The algebra $U^+_q$ and its alternating central extension $U^+_q$

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Abstract

Let $U^+_q$ denote the positive part of the quantized enveloping algebra $U_q(\hat{sl}_2)$. The algebra $U^+_q$ has a presentation involving two generators $W_0$, $W_1$ and two relations, called the $q$-Serre relations. In 1993 I. Damiani obtained a PBW basis for $U^+_q$, consisting of some elements $\{E_{n\delta+\alpha_0}\}_{n=0}^\infty$, $\{E_{n\delta+\alpha_1}\}_{n=0}^\infty$, $\{E_n\}_{n=1}^\infty$. In 2019 we introduced the alternating central extension $U^+_q$ of $U^+_q$. We defined $U^+_q$ by generators and relations. The generators, said to be alternating, are denoted $\{W_{-k}\}_{k=0}^\infty$, $\{W_{k+1}\}_{k=0}^\infty$, $\{G_{k+1}\}_{k=0}^\infty$, $\{\tilde{G}_{k+1}\}_{k=0}^\infty$. Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $U^+_q$ generated by $W_0$, $W_1$. It is known that there exists an algebra isomorphism $U^+_q \rightarrow \langle W_0, W_1 \rangle$ that sends $W_0 \mapsto W_0$ and $W_1 \mapsto W_1$. Via this isomorphism we identify $U^+_q$ with $\langle W_0, W_1 \rangle$. In our main result, we express the Damiani PBW basis elements in terms of the alternating generators. We give the answer in terms of generating functions.

Keywords. Alternating central extension; PBW basis; $q$-Serre relations.

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1 Introduction

We will be discussing the positive part $U^+_q$ of the quantized enveloping algebra $U_q(\hat{sl}_2)$. The algebra $U^+_q$ is associative an infinite-dimensional. It has a presentation involving two generators $W_0$, $W_1$ and two relations, called the $q$-Serre relations:

$[W_0, [W_0, [W_0, [W_1, [W_1, [W_0, W_0]]]]]] = 0,$

$[W_1, [W_1, [W_1, [W_0, W_0]]]] = 0.$

In [7] I. Damiani obtained a Poincaré-Birkhoff-Witt (or PBW) basis for $U^+_q$. The PBW basis elements are denoted

$\{E_{n\delta+\alpha_0}\}_{n=0}^\infty$, $\{E_{n\delta+\alpha_1}\}_{n=0}^\infty$, $\{E_n\}_{n=1}^\infty$. (1)

We will be discussing the generating functions

$E^-(t) = \sum_{n=0}^\infty E_{n\delta+\alpha_0} t^n$, $E^+(t) = \sum_{n=0}^\infty E_{n\delta+\alpha_1} t^n$,

$E(t) = \sum_{n=0}^\infty E_{n\delta} t^n$, $E_{0\delta} = -(q - q^{-1})^{-1}$. 

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In [13] we introduced a type of element in $U_q^+$, said to be alternating. By [13] Lemma 5.11, each alternating element commutes with exactly one of $W_0$, $W_1$, $[W_1, W_0]_q$, $[W_0, W_1]_q$. This gives four types of alternating elements, denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$ 

By [13] Lemma 5.11] the alternating elements of each type mutually commute.

We obtained the alternating elements in the following way. Consider the free algebra $V$ on two generators $x, y$. The standard (linear) basis for $V$ consists of the words in $x, y$. In [10][11], M. Rosso introduced an algebra structure on $V$, called a $q$-shuffle algebra. For $u, v \in \{x, y\}$ their $q$-shuffle product is $u \star v = uv + q^{\langle u, v \rangle}vu$, where $\langle u, v \rangle = 2$ (resp. $\langle u, v \rangle = -2$) if $u = v$ (resp. $u \neq v$). Rosso gave an injective algebra homomorphism $\tilde{\pi}$ from $U_q^+$ into the $q$-shuffle algebra $V$, that sends $W_0 \mapsto x$ and $W_1 \mapsto y$. By [13] Definition 5.2] the map $\tilde{\pi}$ sends

$$W_0 \mapsto x, \quad W_{-1} \mapsto xyx, \quad W_{-2} \mapsto xyxy, \quad \ldots$$
$$W_1 \mapsto y, \quad W_2 \mapsto yxy, \quad W_3 \mapsto yxyxy, \quad \ldots$$
$$G_1 \mapsto yx, \quad G_2 \mapsto yxyx, \quad G_3 \mapsto yxyyxx, \quad \ldots$$
$$\tilde{G}_1 \mapsto xy, \quad \tilde{G}_2 \mapsto xyxy, \quad \tilde{G}_3 \mapsto xyxyx, \quad \ldots$$

In [13] we used $\tilde{\pi}$ to obtain many relations involving the alternating elements; the main relations are listed in Definition 6.1 below and [13] Proposition 8.1. In [13] Section 11] we described how the alternating elements are related to the elements (11).

In [14] we defined an algebra $U_q^+$ by generators and relations in the following way. The generators, said to be alternating, are denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$ 

The relations are the ones in Definition 6.1] By construction there exists a surjective algebra homomorphism $U_q^+ \rightarrow U_q^+$ that sends

$$W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad G_k \mapsto G_k, \quad \tilde{G}_k \mapsto \tilde{G}_k$$
for $k \in \mathbb{N}$. In a moment, we will see that this map is not injective. Denote the ground field by $F$ and let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $F[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $F$. For notational convenience define $z_0 = 1$. In [14] Lemma 3.6, Theorem 5.17] we displayed an algebra isomorphism $\varphi : U_q^+ \rightarrow U_q^+ \otimes F[z_1, z_2, \ldots]$ that sends

$$W_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \quad W_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$
$$G_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \quad \tilde{G}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$
for $n \in \mathbb{N}$. In particular, $\varphi$ sends $W_0 \mapsto W_0 \otimes 1$ and $W_1 \mapsto W_1 \otimes 1$. Following [14] we call $U_q^+$ the alternating central extension of $U_q^+$. 

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In [14] we obtained the following results about the center $\mathcal{Z}$ of $\mathcal{U}_q^+$. By [14, Lemma 5.10] the map $\varphi$ sends $\mathcal{Z} \mapsto 1 \otimes \mathbb{F}[z_1, z_2, \ldots]$. For $n \geq 1$ define

$$Z_n^\vee = \sum_{k=0}^{n} g_k \tilde{g}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k}.$$ 

For notational convenience define $Z_0^\vee = 1$. By [14, Definition 5.5, Proposition 6.2] the subalgebra $\mathcal{Z}$ is generated by $\{Z_n^\vee\}_{n=1}^{\infty}$. By [14, Lemma 5.4], for $n \in \mathbb{N}$ the map $\varphi$ sends $Z_n^\vee \mapsto 1 \otimes z_n^\vee$ where $z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}$. By [14, Corollary 6.3] the elements $\{Z_n^\vee\}_{n=1}^{\infty}$ are algebraically independent.

Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $W_0, W_1$. By [14, Proposition 6.4] there exists an algebra isomorphism $\mathcal{U}_q^+ \rightarrow \langle W_0, W_1 \rangle$ that sends $W_0 \mapsto W_0$ and $W_1 \mapsto W_1$. By [14, Proposition 6.5] the multiplication map

$$\langle W_0, W_1 \rangle \otimes \mathcal{Z} \rightarrow \mathcal{U}_q^+$$

$$w \otimes z \mapsto wz$$

is an algebra isomorphism. By [14, Theorem 10.2] the alternating generators in order

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}$$

(2)

give a PBW basis for $\mathcal{U}_q^+$. We now summarize the main results of the present paper. For the rest of this section, we identify the algebra $\mathcal{U}_q^+$ with $\langle W_0, W_1 \rangle$ via the isomorphism mentioned above. Our goal is to elegantly express the elements (1) in terms of the alternating generators for $\mathcal{U}_q^+$. To accomplish the goal, we first adjust the PBW basis (2) by modifying the ordering as follows. We show that the alternating generators in order

$$\{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$$

(3)

give a PBW basis for $\mathcal{U}_q^+$. This PBW basis induces a basis for $\mathcal{U}_q^+$, in which we will express the elements (1). We give our answer in terms of generating functions. Define

$$W^-(t) = \sum_{n=0}^{\infty} W_{-n} t^n,$$

$$W^+(t) = \sum_{n=0}^{\infty} W_{n+1} t^n,$$

$$G(t) = \sum_{n=0}^{\infty} G_n t^n, \quad \tilde{G}(t) = \sum_{n=0}^{\infty} \tilde{G}_n t^n, \quad G_0 = \tilde{G}_0 = 1.$$

Further define $Z^\vee(t) = \sum_{n \in \mathbb{N}} Z_n^\vee t^n$. By construction

$$Z^\vee(t) = G(q^{-1}t)\tilde{G}(qt) - qtW^-(q^{-1}t)W^+(qt).$$

We obtain the factorization

$$Z^\vee(t) = -(q - q^{-1})\tilde{G}(q^{-1}t)E(\xi t)\tilde{G}(qt),$$

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where \( \xi = -q^2(q - q^{-1})^{-2} \). Using this factorization we obtain

\[
E^-(t) = \mathcal{W}^-(q^{-1} \xi^{-1} t)(\tilde{G}(q^{-1} \xi^{-1} t))^{-1},
\]
\[
E^+(t) = \mathcal{W}^+(q \xi^{-1} t)(\tilde{G}(q \xi^{-1} t))^{-1},
\]
\[
E(t) = -\frac{Z^\vee(\xi^{-1} t)(\tilde{G}(q^{-1} \xi^{-1} t))^{-1}(\tilde{G}(q \xi^{-1} t))^{-1}}{q - q^{-1}}.
\]

The above three equations effectively express the elements \( \mathfrak{H} \) in the basis for \( U_q^+ \) induced by the PBW basis \( \mathfrak{B} \). Using the above three equations and the relations in Definition 6.1, we recover the previously known relations between \( E^\pm(t) \), \( E(t) \).

