A short proof of two identities using techniques of complex numbers

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Abstract. The sweeping development of mathematics is largely due to the introduction of complex numbers. Although complex numbers are just absurd notations for most people, complex numbers play essential roles in engineering fields. When mathematicians began to take an investigation of complex numbers, human beings enter a marvelous world. Complex numbers convince scientists that our world is magical, full of wonderful insights and even miraculous. In this paper, I first review several basic properties of complex numbers. The set of complex numbers is a group not only under the operation of the multiplication but under the operation of addition. Then, I show a visualization of complex numbers through building a bijective connection between complex numbers and points on the complex plane. I also give several alternative expression forms of complex numbers, namely the trigonometric form and the general form. By invoking the arithmetic properties of complex numbers, I prove two trigonometric identities.

1. Introduction
One of the purposes of introducing complex numbers is to obtain a closed algebraic domain containing all of the real numbers[1-5]. An algebraically closed domain can be considered as a set of all polynomials with coefficients of elements from a certain number field [1-3,6-10]. If zeros of each of these polynomials are also in this number field, then such a number field is called an algebraically closed domain. The algebraically closed domain allows us to deal with many problems using polynomials and power series. For example, almost all square matrices of order n with complex coefficients can be diagonalized. Complex numbers play a fundamental role in algebra, geometry, analysis, and other mathematical subjects[1-10]. Because of these fundamental roles, we can deal with practical problems in science and engineering quickly. In this paper, I first make a systematic review of complex numbers. I rigorously study the addition and multiplication of two complex numbers. Then I give a visualization of the complex plane and give the general form of complex numbers. At last, I apply the computational technique of complex numbers to prove two identities.

2. Main works

2.1. Complex numbers
Definition 2.1.1 We define a complex number as an ordered pair of real numbers \((a, b)\), where \(a \in R, b \in R\). Just as \(R\) is used as the standard notation to represent the set of whole real numbers, we use the symbol \(C\) as the set that concludes all complex numbers,
\[
C = \{(a, b): a \in R, b \in R\}.
\]
Definition 2.1.2 We can define two operations, addition and multiplication, of two complex number \( (a, b), (c, d) \), as follows:

Addition: \((a, b) + (c, d) = (a + c, b + d)\),

Multiplication: \((a, b) \cdot (c, d) = (ac - bd, ad + bc)\).

Remark 2.1.3 Both addition and multiplication satisfy commutative and Associative law[2].

For the addition, \((0, 0)\) is zero elements, and \((-a, -b)\) is a negative element of \((a, b)\); \((1, 0)\) is the identity element of multiplication; every non-zero element \((a, b)\) has inverse element \((\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2})\).

Furthermore, both addition and multiplication satisfy distribution property in \(C\):

\[ [(a, b) + (c, d)] \cdot (e, f) = (a, b)(e, f) + (c, d)(e, f) \]

Therefore, the set \(C\) of all complex numbers is a field, called the complex-number field. And a set of real numbers can be viewed as a subset of the set of all complex numbers since the set \(R\) can be written as in the form of,

\[ R = \{(a, 0): a \text{ is a real number}\} \]

2.2 Visualization of complex numbers

When we consider a coordinate system on the plane, a pair of real numbers, \((a, b)\), represents a point on the coordinate plane. We construct a bijection from \(R^2\) to \(C\) as follows:

\[ \phi: R^2 \rightarrow C \]

\[ \phi((a, b)) = (a, b). \]

Thus, the complex number \(z = a + ib\) can be viewed as a point on the plane with first coordination as the x-coordinate and b as the y-coordinate (Figure 1).

![Figure 1. The complex plane](image)

Definition 2.2.1 Given a complex number \((a, b)\), we define its trigonometric form as

\[ r \left( \cos \theta + i \sin \theta \right), \]

where \(r = \sqrt{a^2 + b^2}\), \(\theta\) satisfying that \(\tan(\theta) = \frac{b}{a}\) and \(i^2 = -1\).

Remark 2.2.2 From the definition 2.2.1, we can see that \(a = r \cos \theta\), \(b = r \sin \theta\).

Remark 2.2.3 The parameter \(\theta\) in the definition 2.2.1 is called the argument of \(z\), denoted as \(\theta = \text{Arg}z\). By the periodic property of trigonometric functions, we can see that if \(\theta\) is the argument of \(z\), \(\theta + 2 \cdot k \pi\) is also the argument of \(z\), where \(k\) is any integer. Therefore, a complex number has countable arguments.

Example 2.2.4

The complex number \(z = (1 + \cos \theta, \sin \theta)\) can be written as in trigonometric form,

\[ 2 \cos \left(\frac{\theta}{2}\right) \cos \left(\frac{1}{2} \theta\right), \sin \left(\frac{1}{2} \theta\right). \]
Definition 2.2.5 Given a complex number \( z \), we use the symbol \( \text{Arg} z \) as a unique argument \( \theta \) satisfying \( -\pi < \theta \leq \pi \), and this \( \theta \) is called the main value of \( z \).

Remark 2.2.6 Given a complex number \( z, \text{Arg} z = \text{arg} z + 2k\pi, k \in \mathbb{Z} \), where \( \mathbb{Z} \) denotes the set of all integers.

