On the pro-$p$-Iwahori invariants of supersingular representations of unramified $U(2,1)$

Peng Xu

Abstract

Let $G$ be the unramified unitary group $U(2,1)(E/F)$ defined over a non-archimedean local field $F$ of odd residue characteristic $p$. In this paper, for any supersingular representation of $G$ that contains the Steinberg weight, we prove its pro-$p$-Iwahori invariants, as a right module over the pro-$p$-Iwahori–Hecke algebra of $G$, is not simple.

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1 Introduction

Let $G$ be the unramified unitary group $U(2,1)(E/F)$ defined over a non-archimedean local field $F$ of odd residue characteristic $p$, and $K$ be a maximal compact open subgroup of $G$. Let $I_K$ be the Iwahori subgroup in $K$, and $I_{1,K}$ be the pro-$p$-Sylow subgroup of $I_K$.

In this paper, we prove:

**Theorem 1.1.** *(Theorem 6.1)* Let $\pi$ be a supersingular representation of $G$ containing the Steinberg weight of $K$. Then, we have

$$\dim_{\mathbb{F}_p} \pi^{I_{1,K}} \geq 2.$$ 

Here, an irreducible smooth $\mathbb{F}_p$-representation of $G$ is supersingular, if it is a quotient of certain spherical universal modules, see 2.3 for the precise definition.

Indeed, what we proved in Theorem 6.1 is conceptually stronger than that stated in Theorem 1.1: as a counterpart to a result of Barthel–Livné on $GL_2$ (Theorem 7.2), we propose a naive conjecture (3.2) on the pro-$p$-Iwahori invariants of supersingular representations of $G$, and we prove it for those containing the Steinberg weight.

**Remark 1.2.** Any spherical universal module of $G$ is infinite dimensional ([Xu16, Corollary 4.6]). Therefore, supersingular representations of $G$ containing a given weight do exist. Moreover, according to the philosophy of Breuil and Paškūnas on $GL_2$ ([BP12, Proposition 10.2]), it is indeed very likely that there are infinitely many non-isomorphic supersingular representations of $G$ containing a given weight.

Our motivation of this paper is to understand the pro-$p$-Iwahori invariants of irreducible smooth $\mathbb{F}_p$-representations of $G$. Let $\mathcal{H}(I_{1,K},1) := \text{End}_G(\text{ind}_{I_{1,K}}^G 1)$ be the pro-$p$-Iwahori–Hecke algebra of $G$. For a smooth $\mathbb{F}_p$-representation $\pi$ of $G$, its pro-$p$-Iwahori invariants $\pi^{I_{1,K}}$ is naturally a right module over $\mathcal{H}(I_{1,K},1)$. On one hand, when $\pi$ is an irreducible subquotient of principal series, we understand it well: $\pi^{I_{1,K}}$ is simple over $\mathcal{H}(I_{1,K},1)$ ([KX15, Corollary 4.4]). On the other hand, when $\pi$ is a supersingular representation, we know very little about $\pi^{I_{1,K}}$. The following corollary deduced from Theorem 1.1, which, besides other things, unveils a new feature of them.

**Corollary 1.3.** *(Corollary 6.5)* Let $\pi$ be a supersingular representation of $G$ containing the Steinberg weight of $K$. Then, the space $\pi^{I_{1,K}}$ as a right module over $\mathcal{H}(I_{1,K},1)$, is not simple.
We now review briefly what is known in the literatures (to our knowledge). For a two dimensional continuous irreducible generic $\mathbb{F}_p$-representation of the absolute Galois group $G_{\mathbb{Q}_p^f}$ ($f > 1$), Breuil and Paškūnas have associated it an infinite family of supersingular representations of $GL_2(\mathbb{Q}_p^f)$ ([BP12, Theorem 1.5]), and one of these supersingular representation has been proved by Emerton–Gee–Savitt ([EGS15]) that it will appear globally, which implies that for such a supersingular representation its pro-$p$-Iwahori invariants is equal to the original recipe used to construct it. These in all suggest\(^1\) that the pro-$p$-Iwahori invariants of supersingular representations of $GL_2(F)$, as right modules over the its pro-$p$-Iwahori–Hecke algebra, might not be simple in general, unless $F = \mathbb{Q}_p$.

Our Corollary 1.3 is in a different manner from the result on $GL_2(\mathbb{Q}_p^f)$ ($f > 1$) mentioned above, and the condition of requiring to contain the Steinberg weight, as remarked above, is likely satisfied by infinitely many non-isomorphic supersingular representations of $G$.

The paper is organized as follows. In section 2, we fix notations and review some preliminaries facts that we will use. In section 3, we propose a naive conjecture on the pro-$p$-Iwahori invariants of supersingular representations of $G$. In section 4, we describe the image of the pro-$p$-Iwahori invariants of a maximal compact induction under certain group operators. In section 6, we prove our main results.

