KINKY VORTONS

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Abstract

Cosmic vortons are closed loops of superconducting cosmic strings carrying current and charge. Despite a large number of studies the existence and stability of cosmic vorton solutions is still an open problem. Numerical simulations of the nonlinear field theory are difficult to perform in (3+1)-dimensions, due to the existence of multiple length and time scales. In this paper we study a (2+1)-dimensional analogue of cosmic vortons, which we refer to as kinky vortons, where the cosmic string is replaced by a kink string. Many of the expected qualitative aspects of cosmic vortons transfer to kinky vortons, with the advantage that several approximations used in the study of cosmic vortons can be replaced by exact results. Furthermore, the numerical study of kinky vortons requires less computational resources than cosmic vortons, so a number of issues can be addressed in some depth. The radius of the kinky vorton is determined as a function of the charge and winding number, and it is shown that the chiral limit is a repulsive fixed point. Stability to both axial and non-axial perturbations is demonstrated in the electric and chiral regimes, though surprisingly long lived ringing modes are observed. Kinky vortons which are too magnetic are shown to suffer from a pinching instability, which results in a reduction in the winding number and can convert magnetic into electric solutions.
1 Introduction

Cosmic strings are topological defects which may have formed during a phase transition in the early universe (for a review see [14]). Witten [15] pointed out that if the field of a cosmic string is coupled to another complex scalar field then a non-dissipative current can form to produce a superconducting string. A cosmic vorton [6] is a stable closed loop of superconducting cosmic string which carries both current and charge. The idea is that the current and charge on the string provide a force to balance the string tension and prevent its collapse.

The cosmological consequences of cosmic vortons have been well studied [2] and it has been proposed that they play a role in a number of cosmological phenomena, such as galactic magnetic fields, high energy cosmic rays, gamma ray bursts and baryogenesis. However, the existence and stability of cosmic vortons as classical field theory solutions is still an open problem, and convincing numerical evidence is still lacking. Numerical simulations of the relevant nonlinear field theory are difficult to perform in (3+1)-dimensions and results are limited, mainly due to the existence of multiple length and time scales. Even in the simplest model, which is the global version of Witten’s $U(1) \times U(1)$ model [15], cosmic vorton solutions have not yet been constructed. The only field theory computation to date [10] is in a modified version of this global theory, in which the interaction term between the two complex scalar fields is replaced by a non-renormalizable interaction, designed to make the numerical problem more tractable. Even in this case the vorton solution presented eventually decays, though this may be due to numerical issues.

In this paper we study a (2+1)-dimensional analogue of cosmic vortons, in which the cosmic string is replaced by a kink string. For this reason we name these objects kinky vortons. We shall show that many of the expected qualitative features of cosmic vortons can be demonstrated explicitly for kinky vortons, with the advantage that several approximations used in the study of cosmic vortons can be replaced by exact results. Furthermore, the numerical study of kinky vortons obviously requires less computational resources than cosmic vortons, so a number of issues can be addressed in detail. In particular, the simulations presented in this paper reveal some of the computational difficulties involved and help explain why similar numerical investigations in (3+1)-dimensions have not yet been successfully performed.

Our results include the determination of the radius of a kinky vorton as a function of the charge and winding number, and the demonstration that the chiral limit is a repulsive fixed point. Stability to both axial and non-axial perturbations is exhibited in both the electric and chiral regimes, though surprisingly long lived ringing modes are observed. Kinky vortons which are too magnetic are found to suffer from a pinching instability, which results in a reduction in the winding number allowing conversion of magnetic into electric solutions. Finally, we discuss the implications of our results on kinky vortons to possible future work on cosmic vortons.
The model

The Lagrangian density of our (2+1)-dimensional model is given by

\[ L = \partial_\mu \phi \partial^\mu \phi + \partial_\mu \sigma \partial^\mu \bar{\sigma} - \frac{\lambda_\phi}{4} (\phi^2 - \eta_\phi^2)^2 - \frac{\lambda_\sigma}{4} (|\sigma|^2 - \eta_\sigma^2)^2 - \beta \phi^2 |\sigma|^2 + \frac{\lambda_\sigma}{4} \eta_\sigma^4 \]  

(2.1)

where \( \phi \) and \( \sigma \) are real and complex scalar fields respectively, with \( \eta_\phi, \eta_\sigma, \lambda_\phi, \lambda_\sigma, \beta \) all real positive constants. This Lagrangian density can be obtained from the global version of Witten’s \( U(1) \times U(1) \) model \[15\] by restricting one of the complex scalar fields to be real.

The theory has a global \( \mathbb{Z}_2 \times U(1) \) symmetry and the parameters of the model can be arranged so that in the vacuum the \( \mathbb{Z}_2 \) symmetry is broken, \( \phi = \pm \eta_\phi \neq 0 \), while the \( U(1) \) symmetry remains unbroken, \( |\sigma| = 0 \). For this symmetry breaking pattern there exist kink strings constructed from the \( \phi \) field. If the infinite kink string lies along the \( y \)-axis, then it is given by the solution

\[ \phi = \eta_\phi \tanh \left( \frac{\eta_\phi \sqrt{\lambda_\phi x}}{2} \right), \quad \sigma = 0. \]  

(2.2)

The situation of interest is when a condensate of the \( \sigma \) field carrying current and charge forms in the core of the kink string. For the infinite string given above, such a condensate field takes the form

\[ \sigma = e^{i(\omega t + ky)}|\sigma|, \]  

(2.3)

where \(|\sigma|\) is a function of \( x \) only with \(|\sigma| \to 0 \) as \(|x| \to \infty \). The constant \( k \) describes the rate of twisting of the condensate along the string, though in the literature this is referred to as winding, rather than twisting, so we will stick to this common convention.

A non-zero value of \( \omega \) induces a charge \( Q \) associated with the global \( U(1) \) symmetry, and the winding \( k \) generates a current along the string. It is easy to see that charge and current will have opposite effects on the string, so it is useful to introduce the combination

\[ \chi \equiv \omega^2 - k^2. \]  

(2.4)

In the literature solutions with \( \chi = 0 \) are termed chiral, whereas solutions with \( \chi > 0 \) are referred to as electric and those with \( \chi < 0 \) are called magnetic \[4, 10\].

