Interval observer for Linear Time Invariant (LTI) uncertain systems with state and unknown input estimations

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Abstract. In this paper, the problem of interval observer design for LTI uncertain systems is addressed. The aim is to estimate simultaneously the upper and lower bounds of unmeasurable states and unknown inputs. Based on a set membership approach, the proposed method interest consists in reducing the estimate conservatism assuming that all known and unknown inputs, disturbances, uncertainties and noises are bounded with a priori known bounds. For that, decoupling methods are used to reduce the latter propagation. The estimation problem is led back to a singular model representation. An example is given to illustrate the proposed design method.

Keywords: Uncertain systems, bounded uncertainties, state transformations, unknown input observers, interval observer.

1. INTRODUCTION

Interval observers appeared in the last two decades to estimate in a guaranteed way the admissible unmeasured states for uncertain systems. In the literature, many works has been dedicated to design interval observers assuming that all known and unknown inputs, disturbances, uncertainties and noises are bounded with a priori known bounds. There exists three main approaches to synthesize an interval observer. The first approach is based on the prediction/correction method as the well known Kalman filter approach [1, 2]. For that, the uncertain model knowledge is used to predict the state set at the next time instant and the incompatible state subset is deleted when new measured outputs are available. The second approach [3] performs the first one by considering the Müller theorem [4]. The uncertain system is transformed into two models without considering uncertainty effect, ensuring also the upper and the lower bound estimations of the state at any time. The last method is initially proposed by [5] dealing only with uncertain monotone systems. Two observers, based on the Luenberger observer structure, are computed to guarantee the upper and the lower bounds of the state at any time. This approach is based on the cooperative property [6] of observation errors which is one of the main challenge in interval observer design. Recent works propose new methods to relax this constraint for a larger class of systems [7, 8, 9, 10]. Thereafter, new approaches are proposed in [11, 12, 13, 14, 15] to design interval observers for more general class of uncertain systems.
In presence of unknown inputs, conventional interval observer approach could be applied by considering unknown input as an additional disturbance with a priori known bounds. However, the state and unknown input estimates can be very large and then useless when disturbances, uncertainties and noises are propagated without decoupling a part of their effects. To overcome such problem, two methods were proposed in [16, 17] to design interval observers for LTI systems with noise and unknown input that reduce the jointly estimation conservatism for state and unknown input.

In this context, this paper proposes to extend the results of [16, 17]. The aim of this paper is to jointly estimate the unmeasurable states and the unknown inputs in presence of noises and uncertain parameters. For that, the proposed approach is based on a method developed for a class of nonlinear singular system [18] that can be applied on a large class of LTI systems. Nevertheless, the method proposed in [18] does not consider uncertain parameters and it is not applied in the context of interval observer. Inspired by the methods of [17] and [18], this paper proposes a new interval observer design for a large class of uncertain LTI systems.

This paper is organized as follows. In the section 2, some notations and background commonly used in interval observers theory are recalled. The problem statement and all considered assumptions useful for the interval observer design are introduced. Section 3 is devoted to the observer structure and demonstrations. Section 4 gives an example to illustrate the proposed method.

2. PRELIMINARIES

2.1. NOTATIONS AND BACKGROUND

For two matrices \( A, B \in \mathbb{R}^{n \times m} \) (respectively two vectors \( x_1, x_2 \in \mathbb{R}^m \)), the relation \( A \leq B \) (respectively \( x_1 \leq x_2 \)) should be understood componentwise and define \( A^+ = \max(A, 0) \), \( A^- = A^+ - A \). Given a matrix \( A(\theta) \in \mathbb{R}^{n \times m} \) with \( \theta \in \Theta \subset \mathbb{R}^r \) (respectively a vector \( x_1 \)) and two matrices \( \underline{A}, \overline{A} \in \mathbb{R}^{n \times m} \) (respectively two vectors \( x_1, x_2 \in \mathbb{R}^m \)), we denote the set \( \left[ \underline{A}, \overline{A} \right] = \{ A(\theta) \in \mathbb{R}^{n \times m} \mid A \leq A(\theta) \leq \overline{A}, \forall \theta \in \Theta \} \) (respectively \( \left[ x_1, x_2 \right] = \{ x_1 \in \mathbb{R}^m \mid x_1 \leq x_2 \leq x_1 \} \)). Considering a locally essentially bounded signal \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \), we define the \( L_\infty \) norm of \( u \) by the symbol \( \| u \|_{[t_0, t_1]} \):

\[
\| u \|_{[t_0, t_1]} = \text{ess sup}|u(t)|, t \in [t_0, t_1]
\]

**Lemma 1.** [19] Consider a vector variable \( x \in \mathbb{R}^n \) such that \( \underline{x} \leq x \leq \overline{x} \), \( \underline{x}, \overline{x} \in \mathbb{R}^n \).

(1) Let \( A \in \mathbb{R}^{n \times m} \) be a constant matrix, then

\[
A^+ \underline{x} - A^- \overline{x} \leq A^+ \overline{x} - A^- \underline{x} \tag{1}
\]

(2) Let \( A \in \mathbb{R}^{n \times m} \) be a matrix variable, \( A \leq A \) for some \( \underline{A}, \overline{A} \in \mathbb{R}^{n \times m} \), then

\[
A^+ \underline{x}^+ - A^- \overline{x}^- - \overline{A}^+ \overline{x}^- + \underline{A}^+ \underline{x}^- \leq A^+ \overline{x}^+ - A^- \overline{x}^- - \overline{A}^+ \overline{x}^- + \underline{A}^+ \underline{x}^- \tag{2}
\]

**Proof.** See [19]. \( \square \)

A square matrix \( A \in \mathbb{R}^{n \times n} \) is called Hurwitz if the real part of all of its eigenvalues is strictly negative definite and is said to be Metzler if all its off-diagonal elements are nonnegative.

