Noise Correlations in low-dimensional systems of ultracold atoms

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We derive relations between standard order parameter correlations and the noise correlations in time of flight images, which are valid for systems with long range order as well as low dimensional systems with algebraic decay of correlations. Both Bosonic and Fermionic systems are considered. For one dimensional Fermi systems we show that the noise correlations are equally sensitive to spin, charge and pairing correlations and may be used to distinguish between fluctuations in the different channels. This is in contrast to linear response experiments, such as Bragg spectroscopy, which are only sensitive to fluctuations in the particle-hole channel (spin or charge). For Bosonic systems we find a sharp peak in the noise correlation at opposite momenta that signals pairing correlations in the depletion cloud. In a condensate with true long range order, this peak is a delta function and we can use Bogoliubov theory to study its temperature dependence. Interestingly we find that it is enhanced with temperature in the low temperature limit. In one dimensional condensates with only quasi-long range (i.e. power-law) order the peak in the noise correlations also broadens to a power-law singularity.

I. INTRODUCTION

The ability to trap ultracold atoms in tightly confined tubes formed by an optical lattice has opened the door for the controlled study of one dimensional physics. Such systems have been used to investigate ground state correlations\textsuperscript{1,2} as well as dynamics\textsuperscript{3} and transport\textsuperscript{4} of strongly interacting bosons in one dimension. One dimensional traps of ultracold fermions have also been realized. The ability to control the interactions has been demonstrated using s-wave Feshbach resonances for fermions with spin\textsuperscript{5}, and with p-wave resonances for spin-polarized fermions\textsuperscript{6}.

From the theoretical viewpoint, one dimensional systems provide good starting points to study strong correlation physics. Because of the enhanced quantum fluctuations, continuous symmetries cannot be broken in generic one dimensional systems, and mean field theories fail. Nevertheless the Luttinger liquid framework provides a well developed formalism to treat these systems theoretically (see for example\textsuperscript{7}). In place of a mean field order parameter the tendencies to ordering manifest themselves by slowly decaying algebraic correlations and correspondingly divergent susceptibilities. A diverging susceptibility in a particular channel implies that coupling an array of tubes in the transverse direction would lead to true order in that channel. Thus weakly coupled one dimensional systems provide a possible theoretical route to investigate open questions regarding competing orders in higher dimensions.

But the absence of long range order, which makes one dimensional systems interesting also complicates the ways by which these systems can be probed. Various methods have been proposed and used to probe specific correlations. For example time of flight\textsuperscript{1} as well as interference experiments\textsuperscript{8} can probe single particle correlations. Bragg\textsuperscript{9,10} and lattice modulation\textsuperscript{2} spectroscopies can measure dynamic density correlations. In\textsuperscript{11} we proposed that noise correlations can be used as a highly flexible probe of 1D Fermi systems. In particular this method is sensitive to a wide range of correlations including spin, charge and pairing, and it treats the various channels on an equal footing. These results, obtained using the effective Luttinger liquid theory were confirmed by Luscher et. al. using a numerical simulation of a microscopic model.

In this paper we provide the detailed theory of quantum noise interferometry in one dimensional Fermi systems and extend it to interacting Bose systems.

Atomic shot noise in time of flight imaging was proposed in Ref.\textsuperscript{13} as a probe of many body correlations in systems of ultra cold atoms. Specifically what is measured using this approach is the momentum space correlation function of the atoms in the trap

$$G_{\alpha\alpha'}(k,k') = \langle n_{\alpha k} n_{\alpha' k'} \rangle - \langle n_{\alpha k} \rangle \langle n_{\alpha' k'} \rangle,$$

(1)

where $\alpha, \alpha'$ are spin indices, $k, k'$ are momenta, and $n_{\alpha k}$ and $n_{\alpha' k'}$ are the occupation operators of the corresponding momentum and internal state. This relies on the assumption that the atoms are approximately non-interacting from the time they are released from the trap (even if they were rather strongly interacting in the trap). Here we envision a system of one dimensional tubes, that allows expansion only in the axial direction. To ensure weak interactions during time of flight, the radial confinement in the tubes can be reduced simultaneously with release of the atoms from the global trap.

Recent experiments demonstrated the ability to detect many-body correlations by analysis of the noise correlations\textsuperscript{1} For example long ranged density-wave correlations (induced by external lattice) were seen in a system
of ultracold bosons\cite{14,15} and fermions\cite{10} in deep optical lattices. The sign of the correlation, peak or dip, depends on the statistics as expected, demonstrating that the effect is essentially a generalization of the Hanbury-Brown Twiss effect. Pairing correlations in fermions originating from a dissociated molecular condensate were also observed in experiment\cite{17}. All of these demonstrations involved states with true long range order that are easily related to simple two particle effects such as Hanbury-Brown Twiss or bound state formation. However, one of the main points of Ref. \cite{13} was that the noise correlations can be used to measure more general correlation functions in many-body systems, even in the absence of true long range order (LRO). Here we demonstrate this idea by developing a detailed theory for the noise correlations in one dimensional systems of bosons and fermions.

The analysis is laid out in the following order. In section \ref{sec:3} we consider a model system of spinless fermions in one dimension. To elucidate the connection between noise correlation and ordering tendencies we first assume in section \ref{sec:3A} the presence of true long range order. This allows to provide a clear physical picture for the way order in spin, charge (particle-hole) or pairing (particle-particle) channels translate into sharp features (in this case delta function peaks) in the noise correlations. In section \ref{sec:3B} we shall consider the actual system of interacting fermions in one dimension where only power law order parameter correlations exist (quasi-long range order). This system is characterized by a tendency to charge density wave (CDW) ordering for repulsive interactions, and to superconducting order (SC) for attractive interactions. The naive expectation is that the delta function peaks will be replaced by power-law singularities with a power that reflect the algebraic order parameter correlations. Interestingly we find that such a simple relation exists only if there are order parameter correlations that decay with a sufficiently slow power. This is the case beyond a critical interaction strength (repulsive or attractive). At weak interactions, on the other hand, the situation is more subtle and singular signatures of both the dominant and sub-dominant ordering tendencies are seen in the noise correlations (see Fig. \ref{fig:1},a,c). We compare these results with the information that can be extracted from measuring the static structure factors in the spin and charge channels.

In section \ref{sec:4} we move on to analyze the noise correlations in one dimensional systems of interacting spin-1/2 fermions. In the long wavelength effective field theory the interaction is parameterized by two Luttinger parameters $K_L$ and $K_s$ corresponding to the spin and charge sectors, as well as a backscattering parameter $g_{1\perp}$. The ordering tendencies of this system are summarized in the phase-diagram in Fig. \ref{fig:3} (See \ref{fig:19}). We find that the noise correlations provide direct information on the real space order parameter correlations only for negative backscattering. In this case a spin-gap opens and the system is characterized by competing CDW and SSC correlations, which are revealed by the noise correlations.

In section \ref{sec:5} we turn to interacting Bose systems. We first study the noise correlation within Bogoliubov theory for a condensate with true long range order (section \ref{sec:5A}). The interesting feature here is a peak in the noise correlations \ref{fig:1} at $k + k' = 0$. This can be simply understood as the signature of pairing correlations in the Bogoliubov wave-function, describing quantum depletion of the condensate. We compute the evolution of this signature with increasing temperature. In addition we point out the existence of sharp dips in the noise correlations along the lines $k' = 0$ and $k = 0$, these reflect correlations between the condensate and the quantum depletion cloud. In section \ref{sec:5B} we move on to discuss interacting one dimensional Bose liquids, where the single particle density matrix decays as a power-law with distance. We show how the sharp features in the noise correlation function on the lines $k + k' = 0$, $k = 0$, and $k' = 0$ broaden to power-law singularities.

\section{Spinless fermions}

A system of interacting spinless fermions is perhaps the simplest Fermi system with a non trivial competition of ordering tendencies. Such systems can be implemented in experiments using fully polarized fermionic atoms that interact via an odd angular momentum channel. The interaction strength can be tuned by using a p-wave Feshbach resonance\cite{6}. Another possibility is to use a Bose-Fermi mixture, where the phonons of the bosonic superfluid mediate effective interactions between the fermions\cite{13}.

