Analysis of a Legendre spectral element method (LSEM) for the two-dimensional system of a nonlinear stochastic advection-reaction-diffusion models

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Abstract

In this work, we develop a Legendre spectral element method (LSEM) for solving the stochastic nonlinear system of advection-reaction-diffusion models. The used basis functions are based on a class of Legendre functions such that their mass and diffuse matrices are tridiagonal and diagonal, respectively. The temporal variable is discretized by a Crank–Nicolson finite difference formulation. In the stochastic direction, we also employ a random variable $W$ based on the $Q$–Wiener process. We inspect the rate of convergence and the unconditional stability for the achieved semi-discrete formulation. Then, the Legendre spectral element technique is used to obtain a full-discrete scheme. The error estimation of the proposed numerical scheme is substantiated based upon the energy method. The numerical results confirm the theoretical analysis.

Keywords: Nonlinear system of advection-reaction-diffusion equation, error estimate, spectral element method (SEM), stochastic PDEs, .

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1 Introduction

We consider the stochastic nonlinear system of advection-reaction-diffusion models [1, 2]

\[
\begin{cases}
    du + \left( \xi(x) \nabla u - \nabla \cdot \zeta(x) \nabla u + w_p e_1 f(u, v) \right) dt = dW, & \text{in } D \times (0, T], \\
    dv + \left( \xi(x) \nabla v - \nabla \cdot (\zeta(x) \nabla v) + w_p e_2 f(u, v) \right) dt = dW, & \text{in } D \times (0, T], \\
    dw + \left( \xi(x) \nabla w - \nabla \cdot (\zeta(x) \nabla w) + w_p e_3 f(u, v) + r(x) w \right) dt = dW, & \text{in } D \times (0, T],
\end{cases}
\]

(1.1)

where \( u, v \) and \( w \) denote the concentrations of the main ground substance, aqueous solution electrolyte and microorganism, respectively [1, 2]. In the above model \( r(x) \) is a known function, \( \xi \) is the advection coefficient, \( \zeta \) is the diffusion coefficient, \( e_i \) and \( w_p \) are constant, respectively. Also, \( W \) is a \( Q \)-Wiener process with respect to a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The nonlinear terms are

\[
f(u, v) = g(u, v) = h(u, v) = \frac{u}{\kappa_1 + u} + \frac{v}{\kappa_2 + v}.
\]

Predictions of solute transport in aquifers generally have to rely on mathematical models based on groundwater flow and convection-dispersion equations. The groundwater model is employed to prevent and control the groundwater contaminant with the microbiological technology [2]. Several scholars investigated Eq. (1.1) for example using an improved finite element approach [2], meshless local approaches [3, 4], lattice Boltzmann technique [5], a front-tracking method [6], novel WENO methods [7], or a finite element method [8]. The interested readers can refer to [9, 10] to get more information for Eq. (1.1).

In the past, the groundwater models have been based only on deterministic considerations. In practice, aquifers are generally heterogeneous, i.e., their hydraulic properties (e.g., permeability) change in space. These variations are irregular and characterized by length scales significantly larger than the pore scale. These spatial fluctuations cause the flow variables such as concentration to change in space in an irregular manner. Therefore, a reliable description of the groundwater model can be explained only in a stochastic form [11].

The first stochastic equation can be rewritten as

\[
du(t) = \left( Au(t) + f(u) \right) dt + dW,
\]

(1.2)

where \( -A : \mathcal{D}(-A) \subset H \to H \) is a linear, self-adjoint, positive definite operator where the domain \( \mathcal{D} \) is dense in \( H \) and compactly embedded in \( H \) (i.e., \( L^2(D) \)) and the semigroup \( e^{tA} (t \geq 0) \) is generated by \( -A \). Additionally, we assume that \( f : H \to H \) satisfies the linear growth condition and is twice continuously Frechet differentiable with bounded derivatives up to order 2 [12]. The
initial value $u(0) = u_0$ is deterministic as well. Therefore, \( (1.2) \) has a continuous mild solution \[13\]

\[
u(t) = e^{tA}u_0 + \int_0^te^{(t-s)A}f(u(s))\,ds + \int_0^te^{(t-s)A}dW(s),
\]

where for $t \in [0, T]$ and $u : [0, T] \times D \to H$. Regarding the expected value of the solution, we can assume that $\mathbb{E}\|u(t)\|^2 \leq \infty$. The same mild solutions can be employed for $v$ and $w$.

The deterministic case of Eq. \((1.1)\) has been studied by some scholars for example a new finite volume method \[2\], new Krylov WENO methods \[7\], local radial basis function collocation method \[14\], etc. Also, the SEM is applied to solve some important problems such as the Schrödinger equations \[15\], Pennes bioheat transfer model \[16\], the shallow water equations \[17\], integral differential equations \[18\, 19\, 20\], hyperbolic scalar equations \[21\], predator-prey problem \[22\], some problems in the finance mathematics \[23\, 24\] and so forth.

The main aim of the current paper is to propose a new high-order numerical procedure for solving the two-dimensional system of a nonlinear stochastic advection-reaction-diffusion models. The used technique is based on the modified Legendre spectral element procedure. The coefficient matrix of the employed technique is more well-posed than the traditional Legendre spectral element method. The structure of this article is as follows. In Section \[2\] we propose and analysis the time-discrete scheme. In Section \[3\] we develop the new numerical technique and analysis it. We check the numerical results to solve the considered model in Section \[4\]. Finally, a brief conclusion of the current paper is written in Section \[5\].