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the definition and basic facts about \( U_q^+ \). In Section 4 we recall the PBW basis for \( U_q^+ \) due to Damiani, and give the corresponding reduction rules. In Section 5 we express these reduction rules in terms of the generating functions \( E^\pm(t) \), \( E(t) \). In Section 6 we recall the definition and basic facts about \( U_q^+ \). In Section 7 we express the defining relations for \( U_q^+ \) in terms of the generating functions \( \mathcal{W}^\pm(t) \), \( \mathcal{G}(t) \), \( \tilde{G}(t) \). In Section 8 we obtain a PBW basis for \( U_q^+ \), and give the corresponding reduction rules. In Section 9 we describe the center of \( U_q^+ \) and recall an earlier PBW basis for \( U_q^+ \) and give the corresponding reduction rules.

## 2 Preliminaries

We now begin our formal argument. Throughout the paper, the following notational conventions are in effect. Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and integers \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \). Let \( \mathbb{F} \) denote a field. Every vector space and tensor product discussed in this paper is over \( \mathbb{F} \). Every algebra discussed in this paper is associative, over \( \mathbb{F} \), and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Let \( \mathcal{A} \) denote an algebra. By an automorphism of \( \mathcal{A} \) we mean an algebra isomorphism \( \mathcal{A} \to \mathcal{A} \). The algebra \( \mathcal{A}^{\text{opp}} \) consists of the vector space \( \mathcal{A} \) and the multiplication map \( \mathcal{A} \times \mathcal{A} \to \mathcal{A} \), \( (a, b) \to ba \). By an antiautomorphism of \( \mathcal{A} \) we mean an algebra isomorphism \( \mathcal{A} \to \mathcal{A}^{\text{opp}} \).

We will be discussing generating functions. Let \( \mathcal{A} \) denote an algebra and let \( t \) denote an indeterminate. For a sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements in \( \mathcal{A} \), the corresponding generating function is

\[
a(t) = \sum_{n \in \mathbb{N}} a_n t^n.
\]

The above sum is formal; issues of convergence are not considered. We call \( a(t) \) the generating function over \( \mathcal{A} \) with coefficients \( \{a_n\}_{n \in \mathbb{N}} \). For generating functions \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) and
\[ b(t) = \sum_{n \in \mathbb{N}} b_n t^n \] over \( \mathcal{A} \), their product \( a(t)b(t) \) is the generating function \( \sum_{n \in \mathbb{N}} c_n t^n \) such that \( c_n = \sum_{i=0}^{n} a_i b_{n-i} \) for \( n \in \mathbb{N} \). The set of generating functions over \( \mathcal{A} \) forms an algebra. The following result is readily checked.

**Lemma 2.1.** Let \( \mathcal{A} \) denote an algebra. A generating function \( a(t) = \sum_{n \in \mathbb{N}} a_n t^n \) over \( \mathcal{A} \) is invertible if and only if \( a_0 \) is invertible in \( \mathcal{A} \). In this case \( (a(t))^{-1} = \sum_{n \in \mathbb{N}} b_n t^n \) where \( b_0 = a_0^{-1} \) and for \( n \geq 1 \),

\[ b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k}. \]

**Example 2.2.** Referring to Lemma 2.1 assume that \( a_0 = 1 \). Then

\[
\begin{align*}
    b_0 &= 1, \\
    b_1 &= -a_1, \\
    b_2 &= a_1^2 - a_2, \\
    b_3 &= 2a_1 a_2 - a_1^3 - a_3, \\
    b_4 &= a_1^4 + 2a_1 a_3 + a_2^2 - 3a_1^2 a_2 - a_4.
\end{align*}
\]

**Definition 2.3.** (See [7, p. 299].) Let \( \mathcal{A} \) denote an algebra. A Poincaré-Birkhoff-Witt (or PBW) basis for \( \mathcal{A} \) consists of a subset \( \Omega \subseteq \mathcal{A} \) and a linear order \( < \) on \( \Omega \) such that the following is a basis for the vector space \( \mathcal{A} \):

\[
a_1 a_2 \cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n. \tag{4}
\]

We interpret the empty product as the multiplicative identity in \( \mathcal{A} \).

**Definition 2.4.** We refer to the PBW basis \( \Omega, < \) from Definition 2.3. For any ordered pair \( a, b \) of elements in \( \Omega \) such that \( a > b \), the corresponding reduction rule is the equation that expresses the product \( ab \) as a linear combination of the basis elements from (4). The reduction rule is called trivial whenever \( a, b \) commute.

**Definition 2.5.** Let \( \{z_n\}_{n=1}^{\infty} \) denote mutually commuting indeterminates. Let \( \mathbb{F}[z_1, z_2, \ldots] \) denote the algebra consisting of the polynomials in \( z_1, z_2, \ldots \) that have all coefficients in \( \mathbb{F} \). For notational convenience, define \( z_0 = 1 \).

Throughout the paper, we fix a nonzero \( q \in \mathbb{F} \) that is not a root of unity. Recall the notation

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}.
\]

### 3 The algebra \( U^+_q \)

In this section we recall the algebra \( U^+_q \).

For elements \( X, Y \) in any algebra, define their commutator and \( q \)-commutator by

\[
[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX.
\]

Note that

\[
[X, [X, [X, Y]_q]_q^{-1}] = X^2Y - [3]_q X^2 Y X + [3]_q XY X^2 - Y X^3.
\]
Definition 3.1. (See [9] Corollary 3.2.6.) Define the algebra $U_q^+$ by generators $W_0$, $W_1$ and relations

\begin{align}
[W_0, [W_0, [W_0, W_1]_{q^{-1}}]] &= 0, \\
[W_1, [W_1, [W_1, W_0]_{q^{-1}}]] &= 0.
\end{align}

We call $U_q^+$ the positive part of $U_q(\hat{\mathfrak{sl}}_2)$. The relations (5), (6) are called the $q$-Serre relations.

We mention some symmetries of $U_q^+$.

Lemma 3.2. There exists an automorphism $\sigma$ of $U_q^+$ that sends $W_0 \leftrightarrow W_1$. Moreover $\sigma^2 = id$, where id denotes the identity map.

Lemma 3.3. (See [12] Lemma 2.2.) There exists an antiautomorphism $\dagger$ of $U_q^+$ that fixes each of $W_0$, $W_1$. Moreover $\dagger^2 = id$.

Lemma 3.4. The maps $\sigma$, $\dagger$ commute.

Proof. This is readily checked. \qed

Definition 3.5. Let $\tau$ denote the composition of $\sigma$ and $\dagger$. Note that $\tau$ is an antiautomorphism of $U_q^+$ that sends $W_0 \leftrightarrow W_1$. We have $\tau^2 = id$.

