AUTOMORPHISMS OF CURVES FIXING THE ORDER TWO POINTS OF THE JACOBIAN

INDRANIL BISWAS AND A. J. PARAMESWARAN

Abstract. Let $X$ be an irreducible smooth projective curve, of genus at least two, defined over an algebraically closed field of characteristic different from two. If $X$ admits a nontrivial automorphism $\sigma$ that fixes pointwise all the order two points of $\text{Pic}^0(X)$, then we prove that $X$ is hyperelliptic with $\sigma$ being the unique hyperelliptic involution. As a corollary, if a nontrivial automorphisms $\sigma'$ of $X$ fixes pointwise all the theta characteristics on $X$, then $X$ is hyperelliptic with $\sigma'$ being its hyperelliptic involution.

1. Introduction

Let $Y$ be a compact connected Riemann surface of genus at least two. Assume that there is a nontrivial holomorphic automorphism $\sigma_0 : Y \to Y$ satisfying the condition that for each holomorphic line bundle $\xi$ over $Y$ with $\xi \otimes 2$ trivializable, the pull back $\sigma_0^*\xi$ is holomorphically isomorphic to $\xi$. In [2] it was shown that $Y$ must be hyperelliptic and $\sigma_0$ is the unique hyperelliptic involution (see [2, p. 494, Theorem 1.1]).

We recall that a theta characteristic on $Y$ is a holomorphic line bundle $\theta$ such that $\theta \otimes 2$ is holomorphically isomorphic to the homomorphic cotangent bundle $K_Y$. The group of order two line bundles on $Y$ acts freely transitively on the set of all theta characteristics on $Y$. From this it follows immediately that if an automorphism of $Y$ fixes pointwise all the theta characteristics, then it also fixes pointwise all the order two line bundles on $Y$. Therefore, if $Y$ admits a nontrivial automorphism $\sigma_0'$ that fixes pointwise all the theta characteristics on $Y$, then $Y$ is hyperelliptic and $\sigma_0'$ is its unique hyperelliptic involution.

The proof of Theorem 1.1 in [2] is topological. Here we investigate the corresponding algebraic geometric set-up, where the topological proof of Theorem 1.1 in [2] is no longer valid.

Let $X$ be an irreducible smooth projective curve defined over an algebraically closed field $k$. We will assume that $\text{genus}(X) > 1$ and $\text{char}(k) \neq 2$. We prove the following:

Theorem 1.1. Let 

$$\sigma : X \to X$$

be a nontrivial automorphism that fixes pointwise all the theta characteristics on $X$. Then $X$ is hyperelliptic with $\sigma$ being its unique hyperelliptic involution.

2000 Mathematics Subject Classification. 14H37, 14H40.

Key words and phrases. Curve, automorphism, Jacobian, theta characteristic.
This theorem is proved by showing that if
\[ \sigma' : X \longrightarrow X \]
is a nontrivial automorphism of \( X \) that fixes pointwise all the order two points in \( \text{Pic}^0(X) \), then \( X \) is hyperelliptic with \( \sigma' \) being its unique hyperelliptic involution. (See Lemma 3.1.)

It should be pointed out that Theorem 1.1 is not valid if the assumption that the field \( k \) is algebraically closed is removed. There exists a geometrically irreducible smooth projective real algebraic curve \( Y \) of genus \( g \geq 2 \) which admits a nontrivial involution \( \sigma \) that fixes pointwise all the real points \( \xi \in \text{Pic}^{g-1}(Y) \) with \( \xi \otimes 2 = K_Y \), and \( \text{genus}(Y/\langle \sigma \rangle) \neq 0 \). (The details are in [1].)

2. Automorphisms of polarized abelian varieties

Let \( k \) be an algebraically closed field whose characteristic is different from two. Let \( A \) be an abelian variety defined over \( k \) and \( L \) an ample line bundle over \( A \). For any positive integer \( n \), let

(1) \[ A_n \subset A \]

be the scheme-theoretic kernel of the endomorphism \( A \longrightarrow A \) defined by \( x \mapsto nx \).

**Proposition 2.1.** Let \( \tau : A \longrightarrow A \) be a nontrivial automorphism such that \( \tau^*L = L \otimes L_0 \) for some \( L_0 \in \text{Pic}^0(A) \), and the restriction of \( \tau \) to the subscheme \( A_{n_0} \) (see Eq. (1)) is the identity map for some \( n_0 \geq 2 \).