2 Notations and Preliminaries

2.1 General notations

Let $F$ be a non-archimedean local field of odd residue characteristic $p$, with ring of integers $\mathfrak{o}_F$ and maximal ideal $\mathfrak{p}_F$, and $k_F$ be its residue field of cardinality $q = p^f$. Fix a separable closure $F_s$ of $F$. Let $E$ be the unramified quadratic extension of $F$ in $F_s$. We use similar notations $\mathfrak{o}_E$, $\mathfrak{p}_E$, $k_E$ for analogous objects of $E$. Fix a uniformizer $\varpi_E$ in $E$. We equip $E^3$ with the Hermitian form $h$:

$$h : E^3 \times E^3 \to E, (v_1, v_2) \mapsto v_1^T \beta v_2, v_1, v_2 \in E^3.$$ 

Here, $-$ denotes the non-trivial Galois conjugation on $E/F$, inherited by $E^3$, and $\beta$ is the matrix

$$\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}. $$

\(^1\)We heard of this from Prof. Paškūnas several years ago.
The unitary group \( G \) is defined as:

\[
G = \{ g \in \text{GL}(3,E) \mid h(gv_1,gv_2) = h(v_1,v_2), \forall v_1,v_2 \in E^3 \}.
\]

Let \( B = HN \) (resp, \( B' = HN' \)) be the subgroup of upper (resp, lower) triangular matrices of \( G \), with \( N \) (resp, \( N' \)) the unipotent radical of \( B \) (resp, \( B' \)) and \( H \) the diagonal subgroup of \( G \). Denote an element of the following form in \( N \) and \( N' \) by \( n(x,y) \) and \( n'(x,y) \) respectively:

\[
\begin{pmatrix}
1 & x & y \\
0 & 1 & -\bar{x} \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & x & 0 \\
y & -\bar{x} & 1
\end{pmatrix},
\]

where \((x,y) \in E^2\) satisfies \( x\bar{x} + y + \bar{y} = 0 \). For any \( k \in \mathbb{Z} \), denote by \( N_k \) (resp, \( N'_k \)) the subgroup of \( N \) (resp, \( N' \)) consisting of \( n(x,y) \) (resp, \( n'(x,y) \)) with \( y \in p_k^E \). For \( x \in E^\times \), denote by \( h(x) \) an element in \( H \) of the following form:

\[
\begin{pmatrix}
x & 0 & 0 \\
0 & -\bar{x}x^{-1} & 0 \\
0 & 0 & \bar{x}^{-1}
\end{pmatrix}.
\]

Up to conjugacy, the group \( G \) has two maximal compact open subgroups \( K_0 \) and \( K_1 \), given by:

\[
K_0 = \left( \begin{array}{ccc}
\sigma_E & \sigma_E & \sigma_E \\
\sigma_E & \sigma_E & \sigma_E \\
\sigma_E & \sigma_E & \sigma_E
\end{array} \right) \cap G, \quad K_1 = \left( \begin{array}{ccc}
\sigma_E & \sigma_E & p_E^{-1} \\
p_E & \sigma_E & \sigma_E \\
p_E & p_E & \sigma_E
\end{array} \right) \cap G.
\]

The maximal normal pro-\( p \) subgroups of \( K_0 \) and \( K_1 \) are respectively:

\[
K_0^1 = 1 + \varpi_E M_3(\sigma_E) \cap G, \quad K_1^1 = \left( \begin{array}{ccc}
1 + p_E & \sigma_E & \sigma_E \\
p_E & 1 + p_E & \sigma_E \\
p_E & p_E & 1 + p_E
\end{array} \right) \cap G.
\]

Let \( \alpha \) be the following diagonal matrix in \( G \):

\[
\begin{pmatrix}
\varpi_E^{-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \varpi_E
\end{pmatrix},
\]

and put \( \beta' = \beta \alpha^{-1} \). Note that \( \beta \in K_0 \) and \( \beta' \in K_1 \). We use \( \beta_K \) to denote the unique element in \( K \cap \{ \beta, \beta' \} \).

Let \( K \in \{ K_0, K_1 \} \), and \( K^1 \) be the maximal normal pro-\( p \) subgroup of \( K \). We identify the finite group \( \Gamma_K = K/K^1 \) with the \( k_F \)-points of an algebraic group defined over \( k_F \), denoted also by \( \Gamma_K \) when \( K \) is \( K_0 \), \( \Gamma_K \)
is $U(2,1)(k_E/k_F)$, and when $K$ is $K_1$, $\Gamma_K$ is $U(1,1) \times U(1)(k_E/k_F)$. Let $\mathbb{B}$ (resp, $\mathbb{B}'$) be the upper (resp, lower) triangular subgroup of $\Gamma_K$, and $U$ (resp, $U'$) be its unipotent radical.

The Iwahori subgroup $I_K$ (resp, $I'_K$) and pro-$p$ Iwahori subgroup $I_{1,K}$ (resp, $I'_{1,K}$) in $K$ are the inverse images of $\mathbb{B}$ (resp, $\mathbb{B}'$) and $U$ (resp, $U'$) in $K$. Recall we have the following Bruhat decomposition for $K$:

$$K = I_K \cup I_K \beta K I_K.$$

Put $H_0 = H \cap I_K$, and $H_1 = H \cap I_{1,K}$. As $H_0/H_1 \cong I_K/I_{1,K}$, we will identify the characters of these groups. For a character $\chi$ of $H_0$, i.e., a character of $H_0$, denote by $\chi^s$ the character given by $\chi^s(h) := \chi(\beta_K h \beta_K)$.

Denote by $n_K$ and $m_K$ the unique integers such that $N \cap I_{1,K} = N_{n_K}$ and $N' \cap I_{1,K} = N'_{m_K}$. We have:

$$n_K + m_K = 1.$$

Note that the coset spaces $N_{n_K}/N_{n_K+1}$ and $N'_{m_K}/N'_{m_K+1}$ are indeed groups of order respectively $q^{t_K}$ and $q^{4-t_K}$, for $t_K = 3$ or $1$, depending on $K$ is hyperspecial or not.

