C*-categories, Groupoid Actions, Equivariant KK-theory, and the Baum-Connes Conjecture

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Abstract

In this article we give a characterisation of the Baum-Connes assembly map with coefficients. The technical tools needed are the $K$-theory of C*-categories, and equivariant $KK$-theory in the world of groupoids.

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1 Introduction

Let $G$ be a discrete group. Any given $G$-homotopy-invariant functor from the category of $G$-CW-complexes to the category of spectra has a universal approximation by a generalised $G$-equivariant homology theory. To be specific, we have the following result, proved by Davis and Lück in [5].

Theorem 1.1 Let $\mathcal{E}$ be a $G$-homotopy-invariant functor from the category of proper $G$-CW-complexes to the category of spectra. Then there is a $G$-homotopy-invariant excisive functor $\mathcal{E}^\%$ and a natural transformation $\alpha: \mathcal{E}^\% \to \mathcal{E}$ such that the map

$$\alpha: \mathcal{E}^\%(G/H) \to \mathcal{E}(G/H)$$

is a stable equivalence for every finite subgroup, $H$, of the group $G$.

Further, the pair $(\mathcal{E}^\%, \alpha)$ is unique up to weak equivalence.
Here, a functor $E^%$ is called *excisive* when the collection of functors $X \mapsto \pi_* E^%(X)$ form a $G$-equivariant generalised homology theory. The natural transformation $\alpha: E^% \to E$ is called the *assembly map* associated to the functor $E$.

The above theorem, or rather a slight generalisation, can be used to describe several standard maps that appear in isomorphism conjectures. For example, the map appearing in the Farrell-Jones isomorphism conjecture (see [3]) readily fits into the framework described by Davis and Lück.

Now, let $A$ be a $G$-$C^*$-algebra. For a proper $G$-space $X$, one can define $G$-equivariant $K$-homology groups, $K^G_n(X; A)$, with coefficients in the $G$-$C^*$-algebra $A$. There is a canonical map

$$\beta: K^G_n(X; A) \to K_n(A \rtimes_r G)$$

Here $A \rtimes_r G$ is the reduced crossed product of the $C^*$-algebra $A$ with the group $G$. The map $\beta$ is termed the *Baum-Connes assembly map*.

A proper $G$-CW-complex $EG$ is called a *classifying space for proper actions of $G$* if for a given subgroup $H \leq G$ the fixed point set $EG^H$ is contractible when $H$ is finite, and empty when $H$ is infinite. The *Baum-Connes conjecture* with coefficients in the $G$-$C^*$-algebra $A$ is the assertion that the Baum-Connes assembly map

$$\beta: K^G_n(X; A) \to K_n(A \rtimes_r G)$$

is an isomorphism.

The reader is urged to consult [4] for a full description of the Baum-Connes conjecture and details of some of its geometric and algebraic implications.

In this paper we give a description of the Baum-Connes assembly map at the level of spectra that fits into the framework described by Davis and Lück. In order to use theorem [11], we need to consider groupoids. Actions on spaces naturally lead to groupoids because of the following standard construction.

**Definition 1.2** Let $X$ be a $G$-space. Then we write $\overline{X}$ to denote the category in which the collection of objects is that set $X$, and the morphism sets are defined by writing

$$\text{Hom}(x, y)_{\overline{X}} = \{g \in G \mid xg = y\}$$

Every morphism in the category $\overline{X}$ is invertible, so the category $\overline{X}$ is a groupoid. In this paper we ignore the topology of the space $X$ when considering the groupoid $\overline{X}$.

If $G$ is a groupoid, we define a $G$-$C^*$-algebra to be a functor from the groupoid $G$ to the category of $C^*$-algebras. If $A$ is a $G$-$C^*$-algebra, there is a natural notion of the *reduced crossed product*, $A \rtimes_r G$. When $G$ is a groupoid rather than a group, this reduced crossed product is not a $C^*$-algebra, but rather a more general object called a $C^*$-*category*, as defined in [8].

If $f: G \to H$ is a faithful functor between groupoids, and $A$ is a $H$-$C^*$-algebra, then $A$ can also be considered to be a $G$-$C^*$-algebra, and we have a functorially induced morphism of $C^*$-categories $f_* A \rtimes_r G \to A \rtimes_r H$.

One can define the $K$-theory of $C^*$-categories; see [18] for details. In particular, if $A$ is a $C^*$-category, there is an associated spectrum $K(A)$. The assignment $A \mapsto K(A)$ is functorial.

We are now ready to state the main theorem of this article.
Theorem 1.3 Let $E^\%$ be a $G$-homotopy-invariant excisive functor from the category of proper $G$-CW-complexes to the category of spectra. Suppose we have a natural transformation $\alpha: E^\%(X) \to K(A \rtimes_r X)$ such that the map

$$\alpha: E^\%(G/H) \to K(A \rtimes_r G/H)$$

is a stable equivalence for every finite subgroup, $H$, of the group $G$.

Let $i: X \to G$ be the obvious inclusion functor. Then the composite $i_\ast \alpha: E^\%(X) \to K(A \rtimes_r G)$ is the Baum-Connes assembly map. □

In order to prove the above theorem, we need to develop equivariant $KK$-theory spectra of $C^\ast$-algebras in the world of groupoids. This $KK$-theory must generalise equivariant $KK$-theory for groups, and be related to crossed product $C^\ast$-categories. The bulk of this paper is devoted to the development of such a theory.

We should perhaps comment that Le Gall defines equivariant $KK$-theory for groupoids in [16]. However, Le Gall’s approach is different to ours, and it is not clear to the author how Le Gall’s theory relates to crossed product $C^\ast$-categories. It is a potentially interesting project to compare our theory with Le Gall’s, but not a project we explore in this paper.

2 Preliminaries

Let $\mathbb{F}$ denote either the field of real numbers or the field of complex numbers. Recall that a unital Banach category over the field $\mathbb{F}$ is a category, $\mathcal{A}$, in which every morphism set $\text{Hom}(A, B)_{\mathcal{A}}$ is a Banach space over the field $\mathbb{F}$, composition of morphisms

$$\text{Hom}(B, C)_{\mathcal{A}} \times \text{Hom}(A, B)_{\mathcal{A}} \to \text{Hom}(A, C)_{\mathcal{A}}$$

is bilinear, and the inequality

$$\|xy\| \leq \|x\|\|y\|$$

is satisfied for the norms of composable morphisms $x$ and $y$.

An involution on a Banach category $\mathcal{A}$ is a collection of maps

$$\text{Hom}(A, B)_{\mathcal{A}} \to \text{Hom}(B, A)_{\mathcal{A}}$$

written $x \mapsto x^\ast$ such that:

- $(\alpha x + \beta y)^\ast = \overline{\alpha}x^\ast + \overline{\beta}y^\ast$ for all scalars $\alpha, \beta \in \mathbb{F}$ and morphisms $x, y \in \text{Hom}(A, B)_{\mathcal{A}}$.
- $(xy)^\ast = y^\ast x^\ast$ for all composable morphisms $x$ and $y$.
- $(x^\ast)^\ast = x$ for every morphism $x$.

If $\mathcal{A}$ is a Banach category with involution, an invertible morphism $u$ is called unitary if $u^{-1} = u^\ast$.

The following definition comes from [7] and [19].
Definition 2.1 A unital Banach category with involution is called a unital $C^*$-category if for every morphism $x \in \text{Hom}(A, B)_A$, the product $x^*x$ is a positive element of the Banach algebra $\text{Hom}(A, A)_A$, and the $C^*$-identity

$$\|x^*x\| = \|x\|^2$$

holds.

A non-unital $C^*$-category is a collection of objects and morphisms similar to a unital $C^*$-category except that there need not exist identity morphisms $1 \in \text{Hom}(A, A)_A$.

We should perhaps comment that a non-unital $C^*$-category is not really a category, but rather an object with less structure which might be termed a non-unital category.

If $A$ is a $C^*$-category, each endomorphism set $\text{Hom}(A, A)_A$ is a $C^*$-algebra. Conversely, a $C^*$-algebra can be considered to be a $C^*$-category with one object.

A $C^*$-functor between unital $C^*$-categories is a functor $F: A \to B$ such that each map $F: \text{Hom}(A, B)_A \to \text{Hom}(F(A), F(B))_B$ is linear, and $F(x^*) = F(x)^*$ for each morphism $x$ in the category $A$. We similarly define $C^*$-functors between non-unital $C^*$-categories. It is proved in [19] that any $C^*$-functor is norm-decreasing, and therefore continuous, and if faithful is an isometry. Further, any $C^*$-functor has a closed image.

The category of small $C^*$-categories is formed by taking the (non-unital) graded $C^*$-functors as morphisms.

Example 2.2 The category, $\mathcal{L}(\mathbb{F})$, of all Hilbert spaces and bounded linear operators is a $C^*$-category. The involution is defined by taking adjoints.

A $C^*$-functor $\rho: A \to \mathcal{L}(\mathbb{F})$ is termed a representation of the $C^*$-category $A$. It can be shown (see [7, 19]) that any small $C^*$-category has a faithful, and therefore isometric representation.

For the applications we have in mind in this article it is necessary to look at $C^*$-categories equipped with gradings.

Definition 2.3 A $C^*$-category $A$ is said to be graded if we can write each morphism set $\text{Hom}(A, B)_A$ as a direct sum

$$\text{Hom}(A, B)_A = \text{Hom}(A, B)_0 \oplus \text{Hom}(A, B)_1$$

of morphisms of degree 0 and degree 1 such that for composable morphisms $x$ and $y$ we have the formula

$$\text{deg}(xy) = \text{deg}(x) + \text{deg}(y)$$

Here addition takes place modulo 2.

A $C^*$-functor $F: A \to B$ between graded $C^*$-categories is termed a graded $C^*$-functor if

$$\text{deg}(Fx) = \text{deg}(x)$$

for every morphism $x$ in the category $A$.

A category is called small if the collection of objects is a set. For set-theoretic reasons, one cannot form the category of all $C^*$-categories, whereas the category of small $C^*$-categories does make sense.
As a special case of the above definition, we can speak of *graded* $C^*$-algebras and *morphisms* between graded $C^*$-algebras. The category of small graded $C^*$-categories is formed by taking the graded $C^*$-functors as morphisms.

We can consider an ungraded $C^*$-category to be equipped with the *trivial grading* defined by saying that every morphism is of degree 0. Our attitude is thus to view ungraded $C^*$-categories as special cases of graded $C^*$-categories.

There is a sensible notion of the spatial tensor product, $A \hat{\otimes} B$, of graded $C^*$-categories $A$ and $B$. The objects are pairs, written $A \otimes B$, for objects $A \in Ob(A)$ and $B \in Ob(B)$. The morphism set $\text{Hom}(A \otimes B, A' \otimes B')_A \otimes B$ is a completion of the algebraic graded tensor product $\text{Hom}(A, A')_A \hat{\otimes} \text{Hom}(B, B')_B$. See section 7 of [19] and definition 2.7 of [18] for details.

The main construction in [18] is a functor, $K$, from the category of small graded $C^*$-categories to the category of symmetric $\Omega$-spectra. The spectrum $K(A)$ is called the $K$-theory spectrum associated to the graded $C^*$-category $A$.

We define the $K$-theory group $K_n(A)$ to be the stable homotopy group $\pi_n K(A)$. If $A$ is a graded $C^*$-algebra, the stable homotopy group, we recover from this definition the $K$-theory groups $K_n(A)$ defined in [23, 24]. In particular, when the $C^*$-algebra $A$ is trivially graded, we can obtain the usual definition of $C^*$-algebra $K$-theory in this way.

The $K$-theory of $C^*$-categories has many properties in common with the $K$-theory of $C^*$-algebras. A number of such elementary properties are proved in the article [18] including a version of the Bott periodicity theorem involving Clifford algebras.

**Definition 2.4** Let $p$ and $q$ be natural numbers. Then we define the $(p, q)$-*Clifford algebra*, $\mathbb{F}_{p,q}$, to be the algebra over the field $\mathbb{F}$ generated by elements

\[ \{e_1, \ldots, e_p, f_1, \ldots, f_q\} \]

that pairwise anti-commute and satisfy the formulae

\[ e_i^2 = 1 \quad f_j^2 = -1 \]

The Clifford algebra $\mathbb{F}_{p,q}$ is a graded $C^*$-algebra; the generators themselves are defined to be of degree 1.