It is also convenient to introduce the quadratic coefficients for both fields by the definitions

\[ m_\phi^2 \equiv \frac{\lambda_\phi}{2} \eta_\phi^2, \quad m_\sigma^2 \equiv \frac{\lambda_\sigma}{2} \eta_\sigma^2 + \chi. \]  

(2.5)

Taking into account the form of the condensate (2.3), a simple analysis of the potential term in (2.1), reveals that the condition for the \( \mathbb{Z}_2 \) symmetry to be broken in the vacuum is

\[ \frac{m_\phi^4}{\lambda_\phi} > \frac{m_\sigma^4}{\lambda_\sigma}. \]  

(2.6)

Similarly, for the \( U(1) \) symmetry to remain unbroken in the vacuum requires that the effective mass term (including a contribution from the interaction) for the \( \sigma \) field is positive, which
results in the constraint
\[ \beta > \frac{\lambda_\phi m_\sigma^2}{2m_\phi^2}. \] (2.7)

Note that both conditions (2.6) and (2.7) are identical to those that arise in the cosmic vorton case [10].

For a condensate to form in the string core we require that the solution (2.2) with no condensate is an unstable solution of the field equations
\[ \partial_\mu \partial^\mu \phi + \frac{\lambda_\phi}{2} (\phi^2 - \eta_\phi^2) \phi + \beta \phi |\sigma|^2 = 0, \] (2.8)
\[ \partial_\mu \partial^\mu \sigma + \frac{\lambda_\sigma}{2} (|\sigma|^2 - \eta_\sigma^2) \sigma + \beta \phi^2 \sigma = 0. \] (2.9)

Linearizing these equations around the solution (2.2) with a \( \sigma \) field of the form
\[ \sigma = e^{i\Omega t} e^{i(\omega t + ky)} |\sigma|, \] (2.10)
where again \( |\sigma| \) is a function of \( x \) only, yields the eigenvalue equation
\[ -|\sigma|^" + (\beta \eta_\phi^2 \tanh^2 \left( \frac{\eta_\phi \sqrt{\lambda_\phi x}}{2} \right) - \frac{\lambda_\sigma}{2} \eta_\sigma^2 - \chi) |\sigma| = \Omega^2 |\sigma| \] (2.11)

This is a classic example of a stationary Schrödinger problem where the spectrum is known explicitly [9]. Requiring a negative mode, \( \Omega^2 < 0 \), gives the condition
\[ \beta < \frac{\lambda_\phi m_\sigma^2 (2m_\sigma^2 + m_\phi^2)}{2m_\phi^4}. \] (2.12)

This is similar to the condition derived for cosmic vortons [7, 9], but in that case some approximations must be made, whereas here this is an exact result.

The three conditions (2.6), (2.7), (2.12) provide constraints on the constants of the theory that must be satisfied for some range of the parameter \( \chi \).

### 3 Exact solutions

For an infinite static string, where the condensate field has the form (2.3), the field equations (2.8) and (2.9) reduce to the following nonlinear coupled ordinary differential equations
\[ \phi" = \phi \left( \frac{\lambda_\phi}{2} (\phi^2 - \eta_\phi^2) + \beta |\sigma|^2 \right) \] (3.1)
\[ |\sigma|^" = |\sigma| \left( -\chi + \frac{\lambda_\sigma}{2} (|\sigma|^2 - \eta_\sigma^2) + \beta \phi^2 \right). \] (3.2)

If \( \chi = 0 \) then an exact solution to the above equations can be found if [12, 8]
\[ \left( 2 - \frac{\lambda_\sigma}{\beta} \right) \eta_\sigma^2 = \left( \frac{\lambda_\phi}{2\beta} + \frac{2\beta}{\lambda_\sigma} - 2 \right) \eta_\phi^2. \] (3.3)
For the purposes of the current investigation it is useful if exact solutions are available for a range of values of the parameter $\chi$. This can only be achieved for the choice

$$2\beta = \lambda_\phi = \lambda_\sigma \equiv \lambda. \quad (3.4)$$

With this relation between the constants of the theory a simplification occurs and the two conditions (2.6) and (2.7) become identical and reduce to the condition $m_\phi^2 > m_\sigma^2$. Furthermore, the final condition (2.12) simplifies to $m_\phi^2 < 2m_\sigma^2$, so the whole set of conditions reduces simply to the constraint

$$\frac{1}{\sqrt{2}} m_\phi < m_\sigma < m_\phi. \quad (3.5)$$

The exact solutions to (3.1) and (3.2) are given by

$$\phi = \eta_\phi \tanh \left( x \sqrt{m_\phi^2 - m_\sigma^2} \right), \quad |\sigma| = \sqrt{\frac{2(2m_\sigma^2 - m_\phi^2)}{\lambda}} \sech \left( x \sqrt{m_\phi^2 - m_\sigma^2} \right), \quad (3.6)$$

which are valid for the entire range of $\chi$ satisfying the condition (3.5).

At this stage there are three remaining constants of the theory, that is, $\eta_\phi, \eta_\sigma, \lambda$. Most of the analysis which follows can be extended to arbitrary values of these constants (providing they satisfy (3.5) for some range of $\chi$) but in order to simplify the presentation and to compare with later numerical simulations we shall now fix some specific values. In the penultimate section we shall discuss how other choices for the constants of the theory influence the results.