We will use the following group:

$$L_{q^3} := \{ (x, t) \in k_E^2 \mid x \bar{x} + t + \bar{t} = 0 \},$$

and its central subgroup:

$$L_q := \{ (0, t) \mid t + \bar{t} = 0 \}.$$

Here, the group structure of $L_{q^3}$ is given by

$$(x, t) \cdot (x', t') := (x + x', t + t' - x' \bar{x}).$$

We may identify these groups naturally:

$$L_{n_K} : N_{n_K}/N_{n_K+1} \cong L_{q^{t_K}}$$

and

$$L_{m_K} : N'_{m_K}/N'_{m_K+1} \cong L_{q^{4-t_K}}$$

Here, the elements $x$ and $t$ on the left hand side lie in $\mathfrak{o}_E$.

All the representations of $G$ and its subgroups considered in this paper are smooth over $\overline{\mathbb{F}}_p$. 

5
2.2 The spherical Hecke algebra $\mathcal{H}(K, \sigma)$

Let $K$ be a maximal compact open subgroup of $G$, and $(\sigma, W)$ be an irreducible smooth representation of $K$. As $K^1$ is pro-$p$ and normal, $\sigma$ factors through the finite group $\Gamma_K = K/K^1$, i.e., $\sigma$ is the inflation of an irreducible representation of $\Gamma_K$. Conversely, any irreducible representation of $\Gamma_K$ inflates to an irreducible smooth representation of $K$. We may therefore identify irreducible smooth representations of $K$ with irreducible representations of $\Gamma_K$, and we shall call them weights of $K$ or $\Gamma_K$ from now on.

It is known that $\sigma^{I_1, K}$ and $\sigma^{I_1', K}$ are both one-dimensional, and that the natural composition map $\sigma^{I_1, K} \rightarrow \sigma \rightarrow \sigma^{I_1', K}$ is an isomorphism of vector spaces ([CE04, Theorem 6.12]). Denote by $j_\sigma$ the inverse of the map aforementioned. For $v \in \sigma^{I_1, K}$, we have $j_\sigma(\bar{v}) = v$, where $\bar{v}$ is the image of $v$ in $\sigma^{I_1', K}$. By composition, we view $j_\sigma$ naturally a map in $\text{End}_{F_p}(\sigma)$.

Let $\text{ind}_G^K \sigma$ be the smooth representation of $G$ compactly induced from $\sigma$, i.e., the representation of $G$ with underlying space $S(G, \sigma)$

$$S(G, \sigma) = \{ f : G \rightarrow W \mid f(kg) = \sigma(k) \cdot f(g), \forall k \in K, g \in G, \text{ smooth with compact support} \}$$

and $G$ acting by right translation.

The spherical Hecke algebra $\mathcal{H}(K, \sigma)$ is defined as $\text{End}_G(\text{ind}_G^K \sigma)$. By [BL94, Proposition 5], it is isomorphic to the convolution algebra $\mathcal{H}_K(\sigma)$:

$$\mathcal{H}_K(\sigma) = \{ \varphi : G \rightarrow \text{End}_{F_p}(\sigma) \mid \varphi(kgh) = \sigma(k)\varphi(g)\sigma(h), \forall k, h \in K, g \in G, \text{ smooth with compact support} \}$$

Let $\varphi$ be the function in $\mathcal{H}_K(\sigma)$, supported on $K\alpha K$ and satisfying $\varphi(\alpha) = j_\sigma$. Denote by $T$ the Hecke operator in $\mathcal{H}(K, \sigma)$, which corresponds to the function $\varphi$ via the isomorphism aforementioned between $\mathcal{H}_K(\sigma)$ and $\mathcal{H}(K, \sigma)$. We refer the readers to [Xu16, (4)] for a formula of $T$.

The structure of the algebra $\mathcal{H}(K, \sigma)$ is well-understood, and it is isomorphic to $\mathcal{F}_p[T]$ ([Her11b, Corollary 1.3]).

2.3 Definition of supersingular representations

Let $\sigma$ be a weight of $K$, and $\chi_\sigma$ be the character of $I_K$ for its action on the line $\sigma^{I_1, K}$.

2.3.1 Hecke eigenvalues of principal series for $T$

In this part, we follow along the lines in [BL95] to compute the Hecke eigenvalue of a general principal series of $G$ for $T$. 

6
Lemma 2.1. For a character \( \varepsilon \) of \( B \), the space \( \text{Hom}_G(\text{ind}_K^G \sigma, \text{ind}_B^G \varepsilon) \) is at most one dimensional, and it is non-zero if and only if

\[ \varepsilon |_{H \cap K} = \chi_\sigma^s. \]

Proof. As we have an Iwasawa decomposition \( G = BK \), the argument is identical to that of [BL94, Proposition 23]. \( \square \)

When the condition in Lemma 2.1 is satisfied, the space

\[ \text{Hom}_G(\text{ind}_K^G \sigma, \text{ind}_B^G \varepsilon) \]

is one dimensional, and thus it affords a character of the spherical Hecke algebra \( H(K, \sigma) \).

Proposition 2.2. The eigenvalue of \( T \) on the space \( \text{Hom}_G(\text{ind}_K^G \sigma, \text{ind}_B^G \varepsilon) \) is:

\[ \varepsilon(\alpha) + \sum_{(x,t) \in L_q^* \setminus \Gamma_K} \chi_\sigma(h(t)) \]

Proof. The argument of [Xu14, Proposition 3.24] for \( K_0 \) can be slightly modified to work for any \( K \). \( \square \)

We will say the weight \( \sigma \) is degenerate, if

\[ \sum_{(x,t) \in L_q^* \setminus \Gamma_K} \chi_\sigma(h(t)) \neq 0, \]

Otherwise, we say \( \sigma \) is regular. Note that when \( \chi_\sigma = 1 \), it is trivial to see the above sum is equal to \(-1\).

