THE STABILITY OF SOBOLEV NORMS FOR THE LINEAR WAVE EQUATION WITH UNBOUNDED PERTURBATIONS

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Abstract. In this paper, we prove that the Sobolev norm of solutions of the linear wave equation with unbounded perturbations of order one stay bounded for the all time. The main proof is based on the KAM reducibility of the linear wave equation. To the best of our knowledge, this is the first reducibility result of the linear wave equation with general quasi-periodic unbounded potentials on the torus.

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1. Introduction

In this paper, we consider a linear wave equation with unbounded quasi-periodic perturbations of the form

\begin{equation}
\partial_{tt} u - \partial_{xx} u + m u + W(\omega t) u = 0, \quad t \in \mathbb{R}, \quad x \in T = \mathbb{R}/2\pi \mathbb{Z},
\end{equation}

where $W(\omega t)$ is a pseudo-differential operator of order one, and quasi-periodic in time with frequencies $\omega \in \mathcal{O} := [1, 2]^d$. The mass $m$ is positive. We prove that the Sobolev norms of solutions $(u, u_t)$ of the equation (1.1) are uniformly bounded for a large subset of $\mathcal{O}$. The

Key words and phrases. KAM theory, pseudo-differential operator, Sobolev norms.
The main proof is based on a quantitative reducibility result of the wave equation, in which we construct a bounded and time quasi-periodic transformation on space $H^r_x \times H^r_x$ such that the original equation (1.1) can be transformed into a block diagonal and time independent one.

The problem of estimating the high Sobolev norm of linear partial differential equations has been widely studied. The two remarkable results were obtained by Bourgain [16, 17] for the free Schrödinger equation with time dependent potential

$$i\partial_t u = -\Delta u + V(t, x)u,$$

on the $d$-dimensional torus. Bourgain [16] derived a $\langle t \rangle^c$ upper bound of the Sobolev norm of solutions for smooth and bounded time dependent potentials. When the potential $V$ is analytic and time quasi-periodic, Bourgain [17] proved that the Sobolev norm of solution grows like a power of $\log(t)$. The result obtained in [16] has been extended by Delort [18] and Berti-Maspero citeberti19 to the Zoll manifolds and flat tours. The logarithmic bounds on Sobolev norms in [17] has been extended by Wang [36] to an analytic and bounded time dependent potential on $T$.

However, Bourgain’s original method can only deal with the bounded perturbations, especially the multiplicative potential $V(t, x)$. The first result on the Schrödinger-type equation with unbounded, time dependent perturbation of the form

$$i\partial_t u(t) = Hu(t) + P(t)u,$$

is due to Maspero-Robert [28]. The method in [28] can be applied to the free Schrödinger equation on Zoll manifolds with time smooth perturbations of order $m < 1$, which provided a $\langle t \rangle^c$ upper bound of Sobolev norms of solutions. Based on the pseudo-differential operator technique, Bambusi-Grébert- Maspero-Robert [10] extended their results to more Schrödinger-type equations, including the free Schrödinger on Zoll manifolds with perturbations of order $m < 2$. In the meantime Montolto [30, 32] has independently studied the maximum order of perturbations for the Schrödinger-type equation on $T$. It’s worth mentioning that, based on a delicate Quantum version Nehoroshev theorem [9], Bambusio-Langella-Montalto [8] proved a $\langle t \rangle^c$ upper bound of Sobolev norm of solutions for the free Schrödinger equation on the flat torus with unbounded perturbations of order $m < 2$.

For the Schrödinger-type equation with small time quasi-periodic perturbations of the form

$$i\partial_t u(t) = Hu(t) + \varepsilon P(\omega t)u,$$

the KAM reducibility is a powerful tool to investigate the uniformly boundedness of the solutions in Sobolev space. For the bounded perturbations, we mention the results of Eliasson-Kuksin [19] which proved the reducibility of the Schrödinger equation on $T^d$ and Grébert et al. [24, 25] which proved the reducibility of the quantum harmonic oscillator on $\mathbb{R}^d$. The reducibility results imply that the Sobolev norms of solutions for the equation considered is uniformly bounded. For the unbounded perturbations, there are several papers devoting to the reducibility of some Schrödinger equations, such as the quantum harmonic oscillator [4, 7], duffing oscillator [27], relativistic Schrödinger equation on torus [34] and the free schrödinger equation on Zoll manifolds [22, 23].

Compared with the enormous results of Schrödinger-type equations, there are few results concerning the growth of Sobolev norm of solutions for the wave equation. Estimating the high Sobolev norm of solutions for linear wave equations on compact manifolds is much more subtle than Schrödinger-type equations. Fang-Han-Wang [20] had constructed a small time periodic potential for the wave equation on the torus such that the Sobolev norms of solutions is bounded for all time. While, Bourgain [17] constructed a time periodic potential for the wave equation, which provoke exponential growth of Sobolev norm. In order to avoid such terrible
upper bound, people has to pay more attention to the wave equation with time quasi-periodic perturbation. Naturally, the KAM reducibility becomes one of the main research methods. For the bounded perturbations, we mention the results of Li\cite{26} and Liang\cite{35} which proved the reducibility of wave equation on the torus with small time quasi-periodic multiplicative potential. Maspero\cite{21} proved the reducibility of the wave equation with non-small time quasi-periodic multiplicative potential. For the unbounded perturbations, Montolito\cite{31}, Sun et al.\cite{33} studied the wave equation with some special unbounded perturbations.

It should be mentioned that the unbounded perturbations consider in\cite{31, 33} is in the special form of $V(\omega t)\Delta$, which can be obtained by linearizing some nonlinear equations\cite{32}. People are more concerned with general unbounded perturbations. From the viewpoint of KAM theory, if the order of perturbations is strictly smaller than one, the KAM reducibility for such wave equation is straightforward. If the order of perturbations is equal to one, some serious problems are occurs in the measure estimate in KAM iteration. The similar problems are resolved by Berti-Biasco-Procesi\cite{11}, in which they obtained some quasi-periodic solutions of the Hamiltonian derivative wave equation. In order to estimate the number of non-resonance conditions, they introduced the “quasi-Töplitz” property of the perturbations to get the higher order asymptotic decay estimate of normal frequencies. However, the property of momentum conservation of the equation is indispensable for preserving the “quasi-Töplitz” property in KAM iteration. The property is missing from the wave equation (1.1) in the present paper. Therefore, this paper adopts a completely different method, that is the method of pseudo-differential operator.

For the Schrödinger-type equation with unbounded perturbations, the method of pseudo-differential operator can effectively smoothing the perturbations, so as to avoid a series of difficulties caused by the order of perturbations. However, such skills are almost useless for the wave equation. One of the main reasons is that the wave equation with Hamiltonian form can be seen as a $2 \times 2$ matrix valued, Schrödinger-type equation (2.10). For the matrix-valued pseudo-differential operator, the commutator of two matrix-valued pseudo-differential operators can not gains one derivative. Therefore, we can not transform the whole perturbation in equation (2.10) into a smoothing one. The main novelty of the present paper is that we find a delicate bounded transformation such that the original perturbations $K(\omega t)$ in equation (2.10) can be transformed into a new one $P(\omega t)$ in equation (4.17), where the diagonal part are smoothing operators and the off-diagonal part are still bounded operators. Furthermore, such structure can be maintained in the KAM iteration. Under these conditions we can also get the higher order asymptotic decay estimate of eigenvalues.

**Remark 1.1.** If we have a good control of the matrix decay norm of the operator $e^{iG}$, where $G$ is a self-adjoint, pseudo-differential operator of order $0 < m < \frac{1}{2}$, the method presented in this paper may be extended to the case that the order of perturbations is less than $3/2$. Form the Lemma 3.4 in\cite{32}, we known that the operator $e^{iG}$ is bounded in Sobolev space $\mathcal{H}^r_x$. But the information about its matrix decay norm is missing, which is essential to the KAM iteration.

**Remark 1.2.** In addition, the reducibility problem of wave equation (1.1) in this paper is the cornerstone of some further works. Inspired by\cite{1, 2}, we can use the quantitative reducibility result in this paper to explore the existence of Sobolev, linearly stable, quasi-periodic solutions of the following derivatives wave equations.

- **The Hamiltonian derivatives wave equations with quasi-periodic force**

\[
\partial_t u - \partial_{xx} u + mu + f(\omega t, x, Du) = 0, \quad D = \sqrt{-\partial_{xx} + m}, \quad x \in \mathbb{T}.
\]  

- **The autonomous derivatives wave equations**

\[
\partial_t u - \partial_{xx} u + mu + a(x)f(Du) = 0, \quad D = \sqrt{-\partial_{xx} + m}, \quad x \in \mathbb{T}.
\]
The paper is organized as follows: In section 2, we introduce some important definitions of pseudo-differential operator, so that we can precisely state our main results. In section 3, we introduce some norms of infinite dimensional matrix, such that the KAM iterations in section 5 can be well understood. In section 4, we introduce the the symbolic calculus of pseudo-differential operators in \[32\], such that the diagonal part of the perturbation \(K(\omega t)\) in equation (2.16) can be reduced to a operator of order \(-1\). In section 5, we give a block-diagonal reducibility result for the equation (2.16). In Section 6, we conclude the proof of Theorem 2.7 and the Corollary 2.9.

Notations: In the present paper, we denote the notation \(A \lesssim_s B\) as \(A \leq C(s)B\), where \(C(s)\) depends on the data of the problem, namely the Sobolev index \(s\), the number \(d\) of time frequencies, the diophantine exponent \(\tau > 0\) in the non-resonance conditions, which will be required along the proof.

2. Main result

Given a Function \(f : \mathcal{O} \rightarrow E : \omega \rightarrow f(\omega)\), where \((E, \| \cdot \|_E)\) is Banach space and \(\omega \in \mathcal{O}\). We define the sup-norm and lipschitz semi-norm as

\[
\|f\|_{E,\mathcal{O}}^{\text{sup}} := \sup_{\omega \in \mathcal{O}} \|f(\omega)\|_E, \quad \|f\|_{E,\mathcal{O}}^{\text{lip}} := \sup_{\omega_1,\omega_2 \in \mathcal{O}, \omega_1 \neq \omega_2} \frac{\|f(\omega_1) - f(\omega_2)\|_E}{|\omega_1 - \omega_2|}.
\]

For any \(\gamma > 0\), we define the Lipschitz-norm

\[
\|f\|_{E,\mathcal{O}}^{\gamma} := \|f\|_{E,\mathcal{O}}^{\text{sup}} + \gamma \|f\|_{E,\mathcal{O}}^{\text{lip}}.
\]

For notation convenience, we omit to write the set \(\mathcal{O}\).

2.1. Function space and pseudo-differential operators.

Sobolev space:

For any function \(u(x) \in L^2(T)\), it can be written as

\[
u(x) = \sum_{j \in \mathbb{Z}} \hat{u}_j e^{ij \cdot x}, \quad \hat{u}_j = \frac{1}{2\pi} \int_T u(x) e^{-ij \cdot x} dx.
\]

The Sobolev space \(H^s_x\) is defined by

\[H^s_x := \{ u(x) \in L^2(T) : \|u(x)\|^2_{H^s_x} = \sum_{j \in \mathbb{Z}} |j|^{2s}|\hat{u}_j|^2 < +\infty \},\]

where \(|j| = \max\{|1, |j||\} \}

For any functions \(u(\theta, x) \in L^2(T^d \times T)\), it can be regarded as a \(\theta\)-dependent family of functions \(u(\theta, \cdot) \in L^2(T)\). We can expand in Fourier series as

\[
u(\theta, x) = \sum_{j \in \mathbb{Z}} \hat{u}_j(\theta) e^{ij \cdot x}, \quad \hat{u}_j(\theta) = \frac{1}{2\pi} \int_T u(\theta, x) e^{-ij \cdot x} dx d\theta.
\]

The Sobolev space \(H^s(T^d \times T)\) is defined by

\[H^s(T^d \times T) := \{ u \in L^2(T^d \times T) : \|u\|^2_s = \sum_{(\ell, j) \in \mathbb{Z}^{d+1}} (\ell, j)^{2s}|\hat{u}_j(\ell)|^2 < +\infty \},\]

where \(|\ell, j| = \max\{1, |j|, |\ell|\}.\]
Notation: In the rest of the paper, we fix
\[ s_0 := \left\lfloor \frac{d + 1}{2} \right\rfloor + 1, \]
where for any real number \( x \in \mathbb{R} \), we denote by \( \lfloor x \rfloor \) its integer part.

**Pseudo-differential operators:**

**Definition 2.1.** (Pseudo-differential operators and symbols) Let \( m \in \mathbb{R}, s \geq s_0, \alpha \in \mathbb{N} \), we say that an operator \( A = A(\theta) \) is in the class \( \mathcal{OPS}_{s, \alpha}^m \), if there exists a function \( a = a(\theta, x, \xi) : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R} \to \mathbb{C} \), differentiable \( \beta \) times in the variables \( \xi \), such that
\[ Au(x) = \text{Op}(a)u(x) = \sum_{\xi \in \mathbb{Z}} a(\theta, x, \xi)u(\xi)e^{ix \cdot \xi}, \quad \forall u \in \mathcal{H}_2^0, \]
and
\[ |A|_{m, s, \alpha} := \sup_{|\beta| \leq \alpha} \sup_{\xi \in \mathbb{R}} \| \partial_\xi^\beta a(\theta, x, \xi) \| \mathcal{S}(\xi)^{-m+\beta}. \]

In that case, we say that \( a(\theta, x, \xi) \) is in the class \( \mathcal{S}_{s, \alpha}^m \). The operator \( A \) is said to be a pseudo-differential operator of order \( m \), and the function \( a \) is symbol.

If \( A := A(\lambda) \) is depending in a Lipschitz way on some parameter \( \omega \in \mathcal{O} \subseteq \mathbb{R}^d \), we set
\[ |A|_{m, s, \alpha}^\gamma := \sup_{\omega \in \mathcal{O}} |A|_{m, s, \alpha} + \gamma \sup_{\omega_1, \omega_2 \in \mathcal{O}} \frac{|A(\omega_1) - A(\omega_2)|}{|\omega_1 - \omega_2|}. \]

**Lemma 2.2.** (Lemma 2.13, 2.15 in [14]) Let \( s \geq s_0, m, m' \in \mathbb{R}, \alpha \in \mathbb{N} \).
1. Let \( A := \text{Op}(a) \in \mathcal{OPS}_{s+|m|+\alpha}^m, B := \text{Op}(b) \in \mathcal{OPS}_{s, \alpha}^{m'}, \) then the composition \( AB \) belongs to \( \mathcal{OPS}_{s, \alpha}^{m+m'} \), and
\[ |AB|_{s, m+m', \alpha} \leq |A|_{s, m, \alpha} |B|_{m', s, \alpha} + |A|_{s, m, \alpha} |B|_{m, s, \alpha} + |A|_{s, m, \alpha} |B|_{m, s, \alpha} + |A|_{s, m, \alpha} |B|_{m, s, \alpha}. \]
2. Let \( A := \text{Op}(a) \in \mathcal{OPS}_{s+1}^m, B := \text{Op}(b) \in \mathcal{OPS}_{s+\alpha}^{m'+1}, \) then
\[ AB = \text{Op}(a(\theta, x, \xi)b(\theta, x, \xi)) + \mathcal{R}_{AB}, \quad \mathcal{R}_{AB} \in \mathcal{OPS}_{s, \alpha}^{m+m'+1}, \]
where the Reminder \( \mathcal{R}_{AB} \) satisfies
\[ |\mathcal{R}_{AB}|_{m+m'-1, s, \alpha} \leq |A|_{m, \alpha} |B|_{m', s, \alpha} + |A|_{m, \alpha} |B|_{m', s, \alpha} + |A|_{m, \alpha} |B|_{m', s, \alpha} + |A|_{m, \alpha} |B|_{m', s, \alpha}. \]
3. Let \( A := \text{Op}(a) \in \mathcal{OPS}_{s+1}^m, B := \text{Op}(b) \in \mathcal{OPS}_{s+\alpha}^{m'+2}, \) then
\[ AB = \text{Op}(a(\theta, x, \xi)b(\theta, x, \xi) - i\partial_\xi a(\theta, x, \xi)\partial_\xi b(\theta, x, \xi)) + \mathcal{R}_{AB}, \quad \mathcal{R}_{AB} \in \mathcal{OPS}_{s, \alpha}^{m+m'-2}, \]
where the Reminder \( \mathcal{R}_{AB} \) satisfies
\[ |\mathcal{R}_{AB}|_{m+m'-2, s, \alpha} \leq \frac{1}{2} |A|_{m, s+1} |B|_{m', s, \alpha} + \frac{1}{2} |A|_{m, s+1} |B|_{m', s, \alpha} \]

