Singular vorticity solutions of the incompressible Euler equation via inviscid limits

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Abstract

Singular vorticity solutions of the incompressible 3D-Euler equation are constructed which satisfy the BKM criterion (cf. 4). The construction is done by inviscid limits of (essentially local) vorticity solutions of a family related equations with viscosity parameter, and where a damping potential term ensures the existence of a solution in a cone. Representations of the solution in terms of convolutions with the Gaussian (with variance \( \nu \)) and the first order spatial derivative of the Gaussian lead to an inviscid limit such the original vorticity function solves the Euler equation. This inviscid limit vorticity solution of the incompressible Euler vorticity equation becomes singular at a point of the boundary of a finite domain (the tip of that cone). The construction can be applied locally within compressible Navier Stokes equations systems with degeneracies of viscosity and Lamé viscosity such that some type of turbulent dynamics occurs locally as the Reynold numbers become very large. Some solution branches of the Euler equation loose regularity at any local time while others can be continued.

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1 Singular vorticity of the Euler equation via two hypotheses concerning limits of well-defined solutions of Navier Stokes type equations

We are concerned with singular vorticity solutions of the Cauchy problem of the incompressible Euler equation in dimension \( D = 3 \), i.e., the Cauchy problem for the velocity \( v_i \), \( 1 \leq i \leq D \) of the form

\[
\begin{align*}
\frac{\partial v_i}{\partial t} + \sum_{j=1}^{D} v_j \frac{\partial v_i}{\partial x_j} &= - \frac{\partial p}{\partial x_i}, \\
\sum_{i=1}^{D} \frac{\partial v_i}{\partial x_i} &= 0, \\
v_i(0, x) &= f_i(x), \text{ for } 1 \leq i \leq D,
\end{align*}
\]

(1)

where \( f_i \in H^s(\mathbb{R}^D) \), or \( f_i \in H^s(\mathbb{R}^D) \) for some \( s \geq 0 \), and \( i \in \{1, 2, 3\} \), (i.e., we have finite energy at least). Recall that this equation determines a Cauchy
problem for the vorticity
\[ \omega = \text{curl}(v) = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \]  
\hfill (2)

of the form
\[ \frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \frac{1}{2} \left( \nabla v + \nabla v^T \right) \omega, \]  
\hfill (3)

which has to be solved with the initial data \( \text{curl}(f) \) at time \( t = 0 \). The following considerations can be applied for the domain of a torus \( \mathbb{T}^D \) of dimension \( D = 3 \) and for the whole space of \( \mathbb{R}^3 \), where there are slight differences with respect to the coordinate transformation and with respect to the formulation of the Biot-Savart law. In coordinates we have
\[ \frac{\partial \omega_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial \omega_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \omega_j. \]  
\hfill (4)

The vorticity equation form of the Navier-Stokes equation has an additional viscosity term \( \nu \Delta \omega \) along with viscosity \( \nu > 0 \). As we have (cf. \cite{4})
\[ v(t, x) = \int_{\mathbb{R}^3} K_3(x - y) \omega(t, y) dy, \]  
\hfill (5)

we may interpret (4) as a nonlinear partial integro-differential equation for the vorticity. In case of the torus a corresponding relation may be formulated via Fourier representations. In the following we use the abbreviation
\[ w^f(t, .) := \int_{\mathbb{R}^3} K_3(. - y) w(t, y) dy = \int_{\mathbb{R}^3} K_3(y) w(t, . - y) dy \]  
\hfill (6)

for any function \( w : \mathbb{R}^3 \to \mathbb{R} \) such that (4) is well-defined. If we consider a vorticity Navier-Stokes equation with an additional viscosity operator \( \nu \Delta \) and on a different time scale \( \tau = \rho t \) for some \( \rho > 0 \), then we shall denote the corresponding solution function by \( \omega^{\nu \rho} \), \( 1 \leq i \leq n \), and correspondingly to (4), we write
\[ v^{\nu \rho}(t, x) = \int_{\mathbb{R}^3} K_3(x - y) \omega^{\nu \rho}(\tau, y) dy. \]  
\hfill (7)

Similar relations hold for the torus with an operator defined via Fourier transformations, of course. Here, we mention the convolution rule in order to remind us that derivatives of velocities have a simple relation to derivatives of vorticity. In general, we write \( \omega_{i,j,k} := \frac{\partial^2}{\partial x_j \partial x_k} \omega_i \) as usual, and similarly for velocities. We may also suppress the arguments and write \( w^f \) for a transformation of a function \( w \) as in (6). For smooth functions \( f_i : [0, \infty) \times \Omega \to \mathbb{R}, 1 \leq i \leq 3 \), along with some \( \Omega \subseteq \mathbb{R}^3 \) we consider the ansatz
\[ \omega_i(t, x) = \frac{f_i \left( \sqrt{1 - t} \right) \frac{\pi}{2} \arctan(x) }{1 - t} \]  
\hfill (8)

based on the coordinate diffeomorphism with the domain \( D_1 := [0, 1) \times \mathbb{R}^3 \) for the \( t \)- and \( x \)-coordinates, i.e., the domain of the function \( \omega_i, 1 \leq i \leq n \). Here, we write \( \arctan(x) = (\arctan(x_1), \cdots, \arctan(x_n))^T \). For \( (t, y) = \)
\((t, (1 - t) \frac{2}{\pi} \arctan(x))\) we have \(y_j \in (- (1 - t), (1 - t))\) such that we have a cone image
\[
K_0 = \bigcup_{0 \leq t(\tau) \leq 1} K_{t(\tau)} := \left[0, \infty \right) \times \bigcup_{0 \leq t \leq 1} \left(-1 + t(\tau), 1 - t(\tau)\right)^3 \supset \left[0, \infty \right) \times \mathbb{R}^3.
\]
(9)
in \(\tau\)- and \(y\)-coordinates of the form
\[
(\tau, y) = \left(\frac{t}{\sqrt{1 - t^2}}, (1 - t) \frac{2}{\pi} \arctan(x)\right).
\] (10)
In the following the cone in original time coordinates \(t\) is denoted by \(K_0^t\). Alternatively, and also on (a suitable realization of a) torus we could use a similar diffeomorphism with \((\tau, y) = \left(\frac{t}{\sqrt{1 - t^2}}, (1 - t^2)^{\frac{x}{1 + \sqrt{|x|^2}}}\right)\). As a matter of elegance in notation below, the Cauchy problems considered in transformed coordinates can and are assumed to be extended trivially, i.e., identical to the zero function, outside the cone. This complementary set of the cone \(K_0\) is a subset of the whole domain \(D_\infty := [0, \infty) \times \mathbb{R}^3\). Regular extension with the zero function turns out to be possible for problems of physical interest, i.e., for data of polynomial decay at infinity. This is not essential but it is convenient. We write \(t(\tau)\) and \(x(\tau, y)\) for the components of the reverse. In the following for all \(1 \leq j \leq 3\) the function \((t, x) \rightarrow (1 - t) \frac{2}{\pi} \arctan(x_j)\) is denoted by \(b_j\) (the same notation may be used for the alternative transformation with \((1 - t) \frac{2}{\pi} \arctan(x_j)\), and also be used for the torus). As usual, we denote the derivative of \(b_j\) with respect to the spatial variable \(x_k\) by \(b_{j,k}\) and the derivative of \(b_j\) with respect to \(\tau\) by \(b_{j,\tau}\) etc.. All these terms are products of time functions and spatial functions. For this reason we also denote \(b_j = (1 - t)b_j^t\) etc., extracting the spatial part \(b_j^t\) if this is convenient. Next we consider solution functions \((\tau, y) \rightarrow u_i^\nu(\tau, y)\) of Navier Stokes type equations which approximate \(f_i(\tau, y) = f_i\left(\frac{t}{\sqrt{1 - t^2}}, (1 - t) \frac{2}{\pi} \arctan(x)\right)\) in \((8)\) above for small viscosity \(\nu > 0\). Note that
\[
\frac{\partial \omega_i}{\partial t} = \frac{f_i(\tau, x)}{(1 - t)^2} + \frac{f_{i,\tau}}{1 - t} \frac{dt}{dt} + \sum_{j=1}^{3} \frac{f_{i,j}}{(1 - t)^2} \frac{\partial y_j}{\partial t},
\] (11)
where \(\frac{\partial t}{\partial t} = \frac{1}{\sqrt{1 - t^2}} + \frac{1}{2} \sqrt{1 - t^2} \frac{3}{2} (2t) = \frac{1 - t^2 + t^2}{\sqrt{1 - t^2}} = \frac{1}{\sqrt{1 - t^2(\tau)}}\). Note that
\[
b_{j,\tau} = \frac{\partial y_j}{\partial t} = - \frac{2}{\pi} \arctan(x_j)
\] (12)
are bounded smooth coefficients with bounded derivatives. For the convenience of the reader we derive the equation for
\[
u_i(\tau, y) = f_i(\tau, y) = (1 - t)\omega_i(t, x)
\] (13)
explicitly. If we consider the related vorticity form of the Navier Stokes equation we add a superscript \(\nu > 0\) and write
\[
u_i^\nu(\tau, y) = (1 - t)\omega_i^\nu(t, x),
\] (14)
where $\omega_{\nu}^{i}$, $1 \leq i \leq n$ satisfies the vorticity form of the Navier Stokes equation with viscosity $\nu$, i.e. the Laplacian term $-\nu \Delta \omega$ is added on the side of the equation with the time derivative. We identify for all $\tau, y$ in the considered domain

$$u_i(\tau, y) = u_i^0(\tau, y) = \lim_{\nu \downarrow 0} u_{\nu}^{i}(\tau, y).$$

