Abstract

We establish an equivalent condition to the validity of the Collatz conjecture, using elementary methods. We derive some conclusions and show several examples of our results. We also offer a variety of exercises, problems and conjectures.

1 Introduction

The Collatz conjecture (also known as the 3n + 1 conjecture, Ulam’s conjecture, the Syracuse problem, Kakutani’s problem, Hasse’s algorithm, etc.) was first proposed by Lothar Collatz in 1937 [2]. In terms of the function T(n), defined by

\[ T(n) = \begin{cases} \frac{3n+1}{2}, & n \equiv 1 \text{ mod } 2 \\ \frac{n}{2}, & n \equiv 0 \text{ mod } 2 \end{cases}, \quad n \in \mathbb{N}, \]

(1)

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the conjecture claims that for all natural numbers $n$, there exists a natural number $k$ such that

$$T^{(k)}(n) = T \circ T \circ \cdots \circ T(n) = 1.$$  

For example, we have

$T(3) = 5$, $T(2)(3) = 8$, $T^{(3)}(3) = 4$, $T^{(4)}(3) = 2$, $T^{(5)}(3) = 1$, $T(7) = 11$, $T^{(2)}(7) = 17$, $T^{(3)}(7) = 26$, $T^{(4)}(7) = 13$, $T^{(5)}(7) = 20$, $T^{(6)}(7) = 10$, $T^{(7)}(7) = 5$, $T^{(8)}(7) = 8$, $T^{(9)}(7) = 4$, $T^{(10)}(7) = 2$, $T^{(11)}(7) = 1$.

We define $T^{(\infty)}(n) = 1$.

**Exercise 1** Prove that if $\forall n \in \mathbb{N} \exists k \in \mathbb{N}$ such that $T^{(k)}(n) < n$, then the Collatz conjecture is true. The number $k$ is called the stopping time of $n$.

As of February 2007, the Collatz conjecture has been verified for numbers up to $13 \times 2^{58} = 3,746,994,889,972,252,672$ [7]. However, the general case remains open.

Introducing the total stopping time function $\sigma_\infty(n)$, defined by $\sigma_\infty(1) = 0$ and

$$\sigma_\infty(n) = \inf \left\{ k \in \mathbb{N} \cup \{\infty\} \mid T^{(k)}(n) = 1 \right\}, \quad n \geq 2,$$

we can reformulate the Collatz conjecture as

$$\mathcal{C} = \mathbb{N},$$  

where

$$\mathcal{C} = \{ n \in \mathbb{N} \mid \sigma_\infty(n) < \infty \}.  

From (2), we have

$$\begin{array}{|c|cccccccc|}
  n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \sigma_\infty(n) & 1 & 5 & 2 & 4 & 6 & 11 & 3 \\
\end{array}$$

**Exercise 2** Find $\sigma_\infty(n)$ for $9 \leq n \leq 100$.

Hint: The web page [http://www.numbertheory.org/php/collatz.html](http://www.numbertheory.org/php/collatz.html) contains an implementation which allows the computation of $\sigma_\infty(n)$ for large values of $n$.  

2
One could consider the inverse problem and try to characterize the sets $S_k$, defined by $S_0 = \{1\}$ and

$$S_k = \{n \in \mathbb{N} \mid \sigma_\infty(n) = k\}, \quad k \geq 1.$$  \hfill (4)

The first few $S_k$ are

$$S_1 = \{2\}, \quad S_2 = \{4\}, \quad S_3 = \{8\}, \quad S_4 = \{5, 16\}, \quad S_5 = \{3, 10, 32\}, \quad S_6 = \{6, 20, 21, 64\}, \quad S_7 = \{12, 13, 40, 42, 128\}, \quad S_8 = \{24, 26, 80, 84, 85, 256\}.$$  \hfill (5)