**Theorem 2.5** Let $A$ be a small graded $C^*$-category. Then there is a natural stable equivalence of spectra

\[ \Omega^p K(A) \simeq \Omega^p K(A \hat{\otimes} \mathbb{F}_{p,q}) \]

\[ \square \]

Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a set of small graded $C^*$-categories. Then we can form the product, $\prod_{\lambda \in \Lambda} A_\lambda$. The objects are collections of objects $\{A_\lambda \in \lambda \in Ob(A_\lambda)\}$. The morphism set $\text{Hom}(\{A_\lambda\}, \{B_\lambda\})$ consists of all sets of morphisms $\{x_\lambda \in \text{Hom}(A_\lambda, B_\lambda) \mid \lambda \in \Lambda\}$ such that the supremum $\sup\{|x_\lambda| \mid \lambda \in \Lambda\}$ is finite.

The following result is obvious from the construction of the $K$-theory spectrum in [18].

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2For an alternative construction of the $K$-theory spectrum of a $C^*$-category, see [10].

3See also [1, 3, 2, 13, 12] for further details on this approach to the Bott periodicity theorem, at least for the $K$-theory of $C^*$-algebras.
Proposition 2.6 Let \( \{ A_\lambda \mid \lambda \in \Lambda \} \) be a set of small \( C^* \)-categories. Define \( A \) to be the \( C^* \)-category in which the set of objects is the union \( \bigcup_{\lambda \in \Lambda} \text{Ob}(A_\lambda) \) and the morphism sets are:

\[
\text{Hom}(A, B)_A = \begin{cases} 
\text{Hom}(A, B)_{A_\lambda} & A, B \in \text{Ob}(A_\lambda) \\
\{0\} & A \in \text{Ob}(A_\lambda), B \in \text{Ob}(A_\mu), \ \lambda \neq \mu 
\end{cases}
\]

Then the \( K \)-theory spectra \( K(A) \) and \( K(\prod_{\lambda \in \Lambda} A_\lambda) \) are naturally stably equivalent. \( \square \)

The other main property of \( K \)-theory that we need in this article is a form of stability involving the objects of a \( C^* \)-category.

Definition 2.7 Let \( F, G : A \to B \) be graded \( C^* \)-functors between unital graded \( C^* \)-categories. Then a natural isomorphism between \( F \) and \( G \) consists of a degree 0 unitary morphism \( U_A \in \text{Hom}(F(A), G(A))_B \) for each object \( A \in \text{Ob}(A) \) such that for every morphism \( x \in \text{Hom}(A, B)_A \) the composites \( U_B F(x) \) and \( F(x) U_A \) are equal.

A graded \( C^* \)-functor \( F : A \to B \) between unital \( C^* \)-categories is said to be an equivalence of graded \( C^* \)-categories if there is a graded \( C^* \)-functor \( G : B \to A \) such that the composites \( FG \) and \( GF \) are naturally isomorphic to the identities \( 1_B \) and \( 1_A \) respectively.

Proposition 2.8 Let \( F : A \to B \) be an equivalence of small graded \( C^* \)-categories. Then the induced map \( F_* : K(A) \to K(B) \) is a stable equivalence of \( K \)-theory spectra. \( \square \)

In particular, a small graded unital \( C^* \)-category that is equivalent to a \( C^* \)-algebra has the same \( K \)-theory.

We end our survey of results on the \( K \)-theory of \( C^* \)-categories by indicating one way to define elements of the initial space, \( K(A)_0 \), of the \( K \)-theory spectrum \( K(A) \).

Recall that a right \( A \)-module over a \( C^* \)-category \( A \) is a linear contravariant functor \( \mathcal{E} \) from the category \( A \) to the category of vector spaces. It is similarly possible to define left \( A \)-modules.

We use the notation \( \eta x = \mathcal{E}(x)(\eta) \) to denote the action of a morphism \( x \in \text{Hom}(A, B)_A \) on a vector \( \eta \in \mathcal{E}(A) \).

Definition 2.9 The right \( A \)-module \( \mathcal{E} \) is called a Hilbert \( A \)-module if it is equipped with a collection of bilinear maps \( \langle - , - \rangle : \mathcal{E}(B) \times \mathcal{E}(A) \to \text{Hom}(A, B)_A \) such that:

- For all vectors \( \eta \in \mathcal{E}(B) \), \( \xi, \zeta \in \mathcal{E}(C) \), and morphisms \( x, y \in \text{Hom}(A, C)_A \) we have the formula
  \[
  \langle \eta, \xi x + \zeta y \rangle = \langle \eta, \xi \rangle x + \langle \eta, \zeta \rangle y
  \]

- \( \langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle \)
For each vector $\eta \in E(A)$, the product $\langle \eta, \eta \rangle$ is a positive element of the $C^*$-algebra $\text{Hom}(A, A)$, and is zero only when the vector $\eta$ is zero.

Each vector space $E(A)$ is complete with respect to the norm:

$$\|x\| = \|\langle x, x \rangle \|^{\frac{1}{2}}$$

The collection of maps $\langle - , - \rangle : E(B) \otimes E(A) \to \text{Hom}(A, B)_A$ is called an inner product. If $E$ is a Hilbert module over a $C^*$-category $A$ then each vector space $E(A)$ is a Hilbert module over the $C^*$-algebra $\text{Hom}(A, A)$.

Consider an object $A \in \text{Ob}(A)$. Then we have an associated Hilbert $A$-module $\text{Hom}(-, A)_A$. The space associated to the object $C \in \text{Ob}(A)$ is the morphism set $\text{Hom}(C, A)_A$. The action of the category $A$ is defined by composition of morphisms, and the inner product is defined by the formula

$$\langle x, y \rangle = x^* y$$

There is an obvious notion of the direct sum, $E \oplus F$, of Hilbert $A$-modules $E$ and $F$. We can also define the direct sum of countably many Hilbert $A$-modules; see definition 3.4 of [20].

We refer to two Hilbert $A$-modules, $E$ as isomorphic if there is a natural isomorphism of functors $T : E \to F$ such that

$$\langle \eta, \xi \rangle = \langle T\eta, T\xi \rangle$$

for all vectors $\eta \in E(B)$ and $\xi \in E(A)$.

**Definition 2.10** A Hilbert $A$-module $E$ is called finitely generated and projective if there is a Hilbert $A$-module $E'$ such that the direct sum $E \oplus E'$ is isomorphic to the direct sum of finitely many Hilbert $A$-modules of the form $\text{Hom}(-, A)_A$.

The following result is proved in [20].

**Proposition 2.11** Let $A$ be a trivially graded unital $C^*$-category. Then a finitely generated projective Hilbert $A$-module $E$ defines a canonical element of the initial space of the $K$-theory spectrum: $[E] \in K(A)_0$ \[ \square \]

### 3 Groupoid Actions

Recall that a groupoid is a category in which every morphism is invertible. A group can be viewed as a groupoid with one object. Taking this point of view, if $G$ is a discrete group, a $G$-$C^*$-algebra is a functor from the group $G$, viewed as a category, to the category of $C^*$-algebras. This idea prompts the following definition.

**Definition 3.1** Let $\mathcal{G}$ be a discrete groupoid. Then a $\mathcal{G}$-$C^*$-algebra is a functor from the groupoid $\mathcal{G}$ to the category of $C^*$-algebras.
A $\mathcal{G}$-$C^*$-algebra as defined in [16] is also a $\mathcal{G}$-$C^*$-algebra in the sense of the above definition.

If $A$ is a $\mathcal{G}$-$C^*$-algebra, let us write $A_a$ to denote the $C^*$-algebra associated to an object $a \in Ob(\mathcal{G})$. Then for each morphism $g \in Hom(a, b)_\mathcal{G}$ we have a morphism of $C^*$-algebras $g: A_a \rightarrow A_b$.

A $\mathcal{G}$-$C^*$-algebra $A$ is termed unital if every $C^*$-algebra $A_a$ is unital, and the induced morphisms $g: A_a \rightarrow A_b$ from the groupoid $\mathcal{G}$ all preserve the unit. A $\mathcal{G}$-$C^*$-algebra $A$ is termed graded if every $C^*$-algebra $A_a$ is graded, and the induced morphisms $g: A_a \rightarrow A_b$ from the groupoid $\mathcal{G}$ all preserve the grading.

**Definition 3.2** A $\mathcal{G}$-equivariant map between $\mathcal{G}$-$C^*$-algebras $A$ and $B$ is a natural transformation from the functor $A$ to the functor $B$.

More generally, let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a functor between groupoids, let $A$ be a $\mathcal{G}$-$C^*$-algebra, and let $B$ be an $\mathcal{H}$-$C^*$-algebra. Then an equivariant map $F: A \rightarrow B$ that covers the functor $f$ is a collection of morphisms of $C^*$-algebras $F_a: A_a \rightarrow B_{f(a)}$ such that

$$F_b(gx) = f(g)F_a(x)$$

for every element $x \in A_a$ and morphism $g \in Hom(a, b)_\mathcal{G}$.

If $A$ is a graded $\mathcal{G}$-$C^*$-algebra, and $B$ is a graded $\mathcal{H}$-$C^*$-algebra, we insist that an equivariant map $F: A \rightarrow B$ respects the gradings that are present.

**Example 3.3** Let $f: \mathcal{G} \rightarrow \mathcal{H}$ be a functor between groupoids, and let $A$ be an $\mathcal{H}$-$C^*$-algebra. Then the $\mathcal{H}$-$C^*$-algebra $A$ can also be considered a $\mathcal{G}$-$C^*$-algebra; we associate the $C^*$-algebra $A_{f(a)}$ to the object $a \in Ob(\mathcal{G})$, and the morphism of $C^*$-algebras $f(g): A_{f(a)} \rightarrow A_{f(b)}$ to the morphism $g \in Hom(a, b)_\mathcal{G}$.

The collection of identity maps $1_a: A_{f(a)} \rightarrow A_{f(a)}$ is an equivariant map that covers the functor $f$.

The above example will be important to us later on.

**Definition 3.4** Let $A$ be a $\mathcal{G}$-$C^*$-algebra. Then the convolution category, $\mathcal{G}A$, is the category with the same objects as the groupoid $\mathcal{G}$ in which the morphism set $Hom(a, b)_{\mathcal{G}A}$ consists of all formal sums:

$$x_1g_1 + \cdots + x_ng_n$$

where $x_i \in A_b$ and $g_i \in Hom(a, b)_\mathcal{G}$.

Composition of morphisms in the category $\mathcal{G}A$ is defined by the formula

$$\left( \sum_i x_ig_i \right) \left( \sum_j y_jh_j \right) = \sum_{i,j} x_ig_i(y_j)g_ih_j$$

Further, we have an involution

$$\left( \sum_i x_ig_i \right)^* = \sum_i g_i^{-1}(x_i^*)g_i^{-1}$$

Note that the convolution category $\mathcal{G}A$ is non-unital unless the $\mathcal{G}$-$C^*$-algebra $A$ is unital.
Example 3.5 For a discrete groupoid $\mathcal{G}$, the trivial $\mathcal{G}$-$C^*$-algebra is defined by associating the scalar field $\mathbb{F}$ to each object $a \in \text{Ob}(\mathcal{G})$, and the identity $1: \mathbb{F} \to \mathbb{F}$ to each morphism $g \in \text{Hom}(a, b)_{\mathcal{G}}$.

The morphism set $\text{Hom}(a, b)_{\mathcal{G}}$ in the convolution $C^*$-category $\mathbb{F} \mathcal{G}$ consists of formal sums

$$\lambda_1 g_1 + \cdots + \lambda_n g_n$$

where $\lambda_i \in \mathbb{F}$ and $g_i \in \text{Hom}(a, b)_{\mathcal{G}}$.

Composition and involution are defined by the formulae

$$\left( \sum_i \lambda_i g_i \right) \left( \sum_j \mu_j h_j \right) = \sum_{i,j} \lambda_i \mu_j (g_i h_j)$$

and

$$\left( \sum_i \lambda_i g_i \right)^* = \left( \sum_i \lambda_i g_i^{-1} \right)$$

respectively.

Recall that we define $\mathcal{L}(\mathbb{F})$ to be the $C^*$-category of all Hilbert spaces and bounded linear operators over the field $\mathbb{F}$.

Definition 3.6 A unitary representation of a groupoid $\mathcal{G}$ is a functor $\rho: \mathcal{G} \to \mathcal{L}(\mathbb{F})$ such that $\rho(g^{-1}) = \rho(g)^*$ for every morphism $g \in \text{Hom}(a, b)_{\mathcal{G}}$.

We write $H_a$ to denote the Hilbert space associated to an object $a \in \text{Ob}(A)$. The $C^*$-algebra of bounded linear operators $T: H_a \to H_a$ is denoted $\mathcal{L}(H_a)$.

