Multistage Vertex Cover

Till Fluschnik\textsuperscript{\textdagger}, Rolf Niedermeier, Valentin Rohm, and Philipp Zschoche

Algorithmics and Computational Complexity, Faculty IV, TU Berlin, Germany
\{till.fluschnik,rolf.niedermeier,zschoche\}@tu-berlin.de
valentin.l.rohm@campus.tu-berlin.de

Abstract

Covering all edges of a graph by a minimum number of vertices, this is the NP-hard Vertex Cover problem, is among the most fundamental algorithmic tasks. Following a recent trend in studying dynamic and temporal graphs, we initiate the study of Multistage Vertex Cover. Herein, having a series of graphs with same vertex set but over time changing edge sets (known as temporal graph consisting of various layers), the goal is to find for each layer of the temporal graph a small vertex cover and to guarantee that the two vertex cover sets between two subsequent layers differ not too much (specified by a given parameter). We show that, different from classic Vertex Cover and some other dynamic or temporal variants of it, Multistage Vertex Cover is computationally hard even in fairly restricted settings. On the positive side, however, we also spot several fixed-parameter tractability results based on some of the most natural parameterizations.

1 Introduction

Vertex Cover (VC) asks, given an undirected graph $G$ and an integer $k \geq 0$, whether at most $k$ vertices can be deleted from $G$ such that the remaining graph contains no edge. VC is NP-hard and it is a formative problem of algorithmics and combinatorial optimization. We study a time-dependent, “multistage” version, namely a variant of VC on temporal graphs. A temporal graph $G$ is a tuple $(V, \mathcal{E}, \tau)$ consisting of a set $V$ of vertices, a discrete time-horizon $\tau$, and a set of temporal edges $\mathcal{E} \subseteq \binom{V}{2} \times \{1, \ldots, \tau\}$. Equivalently, a temporal graph can be seen as a vector $(G_1, \ldots, G_\tau)$ of static graphs (layers), where each graph is defined over the same vertex set $V$. Then, our specific goal is to find a small vertex cover $S_i$ for each layer $G_i$ such that the sizes of the symmetric differences $S_i \Delta S_{i+1}$ between the vertex covers $S_i$ and $S_{i+1}$ of every two consecutive layers $G_i$ and $G_{i+1}$ are small. Formally, we thus introduce and study the following problem.

Multistage Vertex Cover (MSVC)

**Input:** A temporal graph $G = (V, \mathcal{E}, \tau)$ and two integers $k \in \mathbb{N}, \ell \in \mathbb{N}_0$.

**Question:** Is there a sequence $S = (S_1, \ldots, S_\tau)$ such that
(i) for all $i \in \{1, \ldots, \tau\}$, it holds that $S_i \subseteq V$ is a size-at-most-$k$ vertex cover for $G_i$, and
(ii) for all $i \in \{1, \ldots, \tau - 1\}$, it holds that $|S_i \Delta S_{i+1}| \leq \ell$?

\textsuperscript{\textdagger}Supported by the DFG, project TORE (NI 369/18).
Throughout this paper we assume that $0 < k < |V|$ because otherwise we have a trivial instance. In our model, we follow the recently proposed multistage \cite{4, 5, 14} view on classical optimization problems on temporal graphs.

In general, the motivation behind a multistage variant of a classical problem such as VERTEX COVER is that the environment changes over time (here reflected by the changing edge sets in the temporal graph) and a corresponding adaptation of the current solution comes with a cost. In this spirit, the parameter $\ell$ in the definition of MSVC allows to model that only moderate changes concerning the solution vertex set may be wanted when moving from one layer to the subsequent one. Indeed, in this sense $\ell$ can be interpreted as a parameter measuring the degree of (non-)conservation \cite{1, 15}.

It is immediate that MSVC is NP-hard as it generalizes VERTEX COVER ($\tau = 1$). We will study its parameterized complexity regarding the problem-specific parameters $k, \tau, \ell$, and some of their combinations, as well as restrictions to temporal graph classes \cite{11}.

**Related Work.** The literature on vertex covering is extremely rich, even when focusing on parameterized complexity studies. Indeed, VERTEX COVER (VC) can be seen as “drosophila” of parameterized algorithmics. Thus, we only consider on VC studies closely related to our setting. First, we mention in passing that VC is studied in dynamic graphs \cite{3, 17} and graph stream models \cite{6}. More importantly for us, Akrida et al. \cite{2} studied a variant of VC on temporal graphs. Their model significantly differs from ours: They want an edge to be covered at least once over every time window of some given size $\Delta$. That is, they define a temporal vertex cover as a set $S \subseteq V \times \{1, \ldots, \tau\}$ such that for every time window of size $\Delta$ and for each edge $e = \{v, w\}$ appearing in a layer contained in the time window it holds that $(v, t) \in S$ or $(w, t) \in S$ for some $t$ in the time window with $(e, t) \in \mathcal{E}$. For their model, they ask whether such an $S$ of small cardinality exists. Note that if $\Delta > 1$, then for some $t \in \{1, \ldots, \tau\}$, the set $S_t := \{v \mid (v, t) \in S\}$ is not necessarily a vertex cover of layer $G_t$. For $\Delta = 1$, each $S_t$ must be a vertex cover of $G_t$. However, in Akrida et al.’s model the size of each $S_t$ as well as the size of the symmetric difference between each $S_t$ and $S_{t+1}$ may strongly vary. They provide several hardness results and algorithms (mostly referring to approximation or exact algorithms, but not to parameterized complexity studies).

A second related line of research, not directly referring to temporal graphs though, studies reconfiguration problems which arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible solutions \cite{13, 16}. Mouawad et al. \cite{19, 20} studied, among other reconfiguration problems, VERTEX COVER RECONFIGURATION which takes as input a graph $G$, two vertex covers $S$ and $T$ of size at most $k$ each, and an integer $\tau$. The goal is to determine whether there is a sequence $(S = S_1, \ldots, S_\tau = T)$ such that each $S_i$ is a vertex cover of size at most $k$. The essential difference to our model is that from one “sequence element” to the next only one vertex may be changed and that the input graph does not change over time. Indeed, there is an easy reduction of this model to ours while the opposite direction is unlikely to hold. This is substantiated by the fact that Mouawad et al. \cite{20} showed that VERTEX COVER RECONFIGURATION is fixed-parameter tractable when parameterized by vertex cover size $k$ while we show W[1]-hardness for the corresponding case of MSVC.

Finally, there is also a close relation to the research on dynamic parameterized problems \cite{1, 18}. Krithika et al. \cite{18} studied DYNAMIC VERTEX COVER where one is given two graphs on the same vertex set and a vertex cover for one of them together with the guarantee that the cardinality of the symmetric difference between the two edge sets is upper-bounded by a parameter $d$. The task
Table 1: Overview on our results. The column headings describe the restrictions on the input and each row corresponds to a parameter. p-NP-hard, PK, and NoPK, abbreviate para-NP-hard, polynomial problem kernel, and no problem kernel of polynomial size unless coNP ⊆ NP/poly. (Obs. 2.5)

| Parameter | General Layers | Tree Layers | One-Edge Layers |
|-----------|----------------|-------------|-----------------|
| | $1 \leq \ell < 2k$ | $\ell \geq 2k$ | $1 \leq \ell < 2k$ | $1 \leq \ell < 2$ |

then is to find a vertex cover for the second graph that is “close enough” (measured by a second parameter) to the vertex cover of the first graph. They show fixed-parameter tractability and a linear kernel with respect to $d$.

Our Contributions. Our results, focusing on the three perhaps most natural parameters, are summarized in Table 1. We highlight a few specific results. Multistage Vertex Cover remains NP-hard even if every layer consists of only one edge; clearly, the corresponding hardness reduction then exploits an unbounded number $\tau$ of time layers. If one only has two layers, however, one of them being a tree and the other being a path, then again Multistage Vertex Cover already becomes NP-hard. MSVC parameterized by solution size $k$ is fixed-parameter tractable if $\ell \geq 2k$, but becomes W[1]-hard if $\ell < 2k$. Considering the tractability results for Dynamic Vertex Cover [18] and Vertex Cover Reconfiguration [20], this seems to be surprising and is our most technical result. Furthermore, in the former case (MSVC parameterized by $k$ with $\ell \geq 2k$) does not admit a problem kernel of polynomial size unless coNP ⊆ NP/poly. If one studies the combined parameter $k + \tau$, however, then besides fixed-parameter tractability in all cases we also obtain polynomial-sized kernels.

2 Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the natural numbers excluding and including zero, respectively. We denote by $A \triangle B := (A \setminus B) \cup (B \setminus A)$ the symmetric difference of two sets $A$ and $B$.

Temporal Graphs. A temporal graph $\mathcal{G}$ is a tuple $(V, \mathcal{E}, \tau)$ consisting of the set of vertices $V$, the set of temporal edges $\mathcal{E}$, and a discrete time-horizon $\tau$. A temporal edge $e$ is an element in $\binom{V}{2} \times \{1, \ldots, \tau\}$. Equivalently, a temporal graph $\mathcal{G}$ is a vector of static graphs $(G_1, \ldots, G_\tau)$, where each graph is defined over the same vertex set $V$. We also denote by $V(\mathcal{G})$, $\mathcal{E}(\mathcal{G})$, and $\tau(\mathcal{G})$ the set of vertices, the set of temporal edges, and the discrete time-horizon of $\mathcal{G}$, respectively. The underlying graph $G_1 = G_1(\mathcal{G})$ of a temporal graph $\mathcal{G}$ is the static graph with vertex set $V(\mathcal{G})$ and edge set $\{e \mid \exists t \in \{1, \ldots, \tau(\mathcal{G})\} : (e, t) \in E\}$. 
Parameterized Complexity Theory. Let $\Sigma$ be a finite alphabet. A parameterized problem $L$ is a subset $L \subseteq \{ (x, k) \in \Sigma^* \times \mathbb{N}_0 \}$. An instance $(x, k) \in \Sigma^* \times \mathbb{N}_0$ is a yes-instance of $L$ if and only if $(x, k) \in L$ (otherwise, it is a no-instance). Two instances $(x, k)$ and $(x', k')$ of parameterized problems $L, L'$ are equivalent if $(x, k) \in L \iff (x', k') \in L'$. A parameterized problem $L$ is fixed-parameter tractable (FPT) if for every input $(x, k)$ one can decide whether $(x, k) \in L$ in $f(k) \cdot |x|^{O(1)}$ time, where $f$ is some computable function only depending on $k$. A parameterized problem $L$ is in XP if for every instance $(x, k)$ one can decide whether $(x, k) \in L$ in time $|x|^{f(k)}$ for some computable function $f$ only depending on $k$. A W[1]-hard parameterized problem is fixed-parameter intractable unless FPT–W[1].

