GENERIC INITIAL IDEALS OF MODULAR POLYNOMIAL INVARINTS

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ABSTRACT. We study the generic initial ideals (gin) of certain ideals that arise in modular invariant theory. For all cases an explicit generating set is known we compute the generic initial ideal of the Hilbert ideal of a cyclic group of prime order for all monomial orders. We also consider the initial ideal of the ideal of transfers for an arbitrary group with respect to the reverse lexicographic order. We show that if this ideal is Borel fixed for a module then it stays Borel fixed for all its submodules. As an incidental result we note that gin respects a permutation of the variables in the monomial order.

1. Introduction

For a homogeneous ideal $I$ in a polynomial ring, its generic initial ideal $gin(I)$ with respect to a term order $>$ is the ideal of initial monomials after a generic change of coordinates. The generic initial ideal measures how close to a $>$-segment ideal $I$ can be made by a linear and invertible substitution. It encodes much information on the combinatorial, geometrical and homological properties of $I$ and the associated variety and plays an important role in the computational aspects of commutative algebra and algebraic geometry. For instance generic initial ideals were used in Hartshorne’s proof of the connectedness of Hilbert schemes. Describing $gin(I)$ is a very difficult task in general and despite their significance, there are relatively few classes of ideals for which generic initial ideals are explicitly computed. We refer the reader to [7] for a survey of results on this matter.

In this paper we study generic initial ideals that arise in invariant theory. We consider a finite dimensional module $V$ of a group $G$ over an infinite field $F$. There is an induced action on the symmetric algebra $F[V] := S(V^*)$ on $V^*$. This is a polynomial algebra $F[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is a basis for $V^*$. A classical object is the ring of invariants $F[V]^G := \{ f \in F[V] \mid \sigma(f) = f \text{ for all } \sigma \in G \}$ which is a graded subalgebra of $F[V]$. The ideal in $F[V]$ generated by homogeneous invariants of positive degree is the Hilbert ideal of $V$ and we denote it by $H(V)$. The Hilbert ideal and its quotient in $F[V]$ often contain information about the invariant ring itself. It plays an important role in finding generators of $F[V]^G$ or obtaining degree bounds for them. When the characteristic of the field divides the order of the group, i.e., $V$ is a modular module, the invariant ring is more complicated and difficult to obtain. Invariants are not known in general even in the simplest modular situation when $G$ is a cyclic group of prime order. For this group we consider the cases where an explicit generating set is known for the Hilbert ideal,
and we compute the generic initial ideals of these Hilbert ideals for all orders. A
natural question one encounters in these computations is how gin changes when the
variables in the monomial order are permuted. Since we did not find a source in
the literature that addresses this question, we include a compact proof of the fact
that gin is also permuted in the same way in Section 4. In Section 3 we consider the
initial ideal of the transfer ideals with respect to the reverse lexicographic order and
show that the Borel fixedness property of this ideal passes to that of submodules.
Case by case analysis and computations of the gin of the modular Hilbert ideals of
a cyclic group of prime order are done in Section 4.

We refer the reader to [8, §4] and to [4,6] for more background on generic initial
ideals and invariant theory, respectively.

2. Permutation of Variables and Gin

In this section we do not consider group actions so we let $S$ (instead of $F[V]$)
denote the polynomial ring $F[x_1, \ldots, x_n]$ in $n$ variables. Let $I$ be a homogeneous
ideal in $S$. We fix a term order $>$ on the set of monomials in $S$. The largest
monomial that appears in a polynomial $f \in S$ is called the initial monomial of $f$
and is denoted by $\text{In}_{>}(f)$. We denote the ideal in $S$ generated by the initial
monomials of elements in $I$ with $\text{In}_{>}(f)$. Let $\pi$ be a permutation of $\{1, 2, \ldots, n\}$. Then $\pi$ induces an isomorphism of $S$ via $\pi(x_i) = x_{\pi(i)}$. Note that this isomorphism
sends monomials to monomials. Let $>_{\pi}$ denote the term order such that

$$\pi(M_1) >_{\pi} \pi(M_2) \text{ if and only if } M_1 > M_2.$$ 

Let $S_d$ denote the $d$-th homogeneous component of $S$ and we consider the $d$-th
exterior power $\wedge^d S_d$ of $S_d$. Recall that an element $m_1 \wedge m_2 \cdots \wedge m_t$, where $m_i$ is
a monomial of degree $d$ with $m_1 > m_2 > \cdots > m_t$, is called a standard exterior monomial
of $\wedge^d S_d$ with respect to $>$. One can order standard exterior monomials
lexicographically: If $m_1 \wedge m_2 \cdots \wedge m_t$ and $w_1 \wedge w_2 \cdots \wedge w_t$ are two standard exterior monomials with respect to $>$, then we set