In the restricted geometry of 1D the Fermi surface consists only of two points, the left (L) Fermi point at $-k_f$ and the right (R) Fermi point at $+k_f$. We introduce the left- and right-moving fields $\psi_L$ and $\psi_R$ through:

\begin{equation}
\psi(x) = e^{-ik_f x} \psi_L + e^{ik_f x} \psi_R \tag{2}
\end{equation}

$\psi_L$ and $\psi_R$ are slowly varying fields, because the rapidly oscillating phase factors $e^{\pm ik_f x}$ have been separated out. The natural order parameters that characterize this system are the charge density and superconducting (pairing) operators:

\begin{align}
O_{CDW} &= \psi_R^\dagger \psi_L = \rho_{2k_f} \tag{3} \\
O_{SC} &= \psi_R \psi_L \tag{4}
\end{align}

The central problem we address in this section concerns the connection between the correlations of these physical order parameters and the observable noise correlation signal.

\subsection{Long range order}

The relation between order parameter correlations and sharp features in the noise correlation function is most
apparent when the system supports true long range order. Of course one dimensional Fermi systems generically do not display spontaneous long range order in either the CDW or SC channels. However long range order can be induced by an external potential (e.g. a superlattice potential) or it can form spontaneously in a weakly coupled array of one dimensional systems. Moreover the intuition gained from the exercise will be useful in approaching the more interesting case of power-law order parameter correlation (quasi long range order), which will be considered in the following sections.

Consider first the case of long range pairing correlations (i.e. SC order). Take \( k \) (\( k' \)) to be near the right (left) Fermi points, and \( q \equiv k - k_F, q' \equiv k' + k_F \) deviations from those Fermi points. It is now useful to write the noise correlations explicitly in terms of the pairing correlations

\[
\langle n_k n_{k'} \rangle \approx \frac{1}{L^2} \int dX dX' dr dr' e^{i(q+q')(X-X')/2} e^{i(q-q')(r-r')/2} \langle \psi_R^\dagger(X + r/2) \psi_L^\dagger(X - r/2) \psi_L(X' - r'/2) \psi_R(X' + r'/2) \rangle = \frac{1}{L^2} \int dX dX' dr dr' e^{i(q+q')(X-X')/2} e^{i(q-q')(r-r')/2} \langle B_{r'}^\dagger(X) B_r^\dagger(X') \rangle
\]

(5)

Here the operator \( B_r(X) \) creates a fermion pair whose constituents are separated by \( r \) and their center of mass coordinate is \( X \). Note that we have dropped terms that undergo rapid \( 2k_F \) spatial oscillations and therefore vanish under integration. The existence of long range pairing correlations implies

\[
\langle B_{r'}^\dagger(X) B_r^\dagger(X') \rangle \rightarrow \Phi(r) \Phi(r')
\]

(6)

at long distances (\( |X - X'| \rightarrow \infty \)). Here \( \Phi(r) = \langle \psi^\dagger(X + r/2) \psi^\dagger(X - r/2) \rangle \) is the translationally invariant pairing wave function. The long range saturation of the pairing correlation function leads to a singular contribution to (5) at \( q = -q' \):\[
\langle n_k n_{-k'} \rangle \sim \delta(q + q') |\Phi(q)|^2
\]

(7)

We thus conclude that the noise correlations in this case are directly related to the long distance limit of the pairing correlations. This is one way to formally justify the mean-field decoupling of the noise correlation function carried out in Ref. [13], which gives:

\[
\langle n_k n_{k'} \rangle = \langle n_k n_{k'} \rangle - \langle n_k n_{k'} \rangle = |\langle \psi_k \psi_{-k} \rangle|^2 \delta(k + k')
\]

(8)

A very similar analysis follows for the case of long range order in the particle-hole channel. Take for example CDW order at the wave-vector \( 2k_F \). In this case we should write the noise correlation function in terms of the density correlations to expose their singular contribution:

\[
\langle n_k n_{k'} \rangle = -\frac{1}{L^2} \int dX dX' dr dr' e^{i(q+q')(X-X')/2} e^{i(q-q')(r-r')/2} \langle \psi_R^\dagger(X + r/2) \psi_L^\dagger(X - r/2) \psi_R^\dagger(X' - r'/2) \psi_L(X' + r'/2) \rangle
\]

(9)

Note the minus sign in front of the integral, which resulted from commuting the Fermi operators to obtain an expression written in terms of density wave correlations. Long range CDW correlations imply

\[
\langle \rho_{2k_F, r}^\dagger(X) \rho_{2k_F, r'}^\dagger(X') \rangle \rightarrow \Xi(r) \Xi(r').
\]

(10)

which leads to a singular contribution to (9) for \( q = q' \):

\[
\langle n_k n_{k'} \rangle \sim -\delta(q - q') |\tilde{\Xi}(q)|^2
\]

(11)

This justifies the corresponding mean field decoupling of the noise correlation function:

\[
\langle n_k n_{k'} \rangle = -|\langle \psi_{k+2k_F} \psi_k \rangle|^2 \delta(k - k')
\]

(12)

Again, as in the SC case, we see that the noise correlations are directly related to the long distance limit of the non decaying order parameter correlation.

We note two obvious distinctions between the cases of order in the particle-particle (pairing) and particle-hole channels. In the former the singular correlations are between particles with opposite momenta and are positive correlations. In the latter case, by contrast, the singular
Correlations are between particles whose momenta differ by $2k_F$ and are negative, that is anti-correlations. To better understand the origin of this effect it is worthwhile to inspect the mean field wave functions, which sustain the respective broken symmetries. For the superconducting order the mean field wave-function is the BCS state representing a pair condensate:

$$\Psi_{BCS} = \prod_k \left[ u_k + v_k \psi_k \psi_k^\dagger \right] |0\rangle$$ (13)

Contrary to a filled Fermi sea the particle number $n_k$ in a specific $k$ point is not definite in this wave function. However, if a particle is found at $k$, there is with certainty another one at $-k$. This implies positive correlation between $n_k$ and $n_{-k}$ as visualized in Fig. 1 (a).

The CDW state on the other hand may be viewed as a condensate of particle-hole pairs on top of the filled Fermi sea. It is then written in a way that exposes the similarity to the BCS state:

$$\Psi_{CDW} = \prod_k \left[ u_k + v_k \psi_k^{\dagger} \psi_{k+2k_F} \right] |FS\rangle$$ (14)

Here $|FS\rangle$ denotes a full fermi sea wavefunction. As in the BCS state, the particle number $n_k$ at a specific $k$ point is not definite. Now however, if one finds a particle at $k > k_F$ there will be with certainty a missing particle (hole) in the Fermi sea at $k - 2k_F$. This implies anti-correlation between $n_k$ and $n_{k-2k_F}$ as visualized in Fig. 1 (b).

**B. Quasi-order**

Because of strong quantum fluctuations, an actual one dimensional Fermi system cannot sustain true long range order such as the SC and CDW orders discussed above. Instead, at zero temperature a critical phase with power law correlations, or quasi long range order, is established. If the power-law decay is sufficiently slow then it is reasonable to expect that it would still make a singular contribution to the integrals in (5) or (9). To calculate the resulting singularity and its dependence on the system parameters we use the low energy Luttinger liquid theory. The basic idea is that the asymptotic low energy and long wave-length properties of the interacting one dimensional Fermi system are captured by a universal harmonic theory:

$$S = \frac{1}{2\pi K} \int dxd\tau \left[ c(\partial_x \Theta)^2 + c^{-1}(\partial_x \Theta)^2 \right]$$

$$= \frac{K}{2\pi} \int dxd\tau \left[ c(\partial_x \Phi)^2 + c^{-1}(\partial_x \Phi)^2 \right]$$ (15)

Here $K$ is the Luttinger parameter, which determines the power-law decay of long range correlations. $c$ is a sound velocity, which we will henceforth set to 1. The bosonization identity, which relates the bosonic fields to the fermion operators is:

$$\psi_{R/L}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\Theta(x)+i\Phi(x)}$$, (16)

with the commutation relation:

$$\{\Theta(x), \Phi(0)\} = \frac{1}{2}\log \frac{\alpha+ix}{\alpha-ix}.$$ (17)

$\alpha$ is an artificial cutoff of the bosonization procedure which must be sent to zero at the end of the calculation. Implicit in the action (15) is also a physical short distance cutoff $x_0$, taken in most cases to be of order $1/k_F$. Here $\Phi$ is the phase field of the Fermi operator. $\Pi = \frac{1}{\pi} \nabla \Theta$ is the smooth ($k \approx 0$) component of the density fluctuation, and by (17), is also canonically conjugate to $\Phi$.

