Quasilinear systems of Jordan block type and the mKP hierarchy

Lingling Xue\textsuperscript{1} and E V Ferapontov\textsuperscript{2,3}\textsuperscript{©}

\textsuperscript{1} Department of Mathematics, Ningbo University, Ningbo 315211, People’s Republic of China
\textsuperscript{2} Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom
\textsuperscript{3} Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, 112, Chernyshevsky Street, Ufa 450077, Russia

E-mail: xuelingling@nbu.edu.cn and E.V.ferapontov@lboro.ac.uk

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Abstract
Hydrodynamic type systems in Riemann invariants arise in a whole range of applications in fluid dynamics, Whitham averaging procedure, differential geometry and the theory of Frobenius manifolds. In this paper we discuss parabolic (Jordan block) analogues of diagonalisable systems. Our main observation is that integrable quasilinear systems of Jordan block type are parametrised by solutions of the modified Kadomtsev–Petviashvili hierarchy. Such systems appear naturally as degenerations of quasilinear systems associated with multi-dimensional hypergeometric functions, in the context of parabolic regularisation of the Riemann equation, as finite-component reductions of hydrodynamic chains, and as hydrodynamic reductions of linearly degenerate dispersionless integrable PDEs in multi-dimensions.

Keywords: parabolic quasilinear systems, commuting flows, conservation laws, mKP hierarchy, hydrodynamic reductions

1. Introduction
Quasilinear systems of the form
\[ u_t = v(u)u_x \] (1)
have been thoroughly investigated in the literature. Here \( u = (u^1, \ldots, u^n)^T \) is a column vector of the dependent variables and \( v \) is a \( n \times n \) matrix. The main emphasis has always been on the
strictly hyperbolic case where the matrix $v$ has real distinct eigenvalues. Under the additional condition that the Haantjes tensor of matrix $v$ vanishes, any such system can be reduced to the diagonal form

$$R_i^t = \lambda^i(R)R_i^x,$$  \hspace{1cm} (2)

$i = 1, \ldots, n$, in specially adapted coordinates $R^1, \ldots, R^n$ known as Riemann invariants. Systems of type (2) govern a wide range of problems in pure and applied mathematics, see e.g. [1, 15, 17]. It was shown by Tsarev [17] that under the so-called semi-Hamiltonian constraint,

$$\left(\frac{\lambda_j}{\lambda_j - \lambda_i}\right)_k = \left(\frac{\lambda_k}{\lambda_k - \lambda_i}\right)_j,$$

system (2) possesses infinitely many conservation laws and commuting flows, and can be solved by the generalised hodograph method (here $i \neq j \neq k$ and low indices indicate differentiation by the variables $R^j$).

In this paper we study quasilinear systems (1) of Jordan block type. More precisely, we assume the existence of special coordinates (which we will also denote $R^1, \ldots, R^n$) where the equations reduce to upper-triangular Toeplitz form

$$R_i^t = MR_i, \quad M = \lambda^0 E + \sum_{i=1}^{n-1} \lambda^i P^i;$$  \hspace{1cm} (3)

here $R = (R^1, \ldots, R^n)^T$, $E$ is the $n \times n$ identity matrix, $P$ is the $n \times n$ Jordan block with zero eigenvalue (note that $P^n = 0$), and $\lambda^0, \lambda^i$ are functions of $R$. Explicitly, a three-component version of system (3) looks like this:

$$\begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} \lambda^0 & \lambda^1 & \lambda^2 \\ 0 & \lambda^0 & \lambda^1 \\ 0 & 0 & \lambda^0 \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x.$$

The main properties of systems (3) can be summarised as follows:

(a) The corresponding matrix $M$ is pointwise of Jordan block type;
(b) The Haantjes tensor of matrix $M$ vanishes.

The vanishing of the Haantjes tensor makes systems (3) natural parabolic analogues of hydrodynamic type systems (2) in Riemann invariants. Note that upper-triangular Toeplitz matrices form a commutative family (cyclic Haantjes algebra in the terminology of [16]). Systems of type (3) appear as degenerations of hydrodynamic type systems associated with multi-dimensional hypergeometric (Lauricella) functions [6], in the context of parabolic regularisation of the Riemann equation [10], and as reductions of hydrodynamic chains and linearly degenerate dispersionless PDEs in 3D [14]. The most well-studied case of system (3) corresponds to the choice $\lambda^0 = R^1, \lambda^1 = 1, \lambda^i = 0, i \geq 2$ [6, 14]. In the three-component case this gives