$$m_1 \wedge m_2 \cdots \wedge m_t > w_1 \wedge w_2 \cdots \wedge w_t$$

if $m_i > w_i$ for the smallest index $i$ with $m_i \neq w_i$. We denote the largest standard exterior monomial in the support of $e \in \wedge^d S_d$ with $\text{In}_{>}(e)$. Standard exterior monomials with respect to $>_{\pi}$ are defined and ordered similarly.

Let $\alpha \in GL_n(F)$. Then $\alpha = (\alpha_{ij})$ induces a degree preserving isomorphism on
$S$ by $x_j \rightarrow \alpha_{1j}x_1 + \alpha_{2j}x_2 + \cdots + \alpha_{nj}x_n$ for $1 \leq j \leq n$. We consider the polynomial ring in the extended set of variables $R = F[x_1, \ldots, x_n, \alpha_{i,j} \mid 1 \leq i, j \leq n]$. We extend $\pi$ to $R$ by letting $\pi(\alpha_{ij}) = \alpha_{\pi(i)\pi(j)}$.

**Lemma 1.** Let $f \in S$ and $\alpha \in GL_n(F)$. Consider $\alpha(f)$ as a polynomial in
$x_1, \ldots, x_n$ with coefficients in $F[\alpha_{ij}]$. Let $M$ be a monomial in $S$ that appears
in $\alpha(f)$ with coefficient $c \in F[\alpha_{ij}]$. Then the coefficient of $\pi(M)$ in $\alpha(f)$ is $\pi(c)$.

**Proof.** Note that $\pi(\alpha(x_j)) = \pi(\sum_{1 \leq i \leq n} \alpha_{ij}x_i) = \sum_{1 \leq i \leq n} \alpha_{\pi(i)j}x_{\pi(i)} = \alpha(x_j)$. Since both $\pi$ and $\alpha$ are ring homomorphisms it follows that $\pi(\alpha(f)) = \alpha(f)$ for all $f \in S$. We write $\alpha(f) = \sum c_k M_k$, where $c_k \in F[\alpha_{ij}]$ and $M_k$ is a monomial in $S$. Then we have

$$\alpha(f) = \pi(\alpha(f)) = \sum \pi(c_k)\pi(M_k).$$
Therefore we get \( \sum c_k M_k = \sum \pi(c_k) \pi(M_k) \). Since \( \pi \) is a permutation of the monomials in \( S \), the assertion of the lemma follows. \( \square \)

**Theorem 2.** Let \( I \) be a homogeneous ideal. Then we have

\[
\text{gin}_{> \pi}(I) = \pi(\text{gin}_{>}(I)).
\]

**Proof.** Consider a homogeneous component \( I_d \) of \( I \) with a basis \( f_1, \ldots, f_m \). Let 
\[
m_{j_1}, \ldots, m_{j_t}
\]
be monomials in \( S \) that appear in \( \alpha(f_1), \ldots, \alpha(f_t) \) with coefficients \( c_1, \ldots, c_t \in F[\alpha_{ij}] \), respectively. Assume that \( c_1 m_{j_1} \wedge \cdots \wedge c_t m_{j_t} \) is a multiple of the standard exterior monomial \( m_1 \wedge \cdots \wedge m_t \) (with respect to \( > \)) in \( \wedge^t S_d \). Call this multiple \( c \in F[\alpha_{ij}] \). Then by the previous lemma, \( \pi(m_{j_1}), \ldots, \pi(m_{j_t}) \) appear in \( \alpha(f_1), \ldots, \alpha(f_t) \) with coefficients \( \pi(c_1), \ldots, \pi(c_t) \in F[\alpha_{ij}] \), respectively. Since ranking of the monomials in \( > \) is preserved in \( >_\pi \) after we apply \( \pi \), it follows that the coefficient of the standard exterior monomial \( \pi(m_1) \wedge \cdots \wedge \pi(m_t) \) (with respect to \( >_\pi \)) in \( \pi(c_1) \pi(m_{j_1}) \wedge \cdots \wedge \pi(c_t) \pi(m_{j_t}) \) is \( \pi(c) \). Therefore we have

\[
\alpha(f_1) \wedge \cdots \wedge \alpha(f_t) = \sum_{m_1 > \cdots > m_t} c(m_1, \ldots, m_t)(m_1 \wedge \cdots \wedge m_t) = \sum_{\pi(m_1) > \cdots > \pi(m_t)} \pi(c(m_1, \ldots, m_t))(\pi(m_1) \wedge \cdots \wedge \pi(m_t)).
\]