The effective action (15) is a free theory in terms of the bosonic fields. It therefore allows to calculate the needed correlation functions using simple gaussian quadrature. The fermion interactions affect the correlation functions only through the Luttinger parameter $K$. For non interacting fermions $K = 1$, $K < 1$ for repulsive interactions and $K > 1$ for attractive interactions. In general $K$ deviates more from $K = 1$ the stronger the interactions. However it is in general not possible to make a precise connection between the microscopic parameters and the Luttinger parameter. Among other things, our analysis points to a way of extracting this parameter from experiments.

For any value of $K$ the system shows either CDW or SC quasi-long range order, which means that the correlation functions decay slow enough to give a divergent susceptibility. The calculation of these correlation functions has been given in numerous places.

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**FIG. 1:** Schematic representation of several types of fluctuations. In these diagrams, $k$ and $k'$ represent the momenta of the atoms and holes, and $q$ and $q'$ the momenta relative to the Fermi points. (a) A pairing fluctuation (or Cooper pair), which is the dominant fluctuation in the SC phase. These fluctuations result in positive noise correlations along $q = -q'$. (b) A particle-hole (p-h) fluctuation associated with a CDW state. This fluctuation results in negative correlations along $q = q'$. (c) Two p-h pairs. This fluctuation results in both positive and negative correlations. Positive correlations for $q = -q'$ and negative for $q = q'$. 
For the long distance behavior of $O_{CDW}$ we have $\langle O_{CDW}(x,\tau)O_{CDW}(0,0)\rangle \sim \cos(2k_f(x^2 + e^2\tau^2)^{-2(1-\alpha_{CDW})/2})$, and for $O_{SC}$ we have $\langle O_{SC}(x,\tau)O_{SC}(0,0)\rangle \sim (x^2 + e^2\tau^2)^{-2(1-\alpha_{SC})/2}$ where the scaling exponents for CDW and SC are

$$\alpha_{CDW} = 2 - 2K,$$

$$\alpha_{SC} = 2 - 2K^{-1}.$$

The susceptibilities correspond to the spatial and temporal Fourier transform of the correlation functions, $\chi(k,\omega) \sim \int dx dt e^{ikx+i\omega\tau} \langle O(x,\tau)O(0,0)\rangle$. These will be divergent at large distances exactly if the scaling exponent of the operator $O(x,\tau)$ is positive. As we can see from Eqs. (18) and (19), $\chi_{CDW}$ diverges at $k = 2k_f$ for $K < 1$, and $\chi_{SC}$ diverges at $k = 0$ for $K > 1$. In this sense of QLRO we say that the system is in the CDW regime for $K < 1$, and in the pairing regime for $K > 1$ [21], as depicted in Fig. 2 e). We note that from Eq. (15) and (16) one can read off the duality mapping: $\theta \leftrightarrow \Phi$, $K \leftrightarrow K^{-1}$, which leaves the action invariant, and maps the CDW regime $0 < K < 1$ onto the SC regime, $1 < K < \infty$.

With this formalism we derive the noise correlations $G(q,q')$, which we show in detail in App. A. We find

$$G(q,q') = \int \frac{e^{i(q,x_1+q'x_3q^3)}(2\pi)^2L}{\mathcal{F}(x_1)\mathcal{F}(x_3)}(A - 1)$$

with

$$\mathcal{F}(x) = \left(\frac{x^2}{x^2 + x^2}\right)^g \frac{1}{\alpha - ix}$$

and

$$A = \left(\frac{x_0^2 + x_1^2}{x_0^2 + x_3^2}\right)^{h_+}$$

The integral in Eq. (20) is over the three spatial variables $x_1$, $x_2$, and $x_3$. The exponents $g$ and $h$ are given by $g = (K + K^{-1} - 2)/4$ and $h = (K - K^{-1})/4$.

Following the discussion of the MFA, we note that this integral 'contains' the equal-time correlations of $O_{CDW}$ and $O_{SC}$. This can be seen by setting $x_1 = 0$ and $x_2 = 0$, which gives the Fourier transform in $q - q'$ of $-\langle O_{CDW}(1)O_{CDW}(2)\rangle$. If we set $x_1 = 0$ and $x_2 = 0$, we obtain the Fourier transform of $\langle O_{SC}(1)O_{SC}(2)\rangle$ in $q + q'$. To discuss this further we introduce the following variables: $z = (x_1 + x_3)/2$, $h_+ = (x_1 + x_3)/2$, and $h_- = (x_1 - x_3)/2$. With this, $\langle n_qn_{q'}\rangle$, is of the form

$$\langle n_qn_{q'}\rangle \sim \int \frac{e^{i(q-q')s+i(q+q')h}}{L} \mathcal{F}(z + h_-)\mathcal{F}^*(z - h_-) \left(\frac{\Lambda^2 + (h_+ + h_-)^2}{\Lambda^2 + (z + h_-)^2}\right)^h \left(\frac{\Lambda^2 + (z + h_-)^2}{\Lambda^2 + (z - h_-)^2}\right)^h.$$  

In the limit $K \ll 1$, we have $h \ll 0$. This enforces the integrand to be negligible except in regions with $z, h_+ \approx 0$. With this, the expression approximately evaluates to

$$\langle n_qn_{q'}\rangle \sim \text{sgn}(2K - 1)|q - q'|^{2K - 1}$$

In the dual limit, $K \gg 1$, we have $h \gg 0$, which enforces $z, h_+ \approx 0$. In this limit the integral approximately evaluates to

$$\langle n_qn_{q'}\rangle \sim \text{sgn}(2K - 1)|q - q'|^{2K - 1}$$

which can also be inferred from duality. These contributions are divergent for $K < 1/2$ and $K > 2$, and turn out to be the dominant contributions in these regimes.

To confirm this expectation, and in order to study the regime $1/2 < K < 2$, we evaluate this integral numerically for different values of $K$. In Fig. 2 we show $G(q,q')$ for $K = 0.4$, $0.8$, $1.25$ and $2.5$. The $K = 0.4$ example shows indeed a power-law divergence of the particle-hole type, which we find throughout the $K < 1/2$ regime. For $K > 2$ we find a result similar to the $K = 2.5$ example, an algebraic divergence of the particle-particle type.
These two regimes are indicated in Fig. 2(e).

In between these two regimes, for $1/2 < K < 2$, we find a regime in which both p-h as well as p-p correlations exist, i.e. we find precursors of the near-by competing order. A simple argument for the qualitative shape of the noise correlations function in this regime is the following: If we consider an interacting 1D Fermi gas and treat the interaction perturbatively, the lowest order contribution would consist of states that contain two p-h pairs, as two fermions have been taken from the Fermi sea and put into the unoccupied states above the Fermi sea (see Fig. 1(c)). Such a state exhibits qualitatively the noise correlations that are observed in Fig. 2(b) and c): positive correlations for $q < 0, q' > 0$ and $q > 0, q' < 0$, and negative correlations for $q > 0, q' > 0$ and $q < 0, q' < 0$. To quantify how this affects the line-shape of the noise correlations we expand $G(q, q')$ to second order in $\hbar$ (see App. B for details). The result is

$$G(q, q') \sim -\hbar^2 \text{sgn}(q)\text{sgn}(q') \min\left(\frac{1}{|q|}, \frac{1}{|q'|}\right).$$

(26)

This expression shows a divergence at $q = q' = 0$, and both particle-particle and particle-hole fluctuations. Higher order terms either enhance particle-particle (for $K > 1$) or particle-hole ($K < 1$) fluctuations. The static structure factor of a BEC has been measured in 14. It corresponds to the instantaneous density-density correlations:

$$S(k) \sim \langle \rho_{-k}\rho_k \rangle.$$  

(30)

For $k \approx 0$, we have

$$S(k \approx 0) \sim K|k|$$  

(31)

and for $k \approx 2nk_f$:

$$S(2nk_f + q) \sim |q|^{2n^2 K^{-1}}.$$  

(32)

Here the first set of peaks appears at $K = 1/2$, the next set of peaks at $K = 1/8$ and so on. This is the power-law divergence that dominates the noise correlations for $K < 1/2$, as discussed in the previous section. Again there is no signature of SC.