In the chiral case, $\chi = 0$, the condition (3.5) reduces to

$$\frac{1}{2} < \left( \frac{\eta_\sigma}{\eta_\phi} \right)^2 < 1. \quad (3.7)$$

To aim for the most generic behaviour possible we set $\eta_\sigma = \sqrt{3} \eta_\phi/2$, so that the value chosen for the ratio which appears in (3.7) is at the midpoint of the allowed range. Finally, the overall scales are normalized by setting $\eta_\phi = 1$ and $\lambda = 2$. In summary, the constants selected in the remainder of this paper (results for more general choices are presented in appendix A) are

$$\eta_\phi = 1, \quad \eta_\sigma = \sqrt{3}/2, \quad \lambda_\phi = 2, \quad \lambda_\sigma = 2, \quad \beta = 1. \quad (3.8)$$

With the above choice the constraint (3.5) reduces to

$$-\frac{1}{4} < \chi < \frac{1}{4} \quad (3.9)$$

hence a reasonable range of electric and magnetic solutions are possible in addition to the chiral case. The infinite string solutions (3.6) simplify to

$$\phi = \tanh \left( \frac{x \sqrt{1 - 4\chi}}{2} \right), \quad |\sigma| = \sqrt{\frac{1 + 4\chi}{2}} \sech \left( \frac{x \sqrt{1 - 4\chi}}{2} \right). \quad (3.10)$$
From these exact solutions the result of adding current \((\chi < 0)\) and charge \((\chi > 0)\) to the condensate can easily be seen. Current clearly quenches the condensate, with the maximum amplitude decreasing as \(\chi\) decreases, until finally the condensate disappears in the limit as \(\chi \to -1/4\), which is the lower edge of the allowed range \((3.9)\). There is also a moderate decrease of the width of the condensate (and kink) as the current is increased. On the other hand, charge enhances the condensate, with the maximum amplitude increasing as \(\chi\) increases. Moreover, the width of the condensate (and kink) increases dramatically as \(\chi\) increases, and becomes delocalized in the limit \(\chi \to 1/4\), which is the upper edge of the allowed range \((3.9)\). Similar quenching and anti-quenching effects are seen in the numerical solutions of infinite cosmic strings carrying a condensate in the \((3+1)\)-dimensional model \([10]\).

4 Kinky vortons

We now turn our attention to kinky vortons, and apply some approximations, in conjunction with the exact results of the previous section, to study circular loops of kink strings carrying a condensate. In the subsequent discussion we will use the terms vortons and cosmic vortons to refer to kinky vortons and their three-dimensional analog respectively.

In polar coordinates the ansatz for a circular loop reads

\[
\phi = \phi(r), \quad \sigma = e^{i(\omega t + N\theta)} |\sigma|(r),
\]

with the boundary conditions \(\phi'(0) = 0\), \(\phi(\infty) = 1\) and \(|\sigma|(0) = 0\), \(|\sigma|(|\infty) = 0\). This describes a vorton of radius \(R\) where \(\phi(R) = 0\). The integer \(N\) is referred to as the winding number. Note that to compare with the straight string the winding rate \(k\) depends upon both \(N\) and the loop radius \(R\) since \(k = N/R\).

The conserved charge \(Q\) of such a field is given by

\[
Q = \frac{1}{2i} \int (\dot{\sigma} \bar{\sigma} - \dot{\bar{\sigma}} \sigma) d^2x = 2\pi \omega \int_0^\infty |\sigma|^2 r dr.
\]

The main aim is to determine the properties of a vorton, and in particular its radius \(R\), as a function of the conserved quantities \(Q\) and \(N\).

If the radius \(R\) is much larger than the width of the condensate (and kink) then a section through the circular loop can be well-approximated by a section through the infinite straight string. This means that approximations of the following form can be used to evaluate a number of required integrals

\[
\int_0^\infty |\sigma|^p r dr \approx R \Sigma_p \equiv R \int_{-\infty}^\infty \left[ \sqrt{\frac{1 + 4\chi}{2}} \text{sech} \left( \frac{x\sqrt{1 - 4\chi}}{2} \right) \right]^p dx,
\]

where the infinite string solution \((3.10)\) has been used to determine the cross-sectional contribution. In particular, we shall later require the exact results

\[
\Sigma_2 = \frac{2(1 + 4\chi)}{\sqrt{1 - 4\chi}}, \quad \Sigma_4 = \frac{2(1 + 4\chi)^2}{3\sqrt{1 - 4\chi}}.
\]
In the analysis of cosmic vortons analogous quantities to $\Sigma_2$ and $\Sigma_4$ are also required, but in that case they can only be computed numerically; although analytical estimates have been given [10] which have a similar form to the exact results (4.4). The main qualitative difference is the appearance of the square root in the denominators in (4.4), which can be attributed to the reduction in the number of spatial dimensions.

Substituting the above result for $\Sigma_2$ into (4.2) yields

$$Q = 4\pi \omega R \frac{(1 + 4\chi)}{\sqrt{1 - 4\chi}}.$$  (4.5)

In the chiral case, $\chi = 0$, then $\omega = k = N/R$, thus (4.5) reveals that for chiral vortons the relationship between the conserved quantities is $Q = 4\pi N$, and is independent of $R$. Thus loops which are initially chiral remain chiral even if the radius changes.

The first goal is to determine the radius $R$ at which the vorton energy is minimized for fixed values of the conserved quantities $Q$ and $N$. One difficulty is that the cross-sectional properties of the string (for example, the amplitude and width of the condensate) vary with the vorton radius. As seen above, this variation is most easily quantified as a function of $\chi$, and is highly implicit as a function of $Q$ and $N$. This is a serious obstacle for cosmic vortons, but the fact that we have exact expressions available makes the problem much more tractable.

Taking the square of equation (4.5) and making the substitution $\omega^2 = \chi + N^2/R^2$ generates the cubic

$$16\chi^3 + 8\left(1 + \frac{2N^2}{R^2}\right)\chi^2 + \left(1 + \frac{8N^2}{R^2} + \frac{Q^2}{4\pi^2 R^2}\right)\chi + \left(N^2 - \frac{Q^2}{16\pi^2}\right) \frac{1}{R^2} = 0.$$  (4.6)

Given the physical properties of the vorton, that is the conserved quantities $Q$ and $N$ together with the radius $R$, this cubic determines $\chi$ (as the root which lies in the interval $(-1/4, 1/4)$).