Since \( \pi \) is a permutation of variables in \( F[\alpha_{ij}] \), \( c(m_1, \ldots, m_t) \) is the zero polynomial if and only if \( \pi(c(m_1, \ldots, m_t)) \) is the zero polynomial. So we have that if \( w_1 \wedge \cdots \wedge w_t \) is the largest exterior monomial (with respect to \( > \)) with the property that there is \( \alpha \in GL_n(F) \) with \( \text{In}_{>\pi}(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge \cdots \wedge w_t \), then \( \pi(w_1) \wedge \cdots \wedge \pi(w_t) \) is the largest exterior monomial (with respect to \( >_\pi \)) such that there exists \( \alpha \in GL_n(F) \) with \( \text{In}_{>\pi}(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = \pi(w_1) \wedge \cdots \wedge \pi(w_t) \). Since \( (\text{gin}_{>\pi})_d \) and \( (\text{gin}_{>})_d \) are generated by \( w_1, \ldots, w_t \) and \( \pi(w_1), \ldots, \pi(w_t) \) (see [8] 4.1.4, 4.1.5), respectively and \( d \) is arbitrary, the result follows. \( \square \)

### 3. Borel fixed ideals and submodules

In this section \( G \) is any group and \( V \) is a finite dimensional \( G \)-module over \( F \). We choose a basis \( e_1, \ldots, e_n \) for \( V \) and let \( W \) be the submodule spanned by \( e_1, \ldots, e_k \). We denote the corresponding bases for the dual spaces \( V^* \) and \( W^* \) with \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_k \), respectively. Dual to the inclusion \( \phi : W \to V \), we have the \( G \)-equivariant surjection \( \phi^* : F[V] = F[x_1, \ldots, x_n] \to F[W] = F[y_1, \ldots, y_k] \) given by \( \phi^*(f)(w) = f(\phi(w)) \) for \( w \in W \) and \( f \in V^* \). We have that \( \phi^*(x_i) = y_i \) for \( 1 \leq i \leq k \) and \( \phi^*(x_i) = 0 \) for \( k+1 \leq i \leq n \). The transfer ideal \( T(V) \) in \( F[V] \) is defined to be the ideal generated by the image of the transfer map \( \text{Tr} : F[V] \to F[V]^G \) with \( f \to \text{Tr}(f) = \sum_{g \in G} g(f) \). The ideal \( T(W) \) in \( F[W] \) is defined similarly. In this section \( > \) denotes the reverse lexicographic order simultaneously on \( F[V] \) and \( F[W] \) with \( x_1 > \cdots > x_n \) and \( y_1 > \cdots > y_k \). Notice that for two monomials \( m_1, m_2 \in F[V] \) we have that \( m_1 > m_2 \) if and only if \( \phi^*(m_1) > \phi^*(m_2) \) provided both \( \phi^*(m_1) \) and \( \phi^*(m_2) \) are non-zero. Also, since \( \phi^* \) is \( G \)-equivariant, it commutes with the transfer map as well and so \( \phi^* \) maps \( T(V) \) into \( T(W) \).

**Lemma 3.** We have \( \text{In}_{>\pi}(\phi^*(f)) = \phi^*(\text{In}_{>}(f)) \) for all \( f \in F[V] \). Moreover, the restriction \( \phi^* : T(V) \to T(W) \) is onto.
Proof. Let \( f \in F[V] \) and let \( m \) denote the initial monomial of \( f \). If there exists \( k+1 \leq i \leq n \) such that \( x_i \) divides \( m \), then every other monomial in \( f \) is divisible by some \( x_j \) for \( i \leq j \leq n \) since we are using the reverse lexicographic order. Since \( \phi^* \) vanishes on the variables \( x_k+1, \ldots, x_n \) it follows that \( \text{In}_{>}(\phi^*(f)) = \phi^*(\text{In}_{>}(f)) = 0 \). If \( m \) is not divisible by any \( x_j \) for \( k+1 \leq j \leq n \), then \( \phi^*(m) \) is the initial monomial of \( \phi^*(f) \) since \( \phi^* \) preserves the ranking between the monomials that are not mapped to zero.

