DISCRIMINANT MODULE AND INTERSECTION THEORY
ON HILBERT SCHEMES OF NODAL CURVES

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ABSTRACT. We study intersection theory on the relative Hilbert scheme of a family of nodal (or smooth) curves, over a base of arbitrary dimension. We introduce an additive group called 'discriminant module', generated by diagonal loci, node scrolls, and twists thereof, and determine the action of the discriminant or big diagonal divisor on this group by intersection. We show that this suffices to determine arbitrary polynomials in Chern classes, in particular Chern numbers, for the tautological vector bundles on the Hilbert schemes, which are closely related to enumerative geometry.

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0. Overview

0.1. Setting. Consider a family of curves given by a flat projective morphism

$$\pi : X \to B$$

over an irreducible base, with fibres

$$X_b = \pi^{-1}(b), b \in B$$

which are irreducible nonsingular for the generic $b$ and at worst nodal for every $b$. For example, $X$ could be the universal family of automorphism-free curves over the appropriate open subset of $\mathcal{M}_g$, the moduli space of Deligne-Mumford stable curves. Many questions in the classical projective and enumerative geometry of this family can be naturally phrased, and in a formal sense solved (see for instance [8]), in the context of the relative Hilbert scheme

$$X^{[m]}_B = \text{Hilb}_m(X/B).$$

This parametrizes length-$m$ subschemes of $X$ contained in fibres of $\pi$, and carries natural tautological vector bundle $\Lambda_m(E)$, associated to any vector bundle $E$ on $X$ (e.g., the relative dualizing sheaf $\omega_{X/B}$). One example of the enumerative questions which may be considered from this viewpoint is the fundamental class in $\mathcal{M}_g$ of the closure of the hyperelliptic locus.

Typically, the geometric questions one wants to consider can be formulated in terms of relative multiple points and multisecants in the family, and the formal solutions involve Chern numbers of the tautological bundles. Thus, turning these formal solutions into meaningful ones requires computing the Chern numbers in question. This problem was stated but, aside from some low-degree cases, left open in [8]. Our main purpose here is to solve this problem in general. More than that, we shall in fact provide a calculus to compute certain images of arbitrary polynomials in the Chern classes of the tautological bundles. In the 'absolute' case $E = \omega_{X/B}$, the computation ultimately reduces these polynomials to polynomials in Mumford’s tautological classes [4] on various boundary strata of $B$. The latter are computed via a conjecture of Witten, proved by Kontsevich [3].
0.2. **Motivation.** Now the framework for our solution is a little different to what is commonly done in similar problems. Rather than compute a suitable intersection ring, we will focus primarily on the (intersection) action of the discriminant or big diagonal $\Gamma^{(m)}$ and its powers. The motivation for this approach comes from a result in [8] called the ‘Splitting principle’. This says that the total Chern class of pullback of a tautological bundle $\Lambda_m(E)$ to the full-flag Hilbert scheme $W^m = W^m(X/B)$, which maps to the degree-$i$ Hilbert schemes $X_B^{[i]}$, $i \leq m$, can be expressed as a simple decomposable polynomial in the (pullbacks of) $\Gamma^{(i)}$, $i \leq m$. The recursive analogue of this result, Cor. 3.5 below, says that the pullback of $c(\Lambda_m(E))$ on the ‘flaglet’ Hilbert scheme $X^{[m,m-1]}B$, parametrizing flags of schemes of lengths $m,m-1$, is a product of $c(\Lambda_{m-1}(E))$ and a polynomial in in discriminants $\Gamma^{(i)}$, $i \leq m$. In order to compute polynomials in the Chern. It follows that if we assume recursively that we have some reasonable way to express polynomials in $c(\Lambda_{m-1}(E))$, say as elements of a ‘Tautological module’ $T_{R}^{m-1}(X/B)$ and want to do the same for $m$, then we need to determine 2 things:

1. Tautological module in degree $m$, $T_{R}^{m}(X/B)$, i.e. a group together with an action of $\Gamma^{(m)}$.
2. Transfer calculus, going from $T_{R}^{m-1}(X/B)$ to $T_{R}^{m}(X/B)$ via the flaglet correspondence $X^{[m,m-1]}B$.

Given these, $T_{R}^{m}(X/B)$ would recursively contain all polynomials in $\Gamma^{(i)}$, $i \leq m$, hence all polynomials in the Chern classes of $\Lambda_m(E)$.

0.3. **Tautological module.** Given a family $X/B$ of nodal (possibly pointed) curves, the associated Tautological Module $T_{R}^{m}(X/B)$ (Definition 2.26) is constructed recursively in $m$, *grosso modo*, as follows (see the body of the paper for details).

- For $m = 1$, it equals $R$, a $\mathbb{Q}$-subalgebra of $H(X)$ containing the relative canonical class $\omega$ as well as any distinguished sections.
- In the general case, we firstly decompose the tautological module according to partitions or 'distributions':

$$T_{R}^{m}(X/B) = \bigoplus_{\mu} T_{R}^{\mu}(X/B)$$

the sum being over all partitions $\mu$ of weight $m$; thus, it suffices to describe each $\mu$ summand. Then, we parametrize the boundary by a union of families $T(\theta)$ associated to the relative nodes $\theta$ of $X/B$, and for each of those let $X^{\theta}/T(\theta)$ be the corresponding family blown up in $\theta$, which is endowed with a pair of distinguished sections denoted $\theta_x, \theta_y$, set $R^{\theta} = R[\theta_x, \theta_y]$, and define firstly the boundary tautological module of type $\mu$ as

$$\partial T_{R}^{\mu} = \bigoplus_{\theta} T_{R^{\theta}}^{\mu}(X^{\theta}/T(\theta))$$
(using recursion, we may assume this defined for $\mu$ of weight $< m$). Then define for $\mu$ of weight $m$,

$$T^\mu_R(X/B) = (TS_\mu(R)) \oplus \left( \bigoplus_{\nu \in \Pi(n) = \mu, \ 0 < j < n} \left( \mathbb{Q}F_j^n \oplus \mathbb{Q}\Gamma^{(m)}F_j^n \right) \otimes \partial T^\nu_R(X/B) \right)$$

in which
- $TS_\mu(R)$ is a purely formal algebraic construct, an appropriate summand of the 'tensor-symmetric' algebra $T(\text{Sym}(R))$,
- $F_j^n$ is a formal symbol (for now), called a 'node scroll',
- $\Gamma^{(m)}$ is the discriminant or big diagonal on $X_B^{[m]}$, for the purpose of the formula just a formal symbol as well,
- $-\Gamma^{(m)}F_j^n$ is called a node section.
- We call the two main summands of (0.3.1) the diagonal and node scroll sectors of the tautological module $T^\mu_R$ and denote them $DT^\mu_R, NT^\mu_R$ and similarly $DT^m_R, NT^m_R$. $NT$ itself splits as $NFT \oplus N\Gamma T$, node scrolls plus node sections.

The above definition is doubly recursive in the sense that modulo the relatively elementary part $DT^m_R$, the remaining part $NT^m_R$ involves tautological modules of lower weight for (boundary) families of lower genus (albeit with more markings). The recursive definition may be replaced by a non-recursive one by working with node polyscrolls, associated to a boundary stratum defined by a collection of nodes rather than a single one.

The tautological module maps to the homology (Chow or ordinary) of the Hilbert scheme, where the diagonal sector maps to cycles living on various diagonal loci (lifted from analogous loci on the symmetric product), and the node scroll sector maps to cycles on certain $\mathbb{P}^1$-bundles which live over the boundary and are exceptional for the cycle map. In particular, a zero-dimensional or 'top degree' element $\alpha \in T^m_R(X/B)$ has a well-defined cycle degree or 'integral' $\int \alpha \in \mathbb{Q}$.

0.4. Discriminant action. Now our first main result, the Tautological module theorem 2.1, describes the action of $\Gamma^{(m)}$, i.e. the $\mathbb{Q}[\Gamma^{(m)}]$-module structure, on the $\mathbb{Q}$-vector space $T^m_R = NT^m_R \oplus DT^m_R$. This structure is an extension

$$0 \to NT^m_R \to T^m_R \to DT^m_R \to 0$$

where the module structure on the quotient $DT^m_R$, unrelated to the singularities, is via standard action of the big diagonal in the cohomology of a symmetric product (which can be modelled by a second-order differential operator); the structure on the submodule $NT^m_R$ is by the standard action (via Grothendieck’s formula) of a section $\Gamma^{(m)}$ on the cohomology of a suitable $\mathbb{P}^1$-bundle (and it therefore anti-triangular with respect to the $NFT \oplus N\Gamma T$ decomposition). It can be described in
terms of discriminant actions of lower weight and lower genus. Also, the 'mixing' part of the action takes \( DT^n_R \) only into the \( NFT \) summand of \( NT^n_R \).

0.5. transfer. As indicated above, the story is completed by the Transfer Theorem 3.3, which computes the transfer (pull-push) operation on \( T^{m-1}_R(X/B) \) via \( X^{[m,m-1]}_B \), viewed as a correspondence between \( X^m_B \) and \( X^{m-1}_B \), showing in particular that it lands in \( T^m_R(X/B) \).

The conjunction of the Splitting Principle, Module Theorem and Transfer Theorem computes all polynomials in the Chern classes, in particular the Chern numbers, of \( \Lambda_m(E) \) as \( \mathbb{Q} \)-linear combinations of tautological classes on \( X^m_B \).

0.6. Applications ? A possible application of this machinery is to compute the fundamental class in \( \overline{\mathcal{M}}_g \) of the locus of curves admitting a \( g^r_d \) for given \( r \) and \( d \), e.g. a \( g^1_2 \) (the hyperelliptics). This work is in progress.

1. Preliminaries

1.1. Staircases. We define a combinatorial function that will be important in computations to follow. Denote by \( Q \) the closed 1st quadrant in the real \((x,y)\) plane, considered as an additive cone. We consider an integral staircase in \( Q \).

Such a staircase is determined by a sequence of points

\[
(0, y_m), (x_1, y_m), (x_1, y_{m-1}), (x_2, y_{m-1}), \ldots, (x_m, y_1), (x_m, 0)
\]

where \( 0 < x_1 < \ldots < x_m, 0 < y_1 < \ldots < y_m \), are integers, and consists of the polygon

\[
B = (-\infty, x_1) \times \{y_m\} \bigcup \{x_1\} \times [y_{m-1}, y_m] \bigcup \{x_1, x_2\} \times \{y_{m-1}\} \bigcup \ldots \bigcup \{x_m\} \times (-\infty, y_1).
\]

The upper region of \( B \) is by definition

\[
R = B + Q = \{(b_1 + u_1, b_2 + u_2) : (b_1, b_2) \in B, u_1, u_2 \geq 0\}
\]

We call such \( R \) a special unbounded polygon. The closure of the complement

\[
S = R^c := \overline{Q \setminus R} \subset Q
\]

has finite (integer) area and will be called a special bounded polygon; in fact the area of \( S \) coincides with the number of integral points in \( S \) that are \( Q \)-interior, i.e. not in \( R \); these are precisely the integer points \((a, b)\) such that \([a, a+1] \times [b, b+1] \subset S \).

Note that we may associate to \( R \) a monomial ideal \( \mathcal{I}(R) \subset \mathbb{C}[x, y] \) generated by the monomials \( x^a y^b \) such that \((a, b) \in R \cap Q \). The area of \( S \) then coincides with \( \text{dim}_C(\mathbb{C}[x, y]/\mathcal{I}(R)) \). It is also possible to think of \( S \) as a partition or Young tableau, with \( x_1 \) many blocks of size \( y_m, \ldots, (x_{i+1} - x_i) \) many blocks of size \( y_{m-i} \).

Fixing a natural number \( m \), we define the basic special bounded polygon associated to \( m \) as

\[
S_m = \bigcup_{i=1}^m \left[0, \left(\frac{m-i+1}{2}\right)\right] \times \left[0, \left(\frac{i+1}{2}\right)\right].
\]

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It has area
\[ \alpha_m = \sum_{i=1}^{m-1} i \left( \frac{m + 1 - i}{2} \right) = \frac{m(m + 2)(m^2 - 1)}{24} \]
and associated special unbounded polygon denoted \( R_m \). Now for each integer \( j = 1, \ldots, m - 1 \) we define a special unbounded polygon \( R_{m,j} \) as follows. Set
\[
P_j = (m - j, -j) \in \mathbb{R}^2,
\]
\[
R_{m,j} = R_m \cup (R_m + P_j) \cup [0, \infty) \times [j, \infty)
\]
(where \( R_m + P_j \) denotes the translate of \( R_m \) by \( P_j \) in \( \mathbb{R}^2 \)). Then let \( S_{m,j} = R_{m,j}^c \) (complement in the 1st quadrant!),
\[
\beta_{m,j} = \text{area}(S_{m,j}),
\]
\[
\beta_m = \sum_{j=1}^{m-1} \beta_{m,j}.
\]
It is easy to see that
\[
(1.1.1) \quad \beta_{m,1} = \left( \frac{m}{2} \right), \beta_{m,2} = \left( \frac{m}{2} \right) + \left( \frac{m-1}{2} \right) - 1, \beta_{m,j} = \beta_{m,m-j},
\]
but we don’t know a closed-form formula for these numbers in general. A few small values are
\[
\beta_{2,1} = \beta_2 = 1, \quad \beta_3 = 6, \quad \beta_4 = 20, \quad \beta_5 = 50, \quad \beta_6 = 105.
\]
For example, for \( m = 5 \) the relevant bounded polygons, viewed as partitions, are
\[
S_{5,1} = 1^{10}, S_{5,2} = 2^61^3, S_{5,3} = 3^5, S_{5,4} = 3^22^2.
\]
Set
\[
J_m = \mathcal{I}(R_m).
\]
For an interpretation of the \( \beta_{m,j} \) as exceptional multiplicities associated to the blowup of the monomial ideal \( J_m \), see §1.6 below.
1.2. **Products, diagonals, partitions.** The intersection calculus we aim to develop is couched in terms certain diagonal-like loci on products, defined in the general case in terms of partitions. To facilitate working with these loci systematically, we now establish some conventions, notations and simple remarks related to partitions. Our view on partitions is influenced by the fact that we will mainly use them to define 'diagonal' conditions, so in particular singleton blocks are essentially insignificant.

By a block partition or b-partition of weight \( m \) and width \( r \) we mean an expression

\[
\{1, \ldots, m\} = \prod I_1 \cdots \prod I_r, \forall I_j \neq \emptyset.
\]

If all blocks \( I_j \) except \( I \) are singletons, we will abbreviate the b-partition \((I,)\) as \((I, m]\). Given a set \( X \) (or an object in a category with products– the modifications for this case are left to the reader), a b-partition \((I,)\) of weight \( m \) defines an ordered 'polydiagonal' subset of the (Cartesian) product \( X^m \), which will be denoted by \( X^{(I,)} \) or \( OD_{(I,)} \) or, if the dependence on \( X \) must be explicated, \( OD_{(I,)}X \): in the set case, identifying \( X^m \) with the set of functions \( \{1, \ldots, m\} \to X \), \( X^{(I,)} = OD_{(I,)} \) consists of the functions constant on each block. It can be identified with \( X^r \).

We may also view \( OD_{(I,)}X \) as a map \( X^r \to X^m \).

The length distribution associated to a b-partition \((I,)\) is the function \( \mu : \mathbb{N} \to \mathbb{Z}_{\geq 0} \) defined by

\[
n \mapsto |\{j : |I_j| = n\}|
\]

This function has weight \( w(\mu) = \sum n \mu(n) = m \) and degree \( d(\mu) = \sum (n-1) \mu(n) \). Any distribution of weight \( m \) comes from a b-partition of weight \( m \). Two b-partitions are said to be equivalent if their distributions are the same or equivalently, if they differ by a permutation of \( 1, \ldots, m \). A distribution is viewed essentially as a collection of block sizes, and will often be specified by specifying the non-singleton block sizes: e.g. \((n)\) for \( n > 1 \) refers to a distribution (of some weight \( m \geq n \)) with unique nonsingleton block of size \( n \). A distribution \( \mu \) defines a polydiagonal

\[
D_\mu = D_{\mu, X} = \prod_n X^{(\mu(n))} \hookrightarrow X^{(w(\mu))}
\]

where \( X^{(k)} \) is the \( k \)th symmetric product. The embedding is defined by repeating an element in the \( n \)th factor, i.e. \( X^{(\mu(n))} \) \( n \) times. When \( X \) has a well-defined dimension \( \dim(X) \), the codimension of \( D_\mu \) in \( X^{(w(\mu))} \) is \( d(\mu) \). As above, \( D_\mu \) may be viewed either as a locus or a map. We will write \( D_{(n),m} \) for \( D_\mu \) where \( \mu \) is the unique distribution of weight \( m \) with unique nonsingleton block of size \( n \). Also, we will denote by \( 1^m \) the unique distribution of weight \( m \) supported on \( \{1\} \), whose associated polyblock diagonal if \( X^{(m)} \) itself.

