Accounting for monopole configurations in Yang-Mills theory in three Euclidean dimensions

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Abstract

A gauge transformation provided by the three eigenfunctions of $B^a(x) \cdot B^b(x)$ (where $B^a(x)$, with $a = 1, 2, 3$, are the non-Abelian magnetic fields) exposes the topological configurations of the Yang-Mills fields. In particular, it gives Dirac monopoles interacting with ‘photons’ and massless charged vector bosons. A magnetic dipole field at each monopole corresponds to infinitesimal translation of the monopole, and provides the functional measure à la collective coordinates. The grand canonical partition function of the monopole plasma is exactly equivalent to a local field theory with certain scalar fields interacting with the Yang-Mills fields. This integrates topological degrees of freedom with perturbation theory.

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1 Introduction

Consider the Yang-Mills quantum field theory in three Euclidean dimensions given by the partition function

\[ Z = \int DA \exp \left( -\frac{1}{2g^2} \int d^3x B^a(x) \cdot B^a(x) \right) \tag{1.1} \]

where \( B^a(x) = \nabla \times A^a(x) - \frac{1}{2} \epsilon^{abc} A^b(x) \times A^c(x), a = 1, 2, 3 \), is the non-Abelian magnetic field in the vector representation of the group \( SO(3) \). The theory is super-renormalizable in a perturbation in the coupling constant \( g \). Therefore we have a good control on the ultraviolet properties of the theory. The perturbation theory has severe infrared divergences. It is expected that the theory is confining, so that there is a mass gap and no infrared divergences. This is supported by lattice gauge theory in a strong coupling expansion and also simulations. We need a qualitative understanding of this phenomenon and quantitative techniques to handle it.

The Georgi-Glashow model in three Euclidean dimensions (GGM), which is a closely related model, is very well understood [1]. (For a review see [2].) Fluctuations about a (presumed) expectation value of the Higgs scalar suggest a massless ‘photon’, massive charged vector bosons and a massive Higgs. But non-perturbative effects change the spectrum of the theory drastically. The ‘dual photon’ acquires a tiny mass and the theory is confining. The reason is a ‘plasma’ of monopoles and anti-monopoles which screens the photon and forms a dipole sheet across a Wilson loop used as a probe of confinement. The action has a stable, topologically non-trivial extremum – the ’t Hooft-Polyakov monopole [3] (playing the role of an instanton here). In a regime of coupling constants, a dilute gas approximation of the plasma can be justified and the effects can be reliably computed.

The situation in case of pure gauge theory is much more complicated. The ’t Hooft-Polyakov ansatz for the Yang-Mills potential is

\[ A^a_i(x) = \epsilon_{iab} x^b 1 - K(r) \frac{r}{r^2} \tag{1.2} \]

where \( i = 1, 2, 3 \) labels the space index and \( a, b = 1, 2, 3 \) label the group indices. Here \( r = \sqrt{x^a x^a} \). As long as \( K(r) = 1 + O(r^2) \) as \( r \to 0 \) and \( K(r) \to 0 \) as \( r \to \infty \), this configuration is also of finite action in pure Yang-Mills theory. But there is no non-trivial classically stable configurations with a finite action. The configuration (1.2) is unstable against an indefinite expansion, as may be checked by a rescaling. For \( K(r) \to 1 \), it is degenerate with the perturbative vacuum \( A = 0 \). We need to handle these long range fluctuations. We need techniques very different from the case of GGM. Nevertheless we may expect (Refs. [2], [4]-[13]) that topological configurations are relevant for the infrared behaviour of the theory.

The first question is whether the configurations (1.2) have any topological significance. They have the telltale effects of a monopole on a large Wilson loop. They contribute a phase proportional to the solid angle subtended by the loop at the monopole (center). But we need to characterize how they are distinct from perturbative fluctuations.

