A Dual Alternating Direction Method of Multipliers for Image Decomposition and Restoration

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Abstract—In this paper, we develop a dual alternating direction method of multipliers (ADMM) for an image decomposition model. In this model, an image is divided into two meaningful components, i.e., a cartoon part and a texture part. The optimization algorithm that we develop not only gives the cartoon part and the texture part of an image but also gives the restored image (cartoon part + texture part). We also present the global convergence and the local linear convergence rate for the algorithm under some mild conditions. Numerical experiments demonstrate the efficiency and robustness of the dual ADMM (dADMM). Furthermore, we can obtain relatively higher signal-to-noise ratio (SNR) comparing to other algorithms. It shows that the choice of the algorithm is also important even for the same model.

Index Terms—Alternating direction method of multipliers, Cartoon and texture, Deblurring, Image decomposition, Inpainting, Total variation.

I. INTRODUCTION

In computer science, digital image processing is the use of computer algorithms to perform image processing on images. As we know, image decomposition and restoration are the basic methods in image and computer vision science. Recently, they have been playing more and more important roles in digit image processing, machine learning, pattern recognition, and biomedical engineering, etc. As for image decomposition, an image can be divided into two parts, i.e., a cartoon part and a texture part. As for image restoration, it includes many types, such as image denoising, image inpainting, image deblurring, and so on. The aim of image decomposition and restoration is to enhance the quality of an image that is degraded by some reason.

The main tasks of this paper are extracting the cartoon and texture components from a degraded image (a blurry and/or missing pixels image) and getting a restored image from the decomposed results. Given an intensity function \( f \) which is an image, the image decomposition is to derive \( f = u + v \), where \( u \) and \( v \) represent the cartoon and texture component of the image \( f \), respectively. In general, the cartoon part \( u \) is an image formed by homogeneous regions and with sharp boundaries, the texture part \( v \) is noise or small scale repeated details. Fig. 1 illustrates an image example (the Weave image in Fig. 2) of cartoon and texture where the distinctions of between the two components can be visually discernable.

![Fig. 1: Image decomposition on Weave image (c) in Fig. 2](image)

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According to some literatures, the first model about image decomposition is the classic image denoising model proposed in [11], i.e., the well-known Rudin, Osher, and Fatemi (ROF) total variation (TV) minimization model for image denoising. The model is given by

\[
\min_{u \in \mathbb{R}^n, \lambda \geq 0} \tau \| \nabla u \|_1 + \lambda \int | f - u |^2,
\]

where the set \( \Omega \) is a domain of \( \mathbb{R}^n \), \( \int_\Omega | \nabla u | \) denotes the TV of \( u \) with the assumption that \( u \) is of bounded variation: \( u \in BV(\Omega) \). \( \int_\Omega | f - u |^2 \) is a fidelity term and \( \lambda > 0 \) is a weight parameter. Although the problem (1) is convex, it is still difficult to solve the functional programming problem. Then we write out the discrete version of the problem (1) as follows

\[
\min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \tau \| \nabla u \|_1 + \| v \|^2_2
\]

s.t. \( u + v = b \),

where \( b \in \mathbb{R}^n \) is a given image, \( \nabla := \begin{pmatrix} \nabla_1 \vspace{1em} \nabla_2 \end{pmatrix} : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \) denotes the discrete version of the first-order derivative operator (see [15] for more details), and \( \tau > 0 \) is a control parameter for the decomposition of \( b \) into the cartoon part \( u \) and the texture part \( v \), \( \| \nabla u \|_1 \) is the TV semi-norm for inducing the cartoon part \( u \in \mathbb{R}^n \), which is defined as

\[
\| \nabla u \|_1 := \sum_{i=1}^n \left( \nabla_1 u_i \right)^2 + \left( \nabla_2 u_i \right)^2 \right)^{1/2}.
\]

It has been pointed out in the seminal book [10] that the \( l_1 \) norm or \( l_2 \) norm does not characterize the oscillatory
approximation to the have overcome this difficulty. For example, motivated by the Euler-Lagrange equation of (3) can not be written out

\[ \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \tau \| \nabla u \|_1 + \| v \|_{-1, \infty} \]

s.t. \( u + v = b, \)

where \( \| v \|_{-1, \infty} \) is the negative semi-norm in Sobolev space for inducing the texture \( v. \) For any \( s \in [1, +\infty), \)

\[ \| v \|_{-1,s} := \inf \{ \| g \|_s \mid v = \text{div} g, \ g \in \mathbb{R}^n \times \mathbb{R}^n \}, \]

where \( \| g \|_s = \left( \sum_{i=1}^n |g_i|^{s} \right)^{\frac{1}{s}}, \) and \( \text{div} := -\nabla^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is the divergence operator. For the details about the operator \( \text{div} \), one may refer to [15].

