Unambiguous Discrimination Between Linearly Dependent States With Multiple Copies

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A set of quantum states can be unambiguously discriminated if and only if they are linearly independent. However, for a linearly dependent set, if $C$ copies of the state are available, then the resulting $C$ particle states may form a linearly independent set and be amenable to unambiguous discrimination. We obtain one necessary and one sufficient condition for the possibility of unambiguous discrimination between $N$ states given that $C$ copies are available and that the single copies span a $D$ dimensional space. These conditions are found to be identical for qubits. We then examine in detail the linearly dependent trine set. The set of $C > 1$ copies of each state is a set of linearly independent lifted trine states. The maximum unambiguous discrimination probability is evaluated for all $C > 1$ with equal a priori probabilities.

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I. INTRODUCTION

Much of the fascination with the information-theoretic properties of quantum systems derives from collective phenomena and processes. On one hand, the information contained in entangled quantum systems is of a collective nature and is central to many intriguing applications of quantum information, such as teleportation and quantum computing. On the other, there are collective operations, such as collective measurements on several quantum systems. Broadly speaking, collective measurements on a set of systems can yield more, or better information than one can obtain by carrying out separate measurements on the individual subsystems, even if these are not entangled. For example, the use of collective measurements is essential for attaining the true classical capacity of a quantum channel\cite{1}, since capacities attained with receivers performing collective measurements on increasingly large strings of signal states are superadditive.

A further illustration of the superiority of collective over individual measurements is the ‘non-locality without entanglement’ discovered by Bennett et al\cite{2}. This refers to the fact that one
can construct a set of orthogonal product states which can only be perfectly distinguished by a collective measurement.

In this paper, we provide a further demonstration of the increased knowledge that can be obtained using collective rather than individual measurements, relating to unambiguous state discrimination\cite{3}. Such measurements can reveal, with zero probability of error, the state of a quantum system, even if the possible states are nonorthogonal. Perfect discrimination between nonorthogonal states is impossible and the price we pay is the non-zero probability of inconclusive results.

It has been established that unambiguous discrimination is possible only for linearly independent states\cite{4}. However, suppose that the possible states form a linearly dependent set, but we have \( C > 1 \) copies of the actual state at our disposal. Unambiguous discrimination is impossible using separate measurements on the individual copies. If, however, the possible \( C \) particle states form a linearly independent set, then unambiguous discrimination will be possible by carrying out a collective measurement on all \( C \) copies.

In section II, we derive one necessary and one sufficient condition for \( N \) states to be amenable to unambiguous discrimination, given that \( C \) copies of the state are available and that the possible single copy states span a finite, \( D \) dimensional space. For qubits \((D = 2)\), these conditions are identical. In section III, we work out in detail a specific example, that of multiple copies of the so-called trine states. The trine set is linearly dependent, although the set comprised of multiple copies of these states is linearly independent for \( C \geq 2 \). Indeed, these states are the lifted trine states recently discussed in a different but related context by Shor\cite{5}. We obtain the maximum discrimination probability for these multi-trine states with equal a priori probabilities and find that it has some curious, unexpected features.

II. BOUNDS ON THE MAXIMUM NUMBER OF DISTINGUISHABLE STATES

Consider the following scenario: a quantum system is prepared in one of the \( N \) pure states \(|\psi_j\rangle\), where \( j = 1, \ldots, N \). These states are nonorthogonal and we would like to determine which state has been prepared. If we are unwilling to tolerate errors, then we should adopt an unambiguous discrimination strategy. Such a measurement will have \( N + 1 \) outcomes; \( N \) of these correspond to the possible states and a further outcome gives inconclusive results. It has been established that the zero errors constraint leads to a nonzero probability of inconclusive results for nonorthogonal states\cite{3}.
Suppose that the $|\psi_j\rangle$ span a $D$ dimensional Hilbert space $\mathcal{H}$. Clearly, $D \leq N$. If $D = N$, then the states are linearly independent. If, on the other hand, $D < N$, then they are linearly dependent. Whether or not the set is linearly independent is crucial, since linear independence is the necessary and sufficient condition for a set of pure states to be amenable to unambiguous discrimination[4].