$$\begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} R^1 & 1 & 0 \\ 0 & R^1 & 1 \\ 0 & 0 & R^1 \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x.$$
In section 2 we classify commuting systems of type (3). Our main observation is that integrable hierarchies of Jordan block type are governed by the modified Kadomtsev-Petviashvili (mKP) hierarchy. Here is a brief summary of our results in this direction. Any two-component hierarchy of type (3) can be parametrised in the form

\[
\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x
\]

where the function \( \psi(R^1, R^2) \) satisfies the Lax equation of the mKP hierarchy,

\[
\psi_2 = \psi_{11} + \rho \psi_1;
\]

here low indices indicate differentiation by \( R^1, R^2 \). Fixing the potential \( \rho(R^1, R^2) \) and varying solution \( \psi \) to the linear PDE (4) we obtain commuting flows of the corresponding hierarchy. Explicitly, this means that if \( \psi, \varphi \) are two solutions of (4) (corresponding to the same \( \rho \)) then the systems

\[
\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_{t_\psi} = \begin{pmatrix} \psi & \psi_1 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x
\]

and

\[
\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_{t_\varphi} = \begin{pmatrix} \varphi & \varphi_1 \\ 0 & \varphi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x
\]

commute with each other (here the corresponding ‘times’ are denoted \( t_\psi \) and \( t_\varphi \), respectively).

Similarly, any three-component hierarchy of type (3) can be parametrised in the form

\[
\begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 & \psi_{11} + w_1 \psi_1 \\ 0 & \psi & \psi_1 \\ 0 & 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x,
\]

where \( w \) solves the mKP equation

\[
4w_{13} + 6w_1^2 w_{11} - w_{1111} - 3w_{22} - 6w_2 w_{11} = 0,
\]

and \( \psi \) satisfies the corresponding Lax equations:

\[
\psi_2 = \psi_{11} + 2w_1 \psi_1, \quad \psi_3 = \psi_{111} + 3w_1 \psi_{11} + \frac{3}{2}(w_2 + w_{11} + w_1^2) \psi_1.
\]

Fixing \( w \) and varying \( \psi \) we obtain commuting flows of the hierarchy. We show that the corresponding conserved densities are governed by the adjoint Lax equations.

In section 3 we demonstrate that systems of Jordan block type naturally occur as hydrodynamic reductions of multi-dimensional linearly degenerate PDEs: the 3D Mikhalev system [12] is used as an illustrating example, see also [14].

### 2. Quasilinear systems of Jordan block type

#### 2.1. Form-invariance

The class of two-component Toeplitz systems (3),

\[
\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_t = \begin{pmatrix} \lambda^0 & \lambda^1 \\ 0 & \lambda^0 \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x,
\]

...
is form-invariant under triangular changes of variables \((R^1, R^2) \leftrightarrow (r^1, r^2)\):

\[
R^1 = F(r^1, r^2), \quad R^2 = G(r^2),
\]

where \(F\) and \(G\) are arbitrary functions of the indicated arguments. Similarly, the class of three-component Toeplitz systems (3),

\[
\begin{pmatrix}
R^1 \\
R^2 \\
R^3
\end{pmatrix}
= 
\begin{pmatrix}
\lambda^0 & \lambda^1 & \lambda^2 \\
0 & \lambda^0 & \lambda^1 \\
0 & 0 & \lambda^0
\end{pmatrix}
\begin{pmatrix}
R^1 \\
R^2 \\
R^3
\end{pmatrix},
\]

is form-invariant under triangular changes of variables \((R^1, R^2, R^3) \leftrightarrow (r^1, r^2, r^3)\):

\[
R^1 = r^1 \left( \frac{\partial^2 G}{\partial r^3} \right) + F(r^2, r^3), \quad R^2 = G(r^2, r^3), \quad R^3 = H(r^3),
\]

where \(F, G\) and \(H\) are arbitrary functions of the indicated arguments. This transformation freedom will be utilised to simplify the classification results. Note that the group preserving the class of diagonal systems (2) is far more narrow, generated by transformations of the form \(R_i = F_i(r^i)\), functions of one variable only.

### 2.2. Commuting flows

Let us recall that commuting flows \(R^*_i = \mu^i(R)R^i\) of the diagonal system (2) are governed by the equations

\[
\mu^i_j = a^{ij}(\mu^j - \mu^i)
\]

where \(a^{ij}(R) = \lambda_j^i/\left(\lambda^j - \lambda^i\right)\) are fixed and solutions \(\mu^i\) to linear system (7) vary [17]. Note that the consistency conditions of system (7),

\[
da^{ij}_k = a^{ij}(a^k + a^h d^j - a^h d^i),
\]

are equivalent to the integrable \(2 + 1\)-dimensional \(n\)-wave system. Thus, commuting flows of diagonal form (2) are governed by the \(n\)-wave hierarchy.

