Cosmology with a new $f(R)$ gravity model in Palatini formalism

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One of the most favourable extensions of General Relativity is the $f(R)$ gravity. $f(R)$ gravity in Palatini formalism can be a realistic alternative to the dark energy problem. In this work we study a recently introduced dark energy $f(R)$ gravity model along with two other models in cosmological perspectives under the Palatini formalism. First, we study the cosmic expansion history of these models with the help of the important cosmographic parameters, such as the Hubble parameter, luminosity distance, effective equation of state etc. This study shows that the new model behaves similarly with the other two models as well as with the $\Lambda$CDM model in some respects in the early or very early phases of the universe. It could predict the present accelerated expansion of the universe somewhat differently from the other models with a peculiar future history of the universe. Within a constrained range of parameters all models show a good agreement with the Union2.1 luminosity distance data. However, the new model shows a quite satisfactory agreement in the whole range of its allowed parameters than that of the other two models. We also obtain cosmological constraints on these models from the Observed Hubble Data. Further, models have been tested by using $Om(z)$ test and statefinder diagnostics. These diagnostics especially, the statefinder diagnostic shows that the evolutionary differences between these models are distinct. The evolutionary trajectories of the new model are completely different from the other two models we have considered.

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I. INTRODUCTION

Einstein’s General Relativity (GR) is the pioneer of modern cosmology. GR is an excellent theory to study the expansion history of the universe. It predicts so many things and they are in excellent agreement with the observations. However, the observations of type Ia supernovae (SNIa) [1, 2] show that the universe is currently at an accelerated expansion phase. This observed result is in contradiction with our expectation from the behavior of ordinary matter and it could not be explained on the basis of GR. One of the best possible explanations of the current accelerated expansion of the universe is to assume the existence of an exotic form of energy in the universe known as dark energy, which has the characteristic of negative pressure. For the simplest explanation of dark energy, initially abandoned the well known Einstein’s cosmological constant $\Lambda$ has been reintroduced as the energy of the vacuum state of the universe $\Lambda$ [3, 4]. In this model the (vacuum) energy density $\rho_\Lambda = \Lambda/8\pi G$ ($c = 1$) is a constant in time and hence the universe evolves with an equation of state $w = -1$. But there are two main issues with this model: problems of the fine tuning and the cosmic coincidence. The problem of fine tuning is related to the huge difference between the observed value of $\rho_\Lambda$ and its calculated value from the concept of zero-point energy fluctuations of QFT. This calculated value is found to be $\sim 10^{121}$ orders of magnitude larger than the observed value if one uses the energy scale as the Planck scale. Even if we use a much lower QCD energy scale, this difference would be $\sim 10^{44}$ orders of magnitude [4]. Thus, this is indeed a severe problem in this field of study. On the other hand, the cosmic coincidence problem is based on the fact that the dark energy density is approximately equal to the dark matter density of the universe today and we do not have a clear answer to the reason behind this equality [5]. These problems motivated the researchers to look for some different approaches to solve the dark energy issue. Moreover, dark energy is attributed to a huge fraction of the universe which is still mysterious to us. This dominant component of the universe not only has a large negative pressure, but also it does not cluster/behave as like the other ordinary matters/energies of the universe do. That is it does not resemble other known matter energy.

Since dark energy is not directly detected experimentally yet, and hence without a confirmed experimental signature, it keeps the window open to other alternative choices without such an enigma. However, there are some recent studies showing possibility of detection of dark energy [6, 7]. Among alternative possibilities to explain the accelerated expansion of the universe is the modification of gravity law in such a way that it behaves as standard GR in strong gravitational regimes and behaves as repulsive force in the low density cosmological scale. For the modification of gravity theory there are two choices: one is to alter the matter part and the other is to modify the curvature part. The modification of the matter energy part basically involves the introduction of a better dark energy model to overcome the drawbacks of the $\Lambda$ model. Analysis done on the SNIa data before some twenty years ago indicates that a time dependent dark energy can give a better fit than the previous cosmological constant model. In

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view of this the dubbed quintessence got significant attention from the researchers, where they used dynamical scenarios of dark energy instead of using \( w \) as a constant in time [8–10]. In another study based on the latest observations, it was found that the dynamical dark energy model is preferred at a 3.5\( \sigma \) significance level [11]. Apart from these, phantom [12, 13], quintom [14, 15], ghost condensates [16, 17] etc. dark energy models have got sufficient attention of the researchers. Further, with an attempt to unify the dark energy and dark matter the Chaplygin gas has been introduced [18, 19]. Similarly, there are some other models also which are introduced as an explanation of dark energy as well as dark matter without modifying the geometry of spacetime [20].

In the other choice, the geometry part of the field equation is modified. The \( f(R) \) gravity is one of such theories in which modification of spacetime curvature takes place by replacing the Ricci curvature scalar in E-H action with a general function of it. For \( f(R) \) gravity, there are two main approaches to obtain the field equations. The first one is the so-called Metric formalism, which is obtained by the variation of the action with respect to the metric. In the second approach, which is known as the Palatini formalism, both the connection and the metric are considered as independent variables. In recent days the Palatini formalism seems to get more importance because of a couple of advantages over the Metric formalism. Here the field equations are a set of second order partial differential equations plus an equation involving connection, which is trivial to solve; in contrast to the 4th order differential equations in metric formalism. Moreover, it was brought to notice by Dolgov and Kawasaki that the fourth order equations in metric formalism suffers from a very serious instability problem [21, 22]. But in the case of Palatini formalism, the field equations being only second order are free from such instabilities [23, 24]. It was also discovered that some \( f(R) \) gravity models in this formalism can not produce a standard matter-dominated era followed by an accelerated expansion [25, 26]. Another important issue with the metric formalism \( f(R) \) gravity models is that some models do not pass the solar system tests. For example, models of the type \( f(R) = R - \beta/R^n, f(R) = \alpha R^n \) etc. do not pass the solar system tests [27]. However, there are several \( f(R) \) gravity models which can easily pass solar system tests and also can describe the matter dominated stage followed by the present accelerating stage of the universe in metric formalism. Some of them are: Starobinsky model [28], Hu-Sawicki model [29] etc. In addition to them, there are some other promising models: a new model defined in [30], another model defined in [31] etc. Apart from this, there is an issue with the correct Newtonian limit also [32, 33]. But in the case of Palatini formalism we do not have to check the solar system tests separately as all \( f(R) \) gravity models pass solar system tests in this formalism. Moreover, in this formalism, the correct Newtonian limit is easily recovered [33]. It was also shown that the selected models in Palatini formalism can produce the sequence of radiation-dominated, matter-dominated and late time accelerating phases successfully [34]. Due to these positive points of the formalism, we shall proceed with the Palatini formalism \( f(R) \) gravity hereafter. For a more details on similar studies in modified gravity readers may follow the Ref.s [35–38].

