COMPARISON OF CUBICAL AND SIMPLICIAL DERIVED FUNCTORS

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Abstract. In this note we prove that the simplicial derived functors introduced by Tierney and Vogel [TV69] are naturally isomorphic to the cubical derived functors introduced by the author in [P09]. We also explain how this result generalizes the well-known fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic.

1. Introduction

In [TV69] Tierney and Vogel for any functor $F: \mathcal{C} \to \mathcal{B}$, where $\mathcal{C}$ is a category with finite limits and a projective class $\mathcal{P}$, and $\mathcal{B}$ is an abelian category, constructed simplicial derived functors and investigated relationships of their theory with other theories of derived functors. Namely, they showed that if $\mathcal{C}$ is abelian and $F$ is additive, then their theory coincides with the classical relative theory of Eilenberg-Moore [EM65], whereas if $\mathcal{C}$ is abelian and $F$ is an arbitrary functor, then it gives a generalization of the theory of Dold-Puppe [DP61]. Besides, they proved that their derived functors are naturally isomorphic to the cotriple derived functors of Barr-Beck ([BB66], [BB69]) if there is a cotriple in $\mathcal{C}$ that realizes the given projective class $\mathcal{P}$.

The key point in the construction of the derived functors by Tierney and Vogel is that using $\mathcal{P}$-projective objects and simplicial kernels, for every $C$ from $\mathcal{C}$ a $\mathcal{P}$-projective pseudosimplicial resolution can be constructed, which is a $C$-augmented pseudosimplicial object in $\mathcal{C}$ and which for a given $C$ is unique up to a presimplicial homotopy.

In [P09] using pseudocubical resolutions instead of pseudosimplicial ones we constructed cubical derived functors for any functor $F: \mathcal{C} \to \mathcal{B}$, where $\mathcal{C}$ is a category with finite limits and a projective class $\mathcal{P}$, and $\mathcal{B}$ is an abelian category. It was shown that if $\mathcal{C}$ is an abelian category, $F$ an additive functor, and $\mathcal{P}$ is closed, then our theory coincides with the theory of Eilenberg-Moore [P09, 4.4]. However, there remained an open question whether the Tierney-Vogel simplicial derived functors and our cubical derived functors are isomorphic in general or not. In this paper we give a positive answer to this question. More precisely, we prove the following

Theorem 1.1. Suppose $\mathcal{C}$ is a category with finite limits, $\mathcal{P}$ a projective class in $\mathcal{C}$ in the sense of [TV69, §2], $\mathcal{B}$ an abelian category, and $F: \mathcal{C} \to \mathcal{B}$ a functor. Let $L_n^\Delta F: \mathcal{C} \to \mathcal{B}$, $n \geq 0$, be the Tierney-Vogel simplicial derived functors of $F$, and $L_n^\Box F: \mathcal{C} \to \mathcal{B}$, $n \geq 0$, the cubical derived functors of $F$. Then there is an isomorphism

$$L_n^\Delta F(C) \cong L_n^\Box F(C), \quad C \in \mathcal{C}, \quad n \geq 0,$$

which is natural in $F$ and in $C$.  

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The main idea of the proof goes back to Barr and Beck [BB69]. The point is that passing to the unique additive extension

$$F_{ad}: \mathbb{ZC} \to \mathcal{B}$$

of the functor $F$, where $\mathbb{ZC}$ denotes the free preadditive category generated by $\mathcal{C}$, one verifies that the Eilenberg-Moore derived functors of $F_{ad}$ (with respect to the class $\mathcal{P}$) restricted to $\mathcal{C}$ are naturally isomorphic to the simplicial derived functors of $F$ on the one hand and to the cubical derived functors of $F$ on the other hand.

The paper is organized as follows. In Section 2 the relative Eilenberg-Moore derived functor theory of additive functors is reviewed from [EM65]. In Section 3 we recall the theory of Tierney-Vogel and prove that the simplicial derived functors of $F: \mathcal{C} \to \mathcal{B}$ are just the Eilenberg-Moore derived functors of $F_{ad}: \mathbb{ZC} \to \mathcal{B}$ restricted to $\mathcal{C}$. Section 4 is devoted to the definition and properties of pseudocubical normalization functor for an idempotent complete preadditive category. Note that the pseudocubical normalization is the main technical tool used in Section 5 to prove that the cubical derived functors of $F: \mathcal{C} \to \mathcal{B}$ are naturally isomorphic to the Eilenberg-Moore derived functors of $F_{ad}: \mathbb{ZC} \to \mathcal{B}$ restricted to $\mathcal{C}$. In the final section we briefly indicate that Theorem 1.1 generalizes the classical fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic.

2. Partially defined Eilenberg-Moore derived functors

The following definitions are well-known.

