GALOIS REPRESENTATIONS AND ORDINARY REDUCTION

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Abstract. We provide conditions on the \( p \)-adic Galois representation of a smooth proper variety over a complete nonarchimedean extension of \( \mathbf{Q}_p \) to have (potentially) good ordinary reduction.

1. Introduction

Let \( K/\mathbf{Q}_p \) be a finite extension with perfect residue field \( k \) of characteristic \( p \), and let \( C \) be a completion of an algebraic closure of \( K \). A smooth proper variety \( X_0/K \) has good reduction if there it admits a smooth proper model \( X_0 \) over \( \mathcal{O}_K \).

Let \( X = X_0 \otimes_{K} \overline{K} \). In this case:

1. The \( G_K \)-representation on \( H^n_\text{ét}(X; \mathbf{Q}_\ell) \) is unramified for all \( n \) and all primes \( \ell \neq p \); moreover,
2. The \( G_K \)-representation \( H^n_\text{ét}(X; \mathbf{Q}_p) \) is crystalline for all \( n \).

In many cases, there is a converse to this result. For instance, when \( X_0 \) is an abelian variety, the Néron-Ogg-Shafarevich theorem (see [ST68]) provides a converse: \( H^1_\text{ét}(X; \mathbf{Q}_\ell) \) is an unramified \( G_K \)-representation for one (hence all) primes \( \ell \neq p \) if and only if \( X_0 \) has good reduction. The \( p \)-adic analogue of this result was proved by Coleman–Iovita ([CI99, Theorem II.4.7]): \( X_0 \) has good reduction if and only if \( H^1_\text{ét}(X; \mathbf{Q}_p) \) is a crystalline \( p \)-adic \( G_K \)-representation. In [CLL17], Chiarellotto, Lazda, and Liedtke obtained analogous statements for certain K3 surfaces.

Let \( A \) be an abelian variety over \( K \) of dimension \( g \) with good reduction, so there exists a smooth proper model \( \mathcal{A} \) over \( \mathcal{O}_K \) whose generic fiber is \( K \). The special fiber \( \mathcal{A}_{sp} \) is an abelian variety over a perfect field of characteristic \( p \). The associated \( p \)-divisible group \( \mathcal{A}_{sp}[p^\infty] \) has height \( 2g \); however, its connected component \( \mathcal{A}_{sp}[p^\infty]^0 \) need not have the same height. The height of the formal group \( \mathcal{A}_{sp}[p^\infty]^0 \) provides a stratification of the moduli space of abelian varieties. The subset with maximal height (namely, \( g \)) is dense in this moduli space; in this case, the abelian variety has ordinary reduction.

In light of the above discussion, it is natural to ask if there is a Galois-theoretic criterion on the étale cohomology of \( A \otimes_K \overline{K} \) which ensures that \( A \) has ordinary reduction. The answer is yes:

**Theorem A.** Let \( A \) be an abelian variety with good reduction, and let \( V \) denote the \( p \)-adic \( G_K \)-representation \( H^n_\text{ét}(A \otimes_K \overline{K}; \mathbf{Q}_p) \). Then \( A \) is ordinary if and only if there is a complete filtration \( F^iV \) of \( V \) which splits, such that the inertia subgroup of \( G_K \) acts trivially on \( F^iV/F^{i+1}V \) for \( i > \dim A \), and by the \( p \)-adic cyclotomic character for \( i \leq \dim A \).

**Remark 1.1.** This result is probably well-known to the experts; however, we were unable to find a reference in the literature, so we include a proof in this paper.
Note that as ordinarity is a condition on the $p$-divisible group of $\mathcal{A}_{sp}$, one only expects such a criterion on the $p$-adic étale cohomology of $A \otimes K \overline{K}$, and not its $\ell$-adic étale cohomology.

Motivated by this result, one can ask if an analogue of Theorem A holds for smooth proper varieties which are not necessarily abelian varieties. In order to do this, one of course needs an appropriate definition of the term “ordinary” for general smooth proper varieties. This was provided by Bloch and Kato in [BK86].

Let $X$ be a smooth proper variety over $K$ with good reduction, and let $\mathcal{X}$ denote a smooth proper model for $X$ over $\mathcal{O}_K$.

**Definition 1.2.** Let $d\Omega^j_{X_{sp}}$ denote the sheaf of exact differentials on $X_{sp}$. We say that $X$ has ordinary reduction if $H^i(\mathcal{X}_{sp}, d\Omega^j_{X_{sp}})$ is 0 for all $i, j$.

In analogy with the condition imposed in Theorem A, we also make the following definition.