4 A PBW basis for $U_q^+$

In [7], Damiani obtained a PBW basis for $U_q^+$ that involves some elements

\begin{align}
\{E_{n\delta+\alpha_0}\}_{n=0}^\infty, & \quad \{E_{n\delta+\alpha_1}\}_{n=0}^\infty, \quad \{E_{n\delta}\}_{n=1}^\infty.
\end{align}

These elements are recursively defined as follows.

\begin{align}
E_{\alpha_0} &= W_0, \\
E_{\alpha_1} &= W_1, \\
E_\delta &= q^{-2}W_1W_0 - W_0W_1,
\end{align}

and for $n \geq 1$,

\begin{align}
E_{n\delta+\alpha_0} &= \frac{[E_{\delta}, E_{(n-1)\delta+\alpha_0}]}{q + q^{-1}}, \\
E_{n\delta+\alpha_1} &= \frac{[E_{(n-1)\delta+\alpha_1}, E_{\delta}]}{q + q^{-1}}, \\
E_{n\delta} &= q^{-2}E_{(n-1)\delta+\alpha_1}W_0 - W_0E_{(n-1)\delta+\alpha_1}.
\end{align}

Proposition 4.1. (See [7] p. 308.) A PBW basis for $U_q^+$ is obtained by the elements (7) in the linear order

\begin{align}
E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < E_\delta < E_{2\delta} < E_{3\delta} < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.
\end{align}

We mention a variation on the formula (10). By [7] p. 307] the following holds for $n \geq 1$:

\begin{align}
E_{n\delta} = q^{-2}W_1E_{(n-1)\delta+\alpha_0} - E_{(n-1)\delta+\alpha_0}W_1.
\end{align}

Recall the antiautomorphism $\tau$ of $U_q^+$, from Definition 3.5.
Lemma 4.2. The antiautomorphism $\tau$ sends $E_{n\delta + \alpha_0} \leftrightarrow E_{n\delta + \alpha_1}$ for $n \in \mathbb{N}$, and fixes $E_{n\delta}$ for $n \geq 1$.

Proof. To verify the first assertion, compare the two relations in (9). To verify the second assertion, compare (10) and (11). \qed

For the PBW basis in Proposition 4.4, the corresponding reduction rules were obtained by Damiani [7, Section 4]. These reduction rules are repeated below using adjusted notation.

By [7] p. 307 the elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

Lemma 4.3. (See [7] p. 307.) For $i, j \in \mathbb{N}$ the following holds in $U_q^+$:

$$E_{i\delta + \alpha_1} E_{j\delta + \alpha_0} = q^2 E_{j\delta + \alpha_0} E_{i\delta + \alpha_1} + q^2 E_{(i+j)\delta}.$$

Lemma 4.4. (See [7] p. 300.) For $i > j \geq 0$ the following hold in $U_q^+$:

(i) Assume that $i - j = 2r + 1$ is odd. Then

$$E_{i\delta + \alpha_0} E_{j\delta + \alpha_0} = q^{-2} E_{j\delta + \alpha_0} E_{i\delta + \alpha_0} - (q^2 - q^{-2}) \sum_{\ell=1}^{r} q^{-2\ell} E_{(j+\ell)\delta + \alpha_0} E_{(i-\ell)\delta + \alpha_0};$$

$$E_{j\delta + \alpha_1} E_{i\delta + \alpha_1} = q^{-2} E_{i\delta + \alpha_1} E_{j\delta + \alpha_1} - (q^2 - q^{-2}) \sum_{\ell=1}^{r} q^{-2\ell} E_{(i-\ell)\delta + \alpha_1} E_{(j+\ell)\delta + \alpha_1}.$$

(ii) Assume that $i - j = 2r$ is even. Then

$$E_{i\delta + \alpha_0} E_{j\delta + \alpha_0} = q^{-2} E_{j\delta + \alpha_0} E_{i\delta + \alpha_0} - q^{-i-1}(q - q^{-1}) E_{(r+j)\delta + \alpha_0}$$

$$- (q^2 - q^{-2}) \sum_{\ell=1}^{r-1} q^{-2\ell} E_{(j+\ell)\delta + \alpha_0} E_{(i-\ell)\delta + \alpha_0};$$

$$E_{j\delta + \alpha_1} E_{i\delta + \alpha_1} = q^{-2} E_{i\delta + \alpha_1} E_{j\delta + \alpha_1} - q^{-i-1}(q - q^{-1}) E_{(r+j)\delta + \alpha_1}$$

$$- (q^2 - q^{-2}) \sum_{\ell=1}^{r-1} q^{-2\ell} E_{(i-\ell)\delta + \alpha_1} E_{(j+\ell)\delta + \alpha_1}.$$

Lemma 4.5. (See [7] p. 304.) For $i \geq 1$ and $j \geq 0$ the following hold in $U_q^+$:

$$E_{i\delta} E_{j\delta} = E_{j\delta} E_{i\delta} + q^{2-2i}(q + q^{-1}) E_{(i+j)\delta + \alpha_0}$$

$$- q^2 (q^2 - q^{-2}) \sum_{\ell=1}^{i-1} q^{-2\ell} E_{(j+\ell)\delta + \alpha_0} E_{(i-\ell)\delta};$$

$$E_{j\delta + \alpha_1} E_{i\delta} = E_{i\delta} E_{j\delta + \alpha_1} + q^{2-2i}(q + q^{-1}) E_{(i+j)\delta + \alpha_1}$$

$$- q^2 (q^2 - q^{-2}) \sum_{\ell=1}^{i-1} q^{-2\ell} E_{(i-\ell)\delta} E_{(j+\ell)\delta + \alpha_1}.$$

We mention an alternative version of Lemma 4.4.
Lemma 4.6. (See [6] Section 2.3 or [12] Lemma 3.5.) The following relations hold in $U_q^+$. For $i \in \mathbb{N},$

$$[E_{(i+1)\delta+\alpha_0}, E_{i\delta+\alpha_0}]_q = 0, \quad [E_{i\delta+\alpha_1}, E_{(i+1)\delta+\alpha_1}]_q = 0.$$  

For distinct $i, j \in \mathbb{N},$

$$[E_{(i+1)\delta+\alpha_0}, E_{j\delta+\alpha_0}]_q + [E_{(i+1)\delta+\alpha_0}, E_{i\delta+\alpha_0}]_q = 0,$$

$$[E_{j\delta+\alpha_1}, E_{(i+1)\delta+\alpha_1}]_q + [E_{i\delta+\alpha_1}, E_{(j+1)\delta+\alpha_1}]_q = 0.$$

We mention an alternative version of Lemma 4.5. For notational convenience define

$$E_0 := -(q - q^{-1}).$$

Lemma 4.7. (See [12] Lemma 3.4.) For $i, j \in \mathbb{N}$ the following hold in $U_q^+$:

$$[E_{i\delta+\alpha_0}, E_{(j+1)\delta}] = [E_{(i+1)\delta+\alpha_0}, E_j] q^2,$$

$$[E_{(j+1)\delta}, E_{i\delta+\alpha_1}] = [E_j, E_{(i+1)\delta+\alpha_1}] q^2.$$

5 Generating functions for $U_q^+$

In the previous section we displayed a PBW basis for $U_q^+$ along with the corresponding reduction rules. In this section we describe these reduction rules using generating functions. We acknowledge that the material in this section is well known to the experts, and readily follows from [8] Section IV] and [4][5]. The material is included for use later in the paper.

Definition 5.1. We define some generating functions in the indeterminate $t$:

$$E^-(t) = \sum_{n \in \mathbb{N}} E_{n \delta+\alpha_0} t^n, \quad E^+(t) = \sum_{n \in \mathbb{N}} E_{n \delta+\alpha_1} t^n,$$

$$E(t) = \sum_{n \in \mathbb{N}} E_{n \delta} t^n. \quad (13)$$

Observe that

$$E^-(0) = W_0, \quad E^+(0) = W_1, \quad E(0) = -(q - q^{-1})^{-1}. \quad (14)$$

Lemma 5.2. For the algebra $U_q^+$,

$$\frac{t[E_{\delta}, E^-(t)]}{q + q^{-1}} = E^-(t) - W_0, \quad \frac{t[E^+(t), E_{\delta}]}{q + q^{-1}} = E^+(t) - W_1. \quad (15)$$

Proof. These equations express the relations [9] in terms of generating functions. \qed

Lemma 5.3. For the algebra $U_q^+$,

$$[W_0, E^+(t)]_q = -qt^{-1}E(t) - \frac{qt^{-1}}{q - q^{-1}},$$

$$[E^-(t), W_1]_q = -qt^{-1}E(t) - \frac{qt^{-1}}{q - q^{-1}}. \quad (17)$$
Proof. The equation (16) (resp. (17)) expresses the relation (10) (resp. (11)) in terms of generating functions.

For the rest of the paper, let $s$ denote an indeterminate that commutes with $t$. By the comment above Lemma 4.3

$$[E(s), E(t)] = 0.$$  \hspace{1cm} \text{(18)}

**Proposition 5.4.** For the algebra $U_q^+$,

$$[E^-(s), E^+(t)]_q = -q \frac{E(s) - E(t)}{s - t}.$$  \hspace{1cm} \text{(19)}

Proof. The equation (19) expresses Lemma 4.3 in terms of generating functions.

**Proposition 5.5.** For the algebra $U_q^+$,

$$0 = \frac{qt - q^{-1}s}{q - q^{-1}} E^-(s) E^-(t) + \frac{qs - q^{-1}t}{q - q^{-1}} E^-(t) E^-(s) - s(E^-(s))^2 - t(E^-(t))^2,$$

$$0 = \frac{qt - q^{-1}s}{q - q^{-1}} E^+(t) E^+(s) + \frac{qs - q^{-1}t}{q - q^{-1}} E^+(s) E^+(t) - s(E^+(s))^2 - t(E^+(t))^2.$$  \hspace{1cm} \text{(20)} \hspace{1cm} \text{(21)}

Proof. These equations express Lemma 4.6 in terms of generating functions.