In summary we have derived the noise correlations of a spinless fermionic LL in this section, and compared it to a MFA result. We found different subregimes with qualitatively different behavior in each of the quasi-phases, summarized in Fig. 2 and discussed how phenomena such as QLRO and competing phases are reflected in the noise correlations.

III. SPIN-1/2 FERMIONS

In this section we discuss the noise correlations of an SU(2)-symmetric Fermi mixture 20.

Our analysis applies to systems of the form

$$H = -i \sum_{i,\sigma} \psi_i^\dagger \sigma \psi_i^\sigma + U \sum_i n_{i,\uparrow} n_{i,\downarrow}$$

(33)

i.e. the 1D Hubbard model, or a mixture in a 1D continuum with a contact interaction:

$$H = \sum_{i,\sigma} \int \psi_i^\dagger \sigma \left( -\nabla^2 \right) \psi_i^\sigma + U \int n_\uparrow(x)n_\downarrow(x)$$

(34)

which can both be realized in experiment.

A. Long range order

Before we turn to the LL picture, we introduce the types of order that occur in this system, and discuss what kind of signature can be expected if they develop long-range order, with similar arguments as in Sect. II A.

As for the spinless fermions we split the spinful fermionic operators into left- and right-moving fields:

$$\psi_{l,1}^\dagger(x) = e^{-ik_f x} \psi_{L,1}/1 + e^{ik_f x} \psi_{R,1}/1.$$  

(35)

We introduce these order parameters 7,21:

$$O_{SS} = \psi_{R,1}^\dagger \psi_{L,1}^\sigma$$

(36)

$$O_{TS} = \psi_{R,1}^\dagger \psi_{L,1}^\sigma$$

(37)

$$O_{SDW} = \psi_{R,1}^\dagger \psi_{L,1}^\sigma$$

(38)

$$O_{CDW} = \psi_{R,1}^\dagger \psi_{L,1}^\sigma.$$  

(39)
charge and a spin sector: the action of any SU(2)-symmetric system separates into a
component of triplet pairing, and therefore describes pairing
between equal spin states. \( O_{SDW} \) is the order parameter of the \( x \) and \( y \) component of the spin density wave order.

Because these order parameters contain both \( R \) and \( L \), as well as \( \uparrow \) and \( \downarrow \) operators, we expect correlations
between \( n_{R,\uparrow} \) and \( n_{L,\downarrow} \), as well as \( n_{R,\downarrow} \) and \( n_{L,\uparrow} \), so we consider two types of correlation functions; correlations
between atoms in the same spin state
\[ G_\uparrow\uparrow(q,q') = \langle n_{\uparrow,q}n_{\uparrow,q'} \rangle - \langle n_{\uparrow,q} \rangle \langle n_{\uparrow,q'} \rangle \]  
and correlations between opposite spins
\[ G_{\uparrow\downarrow}(q,q') = \langle n_{\uparrow,q}n_{\downarrow,q'} \rangle - \langle n_{\uparrow,q} \rangle \langle n_{\downarrow,q'} \rangle. \]

We again use the same convention that \( q \) is the right Fermi point and \( q' \) located near the left Fermi point.

B. Quasi-order

We bosonize these fermionic fields according to:
\[ \psi_{R/L,\uparrow/\downarrow}(x) = \frac{1}{2\sqrt{2\pi\alpha}} e^{\pm i\Theta_{R/L,\uparrow/\downarrow}(x) + \Phi_{R/L,\uparrow/\downarrow}(x)} \]

with the same definitions of \( \Theta_{R/L,\uparrow/\downarrow} \) and \( \Phi_{R/L,\uparrow/\downarrow} \) that we used for spinless fermions. We introduce spin and charge fields according to:
\[ \Theta_{\rho,\sigma} = \frac{1}{\sqrt{2}} (\Theta_{\uparrow} \pm \Theta_{\downarrow}) \]
\[ \Phi_{\rho,\sigma} = \frac{1}{\sqrt{2}} (\Phi_{\uparrow} \pm \Phi_{\downarrow}) \]

Written in terms of \( \Theta_{\sigma} \) and \( \Theta_{\rho}(x) = \Theta_{\rho}(x) - kfx \) the action of any SU(2)-symmetric system separates into a charge and a spin sector:
\[ S = S_{\rho} + S_{\sigma} \]

with:
\[ S_{\rho} = \frac{1}{2\pi K_{\rho}} \int \frac{1}{v_{\rho}} (\partial_{\rho}\Theta_{\rho})^2 + v_{\rho}(\partial_{\rho}\Theta_{\rho})^2 \]

and:
\[ S_{\sigma} = \frac{1}{2\pi K_{\sigma}} \int \frac{1}{v_{\sigma}} (\partial_{\sigma}\Theta_{\sigma})^2 + v_{\sigma}(\partial_{\sigma}\Theta_{\sigma})^2 \]
\[ + \frac{2g_{\perp}}{(2\pi\alpha)^2} \int \cos(\sqrt{8K_{\sigma}}\Theta_{\sigma}) \]

Each of these sectors is characterized by a velocity \( v_{\rho/\sigma} \) and a Luttinger parameter \( K_{\rho/\sigma} \). In addition to the quadratic terms in the action we find a non-linear term, describing backscattering processes in the spin sector, with the prefactor \( g_{1,1} \). If this action is derived from a system with a short-ranged interaction such as \( \delta \) and \( g_{1,1} \), these parameters have the following properties: For repulsive interaction, one finds \( K_{\rho} < 1 \), \( K_{\sigma} > 1 \) and positive backscattering \( g_{1,1} \), for attractive interaction \( K_{\rho} > 1 \), \( K_{\sigma} < 1 \), and negative backscattering \( g_{1,1} < 0 \).

As discussed in \[ \text{[21][22]} \] this sine-Gordon model, can be treated with an RG calculation to identify two limiting cases: The case in which the backscattering term is irrelevant and the system flows towards the non-interacting fixed point \( (K_{\sigma} \rightarrow 1) \), which happens for repulsive interaction, and the case in which the backscattering term is relevant and a spin gap appears \( (K_{\sigma} \rightarrow 0) \), which happens for attractive interaction. In the evaluation of the noise correlation functions we will use these limiting values, \( K_{\sigma} = 1 \) for the gapless phase, \( K_{\sigma} = 0 \) for the spin-gapped regime. One can find the phase diagram of this system in exact analogy to the spinless case, by studying the correlation functions of the order parameters. The scaling exponents of these operators are given by:
\[ \alpha_{SS} = 2 - K_{\rho}^{-1} - K_{\sigma} \]
\[ \alpha_{TS} = 2 - K_{\rho}^{-1} - K_{\sigma}^{-1} \]
\[ \alpha_{SDW} = 2 - K_{\rho} - K_{\sigma}^{-1} \]
\[ \alpha_{CDW} = 2 - K_{\rho} - K_{\sigma} \]

From these expressions one can read off the structure of the phase diagram. In the gapless phase we have \( K_{\sigma} = 1 \), therefore singlet and triplet pairing, as well as SDW and
CDW are algebraically degenerate. For $K_\rho > 1$ we find a TS/SS phase, for $K_\rho < 1$ we obtain a SDW/CW phase. For $K_\sigma \to 0$, both $\alpha_{SDW}$ and $\alpha_{CDW}$ are sent to $-\infty$, whereas $\alpha_{SS}$ and $\alpha_{CDW}$ are now given by $\alpha_{SS} = 2 - K_\rho^{-1}$ and $\alpha_{CDW} = 2 - K_\rho$. Hence, we can distinguish four regimes: For $K_\rho > 2$ we have singlet pairing, for $K_\rho < 1/2$ we get CDW ordering. In between these two values of $K_\rho$ the system shows coexisting orders, that is, both the singlet pairing and the CDW susceptibility are divergent. For $1/2 < K_\rho < 1$ CDW is dominant and SS is subdominant, for $1 < K_\rho < 2$ it is the other way around.