Given a parameterized problem $L$, a kernelization is an algorithm that maps any instance $(x, k)$ of $L$ in time polynomial in $|x| + k$ to an instance $(x', k')$ of $L$ (the problem kernel) such that (i) $(x, k) \in L \iff (x', k') \in L$, and (ii) $|x'| + k' \leq f(k)$ for some computable function $f$ (the size of the problem kernel) only depending on $k$.

We refer to Cygan et al. [7] for more material on parameterized complexity.

General Observations on MSVC. We state some simple but useful observations on Multistage Vertex Cover and its relation to Vertex Cover.

Observation 2.1. Every instance $(\mathcal{G}, k, \ell)$ of MSVC with $k \geq \sum_{i=1}^{r(\mathcal{G})} |E(G_i)|$ is a yes-instance.

Proof. It is easy to see that a graph with $m$ edges always admits a vertex cover of size $m$. Hence, there is a vertex cover $S \subseteq V$ of size $k$ of $G_1(\mathcal{G})$, and hence, $S$ is a vertex cover for each layer. The vector $(S_1, \ldots, S_\tau)$ with $S_i = S$ for all $i \in \{1, \ldots, \tau\}$ is a solution for every $\ell \geq 0$. \hfill \Box

Observation 2.2. Let $(\mathcal{G}, k, \ell)$ be an instance of MSVC. If $(\mathcal{G}, k, \ell)$ is a yes-instance, then there is a solution $S = (S_1, \ldots, S_\tau)$ such that $|S_1| = k$ and $k - 1 \leq |S_i| \leq k$ for all $i \in \{1, \ldots, \tau\}$.

Proof. We first show that there is a solution $S = (S_1, \ldots, S_\tau)$ such that $|S_1| = k$. Towards a contradiction assume that such a solution does not exist and that $S = (S_1, \ldots, S_\tau)$ is a solution such that $|S_1|$ is maximized. Let $i \in \{1, \ldots, \tau\}$ be the maximum layer such that $S_j \subseteq S_{j-1}$, for all $j \in \{2, \ldots, i\}$. If $i = \tau$, then we have that $|S_j| \leq |S_1| < k$ for all $j \in \{1, \ldots, \tau\}$. Hence, we can find subset $X$ of $V \setminus S_1$ such that $(S_1 \cup X, \ldots, S_\tau \cup X)$ is a solution. This contradicts $|S_1|$ being maximized. Now let $i < \tau$. Hence, there is a vertex $v \in S_{i+1} \setminus S_i$. Now we can adjust the solution by adding $v$ to $S_j$ for all $j \in \{1, \ldots, i\}$. This contradicts $|S_1|$ being maximized. Hence, there is a solution $S = (S_1, \ldots, S_\tau)$ such that $|S_1| = k$.

Let $\Sigma$ be the set of solutions such that the first vertex cover is of size $k$. Assume towards a contradiction that all solutions in $\Sigma$ contain a vertex cover smaller than $k - 1$. Let $\Sigma_i \subseteq \Sigma$ be the set of solutions such that for $(S_1, \ldots, S_\tau) \in \Sigma_i$ we have that $|S_j| < k - 1$ and $|S_j| \geq k - 1$ for all $j \in \{1, \ldots, i - 1\}$. Let $i \in \{1, \ldots, \tau\}$ be maximized such that $\Sigma_i \neq \emptyset$. Furthermore, let $S = (S_1, \ldots, S_\tau) \in \Sigma_i$ such that $|S_i|$ is maximized. Hence, there is a vertex $v \in S_{i+1} \setminus S_i$.

(a): Assume that there is a $p \in \{i + 1, \ldots, \tau\}$ such that there is a $w \in S_p \setminus S_{p-1}$ and $S_j \subseteq S_{j-1}$ for all $j \in \{i + 1, \ldots, p - 1\}$. The idea now is to keep $v$ and add $w$ in layer $i$ and then remove $v$ in layer $p$. We can achieve that by simply setting $S_q := S_q \cup \{v, w\}$ for all $q \in \{i, \ldots, p - 1\}$. Note that this is a solution which either contradicts that $|S_1|$ is maximized or that $i$ is maximized.

(b): Now assume that $S_j \subseteq S_{j-1}$ for all $j \in \{i + 1, \ldots, \tau\}$. In this case we take an arbitrary vertex $w \in V \setminus S_i$ and set $S_q := \{v, w\}$ for all $q \in \{i, \ldots, \tau\}$. This contradicts $i$ being maximized. \hfill \Box
There is an algorithm that maps any instance \((G, k)\) of VERTEX COVER in \(\tau \cdot |V(G)|^{O(1)}\) time to an equivalent instance \((\mathcal{G}, k, \ell)\) of MSVC with \(\ell = 0\), where \(\mathcal{G}\) is a sequence of any \(\tau\) subgraphs of \(G\) such that the underlying graph is \(G\).

**Proof.** Construct the temporal graph \(\mathcal{G}\) with sequence \(G_1, \ldots, G_\tau\), where \(G_i = (V, E_i)\) with \(E_i \subseteq E(G)\) for each \(i \in \{1, \ldots, \tau\}\) and \(\bigcup_{i=1}^\tau E_i = E\). We claim that \(I = (G, k)\) is a yes-instance of VERTEX COVER if and only if \(I' = (\mathcal{G}, k, \ell)\) with \(\ell = 0\) is a yes-instance of MSVC.

\((\Rightarrow)\) Let \(S \subseteq V(G)\) be a vertex cover of \(G\) of size \(k\). It is not difficult to see that \((S_1, \ldots, S_\tau)\) with \(S_1 := S\) is a solution to \(I'\) since \(G_1\) is a subgraph of \(G\) for each \(i \in \{1, \ldots, \tau\}\).

\((\Leftarrow)\) Let \((S_1, \ldots, S_\tau)\) with \(S_1 = S_j\) be a solution to \(I'\). We claim that \(S := S_1\) is a solution to \(I\). Suppose not, that is, there is an edge \(e \in E(G)\) such that \(S \cap e = \emptyset\). Since \(\bigcup_{i=1}^\tau E_i = E\), there is an \(i \in \{1, \ldots, \tau\}\) such that \(e \in E_i\). Since \(S = S_1 = S_i\), it follows that \(S_i\) is not a vertex cover for \(G_i\), contradicting the fact that \((S_1, \ldots, S_\tau)\) is a solution to \(I'\). Thus, \(S\) is a solution to \(I\).

**Observation 2.4.** There is a polynomial-time algorithm that maps any instance \((\mathcal{G}, k, \ell)\) of MSVC with \(\ell = 0\) to an equivalent instance \((G_1(\mathcal{G}), k)\) of VERTEX COVER.

**Proof.** Let \((\mathcal{G} = (V, \mathcal{E}, \tau, k, 0)\) be an arbitrary instance of MSVC. Construct the instance \((G_1(\mathcal{G}), k)\) of VERTEX COVER. We claim that \((\mathcal{G}, k, 0)\) is a yes-instance if and only if \((G_1, k)\) is a yes-instance.

\((\Leftarrow)\) Let \(S \subseteq V\) be a vertex cover of size at most \(k\). Since \(S\) is a vertex cover for \(G_1\), \(S\) covers each layer of \(\mathcal{G}\). Hence, \(S_i := S\) for all \(i \in \{1, \ldots, \tau\}\) forms a solution to \((\mathcal{G}, k, 0)\).

\((\Rightarrow)\) Let \((S_1, \ldots, S_\tau)\) be a solution to \((\mathcal{G}, k, 0)\). Clearly, since \(\ell = 0\), we have that \(S_i = S_j\) for all \(i, j \in \{1, \ldots, \tau\}\). It is not difficult to see that \(S := S_1\) is a vertex cover for \(G_1\), and hence the claim follows.

**Observation 2.5.** An instance \((\mathcal{G}, k, \ell)\) of MSVC with \(\ell \geq 2k\) and \(\mathcal{G} = (G_1, \ldots, G_\tau)\) can be decided by deciding each instance of the set \(\{(G_i, k) \mid 1 \leq i \leq \tau\}\) of VERTEX COVER-instances.

**Proof.** For each of the \(\tau\) layers \(G_i\), we can construct an instance of VC of the form \((G_i, k)\). Since \(\ell\) is large enough, we can solve each instance independent of the size of their symmetric differences.

### 3 Hardness On Restricted Inputs

MULTISTAGE VERTEX COVER is NP-hard as it generalizes VERTEX COVER \((\tau = 1)\). In this section we prove that MSVC remains NP-hard on inputs with only two layers one consisting of a path and the other consisting of a tree, and on inputs where every layer consists only of one edge.

**Theorem 3.1.** MULTISTAGE VERTEX COVER is NP-hard even if

(i) \(\tau = 2, \ell = 0\), and the first layer is a path and the second layer is a tree, or

(ii) every layer contains only one edge and \(\ell = 1\).

**Remark 3.2.** Theorem 3.1(i) is tight regarding \(\tau\) since VERTEX COVER (i.e. MSVC with \(\tau = 1\)) on trees is solvable in polynomial time. Theorem 3.1(ii) is tight regarding \(\ell\), because in the case of \(\ell \neq 1\) either Observation 2.3 or Observation 2.5 is applicable.

VERTEX COVER remains NP-complete on cubic Hamiltonian graphs when a Hamiltonian cycle is additionally given in the input \([10]\) we refer to this problem as HAMILTONIAN CUBIC VERTEX COVER (HCVC). To prove Theorem 3.1(i), we give a polynomial-time many-one reduction from HCVC to MSVC with two layers, one being a path, the other being a tree.
Proposition 3.3. There is an algorithm that maps any instance \((G = (V, E), k, C)\) of HCVC in polynomial time to an equivalent instance \((G, k', \ell')\) of MSVC with \(\tau = 2\) and the first layer \(G_1\) being a path and second layer \(G_2\) being a tree.

