A QUANTITATIVE HELLY-TYPE THEOREM: CONTAINMENT IN A HOMOTHET

GRIGORY IVANOV AND MÁRTON NASZÓDI

Abstract. We introduce a new variant of quantitative Helly-type theorems: the minimal “homothetic distance” of the intersection of a family of convex sets to the intersection of a subfamily of a fixed size. As an application, we establish the following quantitative Helly-type result for the diameter. If $K$ is the intersection of finitely many convex bodies in $\mathbb{R}^d$, then one can select $2^d$ of these bodies whose intersection is of diameter at most $(2^d)^{3d/2}\text{diam}(K)$. The best previously known estimate, due to Brazitikos, is $cd^{11/2}$. Moreover, we confirm that the multiplicative factor $cd^{1/2}$ conjectured by Bárány, Katchalski and Pach cannot be improved.

1. Introduction

In [BKP82] (see also [BKP84]), Bárány, Katchalski and Pach proved the following two statements. According to the Quantitative Volume Theorem, if the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of size at most $2^d$ is of volume at most $v(d)$, a constant depending only on $d$. The Quantitative Diameter Theorem states that if the intersection of a family of convex sets in $\mathbb{R}^d$ is of diameter one, then the intersection of some subfamily of size at most $2^d$ is of diameter at most $\delta(d)$, a constant depending only on $d$.

In [BKP82], Bárány, Katchalski and Pach established an upper bound of roughly $d^2$ on $v(d)$ and conjectured that $v(d) \leq (cd)^{d/2}$ holds for a constant $c > 0$. Naszódi confirmed this conjecture in [Nas16] using contact points of the John ellipsoid [Joh14] of the intersection of the family of convex sets. The current best bound, $v(d) \leq (cd)^{3d/2}$, is due to Brazitikos [Bra17].

In [BKP82], the authors obtained a bound on $\delta(d)$ which is exponential in the dimension, and formulated the following conjecture.

Conjecture 1.1 (Bárány, Katchalski, Pach [BKP82]).

$$\delta(d) \leq c\sqrt{d}$$

with a universal constant $c > 0$.

Brazitikos [Bra18] established the first polynomial bound on $\delta(d)$ (see also [Bra16]): using a sparsification result from [BSS14] (see also [Bar14, Lemma 3.1]) related to contact points of John’s ellipsoid, he showed $\delta(d) \leq cd^{11/2}$ with an absolute constant $c > 0$. Recently, Dillon and Soberón [DS20, Theorem 1.2] showed that a fractional version of Conjecture 1.1 holds.

Since $v(1) = \delta(1) = 1$, we will assume that $d \geq 2$ throughout the paper. We use $[n]$ and $\binom{n}{k}$ to denote the sets $\{1, \ldots, n\}$ and the set of all $k$-element subsets of $[n]$, respectively; for a family of sets $\{K_1, \ldots, K_n\}$ and $\sigma \subset [n]$, $K_\sigma$ denotes the intersection $\bigcap_{i \in \sigma} K_i$.

We prove that $v(d) \leq (2d)^{3d}$ and $\delta(d) \leq (2d)^3$.
**Theorem 1.** Let \( \{K_1, \ldots, K_n\} \) be a family of closed convex sets in \( \mathbb{R}^d \) such that their intersection \( K = K_1 \cap \cdots \cap K_n \) is a convex body. Then there is a \( \mu \in \binom{[n]}{\leq 2d+1} \) such that
\[
\text{vol}_d K_\mu \leq (2d)^{3d} \text{vol}_d K \quad \text{and} \quad \text{diam} K_\mu \leq (2d)^d \text{diam} K.
\]

The bound on \( \nu(d) \) is not the best, as both [Nas16] and [Bra17] provide stronger estimates. The method that yields it is new and quite simple as it does not require the use of the John ellipsoid. The bound on \( \delta(d) \), on the other hand, is currently the best.

As we will see, Theorem 1 follows from our main result which concerns a very rough approximation of a convex polytope by the convex hull of \( 2d \) of its well-chosen vertices.