Definition 3.7 A covariant representation of a $\mathcal{G}$-$C^*$-algebra $A$ is a pair $(\rho, \pi)$ consisting of a unitary representation $\rho: \mathcal{G} \to \mathcal{L}(\mathbb{F})$ together with representations $\pi: A_a \to \mathcal{L}(H_a)$ such that

$$\rho(g) \pi(x) = \pi(g x) \rho(g)$$

for every element $x \in A_a$ and morphism $g \in \text{Hom}(a, b)_{\mathcal{G}}$.

Example 3.8 Let $A$ be a $\mathcal{G}$-$C^*$-algebra. Fix an object $a \in \text{Ob}(\mathcal{G})$ and let $a: A_a \to \mathcal{L}(H)$ be a representation of the $C^*$-algebra $A_a$ on a Hilbert space $H$. For each object $b \in \text{Ob}(\mathcal{G})$, let $l^2(a, b)$ be the Hilbert space consisting of sequences $(\eta_g)_{g \in \text{Hom}(a, b)_{\mathcal{G}}}$ in the Hilbert space $H$ such that the series

$$\sum_{g \in \text{Hom}(a, b)_{\mathcal{G}}} \| \eta_g \|^2$$

converges.

A groupoid element $h \in \text{Hom}(b, c)_{\mathcal{G}}$ defines a unitary operator $\rho(h): l^2(a, b) \to l^2(a, c)$ by the formula

$$\rho(h)((\eta_g)_{g \in \text{Hom}(a, b)_{\mathcal{G}}}) = (\eta_{h^{-1} k})_{k \in \text{Hom}(b, c)_{\mathcal{G}}}$$

We thus have a unitary representation, $\rho$, of the groupoid $\mathcal{G}$ defined by mapping the object $b \in \text{Ob}(\mathcal{G})$ to the Hilbert space $l^2(a, b)$, and the morphism $h \in \text{Hom}(b, c)_{\mathcal{G}}$ to the above operator $\rho(h): l^2(a, b) \to l^2(b, c)$.

There are corresponding representations of the $C^*$-algebras $A_b$ defined by writing

$$\pi(x)((\eta_g)_{g \in \text{Hom}(a, b)_{\mathcal{G}}}) = (\alpha(g^{-1}(x)) \eta_g)_{g \in \text{Hom}(a, b)_{\mathcal{G}}}$$

for every morphism $g \in \text{Hom}(a, b)_{\mathcal{G}}$. \hspace{1cm} (9)
It is easy to verify that the formula
\[ \rho(g)\pi(x) = \pi_b(g(x))\rho(g) \]
holds. Therefore the pair \((\rho, \pi)\) is a covariant representation of the \(G\)-\(C^*\)-algebra \(A\).

A covariant representation of the type constructed in the above example is called \textit{regular}.

Associated to a covariant representation \((\rho, \pi)\) we have a linear functor \((\rho, \pi)\star: \mathcal{G} \to \mathcal{L}(\mathbb{F})\) defined by mapping the object \(a\) to the Hilbert space \(H_a\) and the morphism
\[ x_1g_1 + \cdots + x_ng_n \in \text{Hom}(a, b)_{\mathcal{G}} \]
to the bounded linear map
\[ \pi(x_1)\rho(g_1) + \cdots + \pi(x_n)\rho(g_n): H_a \to H_b \]
for any morphism \(f \in \text{Hom}(a, b)_{\mathcal{G}}\), the formula \((\rho, \pi)_\star(f^*) = (\rho, \pi)_\star(f)^*\) holds. We express this formula by saying that the functor \((\rho, \pi)_\star\) respects the involution.

\textbf{Proposition 3.9} Let \(A\) be a unital \(G\)-\(C^*\)-algebra. Then every linear functor \(\alpha: \mathcal{G} \to \mathcal{L}(\mathbb{F})\) that respects the involution takes the form \((\rho, \pi)_\star\) for some covariant representation \((\rho, \pi)\).

\textbf{Proof:} Let \(\alpha: \mathcal{G} \to \mathcal{L}(\mathbb{F})\) be a linear functor that respects the involution. Write \(H_a = \alpha(a)\) for each object \(a \in \text{Ob}(\mathcal{G})\). Then for any morphism \(g \in \text{Hom}(a, b)\) we can define a unitary operator \(\rho(g): H_a \to H_b\) by the formula
\[ \rho(g) = \alpha(1_A, g) \]
where \(1_A\) is the identity element of the \(C^*\)-algebra \(A_b\).

We have a representation \(\pi: A_a \to \mathcal{L}(H_a)\) defined by the formula
\[ \pi(x) = \alpha(x1_a) \]
where \(1_a \in \text{Hom}(a, a)_{\mathcal{G}}\) is the identity morphism. It is easy to verify the formula
\[ \rho(g)\pi(x) = \pi(g(x))\rho(g) \]
and the fact that \((\rho, \pi)_\star = \alpha\). \qed

The above result is also true in the non-unital case; we can prove it by using approximate units for each \(C^*\)-algebra \(A_a\). However, we do not need the non-unital result in this article, and therefore omit the proof.

\textbf{Proposition 3.10} Let \(A\) be a \(G\)-\(C^*\)-algebra. Then we can define a norm on the morphism sets of the convolution category \(\mathcal{G}A\) by the formula
\[ \|\mu\|_{\max} = \sup\{\|(\rho, \pi)_\star(\mu)\| \mid (\rho, \pi) \text{ is a representation of } A\} \]
Proof: Consider a morphism

\[ f = x_1 g_1 + \cdots + x_n g_n \in \text{Hom}(a, b)_{A \rtimes \mathcal{G}} \]

Observe that, for any representation \( \pi \):

\[ \| (\rho, \pi)_* (f) \| \leq \| x_1 \| + \cdots + \| x_n \| \]

Hence the quantity \( \| f \|_{\text{max}} \) must be finite. It is now easy to see that the function \( f \mapsto \| f \|_{\text{max}} \) is a norm so we are done. \( \square \)

Definition 3.11 The \textit{crossed product}, \( A \rtimes \mathcal{G} \), is the Banach category obtained by completing the morphism sets of the convolution category \( A \mathcal{G} \) with respect to the norm \( \| - \|_{\text{max}} \).

The category \( A \rtimes \mathcal{G} \) is equipped with an involution inherited from the convolution category \( A \mathcal{G} \). It is straightforward to verify the following result.

Proposition 3.12 The category \( A \rtimes \mathcal{G} \) is a \( C^* \)-category. \( \square \)

If \( A \) is a graded \( \mathcal{G} \)-\( C^* \)-algebra, the crossed product \( A \rtimes \mathcal{G} \) can be graded by saying that a morphism

\[ \sum_i x_i g_i \in \text{Hom}(a, b)_{A \rtimes \mathcal{G}} \]

has degree \( k \) if the elements \( x_i \in A_b \) all have degree \( k \).

There is another type of crossed product we need to consider. To define it, observe that we can define a norm on the morphism sets of the convolution category \( A \mathcal{G} \) by the formula

\[ \| \mu \|_r = \sup \{ \| (\rho, \pi)_* (\mu) \| \mid (\rho, \pi) \text{ is a regular representation of } A \} \]

Definition 3.13 The \textit{reduced crossed product}, \( A \rtimes_r \mathcal{G} \), is the Banach category obtained by completing the morphism sets of the convolution category \( A \mathcal{G} \) with respect to the norm \( \| - \|_r \).

Proposition 3.14 The category \( A \rtimes_r \mathcal{G} \) is a \( C^* \)-category. \( \square \)

The reduced crossed product \( A \rtimes_r \mathcal{G} \) is a graded \( C^* \)-category when \( A \) is a graded \( \mathcal{G} \)-\( C^* \)-algebra.

If \( G \) is a discrete group we recover from the above definitions the usual crossed product \( C^* \)-algebras \( A \rtimes G \) and \( A \rtimes_r G \).

Recall from example 3.5 that for any groupoid \( \mathcal{G} \) the \textit{trivial} \( \mathcal{G} \)-\( C^* \)-\textit{algebra} is defined by associating the scalar field \( \mathbb{F} \) to each object \( a \in \text{Ob}(\mathcal{G}) \), and the identity \( 1: \mathbb{F} \rightarrow \mathbb{F} \) to each morphism \( g \in \text{Hom}(a, b)_{\mathcal{G}} \).

We have a corresponding convolution category \( \mathbb{F} \mathcal{G} \), and crossed product \( C^* \)-categories \( \mathbb{F} \rtimes \mathcal{G} \) and \( \mathbb{F} \rtimes_r \mathcal{G} \).
Definition 3.15  The crossed product $C^*$-categories $\mathbb{F} \times \mathcal{G}$ and $\mathbb{F} \rtimes \mathcal{G}$ are called the maximal and reduced $C^*$-categories of the groupoid $\mathcal{G}$. We denote them by the symbols $C^*\mathcal{G}$ and $C^*_r\mathcal{G}$ respectively.

The reduced and maximal $C^*$-categories of a groupoid were originally defined without reference to crossed products in [5] and [19] respectively. It is easy to see, using proposition 3.9, that the definitions given in these articles agree with the above definition.

If $G$ is a group, we recover from the above definition the usual maximal and reduced $C^*$-algebras, $C^*G$ and $C^*_rG$, associated to the group $G$.

Proposition 3.16  Let $f: \mathcal{G} \to \mathcal{H}$ be a functor between groupoids. Let $A$ be a $\mathcal{G}$-$C^*$-algebra, $B$ be an $\mathcal{H}$-$C^*$-algebra, and let $F: A \to B$ be an equivariant map covering the functor $f$.

Then we have a functorially induced $C^*$-functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{H}$.

Proof:  We begin by observing that we have an induced functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{H}$ between the convolution categories, defined by writing $F_*(a \times g) = f(a)$ for each object $a \in Ob(A \rtimes \mathcal{G})$ and

$$F_*(x_1g_1 + \cdots + x_ng_n) = F(x_1)f(g_1) + \cdots + F(x_n)f(g_n)$$

for each morphism $x_1g_1 + \cdots + x_ng_n \in Hom(a, b)_{A \rtimes \mathcal{G}}$.

Let $(\rho, \pi)$ be a covariant representation of the $\mathcal{H}$-$C^*$-algebra $B$. Then the pair $(\rho \circ f, \pi \circ F)$ is a representation of the $\mathcal{G}$-$C^*$-algebra $A$.

For any morphism $\mu \in Hom(a, b)_{A \rtimes \mathcal{G}}$ we therefore have the inequality

$$\| (\rho, \pi)_* F_*(\mu) \| \leq \| \mu \|_{\text{max}}$$

Hence $\| F_*(\mu) \|_{\text{max}} \leq \| \mu \|_{\text{max}}$ so the functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{H}$ is continuous. It therefore extends to a $C^*$-functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{H}$.

It is straightforward to check that the $C^*$-functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$ depends functorially on the equivariant map $F$, so we are done. $\square$

If the equivariant map $F: A \to B$ respects the grading, the induced $C^*$-functor $F_*: A \rtimes \mathcal{G} \to B \rtimes \mathcal{H}$ is graded.

The $C^*$-categories $A \rtimes \mathcal{G}$ and $A \rtimes \mathcal{G}$ are not in general equal. To see this, let $G$ be a group, and consider the trivial crossed products $\mathbb{F} \times G = C^*G$ and $\mathbb{F} \rtimes G = C^*_rG$. The group $C^*$-algebras $C^*G$ and $C^*_rG$ are equal if and only if the group $G$ is amenable.

In fact, an example in [3] shows that it is impossible for the assignment $\mathcal{G} \mapsto C^*_r\mathcal{G}$ to be functorial. Thus the analogue of the above proposition for the reduced crossed product is definitely false. We do, however, have the following result.

4See for example [12] for details on amenability of groups. In [1] the idea of amenability for groupoids is analysed, which is of course relevant to the issue we are examining here.
Proposition 3.17 Let $f: \mathcal{G} \to \mathcal{H}$ be a faithful functor between groupoids. Let $A$ be a $\mathcal{G}$-$C^*$-algebra, $B$ be an $\mathcal{H}$-$C^*$-algebra, and let $F: A \to B$ be an injective equivariant map covering the functor $f$.

Then we have a functorially induced $C^*$-functor $F_*: A \rtimes_r \mathcal{G} \to B \rtimes_r \mathcal{H}$.

Proof: As in proposition 3.16 we have an induced functor $F_*: A^G \to B^H$ between the convolution categories. We need to show that this functor is continuous, and so extends to a $C^*$-functor $F_*: A \rtimes_r \mathcal{G} \to B \rtimes_r \mathcal{H}$.

Choose an object $a \in \text{Ob}(\mathcal{G})$ and a representation $\alpha: B_{f(a)} \to \mathcal{L}(H)$. Composition with the map $F$ yields a representation $\alpha F: A_a \to \mathcal{L}(H)$.

