Symmetry structure of multi-dimensional time-fractional partial differential equations

Zhi-Yong Zhang and Jia Zheng

College of Science, Minzu University of China, Beijing 100081, People’s Republic of China

E-mail: zzy@muc.edu.cn and zhengjia2014@muc.edu.cn

Received 27 January 2020, revised 5 April 2021
Accepted for publication 20 April 2021
Published 28 June 2021

Abstract
In this paper, we concentrate on the Lie symmetry structure of a system of multi-dimensional time-fractional partial differential equations (PDEs). Specifically, we first give an explicit prolongation formula involving Riemann–Liouville time-fractional derivative for the Lie infinitesimal generator in multi-dimensional case, and then show that the infinitesimal generator has an elegant structure. Furthermore, we present two simple conditions to determine the infinitesimal generators where one is a system of linear time-fractional PDEs, the other is a system of integer-order PDEs and plays the dominant role in finding the infinitesimal generators. We study three time-fractional PDEs to illustrate the efficiencies of the results.

Keywords: symmetry structure, prolongation formula, Riemann–Liouville fractional derivative, time-fractional partial differential equations

Mathematics Subject Classification numbers: 76M60, 26A33, 35R11.

1. Introduction

The theory of fractional calculus goes back to the Leibniz’s letter to L’Hospital [1]. With the rapid developments and extensive applications in the last several decades, nowadays fractional partial differential equations (PDEs) take an important position in describing the phenomena in the fields such as physics, biology and chemistry, where under certain circumstances integer-order PDEs cannot work well [2–4]. For instance, the anomalous diffusion of tracer particles in complex liquids where superdiffusion and subdiffusion occur is more suitable to be described by the fractional PDE [5]. Fractional-order dynamics of the fractional Bloch–Torrey equation, a generalization of the Bloch–Torrey equation by incorporating a fractional order Brownian model of diffusivity, is observed to fit the signal attenuation in diffusion-weighted images obtained from human articular cartilage and human brain [6]. Consequently, considerable

*Author to whom any correspondence should be addressed.
Recommended by Professor Beatrice Pelloni.
 attentions have been paid to study fractional PDEs and thus a number of effective techniques and methods have been proposed [7–11].

Lie group theory provides widely applicable techniques to study integer-order PDEs, for example, constructing similarity solutions and linearized mappings, investigating integrability, analysing stability and global behaviours of solutions, etc [12–15]. Concerning Lie group theory for fractional PDEs, Buckwar and Luchko first established the invariance of a linear fractional diffusion equation describing subdiffusion in the fractal time random walk [16]

$$\partial_t^\alpha u = D u_{xx}, \quad \alpha > 0,$$

under the scaling transformations $\bar{x} = \lambda x$, $\bar{\tau} = \lambda^{2/\alpha} t$, $\bar{u} = u$, where $\partial_t^\alpha$ denotes the Riemann–Liouville fractional derivative, $\lambda$ is a parameter and $D$ is the constant diffusion coefficient. Consequently, by means of the similarity reduction technique [13], equation (1) was transformed into a fractional ordinary differential equation involving the Erdélyi–Kober differential operator and its solutions were expressed by the generalized Wright functions. Gazizov et al performed Lie symmetry analysis for the time-fractional diffusion equations with variable diffusion coefficient $k(u)$

$$\partial_t^\alpha u = (k(u)u_x)_x, \quad 0 < \alpha \leq 2,$$

in the sense of the Riemann–Liouville and Caputo fractional derivatives respectively. The results showed that the admitted Lie symmetries in both fractional-order cases are narrower than the ones of integer-order case due to the effects of time-fractional derivatives but still exert important roles in constructing exact solutions and studying symmetry properties [17, 18].

Following the established schemata symmetry classifications, symmetry reductions and similarity solutions were performed for numerous scalar fractional PDEs where such fractional PDEs originated from either the descriptions of natural phenomena [4, 8] or the direct deformations from the celebrated mathematical physics equations such as the fifth-order Korteweg-De Vries(KdV) equation, Sharma–Tasso–Olver equation, Harry–Dym equation, etc [19–24]. In [25], Jefferson and Carminati wrote an automated package to compute Lie symmetries of fractional differential equations under an assumption which was shown to be correct in [26]. In addition to the lower-dimensional scalar time-fractional PDEs, in fact, multi-dimensional fractional PDEs also took effective roles in describing abnormal behaviours [4], but only a small number of papers extended Lie group theory to study certain special multi-dimensional time-fractional PDEs. For example, Leo et al employed Lie group theory for $(1 + N)$-dimensional fractional PDEs and studied a scalar fractional diffusion-type equation describing the diffusion of charged particle in a magnetic field [27]. Several coupled time-fractional PDEs artificially deformed from classical PDEs were investigated by Lie group theory and affluent exact solutions were constructed [28–30]. In [31], we used Lie symmetry method for a $(1 + 2)$-dimensional time-fractional biological population model which describes the changes of population density at the concerned region and found several exact solutions. Such results further demonstrate that Lie symmetry method is a powerful technique to study fractional PDEs.

The prerequisite of applying the Lie symmetry method is that the fractional PDEs have affluent symmetries, but the determining system of Lie symmetries for fractional PDEs contains the operations of fractional integral and derivative and integer-order derivative. Moreover, the size of the determining system is very large, thus it is not easy to find solutions of the determining system. Such a dilemma motivates us to explore new techniques to simplify the determining system. Observe that knowing the symmetry structure in advance will greatly drop off the number of the symmetry determining equations and further facilitate the equation solving.
[12, 13]. Quite recently in [26], by analysing the structure of the symmetry determining conditions based on the independence of time-fractional integrals and derivatives, we showed that the infinitesimal generators of Lie symmetries for a scalar time-fractional PDE possess a simple and unified expression. Furthermore, the infinitesimal generators are completely determined by two conditions where one is a system of linear time-fractional PDEs and the other is a system of integer-order PDEs, which makes the Lie symmetries of the scalar fractional PDEs more convenient to be found and also pushes the procedure of finding the Lie symmetries more convenient to be performed with the known solvers of integer-order PDEs. Therefore, as the development of Lie group theory from the lower-dimensional scalar fractional PDEs to the system of multi-dimensional PDEs, it is significant to systematically and profoundly study Lie group theory of multi-dimensional fractional PDEs and make clear the general symmetry information.

In this paper, we further investigate the Lie symmetry structure of the system consisting of \( q \) multi-dimensional time-fractional PDEs with \( k \)th order in the sense of Riemann–Liouville fractional derivative, briefly denoted by

\[
\partial_t^\alpha \mathbf{u} = \mathcal{E}(t, \mathbf{x}, \mathbf{u}^{(k)}), \quad 0 < \alpha < 1,
\]

where \( \mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_q) \) is a smooth vector function involving \( p \) independent variables \( \mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p \) and \( q \) dependent variables \( \mathbf{u} = (u_1, \ldots, u_q) \in \mathbb{R}^q \), together with the derivatives of \( u_i \) with respect to the \( x_i(s = 1, \ldots, q; i = 1, \ldots, p) \) up to some order \( k \), denoted by \( \mathbf{u}^{(k)} = \{u_i^j, s = 1, \ldots, q, |\theta| \leq k\} \) with \( u_i^j = \partial^\theta u_i/\partial x_1^{\theta_1} \cdots \partial x_p^{\theta_p}, \theta = (\theta_1, \ldots, \theta_p) \in \mathbb{Z}_p^+ \) (\( \mathbb{Z}_p^+ \) is the nonnegative integer set) and \( |\theta| = \theta_1 + \cdots + \theta_p \leq k \). In order to facilitate the analysis of the symmetry structure, according to whether the terms in right side of system (2) are independent of \( \mathbf{u} \) and its \( \mathbf{x} \)-derivatives, we rearrange it as the following form

\[
\partial_t^\alpha \mathbf{u} = \mathcal{F}(t, \mathbf{x}, \mathbf{u}^{(k)}) + \mathcal{H}(t, \mathbf{x}), \quad 0 < \alpha < 1,
\]

where \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_p) \) and \( \mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_q) \) are two vector functions, \( \mathcal{E}_i = \mathcal{F}_i + \mathcal{H}_i \), each \( \mathcal{F}_i \) collects all the terms containing at least one element of the set \( \{u_1, \ldots, u_q\} \cup \mathbf{u}^{(k)} \) while the remainders in \( \mathcal{E}_i \) are collected in \( \mathcal{H}_i \) which is only a function of \( t \) and \( \mathbf{x} \).

The main contribution of the paper is to show that the infinitesimal generators of Lie symmetries of the multi-dimensional system (3) have a simple and unified form and are completely determined by two conditions similar as the ones of scalar time-fractional PDE. The two key points of achieving the goals are first to find an explicit prolongation formula of the infinitesimal generator in the multi-dimensional fractional case and then to figure out the structure of \( \mu_\alpha \) in (17). It should be pointed that, compared with the scalar time-fractional PDE, the explicit prolongation formula of the infinitesimal generator for the multi-dimensional fractional system (3) is still not quite clear and the expression of \( \mu_\alpha \) also becomes more complex. Therefore, we start with the prolongation formula involving the Riemann–Liouville fractional derivative and then show the symmetry structure and the determining conditions of Lie symmetries for system (3).

The remainder of the article is outlined as follows: in section 2, after recall the definition and related properties of the Riemann–Liouville fractional derivative, we first give an explicit prolongation formula in the multi-dimensional fractional case and then show the general form of the infinitesimal generators of Lie symmetries for system (3), and finally present two simple conditions to determine the infinitesimal generators. In section 3, we use the results to study three types of time-fractional PDEs. The last section concludes the results.
2. Main results

2.1. Preliminaries

We first review the definition and some related properties of Riemann–Liouville fractional derivative, for details please refer to [2, 3].

**Definition 2.1.** The Riemann–Liouville fractional derivative for a continuous function $u = u(t, x)$ in $[0, b] \times \mathbb{R}$ is defined by

$$\partial_t^\alpha u = \frac{\partial^m u}{\partial t^m} \cdot \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\nu)^{m-\alpha-1} u(\nu, x) d\nu, \quad 0 \leq m-1 < \alpha < m,$$

$$\alpha = m \in \mathbb{Z}_+,$$

where the gamma function is $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

By definition 2.1, for a power function $t^\gamma$, we have

$$\partial_t^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} t^{\gamma - \alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma = \alpha - 1. \end{cases}$$

Then for a function $f = f(x)$ independent of $t$, $\partial_t^\alpha f(t) = f(t)^{-\alpha} / \Gamma(1 - \alpha)$, thus $\partial_t^\alpha f = 0$ if and only if $f = 0$. Let $u = u(t, x)$ and $v = v(t, x)$ be two continuous functions in $[0, b] \times \mathbb{R}$ along with all its $t$-derivatives of $u$. Then the Riemann–Liouville fractional derivatives of their sum and product are listed as follows [2, 3]

$$\partial_t^\alpha (au + bv) = a \partial_t^\alpha (u) + b \partial_t^\alpha (v),$$

$$\partial_t^\alpha (uv) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \partial_t^k u \partial_t^{\alpha-k} v,$$

where $a, b$ are two constants, and the second equality is called the generalized Leibniz rule, hereinafter,

$$\binom{\alpha}{k} = \frac{(-1)^{k-1} \alpha \Gamma(k - \alpha)}{\Gamma(1 - \alpha) \Gamma(k + 1)}.$$

In particular for $v = 1$ and the infinite differentiable function $u = u(t, x)$, by the generalized Leibniz rule in (5) the Riemann–Liouville fractional derivative can be expressed as

$$\partial_t^\alpha u = \sum_{k=0}^{m-1} \binom{\alpha}{k} \frac{\partial^k u}{\partial t^k} \cdot \frac{\Gamma(\gamma + 1 - \alpha)}{\Gamma(\gamma + 1)} t^{\gamma - \alpha}, \quad 0 \leq m-1 < \alpha < m,$$

$$\alpha = m \in \mathbb{Z}_+.$$

Meanwhile, we use $D_t^\alpha$ to denote the fractional total derivative with respect to $t$ and define it as [27]

$$D_t^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{\Gamma(\gamma + 1 - \alpha)}{\Gamma(\gamma + 1 - \alpha)} D_t^k,$$

5189
where \( D_t \) is the total derivative in \( t \) and satisfies \( D_t^0(u) = u, D_t^{t+1} = D_t(D_t^t) \). In the case of \( u = u(t, x) \) with two independent variables \( t \) and \( x \), \( D_t \) is defined as

\[
D_t = \partial_t + u_t \partial_u + u_x \partial_u + u_\tau \partial_u + \cdots.
\]

**Lemma 2.2.** Let \( u = u(t, x) \) and \( v = v(t, x) \) be two infinite differentiable functions. Then \( D_t^\alpha \) satisfies the generalized Leibniz rule

\[
D_t^\alpha (u v) = \sum_{i=0}^{\infty} \binom{\alpha}{i} D_t^i u D_t^{\alpha-i} v,
\]

where the operator \( D_t^{\alpha-i} \) is defined by replacing \( \alpha \) in (7) with \( \alpha - i \).