**Lemma 2.** [6] Given a non-autonomous system described by \( \dot{x}(t) = Ax(t) + B(t) \) where \( A \) is Metzler matrix and \( B(t) \leq 0 \) \((B(t) \geq 0)\). Then \( x(t) \leq 0 \) \((x(t) \geq 0)\) \( \forall t \geq 0 \) provided that \( x(0) \leq 0 \) \((x(0) \geq 0)\)
2.2. HOSM differentiator

Consider a measurement signal \( y(t) = \psi(t) + \gamma(t) \) with \( y(t), \psi(t), \gamma(t) \in \mathbb{R}^q \). \( \psi(t) \) denotes the useful signal assumed to be continuously differentiable and \( \gamma(t) \in L_{\infty} \) represents the measurement noise assumed to be such that \( |\gamma(t)| \leq \gamma \). Then the first derivative of \( \psi(t) \) can be estimated using a HOSM differentiator [20] described by:

\[
\begin{align*}
\dot{q}_0(t) &= v_0(t) \\
v_0(t) &= q_1(t) - \lambda_0 |q_0(t) - y(t)|^{\frac{1}{2}} \text{sign}(q_0(t) - y(t)) \\
\dot{q}_1(t) &= -\lambda_1 \text{sign}(q_0(t) - y(t))
\end{align*}
\]

(3)

where \( \lambda_k, k = 0, 1 \) are positive parameters.

The sufficient conditions allowing the convergence of the estimate \( q_1(t) \) to \( \dot{\psi}(t) \) is related to the appropriate choice of the parameters \( \lambda_0 \) and \( \lambda_1 \), see [21].

**Theorem 1.** [21] Let an input signal \( \psi(t) \) be a continuously differentiable signal and \( \gamma(t) \in L_{\infty} \) then there exist \( 0 \leq T < \infty \) and a constant \( \mu > 0 \) such that for all \( t \geq T \):

\[
|q_1(t) - \dot{\psi}(t)| \leq \mu \|\gamma\|^{\frac{1}{2}}
\]

(4)

**Proof.** see [21] \( \square \)

Based on theorem 1 and the boundedness assumption on the noise, the first derivative error is defined as:

\[
|q_1(t) - \dot{\psi}(t)| \leq \mu \gamma^{\frac{1}{2}}
\]

(5)

The upper and lower bounds for the first derivative of \( \psi(t) \) can be computed based on (5):

\[
\begin{align*}
\overline{\dot{\psi}}(t) &= q_1(t) + \mu \gamma^{\frac{1}{2}} \\
\underline{\dot{\psi}}(t) &= q_1(t) - \mu \gamma^{\frac{1}{2}}
\end{align*}
\]

(6)

**Remark (1).** In [21], the general case is considered in order to estimate the \( s^{th} \) derivative of a noisy signal.

2.3. PROBLEM STATEMENT

Consider an uncertain system described by:

\[
\begin{align*}
\dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + F_1(\theta)d_1(t) + H_1(\theta)v(t) \\
y(t) &= C(\theta)x(t) + D(\theta)u(t) + F_2(\theta)d_2(t) + H_2(\theta)v(t)
\end{align*}
\]

(7)

where \( x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^m \) represent respectively the state, the output and the known input. \( d_1 \in \mathbb{R}^{k_1} \) and \( d_2 \in \mathbb{R}^{k_2} \) denote the state and sensor faults, considered here as two unknown inputs. \( \theta \in \Theta \) defines the unknown parameter vector where \( \Theta = [\underline{\theta}, \overline{\theta}] \) with \( \underline{\theta}, \overline{\theta} \in \mathbb{R}^l \) assumed to be known a priori. \( v \in \mathbb{R}^s \) describes the noise vector such that \( v \in [\underline{v}, \overline{v}] \) where \( \underline{v}, \overline{v} \in \mathbb{R}^s \) are known vectors with \( s = n + p \). Matrices \( A(\theta) \in \mathbb{R}^{n \times n}, B(\theta) \in \mathbb{R}^{n \times m}, F_1(\theta) \in \mathbb{R}^{n \times k_1}, H_1(\theta) \in \mathbb{R}^{n \times s}, C(\theta) \in \mathbb{R}^{p \times n}, D(\theta) \in \mathbb{R}^{p \times m}, F_2(\theta) \in \mathbb{R}^{p \times k_2} \) and \( H_2(\theta) \in \mathbb{R}^{p \times s} \).

This section gives some useful assumptions to design an interval observer which estimates jointly the lower and the upper bounds of the state \( x \) and the unknown inputs \( d_1, d_2 \).

**Assumption 1.** \( \forall \theta \in [\underline{\theta}, \overline{\theta}] \), there exist three matrices \( \Delta X_i, \overline{\Delta X}_i \in \mathbb{R}^{\alpha \times \beta} \) and \( X_i \in \mathbb{R}^{\alpha \times \beta} \) such that \( X_i(\theta) = X_i + \Delta X_i(\theta) \) where \( \Delta X_i(\theta) \in \left[ \Delta X_i, \overline{\Delta X}_i \right] \).
Remark (2). In this assumption, all uncertainties are assumed to be bounded by known bounds and are assumed to have a linear additive form.

Assumption 2. There exist two vectors \( w_1(t) \in \mathbb{R}^{q_1}, w_2(t) \in \mathbb{R}^{q_2} \) and three constant matrices \( G_{11} \in \mathbb{R}^{n \times q_1}, G_{12} \in \mathbb{R}^{n \times q_2}, G_2 \in \mathbb{R}^{p \times q_2} \) such that:

\[
G_{11}w_1(\theta, t) + G_{12}w_2(\theta, t) = \Delta A(\theta)x(t) + F_1(\theta)d_1(t) \quad (8a)
\]
\[
G_{2}w_2(\theta, t) = \Delta C(\theta)x(t) + F_2(\theta)d_2(t) \quad (8b)
\]

Remark (3). The uncertain state distribution matrices \( \Delta A(\theta), \Delta C(\theta) \) provided by assumption 1 and all unknown inputs \( d_1, d_2 \) can be rewritten in a new form to create two unknown input vectors with constant known matrix distributions. This assumption is necessary to decouple a part of unknown inputs and disturbance effects.

