Super-Aging in two-dimensional random ferromagnets

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We study the aging properties, in particular the two-time autocorrelations, of the two-dimensional randomly diluted Ising ferromagnet below the critical temperature via Monte-Carlo simulations. We find that the autocorrelation function displays additive aging $C(t, t_w) = C_{\text{stat}}(t) + C_{\text{dyn}}(t, t_w)$, where the stationary part $C_{\text{stat}}$ decays algebraically. The aging part shows anomalous scaling $C_{\text{dyn}}(t, t_w) = C(t)h(t/w)$, where $h(t)$ is a non-homogeneous function excluding a $t/w$ scaling.

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The phase ordering kinetics in pure systems has attracted much attention in the last years [1]. A common scenario for instance for ferromagnets after a fast quench from above to below the ordering temperature is a continuous domain growth governed by a single length scale that depends algebraically on the time $t_w$ after the quench. The existence of this length scale quite frequently determines also the scaling properties of other dynamical non-equilibrium quantities like the two-time auto-correlation function $C(t, t_w)$, which describes the correlations between the spin configurations at the time $t_w$ after the quench and a later configuration at a time $t + t_w$. It gives rise to what is called simple aging in the context of glassy systems [2]. $C(t, t_w)$ depends for large times $t$ and $t_w$ only on the scaling variable $t/w$. This behavior is rather well established by analytical works in various non-random models [2], and it has been corroborated by a large amount of numerical work [2].

Much less analytical results are available for disordered ferromagnets, where numerical simulations thus play an important role. A recent numerical study of the relaxation dynamics in two-dimensional random magnets [4] found evidence for a power law growth $L(t) \propto t^{1/2}$ of the aforementioned length scale $L(t)$. The dynamical exponent $z$ turned out to depend both on temperature $T$ and disorder strength and to behave as $z \propto 1/T$ at low temperatures $T$. The latter is compatible with activated dynamics of pinned domain walls over logarithmic free energy barriers (rather than power law [2]). The apparent existence of a single length scale that grows algebraically was confirmed by a recent numerical work [5], where it was furthermore claimed that the response function is well described by local scale invariance [7]. In spite of this, the correlation function showed systematic deviations from a simple $t/w$ scaling [5] (although simple aging seems to work well in $d = 3$ [8]). In [8], the autocorrelation was then compared to the scaling form $C(t, t_w) \sim t^{-x} \tilde{c}(t/w)$, which usually holds at a critical point with $x > 0$ [8]. However, a fit of this form to the numerical data obtained in [8] (and to ours as we will report below) yielded negative exponents $x$, which is unphysical. The aim of this paper is to suggest an alternative picture originally applied in the context of aging experiments in glasses [10] and spin glasses [11].

We study the site diluted Ising model (DIM) defined on 2-dimensional square lattice with periodic boundary conditions, and described by the Hamiltonian

$$H = - \sum_{\langle ij \rangle} \rho_i \rho_j s_i s_j$$

where $s_i = \pm 1$ are Ising spins, the $\rho_i$’s are, quenched, identical and independent random variables distributed according to the probability distribution $P(\rho) = \rho^p (1 - \rho)^{1-p}$. Above the percolation threshold $p > p_c$, with $p_c \approx 0.593$ [2], the equilibrium phase diagram is characterized by a critical line $T_c(p)$ (with $T_c(p_c) = 0$) which separates a ferromagnetic phase at low temperature $T$ from a paramagnetic one at high $T$. Here we focus on the relaxation dynamics of this system (1) following a quench in the ferromagnetic phase, $T < T_c(p)$. At the initial time $t = 0$, up and down spins are randomly distributed on the occupied sites when it is suddenly quenched below $T_c(p)$ where it evolves according to Glauber dynamics (corresponding to the heat-bath algorithm) with random sequential update, representing a discretized version of model A dynamics, i.e. for non-conserved order parameter.

In the following we focus on the two-times $t > t_w$ autocorrelation function $C(t, t_w)$ which is defined as

$$C(t, t_w) = \frac{1}{L^2} \sum_i \langle s_i(t + t_w) s_i(t_w) \rangle$$

where $\langle \ldots \rangle$ and $\Rightarrow$ stand for an average over the thermal noise and the disorder respectively and where $L$ is the linear system size. In our simulations, $L = 512$ and $C(t, t_w)$ is obtained by averaging over 50 different disorder configurations. In Fig. we show a plot of $C(t, t_w)$ as a function of the time difference $t$ and for different waiting times $t_w$. These data were obtained for $p = 0.75$ and $T = 0.7 T_c(p = 0.75)$. The data shown on Fig. on a log-log plot suggest a power law behavior defining the off-equilibrium exponent $\lambda$ [12]