**Proof.** Acting the fractional total derivative \( D_t^\alpha \) on the product \( (u v) \) yields

\[
D_t^\alpha (u v) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{t^k}{\Gamma(k + 1 - \alpha)} D_t^k (u v)
\]

\[
= \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{t^k}{\Gamma(k + 1 - \alpha)} \sum_{i=0}^{k} \binom{k}{i} D_t^i u D_t^{k-i} v
\]

\[
= \sum_{i=0}^{\infty} D_t^i u \sum_{k=i}^{\infty} \binom{\alpha}{k} \frac{k!^2}{\Gamma(k + 1 - \alpha)} D_t^{k-i} v
\]

\[
= \sum_{i=0}^{\infty} \binom{\alpha}{i} D_t^i u D_t^{\alpha-i} v,
\]

where the Leibniz rule of integer-order derivative is used in the second step. The proof ends. \( \square \)

### 2.2. Prolongation formula

Given a one-parameter local Lie symmetry group of transformation

\[
\begin{align*}
\tau^* &= t + \epsilon \tau(t, x, u) + O(\epsilon^2), \\
x_i^* &= x_i + \epsilon \xi_i(t, x, u) + O(\epsilon^2), \\
\dot{u}_i &= u_i + \epsilon \eta_i(t, x, u) + O(\epsilon^2),
\end{align*}
\]

with the group parameter \( \epsilon \), which is completely characterized by the infinitesimal generator [12–14]

\[
\mathcal{X} = \tau(t, x, u) \partial_t + \xi_i(t, x, u) \partial_{x_i} + \eta_i(t, x, u) \partial_{u_i},
\]

where the infinitesimals \( \tau = \tau(t, x, u), \) \( \xi_i = \xi_i(t, x, u) \) and \( \eta_i = \eta_i(t, x, u) \) are determined by

\[
\begin{align*}
\tau &= \frac{d\tau^*}{d\epsilon} \big|_{\epsilon=0}, \\
\xi_i &= \frac{dx_i^*}{d\epsilon} \big|_{\epsilon=0}, \\
\eta_i &= \frac{du_i^*}{d\epsilon} \big|_{\epsilon=0},
\end{align*}
\]

i = 1, \ldots, p, s = 1, \ldots, q$. Thus finding Lie symmetry group (9) is equivalent to determine the infinitesimal generator (10), i.e. $\tau, \xi_i$ and $\eta$. Note that here and in the rest of the paper we assume that $0 < \alpha < 1$ and the summation convention for repeated indices is used unless otherwise noted.

Following the Lie invariance criterion for time-fractional PDEs [16, 18, 27], we find that if system (3) is admitted by the Lie symmetry group (9), then the corresponding infinitesimal generator $\mathcal{X}$ in (10) satisfies two conditions

$$
\mathcal{P}^{(\alpha, k)} \mathcal{X} \left( \partial^\alpha u - \mathcal{F} - \mathcal{H} \right) |_{\partial^\alpha u - \mathcal{F} - \mathcal{H} = 0} = 0
$$

and $\tau(t, x, u)|_{t = 0} = 0$, where $|_\Delta$ means that the evaluations work under the condition $\Delta$. $\mathcal{P}^{(\alpha, k)} \mathcal{X}$ denotes the prolongation of the infinitesimal generator $\mathcal{X}$ in (10) and is defined by

$$
\mathcal{P}^{(\alpha, k)} \mathcal{X} = \mathcal{X} + \eta_\theta^0 \frac{\partial}{\partial \eta_\theta^0} + \sum_\theta \eta_\theta^0(t, x, u^{(0)}) \frac{\partial}{\partial u^{(0)}},
$$

with the second summation being over all $\theta = \{\theta_1, \ldots, \theta_p\} \in \mathbb{Z}_p^p$ and $1 \leq |\theta| \leq k$. The coefficient functions $\eta_\theta^0(t, x, u^{(0)})$ are given by the formula [12, 13]

$$
\eta_\theta^0(t, x, u^{(0)}) = D_\theta \left( \eta_i - \tau \partial_t u_i - \sum_{j=0}^p \xi_j u_i^j \right) + \tau \partial_t u_i^0 + \sum_{j=0}^p \xi_j u_i^{0,j},
$$

where $u_i^j = \partial u_i / \partial x_j$ and $u_i^{0,j} = \partial u_i^0 / \partial x_j, D_\theta = D_{\theta_1} \ldots D_{\theta_k}$ is the $|\theta|$-order total derivative operator with respect to $x$ and $D_{\theta_i} = D_{\theta_i}^0$. The symbol $D_{x_i}$ denotes the total derivative with respect to $x_i$.

$$
D_{x_i} = \partial_{x_i} + u_i \partial_{u_i} + u_{ix_i} \partial_{u_{ix_i}} + u_{ix_j} \partial_{u_{ix_j}} + \cdots
$$

Now we give an explicit expression of $\eta_\theta^0$ in the case of $p$ dependent variables and $q$ independent variables.

**Lemma 2.3.** The coefficient function $\eta_\theta^0$ in (12) related to the Riemann–Liouville time-fractional derivative is expressed by

$$
\eta_\theta^0 = D_\theta^\alpha (\eta_\theta - \tau \partial_t u_\theta - \xi_\theta \partial_{x_\theta} u_\theta) + \tau \partial_t^{\alpha+1} u_\theta + \xi_\theta \partial_t^\alpha (\partial_t \xi_\theta),
$$

where $D_\theta^\alpha$ is the fractional total derivative with respect to $t$ and defined by (7).

**Proof.** Extending the transformation group (9) to fractional derivative $\partial^\alpha u_i$ as well as using the series expression (6) of fractional derivative, we find

$$
\eta_\theta^0 = \frac{d}{dt} \left[ \frac{\partial^\alpha}{\partial (t^\alpha)^\beta} u_\theta(t^\alpha, x^\alpha) \right] |_{t = 0}
$$

$$
= \frac{d}{dt} \left[ \sum_{n=0}^\infty \frac{\alpha}{n} t^{\alpha-n} \frac{\Gamma(n+1-\alpha)}{\Gamma(n)} \partial^\alpha u_\theta(t^\alpha, x^\alpha) \right] |_{t = 0}
$$

$$
= \sum_{n=0}^\infty \frac{\alpha}{n} t^{\alpha-n-1} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1-\alpha)} \partial^\alpha u_\theta + \sum_{n=0}^\infty \frac{\alpha}{n} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1-\alpha)} \eta_\theta^{(\alpha, n)},
$$

where $\eta_\theta^{(\alpha, n)}$ is determined by
Lemma 2.4. An explicit expression of time-fractional total derivative $D^\rho$ in terms of Riemann–Liouville fractional integrals and derivatives is given by

\[
\eta_{t(n)} = D^\rho \left( \eta_t - \tau \frac{\partial u_t}{\partial t} - \xi_i \frac{\partial u_t}{\partial x_i} \right) + \tau \frac{\partial^{n+1} u_t}{\partial t^{n+1}} - \xi_i \frac{\partial^{n+1} u_t}{\partial t^{n+1}}.
\]

Then substituting it into (15), we have

\[
\eta^n_t = \frac{\tau}{\Gamma(n)} \sum_{\alpha=0}^{\infty} \binom{\alpha}{n} \frac{\tau^{n-\alpha}}{\Gamma(n-\alpha)} \eta^{\rho^n+\alpha} u_t \frac{\partial^{\rho^n+\alpha} u_t}{\partial \tau^{\rho^n+\alpha}} + \frac{\tau}{\Gamma(n+1)} \sum_{\alpha=0}^{\infty} \binom{\alpha}{n} \frac{\tau^{n-\alpha}}{\Gamma(n+1-\alpha)} \eta^{\rho^n+\alpha} \frac{\partial^{\rho^n+\alpha} u_t}{\partial \tau^{\rho^n+\alpha}} + \xi_i \frac{\partial^{\rho^n+\alpha} u_t}{\partial \tau^{\rho^n+\alpha}}
\]

where property (6) is used repeatedly. The proof ends.

Note that the integer-order prolongation formula in [13] is immediately recovered in the limit case $\alpha \rightarrow 1$ while the $\alpha$th-order one in $(1 + N)$-dimensional case in [27] is obtained by choosing $s = 1$.

**Lemma 2.4.** An explicit expression of time-fractional total derivative $D^\rho_t$ in terms of Riemann–Liouville fractional integrals and derivatives is given by

\[
D^\rho_t(\eta_t) = \frac{\partial^\alpha \eta_t}{\partial \tau^\alpha} + \sum_{i=1}^{q} \left[ \frac{\partial \eta_t}{\partial u_i} \frac{\partial^{\rho} u_i}{\partial \tau^{\rho}} - u_i \frac{\partial^\alpha}{\partial u_i} \left( \frac{\partial \eta_t}{\partial u_i} \right) \right] + \sum_{i=1}^{q} \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^\alpha}{\partial \tau^\alpha} \left( \frac{\partial \eta_t}{\partial u_i} \right) \frac{\partial^{\rho-n} (u_i)}{\partial t^{\rho-n}} + \mu_s,
\]

where

\[
\mu_s = \sum_{n=2}^{\infty} \binom{\alpha}{n} \frac{\tau^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{m_1=1}^{n} \binom{n-1}{m_1} \frac{1}{m_{j+1}} \prod_{j=1}^{n-m_1} \left( n - \sum_{b=1}^{M_j} m_b \right) \\
\times \sum_{k_1=0}^{n} \cdots \sum_{k_q=0}^{n} \prod_{i=1}^{q} k_i^{x_i} \left( \frac{1}{k_i} \right) (-u_i)^{y_i} \frac{\partial^{m_i}}{\partial u_i^{m_i}} (u_i^{1-y_i})
\]

with the indexes satisfying $k_1 + \cdots + k_q = k \geq 2$ and $m_0 + m_1 + m_2 + \cdots + m_q = n$. 

5192
Proof. By means of the generalized Leibniz rule of fractional derivative in (5) and the generalized chain rule of integer-order derivative for a composite function, we obtain

\[ D_t^\alpha(\eta_t) = \sum_{n=0}^{\infty} \left( \frac{\alpha}{n} \right) \frac{\rho^{n-\alpha}}{\Gamma(n+1-\alpha)} D_t^n(\eta_t) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{\alpha}{n} \right) \frac{\rho^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{m_1+\cdots+m_q=0}^n \binom{n}{m_1} \times \prod_{a=1}^{q-1} \left( n - \sum_{b=1}^{a} m_b \right) \frac{\partial^q \eta_t}{\partial \eta_t^q} \left( t, x, u(t_1, x), \ldots, u(t_q, x) \right) \bigg|_{t_1=\cdots=t_q=t} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{\alpha}{n} \right) \frac{\rho^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{m_1+\cdots+m_q=0}^n \binom{n}{m_1} \prod_{j=1}^{q-1} \left( n - \sum_{b=1}^{j} m_b \right) \sum_{k_1=0}^{m_1} \cdots \sum_{k_q=0}^{m_q} \prod_{i=1}^{q} \left( \frac{k_i}{k_i!} \sum_{r_i=0}^{k_i} \frac{(-u_i)^{r_i}}{r_i!} \frac{\partial^{m_i} \left( u_i^{k_i-r_i} \right)}{\partial u_i^{k_i-r_i}} \right) \times \frac{\partial^{m_0}}{\partial u_t^{m_0}} \left( \frac{\partial^q \eta_t}{\partial u_t^q} \right), \tag{18} \]

where \( k_1 + \cdots + k_q = k \) and \( m_0 + m_1 + m_2 + \cdots + m_q = n \).

