Surfaces with prescribed mean curvature in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

In this paper we study rotational surfaces in the space $\mathbb{H}^2 \times \mathbb{R}$ whose mean curvature is given as a prescribed function of their angle function. These surfaces generalize, among others, the ones of constant mean curvature and the translating solitons of the mean curvature flow. Using a phase plane analysis we construct entire rotational graphs, catenoid-type surfaces, and exhibit a classification result when the prescribed function is linear.

1 Introduction

Let us consider a function $\mathcal{H} \in C^1(S^2)$. An oriented surface $\Sigma$ immersed into $\mathbb{R}^3$ is said to be a surface with prescribed mean curvature $\mathcal{H}$ if its mean curvature function $H_{\Sigma}$ satisfies

$$H_{\Sigma}(p) = \mathcal{H}(N_p)$$

for every point $p \in \Sigma$, where $N$ denotes the Gauss map of $\Sigma$. For short, we say that $\Sigma$ is an $\mathcal{H}$-surface. Let us observe that when the function $\mathcal{H}$ is chosen as a constant $H_0$, the surfaces defined by Equation (1.1) are just the surfaces with constant mean curvature (CMC) equal to $H_0$.

The definition of this class of immersed surfaces has its origins on the famous Christoffel and Minkowski problems for ovaloids [Chr, Min]. The existence of prescribed mean curvature ovaloids was studied, among others, by Alexandrov, Pogorelov and Guan-Guan [Ale, Pog, GuGu], while the uniqueness in the Hopf sense has been recently achieved by Gálvez and Mira [GaMi1]. In this fashion, the first author jointly with Gálvez and Mira started to develop the global theory of surfaces with prescribed mean curvature in $\mathbb{R}^3$, taking as motivation the fruitful theory of CMC surfaces, see [BGM1, BGM2]. The first author also proved the existence and uniqueness to the Björling problem [Bue3] and obtained half space theorems for $\mathcal{H}$-surfaces [Bue4].

A relevant case of $\mathcal{H}$-surfaces appears when the function $\mathcal{H}$ depends only on the height of the sphere, i.e., if there exists a one dimensional function $\mathfrak{h} \in C^1([-1,1])$ such that $\mathcal{H}(x) = \mathfrak{h}(\langle x, e_3 \rangle)$ for every $x \in S^2$. In this case, $\mathcal{H}$ is called rotationally symmetric and Equation (1.1) reads as

$$H_{\Sigma}(p) = \mathfrak{h}(\langle N_p, e_3 \rangle) = \mathfrak{h}(\nu(p)),$$

Mathematics Subject Classification: 53A10, 53C42, 34C05, 34C40.

Keywords: Prescribed mean curvature surface, non-linear autonomous system, phase plane analysis.
for every point \( p \in \Sigma \), where \( \nu(p) := \langle N_p, e_3 \rangle \) is the so-called angle function of \( \Sigma \). In this setting, the authors have obtained a classification result for rotational \( \mathcal{H} \)-hypersurfaces in \( \mathbb{R}^{n+1} \) satisfying (1.2), whose prescribed function \( \mathfrak{h} \) is linear, i.e. \( \mathfrak{h}(y) = ay + \lambda \), \( a, \lambda \in \mathbb{R} \), see [BuOr].

Note that for defining Equation (1.2) we only need to measure the projection of a unit normal vector field onto a Killing vector field. Consequently, surfaces obeying (1.2) can be also defined in the product spaces \( \mathbb{M}^2 \times \mathbb{R} \), where \( \mathbb{M}^2 \) is a complete surface, as follows:

**Definition 1.1** Let \( \mathfrak{h} \) be a \( C^1 \) function on \([-1, 1]\). An oriented surface \( \Sigma \) immersed in \( \mathbb{M}^2 \times \mathbb{R} \) has prescribed mean curvature \( \mathfrak{h} \) if its mean curvature function \( H_{\Sigma} \) satisfies

\[
H_{\Sigma}(p) = \mathfrak{h}(\langle \eta_p, \partial_z \rangle),
\]

for every point \( p \in \Sigma \), where \( \eta \) is a unit normal vector field on \( \Sigma \) and \( \partial_z \) is the unit vertical Killing vector field on \( \mathbb{M}^2 \times \mathbb{R} \).

Again, note that \( \nu(p) := \langle \eta_p, \partial_z \rangle \) is the angle function of \( \Sigma \). In analogy with the Euclidean case, we will simply say that \( \Sigma \) is an \( \mathfrak{h} \)-surface.

Observe that if \( \mathfrak{h} \equiv H_0 \in \mathbb{R} \), we recover the theory of CMC surfaces in \( \mathbb{M}^2 \times \mathbb{R} \), which experimented an extraordinary development since Abresch and Rosenberg [AbRo] defined a holomorphic quadratic differential on them, that vanishes on rotational examples. Also, if \( \mathfrak{h}(y) = y \), the \( \mathfrak{h} \)-surfaces arising are the translating solitons of the mean curvature flow, see [Bue1, Bue2, LiMa]. For a general function \( \mathfrak{h} \in C^1([-1, 1]) \) under necessary and sufficient hypothesis, the first author obtained a Delaunay-type classification result in \( \mathbb{M}^2 \times \mathbb{R} \) [Bue5], and a structure-type result for properly embedded \( \mathfrak{h} \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) [Bue6].

Inspired by the fruitful theory of \( \mathcal{H} \)-surfaces in \( \mathbb{R}^3 \), the purpose of this paper is to further investigate the theory of surfaces satisfying (1.3) in the product space \( \mathbb{H}^2 \times \mathbb{R} \). The \( \mathfrak{h} \)-surfaces studied in this paper are motivated by the examples arising in the theory of minimal surfaces and translating solitons in both \( \mathbb{R}^3 \) and \( \mathbb{H}^2 \times \mathbb{R} \), and by \( \mathcal{H} \)-hypersurfaces in \( \mathbb{R}^{n+1} \) where \( \mathcal{H} \in C^1(\mathbb{S}^n) \) is a linear function.

The rest of the introduction is devoted to detail the organization of the paper and highlight some of the main results.

In Section 2 we deduce the formulae that the profile curve of a rotational \( \mathfrak{h} \)-surface in \( \mathbb{H}^2 \times \mathbb{R} \) satisfy. The resulting ODE will be treated as a non-linear autonomous system, and its qualitative study will be carried out by developing a phase plane analysis. From the previous work [BGM2] we compile the main features of the phase plane adapted to the space \( \mathbb{H}^2 \times \mathbb{R} \). In Corollary 2.5 we prove the existence of two unique rotational \( \mathfrak{h} \)-surfaces, \( \Sigma_+ \) and \( \Sigma_- \), intersecting orthogonally the axis of rotation with unit normal \( \pm \partial_z \), respectively.

In Section 3 we prove in Proposition 3.1 the existence of entire, strictly convex \( \mathfrak{h} \)-graphs, called \( \mathfrak{h} \)-bowls in analogy with translating solitons. In Proposition 3.2 we construct properly embedded \( \mathfrak{h} \)-annuli, called \( \mathfrak{h} \)-catenoids for their resemblance to the usual minimal catenoids in \( \mathbb{R}^3 \) and \( \mathbb{H}^2 \times \mathbb{R} \).

In Section 4 we focus on \( \mathfrak{h}_\lambda \)-surfaces, which are defined to be those whose prescribed function is linear, i.e. \( \mathfrak{h}_\lambda(y) := ay + \lambda \), \( a, \lambda \in \mathbb{R} \). The relevance of \( \mathfrak{h}_\lambda \)-surfaces is that they satisfy certain
characterizations that are closely related with the theory of manifolds with density. For instance, \( h_\lambda \)-surfaces have constant weighted mean curvature equal to \( \lambda \) with respect to the density \( e^{\phi} \), where \( \phi(x) = a(x, \partial_z) \), \( \forall x \in \mathbb{H}^2 \times \mathbb{R} \). In particular, if \( \lambda = 0 \) we recover the fact that translating solitons are weighted minimal as pointed out by Ilmanen [Ilm]. Also, \( h_\lambda \)-surfaces are critical points for the weighted area functional, under compactly supported variations preserving the weighted volume (see [BCMR] and [BuOr]).

