ON THE NUMBER OF COUNTABLE SUBDIRECT POWERS OF
FINITE COMMUTATIVE SEMIGROUPS

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Abstract. In 1981/82, Hickin & Plotkin and McKenzie both proved that a finite
group has only countably many non-isomorphic subdirect powers if and only if it
is abelian. In this paper, we prove that a finite commutative semigroup has only
countably many non-isomorphic countable subdirect powers if and only if it is either
a finite abelian group or a null semigroup.

1. Introduction

Let $S$ be a semigroup. For a set $X$, the direct power of $S$ by $X$ is the set $S^X$ of
all functions $X \to S$ with multiplication $(fg)(x) = f(x)g(x)$. When $X = \{ 1, \ldots , n \}$
we can identify $S^X$ with the set of all $n$-tuples under component-wise multiplication.
Similarly, when $X = \mathbb{N} = \{ 1, 2, \ldots \}$, we identify $S^X$ with the set of all infinite sequences
$(s_1, s_2, \ldots )$ under the component-wise multiplication.

A subsemigroup $T$ of a direct power $S^X$ is said to be a subdirect power if $T$ projects
onto each of the $S$-factors, i.e. if $\{ f(x) : f \in T \} = S$ for all $x \in X$.

In 1981/82 Hickin and Plotkin [4] and McKenzie [7] proved results, a special case of
which can be formulated as follows:

Theorem 1.1 (Hickin and Plotkin; McKenzie). A finite group has only countably many
non-isomorphic countable subdirect powers if and only if it is abelian, else it has continuum
many such powers. □

Throughout, by a set being countable we mean that it is finite, or that its cardinality is
equal to $|\mathbb{N}| = \aleph_0$. By a countable subdirect power $T \leq S^X$ we mean that $T$ is countable,
rather than $X$ being countable. The continuum $\mathfrak{c}$ is the cardinality of $\{ 0, 1 \}^\mathbb{N}$.

A natural question then arises as to what the situation is for semigroups. In particular,
is it true that finite commutative semigroups have only countably many non-isomorphic
subdirect powers? This turns out to be very far from being the case, and our main
result gives a complete classification. In its statement, a null semigroup $S$ is one where
there is an element $0 \in S$ such that $st = 0$ for all $s, t \in S$.  

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Main Theorem. A finite commutative semigroup \( S \) has only countably many non-isomorphic subdirect powers if and only if it is either an abelian group or a null semigroup; otherwise \( S \) has continuum many countable subdirect powers.

The rest of the paper is devoted to proving the Main Theorem. To simplify the terminology, we will use the phrase subdirect power type, or SDP-type for short, to mean the number of non-isomorphic countable subdirect powers of \( S \). The considerations split into two parts, depending on the number of idempotents in \( S \). In the case where this number is greater than one, we proceed by showing that \( S \) has SDP-type \( c \) when \( S \) is: (i) the two-element semilattice (Proposition 3.1); (ii) an arbitrary non-trivial semilattice (Proposition 3.5); (iii) an arbitrary commutative semigroup with at least two idempotents (Proposition 4.2). The step from (ii) to (iii) is achieved by reference to the decomposition of a semigroup into its archimedean components. For the case of a single idempotent, it is an easy observation that null semigroups are of countable SDP-type (Proposition 5.1). The remaining considerations are to show that the following are of SDP-type \( c \): (i) \( S \) is nilpotent of class \( > 2 \) (Proposition 5.5); (ii) \( S \) is an ideal extension of an abelian group by a null semigroup (Proposition 6.4).

2. Preliminaries and outline of proof

In this section we introduce the basic concepts and facts concerning commutative semigroups, and explain how the subsequent results combine to prove the Main Theorem.

A commutative semigroup \( S \) is said to be a semilattice if all its elements are idempotents, i.e. \( s^2 = s \) for all \( s \in S \). Alternatively, by letting \( s \leq t \iff st = s \), we can view a semilattice as a partially ordered set in which any pair of elements has a greatest lower bound. In particular, any linearly ordered set can be viewed as a semilattice with the operation \( st = \min(s,t) \). These two definitions are equivalent, and the notion of isomorphism is the same in both interpretations.

A semigroup \( S \) with zero is said to be nilpotent of class \( k \) if \( s_1s_2\ldots s_k = 0 \) for all \( s_1, \ldots, s_k \in S \) but \( s'_1\ldots s'_{k-1} \neq 0 \) for some \( s'_1, \ldots, s'_{k-1} \in S \). The non-trivial null semigroups are precisely nilpotent semigroups of class 2.