**Theorem 2.** Let \( \lambda > 0 \) and let \( Q \subset \mathbb{R}^d \) be a convex body satisfying the inclusion \( Q \subset -\lambda Q \). Then
\[
(1) \text{ there is a subset } Q' \text{ of at most } 2d + 1 \text{ extreme points of } Q \text{ such that } Q \subset -(\lambda + 1)(d + 1) \text{conv} Q',
\]
and
\[
(2) \text{ there is a subset } Q'' \text{ of at most } 2d \text{ extreme points of } Q \text{ such that } Q \subset -(\lambda + 1)(2d^2 + 2d + 1) \text{conv} Q''.
\]

By shifting the origin, one can guarantee \( \lambda = d \) for any convex body \( Q \) in \( \mathbb{R}^d \), see Lemma 3.1. Recall that the polar \( K^\circ \) of a convex set \( K \subset \mathbb{R}^d \) is defined by
\[
K^\circ = \{ p \in \mathbb{R}^d : \langle p, x \rangle \leq 1 \quad \text{for all } x \in K \}.
\]

By a standard polarity (duality) argument, Theorem 2 yields the following.

**Theorem 3.** Let \( \{K_1, \ldots, K_n\} \) be a family of closed convex sets in \( \mathbb{R}^d \) such that their intersection \( K = K_1 \cap \cdots \cap K_n \), is a convex body. Then there is a point \( z \) in the interior of \( K \) such that
\[
(1) \quad (K - z)^\circ \subset -\lambda(K - z)^\circ, \text{ where } 1 \leq \lambda \leq d;
(2) \text{ there is a } \sigma \in \binom{[n]}{\leq 2d+1} \text{ such that } K_\sigma - z \subset -(\lambda + 1)(d + 1)(K - z) \subset -4d^2(K - z);
(3) \text{ there is a } \mu \in \binom{[n]}{\leq 2d} \text{ such that } K_\mu - z \subset -(\lambda + 1)(2d^2 + 2d + 1)(K - z) \subset -8d^3(K - z).
\]

The containments in Theorem 3 immediately yield Theorem 1.

There are several ways to find a point \( z \) in the interior of a convex body \( K \) such that \( (K - z)^\circ \subset -d(K - z)^\circ \), see Lemma 3.1.

We may interpret Theorem 3 as a new type of quantitative Helly-type theorem. First, for a set \( A \) in \( \mathbb{R}^d \), we call \( \widetilde{A} = \mu A + x \) a homothet of \( A \), if \( x \) is a vector in \( \mathbb{R}^d \) and \( \mu \in \mathbb{R}, \mu \neq 0 \). Next, for two convex bodies \( K \) and \( L \) in \( \mathbb{R}^d \) with \( K \subset L \), we define their homothetic distance as the quantity
\[
\inf \{ |\lambda| : K - z \subset L - z \subset \lambda(K - z), z \in \mathbb{R}^d, \lambda \in \mathbb{R} \}.
\]

Note that \( \lambda \) may be negative. This definition is motivated by Grünbaum’s extension of the Banach–Mazur distance to non-symmetric convex bodies [Grü63] (see also [GLM+04, JN11]).

Now, Theorem 3 can be rephrased as finding a subfamily of a family of convex bodies such that the intersection of the subfamily is at a bounded homothetic distance from the intersection of the entire family.

As a lower bound on \( \delta(d) \), we show that the \( \sqrt{d} \) in Conjecture 1.1 cannot be replaced by a lower power of \( d \).

**Theorem 4.** For every \( i \in [n] \), let \( K_i = \{ x \in \mathbb{R}^d : \langle x, u_i \rangle \leq 1 \} \), where \( u_i \) is a unit vector. Then for any \( \sigma \subset [n] \), there is a point in \( K_\sigma \) with norm \( \frac{d}{\sqrt{|\sigma|}} \).
It follows that if the $u_i$ form a sufficiently dense subset of the unit sphere (with a large $n$), then $K = K_{[n]}$ is almost the unit ball, while for any $\sigma \subseteq [n]$ of size $|\sigma| = 2d$, we have that $\text{diam}(K_\sigma) \geq \sqrt{d/2}$.

We mention the following conjecture which is closely related to Theorem 4. It can be found in a different formulation in [Bör04, p.194].

**Conjecture 1.2.** Let $\{u_1, \ldots, u_{2d}\}$ be unit vectors in $\mathbb{R}^d$. There is a point in the set

$$\bigcap_{i=1}^{2d}\{x \in \mathbb{R}^d : \langle u_i, x \rangle \leq 1\}$$

with norm $\sqrt{d}$.