The following is an easy remark.
Lemma 1.1. For a b-partition \((I.)\) with corresponding distribution \(\mu\), the degree of the map \(OD(I.) \rightarrow D_\mu\) is

\[
a(\mu) := \prod_n \mu(n)!
\]

Now, we will need some operations on b-partitions and associated distributions. Let \(u_{a,b}(I.)\) be the b-partition obtained from \((I.)\) by deleting blocks of size \(a, b\) and inserting a block of size \(a + b\); by definition, \(u_{a,b}(I.) = \emptyset\) unless \(I.\) contains blocks of sizes \(a, b\) (two blocks of size \(a\), if \(a = b\)). We let \(u_{a,b}(\mu)\) be the corresponding operation on distributions, which is given by

\[
u_{a,b}(\mu) = \begin{cases} 
\mu - 1_a - 1_b + 1_{a+b}, & \mu \geq 1_a + 1_b \\
\emptyset, & \text{otherwise}
\end{cases}
\]

(1.2.2)

Here \(1_a\) is the indicator (characteristic) function of \(a\).

1.3. **Diagonal operators on tensors.** Let \(R\) be a commutative unitary graded \(\mathbb{Q}\)-algebra and consider the 'tensymmetric' algebra

\[
TS(R) = \bigotimes_{n=1}^\infty \text{Sym}^n(R).
\]

(1.3.1)

Given a simple element \(\alpha.\) in this algebra, we denote by \(\mu(n)\) the degree of its \(n\)th tensor factor. Then \(\mu\) is a length distribution, i.e. a finitely-supported function from the positive integers to the nonnegative integers and \(\alpha.\) can be written as

\[
\alpha. = \bigotimes_n \mu(n) \prod_{i=1}^{\mu(n)} \alpha_{n,i},
\]

where the implicit product is the formal one in \(\text{Sym}^n(R)\) (rather than the one in \(R\), which will be denoted \(\cdot_R\) or \(\prod_R\)). We call \(\mu\) the distribution associated to \(\alpha.\). This yields a grading by distribution:

\[
TS(R) = \bigoplus_{\mu} TS_\mu(R).
\]

We define the **weight** of \(\alpha.\) as that of the associated distribution, i.e. \(w(\alpha.) = \sum n\mu(n)\). Of course, in any simple \(\alpha.\), all but finitely many tensor factors (or 'n-block factors', we shall call them) equal 1. \(\alpha.\) may be viewed as a symmetric element in the 'tensor-tensor' algebra \(\bigotimes_{n=\infty}^1 \bigotimes_{n=\infty}^1 R\), namely

\[
\alpha. = \frac{1}{a(\mu)} \bigotimes_n \left( \sum_{\pi \in \mathfrak{S}_\mu(n)} \bigotimes_i \alpha_{n,\pi(i)} \right)
\]

where \(a(\mu) = \prod_n \mu(n)!\) as above.
For an element \( \theta \in R \), we denote by \([m]_*(\theta) \in TS_{1m}(R)\) the symmetrization of \(\theta.1^{m-1}\), and more generally by \([m]_*(\theta) \in TS_{s,1m-s}\) the symmetrization of \(\theta.s.1^{m-s}\).

There is a weight-preserving ‘projection’

\[
D^\dagger : \text{Sym}^m(R) \to TS(R)
\]

defined as follows. Write \( m = \sum n\mu(n) \) for a distribution \( \mu \) and write an element in \( \text{Sym}^m(R) \) as

\[
\beta = \prod_n \mu(n) \prod_{i=1}^{n} \prod_{k=1}^{R} \beta_{n,i,k}
\]

Then let \( D^\dagger_k(\beta) \) as the average value over all possible representations of \( \beta \). of

\[
\prod_n \mu(n) \prod_{i=1}^{n} (\prod_{k=1}^{R} R\beta_{n,i,k})
\]

where \( \prod_R \) means product in \( R \); and \( D^\dagger = \sum D^\dagger_\mu \). \( D^\dagger_\mu \) is a projection in the sense that it admits a right inverse. This is the natural map

\[
D_\mu : TS_\mu(R) \to \text{Sym}^m(R)
\]

defined by the properties of being multiplicative (for both the tensor and symmetric multiplication, and being given by an element \( \alpha_{n,i} \in R \) in the \( n \)th tensor factor by

\[
D_\mu(\alpha_{n,i}) = \alpha_{n,i}1^{n-1}.
\]

Assembling these together, we get a map

\[
D_*[\mu] = \bigoplus \mu D_\mu[\mu] : TS(R) \to \text{Sym}^m(R)
\]

Often \( R \) will be a graded ring, which naturally induces a gradation on \( TS(R) \), said to be by degree (not to be confused with weight). If \( R \) has top piece \( R^d \) endowed with a linear map \( \int : R^d \to \mathbb{Q} \), extended by zero to \( R \), then \( \int \) extends to \( TS(R) \) by multiplicativity, i.e. for \( \alpha \) decomposable,

\[
\int(\alpha) = \prod_{n,i} \left( \int \alpha_{n,i} \right)
\]

which of course depends only on the degree-\( d \) component of each \( \alpha_{n,i} \) and vanishes if one of these components does.

If \( R \) is \( \mathbb{Q} \)-self-dual, with duality operator \( J_2 \in \text{Sym}^2(R) \) (which corresponds to the identity in \( R \otimes R^* \) under duality), \( D^\dagger_\mu \) also admits a ‘Gysin adjoint’ \( D_{\mu\dagger} \) defined by setting

\[
D_{\mu\dagger}(\alpha_{n,i}) = \alpha_{n,i}J_{n-1}
\]
and extending via multiplicativity, as for $D_\mu$; here $J_{n-1} \in \text{Sym}^{n-1}(R)$ is the natural extension of $J_2$.

**Interpretation:** If $R$ represents some kind of cohomology ring on a space $X$, e.g. the Chow ring on a variety, then such a simple class corresponds to a cohomology class on a polyblock diagonal locus $D_\mu$ in the symmetric product $\text{Sym}(X)$, as in the previous subsection, so each $\alpha_{n,i}$ is considered as living on an $n$-fold diagonal $X \subset \text{Sym}^n(X)$ so in all $\text{TS}(R)$ corresponds to cohomology on all the polyblock diagonals. The map $D_\mu^\dagger$ is the pullback map induced by the inclusions $D_\mu \to \text{Sym}(X)$, while $D_\mu$ is a natural right inverse for it. $D_\mu^\dagger$ is the Gysin map.

1.4. **Discriminant operator.** Our aim now to define a 'discriminant' operation on $\text{TS}(R)$ that corresponds to intersecting with the big diagonal for $X$ smooth. As part of our intersection calculus for Hilbert schemes, we will later derive a formula for intersecting with the discriminant polarization of which this operation will form the 'classical' part.

To this end we first define an operation $u_{n_1, n_2}$ that corresponds to uniting two blocks of sizes $n_1, n_2$, similar to the corresponding definition for polyblocks. The definition is:

\[
\forall n_1 \neq n_2 : u_{n_1, n_2}(\alpha.) = \frac{\mu(n_1)\mu(n_2)}{\mu(n_1 + n_2)} \sum_{i,j} \ldots \otimes \alpha_{n_1+1,n_1,i} \ldots \alpha_{n_1+n_2,\mu(n_1+n_2)}(\alpha_{n_1,i}R\alpha_{n_2,j}) \otimes \ldots \hat{\alpha}_{n_1,i} \ldots \hat{\alpha}_{n_2,j} \ldots
\]

(1.4.4)

\[
u_{n,n}(\alpha.) = \frac{\mu(n)^2}{\mu(2n)} \sum_{i<j} \ldots \otimes \alpha_{2n,1} \ldots \alpha_{2n,\mu(2n)}(\alpha_{n,i}R\alpha_{n,j}) \otimes \ldots \hat{\alpha}_{n,i} \ldots \hat{\alpha}_{n,j} \ldots
\]

In other words (say for $n_1 \neq n_2$): omit in all possible ways one alpha factor from each of the $n_1$ and $n_2$ block subproducts and insert their $R$-product in the $n_1 + n_2$-block subproduct.

Next, define an operation corresponding to 'self-uniting' a block of size $n$: for an element $\omega \in R$, define

\[
u_{n,\omega}(\alpha.) = \sum_i \ldots \otimes \hat{\alpha}_{n,i}(\alpha_{n,i}R\omega) \ldots
\]

(1.4.5)

In other words, omit in all possible ways an alpha factor from the $n$-block subproduct and replace it by its $R$-product with $\omega$. We similarly define for any $\mathbb{Q}$-linear map $g : R \to \mathbb{Q}$,

\[
u_{n,g}(\alpha.) = \sum_i \ldots \otimes \hat{\alpha}_{n,i}g(\alpha_{n,i}) \ldots
\]

(1.4.6)
(this is essentially interior multiplication by \(g\) in position \(n\)). Finally, define the 'discriminant' operator on \(\text{TS}(R)\) by

\[
D\text{sc} = \sum_{n_1 \geq n_2} n_1 n_2 u_{n_1, n_2} - \sum_n \binom{n}{2} u_{n, \omega}.
\]

The motivation for this definition is the following

**Lemma 1.2.** Let \(X\) be a smooth curve with canonical class \(\omega, R = H^1(X)\). Then the action by cup product of the discriminant on polyblock diagonal classes is given by (1.4.7). More precisely, if \(\alpha \in \text{TS}(R)\) is of weight \(m\), then

\[
[D\text{sc}^{(m)}] \cup [D_\bullet(\alpha.)] = [D_\bullet(D\text{sc}(\alpha.))]
\]

where \(D\text{sc}^{(m)}\) is the discriminant in \(X^{(m)}\).

**Proof.** It suffices to prove this on the Cartesian product where \(D\text{sc}^{(m)}\) splits as a sum of simple diagonals pulled back from \(X \times X\), namely \(\sum_{a<b} D_{a,b}\), and \(\alpha\) is replaced by a class \(\alpha_{(I.)}\) on an ordered polyblock diagonal. Then clearly those \(a, b\) in different blocks of sizes \(n_1, n_2\) (the sizes may be different or not) give rise to \(u_{n_1, n_2}\), while those in the same block of size \(n\) give rise to \(u_{n, \omega}\). \(\square\)

This result remains true, in fact, when \(X\) is nodal (as follows, e.g. from the discussion in §2.2, or by an elementary dimension-counting argument). However, it is of little interest in that case because of the lack of geometric meaning of the symmetric products. On the other hand, one of the main ingredients of our intersection calculus, to be developed starting in the next section, is an analogue of the Lemma for Hilbert schemes of families of nodal curves (see 2.16), where the two sides of (1.4.8) are not equal but differ by an 'exceptional' class called a node scroll class.

Now all of the above admits an ordered analogue, with tensor product replacing the symmetric and b-partitions replacing distributions. Thus, we decompose the 'tensor-tensor' product

\[
\text{TT}(R) = \bigotimes_n \bigotimes (I.) R = \bigotimes_{(I.)} \text{TT}_{(I.)}(R)
\]

where \((I.)\) runs over all b-partitions. Operators \(D_{(I.)}\) are defined as above and are assembled together to a map

\[
D_\otimes : \text{TT}(R) \rightarrow T(R)
\]

Likewise, 'union' and 'self-union' operators are defined, hence also the ordered discriminant \(OD\text{sc}^{(m)}\). There is also an analogue of \(u_{n,g}\) as in 1.4.6, defined by

\[
u_{I,g}(\alpha_{(I.)}) = \ldots \alpha_I \cdot g(\alpha_I)\ldots
\]
(and zero whenever \((I.\) does not have a block equal to \(J).\) This may be called interior multiplication by \(g\) in position \(I.\)

1.5. **Norm.** For a line bundle \(L\) on a projective variety \(X\), we denote by \([m]_* (L)\) its norm on the symmetric product \(X^{(m)}\), i.e.

\[
[m]_* (L) = \det \varpi (p_1^* L)
\]

where \(\varpi : X^m \to X^{(m)}\) is the natural \(\Sigma_m\)-quotient. Alternatively, the norm can be defined by the condition that for a Cartier divisor \(D\) on \(X\), the norm of \(D\) is

\[
[m]_* (D) = \varpi_m p_1^* (D)
\]

where \(\varpi_m : X^m \to \text{Sym}^m (X)\) is the symmetrization map and \(p_1 : X^m \to X\) is the projection. Since the linear equivalence class of \([m]_* (D)\) depends only on that of \(D\) by [2], Theorem 1.4, this defines \([m]_*\) on line bundles. In terms of cohomology, the class \([m]_* (D)\) is just the class corresponding to \([D]_{1}^{m-1}\) under the identification of \(H^1 (\text{Sym}^m (X))\) with \(\text{Sym}^m (H^1 (X))\).

Similarly, we set, for \(s \leq m\),

\[
[m]_s^* (D) = \varpi_s (p_1^* (D) \ldots p_s^* (D))
\]

1.6. **Canonical class and half-discriminant.** Let \(X/B\) be a family of smooth curves and \(D^m = Dsc^m\) the big diagonal or discriminant in the relative symmetric product \(X^{(m)}_B\), i.e. \(D^m_\mu \cap X^{(m)}_B\) for \(\mu = (2 \mapsto 1, 1 \mapsto m - 2)\). This is a reduced Cartier divisor, defined locally by the discriminant function which is a polynomial in the elementary symmetric functions of a local parameter of \(X/B\). The associated line bundle \(\mathcal{O}(D^m)\) is always divisible by 2 as line bundle. One way to see this is to note that \(D^m\) is the branch locus of a flat double cover

\[
\epsilon : X_B^{(m)} \to X_B^{(m)}
\]

where \(X_B^{(m)} = X^m_B / \mathfrak{A}_m\) is the 'orientation product', generically parametrizing an \(m\)-cycle together with an orientation. Then \(h\) is defined by

\[
\epsilon_* \mathcal{O}_{X_B^{(m)}} = \mathcal{O}_{X_B^{(m)}} \oplus h^{-1}
\]

Indeed \(\epsilon^* h\) is precisely the (reduced) ramification divisor of \(\epsilon\), which is half of \(\epsilon^* D^m\). In particular, note that \(\epsilon^* h\) is effective. Another way to define \(h\) is by

\[
h = [m]_* (\omega_{X/B}) \otimes \omega_{X_B^{(m)}/B}^{-1}
\]

where \([m]_*\) is the norm defined in §1.5.
2. THE TAUTOGICAL MODULE

In this section we will compute arbitrary powers of the discriminant polarization $\Gamma^m$ on the Hilbert scheme $X^m_B$. The computation will be a by-product of a stronger result determining the (additive) tautological module on $X^m_B$, to be described informally in this introduction, and defined formally in the body of the chapter (see Definition 2.26).

The tautological module

$$T^m = T^m(X/B) \subset A(X^m_B)_q$$

is to be defined as the $\mathbb{Q}$-vector space generated by certain basic tautological classes (as described below). On the other hand, let

$$\mathbb{Q}[\Gamma^m] \subset A(X^m_B)_q$$

be the subring of the Chow ring generated by the discriminant polarization. Then the main result of this chapter is

**Theorem 2.1** (Module Theorem). Under intersection product, $T^m$ is a $\mathbb{Q}[\Gamma^m]$-module; moreover, multiplication by $\Gamma^m$ can be described explicitly.

Because $1 \in T^m$ by definition, this statement includes the nonobvious assertion that

$$\mathbb{Q}[\Gamma^m] \subset T^m;$$

in other words, any polynomial in $\Gamma^m$ is (explicitly) tautological. In this sense, the Theorem includes an 'explicit' (in the recursive sense, at least) computation of all the powers of $\Gamma^m$.

Now the aforementioned basic tautological classes come in two main flavors (plus some subflavors).

(i) The (classes of) (relative) diagonal loci $\Gamma^m_{(n_1,n_2,...)}$: this locus is essentially the closure of the set of schemes of the form $n_1p_1 + n_2p_2 + ...$ where $p_1, p_2,...$ are distinct smooth points of the same (arbitrary) fibre.

More generally, we will consider certain 'twists' of these, denoted $\Gamma_{(n_1,n_2,...)}[\alpha_1, \alpha_2,...]$, where the $\alpha.$ are 'base classes', i.e. cohomology classes on $X$.

(ii) The node classes. First, the node scrolls $F_j^m(\theta)$: these are, essentially, $\mathbb{P}^1$-bundles over an analogous diagonal locus $\Gamma^{(m-n)}_{(n_1)}$ associated to a boundary family $X^m_j$ of $X_B$, whose general fibre can be naturally identified with the punctual Hilbert scheme component $C^m_j$ along the node $\theta$.

Additionally, there are the node sections: these are simply the classes $-\Gamma^m.F$ where $F$ is a node scroll as above (the terminology comes from the fact that $\Gamma^m$ restricts to $O(1)$ on each fibre of a node scroll).
Finally, node scrolls and node sections define correspondence operators, pulling back (tautological) classes from a Hilbert scheme \((X^\theta)^{[m-n]}\).

Effectively, the task of proving Theorem 2.1 has two parts.

(i) Express a product \(\Gamma^{(m)} \cdot \Gamma^{(n)}\) in terms of other diagonal loci and node scrolls, see Proposition 2.16.