An ideal in a semigroup \( S \) is a non-empty set \( I \) such that \( SIS \subseteq I \). The Rees congruence associated with an ideal \( I \) is
\[
\rho_I = \{(s, t) \in S \times S : s = t \quad \text{or} \quad s, t \in I\}.
\]
The Rees quotient \( S/I \) is defined as \( S/\rho_I \). We say that a semigroup \( S \) is an ideal extension of a semigroup \( U \) by a semigroup \( V \) if \( S \) has an ideal \( I \) such that \( I \cong U \) and \( S/I \cong V \).

Let \( S \) be a commutative semigroup. The relation \( \eta \) defined by
\[
snt \iff s^k = tu, t^l = sv \quad \text{for some} \quad k, l \in \mathbb{N}, u, v \in S^i
\]
is a congruence. The equivalence classes of \( \eta \) are called archimedean components of \( S \). If \( e \) and \( f \) are distinct idempotents then \( (e, f) \notin \eta \). Thus, every archimedean component contains at most one idempotent. When \( S \) is finite then in fact every archimedean component contains precisely one idempotent, which is the common idempotent power of all the elements in the component. Furthermore, in this case, each archimedean
component is an ideal extension of a group by a nilpotent semigroup. The quotient $S/\eta$ is a semilattice. In fact $\eta$ is the smallest semilattice congruence on $S$. For a more systematic introduction we refer the reader to [3, Section IV.2].

Direct and subdirect powers of a semigroup $S$ were introduced in the previous section. For $s \in S$ and $X \subseteq S$ we let $\bar{s} := (s, s, \ldots) \in S^N$, and $\Delta_X := \{x : x \in X\}$. Note that $\Delta_S$ is the smallest semilattice congruence on $S$. For a more systematic introduction we refer the reader to [3, Section IV.2].

The outline of the proof of the Main Theorem is as follows. Suppose $S$ is a finite commutative semigroup. If $S$ has more than one idempotent, then SDP-type of $S$ is $c$ by Proposition 4.2. So suppose now that $S$ has only a single idempotent. Then $S$ is a single archimedean component of itself, and is an ideal extension of an (abelian) group $G$ by a nilpotent semigroup $V$. Suppose first that $G$ is trivial, in which case $S \cong S/G \cong V$ is nilpotent. If the class $k$ of $S$ is $\leq 2$, then $S$ is a null semigroup, and has a countable SDP-type by Proposition 5.1; otherwise, when $k \geq 3$, the SDP-type of $S$ is $c$ by Proposition 5.5. Now suppose that $G$ is non-trivial. If $V$ is trivial, then $S \cong G$, and it has a countable SDP-type by Theorem 1.1. Finally, if both $V$ and $G$ are non-trivial, $S$ has SDP-type $c$ by Proposition 6.4, and this completes the proof of the Main Theorem.

3. Semilattices

The purpose of this section is to prove that every non-trivial semilattice $S$ has SDP-type $c$. This is done by exhibiting uncountably many non-isomorphic subdirect powers inside $S^N$. In fact, since this result is a stepping stone towards proving that every commutative semigroup with at least two idempotents has SDP-type $c$, we prove more, i.e. that there are uncountably many such subdirect powers consisting of certain special kinds of elements we call recurring. These are sequences $\bar{s} \in S^N$ that can be obtained from finite sequences $\sigma = (s_1, \ldots, s_k)$ $(k \in \mathbb{N})$ by concatenating infinitely many copies together, i.e. $\bar{s} := (s_1, \ldots, s_k, s_1, \ldots, s_k, s_1, \ldots)$.

We begin with the smallest non-trivial semilattice $L_2 = \{0, 1\}$ under the standard multiplication. The direct power $L_2^N$ is an uncountable semilattice, in which the ordering is component-wise, i.e.

$$(s_1, s_2, \ldots) \leq (t_1, t_2, \ldots) \iff s_i \leq t_i \text{ for all } i = 1, 2, \ldots.$$
So, to prove that \( L_2 \) has SDP-type \( c \) we will find a copy of \( Q \) inside \( L_2^N \). This will imply that \( L_2^N \) contains uncountably many non-isomorphic subsemigroups, and we will turn those subsemigroups into subdirect products by adjoining \( \Delta L_2 \) to each of them. Furthermore, we will be able to accomplish all this by using recurring elements only. Note that the set of all recurring elements is countable. While this fact is not so important here, as we are embedding a fixed countable semilattice, namely \( Q \), the use of recurring elements at this point in the argument will become crucial on Section 4, where it will genuinely ensure the countability of the subdirect products constructed there.