**Definition 2.1.** A preadditive category is a category $\mathcal{A}$ together with the following data:

(i) For any objects $X,Y$ in $\mathcal{A}$, the set of morphisms $\text{Hom}_{\mathcal{A}}(X,Y)$ is an abelian group;
(ii) For any morphisms $f,g: X \to Y$, $h: W \to X$ and $u: Y \to Z$ in $\mathcal{A}$, the following hold

$$(f + g)h = fh + gh, \quad u(f + g) = uf + ug.$$

In other words, a preadditive category is just a ring with several objects in the sense of [M72].

**Definition 2.2.** Let $\mathcal{A}$ be a preadditive category. An augmented chain complex over an object $A \in \mathcal{A}$ (or just a complex over $A$) is a sequence

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} A$$

such that $\partial_n \partial_{n+1} = 0$, $n \geq 0$.

**Definition 2.3.** Let $\mathcal{A}$ be a preadditive category and $\mathcal{P}$ a class of objects in $\mathcal{A}$ (which need not be a “projective class” in any sense). A complex

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} A$$

over $A \in \mathcal{A}$ is said to be $\mathcal{P}$-acyclic if for any $Q \in \mathcal{P}$ the sequence of abelian groups

$$\cdots \to \text{Hom}_{\mathcal{A}}(Q,C_1) \xrightarrow{\partial_0} \text{Hom}_{\mathcal{A}}(Q,C_0) \xrightarrow{\partial_0} \text{Hom}_{\mathcal{A}}(Q,A) \to 0$$

is exact.
Definition 2.4. Let $\mathcal{A}$ be a preadditive category and $\mathcal{P}$ a class of objects in $\mathcal{A}$. A $\mathcal{P}$-resolution of an object $A \in \mathcal{A}$ is a $\mathcal{P}$-acyclic complex

$$
\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A
$$

over $A$ with $P_n \in \mathcal{P}$, $n \geq 0$.

Note that an object $A \in \mathcal{A}$ need not necessarily possess a $\mathcal{P}$-resolution.

There is a comparison theorem for $\mathcal{P}$-resolutions which can be proved using the standard homological algebra arguments (see e.g. [W94, 2.2.7]). More precisely, the following is valid.

Proposition 2.5 (Comparison theorem). Let $P_\ast \longrightarrow A$ be a complex over $A \in \mathcal{A}$ consisting of objects of $\mathcal{P}$, and let $S_\ast \longrightarrow B$ be a $\mathcal{P}$-acyclic complex. Then any morphism $f : A \longrightarrow B$ can be extended to a morphism of augmented chain complexes

$$
P_\ast \xrightarrow{f} A \xrightarrow{f} S_\ast \longrightarrow B.
$$

Moreover, any two such extensions are chain homotopic.

Suppose $\mathcal{A}$ is a preadditive category, $\mathcal{P}$ a class of objects in $\mathcal{A}$, $\mathcal{B}$ an abelian category, $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor, and $\mathcal{A}'$ the full subcategory of those objects in $\mathcal{A}$ which possess $\mathcal{P}$-resolutions. Recall that Proposition 2.5 allows one to construct the left derived functors $L_n F : \mathcal{A}' \longrightarrow \mathcal{B}$, $n \geq 0$, of $F$ with respect to the class $\mathcal{P}$ as follows. If $A \in \mathcal{A}'$, choose (once and for all) a $\mathcal{P}$-resolution $P_\ast \longrightarrow A$ and define

$$
L_n F(A) = H_n(F(P_\ast)), \quad n \geq 0.
$$

Remark 2.6. If $\mathcal{P}$ is a projective class in the sense of [EM65], then $L_n F$, $n \geq 0$, are exactly the derived functors introduced in [EM65, I.3]. Note that in this case $\mathcal{A}' = \mathcal{A}$, i.e., the functors $L_n F$ are defined everywhere.

Further we recall

Definition 2.7 ([EM65, I.2]). Let $\mathcal{A}$ be a preadditive category and $\mathcal{P}$ a class of objects of $\mathcal{A}$. A sequence

$$
X \xrightarrow{f} Y \xrightarrow{g} Z
$$

in $\mathcal{A}$ is said to be $\mathcal{P}$-exact if $gf = 0$ and the sequence of abelian groups

$$
\text{Hom}_\mathcal{A}(P, X) \xrightarrow{f_*} \text{Hom}_\mathcal{A}(P, Y) \xrightarrow{g_*} \text{Hom}_\mathcal{A}(P, Z)
$$

is exact for any $P \in \mathcal{P}$.

Definition 2.8 ([EM65, I.2]). A closure of a class $\mathcal{P}$, denoted by $\mathcal{P}$, is the class of all those objects $Q \in \mathcal{A}$ for which

$$
\text{Hom}_\mathcal{A}(Q, X) \xrightarrow{f_*} \text{Hom}_\mathcal{A}(Q, Y) \xrightarrow{g_*} \text{Hom}_\mathcal{A}(Q, Z)
$$

is exact whenever $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $\mathcal{P}$-exact.
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Clearly, $\mathcal{P} \subseteq \overline{\mathcal{P}}$ and $\mathcal{P}$-exactness is equivalent to $\overline{\mathcal{P}}$-exactness. In particular, $\overline{\mathcal{P}} = \overline{\mathcal{P}}$.