**Remark 2.3.**
- From item 2 in Lemma 2.2, if \( A := \text{Op}(a) \in \mathcal{OPS}_{s+|m|+\alpha+2}^m, B := \text{Op}(b) \in \mathcal{OPS}_{s+|m|+\alpha+2}^{m'}, \) then the commutator \([A, B] := AB - BA \in \mathcal{OPS}_{s, \alpha}^{m+m'-1}, \)
\[ |[A, B]|_{m+m'-1, s, \alpha} \leq |A|_{m, s+|m'|+\alpha+2, \alpha+1} |B|_{m, s+|m'|+\alpha+2, \alpha+1} + |A|_{m, s+|m'|+\alpha+2, \alpha+1} |B|_{m, s+|m'|+\alpha+2, \alpha+1}. \]
- From item 3 in Lemma 2.2, if \( A := \text{Op}(a) \in \mathcal{OPS}_{s+|m|+\alpha+4}^m, B := \text{Op}(b) \in \mathcal{OPS}_{s+|m|+\alpha+4}^{m'}, \) then, the commutator \([A, B] := \text{Op}(-i\{a, b \} + r_{a, b}), \) where \( \{a, b\} = \partial_0 a \partial_x b - \partial_x a \partial_0 b \) and \( \text{Op}(r_{a, b}) \in \mathcal{OPS}_{s, \alpha}^{m+m'-2} \) satisfies
\( |\text{Op}(r.a)^\gamma |_{m+\alpha-2,\alpha} \lesssim m, m^\gamma+\alpha+4, \alpha+2 |\mathcal{B}|^\gamma |_{m, s+|m|+\alpha+4, \alpha+2} + |\mathcal{A}|^\gamma |_{m, s+|m|+\alpha+4, \alpha+2} |\mathcal{B}|^\gamma |_{m, s+|m|+\alpha+4, \alpha+2}. \)

**Adjoint of pseudo-differential operator**

Considering a \( \theta \)-dependent families of pseudo-differential operator \( \mathcal{A}(\theta) \), the symbol of the adjoint operator \( \mathcal{A}^\ast = \text{Op}(a^\ast(\theta, x, \xi)) \) is

\[
a^\ast(\theta, x, \xi) = \sum_{j \in \mathbb{Z}} \hat{a}(\theta, j, \xi - j) e^{ij \cdot x} = \sum_{j \in \mathbb{Z}^d, j \in \mathbb{Z}} \hat{a}(\ell, j, \xi - j) e^{i(j \cdot x + \ell \cdot \theta)}
\]

**Lemma 2.4.** (Lemma 2.16 in [12]) Let \( \mathcal{A} = \text{Op}(a) \in \mathcal{OPS}_s^m + |m|, 0 \), and dependant on the parameters \( \omega \in \mathcal{O} \). Then, the adjoint operator \( \mathcal{A}^\ast \) satisfies

\[
|\mathcal{A}^\ast|^\gamma |_{m, s, 0} \lesssim s, s |\mathcal{A}|^\gamma |_{m, s+|m|, 0}.
\]

**Lemma 2.5.** Let \( \mathcal{A} = \text{Op}(a(\theta, x, \xi)) \in \mathcal{OPS}_s^m + |m|, 2, \alpha \) be a self-adjoint operator and \( \mathcal{G} = \text{Op}(g(\xi)) \) be a real Fourier multiplies of order \( m \) (independent of parameters \( \omega \)). We define a new operator \( \mathcal{B} = \text{Op}(g(\xi) \cdot a(\theta, x, \xi)) \), that

\[
|\mathcal{B}|^\gamma |_{m, s+|m|+2, \alpha} \lesssim s, s |\mathcal{A}|^\gamma |_{m, s+|m|+2, \alpha}.
\]

Also, the operator \( \mathcal{B} - \mathcal{B}^\ast \in \mathcal{OPS}_s^{m-m-1, 0} \) and satisfies

\[
|\mathcal{B} - \mathcal{B}^\ast |^\gamma |_{m, s+|m|+2, \alpha} \lesssim s, s |\mathcal{A}|^\gamma |_{m, s+|m|+2, \alpha}.
\]

**Proof.** The estimate [29] is a direct corollary of Definition 2.1.

For the composition operator \( \mathcal{B} \), one sees

\[
\mathcal{B} = \text{Op}(g(\xi) \cdot a(\theta, x, \xi)) = \text{Op}(a(\theta, x, \xi)) \circ \text{Op}(g(\xi))
\]

and

\[
\mathcal{B}^\ast = \text{Op}^\ast(g(\xi)) \circ \text{Op}^\ast(a(\theta, x, \xi)).
\]

Since the operator \( \mathcal{A} \) is self-adjoint and \( g(\xi) \) is real, one has

\[
\mathcal{B}^\ast = \text{Op}(g(\xi)) \circ \text{Op}(a(\theta, x, \xi)).
\]

From Lemma 2.2 one gets that

\[
\mathcal{B}^\ast = \text{Op}(g(\xi)) \circ \text{Op}(a(\theta, x, \xi)) = \text{Op}(g(\xi) \cdot a(\theta, x, \xi)) + \mathcal{R}_g, \mathcal{A},
\]

and

\[
|\mathcal{B}^\ast - \mathcal{B}|^\gamma |_{m, s+|m|+\alpha} \lesssim s, m^{\gamma+1} |\mathcal{A}|^\gamma |_{m, s+|m|+\alpha} + |\mathcal{G}|^\gamma |_{m, s+|m|+\alpha} + |\mathcal{G}|^\gamma |_{m, s+|m|+\alpha} + |\mathcal{A}|^\gamma |_{m, s+|m|+\alpha}.
\]

For any symbol \( a \in \mathcal{S}^{m}_{s, \alpha} \), we defined the average symbol \( \langle a \rangle_{\theta, x} \) by

\[
\langle a \rangle_{\theta, x} = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} a(\theta, x, \xi) d\theta dx.
\]

Given \( \omega \in \mathbb{R}^d \) and satisfies the non-resonance condition

\[
|\omega \cdot \ell \pm j | \geq \frac{\gamma}{\langle \ell \rangle^\gamma}, \quad \forall (\ell, j) \in \mathbb{Z}^{d+1} \setminus \{0\}.
\]
We define the operator \((\omega \cdot \partial_{\theta} \pm \partial_{\bar{x}})^{-1}\) by setting
\[
(\omega \cdot \partial_{\theta} \pm \partial_{\bar{x}})^{-1}[1] = 0, \quad (\omega \cdot \partial_{\theta} \pm \partial_{\bar{x}})^{-1}((e^{i(\ell \cdot \theta + j \cdot x)}) = \frac{e^{i(\ell \cdot \theta + j \cdot x)}}{i(\omega \cdot \ell \pm j)}, \quad \forall (\ell, j) \in \mathbb{Z}^{d+1}\setminus\{0\}.
\]

**Lemma 2.6.** (Lemma 2.8 in [32]) Given a symbol \(a \in S_{s, \alpha}^m\),
1: \((\alpha^*)^{\theta, x} = (\alpha)^{\theta, x} = ((\alpha)^{\theta, x})^*\).
2: if \(\omega\) satisfies the non-resonance condition \((2.14)\), then
\[
(\omega \cdot \partial_{\theta} \pm \partial_{\bar{x}})^{-1}a = ((\omega \cdot \partial_{\theta} \pm \partial_{\bar{x}})^{-1}a)^*.
\]

We define the operator \(\sqrt{-\Delta}\) as follows, let \(\chi \in C^\infty(\mathbb{R}, \mathbb{R})\) be a cut-off function satisfies
\[
\chi(\xi) := \begin{cases} 
1 & \text{if } |\xi| \geq 1, \\
0 & \text{if } |\xi| \leq \frac{1}{2}.
\end{cases}
\]

We then define the operator \(\sqrt{-\Delta}\) as \(\text{Op}(\chi(\xi))\).

### 2.2. Main result.
Consider the perturbation \(\mathcal{W}(\omega t)\) in equation \((1.1)\), we assume that

**Condition I:** \(\mathcal{W}(\omega t)\) is a real, and self-adjoint linear operator.

**Condition II:** Set the symbol of the pseudo-differential operator \(\mathcal{W}(\omega t)\) as \(w(\theta, x, \xi)\), it satisfies
\[
(\omega)_{\theta, x} = \int_{\mathbb{T}^{d+1}} w(\theta, x, \xi) d\theta d\xi = a(\xi)\{\xi\} + b(\xi),
\]
where \(a(\xi) \in \Gamma = \{a_1, \ldots, a_n\}\), for any \(\xi \in \mathbb{Z}\). Also, there exists an absolute constant \(C\) such that
\[
|b(\xi)| \leq C|\xi|^{1-\varepsilon}, \quad \forall \xi \in \mathbb{Z}, \text{ and } \varepsilon > 0.
\]

In order to state our main results, we rewritten the wave equation \((1.1)\) as new form, by introducing the new variables,
\[
q = D^{\frac{1}{2}}u + iD^{-\frac{1}{2}}\partial_{\bar{x}}u, \quad \bar{q} = D^{\frac{1}{2}}u - iD^{-\frac{1}{2}}\partial_{\bar{x}}u,
\]
where
\[
D = \sqrt{-\Delta} + m.
\]

In the new variables, the equation \((1.1)\) is transformed to
\[
i\partial_t q(t) = Dq(t) + \frac{1}{2}D^{-\frac{1}{2}}\mathcal{W}(\omega t)D^{-\frac{1}{2}}(q(t) + \bar{q}(t)).
\]

Taking its complex conjugate, we can obtain the following matrix valued, Schrödinger-type system
\[
i\partial_t q(t) = H(t)q(t), \quad H(t) = D + \varepsilon K(\omega t),
\]

\[
D = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad K(\omega t) = \begin{pmatrix} \mathcal{K}(\omega t) & \mathcal{K}(\omega t) \\ -\mathcal{K}(\omega t) & -\mathcal{K}(\omega t) \end{pmatrix},
\]

where \(\mathcal{K}(\omega t) = \frac{1}{2}D^{-\frac{1}{2}}\mathcal{W}(\omega t)D^{-\frac{1}{2}}\).

**Theorem 2.7.** Assume the conditions I, II of linear equation \((2.15)\). There exists \(\bar{s} > 0\), such that for any \(s \geq \bar{s}\) there exists \(\epsilon_0 := \epsilon(s, d) > 0\), \(\gamma := \gamma(s, d) > 0\) and \(\bar{S}_s := \bar{S}(s, d) > 0\) with \(0 \leq \bar{S}_s \leq s\) such that if \(\mathcal{W} \in \mathcal{OPS}_{s, 2}^*\) satisfies
\[
|\mathcal{W}|_{1,s,2}^2 \leq \epsilon,
\]
then for any \( \epsilon \in (0, \epsilon_0) \) there exists a cantor like set \( \mathcal{O}_\epsilon \subset \mathcal{O} \) of asymptotically full Lebesgue measure, i.e. 
\[
\lim_{\epsilon \to 0} \text{meas}(\mathcal{O} \setminus \mathcal{O}_\epsilon) = 0,
\]
such that the following hold true. For any \( \omega \in \mathcal{O}_\epsilon \), there exists a liner bounded and invertible operator \( T(\omega t, \omega) \in \mathcal{B} (\mathcal{H}_x^1 \times \mathcal{H}_x^2) \) for any \( 0 \leq r \leq \mathcal{S}_x \), such that the change of coordinates \( \mathbf{q} = T(\omega t, \omega) \mathbf{v} \) conjugates the equation (2.16) to the block diagonal time-independent system 
\[
(2.18) \quad \partial_t \mathbf{v}(t) = \mathbf{H}_0^\infty(t)\mathbf{v}(t), \quad \mathbf{H}_0^\infty(t) = \left( \begin{array}{cc} \mathcal{H}_0^\infty & 0 \\ 0 & -\mathcal{H}_0^\infty \end{array} \right), \quad \mathcal{H}_0^\infty = \text{diag}\{ h_j^\infty | j \in \mathbb{N} \}
\]
h\( h_0^\infty \) is a real number close to \( \sqrt{\mathfrak{m}} \), \( \{ h_j^\infty \}_{j \geq 0} \) are \( 2 \times 2 \) self-adjoint matrices.

**Remark 2.8.** From Lemma 2.2 we known that \( \mathcal{K}(\omega t) \in \mathcal{OSP}_{-3,2}^{-1} \) is a real and self-adjoint operator, and satisfies 
\[
|\mathcal{K}|_{-1, \alpha, \beta} \lesssim \epsilon.
\]

**Corollary 2.9.** For any \( 0 \leq r \leq \mathcal{R}_x \) and \( \omega \in \mathcal{O}_\epsilon \), the solutions \( \mathbf{q}(t, x) := (q(t, x), \tilde{q}(t, x)) \) of equation (2.16) with initial condition \( \mathbf{q}(0, x) := (q(0, x), \tilde{q}(0, x)) \in \mathcal{H}_x^1 \times \mathcal{H}_x^2 \) satisfies 
\[
\| \mathbf{q}(t, x) \|_{\mathcal{H}_x^1 \times \mathcal{H}_x^2} \leq \mathcal{C}_r \| \mathbf{q}(0, x) \|_{\mathcal{H}_x^1 \times \mathcal{H}_x^2},
\]
for some absolute constants \( \mathcal{C}_r > 0 \).