Let $(\rho_\alpha, \pi_\alpha)$ and $(\rho_{\alpha F}, \pi_{\alpha F})$ be the induced regular representations of the convolution categories $B^H$ and $A^G$ respectively, defined as in example 3.8. Write $(B^H)_\alpha$ and $(A^G)_{\alpha F}$ to denote the images of these regular representations. Then there is a faithful $C^*$-functor $F_*: (A^G)_{\alpha F} \to (B^H)_\alpha$ defined in the obvious way.

Since any faithful $C^*$-functor is isometric, we have the identity

$$\|((\rho_\alpha, \pi_\alpha)_*, F_*(\mu))\| = \|((\rho_{\alpha F}, \pi_{\alpha F})_*, (\mu))\|$$

for any morphism $\mu$ in the convolution category $A^G$.

The definition of the norm in a reduced crossed product now gives us the inequality

$$\|F_*(\mu)\|_r \leq \|\mu\|_{\text{max}}$$

and we are done. \hfill \square

Proposition 3.18 Let $A$ be a $\mathcal{G}$-$C^*$-algebra. Then we have a canonical surjective $C^*$-functor $p: A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$.

Proof: Observe that for any morphism $\mu$ in the convolution category we have the inequality

$$\|p_*(\mu)\|_r \leq \|\mu\|_{\text{max}}$$

Hence the identity functor on the convolution category, $1: A^G \to A^G$, extends to a $C^*$-functor $p: A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$.

As we remarked in section 2, any $C^*$-functor has a closed image.\footnote{See corollary 4.9 in [19].} Hence the image $p[\text{Hom}(a, b)_{A \rtimes_r \mathcal{G}}]$ of a morphism set in the $C^*$-category $A \rtimes_r \mathcal{G}$ is a closed subset of the morphism set $\text{Hom}(a, b)_{A \rtimes_r \mathcal{G}}$. However, the image $p[\text{Hom}(a, b)_{A \rtimes_r \mathcal{G}}]$ contains the set $\text{Hom}(a, b)_{A^G}$, which is a dense subset of the space $\text{Hom}(a, b)_{A \rtimes_r \mathcal{G}}$. Therefore the $C^*$-functor $p$ is surjective. \hfill \square

The $C^*$-functor $p: A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$ is natural in the category of faithful functors between groupoids and injective equivariant maps.

4 Equivariant $KK$-theory

We begin our discussion of equivariant $KK$-theory by looking at equivariant Hilbert modules.
Definition 4.1 Let $B$ be a $G$-$C^*$-algebra. Then a $G$-equivariant Hilbert $B$-module is a functor, $E$, from the groupoid $G$ to the category of Banach spaces and invertible bounded linear maps such that:

- The space $E_a$ associated to the object $a \in Ob(G)$ is a Hilbert $B_a$-module.
- For every morphism $g \in Hom(a, b)_G$ we have the formula
  
  \[ g(\eta x) = (g\eta)(gx) \]

for all elements $\eta \in E_a$ and $x \in B_a$.

A $G$-equivariant Hilbert $B$-module $E$ is said to be countably generated if each Hilbert $B_a$-module $E_a$ is countably generated.

If $B$ is a graded $G$-$C^*$-algebra, a $G$-equivariant Hilbert $B$-module $E$ is referred to as graded if each Hilbert $B_a$-module $E_a$ is graded, and the maps $g: E_a \to E_b$ coming from the groupoid $G$ all respect the grading.

Example 4.2 Let $B$ be a $G$-$C^*$-algebra. Then $B$ itself can be considered to be a $G$-equivariant Hilbert module. We have inner products $\langle -, - \rangle: B_a \times B_a \to B_a$ defined by the formula

\[ \langle x, y \rangle = x^* y \]

If $B$ is a graded $G$-$C^*$-algebra, then $B$ is also graded as a $G$-equivariant Hilbert $B$-module.

Definition 4.3 Let $E$ be a $G$-equivariant Hilbert $B$-module. We write $L_G(E)$ to denote the $G$-$C^*$-algebra which associates the $C^*$-algebra $L(E_a)$ to the object $a \in Ob(G)$. The $G$-action is defined by the formula

\[ g(T\eta) = (gT)(g\eta) \]

The $G$-$C^*$-algebra $L_G(E)$ is graded in the obvious way when $E$ is a $G$-equivariant graded Hilbert $B$-module.

For graded $C^*$-algebras $A$ and $B$ we define a graded Hilbert $(A, B)$-bimodule to be a countably generated graded Hilbert $B$-module $F$ equipped with a morphism $\phi: A \to L(F)$. The right $B$-module $F$ is thus also a left $A$-module, with $A$-action:

\[ x\eta = \phi(x)(\eta) \]

We will usually drop explicit mention of the morphism $\phi$ from our notation. Let $E$ be a graded Hilbert $A$-module. Then we can form the algebraic tensor product $E \otimes_A F$: it is the right $B$-module generated by elementary tensors $\eta \otimes \xi$, where $\eta \in E$ and $\xi \in F$, subject to the relation

\[ \eta x \otimes \xi = \eta \otimes x\xi \]

for all elements $x \in A$. The inner tensor product, $E \otimes_A F$, is the completion of the algebraic tensor product $E \otimes_A F$ with respect to the norm defined by the inner product

\[ \langle \eta \otimes \xi, \eta' \otimes \xi' \rangle = \langle \xi, \langle \eta, \eta' \rangle \xi' \rangle \]

See [13] for further details.
Definition 4.4 Let $A$ and $B$ be $G$-$C^*$-algebras. Then a $G$-equivariant graded Hilbert $(A, B)$-bimodule is a $G$-equivariant graded Hilbert $B$-module, $\mathcal{F}$, equipped with a $G$-equivariant map $\phi: A \to \mathcal{L}_G(\mathcal{F})$.

As in the non-equivariant case, we usually drop explicit mention of the morphism $\phi$ from our notation. Observe that if $\mathcal{F}$ is a $G$-equivariant graded Hilbert $(A, B)$-bimodule, each space $\mathcal{F}_a$ is a graded Hilbert $(A_a, B_a)$-bimodule.

We call a $G$-equivariant graded Hilbert $(A, B)$-bimodule countably generated if it is countably generated as a $G$-equivariant graded Hilbert $B$-module.

Example 4.5 Let $A$ and $B$ be graded $G$-$C^*$-algebras and suppose we have a $G$-equivariant map $\phi: A \to B$. According to example 4.2 the $G$-$C^*$-algebra $B$ is itself a graded $G$-equivariant Hilbert $B$-module. Given an element $x \in A_a$ we have an operator $\phi(x): B_a \to B_a$ defined by multiplication. The formula $g(\phi(x)y) = \phi(gx)(gy)$ is satisfied. The map $\phi$ can therefore by considered a $G$-equivariant map $\phi: A \to \mathcal{L}_G(B)$, and the $G$-$C^*$-algebra $B$ is itself a $G$-equivariant Hilbert $(A, B)$-bimodule.

Definition 4.6 Let $E$ be a $G$-equivariant graded Hilbert $A$-module, and let $F$ be a $G$-equivariant graded Hilbert $(A, B)$-bimodule. Then the inner tensor product, $E \hat{\otimes} A F$, is the $G$-equivariant graded Hilbert $B$-module in which the module $(E \hat{\otimes} A F)_a$ is the tensor product $E_a \otimes A_a F_a$. The $G$-action is defined by the formula

$$g(\eta \otimes \xi) = g\eta \otimes g\xi$$

The inner tensor product $E \hat{\otimes} A F$ is countably generated if the $G$-equivariant Hilbert $A$-module $E$ and $G$-equivariant Hilbert $(A, B)$-bimodule $F$ are countably generated.

There is another type of tensor product of Hilbert modules that we will need in our calculations. Recall that if $A$ and $B$ are graded $C^*$-algebras, $E$ is a graded Hilbert $A$-module, and $F$ is a graded Hilbert $B$-module, we can define a Hilbert $A \hat{\otimes} B$-module $E \hat{\otimes} F$. It is the completion of the tensor product of vector spaces $E \otimes F$ with respect to the norm defined by the inner product

$$\langle \eta \otimes \xi, \eta' \otimes \xi' \rangle = \langle \eta, \eta' \rangle \otimes \langle \xi, \xi' \rangle$$

The relevant technical details can be found in [13]. We can define a grading by specifying the degree of elementary tensors:

$$\deg(\eta \otimes \xi) = \deg(\eta) + \deg(\xi)$$

Here addition takes place modulo 2.

Definition 4.7 Let $A$ and $B$ be graded $G$-$C^*$-algebras. Then we define the tensor product, $A \hat{\otimes} B$, to be the $G$-$C^*$-algebra in which the $C^*$-algebra $(A \hat{\otimes} B)_a$ is equal to the tensor product $A_a \hat{\otimes} B_a$. The $G$-action is defined by the formula

$$g(x \otimes y) = gx \otimes gy$$

If $E$ is a $G$-equivariant graded Hilbert $A$-module, and $F$ is a $G$-equivariant graded Hilbert $B$-module, we define the outer tensor product, $E \otimes F$, to be the
$G$-equivariant graded Hilbert $A \otimes B$-module where the module $(\mathcal{E} \otimes \mathcal{F})_a$ is the tensor product $\mathcal{E}_a \otimes \mathcal{F}_a$. The $G$-action is defined by the formula

$$g(\eta \otimes \xi) = g\eta \otimes g\xi$$

The outer tensor product of two countably generated $G$-equivariant Hilbert modules is countably generated.

We need one final definition before we are ready to look at equivariant $KK$-theory.

**Definition 4.8** Let $A$ be a $G$-$C^*$-algebra, and let $\mathcal{E}$ and $\mathcal{E}'$ be $G$-equivariant Hilbert $A$-modules. Then a **bounded operator** $T: \mathcal{E} \to \mathcal{E}'$ is a collection, $T$, of operators $T_a: \mathcal{E}_a \to \mathcal{E}'_a$ such that the norm

$$\|T\| = \sup\{\|T_a\| \mid a \in \text{Ob}(G)\}$$

is finite.

Note that we make no assumptions here concerning equivariance. In the graded case, we say a bounded operator has degree $k$ if each operator $T_a: \mathcal{E}_a \to \mathcal{E}'_a$ has degree $k$.

If $A$ and $B$ are graded $G$-$C^*$-algebras, and $\mathcal{E}$ and $\mathcal{E}'$ are $G$-equivariant graded Hilbert $(A, B)$-bimodules, we call a collection, $T$, of maps $T_a: \mathcal{E}_a \to \mathcal{E}'_a$ a **bounded operator** if it is a bounded operator between $G$-equivariant Hilbert $B$-modules in the sense of the above definition.

Let $T$ be a bounded operator between $G$-equivariant graded Hilbert $(A, B)$-bimodules, and let $x \in A_a$. Then we define the **graded commutator**

$$[x, T] = xT_a - (-1)^{\deg(x)\deg(T)}T_a x$$

This formula only makes sense when the degree of the element $x$ and the operator $T$ are defined. However, we can extend the definition of the graded commutator by requiring it to be linear in each variable.

**Definition 4.9** Let $A$ and $B$ be $G$-$C^*$-algebras. Then a $G$-equivariant Kasparov $(A, B)$-cycle is a pair $(\mathcal{E}, T)$, where $\mathcal{E}$ is a countably generated $G$-equivariant Hilbert $(A, B)$-bimodule, and $T: \mathcal{E} \to \mathcal{E}$ is a bounded operator such that the operators

$$x(T_a - T_a^*) \quad x(T_a^2 - 1) \quad [x, T] \quad x(gT_b - T_ag)$$

are compact for all elements $x \in A_a$ and morphisms $g \in \text{Hom}(b, a)_G$.

In the above definition, the operators defined by the various formulae are just operators between (non-equivariant) Hilbert modules over $C^*$-algebras. We use the standard $C^*$-algebraic notion of such an operator being compact.

An element of equivariant $KK$-theory is a certain equivalence class of equivariant Kasparov cycles.

**Definition 4.10** Let $(\mathcal{E}, T)$ and $(\mathcal{E}', T')$ be $G$-equivariant Kasparov $(A, B)$-cycles. Then the **direct sum** is the Kasparov cycle

$$(\mathcal{E}, T) \oplus (\mathcal{E}', T') = (\mathcal{E} \oplus \mathcal{E}', T \oplus T')$$
Definition 4.11

- A $G$-equivariant Kasparov $(A, B)$-cycle $(\mathcal{E}, T)$ is called degenerate if the operators
\[
x(T_a - T_a^*) \quad x(T_a^2 - 1) \quad [x, T] \quad x(gT_b - T_ag)
\]
are equal to zero for all elements $x \in A_a$ and morphisms $g \in \text{Hom}(b, a)_G$.