If, however, instead of having just one copy of the state, we have $C > 1$ copies, that is, one of the states $|\psi_j\rangle^{\otimes C}$, then there is the possibility that, even if $\{ |\psi_j\rangle \}$ is a linearly dependent set, $\{ |\psi_j\rangle^{\otimes C} \}$ may be linearly independent, making unambiguous discrimination possible. It is of interest to determine the conditions under which this is so. Here, we will obtain two general results relating to the number of states that can be unambiguously discriminated, given that the single copies span a $D$ dimensional space and that $C$ copies of the state are available. Firstly, we will show that the number of states which can be unambiguously discriminated satisfies the inequality

$$N \leq \binom{C + D - 1}{C}.$$  \tag{2.1}

To see why, let us denote by $\mathcal{H}_{SYM}$ the symmetric subspace of $\mathcal{H}^{\otimes C}$. The states $|\psi_j\rangle^{\otimes C}$ are invariant under any permutation of the states of the single copies and thus lie in $\mathcal{H}_{SYM}$. Denoting by $D_{SYM}$ the dimension of $\mathcal{H}_{SYM}$, it can be shown that

$$D_{SYM} = \binom{C + D - 1}{C}. \tag{2.2}$$

The $|\psi_j\rangle^{\otimes C}$ will be linearly dependent if $N$ is greater than the dimension of $\mathcal{H}_{SYM}$. This, together with Eq. (2.2), leads to inequality (2.1), which is a necessary condition for unambiguous discrimination between $N$ states spanning a $D$ dimensional space given $C$ copies of the state.

This bound holds for all pure states. It is tight, in the sense that for all $C, D$, there exists a set of $N$ pure states $\{ |\psi_j\rangle \}$ such that the equality in (2.1) is satisfied and the set $\{ |\psi_j\rangle^{\otimes C} \}$ is linearly independent. To prove this, we make use of the fact that $\mathcal{H}_{SYM}$ is the subspace of $\mathcal{H}^{\otimes C}$ spanned by the states $|\psi\rangle^{\otimes C}$, for all $|\psi\rangle \in \mathcal{H}$. The set of states $\{ |\psi\rangle^{\otimes C} \}$ is linearly dependent. However, every linearly dependent set spanning a vector space $\mathcal{V}$ has a linearly independent subset which is a basis for $\mathcal{V}$. Let $\{ |\psi_j\rangle^{\otimes C} \}$ be such a subset of $\{ |\psi\rangle^{\otimes C} \}$ for $\mathcal{V} = \mathcal{H}_{SYM}$. These states are linearly independent and satisfy the equality in (2.1) since $N = D_{SYM}$.

We now show that any $N$ distinct pure states can be unambiguously discriminated if

$$N \leq C + D - 1. \tag{2.3}$$

Here, the elements of the set $\{ |\psi_j\rangle \}$ are considered distinct iff $|\langle \psi_j | \psi_j' \rangle| < 1 \ \forall \ j \neq j'$. It will suffice to show that if $N = C + D - 1$, then the states $|\psi_j\rangle^{\otimes C}$ are linearly independent. To see why, we
simply note that if this can be shown, then our more general claim will be true as a consequence of the fact that any subset of a linearly independent set is also linearly independent.

To prove that inequality (2.3) is a sufficient condition for unambiguous discrimination, we assume that 
\[ N = C + D - 1 \]
and again make use of the fact that any linearly dependent set has a linearly independent spanning subset. The set \( \{ |\psi_j\rangle \} \) then has a subset of \( D \) linearly independent states, which we shall denote by \( S_{LI}^1 \). Without loss of generality, we can relabel all states according to the index \( j \) in such a way that \( |\psi_j\rangle \in S_{LI}^1 \) for \( j = 1, \ldots, D \).

