Generators of Siegel modular function field of higher genus and level

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Dedicated to the late professor Jun-ichi Igusa

Abstract

For positive integers \( g \) and \( N \), let \( F_N \) be the field of meromorphic Siegel modular functions of genus \( g \) and level \( N \) whose Fourier coefficients belong to the \( N \)th cyclotomic field. We present explicit generators of \( F_N \) over \( F_1 \) in terms of quotient of theta constants, when \( g \geq 2 \) and \( N \geq 3 \).

1 Introduction

Let \( g \) and \( N \) be positive integers. We denote by \( F_N \) the field of meromorphic Siegel modular functions of genus \( g \) and level \( N \) whose Fourier coefficients belong to the \( N \)th cyclotomic field (§2). Then, \( F_1 \) is an algebraic function field of transcendence degree \( g(g+1)/2 \) ([5]), and \( F_N \) is a finite Galois extension of \( F_1 \) with \( \text{Gal}(F_N/F_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \) ([11] or [7]).

In particular, when \( g = 1 \), it is well known that

\[
F_N = \begin{cases} 
\mathbb{Q}(j(\tau)) & \text{if } N = 1, \\
F_1(f_v(\tau) \mid v \in \mathbb{Q}^2 \text{ has exact denominator } N) & \text{if } N \geq 2,
\end{cases}
\]

where \( j(\tau) \) is the elliptic modular function and \( f_v(\tau) \) are Fricke functions ([9] and [10]). Moreover, if \( K \) is an imaginary quadratic field and \( \tau_0 \) is a CM-point then the field

\[
K F_N(\tau_0) = K(f(\tau_0) \mid f \in F_N \text{ which is finite at } \tau_0)
\]

is the ray class field of \( K \) with conductor \( N \) ([9 Chapter 10, §1]). To extend the result to CM-field case, we first need to find generators of \( F_N \) for arbitrary genus \( g \).

When \( g = 2 \), Igusa determined in [2] three generators of \( F_1 \) in terms of Siegel Eisenstein series. Furthermore, when \( g = 3 \), Tsuyumine gave in [13] seven explicit generators of \( F_1 \) which are ratios of Siegel modular forms of weight at most 30. For higher genus and level, although Siegel proved in [12] that every function in \( F_N \) (\( N \geq 3 \)) can be expressed as a ratio of theta constants, it seems to be hard to get a finite number of generators of \( F_N \) explicitly.

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Let \( g \geq 2 \) and \( N \geq 2 \). The purpose of this paper is to investigate explicit generators of \( \mathcal{F}_N \) over \( \mathcal{F}_1 \). To this end we shall first introduce the notion of a primitive Siegel family (Definition 3.1). By using the order formula for one-variable Siegel functions (Proposition 4.1 (v)) we shall also give a concrete example of a primitive Siegel family whenever \( N \neq 2, 4 \) and \((2^g - 1) \nmid N\) (Theorem 5.3). Finally, we shall present explicit generators of \( \mathcal{F}_N \) over \( \mathcal{F}_1 \) as quotient of theta constants developed in [6] when \( N \geq 3 \) (Theorem 6.2).

2 Siegel modular functions

As a preliminary we shall briefly review meromorphic Siegel modular functions.

Let \( g \) be a positive integer. For a commutative ring \( R \) with unity \( 1 \), we denote by
\[
\text{GSp}_{2g}(R) = \{ \alpha \in \text{GL}_{2g}(R) \mid \alpha^T J \alpha = \nu(\alpha) J \text{ with } \nu(\alpha) \in R^\times \},
\]
\[
\text{Sp}_{2g}(R) = \{ \alpha \in \text{GSp}_{2g}(R) \mid \nu(\alpha) = 1 \},
\]
where \( J = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \) and \( \alpha^T \) stands for the transpose of \( \alpha \). Then, it is straightforward that \( \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GL}_{2g}(R) \), where \( A, B, C, D \) are \( g \times g \) block matrices, belongs to \( \text{Sp}_{2g}(R) \) if and only if
\[
A^T D - C^T B = I_g, \quad A^T C = C^T A, \quad B^T D = D^T B.
\]
Furthermore, we observe that if \( \alpha \in \text{GL}_{2g}(R) \) belongs to \( \text{Sp}_{2g}(R) \), then so does \( \alpha^T \) ([11, p. 17]).

In particular, the group
\[
\text{GSp}_{2g}(\mathbb{R})_+ = \{ \alpha \in \text{GSp}_{2g}(\mathbb{R}) \mid \nu(\alpha) > 0 \}
\]
acts on the Siegel upper half-space
\[
\mathbb{H}_g = \{ Z \in M_g(\mathbb{C}) \mid Z^T = Z, \ \text{Im}(Z) \text{ is positive definite} \}
\]
by
\[
\alpha(Z) = (AZ + B)(CZ + D)^{-1} \quad (\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GSp}_{2g}(\mathbb{R})_+, Z \in \mathbb{H}_g).
\]

For a positive integer \( N \), let \( \Gamma(N) \) be the principal congruence subgroup of level \( N \) in \( \text{Sp}_{2g}(\mathbb{Z}) \), namely
\[
\Gamma(N) = \{ \alpha \in \text{Sp}_{2g}(\mathbb{Z}) \mid \alpha \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})} \}.
\]
A holomorphic function \( g : \mathbb{H}_g \to \mathbb{C} \) is called a Siegel modular form of weight \( k \) and level \( N \) \( (k \in \mathbb{Z}) \) if it satisfies
\[
g(\alpha(Z)) = \det(CZ + D)^k g(Z) \text{ for all } \alpha \in \Gamma(N).
\]
When \( g = 1 \), we further require that \( g \) is holomorphic at every cusp. Then it can be written as
\[
g(Z) = \sum_M c(M)e\left(\frac{1}{N}\text{tr}(MZ)\right) \quad (c(M) \in \mathbb{C}),
\]
where $M$ runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(z) = e^{2\pi i z}$ ($z \in \mathbb{C}$). We call the above expression the Fourier expansion of $g$, and call $c(M)$ the Fourier coefficients of $g$.

For a subfield $F$ of $\mathbb{C}$, we denote by

$$\mathcal{M}_k(\Gamma(N), F) = \text{the space of Siegel modular forms of weight } k \text{ and level } N \text{ with Fourier coefficients in } F,$$

$$\mathcal{M}_k(F) = \bigcup_{N=1}^{\infty} \mathcal{M}_k(\Gamma(N), F),$$

$$A_0(\Gamma(N), F) = \text{the field of functions of the form } g/h \text{ with } g \in \mathcal{M}_k(F) \text{ and } h \in \mathcal{M}_k(F) \setminus \{0\} \text{ for the same weight } k, \text{ which are invariant under } \Gamma(N).$$

In particular, we let

$$F_N = A_0(\Gamma(N), \mathbb{Q}(\zeta_N)), \text{ where } \zeta_N = e^{1/N}.$$ 

As is well known, $F_N$ is a Galois extension of $F_1$ with

$$\text{Gal}(F_N/F_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$$

([11 Theorem 8.10] or [7 Proposition 2.2]). More precisely, consider the decomposition

$$\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where $G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \mid \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}$. The action of $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $F_N$ is given as follows: Let $f = g/h$ be an element of $F_N$ for some $g, h \in \mathcal{M}_k(\mathbb{Q}(\zeta_N))$ with Fourier expansions

$$g(Z) = \sum_M c(M)e\left(\frac{1}{N}\text{tr}(MZ)\right) \text{ and } h(Z) = \sum_M d(M)e\left(\frac{1}{N}\text{tr}(MZ)\right).$$

(i) An element $\begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$ of $G_N$ acts on $f$ by

$$f\begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} = \frac{\sum_M c(M)^\sigma e\left(\frac{1}{N}\text{tr}(MZ)\right)}{\sum_M d(M)^\sigma e\left(\frac{1}{N}\text{tr}(MZ)\right)},$$

where $\sigma$ is the automorphism of $\mathbb{Q}(\zeta_N)$ defined by $\zeta_N^\sigma = \zeta_N^\nu$.