- An operator homotopy between $G$-equivariant Kasparov $(A, B)$-cycles $(\mathcal{E}, T)$ and $(\mathcal{E}, T')$ is a norm-continuous path $(\mathcal{E}, T_t)$ of Kasparov cycles such that $T_0 = T$ and $T_1 = T'$.

- Two $G$-equivariant Kasparov $(A, B)$-cycles $(\mathcal{E}_1, T_1)$ and $(\mathcal{E}_2, T_2)$ are called equivalent if there are degenerate Kasparov cycles $(\mathcal{E}_1', T_1')$ and $(\mathcal{E}_2', T_2')$ such that the direct sums $(\mathcal{E}_1, T_1) \oplus (\mathcal{E}_1', T_1')$ and $(\mathcal{E}_2, T_2) \oplus (\mathcal{E}_2', T_2')$ are operator homotopic.

We write $[(\mathcal{E}, T)]$ to denote the equivalence class of a $G$-equivariant Kasparov $(A, B)$-cycle $(\mathcal{E}, T)$, and $KK_G(A, B)$ to denote the set of equivalence classes.

Proposition 4.12 The set $KK_G(A, B)$ is an Abelian group with an operation defined by taking the direct sum of Kasparov cycles.

Proof: It is easy to check that the set $KK_G(A, B)$ is an Abelian semigroup, with identity element $[(\mathcal{E}, T)]$ where $(\mathcal{E}, T)$ is any degenerate $G$-equivariant Kasparov $(A, B)$-cycle.

If $\mathcal{E}$ is a $G$-equivariant Hilbert $B$-module, with grading $\mathcal{E}_a = (\mathcal{E}_a)_0 \oplus (\mathcal{E}_a)_1$, define $\mathcal{E}^{op}$ to be the $G$-equivariant Hilbert $B$-module with the opposite grading.

If we have a $G$-equivariant map $\phi: A \to \mathcal{L}(\mathcal{E})$ we can define a $G$-equivariant map $\phi^{op}: A \to \mathcal{L}(\mathcal{E}^{op})$ by writing $\phi^{op}(x_0 + x_1) = \phi(x_0 - x_1)$ for elements $x_0, x_1 \in A_a$, of degrees 0 and 1 respectively.

Consider a $G$-equivariant Kasparov $(A, B)$-cycle $(\mathcal{E}, T)$. The pair $(\mathcal{E}^{op}, -T)$ is a $G$-equivariant Kasparov $(A, B)$-cycle. We can define an operator homotopy between the Kasparov cycle $(\mathcal{E}, T) \oplus (\mathcal{E}^{op}, -T)$ and the degenerate cycle

\[
\left( \mathcal{E} \oplus \tilde{\mathcal{E}}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

by the formula

\[
G_\theta = \begin{pmatrix} T \cos \theta & \sin \theta \\ \sin \theta & -T \cos \theta \end{pmatrix}, \quad \theta \in [0, \frac{\pi}{2}]
\]

Hence

\[
[(\mathcal{E}, T)] + [(\tilde{\mathcal{E}}, -T)] = 0
\]

and we have proved that the set $KK_G(A, B)$ is an Abelian group. \[\square\]

When $G$ a group, we recover from the above definition the usual equivariant $KK$-theory groups, as defined by Kasparov in [3].

There is a small technical point here that we should mention. It is usual to define the equivariant $KK$-theory groups by looking at an equivalence relation called homotopy on Kasparov cycles, rather than the equivalence relation we have used in our more general definition. However, these relations turn out to be the same; see remark 5.11 (2) in [3].

Let us call a $G$-$C^*$-algebra $A$ *unital* if each $C^*$-algebra $A_a$ has a countable approximate unit.
**Proposition 4.13** Let $A$ and $B$ be $\sigma$-unital $\mathcal{G}$-$C^*$-algebras. The group $KK_\mathcal{G}(A, B)$ is contravariantly functorial in the variable $A$ and covariantly functorial in the variable $B$.

**Proof:** Let $(E, T)$ be a $\mathcal{G}$-equivariant Kasparov $(A, B)$-cycle, and let $F: B \rightarrow B'$ be a $\mathcal{G}$-equivariant map.

Then the $\mathcal{G}$-$C^*$-algebra $B'$ is itself a countably generated $\mathcal{G}$-equivariant graded Hilbert $(B, B')$-bimodule according to example 4.5. We can therefore form the inner tensor product $E \otimes_B B'$. This inner tensor product is a $\mathcal{G}$-equivariant graded Hilbert $(A, B')$-bimodule since we have an $A$-action defined by writing

$$x(\eta \otimes y) = (x\eta) \otimes y$$

where $x \in A_a$, $\eta \in E_a$, and $y \in B_a'$.

Further, there is a bounded operator $T \otimes 1: E \otimes_B B' \rightarrow E \otimes_B B'$ given by the formula

$$(T \otimes 1)(\eta \otimes y) = (T\eta) \otimes y$$

It is easy to check that we have a functorially induced map $F_*: KK_\mathcal{G}(A, B) \rightarrow KK_\mathcal{G}(A, B')$ defined by writing

$$F_*[(E, T)] = [(E \otimes_B B', T \otimes 1)]$$

Now consider a $\mathcal{G}$-equivariant map $G: A' \rightarrow A$. Suppose that the action of the $\mathcal{G}$-$C^*$-algebra $A$ is defined on the Hilbert $B$-module $E$ by the equivariant map $\phi: A \rightarrow L_G(E)$. Then we can form a $\mathcal{G}$-equivariant graded Hilbert $(A', B')$-bimodule $G^*(E)$. The module $G^*(E)$ is equal to the module $E$ as a $\mathcal{G}$-equivariant graded Hilbert $B$-module, and the $A'$-action is defined by the equivariant map $\phi \circ G: A' \rightarrow L_G G^*(E)$.

We have a functorially induced map $G^*: KK_\mathcal{G}(A, B) \rightarrow KK_\mathcal{G}(A', B)$ defined by the formula

$$G^*[(E, T)] = [(G^*(E), T)]$$

$\square$

In [20] the $K$-homology of a $C^*$-algebra $A$ is defined in terms of the ordinary $K$-theory of a ‘dual algebra’ constructed from $A$. We can extend this approach to define the equivariant $KK$-theory groups $KK_\mathcal{G}^{-n}(A, B)$ for $\mathcal{G}$-$C^*$-algebras $A$ and $B$ in terms of the ordinary $K$-theory of some ‘dual $C^*$-category’.

Our definitions and methods here are modeled on the approach to the $KK$-theory of $C^*$-categories in [20].

**Definition 4.14** We write $D_\mathcal{G}(A, B)$ to denote the category of countably generated $\mathcal{G}$-equivariant Hilbert $(A, B)$-bimodules and bounded operators $T: E \rightarrow E'$ such that the operators $[x, T]$ and $x(gT_a - T_bg)$ are compact for all elements $x \in A_a$ and morphisms $g \in Hom(a, b)_G$. We write $KD_\mathcal{G}(A, B)$ to denote the (non-unital) subcategory consisting of bounded operators $T: E \rightarrow E'$ such that the composites $xT_a$ and $T_a x$ are compact operators for all elements $x \in A_a$.

---

We need the $\mathcal{G}$-$C^*$-algebra $B'$ to be $\sigma$-unital for the $\mathcal{G}$-equivariant Hilbert $(B, B')$-module $B'$ to be countably generated.
A $C^*$-ideal, $J$, in a $C^*$-category $A$ is a $C^*$-subcategory such that the composite of a morphism in the category $J$ and a morphism in the category $A$ belongs to the category $J$. One can form the quotient, $A/J$, of a $C^*$-category by a $C^*$-ideal; see [19] for details.

A straightforward calculation tells us that the category $D_G(A, B)$ is a graded $C^*$-category and the subcategory $KD_G(A, B)$ is a $C^*$-ideal. We can therefore form the quotient

$$QD_G(A, B) = D_G(A, B)/KD_G(A, B)$$

The following result is proved in the same way as proposition 4.13.

**Proposition 4.15** Let $A$ and $B$ be $\sigma$-unital $G$-$C^*$-algebras. Then the graded $C^*$-category $QD_G(A, B)$ is contravariantly functorial in the variable $A$ and covariantly functorial in the variable $B$.

The following results are proved in exactly the same way as theorem 4.9 and lemma 4.10 in [20]. We do not repeat the work here.

**Theorem 4.16** Let $A$ and $B$ be $\sigma$-unital $G$-$C^*$-categories. There is a natural isomorphism

$$K_1QD_G(A, B) \cong KK(A, B).$$

**Lemma 4.17** Suppose that $A$ and $B$ be graded $G$-$C^*$-algebras. Let $p, q \in \mathbb{N}$. Then there is a natural isomorphism

$$K_1QD_G(A, B \hat{\otimes} F_{p,q}) \cong K_1(QD_G(A, B) \hat{\otimes} F_{p,q})$$

By the Bott periodicity theorem it therefore makes sense to define further equivariant $KK$-theory groups by the formula

$$KK_G^{p-q}(A, B) = KK_G(A, B \hat{\otimes} F_{p,q})$$

We have natural isomorphisms

$$KK_G^{p-q}(A, B) \cong K_1(QD_G(A, B) \hat{\otimes} F_{p,q}) \cong K_1-(p-q)QD_G(A, B)$$

**Definition 4.18** Let $G$ be a discrete groupoid, and let $A$ and $B$ be $\sigma$-unital $G$-$C^*$-categories. Then we define the $G$-equivariant $KK$-theory spectrum

$$\mathbb{K}K_G(A, B) = \Omega KQD_G(A, B)$$

According to proposition 4.15 the $KK$-theory spectrum $\mathbb{K}K_G(A, B)$ is contravariantly functorial in the variable $A$ and covariantly functorial in the variable $B$. There is another type of functoriality that we need to consider, depending this time on the groupoid $G$.

Let $f: H \rightarrow G$ be a functor between groupoids, and let $A$ be a graded $G$-$C^*$-algebras. Abusing notation, $A$ can also be considered to be a graded $H$-$C^*$-algebra. We associate the graded $C^*$-algebra $A_{f(a)}$ to the object $a \in Ob(G)$ and the morphism $f(g): A_{f(a)} \rightarrow A_{f(a)}$ to the element $g \in Hom(a, b)_H$. 

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Proposition 4.19 There is a functorially induced graded $C^*$-functor $f^* : \mathcal{QD}_G(A,B) \to \mathcal{QD}_H(A,B)$. The $C^*$-functor $f^*$ is natural in the variables $A$ and $B$.

Proof: Consider a $G$-equivariant graded Hilbert $(A,B)$-bimodule $E$. Then we have an $H$-equivariant graded Hilbert $B$-bimodule $f^*(E)$ defined by associating the Hilbert $B_{f(a)}$-module $E_{f(a)}$ to the object $a \in Ob(H)$. The action of the groupoid $H$ is defined by the formula $g_\eta = f(g)\eta$. The action of the $H$-$C^*$-algebra $A$ is the same as the action of the $G$-$C^*$-algebra $A$ on the original bimodule $E$.

Let $E$ and $E'$ be graded Hilbert $(A,B)$-modules, and let $T : E \to E'$ be a bounded operator. Then we have a bounded operator $f^*(T) : f^*(E) \to f^*(E')$ defined by the formula $f^*(T)(\eta) = T\eta$.

If the operator $T$ is a morphism in the category $\mathcal{D}_G(A,B)$, the operator $f^*(T)$ is a morphism in the category $\mathcal{D}_H(A,B)$. We therefore have a functorially induced graded $C^*$-functor $f^* : \mathcal{QD}_G(A,B) \to \mathcal{QD}_H(A,B)$. Naturality of this induced $C^*$-functor in the variables $A$ and $B$ is easy to check. $\square$

In particular, at the level of $K$-theory, we have a map

$$f^* : \mathcal{KK}_G(A,B) \to \mathcal{KK}_H(A,B)$$

Definition 4.20 The induced map

$$f^* : \mathcal{KK}_G(A,B) \to \mathcal{KK}_H(A,B)$$

is called the restriction map.

When $f$ is a group homomorphism, we recover from the above definition the usual restriction maps in equivariant $KK$-theory.

Proposition 4.21 Let $f : H \to G$ be an equivalence of groupoids. Let $A$ and $B$ be unital $G$-$C^*$-algebras. Then the restriction map

$$f^* : \mathcal{KK}_G(A,B) \to \mathcal{KK}_H(A,B)$$

is a stable equivalence of spectra.

