A conformal energy for simplicial surfaces

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Abstract

A new functional for simplicial surfaces is suggested. It is invariant with respect to Möbius transformations and is a discrete analogue of the Willmore functional. Minima of this functional are investigated. As an application a bending energy for discrete thin-shells is derived.

Keywords: Conformal energy, Willmore functional, simplicial surfaces, discrete differential geometry

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1 Introduction

In the variational description of surfaces the following functionals are of primary importance:

- The area $A = \int dA$, where $dA$ is the area element, is preserved by isometries.
- The total Gaussian curvature $G = \int KdA$, where $K$ is the Gaussian curvature, is topological invariant.
- The total mean curvature $M = \int HdA$, where $H$ is the mean curvature, depends on external geometry of the surface.
- The Willmore energy $W = \int H^2dA$ is invariant with respect to Möbius transformations.

Geometric discretizations of the first three functionals for simplicial surfaces are well known. For the area functional it is obvious. The local Gaussian curvature at a vertex $v$ is defined as the angle defect

$$G(v) = 2\pi - \sum \alpha_i,$$

where $\alpha_i$ are the angles of all triangles (see Fig. 2) at vertex $v$. The total Gaussian curvature is a sum over all vertices $G = \sum_v G(v)$. The local mean curvature at an edge $e$ is defined as

$$M(e) = l\theta,$$

where $l$ is the length of the edge and $\theta$ is the angle between the normals to the adjacent faces at $e$ (see Fig. 6). The total mean curvature is the sum over all edges $M = \sum_e M(e)$. These discrete functionals possess the geometric symmetries of the smooth functionals mentioned above.

Until recently a geometric discretization of the Willmore functional was missing. In this paper we introduce a Möbius invariant energy for simplicial surfaces and show that it should be treated as a discrete Willmore energy.

2 Conformal Energy

Let $S$ be a simplicial surface in 3-dimensional Euclidean space with the set of vertices $V$, edges $E$ and (triangular) faces $F$. We define a conformal energy for simplicial surfaces using circumcircles of their faces. Each (internal) edge $e \in E$ is incident to two triangles. Consistent orientation of the triangles naturally induces an orientation of the corresponding circumcircles. Let $\beta(e)$ be the external intersection angle of the circumcircles of the triangles sharing $e$, which is the angle between the tangent vectors of the
oriented circumcircles. Now we are ready to define a new energy functional for simplicial surfaces.

Definition 1 Local conformal (discrete Willmore) energy at a vertex $v$ is given by the sum over all incident edges

$$W(v) = \sum_{e \ni v} \beta(e) - 2\pi.$$ 

The conformal (discrete Willmore) energy of the simplicial surface is the sum over all vertices

$$W(S) = \frac{1}{2} \sum_{v \in V} W(v) = \sum_{e \in E} \beta(e) - \pi \mid V \mid,$$

where $\mid V \mid$ is the number of vertices of $S$.

![Figure 1: Definition of conformal (discrete Willmore) energy](image)

Fig. 1 presents two neighboring circles with their external intersection angle $\beta_i$ as well as a view “from the top” at a vertex $v$ showing all $n$ circumcircles passing through $v$ with the corresponding intersection angles $\beta_1, \ldots, \beta_n$. For simplicity we will consider only simplicial surfaces without boundary.

Note that the energy we defined is obviously invariant with respect to Möbius transformations. This invariance is an important property of the classical Willmore energy defined for smooth surfaces (see below).

Note also that the changing of the orientation of both circles preserves the angle $\beta(e)$. This shows that the energy $W(S)$ is well defined for non-oriented simplicial surfaces as well.

The star $S(v)$ of the vertex $v$ is the subcomplex of $S$ comprised by the triangles incident with $v$. The vertices of $S(v)$ are $v$ and all its neighbors. We call $S(v)$ convex if for any its face $f \in F(S(v))$ the star $S(v)$ lies to one side of the plane of $F$ and strictly convex if the intersection of $S(v)$ with the plane of $f$ is $f$ itself.
Proposition 2  The conformal energy is non-negative

\[ W(v) \geq 0, \]

and vanishes if and only if the star \( S(v) \) is convex and all its vertices lie on a common sphere.