**Proposition 5.6.** For the algebra $U_q^+$,

$$0 = (s - q^2t) E^-(s) E(t) + (q^2 - q^2t - s) E(t) E^-(s) + (q^2 - q^{-2}) t E^-(q^2t) E(t),$$

$$0 = (s - q^2t) E(t) E^+(s) + (q^2 - q^2t - s) E^+(s) E(t) + (q^2 - q^{-2}) t E(t) E^+(q^2t).$$  \hspace{1cm} \text{(22)} \hspace{1cm} \text{(23)}

Proof. These equations express Lemma 4.7 in terms of generating functions.

**Corollary 5.7.** For the algebra $U_q^+$,

$$[W_0, E^-(t)]_q = (q - q^{-1})(E^-(t))^2,$$

$$[W_0, E(t)]_q^2 = (q^2 - q^{-2}) E^-(q^2t) E(t),$$

$$[E^+(t), W_1]_q = (q - q^{-1})(E^+(t))^2,$$

$$[E(t), W_1]_q^2 = (q^2 - q^{-2}) E(t) E^+(q^2t).$$  \hspace{1cm} \text{(24)} \hspace{1cm} \text{(25)} \hspace{1cm} \text{(26)} \hspace{1cm} \text{(27)}

Proof. Set $s = 0$ in Propositions 5.5, 5.6 and evaluate the results using (14).

**Remark 5.8.** Lemmas 5.2, 5.3 and Corollary 5.7 follow from (14), (18) and Propositions 5.4, 5.5, 5.6. Indeed Lemma 5.3 follows from Proposition 5.4 by setting $s = 0$ or $t = 0$, and evaluating the results using (14). Corollary 5.7 follows from Propositions 5.5, 5.6 by the proof of Corollary 5.7. Lemma 5.2 follows from (17), (24), (25) along with (22) at $s = t$. 

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6 The algebra \( U_q^+ \)

In the previous section we discussed the algebra \( U_q^+ \). In this section we discuss its alternating central extension \( U_q^+ \).

**Definition 6.1.** (See [14, Definition 3.1].) Define the algebra \( U_q^+ \) by generators

\[
\{ W_{-k} \}_{k \in \mathbb{N}}, \quad \{ W_{k+1} \}_{k \in \mathbb{N}}, \quad \{ g_{k+1} \}_{k \in \mathbb{N}}, \quad \{ \tilde{g}_{k+1} \}_{k \in \mathbb{N}} \quad (28)
\]

and the following relations. For \( k, \ell \in \mathbb{N} \),

\[
[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{g}_{k+1} - g_{k+1}), \quad (29)
\]

\[
[W_0, g_{k+1}]_q = [\tilde{g}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1}, \quad (30)
\]

\[
[g_{k+1}, W_1]_q = [W_1, \tilde{g}_{k+1}]_q = (q - q^{-1})W_{k+2}, \quad (31)
\]

\[
[W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \quad (32)
\]

\[
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \quad (33)
\]

\[
[W_{-k}, g_{\ell+1}] + [g_{k+1}, W_{-\ell}] = 0, \quad (34)
\]

\[
[W_{-k}, \tilde{g}_{\ell+1}] + [\tilde{g}_{k+1}, W_{-\ell}] = 0, \quad (35)
\]

\[
[W_{k+1}, g_{\ell+1}] + [g_{k+1}, W_{\ell+1}] = 0, \quad (36)
\]

\[
[W_{k+1}, \tilde{g}_{\ell+1}] + [\tilde{g}_{k+1}, W_{\ell+1}] = 0, \quad (37)
\]

\[
[g_{k+1}, g_{\ell+1}] = 0, \quad [\tilde{g}_{k+1}, \tilde{g}_{\ell+1}] = 0, \quad (38)
\]

\[
[\tilde{g}_{k+1}, g_{\ell+1}] + [g_{k+1}, \tilde{g}_{\ell+1}] = 0. \quad (39)
\]

The generators (28) are called alternating. We call \( U_q^+ \) the alternating central extension of \( U_q^+ \). For notational convenience define

\[
g_0 = 1, \quad \tilde{g}_0 = 1. \quad (40)
\]

**Remark 6.2.** The relations in Definition 6.1 resemble some relations involving the \( q \)-Onsager algebra that were found earlier by Baseilhac and Shigechi [3, Definition 3.1]; see also [2].

Next we describe some symmetries of \( U_q^+ \).

**Lemma 6.3.** (See [14, Lemma 3.9].) There exists an automorphism \( \sigma \) of \( U_q^+ \) that sends

\[
W_{-k} \mapsto W_{k+1}, \quad W_{k+1} \mapsto W_{-k}, \quad g_{k+1} \mapsto \tilde{g}_{k+1}, \quad \tilde{g}_{k+1} \mapsto g_{k+1}
\]

for \( k \in \mathbb{N} \). Moreover \( \sigma^2 = \text{id} \).

**Lemma 6.4.** (See [14, Lemma 3.9].) There exists an antiautomorphism \( \dagger \) of \( U_q^+ \) that sends

\[
W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad g_{k+1} \mapsto \tilde{g}_{k+1}, \quad \tilde{g}_{k+1} \mapsto g_{k+1}
\]

for \( k \in \mathbb{N} \). Moreover \( \dagger^2 = \text{id} \).

**Lemma 6.5.** The maps \( \sigma \), \( \dagger \) commute.
Proof. This is readily checked.

**Definition 6.6.** Let \( \tau \) denote the composition of the automorphism \( \sigma \) from Lemma 6.3 and the antiautomorphism \( \dagger \) from Lemma 6.4. Note that \( \tau \) is an antiautomorphism of \( U_q^+ \) that sends

\[
\begin{align*}
W_{-k} &\mapsto W_{k+1}, & W_{k+1} &\mapsto W_{-k}, & G_{k+1} &\mapsto G_{k+1}, & \tilde{G}_{k+1} &\mapsto \tilde{G}_{k+1}
\end{align*}
\]

for \( k \in \mathbb{N} \). We have \( \tau^2 = \text{id} \).

Next we discuss how \( U_q^+ \) is related to \( U_q^+ \).

**Lemma 6.7.** (See [14, Proposition 6.4].) There exists an algebra homomorphism \( \iota : U_q^+ \rightarrow U_q^+ \) that sends \( W_0 \mapsto W_0 \) and \( W_1 \mapsto W_1 \). Moreover, \( \iota \) is injective.

**Lemma 6.8.** The following diagrams commute:

\[
\begin{array}{ccc}
U_q^+ & \xrightarrow{\iota} & U_q^+ \\
\sigma \downarrow & & \sigma \downarrow \\
U_q^+ & \xrightarrow{\iota} & U_q^+
\end{array}
\quad \begin{array}{ccc}
U_q^+ & \xrightarrow{\iota} & U_q^+ \\
\dagger \downarrow & & \dagger \downarrow \\
U_q^+ & \xrightarrow{\iota} & U_q^+
\end{array}
\quad \begin{array}{ccc}
U_q^+ & \xrightarrow{\iota} & U_q^+ \\
\tau \downarrow & & \tau \downarrow \\
U_q^+ & \xrightarrow{\iota} & U_q^+
\end{array}
\]

Proof. Chase the \( U_q^+ \)-generators \( W_0, W_1 \) around each diagram, using Lemmas 3.2, 3.3 and Definition 3.5 along with Lemmas 6.3, 6.4 and Definition 6.6.

\[\Box\]

### 7 Generating functions for \( U_q^+ \)

In Definition 6.4 the algebra \( U_q^+ \) is defined by generators and relations. In this section we describe the defining relations in terms of generating functions.