(15)

Similar vor velocity components: we write $v_{\nu}^{i}$ in order to indicate that these are velocity components of a solution function for an incompressible Navier Stokes equation with viscosity $\nu > 0$. The additional symbol $u_i$ which reminds us that we consider an equation which is trivially extended to the whole domain in transformed coordinates (there is no other reason to introduce this additional symbol but this reminder). We write the transformation in a general form, where the fact that we use a ‘diagonal’ transformation simplifies the equation as indicated by the Kronecker-$\delta$ symbolized by $\delta_{ik}$ for $1 \leq i, k \leq n$. This way it is easy to transfer the following considerations to alternative transformations which are not diagonal but better suited for other domains and general compressible Navier Stokes equation systems considered in the next section. We have

$$\omega_{i,j} = \frac{1}{1 - t} \sum_k u_{i,k} b_{k,j} \delta_{kj},$$

(16)

which leads to

$$\Delta \omega_i = \frac{1}{1 - t} \sum_j \left( \sum_{k,m} u_{i,k,m} b_{k,j} b_{m,j} \delta_{kj} \delta_{mk} \right).$$

(17)

Next the Burgers term

$$\sum_{j=1}^{3} \left( \int_{\mathbb{R}^{3}} K_3(y) \omega_j(t, x - y) dy \right) \frac{\partial \omega_i}{\partial x_j}$$

(18)

becomes

$$\frac{1}{(1 - t)^2} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^{3}} K_3(y) u_j(t, x - y) dy \right) \sum_k u_{i,k} b_{k,j} \delta_{kj},$$

(19)

and the Leray projection term

$$\sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \omega_j$$

(20)

becomes

$$\frac{1}{(1 - t)^2} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^{3}} K_3(y) \sum_k u_{i,k}(t, x - y) b_{k,j} \delta_{jk} dy \right)$$

$$+ \int_{\mathbb{R}^{3}} K_3(y) \sum_k u_{j,k}(t, x - y) b_{k,i} \delta_{ik} dy \right) u_j.$$
Hence, the vorticity form of the Navier Stokes equation in dimension \( D = 3 \)

\[
\frac{\partial \omega^{\nu}_{i}}{\partial t} - \nu \Delta \omega^{\nu}_{i} + \sum_{j=1}^{3} v_{j} \frac{\partial \omega^{\nu}_{i}}{\partial x_{j}} = \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v^{\nu}_{j}}{\partial x_{j}} + \frac{\partial v^{\nu}_{i}}{\partial x_{i}} \right) \omega^{\nu}_{j} \tag{22}
\]

gets, literally written, the form

\[
u \frac{1}{1-t} \sum_{j=1}^{3} \left( \sum_{k,m} u^{\nu}_{i,k,m} b_{j,k} b_{m,j} + \sum_{k,m} u^{\nu}_{i,k} b_{k,j,m} \delta_{k,j} \delta_{m,j} \right) + \nu \frac{1}{t} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^{3}} K_{3}(y) u^{\nu}_{j}(t,x-y) dy \right) \sum_{k} u^{\nu}_{i,k} b_{k,j} \delta_{j} \delta_{k} = \frac{1}{t} \nu \int_{\mathbb{R}^{3}} K_{3}(y) \sum_{k} u^{\nu}_{j,k}(t,x-y) b_{k,i} \delta_{j} d y + \int_{\mathbb{R}^{3}} K_{3}(y) \sum_{k} u^{\nu}_{j,k}(t,x-y) b_{k,i} \delta_{j} d y \quad (23)
\]

Using \( b_{i} = (1-t) b^{\nu}_{i} \) and similar relations for spatial derivatives and ordering terms a bit we get

\[
u \frac{1}{1-t} \sum_{j=1}^{3} u^{\nu}_{i,j} b_{j,t} + \nu \frac{1}{t} \int_{\mathbb{R}^{3}} K_{3}(y) u^{\nu}_{j}(t,x-y) dy \sum_{k} u^{\nu}_{i,k} b_{k,j} \delta_{j} \delta_{k} = \frac{1}{t} \nu \int_{\mathbb{R}^{3}} K_{3}(y) \sum_{k} u^{\nu}_{j,k}(t,x-y) b_{k,i} \delta_{j} d y + \int_{\mathbb{R}^{3}} K_{3}(y) \sum_{k} u^{\nu}_{j,k}(t,x-y) b_{k,i} \delta_{j} d y \quad (24)
\]

The additional terms with the coefficient \( \nu \) correspond to an additional viscosity term \( \nu \Delta \omega \) along with viscosity \( \nu > 0 \) of the vorticity form of the Navier Stokes equation. As we have (cf. [4])

\[
u \omega^{\nu}(t,x) = \int_{\mathbb{R}^{3}} K_{3}(x-\tau,y) \omega^{\nu}(t,y) dy, \quad \text{where } K_{3}(x) h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \tag{25}
\]

the velocity data \( v^{\nu}_{i} \) of the original Navier Stokes equation can be recovered from the vorticity equation.

Simplifying a bit, In the \( x \)- and \( y \)-coordinates and for \( u_{i}(\tau,y) = (1-t)\omega_{i}(t,x) \) on the domain \( D_{\infty} \) (trivially extended by values zero outside the image cone) the vorticity form of the Navier Stokes equation in dimension \( D = 3 \) (cf. [3])
gets the form (using the fact that we have a diagonal transformation)

\[ u_i^{\nu, \tau} + \sqrt{1 - t(\tau)} \sum_j b_{i,j} u_j^{\nu, \tau} + \sqrt{1 - t(\tau)} \sum_j b_{i,j}^{\nu, f} u_{i,k}^{\nu, f} + \frac{\sqrt{1 - t^2(\tau)}}{1 - t} u_i^{\nu, \tau} \]

\[ - \nu \sqrt{1 - t(\tau)} \sum_j b_{i,j} b_{j, \tau} u_j^{\nu, \tau} - \nu \sqrt{1 - t(\tau)} \sum_j b_{i,j, \tau}^{\nu, f} u_j^{\nu, \tau} \]

\[ = \sqrt{1 - t(\tau)} \sum_j \frac{1}{2} \left( b_{i,j}^{\nu, f} u_{i,j}^{\nu, f} + b_{i,i}^{\nu, f} u_{i,j}^{\nu, f} \right) u_j^{\nu, \tau}. \]  

(26)

Note that the potential term has a not a real singularity as \( \sqrt{1 - t} = (1 + t) \sqrt{1 - t^2} \), and for the function \( t(\tau) \) we have \( t([0, \infty)) \subseteq [0, 1) \) such that all other coefficients are bounded with respect to time. Note furthermore, that spatial coefficient notation with upper script \( s \) is applied for the nonlinear terms in order to observe that the coefficients are bounded.

We emphasize again that we identify the functions \( u_i^{\nu, \tau}, 1 \leq i \leq n \), which are defined via the transformation above on a cone, with their trivial extensions \( \overline{\nu}_i : D_\infty \to \mathbb{R}, 1 \leq i \leq 3 \) outside the cone, i.e., we define

\[ u_i^{\nu}(\tau, y) = \overline{\nu}_i(\tau, y) \equiv 0 \text{ on } D_\infty \setminus K_0. \]  

(27)

Similarly for the viscosity limit functions \( u_i^0 \) where \( \nu \downarrow 0 \) in the function \( u_i^{\nu}, 1 \leq i \leq n \). For a specific parameter \( \nu \) a solution of (26) is denoted by \( u_i^{\nu, *}, 1 \leq i \leq 3 \) in the respective coordinates, such that

\[ u_i^{\nu}(\tau, y) = u_i^{\nu, *}(t, x) = u_i^{\nu, *}(t(\tau), x(\tau, y)) \]  

(28)

on the domains \( D_\infty \), i.e., the domain with respect to \( \tau \)-coordinates, and \( D_1 \), i.e., the corresponding domain with respect to \( t \)-coordinates (cf. definitions below). In the following the symbol \( \lim \) refers to the limit superior. We have

**Theorem 1.1.** For all \( \nu > 0 \) there exists a local regular bounded solution \( u_i^{\nu, *}, 1 \leq i \leq 3 \) : \([0, 1) \times \mathbb{R}^3 \to \mathbb{R} \) corresponding to a global bounded solution \( u_i^{\nu}, 1 \leq i \leq 3 \) : \([0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) of (26), and such that there is a well-defined viscosity limit function in \( C^{1,2} \) in transformed coordinates as \( \nu \downarrow 0 \). Furthermore, if for some \( y \in \mathbb{R}^3 \)

\[ \lim_{\nu \downarrow 0} \lim_{\tau \to \infty} u_i^{\nu}(\tau, y) = \lim_{\nu \downarrow 0} \lim_{\tau \to \infty} u_i^{\nu, *}(t(\tau), x(\tau, y)) = c \neq 0 \]  

(29)

for some \( 1 \leq i \leq 3 \) and some constant \( c \neq 0 \), then the function \( \omega_i, 1 \leq i \leq 3 \) (trivially extended outside the cone \( K_0) \) with

\[ \omega_i(t, x) := \lim_{\nu \downarrow 0} u_i^{\nu, *}(t(\tau), x(\tau, y)) \frac{1 - t}{1 - t}, \quad 1 \leq i \leq 3, \]  

(30)

considered as a function in \( t \)- and \( x \)-coordinates is well-defined on \( D_1 \), and satisfies the incompressible vorticity form of the Euler equation, and this function \( \omega_i, 1 \leq i \leq n \) becomes singular at the tip \( (1, 0) \) of the cone \( K_0) \).