(ii) An element $\tilde{\gamma}$ of $\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on $f$ by

$$f\tilde{\gamma} = f \circ \gamma,$$

where $\gamma$ is any preimage of $\tilde{\gamma}$ under the reduction $\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2}\}$.  

3
3 Primitive Siegel families

In this section we shall define a primitive Siegel family on which GSp\(_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \cong \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)\) acts in a natural way. By making use of certain members of a primitive Siegel family, we shall construct Siegel modular function fields for various congruence subgroups of Sp\(_{2g}(\mathbb{Z})\).

Let \( g \) and \( N \) be positive integers such that \( N \geq 2 \). Let

\[
\mathcal{I}_N = \{ \mathbf{v} \in \mathbb{Q}^{2g} | N \text{ is the smallest positive integer so that } N\mathbf{v} \in \mathbb{Z}^{2g} \}.
\]

**Definition 3.1.** We call a family \( \{f_{\mathbf{v}}(Z)\}_{\mathbf{v} \in \mathcal{I}_N} \) a *Siegel family* (indexed by \( \mathcal{I}_N \)) if it satisfies the following properties:

1. **(S1)** Every \( f_{\mathbf{v}}(Z) \) belongs to \( \mathcal{F}_N \).
2. **(S2)** The group GSp\(_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \cong \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)\) acts on the family as

\[
 f_{\mathbf{v}}(Z)^\alpha = f_{\alpha^T\mathbf{v}}(Z) \quad (\alpha \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}).
\]

Moreover, we say that the family is *primitive* if

3. **(S3)** \( f_{\mathbf{v}} = f_{\mathbf{v}'} \) if and only if \( \mathbf{v} \equiv \pm \mathbf{v}' \pmod{\mathbb{Z}^{2g}} \).

**Remark 3.2.** Note that if \( \{f_{\mathbf{v}}(Z)\}_{\mathbf{v} \in \mathcal{I}_N} \) is a Siegel family, then so is \( \{f_{\mathbf{v}}(Z)^n\}_{\mathbf{v} \in \mathcal{I}_N} \) for every nonzero integer \( n \).

Let \( \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2g}\} \) be the standard basis for \( \mathbb{R}^{2g} \), and let

\[
\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{2g} \quad \text{and} \quad \mathbf{f} = \mathbf{e}_1 + \cdots + \mathbf{e}_g.
\]

**Proposition 3.3.** If \( \{f_{\mathbf{v}}(Z)\}_{\mathbf{v} \in \mathcal{I}_N} \) is a primitive Siegel family, then we have

\[
\mathcal{F}_N = \mathcal{F}_1 \left( f_{(1/N)e_1}(Z), f_{(1/N)e_2}(Z), \ldots, f_{(1/N)e_{2g}}(Z), f_{(1/N)e}(Z) \right). \tag{1}
\]

**Proof.** Let \( E \) be the field on the right-hand side of (1) which is a subfield of \( \mathcal{F}_N \) by (S1). Suppose that an element \( \alpha = [a_{ij}] \) of GSp\(_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \) leaves \( E \) fixed elementwise. Let \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2g} \) be the row vectors of \( \alpha \). We then derive by (S2)

\[
 f_{(1/N)e_j}(Z) = f_{(1/N)e_j}(Z)^\alpha = f_{(1/N)\alpha^T e_j}(Z) = f_{(1/N)r_j^T}(Z) \quad (j = 1, 2, \ldots, 2g).
\]

Thus we obtain \( (1/N)e_j \equiv \pm (1/N)r_j^T \pmod{\mathbb{Z}^{2g}} \) by (S3), and hence

\[
\alpha \equiv \begin{bmatrix}
\pm 1 & 0 & \cdots & 0 \\
0 & \pm 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \pm 1
\end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})}.
\]
And, since
\[
f_{(1/N)e}(Z) = f_{(1/N)e}(Z)^\alpha = f_{(1/N)\alpha^T e}(Z)
\]
by (S2), we get by (S3)
\[
\begin{bmatrix}
a_{11} \\
a_{22} \\
\vdots \\
a_{2g \times 2g}
\end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pmod{N \cdot \mathbb{Z}^2}.
\]
Hence \(\alpha\) represents the identity element of \(\text{Gal}(F_N/F_1)\), which proves \(F_N = E\), as desired. \(\blacksquare\)

We further consider the following congruence subgroups of \(\text{Sp}_{2g}(\mathbb{Z})\):
\[
\Gamma^1(N) = \left\{ \alpha \in \text{Sp}_{2g}(\mathbb{Z}) \mid \alpha \equiv \begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},
\]
\[
\Gamma_1(N) = \left\{ \alpha \in \text{Sp}_{2g}(\mathbb{Z}) \mid \alpha \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\}.
\]

Let \(F^1_N(\mathbb{Q})\) and \(F_{1,N}(\mathbb{Q})\) be the subfields of \(F_N\) consisting of functions with rational Fourier coefficients which are invariant under \(\Gamma^1(N)\) and \(\Gamma_1(N)\), respectively.

**Proposition 3.4.** Let \(\{f_\nu(Z)\}_{\nu \in \mathcal{I}_N}\) be a primitive Siegel family. Then,

(i) \(F^1_N(\mathbb{Q}) = F_1(f_{(1/N)e_1}(Z), \ldots, f_{(1/N)e_g}(Z), f_{(1/N)\mathfrak{r}}(Z))\).

(ii) \(F_{1,N}(\mathbb{Q}) = \mathbb{Q}\left(f(NZ), f_{(1/N)e_1}(NZ), \ldots, f_{(1/N)e_g}(NZ), f_{(1/N)\mathfrak{r}}(NZ) \mid f(Z) \in F_1\right)\).

**Proof.**

(i) Let \(L = F_1(f_{(1/N)e_1}(Z), \ldots, f_{(1/N)e_g}(Z), f_{(1/N)\mathfrak{r}}(Z))\). For any \(\gamma \in \Gamma^1(N)\) and \(\mathfrak{r} \in (1/N)\mathbb{Z}^g\), we see that
\[
\gamma^T \begin{bmatrix} \mathfrak{r} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \begin{bmatrix} \mathfrak{r} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \mathfrak{r} \\ 0 \end{bmatrix} \pmod{\mathbb{Z}^2}.
\]
This implies that any function in \(L\) is modular for \(\Gamma^1(N)\). Moreover, since
\[
\begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} \mathfrak{r} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \mathfrak{r} \\ 0 \end{bmatrix} \pmod{\mathbb{Z}^2}
\]
for all \(\nu \in (\mathbb{Z}/NZ)^\times\), every function in \(L\) has rational Fourier coefficients. Thus we reach the inclusion \(L \subseteq F^1_N(\mathbb{Q})\).