The cubic (4.6) allows the variation of $\chi$ with the radius $R$ to be easily studied. First of all, the cubic confirms that the chiral case is a fixed point, that is, if $Q = 4\pi N$ then $\chi = 0$ for all $R$. Also, as $R \to \infty$ the final term in (4.6) tends to zero, thus $\chi \to 0$, demonstrating that if a vorton is created with a sufficiently large radius it will be arbitrarily close to chiral. Early studies of cosmic vortons suggested that the chiral limit might be an attractor [6] but more recent investigations [10] have demonstrated that the chiral limit is a repulsive fixed point not an attractor. For kinky vortons this can easily be seen from (4.6) as follows. The electric regime is given by $Q > 4\pi N$, in which case the final term in (4.6) is negative and $\chi > 0$. For large enough $R$ then $\chi \approx 0$, but as $R$ decreases the modulus of the final term in (4.6) increases and therefore so does $\chi$, moving away from the chiral limit. Similarly, if $Q < 4\pi N$, then all coefficients in the cubic (4.6) are positive thus $\chi < 0$ and this is the magnetic regime. As $R$ decreases then again the modulus of the final term in (4.6) increases and therefore $\chi$ becomes more negative, again moving away from the chiral limit. This is illustrated graphically in Figure 1 where curves are plotted showing the value of $\chi$ as $R$ decreases for $Q = 1500$ and $N = 250, 200, 150, 100, 50, 10$ (left to right). In each case as $R$ decreases the curve moves away from the chiral line $\chi = 0$. Each of these curves terminates at the energy minimizing radius, which we now discuss.
Figure 1: The variation of $\chi$ as the vorton loop radius $R$ decreases for $Q = 1500$ and $N = 250, 200, 150, 100, 50, 10$ (left to right). Note that as $R$ decreases each curve moves away from the chiral line $\chi = 0$. Each curve terminates at the energy minimizing radius.

Using the cross-sectional formulae (4.4) and similar expressions for the cross-sectional integrals of powers of $\phi$, $\phi'$ and $|\sigma'|$ the vorton energy is found to be

$$E = \frac{4\pi}{\sqrt{1-4\chi}} \left( \frac{R}{3} \left[ \frac{5}{2} - 5\chi + 4\chi^2 \right] + \frac{1}{R} \left[ N^2(1 + 4\chi) + \left( \frac{Q}{4\pi} \right)^2 \frac{(1 - 4\chi)}{(1 + 4\chi)} \right] \right).$$  \hspace{1cm} (4.7)$$

For given values of the conserved quantities $Q$ and $N$, this defines the energy as a function of the vorton radius $R$, where $\chi$ is again given by (4.6). Using this formula the value of $R$ at which this energy is minimized can be calculated. This is plotted in Figure 2 for $1 \leq N \leq 300$ and the three values $Q = 2000$ (dashed curve), $Q = 1500$ (solid curve) and $Q = 500$ (dotted curve).

The energy formula (4.7) together with the cubic (4.6) can be used to show that if either $Q = 0$ or $N = 0$ then the energy minimizing radius is $R = 0$. Thus vortons require both current and charge, in agreement with the results for cosmic vortons [10].

The above analysis for the vorton radius as a function of $Q$ and $N$ can be compared with the results of numerical simulations of the full nonlinear field theory. Restricting to radially symmetric fields of the form (4.1) the energy can be written as

$$E = 2\pi \int_0^\infty \left\{ \phi'^2 + |\sigma|^2 + \frac{1}{2} (\phi'^2 - 1)^2 + |\sigma|^2 \left( \phi'^2 + \frac{N^2}{r^2} + \frac{1}{2} |\sigma|^2 - \frac{3}{4} \right) \right\} r \, dr + \frac{Q^2}{2\pi} \int_0^\infty |\sigma|^2 r \, dr.$$  \hspace{1cm} (4.8)$$
Figure 2: The energy minimizing radius $R$ as a function of $N$ for the three values $Q = 2000$ (dashed curve), $Q = 1500$ (solid curve), and $Q = 500$ (dotted curve). The results of a field theory energy minimization are presented as filled circles for $Q = 1500$ and open circles for $Q = 2000$. We note that the solution with $Q = 1500$, $N = 10$ and $R = 50$ is the smallest radius solution we have been able to find. Vorton solutions will only exist for points on each of these curves down to a minimal value of $R$ due to the splitting instability.
For fixed values of $N$ and $Q$ this energy has been minimized using a simulated annealing algorithm. The resulting loop radius for the four values $N = 10, 100, 200, 300$ are presented in Figure 2 as the filled circles for $Q = 1500$ and the open circles for $Q = 2000$. From these, and other similar results, it can be seen that the analytic calculation compares well with the numerical results, when a vorton solution exists. However, if $Q$ and $N$ are not sufficiently large then it appears that no vorton solution exists. For example, for $Q = 500$ (corresponding to the dotted curve in Figure 2) no solutions were found for any value of $N$ and for $Q = 1500$ no solution was found with $N < 10$. This is because of a splitting instability where the effective potential trapping the condensate on the kink loop is not strong enough to prevent a separation in which the condensate radiates away to infinity leaving behind a bare kink loop that collapses. This instability can not be seen in the analytic calculation, which assumes that the condensate is trapped on the kink loop, but is clearly evident in the numerical energy minimization process. In principle, it should be possible to numerically map out the region of the $(Q, N)$-plane in which vortons exist. We have not performed this computation due to the fact that it requires significant calculation to determine the boundary of instability since it can take a long time to develop when close to the critical value.

5 Field theory dynamics

In this section we describe the results of numerical simulations of the dynamical field theory equations (2.8) and (2.9). The numerical algorithm is an explicit finite difference scheme which is second order accurate in both the space and time derivatives. For most simulations the lattice spacing is taken to be $\Delta x = 0.5$, although for some simulations this is reduced to $\Delta x = 0.25$ when greater resolution is required. The time step is $\Delta t = 0.1$ for all the simulations presented. Neumann boundary conditions are applied and the number of grid points ranges from $401 \times 401$ to $2801 \times 2801$ depending upon the size of the vorton that is being simulated.