3. **Linear operators**

3.1. **Matrix representation of linear operator.** Consider a linear operator \( \mathcal{A} : L^2(\mathbb{T}) \mapsto L^2(\mathbb{T}) \), it action on a function \( \mathbf{u} \in L^2(\mathbb{T}) \) as 
\[
\mathcal{A}[\mathbf{u}] = \sum_{j, k \in \mathbb{Z}} A_{j}^k \hat{u}_j e^{ijx},
\]
where 
\[
A_{j}^k = \frac{1}{2\pi} \int_{\mathbb{T}} \mathcal{A}[e^{ijx}] e^{-ikx} dx, \quad \forall j, k \in \mathbb{Z}.
\]
Also, we can identify the linear operator \( \mathcal{A} \) as a infinite-dimensional block matrix 
\[
(\mathcal{A}_\alpha^\beta)_{\alpha, \beta \in \mathbb{N}}
\]
where 
\[
[\mathcal{A}]_{\alpha}^\beta = (A_{j}^k)_{|j| = \alpha, |k| = \beta}.
\]
Note that the matrix \( [\mathcal{A}]_{\alpha}^\beta \) is a linear operator from \( \mathcal{E}_\alpha \) to \( \mathcal{E}_\beta \), where 
\[
\mathcal{E}_\alpha := \text{Span}\{ e^{ijx}, j \in \mathbb{Z}, |j| = \alpha \}.
\]
We also consider a smooth \( \theta \)- dependent families of linear operator \( \theta \mapsto \mathcal{A}(\theta), \mathcal{T} \mapsto \mathcal{B}(L^2(\mathbb{T})) \), which can be expanded in Fourier series as 
\[
(3.2) \quad \mathcal{A}(\theta) := \sum_{\ell \in \mathbb{Z}^d} \hat{A}(\ell) e^{i\ell \cdot \theta}, \quad \hat{A}(\ell) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathcal{A}(\theta) e^{-i\ell \cdot \theta} d\theta.
\]
It action on a function \( u \in L^2(\mathbb{T}^d \times \mathbb{T}) \) as 
\[
(3.3) \quad \mathcal{A}(\theta)[u(\theta, x)] = \sum_{j, k \in \mathbb{Z}} \hat{A}(\ell - \ell')^j \hat{u}_j(\ell') e^{i\ell \cdot \theta} e^{ijx}.
\]
Remark 3.1. From the matrix representation (3.3), we can regard the linear operator \( A \) as a pseudo-differential operator \( \mathcal{O}(a(\theta, x, \xi)) \), for any \( \xi = j \in \mathbb{Z} \), one has
\[
a(\theta, x, j) = \sum_{k \in \mathbb{Z}} A_k^j(\theta)e^{i(k-j)x}.
\]

**Definition 3.2. (Matrix block decay norm)** Let \( A \) be a \( \theta \)-dependent linear operator, \( \mathcal{O}(\theta) : \mathbb{T}^d \to \mathcal{B}(L^2(\mathbb{T})) \). Given \( s \geq 0 \), we say that \( A \in \mathcal{M}_s \), if and only if
\[
|A|_s := \sup_{\alpha \in \mathbb{N}} \left( \sum_{\ell, \beta \in \mathbb{N}} (\ell, \beta - \alpha)^{2s} \| \hat{A}(\ell) \beta \|^2 \| \| \|^2 \right)^{1/2} < +\infty.
\]
\( \| \cdot \|_0 \) stands for \( L^2 \) operator norm. If the operator \( A \) is Lipschitz depending on the parameters \( \omega \in \mathcal{O} \). For any \( \gamma \geq 0 \), we claim that \( A(\omega) \in \text{Lip}(\mathcal{O}, \mathcal{M}_s) \), if and only if
\[
|A|_{\gamma, \mathcal{O}} := \sup_{\omega \in \mathcal{O}} |A(\omega)|_s + \gamma \sup_{\omega_1, \omega_2 \in \mathcal{O}} \| A(\omega_1) - A(\omega_2) \|_s < +\infty
\]

**Remark 3.3.** If the linear operator \( A \) is independent of \( \theta \) and has the block matrix representation (3.1), the matrix decay block norm becomes
\[
|A|_s := \sup_{\alpha \in \mathbb{N}} \left( \sum_{\beta \in \mathbb{N}} (\beta - \alpha)^{2s} \| \hat{A}(\ell) \beta \|^2 \| \| \|^2 \right)^{1/2}.
\]

**Lemma 3.4.** 1: Let \( s \geq s_0 \), and \( A \in \mathcal{M}_s, u \in \mathcal{H}^s \). Then,
\[
\| Au \|_s \leq \| A \|_s \| u \|_{s_0} + |A|_{s_0} \| u \|_s.
\]
2: Let \( s \geq s_0 \), and \( A, B \in \mathcal{M}_s \). Then,
\[
|AB|_s \leq \| A \|_s \| B \|_{s_0} + |A|_{s_0} \| B \|_{s_0}.
\]
3: Let \( s \geq s_0 \), and \( A \in \mathcal{M}_s \). There exists a constant \( C(s) > 0 \), such that for any integer \( n \geq 1 \),
\[
|A^n|_s \leq C(s)^{n-1} (|A|_{s_0})^{n-1} |A|_s.
\]
4: Let \( s \geq s_0 \), and \( \Phi := \text{exp}(A) \) with \( A \in \mathcal{M}_s, \| A \|_{s_0} \leq 1 \). Then,
\[
|\Phi - \text{Id}|_s \leq \| A \|_s.
\]
5: Items 1 - 4 hold, replacing \( \| \cdot \|_s \) by \( \| \cdot \|_\gamma \) and \( \| A \|_s \) by \( \| A \|_\gamma \).

**Proof.** The proof is similar to Lemmata 2.7, 2.8 in [29]. \( \square \)

Given \( N \in \mathbb{N} \), we define a smoothing operator \( \Pi_N A \) for any operator \( A \) with block-matrix representation (3.1),
\[
[\Pi_N A(\ell)]_{\beta, \alpha} = \begin{cases} \hat{A}(\ell)_{\beta, \alpha}, & \text{if } |\ell| < N, \\ 0, & \text{otherwise}. \end{cases}
\]

**Lemma 3.5.** For any \( s, \alpha > 0 \), the operator \( \Pi_N^s := \text{Id} - \Pi_N^s \) satisfies
\[
|\Pi_N^s A|_s \leq N^{-\alpha} |A|_{s+\alpha}, \quad |\Pi_N^s A|_\gamma \leq N^{-\alpha} |A|_{s+\alpha}.
\]

From the Definition 3.2. and Remark 3.1, we can state a link between the pseudo-differential norms and matrix block decay norms.

**Lemma 3.6.** Let \( s \geq s_0 \) and \( A \in \text{OPS}^0_{s_0} \), one has
\[
|A|_s \leq |A|_{s_0} \gamma
\]
Lemma 3.7. (Lemma A.2 in [23]) Let $A \in B(H_s^r)$ for $r \geq 0$ with $|A|_{r+s_0} \leq +\infty$, then, one has
\[
\sup_{\theta \in T_d} \|A(\theta)\|_{B(H_s^r)} \lesssim |A|_{r+s_0}.
\]

Definition 3.8. Let $A$ be a $\theta$-dependent linear operator. Given $s \geq s_0$ and $\rho > 0$, we say that $A \in \mathcal{M}_{s,\rho}$ if
\[
(D)^{\sigma}A, \quad (D)^{\sigma}A(D)^{\sigma} \in \mathcal{M}_{s}, \quad \forall \sigma \in \{0,\pm \rho\}.
\]

We can endow the $\mathcal{M}_{s,\rho}$ with the norm
\[
\|A\|_{s,\rho} := |(D)^{\sigma}A|_s + |A(D)^{\sigma}|_s + \sum_{\sigma \in \{\pm \rho,0\}} |(D)^{\sigma}A(D)^{-\sigma}|_s.
\]

If the operator $A$ is Lipschitz depending on the parameters $\omega \in \mathcal{O}$, we say that $A(\omega) \in \text{Lip}(\mathcal{O},\mathcal{M}_{s,\rho})$. For any $\gamma > 0$, we endow it with the norm
\[
\|A\|_{s,\rho}^{\gamma,\mathcal{O}} := \sup_{\omega \in \mathcal{O}} \|A(\omega)\|_{s,\rho} + \gamma \sup_{\omega_1,\omega_2 \in \mathcal{O}, \omega_1 \neq \omega_2} \frac{\|A(\omega_1) - A(\omega_2)\|_{s,\rho}}{|\omega_1 - \omega_2|}.
\]

Remark 3.9. For any $\sigma \in \mathbb{R}$, the operator $(D)^{\sigma}$ is defined by $(D)^{\sigma}e^{iyx} = (j)^{\sigma}e^{iyx}$.

Definition 3.10. Let $A$ be a $\theta$-dependent linear operator. Given $s \geq s_0$ and $\rho > 0$, we say that $A \in \mathcal{L}_{s,\rho}$ if
\[
(D)^{\sigma}A(D)^{-\sigma} \in \mathcal{M}_{s}, \quad \forall \sigma \in \{0,\pm \rho\}.
\]

We can endow the $\mathcal{L}_{s,\rho}$ with the norm
\[
\|A\|_{\rho} := \sum_{\sigma \in \{\pm \rho,0\}} |(D)^{\sigma}A(D)^{-\sigma}|_s.
\]

If the operator $A$ is Lipschitz depending on the parameters $\omega \in \mathcal{O}$, we say that $A(\omega) \in \text{Lip}(\mathcal{O},\mathcal{L}_{s,\rho})$. For any $\gamma > 0$, we endow it with the norm
\[
\|A\|_{\rho,\gamma,\mathcal{O}} := \sup_{\omega \in \mathcal{O}} \|A(\omega)\|_{\rho} + \gamma \sup_{\omega_1,\omega_2 \in \mathcal{O}, \omega_1 \neq \omega_2} \frac{\|A(\omega_1) - A(\omega_2)\|_{\rho}}{|\omega_1 - \omega_2|}.
\]

Lemma 3.11. Let $\rho > 0$ and $s \geq s_0$. Assume that $A \in \mathcal{M}_{s,\rho}$ and $B \in \mathcal{L}_{s,\rho}$. Then, the following assertions hold true.
1: For any $r \in [0, s - s_0]$ and $\theta \in T_d$, the operator $e^{iB(\theta)} \in B(H_s^r)$ with the standard operator norm uniformly bounded in $\theta$.
2: The commutator $i[B, A] := i(BA - AB)$ belongs to $\mathcal{M}_{s,\rho}$ and satisfies
\[
\|[i[B, A]]\|_{s,\rho} \lesssim \|A\|_{s,\rho}\|B\|_s + \|A\|_{s,\rho}\|B\|_s.
\]
3: The operator $e^{iB}Ae^{-iB}$ belongs to $\mathcal{M}_{s,\rho}$ and with the quantitative bounds
\[
\|e^{iB}Ae^{-iB}\|_{s,\rho} \leq e^{2C_r\|B\|_s} \|A\|_{s,\rho}.
\]

The analogous assertions hold true, if $A \in \text{Lip}(\mathcal{O},\mathcal{M}_{s,\rho})$ and $B \in \text{Lip}(\mathcal{O},\mathcal{M}_s)$.

Proof. The item 1 is a direct corollary of Lemmata 3.4, 3.7 and Definition 3.8.
For the commutator $i[B, A] := i(BA - AB)$, the following inequalities hold (here $\sigma := \pm \rho, 0$)

$$
(3.21) \quad |(D)^{\sigma} AB|_s \lesssim |(D)^{\rho} A|_s |B|_s + |(D)^{\rho} A|_s |B|_s,
$$

$$
|AB(D)^{\sigma}|_s \lesssim |A(D)^{\rho}|_s |B(D)^{\rho}|_s + |A(D)^{\rho}|_s |B(D)^{\rho}|_s,
$$

$$
|(D)^{\sigma} AB(D)^{\rho}|_s \lesssim |(D)^{\rho} A(D)^{\rho}|_s |B(D)^{\rho}|_s + |(D)^{\rho} A(D)^{\rho}|_s |B(D)^{\rho}|_s,
$$

the same inequalities hold for $BA$. Thus, one can get the following bounds

$$
(3.22) \quad \|i[A, B]\|_{s, \rho} \lesssim \|A\|_{s, \rho} \|B\|_{s, \rho} + \|A\|_{s, \rho} \|B\|_{s, \rho}.
$$

For the operator $e^{iB} A e^{-iB}$, one has

$$
(3.23) \quad e^{iB} A e^{-iB} := A + i[B, A] + \frac{i[B, i[B, A]]}{2!} + \ldots.
$$

From (3.19), one gets

$$
\|e^{iB} A e^{-iB}\|_{s, \rho} \leq \|A\|_{s, \rho} + \|i[B, A]\|_{s, \rho} + \frac{\|i[B, i[B, A]]\|}{2!} + \ldots
$$

$$
(3.24) \quad \leq \|A\|_{s, \rho} + 2\|B\|_{s, \rho} \|B\|_{s, \rho} + \frac{2\|B\|_{s, \rho} \|B\|_{s, \rho}}{2!} + \ldots
\leq \|A\|_{s, \rho} e^{2\|B\|_{s, \rho}}
$$

\[\square\]

**Lemma 3.12.** Let $s \geq s_0, \rho > 0$ and $A \in \mathcal{OPS}_{s+\rho, 0}^\rho$, one has

$$
\|A\|_{s, \rho}^\gamma \lesssim \|A\|_{s, \rho}^\gamma\|A\|_{s, \rho}^\gamma.
$$

Also, if $A \in \mathcal{OPS}_{s+\rho, 0}^0$, one has

$$
\|A\|_{s, \rho}^\rho \lesssim \|A\|_{s, \rho}^\rho\|A\|_{s, \rho}^\rho.
$$

**Proof.** The proof is a direct corollary of Lemma 3.6 and Definitions 3.8 and 3.10 \[\square\]

### 3.2. The real and self-adjoint operators.

**Definition 3.13.** (1) Given a $\theta$-dependent operator $A(\theta) : \mathbb{T}^d \mapsto \mathcal{B}(L^2(\mathbb{T}))$, we define its conjugate operator $\hat{A}$ by $\hat{A}u = \overline{A u}$. The conjugate operator $\hat{A}$ has the matrix representation

$$
\hat{A}(\theta)_{ij} = (A(\theta)_{ji})^*, \quad \forall i, j \in \mathbb{Z}, \ \theta \in \mathbb{T}^d.
$$

(2) Given a $\theta$-dependent operator $A(\theta) : \mathbb{T}^d \mapsto \mathcal{B}(L^2(\mathbb{T}))$, we define its adjoint operator $A^*(\theta)$ by

$$
\int_{\mathbb{T}} A(\theta)[u] \cdot \overline{v} dx = \int_{\mathbb{T}} u \cdot A^*(\theta)[v] dx.
$$

The adjoint operator $A^*$ has the matrix representation

$$
A^*(\theta)_{ij} = \hat{A}(\theta)_{ij}^*, \quad \forall i, j \in \mathbb{Z}, \ \theta \in \mathbb{T}^d.
$$

**Lemma 3.14.** (1) A linear operator $A(\theta)$ is real, if it maps real functions to real and $\hat{A}(\theta) = \overline{A(\theta)}$. Then, for any $i, j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}, \theta \in \mathbb{T}^d$, the following assertions hold true,

$$
(3.25) \quad A(\theta)_{ij} = (A(\theta)_{ji})^* \iff \hat{A}(\theta)_{ij} = \overline{A(\theta)_{ji}} \iff [\hat{A}(\theta)]_{ij}^\alpha = \overline{[A(\theta)]_{ij}}^\beta.
$$

(2) A linear operator $A(\theta)$ is self-adjoint, if $A(\theta) = A^*(\theta)$. Then, for any $i, j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}, \theta \in \mathbb{T}^d$, the following assertions hold true,

$$
(3.26) \quad A(\theta)_{ij} = (A(\theta)_{ji})^* \iff \hat{A}(\theta)_{ij} = \overline{A(\theta)_{ji}} \iff [\hat{A}(\theta)]_{ij}^\alpha = ([A(\theta)]_{ij}^\beta)^*.
$$
Lemma 3.15. Let $A = \text{Op}(a) \in \mathcal{OPS}^m_{s,\alpha}$. Then, $A$ is real if and only if $a(\theta, x, \xi) = \overline{a(\theta, x, -\xi)}$.

3.3. $2 \times 2$ operator matrix. In this section, we will describe a special structure of operator matrix.

\begin{equation}
A = \begin{pmatrix}
A^d & A^a \\
-\overline{A^a} & -A^d
\end{pmatrix}
\end{equation}

It needs emphasize that the diagonal operator $A^d$ and anti-diagonal operator $A^a$ have different symmetry properties and matrix decay properties. For the details, we show it in the following definition.

Definition 3.16. Given a $2 \times 2$ operator matrix $A$ of the form (5.12), and $\rho, o \in \mathbb{R}, s \geq s_0$, we say that $A$ belongs to $\mathcal{N}_s(\rho, o)$ if

\begin{equation}
[A^d]^* = A^d, \quad [A^a]^* = \overline{A^a}
\end{equation}

and

\begin{equation}
A^d \in \mathcal{M}_{s,\rho}, \quad A^a \in \mathcal{M}_{s,o}
\end{equation}

We can endow the $\mathcal{N}_s(\rho, o)$ with the norm

\begin{equation}
\|A\|_{s,\rho,o} := |\langle D \rangle^s A^d|_s + |\langle D \rangle^s \langle D \rangle^o A^a|_s + |\langle D \rangle^s \langle D \rangle^o A^a|_s + |\langle D \rangle^s \langle D \rangle^o A^a|_s
\end{equation}

\begin{equation}
+ \sum_{\sigma \in \{\pm, 0\}} \sup_{\delta \in \{d,a\}} |\langle D \rangle^\sigma A^d \langle D \rangle^{-\sigma}|_s.
\end{equation}

If the operator $A^d$ and $A^a$ are Lipschitz depending on the parameters $\omega \in \mathcal{O}$, we say that that $A(\omega) \in \text{Lip}(\mathcal{O}, \mathcal{N}_s(\rho, o))$. For any $\gamma > 0$, we endow it with the norm

\begin{equation}
\|A\|_{\gamma,\rho,o} := \sup_{\omega \in \mathcal{O}} \|A(\omega)\|_{s,\rho,o} + \gamma \sup_{\omega_1, \omega_2 \in \mathcal{O}} \frac{\|A(\omega_1) - A(\omega_2)\|_{s,\rho,o}}{\|\omega_1 - \omega_2\|}.
\end{equation}

Remark 3.17.