By means of the Faà di Bruno formula for the \( m_t \)-th order \( t \)-derivative of \( u_t^{k_i-r_i} \) and the direct computations, we find

\[ \sum_{r_i=0}^{k_i} \left( \frac{k_i}{r_i!} \sum_{r_i=0}^{k_i} \frac{(-u_i)^{r_i}}{r_i!} \frac{\partial^{m_i} \left( u_i^{k_i-r_i} \right)}{\partial u_i^{k_i-r_i}} \right) = \sum_{r_i=0}^{k_i} \left( \frac{k_i}{r_i!} \sum_{r_i=0}^{k_i} \frac{(-u_i)^{r_i}}{r_i!} \frac{\partial^{m_i} \left( u_i^{k_i-r_i} \right)}{\partial u_i^{k_i-r_i}} \right) \]

\[ \times \left( \frac{(m_i)!}{a_1! \cdots a_m_i!} \prod_{j=1}^{m_i} \left( \frac{\partial^j u_i}{\partial u_i^j} \right)^{a_j} \right), \tag{19} \]

where \( a_j \) are nonnegative integers, the second sum works on \( a_1 + \cdots + a_m_i = a \leq (k_i - r_i) \) and \( a_1 + 2a_2 + \cdots + m_i a_{m_i} = m_i \). Moreover, each term in (19) is homogeneous in \( u_t \) and its \( t \)-derivatives and the total degree is \( k_t \). Thus to isolate the linear terms in \( u_t \) and its \( t \)-derivatives form (18) is equivalent to search for the terms with the indexes satisfying \( k = k_1 + \cdots + k_q \leq 1 \). The solutions of this inequality are divided into \( q + 1 \) cases where the first one is \( k_i = 0 \) for each \( i \) \( \in \mathbb{Z}_+^q \), the other \( q \) cases are \( k_i = 1 \) for some \( i \) \( \in \mathbb{Z}_+^q \) and \( k_j = 0 \) for all other \( j \neq i \) \( \in \mathbb{Z}_+^q \), where \( \mathbb{Z}_+^q \) denotes the set of positive integer not greater than \( q \).

Case I. For each \( i \) \( \in \mathbb{Z}_+^q \), \( k_i = 1 \) means \( k = 0 \) and requires \( m_i = 0, m_0 = n \) for all \( i \) \( \in \mathbb{Z}_+^q \), otherwise \( \partial^{m_i} \left( u_i^{k_i-r_i} \right) / \partial u_i^{k_i-r_i} = 0 \). Thus this case corresponds to the term \( P_1 = \partial^q \eta_t / \partial t^q \), where property (6) is used.

Case II. Consider the last \( q \) cases. We fix \( i = 1 \) and all \( k_j = 0 \) with \( j \neq i \) \( \in \mathbb{Z}_+^q \). Then similar as case I, for each \( j \neq i, k_j = 0 \) requires \( m_j = 0 \) and then \( k_i = k = 1, m_0 + m_i = n, \) where \( m_i \in [1, n] \) is a positive integer since \( m_i \geq k_i = 1 \). Meanwhile, the value of \( r_i \) associated with \( k_i \) is divided into two cases \( k_i = r_i = 1 \) and \( k_i = 1, r_i = 0 \). In the former
case, since $u_{i}^{k-i} = u_{i}^{0} = 1$ and $m_{i} \geq 1$, then $\partial^{m_{i}}(u_{i}^{k-i})/\partial \partial^{m_{i}} = 0$ and thus all terms vanish.

For the latter case, the terms with the given $i$ are collected as

$$
P_{i}^{I} = \sum_{n=0}^{\infty} \left( \frac{\alpha}{n} \right) \frac{\mu^{n-\alpha}}{\Gamma(n + 1 - \alpha)} \sum_{m=1}^{n} \left( \frac{\partial^{m} u_{i}}{\partial \partial^{m}} \right) \frac{\partial^{n-m} \left( \partial \partial^{m} \right)}{\partial \partial^{n-m}} \left( \frac{\partial \eta_{i}}{\partial u_{i}} \right)
$$

where property (6) and the generalized Leibniz rule in (5) are used in the third and last steps respectively.

Therefore, separating the term $P_{i}$ in case I and the terms $\sum_{i=1}^{q} P_{i}^{I}$ in case II from $D_{\tau}^{r}(\eta_{i})$, we obtain the explicit expression of $D_{\tau}^{r}(\eta_{i})$ in (16). It completes the proof.

By lemmas 2.3 and 2.4, we give an explicit expression of $\eta_{i}^{\mu}$.

**Theorem 2.5.** An explicit expression of the coefficient function $\eta_{i}^{\mu}$ is given by

$$
\eta_{i}^{\mu} = \frac{\partial^{r} \eta_{i}}{\partial \tau^{r}} + \sum_{i=1}^{q} \left[ \frac{\partial \eta_{i}}{\partial u_{i}} \frac{\partial^{r} u_{i}}{\partial \tau^{r}} - u_{i} \frac{\partial^{r} \eta_{i}}{\partial \tau^{r}} \left( \frac{\partial \eta_{i}}{\partial u_{i}} \right) \right]
$$

$$
+ \sum_{i=1}^{q} \sum_{k=1}^{\infty} \left( \frac{\alpha}{k} \right) \frac{\partial^{k} \left( \frac{\partial \eta_{i}}{\partial u_{i}} \right)}{\partial \tau^{k}} \frac{\partial \delta_{i}^{-k}(u_{i})}{\partial \tau^{k}} - \sum_{k=0}^{\infty} \left( \frac{\alpha}{k+1} \right) D_{\tau}^{k+1}(\tau) \frac{\partial \delta_{i}^{-k}(u_{i})}{\partial \tau^{k}}
$$

$$
- \sum_{k=0}^{\infty} \left( \frac{\alpha}{k} \right) D_{\tau}^{k}(\xi_{i}) \frac{\partial \delta_{i}^{-k}(u_{i})}{\partial \tau^{k}} + \mu_{s},
$$

where $\mu_{s}$ is given by (17).

**Proof.** Using the generalized Leibniz rule for $D_{\tau}^{r}$ in lemma 2.2, we have

$$
D_{\tau}^{r}(\xi_{i}, u_{i}) = \xi_{i} D_{\tau}^{r}(\partial_{i}, u_{i}) + \sum_{k=1}^{\infty} \left( \frac{\alpha}{k} \right) D_{\tau}^{k}(\xi_{i}) D_{\tau}^{r-k}(\partial_{i}, u_{i}),
$$

$$
D_{\tau}^{r}(\tau, u_{i}) = \sum_{k=0}^{\infty} \left( \frac{\alpha}{k} \right) D_{\tau}^{k}(\tau) D_{\tau}^{r-k+1}(u_{i})
$$

$$
= \tau D_{\tau}^{r+1}(u_{i}) + \sum_{k=1}^{\infty} \left( \frac{\alpha}{k} \right) D_{\tau}^{k}(\tau) D_{\tau}^{r-k+1}(u_{i})
$$

$$
= \tau D_{\tau}^{r+1}(u_{i}) + \sum_{k=0}^{\infty} \left( \frac{\alpha}{k+1} \right) D_{\tau}^{k+1}(\tau) D_{\tau}^{r-k}(u_{i}).
$$
Then inserting (16) and (21) into (14) yields (20) since \( D_t^{\alpha-k}(u_s) = \partial_{\tau}^{\alpha-k}(u), D_t^{\alpha-k}(\partial_s u_s) = \partial_{\tau}^{\alpha-k}(\partial_s u_s) \). It completes the proof. \( \square \)

The expression of \( \eta''_s \) in (20) includes the previous prolongation formulas in [22, 28–30] as special cases and also revises some inaccurate expressions of \( \mu_s \). In particular, for the \((1+1)-\)dimensional case, i.e. \( x_1 = x, u_1 = u, \eta_s = \eta \), then formula (20) becomes [26]

\[
\eta'' = \partial_t^\alpha \eta + [\eta_s - \alpha D_t(\tau)] \partial_t^\alpha u - u \partial_t^\alpha(\eta_s) + \mu - \sum_{k=1}^{\infty} \left( \begin{array}{c} \alpha \\ k \end{array} \right) D_t^k(\xi) \partial_t^{\alpha-k}(u_s)
+ \sum_{k=1}^{\infty} \left[ \left( \begin{array}{c} \alpha \\ k \end{array} \right) \partial_t^k(\eta_s) - \left( \begin{array}{c} \alpha \\ k+1 \end{array} \right) D_t^{k+1}(\tau) \right] \partial_t^{\alpha-k}(u),
\]

where

\[
\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \sum_{r=0}^{n-1} \left( \begin{array}{c} \alpha \\ n \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \frac{1}{k!} \frac{r^{n-\alpha-u-1}}{\Gamma(n+1-\alpha)} \frac{\partial^{n-m} u_{x_1}^{\alpha-m}}{\partial u^r}.
\]

Therefore, by means of coefficients formulas (13) and (20), we get an explicit expression of the prolongation \( Pr(\alpha,k) X' \) in (12) for the multi-dimensional fractional case. Then the procedure for searching the infinitesimals \( \tau, \xi_i \) and \( \eta_s \) in (10) is similar as the one of integer-order PDEs. Thus substituting the formula (12) into condition (11) and then annihilating to zero first the coefficients of time-fractional integrals and derivatives of \( u \) and followed by the coefficients of integer-order \( x \)-derivatives of \( u \), we obtain an over-determined system for \( \tau, \xi_i \) and \( \eta_s \) which includes integer-order derivative and fractional integral and derivative operations. Then solving the determining system together with the condition \( \tau(t,x,u)|_{t=0} = 0 \) gives the infinitesimal generator (10).

### 2.3. Symmetry structure

It is well-known that finding solutions of the determining system of Lie symmetry is a rather labour-consuming task for integer-order PDEs [12–14], not to mention herein the system involving fractional integral and derivative operations. Thus in this section, we analyse the symmetry structure of system (3) and show that the infinitesimal generator \( X' \) in (10) has a simple and unified expression. Such a scenario of knowing the structure of infinitesimal generator in advance will simplify the symmetry determining system largely. Note that in the procedure of searching for Lie symmetries, all time-fractional integrals and derivatives arising in the prolongation formulas are considered as independent variables.

**Lemma 2.6.** Let \( \mu_s \) be given in (17). A necessary and sufficient condition for \( \mu_s = 0 \) is that \( \eta_s = \eta_s(t,x,u) \) is linear in \( u_t \).

**Proof.** By the expression of \( \mu_s \) in (17), if \( \eta_s \) is linear in \( u_t \), then \( \partial^k \eta_s/\partial u_1^{k_1} \ldots \partial u_q^{k_q} = 0 \) with \( k = k_1 + \ldots + k_q \geq 2 \) and thus \( \mu_s = 0 \). The sufficiency holds. Next we prove necessity, i.e. prove \( \partial^k \eta_s/\partial u_t^k = \partial^2 \eta_s/\partial u_t \partial u_t = 0 (i \neq l) \). By the proof of lemma 2.4, we find that all terms in \( \mu_s \) are nonlinear in \( u_t \) and their \( t \)-derivatives.

Consider the term \( (\partial_t u_t)^2 \) which occurs uniquely for \( m_i = 2, m_j = 0 (j \neq i), i, j = 1, \ldots, q \). Thus we separate the case \( m_i = 2 \) from \( \mu_s \) and rewrite \( \mu_s \) as the following form
we further arrange (24) as the form

\[ \mu_3 = \sum_{n=2}^{\infty} \left( \frac{\alpha}{n} \right) \frac{n^{\alpha}}{n+1-n} \sum_{r_j=0}^{2} \left[ \frac{1}{2!} \binom{2}{r_j} \left( -u \right)^{n-1} \frac{\partial^2}{\partial u_1^2} \left( u_1^{n-1} \right) \right] \times \frac{\partial^{n-2}}{\partial u_1^{n-2}} \left( \frac{\partial^2 }{\partial u_1^2} \right) + \text{remainder} \]

\[ = \left[ \sum_{n=2}^{\infty} \left( \frac{\alpha}{n} \right) \frac{n^{\alpha}}{n+1-n} \frac{\partial^{n-2}}{\partial u_1^{n-2}} \left( \frac{\partial^2 }{\partial u_1^2} \right) \right] (\partial \eta_i)^2 + \text{remainder.} \tag{22} \]