**Proposition 3.1.** Let \( L_2 = \{0, 1\} \) be the two-element semilattice, and let \( Q \) be the linearly ordered set of rational numbers.

(i) For any two recurring elements \( \overline{\alpha}, \overline{\beta} \in L_2^N \) with \( \overline{\alpha} < \overline{\beta} \) there exists a recurring element \( \overline{\gamma} \in L_2^N \) such that \( \overline{\alpha} < \overline{\gamma} < \overline{\beta} \).

(ii) \( L_2^N \) contains a subsemilattice isomorphic to \( Q \) consisting of recurring elements.

(iii) \( L_2^N \) contains uncountably many non-isomorphic countable subdirect powers, each of which consists of recurring elements, is a linearly ordered set, and contains \( \overline{0} \) and \( \overline{1} \).

(iv) The SDP-type of \( L_2 \) is \( c \).

**Proof.**

(i) Observe that for any finite sequence \( \sigma \), we have \( \overline{\sigma} = \overline{\tau} \), where \( \tau \) is any finite concatenation of copies of \( \sigma \). Therefore, we may assume without loss that \( \alpha \) and \( \beta \) have the same length, say \( \alpha = (a_1, \ldots, a_k) \), \( \beta = (b_1, \ldots, b_k) \). The assumption \( \overline{\alpha} < \overline{\beta} \) means that \( a_i \leq b_i \) for all \( i = 1, \ldots, k \), and that at least one inequality is strict, say \( a_j = 0 \), \( b_j = 1 \). Now for

\[
\gamma := (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_k, a_1, \ldots, a_{j-1}, 1, a_{j+1}, \ldots, a_k),
\]

we have \( \overline{\alpha} < \overline{\gamma} < \overline{\beta} \).

(ii) This follows by a standard argument: start with \( \overline{0}, \overline{1} \), and iteratively insert a recurring element (whose existence is guaranteed by (i)) between any two points previously constructed. In the limit, this will create a countable, dense linear order with two end-points \( \overline{0}, \overline{1} \). Removing the endpoints leaves us with a copy of \( Q \).

(iii) The copy of \( Q \) identified in (ii) contains copies of all countable linear orders, of which there are uncountably many up to isomorphism. Adding back \( \overline{0}, \overline{1} \) to each of them yields the required family of subdirect powers.

(iv) This is an immediate consequence of (iii). □

We now turn our attention to an arbitrary finite, non-trivial semilattice \( S \). The idea here is to use the uncountably many subdirect powers guaranteed by Proposition 3.1, and ‘insert’ each of them in a suitable part of the diagonal copy \( \Delta S \) of \( S \), to yield uncountably many subdirect powers of \( S \) in \( S^N \), again with additional desired properties to facilitate the proof of Proposition 4.2 in the next section. We now describe the actual construction embodying the above idea.

**Construction 3.2.** Let \( S \) be a finite, non-trivial semilattice. Let \( 0 \) denote the smallest element of \( S \), which must exist because \( S \) is finite. Also, let \( e \) denote an arbitrary minimal non-zero element of \( S \); this must exist because \( S \) is finite and non-trivial. The
Figure 1. The construction of \( \tilde{P} \) from \( S \) and \( P \).

The set \( \{0, e\} \) is a two element subsemilattice of \( S^N \); with a slight abuse of notation let us denote it by \( L_2 \). Now, for a subsemilattice \( P \) of \( L_2^N \) which contains \( \emptyset \) and \( e \), let us define \( \tilde{P} := P \cup \Delta_S \). Intuitively, \( \tilde{P} \) is obtained by taking the semilattice \( S \), and replacing the two element subsemilattice \( L_2 = \{0, e\} \) by an interval isomorphic to \( P \); see Figure 1.

