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Refinement of the Jensen integral inequality

DOI 10.1515/math-2016-0020
Received September 7, 2015; accepted December 28, 2015.

Abstract: In this paper we give a refinement of Jensen’s integral inequality and its generalization for linear functionals. We also present some applications in Information Theory.

Keywords: Convex functions, Jensen’s inequality, f-divergences

MSC: 26D15, 94A17

1 Introduction

Let \( C \) be a convex subset of the linear space \( X \) and \( f \) be a convex function on \( C \). If \( p = (p_1, \ldots, p_n) \) is probability sequence and \( x = (x_1, \ldots, x_n) \in C^n \), then

\[
f \left( \sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i f(x_i)
\]

is well known in the literature as Jensen’s inequality.

The Lebesgue integral version of the Jensen inequality is given below:

**Theorem 1.1.** Let \((\Omega, \Lambda, \mu)\) be a measure space with \( 0 < \mu(\Omega) < \infty \) and let \( \phi : I \rightarrow \mathbb{R} \) be a convex function defined on an open interval \( I \) in \( \mathbb{R} \). If \( f : \Omega \rightarrow I \) is such that \( f, \phi \circ f \in L(\Omega, \Lambda, \mu) \), then

\[
\phi \left( \frac{1}{\mu(\Omega)} \int f d\mu \right) \leq \frac{1}{\mu(\Omega)} \int \phi(f) d\mu.
\]

In case when \( \phi \) is strictly convex on \( I \) one has equality in (2) if and only if \( f \) is constant almost everywhere on \( \Omega \).

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic mean-geometric mean inequality, the Hölder and Minkowski inequalities, the Ky Fan inequality etc. can be obtained as particular cases of it.

There is an extensive literature devoted to Jensen’s inequality concerning different generalizations, refinements, counterparts and converse results, see, for example [1–9].

In this paper we give a refinement of Jensen’s integral inequality and its generalization for linear functionals. We also present some applications in Information Theory for example for Kullback-Leibler, total variation and Karl Pearson \( \chi^2 \)-divergences etc.

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2 Main results

Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and \(L(\Omega, \Lambda, \mu) = \{ f : \Omega \to \mathbb{R} : f \text{ is } \mu \text{-measurable and } \int_\Omega f(t) d\mu(t) < \infty \}\) be a Lebesgue space. Consider the set \(\mathcal{S} = \{ \omega \in \Lambda : \mu(\omega) \neq 0 \text{ and } \mu(\Omega \setminus \omega) \neq 0 \}\) and \(\phi : (a, b) \to \mathbb{R}\) be a convex function defined on an open interval \((a, b)\). If \(f : \Omega \to (a, b)\) is such that \(f, \phi \circ f \in L(\Omega, \Lambda, \mu)\), then for any set \(\omega \in \mathcal{S}\), define the functional as

\[
F(\phi, f; \omega) = \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int_\omega f d\mu \right) + \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int f d\mu \right).
\]

We give the following refinement of Jensen's inequality.

**Theorem 2.1.** Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and let \(\phi : (a, b) \to \mathbb{R}\) be a convex function defined on an open interval \((a, b)\). If \(f : \Omega \to (a, b)\) is such that \(f, \phi \circ f \in L(\Omega, \Lambda, \mu)\), then for any set \(\omega \in \mathcal{S}\) we have

\[
\phi \left( \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right) \leq F(\phi, f; \omega) \leq \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu.
\]

**Proof.** As for any \(\omega \in \mathcal{S}\) we have

\[
\phi \left( \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right) = \phi \left[ \frac{\mu(\omega)}{\mu(\Omega)} \left( \frac{1}{\mu(\omega)} \int_\omega f d\mu \right) + \frac{\mu(\omega)}{\mu(\Omega)} \left( \frac{1}{\mu(\omega)} \int f d\mu \right) \right].
\]

Therefore by the convexity of the function \(\phi\) we get

\[
\phi \left( \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right) \leq \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int_\omega f d\mu \right) + \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int f d\mu \right) = F(\phi, f; \omega),
\]