The assumption in (26) of the preceding theorem 1.1 can be weaken if we consider an additional local time-transformation of the Euler equations. For \( \rho > 0 \) we consider a family of coordinate diffeomorphisms from the domain
Conclusion holds if for some $1 \leq \nu$ where $L_{\infty}$ is dense in $\nu$. For some regular function $u(D)$ we may consider such data. For some constant $\rho > 0$ we have $|u(\rho, y)| = c \neq 0$ for some $1 \leq i \leq 3$ and some constant $c \neq 0$ which implies the existence of a singular solution to the vorticity equation.

**Theorem 1.2.** Assume that for all $\nu > 0$ and $\rho > 0$ there exists a local regular bounded solution $u^{\rho,\nu}$, $1 \leq i \leq 3 : D_\rho \rightarrow \mathbb{R}$ which corresponds to a global bounded solution $u^{\rho,\nu}$, $1 \leq i \leq 3 : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ of the equation (26) written in $\tau^\rho$- and $y^\rho$-coordinates. Then for each element $f_i(0,\cdot)$, $1 \leq i \leq 3$ of a set of data which is dense in $[L^2(\mathbb{R}^3)]^3$ there exists a size $\rho > 0$ such that the function $\omega_i$, $1 \leq i \leq 3$ with

$$\omega_i(t, x) = \lim_{\nu \rightarrow 0} u_i^{\rho,\nu}(t(\rho), x(\rho, y)) \quad \rho - t$$

satisfies the incompressible vorticity form of the Euler equation on $D_\rho$, and is singular at the tip of the cone $K_0^{\rho,t}$, i.e., at $(\rho,0)$.

**Remark 1.3.** The connection to the preceding theorem is that for elements of a dense set of data in $L^2$ there some $\rho > 0$ such that there is a viscosity limit $\nu \downarrow 0$ in transformed coordinates such that for some $y^\rho \in \mathbb{R}^3$

$$\lim_{\nu \rightarrow 0} |u_i^{\rho,\nu}(\tau(\rho), x^\rho)| = \lim_{\nu \rightarrow 0} |u_i^{\rho,\nu}(t(\rho), x(\rho, y))| = c \neq 0$$

for some $1 \leq i \leq 3$ and some constant $c \neq 0$ which implies the existence of a singular solution to the vorticity equation.

**Proof.** As the functions space $C_0^\infty$ of smooth data which decay to zero at spatial infinity is dense in $L^2$ we may consider such data. For some constant $c > 0$ the conclusion holds if for some $1 \leq i \leq 3$ we have

$$\lim_{\nu \rightarrow 0} \lim_{\tau \rightarrow \infty} |u_i^{\rho,\nu}(\tau(\rho), y)| \leq \lim_{\nu \rightarrow 0} \lim_{\tau \rightarrow \rho} |u_i^{\rho,\nu}(t(\rho), x(\rho, y))| = c \neq 0$$

for some regular function $u_i^{\rho,\nu}$ which satisfies the limit equation $(\nu \downarrow 0)$ for $u_i^{\rho,\nu}, 1 \leq i \leq n$ considered below. If there is some $c \neq 0$ and $y \in \mathbb{R}^3$ where $f_i(0,y) = c \neq 0$ for some $1 \leq i \leq n$, then local time analysis of the Navier Stokes equation then shows that for small $\rho > 0$ we have $|u_i^{\rho}(\rho, y)| = c \neq 0$ independently of $\nu > 0$. This implies that there are singular solutions for a dense set of data as described. The details of the local analysis of the equation for $u_i^{\rho,\nu}, 1 \leq i \leq n$ and the viscosity limit are considered in the last section.
2 Vorticity singularities and local turbulence in compressible Navier Stokes equation systems

We consider the dynamic relevance of vorticity solutions of the Euler equation as inviscid limits of solutions of the incompressible Navier Stokes equation system. The incompressible Navier Stokes equation is in turn a simplification of the physically more realistic standard compressible Navier Stokes equation systems for the dynamic variables $\rho_m$ (mass density) $v = (v_1, \cdots, v_n)^T$ (velocity field), $p$ (pressure) and $T$ (temperature) of the form

\[
\begin{align*}
\frac{\partial \rho_m}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i} (\rho_m v_i) &= \frac{\partial p_m}{\partial t} + \sum_{i=1}^3 v_i \frac{\partial}{\partial x_i} \rho_m + \rho_m \text{div} v = 0 \\
\rho_m \frac{\partial v_i}{\partial t} - \mu \Delta v_i + \rho_m (v \cdot \nabla) v &= -\nabla p + \mu' \nabla (\nabla \cdot v) + \rho_m f_i, \\
\frac{\partial (\rho_m e)}{\partial t} + \nabla \cdot ((\rho_m e + p) v) &= \nabla \cdot (\Sigma \cdot v) + \rho_m f \cdot v + \nabla \cdot (\lambda \nabla T) \\
\rho_m \frac{R}{\gamma} T &= p \\
v(0, \cdot) &= h, \quad T(0, x) = T_0(x), \quad p(0, x) = p_0(x), \quad \rho_m(0, x) = \rho_{m0}(x),
\end{align*}
\]

where the energy $e$ is determined by the equation

\[
e = u + \frac{1}{2} |v|^2, \quad \text{along with } u = \frac{R}{(\gamma - 1)} T,
\]

and where the latter equation determines the inner energy $u$ (according to Boyle-Mariott) and where $\gamma$ is an adiabatic constant and $R$ is a gas constant. The additional constants or functions $M$ and $\lambda$ are exogenous as is the force $f$.

Natural physical constraints are $T_0 > 0$, $p_0 > 0$, $\rho_{m0} > 0$ and regular (smooth) data $h = (h_1, \cdots, h_n)^T$ of polynomial decay of any order. It is common sense that turbulence is related to high Reynolds numbers $Re \sim \frac{u}{\nu}$ at certain points of space-time. Singularities of solutions are mainly caused by degeneracies of the mass density $\rho_m$ or of the Lamé-viscosity $\mu$. Depending on the material this viscosity may be considered to be a constant or itself dependent on space and time. Similar for $\mu'$. For variable Lamé viscosity $\mu$ we define the set $D_e$ of real degeneracies to be the set of all points which have a degenerate cylinder neighborhood in $D_e$, i.e., all space-time points $(t, x) \in [0, \infty) \times \mathbb{R}^3$ which satisfy $\mu(t, y) = 0$ and such that $(t_0, t_1) \times B_\epsilon(x) \subseteq D_e$ for some $\infty > t_1 - t_0 > 0$ and some ball $B_\epsilon(x)$ of radius $\epsilon > 0$ around $x$ we have $\mu(t, y) = 0$. If there is no such space time point then $D_e$ is the empty set. Otherwise we say that $D_e$ is the real set of degeneracies of \(37\). Turbulence related to high Reynolds numbers can only be caused by small $\mu$ or high values of mass density $\rho$. For large mass density the second equation in \(37\), i.e., the essential equation for the dynamics, reduces to an Navier Stokes equation equation with force term which is dynamically as stable as the force term allows. As we have no mathematical theory of turbulence we may use 'weak' notions of turbulence which are strictly compatible with standard qualitative description. Next we argue that a) the singular inviscid limit vorticity of Euler equations is essentially a local weak concept of turbulence (compatible with the description of turbulence in \(38\)), and b) if there is real set of degeneracies $D_e \neq \emptyset$ of \(37\) which admits a regular
solution on the complementary set, then there is a solution branch of (37) with at least one vorticity singularity in each connected component of \( D_e \). As a qualitative description of turbulence we consider paragraph 31 of [3]. The list of notions of the concept of turbulence associated to high Reynold numbers given there consists of 

- \( \alpha \): an extraordinary irregular and disordered change of velocity at every point of a space-time area,
- \( \beta \): the velocity fluctuates around a mean value,
- \( \gamma \): the amplitudes of the fluctuations are not small compared to the magnitude of the velocity itself in general,
- \( \delta \): the fluctuations of the velocity can be found at a fixed point of time,
- \( \epsilon \): the trajectories of fluid particles are very complicated causing a strong mixture of the fluid.

A weak quantitative concept of turbulence is a list of properties of possible solutions of a Navier Stokes equation or an Euler equation such that instances of this list can be subsumed by the notions in the list \( \alpha \) to \( \epsilon \) on a real set of space time points.

The proof of the theorem 2.1 follows from the construction of local vorticity singularities and some additional observations of the next section.

**Theorem 2.1.** Assume that the compressible Navier Stokes equation system in (37) has a real set of degeneracies \( D_e \) in the sense defined above. Assume furthermore that the Cauchy problem for (37) has a global regular solution outside the set \( D_e \) of degeneracies with strictly positive mass density. Then in each connected component of \( D_e \) there is at least one singular point of vorticity, where each such singular point is a weak concept of turbulence.

Singular vorticity in space-time sets of real degeneracies is closely related to the local analysis which is needed to prove the hypothesis of theorem 1.2 which is indeed the next step.