Below we demonstrate that commuting flows of Toeplitz type (3) are governed by the mKP hierarchy. In a sense, Toeplitz systems and the underlying mKP hierarchy can be viewed as parabolic degenerations of hyperbolic systems in Riemann invariants and the underlying \(n\)-wave hierarchy, respectively.

#### 2.2.1. Two-component case

Consider two upper-triangular systems of Toeplitz type,

\[
\begin{pmatrix}
R^1 \\
R^2
\end{pmatrix}_t = \begin{pmatrix}
\lambda^0 & \lambda^1 \\
0 & \lambda^0
\end{pmatrix}
\begin{pmatrix}
R^1 \\
R^2
\end{pmatrix}_x,
\]

\[
\begin{pmatrix}
R^1 \\
R^2
\end{pmatrix}_y = \begin{pmatrix}
\mu^0 & \mu^1 \\
0 & \mu^0
\end{pmatrix}
\begin{pmatrix}
R^1 \\
R^2
\end{pmatrix}_x.
\]

Direct calculation shows that their commutativity (compatibility) conditions, that is, \(R^1_t = R^2_t\) and \(R^2_t = R^1_t\), are equivalent to the following two conditions:

\[
\frac{\lambda^0}{\lambda^1} = \frac{\mu^0}{\mu^1},
\]

\[
\frac{\lambda^0_2 - \lambda^1_1}{\lambda^1} = \frac{\mu^0_2 - \mu^1_1}{\mu^1}.\]
Let us introduce the notation $\lambda_0 = m$, $\lambda_2 - \lambda_1 = 1$, $\lambda_1 = \rho$; note that the quantities $m, \rho$ are shared by all commuting flows. If $m \neq 0$, using symmetry (5) one can set $m = 1$. Denoting $\lambda_0 = \psi$ we obtain $\lambda_1 = \psi_1, \lambda_2 = \psi_11 + \rho \psi_1$. Thus, members of the commuting hierarchy can be parametrised in the form

$$
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \begin{pmatrix}
\psi & \psi_1 \\
0 & \psi
\end{pmatrix}
\begin{pmatrix}
R_1 \\
R_2
\end{pmatrix}_x,
$$

(8)

where $\psi$ solves the Lax equation of the mKP hierarchy,

$$
\psi_2 = \psi_11 + \rho \psi_1.
$$

Here the potential $\rho$ is fixed and the solution $\psi$ of the Lax equation varies. The choice $\rho = 0, \psi = R_1$ gives rise to the system

$$
\begin{pmatrix}
R_1 \\
R_2
\end{pmatrix}_t = \begin{pmatrix}
R_1 & 1 \\
0 & R_1
\end{pmatrix}
\begin{pmatrix}
R_1 \\
R_2
\end{pmatrix}_x
$$

whose commuting flows are parametrised by solutions to the heat equation [6, 14].

**Remark 1.** Conservation laws of system (8) are relations of the form $\phi_t = \Omega_x$ which hold identically modulo (8); here the functions $\phi(R_1, R_2) = \Omega(R_1, R_2)$ are the conserved density and flux, respectively. To obtain equations for $\phi$ and $\Omega$ we represent system (8) in the form

$$
R_t = M_2(\psi)R_x
$$

where

$$
M_2(\psi) = \begin{pmatrix}
\psi & \psi_1 \\
0 & \psi
\end{pmatrix}.
$$

The evaluation of the conservation law $\phi_t = \Omega_x$ modulo (8) and comparison of the coefficients at $R_1$ and $R_2$ gives

$$
\nabla \Omega = \nabla \phi M_2(\psi),
$$

where $\nabla \Omega$ and $\nabla \phi$ are the gradient (row) vectors. In components, this gives

$$
\Omega_1 = \psi \phi_1, \quad \Omega_2 = \psi \phi_2 + \psi_1 \phi_1,
$$

and the elimination of $\Omega$ results in the adjoint Lax equation for the conserved density $\phi$:

$$
\phi_2 = -\phi_11 + \rho \phi_1.
$$

Let us note that the flux $\Omega$ can be interpreted as the squared eigenfunction potential:

$$
\Omega = \Omega(\psi, \phi) = \int \psi \phi_1 dR_1 + [\psi \phi_2 + \psi_1 \phi_1] dR_2.
$$