$$C(t, t_w) \propto t^{-\lambda/2} \quad t \gg t_w$$

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which is, as we can see on Fig. 2, weakly dependent on $t_w$. We have checked that our simulations reproduce the well known values for the pure case, with $z_{\text{pure}} = 2$ \cite{Paul_87} and $\lambda_{\text{pure}} = 1.25$ conjectured to be exact in Ref. \cite{Paul_87}. Fig. 2 shows a plot of $\lambda$ as a function of $C(T,T_c)/(p)$ for $p = 0.75, 0.8$ and $p = 0.89$. As we can see, $\lambda(T,p)$ depends rather weakly on $T$ (in constrast with $z$) and $p$. Besides, the obtained values violate the lower bound proposed in Ref. \cite{Paul_87} $\lambda \geq d/2$. Such a violation was also obtained analytically for random field XY model in $d = 2$ \cite{Sokal_89}.

We now focus on the scaling form of $C(t,t_w)$ as a function of both times $t, t_w$. For non-disordered ferromagnets with purely dissipative dynamics, one expects that $C(t,t_w)$ depends only on the ratio $\ell(t)/\ell(t_w)$, i.e.

$$C(t,t_w) = F_{\text{pure}}(\ell(t)/\ell(t_w))$$

(4)

with $\ell(t) \propto t^{1/2}$. This has been corroborated by numerical simulations \cite{Paul_90} as well as analytical results in exactly solvable limits \cite{Paul_87, Sokal_89}. As shown by Paul et al. \cite{Paul_87}, a power law domain growth is also observed for the present disordered system, which suggests to plot, here also, $C(t,t_w)$ as a function of $t/t_w$ : this is depicted in Fig. 3a. Here one sees that this scaling form does not allow for a good collapse of the curves for different $t_w$. The deviation from this scaling form is indeed systematic and we have checked that the disagreement with such a scaling persists even for larger waiting time $t_w$. We have also obtained that simulations for other values of $p$ and $T/T_c(p)$ show the same deviations from $t/t_w$ scaling.

In Ref. \cite{Paul_87} the random bond ferromagnet, which is expected to display qualitatively the same behavior as the DIM, the autocorrelation was compared to the scaling form

$$C(t,t_w) = t^{-z(T,p)\zeta(t/t_w)}$$

(5)
which works well for critical dynamics [9] as well as some disordered systems such as spin-glasses in dimension $d = 3$ [10] or an elastic line in random media [17] with a positive exponent $x(T, p) > 0$. However, in Ref. [9] a negative value for $x(T, p)$ was obtained by fitting the data for $C(t, t_w)$ to (3). We also get a good data collapse for our data, as shown in Fig. 4 when using a negative exponent $x$. The best collapse according to Eq. (5) is obtained for $x = -0.04 < 0$ (for $p = 0.75$ and $T = 0.7 T_c$).

The fact that $x < 0$ would mean that $C(y t_w, t_w)$ grows without bounds when $t_w \to \infty$ (keeping $y \gg 1$ fixed), which is unphysical. This implies that Eq. (5) is not the correct scaling form for $C(t, t_w)$, for which reason we search for an alternative picture. First we point out that as $t_w$ increases $C(t, t_w)$ clearly displays the formation of a plateau (see Fig. 1). This suggests an additive structure, as expected in the ferromagnetic phase (and in contrast to the multiplicative scaling found at $T_c(p)$ in random ferromagnets [18]):

$$C(t, t_w) = C_{eq}(t) + C_{ag}(t, t_w)$$

such that $\lim_{t \to \infty} C_{st}(t) = 0$ and $\lim_{t \to \infty} \lim_{t_w \to \infty} C(t, t_w) = M_{eq}$ where $M_{eq}$ is the equilibrium magnetization.

We first focus on the stationary component $C_{st}(t)$ in Eq. (6), for which analytical predictions exist relying on droplet models [10]. To study this part numerically, we first equilibrate the system using Swendsen-Wang algorithm [20] and then let the system evolve according to Glauber dynamics starting with such an equilibrated initial configuration. We denote $C_{eq}(t, t_w)$ the (equilibrium) autocorrelation function [2] computed using this protocol and we have checked that it is indeed independent of $t_w$, $C_{eq}(t, t_w) \equiv C_{eq}(t)$. In the inset of Fig. 4 we plot $C_{eq}(t)$ as a function of $t$ for $p = 0.75$ and $T = 0.7 \cdot T_c$.

