Extended Legendrian Dualities Theorem in Singularity Theory

Haiming Liu *†,† and Jiajing Miao †

School of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, China; jiajing0407@126.com
* Correspondence: haiming0626@126.com
† These authors contributed equally to this work.

Abstract: In this paper, we find some new information on Legendrian dualities and extend them to the case of Legendrian dualities for continuous families of pseudo-spheres in general semi-Euclidean space. In particular, we construct all contact diffeomorphic mappings between the contact manifolds and display them in a table that contains all information about Legendrian dualities.

Keywords: Legendrian dualities; Legendrian singularity theory; semi-Euclidean space

MSC: 53A35; 58C25

1. Introduction

Singularity theory is a young branch of analysis that currently occupies a central place in mathematics. It is a wide-ranging generalization of the theory of the minima and maxima of functions and a descendant of differential calculus. Whitney firstly noticed this field and Mather set its foundations by the actions of group theory, which is a powerful tool to study symmetry [1,2]. Arnold and Zakalyukin developed the theory of singularity from the viewpoints of symplectic geometry and contact geometry [3]. Since most of the important properties of submanifolds are characterized and distinguished by their singularities, it is important to deal with singularities. However, there are many difficulties in doing this, because the usual mathematical tools fail at the singularity. With the rapid development of singularity theory, it has acted as a microscope to observe the geometric and topological properties of submanifolds near the singularities. Many mathematicians have contributed to this field, including Thom, Porteous, Bruce, Giblin, Izumiya, Romer Fuster, Tari, Pei and Chen, etc. [2,3]. They have considered geometric and topological properties caustics and wavefronts, including evolute, parallel curve, pedal curve, symmetry sets, Gauss map, focal surface, parallel surface, umbilic, foliations, etc. There are two typical applications of singularity theory in symmetry. The first one is that Gutsu gave the notion of the simple symmetric function of germs at a critical point, which are reducible to normal forms by the action of the group of symmetric diffeomorphisms in [3]. The normal forms of the simple symmetric function germs are classified in [3], where more details on many questions of the theory of symmetric critical points can be found. Another interesting application of singularity to symmetry is the singularities of symmetry sets. A symmetric set of a curve (respectively, surface) arises as the locus of centers of circles (respectively, spheres), which have contact with the curve (respectively, surface) in two places. A local version of a symmetric set can be found in [4], where the authors give many examples and trace symmetric sets by using a computer. The main results indicate that there are many simple singularities in symmetric sets, which can be detected by the powerful tool of the theory of singularity. For example, the symmetric sets of quartic curves consist of many cusps. It is incredible that these quartic curves are not closed, but the parts of their symmetric sets would not be affected by closing them up. Hereafter, we focus on the Legendrian duality theorem, which is one of the most important results in singularity theory. It has been an important tool to study the geometric properties of degenerate submanifolds. In this paper,
we find some new information on Legendrian dualities and extend them to the case of Legendrian dualities for continuous families of pseudo-spheres in general semi-Euclidean space.

In 2007, Izumiya showed four Legendrian dualities between pseudo-spheres in Minkowski space [5]. Then, he and his coauthors extended them to the cases of semi-Euclidean spaces with general index [6], one-parameter families of pseudo-spheres in Lorentz–Minkowski space [7], and the spherical Legendrian duality [8]. It is well known that Legendrian dualities provide a new way of constructing frames from the viewpoints of contact geometry and Legendrian singularity theory, which have been widely used for studying the geometric properties of curves, surfaces and other submanifolds with singularities in Euclidean and pseudo-Euclidean spaces. These dualities have become a core tool for studying the geometric and topological properties of submanifolds with singularities. Some typical applications are the research of submanifolds with singularities. In this paper, we extend the theorem of Legendrian dualities to the case of Legendrian dualities for continuous families of pseudo-spheres in general semi-Euclidean space and display them in a Table 1 on Legendrian dualities. In particular, we calculate some new information. We construct all contact diffeomorphisms among the contact manifolds and display them in a Table 1 on Legendrian dualities. In particular, we calculate some new information. We construct all contact diffeomorphisms among the contact manifolds and display them in a Table 1 on Legendrian dualities. In particular, we calculate some new information. We construct all contact diffeomorphisms among the contact manifolds and display them in a Table 1 on Legendrian dualities. In particular, we calculate some new information. We construct all contact diffeomorphisms among the contact manifolds and display them in a Table 1 on Legendrian dualities. In particular, we calculate some new information. We construct all contact diffeomorphisms among the contact manifolds and display them in a Table 1 on Legendrian dualities.
Table 1. A table on Legendrian dualities.

| $\Delta_1$ | $\Delta_2^+$ | $\Delta_2^-$ | $\Delta_3^+$ | $\Delta_3^-$ | $\Delta_4^+$ | $\Delta_4^-$ | $\Delta_5^+(\theta)$ | $\Delta_5^-(\theta)$ | $\Delta_6^+(\theta)$ | $\Delta_6^-(\theta)$ | $\Delta_7^+(\theta)$ | $\Delta_7^-(\theta)$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|----------------|----------------|----------------|----------------|-------------|
| $\text{id}$ | $\mathcal{C}_{12}^+$ | $\mathcal{C}_{13}^+$ | $\mathcal{C}_{14}^+$ | $\mathcal{C}_{15}^+$ | $\mathcal{C}_{12}^-$ | $\mathcal{C}_{13}^-$ | $\mathcal{C}_{14}^-$ | $\mathcal{C}_{15}^-$ | $\mathcal{C}_{16}^+$ | $\mathcal{C}_{16}^-$ | $\mathcal{C}_{17}^+$ | $\mathcal{C}_{17}^-$ | $\mathcal{C}_{18}^+$ | $\mathcal{C}_{18}^-$ |

In Section 2, we give some basic concepts. In Section 3, we extend the theorem of Legendrian dualities to the case of Legendrian dualities for continuous families of pseudospheres in general semi-Euclidean space. To do this, we construct some new contact diffeomorphisms among the contact manifolds. In Section 4, we give two applications of the extended Legendrian dualities theorem. In Section 5, we summarize this paper.

2. Basic Notions on Legendrian Dualities in Semi-Euclidean Space

In this section, we introduce some basic notions of Legendrian dualities in semi-Euclidean $(n + 1)$-space with index $r$. Let $\mathbb{R}^{n+1} = \{ (a_1, a_2, \cdots, a_{n+1}) \mid a_1 \in \mathbb{R}, \tau = 1, 2, \ldots, n+1 \}$ be an $(n + 1)$-dimensional vector space. For any vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$ in $\mathbb{R}^{n+1}$, the pseudo scalar product of $\alpha$ and $\beta$ is defined by

$$\langle \alpha, \beta \rangle_r = - \sum_{r=1}^{n+1} a_{r} \beta_{r} + \sum_{r=1}^{n+1} a_{r} \beta_{r}. $$

We call $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_r)$ a semi-Euclidean $(n + 1)$-space with index $r$ and denote it by $\mathbb{R}^{n+1}_r$. A non-zero vector $a \in \mathbb{R}^{n+1}_r$ is called spacelike, null or timelike if $\langle a, a \rangle_r > 0$, $\langle a, a \rangle_r = 0$ or $\langle a, a \rangle_r < 0$, respectively. The norm of the vector $v \in \mathbb{R}^{n+1}_r$ is defined by $\| v \| = \sqrt{\langle v, v \rangle_r}$. We now define the $n$-hyperbolic space with index $r - 1$ by $\mathbb{H}^{n}_{r-1}(-c^2) = \{ v \in \mathbb{R}^{n+1}_r \mid \langle v, v \rangle_r = -c^2 \}$, a unit pseudo $n$-sphere with index $r$ by $S^n_r(c^2) = \{ v \in \mathbb{R}^{n+1}_r \mid \langle v, v \rangle_r = c^2 \}$; and the open nullcone by $\Lambda^n = \{ v \in \mathbb{R}^{n+1}_r \setminus \{0\} \mid \langle v, v \rangle_r = 0 \}$ for any real number $c$. We also need some basic notions of contact geometry. A $(2n+1)$-dimensional manifold $E$ with a contact structure $K$ is called a contact manifold and is denoted by $(E, K)$. Let $\mathcal{L}$ be an $n$-dimensional submanifold of $E$; if the tangent space of $\mathcal{L}$ at any point $p$ is a subspace of $K_p$, we call $\mathcal{L}$ a Legendrian submanifold. If the fibers of the fiber bundle $\pi : E \rightarrow N$ are Legendrian submanifolds of $E$, we call $\pi$ a Legendrian fibration.
For our purpose, we should consider the following extended Legendrian dualities in general semi-Euclidean space.