So far a few models are tested in Palatini formalism to study the expansion history of the universe. Here in this work we try to explain the nature of past, present and future acceleration/deceleration of the universe with the help a new \( f(R) \) gravity model [30] in comparison with the power law model of \( f(R) \) gravity and a model of the form \( f(R) = R + \alpha R^n \). The reason behind choosing these models are as follows. The power law model is a very simple extension of GR. Although this model does not behave well in the solar system regime in metric formalism, in Palatini formalism it is free from such issues. In a very recent study, the polarization modes of gravitational waves were studied in this model and for a special case of the model, it behaves very uniquely in terms of gravitational wave polarization modes than from other \( f(R) \) gravity models [39]. The model of the form \( f(R) = R + \alpha R^n \) contains the Starobinsky’s inflation model [40] as a subset. Apart from this, some previous studies on such types of models show significant results [41, 42]. Finally, the new \( f(R) \) gravity model defined in Ref. [30] is not studied in cosmological perspectives yet. For the rest of this paper, we shall use the term “Gogoi-Goswami model” to denote this new model specifically. It should be pointed out that in this study the first two models are considered as the standard references for the comparative analysis of the new model results as these two models are studied many times in the similar context. For instance, in Ref. [43], the power law model of \( f(R) \) gravity has been used and it is shown that such models are able to explain the current acceleration of the universe with a suitable choice of the model parameters. Authors have also introduced a new Lagrangian which is a function of the inverse of sinh(\( R \)) and can effectively show the current acceleration of the universe. In another study, the power law gravity model was used in the Friedmann-Robertson-Walker metric and it was found that first-order nonlinear gravity can survive SNIa and baryon oscillation tests [44]. In a recent study, polynomial \( f(R) \) gravity is considered in dynamical system approach, which showed that the Starobinsky \( f(R) \) gravity in the Jordan frame and Einstein frame are not physically equivalent [45]. Being motivated from these results, in this work, we have considered our new dark energy \( f(R) \) gravity model, as mentioned above which has not been studied earlier in cosmological perspectives. Therefore, we believe, a study of this model in this direction will provide some useful insights to its behaviour and viability in the respective areas. Apart from this, the study will also provide a constraint on this new model from the available observational data. We shall also study the behaviour of the models in the \( Om(z) \) test and the statefinder diagnostics, and then will compare the results to see the viabilities and uniqueness of this new model.

The rest of the paper is organized as follows. In section II, we briefly explain the derivation of field equations in Palatini formalism and find out the connection for the \( f(R) \) gravity models in this formalism. In section III, we define the basic mathematical formulations required for the study along with the necessary physical and cosmographical parameters. In section IV, we define the \( f(R) \) gravity models used in this study and then calculate the cosmographical parameters for these models. Also in this section the numerically calculated cosmographical parameters are compared with the available data and corresponding
In terms of this conformal metric to \( g \) mentioned above. To find the solution of Eq. (5), we define a new metric \( h \) hence the gravitational fields are described by the metric and also by the proposed \( f \) where \( \Gamma \) in independent connection turns out to be the Levi-Civita one. This is due to the dynamical feature of the connection \( \Gamma \) in the GR limit this equation becomes the definition of the Levi-Civita connection, i.e. in the Palatini formalism for GR, the action (1) \[46\]. Moreover, since the connection is dynamical, we do not need to assume any priori symmetry in its lower indices \[48\]. It is to be noted that when \( f(R) \) is the energy-momentum tensor as given by \[46\]. However, for more complicated or higher order Ricci type Lagrangians, one needs to assume a priori that the connection variable is symmetric if a reasonable field equation for the connection is expected \[50\].

In the Palatini approach, the metric \( g_{\mu
u} \) and the torsion-free connection \( \Gamma^\alpha_{\mu
u} \) are treated as independent variables. However, the matter action \( S_m \) is assumed to not depend on the independent connection, but depend only on the metric and the matter fields. This is the main feature of the Palatini formalism as this assumption is required to derive Einstein’s equations from the action (1) \[46\]. Moreover, since the connection is dynamical, we do not need to assume any priori symmetry in its lower indices \[48\]. Thus, while we vary the action to obtain the field equations we consider action (1) \[46\].

II. PALATINI FORMALISM OF \( f(R) \) GRAVITY

The simplest class of modified gravity models is the \( f(R) \) gravity models in which Einstein gravity is modified by replacing the Ricci curvature scalar \( R \) by an arbitrary curvature function \( f(R) \). The action for the \( f(R) \) gravity models in the Jordan frame is given by \[46–48\]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \Psi),
\]

where \( \kappa^2 = 8\pi G \), \( S_m \) is the standard action for matter fields and \( \Psi \) represents the matter fields collectively. Throughout this paper we shall work only in the Jordan frame. The approach to the Einstein frame action can be found in \[49\].

In the Palatini approach, the metric \( g_{\mu\nu} \) and the torsion-free connection \( \Gamma^\alpha_{\mu\nu} \) are treated as independent variables. However, the matter action \( S_m \) is assumed to not depend on the independent connection, but depend only on the metric and the matter fields. This is the main feature of the Palatini formalism as this assumption is required to derive Einstein’s equations from the action (1) \[46\]. Moreover, since the connection is dynamical, we do not need to assume any priori symmetry in its lower indices \[48\]. 

Thus, while we vary the action to obtain the field equations we consider \( \Gamma^\alpha_{\mu\nu} \neq \Gamma^\alpha_{\nu\mu} \). However, for more complicated or higher order Ricci type Lagrangians, one needs to assume a priori that the connection variable is symmetric if a reasonable field equation for the connection is expected \[50\].

Now varying the action (1) with respect to the metric \( g_{\mu\nu} \) we get \[46, 48\],

\[
R_{(\mu\nu)} f_R - \frac{1}{2} g_{\mu\nu} f(R) = \kappa^2 T_{\mu\nu},
\]

where \( f_R \) denotes the derivatives of \( f(R) \) with respect to \( R \), \( (\mu\nu) \) represents symmetrization over the indices \( \mu \) and \( \nu \), and \( T_{\mu\nu} \) is the energy-momentum tensor as given by

\[
T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}.
\]

It is to be noted that when \( f(R) = R \) the Eq. (2) yields Einstein’s equations, i.e. in the limit of \( f(R) = R \) the Palatini formalism leads to GR.

Again varying the action with respect to \( \Gamma^\alpha_{\mu\nu} \), we get,

\[
\nabla_\alpha \left[ f_R \sqrt{-g} g^{\mu\nu} \right] - \frac{1}{2} \nabla_\sigma \left[ f_R \sqrt{-g} g^{\mu\nu} \right] \delta_\alpha^\sigma - \frac{1}{2} \nabla_\sigma \left[ f_R \sqrt{-g} g^\sigma_{\mu\nu} \right] \delta_\alpha^\mu = 0,
\]

where \( \nabla_\mu \) represents the covariant derivative. By taking the trace it is easy to show that the Eq. (4) is equivalent to

\[
\nabla_\alpha \left[ f_R \sqrt{-g} g^{\alpha\mu} \right] = 0.
\]

In the GR limit this equation becomes the definition of the Levi-Civita connection, i.e. in the Palatini formalism for GR, the independent connection turns out to be the Levi-Civita one. This is due to the dynamical feature of the connection \( \Gamma^\alpha_{\mu\nu} \) as mentioned above. To find the solution of Eq. (5), we define a new metric

\[
h_{\mu\nu} = g_{\mu\nu} f_R.
\]

In terms of this conformal metric to \( g_{\mu\nu} \), the Eq. (5) transformed into

\[
\nabla_\alpha \left[ \sqrt{-h} h^{\alpha\mu} \right] = 0.
\]

This equation gives the definition of the Levi-Civita connection of \( h_{\mu\nu} \) and the algebraic solution of the equation leads to the connection of the from:

\[
\Gamma^\alpha_{\mu\nu} = \frac{h^{\alpha\sigma}}{2} \left( \partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} \right).
\]

In terms of \( g_{\mu\nu} \) it can be expressed as

\[
\Gamma^\alpha_{\mu\nu} = \left\{ \begin{array}{c} \alpha \\ \mu \\ \nu \end{array} \right\} + \frac{1}{2f_R} \left( \delta_\alpha^\mu \partial_\mu f_R + \delta_\alpha^\nu \partial_\nu f_R - g^{\alpha\sigma} g_{\mu\nu} \partial_\sigma f_R \right),
\]

where \( \left\{ \begin{array}{c} \alpha \\ \mu \\ \nu \end{array} \right\} \) is the Christoffel symbol of 2nd kind of the metric \( g_{\mu\nu} \). It is clear from this equation that the connection, and hence the gravitational fields are described by the metric and also by the proposed \( f(R) \) function.

results are discussed. We constrain these models using the observed Hubble data in section V. We used the Om(z) test and statefinder diagnostics in section VI to study the cosmological behaviours of these models. Finally we conclude the paper with a concise discussion of the results in section VII.
III. COSMOLOGICAL EQUATIONS