(ii) For each node \(\theta\) and associated \((\theta\text{-normalized})\) boundary family \(X^\theta_T\), determine a series of explicit line bundles \(E^\theta_j(\theta), j = 1, \ldots, n\) on the relative Hilbert scheme \((X^\theta_T)^{[m-n]}\) together with an identification

\[ F^\theta_j(\theta) \simeq \mathbb{P}(e^\theta_j(\theta) \oplus e^\theta_{j+1}(\theta)), \]

such that the restriction of the discriminant polarization \(-\Gamma^{(m)}\) on \(F^\theta_j(\theta)\) becomes the standard \(\mathcal{O}(1)\) polarization on the projectivized vector bundle. This is just the Node Scroll Theorem of [6]. In fact, it transpires that \(e^\theta_j(\theta)\) is just the sum of the polarization \(\Gamma^{[m-n]}\) and a suitable base divisor, that is itself a tautological class in the sense of Mumford. It then follows easily that the restriction of an arbitrary power \((\Gamma^{(m)})^k\) on \(F\) can be easily and explicitly expressed in terms of tautological classes on Hilbert schemes of lower degree on boundary families (which also have lower genus, in the stable case) (see Theorem 2.19).

2.1. The small diagonal. We begin our study of diagonal-type loci and their intersection product with the discriminant polarization with the smallest such locus, i.e., the small diagonal. In a sense this is actually the heart of the matter, which is hardly surprising, considering as the small diagonal is in the 'most special' position vis-a-vis the discriminant. The next result is in essence a corollary to the Blowup Theorem of [6].

Let \(\Gamma^{(m)} \subset X_B^{[m]}\) be the small diagonal, which parametrizes schemes with 1-point support, and which is the pullback of the small diagonal

\[ D^{(m)} \simeq X \subset X_B^{(m)}. \]

This corresponds to the distribution \(\mu\) with the unique nonzero value \(\mu(m) = 1\). The restriction of the cycle map yields a birational morphism

\[ \epsilon_m : \Gamma^{(m)} \to X \]

which is an isomorphism except over the nodes of \(X/B\). For the remainder of the paper, we fix a covering system of boundary data \(\{(T, \delta, \theta)\}\) as in [6]. and focus on its typical node \(\theta\). Thus, \(\theta\) is a relative node of \(X/B\), \(\delta : T \to B\) is a generically finite surjective map onto a boundary component, and \(X^\theta_T\) is the blowup of \(X \times_B T\) in \(\theta \times_B T\). Let

\[ J^{\theta_i} = \bigcap_i J^{\theta_i}_m \subset \mathcal{O}_X \]
be the ideal sheaf whose stalk at each fibre node \( \theta_i \) is locally of type \( J_m \) as in §1.1. Note that \( J^\theta_m \) is well-defined independent of the choice of local parameters and independent as well of the ordering of the branches at each node, hence makes sense and is globally defined on \( X \).

**Proposition 2.2.** Via \( \epsilon_m, \Gamma_m \) is equivalent to the blow-up of \( J^\theta_m \). If \( \mathcal{O}_{\Gamma_m}(1) \) denotes the canonical blowup polarization, we have

\[
\mathcal{O}_{\Gamma_m}(-\Gamma_m) = \omega_{X/B}^{\frac{m}{2}} \otimes \mathcal{O}_{\Gamma_m}(1).J.
\]

Furthermore, if \( X \) is smooth at a node \( \theta \), then \( \Gamma_m \) has multiplicity \( \min(i, m - i) \) along the corresponding divisor \( C^m_m - \{Q_i^m, Q^m_{i+1}\} \) for \( i = 1, ..., m - 1 \). In particular, \( \Gamma_m \) is smooth along \( \bigcup (C^m_m - Q^m_m + Q^m_{m-1}) \).

**Proof.** We may work with the ordered versions of these objects, then pass to \( \mathcal{S}_m \)-invariants. We first work locally over a neighborhood of a point on \( \theta_m \in X^m_B \) where \( \theta \) is a fibre node. Because blowing up and the Hilbert scheme are both compatible with base-change, we may then assume \( X \) is a smooth surface and \( X/B \) is given by \( xy = t \). Then the ideal of \( OD^m \) is generated by \( G_1, ..., G_m \) and \( G_1 \) has the Van der Monde form \( v_x^m \), while the other \( G_i \) are given by [6], §6. We try to restrict the ideal of \( OD^m \) on the small diagonal \( OD_{(m)} \). To this end, note to begin with the natural map

\[
\mathcal{I}_{OD^m} \to \omega^{\frac{m}{2}}, \omega := \omega_{X/B}.
\]

Indeed this map is clearly defined off the singular locus of \( X^m_B \), hence by reflexivity of \( \mathcal{I}_{OD^m} \) extends everywhere, hence moreover factors through a map

\[
\mathcal{I}_{OD^m}.OD_{(m)} = \mathcal{I}_{OD^m} \otimes \mathcal{O}_{OD_{(m)}}/(\text{torsion}) \to \omega^{\frac{m}{2}}.
\]

To identify the image, note that

\[
(x_i - x_j)|_{OD_{(m)}} = \frac{dx}{x},
\]

and \( \eta = \frac{dx}{x} = -\frac{dy}{y} \) is a local generator of \( \omega \) along \( \theta \). Therefore

\[
G_1|_{OD_{(m)}} = x^{\frac{m}{2}} \eta^{\frac{m}{2}}.
\]

From [6], loc. cit., we then deduce

\[
G_i|_{\Gamma_m} = x^{\frac{m-i+1}{2}} y^{\frac{i}{2}} \eta^{\frac{m}{2}}, \quad i = 1, ..., m.
\]

Since \( G_1, ..., G_m \) generate the ideal \( I_{OD^m} \) along \( \theta \), it follows that over a neighborhood of \( \theta \), we have

\[
I_{OD^m}.OD_{(m)} \simeq J^\theta_m \otimes \omega^{\frac{m}{2}}.
\]

This being true for each node, it is also true globally. Consequently, passing to the \( \mathcal{S}_m \)-quotient, we also have

\[
I_{D^m}.D_{(m)} \simeq J^\theta_m \otimes \omega^{\frac{m}{2}}.
\]
Then pulling back to \(X[m]_B\) we get (2.1.1).

Finally, it follows from the above, plus the explicit description of the model \(H_m\), that, along the 'finite' part \(C_i^m - Q_{i+1}^m\), \(\Gamma(m)\) has equation \(x^{m-i} - uy^i\) where \(u\) is an affine coordinate on \(C_i^m - Q_{i+1}^m\), from which our last assertion follows easily.  

Now it follows from the Proposition that, given a node \(\theta\) of \(X/B\), the pullback ideal of \(\mathcal{J}_{\theta}^m\) on \(\Gamma = D(m)\) is an invertible ideal supported on the inverse image of \(\theta\), i.e. \(\bigcup_{i=1}^{m-1} C_i^m(\theta)\); we denote this ideal by \(\mathcal{O}_{\Gamma}(1)\mathcal{J}_{\theta}^m\) or \(\mathcal{O}_{\Gamma}(-e_{\theta}^m)\). It must not be confused with the pullback of the reduced ideal of \(\theta\).

**Proposition 2.3.** We have

\[
e^\theta_m = \sum_{i=1}^{m-1} \beta_{m,i} C_i^m(\theta)
\]

where the \(\beta_{m,i}\) are as in §0.

**Proof.** We may fix \(\theta\) and work locally with the universal family \(xy - t\). Clearly the support of \(e_m\) is \(C^m = \bigcup_{i=1}^{m-1} C_i^m\), so we can write

\[
e_m = \sum_{i=1}^{m-1} b_{m,i} C_i^m
\]

and we have

\[-e_m^2 = \deg(\mathcal{O}(1) e_m) = \sum_{i=1}^{m-1} b_{m,i} =: b_m.
\]

Now the general point on \(C_i^m\) corresponds to an ideal \((x^{m-i} + ay^i), a \in \mathbb{C}^*\) and the rational function \(x^{m-i}/y^i\) restricts to a coordinate on \(C_i^m\). It follows that if \(A_i \subset X\) is the curve with equation \(f_i = x^{m-i} - ay^i\) for some constant \(a \in \mathbb{C}^*\), then its proper transform \(\tilde{A}_i\) meets \(C^m\) transversely in the unique point \(q \in C_i^m\) with coordinate \(a\), so that

\[A_i e_m = b_{m,i}.
\]

Thus, setting \(J_{m,i} = J_m + (f_i)\) we get following characterization of \(b_{m,i}\):

\[b_{m,i} = \ell(\mathcal{O}_X/J_{m,i}).
\]

To compute this, we start by noting that a cobasis \(B_m\) for \(J_m\), i.e. a basis for \(\mathcal{O}_X/J_m\) is given by the monomials \(x^a y^b\) where \((a, b)\) is an interior point of the polygon \(S_m\) as in §0; equivalently, the square with bottom left corner \((a, b)\) lies in \(R_m\). Then a cobasis \(B_{m,i}\) for \(J_{m,i}\) can be obtained by starting with \(B_m\) and eliminating

- all monomials \(x^a y^b\) with \(b \geq i\);
- for any \( j \) with \( \binom{j}{2} \geq i \), all monomials that are multiples of \( x^{(m+1-j) + m-i} y^{\binom{j}{2} - i} \), the latter of course comes from the relations
\[
x^{(m+1-j)} y^{\binom{j}{2}} \equiv 0 \mod J_m, \quad x^{(m+1-j) + m-i} y^{\binom{j}{2} - i} \equiv x^{(m+1-j)} y^{\binom{j}{2}} \mod f_i.
\]
Graphically, this cobasis corresponds exactly to the polygon \( S_{m,i} \) in \( \S 0 \), hence
\[
b_{m,i} = \beta_{m,i}, b_m = \beta_m; \quad \square
\]

**Corollary 2.4.** Suppose \( B \) is 1-dimensional. With the above notations, we have
\[
(2.1.4) \quad \epsilon_m^2 = -\sigma \beta_m,
\]
where \( \sigma \) is the number of nodes of \( X/B \);

\[
(2.1.5) \quad \Gamma^{(m)} \Gamma_{(m)} = \sum_{\theta, i} \beta_{m,i} C^m_i (\theta) - \left( \frac{m}{2} \right) [\omega_{X/B}];
\]

\[
(2.1.6) \quad \int_{\Gamma_{(m)}} (\Gamma^{(m)})^2 = -\sigma \beta_m + \left( \frac{m}{2} \right)^2 \omega^2_{X/B}.
\]

**Remark 2.5.** The components \( C^m_i (\theta), i = 1, \ldots, m-1 \) of \( \epsilon_m \) are \( \mathbb{P}^1 \)-bundles over \( \theta \) and are special cases of the node scrolls, encountered in the previous section, and which will be further discussed in \( \S 2.3 \) below. The coefficients \( \beta_{m,i} \) play an essential role in our intersection calculus.

For the remainder of the paper, we set
\[
\omega = \omega_{X/B}.
\]
We will view this interchangeably as line bundle or divisor class.

**2.2. Monoblock and polyblock diagonals: ordered case.** Returning to our family \( X/B \) of nodal curves, we now begin extending the results of \( \S 2.1 \) to the more general diagonal loci as defined above, first for those that live over all of \( B \), and subsequently for loci associated to the boundary. We recall the ordered polyblock diagonal loci \( OD_{(I)} = OD_{(I),X/B} \) discussed in \( \S 1.2 \). Here we will use this notation to refer to the appropriate loci in the relative Cartesian product \( X^m_B \). In particular, we have the ordered monoblock diagonal
\[
(2.2.1) \quad OD^m_{I,X/B} = OD_I = \subset X^m_B,
\]
and the big diagonal
\[
(2.2.2) \quad OD^m = \sum_{1 \leq a < b \leq m} OD^m_{a,b}.
\]
Similar loci exist in the ordered Hilbert scheme:

\[(2.2.3)\]  
\[\Gamma_I = \Gamma_I^{[m]} := oc^{-1}(OD_I) \subset X_B^{[m]}\]

Note that \(OD_I\), hence \(\Gamma_I\), are defined locally near a node by equations

\[(2.2.4)\]  
\[x_i - x_j = 0 = y_i - y_j, \forall i, j \in I.\]

Generally, for any \(b\)-partition \((I.) = (I_1, ..., I_r) \subset [1, m]\), we have an analogous locus (ordered polyblock diagonal)

\[(2.2.5)\]  
\[\Gamma_{I_1|...|I_r} = \Gamma_{I_1|...|I_r}^{[m]} \subset X_B^{[m]}\]

and note that

\[(2.2.6)\]  
\[\Gamma_{I_1|...|I_r} = \Gamma_I \cap ... \cap \Gamma_I,\]

(transverse intersection). Also

\[(2.2.7)\]  
\[\Gamma_{(I.)} = oc^{-1}(OD_{(I.)})\]

where \(OD_{(I.)} \subset X_B^{m}\) is the analogous polyblock diagonal. We may view \(\Gamma_{(I.)}\) as an operator

\[\Gamma_{(I.)}[] : TT_{(I.)}(R) \rightarrow H(X_B^{[m]}),\]

\[(\alpha.) \mapsto \Gamma_{(I.)} \cap oc^*(\alpha.).\]

Thus, the values of \(\Gamma_{(I.)}[]\) are homology rather than cohomology classes. However, their \(\mathfrak{S}_m\)-symmetrized versions will descend to the (unordered) Hilbert scheme \(X_B^{[m]}\), which is typically smooth, so the distinction between homology and cohomology will not matter.

Now our first goal is to determine the action of discriminant operator on a monoblock diagonal cycle, i.e. to determine the intersection cycle \(\Gamma^{[m]}_I \cap \Gamma_I\). In this computation, a key technical question is to determine the part of \(OD_I\) and \(\Gamma_I\) over the boundary of \(B\), or at least its irreducible components. Thus for each boundary datum \((\theta, T, \delta)\), with the associated map \(\phi : X_T^\theta \rightarrow X\), we need to determine \((\phi^m)^*(OD_I)\) and its inverse image in \((X_T^\theta)^{[m]}\) which we call the \((\theta, T, \delta)\) boundary of \(\Gamma_I\). A priori, it is clear that any difference between the answers in Sym and Hilb will have to do with node-supported loci, i.e. node scrolls. Thus to state the answer, we recall from [6] the ordered node scrolls \(OF_{jI}^\theta\), which is the portion of \(\varpi^{-1}(F_j^\theta, n = |I|)\), where the \(n\) points coalesced in \(\theta\) lie in the \(I\)-indexed coordinates. This maps to \((X_T^\theta)^{[1,m]\backslash I}\) (i.e. a copy of \((X_T^\theta)^{[m-n]}\) indexed by \(\{1,...,m\} \backslash I\)) which, locally near \(\theta_x \cup \theta_y\) breaks up into branches \(OF_{jI,K_x,K_y}(\theta)\), depending on which points lie in the \(x\) or \(y\)-branches, denoted \(X', X''\) respectively.

**Lemma 2.6.** Set-theoretically, the \((\theta, T, \delta)\) boundary of \(\Gamma_I\) is the union of the following loci, each one itself a union of irreducible components:
(i) for each index-set K, \([1, m] \supset K \supset I\), a locus \(\tilde{\Theta}_{K/I}\), mapping birationally to its image \(\Theta_{K/I} \subset OD_I\);
(ii) for each \(K \subset I^c = [1, m] \setminus I\), ditto;
(iii) for each \(K\) straddling \(I\) and \(I^c\), and each \(j = 1, ..., |I| - 1\), a component \(OF_j^{K-I|K^e-I}(\theta) \subset OF_I(\theta)\) projecting as \(\mathbb{P}^1\)-bundle to its image in \((X^\theta_I)^{|m-I|}\), which lies over \((X')^{K-I} \times_T (X'')^{K^e-I} =: (X^\theta_I)^{K-I|K^e-I} \subset (X^\theta_I)^{|m-I|}\).

**Proof.** The loci of type (i), (ii) are clearly there and any other component must occur at the boundary. Hence, we may fix a node \(\theta\) and work locally over a neighborhood of \(\theta\) in \(X\). The main point is first to determine the boundary of \(OD_I\) (in the symmetric product). But this is easily determined as in the \(\Theta\) decomposition of [6] §4: the boundary is given locally by

\[
\bigcup_{K \subset [1, m]} OD_I \cap \Theta_K.
\]

Set \(\Theta_{K/I} = OD_I \cap \Theta_K\). To describe these, there are 3 cases depending on \(K\):

(i) if \(I \subset K\), then
\[
\Theta_{K/I} = (X')^{K-I} \times (X'')^{K^e};
\]
(ii) if \(I \subset K^e\), then
\[
\Theta_{K/I} = (X')^K \times (X'')^{K^e/I};
\]
(iii) otherwise, i.e. if \(I\) straddles \(K\) and \(K^e\), then
\[
\Theta_{K/I} = \{y_i = 0, \forall i \in K \cup I, x_i = 0, \forall i \in K^c \cup I\}
= (X')^{K-I} \times (X'')^{K^e-I} \times 0^I =: X^{K-I|K^e-I}
\]
(to specify the special value \(s \in B\), a subscript \(s\) may subsequently be added in the above).