**Definition 1.3.** A $G_K$-representation $V$ is ordinary if there is a finite filtration by $G_K$-stable vector spaces $F_iV$ such that the inertia subgroup of $G_K$ acts on $F_iV/F_{i+1}V$ by some power $\chi^{n_i}$ of the cyclotomic character.

The main goal of this paper is to prove the following analogues of Theorem A:

**Theorem B.** Let $X_0$ be a smooth proper variety over $K$ with a smooth proper model $\mathcal{X}$ over $\mathcal{O}_K$ such that $H^\ast_{\text{cris}}(\mathcal{X}_{sp}/\mathcal{O}(k))$ and $H^\ast(\mathcal{X}_{sp}, W\Omega^\ast_{X_{sp}})$ are torsion-free. The étale cohomology $H^\ast_{\text{ét}}(X_{sp}; \mathbb{Q}_p)$ is an ordinary $G_K$-representation whose associated filtration (from Definition 1.3) splits for all $i$ if and only if $X_0$ has ordinary reduction.

**Theorem C.** Let $X_0$ be a K3 surface over $K$ with good reduction. The étale cohomology $H^2_{\text{ét}}(X; \mathbb{Q}_p)$ is an ordinary $G_K$-representation whose associated filtration (from Definition 1.3) splits if and only if $X_0$ has ordinary reduction.

**Remark 1.4.** Combined with [Mat14], we obtain conditions on the $G_K$-representation associated to a K3 surface guaranteeing potentially good ordinary reduction.

We will deduce these results from the following more general theorem.

**Theorem D.** Suppose $X_0$ is a smooth proper variety over $K$ with good reduction such that the special fiber of some model $\mathcal{X}$ of $X_0$ over $\mathcal{O}_K$ has torsion-free crystalline cohomology. If $H^n_{\text{ét}}(X; \mathbb{Q}_p)$ is an ordinary $G_K$-representation whose filtration (from Definition 1.3) splits, then the Newton polygon for $\mathcal{X}_{sp}$ has integer slopes.

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# 2. Background

## 2.1. Proving Theorem A

Our goal is to prove Theorem A which we will recall here for the reader’s convenience.

**Theorem A.** Let $A$ be an abelian variety with good reduction, and let $V$ denote the $p$-adic $G_K$-representation $H^n_{\text{ét}}(A \otimes K \overline{K}; \mathbb{Q}_p)$. Then $A$ is ordinary iff there is a
complete filtration $F^i V$ of which splits, such that the inertia subgroup of $G_K$ acts trivially on $F^i V / F^{i+1} V$ for $i > \dim A$, and by the $p$-adic cyclotomic character for $i \leq \dim A$.

\textbf{Proof.} In this case, we only have to consider the case $n = 1$. Assume that $A$ is ordinary. Since $A$ has good reduction, we lift it to a smooth proper model $\mathcal{A}$ over $\mathfrak{O}_K$. Let $\mathcal{A}[p^\infty]$ denote the $p$-divisible group of its special fiber. The connected-étale sequence runs

$$0 \to \mathcal{A}[p^\infty] \to \mathcal{A}[p^\infty] \to \mathcal{A}[p^\infty]^{\text{ét}} \to 0;$$

taking the rational $p$-adic Tate module gives

$$0 \to V_p(\mathcal{A}[p^\infty]^0) \to V_p(\mathcal{A}[p^\infty]) \to V_p(\mathcal{A}[p^\infty]^{\text{ét}}) \to 0.$$

The $G_K$-representation $\mathcal{V}_p(\mathcal{A}[p^\infty]^{\text{ét}})$ is unramified, while the inertia subgroup of $G_K$ acts on $V_p(\mathcal{A}[p^\infty])$ by a direct sum of copies of the $p$-adic cyclotomic character. By \cite{CLL17} Theorem 2.4, we know that as $G_K$-representations, we have

$$H^k_{\text{ét}}(A \otimes_K \overline{K}, \mathcal{Q}_p) \cong H^k_{\text{ét}}(\mathcal{A} \otimes_{\mathcal{O}_K} \overline{K}, \mathcal{Q}_p).$$

Note that $V_p(A) = H^1_{\text{ét}}(A \otimes_K \overline{K}, \mathcal{Q}_p)$, so $V_p(\mathcal{A}[p^\infty])$ can be identified with $V_p(A) =: V$ as $G_K$-representations. As $A$ is ordinary, we know that $\mathcal{A}[p^\infty]$ is an extension of an étale $p$-divisible group by a multiplicative $p$-divisible group; taking the rational Tate module of the resulting filtration on $\mathcal{A}[p^\infty]$ gives a filtration on $V$ (with $F^\dim AV = V_p(\mathcal{A}[p^\infty]^0)$ satisfying the conditions listed in Theorem A.