Let us now consider the sets \( S_{LI}^r = \{ |\psi_j\rangle^\otimes r | j = 1, \ldots, D + r - 1 \}, \) for \( r = 1, \ldots, C \). Notice that \( S_{LI}^1 \) accords with our previous definition and that \( S_{LI}^C = \{ |\psi_j\rangle^\otimes C \} \). We will use induction to prove that \( \{ |\psi_j\rangle^\otimes C \} \) is linearly independent. The set \( S_{LI}^1 \) is linearly independent by definition. We will show that if \( S_{LI}^{r-1} \) is linearly independent, then so is \( S_{LI}^r \). To do this, we shall require the following:

**Lemma.** Let \( \{ |\chi_k\rangle \} \in \mathcal{H} \) and \( \{ |\phi_k\rangle \} \in \mathcal{H}' \) be sets of distinct, normalised state vectors which have equal cardinality. Consider any normalised states \( |\chi\rangle \in \mathcal{H} \) and \( |\phi\rangle \in \mathcal{H}' \) such that \( |\chi\rangle \) is distinct from all elements of \( \{ |\chi_k\rangle \} \). If the set \( \{ |\phi_k\rangle \} \) is linearly independent, then so is the set \( \{ |\phi_k\rangle \otimes |\chi_k\rangle \} \cup \{ |\phi\rangle \otimes |\chi\rangle \} \).

A proof of this is given in the appendix. The linear independence of \( S_{LI}^{r-1} \) can be seen to imply that of \( S_{LI}^r \) if we make the identifications:

\[
\begin{align*}
\{ |\phi_k\rangle \} &= S_{LI}^{r-1}, \quad (2.4) \\
\{ |\chi_k\rangle \} &= \{ |\psi_j\rangle | j = 1, \ldots, D + r - 2 \}, \quad (2.5) \\
|\phi\rangle &= |\psi_{D+r-1}\rangle^\otimes r-1, \quad (2.6) \\
|\chi\rangle &= |\psi_{D+r-1}\rangle, \quad (2.7)
\end{align*}
\]

for \( r = 2, \ldots, C \). Thus, the set \( S_{LI}^C = \{ |\psi_j\rangle^\otimes C \} \) is linearly independent and this completes the proof. We have shown that (2.3) is a sufficient condition for unambiguous discrimination between \( N \) states spanning a \( D \) dimensional space given \( C \) copies of the state.

Like the necessary condition in (2.1), this bound is the tightest we can obtain using \( N, C \) and \( D \) alone, in the sense that for all values of these parameters which do not satisfy (2.3), there exists a set of states \( \{ |\psi_j\rangle^\otimes C \} \) which is linearly dependent. To prove this, suppose that for \( j = 1, \ldots, D \), the \( |\psi_j\rangle \) are linearly independent and that for \( j = D+1, \ldots, N \), \( |\psi_j\rangle = a_j |\psi_{D-1}\rangle + b_j |\psi_D\rangle \), for some complex coefficients \( a_j, b_j \). If (2.3) is not satisfied, then \( N \geq C + D \) and the subspace spanned by \( |\psi_{D-1}\rangle \) and \( |\psi_D\rangle \) contains at least \( C + 2 \) states in the set \( \{ |\psi_j\rangle \} \). We will now see that the set of
states \(\{|\psi_j\rangle^\otimes C\}_{j = D-1, D, \ldots, N}\) is linearly dependent. For \(j = D-1, \ldots, N\), the \(|\psi_j\rangle\) all lie in the same two dimensional subspace, so that the corresponding \(C\)-fold copies \(|\psi_j\rangle^\otimes C\) lie in the symmetric subspace of \(C\) qubits, which, from Eq. (2.2), is \(C + 1\) dimensional. It follows that if there are at least \(C + 2\) of these states, they must be linearly dependent. This implies that the entire set of \(C\)-fold copies \(\{\{|\psi_j\rangle^\otimes C\}_{j = 1, \ldots, N}\}\) is linearly dependent.

The necessary and sufficient conditions, (2.1) and (2.3), for the linear independence of \(C\) copies of \(N\) states, with single copy Hilbert space dimension \(D\) are thus the most complete statements that can be made about the possibility of unambiguous discrimination given only these three parameters. These two bounds are also, in general, different from each other, which implies that for a particular set of states, additional, more detailed information about the set may be useful.

However, this is not the case for \(D = 2\). For the case of qubits, these bounds are identical and equal to \(C + 1\). The necessary and sufficient condition for the possibility of unambiguous discrimination between \(N\) pure, distinct states of a qubit, given \(C\) copies of the state, is then

\[
N \leq C + 1.
\] (2.8)

The generality of this result is quite remarkable, since it is completely independent of the actual states involved. These will, however, have a strong bearing on the maximum probability of success.