For a configuration with many monopoles and anti-monopoles in GGM the topological features can be located by the zeroes of the Higgs field and its behaviour in the neighborhood [14, 15]. In Ref. [16] it has been proposed that for the Yang-Mills field of (1.2),
the topological aspects are characterized by the degeneracies in eigenvalues of the gauge invariant symmetric matrix field, \( S_{ij}(x) = \sum_a B^a_i(x) B^a_j(x) \),

\[
S_{ij}(x)\xi^A_j(x) = \lambda^A(x)\xi^A_i(x), \quad A = 1, 2, 3
\]  

(1.3)

The (anti-)monopoles can be located at points where the three eigenvalues become degenerate. At such points one of the eigenfunctions, say \( A = 3 \), has a 'radial' behaviour.

We compare and contrast this proposal with the Abelian projection proposal of t'Hoof [15]. There a composite scalar transforming in the adjoint representation of the gauge group is formed out of the gauge field and used in place of the fundamental Higgs of the GGM. The location of the zeroes depend on the composite chosen, though the net 'monopole charge' is an invariant. In contrast we are using a gauge invariant composite (1.3) for locating the topological aspects. In spite of this difference there is a direct connection with the abelian projection proposal. We also have a closely related entity, \( s_{ab}(x) = B^a_i(x) \cdot B^b_j(x) \),

\[
B^a_i(x) \cdot B^b_j(x)\xi^A_b = \lambda^A(x)\xi^A_a
\]

(1.4)

which transforms homogeneously in the symmetric tensor representation of gauge group \( SO(3) \). Its eigenvalues are the same as of (1.3) and hence gauge invariant. Each of the three eigenfunctions are in the adjoint representation. One of these, say \( A = 3 \), plays the role of the scalar composite of t'Hoof.

We have located the monopoles at points where the eigenvalues in (1.3) are triply degenerate. We may expect that a more generic singularity is a double degeneracy as analysed by t'Hoof in the context of abelian projection [15]. In Ref. [18] it has been argued that the generic configuration has half-monopoles joined by (presumably) \( Z_2 \) strings. The triple degeneracy is the extreme limit where the two half monopoles are collapsed on each other.

(It is likely that our criterion [16, 17] gives a complete characterization of the topological singularities of the Yang-Mills fields in 3-dimensions.) In this paper we restrict only to the point singularities i.e. the (anti-)monopoles. It is possible that very long \( Z_2 \) strings are the dominant configurations. This more general case can also be handled by our techniques. This will be illustrated in case of Yang-Mills theory in 3+1 dimensions elsewhere [19].

Our choice for locating the topological aspects has an added advantage. As \( s_{ab}(x) \) is a symmetric (real) matrix, the three eigenfunctions \( \xi^A_a(x) \) in (1.4) (after normalization) form an orthonormal set and give an \( SO(3) \) matrix which can be used for a local gauge transformation. Note that a given Yang-Mills field uniquely prescribes this gauge transformation (up to a trivial ambiguity in the choice of the eigenvectors).

At points of degeneracy of the eigenvalues the gauge transformation is singular. In case of the 't Hooft-Polyakov ansatz (1.2), the \( SO(3) \) matrix takes the form (\( \hat{\theta}, \hat{\phi}, \hat{r} \)) i.e. the unit vectors of the spherical coordinate system [3, 14, 17]. Even though the Yang-Mills configuration (1.2) looks innocuous the transformed gauge field has singularities. Indeed, the third component (in group space) is precisely the Dirac vector potential of a monopole. The Dirac string along the third direction is due to an arbitrary choice of the ‘curling' eigenvectors \( \hat{\theta}, \hat{\phi} \) among the degenerate eigenvectors. Thus our gauge transformation which is dictated by the gauge configuration itself, highlights the topological aspects of the given gauge field.

The value of a singular gauge configuration has been realized early in attempts (Refs. [4]-[13]) to handle topological configurations beyond a semiclassical approximation. There
is extensive work using the maximal abelian gauge. Our method of using the eigenfunctions of Eq. (1.4) has many technical advantages as seen below.

Even though our gauge transformation has singularities, it is an $SO(3)$ matrix at each $x$. Therefore, even though the transformed gauge potential has singularities, the transformed non-abelian field strength, which was finite to begin with, remains finite everywhere. This is the reason why the Dirac string does not contribute a singular term to the action. This is shown explicitly in Ref. [20].