It is clear from (4) that $2^k \in S_k \quad \forall k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In terms of the sets $S_k$, the Collatz conjecture reads

$$\bigcup_{k=0}^{\infty} S_k = \mathbb{N}. \quad \text{(C2)}$$

**Exercise 3** Compute $S_k$ for $9 \leq k \leq 100$.

*Hint: Consider the inverse map $T^{-1} : \mathbb{N} \to \mathbb{P}(\mathbb{N})$, given by

$$T^{-1}(n) = \begin{cases} \{2n\}, & n \equiv 0, 1 \pmod{3} \\ \{2n, \frac{1}{3}(2n - 1)\}, & n \equiv 2 \pmod{3} \end{cases}$$

The sequence of natural numbers $\{x_n^{(m)}, \quad n \geq 0\}$, defined by $x_0^{(m)} = m$ and

$$x_{n+1}^{(m)} = T(x_n^{(m)}), \quad 0 \leq n,$$  \hfill (6)

is called the trajectory or forward orbit of $m \in \mathbb{N}$. From (2), we have

$$\{x_n^{(2)}\} = \{2, 1\}, \quad \{x_n^{(3)}\} = \{3, 5, 8, 4, 2, 1\}, \quad \{x_n^{(4)}\} = \{4, 2, 1\},$$

$$\{x_n^{(5)}\} = \{5, 8, 4, 2, 1\}, \quad \{x_n^{(6)}\} = \{6, 3, 5, 8, 4, 2, 1\},$$

$$\{x_n^{(7)}\} = \{7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}, \quad \{x_n^{(8)}\} = \{8, 4, 2, 1\}.$$**

**Exercise 4** Find $\{x_n^{(m)}\}$ for $9 \leq m \leq 100$.

Using the sequences $\{x_n^{(m)}\}$ we can restate Collatz’s conjecture as

$$\bigcap_{m=2}^{\infty} \{x_n^{(m)}\} = \{2, 1\}. \quad \text{(C3)}$$
We can also consider higher order recurrences, i.e., instead of (6), use

\[ x^{(m)}_{n+i} = T^{(i)}(x^{(m)}_n), \quad 0 \leq n, \]

where

\[ T^{(i)}(x) = f_{i,j}(x), \quad \text{if } x \equiv j \pmod{2^i}, \quad 0 \leq j \leq 2^i - 1. \quad (7) \]

For \( i = 1, 2, 3 \), we have

\[ f_{1,0}(x) = \frac{x}{2}, \quad f_{1,1}(x) = \frac{3x + 1}{2}, \]
\[ f_{2,0}(x) = \frac{x}{4}, \quad f_{2,1}(x) = \frac{3x + 1}{4}, \quad f_{2,2}(x) = \frac{3x + 2}{4}, \quad f_{2,3}(x) = \frac{9x + 5}{4}, \]
\[ f_{3,0}(x) = \frac{x}{8}, \quad f_{3,1}(x) = \frac{9x + 7}{8}, \quad f_{3,2}(x) = \frac{3x + 2}{8}, \quad f_{3,3}(x) = \frac{9x + 5}{8}, \]
\[ f_{3,4}(x) = \frac{3x + 4}{8}, \quad f_{3,5}(x) = \frac{3x + 1}{8}, \quad f_{3,6}(x) = \frac{9x + 10}{8}, \quad f_{3,7}(x) = \frac{27x + 19}{8}. \]

**Exercise 5** Prove that if the sequence \( \{3k + 4\} \subset \mathbb{C} \), then the Collatz conjecture is true.

In terms of (7), the Collatz conjecture reads

\[ \forall n \in \mathbb{N} \exists m \in \mathbb{N} \text{ such that } n \equiv k \pmod{2^m} \text{ and } \frac{dx}{dx} f_{m,k}(x) < 1. \quad (C4) \]

For example, we have 11 \( \equiv 11 \pmod{2^5} \) and

\[ f_{5,10}(x) = \frac{3x + 2}{32}, \quad f_{5,11}(x) = \frac{27x + 23}{32}. \]