\[(1) \mathbb{H}^n_{\pi -1}(-1) \times S^n_r(1) \ni \Delta_1 = \{(n, b) \mid \langle n, b \rangle_r = 0\},\]

\[\langle \eta \rangle_{11} = \langle \langle \pi \rangle_{11} \rangle \in \text{Ext}_{11}, \eta_{12} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{12}.\]

\[(2) \mathbb{H}^n_{\pi -1}(-1) \times \Lambda^n \ni \Delta_2 = \{(n, b) \mid \langle n, b \rangle_r = \pm 1\},\]

\[\langle \eta \rangle_{21} = \langle \langle \pi \rangle_{21} \rangle \in \text{Ext}_{21}, \eta_{22} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{22}.\]

\[(3) \mathbb{H}^n_{\pi -1}(-1) \times S^n_r(1) \ni \Delta_3 = \{(n, b) \mid \langle n, b \rangle_r = \pm 1\},\]

\[\langle \eta \rangle_{31} = \langle \langle \pi \rangle_{31} \rangle \in \text{Ext}_{31}, \eta_{32} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{32}.\]

\[(4) \Lambda^n \times \Lambda^n \ni \Delta_4 = \{(n, b) \mid \langle n, b \rangle_r = \pm 2\},\]

\[\langle \eta \rangle_{41} = \langle \langle \pi \rangle_{41} \rangle \in \text{Ext}_{41}, \eta_{42} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{42}.\]

\[(5) \mathbb{H}^n_{\pi -1}(-1) \times S^n_r(\cos^2 \theta) \ni \Delta_5 = \{(n, b) \mid \langle n, b \rangle_r = \pm \sin \theta\},\]

\[\langle \eta \rangle_{51} = \langle \langle \pi \rangle_{51} \rangle \in \text{Ext}_{51}, \eta_{52} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{52}.\]

\[(6) \mathbb{H}^n_{\pi -1}(-1) \times S^n_r(\sin^2 \theta) \ni \Delta_6 = \{(n, b) \mid \langle n, b \rangle_r = \pm 2\sin \theta\},\]

\[\langle \eta \rangle_{61} = \langle \langle \pi \rangle_{61} \rangle \in \text{Ext}_{61}, \eta_{62} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{62}.\]

\[(7) \Lambda^n \times \Lambda^n \ni \Delta_7 = \{(n, b) \mid \langle n, b \rangle_r = \pm (\sin \theta + \cos \theta)\},\]

\[\langle \eta \rangle_{71} = \langle \langle \pi \rangle_{71} \rangle \in \text{Ext}_{71}, \eta_{72} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{72}.\]

\[(8) \Lambda^n \times \Lambda^n \ni \Delta_8 = \{(n, b) \mid \langle n, b \rangle_r = \pm (\sin \theta + 1)\},\]

\[\langle \eta \rangle_{81} = \langle \langle \pi \rangle_{81} \rangle \in \text{Ext}_{81}, \eta_{82} = \langle \langle n, d \rangle \rangle \in \text{Ext}_{82}.\]

where

\[\pi_{11}(n, b) = n, \pi_{12}(n, b) = b, \pi_{11}^+(n, b) = n, \pi_{12}^+(n, b) = b(\alpha = 2, 3, 4),\]

\[\langle \langle n, b \rangle \rangle_r = -\sum_{i=1}^{r} b_i d n_i + \sum_{i=r+1}^{\alpha+1} b_i d n_i, \langle n, d \rangle_r = -\sum_{i=1}^{r} n_i d b_i + \sum_{i=r+1}^{\alpha+1} n_i d b_i\]

are 1-forms on \(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\). It is easy to know that \(\eta_{11}^{-1}(0)\) and \(\eta_{12}^{-1}(0)\) define the same tangent hyperplane on \(\Delta_1\), denoted by \(K_1\). For the same reason, \((\eta_{21}^{-1})^{-1}(0)\) and \((\eta_{22}^{-1})^{-1}(0)\) de-
fine the same tangent hyperplane on $\Delta^\pm_\beta$, denoted by $K^\pm_\beta(\alpha = 2, 3, 4)$. Note that $\pi|_{(\alpha\beta)^1}(n, b) = (n, \pi[\theta]_{(\alpha\beta)^1}(n, b) = b(\eta[\theta]_{(\alpha\beta)^1}^{-1}(0))$ and $(\eta[\theta]_{(\alpha\beta)^2})^{-1}(0)$ also define the same tangent hyperplane $K[\theta]_{(\alpha\beta)^1}(\alpha, \beta = 1, 2, 3, 4; \alpha < \beta)$.

3. Extended Legendrian Dualities Theorem

In this section, we extend the theorem of Legendrian dualities to the case of Legendrian dualities for continuous families of pseudo-spheres in general semi-Euclidean space. We can obtain the following extended theorem.

Theorem 1. $(\Delta_1, K_1), (\Delta_2, K_2), (\Delta_3, K_3), (\Delta_4, K_4), (\Delta_{12}, K_{12}), (\Delta_{13}, K_{13}), (\Delta_{14}, K_{14}), (\Delta_{23}, K_{23}), (\Delta_{24}, K_{24}), (\Delta_{34}, K_{34})$ are contact manifolds such that $\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}, \pi_{123}, \pi_{124}, \pi_{134}, \pi_{234}$ are contact fibrations. Moreover, the above manifolds are contact diffeomorphic to each other.

Proof. First, we consider $(\Delta_1, K_1), (\Delta_2, K_2), (\Delta_3, K_3)(\alpha = 2, 3, 4)$. By definition, we can show that $\Delta_1$ and $\Delta_{\alpha 1}^\pm, \alpha = 2, 3, 4$, are smooth submanifolds in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, and all of $\pi_{12}$ and $\pi_{12}, \beta = 1, 2$, are smooth Legendrian fibrations. It also follows from the definition of $\theta_{\alpha\beta}$ that each fiber of $\pi_{\alpha\beta}$ is an integral submanifold of $K^\pm_\beta, \alpha = 1, 2, 3, 4$. In [6], it was shown that $(\Delta_1, \eta_{11}^1(0))$ is a contact manifold. We need to prove that $(\Delta_{\alpha 1}^\pm, K_{\alpha 1}^\pm, \alpha = 2, 3, 4$, are contact manifolds. To do this, we can prove that they are contact diffeomorphic to $(\Delta_1, \eta_{11}^1(0))$. We construct smooth mappings $L^\pm_{1a}: \Delta_1 \rightarrow \Delta^\pm_\alpha$ and their inverse mappings $L^\pm_{a1}: \Delta^\pm_\alpha \rightarrow \Delta_1$ as follows:

$$L^\pm_{12}(n, b) = (n, \pm n + b), \quad L^\pm_{23}(n, b) = (n, \mp n + b),$$
$$L^\pm_{13}(n, b) = (n, \mp b), \quad L^\pm_{31}(n, b) = (n, \mp b),$$
$$L^\pm_{14}(n, b) = (n, \pm b, \mp n + b), \quad L^\pm_{41}(n, b) = (1/2)(n \mp b, \pm n + b).$$