For the coefficient of \((\partial \eta_i)^2\) in (22), let \(\lambda = n-2\), then it becomes

\[ \sum_{\lambda=0}^{\infty} \left( \frac{\alpha}{\lambda + 2} \right) \left( \lambda + 2 \right) \frac{\lambda^{\alpha-2+\alpha}}{\lambda (\lambda + 3 - \alpha) \partial^\lambda \left( \frac{\partial^2 }{\partial u_1^2} \right)} \]

\[ = \frac{1}{2} \alpha (\alpha - 1) \sum_{\lambda=0}^{\infty} \left( \frac{\alpha - 2}{\lambda} \right) \frac{\lambda^{\alpha-2+\alpha}}{\lambda (\lambda + 3 - \alpha) \partial^\lambda \left( \frac{\partial^2 }{\partial u_1^2} \right)} \]

\[ = \frac{1}{2} \alpha (\alpha - 1) \partial^{n-2} \left( \frac{\partial^2 }{\partial u_1^2} \right), \tag{23} \]

where property (6) is used. By the uniqueness of \((\partial \eta_i)^2\), annihilating its coefficient to zero yields \(\partial^{n-2} (\partial^2 \eta_i / \partial u_1^2) = 0\) which means \(\partial^2 \eta_i / \partial u_1^2 = C(x, u) r^{n-3}\) with an undetermined function \(C(x, u)\). Then further splitting the case \(k = k_i = 2\) from \(\mu_3\), which implies \(k_j = m_j = 0\) for \(j \neq i\), thus we rewrite \(\mu_3\) in the form

\[ \mu_3 = C(x, u) \left\{ \sum_{n=3}^{\infty} \left( \frac{\alpha}{n} \right) \frac{n^{\alpha}}{n+1-n} \sum_{m_i=3}^{n} \left( \frac{n}{m_i} \right) \right. \]

\[ \times \left[ \sum_{r_j=0}^{2} \left[ \frac{1}{2!} \binom{2}{r_j} \left( -u \right)^{n-1} \frac{\partial^m_0}{\partial u_i^m} \left( u_1^{n-1} \right) \right] \frac{\partial^m_0}{\partial u_i^m} \left( r^{n-3} \right) \right] + \text{remainder} \]

\[ = C(x, u) \left[ \sum_{n=3}^{\infty} \left( \frac{\alpha}{n} \right) \frac{n^{\alpha}}{n+1-n} \sum_{m_i=3}^{n} \left( \frac{n}{m_i} \right) \frac{\partial^m_0}{\partial u_i^m} \left( r^{n-3} \right) \right] \]

\[ \times \left[ \sum_{k_i=1}^{m_i-1} \left( \frac{m_i-1}{k_i} \right) \frac{\partial^k_1 u_i}{\partial u_i^k} \frac{\partial^m_0}{\partial u_i^m} \right] + \text{remainder.} \tag{24} \]

Then by interchanging the order of summations and adopting the technique used in (23), we further arrange (24) as the form

\[ \mu_3 = C(x, u) \left[ \sum_{j=3}^{\infty} \sum_{n=3}^{\infty} \left( \frac{\alpha}{n} \right) \frac{n^{\alpha}}{n+1-n} \left( \frac{n}{j} \right) \frac{\partial^m-j}{\partial u_i^{m-j}} \left( r^{n-3} \right) \right] \]

\[ \times \left[ \sum_{k_i=1}^{m_i-1} \left( \frac{m_i-1}{k_i} \right) \frac{\partial^k_1 u_i}{\partial u_i^k} \frac{\partial^m_0}{\partial u_i^m} \right] + \text{remainder} \]
\[ C(x, u) \left[ \sum_{j=3}^{\infty} \binom{\alpha}{j} \sum_{k=0}^{\infty} \binom{\alpha - j}{k} \frac{\mu^{k+j-\alpha}}{(k+j+1-\alpha)\Gamma(k+j+1-\alpha)} \frac{\partial^k}{\partial \mu^k} \left( \mu^{\alpha-3} \right) \right] \]

\[ \times \left[ \sum_{k=1}^{m_i-1} \binom{m_i-1}{k} \partial_x^k u \partial_t^{m_i-k} u \right] + \text{remainder} \]

\[ = C(x, u) \sum_{j=3}^{\infty} \binom{\alpha}{j} \partial_x^{\alpha-j} \left( \mu^{\alpha-3} \right) \left[ \sum_{k=1}^{m_i-1} \binom{m_i-1}{k} \partial_x^k u \partial_t^{m_i-k} u \right] + \text{remainder. (25)} \]

By the uniqueness of nonlinear terms \( \partial_x^2 u_1 \partial_t^{m_1-2} u_1 \) with \( m_i \neq 2 \) in \( \mu_i \), we obtain the coefficients \( C(x, u) = 0 \) and then \( \partial^2 \eta_i / \partial u_1^2 = 0 \). Thus in \( \mu_i \), the terms \( \partial^j \eta_i / \partial u_1^j (k \geq 2) \) and their derivatives vanish while the remaining terms take the form \( \partial^k \eta_i / \partial u_1^k \partial u_2^k \ldots \partial u_q^k \) with nonnegative integer \( k_i \leq 1, i = 1, \ldots , q \).

Next consider the cross derivative terms \( \partial^2 \eta_i / \partial u_i \partial u_1 (i \neq 1) \). It corresponds to \( k = 2, k_i = k_1 = k_2 = 0 \) with \( d(\neq 1, 1) = 1, \ldots , q \), thus \( m_i \geq 1, m_j \geq 1 \) and \( m_k = 0 \), otherwise \( \mu_i \) vanishes identically. Then one has

\[ \left[ \sum_{i=0}^{1} \binom{1}{i} (-u_0)^i \frac{\partial u_0}{\partial u_0} \left( u_1^{1-i} \right) \right] \left[ \sum_{j=0}^{1} \binom{1}{j} (-u_0)^j \frac{\partial u_0}{\partial u_0} \left( u_1^{1-j} \right) \right] \]

which are unique in \( \mu_i \).

We first consider \( m_i = m_j = 1 \), i.e. the term \( \partial_1 u_0, \partial_2 u_1 \). Then \( m_0 = n - 2 \) and the coefficient of \( \partial_1 u_0, \partial_2 u_1 \) is

\[ \sum_{n=2}^{\infty} \binom{\alpha}{n} \frac{n(n-1)\mu^{n-\alpha}}{\Gamma(n+1-\alpha)} \partial_1^{n-2} \left( \frac{\partial^2 \eta_i}{\partial u_0 \partial u_1} \right) = \alpha(\alpha-1)\partial_1^{\alpha-2} \left( \frac{\partial^2 \eta_i}{\partial u_0 \partial u_1} \right). \]

(26)

where the technique adopted in (23) is used again. Then by solving \( \partial_1^{\alpha-2} \left( \frac{\partial^2 \eta_i}{\partial u_0 \partial u_1} \right) = 0 \), we get \( \partial^2 \eta_i / \partial u_0 \partial u_1 = B(x, u) \mu^{\alpha-3} \) with an undetermined function \( B(x, u) \).

Secondly, consider the case \( m_i = m_j = 2 \) which corresponds to the term \( \partial^2 u_1, \partial^2 u_2 \). Then \( m_0 = n - 4 \), and the coefficient of \( \partial_1^2 u_0, \partial_2^2 u_1 \) is

\[ \frac{1}{4} \alpha(\alpha-1)(\alpha-2)(\alpha-3) \mu^{\alpha-4} \left( \frac{\partial^2 \eta_i}{\partial u_0 \partial u_1} \right) \]

\[ = \frac{1}{4} \alpha(\alpha-1)(\alpha-2)(\alpha-3) B(x, u) \mu^{\alpha-4} \left( \mu^{\alpha-3} \right) \]

\[ = \frac{1}{4} \alpha(\alpha-3) \Gamma(\alpha+1) B(x, u). \]

Thus by the uniqueness of \( \partial^2 u_1, \partial^2 u_2 \) we obtain \( B(x, u) = 0 \) and then for all \( i \neq 1 \), \( \partial^2 \eta_i / \partial u_0 \partial u_1 = 0 \). It completes the proof.

**Theorem 2.7.** If the infinitesimal generator \( X \) given in (10) leaves system (3) invariant, then \( X \) must take the form

\[ X = (\chi_2 I^2 + \chi_1 I) \partial_t + \xi(x) \partial_{\omega} + \eta_t(t, x, u) \partial_{\omega}, \]

(27)
where $\chi_1$ and $\chi_2$ are arbitrary constants, $\eta_\alpha = \eta_\alpha(t, x, u)$ is given by
\begin{equation}
\eta_\alpha = [g_\alpha(x) + \gamma(2\chi_2t + \chi_1)]u_t + \sum_{i \neq \alpha} f_i(x)u_i + h_i(t, x),
\end{equation}
where $g_\alpha(x), f_i(x)$ and $h_i(t, x)$ are undetermined smooth functions of their arguments respectively, and the constant $\gamma$ satisfies
\begin{equation}
\gamma = \begin{cases} 
0, & \chi_2 = 0, \\
\frac{1}{2}(\alpha - 1), & \chi_2 \neq 0.
\end{cases}
\end{equation}

Proof. We show the theorem by analysing the structure of equation (11) on the space $(\partial^\alpha u^{(k)}, \partial^\alpha u^{(k-1)}, \ldots)$. Thus expanding equation (11) on the solution space of system (3) yields
\begin{equation}
\eta''_\alpha - \tau(t, x, u)(F_t + H_t) - \xi(t, x, u)(F_x + H_x) - \sum_{j=1}^q \sum_{\theta} \eta''_\theta(t, x, u^{(k)}) \partial_{\theta}^j(F) = 0,
\end{equation}
where $\eta''_\alpha$ takes the form
\begin{align*}
\eta''_\alpha &= \frac{\partial^\alpha \eta_\alpha}{\partial t^\alpha} + \sum_{i=1}^q \left[ \frac{\partial \eta_\alpha}{\partial u_i}(F_t + H_t) - u_i \frac{\partial \eta_\alpha}{\partial u_i} \right] \\
&\quad + \sum_{j=1}^q \sum_{k=1}^{\infty} \binom{\alpha}{k} \partial^k \left( \frac{\partial \eta_\alpha}{\partial u_i} \right) \partial^{\alpha-k}(u_i) - \alpha D_t(\tau)(F_t + H_t) \\
&\quad - \sum_{k=1}^{\infty} \left( \binom{\alpha}{k+1} D^{k+1}_t(\tau) \partial^{\alpha-k}(u_i) - \sum_{k=1}^{\infty} \binom{\alpha}{k} D^k_t(\xi_t) \partial^{\alpha-k} \left( \frac{\partial u_i}{\partial x_j} \right) + \mu, \right)
\end{align*}
and $\eta''_\theta(t, x, u^{(k)})$ are given by formula (13).

Then substituting (13) and (31) into equation (30) and vanishing the coefficients of $\partial^{\alpha-k}(\partial u_i/\partial x_j)$, one obtains
\begin{equation}
\binom{\alpha}{k} D^k_t(\xi_t) = 0, \quad k = 1, 2, \ldots.
\end{equation}
Since equation (32) work for $k = 1, 2, \ldots$, then for $k = 1$ we have
\begin{equation}
D_t(\xi) = \frac{\partial \xi}{\partial t} + \sum_{j=1}^q \frac{\partial \xi}{\partial u_j} \frac{\partial u_j}{\partial t} = 0,
\end{equation}
which implies $\partial \xi/\partial t = \partial \xi/\partial u_j = 0$, i.e. $\xi = \xi(x)$. Then equation (32) with $k \geq 2$ hold identically.

Next consider $\tau = \tau(t, x, u)$. We claim that in system (3), for each $x_i$, there exists at least one $u_i$ such that $\partial u_i/\partial x_j$ or its higher order $x$-derivatives occur. If not, $x_i$ can be regarded as a parameter variable and system (3) involves $p-1$ independent variables, which is contradictory. Thus we assume that such type of derivative in $F$ has the maximal order $|\vartheta|$ and takes the form $u^\vartheta$ with $\vartheta = (\vartheta_1, \ldots, \vartheta_p) \in \mathbb{Z}_+^p$ satisfying $1 \leq |\vartheta| \leq k$. Observe that in equation (30) the derivative $\partial u_j^\vartheta$ uniquely exists in $\eta''_\theta(t, x, u^{(k)})$ given by (13), where $\vartheta = (\vartheta_1, \ldots, \vartheta_{i-1}, \vartheta_i - 1,$
\(\vartheta_{i+1}, \ldots, \vartheta_p\). More precisely, the term \(\partial_{u_j^i}\varrho\) appears uniquely in \(D_{\varrho}(\tau\partial_{\varrho}u_j)\) and its coefficient is \(\partial_{\varrho}(F)D_{\varrho}\tau\) which should be vanished, i.e.

\[
D_{\varrho}\tau = \frac{\partial\tau}{\partial x_i} + \sum_{j=1}^{q} \frac{\partial u_j}{\partial x_i} \frac{\partial\tau}{\partial u_j} = 0,
\]

since \(\partial_{\varrho}(F) \neq 0\) by the claim. Note that by direct computations the term \(\tau\partial_{u_j^i}\varrho\) in (13) vanishes identically. Thus we obtain \(\partial_{u_j^i}\tau = \partial_{\varrho}\tau = 0\), i.e. \(\tau = \tau(t)\). Then the prolongation formula (13) is simplified to

\[
\eta^0_j(t, x, u^{(i)}) = D_\theta \left( \eta_j - \sum_{i=0}^{p} \xi_i u_j^i \right) + \sum_{i=0}^{p} \xi_i u_j^i.
\]  

(33)

With the above simplifications, condition (30) becomes

\[
\begin{align*}
\partial^\alpha_{\varrho} \eta_s + \sum_{i=1}^{q} \left[ \frac{\partial \eta_s}{\partial u_i}(F_i + H_i) - u_i \partial^\alpha_{\varrho} \left( \frac{\partial \eta_s}{\partial u_i} \right) \right] + \sum_{i=1}^{q} \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) \\
- \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) + \mu_s - \alpha D_\theta(\tau)(F_i + H_i) \\
- \tau(t)(F_i + H_i) - \xi_s(x)(F_{i+1} + H_{i+1}) - \sum_{j=0}^{q} \eta^0_j(t, x, u^{(i)}) \frac{\partial \eta_j}{\partial \varrho}(F_j) = 0,
\end{align*}
\]  

(34)

where \(\eta^0_j(t, x, u^{(i)})\) are given by (33).