Thus,

\[ x^{(11)}_5 = f_{5,11}(11) = 10, \quad x^{(11)}_{10} = f_{5,10}(10) = 1. \]

The literature on the Collatz conjecture is vast and growing rapidly. Rather than attempting to cover it, we refer the reader to the excellent survey papers [5] and [6].
2 Representation of natural numbers

Let the sets $\Lambda_m$ be defined by $\Lambda_m = \{2^m\}, \quad 0 \leq m \leq 3$ and

$$\Lambda_m = \left\{ n \in \mathbb{N} \mid n = \frac{2^m}{3^l} - \sum_{k=1}^{l} \frac{2^b_k}{3^k} \right\}, \quad m \geq 4,$$

for some $m, l, b_1, \ldots, b_l \in \mathbb{N}_0$, with

$$0 \leq l \leq m - 3 \quad \text{and} \quad 0 \leq b_1 < b_2 < \cdots < b_l \leq m - 4.$$  

The first few $\Lambda_m$ are

$$\Lambda_4 = \left\{ \frac{2^4}{3^0}, \frac{2^4}{3^1} - \frac{2^0}{3^1} \right\},$$

$$\Lambda_5 = \left\{ \frac{2^5}{3^0}, \frac{2^5}{3^1} - \frac{2^1}{3^1}, \frac{2^5}{3^2} - \left( \frac{2^0}{3^1} + \frac{2^1}{3^2} \right) \right\},$$

$$\Lambda_6 = \left\{ \frac{2^6}{3^0}, \frac{2^6}{3^1} - \frac{2^0}{3^1}, \frac{2^6}{3^2} - \frac{2^2}{3^1}, \frac{2^6}{3^3} - \left( \frac{2^1}{3^1} + \frac{2^2}{3^2} \right) \right\}. $$

Using the $(l + 2)$–tuple $(l, b_1, \ldots, b_l, m)$ to represent the number $\frac{2^m}{3^l} - \sum_{k=1}^{l} \frac{2^b_k}{3^k}$, we can write

$$\Lambda_4 = \{(0, 4), (1, 0, 4)\}, \quad \Lambda_5 = \{(0, 5), (1, 1, 5), (2, 0, 1, 5)\},$$

$$\Lambda_6 = \{(0, 6), (1, 0, 6), (1, 2, 6), (2, 1, 2, 6)\},$$

$$\Lambda_7 = \{(0, 7), (1, 1, 7), (1, 3, 7), (2, 0, 3, 7), (2, 2, 3, 7)\},$$

$$\Lambda_8 = \{(0, 8), (1, 0, 8), (1, 2, 8), (1, 4, 8), (2, 1, 4, 8), (2, 3, 4, 8)\}. $$

(8)

Exercise 6 Compute $\Lambda_m$ for $9 \leq m \leq 100$.

Hint: (a) If $(v_1, v_2, \ldots, v_{l+2}) \in \Lambda_m$, then $(v_1, v_2 + 1, \ldots, v_{l+2} + 1) \in \Lambda_{m+1}$.

(b) $(1, 0, 2m) \in \Lambda_{2m}$ for all $m \geq 2$.

Comparing (5) with (8), it seems that $S_m = \Lambda_m$. The next results will show this to be true.

Lemma 7 For all $m \in \mathbb{N}_0$, we have

$$T(\Lambda_{m+1}) \subset \Lambda_m.$$  

(9)
Proof. Let $n \in \Lambda_{m+1}$. Then,
\[
n = \frac{2^{m+1}}{3^l} - \sum_{k=1}^{l} \frac{2^{b_k}}{3^k},
\]
and
\[
T(n) = \frac{2^m}{3^l} - \sum_{k=1}^{l} \frac{2^{b_k-1}}{3^k},
\]
if $b_1 > 0$ ($n$ even) or
\[
T(n) = \frac{1}{2} \left( \frac{2^{m+1}}{3^l-1} - \sum_{k=1}^{l} \frac{2^{b_k}}{3^{k-1}} + 1 \right) = \frac{1}{2} \left( \frac{2^{m+1}}{3^l-1} - \sum_{k=2}^{l} \frac{2^{b_{k-1}}}{3^{k-1}} \right)
= \frac{1}{2} \left( \frac{2^{m+1}}{3^l-1} - \sum_{k=1}^{l-1} \frac{2^{b_{k+1}}}{3^k} \right) = \frac{2^m}{3^l-1} - \sum_{k=1}^{l-1} \frac{2^{b_{k+1}-1}}{3^k},
\]
if $b_1 = 0$ ($n$ odd). In either case, $T(n) \in \Lambda_m$. □