**Definition 7.1.** (See [14, Definition A.1].) We define some generating functions in the indeterminate \( t \):

\[
\begin{align*}
W^{-}(t) &= \sum_{n \in \mathbb{N}} W_{-n} t^n, & W^{+}(t) &= \sum_{n \in \mathbb{N}} W_{n+1} t^n, \\
G(t) &= \sum_{n \in \mathbb{N}} G_n t^n, & \tilde{G}(t) &= \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.
\end{align*}
\]

Observe that

\[
W^{-}(0) = W_0, \quad W^{+}(0) = W_1, \quad G(0) = 1, \quad \tilde{G}(0) = 1.
\]

We now give the relations (29)–(39) in terms of generating functions.
Lemma 7.2. (See [14] Lemmas A.2, A.3.) For the algebra $\mathcal{U}_q^+$,

$$[\mathcal{W}_0, \mathcal{W}^+(t)] = [\mathcal{W}^-(t), \mathcal{W}_1] = (1 - q^{-2})t^{-1}(\tilde{G}(t) - G(t)), \quad (41)$$
$$[\mathcal{W}_0, \mathcal{G}(t)]_q = [\tilde{G}(t), \mathcal{W}_0]_q = (q - q^{-1})\mathcal{W}^-(t), \quad (42)$$
$$[\mathcal{G}(t), \mathcal{W}_1]_q = [\mathcal{W}_1, \tilde{G}(t)]_q = (q - q^{-1})\mathcal{W}^+(t), \quad (43)$$
$$[\mathcal{W}^-(s), \mathcal{W}^-(t)] = 0, \quad [\mathcal{W}^+(s), \mathcal{W}^+(t)] = 0, \quad (44)$$
$$[\mathcal{W}^-(s), \mathcal{W}^+(t)] + [\mathcal{W}^+(s), \mathcal{W}^-(t)] = 0, \quad (45)$$
$$s[\mathcal{W}^-(s), \mathcal{G}(t)] + t[\mathcal{G}(s), \mathcal{W}^-(t)] = 0, \quad (46)$$
$$s[\mathcal{W}^-(s), \tilde{G}(t)] + t[\tilde{G}(s), \mathcal{W}^-(t)] = 0, \quad (47)$$
$$s[\mathcal{W}^+(s), \mathcal{G}(t)] + t[\mathcal{G}(s), \mathcal{W}^+(t)] = 0, \quad (48)$$
$$s[\mathcal{W}^+(s), \tilde{G}(t)] + t[\tilde{G}(s), \mathcal{W}^+(t)] = 0, \quad (49)$$
$$[\mathcal{G}(s), \mathcal{G}(t)] = 0, \quad [\mathcal{G}(s), \tilde{G}(t)] = 0, \quad (50)$$
$$[\tilde{G}(s), \mathcal{G}(t)] + [\mathcal{G}(s), \tilde{G}(t)] = 0. \quad (51)$$

8 A PBW basis for $\mathcal{U}_q^+$

In [14, Theorem 10.2] a PBW basis for $\mathcal{U}_q^+$ is obtained from the alternating generators in a certain linear order; see Appendix A below. In the present section we modify the linear order to get a new PBW basis for $\mathcal{U}_q^+$ that is better suited to our purpose. For the new PBW basis we display the corresponding reduction rules.

Definition 8.1. Let $L$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\{\mathcal{W}_{i-}\}_{i \in \mathbb{N}}, \{\mathcal{G}_{j-}\}_{j \in \mathbb{N}}$. Let $R$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\{\mathcal{W}_{k-}\}_{k \in \mathbb{N}}, \{\tilde{G}_{k-}\}_{k \in \mathbb{N}}$.

Lemma 8.2. The following (i)–(iii) hold for the subalgebras $L$ and $R$:

(i) a PBW basis for $L$ is obtained by the elements $\{\mathcal{W}_{i-}\}_{i \in \mathbb{N}}, \{\mathcal{G}_{j-}\}_{j \in \mathbb{N}}$ in any linear order such that $\mathcal{W}_{i-} < \mathcal{G}_{j-}$ for $i, j \in \mathbb{N}$;

(ii) a PBW basis for $R$ is obtained by the elements $\{\tilde{G}_{k-}\}_{k \in \mathbb{N}}, \{\mathcal{W}_{\ell-}\}_{\ell \in \mathbb{N}}$ in any linear order such that $\tilde{G}_{k-} < \mathcal{W}_{\ell-}$ for $k, \ell \in \mathbb{N}$;

(iii) the multiplication map $L \otimes R \to \mathcal{U}_q^+$

$$l \otimes r \mapsto lr$$

is an isomorphism of vector spaces.

Proof. We refer to Appendix A.

(i) By Lemma [14.1] and the third displayed equation in Lemma [14.3]

(ii) By Lemma [14.1] and the last displayed equation in Lemma [14.3]

(iii) By Lemma [14.1] and (i), (ii) above.
Recall from Lemma 6.3 the automorphism $\sigma$ of $U_q^+$.  

**Lemma 8.3.** The automorphism $\sigma$ sends $L \leftrightarrow R$.  

**Proof.** By Lemma 6.3 and Definition 8.1. 

**Lemma 8.4.** The following (i), (ii) hold for the subalgebras $L$ and $R$:  

(i) a PBW basis for $L$ is obtained by the elements $\{G_{i+1}\}_{i \in \mathbb{N}}, \{W_{-j}\}_{j \in \mathbb{N}}$ in any linear order such that $G_{i+1} < W_{-j}$ for $i, j \in \mathbb{N}$;  

(ii) a PBW basis for $R$ is obtained by the elements $\{W_{k+1}\}_{k \in \mathbb{N}}, \{\tilde{G}_{\ell+1}\}_{\ell \in \mathbb{N}}$ in any linear order such that $W_{k+1} < \tilde{G}_{\ell+1}$ for $k, \ell \in \mathbb{N}$.  

**Proof.** (i) Apply $\sigma$ to the PBW basis for $R$ given in Lemma 8.2(ii), and use Lemmas 6.3, 8.3.  

(ii) Apply $\sigma$ to the PBW basis for $L$ given in Lemma 8.2(i), and use Lemmas 6.3, 8.3.  

**Proposition 8.5.** A PBW basis for $U_q^+$ is obtained by its alternating generators in any linear order $<$ such that  

$$G_{i+1} < W_{-j} < W_{k+1} < \tilde{G}_{\ell+1} \quad i, j, k, \ell \in \mathbb{N}. \quad (52)$$  

**Proof.** By Lemma 8.2(iii) and Lemma 8.4.  

For the PBW basis in Proposition 8.5, the nontrivial reduction rules are a consequence of the following result.  

**Lemma 8.6.** For the algebra $U_q^+$, we have  

$$W^+(s)W^-(t) = W^-(t)W^+(s) + (1 - q^{-2}) \frac{G(s)\tilde{G}(t) - G(t)\tilde{G}(s)}{s - t},$$  

$$\tilde{G}(s)\tilde{G}(t) = G(t)\tilde{G}(s) + (1 - q^2)st \frac{W^-(t)W^+(s) - W^-(s)W^+(t)}{s - t}$$  

and also  

$$W^-(s)\tilde{G}(t) = q^{-1}\frac{(qs - q^{-1}t)\tilde{G}(t)W^-(s) - (q - q^{-1})t\tilde{G}(s)W^-(t)}{s - t},$$  

$$W^+(s)G(t) = q\frac{(q^{-1}s - qt)G(t)W^+(s) + (q - q^{-1})tG(s)W^+(t)}{s - t},$$  

$$\tilde{G}(s)W^-(t) = q^{-1}\frac{(q^{-1}s - qt)W^-(t)\tilde{G}(s) + (q - q^{-1})sW^-(s)\tilde{G}(t)}{s - t},$$  

$$\tilde{G}(s)W^+(t) = q\frac{(qs - q^{-1}t)W^+(t)\tilde{G}(s) - (q - q^{-1})sW^+(s)\tilde{G}(t)}{s - t}.$$  

**Proof.** For each of the above equations, either the equation or its $\sigma$-image is listed in Lemma 14.2.  

Next we give the nontrivial reduction rules for the PBW basis in Proposition 8.5.
Proposition 8.7. The following relations hold in $\mathcal{U}_q^+$. For $i, j \in \mathbb{N}$,

\[
\mathcal{W}_{i+1} \mathcal{W}_{-j} = \mathcal{W}_{-j} \mathcal{W}_{i+1} + q^{-1}(q - q^{-1}) \sum_{\ell=0}^{\min(i,j)} (G_{i+j+1-\ell} \tilde{G}_\ell - G_\ell \tilde{G}_{i+j+1-\ell}),
\]

\[
\tilde{G}_{i+1} \mathcal{G}_{j+1} = \mathcal{G}_{j+1} \tilde{G}_{i+1} + q(q - 1) \sum_{\ell=0}^{\min(i,j)} (\mathcal{W}_{\ell-i-j-1} \mathcal{W}_{\ell+1} - \mathcal{W}_{-\ell} \mathcal{W}_{i+j+2-\ell}),
\]

and also

\[
\mathcal{W}_{-i} \tilde{G}_{j+1} = \mathcal{G}_{j+1} \mathcal{W}_{-i} + q^{-1}(q - q^{-1}) \sum_{\ell=0}^{\min(i,j)} (G_{\ell} \mathcal{W}_{\ell-i-j-1} - G_{i+j+1-\ell} \mathcal{W}_{-\ell}),
\]

\[
\mathcal{W}_{i+1} \mathcal{G}_{j+1} = \mathcal{G}_{j+1} \mathcal{W}_{i+1} + q(q - 1) \sum_{\ell=0}^{\min(i,j)} (G_{i+j+1-\ell} \mathcal{W}_{\ell+1} - G_\ell \mathcal{W}_{i+j+2-\ell}),
\]

\[
\tilde{G}_{i+1} \mathcal{W}_{-j} = \mathcal{W}_{-j} \tilde{G}_{i+1} + q^{-1}(q - q^{-1}) \sum_{\ell=0}^{\min(i,j)} (\mathcal{W}_{\ell-i-j-1} \tilde{G}_\ell - \mathcal{W}_{-\ell} \tilde{G}_{i+j+1-\ell}),
\]

\[
\tilde{G}_{i+1} \mathcal{W}_{j+1} = \mathcal{W}_{j+1} \tilde{G}_{i+1} + q(q - 1) \sum_{\ell=0}^{\min(i,j)} (\mathcal{W}_{\ell+i+j+1-\ell} \mathcal{W}_{i+j+2-\ell} - \mathcal{W}_{i+j+2-\ell} \tilde{G}_\ell).
\]

Proof. These relations are obtained by unpacking the equations in Lemma 8.6.