Proof. Let \(e \in E(C)\) be some edge of \(C\), and let \(P = C - \{e\}\) be the Hamiltonian path obtained from \(C\) when removing \(e\). Let \(E_1 = E(P)\), and \(E_2 = E \setminus E(P)\). Set initially \(G_1 = (V, E_1)\) and \(G_2 = (V, E_2)\). Note that \(G_1\) is a path and \(G_2\) is the disjoint union of \(|V|/2 - 2\) paths of length one and one path of length three. Add two special vertices \(z, z'\), and connect \(z\) in \(G_1\) with one endpoint of \(P\) and with \(z'\) in \(G_2\) with exactly one vertex of each connected component and with \(z'\). Set \(k' = k + 1\) and \(\ell' = 0\). We claim that \((G = (V, E), k, C)\) is a yes-instance if and only if \((G, k', \ell')\) is a yes-instance.

\((\Rightarrow)\) Let \(S'\) be a vertex cover of \(G\) of size at most \(k\). We claim that \(S' := S \cup \{z\}\) is a vertex cover for each \(G_1\) and \(G_2\). Observe that \(G_1[E_1] \text{ and } G_2[E_2]\) are subgraphs of \(G\), and hence all edges are covered by \(S'\). Moreover, all the edges in \(G_1 - E_i, i \in \{1, 2\}\), are incident with \(z\) and hence covered by \(S'\).

\((\Leftarrow)\) Let \((S_1, S_2)\) be a minimal solution to \((G, k', \ell')\). Assume that \(z \in S',\) since the edge \(\{z, z'\}\) is present in both \(G_1\) and \(G_2\), and exchanging \(z\) in \(z'\) does not cover less edges. Moreover, we can assume that not both \(z\) and \(z'\) are in \(S'\) due to the minimality of \(S'\). Let \(S := S' \setminus \{z\}\). Observe that \(S\) covers all edges in \(E_1 \cup E_2\) and hence, \(S\) forms a vertex cover of \(G\) of size at most \(k' = k' - 1\).

In order to prove Theorem 3.1(ii), we give a polynomial-time many-one reduction from VERTEX COVER to MSVC.

Proposition 3.4. There is an algorithm that maps any instance \((G = (V, E), k)\) of VERTEX COVER in polynomial time to an equivalent instance \((G, k', \ell')\) of MSVC where \(\ell' = 1\) and every layer \(G_i\) contains only one edge.

Proof. Let the edges of \(G\) be enumerated \(E = \{e_1, \ldots, e_m\}\). Set \(\tau = m\). Set \(V' = V \cup W\), where \(W = \{w_1, \ldots, w_{2\tau}\}\). Set \(G_{2i-1} = (V', \{e_i\})\) and \(G_{2i} = (V', \{w_i, w_{i+\tau}\})\) for each \(i \in \{1, \ldots, \tau\}\). Set \(k' = k + 1\) and \(\ell' = 1\). We claim that \((G = (V, E), k)\) is a yes-instance of VERTEX COVER if and only if \((G, k', \ell')\) is a yes-instance of MSVC.

\((\Rightarrow)\) Let \(S\) be a vertex cover of \(G\) of size at most \(k\). Set \(S_{2i-1} := S\), and \(S_{2i} := S \cup \{w_i\}\) for all \(i \in \{1, \ldots, \tau\}\). Clearly, \(S_i\) is a vertex cover of \(G_i\) for all \(i \in \{1, \ldots, 2\tau\}\) of size at most \(k' = k + 1\). Moreover, by construction \(|S_i \setminus S_{i+\tau}| \leq 1\) for all \(i \in \{1, \ldots, 2\tau - 1\}\). Hence, \((S_1, \ldots, S_{2\tau})\) forms a solution to \((G, k', \ell')\).

\((\Leftarrow)\) Let \(S = (S_1, \ldots, S_{2\tau})\) be a solution to \((G, k', \ell')\). Observe that \(|\bigcup_i S_i| \leq k + \tau\). We know that \(|W \cap \bigcup_i S_i| \geq \tau\). It follows that there are at most \(k\) vertices covering all edges in the layers \(G_{2i-1}\), that is \(G\). \(\square\)

Theorem 3.1 now follows from Propositions 3.3 and 3.4.

4 Parameter Vertex Cover Size

In this section, we study the parameter size \(k\) of the vertex cover of each layer for MSVC. VERTEX COVER and VERTEX COVER RECONFIGURATION [20] when parameterized by the vertex cover size are fixed-parameter tractable. We prove that this is no longer true for MSVC (unless \(\text{FPT} = \text{W}[1]\)).
Theorem 4.1. \textsc{Multistage Vertex Cover} parameterized by $k$ is in XP and $W[1]$-hard.

We first show the XP-algorithm (Section 4.2) and then prove $W[1]$-hardness (Section 4.8).

4.2 An XP-Algorithm

In this section, we prove the following.

Proposition 4.3. Every instance $(\mathcal{G}, k, \ell)$ of \textsc{Multistage Vertex Cover} can be decided in $O(\tau(\mathcal{G}) \cdot |V(\mathcal{G})|^{2k+1})$ time.

In a nutshell, we first consider for each layer all subsets of vertices of size at most $k$ that form a vertex cover. Second, we find a sequence of vertex covers for all layers such that the sizes of the symmetric differences for every two consecutive solutions is at most $\ell$. We show that the second step can be solved via computing a directed source-sink path in a helper graph that we call configuration graph.

Definition 4.4. Given a temporal graph $\mathcal{G}$, the $(k, \ell)$-configuration graph of $\mathcal{G}$ is the graph $D = (V = V_1 \cup \cdots \cup V_\tau \cup \{s, t\}, A, \gamma)$ equipped with a function $\gamma : V \rightarrow \{V' \subseteq V(\mathcal{G}) \mid |V'| \leq k\}$ such that

(i) for every $i \in \{1, \ldots, \tau(\mathcal{G})\}$, it holds true that $\gamma$ is a vertex cover of $G_i$ of size at most $k$ if and only if there is a vertex $v \in V_i$ with $\gamma(v) = S_i$,
(ii) there is an arc from $v$ to $w$, $v, w \in V$, if and only if $v \in V_i$, $w \in V_{i+1}$, and $|\gamma(v) \Delta \gamma(w)| \leq \ell$, and
(iii) there is an arc $(s, v)$ for all $v \in V_1$ and an arc $(v, t)$ for all $v \in V_\tau$.

Note, that a similar idea leads to fixed-parameter tractability of \textsc{Vertex Cover Reconfiguration} [20]. However, in the multistage setting we show an XP-algorithm regarding $k$ to construct the configuration graph.

Lemma 4.5. The $(k, \ell)$-configuration graph of a temporal graph $\mathcal{G}$ with $n$ vertices and time horizon $\tau$ (i) can be constructed in $O(\tau \cdot n^{2k+1})$ time, and (ii) contains at most $\tau \cdot 2n^k + 2$ vertices and $(\tau - 1)n^{2k} + 2n^k$ arcs.

Proof. Compute the set $S = \{V' \subseteq V(\mathcal{G}) \mid k - 1 \leq |V'| \leq k\}$ in $O(n^k)$ time. For each layer $G_i$ and each set $S \in \mathcal{S}$, check in $O(|E(G_i)|)$ time whether $S$ is a vertex cover for $G_i$. Let $S_i \subseteq \mathcal{S}$ denote the set of vertex covers of size $k - 1$ or $k$ of layer $G_i$. For each $S \in S_i$, add a vertex $v$ to $V_i$ and set $\gamma(v) = S$. Hence, we can construct the vertex set $V$ of size $\tau \cdot 2n^k + 2$ from $D$ in $O(n^{k+2} \cdot \tau)$ time. For every $i \in \{1, \ldots, \tau - 1\}$, $v \in V_i$ and $w \in V_{i+1}$ check whether $|\gamma(v) \Delta \gamma(w)| \leq \ell$ in $O(k)$ time, if so, add the arc $(v, w)$. The latter step can be done in $O(n^{2k+1} \cdot (\tau - 1))$ time. Finally, add the vertices $s$ and $t$, and the arc $(s, v)$ for each $v \in V_1$ and the arc $(v, t)$ for each $v \in V_\tau$ in $O(n^k)$ time. The finishes the construction of $D$. \hfill $\Box$

Lemma 4.6. MSVC-instance $(\mathcal{G}, k, \ell)$ is a yes-instance if and only if there is an s-t path in the $(k, \ell)$-configuration graph $D$ of $\mathcal{G}$.

Proof. $(\Rightarrow)$ Let $(S_1, \ldots, S_\tau)$ be a solution to $(\mathcal{G}, k, \ell)$. By Observation 2.2, we can assume without loss of generality that $k - 1 \leq |S_i| \leq k$, for all $i \in \{1, \ldots, \tau\}$. Hence for each $S_i$, there is a $v_i \in V_i$ such that $\gamma(v_i) = S_i$, for all $i \in \{1, \ldots, \tau\}$. Note that the arc $(v_i, v_{i+1})$ is contained in $A$ for
each \( i \in \{1, \ldots, \tau - 1\} \) since \( |\gamma(v_i) \triangle \gamma(v_{i+1})| = |S_i \triangle S_{i+1}| \leq \ell \). Hence, \( P = \{v_1, \ldots, v_{\tau}\} \cup \{s, t\}, (s, v_1) \cup (v_\tau, t) \cup \bigcup_{i=1}^{\tau-1} (v_i, v_{i+1}) \) is an \( s \)-\( t \) path in \( D \).

(\Rightarrow) Let \( P = \{v_1, \ldots, v_{\tau}\} \cup \{s, t\}, (s, v_1) \cup (v_\tau, t) \cup \bigcup_{i=1}^{\tau-1} (v_i, v_{i+1}) \) be an \( s \)-\( t \) path in \( D \). We claim that \( (\gamma(v_i))_{i=1}^{\tau} \) forms a solution to \((\mathcal{G}, k, \ell)\). First, note that for all \( i \in \{1, \ldots, \tau\} \), \( \gamma(v_i) \) is a vertex cover for \( G_i \) of size at most \( k \). Moreover, for all \( i \in \{1, \ldots, \tau - 1\} \), \( |\gamma(v_i) \triangle \gamma(v_{i+1})| \leq \ell \) since the arc \((v_i, v_{i+1})\) is present in \( D \). This finishes the proof. \( \square \)

We are ready to prove Proposition 4.3.