### 3 Analysis of the regular factors \( u_{i}^{\rho,\nu}, 1 \leq i \leq n \) of the vorticity function and their viscosity limit \( \nu \downarrow 0 \)

Next we consider the local analysis of the regular part \( u_{i}^{\rho,\nu}, 1 \leq i \leq n \) of the vorticity functions and the viscosity limit \( \nu \downarrow 0 \). Note that our considerations imply a positive answer to the question in (5.2) on p. 168 of [4]. In order to achieve this we first describe a local scheme for the functions \( u_{i}^{\rho,\nu}, 1 \leq i \leq n \) which corresponds to a global scheme for the functions \( u_{i}^{\rho,\nu}, 1 \leq i \leq n \) and show the existence of uniformly bounded solutions of the equation of the form (26).

We note that we do not need global existence arguments for the Navier Stokes equation for our argument, since the the global solution of (26) is a local equation in original time coordinates, and it is not that hard to get regular upper bounds for solutions of local equations. More precisely, we can prove global existence for the equivalent equations in \( (\tau^\rho, y^\rho) \)-coordinates or local regular existence for the equations in \( (t, y^\rho) \)-coordinates with original time coordinates, where we want the original time interval to become small for \( \rho \) small. In addition to the existence result we need the verification of the fact that the regular factor \( u_{i}^{\rho} \) of the vorticity \( \omega \) is not zero at the crucial point \( (\rho, 0) \) at the tip of the cone (in original time coordinates) where we want to detect the singularity. This requires

- a) that the solution function \( u_{i}^{\rho,\nu}, 1 \leq i \leq n \) does not change too much on the time interval \( [0, \rho] \) with respect to original time \( t \) (independently of the viscosity
\(\nu > 0\) and b) that there is an upper bound of the solution of the equation \[52\] below in transformed coordinates \((\tau^\rho, y^\rho)\) in a pointwise and strong sense, and which is again independent of the viscosity. We emphasize that this is essentially a local upper bound in original time coordinates.

The requirement a) can be verified by local contraction results. Furthermore, for part b) we do not have to rely on arguments concerning upper bounds for the incompressible Navier Stokes equation in strong norms, because bα we have a correspondence to a local Navier stokes equation, and bβ) the Navier Stokes type equation considered here has a potential term with 'the right sign', i.e., a term which causes a damping effect - which naturally leads to regular solutions and global upper bounds in strong norms exploiting some spatial effects of the operator. Again we emphasize that for the argument of this paper we need local regular existence result only, and we get these a fortiori for the regular factor \(u^\rho,\nu\), \(1 \leq i \leq n\) as we have them for the original velocity function \(v_i, 1 \leq i \leq n\). Note that local existence in original \(t\)-coordinates corresponds to global existence in \(\tau\)-coordinates. Next, we define a global scheme for the equation in \((\tau^\rho, y^\rho)\)-coordinates. For the convenience of the reader we write down this equation in detail. For transformed coordinates

\[
(\tau^\rho, y^\rho) = \left( \frac{t}{\sqrt{1 - \frac{t^2}{\rho^2}}}, y^\rho \right) = \left( \frac{t}{\sqrt{\rho^2 - t^2}}, (\rho - t) \frac{2}{\pi} \arctan(x) \right)
\]

we define for \(t \in [0, \rho)\)

\[
\omega_{i,\nu}^\rho(t, x) = \frac{u_{i,\nu}^\rho(\tau^\rho, y^\rho)}{\rho - t}.
\]

Again we use the abbreviations

\[
b_i^\rho = y_i^\rho, \quad b_i^\rho = (\rho - t) b_i^{\rho,s}.
\]

For the convenience of the reader we derive the equation for the function \(u_{i,\nu}^\rho, 1 \leq i \leq n\) explicitly. We have

\[
\frac{\partial \omega_{i,\nu}^\rho}{\partial t} = \frac{u_{i,\nu}^\rho(\tau, x)}{(\rho - t)} + \frac{u_{i,\nu}^\rho}{\rho - t} \frac{d\tau^\rho}{dt} + \sum_{j=1}^{3} \frac{u_{i,j}^\rho}{\rho - t} \frac{\partial b_j^\rho}{\partial t}.
\]

where \(\frac{\partial \tau^\rho}{\partial t} = \frac{1}{\sqrt{\rho^2 - t^2}} + t \frac{1}{2} \sqrt{\rho^2 - t^2} \cdot (-2t) = \frac{\rho^2 - t^2 + t^2}{\sqrt{\rho^2 - t^2}} = \frac{\rho^2}{\sqrt{\rho^2 - t^2}}\). Note that

\[
b_{j,t}^\rho = \frac{\partial y_j^\rho}{\partial t} = -\frac{2}{\pi} \arctan(x_j)
\]

are bounded smooth coefficients with bounded derivatives on the domain where the functions are supported. Next we have

\[
\omega_{i,j}^{\rho,\nu} = \frac{1}{\rho - t} u_{i,j}^{\rho,\nu} b_{j,j}^\rho,
\]

which leads to

\[
\Delta \omega_{i,\nu}^\rho = \frac{1}{\rho - t} \sum_j (u_{i,j,j}^{\rho,\nu} b_{j,j,j}^\rho b_{j,j}^\rho + u_{i,j}^{\rho,\nu} b_{j,j}^\rho).
\]
Next the Burgers term
\[
\sum_{j=1}^{3} \left( \int_{\mathbb{R}^3} K_3(y) \omega_{j}^{\rho,\nu}(t, x - y) \, dy \right) \frac{\partial \omega_{j}^{\rho,\nu}}{\partial x_j}
\]
becomes
\[
\frac{1}{(\rho - t)^2} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^3} K_3(y) u_{i,j}^{\rho,\nu}(t, x - y) \, dy \right) u_{i,j}^{\rho,\nu},
\]
and the Leray projection term
\[
\sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial u_{i,j}^{\rho,\nu}}{\partial x_j} + \frac{\partial u_{j,j}^{\rho,\nu}}{\partial x_i} \right) \omega_{j}^{\rho,\nu}
\]
becomes
\[
\frac{1}{(\rho - t)^2} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^3} K_3(y) u_{i,j}^{\rho,\nu}(t, x - y) \, dy \right) u_{i,j}^{\rho,\nu},
\]
Hence the vorticity form of the Navier Stokes equation in dimension \( D = 3 \)
\[
\frac{\partial \omega_i}{\partial t} - \nu \Delta \omega_i + \sum_{j=1}^{3} \nu \frac{\partial \omega_i}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial u_{i,j}^{\rho,\nu}}{\partial x_j} + \frac{\partial u_{j,j}^{\rho,\nu}}{\partial x_i} \right) \omega_{j}^{\rho,\nu}
\]
in (50) gets, literally written, the form
\[
\frac{u_{i,j}^{\rho,\nu}(\tau, x)}{(\rho - t)^2} + \frac{u_{i,j}^{\rho,\nu}}{\rho - t} \sqrt{\rho^2 - t^2} + \sum_{j=1}^{3} \frac{u_{i,j}^{\rho,\nu}}{(\rho - t)} \frac{\partial b_{l}^{\rho,\nu}}{\partial t} - \nu \frac{1}{\rho - t} \sum_{j} \left( u_{i,j,j,j} b_{l,j}^{\rho,\nu} + u_{i,j} b_{l,j,j} \right)
+ \frac{1}{(\rho - t)^2} \sum_{j=1}^{3} \left( \int_{\mathbb{R}^3} K_3(y) u_{i,j}^{\rho,\nu}(t, x - y) \, dy \right) u_{i,j}^{\rho,\nu} b_{l,j,j}
+ \frac{1}{(\rho - t)^2} \sum_{j=1}^{3} \frac{1}{2} \left( \int_{\mathbb{R}^3} K_3(y) u_{i,j}^{\rho,\nu}(t, x - y) b_{l,j,j} \, dy \right) u_{j}^{\rho,\nu}.
\]
Abbreviating \( \mathcal{P}^{\rho,\nu} = \frac{1}{\rho^2 - t^2} \), and using \( b_{l}^{\rho} = (\rho - t) b_{l}^{\rho,s} \) the equation takes the form
\[
\frac{u_{i,j}^{\rho,\nu}}{(\rho - t)^2} - \nu \mathcal{P}^{\rho,\nu} \sum_{j} b_{l,j,j} b_{l,j} u_{i,j}^{\rho,\nu} - \nu \mathcal{P}^{\rho,\nu} \sum_{j,k} b_{l,j,j} u_{i,j}^{\rho,\nu} + \mathcal{P}^{\rho,\nu} \sum_{j} b_{l,j} u_{i,j}^{\rho,\nu}
+ \mathcal{P}^{\rho,\nu} \sum_{j} b_{l,j} u_{i,j}^{\rho,\nu} + \frac{1}{\rho^2 - t^2} \mathcal{P}^{\rho,\nu} \sum_{j} b_{l,j} u_{i,j}^{\rho,\nu}
+ \mathcal{P}^{\rho,\nu} \sum_{j} \frac{1}{2} \left( b_{l,j} u_{i,j}^{\rho,\nu} + b_{l,j} u_{i,j}^{\rho,\nu} \right) u_{j}^{\rho,\nu}.
\]
The first line of equation (52) represents a parabolic linear scalar operator with bounded regular coefficients which depend on space and time. The first term in the second line of equation (52) represents a Burgers coupling term in vorticity form with space- and time-dependent coefficients. The second term in the second line of equation (52) represents a potential term which has the right sign in order to have a damping effect. This potential term is the only term where coefficients do not become small for small $\rho > 0$ and small time $t < \rho$, i.e.,

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^2} \left( \frac{\sqrt{\rho^2 - \tau^2(\tau)}}{\rho - t(\tau)} \right)^3 \neq 0,$$

while for all other coefficients we have this behavior, i.e., $\lim_{\rho \downarrow 0} \rho^2 b_{k,j}^{l,s} = 0$ etc. Note that

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^2} \left( \frac{\sqrt{\rho^2 - \tau^2(\tau)}}{\rho - t(\tau)} \right)^3 = 1,$$

for all $\rho > 0$ such that we have a finite potential term for all $\rho > 0$. Furthermore note that the term next to the limit in (54) can be written as $\sqrt{\rho - t(\rho + t)}$, and, hence, has an uniform upper bound which holds for all time. More precisely, as the latter term is $\sqrt{1 - (t/\rho)^2} \sqrt{1 + (t/\rho)^2}$, we surely have the upper bound $\sqrt{2}$ for all time, and as we integrate this upper bound over a small time horizon we conclude that we get a small change of the the initial data if $\rho$ becomes small.