2 Temporal discretization

First of all, we briefly review some important notations used in the paper. Considering $\Omega \subset \mathbb{R}^d$, we define the following functional spaces

$$L^2(\Omega) = \left\{ f : \int_{\Omega} f^2\,d\Omega < \infty \right\},$$

$$H^1(\Omega) = \{ f \in L^2(\Omega), \ \nabla f \in L^2(\Omega) \},$$

$$H^1_0(\Omega) = \{ f \in H^1(\Omega), \ f|_{\partial \Omega} = 0 \},$$

$$H^k(\Omega) = \{ f \in L^2(\Omega), \ D^\beta f \in L^2(\Omega) \ for \ all \ |\beta| \leq k \},$$

and the derivative

$$D^\alpha f = \left( \frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1}} \right) \left( \frac{\partial^{\alpha_2} f}{\partial x_2^{\alpha_2}} \right) \ldots \left( \frac{\partial^{\alpha_p} f}{\partial x_p^{\alpha_p}} \right), \quad |\alpha| = \sum_{i=1}^p \alpha_i.$$
The corresponding inner products for $L^2(\Omega)$ and $H^1(\Omega)$ are as follows

$$(f, g) = \int_\Omega f(x)g(x)d\Omega, \quad (f, g)_1 = (f, g) + (\nabla f, \nabla g),$$

and the associated norms are

$$\|f\|_{L^2(\Omega)} = (f, f)^{\frac{1}{2}}, \quad \|f\|_{H^1(\Omega)} = (f, f)_1^{\frac{1}{2}}, \quad |f|_1 = (\nabla f, \nabla f)^{\frac{1}{2}}.$$

Furthermore, associated norm for the space $H^m$ is as

$$\|f\|_{H^m(\Omega)} = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

To discretize the time variable, we define

$$t_n = n\tau, \quad \forall \ n = 0, 1, \ldots, N,$

where $\tau = T/N$ is the step size. We introduce additionally

$$v^{n-\frac{1}{2}} = v(x, y, t_{n-\frac{1}{2}}) = \frac{1}{2} (v^n + v^{n-1}), \quad \delta_tv^{n-\frac{1}{2}} = \frac{1}{\tau} (v^n - v^{n-1}), \quad v^n = v(x, y, t_n).$$

The Crank-Nicolson scheme for problem (1.1) is as follows

$$\begin{cases}
\frac{\partial u^{n-\frac{1}{2}}}{\partial t} + \xi(x)\nabla u^{n-\frac{1}{2}} - \nabla \cdot (\zeta(x)\nabla u^{n-\frac{1}{2}}) + w_pe_1f(u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}}) = \ddot{W}, \\
\frac{\partial v^{n-\frac{1}{2}}}{\partial t} + \xi(x)\nabla v^{n-\frac{1}{2}} - \nabla \cdot (\zeta(x)\nabla v^{n-\frac{1}{2}}) + w_pe_2f(u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}}) = \ddot{W}, \\
\frac{\partial w^{n-\frac{1}{2}}}{\partial t} + \xi(x)\nabla w^{n-\frac{1}{2}} - \nabla \cdot (\zeta(x)\nabla w^{n-\frac{1}{2}}) + w_pe_3f(u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}}) + r(x)w^{n-\frac{1}{2}} = \ddot{W},
\end{cases} \quad (2.1)$$
where $C$ is a positive constant such that $|R^i_0|$ and $|R^i_B| \leq C\tau^2$. Discretizing relation (2.1) yields

$$
\begin{align*}
\frac{u^n - u^{n-1}}{\tau} &+ \xi(x) \left[ \nabla u^n + \nabla u^{n-1} \right] - \nabla \cdot \left[ \zeta(x) \left( \frac{\nabla u^n + \nabla u^{n-1}}{2} \right) \right] + w_p e_1 f \left( u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}} \right) = \frac{W^n - W^{n-1}}{\tau}, \\
\frac{v^n - v^{n-1}}{\tau} &+ \xi(x) \left[ \nabla v^n + \nabla v^{n-1} \right] - \nabla \cdot \left[ \zeta(x) \left( \frac{\nabla v^n + \nabla v^{n-1}}{2} \right) \right] + w_p e_2 f \left( u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}} \right) = \frac{W^n - W^{n-1}}{\tau}, \\
\frac{w^n - w^{n-1}}{\tau} &+ \xi(x) \left[ \nabla w^n + \nabla w^{n-1} \right] - \nabla \cdot \left[ \zeta(x) \left( \frac{\nabla w^n + \nabla w^{n-1}}{2} \right) \right] + w_p e_3 f \left( u^{n-\frac{1}{2}}, v^{n-\frac{1}{2}} \right) + r(x) \left[ \frac{w^n + w^{n-1}}{2} \right] = \frac{W^n - W^{n-1}}{\tau},
\end{align*}
$$

or

$$
\begin{align*}
&u^n + \frac{\tau}{2} \xi(x) \nabla u^n - \frac{\tau}{2} \nabla \cdot \left[ \zeta(x) \nabla u^n \right] + W^n \\
&= u^{n-1} - \frac{\tau}{2} \xi(x) \nabla u^{n-1} + \frac{\tau}{2} \nabla \cdot \left[ \zeta(x) \nabla u^{n-1} \right] - \tau w_p e_1 f \left( u^{n-1}, v^{n-1} \right) + W^{n-1}, \\
v^n + \frac{dt}{2} \xi(x) \nabla v^n - \frac{\tau}{2} \nabla \cdot \left[ \zeta(x) \nabla v^n \right] + W^n \\
&= v^{n-1} - \frac{dt}{2} \xi(x) \nabla v^{n-1} + \frac{dt}{2} \nabla \cdot \left[ \zeta(x) \nabla v^{n-1} \right] - dt w_p e_2 f \left( u^{n-1}, v^{n-1} \right) + W^{n-1}, \\
&\left( 1 + \frac{dt}{2} r(x) \right) w^n + \frac{dt}{2} \xi(x) \nabla w^n - \frac{dt}{2} \nabla \cdot \left[ \zeta(x) \nabla w^n \right] + W^n \\
&= \left( 1 - \frac{dt}{2} r(x) \right) w^{n-1} - \frac{dt}{2} \xi(x) \nabla w^{n-1} + \frac{dt}{2} \nabla \cdot \left[ \zeta(x) \nabla w^{n-1} \right] - dt w_p e_3 f \left( u^{n-1}, v^{n-1} \right) + W^{n-1}
\end{align*}
$$

The vector-matrix configuration of Eq. (2.3) is

$$
H_1 U^n + \frac{\tau}{2} I \nabla U^n - \frac{\tau}{2} I \nabla \cdot \zeta(x) \nabla U^n + W^n = H_2 U^{n-1} - \frac{\tau}{2} I \nabla U^{n-1} + \frac{\tau}{2} I \nabla \cdot \zeta(x) \nabla U^{n-1} - \tau N F \left( U^{n-1} \right) + W^{n-1},
$$

where $I$ is the identity matrix and

$$
H_1 = \text{diag} \left( 1, 1, 1 + \frac{\tau}{2} r(x) \right), \quad H_2 = \text{diag} \left( 1, 1, 1 - \frac{\tau}{2} r(x) \right), \quad N = \text{diag} \left( w_p e_1, w_p e_2, w_p e_3 \right),
$$

and also the unknown vector is $U = (u, v, w)$. 