Note that if a preadditive category $\mathcal{A}$ has a terminal object, then any $\mathcal{P}$-resolution is a $\overline{\mathcal{P}}$-resolution as well. This together with 2.5 implies the following

**Proposition 2.9.** Let $\mathcal{A}$ be a preadditive category with a terminal object, $\mathcal{P}$ a class of objects in $\mathcal{A}$, $\mathcal{B}$ an abelian category, $F: \mathcal{A} \to \mathcal{B}$ an additive functor, and $A$ an object in $\mathcal{A}$ which possesses a $\mathcal{P}$-resolution. Then there is a natural isomorphism

$$L^n_\mathcal{P} F(A) \cong L^n_{\overline{\mathcal{P}}} F(A), \quad n \geq 0.$$  

3. Simplicial derived functors and Eilenberg-Moore derived functors

In this section we briefly review the construction of simplicial derived functors from [TV69, §2] and show that they can be obtained as derived functors of an additive functor.

Let us recall the following definitions.

**Definition 3.1.** A presimplicial object $S$ in a category $\mathcal{C}$ is a family of objects $(S_n \in \mathcal{C})_{n \geq 0}$ together with morphisms

$$\partial_i : S_n \to S_{n-1}, \quad n \geq 1, \quad 0 \leq i \leq n,$$

in $\mathcal{C}$ satisfying the presimplicial identities

$$\partial_i \partial_j = \partial_{j-1} \partial_i, \quad i < j.$$

**Definition 3.2.** Let $S$ be a presimplicial object in a preadditive category $\mathcal{A}$. The unnormalized chain complex $K(S)$ associated to $S$ is defined by

$$K(S)_n = S_n, \quad n \geq 0,$$

$$\partial = \sum_{i=0}^n (-1)^i \partial_i : K(S)_n \to K(S)_{n-1}, \quad n > 0.$$

The presimplicial identities imply that $\partial^2 = 0$.

Now let $\mathcal{C}$ be a category with finite limits, $\mathcal{P}$ a projective class in $\mathcal{C}$ in the sense of [TV69, §2], $\mathcal{B}$ an abelian category, and $F: \mathcal{C} \to \mathcal{B}$ a functor. The simplicial derived functors $L^n_\Delta F$ of $F$ with respect to the class $\mathcal{P}$ are defined as follows. For any object $C \in \mathcal{C}$, choose (once and for all) a $\mathcal{P}$-projective presimplicial resolution

$$S \to C$$

of $C$ (i.e., a $\mathcal{P}$-exact presimplicial object $S$ augmented over $C$ with $S_n \in \mathcal{P}, \quad n \geq 0$) and define

$$L^n_\Delta F(C) = H_n(K(F(S))), \quad n \geq 0.$$  

By the comparison theorem for projective presimplicial resolutions [TV69, (2.4) Theorem], the objects $L^n_\Delta F(C)$ are well-defined and functorial in $F$ and $C$.

We will now show that the derived functors $L^n_\Delta F$ can be obtained as derived functors of some additive functor. First recall
**Lemma 3.3.** Let $S ightarrow S_{-1}$ be an augmented presimplicial set. Suppose that $\partial_0 : S_0 \rightarrow S_{-1}$ is surjective and the following extension condition holds: For any $n \geq 0$ and any collection of $n+2$ elements $x_i \in S_n$, $0 \leq i \leq n+1$, satisfying

$$\partial_i x_j = \partial_{j-1} x_i, \quad 0 \leq i < j \leq n+1,$$

there exists $x \in S_{n+1}$ such that $\partial_i x = x_i$, $0 \leq i \leq n+1$. Then the augmented chain complex

$$K(\mathbb{Z}[S]) \xrightarrow{\partial_0} \mathbb{Z}[S_{-1}]$$

is chain contractible ($\mathbb{Z}[X]$ denotes the free abelian group generated by $X$). In particular, it has trivial homology in each dimension.

The proof is standard (one constructs inductively a presimplicial contraction).

**Example 3.4.** Let $S ightarrow C$ be a $\mathcal{P}$-projective presimplicial resolution of $C$ and suppose $Q \in \mathcal{P}$. Then the augmented presimplicial set

$$\text{Hom}_\mathcal{P}(Q, S) ightarrow \text{Hom}_\mathcal{P}(Q, C)$$

satisfies the conditions of Lemma 3.3. In particular, the homologies of the augmented chain complex

$$K(\mathbb{Z}[\text{Hom}_\mathcal{P}(Q, S)]) \rightarrow \mathbb{Z}[\text{Hom}_\mathcal{P}(Q, C)]$$

vanish.