Now, let \(\alpha = [a_{ij}]\) be an element of \(\text{Sp}_{2g}(\mathbb{Z}/NZ)/\{\pm I_{2g}\}\) which leaves \(L\) fixed. Let \(\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_{2g}\) be the row vectors of \(\alpha\). Since
\[
f_{(1/N)e_j}(Z) = f_{(1/N)e_j}(Z)^\alpha = f_{(1/N)\alpha^T e_j}(Z) \quad (j = 1, \ldots, g)
\]
by (S2), we attain by (S3)
\[
e_j \equiv \pm \alpha^T e_j \equiv \pm \mathfrak{r}_j \pmod{N \cdot \mathbb{Z}^2} \quad (j = 1, \ldots, g).
\]
This gives
\[
\alpha \equiv \begin{pmatrix}
\pm 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \pm 1 & 0 & \cdots & 0 \\
* & \cdots & * & \cdots & * & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
* & \cdots & * & \cdots & * & \cdots
\end{pmatrix} (\text{mod } N \cdot M_{2g}(\mathbb{Z})).
\]

And, since
\[
f_{(1/N)\tilde{f}}(Z) = f_{(1/N)\tilde{f}}(Z)^\alpha = f_{(1/N)\alpha^T\tilde{f}}(Z)
\]
by (S2), we get by (S3) that
\[
\begin{pmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \equiv \pm \alpha^T
\begin{pmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \equiv \pm \begin{pmatrix}
1 & \cdots & 0 & \cdots & * & \cdots \\
\vdots & \ddots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & \pm 1 & 0 & \cdots & 0 \\
* & \cdots & * & \cdots & * & \cdots \\
\vdots & \ddots & \vdots & \ddots & \cdots & \vdots \\
* & \cdots & * & \cdots & * & \cdots
\end{pmatrix} \equiv \pm \begin{pmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix} (\text{mod } N \cdot \mathbb{Z}^{2g}).
\]

Thus we achieve
\[
\alpha \equiv \pm \begin{pmatrix}
I_g & O_g \\
* & *
\end{pmatrix} (\text{mod } N \cdot M_{2g}(\mathbb{Z})).
\]

It then follows from the fact \(\alpha \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}\) that
\[
\alpha \equiv \pm \begin{pmatrix}
I_g & O_g \\
* & I_g
\end{pmatrix} (\text{mod } N \cdot M_{2g}(\mathbb{Z})),
\]
and so \(\alpha\) leaves \(\mathcal{F}_N^1(\mathbb{Q})\) fixed. This implies that
\[
\text{Gal}(\mathcal{F}_N/L) \subseteq \text{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})),
\]
and hence we get the converse inclusion \(L \supseteq \mathcal{F}_N^1(\mathbb{Q})\). Therefore we conclude that \(L = \mathcal{F}_N^1(\mathbb{Q})\).

(ii) Let \(R = \mathbb{Q}(f(NZ), f_{(1/N)e_1}(NZ), \ldots, f_{(1/N)e_g}(NZ), f_{(1/N)\tilde{f}}(NZ) | f(Z) \in \mathcal{F}_1)\). One can readily check that in \(\text{Sp}_{2g}(\mathbb{R})\)
\[
\Gamma^1(N) = \gamma \Gamma_1(N) \gamma^{-1}, \quad \text{where } \gamma = \begin{pmatrix}
\sqrt{N}I_g & O_g \\
O_g & (1/\sqrt{N})I_g
\end{pmatrix}.
\]
Thus we have the isomorphism
\[
\mathcal{F}_N^1(\mathbb{Q}) \rightarrow \mathcal{F}_{1,N}(\mathbb{Q})
\]
\[
f(Z) \mapsto (f \circ \gamma)(Z) = f(NZ).
\]
This proves \(R = \mathcal{F}_{1,N}(\mathbb{Q})\).
4 Theta constants

In this section, we shall give a concrete example of a Siegel family in terms of ratios of theta constants. Furthermore, we shall develop a useful lemma for later sections concerning primitivity of a Siegel family.

Let \( N \geq 2 \). For a vector \( \mathbf{v} = \begin{bmatrix} r \\ s \end{bmatrix} \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2 \), the Siegel function \( g_\mathbf{v}(\tau) \) is defined on the complex upper half-plane \( \mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \) by

\[
g_\mathbf{v}(\tau) = -q^{(1/2)B_2(r)}e(s(r-1)/2)(1-q^r e(s)) \prod_{n=1}^{\infty} (1-q^n r e(s))(1-q^n r e(-s)),
\]

where \( B_2(r) = r^2 - r + 1/6 \) is the second Bernoulli polynomial and \( q = e(\tau) \). It has no zeros nor poles on \( \mathbb{H} \).

For a real number \( x \), let \( \langle x \rangle \) be the fractional part of \( x \) in the interval \([0, 1)\).

**Proposition 4.1.** We have the following properties of Siegel functions:

(i) \( g_\mathbf{v}(\tau)^{12N} \) depends only on \( \pm \mathbf{v} \pmod{\mathbb{Z}^2} \).

(ii) \( g_\mathbf{v}(\tau)^{12N} \) belongs to \( \mathcal{F}_N \).

(iii) Each \( \alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \) acts on the function \( g_\mathbf{v}(\tau)^{12N} \) by

\[
(g_\mathbf{v}(\tau)^{12N})^\alpha = g_{\alpha \mathbf{v}}(\tau)^{12N}.
\]

(iv) Let \( n \) be a nonzero integer. Then \( g_\mathbf{v}(\tau)^{12Nn} = g_{\mathbf{v}'}(\tau)^{12Nn} \) if and only if \( \mathbf{v} \equiv \pm \mathbf{v}' \pmod{\mathbb{Z}^2} \).

(v) \( \text{ord}_q g_\mathbf{v}(\tau) = (1/2)B_2(\langle r \rangle) \).

**Proof.** See [8, §2.1] and [4, Example 3.1]. \( \square \)

**Remark 4.2.** So, for any nonzero integer \( n \), \( \{g_\mathbf{v}(\tau)^{12Nn}\}_{\mathbf{v} \in \mathcal{I}_N} \) is a Siegel family (when \( g = 1 \) which is also called a Fricke family ([8, p. 32–33])). Moreover, it is primitive.

Now, let \( g \) be a positive integer. For \( \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix} \in \mathbb{R}^{2g} \), we denote by

\[
\mathbf{v}_u = \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix}, \quad \mathbf{v}_l = \begin{bmatrix} v_{g+1} \\ \vdots \\ v_{2g} \end{bmatrix}, \quad \langle \mathbf{v} \rangle = \begin{bmatrix} \langle v_1 \rangle \\ \vdots \\ \langle v_{2g} \rangle \end{bmatrix}.
\]

The theta constant \( \theta_\mathbf{v}(Z) \) is defined by the following infinite series

\[
\theta_\mathbf{v}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}^g} e\left( \frac{1}{2}(\mathbf{n} + \mathbf{v}_u)^T Z (\mathbf{n} + \mathbf{v}_u) + (\mathbf{n} + \mathbf{v}_u)^T \mathbf{v}_l \right) \quad (Z \in \mathbb{H}_g).
\]
Igusa ([3, Theorem 2]) showed that \( \theta_v(Z) \) is identically zero if and only if \( \langle v \rangle \) belongs to the set
\[
S_- = \{ a \in \{0, 1/2\}^{2g} \mid e(2a^T_a a) = -1 \}.
\]

Now, let
\[
S_+ = \{0, 1/2\}^{2g} \setminus S_- = \{ b \in \{0, 1/2\}^{2g} \mid e(2b^T_b b) = 1 \}.
\]

One can then readily show that
\[
|S_-| = 2^{g-1}(2^g - 1) \quad \text{and} \quad |S_+| = 2^{g-1}(2^g + 1).
\]

Recently, Koo et al. defined the function for \( v \in \mathcal{I}_N \)
\[
\Theta_v(Z) = 2^{4N} e (-2^g N(2^g - 1)(2^g + 1)v^T_u v) \prod_{a \in S_-} \theta_{a-v}(Z)^{4N(2^g + 1)} \prod_{b \in S_+} \theta_{b}(Z)^{4N(2^g - 1)} \quad (Z \in \mathbb{H}_g)
\]
as a quotient of theta constants.