**Lemma 3.3.** \( \tilde{P} \) is a subdirect power of \( S \) in \( S^N \).

**Proof.** Each of \( P \) and \( \Delta_S \) is a subsemigroup of \( S^N \). Now consider arbitrary \( \pi \in P \) and \( \sigma = s \in \Delta_S \). If \( s \geq e \) then \( se = es = e \), and hence \( \sigma \pi = \pi \sigma = \pi \in \tilde{P} \). Similarly, if \( s \not\geq e \), then \( se = es = 0 \) because of minimality of \( e \), and hence \( \sigma \pi = \pi \sigma = \emptyset \in \tilde{P} \). It follows that \( \tilde{P} \) is a subsemigroup, and the elements from \( \Delta_S \) ensure it is a subdirect power. \( \square \)

**Lemma 3.4.** If \( P_1 \) and \( P_2 \) are non-isomorphic infinite subsemilattices of \( L_2^N \) containing \( \emptyset \) and \( e \), then \( \tilde{P}_1 \not\sim \tilde{P}_2 \).

**Proof.** For an element \( \sigma \in \tilde{P}_i \) (\( i = 1, 2 \)), let \( \sigma^\downarrow := \{ \tau \in \tilde{P}_i : \tau \leq \sigma \} \), the principal ideal generated by \( \sigma \), and \( \sigma^\uparrow := \{ \tau \in \tilde{P}_i : \tau \geq \sigma \} \), the principal filter generated by \( \sigma \). In what follows, it will be clear from context whether these sets are meant to be taken in \( \tilde{P}_1 \) or \( \tilde{P}_2 \). Note that

\[
\sigma^\downarrow = P_i, \quad \sigma^\uparrow = \{ s \in S : s \geq e \}.
\]  

Now suppose that there is an isomorphism \( f : \tilde{P}_1 \rightarrow \tilde{P}_2 \). From \( \emptyset \) it follows that, in each \( \tilde{P}_i \), the element \( e \) is the unique element whose principal ideal is infinite, and whose principal filter has size \( |\{ s \in S : s \geq e \}| \). Therefore we must have \( f(e) = e \). But then

\[
f(P_1) = f(\sigma^\downarrow) = f(\sigma^\uparrow) = \sigma^\downarrow = P_2,
\]

contradicting the assumption \( P_1 \not\sim P_2 \), and proving the lemma. \( \square \)

**Proposition 3.5.** Let \( S \) be a non-trivial finite semilattice. Then \( S^N \) contains continuum many non-isomorphic subdirect powers of \( S \), each of which consists of recurring elements and contains the diagonal copy \( \Delta_S \) of \( S \). In particular, \( S \) has SDP-type \( c \).

**Proof.** The result follows by taking \( c \) many subdirect powers of \( L_2 \) guaranteed by Proposition 3.1, applying the above construction to each of them, and appealing to Lemmas 3.3 and 3.4. \( \square \)
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Remark 3.6. Had our sole aim been to prove that non-trivial semilattices have SDP-type $c$, we could have accomplished it much more rapidly than the preceding argument by using Birkhoff’s Subdirect Representation Theorem; e.g. see [8, Theorem 4.44]. According to this theorem, every semilattice is a subdirect product of subdirectly irreducible semilattices. As is well known, the two-element semilattice $L_2$ is in fact the only subdirectly irreducible semilattice (e.g. see [8, Corollary 2 (i), p. 172]), and this proves Proposition 3.1 (iv). One can then deploy a variety of constructions, one of them being our Construction 3.2, to extend this to every finite semilattice having SDP-type $c$. However, we do need the extra information contained in Proposition 3.5 to facilitate the argument in the next section.

4. Commutative semigroups with more than one idempotent

In this section we prove that the SDP-type of every finite commutative semigroup with at least two idempotents is $c$.

Let $S$ be such a semigroup, and let $E = E(S)$ be its semilattice of idempotents. Since $S$ is finite, each archimedean component of $S$ contains precisely one idempotent from $E$. Therefore we can index the archimedean components of $S$ as $A_e$, for $e \in E$, where $e \in A_e$.

Since $|E| > 1$, by Proposition 3.5 the semilattice $E^\mathbb{N}$ contains uncountably many non-isomorphic subdirect powers of $E$, each of which contains the diagonal copy of $E$, and consists of recurring elements. To prove that $S$ has SDP-type $c$ we will associate to each such subdirect power of $E^\mathbb{N}$ a subdirect power of $S^\mathbb{N}$, and prove that they are all non-isomorphic.