- The symmetry properties of the space $\mathcal{N}_s(\rho, o)$ is equivalent to ask that the operator matrix $A$ is the Hamiltonian vector field of a real valued quadratic Hamilton. For the details, we refer to [29, 31].

- The decay properties (3.29) of the diagonal operator $A^d$ is essential to the measure estimate in section 4, we will show that the decay properties of the diagonal operator can be maintained in KAM iteration.

Lemma 3.18. Let $\alpha > 0$ and $s \geq s_0$. Assume that $A \in \mathcal{N}_s(\rho, o)$ and $B \in \mathcal{N}_s(\rho, o)$. Then $i[B, A]$ belongs to $\mathcal{N}_s(\rho, o)$ and satisfies

\begin{equation}
\|i[B, A]\|_{s,\rho,o} \lesssim \|A\|_{s,\rho,o} \|B\|_{s,\rho,o} + \|A\|_{s,\rho,o} \|B\|_{s,\rho,o}
\end{equation}

If $A \in \text{Lip}(\mathcal{O}, \mathcal{N}_s(\rho, o))$ and $B \in \text{Lip}(\mathcal{O}, \mathcal{N}_s(\rho, o))$. Then $i[B, A]$ belongs to $\text{Lip}(\mathcal{O}, \mathcal{N}_s(\rho, o))$ and satisfies

\begin{equation}
\|i[B, A]\|_{s,\rho,o} \lesssim \|A\|_{s,\rho,o} \|B\|_{s,\rho,o} + \|A\|_{s,\rho,o} \|B\|_{s,\rho,o}.
\end{equation}

Proof. The proof is similar to Lemma 2.22 in [21] and Lemma [3.11].
Lemma 3.19. Let \( \rho > 0 \) and \( s \geq s_0 \). Assume that \( A \in \mathcal{N}_s(\rho, 0) \) and \( B \in \mathcal{N}_s(\rho, \rho) \). Then, the following assertions hold true.

1: For any \( r \in [0, s] \) and \( \theta \in \mathbb{T}^d \), the operator \( e^{iB(\theta)} \in \mathcal{B}(\mathcal{H}_x^r \times \mathcal{H}_x^r) \) with the standard operator norm uniformly bounded in \( \theta \).

2: The operator \( e^{iB}Ae^{-iB} \) belongs to \( \mathcal{N}_s(\rho, 0) \) and \( e^{iB}Ae^{-iB} = A \) belongs to \( \mathcal{N}_s(\rho, \rho) \) with the quantitative bounds

\[
\| e^{iB}Ae^{-iB} \|_{s, \rho, 0} \leq 2C_s e^{2C_s \| B \|_{s, \rho, \rho}} \| A \|_{s, \rho, 0},
\]

\[
\| e^{iB}Ae^{-iB} - A \|_{s, \rho, 0} \leq 2C_s e^{2C_s \| B \|_{s, \rho, \rho}} \| A \|_{s, \rho, 0} \| B \|_{s, \rho, \rho}.
\]

The analogous assertions hold true, if \( A \in \text{Lip}(O, \mathcal{N}_s(\rho, 0)) \) and \( B \in \text{Lip}(O, \mathcal{N}_s(\rho, \rho)) \).

Proof. The proof is similar to Lemma 2.23 in \([21]\) and Lemma 3.11.

4. Regularization procedure

The goal of this section is smoothing the diagonal part of the \( 2 \times 2 \) operator matrix \( K(\omega t) \) in equation (2.10). In the following lemmas, we will conjugate the diagonal parts of the perturbation into smoothing operators, while the anti-diagonal parts remain bounded operators. The results of this section is essential to the KAM iteration in the following section. For any \( \gamma \in (0, 1) \) and \( \tau_0 > d \), we introduce the set

\[
\mathcal{O}_\gamma = \{ \omega \in \mathcal{O} : |\omega \cdot \ell + j| \geq \frac{\gamma}{\ell \tau_0}, \quad \forall (\ell, j) \in \mathbb{Z}^{d+1} \setminus \{0\} \}.
\]

Lemma 4.1. For any \( \omega \in \mathcal{O}_\gamma \), and symbol \( k \in \mathcal{S}^{0}_{s-7, 2} \) with \( k = k^* \), there exists a symbol \( g \in \mathcal{S}^0_{s-\tau_0-5, 2} \), with \( g = g^* \), such that

\[
k - \langle k \rangle_{\theta, x} - \omega \cdot \partial_\theta g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi) \in \mathcal{S}^{-1}_{s-\tau_0-7, 2}.
\]

Furthermore, one has

\[
|\text{Op}(g)|_{0, s-\tau_0-5, 2} \lesssim \frac{1}{\gamma} |\text{Op}(k)|_{0, s, 2}^2
\]

and

\[
|\text{Op}(k - \langle k \rangle_{\theta, x} - \omega \cdot \partial_\theta g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi))|_{1, s-\tau_0-7, 2} \lesssim \frac{1}{\gamma} |\text{Op}(k)|_{0, s, 2}.
\]

Proof. Taking a cut-off function \( \chi_1(\xi) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \), it is satisfying

\[
\chi_1(\xi) = 0, \quad \forall |\xi| \leq \frac{3}{2} \quad \text{and} \quad \chi_1(\xi) = 1, \quad \forall |\xi| \geq 2.
\]

For the symbol (1.2), one has

\[
k - \langle k \rangle_{\theta, x} - \omega \cdot \partial_\theta g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi)
\]

(4.5)

\[= \chi_1(k - \langle k \rangle_{\theta, x}) - \omega \cdot \partial_\theta g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi) + (1 - \chi_1)(k - \langle k \rangle_{\theta, x}).
\]

From the Lemma 2.5 we know that \( (1 - \chi_1)(k - \langle k \rangle_{\theta, x}) \in \mathcal{S}^{-1}_{s, 2} \) and

\[
|\text{Op}((1 - \chi_1)(k - \langle k \rangle_{\theta, x}))|_{1, s, 2} \lesssim |\text{Op}(k)|_{0, s, 2}.
\]

(4.6)

Our goal is determine a symbol \( g \), such that

\[
\chi_1(k - \langle k \rangle_{\theta, x} - \omega \cdot \partial_\theta g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi) \in \mathcal{S}^{-1}_{s-\tau_0-5, 2}.
\]
Since we require that \( g = g^* \), we look for a symbol of the form \( g = \frac{2 + q^*}{2} \). From (4.3), one has

\[
\chi_1(k - \langle k \rangle_{\theta, x}) - \omega \cdot \partial_0 g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi) = \chi_1(k - \langle k \rangle_{\theta, x}) - \omega \cdot \partial_0 q - \partial_x q \cdot \frac{\xi}{|\xi|} \chi(\xi)
\]

\[
+ (\omega \cdot \partial_0 - \frac{\xi}{|\xi|} \chi(\xi) \cdot \partial_x)[\frac{q^* - q}{2}]
\]

Next, we look for a symbol \( q \) that satisfies

\[
\chi_1(k - \langle k \rangle_{\theta, x}) - \omega \cdot \partial_0 q - \partial_x q \cdot \frac{\xi}{|\xi|} \chi(\xi) = 0.
\]

For any \( \omega \in \mathcal{O}_\gamma \), the symbol \( q \) defined as

\[
q(\theta, x, \xi, \omega) := (\omega \cdot \partial_0 + \partial_x)^{-1}[k(\theta, x, \xi) - \langle k \rangle_{\theta, x}(\xi)]\chi_1^+(\xi)
\]

\[
+ (\omega \cdot \partial_0 - \partial_x)^{-1}[k(\theta, x, \xi) - \langle k \rangle_{\theta, x}(\xi)]\chi_2(\xi),
\]

where \( \chi_1^+(\xi) := \chi_1(\xi)1_{\xi > 0}, \chi_1^-(\xi) := \chi_1(\xi)1_{\xi \leq 0} \). \( 1_{\xi > 0} \) (resp. \( 1_{\xi \leq 0} \)) is the characteristic function of the set \( \{ \xi \in \mathbb{R} : \xi > 0 \} \) (resp. \( \{ \xi \in \mathbb{R} : \xi \leq 0 \} \)).

It is easy to verify that \( \chi_1^+(\xi), \chi_1^-(\xi) \) are \( C^\infty \). From the non-resonance condition (4.11) and Definition 2.1, we known that \( (\omega \cdot \partial_0 \pm \partial_x)^{-1}[k(\theta, x, \xi) - \langle k \rangle_{\theta, x}(\xi)] \in \mathcal{S}_{0, s - \tau_0 - 3, 2}^0 \),

\[
|\text{Op}(k - \langle k \rangle_{\theta, x}(\xi) - k(\theta, x, \xi))|_{0,s-\tau_0-3,2}^6 \leq \frac{1}{\gamma} |\text{Op}(k)|_{0,s,2}^6.
\]

and

\[
|\text{Op}(q)|_{0,s-\tau_0-3,2}^6 \lesssim_s \frac{1}{\gamma} |\text{Op}(k)|_{0,s,2}^6.
\]

Note that \( k = k^* \), from Lemmata 2.5, 2.6 one gets

\[
|\text{Op}(q) - \text{Op}(q^*)|_{-1,s-\tau_0-5,2}^6 \lesssim_s |\text{Op}(k - \langle k \rangle_{\theta, x} - \langle k \rangle_{\theta, x}(\xi))|_{0,s-\tau_0-3,2}^6
\]

\[
\lesssim_s \frac{1}{\gamma} |\text{Op}(k)|_{0,s,2}^6
\]

Hence, one gets

\[
|\text{Op}(\omega \cdot \partial_0 - \frac{\xi}{|\xi|} \chi(\xi) \partial_x)[\frac{q^* - q}{2}]|_{-1,s-\tau_0-6,2}^6 \lesssim_s \frac{1}{\gamma} |\text{Op}(k)|_{0,s,2}^6.
\]

Combined (4.12), (4.13) and (4.14), there exists a symbol \( g = \frac{2 + q^*}{2} \) such that

\[
k - \langle k \rangle_{\theta, x} + \omega \cdot \partial_0 g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi) = (1 - \chi_1)(k - \langle k \rangle_{\theta, x}) + (\omega \cdot \partial_0 - \frac{\xi}{|\xi|} \chi(\xi) \partial_x)[\frac{q^* - q}{2}],
\]

\[
|\text{Op}(k - \langle k \rangle_{\theta, x} + \omega \cdot \partial_0 g - \partial_x g \cdot \frac{\xi}{|\xi|} \chi(\xi))|_{-1,s-\tau_0-6,2}^6 \leq |\text{Op}((1 - \chi_1)(k - \langle k \rangle))|_{1,s,2}^6
\]

\[
+ |\text{Op}(\omega \cdot \partial_0 - \frac{\xi}{|\xi|} \chi(\xi) \partial_x)[\frac{q^* - q}{2}]|_{1,s-\tau_0-6,2}^6
\]

\[
\lesssim_s \frac{1}{\gamma} |\text{Op}(k)|_{0,s,2}^6,
\]

and

\[
|\text{Op}(q)|_{0,s-\tau_0-5,2}^6 \lesssim_s \frac{1}{\gamma} |\text{Op}(k)|_{0,s-3,2}^6.
\]
Lemma 4.2. Consider the linear equation (2.16) and assume conditions I, II. If the frequency vector $\omega \in O_\gamma$, there exists a time dependent change of coordinates

$$[q(t), \bar{q}(t)]^T = [e^{-i \mathcal{G}(\omega t, \omega)} v(t), e^{i \mathcal{G}(\omega t, \omega)} \bar{v}(t)]^T,$$

where

$$\mathcal{G}(\omega t, \omega) = \text{Op}(g(\theta, x, \xi, \omega)) \in OPS_{s-\tau_0-5,2}^0,$$

that conjugates equation (2.16) to

$$(4.17) \quad i\partial_t v(t) = \tilde{H}(t)v(t), \quad \tilde{H}(t) = H_0 + P(\omega t, \omega),$$

where

$$H_0 = D + [K], \quad [K] = \begin{pmatrix} \text{Op}((k)_{\theta,x}) & 0 \\ 0 & -\text{Op}((k)_{\theta,x}) \end{pmatrix}$$

and

$$P(\omega t, \omega) \in \text{Lip}(O, N_{s-\tau_0-11}(1,0)).$$

Also, there exists a constant $C_\gamma$ depending on $s$ such that

$$\|P(\omega t, \omega)\|_{s-\tau_0-11,1,0} \leq \frac{1}{\gamma} e^{2 \mathcal{G}_{0}} |K|_{s-3,0}^\gamma.$$

Proof. First of all, we split the diagonal operator $D$ in (2.16) into two parts, that is

$$(4.19) \quad D = B + Z, \quad B = \begin{pmatrix} \sqrt{-\Delta} & 0 \\ 0 & -\sqrt{-\Delta} \end{pmatrix}, \quad Z = \begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix}.$$

From Lemma 8.6 in [34], we known that the operator $Z$ is a real Fourier multiplies of order $-1$. Also, we divided the perturbation $K(\omega t)$ into two parts, that is

$$(4.20) \quad K = K_1 + K_2, \quad K_1 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}.$$

Through the transformation $q(t) = e^{-i \mathcal{G}(\omega t, \omega)} v(t)$, where

$$(4.21) \quad e^{-i \mathcal{G}(\omega t, \omega)} = \begin{pmatrix} e^{-i \mathcal{G}(\omega t, \omega)} & 0 \\ 0 & e^{i \mathcal{G}(\omega t, \omega)} \end{pmatrix},$$

we can conjugated the equation (2.16) to

$$(4.22) \quad i\partial_t v(t) = \tilde{H}(t)v(t),$$
\[
\bar{H}(t) = e^{G(t)G} H(t) e^{-G(t)G} - \int_0^1 e^{isG} G e^{-isG} ds
\]
\[
= B + Z + i[G, B] + K_1 - \bar{G}
\]
\[
= i \int_0^1 (1-s) e^{isG} [G, \bar{G}] e^{-isG} ds
\]
\[
+ i \int_0^1 e^{isG} [G, K_1] e^{-isG} ds
\]
\[
+ i \int_0^1 e^{isG} [G, Z] e^{-isG} ds
\]
\[
+ i \int_0^1 e^{isG} [G, i[B, B]] e^{-isG} ds
\]
\[
+ e^{iG} K_2 e^{-iG}
\]

The main goal is determined a $2 \times 2$ operator matrix $\mathbf{G}$, where

\[
\mathbf{G} = \begin{pmatrix}
\mathcal{G}(\omega t, \omega) & 0 \\
0 & -\mathcal{G}(\omega t, \omega)
\end{pmatrix},
\]

\[
\mathcal{G}(\omega t, \omega) = \text{Op}(g(\theta, x, \xi, \omega)),
\]

such that $i[G, B] + K_1 - \bar{G}$ becomes a smoothing operator matrix. From (4.21), one gets

\[
i[G, B] + K_1 - \bar{G} = \begin{pmatrix}
i[G, \sqrt{\Delta}] - \omega \cdot \partial \mathcal{G} + \mathcal{K} & 0 \\
0 & i[\bar{\mathcal{G}}, \sqrt{-\Delta}] + \omega \cdot \partial \bar{\mathcal{G}} - \mathcal{K}
\end{pmatrix},
\]

