k-1}(x_0, -(z^k_0)^k))_j = ((x_0 A^k_{O(s)} - (z^k_0)^k) F^k (A^{k-1}_{k(t)})^T)_j \neq 0, $$

which gives a nonzero component of $(x_0 A^k_{O(s)} - (z^k_0)^k) F^k$ corresponding to the edge that precedes edge $e$ on the $j$th path from $s$ to $t$. This shows that the components of $(x_0 A^k_{O(s)} - (z^k_0)^k) F^k$ corresponding to the edges in $I(tail(e))$ are not all zero. On the other hand, a component of $k_e$ can possibly be nonzero if and only if it corresponds to an edge in $I(tail(e))$. Therefore, the solution space of $k_e$ in (82) is an $F^{|I(tail(e))| - 1}$-dimensional subspace.

By Lemma 13 for each $(x, z^k) \in \Sigma^k_{L}$, $z^k$ must match an error pattern $\rho$ with $|\rho| = d_t - 1 - |L|$ and $e \notin \rho$. Since there are totally $n_s + k - 1$ edges in $G_k^t$ excluding $e$, there are $\binom{n_s + k - 1}{d_t - 1 - |C|}$ error patterns with size $d_t - 1 - |L|$.

Consider an error pattern $\rho$ with $|\rho| = d_t - 1 - |L|$ and $e \notin \rho$. By Lemma 15, if $(x_0, z^k_0) \in \Sigma^k_{L}$ with $z^k_0 \in \rho^*$, all $(x, z^k) \in \Sigma^k_{L}$ with $z^k \in \rho^*$ can be expressed as $(\alpha x_0, \alpha z^k_0)$ with nonzero $\alpha \in \mathbb{F}$. Since we obtain the same solutions of $k_e$ in (82) when $x_0$ and $z^k_0$ are replaced by $\alpha x_0$ and $\alpha z^k_0$, respectively, for a particular pattern $\rho$, we only need to consider one $(x_0, z^k_0) \in \Sigma^k_{L}$ with $z^k_0 \in \rho^*$.

Upon considering all error patterns $\rho$ with $|\rho| = d_t - 1 - |L|$ and $e \notin \rho$, we conclude that there exist at most $\binom{n_s + k - 1}{d_t - 1 - |C|} q^{|I(tail(e))| - 1}$ values of $k_e$ not satisfying (76) for some $(x, z^k) \in \Sigma^k_{L}$. 

Considering the worst case that for all $t \in T$, edge $e$ is on an edge-disjoint path from the source node $s$ to sink
node \( t \), and considering all the index set \( L \) with \( 0 \leq |L| \leq d_t - 1 \) and \( j \notin L \) for each sink node \( t \), we have at most
\[
\sum_{t \in T} \sum_{l=0}^{d_t-1} \binom{r_t - 1}{l} \binom{n_s + k - 1}{d_t - 1 - l} q^{l |\text{tail}(e)|-1} = \sum_{t \in T} \binom{r_t + n_s + k - 1}{d_t - 1} q^{l |\text{tail}(e)|}-1
\]

(85)
\[
\leq \sum_{t \in T} \binom{r_t + |\mathcal{E}| - 2}{d_t - 1} q^{l |\text{tail}(e)|}-1
\]

(86)

vectors that cannot be chosen as \( \mathbf{k}_c \). Note that (86) is justified because \( 0 \leq k - 1 < |\mathcal{E}| - n_s \). Since \( q > \sum_{t \in T} (r_t + |\mathcal{E}| - 2) \), there exists a choice of \( \mathbf{k}_c \) such that for all \( L \) satisfying (2) and \( j \notin L \), all \( x \) satisfying (3), and all \( z^k \) satisfying (4) for \( i = k \), (48) holds for \( i = k \). Together with the other cases (where the choice of \( \mathbf{k}_c \) is immaterial), we have proved the existence of a \( \mathbf{k}_c \) such that the feasible condition holds for \( i = k \).

VI. CONCLUDING REMARKS

This work, together with the previous work [15], gives a framework for coherent network error correction. The work [15] characterizes the error correction/detection capability of a general transmission system with network coding being a special case. The problems concerned here are the coding bounds and the code construction for network error correction.

In this work, refined versions of the Hamming bound, the Singleton bound and the Gilbert-Varshamov bound for network error correction have been presented with simple proofs based on the distance measures developed in [15]. These bounds are improvements over the ones in [6], [10], [11] for the linear network coding case. Even though these bounds are stated based on the Hamming weight as the weight measure on the error vectors, they can also be applied to the weight measures in [12], [16], [17] because of the equivalence relation among all these weight measures (See [15], [28]).

Like the original version of the Singleton bound [6], [10], the refined Singleton bound for linear network codes proved in this paper continues to be tight. Two different construction algorithms have been presented and both of them can achieve the refined Singleton bound. The first algorithm finds a codebook based on a given set of local encoding kernels, which simply constructs an MDS code when the problem setting is the classical case. The second algorithm constructs a set of of local encoding kernels based on a given classical error-correcting code satisfying a certain minimum distance requirement by recursively choosing the local encoding kernels that preserve the required minimum distance properties.

There are many problems to be solved towards application of network error correction. Our algorithms require a large field size to guarantee the existence of network codes with large minimum distances. One future work is to consider how to relax this field size requirement. Fast decoding algorithms of network error-correcting codes are also desired. Moreover, network error correction in cyclic networks is sill lack of investigation.
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