In the large scale the universe is looking homogeneous at every point and isotropic in all directions. So, as a simplest model we may consider the homogeneous and isotropic flat universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

\[ ds^2 = -dt^2 + a^2(t) \, dr^2, \]

where we used the metric convention as \((- , +, +, +)\) and \(a(t)\) is the cosmological scale factor. Similarly for an ideal situation we consider that the universe is filled with a perfect fluid which is the source of curvature. If \(\rho\) is the energy density and \(p\) is the pressure of such fluid, then its energy-momentum tensor can be written as

\[ T^\mu_\nu = diag(-\rho, p, p, p). \]

For our universe define Eqs. (10) and (11), we may obtain the generalized Friedmann equation from the Eqs. (2) and (9), which can expressed in terms of redshift \(z\) as \([51]\)

\[ \frac{H^2}{H_0^2} = \frac{3 \Omega_{m0} (1 + z)^3 + 6 \Omega_{r0} (1 + z)^4 + f(R)}{6 f_R \zeta^2}, \]  

where

\[ \zeta = 1 + \frac{9 f_{RR} H_0^2 \Omega_{m0} (1 + z)^3}{2 f_R R f_{RR} - f_R}. \]  

Here \(\Omega_{m0}\) and \(\Omega_{r0}\) are the present day value of the matter and radiation density parameters respectively. The trace of the field Eq. (2) gives:

\[ R f_R - 2 f(R) = \kappa^2 T. \]  

Using the energy-momentum tensor given in Eq. (11), this equation can be written as

\[ R f_R - 2 f(R) = -\kappa^2 \rho(1 - 3w), \]  

where \(w = p/\rho\) is the cosmological equation of state. For the radiation dominated universe \(w = 1/3\) and hence the above equation reduces to such a universe as

\[ R f_R - 2 f(R) = 0. \]  

This equation can be directly obtained from the Eq. (14) as \(T = 0\) for the radiation dominated universe. However, we will not consider this equation further as it creates an inconsistent situation for the field equations for most of the \(f(R)\) models \([46]\). As the radiation dominated era of the universe has insignificant contribution to the present stage of the universe it will not affect our result. On the other hand for the matter dominated universe \(w = 0\) and if we take \(\rho_m = \rho_{m0} (1 + z)^3\) and then substituting \(\kappa^2 \rho_{m0} = 3 H_0^2 \Omega_{m0}\), we may write the Eq. (15) for the matter dominated universe in the form:

\[ R f_R - 2 f(R) = -3 H_0^2 \Omega_{m0} (1 + z)^3. \]  

This equation will provide us a condition to restrict the parameter of a given \(f(R)\) model.

Apart from the Hubble parameter \(H(z)\), which is given by the Eq. (12), other two important cosmographic parameters to study the expansion history of the universe are effective equation of state \(\omega_{eff}(z)\) and deceleration parameter \(q(z)\). These parameters can be studied from the following equations along with the Eq. (12) as given by \([46, 51]\)

\[ \omega_{eff}(z) = -1 + \frac{2 (1 + z)}{3 H(z)} H'(z), \]  

\[ q(z) = \frac{1 + z}{H(z)} H'(z) - 1, \]  

where a prime denotes differentiation with respect to \(z\). Another important parameter is the distance modulus \(D_m(z)\). It is possible to express the distance modulus in terms of an \(f(R)\) gravity model in the following way:

\[ D_m = 5 \log_{10} \left( \frac{\sqrt{3} H_0^2 \Omega_{m0} (2 f(R_1) - R_1 f'(R_1))}{\sqrt{3} H_0^2 \Omega_{m0}} \int_{R_0}^{R_1} \frac{69288.2 H_0^2 \Omega_{m0} (f(R) - R f''(R))}{H(R) (H_0^2 \Omega_{m0} (2 f(R) - R f'(R)))^{2/3}} dR \right), \]  

where \(R_0, R_1\) are the present and future scale factors respectively.
where $R_1$ is some arbitrary curvature in the past for some fixed value of $z$. $R_0$ is the present value of curvature at $z = 0$. In the above expression, $H$ is a function of $R$ only. For the rest part of the study, we will use $\Omega_{m0} = 0.315$, and $\Omega_{r0} = 0.000053$ if not specified. We consider $H_0 = 68.02$ km s$^{-1}$ Mpc$^{-1}$ [52], which is very close to recent Planck’s result [53]. We shall use this expression to compare the models with the Union2.1 data [54].

Since, the forms of different models might increase the complexity of obtaining an analytical solution of $R$ in terms of $z$, we may not obtain some parameters like $H$ as a function of $z$ directly. To solve this issue, we shall express $H$, $\omega_{eff}$ etc. as a function of $R$. To do so, we shall use the following expressions.

For the rest part of the study, we will use $\Omega_{m0} = 0.315$, and $\Omega_{r0} = 0.000053$ if not specified. We consider $H_0 = 68.02$ km s$^{-1}$ Mpc$^{-1}$ [52], which is very close to recent Planck’s result [53]. We shall use this expression to compare the models with the Union2.1 data [54].

Using the field equations and cosmological equations as discuss in the previous sections, in this section we will study the evolution of universe and consequent cosmological implications of the power law model, model of the type $R + \alpha R^n$ and the Gogoi-Goswami model as follows:

### IV. $f(R)$ Gravity Models and Properties of Cosmic Evolution

#### A. Power Law Model

The general power law $f(R)$ gravity model is given by

$$f(R) = \lambda R^n,$$  \hspace{1cm} (24)

where $\lambda$ and $n$ are model parameters. Using the Eq. (17) on this model for $z = 0$, we can find out the restrictions on the model parameters as given by

$$\lambda = -\frac{3 H_0^2 \Omega_{m0}}{(n - 2) R_0^n}.$$  \hspace{1cm} (25)

It is clear from this expression that the model parameter $n$ should have values other than 2 to avoid the singular value of the parameter $\lambda$. In fact it is very easy to show that $n = 2$ is the trivial solution of the Eq. (16) for this model. That is for the value of $n = 2$, the power-law model specifies the radiation dominated phase of the universe. As the parameter $\lambda$ depends upon the value of $R_0$ also, so to find the expression for $R_0$ we proceed as follows:

For the model (24) the Eq. (13) gives:

$$\zeta = 1 - \frac{3}{2} \frac{n - 1}{n} \left( \frac{R_0}{R} \right)^n (1 + z)^3.$$  \hspace{1cm} (26)

For $z = 0$, the value of $\zeta$ depends only on the parameter $n$ as given by

$$\zeta_0 = 1 - \frac{3}{2} \frac{n - 1}{n}.$$  \hspace{1cm} (27)

It is seen that $\zeta$ becomes unity and hence it is $f(R)$ model independent when the power-law model takes the value $n = 1$. Thus this sets another restriction on the parameter $n$ of the model. In fact, this value of $n$ corresponds to GR. Finally setting $z = 0$ in the Eq. (12) and using the Eq. (27) we can find out the expression for $R_0$ for the power law model as

$$R_0 = -\frac{3 (3 - n)^2 H_0^2 \Omega_{m0}}{2n [(n - 3) \Omega_{m0} + 2 (n - 2) \Omega_{r0}]}.$$  \hspace{1cm} (28)
This equation also sets another restriction on the parameter $n$ that its value should be other than 3, otherwise $R_0$ will vanish.

To find the expression of the scalar curvature function $R$ in terms of redshift $z$ we solve the Eq. (17) using the expression (25) for the parameter $\lambda$, which leads to a compact expression of $R$ in the form:

$$R(z) = R_0 (1 + z)^{\frac{2}{n}}.$$  

(29)

In view of this equation, from Eqs. (26) and (27) it is clear that $\zeta = \zeta_0$. So the parameter $\zeta$ is independent of cosmological evolution for a given value of the parameter $n$ of the power-law model. Fig. 1 shows the variation of $R$ with respect to $z$ for three values of $n$ as given by the Eq. (29). It is seen that the slope of the scalar curvature $R$ increases in this model when $n$ approaches the GR limit. Also the slope changes significantly for the small variation of the value of $n$.