Take the operator $i[G, \sqrt{-\Delta}] - \omega \cdot \partial \mathcal{G} + \mathcal{K} - \text{Op}((\langle k \rangle_{\theta, x})$ as $\mathcal{P}_1$, from Lemma 2.2 and Remark 2.3, one gets

\[
\mathcal{P}_1 + \text{Op}((\langle k \rangle_{\theta, x}) = \text{Op}\left\{ g, |\xi| \xi |\mathcal{G}(\xi)| \right\} + i \text{Op}(r_g, |\xi| \xi |\mathcal{G}(\xi)|) + \text{Op}(-\omega \cdot \partial \mathcal{G}) + \text{Op}(k)
\]
\[
= \text{Op}(-\partial \mathcal{G} \cdot |\xi| \xi |\mathcal{G}(\xi)| - \omega \cdot \partial \mathcal{G} + k - \langle k \rangle_{\theta, x}) + \text{Op}(-\partial \mathcal{G} \cdot |\xi| \xi |\mathcal{G}(\xi)|)
\]
\[
+ i \text{Op}(r_g, |\xi| \xi |\mathcal{G}(\xi)|) + \text{Op}((\langle k \rangle_{\theta, x}).
\]

From Lemma 4.1, there exists a symbol $g \in S^0_{s-\tau_0-5,2}$ with $g = g^*$, such that

\[
|\text{Op}(k - \langle k \rangle) - \omega \cdot \partial \mathcal{G} - \partial \mathcal{G} \cdot \frac{|\xi|}{|\xi|} \mathcal{G}(\xi)| \leq \frac{1}{\gamma} |k|_{s, \tau_0-5,2}.
\]

Since $\partial \mathcal{G}(\xi) = 0$ for $|\xi| \geq 1$, from Lemma 2.5 one gets

\[
|\text{Op}(-\partial \mathcal{G} \cdot |\xi| \xi |\mathcal{G}(\xi)|) | \leq \frac{1}{\gamma} |\text{Op}(g)|_{s-\tau_0-5,2},
\]

From Lemma 2.2 and Remark 2.3, one gets

\[
|\text{Op}(r_g, |\xi| \xi |\mathcal{G}(\xi)|) | \leq \frac{1}{\gamma} |\text{Op}(g)|_{s-\tau_0-5,2}.
\]

Combining the results of (4.33), (4.34), (4.35) and (4.36), from Lemma 4.12, we can get

\[
|\mathcal{P}_1 \mathcal{P}_1^d|_{s-\tau_0-11,1} \leq \frac{1}{\gamma} |\mathcal{P}_1 \mathcal{P}_1^d|_{s-\tau_0-10,0} \leq \frac{1}{\gamma} |\text{Op}(k)|_{s-\tau_0-3,2}.
\]

Since the operators $\mathcal{G}, \sqrt{-\Delta}$ are self-adjoint, the operator $\mathcal{P}_1^d$ is also self-adjoint. Note that the operators $\sqrt{-\Delta}, \mathcal{K}$ are real, then

\[
-\mathcal{P}_1^d = i[\mathcal{G}, \sqrt{-\Delta}] + \omega \cdot \partial \mathcal{G} - \mathcal{K} + \text{Op}((\langle k \rangle)).
\]
Finally, (4.24) can be rewritten as $H_0 + P_1$, where

$$H_0 = B + Z + [K],\quad P_1 = \begin{pmatrix} B^d & 0 \\ 0 & -B^d \end{pmatrix}.$$  

From (4.36), one gets $P_1 \in \mathcal{N}_{s,-\tau_0 - 11}(1,0)$ and

$$\|P_1\|_{s,-\tau_0 - 11,0} \lesssim s \frac{1}{\gamma} \|\text{Op}(k)[\gamma,0,s,-3,2].$$

For the notational convenience, we rename $P_{25-29}$ as $P_2 - P_6$. The estimates of (4.25)–(4.28) are similar, so we take (4.26) as an example. From (4.20) and (4.30), one sees that $P_3 = \int_0^1 i e^{i\mathcal{G}[G,K]e^{-i\mathcal{G}}} ds$, where

$$i e^{i\mathcal{G}[G,K]}e^{-i\mathcal{G}} = \begin{pmatrix} e^{-i\mathcal{G}(\omega t,\omega)} & 0 \\ 0 & e^{i\mathcal{G}(\omega t,\omega)} \end{pmatrix} \begin{pmatrix} i[\mathcal{G},K] & 0 \\ 0 & i[\mathcal{G},\mathcal{K}] \end{pmatrix} \begin{pmatrix} e^{i\mathcal{G}(\omega t,\omega)} & 0 \\ 0 & e^{-i\mathcal{G}(\omega t,\omega)} \end{pmatrix}$$

From Lemma 2.2 and Definitions 3.8 and 3.10, one sees that $\mathcal{G} \in \mathcal{L}_{s,-\tau_0 - 11,1}$

$$\|\mathcal{G}\|_{s,-\tau_0 - 0} \lesssim s |\mathcal{G}|_{0,s,-5,2} \lesssim s \frac{1}{\gamma} |\mathcal{K}|_{0,s,-3,2}$$

and

$$\|\mathcal{G},\mathcal{K}\|_{s,-\tau_0 - 1,1} \lesssim s |\mathcal{G}|_{1,-s,-7,1} \lesssim s \frac{1}{\gamma} (|\mathcal{K}|_{0,s,-3,2})^2.$$

From Lemma 3.11, one gets

$$\|ie^{-i\mathcal{G}[G,K]}e^{i\mathcal{G}}\|_{s,-\tau_0 - 9,1} \lesssim \frac{1}{\gamma} e^{2\mathcal{G}[\mathcal{G},\mathcal{K}]_{0,s,-3,2}}.$$ 

Since $ie^{-i\mathcal{G}[G,K]}e^{i\mathcal{G}}$ is a self-adjoint operator, and $-ie^{-i\mathcal{G}[G,K]}e^{i\mathcal{G}} = ie^{i\mathcal{G}[G,K]}e^{-i\mathcal{G}}$, we can get $P_3 \in \mathcal{N}_{s,-\tau_0 - 6}(1,0)$ and

$$\|P_3\|_{s,-\tau_0 - 6,1,0} \lesssim \frac{1}{\gamma} e^{2\mathcal{G}[\mathcal{G},\mathcal{K}]_{0,s,-3,2}} |\mathcal{K}|_{0,s,-3,2}^2.$$

Repeat the estimate procedure of $P_3$ for $P_2, P_4, P_5$, one can get $P_2, P_4, P_5 \in \mathcal{N}_{s,-\tau_0 - 11}(1,0)$ and

$$\|P_2 + P_4 + P_5\|_{s,-\tau_0 - 11,1,0} \lesssim \frac{1}{\gamma} e^{2\mathcal{G}[\mathcal{G},\mathcal{K}]_{0,s,-3,2}} |\mathcal{K}|_{0,s,-3,2}.$$ 

For the operator matrix $P_6$, we can get

$$P_6 = e^{i\mathcal{G}K}e^{-i\mathcal{G}}$$

$$= \begin{pmatrix} e^{-i\mathcal{G}(\omega t,\omega)} & 0 \\ 0 & e^{i\mathcal{G}(\omega t,\omega)} \end{pmatrix} \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \begin{pmatrix} e^{i\mathcal{G}(\omega t,\omega)} & 0 \\ 0 & e^{-i\mathcal{G}(\omega t,\omega)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -e^{-i\mathcal{K}}\mathcal{G}e^{-i\mathcal{G}} \\ -e^{-i\mathcal{K}}\mathcal{G}e^{i\mathcal{G}} & 0 \end{pmatrix}$$

Take $e^{-i\mathcal{G}K}e^{-i\mathcal{G}}$ as $P_6^a$, from Lemma 3.4 and 3.6, we can get

$$|P_6|_{s,-\tau_0 - 6} \lesssim e^{2\mathcal{G}[\mathcal{G},\mathcal{K}]_{0,s,-3,2}} |\mathcal{K}|_{0,s,-3,2}.$$
Since $[P_6^0]^* = e^{i\theta_0} Ke^{i\theta_0} = \overline{P_6^0}$, we see that $P_6 \in \mathcal{N}_{s-\tau_0}(1,0)$ and
\begin{equation}
\|P_6\|_{s-\tau_0,0} \leq e^{2\omega |K|_0,s-3.2} |K|_0,s-3.2.
\end{equation}

Summing up $P_1$ to $P_6$, from (4.38), (4.43), (4.44) and (4.47), one see that $P \in \mathcal{N}_{s-\tau_0-11}(1,0)$ and
\begin{equation}
\|P\|_{s-\tau_0-11,0} \lesssim e^{-2\omega |K|_0,s-3.2}
\end{equation}
\hfill \Box

Remark 4.3.

- For the non-resonance set $O_\gamma$, it is easy to verified that
\begin{equation}
\text{meas}(O \setminus O_\gamma) \leq C\gamma
\end{equation}
for an absolute constant.
- Take $V^\pm(\omega t,\omega) := e^{\pm e^{i\theta_0} G(\omega t,\omega)}$, from Lemma 3.19 the operator $V^\pm(\omega t,\omega)$ belongs to $\mathcal{B}(\mathcal{H}_k \times \mathcal{H}_k)$ with standard operator norm for any $0 \leq r \leq s - \tau_0 - 11 - s_0$.

5. KAM iteration

After the preliminary transformation in the previous section, we can get the following equation
\begin{equation}
i\partial_t v(t) = \widetilde{H}^0(t)v(t), \quad \widetilde{H}^0(t) = H^0_0 + P_0(\omega t,\omega),
\end{equation}
where
\begin{equation}
P^0 = \tilde{P}, \quad H^0_0 = H_0 = \begin{pmatrix} H_0 & 0 \\ 0 & -H_0 \end{pmatrix}, \quad H_0 = \text{diag}\{h_j^0 \mid j \in \mathbb{N}\},
\end{equation}
and $h_j^0$ is linear operator from $E_j$ to $E_j$. To be more precise, for $j \in \mathbb{N}^+$ and $a \in \{-1,1\}$, one has
\begin{equation}
h_j^0 = \begin{pmatrix} \lambda_{j,1}^0 & 0 \\ 0 & \lambda_{j,-1}^0 \end{pmatrix}, \quad \lambda_{j,a}^0 = (j^2 + m)^{\frac{1}{2}} + (a)(aj),
\end{equation}
and $h_j^0 = \sqrt{m} + (a)\theta_x(0)$.

5.1. General step of KAM iteration. In this section, we are going to perform an KAM iteration reducibility scheme for the linear equation (5.1). The main goal is to block-diagonalize the linear equation (5.1), and the key is to constantly square the size of the perturbation.

In the following, we show the outline of $k$th KAM iteration. For notional convenience, in the subsection, we drop the index $n$ and write $+$ instead of $n + 1$.

Through a transformation $v = e^{-i\overline{U}^+}v^+$, where
\begin{equation}
U = \begin{pmatrix} U^d & -U^a \\ U^a & U^d \end{pmatrix}, \quad [U^d]^* = U^d, \quad [U^a]^* = \overline{U^a},
\end{equation}
the equation $i\partial_t v(t) = \overline{H}(t)v(t)$ can be conjugated into
\begin{equation}
i\partial_t v^+ = \overline{H}^+(t)v^+,
\end{equation}
where
\begin{align}
\overline{H}^+(t) &= e^{i\overline{U}^+(\omega t,\omega)}\overline{H}(t)e^{-i\overline{U}^+(\omega t,\omega)} - \int_0^1 e^{is\overline{U}^+(\omega t,\omega)}\overline{U}^+e^{-is\overline{U}^+(\omega t,\omega)} ds, \\
\overline{H}^+ &= H_0 + i[U^+,H_0] + P - \overline{U}^+ + R,
\end{align}
Denoting the divergence of the KAM iteration, the second-order Melnikov conditions are required to be imposed.

\[ \omega \cdot \partial_0 U^+ = i[U^+, H_0] + \Pi_N P - [P], \]

where

\[ |P| = \begin{pmatrix} [P^d] & 0 \\ 0 & -[P^d] \end{pmatrix}, \quad |P^d| = \text{diag}([\tilde{P}^d(0)]_j | j \in \mathbb{N}). \]

The new Hamiltonian becomes \( H^+(t) = H_0^+ + P^+ \), where \( P^+ = R + \Pi_N P \) and

\[ H^+_0 = \begin{pmatrix} \mathcal{H}_0^+ & 0 \\ 0 & -\mathcal{H}_0^+ \end{pmatrix}, \quad \mathcal{H}_0^+ = \text{diag}\{h^+_j | j \in \mathbb{N}\} = \text{diag}\{h_j + [\tilde{P}^d(0)]_j | j \in \mathbb{N}\}. \]

In order to get a nice solution of the homological equation (5.10) and ensure the convergence of the KAM iteration, the second-order Melnikov conditions are required to be imposed. Denoting \( \lambda_{j,a}, a \in \{1, -1\} \) as the eigenvalues of the block \( h_j \), we choose the frequency vector \( \omega \) from the following set

\[ O_+^\gamma := \{ \omega \in \mathcal{O}_\gamma : |\omega \cdot \ell + \lambda_{j,a} - \lambda_{j,a'}| \geq \frac{\gamma(i - j)}{N^\tau}, \quad \forall (\ell, i, j) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \quad (\ell, i, j) \neq (0, j, j), \]

\[ |\ell| \leq N, \quad \text{and} \quad \{a, a'\} \in \{-1, 1\}, \quad |\omega \cdot \ell + \lambda_{j,a} + \lambda_{j,a'}| \geq \frac{\gamma(i + j)}{N^\tau}, \quad \forall (\ell, i, j) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \quad |\ell| \leq N, \quad \{a, a'\} \in \{-1, 1\}. \]

**Lemma 5.1. (Homological Equation)** Assume that \( P(\omega, t) \in \text{Lip}(\mathcal{O}_\gamma, \mathcal{N}_*(1, 0)) \), and

\[ \max_{\omega \in \mathcal{O}_\gamma} \frac{\|\Delta_\omega h_j\|_0}{\|\Delta \omega\|} \leq C \]

for an absolute constant. For any \( \omega \in \mathcal{O}_+^\gamma \), there exists a solution \( U^+ \) solve the equation (5.10), which belong to \( \text{Lip}(O_+^\gamma, \mathcal{N}_*(1, 1)) \) with the quantitative bound

\[ \|U^+\|_{\gamma, 1,1} \lesssim \frac{N^{2r+1}}{\gamma} \|P\|_{\gamma, 1,0}. \]

**Proof.** The homological equation (5.10) is split in the two equation

\[ -i\omega \cdot \partial_0 U^d + [\mathcal{H}_0, U^d] + i\Pi_N P^d = i[P^d], \]

\[ -i\omega \cdot \partial_0 U^a + \mathcal{H}_0 U^a + U^a \mathcal{H}_0 + i\Pi_N P^a = 0. \]

Considering the block matrix representation and expanding the Fourier series in time, for any \( \ell \in \mathbb{Z}^d, |\ell| \leq N \) and \( i, j \in \mathbb{N} \), we get

\[ \omega \cdot \ell [\tilde{U}^d(t)]_i + [h_i, \tilde{U}^d(t)]_i - [\tilde{U}^d(t)]_i h_j = -i[\tilde{P}^d(t)]_j - [P^d]_j, \]

\[ \omega \cdot \ell [\tilde{U}^a(t)]_i + [h_i, \tilde{U}^a(t)]_i + [\tilde{U}^a(t)]_i h_j = -i[\tilde{P}^d(t)]_j. \]

Taking \( I \) as unite matrix, one see that, for any \( i, j \in \mathbb{N} \),

\[ \text{spec}((\omega \cdot (I + h_i) = \{\omega \cdot \ell + \lambda_{i,a} | a \in \{1, -1\}\}). \]
Since the block $h_j$ is self-adjoint, one sees
$$\text{spec}(h_j) = \text{spec}(\overline{h}_j) = \{ \omega \cdot \ell + \lambda_{j,a} \mid a \in \{1, -1\} \}.$$  

From Lemma 7.2 one again get immediately that
$$\| [U^d(\ell)]^i_0 \|_0 \lesssim \frac{N^r}{\gamma} \| [\overline{P}^d(\ell)]^i_0 \|_0. \tag{5.20}$$

Let $\omega_1, \omega_2 \in O^\gamma$, for any function $f = f(\omega)$, we write $\Delta_\omega f = f(\omega_1) - f(\omega_2)$. For the equation \(5.18\), we can get that
$$\omega \cdot \ell \Delta_\omega [U^d(\ell)]^i_0 + h_j \Delta_\omega [U^d(\ell)]^i_0 - \Delta_\omega [U^d(\ell)]^i_0 h_j = -(\Delta_\omega \cdot \ell) [U^d(\ell)]^i_0(\omega_1) - \Delta_\omega h_j [U^d(\ell)]^i_0(\omega_1)$$
$$+ [U^d(\ell)]^i_0(\omega_1) \Delta_\omega h_j - i\Delta_\omega [\overline{P}^d(\ell)]^i_0 - i\Delta_\omega [P^d]^i_0 \tag{5.21}$$

From Lemma 7.2 again, we can get
$$\| \Delta_\omega [U^d(\ell)]^i_0 \|_0 \lesssim \frac{N^r}{\gamma} \| \Delta_\omega [\overline{P}^d(\ell)]^i_0 \|_0 + \frac{N^{r+1}}{\gamma^2} \| [\overline{P}^d(\ell)]^i_0 \|_0 |\Delta_\omega| \tag{5.22}$$

Thus, \(5.20, 5.22\) imply that
$$\| U^d \|_0^\gamma \lesssim \frac{N^r}{\gamma} |P^d|^\gamma. \tag{5.23}$$

Also, the norms of $\| (D)U^d \|_0^\gamma$, $\| (D)U^d(\ell) \|_0^\gamma$, $\| (D)^\sigma U^d(\ell)^{-\sigma} \|_0^\gamma$, $(\sigma = \pm 1)$ can be bounded by the same norms of $P^d$.