### III. DISCRIMINATION BETWEEN MULTI-TRINE STATES

**A. Trine and lifted trine states**

Having discussed in the preceding section the conditions under which unambiguous discrimination between multiple copies of linearly dependent states is possible, let us examine in detail one particular example, that of the so-called ‘trine’ set. Consider a qubit whose two dimensional Hilbert space is denoted by \(H_2\). Let \(\{|x\rangle, |y\rangle\}\) be an orthonormal basis for \(H_2\). Then the following states form the trine set:

\[
|t_1\rangle = |y\rangle,
\]
\[
|t_2\rangle = \frac{1}{2}(-|y\rangle + \sqrt{3}|x\rangle),
\]
\[
|t_3\rangle = \frac{-1}{2}(|y\rangle + \sqrt{3}|x\rangle).
\] (3.1) (3.2) (3.3)

These states are clearly linearly dependent and so cannot be unambiguously discriminated at the level of one copy. Given only a single copy, we must tolerate a nonzero error probability in any
attempt to distinguish between these states. If they have equal a priori probabilities of 1/3, then the minimum error probability is also equal to 1/3. The optimum such measurement has recently been carried out in the laboratory, where the trine set was implemented as a set of nonorthogonal optical polarisation states. Applications of the trine set and optimal measurements to quantum key distribution are discussed in [10].

The trine set may be regarded as a special case of a more general set of states having the same 3-fold rotational symmetry, but also having a component in a third direction, which exists in a larger, 3 dimensional Hilbert space $\mathcal{H}_3 \supset \mathcal{H}_2$. Let this third dimension be spanned by the vector $|z\rangle$ orthogonal to both $|x\rangle$ and $|y\rangle$. This generalised trine set may be written as

$$|T_j(\lambda)\rangle = \lambda |z\rangle + \sqrt{1 - \lambda^2} |t_j\rangle,$$

for some real parameter $\lambda \in [0, 1]$ known as the lift parameter. When $\lambda = 0$, the $|T_j(\lambda)\rangle$ are just the coplanar trine states. If, however, $\lambda > 0$, then the states are lifted out of the plane and are linearly independent for $\lambda \neq 0, 1$. These are known as lifted trine states. As the lift parameter is increased, the states become increasingly distinct until $\lambda = 1/\sqrt{3}$, at which point they are orthogonal. Increasing $\lambda$ further serves to draw the three states closer to $|z\rangle$ axis until $\lambda = 1$, at which point $|T_j(\lambda)\rangle = |z\rangle$.

In this section, we show that the set of $C$-fold copies of the trine set, which we refer to as a multi-trine set, may be represented as a lifted trine set. We will use this, together with the fact that the maximum discrimination probability for lifted-trine states can be derived exactly, to determine the maximum discrimination probability for multiple copies of the trine states for equal a priori probabilities.

To show that the states $|t_j\rangle \otimes C$ are lifted trine states, we will make use of the fact that the states $|\tau_j(\lambda)\rangle = |T_j(\lambda)\rangle \otimes |t_j\rangle$, for $\lambda \in [0, 1)$, are also lifted trine states, with a different, nonzero lift parameter. To see this, let us define the three orthogonal states:

$$|X\rangle = \sqrt{\frac{2}{1 + \lambda^2}} \left( \lambda |z\rangle \otimes |x\rangle - \frac{\sqrt{1 - \lambda^2}}{2} (|x\rangle \otimes |y\rangle + |y\rangle \otimes |x\rangle) \right),$$

$$|Y\rangle = \sqrt{\frac{2}{1 + \lambda^2}} \left( \lambda |z\rangle \otimes |y\rangle - \frac{\sqrt{1 - \lambda^2}}{2} (|x\rangle \otimes |x\rangle - |y\rangle \otimes |y\rangle) \right),$$

$$|Z\rangle = \frac{1}{\sqrt{2}} (|x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle).$$
Then the $|\tau_j(\lambda)\rangle$ may be written as

\begin{align}
|\tau_1(\lambda)\rangle &= L|Z\rangle + \sqrt{1-L^2}|Y\rangle, \\
|\tau_2(\lambda)\rangle &= L|Z\rangle + \frac{\sqrt{1-L^2}}{2}(-|Y\rangle + \sqrt{3}|X\rangle), \\
|\tau_3(\lambda)\rangle &= L|Z\rangle - \frac{\sqrt{1-L^2}}{2}(|Y\rangle + \sqrt{3}|X\rangle),
\end{align}