**Proof of Proposition 4.3.** First, compute the configuration graph \( D \) of the instance \((\mathcal{G} = (V, E, \tau), k, \ell)\) of **Multistage Vertex Cover** in \( O(\tau \cdot |V|^{2k+1}) \) time (Lemma 4.5(i)). Then, find an \( s \)-\( t \) path in \( D \) with a breath-first search in \( O(\tau \cdot |V|^{2k}) \) time (Lemma 4.5(ii)). If an \( s \)-\( t \) path is found, then return **yes**, otherwise return **no** (Lemma 4.6). \( \square \)

**Remark 4.7.** The reason why the algorithm behind Proposition 4.3 is an XP-algorithm and not an FPT-algorithm regarding \( k \) for **Multistage Vertex Cover** is because we do not have a better upper bound on the number of vertices in the \((k, \ell)\)-configuration graph for \( \mathcal{G} \) than \( O(\tau(\mathcal{G}) \cdot |V(\mathcal{G})|^{k}) \). This is due to the fact that we check for each subset of \( V(\mathcal{G}) \) of size \( k \) or \( k+1 \) whether it is a vertex cover in some layer.

This changes if we consider **Minimal Multistage Vertex Cover** where we additionally demand the \( i \)-th set in the solution to be a minimal vertex cover for the layer \( G_i \). Here, we can enumerate for each layer \( G_i \) all minimal vertex covers of size at most \( k \) (and hence all candidates for the \( i \)-th set of the solution) with the folklore search-tree algorithm for vertex cover (see e.g. [8]). This leads to \( O(2^k \tau(\mathcal{G})) \) many vertices in the \((k, \ell)\)-configuration graph (for **Minimal Multistage Vertex Cover**) and thus to fixed-parameter tractability of **Minimal Multistage Vertex Cover** parameterized by the vertex cover size \( k \).

However, it is not likely (unless \( \text{FPT} = \text{W}[1] \)) that one can improve substantially the algorithm behind Proposition 4.3 as we show next.

## 4.8 Fixed-parameter Intractability

In this section we show that **MSVC** is \( \text{W}[1] \)-hard when parameterized by \( k \). This hardness result is established by the following parameterized reduction from the \( \text{W}[1] \)-complete [8] **CLIQUE** problem, where, given an undirected graph \( G \) and a positive integer \( k \), the question is whether \( G \) contains a clique of size \( k \) (that is, \( k \) vertices that are pairwise adjacent).

**Proposition 4.9.** There is an algorithm that maps any instance \((G, k)\) of **CLIQUE** in polynomial time to an equivalent instance \((\mathcal{G}, k', \ell)\) of **MSVC** with \( k'=2\binom{k}{2}+k+1 \), \( \ell=2 \), and each layer of \( \mathcal{G} \) being a tree.

The proof of Proposition 4.9 is deferred to the end of this section. We construct an instance of **Multistage Vertex Cover** from an instance of **CLIQUE** as follows (see Figure 1 for an illustrative example).

**Construction 4.10.** Let \((G = (V, E), k)\) be an instance of **CLIQUE** with \( m = |E| \) and \( E = \{e_1, \ldots, e_m\} \). Let

\[
K = \binom{k}{2}, \quad k' = 2K + k + 1, \quad \text{and} \quad \kappa = K + k + 3.
\]
We construct a temporal graph $G = (V', E, \tau)$ as follows. Let $V'$ be initially $V \cup E$ (note that $E$ simultaneously describes the edge set of $G$ and a vertex subset of $G$). We add the following vertex sets

$$U^t = \{u^i_j \mid j \in \{1, \ldots, K\}, t \in \{1, \ldots, \kappa + 1\} \text{ and } C = \{c_1, \ldots, c_{2m+1}\}.$$

Let $E$ be initially empty. We extend the set $V'$ and define $E$ through the $\tau = 2mK + 1$ layers we construct in the following.

1. In each layer $G_i$ with $i$ being odd, make $c_i$ the center of a star with $k^i + 1$ leaves.
2. In each layer $G_{2mj + 1}$, $j \in \{0, \ldots, \kappa\}$, make each vertex in $U^{j+1}$ the center of a star with $k^i + 1$ leaves.
3. For each $j \in \{0, \ldots, \kappa - 1\}$, in each layer $G_{2mj + i}$ with $i \in \{1, \ldots, 2m + 1\}$, make $u^{j+1}_x$ adjacent to $u^{j+2}_x$ for each $x \in \{1, \ldots, K\}$. 
4. For each $i$ being even, add the edge $\{e_i, c_{i+1}\}$ to $G_i$ and to $G_{i+1}$.
5. For each $j \in \{0, \ldots, \kappa - 1\}$, for each $i \in \{1, \ldots, m\}$, in $G_{2mj + 2i}$, make $c_{j2m+2i}$ adjacent with $e_i = \{v, w\}$, $v$, and $w$.

This finishes the construction of $G$.

The construction essentially repeats the same gadget (which we call phase) $\kappa$ times where the layer $2m \cdot i + 1$ is simultaneously last layer of phase $i$ and the first layer of phase $i + 1$. In the beginning of phase $i$, a solution must contain the vertices of $U^i$. The idea now is that during phase $i$ one has to exchange the vertices of $U^i$ with the vertices of $U^{i+1}$.

It is not difficult to see that the instance in Construction 4.10 can be computed in polynomial time. Hence, it remains to prove the equivalence stated in Proposition 4.9. We prepare the proofs of the forward and backward direction in the following Sections 4.11.1 and 4.12.1, respectively.

Remark 4.11. We can turn the instance $G, k^i, \ell$ computed by Construction 4.10 into an equivalent instance $(G', k'^i, \ell)$ where each layer is a tree as follows. Set $k'^i = k^i + 1$. Add a vertex $x$ to $G$. In each layer of $G$, make $x$ a star with $k'^i + 1$ vertices and connect $x$ with exactly one vertex of each connected component. Note that in every solution, $x$ is contained in a vertex cover for each layer in $G'$.
4.11.1 Forward direction

The forward direction of Proposition 4.9 is—in a nutshell—as follows: If \( V' \cup E' \) with \( V' \subseteq V \) and \( E' \subseteq E \) correspond to the vertex set and edge set of a clique of size \( k \), then there are \( K \) layers in each phase covered by \( V' \cup E' \). Hence, having \( K \) layers where no vertices from \( C \) have to be exchanged, in each phase \( t \) we can exchange all vertices from \( U^t \) to \( U^{t+1} \). Starting with set \( S_1 = U^1 \cup V' \cup E' \cup \{c_1\} \) then yields a solution.

Lemma 4.12. Let \((G, k)\) be an instance of CLIQUE and \((\mathcal{G}, k^l, \ell)\) be the instance of MULTISTAGE VERTEX COVER resulting from Construction 4.10. If \((G, k)\) is a yes-instance, then \((\mathcal{G}, k^l, \ell)\) is a yes-instance.

Proof. Let \( G_K = (V', E') \) be the clique of size \( k \) in \( G \). We construct a solution \( S = (S^1_1, \ldots, S^{k}_{2m}, S^1_{2m+1}, S^{k}_{2m+1} = S^1_{k+1}) \) for \((G, k^l, \ell)\) in the following way. For each \( t \in \{1, \ldots, \kappa + 1\} \) we set \( S^1_t = V' \cup E' \cup U^t \cup \{c_1(t-2m+1)\} \) which is a vertex cover of size \( k^l \) for \( G(t-1)_{2m+1} \).

Now, for each \( t \in \{1, \ldots, \kappa\} \), we iteratively construct vertex covers for the layers \((t-1)2m + 2 \) until \( 2m \) in the following way. Let \( T := (t-1) \cdot 2m \). Let \( i \in \{1, \ldots, 2m - 1\} \), and assume the set \( S_i^t \) is already constructed and is a vertex cover for \( G_{T+i} \) (this is possible due to the definition of \( S_i^t \))

Case: \( i \) is odd. We know that \( c_{T+i} \in S_i^t \). If \( (S^t_i \setminus \{c_{T+i}\}) \cup \{c_{T+i+2}\} \) is a vertex cover for \( G_{T+i+1} \), then we set \( S^t_{i+1} := (S^t_i \setminus \{c_{T+i}\}) \cup \{c_{T+i+2}\} \). Otherwise we set \( S^t_{i+1} := (S^t_i \setminus \{c_{T+i}\}) \cup \{c_{T+i+1}\} \). In both cases \( S^t_{i+1} \) is a vertex cover for \( G_{T+i+1} \) and either \( S^t_{i+1} \cap C = \{c_{T+i+1}\} \) or \( S^t_{i+1} \cap C = \{c_{T+i+2}\} \).

Case: \( i \) is even. We know that \( c_{T+i} \) or \( c_{T+i+1} \) is in \( S_i^t \). If \( c_{T+i} \in S_i^t \), then we set \( S^t_{i+1} := (S^t_i \setminus \{c_{T+i}\}) \cup \{c_{T+i+1}\} \) which is a vertex cover for \( G_{T+i+1} \). If \( c_{T+i+1} \in S_i^t \), then \( S_i^t \) is already a vertex cover for \( G_{T+i+1} \) and vertices in \( V' \cup E' \) cover all the edges incident with \( c_{T+i} \) in the graph \( G_{T+i} \). In this case we say that \( G_K \) covers the layer \( T + i \) and set \( S^t_{i+1} := (S^t_i \setminus \{u_{j}\}) \cup \{u_{j+1}\} \), where \( u_j \) is an arbitrary vertex in \( S_i^t \cap U^t \).