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2.1 Error analysis of the semi-discrete formulation

Theorem 2.1. If $\mathcal{U}^n \in H_0^1(\Omega)$, then relation (2.4) will be unconditionally stable.

Proof. Let $\zeta(x)$ and $\xi(x) \in L^2(\Omega)$. We want to find $\mathcal{U}^n \in H_0^1(\Omega)$ such that

$$H_1(\mathcal{U}^n, \chi) + \frac{\tau}{2} I \left( \zeta(x) \nabla \mathcal{U}^n, \nabla \chi \right) - \frac{\tau}{2} I \left( \mathcal{U}^n, \frac{\partial}{\partial x} \chi \right) - \frac{\tau}{2} I \left( \mathcal{U}^n, \frac{\partial}{\partial y} \chi \right) + (W^n, V)$$

$$= H_2(\mathcal{U}^{n-1}, \chi) - \frac{\tau}{2} I \left( \zeta(x) \nabla \mathcal{U}^{n-1}, \nabla \chi \right) + \frac{\tau}{2} I \left( \mathcal{U}^{n-1}, \frac{\partial}{\partial x} \chi \right)$$

$$+ \frac{\tau}{2} I \left( \mathcal{U}^{n-1}, \frac{\partial}{\partial y} \chi \right) - \tau N \left( F, \chi \right) + (W^{n-1}, V) \quad \forall \chi \in H_0^1(\Omega). \quad (2.6)$$

Let $\tilde{\mathcal{U}}^n$ be an approximate solution of $\mathcal{U}^n$, then

$$H_1\left( \tilde{\mathcal{U}}^n, \chi \right) + \frac{\tau}{2} I \left( \zeta(x) \nabla \tilde{\mathcal{U}}^n, \nabla \chi \right) - \frac{\tau}{2} I \left( \tilde{\mathcal{U}}^n, \frac{\partial}{\partial x} \chi \right) - \frac{\tau}{2} I \left( \tilde{\mathcal{U}}^n, \frac{\partial}{\partial y} \chi \right) + (W^n, V)$$

$$= H_2\left( \tilde{\mathcal{U}}^{n-1}, \chi \right) - \frac{\tau}{2} I \left( \zeta(x) \nabla \tilde{\mathcal{U}}^{n-1}, \nabla \chi \right) + \frac{\tau}{2} I \left( \tilde{\mathcal{U}}^{n-1}, \frac{\partial}{\partial x} \chi \right)$$

$$+ \frac{\tau}{2} I \left( \tilde{\mathcal{U}}^{n-1}, \frac{\partial}{\partial y} \chi \right) - \tau N \left( \tilde{F}, \chi \right) + (W^{n-1}, V) \quad \forall \chi \in H_0^1(\Omega), \quad (2.7)$$

where $\tilde{F} = F(\tilde{\mathcal{U}})$. Subtracting Eq. (2.7) for Eq. (2.6), results

$$H_1(\Psi^n, \chi) + \frac{\tau}{2} I \left( \zeta(x) \nabla \Psi^n, \nabla \chi \right) - \frac{\tau}{2} I \left( \Psi^n, \frac{\partial}{\partial x} \chi \right) - \frac{\tau}{2} I \left( \Psi^n, \frac{\partial}{\partial y} \chi \right)$$

$$= H_2(\Psi^{n-1}, \chi) - \frac{\tau}{2} I \left( \zeta(x) \nabla \Psi^{n-1}, \nabla \chi \right) + \frac{\tau}{2} I \left( \Psi^{n-1}, \frac{\partial}{\partial x} \chi \right)$$

$$+ \frac{\tau}{2} I \left( \Psi^{n-1}, \frac{\partial}{\partial y} \chi \right) - \tau N \left( F - \tilde{F}, \chi \right), \quad \forall \chi \in H_0^1(\Omega), \quad (2.8)$$

where

$$\Psi^n = E[\mathcal{U}^n - \tilde{\mathcal{U}}^n].$$

Setting $\chi = \Psi^n$ in Eq. (2.8) yields

$$H_1(\Psi^n, \Psi^n) + \frac{\tau}{2} I \left( \zeta(x) \nabla \Psi^n, \nabla \Psi^n \right) - \frac{\tau}{2} I \left( \Psi^n, \frac{\partial}{\partial x} \Psi^n \right) - \frac{\tau}{2} I \left( \Psi^n, \frac{\partial}{\partial y} \Psi^n \right)$$

$$= H_2(\Psi^{n-1}, \Psi^n) - \frac{\tau}{2} I \left( \zeta(x) \nabla \Psi^{n-1}, \nabla \Psi^n \right) + \frac{\tau}{2} I \left( \Psi^{n-1}, \frac{\partial}{\partial x} \Psi^n \right)$$

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\[ + \frac{\tau}{2} I \left( \Psi^{n-1}, \frac{\partial}{\partial y} \chi \right) - \tau N \left( F - \tilde{F}, \Psi^n \right). \] (2.9)

Applying the Cauchy-Schwarz inequality for Eq. (2.9), results

\[ \|H_1\| \|\Psi^n\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\zeta(x)\| \|\nabla \Psi^n\|^2_{L^2(\Omega)} \leq \frac{\tau}{2} (\Psi^n, \frac{\partial}{\partial x} \Psi^n) + \frac{\tau}{2} (\Psi^n, \frac{\partial}{\partial y} \Psi^n) \]

\[ + \|H_2\| \|\Psi^n\|_{L^2(\Omega)} \|\Psi^{n-1}\|_{L^2(\Omega)} + \frac{\tau}{2} \|\zeta(x)\| \|\nabla \Psi^n\|_{L^2(\Omega)} \|\nabla \Psi^{n-1}\|_{L^2(\Omega)} \]

\[ + \frac{\tau}{2} (\Psi^{n-1}, \frac{\partial}{\partial y} \Psi^n) + \frac{\tau}{2} (\Psi^{n-1}, \frac{\partial}{\partial x} \Psi^n) - \tau N \left( F - \tilde{F}, \Psi^n \right). \]

There exists constant \( C \) such that

\[ \|H_2\|, \|H_3\| \leq C, \] (2.10)

and

\[ \|F - \tilde{F}\| \leq L\Psi^{n-1}. \] (2.11)