Also for any \(\omega \in \mathcal{S}\) and by the Jensen inequality we have

\[
F(\phi, f; \omega) = \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int_\omega f d\mu \right) + \frac{\mu(\omega)}{\mu(\Omega)} \phi \left( \frac{1}{\mu(\omega)} \int f d\mu \right) \\
\leq \frac{1}{\mu(\Omega)} \int_\omega \phi(f) d\mu + \frac{1}{\mu(\Omega)} \int \phi(f) d\mu \\
= \frac{1}{\mu(\Omega)} \int \phi(f) d\mu.
\]

From (5) and (6) we have (4). \(\square\)

**Remark 2.2.** We observe that the inequality (4) can be written in an equivalent form as

\[
\inf_{\omega \in \mathcal{S}} F(\phi, f; \omega) \geq \phi \left( \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right)
\]

and

\[
\frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu \geq \sup_{\omega \in \mathcal{S}} F(\phi, f; \omega).
\]

**Remark 2.3.** If \(\emptyset, \Omega \in \mathcal{S}\) and if we take \(\omega = \emptyset\) or \(\omega = \Omega\), then we have \(F(\phi, f; \omega)\) is equal to the left hand side of (2). In this case (5) holds trivially.
Particularly Riemann integral version can be given as:

**Corollary 2.4.** Let \( \phi : [a, b] \to \mathbb{R} \) be a convex function defined on the interval \([a, b]\). If \( f : [c, d] \to [a, b] \), \( p : [c, d] \to \mathbb{R}^+ \) are such that \( f, fp \) and \( (\phi \circ f)p \) are all integrable on \([c, d]\), then we have

\[
\inf_{x \in (c, d)} \left[ \frac{x - c}{d - c} \phi \left( \frac{1}{x - c} \int_c^x p(t)f(t)dt \right) + \frac{d - x}{d - c} \phi \left( \frac{1}{d - x} \int_x^d p(t)f(t)dt \right) \right] \geq \phi \left( \frac{1}{d - c} \int_c^d p(t)f(t)dt \right).
\]

As a simple consequence of Theorem 2.1 we can obtain refinement of Hermite-Hadamard inequality:

**Corollary 2.5.** If \( \phi : [a, b] \to \mathbb{R} \) is a convex function defined on the interval \([a, b]\), then for any \([c, d] \subseteq [a, b]\) we have

\[
\phi \left( \frac{d + c}{2} \right) \leq \inf_{x \in [c, d]} \left[ \frac{x - c}{d - c} \phi \left( \frac{x + c}{2} \right) + \frac{d - x}{d - c} \phi \left( \frac{d + x}{2} \right) \right].
\]

\[
\frac{1}{d - c} \int_c^d \phi(t)dt \geq \sup_{x \in [c, d]} \left[ \frac{x - c}{d - c} \phi \left( \frac{x + c}{2} \right) + \frac{d - x}{d - c} \phi \left( \frac{d + x}{2} \right) \right].
\]

### 3 Further generalization

Let \( E \) be a nonempty set, \( \mathcal{A} \) be an algebra of subsets of \( E \), and \( L \) be a linear class of real-valued functions \( f : E \to \mathbb{R} \) having the properties:

- **L1**: \( f, g \in L \Rightarrow (\alpha f + \beta g) \in L \) for all \( \alpha, \beta \in \mathbb{R} \);
- **L2**: \( \mathbf{1} \in L \), i.e., if \( f(t) = 1 \) for all \( t \in E \), then \( f \in L \);
- **L3**: \( f \in L, E_1 \in \mathcal{A} \Rightarrow f \chi_{E_1} \in L \),

where \( \chi_{E_1} \) is the indicator function of \( E_1 \). It follows from \( L_2, L_3 \) that \( \chi_{E_1} \in L \) for every \( E_1 \in \mathcal{A} \).

A positive isotonic linear functional \( A : L \to \mathbb{R} \) is a functional satisfying the following properties:

- **A1**: \( A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \) for \( f, g \in L, \alpha, \beta \in \mathbb{R} \);
- **A2**: \( f \in L, f(t) \geq 0 \) on \( E \Rightarrow A(f) \geq 0 \);

It follows from \( L_3 \) that for every \( E_1 \in \mathcal{A} \) such that \( A(\chi_{E_1}) \geq 0 \), the functional \( A_{E_1} \) is defined for a fixed positive isotonic linear functional \( A \) as \( A_{E_1}(f) = \frac{A(f, \chi_{E_1})}{A(\chi_{E_1})} \), for all \( f \in L \), with \( A(\mathbf{1}) = 1 \). Furthermore, we observe that

\[
A(\chi_{E_1}) + A(\chi_{E_1} \setminus E_1) = 1,
\]

\[
A(f) = A(f, \chi_{E_1}) + A(f, \chi_{E_1} \setminus E_1). \tag{7}
\]

Jessen (see [10, p-47]) gave the following generalization of Jensen’s inequality for convex functions.