We have used the singular gauge transformation to motivate the action in the form to expose the topological degrees of freedom [20]. With this choice of gauge, the transformed non-abelian magnetic fields $B_1, B_2, B_3$ are mutually orthogonal. However, we do not want to be bound to this gauge except close to the locations of monopoles (where we want all Dirac monopoles to be in the color direction $A = 3$ and all Dirac strings to be along the $z$-axis). We can therefore continue using the Faddeev-Popov technique (eliminating the gauge zero-modes of the fields by gauge-fixing) and keep the successes of renormalized perturbation theory.

We have abelianized the topological aspects by having all monopoles in one colour direction. Also, we have a linear superposition of the topological configurations, to make any computation possible. Thus our action to include the topological degrees explicitly is

$$S = \int d^3x \left( \frac{1}{2} \left( -\nabla \Phi + \nabla \times a + iW^+ \times W^- \right)^2 + |D[A + a] \times W^-|^2 \right)$$

where we have ignored the Dirac string contribution as justified in Ref. [20]. Here

$$\Phi(x) = \sum_m q_m \frac{1}{|x - x_m|}$$

is the magnetic potential due to monopoles and anti-monopoles at $x_m$ of (quantized) charges $q_m$. Also

$$D[A] = \nabla - iA$$

is the Abelian covariant derivative. In addition there are the gauge fixing and ghost terms which will not be handled explicitly in this paper.

Thus we have Dirac monopoles at arbitrary points with ‘photons’ and charged massless vector bosons $W^\pm$ scattering off them. Even though the monopole field is singular, the action is rendered finite by the singular boundary conditions of $W$ at the location of the monopoles. In fact requiring the action be finite selects out the boundary conditions for the charged vector mesons.

$$W(x) \to (\hat{\phi}_m - i\hat{\theta}_m) \frac{e^{i\phi_m}}{\sqrt{2|x - x_m|}} + O(|x - x_m|). \quad (1.8)$$

where $\phi_m, \theta_m$ are the spherical coordinates centered at $x_m$.

We now point out how the topological aspects change the contribution to the partition function. In the action (1.5) consider terms quadratic in $W$, setting the ‘photon’ fluctuations $a$ to zero. This has precisely the form $W^+ H W^-$ where $H$ is the non-relativistic
Hamiltonian (with unit mass) for a charged vector boson interacting with Dirac monopoles at fixed positions \([20]\). This includes an anomalous magnetic moment \(g = 2\) interaction, which is a consequence of non-abelian gauge invariance. This Hamiltonian has zero modes \([20]\) of the form \(D \Lambda\) for an arbitrary function \(\Lambda(x)\) where \(D\) is the covariant derivative [1.7]. This is just a reflection of the local gauge invariance. With any gauge fixing for \(W\), these zero modes are absent. In case of one monopole, it has been shown in Ref [20] that there are no other zero modes. Thus (after a gauge fixing) we just get a contribution \((\text{Det} H)^{-1}\). This determinant involving the background of the Dirac monopoles depends on the scattering phase shifts and is obviously different from that of free vector bosons. In this way the monopole configurations of the Yang-Mills theory give a contribution distinct from the perturbation theory.

2 Local field theory of interactions of monopole charges and electric currents in three Euclidean dimensions

Consider the free Maxwell theory in three Euclidean dimensions. The partition function is

\[
Z = \int Da \exp \left( -\frac{1}{2g^2} \int d^3x (\nabla \times a(x))^2 \right)
\]  

(2.1)

Using an auxiliary field \(b\) we write

\[
Z = \int Db Da \exp \left( \int d^3x \left( -\frac{g^2}{2} b(x)^2 + i b(x) \cdot \nabla \times a(x) \right) \right)
\]  

(2.2)

Note the presence of \(i = \sqrt{-1}\) in the term linear in \(a\) in the exponent. Now the integration over \(a\) gives a constraint:

\[
Z = \int Db \prod_x \delta(\nabla \times b(x)) \exp \left( -\frac{g^2}{2} \int d^3x b(x)^2 \right)
\]  

(2.3)

The constraint has the solution \(b(x) = \nabla \chi(x)\). Therefore

\[
Z = \int D\chi \exp \left( -\frac{g^2}{2} \int d^3x (\nabla \chi(x))^2 \right)
\]  

(2.4)

describing a free massless scalar. This field has the interpretation as the ‘dual’ photon. In three dimensions the ‘photon’ has only one transverse degree of freedom which is described by the scalar \(\chi\).