Remark 2.3. The value of the sum \( \sum_{(x,t) \in L_q^* \setminus \Gamma_K} \chi_\sigma(h(t)) \) appearing in Proposition 2.2 is known, and the readers are referred to [KX15, Appendix A] for a full list of it. In terms of loc.cit., the degenerate case will only happen in two situations:

1). \( \chi_\sigma \) is of the trivial type.
2). \( \chi_\sigma \) is hybrid and \( K \) is hyperspecial.

When \( \sigma \) is degenerate, we have (loc.cit):

\[ \sum_{(x,t) \in L_q^* \setminus \Gamma_K} \chi_\sigma(h(t)) = -\chi_\sigma(h(t)). \]
2.3.2 The Hecke operator $T_{\sigma}$

On account of Proposition 2.2, we consider $T_{\sigma} \in \mathcal{H}(K, \sigma)$ given by:

$$T_{\sigma} := T - \sum_{(x,t) \in L_{q^4 - 1}^\times} \chi_\sigma(h(t))$$

Definition 2.4. An irreducible smooth representation $\pi$ of $G$ is called supersingular if it is a quotient of $\text{ind}_K^G \sigma/(T_{\sigma})$, for some weight $\sigma$ of $K$.

Remark 2.5. Recall from [AHHV17, III, Part A], an irreducible admissible smooth representation $\pi$ of a $p$-adic reductive group $G$ is supersingular, if it contains a weight $\sigma$ (w.r.t a special parahoric subgroup $K$), satisfying that the space $\text{Hom}_G(\text{ind}_K^G \sigma, \pi)$ admits a supersingular Hecke eigenvalue for the center $Z_G(\sigma)$ of the spherical Hecke algebra $\mathcal{H}(K, \sigma)$. Here, the assumption of admissibility, as far as we understand, can be replaced by the existence of Hecke eigenvalue. In our case $G = U(2,1)$,

a). the existence of Hecke eigenvalue is proved in [Xu18].

b). a character of $\mathcal{H}(K, \sigma)$ is supersingular if and only if it kills the maximal ideal $(T_{\sigma})$ ([AHHV17, III 4, Corollary]).

2.4 The space $(\text{ind}_K^G \sigma)^{I_{1,K}}$

Let $\sigma$ be a weight of $K$. Fix a non-zero vector $v_0 \in \sigma^{I_{1,K}}$. Let $f_n$ be the function in $(\text{ind}_K^G \sigma)^{I_{1,K}}$, supported on $K\alpha^{-n}I_{1,K}$, such that

$$f_n(\alpha^{-n}) = \begin{cases} \beta_K \cdot v_0, & n > 0, \\ v_0 & n \leq 0. \end{cases}$$

Then, we have the following ([Xu16, Lemma 3.5])

Lemma 2.6. The set of functions $\{f_n \mid n \in \mathbb{Z}\}$ consists of a basis of the $I_{1,K}$-invariants of $\text{ind}_K^G \sigma$.

3 $\dim_{\mathbb{F}_p} \pi^{I_{1,K}} \geq 2$ for supersingular $\pi$: a conjecture

3.1 Some motivational remarks

The simple modules of the pro-$p$-Iwahori–Hecke algebra $\mathcal{H}(I_{1,K}, 1)$ of $G$ were classified in [KX15], and they are either one or two dimensional. The so-called simple supersingular modules are all one-dimensional characters,
and all non-supersingular simple modules arise from the $I_{1,K}$-invariants of irreducible subquotients of principal series. The goal of this paper is to prove that the subspace of $I_{1,K}$-invariants of certain supersingular representation $\pi$ is at least two dimensional.

To address our result, we remind the readers of a simple but important fact from classical case (i.e., complex case). The following is [BH06, Proposition 4.3, (2)]:

**Proposition 3.1.** For a locally profinite group $\mathcal{G}$ and a compact open subgroup $K$, the process $\pi \mapsto \pi^K$ induces a bijection between the following two sets:

a). equivalence classes of irreducible smooth representations $\pi$ of $\mathcal{G}$ such that $\pi^K \neq 0$.

b). isomorphic classes of simple $\mathcal{H}(K,1)$-modules.

As far as we know, the pro-$p$-Iwahori invariants of supersingular representations are only fully understood for the group $GL_2(Q_p)$ ([Bre03]) and some closely related cases, and in these cases the pro-$p$-Iwahori invariants are still simple modules; beyond $GL_2(Q_p)$, very little is known. For the group $GL_2(Q_p^f)$ ($f > 1$), certain supersingular representation constructed by Breuil and Paskunas in [BP12] has been proved in the recent work of Emerton–Gee–Savitt ([EGS15]) that it appears in the cohomology of a suitable Shimura curve, and that its pro-$p$-Iwahori invariants matches the original recipe used to construct it. To our knowledge, that is the only case beyond $GL_2(Q_p)$ for a single supersingular representation its pro-$p$-Iwahori invariants is known. In all, these work provide the first examples that the analogue of Proposition 3.1 above fail in the $p$-modular representation theory.

Our main input (Theorem 6.1) proved in this paper, combined something natural, that is the pro-$p$-Iwahori invariants of an admissible supersingular representation only admits supersingular subquotient $^2$, shows that the analogue of Proposition 3.1 fails wildly for supersingular representations of $G$, or possibly always fails.

### 3.2 $\dim_{\mathbb{F}_p} \pi^{I_{1,K}} \geq 2$ for any supersingular $\pi$?

We propose the following:

**Conjecture 3.2.** Let $\sigma$ be a weight of $K$. Then, for any irreducible quotient $\pi$ of $\text{ind}^G_K \sigma/(T_\sigma)$, the images of the functions $f_0$ and $f_1$ in $\pi$ are not proportional.