Observe that the clique \( G_K \) covers in each phase \( K \) even layers. Hence, we replace, during the phase \( t \in \{1, \ldots, \kappa\} \) (that is from layer \((t-1)2m + 1 \) to \( 2m + 1 \)), the vertices \( U^t \) with the vertices \( U^{t+1} \). This also implies that the symmetric difference of two consecutive sets in \( S \) is exactly \( 2 = \ell \). It follows that \( S \) is a solution for \((\mathcal{G}, k^l, \ell)\). \( \square \)

4.12.1 Backward direction

In this section we prepare the proof of the backward direction for the proof of Proposition 4.9. We first show that if an instance of MULTISTAGE VERTEX COVER computed by Construction 4.10 is a yes-instance, then it is safe to assume that neither two vertices are deleted from nor added to a vertex cover in a consecutive step (we refer to these solutions as smooth, see Definition 4.14). Moreover, a vertex from \( C \) is only exchanged with another vertex from \( C \) and, at any time, there is exactly one vertex from \( C \) contained in the solution (similarly to the constructed solution in Lemma 4.12). We call these solutions one-centered (Definition 4.16). We then prove that there must be a phase \( t \) for any one-centered solution that is deleting at least \( k \) times a vertex from “past” sets \( U_t \), \( t \leq t \). This at hand, we prove that such a phase witnesses a clique of size \( k \).

That a solution needs to contain at least one vertex from \( C \) at any time follows immediately from the fact that there is either an edge between two vertices in \( C \) or there is a vertex in \( C \) which is the center of a star with \( k^l + 1 \) leaves.

Observation 4.13. Let \((\mathcal{G}, k^l, \ell)\) from Construction 4.10 be a yes-instance. Then for each solution \((S_1, \ldots, S_\tau)\) it holds true that \( |S_i \cap C| \geq 1 \) for all \( i \in \{1, \ldots, \tau(\mathcal{G})\} \).
In the remainder of this section we denote which vertices are exchanged from set \( S_{i-1} \) to next set \( S_i \) in a solution \( S \) by
\[
S_{i-1} \otimes S_i := (S_{i-1} \setminus S_i, S_i \setminus S_{i-1}).
\]
In case of \( S_{i-1} \setminus S_i \) or \( S_i \setminus S_{i-1} \) being of size at most one we will omit the set braces.

**Definition 4.14.** A solution \( S = (S_1, \ldots, S_\tau) \) for \((G, k', \ell)\) from Construction 4.10 is smooth if for all \( i \in \{2, \ldots, \tau\} \) we have \( S_{i-1} \otimes S_i = (a, b) \), that is, \(|S_{i-1} \setminus S_i| \leq 1\) and \(|S_{i-1} \setminus S_i| \leq 1\).

**Observation 4.15.** Let \((G, k', \ell)\) from Construction 4.10 be a yes-instance. Then there is a smooth solution \((S_1, \ldots, S_\tau)\).

**Proof.** By Observation 2.1, we know that there is a solution \( S = (S_1, \ldots, S_\tau) \) such that \(|S_i| = k'\) and \(k' - 1 \leq |S_i| \leq k'\) for all \( i \in \{1, \ldots, \tau\} \). Hence, for all \( i \in \{2, \ldots, \tau\} \) it holds true that \(|S_i| - |S_{i-1}|| \leq 1\). It follows that \(|S_{i-1} \setminus S_i| \leq 1\) and \(|S_{i-1} \setminus S_i| \leq 1\), and thus, \( S \) is a smooth solution. \( \square \)

Our goal is to prove the existence of the following type of solutions.

**Definition 4.16.** A smooth solution \( S = (S_1, \ldots, S_\tau) \) for \((G, k', \ell)\) from Construction 4.10 is one-centered if (i) for all \( i \in \{1, \ldots, \tau\} \) we have \(|S_i \cap C| = 1\), and (ii) for all \( i \in \{2, \ldots, \tau\} \) and \( S_{i-1} \otimes S_i = (a, b) \) we have that \( a \in C \iff b \in C \).

We now show that there are solutions where \( c_1 \) is the only vertex from \( C \) in the first set of the solution.

**Lemma 4.17.** Let \((G, k', \ell)\) from Construction 4.10 be a yes-instance. Then there is a smooth solution \((S_1, \ldots, S_\tau)\) such that \( S_1 \cap C = \{c_1\} \).

**Proof.** Suppose not, that is, due to Observation 4.13 in every smooth solution, the first vertex cover \( S_1 \) contains at least two vertices from \( C \). Let \( \Sigma \) be the set of smooth solutions with \(|S_1 \cap C|\) being minimal, where \( S_1 \) is the first vertex cover. Let \( S = (S_1, \ldots, S_\tau) \in \Sigma \) be a smooth solution such that the smallest \( c_i \in S_1 \setminus \{c_1\} \) is maximal regarding \( i \). Let \( S' = (S_1', \ldots, S_\tau') \) be initially \( S \).

Suppose there is \( j \in \{1, \ldots, i - 1\} \) such that \( S_j \otimes S_{j+1} = (c_i, a) \), and let \( j' \) be the smallest among them. Then, set \( S_q' := S_q \setminus \{c_i\} \) for all \( q \in \{1, \ldots, j'\} \) to get a feasible solution contradicting the minimality of \( S \) regarding \(|S_1 \cap C|\). Note that \( S_{j' - 1} \otimes S_{j'} = (\emptyset, a) \), being feasible since \(|S_{j' - 1}| \leq k - 1\).

Hence, suppose there is no such \( j \). If \( S_1 \setminus \{c_1\} \) is a vertex cover of layer \( G_i \), then setting \( S_q' := S_q \setminus \{c_i\} \), for all \( q \in \{1, \ldots, p\} \) with \( c_i \in S_q \) for all \( q \in \{1, \ldots, p\} \) and \( p \) being maximal, yields a feasible solution contradicting the minimality of \( S \) regarding \(|S_1 \cap C|\).

Hence, suppose there is no such \( j \) and \( c_i \in S_i \). Let \( S_{i-1} \otimes S_i = (a, b) \) for some \( a, b \) (each being possibly the empty set). Then for all \( q \in \{1, \ldots, i - 1\} \):

If \( b = c_r \) with \( r < i \): Set \( S_q' := S_q \setminus \{c_i\} \) and \( S_p' := (S_p \setminus \{b\}) \) (i.e. \( S_{i-1}' \otimes S_i' = (a, c_i) \)) for all \( i \leq p \leq p' \) with \( b \in S_p \) for all \( i \leq p \leq p' \) and \( p' \) maximal.

Otherwise: Set \( S_q' := (S_q \setminus \{c_i\}) \cup \{b\} \) (note that \( S_i' = S_i' \) and hence \( S_{i-1}' \otimes S_i' = (a, c_i) \)). Note that if there is \( 1 < p < i - 1 \) with \( S_{p-1}' \otimes S_p = (b, x) \) or \( S_{p-1}' \otimes S_p = (a, x) \), we get \( S_{p-1}' \otimes S_p = (\emptyset, x) \) and \( S_{p-1}' \otimes S_p = (x, \emptyset) \), respectively.

If \( b = c_r \) with \( r < i \) (in this case, \( b \) is not contained in \( S_1' \)), or \( b \notin C \), this is contradicting the minimality of \( S \) regarding \(|S_1 \cap C|\). If \( b = c_r \) with \( r > i \), this is contradicting \( c_i \) to be maximal regarding \( i \). \( \square \)
Next, we show that there are solutions such that whenever we remove a vertex in $C$ from the vertex cover then we simultaneously add another vertex from $C$ to the vertex cover. Formally, we prove the following.

**Lemma 4.18.** Let $(G, k^l, \ell)$ from **Construction 4.10** be a yes-instance. Then there is a smooth solution $(S_1, \ldots, S_r)$ such that $S_1 \cap C = \{c_1\}$ and for all $i$ with $S_{i-1} \otimes S_i = (a, c)$ and $c \in C$ holds $a \in C$.

**Proof.** Suppose not, that is, for every smooth solution $(S_1, \ldots, S_r)$ there is an $i$ with $S_{i-1} \otimes S_i = (a, c)$ and $c \in C$ and $a \notin C$. Let $\Sigma$ be the non-empty (due to **Lemma 4.17**) set of smooth solutions $(S_1, \ldots, S_r)$ with $|S_1 \cap C| = 1$. Let $\Sigma' \subset \Sigma$ be the set of smooth solutions that maximizes $i$ with $S_{i-1} \otimes S_i = (a, c_q)$ with $c_q \in C$ and $a \notin C$. Among those solutions, consider $S = (S_1, \ldots, S_r) \in \Sigma'$ to be the one with $q$ being maximal. Note that due to **Observation 4.13**, $|S_{i-1} \cap C| \geq 1$. Let $S_i' := S_i$ for all $i \in \{1, \ldots, r\}$.

**Case:** $i > 1$ is odd. Since $c_i$ is the center of a star in layer $i$, $c_i$ has to be in $S_i$.

**Subcase:** $q < i$. Set $S_i' := (S_i \setminus \{c_i\})$ (i.e. $S_i' \otimes S_i' = (a, \emptyset)$) for all $i \leq j \leq q'$ with $c_q \in S_j$ for all $i \leq j \leq q'$ and $q'$ being maximal. It follows that $(S_1', \ldots, S_r')$ is again a feasible smooth solution contradicting $i$ being maximal.

**Subcase:** $q = i$. Then $c_{i-1} \in S_{i-1}$ since the edge $\{c_{i-1}, c_i\}$ has to be covered. Set $S_p' := (S_p \setminus \{c_i\}) \cup \{a\}$ (i.e. $S'_i \otimes S_i' = (c_{i-1}, c_q)$) for all $p \in \{i, \ldots, j\}$, where $j > i$ is minimal such that $S_j \otimes S_{j+1} = (c_{j-1}, x)$, or $\tau$ if such a $c_j$ does not exist. If there is a minimal $j > i$ such that $S_j \otimes S_{j+1} = (c_{j-1}, x)$, set $S_p' := (S_p \setminus \{a\})$ (i.e. $S'_j \otimes S_j' = (a, x)$) for all $j \leq p \leq q$ with $a \in S_p$ for all $j \leq p \leq q$ and $q$ being maximal. Suppose that in between $i$ and $j$, there are $j_1$ and $j_2$ such that $S_{j_1-1} \otimes S_{j_1} = (y, a)$ and $S_{j_2-1} \otimes S_{j_2} = (a, y')$. Note that $S_{j_1-1}' \otimes S_{j_1}' = (y, \emptyset)$ and $S_{j_1-1}' \otimes S_{j_1}' = (\emptyset, y')$. It follows that $(S_1', \ldots, S_r')$ is again a feasible smooth solution contradicting $i$ being maximal.