For the converse, suppose that the $p$-adic $G_K$-representation $H^1_{\text{ét}}(A \otimes_K \overline{K}, \mathcal{Q}_p) =: V$ satisfies the conditions in Theorem A. Then $V$ contains a $G_K$-stable $\mathcal{Z}_p$-lattice $\Lambda := H^1_{\text{ét}}(A \otimes_K \overline{K}, \mathcal{Z}_p)$. Under the correspondence of \cite{BC09} Proposition 7.2.2, this lattice comes from the $p$-divisible group $A[p^\infty]$. Since $A$ has good reduction, this can be lifted to a $p$-divisible group over $\mathfrak{O}_K$.

Let $W \subseteq V$ denote the $\dim A$-th step of the filtration on $V$. We get a $G_K$-stable $\mathcal{Z}_p$-lattice $W \cap \Lambda$ of $W$ which is contained inside $\Lambda$. Again referring to \cite{BC09} Proposition 7.2.2, we can find a $p$-divisible subgroup $\mathcal{G}$ of $A[p^\infty]$. This in turn can be lifted to a subgroup $\mathfrak{G}$ of the $p$-divisible group over $\mathfrak{O}_K$. By our assumptions on $V$, the associated $G_K$-representation itself admits a complete split filtration, with the inertia subgroup of $G_K$ acting on each quotient by the $p$-adic cyclotomic character. Thereby identifying $\mathfrak{G}$ with a multiplicative $p$-divisible group, we find that the special fiber $\mathfrak{G}_{sp}$ is an ordinary abelian variety, as desired. \hfill $\square$

2.2. Breuil–Kisin modules. In order to prove Theorem D we will need to recall some of the theory of Breuil–Kisin modules. Let $\mathfrak{S} = W(k)[[T]]$. There is a Frobenius $\varphi$ on $\mathfrak{S}$ defined by the usual Frobenius on $W(k)$ and the map $T \mapsto T^p$. The map $\mathfrak{S} \to \mathfrak{O}_K$ sending $T$ to a uniformizer $\pi$ has kernel generated by $E(T)$, the Eisenstein polynomial for $\pi$.

\textbf{Definition 2.1.} A Breuil–Kisin module $M$ is a $\mathfrak{S}$-module along with an isomorphism

$$\varphi_M : M \otimes_{\mathfrak{S}} \mathfrak{S} \left[ \frac{1}{E} \right] \simeq M \left[ \frac{1}{E} \right].$$

\textbf{Example 2.2} (Tate twists). Let $\mathfrak{S}(1)$ denote the Breuil–Kisin module with underlying $\mathfrak{S}$-module $\mathfrak{S}$ and Frobenius $\varphi_{\mathfrak{S}(1)}(x) = \frac{n}{\pi^i} \varphi(x)$, with $u$ some explicit unit in $\mathfrak{S}$. We write $\mathfrak{S}(n) = \mathfrak{S}(1)^{\otimes n}$. 


Remark 2.3. In analogy with Definition 1.3, we say that a Breuil–Kisin module $(M, \varphi_M)$ is ordinary if there is a filtration by submodules $(F^i M, \varphi_M |_{F^i M})$ such that each successive quotient is a finite free Breuil–Kisin module of rank 1.

We will need the following result (see [BMS16 Theorem 4.4]):

Theorem 2.4. There is a fully faithful tensor functor $M$ from $\mathbb{Z}_p$-lattices $\Lambda$ in crystalline $G_K$-representations $V$ to finite free Breuil–Kisin modules, characterized by the property that there is a $\varphi, G_{K_{\infty}}$-equivariant identification

$$M(\Lambda) \otimes_{\mathbb{S}} W(C^\flat) \simeq \Lambda \otimes_{\mathbb{Z}_p} W(C^\flat).$$

Example 2.5. We have $M(\mathbb{Z}_p(n)) \simeq \mathbb{S}\{n\}$.

The functor of Theorem 2.4 is not generally exact. However, we have:

Lemma 2.6. Let $V$ be an ordinary crystalline $G_K$-representation, and let $\Lambda$ be a $G_K$-stable $\mathbb{Z}_p$-lattice in $V$. Then $M(\Lambda)$ is an ordinary Breuil–Kisin module.

Proof. Let $F^i V$ be the filtration on $V$ (from Definition 1.3). Then $F^i V \cap \Lambda =: F^i \Lambda$ forms a filtration of $G_K$-stable $\mathbb{Z}_p$-lattices inside $\Lambda$. As $W(C^\flat)$ is torsion-free, the characterization of $M(\Lambda)$ from Theorem 2.4 proves that the rank of $M(\Lambda)$ as a $\mathbb{S}$-module is the rank of $\Lambda$ as a $\mathbb{Z}_p$-lattice. It follows that $M(F^i \Lambda)/M(F^{i+1} \Lambda)$ is a finite free Breuil–Kisin module of rank 1.