Now suppose again that $\mathcal{C}$ is a category with finite limits, $\mathcal{P}$ a projective class in $\mathcal{C}$, $\mathcal{B}$ an abelian category, and $F : \mathcal{C} \rightarrow \mathcal{B}$ a functor. Let $\mathbb{Z}\mathcal{C}$ denote the free preadditive category generated by $\mathcal{C}$ [M72, §1], i.e., the objects of $\mathbb{Z}\mathcal{C}$ are those of $\mathcal{C}$, and for any objects $C$ and $D$ in $\mathcal{C}$, $\text{Hom}_{\mathbb{Z}\mathcal{C}}(C, D)$ is the free abelian group generated by $\text{Hom}_\mathcal{C}(C, D)$. The composition of morphisms in $\mathbb{Z}\mathcal{C}$ is induced by that in $\mathcal{C}$. Clearly, $\mathcal{C}$ is a subcategory of $\mathbb{Z}\mathcal{C}$. Further, since the category $\mathcal{B}$ is abelian (and therefore additive), the functor $F : \mathcal{C} \rightarrow \mathcal{B}$ can be uniquely extended to an additive functor

$$F_{ad} : \mathbb{Z}\mathcal{C} \rightarrow \mathcal{B}.$$ 

The following proposition relates the simplicial derived functors of $F$ to the Eilenberg-Moore derived functors of $F_{ad}$.

**Proposition 3.5.** Let $\mathcal{C}$ be a category with finite limits, $\mathcal{P}$ a projective class in $\mathcal{C}$, $\mathcal{B}$ an abelian category, and $F : \mathcal{C} \rightarrow \mathcal{B}$ a functor. Then:

(i) For any $\mathcal{P}$-projective presimplicial resolution $S \rightarrow C$, the augmented chain complex

$$K(S) \rightarrow C$$

in $\mathbb{Z}\mathcal{C}$ is a $\mathcal{P}$-resolution of $C$ in the sense of Definition 2.4.

(ii) For any $C \in \mathcal{C}$, there is a natural isomorphism

$$L^n F(C) \cong L^n_{\mathcal{P}} F_{ad}(C), \quad n \geq 0.$$
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Proof. The first claim immediately follows from 3.4 and the definition of $Z_C$. The second claim is a consequence of the first one and the definition of $F_{ad}$. Indeed, if $S \rightarrow C$ is a $\mathcal{P}$-projective presimplicial resolution of $C$, then we have

$$L_n^\Delta F(C) = H_n(K(F(S))) = H_n(F_{ad}(K(S))) = L_n^\mathcal{P} F_{ad}(C).$$

□

Remark 3.6. Proposition 3.5 is essentially due to Barr and Beck [BB69, §5]. More precisely, in the case when the projective class $\mathcal{P}$ comes from a cotriple (see [TV69, §3]) the above statement is proved in [BB69, §5]. (The cotriple derived functor theory of Barr-Beck is a special case of the Tierney-Vogel theory [TV69 §3].) Thus 3.5 is a simple generalization of the result of Barr and Beck.

4. Pseudocubical objects in idempotent complete preadditive categories

Definition 4.1 ([P09, 2.2]). A pseudocubical object $X$ in a category $\mathcal{C}$ is a family of objects $(X_n \in \mathcal{C})_{n \geq 0}$ together with face operators

$$\partial_i^0, \partial_i^1: X_n \rightarrow X_{n-1}, \quad 1 \leq i \leq n,$$

and pseudodegeneracy operators

$$s_i: X_{n-1} \rightarrow X_n, \quad 1 \leq i \leq n,$$

satisfying the pseudocubical identities

$$\partial_i^\alpha \partial_j^\varepsilon = \partial_{j-1}^\varepsilon \partial_i^\alpha \quad i < j, \quad \alpha, \varepsilon \in \{0, 1\},$$

and

$$\partial_i^\alpha s_j = \begin{cases} s_{j-1} \partial_i^\alpha & i < j, \\ \text{id} & i = j, \\ s_j \partial_i^{\alpha-1} & i > j, \end{cases}$$

for $\alpha \in \{0, 1\}$.

Important examples of pseudocubical objects appear in a natural way: Let $\mathcal{C}$ be a category with finite limits and $\mathcal{P}$ a projective class in $\mathcal{C}$. Then for any object $C \in \mathcal{C}$, there is a $\mathcal{P}$-exact augmented pseudocubical object $X \rightarrow C$ with $X_n \in \mathcal{P}$, $n \geq 0$, called $\mathcal{P}$-projective pseudocubical resolution of $C$ (see [P09, §3] for details).