In this section we obtain our two main results in which we achieve a classification of complete rotational \( h_\lambda \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \). First, we prove that we can reduce to study the case \( h_\lambda(y) = y + \lambda, \lambda > 0 \). If such \( h_\lambda \)-surfaces intersect the axis of rotation we get:

**Theorem 1.2** Let be \( \Sigma_+ \) and \( \Sigma_- \) the complete, rotational \( h_\lambda \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) intersecting the rotation axis with upwards and downwards orientation respectively. Then:

1. For \( \lambda > 1/2 \), \( \Sigma_+ \) is properly embedded, simply connected and converges to the flat CMC cylinder \( C_\lambda \) of radius \( \text{arg tanh} \left( \frac{1}{2\lambda} \right) \). Moreover:
   1.1. If \( \lambda > \sqrt{2}/2 \), \( \Sigma_+ \) intersects \( C_\lambda \) infinitely many times.
   1.2. If \( \lambda = \sqrt{2}/2 \), \( \Sigma_+ \) intersects \( C_\lambda \) a finite number of times and is a graph outside a compact set.
   1.3. If \( \lambda < \sqrt{2}/2 \), \( \Sigma_+ \) is a strictly convex graph over the disk in \( \mathbb{H}^2 \) of radius \( \text{arg tanh} \left( \frac{1}{2\lambda} \right) \).
2. For \( \lambda \leq 1/2 \), \( \Sigma_+ \) is an entire, strictly convex graph.
3. For \( \lambda > \sqrt{5}/2 \), \( \Sigma_- \) is properly immersed (with infinitely many self-intersections), simply connected and has unbounded distance to the rotation axis.
4. For \( \lambda \leq \sqrt{5}/2 \), \( \Sigma_- \) is an entire graph. Moreover, if \( \lambda = 1 \), \( \Sigma_- \) is a horizontal plane (hence minimal and flat), and if \( \lambda \neq 1 \), \( \Sigma_- \) has positive Gauss-Kronecker curvature.

For rotational \( h_\lambda \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) non-intersecting the axis of rotation, we prove the following:

**Theorem 1.3** Let \( \Sigma \) be a complete, rotational \( h_\lambda \)-surface in \( \mathbb{H}^2 \times \mathbb{R} \) non-intersecting the rotation axis. Then, \( \Sigma \) is properly immersed and diffeomorphic to \( S^1 \times \mathbb{R} \). Moreover,

1. If \( \lambda > 1/2 \), then:
   1.1. either \( \Sigma \) is the CMC cylinder \( C_\lambda \) of radius \( \text{arg tanh} \left( \frac{1}{2\lambda} \right) \), or
   1.2. one end converges to \( C_\lambda \) with the same asymptotic behavior as in item 1 in Theorem 1.2, and:
      a) If \( \lambda > \sqrt{5}/2 \), the other end of \( \Sigma \) has unbounded distance to the rotation axis and self-intersects infinitely many times.
      b) If \( \lambda \leq \sqrt{5}/2 \), the other end is a graph outside a compact set.
2. If \( \lambda \leq 1/2 \), then both ends are graphs outside compact sets.
2 A phase plane analysis of rotational \( \mathfrak{h} \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \)

In the development of this section, we regard \( \mathbb{H}^2 \times \mathbb{R} \) as a submanifold of \( \mathbb{R}^3 \times \mathbb{R} \) endowed with the metric \(+, +, −, +\). Consider a regular curve parametrized by arc-length \( \alpha(s) = (x_1(s), 0, x_3(s), z(s)) \subset \mathbb{H}^2 \times \mathbb{R}, x_1(s) > 0, s \in I \subset \mathbb{R} \), contained in a vertical plane passing through the point \((0, 0, 1, 0)\), and rotate it around the vertical axis \{((0,0,1)) \times \mathbb{R} \}. Since \( x_1^2(s) - x_3^2(s) = -1 \), there exists a \( C^1 \) function \( x(s) > 0 \) such that

\[
\alpha(s) = (\sinh(x(s)), 0, \cosh(x(s)), z(s)).
\]

For saving notation, we will simply denote \( \alpha(s) \) by \((x(s), z(s))\). Now, note that the image of \( \alpha(s) \) under this 1-parameter group of rotations generates an immersed, rotational surface \( \Sigma \) in \( \mathbb{H}^2 \times \mathbb{R} \) parametrized by

\[
\psi(s, \theta) = (\sinh(x(s)) \cos \theta, \sinh(x(s)) \sin \theta, \cosh(x(s)), z(s)) : I \times (0, 2\pi) \to \mathbb{H}^2 \times \mathbb{R},
\]

where \( \psi(s, \theta) \) is given by \( \nu(\psi(s, \theta)) = x'(s), \forall s \in I \). Moreover, the principal curvatures on \( \Sigma \) are

\[
\kappa_1 = \kappa_\alpha = x'z'' - x''z', \quad \kappa_2 = \frac{z'}{\tanh x},
\]

where \( \kappa_\alpha \) stands for the geodesic curvature of the profile curve \( \alpha(s) \). Thus, the mean curvature of \( \Sigma \) is easily related to the coordinates of \( \alpha(s) \) and bearing in mind that \( x'^2 + z'^2 = 1 \), we get that the function \( x \) is a solution of the autonomous second order ODE

\[
x'' = \frac{1 - x'^2}{\tanh x} - 2\varepsilon H_\Sigma \sqrt{1 - x'^2}, \quad \varepsilon = \text{sign}(z'), \tag{2.3}
\]

on every subinterval \( J \subset I \) where \( z'(s) \neq 0 \) for all \( s \in J \).

Now, assume that \( \Sigma \) is an \( \mathfrak{h} \)-surface for some \( \mathfrak{h} \in C^1([-1,1]) \), that is, by using (1.3) we get

\[
H_\Sigma(\psi(s, \theta)) = \mathfrak{h}(x'(s)).
\]

Hence, after the change \( y = x' \), we can rewrite (2.3) as the following autonomous ODE system

\[
\left( \begin{array}{c} x \\ y \end{array} \right)' = \left( \begin{array}{c} 1 - y^2 \\ \frac{y}{\tanh x} - 2\varepsilon \mathfrak{h}(y) \sqrt{1 - y^2} \end{array} \right) =: F_\varepsilon(x, y). \tag{2.4}
\]

At this point, we study this system by using a phase plane analysis as the first author did in [BGM2]. Specifically, the phase plane of (2.4) is defined as the half-strip \( \Theta_\varepsilon := (0, \infty) \times (-1, 1) \), with coordinates \((x, y)\) denoting, respectively, the distance to the rotation axis and the angle function of \( \Sigma \). The solutions \( \gamma(s) = (x(s), y(s)) \) of system (2.4) are called orbits, and the equilibrium points are the points \( e_\varepsilon^x = (x_0^\varepsilon, y_0^\varepsilon) \in \Theta_\varepsilon \) such that \( F_\varepsilon(x_0^\varepsilon, y_0^\varepsilon) = 0 \), which must lie in the axis \( y = 0 \) according to system (2.4).

Next, we compile some features that can be derived from the study of the phase plane.

**Lemma 2.1** In the above conditions, the following properties hold:
1. If \( \varepsilon h(0) > 0 \), there is a unique equilibrium \( e_0^\varepsilon = (x_0^\varepsilon, 0) \) of \( (2.4) \) in \( \Theta_{\varepsilon} \) given by
\[
x_0^\varepsilon = \text{arg tanh} \left( \frac{1}{2\varepsilon h(0)} \right).
\]
This equilibrium generates the right circular cylinder \( S^1(x_0^\varepsilon) \times \mathbb{R} \) of constant mean curvature \( h(0) \) and vertical rulings.

2. The Cauchy problem associated to \( (2.4) \) for the initial condition \( (x_0, y_0) \in \Theta_{\varepsilon} \) has local existence and uniqueness. Thus, the orbits provide a foliation by regular \( C^1 \) curves of \( \Theta_{\varepsilon} \) (or of \( \Theta_{\varepsilon} - \{e_0^\varepsilon\} \), in case some \( e_0^\varepsilon \) exists).

3. The instants \( s_0 \in J \) such that \( \kappa_\alpha(s_0) = 0 \) are the ones for which \( x''(s_0) = y'(s_0) = 0 \), i.e. those such that \( (x(s_0), y(s_0)) \in \Gamma_{\varepsilon} := \Theta_{\varepsilon} \cap \{ x = \Gamma_{\varepsilon}(y) \} \), where
\[
\Gamma_{\varepsilon}(y) = \text{arg tanh} \left( \frac{\sqrt{1 - y^2}}{2\varepsilon h(y)} \right).
\]
Moreover, the curve \( \Gamma_{\varepsilon} \) is empty when \( \varepsilon h(y) \leq 0 \).