Let $\phi: S \to E$ be the natural epimorphism, specifically $\phi(s) = e$ when $s \in A_e$. Extend $\phi$ in a component-wise manner to epimorphisms $S^k \to E^k$ ($k \in \mathbb{N}$) and $S^\mathbb{N} \to E^\mathbb{N}$; these mappings will also be denoted by $\phi$ by a slight abuse of notation. Now, for a subset $U \subseteq E^\mathbb{N}$ consisting of recurring elements, let

$$\hat{U} = \{ \sigma \in S^k, \ k \in \mathbb{N}, \phi(\sigma) \in U \}.$$  

Intuitively, we can think of $\hat{U}$ as follows: take in turn every $\tau = (e_1, \ldots, e_k) \in E^k$ ($k = 1, 2, \ldots$) such that $\tau \in U$, and replace each $e_i$ by all possible elements of its archimedean component $A_{e_i}$. For each $\sigma \in S^k$ thus obtained, $\sigma$ is an element of $\hat{U}$.

Lemma 4.1. If $U$ is a subsemilattice of $E^\mathbb{N}$ and consists of recurring elements, then $\hat{U}$ is a subsemigroup of $S^\mathbb{N}$, with $U$ its semilattice of idempotents.

Proof. Let $\sigma, \tau \in \hat{U}$, with $\phi(\sigma), \phi(\tau) \in U$. By replacing $\sigma$ and $\tau$ by appropriate concatenations, we can assume without loss that they have equal lengths. But then

$$\phi(\sigma\tau) = \phi(\sigma)\phi(\tau) = \phi(\sigma)\phi(\tau) \in U,$$

because $U$ is a subsemilattice. It follows that $\phi(\sigma)\phi(\tau) = \phi(\sigma\tau) \in U$, and so $\hat{U}$ is a subsemigroup. An element $\sigma \in \hat{U}$ is an idempotent if and only if all its entries are idempotents, which is equivalent to $\phi(\sigma) = \phi(\sigma) \in U$. \qed

Proposition 4.2. Every finite commutative semigroup with at least two idempotents has SDP-type $c$. 

Proof. Let $S$ be such a semigroup, and retain all the notation from above. By Proposition 3.5 there exists a family $U_i$ ($i \in I$) of $c$ many pairwise non-isomorphic subdirect powers of $E$ in $E^{\mathbb{N}}$ consisting of recurring elements. By definition, each $\hat{U}_i$ also consists of recurring elements, and hence is countable. Also, each $\hat{U}_i$ is a subdirect power of $S$, because in our construction of $\hat{U}_i$ each occurrence of some $e \in E$ in some tuple in $U_i$ is at some point replaced by all elements from its archimedean component, and the archimedean components partition $S$. Finally, the semigroups $\hat{U}_i$ ($i \in I$) are pairwise non-isomorphic by Lemma 4.1, because $U_i$ can be recovered from $\hat{U}_i$ as its semilattice of idempotents. □

5. Nilpotent semigroups

We now turn our attention to nilpotent semigroups. The status of those of nilpotency class 2, i.e. of null semigroups, is easy to resolve:

**Proposition 5.1.** Every finite null semigroup has SDP-type $\aleph_0$.

**Proof.** A null semigroup is determined up to isomorphism by its size, and so there are only countably many countable null semigroups. □

For the rest of this section we will be concerned with nilpotent semigroups $S$ of class greater than 2. It turns out that we do not need to assume commutativity in our considerations. For two elements $s, t \in S$ we say that $t$ is a (left) divisor of $s$ if $s = tu$ for some $u \in S$. Denote by $\text{Div}_S(s)$ the set of all divisors of $s$.

Now assume that $S$ is finite, and let its nilpotency class be $k > 2$. We aim to show that the SDP-type of $S$ is $c$. To this end we introduce a construction which will yield a family of continuum many non-isomorphic countable subdirect powers of $S$.

**Construction 5.2.** Let $S$ be a finite nilpotent semigroup of nilpotency class $k > 2$. Thus there exists $x = s_1 \ldots s_{k-1} \neq 0$, with $s_1, \ldots, s_{k-1} \in S$, and we must have $xS = Sx = \{0\}$. Keeping in mind that $k > 2$, let $y := s_1 \ldots s_{k-2}$, so that $y \in \text{Div}_S(x)$.

Now consider $S^{\mathbb{N}}$. It is an uncountable nilpotent semigroup of nilpotency class $k > 2$. Thus there exists $x = s_1 \ldots s_{k-1} \neq 0$, with $s_1, \ldots, s_{k-1} \in S$, and we must have $xS = Sx = \{0\}$. Keeping in mind that $k > 2$, let $y := s_1 \ldots s_{k-2}$, so that $y \in \text{Div}_S(x)$.