For larger transformed time $\tau$ we have to take spatial effect into account. Note that in original time coordinates we have to cancel the $\frac{\partial \rho}{\partial t}$-term such that we get from (52) and with $b_i^0 = (\rho - t) b_i^{l,s}$ the equation

$$u_i^{0,\nu,*} - \nu \sum_j b_j^{\nu} b_j^{\rho,\nu,*} u_i^{0,\nu,*} - \nu \sum_{j,k} b_{j,k}^{\rho,\nu,*} u_{i,j,k}^{0,\nu,*} + \sum_j b_j^{\rho} b_i^{0,\nu,*} + \frac{1}{\rho - t} u_i^{0,\nu,*} = 0.$$  

We observe that we have a strong damping term which makes local regular upper bounds more than plausible. However it is not too strong in the sense, that this does not imply that the value function becomes zero at $(\rho, 0)$, because the function is supported on the cone $K_t^\rho = \{0 \leq t \leq \rho \} \times (0, \rho + t, \rho - t)^3$, and then the representation in terms of the fundamental solution $G_\rho$ of the equation

$$u_i^{0,\nu,*} - \nu \sum_j b_j^{\rho} b_j^{\rho,\nu,*} u_i^{0,\nu,*} - \nu \sum_{j,k} b_{j,k}^{\rho,\nu,*} u_{i,j,k}^{0,\nu,*} + \sum_j b_j^{\rho,\nu,*} u_i^{0,\nu,*} = 0,$$

supported on the cone $K_t^\rho$ and zero elsewhere (note that we have existence on the cone via transformation of the Gaussian) has the form

$$u_i^{0,\nu,*}(t, y) = -\int_{K_t^\rho} \left( \sum_j b_j^{\rho} b_j^{\rho,\nu,*} u_j^{0,\nu,*} \right) (s, z) G_\rho(t, y; s, z) dz ds$$

$$+ \int_{K_t^\rho} \frac{1}{\rho^2} u_i^{0,\nu,*}(s, z) G_\rho(t, y; s, z) dz ds$$

$$= \int_{K_t^\rho} \left( \sum_j \frac{1}{\rho^2} (b_j^{\rho} b_j^{\rho,\nu,*} + b_j^{\rho,\nu,*} b_j^{\rho,\nu,*}) u_j^{0,\nu,*} \right) (s, z) G_\rho(t, y; s, z) dz ds,$$
and this support on a cone leads to appropriate upper and lower bounds of the damping term (as we shall observe below). In any of the two variations of argument for local time \( t \) and all time \( \tau \) the damping term dominates potential growth terms if data become large in a time-discretized scheme. Furthermore, as the support of the value functions \( u_i^{\rho,\nu} \), \( 1 \leq i \leq 3 \) is the cone \( K_0^\nu \) we shall observe that the damping effect does not only dominate any growth which may due to other terms, but has also an upper bound which becomes small compared to nonzero initial data \( u_i^{\rho,\nu}(0,0) = c \neq 0 \) in order to obtain that \( u_i^{\rho,\nu,\ast}(\rho,0) \neq 0 \) and \( u^{\rho,\nu}(\rho,0) = u_i^{\rho,\nu,\ast}(\rho,0) \neq 0 \) in the viscosity limit \( \nu = 0 \) such that we detect a singularity. The third term in the second line of equation (52) represents a vorticity form of the Leray projection term, again with space and time dependent coefficients. We mention that in \( [1] \) we use a the transformation considered in order to obtain a global existence scheme for the incompressible Navier Stokes equation as the damping term is quite useful in order to construct global upper bounds of the solution, but we only need a local existence result here. The program of proof is then as follows:

i) for \( \rho > 0 \) small we obtain Banach contraction results for an iterative series of linear approximating equations converging to the nonlinear equation for \( u_i^{\rho,\nu,\ast} \), \( 1 \leq i \leq 3 \) on a local time interval corresponding to a solution \( u_i^{\rho,\nu} \), \( 1 \leq i \leq 3 \) on a global time interval which are both the regular parts of the the vorticity \( \omega_i^\nu \), \( 1 \leq i \leq 3 \) on a time interval \([0,\rho]\) respectively;

ii) the contraction result of item i) which is a local existence result in time \( t \)-coordinates which corresponds to a global existence result in time \( \tau \)-coordinates, leads to a global upper bound for the value functions \( u_i^{\rho,\nu} \), \( 1 \leq i \leq 3 \). This global upper bound and global solution for the damped equation (which is local in original time coordinates) also exists in the viscosity limit \( \nu \downarrow 0 \);

iii) given regular data \( f \) such that for some constants \( C, c > 0 \) we have \( |f|_{H^2} \leq C \) and \( \max_{1 \leq i \leq 3} u_i^{\rho,\nu}(0,0) = f_i(0,0) = c \neq 0 \) for some index \( i_0 \) and some small \( \rho > 0 \) the maximum increment has an upper bound \( \max_{1 \leq i \leq 3} |\delta u_i^{\rho,\nu}(\rho,0)| \leq \frac{1}{2}C \), which holds also in the viscosity limit \( \nu \downarrow 0 \). Hence the regular part \( u_i^\nu \), \( 1 \leq i \leq n \) of the vorticity \( \omega_i \) at the crucial point \((\rho,0)\) is not equal to zero. Here the increment is \( \delta u_i^{\rho,\nu}(\rho,0) = u_i^{\rho,\nu}(\rho,0) - \delta u_i^{\rho,\nu}(0,0) \). This leads to the conclusion of singular solutions.

Next we consider each item in detail. Ad i), for the function \( u_i^{\rho,\nu,q+1} \), \( 1 \leq i \leq n \) with an iteration index \( q + 1 \geq 0 \), and with \( u_i^{\rho,\nu,-1} = f_i(0,0) \) for \( q = 0 \) we consider linearized equations of the form

\[
\begin{align*}
&u_{i,\tau}^{\rho,\nu,q+1} - \nu \overline{\rho}^\nu \sum_j b_{j,i,j}^{\rho,\nu} u_{i,j,j}^{\rho,\nu,q+1} - \nu \overline{\rho}^\nu \sum_j b_{j,i,j}^{\rho,\nu} u_{i,j,j}^{\rho,\nu,q+1} \\
&+ \overline{\rho}^\nu \sum_j b_{j,i,i}^{\rho,\nu} u_{i,i,i}^{\rho,\nu,q+1} + \overline{\rho}^\nu \sum_j b_{j,i,i}^{\rho,\nu} u_{i,i,i}^{\rho,\nu,q} u_{i,i,i}^{\rho,\nu,q} + \frac{1}{\nu} \sqrt{\overline{\rho}^\nu - \nu^\tau} u_{i,i,i}^{\rho,\nu,q+1} \\
&= \overline{\rho}^\nu \sum_j \frac{1}{2} \left( b_{j,i,i}^{\rho,\nu} u_{j,i,i}^{\rho,\nu} + b_{j,i,i}^{\rho,\nu} u_{j,i,i}^{\rho,\nu} \right) u_{j,i,i}^{\rho,\nu,q}.
\end{align*}
\]

Note that we take information from the previous iteration step only for the nonlinear terms. The reason is that the damping term with iteration index \( q + 1 \)
Theorem 3.1. We can use the following local contraction result.