We now turn to consider \(\eta_s = \eta_s(t, x, u)\). Observe that integer-order \(t\)-derivatives of \(u_i\) in equation (34) uniquely occur in \(\mu_s\) while \(\partial^{\alpha-k}_{\varrho}(u_i)\) with integers \(k > 1\) are fractional integrals of \(u_i\). Thus by considering the structure of \(\mu_s\) and lemma 2.6, we obtain

\[
\eta_s = \sum_{i=1}^{q} r_i(t, x) u_i + h_s(t, x).
\]  

(35)

where \(r_i(t, x)\) and \(h_s(t, x)\) are undetermined functions.

Next we further separate equation (34) with respect to time-fractional integrals and derivatives of \(u_i\) and get

\[
\begin{align*}
\partial^\alpha_{\varrho} \eta_s + \sum_{i=1}^{q} \left[ \frac{\partial \eta_s}{\partial u_i}(F_i + H_i) - u_i \frac{\partial^\alpha_{\varrho}}{\partial u_i} \left( \frac{\partial \eta_s}{\partial u_i} \right) \right] + \sum_{i=1}^{q} \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) \\
- \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) + \mu_s - \alpha D_\theta(\tau)(F_i + H_i) \\
- \tau(t)(F_i + H_i) - \xi_s(x)(F_{i+1} + H_{i+1}) - \sum_{j=0}^{q} \eta^0_j(t, x, u^{(i)}) \frac{\partial \eta_j}{\partial \varrho}(F_j) = 0,
\end{align*}
\]  

(36a)

\[
\begin{align*}
\partial^\alpha_{\varrho} \eta_s + \sum_{i=1}^{q} \left[ \frac{\partial \eta_s}{\partial u_i}(F_i + H_i) - u_i \frac{\partial^\alpha_{\varrho}}{\partial u_i} \left( \frac{\partial \eta_s}{\partial u_i} \right) \right] + \sum_{i=1}^{q} \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) \\
- \sum_{k=1}^{\infty} \left( \begin{array}{c}
\alpha \\
k + 1
\end{array} \right) D^{k+1}_{\varrho}(\tau) \partial^{\alpha-k}_{\varrho}(u_i) + \mu_s - \alpha D_\theta(\tau)(F_i + H_i) \\
- \tau(t)(F_i + H_i) - \xi_s(x)(F_{i+1} + H_{i+1}) - \sum_{j=0}^{q} \eta^0_j(t, x, u^{(i)}) \frac{\partial \eta_j}{\partial \varrho}(F_j) = 0,
\end{align*}
\]  

(36b)

\[
\begin{align*}
\begin{align*}
\begin{align*}
\begin{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\]  

(36c)

5199
which hold for $k = 1, 2, \ldots$. Since equation (36b) holds for $k = 1, 2, \ldots$, then for $k = 1$, together with (35) one has
\[
\frac{\partial^2 \eta_1}{\partial t^2} + \frac{\partial}{\partial t} \frac{\partial r_1}{\partial t} = 0 \quad \text{with } i \neq s, \text{ thus }
\eta_1 = r_s(t, x)u_s + \sum_{i \neq s} f_i(x)u_i + h_i(t, x),
\]
(37)
where $f_i(x)$ are undetermined functions.

With such results equation (36b) with integers $k \geq 2$ hold identically. Next we consider equation (36a) to further find explicit expressions of $\tau$ and $\eta_i$. Since $\tau = \tau(t)$, we isolate the case $k = 1$ from equation (36a) and divide them into two parts
\[
\frac{\partial^2 \eta_1}{\partial \tau \partial u_1} - \frac{1}{2}(\alpha - 1)\tau'' = 0, \quad k = 1,
\]
\[
\frac{\partial^k}{\partial \tau^k} \left( \frac{\partial \eta_1}{\partial u_1} \right) - \frac{\alpha - k}{k + 1} \tau^{k+1} = 0, \quad k \geq 2. \quad (38)
\]
We use the partial derivative on $\tau$ in order to write the second part of equations uniformly. Then inserting the first equation into the second ones gives
\[
\left( \frac{\alpha - 1}{2} - \frac{\alpha - k}{k + 1} \right) \frac{d^{k-1}}{d\tau^{k-1}}(\tau'') = 0, \quad k = 2, 3, \ldots \quad (39)
\]
In particular, for $k = 2$, equation (39) give $d^3\tau(t)/dt^3 = 0$. Together with the condition $\tau(t)|_{t=0} = 0$ we obtain $\tau = \chi_1 t + \chi_2 t^2$, where $\chi_1$ and $\chi_2$ are two arbitrary constants. By means of the above results, solving the first equation in system (38) yields two different cases.

(a) One is $\tau'' = 0$, i.e. $\chi_2 = 0$, then $\partial^2 \eta_1/\partial \tau \partial u_1 = 0$. By considering (37), we obtain $\tau = \chi_1 t$ and $\partial r_s(t, x)/\partial t = 0$. Thus
\[
\eta_1 = g_s(x)u_s + \sum_{i \neq s} f_i(x)u_i + h_i(t, x).
\]
(b) The other is $\tau'' \neq 0$, i.e. $\chi_2 \neq 0$. From (37) and the first equation in system (38) we obtain
\[
r_i(t, x) = (\alpha - 1)\tau'/2 + g_i(x),
\]
then
\[
\eta_1 = \left[ g_s(x) + \frac{1}{2}(\alpha - 1)\tau' \right] u_s + \sum_{i \neq s} f_i(x)u_i + h_i(t, x).
\]

Finally, we collect the above two expressions of $\eta_i$ as a unified form
\[
\eta_i = \left[ g_s(x) + \gamma \tau' \right] u_s + \sum_{i \neq s} f_i(x)u_i + h_i(t, x) \quad \text{where } \gamma = (\alpha - 1)/2 \quad \text{for } \chi_2 \neq 0 \text{ and } \gamma = 0 \quad \text{for } \chi_2 = 0,
\]
the functions $g_s(x), f_i(x)$ and $h_i(t, x)$ are determined by equation (36c). It completes the proof. \qed

2.4. Determining conditions

By means of the symmetry structure of system (3), we show that the Lie symmetries of system (3) are determined by two elegant conditions which provide a possibility to use the known computer programs of integer-order PDEs to solve the symmetry determining equations of multi-dimensional time-fractional PDEs.
Theorem 2.8. Following the above notations, the Lie symmetries of system (3) are completely determined by

\[
\frac{\partial^q h_i(t, x)}{\partial t^q} + \sum_{i=1}^{q} \frac{\partial h_i}{\partial u_i} \mathcal{H}_i - \alpha D_t^\tau(\tau) \mathcal{H}_i - \tau \frac{\partial \mathcal{H}_i}{\partial t} - \xi \frac{\partial \mathcal{H}_i}{\partial x_i} \\
- \sum_{j=1}^{q} \sum_{u_i^j \in J_i} D_t^\tau(h_j(t, x)) \frac{\partial h_j^i(F_i)}{\partial t} = 0,
\]

where \( I_s = \{ \text{all terms in } F_s \} \), \( J_s = \{ \text{all terms in } F_s \text{ which are linear in } u_i^j \} \) and \( I_s \setminus J_s = \{ \text{the terms contained in } I_s \text{ not in } J_s \} \).

Proof. By the proof of theorem 2.7, equation (36c) becomes

\[
\frac{\partial^q \eta_j}{\partial t^q} + \sum_{i=1}^{q} \left[ \frac{\partial \eta_j}{\partial u_i} (F_i + \mathcal{H}_i) - u_i \frac{\partial}{\partial t} \left( \frac{\partial \eta_j}{\partial u_i} \right) \right] - \alpha D_t^\tau(\tau) (F_i + \mathcal{H}_i) \\
- \tau \left( \frac{\partial \mathcal{H}_i}{\partial t} + \frac{\partial h_i}{\partial t} \right) - \xi \left( \frac{\partial \mathcal{H}_i}{\partial x_i} + \frac{\partial h_i}{\partial x_i} \right) - \sum_{j=1}^{q} \sum_{u_i^j \in J_s} \eta_j^i \frac{\partial h_j^i(F_i)}{\partial t} = 0, \tag{41}
\]

where \( \tau = \lambda \tau^2 + \chi x \), \( \xi = \xi_i(x) \) and \( \eta_j \) is given by (28). Moreover, equations (32), (36a) and (36b) hold identically with the given \( \tau, \xi_i \) and \( \eta_j \) in theorem 2.7, thus Lie symmetries of system (3) are uniquely determined by equation (41).

On the space \((t, x, u)\), one has

\[
\frac{\partial^q \eta_j}{\partial t^q} + \sum_{i=1}^{q} u_i \frac{\partial}{\partial t^q} \left( \frac{\partial \eta_j}{\partial u_i} \right) = \frac{\partial}{\partial t^q} \left( \eta_j - \sum_{i=1}^{q} u_i \frac{\partial \eta_j}{\partial u_i} \right) = \frac{\partial}{\partial t^q} h_j(t, x).
\]

Let \( I_s = \{ \text{all terms in } F_s \} \) and \( J_s = \{ \text{all terms in } F_s \text{ which are linear in } u_i^j \} \), the set \( I_s \setminus J_s = \{ \text{the terms contained in } I_s \text{ not in } J_s \} \). By considering whether the terms involve \( u \) and its \( x \)-derivatives or not, we separate equation (41) into two parts given in (40). The proof ends.

Theorem 2.8 shows that the symmetry determining equations of system (3) can be divided into two parts: a system of integer-order PDEs in \( \tau, \xi_i \) and \( \eta_j \) and a system of linear time-fractional PDEs in \( h_j(t, x) \). Moreover, for the most of time-fractional PDEs, the second condition in (40) ‘almost’ completely determines the admitted Lie symmetries while the first one either holds automatically or is used to check the final results.

Following the above theoretical preparations, we formulate the procedure of finding Lie symmetries of system (3) as the following three steps:

Step 1. Assume system (3) is admitted by the infinitesimal generator (10), then by theorem 2.7, the infinitesimals \( \tau, \xi_i \) and \( \eta_j \) are directly assumed to be the explicit forms (27).
Step 2. Finding the two conditions to determine the Lie symmetry. By theorem 2.8, first write down the expressions $H_i, F_i$ and the sets $I_i, J_i, I_i \setminus J_i$, then obtain the two determining conditions given by (40).

Step 3. Further separation of the second condition in system (40) with respect to $u$ and its $x$-derivatives to get the symmetry determining system about $\tau, \xi, \eta$, then together with the first condition, solve the system to get the infinitesimal generator (10).

It should be mentioned that in step 3 the separation of the second condition in system (40) is more deeper than the one for integer-order PDEs where the former one is divided about $u$ and its $x$-derivatives because the prescribed infinitesimals $\tau, \xi, \eta$ in theorem 2.7 are independent of $u$ and its $x$-derivatives, while the latter one is done only with respect to $x$-derivatives of $u$. Thus the separation in step 3 will generate a more simplified symmetry determining system.

3. Three examples

We consider three examples to illustrate the efficiencies and applications of our results. In the first subsection we will adopt two methods to look for Lie symmetries of the time-fractional generalized Zakharov–Kuznetsov equation (42) in order to show the efficiencies of our results while in next two subsections we directly use our method for the other two examples.