Assumption 3.

\[
\text{rank} \left( \begin{bmatrix} F^+_i & 0 & 0 & F^-_i \\ 0 & F^+_i & F^-_i & 0 \\ 0 & F^-_i & F^+_i & 0 \\ F^-_i & 0 & 0 & F^+_i \end{bmatrix} \right) = 4k_i, \ \forall i = 1, 2 \quad (9)
\]

Remark (4). Based on \cite{17, 22}, this assumption is necessary in order to estimate the upper and lower bounds of the unknown inputs \( d_1(t) \) and \( d_2(t) \) from equations (8a)-(8b). An inversion method is applied to estimate the unknown inputs considering the uncertainties of \( F_i(\theta) \) with \( i = 1, 2 \). The lemma 1 allows to transform the uncertain equations (8a-8b) into a determinist form such that the inversion is feasible if assumption 3 holds.

Under Assumption 1 and Assumption 2 and based on equalities (8a)-(8b), the system (7) can be rewritten as:

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + B(\theta)u(t) + H_1(\theta)v(t) + G_{11}w_1(\theta, t) + G_{12}w_2(\theta, t) \\
y(t) &= Cx(t) + D(\theta)u(t) + H_2(\theta)v(t) + G_2w_2(\theta, t)
\end{aligned} \quad (10)
\]

Assumption 4.

\[
\begin{align*}
\text{rank}(CG_{11}) &= \text{rank}(G_{11}) = q_1, \quad q_1 < n \\
\text{rank} \left( \begin{bmatrix} CG_{11} & G_2 \end{bmatrix} \right) &= q_1 + q_2, \quad q_1 + q_2 < p
\end{align*} \quad (11)
\]

Remark (5). This assumption is standard in the literature \cite{16, 17, 22} for observer design, especially for the unknown input estimation. From this assumption, a non-singular matrix can be computed to decouple the effects of \( w_1 \) and secondly, a joint estimation of the state \( x \) and the unknown input \( w_2 \) is achieved by considering an augmented state vector.

A non-singular state transformation is applied on (10) in order to decompose the state in two parts. One part is still affected by the unknown input \( w_1 \) whereas the second is decoupled from unknown input effect:

\[
x(t) = M_0z(t), \quad M_0 = [G_{11}, G], \quad G \in \mathbb{R}^{n \times (n-q_1)} \quad (12)
\]

Then the system (10) becomes:

\[
\begin{aligned}
\dot{z}(t) &= A_0z(t) + B_0(\theta)u(t) + H_0(\theta)v(t) + G_{01}w_1(\theta, t) + G_{02}w_2(\theta, t) \\
y(t) &= C_0z(t) + D(\theta)u(t) + H_2(\theta)v(t) + G_{2}w_2(\theta, t)
\end{aligned} \quad (13)
\]

where

\[
A_0 = M_0^{-1}AM_0 = \begin{bmatrix} A_{01} \\ A_{02} \end{bmatrix}, B_0(\theta) = M_0^{-1}B(\theta) = \begin{bmatrix} B_{01}(\theta) \\ B_{02}(\theta) \end{bmatrix}, H_0(\theta) = M_0^{-1}H_1(\theta) = \begin{bmatrix} H_{01}(\theta) \\ H_{02}(\theta) \end{bmatrix},
\]

\[
G_{01} = M_0^{-1}G_{11} = \begin{bmatrix} I_{q_1} \\ 0_{(n-q_1) \times q_1} \end{bmatrix}, G_{02} = M_0^{-1}G_{12} = \begin{bmatrix} G_{10} \\ G_{02} \end{bmatrix}, C_0 = CM_0.
\]
Now, keeping only the part of the dynamics from $z$ decoupled from the unknown input $w_1$ effect, the system is now rewritten by considering the singular representation of (13):

$$\begin{align*}
E \hat{z}(t) &= A_{02}z(t) + B_{02}(\theta)u(t) + H_{02}(\theta)v(t) + \tilde{G}_{02}w_2(\theta, t) \\
y(t) &= C_0 z(t) + D(\theta)u(t) + H_2(\theta)v(t) + G_2 w_2(\theta, t)
\end{align*}$$

(14)

where $E = [0_{(n-q_1)\times q_1} \quad I_{(n-q_1)}]$. Finally, augmenting the state $z$ with the unknown input $w_2$, the system (14) is rewritten:

$$\begin{align*}
\hat{E} \hat{z}(t) &= \hat{A} \hat{z}(t) + \hat{B}(\theta)u(t) + \hat{H}_1(\theta)v(t) \\
y(t) &= \hat{C} \hat{z}(t) + D(\theta)u(t) + H_2(\theta)v(t)
\end{align*}$$

(15)

where $\hat{z}(t) = \begin{bmatrix} z(t) \\ w_2(\theta, t) \end{bmatrix}$, $\hat{E} = [E \quad 0_{(n-q_1)\times q_2}]$, $\hat{A} = [A_{02} \quad \tilde{G}_{02}]$, $\hat{C} = [C_0 \quad G_2]$, $\hat{B}(\theta) = B_{02}(\theta)$, $\hat{H}_1(\theta) = H_{02}(\theta)$.

**Assumption 5.** The pair $(\hat{T}, \hat{A}, \hat{C})$ is observable (or at least detectable) where $T \in \mathbb{R}^{(n+q_2)\times(n-q_1)}$ is a matrix chosen such that $[T \quad F][\hat{E} \quad \hat{C}] = I_{n+q_2}$ with $F$ a matrix of appropriate dimension.