4. The axis \( y = 0 \) and the curve \( \Gamma_{\varepsilon} \) divide \( \Theta_{\varepsilon} \) into connected components, called monotonicity regions, where the coordinates \( x(s) \) and \( y(s) \) of every orbit are strictly monotonous. Moreover, at each of these regions, we have
\[
\text{sign}(\kappa_1) = \text{sign}(-\varepsilon y'(s)), \quad \text{sign}(\kappa_2) = \text{sign}(\varepsilon).
\]

Let us focus now on the behavior of the orbits of system \( (2.4) \) in more detail. Firstly, note that we can view such orbits as vertical graphs \( y = y(x) \) where \( y \neq 0 \) (i.e. where \( x' \neq 0 \)), so the chain rule yields
\[
y'(s) = \frac{dy}{ds} = \frac{dy}{dx} \frac{dx}{ds} = y'(x)y(x) = \frac{1 - y(x)^2}{\text{tanh}x} - 2\varepsilon h(y(x))\sqrt{1 - y(x)^2}.
\]
Since at each monotonicity region the sign of the quantity \( y(x)y'(x) \) is constant, the behavior of the orbit passing through some \( (x_0, y_0) \in \Theta_{\varepsilon} \) is determined by the signs of \( y_0 \) and \( \Gamma_{\varepsilon}(y_0) - x_0 \) (whenever \( \Gamma_{\varepsilon}(y_0) \) exists). We detail it next:

**Lemma 2.2** In the above conditions, the behavior of the orbit of \( (2.4) \) passing through a given point \( (x_0, y_0) \in \Theta_{\varepsilon} \) such that \( \Gamma_{\varepsilon}(y_0) \) exists is described as follows:

1. If \( x_0 > \Gamma_{\varepsilon}(y_0) \) (resp. \( x_0 < \Gamma_{\varepsilon}(y_0) \)) and \( y_0 > 0 \), then \( y(x) \) is strictly decreasing (resp. increasing) at \( x_0 \).
2. If \( x_0 > \Gamma_{\varepsilon}(y_0) \) (resp. \( x_0 < \Gamma_{\varepsilon}(y_0) \)) and \( y_0 < 0 \), then \( y(x) \) is strictly increasing (resp. decreasing) at \( x_0 \).
3. If \( y_0 = 0 \), then the orbit passing through \( (x_0, 0) \) is orthogonal to the \( x \) axis.
4. If \( x_0 = \Gamma_{\varepsilon}(y_0) \), then \( y'(x_0) = 0 \) and \( y(x) \) has a local extremum at \( x_0 \).
The following result discusses how an orbit \( \gamma(s) = (x(s), y(s)) \in \Theta_\varepsilon \) behaves when \( x(s) \to \infty \).

**Proposition 2.3** Let \( \gamma(s) = (x(s), y(s)) \) be an orbit in \( \Theta_\varepsilon \) such that \( (x(s), y(s)) \to (\infty, y_0) \) when \( |s| \to \infty \), with \( y_0 \in (-1, 1) \). Then, \( 2\varepsilon \eta(y_0) = \sqrt{1 - y_0^2} \).

**Proof:** Suppose that \( \gamma(s) \in \Theta_\varepsilon \) satisfies \( x(s) \to \infty \) and \( y(s) \to y_0 \in (-1, 1) \) when \( |s| \to \infty \). Then, there exists \( s_0 \in \mathbb{R} \) such that for every \( s \in \mathbb{R} \) satisfying \( |s| > |s_0| \), \( \gamma(s) \) is strictly contained in some monotonicity region and so does not intersect the axis \( y = 0 \). Thus, \( \gamma(s) \) can be written as a vertical graph \( y = y(x) \) satisfying \( y(x) \to y_0 \) and \( y'(x) \to 0 \) as \( x \to \infty \), and substituting into Equation (2.7) we get

\[
1 - y_0^2 - 2\varepsilon \eta(y_0)\sqrt{1 - y_0^2} = 0
\]

concluding the result. \( \square \)

Additionally, we highlight that the possible endpoints of an orbit are restricted as shown in [BGM2, Theorem 4.1, pp. 13-14].

**Lemma 2.4** No orbit in \( \Theta_\varepsilon \) can converge to some point of the form \((0, y)\) with \(|y| < 1\).

From this result, we conclude that in case that an orbit converges to the axis \( x = 0 \), it must do it to the points \((0, \pm 1)\). However, we point out that the existence of an orbit with endpoint at \((0, \pm 1)\) can not be guaranteed by solving the Cauchy problem since system (2.4) has a singularity at the points with \( x_0 = 0 \). In this case, we can ensure the existence of such an orbit by using the work of Gálvez and Mira [GaMi2] in which they solved the Dirichlet problem for radial solutions of an arbitrary fully nonlinear elliptic PDE. Therefore, it is ensured the existence of rotational \( \eta \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) intersecting the rotation axis in an orthogonal way (see Section 3 in [Bue5] for further details). Furthermore, this fact derives the following result in the phase plane \( \Theta_\varepsilon \).

**Corollary 2.5** Let be \( \varepsilon, \delta \in \{-1, 1\} \) such that \( \varepsilon \eta(\delta) > 0 \) and denote \( \sigma = \text{sign}(\delta) \). Then, there exists a unique orbit \( \gamma_\sigma \) in \( \Theta_\varepsilon \) that has \((0, \delta) \in \Theta_\varepsilon \) as an endpoint. There is no such an orbit in \( \Theta_{-\varepsilon} \).

The unique rotational \( \eta \)-surface associated to the orbit \( \gamma_\sigma \), that intersects orthogonally the axis of rotation at some point having unit normal \( \delta \partial_\sigma \), will be denoted by \( \Sigma_\sigma \).

### 3 Existence of \( \eta \)-bowls and \( \eta \)-catenoids in \( \mathbb{H}^2 \times \mathbb{R} \)

This section is devoted to construct examples of properly embedded rotational \( \eta \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \), under some additional assumptions over the function \( \eta \in C^1([-1, 1]) \). For this purpose, we follow the ideas compiled in Section 3 in [BGM2].

Firstly, we show the existence of entire, strictly convex \( \eta \)-graphs in \( \mathbb{H}^2 \times \mathbb{R} \). Recall that there exists an entire graph with CMC equal to \( H_0 \in \mathbb{R} \) if and only if \( |H_0| \leq 1/2 \). In addition, there exists a sphere with CMC equal to \( H_0 \) if and only if \( |H_0| > 1/2 \).
Regarding $h$-surfaces, a necessary and sufficient condition to ensure the existence of an $h$-sphere in $\mathbb{H}^2 \times \mathbb{R}$ is that $h$ must be an even function satisfying
\[
2|h(y)| > \sqrt{1 - y^2}
\]
for every $y \in [-1, 1]$ (see Proposition 3.6 and Theorem 4.1 in [Bue5]). Furthermore, under such hypotheses over $h$, we must take into account that the existence of an $h$-sphere in $\mathbb{H}^2 \times \mathbb{R}$ forbids the existence of entire vertical $h$-graphs in $\mathbb{H}^2 \times \mathbb{R}$, since this fact would yield to a contradiction with the maximum principle. Therefore, we construct the announced $h$-graphs by assuming the failure of inequality (3.1).

**Proposition 3.1** Let $h$ be a $C^1$ function on $[-1, 1]$, and suppose that there exists $y_0 \in (0, 1]$ (resp. $y_0 \in [-1, 0]$) such that $2h(y_0) = \sqrt{1 - y_0^2}$. Then, there exists an upwards-oriented (resp. downwards-oriented) entire rotational $h$-graph $\Sigma$ in $\mathbb{H}^2 \times \mathbb{R}$. Moreover:

1. either $\Sigma$ is a horizontal plane,

2. or $\Sigma$ is a strictly convex graph.

**Proof:** If $y_0 = 1$ (resp. $y_0 = -1$), then $h(1) = 0$ (resp. $h(-1) = 0$), and we choose the surface $\Sigma$ as the horizontal plane $\mathbb{H}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$, which is minimal. Then, by considering $\Sigma$ with upwards orientation (resp. downwards orientation), its angle function is $\nu \equiv 1$ (resp. $\nu \equiv -1$), and so from (1.3) the result is trivial.

Now, suppose that $2h(y_0) = \sqrt{1 - y_0^2}$ holds for some $y_0 \in [0, 1)$, assume $h(1) > 0$ and define $y_0 := \max\{y \in (0, 1) : 2h(y) = \sqrt{1 - y^2}\}$. Note that $y_0$ is well defined since $h(1) > 0$, and by continuity $2h(y) > \sqrt{1 - y^2}$ if $y \in (y_0, 1]$. Hence, the horizontal graph $\Gamma_1 = \Theta_1 \cap \Gamma_1(y)$ defined by (2.5) has one connected component when we restrict $\Gamma_1(y)$ to $(y_0, 1]$. Moreover, it satisfies $\Gamma_1(1) = 0$ and $\Gamma_1(y) \to \infty$ as $y \to y_0$.

Let us take $\Lambda := \{(x, y) \in \Theta_1 : y > y_0\}$, and define $\Lambda_+ := \{(x, y) \in \Lambda : x > \Gamma_1(y)\}$ and $\Lambda_- := \{(x, y) \in \Lambda : x < \Gamma_1(y)\}$. Note that $\Lambda \setminus \Gamma_1$ is divided into two connected components $\Lambda_+$ and $\Lambda_-$, which are precisely monotonicity regions of $\Theta_1$ because of item 4 in Lemma 2.1. Indeed, by Lemma 2.2 each orbit $y = y(x)$ in $\Lambda_+$ (resp. $\Lambda_-$) satisfies that $y'(x) < 0$ (resp. $y'(x) > 0$).