**Theorem 3.1.** Let \( L \) satisfy \( L_1 \) and \( L_2 \) on a nonempty set \( E \), and assume that \( \phi : [a, b] \to \mathbb{R} \) be a continuous convex function. If \( A \) is linear positive functional with \( A(\mathbf{1}) = 1 \), then for all \( f \in L \) such that \( \phi(f) \in L \) we have \( A(f) \in [a, b] \) and

\[
\phi(A(f)) \leq A(\phi(f)). \tag{8}
\]
The following refinement of (8) holds.

**Theorem 3.2.** Under the above assumptions, if $\phi : [a, b] \to \mathbb{R}$ is a continuous convex function, then

$$\phi(A(f)) \leq \overline{D}(A, f, \phi; E_1) \leq A(\phi(f));$$

(9)

where

$$\overline{D}(A, f, \phi; E_1) = A(\mathcal{X}_{E_1})\phi \left( \frac{A(f, \mathcal{X}_{E_1})}{A(\mathcal{X}_{E_1})} \right) + A(\mathcal{X}_{E\setminus E_1})\phi \left( \frac{A(f, \mathcal{X}_{E\setminus E_1})}{A(\mathcal{X}_{E\setminus E_1})} \right)$$

for all non empty set $E_1 \in \Omega$ such that $0 < A(\mathcal{X}_{E_1}) < 1$

**Proof.** Since

$$\overline{D}(A, f, \phi; E_1) = A(\mathcal{X}_{E_1})\phi \left( \frac{A(f, \mathcal{X}_{E_1})}{A(\mathcal{X}_{E_1})} \right) + A(\mathcal{X}_{E\setminus E_1})\phi \left( \frac{A(f, \mathcal{X}_{E\setminus E_1})}{A(\mathcal{X}_{E\setminus E_1})} \right)$$

i.e. $\overline{D}(A, f, \phi; E_1) = A(\mathcal{X}_{E_1})\phi(A_{E_1}(f)) + A(\mathcal{X}_{E\setminus E_1})\phi(A_{E\setminus E_1}(f))$.

Using the inequality (8) we obtain

$$\overline{D}(A, f, \phi; E_1) \leq A(\phi(f); \mathcal{X}_{E_1}) + A(\phi(f); \mathcal{X}_{E\setminus E_1}) = A(\phi(f)).$$

(10)

This proves the second inequality in (9).

The first inequality follows by using definition of convex function and identity (7).

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4 Applications for Csiszár divergence measures

Let $(\Omega, \Lambda, \mu)$ be a probability measure space. Consider the set of all density functions on $\mu$ to be $S := \{ p | p : \Omega \to \mathbb{R}, p(s) > 0, \int_\Omega p(s)d\mu(s) = 1 \}$.

Csiszár introduced the concept of $f$-divergence for a convex function $f : (0, \infty) \to (-\infty, \infty)$ (cf. [11], see also [12]) by

$$I_f(q, p) = \int_{\Omega} p(s) f\left( \frac{q(s)}{p(s)} \right) d\mu(s), \; p, q \in S.$$  

By appropriately defining the convex function $f$, various divergences can be derived. We give some important $f$-divergences, playing a significant role in Information Theory and Statistics.