(3.8)

(3.9)

(3.10)

where the parameter $L$ is

\begin{equation}
L = \sqrt{\frac{1-\lambda^2}{2}}. \tag{3.11}
\end{equation}

Comparison of the $|\tau_j(\lambda)\rangle$ with the $|T_j(\lambda)\rangle$ shows that they are indeed lifted trine states, with lift parameter $L$, given by Eq. (3.11). Also, for all $\lambda \in [0,1)$, $L \neq 0,1$ and the $|\tau_j(\lambda)\rangle$ are linearly independent.

We can now use simple induction to show that the states $|t_j\rangle^C$ are lifted trine states. In the above argument, if we let $|T_j(\lambda)\rangle = |t_j\rangle$, i.e., take $\lambda = 0$, then we find that $|\tau_j(\lambda)\rangle = |t_j\rangle^2$ and that these are lifted trine states with lift parameter $1/\sqrt{2}$. For the inductive step, we can say that if $|t_j\rangle^C-1$ is a set of lifted trine states with lift parameter $L_{C-1}$, then so is the set $|t_j\rangle^C$, with some lift parameter $L_C$. It follows from Eq. (3.11) that these lift parameters for successive values of $C$ obey the recurrence relation

\begin{equation}
L_C = \sqrt{\frac{1-L_{C-1}^2}{2}}, \tag{3.12}
\end{equation}

with the boundary condition $L_1 = 0$. The solution is

\begin{equation}
L_C = \left[\frac{1}{3} \left(1 - \left(-\frac{\lambda}{2}\right)^{C-1}\right)\right]^{1/2}. \tag{3.13}
\end{equation}

So, the states $|t_j\rangle^C$ are lifted trine states with lift parameter given by Eq. (3.13).

**B. Discrimination between lifted trine states**

To determine the maximum discrimination probability for the lifted trine set with equal a priori probabilities, we make use of the theorem in [11] which gives the maximum discrimination probability for equally probable, linearly independent symmetrical states.

A set of $N$ linearly independent symmetric states can be expressed as

\begin{equation}
|\psi_j\rangle = \sum_{k=0}^{N-1} c_k e^{2\pi i j k / N} |u_k\rangle, \tag{3.14}
\end{equation}
where \( \sum_{k=0}^{N-1} |c_k|^2 = 1, c_k \neq 0 \) and \( \langle u_{k'} | u_k \rangle = \delta_{k,k'} \). The maximum discrimination probability is

\[
P_{\text{max}} = N \times \min_k |c_k|^2. \tag{3.15}
\]

For the lifted trine states, we define the following orthogonal states:

\[
|u_0\rangle = |z\rangle, \tag{3.16}
\]

\[
|u_1\rangle = \frac{e^{\frac{2\pi}{3}i}}{\sqrt{2}}(|x\rangle + i|y\rangle), \tag{3.17}
\]

\[
|u_2\rangle = \frac{e^{\frac{5\pi}{6}i}}{\sqrt{2}}(|x\rangle - i|y\rangle). \tag{3.18}
\]

In terms of these states, one can easily verify that the lifted trine states have the form

\[
|T_1(\lambda)\rangle = \lambda |u_0\rangle + \sqrt{\frac{1 - \lambda^2}{2}} \left( e^{\frac{2\pi}{3}i} |u_1\rangle + e^{\frac{4\pi}{3}i} |u_2\rangle \right), \tag{3.19}
\]

\[
|T_2(\lambda)\rangle = \lambda |u_0\rangle + \sqrt{\frac{1 - \lambda^2}{2}} \left( e^{\frac{4\pi}{3}i} |u_1\rangle + e^{\frac{8\pi}{3}i} |u_2\rangle \right), \tag{3.20}
\]

\[
|T_3(\lambda)\rangle = \lambda |u_0\rangle + \sqrt{\frac{1 - \lambda^2}{2}} (|u_1\rangle + |u_2\rangle). \tag{3.21}
\]