Consider now the interaction between sources \(j\) of Dirac monopoles and electric currents \(J\). This is described by the partition function

\[
Z = \exp \left( \int d^3x d^3y \left( -\frac{1}{2g^2} j(x) D(x - y) j(y) + i j(x) A_i(x - y) J_i(y) - \frac{g^2}{2} J_i(x) D_{ij}(x - y) J_j(y) \right) \right)
\]  

(2.5)
\[ D(x-y) \text{ and } D_{ij}(x-y) \text{ are the free propagators for massless scalar (monopole) and massless vector fields in coordinate space.} \]

\[ A(x-y) \text{ is the Dirac vector potential of a monopole, here playing the role of the propagator connecting monopole charges to electric currents.} \]

\[ \text{The partition function in (2.5) describes the following local quantum field theory involving monopole charges, photons and electric currents:} \]

\[ Z = \int DaD\chi \exp \left( \int d^3x \left( -\frac{1}{2}(\partial_3 \chi(x))^2 - \frac{1}{2}(\hat{n}_3 \times (\nabla \times a(x)))^2 + i\partial_3 \chi(x) \hat{n}_3 \cdot \nabla \times a(x) + igJ_i(x)a_i(x) + ig^{-1}j(x)\chi(x) \right) \right) \quad (2.6) \]

To see this, we use a gauge-fixing term for \( a(x) \) corresponding to the Feynman gauge and go over to the Fourier space. [It is useful to employ the identity]

\[ (\hat{n}_3 \times (\nabla \times a))^2 = (\nabla \times a)^2 - (\hat{n}_3 \cdot \nabla \times a)^2 = (\nabla \times a)^2 - ((\hat{n}_3 \times \nabla) \cdot a)^2 \quad (2.7) \]

then we obtain

\[ Z = \int Da(k)D\chi(k) \exp \left( \int d^3k \left( -\frac{1}{2}\chi(-k)k_3^2\chi(k) - \frac{1}{2}a_i(-k)(k^2\delta_{ij} - k_3k_3\delta_{ij})a_j(k) + i\chi(-k)k_3k_3a_i(k) + igJ_i(-k)a_i(k) + ig^{-1}j(-k)\chi(k) \right) \right) \quad (2.8) \]

where \( k_\perp = \hat{n}_3 \times k \). On performing the functional integration over \( a(k) \) and \( \chi(k) \), one gets a quadratic form in the currents. This is found to be the same as that in Eq. (2.5) in Fourier space, on using the momentum space representation of \( A_i(x-y) \) given in Eq. (A.9).

Eq. (2.6) has the ‘photon’ quanta described simultaneously by the dual fields, the usual vector potential \( a \) and the dual scalar \( \chi \). This is the analogue of the two potential formalism of Zwanziger [21] for quantum electrodynamics of monopoles and charges in three Euclidean dimensions. Note the characteristic appearance of \( \hat{n}_3 \cdot \nabla \times A \) in the terms involving the photon and the dual photon. Also only the \( \partial_3 \) derivative of \( \chi \) is appearing. Similar terms will appear in Sec. 4.

### 3 Functional measure in the presence of monopole configurations

We now address the issue of the functional measure in the presence of monopoles. In the case of a semiclassical quantization about (say) an instanton, the standard technique to obtain the measure is the collective coordinate method. The quadratic terms in fluctuations about an instanton have zero modes related to translation and other continuous symmetries of the theory that are broken by the choice of position and other degrees of freedom of the
instanton. The fluctuations which translate the instanton (for example) are replaced by an integration over the position of the instanton using the Faddeev-Popov trick.