The noise correlations can be calculated in the same way as described for the spinless case. We obtain analogous expressions to Eq. (20), in which the exponents $g$ and $h$ are replaced by:

$$g_{11/11} = (K_\rho + K_\rho^{-1} + K_\sigma + K_\sigma^{-1} - 4)/8 \quad (53)$$

and:

$$h_{1\uparrow\downarrow} = (K_\rho - K_\rho^{-1})/8 + (K_\sigma - K_\sigma^{-1})/8 \quad (54)$$

$$h_{1\uparrow\uparrow} = (K_\rho - K_\rho^{-1})/8 - (K_\sigma - K_\sigma^{-1})/8 \quad (55)$$

In order to understand in what regimes of the phase diagram we expect algebraic divergencies, we again consider the equal-time correlation functions of the operators (20)–(29). In momentum space, these correlation functions scale as $|q|^{-\alpha}$, where $\alpha$ is the corresponding scaling exponent, given in (40)–(42). If the system is in the gapless phase (i.e. $K_\sigma = 1$), these correlation functions never exhibit an algebraic divergence, and we should expect to find coexisting fluctuations throughout the phase diagram for $g_{1,\perp} > 0$, similar to the regime $1/2 < K < 2$ for spinless fermions. If the system is in the spin-gapped phase, we find the following behavior: Both TS and SDW fluctuations are frozen out, i.e. only short-ranged, whereas SS and CDW are increased by 1, compared to the gapless phase, because $K_\sigma \to 0$. We therefore expect algebraic divergencies for $K_\rho < 1$ in the $\uparrow\uparrow$ channel, and for $K_\rho > 1$ in the $\uparrow\downarrow$ channel.

A numerical study of the noise correlations confirms these expectations: We indeed find coexisting fluctuations in the gapless phase for any value of $K_\rho$. Furthermore, since the expressions (51) and (55) become identical for $K_\sigma = 1$, we find that $G_{1\uparrow\downarrow}(q, q') = G_{1\uparrow\uparrow}(q, q')$ in this regime. This is a manifestation of the degeneracy (at the algebraic level) of triplet and singlet pairing for $K_\rho > 1$, and of spin density and charge density wave ordering for $K_\rho < 1$, as discussed in (21). For the spin-gapped phase ($K_\sigma \to 0$) this symmetry is broken and $G_{1\uparrow\downarrow}$ and $G_{1\uparrow\uparrow}$ behave qualitatively different. We find the following behavior: $G_{1\uparrow\downarrow}(q, q')$ shows an algebraic divergence of the p-p type for $K_\rho < 1$, as expected from the equal-time correlation function of the CDW order parameter, an algebraic cusp for $1 < K_\rho < 2$, and no ordering for $K_\rho > 2$. $G_{1\uparrow\uparrow}(q, q')$ behaves in a complementary way: an algebraic divergence of the p-p type is found for $K_\rho > 1$, an algebraic cusp for $1/2 < K_\rho < 1$ and no ordering below that. In particular, for $K_\rho$ in the vicinity of 1, we find coexisting orders, as we demonstrate in Fig. 4. This is particularly clear if we consider the noise correlations of the total density $G_{1\uparrow\downarrow}(q, q')$ in this regime. In Fig. 4 (g)–i) we clearly see the coexistence of pairing and CDW ordering.

To understand this behavior further in this limit, we use the same argument as for the $K < 1/2$ and $K > 2$ regimes for spinless fermions. We re-write ($n_{1,q}n_{1,q'}$) and $\langle n_{1,q}n_{1,q'} \rangle$ in the same way as (29), where $g$ and $h$ need to be replaced by $g_{11/11}$ and $h_{1\uparrow\uparrow}$, respectively. In the limit $K_\sigma \to 0$ the arguments that lead to the scaling behavior (21) and (25) become exact, and we obtain

$$\langle n_{1,q}n_{1,q'} \rangle \sim \text{sgn}(K_\rho - 1)|q - q'|^{K_\rho - 1} \quad (56)$$

$$\langle n_{1,q}n_{1,q'} \rangle \sim \text{sgn}(K_\rho^{-1} - 1)|q + q'|^{K_\rho^{-1} - 1}. \quad (57)$$

These expressions show exactly the structure that was found numerically: algebraic divergencies for $K_\rho < 1$ ($K_\rho > 1$) in the $\uparrow\downarrow$ ($\uparrow\uparrow$) channel, and an algebraic cusp for $1 < K_\rho < 2$ ($1/2 < K_\rho < 1$).

IV. BOSONS

We turn to address the noise correlations in bosonic systems with either long range or quasi long-range order in the off diagonal density matrix $\langle b^\dagger(x)b(0) \rangle$. As in the fermion case we shall start with the case of true long range order, relevant to three dimensional systems at temperature $T < T_c$. This analysis is also relevant for
lower dimensional systems if they are sufficiently weakly interacting. Then the fact that the off diagonal density matrix decays as a power law is unnoticeable in a condensate of realistic size. We shall discuss in some detail the possibility of observing the pairing correlations associated with quantum depletion and show how such measurements would depend on the temperature. Then we move on to derive the noise correlations in a one dimensional Bose system at zero temperature, taking into account the power-law behavior of the correlations. As in the case of Fermions we use the effective Luttinger liquid theory, which correctly accounts for the singular contributions to the noise correlations due to the long distance power-law behavior of the off diagonal density matrix.

A. Bose-Einstein condensate with true ODLRO

Our starting point for analysis of the noise correlations is the Hamiltonian of a weakly interacting Bose gas with contact interactions

\[
H = \sum_k \epsilon_k a_k^\dagger a_k + \frac{U}{2V} \sum_k a_k^\dagger a_k^\dagger a_p a_p a_k.
\]

(58)

Here \(\epsilon_k = k^2/2m\) is the free particle dispersion with \(m\) and \(U = 4\pi a_s/m\) the s-wave scattering length. To compute the correlations in the condensed phase we apply the standard Bogoliubov theory (see e.g. [25]). As usual, the operators \(a_k^\dagger\) and \(a_k\) are replaced by a number representing the condensate amplitude \(\sqrt{N_0}\), while the other modes are treated as fluctuations and expanded to quadratic order. The effective Bogoliubov Hamiltonian is then given by

\[
H_B = \sum_k (\epsilon_k + U\rho_0) a_k^\dagger a_k + \frac{U\rho_0}{2} \sum_k a_k^\dagger a_{-k}^\dagger a_{-k} a_k,
\]

(59)

where \(\rho_0 = N_0/V\) is the condensate density. This Hamiltonian is diagonalized by the Bogoliubov transformation

\[
a_k = u_k \alpha_k + v_k \alpha_k^\dagger \quad \text{with} \quad u_k^2 = (\omega_k + \epsilon_k + U\rho_0)/2\omega_k, \quad v_k^2 = (-\omega_k + \epsilon_k + U\rho_0)/2\omega_k, \quad \text{and} \quad \omega_k = \sqrt{\epsilon_k^2 + 2U\rho_0}.
\]

The structure of the ground state wave function in the Bogoliubov approximation is given by

\[
|\Psi_B\rangle \sim \exp\left(\sqrt{N_0} a_0^\dagger + \sum_{k \neq 0} (v_k/\omega_k) a_{-k}^\dagger a_k\right) |0\rangle.
\]

(60)

Like the BCS state [13], the Bogoliubov wave-function describes perfectly correlated pairs of particles at momenta \(k\) and \(-k\), which suggests the appearance of pairing correlations in the noise.