**Remark (6).** Assumption 5 is necessary to guarantee the stability of the interval observer proposed in the sequel. The choice of matrices $T$ and $F$ is guaranteed if the matrix $[\hat{E} \quad \hat{C}]$ is full column rank. This last condition holds if Assumption 4 holds also.

**3. MAIN RESULT**

**3.1. UNKNOWN INPUT INTERVAL OBSERVER**

In this subsection, an interval observer based on Assumption 1-5 is proposed to estimate the state of system (15). For that, a non-singular state transformation $\hat{z}(t) = M_1 r(t)$, $M_1 \in \mathbb{R}^{(n+q_2)\times(n+q_2)}$ is applied such that the system (15) is rewritten as:

$$\begin{align*}
\hat{E} M_1 \dot{r}(t) &= \hat{A} M_1 r(t) + (\hat{B} + \Delta \hat{B}(\theta))u(t) + \hat{H}_1(\theta)v(t) \\
y(t) &= \hat{C} M_1 r(t) + (D + \Delta D(\theta))u(t) + H_2(\theta)v(t)
\end{align*}$$

(16)

Then, the interval observer for (16) is now introduced as follows:

$$\begin{align*}
\dot{\xi}_1(t) &= M_1^{-1} N M_1 \xi_1(t) + M_1^{-1} J y(t) + M_1^{-1} K u(t) - \overline{W}_1 \\
\tau(t) &= \xi_1(t) + M_1^{-1} F y(t) - M_1^{-1} F D u(t) - \overline{W}_2 \\
\dot{\xi}_2(t) &= M_1^{-1} N M_1 \xi_2(t) + M_1^{-1} J y(t) + M_1^{-1} K u(t) + \overline{W}_1 \\
\tau(t) &= \xi_2(t) + M_1^{-1} F y(t) - M_1^{-1} F D u(t) + \overline{W}_2 
\end{align*}$$

(17a, 17b)

where

$$\begin{align*}
\overline{W}_1 &= \overline{W}_1^{+} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{+} - \overline{W}_1^{+} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} - \overline{W}_1 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{+} + \overline{W}_1 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \\
\overline{W}_2 &= \overline{W}_2^{+} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{+} - \overline{W}_2^{+} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} - \overline{W}_2 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{+} + \overline{W}_2 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}
\end{align*}$$
with
\[
\begin{align*}
\tilde{W}_1 &= [\tilde{W}_{11} \tilde{W}_{12}] \\
\tilde{W}_{11} &= M_1^{-1}(J\Delta D(\theta) - T\Delta \tilde{B}(\theta)) \\
\tilde{W}_{12} &= M_1^{-1}(JH_2(\theta) - TH_1(\theta)) \\
\tilde{W}_2 &= [\tilde{W}_{21} \tilde{W}_{22}] \\
\tilde{W}_{21} &= M_1^{-1}F\Delta D(\theta) \\
\tilde{W}_{22} &= M_1^{-1}FH_2(\theta)
\end{align*}
\]

Here, \( M_1, N, J, K, F, \) and \( T \) are unknown matrices of appropriate dimension such that the proposed methodology is able to be determined.

In the sequel, we denote the upper and lower bounds of observation errors by:

\[
\begin{align*}
\varepsilon_1(t) &= \xi_1(t) - M_1^{-1}T\tilde{E}M_1r(t) \\
\varepsilon_2(t) &= \xi_2(t) - M_1^{-1}T\tilde{E}M_1r(t) \\
e_1(t) &= \tilde{e}(t) - r(t) \\
e_2(t) &= \tau(t) - r(t)
\end{align*}
\]

**Theorem 2.** Consider \( \xi_0 \in \mathbb{R}^{n+q_2} \), \( \tau_0 \in \mathbb{R}^{n+q_2} \) such that \( \xi_0 \leq r(0) \leq \tau_0 \). Under Assumptions 1-5, there exist two non-singular constant matrices \( N \in \mathbb{R}^{(n+q_2) \times (n+q_2)} \), \( M_1 \in \mathbb{R}^{(n+q_2) \times (n+q_2)} \) such that \( M_1^{-1}NM_1 \) is Hurwitz and Metzler. If the following equalities are satisfied:

\[
\begin{align*}
\varepsilon_1(0) &\leq 0, \varepsilon_2(0) \geq 0 \\
NT\tilde{E} + J\tilde{C} - T\tilde{A} &= 0 \\
K &= T\tilde{B} - JD \\
\begin{bmatrix} T & F \end{bmatrix} \begin{bmatrix} \tilde{E} \\ \tilde{C} \end{bmatrix} &= I_{n+q_2}
\end{align*}
\]

Then \( r(t) \leq r(t) \leq \tau(t) \forall t \geq 0 \). The system (17) is an unknown input interval observer for (16).

**Proof.** The proof of the Theorem 2 is organized in two steps. The first step shows the stable cooperative dynamic property of (18a) and (18b). The second step ensures the negativity property of (18c) and the positivity property of (18d) for all \( t \geq 0 \).