Lemma 8 For all $m \in \mathbb{N}_0$, we have
\[
\Lambda_m \subset S_m. \tag{10}
\]

Proof. We use induction on $m$. The case of $m = 0$ is clearly true, since $\Lambda_0 = \{1\} = S_0$.

Assuming (10) to be true for $m$, let $n \in \Lambda_{m+1}$. From (9) we have $T(n) \in \Lambda_m$ and therefore $\sigma_\infty[T(n)] = m$. Thus, $\sigma_\infty(n) = m+1$ and the result follows. □

Exercise 9 Show that
\[
T(n) = \left( n + \frac{1}{2} \right) \sin^2 \left( \frac{\pi}{2} n \right) + \frac{n}{2}. \tag{11}
\]

The other inclusion is also true.

Theorem 10 For all $m \in \mathbb{N}_0$,
\[
S_m \subset \Lambda_m.
\]
Proof. Clearly, \( S_m = \{2^m\} = \Lambda_m, \ 0 \leq m \leq 3 \).

Let \( m \geq 4 \) and \( s \in S_m \). Using (11) we can write the recurrence (6) as

\[
x^{(s)}_{n+1} = \left[ x^{(s)}_n + \frac{1}{2} \right] \theta_n + \frac{x^{(s)}_n}{2}, \quad x^{(s)}_0 = s,
\]

where

\[
\theta_n = \sin^2 \left( \frac{\pi}{2} x^{(s)}_n \right), \quad \text{i.e.,} \quad x^{(s)}_n \equiv \theta^{(s)}_n \quad \text{mod} \ 2.
\]

Assuming \( \{\theta_n\} \) to be a known sequence, the solution of (12) is

\[
x^{(s)}_n = 2^{-n} \prod_{j=0}^{n-1} \left( 2\theta^{(s)}_j + 1 \right) \left( s + \sum_{k=0}^{n-1} \frac{2^k \theta^{(s)}_k}{\prod_{j=0}^k \left( 2\theta^{(s)}_j + 1 \right)} \right),
\]

or using (13)

\[
x^{(s)}_n = 2^{-n} \Theta^{(n-1)} \left( s + \sum_{k=0}^{n-1} \frac{2^k \theta^{(s)}_k}{3\Theta^{(k)}} \right),
\]

with

\[
\Theta (x) = \sum_{j=0}^x \theta^{(s)}_j.
\]

Setting \( n = m \) and solving for \( s \) in (14), we obtain \( x^{(s)}_m = 1 \)

\[
s = \frac{2^m}{3\Theta^{(m-1)}} - \sum_{k=0}^{m-1} \frac{2^k \theta^{(s)}_k}{3\Theta^{(k)}}.
\]

Let \( l = \Theta (m-1) \). From (13) and (15), we see that \( \Theta (x) \) is a step function with unit jumps at \( x = b_1, b_2, \ldots, b_l, m \), where \( 0 \leq b_1 < b_2 < \cdots < b_l < m \).