**Proposition 4.3.** We get the following properties of \( \Theta_v(Z) \):

(i) \( \Theta_v(Z) \) depends only on \( \pm v \) (mod \( \mathbb{Z}^{2g} \)).

(ii) \( \Theta_v(Z) \) belongs to \( \mathcal{F}_N \).

(iii) For every \( \alpha \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/(\pm I_{2g}) \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \), it satisfies \( \Theta_v(Z)^\alpha = \Theta_{\alpha^T v}(Z) \).

**Proof.** See [6, Lemma 4.4 and Proposition 4.5]. \( \blacksquare \)

**Remark 4.4.**

(i) \( \{ \Theta_v(Z) \}_{v \in \mathcal{I}_N} \) becomes a Siegel family satisfying (S1) and (S2).

(ii) If \( g = 1 \), then one can obtain by using Jacobi’s triple product identity that
\[
\Theta_v(\tau) = g_v(\tau)^{12N} \quad (\tau \in \mathbb{H}_1)
\]
([6, Remark 4.3]). This shows that \( \Theta_v \) is a multivariable generalization of the Siegel function \( g_v \).

(iii) One can verify that \( \Theta_v(Z) \) becomes identically zero when \( N = 2 \).

For \( \tau_1, \ldots, \tau_g \in \mathbb{H}_1 \), by \( \text{diag}(\tau_1, \ldots, \tau_g) \) we mean the \( g \times g \) diagonal matrix with diagonal entries \( \tau_1, \ldots, \tau_g \), that is
\[
\text{diag}(\tau_1, \ldots, \tau_g) = \begin{bmatrix}
\tau_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \tau_g
\end{bmatrix}.
\]

Note that \( \text{diag}(\tau_1, \ldots, \tau_g) \) belongs to \( \mathbb{H}_g \).
Lemma 4.5. We have the relation

\[ \theta_\nu(\text{diag}(\tau_1, \ldots, \tau_g)) \quad (\tau_1, \ldots, \tau_g \in \mathbb{H}_1) \]

\[ = \left\{ \begin{aligned}
\prod_{k=1}^g \xi_k & \left[ g_{1/2-v_k} \right] (\tau_k)^g_{1/2} = 1/2 \\
0 & \text{otherwise},
\end{aligned} \right. \]

where \( \xi_k = e((2v_kv_{k+g} + v_k - v_{k+g})/4) \).

Proof. See [1] Example 4.3.

For each \( k = 1, \ldots, 2g \), let

\[ n_{k,0} = |\{ a \in S_- \mid a_k = 0 \}| \quad \text{and} \quad n_{k,1/2} = |\{ a \in S_- \mid a_k = 1/2 \}|. \]

One can readily see from (3) that

\[ n_{k,0} = 2^{2g-2} - 2^{g-1} \quad \text{and} \quad n_{k,1/2} = 2^{2g-2} \quad (k = 1, \ldots, 2g). \] (5)

Since these values do not depend on \( k \), we simply write \( n_0 \) and \( n_{1/2} \) in place of \( n_{k,0} \) and \( n_{k,1/2} \), respectively.

Lemma 4.6. Let \( g \geq 2 \) and \( n \) be a nonzero integer, and let \( v = \begin{bmatrix} v_1 \\ \vdots \\ v_{2g} \end{bmatrix} \), \( v' = \begin{bmatrix} v'_1 \\ \vdots \\ v'_{2g} \end{bmatrix} \in \mathcal{I}_N. \)

Assume that \( \Theta_\nu(Z)^n = \Theta_\nu'(Z)^n \) and \( \begin{bmatrix} \langle v_k \rangle \\ \langle v_{k+g} \rangle \end{bmatrix} \not\subseteq \{0, 1/2\}^2 \) for all \( k = 1, \ldots, g \). Then we attain

\[ \begin{bmatrix} \langle v'_k \rangle \\ \langle v'_{k+g} \rangle \end{bmatrix} \not\subseteq \{0, 1/2\}^2 \quad \text{for all} \quad k = 1, \ldots, g \]

and

\[ n_0 B_2(1/2 + v_k) + n_{1/2} B_2(\langle v'_k \rangle) = n_0 B_2(1/2 + v'_k) + n_{1/2} B_2(\langle v'_k \rangle) \quad \text{for each} \quad k = 1, \ldots, 2g. \] (6)

Proof. Since \( \Theta_\nu(Z)^n = \Theta_\nu'(Z)^n \), we get

\[ \Theta_\nu(\text{diag}(\tau_1, \ldots, \tau_g))^n = \Theta_\nu'(\text{diag}(\tau_1, \ldots, \tau_g))^n \quad (\tau_1, \ldots, \tau_g \in \mathbb{H}_1). \]

It then follows from the definition (2) that

\[ \prod_{\mathbf{a} \in S_-} \theta_\mathbf{a-\nu}(\text{diag}(\tau_1, \ldots, \tau_g))^n = \prod_{\mathbf{a} \in S_-} \theta_\mathbf{a-\nu}^n(\text{diag}(\tau_1, \ldots, \tau_g))^n, \]

where \( \doteq \) stands for the equality up to a root of unity. Here we note that if \( g \geq 2 \) then for any \( x \in \{0, 1/2\}^2 \) and \( k = 1, \ldots, g \), there exists \( \mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \in S_- \) such that \( \begin{bmatrix} \langle a_k \rangle \\ \langle a_{k+g} \rangle \end{bmatrix} = x \). Hence we know by Lemma 4.5 that

\[ \begin{bmatrix} \langle v'_k \rangle \\ \langle v'_{k+g} \rangle \end{bmatrix} \not\subseteq \{0, 1/2\}^2 \quad \text{for all} \quad k = 1, \ldots, g \]
and
\[ \prod_{a \in \mathcal{S}} \prod_{k=1}^{g} g \left( \frac{1}{2-a_k+v_k}, \frac{1}{2-a_k+v_k+g} \right)^{(\tau_k)^{4(2^g+1)n}} \equiv \prod_{a \in \mathcal{S}} \prod_{k=1}^{g} g \left( \frac{1}{2-a_k+v'_k}, \frac{1}{2-a_k+v'_k+g} \right)^{(\tau_k)^{4(2^g+1)n}}. \]

Comparing the orders with respect to \(e(\tau_k)\) by using Proposition 4.4 (v), we obtain
\[ \sum_{a \in \mathcal{S}} B_2(\langle 1/2 - a_k + v_k \rangle) = \sum_{a \in \mathcal{S}} B_2(\langle 1/2 - a_k + v'_k \rangle) \quad \text{for each } k = 1, \ldots, g. \] (7)

On the other hand, acting \( \begin{bmatrix} O_g & I_g \\ -I_g & O_g \end{bmatrix}^T \in \text{Sp}_{2g}(\mathbb{Z}) \) on both sides of \( \Theta_v(Z)^n = \Theta_{v'}(Z)^n \), we have by Proposition 4.3 (iii) that
\[ \Theta \begin{bmatrix} v \cr v' \end{bmatrix}(Z)^n = \Theta \begin{bmatrix} v' \cr v \end{bmatrix}(Z)^n. \]

In exactly the same way as the first part of this proof, one can also achieve
\[ \sum_{a \in \mathcal{S}} B_2(\langle 1/2 - a_k + v_k + g \rangle) = \sum_{a \in \mathcal{S}} B_2(\langle 1/2 - a_k + v'_k + g \rangle) \quad \text{for each } k = 1, \ldots, g. \] (8)

Now, (7) and (8) yield the formula (6).

\[ \Box \]

5 Primitivity of the Siegel family \( \{\Theta_v(Z)\}_{v \in \mathcal{I}_N} \)

Assume that
\[ g \geq 2, \ N \neq 1, 2, 4 \ (2^g - 1) \mid N. \] (9)

In this section we shall prove that the Siegel family \( \{\Theta_v(Z)^n\}_{v \in \mathcal{I}_N} \) is primitive for every nonzero integer \( n \).