Now is an elementary check that the loci of type (i) and (ii) are precisely the irreducible components of the special fibre of \(OD_I\), while the union of the loci \(\Theta_{K/I}\) of type (iii) coincides with the intersection of \(OD_I\) with the fundamental locus (=image of exceptional locus) of the ordered cycle map \(oc_m\), i.e. the locus of cycles containing the node with multiplicity \(> 1\). Also, each \(\Theta_{K/I}\) of type (iii) is of codimension 2 in \(OD_I\). On the other hand, each such \(\Theta_{K/I} = X^{K-I|K^e-I}\) is just a component of the inverse image in \(X^m_R\) of the locus denoted \(X^{(a,b)}\) in [6], §5, where \(a = |K - I|, b = |K^e - I|\), and therefore by that Lemma, the ordered cycle map over it is a union of \(\mathbb{P}^1\) bundles, viz

\[
(2.2.8) \quad oc_m^{-1}(X^{K-I|K^e-I}) = \bigcup_{j=1}^{|I|-1} OF_j^{K-I|K^e-I}
\]

where \(OF_j^{K-I|K^e-I}\) is the pullback of \(F_j^{(m-a-b,a)b}\) over \(X^{K-I|K^e-I}\), which is a \(\mathbb{P}^1\) bundle with fibre \(C_j\). This concludes the proof. \(\square\)
Notice that, given disjoint index-sets $K_1, K_2$ with $K_1 \bigsqcup K_2 = I^c$, the number of straddler sets $K$ such that $K - I = K_1, K^c - I = K_2$ is precisely $2^n - 2$ (i.e. the number of proper nonempty subsets of $I$). Thus, a given $OF_j^{I_1:K_1 I_2:K_2}$ will lie on this many components of $\hat{\Theta}$. This however is a completely separate issue from the multiplicity of $OF_j^{I_1:K_1 I_2:K_2}$ in the intersection cycle $\Gamma^{[m]}_I$, which has to do with the blowup structure and will be determined below.

From the foregoing analysis, we can easily compute the intersection of a monoblock diagonal cycle with the discriminant polarization, as follows. We will fix a covering system of boundary data $(T_s, \delta_s, \theta_s)$, and recall that each datum must be weighted by $\frac{1}{\deg(\delta_s)}$.

**Proposition 2.7.** We have an equality of divisor classes on $\Gamma_I$:

$$\Gamma^{[m]}_I = \sum_{i \neq j} \Gamma_{I \setminus \{i,j\}} + |I| \sum_{i \not\in I} \Gamma_{I \cup \{i\}} - \left(\frac{|I|}{2}\right) p^*_{\min(I)} \omega + \sum_s \frac{1}{\deg(\delta_s)} \sum_{j=1}^{\frac{|I|-1}{2}} \beta_{I \setminus \{i\},j} \delta_{s,j}^I OF_j^I(\theta_s),$$

where $I \setminus \{i,j\}$ and $I \cup \{i\}$ denote the evident diblock partition and uniblock, respectively, the 4th term denotes the class of the image of the node scroll on $\Gamma_I$, $OF_j^I(\theta) = \sum_{K_1 \bigsqcup K_2 = I^c} OF_j^{I_1:K_1 I_2:K_2}(\theta)$, and $\delta_{s,j}^I$ is the natural map of the latter to $\Gamma_I \subset X^{[m]}_B$; precisely put, the line bundle on $\Gamma_I$ given by $O_{\Gamma_I}(\Gamma^{[m]}_I) \otimes p^*_{\min(I)} (\omega^{[I]})$ is represented by an effective divisor comprising the 1st, 2nd and 4th terms of the RHS of (2.2.9).

**Proof.** To begin with, the asserted equality trivially holds away from the exceptional locus of $OC_m$, where the 1st, second and third summands come from components $\Gamma_{i,j}$ of $\Gamma^{[m]}_I$ having $|I \cap \{i,j\}| = 0, 1, 2$, respectively.

Next, both sides being divisors on $\Gamma_I$, it will suffice to check equality away from codimension 2, e.g. over a generic point of each (boundary) locus $(X_B^2)^{K-I K^{c} - I}$. But there, our cycle map $OC_m$ is locally just $OC_r \times \text{iso}, r = |I|$, with

$$\Gamma^{[m]}_I \sim \Gamma^{[r]} + \sum_{\{i,j\} \not\subset I} \Gamma_{i,j}.$$ 

We are then reduced to the case of the small diagonal, discussed in §2.1. \hfill \Box

Now this result immediately implies an analogous one for the operator $\Gamma_I[]$ whose arguments, as cohomology classes, can be represented by cycles in generic position. This can be written compactly using the ‘formal discriminant’ operator’ of §0, Namely
Corollary 2.8. Notations as above,
\[ \Gamma^{[m]} \cdot \Gamma_{\bar{\alpha}} = \Gamma_{\bar{\alpha}}[\text{ODsc}^{[m]}(\alpha_{\cdot})] \]
\[ (2.2.10) \]
\[ + \sum_s \frac{1}{\deg(\delta_s)} \sum_{j=1}^{[I]-1} \beta_{|I|,j}^{\ell} \delta_{s,j}^\ell \cdot O_{F_j}(\theta_s)[u_{I,f}(\alpha_{\cdot})], \alpha_{\cdot} \in TS_I(R) \]

where \( u_{I,f} \) is as in (1.4.9) where \( g(\alpha) = \int \alpha \).

We note that the integral above means the degree (which depends only on the codimension-0 part of \( \alpha \)). Also, we are viewing \( H^*(X) \subset H^*(X^\theta) \) via pullback.

Our next goal is to extend the foregoing result from the monoblock to the polyblock case—still in the ordered setting. While the extension in question is in principle straightforward, it is a bit complicated to describe. Again, a key issue is to describe the boundary of a polyblock diagonal locus \( OD_{(I)} \) in terms of the \( \Theta \) decomposition of [6], §4.. Fix a boundary datum \( (T, \delta, \theta) \). To simplify notations, we will assume, losing no generality, that the partition \( I \) is (relatively to a block \( I_{\ell} \) of \( I \)), and \( I_{\ell} \) is a straddler block for \( K \), if \( I_{\ell} \) meets both \( K \) and \( K^c \). The straddler number \( \text{strad}_{(I)}(K) \) of \( K \) w.r.t. \((I)\) is the number of straddler blocks \( I_{\ell} \). The straddler portion of \((I)\) relative to \( K \) is by definition the union of all straddler blocks, i.e.
\[ (2.2.11) \]
\[ s_K(I_{\ell}) = \bigcup_{I_{\ell} \cap K \neq \emptyset} I_{\ell}. \]

The \( x \)- (resp. \( y \))-portion of \((I)\) (relative to \( K \), of course) are by definition the partitions
\[ (2.2.12) \]
\[ x_K(I_{\ell}) = \{ I_{\ell} : I_{\ell} \subset K \}, y_K(I_{\ell}) = \{ I_{\ell} : I_{\ell} \subset K^c \}. \]

Finally the multipartition data associated to \((I)\) w.r.t. \( K \) are
\[ (2.2.13) \]
\[ \Phi_K(I_{\ell}) = (s_K(I_{\ell}) : x_K(I_{\ell}) | y_K(I_{\ell})). \]

In reality, this is a partition broken up into 3 parts: the nodebound part \( s_K(I_{\ell}) \), a single block, plus 2 at large parts, an \( x \) part and a \( y \) part. As before, we set
\[ (2.2.14) \]
\[ X^{\Phi_K(I_{\ell})} = (X')^{x_K(I_{\ell})} \times (X^\prime)_{y_K(I_{\ell})} \]

and equip it as before with the map to \( X^m_\alpha \) obtained by inserting the node \( \theta \) at the \( s_K(I_{\ell}) \) positions. Now the analogue of Lemma 2.6 is the following

**Lemma 2.9.** For any partition \((I)\) and boundary datum \((T, \delta, \theta)\), the corresponding boundary portion of \( \Gamma_{(I)} \) is
\[ (2.2.15) \]
\[ \bigcup_{\text{strad}_{(I)}(K)=0} \hat{\Theta}_{K_{(I)}(T, \delta, \theta)} \cup \bigcup_{\ell} \left[ I_{\ell} \right] \bigcup_{I' \cdot |I'| = I_{\ell} \setminus I_{\ell}} \bigcup_{j=1}^{[I]-1} O_{F_j}(I_{\ell} : I'_{\ell}, |I'|)(\theta) \]
Proof. Now, one can easily verify

(2.2.16) \[ OD(I) \cap \Phi_K = X^{\Phi_K(I)} =: \Theta_{K,I} \]

so that

(2.2.17) \[ OD(I) \cap X_0^m = \bigcup_{K \subset [1,m]} \Theta_{K,I}. \]

Now, an elementary observation is in order. Clearly, the codimension of \( OD(I) \) in \( X_B^m \) is \( \sum (|I| - 1) \), and this also equals the codimension of \( OD(I) \cap X_0^m \) in \( X_0^m \). On the other hand, we have

(2.2.18) \[
\dim(\Theta_{K,I}) = m - \left( \sum_{I_\ell \text{ nonstraddler rel} K} (|I_\ell| - 1) + \sum_{I_\ell \text{ straddler rel} K} |I_\ell| \right) = m - \sum I_\ell (|I_\ell| - 1) - \text{strad}(I)(K).
\]

It follows that

- the index-sets \( K \) such that \( \Theta_{K,I} \) is a component of the boundary \( OD(I) \cap (X_B^m)^m \) are precisely the nonstraddlers;
- those \( K \) such that \( \Theta_{K,I} \) is of codimension 1 in the special fibre are precisely those of straddle number 1 (unistraddlers).

Next, what are the preimages of these loci upstairs in the ordered Hilbert scheme \( X_B^m \)? They can be analyzed as in the monoblock case:

- if \( K \) is a nonstraddler, a general cycle parametrized by \( \Theta_{K,I} \) is disjoint from the node, so there will be a unique component \( \Theta_{K,I} \subset oc_m^{-1}(\Theta_{K,I}) \) dominating \( \Theta_{K,I} \);
- if \( K \) is a unistraddler (straddle number = 1), the dominant components of \( oc_m^{-1}(\Theta_{K,I}) \) will be the \( \mathbb{P}^1 \)-bundles \( F_{j_\ell}^{\Phi_K(I)} \), \( j = 1, \ldots, s_K(I) - 1 \); note that if \( I_\ell \) the unique block making \( K \) a straddler, then \( \Phi_K(I_\ell) = (I_\ell : x_K(I_\ell)y_K(I_\ell)) \); moreover as \( K \) runs through all unistraddlers, \( \Phi_K(I) \) runs through the date consisting of a choice of block \( I_\ell \) plus a partition of the set of remaining blocks in two \( (x- \text{ and } y-) \) blocks;
- because all fibres of \( oc_m \) are at most 1-dimensional, while every component of the boundary is of codimension 1 in \( \Gamma(I) \), no index-set \( K \) with straddle number \( \text{strad}(I)(K) > 1 \) (i.e. multistraddler) can contribute a component to that special fibre.

This completes the proof. \qed

What the Lemma means is that the analysis leading to Proposition 2.7 extends with no essential changes to the polyblock case, and therefore the natural analogue of that Proposition holds. This is the subject of the next Corollary which
for convenience will be stated in operator form. The statement is nearly identical to the monoblock case, except that the node scrolls appearing will themselves contain a polydiagonal conditions on the variable points on $X^\theta$. We will write $\Gamma_{(I),Y}$ to indicate the appropriate polydiagonal locus associated to a given family $Y$ (e.g. $Y = X/B, X^\theta_T/T$ etc.) then define

$$OF^{I_\ell/I_j}(\theta) = OF^{I_\ell/I_j}(\theta) \circ \Gamma_{(I)} \circ \text{ODsc}^{(m)}$$

(2.2.19)

As above, the pullback via $X^\theta_T \to X$ gives an inclusion $H^* \to H^*(X^\theta_T)$ so for any subring $R \subset H^*(X)$ containing $\omega$ the operator

$$OF^{I_\ell/I_j}(\theta) \circ \Gamma_{(I)} \circ \text{ODsc}^{(m)} \subset (X^\theta_T)^{\lceil m \rceil}$$

(2.2.19)

is defined.

**Corollary 2.10.** (i) For any block partition $I. = I_1|...|I_r$ on $[1,m]$, we have an equality of operators

$$\Gamma^{[m]} \circ \Gamma_{(I)} \circ \text{ODsc}^{(m)} = \Gamma_{(I)} \circ \text{ODsc}^{(m)}$$

(2.2.20)

$$+ \sum_s \frac{1}{\deg(\delta_s)} \sum_{\ell=1}^{\lceil |I_\ell| \rceil -1} \sum_{j=1}^{\beta|I_\ell|,j} \delta_s, \delta^{I_\ell} \circ OF^{I_\ell/I_j}(\theta) \circ u_{I_\ell,j} \circ \Gamma_{(I_\ell)} \circ \text{ODsc}^{(m)}$$

where $u_{I_\ell,j}$ is interior multiplication by $\int_X$ as in (1.4.9). □

### 2.3. Monoblock and polyblock diagonals: unordered case.

We need the analogues of the formulae of the latter section in the (unordered) Hilbert scheme. These are essentially straightforward, and may be obtained from the ordered versions using push-forward by the symmetrization map $\varpi_m$. We begin with the monoblock case. Recall first the the monoblock (unordered) diagonal operator $\Gamma_{(n)}$ which may be defined for $n > 1$ as

$$\Gamma_{(n)}[\alpha.] = \frac{1}{(m-n)!} \varpi_m \circ \Gamma_{(I)}[\alpha.]$$

Generally for a distribution $\mu$ the polyblock diagonal operators $\Gamma_{\mu}$ can be defined similarly by

$$\Gamma_{\mu}[\alpha.] = \frac{1}{a(\mu) \varpi_m \circ \Gamma_{(I)}[\alpha.]$$

where $(I.)$ is any $b$-partition with distribution $\mu$ and $a(\mu) = \prod_{n} \mu(n)!$ is the degree of the restricted symmetrization map $\Gamma_{(I)} \to \Gamma_{\mu}$. We will often specify a distribution by specifying only its non-singleton blocks. Thus $\Gamma_{(n),m}$ or $\Gamma(n)$ for $n > 1$ is short for $\Gamma_{\mu}$ with $\mu$ of weight $m$ with $\mu(n) = 1, \mu(1) = m - n$; similarly for $\Gamma_{(n)\mu',...}$. Note the following elementary facts:
(i)
(2.3.1) \( \varpi_{m*}(\Gamma^{[m]} \cdot \Gamma_I) = \Gamma^{(m)} \cdot \varpi_{m*} \Gamma_I \)
(projection formula, because \( \varpi_{m*}(\Gamma^{(m)}) = \Gamma^{[m]} \); NB \( \varpi \) is ramified over the support of \( \Gamma^{(m)} \), still no factor of 2 in \( \varpi_{m*}(\Gamma^{(m)}) \), by our definition of \( \Gamma^{(m)} \) as 1/2 its support);
(ii)
(2.3.2) \( \varpi_{m*}(\Gamma_I[\alpha]) = (m - n)!\Gamma_{(n)}[\alpha], n = |I| > 1; \)
(iii)
(2.3.3) \( \varpi_{m*}(\Gamma_{I[i,j]}) = \begin{cases} (m - n - 2)!\Gamma_{(n|2)}, & n \neq 2; \\ 2(m - n - 2)!\Gamma_{(2|2)}, & n = 2, \\ (1 + \delta_{2,n})(m - n - 2)!\Gamma_{(n|2)}, & \forall n \end{cases} \)
(\( \delta \) = Kronecker delta) here \( \Gamma_{(n|2)} \) is the diagonal locus corresponding to the distribution (of weight \( m \)) with blocks of sizes \( n, 2 \) plus singletons;
(iv)
(2.3.4) \( \varpi_{m*}(\Gamma_{I[1]}[\alpha]) = (m - n - 1)!\Gamma_{(n+1)}; \)
(v)
(2.3.5) \( \varpi_{m*}(OF_{j}^{I,K-I|K^c-I}(\theta)) = a!b!F_{j}^{(n|a|b)}(\theta), \ a = |K - I|, b = |K^c - I| = m - n - a \)
where we recall that \( F_{j}^{(n|a|b)}(\theta) \) is the unordered analogue of the node scroll \( F_{j}^{(I,K'|K^c)} \); moreover the number of distinct subsets \( K - I \) with \( a = |K - I| \), for fixed \( I \) and \( a \), is \( \binom{m - n}{a} \) which easily implies that the push-forward, properly weighted, of the total of the ordered node scrolls by symmetrization equals the total of the unordered node scrolls.