Therefore, we can rewrite (16) as

\[
s = \frac{2^m}{3^l} - \sum_{k=1}^l \frac{2^{b_k}}{3^k}.
\]

Finally, since \( x^{(s)}_{m-3} = 8, x^{(s)}_{m-2} = 4, x^{(s)}_{m-1} = 2 \) and \( x^{(s)}_m = 1 \), the penultimate jump must occur before or at \( x = m - 4 \). Thus, \( b_l \leq m - 4 \) and \( l = \Theta (m-1) \leq m - 3 \). ■
Corollary 11 The Collatz conjecture is true if and only if every natural number \( n \) can be represented in the form

\[
n = \frac{2^m}{3^l} - \sum_{k=1}^{l} \frac{2^{b_k}}{3^k} \quad \text{(C5)}
\]

for some \( m, l, b_1, \ldots, b_l \in \mathbb{N}_0 \), with

\[
0 \leq l \leq m - 3 \quad \text{and} \quad 0 \leq b_1 < b_2 < \cdots < b_l \leq m - 4.
\]

Corollary 11 is not a proof of the Collatz conjecture, but it provides a lot of information on the set \( \mathcal{C} \) and the function \( \sigma_\infty(n) \). When \( l = 0 \), we recover the known fact that \( 2^m \in S_m, \forall m \in \mathbb{N}_0 \). For \( l = 1 \), we have the following result.

Lemma 12 For all \( m \in \mathbb{N} \), we have

\[
\begin{align*}
\frac{2^m}{3} & - \frac{2^{2k}}{3} \in S_m, \quad 0 \leq k \leq \frac{m - 4}{2}, \quad m \; \text{even}, \\
\frac{2^m}{3} & - \frac{2^{2k+1}}{3} \in S_m, \quad 0 \leq k \leq \frac{m - 5}{2}, \quad m \; \text{odd}.
\end{align*}
\]

Proof. Let \( n \in \Lambda_m \), with \( l = 1 \). We have

\[
n = \frac{2^m - 2^{b_1}}{3} = 2^{b_1} \times \frac{2^{m-b_1} - 1}{3}, \quad 0 \leq b \leq m - 4.
\]

Thus, \( 2^{m-b_1} \equiv 1 \pmod{2} \) and therefore \( m - b_1 \equiv 0 \pmod{2} \). Considering the cases \( m \) even and \( m \) odd, the result follows. \( \blacksquare \)

When \( l = 2 \), the situation is slightly more complicated. To simplify matters, we restrict ourselves to the case of \( n \) being odd.

Proposition 13 For all \( m \geq 5 \), with \( m \neq 6, 8 \), we have

\[
\begin{align*}
\frac{2^m}{3^2} & - \frac{2^0}{3} - \frac{2^{m-2-6k}}{3^2} \in S_m, \quad 1 \leq k \leq \frac{m - 2}{6}, \quad m \geq 10 \; \text{even}, \\
\frac{2^m}{3^2} & - \frac{2^0}{3} - \frac{2^{m-4-6k}}{3^2} \in S_m, \quad 0 \leq k \leq \frac{m - 4}{6}, \quad m \geq 5 \; \text{odd}.
\end{align*}
\]
**Proof.** Let $n \in \Lambda_m$, odd, with $l = 2$. Then,

$$n = \frac{2^m - 3 - 2^{b_2}}{9}$$

and therefore

$$2^m - 2^{b_2} = 2^{b_2} \times (2^{m-b_2} - 1) \equiv 3 \pmod{9}.$$  

Considering all possible cases, we have

1) $2^{b_2} \equiv 1 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 3 \pmod{9}$, which implies

$$b_2 \equiv 0 \pmod{6}, \quad m - b_2 \equiv 2 \pmod{6}.$$  

2) $2^{b_2} \equiv 2 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 6 \pmod{9}$, which implies

$$b_2 \equiv 1 \pmod{6}, \quad m - b_2 \equiv 4 \pmod{6}.$$  

3) $2^{b_2} \equiv 4 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 3 \pmod{9}$, which implies

$$b_2 \equiv 2 \pmod{6}, \quad m - b_2 \equiv 2 \pmod{6}.$$  

4) $2^{b_2} \equiv 5 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 6 \pmod{9}$, which implies