Define the two endomorphisms

\[ f_\pm := \text{Id}_A \pm \tau : A \longrightarrow A \]

Let \( A_+ \) (respectively, \( A_- \)) be the image of \( f_+ \) (respectively, \( f_- \)). Then

1. \( n_0 = 2 \).
2. \( \tau^2 = \tau \circ \tau \) is the identity automorphism of \( A \).
3. The natural homomorphism

(2) \[ \beta : A_+ \times A_- \longrightarrow A \]

defined by the inclusions of \( A_+ \) and \( A_- \) in \( A \) is an isomorphism.
4. The pull back \( \beta^*L \) is of the form \( p_+^*L_+ \otimes p_-^*L_- \), where \( p_+ \) (respectively, \( p_- \)) is the projection of \( A_+ \times A_- \) to \( A_+ \) (respectively, \( A_- \)).

**Proof.** A proof of statement (1) is given in [4, p. 207, Theorem 5]. See [3, p. 120, Corollary 1.10] for a proof under the assumption that \( k \) is the field of complex numbers.

To prove statement (2), we will show that the restriction of \( \tau^2 \) to \( A_+ \) is the identity map. Take any point \( x \in A_+ \). Then \( \tau(2x) = 2x \) because \( 2x \in A_2 \). Hence \( \tau(x) = x' + x \) for some \( x' \in A_2 \). Thus

\[ \tau(\tau(x)) = \tau(x' + x) = \tau(x') + \tau(x) = x' + (x' + x) = x. \]
Consequently, the restriction of \( \tau^2 \) to \( A_4 \) is the identity map. Now statement (2) follows from statement (1).

To prove statement (3), consider the composition homomorphism

\[
A \xrightarrow{f_+ \times f_-} A_+ \times A_- \xrightarrow{\beta} A,
\]

where \( \beta \) is the homomorphism in Eq. (2). It coincides with the endomorphism of \( A \) defined by \( x \mapsto 2x \). We also note that \( A_2 \subset \ker(f_+ \times f_-) \). Hence

\[
(\ker(\beta \circ (f_+ \times f_-)) \subset \ker(f_+ \times f_-).
\]

Since \( \tau^2 = \text{Id}_A \), the composition \( f_+ \circ f_- \) is the zero homomorphism. Hence \( \dim(A_+ \times A_-) \leq \dim A \). Now from Eq. (3) it follows that \( \beta \) is an isomorphism.

To prove statement (4), let

\[
\phi_{\beta^* L} : A_+ \times A_- \longrightarrow \text{Pic}^0(A_+ \times A_-) = \text{Pic}^0(A_+) \times \text{Pic}^0(A_-)
\]

be the homomorphism that sends any \( k \)-rational point \( x \in A_+ \times A_- \) to the line bundle \( (t_x^* \beta^* L) \otimes \beta^* L^* \), where \( t_x \) is the translation map of \( A_+ \times A_- \) defined by \( y \mapsto y + x \); see [1] p. 131, Corollary 5] for a precise definition of the morphism \( \phi_{\beta^* L} \). Let

\[
\tau' := \text{Id}_{A_+} \times (-\text{Id}_{A_-})
\]

be the automorphism of \( A_+ \times A_- \). We note that the isomorphism \( \beta \) in Eq. (2) takes \( \tau \) to \( \tau' \).

Let

\[
\hat{\tau}' := \text{Id}_{\text{Pic}^0(A_+)} \times (-\text{Id}_{\text{Pic}^0(A_-)})
\]

be the automorphism of \( \text{Pic}^0(A_+) \times \text{Pic}^0(A_-) = \text{Pic}^0(A_+) \times \text{Pic}^0(A_-) \). Since \( \tau^* L = L \otimes L_0 \) for some \( L_0 \in \text{Pic}^0(A) \), the following diagram is commutative

\[
\begin{array}{ccc}
A_+ \times A_- & \xrightarrow{\phi_{\beta^* L}} & \text{Pic}^0(A_+) \times \text{Pic}^0(A_-) \\
\downarrow{\tau'} & & \downarrow{\hat{\tau}'} \\
A_+ \times A_- & \xrightarrow{\phi_{\beta^* L}} & \text{Pic}^0(A_+) \times \text{Pic}^0(A_-)
\end{array}
\]

Therefore, the homomorphism \( \phi_{\beta^* L} \) takes the subgroup \( A_+ \) (respectively, \( A_- \)) of \( A_+ \times A_- \) to the subgroup \( \text{Pic}^0(A_+) \) (respectively, \( \text{Pic}^0(A_-) \)) of \( \text{Pic}^0(A_+) \times \text{Pic}^0(A_-) \). Now from the injectivity of the homomorphism

\[
\text{NS}(A_+ \times A_-) \longrightarrow \text{Hom}(A_+ \times A_-, \text{Pic}^0(A_+) \times \text{Pic}^0(A_-))
\]

de\( efined \ by \( \xi \mapsto \phi_{\xi} \) it follows immediately that the Néron–Severi class of \( \beta^* L \) coincides with that of some line bundle of the form \( p_+^* L_+ \otimes p_-^* L_- \) (see [1] p. 178] for the injectivity of the above homomorphism). Therefore, statement (4) follows using the fact that \( \text{Pic}^0(A_+) \times \text{Pic}^0(A_-) = \text{Pic}^0(A_+ \times A_-) \). This completes the proof of the proposition. \( \square \)
3. AUTOMORPHISMS AND THETA CHARACTERISTICS

Let $X$ be an irreducible smooth projective curve, of genus at least two, defined over the field $k$.