The proof of this proposition is based on the following elementary lemma.

Lemma 3  Let \( P \) be a (not necessarily planar) \( n \)-gon with external angles \( \beta_i \). Choose a point \( P \) and connect it to all vertices of \( P \). Let \( \alpha_i \) be the angles (see Fig.2) of the triangles at the tip \( P \) of the obtained pyramid. Then

\[ \sum_{i=1}^{n} \beta_i \geq \sum_{i=1}^{n} \alpha_i, \]

and the equality holds if and only if \( P \) is planar and convex\(^1\) and the vertex \( P \) lies inside \( P \).

![Figure 2: Proof of Lemma 3](image)

Proof. Let us denote by \( \gamma_i \) and \( \delta_i \) the angles of the cite-triangles at the vertices of \( P \) (see Fig.2). The claim of Lemma 3 follows from adding over all \( i = 1, \ldots, n \) two obvious (in)equalities

\[ \beta_{i+1} \geq \pi - (\gamma_{i+1} + \delta_i), \]

\[ \pi - (\gamma_i + \delta_i) = \alpha_i. \]

All inequalities become equalities only in the case when \( P \) is planar, convex and contains \( P \). Note that some of the external angles \( \beta_i \) may vanish. As a corollary we obtain a polygonal version of Fenchel’s theorem [6].

Corollary 4

\[ \sum_{i=1}^{n} \beta_i \geq 2\pi. \]

\(^1\)The obtained pyramid is convex in this case. Note that we distinguish convex and strictly convex polygons (and pyramids). Some of the angles \( \beta_i \) of a convex polygon may vanish. The corresponding cite-triangles of the pyramid lie in one plane.
Proof. For a given $P$ choose the point $P$ varying on a straight line encircled by $P$. There always exist points $P$ so that the star at $P$ is not strictly convex, and thus $\sum \alpha_i \geq 2\pi$.

Proof of Proposition 3. The claim of Proposition 3 is invariant with respect to Möbius transformations. Applying a Möbius transformation $M$ which maps the vertex $v$ to infinity, $M(v) = \infty$, one observes that all the circles passing through $v$ become straight lines and we arrive at the geometry shown in Fig.2 with $P = M(\infty)$. Now the claim follows immediately from Corollary 4.

Theorem 5 Let $S$ be a simplicial surface. Then

$$W(S) \geq 0,$$

and the equality holds if and only if $S$ is a (part of a) convex polyhedron inscribed in a sphere.

Proof. Only the second statement needs to be proven. Due to Proposition 2 the equality $W(S) = 0$ implies that all vertices and edges of $S$ are convex (but not necessarily strictly convex). Deleting the edges which separate triangles lying in one plane one obtains a polyhedral surface $S_P$ with circular faces and all strictly convex vertices and edges. Proposition 2 implies that for every vertex $v$ there exists a sphere $S_v$ with all vertices of the star $S(v)$ lying on it. For any edge $(v_1, v_2)$ of $S_P$ two neighboring spheres $S_{v_1}$ and $S_{v_2}$ share two different circles of their common faces. This implies $S_{v_1} = S_{v_2}$ and finally the coincidence of all the spheres $S_v$.

Discrete conformal energy $W$ defined above is a discrete analogue of the Willmore energy for smooth surfaces [14].

$$W(S) = \frac{1}{4} \int_S (k_1 - k_2)^2 dA = \int_S H^2 dA - \int_S K dA.$$

Here $dA$ is the area element, $k_1, k_2$ the principal curvatures, $H = \frac{1}{2}(k_1 + k_2)$ the mean curvature, $K = k_1k_2$ the Gaussian curvature of the surface. Here we prefer a definition for $W$ with a Möbius invariant integrand. It differs from the one in the introduction by a topological invariant.

