Enhancement of intermittency in superfluid turbulence

Laurent Boué, Victor L’vov, Anna Pomyalov, Itamar Procaccia
Department of Chemical Physics, Weizmann Institute of Science, Rehovot 76100, Israel
(Dated: May 2, 2014)

We consider the intermittent behavior of superfluid turbulence in $^4$He. Due to the similarity in the nonlinear structure of the two-fluid model of superfluidity and the Euler and Navier-Stokes equations one expects the scaling exponents of the structure functions to be the same as in classical turbulence for temperatures close to the superfluid transition $T_s$ and also for $T \ll T_s$. This is not the case when mutual friction becomes important. Using shell model simulations, we propose that for an intermediate regime of temperatures, such that the density of normal and superfluid components are comparable to each other, there exists a range of scales in which the effective exponents indicate stronger intermittency. We offer a bridge relation between these effective and the classical scaling exponents. Since this effect occurs at accessible temperatures and Reynolds numbers, we propose that experiments should be conducted to further assess the validity and implications of this prediction.

Introduction. Non-equilibrium systems are often characterized by relatively quiescent periods which are interrupted by rare but violent changes of the physical observables. This phenomenon which is referred to as “intermittency” is very generic as it appears in many different stationary random processes.

It is believed [1], that highly excited hydrodynamic turbulence possesses universal statistics in the inertial interval of scales $L \gg r \gg \eta$. Here $L$ is the of energy input scale, e.g. due to instabilities of high velocity flows, and $\eta$ is the viscous energy dissipation scale. The intermittent behavior of space homogeneous fully developed turbulence of incompressible fluids can be studied in terms of the velocity structure functions $S_p(r)$

$$S_p(r) \equiv \langle |v(R, t) - v(R + r, t)|^p \rangle \propto r^{\xi_p}. \tag{1}$$

Here $(..)$ denotes a time averaging. In the inertial interval, the functions $S_p(r)$ are scale invariant and they are characterized by the $p$-order scaling exponents $\xi_p$.

Accepting Richardson’s idea of a step-by-step energy cascade in the inertial interval and assuming that the energy flux over the scales $\varepsilon$ is the only relevant parameter, Kolmogorov proposed from dimensional reasoning that $S_p^{K41}(r) \simeq (\varepsilon r)^{p/3}$ i.e. $\xi_p \equiv \xi_p^{K41} = p/3$. In particular, this means that $S_p(r)/|S_2(r)|^{p/2}$ should be $r$-independent, like in Gaussian statistics. However this scenario is not borne out by experiments [1]: the observed values $\xi_p^{exp}$ for $p > 3$ deviate down from $p/3$ and there atios $S_p(r)/|S_2(r)|^{p/2}$ increase with $r$. Therefore the probability of observing large values of small scale velocity fluctuations (that mainly contribute to $S_p(r)$ with large $p$) significantly exceeds its normal K41 level, i.e. the small scale turbulent flow is strongly intermittent. In spite of fifty years efforts intermittency still requires deeper understanding. In particular there is no analytical framework to compute $\xi_p(p)$.

Turbulence in superfluid $^4$He and $^4$He has emerged as a hot topic of interest merging the traditional fluid mechanics with the low-temperature physics. Unlike classical fluids, where the circulation around the vortices is a dynamical variable, in superfluids it is quantized to integer values of $\kappa = \pi h/\rho_\lambda$. Here $h$ is the Plank’s constant and $\rho_\lambda$ is the mass of a superfluid particle. This leads to the appearance of an additional length scale $\ell$, the mean inter-vortex distance. For motions with scales $r \gg \ell$ superfluids can be described as two inter-penetrating entities with two different velocities: $\nu_s$ (inviscid superfluid component) and $\nu_n$ (normal component), and temperature dependent densities $\rho_s(T)$ and $\rho_n(T) = \rho - \rho_s(T)$ interacting via the so-called mutual friction force [2].

One of the most interesting questions concerns the similarities and differences between superfluid and classical turbulence. The pioneering experiments of Mau- rer and Tabeling suggest that intermittency would also be present in turbulent superfluid $^4$He [3]. This crucial observation implies that the existence of intermittency is not necessarily related to the exact structure of the Navier-Stokes equations and thus superfluid turbulence offers an additional way to investigate its physical mechanisms. Unfortunately, intermittency in superfluids has received far less attention than in classical turbulence [4].