---

$^2$This is indeed a theorem of Ollivier–Vigneras [OV17] now, if one assumes admissibility.
We end this part by some general but more related comments.

1). The images of the functions $f_0$ and $f_1$ are linearly independent in $\text{ind}_G^K \sigma/(T_\sigma)$ ([Xu17, Corollary 2.10]).

2). An analogue of Conjecture 3.2 for $GL_2$ holds and is indeed implicitly verified in [BL94]. See Theorem 7.2 in Section 7.

3). Our strategy to prove Conjecture 3.2 is naturally by contradiction. Assume there is some $c \in \overline{F}_p^\times$ such that

$$\text{the image of } f_0 + c \cdot f_1 \text{ in } \pi \text{ is zero.}$$

Then starting from such a function we proceed to generate more functions whose images in $\pi$ are still zero, and when $\sigma$ is the Steinberg weight we find a contradiction after we take some care in the process.

4 The images of $(\text{ind}_G^K \sigma)^{I_{1,K}}$ under $S_K$ and $S_-$

For a smooth representation $\pi$ of $G$, we have introduced two partial linear maps $S_K$ and $S_-$ in [Xu17, subsection 4.3] as follows:

$$S_K : \pi^{N_{m_K}} \to \pi^{N_{n_K}},$$
$$v \mapsto \sum_{u \in N_{n_K}/N_{n_K}+1} u\beta_K v.$$

$$S_- : \pi^{N_{n_K}} \to \pi^{N_{m_K}},$$
$$v \mapsto \sum_{u' \in N_{m_K}/N_{m_K}+1} u'\beta_K^{-1} v.$$

We have proved these two maps satisfy the following nice property:

**Proposition 4.1.** If $v \in \pi^{I_{1,K}}$, then it is the same for $S_K v$ and $S_- v$.

*Proof.* This is [Xu17, (2) of Proposition 4.10]. \hfill \Box

Our main goal in this section is to apply the maps $S_K$ and $S_-$ to the space $(\text{ind}_G^K \sigma)^{I_{1,K}}$ and determine explicitly the images. It is done in the following Proposition.

**Proposition 4.2.** We have:

1. For $n \geq 1$,
$$S_K f_n = f_{-n}, \quad S_- f_n = c_- f_n.$$

Here, the constant $c_-$ is given by:
$$c_- = \sum_{(x,t) \in L_q^{L_{\chi}}} \chi_\sigma(h(t)).$$
Here, the constant $d_n$ $(n \geq 1)$ is given by:

$$d_n = \sum_{(x,t) \in L_{\mathcal{H}^n}^\times} \chi_\sigma((h(t)),$$

and the constant $d_0$ is equal to:

$$d_0 = \begin{cases} -\chi_\sigma(h(1)), & \text{if } \sigma \cong \text{ a twist of the Steinberg weight;} \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** We will prove $S_K f_n = f_n$ for $n \geq 1$ and $S_- f_n = f_{n+1}$ for $n \geq 0$ at first.

For $n \geq 1$, the support of the function $S_K f_n$ is contained in:

$$K \alpha^{-n} I_{1,K} \beta_K N_{m_K} = K \alpha^n I_{1,K}.$$

Then, by Proposition 4.1 and [Xu16, Remark 3.8], the function $S_K f_n$ is proportional to $f_n$. We compute:

$$S_K f_n(\alpha^n) = \sum_{u \in N_{m_K}/N_{m_K}^{n+1}} f_n(\alpha^n u \beta_K) = f_n(\alpha^n \beta_K) = v_0,$$

where we note that $\alpha^n u \beta_K \in K \alpha^n I_{1,K}$, for $u \in N_{m_K} \setminus N_{m_K}^{n+1}$ ([Xu17, Lemma 3.7 (3)]). Hence, we have verified $S_K f_n = f_n$ for $n \geq 1$.

For $n \geq 0$, the support of the function $S_- f_n$ is contained in

$$K \alpha^n I_{1,K} \beta_K \alpha^{-1} N_{m_K}' = K \alpha^{-(n+1)} I_{1,K}.$$

By Proposition 4.1 and [Xu16, Remark 3.8] again, the function $S_- f_n$ is proportional to $f_{n+1}$. We compute:

$$S_- f_n(\alpha^{-(n+1)}) = \sum_{u' \in N_{m_K}'/N_{m_K}'^{n+1}} f_n(\alpha^{-(n+1)} u' \alpha \beta_K) = \beta_K v_0,$$

where we note that $\alpha^{-(n+1)} u' \alpha \beta_K \in K \alpha^{n+1} I_{1,K}$, for $u' \in N_{m_K}' \setminus N_{m_K}'^{n+1}$ ([Xu17, Lemma 3.7 (2), (3)]). Thus, we have verified $S_- f_n = f_{n+1}$, for $n \geq 0$.