Let us mention two important properties of the Willmore energy:

- $W(S) \geq 0$ and $W(S) = 0$ if and only if $S$ is the round sphere.

- $W(S)$ (and the integrand $(k_1 - k_2)^2 dA$) is Möbius invariant [11,14].

Whereas the first claim almost immediately follows from the definition, the second one is a non-trivial property. We have shown that the same properties hold for the discrete energy $W$; in the discrete case the Möbius invariance is built into the definition and the non-negativity of the energy is non-trivial.
In the same way one can define conformal (Willmore) energy for simplicial surfaces in Euclidean spaces of higher dimension and space forms.

The discrete conformal energy is well defined for polyhedral surfaces with circular faces (not necessarily simplicial).

3 Computation of the Energy

Consider two triangles with a common edge. Let \( a, b, c, d \in \mathbb{R}^3 \) be their other edges oriented as in Fig.3. Identifying vectors in \( \mathbb{R}^3 \) with imaginary quaternions \( \text{Im} \mathbb{H} \) one obtains for the quaternionic product

\[
ab = - < a, b > + a \times b,
\]

where \( < a, b > \) and \( a \times b \) are the scalar and vector products in \( \mathbb{R}^3 \).

![Figure 3: Formula for the angle between circumcircles](image)

**Proposition 6** The external angle \( \beta \in [0, \pi] \) between the circumcircles of the triangles in Fig.3 is given by one of the equivalent formulas:

\[
\cos(\beta) = - \frac{\text{Re } q}{|q|} = - \frac{\text{Re } abcd}{|abcd|} = \frac{< a, c > < b, d > - < a, b > < c, d > - < b, c > < d, a >}{|a| |b| |c| |d|}.
\]

Here \( q = ab^{-1}cd^{-1} \) is the cross-ratio of the quadrilateral.

**Proof.** Since \( \text{Re } q, |q| \) and \( \beta \) are Möbius invariant it is enough to prove the first formula for the planar case \( a, b, c, d \in \mathbb{C} \), mapping all four vertices to a plane by a Möbius transformation. In this case \( q \) becomes the classical complex cross-ratio. Considering the arguments \( a, b, c, d \in \mathbb{C} \) one easily arrives at \( \beta = \pi - \arg q \). The second representation follows from the identity \( b^{-1} = -b/|b| \) for imaginary quaternions. Finally applying (1) we obtain

\[
\text{Re } abcd = < a, b > < c, d > - < a \times b, c \times d > = < a, b > < c, d > + < b, c > < d, a > - < a, c > < b, d >.
\]
4 Minimizing Discrete Conformal Energy

Similarly to the smooth Willmore functional $W$, minimizing the discrete conformal energy $W$ makes the surface as round as possible.

Let us denote by $S$ the combinatorial data of $S$. The simplicial surface $S$ is called a geometric realization of the abstract simplicial surface $S$.

**Definition 7** Critical points of $W(S)$ are called simplicial Willmore surfaces. The conformal (Willmore) energy of an abstract simplicial surface is the infimum over all geometric realizations

$$W(S) = \inf_{S \in S} W(S).$$

Kevin Bauer implemented the proposed conformal functional with the Brakke’s evolver [3] and did some minimization numerical experiments. Examples of those are presented in Fig. 4. In the first and the second lines the initial configurations and the corresponding Willmore surfaces minimizing the conformal energy respectively are shown. Let us call the gradient flow of the energy $W$ the discrete Willmore flow. By this flow the energy of the first simplicial sphere decreases to zero and the surface evolves into a convex polyhedron with all the vertices lying on a sphere. The abstract simplicial surface of the central example is different and we obtain a simplicial Willmore sphere with positive conformal energy. The third example is
a simplicial projective plane. The initial configuration is made from squares divided into triangles (cf. [12]). We see that the minimum is close to the smooth Boy surface known to minimize [11] the Willmore energy for projective planes.