The aim of this Letter is to compare intermittency in $^4$He-superfluid turbulence with turbulence in normal fluids. An important parameter characterizing superfluid turbulence is the ratio $\rho_s/\rho_n$. For $T$ slightly below the phase transition temperature $T_s$, when $\rho_s/\rho_n \ll 1$, one can neglect the presence of the superfluid component and the statistics of turbulent superfluid $^4$He is expected to be close to that of classical fluids. We also expect similar inertial range behavior of classical and superfluid turbulence for $T \ll T_s$, when $\rho_n \ll \rho_s$, due to the inconsequential role played by the normal component, see, e.g. [4]. Moreover (and less trivially) the intermittent scaling exponents $\xi_p$ have to be the same in classical and low-temperature superfluid turbulence. This is because the nonlinear structure of the equation for the superfluid component is the same as the Euler equation, and dissipative mechanisms are believed to be irrelevant.
Based on the above reasoning, we focus in on the range of $T \simeq (0.9 \div 0.8) T_L$, where $\rho_s \sim \rho_n$ and compare it to the high and low $T$ limits, when $\rho_s/\rho = 0.1$ and 0.9, respectively. Our main tool is numerical simulations of superfluid turbulence in the two-fluid model [2]. We utilize a shell-model approximation, which is known to provide accurate scaling exponents in the classical case when the parameters are cleverly chosen (see, e.g. [5-7]). In particular, shell models with the right parameters exhibit anomalous scaling exponents $\xi_{\text{num}}$ that agree well with experimental values $\xi_{\text{exp}}$: $\xi_{2}\num = 0.72$ vs. $\xi_{2}\exp = 0.70$; $\xi_{4}\num = 1.26$ vs. $\xi_{4}\exp = 1.28$, etc.

The main result of the Letter is as follows: For temperatures such that $\rho_s/\rho \lesssim 0.1$ and $\rho_s/\rho \gtrsim 0.9$, the scaling behavior of superfluid turbulence is practically the same as in classical fluids. However, for temperatures such that $\rho_s/\rho \simeq 0.5 \div 0.75$, we discovered the emergence of a wide (up to 3 decades) interval of scales characterized by substantially different effective scaling exponents $\tilde{\xi} \neq \xi_{\text{num}}$. For example, when $\rho_s \approx \rho_n$ the value of $\xi_2$ diminishes from its intermittent value $\xi_{2}\num \approx 0.72$ down to $\xi_2 \approx 0.67$, close to the K41 value $\xi_{2}\K41 = 2/3$. At the same time, the exponents $\tilde{\xi}_p$ for $p > 3$ deviate further from the K41 values: $\tilde{\xi}_p < \xi_{p}\num < \xi_{p}\K41$. This fact can be considered as enhancement of intermittency in superfluid turbulence compared to the classical case.

In addition, based on numerical observations that the probability distribution function (PDF) of the normalized shell energy is temperature independent, we suggest the relation between effective and classical exponents:

$$\tilde{\xi}_p = \xi_{p}\num - \frac{p}{2} (\xi_{2}\num - \xi_2) \approx \xi_{p}\num - 0.026p,$$  \hspace{1cm} (2)

which is in good agreement with observed values of $\tilde{\xi}_p$.

**Numerical procedure.** Shell models of superfluid turbulence [8, 9] are simplified caricatures of the Navier-Stokes and Euler equations in $k$-representation:

$$\left[ \frac{d}{dt} + \nu_k k^2 \right] u^n_k = \text{NL}[u^n_k] + F^n_k + F^n_{\text{m}},$$  \hspace{1cm} (3a)

$$\left[ \frac{d}{dt} + \nu_k k^2 \right] u^s_k = \text{NL}[u^s_k] - F^s_k + F^s_{\text{m}},$$  \hspace{1cm} (3b)

$$\text{NL}[u_m] = i \left( \alpha k_{m+1} u_{m+2} u_{m+1} + \frac{c k_{m-1} u_{m-2} u_{m-1}}{k_m} \right),$$  \hspace{1cm} (3c)

$$F^s_{m} = \alpha_n \Omega_s (u^s_{m} - u^n_{m}), \hspace{1cm} \Omega^s_{m} = \sum_m k_{m}^2 |u^s_{m}|^2.$$  \hspace{1cm} (3d)

They mimic the statistical behavior of $k$-Fourier components of the turbulent superfluid and normal velocity fields in the entire shell of wave vectors $k_m < k < k_{m+1}$ by complex shell velocity $u^s_{m}$. The shell wave numbers are chosen as a geometric progression $k_m = k_0 \lambda^m$, where $m = 1, 2, \ldots, M$ are the shell indexes, and we have used the shell-spacing parameter $\lambda = 2$ and $k_0 = 1/16$. In order to resolve a large range of scales we used a total of $M = 36$ shells. This allows us to clearly resolve subregions with different scaling behavior.