It is very difficult to get a numerical solution because the Euler-Lagrange equation of (3) can not be written out directly. There are many image decomposition models which have overcome this difficulty. For example, motivated by the approximation to the \( l_\infty \) norm of \( |g|_i = \sqrt{(g_1)_i^2 + (g_2)_i^2}, \)

\( g = (g_1, g_2) \in \mathbb{R}^n \times \mathbb{R}^n, \) and

\[ \| g \|_\infty = \| \sqrt{g_1^2 + g_2^2} \|_\infty = \lim_{s \to \infty} \| \sqrt{g_1^2 + g_2^2} \|_s, \]

the following convex minimization problem is used in [4].

\[ \min_{u \in \mathbb{R}^n, g \in \mathbb{R}^n \times \mathbb{R}^n} \tau \| \nabla u \|_1 + \frac{1}{2} \| u + \text{div} g - b \|_2^2 + \mu \| g \|_s, \]

where \( u \) is the cartoon part of an image, \( v = \text{div} g \) is the texture part of an image, \( s \geq 1, \) and \( \tau > 0, \mu > 0 \) are two control parameters. The first and third terms are penalty terms, and the second term satisfies \( b \approx u + \text{div} g. \) This is the first practical method to overcome the difficulty. In [4], the gradient method is employed to solve the Euler-Lagrange equation of (3). It has also been demonstrated that the solution of (3) can be used for texture discrimination and segmentation.

In [17], Osher et al. proposed an alternative model by replacing \( \| v \|_{-1, \infty} \) in the problem (3) with \( \| v \|_{-1,2}. \) Then the model is

\[ \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \tau \| \nabla u \|_1 + \| v \|_{-1,2} \]

s.t. \( u + v = b. \)

In [3], Aujol et al. studied a constrained optimization model for image decomposition, that is,

\[ \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \| \nabla u \|_1 + \frac{1}{2\sigma} \| b - u - v \|_2^2 \]

s.t. \( \| v \|_{-1, \infty} \leq \mu, \)

where \( \sigma \) and \( \mu \) are two positive parameters to decompose the image \( b \) into the cartoon part \( u \) and texture part \( v. \) Yin et al. [19] studied a second-order cone programming problem, and the problem (3) is a special case of this problem. An overview of image and signal decomposition by using sparsity and morphological diversity was given in [18].

As for image restoration, it is often formulated as an inverse problem. For a degraded image \( b, \) we recover an unknown clean image \( x \) from \( b \) such that

\[ b \approx H x, \]

where the degradation operation \( H : \mathbb{R}^n \to \mathbb{R}^n \) is a linear operator. \( H x \) may be contaminated by some noises, such as the additive noise (for example, Gaussian noise, or impulse noise, i.e., \( b = H x + \varepsilon \)). Poisson noise and other multiplicative noise. \( H \) can be an identity operator (for denoising), a convolution operator (for deblurring), a projection operator (for inpainting) or the mixing of these operators. When we decompose an image \( x \) into two parts, i.e., the cartoon part \( u \) and the texture part \( v, \) it satisfies

\[ b \approx H(u + v) = H(u + \text{div} g). \]

In this paper, we study an image decomposition and reconstruction model for images with degradations. The optimization problem to be considered can be explicitly written as

\[ \min_{u \in \mathbb{R}^n, g \in \mathbb{R}^n \times \mathbb{R}^n} \tau \| \nabla u \|_1 + \frac{1}{2} \| H(u + \text{div} g) - b \|_2^2 + \mu \| g \|_s, \]

where \( \tau > 0 \) and \( \mu > 0 \) are the trade-off parameters between the cartoon part \( u \) and the texture part \( \text{div} g, \) respectively; and \( s \geq 1. \) Actually only the cases of \( s = 1, s = 2 \) and \( s = \infty \) are considered in the numerical experiments (see Section III for details).

There are some algorithms to solve problem (6). The alternating direction method of multipliers (ADMM) [13] and the ADMM with Gaussian back substitution in [2] are often used methods. Most of the ADMM type methods are applied to the primal problem (pADMM). The proximal gradient (forward-backward splitting) method in [12] is a special gradient descent method, which is mainly used to solve the optimization problem of non-differentiable objective function. Furthermore, the alternating minimization (AM) based method [14] and the partial differential equation (PDE) based method [4] have also been used to solve this kind of problem. Among these algorithms, by and large, the pADMM is relatively most effective and robust. However, for some problems the pADMM is less efficient which also affect to some extent the quality of the image decomposition and restoration.

The rest of this paper is organized as follows. In Section II, we develop a dual ADMM (dADMM) for a more general model problem. Next, some details about the algorithm are explained. We also present the global convergence and the local linear convergence rate of the algorithm, respectively. Some numerical experiments for image decomposition and restoration are presented in Section III. Finally, we give the conclusion in Section IV.

A. Preliminaries

In this subsection, we summarize some notations of convex optimization which will be used in the subsequent analysis.
For a given proper convex function \( f : \mathbb{R}^n \to (-\infty, +\infty] \), the proximal mapping \( \text{Prox}_{\sigma f}(\cdot) \) of \( f \) with positive parameter \( \sigma \) is defined by
\[
\text{Prox}_{\sigma f}(x) := \arg \min_{u \in \mathbb{R}^n} \{ f(u) + \frac{1}{2\sigma} \|u - x\|^2 \}, \quad \forall x \in \text{dom} f,
\]
where \( \text{dom} f \) is defined as \( \{ x \in \mathbb{R}^n \mid f(x) < +\infty \} \).