To study the stability of a vorton to axially symmetric perturbations we wish to consider its evolution from an initial state in which the radius differs from that of the optimal value. Of course, the initial field must have the correct charge $Q$ and winding $N$ of the vorton we wish to study. To create such an initial condition for a circular loop with an initial radius $R_0$, we use the radially symmetric ansatz (4.1) with the required value of $N$ and where the fields through a section of the loop are approximated by the infinite straight string fields (3.10). Let $\chi_0$ be the value of $\chi$ used in the initial conditions, which is not that of the original vorton to be investigated, but is obtained by substituting the required values of $Q$ and $N$, together with the initial radius $R_0$, into the cubic (4.6). The initial frequency $\omega_0$ is then obtained as

$$\omega_0 = \sqrt{\chi_0 + \left(\frac{N}{R_0}\right)^2},$$

where we have made use of the formula (2.4).

Vortons with a small loop radius require less computational resources to study than larger vortons, simply because fewer grid points are required to fit the vorton inside the simulation region. From Figure 2 and recalling the earlier comments that large values of $Q$ appear to be necessary to avoid the splitting instability, it can be seen that the strongly electric regime
Figure 3: The radius of an electric vorton for $0 \leq t \leq 50000$. The charge and winding are $Q = 1500$ and $N = 10$, corresponding to an optimal radius of $R = 50$, but the initial condition has $R = R_0 = 70$. The inset is a blow-up of this plot for early times $0 \leq t \leq 2000$. It can be seen that the radius oscillates around the optimal value but the amplitude decreases extremely slowly.

Figure 4: An electric vorton at $t = 0, 8000, 20000$. Dark (red) regions are where $\phi^2 \leq 0.2$. The charge and winding number are $Q = 1500$ and $N = 10$, corresponding to an optimal radius of $R = 50$. The initial condition is a circular loop of radius $R = 50$ which has been squashed by 10% along the $y$-axis and stretched by the same factor along the $x$-axis.
Figure 5: The radius along the positive $x$-axis of an electric vorton for $0 \leq t \leq 20000$. The charge and winding are $Q = 1500$ and $N = 10$, corresponding to an optimal radius of $R = 50$, but the initial condition is an ellipse. The inset is a blow-up of this plot for early times $0 \leq t \leq 2000$. Long-lived ringing modes are observed once again.

is the easiest to study. In particular, there is a solution with $Q = 1500$ and $N = 10$, with a corresponding value of $\chi = 0.193$ and a radius $R = 50$, which is the smallest vorton that appears to exist in this theory. Field theory simulations of this vorton using the correct initial radius $R_0 = 50$ confirm that the vorton radius remains constant throughout the simulation. To test stability we consider an initial condition in which the radius is $R_0 = 70$ and therefore is substantially larger than the correct value for the given charge and winding. The evolution of the radius with time is displayed in Figure 3. The inset shows the evolution of the radius for earlier times $0 \leq t \leq 2000$, when only a few oscillations of the radius have taken place. This confirms that the radius oscillates around the optimal value $R = 50$ but reveals that the amplitude of the oscillation decreases extremely slowly due to radiation. The full plot extends to very large times $0 \leq t \leq 50000$ and reveals the decaying amplitude and approach to the optimal radius, but even after over a hundred oscillations the amplitude is still a reasonable fraction of the initial perturbation. The ringing modes are surprisingly long lived and the simulations require a large number of time steps (in this example half a million) to observe the approach to equilibrium.

We have performed a number of simulations similar to the one described above, and all suggest that when vorton solutions exist they are stable to axially symmetric perturbations. In order to test stability to general perturbations we require an initial condition which breaks the axial symmetry, but preserves the charge $Q$ of the unperturbed solution. Such an initial condition is obtained by taking a vorton solution and performing the combined
scaling transformation \( x \mapsto x\mu \) and \( y \mapsto y/\mu \), which clearly leaves the charge \( Q \) invariant and converts the circular vorton into an ellipse. In the study of cosmic vortons with a modified interaction term, elliptic initial conditions were also considered, but in that case it appears that these were required to be fine-tuned to ensure an initially homogeneous current distribution. As we shall see, our vortons are more robust and do not require any fine-tuning of the initial elliptic loop.

As an illustration, we again consider the electric vorton with parameters \( Q = 1500 \) and \( N = 10 \) yielding the radius \( R = 50 \). The initial condition has squashing parameter \( \mu = 1.1 \) to produce the ellipse displayed in Figure 4A. Dark regions are where \( \phi^2 \leq 0.2 \), and hence indicate the position of the kink string. The vorton evolves as a rotating and oscillating ellipse, with the initial elliptical shape still apparent even at the much later time \( t = 20000 \) Figure 4C. It is expected that the eccentricity of the initial loop decreases with time so that the unperturbed circular vortex is eventually recovered. However, as with the axially symmetric perturbation, there are extremely long-lived ringing modes. In Figure 5 we plot the radius along the positive \( x \)-axis as a function of time, to demonstrate the slow approach towards equilibrium.

The field theory dynamics described so far has been concerned with strongly electric vortons, but similar results have also been obtained for chiral vortons. As an example, Figure 6 displays the evolution \((t = 0, 5000, 10000)\) of a squashed chiral vorton. The unperturbed vorton has \( Q = 1500 \) and \( N = 119 \) giving a radius \( R = 185 \) and \( \chi \approx 0 \). The elliptic perturbation is given by a squashing parameter \( \mu = 1.1 \). The chiral vorton is substantially larger than the strongly electric vorton studied above with the same charge \( Q \). Recall that as \( \chi \) decreases the kink and condensate width both decrease, so that the ratio of the vorton radius to kink width is significantly larger in the chiral case than in the strongly electric regime. Chiral vortons are certainly well inside the thin ring limit. This is illustrated in Figure 6 where the dark region is where \( \phi^2 \leq 0.5 \). Note that this value of 0.5 is larger than the value 0.2 used in the plots of the electric vorton, and therefore makes the chiral vorton appear thicker than it would have done if the same value as earlier had been used. However, for the chiral vorton a much bigger grid is required and the results in Figure 6 correspond to a grid containing \( 1001 \times 1001 \) grid points, but with the same lattice spacing \( \Delta x = 0.5 \). These results confirm the stability of chiral vortons and also demonstrate that significant computing resources are required even to study kinky vortons in the chiral regime and that similar studies of cosmic vortons would be prohibitive. This helps to explain why numerical studies of cosmic vortons are currently limited and the only solutions found to date are in a theory with a modified interaction term designed to reduce these difficulties. We shall make some further remarks on this topic later.