We can also construct smooth mappings $L^\pm_{a b}: \Delta^\pm_a \rightarrow \Delta^\pm_b$ and their inverse mappings $L^\pm_{b a}: \Delta^\pm_b \rightarrow \Delta^\pm_a, \alpha, \beta = 2, 3, 4$, as follows:

$$L^\pm_{23}(n, b) = (n, \mp b, \mp n + b), \quad L^\pm_{32}(n, b) = (n, \pm b, \mp n),$$
$$L^\pm_{24}(n, b) = (n, \mp 2n + b), \quad L^\pm_{42}(n, b) = (1/2)(n \mp b, \pm n),$$
$$L^\pm_{34}(n, b) = (n, \mp n + 2b), \quad L^\pm_{43}(n, b) = (1/2)(\pm n + b).$$

Therefore, they are diffeomorphisms. We should check that they are contact diffeomorphisms. Taking $L^\pm_{13}$, for example, we have

$$L^\pm_{13} \eta^\pm_{31}(n, b) = \eta^\pm_{31}(n, \pm b, b),$$
$$= (d(n \pm b), b)|_{\Delta_1}$$
$$= \langle d(n, b)|_{\Delta_1}, \pm \langle db, w) \rangle_{\Delta_1}$$
$$= \langle d(n, b) \rangle_{\Delta_1}$$
$$= \eta_{11}.$$

This indicates that each of $(\Delta_{31}^\pm, K_{31}^\pm)$ is a contact manifold and each of $L^\pm_{13}$ is a contact diffeomorphism. By similar calculations, $(\Delta_{a 1}^\pm, K_{a 1}^\pm, \alpha = 2, 4$, are contact manifolds.
Second, we consider \((\Delta_1, K_1)\) and \((\Delta^\pm_{n\beta}(\theta), K^\pm_{(\theta)_{(n\beta)})}(a, \beta = 1, 2, 3, 4; \alpha < \beta)\). We can construct \(L^\pm_{1(a\beta)} : \Delta^\pm_{n\beta}(\theta) \mapsto \Delta^\pm_{n\beta}(\theta)\) and their converse mappings \(L^\pm_{1(a\beta)} : \Delta^\pm_{n\beta}(\theta) \mapsto \Delta_1\) with \(dL^\pm_{(12)}(K^\pm_{(\theta)_{(12)})} = K_1\). For any \((n, b) \in \Delta_1\), we have

\[
\langle \mp \sin \theta b + b, \mp \sin \theta n + b \rangle_r = -\sin^2 \theta + 1 = \cos^2 \theta
\]

and

\[
(n, \mp \sin \theta n + b, \mp \sin \theta n + b)_r = \pm \sin \theta.
\]

Therefore, we find that \(L^\pm_{1(12)}(\Delta_1) \subset \Delta^\pm_{12}(\theta)\). For any \((n, b) \in \Delta^\pm_{12}(\theta)\), we also have

\[
\langle \mp \sin \theta n + b, \mp \sin \theta n + b \rangle_r = -\sin^2 \theta + 2\sin^2 \theta + \cos^2 \theta = 1
\]

and \((n, \pm \sin \theta n + b)_r = \mp \sin \theta \pm \sin \theta = 0\). Therefore, we have \(L^\pm_{1(12)}(\Delta^\pm_{12}(\theta)) \subset \Delta_1\). Thus, we find that \(L^\pm_{1(12)} \circ L^\pm_{1(12)} \mid_{\Delta^\pm_{12}(\theta)} = \text{id}_{\Delta^\pm_{12}(\theta)}\) and \(L^\pm_{1(12)} \circ L^\pm_{1(12)} \mid_{\Delta_1} = \text{id}_{\Delta_1}\). We also have

\[
(L^\pm_{1(12)})_\theta \mid_{11} = \langle dn, \pm \sin \theta n + b \rangle_r \mid_{\Delta^\pm_{12}(\theta)} = \langle dn, b \rangle_r \mid_{\Delta^\pm_{12}(\theta)} = \eta \mid_{(12)(1)1}\n
\]

Therefore, \(K^\pm_{(\theta)_{(12)}}\) is a contact structure on \(\Delta^\pm_{12}(\theta)\) such that \(L^\pm_{1(12)}\) is a contact diffeomorphism. Then, for other cases, we consider smooth mappings \(L^\pm_{1(a\beta)} : \Delta^\pm_1 \mapsto \Delta^\pm_{n\beta}(\theta)\) and their converse mappings \(L^\pm_{1(a\beta)} : \Delta^\pm_{n\beta}(\theta) \mapsto \Delta_1\). Moreover, we have to consider some mappings \(L^\pm_{1(a\beta)}(\gamma\delta) : \Delta^\pm_{n\beta} \mapsto \Delta^\pm_{r\delta} (\alpha, \beta, \gamma, \delta = 1, 2, 3, 4; \alpha < \beta, \gamma < \delta)\).

We only prove that \(L^\pm_{1(12)(14)}\) is a contact diffeomorphism as an example. For any \((n, b) \in \Delta^\pm_{12}(\theta)\), we have

\[
L^\pm_{1(12)(14)} \eta \mid_{(12)(14)}(n, b) = \eta \mid_{(12)(14)}(1 + \sin^2 \theta) n \pm \sin \theta b, b
= \langle d(1 + \sin^2 \theta) n \pm \sin \theta b, b \rangle_r \mid_{\Delta^\pm_{12}}
= (1 + \sin^2 \theta) \langle dn, b \rangle_r \mid_{\Delta^\pm_{12}} \pm \sin \theta \langle db, b \rangle_r \mid_{\Delta^\pm_{12}}
= (1 + \sin^2 \theta) \langle dn, b \rangle_r \mid_{\Delta^\pm_{12}} \pm \sin \theta \langle db, b \rangle_r \mid_{\Delta^\pm_{12}}
= (1 + \sin^2 \theta) \eta \mid_{(12)(14)}.
\]

This means that \(\Delta^\pm \mid_{1(14)}, K^\pm_{(\theta)_{(14)}}\) is a contact manifold such that \(L^\pm_{1(12)(14)}\) is a contact diffeomorphism. Finally, for other cases, we have a similar calculation, so that \(\Delta^\pm_{1(14)}(\theta), K^\pm_{(\theta)_{(14)}}(\alpha = 1, 2, 3; \beta = 2, 3, 4; \alpha < \beta)\) are contact manifolds. We remark that one has to construct the expressions of the contact diffeomorphisms \(L^\pm_{1(a\beta)} : \Delta_1 \mapsto \Delta^\pm_{n\beta}(\theta)\) and their converse mappings \(L^\pm_{1(a\beta)} : \Delta^\pm_{n\beta}(\theta) \mapsto \Delta_1\) as follows:

\[
L^\pm_{1(12)}(n, b) = (n, \mp \sin \theta n + b),
L^\pm_{1(12)}(n, b) = (n, \mp \sin \theta n + b),
L^\pm_{1(13)}(n, b) = (n, \mp \sin \theta b, b),
L^\pm_{1(13)}(n, b) = (n, \mp \sin \theta b, b),
L^\pm_{1(14)}(n, b) = (n \pm \sin \theta b, \mp \sin \theta n + b),
L^\pm_{1(14)}(n, b) = (n \pm \sin \theta b, \mp \sin \theta n + b),
L^\pm_{1(23)}(n, b) = (n \pm \sin \theta b, \mp \sin \theta n + b),
L^\pm_{1(23)}(n, b) = (n \pm \sin \theta b, \mp \sin \theta n + b),
\]
Moreover, we have to construct the expressions of $L_{1(24)} \pm (n, b) \rightarrow (n \pm \sin \theta, n \pm \theta b)$, $L_{1(24)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm n + b)$, $L_{1(34)} \pm (n, b) = (n \pm b, \mp \sin \theta n + b)$, and $L_{1(34)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm n + b)$.