In [P09] we use the normalized chain complex of a pseudocubical object in an abelian category to define the cubical derived functors. (Note that the normalized chain complex of a cubical object in an abelian category was introduced by Świątek in [Ś75].) Below we recall the definition and some properties of the normalized chain complex of a pseudocubical object in the general setting of idempotent complete preadditive categories. These are needed to prove a cubical analog of Proposition 3.5 in the next section.
**Definition 4.2.** A preadditive category $\mathcal{A}$ is said to be idempotent complete if any idempotent $p : E \to E$ in $\mathcal{A}$ (i.e., $p^2 = p$) has a kernel. That is, there is a morphism $i : \text{Ker}(p) \to E$ with $pi = 0$, and for any morphism $f : F \to E$, satisfying $pf = 0$, there is a unique morphism $g : F \to \text{Ker}(p)$ such that $ig = f$.

The following two propositions are well known (see e.g. [K78]).

**Proposition 4.3.** Let $\mathcal{A}$ be an idempotent complete preadditive category and $p : E \to E$ an idempotent in $\mathcal{A}$. Then there is a diagram

$$
\begin{array}{ccc}
\text{Ker}(p) & \xrightarrow{i_1} & E \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
\text{Ker}(1 - p) & \xleftarrow{i_2} & E
\end{array}
$$

such that

$$
\pi_1 i_1 = 1, \quad \pi_2 i_2 = 1,
\pi_1 i_2 = 0, \quad \pi_2 i_1 = 0,
\pi_2 i_1 = 1 - p, \quad \pi_2 i_2 = p.
$$

In particular, the coproduct $\text{Ker}(p) \oplus \text{Ker}(1 - p)$ exists in $\mathcal{A}$ and is isomorphic to $E$.

**Proposition 4.4.** Let $\mathcal{A}$ be a preadditive category. Then there exists an idempotent complete preadditive category $\widetilde{\mathcal{A}}$ and a full additive embedding $\varphi : \mathcal{A} \to \widetilde{\mathcal{A}}$ satisfying the following universal property: For any idempotent complete preadditive category $\mathcal{D}$ and an additive functor $\psi : \mathcal{A} \to \mathcal{D}$, there is an additive functor $\psi' : \widetilde{\mathcal{A}} \to \mathcal{D}$ which makes the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \widetilde{\mathcal{A}} \\
\downarrow \psi & & \downarrow \psi' \\
\mathcal{D} & & \mathcal{D}
\end{array}
$$

commute up to a natural equivalence, and which is unique up to a natural isomorphism.

Let $X$ be a pseudocubical object in an idempotent complete preadditive category $\mathcal{D}$.

**Definition 4.5.** The unnormalized chain complex $C(X)$ associated to $X$ is defined by

$$
C(X)_n = X_n, \quad n \geq 0,
$$

$$
\partial = \sum_{i=1}^{n} (-1)^i (\partial^1_i - \partial^0_i) : C(X)_n \to C(X)_{n-1}, \quad n > 0.
$$

The pseudocubical identities show that $\partial^2 = 0$. Moreover, they imply that the morphisms

$$
\sigma^X_n = (1 - s_1 \partial^1_1) \cdots (1 - s_n \partial^1_n) : X_n \to X_n, \quad n \geq 0, \quad (\sigma_0 = 1)
$$

are idempotents and form an endomorphism of the chain complex $C(X)$. We denote this endomorphism by

$$
\sigma^X : C(X) \to C(X).
$$
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Since \((\sigma^X)^2 = \sigma^X\) and the category \(\mathcal{D}\) is idempotent complete, the chain map \(\sigma^X\) has a kernel \(\text{Ker} \sigma^X\) in the category of non-negative chain complexes in \(\mathcal{D}\). Furthermore, by [4.3] there is a diagram in the category of chain complexes

\[
\text{Ker}(\sigma^X) \xrightarrow{\pi_1} C(X) \xrightarrow{\pi_2} \text{Ker}(1 - \sigma^X)
\]

such that

\[
\begin{align*}
\pi_1 i_1 &= 1, \quad \pi_2 i_2 = 1, \\
\pi_1 i_2 &= 0, \quad \pi_2 i_1 = 0, \\
i_1 \pi_1 &= 1 - \sigma^X, \quad i_2 \pi_2 = \sigma^X.
\end{align*}
\]

Definition 4.6. Let \(X\) be a pseudocubical object in an idempotent complete preadditive category \(\mathcal{D}\). The chain complex \(\text{Ker}(1 - \sigma^X)\), denoted by \(N(X)\), is called the normalized chain complex of \(X\).

Remark 4.7. If \(\mathcal{D}\) is an abelian category, then \(N(X)\) admits the following description:

\[
N(X)_0 = X_0, \quad N(X)_n = \bigcap_{i=1}^n \text{Ker}(\partial^i_1), \quad n > 0,
\]

\[
\partial = \sum_{i=1}^n (-1)^{i+1} \partial^i_1 : N(X)_n \to N(X)_{n-1}, \quad n > 0.
\]

Thus in the abelian case one does not need pseudodegeneracies to define \(N(X)\).