Next, the Moreau identity is given by
\[
\text{Prox}_{\sigma f}(x) + \sigma \text{Prox}_{f/\sigma}(x/\sigma) = x.
\]

For a given function \( f : \mathbb{R}^n \to [-\infty, +\infty] \), the conjugate of \( f \) is defined as
\[
f^*(y) = \sup_{x \in \text{dom} f} \{ \langle y, x \rangle - f(x) \}.
\]
The function \( f^* \) is closed and convex (even when \( f \) is not). For more details, one may refer to [20].

## II. The Algorithm

In this section, we first introduce a general optimization problem. Then the dADMM is applied to solve the problem. Based on the convergence result of the semi-proximal ADMM which was summarized in [5] and the linear convergence rate result which was established in [6], we will establish the global convergence and the local linear convergence rate of our algorithm.

Now we consider a general optimization problem as the following form
\[
\min_{x \in X, y \in Y} \frac{1}{2} \|Ax + By - b\|^2 + p(x) + q(y), \tag{7}
\]
where \( X, Y \) and \( Z \) are three finite-dimensional real Euclidean spaces each equipped with an inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \| \cdot \| \), \( b \in Z \) is a given variable, \( A : X \to Z \) and \( B : Y \to Z \) are two linear operators, \( p(x) \) and \( q(y) \) are two closed proper convex functions (which may be nonsmooth).

It is quite clear that the problem (6) is a special case of the problem (7) with the variables \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \times \mathbb{R}^n \), \( b \in \mathbb{R}^n \), the linear operators \( A = H \) and \( B = H \cdot \text{div} \), the functions \( p(\cdot) = \|\nabla v(\cdot)\|_1 \) and \( q(\cdot) = \mu \|\cdot\|_s \).

By introducing a slack variable \( z \), the optimization problem (7) is equivalent to
\[
\min_{x \in X, y \in Y, z \in Z} \frac{1}{2} \|z\|^2 + p(x) + q(y) \quad \text{s.t.} \quad Ax + By - b = z. \tag{P}
\]
The Lagrangian function associated with problem (P) is given by
\[
l(z, x, y; u) = \frac{1}{2} \|z\|^2 + p(x) + q(y) + \langle u, Ax + By - z - b \rangle.
\]

Then
\[
\begin{align*}
\max & \inf_{u \in Z} l(z, x, y; u) \\
&= \max \{ \inf_{z} \left\{ \frac{1}{2} \|z\|^2 - \langle u, z \rangle \right\} + \inf_{x} \{ p(x) + \langle A^* u, x \rangle \} \\
&\quad + \inf_{y} \{ q(y) + \langle B^* u, y \rangle \} - \langle u, b \rangle \} \\
&= \max \left\{ -\frac{1}{2} \|u\|^2 - \sup_{x} \left\{ -\langle A^* u, x \rangle - p(x) \right\} \right\} \\
&\quad - \sup_{y} \left\{ \langle -B^* u, y \rangle - q(y) \right\} - \langle u, b \rangle \\
&= \max \left\{ -\frac{1}{2} \|u\|^2 - p^*(-A^* u) - q^*(B^* u) - \langle u, b \rangle \right\}.
\end{align*}
\]
Thus the dual of problem (P) can be explicitly written as
\[
\min_{u \in Z} \frac{1}{2} \|u\|^2 + p^*(-A^* u) + q^*(-B^* u) + \langle u, b \rangle. \tag{8}
\]

By introducing two slack variables \( v \) and \( w \), (8) can be equivalently rewritten as
\[
\min_{u \in Z, v \in X, w \in Y} \frac{1}{2} \|u\|^2 + \langle u, b \rangle + p^*(v) + q^*(w) \quad \text{s.t.} \quad -A^* u - v = 0, \\
-B^* u + w = 0. \tag{D}
\]

Then the augmented Lagrangian function associated with problem (D) is given by
\[
L_\sigma(u, v, w; x, y) = \frac{1}{2} \|u\|^2 + \langle u, b \rangle + p^*(v) + q^*(w) + \langle x, -A^* u - v \rangle + \langle y, -B^* u + w \rangle \\
+ \frac{\sigma}{2} \|A^* u + v\|^2 + \frac{\sigma}{2} \|B^* u + w\|^2.
\]

Now we describe the algorithmic framework of the dADMM as follows.