Now, from Corollary 2.5 it is known that there exists a unique orbit $\gamma_+$ in $\Theta_1$ with $(0, 1)$ an endpoint. Additionally, by the aforementioned monotonicity properties and Lemma 2.2 it is clear that $\gamma_+$ is globally contained in $\Lambda_+$. Thus, $\gamma_+$ can be globally defined by a graph $y = f(x)$, where $f \in C^1([0, \infty))$ satisfies $f(0) = 1$, $f(x) \to y_1 \geq y_0$ as $x \to \infty$, and $f'(x) < 0$ for all $x > 0$. As a matter of fact, Proposition 2.3 ensures us that $y_1 = y_0$ (see Figure 1, left).

Consequently, the $h$-surface $\Sigma_+$ generated by the orbit $\gamma_+$ is an entire rotational graph in $\mathbb{H}^2 \times \mathbb{R}$. It remains to prove that $\Sigma_+$ is strictly convex. On the one hand, since $\gamma_+$ is totally contained in $\Theta_1$, hence $\varepsilon = 1$, we deduce by (2.6) that $\kappa_2$ of $\Sigma_+$ is everywhere positive. On the other hand, $\gamma_+$ is totally contained in $\Lambda_+$, so (2.6) implies that $\kappa_1$ of $\Sigma_+$ is also everywhere positive concluding the proof of this case.
To finish, note that the case $h(-1) > 0$, $y_* \leq 0$ is treated analogously; and the two remaining cases, $h(1) < 0$, $y_* \geq 0$ and $h(-1) < 0$, $y_* \leq 0$, can be reduced to the previous ones by changing the orientation.

These $h$-surfaces will be called $h$-bowls, in analogy with the theory of self-translating solitons of the mean curvature flow (see [Bue1, Bue2, LiMa]) that we extend with the previous result. See Figure 1, right, for a graphic of an $h$-bowl in $\mathbb{H}^2 \times \mathbb{R}$.

Secondly, we study the existence of catenoid-type rotational $h$-surfaces under appropriate conditions for the prescribed function $h$.

**Proposition 3.2** Let $h$ be a $C^1$ function on $[-1, 1]$, and suppose that $h \leq 0$ and $h(\pm 1) = 0$. Then, there exists a one-parameter family of properly embedded, rotational $h$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ of strictly negative extrinsic curvature at every point, and diffeomorphic to $S^1 \times \mathbb{R}$. Each example is a bi-graph over $\mathbb{H}^2 - D_{\mathbb{H}^2}(x_0)$, where $D_{\mathbb{H}^2}(x_0) = \{x \in \mathbb{H}^2 : |x|_{\mathbb{H}^2} < x_0\}$, for some $x_0 > 0$.

**Proof:** Let $\Sigma(x_0)$ be the rotational $h$-surface generated by the arc-length parametrized curve $\alpha(s) = (x(s), z(s))$ with the following initial conditions

$$x(0) = x_0, \ z(0) = 0, \text{ and } z'(0) = 1, \text{ with } x_0 > 0.$$ 

Then, the orbit $\gamma(s) = (x(s), y(s))$ of system (2.4) associated to $\alpha(s)$ passes through the point $(x_0, 0)$ at $s = 0$. Moreover, $\gamma$ is contained in $\Theta_1$ around such a point, that is, $\varepsilon = 1$.

Observe that the curve $\Gamma_1$ given by (2.5) does not exist because of the assumption $h \leq 0$. Consequently, by item 4 in Lemma 2.1 there are two monotonicity regions of $\Theta_1$ given by $\Lambda_+ := \{(x, y) \in \Theta_1 : y > 0\}$ and $\Lambda_- := \{(x, y) \in \Theta_1 : y < 0\}$. Then, from (2.4) we know that $\gamma$ satisfies $x' > 0$ and $y' > 0$ in $\Lambda_+$, and $x' < 0$ and $y' > 0$ in $\Lambda_-$; see Figure 2, left.

Let us prove now that $\gamma$ must be a proper arc strictly contained in $\Theta_1$ satisfying $\gamma(s) \to (\infty, \pm 1)$ as $s \to \pm \infty$. First note that, the assumption $h \leq 0$ implies that given $y_0 \in (-1, 1)$, the equation $2h(y_0) = \sqrt{1 - y_0^2}$ has no solutions, so from Proposition 2.3 $\gamma$ cannot satisfy $\gamma(s) \to (\infty, y_0)$ as $s \to \pm \infty$. Second, note that $\gamma$ cannot approach the set $\partial \Theta_1$ since $\gamma$ is a proper arc.
\(\pm \infty\). Second, since \(h \in C^1\) and \(h(\pm 1) = 0\), from the uniqueness of the Cauchy problem associated to (2.4), the curve \((s, \pm 1), s > 0\), is a solution to (2.4) which corresponds to a horizontal plane \(\mathbb{H}^2 \times \{t_0\}, t_0 \in \mathbb{R}\), endowed with \(\pm \partial_z\) as unit normal, that is, \(\gamma(s)\) cannot satisfy \(\gamma(s_0) = (x_0, \pm 1)\) for some \(s_0 \in \mathbb{R}\). Finally, it remains to show that \(\gamma(s)\) cannot converge to some \((x_0, \pm 1)\), \(x_0 > 0\), as \(s \to \pm \infty\). Otherwise, there would exist \(s_0 \in \mathbb{R}\) such that for \(|s| > |s_0|\), \(x(s)\) is a monotonous function satisfying \(x(s) \to x_0\) as \(|s| \to \infty\). Thus, the mean value theorem ensures us that \(x'(s) = y(s) \to 0\), which is a contradiction with the fact that \(y(s) \to \pm 1\).

Thus, \(\Sigma(x_0)\) is a bi-graph in \(\mathbb{H}^2 \times \mathbb{R}\) over \(\Omega(x_0) := \mathbb{H}^2 - \mathbb{D}_{\mathbb{H}^2}(x_0)\), with the topology of \(S^1 \times \mathbb{R}\). Indeed, \(\Sigma(x_0) = \Sigma_1 \cup \Sigma_2\) where both \(\Sigma_i\) are graphs over \(\Omega(x_0)\) with \(\partial \Sigma_i = \partial \Omega(x_0)\) and \(\Sigma_i\) meets the horizontal plane \(\mathbb{H}^2 \times \{0\}\) in an orthogonal way along \(\partial \Sigma_i\) (see Figure 2 right).

It remains to prove that the extrinsic curvature of \(\Sigma(x_0)\) is strictly negative. By (2.4) we get \(y'(s) > 0\) for all \(s\), so from (2.6) we derive \(\kappa_1 < 0\) and \(\kappa_2 > 0\) at every \(p \in \Sigma\).

This one-parameter family of rotational \(h\)-surfaces is a generalization of the usual minimal catenoids in \(\mathbb{H}^2 \times \mathbb{R}\), and this is the reason for calling them \(h\)-catenoids. As happens for the minimal catenoids, the \(h\)-catenoids are parametrized by their necksizes, i.e. the distance of their waists to the axis of rotation.

4 Classification of rotational \(h\)-surfaces with linear prescribed mean curvature

Our aim in this section is to classify the rotational examples of the following class of \(h\)-surfaces:

**Definition 4.1** An oriented surface \(\Sigma\) immersed in \(\mathbb{H}^2 \times \mathbb{R}\) is an \(h_\lambda\)-surface if its mean curvature function \(H_{\Sigma}\) satisfies

\[
H_{\Sigma}(p) = h_\lambda(\nu(p)) = a\nu(p) + \lambda, \quad \forall p \in \Sigma, \quad a, \lambda \in \mathbb{R}.
\]  

(4.1)

Note that if \(a = 0\), then we are studying surfaces with constant mean curvature equal to \(\lambda\). Also, if \(\lambda = 0\) the \(h_\lambda\)-surfaces are translating solitons of the mean curvature flow, see [Bue1, Bue2, LiMa].
Hence, we suppose that \( a, \lambda \) are not null in order to avoid these cases. After a homothety in \( \mathbb{H}^2 \times \mathbb{R} \) we can suppose \( a = 1 \) in Equation (4.1). Moreover, if \( \Sigma \) is an \( \mathfrak{h}_\lambda \)-surface, then \( \Sigma \) with its opposite orientation is an \( \mathfrak{h}_{-\lambda} \)-surface. Therefore, we will assume \( \lambda > 0 \) without losing generality. In particular this implies that \( \varepsilon \mathfrak{h}(0) > 0 \) if and only if \( \varepsilon = 1 \), and consequently the equilibrium \( e_0 = (\text{arg tanh}(\frac{1}{2\varepsilon \mathfrak{h}(0)}), 0) \) can only exist in \( \Theta_1 \). The CMC vertical cylinder generated by \( e_0 \) will be denoted by \( C_\lambda \).