This object has appeared before in the context of binary Darboux transformations and squared eigenfunction symmetries of the mKP hierarchy, see e.g. [4, 11, 13], as well as section 2.2.4, for the general formulae.
2.2.2. Three-component case. Direct calculation shows that the compatibility of three-component systems of the form

\[
\begin{pmatrix}
R^1 \\
R^2 \\
R^3
\end{pmatrix}_t = \begin{pmatrix}
\lambda^0 & \lambda^1 & \lambda^2 \\
0 & \lambda^0 & \lambda^1 \\
0 & 0 & \lambda^0
\end{pmatrix},
\begin{pmatrix}
R^1 \\
R^2 \\
R^3
\end{pmatrix}_x = \begin{pmatrix}
\mu^0 & \mu^1 & \mu^2 \\
0 & \mu^0 & \mu^1 \\
0 & 0 & \mu^0
\end{pmatrix},
\begin{pmatrix}
R^1 \\
R^2 \\
R^3
\end{pmatrix}_y = \begin{pmatrix}
\mu^0 & \mu^1 & \mu^2 \\
0 & \mu^0 & \mu^1 \\
0 & 0 & \mu^0
\end{pmatrix}
\]

is equivalent to the following six conditions (we assume \(\lambda^1 \neq 0\) and \(\mu^1 \neq 0\)):

\[
m \equiv \frac{\lambda^0}{\lambda^1} = \frac{\mu^0}{\mu^1},
\]

(9)

\[
\rho \equiv \frac{\lambda^0_2 - \lambda^1_1}{\lambda^1} = \frac{\mu^0_2 - \mu^1_1}{\mu^1},
\]

(10)

\[
p \equiv \frac{\lambda^0_2 - m \lambda^2}{\lambda^1} = \frac{\mu^0_2 - m \mu^2}{\mu^1},
\]

(11)

\[
s \equiv \frac{\lambda^1_2 - \lambda^2_1 - p \lambda^2}{\lambda^1} = \frac{\mu^1_2 - \mu^2_1 - p \mu^2}{\mu^1},
\]

(12)

\[
r \equiv \frac{\lambda^1_3 - \lambda^2_1 + (p - \rho) \lambda^2}{\lambda^1} = \frac{\mu^1_3 - \mu^2_1 + (p - \rho) \mu^2}{\mu^1},
\]

(13)

\[
h \equiv \frac{\lambda^1_1 - \lambda^2_2 - (s + r) \lambda^2}{\lambda^1} = \frac{\mu^1_3 - \mu^2_2 - (s + r) \mu^2}{\mu^1},
\]

(14)

Note that (9) and (10) arise from the coefficients of \((R^2)_x^2\), while (11)–(14) are the coefficients of \(R^2_i R^i\). These conditions can be reduced to the mKP equation and its Lax pair as follows. Setting \(R^3 = \text{const}\) we first reduce three-component case to the two-component case. Thus, by induction (or from (9) and (10)), we can assume \(m = 1\), \(\lambda^0 = \psi\), \(\lambda^1 = \psi_1\) where \(\psi_2 = \psi_{11} + \rho \psi_1\),

(15)

so that it remains to reconstruct \(\lambda^2\) (we will only be working with the \(\lambda\)-part of the commutativity conditions (9)–(14)). Equations (11) and (12) imply

\[
\lambda^2 = \psi_2 - p \psi_1,
\]

and \(s = p_1 + p(p - \rho)\), respectively. Then (13) yields

\[
\psi_3 = \psi_{11} + (2p - \rho) \psi_{11} + [r + (p - \rho)^2 + p_1] \psi_1.
\]

(16)

Finally, it follows from (14) that

\[
(2p - \rho) \psi_{11} - (q^2 p - h + q_2 + q q_1 + 2 q p_1 + r_1) \psi_1 = 0,
\]