### The dADMM for problem (D):

Given \( \varepsilon > 0 \). Let \( \tau \in (0, (1 + \sqrt{5})/2) \) be a scalar parameter, \( \sigma > 0 \) be an arbitrary parameter, and choose \( u_0, v_0, w_0, x_0, y_0 \). For \( k = 0, 1, 2, \ldots \), perform the following steps in each iteration:

#### Step 1. Compute the \( u, v, w \):

\[
(I_{Z} + \sigma AA^* + \sigma BB^*)u^{k+1} = Ax^k + By^k - b \\
- \sigma Au^k - \sigma Bu^k,
\]

\[
\begin{bmatrix}
u^{k+1} \\
w^{k+1}
\end{bmatrix} = \begin{bmatrix}
\text{Prox}_{p^*/\sigma}(\frac{\varepsilon}{\sigma} - A^* u^{k+1}) \\
\text{Prox}_{q^*/\sigma}(\frac{\varepsilon}{\sigma} - B^* w^{k+1})
\end{bmatrix},
\]

#### Step 2. Update \( x, y \):

\[
x^{k+1} = x^k + \tau \sigma (-A^* u^{k+1} - w^{k+1}), \\
y^{k+1} = y^k + \tau \sigma (-B^* w^{k+1} - u^{k+1}),
\]

#### Step 3. If a termination criterion is not satisfied, go to Step 1.

Then we describe the details of how to solve the \( u \)-subproblem and \((v, w)\)-subproblem of the algorithm.
• $u$- subproblem:
\[
\bar{u} = \arg \min_{u \in \mathbb{Z}} \frac{1}{2} \|u\|^2 + \langle u, b \rangle - \langle x, A^*u + v \rangle
- \langle y, B^*u + w \rangle + \frac{\sigma}{2} \|A^*u + v\|^2
+ \frac{\sigma}{2} \|B^*u + w\|^2.
\]

which is equivalent to solve the following linear system
\[
(I_Z + \sigma AA^* + \sigma BB^*)u = Ax + By - b - \sigma Av - \sigma Bw.
\]

Proof. We can prove the conclusion of this theorem based on Theorem B.1 in \cite{5} directly. Thus, we omit the details here.

Before going to the statement of the following theorem, we denote $m := (u, v, w, x, y)$ with $u \in \mathbb{Z}, v, x \in \mathcal{X}$ and $w, y \in \mathcal{Y}$, and $\mathcal{M} := \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y}$. Besides, define the KKT (Karush-Kuhn-Tucker) mapping \cite{22} $R : \mathcal{M} \rightarrow \mathcal{M}$ as
\[
R(m) := \begin{bmatrix}
u + b - Ax - By \\
v - \text{Prox}_{P}(w + x) \\
w - \text{Prox}_{Q}(w + y) \\
-A^*u - v \\
-B^*u - w
\end{bmatrix}, \quad \forall m \in \mathcal{M}.
\]

If there exists $(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})$ which satisfies $R(m) = 0$, then $(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})$ is called a KKT point for the problem (D).

Define a self-adjoint linear operator $\mathcal{P}$ as below
\[
\mathcal{P} := \text{Diag}(I_Z, \sigma I_{X \times Y}, (\tau \sigma)^{-1}I_{X \times Y}) + s_r \sigma \Gamma^*,
\]

where
\[
s_r := \frac{5 - \tau - 3 \min\{\tau, \tau^{-1}\}}{4}, \quad \forall \tau \in (0, \infty),
\]

and $\text{Diag}(\cdot, \cdot, \cdot)$ is a $3 \times 3$ block diagonal linear operator. Note that $1/4 \leq s_r \leq 5/4, \forall \tau \in (0, (1 + \sqrt{5})/2)$, and let $\Gamma : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{M}$ be a linear operator such that its adjoint $\Gamma^*$ satisfies
\[
\Gamma^*(m) = \begin{bmatrix}
-A^*u - v \\
-B^*u - w
\end{bmatrix}, \quad \forall m \in \mathcal{M}.
\]

Then we denote $\|m\|_P := \sqrt{\langle m, Pm \rangle}$ and $\text{dist}_P(m, \Omega) := \inf_{\omega \in \Omega} \|m - \omega\|_P$ for any $m \in \mathcal{M}$ and any set $\Omega \subseteq \mathcal{M}$.

**Theorem II.2.** We consider the situation of $S = 0$ and $T = 0$ in \cite{10}, Corollary 1). Let $\tau \in (0, (1 + \sqrt{5})/2)$. Assume that the solution set $\Omega = \{m := (\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})\}$ is nonempty. Suppose that the mapping $R : \mathcal{M} \rightarrow \mathcal{M}$ is piecewise polyhedral. Then there exist a constant $\hat{\eta} > 0$ such that the infinite sequence \{(\bar{u}^k, \bar{v}^k, \bar{w}^k, \bar{x}^k, \bar{y}^k)\} generated from the dADMM satisfies that for all $k \geq 1$,
\[
\text{dist}(m^k, \hat{\Omega}) \leq \hat{\eta} \|R(m^k)\|_2,
\]
\[
\text{dist}_P(m^{k+1}, \hat{\Omega}) \leq \hat{\mu} \text{dist}_P(m^k, \hat{\Omega}),
\]

where $0 < \hat{\mu} < 1$ is a constant parameter.