Analytic and numerical work [10] on infinite cosmic strings carrying current and charge has shown that if the string is strongly magnetic \((\chi \ll 0)\) then it will develop a pinching instability, in which the condensate field is driven to zero at some point on the string and the field partially unwinds to produce a string with a lower winding number \( N \). Based on this result it has been conjectured that a magnetic vorton with \(|\chi| > \chi_c\) will excite the unstable pinching mode, reduce its winding number and recover as a vorton which is less magnetic,
or possibly even electric. We shall now investigate the dynamics of magnetic kinky vortons and demonstrate this process has a significant impact on the dynamics.

An analysis of infinite kink strings to a pinching instability is presented in appendix B and suggests that such an instability will be present if \( \chi < -0.1 \). This should provide a reasonable estimate for the value at which such an instability emerges for kinky vortons, but is not expected to be exact as it neglects the curvature of the vorton loop. In fact the numerical simulations described below reveal an instability when \( \chi \approx -0.08 \), which is in reasonable agreement with the analytic calculation.

A magnetic vorton will have a larger radius and a smaller width than its chiral counterpart so even larger grid sizes must be used to simulate the magnetic regime. The following simulations of magnetic vortons used a grid containing 2801 \( \times \) 2801 points and a reduced lattice spacing of \( \Delta x = 0.25 \), with 80000 time steps per simulation. Once again it is worth pointing out that analogous simulations in (3+1)-dimensions would require very considerable computing resources.

First of all we shall consider a magnetic vorton which is expected to be stable. If \( Q = 1500 \) and \( N = 159 \) then \( R = 213 \) and \( \chi = -0.04 \), which should be within the stable regime. The dashed curve in Figure 7 shows that the radius oscillates with a small amplitude around the initial value, suggesting that this magnetic vorton is indeed stable.

For the values \( Q = 1500 \) and \( N = 214 \) the vorton radius is \( R = 245 \) and \( \chi = -0.08 \), which is expected to be close to the onset of the pinching instability. The solid curve in Figure 7 displays the radius as a function of time for this vorton. It can be seen that the radius decreases significantly towards the start of the evolution and subsequently oscillates around a much lower value. This is precisely the behaviour expected of an unstable magnetic vorton which partially unwinds and recovers to a vorton with a lower value of \( N \). This
Figure 7: The radius as a function of time for two initially magnetic vortons. The dashed curve corresponds to a vorton with initial parameters $Q = 1500$ and $N = 159$ giving $R = 213$ and $\chi = -0.04$. This vorton is stable and the radius oscillates around the initial value with a small amplitude. The solid curve is for a vorton with initial parameters $Q = 1500$ and $N = 214$ giving $R = 245$ and $\chi = -0.08$. This vorton is unstable and the radius decreases significantly near the start of the evolution and then oscillates around the lower value of $R = 167$. 
Figure 8: The dark (red) region is where $\Re(\sigma) > 0.05$, plotted at the times $t = 0, 1000, 8000$. Initially it can be seen that the winding number is $N = 214$ but this has reduced to $N = 98$ in the two subsequent plots.

Interpretation is confirmed in Figure 8 where we plot the dark region where $\Re(\sigma) > 0.05$ at times $t = 0, 1000, 8000$. The advantage of displaying the quantity $\Re(\sigma)$ is that not only can the location of the condensate (and hence the vorton radius) be observed but so can the winding number $N$, since it corresponds to the number of dashes contained in the vorton loop. Counting the number of dashes in Figure 8A confirms that initially $N = 214$. Counting the number of dashes in Figure 8B and Figure 8C reveals that each configuration has a winding number $N = 98$. A vorton with $Q = 1500$ and $N = 98$ has a radius $R = 167$ and this is consistent with the radius around which the solid curve in Figure 7 oscillates at later times. This vorton has $\chi = 0.03$ and is therefore electric. These results confirm that the initial magnetic vorton is unstable and partially unwinds to convert to a stable electric vorton. Note that the pinching instability can only be excited if the axial symmetry of the vorton is broken but this is provided by the small perturbation produced by the numerical grid and in particular the boundary. This can be seen by the square deformation of the vorton in Figure 8C.

6 Parameter values and electric instability

The field theory simulations discussed in the previous section illustrate that the computational resources required to study strongly electric vortons are not as substantial as for chiral or magnetic vortons. This is relevant for the study of cosmic vortons, where simulations in (3+1)-dimensions are close to the limit of current feasibility. It is more likely that cosmic vortons can be constructed numerically if they exist in the strongly electric regime. In this section we shall discuss how the existence of strongly electric kinky vortons can depend crucially on the specific parameter values of the model.
Recall from earlier, that vortons do not exist if they are too electric because of a splitting instability, where the effective potential trapping the condensate on the kink loop is not strong enough to prevent a separation in which the condensate radiates away to infinity. The only current numerical construction [10] of cosmic vortons is in a modified model where the interaction term $|\phi|^2|\sigma|^2$ is replaced by a non-renormalizable interaction term $|\phi|^6|\sigma|^2$. The effect of this modification is to increase the width of the effective trapping potential and prevent the separation of the condensate and the string, thereby allowing solutions which are more electric (and therefore of a smaller radius) than in the original theory. Below we demonstrate that the effective trapping potential, and its dependence on $\chi$, can vary with the choice of the parameters in the Lagrangian. In particular, the parameters used in this paper (chosen to allow explicit exact infinite string solutions carrying current and charge) produce a trapping potential which does not decrease as $\chi$ increases, in contrast to some other choices. This suggests that the parameter values used have allowed the construction of strongly electric vortons that may not exist for some other parameter values.