We proceed to prove $S_- f_n = c_- f_n$ for $n \geq 1$; actually we will determine the value of $c_-\alpha$ explicitly. The support of the function $S_- f_n$ is contained in

$$K \alpha^{-n} I_{1,K} \alpha \beta_K N_{m_K}' \subseteq K \alpha^{n-1} I_{1,K} \cup K \alpha^n I_{1,K},$$

where the inclusion follows from [Xu17, Lemma 3.7 (1), (3)]. We conclude that $S_- f_n \in \langle f_{-(n-1)}, f_n \rangle$ by Proposition 4.1 and [Xu16, Remark 3.8]. We compute:

$$S_- f_n(\alpha^{-n}) = \sum_{u' \in N_{m_K}'/N_{m_K}'^{n+1}} f_n(\alpha^{-n} u' \alpha \beta_K) = \sum_{u' \in N_{m_K}'/N_{m_K}'^{n+1}} v_0 = 0.$$
It remains to compute $S_f(-n)$:

$$S_f(-n) = \sum_{u \in \mathbb{N}_{m_K} / \mathbb{N}_{m_K+1}} f_n(\alpha^n u' \alpha \beta_K).$$

Note that $\alpha^n u' \alpha \beta_K \in K \alpha^{n-1} I_{1,K}$ for $u' \in \mathbb{N}_{m_K+1}$, and we are reduced to

$$S_f(-n) = \sum_{u' \in \mathbb{N}_{m_K'} \setminus \mathbb{N}_{m_K+1}} f_n(\alpha^n u' \alpha \beta_K).$$

For a $u' = n'(\ast, \omega^{nk} t^l)$ for some $t \in \mathfrak{O}_E^k$, we have (using [Xu16, (1)])

$$\alpha^n u' \alpha \beta_K = n(\ast, \omega^{2n-1+nk} t^{-1}) h(t) \alpha^n n'(\ast, \omega^{nk} t^{-1}).$$

Thus, we immediately get:

$$S_f(-n) = (\sum_{(x,t) \in L_{q^{-1}K}} \chi_{\sigma(h(t))}) \beta_K v_0,$$

here we have identified the group $\mathbb{N}_{m_K'} / \mathbb{N}_{m_K+1}$ with $L_{q^{-1}K}$, via the map $L_{m_K}$ in the introduction.

Hence, we get

$$c_\ast = \sum_{(x,t) \in L_{q^{-1}K}} \chi_{\sigma(h(t))}.$$

We move to deal with the last statement: $S_K f_n = d_n f_n$, for $n \geq 0$. The support of the function $S_K f_n$ is contained in

$$K \alpha^n I_{1,K} \beta_K N_{n_K} \subseteq K \alpha^n K$$

By [Xu16, Remark 3.8], we get:

- when $n = 0$, $S_K f_0 = \langle f_0 \rangle$;
- when $n > 0$, $S_K f_n = \langle f_n, f_n \rangle$.

We consider the second case at first. Assume $n > 0$. We compute:

$$S_K f_n(\alpha^n) = \sum_{u \in \mathbb{N}_{n_K} \setminus \mathbb{N}_{n_K+1}} f_n(\alpha^n u \beta_K) = \sum_{u \in \mathbb{N}_{n_K} \setminus \mathbb{N}_{n_K+1}} \beta_K v_0 = 0.$$

Next, we compute $S_K f_n(\alpha^n)$:

$$S_K f_n(\alpha^n) = \sum_{u \in \mathbb{N}_{n_K} \setminus \mathbb{N}_{n_K+1}} f_n(\alpha^n u \beta_K).$$

Note that $\alpha^n u \beta_K \in K \alpha^n N_{n_K}$, for $u \in \mathbb{N}_{n_K+1}$. We are thus reduced to:

$$S_K f_n(\alpha^n) = \sum_{u \in \mathbb{N}_{n_K} \setminus \mathbb{N}_{n_K+1}} f_n(\alpha^n u \beta_K).$$

For $u = n(\ast, \omega^{nk} t^l)$, for some $t \in \mathfrak{O}_E^k$, we have (using [Xu16, (1)])

$$\alpha^n u \beta_K = n'(\ast, \omega^{2n+1+nk} t^{-1}) h(t) \alpha^n n'(\ast, \omega^{nk} t^{-1}).$$

Thus, we get

$$S_K f_n(\alpha^n) = (\sum_{(x,t) \in L_{q^{-1}K}} \chi_{\sigma(h(t))}) v_0,$$

and we are reduced to
here we have identified the group $N_{n_K}/N_{n_K+1}$ with $L_{q^K}$, via the map $L_{n_K}$ in the introduction. Hence, we get:

$$d_n = \sum_{(x,t) \in L_{q^K}} \chi_\sigma((h(t))$$

**Remark 4.3.** The exact values of $c_-$ and $d_n$ ($n \geq 1$) depend on the nature of the character $\chi_\sigma$, and they have already been computed explicitly in [KX15, Appendix A]. Note that $c_-$ is the same sum we have used in subsection 2.3.

We still need to compute the constant $d_0$ appearing in $S_K f_0 = d_0 f_0$. By definition, the constant $d_0$ is determined by

$$\sum_{u \in N_{n_K}/N_{n_K+1}} u\beta_K v_0 = d_0 v_0. \tag{1}$$

We recall some stuff from [KX15, section 5]:

1). (Definition 5.2 of loc.cit)
   To any character $\chi$ of $H_0/H_1$, a subset $J_K(\chi) \subset \{s\}$ is attached.
2). (Definition 5.3 of loc.cit)
   For any subset $J \subset J_K(\chi)$, one defines a character $M_{\chi,J}$ of the finite Hecke algebra $H_{\Gamma_K} := \text{End}_{\Gamma_K}(\text{Ind}_{U}^1 \{1\})$.  
3). (Proposition 5.4 of loc.cit)
   Every simple module of the algebra $H_{\Gamma_K}$ is isomorphic to $M_{\chi,J}$ for some character $\chi$ of $H_0/H_1$ and some $J \subset J_K(\chi)$.  
4). (Proposition 5.5 of loc.cit)
   The functor $\sigma \to \sigma^U$ gives a bijection of isomorphism classes of irreducible representations of $\Gamma_K$ and isomorphism classes of simple right $H_{\Gamma_K}$-modules.