9 The center of $\mathcal{U}_q^+$

Earlier in this paper we discussed the generating functions $E^\pm(t), E(t)$ for $\mathcal{U}_q^+$ and the generating functions $\mathcal{W}^\pm(t), \mathcal{G}(t), \tilde{\mathcal{G}}(t)$ for $\mathcal{U}_q^+$. In the next section, we will investigate how $E^\pm(t), E(t)$ are related to $\mathcal{W}^\pm(t), \mathcal{G}(t), \tilde{\mathcal{G}}(t)$. In the present section, we prepare for this investigation with some remarks about the center $\mathcal{Z}$ of $\mathcal{U}_q^+$.

Definition 9.1. (See [14, Definition 5.1].) For $n \geq 1$ define

\[
\mathcal{Z}_n = \sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}.
\]

(53)

For notational convenience define $\mathcal{Z}_0 = 1$.

Next, we interpret Definition 9.1 in terms of generating functions.

Definition 9.2. Define the generating function

\[
\mathcal{Z}^\vee(t) = \sum_{n \in \mathbb{N}} \mathcal{Z}_n^\vee t^n.
\]
Lemma 9.3. (See [14, Lemma A.8].) We have
\[ Z^\vee(t) = \mathcal{G}(q^{-1}t)\tilde{\mathcal{G}}(qt) - qt\mathcal{W}^-(q^{-1}t)\mathcal{W}^+(qt). \] (54)

Lemma 9.4. (See [14, Lemma 5.2 and Proposition 8.3].) For \( n \geq 1 \) we have \( Z_n^\vee \in \mathcal{Z} \). Moreover \( Z_n^\vee \) fixed by \( \sigma \) and \( \dagger \) and \( \tau \).

Definition 9.5. Let \( \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \) denote the subalgebra of \( \mathcal{U}_q^+ \) generated by \( \mathcal{W}_0, \mathcal{W}_1 \).

Proposition 9.6. (See [14, Section 6].) For the algebra \( \mathcal{U}_q^+ \) the following (i)–(iii) hold:

(i) there exists an algebra isomorphism \( \mathcal{U}_q^+ \to \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \) that sends \( \mathcal{W}_0 \mapsto \mathcal{W}_0 \) and \( \mathcal{W}_1 \mapsto \mathcal{W}_1 \);

(ii) there exists an algebra isomorphism \( \mathbb{F}[z_1, z_2, \ldots] \to \mathcal{Z} \) that sends \( z_n \mapsto Z_n^\vee \) for \( n \geq 1 \);

(iii) the multiplication map
\[ \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \to \mathcal{U}_q^+ \]
\[ w \otimes z \mapsto wz \]
is an isomorphism of algebras.

Note that the isomorphism in Proposition 9.6(i) is induced by the map \( \iota \) from Lemma 6.7. We emphasize a few points.

Corollary 9.7. For the algebra \( \mathcal{U}_q^+ \) the following (i)–(iii) hold:

(i) the algebra \( \mathcal{U}_q^+ \) is generated by \( \mathcal{W}_0, \mathcal{W}_1, \mathcal{Z} \);

(ii) the elements \( \{Z_n^\vee\}_{n=1}^{\infty} \) are algebraically independent and generate \( \mathcal{Z} \);

(iii) everything in \( \mathcal{Z} \) is fixed by \( \sigma \) and \( \dagger \) and \( \tau \).

Proof. (i) By Proposition 9.6(iii).

(ii) By Proposition 9.6(ii).

(iii) By (ii) above and Lemma 9.4. \( \square \)

10 Comparing the generating functions for \( \mathcal{U}_q^+ \) and \( \mathcal{U}_q^+ \)

In this section we investigate how the generating functions \( E^\pm(t), E(t) \) for \( \mathcal{U}_q^+ \) are related to the generating functions \( \mathcal{W}^\pm(t), \mathcal{G}(t), \tilde{\mathcal{G}}(t) \) for \( \mathcal{U}_q^+ \).

Throughout this section, we identify \( \mathcal{U}_q^+ \) with \( \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \) via the map \( \iota \) from Lemma 6.7. For notational convenience define
\[ \xi = -q^2(q - q^{-1})^{-2}. \] (55)
Proposition 10.1. For the algebra $\mathcal{U}_q^+$,

\begin{align*}
\mathcal{W}^-(t) &= E^-(q\xi t)\tilde{G}(t) = \tilde{G}(t)E^-(q^{-1}\xi t), \quad (56) \\
\mathcal{W}^+(t) &= E^+(q^{-1}\xi t)\tilde{G}(t) = \tilde{G}(t)E^+(q\xi t). \quad (57)
\end{align*}

Proof. The equation on the left in (56) (resp. (57)) is equation (15) (resp. equation (14)) in [15], expressed in terms of generating functions. Using the antiautomorphism $\tau$ we get the equations on the right in (56), (57).

In the next two results we give some consequences of Proposition 10.1.

Proposition 10.2. For the algebra $\mathcal{U}_q^+$,

\begin{align*}
\tilde{G}(t)\mathcal{W}_0 &= \left(q^{-2}\mathcal{W}_0 + q^{-1}(q - q^{-1})E^-(q\xi t)\right)\tilde{G}(t), \quad (58) \\
\tilde{G}(t)\mathcal{W}_1 &= \left(q^2\mathcal{W}_1 - q(q - q^{-1})E^+(q^{-1}\xi t)\right)\tilde{G}(t) \quad (59)
\end{align*}

and also

\begin{align*}
\mathcal{W}_0\tilde{G}(t) &= \tilde{G}(t)\left(q^2\mathcal{W}_0 - q(q - q^{-1})E^-(q^{-1}\xi t)\right), \quad (60) \\
\mathcal{W}_1\tilde{G}(t) &= \tilde{G}(t)\left(q^{-2}\mathcal{W}_1 + q^{-1}(q - q^{-1})E^+(q\xi t)\right). \quad (61)
\end{align*}

Proof. By (42), (43) we have

\begin{align*}
[\tilde{G}(t), \mathcal{W}_0]_q &= (q - q^{-1})\mathcal{W}^-(t), \quad [\mathcal{W}_1, \tilde{G}(t)]_q = (q - q^{-1})\mathcal{W}^+(t).
\end{align*}

In these equations, eliminate $\mathcal{W}^-(t)$ and $\mathcal{W}^+(t)$ using Proposition 10.1 and simplify the result.

Proposition 10.3. For the algebra $\mathcal{U}_q^+$,

\begin{align*}
\mathcal{G}(t) &= \left(q^2tE^-(q\xi t)E^+(q^{-1}\xi t) - (q - q^{-1})E(q\xi t)\right)\tilde{G}(t) \quad (62) \\
&= \left(tE^+(q^{-1}\xi t)E^-(q\xi t) - (q - q^{-1})E(q^{-1}\xi t)\right)\tilde{G}(t) \quad (63) \\
&= \tilde{G}(t)\left(q^2tE^-(q^{-1}\xi t)E^+(q\xi t) - (q - q^{-1})E(q\xi t)\right) \quad (64) \\
&= \tilde{G}(t)\left(tE^+(q\xi t)E^-(q^{-1}\xi t) - (q - q^{-1})E(q^{-1}\xi t)\right). \quad (65)
\end{align*}

Proof. We first show (62). By (41),

\begin{align*}
\mathcal{W}^-(t)\mathcal{W}_1 - \mathcal{W}_1\mathcal{W}^-(t) &= (1 - q^{-2})t^{-1}(\tilde{G}(t) - \mathcal{G}(t)).
\end{align*}

In this equation, eliminate $\mathcal{W}^-(t)$ using the equation on the left in (56). Evaluate the resulting equation using (59). In the resulting equation, eliminate $[E^-(q\xi t), \mathcal{W}_1]_q$ using (17). The resulting equation becomes (62) after simplification. We have shown (62). The right-hand sides of (62), (63) are equal by Proposition 10.1 so (63) holds. Using $\tau$ we obtain (64), (65).
Remark 10.4. The above Propositions 10.1, 10.3 are variations on [1, Proposition 5.18] and [1, Proposition 5.20].