Putting these together with Proposition 2.7, we conclude
(2.3.6) \( \Gamma^{(m)} \cdot \Gamma_{(n)} \sim \)
\[
\frac{1 + \delta_{2,n}}{2} \Gamma_{(n|2)} + n\Gamma_{(n+1)} - \binom{n}{2} \Gamma_{(n)}[\omega] + \sum_{s} \frac{1}{\deg(\delta_{s})} \sum_{a=0}^{m-n} \sum_{j=1}^{n-1} \beta_{n,j} \delta_{s,j}^{n} F_{j}^{(n|a|m-n-a)}(\theta_{s}).
\]
Set
\[
\sum_{a=0}^{m-n} F_{j}^{(n|a|m-n-a)} = F_{j}^{n,m}
\]
(when \( m \) is understood, we will denote this by \( F_{j}^{n} \)). We note that in this sum, the first 3 terms in (2.3.6) match up exactly with (1.4.7), where the first term corresponds to uniting two singleton blocks and the second to uniting a singleton block with a non-singleton block of size \( n \).
block with the $n$-block. Therefore the formula may be extended to the twisted case and written more compactly as follows

**Proposition 2.11.** For any monoblock diagonal $\Gamma(n)$, $n > 1$, we have

\[
\Gamma(m) \cdot \Gamma(n) \sim \Gamma(n) \circ \text{Dsc}^{(m)} + \sum_{s} \frac{1}{\text{deg}(\delta_s)} \sum_{j=1}^{n-1} \beta_{n,j} \delta_{s,j}^{n} F_{j}^{m}(\theta_s) \circ u_{n,j}. \quad \Box
\]

When $n = 2$, $\Gamma(n)$ is just $2\Gamma(m)$, hence

**Corollary 2.12.**

\[
(\Gamma(m))^{2} \sim \frac{1}{2} \Gamma(2) - \Gamma(3) - \Gamma(m)[\omega] + \sum_{s} \frac{1}{\text{deg}(\delta_s)} \frac{1}{2} \delta_{s,j}^{2} F^{2,m}(\theta_s). \quad \Box
\]

Here and elsewhere, we denote by $(n \rightarrow k)$ the distribution $\mu$ with $\mu(n) = k$ and zeros elsewhere; when $k = 1$ it will be omitted. Also $(\mu_1|\mu_2|\ldots)$ denoted the sum as functions $\mu_1 + \mu_2 + \ldots$.

**Corollary 2.13.** We have

\[
\Gamma(m) \cdot \Gamma(2) \circ u_{2,\omega} = \Gamma(2) - 2 \Gamma(3) \circ u_{2,\omega} + \sum_{s} \frac{3}{\text{deg}(\delta_s)} \sum_{a=0}^{m-3} \delta_{s,1}^{3} (F_{1}^{3,m}(\theta_s) + F_{2}^{3,m}(\theta_s)).
\]

**Corollary 2.14.** We have for $m = 2$:

\[
(\Gamma^{(2)})^{k} = \Gamma^{(2)} \circ u_{2,(-\omega)^{k-1}} + \sum_{s} \frac{1}{\text{deg}(\delta_s)} \frac{1}{2} \delta_{s,1}^{2} (\Gamma^{(2)})^{k-2} F_{1}^{2,2}(\theta_s), \quad k \geq 3;
\]

if $m = 2$, $\dim(B) = 1$,

\[
\int_{x_B^{[2]}} \Gamma^{(2)} = \frac{1}{2} \omega^2 - \frac{1}{2} \sigma, \quad \sigma = |\{\text{singular values}\}|;
\]

for $m = 3$

\[
(\Gamma^{(3)})^{3} = -4 \Gamma^{(3)}[\omega] + \Gamma^{(3)}[\omega^2]
\]

\[
+ \sum_{s} \frac{1}{\text{deg}(\delta_s)} (3\delta_{s,1}^{3} F_{1}^{3,3}(\theta_s) + \delta_{s,2}^{3} F_{2}^{3,3}(\theta_s)) + \frac{1}{2} \delta_{s,1}^{2} \Gamma^{(3)}(F_{1}^{2,3}(\theta_s) + F_{2}^{2,3}(\theta_s))
\]

[We have used the elementary fact that $\omega \cdot \theta_s = 0$, hence $\omega \cdot F_{2}^{2,3}(\theta_s) = 0, \forall i > 0$, because this node scroll maps to $\theta_s$, more precisely to $2[\theta_s] \subset X_B^{(2)}$.]

To simplify notation we shall henceforth denote $\frac{1}{\text{deg}(\delta_s)} \sum_{s} F_{\bullet}^{\ell}(\theta_s)$ simply as $F_{\bullet}^{\ell}$. 

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Example 2.15. This is presented here mainly as a check on some of the coefficients in the formulas above. For \( X = \mathbb{P}^1 \), \( X^{(m)} = \mathbb{P}(H^0(\mathcal{O}_X(m))) = \mathbb{P}^m \), and the degree of \( \Gamma^{(m)}_{(n)} \) is \( n(m - n + 1) \). Indeed this degree may be computed as the degree of the degeneracy locus of a generic map \( n\mathcal{O}_X \to P^{n-1}_X(\mathcal{O}_X(m)) \) where \( P^k_X \) denotes the \( k \)-th principal parts or jet sheaf. It is not hard to show that \( P^{n-1}_X(\mathcal{O}_X(m)) \cong n\mathcal{O}_X(m - n + 1) \).

For example, \( \Gamma^{(3)}_{(2)} \) is a quartic scroll equal to the tangent developable of its cuspidal edge, i.e. the twisted cubic \( \Gamma^{(3)}_{(3)} \). The rulings are the lines \( L_p = \{ 2p + q : q \in X \} \), tangent to the \( \Gamma^{(3)}_{(3)} \), each of which has class \( -\frac{1}{2}\Gamma^{(3)}[\omega] \). Therefore by Corollary 2.12, the self-intersection of \( \Gamma^{(3)} \) in \( \mathbb{P}^3 \) (or half the intersection of \( \Gamma^{(3)} \) with \( \Gamma^{(3)}_{(2)} \), as a class on \( \Gamma^{(3)}_{(2)} \)) is represented by \( \Gamma^{(3)}_{(3)}[\omega] \).

Next we extend Proposition 2.11 to the polyblock case, in other words work out the unordered analogue of Corollary 2.10. Consider a distribution \( \mu \) of weight \( m \) and associated polyblock diagonal loci and operators \( \Gamma_{\mu}, \Gamma_{\mu}, \Gamma_{\mu}[] \) where, e.g. \( \Gamma_{\mu}[:] : TS_{\mu}(R) \to H.(X_B^{[m]}) \)

Now the node scroll \( F^n_j(\theta) \) (see the next section for more detail) is a \( \mathbb{P}^1 \)-bundle over \( (X_\theta)^{[m-n]} \), whence operators, for any distribution \( \nu \) of weight \( m - n \):

\[
F^n_j(\theta)[] : TS_{\nu}(R) \to H.(X_B^{[m]})
\]

\[
\alpha \mapsto [F^n_j(\theta)] \cap p^*_{[m-n]} \circ \Gamma_{\nu,X_\theta} \circ \phi^*(\alpha.)
\]

where \( \phi : X_\theta \to X \) is the natural map. Clearly, given a distribution \( \mu \) of weight \( m \), the \( \nu \)-s corresponding to it via the unordered analogue of Corollary 2.10 will have the form \( \nu = \mu - 1_n \) with \( \mu(n) \geq 1 \).

Now the following result follows directly from Corollary 2.10 by adjusting for the degrees of the various symmetrization maps.

**Proposition 2.16.** For a distribution \( \mu \) of weight \( m \), we have an equality of operators

\[
\Gamma^{(m)}_\mu[:] = \Gamma_{\mu} \circ \text{Dsc}^{(m)} + \sum_{\theta} \sum_n \frac{1}{n} \sum_{j=1}^{n-1} \beta_{n,j} F^n_j(\theta)[-] \circ u_{n,j}
\]
Example 2.17.  
(2.3.16)  
\[ \Gamma^{(m)} \circ (2\to3) = \frac{3}{2} \Gamma^{(2\to3)} \circ u_{1,1} + 2 \Gamma^{(4)} \circ u_{2,2} + \Gamma^{(3\to2)} \circ u_{2,1} - \Gamma^{(2\to2)} \circ u_{2,\omega} + \frac{1}{2} F^{2,m}_1 \circ u_{2,f} \]

Example 2.18.  
(2.3.17)  
\[ (\Gamma^{(m)})^3 = \frac{3}{4} \Gamma^{(2\to3)} + 4 \Gamma^{(4)} + \frac{3}{2} \Gamma^{(3\to2)} - \Gamma^{(2\to2)} \circ u_{2,\omega} - 4 \Gamma^{(3)} \circ u_{3,\omega} \]

Now to compute the Tautological Module, which is yet to be defined but must certainly include the \( \Gamma_{\mu} \), we must be able to ‘iterate’ (2.3.15), that is, multiply \( \Gamma_{\mu} \) or equivalently, the node scrolls \( F^{n}_{\mu} \theta \cdot \) with powers of \( \Gamma^{(m)} \). This is taken up in the next section.

2.4. Polarized node scrolls. Before taking up the node scrolls we mention an elementary analogue. Suppose the family \( X/B \) is generically finite onto a component the locus of schemes having length at least \( n \) at \( \theta \), while \( p_{[m-n]} \) is a \( \mathbb{P}^1 \)-bundle projection. Note that \( F^{n}_{\mu} \theta \cdot \) defines an operator

(2.4.2)  
\[ F^{n}_{\mu} \theta \cdot \downarrow \quad p_{[m-n]} \]

where \( p_{[m]} \) is generically finite onto a component the locus of schemes having length at least \( n \) at \( \theta \), while \( p_{[m-n]} \) is a \( \mathbb{P}^1 \)-bundle projection. Note that \( F^{n}_{\mu} \theta \cdot \) defines an operator

\[ H : (X^{[m-n]}_T) \to H(X^{[m]}_{\beta} \cdot \), \]

(2.4.3)  
\[ \beta \mapsto p_{[m]} \cdot p_{[m-n]}(\beta) \]

We will however view \( F^{n}_{\mu} \theta \cdot \) as acting just on the tautological module \( T^{m-n}(X^{\theta}_{T}) \), which we may assume defined by induction on Hilb degree (more on this shortly). We will call \( F^{n}_{\mu} \theta \cdot [\beta] \) for \( \beta \in T^{m-n}(X^{\theta}_{T}) \) a twisted node scroll class (of Hilb degree \( m \) on \( X/B \)). The polarized structure of the node scroll \( F^{m,m}_{\mu} \theta \cdot \), refers to its description as projectivization of a rank-2 vector bundle (in fact, a direct sum of two explicit line bundles) on the degree-(\( m - n \)) Hilbert scheme \( (X^{\theta}_{T})^{[m-n]} \), with the property that the associated \( \mathcal{O}(1) \) relative polarization coincides with \(-p_{[m]}(\Gamma^{(m)}) \). This was worked out in [6] and can be described as follows.
Fix a boundary family $X^\theta_T$ and let $\theta_x, \theta_y$ be the sections of $X^\theta_T$ mapping to the node $\theta$, and let

$$\psi_x = \theta_x^*(\omega_{X^\theta_T/T}),$$

considered as a line bundle on $T$ (and by pullback, on any space mapping to $T$). As in $\S$1.5.10, let $[h]\cdot L$ be the $k$-th norm associated to a line bundle $L$ on $X$ (which is a line bundle on the $k$-th symmetric product of $X$). Then set

$$D_j^n(\theta) = \left(\frac{n-j+1}{2}\right)\psi_x + \left(\frac{j}{2}\right)\psi_y - (n-j+1)[m-n]_s\theta_x - j[m-n]_s\theta_y$$

(2.4.4)

(confusing divisors and line bundles on $(X^\theta_T)^{[m-n]}$). The the Node Scroll theorem of [6] yields an isomorphism

$$F_j^n(\theta) \simeq \mathbb{P}(\mathcal{O}(D_j^n(\theta)) \oplus \mathcal{O}(D_{j+1}^n(\theta)))$$

under which

$$-p^*_m(\Gamma^{(m)}) + p^*_m(\Gamma^{(m-n)}) \leftrightarrow \mathcal{O}(1).$$

To make use of this, set

$$e_j^n(\theta) = [D_j^n(\theta)] - \Gamma^{(m-n)} \in A^1((X^\theta_T)^{[m-n]}).$$

Of course, $\Gamma^{(m-n)} = 0$ if $m-n \leq 1$. Thus, this term begins to appear only for $m \geq 4$. We will identify this class with its pullback on $F_j^n(\theta)$. Also, set formally

$$s_k(a, b) = a^{k-1} + a^{k-2}b + \ldots + b^{k-1}(= \frac{a^k - b^k}{a - b})$$

(2.4.6)

Thus,

$$s_k(e_j, e_{j+1}) = \frac{e_{j+1}^k - e_j^k}{-(n-j)\psi_x + j\psi_y + \theta_x - \theta_y}.$$  

(2.4.7)

Then the Node Scroll Theorem plus the usual relation of Chern and Segre classes yield immediately

**Theorem 2.19.** For any twisted node scroll class $F_j^n(\theta)[\beta]$, we have

$$(-\Gamma^{(m)}\ell)F_j^n(\theta)[\beta] = (-\Gamma^{(m)})F_j^n(\theta)[s_{\ell-1}(e_j, e_{j+1})\beta] - F_j^n(\theta)[e_je_{j+1}s_{\ell-2}(e_j, e_{j+1})\beta]$$

(2.4.8)

The class $-\Gamma^{(m)}F_j^n(\theta)$, called a node section, projects with degree 1 to $(X^\theta_T)^{[m-n]}$. Evaluating the rest of the RHS of 2.4.8 involves, essentially, the tautological module in lower degree and, in case $X/B$ is a family of stable curves, lower genus as well, albeit for a family of pointed curves $X^\theta_T$, with distinguished sections $\theta_x, \theta_y$. To evaluate the terms involving these, we may note the following elementary formulas, in which $\theta$ denotes any section and $\psi = \pi_*(\omega_{\theta})$:

$$\theta^r = (-\psi)^{r-1}\theta, r \geq 1; \theta_x\theta_y = 0;$$

(2.4.9)
\begin{equation}
([k]_s \theta)^t = \sum_{s=1}^{\min(k,t)} (s^t - (s - 1)^t)(-\psi)^{t-s}[k]_s^s(\theta)
\end{equation}

where we recall, cf. (1.5.11), that $[k]_s^s(\theta)$ denote the symmetrization of $\theta^{x,s}$ and its pullback on the Hilbert scheme.

(\text{proof of (2.4.10)}): clearly,

$$\sum_{s=1}^{\min(k,t)} (s^t - (s - 1)^t)(-\psi)^{t-s}[k]_s^s(\theta)$$

To evaluate the numerical coefficient, say $a_s$, note that

$$a_1 + \ldots + a_s = \sum_{r_1, \ldots, r_s} t = s^t,$$

hence $a_s = s^t - (s - 1)^t$. The pullback of (2.4.10) on a polyblock diagonal $\Gamma_\nu$ is given by the $D^\dagger$ operator defined in §1.3, viz.

(2.4.11)

$$\Gamma_\nu([k]_s \theta)^t = \sum_{s=1}^{\min(k,t)} (s^t - (s - 1)^t)(-\psi)^{t-s}\Gamma_\nu[D^\dagger([k]_s^s(\theta))]$$

Similarly, on the operator level,

(2.4.12)

$$\Gamma_\nu([k]_s \theta)^t[\beta] = \sum_{s=1}^{\min(k,t)} (s^t - (s - 1)^t)(-\psi)^{t-s}\Gamma_\nu[D^\dagger([k]_s^s(\theta)).\beta)]$$

for $\beta \in TS_\nu(H^\nu(X^\theta_T))$ (where the $.\beta$ means formal symmetric multiplication). In particular, using the inductive case of Theorem 2.1, it follows that the bracketed expressions appearing in Theorem 2.19 are all tautological classes, therefore

**Corollary 2.20.** Notations as above, $(-\Gamma^{(m)}).\ell F^n_j(\theta)[\beta]$ is a twisted node scroll class.

**Example 2.21.** We have

$$F_1^{(2,3)}(\theta) = \mathbb{P}_X^\theta(\mathcal{O}(-2\theta_x - \theta_y) \oplus \mathcal{O}(-\theta_x - 2\theta_y))$$