It is straightforward to compute the noise correlations \(G(k,k')\) for \(k,k' \neq 0\). Because the Bogoliubov Hamiltonian [59] is quadratic we can use Wick’s theorem to decouple the four point function

\[
G(k,k') = \langle n_k \rangle \langle 1 + n_k \rangle \delta_{kk'} + |\langle a_{-k}^\dagger a_k^\dagger \rangle|^2 \delta_{k,-k'}.
\]

(61)

where the expectation values correspond to thermal averages, and \(n_k = a_k^\dagger a_k\). A bit more care is needed if either \(k = 0\) or \(k' = 0\) because the quadratic hamiltonian describes only the fluctuations in the depletion cloud, not in the condensate number. To obtain the fluctuations in the condensate within Bogoliubov theory, we use the conservation of total particle number, which implies that fluctuations in the condensate number are exactly minus those of the depletion cloud. In other words we substitute \(n_0 = N - \sum_{k \neq 0} n_k\) for the condensate particle number operator. Then we may use [59] to compute the noise correlation between points \(k = 0\) and \(k'\). Putting it all together we get the general expression for the noise correlations:

\[
G(k,k') = g_k \delta_{k,-k'} + f_k \delta_{kk'} - h_k (\delta_{k0} + \delta_{k'0}) + \delta_{k0} \delta_{k'0} \sum_q h_q \tag{62}
\]

where

\[
g_k = \frac{u_k^2 v_k^2}{4} (1 + 2 \langle n_{a,k} \rangle)^2
\]

\[
f_k = g_k + \langle n_{a,k} \rangle (1 + \langle n_{a,k} \rangle)
\]

\[
h_k = 2g_k + \langle n_{a,k} \rangle (1 + \langle n_{a,k} \rangle)
\]

(63)

Here \(\langle n_{a,k} \rangle = \langle a_k^\dagger a_k \rangle = \langle \exp(\omega_k/T) - 1 \rangle^{-1}\) is the quasiparticle number distribution.

At zero temperature each term in [62] has a simple physical interpretation. We already noted that the first term manifests the pairing correlations present in the quantum depletion described by the Bogoliubov wave function [60]. The second term is a positive correlation due to boson bunching at a point in \(k\)-space. The dips at \(G(k,0)\) reflect the fact that an extra atom found at \(k\) in the quantum depletion cloud corresponds to a pair of atoms, now missing from the condensate. Finally the positive peak at \(G(0,0)\) appears because extra atoms in the condensate must always come in pairs, annihilated from the depletion cloud. That is, if we find an extra atom in the condensate, then we are sure to find another extra atom in it.

![FIG. 5: Noise correlation of pairs \(G(k,-k)\) of a BEC, as a function of \(k\xi\), for different ratios of \(\lambda_T/\xi\).](image)
The evolution of the peaks with temperature and their momentum dependencies are controlled by the ratio of two natural length scales of the problem. One is the healing length of the condensate which is determined by interactions $\xi = 1/\sqrt{2mU/\rho_0}$. The other is the thermal wavelength $\lambda_T = 1/\sqrt{2mT}$. The combination $u_k^2v_k^2$ is of course temperature independent and may be expressed using the healing length alone as $[8(\xi k)^2 + 4(\xi k)^4]^{-1}$. The temperature dependence arises from the ratio $\omega_k/T$, which appears in the distribution function, and may be expressed as $(\lambda_T/\xi)^2 (k\xi)\sqrt{1 + (k\xi)^2}$. Note that the dimensionless ratio $(\lambda_T/\xi)^2$ is the same as the ratio $\mu/T$.

An interesting correlation to observe is the pairing correlation at $k' = -k$, which can be written explicitly using (62) and (63) as

$$G(k, -k) = \frac{\coth^2 \left( \frac{\mu}{T} (k\xi) \sqrt{1 + (k\xi)^2} \right)}{8(\xi k)^2 + 4(\xi k)^4}$$

This function is plotted on a log-log scale in Fig. 5. The behavior of the peak weight at small relative momentum $k\xi << 1$ has a very simple form. First, at very low temperatures, such that $(\mu/T)k\xi >> 1$ we have $G(k, -k) \approx 1/[8(k\xi)^2]$. This is the power law seen for $T = 0$ in Fig. 5. At higher temperature or sufficiently small momentum such that $(\mu/T)k\xi << 1$ we have $G(k, -k) \approx (T/\mu)^2/[2(k\xi)^4]$, which is seen in the other curves in the same figure.

It is interesting to note that the pairing correlations are substantially enhanced with temperature. This seems surprising given that the origin of the pairing is the quantum depletion in the ground state [60]. The effect may be interpreted as Bose enhancement of paired thermal fluctuations. However, we should also note that the overall noise level is also growing with temperature, that is the local (in $k$) particle number fluctuation $G(k, k)$ is increasing even more steeply with $T$. In Fig. 6 we plot the pair correlation normalized by the local number fluctuation

$$P(k) = \frac{G(k, -k)}{G(k, k)}$$

Clearly, for $T = 0$ (i.e. $\lambda_T/\xi \rightarrow \infty$) we have $P(k) = 1$ because in the ground state [60] the number fluctuations always come in opposite momentum pairs the particle number at $k$ and $-k$ must be identical, and hence also the correlations $G(k, -k)$ and $G(k, k)$ are equal. At finite temperature the pairing correlations are suppressed compared to the local (in $k$) number fluctuation. The ratio remains 1 at very high momentum because at $\omega_k >> T$ thermal occupation (which is exponentially suppressed is negligible compared to the quantum depletion. More interesting is the fact that $P(k)$ approaches 1 also in the limit of small relative momenta, which may again be a signature of Bose enhancement of the pairing fluctuations.

B. Quasi-condensate

In this section we discuss the noise correlations for a LL of bosons. We use Haldane’s representation [26] of a bosonic operator, defined as:

$$b(x) = [\rho_0 + \Pi(x)]^{1/2} \sum_m e^{2im\Theta(x)} e^{i\Phi(x)}$$

The fields $\Theta(x)$, $\Phi(x)$ and $\Pi(x)$ are defined in the same way as for fermionic LLs. $\rho_0$ is the average density. Note that now the sum is over the even harmonics, $2m$, and not the odd ones, $2m + 1$, that are used to represent a fermionic operator. The action of the system can be written as:

$$S \sim \frac{1}{2\pi K} \int d^2x \partial_\mu \theta \partial^\mu \theta \sim \frac{K}{2\pi} \int d^2x \partial_\mu \Phi \partial^\mu \Phi$$

As for fermionic systems, these representations are just quadratic in the fields, therefore all correlation functions reduce to Gaussian integrals.

For small momenta $k$ and $k'$ the noise correlations are

$$G(k, k') \sim \rho_0^2 \int e^{ikx_1+ik'x_3} F(x_12) F(x_34) (A_h - 1)$$

which we derive in detail in App. $F$ and $A$ are of similar form as before, with $g = 1/(4K)$ and $h = -1/(4K)$. This expression can be numerically evaluated for a finite system, with the replacement $x \rightarrow L \sin(2\pi x/L)/2\pi$. In Fig. 7 we show $G(k, k')$, plotted for different values of $K$.

For large values of $K$, the LL results resemble qualitatively the BEC result: The noise correlation function shows both a sharp pairing and bunching contribution which are equal in magnitude, as well as sharp quasi-condensate contributions along $(k, 0)$ and $(0, k')$. In addition there is a large peak at $(k = 0, k' = 0)$, (invisible in the representation in Fig. 7). As we reduce $K$ to smaller values (corresponding to larger repulsive interactions between the bosons), the quasi-condensate gets visibly broadened. This is due to the fact that there is no

![Figure 6: Pair correlation function $P(k) = G(k, -k)/G(k, k)$, as function of $k\xi$, for different ratios of $\lambda_T/\xi$.](image-url)
true condensate in 1D: The occupation in k-space is not peaked as a δ-function, but only an algebraic divergence, which gets broader for smaller values of K. As a consequence, also the pairing peak gets broadened, because the distribution of the total momentum of the pairs that are created from the condensate gets broadened. Furthermore, the overlap with the broad quasi-condensate dips diminishes the magnitude of the pairing peaks for smaller values of K, and eventually, for K = 1, which is the Tonks-Girardeau limit, the pairing peak is entirely suppressed.

