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2013

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PEELING THE GRID*
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Abstract. Consider the set of points formed by the integer \( n \times n \) grid and the process that in each iteration removes from the point set the vertices of its convex hull. Here, we prove that the number of iterations of this process is \( O(n^{4/3}) \); that is, the number of convex layers of the \( n \times n \) grid is \( \Theta(n^{4/3}) \).

Key words. grid, peeling, convex hull

AMS subject classifications. 52A10, 52A22, 52C05

DOI. 10.1137/120892660

1. Introduction. For many algorithms, the worst case behavior is rarely encountered in practice. This is because the worst case behavior might require a degenerate and convoluted input. To address this gap between the worst case analysis and a real world behavior, a considerable amount of research has been spent on analyzing algorithms and discrete geometric structures under certain assumptions on the input, including (i) realistic input models [5], (ii) fatness [1], (iii) randomness, etc.

Random points. There is a significant amount of work on the geometric behavior of random point sets [10, 9, 12, 2, 8, 7]. The question of how the Voronoi diagram or the convex hull of a point set randomly generated inside a convex domain behaves has received considerable attention. In particular, it is known that for a set of \( n \) points chosen uniformly in the unit square, the expected complexity of the convex hull is \( O(\log n) \), and \( O(n^{1/3}) \) if the domain is a disk (this bound holds for any convex shape).

Grid points. Surprisingly, the known results on uniformly sampled points match the results known for the grid point set. For example, the number of vertices of the convex hull of any subset of the \( \sqrt{n} \times \sqrt{n} \) grid is \( O(n^{1/3}) \), which matches the bound for the random points. This phenomenon holds for many similar scenarios; see the survey by Bárány [2].

Convex layers. The decomposition of a point set into convex layers is one possible way to measure the depth of a point inside the point set. Formally, the convex depth of a point \( p \) in a point set \( P \) is \( d_p(P) = 1 \) if \( p \) is a vertex of the convex hull of \( P \), and it is \( d_p(P) = 1 + d_p(P \setminus V(CH(P))) \) otherwise, where \( CH(P) \) denotes the convex hull of \( P \) and \( V(CH(P)) \) denotes the set of its vertices.\(^1\) This partitions the point set into convex layers, as depicted in Figure 1. In particular, if the points rise out of physical measurements (that might contain noise), a point with large convex depth is unlikely to be an outlier. This is one possible definition of robust statistics for points, although this definition has its limitations; see [11] for details. In particular, Chazelle

\(^*\)Received by the editors September 25, 2012; accepted for publication February 4, 2013; published electronically April 10, 2013.

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\(^1\)A point of \( P \) is a vertex of the convex hull only if it is a corner of the convex hull. Formally, \( p \) is a vertex of the convex hull of \( P \) if \( CH(P) \neq CH(P \setminus \{p\}) \).
[3] provided an $O(n \log n)$ time algorithm for computing all the convex layers for a set of points in the plane.

For a set of $n$ points picked uniformly inside a bounded convex domain in $\mathbb{R}^d$, it is known that the expected number of convex layers is $\Theta(n^{2/(d+1)})$ [4].

Our results. In this paper, we prove that the number of convex layers of the $n \times n$ grid is $\Theta(n^{4/3})$. This bound is quite surprising—indeed, as demonstrated by Figure 2, the peeling process starts out quite slowly, the first three layers having 4, 8, and 8 vertices (independent of the value of $n$), respectively. A priori, it is not clear why this process accelerates and contains more vertices. Furthermore, the maximum number of vertices in convex position in an $n \times n$ grid is $O(n^{2/3})$ (this is well known; see Lemma 2.1). Namely, somewhat surprisingly, a constant fraction of the layers have asymptotically maximum size. Our result matches the known result for random points. Note that although the bounds are similar, the proof for the random point set does not carry over to the grid case.

We also observe that the number of convex layers is $\Omega(n^2)$ if the grid of $n \times n$ points is allowed to be nonuniform (instead of using the integer grid above). Naturally, in this construction, every point is on two lines, where each has $n$ points.

2. Peeling the grid. Let $P_0 = G_n = \{1, \ldots, n\}^2$ be the $n \times n$ integer grid. In the $i$th iteration, consider the convex hull $C_i = \text{CH}(P_{i-1})$ for $i = 1, \ldots$. Let $V_i$ be the set of vertices of $C_i$. Naturally, we consider a grid point to be a vertex only if it is a corner of the convex hull, and, as such, grid points falling in the middle of edges of $C_i$ are not in $V_i$. Now, let $P_i = P_{i-1} \setminus V_i$. In other words, we start with the $n \times n$ grid and peel away the vertices of the convex hull, and we repeat this process until all the grid points of $G_n$ are removed. Let $\tau(n)$ be the number of iterations until $P_i$ is an empty set. Here we are interested in the behavior of $\tau(n)$. See Figure 2 for an example of what the generated polygons look like.