**Lemma 5.1.** With the assumption (9), let \( n \) be any nonzero integer, and let \( v, v' \in \mathcal{I}_N \) such that
\[ \begin{bmatrix} \langle v_k \rangle \\ \langle v_{k+g} \rangle \end{bmatrix} \notin \{0, 1/2 \}^2 \text{ for all } k = 1, \ldots, g. \] If \( \Theta_v(Z)^n = \Theta_{v'}(Z)^n \), then we have \( v \equiv \pm v' \pmod{\mathbb{Z}^{2g}} \).

**Proof.** We may assume by Proposition 4.3 (i) that
\[ 0 \leq v_k, v'_k < 1 \quad (k = 1, \ldots, 2g). \] (10)

For simplicity, let
\[ V_k = Nv_k \quad \text{and} \quad V'_k = Nv'_k \quad (k = 1, \ldots, 2g). \]

First, we shall show that \( V_k \equiv \pm V'_k \pmod{N} \) for each \( k = 1, \ldots, 2g \) via the following four steps.

(i) Let \( k \) be an index \( (1 \leq k \leq 2g) \) such that \( 0 \leq v_k, v'_k < 1/2 \). Since we are assuming \( \Theta_v(Z)^n = \Theta_{v'}(Z)^n \), we get by Lemma 4.6 and (10)
\[ n_0B_2(1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(1/2 + v'_k) + n_{1/2}B_2(v'_k). \]
By multiplying both sides by $4N^2$ we establish

$$n_0 \{(N + 2V_k)^2 - 2N(N + 2V_k) + 2N^2/3\} + n_{1/2}(4V_k^2 - 4NV_k + 2N^2/3)$$

$$= n_0 \{(N + 2V_k')^2 - 2N(N + 2V_k') + 2N^2/3\} + n_{1/2}(4V_k'^2 - 4NV_k + 2N^2/3),$$

from which we attain

$$(V_k - V_k') \{(n_0 + n_{1/2})(V_k + V_k') - n_{1/2}N\} = 0.$$ 

If $V_k \neq V_k'$, then we achieve by (5)

$$V_k + V_k' = \frac{n_{1/2}}{n_0 + n_{1/2}} N = \frac{2^{g-1}}{2^g - 1} N,$$

which is not an integer by the assumption $(2^g - 1) \mid N$. Thus we must have $V_k = V_k'$.

(ii) Let $k$ be an index $(1 \leq k \leq 2g)$ such that $0 \leq v_k < 1/2$ and $1/2 \leq v_k' < 1$. We have by Lemma 4.6 and (10)

$$n_0B_2(1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(-1/2 + v_k') + n_{1/2}B_2(v_k').$$

We then see that

$$(V_k + V_k' - N) \{(n_0 + n_{1/2})(V_k - V_k') + n_0N\} = 0.$$ 

If $V_k \neq N - V_k'$, then we derive by (5)

$$V_k - V_k' = -\frac{n_0}{n_0 + n_{1/2}} N = -\frac{(2^{g-1} - 1)}{2^g - 1} N,$$

which is not an integer again by the assumption $(2^g - 1) \mid N$. Hence we should have $V_k = N - V_k'$.

(iii) Let $k$ be an index $(1 \leq k \leq 2g)$ such that $1/2 \leq v_k < 1$ and $0 \leq v_k' < 1/2$. In a similar way to (ii), one can also show that $V_k = N - V_k'$.

(iv) Let $k$ be an integer $(1 \leq k \leq 2g)$ such that $1/2 \leq v_k, v_k' < 1$. We deduce by Lemma 4.6 and (10)

$$n_0B_2(-1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(-1/2 + v_k') + n_{1/2}B_2(v_k').$$

And, we get

$$(V_k - V_k') \{(n_0 + n_{1/2})(V_k + V_k') - (2n_0 + n_{1/2})N\} = 0.$$ 

If $V_k \neq V_k'$, then it follows from (5) that

$$V_k + V_k' = \frac{2n_0 + n_{1/2}}{n_0 + n_{1/2}} N = N + \frac{2^{g-1} - 1}{2^g - 1} N,$$

which is not an integer by the assumption $(2^g - 1) \mid N$. Therefore we are forced to have $V_k = V_k'$. 

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Second, we shall justify that 

\[ V_k \equiv V'_k \pmod{N} \]

for all \( k = 1, \ldots, 2g \), or \( V_k \equiv -V'_k \pmod{N} \) for all \( k = 1, \ldots, 2g \). Since the exact denominator of the vector \( v \) is \( N \) which is \( \geq 3 \) and \( \neq 4 \) by the assumption, one can take an index \( j \) (\( 1 \leq j \leq 2g \)) such that

\[ \text{the exact denominator of } v_j \text{ is } \geq 3 \text{ and } \neq 4. \] (11)

In particular, we have

\[ V_j \not\equiv -V_j \pmod{N}. \] (12)

Here, we may assume that

\[ 1 \leq j \leq g \]

because the action of \( \begin{bmatrix} O_g & I_g \\ -I_g & O_g \end{bmatrix}^T \in \text{Sp}_{2g}(\mathbb{Z}) \) on both sides of \( \Theta_v(Z)^n = \Theta_{v'}(Z)^n \) (if necessary) yields

\[ \Theta_v[-v_j](Z)^n = \Theta_v'[-v'_j](Z)^n \]

by Proposition \[13\](iii). Let \( i \) be an arbitrary index (\( 1 \leq i \leq 2g \)) such that \( i \neq j \). If \( \langle v_i + v_j \rangle, \langle v_i - v_j \rangle \in \{0, 1/2\} \), then we get \( v_j \in \{0, 1/4, 1/2, 3/4\} \), which contradicts (11). So, we must have

\[ \langle v_i + v_j \rangle \neq 0, 1/2 \quad \text{or} \quad \langle v_i - v_j \rangle \neq 0, 1/2. \]

Now, there are four possible cases:

(C1) \( 1 \leq i \leq g \) and \( \langle v_i + v_j \rangle \neq 0, 1/2. \)

(C2) \( 1 \leq i \leq g \) and \( \langle v_i - v_j \rangle \neq 0, 1/2. \)

(C3) \( g + 1 \leq i \leq 2g \) and \( \langle v_i + v_j \rangle \neq 0, 1/2. \)

(C4) \( g + 1 \leq i \leq 2g \) and \( \langle v_i - v_j \rangle \neq 0, 1/2. \)

For each \( 1 \leq r, s \leq g \), let \( E_{rs} \) be the \( g \times g \) matrix with 1 at the entry \((r, s)\) and zeros everywhere else, and let

\[ E'_{rs} = \begin{cases} E_{rs} + E_{sr} & \text{if } r \neq s, \\ E_{rr} & \text{if } r = s. \end{cases} \]