By simplification we have

\[ \|H_1\| \|\Psi^n\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\zeta(x)\| \|\nabla \Psi^n\|^2_{L^2(\Omega)} \leq \frac{\tau}{2} (\Psi^n, \frac{\partial}{\partial x} \Psi^n) + \frac{\tau}{2} (\Psi^n, \frac{\partial}{\partial y} \Psi^n) \]

\[ + \|H_2\| \|\Psi^n\|_{L^2(\Omega)} \|\Psi^{n-1}\|_{L^2(\Omega)} + \frac{\tau}{2} \|\zeta(x)\| \|\nabla \Psi^n\|_{L^2(\Omega)} \|\nabla \Psi^{n-1}\|_{L^2(\Omega)} \]

\[ + \tau L \|N\| \|\Psi^{n-1}\|_{L^2(\Omega)} \|\Psi^n\|_{L^2(\Omega)}. \]

So, from the following assumption and the definition of matrices \( H_1 \) and \( H_2 \), we have

\[ \|H_2\| \leq \|H_1\|. \]

Now, we can get

\[ \frac{1}{2} \|H_1\| \|\Psi^n\|^2_{L^2(\Omega)} + \frac{\tau}{4} \|\zeta(x)\| \|\nabla \Psi^n\|^2_{L^2(\Omega)} \]

\[ \leq \frac{1}{2} \|H_1\| \|\Psi^{n-1}\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\zeta(x)\| \|\nabla \Psi^{n-1}\|^2_{L^2(\Omega)} \]

\[ + \frac{C_1 L \tau}{2} \|\Psi^n\|^2_{L^2(\Omega)} + \frac{C_2 L \tau}{2} \|\Psi^n\|^2_{L^2(\Omega)}. \]

Using the below relation

\[ \|\Psi^n\|^2_{H_w(\Omega)} = \|H_1\| \|\Psi^n\|^2_{L^2(\Omega)} + \frac{1}{2} \|\zeta(x)\| \|\nabla \Psi^n\|^2_{L^2(\Omega)}, \]
Eq. (2.12) is changed to
\[
\|\Psi^n\|^2_{H_w(\Omega)} \leq \|\Psi^{n-1}\|^2_{H_w(\Omega)} + \frac{C_1L\tau}{\|\zeta(x)\|} \|\Psi^n\|^2_{H_w(\Omega)} + \frac{C_2L\tau}{\|\zeta(x)\|} \|\Psi^{n-1}\|^2_{H_w(\Omega)}.
\]  
(2.13)

By summing Eq. (2.13) for \(j\) from 0 to \(n\), gives
\[
\sum_{m=1}^{n} \|\Psi^m\|^2_{H_w(\Omega)} \leq \sum_{m=1}^{n} \|\Psi^{m-1}\|^2_{H_w(\Omega)} + \frac{C_1L\tau}{\|\zeta(x)\|} \sum_{m=1}^{n} \|\Psi^m\|^2_{H_w(\Omega)} + \frac{C_2L\tau}{\|\zeta(x)\|} \sum_{m=1}^{n} \|\Psi^{m-1}\|^2_{H_w(\Omega)}.
\]

Thus, we have
\[
\|\Psi^n\|^2_{H_w(\Omega)} \leq \|\Psi^0\|^2_{H_w(\Omega)} + \frac{2CL\tau}{\|\zeta(x)\|} \sum_{m=1}^{n} \|\Psi^m\|^2_{H_w(\Omega)} \tag{2.14}
\]

Considering Gronwall’s inequality for Eq. (2.14) yields
\[
\|\Psi^n\|^2_{H_w(\Omega)} \leq \left(\|\Psi^0\|^2_{H_w(\Omega)}\right) \exp\left(\frac{2CLn\tau}{\|\zeta(x)\|}\right) \leq C \|\Psi^0\|^2_{H_w(\Omega)}.
\]

So, we have
\[
\|\Psi^n\|^2_{L^2(\Omega)} \leq \|\Psi^n\|^2_{H_w(\Omega)} \leq C \|\Psi^0\|^2_{H_w(\Omega)}.
\]

\[\square\]

**Theorem 2.2.** The convergence order of relation (2.4) is \(O(\tau^2)\).

**Proof.** Let us assume \(u^n, U^n \in H^1_0(\Omega)\). We set
\[
X^n = \mathbb{E}[u^n - U^n] \quad n \geq 1,
\]
where \(X^0 = 0\). Then, we have
\[
H_1X^n + \frac{\tau}{2} I \nabla X^n - \frac{\tau}{2} I \nabla \cdot \zeta(x) \nabla X^n =
\]
\[
H_2X^{n-1} - \frac{\tau}{2} I \nabla X^{n-1} + \frac{\tau}{2} I \nabla \cdot \zeta(x) \nabla X^{n-1} + \tau R - \tau N \left(F^{n-1} - \tilde{F}^{n-1}\right).
\]  
(2.15)

According to the Crank-Nicolson idea, we have
\[
|R| \leq C_1\tau^2.
\]
Similar to Theorem 2.1, we obtain
\[
\|X^n\|_{H^w(\Omega)}^2 \leq \|X^0\|_{H^w(\Omega)}^2 + \frac{2L^\tau}{\|\zeta(x)\|} \sum_{m=1}^n \|X^m\|_{H^w(\Omega)}^2 + \max_{1 \leq m \leq n} \|R\|_{L^2(\Omega)}^2 \\
\leq \left\{ \max_{1 \leq m \leq n} \|R\|_{L^2(\Omega)}^2 \right\} \exp\left( \frac{2Ln^\tau}{\|\zeta(x)\|} \right) \leq C^2 \tau^2 \\
\leq \left\{ \max_{1 \leq m \leq n} \|R\|_{L^2(\Omega)}^2 \right\} \exp\left( \frac{2Ln^\tau}{\|\zeta(x)\|} \right) \\
\leq \exp\left( \frac{2Ln^\tau}{\|\zeta(x)\|} \right) C_1 \tau^2 \leq C \tau^2.
\]
which completes the proof. 

3 Error estimation for full-discrete plane

In this section, we employ a new class of Legendre polynomial functions which were developed in [25].