For the second statement let \( g \in T(W) \). We write \( g = \sum g_i \text{Tr}(h_i) \) with \( g_i, h_i \in F[W] \). Since \( \text{Tr} \) is a linear map, we may assume that \( g_i, h_i \) are monomials. Then there are unique monomials \( g'_i, h'_i \in F[V] \) such that \( \phi^*(g'_i) = g_i \) and \( \phi^*(h'_i) = h_i \). So we get \( \phi^*(\sum g'_i \text{Tr}(h'_i)) = g \) since \( \phi^* \) is a ring homomorphism and it commutes with the transfer map. \( \square \)

The subgroup of \( GL_i(F) \) consisting of upper triangular matrices is called the Borel subgroup and we denote it by \( B_\ell \). Note that the subgroups \( B_n \) and \( B_k \) act naturally on \( F[V] \) and \( F[W] \), respectively. We show that if the initial ideal of \( T(V) \) is Borel fixed, i.e., closed under the action of the Borel subgroup, then this property passes to the initial ideal of \( T(W) \). We also prove a partial converse.

Theorem 4. Assume the convention of this section. If \( \text{In}_{>}(T(V)) \) is Borel fixed, then \( \text{In}_{>}(T(W)) \) is also Borel fixed. Moreover, if \( \text{In}_{>}(T(V)) \) is Borel fixed, then \( \text{In}_{>}(T(W)) \) is Borel fixed in \( F[V]/(x_{k+1}, \ldots, x_n) \).

Proof. Assume that \( \text{In}_{>}(T(V)) \) is Borel fixed and let \( m \in \text{In}_{>}(T(W)) \) be a monomial. By the previous lemma, there exists \( m' \in \text{In}_{>}(T(V)) \) such that \( \phi^*(m') = m \). Let \( \alpha \in B_k \). We extend \( \alpha \) to \( \alpha^* \in GL_n(F) \) by setting \( \alpha_{i,i} = 1 \) for \( k+1 \leq i \leq n \) and \( \alpha_{i,j} = 0 \) if \( i \neq j \) and one of \( i \) or \( j \) is at least \( k+1 \). Then \( \alpha^* \) lies in \( B_n \).

Note also that since \( m' \) is not divisible by any of the variables \( x_{k+1}, \ldots, x_n \) we have \( \alpha(\phi^*(m')) = \phi^*(\alpha^*(m')) \). Equivalently, \( \alpha(m) = \phi^*(\alpha^*(m')) \). But since \( \text{In}_{>}(T(V)) \) is Borel fixed, we get \( \phi^*(\alpha^*(m')) \in \phi^*(\text{In}_{>}(T(V))) \). Therefore, by the previous lemma we have \( \alpha(m) \in \text{In}_{>}(T(W)) \) giving that \( \text{In}_{>}(T(W)) \) is Borel fixed.

The second assertion follows from the isomorphism \( W^* \cong V^*/<x_{k+1}, \ldots, x_n> \) of \( G \)-modules that induces a \( G \)-equivariant ring isomorphism between \( F[W] \) and \( F[V]/(x_{k+1}, \ldots, x_n) \). \( \square \)

4. Gin of modular Hilbert ideals

In this section \( G \) denotes a cyclic group of prime order \( p \) and we assume that the characteristic of \( F \) is also \( p \). There are exactly \( p \) indecomposable \( G \)-modules \( V_1, \ldots, V_p \) over \( F \) and each indecomposable module \( V_n \) is afforded by a Jordan block of dimension \( n \) with 1’s on the diagonal. If \( V \) is a direct sum of indecomposable modules \( V_{n_1}, \ldots, V_{n_k} \), then we write \( V = \sum_{1 \leq j \leq k} V_{n_j} \). In the sequel we consider the action of the Borel subgroup on \( F[V] \). We always assume that this action is compatible with the monomial order, i.e., the action of a matrix in the Borel subgroup on the \( i \)-th variable in the monomial order is given by the \( i \)-th column of the matrix. We compute all generic initial ideals of \( H(V) \) for all cases for which an explicit generating set for \( H(V) \) is known.