Now consider $S^{\mathbb{N}}$. It is an uncountable nilpotent semigroup of class $k$. Define

$$
\sigma(i, s) := (0, \ldots, 0, s, 0, 0, \ldots) \quad (i \in \mathbb{N}, \ s \in S),
$$

$$
\chi(i, j) := (0, \ldots, 0, y, x, \ldots, x, 0, 0, \ldots) \quad (i, j \in \mathbb{N}).
$$

For an infinite subset $M = \{m_1, m_2, \ldots\} \subseteq \mathbb{N}$, where $m_1 < m_2 < \ldots$, let

$$
T_M := \{\sigma(i, s) : i \in \mathbb{N}, \ s \in S\} \cup \{\chi(i, j) : i, j \in \mathbb{N}, \ 1 \leq j \leq m_1\}.
$$

The only non-zero products of elements from $T_M$ are:

$$
\sigma(i, s)\sigma(i, t) = \sigma(i, st) \quad (\text{if } st \neq 0),
$$

$$
\sigma(i, s)\chi(i, j) = \sigma(i, sy) \quad (\text{if } sy \neq 0),
$$

$$
\chi(i, j)\sigma(i, s) = \sigma(i, ys) \quad (\text{if } ys \neq 0),
$$

$$
\chi(i, j)\chi(i, l) = \sigma(i, y^2) \quad (\text{if } y^2 \neq 0).
$$
We then immediately conclude:

Lemma 5.3. $T_M$ is a subdirect power of $S$ in $S^\mathbb{N}$. □

Lemma 5.4. For $s \in S \setminus \{0\}$, $i \in \mathbb{N}$, $1 \leq j \leq m_i$, we have

$$|\text{Div}_{T_M}(\sigma(i, s))| = \begin{cases} |\text{Div}(s)| & \text{if } y \notin \text{Div}_S(s), \\ |\text{Div}(s)| + m_i & \text{if } y \in \text{Div}_S(s), \end{cases}$$

and

$$|\text{Div}_{T_M}(\chi(i, j))| = 0.$$

Proof. From (2) we see that $\sigma(i, s)$ certainly has divisors $\sigma(i, t)$, where $t \in \text{Div}_S(s)$. Furthermore, by (1), it will also have all $\chi(i, j)$ ($j = 1, \ldots, m_i$) as divisors, provided $y \in \text{Div}_S(s)$. Finally no $\chi(i, j)$ is a product of two elements of $T_M$, i.e. it has no divisors. □

Proposition 5.5. A finite nilpotent semigroup of class greater than 2 has SDP-type $c$.

Proof. Let $S$ be such a semigroup, with all the notation as in the preceding discussion. Let $n := |S|$. The semigroups $T_M$, where $M$ is an infinite subset of $\{n, 2n, 3n, \ldots\}$, provide a family of continuum many countable subdirect powers by Lemma [5.3]. We prove that they are pairwise non-isomorphic. To this end, let $M$ and $P$ be two distinct subsets of $\{n, 2n, \ldots\}$. Let the elements of $M$ be $m_1 < m_2 < \ldots$, and let those of $P$ be $p_1 < p_2 < \ldots$. Without loss assume that $m_i \notin P$ for some $i$. By Lemma [5.4] the element $\sigma(i, x)$ has $|\text{Div}_S(x)| + m_i$ divisors, because $y \in \text{Div}_S(x)$ by choice of $x$ and $y$. We claim that no non-zero element of $T_P$ has this number of divisors. By Lemma [5.4], possible numbers of divisors for such an element are $0$, $|\text{Div}_S(s)|$ and $|\text{Div}_S(s)| + p_j$, where $s \in S \setminus \{0\}$, $j \in \mathbb{N}$. Since $n = |S|$, we have

$$|\text{Div}_S(x)| + m_i > n > |\text{Div}_S(s)| \geq 0.$$ 

Also, remembering that $m_i, p_j \in \{n, 2n, \ldots\}$ and $m_i \neq p_j$, we have

$$|m_i - p_j| \geq n > |\text{Div}_S(s)| - |\text{Div}_S(x)|,$$

which implies

$$|\text{Div}_S(x)| + m_i \neq |\text{Div}_S(s)| + p_j.$$

Therefore $T_M$ and $T_P$ cannot be isomorphic, and the proposition is proved. □

6. Ideal Extensions of Groups by Nilpotent Semigroups

In this section we prove that every finite ideal extension of a non-trivial group by a non-trivial nilpotent semigroup has SDP-type $c$. Given such a semigroup, we again will detail a construction similar to the nilpotent case in order to do this. Yet again, there is no need to assume commutativity.