Lemma 3.1. [25] Consider the following relations
\[
\psi_k(x) = \gamma_k (L_k(x) - L_{k+2}(x)), \quad (3.1)
\]
in which \( \gamma_k = (4k + 6)^{-\frac{1}{2}} \) and \( L_k(x) \) are the Legendre polynomials. Let us denote
\[
a_{jk} = \int_{-1}^1 \frac{d\psi_k(x)}{dx} \frac{d\psi_j(x)}{dx} dx, \quad b_{jk} = \int_{-1}^1 \psi_k(x) \psi_j(x) dx. \quad (3.2)
\]
Then
\[
a_{jk} = \begin{cases} 
1, & k = j, \\
0, & k \neq j,
\end{cases} \quad b_{jk} = b_{kj} = \begin{cases} 
\gamma_k \gamma_j \left( \frac{2}{2j + 1} + \frac{2}{2j + 5} \right), & k = j, \\
-\gamma_k \gamma_j \frac{2}{2k + 1}, & k = j + 2, \\
0, & \text{Otherwise.}
\end{cases} \quad (3.3)
\]
The SEM as a combination of the finite element method and spectral polynomials has been developed by Patera [26]. By dividing the computational region into \( N_e \) non-overlapping elements
\( \Omega_e \)

\[ \Omega = \bigcup_{e=1}^{N_e} \Omega_e, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j. \]

Now, we define the following projection operator.

\[ \mathcal{P}_h^1 : H^1_0(\Omega) \to V_N^0, \tag{3.4} \]

where

\[ (\nabla (u - \mathcal{P}_h^1 u), \nabla v) = 0, \quad u \in H^1_0(\Omega), \quad \forall v \in V_N^0, \tag{3.5} \]

and \( V_N^0 \) is the spectral element approximation space

\[ V_N^0 = \{ w \in H^1_0(\Omega) : w|_{\Omega_s} \in \mathbb{P}_N(\Omega), \ s = 1, 2, \ldots, n_s \}. \tag{3.6} \]

**Lemma 3.2.** [27] Let \( u \in H^v \ (v \geq 1) \), therefore

\[ \| u - \mathcal{P}_h^1 u \| \leq C \left[ \sum_{k=1}^{n_s} h_k^{2(\min(N_{k+1,v})-1)} N_k^{2(1-v)} \| u \|_v^2 \right]^{\frac{1}{2}}. \tag{3.7} \]

In the special cases \( N_k = N \) and \( h \leq h_k \leq c'h \) we get

\[ \| u - \mathcal{P}_h^1 u \| \leq Ch_k^{(\min(N+1,v)-1)} N^{1-v} \| u \|_v. \tag{3.8} \]

We aim to find a \( \mathbf{u}^n \in \omega^d_e \) such that

\[
H_1(\mathbf{u}^n, \chi) + \frac{\tau}{2} I(\zeta(x) \nabla \mathbf{u}^n, \nabla \chi) - \frac{\tau}{2} I\left(\mathbf{u}^n, \frac{\partial}{\partial x} \chi\right) - \frac{\tau}{2} I\left(\mathbf{u}^n, \frac{\partial}{\partial y} \chi\right) = (W^n, V)
\]

\[
= H_2(\mathbf{u}^{n-1}, \chi) - \frac{\tau}{2} I(\zeta(x) \nabla \mathbf{u}^{n-1}, \nabla \chi) + \frac{\tau}{2} I\left(\mathbf{u}^{n-1}, \frac{\partial}{\partial x} \chi\right) + \frac{\tau}{2} I\left(\mathbf{u}^{n-1}, \frac{\partial}{\partial y} \chi\right) + \tau N(\mathbf{F}^{n-1}, \chi) + \tau (\mathbf{R}^h_n, \chi) + (W^{n-1}, V) \quad \chi \in H^1_0(\Omega). \tag{3.9}
\]

The spectral element formulation is: find a \( \mathbf{u}^n_h \in \omega^d_i \) such that

\[
H_1(\mathbf{u}^n_h, \chi_h) + \frac{\tau}{2} I(\zeta(x) \nabla \mathbf{u}^n_h, \nabla \chi_h) - \frac{\tau}{2} I\left(\mathbf{u}^n_h, \frac{\partial}{\partial x} \chi_h\right) - \frac{\tau}{2} I\left(\mathbf{u}^n_h, \frac{\partial}{\partial y} \chi_h\right) = (W^n, V)
\]

\[
= H_2(\mathbf{u}^{n-1}_h, \chi_h) - \frac{\tau}{2} I(\zeta(x) \nabla \mathbf{u}^{n-1}_h, \nabla \chi_h) + \frac{\tau}{2} I\left(\mathbf{u}^{n-1}_h, \frac{\partial}{\partial x} \chi_h\right) + \frac{\tau}{2} I\left(\mathbf{u}^{n-1}_h, \frac{\partial}{\partial y} \chi_h\right) + \tau N(\mathbf{F}^{n-1}, \chi_h) + (W^{n-1}, V) \quad \forall \ \chi_h \in H^1_0(\Omega). \tag{3.10}
\]
Lemma 3.3. Let
\[(\mathcal{G}^n_{r,d}, \chi_r) = (p^n h \hat{u}^n_r - \hat{u}^n_r, \chi_r) + \tau A_2 \left( p^n h \hat{u}^n_r - \hat{u}^n_r, \frac{\partial}{\partial x} \chi_r \right) + \tau A_3 \left( p^n h \hat{u}^n_r - \hat{u}^n_r, \frac{\partial}{\partial y} \chi_r \right) \tag{3.11} \]

Then, we have
\[
\|\mathcal{G}^n_{r,d}\|_{L^2(\Omega)} \leq CN^{1-\nu}.
\]

Proof. Eq. (3.11) is changed to
\[
\left( \mathcal{G}^n_{r,d}, \chi_r \right) = \left( p^n h \hat{u}^n_r - \hat{u}^n_r, \chi_r \right) - \tau A_2 \left( \frac{\partial}{\partial x} \left( p^n h \hat{u}^n_r - \hat{u}^n_r \right), \chi_r \right) - \tau A_3 \left( \frac{\partial}{\partial y} \left( p^n h \hat{u}^n_r - \hat{u}^n_r \right), \chi_r \right)
\]

From the above relation, by setting \( \chi_r = Y^n_{r,d} \) we have
\[
\|\mathcal{G}^n_{r,d}\|_{L^2(\Omega)} \leq \left\| p^n h \hat{u}^n_r - \hat{u}^n_r \right\|_{L^2(\Omega)} + \tau A_2 \left\| \frac{\partial}{\partial x} \left( p^n h \hat{u}^n_r - \hat{u}^n_r \right) \right\|_{L^2(\Omega)} + \tau A_3 \left\| \frac{\partial}{\partial y} \left( p^n h \hat{u}^n_r - \hat{u}^n_r \right) \right\|_{L^2(\Omega)}
\]

which concludes the proof.