where \(q = p - \rho\), which allows us to set

\[
\rho = 2p, \quad h = p^3 - rp - p_2 - p p_1 + r_1.
\]
This reduces equations (15) and (16) to the form
\[ \psi_2 = \psi_{11} + 2p \psi_1 \]
and
\[ \psi_3 = \psi_{111} + 3p \psi_{11} + [2p_1 + p^2 + r] \psi_1, \]
respectively. Their compatibility, i.e., the condition \( \psi_{23} = \psi_{32} \), leads to
\[ (p_{11} - 2p_1 + 2r_1 - 3p_2)\psi_{11} = (2p_1^2 - 2p_1^2 p_1 + 2rp_1 - 2p_1 p_2 - r_1 + r_2 + 2p_1 + 2p_2 - 2p_3) \psi_1, \]
from which we obtain the following two equations:
\[ p_{11} - 2p_1 + 2r_1 - 3p_2 = 0, \tag{17} \]
\[ 2p_1^2 + 2(r - p_1^2)p_1 - 2p_1 p_2 - r_1 + r_2 + 2p_1 + 2p_2 - 2p_3 = 0. \tag{18} \]
In order to solve (17) for \( r \) we introduce the potential variable \( w \) such that \( p = w_1 \). Then integrating (17) gives
\[ r = \frac{1}{2}w_1^2 - \frac{1}{2}w_{11} + \frac{3}{2}w_2. \]
Finally, from (18) we obtain
\[ 4w_{13} + 6w_1^2 w_{11} - w_{1111} - 3w_{22} - 6w_2 w_{11} = 0, \tag{19} \]
which is the potential mKP equation. Note that the mKP hierarchy was introduced in [5, 7]. The generalised Miura transformation connecting KP and mKP equations was constructed in [7], see also [8] for the first classification results of integrable equations in \( 2 + 1 \) dimensions. Exact solutions of mKP equation were constructed in [2, 3, 9], see also references therein.

To summarise, members of the three-component commuting hierarchy can be parametrised in the form
\[ \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 & \psi_{11} + w_1 \psi_1 \\ 0 & \psi & \psi_1 \\ 0 & 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x, \tag{20} \]
where \( w \) satisfies the mKP equation (19) and \( \psi \) solves the corresponding Lax equations (15) and (16):
\[ \psi_2 = \psi_{11} + 2w_1 \psi_1, \quad \psi_3 = \psi_{111} + 3w_1 \psi_{11} + \frac{3}{2}(w_2 + w_{11} + w_1^2) \psi_1. \tag{21} \]
Fixing \( w \) and varying \( \psi \) we obtain commuting flows of the hierarchy.

**Example 1.** Set \( w = 0 \), then equations for \( \psi \) and \( \lambda^0, \lambda^1, \lambda^2 \) become
\[ \psi_2 = \psi_{11}, \quad \psi_3 = \psi_{111}, \quad \lambda^0 = \psi, \quad \lambda^1 = \psi_1, \quad \lambda^2 = \psi_{11}. \]
We can choose
\[ \psi = R^1, \quad \lambda^0 = R^1, \quad \lambda^1 = 1, \quad \lambda^2 = 0, \]
or
\[ \psi = e^{iR^1 + i^2 R^2 + i^3 R^3}, \quad \lambda^0 = \psi, \quad \lambda^1 = k \psi, \quad \lambda^2 = k^2 \psi, \]
where \( k \) is an arbitrary constant (the former solution was considered in [14]).

**Remark 2.** Conservation laws of system (20) are relations of the form \( \phi_t = \Omega_x \) which hold identically modulo (20). Representing system (20) in the form
\[ R_t = M_3(\psi) R_x \]
where
\[ M_3(\psi) = \begin{pmatrix} \psi & \psi_1 & \psi_11 + w_1 \psi_1 \\ 0 & \psi & \psi_1 \\ 0 & 0 & \psi \end{pmatrix} \]
and evaluating the conservation law \( \phi_t = \Omega_x \) modulo (20) we obtain
\[ \nabla \Omega = \nabla \phi M_3(\psi) \]
or, in components,
\[ \Omega_1 = \psi \phi_1, \quad \Omega_2 = \psi \phi_2 + \psi_1 \phi_1, \quad \Omega_3 = \psi \phi_3 + \psi_1 \phi_2 + (\psi_11 + w_1 \psi_1) \phi_1. \]
(22)

The elimination of \( \Omega \) results in the adjoint Lax equations for the conserved density \( \phi \):
\[ \phi_2 = -\phi_11 + 2 w_1 \phi_1, \quad \phi_3 = \phi_111 - 3 w_1 \phi_11 + \frac{3}{2}(w_2 - w_11) \phi_1. \]

Again, the flux \( \Omega \) is the corresponding squared eigenfunction potential:
\[ \Omega = \Omega(\psi, \phi) = \int \psi \phi_1 dR^1 + [\psi \phi_2 + \psi_1 \phi_1] dR^2 + [\psi \phi_3 + \psi_1 \phi_2 + (\psi_11 + w_1 \psi_1) \phi_1] dR^3. \]

Note that the existence of an infinity of conservation laws of system (20) is yet another demonstration of its hydrodynamic integrability.