Construction 6.1. Let $S$ be a finite semigroup such that its minimal ideal is a non-trivial group $G$, and the quotient $S/G$ is nilpotent of class $k > 1$. Thus we have $s_1 s_2 \cdots s_k \in G$ for all $s_1, \ldots, s_k \in S$, but there exist some $s'_1, \ldots, s'_{k-1} \in S$ such that

$$x := s'_1 \cdots s'_{k-1} \notin G.$$
Notice that $xS \cup Sx \subseteq G$. In particular, $x^t \in G$ for all $t \geq 2$. Denote by $e$ the identity element of $G$, which is also the unique idempotent of $S$. Let $x := ex \in G$, and fix an arbitrary $g \in G$ with $g \neq x$.

The direct power $S^N$ is again an ideal extension of a group by a $k$-nilpotent semigroup. Its minimal ideal is the group $G^N$, but the quotient $S^N/G^N$ is not the same as $(S/G)^N$.

We want to exhibit a family of continuum many countable subdirect powers of $S$ in $S^N$. To this end, for any infinite subset $M = \{m_1, m_2, \ldots \} \subseteq \mathbb{N}$, with $m_1 < m_2 < \ldots$, define

$$W_M := G^\infty \cup \Delta_S \cup U_M,$$

where

$$G^\infty := \{(g_1, g_2, \ldots) \in G^N : g_{p+1} = g_{p+2} = \ldots \text{ for some } p \geq 0\},$$

$$\Delta_S := \{(s, s, \ldots) : s \in S\},$$

$$U_M := \{(e, \ldots, e, g, x, \ldots, x, x, x, \ldots) : p \in \mathbb{N}, \ 1 \leq q \leq m_p\}.$$

**Lemma 6.2.** With the notation as above, each $W_M$ is a countable subdirect power in $S^N$.

**Proof.** That $W_M$ is a subsemigroup of, and subdirect power in, $S^N$ follows from the following facts:

- $G^\infty \leq G^N$;
- $\Delta_S$ is a subdirect power in $S^N$;
- $G^\infty \cup \Delta_S \leq S^N$;
- $W_M U_M \cup U_M W_M \subseteq G^\infty$.

Countability is obvious.\[\square\]

To demonstrate that certain $W_M$ are pairwise non-isomorphic, we will use roots, defined as follows. Let $m := |G|$. Recall that $h^m = e$ and $h^{m+1} = h$ for all $h \in G$, and hence $\sigma^m = \tau$ and $\sigma^{m+1} = \sigma$ for all $\sigma \in G^N$. For $\sigma \in G^\infty$ let

$$m+\sqrt[m]{\sigma} := \{\tau \in W_M \setminus G^\infty : \tau^{m+1} = \sigma\}.$$

**Lemma 6.3.** Suppose $|S| = n$, and let $M$ be an infinite subset of $\{n+1, n+2, \ldots \}$. With the notation as above, we have

$$\{m+\sqrt[m]{\sigma} : \sigma \in W_M\} \cap \{n+1, n+2, \ldots \} = M.$$

**Proof.** The lemma is proved by computing the $(m+1)$st powers of all elements of $W_M \setminus G^\infty = (\Delta_S \setminus \Delta_G) \cup U_M$.

First, however, consider the element $x^{m+1}$. Since $m \geq 2$, we have $x^{m+1} \in G$, and so

$$x^{m+1} = ex^{m+1} = exex^m = \ldots = (ex)^{m+1} = x^{m+1}.$$

Now, for $\sigma \in \Delta_S \setminus \Delta_G$ we have $\sigma^{m+1} \in \Delta_S$, whereas for

$$\sigma = (e, \ldots, e, g, x, \ldots, x, x, x, \ldots) \in U_M$$
we have, using (6),
\[ \sigma^{m+1} = (e, \ldots, e, g, \mathcal{G}, \mathcal{G}, \ldots). \]
(8)

Note that this last element is never in \( \Delta_S \), because \( g \neq \mathcal{G} \). From this, it first of all follows that the elements in \( \Delta_S \setminus \Delta_G \) have \( \leq n \) roots. And then it follows that the elements having more than \( n \) roots are precisely those appearing in (8). The number of roots this element has is precisely the number of possible values of \( q \) in (7), and this is equal to \( m_p \).

Proposition 6.4. Every finite semigroup which is an ideal extension of a non-trivial group by a non-trivial nilpotent semigroup has SDP-type \( \omega \).