**Proof.** By using the Corollary 1 in \cite{10}, the desired conclusions of this theorem can be obtained readily. So we omit the details here. \hfill \square

**III. Numerical Experiments**

In this section, we perform the numerical experiments for the dADMM to solve the problem (D) with different choices of $A$. More specifically, we test four cases in this section:

1. $A = I$, where $I$ is a identity matrix;
2. $A = S$, where $S$ is a blurring matrix;
3. $A = K$, where $K$ is a binary matrix which indicates the missing pixels by zero entries;
4. $A = KS$, where $KS$ is the composition of a binary matrix and a blurring matrix.

All of the experiments were implemented in MATLAB R2018a x64 on a PC with an Intel i5-8600K 3.6 GHz processor and 8GB memory. In order to measure the quality of the image decomposition, it is assumed that the cartoon part and the texture part of an image are uncorrelated according to the paper \cite{3}. The correlation between the cartoon part $u$ and the texture part $v$ is computed by
\[
\text{Corr}(u, v) := \frac{\text{cov}(u, v)}{\sqrt{\text{var}(u)\text{var}(v)}}.
\]
where \( \text{var}(\cdot) \) and \( \text{cov}(\cdot, \cdot) \) refer to the sample variance and covariance of given data, respectively. We use the PSNR value which is defined by

\[
\text{PSNR} = 10 \log_{10} \left( \frac{I_{\text{max}}^2}{\text{MSE}} \right),
\]

to measure the performance of image restoration, where \( I_{\text{max}} \) is the maximum intensity of the original image, and MSE is defined by

\[
\text{MSE} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} [I(i, j) - N(i, j)]^2,
\]

with \( I \) being a noise-free \( m \times n \) monochrome image and \( N \) being the noisy approximation image of \( I \).

The images used in this section are displayed in Fig. 2. Note that the Mixed image \( (c) \) in Fig. 2 is derived by combining (a) and (b) with the ratio of 6:4.

We denote

\[
R_P = \frac{\| u + b - Ax - By \|_2}{1 + \| A \|_2},
\]

\[
R_D = \frac{\| A^T u + v \|_2 + \| B^T u + w \|_2}{1 + \| A \|_2},
\]

\[
R_C = \frac{\| v - \text{Prox}_{\rho \cdot} (v + x) \|_2 + \| w - \text{Prox}_{\rho \cdot} (w + y) \|_2}{1 + \| A \|_2}.
\]

The numerical experiments are terminated if the stopping criterion

\[
\text{Tol} = \max \{ R_P, R_D, R_C \} \leq 10^{-3}
\]

is satisfied or the maximal iterations reaches 70. This stopping criterion is also used for the other two algorithms (ADME and ADMGB) in Fig. 2. For all the experiments, the initial iteration point is set to be zero.

**Remark 1.** Some elements of the texture part \( v \in \mathbb{R}^{n \times m} \) are less than zero in the experiments. We normalize the matrix \( v \) in order to show the texture part of the image clearly. The step of normalization is defined by

\[
v_{ij} = \frac{v_{ij} - v_{\text{min}}}{v_{\text{max}} - v_{\text{min}}}, \quad \forall i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

where \( v_{\text{min}} = \min_{1 \leq i \leq n, 1 \leq j \leq m} \{ v_{ij} \} \) and \( v_{\text{max}} = \max_{1 \leq i \leq n, 1 \leq j \leq m} \{ v_{ij} \} \).

**A. Case 1: \( A = I \)**

We consider the case of \( A = I \) in this subsection, i.e., a image will be decomposed into a clean and an additive noise images. For the images \( f \) and \( g \) in Fig. 2 we implement the ADME, ADMGB and dADMM in this case. For the ADME and ADMGB, we take the parameters as \( \tau = 1 \times 10^{-1}, \mu = 1 \times 10^{-3} \) and \( \beta_1 = \beta_2 = \beta_3 = 10. \) For the dADMM, we set the parameters as \( \tau = 1 \times 10^{-1}, \mu = 3 \times 10^{-2} \) and \( \sigma = 0.8. \)

In Fig. 3, the decomposed results (the cartoon part and the texture part) of the images \( f \) and \( g \) with different values of \( s \) are displayed. We can hardly see the difference with different values of \( s \) from Fig. 3 directly. In order to measure the quality of the image decomposition, the correlation between the cartoon and the texture is computed. The corresponding results are displayed in Fig. 4. The results in Fig. 4 show a tiny difference of different \( s \), which coincides with the conclusion from Fig. 4. These results show that the effectiveness of the optimization problem (5) is not sensitive to different values of \( s \).

Next, we compare the dADMM with the other two algorithms (the ADME and ADMGB in [2]) on image decomposition. For this comparison, we focus on the images \( (c) \) and \( (g) \) in Fig. 2. The decomposed results about the image \( (c) \) in Fig. 2 are showed in Fig. 5 and the variations of the correlation values of the images \( (c) \) and \( (g) \) are displayed in Fig. 6.