Assume that for some finite constant \( C > 0 \), we have \( u^{i,\nu}_{1} \), \( 1 \leq i \leq n \). An alternative argument (which we consider in more detail in the article about auto-controlled schemes) is that we have a semi-group property for the vorticity equations, and the damping term ensures that boundedness of local solutions is inherited. Anyway, it suffices to consider the iteration scheme locally on the time intervals \([l - 1, l]\) with time step size \( 1 = l - (l - 1) \). Note that the damping term has iteration index \( q + 1 \) as this term has the right sign and serves for a smaller contraction constant in general. Recall that we identified the functions \( u^{i,\nu}_{l} \) with the trivially spatially extended functions to the whole domain \([l - 1, l]\times \mathbb{R}^{3}\) (equal to zero outside the cone). Similarly, we tacitly understand the time-local functions \( u^{i,\nu}_{l} \) on the time interval \([l - 1, l]\) and their iterative approximations \( u^{i,\nu,l,q+1}_{l} \) are understood to be trivially extended to the whole domain of \([l - 1, l]\times \mathbb{R}^{3}\). At each time step having computed \( u^{i,\nu,l-1}_{l}(l - 1, \cdot) \) at the beginning of time step \( l \) we compute a series \( u^{i,\nu,l,q}_{l}, 1 \leq i \leq 3 \) on the domain \([l - 1, l]\times \mathbb{R}^{3}\) iteratively, which satisfy the Cauchy problems

\[
\begin{align*}
& u^{i,\nu,l,q+1}_{l} - \nu \sum_{j} b^{i,j}_{j,j} u^{i,\nu,l,q+1}_{l} + \nu \sum_{j} b^{i,j}_{j,j} u^{i,\nu,l,q+1}_{l} \\
& + \sum_{j} k^{i,j}_{j,j} u^{i,\nu,l,q+1}_{l} + \sum_{j} k^{i,j}_{j,j} f^{i,\nu,l,q}_{l} u^{i,\nu,l,q}_{l} + \frac{1}{\rho} \sum_{j} \frac{\rho^{i,j}_{j,j}^{\nu,l}(\tau)}{\rho_{l+1}} u^{i,\nu,l,q+1}_{l} \\
& = \sum_{j} \frac{1}{2} \left( b^{i,j}_{j,j} u^{i,\nu,l,q}_{l} + b^{i,j}_{j,j} f^{i,\nu,l,q}_{l} + b^{i,j}_{j,j} u^{i,\nu,l,q}_{l} \right) u^{i,\nu,l,q}_{l}, \\
& u^{i,\nu,l,q+1}_{l}(l - 1, \cdot) = u^{i,\nu,l-1}_{l}(l - 1, \cdot).
\end{align*}
\]

In order to get local regular existence of solutions

\[
u^{1,\nu,l}_{l} = u^{1,\nu,l-1}_{l}(l - 1, \cdot) + \sum_{m=1}^{\infty} \delta u^{1,\nu,l,m}_{l},
\]

for \( 1 \leq i \leq n \) and on the domain \([l - 1, l]\times \mathbb{R}^{3}\), where

\[
\delta u^{1,\nu,l,m}_{l} = u^{1,\nu,l,m}_{l} - u^{1,\nu,l,m-1}_{l}
\]

we can use the following local contraction result.

**Theorem 3.1.** Given \( l \geq 1 \) and \( u^{i,\nu,l-1}_{l}(l - 1, \cdot) \in H^{m} \cap C^{0}_{\infty} \) for some \( m \geq 2 \) assume that for some finite constant \( C > 0 \)

\[
\left| u^{i,\nu,l-1}_{l}(l - 1, \cdot) \right|_{H^{m}} := \sum_{|\alpha| \leq m} \left| D^{\alpha} u^{i,\nu,l-1}_{l}(l - 1, \cdot) \right|_{L^{2}} \leq C.
\]

Then there is a \( \rho > 0 \) such that

\[
\sup_{\sigma \in [l - 1, l]} \left| \delta u^{i,\nu,l,0}_{l}(\sigma, \cdot) \right|_{H^{m}} \leq \frac{1}{4}.
\]
and such that for all \( q \geq 1 \)

\[
\sup_{\sigma \in [t-1, t]} |\delta u^{o,\nu,l,q}_i(\sigma,.)|_{H^m} \leq \frac{1}{4} \sup_{\sigma \in [t-1, t]} |\delta u^{o,\nu,l,q-1}_i(\sigma,.)|_{H^m},
\]

(65)

A similar result holds for the function \( u^{o,\nu,*}_i, 1 \leq i \leq n \) on the time interval \([0, \rho]\). It is essential to have a contraction result for the \( H^2 \) function space, because this space includes the Hölder continuous functions such that standard regularity theory leads to the existence of classical local solutions.

**Proof.** We did similar estimates elsewhere, and do not repeat here all details, but sketch the reasoning and emphasize some features which are special for the equations for \( u^{o,\nu,l}_i, 1 \leq i \leq 3 \). Given time step \( \tau \geq 1 \) for each iteration step \( q \) the functional increments \( \delta u^{o,\nu,l,q}_i, 1 \leq i \leq 3 \) have zero initial data at time \( \tau = l - 1 \), and satisfy the equation

\[
\begin{align*}
\delta u^{o,\nu,l,q+1}_i &= -\nu_p \delta u^{o,\nu,l,q+1}_i + \frac{1}{\rho} \sqrt{\frac{\rho^2 - t^2(\tau)}{\rho^2 - t^2}} \delta u^{o,\nu,l,q}_i \\
&= -\nu_p \delta u^{o,\nu,l,q+1}_i + \frac{1}{\rho} \sqrt{\frac{\rho^2 - t^2(\tau)}{\rho^2 - t^2}} \delta u^{o,\nu,l,q}_i \\
&= -\nu_p \delta u^{o,\nu,l,q+1}_i + \frac{1}{\rho} \sqrt{\frac{\rho^2 - t^2(\tau)}{\rho^2 - t^2}} \delta u^{o,\nu,l,q}_i.
\end{align*}
\]

(66)

There are classical representations of the solutions in terms of the fundamental solution \( p^{o,l} \) of the equation

\[
\begin{align*}
p^{o,l}_i - \nu_p \sum_j b^{o}_j \delta p^{o,l}_j - \nu_p \sum_j b^{o}_j \delta p^{o,l}_j \\
&= -\frac{1}{\rho} \sqrt{\frac{\rho^2 - t^2(\tau)}{\rho^2 - t^2}} \delta p^{o,l}_i.
\end{align*}
\]

(67)

Note that the density in spatial variable \( y \) is supported on a cone and assumed to be trivially extended outside, and that in original spatial coordinates \( x \) it is supported on the whole domain. In these original spatial variables the density has only time dependent coefficients. Using this observation it is straightforward to observe that the density and its first order spatial derivatives satisfies classical local a priori estimates of the form

\[
|p^{o,l}(\tau, x; s, y)| \leq \frac{C}{(\tau - s)^\alpha |x - y|^{n-2\alpha}}
\]

(68)

for some \( \alpha \in (0, 1) \), and

\[
|p^{o,l}_j(\tau, x; s, y)| \leq \frac{C}{(\tau - s)^\alpha |x - y|^{n+1-2\alpha}}
\]

(69)

for some \( \alpha \in (\frac{1}{2}, 1) \) such that we have local \( L^1 \) integrability. Since all coefficients of the spatial terms are bounded functions times a factor \( \rho \), they become
small for a small time horizon $\rho > 0$ (which is the time horizon of the related problem in original time coordinates $t$), and, hence, the effect of the density becomes small. The damping term can be put on the 'left side', and the increments on the right side can be extracted by use of Young inequalities. Here integrals are splitted in local integrals in a ball around weak singularities and their complements. Note that in dimension $D = 3$ the kernel in the definition of the Biot-Savart law is locally $L^1$ and is $L^2$ outside a ball containing the weak singularity. The reader may consult related papers where this has been carried out in similar situations

Ad ii), by induction on the time step number $l$ and by the semi-group property it for the first statement in item ii) it suffices to show that for each $l \geq 1$ the local limit function

$$u^{\rho, \nu, l}_i = u^{\rho, \nu, l, 0}_i + \sum_{q=1}^{\infty} \delta u^{\rho, \nu, l, q}_i,$$  \hspace{1cm} (70)$$

-which is just the restriction of the function $u^{\rho, \nu, 1}_i$, $1 \leq i \leq n$ to the time interval $[l-1, l]$ with respect to time $\tau$-coordinates- satisfies the Cauchy problem

$$
\begin{cases}
\rho \mu \sum_{j} b^{\rho}_{j,j} u^{\rho, \nu, l}_i,
+ \rho \sum_{j} b^{\rho}_{j,t} u^{\rho, \nu, l}_i + \frac{1}{\rho \sqrt{\rho^2 - \rho^2(\tau)}} u^{\rho, \nu, l}_i,
\end{cases}
$$

$$u^{\rho, \nu, l}_i(l - 1, \cdot) = u^{\rho, \nu, 1}_i(l - 1, \cdot).$$  \hspace{1cm} (71)

Define

$$u^{\rho, \nu, l,q}_i = u^{\rho, \nu, l, 0}_i + \sum_{m=1}^{q} \delta u^{\rho, \nu, l,m}_i.$$  \hspace{1cm} (72)

Plugging in the equation in (71), we get

$$u^{\rho, \nu, l,q}_i - \rho \mu \sum_{j} b^{\rho}_{j,j} u^{\rho, \nu, l,q}_i - \rho \sum_{j} b^{\rho}_{j,t} u^{\rho, \nu, l,q}_i + \frac{1}{\rho \sqrt{\rho^2 - \rho^2(\tau)}} u^{\rho, \nu, l,q}_i,$$

$$+ \rho \sum_{j} b^{\rho,s}_{j,j} u^{\rho, \nu, l,q}_i,
+ \rho \sum_{j} b^{\rho,s}_{j,t} u^{\rho, \nu, l,q}_i + \frac{1}{\rho \sqrt{\rho^2 - \rho^2(\tau)}} u^{\rho, \nu, l,q}_i,$$

$$+ \rho \sum_{j} \frac{1}{2} \sum_{k} b^{\rho,s}_{j,k} u^{\rho, \nu, l,q}_i,$$

$$= \rho \sum_{j} b^{\rho,s}_{j,j} \delta u^{\rho, \nu, l,q}_i u^{\rho, \nu, l,q}_i$$

$$+ \rho \sum_{j} \frac{1}{2} \sum_{k} \delta u^{\rho, \nu, l,q}_i.$$  \hspace{1cm} (73)