3.1. Time-fractional generalized Zakharov–Kuznetsov equation

The first example is the time-fractional generalized Zakharov–Kuznetsov equation

$$\partial_\alpha^\mu u + \rho u_x + u_{xxx} + u_{yyy} = 0, \quad (42)$$

where $\rho$ is a nonzero constant and $u = u(t, x, y)$. Equation (42) with $\alpha = \rho = 1$ is the Zakharov–Kuznetsov equation which describes weakly nonlinear ion-acoustic wave in a strongly magnetized lossless plasma in two dimensions [32].

We assume that equation (42) is admitted by a one-parameter local Lie symmetry group with the infinitesimal generator

$$X = \tau \partial_t + \xi \partial_x + \psi \partial_y + \eta \partial_u,$$

where $\xi, \tau, \psi$ and $\eta$ are smooth functions of $t, x, y$ and $u$ respectively.

3.1.1. The original method. The Lie infinitesimal criterion for equation (42) gives

$$\text{Pr}_3 X \left( \partial_\alpha^\mu u + \rho u_x + u_{xxx} + u_{yyy} \right) \big|_{(42)} = 0, \quad (43)$$

where $\text{Pr}_3$ is given by (12) with $k = 3$. Specifically, expanding condition (43) yields

$$\eta^\alpha = \eta^\alpha + \eta^{\alpha-1} u_x \eta^1 + \eta^\alpha u_x + \eta^{xxx} + \eta^{yyy} = 0, \quad (44)$$

where $\eta^\alpha$ is formulated by (20) while $\eta^1$ and $\eta^{xxx}, \eta^{yyy}$ are expressed by (13).

First assume $\mu = 0$ in $\eta^\alpha$. Then inserting (20) into condition (44) and separating it with respect to different time-fractional integrals and derivatives of $u$, one has

$$\binom{\alpha}{k} D_x^k (\xi) = \binom{\alpha}{k} D_x^k (\psi) = 0, \quad k = 1, 2, \ldots,$$

$$\binom{\alpha}{k} \frac{\partial^k}{\partial t^k} \left( \frac{\partial \eta}{\partial u} \right) - \binom{\alpha}{k+1} D_x^{k+1} (\tau) = 0, \quad k = 1, 2, \ldots.$$
3.1.2. Our method. By theorem 2.7, we directly assume

$$\tau = \chi_2 t^2 + \chi_1 t, \quad \xi = \xi(x, y), \quad \psi = \psi(x, y), \quad \eta = [g(x, y) + \gamma(2\chi_2 t + \chi_1)]u + h(t, x, y),$$

where $\xi$, $\psi$, and $g = g(x, y), h = h(t, x, y)$ are smooth undetermined functions. By theorem 2.8, $\mathcal{H}_1 = 0, \mathcal{F}_1 = -u^\prime u_x - u_{xxx} - u_{xxy}, I_1 = \{u^\prime u_x, u_{xxx}, u_{xxy}\}, J_1 = \{u_{xxx}, u_{xxy}\},$ then the two conditions given in (40) become

$$\frac{\partial^\alpha \eta}{\partial t^{\alpha}} + h_{xxx} + h_{xxy} = 0, \quad (46a)$$

$$[g(x, y) + (\gamma - \alpha)(2\chi_2 t + \chi_1)](-u^\prime u_x - u_{xxx} - u_{xxy}) + (\eta^{xxx} - h_{xxx}) + (\eta^{xxy} - h_{xxy}) + \rho u^\prime u_x \eta + u^\prime \eta^x = 0, \quad (46b)$$

where $h = h(t, x, y), \eta^i$ and $\eta^{xxx}, \eta^{xxy}$ are given by

$$\eta^x = D_x(\eta - \xi u_x - \psi u_y) + \xi u_{xx} + \psi u_{xy},$$

$$\eta^{xxx} = D_x D_x^2(\eta - \xi u_x - \psi u_y) + \xi u_{xxx} + \psi u_{xxy},$$

$$\eta^{xxy} = D_x D_x^2(\eta - \xi u_x - \psi u_y) + \xi u_{xxy} + \psi u_{xy},$$

which are more simpler than the usual ones (13) since they do not contain function $\tau$.

Substituting them into equation (46b) and separating it with respect to different powers of $u$ and its derivatives, we obtain

$$\xi_x = \psi_y = g_x = g_y = h_x = \rho h = 0,$$

$$\alpha(2\chi_2 + \chi_1) - 3\xi_x = 0,$$

$$2\psi_y + \xi_x - \alpha(2\chi_2 t + \chi_1) = 0,$$

$$\rho g - \xi_x + (\alpha + \gamma \rho)(2\chi_2 t + \chi_1) = 0,$$  \( (47) \)

which are integer-order linear PDEs and very easy to be solved. Solving the system gives $\chi_2 = \gamma = h = 0$ and

$$\tau = \chi_1 t, \quad \xi = \frac{1}{3} \alpha \chi_1 x + c_1, \quad \psi = \frac{1}{3} \alpha \chi_1 y + c_2, \quad \eta = -\frac{2}{3 \rho} \alpha \chi_1 u. \quad (48)$$

Observe that system (47) obtained by separating equation (46b) with respect to $u$ together with $x$- and $y$-derivatives of $u$ completely determines the Lie symmetries of equation (42) and solutions (48) automatically satisfy equation (46a) since $h = 0$. 

5203
The direct role of the infinitesimal operator is to reduce the PDEs into lower-dimensional PDEs. Before the performance, we recall the two variables Erdélyi–Kober fractional differential operator

\[ (K_{\delta, \sigma}^{\alpha, \alpha} U) (\omega, \theta) = \int_0^\infty (u - 1)^{\alpha-1} u^{-(\alpha+\alpha)} U \left( \omega u^{\frac{1}{\alpha}}, \theta u^{\frac{1}{\sigma}} \right) du, \quad \alpha > 0, \]

in order to present an elegant expression for the reduced equations, where

\[ (K_{\delta, \sigma}^{\alpha, \alpha} U) (\omega, \theta) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\infty (u - 1)^{\alpha-1} u^{-(\alpha+\alpha)} U \left( \omega u^{\frac{1}{\alpha}}, \theta u^{\frac{1}{\sigma}} \right) du, & \alpha > 0, \\ U(\omega, \theta), & \alpha = 0. \end{cases} \]

Note that in the case of single independent variable operator (49) becomes the classical Erdélyi–Kober fractional differential operator [3]. In what follows, we give an explicit procedure of constructing the reduced equation by one infinitesimal generator while for other two ones as well as the two examples below we directly list the reduced equations and similarity solutions without details.

**Proposition 3.1.** By the infinitesimal generator \( X = \iota \partial_t + \alpha x/3 \partial_x + \alpha y/3 \partial_y - 2\alpha/(3\rho) u \partial_u \), we reduce equation (42) to the form

\[ \left( P_x^\frac{1}{\alpha-\alpha} U \right) (z_1, z_2) = U_{z_1} + \frac{\partial^3 U}{\partial z_1^3} + \frac{\partial^3 U}{\partial z_1 \partial z_2^2}, \]

where the similarity variables are \( z_1 = x t^{-\alpha/3}, z_2 = y t^{-\alpha/3} \) and \( U(z_1, z_2) = u^{2\alpha/3\rho} \).

**Proof.** The first step is to find the similarity variable \( I = I(t, x, y, u) \) by solving the linear PDE \( X(I) = 0 \). The corresponding characteristic equations are

\[ \frac{dx}{\alpha} = \frac{dy}{\alpha} = \frac{du}{-\rho u}, \]

which gives the similarity variables \( z_1 = x t^{-\alpha/3}, z_2 = y t^{-\alpha/3} \) and \( U(z_1, z_2) = u^{2\alpha/3\rho} \). Then by the chain rule for integer-order derivative, we get

\[ u_x = t^{-\frac{\alpha}{3}(\frac{1}{\alpha}+1)} \frac{\partial U}{\partial z_1}, \quad u_{xxx} = t^{-\frac{\alpha}{3}(\frac{1}{\alpha}+3)} \frac{\partial^3 U}{\partial z_1^3}, \quad u_{yy} = t^{-\frac{\alpha}{3}(\frac{1}{\alpha}+1)} \frac{\partial^3 U}{\partial z_1 \partial z_2^2}. \]

Next we consider the time-fractional derivative \( \partial_t^\alpha u \) with \( 0 < \alpha < 1 \). Inserting the above similarity variables into the Riemann–Liouville fractional derivative in definition 2.1, \( \partial_t^\alpha u \) can be written as

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \int_0^t (t - s)^{-\alpha} s^{\frac{\alpha}{\alpha}} U \left( x s^{-\frac{\alpha}{\alpha}}, y s^{\frac{\alpha}{\alpha}} \right) ds. \]
Let \( v = t/s, \, ds = -t/\nu^2 \, dv \), then we transform \( \partial^\nu u \) in (53) into the following form

\[
\frac{\partial^\nu u}{\partial t^\nu} = \frac{\partial}{\partial t} \left[ t^{1-\frac{2\alpha}{\rho}} \right] \left[ \frac{\alpha}{\Gamma(1-\alpha)} \int_1^{\infty} (v-1)^{-\alpha} v^{-\left(2-\frac{2\alpha}{\rho}\right)} U \left( xv^{\frac{\rho}{2}}, yv^{\frac{\rho}{2}} \right) \, dv \right].
\] (54)

Substituting the Erdélyi–Kober fractional differential operator in (49) into equation (54), we arrive at a compact expression

\[
\frac{\partial^\nu u}{\partial t^\nu} = \frac{\partial}{\partial t} \left[ t^{1-\frac{2\alpha}{\rho} - \alpha} \left( K_{\frac{1}{2}, \frac{1}{2}}^{1-\alpha} \right) \right] (z_1, z_2).
\] (55)

Observe that

\[
t \frac{\partial}{\partial t} U(z_1, z_2) = -\frac{\alpha}{3} t^{\frac{\alpha}{2}} \left[ x \frac{\partial}{\partial z_1} U(z_1, z_2) + y \frac{\partial}{\partial z_2} U(z_1, z_2) \right]
\]

\[
= -\frac{\alpha}{3} z_1 \frac{\partial}{\partial z_1} U(z_1, z_2) - \frac{\alpha}{3} z_2 \frac{\partial}{\partial z_2} U(z_1, z_2).
\]

Then we convert equation (55) into the form

\[
\frac{\partial^\nu u}{\partial t^\nu} = t^{1-\frac{2\alpha}{\rho} - \alpha - 1} \left[ 1 - \frac{2\alpha}{\rho} + \alpha - \frac{\alpha}{3} \left( \frac{z_1}{\partial z_1} + \frac{z_2}{\partial z_2} \right) \right] \left( K_{\frac{1}{2}, \frac{1}{2}}^{1-\alpha} \right) (z_1, z_2)
\]

\[
= t^{1-\frac{2\alpha}{\rho} - \alpha} \left( P_{\frac{1}{\rho}, \frac{1}{\rho}}^{1-\alpha, 0} \right) \left( z_1, z_2 \right). \] (56)

Finally, inserting expressions (52) and (56) into equation (43), we obtain the reduced equation (50). The proof ends.

Following the above procedure, we find that the similarity variables of the infinitesimal generator \( X = \partial_t \) are \( z_1 = t, \, z_2 = y \) and \( U(z_1, z_2) = u(t, x, y) \). Then equation (42) is converted to the form \( \partial^\nu U(z_1, z_2)/\partial z_1^\nu = 0 \). By solving it and returning to original variables we find a solution of equation (42) in the form \( u(t, x, y) = f(y)^{\rho-1} \) with an arbitrary function \( f(y) \).

Similarly, with the infinitesimal generator \( X = \partial_x \), equation (42) is reduced to

\[
\frac{\partial^\nu U}{\partial z_1^\nu} = U^\nu U_{z_2} + U_{z_1 z_2},
\]

where \( z_1 = t, \, z_2 = x \) and \( U = u(t, x, y) \).

3.2. Time-fractional Hirota–Satsuma coupled KdV equations

Consider the time-fractional Hirota–Satsuma coupled KdV equations

\[
\partial_t^\nu u = uu_x + \nu v_x + u_{xxx},
\]

\[
\partial_t^\nu v = -uv_x - 2v_{xxx}, \] (57)

whose Lie symmetry analysis has been performed in [28]. Here we directly use our results to find Lie symmetries of system (57).