2.2.3. **Four-component case.** Omitting details of calculations we present the final result: four-component commuting flows of Toeplitz type (3) can be parametrised in the form
\[ \begin{pmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \end{pmatrix} = \begin{pmatrix} \psi & \psi_1 & \psi_11 + w_1 \psi_1 & \psi_111 + 2 w_1 \psi_11 + \frac{1}{2}(w_2 + 3 w_11 + w_1^2) \psi_1 \\ 0 & \psi & \psi_1 & \psi_11 + w_1 \psi_1 \\ 0 & 0 & \psi & \psi_1 \\ 0 & 0 & 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \end{pmatrix}, \]
(23)

where \( w \) solves the first three equations of the mKP hierarchy,
\[ 4w_{13} + 6w_1^2 w_{11} - w_{111} - 3w_{22} - 6w_2 w_{11} = 0, \]
\[ 2w_{23} = 3w_{14} - w_{1112} + (3w_1^2 - 3w_2)w_{12} + 6w_1 w_{11} w_2 + (w_{111} - 4w_3 - 2w_1^3)w_{11}. \]
The same formula for component ones. This means that when studying \( \lambda \) this leaves only one unknown, namely, \( \psi \) and \( w \). Fixing \( w \) and varying \( \psi \) we obtain commuting flows of the hierarchy.

### 2.2.4. Multi-component case.

Formulae (8), (20), (23) for \( n = 2, 3, 4 \) suggest a general recursive formula for commuting flows of Toeplitz type (3). Note that elements of the matrix \( M_4(\psi) \) from (23),

\[
\begin{align*}
\lambda^0 & = \psi, \quad \lambda^1 = \psi_1, \quad \lambda^2 = \psi_{11} + w_1 \psi_1, \\
\lambda^3 & = \psi_{111} + 2 w_1 \psi_{11} + \frac{1}{2} (w_1^2 + w_2 + 3 w_{11}) \psi_1,
\end{align*}
\]

satisfy the relations

\[
\lambda^0 = \psi, \quad \lambda^1 = \psi_1, \quad \lambda^2 + w_1 \lambda^1 = \psi_2, \quad \lambda^3 + w_1 \lambda^2 + w_2 \lambda^1 = \psi_3.
\]

This recurrence generalises to the general \( n \)-component case: the matrix elements of \( M_n(\psi) \) can be obtained from the triangular linear system

\[
\lambda^0 = \psi, \quad \lambda^1 = \psi_1, \quad \lambda^k + w_1 \lambda^{k-1} + w_2 \lambda^{k-2} + \cdots + w_{k-1} \lambda^1 = \psi_k, \quad 2 \leq k \leq n - 1.
\]

(24)

The structure of commuting flows suggests an inductive approach to the proof of formula (24), indeed, setting \( R^0 = \text{const} \) in the \( n \)-component commuting flows we obtain \( (n - 1) \)-component commuting flows, thus, all \( (n - 1) \)-component commutativity conditions are part of the \( n \)-component ones. This means that when studying \( n \)-component commuting flows we can use the same formulae for \( \lambda^0, \ldots, \lambda^{n-2} \) as in the \( (n - 1) \)-component case (inductive assumption). This leaves only one unknown, namely, \( \lambda^{n-1} \). Still, we were not able to find a general procedure to reconstruct \( \lambda^{n-1} \) that would work for every \( n \).

There is however an alternative approach to the general \( n \)-component case coming from the theory of binary Darboux transformations and squared eigenfunction symmetries of the mKP hierarchy [4, 11, 13]. Given a Lax eigenfunction \( \psi \) and the adjoint eigenfunction \( \phi \) that satisfy