Next, we recall the construction of cubical derived functors from [P09, §3]. Let \(\mathcal{C}\) be a category with finite limits, \(\mathcal{P}\) a projective class in \(\mathcal{C}\), \(\mathcal{B}\) an abelian category, and \(F : \mathcal{C} \to \mathcal{B}\) a functor. Then the cubical derived functors \(L^\Box_n F\) of \(F\) with respect to the class \(\mathcal{P}\) are defined as follows. For any object \(C \in \mathcal{C}\), choose (once and for all) a \(\mathcal{P}\)-projective pseudocubical resolution

\[
X \to C
\]

of \(C\) and define

\[
L^\Box_n F(C) = H_n(N(F(X))), \quad n \geq 0.
\]

The comparison theorem for precubical resolutions [P09, 3.3] and the homotopy invariance of the functor \(N\) [P09, 3.6] imply that the objects \(L^\Box_n F(C)\) are well-defined and functorial in \(F\) and \(C\).

Note that one cannot use the unnormalized chain complex \(C(X)\) instead of \(N(X)\) to define the cubical derived functors [P09, 3.8].

The following lemma is the main technical tool for proving a cubical analog of Proposition 3.5.

Lemma 4.8. Let \(F : \mathcal{D} \to \mathcal{D}'\) be an additive functor between idempotent complete preadditive categories. Then for any pseudocubical object \(X\) in \(\mathcal{D}\), there is a natural isomorphism

\[
F(N(X)) \cong N(F(X))
\]

of chain complexes in \(\mathcal{D}'\).

Proof. Applying the additive functor \(F\) to the diagram

\[
\text{Ker}(\sigma^X) \xrightarrow{\pi_1} C(X) \xrightarrow{\pi_2} \text{Ker}(1 - \sigma^X) = N(X),
\]
we get a digram in \( \mathcal{D}' \)

\[
\begin{array}{c}
F(\text{Ker}(\sigma^X)) \xrightarrow{F(\pi_1)} C(F(X)) \xrightarrow{F(\pi_2)} F(N(X))
\end{array}
\]

whose morphisms satisfy the following identities:

- \( F(\pi_1)F(i_1) = 1, \) \( F(\pi_2)F(i_2) = 1, \)
- \( F(\pi_1)F(i_2) = 0, \) \( F(\pi_2)F(i_1) = 0, \)
- \( F(i_1)F(\pi_1) = 1 - F(\sigma^X), \)
- \( F(i_2)F(\pi_2) = F(\sigma^X). \)

Besides, it follows from the additivity of \( F \) that \( F(\sigma^X) = \sigma^{F(X)} \), and hence we obtain

\[
F(i_2)F(\pi_2) = \sigma^{F(X)}.
\]

This finally implies that

\[
F(N(X)) \cong \text{Ker}(1 - \sigma^{F(X)}) = N(F(X)).
\]

### 5. Cubical derived functors and Eilenberg-Moore derived functors

Let \( \mathcal{C} \) be a category with finite limits, \( \mathcal{P} \) a projective class, \( \mathcal{B} \) an abelian category, and \( F: \mathcal{C} \to \mathcal{B} \) a functor. In this section we prove that for any object \( C \in \mathcal{C} \), there is a natural isomorphism

\[
L^\mathcal{P}_n F(C) \cong L^\mathcal{P}_n F_{ad}(C), \quad n \geq 0.
\]

This together with \( 3.5 \) obviously implies Theorem 1.1.

The proof of this isomorphism is similar to that of \( 3.5 \). However, things become a little bit complicated in the cubical setting as we have to consider normalized chain complexes in order to get the “right” homology.

**Proposition 5.1.** Suppose \( \mathcal{A} \) is a preadditive category, \( \mathcal{P} \) a class of objects in \( \mathcal{A} \), \( \mathcal{B} \) an abelian category, and \( F: \mathcal{A} \to \mathcal{B} \) an additive functor. Suppose further that \( \mathcal{P} \) in the idempotent completion \( \mathcal{A} \), and \( \tilde{F}: \mathcal{A} \to \mathcal{B} \) the extension of \( F \). Then for any \( A \in \mathcal{A} \) which possesses a \( \mathcal{P} \)-resolution, there is a natural isomorphism

\[
L^\mathcal{P}_n F(A) \cong L^n_{\mathcal{P}} \tilde{F}(A), \quad n \geq 0.
\]

**Proof.** Since \( \mathcal{A} \) has a zero object, any \( \mathcal{P} \)-resolution in \( \mathcal{A} \) is a \( \mathcal{P} \)-resolution in \( \mathcal{A} \). The rest follows from \( 2.9 \). \( \square \)