Assume that a local Lie symmetry group with the infinitesimal generator

\[
X = \tau \partial_t + \xi \partial_x + \eta \partial_y + \phi \partial_z,
\]

5205
where $\xi, \tau, \eta$ and $\phi$ are arbitrary smooth functions of $t, x, u$ and $v$ respectively, leaves system (57) invariant. Then by theorem 2.7, we get $\tau = \chi_2 t^2 + \chi_1 t, \xi = \xi(x),$

$$\eta = [g_1(x) + \gamma(2\chi_2 t + \chi_1)] u + f_2(x)v + h_1(t, x),$$

and

$$\phi = [g_2(x) + \gamma(2\chi_2 t + \chi_1)] v + f_1(x)u + h_2(t, x).$$

By theorem 2.8, one has

$$\mathcal{H}_1(t, x) = 0, \quad \mathcal{F}_1 = uu_x + vv_x + u_{xxx},$$

$$I_1 = \{uu_x, vv_x, u_{xxx}\}, \quad J_1 = \{u_{xxx}\}, \quad I_1 \setminus J_1 = \{uu_x, vv_x\};$$

$$\mathcal{H}_2(t, x) = 0, \quad \mathcal{F}_2 = -uv_x - 2v_{xxx},$$

$$I_2 = \{uv_x, uu_x, v_{xxx}\}, \quad J_2 = \{v_{xxx}\}, \quad I_2 \setminus J_2 = \{uv_x, uu_x\}.$$

Then the two conditions in (40) for the first equation in system (57) become

$$\partial^\alpha h_1 = \beta_j \partial_j^3 h_1 = 0,$$

$$[g_1(x) + (\gamma - \alpha)(2\chi_2 t + \chi_1)] \mathcal{F}_1 + f_2(x) \mathcal{F}_2$$

$$- (\partial_x \eta - \eta') - (v_x \phi + v\phi') = 0,$$

(58)

and for the second equation become

$$\partial^\alpha h_2 = \delta_3 \partial_3 h_2 = 0,$$

$$[g_2(x) + (\gamma - \alpha)(2\chi_2 t + \chi_1)] \mathcal{F}_2 + f_1(x) \mathcal{F}_1 + 2 \left(\partial_x \phi - \partial_3 h_2 \right) + (v_x \eta + u\phi') = 0,$$

(59)

where $h_i = h_i(t, x), \eta', \phi'$ and $\eta_{xxx}, \phi_{xxx}$ are given by

$$\eta' = D(\eta - \xi u_x) + \xi u_{xx}, \quad \eta_{xxx} = D(\eta - \xi u_x) + \xi u_{xxx},$$

$$\phi' = D(\phi - \xi v_x) + \xi v_{xx}, \quad \phi_{xxx} = D(\phi - \xi v_x) + \xi v_{xxx}.$$

Inserting the above coefficient functions into systems (58) and (59) and annihilating the coefficients of different powers of $u, v$ and their derivatives to zero, we obtain

$$f_1 = f_2 = h_1 = h_2 = 0, \quad g_1' = 0, \quad g_1 = g_2,$$

$$\alpha(2\chi_2 t + \chi_1) - 3 \xi' = 0,$$

$$(\alpha + \gamma)(2\chi_2 t + \chi_1) + g_1 - \xi' = 0.$$  (60)

Solving the system yields

$$\tau = \chi_1 t, \quad \xi = \frac{1}{3} \alpha \chi_1 x + c_1, \quad \eta = -\frac{2}{3} \alpha \chi_1 u, \quad \phi = -\frac{2}{3} \alpha \chi_1 v,$$

which is the same as the results in [28]. However, the symmetry determining system (60) is very simple and easy to be solved.
Next we use the infinitesimal operators to construct reduced equations. By the infinitesimal generator \( X = \partial_t \), we reduce system (57) to the form
\[
\frac{\partial^\alpha U(\zeta)}{\partial \zeta^\alpha} = 0, \quad \frac{\partial^\alpha V(\zeta)}{\partial \zeta^\alpha} = 0, \tag{61}
\]
where the similarity variables are \( \zeta = t, U(\zeta) = u(t, x) \) and \( V(\zeta) = v(t, x) \). Solving system (61) gives one solution of system (57) in the form \( u(t, x) = C_1 t^{\alpha-1}, v(t, x) = C_2 t^{\alpha-1} \), where \( C_1 \) and \( C_2 \) are integral constants.

Similarly, induced by the operator \( X = \partial_t + \alpha/3x\partial _x - 2\alpha/3u\partial_u - 2\alpha/3v\partial_v \), system (57) is reduced to
\[
\left( P_3^{1-\alpha/3} U \right)(\zeta) = U(\zeta)U'(\zeta) + V(\zeta)h'(\zeta) + U'''(\zeta), \tag{62a}
\]
\[
\left( P_3^{1-\alpha/3} V \right)(\zeta) = -U(\zeta)V'(\zeta) - 2V'''(\zeta), \tag{62b}
\]
where the similarity variables are \( \zeta = xt^{-\alpha/3}, U(\zeta) = u^{2\alpha/3} \) and \( V(\zeta) = v^{2\alpha/3} \).

### 3.3. Time-fractional nonlinear telegraph equations

The third example is the time-fractional nonlinear telegraph equations with variable coefficients
\[
\begin{align*}
\partial_t^\alpha u &= v, \\
\partial_t^\beta v &= P(u)u_t + G(u),
\end{align*} \tag{62}
\]
where \( P(u) \) and \( Q(u) \) are two smooth nonzero functions of \( u \) which make system (62) nonlinear. In what follows, we will perform a Lie symmetry classification of system (62) which means first to classify the functions \( P(u) \) and \( Q(u) \) making system (62) admit the extended symmetries and then to determine the Lie symmetries.

To simplify our calculations, we use an equivalent transformation of system (62) given by
\[
t^* = t, \quad x^* = \beta_3 x + \beta_4, \quad u^* = \beta_1 u, \quad v^* = \beta_2 v, \quad P^*(u^*) = \frac{\beta_3 \beta_5 P(u)}{\beta_1}, \quad G^*(u^*) = \frac{\beta_2 G(u)}{\beta_1},
\]
(63)
where nonzero constants \( \beta_i \) satisfy \( \beta_1 \beta_2 = \beta_3 \neq 0 \). Transformation (63) maps system (62) into the same form. Its main role is that in the procedure of Lie symmetry classification for system (62), scalings of \( P(u) \) and \( G(u) \) do not affect the final classified results. For example, if \( P(u) = \beta_1 u^2 + \beta_2 G(u) = \beta_3 u \), we can assume \( P(u) = u^2 + \beta_2, G(u) = u \).

Assume that a local Lie symmetry group with the infinitesimal generator
\[
X = \tau \partial_t + \xi \partial_x + \eta \partial_u + \phi \partial_v, \tag{64}
\]
where the infinitesimals \( \xi, \tau, \eta \) and \( \phi \) are undetermined smooth functions of \( t, x, u \) and \( v \) respectively, leaves system (62) invariant. Then by theorem 2.7, \( \tau = \chi_1 t^2 + \chi_1 t, \xi = \xi(t, x) \),
\[
\eta = [g_1(x) + \gamma(2\chi_2 t + \chi_1)]u + f_2(x)v + h_1(t, x),
\]
and
\[
\phi = [g_2(x) + \gamma(2\chi_2 t + \chi_1)]v + f_1(x)u + h_2(t, x),
\]
5207
where, hereinafter, $f_i(x), g_i(x)$ and $h_i(t, x)$ with $i = 1, 2$ are undetermined functions. Moreover, by theorem 2.8 for system (62), one has $H_1(t, x) = H_2(t, x) = 0$, $F_1 = v_x$, $F_2 = P(u)u_x + G(u)$.

We first consider the first equation in system (62). By theorem 2.8, $I_1 = \{v_x\}$ and $J_1 = \{v_x\}$, then the set $I_1 \backslash J_1$ is empty and we obtain

$$\frac{\partial^n h_i(t, x)}{\partial t^n} - \frac{\partial h_2(t, x)}{\partial x} = 0,$$

$$[g_1(x) + (\gamma - \alpha)(2\chi_2t + \chi_1)] v_x = \left(\phi^* - \frac{\partial h_2(t, x)}{\partial x}\right) + f_2(x)[P(u)u_x + G(u)] = 0,$$  \hspace{1cm} (65)

where $\phi^*$ is expressed by

$$\phi^* = [g_2(x) + \gamma(2\chi_2t + \chi_1)] v_x + g'_2(x)v + f'_1(x)u + f_1(x)u_x$$

$$+ \frac{\partial h_2(t, x)}{\partial x} - \xi'(x)v_x.$$ \hspace{1cm} (66)

Inserting (66) into the second equation in system (65) and separating it with respect to $v, v_x$ and $u_x$, we obtain $g_2(x) = c_1$, and

$$f_2(x)P(u) - f_1(x) = 0,$$

$$f_2(x)G(u) - f_1'(x)u = 0,$$

$$\xi'(x) - \alpha(2\chi_2t + \chi_1) + g_1(x) - c_1 = 0,$$ \hspace{1cm} (67)

where, here and below, $c_j$ are arbitrary constants, $j = 1, 2, 3$.

Observe that if $f_1(x)f_2(x) \neq 0, P(u)$ is a constant and $G(u)$ is either linear in $u$ or a constant, which is contradict with nonlinear system (62). Thus $f_1(x)f_2(x) = 0$. Since $f_2(x)P(u) = f_1(x)$ and $P(u) \neq 0$, thus $f_1(x) = 0$ implies $f_2(x) = 0$ and vice versa, i.e. $f_1(x) = f_2(x) = 0$. Then by dividing the last equation in system (67) with respect to $t$ and reconsidering (65), we obtain the symmetry determining equations for the first equation in system (62)

$$\frac{\partial^n h_1(t, x)}{\partial t^n} - \frac{\partial h_2(t, x)}{\partial x} = 0,$$

$$\chi_2 = -\alpha\chi_1 + g_1(x) - c_1 + \xi'(x) = 0.$$ \hspace{1cm} (68)

Then by the above analysis the infinitesimals in (64) are simplified to

$$\tau = \chi_1 t, \quad \xi = \xi(x), \quad \eta = g_1(x)u + h_1(t, x), \quad \phi = c_1v + h_2(t, x).$$

Now we turn to the second equation in system (62). Since system (62) is nonlinear, then either $P(u)$ is a nonconstant function or $Q(u)$ is a nonlinear function or both of them. In order to facilitate symmetry classification, we divide $P(u)$ and $G(u)$ into two parts respectively, set $P(u) = \hat{P} + \beta, G(u) = \lambda u + \hat{G}$, where $\lambda$ and $\beta$ are two constants, $\hat{P} = \hat{P}(u)$ is a function without containing constant term and $\hat{G} = \hat{G}(u)$ is a nonlinear function without containing the linear term $\lambda u$.

With the above assumptions, $I_2 = \{u, u_x, \hat{P}u_x, \hat{G}\}$, $J_2 = \{u, u_x\}$, $I_2 \backslash J_2 = \{\hat{P}u_x, \hat{G}\}$ and $J_2 = \{\hat{P}u_x + \hat{G} + \beta u_x + \lambda u, \hat{G}\}$. By theorem 2.8, the two determining conditions for the second equation in system (62) are

$$\frac{\partial^n h_2(t, x)}{\partial t^n} - \beta \frac{\partial h_1(t, x)}{\partial x} - \lambda h_1(t, x) = 0,$$ \hspace{1cm} (69a)
Equating the coefficients of \( \text{tem} (62) \) with \( \text{Nonlinearity} \) implies that where system (68) is used. We start with equation (70a) to classify the pairs \((g, P)\) since \(\hat{h} = \text{Proposition 3.2.} \)

Inserting them into equation (69a) yields

\[
\text{constant, then solving the first equation in system (68) gives } h_2(t, x) = C x t^{-\alpha} / \Gamma(1 - \alpha) + k(t).
\]

Equating the coefficients of \( x \) to zero yields \( C = 0 \) and then \( k(t) = c_2 x t^{-\alpha} \). Thus \( h_1(t, x) = 0 \) and \( h_2(t, x) = c_2 x t^{-\alpha} \).

Then inserting \( \eta \) and \( \eta' \) into equation (69b) and separating it with respect to \( u_x \) yields

\[
(\alpha - \alpha^1_1) \left( \hat{P} u_x + \hat{G} + \beta u_x + \lambda u \right) - \lambda \left( \eta - h_1(t, x) \right)
\]

where \( \eta' \) is given by

\[
\eta' = \left[ g_1(x) - \xi'(x) \right] u_x + g'_1(x) u + \frac{\partial h_1(t, x)}{\partial x}.
\]

We claim that if \( h_1(t, x) = 0 \), then \( h_1(t, x) = \mathcal{C} \) is a constant, then solving the first equation in system (68) gives \( h_2(t, x) = c_2 x t^{-\alpha} / \Gamma(1 - \alpha) + k(t) \).