Take

\[ \alpha = \begin{cases} \begin{bmatrix} I_g + E_{ij} & O_g \\ O_g & I_g - E_{ji} \end{bmatrix}^T & \text{for (C1)}, \\ \begin{bmatrix} I_g - E_{ij} & O_g \\ O_g & I_g + E_{ji} \end{bmatrix}^T & \text{for (C2)}, \\ \begin{bmatrix} I_g & O_g \\ E'_{i-gj} & I_g \end{bmatrix} & \text{for (C3)}, \\ \begin{bmatrix} I_g & O_g \\ -E'_{i-gj} & I_g \end{bmatrix} & \text{for (C4)}. \end{cases} \]
which belongs to $\text{Sp}_{2g}(Z)$. Acting $\alpha$ on both sides of $\Theta_v(Z)^n = \Theta_{v'}(Z)^n$, we obtain by Proposition 4.3 (iii)

$$\Theta_u(Z)^n = \Theta_{u'}(Z)^n,$$

(13)

where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_{2g} \end{bmatrix}$, $u' = \begin{bmatrix} u'_1 \\ \vdots \\ u'_{2g} \end{bmatrix} \in \mathcal{I}_N$ such that

$$(u_k, u'_k) = \begin{cases} (v_k, v'_k) & \text{if } k \neq i,j+g, \\
(v_i + v_j, v'_i + v'_j) & \text{if } k = i, \text{ for (C1),} \\
(-v_i + v_j + v'_i + v'_j) & \text{if } k = j + g \\
(v_i, v'_i) & \text{if } k = j + g, \text{ for (C2),} \\
(v_i + v_j + v'_i + v'_j) & \text{if } k \neq i,j+g, \\
(v_i, v'_i) & \text{if } k = j + g, \text{ for (C3),} \\
(v_i + v_j + v'_i + v'_j) & \text{if } k \neq i,j+g, \\
(v_i, v'_i) & \text{if } k = j + g, \text{ for (C4).} \
\end{cases}$$

Here, we observe by (11) that

$$\begin{bmatrix} \langle u_k \rangle \\ \langle u_{k+g} \rangle \end{bmatrix} \not\in \{0,1/2\}^2 \text{ for all } k = 1, \ldots, g.$$  

Thus we deduce by applying the first part of the proof to the equality (13) that

$$\begin{cases} V_i + V_j \equiv \pm(V'_i + V'_j) \pmod{N} \text{ for (C1) and (C3),} \\
V_i - V_j \equiv \pm(V'_i - V'_j) \pmod{N} \text{ for (C2) and (C4).} 
\end{cases}$$

Now, remember that

$$V_i \equiv \pm V'_i \pmod{N} \text{ and } V_j \equiv \pm V'_j \pmod{N}$$

derived in the first part of the proof. For simplicity, let (A) represent the cases (C1) and (C3), and let (B) represent the cases (C2) and (C4). We then have eight possibilities for each case of (A) and (B):

(A1) $V_i + V_j \equiv V'_i + V'_j \pmod{N}$, $V_i \equiv V'_i \pmod{N}$, $V_j \equiv V'_j \pmod{N}$.

(A2) $V_i + V_j \equiv V'_i + V'_j \pmod{N}$, $V_i \equiv V'_i \pmod{N}$, $V_j \equiv -V'_j \pmod{N}$.

(A3) $V_i + V_j \equiv V'_i + V'_j \pmod{N}$, $V_i \equiv -V'_i \pmod{N}$, $V_j \equiv V'_j \pmod{N}$.

(A4) $V_i + V_j \equiv V'_i + V'_j \pmod{N}$, $V_i \equiv -V'_i \pmod{N}$, $V_j \equiv -V'_j \pmod{N}$.
(A5) \( V_i + V_j \equiv -(V'_i + V'_j) \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(A6) \( V_i + V_j \equiv -(V'_i + V'_j) \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

(A7) \( V_i + V_j \equiv -(V'_i + V'_j) \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(A8) \( V_i + V_j \equiv -(V'_i + V'_j) \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

(B1) \( V_i - V_j \equiv V'_i - V'_j \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(B2) \( V_i - V_j \equiv V'_i - V'_j \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

(B3) \( V_i - V_j \equiv V'_i - V'_j \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(B4) \( V_i - V_j \equiv V'_i - V'_j \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

(B5) \( V_i - V_j \equiv -(V'_i - V'_j) \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(B6) \( V_i - V_j \equiv -(V'_i - V'_j) \pmod{N} \), \( V_i \equiv V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

(B7) \( V_i - V_j \equiv -(V'_i - V'_j) \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv V'_j \pmod{N} \).

(B8) \( V_i - V_j \equiv -(V'_i - V'_j) \pmod{N} \), \( V_i \equiv -V'_i \pmod{N} \), \( V_j \equiv -V'_j \pmod{N} \).

One can then readily check that (A2), (A7), (B2) and (B7) contradict (12). Furthermore, (A4) contradicts \( \langle v_i + v_j \rangle \neq 0, 1/2 \), and (B4) contradicts \( \langle v_i - v_j \rangle \neq 0, 1/2 \). Other ten cases yield

\[
\begin{cases}
V_i \equiv V'_i \pmod{N} \text{ and } V_j \equiv V'_j \pmod{N}, & \text{or} \\
V_i \equiv -V'_i \pmod{N} \text{ and } V_j \equiv -V'_j \pmod{N}.
\end{cases}
\]

Therefore, we claim by (12) that

\[
\begin{cases}
V_k \equiv V'_k \pmod{N} \text{ for all } k = 1, \ldots, 2g, & \text{or} \\
V_k \equiv -V'_k \pmod{N} \text{ for all } k = 1, \ldots, 2g,
\end{cases}
\]

and hence \( v \equiv \pm v' \pmod{\mathbb{Z}^{2g}} \), as desired. \( \square \)

**Lemma 5.2.** With the assumption (9), let \( n \) be any nonzero integer, and let \( v, v' \in \mathcal{I}_N \) such that \( \begin{bmatrix} \langle v_{k_i} \rangle \\ \langle v_{k_i+g} \rangle \end{bmatrix} \in \{0, 1/2\}^2 \) for some \( 1 \leq k_1, \ldots, k_m \leq g \). If \( \Theta_v(Z)^n = \Theta_{v'}(Z)^n \), then we have \( v \equiv \pm v' \pmod{\mathbb{Z}^{2g}} \).

**Proof.** As in Lemma 5.1, we take an index \( j \) \( (1 \leq j \leq g) \) such that the exact denominator of \( v_j \) is \( \geq 3 \) and \( \neq 4 \). Then we derive

\( \langle v_{k_i} + v_j \rangle \neq 0, 1/2 \) \ for all \( i = 1, \ldots, m \).