\[
\psi_k = (L^k)_{\geq 1} \psi, \quad \phi_k = -\partial^{-1} \left( (L^k)_{\geq 1} \right)^* \partial \phi, \quad 1 \leq k \leq n,
\]
one can construct the squared eigenfunction potential
\[
\Omega(\psi, \phi) = \int \psi_1 dR_1 + [\psi_2 + \psi_1 \phi_1] dR_2 + [\psi_2 + \psi_1 \phi_2 + (\psi_1 + w_1 \psi) \phi_1] dR_3 + \cdots,
\]
where the coefficient at \(dR^k\) is given by the explicit formula ([13], lemma 4.3):
\[
\text{Res}(\partial^{-1} \phi_1 (L^k)_{\geq 1} \psi \partial^{-1}), \quad 1 \leq k \leq n.
\]
Here \(\partial = \partial_{R^0}\),
\[
L = \partial + w_1 + \frac{1}{2} (w_2 - w_1 - w_1^2) \partial^{-1} + \cdots
\]
is the pseudo-differential Lax operator of the mKP hierarchy, \(L_{\geq 1}\) means the differential part of a pseudo-differential operator, and \(\ast\) denotes the adjoint operator. One has the identity
\[
\nabla \Omega(\psi, \phi) = \nabla \phi M_n(\psi)
\]
where \(M_n(\psi)\) is an upper-triangular Toeplitz matrix depending on the eigenfunction \(\psi\). Thus, \(\phi\) and \(\Omega = \Omega(\psi, \phi)\) can be interpreted as the density and the flux of the conservation law \(\phi_t = \Omega_x\) of the corresponding system
\[
R_t = M_n(\psi) R_x.
\]
Since this system possesses infinitely many conserved densities (parametrised by adjoint eigenfunctions \(\phi\)), it is hydrodynamically integrable, and therefore must possess infinitely many commuting flows as well.

**Example 2.** For the zero solution of mKP hierarchy, namely \(w = 0\), equations for the eigenfunction \(\psi\) become
\[
\psi_k = \partial^k \psi, \quad 1 \leq k \leq n.
\]
(25)
Taking linearly independent solutions \(\psi^j (j \geq 1)\) we obtain commuting systems
\[
R_j = M_n(\psi^j) R_x, \quad \lambda^k = \partial^k \psi^j, \quad 0 \leq k \leq n.
\]
(26)
In particular, polynomial solutions of (25),
\[
\psi^1 = R^1, \quad \psi^2 = R^2 + \frac{1}{2} (R^1)^2, \quad \psi^3 = R^3 + R^2 R^1 + \frac{1}{3!} (R^1)^3, \ldots
\]
lead to systems (26) considered in [6].

2.3. Generalised hodograph method

Solutions to system (3) can be obtained by the following recipe which is analogous to the generalised hodograph method of Tsarev [17]. Let
\[
R_t = \left( \mu^0 E + \sum_{i=1}^{n-1} \mu^i P^i \right) R_x
\]
be a commuting flow of system (3). Then the matrix equation
\[ \mu^0 E + \sum_{i=1}^{n-1} \mu^i P^i = Ex + \left( \lambda^0 E + \sum_{i=1}^{n-1} \lambda^i P^i \right) t \]
defines an implicit solution of (3). In components, this is equivalent to \( n \) implicit relations
\[ \mu^0 = x + \lambda^0 t, \quad \mu^1 = \lambda^1 t, \ldots, \quad \mu^n = \lambda^n t. \]

3. Jordan type reductions of linearly degenerate PDEs

It is remarkable that, although for ‘strongly nonlinear’ PDEs such as the dispersionless KP/Toda equations, the Jordan type reductions do not occur, they naturally arise in the context of multi-dimensional linearly degenerate PDEs (such as linearly degenerate systems of hydrodynamic type, Monge–Ampére equations, etc). Below we illustrate this phenomenon for the 3D Mikhailov system [12].

3.1. 3D Mikhailov system

Here we consider the system
\[ u_t = v_y + uv_x - vu_x, \quad v_t = u_y. \] (27)

Two-component hydrodynamic reductions (of Jordan block type) of system (27) are exact solutions of the form
\[ u = u(R^1, R^2), \quad v = v(R^1, R^2) \] (28)
where the variables \( R^1, R^2 \) satisfy a pair of commuting \( 2 \times 2 \) systems (3.1):
\[ \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x, \quad \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_y = \begin{pmatrix} \varphi & \varphi_1 \\ 0 & \varphi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x, \]
with
\[ \psi_2 = \psi_1 + \rho \psi_1, \quad \varphi_2 = \varphi_1 + \rho \varphi_1. \]

All such reductions can be described explicitly. Direct calculation shows that \( u \) and \( v \) must be polynomial in \( R^1 \) of degree 2 and 4, respectively:
\[ u = -\frac{1}{4} a(R^1)^2 + b R^1 + c, \quad v = -\frac{1}{2} u^2 + au + d, \]
where the coefficients \( a, b, c, d \) are functions of \( R^2 \) satisfying a single relation \( d' + b^2 + cd' = 0 \) (prime denotes differentiation by \( R^2 \)). The functions \( \varphi, \psi \) and \( \rho \) are expressed in terms of \( u \) by the formulae
\[ \varphi = -u + a, \quad \psi = -v + a \varphi, \quad \rho = \frac{2u_2 - a'}{2u_1}. \]

A particular choice \( a = 0, b = -1, c = 0, d = -R^2 \) leads to
\[ u = -R^1, \quad v = -R^2 - \frac{1}{2}(R^1)^2, \quad \varphi = R^1, \quad \psi = R^2 + \frac{1}{2}(R^1)^2, \quad \rho = 0. \]
the case considered in [14].