**Subcase:** $q > i$. Then $c_i \in S_{i-1}$. Let $S_{q-1} \otimes S_q = (b, d)$.

Suppose that $d = c_p$ with $p < q$, then set $S_j' := S_j \setminus \{c_q\}$ (i.e. $S_j' \otimes S_j' = (a, \emptyset)$) for all $i \leq j < q$, and $S_q = (S_q \setminus \{c_q\}) \cup \{c_q\}$ (i.e. $S_q' \otimes S_q' = (b, c_q)$) and $S_j' = (S_j \setminus \{d\}) \cup \{c_q\}$ for all $q < j \leq q'$ with $d \in S_j$ for all $q \leq j \leq q'$ and $q'$ being maximal.

Suppose that $d \notin C$ or if $d = c_p$, then $p > q$. Then set $S_j' := (S_j \setminus \{c_q\}) \cup \{d\}$ (i.e. $S_{q-1} \otimes S_q' = (a, d)$) for all $i \leq j < q$ and $S_j' := S_j \cup \{c_q\}$ (i.e. $S_{q-1} \otimes S_q' = (b, c_q)$) or $S_q' \otimes S_q' = (b, \emptyset)$) for all $q \leq j \leq q'$ with $c_q \in S_j$ for all $q \leq j \leq q'$ and $q'$ being maximal.

Then $(S_1', \ldots, S_r')$ is a feasible solution contradicting either $i$ being maximal ($d \notin C$, or $d = c_p$ with $p < q$) or $q$ being maximal ($d = c_p$ with $p > q$).

**Case:** $i > 1$ is even. Then $c_{i-1} \in S_{i-1}$ and $c_q \in \{c_i, c_{i+1}\}$. Set $S_j' := (S_j \setminus \{c_{i-1}\}) \cup \{a\}$ (i.e. $S_{i-1} \otimes S_i' = (c_{i-1}, c_q)$) for all $i \leq j \leq q'$ with $c_{i-1} \in S_j$ for all $i \leq j \leq q'$ and $q'$ being maximal. Then $(S_1', \ldots, S_r')$ is a feasible solution contradicting $i$ being maximal. □

Combining **Observation 4.13** and **Lemma 4.18**, we can assume that given a yes-instance, there is a solution which is one-centered.

**Corollary 4.19.** Let $(G, k^l, \ell)$ from **Construction 4.10** be a yes-instance. Then, there is a solution $S$ which is one-centered.
In the remainder of this section, for each \( t \in \{1, \ldots, \kappa + 1\} \) let the union of all \( U^t \) be denoted by
\[
\bar{U}_t = \bigcup_{i=1}^{t} U^i.
\]

We introduce further notation regarding a one-centered solution \( S := (S_1^1, \ldots, S_{2m+1}^1, S_2^2, \ldots, S_{2m+1}^\kappa, \ldots, S_1^\kappa, S_{2m+1}^\kappa) \) for \((G, k^1, \ell)\). Here, \( S_i^t \) is the \( i \)-th set of the phase \( t \) and thus the \((2m(t-1)+i)\)-th set of \( S \). The set
\[
Y_i^1 := \{ e_j \in S_i^t \cap E \mid 2j \geq i \}
\]
is the set of vertices \( e_j \) from \( E \) in \( S_i^t \) such that the corresponding layer for \( e_j \) in phase \( t \) is not before the layer \( i \) in phase \( t \). The set
\[
F_i^t := \{ j > i \mid S_{j-1}^t \ominus S_j^t = (u, b) \text{ with } u \in \bar{U}_t \}
\]
is the set of layers from \( G \) in phase \( t \) where a vertex from \( \bar{U}_t \) is not carried over to the next layer’s vertex cover. We now show that there is a phase \( t \) where \( |F_i^t| \geq K \).

**Lemma 4.20.** Let \( S = (S_1^1, \ldots, S_{2m+1}^1, S_2^2, \ldots, S_{2m+1}^\kappa, \ldots, S_1^\kappa, S_{2m+1}^\kappa) \) be a one-centered solution to \((G, k^1, \ell)\) from Construction 4.10 being a yes-instance. Then, there is a \( t \in \{1, \ldots, \kappa\} \) such that \( |F_i^t| \geq K \).

**Proof.** Suppose not, that is, for all \( t \in \{1, \ldots, \kappa\} \) it holds true that \( |F_i^t| < K \). Then, for each \( i \in \{2, \ldots, \kappa+1\} \), we have that \( |S_i^t \cap \bar{U}_{i-1}| \geq i - 1 \). Since \( S \) is a solution, we know that \( U^{\kappa+1} \subseteq S_1^{\kappa+1} \) and hence \( |S_1^{\kappa+1} \cap U^{\kappa+1}| = K \). Thus, we have that
\[
|S_1^{\kappa+1}| \geq |S_1^{\kappa+1} \cap U^{\kappa+1}| + |S_1^{\kappa+1} \cap \bar{U}_\kappa| \geq K + \kappa - 1 = 2K + k + 2 > k^1.
\]
contradicting \( S \) being a solution.

In the remainder of this section the value
\[
f_i^t := |S_i^t \cap \bar{U}_{\kappa+1}| - K
\]
describes the number of vertices in \( \bar{U}_{\kappa+1} \) which we could remove from \( S_i^t \) such that \( S_i^t \) is still a vertex cover for \( G_{2m(t-1)+i} \) (the \( i \)-th layer of phase \( t \)). Observe, that \( f_i^t \geq 0 \) for all \( i \in \{1, \ldots, 2m + 1\} \) and \( t \in \{1, \ldots, \kappa\} \), because we need in each layer exactly \( K \) vertices from \( \bar{U}_{\kappa+1} \) in the vertex cover.

We now derive an invariant which must be true in each phase.

**Lemma 4.21.** Let \( S = (S_1^1, \ldots, S_{2m+1}^1, S_2^2, \ldots, S_{2m+1}^\kappa, \ldots, S_1^\kappa, S_{2m+1}^\kappa) \) be a one-centered solution to \((G, k^1, \ell)\) from Construction 4.10 being a yes-instance. Then, for all \( t \in \{1, \ldots, \kappa\} \) and \( i \in \{1, \ldots, 2m + 1\} \), it holds true that \( |F_i^t| - |Y_i^t| \leq f_i^t \).

**Proof.** Define \( \varepsilon_i = |F_i^t| - |Y_i^t| - f_i^t \) for all \( i \in \{1, \ldots, 2m + 1\} \). We show that \( \varepsilon_i - \varepsilon_{i-1} \geq 0 \) for all \( i \in \{1, \ldots, 2m + 1\} \). Since \( S \) is one-centered, in Table 2 all relevant tuples for \( S_{i-1}^t \ominus S_i^t \) are shown.

Now assume towards a contradiction that there is a \( j \in \{1, \ldots, 2m + 1\} \) such that \( \varepsilon_j > 0 \). Since \( \varepsilon_i - \varepsilon_{i-1} \geq 0 \) for all \( i \in \{1, \ldots, 2m + 1\} \), we have that \( \varepsilon_{2m+1} > 0 \iff |F_{2m+1}^t| - |Y_{2m+1}^t| > f_{2m+1}^t \).

By definition, we have that \( |F_{2m+1}^t| = 0 \) and \( |Y_{2m+1}^t| = 0 \). Moreover, since \( S \) is a solution and each vertex cover needs at least \( K \) vertices from \( \bar{U}_t \), we have that \( f_{2m+1}^t \geq 0 \). It follows that \( 0 = |F_{2m+1}^t| - |Y_{2m+1}^t| > f_{2m+1}^t \geq 0 \), yielding a contradiction.
Next, we prove that in a phase $t$ with $|F_t^i| \geq K$, there are exactly $k$ vertices from $V$ contained in the union of the vertex covers of phase $t$.

**Lemma 4.22.** Let $S = (S_1^1, \ldots, S_{2m+1}^1 = S_2^2, \ldots, S_{2m+1}^m, \ldots, S_{2m+1}^2)$ be a one-centered solution to $(G, k', \ell)$ from Construction 4.10 being a yes-instance, and let $t \in \{1, \ldots, \kappa\}$ be such that $|F_t^i| \geq K$. Then, $|\bigcup_{i=1}^{2m+1} S_t^i \cap V| \leq k$.

**Proof.** From Lemma 4.21, we know that $|Y_t^i| \geq K - f_t^i$. Let $|Y_t^i| = K - f_t^i + \lambda$ for some $\lambda \in \mathbb{N}_0$, and $\varepsilon_i = |F_t^i| - |Y_t^i| - f_t^i$, for all $i \in \{1, \ldots, 2m+1\}$.

We now show that at most $\lambda$ layers where we exchange a vertex currently in the vertex cover with a vertex in $V$. Let $i \in \{2, \ldots, 2m+1\}$ such that $S_{t-1}^i \triangle S_t^i = (a, v)$ with $v \in V$. From Table 2 (recall that one-centered solutions are smooth), we know that $\varepsilon_i \geq \varepsilon_{i-1} + 1$. Assume towards a contradiction that there are $\lambda + 1$ many of these exchanges. Then, there is a $j \in \{1, \ldots, 2m+1\}$ such that

$$\varepsilon_j \geq \varepsilon_1 + \lambda + 1 = |F_t^i| - |Y_t^i| - f_t^i + \lambda + 1 \geq K - (K - f_t^i + \lambda) - f_t^i + \lambda + 1 \geq 1$$

$$\iff |F_t^j| - |Y_t^j| > f_t^j.$$  

This contradicts the invariant of Lemma 4.21.

In the beginning of the phase $t$, we have at most $k - \lambda$ vertices from $V$ in the vertex cover, because $|S_t^i \cap V| \leq K + k - |Y_t^i| - f_t^i = K + k - (K - f_t^i + \lambda) - f_t^i = k - \lambda$. Since there at most $\lambda$ many exchanges $S_{t-1}^i \triangle S_t^i = (a, v)$ where $v \in V$ and $i \in \{2, \ldots, 2m+1\}$, we know that the vertex set $\bigcup_{i=1}^{2m+1} S_t^i \cap V$ is of size at most $k$.