In addition, we need to construct the expressions of $L_{2(\alpha\beta)} : \Delta_{2} \pm \rightarrow \Delta_{2} \pm (\theta)$ and their converse mappings $L_{3(\alpha\beta)} : \Delta_{3} \pm (\theta) \rightarrow \Delta_{3} \pm$. They are contact diffeomorphisms and are denoted, respectively, by

$L_{2(12)} \pm (n, b) = (n, \mp(1 + \sin \theta)n + b)$,
$L_{2(12)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm n + b)$,
$L_{2(13)} \pm (n, b) = ([1 - \sin \theta]n + w, \mp n + b)$,
$L_{2(13)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm n + \mp (1 - \sin \theta)b)$,
$L_{2(14)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm 1 \mp (1 + \sin \theta)n + b)$,
$L_{2(14)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \mp n + \mp (1 - \sin \theta)b)$,
$L_{2(22)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm 2n + b)$,
$L_{2(22)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \mp n + b)$,
$L_{2(34)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \pm n + b)$,
$L_{2(34)} \pm (n, b) = \frac{1}{\pm \sin \theta} (n \mp \sin \theta b, \mp n + (1 + \sin \theta)n)$.

Furthermore, we also need to construct the expressions of $L_{3(\alpha\beta)} : \Delta_{3} \pm \rightarrow \Delta_{3} \pm (\alpha\beta)$ and their converse mappings $L_{3(\alpha\beta)} : \Delta_{3} \pm (\theta) \rightarrow \Delta_{3} \pm$. They are contact diffeomorphisms and are denoted, respectively, by

$L_{3(12)} \pm (n, b) = [n \mp b, \mp \sin \theta n + (1 + \sin \theta)b]$,
$L_{3(12)} \pm (n, b) = [1 + \sin \theta]n \mp b, \mp \sin \theta n + b]$,
$L_{3(13)} \pm (n, b) = [n \mp (1 - \sin \theta)b, b, b]$,
$L_{3(13)} \pm (n, b) = [n \mp (1 - \sin \theta)b, b, b]$,
$L_{3(14)} \pm (n, b) = [n \mp (1 - \sin \theta)b, \mp \sin \theta n + (1 + \sin \theta)b]$,
$L_{3(14)} \pm (n, b) = \frac{1}{\pm \sin \theta} ([1 + \sin \theta]n \pm (1 - \sin \theta)b, \mp \sin \theta n + b]$,
$L_{3(23)} \pm (n, b) = [n \mp (1 - \sin \theta)b, b, b)$,
$L_{3(23)} \pm (n, b) = [n \mp (1 - \sin \theta)b, b, b)$,
$L_{3(24)} \pm (n, b) = [n \mp (1 - \sin \theta)b, \mp n + b]$,
$L_{3(24)} \pm (n, b) = [n \mp (1 - \sin \theta)b, \mp n + b]$,
$L_{3(34)} \pm (n, b) = [n, \mp \sin \theta n + (1 + \sin \theta)b]$,
$L_{3(34)} \pm (n, b) = [n, \mp \sin \theta n + (1 + \sin \theta)b]$.

Moreover, we have to construct the expressions of $L_{4(\alpha\beta)} : \Delta_{4} \pm \rightarrow \Delta_{4} \pm (\alpha\beta)$ and their converse mappings $L_{4(\alpha\beta)} : \Delta_{4} \pm (\theta) \rightarrow \Delta_{4} \pm$. They are contact diffeomorphisms and are denoted, respectively, by

$L_{4(12)} \pm (n, b) = ([1/2] (n \mp b), \pm (1/2) (1 - \sin \theta) n + (1 + \sin \theta)b]$,
$L_{4(12)} \pm (n, b) = (1 + \sin \theta) n \mp b, \mp (1 - \sin \theta) n + b]$,
$L_{4(13)} \pm (n, b) = [(1/2) (1 + \sin \theta)n - (1/2) (1 \mp \sin \theta) b, (1/2) (\mp n + b)]$,
$L_{4(13)} \pm (n, b) = [n \mp (1 + \sin \theta)b, \mp n + (1 + \sin \theta)b]$,
$L_{4(14)} \pm (n, b) = (1/2) [1 \mp \sin \theta]n \mp (1 - \sin \theta)b, \pm (1 \mp \sin \theta) n + (1 + \sin \theta)b]$,
\[ \mathcal{L}_{(14)}^{\pm}(n, b) = \frac{1}{\sin^2 \theta + 1} \left[ (1 + \sin \theta) n \pm (1 - \sin \theta) b, \mp (1 - \sin \theta) n + (1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(23)}^{\pm}(n, b) = \frac{1}{2} \left[ (1 + \sin \theta) n \mp (1 - \sin \theta) b, \pm (1 - \cos \theta) n \pm (1 + \cos \theta) b \right], \]
\[ \mathcal{L}_{(23,4)}^{\pm}(n, b) = \frac{1}{\sin \theta \cos \theta} \left[ (1 + \cos \theta) n \pm (1 - \sin \theta) b, \mp (1 - \cos \theta) n \pm (1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(24)}^{\pm}(n, b) = \left[ (1/2)(1 + \sin \theta) n \mp (1/2)(1 - \sin \theta) b, b \right], \]
\[ \mathcal{L}_{(24,34)}^{\pm}(n, b) = \frac{1}{\sin \theta \cos \theta} \left[ 2n \pm (1 - \sin \theta) b, (1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(34)}^{\pm}(n, b) = \left[ n, \pm (1/2)(1 - \sin \theta) n + (1/2)(1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(34,2)}^{\pm}(n, b) = \frac{1}{\sin \theta \cos \theta} \left[ (1 + \sin \theta) n, \mp (1 - \sin \theta) n \mp 2b \right]. \]

Last but not least, we construct the expressions of \( \mathcal{L}_{(a_1 a_2)}^{\pm}(\theta) \): \( \Delta_{n_1 \delta_1}^{\pm}(\theta) \rightarrow \Delta_{n_2 \delta_2}^{\pm}(\theta) \) and their converse mappings

\[ \mathcal{L}_{(a_1 a_2)}^{\pm}(\theta) : \Delta_{n_1 \delta_1}^{\pm}(\theta) \rightarrow \Delta_{n_2 \delta_2}^{\pm}(\theta) \quad (a, b, \gamma, \delta = 1, 2, 3, 4, a < \beta, \gamma < \delta). \]