### 2. Proof of Theorem 2

Clearly, there is a largest volume simplex in $Q$, whose vertices are extreme points of $Q$. Let $S$ be such a simplex. It is well known that

$$-d(S - v) + v \supseteq Q, \tag{2.1}$$

where $v$ is the centroid of $S$. If $v$ coincides with the origin, then by this inclusion, there is nothing to prove. Thus, we assume that $v$ is not origin.

To prove assertion (1), we set $\ell$ to be the ray in $\mathbb{R}^d$ emanating from the origin in the direction $-v$, and let $y$ be the point of intersection of $\ell$ with the boundary of $Q$. By Carathéodory’s theorem, there are extreme points $q_1, \ldots, q_d$ of $Q$ on a support hyperplane to $Q$ at $y$ such that $y \in \text{conv}\{q_1, \ldots, q_d\}$.

We set $Q'$ to be the union of the vertex set of $S$ and the set $\{q_1, \ldots, q_d\}$. Since $v \in Q$ and by $Q \subseteq -\lambda Q$, one has $-v \in \lambda Q$, which, by the choice of $Q'$, yields $-v \in \lambda \text{conv} Q'$, that is,

$$v \in -\lambda \text{conv} Q'. \tag{2.2}$$

Consequently,

$$Q \subseteq -d(S - v) + v \subseteq -d(\text{conv} Q' - v) + v = -d \text{conv} Q' + (d + 1)v \subseteq - (\lambda + 1)(d + 1) \text{conv} Q'. \tag{2.3}$$

This completes the proof of assertion (1) of the theorem.

We proceed with assertion (2) of the theorem. Our goal is to find a vertex of $S$ that can be omitted.

Consider the ray in $\mathbb{R}^d$ emanating from $v$ in the direction $v$, and let $q$ denote the intersection point of this ray and the boundary of $S$. Let $q$ be in a facet $F$ of $S$. Denote the simplex $\text{conv}\{v, F\}$ by $S_1$ and its centroid by $w$. Clearly, $S \subseteq (d + 1)(S_1 - w) + w = (d + 1)S_1 - dw$, and hence,

$$-d(S - v) + v = -dS + (d + 1)v \subseteq -d((d + 1)S_1 - dw) + (d + 1)S_1 \subseteq$$

$$-d(d + 1)S_1 + d^2w + (d + 1)(-d(S_1 - w) + w) =$$

$$-2d(d + 1)S_1 + (2d^2 + 2d + 1)w.$$

Thus, by (2.1), we have

$$Q \subseteq -2d(d + 1)S_1 + (2d^2 + 2d + 1)w. \tag{2.3}$$

Let $\ell_2$ be the ray in $\mathbb{R}^d$ emanating from the origin in the direction $-w$, and let $y_2$ be the point of intersection of $\ell$ with the boundary of $Q$. By Carathéodory’s theorem, there are extreme points $p_1, \ldots, p_d$ of $Q$ on a support hyperplane to $Q$ at $y_2$ such that $y_2 \in \text{conv}\{p_1, \ldots, p_d\}$.

We set $Q''$ to be the union of the vertex set of $F$ and the set $\{p_1, \ldots, p_d\}$, and claim that $v \in \text{conv} Q''$. Indeed, consider the simplex $\text{conv}\{o, F\}$. By construction, $S_1 \subseteq \text{conv}\{o, F\}$. It follows that the ray emanating from the origin in the direction of $w$ intersects the facet $F$ of $S$. Thus, the origin is in $\text{conv} Q''$, and hence, $v \in \text{conv} Q''$, see Figure 1.
Since \( q \in \text{conv } Q'' \), and \( v \) is on the line segment \([o, q]\), we have \( v \in \text{conv } Q'' \). It follows that

\[
S_1 \subset \text{conv } Q''.
\]

Again, since \( Q \subset -\lambda Q \), we have \(-w \in \lambda Q\), which, by the choice of \( Q'' \), yields \(-w \in \lambda \text{conv } Q''\), that is,

\[
w \in -\lambda \text{conv } Q''.
\]

Finally, it follows from (2.3), (2.4) and (2.5) that

\[
Q \subseteq -2d(d+1) \text{conv } Q'' - \lambda(2d^2 + 2d + 1) \text{conv } Q'' \subseteq -(\lambda + 1)(2d^2 + 2d + 1) \text{conv } Q''.
\]

This completes the proof of Theorem 2.