Three-component reductions of Jordan block type can be sought in the form

\[
R_t = \begin{pmatrix}
\psi & \psi_1 + w_1 \psi

0 & \psi_1

0 & 0
\end{pmatrix}
R_{x},
R_y = \begin{pmatrix}
\varphi & \varphi_1 + w_1 \varphi

0 & \varphi_1

0 & \varphi
\end{pmatrix}
R_{x},
\]

where \( w \) satisfies the mKP equation (19) and \( \psi, \varphi \) are two solutions of the corresponding Lax equation (21). In this case the formulae become more complicated. Direct calculation shows that \( u \) and \( v \) must be polynomial in \( R^1, R^2 \):

\[
u = \frac{1}{2} u^2 + au - \frac{4}{27} a''(R^2)^3 - b\gamma R^2 + \zeta,
\]

where \( b = \gamma - \frac{2}{3} a'dR^2 \) and \( a, \alpha, \beta, \gamma, \zeta \) are functions of \( R^1 \) with the condition \( \zeta' + a' \beta + \frac{3}{2} \alpha \gamma = 0 \). The functions \( \varphi, \psi \) and \( w \) are as follows:

\[
\varphi = -u + a, \quad \psi = -v + a\varphi,
\]

\[
w = -\frac{a'}{6b}(R^1)^2 + \frac{1}{18b} R^1 \left( 4a''(R^2)^3 - 12\gamma/R^2 - 9\alpha \right) + f,
\]

where \( f \) is a function of \( R^2 \) and \( R^3 \) which satisfies the equation

\[
\frac{1}{2} (3\gamma - 2a'dR^2)^2 f_2 = x^4(R^2)^4 + x^3(R^2)^3 + x^2(R^2)^2 + x^1(R^2) + 3x^0.
\quad (29)
\]

Here the coefficients \( x^i \) are functions of \( R^1 \) defined as

\[
x^1 \equiv \frac{8}{27} a''d' - \frac{2}{9} a'n^2, \quad x^3 \equiv \frac{4}{3} d''\gamma' - \frac{4}{9} a''\gamma - \frac{4}{3} \gamma''d',
\]

\[
x^2 \equiv \alpha a'' - 2\alpha'd' + 2\gamma''\gamma - 2\gamma'^2, \quad x^1 \equiv 3a'\gamma - 2\beta'a' - 3\alpha\gamma' + a^2,
\]

\[
x^0 \equiv \beta'\gamma - \frac{1}{2} a'\gamma - \frac{3}{8} a^2.
\]

If \( a' = 0 \), then (29) gives

\[
f = \frac{4}{27} (R^2)^3 \left( \frac{\gamma'}{\gamma} \right)' + \frac{1}{3} (R^2)^2 \left( \frac{\alpha}{\gamma} \right)' + R^2 \left( \frac{2\beta'}{3\gamma} \frac{\alpha^2}{4\gamma^2} \right) + \eta,
\]

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otherwise,
\[
    f = \frac{(R^2)^3}{6a^2} x^4 + \frac{(R^2)^2}{4a^3} (a' x^3 + 3\gamma x^4) + \frac{R^2}{8a^4} (4a'^2 x^2 + 12a' \gamma x^3 + 27\gamma^2 x^4)
\]
\[
    + \frac{1}{16a^5b} (16a'^2 x^0 + 8a'^3 \gamma x^1 + 12a'^2 \gamma^2 x^2 + 18a' \gamma^3 x^3 + 27\gamma^4 x^4)
\]
\[
    + \ln b \frac{\eta}{8a^3} (4a'^3 x^1 + 12a'^2 \gamma x^2 + 27a' \gamma^3 x^3 + 54\gamma^4 x^4) + \eta;
\]

here \( \eta \) is an extra arbitrary function of \( R^3 \).

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ORCID iDs

E V Ferapontov © https://orcid.org/0000-0001-9025-9478

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