Similarly to the Navier Stokes equation, the NL term in Eq. (3c) is quadratic in velocities, proportional to $k$ and conserves (in the force-less, inviscid limit) the kinetic energy $E = \frac{1}{2} \sum_m |u^s_{m}|^2$, provided that $a + b + c = 0$. We used the Sabra version [6] of NL with the traditional (and physically motivated choice) $b = c = -a/2$, which ensures [5, 6] that $\xi_{p}\num$ are close to $\xi_{p}\exp$.

In Eq. (3a), $\nu_n$ is the kinematic viscosity of normal component. It is known that vortex reconnections act as an energy sink in superfluids, which we accounted for (admittedly in a rough manner) by adding in Eq. (3b) an effective superfluid viscosity $\nu_s \ll \nu_n$. The details of this complicated and otherwise still not fully understood dissipative process go beyond the scope of this Letter. In simulations we used $\nu_s = 10^{-12}$ and $\nu_n = 10^{-10}$.

The coupling terms $F^s_{m}$ were applied only for $m = 1, 2$. Eqs (3) were solved using 4-th order Runge-Kutta method. The chosen values of $\rho_s/\rho$ and corresponding values of $T/T_L$ and $\alpha$ (interpolated using the experimental data in [10]) are shown below:

| $\rho_s/\rho$ | T/T_L | $\alpha$ | $\alpha_s/\rho_s$ |
|-----------|------|--------|-----------------|
| 0.1 | 0.25 | 0.4 | 0.5 | 0.6 | 0.65 | 0.7 | 0.75 | 0.9 |
| 0.99 | 0.97 | 0.93 | 0.90 | 0.87 | 0.84 | 0.82 | 0.79 | 0.677 |
| $\tilde{\xi}_2$ | 0.1 | 0.51 | 0.314 | 0.25 | 0.22 | 0.175 | 0.155 | 0.14 | 0.07 |
| $\alpha_s/\rho_s$ | 1.21 | 0.68 | 0.52 | 0.49 | 0.50 | 0.52 | 0.54 | 0.67 |

**Results and discussions.** First of all, we study the scaling behavior of the velocity structure functions (1), which in the shell models can be written as

$$S^s_m(n)(m) \equiv \langle |u^s_{m+1} - u^s_{m}|^n \rangle, \hspace{1cm} C_p(k_m) \equiv k_{m}\num S_p(k_m).$$  \hspace{1cm} (4)

We defined for convenience the structure functions $C_p(k_m)$ compensated by the classical anomalous scaling.

Figure 1A shows $C_2(k_m)$. The almost horizontal lines for $\rho_s/\rho = 0.9$ and 0.1, indicate the usual intermittent scaling behavior. However for $\rho_s/\rho = 0.5$ one detects the emergence of a wide enhancement interval with the energy accumulations and where the scaling exponent is close to $\tilde{\xi}_2 \simeq 2/3$ [11]. This is clearly seen in the inset, where $S_2(k_m)$ is compensated with K41 scaling. At first glance, the scaling with $\tilde{\xi}_2 \simeq \xi_{2}\K41 = 2/3$ can be considered as the normal K41 scaling in superfluids. In this case one would expect that the PDF of normalized energy in $m$-shell $P^s_m \equiv \langle |u^s_{m+1/2}/S^s_m(k_m)| \rangle$ should collapse for all $m$. This is, however, NOT the case here. Figure 1B shows that the PDFs for different $m$ are in fact very different from each other. The much longer tails associated
with larger values of $k_m$ reveal strong intermittency effects. This is clearly seen in Fig. 1C where $C_{4}^n(k_m)$ is almost flat for $\rho_s/\rho = 0.9$ and 0.1, but show a different scaling for $\rho_s/\rho = 0.5$ with an effective exponent $\xi_4 \approx 1.21 < \xi_4^{num} = 1.256 < \xi_4^{K41} = 4/3$. The next order structure functions, $S_0(k)$, $S_2(k)$, ... demonstrate the same type of behavior where the effective exponents $\xi_p$ deviate even further away from the K41 values than the intermittency exponents $\xi_p^{num}$ in classical fluids.