We have a comment. The generating function $\tilde{G}(t)$ is invertible by Lemma 2.1 and $\tilde{G}_0 = 1$.

Lemma 10.5. For the algebra $U_q^+$,

$$( \tilde{G}(t) )^{-1} W^{-}(t) = W^{-}(q^{-2}t)(\tilde{G}(q^{-2}t))^{-1},$$

$$W^{+}(t) = W^{+}(q^2t)(\tilde{G}(q^2t))^{-1}.$$  \hspace{1cm} (66) \hspace{1cm} (67)

Proof. To get (66), compare the two equations in (56). To get (67), compare the two equations in (57).

11 A factorization of $Z^\vee(t)$

Recall the generating function $Z^\vee(t)$ from Definition 9.2 and Lemma 9.3. In this section we obtain a factorization of $Z^\vee(t)$.

Throughout this section we identify $U_q^+$ with $\langle W_0, W_1 \rangle$ via the map $i$ from Lemma 6.7.

Proposition 11.1. For the algebra $U_q^+$ we have

$$Z^\vee(t) = -(q - q^{-1})\tilde{G}(q^{-1}t)E(\xi t)\tilde{G}(qt),$$

where we recall $\xi = -q^2(q - q^{-1})^{-2}$.

Proof. Consider the terms on the right in (54). By (64),

$$\tilde{G}(q^{-1}t) = \tilde{G}(q^{-1}t)\left( qt E^{-}(q^{-2}\xi t)E^{+}(\xi t) - (q - q^{-1})E(\xi t) \right).$$

By Proposition 10.1

$$W^{-}(q^{-1}t) = \tilde{G}(q^{-1}t)E^{-}(q^{-2}\xi t), \quad W^{+}(qt) = E^{+}(\xi t)\tilde{G}(qt).$$

Evaluating the right-hand side of (54) using (69), (70) we routinely obtain (68).

Next, we give some consequences of Proposition 11.1

Definition 11.2. For notational convenience, define

$$E^\vee(t) = -(q - q^{-1})E(t).$$

Corollary 11.3. For the algebra $U_q^+$ we have

$$E^\vee(t) = (\tilde{G}(q^{-1}t)^{-1})Z^\vee(\xi^{-1}t)(\tilde{G}(q\xi^{-1}t))^{-1}.$$  \hspace{1cm} (72)

Proof. Rearrange the terms in (68).
Corollary 11.4. For the algebra $\mathcal{U}_q^+$, 

$$[\tilde{G}(s), E^\vee(t)] = 0.$$ 

Proof. The generating function $\tilde{G}(s)$ commutes with each factor on the right in (72). □

Corollary 11.5. For the algebra $\mathcal{U}_q^+$, 

$$[\tilde{G}_{k+1}, E_{nd}] = 0 \quad k, n \in \mathbb{N}.$$ 

Proof. By Corollary 11.4 □

Corollary 11.6. The generating function $Z^\vee(t)$ is equal to each of 

$$\tilde{G}(q^{-1}t)E^\vee(\xi t)\tilde{G}(qt), \quad E^\vee(\xi t)\tilde{G}(q^{-1}t)\tilde{G}(qt), \quad \tilde{G}(q^{-1}t)\tilde{G}(qt)E^\vee(\xi t), \quad \tilde{G}(qt)\tilde{G}(q^{-1}t)E^\vee(\xi t).$$ 

Proof. Evaluate (68) using (71) along with Corollary 11.5 and the equation on the right in (38). □

12 Expressing $E^\pm(t)$, $E(t)$ in terms of $\mathcal{W}^\pm(t)$, $G(t)$, $\tilde{G}(t)$

In this section, we continue to discuss the generating functions $E^\pm(t)$, $E(t)$ for $\mathcal{U}_q^+$ and $\mathcal{W}^\pm(t)$, $G(t)$, $\tilde{G}(t)$ for $\mathcal{U}_q^+$. We first express $E^\pm(t)$, $E(t)$ in terms of $\mathcal{W}^\pm(t)$, $G(t)$, $\tilde{G}(t)$. We then use these expressions to recover the results about $E^\pm(t)$, $E(t)$ from Section 5.

Throughout this section we identify $\mathcal{U}_q^+$ with $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ via the map $\iota$ from Lemma 6.7.

To simplify our calculations, we use the following change of variables involving $G(t)$, $Z^\vee(t)$.

Lemma 12.1. For the algebra $\mathcal{U}_q^+$,

$$G(t) = Z^\vee(qt)(\tilde{G}(q^2t))^{-1} + q^2t\mathcal{W}^-(t)\mathcal{W}^+(q^2t)(\tilde{G}(q^2t))^{-1}.$$ 

Proof. Solve (54) for $G(t)$. □

Theorem 12.2. For the algebra $\mathcal{U}_q^+$,

$$E^-(t) = \mathcal{W}^-(q^{-1}\xi^{-1}t)(\tilde{G}(q^{-1}\xi^{-1}t))^{-1}, \quad (73)$$

$$E^+(t) = \mathcal{W}^+(q\xi^{-1}t)(\tilde{G}(q\xi^{-1}t))^{-1}, \quad (74)$$

$$E(t) = -\frac{Z^\vee(\xi^{-1}t)(\tilde{G}(q^{-1}\xi^{-1}t))^{-1}(\tilde{G}(q\xi^{-1}t))^{-1}}{q - q^{-1}}. \quad (75)$$

Proof. To get (73), replace $t$ by $q^{-1}\xi^{-1}t$ in the equation on the left in (56). To get (74), replace $t$ by $q\xi^{-1}t$ in the equation on the left in (57). To get (75), replace $t$ by $\xi^{-1}t$ in Corollary 11.6. □
In Section 5, we gave some relations involving $E^\pm(t)$, $E(t)$. Our next goal is to recover these relations using Theorem 12.2. In order to make use of Theorem 12.2, we display some equations involving $(\tilde{G}(t))^{-1}$.

**Proposition 12.3.** For the algebra $\mathcal{U}_q^+$,\[ (\tilde{G}(s))^{-1}\tilde{G}(t) = \tilde{G}(t)(\tilde{G}(s))^{-1}, \quad (\tilde{G}(s))^{-1}(\tilde{G}(t))^{-1} = (\tilde{G}(t))^{-1}(\tilde{G}(s))^{-1}, \] (76)\[ (\tilde{G}(s))^{-1}W^-(t) = \frac{q(s-t)W^-(t)(\tilde{G}(s))^{-1} - (q - q^{-1})sW^-(q^{-2}s)(\tilde{G}(s))^{-1}(\tilde{G}(q^{-2}s))^{-1}\tilde{G}(t)}{q^{-1}s - qt}, \] (77)\[ (\tilde{G}(s))^{-1}W^+(t) = \frac{q^{-1}(s-t)W^+(t)(\tilde{G}(s))^{-1} + (q - q^{-1})sW^+(q^2s)(\tilde{G}(s))^{-1}(\tilde{G}(q^2s))^{-1}\tilde{G}(t)}{qs - q^{-1}t}, \] (78)\[ (\tilde{G}(s))^{-1}Z^\vee(t) = Z^\vee(t)(\tilde{G}(s))^{-1}. \] (79)

**Proof.** The equations in (76) follow from the equation on the right in (50). To obtain (77), start with the fifth displayed equation in Lemma 8.6. In this equation, multiply each term on the left by $(\tilde{G}(s))^{-1}$ and on the right by $(\tilde{G}(s))^{-1}$. In the resulting equation, eliminate $(\tilde{G}(s))^{-1}W^-(s)$ using (66) and then solve for $(\tilde{G}(s))^{-1}W^-(t)$ to get (77). To obtain (78), start with the last displayed equation in Lemma 8.6. In this equation, multiply each term on the left by $(\tilde{G}(s))^{-1}$ and on the right by $(\tilde{G}(s))^{-1}$. In the resulting equation, eliminate $(\tilde{G}(s))^{-1}W^+(s)$ using (67) and then solve for $(\tilde{G}(s))^{-1}W^+(t)$ to get (78). Equation (79) holds since $Z^\vee(t)$ is central. \[ \Box \]

Line (18) and Propositions 5.4, 5.5, 5.6 contain some relations involving $E^\pm(t)$, $E(t)$. These relations can be recovered using Theorem 12.2 along with Lemmas 8.6, 10.5, 12.1 and Proposition 12.3. The calculations are routine and omitted. Lemmas 5.2, 5.3 and Corollary 5.7 can be obtained using Remark 5.8. They can also be obtained using Theorem 12.2 along with Proposition 12.3 and the following results.