The bounds control of $U^a$ is more delicate. From Lemma 7.2 we can get that
$$\| [U^a(\ell)]^i_0 \|_0 \lesssim \frac{N^r}{\gamma(i + j)} \| [\overline{P}^a(\ell)]^i_0 \|_0. \tag{5.24}$$

We also need control the bounds of $[\overline{(D)}U^a(\ell)]^i_0$, $[\overline{(D)}U^a(\ell)]^i_0$, and $[\overline{(D)^\sigma U^a(\ell)}]^{-\sigma}^i_0$, $(\sigma = \pm 1)$. Considering the term $[\overline{(D)}U^a(\ell)]^i_0$, and applying Lemma 7.2 again, one gets
$$\| [\overline{(D)}U^a(\ell)]^i_0 \|_0 = \| (\iota) [\overline{U}^a(\ell)]^i_0 \|_0 \lesssim \frac{N^r}{\gamma(i + j)} \| [\overline{P}^a(\ell)]^i_0 \|_0 \tag{5.25}$$
$$\lesssim \frac{N^r}{\gamma} \| [\overline{P}^a(\ell)]^i_0 \|_0.$$

The similar bounds hold for $[\overline{(D)}U^a(\ell)]^i_0$ and $[\overline{(D)^\sigma U^a(\ell)}]^{-\sigma}^i_0$. Applying the difference operator $\Delta_\omega$ to equation \(5.19\), one gets
$$\omega \cdot \ell \Delta_\omega [U^a(\ell)]^i_0 + h_j \Delta_\omega [U^a(\ell)]^i_0 + \Delta_\omega [U^a(\ell)]^i_0 \overline{T}_j = -(\Delta_\omega \cdot \ell) [U^a(\ell)]^i_0(\omega_1) - \Delta_\omega h_j [U^a(\ell)]^i_0(\omega_1)$$
$$+ [U^a(\ell)]^i_0(\omega_1) \Delta_\omega h_j - i\Delta_\omega [\overline{P}^a(\ell)]^i_0 - i\Delta_\omega [P^a(\ell)]^i_0 \tag{5.26}$$

Applying Lemma 7.2 again, we can get
$$\| \Delta_\omega [U^a(\ell)]^i_0 \|_0 \lesssim \frac{N^r}{\gamma(i + j)} \| \Delta_\omega [\overline{P}^a(\ell)]^i_0 \|_0 + \frac{N^{r+1}}{\gamma^2(i + j)^2} \| [\overline{P}^d(\ell)]^i_0 \|_0 |\Delta_\omega| \tag{5.27}$$

The similar bounds hold for $\Delta_\omega [\overline{(D)}U^a(\ell)]^i_0$, $\Delta_\omega [\overline{(D)}U^a(\ell)]^i_0$ and $\Delta_\omega [\overline{(D)^\sigma U^a(\ell)}]^{-\sigma}^i_0$, $(\sigma = \pm 1)$.
Finally, (5.23), (5.24), (5.25), (5.27) and Definition 3.16 imply that

\[ \| U^+ \|_{s,1,1} \lesssim \frac{N^{2\tau+1}}{\gamma} \| P \|_{s,1,0}^\gamma. \]  

\[ \Box \]

Lemma 5.2. *(The New Perturbation)* Fix \( s \geq s_0 \) and \( b > 0 \). Let \( P(\omega t, \omega) \in \text{Lip}(O_\gamma, N_{s+b}(1,0)) \).
Assume (5.14) and, for some fixed constant \( C_s \),

\[ C_s \frac{N^{2\tau+1}}{\gamma} \| P \|_{s_0,1,0} \leq \frac{1}{2}. \]

Then, \( P^+ = \Pi_N P + R \) is defined on \( O^+_\gamma \) and satisfies the quantitative bounds

\[ \| P^+ \|_{s,1,0} \lesssim N^{-b} \| P \|_{s+b,1,0} \| P \|_{s,1,0} \| P \|_{s,1,0}^\gamma \]  

\[ \| P^+ \|_{s+b,1,0} \leq C(s+b) \| P \|_{s+b,1,0}. \]

**Proof.** From homological equation (5.10), the operator \( R \) defined in (5.8) can be rewritten as

\[ R = i \int_0^1 (1-s)e^{isU^+} [U^+, P] - \Pi_N P] e^{-isU^+} ds + i \int_0^1 e^{isU^+} [U^+, P] e^{-isU^+} ds. \]

From Lemmata 3.18, 3.19, 3.1 one gets

\[ \| i[U^+, P] \|_{s,1,1} \lesssim \frac{N^{2\tau+1}}{\gamma} \| P \|_{s,1,0} \| P \|_{s,1,0}^\gamma \]  

and

\[ \| e^{isU^+} [U^+, P] e^{-isU^+} \|_{s,1,1} \leq \| i[U^+, P] \|_{s,1,1} + \| i[U^+, i[U^+, P]] \|_{s,1,1} + \cdots \]

\[ \leq \frac{C_s N^{2\tau+1}}{\gamma} \| P \|_{s,1,0} \| P \|_{s,1,0} \| P \|_{s,1,0}^\gamma \left( \| P \|_{s,1,0} \| P \|_{s,1,0}^\gamma \right)^2 + \cdots \]

\[ \leq e^{C_s N^{2\tau+1}} \frac{N^{2\tau+1}}{\gamma} \| P \|_{s,1,0} \| P \|_{s,1,0} \| P \|_{s,1,0}^\gamma. \]

The same bounds hold for \( \| [U^+, P] - \Pi_N P \|_{s,1,1} \) and \( \| e^{isU^+} [U^+, P] - \Pi_N P] e^{-isU^+} \|_{s,1,1} \).

From Lemma 3.3 and Definition 3.16 one gets

\[ \| \Pi_N P \|_{s,1,0} \lesssim N^{-b} \| P \|_{s,1,0} \]  

Finally, (5.34) and (5.35) imply (5.30). By (5.38), the estimate of (5.31) can be obtained in the same way. \[ \Box \]

5.2. Iterative procedure. The KAM iteration is start with the linear equation (5.1). The iteration objects are construct in Lemmata 5.1, 5.2 by setting for \( n \geq 0 \)

\[ i\partial_t \nu = \tilde{H}^n(t) \nu, \quad \tilde{H}^n(t) := H_0^n(\omega) + P^n(\omega, \omega), \quad U := U^n(\omega, \omega), \]

\[ R := R^n(\omega, \omega), \quad h_j := h^n_j, \quad \lambda_j := \lambda^n_j. \]

We define

\[ N_{-1} := 1, \quad N_n := N^n_0(\omega), \quad \forall n \geq 0, \]
and for $\tau_0 > 0$, $\tau > 0$, we define the constants
\begin{equation}
(5.38) \quad s := s - \tau_0 - 11, \quad a := 6\tau + 4 \quad b = a + 1, \quad s - b > s_0.
\end{equation}
Also, we assume that
\begin{equation}
(5.39) \quad \varepsilon := \max\{\varepsilon, \|P\|_{\gamma, s, 1, 0}\}.
\end{equation}

**Theorem 5.3. (KAM Reducibility)** Let $\gamma \in (0, 1)$ and $s$ satisfies \((5.38)\). There exists $N_0 := N_0(s, \tau, d) \in \mathbb{N}$ large enough and $\delta_0 := \delta_0(s, \tau, d) \in (0, 1)$ such that if
\begin{equation}
(5.40) \quad \varepsilon \gamma^{-1} \leq \delta_0
\end{equation}
then:

**I:** Seeing $O_\gamma^0$ as $O_\gamma$ in \((5.1)\), we can recursively defined for $n \geq 0$ and $a, a' \in \{1, -1\}$,
\begin{equation}
(5.41) \quad O_\gamma^{n+1} := \left\{ \omega \in O_\gamma^n : |\omega \cdot \ell + \lambda_{i,a}^n| \geq \frac{\gamma(j + j)}{N_n^\ell}, \forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}, \right.
\end{equation}
\begin{align}
&|\ell| \leq N_n, \text{ and } |\omega \cdot \ell + \lambda_{i,a}^n + \lambda_{j,a'}^n| \geq \frac{\gamma(j + j)}{N_n^\gamma} \\
&\forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}, (\ell, i, j) \neq (0, j, j), |\ell| \leq N_n \}.
\end{align}

**II:** There exists an operator matrix $U^n(\omega, \omega) \in \text{Lip}(O_\gamma^n, N_3(1, 1))$ and satisfies
\begin{equation}
(5.42) \quad \|U^n\|^2_{\gamma, s-b, 1, 1} \leq \|P^0\|^2_{\gamma, s, 1, 0} \gamma^{-1} N_{n-1}^{2^p + 1} N_{n-2}^{-a}.
\end{equation}

The change of coordinate $e^{-1U^n}$ to $H^{n-1}$ to $H^n := H^n_0 \omega + P^n(\omega, \omega)$ such that

**III:** The operator $H^n_0$ is block diagonal, self-adjoint and time independent, where
\begin{equation}
(5.43) \quad H^n_0 = \left( \begin{array}{cc} H^n_0 & 0 \\ 0 & -H^n_0 \end{array} \right), \quad H^n_0 = \text{diag}(h^n_j | j \in \mathbb{N}),
\end{equation}
and the block $h^n_j$ is defined over $O_\gamma$, satisfies
\begin{equation}
(5.44) \quad \|h^n_j - h^{n-1}_j\|^2 \leq N_{n-2}^{-a} \varepsilon j^{-1}.
\end{equation}

**IV:** The new perturbation $P^n$ belongs to $\text{Lip}(O_\gamma^n, N_3(1, 1))$ and fulfils
\begin{equation}
(5.45) \quad \|P^n\|^2_{\gamma, s, 1, 0} \leq \|P^0\|^2_{\gamma, s, 1, 0} N_{n-1}^{-a}.
\end{equation}
\begin{equation}
(5.46) \quad \|P^n\|^2_{\gamma, s-b, 1, 0} \leq \|P^0\|^2_{\gamma, s, 1, 0} N_{n-1}^{-a}.
\end{equation}

**Proof.** We prove these assertions by inductive. It’s easy to verify that the properties in items I – IV hold true for $n = 0$. Let us suppose that the statements hold true for a fixed $n \in \mathbb{N}$ and define the set $O^{n+1}_\gamma$ in item I. We prove these assertions also hold true for $n + 1$.

In order to apply Lemmata 5.1, 5.2 we need check the assumptions in the two Lemmata. For the assumption in Lemma 5.1 one has
\begin{align}
\max_{\omega \in \mathcal{O}} \frac{\|\Delta \cdot h^n_j\|_{\omega}}{|\Delta \omega|} & \leq \sum_{m=1}^{n} \frac{\|h^n_j - h^{m-1}_j\|_{\omega}}{|\Delta \omega|} \frac{1}{\gamma} + \max_{\omega \in \mathcal{O}} \frac{\|\Delta \cdot h^0_j\|_{\omega}}{|\Delta \omega|} \\
& \leq \frac{1}{\gamma} \sum_{m=1}^{n} \frac{\|P^n\|_{\gamma, s-b, 1, 0}}{j} + |\text{Op}(\langle k \rangle)|_{\text{lip}} \\
& \leq \frac{1}{\gamma} \cdot j \sum_{m=1}^{n} N_{n-1}^{-a} \varepsilon + C \varepsilon \frac{1}{\gamma} \\
& \leq C.
\end{align}
For the assumption in Lemma 5.2 if $N_0$ is sufficient large, one has

\[
C_s \frac{N^{2\tau+1}}{\gamma} \| P^n \|_{s_0,1,0}^\gamma \leq C_s \frac{N^{2\tau+1}}{\gamma} N^{-a}_{n-1} \| P^0 \|_{s_1,1,0}^\gamma \leq \frac{1}{2} \gamma^{-1} \varepsilon \leq \frac{1}{2},
\]

(5.48)
since $\frac{3}{2}(2\tau + 1) - a < 0$.

Now, we can apply Lemma 5.1 with $P := P^n$ and $U^+ := U^{n+1}$,

\[
\| U^{n+1} \|_{s-b,1,1}^\gamma \leq \frac{N^{2\tau+1}}{\gamma} \| P^n \|_{s-b,1,0}^\gamma \leq \| P^0 \|_{s_1,1,0}^\gamma N^{-a}_{n-1} \frac{N^{2\tau+1}}{\gamma}.
\]

(5.49)

So, the item II for $n + 1$ is valid.

Furthermore, from Lemma 5.2 one can gets

\[
\| P^{n+1} \|_{s,1,0}^\gamma \leq C(s) \| P^n \|_{s,1,0}^\gamma \leq C(s) N_{n-1} \leq N_n,
\]

and

\[
\| P^{n+1} \|_{s-b,1,0}^\gamma \leq \| P^n \|_{s-b,1,0}^\gamma + \frac{N^{2\tau+1}}{\gamma} \| P^n \|_{s-b,1,0}^\gamma \| P^n \|_{s_0,1,0}^\gamma \leq \| P^0 \|_{s_1,1,0}^\gamma N^{-a}_{n-1} \frac{N^{2\tau+1}}{\gamma} \| P^n \|_{s,1,0}^\gamma
\]

(5.51)

provided

\[
2C(s) N_{n-1}^{-a} \leq 1, \quad 2C(s) \frac{N^{a+2\tau+1}}{\gamma} N_{n-1}^{-2a} \varepsilon \leq 1.
\]

(5.52)

The inequality (5.52) can be verified by (5.38), by taking $N_0$ larger enough and $\delta$ small enough. So, the item IV is valid for $n + 1$. \qed

**Corollary 5.4.** Let $s - \tau_0 - 11 := s > s_0 + b$, and $r \in [0, s - b - s_0]$. $\forall \omega \in \bigcap_{n=0}^\infty O_n^\gamma$, the sequence of transformation

\[
\tilde{\Phi}_n(\theta, \omega) := \Phi_n \cdots \Phi_2 \Phi_1, \quad \Phi_n := e^{-U^n},
\]

(5.53)
is convergence in $\| \cdot \|_{B(H_2^\gamma \times H_2^\gamma)}$ to an invertible linear operator $\tilde{\Phi}_\infty$, that fulfilling

\[
\sup_{\theta \in T^d} \| \tilde{\Phi}_\infty^\gamma(\theta) - \text{Id} \|_{B(H_2^\gamma \times H_2^\gamma)}^\gamma \leq C \varepsilon^{-1} \gamma^{-1}.
\]

(5.54)

**Proof.** The convergence of the transformations is a standard argument, we skip the details. \qed

**Corollary 5.5.** For all $j \in \mathbb{N}$ and $\omega \in O_\gamma$, the self-adjoint block $\{h_j^n\}_{n \geq 0}$ is convergence in $\| \cdot \|_{0}$ to a block $h_j^\infty$, which is fulfills

\[
\| h_j^\infty - h_j^0 \| \leq 2 \varepsilon^{-1}.
\]