2.1. A lower bound on $\tau(n)$. The following is well known, and we include a proof for the sake of completeness.

Lemma 2.1. Given any convex set $C$ in the plane, it can have at most $O(n^{2/3})$ vertices of $G_n$.

Proof. Consider a convex set $C$ such that all its vertices are points of $G_n$. The perimeter of $C$ is at most $4n$. The number of edges of the convex hull of $C$ of length at least (or equal to) $\mu$ is at most $4n/\mu$. The number of edges having length smaller than $\mu$ is bounded by the number of integer points of distance at most $\mu$ from the
origin, and this number is bounded by $(2\mu + 1)^2 = O(\mu^2)$. As such, the number of vertices of $C$ is at most $O(n/\mu + \mu^2)$. Setting $\mu = \lfloor n^{1/3} \rfloor$ then implies the claim.

As such, $|V_i| = O(n^{2/3})$, which implies immediately that $\tau(n) \geq n^2 / \max_i |V_i| = \Omega(n^{4/3})$.

2.2. An upper bound on $\tau(n)$. An integer vector $(x, y)$ is called primitive if gcd($x, y$) = 1. For an integer $\mu$, let $V_{\mu}$ be the set of all primitive nonzero integer vectors $(x, y)$, where $0 \leq y < x \leq \mu$. The following is well known, and we sketch a proof for the sake of completeness.

Lemma 2.2. We have $|V_\mu| \geq c\mu^2$ for some constant $c > 0$.

Proof. For a fixed $x$, consider the vectors $(x, y)$ in $V_\mu$, such that $y < x$, and gcd($x, y$) = 1. The number of such vectors is the number of integer values of $y$ that are relative prime to $x$, and this number is Euler’s totient function $\phi(x)$. As such, $|V_\mu| \geq \sum_{i=1}^{\mu} \phi(i) \geq c\mu^2$. The last step follows from known bounds; see [6].

In the following, we pick $\mu$ to be smaller than $n/4$, and $n$ is sufficiently large.

For every vector $v \in V_\mu$, consider the set $L_v$ of all lines having direction $v$ that intersect the grid points $G_n$. Every line in $L_v$ contains at most $1 + \lfloor (n - 1)/v_x \rfloor$ points of the grid (and most lines in this family contain at least $\lfloor (n - 1)/v_x \rfloor$ points of the grid (the only problematic lines are those that have short intersection with the square $[1, n]^2$ because of the corners)).

Claim 2.3. For $n > 10$, $\mu < n/4$, and $v \in V_\mu$, we have that $|L_v| \leq 4n\mu$.

Proof. A line $\ell \in L_v$ that intersects $G_n$ has an intersection of length at least $n$ with the enlarged square $[1, 2n]^2$. Specifically, the projection of the intersection on the $x$ axis has length at least $n$. Since $\ell$ has direction $v$ and it contains a grid point, it follows that it has grid points on it that are of distance $||v||$ from each other. On the projection, the distance between these points is $v_x$. As such, this intersection contains at least $1 + |n/v_x| \geq n/\mu$ points of the grid $G_{2n}$ on it. In particular, the number of such lines can be at most $4n^2/(n/\mu) = 4n\mu$.

Since the lines of $L_v$ cover all the grid points of $G_n$ and the vertices of $C_i$ are grid points, it follows that $L_v$ always contains two lines that are tangent to $C_i$. If these two tangent lines intersect $\partial C_i$ along a nonempty edge, then $v$ is active at iteration $i$ (i.e., $v$ is not active if the two tangents touch $C_i$ at a vertex).
Fig. 3. (a) An active direction $v$ and the set of lines $L_v$. (b) An inactive iteration for $v$. (c) The next iteration—the two “old” tangent lines no longer intersect the current convex layer.

In the following, we slightly abuse notation and use $L_v \cap C_i$ to denote the set of all lines of $L_v$ that have nonempty intersection with $C_i$.