First, we announce two technical results that will be useful in the sequel. The first one was originally proved by López in [Lop] for surfaces in \( \mathbb{R}^3 \) whose mean curvature is given by Equation (4.1).

**Lemma 4.2** There do not exist closed \( \mathfrak{h}_\lambda \)-surfaces in \( \mathbb{H}^2 \times \mathbb{R} \).

**Proof:** Arguing by contradiction, suppose that \( \Sigma \) is a closed \( \mathfrak{h}_\lambda \)-surface in \( \mathbb{H}^2 \times \mathbb{R} \). If \( h : \Sigma \to \mathbb{R} \) denotes the height function of \( \Sigma \), it is known that the Laplace-Beltrami operator \( \Delta_{\Sigma} \) of \( h \) is \( \Delta_{\Sigma} h = 2H_{\Sigma} \langle \eta, \partial_z \rangle \). Since \( \Sigma \) is an \( \mathfrak{h}_\lambda \)-surface, we get

\[
\Delta_{\Sigma} h = 2\langle \eta, \partial_z \rangle^2 + 2\lambda \langle \eta, \partial_z \rangle.
\]

We integrate this equation in \( \Sigma \). By the divergence theorem and since \( \partial_{\Sigma} = \emptyset \), we have

\[
0 = \int_{\Sigma} \langle \eta, \partial_z \rangle^2 d\Sigma + \lambda \int_{\Sigma} \langle \eta, \partial_z \rangle d\Sigma.
\]

The second integral is zero by the divergence theorem, since the constant vector field \( \partial_z \) has zero divergence. So, the first integral vanishes, that is, \( \Sigma \) is contained in a cylindrical surface of the form \( \beta \times \mathbb{R} \), \( \beta \subset \mathbb{H}^2 \) being a curve, which contradicts that \( \Sigma \) is compact. \( \square \)

The second result forbids the existence of closed orbits in the phase plane of system (2.4) for some prescribed functions \( \mathfrak{h} \). It follows from Bendixson-Dulac theorem, a classical result which appears in most textbooks on differential equations; see e.g. [ADL].

**Theorem 4.3** Let \( \mathfrak{h} \) be a \( C^1 \) function on \([-1, 1]\) such that \( \mathfrak{h}'(y) \neq 0, \forall y \in (-1, 1) \). Then, there do not exist closed orbits in \( \Theta_\varepsilon \).

**Proof:** Let us write system (2.4) as

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}' = \begin{pmatrix}
1 - y^2 \\
\tanh x - 2\varepsilon \mathfrak{h}(y) \sqrt{1 - y^2}
\end{pmatrix} = \begin{pmatrix}
P(x, y) \\
Q(x, y)
\end{pmatrix},
\]

and define the function \( \alpha : \Theta_\varepsilon \to \mathbb{R} \) and the vector field \( V : \Theta_\varepsilon \to \Theta_\varepsilon \) as

\[
\alpha(x, y) = \frac{\sinh x}{\sqrt{1 - y^2}}, \quad V(x, y) = \alpha(x, y)(P(x, y), Q(x, y)).
\]
Arguing by contradiction, suppose that there exists some closed orbit $\gamma$ in $\Theta_\varepsilon$ and name $\Omega$ to its inner region. A simple computation yields $\text{div} V = (\alpha P)_x + (\alpha Q)_y = -2\varepsilon h'(y) \sinh x$, which has constant sign since $x > 0$ in $\Theta_\varepsilon$ and $h'(y) \neq 0$. Therefore, the divergence theorem in $\Omega$ yields

$$0 \neq \int_{\Omega} \text{div} V = \int_{\gamma} \langle V, n_\gamma \rangle = 0,$$

where $n_\gamma$ is the unit normal to the curve $\gamma$. Recall that the last integral vanishes since $V$ is everywhere tangent to $\gamma$. This contradiction proves the result. \(\square\)

In particular, the prescribed function $h_\lambda(y) = y + \lambda$, $\lambda > 0$ lies in the hypothesis of Theorem 4.3, hence the phase plane $\Theta_\varepsilon$ of system (2.4) for $h_\lambda$-surfaces does not have closed orbits.

Now, suppose that $\Sigma$ is a rotational $h_\lambda$-surface generated by an arc-length parametrized curve $\alpha(s) = (x(s), z(s))$. Then, (1.3) and (2.2) yields

$$2H_\Sigma = 2(x' + \lambda) = x'z'' - x''z' + \frac{z'}{\tanh x}. \quad (4.2)$$

Our first goal is to study the structure of the orbits around $e_0 = (\text{arg tanh} (\frac{1}{\sqrt{2} \lambda}), 0)$. Recall that $e_0$ exists if and only if $\lambda > 1/2$; otherwise, $\text{arg tanh}$ is not well defined. The linearized system of (2.4) at $e_0$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 - 4\lambda^2 & -2 \end{pmatrix}, \quad (4.3)$$

whose eigenvalues are

$$\mu_1 = -1 + \sqrt{2 - 4\lambda^2}, \quad \text{and} \quad \mu_2 = -1 - \sqrt{2 - 4\lambda^2}.$$

From standard theory of non-linear autonomous systems we derive:

**Lemma 4.4** In the above conditions, we have:

- **If** $\lambda > \sqrt{2} / 2$, then $\mu_1$ and $\mu_2$ are complex conjugate with negative real part. Thus, $e_0$ has an inward spiral structure, and every orbit close enough to $e_0$ converges asymptotically to it spiraling around infinitely many times.

- **If** $\lambda = \sqrt{2} / 2$, then $\mu_1 = \mu_2 = -1$. Thus, $e_0$ is an asymptotically stable improper node, and every orbit close enough to $e_0$ converges asymptotically to it, maybe spiraling around a finite number of times.

- **If** $0 < \lambda < \sqrt{2} / 2$, then $\mu_1$ and $\mu_2$ are different and real. Thus, $e_0$ is an asymptotically stable node and has a sink structure, hence every orbit close enough to $e_0$ converges asymptotically to it directly, i.e. without spiraling around.

Now we stand in position to prove Theorem 1.2.
Proof of Theorem 1.2: Note that the behavior of the orbits in each phase plane Θ_ε depends on the curve Γ_ε and the monotonicity regions generated by it. Consequently, we analyze three different cases for λ: λ > \sqrt{5}/2, λ = \sqrt{5}/2 and 0 < λ < \sqrt{5}/2.

Case λ > \sqrt{5}/2

Let us assume λ > \sqrt{5}/2. For ε = 1, the curve Γ_1 given by (2.5) is a compact connected arc in Θ_1 joining (0, 1) and (0, −1), whereas for ε = −1, since h is positive, the curve Γ_{−1} does not exist in Θ_{−1}. As a matter of fact, by item 4 in Lemma 2.1 we know that there are four monotonicity regions in Θ_1 which will be denoted by Λ_1, ..., Λ_4, and there are only two monotonicity regions in Θ_{−1} which will be denoted by Λ_+ and Λ_. See Figure 3.

![Figure 3: The phase planes Θ_1 and Θ_{−1} for λ > \sqrt{5}/2 with their monotonicity regions and the direction of the motion of the orbits at each of them.](image)

Now, by using Corollary 2.5 it is clear that there exists a unique orbit γ_+ (resp. γ_−) in Θ_1 with (0, 1) (resp. (0, −1)) as an endpoint.

On the one hand, let us study the behavior of γ_+. Firstly, we can suppose that such an orbit satisfies γ_+(0) = (0, 1) in Θ_1, i.e., it generates an arc-length parametrized curve α_+(s) intersecting orthogonally the rotation axis with upwards oriented unit normal at s = 0. Because of the monotonicity properties, γ_+(s) is strictly contained in the region Λ_1 for s > 0 small enough. However, the orbit γ_+ cannot stay forever in Λ_1, otherwise γ_+ would be globally defined by a graph y = f(x) such that

\[ f(0) = 1, \quad f'(x) < 0 \quad \forall x > 0 \quad \text{and} \quad \lim_{x \to \infty} f(x) = c \in [0, 1). \]

This contradicts Proposition 2.3 since λ > \sqrt{5}/2 and hence 2h_λ(y) = 2(y + λ) > \sqrt{1 - y^2}. Thus, γ_+(s) intersects the axis y = 0 in an orthogonal way at a point \(x_+, 0\) with \(x_+ > \arg \tanh(\frac{1}{2\lambda})\) at some finite instant \(s_+ > 0\).