![Plot of scalar curvature $R$ versus redshift $z$ for different values of $n$.](image)

**FIG. 1:** Behaviour of scalar curvature $R$ with respect to redshift $z$ for three different values of the parameter $n$ of the $f(R)$ gravity power law model.

From the generalized Friedmann Eq. (12) the Hubble parameter $H(z)$ can be expressed for the power law model (24) as

$$H(z) = \left[ -\frac{2nR_0}{3(3-n)^2\Omega_m0} \left\{ (n-3)\Omega_m0 (1+z)^{\frac{2}{n}} + 2(n-2)\Omega_r0 (1+z)^{(\frac{n+3}{n})} \right\} \right]^\frac{1}{2}.$$  

(30)

As already mentioned, it is also seen from this equation that the value of the model parameter $n$ must be different from 3 to avoid the singular value of the Hubble parameter to be predicted by this model. Varying this equation for $H(z)$ with respect to $z$ we get,

$$H'(z) = \frac{-nR_0}{3(3-n)^2\Omega_m0H} \left[ \frac{3(n-3)}{n}\Omega_m0 (1+z)^{\frac{2}{n}-1} + \frac{2(n-2)}{n} (n+3)\Omega_r0 (1+z)^{\frac{2}{n}} \right].$$  

(31)

Expressions of $H(z)$ and $H'(z)$ given by Eqs. (30) and (31) respectively for the power law model can be used to study the other cosmological parameters, like the effective equation of state given by the Eq. (18) and the deceleration parameter given in the Eq. (19). To understand the behaviour of the Hubble parameter $H(z)$ in terms of $z$ for different values of $n$, we have plotted $H(z)$ with respect to $z$ for three different values of $n$ on the left plot of Fig. 2. To choose the reliable values of $n$ we have used the four sets of available $H(z)$ data, viz., HKP data, SVJ05 data [55], SJVKS10 data [56] and GCH09 data [57]. Although the SVJ05 data set has been replaced already by SJVKS10 data, we have used this data set for reference only [58]. Moreover, in this plot $\Lambda CDM$ model prediction is also shown. The comparison shows that for smaller values of $n$, the theoretical curves deviate from the observational data at higher redshift regions ($z > 0.5$). The values of $n = 1.38$ and $n = 1.90$ show good agreement with the observational $H(z)$ data. Again $n = 1.90$ shows very close behaviour with the $\Lambda CDM$ model up to the redshift limit $z \leq 1$. However, we used the value $n = 1.25$ as the lower limit value of $n$ for the upper limit $H(z)$ values at higher redshift regions in this model.

We have used Eq. (20) to evaluate the distance modulus $D_m(z)$ for the model using the above mentioned three constrained values of $n$. The obtained results are compared with the Union2.1 data [54] and with the $\Lambda CDM$ model prediction as shown in the right plot of Fig. 2. It is observed that for $n = 1.90$ the model shows a very good resemblance with the experimental data as well as with the $\Lambda CDM$ model. With decrease in the value of $n$ the model slightly deviates from the $\Lambda CDM$ model as expected. It is because the model approaches GR for $n \to 1$ without a cosmological constant. Further, it should be noted that for the other two values of $n$ also, i.e. of $n = 1.38$ and 1.25 the power law model predicted values of $D_m(z)$ lay well within the experimental data.
\( \omega \) observed that the curve of \( \omega \) plot it is seen that the power law model based \( \omega \) model’s predictions of these two cosmological parameters are also plotted in the corresponding plots. From the calculated with respect to \( z \) and (31). The results of these calculations are plotted respectively on the left and right panels of Fig. 3. Further, the \( \Lambda CDM \) model’s predictions of these two cosmological parameters by the \( \Lambda CDM \) model predictions are also shown.

\[ H(z) (\text{km s}^{-1} \text{Mpc}^{-1}) \]

\[ D_m(z) \]

\[ \omega_{\text{eff}} \]

\[ q(z) \]

FIG. 2: Behaviours of Hubble parameter \( H(z) \) versus redshift \( z \) (left panel) and distance modulus \( D_m(z) \) versus \( z \) (right panel) for three different values of the parameter \( n \) of \( f(R) \) gravity power law model. The three values of \( n \) are first obtained by fitting with four sets of observed data in the \( H(z) \) plot. These values of \( n \) are then used to fit the Union2.1 data [54] in the \( D_m(z) \) plot. In both plots the corresponding \( \Lambda CDM \) model predictions are also shown.

FIG. 3: Variations of \( \omega_{\text{eff}}(z) \) versus \( z \) (left plot) and deceleration parameter \( q(z) \) versus \( z \) (right plot) for three constrained values of the parameter \( n \) of \( f(R) \) gravity power law model. The predictions of these two cosmological parameters by the \( \Lambda CDM \) model are also shown in the corresponding plots.

The effective equation state \( \omega_{\text{eff}}(z) \) and the deceleration parameter \( q(z) \) given by Eqs. (18) and (19) respectively are calculated with respect to \( z \) for the three constrained values of the parameter \( n \) of \( f(R) \) gravity power law model, using the Eqs. (30) and (31). The results of these calculations are plotted respectively on the left and right panels of Fig. 3. Further, the \( \Lambda CDM \) model’s predictions of these two cosmological parameters are also plotted in the corresponding plots. From the \( \omega_{\text{eff}} \) versus \( z \) plot it is seen that the power law model based \( \omega_{\text{eff}} \) deviates significantly from \( \Lambda CDM \) in the present scenario. However, it is observed that the curve of \( \omega_{\text{eff}} \) approaches \( \Lambda CDM \) with an increase in the value of \( n \) from 1.25 to 1.90 for the present epoch. But, for the higher values of \( z \), the curve with \( n = 1.90 \) hardly mimics \( \Lambda CDM \). One can see that, for \( n = 1.90 \) the universe does not have a radiation dominated and matter dominated phase of evolution. It has mostly the accelerating phase including the late time one, but which is not as much as expected. We see the similar trend from the \( q \) versus \( z \) plot. Here it is interesting to see that for \( n = 1.25 \) and 1.38 the power law model does not predict any accelerating phase throughout the history of the universe. These observations indicate that the power law model can not efficiently mimic the \( \Lambda CDM \) model in the early universe for large values of \( n \) and the current universe for small values of \( n \). For a more detailed observation in this direction, we use the \( Om(z) \) test and the statefinder diagnostics in section VI.

B. \( f(R) = R + \alpha R^n \) model

This model contains two constant parameters \( \alpha \) and \( n \). Here the parameter \( \alpha \) is not dimensionless and its dimension depends on the value of \( n \) used. Accordingly, for the ease of analysis we have to make \( \alpha \) dimensionless. For this purpose we rewrite the...
model as
\[ f(R) = R + \alpha H_0^{2(1-n)} R^n. \] (32)

Now using the Eq. (17) for this model in the limit \( z \to 0 \), the parameter \( \alpha \) can be expressed as
\[ \alpha = \frac{H_0^{2n-2} R_{\text{m0}}^{-n} (R_0 - 3 H_0^2 \Omega_{\text{m0}})}{n - 2}. \] (33)

Similarly, we can use the Eq. (12) in the limit \( z \to 0 \) to get the following expression for this model:
\[ \frac{2R_0 (\alpha H_0^2 n R_{\text{m0}}^n + R_0 H_0^{2n}) (R_0 H_0^{2n} - \alpha H_0^2 (n - 2) n R_{\text{m0}}^n)^2 (H_0^2 (3\Omega_{\text{m0}} + 2\Omega_{\text{om}}) + R_0 + \alpha H_0^2 R_{\text{m0}}^n)}{3 (\alpha n H_0^{2n+3} R_{\text{m0}}^n (9 H_0^2 (n - 1) \Omega_{\text{m0}} + 2(n - 3) R_0) + 2\alpha^2 H_0^2 (n - 2) n^2 R_{\text{m0}}^n - 2 R_0^2 H_0^{4n+1})^2} = 1. \] (34)

Using Eq. (33) in the above expression, we can solve this expression for \( R_0 \) numerically. One can see that the complexity of the above equation makes it difficult to have an analytical solution of the present background curvature \( R_0 \). For this model, the relation between \( z \) and \( R \) can be obtained from the Eq. (17) as given by,
\[ z = \sqrt[3]{H_0^4 \Omega_{\text{m0}}^2 (R - \alpha (n - 2) H_0^{2n} R^n)} - 1. \] (35)

From this equation we calculate numerically the values of \( z \) for a range of values of \( R \) for different values of the model parameter \( n \) together with the solutions of \( R_0 \) obtained by using the Eqs. (33) and (34) for the corresponding values of \( n \) to see the variation pattern of \( R \) with respect to \( z \) in this model. The results of this calculation is shown in Fig. 4. In this case, the scenario is opposite to the case of the power law model. Here, with increase in the value of the parameter \( n \), the slope of the curve increases, predicting deviations from the GR. For smaller values of \( n \), the curves have smaller slopes and the differences are again significant as like in the case of the power law model.