Then inserting \( \eta \) and \( \eta' \) into equation (69b) and separating it with respect to \( u_x \) yields

\[
2(\hat{P} + \beta) [c_1 - g_1(x)] - \hat{P} \left[ g_1(x) u + h_1(t, x) \right] = 0,
\]

where system (68) is used. We start with equation (70a) to classify the pairs \((P(u), G(u))\) and then to determine the corresponding Lie symmetries. First consider \( P \neq 0 \). Then equation (70a) implies that \( g_1(x) \) and \( h_1(t, x) \) are constants respectively, and thus \( h_1(t, x) = 0 \) and \( h_2(t, x) = c_2 x t^{-\alpha} \) by the claim. Let \( g_1(x) = \omega \) be a constant. If \( \omega = 0 \), equation (70a) gives \( c_1 = 0 \), then we find that \( P(u) \) is arbitrary and \( G(u) = 0 \) from system (70), which is contradict with nonzero \( G(u) \). While for \( \omega 
eq 0 \), solving equation (70a) gives \( \hat{P} = u^{2(\alpha - \omega)/\omega} - \beta \), which implies \( \beta = 0 \) since \( \hat{P} \) is independent of constant term, i.e. \( \hat{P} = u^{2(\alpha - \omega)/\omega} \).

Furthermore, from the second equation in system (68), we find \( \xi(x) = (\alpha \chi_1 - \omega + c_1)x + c_2 \) and then equation (70b) becomes

\[
\hat{G} (c_1 - \alpha \chi_1) + u [\lambda \chi_1 - \lambda g_1(x) - \beta g'_1(x) - \lambda \alpha \chi_1] - \hat{P} \left[ g'_1(x) u + \frac{\partial h_1(t, x)}{\partial x} \right] - \hat{G} \left[ g_1(x) u + h_1(t, x) \right] = 0,
\]

where \( \xi(x) = (\alpha \chi_1 - \omega + c_1)x + c_2 \) and then equation (70b) becomes

\[
\hat{G} (c_1 - \alpha \chi_1) + u [\lambda \chi_1 - \lambda g_1(x) - \alpha \chi_1] - \omega u \hat{G} = 0,
\]

which gives \( \hat{G} = u^{(\alpha_1 - \alpha \chi_1)/\omega} \) and \( \omega = 0 \) since \( \hat{G} \) does not contain the term \( u \). Therefore, system (62) with \( P(u) = u^{2(\chi_1 - \omega)/\omega}, G(u) = u^{(\alpha_1 - \alpha \chi_1)/\omega} \) has a Lie symmetry with the infinitesimal generator

\[
X = \chi_1 \partial_x + (\alpha \chi_1 - \omega + c_1)x + c_2 \partial_t + \omega u \partial_u + \left( c_1 u v + c_2 t^{-\alpha - 1} \right) \partial_v.
\]

If \( \hat{P} = 0 \), then \( \beta \neq 0 \) since \( P(u) \neq 0 \). We find \( g_1(x) = c_1 \) from equation (70a) and \( h_1(t, x) = 0 \) by the claim and equation (70b). Under such conditions, equation (70b) becomes equation (71) with \( \omega = c_1 \), which is a particular case of \( \hat{P} \neq 0 \).

We summarize the above Lie symmetry classifications of system (62) as the following proposition.

**Proposition 3.2.** The Lie symmetries admitted by the time-fractional nonlinear telegraph equation (62) are classified as follows:
(a) For arbitrary functions $P(u)$ and $G(u)$, system (62) is admitted by $X = \partial_t$.

(b) For $P(u) = u^{2\alpha/\omega - 2}$, $G(u) = u^{(c_1-\alpha\chi_1)/\omega}$, system (62) is admitted by

$$X = \chi t \partial_t + [(\alpha \chi_1 - \omega + c_1) x + c_3] \partial_x + \omega u \partial_u + (c_1 v + c_2 t^{\alpha-1}) \partial_v.$$

Note that the cases of symmetry classification for fractional nonlinear telegraph equation (62) decrease significantly compared with the ones of integer-order case in [33], the reason for such phenomenon is that the fractional derivative greatly affects the symmetry properties of fractional PDEs.

With the infinitesimal generators in proposition 3.2, we perform symmetry reductions for system (62) as the following three cases.

(a) For arbitrary functions $P(u)$ and $G(u)$, system (62) is admitted by the infinitesimal generator $X = \partial_t$. Similar as the case for system (57), we obtain a solution of system (62) $u(t, x) = C_1 t^{-1}, v(t, x) = C_2 t^{-1}$, where $C_1$ and $C_2$ are integral constants.

(b) For $P(u) = u^{2\alpha/\omega - 2}$, $G(u) = u^{(c_1-\alpha\chi_1)/\omega}$, induced by the infinitesimal generator $X = \partial_t + c_2 t^{\alpha-1} \partial_v$, system (62) is reduced to

$$\frac{\partial^\alpha U(\zeta)}{\partial \zeta^\alpha} = C_2 s^{\alpha-1}, \quad \frac{\partial^\alpha V(\zeta)}{\partial \zeta^\alpha} = U(\zeta)^{1/(c_1-\alpha\chi_1)}, \tag{72}$$

where the similarity variables are $\zeta = t, U(\zeta) = u(t, x)$ and $V(\zeta) = v(t, x) - x t^{\alpha-1}$. Then by solving the reduced equation (72) and changing to the original variables, we obtain a particular solution of system (62)

$$u(t, x) = \frac{c_2 \Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1}, \quad v(t, x) = x t^{\alpha-1} + \left[ \frac{c_2 \Gamma(\alpha)}{\Gamma(2\alpha)} \right]^\alpha \frac{\Gamma((2\alpha - 1)\vartheta + 1)}{\Gamma((2\alpha - 1)\vartheta + \alpha + 1)} j^{(2\alpha-1)\vartheta+\alpha},$$

where $\vartheta = (c_1 - \alpha\chi_1)/\omega$.

(c) For $P(u) = u^{2\alpha/\omega - 2}$, $G(u) = u^{(c_1-\alpha\chi_1)/\omega}$, by the infinitesimal generator $X = \chi t \partial_t + (c_1 + \chi_1 \alpha - \omega)x \partial_x + \omega u \partial_u + c_1 v \partial_v$, system (62) is reduced to

$$\left( P \frac{1 + \frac{\alpha - \omega}{\chi_1}}{1 + \frac{\alpha - \omega}{\chi_1} \chi_1} U \right)(\zeta) = V'(\zeta),$$

$$\left( P \frac{1 + \frac{\alpha - \omega}{\chi_1}}{1 + \frac{\alpha - \omega}{\chi_1} \chi_1} V \right)(\zeta) = U(\zeta)U'(\zeta) + U^{(c_1-\alpha\chi_1)/\omega}(\zeta),$$

where $\zeta = x t^{-(c_1+\alpha\chi_1-\omega)/\chi_1}, U(\zeta) = u(t, x) t^{-\omega/\chi_1}$ and $V(\zeta) = v(t, x) t^{-c_1/\chi_1}$.

4. Conclusions

We deeply study the Lie group theory for the system of multi-dimensional time-fractional PDEs and give an explicit formula of the prolongation involving Riemann–Liouville fractional derivative. Moreover, we show that the infinitesimal generators of Lie symmetries for system (3) have an elegant structure and are completely determined by two elegant conditions. Our results pave a simple way for acquiring symmetry information of multi-dimensional time-fractional PDEs (3) and also motivate the automatic implementation of searching for Lie symmetries with lower complexity of computation.
Conflict of interest

None.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China (No. 11671014).

ORCID iDs

Zhi-Yong Zhang © https://orcid.org/0000-0003-4416-4798

References

[1] Leibniz G W 1962 Mathematische Schriften (Hildesheim: Georg Olms Verlagshandlung)
[2] Podlubny I 1999 Fractional Differential Equations (San Diego, California: Academic)
[3] Kiryakova V 1994 Generalized Fractional Calculus and Applications (Pitman Research Notes in Mathematics vol 301) (London: Longmans Green)
[4] Sun H, Zhang Y, Baleanu D, Chen W and Chen Y 2018 A new collection of real world applications of fractional calculus in science and engineering Commun. Nonlinear Sci. Numer. Simul. 64 213–31
[5] Metzler R, Jeon J-H, Cherstvy A G and Barkai E 2014 Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking Phys. Chem. Chem. Phys. 16 24128–64
[6] Magin R L, Abdullah O, Baleanu D and Zhou X J 2008 Anomalous diffusion expressed through fractional order differential operators in the Bloch–Torrey equation J. Magn. Reson. 190 255–70
[7] Kilbas A A, Srivastava H M and Trujillo J J 2006 Theory and Application of Fractional Differential Equations vol 204 (Netherlands: Elsevier Science Limited)
[8] Guo B L, Pu X K and Huang F H 2015 Fractional Partial Differential Equations and Their Numerical Solutions (Singapore: World Scientific)
[9] Luchko Y and Gorenflo R 1999 An operational method for solving fractional differential equations with the Caputo derivatives Acta Math. Vietnam. 24 207–33 http://journals.math.ac.vn/acta/pdf/9902207.pdf
[10] Bhrawy A H and Zaky M A 2015 A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations J. Comput. Phys. 281 876–95
[11] Chen J, Liu F and Anh V 2008 Analytical solution for the time-fractional telegraph equation by the method of separating variables J. Math. Anal. Appl. 338 1364–77
[12] Bluman G W, Cheviakov A F and Anco S C 2010 Applications of Symmetry Methods to Partial Differential Equations (New York: Springer)
[13] Olver P J 1993 Applications of Lie Groups to Differential Equations (New York: Springer)
[14] Ovsyannikov L V 1982 Group Analysis of Differential Equations (New York: Academic)
[15] Hereman W 1997 Review of symbolic software for Lie symmetry analysis Math. Comput. Modelling 25 115–32
[16] Buckwar E and Luchko Y 1998 Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations J. Math. Anal. Appl. 227 81–97
[17] Gazizov R K, Kasatkin A A and Lukashchuk S Y 2009 Symmetry properties of fractional diffusion equations Phys. Scr. T136 014016
[18] Gazizov R K, Kasatkin A A and Lukashchuk S Y 2007 Continuous transformation groups of fractional-order differential equations Vestnik USATU 9 125–35 (in Russian)
[19] Wang G-W and Xu T-Z 2014 Invariant analysis and exact solutions of nonlinear time fractional Sharma–Tasso–Olver equation by Lie group analysis Nonlinear Dyn. 76 571–80
[20] Liu H 2013 Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations Stud. Appl. Math. 131 317–30

5211
[21] Huang Q and Zhidanov R 2014 Symmetries and exact solutions of the time fractional Harry–Dym equation with Riemann–Liouville derivative Physica A 409 110–8
[22] Singla K and Gupta R K 2016 On invariant analysis of some time fractional nonlinear systems of partial differential equations. J. Math. Phys. 57 101504
[23] Zhang Y, Mei J and Zhang X 2018 Symmetry properties and explicit solutions of some nonlinear differential and fractional equations Appl. Math. Comput. 337 408–18
[24] Chen C and Jiang Y-L 2017 Lie group analysis and invariant solutions for nonlinear time-fractional diffusion-convection equations Commun. Theor. Phys. 68 295–300
[25] Jefferson G F and Carminati J 2014 FracSym: automated symbolic computation of Lie symmetries of fractional differential equations Comput. Phys. Commun. 185 430–41
[26] Zhang Z-Y 2020 Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation Proc. R. Soc. A 476 20190564
[27] Leo R A, Sicuro G and Tempesta P 2017 A foundational approach to the Lie theory for fractional order partial differential equations Fract. Calc. Appl. Anal. 20 212–31
[28] Sahadevan R and Prakash P 2017 On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations Chaos Solitons Fractals 104 107–20
[29] Sahoo S and Saha Ray S 2018 The conservation laws with Lie symmetry analysis for time fractional integrable coupled KdV-mKdV system Int. J. Non-Linear Mech. 98 114–21
[30] Dorjgotov K, Ochiai H and Zunderiya U 2018 Lie symmetry analysis of a class of time fractional nonlinear evolution systems Appl. Math. Comput. 329 105–17
[31] Zhang Z-Y and Li G-F 2020 Lie symmetry analysis and exact solutions of the time-fractional biological population model Physica A 540 123134
[32] Zakharov V E and Kuznetsov E A 1974 Three-dimensional solitons Sov. Phys.-JETP 39 285–6
[33] Bluman G W, Chaolu T and Sahadevan R 2005 Local and nonlocal symmetries for nonlinear telegraph equations J. Math. Phys. 46 1–12