The minimization of the conformal energy for simplicial spheres is related to a classical result of Steinitz [13] who has shown that there exist abstract simplicial 3-polytopes without geometric realizations all vertices of which belong to a sphere. We call these combinatorial types non-inscribable.

The non-inscribable examples of Steinitz are constructed as follows [8]. Let $S$ be an abstract simplicial sphere with vertices colored in black and white. Denote the sets of white and black vertices by $V_w$ and $V_b$ respectively, $V = V_w \cup V_b$. Assume that the number of black vertices does not exceed the number of white vertices, $|V_w| \geq |V_b|$, and there are no edges connecting two white vertices and there are edges connecting black vertices. It is easy to see that $S$ with these properties cannot be inscribed in a sphere. Indeed, assume that we have constructed such inscribed convex polyhedron. Then the equality of the intersection angles at both ends of an edge (see left Fig.1) implies

$$2\pi |V_b| \geq \sum_{e \in E} \beta(e) \geq 2\pi |V_w|,$$

and the equalities hold only if all edges connect vertices of different color. The obtained contradiction to the assumed inequality implies the claim.

To construct abstract polyhedra with $|V_w| \geq |V_b|$ and with some edges connecting black points, take a polyhedron $P$ whose number of vertices does not exceed the number of faces $|\hat{F}| \geq |\hat{V}|$. Color all the vertices in black, add white vertices at the faces and connect them to all black vertices of a face. We obtain a polyhedron with black (original) edges and $|V_w| = |\hat{F}| \geq |V_b| = |\hat{V}|$. The example with minimal possible number of vertices $|V| = 11$ is shown in Fig.5. The starting polyhedron $P$ here are two tetrahedra identified along a common face: $\hat{F} = 6, \hat{V} = 5$.

Hodgson, Rivin and Smith [9] found a characterization of inscribable combinatorial types, based on a transfer to the Klein model of hyperbolic 3-space. It is not clear whether there exist non-inscribable examples of non-Steinitz type.

Numerical experiments lead us to the following

**Conjecture 8** The conformal energy of simplicial Willmore spheres is quantized

$$W = 2\pi N, \quad N \in \mathbb{N}.$$ 

Note that this claim belongs to differential geometry of discrete surfaces. It would be interesting to find a (combinatorial) meaning of the integer $N$. Compare also with the famous classification of smooth Willmore spheres by Bryant [5], who has shown that the energy of Willmore spheres is quantized $W = 4\pi N, \quad N \in \mathbb{N}$. 

8
The discrete Willmore energy is defined for the ambient spaces ($\mathbb{R}^n$ or $S^n$) of any dimension. This leads to combinatorial Willmore energies

$$W_n(S) = \inf_{S \in \mathcal{S}} W(S), \quad S \subset S^n,$$

where the infimum is taken over all realizations in the $n$–dimensional sphere. Obviously these numbers build a non-increasing sequence $W_n(S) \geq W_{n+1}(S)$ which becomes constant for sufficiently large $n$.

Complete understanding of non-inscribable simplicial spheres is an interesting mathematical problem. However the phenomenon of existence of such spheres might be seen as a problem in using of the conformal functional for applications in computer graphics, such as fairing of surfaces. Fortunately the problem disappears just after one refinement step: all simplicial spheres become inscribable. Let $\mathcal{S}$ be an abstract simplicial sphere. Define its refinement $\mathcal{S}_R$ as follows: split every edge of $\mathcal{S}$ in two by putting additional vertices and connect these new vertices sharing a face of $\mathcal{S}$ by additional edges.