They are contact diffeomorphisms and are denoted, respectively, by

\[ \mathcal{L}_{(12)}^{\pm}(n, b) = \left[ (1 + \sin^2 \theta) n \mp \sin \theta b, b \right], \]
\[ \mathcal{L}_{(14)}^{\pm}(n, b) = \left[ (1/1 + \sin^2 \theta)(n \mp \sin \theta b), b \right], \]
\[ \mathcal{L}_{(12)}^{\pm}(n, b) = \left[ (1 + \sin^2 \theta) n \mp \sin \theta b \pm \sin \theta n + b \right], \]
\[ \mathcal{L}_{(14)}^{\pm}(n, b) = \left[ (1/1 + \sin^2 \theta)(n \mp \sin \theta b) \mp \sin \theta n + b \right], \]
\[ \mathcal{L}_{(23)}^{\pm}(n, b) = \left[ (1 + \sin^2 \theta) n \pm \sin \theta b \pm (\sin \theta - \cos \theta) n + b \right], \]
\[ \mathcal{L}_{(23)}^{\pm}(n, b) = \left[ (1 + \sin^2 \theta) n \pm \sin \theta b, (\mp \sin \theta \pm \cos \theta) n + (1 + \sin^2 \theta) b \right], \]
\[ \mathcal{L}_{(24)}^{\pm}(n, b) = \left[ (1 + \sin^2 \theta) n \mp \sin \theta b, (\pm \sin \theta \mp 1) n + b \right], \]
\[ \mathcal{L}_{(24,34)}^{\pm}(n, b) = \left[ (1 + \sin \theta) n \pm \sin \theta b \pm b \right], \]
\[ \mathcal{L}_{(34,2)}^{\pm}(n, b) = \left[ (1/1 + \sin \theta) n \mp \sin \theta b, b \right], \]
\[ \mathcal{L}_{(13)}^{\pm}(n, b) = \left[ (1/1 + \sin^2 \theta) (\pm \sin \theta n + b) \right], \]
\[ \mathcal{L}_{(13)}^{\pm}(n, b) = \left[ n, (1/1 + \sin^2 \theta) (\pm \sin \theta n + b) \right], \]
\[ \mathcal{L}_{(13)}^{\pm}(n, b) = \left[ n, \mp \cos \theta n + (1 + \sin \theta \cos \theta) b \right], \]
\[ \mathcal{L}_{(13)}^{\pm}(n, b) = \left[ (n, 1/1 + \sin \theta \cos \theta) (\pm \cos \theta n + b) \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \mp n + (1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \mp n + (1 + \sin \theta) b \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \pm b (1 - \sin \theta), \mp n \sin \theta + (1 + \sin^2 \theta) b \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \pm b (1 - \sin \theta), \mp n \sin \theta + (1 + \sin^2 \theta) b \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \pm b (1 - \sin \theta), \mp n \sin \theta + (1 + \sin^2 \theta) b \right], \]
\[ \mathcal{L}_{(13,2)}^{\pm}(n, b) = \left[ n \pm b (1 - \sin \theta), \mp n \sin \theta + (1 + \sin^2 \theta) b \right], \]
where \( n = (1 + \cos \theta) \mathbf{n} + \mathbf{b}(\mp \sin \theta \pm 1) \), \( W = (\mp \sin \theta \pm \cos \theta) \mathbf{n} + (1 + \sin^2 \theta) \mathbf{b} \), and \( \text{id} \) is an identity map. This completes the proof. \( \square \)

4. Applications

In this section, we give two applications of the extended Legendrian dualities theorem. We focus on an open nullcone defined by

\[
\Lambda^n = \{ x \in \mathbb{R}^{n+1} \setminus \{ 0 \} \mid \langle x, x \rangle = 0 \}.
\]

It is well known that one of the difficulties in the study of the Lorentzian hypersurface in an open nullcone is that it is impossible to obtain the normal vector of the hypersurface from its tangent space by using a pseudo-wedge operation since the induced metric on the open nullcone is degenerate. As the first application of the extended Legendrian dualities, we will construct one of the most important normal vectors by the extended Legendrian duality theorem in order to solve this question. Furthermore, we construct the nullcone Gaussian image, the anti-de Sitter Gaussian image and the pseudo-sphere Gaussian image of the Lorentzian hypersurface in the open nullcone by the extended Legendrian duality theorem. One of our results (cf. Proposition 1) indicates that there are three kinds of totally umbilic hypersurfaces in the nullcone. A naturally interesting question is whether there are relationships among these totally umbilic hypersurfaces. As the second application of the extended Legendrian dualities, we will establish the relations among these totally umbilic hypersurfaces.

Let \( \mathcal{X}^t : U \rightarrow \Lambda^n \) be a timelike embedding for an open subset \( U \subset \mathbb{R}^{n-1} \). We denote that \( M = \mathcal{X}^t(U) \) and identify \( M \) with \( U \) through the embedding \( \mathcal{X}^t \). \( M = \mathcal{X}^t(U) \) is called a Lorentzian hypersurface. The metric on the open nullcone is degenerate, so that we cannot construct the normal vector of the Lorentzian hypersurface by using a pseudo-wedge operation. To deal with this difficulty, we employ the extended Legendrian duality theorem. We consider the \( \Lambda^n \times \Lambda^n \supset \Delta_4 \) duality and define Legendrian embedding

\[
\mathcal{L}_4 : U \rightarrow \Delta_4, \quad \mathcal{L}_4(u) = (\mathcal{X}(u), \mathcal{X}^n(u)).
\]

One can check that \( \langle d\mathcal{X}(u), \mathcal{X}^n(u) \rangle = 0 \). This indicates that \( \mathcal{X}^n(u) \) lies on the normal space \( N_p M = \mathcal{X}(U) \) at \( p = \mathcal{X}(u) \). Since \( N_p M \) is locally isomorphic to the Lorentzian plane and \( \langle \mathcal{X}(u), \mathcal{X}^n(u) \rangle = -2 \), \( \mathcal{L}_4 \) is the unique Legendrian lift of \( M = \mathcal{X}(U) \). We call \( \mathcal{X}^n(u) \) the nullcone normal vector field of Lorentzian hypersurface \( M = \mathcal{X}(U) \) at \( \mathcal{X}(u) \).

We consider the diffeomorphism \( \mathcal{L}_{41} : \Delta_4^- \rightarrow \Delta_1 \), with \( \mathcal{L}_{41} : (v, w) = \left( \frac{v+w}{2}, \frac{v-w}{2} \right) \).

One can obtain a Legendrian submanifold \( \mathcal{L}_1 : U \rightarrow \Delta_1 \) by \( \mathcal{L}_1(u) = \mathcal{L}_{41} \circ \mathcal{L}_4(u) \). In particular, let \( \mathcal{L}_1(u) = (\mathcal{X}^t(u), \mathcal{X}^s(u)) \), then

\[
\mathcal{X}^t(u) = \frac{\mathcal{X}(u) + \mathcal{X}^n(u)}{2}, \quad \mathcal{X}^s(u) = \frac{\mathcal{X}(u) - \mathcal{X}^n(u)}{2}.
\]

We call \( \mathcal{X}^t(u) \) and \( \mathcal{X}^s(u) \) anti-de Sitter normal vector field and pseudo-sphere normal vector field of Lorentzian hypersurface \( M = \mathcal{X}(U) \) at \( \mathcal{X}(u) \), respectively. Since \( \mathcal{X}(u) \) and \( \mathcal{X}^n(u) \) are linearly independent null vectors and \( \mathcal{X}(u) \) is a Lorentzian hypersurface, we obtain a basis

\[
\mathcal{X}(u), \mathcal{X}^n(u), \mathcal{X}_{t1}(u), \ldots, \mathcal{X}_{tn-1}(u)
\]

of \( T_p \mathbb{R}^{n+1}_2 \), where \( p = \mathcal{X}(u) \). We call \( \mathcal{X}^n : U \rightarrow \Lambda^n \) the nullcone Gaussian image, \( \mathcal{X}^t : U \rightarrow H^1_1(-1) \) the anti-de Sitter Gaussian image, and \( \mathcal{X}^s : U \rightarrow S^2_2(1) \) the pseudo-sphere Gaussian image of the Lorentzian hypersurface \( M = \mathcal{X}(U) \). One can define the following three linear transformations, which are shape operators. We, respectively, call \( S^n_p[\theta](u) = -d\mathcal{X}^n(u) : T_p M \rightarrow T_p M \) the nullcone shape operator, \( S^t_p[\theta](u) = -d\mathcal{X}^t(u) : T_p M \rightarrow T_p M \) the anti-de Sitter shape operator, and \( S^s_p[\theta](u) = -d\mathcal{X}^s(u) : T_p M \rightarrow T_p M \) the pseudo-sphere shape operator, where \( p = \mathcal{X}(u) \).
We, respectively, denote the eigenvalues of $S^n_p[\theta](u)$ by $k_n[\theta](p)$, $S^n_p[\theta](u)$ by $k_1[\theta](p)$ and $S^n_p[\theta](u)$ by $k_3[\theta](p)$, which we call null principal curvature, anti-de Sitter principal curvature and pseudo-sphere principal curvature. One can check that

$$k_1[\theta](p) = \frac{k_n[\theta](p) - 1}{2}, \quad k_3[\theta](p) = -\frac{k_n[\theta](p) - 1}{2}.$$ 

We, respectively, define nullcone Gaussian curvature by $K_n[\theta](u_0) = det S^n_p[\theta](u_0)$, anti-de Sitter Gaussian curvature by $K_1[\theta](u_0) = det S^n_p[\theta](u_0)$, and pseudo-sphere Gaussian curvature by $K_2[\theta](u_0) = det S^n_p[\theta](u_0)$ at $p_0 = X(u_0)$. If $K_n[\theta](u) = 0$, then $p = X(u)$ is called a nullcone parabolic point. If $S^n[\theta](p) = k_n[\theta]id_{F,M}$, we call $p = X(u)$ a nullcone umbilic point. A hypersurface $M$ is called a totally nullcone umbilic hypersurface if every point on $M$ is nullcone umbilic. We define a hypersurface by $NH(n,c) = \Lambda^n \cap HP(n,c)$, which can be taken as the model of the totally nullcone umbilic hypersurface in the nullcone. We summarize the classifications of totally nullcone umbilic hypersurfaces in the nullcone in Table 2.