4.1. The monomial cases: \( lV_2 + mV_3 \) and \( V_4 \). For \( V = lV_2 + mV_3 \), we identify \( F[V] \) with \( F[x_1, y_1, z_r, 1 \leq i, j \leq l + m, \ l + 1 \leq r \leq l + m] \). For \( 1 \leq i \leq l \)
\( \{ x_i, y_i \} \) spans a copy of \( V^*_2 \) and \( \{ x_i, y_i, z_i \} \) spans a copy of \( V^*_4 \) for \( 1 \leq i \leq m + l \).

In \[9\] 2.6, it is shown that \( H(I(V_2 + M V_3)) \) is generated by

\[
L = \{ x_i, y_i^l \mid 1 \leq i \leq l \} \cup \{ x_i, y_i y_j, z_i^l \mid 1 + l \leq i, j \leq m + l, i \leq j \}.
\]

**Proposition 5.** There are \((2l + 3m)!\) generic initial ideals of \( H(I(V_2 + M V_3)) \). Each of them is generated by \( \pi(L) \) for some permutation \( \pi \) of the variables in \( F[I(V_2 + M V_3)] \).

**Proof.** Let \( \theta \) be a monomial order with \( x_{m+l} > \cdots > x_1 > y_{m+l} > \cdots > y_1 > z_{m+l} > \cdots > z_{l+1} \) and let \( \theta \in F_{2l+3m} \). Let \( J \) denote the ideal generated by the subset \( L' = \{ x_i \mid 1 \leq i \leq l+m \} \cup \{ y_i y_j \mid 1 + l \leq i, j \leq m + l, i \leq j \} \) of \( L \). Notice that \( J \) is a strongly stable ideal. It follows that \( \theta \) sends \( J \) into itself. On the other hand the remaining generators of \( H(V_2 + M V_3) \) in \( L \setminus L' \) are pure powers of variables of degree \( p \) and since we are in characteristic \( p \), \( \theta \) sends a \( p \)-th power of a variable to a combination of \( p \)-th powers of variables. But such a combination is in \( H(I(V_2 + M V_3)) \) as well since \( H(I(V_2 + V_3)) \) contains all \( p \)-th powers of variables. It follows that \( H(I(V_2 + M V_3)) \) is Borel fixed and so we have \( \text{gin}_{\theta}(H(I(V_2 + M V_3))) = H(I(V_2 + M V_3)) \) by \[2\] 1.8. The final assertion of the proposition follows from Theorem 2. \( \square \)

For \( V = V_4 \), we identify \( F[V_4] \) with \( F[x_1, x_2, x_3, x_4] \). From \[9\] 3.2 we get that \( H(V_4) \) is generated by

\[
M = \{ x_1, x_2^2, x_2 x_3^{p-3}, x_3^{p-1}, x_4^p \}.
\]

**Proposition 6.** There are \( 4! \) generic initial ideals of \( H(V_4) \). Each of them is generated by \( \pi(M) \) for some permutation \( \pi \) of the variables in \( F[V_4] \).

**Proof.** Let \( \theta \) be a monomial order such that \( x_1 > x_2 > x_3 > x_4 \) and let \( \theta \in F_4 \). Note that the ideal generated by the subset \( M' = \{ x_1, x_2^2, x_2 x_3^{p-3}, x_3^{p-1} \} \) is strongly stable and hence \( \theta \) sends this ideal into itself. Furthermore, the only other generator is a pure \( p \)-th power of a variable. Since \( \theta \) sends a \( p \)-th power of a variable to a combination of \( p \)-th powers of variables and all \( p \)-th powers of variables are contained in \( H(V_4) \), it follows that \( H(V_4) \) is Borel fixed. So \( \text{gin}_{\theta}(H(V_4)) = H(V_4) \), by \[2\] 1.8. The final assertion of the proposition follows from Theorem 2 as in the previous case. \( \square \)

4.2. The non-monomial case: \( V_5 \). We identify \( F[V_5] \) with \( F[x_1, x_2, x_3, x_4, x_5] \). In \[9\] 4.1 it is shown that

\[
T = \{ x_1, x_2^2, x_3^2 - 2 x_2 x_4 - x_2 x_3 x_4, x_2 x_3^{p-4}, x_3 x_4^{p-3}, x_4^{p-1}, x_5^p \}
\]

is a generating set for \( H(V_5) \) for \( p > 5 \). We denote the generators of \( H(V_5) \) in \( T \) with \( f_i \) for \( 1 \leq i \leq 8 \) with \( f_1 = x_1 \) and \( f_8 = x_5^p \). Let \( \alpha = (\alpha_{ij}) \) be an element in \( F_5 \). Define \( C = \alpha_{33}^2 (2 \alpha_{23} \alpha_{33} - 2 \alpha_{22} \alpha_{34} - \alpha_{22} \alpha_{33}) \) and \( D = \alpha_{33}^2 (-2 \alpha_{22} \alpha_{44}) \). We describe a generating set for \( \alpha(H(V_5)) \).