Consequently, if the boundary is finite,

$$(-\Gamma^{(3)})^2 F_1^{(2,3)} = -6.$$ 

Note that in the 'extreme' case $m = n$, the $e^n_j(\theta)$ and the node scroll $F^n_j(\theta)$ live on the base itself $T$ of the boundary datum and we have

$$e^n_j(\theta) = \binom{m - j + 1}{2}\psi_x + \binom{j}{2}\psi_y.$$
Example 2.22. For $m = n = 2$, $F = F^2_1(\theta) = \mathbb{P}(\psi_x \oplus \psi_y)$, we have
\begin{equation}
(-\Gamma^{(2)})^{k} F = (\psi_x^{k-1} + \psi_x^{k-2} \psi_y + \ldots + \psi_y^{k-1})(-\Gamma^{(2)}) - \psi_x \psi_y (\psi_x^{k-2} + \psi_x^{k-3} \psi_y + \ldots + \psi_y^{k-2}).
\end{equation}
In particular, for $k = \dim(B) = \dim(F) = 1 + \dim(T)$, which is when the class becomes 0-dimensional, we have for its degree
\begin{equation}
(-\Gamma^{(2)})^{k} F = \int_T (\psi_x^{k-1} + \psi_x^{k-2} \psi_y + \ldots + \psi_y^{k-1}).
\end{equation}
Note that if $B = \overline{\mathcal{M}}_g$ and $T = \overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1}$, $1 \leq i \leq g/2$ (the usual $i$-th boundary component), only one summand contributes to the latter integral, which reduces to
\begin{equation}
\int_{\overline{\mathcal{M}}_i} \psi_x^{3i-2} \int_{\overline{\mathcal{M}}_{g-i}} \psi_y^{3(g-i)-2}
\end{equation}
Note that (2.4.16) and (2.3.11) together imply
\begin{corollary}
(i) The powers of the polarization on $X^{(2)}_B$ are
\begin{equation}
(-\Gamma^{(2)})^{k} = -\Gamma[\omega^{k-1}] + \frac{1}{2} \sum_s \delta_s s ((\psi_x^{k-3} + \psi_x^{k-4} \psi_y + \ldots + \psi_y^{k-3})(-\Gamma^{(2)}) - \psi_x \psi_y (\psi_x^{k-4} + \psi_x^{k-5} \psi_y + \ldots + \psi_y^{k-4}))
\end{equation}
(ii) The image of the latter class on the symmetric product $X^{(2)}_B$ equals
\begin{equation}
-\Gamma[\omega^{k-1}] + \frac{1}{2} \sum_s \delta_s s ((\psi_x^{k-3} + \psi_x^{k-4} \psi_y + \ldots + \psi_y^{k-3})
\end{equation}
(iii) The image of the latter class on $B$ equals $-\kappa_{k-2}$.
\end{corollary}
\begin{proof}
(i) has been proved above; (ii) follows because in the last summation in (2.4.18), the terms without $\Gamma^{(2)}$, i.e. the twisted node scroll, collapses under the cycle map to $X^{(2)}_B$; (iii) follows similarly. \hfill \Box
\end{proof}
Remark 2.24. It is interesting to compare the above boundary term with the boundary term in Mumford’s formula [4] for the Chern character of the Hodge bundle; our $\psi_x, \psi_y$ are his $K_1, K_2$, and $\psi_x \oplus \psi_y$ is the conormal bundle to $\theta$ in $X$; so our term is essentially the Segre class of $\theta$ in $X$, while Mumford’s term is a Todd class of the same.
Example 2.25. $m = 3, n = 2, \dim(B) = 1$:
\begin{equation}
(-\Gamma^{(3)})^{2} F^{(2;3)}_1(\delta) = -6
\end{equation}
(see Example 2.21). Consequently, in view of Corollary 2.14, we conclude

\[(2.4.21)\]

\[
\int_{X^4_B} (\Gamma^{(3)})^4 = 13\omega^2 - 9\sigma
\]

(recall that each \(F^{(3,3)}_i, i = 1, 2\) is a line with respect to the discriminant polarization \(-\Gamma^{(3)}\)).

2.5. **Tautological module.** We are now in position to give the formal (recursive) definition of the tautological module \(T^m(X/B)\) and the proof of Theorem 2.1.

**Definition 2.26.** Given a cohomology theory \(H\) and a \(\mathbb{Q}\)-subalgebra \(R \subset H^*(X)\) containing the canonical class \(\omega\), the tautological module \(T^m_R(X/B)\) is the \(R\)-submodule of \(H^*(X^m_B)\) generated by

(i) the twisted polyblock diagonal classes \(\Gamma_{\mu}[\alpha.], \alpha. \in TS_{\mu}(H^*(X))\), \(w(\mu) = m\);

(ii) the direct images on \(X^m_B\) the twisted node scroll classes \(F^{n}(\theta)[\beta]\) and the twisted node scroll sections \(-\Gamma^m(F^{n}(\theta)[\beta]\) as \((T, \delta, \theta)\) ranges over a fixed covering system of boundary data for the family \(X/B\), \(\beta \in T^m_{R-n}(X^m_B)\) and \(2 \leq n \leq m\).

For the default choice \(R = \mathbb{Q}[\omega]\), we denote \(T^m_R\) by \(T^m\).

**Proof of Theorem 2.1.** We wish to compute the product of a tautological class \(c\) by \(\Gamma^m\). If \(c\) is a (twisted) diagonal class \(\Gamma_{\mu}[\alpha.],\) this is clear from Proposition 2.16. If \(c\) is a twisted node scroll class \(F^n_{\theta}[\alpha.],\) it is obvious. Finally if \(c\) is a node scroll section \(-\Gamma^m(F^n_{\theta}[\alpha.],\) it is clear from the case \(\ell = 2\) of Theorem 2.19. \(\square\)

**Remark 2.27.** In the important special case of computing a power \((\Gamma^m)^k\) it is probably more efficient not to proceed by simple recursion, but rather to apply just Proposition 2.16 repeatedly to express \((\Gamma^m)^k\) in terms of twisted diagonals plus classes \((\Gamma^m)^t, F\) for various \(t\)’s and various \(F\)’s; then each of the latter classes can be computed at once using Theorem 2.19.

3. **Tautological transfer and Chern numbers**

In this chapter we will complete the development of our intersection calculus. First we study the transfer operation \(\tau_m\), taking cycles on \(X^m_B\) to cycles (of dimension 1 larger) on \(X^m_{B}\), via the flaglet Hilbert scheme \(X^m_{B}\). In the Transfer Theorem 3.3 we will show in fact that for any tautological class \(u\) on \(X^m_B\), the image \(\tau_m(u)\) is a simple linear combination of basic tautological classes on \(X^m_B\). We then review a splitting principle established in [8], which expresses the Chern classes of the tautological bundle \(\Lambda_m(E)\), pulled back on \(X^m_{B}\), in terms of those of \(\Lambda_{m-1}(E)\), the discriminant polarization \(\Gamma^m\), and base classes. Putting
this result together with the Module Theorem and the Transfer Theorem yields the calculus for arbitrary polynomials in the Chern classes of $\Lambda_m(E)$.

3.1. **Flaglet geometry and the transfer theorem.** In this section we study the $(m, m-1)$ flag (or 'flaglet') Hilbert scheme, which we view as a correspondence between the Hilbert schemes for lengths $m$ and $m-1$ providing a way of transporting cycles, especially tautological ones, between these Hilbert schemes. We will make strong, chapter-verse use of the results of [9].

Thus let

$$X_B^{[m,m-1]} \subset X[m] \times_B X^{[m-1]}$$

denote the flag Hilbert scheme, parametrizing pairs of schemes $(z_1, z_2)$ satisfying $z_1 \supset z_2$. This comes equipped with a (flag) cycle map

$$c_{m,m-1} : X_B^{[m,m-1]} \to X_B^{(m,m-1)},$$

where $X_B^{(m,m-1)} \subset X_B^{(m)} \times_B X_B^{(m-1)}$ is the subvariety parametrizing cycle pairs ($c_m \geq c_{m-1}$). Note that the normalization of $X_B^{(m,m-1)}$ may be identified with $X_B^{(m-1)} \times_B X$; however the normalization map, though bijective, is not an isomorphism. Note also that we also have an ordered version $X_B^{[m,m-1]}$, with its own cycle map

$$oc_{m,m-1} : X_B^{[m,m-1]} \to X^m_B.$$ 

In addition to the obvious projections

$$X_B^{[m,m-1]} \xymatrix{ \ar@{^{(}->}[r]_{p_m} & \ar@{^{(}->}[l]^{p_{m-1}} X_B^{[m-1]}},$$

with respective generic fibres $m$ distinct points (corresponding to removing a point from a given $m$-tuple) and a generic fibre of $X/B$ (corresponding to adding a point to a given $m-1$-tuple), $X_B^{[m,m-1]}$ admits a natural map

$$(3.1.2) \quad a : X_B^{[m,m-1]} \to X,$$

$$\left(z_1 \supset z_2\right) \mapsto \text{ann}(z_1/z_2)$$

(identifying $X$ with the Hilbert scheme of colength-1 ideals). Therefore $X_B^{[m,m-1]}$ admits a 'refined cycle map' (factoring the flag cycle map)

$$(3.1.3) \quad c : X_B^{[m,m-1]} \to X \times_B X_B^{(m-1)}$$

$$c = a \times (c_{m-1} \circ p_{m-1}).$$

Now in [9] (Theorem 5 et seq., especially Construction 5.4 p.442) we worked out a complete model for $X_B^{[m,m-1]}$, locally over $X_B^{[m,m-1]}$. Let

$$(3.1.4) \quad H_m \subset X_B^{(m)} \times_B \tilde{C}_B^{[u,v]} \subset X_B^{(m)} \times \mathbb{P}^{m-1}_Z,$$

$$(3.1.5) \quad H_{m-1} \subset X_B^{(m-1)} \times_B \tilde{C}_B^{[u',v']} \subset X_B^{(m-1)} \times \mathbb{P}^{m-2}_{Z'}.$$
be respective local models for $X_{B}^{[m]}$, $X_{B}^{[m-1]}$ as constructed in §1 above, with coordinates as indicated. Consider the subscheme

$$H_{m,m-1} \subset H_{m} \times_{B} H_{m-1} \times_{X_B^{(m)}} X_{B}^{(m,m-1)}$$

defined by the equations

$$u'_{i}v_{i} = (\sigma_{i}^{x} - \sigma_{i}'^{x})u_{i}v'_{i}, \quad v'_{i}u_{i+1} = (\sigma_{i}^{y} - \sigma_{i}'^{y})v_{i+1}u'_{i}, \quad 1 \leq i \leq m - 2$$

or alternatively, in terms of the $Z$ coordinates,

$$Z_{i}Z'_{j} = (\sigma_{i}^{x} - \sigma_{i}'^{x})Z_{i+1}Z'_{j-1}, \quad i + 1 \leq j \leq m - 1$$

$$= (\sigma_{i}^{y} - \sigma_{i}'^{y})Z_{i}Z'_{j+1, l} \quad 1 \leq j \leq i - 2.$$ 

To ‘explain’ these relations in part, note that in the ordered model over $X_{B}^{m}$, we have

$$\sigma_{i}^{x} - \sigma_{i}'^{x} = x_{m}, \quad \sigma_{i}^{y} - \sigma_{i}'^{y} = y_{m}$$

and then the analogue of (3.1.8) for the $G$ functions is immediate from the definition of these in [6], §4. Then the result of [9], Thm. 5, is that $H_{m,m-1}$, with its map to $X_{B}^{(m,m-1)}$ is isomorphic to a neighborhood of the special fibre over $(mp, (m - 1)p)$ of the flag Hilbert scheme $X_{B}^{[m,m-1]}$. In fact the result of [9] is even more precise and identifies $H_{m,m-1}$ with a subscheme of $H_{m} \times_{B} H_{m-1}$ and even of $H_{m-1} \times_{B} \tilde{C}^{m} \times_{B} X$, where the map to $X$ is the annihilator map $a$ above.

As noted in [9], Thm 5, the special fibre of the flag cycle map on $H_{m,m-1}$, aka the punctual flag Hilbert scheme, is a normal-crossing chain of $\mathbb{P}^{1}$’s:

$$C_{m,m-1} = \tilde{C}_{1}^{m} \cup \tilde{C}_{1}^{m-1} \cup \tilde{C}_{2}^{m} \cup \ldots \cup \tilde{C}_{m-2}^{m} \cup \tilde{C}_{m-1}^{m} \subset C_{m} \times C_{m-1}^{m}.$$ 

where the embedding is via

$$\tilde{C}_{i}^{m} \rightarrow C_{i}^{m} \times \{Q_{i}^{m-1}\}, \quad \tilde{C}_{i}^{m-1} \rightarrow \{Q_{i+1}^{m}\} \times C_{i}^{m-1}$$

and in particular,

$$\tilde{C}_{i}^{m} \cap \tilde{C}_{i}^{m-1} = \{(Q_{i+1}^{m}, Q_{i}^{m-1})\}, \quad \tilde{C}_{i}^{m-1} \cap \tilde{C}_{i+1}^{m} = \{(Q_{i+1}^{m}, Q_{i+1}^{m-1})\}$$

where $Q_{i}^{m} = (x_{m-i+1}, y_{i})$ as usual.

**Theorem 3.1.** The cycle map $c_{m,m-1}$ exhibits the flag Hilbert scheme $X_{B}^{[m,m-1]}$ as the blow-up of the sheaf of ideals $\mathcal{I}_{D_{m,m-1}} := \mathcal{I}_{D_{m-1}} \mathcal{I}_{D_{m}}$ on $X_{B}^{(m,m-1)}$.

We shall not really need this result, just the explicit constructions above, so we just sketch the proof, which is analogous to that of the Blowup Theorem of [6]. To begin with, it is again sufficient to prove the ordered analogue of this result, for the ‘ordered flag cycle map’

$$c_{m,m-1}^{[m,m-1]} : X_{B}^{(m,m-1)} \rightarrow X_{B}^{m}.$$
Here \( X_B^{[m,m-1]} \) is embedded as a subscheme of \( X_B^{[m]} \times X_B^{[m-1]} \times_B X \), and we have already observed that as such, it satisfies the equations (3.1.8).

Now we will use the following construction. Let \( I_1, I_2 \) be ideals on a scheme \( Y \). Then the surjection of graded algebras
\[
\bigoplus_n I_1^n \otimes \bigoplus_n I_2^n \to \bigoplus_n (I_1 I_2)^n
\]
yields a closed immersion
\[
\text{Bl}_{I_1 I_2} Y \hookrightarrow \text{Bl}_{I_1} Y \times_Y \text{Bl}_{I_2} Y;
\]
the latter is in turn a subscheme of the Segre subscheme
\[
\mathbb{P}(I_1) \times_Y \mathbb{P}(I_2) \subset \mathbb{P}(I_1 \otimes I_2).
\]
In our case, the Blowup Theorem of [6] allows us to identify
\[
OH_m \approx \text{Bl}_{I_{OD,m}} X_B^m, \ OH_{m-1} \times_B X \approx \text{Bl}_{I_{OD,m-1} X_B^m} X_B^m
\]
(where \( OH_m = H_m \times_{X_B^m} X_B^m \) etc.), whence an embedding
\[
\text{Bl}_{I_{OD,m-1}} X_B^m \to OH_{m-1} \times_B X
\]
As observed above, the generators \( G_i \cdot G'_j \) satisfy the analogues of the relations (3.1.8), so the image is actually contained in \( OH_{m,m-1} \), so we have an embedding
\[
\text{Bl}_{I_{OD,m-1}} X_B^m \to OH_{m,m-1}.
\]
We are claiming that this is an isomorphism. This can be verified locally, as in the proof of the Blowup Theorem in [6].

One consequence of the explicit local model for \( X_B^{[m,m-1]} \) is the following

**Corollary 3.2.**

(i) The projection \( q_{m-1} \) is flat, with 1-dimensional fibres;

(ii) \( \text{Let } z \in X_B^{[m-1]} \) be a subscheme of a fibre \( X_s \), and let \( z_0 \) be the part of \( z \) supported on nodes of \( X_s \), if any. Then if \( z_0 \) is principal (i.e. Cartier) on \( X_s \), the fibre \( q_{m-1}^{-1}(z) \) is birational to \( X_s \) and its general members are equal to \( z_0 \) locally at the nodes.

**Proof.** (i) is proven in [8], and also follows easily from our explicit model \( H_{m,m-1} \). As for (ii), we may suppose, in the notation of [9], that \( z_0 \) is of type \( I_i^0(a) \). Now if \( z' \in q_{m-1}^{-1}(z) \), then the part \( z'_0 \) of \( z' \) supported on nodes must have length \( n \) or \( n+1 \). In the former case \( z'_0 = z_0 \), while in the latter case \( z'_0 \) must equal \( Q_{i+1}^{n+1} \) by [9], Thm. 5 p. 438, in which case \( z' \) is unique, hence not general.

Next we define the fundamental transfer operation. Essentially, this takes cycles from \( X_B^{[m-1]} \) to \( X_B^{[m]} \), but we also allow the additional flexibility of twisting
by base classes via the \(m\)-th factor. Thus the **tautological transfer map** \(\tau_m\) is defined by

\[
\tau_m : A.(X_B^{[m-1]}) \otimes A.(X) \rightarrow A.(X_B^m)_Q,
\]

\[
\tau_m = q_{m*}(q_{m-1}^{*} \otimes a^*).
\]

Note that this operation raises dimension by 1 and preserves codimension. Suggestively, and a little abusively, we will write a typical decomposable element of the source of \(\tau_m\) as \(\gamma \beta(m)\) where \(\gamma \in A.(X_B^{[m-1]}), \beta \in A.(X)\). The following result which computes \(\tau_m\) is a key to our inductive computation of Chern numbers.