**Proposition 2.** Suppose that $M = X(U)$ is a totally nullcone umbilic hypersurface with constant $k_n[\theta](p) = k$ in the nullcone; then, one can obtain the following classifications in Table 2.

**Table 2.** Classifications of totally nullcone umbilic hypersurfaces in nullcone.

| Conditions | Constant Normal Vector | Classifications |
|-----------|------------------------|-----------------|
| $k_n[\theta] < 0$ | $n = \frac{1}{\sqrt{-k_n[\theta]}}(k_n[\theta]X(u) + X^n(u)) \in S^n_2$ | $M$ is a subset of $NH(n,1/\sqrt{-k_n[\theta]})$ |
| $k_n[\theta] = 0$ | $n = X^n(u) \in \Lambda^n$ | $M$ is a subset of $NH(n,-1)$ |
| $k_n[\theta] > 0$ | $n = \frac{1}{\sqrt{k_n[\theta]}}(k_n[\theta]X(u) + X^n(u)) \in H^n_1$ | $M$ is a subset of $NH(n,-1/\sqrt{k_n[\theta]})$ |

Proposition 1 indicates that there are three kinds of totally umbilic hypersurfaces in the nullcone. A natural question is how to establish the relations among these totally umbilic hypersurfaces. As the second application of extended Legendrian dualities, we try to solve this question. We define $\Delta_{43}[\theta] = \Delta_{34}[\frac{1}{2} - \theta], K_{43}[\theta] = K_{34}[\frac{1}{2} - \theta], \pi[\theta]_{(43)1} = \pi[\frac{1}{2} - \theta]_{(34)1}$, where $i = 1, 2$. In particular, we consider the following double fibration:

$$(10^*) (a) \Lambda^n \times S^n_2(\sin^2 \theta) \supset \Delta_{43}[\theta] = \{ (v, w) \mid (v, w) = -((\cos \theta + 1)) \},$$

$$(b) \pi[\theta]_{(43)1} : \Delta_{43}[\theta] \longrightarrow \Lambda^n, \quad \pi[\theta]_{(43)2} : \Delta_{43}[\theta] \longrightarrow S^n_2(\sin^2 \theta),$$

$$(c) \eta[\theta]_{(43)1} = \langle dv, w \rangle_{\Delta_{43}[\theta]}, \eta[\theta]_{(43)2} = \langle v, dw \rangle_{\Delta_{43}[\theta]}.$$ 

By Theorem 1, $(\Delta_{43}[\theta], K[\theta]_{43})$ is a contact manifold. One can prove that

$$L_{4(43)}^- : \Delta_{4}^- \rightarrow \Delta_{43}[\theta], \quad L_{4(43)}^+(v, w) = \left( v, \frac{1}{2}((\cos \theta - 1)v + (\cos \theta + 1)w) \right)$$

is a contact diffeomorphism. We define a map $N^p_n[\theta] : U \rightarrow S^n_2(\sin^2 \theta)$ by

$$N^p_n[\theta](u) = \frac{1}{2}((\cos \theta - 1)X(u) + (\cos \theta + 1)X^n(u)).$$

Furthermore, we define

$$L_{43}[\theta] : U \rightarrow \Delta_{43}[\theta] \subset \Lambda^n \times S^n_2(\sin^2 \theta)$$

by $\mathcal{L}_{43}[\theta](\mathbf{u}) = (\mathbf{X}(\mathbf{u}), \mathbf{N}_n^\theta(\mathbf{u})))$. Since $\mathcal{L}_{43}[\theta](\mathbf{u}) = \mathcal{L}_{41/(43)} \circ \mathcal{L}_4(\mathbf{u})$, $\mathcal{L}_{43}[\theta](\mathbf{u})$ is a Legendrian embedding. Therefore,

$$
(\mathbf{d}\mathbf{X}(\mathbf{u}), \mathbf{N}_n^\theta(\mathbf{u})) = \mathcal{L}_{43}[\theta](\mathbf{u})^*\eta[\mathbf{Y}]_{(43)} = 0.
$$

This means that $\mathbf{N}_n^\theta(\mathbf{u})$ is a normal vector of $M = \mathbf{X}(\mathbf{U})$ at $\mathbf{p} = \mathbf{X}(\mathbf{u})$. We call it a $\theta$-pseudo sphere Gaussian map. One can obtain the following relations among these Gaussian maps in Table 3.

Table 3. Relations among Gaussian maps of Lorentzian hypersurfaces in nullcone.

| Conditions | Relations among Gaussian Maps |
|------------|-------------------------------|
| $\theta = 0$ | $\mathbf{N}_n^\theta(\mathbf{u}) = \mathbf{X}(\mathbf{u})$ |
| $\theta \in (0, \frac{\pi}{2})$ | $\mathbf{N}_n^\theta(\mathbf{u}) = \frac{1}{4}((\cos \theta - 1)\mathbf{X}(\mathbf{u}) + (\cos \theta + 1)\mathbf{X}^n(\mathbf{u}))$. |
| $\theta = \frac{\pi}{2}$ | $\mathbf{N}_n^\theta(\mathbf{u}) = \mathbf{X}^n(\mathbf{u})$ |

We define a new model hypersurface in the nullcone by

$$
\mathbf{N}(\mathbf{n}, -(\cos \theta + 1)) = \Lambda^n \cap HP(\mathbf{n}(-(\cos \theta + 1))
$$

The following proposition indicates that there is a new kind of interesting geometry where $\mathbf{N}(\mathbf{n}, -(\cos \theta + 1))$ can be seen as a totally umbilic hypersurface in the nullcone.

**Proposition 3.** Let $\mathbf{X} : \mathbf{U} \longrightarrow \Lambda^n$ be a Lorentzian hypersurface, and then $\mathbf{N}_n^\theta(\mathbf{u})$ is a constant vector if and only if $\mathbf{X}(\mathbf{U})$ is a subset of $\mathbf{N}(\mathbf{n}, -(\cos \theta + 1))$, where $\mathbf{n} \in S_2^\theta(\sin^2 \theta)$.