**Step 1:** Using the derivative of expressions (18a) and (18b), we get:

\[
\begin{align*}
\dot{\varepsilon}_1(t) &= M_1^{-1}NM_1\varepsilon_1(t) + M_1^{-1}(NT\tilde{E} + J\tilde{C} - T\tilde{A})M_1r(t) + W_1 - \overline{W}_1 \\
\dot{\varepsilon}_2(t) &= M_1^{-1}NM_1\varepsilon_2(t) + M_1^{-1}(NT\tilde{E} + J\tilde{C} - T\tilde{A})M_1r(t) + W_1 + \overline{W}_1
\end{align*}
\]

where

\[
W_1 = \begin{cases} \\
M_1^{-1}(J(D + \Delta D(\theta)) + K - T(B + \Delta \tilde{B}(\theta))u(t) \\
+ M_1^{-1}(JH_2(\theta) - TH_1(\theta))v(t)
\end{cases}
\]

It is trivial to prove with the Lemma 1 and the condition (19c) that \( W_1 - \overline{W}_1 \leq 0 \) and \( W_1 + \overline{W}_1 \geq 0 \), \( \forall t \geq 0 \). Using the same Lemma, there exist \( \xi_1(0) \leq M_1^{-1}T\tilde{E}M_1r(0), \xi_2(0) \geq M_1^{-1}T\tilde{E}M_1r(0) \) provided by \( \xi_0 \leq r(0) \leq \tau_0 \) and it follows that \( \varepsilon_1(0) \leq 0, \varepsilon_2(0) \geq 0 \). Considering the property (19b) satisfied then (20a) and (20b) are rewritten as:

\[
\begin{align*}
\dot{\varepsilon}_1(t) &= M_1^{-1}NM_1\varepsilon_1(t) + W_1 - \overline{W}_1 \\
\dot{\varepsilon}_2(t) &= M_1^{-1}NM_1\varepsilon_2(t) + W_1 + \overline{W}_1
\end{align*}
\]

From previous results and considering \( M_1^{-1}NM_1 \) Hurwitz and Metzler, all conditions for the Lemma 2 hold. Then, we deduce that (21a) and (21b) are stable cooperative dynamical systems.
**Step 2:** To prove $e_1(t) \leq 0$ and $e_2(t) \geq 0 \forall t \geq 0$, we combine the second equation of (17a) with (18a) and the second equation of (17b) with (18b):

$$\begin{align*}
    \xi(t) &= E_1(t) + M_1^{-1} \begin{bmatrix} T & F \end{bmatrix} \begin{bmatrix} \dot{E} \\ C \end{bmatrix} M_1 r(t) + W_2 - \overline{W}_2 \\
    \nu(t) &= E_2(t) + M_1^{-1} \begin{bmatrix} T & F \end{bmatrix} \begin{bmatrix} \dot{E} \\ C \end{bmatrix} M_1 r(t) + W_2 + \overline{W}_2
\end{align*} \tag{22a}$$

where $W_2 = M_1^{-1} F \Delta D(\theta) u(t) + M_1^{-1} F \dot{H}_2(\theta) v(t)$

Under Assumption 4, there exist $T \in \mathbb{R}^{(n+q_1) \times (n-q_1)}$ and $F \in \mathbb{R}^{(n+q_2) \times p}$ such that $[ T \quad F ] \begin{bmatrix} \dot{E} \\ C \end{bmatrix} = I_{n+q_2}$. Equations (22a) and (22b) are now defined as:

$$\begin{align*}
    e_1(t) &= E_1(t) + W_2 - \overline{W}_2 \\
    e_2(t) &= E_2(t) + W_2 + \overline{W}_2
\end{align*} \tag{23a}$$

Lemma 1 allows us to prove $(W_2 - \overline{W}_2) \leq 0$ and $(W_2 + \overline{W}_2) \geq 0$. Finally, with the previous results proposed in the first step, we obtain $e_1(t) \leq 0$ and $e_2(t) \geq 0$, $\forall t \geq 0$. \hspace{1cm} \Box

**Remark (7).** As mentioned in Theorem 2, an unknown input interval observer is designed if there exist two non-singular constant matrices $N \in \mathbb{R}^{(n+q_1) \times (n-q_1)}$, $M_1 \in \mathbb{R}^{(n+q_2) \times (n+q_2)}$ such that $M_1^{-1} N M_1$ is Hurwitz and Metzler. A practical method to compute $N$, $M_1$ and $J$ is based on Assumption 5 and on the equations (19b),(19d). From (19d), we get $T \dot{E} = I - F \dot{C}$. Using the previous expression in (19b), an expression of $N$ is chosen such that : $N = T \tilde{A} - L \dot{C}$ with $L = J - NF$. Finally, from Assumption 5, it is shown that there exists a Hurwitz matrix $N$. Using a methodology that leads to get the Metzler and Hurwitz properties for the matrix $M_1^{-1} N M_1$ as described in [8], $M_1$ is computed and $J$ is deduced from $J = NF + L$.

### 3.2. States and Unknown Inputs Estimation

**Corollary 1.** Assuming that Lemma 1 and Theorem 2 hold, there exist $\forall t \geq 0$, $\xi(t), \nu(t) \in \mathbb{R}^n$ and $w_1(\theta, t), w_2(\theta, t) \in \mathbb{R}^n$ such that:

$$\begin{align*}
    \begin{cases}
        \xi(t) \leq x(t) \leq \nu(t) \\
        w_2(\theta, t) \leq w_2(\theta, t) \leq \overline{w}_2(\theta, t)
    \end{cases}
\end{align*} \tag{24}$$

Proof. If Theorem 2 holds then $\nu(t) \leq r(t) \leq \nu(t)$. Recalling that $\tilde{z}(t) = M_1 r(t)$ where $\tilde{z}(t) = \begin{bmatrix} z(t) \\ w_2(\theta, t) \end{bmatrix}$ and $x(t) = M_0 \tilde{z}(t)$, it is shown under Lemma 1 that:

$$\begin{align*}
    \begin{cases}
        \tilde{x}(t) = M^T \tilde{r}(t) - M^T \nu(t) \\
        \tilde{z}(t) = M^T \tilde{r}(t) - M^T \nu(t)
    \end{cases}
\end{align*} \tag{25}$$

where $\tilde{z}(t) = \begin{bmatrix} x(t) \\ w_2(\theta, t) \end{bmatrix}$, $M = \tilde{M}_0 M_1$ with $\tilde{M}_0 = \begin{bmatrix} M_0 & 0 \\ 0 & I_{q_2} \end{bmatrix}$. \hspace{1cm} \Box