Remark 2.2.6 We do not take into consideration the argument of 0.

We can also think of the complex number \( (a, b) \) as a vector in \( \mathbb{R}^2 \). Mathematically, we take complex numbers and vectors as synonyms.

From this, we can see that the addition of two complex numbers coincides with addition of corresponding vectors in \( \mathbb{R}^2 \). Taking two different non-zero vectors \( z_1 \) and \( z_2 \) both starting at the origin, by the law of parallelogram, we can obtain a new vector by addition of vectors \( z_1 \) and \( z_2 \), which is shown in Figure 2.

![Figure 2. Addition and subtraction of two complex numbers](image)

2.3. General form of Complex numbers

Definition 2.3.1 A complex number \( z \) is written in the form of \( z = x + i \cdot y \), where \( x, y \) are two real numbers and \( i = \sqrt{-1} \).

Remark 2.3.2 We use the symbol \( \mathbb{C} \) as the set of all complex numbers. That is, \( \mathbb{C} = \{z, z = x + iy, x, y \in \mathbb{R}\} \).

For a given complex number \( z = x + i \cdot y \), where \( x, y \) are two real numbers, we say the real number \( x \) is the real part, and \( y \) is the imaginary part of the complex number \( z \). We use the symbols \( \text{Re}(z) \) and \( \text{Im}(z) \) to denote the real and imaginary part. That is, \( x = \text{Re}(z), y = \text{Im}(z) \).

If a complex \( z = x + i \cdot y \), where \( x, y \) are two real numbers, whose real part is zero and \( y \neq 0 \), we say \( z \) is a pure imaginary; If a complex \( z = x + i \cdot y \), where \( x, y \) are two real numbers, whose imaginary part is zero, we say the complex number \( z \) is a real.

Arithmetic operations of complex numbers

Definition 2.3.3 (Addition of complex numbers)

For any two complex numbers, \( z_1 = x_1 + i \cdot y_1, z_2 = x_2 + i \cdot y_2 \), where \( x_i, y_i (i = 1, 2) \) are real numbers, we define the addition of these two complex numbers as follows:

\[
z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)
\]

Definition 2.3.4 (Multiplication of complex numbers)

For any two complex numbers, \( z_1 = x_1 + i \cdot y_1, z_2 = x_2 + i \cdot y_2 \), where \( x_i, y_i (i = 1, 2) \) are real numbers, we define the multiplication of these two complex numbers as follows:

\[
z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1)
\]

Definition 2.3.4 (Complex conjugation)

For a given complex number \( z = x + i \cdot y \), where \( x, y \) are two real numbers, we define its complex conjugation \( x - iy \) as \( x - iy \).

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**Arithmetic operations of complex numbers**

**Definition 2.3.3** (Addition of complex numbers)

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\[
z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)
\]

**Definition 2.3.4** (Multiplication of complex numbers)

For any two complex numbers, \( z_1 = x_1 + i \cdot y_1, z_2 = x_2 + i \cdot y_2 \), where \( x_i, y_i (i = 1, 2) \) are real numbers, we define the multiplication of these two complex numbers as follows:

\[
z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1)
\]

**Definition 2.3.4** (Complex conjugation)

For a given complex number \( z = x + i \cdot y \), where \( x, y \) are two real numbers, we define its complex conjugation \( x - iy \) as \( x - iy \).
Theorem 2.3.5 (Properties of complex conjugation)

\[ z + w = \overline{z} + \overline{w}, \]
\[ z \cdot w = \overline{z} \cdot \overline{w}, \]
\[ \Re(z) = \frac{1}{2} (z + \overline{z}), \]
\[ \Im(z) = \frac{1}{2i} (z - \overline{z}). \]

Two Identities.

\[ \sum_{k=0}^{n} \cos k\theta = \frac{\sin \frac{\theta}{2} + \sin \left( n + \frac{1}{2} \right) \theta}{2 \sin \frac{\theta}{2}}, \]
\[ \sum_{k=1}^{n} \sin k\theta = \frac{\cos \frac{\theta}{2} - \cos \left( n + \frac{1}{2} \right) \theta}{2 \sin \frac{\theta}{2}}. \]

First, by summation formula of geometric series, we have

\[ \sum_{k=0}^{n} e^{ik\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{\sin \frac{n}{2} \theta}{\sin \frac{\theta}{2}} \cdot e^{\frac{i\theta}{2}}. \]

Thus we can get

\[ \sum_{k=0}^{n} \cos k\theta = \Re \left[ \sum_{k=0}^{n} e^{ik\theta} \right], \]
\[ \sum_{k=0}^{n} \sin k\theta = \Im \left[ \sum_{k=0}^{n} e^{ik\theta} \right]. \]

3. Conclusion

In this article, we discuss and explain in details the arithmetic and geometric properties of complex numbers. The arithmetic properties of complex numbers make it easy to compute some real number identities. In details, it is easy to give a closed form for the summation of trigonometric functions by invoking arithmetic properties of complex numbers. The geometric properties of complex numbers bridge the gap between the algebra and geometry. In detail, we can view a summation of two vector on the plane as a summation of two complex numbers. In the future, I will report my progress on holomorphic functions.

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