On the other hand, for the orbit γ_− we assume that γ_−(0) = (0, −1) in Θ_1, that is, it generates an arc-length parametrized curve α_−(s) intersecting orthogonally the axis of rotation with downwards oriented unit normal at s = 0. By an analogous reasoning, we can assert that γ_−(s) intersects y = 0 orthogonally at \(x_−, 0\) with \(x_− > \arg \tanh(\frac{1}{2\lambda})\) at some finite instant \(s_− < 0\).

Now, we prove that \(x_+ < x_−\) arguing by contradiction. First, see that if \(x_+ = x_− = \bar{x}\), by uniqueness of the Cauchy problem the orbits γ_+ and γ_− could be smoothly glued together constructing a larger orbit \(\tilde{γ}\) which would be a compact arc joining the points (0, 1), \((\bar{x}, 0)\) and \((0, −1)\) and so the rotational h_λ-surface generated would be a rotational h-sphere, which is impossible because of Lemma 4.2. Additionally, if \(x_+ > x_−\), it would mean that γ_− intersect the axis y = 0 at
the left-hand side of $\gamma_+$, and consequently the only possibility for $\gamma_-$ would be to enter the region $\Lambda_1$, later $\Lambda_4$ and after that $\Lambda_3$. In any case, $\gamma_-$ cannot converge to any point $(0, y)$, $|y| < 1$ in virtue of Lemma 2.4. Repeating this process, and since $\gamma_-$ cannot self-intersect nor converge to a closed orbit by Theorem 4.3, $\gamma_-$ finishes converging asymptotically to the equilibrium $e_0$ as $s \to -\infty$, spiraling around infinitely many times. This is a contradiction with the inward spiral structure of $e_0$, since this orbit would tend to escape from $e_0$ when $s$ increases. See Figure 4.

Figure 4: The phase plane $\Theta_1$ for $\lambda > \sqrt{5/2}$ with the configuration of the orbits $\gamma_+$ and $\gamma_-$, plotted in red and orange, respectively, until they intersect the axis $y = 0$.

Let us continue by analyzing the global behavior of both orbits. Firstly, when $\gamma_+$ passes through $(x_+, 0)$, it enters to $\Lambda_2$ but cannot intersect $\gamma_-$, so $\gamma_+$ has to enter to $\Lambda_3$. After that, due to the monotonicity properties and Lemma 2.4 we deduce that $\gamma_+$ has to enter to $\Lambda_4$ and intersect $\Gamma_1$. Since $\gamma_+$ cannot self-intersect nor converge to a limit closed orbit in $\Theta_1$ in virtue of Theorem 4.3, the only possibility for $\gamma_+$ is to repeat this behavior and eventually converge asymptotically to the equilibrium $e_0$ (see the plot of $\gamma_+$ in Figure 5 top-left). Furthermore, since $\lambda > \sqrt{2}/2$, $\gamma_+$ spirals around $e_0$ infinitely many times.

In this way, $\gamma_+$ generates the curve $\alpha_+(s) = (x_+(s), z_+(s))$ satisfying: the $x_+(s)$-coordinate is bounded by the value $x_+$ and converges to $\operatorname{arg\tanh}(\frac{1}{\lambda})$; and the $z_+(s)$-coordinate is strictly increasing since $\gamma_+ \subset \Theta_1$, hence $z_+(s) > 0$. Then, $\alpha_+(s)$ is an embedded curve that converges to the line $x = \operatorname{arg\tanh}(\frac{1}{\lambda})$ intersecting it infinitely many times. Therefore, after rotating such a curve around the rotation axis, we derive that the generated surface $\Sigma_+$ is a properly embedded, simply connected $h_\lambda$-surface that converges to the CMC cylinder $C_\lambda$ intersecting it infinitely many times. See $\Sigma_+$ in Figure 5 top-right.

Now we focus on $\gamma_-$, which intersects the axis $y = 0$ at the point $\gamma_-(s_-) = (x_-, 0)$ at some finite instant $s_- < 0$. So, when the parameter $s < s_-$ decreases, $\gamma_-$ enters to $\Lambda_1$. Bearing in mind that $\gamma_+$ and $\gamma_-$ cannot intersect, from the monotonicity properties we deduce that $\gamma_-$ has as endpoint $\gamma_-(s_1) = (x_1, 1)$ with $0 < x_1 < x_-$ and $s_1 < s_-$. Then, $\gamma_-$ generates the curve $\alpha_-(s) = (x_-(s), z_-(s))$ satisfying: $x_-(s_1) = x_1$ and $x'_-(s_1) = 1$, so from (4.2) we get $z''_-(s_1) > 0$, i.e., the height of $\alpha_-$ reaches a minimum at $s_1$. If we name $\Sigma_-$ to the $h_\lambda$-surface associated to $\gamma_-$ and generated by rotating $\alpha_-(s)$, the image of the points $\alpha_-(s_1)$ under such rotation corresponds to points on the boundary of $\Sigma_-$ having unit normal $\partial_2$.

Therefore, for $s < s_1$, $s$ close enough to $s_1$, the height function $z_-(s)$ of $\alpha_-(s)$ is decreasing, i.e. $z'_-(s) < 0$. Consequently, $\alpha_-(s)$ for $s < s_1$ close enough to $s_1$ generates an orbit in $\Theta_1$ (since $\varepsilon = \operatorname{sign}(z'_-) = -1$), which will be named $\gamma_-$ for saving notation. Hence, the orbit $\gamma_-(s)$ continues
from $\Theta_1$ to $\Theta_{-1}$ as $s$ decreases from $s = s_1$. At this point, the continuation of $\gamma_-$ between the phase planes $\Theta_{\pm 1}$ has to be understood as the extension of $\Sigma_-$ by solving the Cauchy problem for rotational vertical $h_\lambda$-graphs having the same vertical unit normal.

Hence, $\gamma_-(s)$ belongs to $\Theta_{-1}$ for $s < s_1$ close enough to $s_1$ and lies in the region $\Lambda_-$. Once again, by monotonicity, $\gamma_-$ must intersect the axis $y = 0$ orthogonally and enter to $\Lambda_-$. As $\gamma_-$ cannot stay forever in $\Lambda_-$ with $x_-(s) \to \infty$ for $s \to -\infty$, we derive that there exists $s_2 < s_1$ such that $\gamma_-(s_2) = (x_2, -1)$ (see the plot of $\gamma_-$ in Figure 5 left). Repeating this process indefinitely, we construct a complete, arc-length parametrized curve $\alpha_-(s)$ with infinitely many self-intersections, whose height function increases and decreases until reaching the rotation axis. Hence, the generated rotational $h_\lambda$-surface $\Sigma_-$ is properly immersed (with self-intersections) and simply connected. See Figure 5, bottom right.

**Case $\lambda = \sqrt{5}/2$**

Assume that $\lambda = \sqrt{5}/2$. For $\varepsilon = 1$, $\Gamma_1$ is formed by two connected arcs $\Gamma^+_1$, each of them having the point $(0, \pm 1)$ as endpoint respectively, and both having the line $y = -2/\sqrt{5}$ as an asymptote. From item 4 in Lemma 2.1 we find five monotonicity regions in the phase plane $\Theta_1$ denoted by $\Lambda_1, \ldots, \Lambda_5$ (see Figure 6 left). For $\varepsilon = -1$, the phase plane $\Theta_{-1}$ is exactly the same that in the previous case $\lambda > \sqrt{5}/2$.

From Corollary 2.5 we can assert that there exists a unique orbit $\gamma_+$ (resp. $\gamma_-$) in $\Theta_1$ with $(0, 1)$ (resp. $(0, -1)$) as an endpoint.

Regarding the orbit $\gamma_+$, it converges to $e_0 = (\text{argtanh} \frac{1}{\sqrt{5}}, 0)$ as $s \to \infty$ spiraling around it.
infinitely many times in the same fashion as the orbit $\gamma_+$ studied in the previous case $\lambda > \sqrt{5}/2$, see Figure 6 left. Consequently, its corresponding $h_\lambda$-surface $\Sigma_+$ has the same behavior as the one shown in Figure 5, top right.

Now, consider the orbit $\gamma_-$ such that $\gamma_-(0) = (0, -1)$. It is clear that $\gamma_-(s)$ is totally contained in $\Lambda_3$ and when the parameter $s$ tends to $-\infty$, the orbit $\gamma_-(s)$ converges to the line $y = -2/\sqrt{5}$. Note that $\gamma_-(s)$ cannot converge to other line $y = y_0$, $y_0 \in (-1, -2/\sqrt{5})$ in virtue of Proposition 2.3 (see the plot of $\gamma_-$ in Figure 6 left). Therefore, the generated $h_\lambda$-surface $\Sigma_-$ is an entire, strictly convex graph whose angle function tends to the value $-2/\sqrt{5}$. See Figure 6 right.