Remark 2.7. If the filtration on $V$ splits, then so does the filtration on $M(\Lambda)$. Indeed, there is a canonical map $M(F^i \Lambda)/M(F^{i+1} \Lambda) \to M(F^i \Lambda/F^{i+1} \Lambda)$; by functoriality, we get a map $M(F^i \Lambda/F^{i+1} \Lambda)$ which splits the exact sequence defining $M(F^i \Lambda)/M(F^{i+1} \Lambda)$.

3. The proof of Theorem D

We recall Theorem D for the reader’s convenience.

Theorem D. Suppose $X_0$ is a smooth proper variety over $K$ with good reduction such that the special fiber of some model $\mathcal{X}$ of $X_0$ over $\mathfrak{O}_K$ has torsion-free crystalline cohomology. If $H^r_{\text{ét}}(X; \mathbb{Q}_p)$ is an ordinary $G_K$-representation whose filtration splits, then the Newton polygon for $\mathcal{X}_{\text{sp}}$ has integer slopes.

Proof. Suppose that $X_0$ is as above, and that $V = H^r_{\text{ét}}(X; \mathbb{Q}_p)$. Since $H^r_{\text{cris}}(\mathcal{X}_{\text{sp}}/W(k))$ is torsion-free, we learn from [BMS16 Theorem 1.1(ii)] and [CLL17 Theorem 2.4] that $H^r_{\text{ét}}(X; \mathbb{Z}_p) =: T$ is a $G_K$-stable $\mathbb{Z}_p$-lattice inside $V$. The discussion preceding [BMS16 Theorem 1.4] implies that

$$M(T) \otimes_{\mathbb{S}} W(k) = H^r_{\text{cris}}(\mathcal{X}_{\text{sp}}/W(k))$$

(1)

Suppose $V$ is an ordinary $G_K$-representation whose filtration splits. Then by Lemma 2.6, Remark 2.7, and Equation (1), we find that $H^r_{\text{cris}}(\mathcal{X}_{\text{sp}}/W(k))$ is a $F$-crystal over $k$ which splits into a direct sum of rank one $F$-crystals. By the Dieudonné–Manin classification, rank one $F$-crystals are of the form $M_{r/\lambda} = W(k)\langle T \rangle/(T = p^\lambda)$ for $r \in \mathbb{Z}_{\geq 0}$. In particular, the Newton polygon for $\mathcal{X}_{\text{sp}}$ has integer slopes, as desired.

Remark 3.1. The slopes of the Hodge polygon of a smooth proper variety are always integers.
Proof of Theorem B. This follows from Theorem D and [BK86, Proposition 7.3(7)]. □

In order to prove Theorem C, we will need the following result from [Kat83]:

**Proposition 3.2.** Let \( X_0 \) be a smooth proper variety over \( K \) with good reduction. Let \( X \) be a lift of \( X_0 \) to \( \mathcal{O}_K \). Suppose that \( H^*_{\text{cris}}(X_{\text{sp}}/W(k)) \) is torsion-free. Then \( X_0 \) is ordinary if and only if the following conditions are satisfied:

- the Hodge–de-Rham spectral sequence \( E_1^{s,t} = H^t(X_{\text{sp}}; \Omega^s_{X_{\text{sp}}/k}) \Rightarrow H^{s+t}_{\text{dR}}(X_{\text{sp}}/k) \) collapses at the \( E_1 \)-page.
- the Newton and Hodge polygons of \( H^*_{\text{cris}}(X_{\text{sp}}/W(k)) \) coincide.

**Proof of Theorem C.** It is a general fact about K3 surfaces that \( H^*_{\text{cris}}(X_{\text{sp}}/W(k)) \) is torsion-free. By [Lie14, Proposition 2.5], the conditions of Theorem D are satisfied for every K3 surface having good reduction. By Theorem D, we find that the slopes of the Newton polygon of \( X_0 \) must be integers. Moreover, our hypotheses on \( X_0 \) imply that it cannot be a supersingular K3 surface. By the Artin–Mazur classification of the heights of K3 surfaces, we conclude that the slopes of the Newton polygon of \( X_0 \) can be integers if and only if the height of \( X_0 \) is 1; this implies that the Hodge and Newton polygons coincide. By Proposition 3.2, we conclude that \( X_0 \) has potentially good ordinary reduction, as desired. □

**Remark 3.3.** The same argument gives another proof of Theorem A.

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