(5.55)
Proof. The convergence of the block is standard. For the bound \((5.1)\), from \((5.44)\), one gets
\[
\|h_j^n - h_j^0\|_0^2 \leq \sum_{n=1}^{\infty} \|h_j^n - h_j^{n-1}\|_0^2
\]
(5.56)
\[
\leq \sum_{n=1}^{\infty} N_{n-2}^{-\alpha}j^{-1} \leq 2\varepsilon j^{-1},
\]
by taking \(N_0\) large enough. \(\square\)

5.3. **Measure estimate.** Set the eigenvalues of block \(h_j^\infty\) as \(\{\lambda_{j,a}^\infty\}_{a \in \{1, -1\}}\), we define the set
\[
O_{2\gamma}^\infty := \left\{ \omega \in O_\gamma : |\omega \cdot \ell + \lambda_{i,a}^\infty + \lambda_{j,a'}^\infty| \geq \frac{\gamma(i+j)}{(\ell)^{\tau}}, \quad \forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N} \right\}
\]
(5.57)
\[a, a' \in \{1, -1\}, \quad \text{and} \quad |\omega \cdot \ell + \lambda_{i,a}^\infty - \lambda_{j,a'}^\infty| \geq \frac{\gamma(i-j)}{(\ell)^{\tau}}, \quad \forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}, \quad (\ell, i, j) \neq (0, j, j), \quad a, a' \in \{1, -1\}\right\}.
\]

**Lemma 5.6.** One has
\[
O_{2\gamma}^\infty \subseteq \bigcap_{n=0}^{\infty} O_\gamma^n.
\]

**Proof.** It’s suffice to show that for any \(n \geq 0\), \(O_{2\gamma}^\infty \subseteq O_\gamma^n\). From the definition of \(O_{2\gamma}^\infty\), one sees \(O_{2\gamma}^\infty \subseteq O_\gamma^0\). For any \(n > 0\), from Theorem \((5.3)\) and Lemma \((7.1)\), one sees that
\[
|\lambda_{j,a}^\infty - \lambda_{j,a'}^\infty| \leq \|h_j^n - h_j^0\|_0 \leq \sum_{m=n+1}^{\infty} \|h_j^n - h_j^{m-1}\|_0
\]
(5.58)
\[
\leq \sum_{m=n+1}^{\infty} N_{m-2}^{-\alpha}j^{-1} \leq 2N_{n-1}^{-\alpha}j^{-1}.
\]
If \(\omega \in O_{2\gamma}^\infty\), for any \((\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}\), \((\ell, i, j) \neq (0, j, j)\), \(a, a' \in \{1, -1\}\) and \(|\ell| \leq N_{n-1}\), one gets
\[
|\omega \cdot \ell + \lambda_{i,a}^\infty - \lambda_{j,a'}^\infty| \geq |\omega \cdot \ell + \lambda_{i,a}^\infty - \lambda_{j,a'}^\infty| - |(\lambda_{i,a}^\infty - \lambda_{i,a}^\infty) - (\lambda_{j,a}^\infty - \lambda_{j,a}^\infty)|
\]
(5.59)
\[
\geq 2\frac{\gamma(i-j)}{(\ell)^{\tau}} - \frac{4\varepsilon}{jN_{n-1}^a}
\]
\[
\geq \frac{\gamma(i-j)}{N_{n-1}^a}.
\]
The last inequality holds true, because \(4\varepsilon \gamma^{-1}N_{n-1}^a \leq |i-j|jN_{n-1}^a\).

Also, if \(\omega \in O_{2\gamma}^\infty\), for any \((\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}\), \(a, a' \in \{1, -1\}\) and \(|\ell| \leq N_{n-1}\), one gets
\[
|\omega \cdot \ell + \lambda_{i,a}^\infty + \lambda_{j,a'}^\infty| \geq |\omega \cdot \ell + \lambda_{i,a}^\infty + \lambda_{j,a'}^\infty| - |(\lambda_{i,a}^\infty - \lambda_{i,a}^\infty) + (\lambda_{j,a}^\infty - \lambda_{j,a}^\infty)|
\]
(5.60)
\[
\geq 2\frac{\gamma(i+j)}{(\ell)^{\tau}} - \frac{4\varepsilon}{jN_{n-1}^a}
\]
\[
\geq \frac{\gamma(i+j)}{N_{n-1}^a}.
\]
The last inequality holds true, because \(4\varepsilon \gamma^{-1}N_{n-1}^a \leq |i+j|jN_{n-1}^a\). Finally, \((5.59)\) and \((5.60)\) imply that \(O_{2\gamma}^\infty \subseteq O_\gamma^n\). \(\square\)
**Lemma 5.7.** Fix \( \ell \in \mathbb{Z}^d \setminus \{0\} \), and let \( \mathcal{O} \ni \omega \mapsto h(\omega) \in \mathbb{R} \) be a Lipschitz function fulfilling \( \sup_{\omega \in \mathcal{O}} \frac{|D_h(\omega)|}{|x|} \leq \frac{1}{2} \). Define \( f(\omega) = \omega \cdot \ell + h(\omega) \). Then for any \( \sigma > 0 \), the measure of the set \( \mathcal{R} := \{ \omega \in \mathcal{O} | |f(\omega)| \leq \sigma \} \) satisfies the upper bound

\[
\text{meas}(\mathcal{R}) \leq 2 \frac{\sigma}{|\ell|}.
\]

**Proof.** Fix \( \ell \in \mathbb{Z}^d \setminus \{0\} \); we write \( \omega := \frac{1}{|\ell|} \cdot r + \omega_1, \omega_1 \in \mathbb{R} \) and \( \omega_1 \cdot \ell = 0 \), then

\[
f(\omega) := f(s) = |\ell|r + h(\omega(r)).
\]

We can obtain

\[
|f(r_1 - f(r_2))| \geq (|\ell| - \frac{1}{2})(r_1 - r_2) \geq \frac{|\ell|}{2}(r_1 - r_2),
\]

such that

\[
\text{meas}\{ r \in \mathbb{R} | |f(s)| \leq \sigma \} \leq 2 \frac{\sigma}{|\ell|}.
\]

From the Fubini theorem, we can obtain \((5.61)\). \(\square\)

For any \( i \in \mathbb{Z} \), we known that \( \langle k \rangle_{\theta,x}(i) = \text{Op}((k))_i \). Thus, from the condition II and Definition 2.1, one has

\[
\langle k \rangle_{\theta,x}(i)(\omega) = \frac{\langle w \rangle_{\theta,x}(i)(\omega)}{\sqrt{i^2 + m}} = c^*(i, \omega) + b^*(i, \omega)
\]

where \( c^* \in \Gamma^* := \{ c_1^*, \ldots, c_q^* \} \). Also, there exist an absolute positive constant \( C \), such that \( |b^*(i, \omega)| \leq \frac{C}{\sqrt{i^2 + m}} \). Take the set \( \Gamma \) as \( \{1, \ldots, q\} \), we can define the set

\[
\overline{\mathcal{O}}_{\gamma_0} := \{ \omega \in \mathcal{O} : |\omega \cdot \ell + j + c_a^* + c_{a'}^*| \geq \frac{\gamma_0(j)}{\langle \ell \rangle_{\gamma_0}}, \forall (\ell, j) \in \mathbb{Z}^{d+1} \setminus \{0\}, a, a' \in \Gamma \}.
\]

**Lemma 5.8.** Let \( 0 < \gamma_0 < \frac{1}{4}, \tau_0 > d \), one has

\[
\text{meas}(\mathcal{O} \setminus \overline{\mathcal{O}}_{\gamma_0}) \leq C \gamma_0,
\]

where \( C \) is a positive constant depending on \( q \).

**Proof.** If \( j \neq 0 \) and \( \ell = 0 \), we known that the bound in \((5.65)\) hold true.

If \( j = 0, \ell \neq 0 \), from Lemma 5.7, the set \( \mathcal{R}_{a,a'}^{\ell,0} := \{ \omega \in \mathcal{O} | |\omega \cdot \ell + c_a^* + c_{a'}^*| \leq \frac{\gamma_0(j)}{\langle \ell \rangle_{\gamma_0}} \} \) fulfills

\[
\text{meas}(\mathcal{R}_{a,a'}^{\ell,0}) \leq \frac{4 \gamma_0}{\langle \ell \rangle_{\gamma_0 + 1}},
\]

Let \( \mathcal{R}_1 = \bigcup_{a,a' \in \Gamma} \mathcal{R}_{a,a'}^{\ell,0} \), one has

\[
\text{meas}(\mathcal{R}_1) \leq \sum_{\ell \in \mathbb{Z}_d} \sum_{a,a' \in \Gamma} \frac{4 \gamma_0}{\langle \ell \rangle_{\gamma_0 + 1}} \leq \sum_{\ell \in \mathbb{Z}_d} \frac{4 \gamma_0}{\langle \ell \rangle_{\gamma_0 + 1}} \leq C_1(p) \gamma_0.
\]

If \( j \neq 0, \ell \neq 0 \) and \( |j| \geq 8|\ell| \), one has

\[
|\omega \cdot \ell + j + c_a^* + c_{a'}^*| \geq |j + c_a^* + c_{a'}^*| - |\omega \cdot \ell| \geq \frac{1}{2}|j| - |\omega \cdot \ell| \geq \frac{1}{4}|j| \geq \frac{\gamma_0(j)}{\langle \ell \rangle_{\gamma_0 + 1}}
\]
Then, consider the case \( 1 \leq |j| < 8\ell \). For fixed \( \ell, j \), we defined the set \( R_{a,a'}^{\ell,j} := \{ \omega \in \mathcal{O} \mid \omega \cdot \ell + j + c_a \pm c_{a'} \leq \frac{\gamma_0}{\ell} \} \). Applying the Lemma \( 5.7 \) again, one gets

\[
\text{meas}(R_{a,a'}^{\ell,j}) \leq \frac{4\gamma_0}{\ell}.
\]

Let \( R_2 = \bigcap_{\ell \in \mathbb{Z}^d, |j| \leq 8\ell} R_{a,a'}^{\ell,0} \), one has

\[
\text{meas}(R_2) \leq \sum_{\ell \in \mathbb{Z}^d} \sum_{|j| \leq 8\ell} 4\gamma_0 \frac{\gamma_0}{\ell} \leq \sum_{\ell \in \mathbb{Z}^d} \sum_{|j| \leq 8\ell} \frac{4\gamma_0^2j}{\ell} \frac{j}{\ell} + \gamma_0 \frac{j}{\ell} \gamma_0 + 1
\]

\[
\leq \sum_{\ell \in \mathbb{Z}^d} \frac{32\ell^2}{\ell} \gamma_0 \leq C_2(q) \gamma_0 \tag{5.70}
\]

One sees that \( \mathcal{O} \setminus \tilde{O}_\gamma \subseteq \mathcal{R}_1 \cup \mathcal{R}_2 \), which finished the proof.

For any \( j \in \mathbb{Z} \), and \( a, a' \in \{1, -1\} \), we take

\[
d_{j,a} := \sqrt{j^2 + m + c^*(a_j)} = j + \frac{c(m,j)}{j} + c^*(a_j),
\]

and define the set

\[
\tilde{O}_{\gamma_j} := \{ \omega \in \tilde{O}_\gamma \mid |\omega \cdot \ell + d_{i,a} + d_{j,a'}| \geq \frac{\gamma_j(i+j)}{\ell} \},
\]

\[
\forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}
\]

\[
a, a' \in \{1, -1\}, \quad \text{and} \quad |\omega \cdot \ell + d_{i,a} - d_{j,a'}| \geq \frac{\gamma_j(i-j)}{\ell},
\]

\[
\forall (\ell, i, j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}, \quad (\ell, i, j) \neq (0, j, j), \quad a, a' \in \{1, -1\} \}. \tag{5.71}
\]

**Lemma 5.9.** Let \( 0 < \gamma_1 \leq \frac{C_1}{2} \) and \( \tau_1 > \tau_0 + d \), one has

\[
\text{meas}(\tilde{O}_\gamma \setminus \tilde{O}_{\gamma_j}) \leq C \frac{\gamma_1}{\gamma_0} \tag{5.72}
\]

**Proof.** We define the set

\[
U_{\ell,i,j} = \{ \omega \in \tilde{O}_\gamma \mid |\omega \cdot \ell + d_{i,a} - d_{j,a'}| \leq \frac{\gamma_1(i-j)}{\ell} \}, \quad \forall a, a' \in \{1, -1\} \}, \tag{5.73}
\]

and

\[
V_{\ell,i,j} = \{ \omega \in \tilde{O}_\gamma \mid |\omega \cdot \ell + d_{i,a} + d_{j,a'}| \leq \frac{\gamma_1(i+j)}{\ell} \}, \quad \forall a, a' \in \{1, -1\} \}. \tag{5.74}
\]

Let \( U := \bigcup_{(\ell,i,j) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}} U_{\ell,i,j} \) and \( V := \bigcup_{(\ell,i,j) \notin (0,j,j)} V_{\ell,i,j} \), one has

\[
\tilde{O}_\gamma \setminus \tilde{O}_{\gamma_j} \subseteq U \cup V.
\]

We consider the measure estimate of set \( P \), estimating the measure of set \( Q \) is relatively simple.

**Case 1:** If \( \ell = 0 \) and \( i \neq j \), one has

\[
|d_{i,a} - d_{j,a'}| \geq \frac{1}{2} |i-j| \geq \gamma_1(i-j). \tag{5.75}
\]
Case 2: If \( \ell \neq 0 \) and \( i = j \), one has
\[
|\omega \cdot \ell + c^*(a_j) - c^*(a'_j)| \geq \frac{\gamma_0}{\langle \ell \rangle^{\gamma_0}} \geq \frac{\gamma_1}{\langle \ell \rangle^{\gamma_1}}.
\]

Case 3: If \( \ell \neq 0, i \neq j \) and \( |i - j| > 8|\ell| \), one can obtain
\[
|\omega \cdot \ell + d_i + d_j + d_j| \geq \frac{1}{2}|i - j| - |\omega \cdot \ell| \geq \frac{1}{4}|i - j| \geq \frac{\gamma_1(i - j)}{\langle \ell \rangle^{\gamma_1}}.
\]

Case 4: Let \( |i - j| \leq 8|\ell| \) and \( i < j \), we assume that
\[
\langle i \rangle(i - j) \geq \frac{4m(\ell)^{\gamma_0}}{\gamma_0},
\]
then
\[
|\omega \cdot \ell + \lambda_{i,a} + \lambda_{j,a'}| \geq |\omega \cdot \ell + i - j + c_{i,a} - c_{j,a'}| - \frac{2m}{\langle i \rangle}
\]
\[
\geq \frac{\gamma_0(i - j)}{\langle \ell \rangle^{\gamma_0}} - \frac{2m}{\langle i \rangle}
\]
\[
\geq \frac{\gamma_0(i - j)}{2\langle \ell \rangle^{\gamma_0}}.
\]

Therefore, we restrict ourself to the case \( i < j \) and \( \langle i \rangle(i - j) \leq \frac{4m(\ell)^{\gamma_0}}{\gamma_0} \). The same arguments can be extended to the symmetric case \( j > i \) and \( \langle j \rangle(i - j) \leq \frac{4m(\ell)^{\gamma_0}}{\gamma_0} \).