**Lemma 7.** Assume the convention of the previous paragraph and that \( p > 5 \). Then \( \alpha(H(V_5)) \) is generated by

\[
T' := \{ x_1, x_2^2, x_3^2 + C x_2 x_3 + D x_2 x_4, x_2 x_3 x_4, x_2 x_3^{p-4}, x_3 x_4^{p-3}, x_4^{p-1}, x_5^p \}.
\]

**Proof.** For \( 1 \leq i \leq 8 \), let \( J_i \) denote the ideal generated by \( \alpha(f_1), \ldots, \alpha(f_i) \). Since \( \alpha \) is a ring homomorphism, \( \alpha(H(V_5)) \) is generated by \( \alpha(f_1), \ldots, \alpha(f_8) \) and so we have \( \alpha(H(V_5)) = J_8 \). We also denote \( x_3^2 + C x_2 x_3 + D x_2 x_4 \) with \( f_3' \).
Note that, since α sends \( x_1 \) to a multiple of \( x_1 \) and \( x_2 \) to a linear combination of \( x_1 \) and \( x_2 \) we have that \( J_2 = (x_1, x_2^2) \). On the other hand direct computation gives \( \alpha(x_3^2 - 2x_2x_3 - x_2x_3) = (\sum_{1 \leq i \leq 4} \alpha_3 x_i)^2 - 2(\sum_{1 \leq i \leq 2} \alpha_2 x_i)(\sum_{1 \leq i \leq 4} \alpha_4 x_i) - (\sum_{1 \leq i \leq 2} \alpha_2 x_i)(\sum_{1 \leq i \leq 3} \alpha_3 x_i) \equiv \alpha_3 f_3 \mod J_2. \) It follows that \( J_3 = (x_1, x_2, f_3) \).

We finish the proof by showing that \( \alpha(f_i) \) is a scalar multiple of \( f_i \) modulo \( J_{i-1} \) for \( 4 \leq i \leq 8 \). This gives \( J_i = (f_i) + J_{i-1} \) for \( 4 \leq i \leq 8 \) and hence \( J_8 = \alpha(H(V)) \) is generated by \( T' \).

Note that \( x_2x_3^2 = x_2f_1 - x_2(Cx_3 + Dx_4) \in J_3 \). Therefore, since \( x_1, x_2^2 \in J_3 \) as well, we have \( \alpha(f_4) = \alpha(x_2x_3x_4) \equiv \alpha_2 \alpha_3 \alpha_4 (x_2x_3x_4) \mod J_3 \). We also have \( \alpha(f_5) = \alpha(x_2x_4^{p-4}) \equiv \alpha_2 \alpha_3 \alpha_4 (x_2x_4^{p-4}) \equiv \alpha_2 \alpha_4 (x_2^{p-4}) \mod J_4 \), where the first equivalence uses \( x_1 \in J_4 \) and the second equivalence uses that \( x_2^2, x_2x_3, x_2x_4 \in J_4 \) and \( p > 5 \). To compute \( \alpha(f_6) \) we note the identities \( x_3^3 = x_3f_1 - Cx_3^2 - Dx_2x_3x_4 \) and \( x_2x_4^{p-4} = x_2^{p-4}f - Cx_4^{p-5}(x_2x_3x_4) - Dx_4(x_2x_4^{p-4}) \) which give that \( x_3^3, x_3^2x_4^{p-4} \in J_5 \). So

\[
\alpha(x_3^4) \equiv (\alpha_2 \alpha_3 \alpha_4)(x_2^2, x_2x_3, x_2x_4) x_4^{p-4} \equiv J_5 \]

where the first two equivalences use \( x_1, x_2^2 \in J_5 \) and we have the third equivalence because \( x_2x_3^2, x_2x_3x_4, x_2x_4^{p-4} \in J_5 \). The final equivalence follows because \( x_2^2, x_2x_3x_4, x_3^3, x_2x_4^{p-4} \in J_5 \) and \( p > 5 \). For \( f_7 \) we have

\[
\alpha(x_4^{p-1}) = (\alpha_1 \alpha_2 x_1 + \alpha_2 \alpha_3 x_2 + \alpha_4 x_4^{p-1}) \equiv (\alpha_2 \alpha_3 x_2 + \alpha_4 x_4^{p-1}) \mod J_6,
\]

where the first equivalence uses that \( x_1, x_2, x_2x_3, x_2x_4, x_2x_4^{p-4} \in J_6 \) and the second one uses that \( x_3^3, x_3x_4^{p-3}, x_3x_4^{p-4} \in J_6 \). Finally \( \alpha(f_8) = \alpha_5 x_5^p \mod J_7 \) because \( x_5 