Theorem 3.4. Let \( \hat{u}_h^n \) and \( u_h^n \) be solutions of (3.9) and (3.10), respectively. Then
\[
\|\mathbb{E}[\hat{u}_h^n - u_h^n]\|_{L^2(\Omega)} \leq C(\tau^2 + N^{1-\nu}). \tag{3.12}
\]

Proof. Defining \( Z^n := \mathbb{E}[u^n - U_h^n] \) and subtracting (3.10) from (3.9) give rise to
\[
H_1(Z^n, v) = \frac{\tau}{2} I \left( \zeta(x) \nabla Z^n, \nabla \chi \right) - \frac{\tau}{2} I \left( Z^n, \frac{\partial}{\partial x} \chi \right) - \frac{\tau}{2} I \left( Z^n, \frac{\partial}{\partial y} \chi \right)
\]

Then, we define \( \omega_{h}^{1,n} := \mathbb{E}[P_i^n U^n - U^n_h] \) and \( n_{h}^{1,n} := \mathbb{E}[U^n - P_i^n U^n] \), then
\[
H_1(\omega_{h}^{1,n}, \chi) + \frac{\tau}{2} I \left( \zeta(x) \nabla \omega_{h}^{1,n}, \nabla \chi \right) - \frac{\tau}{2} I \left( \omega_{h}^{1,n}, \frac{\partial}{\partial x} \chi \right) - \frac{\tau}{2} I \left( \omega_{h}^{1,n}, \frac{\partial}{\partial y} \chi \right)
\]
\[ \begin{align*}
&= H_2 (\varphi_{h}^{1,n-1}, \chi) - \frac{\tau}{2} I (\zeta(x) \nabla \varphi_{h}^{1,n-1}, \nabla \chi) \\
&+ \frac{\tau}{2} I \left( \varphi_{h}^{1,n-1}, \frac{\partial}{\partial x} \chi \right) + \frac{\tau}{2} I \left( \varphi_{h}^{1,n-1}, \frac{\partial}{\partial y} \chi \right) \\
&- \tau N (F_{h}^{n-1} - \bar{F}_{h}^{n-1}, \chi) + \tau (R_{i}^{n}, \chi) - H_1 (A_{h}^{1,n}, \chi) \\
&+ \frac{\tau}{2} I \left( A_{h}^{1,n}, \frac{\partial}{\partial x} \chi \right) + \frac{\tau}{2} I \left( A_{h}^{1,n}, \frac{\partial}{\partial y} \chi \right) + H_2 (A_{h}^{1,n-1}, \chi) \\
&+ \frac{\tau}{2} I \left( A_{h}^{1,n-1}, \frac{\partial}{\partial x} \chi \right) + \frac{\tau}{2} I \left( A_{h}^{1,n-1}, \frac{\partial}{\partial y} \chi \right), \quad \forall \chi \in H_0^1(\Omega).
\end{align*} \]

Thus, by assuming
\[ (\Phi_{h}^{1,n}, \chi) = -H_1 (A_{h}^{1,n}, \chi) + \frac{\tau}{2} I \left( A_{h}^{1,n}, \frac{\partial}{\partial x} \chi \right) + \frac{\tau}{2} I \left( A_{h}^{1,n}, \frac{\partial}{\partial y} \chi \right) \]
\[ + H_2 (A_{h}^{1,n-1}, \chi) + \frac{\tau}{2} I \left( A_{h}^{1,n-1}, \frac{\partial}{\partial x} \chi \right) + \frac{\tau}{2} I \left( A_{h}^{1,n-1}, \frac{\partial}{\partial y} \chi \right), \quad \forall \chi \in H_0^1(\Omega). \]

we have
\[ \begin{align*}
H_1 (\varphi_{h}^{1,n}, \chi) &+ \frac{\tau}{2} I (\zeta(x) \nabla \varphi_{h}^{1,n}, \nabla \chi) - \frac{\tau}{2} I (\varphi_{h}^{1,n}, \frac{\partial}{\partial x} \chi) - \frac{\tau}{2} I (\varphi_{h}^{1,n}, \frac{\partial}{\partial y} \chi) \\
&= H_2 (\varphi_{h}^{1,n-1}, \chi) - \frac{\tau}{2} I (\zeta(x) \nabla \varphi_{h}^{1,n-1}, \nabla \chi) + \frac{\tau}{2} I (\varphi_{h}^{1,n-1}, \frac{\partial}{\partial x} \chi) \\
&+ \frac{\tau}{2} I (\varphi_{h}^{1,n-1}, \frac{\partial}{\partial y} \chi) - \tau N (F_{h}^{n-1} - \bar{F}_{h}^{n-1}, \chi) + \tau (R_{i}^{n}, \chi) + (\Phi_{h}^{1,n}, \chi), \quad \forall \chi \in H_0^1(\Omega).
\end{align*} \]

Setting \( \chi_{r} = \chi_{r,d}^{n} \), gives
\[ \begin{align*}
H_1 (\varphi_{h}^{1,n}, \varphi_{h}^{1,n}) + \frac{\tau}{2} I (\zeta(x) \nabla \varphi_{h}^{1,n}, \nabla \varphi_{h}^{1,n}) - \frac{\tau}{2} I (\varphi_{h}^{1,n}, \frac{\partial}{\partial x} \varphi_{h}^{1,n}) - \frac{\tau}{2} I (\varphi_{h}^{1,n}, \frac{\partial}{\partial y} \varphi_{h}^{1,n}) \\
&= H_2 (\varphi_{h}^{1,n-1}, \varphi_{h}^{1,n}) - \frac{\tau}{2} I (\zeta(x) \nabla \varphi_{h}^{1,n-1}, \nabla \varphi_{h}^{1,n}) + \frac{\tau}{2} I (\varphi_{h}^{1,n-1}, \frac{\partial}{\partial x} \varphi_{h}^{1,n}) \\
&+ \frac{\tau}{2} I (\varphi_{h}^{1,n-1}, \frac{\partial}{\partial y} \varphi_{h}^{1,n}) - \tau N (F_{h}^{n-1} - \bar{F}_{h}^{n-1}, \varphi_{h}^{1,n}) + \tau (R_{i}^{n}, \varphi_{h}^{1,n}) + (\Phi_{h}^{1,n}, \varphi_{h}^{1,n})
\end{align*} \]
Thus, we can write

\[
\frac{1}{2} \left\| H_1 \right\| \left\| \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 + \frac{\tau}{2} I \left\| \zeta(x) \right\| \left\| \nabla \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 \\
\leq \frac{\tau}{2} I \left( \omega_h^{1,n}, \frac{\partial}{\partial x} \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n}, \frac{\partial}{\partial y} \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n}, 0 \right) + \frac{\tau}{2} I \left( \omega_h^{1,n}, \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n} \right) + \frac{\tau}{2} I \left( \omega_h^{1,n} \right)
\]