(i) The class of $\chi$-divergences: The $f$-divergences, in this class, are generated by the family of functions

$$f_\alpha(u) = |u - 1|^{\alpha}, \; u \geq 0 \text{ and } \alpha \geq 1.$$  

$$I_{f_\alpha}(q, p) = \int_{\Omega} p^{1-\alpha}(s) |q(s) - p(s)|^{\alpha} d\mu(s).$$

For $\alpha = 1$, it gives the total variation distance,

$$V(q, p) = \int_{\Omega} |q(s) - p(s)| d\mu(s).$$

For $\alpha = 2$, it gives the Karl Pearson $\chi^2$-divergence,

$$I_{\chi^2}(q, p) = \int_{\Omega} \frac{[q(s) - p(s)]^2}{p(s)} \; d\mu(s).$$

(ii) $\alpha$-order Rényi entropy : For $\alpha > 1$ let

$$f(t) = t^\alpha, \; t > 0.$$
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Then $I_f$ gives $\alpha$-order entropy

$$D_\alpha(q, p) = \int_{\Omega} q^\alpha(s) \ p^{1-\alpha}(s) \, d\mu(s).$$

(iii) Harmonic distance: Let

$$f(t) = -\frac{2t}{1 + t}, \quad t > 0.$$  

Then $I_f$ gives Harmonic distance

$$D_H(q, p) = \int_{\Omega} \frac{2 \ p(s) q(s)}{p(s) + q(s)} \, d\mu(s).$$

(iv) Kullback-Leibler: Let

$$f(t) = t \ \log t, \quad t > 0.$$  

Then $f$-divergence functional give rise to Kullback-Leibler distance [13]

$$D_{KL}(q, p) = \int_{\Omega} q(s) \ \log \left( \frac{q(s)}{p(s)} \right) \, d\mu(s).$$

One parametric generalization of the Kullback-Leibler [13] relative information was studied in a different way by Cressie and Read [14].

(v) Jeffreys divergence: Let

$$f(t) = \frac{1}{2} \log t, \quad t > 0.$$  

Then $f$-divergence functional give Jeffreys divergence

$$J(q, p) = \int_{\omega} \frac{1}{2} \log \left( \frac{p(s)}{q(s)} \right) \, d\mu(s).$$

(vi) The Dichotomy class: This class is generated by the family of functions $g_\alpha : [0, \infty) \to \mathbb{R},$

$$g_\alpha(u) = \begin{cases} 
    u - 1 - \log u, & \alpha = 0 \\
    \frac{1}{\alpha(1-\alpha)} \left[ \alpha u + 1 - \alpha - u^\alpha \right], & \alpha \in \mathbb{R}\setminus\{0, 1\}; \\
    1 - u + u \log u, & \alpha = 1.
\end{cases}$$  \hspace{1cm} (11)

This class gives, for particular values of $\alpha$, some important divergences. For instance, for $\alpha = \frac{1}{2}$ it provides a distance, namely, the Hellinger distance.

There are various other divergences in Information Theory and Statistics such as Arimoto-type divergences, Matusita’s divergence, Puri-Vince divergences etc. (cf. [15], [16]) used in various problems in Information Theory and statistics. An application of Theorem 1.1 is the following result given by Csiszár and Korner (cf. [17]).

**Theorem 4.1.** Let $f : [0, \infty) \to \mathbb{R}$ be a convex function and $p, q$ be positive functions from $S$. Then the following inequality is valid,

$$I_f(q, p) \geq f(1).$$  \hspace{1cm} (12)

**Theorem 4.2.** Let $f : [0, \infty) \to \mathbb{R}$ be a convex function, then for any $p$ and $q$ in $S$ we have:

$$I_f(q, p) \geq \mu(\omega) \ f \left( \frac{1}{\mu(\omega)} \int_{\omega} q(s) \, d\mu(s) \right) + \mu(\omega) \ f \left( \frac{1}{\mu(\omega)} \int_{\omega} q(s) \, d\mu(s) \right) \geq f(1).$$  \hspace{1cm} (13)

**Proof.** By substituting $\phi(s) = f(s)$, $f(s) = \frac{q(s)}{p(s)}$ and $d\mu(s) = p(s) \, d\mu(s)$ in Theorem 2.1, we deduce (13). \hfill \Box
Proposition 4.3. Let $p, q \in S$, then we have
\[ V(q, p) \geq 2 \sup_{\omega \in \mathcal{S}} \left| \int q(s) d\mu(s) - \mu(\omega) \right| \geq 0. \] (14)

Proof. By putting $f(x) = |x - 1|$ for all $x \geq 0$ in Theorem 4.2 we get (14).