**Corollary 5.2.** Assume that \( \mathcal{C} \) ia a category with finite limits, \( \mathcal{P} \) a projective class in \( \mathcal{C} \), \( \mathcal{B} \) an abelian category, and \( F: \mathcal{C} \to \mathcal{B} \) a functor. Assume further that \( \tilde{F}_{ad}: \mathcal{ZC} \to \mathcal{B} \) is the extension of \( F_{ad}: \mathcal{ZC} \to \mathcal{B} \) to the idempotent completion \( \mathcal{ZC} \), and \( \mathcal{P} \) the closure of \( \mathcal{P} \) in \( \mathcal{ZC} \). Then for any object \( C \in \mathcal{C} \), there is a natural isomorphism

\[
L^n_{\mathcal{P}} F_{ad}(C) \cong L^n \tilde{F}_{ad}(C), \quad n \geq 0.
\]
Next we state the following technical lemma.

**Lemma 5.3.** Let $X \to X_{-1}$ be an augmented pseudocubical set. Suppose that $\partial : X_0 \to X_{-1}$ is surjective and the following conditions hold:

(i) For any $x, y \in X_0$, satisfying $\partial x = \partial y$, there exists $z \in X_1$ such that $\partial^0_1 z = x$ and $\partial^1_1 z = y$.
(ii) For any $n \geq 1$ and any collection of $2n+2$ elements $x_\epsilon^i \in X_n$, $1 \leq i \leq n+1$, $\epsilon \in \{0, 1\}$, satisfying

$$\partial^{\epsilon}_i x^i_\epsilon = \partial^{\epsilon}_{i-1} x^i_\epsilon, \quad 1 \leq i < n+1, \quad \alpha, \epsilon \in \{0, 1\},$$

there exists $x \in X_{n+1}$, such that

$$\partial^{\epsilon}_i x = x^i_\epsilon, \quad 1 \leq i \leq n+1, \quad \epsilon \in \{0, 1\}.$$

Then the augmented normalized chain complex

$$N(\mathbb{Z}[X]) \to \mathbb{Z}[X_{-1}]$$

is chain contractible. In particular, it has trivial homology in each dimension.

We omit the routine details of the proof here. Note only that the main idea is to construct inductively a precubical homotopy equivalence between $X$ and the constant cubical object determined by $X_{-1}$ and then use the homotopy invariance of the functor $N$ [P09, 3.6].

**Example 5.4.** Let $X \to C$ be a $\mathcal{P}$-projective pseudocubical resolution of $C$ and suppose $Q \in \mathcal{P}$. Then the augmented pseudocubical set

$$\text{Hom}_{\mathcal{P}}(Q, X) \to \text{Hom}_{\mathcal{P}}(Q, C)$$

satisfies the conditions of 5.3. In particular, the homologies of the augmented chain complex

$$N(\mathbb{Z}[\text{Hom}_{\mathcal{P}}(Q, X)]) \to \mathbb{Z}[\text{Hom}_{\mathcal{P}}(Q, C)]$$

vanish.

We are now ready to prove the main result of this section.

**Proposition 5.5.** Let $\mathcal{C}$ be a category with finite limits, $\mathcal{P}$ a projective class in $\mathcal{C}$, $\mathcal{B}$ an abelian category, and $F : \mathcal{C} \to \mathcal{B}$ a functor. Then:

(i) For any $\mathcal{P}$-projective pseudocubical resolution $X \to C$, the augmented chain complex

$$N(X) \to C$$

in the category $\mathcal{Z}C$ is a $\overline{\mathcal{P}}$-resolution of $C$ in the sense of 2.4 ($\overline{\mathcal{P}}$ is the closure of $\mathcal{P}$ in $\mathcal{Z}C$).

(ii) For any $C \in \mathcal{C}$, there is a natural isomorphism

$$\mathbf{L}_n^P F(C) \cong \mathbf{L}_n^{\overline{\mathcal{P}}} F_{\text{ad}}(C), \quad n \geq 0.$$

**Proof.** For all $n \geq 0$, $N(X)_n \in \overline{\mathcal{P}}$ since $N(X)_n$ is a retract of $X_n$ and $\overline{\mathcal{P}}$ is closed under retracts. Further, by 4.8 one has a natural isomorphism of augmented chain complexes

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{Z}C}(Q, N(X)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{Z}C}(Q, C) \\
\downarrow \cong & & \downarrow \text{id} \\
N(\mathbb{Z}[\text{Hom}_{\mathcal{P}}(Q, X)]) & \to & \mathbb{Z}[\text{Hom}_{\mathcal{P}}(Q, C)]
\end{array}$$
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for any $Q \in \mathcal{P}$. It follows from [5.4] that the lower chain complex is acyclic and thus so is the upper one. Consequently, the augmented chain complex $N(X) \rightarrow C$ in $\mathbb{Z} \mathcal{C}$ is $\mathcal{P}$-acyclic or, equivalently, $\mathcal{P}^{\infty}$-acyclic. This completes the proof of the first claim.