We first consider the case $x_2^3 > x_2 x_4$. Note then the set $A$ consists of $\text{In}_{>}(f_i)$ for $1 \leq i \leq 10$, $i \neq 4$ ($f_4 = C^{-1}(x_4 f_3 - f_{10})$ and $\text{In}_{>}(f_4)$ is divisible by $\text{In}_{>}(f_3)$). Therefore it remains to show that this set of polynomials satisfy the Buchberger criterion. That is, the $S$-polynomial of any pair of polynomials $f_i,f_j$ with $1 \leq i,j \leq 10$ and $i,j \neq 4$ reduces to zero. Since the $S$-polynomial of two monomials is zero, it suffices to consider the $S$-polynomials involving either $f_3$ or $f_{10}$. We go through the pairs and write the polynomials in the order they appear in the polynomial division: $S(f_2,f_3)$ reduces to zero via $f_2,f_3,f_9,f_{10}$. Both $S$-polynomials $S(f_3,f_5)$ and $S(f_5,f_6)$ reduce to zero via $f_{10}$. The $S$-polynomial $S(f_3,f_5)$ reduces to zero via $f_9,f_3$ and $f_{10}$ and the $S$-polynomial $S(f_3,f_{10})$ reduces to zero via $f_2,f_3,f_9,f_{10}$. The $S$-polynomials $S(f_5,f_{10})$, $S(f_6,f_{10})$, $S(f_7,f_{10})$ reduce to zero at one step each via $f_5$. Finally, $S(f_6,f_{10})$ reduces to zero via $f_6$ and $f_{10}$. We have considered all pairs whose initial terms are not relatively prime. So the proof for this case is complete because the $S$-polynomial of two polynomials that have relatively prime initial terms reduces to zero.

Now we consider the case $x_2 x_4 > x_3^3$. Define $f_{11} = x_4^{-6} f_{10} - D f_5 = x_4^3 x_4^{-5}$. Since $D$ is a non-zero scalar, from this equality we have that $f_i$ for $1 \leq i \leq 11$ and $i \neq 4,5$ also generate $\alpha(H(V_5))$. Since the set $B$ consists of $\text{In}_{>}(f_i)$ for $1 \leq i \leq 11$ and $i \neq 4,5$, it remains to show that this set satisfies the Buchberger criterion. As in the previous case we just need to check pairs whose initial terms are not relatively prime and one of the polynomials in the pair is not monomial. We note that the reductions of the $S$-polynomials involving $f_3$ in the previous case carry over to this case except $S(f_3,f_{10})$. To see this, first note that the missing initial monomial $\text{In}_{>}(f_3)$ is not used in these reductions. Also the initial monomials of $f_i$ for $1 \leq i \leq 10$ do not change with the change of the order except $f_{10}$. But in these reductions $f_{10}$ is not used in the polynomial division except at the last step. So the last non-zero remainder is a multiple of $f_{10}$ giving that this last polynomial division is also a reduction in the second order as well. Finally, $S(f_3,f_{11})$ reduces to zero via $f_9,f_3,f_{10}$. We finish the proof by checking the pairs involving $f_{10}$. We have that $S(f_3,f_{10})$ reduces to zero via $f_9,f_{10}$. The $S$-polynomial $S(f_2,f_{10})$ reduces to zero via $f_3,f_{10}$ and the $S$-polynomial $S(f_7,f_{10})$ reduces to zero via $f_6$. Both $S(f_6,f_{10})$ and $S(f_{10},f_{11})$ reduce to zero via $f_9$.

**Theorem 9.** Assume that $p > 5$. There are $2(5!)$ generic initial ideals of $H(V_5)$. Each of them is generated by $\pi(A)$ or $\pi(B)$, where $\pi$ is a permutation of the variables in $F[V_5]$.