Now, we have for $m \geq 2$ some $\rho$ such that

$$\sup_{\tau \in [l-1, l]} |\delta u^{\rho, \nu, l,q}_i(\tau, \cdot)|_{H^m \cap C^0} = 0,$$  \hspace{1cm} (74)
and this implies that for such a $\rho$ we have
\[
\sup_{\tau \in [|l - 1|]} \left| \delta u_{\rho,l}^{i,j}(\tau,.) \right|_{H_m \cap C_m} \downarrow 0. \tag{75}
\]
Hence, the limit $q \uparrow \infty$ of (25) satisfies the Cauchy problem in (23) pointwise and is, hence, a classical solution. Next it is the potential term which has the right sign in order to ensure the existence of a uniform global upper bound of the solution of (26) (it has a similar role as the control function in a controlled scheme which we considered elsewhere). Now as we have a local classical solution $u_{\rho,l}^{i,j}$, $1 \leq j \leq 3$, we may consider the fundamental solution
\[
\rho,\nu,l \quad \text{of the time local Navier Stokes type equation with bounded regular functions}
\]
where we include the Burgers term (we added the superscript $B$ to indicate this). We can do this because we know the local solution $u_{\rho,l}^{i,j}$, $1 \leq i \leq 3$, and therefore we know $u_{\rho,l}^{i,j}$. $1 \leq i \leq 3$. Note that for each time step $l \geq 1$ the fundamental solution $p_{\rho,l}^{i,j}$ may be considered to be defined on the extended domain $D_l \times D_l$ with $(\sigma, z), (r, y) \in D_l = [|l - 1|, l] \times R^3$ and $\sigma > r$. Note again that the equation is considered to be trivially (values zero) extended outs side the cone $K_0 \cap D_l$. Likewise the density has support on the latter cone and is zero outside. We emphasize that $u_{\rho,l}^{i,j}$ is known from local existence at time step $l$ of the scheme, such that we may determine a time-local fundamental solution in order get a useful representation of $u_{\rho,l}^{i,j}$, $1 \leq i \leq 3$ itself. For $\tau \in [|l - 1|]$ we have
\[
u,\nu,l \quad \text{of the density equation in (76)}
\]
\[
\left( 1 \right) (r, y) \times \right) ^{\rho,l,B}(\tau; r, y)dydr.
\]
Note the integral over $R^3$ in the latter equation which is legitimate as we consider the time horizon of the original domain, and observe that the factor
\[
\rho,\nu,l \quad \text{small, such that for a small time horizon of the original time problem we have}
\]
\[
\rho,\nu,l \quad \text{which is close to the identity. Hence, assuming inductively for some point $x$ that $|u_{\rho,l}^{i,j}(l - 1, x)| \in [C]$.}
\]
\[
\text{Hence, we get}
\]
\[
\left. \sup_{z \in R^3} |u_{\rho,l}^{i,j}(l - 1, z)| \right| \leq C \Rightarrow \left. \sup_{z \in R^3} |u_{\rho,l}^{i,j}(l, z)| \right| \leq C. \tag{79}
\]
\[
\rho,\nu,l \quad \text{of the density equation in (76) become small as $\rho > 0$ becomes small, such that for a small time horizon of the original time problem we have}
\]
\[
\rho,\nu,l \quad \text{which is close to the identity. Hence, assuming inductively for some point $x$ that $|u_{\rho,l}^{i,j}(l - 1, x)| \in [C]$.}
\]
Inductively we get a global upper bound $C > 0$ (independent of the time step number $l \geq 1$) for the value functions $u^{\rho,\nu,l}$, and for the function $\omega_l$ in original coordinates. Furthermore, there is an upper bound which is independent of the viscosity $\nu > 0$ by construction and by the damping of the potential term. Note that we get a similar upper bound for spatial derivatives $D_\alpha^{\nu,\nu,l}$ inductively for $0 \leq |\alpha| \leq m$ if the data $h_i$ have sufficient regularity and $\max_{0 \leq |\alpha| \leq m} |D_\alpha^{\nu,l}(\cdot)| \leq C$. Closing step ii) we observe that the contraction result holds also in the viscosity limit $\nu \downarrow 0$. This is due to the fact that the $L^1$-, $L^2$- and $H^1$-norms of the density used in the contraction argument are independent of the size of the viscosity constant $\nu > 0$. Let us have a closer look at this. In original spatial coordinates $x$ we have an operator with an uniformly elliptic part, such that the classical Levy expansion can be applied, and such that we have the classical a priori estimates, which are preserved under the back transformation $x \to y$. Note that the classical Levy expansion of the fundamental solution for the equation in $x$-coordinates is with coefficients which depend only on time. Especially we have estimates

$$|p^{\rho,l}(x; s, y)| \leq \frac{C}{\sqrt{\nu(t-s)}} \exp \left( -\frac{\lambda(x - y)^2}{\nu(t-s)} \right)$$

(80)

for some constants $C > 0$ and $\lambda > 0$ and which are independent of $\nu > 0$. For $z = \frac{(x-y)}{\sqrt{\nu}}$ we have $-dy = \sqrt{\nu}dz$

$$\int_{\mathbb{R}^n} |p^{\rho,l}(x; s, y)| dy \leq \int_{\mathbb{R}^n} \frac{C}{\sqrt{\nu(t-s)}} \exp \left( -\frac{\lambda z^2}{(t-s)} \right) \sqrt{\nu} \, dz$$

(81)

for some finite constant $\tilde{C}$ independent of $\nu$. Note that for local contraction result the $L^2$-norm is only used in the complement of a ball around $x$, and then a similar observation holds. For the first order spatial derivatives of the fundamental solution we have

$$\int_{\mathbb{R}^n} |p_{x}^{\rho,l}(x; s, y)| dy \leq \int_{\mathbb{R}^n} \frac{C}{\nu(t-s)^{1/2}} \exp \left( -\frac{\lambda z^2}{(t-s)} \right) \sqrt{\nu} \, dz$$

(82)

for some finite constant $\tilde{C}_2$ independent of $\nu$. Hence in any case we get local $L^1$ estimates for the fundamental solutions which are independent of the viscosity $\nu$ (details in the proof of the corollary below). Furthermore, we observe that in dimension $D = 3$ we get $L^2$-upper bounds in the complement of the ball around weak singularities which are independent of the viscosity constant $\nu$. Hence in the contraction result we can apply the Young inequalities where we get estimates which are uniform with respect to $\nu$ and hold in the viscosity limit $\nu \downarrow 0$. Since the damping term is preserved we get a global solution branch and a global upper bound (independent of time $\tau$). Hence, we have
Corollary 3.2. The contraction result in theorem 3.1 for the functions \( u_i^{\rho,\nu}\), hold in the viscosity limit \( \nu \downarrow 0 \).

It follows that the equation in (52) for the function \( u_i^{\rho,\nu} \), \( 1 \leq i \leq n \) is satisfied for all \( \nu > 0 \) with a uniform bound on the Laplacian which vanishes in the limit \( \nu \downarrow 0 \) as we have the contraction space \( H^m \cap C^m \) for \( m = 2 \). Hence the viscosity limit function \( u_i^{\rho}\), \( 1 \leq i \leq n \) is a regular factor of a local solution of the incompressible Euler equation in local original time \( t \in [0, \rho] \). An alternative argument for item ii) can be found in \([2]\).

Finally, ad iii), as we have a local solution \( u_i^{\rho,\nu,*} \), \( 1 \leq i \leq n \) corresponding to a global solution \( u_i^{\rho,\nu} \), \( 1 \leq i \leq n \) of \([3]\) with upper bound

\[
\sup_{\tau \in [0, \infty)} |u_i^{\rho,\nu}(\tau, \cdot)|_{H^m \cap C^m} \leq C
\]

for some constant \( C > 0 \) which is independent of time \( \tau \) we may consider the fundamental solution \( p^{\rho,\nu} \), \( \rho, \nu \) to the viscous limit \( \nu \downarrow 0 \) for all \( \delta u_i \) increments as we labored more on the latter function, we consider this one next. For the \( \nu \) hold in the viscosity limit (or, respectively, we may consider the fundamental solution \( p^{\rho,\nu}\) or for related increments \( \nu \leq \nu^* \) for all \( \delta u_i \).

\[
\frac{\partial \omega_i}{\partial t} - \nu \Delta \omega_i = 0.
\]