Now we add some additive noise to the image \( (e) \) in Fig. 2, which is \( b = (u + v) + \varepsilon \), where the additive noise \( \varepsilon \) is Gaussian noise. The noised image \( b \) is decomposed by the model (5) with three algorithms. In this experiment, the Gaussian white noise \( \varepsilon \) is generated by MATLAB function `imnoise(I, ’gaussian’, 0, 0.1)`. The results are displayed in Fig. 7 and Fig. 8 respectively. From Fig. 7 we note that we can obtain a better cartoon part and a more clear texture part by the dADMM than the other two algorithms (ADME and ADMGB). We plot the variations of \( \text{Corr}(u, v) \) with respect to the iterations in Fig. 8. It shows that dADMM can reach a relatively lower correlation value with fewer iterations than the other two algorithms.

We can see that the dADMM has a better performance than the other two algorithms (i.e., ADME and ADMGB in [2]) in the experiments in case of \( A = I \).

**B. Case 2: \( A = S \)**

In this case, \( S \) is a blurring matrix, i.e., \( A \) in (7) is a convolution operator. We test both the Out-of-focus blur and the Gaussian blur in this subsection with the images \( (a) \) and \( (d) \) in Fig. 2 and compare the performances of the algorithms (the ADME, ADMGB and dADMM). To implement the ADME and ADMGB, we take the parameters as \( \tau = 5 \times 10^{-5}, \mu = 1 \times 10^{-5} \) and \( \beta_1 = \beta_2 = \beta_3 = 1 \times 10^{-2}. \) To implement the dADMM, we set the parameters as \( \tau = 8 \times 10^{-6}, \mu = 4 \times 10^{-4} \) and \( \sigma = 2 \times 10^{2}. \)

The numerical results of the ADME, ADMGB and dADMM on the images \( (a) \) and \( (d) \) with different blur kernels are showed in Table 1. In the table, “Gaussian(40,40)” means the blur kernel is generated by MATLAB function `fspecial(’gaussian’, 40, 40), and “Out-of-focus(40)” means the Out-of-focus blur kernel with a radius of 40 given by MATLAB function `fspecial(’disk’, 40).`

In Table 1 we report the detailed numerical results for the ADME, ADMGB and dADMM in the image decompositions and reconstruction problems. One can observe from Table 1 that the dADMM takes less time and less iterations to get a higher PSNR value of the restored image than the ADME and ADMGB for most of the tested examples. And we display the decomposed results of the cartoon parts, the texture parts and the restored images of “Weave” image \( (d) \) in Fig. 2 in the case of “Out-of-focus(40)” in Table 1 by two algorithms (ADMGB...
Fig. 2: Testing images: (a) $512 \times 512$ Lena image, (b) $512 \times 512$ Wool image, (c) combined $512 \times 512$ Lena and Wool image (denoted Mixed image), (d) $768 \times 1024 \times 3$ Weave image, (e) $576 \times 787 \times 3$ Barbara_RGB image, (f) $512 \times 512$ a part of Barbara image, (g) $393 \times 635 \times 3$ Brick image, (h) $1024 \times 1024 \times 3$ Wood image.

Fig. 3: Image decomposition on clean images ((f) and (g) in Fig. 2, respectively) with different values of $s$. From left to right: the cartoon part, the texture part of (f) and (g), respectively. The top row: $s = 1$. The center row: $s = 2$. The bottom row: $s = \infty$. 
TABLE I: Image decomposition on the images with blurry: “Iter” – the number of iterations; “Tol” – tolerance for the stopping criterion; “Time” – computing time (in seconds); “PSNR” – the PSNR value between the blurred (restored) image and the original image, respectively. And a=“ADME”; b=“ADMGB”; c=“dADMM”.

| Image       | Blur          | Iter (a) | Tol (a) | Time (a) | PSNR (a) |
|-------------|---------------|----------|---------|----------|----------|
| Lena        | Gaussian(20,20)| 22.25    | 3.0e-1  | 9.8e-4   | 5.19     | 30.41    |
|             | Gaussian(30,30)| 20.61    | 2.1e-1  | 8.6e-4   | 5.20     | 28.23    |
|             | Out-of-focus(20)| 19.95    | 1.0e-2  | 9.5e-4   | 5.48     | 29.80    |
|             | Out-of-focus(30)| 18.48    | 2.7e-2  | 9.2e-4   | 5.20     | 27.76    |
| Weave       | Gaussian(40,40)| 19.60    | 2.2e-1  | 9.9e-4   | 77.04    | 66.73    |
|             | Gaussian(50,50)| 19.44    | 1.9e-1  | 9.8e-4   | 78.65    | 64.16    |
|             | Out-of-focus(40)| 19.30    | 4.3e-2  | 9.9e-4   | 75.86    | 49.23    |
|             | Out-of-focus(55)| 19.10    | 3.8e-2  | 9.9e-4   | 76.89    | 49.15    |

Fig. 4: Variations of Corr($u,v$) with respect to the iterations for the image (f) Barbara in Fig. 2 with $s = 1$, $s = 2$, and $s = \infty$.

and dADMM) in Fig. 2. The numerical performances indicate that the dADMM is a robust, high-performance algorithm for the optimization problem (6).