We now point out that even though we are not doing an expansion about a saddle point of the action, we can adopt the same strategy. The action in Eq. (1.5) contains the combination $-\nabla \Phi + \nabla \times a$. Given a magnetic potential $\Phi$ corresponding to a configuration of monopoles and anti-monopoles of various (quantized) charges, consider another where all but one (anti-)monopole, say at $x_m$, is displaced by an infinitesimal amount $\delta_j$ in the $j$th direction. The difference in the magnetic potential $\Phi$ of the two configurations is precisely that of a magnetic dipole of moment $q_m \delta_j$. The corresponding magnetic field $-q_m \delta_j \nabla \phi$ can be obtained from a vector potential. This means that the mode of the ‘photon’ $a$ corresponding to such a dipole can be treated as a mode which displaces the monopole at $x_m$ in the $j$th direction. We can therefore eliminate these modes from $a$ and replace them with integration over the positions of the monopoles.

Let us now directly check that these dipole modes of $a$ are of the form $\partial_j A(x - x_m)$. The Dirac potential of a monopole located at the origin, with the Dirac string along the $z$-direction, is given by Eq. (A.1). First consider the derivative in the $z$-direction:

$$\frac{\partial A(x)}{\partial x_3} = \hat{n}_3 \times \frac{r}{r^3}$$

(3.1)

[This follows from Eq. (A.2), Eq. (A.4) and $\partial_3 g = 1/r$.] This is precisely the vector potential of a dipole at the origin, and so the curl of it gives the magnetic field of a unit dipole moment in the $z$-direction. But the situation is a little more involved for the derivatives along (say) the $x$ direction. The reason is that the Dirac string along the $z$ direction is now infinitesimally displaced in the transverse direction and its effects persist in the vector potential. Consider the vector potential of a monopole with the Dirac string in the $x$ direction. It differs from (A.1) by a gauge transformation. Its derivative in the $x$ direction gives the standard dipole potential. Thus the $x$ and $y$ derivatives of (A.1) also give the dipole potential but in a non-standard gauge.

We insert a unit factor in the partition function for each (anti-) monopole:

$$1 = \int d^3 x_m \prod_{j=1,2,3} \delta(\int d^3 x \partial_j A(x - x_m) \cdot a(x)) \int d^3 x \partial_p \partial_q A(x - x_m) \cdot a(x)$$

(3.2)

Here $\partial_p$ stands for derivative with respect to the $p$th component of $x_m$ and $|M_{pq}|$ stands for the determinant of a $3 \times 3$ matrix $M_{pq}$. The constraint is independent of the charge of the monopole. We can use the BRST techniques to handle these constraints. We have thus obtained integration over the positions of the monopoles from the functional measure, à la the collective coordinate method.

4 Effects of the monopole plasma from a local action

A major reason for the success [1] in understanding the GGM is that the grand canonical partion function of the monopole plasma could be handled. We need to know the effects of an arbitrary number of monopoles and anti-monopoles at arbitrary locations on an external probe. In order to do this in the present case we use the first order formalism. Note
that the first order formalism is as good as the usual second order formalism for carrying out a renormalized perturbation theory. For explicit Feynman rules and diagrammatic calculations (in the 3+1-dimensional context) see Ref. [22].

In the first order formalism, our partition function takes the form (as in Eq. (2.2))

\[
Z = \int D\lambda D\Phi D\sigma D\psi \exp \left( \int d^4x \left( \frac{1}{2} \sigma^2 - i \sigma \cdot \Phi + \frac{1}{2} \lambda^2 + \frac{1}{2} \psi^2 \right) \right)
\]

(4.1)

There are also the gauge fixing and the ghost terms, which we have not written explicitly.