To study the trapping potential we introduce the normalized interaction

$$\Gamma(\chi) = \frac{\int_{-\infty}^{\infty} \phi^2|\sigma|^2 \, dx}{\int_{-\infty}^{\infty} \eta_\phi^2|\sigma|^2 \, dx} \tag{6.1}$$

which we refer to as the splitting parameter. In this definition $\phi$ and $\sigma$ are the infinite straight kink and condensate fields respectively, for a given value of $\chi$. By construction we have that $0 < \Gamma < 1$ and the larger the value of $\Gamma$ the weaker the trapping potential, since the condensate samples less of the kink field away from its vacuum value. This is why we refer to $\Gamma$ as the splitting parameter, since larger values of $\Gamma$ suggest a greater tendency to instability via splitting.

For the parameter values (3.8) used in this paper, the exact solution can be used to compute $\Gamma(\chi)$ explicitly, yielding the result $\Gamma(\chi) = 1/3$, independent of $\chi$. This is a special situation and is not generic. As an example, consider the parameter values used in the study of cosmic vortons in [10], namely

$$\eta_\phi = 1, \quad \eta_\sigma = \frac{1}{2}, \quad \lambda_\phi = \frac{3}{2}, \quad \lambda_\sigma = 10, \quad \beta = \frac{3}{2}. \tag{6.2}$$

For this parameter set there are no exact solutions available and hence the fields and $\Gamma(\chi)$ must be computed numerically. The result is displayed as the solid curve in Figure 9 for the allowed range of $\chi$ (for comparison the dashed line is a numerical computation for the parameter set (3.8) and is a test on the numerical accuracy since the exact result $\Gamma(\chi) = 1/3$ is known in this case). Figure 9 reveals that $\Gamma(\chi)$ increases with $\chi$ for the parameters (6.2), and suggests that the splitting instability is more severe for this parameter set, hence making it more difficult to find strongly electric (and therefore small radius) vortons. We have been able to compute some kinky vortons with this parameter set but the solutions indeed appear to exist only in a regime which is much closer to the chiral limit. This fact, together with the absence of exact solutions, makes it a more difficult problem to study in detail.

As mentioned above, one approach to partially overcome the splitting instability is to modify the power in the interaction term [10]. However, the above results suggest that an
alternative possibility is to find a more favourable region of parameter space. The parameter set (3.8) appears to be a better choice than (6.2) but there is no reason to believe that the parameters which allow exact solutions are the best choice from this point of view. As a further example, consider the parameter set

$$\eta_\phi = 1, \quad \eta_\sigma = 1, \quad \lambda_\phi = 3, \quad \lambda_\sigma = 2, \quad \beta = 2.$$  

The associated splitting parameter $\Gamma(\chi)$ is shown as the dotted curve in Figure 9 for the allowed range of $\chi$. In this case $\Gamma(\chi)$ decreases as $\chi$ increases, suggesting that the parameters (6.3) could be an improvement over the values (3.8) in searching for small electric vortons. This issue, and particularly its implications for cosmic vortons, is currently under investigation.

7 Conclusion

In this paper we have introduced and studied kinky vortons, which are (2+1)-dimensional analogues of cosmic vortons. The lower dimensional system allows some exact results to be obtained and greatly simplifies both the analytic and numerical treatment. We find that there are remarkable similarities between kinky and cosmic vortons, and that many of the expected properties of cosmic vortons can be realized in the kinky vorton system.
Our study of kinky vortons has revealed some of the difficulties that are responsible for the current limited results on cosmic vortons and has suggested some possible approaches to tackle these issues. Investigations of cosmic vortons are currently in progress which make use of the results presented here.

Most studies of cosmic vortons apply a string approximation (see [3] and references therein) which should be valid in the thin ring limit, where the vorton radius is much larger than the cosmic string width. The simulations of kinky vortons suggests that this will provide a good approximation providing the vorton indeed exists and is stable. However, the existence and stability appear to require a detailed analysis of the full field theory equations, so some caution may need to be applied when using results derived within the string approximation.

It is perhaps worth pointing out that vorton-like objects occur as classical solutions in several field theories describing condensed matter systems [1, 5, 13], so in the future it may be possible to study these objects experimentally within this setting.

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Note Added

During the final stages of the preparation of this paper an interesting preprint [11] appeared on the arXiv containing some numerical results on the construction of cosmic vortons. However, the cosmic vortons in [11] are in a very different regime to those discussed in this paper. In particular, the vorton radius is almost equal to the cosmic string width, so they are far from the thin ring limit. Indeed, only very low winding numbers, $N \leq 5$, could be computed and it was acknowledged that the construction of vortons in the thin ring limit remains a numerical challenge. The vortons in [11] were constructed by perturbing away from the sigma model limit $\lambda_\phi = \lambda_\sigma \to \infty$ and $\beta \to \infty$ and it was noted that such vorton solutions are essentially the Skyrmion solutions obtained by the current authors [1] in a related model of a two-component Bose-Einstein condensate.

Appendix A

In this appendix we provide some formulae for kinky vortons for general values of the parameters $\lambda$, $\eta_\phi$ and $\eta_\sigma$, while maintaining the condition $\lambda = \lambda_\phi = \lambda_\sigma = 2\beta$. If we define
\[ \alpha = (\eta/\eta_0)^2 \text{ and } \tilde{\chi} = \chi/m_\phi^2, \text{ then the constraint (3.5) becomes} \]
\[ \frac{1}{2} - \alpha < \tilde{\chi} < 1 - \alpha, \quad (A.1) \]
and the solutions (3.6) can be written as
\[ \phi = \eta_0 \tanh \left[ m_\phi x \sqrt{1 - \alpha - \tilde{\chi}} \right], \]
\[ |\sigma| = \sqrt{2} \eta_0 \sqrt{\alpha - \frac{1}{2} + \tilde{\chi} \sech \left[ m_\phi x \sqrt{1 - \alpha - \tilde{\chi}} \right]}. \quad (A.2) \]