$$b_2 \equiv 5 \pmod{6}, \quad m - b_2 \equiv 4 \pmod{6}.$$  

5) $2^{b_2} \equiv 7 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 3 \pmod{9}$, which implies

$$b_2 \equiv 4 \pmod{6}, \quad m - b_2 \equiv 2 \pmod{6}.$$  

6) $2^{b_2} \equiv 8 \pmod{9}$ and $2^{m-b_2} - 1 \equiv 6 \pmod{9}$, which implies

$$b_2 \equiv 3 \pmod{6}, \quad m - b_2 \equiv 4 \pmod{6}.$$  

Thus, for $m$ even we shall have $m - b_2 \equiv 2 \pmod{6}$ or $b_2 \equiv m - 2 \pmod{6}$ and for $m$ odd we need $m - b_2 \equiv 4 \pmod{6}$ or $b_2 \equiv m - 4 \pmod{6}$, with $1 \leq b_2 \leq m - 4$. Writing $b_2$ in terms of $m$, the result follows. $\blacksquare$

From Corollary [11] we can also get an idea of how the total stopping time $\sigma_\infty(n)$ behaves if the Collatz conjecture is true. Solving for $m$ in (C5) we have

$$m = \frac{1}{\ln(2)} \ln \left(3^l n + 3^l \sum_{k=1}^{l} \frac{2^{b_k}}{3^k} \right).$$
In other words, \( \sigma_\infty(n) \) lies on the family of parametric curves

\[
\frac{1}{\ln(2)} \ln(3^i n + j), \quad i, j \in \mathbb{N}_0, \quad i \leq j.
\]

For example, we have

| \( n \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- |
| \( \sigma_\infty(n) \) | 1 | 5 | 2 | 4 | 6 | 11 | 3 |
| \( (i, j) \) | (0, 0) | (2, 5) | (0, 0) | (1, 1) | (2, 10) | (5, 347) | (0, 0) |

**Exercise 14** Prove that

\[
\frac{\ln(n)}{\ln(2)} \leq \sigma_\infty(n) \quad \forall n \in \mathbb{N}.
\]

**2.0.1 Binary sequences**

Another approach is to study the sequence \( \{ \theta_k(n), k \geq 0 \} \), which contains a wealth of information.

**Definition 15** Let \( \tau : \mathcal{C} \to \mathbb{N} \) be defined by

\[
\tau(n) = \sum_{k=0}^{\sigma_\infty(n)} \theta_k(n) 2^k.
\]  

(17)

For example, we have

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \tau(n) \) | 1 | 2 | 35 | 4 | 17 | 70 | 2199 | 8 |

Clearly, \( \tau(2^n) = 2^n \), \( \forall n \in \mathbb{N}_0 \).

**Exercise 16** Find \( \tau(n) \) for \( 9 \leq n \leq 100 \).

Let’s study the image of \( \Lambda_m \) by \( \tau \). We have

\[
\tau(\Lambda_0) = \{1\}, \quad \tau(\Lambda_1) = \{2\}, \quad \tau(\Lambda_2) = \{4\}, \quad \tau(\Lambda_3) = \{8\},
\]

\[
\tau(\Lambda_4) = \{16, 17\}, \quad \tau(\Lambda_5) = \{32, 34, 35\}, \quad \tau(\Lambda_6) = \{64, 65, 68, 70\}, \quad \tau(\Lambda_7) = \{128, 130, 136, 137, 140\}, \quad \tau(\Lambda_8) = \{256, 257, 260, 272, 274, 280\}.
\]
Exercise 17  Let
\[ \Lambda_m = \left\{ \lambda_1^{(m)}, \ldots, \lambda_{N_m}^{(m)} \right\}, \]
where \( N_m = \#\Lambda_m \) denotes the number of elements in the set \( \Lambda_m \). Prove that \( \forall m \in \mathbb{N}_0 \) there exist a sequence
\[ 2^m > \alpha_1^{(m)} \geq \cdots \geq \alpha_{N_m}^{(m)} = 0, \tag{19} \]
such that
\[ \tau \left[ \lambda_k^{(m)} \right] = 2^m + \alpha_k^{(m)}. \]