**Corollary 2.** Considering corollary 1 and Theorem 2 hold, there exist $\forall t \geq 0$, $d_1(t), d_1(t) \in \mathbb{R}^{k_1}$ and $d_2(t), d_2(t) \in \mathbb{R}^{k_2}$ such that:

$$\begin{align*}
    \begin{cases}
        d_1(t) \leq d_1(t) \leq \overline{d}_1(t) \\
        d_2(t) \leq d_2(t) \leq \overline{d}_2(t)
    \end{cases}
\end{align*} \tag{26}$$
Proof. Step 1: This part is devoted to prove that \( d_1(t) \leq d_1(t) \leq \overline{d_1}(t) \). The differentiation of \( y \) in (10) leads to:

\[
\dot{y}(t) = C\dot{x}(t) + D(\theta)\dot{u}(t) + H_2(\theta)\dot{v}(t) + G_2\dot{w}_2(\theta, t)
\]  

(27)

Injecting the expression of \( \dot{x} \) described in (10), we get:

\[
\dot{y}(t) = CAx(t) + CB(\theta)u(t) + CH_1(\theta)v(t) + CG_{11}w_1(t) + CG_{12}w_2(\theta, t) + D(\theta)\dot{u}(t) + H_2(\theta)\dot{v}(t) + G_2\dot{w}_2(\theta, t)
\]  

(28)

Under Assumption 4, there exists a non-singular output transformation \( \tilde{M} = \begin{bmatrix} CG_{11} & G_2 & R \end{bmatrix} \), \( R \in \mathbb{R}^{p \times (p-q_1-q_2)} \) such that \( \tilde{M}^{-1} = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 \end{bmatrix} \) with \( \tilde{M}_1 \in \mathbb{R}^{q \times p} \), \( \tilde{M}_2 \in \mathbb{R}^{(p-q_1) \times p} \). Multiplying (28) by \( \tilde{M}_1^{-1} \) and isolating the expression \( w_1(t) \) in the latter, we get:

\[
w_1(t) = \psi(t) - \tilde{M}_1 CAx(t) - \tilde{M}_1 CB(\theta)u(t) - \tilde{M}_1 CH_1(\theta)v(t) - \tilde{M}_1 CG_{12}w_2(\theta, t) - \tilde{M}_1 D(\theta)\dot{u}(t)
\]  

(29)

where \( \psi(t) = \tilde{M}_1 y(t) - \tilde{M}_1 H_2(\theta)v(t) \).

To reduce the conservatism, the augmented state \( \tilde{x} \) defined in (25) is used in (29):

\[
w_1(t) = \psi(t) - S r(t) - \tilde{M}_1 CB(\theta)u(t) - \tilde{M}_1 CH_1(\theta)v(t) - \tilde{M}_1 D(\theta)\dot{u}(t)
\]

(30)

where \( S = \tilde{M}_1 C \begin{bmatrix} A & G_{12} \end{bmatrix} M \) with \( M \) defined in (25).

From Lemma 1 and equation (30), the expressions of \( \overline{w_1} \) \( \overline{w_1} \) are deduced.

Remark (8). For the brevity of presentation, the expressions of \( \overline{w_1} \) \( \overline{w_1} \) are not detailed but can be easily obtained.

Remark (9). \( \psi \) and \( \tilde{M}_1 H_2(\theta)v \) are respectively assumed to be the useful signal and the noise effect (see part 2.2 for further details). An HOSM differentiator is computed to estimate the first derivative of \( \psi \) \( \psi \) and \( v \) in (30).

Applying Lemma 1, the equation (8a) is rewritten as:

\[
S_1 \ddot{d}_1(t) = P_1(t)
\]

(31)

where

\[
S_1 = \begin{bmatrix} F_1^+ & 0 & 0 & F_1^- \\ 0 & F_1^+ & F_1^- & 0 \\ 0 & F_1^- & F_1^+ & 0 \\ F_1^- & 0 & 0 & F_1^+ \end{bmatrix}, \quad \ddot{d}_1(t) = \begin{bmatrix} \ddot{d}_1(t)^+ \\ \ddot{d}_1(t)^- \\ \ddot{d}_1(t)^+ \\ \ddot{d}_1(t)^- \end{bmatrix}, \quad P_1(t) = \begin{bmatrix} P_{11}(t) \\ P_{12}(t) \\ P_{13}(t) \\ P_{14}(t) \end{bmatrix}
\]

with

\[
P_{11}(t) = G_{11}^T \overline{w_1}(t) + G_{12}^T \overline{w_2}(t) + \Delta A^+ \overline{x}(t)^- + \Delta A^- \overline{x}(t)^+ \\
P_{12}(t) = G_{11}^T \overline{w_1}(t) + G_{12}^T \overline{w_2}(t) + \Delta A^+ \overline{x}(t)^+ + \Delta A^- \overline{x}(t)^- \\
P_{13}(t) = G_{11}^T \overline{w_1}(t) + G_{12}^T \overline{w_2}(t) + \Delta A^+ \overline{x}(t)^+ + \Delta A^- \overline{x}(t)^+ \\
P_{14}(t) = G_{11}^T \overline{w_1}(t) + G_{12}^T \overline{w_2}(t) + \Delta A^+ \overline{x}(t)^+ + \Delta A^- \overline{x}(t)^-
\]