Proof. For such a semigroup \( S \) of order \( n \), letting \( M \) range over all infinite subsets of \( \{n+1, n+2, \ldots\} \) yields continuum many semigroups \( W_M \). They are all subdirect powers of \( S \) by Lemma 6.2, and are pairwise non-isomorphic by Lemma 6.3.

\( \square \)

7. Concluding remarks

We have seen that unlike in groups, where having SDP-type \( \aleph_0 \) is equivalent to being commutative by the results of Hickin and Plotkin [4] and McKenzie [7] (Theorem 1.1), for semigroups this is not the case: ‘most’ commutative semigroups in fact have uncountable SDP-type according to our Main Theorem. A natural question arises to classify all semigroups with countable SDP-type. In particular, one might wonder whether such semigroups must be commutative. This, however, is easily seen not to be the case: semigroups of left or right zeros (i.e. semigroups satisfying \( xy = x \) or \( xy = y \) for all \( x, y \)) are non-commutative, but are easily seen to have countable SDP-type, by an argument analogous to the proof for null semigroups (Proposition 5.1). More generally, the same holds for rectangular bands, i.e. semigroups of the form \( S = I \times J \), where \( I \) and \( J \) are any sets, and multiplication is given by \( (i, j)(k, l) = (i, l) \). Clearly, the isomorphism type of a rectangular band is completely determined by the cardinalities \( |I| \) and \( |J| \). An alternative description of rectangular bands is that they are precisely the semigroups satisfying \( x^2 = x \) and \( xyz = xz \) for all \( x, y, z \) [5, Theorem 1.1.3]. It follows that subsemigroups and direct products of rectangular bands are again rectangular bands. Combining these observations together we obtain:

Proposition 7.1. Every finite rectangular band has SDP-type \( \aleph_0 \).

\( \square \)

Note that a rectangular band \( S = I \times J \) is commutative if and only if \( |I| = |J| = 1 \).

One might also wonder whether the direct product of two finite semigroups of SDP-type \( \aleph_0 \) also necessarily has SDP-type \( \aleph_0 \). However, again, this is not the case: the direct product of a non-trivial abelian group \( G \) by a non-trivial null semigroup is a commutative semigroup and is an extension of \( G \) by a (larger) null semigroup, and hence has SDP-type \( \omega \) by the Main Theorem.

As we mentioned in the Introduction, both Hickin and Plotkin [4] and McKenzie [7] prove results that are more general than Theorem 1.1. In particular, McKenzie’s result
states that if $G$ is a non-abelian group of any cardinality, and if $\kappa \geq |G|$ is an infinite cardinal, then $G$ has $2^{\kappa}$ non-isomorphic subdirect powers of cardinality $\kappa$. We may ask:

**Question 7.2.** Is it true that for every semigroup $S$, the number of subdirect powers of $S$ of cardinality $\kappa \geq |S|$ is either $\kappa$ or $2^{\kappa}$?

The above is not the case for unary algebras, as was shown in [10, Proposition 5.3]. Specifically, for the monounary algebra $(\{0, 1, 2\}, f)$ where $f(x) = \max(x - 1, 0)$, the number of non-isomorphic subdirect powers of cardinality $\kappa \geq \aleph_0$ is equal to the number of cardinals $\beta \leq \kappa$. Unary algebras can of course be viewed as monoids acting on sets, and the main result of [10] can be interpreted as a classification of finite monoid acts of SDP-type $\aleph_0$. No further classifications are known to the best of our knowledge, and situation for rings, associative and Lie algebras, loops and lattices seems to be particularly worth investigating.

The question of the number of countable subdirect powers for an algebra is related to the notion of boolean separation. We say that an algebraic structure $A$ is boolean separating if $A^{B_1} \cong A^{B_2}$ implies $B_1 \cong B_2$ for any boolean algebras $B_1, B_2$. Here $A^B$ denotes the boolean power of the algebra $A$ by a boolean algebra $B$; see [2, Section IV.5]. Since boolean powers are subdirect powers and since there are uncountably many countable boolean algebras, it follows that every boolean separating finite algebra has SDP-type $\aleph_0$. Finite boolean separating groups $G$ have been classified in [1]. In particular, when $G$ is subdirectly irreducible, it is boolean separating if and only if it is non-abelian [6].

Motivated by this, and by our work in this paper, we pose:

**Problem 7.3.** Classify finite boolean separating (commutative) semigroups.

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