**Lemma 12.4.** For the algebra $\mathcal{U}_q^+$:\[ W_0\tilde{G}(t) = q^{-2}\tilde{G}(t)W_0 + (1 - q^{-2})W^-(t), \]
\[ W_0W^-(t) = W^-(t)W_0, \]
\[ W^+(t)W_0 = W_0W^+(t) + (1 - q^{-2})t^{-1}(\tilde{G}(t) - \tilde{G}(t)), \]
\[ \tilde{G}(t)W_0 = q^{-2}W_0\tilde{G}(t) + (1 - q^{-2})W^-(t) \]
and\[ W_1\tilde{G}(t) = q^2\tilde{G}(t)W_1 + (1 - q^2)W^+(t), \]
\[ W_1W^+(t) = W^+(t)W_1 + (1 - q^2)t^{-1}(\tilde{G}(t) - \tilde{G}(t)), \]
\[ W^+(t)W_1 = W_1W^+(t), \]
\[ \tilde{G}(t)W_1 = q^2W_1\tilde{G}(t) + (1 - q^2)W^+(t), \]
Proof. Use \((11) - (14)\).

**Corollary 12.5.** For the algebra \(U_q^+\),

\[
(\tilde{\mathcal{G}}(t))^{-1}W_0 = q^2W_0(\tilde{\mathcal{G}}(t))^{-1} - q(q - q^{-1})W^{-}(q^{-2}t)(\tilde{\mathcal{G}}(q^{-2}t))^{-1}(\tilde{\mathcal{G}}(t))^{-1}, \quad (80)
\]

\[
(\tilde{\mathcal{G}}(t))^{-1}W_1 = q^{-2}W_1(\tilde{\mathcal{G}}(t))^{-1} + q^{-1}(q - q^{-1})W^+(q^2t)(\tilde{\mathcal{G}}(q^2t))^{-1}(\tilde{\mathcal{G}}(t))^{-1}. \quad (81)
\]

Proof. Set \(s = t'\) and \(t = 0\) in \((77), (78)\). Evaluate the results using \(W^-(0) = W_0\) and \(W^+(0) = W_1\) and \(\tilde{\mathcal{G}}(0) = 1\). \(\square\)

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# 14 Appendix A: An earlier PBW basis for \(U_q^+\)

In [14], Theorem 10.2] we gave a PBW basis for \(U_q^+\). In the present section we recall this PBW basis, and give the corresponding reduction rules.

**Lemma 14.1.** \(\text{(See [14], Theorem 10.2.)}\) A PBW basis for \(U_q^+\) is obtained by its alternating generators in any linear order \(<\) such that

\[
W_{-i} < \mathcal{G}_{j+1} < \tilde{\mathcal{G}}_{k+1} < W_{\ell+1} \quad i, j, k, \ell \in \mathbb{N}. \quad (82)
\]

For the above PBW basis, the nontrivial reduction rules are a consequence of the following result.

**Lemma 14.2.** \(\text{(See [14], Lemma A.6.)}\) For the algebra \(U_q^+\) we have

\[
W^+(s)W^{-}(t) = W^{-}(t)W^+(s) + (1 - q^{-2})\frac{\mathcal{G}(s)\tilde{\mathcal{G}}(t) - \mathcal{G}(t)\tilde{\mathcal{G}}(s)}{s - t},
\]

\[
\mathcal{G}(s)\mathcal{G}(t) = \mathcal{G}(t)\mathcal{G}(s) + (1 - q^2)st\frac{W^{-}(t)W^+(s) - W^{-}(s)W^+(t)}{s - t}
\]

and also

\[
\mathcal{G}(s)W^{-}(t) = q\frac{(qs - q^{-1}t)W^{-}(t)\mathcal{G}(s) - (q - q^{-1})sW^{-}(s)\mathcal{G}(t)}{s - t},
\]

\[
W^+(s)\mathcal{G}(t) = q\frac{(q^{-1}s - qt)\mathcal{G}(t)W^+(s) + (q - q^{-1})t\mathcal{G}(s)W^+(t)}{s - t},
\]

\[
\tilde{\mathcal{G}}(s)W^{-}(t) = q^{-1}\frac{(q^{-1}s - qt)W^{-}(t)\tilde{\mathcal{G}}(s) + (q - q^{-1})sW^{-}(s)\tilde{\mathcal{G}}(t)}{s - t},
\]

\[
W^+(s)\tilde{\mathcal{G}}(t) = q^{-1}\frac{(qs - q^{-1}t)\tilde{\mathcal{G}}(t)W^+(s) - (q - q^{-1})t\tilde{\mathcal{G}}(s)W^+(t)}{s - t}
\]

Next we give the nontrivial reduction rules for the PBW basis in Lemma 14.1.
Lemma 14.3. For the algebra $\mathcal{U}_q^+$ the following hold for $i, j \in \mathbb{N}$:

$$W_{i+1}W_j = W_j W_{i+1} + q^{-1}(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (G_{i+j+1-\ell} \tilde{G}_\ell - G_\ell \tilde{G}_{i+j+1-\ell}),$$

$$\tilde{G}_{i+1} \tilde{G}_{j+1} = G_{j+1} \tilde{G}_{i+1} + q(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (W_{\ell-i-j-1} W_{\ell+1} - W_{-\ell} W_{i+j+2-\ell}),$$

and

$$G_{i+1}W_j = W_j G_{i+1} + q(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (W_{-\ell} G_{i+j+1-\ell} - W_{-\ell-i-j-1} G_\ell),$$

$$W_{i+1}G_{j+1} = G_{j+1} W_{i+1} + q(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (G_{i+j+1-\ell} W_{\ell+1} - G_\ell W_{i+j+2-\ell}),$$

$$\tilde{G}_{i+1}W_j = W_j \tilde{G}_{i+1} + q^{-1}(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (W_{\ell-i-j-1} \tilde{G}_\ell - W_{-\ell} \tilde{G}_{i+j+1-\ell}),$$

$$W_{i+1} \tilde{G}_{j+1} = \tilde{G}_{j+1} W_{i+1} + q^{-1}(q^{-1}) \sum_{\ell=0}^{\min(i,j)} (\tilde{G}_\ell W_{i+j+2-\ell} - \tilde{G}_{i+j+1-\ell} W_{\ell+1}).$$

Proof. These relations are obtained by unpacking the equations in Lemma 14.2.

References

[1] P. Baseilhac. The alternating presentation of $U_q(\hat{g}l_2)$ from Freidel-Maillet algebras. *Nuclear Phys. B* 967 (2021) 115400; [arXiv:2011.01572](https://arxiv.org/abs/2011.01572).

[2] P. Baseilhac and K. Koizumi. A new (in)finite dimensional algebra for quantum integrable models. *Nuclear Phys. B* 720 (2005) 325–347; [arXiv:math-ph/0503036](https://arxiv.org/abs/math-ph/0503036).

[3] P. Baseilhac and K. Shigechi. A new current algebra and the reflection equation. *Lett. Math. Phys.* 92 (2010) 47–65; [arXiv:0906.1482v2](https://arxiv.org/abs/0906.1482v2).

[4] J. Beck. Braid group action and quantum affine algebras. *Commun. Math. Phys.* (1994) 555–568.

[5] J. Beck, V. Chari, A. Pressley. An algebraic characterization of the affine canonical basis. *Duke Math. J.* (1999) 455–487; [arXiv:math/9808060](https://arxiv.org/abs/math/9808060).

[6] V. Chari and A. Pressley. Quantum affine algebras. *Commun. Math. Phys.* 142 (1991) 261–283.

[7] I. Damiani. A basis of type Poincare-Birkoff-Witt for the quantum algebra of $\hat{sl}_2$. *J. Algebra* 161 (1993) 291–310.
[8] J. Ding and I. B. Frenkel. Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(n))$. Comm. Math. Phys. 156 (1993) 277–300.

[9] G. Lusztig. Introduction to quantum groups. Progress in Mathematics, 110. Birkhauser, Boston, 1993.

[10] M. Rosso. Groupes quantiques et algèbres de battage quantiques. C. R. Acad. Sci. Paris 320 (1995) 145–148.

[11] M. Rosso. Quantum groups and quantum shuffles. Invent. Math 133 (1998) 399–416.

[12] P. Terwilliger. Using Catalan words and a $q$-shuffle algebra to describe a PBW basis for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. J. Algebra 525 (2019) 359–373; arXiv:1806.11228.

[13] P. Terwilliger. The alternating PBW basis for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. J. Math. Phys. 60 (2019) 071704; arXiv:1902.00721.

[14] P. Terwilliger. The alternating central extension for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. Nuclear Phys. B 947 (2019) 114729; arXiv:1907.09872.

[15] P. Terwilliger. The compact presentation for the alternating central extension of the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. Preprint; arXiv:2011.02463.

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