One can verify that these expressions are of the form (3.14) if we take the coefficients \( c_k \) to be

\[
c_0 = \lambda, \tag{3.22}
\]

\[
c_1 = c_2 = \sqrt{\frac{1 - \lambda^2}{2}}. \tag{3.23}
\]

Making use of these expressions and employing (3.15), we find that the maximum discrimination probability for the lifted trine states is

\[
P_{\text{max}} = 3 \times \min \left( \lambda^2, \frac{1 - \lambda^2}{2} \right). \tag{3.24}
\]

The behaviour of \( P_{\text{max}} \) as a function of \( \lambda \) is illustrated in figure (1). For \( 0 \leq \lambda \leq 1/\sqrt{3} \), \( P_{\text{max}} = 2\lambda^2 \), which increases monotonically to 1 until the orthogonality point \( \lambda = 1/\sqrt{3} \). For \( 1/\sqrt{3} \leq \lambda \leq 1 \), \( P_{\text{max}} = (3/2)(1 - \lambda^2) \), which decreases monotonically, reaching zero when \( \lambda = 1 \), at which point all three states are identical.

We have shown how to calculate the maximum discrimination probability for lifted trine states. We will now see how these results can be used to obtain the maximum discrimination probability for multiple copies of the trine states.
FIG. 1: Maximum probability $P_{\text{max}}$ of unambiguous discrimination between lifted trine states as a function of the lift parameter $\lambda$. For $\lambda = 0, 1$, the states are linearly dependent and so unambiguous discrimination is impossible. However, at $\lambda = 1/\sqrt{3} \approx 0.557$, the states are orthogonal and can be discriminated with unit probability.

C. Discrimination between multi-trine states

We are now in a position to calculate the maximum discrimination probability for the states $|t_j \rangle ^{\otimes C}$. It follows from (2.8) that the necessary and sufficient condition for unambiguous discrimination is that $C \geq 2$. Making use of (3.13) and (3.24), we see that the maximum discrimination probability is

$$P_{\text{max}}(|t_j \rangle ^{\otimes C}) = 3 \times \min \left( L_C^2, \frac{1 - L_C^2}{2} \right)$$

$$= 3 \times \min \left( L_C^2, L_{C+1}^2 \right)$$

$$= \min \left( 1 - \left( \frac{-1}{2} \right)^{C-1}, 1 - \left( \frac{-1}{2} \right)^C \right).$$

(3.25)
It is quite straightforward to show that the smaller of these two terms is determined solely by whether \( C \) is even or odd and we find

\[
P_{\text{max}}(|t_j\rangle \otimes C) = \begin{cases} 
1 - 2^{-C} & : \text{even } C \\
1 - 2^{-(C-1)} & : \text{odd } C.
\end{cases}
\] (3.26)

Some interesting observations can be made about this result. Firstly, the minimum probability of inconclusive results, given by \( 1 - P_{\text{max}}(|t_j\rangle \otimes C) \), decreases exponentially with \( C \), with even and odd cases considered separately. However, Eq. (3.26) has the peculiar, unexpected property that \( P_{\text{max}}(|t_j\rangle \otimes C) = P_{\text{max}}(|t_j\rangle \otimes C+1) \) for even \( C \). That is, adding another copy to an even number of copies does not increase the maximum discrimination probability. This behaviour provides an interesting exception to the trend observed in state estimation/discrimination that the more copies we have of the state, the better we can determine it [3].

One further curious feature of the maximum discrimination probability in Eq. (3.26) is that it can be attained by carrying out collective discrimination measurements only on pairs of copies of the state. Suppose that \( C \) is even; if it is not, we can, in view of the above property of \( P_{\text{max}}(|t_j\rangle \otimes C) \), simply discard one of the copies. We divide the set of copies into \( C/2 \) pairs and carry out an optimal discrimination measurement on each pair. The probability of success for one pair is \( P_{\text{max}}(|t_j\rangle \otimes 2) \). The success probability for all \( C \) copies by this method is simply the probability that not all of the \( C/2 \) pairwise measurements give inconclusive results, which is simply \( 1 - [1 - P_{\text{max}}(|t_j\rangle \otimes 2)]^{C/2} \). Into this we insert \( P_{\text{max}}(|t_j\rangle \otimes 2) = 3/4 \), which is the special case of (3.26) for \( C = 2 \) and obtain the general maximum discrimination probability in Eq. (3.26). The ability to do optimum discrimination for this ensemble with only pairwise discrimination measurements is clearly convenient from a practical perspective.