By 4) above, we write our $\sigma$ as $\sigma_{\chi,J}$ such that:

$$\sigma^U \cong M_{\chi,J},$$

for some $J \subset J_K(\chi_\sigma)$. Then, by comparing (1) and the right action of $H_{\Gamma_K}$ ([KX15, 3.1, (1)]) on $\sigma^U$, we see immediately that

$$d_0 = M_{\chi,J}(T_{\beta_K}),$$

where $T_{\beta_K}$ is the Hecke operator in $H_{\Gamma_K} \hookrightarrow H(I_{1,K},1)$ which corresponds to the double coset $I_{1,K} \beta_K I_{1,K}$. By the identification in [KX15, Proposition 5.7], our statement for the value of $d_0$ now follows from the lists in Definition 5.3 of loc.cit:

$$d_0 = \begin{cases} -\chi_\sigma(h(t)), & \text{if } \sigma \cong \text{a twist of the Steinberg weight} \\ 0, & \text{otherwise} \end{cases}$$
Here, we note that the element $\beta_K$ is different from a normalized one (i.e., with determinant 1) used in [KX15] by exactly the diagonal matrix $h(t)$.

**Remark 4.4.** Our argument for the value of $d_0$ in the Proposition is nearly formal and works for any $\sigma$. But we point out:

1. It is trivial to see $d_0 = 0$ if $\sigma$ is a character.
2. In the only case that $d_0$ is non-zero, i.e., $\sigma$ is a twist of the Steinberg weight, we can work it out by hand without referring to [KX15].

## 5 $\overline{f_1} \neq 0$ in the degenerate case

When the weight $\sigma$ is degenerate, the Hecke operator $T_\sigma$ is different from $T$ (Definition 2.4). Because of that, we may verify with ease that the image of the function $f_1$ in a supersingular $\pi$ is non-zero.

**Proposition 5.1.** Assume $\sigma$ is degenerate, and $\pi$ is an irreducible quotient of $\text{ind}_{G_K}^{G}\sigma/(T_\sigma)$. Then, we have $\overline{f_1} \neq 0$.

**Proof.** Recall that when $\sigma$ is degenerate, we have

$$T_\sigma = T + \chi_\sigma(h(t)).$$

By the assumption, the image of the function $T_\sigma f_0$ in $\pi$ is zero. By [Xu16, Proposition 3.6], we have

$$T_\sigma f_0 = f_{-1} + \lambda_{\beta_K,\sigma} f_1 + \chi_\sigma(h(t)) f_0,$$

where $\lambda_{\beta_K,\sigma}$ is some constant ([Xu17, Remark 2.1]). If the image of $f_1$ in $\pi$ is zero, it is the same for the function $f_{-1}$, as by definitions (or by (1) of Proposition 4.2) we have:

$$f_{-1} = S_K f_1.$$

In all, if $\overline{f_1} = 0$, then we get $\overline{\chi_\sigma(h(t)) f_0} = 0$, i.e., $\overline{f_0} = 0$, as $\chi_\sigma(h(t)) \neq 0$. We get a contradiction, as the function $f_0$ generates $\text{ind}_{K}^{G}\sigma$, and its image in the irreducible representation $\pi$ can not be zero. □

**Remark 5.2.** We have indeed verified that the image of $f_1$ is non-zero in any irreducible quotient of $\text{ind}_{K}^{G}\sigma/(T - \lambda)$ with $\lambda \neq 0$.

**Remark 5.3.** For a supersingular representation $\pi$ containing a regular weight $\sigma$, we believe $\overline{f_1}$ is still non-zero in $\pi$. But it seems challenging (to the author) to prove it at this stage.
6 Proof of Theorem 1.1 and Corollary 1.3

**Theorem 6.1.** Assume $\sigma$ is the Steinberg weight. Then Conjecture 3.2 holds.

**Proof.** Assume Conjecture 3.2 fails, and there is a non-zero $c \in \mathbb{F}_p^\times$ such that:

the image $f(c)$ of the function $f(c) := f_0 + c \cdot f_1$ in $\pi$ is zero.

Consider the function $f'(c) := S_K f(c)$, which by Proposition 4.2 is equal to

$$-f_0 + c \cdot f_{-1},$$

where we note that $d_0 = -1$ in the current case.

Next, we consider the function $f''(c) := S_{-} f'(c)$, which, by Proposition 4.2 again, is equal to

$$-f_1 + c \cdot f_2$$

Now, by [Xu16, Corollary 3.11], we have $T f_1 = f_2$ \footnote{Recall we have proved in loc. cit that $T f_m = c_m f_m + f_{m+1}$ for any $m \geq 1$, even the value of the constant $c_m$ is not recorded explicitly there; it is zero if $\dim_{\mathbb{F}_p} \sigma > 1$ (using [Xu17, Remark 2.1, Lemma 3.7]); when $\sigma$ is a character, it is equal to $\sum_{(x,t) \in L_{q-1}} \chi_\sigma(h(t))$ (Remark 2.3).}. Recall that in the current case $T_{\sigma} = T + 1$ (Definition 2.4). Putting all these together, we see

$$f''(c) = (-c - 1) f_1 + c \cdot (T + 1) f_1$$

As $f''(c) = (T + 1) f_1 = 0$, we get

$$(-c - 1) f_1 = 0.$$

Thus, we must have $c = -1$, as $f_1 \neq 0$ (Proposition 5.1).

Now we return to consider the function $S_{-} f(-1)$, which, by Proposition 4.2, is equal to

$$f_1 + f_1 = 2 f_1,$$

where we note that in the current case $c_{-}$ is clearly equal to $-1$. We then get a contradiction, as $2 f_1 \neq 0$ in $\pi$ ($p \neq 2$ !).