Finally, under Assumption 3, the matrix \( S_1 \) is full column rank and it follows that \( \ddot{d}_1(t) = (S_1^T S_1)^{-1} S_1^T P_1(t) \). The upper and the lower bounds for the unknown input \( d_1 \) are obtained by \( \ddot{d}_1(t) = \ddot{d}_1(t)^+ - \ddot{d}_1(t)^- \) and \( \overline{d}_1(t) = \overline{d}_1(t)^+ - \overline{d}_1(t)^- \).
Step 2: To prove that \( d_2(t) \leq d_2(t) \leq \bar{d}_2(t) \), we consider that step 1 holds. Using Lemma 1, the equation (8b) is rewritten as:

\[
S_2 \tilde{d}_2(t) = P_2(t)
\]

where

\[
S_2 = \begin{bmatrix}
F_2^+ & 0 & 0 & F_2^- \\
0 & F_2^+ & F_2^- & 0 \\
0 & F_2^- & F_2^+ & 0 \\
F_2^- & 0 & 0 & F_2^+
\end{bmatrix}, \quad \tilde{d}_2(t) = \begin{bmatrix}
d_2(t)^+ \\
d_2(t)^- \\
d_2(t)^- \\
d_2(t)^+
\end{bmatrix}, \quad P_2(t) = \begin{bmatrix}
G_2^+ w_2(t) + \Delta \pi(t)^+ + \Delta \pi(t)^- \\
G_2^- w_2(t) + \Delta \pi(t)^+ + \Delta \pi(t)^- \\
G_2^- w_2(t) + \Delta \pi(t)^+ + \Delta \pi(t)^- \\
G_2^- w_2(t) + \Delta \pi(t)^+ + \Delta \pi(t)^-
\end{bmatrix}
\]

Under Assumption 3, the matrix \( S_2 \) is full column rank and it follows that \( \tilde{d}_2(t) = (S_2^T S_2)^{-1} S_2^T P_2(t) \). The upper and the lower bounds for the unknown input \( d_2 \) are obtained by \( \underline{d}_2(t) = d_2(t)^+ - \tilde{d}_2(t)^- \) and \( \bar{d}_2(t) = \bar{d}_2(t)^+ - \bar{d}_2(t)^- \).

4. EXAMPLE

4.1. MODEL DESCRIPTION

To illustrate the proposed approach, consider the nonlinear model of a single-flexible joint robot coupled with a DC motor described by:

\[
\begin{align*}
L \frac{d^2 i(t)}{dt^2} + Ri(t) &= V(t) - K_e \dot{\theta}_m(t) \\
J_m \theta_m(t) + k(\theta_m(t) - \theta_i(t)) + B \dot{\theta}_m(t) &= K_i i(t) \\
J_i \dot{\theta}_i(t) + k(\theta_i(t) - \theta_m(t)) + mgh \sin(\theta_i(t)) &= 0
\end{align*}
\]

(33)

where \( i, V, \theta_m, \dot{\theta}_m \) and \( \ddot{\theta}_m \) are respectively the current, the tension, the angular position, the angular velocity and the angular acceleration of the motor. \( \theta_i, \dot{\theta}_i \) and \( \ddot{\theta}_i \) are respectively the angular position, the angular velocity and the angular acceleration of the link.

For the simulation, we consider \( k \) unknown such that \( k = k_0 + \Delta k \) with \( \Delta k \in [-\Delta k, \Delta k] = \theta \). The measurement output is defined as \( y(t) = v(t) + v(t) \), where \( v(t) = \begin{bmatrix} \dot{\theta}_m(t) & (\theta_m(t) - \theta_i(t)) & i(t) \end{bmatrix}^T \) and the measurement noise \( v(t) \) belongs in \( [-\nu, \nu] \) with \( \nu = 0.03I_{5 \times 1} \). In the sequel, we consider \( d_1(t) = \sin(\theta_1(t)) \) and \( d_2(t) = f_\sigma(t) \) where \( f_\sigma(t) \) is the fault located on the current sensor.

4.2. UNKNOWN INPUT INTERVAL OBSERVER DESIGN

Under Assumption 1-2, the system (33) can be rewritten under the form described by (10) defining the state \( x(t) \), the input \( u(t) \) and the unknown inputs \( w_1(\theta, t), w_2(\theta, t) \) by

\[
\begin{align*}
x(t) &= \begin{bmatrix} \theta_m(t) & \theta_i(t) & i(t) & \dot{\theta}_m(t) & \dot{\theta}_i(t) \end{bmatrix}^T \\
u(t) &= V(t) \\
w_1(\theta, t) &= \frac{mgh \sin(\theta_i(t))}{J_m} \\
w_2(\theta, t) &= \begin{bmatrix} -\Delta k(\dot{\theta}_i(t) - \theta_m(t)) - \frac{mgh \sin(\theta_i(t))}{J_m} \\
\end{bmatrix}
\end{align*}
\]

(34)
Table 1. Simulation parameters table.

| Parameter                          | Value         | Parameter                          | Value         |
|-----------------------------------|---------------|-----------------------------------|---------------|
| Inductance \((H)\)                | \(L\) = 1     | Rotor inertia \((kg.m^2)\)        | \(J_m\) = 3.7 \times 10^{-3} |
| Resistance \((Ohm)\)              | \(R\) = 0.5   | Link inertia \((kg.m^2)\)         | \(J_l\) = 9.3 \times 10^{-3} |
| Motor gain \((V.rad^{-1}.s^{-1})\) | \(K_e\) = 0.1 | Stiffness \((M.m.rad^{-1})\)     | \(k\) = 0.17  |
| Motor gain \((N.m.A^{-1})\)       | \(K_i\) = 0.1 | Stiffness \((M.m.rad^{-1})\)     | \(\Delta k\) = 5 \times 10^{-2} |
| Link mass \((kg)\)                | \(m\) = 0.21  |                                    |               |
| Earth gravity \((m.s^{-2})\)      | \(g\) = 9.81  | Center of mass \((m)\)            | \(h\) = 0.31  |
| Viscous friction \((M.m.s.rad^{-1})\) | \(B\) = 4.6 \times 10^{-2} |