In agreement with previous analytical predictions [19],

$$C(t, t_w) = C_{eq}(t) + M_{eq}^2$$

These data can be nicely fitted to $C_{eq}(t) = C_{st}(t) + M_{eq}^2$ with $M_{eq}^2 = 0.925$ and a power law behavior

$$C_{st}(t) \propto t^{-\eta(T, p)}$$

This is depicted in Fig. 4, where we show a plot of $C_{st}(t)$ as a function of $t$ for $p = 0.75$ and $T = 0.7 T_c$, for these parameters, one finds $\eta = 0.40(2)$.

We now come to the aging part $C_{ag}(t, t_w)$ in (6), by first noticing that simple aging also does not hold for $C_{ag}(t, t_w)$. Inspired by a picture originally suggested in the context of aging experiments in glasses [10] and spin glasses [11], and also occurring within the analytical solution of the non-equilibrium dynamics mean field spin glasses [2], we use a form that generalizes Eq. (4):

$$C_{ag}(t, t_w) \simeq C(h(t)/h(t_w))$$

A widely used form for $h(u)$, which we choose here, is $h(u) = \exp[u^{1-\mu}/(1 - \mu)]$ where $\mu$ allows to interpolate between super-aging ($\mu > 1$) and sub-aging ($\mu < 1$) via simple aging ($\mu = 1$). In Fig. 5, we show that this

![Figure 4](https://example.com/fig4.png)

**FIG. 4:** $C_{st}(t)$ plotted in the double logarithmic scale as a function of $t$. The line corresponds to $\eta = 0.4$ in Eq. (7). Inset: $C_{eq}(t)$ as function of $t$. Data set is obtained for $p = 0.75$ and $T = 0.7 \cdot T_c$.

![Figure 5](https://example.com/fig5.png)

**FIG. 5:** a) $C_{eq}(t, t_w)$ plotted as a function of $h(t)/h(t_w)$ with $h(t) = \exp[t^{1-\mu}]/(1 - \mu)$ for different $t_w = 31, 100, 316, 1000$, with $\mu = 1.035$. Data set is obtained for $p = 0.75$ and $T = 0.7 \cdot T_c$. b) $C(t, t_w)$ plotted as a function of $h(t)/h(t_w)$ with $\mu = 1.042$. Data set is obtained for $p = 0.75$ and $T = 0.5 \cdot T_c$.
form with $\mu = 1.035$ allows for a nice collapse of the curves presented in Fig. 4 for different $t_w$, corresponding to $p = 0.75$ and $T = 0.7T_c$. We point out that a good data collapse is also obtained (with the same exponent $\mu$) when $C_{st}(t)$ is not subtracted. In Fig. 5 we show a plot of $C(t,t_w)$ as a function of $b(t)/b(t_w)$ for $p = 0.75$ and $T = 0.5T_c$. For this temperature, the best data collapse is obtained for a larger value of $\mu = 1.042$, which suggests that $\mu$ is a decreasing function of $T$ (and one expects $\mu \rightarrow 1$ for $T \rightarrow T_c$).

In Ref. [21], such a super-aging behavior – with comparable values of $\mu$ – was also observed in the 4$d$ Edwards Anderson spin glass. There it was argued that super-aging is consistent with a growth law $t(L)$ of the form

$$t(L) \simeq \tau_0 L^{z_c} \exp \left[ \Theta(T) L^{\psi}/T \right]$$  \hspace{1cm} (9)$$

where $z_c$ is the dynamic critical exponent (and here $z_c = 2.1667(5)$ [22]), $\psi$ the barrier exponent and $\Theta(T)$ a typical free energy scale (vanishing at $T_c$). If one assumes $b(t) = L(t)$ in Eq. 8 with $t(L)$ as in Eq. 9, then one can identify $\psi/z_c = (\mu - 1)$ [21]. In our case this would give a $T$-dependent barrier exponent $\psi$ (see Fig. 4). In addition, the values obtained for $\psi$ from that relation are different from the exact value $\psi = 1/4$ [3].

To conclude, we have performed a detailed numerical study of the autocorrelation function during the coarsening dynamics of diluted Ising ferromagnets in dimension $d = 2$. Our data show clear deviations from a simple $t/t_w$ scaling, which were also observed in a recent work on a random ferromagnet in $d = 2$ [3]. However, attempts to fit the data to the simple scaling form as in Eq. 5 leads, as in Ref. [3], to $x < 0$, which is unphysical. Here we proposed an alternative way of describing the dynamics in terms of super-aging; this allows for a consistent description of the autocorrelation function in disordered ferromagnets. If our results reflect correctly the asymptotic scaling behavior of the autocorrelation in 2$d$ disordered ferromagnets one would thus conclude that it is not accurately described by local scale invariance as in Ref. [23]. With regards to a recent experimental study of super-aging in spin glasses [24], it would be interesting to understand whether the super-aging behavior we find is related to the choice of the initial conditions for the dynamics.

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