From (18), we have
\[
\begin{array}{c|cccccccc}
   m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
   \alpha_1^{(m)} & 0 & 0 & 0 & 1 & 3 & 6 & 12 & 24 \\
   \alpha_2^{(m)} & 0 & 0 & 0 & 2 & 4 & 9 & 18 \\
\end{array}
\]

It follows from (19) that
\[ \#\Lambda_m \leq \alpha_1^{(m)} + 1, \quad \forall m \geq 0. \]

Using (16), we can define an inverse function for \( \tau(n) \).

Definition 18  Let \( \phi : \mathbb{N} \to \mathbb{Q} \) be defined by
\[ \phi(n) = \frac{2^m}{3^{\Phi(m-1)}} - \sum_{k=0}^{m-1} 2^k \beta_k^{(n)} \]

where \( \beta_{m-1}^{(n)} \cdots \beta_1^{(n)} \beta_0^{(n)} \) is the binary representation of \( n \), i.e.,
\[ \sum_{k=0}^{m-1} 2^k \beta_k^{(n)} = n \quad \text{and} \quad \Phi(x) = \sum_{j=0}^{x} \beta_j^{(n)}. \]

For example, we have
\[
\begin{array}{c|cccccccc}
   n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
   \phi(n) & 1 & 2 & \frac{7}{3} & 4 & \frac{5}{3} & \frac{7}{3} & -\frac{1}{3} & 8 \\
\end{array}
\]

Exercise 19  Find \( \phi(n) \) for \( 9 \leq n \leq 100 \).
It follows from Theorem 10 that 
\[ \phi \circ \tau (n) = n, \quad \forall n \in \mathbb{N}, \]
while (19) implies that 
\[
S_m = \phi \left( \left\{ 2^m, \ldots, 2^m + \alpha_0^{(m)} \right\} \right) \cap \mathbb{N}, \quad \forall m \geq 0.
\]

In terms of \( \phi \), the Collatz conjecture reads 
\[
\mathbb{N} \subset \phi (\mathbb{N}). \quad \text{(C6)}
\]

With (C6), we finally reach a statement equivalent to the Collatz conjecture, which is independent of the original formulation in terms of \( T(x) \). Although we have not succeeded in proving (C6), we hope that studying the function \( \phi(n) \) will shed new light on the Collatz problem.

3 Further problems

In the spirit of the Monthly, we offer a series of problems to the curious reader. Those labeled "Exercise" are relatively easy to prove, "Problems" denote results strongly supported by numerical evidence and "Conjectures" are those that we would really wish to prove, but that may turn out to be false.

**Conjecture 20** Prove that 
\[
\sigma_\infty(n) \leq \delta(n) \ln(n) \quad \forall n \geq 2,
\]
where \( \delta(n) \) is a slowly varying function, which might be eventually constant.

**Definition 21** The Abby-Normal numbers (\( \mathcal{AN} \) numbers). Let the scaled total stopping time \( \gamma(n) \) be defined by 
\[
\gamma(n) = \frac{\sigma_\infty(n)}{\ln(n)}, \quad n \geq 2.
\]

We say that \( a_k \) is the \( k \)-th \( \mathcal{AN} \) number if 
\[
\gamma(a_k) = \max \{ \gamma(n) \mid a_{k-1} \leq n \leq a_k \}, \quad k \geq 1
\]
with \( a_0 = 2 \). In other words, \( \{ \gamma(a_k) \} \) is an increasing sequence of sharp lower bounds for the function \( \delta(n) \) defined in (20).
Exercise 22  Show that

|   | 1   | 2   | 3   | 4   | 5    | 6    | 7    |
|---|-----|-----|-----|-----|------|------|------|
| \(a(n)\) | 3   | 7   | 9   | 27  | 230,631 | 626,331 | 837,799 |
| \(\gamma(n)\) | 4.55 | 5.65 | 5.92 | 21.24 | 22.51 | 23.90 | 24.12 |