Figure 6: Left: the phase plane $\Theta_1$ for $\lambda = \sqrt{5}/2$ and the orbits $\gamma_+$ and $\gamma_-$, plotted in red and orange, respectively. Right: the corresponding rotational $h_\lambda$-surface $\Sigma_-$. 

Case $\lambda < \sqrt{5}/2$

We begin by analyzing the behavior of $\Gamma_\varepsilon$ in $\Theta_\varepsilon$. Recall that the equilibrium $e_0$ exists in $\Theta_1$ if and only if $\lambda > 1/2$. Additionally, we define the candidates of asymptotes for $\Gamma_\varepsilon$ as:

$y_0^+ := \frac{1}{5}(-4\lambda + \sqrt{5 - 4\lambda^2}), \quad y_0^- := \frac{1}{5}(-4\lambda - \sqrt{5 - 4\lambda^2}).$

Let us distinguish further cases of $\lambda$:

1. If $\lambda > 1$, then $\Gamma_1$ is a disconnected arc having two connected components $\Gamma_1^+$ and $\Gamma_1^-$, with $\Gamma_+$ (resp. $\Gamma_-$) having the point $(0, 1)$ (resp. $(0, -1)$) as an endpoint and the line $y = y_0^+$ (resp. $y = y_0^-$) as asymptote; see Figure 7. The curve $\Gamma_-1$ does not exist in $\Theta_-1$ as in the previous cases.

Figure 7: The phase plane $\Theta_1$ for some $\lambda \in (1, \sqrt{5}/2)$. 

2. If $\lambda = 1$, then $\Gamma_1$ is a connected arc having $(0, 1)$ as endpoint and the line $y = y^+_0 = -3/5$ as asymptote. The line $y = y^-_0 = -1$ does not appear. The curve $\Gamma_{-1}$ does not exist in $\Theta_{-1}$.

3. If $\lambda < 1$, then $\Gamma_1$ is a connected arc having $(0, 1)$ as endpoint and the line $y = y^+_0$ as asymptote. Moreover, $y^+_0 \geq 0$ if and only if $\lambda \leq 1/2$. The curve $\Gamma_{-1}$ is a connected arc having $(0, -1)$ as endpoint and the line $y = y^-_0$ as an asymptote; see Figure 8.

Figure 8: Top left: the phase plane $\Theta_1$ for $\lambda > 1/2$. Bottom left: the phase plane $\Theta_1$ for $\lambda \leq 1/2$. Right: the phase plane $\Theta_{-1}$ for $\lambda < 1$.

Now, we study the behavior of the orbits. Once again, the existence of the orbit $\gamma_+(s)$ in $\Theta_1$ such that $\gamma(0) = (0, 1)$ follows from Corollary 2.5. Then, if we suppose that $\lambda \leq 1/2$, $\gamma_+(s)$ stays in $\Lambda^+_1$ as it converges to the line $y = y^+_0$, and so $\Sigma_+$ is an entire, strictly convex graph. Otherwise, i.e., if $\lambda > 1/2$, the equilibrium $e_0$ exists. Since no closed orbit exists in virtue of Theorem 4.3, $\gamma_+$ converges to $e_0$ as $s \to \infty$, and its behavior is detailed in Lemma 4.4. Consequently, $\Sigma_+$ is a properly embedded, simply connected $h_\lambda$-surface and:

- If $\lambda > \frac{\sqrt{2}}{2}$, $\Sigma_+$ intersects $C_\lambda$ infinitely many times.
- If $\lambda = \frac{\sqrt{2}}{2}$, $\Sigma_+$ intersects $C_\lambda$ a finite number of times.
- If $\lambda < \frac{\sqrt{2}}{2}$, $\Sigma_+$ is a strictly convex graph contained in the solid cylinder bounded by $C_\lambda$ and converging asymptotically to it.

For the study of the orbit $\gamma_-(s)$ such that $\gamma_-(0) = (0, -1)$ we have to distinguish between the cases $\lambda > 1$, $\lambda = 1$, $\lambda < 1$. This discussion will deeply influence the outcome of Corollary 2.5:

1. If $\lambda > 1$, then $h_\lambda(-1) > 0$ and $\gamma_-(s)$ lies in $\Theta_1$.
2. If $\lambda = 1$, then $h_\lambda(-1) = 0$ and $\gamma_-(s)$ does not exist in either $\Theta_1$ or $\Theta_{-1}$.
3. If \( \lambda < 1 \), then \( h_\lambda(-1) < 0 \) and \( \gamma_-(s) \) lies in \( \Theta_{-1} \).

If \( \lambda = 1 \), horizontal minimal planes downwards oriented are \( h_1 \)-surfaces. Consequently, the uniqueness of the Cauchy problem of (2.4) extends to the line \( y = -1 \) and so no orbit can have and endpoint at this line.

If \( \lambda \neq 1 \), by monotonicity and Proposition 2.3, the only possibility for \( \gamma_-(s) \) is to converge to the line \( y = y_0 \), and so \( \Sigma_- \) is a downwards oriented, strictly convex, entire graph. For \( \lambda > 1 \), the height of \( \Sigma_- \) tends to minus infinity; for \( \lambda < 1 \), the height of \( \Sigma_- \) tends to infinity.

This concludes the classification of the rotational \( h_\lambda \)-surfaces that intersect the axis of rotation. \( \square \)

To finish, we prove Theorem 1.3.

Proof of Theorem 1.3: First, the equilibrium \( e_0 = (\text{argtanh}\frac{1}{2\lambda}, 0) \) exists if and only if \( \lambda > \frac{1}{2} \). This equilibrium generates the cylinder \( C_\lambda \) with CMC equal to \( \lambda \) and vertical rulings. For the remaining \( h_\lambda \)-surfaces, we distinguish again three cases depending on \( \lambda \). Take into account that the structure of the phase planes \( \Theta_1 \) and \( \Theta_{-1} \) has been just studied in the previous proof. See Figure 3 for \( \lambda > \sqrt{5}/2 \), Figure 6 for \( \lambda = \sqrt{5}/2 \) and Figures 7 and 8 for \( \lambda < \sqrt{5}/2 \).

Case \( \lambda > \sqrt{5}/2 \)

Let us take \( x_0 > \text{argtanh}\left(\frac{1}{2\lambda}\right) \) and let \( \gamma(s) \) be the orbit in \( \Theta_1 \) passing through the point \((x_0, 0)\) at the instant \( s = 0 \). For \( s < 0 \), \( \gamma \) has as endpoint some \((x_1, 1)\), \( x_1 > 0 \) (if \( x_1 = 0 \), \( \gamma = \gamma_+ \)), and for \( s > 0 \) either converges to \( e_0 \) as \( s \to \infty \) or has another endpoint of the form \((x_2, -1)\), \( x_2 > 0 \) (again, if \( x_2 = 0 \), \( \gamma = \gamma_- \)). In the second case, the orbit \( \gamma \) continues in \( \Theta_{-1} \) as a compact arc and then goes again in \( \Theta_1 \); see Figure 9, left. After a finite number of iterations, the orbit \( \gamma \) eventually converges to \( e_0 \) spiraling around it infinitely many times.
This configuration ensures us that the $h_\lambda$-surface generated by $\gamma$ is properly immersed, non-embedded, and diffeomorphic to $S^1 \times \mathbb{R}$. One end converges to $C_\lambda$ intersecting it infinitely many times, and the other end has unbounded distance to the axis of rotation, looping and self-intersecting infinitely many times (see Figure 9, right).

**Case $\lambda = \sqrt{5}/2$**

Firstly, let us fix some $x_0 > \text{arg tanh}(\frac{1}{\sqrt{5}})$ and consider the orbit $\gamma_1(s)$ passing through $(x_0, 0)$ at $s = 0$. For $s < 0$, $\gamma_1(s)$ is contained in $\Lambda_1$, so it satisfies $\gamma_1(s_1) = (x_1, 1)$ as endpoint for some $s_1 < 0$. Hence, for $s < s_1$, $\gamma_1(s)$ lies in $\Theta_{-1}$ and is a compact arc whose other endpoint is located at some $\gamma_1(s_2) = (x_2, -1)$. Finally, for $s < s_2$, $\gamma_1(s)$ lies in the monotonicity region $\Lambda_3$ in $\Theta_1$ and stays there as it converges to the line $y = -2/\sqrt{5}$ as $s \to -\infty$; see Figure 10, top left, the red orbit.

For $s > 0$, $\gamma_1(s)$ enters inside $\Lambda_4$, goes inside $\Lambda_4$ and intersects $y = 0$ for the second time in some $(\tilde{x}_0, 0)$ with $\tilde{x}_0 > 0$. It is clear that when $x_0$ increases, then $\tilde{x}_0$ decreases and so $\tilde{x}_0 \to x_\infty \geq 0$ as $x_0 \to \infty$. Also, note that $\gamma_1(s)$ stays always above the line $y = -2/\sqrt{5}$ since the minimum of its $y(s)$-coordinate is at the intersection of $\gamma_1(s)$ with $\Gamma_1$. In this setting, we claim that $x_\infty > 0$.