Let

\[
\beta = \begin{bmatrix} I_g + \sum_{i=1}^m E_{k,i} & O_g \\ O_g & I_g - \sum_{i=1}^m E_{j,k_i} \end{bmatrix}^T
\]

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which belongs to \( \text{Sp}_{2g}(\mathbb{Z}) \). Acting \( \beta \) on both sides of \( \Theta_{\nu}(\mathbb{Z})^n = \Theta_{\nu'}(\mathbb{Z})^n \), we get

\[
\Theta_{w}(\mathbb{Z})^n = \Theta_{w'}(\mathbb{Z})^n,
\]

where \( w = \begin{bmatrix} w_1 \\ \vdots \\ w_{2g} \end{bmatrix}, \ w' = \begin{bmatrix} w'_1 \\ \vdots \\ w'_{2g} \end{bmatrix} \in I_N \) such that

\[
(w_k, w'_k) = \begin{cases} 
(v_k, v'_k) & \text{if } k \neq k_1, \ldots, k_m; j + g, \\
(v_k + v_j, v'_k + v'_j) & \text{if } k = k_i (i = 1, \ldots, m), \\
v_{j+g} - \sum_{i=1}^{m} v_{k_i+g}, v'_{j+g} - \sum_{i=1}^{m} v'_{k_i+g} & \text{if } k = j + g.
\end{cases}
\]

Note that

\[
\begin{bmatrix} \langle w_k \rangle \\ \langle w_{k+g} \rangle \end{bmatrix} \not\in \{0,1/2\}^2 \text{ for all } k = 1, \ldots, g.
\]

Then we obtain by applying Lemma 5.1 to the equality (14) that \( w \equiv \pm w' \) (mod \( \mathbb{Z}^{2g} \)), and hence we conclude that \( v \equiv \pm v' \) (mod \( \mathbb{Z}^{2g} \)). \( \Box \)

By Proposition 4.3, Lemmas 5.1 and 5.2 we establish the following result as mentioned in the beginning of this section.

**Theorem 5.3.** Assume that \( g \geq 2, N \neq 1, 2, 4 \) and \( (2^g - 1) \nmid N \). Then, the Siegel family \( \{\Theta_{\nu}(\mathbb{Z})^n\}_{\nu \in I_N} \) is primitive for every nonzero integer \( n \).

### 6 Explicit generators of Siegel modular function fields

By improving Lemmas 5.1 and 5.2 in some special cases we shall obtain our main result on generators of Siegel modular function fields for various congruence subgroups.

**Lemma 6.1.** Let \( n \) be any nonzero integer and \( \nu \in I_N \). Assume that

\[
g \geq 2 \text{ and } N \geq 3.
\]

(i) If \( \Theta_{\nu}(\mathbb{Z})^n = \Theta_{(1/N)\nu}(\mathbb{Z})^n \), then we have \( \nu \equiv \pm (1/N)\nu \) (mod \( \mathbb{Z}^{2g} \)).

(ii) If \( \Theta_{\nu}(\mathbb{Z})^n = \Theta_{(1/N)e}(\mathbb{Z})^n \), then we have \( \nu \equiv \pm (1/N)e \) (mod \( \mathbb{Z}^{2g} \)).

(iii) If \( \Theta_{\nu}(\mathbb{Z})^n = \Theta_{(1/N)e_j}(\mathbb{Z})^n \) for \( 1 \leq j \leq 2g \), then we have \( \nu \equiv \pm (1/N)e_j \) (mod \( \mathbb{Z}^{2g} \)).

**Proof.** (i) By Proposition 4.3 (i) we may assume that \( 0 \leq v_k < 1 \) for all \( k = 1, \ldots, 2g \). We shall first show that

\[
v_k \equiv \begin{cases} 
\pm 1/N \text{ (mod } \mathbb{Z} \text{)} & \text{if } 1 \leq k \leq g \text{ and } N \neq 2^g - 1, \\
\pm 1/N \text{ or } 1/2 \pm 1/2N \text{ (mod } \mathbb{Z} \text{)} & \text{if } 1 \leq k \leq g \text{ and } N = 2^g - 1, \\
0 \text{ (mod } \mathbb{Z} \text{)} & \text{if } g + 1 \leq k \leq 2g
\end{cases}
\]

via the following four steps.
(a) Let \( k \) be an index \((1 \leq k \leq g)\) such that \( 0 \leq v_k < 1/2 \). Since we are assuming \( \Theta_{v}(Z)^n = \Theta_{(1/N)v}(Z)^n \), we get by Lemma 4.6 and the assumption \( N \geq 3 \)

\[
n_0B_2(1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(1/2 + 1/N) + n_{1/2}B_2(1/N).
\]

We then see by (5)

\[
v_k = \frac{1}{N} \quad \text{or} \quad \frac{2^{g-1}}{2^g - 1} - \frac{1}{N}.
\]

If \( v_k = \frac{2^{g-1}}{2^g - 1} - \frac{1}{N} \), then we obtain by the facts \( Nv_k \in \mathbb{Z} \) and \( v_k < 1/2 \) that

\[
(2^g - 1) | N \quad \text{and} \quad N < 2(2^g - 1).
\]

Thus we must have

\[
v_k = \begin{cases} 
1/N & \text{if } N \neq 2^g - 1, \\
1/N \text{ or } 1/2 - 1/2N & \text{if } N = 2^g - 1.
\end{cases}
\]

(b) Let \( k \) be an index \((1 \leq k \leq g)\) such that \( 1/2 \leq v_k < 1 \). We attain by Lemma 4.6 and the assumption \( N \geq 3 \)

\[
n_0B_2(-1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(1/2 + 1/N) + n_{1/2}B_2(1/N).
\]

We then derive by (5)

\[
v_k = 1 - \frac{1}{N} \quad \text{or} \quad \frac{2^{g-1} - 1}{2^g - 1} + \frac{1}{N}.
\]

Suppose \( v_k = \frac{2^{g-1} - 1}{2^g - 1} + \frac{1}{N} \). Then we achieve by the fact \( Nv_k \in \mathbb{Z} \) and \( v_k \geq 1/2 \) that

\[
(2^g - 1) | N \quad \text{and} \quad N \leq 2(2^g - 1).
\]

If \( N = 2(2^g - 1) \), we have \( v_k = 1/2 \) and so \( \langle v_k \rangle = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \) by (c). It contradicts Lemma 4.6. Hence we should have

\[
v_k = \begin{cases} 
1 - 1/N & \text{if } N \neq 2^g - 1, \\
1 - 1/N \text{ or } 1/2 + 1/2N & \text{if } N = 2^g - 1.
\end{cases}
\]

(c) Let \( k \) be an index \((g + 1 \leq k \leq 2g)\) such that \( 0 \leq v_k < 1/2 \). We deduce by Lemma 4.6

\[
n_0B_2(1/2 + v_k) + n_{1/2}B_2(v_k) = n_0B_2(1/2) + n_{1/2}B_2(0).
\]

It then follows from (5) that

\[
v_k = 0 \quad \text{or} \quad \frac{2^{g-1}}{2^g - 1}.
\]

Since \( v_k < 1/2 \), we must take \( v_k = 0 \).
(d) Let $k$ be an index $(g + 1 \leq k \leq 2g)$ such that $1/2 \leq v_k < 1$. We see by Lemma 4.6 that
\[
B_2(-1/2 + v_k) + n_1/2B_2(v_k) = B_2(1/2) + n_1/2B_2(0).
\]

We then claim by (5) that
\[
v_k = 1 \quad \text{or} \quad \frac{2^g - 1}{2^g - 1}
\]
which is impossible because $1/2 \leq v_k < 1$. Therefore this case cannot happen.