The associated distribution matrices to (10) are now given by:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-k_0 \frac{d}{d_1} & k_0 \frac{d}{d_1} & -\frac{R}{K_m} & -\frac{B}{K_m} & 0 \\
-k_0 \frac{d}{d_1} & -k_0 \frac{d}{d_1} & 0 & 0 & 0
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
\frac{1}{L} \\
0
\end{bmatrix},
G_{11} = \begin{bmatrix}
0 \\
0 \\
-1 \\
0
\end{bmatrix},
G_{12} = \begin{bmatrix}
0 \\
0 \\
-1 \\
0
\end{bmatrix},
G_2 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix},
H_1 = 0_{n\times s}, H_2 = \begin{bmatrix} 0_{p\times n} & I_p \end{bmatrix}
\]

The LTI model, proposed above, allows us to consider that Assumption 4 holds. Then, there exists a non-singular state transformation \(M_0\) such that the system (10) is rewritten under the singular form presented in (15):

\[
M_0 = \begin{bmatrix}
0 & 0.5883 & -0.2215 & -0.2243 & -0.9209 \\
0 & 0.8645 & 0.7880 & -0.5945 & -0.8970 \\
0 & -0.1439 & 0.9087 & 0.8102 & -0.8687 \\
-1 & 0.8784 & 0.0617 & -0.7832 & 0.2541 \\
0 & 0.9596 & 0.0512 & 0.0907 & 0.5007
\end{bmatrix}
\]  \hspace{1em} (35)

It follows that the matrix \(\begin{bmatrix} \hat{E} \\ \hat{C} \end{bmatrix}\) is full column rank. Hence, \(T\) and \(F\) are computed from (19d) and it is deduced that the pair \((T\hat{A}, \hat{C})\) is detectable. Hence, Assumption 5 holds and a pole placement \(\{ -0.1 , -0.3 , -0.6 , -1 , -1.3 , -1.7 , -2 \}\) is used to compute the matrix \(N\) as in remark (7). From this result, the expressions of the matrices \(J\) and \(M_1\) are determined. Finally, we compute \(K\) such that the theorem 2 holds.

For the estimation of the unknown input \(d_t\), the output transformation \(\tilde{M}\) is chosen as

\[
\tilde{M} = \begin{bmatrix}
0 & 0 & 0 & -0.9538 & 0.9892 \\
0 & 0 & 0 & 0.2093 & 0.7393 \\
0 & 0 & 1.0000 & 0.3085 & 0.4540 \\
-1.0000 & 0 & 0 & 0.0078 & 0.0514 \\
0 & 0.3978 & 0 & 0.5060 & -0.0508
\end{bmatrix}
\]  \hspace{1em} (36)
Remark (10). For the brevity of presentation, all matrices are not detailed but could be easily deduced from $M_0$. The matrix $M_1$ is computed here to get the cooperative property by the diagonalization of the matrix $N$.

4.3. SIMULATIONS AND RESULTS
For the simulation, we consider the known input as:

$$u(t) = \begin{cases} 
-5\sin(t) & \forall \ t < 10\text{s} \\
-5\sin(t) - 3\sin(3(t - 10)) & \forall \ t \geq 10\text{s} 
\end{cases} \quad (37)$$

The fault sensor is assumed to be unknown. For the simulation, we consider

$$f(t) = \begin{cases} 
0 & \forall \ t < 15\text{s} \\
2\sin\left(\frac{t-15}{2}\right) + 0.6\cos\left(\frac{t}{2} - 14\right) & \forall \ t \geq 15\text{s} 
\end{cases} \quad (38)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{simulation_results}
\caption{Interval estimation of the state $x$}
\end{figure}

It is considered that $\pi(0) = \left[ \begin{array}{ccc} \pi & \pi & \pi & 1 \end{array} \right]^T$, $\bar{x}(0) = -\pi(0)$, $\overline{\omega_2}(0) = \left[ \begin{array}{c} 44 \\ 1 \end{array} \right]^T$, $\underline{\omega_2}(0) = -\pi(0)$. The initial conditions of the interval observer are given by $\xi_1(0) = \overline{\omega_2}(0) - \overline{\omega_2}(0)$ and $\xi_2(0) = \overline{W}^+\xi(0) - \overline{W}^-\xi(0)$ where $\overline{W} = M_1^{-1}T\hat{E}M_0^{-1}$ and $\hat{E}$ defined in (25).

To estimate the unknown input $d_1(t)$, the upper and the lower bounds of $\hat{y}(t)$ are determined by the first order HOSM differentiator. The parameters are chosen as $\lambda_0 = 100$, $\lambda_1 = 12$ and the parameter $\mu$ is consider as $\mu = 17.3$ for each output. The sample time for the simulation is $T_e = 1\text{ms}$. The upper and lower bounds of the state, presented in figure 1, converge asymptotically to a tight interval showing the efficiency of the decoupling method proposed in this paper to reduce the conservatism caused by noises, unknown inputs and other disturbances.
The convergence speed is fixed by the interval observer design with the pole placement. The figure 2 is dedicated to the upper and lower bound estimations of unknown inputs $d_1$ and $d_2$. The interval estimation of the unknown input $d_2$ converge asymptotically to a tight interval given, in this case, by the interval observer properties. The width of the interval estimation for $d_1$ is proportional to state $r$, the noise $v$, the input $u$ and the uncertain parameters $\theta$.

**Figure 2.** Interval estimation of the unknown inputs $d_1$ and $d_2$

5. CONCLUSION

A methodology to estimate jointly the unmeasurable states and unknown inputs estimation for uncertain LTI systems is proposed and is shown in this paper. The pertinence of the proposed approach has been illustrated on an academic example. The results obtained are significant to show the pertinence of the proposed approach. Indeed, the uncertainty, disturbance and noise propagation is reduced, decreasing significantly the conservatism of the proposed method.

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