Using these formulae one can show that
\[ Q = \frac{16\pi R_\omega (\alpha - \frac{1}{2} + \tilde{\chi})}{\lambda \sqrt{1 - \alpha - \tilde{\chi}}}, \quad (A.3) \]
and \( \tilde{\chi} \) satisfies the cubic
\[ 16\tilde{\chi}^3 + \left[ 32 \left( \alpha - \frac{1}{2} \right) + \frac{16N^2}{R^2} \right] \tilde{\chi}^2 + \left[ 16 \left( \alpha - \frac{1}{2} \right)^2 + 32 \left( \alpha - \frac{1}{2} \right) \frac{N^2}{R^2} + \left( \frac{Q\Lambda}{4\pi \hat{R}} \right)^2 \right] \tilde{\chi} \]
\[ + \left[ 16 \left( \alpha - \frac{1}{2} \right)^2 N^2 + (\alpha - 1) \left( \frac{Q\Lambda}{4\pi} \right)^2 \right] \frac{1}{\hat{R}^2} = 0 \quad (A.4) \]

where \( \hat{R} = m_\phi R \). The energy is then given by
\[ E = \frac{2\pi \eta_0^2}{\sqrt{1 - \alpha - \tilde{\chi}}} \left( \frac{\hat{R}}{3} \left[ 2(2\alpha+1)(2-2\alpha-\tilde{\chi})+4\tilde{\chi}^2 \right] + \frac{1}{\hat{R}} \left[ 4N^2 \left( \alpha - \frac{1}{2} + \tilde{\chi} \right) + \frac{1-\alpha-\tilde{\chi}}{\alpha - \frac{1}{2} + \tilde{\chi}} \left( \frac{Q\Lambda}{8\pi} \right)^2 \right] \right). \quad (A.5) \]

For the values \( \eta_0 = 1, \lambda = 2 \) and \( \alpha = 3/4 \), as used in the rest of this paper, these expressions reproduce those presented earlier.

**Appendix B**

In this appendix we provide an analysis to estimate the value of \( \chi \) below which a pinching instability is expected to exist. This is a lower dimensional analogue of a similar calculation performed for cosmic vortons in [10].

The instability is studied for a straight infinite kink string in the magnetic regime. Using the Lorentz invariance of the theory we can restrict to the case \( \omega = 0 \) and therefore \( \chi = -k^2 \).

The task is to consider the stability of the kink string fields (3.10)
\[ \phi_0 = \tanh \left( \frac{x\sqrt{1-4\chi}}{2} \right), \quad \sigma_0 = e^{iky} \sqrt{\frac{1+4\chi}{2}} \sech \left( \frac{x\sqrt{1-4\chi}}{2} \right). \quad (B.1) \]
An exact analysis of the stability of this solution would require a numerical approach, so to make analytic progress we consider a perturbation of the condensate field of the form

$$\sigma = \sigma_0 + |\sigma_0|\nu,$$

where $\nu(y, t)$ is a complex function of $y$ and $t$ but is independent of $x$. There are no exact negative modes with the above specified $x$ dependence, as can be seen from the linearized equation

$$|\sigma_0|(\partial_t^2 \nu - \partial_y^2 \nu) - \nu|\sigma_0|'' + \nu|\sigma_0|(2|\sigma_0|^2 - \frac{3}{4} + \phi_0^2) + \nu|\sigma_0|\sigma_0'' = 0. \quad (B.3)$$

However, we can consider integrating this equation over the string cross-section by multiplying by $|\sigma_0|$ and integrating over $x$. This produces the equation

$$\Sigma_2(\partial_t^2 \nu - \partial_y^2 \nu) + \nu(\chi \Sigma_2 + \Sigma_4) + \nu\Sigma_4 e^{2iky} = 0,$$

and we can now search for negative modes of this equation of the form

$$\nu(y, t) = e^{\Lambda t} e^{iky} u(y), \quad (B.5)$$

where $u(y)$ is a complex function of $y$ and we require that $\Lambda^2 > 0$ so that $\Lambda$ is real with a positive value corresponding to the required negative mode. The eigenvalue equation for $u(y)$, obtained by substituting (B.5) into (B.4), is

$$-u'' - 2iku' + (u + \bar{u})\Sigma + \Lambda^2 u = 0, \quad (B.6)$$

where we have defined $\Sigma = \Sigma_4/\Sigma_2$. Note that an equivalent method to derive this equation is to substitute the perturbation (B.2) into the expression for the energy and integrate over the string cross-section, then require negative modes for the resulting effective energy.

Splitting $u$ into real and imaginary parts as $u = u_1 + iu_2$ gives the coupled equations

$$-u_1'' + 2ku_2' + (2\Sigma + \Lambda^2)u_1 = 0, \quad -u_2'' - 2ku_1' + \Lambda^2u_2 = 0. \quad (B.7)$$

Expanding in terms of fourier modes $u_1 = a_1 \cos(py)$ and $u_2 = a_2 \sin(py)$ yields the equation

$$\left(\begin{array}{cc} p^2 + 2\Sigma + \Lambda^2 & 2kp \\ 2kp & p^2 + \Lambda^2 \end{array}\right) \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) = 0. \quad (B.8)$$

A non-zero solution exists only if the determinant vanishes, which gives the condition

$$\Lambda^2 = -(p^2 + \Sigma) \pm \sqrt{\Sigma^2 - 4\chi p^2}. \quad (B.9)$$

Requiring a solution with $\Lambda^2 > 0$ produces the constraint

$$p^2 + 2\Sigma + 4\chi < 0, \quad (B.10)$$

and hence

$$\chi < -\frac{\Sigma_4}{2\Sigma_2}. \quad (B.11)$$

Using the exact expressions (4.4) this constraint is equivalent to $\chi < -\frac{1}{10}$.

The above analysis has neglected the curvature of the vorton and also integrated over the string cross-section, rather than solving the full linearized equations, but it is expected to give a reasonable estimate of the critical value of $\chi$ below which the magnetic instability appears.

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