As a result

\[
2\omega \leq \left\| A_1 \right\| \leq \omega, \quad \left\| A_2 \right\|, \left\| A_3 \right\| \leq C. \tag{3.13}
\]

Applying the definition

\[
\left\| \omega_h^{1,n} \right\|_{H_w(\Omega)}^2 := \left\| \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \tau \omega \left\| \nabla \omega_h^{1,n} \right\|_{L^2(\Omega)}^2.
\]

Eq. (3.14) can be written as

\[
\left\| \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 + \frac{\tau}{2} \left\| \zeta(x) \right\| \left\| \nabla \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 \leq \left\| H_1 \right\| \left\| \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 + \frac{\tau}{2} \left\| \zeta(x) \right\| \left\| \nabla \omega_h^{1,n} \right\|_{L^2(\Omega)}^2 + C_T \left\| R^n_t \right\|_{L^2(\Omega)}^2 + C_T \left\| \Phi^{i,n}_h \right\|_{L^2(\Omega)}^2.
\]
Now, from the above Eq., we have

$$\sum_{m=1}^{n} \| \varpi_{h}^{1,m} \|_{L^2(\Omega)}^2 + \frac{T}{2} \| \zeta(\mathbf{x}) \| \sum_{m=1}^{n} \| \nabla \varpi_{h}^{1,m} \|_{L^2(\Omega)}^2$$

\[ \leq \sum_{m=1}^{n} \| \varpi_{h}^{1,m-1} \|_{L^2(\Omega)}^2 + \frac{T}{2} \| \zeta(\mathbf{x}) \| \sum_{m=1}^{n} \| \nabla \varpi_{h}^{1,m-1} \|_{L^2(\Omega)}^2 \]

\[ + \frac{C_{1}L_T^2}{2\| \zeta(\mathbf{x}) \|} \sum_{m=1}^{n} \| \varpi_{h}^{1,m} \|_{L^2(\Omega)}^2 + \frac{C_{2}L_T}{2\| \zeta(\mathbf{x}) \|} \sum_{m=1}^{n} \| \varpi_{h}^{1,m-1} \|_{L^2(\Omega)}^2 \]

\[ + C_T \sum_{m=1}^{n} \| R_{t}^{m} \|_{L^2(\Omega)}^2 + C_T \sum_{m=1}^{n} \| \Phi_{h}^{1,m} \|_{L^2(\Omega)}^2. \]

By engaging the Gronwall lemma, the above relation can be rewritten as

$$\| \varpi_{h}^{1,n} \|_{H_{w}(\Omega)}^2 \leq \frac{C L_T}{\| \zeta(\mathbf{x}) \|} \sum_{m=1}^{n} \| \varpi_{h}^{1,m} \|_{L^2(\Omega)}^2 + C_T \sum_{m=1}^{n} \| R_{t}^{m} \|_{L^2(\Omega)}^2 + C_T \sum_{m=1}^{n} \| \Phi_{h}^{1,m} \|_{L^2(\Omega)}^2$$

\[ \leq \frac{C L_T}{\| \zeta(\mathbf{x}) \|} \sum_{m=1}^{n} \| \varpi_{h}^{1,m} \|_{L^2(\Omega)}^2 + C n \tau \| R_{t}^{m} \|_{L^2(\Omega)}^2 + C n \tau \| \Phi_{h}^{1,m} \|_{L^2(\Omega)}^2 \]

\[ \leq \left[ C n \tau \| R_{t}^{m} \|_{L^2(\Omega)}^2 + C n \tau \| \Phi_{h}^{1,m} \|_{L^2(\Omega)}^2 \right] \exp \left( \frac{C L n \tau}{\| \zeta(\mathbf{x}) \|} \right) \]

\[ \leq \left[ C \tau^2 + C n \tau (\tau^2 + N^{-1-v}) \right] \exp \left( \frac{C L_T}{\| \zeta(\mathbf{x}) \|} \right) \]

\[ \leq C (\tau^2 + N^{-1-v})^2. \]

Thus we have

$$\| \varpi_{h}^{1,n} \|_{L^2(\Omega)}^2 \leq \| \varpi_{h}^{1,n} \|_{H_{w}(\Omega)} \leq C (\tau^2 + N^{-1-v}). \quad (3.16)$$

4 Numerical discussions

Here, we provide two numerical examples to check the accuracy and the efficiency of the proposed numerical procedure. In both cases, in order to estimate the expected value $M = 1000$ independent random variables are used.
4.1 Test problem 1

For the first example, we study the following numerical example with $\Omega = [0, 1] \times [0, 1]$ as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) - D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 0.6 \varpi_p \frac{uv}{(1 + u)(v + 2)} &= f(x, y, t) + dW, \\
\frac{\partial v}{\partial t} + \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) - D \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + 0.6 \varpi_p \frac{uv}{(1 + u)(v + 2)} &= g(x, y, t) + dW, \\
\frac{\partial w}{\partial t} + \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) - D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 0.6 \varpi_p \frac{uv}{(1 + u)(v + 2)} + 2w &= h(x, y, t) + dW,
\end{align*}
\]

(4.1)

Figure 1: The computational error of expected value of the solution as a function of different number of basis functions (left panel $\tau = 10^{-3}$ and right panel $\tau = 10^{-4}$) for test problem 1.

where the diffusion coefficient is $D = 10^{-3}$ and zero Dirichlet boundary conditions are applied. The initial conditions are

\[
u_0 = v_0 = w_0 = \sin(\pi x) \sin(\pi y).\]

(4.2)