Consider the gradient and curl parts of \( b \):

\[
b = \nabla \chi + \nabla \times c
\]

(4.2)

Now the part of the partition function involving \( b \) takes the form,

\[
Z = \int D\lambda D\Phi D\sigma D\psi D\chi Dc \exp \left( \int d^4x \left( - \frac{g^2}{2} \left( \nabla \chi \right)^2 + \left( \nabla \times c \right)^2 \right) + \chi(x) \nabla \cdot (W^+ \times W^-) + i(\nabla \times c) \cdot (\nabla \times \psi + iW^+ \times W^-) - i \sum_m q_m \chi(x_m)
\]

(4.3)

Here we have used

\[
\nabla^2 \Phi(x) = - \sum_m q_m \delta(x - x_m)
\]

(4.4)

The gradient part \( \chi \) is the ‘dual photon’ as discussed in Sec. 2. In Eq. (4.3), \( \chi \) couples to the monopoles locally as the dual photon potential should. If we consider only monopoles and anti-monopoles of unit charge as in \( [1] \), summing over these charges we get \( \cos \chi(x) \). A sum over arbitrary number of monopoles and anti-monopoles exponentiates this into a new term in the action \( [1] \). This gives a mass to the ‘dual photon’. However the situation is more complicated in our case.

We also have to handle the interactions of the monopoles with the massless charged vector bosons. It is possible to adopt the techniques of Sec. 2 with some modification to obtain the analogue of the two potential formulation. But we adopt a different procedure here. Our main purpose here is to represent net effects of the monopole plasma by a local field theory.

The interaction of a monopole at \( x_m \) with an electric current at \( x \) can be put in the convenient form

\[
i \int d^3x A(x - x_m) \cdot J(x) = i 4\pi \int d^3x (\partial_5 \nabla^2)^{-1}(x - x_m) \hat{n}_3 \cdot \nabla \times J(x)
\]

(4.5)

where we used Eq. (A.6). Here the specific form of the (hermitean) current \( J \) will not matter. [In Eq. (4.1), \( J = i(\sigma^+ \times W^- - \sigma^- \times W^+) \).] Now, Eq. (4.5) will have a sum over
m corresponding to monopole charges $q_m$ in the present case. The resulting expression can be put in a local form using auxiliary scalars $\phi$ and $\psi$:

$$\int D\phi D\psi \exp \left( i \int d^3x \psi(x) \partial_3 \nabla^2 \phi(x) + i \sum_m q_m \phi(x_m) + i 4\pi \int d^3x \psi(x) \hat{n}_3 \cdot \nabla \times J(x) \right) \quad (4.6)$$

[To see this, write the second term in (4.6) as $i \sum_m q_m \int d^3x \phi(x) \delta(x - x_m)$]. Integration over $\phi(x)$ then gives a constraint on $\psi(x)$. Thus the scalar $\phi$ couples locally to the monopoles just as $\chi$ in Eq. (4.3).

Now we consider the effects of summing over the monopole charges $q_m = \pm 1, \pm 2$ etc. In GGM [1], the fugacity for each monopole meant that it is sufficient to consider only $q_m = \pm 1$. Now there is no separate fugacity for each monopole. Nonetheless we expect the generic configuration will have $q_m = \pm 1$, with higher charges resulting from a merging of monopoles with a loss of entropy. Summing over only $q_m = \pm 1$ we get $\cos(\phi(x) - \chi(x))$ in the action. The combination $\phi(x) - \chi(x)$ plays the role of $\phi(x)$ of the GGM.

We now address the constraints due to the collective coordinates (3.2). These are non-local constraints on the ‘photon’. We make them local by introducing an auxiliary scalar $\eta(x)$:

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int D\eta \prod_x \delta(\partial_3 \nabla^2 \eta(x) + 4\pi \hat{n}_3 \cdot \nabla \times a(x))$$

$$\times \prod_{m=1}^N \int d^3x_m \prod_{j=1,2,3} \delta(\partial_j \eta(x_m)) |\partial_p \partial_q \eta(x_m)|$$

(4.7)

[To check this, define $\eta(x) = \int d^3y A(y - x) \cdot a(y)$, use Eq. (A.6) and operate on $\eta(x)$ by $\partial_3 \nabla^2$.] Eq. (4.7) means that the auxiliary scalar $\eta$ has an extremum at the locations of the (anti-)monopoles, and $|\partial_p \partial_q \eta|$ is the determinant of the quadratic fluctuations at these extrema.