Proposition 4.4. For any $p, q \in S$,
\[ I_{\chi^2}(q, p) \geq \sup_{\omega \in \mathcal{S}} \left\{ \left( \frac{\int_{\omega} q(s) d\mu(s) - \mu(\omega)}{\mu(\omega)(1 - \mu(\omega))} \right)^2 \right\} \geq 4 \sup_{\omega \in \mathcal{S}} \left\{ \left( \int_{\omega} q(s) d\mu(s) - \mu(\omega) \right)^2 \right\} \geq 0. \] (15)

Proof. By making use of the function $f(x) = (t - 1)^2$ in Theorem 4.2 we get
\[ \int_{\Omega} p(s) \left( \frac{q(s)}{p(s)} - 1 \right)^2 d\mu(s) \geq \sup_{\omega \in \mathcal{S}} \left\{ \mu(\omega) \left( \frac{1}{\mu(\omega)} \int_{\omega} q(s) d\mu(s) - 1 \right)^2 + \mu(\omega) \left( \frac{1}{\mu(\omega)} \int q(s) d\mu(s) - 1 \right)^2 \right\} \geq 0. \]

Since by Arithmetic-Geometric mean inequality we have
\[ \mu(\omega)(1 - \mu(\omega)) \leq \frac{1}{4} \left( \mu(\omega) + (1 - \mu(\omega)) \right)^2 = \frac{1}{4}, \]
we have
\[ \left( \frac{\int_{\omega} q(s) d\mu(s) - \mu(\omega)}{\mu(\omega)(1 - \mu(\omega))} \right) \geq 4 \left( \int_{\omega} q(s) d\mu(s) - \mu(\omega) \right)^2 \geq 0. \]

Proposition 4.5. For any $p, q \in S$, we have:
\[ D_{KL}(q, p) \geq \ln \left[ \left( \frac{1 - \int_{\omega} q(s) d\mu(s)}{1 - \mu(\omega)} \right)^{1 - \int_{\omega} q(s) d\mu(s)} \cdot \left( \frac{\int_{\omega} q(s) d\mu(s)}{\mu(\omega)} \right)^{\int_{\omega} q(s) d\mu(s)} \right] \geq 0. \] (16)

Proof. By putting $f(t) = t \ln(t)$ in Theorem 4.2 one can get first inequality in (16).

To prove the second inequality, we utilize the inequality between the geometric mean and harmonic mean,
\[ x^\alpha y^{1-\alpha} \geq \frac{1}{\frac{x}{\alpha} + \frac{y}{1-\alpha}}, \quad x, y, \alpha \in [0, 1]. \]

we have for
\[ x = \int_{\omega} q(s) d\mu(s), \quad y = \frac{1 - \int_{\omega} q(s) d\mu(s)}{1 - \mu(\omega)} \quad \text{and} \quad \alpha = \int_{\omega} q(s) d\mu(s) \quad \text{that} \]
\[ \left( \frac{1 - \int_{\omega} q(s) d\mu(s)}{1 - \mu(\omega)} \right)^{1 - \int_{\omega} q(s) d\mu(s)} \cdot \left( \frac{\int_{\omega} q(s) d\mu(s)}{\mu(\omega)} \right)^{\int_{\omega} q(s) d\mu(s)} \geq 1, \]
for any $\omega \in \mathcal{S}$, which implies the second inequality in (16).
Proposition 4.6. For any $p, q \in S$, we have:

$$J(q, p) \geq \ln \left( \sup_{\omega \in \mathcal{G}} \left\{ \left[ \frac{(1 - \int_{\omega} q(s)d\mu(s))\mu(\omega)}{(1 - \mu(\omega))\int_{\omega} q(s)d\mu(s)} \right]^{(\mu(\omega) - \int_{\omega} q(s)d\mu(s))} \right\} \right)$$

\[ \geq \sup_{\omega \in \mathcal{G}} \left( \frac{(\mu(\omega) - \int_{\omega} q(s)d\mu(s))^{2}}{\int_{\omega} q(s)d\mu(s) + \mu(\omega) - 2\int_{\omega} q(s)d\mu(s)\mu(\omega)} \right) \geq 0. \tag{17} \]