C. Case 3: $A = K$

This subsection is devoted to the more difficult part of image decomposition and restoration on an image with missing pixels. For a binary matrix $K \in \mathbb{R}^{m \times n}$ and a clean image $M \in \mathbb{R}^{m \times n}$, The degraded image $D = K \circ M$, where the operator $\circ$ denotes the Hadamard product, i.e., $D_{i,j} = K_{i,j} \times M_{i,j} \quad (i = 1, \ldots, m; \ j = 1, \ldots, n)$. The images (f) and (h) in Fig. 2 are used in this subsection. And $K$ is set to be a $512 \times 512$ matrix and a $1024 \times 1024$ matrix for the image (f) and (h), respectively.

TABLE II: Image decomposition and restoration on the images with missing pixels: “Iter” – the number of iterations; “Tol” – tolerance for the stopping criterion; “Time” – computing time (in seconds); “PSNR” – the PSNR value between the restored image and the original image, respectively. And a=“ADMGB”; b=“dADMM”.

| Image   | PSNR (a) | Iter (a) | Tol (a) | Time (a) | PSNR (a) |
|---------|----------|----------|---------|----------|----------|
| Barbara | 14.13    | 70/39    | 7.5e-3 | 9.3e-4   | 4.95     | 29.34/30.7 |
| Wood    | 14.81    | 70/39    | 2.1e-2 | 9.0e-4   | 97.63    | 24.24/25.25 |

Fig. 5: Image decomposition on the Mixed image (c) in Fig. 2 with $A = I$. The first column: the cartoon part. The second column: the texture part. From top to bottom are the decomposed results by the ADME, ADMGB, and dADMM.

We compare the performances of the ADMGB and dADMM in this case. The parameters of the ADMGB are set as $\tau = 1 \times 10^{-2}$, $\mu = 5 \times 10^{-3}$ and $\beta_1 = \beta_2 = \beta_3 = 5 \times 10^{-2}$. To implement the dADMM, we take $\tau = 4 \times 10^{-3}$, $\mu = 1 \times 10^{-3}$, and $\sigma = 3 \times 10^{3}$. Finally, the PSNR values with respect to the iterations for the images (f) and (h) in Fig. 2 with missing pixels are showed in Fig. 10. We display the decomposed images and the restored images of two algorithms (ADMGB and dADMM) for the images (f) and (h) in Fig. 2 in Fig. 11.

From Fig. 10 and Table II we can see that the PSNR value
Fig. 6: Variations of Corr$(u, v)$ with respect to the iterations for *Mixed* image (c) and *Brick* image (g) in Fig. 2.

Fig. 7: Image decomposition on the noised image (e) in Fig. 2 with $A = I$. From top to bottom are the noised image, cartoon parts and texture parts. From left to right are the decompositions by the ADME, ADMGB and dADMM.

Fig. 8: Variations of Corr$(u, v)$ with respect to the iterations for the ADME, ADMGB and dADMM on the image (e) in Fig. 2 with Gaussian noise.

Fig. 9: Image decomposition and restoration on *blurred* image with “Out-of-focus(40)” about image (d) in Fig. 2. From top to bottom are the blurred image, cartoons, textures and restored images. From left to right are the results obtained by the ADMGB and dADMM, respectively.

Fig. 10: Variations of the PSNR values with respect to the iterations for the ADMGB and dADMM on the images (f) and (h) in Fig. 2 with missing pixels.
Fig. 11: Image decomposition and restoration on images with missing pixels. From top row to bottom row are degraded images, cartoon parts, texture parts and restored images. 512 \times 512 Barbara image with missing pixels, decomposition and restoration image by the ADMGB and dADMM on first two column respectively. 1024 \times 1024 \times 3 Wood image with missing pixels, decomposition and restoration image by the ADMGB and dADMM on last two column respectively.

of the reconstructed image generated by the dADMM is higher than that generated by the ADMGB. Moreover, we can get a better PSNR value with less time and less iterations by the dADMM.

D. Case 4: \(A = KS\)

In this subsection, we solve the problem (7) with \(A = KS\), where \(S\) is a blurring matrix, and \(K\) is a binary matrix. That is, we consider decomposing images with both blurry and missing pixels. The images (a) and (h) in Fig. 2 are used in this part. The blurring matrices \(S\) (the Out-of-focus blur and the Gaussian blur) in Table I are used as blurring matrix, and \(K\) is set to be a 512 \times 512 matrix and a 1024 \times 1024 matrix for image (f) and (h), respectively.