**Lemma 3.1.** Let

$$\sigma : X \longrightarrow X$$

be a nontrivial automorphism of $X$ that fixes pointwise all the order two points $\text{Pic}^0(X)_2 \subset \text{Pic}^0(X)$. then $X$ is hyperelliptic with $\sigma$ being its unique hyperelliptic involution.

**Proof.** Let $\text{Pic}^d(X)$ denote the moduli space of line bundles over $X$ of degree $d$. Let $g$ denote the genus of $X$. On $\text{Pic}^{g-1}(X)$, we have the theta divisor $\Theta$ given by the locus of the line bundles admitting nontrivial sections. Fix a $k$–rational point $x_0 \in X$. Let $L$ be the pull back of the line bundle $O_{\text{Pic}^{g-1}(X)}(\Theta)$ by the morphism $\text{Pic}^0(X) \longrightarrow \text{Pic}^{g-1}(X)$ that sends any $\xi$ to $\xi \otimes O_X((g-1)x_0)$.

Let $\tau : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$ be the automorphism defined by $\xi \longmapsto \sigma^*\xi$. This $\tau$ satisfies the conditions in Proposition 2.1. Hence $\tau$ is an involution (see Proposition 2.1(2)). This implies that $\sigma$ is an involution.

A hyperelliptic smooth projective curve $Y$ of genus at least two admits a unique involution $\sigma_Y$ such that $\text{genus}(Y/\langle \sigma_Y \rangle) = 0$. Therefore, to complete the proof of the lemma it suffices to show that $\text{genus}(X/\langle \sigma \rangle) = 0$. We note that the theta divisor $\Theta$ on $\text{Pic}^{g-1}(X)$ is irreducible. Indeed, it is the image of $\text{Sym}^{g-1}(X)$ by the obvious map. Also, $h^0(O_{\text{Pic}^{g-1}(X)}(\Theta)) = 1$ because $\Theta$ defines a principal polarization.

On the other hand, any ample hypersurface of the form $(A_+ \times D_-) \cup (D_+ \times A_-)$ on $A_+ \times A_-$ is never irreducible unless at least one of $A_+$ and $A_-$ is a point; here $D_+$ (respectively, $D_-$) is a hypersurface on $A_+$ (respectively, $A_-$). Therefore, from statement (4) of Proposition 2.1 and the irreducibility of $\Theta$ we conclude that either $\dim A_+ = 0$ or $\dim A_- = 0$. But $\dim A_- = \text{genus}(X) - \text{genus}(X/\langle \sigma \rangle)$, and $\dim A_+ = \text{genus}(X/\langle \sigma \rangle)$. Since $\text{genus}(X) > \text{genus}(X/\langle \sigma \rangle)$, we now conclude that $\text{genus}(X/\langle \sigma \rangle) = 0$. This completes the proof of the lemma.

A line bundle $\theta$ is called a *theta characteristic* of $X$ if $\theta^{\otimes 2}$ is isomorphic to the canonical line bundle $K_X$ of $X$. The space of theta characteristics on $X$ is a principal homogeneous space for $\text{Pic}^0(X)_2$. Therefore, if an automorphism $\sigma$ of $X$ fixes pointwise all the theta characteristics on $X$, then $\sigma$ fixes $\text{Pic}^0(X)_2$ pointwise. Consequently, the following theorem is deduced from Lemma 3.1.

**Theorem 3.2.** Let $\sigma : X \longrightarrow X$ be a nontrivial automorphism that fixes pointwise all the theta characteristics on $X$. Then $X$ is hyperelliptic with $\sigma$ being its unique hyperelliptic involution.

**References**

[1] Biswas, I., Gadgil, S.: Real theta characteristics and automorphisms of a real curve. Preprint (2007)
[2] Biswas, I., Gadgil, S., Sankaran, P.: On theta characteristics of a compact Riemann surface. Bull. Sci. Math. 131, 493–499 (2007)

[3] Lange, H., Birkenhake, C.: Complex abelian varieties. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 1992

[4] Mumford, D.: Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Oxford University Press, London, 1970

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: param@math.tifr.res.in