Proof: By proposition 2.8 it suffices to show that the $C^*$-functor

$$f^* : \mathcal{QD}_G(A,B) \to \mathcal{QD}_H(A,B)$$

is an equivalence of $C^*$-categories. Let $g : G \to H$ be a functor such that we have natural isomorphisms $F : fg \to 1_G$ and $G : gf \to 1_H$ respectively.

Consider an object $a \in Ob(G)$. Then we have a morphism $F_a \in Hom(fg(a),a)_G$ determined by the natural isomorphism $F$. If $E$ is a $G$-equivariant Hilbert $B$-module, the action of the groupoid $G$ gives us a unitary operator

$$F_a : g^*f^*(E) \to E$$

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It is easy to check that we have an induced natural isomorphism

$$F_*: g^*f^* \to 1_{QD(\mathcal{G}, B)}$$

Similarly we obtain a natural isomorphism

$$G_*: f^*g^* \to 1_{QD(\mathcal{H}, B)}$$

Therefore the $C^*$-functor $f^*: QD(\mathcal{G}, B) \to QD(\mathcal{H}, B)$ is an equivalence of $C^*$-categories and we are done. 

\[\Box\]

5 Descent

There is a canonical descent map from the equivariant $KK$-theory of $\mathcal{G}$-$C^*$-algebras to the $KK$-theory of the associated crossed product $C^*$-categories. Before we construct it, we need to review the $KK$-theory of $C^*$-categories, as defined in [20].

The $KK$-theory of $C^*$-categories is constructed by considering certain operators between countably generated Hilbert modules.

We begin by making the notion of countably generated precise in this context. Let $\mathcal{B}$ be a $C^*$-category. Recall that a right $\mathcal{B}$-module, $\mathcal{E}$, is said to be countably generated if there is a countable set

$$\Omega \subseteq \bigcup_{A \in Ob(\mathcal{B})} \mathcal{E}(A)$$

such that for each object $A \in Ob(\mathcal{B})$, every element of the vector space $\mathcal{E}(A)$ is a finite linear combination of elements of the form $\eta x$, where $x \in Hom(A, B)_{\mathcal{B}}$ and $\eta \in \Omega \cap \mathcal{E}(B)$.

**Definition 5.1** A Hilbert $\mathcal{B}$-module $\mathcal{E}$ is countably generated if there is a countably generated right $\mathcal{B}$-module $\mathcal{E}_0$ such that the space $\mathcal{E}_0(A)$ is a dense subset of the space $\mathcal{E}(A)$ for every object $A \in Ob(\mathcal{B})$.

The countable set $\Omega$ which generates the right $\mathcal{A}$-module $\mathcal{E}_0$ is referred to as a generating set for the Hilbert $\mathcal{A}$-module $\mathcal{E}$.

Note that the above definition is the same as that in [19], [20] but differs from that of [19].

When the $C^*$-category $\mathcal{A}$ is not unital, the Hilbert $\mathcal{A}$-modules $Hom(-, A)_{\mathcal{A}}$ are not countably generated in general.

**Definition 5.2** A $C^*$-category $\mathcal{A}$ is called $\sigma$-unital if each $C^*$-algebra $Hom(A, A)_{\mathcal{A}}$ has a countable approximate unit.

When $\mathcal{A}$ is a $\sigma$-unital $C^*$-category, it is clear that the Hilbert $\mathcal{A}$-modules $Hom(-, A)_{\mathcal{A}}$ are all countably generated.

**Definition 5.3** Let $\mathcal{B}$ be a graded $C^*$-category. Then a Hilbert $\mathcal{B}$-module $\mathcal{E}$ is called graded if each space $\mathcal{E}(A)$ admits decompositions $\mathcal{E}(A) = \mathcal{E}(A)_0 \oplus \mathcal{E}(A)_1$ into vectors of degree 0 and vectors of degree 1 such that
\[
\deg(\eta x) = \deg(\eta) + \deg(x) \text{ for all vectors } \eta \in \mathcal{E}(B) \text{ and morphisms } x \in \text{Hom}(A, B)_B
\]
\[
\deg(\langle \eta, \xi \rangle) = \deg(\eta) + \deg(\xi) \text{ for all vectors } \eta \in \mathcal{E}(B) \text{ and } \xi \in \mathcal{E}(A).
\]

Here all addition takes place modulo 2.

**Definition 5.4** Let \( \mathcal{E} \) and \( \mathcal{F} \) be Hilbert modules over a \( C^* \)-category, \( B \). Then an **operator** \( T: \mathcal{E} \to \mathcal{F} \) is a collection of maps \( T_A: \mathcal{E}(A) \to \mathcal{F}(A) \) such that there are maps \( T_A^*: \mathcal{F}(A) \to \mathcal{E}(A) \) with the property
\[
\langle \eta, T_A \xi \rangle = \langle T_A^* \xi, \eta \rangle
\]
for all vectors \( \eta \in \mathcal{F}(B) \) and \( \xi \in \mathcal{E}(A) \).

It is shown in [19] that an operator \( T: \mathcal{E} \to \mathcal{F} \) is a natural transformation, each map \( T_A: \mathcal{E}(A) \to \mathcal{F}(A) \) is bounded and linear, and the collection of maps \( T_A^* \) defines an operator \( T^* \). The operator \( T^* \) is called the **adjoint** of the operator \( T \).

An operator \( T \) is called **bounded** if the norm
\[
\|T\| = \sup \{\|T_A\| \mid A \in \text{Ob}(B)\}
\]
is finite. The adjoint of a bounded operator is bounded.

If \( B \) is a graded \( C^* \)-category, we write \( \mathcal{L}(B) \) to denote the category of all countably generated graded Hilbert \( B \)-modules and bounded linear operators. It can be shown (see [19]) that the category \( \mathcal{L}(B) \) is a \( C^* \)-category. Moreover, it is a graded \( C^* \)-category; we define the **degree** of \( T \) by the formula
\[
\deg(T \eta) = \deg(T) + \deg(\eta)
\]

Of course, addition in the above formula takes place modulo 2.

**Definition 5.5** Let \( A \) and \( B \) be graded \( C^* \)-categories. Then a **graded Hilbert \((A, B)\)-bimodule** is a graded \( C^* \)-functor \( \mathcal{E}: A \to \mathcal{L}(B) \).

For each object \( A \in \text{Ob}(A) \) we write \( \mathcal{E}(\cdot, A) \) to denote the corresponding graded Hilbert \( B \)-module. Given another object \( B \in \text{Ob}(B) \) we have a vector space \( \mathcal{E}(B, A) \). For each morphism \( x \in \text{Hom}(A, B)_A \) there is a bounded operator \( x: \mathcal{E}(\cdot, A) \to \mathcal{E}(\cdot, B) \).

**Definition 5.6** A **bounded operator** \( T: \mathcal{E} \to \mathcal{E}' \) between graded Hilbert \((A, B)\)-bimodules \( \mathcal{E} \) and \( \mathcal{E}' \) is a collection, \( T \), of operators \( T_A: \mathcal{E}(\cdot, A) \to \mathcal{E}'(\cdot, A) \) such that the norm
\[
\|T\| = \sup \{\|T_A\| \mid A \in \text{Ob}(A)\}
\]
is finite.

We say that the operator \( T \) has degree \( k \) if each operator \( T_A: \mathcal{E}(\cdot, A) \to \mathcal{E}'(\cdot, A) \) has degree \( k \).
Note that we make no assumptions here concerning naturality. If $T$ is a bounded operator between graded Hilbert $(\mathcal{A}, \mathcal{B})$-bimodules, and $x \in \text{Hom}(\mathcal{A}, \mathcal{A}')_A$ is a morphism in the $C^*$-category $\mathcal{A}$, let us define the graded commutator by the formula

$$[x, T] = xT_A - (-1)^{\deg(x) \cdot \deg(T)} T_{A'} x$$

The above formula only makes sense when the degree of the morphism $x$ and the operator $T$ are defined. However, we can extend the definition of the graded commutator by requiring it to be linear in each variable.

**Definition 5.7** Let $\mathcal{B}$ be a $C^*$-category, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{B}$-modules. Then a rank one operator $T: \mathcal{E} \to \mathcal{F}$ is an operator of the form

$$\zeta \mapsto \eta \langle \xi, \zeta \rangle$$

for elements $\eta \in \mathcal{F}$ and $\xi \in \mathcal{E}$. We write this operator $\eta \langle \xi, - \rangle$. A compact operator is a norm-limit of finite linear combinations of rank one operators.

**Definition 5.8** We write $\mathcal{D}(\mathcal{A}, \mathcal{B})$ to denote the category of graded Hilbert $(\mathcal{A}, \mathcal{B})$-bimodules and bounded operators $T: \mathcal{E} \to \mathcal{E}'$ such that the graded commutator $[x, T]$ is compact for all morphisms $x \in \text{Hom}(\mathcal{A}, \mathcal{A}')_A$. We write $\mathcal{KD}(\mathcal{A}, \mathcal{B})$ to denote the (non-unital) subcategory consisting of bounded operators $T: \mathcal{E} \to \mathcal{E}'$ such that the composites $xT_A$ and $T_{A'} x$ are compact for all morphisms $x \in \text{Hom}(\mathcal{A}, \mathcal{A}')_A$.

The category $\mathcal{D}(\mathcal{A}, \mathcal{B})$ is a graded $C^*$-category and the subcategory $\mathcal{KD}(\mathcal{A}, \mathcal{B})$ is a $C^*$-ideal. We can therefore form the quotient $\mathcal{QD}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{A}, \mathcal{B})/\mathcal{KD}(\mathcal{A}, \mathcal{B})$.

**Definition 5.9** Let $\mathcal{A}$ and $\mathcal{B}$ be small $\sigma$-unital graded $C^*$-categories. We define the $KK$-theory spectrum

$$\mathbb{K}K(\mathcal{A}, \mathcal{B}) = \Omega \mathbb{K} \mathcal{QD}(\mathcal{A}, \mathcal{B})$$

It is proved in [20] that if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras, the usual $KK$-theory groups, as defined by Kasparov in [13], can be recovered as the stable homotopy groups of the spectrum $\mathbb{K}K(\mathcal{A}, \mathcal{B})$.

The following results are proved in [20].

**Proposition 5.10** The $C^*$-category $\mathcal{QD}(\mathcal{A}, \mathcal{B})$ is contravariantly functorial in the variable $\mathcal{A}$ and covariantly functorial in the variable $\mathcal{B}$. ⊓⊔

Hence the $KK$-theory spectrum $\mathbb{K}K(\mathcal{A}, \mathcal{B})$ is contravariantly functorial in the variable $\mathcal{A}$ and covariantly functorial in the variable $\mathcal{B}$.

**Proposition 5.11** Let $\mathcal{B}$ be a small $\sigma$-unital graded $C^*$-category. Then the spectra $\mathbb{K}K(\mathcal{F}, \mathcal{B})$ and $\mathbb{K}(\mathcal{B})$ are naturally stably equivalent. ⊓⊔

The main property of $KK$-theory we need in this article is a special case of the Kasparov product.
Proposition 5.12 Let $A$ and $B$ be $\sigma$-unital $C^*$-categories. Then we have a product

$$K(A) \otimes KK(A, B) \to K(B)$$

This product agrees with the Kasparov product when $A$ and $B$ are $C^*$-algebras. It is natural in the variable $B$ in the obvious sense, and natural in the variable $A$ in the sense that for a $C^*$-functor $F: A \to A'$ we have an induced commutative diagram

\[
\begin{array}{ccc}
K(A) \otimes KK(A, B) & \to & K(B) \\
F^* & \downarrow & \\
K(A') \otimes KK(A', B) & \to & K(B)
\end{array}
\]

\[\Box\]

Proposition 2.11 gives us the following special case.

Proposition 5.13 Let $A$ and $B$ be trivially graded unital $C^*$-categories, and let $E$ be a finitely generated projective Hilbert $A$-module. Then we have a natural map of spectra

$$[E] \otimes KK(A, B) \to K(B)$$

Given a $C^*$-functor $F: A \to A'$ we have a commutative diagram

\[
\begin{array}{ccc}
KK(A, B) & \to & KB \\
F^* & \downarrow & \\
KK(A', B) & \to & KB
\end{array}
\]

\[\Box\]

We are now ready to construct the descent map relating equivariant $KK$-theory to crossed products.