**Theorem 3.3. (Tautological transfer)** \(\tau_m\) takes tautological classes on \(X_B^{[m-1]}\) to tautological classes on \(X_B^m\). More specifically we have, for any class \(\beta \in A.(X)\):

(i) for any twisted polyblock diagonal class \(\Gamma_{\mu}[\alpha.], \alpha \in TS_{\mu}(H.(X)), w(\mu) = m-1, \)

\[
\tau_m(\Gamma_{\mu}[\alpha.]) = \Gamma_{\mu+1}[\alpha.],\beta
\]

where \(1_1\) is the distribution of weight 1 and support \(\{1\}\) and \(\alpha.\beta\) is the firmal symmetric multiplication:

(ii) for any twisted node scroll class \(F[\alpha] = F_j^{(n,m-1)}(\theta)[\alpha], \alpha \in T^{m-n-1}(X^\theta), \)

\[
\tau_m(F[\alpha] \beta(m)) = F_j^{n,m}[\alpha.\beta]
\]

(iii) for any twisted node section \(\Gamma^{(m-1)}.F[\alpha]\) with \(F[\alpha]\) as above,

\[
\tau_m(-\Gamma^{(m-1)}.F[\alpha] \beta(m)) = (\int_X \beta)F_j^{n+1,m}[\alpha] + (\Gamma^{(m)})F_j^{n,m}[\alpha.\beta]
\]

\[
= (\int_X \beta)F_j^{n+1,m}[\alpha] + (\Gamma^{(m)})F_j^{n,m}[\alpha.\beta]
\]

**Proof.** Part (i) is obvious. As for Part (ii), the flatness of \(q_{m-1}\) allows us to work over a general \(z \in F\) and then Corollary 3.2, (ii) allows us to assume that the added point is a general point on the fibre \(X_s\), which leads to (3.1.17).

As for (iii), we recall Corollary 7.8 of [6], which states that on \(F_j^{n,m-1}(\theta)\), we have

\[
-\Gamma^{(m-1)} \sim Q_j^{n,m-1} + p_{[m-n-1]}^*(D_j^{n,m-1}) \sim Q_j^{n,m-1} + p_{[m-n-1]}^*(D_j^{n,m-1}).
\]

Similarly, on \(F_j^{n,m}(\theta)\), we have

\[
-\Gamma^{(m)} \sim Q_j^{n,m} + p_{[m-n]}^*(D_j^{n,m}) \sim Q_j^{n,m} + p_{[m-n]}^*(D_j^{n,m}).
\]

Note that the transfer \(\tau_m\) takes \(D_j^{n,m-1}\) to \(D_j^{n,m}, \forall j\). Therefore, it will suffice to prove that

\[
\tau_m(Q_j^{n,m-1} \beta(m)) = (\int_X \beta)F_j^{n+1,m}[\alpha] + Q_j^{n,m}[\alpha.\beta]
\]

and similarly for \(j\), which case is similar (see below). It will suffice to prove this without the \(\alpha, \beta\) twisting.
To this end, note that, with \( Q = Q_{j+1}^{n,m-1} \), \( q_{m-1}^*Q \) splits in two parts, depending on whether the point \( w \) added to a scheme \( z \in Q \) is in the off-node or nodebound portion of \( z \). It is easy to see that the first part gives rise to the 2nd term in the RHS of (3.1.19).

The analysis of the other part, which leads to the first summand in the RHS of (3.1.19) is a bit more involved. Essentially, what has to be proved in the case at hand is that \( F_{j+1}^{m+1,m} \) appears with coefficient equal to 1. To begin with, it is easy to see that we may assume \( m = n + 1 \), in which case \( F \) is just \( \mathbb{P}^1 \), namely \( C_j^{m-1} \).

Now as in (3.1.9),, the special fibre of the cycle map on \( q_{m-1}^*(C_j^{m-1}) \), as a set, is given by \( \hat{C}_j^m \cap \hat{C}_j^{m-1} \cup \hat{C}_j^m \) and this coincides as a set with \( \Delta^{(m)} \cdot q_{m-1}^*(C_j^{m-1}) \). As \( \hat{C}_j^m \) collapses under \( q_m \), the proof will be complete if we can show that the multiplicity of \( C_j^m \) and \( C_j^{m+1} \) on \( \Delta^{(m)} \cdot q_{m-1}^*(C_j^{m-1}) \) are both equal to \( n = m - 1 \). We will do this for \( \hat{C}_j^{m+1} \) as the case of \( \hat{C}_j^m \) is similar and only notationally more cumbersome.

In that case, our assertion will be an elementary consequence of the equations on p. 440, l. 9-14 of [9], describing the local model \( H_{m,m-1} \), as well as those on p. 433, describing the analogous local model \( H_m \), to which equations we will be referring constantly in the remainder of the present proof. Note that \( c_{m-i} \) (resp. \( b_{i-1} \)) plays the role of the affine coordinate \( u_i/v_i \) (resp. \( u_{i-1}'/v_{i-1}' \)). Also our \( j + 1 \) is the \( i \) there. We work on \( q_{m-1}^*(C_j^{m-1}) \). Now to complete the proof, it will suffice to prove

**Claim**: Over a neighborhood of \( Q \), \( q_{m-1}^*(Q) \) contains \( \hat{C}_j^{m+1} \) with multiplicity 1.

To see this note that the defining equations of \( C_i^{m-1} \) on \( X_B^m \) are given by setting all \( a_k \) and \( d_k \) as well as \( c_{m-i} \) to zero. By loc. cit. p.433 l.9, this implies that we have \( b_1 = ... = b_{i-2} = 0 \) on \( q_{m-1}^*(C_j^{m-1}) \) as well. At a general point of \( C_j^{m-1} \), \( c_{m-i} \) is nonzero. Therefore we may consider \( c_{m-i} \) as a unit. By loc. cit. p.440, eq. (15), we conclude \( a_{m-i} = 0 \). From this we see easily that all \( a_k = d_k = 0 \) except \( d_{i-1} \), which is a local equation for \( \hat{C}_j^{m+1} \), while \( b_{i-1} \) is a coordinate along \( C_j^{m-1} \) having \( Q_{j+1}^{m-1} \) as its unique zero. Now by p.440 l. 14, \( b_{i-1} \) and \( d_{i-1} \) differ by the multiplicative unit \( -c_{m-i} \), therefore \( b_{i-1} \) generically cut out exactly \( \hat{C}_j^{m+1} \), which proves our Claim.

### 3.2. Full-flag transfer and Chern numbers

We are now ready to tackle the computation of Chern numbers, and in fact all polynomials in the Chern classes of the tautological bundle on the relative Hilbert scheme \( X_B^m \). The computation is based on passing from \( X_B^m \) to the corresponding full-flag Hilbert scheme \( W = W^m(X/B) \) studied in [8] and a **diagonalization theorem** for the total Chern class of (the pullback of) a tautological bundle on \( W \), expressing it either as a simple (factorable) polynomial in diagonal classes induced from the various \( X_B^n \), \( n \leq m \), plus base classes, or, more conveniently, as the product of the Chern class of a smaller tautological bundle and a diagonal class. Given this, we can compute
Chern numbers essentially by repeatedly applying the transfer calculus of the last section.

We start by reviewing some results from [8]. Let

\[ W^m = W^m(X/B) \xrightarrow{\pi(m)} B \]

denote the relative flag-Hilbert scheme of \( X/B \), parametrizing flags of subschemes

\[ z = (z_1 < ... < z_m) \]

where \( z_i \) has length \( i \) and \( z_m \) is contained in some fibre of \( X/B \). Let

\[ w^m : W^m \to X_B^{[m]}, w^{m,i} : W^m \to X_B^{[i]} \]

be the canonical (forgetful) maps. Let

\[ a_i : W^m \to X \]

be the canonical map sending a flag \( z \) to the 1-point support of \( z_i/z_{i-1} \) and

\[ a^m = \prod a_i : W^m \to X_B^m \]

their (fibred) product, which might be called the ‘ordered cycle map’. Let

\[ \mathcal{I}_m < \mathcal{O}_{X_B^{[m]} \times_B X} \]

be the universal ideal of colength \( m \). For any coherent sheaf on \( X \), set

\[ \lambda_m(E) = p_X^{[m],*}(p_X^*(E) \otimes (\mathcal{O}_{X_B^{[m]} \times_B X}/\mathcal{I}_m)) \]

These are called the tautological sheaves associated to \( E \); they are locally free if \( E \) is. Abusing notation, we will also denote by \( \lambda_m(E) \) the pullback of the tautological sheaf to appropriate flag Hilbert schemes mapping naturally to \( X_B^{[m]} \), such as \( W^m \) or \( X_B^{[m,m-1]} \). With a similar convention, set

\[ (3.2.1) \Delta^{(m)} = \Gamma^{(m)} - \Gamma^{(m-1)}. \]

The various tautological sheaves form a flag of quotients on \( W^m \):

\[ (3.2.2) \ldots \lambda_{m,i}(E) \to \lambda_{m,i-1}(E) \to \ldots \]

This flag makes possible a simple formula for the total Chern class of the tautological bundles, namely the following diagonalization theorem ([8], Cor. 3.2):

**Theorem 3.4.** The total Chern class of the tautological bundle \( \lambda_m(E) \) is given by

\[ (3.2.3) c(\lambda_m(E)) = \prod_{i=1}^m c(a_i^*(E)(-\Delta^{(i)})) \]

An analogue of this, more useful for our purposed, holds already on the flaglet Hilbert scheme. It can be proved in the same way, or as an easy consequence of Thm 3.4.
**Corollary 3.5.** We have an identity in \( A(X_B^{[m,m-1]})_\mathbb{Q} \):

\[
(3.2.4) \quad c(\lambda_m(E)) = c(\lambda_{m-1}(E))c(a_m^*(E)(-\Delta^{(m)})).
\]

**Proof.** By Theorem 3.4, the RHS and LHS pull back to the same class in \( W^m \). As the projection \( W^m \to X_B^{[m,m-1]} \) is generically finite, they agree mod torsion. \( \square \)

Motivated by this result we make the following definition.

**Definition 3.6.** Let \( R \) be a \( \mathbb{Q} \)-subalgebra of \( A(X) \) containing 1 and the canonical class \( \omega \). The Chern tautological ring on \( X_B^{[m]} \), denoted

\[
TC^m_R = TC^m_R(X/B),
\]

is the \( R \)-subalgebra of \( A(X_B^{[m]})_\mathbb{Q} \) generated by the Chern classes of \( \lambda_m(E) \) and the discriminant class \( \Gamma^{(m)} \).

**Remark 3.7.** If \( E \) is a line bundle, then it is easy to see from Theorem 3.4 that

\[
c_1(\lambda_m(E)) = mc_1(E) - \Gamma^{(m)}.
\]

The following is the main result of this paper.

**Theorem 3.8.** There is a computable inclusion

\[
(3.2.5) \quad TC^m_R \to T^m_R.
\]

More explicitly, any polynomial in the Chern classes of \( \lambda_m(E) \), in particular the Chern numbers, can be computably expressed as a linear combination of standard tautological classes: twisted diagonal classes, twisted node scrolls, and twisted node sections.

**Proof.** For \( m = 1 \) the statement is essentially vacuous. For \( m = 2 \) it is a consequence of the Module Theorem 2.1. For general \( m \), we assume inductively the result is true for \( m - 1 \). Given any polynomial \( P \) in the Chern classes of \( \lambda_m(E) \), Corollary 3.5 implies that we can write its pullback on \( X_B^{[m,m-1]} \) as a sum of terms of the form \( P^*_{X_B^{[m-1]}}(Q)(\Gamma^{(m)})^k.S \) where \( Q \in TC^{m-1}_R \). By induction, \( Q \in T^{m-1}_R \), so by the Transfer Theorem 3.3, \( \tau_m(Q) \in T^m_R \). By the projection formula and the Module Theorem 2.1, it follows that \( P \in T^m_R \). \( \square \)

**Remark 3.9.** This result suggests the natural question: is \( T^m_R \) a ring? more ambitiously, is the inclusion \( TC^m_R \to T^m_R \) an equality?

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4. Appendix: Examples

4.1. **Trisecants to one space curve.** If $X$ is a smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^3$, the virtual degree of its trisecant scroll, i.e. the virtual number of trisecant lines to $X$ meeting a generic line, is given by $c_3(L(\mathcal{O}_X(1)))$, which can be easily computed to be

$$c_3(L(\mathcal{O}_X(1))) = \frac{1}{6}(2d^3 - 12d^2 + 16d - 3d(2g - 2) + 6(2g - 2))$$

4.2. **Trisecants in a pencil.** With $X/B$ as above ($B$ a smooth curve), suppose

$$f : X \to \mathbb{P}^{2m-1}$$

is a morphism. One, quite special, class of examples of this situation arises as what we call a *generic rational pencil*; that is, generally, the normalization of the family of rational curves of fixed degree $d$ in $\mathbb{P}^r$ (so $r = 2m - 1$ here) that are incident to a generic collection $A_1, \ldots, A_k$ of linear spaces, with

$$(r + 1)d + r - 4 = \sum (\text{codim}(A_i) - 1);$$

see [7] and references therein, or [5] for an 'executive summary'. While our result seems new in this case, we note that it applies to curves of arbitrary genus.

Returning to the general situation, one expects a finite number $N_m$ of curves $f(X_b)$ to admit an $m$-secant $(m-2)$-plane, and this number can be evaluated as follows. Let $G = G(m-1, 2m)$ be the Grassmannian of $(m-2)$-planes in $\mathbb{P}^{2m-1}$, with rank-$m+1$ tautological subbundle $S$, and let $L = f^*\mathcal{O}(1)$. Then

$$m!N_m = \int_{W^m \times G} c_{m+1}(S^* \boxtimes L_m \lambda_m(L))$$

$$= \int_{W^m \times G} c_{m+1}(S^*(L(1)))c_{m+1}(S^*(L(2) - \Delta(2))) \cdots c_{m+1}(S^*(L(m) - \Delta(m)))$$

$$= \int_{W^m \times G} \prod_{i=1}^{m+1} \left( \sum_{j=0}^{m+1} \binom{m+1}{j} c_{m+1-j}(S^*(L^{(i)} - \Delta^{(i)})) \right)$$

$$= \sum_{\nu \cdot \omega} \int_{W^m} c_{u_1, \ldots, u_k} \cdots \cdot \cdots c_{w_1, \ldots, w_k} \cdots$$

where $c_{u, v, \cdots} = c_v c_w \cdots$, and, applied to $S^*$, represents the condition that an $(m-2)$-plane in $\mathbb{P}^{2m-1}$ meet a generic collection of planes of respective dimensions $u, v, \cdots$. Note that only terms with $j_1 + \cdots + j_k \leq k + 1, \forall k$, can contribute.

By the intersection calculus developed above, this number can be computed in terms of the characters

$$b = L^2, d = \deg_{\nu}(L), \omega^2, \sigma, \omega, L, \deg_{\nu}(\omega) = 2g - 2, g = \text{fibre genus};$$

in the generic rational pencil case, all these characters can be computed by recursion on $d$.

Suppose now that $m = 3$, where the only relevant $(j)$ are

$$(2, 1, 1), (1, 1, 2), (2, 0, 2), (1, 2, 1), (1, 0, 3), (0, 3, 1), (0, 2, 2), (0, 1, 3), (0, 0, 4).$$
In each of these cases, it is easy to see that the $G$ integral evaluates to 1. The $W$ integrals may be evaluated by the calculus developed above. The general procedure is to proceed inductively, each time transferring the leftmost factor from the $X^{(i)}_B$ from whence it came to $X^{(i+1)}_B$. We will repeatedly be using Corollaries 2.12 and 2.14, as well as standard projection formulas (for the symmetrization map). After the first transfer (to $X^{(2)}_B$), we will treat the resulting term as polynomial in $\Gamma^{(3)}$ and break things up according to the power of $\Gamma^{(3)}$ involved. The main multiplication rules to be used are the following. We will use the notation $[\alpha,...]$, for a base class $\alpha$, in place of $\Gamma^{(3)}[\alpha,...]$, where (1.) is a trivial (singleton blocks only) distribution. We also recall that $\Gamma^{(3)}[\alpha]$ is short for $\Gamma^{(3)}[\alpha,1]$.

**Multiplication rules**

(i)

$[\alpha,\beta,\gamma] \Gamma^{(3)} = 2(\Gamma^{(3)}[\alpha,\beta,\gamma] + \Gamma^{(3)}[\alpha,\gamma,\beta] + \Gamma^{(3)}[\beta,\gamma,\alpha])$

(ii)

$\Gamma^{(3)}[\alpha,\beta] \Gamma^{(3)} = \Gamma^{(3)}[\alpha] - \Gamma^{(3)}[\alpha,\beta]$

The detailed computation follows.

(2.1.1) that is, $L^{2}(L_{(2)} - \Delta^{(2)})(L_{(3)} - \Delta^{(3)})$. First,

$\tau_{2}(L^{2}_{(1)}(L_{(2)} - \Delta^{(2)})) = [L^{2}, L] - 2\Gamma^{(2)}[L^{2}]$.

Next,

$\Gamma^{(3)}[L^{2}, L] + 2\Gamma^{(3)}[L^{2}] = [L^{2}, L^{(2)}] - 2\Gamma^{(3)}[L^{2}, L] = bd^{2} - bd$,

$\Gamma^{(3)}[L^{2}, L] + 2\Gamma^{(3)}[L^{2}] = 4\Gamma^{(3)}[L^{2}, L] - 2\Gamma^{(3)}[L^{2}] = 2bd - 2b$

Thus the total is $bd^{2} - 3bd + 2b = b(d - 1)(d - 2)$.