Let us prove the second claim. By [5.2] it suffices to get a natural isomorphism

$$L_n^\mathcal{P} F(C) \cong L_n \overline{F}_{ad}(C).$$

Choose any $\mathcal{P}$-projective pseudocubical resolution $X \rightarrow C$. The first claim together with [4.8] gives

$$L_n^\mathcal{P} F(C) = H_n(N(F(X))) = H_n(\overline{F}_{ad}(X)) \cong H_n(\overline{F}_{ad}(N(X))) = L_n \overline{F}_{ad}(C).$$

Clearly, Theorem 1.1 is an immediate consequence of [5.5] and [5.5].

6. Connection with topology

In this section we briefly explain that Theorem 1.1 generalizes the well-known fact that the cubical and simplicial singular homologies of a topological space are naturally isomorphic. For the definition and basic properties of the cubical singular homology see [M80].

Let $\text{Top}$ denote the category of topological spaces, and let $\Delta^n, n \geq 0$, be the standard $n$-simplex. The class $\mathcal{P}_{\Delta}$ of all possible disjoint unions of standard simplices is a projective class in $\text{Top}$ in the sense of [TV69, §2]. (Moreover, in fact, it comes from a cotriple [BB69, (10.2)].) Indeed, for any space $Y$, the map

$$\bigsqcup_{\Delta^n \rightarrow Y} \Delta^n \rightarrow Y,$$

where the disjoint union is taken over all possible continuous maps $\Delta^n \rightarrow Y, n \geq 0$, is a $\mathcal{P}_{\Delta}$-epimorphism. Consider the functor

$$F: \text{Top} \rightarrow \text{Ab}, \quad F(Y) = H^n_{\mathcal{P}_{\Delta}}(Y, A) = \mathbb{Z}[\pi_0 Y] \otimes A,$$

where $\text{Ab}$ is the category of abelian groups, $H^n_{\mathcal{P}_{\Delta}}(Y, A)$ the simplicial singular homology of $Y$ with coefficients in an abelian group $A$, and $\pi_0 Y$ the set of path components of $Y$. It follows from [BB69, (10.2)] and [TV69, (3.1) Theorem] that there is a natural isomorphism

$$L_n^\Delta F(Y) \cong H_n^{\Delta^n}(Y, A), \quad n \geq 0,$$

where the simplicial derived functors are taken with respect to the projective class $\mathcal{P}_{\Delta}$ (cf. [R69], [R72]). We sketch the proof of this natural isomorphism along the lines of [BB69, (10.2)]. The standard cosimplicial object $\Delta^\bullet$ gives rise to an augmented simplicial functor

$$F: \Delta^\bullet \rightarrow F, \quad F_n(Y) = \mathbb{Z}[\text{Hom}_{\text{Top}}(\Delta^n, Y)] \otimes A.$$

Further, suppose $S^\bullet \rightarrow Y$ is a $\mathcal{P}_{\Delta}$-projective presimplicial resolution of $Y$. Evaluating $F$ on $S^\bullet$ yields a bipresimplicial abelian group. It is easily seen that both resulting spectral sequences collapse at $E^2$. Finally, playing these two spectral sequences against each other gives the desired isomorphism.
Similarly, one can describe the cubical singular homologies $H^n_{\Box}(Y, A)$ as cubical derived functors of the functor $F(Y) = \mathbb{Z}[\pi_0 Y] \otimes A$. For this one uses the class $\mathcal{P}_{\Box}$ consisting of all possible disjoint unions of standard cubes. The class $\mathcal{P}_{\Box}$ is a projective class in $\text{Top}$ and there is a natural isomorphism

$$L^n_{\Box}F(Y) \cong H^n_{\Box}(Y, A), \quad n \geq 0,$$

where the cubical derived functors are taken with respect to $\mathcal{P}_{\Box}$. The proof of this isomorphism is technically a little bit complicated compared to its simplicial counterpart as one has to consider spectral sequences of bipseudocubical objects and take care of the normalizations.

Note that the class $\mathcal{P} = \mathcal{P}_{\Delta} \cup \mathcal{P}_{\Box}$ is also a projective class in $\text{Top}$. Obviously, the simplicial derived functors with respect to the class $\mathcal{P}_{\Delta}$ are naturally isomorphic to the simplicial derived functors with respect to $\mathcal{P}$. On the other hand, the cubical derived functors with respect to the class $\mathcal{P}_{\Box}$ are naturally isomorphic to the cubical derived functors with respect to $\mathcal{P}$. Thus, by 1.1 there is a natural isomorphism

$$L^n_{\Delta}F(Y) \cong L^n_{\Box}F(Y), \quad n \geq 0,$$

for any topological space $Y$, i.e.,

$$H^n_{\Delta}(Y, A) \cong H^n_{\Box}(Y, A), \quad n \geq 0.$$

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