Arguing by contradiction, suppose that $x_\infty = 0$ and consider an orbit $\sigma(s)$ such that $\sigma(0)$ lies in $\Lambda_4$ and is located below the line $y = -2/\sqrt{5}$. Then, for $s > 0$ in virtue of Proposition 2.4 and by monotonicity, $\sigma(s)$ has to reach the axis $y = 0$ at a point $(\hat{r}, 0)$ for some finite instant. Due to the definition of $x_\infty$ and the assumption $x_\infty = 0$, there exists $r > e_0$ and an orbit $\gamma$ such that $\gamma(0) = (r, 0)$ and $\gamma(s) = (\hat{r}, 0)$ with $0 < \hat{r} < \hat{r}_1$, for some $s_1 > 0$. Hence, $\sigma$ and $\gamma$ intersect each other, which is a contradiction with the uniqueness of the Cauchy problem.

Secondly, let be $r_0 > 0$ and take $\gamma_2(s)$ the orbit such that $\gamma_2(0) = (r_0, -2/\sqrt{5})$. For $s < 0$, $\gamma_2(s)$ lies in $\Lambda_4$, intersects the curve $\Gamma_1^+$, enters $\Lambda_3$ and converges to the line $y = -2/\sqrt{5}$. For $s > 0$, $\gamma_2(s)$ lies in $\Lambda_4$ until intersecting $y = 0$ at some $(\tilde{r}_0, 0)$. Moreover, as $r_0$ increases $\tilde{r}_0$ also increases, and so $\tilde{r}_0 \to r_\infty$ as $r_0 \to \infty$. In particular, $r_\infty \leq x_\infty$. For $s \to \infty$, $\gamma_2(s)$ converges asymptotically to $e_0$, spiraling around it infinitely many times; see Figure 10, top left, the blue orbit.

Finally, take some $\xi_0 \in [r_\infty, x_\infty]$ and let $\gamma_3(s)$ be the orbit such that $\gamma_3(0) = (\xi_0, 0)$. For $s > 0$ it is clear that $\gamma_3(s)$ converges asymptotically to $e_0$, spiraling around it infinitely many times. Because of how $r_\infty$ and $x_\infty$ have been defined, for $s < 0$ the orbit $\gamma_3(s)$ cannot intersect $\Gamma_1^-$ nor intersect $y = -2/\sqrt{5}$. Thus, the only possibility for $\gamma_3(s)$ is to converge to the line $y = -2/\sqrt{5}$ with strictly decreasing $y(s)$-coordinate; see Figure 10, top left, the purple orbit.

Thus, each orbit $\gamma_i$, $i = 1, 2, 3$ generates a properly immersed $h_\lambda$-surface $\Sigma_i$, diffeomorphic to $S^1 \times \mathbb{R}$, with one end converging asymptotically to the CMC cylinder $C_\lambda$ and the other being a graph outside a compact set. Moreover, $\Sigma_1$ is non-embedded, while $\Sigma_2$ and $\Sigma_3$ have monotonous height and in particular are embedded; see Figure 10, bottom.
Case $\lambda < \sqrt{5}/2$

In this final case, we also distinguish between values of $\lambda$. Note that the equilibrium $e_0 = (\text{arg tanh}(\frac{1}{\sqrt{2}}), 0)$ exists if and only if $\lambda > 1/2$. Again, we define $y_0^+ := \frac{1}{5}(-4\lambda + \sqrt{5 - 4\lambda^2})$, and $y_0^- := \frac{1}{5}(-4\lambda - \sqrt{5 - 4\lambda^2})$.

1. Case $1 \leq \lambda < \sqrt{5}/2$. The structure of the phase plane $\Theta_1$ (resp. $\Theta_{-1}$) is the same as the one in Figure 7 (resp. Figure 3, right). Recall that $y_0^- = -1$ in $\Theta_1$ for $\lambda = 1$ and there are only four monotonicity regions. The same reasoning as in the case $\lambda = \sqrt{5}/2$ ensures us that we can construct three types of orbits (see Figure 10) and also the points $x_{\infty}$ and $r_{\infty}$:

- $\gamma_1(s)$ such that $\gamma_1(0) = (x_0, 0)$ with $x_0 > \text{arg tanh}(\frac{1}{\sqrt{2}})$ and $\gamma_1(s) \to e_0$ as $s \to \infty$.
- $\gamma_2(s)$ such that $\gamma_2(0) = (r_0, y_0^+)$ with $r_0 > 0$, $\gamma_2(s) \to e_0$ as $s \to \infty$ and $\gamma_2(s) \to y_0^-$ as $s \to -\infty$.
- $\gamma_3(s)$ such that $\gamma_3(0) = (\xi_0, 0)$ with $\xi_0 \in [r_{\infty}, x_{\infty}]$, $\gamma_3(s) \to e_0$ as $s \to \infty$ and $\gamma_3(s) \to y_0^+$ as $s \to -\infty$.

Again, each orbit $\gamma_i$, $i = 1, 2, 3$ generates a properly immersed $h_\lambda$-surface $\Sigma_i$ that is diffeomorphic to $S^1 \times \mathbb{R}$, with one end converging asymptotically to the CMC cylinder $C_\lambda$ and the other being a graph outside a compact set. Moreover, $\Sigma_1$ self-intersects, while $\Sigma_2$ and $\Sigma_3$ have monotonous height and in particular are embedded.
2. Case $1/2 < \lambda < 1$. In this case, the structure of $\Theta_1$ and $\Theta_{-1}$ is shown in Figure 8 top left and right respectively. There are also three kind of orbits $\gamma_1(s)$, $\gamma_2(s)$, $\gamma_3(s)$ in $\Theta_1$, and $\gamma_1(s)$ and $\gamma_3(s)$ behave as shown in Figure 10. The only difference here is that the orbit $\gamma_2(s)$ intersects the line $y = -1$ at a finite point as $s$ decreases. Then, $\gamma_2(s)$ enters to $\Theta_{-1}$ and converges to the line $y = y_0^-$ as $s \to -\infty$.

The corresponding $h_\lambda$-surfaces are also similar to the ones constructed in the previous case.

3. Case $\lambda \leq 1/2$. In this case, the structure of $\Theta_1$ and $\Theta_{-1}$ is shown in Figure 8 bottom left and right respectively. In particular, no equilibrium point exists and the behavior of the orbits is different from the previous cases.

![Figure 11: Top: the phase planes $\Theta_1$ and $\Theta_{-1}$ for $\lambda = 1/3$. Bottom: the two corresponding $h_\lambda$-surfaces for $\lambda = 1/3$.](image)

First, let be $x_0 > 0$ and $\gamma_1(s)$ the orbit in $\Theta_1$ such that $\gamma_1(0) = (x_0, 0)$. For $s > 0$, $\gamma_1(s)$ enters to $\Lambda_3^+$, intersects $\Gamma_1$ and then lies in $\Lambda_1^+$ converging to the line $y = y_0^+$. For $s < 0$, the orbit $\gamma_1(s)$ lies in $\Lambda_2^+$ and has some $\gamma_1(s_0) = (x_1, -1)$ as endpoint. Thus, $\gamma_1(s)$ for $s < s_0$ lies in $\Lambda_2^-$ and stays there converging to the line $y = y_0^-$ as $s \to -\infty$. See Figure 11, the orbit in red.

Lastly, let be $r_0 > 0$ and $\gamma_2(s)$ the orbit in $\Theta_{-1}$ such that $\gamma_2(0) = (r_0, 0)$. For $s < 0$, $\gamma_2(s)$ enters the region $\Lambda_3^-$ and ends up converging to the line $y = y_0^-$ as $s \to -\infty$. For $s > 0$, $\gamma_2(s)$ enters the region $\Lambda_1^-$ and has as endpoint some $\gamma_2(s_0) = (r_1, 1)$. Then, $\gamma_2(s)$ for $s > s_0$ lies in $\Lambda_1^+$ and stays there as it converges to the line $y = y_0^+$ as $s \to \infty$. See Figure 11, the orbit in purple.

Again, we have two distinct $h_\lambda$-surfaces $\Sigma_1$ and $\Sigma_2$ generated by the orbits $\gamma_1$ and $\gamma_2$. Each $\Sigma_i$
is properly immersed and diffeomorphic to $S^1 \times \mathbb{R}$, and both ends are graphs outside compact sets. Moreover, $\Sigma_1$ is embedded, while $\Sigma_2$ self-intersects; see Figure 11, bottom.

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