Imprints of the Quantum World in Classical Mechanics

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Abstract The imprints left by quantum mechanics in classical (Hamiltonian) mechanics are much more numerous than is usually believed. We show that the Schrödinger equation for a nonrelativistic spinless particle is a classical equation which is equivalent to Hamilton’s equations. Our discussion is quite general, and incorporates time-dependent systems. This gives us the opportunity of discussing the group of Hamiltonian canonical transformations which is a non-linear variant of the usual symplectic group.

Keywords Quantization · Schrödinger’s equation · Hamiltonian flows · Symplectic covariance of Weyl calculus · Stone’s theorem

1 Introduction

“Where did that [the Schrödinger equation] come from? Nowhere. It came out of the mind of Schrödinger, invented in his struggle to find an understanding of the experimental observations in the real world.” (Richard Feynman in [5].)

Similar statements abound in the physical literature; they are found in both introductory and advanced text on quantum mechanics, and we can read them on the web in various blogs and forums. However, they are strictly speaking not true; already in 1966 Nelson [22] showed that Schrödinger’s equation could be derived from
Newtonian mechanics; he however introduced some extra physical assumptions (stochasticity); also see the more recent paper by Hall and Reginatto [13], who use the uncertainty principle. We also mention, because of its historical interest, Feynman’s non-rigorous approach in [4] based on the Lagrangian formulation of mechanics (see Derbes’ comments [3] of Feynman’s paper).

In the present paper we will show that one can \textit{mathematically derive rigorously} from Hamiltonian mechanics, the Schrödinger equation

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i\hbar\frac{\partial \psi}{\partial t} = H(x, -\hbar \nabla x, t)\psi
\]

and this without any recourse to any physical ‘quantum-mechanical’ hypotheses. In fact we will show that there is a surprising one-to-one and onto correspondence between Hamiltonian flows and the quantum evolution group, which only becomes apparent if one uses a deep property of symplectic covariance together with Stone’s theorem on one-parameter groups of unitary operators.

Schrödinger [27] was led to his equation from his knowledge of the classical Hamilton-Jacobi approach which has a close connection with the eikonal of classical wave theory. His original proposal was not firmly based on rigorous mathematics as he himself acknowledged but his intuition was correct. Attempts to provide a detailed relationship between classical and quantum mechanics has remained somewhat of a problem because we have been left with two widely different mathematical formulations, one involving particles evolving under dynamical laws in a phase space, the other involving operators and waves operating in a Hilbert space. This leaves the impression that there are two very different worlds, the classical and the quantum. However it is clear that we do not inhabit two different worlds and so we are left with the puzzle as how to relate the two formalisms.

Already in studying the properties of light, we see in one phenomena both aspects. Geometric optics gives us light rays travelling in straight lines, while wave optics gives us interference and diffraction. In this phenomena we see the essential mathematics emerging. Ray optics emerges from symplectic geometry, while wave optics emerges from the geometry of the covering group of the symplectic group, \(\text{Sp}(2n, \mathbb{R})\), namely the metaplectic group, \(\text{Mp}(2n, \mathbb{R})\). Ray tracing involves symplectic flows, while the wave evolution is identified with the metaplectic flows. These are not separate flows but deeply related as the metaplectic flows can be mathematically ‘lifted’ from symplectic flows and the metaplectic flows can be projected onto symplectic flows.

The fact that classical mechanics has the same mathematical structure as ray optics, namely the symplectic geometry, suggest that quantum mechanics in its wave description could be related to such a covering structure. Indeed if we restrict ourselves to quadratic Hamiltonians, we find that the metaplectic flow is determined exactly by Schrödinger’s equation, in which the classical Hamiltonian is replaced by the Weyl Hamiltonian, \(H \leftrightarrow \hat{H}\). Unfortunately the generalisation to all physically relevant Hamiltonians has always floundered on the mathematical difficulties involved in investigating the covering properties of general group of canonical transformations which involve non-linear transformations. However such difficulties should not force us into a ‘two world’ situation.