4.22.1 Proof of Proposition 4.9

**Proof of Proposition 4.9.** Let $(G, k)$ be an instance of CLIQUE and $(G, k', \ell)$ be the instance of MSVC resulting from Construction 4.10. Observe that Construction 4.10 runs in polynomial time. We prove that $(G, k)$ is a yes-instance of CLIQUE if and only if $(G, k', \ell)$ is a yes-instance of MSVC.

$(\Rightarrow)$ It follows from Lemma 4.12 that $(G, k', \ell)$ is a yes-instance if $(G, k)$ is a yes-instance.

$(\Leftarrow)$ Let $(G, k', \ell)$ be a yes-instance. From Corollary 4.19 it follows that there is a one-centered solution $S = (S_1^1, \ldots, S_{2m+1}^1 = S_2^2, \ldots, S_{2m+1}^m, \ldots, S_{2m+1}^2)$ for $(G, k', \ell)$. By Lemma 4.20, there is a $t \in \{1, \ldots, \kappa\}$ such that $|F_t^i| \geq K$. By Lemma 4.22, we know that $|\bigcup_{i=1}^{2m+1} S_t^i \cap V| \leq k$. Now we
identify the clique of size \( k \) in \( G \). Since \( |F_t^i| \geq K \), we know that, by Construction 4.10, at least \( K = \binom{k}{2} \) layers are covered by vertices in \( V \cup E \cup U_{k+1} \cup \{v_{2j+1} \mid j \in \{1, \ldots, m\}\} \) in phase \( t \). Note that each of these layers correspond to an edge \( e = \{v, w\} \) in \( G \) and that we need in particular the vertices \( v \) and \( w \) in the vertex cover. Since we have at most \( k \) vertices in \( \bigcup_{i=1}^{2m+1} S_t^i \cap V \), these vertices induce a clique of size \( k \) in \( G \).

Finally, following Remark 4.11, we can turn each layer into a tree preserving equivalence. The \( W[1] \)-hardness of CLIQUE \([8] \) regarding \( k \) and that \( k' \in O(k^2) \) then finishes the proof. \( \square \)

5 On Efficient Data Reduction

In this section, we study the possibility of effective data reduction for MSVC when parameterized by \( k, \tau \), and \( k+\tau \), that is, the possible existence of problem kernels of polynomial size. We prove that unless \( \text{coNP} \subseteq \text{NP/poly} \), MSVC admits no problem kernel of size polynomial in \( k \) (Section 5.1). Yet, when combining \( k \) and \( \tau \), we prove a problem kernel of size \( O(k^2 \tau) \) (Section 5.8). Moreover, we prove a problem kernel of size \( 5\tau \) when each layer consists of only one edge (Section 5.12). Recall that MSVC is para-NP-hard regarding \( \tau \) even if each layer is a tree.

5.1 No problem kernel of size polynomial in \( k \)

We prove that if (i) each layer consists only of one edge and \( \ell \geq 1 \), and (ii) if each layer is planar and \( \ell \geq 2k \), MSVC admits no kernel of size polynomial in \( k \) unless \( \text{coNP} \subseteq \text{NP/poly} \). Recall that fixed-parameter tractability of MSVC regarding \( k \) holds true in case of (i) (see Observation 2.5), while we left open whether it also holds true in case (ii).

**Theorem 5.2.** Unless \( \text{coNP} \subseteq \text{NP/poly} \), MSVC admits no polynomial kernel when parameterized by \( k \), even if (i) each layer consists of one edge and \( \ell = 1 \), or (ii) each layer is planar and \( \ell \geq 2k \).

We prove Theorem 5.2 using AND-compositions.

**Definition 5.3.** An AND-composition for a parameterized problem \( L \) is an algorithm that given \( p \) instances \( (x_1, k), \ldots, (x_p, k) \) of \( L \), computes in time polynomial in \( \sum_{i=1}^p |x_i| \) an instance \( (y, k') \) of \( L \) such that (i) \( (y, k') \in L \) if and only if \( (x_i, k) \in L \) for all \( i \in \{1, \ldots, p\} \), and (ii) \( k' \) is polynomially upper-bounded in \( k \).

The following is the crucial connection to polynomial kernelization.

**Theorem 5.4** (Drucker \([9] \)). If a parameterized problem whose unparameterized version is NP-hard admits an AND-composition, then \( \text{coNP} \subseteq \text{NP/poly} \).

Note that \( \text{coNP} \subseteq \text{NP/poly} \) implies a collapse of the polynomial-time hierarchy to its third level \([21] \).

In the proof of Theorem 5.2(ii), we use the following.

**Construction 5.5.** Let \( (G_1 = (V, E_1, \tau_1), k, \ell), \ldots, (G_p = (V, E_p, \tau_p), k, \ell) \) be \( p \) instances of MSVC where each layer consists of one edge and \( \ell = 1 \). We construct an instance \( (G = (V, E, \tau), k, \ell) \) of MSVC as follows. Denote by \( (G_{i1}^1, \ldots, G_{i\tau_i}^i) \) the sequence of layers of \( G_i \). Initially, let \( G \) be the temporal graph with layer sequence \( ((G_{i1}^1)_{1 \leq i \leq \tau_i})_{1 \leq i \leq p} \). Next, for each \( i \in \{1, \ldots, p-1\} \), insert between \( G_{i\tau_i}^i \) and \( G_{i+1}^{i+1} \) the sequence \( (H_{i1}^i, H_{i2}^i, \ldots, H_{2k}^i) = (G_{i\tau_i}^i, G_{i1}^{i+1}, \ldots, G_{i1}^{i+1}) \). This finishes the construction. Note that \( \tau = 2k(p-1) + \sum_{i=1}^p \tau_i \).
Construction 5.5 gives an AND-composition used in the proof of Theorem 5.2(i).

Proposition 5.6. MSVC where each layer consists of one edge and \( \ell = 1 \) AND-composes into itself parameterized by \( k \).

Proof. We AND-compose MSVC where each layer consists of one edge. Let \( I_1 = (G_1 = (V, E_1, \tau_1), k, \ell), \ldots, I_p = (G_p = (V, E_p, \tau_p), k, \ell) \) be \( p \) instances of MSVC with \( \ell = 1 \) where each layer consists of one edge. Apply Construction 5.5 to obtain instance \( I = (G = (V, G, \tau), k, \ell) \) of MSVC. We claim that \( I \) is a yes-instance if and only if \( I_i \) is a yes-instance for all \( i \in \{1, \ldots, p\} \).

(\( \Rightarrow \)) If \( I \) is a yes-instance, then for each \( i \in \{1, \ldots, p\} \), the subsequence of the solution restricted to the layers \( (G_j^i)_{1 \leq j \leq \tau_i} \) forms a solution to \( I_i \).

(\( \Leftarrow \)) Let \( (S_1^i, \ldots, S_{\tau_i}^i) \) be a solution to \( I_i \) for each \( i \in \{1, \ldots, p\} \). Clearly, \( (S_1^i, \ldots, S_{\tau_i}^i) \) forms a solution to the layers \( (G_j^i)_{1 \leq j \leq \tau_i} \). For \( H_1^i \), let \( T_1^i = S_{\tau_i}^i \setminus \{v\} \) for some \( v \) such that the unique edge of \( H_1^i \) is still covered. Next, set \( T_2^i = T_1^i \cup \{w\} \), where \( w \in S_{\tau_i+1}^i \) with \( w \) being incident with the unique edge of \( H_2^i \). Now, over the next \( 2k-2 \) layers, transform \( T_2^i \) into \( S_{\tau_i+1}^i \) by first removing layer by layer the vertices in \( T_2^i \setminus S_{\tau_i+1}^i \) (at most \( k-1 \) many vertices), and then layer by layer add the vertices in \( S_{\tau_i+1}^i \setminus T_2^i \) (again, at most \( k-1 \) vertices). This forms a solution to \( I \). \( \square \)

Turning a set of input instances of VERTEX COVER on planar graphs (being equivalent to MSVC with one layer being a planar graph) into a sequence gives an AND-composition used in the proof of Theorem 5.2(ii).

Proposition 5.7. MSVC with one layer being a planar graph AND-composes into MSVC with \( \ell \geq 2k \) and each layer being parameterized by \( k \).

Proof. We AND-cross-compose VC on planar graphs into MSVC with \( \ell \geq 2k \). Let \( (G_1, k), \ldots, (G_p, k) \) be \( p \)-instances of VC. Construct a temporal graph \( G \) with layers \( (G_1, \ldots, G_p) \). Set \( \ell = 2k \). This finishes the construction. It is not difficult to see that \( (G, k, \ell) \) is a yes-instance of MSVC if and only if \( (G_i, k) \) is a yes-instance of VC for all \( i \in \{1, \ldots, p\} \). \( \square \)

Proof of Theorem 5.2. Together with Theorem 5.4, Propositions 5.6 and 5.7 prove Theorem 5.2(i) and (ii), respectively. Recall that MSVC where each layer consists of one edge (Theorem 3.1) and VERTEX COVER on planar graphs [12] are NP-hard. \( \square \)

5.8 A problem kernel of size \( O(k^2 \tau) \)

MSVC remains NP-hard for \( \tau = 2 \), even if each layer is a tree (Theorem 3.1). Moreover, MSVC does not admit a problem kernel of size polynomial in \( k \), even if each layer consists of one edge (Theorem 5.2). Yet, when combining both parameters we obtain a problem kernel of cubic size.

Theorem 5.9. There is an algorithm that maps any instance \( (G = (V, E, \tau), k, \ell) \) of MSVC in time \( O(|V|^2 \tau) \) to an instance \( (G', k, \ell) \) of MSVC with at most \( 2k^2 \tau \) vertices and \( k^2 \tau \) temporal edges.

In the proof of Theorem 5.9, we apply three polynomial-time data reduction rules. These reduction rules can be understood as the temporal variants of the folklore reduction rules for VERTEX COVER. Our first reduction rule is immediate.
Reduction Rule 1 (Isolated vertices). If there is some vertex \( v \in V \) such that \( e \cap v = \emptyset \) for all \( e \in E(G_1) \), then delete \( v \).