3. Proof of Theorem 3

Recall that the centroid \( c(K) \) of a body \( K \subset \mathbb{R}^d \) is defined by

\[
c(K) = \frac{1}{\text{vol}_d K} \int_K x \, dx.
\]

**Lemma 3.1.** Let \( K \) be a convex body in \( \mathbb{R}^d \). Then the inclusion \((K - z)^\circ \subseteq -d(K - z)^\circ\) holds if \( z \) is

1. the centroid of \( K \);
2. the Santaló point of \( K \);
3. the center of John ellipsoid of \( K \);
4. the center of Löwner ellipsoid of \( K \).

We note that other “centers” may be found using [GLM+04, Theorem 5.1].

**Proof.** It is well known (see [Grü60, Grü63]) that if \( c(K) \) is the origin, then \( K \subseteq -dK \). By taking the polar of this containment, we obtain assertion (1).

Recall that the Santaló point of a convex body \( K \subset \mathbb{R}^d \) is the unique point \( z \) minimizing \( \text{vol}_d (K - z) \text{vol}_d (K - z)^\circ \). Next, (2) follows from [AS17, Proposition D.2] which states that for any convex body \( K \) whose Santaló point is the origin, the centroid of \( K^\circ \) is the origin.

Recall that if \( E \) is the John ellipsoid of \( K \), that is, the largest volume ellipsoid in \( K \), and \( E \) is centered at \( z \), then \( K \subseteq d(E - z) + z \). A similar statement holds for the Löwner ellipsoid of \( K \), that is, the smallest volume ellipsoid containing \( K \). These containments then easily yield assertions (3) and (4), we leave the details to the reader. \(\square\)

**Proof of Theorem 3.** By Lemma 3.1, there is a point \( z \) in \( K \) such that the inclusion \((K - z)^\circ \subseteq -\lambda(K - z)^\circ\) holds with \( 1 \leq \lambda \leq d \). Fix such \( z \) and set \( Q = (K - z)^\circ \).
Since the polar of the intersection of convex bodies containing the origin is the convex hull of the polars of these bodies, each extreme point of $Q$ belongs to a set $(K_i - z)\circ$ for some $i \in [n]$. That is, if $p$ is an extreme point of $Q$, then there exists $i$ such that

$$K_i - z \subset \{ x \in \mathbb{R}^d : \langle p, x \rangle \leq 1 \}.$$ 

We will say in this case that $i$ corresponds to $p$.

Let $Q'$ and $Q''$ be as in the assertions of Theorem 2. For every $p \in Q'$, we find one index $i \in [n]$ that corresponds to $p$, and set $\sigma$ to be the set of these indexes. Clearly, $K_\sigma$ satisfies assertion (2) of Theorem 3. Analogously, for every $p \in Q''$, we find one index $i \in [n]$ that corresponds to $p$, and set $\mu$ to be the set of these indexes. Clearly, $K_\mu$ satisfies assertion (3) of Theorem 3. $\square$

4. Lower bound for diameter

In this section, we prove Theorem 4. The result follows from the following lemma due to K. Ball and M. Prodromou.

**Lemma 4.1** ([BP09], Theorem 1.4). Let vectors $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ satisfying $\sum_1^n v_i \otimes v_i = \text{Id}$. Then for any positive semi-definite operator $T : \mathbb{R}^d \to \mathbb{R}^d$, there is a point $p$ in the intersection of the strips $\{ x \in \mathbb{R}^d : |\langle x, v_i \rangle | \leq 1 \}$ satisfying $\langle p, Tp \rangle \geq \text{trace} T$.