In contrast to the broadening of the pairing peak and the quasi-condensate dips, the bunching peak is a δ-function for all values of K. This arises because we have |g| = |h|, and therefore the integrand in Eq. (C6) does not fall off in one direction. This can be understood from the integral expression (C6). We introduce the variables \( x = (x_{12} - x_{34})/2, z = (x_{12} + x_{34})/2 \) and \( y = x_{23} + z \), and rearrange the integral (C4): \( \langle \hat{n}_k \hat{n}_{k'} \rangle = \int e^{i\Delta k z + ikz} \mathcal{F}(z + y) \mathcal{F}(z - y) \hat{A} \). Here we introduced the definitions \( \Delta k = k - k', k = k + k' \), and

\[
\hat{A} = \left( \frac{x_0^2 + (x + z)^2}{x_0^2 + (x - z)^2} \right) \left( \frac{x_0^2 + (x + y)^2}{x_0^2 + (x - y)^2} \right)^n. \tag{69}
\]

From this, it is clear that for \( x \to \infty \) the integrand approaches 1, and does not fall off to zero, giving rise to a δ-function.

Finally, we discuss the transition of a 1D bosonic superfluid to a Mott insulator, which can occur if there is a lattice potential present that is commensurate to the density of the superfluid. For \( k, k' \approx 0 \) and for \( K \to 0 \), \( \hat{A} \) approaches 1, as can be seen from (69). In this limiting case, the integral of \( x \) becomes \( \delta_{\Delta k} = \delta_{k,k'} \). The remainder of the integral can be evaluated to be \( (\langle n_k \rangle)^2 \) for \( k' = k \), so the entire singular contribution is given by \( (\langle n_k \rangle)^2 \delta_{k,k'} \).

Therefore, for \( K \to 0 \), which describes the Mott insulator transition, we find \( \hat{G}(k,k') \to (\langle n_k \rangle)^2 \delta_{k,k'} - (\langle n_k \rangle \langle n_k' \rangle) \), as for the higher dimensional case [13]. By using the higher modes of Haldane’s representation, we can determine the behavior of \( G(k,k') \) for \( k \approx 2mk \) and \( k' \approx 2mk' \). We use the expression \( b = \sqrt{\rho_0 e^{2in\theta} e^{i\phi}} \), and we get: \( \langle n_k \rangle = \rho_0 \int d\varphi_1 e^{ikx_1 \varphi_1} \mathcal{F}_n(x_1) \), with \( g \) given by \( g = 1/4K + n^2K \). For \( \langle n_k n_{k'} \rangle \) we obtain \( \langle n_k n_{k'} \rangle = \rho_0^2 \int e^{ikx_1 + ikx_3} \mathcal{F}_n(x_1, x_3) \mathcal{F}_{m}(x_3) \hat{A} \). \( \hat{A} \) is defined as before, with \( h \) now given by \( h = -nmK - 1/4K \). We can now go through the same steps that were used to identify the δ-function for \( k, k' \approx 0 \). We find that for any \( n \) and \( m \), the noise correlation function approaches the peaked shape that was found for \( k, k' \approx 0 \) in the limit \( K \to 0 \). This coincides with the result found in [13], in which an ansatz of the form \(|MI\rangle = \prod b_i^\dagger |0\rangle \) was used to derive this result.

V. CONCLUSIONS

We have investigated the nature of noise correlations in systems of ultra cold atoms, that do not necessarily support long range order in any order parameter. The essential difference between noise correlations, which may be measured in time of flight experiments [14, 15, 16, 17], and standard order parameter correlations, is that the noise correlations are non local in real space. In other words, they are not a Fourier transform of a two point correlation function in real space. In this paper we investigated, the general connections, that nevertheless exist between these two types of correlations in many-body systems. We then derived the singular contributions to the noise correlation function for several ultra cold boson and fermion systems of interest. Our focus was on low dimensional systems, which display power law order-parameter correlations.

For Fermi systems, we showed in sec 11A, that true long range order in a particular order parameter (spin, density or pairing) leads to a delta-function contribution to the noise correlations. It is tempting to assume that if there are power-law decaying correlations in the same order parameter, then they would contribute corresponding algebraic singularities to the noise. However, we showed that this is only the case provided the algebraic decay is sufficiently slow. For example in a system of one dimensional spinless Fermions, the noise correlations appear similar to the mean field result, albeit with power-law peaks, only for \( K < 1 \) (strong repulsive interaction) or \( K > 2 \) (strong attractive interaction). In
the intermediate regime $2 > K > 1/2$, they display a unique structure that reveals both the dominant and sub-dominant order parameter correlations.

In a system of spinfull fermions, on the other hand, the noise correlations display significant singular contributions only in the spin-gapped phases (i.e. for negative backscattering). In this regime they provide information on both the singlet-pairing and density wave correlations.

In all cases we showed, that the noise correlations treat the particle-particle (pairing) and particle-hole (spin or density) channels on the same footing. Noise correlations are thus sensitive, and can easily distinguish between the dual order parameters. In general, pairing correlations manifest as positive noise correlations (i.e. peaks), while spin and charge correlations appear as anti-correlations in the noise (dips). This is in marked contrast to external probes, such as Bragg scattering, which couple only to spin or particle densities (particle-hole channel).

When considering Bose systems in section [IV] we first treated a condensate with true off diagonal long range order using Bogoliubov theory. The most interesting feature in this case is a correlation peak $\sim \delta(k+k')$, which is a direct manifestation of the pairing correlations in the quantum depletion cloud. We find, somewhat counter intuitively, that these correlations are enhanced with increasing temperature at low temperatures. It should be noted however that the "normalized" pairing correlation, that is, relative to the number fluctuation on the $k$-point, is indeed suppressed with temperature. Additional peaks appear due to correlations between the condensate number and the number of particles in the depletion cloud. These appear on the lines $k = 0$ and $k' = 0$.

Finally we used the quantum hydrodynamic (Luttinger liquid)description [20] of the Bose liquid, to address one dimensional Bose systems at $T = 0$. For this system, characterized by algebraic decay of off diagonal order, the pairing correlation in the noise also converts to a power-law singularity.

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APPENDIX A

In this appendix we derive Eq. [20] using bosonization. We write $\langle n_q n_{q'} \rangle$ and $\langle n_q \rangle$ as

$$\langle n_q n_{q'} \rangle = \frac{1}{L^2} \int d^4 x_1 e^{i q x_{12} + i q' x_{34}} \langle \psi_R^\dagger(1) \psi_R(2) \psi_L^\dagger(3) \psi_L(4) \rangle \quad (A1)$$

$$\langle n_q \rangle = \frac{1}{L} \int d^4 x_1 e^{i q x_{12}} \langle \psi_R^\dagger(1) \psi_R(2) \rangle \quad (A2)$$

$q$ is the momentum relative to the right Fermi point, $q = k - k_F$, and $q'$ the one relative to the left Fermi point, $q' = k' + k_F$, as before. We use the notation 'L' for $x_1$, etc., and $x_{12}$ for $x_1 - x_2$, etc. An analogous expression holds for $\langle n_{q'} \rangle$, with 'R' replaced by 'L'.