Theorem 5.14 Let $A$ and $B$ be $\sigma$-unital graded $\mathcal{G}$-$C^*$-algebras. Then we have a canonical graded $C^*$-functor

$$D: QD_G(A, B) \to QD(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$$

The $C^*$-functor $D$ is natural in the variables $A$ and $B$ in the obvious sense, and natural in the variable $\mathcal{G}$ in the sense that given a faithful functor $f: \mathcal{H} \to \mathcal{G}$ between groupoids we have a commutative diagram

\[
\begin{array}{ccc}
QD_G(A, B) & \to & QD(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G}) \\
\downarrow & \downarrow & \\
QD_H(A, B) & \to & QD(A \rtimes_r \mathcal{H}, B \rtimes_r \mathcal{H})
\end{array}
\]
\textbf{Proof:} Let $\mathcal{E}$ be a countably generated $\mathcal{G}$-equivariant graded Hilbert $(A, B)$-bimodule. We begin our construction by associating to $\mathcal{E}$ a countably generated graded Hilbert $(A \ltimes \mathcal{G}, B \ltimes \mathcal{G})$-bimodule $D(\mathcal{E})$.

Fix an object $a \in Ob(\mathcal{G})$. Then for each object $b \in Ob(\mathcal{G})$ there is a vector space

$$D(\mathcal{E})_0(b, a) = \{ \eta_1 g_1 + \cdots + \eta_n g_n \mid \eta_i \in \mathcal{E}_a, g_j \in \text{Hom}(b, a)_\mathcal{G} \}$$

We can define a linear contravariant functor $D(\mathcal{E})_0(-, a)$ from the convolution category $B\mathcal{G}$ to the category of vector spaces. The object $b \in Ob(\mathcal{G})$ is mapped to the vector space $D(\mathcal{E})_0(b, a)$. The action of the category $B\mathcal{G}$ is defined by the formula\footnote{Note the similarity with the composition law in the convolution category.}

$$\left( \sum_i \eta_i g_i \right) \left( \sum_j x_j h_j \right) = \sum_{i,j} \eta_j g_i(x_j) g_i h_j$$

There is an inner product

$$D(\mathcal{E})_0(c, a) \times D(\mathcal{E})_0(b, a) \to \text{Hom}(b, c)_{B \ltimes \mathcal{G}}$$

defined by the formula

$$\langle \sum_i \eta_i g_i, \sum_j \xi_j h_j \rangle = \sum_{i,j} g_j^{-1}(\langle \eta_i, \xi_j \rangle) g_i^{-1} h_j$$

Completing the spaces $D(\mathcal{E})_0(b, a)$ with respect to the norms defined by the above inner products we obtain a Hilbert $B \ltimes \mathcal{G}$-module $D(\mathcal{E})(-, a)$. This Hilbert module can be graded by saying that the sum $\sum_i \eta_i g_i$ has degree $k$ if each vector $\eta_i \in \mathcal{E}_a$ has degree $k$.

We can define a graded $C^*$-functor $D(\mathcal{E}) : A \ltimes \mathcal{G} \to \mathcal{L}(B \ltimes \mathcal{G})$ by mapping the object $a \in Ob(\mathcal{G})$ to the Hilbert $B \ltimes \mathcal{G}$-module $D(\mathcal{E})(-, a)$. The action of the $C^*$-category $A \ltimes \mathcal{G}$ is defined by the formula

$$\left( \sum_i x_i g_i \right) \left( \sum_j \eta_j h_j \right) = \sum_{i,j} x_i g_i(\eta_j) g_i h_j$$

The $C^*$-functor $D(\mathcal{E})$ is the desired graded Hilbert $(A \ltimes \mathcal{G}, B \ltimes \mathcal{G})$-bimodule associated to the graded Hilbert $(A, B)$-bimodule $\mathcal{E}$.

Suppose we have a bounded operator $T : \mathcal{E} \to \mathcal{E}'$ between $\mathcal{G}$-equivariant graded Hilbert $(A, B)$-bimodules $\mathcal{E}$ and $\mathcal{E}'$. Then we have an operator $D(T) : D(\mathcal{E}) \to D(\mathcal{E}')$ defined by the formula

$$D(T) \left( \sum_i \eta_i g_i \right) = \sum_i (T \eta_i) g_i$$

The degree of the operator $D(T)$ is the same as that of the operator $T$. A straightforward calculation verifies that if $T$ is a morphism in the category $D_\mathcal{G}(A, B)$, the operator $D(T)$ is a morphism in the category $D(A \ltimes \mathcal{G}, B \ltimes \mathcal{G})$.  

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Similarly, if the operator \( T \) is a morphism in the category \( \mathcal{KD}_\mathcal{G}(A, B) \), the operator \( D(T) \) is a morphism in the category \( \mathcal{KD}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G}) \). We thus have a graded \( \mathcal{C}^* \)-functor
\[
D: \mathcal{QD}_\mathcal{G}(A, B) \to \mathcal{QD}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})
\]
The desired naturality properties are easy to check. \( \square \)

We therefore have a natural map of spectra
\[
D: \mathcal{KK}_\mathcal{G}(A, B) \to \mathcal{KK}(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})
\]
If \( G \) is a discrete group, the above map of spectra induces a map of \( KK \)-theory groups \( D: \text{KK}_\mathcal{G}(A, B) \to \text{KK}(A \rtimes_r G, B \rtimes_r G) \). This induced map is the same as the descent map defined by Kasparov in [14].

A similar construction is possible if we look at full crossed products; we do not need the details here.

6 Assembly

Let \( G \) be a discrete group, and let \( X \) be a \( G \)-space. Let us assume that the group \( G \) acts on the space \( X \) on the right, so the \( \mathcal{C}^* \)-algebra \( \mathcal{C}_0(X) \) is a \( G \)-\( \mathcal{C}^* \)-algebra, with \( G \)-action defined by the formula
\[
(g\varphi)(x) = \varphi(xg)
\]

**Definition 6.1** Let \( X \) be a \( G \)-\( CW \)-complex. Then \( X \) is called a **proper \( G \)-\( CW \)-complex** if for every point \( x \in X \) the isotropy group
\[
G_x = \{ x \in X \mid xg = x \}
\]
is finite.

Note that a \( G \)-\( CW \)-complex is a proper \( G \)-space in the usual sense (see for example [21]) if and only if it is a proper \( G \)-\( CW \)-complex according to the above definition.

Recall that a \( G \)-space \( X \) is called **\( G \)-compact** if the quotient \( X/G \) is compact. Observe that a \( G \)-compact proper \( G \)-\( CW \)-complex must be locally compact. Any proper \( G \)-\( CW \)-complex is the direct limit of its \( G \)-compact subspaces.

**Definition 6.2** Let \( A \) be a \( \sigma \)-unital \( G \)-\( C^* \)-algebra, and let \( X \) be a proper \( G \)-\( CW \)-complex. Then we define the **\( G \)-equivariant \( K \)-homology spectrum** of \( X \) with coefficients in \( A \) to be the direct limit
\[
\mathcal{K}^G_{\text{hom}}(X; A) = \lim_{\longrightarrow} \mathcal{KK}_\mathcal{G}(\mathcal{C}_0(K), A)
\]

Note that the \( C^* \)-algebra \( \mathcal{C}_0(K) \) is \( \sigma \)-unital when \( K \) is a \( G \)-compact proper \( G \)-\( CW \)-complex. According to theorem [5.14] we have a natural map
\[
D: \mathcal{KK}_\mathcal{G}(\mathcal{C}_0(K), A) \to \mathcal{KK}(\mathcal{C}_0(K) \rtimes_r G, A \rtimes_r G)
\]
The $C^*$-algebra $C_0(K) \rtimes_r G$ is itself a finitely generated projective Hilbert $C_0(K) \rtimes_r G$-module. Let us label this module $\mathcal{E}_K$. Then by corollary 5.13 we have an induced map

$$[\mathcal{E}_K] : \mathcal{K}(C_0(K) \rtimes_r G, A \rtimes_r G) \to \mathcal{K}(A \rtimes_r G)$$

Composing these two maps and taking the direct limit we obtain a map

$$\beta : \mathcal{K}_{\text{hom}}(X; A) \to \mathcal{K}(A \rtimes_r G)$$

**Definition 6.3** The map $\beta$ is called the *Baum-Connes assembly map* with *coefficients* in the $G$-$C^*$-algebra $A$.

The assembly map for the Baum-Connes conjecture with coefficients is described in section 9 of [4]. The above definition is simply a version of the standard definition at the level of spectra.

**Definition 6.4** Let $G$ be a discrete group. Then a proper $G$-CW-complex $EG$ is called a *classifying space for proper actions of $G$* if for a given subgroup $H \leq G$ the fixed point set $EG^H$ is contractible if $H$ is finite, and empty if $H$ is infinite.

Note that the classifying space $EG$ always exists, and is unique up to $G$-homotopy equivalence. For details, see [4].

**Definition 6.5** A group $G$ is said to satisfy the *Baum-Connes conjecture with coefficients* in the $G$-$C^*$-algebra $A$ if the Baum-Connes assembly map

$$\beta : \mathcal{K}_{\text{hom}}(EG; A) \to \mathcal{K}(A \rtimes_r G)$$

is a stable equivalence of spectra.

Again, the reader can consult [4] for further details. Note that if $G$ is a finite group, we can take the classifying space $EG$ to be a single point, and the Baum-Connes conjecture for $G$ is true for trivial reasons.

As we explained in the introduction, the main purpose of this article is to generalise the Baum-Connes assembly map in such a way that it fits into the picture described by the following result.

**Theorem 6.6** Let $\mathcal{E}$ be a $G$-homotopy-invariant functor from the category of proper $G$-CW-complexes to the category of spectra. Then there is a $G$-homotopy-invariant excisive functor $\mathcal{E}^\%$ and a natural transformation $\alpha : \mathcal{E}^\% \to \mathcal{E}$ such that the map

$$\alpha : \mathcal{E}^\%(G/H) \to \mathcal{E}(G/H)$$

is a stable equivalence for every finite subgroup, $H$, of the group $G$.

Further, the pair $(\mathcal{E}^\%, \alpha)$ is unique up to weak equivalence.

The above result is a special case of theorem 6.3 in [5]. We call the map $\alpha : \mathcal{E}^\% \to \mathcal{E}$ the *equivariant assembly map* associated to the functor $\mathcal{E}$.

**Definition 6.7** Let $X$ be a $G$-space. Then we write $\underline{X}$ to denote the groupoid in which the collection of objects is that set $X$, and the morphism sets are defined by writing

$$\text{Hom}(x, y)_{\underline{X}} = \{g \in G \mid xg = y\}$$
We regard the groupoid $\mathcal{X}$ as a discrete groupoid; even though there is a topology inherited from the space $X$, we do not want to take this information into account. If $f: X \rightarrow Y$ is an equivariant map between $G$-spaces we have an induced faithful functor $f^*: \mathcal{X} \rightarrow \mathcal{Y}$.

In the literature the groupoid $\mathcal{X}$ is often referred to as the crossed product of $X$ by $G$. We do not adopt this terminology here for fear of confusion with the other crossed products that are present.

There is an obvious natural faithful functor $i: \mathcal{X} \rightarrow G$. If $A$ is a $G$-$C^*$-algebra, then $A$ can also be regarded as a $\mathcal{X}$-$C^*$-algebra; we associate the $C^*$-algebra $A$ to each object of the groupoid $\mathcal{X}$, and the morphism $g: A \rightarrow A$ to the element $g \in \text{Hom}(x,y)_{\mathcal{X}} \subseteq G$.

Now let $X$ be a proper $G$-$CW$-complex. Let $K$ be a $G$-compact subspace of $X$. Then we have an induced restriction map $i^*: KK_G(C_0(K), A) \rightarrow KK_X(C_0(K), A)$.

By proposition 5.14 there is a natural map

$$D: KK_X(C_0(K), A) \rightarrow KK(C_0(K) \rtimes_r \mathcal{X}, A \rtimes_r \mathcal{X})$$

Definition 6.8 Let $x \in X$. We write $\mathcal{E}_K(x)$ to denote the set of collections

$$\{\eta_y \in \text{Hom}(x,y)_{C_0(K) \rtimes_r \mathcal{X}} \mid y \in X\}$$

such that the formula

$$\eta_y g = \eta_z$$

is satisfied for all elements $g \in G$ such that $yg = z$.

The collection of spaces $\mathcal{E}_K(x)$ is a Hilbert $C_0(K) \rtimes_r \mathcal{X}$-module. The $C_0(K) \rtimes_r \mathcal{X}$-action is defined by composition of morphisms. The inner product is defined by the formula

$$\langle \{\eta_y\}, \{\xi_y\}\rangle = \eta^*_y \xi_y$$

for any point $y \in X$.

The Hilbert module $\mathcal{E}_K$ is not in general finitely generated and projective. However, we do have the following result.

Proposition 6.9 The Hilbert $C_0(K) \rtimes_r \mathcal{X}$-module $\mathcal{E}_K$ defines a $K$-theory element $[\mathcal{E}_K] \in K(C_0(K) \rtimes_r \mathcal{X})_0$.