We implement two algorithms (the ADMGB and dADMM) in this part. For the ADMGB, we take \(\tau = 5 \times 10^{-5}\), \(\mu = 1 \times 10^{-5}\) and \(\beta_1 = \beta_2 = \beta_3 = 1 \times 10^{-2}\). And for the dADMM, we take \(\tau = 5 \times 10^{-3}\), \(\mu = 3 \times 10^{-3}\) and \(\sigma = 3 \times 10^{3}\). All results in this case are displayed in Table III and Fig. 12. And the result images (cartoon parts, texture parts and restored images) for Wood image with “Out-of-focus(20)” plus missing pixels are showed in Fig. 12. We report in Table III the detailed numerical results for the ADMGB and dADMM on the image decomposition and restoration of the images with blurry and missing pixels. It can be observed from Table III that the PSNR
TABLE III: Image decomposition and restoration of the images with blurry and missing pixels: “Iter” – the number of iterations; “Tol” – tolerance for the stopping criterion; “Time” – computing time (in seconds); “PSNR$^{0\text{th}}$” (“PSNR”) – the PSNR value between the blurred (restored) image and the original image, respectively. And $a=$“ADMGB”; $b=$ “dADMM”.

| Image                  | Blur + Missing pixels | PSNR$^{0\text{th}}$ | Iter (a | b) | Tol (a | b) | Time (a | b) | PSNR (a | b) |
|------------------------|-----------------------|----------------------|--------|-----|--------|-----|--------|-----|--------|-----|
| Lena                   | Gaussian(15,15) + Missing pixels | 11.54 | 70 | 16 | 4.9e-2 | 7.5e-4 | 16.20 | 2.77 | 24.88 | 26.44 |
| Lena                   | Gaussian(20,20) + Missing pixels | 11.47 | 70 | 15 | 5.6e-2 | 8.2e-4 | 16.22 | 2.62 | 25.48 | 26.20 |
| Lena                   | Out-of-focus(10) + Missing pixels | 12.98 | 70 | 15 | 5.3e-2 | 7.8e-4 | 15.43 | 2.59 | 25.76 | 26.76 |
| Lena                   | Out-of-focus(15) + Missing pixels | 11.38 | 70 | 15 | 5.8e-2 | 7.2e-4 | 14.85 | 2.56 | 24.05 | 25.67 |
| Wood                   | Gaussian(20,20) + Missing pixels | 12.70 | 70 | 15 | 8.6e-2 | 9.1e-4 | 423.97 | 101.35 | 18.54 | 20.93 |
| Wood                   | Gaussian(30,30) + Missing pixels | 12.48 | 70 | 15 | 8.5e-2 | 9.1e-4 | 441.78 | 100.35 | 17.75 | 19.66 |
| Wood                   | Out-of-focus(15) + Missing pixels | 12.58 | 70 | 15 | 6.8e-2 | 9.1e-4 | 395.44 | 100.55 | 17.48 | 19.23 |
| Wood                   | Out-of-focus(20) + Missing pixels | 12.43 | 70 | 15 | 7.2e-2 | 9.0e-4 | 395.85 | 99.96 | 17.04 | 18.35 |

Fig. 12: Image decomposition and restoration on the image with blurry and missing pixels (Wood image in Fig. 2 with blurry (“Out-of-focus(20)”) and missing pixels). From top to bottom are the degraded image, cartoon parts, texture parts and restored images, respectively. Left column: Decomposition by the ADMGB. Right column: Decomposition by the dADMM. Values by the dADMM are higher than those by the ADMGB. Moreover, the dADMM is obviously faster than the ADMGB.

Now we take a further look at the variations of the PSNR values and the KKT residuals with respect to the iterations. Fig. 13 shows the variations of the PSNR values and the KKT residuals with respect to the iterations for the ADMGB and dADMM for (h) in Fig. 2 with blurry (“Out-of-focus(20)”) and missing pixels, respectively. Firstly, for the ADMGB, the PSNR value can hardly improve after the first several iterations, while for the dADMM, the PSNR value increase all the time though it increases very slow at the later stage. Secondly, the KKT residuals by the dADMM is obviously smaller than that by the ADMGB. Note that the reason lies in the fact that several variables are introduced for the ADMGB which directly lead to a smaller iteration step size than the dADMM.

IV. CONCLUSION

In this paper, we developed the dADMM to solve the image decomposition and restoration problem with blurry and/or missing pixels. The global convergence and the local linear convergence rate of the algorithm were also given. The numerical simulation results also demonstrated that our proposed algorithm is robust and efficient, and can obtain relatively more higher SNRs for various image decomposition and restoration problems. Thus, it shows that even for the same model problem, the choice of the algorithm is also important.

APPENDICES

Note that to compute Prox$\sigma_p(\cdot)$ is equivalent to solve the following optimization problem

$$\bar{x} = \arg\min_x \sigma \|\nabla x\|_1 + \frac{1}{2} \|x - y\|_2^2.$$  \hspace{1cm} (10)

Therefore, we adopt the algorithm in [7] to solve problem (10). For more detailed information, one may see [7].

Furthermore, to compute Prox$\sigma_q(\cdot)$ is equivalent to solve the following optimization problem

$$\bar{x} = \arg\min_x \sigma \|x\|_s + \frac{1}{2} \|x - y\|_2^2,$$

where $\sigma > 0$.

- If $s = 1$,

$$\bar{x} = y - \max\{\min\{y, \sigma\}, -\sigma\}.$$
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Fig. 13: From top to bottom are the variations of the PSNR values and KKT residuals with respect to the iterations for the ADMGB and dADMM for (h) in Fig. 2 with blurry (“Out-of-focus(20)”) and missing pixels, respectively.

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