### IV. DISCUSSION

It is impossible to discriminate unambiguously between a set of linearly dependent states. If, however, we have access to more than one copy belonging to such a set, then the compound states may be linearly independent and thus amenable to unambiguous discrimination. This is the possibility we explored in this paper.

It is natural to search for any general limitations on the extent to which this is achievable. The most natural parameters to consider are \( D \), the dimension of the Hilbert space of a single copy, \( C \), the number of copies and \( N \), the number of states. We derived one necessary and one sufficient condition, respectively (2.1) and (2.3), for \( N \) states to be amenable to unambiguous discrimination.
for fixed $C$ and $D$. These conditions were shown to be identical for $D = 2$ and combining them, which gives (2.8), solves the problem completely for qubits.

We then worked out in detail the specific example of unambiguous discrimination between $C \geq 2$ copies of the trine states. We showed how such multi-trine states can be interpreted as lifted trine states, for which the maximum unambiguous discrimination probability can be calculated exactly. We also found that if $C$ is even, then adding a further copy, strangely, fails to increase the maximum discrimination probability. Also, we described how the optimum measurement for arbitrary $C \geq 2$ can be carried out by performing discrimination measurements only on pairs of copies.

We conclude with an observation regarding the related subject of probabilistic cloning. It was established by Duan and Guo [12] that a set of quantum states can be probabilistically copied exactly if and only if they are linearly independent. This result is rigorously correct for $1 \rightarrow M$ cloning. If, however, $1 < C < M$ copies of the state are initially available, then sometimes $C \rightarrow M$ cloning will be possible for linearly dependent sets. A sufficient condition is the linear independence of the states $|\psi_j\rangle^{\otimes C}$. When this is so, probabilistic exact cloning may be accomplished, for example, by carrying out an unambiguous discrimination measurement to determine the state then manufacturing $M$ copies of the state.

**Appendix: Proof of Lemma**

**Proof:** We prove this by contradiction. If the set $\{|\phi_k\rangle \otimes |\chi_k\rangle\} \cup (|\phi\rangle \otimes |\chi\rangle)$ is linearly dependent, then there exist coefficients $b$ and $b_k$ such that

$$b|\phi\rangle \otimes |\chi\rangle + \sum_k b_k|\phi_k\rangle \otimes |\chi_k\rangle = 0,$$

(A.1)

where not all of the coefficients in $\{b, b_k\}$ are zero. In fact, we can show that at least two of the $b_k$ are nonzero. If only one of the $b_k$ were nonzero, then the corresponding $|\chi_k\rangle$ would be equal to either $|\chi\rangle$ (up to a phase) or the zero vector, depending on whether or not $b = 0$. The latter possibility contradicts the premises of the lemma (normalisation). The former does also, since it would imply that for the nonzero $b_k$, $|\chi_k\rangle$ is not distinct from $|\chi\rangle$.

The set $\{|\phi_k\rangle\}$ is linearly independent, so there exists a set of reciprocal states $\{\tilde{\phi}_k\} \in \mathcal{H}'$ such that $\langle \tilde{\phi}_k | \phi_k \rangle = \langle \tilde{\phi}_k | \phi_k \rangle \delta_{kk'}$ and $\langle \tilde{\phi}_k | \phi_k \rangle \neq 0 \ \forall \ k$. Acting on Eq. (A.1) throughout with $\langle \tilde{\phi}_k | \otimes 1$ gives

$$b\langle \tilde{\phi}_k | \phi \rangle |\chi\rangle + b_k \langle \tilde{\phi}_k | \phi_k \rangle |\chi_k\rangle = 0 \ \forall \ k.$$

(A.2)

The fact that at least two of the $b_k$ are nonzero implies that the corresponding $|\chi_k\rangle$ will be indistinct, contradicting the premise. This completes the proof.
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