For the collective coordinates we can as well require the ‘photon’ fluctuations to satisfy the constraint $\int d^3x \partial_j A(x_m - x) \cdot a(x) = v_j$ for any chosen vector $v$. This allows a smoothening of the constraint $\delta(\partial_j \eta(x_m))$ as in the Faddeev-Popov procedure. Also for the determinant we can use ghost fields.

In this paper we developed a procedure to rewrite the Yang-Mills quantum field theory in three Euclidean dimensions to formally include the monopole configurations. The effects can be presented as a local action with some new scalar fields. The connection to the two-potential formulation of Zwanziger [21] is made. Our aim is to obtain a reliable scheme of calculations which integrates renormalised perturbation theory for the ultraviolet behaviour with effects of topological degrees for the infrared behaviour.

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A Appendix

In this Appendix, we represent the Dirac vector potential of a monopole in a more amenable form. The Dirac potential of a monopole located at the origin has the form

\[ \vec{A}(x) = \frac{\phi}{r(1 + \cos \theta)} \hat{n}_3 \times \frac{\hat{r}}{r + x_3} \] (A.1)

with the Dirac string along the negative z-direction. Here \( \hat{n}_3 \) is the unit vector along the \( \hat{z} \) direction. [For checking this and other results below, a useful formula is \( \hat{n}_3 = \cos \theta \hat{r} - \sin \theta \hat{\theta} \).

Let us write

\[ \vec{A}(x) = \hat{n}_3 \times \vec{c} \] (A.2)

where the vector field \( \vec{c} \) is undetermined up to addition of a vector in the 3-direction. We choose

\[ \vec{c} = \frac{\hat{r} + \hat{n}_3}{r + x_3} \] (A.3)

so that \[ \vec{c} = \nabla g, \quad g = \ln(r + x_3) \] (A.4)

Note that \( \partial_3 g = 1/r \), and so \( \partial_3 \nabla^2 g = -4\pi \delta(x) \). Thus the Dirac potential at \( x \) due to a monopole at \( x' \) can be expressed in terms of the Green function for the operator \( \partial_3 \nabla^2 \):

\[ \vec{A}(x - x') = \hat{n}_3 \times \nabla \ln(|x - x'| + x_3 - x'_3) \] (A.5)
\[ = -4\pi \hat{n}_3 \times \nabla (\partial_3 \nabla^2)^{-1}(x - x') \] (A.6)

[An alternative form of the Dirac potential is \( \vec{A} = -\phi(1/r) \cot \theta = -\hat{n}_3 \times \hat{r}(x_3/\rho^2) \), with the Dirac strings along the \( \pm z \) directions. Here \( \rho^2 = x_1^2 + x_2^2 \). In this case, we choose \( \vec{c} = (r\hat{n}_3 - x_3\hat{r})/\rho^2 \) in Eq. (A.2). Then \( \vec{c} = \nabla g \) and \( \partial_3 g = 1/r \) continue to hold, but with \( \rho = (1/2) \ln((r + x_3)/(r - x_3)) \).

The result given in Eq. (A.6) can also be seen by going over to the Fourier space. For the potential of Eq. (A.1),

\[ \nabla \times \vec{A} = \frac{\hat{r}}{r^2} + \hat{n}_3 4\pi \delta(x_1) \delta(x_2) \theta(-x_3) \] (A.7)

Taking the Fourier transform, we get

\[ \vec{k} \times \tilde{\vec{A}}(\vec{k}) = -4\pi \frac{k}{k^2} + \frac{4\pi}{k_3} \hat{n}_3 \] (A.8)

(The Fourier transform of the theta function can be obtained using \( d\theta(x)/dx = \delta(x) \).) We now evaluate \( \vec{k} \times \) both sides and use \( \vec{k} \cdot \vec{A} = 0 \) (since \( \nabla \cdot \vec{A} = 0 \)) to obtain

\[ \tilde{\vec{A}}(\vec{k}) = \frac{4\pi \hat{n}_3 \times \vec{k}}{k_3 k^2} \] (A.9)

This agrees with Eq. (A.6).
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