Another important observation in Fig. 1B is that the PDFs are practically temperature independent. Then $\rho_s/\rho$ regions with corresponding wavenumbers. Panel B: Demonstration of the temperature-independence of the PDF's for shells $m = 14$ and $m = 20$. Panel C: The normalized compensated $C_m^{n,a}(k_m)$. Inset: The collapse of $k^2 S_4(k)/[S_2(k)]^2$, compensated using classical exponents $\delta \equiv 2\xi^{num} - \xi^{num} \approx 0.184$. The horizontal black dashed lines serve to guide the eye only. Color code and line labeling is the same in all panels. Vertical dot-dashed lines in Panels A and C denote the intermittence interval and its extent as $k^{256} \approx 10^{-1}$.

To quantify the intermittency enhancement we display in Fig. 2 the $T$- and $\rho_s/\rho$-dependence of $\xi_2$ together with the dimensionless magnitude $\Psi_2$ of this effect, defined as $\Psi_2 \equiv \log_2(\max_m C_2(k_m)/(C_2(k_m)))$. Here $\langle C_2(k_m) \rangle$ is the mean value of $C_2(k_m)$ on the plateau. Denote by $m_\pm$ the left and right edges of the enhancement interval and its extent as $k_{m_+}/k_{m_-} = 2^{\Delta m}$, $\Delta m \equiv (m_+ - m_-)$. Then, using data in Fig. 2 and computing the ratios $\Psi_2(T)/[\xi^{num}_2 - \xi_2(T)] \equiv \Delta m$ for all $T$, we found that $\Delta m \approx 8 \times 10$ practically independent of $T$. In other words the extent of the enhancement interval $k_{m_+}/k_{m_-} = 2^{8} \approx 250 \div 1000$.

As expected, significant intermittency enhancement takes place for $\rho_s \sim \rho_n$ [or more precisely $\rho_s \simeq (0.4 \div 0.8)\rho$ - see Fig. 2] when the two-fluid model (3) differs mostly from a single classical fluid. Normal and superfluid components in Eqs. (3) are coupled by the mutual friction force $F_m \propto \alpha (u_m - u_m^s)$ which causes the energy exchange between normal and superfluid components and the energy dissipation. Thus these two phenomena are the only possible reasons for the observed intermittency enhancement and both disappear in case of full velocity locking $u_m^s(t) = u_m(t)$. Velocity locking can be statistically quantified by the cross-correlation coefficient of normal and superfluid velocities: $k_m \equiv \langle \{u_m u_m^s + u_m^s u_m\}\rangle /\langle \{u_m u_m + u_m^s u_m\}\rangle$. For $K(k_m) = 0$, the superfluid and normal components are statistically independent subsystems, while for $K(k_m) = 1$ they are completely locked to each other. Clearly only for $0 < K(k_m) < 1$ we can expect intermittency enhancement. A result of [13] for the correlation coefficients can be reformulated for shell models as: $K_m \equiv 2[\alpha_0 S_{2,m} + \alpha S_{2,m}^2]/[S_{2,m}^2 + S_{2,m}]$.

Here $S_{2,m}^2 \equiv S_2^2(k_m)$. Analysis of this equation shows that the $v_m^s$ and $v_m^s$ are locked when $k_m \eta \ll 1$ and be...
come uncorrelated when $k_m \simeq (\alpha \rho/\rho_n)^{3/2}/\eta$. As one sees in the Table, for all $T$ in our simulation $\alpha \rho/\rho_n \simeq 0.5 \div 1.2$. Therefore one should expect velocity locking for all $T$ and for all $k_m \eta \lesssim 1$ including the enhancement range. This prediction agrees with our results.

This means that the origin for the variations of the scaling is not in the enhancement range but rather in the vicinity of the viscous cutoff, where $k_s \simeq k_T$. This conclusion is strongly supported by our simulations with different $T$, $M = 28 \div 46$ and $\nu_m/\nu_s \simeq 10^2 \div 10^6$, in all of which the extent of the enhancement interval remains unchanged, $\Delta m \simeq 8 \div 10$, and always $k_s \simeq k_T$.