For VERTEX COVER when asking for a vertex cover of size \( q \), there is the well-known reduction rule dealing with high-degree vertices: If there is a vertex \( v \) of degree larger than \( q \), then delete \( v \) and its incident edges and decrease \( q \) by one. For MSVC a high-degree vertex can only appear in some layers, and hence deleting this vertex is in general not correct. However, there is a temporal variant of the high-degree rule as follows.

Reduction Rule 2 (High degree). If there exists a vertex \( v \) such that there is an inclusion-maximal subset \( J \subseteq \{1, \ldots, \tau\} \) such that \( \deg_{G_1}(v) > k \) for all \( i \in J \), then add a vertex \( w_v \) to \( V \) and for each \( i \in J \), remove all edges incident to \( v \) in \( G_1 \), and add the edge \( \{v, w_v\} \).

Lemma 5.10. Reduction Rule 2 is correct and exhaustively applicable in \( O(|V|^2 \tau) \) time.

Proof. (Correctness) Let \( I \) be an instance, and let \( I' \) be the instance obtained from \( I \) applying Reduction Rule 2 with vertex \( v \) and index set \( J \). We prove that \( I \) is a \texttt{yes}-instance if and only if \( I' \) is a \texttt{yes}-instance.

\( (\Rightarrow) \) Let \( (S_1, \ldots, S_\tau) \) be a solution to \( I \). Observe that for all \( i \in J \), \( \deg_{G_1}(v) > k \) and hence \( v \in S_i \). It follows that \( (S_1, \ldots, S_\tau) \) is a solution to \( I' \).

\( (\Leftarrow) \) Let \( (S'_1, \ldots, S'_\tau) \) be a solution to \( I' \). Observe that for each \( i \in J \), \( S'_i \cap \{v, w_v\} \neq \emptyset \). Set \( S_i := (S'_i \setminus \{w_v\}) \cup \{v\} \) for all \( i \in J \). For each \( i \in \{1, \ldots, \tau\} \setminus J \), set \( S_i := S'_i \) if \( w_v \notin S'_i \), and \( (S'_i \setminus \{w_v\}) \cup \{v\} \) otherwise. Note that \( |S_i| \leq |S'_i| \) for all \( i \in \{1, \ldots, \tau\} \), and that \( |S_i \Delta S_{i+1}| \leq \ell \) for all \( i \in \{1, \ldots, \tau - 1\} \). It follows that \( (S_1, \ldots, S_\tau) \) is a solution to \( I \).

(Running time) For each vertex, we count the number of edges in each layer. If there are more than \( k \) edges in one layer, then we remember the index of the layer. For each layer, we compute for each vertex the degree and make the modification. Once for some \( v \) vertex \( w_v \) is introduced, we add a pointer from \( v \) to \( w_v \), and add the edge \( \{v, w_v\} \) in subsequent layers when needed. Hence, in each layer we touch each edge at most twice, yielding \( O(|V|^2) \) time per layer.

Similarly as in the reduction rules for VERTEX COVER, we now count the number of edges in each layer: If more than \( k^2 \) edges are contained in one layer, then no set of \( k \) vertices each of degree at most \( k \) can cover more than \( k^2 \) edges.
Reduction Rule 3 (no-instance). If none of Reduction Rules 1 and 2 is applicable and there is a layer with more than $k^2$ edges, then output a trivial no-instance.

We are ready to prove that when none of the Reduction Rules 1 to 3 can be applied, the instance is either a no-instance or has "few" vertices and temporal edges.

Lemma 5.11. Let $(G, k, \ell)$ be an instance of MSVC such that none of Reduction Rules 1 to 3 is applicable. Then $G$ consists of at most $2k^2\tau(G)$ vertices and $k^2\tau(G)$ temporal edges.

Proof. Let none of Reduction Rules 1 and 2 be applicable. For each layer, it holds true that there is no isolated vertex and no vertex of degree larger than $k$. Due to Reduction Rule 3, either we deal with a trivial no-instance or each layer consists of at most $k^2$ edges, accounting to at most $k^2\tau$ temporal edges in $G$. Consequently, due to Reduction Rule 1, there are at most $2k^2\tau$ vertices in $G$.

We are ready to prove the main result of this section.

Proof of Theorem 5.9. Apply Reduction Rules 1 to 3 exhaustively in $O(|V|^2\tau)$ time to obtain an equivalent instance $(G', k, \ell)$. Due to Lemma 5.11, $G'$ consists of at most $2k^2\tau$ vertices and at most $k^2\tau$ temporal edges.

5.12 A problem kernel of size $5\tau$

MSVC when each layer is a tree does not admit a problem kernel of any size in $\tau$ unless $P = NP$. Yet, when each layer consists of only one edge, then MSVC admits a problem kernel of size linear in $\tau$.

Theorem 5.13. Let $(G = (V, E, \tau), k, \ell)$ be an instance of MSVC where each layer consists of one edge. Then we can compute in $O(|V| \cdot \tau)$ time an instance $(G', k, \ell)$ of size at most $5\tau$.

Proof. Observe that we can immediately output a trivial yes-instance if $k \geq \tau$ (Observation 2.1) or $\ell \geq 2$ (Observation 2.5). Hence, assume that $k \leq \tau - 1$ and $\ell \leq 1$. Apply Reduction Rule 1 exhaustively on $(G, k, \ell)$ to obtain $(G', k, \ell)$. Since there are $\tau$ edges in $G$, there are at most $2\tau$ vertices in $G'$. It follows that the encoding length of $(G', k, \ell)$ is at most $5\tau$.

6 Conclusion

We introduced Multistage Vertex Cover, proved it to be NP-hard even on restricted inputs, and studied its parameterized complexity regarding the natural parameters $k$, $\ell$, and $\tau$ (each given as input). We leave open whether MSVC parameterized by $k$ is fixed-parameter tractable when each layer consists of only one edge (see Table 1). Moreover, it is open whether MSVC remains NP-hard on two layers each being a path (that is, strengthening Theorem 3.1(i)).

References

[1] Faisal N. Abu-Khzam, Judith Egan, Michael R. Fellows, Frances A. Rosamond, and Peter Shaw. On the parameterized complexity of dynamic problems. Theoretical Computer Science, 607:426–434, 2015.
[2] Eleni C. Akrida, George B. Mertzios, Paul G. Spirakis, and Viktor Zamaraev. Temporal vertex cover with a sliding time window. In Proceedings of 45th International Colloquium on Automata, Languages, and Programming (ICALP 2018), volume 107 of LIPIcs, pages 148:1–148:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[3] Josh Alman, Matthias Mnich, and Virginia Vassilevska Williams. Dynamic parameterized problems and algorithms. In Proceedings of 44th International Colloquium on Automata, Languages, and Programming (ICALP 2017), volume 80 of LIPIcs, pages 41:1–41:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

[4] Evripidis Bampis, Bruno Escoffier, Michael Lampis, and Vangelis Th. Paschos. Multistage matchings. In Proceedings of 16th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2018), volume 101 of LIPIcs, pages 7:1–7:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[5] Evripidis Bampis, Bruno Escoffier, and Alexandre Teiller. Multistage knapsack. CoRR, abs/1901.11260, 2019.

[6] Rajesh Chitnis, Graham Cormode, Hossein Esfandiari, MohammadTaghi Hajiaghayi, Andrew McGregor, Morteza Monemizadeh, and Sofya Vorotnikova. Kernelization via sampling with applications to finding matchings and related problems in dynamic graph streams. In Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2016), pages 1326–1344. SIAM, 2016.

[7] Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.

[8] Rodney G. Downey and Michael R. Fellows. Parameterized Complexity. Monographs in Computer Science. Springer, 1999.

[9] Andrew Drucker. New limits to classical and quantum instance compression. SIAM Journal on Computing, 44(5):1443–1479, 2015.

[10] Herbert Fleischner, Gert Sabidussi, and Vladimir I. Sarvanov. Maximum independent sets in 3- and 4-regular hamiltonian graphs. 310(20):2742–2749, 2010.

[11] Till Fluschnik, Hendrik Molter, Rolf Niedermeier, and Philipp Zschoche. Temporal graph classes: A view through temporal separators. In Proceedings of the 44th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2018), volume 11159 of Lecture Notes in Computer Science, pages 216–227. Springer, 2018.

[12] M. R. Garey and David S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

[13] Parikshit Gopalan, Phokion G Kolaitis, Elitza Maneva, and Christos H Papadimitriou. The connectivity of boolean satisfiability: computational and structural dichotomies. SIAM Journal on Computing, 38(6):2330–2355, 2009.
Anupam Gupta, Kunal Talwar, and Udi Wieder. Changing bases: Multistage optimization for matroids and matchings. In Proceedings of 41st International Colloquium on Automata, Languages, and Programming (ICALP 2014), volume 8572 of LNCS, pages 563–575. Springer, 2014.

Sepp Hartung and Rolf Niedermeier. Incremental list coloring of graphs, parameterized by conservation. Theoretical Computer Science, 494:86–98, 2013.

Takehiro Ito, Erik D Demaine, Nicholas JA Harvey, Christos H Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno. On the complexity of reconfiguration problems. Theoretical Computer Science, 412(12-14):1054–1065, 2011.

Yoichi Iwata and Keigo Oka. Fast dynamic graph algorithms for parameterized problems. In Proceedings of 12th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2014), volume 8503 of Lecture Notes in Computer Science, pages 241–252. Springer, 2014.

R. Krithika, Abhishek Sahu, and Prafullkumar Tale. Dynamic parameterized problems. Algorithmica, 80(9):2637–2655, 2018.

Amer Mouawad, Naomi Nishimura, Venkatesh Raman, and Sebastian Siebertz. Vertex cover reconfiguration and beyond. Algorithms, 11(2):20, 2018.

Amer E Mouawad, Naomi Nishimura, Venkatesh Raman, Narges Simjour, and Akira Suzuki. On the parameterized complexity of reconfiguration problems. Algorithmica, 78(1):274–297, 2017.

Chee-Keng Yap. Some consequences of non-uniform conditions on uniform classes. Theoretical Computer Science, 26:287–300, 1983.