From the results obtained by Eric Roosendaal [8], it follows that 6, 649, 279, 8, 400, 511, 11, 200, 681, 15, 733, 191, 63, 728, 127, 3, 743, 559, 068, 799, 100, 759, 293, 214, 567, are possible \(\mathcal{AN}\) numbers. We have

\[\gamma(100, 759, 293, 214, 567) \approx 35.17.\]

Exercise 23  Find all \(\mathcal{AN}\) numbers in the interval \([1,000,000, 100,759, 293, 214, 567]\).

Conjecture 24  Prove that there exist infinitely many \(\mathcal{AN}\) numbers.

Problem 25  Let \(V_m\) be the \(m\)-vector

\[V_m = \left[ x_0^{(m)}, \ldots, x_{\sigma_{\infty}(m)}^{(m)} \right],\]

and \(L : \mathbb{R}^N \to \mathbb{R}^N\) the linear operator defined by

\[L \left( [v_1, v_2, \ldots v_N] \right) = [v_2, v_3, \ldots v_N, v_1].\]

Let \(\theta(m)\) be the angle between \(V_m\) and \(L(V_m)\). Prove that

(i) \[\frac{1}{2} < \cos \theta(m) < \frac{7}{8}, \quad \forall m \in \mathbb{N}.\]

(ii) \[\frac{3}{4} < \cos \theta(m) < \frac{7}{8}, \quad \forall m \equiv 1(2).\]

(iii) \[\lim_{k \to \infty} \cos \theta(a_k) = \frac{7}{8}.\]

Hint: See [4].
**Problem 26** Prove that:

(i) $\forall k \geq 6 \exists m_k \in S_k$ such that $m_k + 1 \in S_k$.

(ii) $\forall k \geq 7 \exists m_k \in S_k$ such that $m_k + 2 \in S_k$.

(iii) $\forall l \in \mathbb{N} \exists K \in \mathbb{N}$ such that $\forall k \geq K \exists m_k \in S_k$ such that $m_k + l \in S_k$.

*Hint: See [3].*

**Exercise 27** Prove that

$$\lim_{k \to \infty} \frac{\#S_{k+1}}{\#S_k} = \frac{4}{3}.$$

**Exercise 28** Let

$$\zeta(m) = \# \{2 \leq k \leq m \mid \sigma_\infty(k) = \sigma_\infty(k - 1)\}.$$

Show that

$$\zeta(m) = 0, \ 2 \leq m \leq 12, \ \zeta(m) = 1, \ 13 \leq m \leq 14, \ \zeta(m) = 2, \ 15 \leq m \leq 18, \ \zeta(m) = 3, \ 19 \leq m \leq 20.$$

**Conjecture 29** Prove that

$$\lim_{m \to \infty} \frac{\zeta(m)}{m} = \frac{1}{2}.$$

**Problem 30** Prove that

$$\alpha_1^{(m)} = \lfloor 3 \times 2^{m-5} \rfloor, \ m \geq 0 \quad (21)$$

and

$$\alpha_2^{(m)} = \lfloor 605 \times 2^{m-13} \rfloor, \ m \geq 0,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

**Problem 31** Let

$$S_m = \{s^{(m)}_1, \ldots, s^{(m)}_{N_m}\}, \ N_m = \#S_m,$$

with

$$2^m = s^{(m)}_1 > s^{(m)}_2 > \cdots > s^{(m)}_{N_m}.$$

Prove that

$$\phi(2^m + 2^{2k}) = s^{(m)}_{k+2}, \ 0 \leq k \leq \frac{m-4}{2}, \ m \geq 4, \ m \equiv 0 (\text{mod } 2)$$

$$\phi(2^m + 2^{2k+1}) = s^{(m)}_{k+2}, \ 0 \leq k \leq \frac{m-5}{2}, \ m \geq 5, \ m \equiv 1 (\text{mod } 2).$$
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