FIG. 4: Variation pattern of \( R \) with respect to \( z \) for three different values of the parameter \( n \) of the \( f(R) = R + \alpha H_0^{2(1-n)} R^n \) model.

Again, using the Eq. (12) we may write the Hubble parameter \( H \) for this model as
\[ H = \frac{H_0 \sqrt{\alpha n} H_0^{2-2n} R^n + \frac{R}{R_0} + 3(z + 1)^3 (\Omega_{\text{m0}} + 2(z + 1)\Omega_{\text{om}})}{\sqrt{6} \sqrt{\alpha n} H_0^{2-2n} R^n + R \left( 1 - \frac{9 \alpha (n - 1) (z + 1)^3 H_0^{2n+4}\Omega_{\text{m0}} R^n}{2 (\alpha n R_{\text{m0}}^n + R H_0^{2n})(R H_0^{2n} - \alpha H_0^2 (n - 2) n R^n)} \right)}. \] (36)

Now, using the expression for \( z \) in this equation it is possible to get \( H \) as a function of the scalar curvature \( R \) from this equation. Finally, all these expressions can be used to study the cosmological behaviours of this model. With an objective similar to the power law model, we have plotted \( H(z) \) with respect to \( z \) for this model also and compared the model predictions with the HKP data, SVJ05 data, SIVKS10 data and GCH90 data as shown in the left plot Fig. 5. This plot shows that the model is in good agreement with the observational data for the parameter around \( n = 0.4 \). With an increase in the value of \( n \) from 0.1, we observe that the model deviates from \( \Lambda \text{CDM} \) model in a significant manner. So, these observations predict that the model can be a good alternative of \( \Lambda \text{CDM} \) model for smaller \( n \) values. Moreover, \( n = 0.1 \) and 0.7 can be considered as lower and upper limits of values of \( n \) corresponding to the lower and upper limits of values of observed \( H(z) \).
Also we use Eq. (20) as in case of the power law model to calculate numerically the distance modulus for this model using the above three constrained values of \( n \) and results are shown in the right plot of Fig. 5. We compare these results with the Union2.1 data and we see that the model is in good agreement with the observational Union2.1 data for all three values of \( n \), i.e. \( n = 0.1, 0.4 \) and \( 0.7 \). However, it is clear that the model shows a better outcome for \( n = 0.1 \). With the increase in the value of \( n \) from 0.1, we see a tendency of deviation of the model prediction from the observational Union2.1 data.

![Graph showing variation of Hubble parameter \( H(z) \) and distance modulus \( D_m(z) \) with respect to \( z \) for three different values of the parameter \( n \) of \( f(R) = R + \alpha H_0^2(1-n) R^n \) model. The predictions of these two cosmological parameters by the \( \Lambda CDM \) model are also shown.](image)

**FIG. 5:** Variation of Hubble parameter \( H(z) \) (left panel) and distance modulus \( D_m(z) \) (right panel) with respect to \( z \) for three different values of the parameter \( n \) of \( f(R) = R + \alpha H_0^2(1-n) R^n \) model. Similar to the case of power law model, the three values of \( n \) are first obtained by fitting with four sets of observed data in the \( H(z) \) plot. These values of \( n \) are then used to fit the Union2.1 data [54] in the \( D_m(z) \) plot. In both plots the corresponding \( \Lambda CDM \) model predictions are also shown.

On the left panel of Fig. 6, we have plotted the \( \omega_{eff}(z) \) versus \( z + 1 \) plot for this model. Unlike the power law model, for this case, we see that the model can mimic \( \Lambda CDM \) model in a better way. The deviations are again larger for \( n = 0.7 \) from the \( \Lambda CDM \) model mostly within the region of small values of \( z \). Around \( n = 0.1 \), the model predicts the behaviour of the \( \Lambda CDM \) model in a comparatively better form. To be more specific, in this model the universe can have all the three phases of evolution similar to the \( \Lambda CDM \) model, i.e. the universe in this model starts from the radiation dominated phase, covers the matter dominated phase and finally ends with late time accelerating phase. These results are in a good agreement with the previous results of the model [42]. Similar results are obtained for the deceleration parameter \( q(z) \) versus redshift \( z \) calculations as predicted by the model (see the right plot of Fig. 6).

![Graph showing pattern of variations of \( \omega_{eff}(z) \) and deceleration parameter \( q(z) \) with respect to \( z \) for three constrained values of the parameter \( n \) of \( f(R) = R + \alpha H_0^2(1-n) R^n \) model. The predictions of these two cosmological parameters by the \( \Lambda CDM \) model are also shown.](image)

**FIG. 6:** Pattern of variations of \( \omega_{eff}(z) \) (left panel) and deceleration parameter \( q(z) \) (right panel) with respect to \( z \) for three constrained values of the parameter \( n \) of the \( f(R) = R + \alpha H_0^2(1-n) R^n \) model. The predictions of these two cosmological parameters by the \( \Lambda CDM \) model are also shown in the corresponding plots.

### C. The Gogoi-Goswami model

Most recently introduced the Gogoi-Goswami model of \( f(R) \) gravity is given by [30]

\[
f(R) = R - \frac{\alpha}{\pi} R_c \cot^{-1} \left( \frac{R_c^2}{R^2} \right) - \beta R_c \left[ 1 - \exp \left( -\frac{R}{R_c} \right) \right],
\]

(37)
where \( \alpha \) and \( \beta \) are two dimensionless positive constants. \( R_c \) is a characteristic curvature constant having dimensions same as curvature scalar \( R \). This model was first defined in [30], where the model was studied in the metric formalism to see the properties of gravitational waves in it and the scalar degrees of freedom extensively. The model passes the solar system tests in the metric formalism and can mimic the \( \Lambda \text{CDM} \) model at large curvatures. Another important feature of this model is that it has two correction factors having different significant contributions to its behaviours as seen from the results studied in [30]. Here, we shall focus on the Palatini formalism to see the cosmological implications of the model. For mathematical simplicity, we shall assume that the characteristic curvature constant \( R_c \) is equal to the background curvature of the present epoch. So, hereafter, we will use \( R_c = R_0 \). For this model, in the limit \( z \to 0 \), Eq. (17) gives,

\[
\beta = -\frac{e \left( 6 \pi H_0^2 \Omega m_0 + \pi (\alpha - 2) R_0 - 2 \alpha R_0 \right)}{2(2e - 3) \pi R_0},
\]  
(38)

and Eq. (12) in the limit \( z \to 0 \) becomes,

\[
\frac{\pi R_0^3 (e(\pi - 2\alpha) - 2\pi\beta)^2 (e(\pi - \alpha) - \pi\beta)}{6 (2H_0 R_0 (e(\pi - 2\alpha) - 2\pi\beta)(e(\pi - \alpha) - \pi\beta) - 9e\pi H_0^2 (e(\alpha + \pi\beta) \Omega m_0)^2) = 1.
\]  
(39)