Proof. By putting $f(x) = (x - 1)\ln(x), x > 0$ in Theorem 4.2 we have

\[ \int_{\omega} p(s) \left( \frac{q(s)}{p(s)} - 1 \right) \ln \left( \frac{q(s)}{p(s)} \right) d\mu(s) \geq \sup_{\omega \in \mathcal{G}} \left( \mu(\omega) \left( \int_{\omega} q(s)d\mu(s) \right) - 1 \right) \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \]

\[ + \mu(\tilde{\omega}) \left( \int_{\tilde{\omega}} q(s)d\mu(s) - 1 \right) \ln \left( \frac{\int_{\tilde{\omega}} q(s)d\mu(s)}{\mu(\tilde{\omega})} \right) \]

\[ = \sup_{\omega \in \mathcal{G}} \left( \left( \int_{\omega} q(s)d\mu(s) - \mu(\omega) \right) \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \right. \]

\[ + \left. \left( \int_{\omega} q(s)d\mu(s) - \mu(\tilde{\omega}) \right) \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\tilde{\omega})} \right) \right) \]

that is

$$J(q, p) \geq \sup_{\omega \in \mathcal{G}} \left( \left( \mu(\omega) - \int_{\omega} q(s)d\mu(s) \right) \ln \left( \frac{1 - \int_{\omega} q(s)d\mu(s)}{1 - \mu(\omega)} \right) \right. \]

\[ - \left. \left( \mu(\omega) - \int_{\omega} q(s)d\mu(s) \right) \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \right) \]

proving the first inequality in (17).

Utilizing the elementary inequality for positive numbers, 

$$\ln b - \ln a \geq \frac{2}{a + b}, \quad a, b > 0$$

we have

$$\left( \mu(\omega) - \int_{\omega} q(s)d\mu(s) \right) \left[ \ln \left( \frac{1 - \int_{\omega} q(s)d\mu(s)}{1 - \mu(\omega)} \right) \right. \]

\[ - \left. \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \right] \]

\[ = \left( \mu(\omega) - \int_{\omega} q(s)d\mu(s) \right) \ln \left( \frac{1 - \int_{\omega} q(s)d\mu(s)}{1 - \mu(\omega)} \right) \]

\[ - \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \]

\[ = \frac{\mu(\omega) - \int_{\omega} q(s)d\mu(s)}{\mu(\omega)(1 - \mu(\omega))} \bigg[ 1 - \int_{\omega} q(s)d\mu(s) \bigg] \ln \left( \frac{1 - \int_{\omega} q(s)d\mu(s)}{1 - \mu(\omega)} \right) \]

\[ - \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \]

\[ \geq \frac{\mu(\omega) - \int_{\omega} q(s)d\mu(s)}{\mu(\omega)(1 - \mu(\omega))} \bigg[ 1 - \int_{\omega} q(s)d\mu(s) \bigg] \left( 1 - \int_{\omega} q(s)d\mu(s) \right) \ln \left( \frac{1 - \int_{\omega} q(s)d\mu(s)}{1 - \mu(\omega)} \right) \]

\[ - \ln \left( \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \right) \]

\[ = \frac{(\mu(\omega) - \int_{\omega} q(s)d\mu(s))^{2}}{\mu(\omega)(1 - \mu(\omega))} \left( 1 - \int_{\omega} q(s)d\mu(s) \right) + \frac{\int_{\omega} q(s)d\mu(s)}{\mu(\omega)} \]

\[ = \frac{2(\mu(\omega) - \int_{\omega} q(s)d\mu(s))^{2}}{\int_{\omega} q(s)d\mu(s) + \mu(\omega) - 2\int_{\omega} q(s)d\mu(s)\mu(\omega)} \geq 0, \]

for each $\omega \in \mathcal{G}$, giving the second inequality in (17).
Proposition 4.7. For any \( p, q \in S \), we have:

\[
D_\alpha(q, p) \geq \sup_{\omega \in \Theta} \left[ (\mu(\omega))^{1-\alpha} \left( \int_\Theta q(s)d\mu(s) \right)^\alpha + (1 - \mu(\omega))^{1-\alpha} \left( 1 - \int_\Theta q(s)d\mu(s) \right)^\alpha \right] \geq 1. \quad (18)
\]

Proof. By putting \( f(x) = x^\alpha \) for \( \alpha > 1, \ x > 0 \), in Theorem 4.2 we get the required inequalities. \( \square \)

Acknowledgement: The authors express their sincere thanks to the referees for their careful reading of the manuscript and very helpful suggestions that improved the manuscript.

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