(1.1.2) We treat this as a polynomial in $\Gamma^{(3)}$. The terms are as follows. Degree 0:

$4\Gamma^{(3)}[L^{2}, L] + 4\Gamma^{(3)}[L, \omega, L] + [L^{2}, L] - 2\Gamma^{(3)}[L, L^{2}] = 2bd + 2dL.\omega + bd^{2} - bd$

$= bd + bd^{2} + 2dL.\omega$

Degree 1: similarly

$-2\tau_{3}(2\Gamma^{(2)}[L^{2}] + 2\Gamma^{(2)}[L, \omega] + [L^{2}]L_{(3)} - 2\Gamma^{(2)}[L]L_{(3)})\Gamma^{(3)} =$

$-2(2\Gamma^{(3)}[L^{2}] + 2\Gamma^{(3)}[L, \omega] + [L^{(3)}] - 2\Gamma^{(3)}[L, L])\Gamma^{(3)} =$

$-2(2\Gamma^{(3)}[L^{2}] + 4\Gamma^{(3)}[L, \omega] + 6\Gamma^{(3)}[L^{2}, L] - 2\Gamma^{(3)}[L^{2}] + 2\Gamma^{(3)}[L, \omega, L]) =$

$-2(2b + 2L.\omega + 3bd - 2b + dL.\omega) = -2(2L.\omega + 3bd + dL.\omega)$

Degree 2:

$(L^{(2)} - 2\Gamma^{(3)}[L])(\Gamma^{(3)})^{2} = (2\Gamma^{(3)}[L^{2}] + 4\Gamma^{(3)}[L, L] - 2\Gamma^{(3)}[L] + 2\Gamma^{(3)}[L, \omega, L])\Gamma^{(3)} =$

$2\Gamma^{(3)}[L^{2}] + 4\Gamma^{(3)}[L, L] - 4\Gamma^{(3)}[L, \omega, L] + 6\Gamma^{(3)}[L, \omega, L] + 2\Gamma^{(3)}[L, \omega, L] = 6b - 2dL.\omega + 8L.\omega$

Total: $-5bd + bd^{2} + 6b - 2dL.\omega + 4L.\omega$

(2.0.2) This case is similar to (1.1.2) and easier. The result is

$-2bd - b(2g - 2) + 2b$.

(1.2.1) This is again quite similar to the (2.1.1) case treated above, and yields

$(bd - 2b - L.\omega)(d - 2)$. 

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Again we consider this as a polynomial in $\Gamma^{(3)}$ and compute term by term. For degree 0 we have

$$\tau_3(6\Gamma^{(2)}[L](L^{(3)})^2 - 6\Gamma^{(2)}[L, \omega]L^{(3)}) = 3bd - 3dL, \omega$$

Degree 1:

$$-3([L, L^2] + 4\Gamma^{(3)}[L, L] - 2\Gamma^{(3)}[\omega, L])^{(3)} =$$

$$-3(2\Gamma^{(3)}[L, L^2] + 2\Gamma^{(3)}[L^2, L] + 4\Gamma^{(3)}[L^2] - 4\Gamma^{(3)}[\omega, L, L] - 2\Gamma^{(3)}[\omega, L]) =$$

$$-3(3b + 3b(2g - 2) - 3dL - 3dL, \omega - \omega, L)$$

For degree 2:

$$3([L^{(2)}) + 2\Gamma^{(3)}[L])([\Gamma^{(3)}]^2) = 3(2\Gamma^{(3)}[L^2] + 4\Gamma^{(3)}[L, L] + 2\Gamma^{(3)}[L] - 2\Gamma^{(3)}[\omega, L])^{(3)} =$$

$$3(2\Gamma^{(3)}[L^2] + 4\Gamma^{(3)}[L^2] - 4\Gamma^{(3)}[\omega, L, L] + 6\Gamma^{(3)}[\omega, L] - 2\Gamma^{(3)}[\omega, L]) =$$

$$3(2b + 4b - 2dL - 6L, L - 2L, \omega, L)$$

For degree 3: by Corollary 2.14, we get

$$24\omega, L - d\omega^2$$

Summing up, we get

$$\begin{align*}
-3bd - 3dL, \omega + 6b + 6L, \omega - d\omega^2
\end{align*}$$

Expanding, we get

$$\tau_2((L^{(2)} - \Delta^{(2)})^3) = -6\tau^{(2)}[L^2] - 6\Gamma^{(2)}[\omega, L] - 2(\Gamma^{(2)})^3 = -3b - 3L, \omega - (\omega^2 - \sigma)$$

Since this is a point cycle, multiplying by $L_{(3)} - \Delta^{(3)}$ or $L_{(3)} - \Gamma^{(3)}$ again just multiplies the coefficient by $d - 2$, for a total of $(-3b - 3L, \omega - (-\sigma + \omega^2))(d - 2)$.

Again working degree by degree in $\Gamma^{(3)}$, we get:

Degree 0:

$$\tau_3(2\tau_2((L^{(2)} - 2L).\Gamma^{(2)} + (\Gamma^{(2)} - 2L).\Gamma^{(2)})L^{(3)}) + \tau_3(\tau_2(L_{(2)} - 2L).\Gamma^{(2)} + (\Gamma^{(2)} - 2L).\Gamma^{(2)})L^{(3)}) =$$

$$4\Gamma^{(3)}[L^2, L] + 8\Gamma^{(3)}[\omega, L, L] + [\omega^2 - \sigma, L] - 4\Gamma^{(3)}[L, L^2] - 2\Gamma^{(3)}[\omega, L^2] =$$

$$4dL, L + 2d(\omega^2 - \sigma) - b(2g - 2)$$

Degree 1:

$$-2\tau_3(\tau_2((L^{(2)} - 2L).\Gamma^{(2)} + (\Gamma^{(2)} - 2L).\Gamma^{(2)})L^{(3)}) + \tau_3(\tau_2(L_{(2)} - 2L).\Gamma^{(2)} + (\Gamma^{(2)} - 2L).\Gamma^{(2)})L^{(3)}) =$$

$$-2(2\Gamma^{(3)}[L^2] + 4\Gamma^{(3)}[\omega, L] + [\omega^2 + \sigma] + [L^2, L] - 4\Gamma^{(3)}[L, L] - 2\Gamma^{(3)}[\omega, L])^{(3)} =$$

$$-2(2b + 4\omega, L + 2(\omega^2 - \sigma) + 2bd - 4b + 2dL - 2L, \omega, L) =$$

$$4b - 4L, L - 4(\omega^2 - \sigma) - 4bd - 4dL, \omega - 4dL, \omega$$

Degree 2:

$$((L^2) - 4\Gamma^{(3)}[L] - 2\Gamma^{(3)}[\omega] + F^{(2,1)(0)} + F^{(2,0)(1)})(\Gamma^{(3)})^2 =$$

$$(4\Gamma^{(3)}[L^2] + 2\Gamma^{(3)}[L, L^2] - 4\Gamma^{(3)}[L, L] + 4\Gamma^{(3)}[\omega, L] - 2\Gamma^{(3)}[\omega, L] + 2\Gamma^{(3)}[\omega^2])^{(3)} =$$

$$6(4 + 2b - (2g - 2) + 12L, L + 4L, \omega + 6\omega^2 + 2\omega^2 - 6\sigma =$$

$$6b - b(2g - 2) + 16L, L + 8\omega^2 - 6\sigma)$$

The total is

$$-2bd + 10b + 12L, L + 4\omega^2 - 2\sigma - b(2g - 2)$$

Again we consider this as polynomial in $\Gamma^{(3)}$. The terms are:

Degree 0:

$$\tau_3((L^{(2)} - \Gamma^{(2)})(3L^{(3)}\Gamma^{(2)} + 3L^{(3)}\Gamma^{(2)})^2) = 6\Gamma^{(3)}[L, L^2] + 6\Gamma^{(3)}[\omega, L^2] - 6\Gamma^{(3)}[\omega, L, L] - 3(\omega^2 - \sigma), L =$$

$$3db + 3b(2g - 2) - 3dL, L - 3d(\omega^2 - \sigma)$$
degree 0:

\[-3\tau_3((L(2) - \Gamma(2))(L(3) + \Gamma(3))^2)\Gamma(3) =
3(-[L, L^2] + 2\Gamma(3)[1, L^2] - 4\Gamma(3)[L, L] + 4\Gamma(3)[\omega, L] - 2\Gamma(3)[\omega, L] + [\omega^2 - \sigma])\Gamma(3) =
3(-2bd - b(2g - 2) - 2b + 2d\omega, L + 2d\omega^2 + 2\omega, L + 2(\omega^2 - \sigma))\]

degree 1:

\[-3\tau_3((L(2) - \Gamma(2))(L(3) + \Gamma(3))^2)\Gamma(3)^3 =
3([L, L] - 2\Gamma(3)[1, L] + 2\Gamma(3)[L] + 2\Gamma(3)[\omega])\Gamma(3)^3 =
3((2\Gamma(3)[L^2] + 4\Gamma(3)[L, L] + 2\Gamma(3)[\omega, L] - 2\Gamma(3)[L, L] - 2\Gamma(3)[L, L] + 2\Gamma(3)[L, L] + 2\Gamma(3)[\omega, L] + 6\Gamma(3)[L, L] - 2\Gamma(3)[L, L] - 6\Gamma(3)[L, L] - 2\Gamma(3)[\omega^2] - 6\Gamma(3)[\omega^2]) = 3(6b - 2dL, \omega - d\omega^2 - 8\omega^2)\]

degree 2:

\[-3\tau_3((L(2) - \Gamma(2))(L(3) + \Gamma(3))^3) = -(L - 2\Gamma(3))(\Gamma(3))^3 =
(\text{using 2.14 and 2.25})\]

\[= 12L, \omega - d\omega^2 + 26\omega^2 + 2(-6\sigma - 3\sigma) + d\sigma\]

Total:

\[-3db - 3d\sigma, L - d\omega^2 + 4d\sigma + 12b + 18L, L + 8\omega^2 - 24\sigma\]

(0,0,4) Proceeding as above, we get: degree 0:

\[\tau_3(12(\Gamma(2))^2(L(3))^2 + 8(\Gamma(2))^3L(3)) = -12\Gamma(3)[\omega, L^2] + 8\Gamma(3)[\omega^2, L] + 4[\omega^2 - \sigma, L] =
-6b(2g - 2) + 4d\omega^2 + 4d(\omega^2 - \sigma).\]

degree 1:

\[-4\tau_3((\Gamma(2) + L(3))^3)\Gamma(3) = -4\tau_3(3\Gamma(3)L(3)^2 + 3\Gamma(3)^2L(3) + (\Gamma(2)^3)\Gamma(3) =
-4(6\Gamma(3)[1, L^2] - 6\Gamma(3)[\omega, L] + [\omega^2 - \sigma])\Gamma(3) =
-4(6\Gamma(3)[L^2] - 6\Gamma(3)[\omega, L^2] - 6\Gamma(3)[\omega, L] + 6\Gamma(3)[\omega^2, L] + 2[\omega^2 - \sigma]) =
-24b + 12b(2g - 2) + 24\omega, L - 8\omega^2 + 8\sigma.\]

degree 2:

\[+6\tau_3((L(3) + \Gamma(3))^2)\Gamma(3)^2 = 6\tau_3(L(3)^2 + 2L(3, \Gamma(2) + (\Gamma(2)^2)\Gamma(3)^2 =
6([L^2] + 4\Gamma(3)[L] - 2\Gamma(3)[\omega] + \frac{1}{2}(F_1^{(2,1,0)} + F_1^{(2,0,1)}))\Gamma(3)^2 =
6(4\Gamma(3)[L^2] + 2\Gamma(3)[1, L^2] - 4\Gamma(3)[L, L] - 4\Gamma(3)[L, L] - 2\Gamma(3)[\omega, L] + 2\Gamma(3)[\omega^2] + \frac{1}{2}(F_1^{(2,1,0)} + F_1^{(2,0,1)}))\Gamma(3)^3 =
6(4\Gamma(3)[L^2] + 2\Gamma(3)[L^2] - 2\Gamma(3)[\omega, L] - 12\Gamma(3)[L, L] - 4\Gamma(3)[L, L] + 6\Gamma(3)[\omega^2] + 2\Gamma(3)[\omega^2] - 3\sigma) =
6(4b + 2b - b(2g - 2) - 12L, \omega - 4L, \omega + 6\omega^2 + 6\omega^2 + 3\sigma) = 6(6b - b(2g - 2) - 16L, \omega + 8\omega^2 - 3\sigma)\]

degree 3:

\[-4\tau_3((\Gamma(2) + L(3))^3)\Gamma(3)^3 = -4(2\Gamma(3)[L])\Gamma(3)^3 =
-4(-24\Gamma(3)[\omega, L] + 2\Gamma(3)[\omega^2, L] + 2(\Gamma(3))^4) = -4(-24\omega, L + d\omega^2 + 2(12\Gamma(3)[\omega^2] + \Gamma(3)[\omega^2] - 6\sigma - 3\sigma) =
-4(-24\omega, L + d\omega^2 - d\sigma + 26\omega^2 - 18\sigma)\]

degree 4:

\[\tau_3(\tau_2(1))(\Gamma(3)^4 = 6(13\omega^2 - 9\sigma)\]

Total:

\[12b + 24\omega, L + 14\omega^2\]
4.3. **Double points.** Let \( X / B \) be an arbitrary nodal family and \( f : X \to P^n \) a morphism. Consider the relative double points of \( f \), i.e. double points on fibres. This locus is given on \( X^{[2]}_B \) as the degeneracy locus of a bundle map

\[ \phi : (n + 1)O \to \Lambda_2(L), \ L := f^*O(1). \]

By Porteous, the virtual fundamental class of this locus is given by the Segre class \( s_0(\Lambda_2(L)^* \right) \), which equals

\[ (4.3.1) \quad \sum_{i = 0}^{n} (L_1)^{n-i}(L_2 - \Gamma)^i, \Gamma = \Gamma^{[2]}. \]

The powers of \( \Gamma \) can be evaluated using Corollary 2.23. Pushing the result down to \( X^{[2]}_B \) for simplicity yields

\[ \sum_{i = 0}^{n} L_1^{n-i}L_2 - \sum_{i = 0}^{n} L_1^{n-i}(\Gamma^{[\omega^j,j]} + \sum_{s,k} 1 \psi^j_{\psi_y^k})L_i - \] (4.3.5)

To describe the direct image of this on \( B \), we need some notation. Recall that \( \kappa_j = \pi_*(\omega^{j+1}) \). Extending this, we may set

\[ (4.3.2) \quad \kappa_j(L) = \pi_*(L^{j+1}), \kappa_{k,j}(L, M) = \pi_*(L^{j+1}M^{j+1}). \]

Note that in our case \( \kappa_j(L) \) may be interpreted as the class of the locus of curves meeting a generic \( P^n - j \). Also, for each boundary datum \( (T_s, \delta_s, \theta_s) \), \( T_s \) admits a map to \( P^n \) via either the \( x \) or \( y \)-section (the two maps are the same), via which we can pull back \( L / \) which corresponds to the locus of boundary curves whose node \( \delta_i \) meets \( P^n - j \). Then pushing the above down to \( B \) yields

\[ (4.3.3) \quad 2m_{2,B} = (-1)^n \sum_{i = 1}^{n-k} \kappa_i(L) - \sum_{i = 1}^{n-k} (\Gamma^{[\omega^j,j]} + \sum_{s,k} 1 \psi^j_{\psi_y^k})c_{n-i}(T_Y) \]

(4.3.4)

More generally, for any smooth variety \( Y \) of dimension \( n \) and map \( f : X \to Y \), one can use the double-point formula of [8], Th. 3.3ter, p. 1208, to evaluate the class of the double-point locus in \( X^{[2]}_B \) in terms of the diagonal class \( \Delta_Y \) on \( Y \times Y \) as

\[ 2m(f)X^{[2]}_B = (f^2)^*(\Delta_Y) - \sum_{i \geq 1} (-\Gamma^{[\omega^j,j]} + \sum_{s,k} 1 \psi^j_{\psi_y^k})c_{n-i}(T_Y) \]

(4.3.5)

Applying this to (4.3.4), we note that \( L \) pulls back to \( \omega \), which meets each \( \theta_i \) trivially, so we obtain

\[ 2mX^{[2]}_B(f) = \sum (-1)^k \lambda_k [\omega^1, \omega^{g-1-k}] + \sum_{i \geq \max(1,k)} (-1)^k \lambda_k [\omega^{g-2-k}] \]

(4.3.6)
Multiplying by $\omega_1$ and projecting to $B$ we obtain (compare [4. §7]):

\[(4.3.7) \quad 2(2g-2)m_2 = \sum (-1)^{k} \lambda_k \kappa_{i-1} \kappa_{g-2-i-k} + (2^{g-1} - 1) \kappa_{g-2} + \sum_{k=1}^{g-1} (-1)^{k} 2^{g-1-k} \lambda_k \kappa_{g-2-k}\]

where we set $\kappa_0 = 2g - 2$ for simplicity. This formula is correct over the locus $\mathcal{M}_0 \cup \Delta^0_0$ of curves with at most 1 nonseparating node, but breaks down over the curves with a separating node or a separating pair of nodes, because there naive notion of canonical curve in $\overline{P}^{g-1}$ is ill-behaved and requires substantial modification. This work is currently in progress.

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