Using the value of \( \beta \) given in the above expression we can solve for the present background curvature \( R_0 \) numerically. Now, \( z - R \) relation can be easily obtained using the Eq. (17), which is

\[
z = \frac{3^{2/3} \sqrt{H_0^4 \Omega m_0^2 \left( -2\alpha R_0 \cot^{-1} \left( \frac{R_0^2}{R^2} \right) + \frac{2\alpha R_0^2 R_0^3}{R^2 + R_0^2} + \pi R_0 \beta e^{-\frac{R_0^2}{R^2}} \left( R + 2R_0 \right) + \pi (R - 2\beta R_0) \right)} - 3 \sqrt{\pi H_0^2 \Omega m_0}}{3 \sqrt{\pi H_0^2 \Omega m_0}}.
\]  
(40)

The above expression shows that it is difficult to obtain an analytical solution of the Ricci scalar \( R \) in terms of \( z \). Therefore, we will express the cosmographic parameters as functions of \( R \) as mentioned earlier. The numerically calculated \( z - R \) relations for three different values of the model parameter \( \alpha = 0.05, 0.07 \) and 0.10 are shown in Fig. 7. These three values of \( \alpha \) are taken from the region allowed by the parameter space analysis of the model [30]. Although the differences of the curvatures of \( R \) for these three values of \( \alpha \) are very small, we see that the slope of the curvature is smaller for smaller values of \( \alpha \), similar to the case of the previously discussed Starobinsky type model.

![FIG. 7: Plot of \( R/R_0 \) versus \( z \) for three different values of the parameter \( \alpha \) of the Gogoi-Goswami model.](image)

For this model, the expression for \( H \) is given by

\[
H = \sqrt{-\frac{H_0^2}{\frac{\pi e^\frac{R_0^2}{R^2} \left( R + 2R_0 \right) + \pi R_0 \beta e^{-\frac{R_0^2}{R^2}} \left( R + 2R_0 \right) + \pi (R - 2\beta R_0) \right)} + \frac{9\pi R_0^2 e^\frac{R_0^2}{R^2} \left( R + 2R_0 \right)(z+1)^2 \Omega m_0 \left( \pi \beta \left( R + 2R_0 \right) \right)^2 - 2\alpha e^{-\frac{R_0^2}{R^2} \left( R + 3R_0 \right)^2} + 6(z+1)H_0^2 \Omega m_0 + 6(z+1)^4 \Omega m_0}}{(\pi e^\frac{R_0^2}{R^2} \left( R + 2R_0 \right) - 2\alpha e^{-\frac{R_0^2}{R^2} \left( R + 3R_0 \right)^2})} - \frac{\pi R_0 \beta (R + 2R_0)}{\pi R_0 \beta (R + 2R_0)} \left( \frac{\pi e^\frac{R_0^2}{R^2} \left( R + 2R_0 \right) - 2\alpha e^{-\frac{R_0^2}{R^2} \left( R + 3R_0 \right)^2}}{\pi R_0 \beta (R + 2R_0)} \right)}\right)
\]  
(41)

Now, using Eq. (40) in (41) we can obtain \( H \) as a function of \( R \) instead of \( z \). With an intention similar to the case of other two models, on the left panel of Fig. 8 we have plotted \( H(z) \) versus \( z \) for this model with the model parameter \( \alpha = 0.05, 0.07 \) and
0.10 as used for the \( z - R \) plot and compared with the experimental HKP data, SVJ05 data, SJVKS10 data and GCH09 data. In this case also, the model shows good behaviour with the experimental data as well as with the prediction of the CDM model. It is seen that the differences in \( z - H \) curves for all three used values of \( \alpha \) are almost negligible. However, the values of \( H(z) \) for \( \alpha = 0.05 \) are slightly greater than those corresponding to \( \alpha = 0.07 \) and 0.10, especially in the region \( z \leq 1 \).

FIG. 8: Hubble parameter \( H(z) \) versus \( z \) (left) and distance modulus \( D_m \) versus \( z \) (right) plots for different values of parameter \( \alpha \) of the Gogoi-Goswami model. The three values of \( \alpha \) are taken from the parameter space analysis of the model [30] to fit with four sets of observed data in the \( H(z) \) plot. These values of \( \alpha \) are then used to fit the Union2.1 data [54] in the \( D_m(z) \) plot. As in the case of other models, in both plots the corresponding CDM model predictions are also shown.

Numerically calculated values of the distance modulus \( D_m(z) \) from the Eq. (20) for the Gogoi-Goswami model for the said values of the parameter \( \alpha \) are shown on the right panel of Fig. 8. It is seen that all these three values of \( \alpha \) show a good agreement with the observational Union2.1 data and with the CDM prediction as expected from the previous two plots. This shows a better suitability of the parameter of the model to the experimental data in comparison to other two models.

Variations of \( \omega_{eff}(z) \) and \( q(z) \) with respect to \( z + 1 \) for the Gogoi-Goswami model are shown in the left and right plots respectively of Fig. 9 for the same set of values of \( \alpha \) used above. We see that in the near future at around \( z = -1 \), \( \omega_{eff} \) approaches zero for this model showing large deviations from the CDM model. However, in the present universe, the model mimics the results of CDM model and then with a slight deviation it follows the trend of the CDM model towards the past. Finally in the early universe, the model behaviour is totally identical to the CDM model. That is, in the early time, the universe in this model starts from the radiation dominated phase and then as time passes it attains the matter dominated phase in exactly the same way of the CDM model. Towards the present time, the universe shows late time accelerating phase with a slight deviation from the CDM model and in the near future, the universe is crossing the matter dominated phase towards the radiation dominated phase and finally ends near the matter dominated phase. This part of future time (\( z < 0 \) region) behaviour of the model is unique and does not match with the evolution phase of the universe during this period in the CDM model. Similar results hold for the deceleration parameter \( q(z) \) versus redshift \( z \) plot also as seen from the right panel of Fig. 9.

FIG. 9: \( \omega_{eff}(z) \) (left) and deceleration parameter \( q(z) \) (right) variations with respect to redshift \( z \) obtained for the Gogoi-Goswami model for three different values of the model parameter \( \alpha \). As in the previous cases the predictions of these two cosmological parameters by the CDM model are also shown in the corresponding plots.
V. CONSTRAINTS ON THE MODELS FROM OBSERVED HUBBLE DATA

Although in the last section we have used four sets of Hubble parameter data to constrained the parameters of the models within a possible range of viable parameter space, in this section we use all possible Observed Hubble Data (OHD) available to us, shown in the Table I to get a knowledge on the feasible parameter space by constraining the models in a comprehensive way. This will enable us to see or predict the behaviour of the model for the reliable parameter allowed or constrained by OHD. Here, we shall use the OHD dataset to fit the models considered in this study and to see the goodness of the fitting we shall implement the $\chi^2$ minimization method defined as

$$\chi^2 = \sum_i \frac{[H_{th}(z_i|p) - H_{obs}(z_i)]^2}{\sigma^2(z_i)},$$

(42)

where $H_{th}(z_i|p)$ is the theoretical value of the Hubble parameter $H$ at a specific redshift $z_i$ and for parameters $p$ depend on the $f(R)$ models, $H_{obs}(z_i)$ are the OHD, and $\sigma(z_i)$ is the uncertainty of each $H_{obs}(z_i)$ as obtained in OHD. Here we assume that each measurement in $H_{obs}(z_i)$ is independent.

![Diagram](image.png)

FIG. 10: $\chi^2$ fitting to the OHD dataset obtained for the power law model, $R + \alpha H_0^{(1-n)} R^n$ model and Gogoi-Goswami model with the model parameters $n = 1.4, \alpha = 0.1$ and $\Omega_m = 0.3$. The best fit value of $n$ almost agrees with the earlier best fit value of $n = 1.38$ used in the previous section for this model.