Using the bosonization expression in Eq. [16] for $\langle n_q \rangle$ we obtain

$$\langle n_q \rangle = \frac{1}{L^2} \int d^4 x_1 e^{i q x_{12}} \langle \psi_R^\dagger(1) \psi_R(2) \rangle \quad (A3)$$

and a similar expression for $n_{q'}$ with $\Theta(2) \rightarrow -\Theta(2)$ and $-\Theta(1) \rightarrow \Theta(1)$. By using $e^A e^B = e^{A+B} e^{|A,B|/2}$ we get:

$$\langle n_q \rangle = \int d^4 x_1 e^{iq x_{12}} \langle \psi_R^\dagger(1) \psi_R(2) \rangle \quad (A4)$$

The commutator between the fields $\Theta(x)$ and $\Phi(x)$ is in Eq. [17] Next we use $\langle e^A \rangle = \langle e^{|A|} \rangle$. Here it is necessary to impose a short distance cut-off $x_0$ on the interaction, that is, $K$ needs to depend on the momentum and has to fall off exponentially to 1 for momenta of the order of $1/x_0$. The expression that we use is

$$\langle (\Theta(x_0) - \Theta(0))^2 \rangle = \frac{K}{2} \log \left( \frac{x_0^2 + x^2}{x_0^2} \right) + \frac{1}{2} \log \frac{\alpha^2 + x^2}{\alpha^2}$$

An analogous expression holds for $\Phi(x)$ with $K$ replaced by $K^{-1}$. The cut-off $x_0$ on the interactions will stay finite throughout the calculation and can be interpreted as an effective bandwidth. If we apply this to $\langle n_q n_{q'} \rangle$, we obtain

$$\langle n_q n_{q'} \rangle = \frac{1}{2\pi} \int d^4 x_1 e^{iq x_{12}} \mathcal{F}^\dagger(\chi_{12}) \quad (A5)$$

where we introduced

$$\mathcal{F}(x) \equiv \left( \frac{x_0^2}{x_0^2 + x^2} \right)^g \frac{1}{\alpha - ix} \quad (A6)$$

with $g$ given by $g = (K + K^{-1} - 2)/4$.

We can evaluate $\langle n_q n_{q'} \rangle$ along the same lines. We again use Eq. [16] rearrange the exponents in the same way as we did for $\langle n_q \rangle$, while keeping track of the non-vanishing commutators between them, and take the expectation value, to obtain

$$\langle n_q n_{q'} \rangle = \int \frac{e^{iq x_{12} + iq' x_{34}}}{(2\pi)^2 L^2} e^{-\langle (\Theta(1) - \Theta(2) - \Theta(3) + \Theta(4))^2 \rangle/2}$$

$$\times \left[ e^{-\langle (\Phi(1) - \Phi(3) + \Phi(4))^2 \rangle/2} + e^{\langle (\Phi(1) - \Phi(2))^2 \rangle/2} \right] \mathcal{F}(x_{12}) \mathcal{F}^\dagger(x_{34}) \quad (A7)$$

This can be evaluated to

$$\langle n_q n_{q'} \rangle = \int \frac{e^{iq x_{12} + iq' x_{34}}}{(2\pi)^2 L^2} \mathcal{F}(x_{12}) \mathcal{F}^\dagger(x_{34}) \mathcal{A} \quad (A8)$$
The integration in this expression is over the three spatial variables $x_{12}$, $x_{23}$ and $x_{34}$. $A$ is defined as

$$A = \left( \frac{x_0^2 + x_{14}^2(x_0^2 + x_{23}^2)}{x_0^2 + x_{13}^2(x_0^2 + x_{24}^2)} \right)^2.$$ (A9)

The exponent $h$ is given by $h = (K-K^{-1})/4$. Combining the expressions that we derived for $\langle n_q \rangle$ and $\langle n_q n_{q'} \rangle$ we get for $G(q, q')$:

$$G(q, q') = \int \frac{e^{iqx_{12} + iq'x_{34}}}{(2\pi)^2 L} F(x_{12}) F^*(x_{34})(A - 1)$$

which is Eq. 20.

**APPENDIX B**

In this appendix we expand $G(q, q')$ to second order in the exponent $h$. We show that the first order term vanishes, and the second order term gives Eq. 20. The first order term is given by

$$h \int dx_{23} \log \left( \frac{x_0^2 + s(x_{14})^2(x_0^2 + s(x_{23})^2)}{x_0^2 + x_{13}^2(x_0^2 + x_{24}^2)} \right)$$ (B1)

where $s(x)$ is defined as $s(x) = L/2\pi \sin(2\pi x/L)$. Here we explicitly kept the expression for a finite-size system. Since the integral separates into the sum of two terms of the form $\pm \int dx_{12} \log(x_{0}^2 + s(x)^2)$, and each of these four terms integrates to the same value (as can be seen by shifting the integration variable, and using the periodicity of $\sin(2\pi x/L)$), the entire expression integrates to zero. The second order term is given by

$$\frac{h^2}{2} \int dx_{23} \log \left( \frac{(x_0^2 + x_{14}^2)(x_0^2 + s(x_{23})^2)}{(x_0^2 + x_{13}^2)(x_0^2 + x_{24}^2)} \right)^2,$$ (B2)

where we left out the cut-off and the finite-size representation for notational convenience. It can be checked that these will ensure the following expressions and manipulations to be well-defined.

Expression (B2) corresponds to a sum of integrals of two types: $\int dx \log^2(x^2)$ and $f(r) = \int dx x^2 \log(x^2 + r^2)$, as can be seen by expanding the square in (B2). The first integral merely provides terms that cancel divergent terms of the second integral type in the limit $L \to \infty$. The second term is given by $f(r) = C_1 + C_2 |r|$, with $C_1$ and $C_2$ some constants. This can easily be checked by taking derivatives, and by observing that $f(r)$ is even.

With this expression for the various integrals of the type $f(r)$, that we get from expanding the square in (B2), we get for $G(q, q')$:

$$G(q, q') \sim h^2 \int \frac{e^{iqx_{12} + iq'x_{34}}}{x_{12}} \frac{e^{iqx_{12} + iq'x_{34}}}{x_{34}} \left| x_{12} + x_{34} \right| \left| x_{12} - x_{34} \right|$$

To evaluate this integral we divide the integration range into eight sectors. As an example we treat the case given by: $x_{12} > 0$, $x_{34} > 0$, $x_{12} - x_{34} > 0$. (The other sectors are characterized by similar sets of inequalities.) For this part of the integration range we get an expression of the form:

$$-2 \int_0^\infty dx_{12} e^{iqx_{12}} \frac{1}{x_{12} |q'|} (e^{q'x_{12}} - 1)$$ (B4)

The other sectors of the integration range give similar expressions. These can be grouped into a sum of integrals that contain integrals of the form $\int dx/x \exp(ipx) = i\pi \text{sgn}(p)$, with different combinations of $q$ and $q'$ in the exponent. With that we get for $G(q, q')$ an expression which can be written as

$$G(q, q') \sim -h^2 \text{sgn}(q) \text{sgn}(q') \left( \frac{1}{|q|} - \frac{1}{|q'|} \right),$$ (B5)

which is Eq. 20.

**APPENDIX C**

We calculate $\langle n_k \rangle$ for $k \approx 0$ mode, for which the Bose operator is given by $b \sim \sqrt{\rho_0} e^{i\Phi}$. For $\langle n_k \rangle$ we find:

$$\langle n_k \rangle \sim \rho_0 \int \frac{dx_{12} e^{ikx_{12}} e^{-\frac{1}{2}((\Phi(2) - \Phi(1))^2)}}$$ (C1)

For the evaluation of the expectation value $\langle (\Phi(2) - \Phi(1))^2 \rangle$ we use a slightly different cut-off procedure than for the fermions, in particular:

$$\langle (\Phi(2) - \Phi(1))^2 \rangle = \frac{1}{2K} \log \frac{x_0^2 + x_{12}^2}{x_0^2}. \tag{C2}$$

With that we find $\langle n_k \rangle \sim \rho_0 \int \frac{dx_{12} e^{ikx_{12}} F(x_{12})}{x_{12}}$, where we defined:

$$F(x) = \left( \frac{x_0^2}{x_0^2 + x^2} \right)^g$$ (C3)

The exponent $g$ is given by $g = 1/4K$. Next we evaluate the expectation value $\langle n_k n_{k'} \rangle$ along the same lines. We obtain:

$$\langle n_k n_{k'} \rangle \sim \rho_0^2 \int e^{ikx_{12} + ikx_{34}} F(x_{12}) F(x_{34}) A \tag{C4}$$

$A$ is defined in the same as in Eq. (A9), but $h$ is now given by $h = -1/4K$. We combine these expressions to get the correlation function $G(k, k')$:

$$G(k, k') \sim \rho_0^2 \int e^{ikx_{12} + ikx_{34}} F(x_{12}) F(x_{34})(A - 1) \tag{C5}$$
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