**Claim 2.4.** If $v$ is not active at iteration $i$, then $|L_v \cap C_{i+1}| \leq |L_v \cap C_i| - 2$.

**Proof.** If $v$ is not active at iteration $i$, then a tangent $\ell$ to $C_i$ from $L_v$ intersects $C_i$ only at a vertex. But this vertex is being removed from the point set when computing $P_{i+1}$. In particular, the line $\ell$ no longer intersects $C_{i+1}$. The same argument also applies to the other tangent. This is demonstrated in Figure 3.

**Claim 2.5.** Throughout the process, for a vector $v \in \mathcal{V}_\mu$, it can be inactive in at most $2n\mu$ iterations.

**Proof.** Every time $v$ is not active, the number of lines of $L_v$ that intersect the active convex hull decreases by two, by Claim 2.4. By Claim 2.3 there are at most $4n\mu$ lines in the set $L_v$, and, as such, this can happen at most $4n\mu/2$ times.

If the process continues for more than $M = 4n\mu$ iterations, then every vector in $\mathcal{V}_\mu$ is active in at least half of the iterations. In particular, if $n_i$ is the number of active directions at iteration $i$, then we have that

$$\alpha = \sum_{i=1}^{M} n_i \geq 2n\mu |\mathcal{V}_\mu| \geq 2cn\mu^3$$

by Lemma 2.2.

Observe that if $n_i$ vectors are active at the $i$th iteration, then the convex hull of $C_i$ has at least $2n_i$ edges (and thus vertices) at iteration $i$. As such, if we set $\mu = \left[ \frac{n^{1/3} \epsilon^{1/3}}{c^{1/3}} \right] = \Theta\left( \frac{1}{\epsilon} \right)$, we have that the total number of vertices of the convex hulls in the first $M$ iterations is at least

$$2\alpha \geq 4cn\mu^3 \geq 4n^2,$$
Fig. 4. A point set where the peeling process requires $\Omega(n^2)$ steps.

which is a contradiction, as the initial grid set has at most $n^2$ points. We conclude
that the algorithm must terminate after $M = 4n\mu = O(n^{4/3})$ iterations. We have
thus proved the following.

**Theorem 2.6.** Starting with the grid $G_n$, consider the process that repeatedly
removes the convex-hull vertices of the current set of vertices. This process takes
$\Theta(n^{4/3})$ steps.

3. Lower bound of $\Omega(n^2)$ for a nonuniform grid. This section is devoted
to describing a set $M$ of $n^2$ points in the plane where the peeling process takes $\Omega(n^2)$
steps. For simplicity assume that $n = 2k$ for some integer $k$.

Take a collection of $k$ squares $S_1, \ldots, S_k$, where $S_i$ has length of its side $3^i$ and
the squares are positioned such that their centers coincide with the origin. Let $L$ be
the set of $4k$ lines that are obtained by extending the segments of the squares into
lines. Finally, let $M$ be the set of all intersections of lines in $L$. Notice that each line
contains $2k$ points and that $|L| = 4k^2 = n^2$. See Figure 4.

Let the peeling process partition $M$ into convex sets $C_1, C_2, \ldots$.

**Claim 3.1.** For every $C_i$ there exists $S_j$ such that $C_i \subseteq S_j$.

**Proof.** Let $j$ be the largest index such that $C_i \cap S_j \neq \emptyset$. Notice that $C_i$
are centrally symmetric as $M$ is centrally symmetric and that this property is preserved
by the peeling process. If $|C_i \cap S_j| = 4$, then $C_i \cap S_j$ are the four corners of $S_j$, and
thus $|C_i| = 4$ as $C_i$ is strictly convex. Hence $|C_i \cap S_j| = 8$ and $C_i$ contains points on
both vertical and horizontal lines of $S_j$ in every quadrant. Let $D$ be the square with
corners being intersections on the axis and $CH(S_j)$. See Figure 4 on the left. Notice
that $S_i \subset D \subset CH(C_i \cap S_j)$ for every $l < j$. Therefore $C_i = C_i \cap S_j \subseteq S_j$.

The previous claim implies that $|C_i| \leq 8$ for every $i$. Hence the peeling process
needs at least $n^2/8 = \Omega(n^2)$ steps.

4. Conclusions. The most natural task left by our work is to prove similar
bounds in higher dimensions. This seems quite challenging, and we leave it as an
open problem for further research.

Let us also note for the interested reader that, according to experiments, the
layers in the peeling process are getting close to circles as the process is advancing.

**Acknowledgments.** We thank Robert Jamison for many useful discussions on
this and related problems. We would also like to thank Imre Bárány for providing
relevant information.
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