To make the expressions comparatively simple for the calculation of $H_{th}(z_i|p)$ values for the models, we have neglected the radiation component contribution in the OHD as the associated values of $z$ are not very high. After this simplification, for the power law model one can see from the previous section that the Hubble parameter becomes independent of the term $\Omega_m$. So the set of parameters considered for this model are: $p = (H_0, n)$. And from the best fit, we have obtained for the power law model that $n = 1.4$ and $H_0 = 59.8$ km/s/Mpc. This best value of $n$ almost agrees with the earlier best fit value of $n = 1.38$ used in the previous section for this model. For the model $f(R) = R + \alpha H_0^{(1-n)} R^n$, the set of parameters considered are: $p = (H_0, \alpha, \Omega_m)$. In this case, we have obtained the best fit values as $n = 0.10, \Omega_m = 0.24$ and $H_0 = 70$ km/s/Mpc. The values of $\Omega_m$ and $H_0$ found in this best fit are close to the results obtained in [42, 59]. Similarly, for the Gogoi-Goswami model, we have considered $p = (H_0, \alpha, \Omega_m)$ and the best fit values are found as $n = 0.1, \Omega_m = 0.30$ and $H_0 = 68.6$ km/s/Mpc. Here the best fitted values of $\Omega_m$ and $H_0$ are relatively near to their values found in [52] and in recent Planck’s results [53]. Fig. 10 shows the best fit plots to the OHD for the all three models.

VI. DIAGNOSTICS OF MODELS

In this section we analyze how much our models’ cosmological behaviours are different from the $\Lambda$CDM model as well as from each other by using two most effective diagnostics analyses, which are the $\Omega_m(z)$ test and statefinder diagnostic as follows.

A. $\Omega_m(z)$ diagnostic

The $\Omega_m(z)$ diagnostic is a kind of test, which can be used to discriminate between different cosmological or dark energy models from the $\Lambda$CDM model. This diagnostic was first introduced in [69] and subsequently used extensively by different
authors for the said purpose. This diagnostic parameter is defined by

\[ \Omega_m(z) = \frac{E^2(z) - 1}{(1 + z)^3 - 1} \]  

(43)

where \( E(z) = H(z)/H_0 \) is the dimensionless Hubble expansion rate. Being negligible, if we ignore the radiation component at very low redshift, we can write the Friedmann equation for the \( \Lambda \)CDM model as

\[ H(z)^2 = H_0^2 \left( \Omega_m(1 + z)^3 + 1 - \Omega_m \right). \]  

(44)

This shows that if any cosmological model behaves exactly as the \( \Lambda \)CDM model, then the \( \Omega_m(z) \) parameter of that model should be exactly equal to \( \Omega_m \). Thus, this diagnostic can be used to compare and explicitly illustrate the difference between different \( f(R) \) gravity cosmological models and the \( \Lambda \)CDM model. For the \( f(R) \) gravity models considered in this paper, we have plotted the evolution of \( \Omega_m(z) \) with respect to \( z \) in Fig. 11. For the power law model, the \( \Omega_m(z) \) function shows large deviations in the present universe and it approaches the \( \Lambda \)CDM model in the early universe. However, it is seen that the model can not completely mimic the \( \Lambda \)CDM model for a long time in the early universe as mimicking is only momentary in nature. The model parameter \( n = 1.90 \) coincides the \( \Lambda \)CDM model just before \( z = 1 \) and then starts deviating from it. This coinciding point moves towards the higher \( z \) values with the decreasing value of \( n \) (see left plot of Fig. 11).

For the model of the type \( f(R) = R + \alpha H_0^{2(1-n)} R^n \), \( n = 0.1 \) shows the smallest deviations from the \( \Lambda \)CDM model and starts mimicking it in the early universe. However, \( n = 0.4 \) and \( n = 0.7 \) show comparatively large deviations from the \( \Lambda \)CDM model in the present universe, which decreases towards the early universe. So, in conclusion, we can say that this model can be indistinguishable from the \( \Lambda \)CDM model practically in the early universe for smaller values of \( n \) (see middle plot of Fig. 11).

On the right panel of Fig. 11, it is seen that the Gogoi-Goswami model is practically indistinguishable from the \( \Lambda \)CDM model in the early universe for all allowed values of the parameter \( \alpha \). Although the model parameter \( \alpha = 0.05 \) shows closer results to the \( \Lambda \)CDM model in the near past scenario, all the three values of it are seen to be indistinguishable in the early universe. As a whole, the \( \Omega_m(z) \) function sharply decreases in the low redshift regime and crosses the \( \Lambda \)CDM model. After that, it starts following the \( \Lambda \)CDM model towards higher redshifts and becomes indistinguishable towards the early universe.

It is to be noted that these results are in agreement with previous results obtained from the behaviours \( \omega_{eff}(z) \) and \( q(z) \) with respect to \( z \) for all three \( f(R) \) gravity models.

| \( z \)   | \( H(z) \)   | Method    | Reference |
|---------|--------------|-----------|-----------|
| 0.0708  | 69.0 ± 19.68 | DGAM [60] |           |
| 0.09    | 69.0 ± 12.0  | DGAM [61] |           |
| 0.12    | 68.6 ± 26.2  | DGAM [60] |           |
| 0.17    | 83.0 ± 8.0   | DGAM [55] |           |
| 0.179   | 75.0 ± 4.0   | DGAM [62] |           |
| 0.199   | 75.0 ± 5.0   | DGAM [62] |           |
| 0.20    | 72.9 ± 29.6  | DGAM [60] |           |
| 0.240   | 79.69 ± 2.65 | RBAOM [57]|           |
| 0.27    | 77.0 ± 14.0  | DGAM [55] |           |
| 0.28    | 88.8 ± 36.6  | DGAM [60] |           |
| 0.35    | 84.4 ± 7.0   | RBAOM [63]|           |
| 0.352   | 83.0 ± 14.0  | DGAM [62] |           |
| 0.3802  | 83.0 ± 13.5  | DGAM [64] |           |
| 0.4     | 95 ± 17.0    | DGAM [55] |           |
| 0.4004  | 77.0 ± 10.2  | DGAM [64] |           |
| 0.4247  | 87.1 ± 11.2  | DGAM [64] |           |
| 0.43    | 86.45 ± 3.68 | RBAOM [57]|           |
| 0.44    | 82.6 ± 7.8   | RBAOM [65]|           |
| 0.4497  | 92.8 ± 12.9  | DGAM [64] |           |

TABLE I: Currently available observed Hubble dataset. Here \( H(z) \) is in units of km/s/Mpc, DGAM stands for “Differential Galactic Ages Method” and RBAOM for “Radial BAO Method”.
The deceleration parameter \( \Lambda \) and Gogoi-Goswami model in Fig. 12. It is seen that for the power law model, the parameter \( r \) behaves differently and cannot mimic the \( \Lambda \)CDM model behaviour. Moreover, for \( n = 1.25 \) and \( n = 1.38 \) it crosses the \( \Lambda \)CDM model at two distinct points in the early universe and the model starts showing huge deviations just after or before these two points. On the other hand, for the case \( n = 1.90 \), the model curve has less slope or inclination in comparison to the two other cases and is expected to intersect the \( \Lambda \)CDM model in such a scenario. Similarly, in case of the Gogoi-Goswami model, we observe that for small values of \( n \) the model shows maximum deviations from the \( \Lambda \)CDM model in the present or near future. The model shows maximum deviations from the \( \Lambda \)CDM model in the near past and then again starts approaching the \( \Lambda \)CDM model in the early universe. So, in the present universe or in the very early universe, the model can mimic the \( \Lambda \)CDM model and it might be difficult to differentiate it from the \( \Lambda \)CDM model in such a scenario. Similarly, in case of the Gogoi-Goswami model, we observe that the model shows maximum deviations from the \( \Lambda \)CDM model in the present universe and in the near future or early universe it shows similar behaviour as that of the \( \Lambda \)CDM model. So in the early universe, the model cannot be efficiently differentiated from the \( \Lambda \)CDM.