**Remark 6.2.** Note that in Theorem 6.1 we don’t put any restriction on the field $F$, e.g., it is not necessary to be characteristic 0. But we do have used the assumption $p \neq 2$ !
Remark 6.3. By Theorem 6.1, Theorem 1.1 follows immediately. As the kernel of any character of $G$ contains the group $I_{1,K}$, Theorem 1.1 can be slightly extended to any supersingular representation of $G$ that contains a twist of the Steinberg weight by a character extendable to $G$.

Remark 6.4. We remark briefly the cases beyond Theorem 6.1:

a). For a supersingular representation $\pi$ containing a regular weight $\sigma$, we can also prove $f_0$ and $f_1$ are not proportional in $\pi$, if we know $f_1 \neq 0$ in $\pi$.

b). It seems the strategy we use to prove Theorem 6.1 might not simply work for a weight like the trivial character of $K$. But it seems one may still be able to conclude an analogue of Theorem 1.1 for the trivial weight case from Theorem 1.1, by changing of weight in this setting (see [Her11a, Example 6.14], or [Sch14, Corollary 2.17]).

Corollary 6.5. Let $\pi$ be a supersingular representation of $G$ containing the Steinberg weight of $K$. Then, the space $\pi^{I_{1,K}}$ as a right module over $\mathcal{H}(I_{1,K},1)$, is not simple.

Proof. As simple modules of the pro-$p$-Iwahori–Hecke algebra $\mathcal{H}(I_{1,K},1)$ are finite dimensional, for our purpose we may assume $\dim \pi^{I_{1,K}} < \infty$. By [KX15, 4.2], any supersingular module of $\mathcal{H}(I_{1,K},1)$ is a character. Thanks to Ollivier–Vignéras [OV17, Theorem 5.3, Remark 5.2 (e)], we know the pro-$p$-Iwahori invariants of an admissible supersingular representation only admits supersingular subquotients. Now the corollary follows from Theorem 6.1. □

7 Appendix A: some remarks on $GL_2(F)$

In this appendix, we point out that an analogue of Conjecture 3.2 is true for $GL_2(F)$, due to Barthel–Livné. Our notations in this appendix are independent from other parts of this paper.

Let $F$ be a non-archimedean local field, with ring of integers $\mathfrak{o}_F$ and maximal ideal $p_F$, and let $k_F$ be its residue field of characteristic $p$. Let $G = GL_2(F)$, $K = GL_2(\mathfrak{o}_F)$ the maximal compact open subgroup of $G$, $Z = F^\times$ the center of $G$. Let $I$ be the standard Iwahori subgroup of $G$, and $I_1$ be the pro-$p$-Sylow subgroup of $I$. Let $K_1$ be the first congruence subgroup of $K$, so that $K/K_1 \cong GL_2(k_F)$. Fix a uniformizer $\varpi$ in $F$. Denote by $\alpha$ and $\beta$ respectively the following two matrices:

\[\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}\]

\footnote{We would like to thank Karol Kozioł for pointing out this.}
Put $\gamma = \alpha \beta$. Note that $\gamma^2 = \varpi I_2$ and $\gamma$ normalizes $I_1$.

Let $\sigma$ be an irreducible representation of $GL_2(k_F)$ over $\overline{F}_p$, inflated to $K = GL_2(o_F)$ via the isomorphism $K/K_1 \cong GL_2(k_F)$. Conversely, any irreducible smooth representation of $K$ over $\overline{F}_p$ arises from such an inflation. Nowadays, such a representation $\sigma$ is usually called a weight of $GL_2(k_F)$ or $K$. We extend $\sigma$ to a representation of $KZ$ by requiring $\varpi$ acts trivially.

An irreducible smooth $\overline{F}_p$-representation of $G$ is called supersingular if it is a quotient of $\text{ind}_{KZ}^G \sigma / (T)$, for a weight $\sigma$ of $K$, and for certain Hecke operator $T$ in the spherical Hecke algebra $\mathcal{H}(KZ, \sigma)$ ([BL94, 3.1]).

Fix a non-zero vector $v_0 \in \sigma I_1$. For $n \in \mathbb{Z}$, let $\psi_n$ be the function in $(\text{ind}_{KZ}^G \sigma)^{I_1}$, supported on $KZ \alpha^{-n} I_1$, such that

$$\psi_n(\alpha^{-n}) = \begin{cases} 
\beta \cdot v_0, & n > 0, \\
v_0, & n \leq 0.
\end{cases}$$

**Remark 7.1.** Up to a scalar, the functions $\psi_n$ defined above coincide that in [BL94, section 4]. One can simply check

$$\psi_1 = \gamma \cdot \psi_0$$

Now we are ready to state the following analogue of Conjecture 3.2 for $GL_2(F)$, due to Barthel–Livné.

**Theorem 7.2.** Let $\sigma$ be a weight of $K$. Then, the images of $\psi_0$ and $\psi_1$ in any irreducible quotient of $\text{ind}_{KZ}^G \sigma / (T)$ are **not** proportional.

**Proof.** The is indeed verified in the argument of [BL94, Lemma 35].

**Remark 7.3.** Note that we can not deduce an analogue of Corollary 6.5 from Theorem 7.2, as supersingular modules of $GL_2(F)$ are all two-dimensional ([Vig94]). Note also that, when $F = \mathbb{Q}_p$, a main input in [Bre03] is to prove the functions $\psi_0$ and $\psi_1$ together give a basis of the $I_1$-invariants of $\text{ind}_{KZ}^G \sigma / (T)$, which fails for any other $F$.

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Einstein Institute of Mathematics, HUJI, Jerusalem, 9190401, Israel

E-mail address: Peng.Xu@mail.huji.ac.il