**Proof.** If $\mathbf{N}_n^\theta(\mathbf{u}) = \mathbf{n}$ is a constant vector, then

$$
\langle \mathbf{X}(\mathbf{u}), \mathbf{n} \rangle = \langle \mathbf{X}(\mathbf{u}), \mathbf{N}_n^\theta(\mathbf{u}) \rangle = -(\cos \theta + 1).
$$

This means that

$$
\mathbf{X}(\mathbf{U}) \subset \mathbf{N}(\mathbf{n}, -(\cos \theta + 1)).
$$

Therefore, $M$ is a subset of $\mathbf{N}(\mathbf{n}, -(\cos \theta + 1))$. Conversely, if

$$
M \subset \mathbf{N}(\mathbf{n}, -(\cos \theta + 1)),
$$

where $\mathbf{n} \in S_2^\theta(\sin^2 \theta)$. Since $\mathbf{n}$ is a normal vector of $M$, there are real numbers $a$ and $b$ such that $\mathbf{n} = a\mathbf{X}(\mathbf{u}) + b\mathbf{X}^n(\mathbf{u})$ and $\sin^2 \theta = -4ab$. By definition, we obtain

$$
-(\cos \theta + 1) = \langle \mathbf{X}(\mathbf{u}), \mathbf{n} \rangle = -2b.
$$

Therefore, $b = \frac{1}{2}(\cos \theta + 1)$ and $a = \frac{1}{2}(\cos \theta - 1)$. This indicates that $\mathbf{n} = \mathbf{N}_n^\theta(\mathbf{u})$. 

We define $\Delta_{42}^\theta(\mathbf{v}) = \Delta_{24}^\theta(\mathbf{v})$, $K_{42}^\theta(\mathbf{v}) = K_{24}^\theta(\mathbf{v})$, $\pi[\theta]_{(24)}^\theta = \pi[\theta]_{(42)}^\theta$. In particular, we consider the following double fibration:

$$(\alpha) H^n_1(-\sin^2 \theta) \times \Lambda^n \supset \Delta_{42}^\theta(\mathbf{v}) = \{ (\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = -(\cos \theta + 1) \},$$

$$(\beta) \pi[\theta]_{(42)}^\theta : \Delta_{42}^\theta(\mathbf{v}) \longrightarrow H^n_1(-\sin^2 \theta), \pi[\theta]_{(42)}^\theta : \Delta_{42}^\theta(\mathbf{v}) \longrightarrow \Lambda^n,$$

$$(\gamma) \eta[\theta]_{(42)}^\theta : \langle d\mathbf{v}, d\mathbf{w} \rangle |_{\Delta_{42}^\theta(\mathbf{v})} = \langle \mathbf{v}, d\mathbf{w} \rangle |_{\Delta_{42}^\theta(\mathbf{v})}.$$

By Theorem 1, $(\Delta_{42}^\theta, K_{42}^\theta(\mathbf{v}))$ is a contact manifold. One can check that

$$
\mathcal{L}_{4(42)}^\theta : \Delta_4 \rightarrow \Delta_{42}^\theta, \mathcal{L}_{4(42)}^\theta(\mathbf{v}, \mathbf{w}) = \left( \frac{1}{2}((1-\cos \theta)\mathbf{v} + (1+\cos \theta)\mathbf{w}), \mathbf{w} \right)
$$

by $\mathcal{L}_{4(42)}^\theta(\mathbf{v}, \mathbf{w}) = (\mathbf{X}(\mathbf{v}), \mathbf{N}_{42}^\theta(\mathbf{v}, \mathbf{w})))$. Since $\mathcal{L}_{4(42)}^\theta(\mathbf{v}, \mathbf{w}) = \mathcal{L}_{4/(42)}^\theta \circ \mathcal{L}_4(\mathbf{v}, \mathbf{w})$, $\mathcal{L}_{4(42)}^\theta(\mathbf{v}, \mathbf{w})$ is a Legendrian embedding. Therefore,
is a contact diffeomorphism. We define a map \( N_n^4[\theta] : U \to H^n_1(−\sin^2 \theta) \) by

\[
N_n^4[\theta](u) = \frac{1}{2}((1 - \cos \theta)X(u) + (1 + \cos \theta)X^n(u)).
\]

Furthermore, we define \( L_{42}[\theta] : U \to \Delta^-_{42}[\theta] \subset H^n_1(−\sin^2 \theta) \times \Lambda^n \) by

\[
L_{42}[\theta](u) = (N_n^4[\theta](u), X(u)).
\]

Since \( L_{42}[\theta](u) = L_{4(42)}^{-} \circ L_{4}(u) \), \( L_{42}[\theta](u) \) is a Legendrian embedding. Therefore,

\[
(\alpha N_n^4[\theta](u), X(u)) = L_{42}[\theta](u)^*\eta[\theta] = 0.
\]

This indicates that \( N_n^4[\theta](u) \) can be seen as a normal vector of \( M = X(U) \) at \( p = X(u) \). We call \( N_n^4[\theta](u) \) a \( \theta \)-hyperbolic pseudo-sphere Gaussian map. One can also obtain the following relations among Gaussian maps in Table 4.

Table 4. Relations among Gaussian maps of Lorentzian hypersurfaces in nullcone.

| Conditions | Relations among Gaussian Maps |
|------------|-------------------------------|
| \( \theta = 0 \) |
| \( \theta \in (0, \frac{\pi}{2}) \) | \( N_n^4[\theta](u) = X^n(u) \) |
| \( \theta = \frac{\pi}{2} \) | \( N_n^4[\frac{\pi}{2}](u) = X(u) \) |

According to the results in Tables 3 and 4, we establish the relations among different kinds of geometries of Lorentzian hypersurfaces in the nullcone.

**Proposition 4.** Let \( X : U \to \Lambda^n \) be a Lorentzian hypersurface; then, \( N_n^4[\theta](u) \) is a constant vector if and only if \( X(U) \) is a subset of \( NH(n, -(\cos \theta + 1)) \), where \( n \in H^n_1(−\sin^2 \theta) \).

**Proof.** If \( N_n^4[\theta](u) = n \) is a constant vector, then

\[
\langle X(u), n \rangle = \langle X(u), N_n^4[\theta](u) \rangle = -(\cos \theta + 1).
\]

This means that

\[
X(U) \subset NH(N_n^4[\theta](u), -(\cos \theta + 1)).
\]

Therefore, \( M \) is a subset of \( NH(N_n^4[\theta](u), -(\cos \theta + 1)) \). Conversely, if

\[
M \subset NH(N_n^4[\theta](u), -(\cos \theta + 1)),
\]

where \( n \in H^n_1(−\sin^2 \theta) \). Since \( n \) is a normal vector of \( M \), there are real numbers \( c \) and \( d \) such that \( n = cX(u) + dX^n(u) \) and \(-4cd = -\sin^2 \theta \). By definition, we obtain

\[
-(\cos \theta + 1) = \langle X(u), n \rangle = -2d.
\]

Therefore, \( d = \frac{1}{2}(1 + \cos \theta) \) and \( c = \frac{1}{2}(1 - \cos \theta) \). This indicates that \( n = N_n^4[\theta](u) \). \( \square \)

5. Conclusions

This paper deals with an interesting question of Legendrian dualities for continuous families of pseudo-spheres in semi-Euclidean space. We construct all contact diffeomorphisms among the contact manifolds and display them in a table of Legendrian dualities. We also extend the theorem of Legendrian dualities to the case of Legendrian dualities for continuous families of pseudo-spheres in general semi-Euclidean space. Finally, we give two applications of the extended Legendrian duality theorem.
As a future work, we plan to proceed to study some applications of Legendrian dualities combined with singularity theory and submanifold theory, etc., in [13–17,26–34], to obtain new results and theorems. Furthermore, we will explore some new geometric properties of Lie groups based on the results in [35–38].

**Author Contributions:** Conceptualization, H.L.; Writing—Original Draft Preparation, H.L.; Calculations, J.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by the Natural Science Foundation of Heilongjiang Province of China, grant No. LH2021A020.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Data sharing not applicable.