The evolution trajectories of \((r, s)\) pair in \( r - s \) plane and \( r - q \) plane for these three models are shown in Fig.s 13, 14 and 15 respectively. For the power law model, for \( n = 1.25 \) and \( n = 1.38 \), the evolution trajectory starts from \((r, s) = (1, 0)\),...
FIG. 12: Statefinder parameter $r$ versus $z$ for the $f(R)$ gravity power law model (left), $f(R) = R + \alpha H_0^{2(1-n)} R^n$ model (middle) and Gogoi-Goswami model (right).

FIG. 13: Evolution of the statefinder parameter $r$ with respect to the parameter $s$ (left) and evolution of $r$ with the deceleration parameter $q$ (right) for the $f(R)$ gravity power law model.

the statefinder pair of the $\Lambda$CDM model and moves in a similar path towards the present universe at which the deviation from each other becomes significant. However, for $n = 1.90$ the trajectory is completely different from the previous two. Here the trajectory covers a different path and never moves near the point $(r, s) = (1, 0)$. Similarly, in the $r – q$ plane trajectories for $n = 1.25$ and $n = 1.38$ follow a similar trend. For $n = 1.90$, the evolution of trajectory in the $r – q$ plane is different from $n = 1.25$ and $n = 1.38$.

For the $f(R) = R + \alpha H_0^{2(1-n)} R^n$ model, the evolution trajectories of $r – s$ start from the statefinder point of the $\Lambda$CDM model and then covering a distinct path in the $r – s$ plane they again meet the statefinder point of the $\Lambda$CDM model as shown in Fig. 14. For $n = 0.1$, the statefinder curve covers a small area in comparison to the cases with $n = 0.4$ and $n = 0.7$. The statefinders show that there are obvious fluctuations in the $r$ parameter and it can be a good measure to differentiate between different $n$ values as well as other models. The $r – q$ plane shows the variation of the statefinder parameter $r$ with respect to the

FIG. 14: Evolution of the statefinder parameter $r$ with respect to the parameter $s$ (left) and evolution of $r$ with the deceleration parameter $q$ (right) for the $f(R) = R + \alpha H_0^{2(1-n)} R^n$ model.
deceleration parameter $q$. Similar to the $r - s$ plane, one can see that the evolution trajectories in $r - q$ plane mimic the $\Lambda$CDM model in the early universe and towards the present and near future universe. So, before $q = 0.4$ and after $q = -0.4$ the $r - q$ plane trajectories may not effectively differentiate the model from the $\Lambda$CDM model.

Similarly, for the Gogoi-Goswami model, the $r - s$ and $r - q$ evolution trajectories are shown in Fig. 15. It is to be mentioned here that for a small change in the value of $z$ the statefinder parameters especially, $s$ shows higher fluctuation of its value and both of them give higher values for a given $z$ in this model in comparison to the other two models. Hence, for this model the trajectories in the $r - s$ plane are shown only for a small region from around the present universe to the near future. Figure shows that the evolution trajectories arrive from different directions and cross the statefinder point of the $\Lambda$CDM model in a future point and then again covers a wide path to attain the point $(2, 1)$, where all the curves converge together. From this point onwards, the trajectories move to the statefinder point of the $\Lambda$CDM model as a final destination. Unlike for the previous models, here the trajectories cover a different pattern which enables to differentiate it from other models in terms of statefinder parameters. The $r$ fluctuations in the trajectories show that the evolution of $r$ might be useful to differentiate the model behaviour. On the right panel of Fig. 15, we have plotted the statefinder parameter $r$ versus deceleration parameter $q$ evolution trajectories. From this plot one can see the unique behaviour of the model. The trajectories start moving in the positive $r$ side initially in the early universe. After a long journey, the trajectories move to the negative $r$ side and start moving in the opposite direction and finally at the end, they again move to the positive $r$ side and return to the initial point mimicking the $\Lambda$CDM model. Due to such unique behaviours of the model in $r - s$ and $r - q$ plane, it is possible to differentiate the model easily from the $\Lambda$CDM model, power law type model and model of the type $f(R) = R + \alpha H_0^{2(1-n)} R^n$.

VII. CONCLUSIONS

We have investigated the evolution of the universe with the help of cosmographic parameters like Hubble parameter, deceleration parameter, effective equation of state etc. as predicted by three $f(R)$ gravity models, viz., (i) the power law $f(R)$ gravity model $\xi R^n$, (ii) $f(R) = R + \alpha H_0^{2(1-n)} R^n$ model and (iii) the Gogoi-Goswami model for different values of the parameters of these models. As the Hubble parameter determines the expansion rate of the universe, we have plotted $H(z)$ versus $z$ for these models to check their behaviours in expansion rate at different stages of the universe. We have seen that for the considered values of the model parameters, the models show the consistency with the HKP data, SVJ05 data, SJVKS10 data and GCH09 data. Similarly, we have also calculated distance modulus for the models with the considered model parameter values sets and see that of this model has not been studied till now. So undoubtedly, this is the first study dealing with the cosmological perspectives of the model. The $H(z)$ versus $z$ plots and distance modulus versus $z$ plots show that for the parameter $n = 0.1$, the model behaves in a better manner with the respective experimental results. Later, analysis with OHD data suggests that the model can show a good fitting for $n = 0.1$ with $H_0 = 70 \text{ km/s/Mpc}$ and $\Omega_{m0} = 0.24$, which is in accordance with the previous results. The third model we have considered here is a very new dark energy $f(R)$ gravity model and the cosmological behaviour of this model has not been studied till now. So undoubtedly, this is the first study dealing with the cosmological perspectives of the model. The $H(z)$ versus $z$ plots and distance modulus versus $z$ plots for this model show comparatively good behaviour in comparison to the previous two models. It is seen from these plots that for a smaller value of the parameter $\alpha$ the model behaves in a better way with the observed data, although there is no significant difference of the results obtained for the allowed range of
values $\alpha$. Here, we see that for $\alpha = 0.05$, the model predictions are closer to that of the $\Lambda$CDM model. The OHD best fit values for the model are found to be $\alpha = 0.1$, $\Omega_{m0} = 0.30$ and $H_0 = 68.6 \text{ km/s/Mpc}$ which are in good agreement with the previous results.

For more detailed information on the models, we have performed the $Om(z)$ test on the models. We see that the first model is significantly different from the $\Lambda$CDM model, whereas the second and the third model can mimic the $\Lambda$CDM model in the early universe. However, as a whole, for the second model i.e. the $f(R) = R + \alpha R_0^{2(1-n)} R^n$ model, the $Om(z)$ function gives higher values than that of the $\Lambda$CDM model. But in the Gogoi-Goswami model, the $Om(z)$ function initially starts with a higher value than that of the $\Lambda$CDM model, then drops below the $Om(z)_{ACDM}$ and finally starts approaching the $\Lambda$CDM model in the early universe. This behaviour suggests that the Gogoi-Goswami model behaves differently than the first and the second model. Similar results are obtained in the statefinder analysis also. We have seen that the evolution trajectories of the statefinder parameters behave uniquely in the Gogoi-Goswami model and the model can be easily differentiated from the previous two models with the help of statefinder parameters. Thus, it is seen that although the $\Lambda$CDM can be reconstructed by a huge class of extended gravity models with the help of model parameters, usually the models can be differentiated from each other by these two most effective diagnostic tests. Also, one may consider the Bayesian method for the Bayesian information criteria for a better comparison of the dark energy models including modified gravity models [74]. We leave this method of study as a future scope of our works. Moreover, to comment more about the viabilities of such models, one may consider the studies of inflationary epoch, large scale structure etc. from such models [75]. So we believe that another study in this perspective will shed more light to the properties and behaviour of the new model.

Thus from our study, we conclude that the Palatini Gogoi-Goswami model can be a good alternative to the $\Lambda$CDM model and can result in more interesting outcomes in predicting the fate and evolution of the universe. In the near future this model can be used in the metric formalism as well as in the Palatini formalism to test it in the context of various observational astrophysical and cosmological data.

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