Suppose that $v_i \neq v_j \pmod{Z}$ for some $1 \leq i < j \leq g$. Acting the matrix
\[
\begin{bmatrix}
I_g & O_g \\
E_{1i} & I_g
\end{bmatrix}
\]
on both sides of $\Theta_{v}(Z)^n = \Theta_{(1/N)f}(Z)^n$, we deduce by Proposition 4.3 (iii) that
\[
\Theta_{u}(Z)^n = \Theta_{(1/N)f}(Z)^n,
\]
where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_{2g} \end{bmatrix} \in \mathcal{I}_N$ with $u_{g+1} \equiv v_i - v_j \neq 0 \pmod{Z}$. On the other hand, we see from the above claim that $u_{g+1} \equiv 0 \pmod{Z}$, which gives a contradiction. Hence we have
\[
v = \begin{cases} 
\pm(1/N)f \pmod{Z^{2g}} & \text{if } N \neq 2^g - 1, \\
\pm(1/N)f \text{ or } (1/2 \pm 1/2N)f \pmod{Z^{2g}} & \text{if } N = 2^g - 1.
\end{cases}
\]

Now, assume that $N = 2^g - 1$ and $v \equiv (1/2 \pm 1/2N)f \pmod{Z^{2g}}$. If $N = 3$ then $\pm 1/N \equiv 1/2 \mp 1/(2N) \pmod{Z}$. So we further assume that $N \neq 3$. Acting the matrix
\[
\alpha = \begin{bmatrix}
I_g & O_g \\
E_{1i} & I_g
\end{bmatrix}
\]
for any $1 \leq i < j \leq g$ on both sides of $\Theta_{v}(Z)^n = \Theta_{(1/N)f}(Z)^n$, we obtain
\[
\Theta_{\alpha^T v}(Z)^n = \Theta_{(1/N)\alpha^T f}(Z)^n.
\]

Here we observe that $(\alpha^T v)_{g+1} \equiv \pm 1/N \pmod{Z}$ and $((1/N)\alpha^T f)_{g+1} \equiv 2/N \pmod{Z}$. Meanwhile, one can show by Lemma 4.6 that
\[
((1/N)\alpha^T f)_{g+1} \equiv \pm 1/N \quad \text{or} \quad 1/2 \pm 1/2N \pmod{Z},
\]
which is a contradiction. Therefore, we conclude $v \equiv (1/N)f \pmod{Z^{2g}}$.

(ii) Acting $\begin{bmatrix} I_g & O_g \\ -I_g & I_g \end{bmatrix} \in Sp_{2g}(Z)$ on both sides of $\Theta_{v}(Z)^n = \Theta_{(1/N)e}(Z)^n$, we get
\[
\Theta_{[-v + v_1]}(Z)^n = \Theta_{(1/N)f}(Z)^n.
\]
Thus we obtain by (i)\[
\begin{bmatrix}
\mathbf{v}_u \\
-\mathbf{v}_u + \mathbf{v}_t
\end{bmatrix} \equiv \pm (1/N)\mathbf{f} \pmod{\mathbb{Z}^{2g}},
\]
from which it follows that \(\mathbf{v} \equiv \pm (1/N)\mathbf{e} \pmod{\mathbb{Z}^{2g}}\).

(iii) Let\[
\mathcal{A} = \begin{cases}
I_g + \sum_{1 \leq i \leq g, i \neq j} E_{ij} & \text{when } 1 \leq j \leq g, \\
I_g + \sum_{1 \leq i \leq g, i \neq j} E_{ij-g} & \text{when } g + 1 \leq j \leq 2g,
\end{cases}
\]
and let\[
\alpha = \begin{bmatrix}
A & O_g \\
O_g & (A^T)^{-1}
\end{bmatrix}^T \quad \text{if } 1 \leq j \leq g,
\]
\[
\begin{bmatrix}
O_g \\
-(A^T)^{-1} & A
\end{bmatrix}^T \quad \text{if } g + 1 \leq j \leq 2g
\]
which belongs to \(\text{Sp}_{2g}(\mathbb{Z})\). Acting \(\alpha\) on both sides of \(\Theta_{\mathbf{v}}(Z)^n = \Theta_{(1/N)e_j}(Z)^n\), we claim by Proposition 4.3 (iii) that\[
\Theta_{\alpha^*\mathbf{v}}(Z)^n = \Theta_{(1/N)\alpha^*e_j}(Z)^n = \Theta_{(1/N)f}(Z)^n.
\]
Thus we derive (iii) by utilizing (i).

\textbf{Theorem 6.2.} Assume that \(g \geq 2\) and \(N \geq 3\). Then, for any nonzero integer \(n\), we have
\[
\mathcal{F}_N = \mathcal{F}_1(\Theta_{(1/N)e_1}(Z)^n, \ldots, \Theta_{(1/N)e_{2g}}(Z)^n, \Theta_{(1/N)e_j}(Z)^n),
\]
\[
\mathcal{F}_N^1(\mathbb{Q}) = \mathcal{F}_1(\Theta_{(1/N)e_1}(Z)^n, \ldots, \Theta_{(1/N)e_j}(Z)^n, \Theta_{(1/N)f}(Z)^n),
\]
\[
\mathcal{F}_{1,N}(\mathbb{Q}) = \mathbb{Q}(f(NZ), \Theta_{(1/N)e_1}(NZ)^n, \ldots, \Theta_{(1/N)e_j}(NZ)^n, \Theta_{(1/N)f}(NZ)^n | f(Z) \in \mathcal{F}_1).
\]

\textbf{Proof.} Let \(\{f_{\mathbf{v}}(Z)\}_{\mathbf{v} \in \mathcal{I}_N}\) be a Siegel family. Note that in Propositions 3.3 and 3.4 we only require the family \(\{f_{\mathbf{v}}(Z)\}_{\mathbf{v} \in \mathcal{I}_N}\) to satisfy the property
\[
f_{\mathbf{v}}(Z) = f_{\mathbf{v}'}(Z) \iff \mathbf{v} \equiv \pm \mathbf{v}' \pmod{\mathbb{Z}^{2g}}
\]
when \(\mathbf{v}' = \mathbf{e}, \mathbf{f}\) or \(\mathbf{e}_j\) \((1 \leq j \leq 2g)\). Hence the theorem follows from Lemma 6.1.

\textbf{Example 6.3.} In particular, let \(g = 2\). As is well known, we have\[
\mathcal{F}_1 = \mathbb{Q}\left(\frac{E_4(Z)E_6(Z)}{E_{10}(Z)}, \frac{E_6(Z)^2}{E_{12}(Z)}, \frac{E_4(Z)^5}{E_{10}(Z)^2}\right),
\]
where \(E_{2k}(Z)\) is the Siegel Eisenstein series of weight \(2k\) (\[2\] Theorem 3) or \([5\text{ §10 Proposition 3}]\). Thus, if \(N \geq 3\), then we deduce by Theorem 6.2 that
\[
\mathcal{F}_N = \mathbb{Q}\left(\frac{E_4(Z)E_6(Z)}{E_{10}(Z)}, \frac{E_6(Z)^2}{E_{12}(Z)}, \frac{E_4(Z)^5}{E_{10}(Z)^2}, \Theta_{[1/N]}(Z), \Theta_{[0/0]}(Z), \Theta_{[0/1]}(Z), \Theta_{[1/0]}(Z)\right),
\]
\[
\begin{pmatrix}
0 & 1/N \\
0 & 1/N \\
1/N & 1/N
\end{pmatrix}
\begin{pmatrix}
\Theta (Z), \\
\Theta (Z), \\
\Theta (Z)
\end{pmatrix},
\]

\[
\mathcal{F}_1(N) = \mathbb{Q} \left( \frac{E_4(Z) E_6(Z)}{E_{10}(Z)}, \frac{E_6(Z)^2}{E_{12}(Z)}, \frac{E_4(Z)^5}{E_{10}(Z)^2}, \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (Z), \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (Z), \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (Z) \right),
\]

\[
\mathcal{F}_{1,N}(Q) = \mathbb{Q} \left( \frac{E_4(NZ) E_6(NZ)}{E_{10}(NZ)}, \frac{E_6(NZ)^2}{E_{12}(NZ)}, \frac{E_4(NZ)^5}{E_{10}(NZ)^2}, \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (NZ), \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (NZ), \Theta \left( \begin{pmatrix}
1/N \\
0 \\
0
\end{pmatrix} \right) (NZ) \right).
\]

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