**Proof of Theorem 4.** We prove a bit stronger statement. We will find a point with the desired large norm in the subset

$$K'_\sigma = \bigcap_{i \in \sigma} \{ x : |\langle u_i, x \rangle | \leq 1 \}$$

of $K_\sigma$. If $\{u_i : i \in \sigma\}$ does not span the space, then $K'_\sigma$ is unbounded. Thus, we assume $\{u_i : i \in \sigma\}$ spans $\mathbb{R}^d$. Consider $A = \sum_{i \in \sigma} u_i \otimes u_i$. Since the vectors span the space, $A$ is positive definite. Using Lemma 4.1 with $v_i = A^{-1/2} u_i, i \in \sigma$, and $T = A^{-1}$, we find a point $p$ in

$$\bigcap_{i \in \sigma} \{ x : |\langle v_i, x \rangle | \leq 1 \}.$$ 

such that

$$\langle p, A^{-1} p \rangle \geq \text{trace} A^{-1}.$$ 

Denote $q = A^{-1/2} p$. Then, by the choice of $p$, $1 \geq |\langle p, A^{-1/2} u_i \rangle | = \langle A^{-1/2} p, u_i \rangle = |\langle q, u_i \rangle |$.

That is, $q \in K'_\sigma$. On the other hand,

$$|q|^2 = \langle A^{-1/2} p, A^{-1/2} p \rangle = \langle p, A^{-1} p \rangle \geq \text{trace} A^{-1}.$$

Finally, since $\text{trace} A = |\sigma|$ and by the Cauchy–Schwarz inequality, one sees that $\text{trace} A^{-1}$ is at least $\frac{d^2}{|\sigma|}$. Thus, $|q| \geq \frac{d}{\sqrt{|\sigma|}}$. $\square$

**References**

[AS17] Guillaume Aubrun and Stanislaw J. Szarek, *Alice and Bob Meet Banach: The interface of Asymptotic Geometric Analysis and Quantum Information Theory*, vol. 223, American Mathematical Soc., 2017.

[Bar14] Alexander Barvinok, *Thrifty approximations of convex bodies by polytopes*, International Mathematics Research Notices 2014 (2014), no. 16, 4341–4356.

[BKP82] Imre Bárány, Meir Katchalski, and Janos Pach, *Quantitative Helly-type theorems*, Proceedings of the American Mathematical Society 86 (1982), no. 1, 109–114.

[BKP84] , *Helly’s theorem with volumes*, The American Mathematical Monthly 91 (1984), no. 6, 362–365.

[Bör04] Károly Böröczky, Jr., *Finite packing and covering*, Cambridge Tracts in Mathematics, vol. 154, Cambridge University Press, Cambridge, 2004.
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[BP09] Keith Ball and Maria Prodromou, A sharp combinatorial version of Vaaler’s theorem, Bulletin of the London Mathematical Society 41 (2009), no. 5, 853–858.

[Bra16] Silouanos Brazitikos, Quantitative Helly-type theorem for the diameter of convex sets, Discrete & Computational Geometry 57 (2016), no. 2, 494–505.

[Bra17] Silouanos Brazitikos, Brascamp–Lieb inequality and quantitative versions of Helly’s theorem, Mathematika 63 (2017), no. 1, 272–291.

[Bra18] ———, Polynomial estimates towards a sharp Helly-type theorem for the diameter of convex sets, Bulletin of the Hellenic mathematical society 62 (2018), 19–25.

[BSS14] Joshua Batson, Daniel A. Spielman, and Nikhil Srivastava, Twice-Ramanujan sparsifiers, SIAM Rev. 56 (2014), no. 2, 315–334. MR 3201185

[DS20] Travis Dillon and Pablo Soberón, A m\’elange of diameter helly-type theorems, arXiv preprint arXiv:2008.13737 (2020).

[GLM⁺04] Yehoram Gordon, A. E. Litvak, Mathieu Meyer, Alain Pajor, et al., John’s decomposition in the general case and applications, J. Differential Geom. 68 (2004), no. 1, 99–119.

[Grü60] Branko Grünbaum, Partitions of mass-distributions and of convex bodies by hyperplanes., Pacific Journal of Mathematics 10 (1960), no. 4, 1257–1261.

[Grü63] ———, Measures of symmetry for convex sets, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 233–270.

[JN11] C. Hugo Jiménez and Márton Naszódi, On the extremal distance between two convex bodies, Israel J. Math. 183 (2011), 103–115.

[Joh14] Fritz John, Extremum problems with inequalities as subsidiary conditions, Traces and emergence of nonlinear programming, Springer, 2014, pp. 197–215.

[Nas16] Márton Naszódi, Proof of a conjecture of Bárány, Katchalski and Pach, Discrete & Computational Geometry 55 (2016), no. 1, 243–248.

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