Next we have to establish upper bounds for the increments \( \delta u_i^{\rho,\nu,*} \), \( 1 \leq i \leq n \) or for related increments \( \delta u_i^{\rho,\nu} \), \( 1 \leq i \leq n \). The estimations are similar, and as we labored more on the latter function, we consider this one next. For the increments \( \delta u_i^{\rho,\nu} \) we have the classical representation

\[
\delta u_i^{\rho,\nu}(\tau, y) = -\int_{K_0^*} \mathcal{P}^\rho \sum_j b_{i,j}^\rho u_{i,j}^{\rho,\nu} p^{\rho,\nu}(\tau, y; s, z) dz ds
\]

\[
- \int_{K_0^*} \left( \mathcal{P}^\rho \sum_j b_{i,j}^\rho u_{i,j}^{\rho,\nu} \nabla p^{\rho,\nu}(\tau, y; s, z) \right) dz ds
\]

\[
- \int_{K_0^*} \left( \mathcal{P}^\rho \sum_j \frac{1}{2} \left( b_{i,j}^\rho u_{i,j}^{\rho,\nu} + b_{i,j}^\rho u_{j,i}^{\rho,\nu} \right) \nabla p^{\rho,\nu}(\tau, y; s, z) \right) dz ds
\]

\[
+ \int_{K_0^*} f_i(0, y) p^{\rho,\nu}(\tau, y; s, z) dz ds - f_i(0, y).
\]

We may estimate this integral in original time \( t \)-coordinates, but keep \( y = (\rho - \tau)x \) coordinates. We denote the fundamental solution by

\[
p^{\rho,\nu}(t, x; s, y) = p^{\rho,\nu,t}(t; x; s, y)
\]

and

\[
u_i^{\rho,\nu,t}(t, y) := u_i^{\rho,\nu}(\tau, y),
\]

and

\[
\mathcal{P}^{\rho,t} = \frac{1}{\rho^2} \sqrt{\rho^2 - t^2} = \frac{1}{\rho^2} \sqrt{\rho^2 - \rho^2(\tau)} = \mathcal{P}^{\rho}.
\]
and, similarly, we denote \( b_{i,t}^{\rho}(t, y) = b_{i,t}^{\rho}(\tau, y) \). It is clear that spatial derivatives of the latter function remain unchanged. Furthermore, observe that

\[
b_{i,t}^{\rho} = -\frac{2}{\pi} \arctan(y_i),
\]

such that the transformed coefficient \( b_{i,t}^{\rho}(t, y) := b_{i,t}^{\rho}(\tau, y) \) remains identical.

We get for any \( t \in [0, \rho] \) and \( y \)

\[
|\delta u_{i,\nu}^{\rho,\nu,t}(t, y)| = |\delta u_{i,\nu}^{\rho,\nu}(\tau, y)| \\
|\int_{K_0^{\rho,t}} \pi^{\rho,t} \sum_j b_{j,i}^{\rho,t} y_{i,j}^{\rho,\nu,t} p_{\rho,\nu,t}(t, y; s_0, z) dz \sqrt{\rho^2 - s_0^2} ds_0 | \\
+ |\int_{K_0^{\rho,t}} \left( \pi^{\rho,t} \sum_j \frac{1}{2} \sum_k \left( b_{j,i,k}^{\rho,t} u_{i,j,k}^{\rho,\nu,t} + b_{j,i}^{\rho,t} f_{i,\nu}^{\rho,\nu,t} \right) u_{i,k}^{\rho,\nu,t} \right) \times \\
p_{\rho,\nu,t}(t, y; s_0, z) \sqrt{\rho^2 - s_0^2} ds_0 | \\
+ |\int_{K_0^{\rho,t}} f_i(0, y) p_{\rho,\nu,t}(t, y; s_0, z) ds_0 - f_i(0, y)|,
\]

where we use the measure change

\[
d\tau = d\tau dt = \frac{\rho^2}{\sqrt{\rho^2 - t^2}} dt.
\]

Here the cone \( K_0^{\rho,t} \) is the cone in original coordinates where the time \( t \) is in the interval \([0, \rho]\). Hence, all terms go obviously to zero as \( \rho \downarrow 0 \) except for the damping term for which seams less obvious. Let us have a closer look. We have

\[
\left|\int_{K_0^{\rho,t}} \left( \frac{1}{\rho^2 s_0} \int_{\rho^2 - s_0^2}^{\rho^2} u_{i,k}^{\rho,\nu,t} \right) p_{\rho,\nu,t}(t, y; s_0, z) \sqrt{\rho^2 - s_0^2} ds_0 \right|
\]

\[
= \left|\int_{K_0^{\rho,t}} \frac{1}{\rho^2 s_0} u_{i,k}^{\rho,\nu,t} p_{\rho,\nu,t}(t, y; s_0, z) dy ds_0 \right|
\]

\[
\leq \left|\int_{K_0^{\rho,t}} \frac{1}{\rho^2 s_0} C dy ds_0 \right| \leq \frac{C}{\rho}s_0,
\]
time first
\[
\left| \int_{K_{\rho}^{*}} \frac{1}{\rho} \delta \omega \, dyds \right| = \lim_{\rho \uparrow \rho_{0}} \left| \int_{|y| \leq \rho} \frac{1}{\rho} \delta \omega \, dyds \right|
\]
\[
\leq \lim_{\rho \uparrow \rho_{0}} \left| \int_{|y| \leq \rho} (\ln(\rho) - \ln(\rho_{0})) dyds \right| \downarrow 0 \text{ as } \rho \downarrow 0.
\]

Hence if \( u_{i}^{\rho,\nu}(0,0) = c \neq 0 \) for some \( 1 \leq i \leq 3 \) and regular data \( u_{i}^{\rho,\nu} \in H^{m} \cap C^{m} \) for \( m \geq 2 \), then there is a small time horizon \( \rho > 0 \) such that
\[
\left| \delta u_{i}^{\rho,\nu}(\rho,0) \right| \leq \frac{c}{2},
\]
and it follows that the vorticity \( \omega_{i} \) has a singular point at \((\rho,0)\).

The local analysis can be transferred to the compressible Navier Stokes equation in (31) considering inviscid limits \( \mu, \mu' \downarrow 0 \). The extension is quite straightforward on a domain of real degeneracies (in the sense explained above) in the region of incompressibility, i.e., in the region \( D_{\epsilon} \cap \{(t,x)| \text{div}(t,x) = 0\} \) (if this subset of real degeneracies is not empty). Otherwise the scalar densities in the local analysis above have to be replaced by fundamental solutions of linear parabolic systems which are coupled iteratively with transport equations of first order. The extension of the local analysis is then straightforward if it is ensured that the mass density is strictly positive almost surely. If the data are all strictly positive this seems to be true, but this will be considered elsewhere.

Next, arguing for a) of the preceding section we first remark that an attitude towards singularities which may be dubbed horror singulari, may often be a hindrance in studying certain classes of solutions which may be otherwise an important guide for further analysis- such as the classical singular solutions of the Einstein field equations may be an important guide for further investigation of gravity. A similar point of view with respect to singular solutions of equations in fluid mechanics seems appropriate. Sometimes it is thought that the solution is only local if there is a singularity. However, even the function \( t \rightarrow \frac{2n}{t - t_{0}} \) is a solution of the ODE \( \dot{x} = x^{2} \), \( x(0) = x_{0} = 1 \) on the domain \( (0, \infty) \setminus \{1\} \), although we loose uniqueness after time \( t = 1 \). Singularities may be artificial but not more then other aspects of our description of nature to our best knowledge. On the other hand the existence of singularities is a rather weak insight compared to what may be expected of a quantitative theory of turbulence. Next we argue that the existence of a vorticity singularity as constructed is a weak concept of turbulence in the sense that local solutions in the neighborhood of such a singularity exemplify the mentioned list of notions of turbulence in [3]. Assume that we have a local vorticity solution \( \omega_{i}(t,y) = \frac{f_{i}(t,y)}{\rho} \) on a domain \( [t_{i}, t_{c}] \times \mathbb{R}^{3} \) with a regular function \( f_{i} \) which becomes singular at \( (1, y_{0}) \) as \( f_{i}(t_{0}, y_{0}) \neq 0 \) for \( 1 \in (t_{i}, t_{c}) \) for some \( 1 \leq i \leq 3 \). Then \( \omega_{i}(\cdot, y_{0}) \) changes sign at \( t = 1 \) which implies extreme changes of the velocity magnitude in the neighborhood of the singularity compatible with notion a) of the description above. Second, the velocity integral \( v_{i} = \int K_{3i} \omega_{i} \) may imply that although we have extreme changes
of velocity across \( t = 1 \) we may have \( v_1(1, y_0) = 0 \), i.e., a stationary point at \( (1, y_0) \) and this is compatible with \( \beta \) of that list. Third, as there may be a stationary point of velocity zero at the singular point of vorticity \( (1, y_0) \) the amplitudes of the vorticity may be large compared to the velocity itself, and this is compatible with \( \gamma \) of Landau's list. Fourth, even as we freeze time time \( t_1 \) close to \( t = 1 \) small changes of \( f_i(t_1, y) \) can lead to large changes with proportionality factor \( \frac{1}{t_1-t} \) of the vorticity, which is compatible with \( \delta \) of the list above. Finally the extreme changes of velocity near time \( t = 1 \) is compatible with complicated trajectories and strong mixture as noted in \( \epsilon \) above. The latter point exemplifies the weakness of the singularity concept of turbulence, as the notion of strong mixture and related notions can be studied quantitatively by bifurcation theory. Finally we mention that the technique introduced in this article can be used to show that Navier Stokes equation models with \( L^2 \)-force term can have singular solutions (cf. [2]).

Remark 3.4. There has been a lot of research on singularity criteria of solutions for the Euler equation. The preceding text is an independent research with result which has not be covered elsewhere. As a non-specialist of the Euler equation I do not have the competence to choose and comment the huge amount of literature which has a content which is beyond that of the book of Majda and Bertozzi cited in [4] below.

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