**Proof.** Let $\alpha = (\alpha_{ij}) \in B_5$. If $C = 0$, then from Lemma 7 we get that the monic generators of $\text{In}_{>}(\alpha(H(V_5)))$ of degree at most two are $x_1, x_2^2, x_3^2$ if $x_3^2 > x_2 x_4$ and are $x_1, x_2^2, x_2 x_4$ otherwise. Note that $x_2 x_3 \notin \text{In}_{>}(\alpha(H(V_5)))$ in both cases and so $\text{In}_{>}(\alpha(H(V_5)))$ fails to be Borel fixed because either $x_3^2$ or $x_2 x_4$ lies in $\text{In}_{>}(\alpha(H(V_5)))$. It follows that $\text{gin}_{>}(H(V_5)) \neq \text{In}_{>}(\alpha(H(V_5)))$ if $C = 0$ because generic initial ideals are always Borel fixed, see for instance [6] §15. On the other hand, by the previous lemma all other members of $B_5$ generate the same initial ideal. But it is a standard fact that at least one element in $B_5$ generates the generic initial ideal ([6] 15.18) and so $\text{gin}_{>}(H(V_5)) = \text{In}_{>}(\alpha(H(V_5)))$ for $\alpha \in B_5$ satisfying $C \neq 0$. This initial ideal is generated by $A$ or $B$ depending how $>\text{compares} x_2 x_4$ and $x_3^2$. The final assertion of the proposition follows from Theorem 2.
Remark 10. (1) The computation for \( p = 5 \) follows along the same lines as follows. MAGMA [3] computation gives that, for \( \alpha \in \mathbb{F}_5 \), \( \alpha(H(V_5)) \) is generated by \( \{x_1, x_2, x_2^2 + C x_2 x_3 + D x_2 x_4, x_2 x_3 x_4, C_0 x_2^2 x_4^2 + D_0 x_3 x_4^2, x_2^2 x_3^2, x_2^4, x_3^2, x_3^4, x_4^5\} \), where \( C_0 = (4 \alpha_3 - 1) \alpha_3^2 + 2 \alpha_2 + \alpha_0 \) and \( D_0 = \alpha_3^2 \). The degree two component of \( \text{In}_{>}(\alpha(H(V_5))) \) is not Borel fixed unless \( C \neq 0 \) and \( C_0 \neq 0 \) and for \( C, C_0 \neq 0 \), \( \text{In}_{>}(\alpha(H(V_5))) \) is generated by \( A' = \{x_1, x_2, x_2 x_3, x_2 x_4^2, x_3^2 x_4, x_3 x_4^2, x_4^4, x_5^5\} \).

It follows, as before that there are 5! generic initial ideals of \( H(V_5) \). Each of them is generated by \( \pi(A') \), where \( \pi \) is a permutation of the variables in \( F[V_5] \).

(2) The identity matrix satisfies that \( C \neq 0 \). Therefore from the proof of the theorem we get that \( \text{gin}_{>}(H(V_5)) = \text{In}_{>}(H(V_5)) \) for all monomial orders.

References

[1] William W. Adams and Philippe Loustaunau. An introduction to Gröbner bases, volume 3 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1994.
[2] A. M. Bigatti, A. Conca, and L. Robbiano. Generic initial ideals and distractions. Comm. Algebra, 33(6):1709–1732, 2005.
[3] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
[4] H. E. A. Eddy Campbell and David L. Wehlau. Modular invariant theory, volume 139 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2011. Invariant Theory and Algebraic Transformation Groups, 8.
[5] Harm Derksen and Gregor Kemper. Computational invariant theory, volume 130 of Encyclopaedia of Mathematical Sciences. Springer, Heidelberg, enlarged edition, 2015. With two appendices by Vladimir L. Popov, and an addendum by Norbert A’Campo and Popov, Invariant Theory and Algebraic Transformation Groups, VIII.
[6] David Eisenbud. Commutative algebra, volume 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
[7] Mark L. Green. Generic initial ideals. In Six lectures on commutative algebra (Bellaterra, 1996), volume 166 of Progr. Math., pages 119–186. Birkhäuser, Basel, 1998.
[8] Jürgen Herzog and Takayuki Hibi. Monomial ideals, volume 260 of Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2011.
[9] Müfit Sezer and R. James Shank. On the coinvariants of modular representations of cyclic groups of prime order. J. Pure Appl. Algebra, 205(1):210–225, 2006.

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