**Proposition 9** The refined simplicial sphere $\mathcal{S}_R$ is inscribable, and thus $W(\mathcal{S}_R) = 0$.

**Proof.** Koebe’s theorem (see, for example, [15, 2]) claims that every abstract simplicial sphere $\mathcal{S}$ can be realized as a convex polyhedron $S$ all edges of which touch a common sphere $S^2$. Starting with this realization $S$ it is easy to construct a geometric realization $S_R$ of the refinement $\mathcal{S}_R$ inscribed in $S^2$. Indeed choose the touching points of the edges of $S$ with $S^2$ as additional vertices of $S_R$ and project the original vertices of $S$ (which lie outside of the sphere $S^2$) to $S^2$. One obtains a convex simplicial polyhedron $S_R$ inscribed in $S^2$. 
Another interesting variational problem with the conformal energy is the optimization of triangulations of a given simplicial surface. Here one fixes the vertices and chooses an equivalent triangulation (abstract simplicial surface $S$) minimizing the conformal functional. The minimum

$$W(V) = \min_{S \supseteq S} W(S)$$

yields “an optimal” triangulation for a given vertex data. In the case of $S^2$ this optimal triangulation is classical.

**Proposition 10** Let $S$ be a simplicial surface with all vertices $V$ on a two dimensional sphere $S^2$. Then $W(S) = 0$ if and only if it is the Delaunay triangulation on the sphere, i.e. $S$ is the boundary of the convex hull of $V$.

In differential geometric applications like numerical minimizing the Willmore energy of smooth surfaces (cf. [13]) it is not natural to preserve the triangulation by minimizing the energy, and one should also change the combinatorial type decreasing the energy.

Discrete conformal energy $W$ is not only a discrete analogue of the Willmore energy. One can show that it approximates the smooth Willmore energy although the smooth limit is very sensitive to the refinement method and should be chosen in a very special way. A computation which will be published elsewhere shows that if one chooses the vertices of a curvature line net of a smooth surface $S$ for the vertices of $S$ and triangularizes it then $W(S)$ converges to $W(S)$ by natural refinement. On the other hand the infinitesimal equilateral triangular lattice gives in the limit $3/2$ times larger energy. Possibly the minimization of the discrete Willmore energy with the vertices on the smooth surface could be used for computation of the curvature line net. We are going to investigate this interesting and complicated phenomenon.

### 5 Bending of Simplicial Surfaces

An accurate model for bending of discrete surfaces is important for modelling in virtual reality.

Let $S_0$ be a thin shell and $S$ its deformation. The bending energy of smooth thin shells is given by the integral [7]

$$E = \int (H - H_0)^2 dA,$$

where $H_0$ and $H$ are the mean curvatures of the original and deformed surface respectively. For $H_0 = 0$ it reduces to the Willmore energy.
To derive the bending energy for simplicial surfaces let us consider the limit of fine triangulation, i.e. of small angles between the normals of neighboring triangles. Consider an isometric deformation of two adjacent triangles. Let $\theta$ be the complement of the dihedral angle of the edge $e$, or, equivalently, the angle between the normals of these triangles (see Fig.6) and $\beta(\theta)$ the external intersection angle between the circumcircles of the triangles (see Fig.11) as a function of $\theta$.

**Proposition 11** Assume that the circumcenters of the circumcircles of two adjacent triangles do not coincide. Then in the limit of small angles $\theta \to 0$ the angle $\beta$ between the circles behaves as follows:

$$\beta(\theta) = \beta(0) + \frac{l}{L} \theta^2 + o(\theta^3).$$

Here $l$ is the length of the edge and $L \neq 0$ is the distance between the centers of the circles.

This proposition and our definition of conformal energy for simplicial surfaces motivate to suggest

$$E = \sum_{e \in E} \frac{l}{L} \theta^2$$

for the bending energy of discrete thin-shells.

![Figure 6: To definition of the bending energy for simplicial surfaces](image)

In [4, 7] similar representations for the bending energy of simplicial surfaces were found empirically. They were demonstrated to give convincing simulations and good comparison with real processes. In [7] the distance between the barycenters is used for $L$ in the energy expression but possible numerical advantages in using circumcenters are indicated.

Using the Willmore energy and Willmore flow is a hot topic in computer graphics. Applications include fairing of surfaces and surface restoration. We hope that our conformal energy will be useful for these applications and plan to work on them.

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