**Acknowledgments:** The authors would like to thank the reviewers for their careful reading and useful comments. The first author would like to thank Donghe Pei and Liang Chen for their good suggestions.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Whitney, H. On Singularities of mappings of Euclidean space I: Mappings of the plane in the plane. *Ann. Math.* 1955, 62, 374–410. [CrossRef]
2. Izumiya, S.; Romero Fuster, M.D.C.; Ruas, M.A.S.; Tari, F. Differential Geometry from a Singularity Theory Viewpoint; World Scientific: Singapore, 2016.
3. Arnol’d, V.I.; Gusein-zade, S.M.; Varchenko, A.N. Singularities of Differentiable Maps; Birkhäuser: Basel, Switzerland, 1988.
4. Bruce, J.W.; Giblin, P.J. *Curves and Singularities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1992.
5. Izumiya, S. Timelike hypersurfaces in de Sitter space and Legendrian singularities. *J. Math. Sci.* 2007, 144, 3789–3803. [CrossRef]
6. Chen, L.; Izumiya, S. A mandala of Legendrian dualities for pseudo-spheres in semi-Euclidean space. *Proc. Jpn. Acad. A-Math.* 2009, 85, 49–54. [CrossRef]
7. Izumiya, S.; Yildirim, H. Extensions of the mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space. *Topol. Appl.* 2012, 159, 545–554. [CrossRef]
8. Nagai, T. The Gauss map of a hypersurface in Euclidean sphere and the spherical Legendrian duality. *Topol. Appl.* 2012, 159, 509–518. [CrossRef]
9. Li, Y.L.; Liu, S.Y.; Wang, Z.G. Tangent developables and Darboux developables of framed curves. *Topol. Appl.* 2021, 301, 107526. [CrossRef]
10. Izumiya, S. Legendrian dualities and spacelike hypersurfaces in the lightcone. *Moscow Math. J.* 2009, 9, 325–357. [CrossRef]
11. Wang, Y.Q.; Pei, D.H.; Cui, X.P. Pseudo-spherical normal Darboux images of curves on a lightlike surface. *Math. Methods Appl. Sci.* 2017, 40, 7151–7161. [CrossRef]
12. Zhou, K.; Wang, Z. Pseudo-spherical Darboux images and lightcone images of principal-directional curves of nonlightlike curves in Minkowski 3-space. *Math. Methods Appl. Sci.* 2020, 43, 35–77. [CrossRef]
13. Chen, L.; Izumiya, S.; Pei, D.H. Timelike hypersurfaces in anti-de Sitter space from a contact viewpoint. *J. Math. Sci.* 2014, 199, 629–644. [CrossRef]
14. Chen, L.; Izumiya, S. Singularities of Anti de Sitter torus Gauss maps. *Bull. Braz. Math. Soc.* 2010, 41, 37–61. [CrossRef]
15. Chen, L.; Izumiya, S.; Pei, D.; Saji, K. Anti de Sitter horospherical flat timelike surfaces. *Sci. China Math.* 2014, 57, 1841–1866. [CrossRef]
16. Chen, L.; Takahashi, M. Dualities and evolutes of fronts in hyperbolic 2-space and de Sitter 2-space. *J. Math. Anal. Appl.* 2015, 437, 133–159. [CrossRef]
17. Pei, D.H.; Wang, Y.Q. Spacelike submanifolds of codimension two in anti-de Sitter space. *Appl. Anal.* 2019, 98, 1–16. [CrossRef]
18. Li, Y.L.; Wang, Z.G. Lightlike tangent developables in de Sitter 3-space. *J. Geom. Phys.* 2021, 164, 104188. [CrossRef]
19. Li, Y.L.; Zhu, Y.S.; Sun, Q.Y. Singularities and dualities of pedal curves in pseudo-hyperbolic and de Sitter space. *Int. J. Geom. Methods Mod. Phys.* 2021, 18, 2150008. [CrossRef]
20. Liu, H.M.; Miao, J.J. Geometric invariants and focal surfaces of spacelike curves in de Sitter space from a caustic viewpoint. *AIMS Math.* 2021, 6, 3177–3204. [CrossRef]
21. Liu, H.M.; Miao, J.J. Singularities of timelike Anti-de Sitter Gauss images of spacelike hypersurfaces in Anti-de Sitter n-space. *Sci. Sin. Math.* 2010, 40, 813–826.
22. Liu, H.M.; Pei, D.H. Lightcone dual surfaces and hyperbolic dual surfaces of spacelike curves in de Sitter 3-space. *J. Nonlinear Sci. Appl.* 2016, 9, 2563–2576. [CrossRef]
23. Liu, H.M.; Pei, D.H. Legendrian dualities between spherical indicatrixes of curves and surfaces according to Bishop frame. *J. Nonlinear Sci. Appl.* 2016, 9, 2875–2887. [CrossRef]

24. Wang, Y.Q.; Chang, Y.; Liu, H.M. Singularities of helix surfaces in Euclidean 3-space. *J. Geom. Phys.* 2020, 156, 103781. [CrossRef]

25. Liu, H.M.; Miao, J.J. Geometric Properties of Lorentzian Hypersurfaces in Open Nullcone. *Math. Pract. Theory* 2021, 50, 1–6.

26. Li, Y.L.; Dey, S.; Pahan, S.; Ali, A. Geometry of conformal $\eta$-Ricci solitons and conformal $\eta$-Ricci almost solitons on Paracontact geometry. *Open Math.* 2022, 20, 1–20. [CrossRef]

27. Li, Y.L.; Ganguly, D.; Dey, S.; Bhattacharyya, A. Conformal $\eta$-Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math.* 2022, 7, 5408–5430. [CrossRef]

28. Li, Y.L.; Alkhaldi, A.H.; Ali, A.; Laurian-Ioan, F. On the Topology of Warped Product Pointwise Semi-Slant Submanifolds with Positive Curvature. *Mathematics* 2021, 9, 3156. [CrossRef]

29. Li, Y.L.; Ali, A.; Mofarreh, F.; Alluhaibi, N. Homology groups in warped product submanifolds in hyperbolic spaces. *J. Math.* 2021, 2021, 8554738. [CrossRef]

30. Li, Y.L.; Ali, A.; Ali, R. A general inequality for CR-warped products in generalized Sasakian space form and its applications. *Adv. Math. Phys.* 2021, 2021, 5777554. [CrossRef]

31. Li, Y.L.; Lone, M.A.; Wani, U.A. Biharmonic submanifolds of Kähler product manifolds. *AIMS Math.* 2021, 6, 9309–9321. [CrossRef]

32. Li, Y.L.; Abolarinwa, A.; Azami, S.; Ali, A. Yamabe constant evolution and monotonicity along the conformal Ricci flow. *AIMS Math.* 2022, 7, 12077–12090. [CrossRef]

33. Li, Y.L.; Ali, A.; Mofarreh, F.; Abolarinwa, A.; Ali, R. Some eigenvalues estimate for the $\phi$-Laplace operator on slant submanifolds of Sasakian space forms. *J. Funct. Space* 2021, 2021, 6195939.

34. Yang, Z.C.; Li, Y.L.; Erdogdu, M.; Zhu, Y.S. Evolving evolutoids and pedaloids from viewpoints of envelope and singularity theory in Minkowski plane. *J. Geom. Phys.* 2022, 104513,1–23. [CrossRef]

35. Liu, H.; Miao, J.; Li, W.; Guan, J. The sub-Riemannian limit of curvatures for curves and surfaces and a Gauss-Bonnet theorem in the rototranslation group. *J. Math.* 2021, 2021, 9981442. [CrossRef]

36. Liu H.M.; Miao J.J. Gauss-Bonnet theorem in Lorentzian Sasakian space forms. *AIMS Math.* 2021, 6, 8772–8791. [CrossRef]

37. Liu H.M.; Guan J.Y. The sub-Riemannian limit of curvatures for curves and surfaces and a Gauss-Bonnet theorem in the group of rigid motions of Minkowski plane with general left-invariant metric. *J. Funct. Space* 2021, 2021, 1431082.

38. Li, W.Z.; Liu H.M. Gauss-Bonnet Theorem in the Universal Covering Group of Euclidean Motion Group $E(2)$ with the General Left-Invariant Metric. *J. Nonlinear Math. Phys.* 2022, 2022, 1–32. [CrossRef]