From Lemma [5.7], we known that for any \( \ell \neq 0 \)
\[
\text{meas}(U^{i,j}) \leq \frac{8\gamma_1(i - j)}{\langle \ell \rangle^{\gamma_1}}
\]

Now, we define the index set of \((\ell, i, j)\), that is
\[
E := \{ |i - j| \leq 8|\ell| \} \bigcap \left( \{ i \leq j, \langle i \rangle(i - j) \leq \frac{4m(\ell)^{\gamma_0}}{\gamma_0} \} \bigcup \{ j \leq i, \langle j \rangle(i - j) \leq \frac{4m(\ell)^{\gamma_1}}{\gamma_0} \} \right).
\]
Since \( U = \bigcup_{(\ell, i, j) \in E} U^{i,j} \), from (5.80), one gets
\[
\text{meas}(U) \leq \sum_{(i, j) \in E} \text{meas}(U^{i,j})
\]
\[
\leq 16\gamma_1 \sum_{\ell \neq 0} \sum_{i < j} \frac{|i - j| \langle i \rangle(i - j) \leq \frac{4m(\ell)^{\gamma_0}}{\gamma_0} |i - j| \leq 8|\ell|} {\langle \ell \rangle^{\gamma_1 + 1}}
\]
\[
\leq 16\gamma_1 \sum_{\ell \neq 0} \sum_{k \leq 8|\ell|} \frac{1}{\langle \ell \rangle^{\gamma_1 + 1}}
\]
\[
\leq 64m\gamma_1 \sum_{\ell \neq 0} \sum_{k \leq 8|\ell|} \frac{1}{\langle \ell \rangle^{\gamma_1 + 1}}
\]
\[
\leq 512m\gamma_1 \sum_{\ell \neq 0} \frac{1}{\langle \ell \rangle^{\gamma_1 + 1}}
\]
\[
\leq C\gamma_1 \gamma_0.
\]
provided \( \tau > d + \tau_0 \). The same computation hold for the set \( \mathcal{V} \). Hence, we conclude the estimate \((5.72)\).

From the Lemma \(7.1\) and Corollary \(5.4\) for any \( j \in \mathbb{N} \) and \( a \in \{1, -1\} \), the final eigenvalues \( \lambda^\infty_{j,a} \) fulfils

\[
\lambda^\infty_{j,a} := \lambda^\infty_{j,a} + \varepsilon^\infty_{j,a}(\omega) = \sqrt{j^2 + m^2 + \langle k \rangle_{\theta,a}(a_j) + \varepsilon^\infty_{j,a}(\omega)} = j + \frac{c(m_j)}{\sqrt{j}} + c^*(a_j) + b^*(a_j) + \varepsilon^\infty_{j,a}(\omega),
\]

where

\[
|b^*(a_j)|^\gamma \leq \frac{c(\gamma)}{\sqrt{j}} \varepsilon^\infty_{j,a}(\omega)^\gamma \leq \frac{2c}{\sqrt{j}}.
\]

Take \( \rho := \min\{1, c\} \), one gets

\[
(5.83) \quad |b^*(a_j)|^\gamma + \varepsilon^\infty_{j,a}(\omega)^\gamma \leq \frac{c_1}{\sqrt{j}}, \quad c_1 = 2 + c.
\]

**Lemma 5.10.** Let \( 0 < \gamma < \frac{21}{5} \) and \( \tau > \max\{d + \frac{2d}{\rho} - \frac{1}{\rho}, d + \frac{2d}{\rho} - 1\} \), one has that

\[
(5.84) \quad \text{meas}(\overline{\mathcal{O}}_{\gamma} \setminus \mathcal{O}^\infty_{2\gamma}) \leq C\gamma.
\]

**Proof.** We define the set

\[
(5.85) \quad \mathcal{P}^{\ell,i,j} = \{ \omega \in \overline{\mathcal{O}}_{\gamma_0} \mid |\omega \cdot \ell + \lambda^\infty_{\ell,a} - \lambda^\infty_{j,a}| \leq \frac{2\gamma(i - j)}{(\ell)^\tau}, \forall a, a' \in \{1, -1\} \},
\]

and

\[
(5.86) \quad \mathcal{Q}^{\ell,i,j} = \{ \omega \in \overline{\mathcal{O}}_{\gamma_0} \mid |\omega \cdot \ell + \lambda^\infty_{\ell,a} + \lambda^\infty_{j,a'}| \leq \frac{2\gamma(i + j)}{(\ell)^\tau}, \forall a, a' \in \{1, -1\} \}.
\]

Let

\[
\mathcal{P} := \bigcup_{(\ell,i,j) \in \mathbb{Z}^2 \times \mathbb{N} \atop (\ell,i,j) \neq (0,j,j)} \mathcal{P}^{\ell,i,j} \quad \text{and} \quad \mathcal{Q} := \bigcup_{(\ell,i,j) \in \mathbb{Z}^2 \times \mathbb{N} \atop (\ell,i,j) \neq (0,j,j)} \mathcal{Q}^{\ell,i,j},
\]

one gets

\[
\overline{\mathcal{O}}_{\gamma_0} \setminus \mathcal{O}^\infty_{2\gamma} \subseteq \mathcal{P} \bigcup \mathcal{Q}.
\]

We focus on the measure estimate of set \( \mathcal{P} \), it’s relatively simple to estimate the measure of set \( \mathcal{Q} \).

**case 1:** If \( \ell = 0 \) and \( i \neq j \), one has

\[
|\lambda^\infty_{i,a} + \lambda^\infty_{j,a'}| \geq \frac{1}{2}|i - j| \geq 2\gamma(i - j).
\]

**case 2:** If \( \ell \neq 0 \) and \( i = j \), let \( \langle j \rangle^\rho > c(\varepsilon, \gamma_0)(\ell)^{\gamma_0} \), one has

\[
|\omega \cdot \ell + \lambda^\infty_{j,a} + \lambda^\infty_{j,a'}| \geq |\omega \cdot \ell + c^*(a_j) - c^*(a'j)| - \frac{2c_1\varepsilon}{\langle j \rangle^\rho} \geq \frac{\gamma_0}{(\ell)^{\gamma_0}} - \frac{2c_1\varepsilon}{(\langle j \rangle^\rho)^\rho} \geq \frac{\gamma_0}{2(\ell)^{\gamma_0}}.
\]

Let \( \mathcal{P}_1 := \bigcup_{(\ell,i,j) \in \mathbb{Z}^2 \times \mathbb{N} \atop (\ell,i,j) \neq (0,j,j)} \mathcal{P}^{\ell,i,j} \), from \((5.2)\), one has

\[
\mathcal{P}_1 = \bigcup_{(\ell,i,j) \neq (0,j,j)} \mathcal{P}^{\ell,i,j},
\]

and
From Lemma [5.7] for any $\ell \neq 0$, one gets

\[(5.89) \quad \text{meas}(P^{\ell, i, j}) \leq \frac{16\gamma}{\ell^{\tau+1}}.\]

Then

\[(5.90) \quad \text{meas}(P_i) \leq \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \sum_{(j)^{\rho} \leq c(\varepsilon, \gamma_0)\ell} \frac{16\gamma}{\ell^{\tau+1}} \leq 16\gamma(c(\varepsilon, \gamma_0))^{\frac{1}{2}} \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{1}{\ell^{\tau+1}} \leq \tilde{c}(\varepsilon, \gamma_0)\gamma,
\]

provided $\tau - \frac{m}{\rho} + 1 > d$.

**case 3**: If $\ell = 0$ and $i \neq j$. Let $|i - j| \geq 8|\ell|$, we can get

\[(5.91) \quad |\omega \cdot \ell + \lambda_{i,a}^{\infty} + \lambda_{j,a'}^{\infty}| \geq \frac{1}{2}|i-j| - |\omega \cdot \ell| \geq \frac{1}{4}|i-j| \geq \frac{2\gamma(i-j)}{\ell^\tau}.
\]

**case 4**: Let $|i - j| \leq 8|\ell|$ and $i < j$, we assume that

\[(5.92) \quad (i)^{\rho}(i-j) \geq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}
\]

then

\[(5.93) \quad |\omega \cdot \ell + \lambda_{i,a}^{\infty} + \lambda_{j,a'}^{\infty}| \geq |\omega \cdot \ell + d_{i,a} - d_{j,a'}| - \frac{2c_1\varepsilon}{\langle i \rangle^{\rho}} \geq \gamma_1(i-j) - \frac{2c_1\varepsilon}{\langle i \rangle^{\rho}} \geq \frac{\gamma_1(i-j)}{2(\ell)^{\tau_1}}.
\]

Therefore, we restrict ourself to the case $i < j$ and $(i)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}$. The same arguments can be extended to the symmetric case $j < i$ and $(j)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}$. From Lemma [5.7] we known that for any $\ell \neq 0$ and $i \neq j$

\[(5.94) \quad \text{meas}(P^{\ell, i, j}) \leq \frac{16\gamma(i-j)}{(\ell)^{\tau+1}}.
\]

Now, we define the index set of $(\ell, i, j)$, that is

\[\mathcal{T} := \{i-j| \leq 8|\ell|\} \bigcap \{i < j, (i)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}\} \bigcup \{j < i, (j)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}\} \bigcup \{j < i, (j)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}\} \bigcup \{j < i, (j)^{\rho}(i-j) \leq c(\varepsilon, \gamma_1)(\ell)^{\tau_1}\}.
\]
Let $\mathcal{U}_2 = \bigcup_{(i,j) \in \mathcal{T}}$, one gets
\[
\begin{align*}
\text{meas}(\mathcal{P}_2) & \leq \sum_{(i,j) \in \mathcal{T}} \text{meas}(\mathcal{P}^{i,j}) \\
& \leq 16 \gamma \sum_{\ell \neq 0} \sum_{j=i}^{k} \sum_{k \leq s(\ell)} \frac{(i-j)^k}{(\ell)^{r+1}} \\
& \leq 16 \gamma \sum_{\ell \neq 0} \sum_{j=i}^{k} \sum_{k \leq s(\ell)} \frac{k^{1+\frac{1}{p}}}{(\ell)^{r+1}} \\
& \leq 16 \gamma \sum_{\ell \neq 0} \sum_{j=i}^{k} \sum_{k \leq s(\ell)} \frac{1}{(\ell)^{r+1}} \\
& \leq 16 \cdot 8^{1+\frac{1}{p}} (c(\varepsilon, \gamma_1))^{\frac{3}{2}} \gamma \sum_{\ell \neq 0} \frac{1}{(\ell)^{r+1}} \\
& \leq \tilde{c}(\varepsilon, \gamma_1) \gamma,
\end{align*}
\tag{5.95}
\]
provided $\tau - \frac{d}{\rho} + \frac{1}{\rho} > d$. The bounds (5.90) and (5.95) imply that
\[
\text{meas}(\mathcal{P}) \leq \tilde{c}(\varepsilon, \gamma_0, \gamma_1) \gamma.
\]

The same computation hold for the set $\mathcal{Q}$. Hence, we conclude the measure estimate (5.84). \hfill \square

**Proposition 5.11.** One has
\[
\text{meas}(\mathcal{O} \setminus \mathcal{O}_{2\gamma}^\infty) \leq \bar{C} \gamma^{\frac{1}{2}}.
\tag{5.96}
\]

**Proof.** From the definitions of sets $\mathcal{O}, \mathcal{O}_{\gamma}, \mathcal{O}_{\gamma_0}, \mathcal{O}_{\gamma_1}, \mathcal{O}_{2\gamma}^\infty$, one gets
\[
\mathcal{O} \setminus \mathcal{O}_{2\gamma}^\infty = (\mathcal{O} \setminus \mathcal{O}_{\gamma}) \bigcup (\mathcal{O}_{\gamma} \setminus \mathcal{O}_{\gamma_0}) \bigcup (\mathcal{O}_{\gamma_0} \setminus \mathcal{O}_{\gamma_1}) \bigcup (\mathcal{O}_{\gamma_1} \setminus \mathcal{O}_{2\gamma}^\infty).
\tag{5.97}
\]

Let $\gamma_0 = \gamma^\frac{1}{2}$ and $\gamma_1 = \gamma^\frac{3}{2}$, from Lemmata 5.8, 5.9, 5.10, we can get
\[
\text{meas}(\mathcal{O} \setminus \mathcal{O}_{2\gamma}^\infty) \leq C_0 \gamma + C_1 \gamma + C_2 \gamma + C_3 \gamma \leq \bar{C} \gamma^{\frac{1}{2}},
\tag{5.98}
\]
where $\bar{C} := 4 \cdot \max\{C_0, C_1, C_2, C_3\}$. \hfill \square

6. **Proof of main Theorem 2.7 and Corollary 2.9**

We define the composition operator
\[
A(\theta, \omega) := \tilde{\Phi}_\infty(\theta, \omega) \circ V(\theta, \omega)
\tag{6.1}
\]
where $V(\theta, \omega)$ is defined in Remark 4.3 and $\tilde{\Phi}_\infty(\theta, \omega)$ is defined in Corollary 5.4. We also define the constants
\[
\bar{s} := s_0 + \tau_0 + 11 + b
\]
and for any $s > \bar{s}$, we define
\[
\mathfrak{R}_s := s - \tau_0 - 11 - b - s_0,
\]
where we recall the definitions in (5.38). From Lemmata 5.7, 5.19, 4.2 and Theorem 5.3, one gets that for $\varepsilon \gamma^{-1} \leq \delta_s$, for any $\theta \in \mathbb{T}^d$ and $\omega \in \mathcal{O}_{2\gamma}^\infty$, the maps $A^\pm(\theta, \omega)$ are bounded and invertible with
\[
A^\pm(\theta, \omega) : (\mathcal{H}_x^\tau \times \mathcal{H}_x^\tau) \to (\mathcal{H}_x^\tau \times \mathcal{H}_x^\tau),
\tag{6.2}
\]
for any $0 \leq r \leq R_s$.

Also, for any $\omega \in \mathcal{O}_{2\gamma}^\infty$, by the change of variables $q := A(\omega t)v$, the Cauchy problem
\[
\begin{aligned}
&i\partial_t q(t) = H(t)q(t), \\
&q(0, x) = (q(0, x), \bar{q}(0, x)),
\end{aligned}
\]
is transformed into
\[
\begin{aligned}
&i\partial_t v(t) = H_0^\infty v(t), \\
&v(0, x) = (v(0, x), \bar{v}(0, x)),
\end{aligned}
\]
where the operator $H_0^\infty = \begin{pmatrix} H_0^\infty & 0 \\ 0 & -H_0^\infty \end{pmatrix}$ is defined in Corollary 5.5. Then, we can consider the Cauchy problem
\[
\begin{aligned}
&i\partial_t v(t) = H_0^\infty v(t), \\
&v_0(x) = v(0, x).
\end{aligned}
\]
Since the operator $H_0^\infty$ is block-diagonal and self-adjoint, we can verified that
\[
\partial_t \|v(t, x)\|_{H_r^\gamma}^2 = -(i(H_0^\infty - (H_0^\infty)^*) (D)^r v, (D)^r v) = 0,
\]
which implies that
\[
\|v(t, x)\|_{H_r^\gamma} = \|v(0, x)\|_{H_r^\gamma}.
\]
By (6.2) and (6.4), we can get
\[
\|q(0, x)\|_{H_r^\gamma} \lesssim_r \|q(t, x)\|_{H_r^\gamma} \lesssim_r \|q(0, x)\|_{H_r^\gamma}.
\]
Set $\gamma = \varepsilon^a$, $0 < a < 1$ and $\mathcal{O}_\varepsilon = \mathcal{O}_{2\gamma}^\infty$, the Proposition 5.11 implies that
\[
\lim_{\varepsilon \to 0} \text{meas}(\mathcal{O}_\varepsilon \setminus \mathcal{O}) = 0.
\]

7. Appendix

7.1. Properties of self-adjoint matrix.

In this section, we recall some well known facts about self-adjoint operator in the finite dimension Hilbert space $\mathcal{H}$. Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension $n$ equipped by the inner product $(,)_\mathcal{H}$. For any self-adjoint operator $A$, we order its eigenvalues as $\text{spec}(A) := \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$.

**Proposition 7.1.** (Weyl’s Perturbation Theorem) [13], Theorem III.2.1) Let $A$ and $B$ be self-adjoint matrices. Then
\[
|\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|_0, \ \forall k \in 1, \cdots, n.
\]

**Proposition 7.2.** [13], Theorem VII.2.8) Let $A$ and $B$ be self-adjoint matrices, and let $\delta = \text{dist}(\sigma(A), \sigma(B))$. Then the solution $X$ of the equation $AX - XB = Y$ satisfies the inequality
\[
\|X\|_0 \leq \frac{C}{\delta}\|Y\|_0.
\]

Acknowledgements

The work is supported by the Jiangsu Province Postdoctoral Foundation (No.2021K163B), China Postdoctoral Foundation (No.2021M692717), and the National Natural Science Foundation of China (No.12101542).
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