We assume that the right hand sides are

\[
\begin{align*}
f(x, y, t) &= (\pi - 5)e^{-5t}\cos(\pi x)\sin(\pi y) \\
&\quad + \pi e^{-5t}\cos(\pi y)\sin(\pi x) + 2D\pi^2 e^{-5t}\sin(\pi x)\sin(\pi y) \\
&\quad + 3e^{-10t}\sin(\pi x)^3\sin(\pi y)^3 \left[ 5 \left( e^{-2t}\sin(\pi x)\sin(\pi y) + 2 \right) \left( e^{-5t}\sin(\pi x)\sin(\pi y) + 1 \right) \right]^{-1}.
\end{align*}
\]

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Figure 2: The computational error of expected value of the solution as a function of different number of basis functions (left panel $N = 15$ and right panel $N = 25$) for test problem 1.

\[
g(x, y, t) = e^{-2t} \left[ \pi \cos(\pi x) \sin(\pi y) - 2e^{-2t} \sin(\pi x) \sin(\pi y) + \pi \cos(\pi y) \sin(\pi x) + 2D\pi^2 \sin(\pi x) u \sin(\pi y) \right] \\
+ e^{-10t} \sin(\pi x)^2 \sin(\pi y)^3 \left[ 10 \left(e^{-2t} \sin(\pi x) \sin(\pi y) + 2 \right) \left(e^{-5t} \sin(\pi x) \sin(\pi y) + 1 \right) \right]^{-1}.
\]

\[
h(x, y, t) = (\pi - 1) e^{-3t} \cos(\pi x) \sin(\pi y) + \pi e^{-3t} \cos(\pi y) \sin(\pi x) \\
+ e^{-3t} \sin(\pi x) \left( \pi \cos(\pi y) + 2D\pi^2 \sin(\pi y) \right) \\
+ 4e^{-10t} \sin(\pi x)^3 \sin(\pi y)^3 \left[ 5 \left(e^{-2t} \sin(\pi x) \sin(\pi y) + 2 \right) \left(e^{-5t} \sin(\pi x) \sin(\pi y) + 1 \right) \right]^{-1}.
\]

In the deterministic case, the exact solution is

\[
u(x, y, t) = \exp(-5t)\rho(x, y), \quad v(x, y, t) = \exp(-2t)\rho(x, y), \quad w(x, y, t) = \exp(-3t)\rho(x, y),
\]

where $\rho(x, y) = \sin(\pi x) \sin(\pi y)$. In order to estimate the computational error, we use the reference solution with $N = 30$ basis function. The developed LSEM method is used to approximate the expected value of the solution. In this example, we consider the summations of three computational errors with respect to $u$, $v$, and $w$ at $T = 1$ where the results are shown in Figure 1 for different numbers of basis functions. As shown a noticeable error reduction has been achieved which indicates the method efficiency. We also estimated the solution for two different time steps, i.e., $\tau = 10^{-3}$ and $\tau = 10^{-4}$. The computational error of expected value of the solution as a function of different number of basis functions (left panel $N = 15$ and right panel $N = 25$) has been depicted in Figure 2 for test problem 1. The results show that as we expected smaller time steps gives rises to better error convergence.
Figure 3: The computational error of expected value of the solution as a function of different number of basis functions for test problem 2.

| tau  | $N = 10$          | $N = 20$          | CPU time(s) |
|------|-------------------|-------------------|-------------|
|      | $L_\infty$ $C_1$-order | $L_\infty$ $C_1$-order |             |
| 1/32 | $1.2863 \times 10^{-3}$  | $1.2902 \times 10^{-3}$  | 0.25        |
| 1/64 | $3.3204 \times 10^{-4}$  | $3.3304 \times 10^{-4}$  | 0.39        |
| 1/128| $8.3715 \times 10^{-5}$  | $8.3968 \times 10^{-5}$  | 1.5         |
| 1/256| $5.2520 \times 10^{-6}$  | $2.1037 \times 10^{-5}$  | 34          |
| 1/512| $1.3150 \times 10^{-6}$  | $5.2610 \times 10^{-6}$  | 65          |

Table 1 and Figure 2 confirm the theoretical results as the computational convergence order of the proposed scheme is closed to the theoretical convergence order.

4.2 Test problem 2

In this second numerical example, we consider a sophisticated example. The initial conditions for the considered example are based on the delta function and $\Omega = [0,1] \times [0,1]$. In fact, since the delta function is a discontinuous function, the initial condition is not smooth. We solve this case of groundwater model [2] using the proposed numerical procedure. We investigate the following
Figure 4: The evolution of the solution (here \( u \)) for test problem 2.
\[
\begin{align*}
\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial u}{\partial y} - D \frac{\partial^2 u}{\partial y^2} + 0.6 \omega_p \frac{uv}{(1 + u)(v + 2)} &= dW, \\
\frac{\partial v}{\partial t} + \mu \frac{\partial v}{\partial x} - D \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial v}{\partial y} - D \frac{\partial^2 v}{\partial y^2} + 0.6 \omega_p \frac{uv}{(1 + u)(v + 2)} &= dW, \\
\frac{\partial w}{\partial t} + \mu \frac{\partial w}{\partial x} - D \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial w}{\partial y} - D \frac{\partial^2 w}{\partial y^2} + 0.6 \omega_p \frac{uv}{(1 + u)(v + 2)} + 2w &= dW.
\end{align*}
\]

(4.3)

In this advection-diffusion equation, the advection coefficients are \(\mu = [1, 1]\) the diffusion coefficient is \(D = 10^{-4}\), and zero Dirichlet boundary conditions are applied. The groundwater model is a system of nonlinear equations that it explains how to remove pollutants of groundwater \([2]\). Now, we consider two initial conditions that they are near to the real world problems as

\[ u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = x(1 - x)y(1 - y), \quad (4.4) \]

and

\[ u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = \delta(0, 0). \quad (4.5) \]

Relations \((4.3)\) and \((4.4)\) are respectively smooth and nonsmooth initial data. We apply the developed technique to approximate the solution and the physical phenomena (here \(u\)) using the nonsmooth initial condition. Figure 4 illustrates the expected value of the solution for the second test problem during different computational time.

5 Conclusion

The current article presents a new Legendre spectral element technique for solving the stochastic nonlinear system of advection-reaction-diffusion equations. The main advantage of the proposed numerical procedure is that the derived mass and diffuse matrices have tridiagonal and diagonal forms, respectively. We proved that the full-discrete scheme is unconditionally stable and convergent. The computational results confirm the capability of the present scheme and the theoretical concepts in our investigation.

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