Proof: Given a point $x \in X$, let us write $Or(x)$ to denote the orbit of $x$ in the space $X$ with respect to the action of the group $G$. Observe that $\text{Hom}(x,y)_{C_0(K) \rtimes_r \mathcal{X}} = \{0\}$ unless $Or(x) = Or(y)$. Let $\mathcal{X}|_{Or(x)}$ denote the full subcategory of the groupoid $\mathcal{X}$ in which the objects are the points of the orbit $Or(x)$. The $K$-theory spectrum $K(C_0(K) \rtimes_r \mathcal{X}|_{Or(x)})$ is naturally equivalent to the spectrum

$$K \left( \prod_{Or(x) \in X/G} C_0(K) \rtimes_r \mathcal{X}|_{Or(x)} \right)$$

See proposition 4.19.
by proposition 2.6.
Let $E|_{Or(x)}$ be the Hilbert $C_0(K) \rtimes_r X|_{Or(x)}$-module defined by restricting the Hilbert module $E_K$. Then the Hilbert module $E|_{Or(x)}$ is isomorphic to the module $Hom(-, x)_{C_0(K) \rtimes_r X|_{Or(x)}}$ and so is finitely generated and projective.
Hence the direct sum
$$\prod_{Or(x) \in X/G} E|_{Or(x)}$$
is a finitely generated projective Hilbert module over the $C^*$-category $\prod_{Or(x) \in X/G} C_0(K) \rtimes r X|_{Or(x)}$. It therefore defines a $K$-theory element $[E_K] \in \mathbb{K}(C_0(K) \rtimes_r X)$ as required.

The $K$-theory element $[E_K]$ has the following naturality property.

**Proposition 6.10** Let $f: (X, K) \to (Y, L)$ be a map of pairs of $G$-spaces, where the subspaces $K$ and $L$ are $G$-compact. Let
$$f^*: C_0(L) \rtimes_r X \to C_0(K) \rtimes_r X \quad f_*: C_0(L) \rtimes_r X \to C_0(L) \rtimes_r Y$$
be the obvious induced maps. Then there is a $K$-theory element $[\theta] \in \mathbb{K}(C_0(L) \rtimes_r X)$ such that $f_*[\theta] = [E_L]$ and $f^*[\theta] = [E_K]$.

**Proof:** Let $x \in X$. Let $\theta(x)$ to denote the set of collections
$$\{\eta_y \in Hom(x, y)_{C_0(K) \rtimes_r X} \mid y \in X\}$$
such that the formula
$$\eta_y g = \eta_z$$
is satisfied for all elements $g \in G$ such that $yg = z$.

The collection of spaces $\theta(x)$ is a Hilbert $C_0(L) \rtimes_r X$-module. The $C_0(L) \rtimes_r X$-action is defined by composition of morphisms. The inner product is defined by the formula
$$\langle \{\eta_y\}, \{\xi_y\} \rangle = \eta_y^* \xi_y$$
for any point $y \in X$.

As in proposition 5.9 the Hilbert $C_0(L) \rtimes_r X$-module $\theta$ defines a $K$-theory element $[\theta] \in \mathbb{K}(C_0(L) \rtimes_r X)$. The formulae $f_*[\theta] = [E_L]$ and $f^*[\theta] = [E_K]$ are easy to check.

Now, proposition 5.12 gives us a map
$$[E_K] \wedge: \mathbb{K}K(C_0(K) \rtimes_r X, A \rtimes_r X) \to \mathbb{K}(A \rtimes_r X)$$
We can form the composite
$$\gamma_K: \mathbb{K}K_G(C_0(K), A) \to \mathbb{K}(A \rtimes_r X)$$
and take direct limits to obtain a map
$$\gamma: \mathbb{K}K_{hom}(X; A) \to \mathbb{K}(A \rtimes_r X)$$
Lemma 6.11  The map $\gamma: \mathbb{K}_\text{hom}^G(X; A) \to \mathbb{K}(A \rtimes_r X)$ is natural for proper $G$-CW-complexes $X$.

Proof:  Let $f: (X, K) \to (Y, L)$ be a map of pairs of $G$-spaces, where the subspaces $K$ and $L$ are $G$-compact. Then by proposition 6.10 and naturality of the Kasparov product and descent map, we have a commutative diagram

$$
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(K), A) \\ \downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(G/H), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(G/H) \rtimes_r H, A \rtimes_r H) \\
\downarrow \left[\xi_{G/H}\right]^\wedge \\
\mathbb{K}(A \rtimes_r H)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\ \downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r X, A \rtimes_r X) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r Y, A \rtimes_r Y) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
$$

Taking direct limits, the desired result follows. 

Lemma 6.12  Let $H$ be a finite subgroup of $G$. Then the map $\gamma: \mathbb{K}_\text{hom}^G(G/H; A) \to \mathbb{K}(A \rtimes_r G/H)$ is a stable equivalence of spectra.

Proof:  Let $i: H \to G$ be the inclusion homomorphism. Then by proposition 4.21 the map $\gamma$ is equivalent to the composite

$$
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(G/H), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(G/H), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(G/H) \rtimes_r H, A \rtimes_r H) \\
\downarrow \left[\xi_{G/H}\right]^\wedge \\
\mathbb{K}(A \rtimes_r H)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r X, A \rtimes_r X) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r Y, A \rtimes_r Y) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
$$

Let $j: \mathbb{F} \to C_0(G/H)$ be the inclusion defined by writing $j(\lambda)([1]) = \lambda$ and $j(\lambda)(x) = 0$ if $x \neq [1]$. Let $+$ denote the one point topological space. Then $\mathbb{F} = C_0(+) and we have a commutative diagram:

$$
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(G/H), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(G/H), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(G/H) \rtimes_r H, A \rtimes_r H) \\
\downarrow \left[\xi_{G/H}\right]^\wedge \\
\mathbb{K}(A \rtimes_r H)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r X, A \rtimes_r X) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
\begin{array}{c}
\mathbb{K}\mathbb{K}_G(C_0(L), A) \\
\downarrow i^* \\
\mathbb{K}\mathbb{K}_H(C_0(L), A) \\
\downarrow D \\
\mathbb{K}\mathbb{K}(C_0(L) \rtimes_r Y, A \rtimes_r Y) \\
\downarrow \\
\mathbb{K}(A \rtimes_r Y)
\end{array}
\begin{array}{c}
\mathbb{K}(A \rtimes_r H) \\
\downarrow 1 \\
\mathbb{K}(A \rtimes_r H)
\end{array}
$$
A straightforward calculation tells us that the composite
\( j^* i^*: KK_G(C_0(G/H), A) \to KK_H(C_0(+), A) \) is a stable equivalence of spectra. The composite map on the right, \( \beta: KK_H(C_0(+), A) \to KK(A \rtimes_r H) \), is the Baum-Connes assembly map. Since the group \( H \) is finite, the space + is a model for the classifying space \( EH \) and the map \( \beta \) is a stable equivalence of spectra. Hence the map
\[
\gamma: KK^G_{\text{hom}}(G/H; A) \to KK(A \rtimes_r G/H)
\]
is a stable equivalence as claimed. \( \square \)

The canonical functor \( i: \overline{X} \to G \) gives us an induced map
\[
i_*: KK(A \rtimes_r \overline{X}) \to KK(A \rtimes_r G)
\]
by proposition \( \text{3.16} \).

**Lemma 6.13** The composite map \( i_* \gamma: KK^G_{\text{hom}}(X; A) \to KK(A \rtimes_r G) \) is the Baum-Connes assembly map.

**Proof:** Let \( K \) be a \( G \)-compact subspace of \( X \). The naturality properties of the various descent maps and products give us a commutative diagram
\[
\begin{array}{ccc}
KK_G(C_0(K), A) & \xrightarrow{D} & KK(C_0(K) \rtimes_r G, A \rtimes_r G) \\
\downarrow{\iota^*} & & \downarrow{[\mathcal{E}_K] \wedge}
\downarrow{[\mathcal{E}_K] \wedge} & \downarrow{i_*}

\end{array}
\]

Here the top row is the Baum-Connes assembly map, and the composite \( [\mathcal{E}_K] \wedge \circ D \circ \iota^* \) is the map \( \gamma \). Taking direct limits, the desired result follows. \( \square \)

Observe that the equivariant \( K \)-homology functor \( KK^G_{\text{hom}}(-, A) \) is \( G \)-homotopy-invariant and excisive. We can therefore use the above three lemmas to apply theorem \( \text{6.6} \) to the study of the Baum-Connes assembly map. We immediately obtain the following result.

**Theorem 6.14** Let \( \mathcal{E}^K \) be a \( G \)-homotopy-invariant excisive functor from the category of proper \( G \)-CW-complexes to the category of spectra. Suppose we have a natural transformation \( \alpha: \mathcal{E}^K(X) \to KK(A \rtimes_r \overline{X}) \) such that the map
\[
\alpha: \mathcal{E}^K(G/H) \to KK(A \rtimes_r G/H)
\]
is a stable equivalence for every finite subgroup, \( H \), of the group \( G \).

Then up to stable equivalence the composite \( \iota_* \alpha: \mathcal{E}^K(X) \to KK(A \rtimes_r G) \) is the Baum-Connes assembly map. \( \square \)

We conclude by using the above result to give an alternative description of the Baum-Connes assembly map. To formulate it, we need to introduce one more piece of machinery from \([6]\).
**Definition 6.15** Let $G$ be a discrete group. Then we define the *orbit category*, $\text{Or}(G)$, to be the category in which the objects are $G$-spaces, $G/H$, where $H$ is a subgroup of $G$, and the morphisms are $G$-equivariant maps.

An $\text{Or}(G)$-spectrum is a functor from the category $\text{Or}(G)$ to the category of symmetric spectra. Our main example of an $\text{Or}(G)$-spectrum is defined by writing

$$E(G/H) = \mathbb{K}(A \rtimes_r G/H)$$

where $A$ is a given $G$-$C^*$-algebra.

The following result is proved in [5].

**Theorem 6.16** Let $E$ be an $\text{Or}(G)$-spectrum. Then there is a $G$-homotopy-invariant excisive functor, $E^\%$, from the category of $G$-CW-complexes to the category of spectra such that $E^\%(G/H) = E(G/H)$ whenever $H$ is a subgroup of $G$.

Further, given a functor $F$ from the category of $G$-CW-complexes to the category of spectra, there is a natural transformation

$$\beta: (F|_{\text{Or}(G)})^\% \to F$$

such that the map

$$\beta: (F|_{\text{Or}(G)})^\%(G/H) \to F(G/H)$$

is a stable equivalence for every subgroup, $H$, of the group $G$. \hfill \Box

**Theorem 6.17** Consider the $\text{Or}(G)$-spectrum

$$E(G/H) = \mathbb{K}(A \rtimes_r G/H)$$

Let $X$ be a path-connected space, and let $c: X \to +$ be the constant map. Then up to stable equivalence the induced map

$$c_*: E^\%(X) \to E^\%(+)$$

is the Baum-Connes assembly map.

**Proof:** Consider the functor

$$F: X \mapsto \mathbb{K}(A \rtimes_r X)$$

Then $F|_{\text{Or}(G)} = E$. Let $c: X \to \pi_0(X)$ be the map defined by sending a point of a topological space $X$ to its path-component. By theorem 6.16 we have a commutative diagram

$$\begin{array}{ccc}
E^%(X) & \xrightarrow{\beta} & E(X) \\
\downarrow c_* & & \downarrow c_* \\
E^%\pi_0(X) & \xrightarrow{\beta} & E\pi_0(X)
\end{array}$$

where the map $\beta: E^%(G/H) \to E(G/H)$ is a stable equivalence whenever $H$ is a subgroup of $G$. \hfill \Box

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If the space $X$ is path-connected, by theorem 6.14 the composite $c_*\beta$ is equivalent to the Baum-Connes assembly map. Observe that the map $\beta: E\%(+) \to E(+)\text{ is a stable equivalence.}$

Hence the map $c_*: E\%(X) \to E\%(+)\text{ is equivalent to the Baum-Connes assembly map as claimed.}$

In particular, it follows from the above result that the map

$$c_*: E\%(EG) \to E\%(+)$$

is the Baum-Connes assembly map. This fact is already stated in [5] for the Baum-Connes assembly map \textit{without} coefficients. An alternative proof of the above result, at least in the coefficient-free case, can be found in [8].

In the coefficient-free case, the above theorem is used in [17] to explicitly calculate the rationalisations of the $K$-theory groups $K_n C^*_r (G)$ when $G$ is a discrete group that satisfies the Baum-Connes conjecture. Taking coefficients into account, a similar calculation should be possible to calculate the rationalisations of the groups $K_n (A \rtimes_r G)$.

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