One concludes that a key role should be played by the energy dissipation due to the mutual friction $D^{\text{MF}}_m = \alpha \rho \Omega |u^m_\nu - u^s_\nu|^2$, as follows from Eqs. (3). Clearly, $D^{\text{MF}}_m \rightarrow 0$ for $T \rightarrow T_\Lambda$ (because $\rho_s \rightarrow 0$) and for $T \rightarrow 0$ (because $\alpha \rightarrow 0$) and has a maximum at $T \simeq 0.80 \div 0.85$, i.e. in the range, where the intermittency enhancement is observed. Moreover, in this range, the total mutual friction dissipation $\sum_m D^{\text{MF}}_m$ is close to (or even larger than) the total viscous dissipation $\nu_s \sum_m k^2_s \omega^2_\nu$. Importantly, $D^{\text{MF}}_m$ has a maximum at $k_m$, close to that of the viscous dissipation. As a result, for $T \simeq (0.80 \div 0.85)/T_\Lambda$, the mutual friction between superfluid and normal components markedly drives the system away from the simple one-fluid behavior primarily around $k \sim k_s$, leading to a sharper decay of the energy spectrum $s^2 m^{-2}$. In turn, this may give rise to a bottleneck effect leading to an energy accumulation near $k_s$. This phenomenon shares similarities with the bottleneck in classical fluids [14] but differs in important aspects: e.g. in classical fluids the width of the energy accumulation range is almost independent of underlying mechanism and only its magnitude varies. In our case, however, the extent of the enhancement interval is much larger, about 3 decades, compared to one decade in classical fluids. A possible reason for such an extended enhancement interval is the energy transfer from superfluid to normal component, caused by mutual friction, and it is also notable in the same $T$ range.

A comprehensive study into the quantitative aspects of this enhancement of intermittency is beyond the scope of the Letter. It will require an extensive analysis of the time dependencies of $u^s_\nu(t)$ and $u^m_\nu(t)$ in order to clarify the role of the mutual friction on the statistics of outliers; the main contributors to the fat tails of the PDFs which are responsible for the intermittency [15].

**Summary.** Our shell-model simulations of the large-scale superfluid turbulence in $^4$He indicate that for high-and low-$T$-limits, when $\rho_s < 0.1 \rho$ and $\rho_s > 0.9 \rho$ anomalous scaling exponents of superfluid turbulence are practically the same as their classical counterparts, as follows from our theoretical reasoning and agrees with numerical and laboratory observations [4]. The shell-model simulations have allowed us to uncover considerable $T$-dependent variations of the apparent scaling over the enhancement interval of scales about 3 decades when $\rho_s \simeq \rho_n$. This scaling can be characterized by the effective scaling exponents $\xi_\nu(T)$ that deviate further from the normal K41 values as compared to the exponents of classical fluids. Furthermore if the entire inertial interval is shorter than the enhancement interval, only the effective scaling is observed. We also suggested Eq. (2) relating $\xi_\nu(T)$ to $\xi_\nu$.

While our interpretation indicates the existence of a form of the energy bottleneck in the two-fluid system, the phenomenon definitely calls for further investigation. DNS of superfluid turbulence oriented towards the study of the predicted $T$-dependent scaling over an interval of about three decades is nowadays realistic and appears as particularly timely. Even more importantly, the temperature range $T \simeq (1.7 \div 1.9) K$, where we forecast a change in the scaling behavior is accessible to laboratory experiments. We hope that the results presented in this Letter will inspire new numerical and laboratory studies of intermittency in superfluid turbulence.

This work is supported by the EU FP7 Microkelvin program (Project No. 228464) and by the U.S. Israel Binational Science Foundation.

[1] U. Frisch, Turbulence, the legacy of A.N. Kolomogorov, Cambridge Univ. Press, 1995.
[2] L. Landau, Phys. Rev. 60 356(1941).
[3] J. Maurer and P. Tabeling, Europhys. Lett. 43, 29 (1998).
[4] J. Salort, B. Chabaud, E.Lévéque and P.-E. Roche, J. Phys.: Conf. Ser. 318 042014 (2011); Europhys. Lett. 97 34006 (2012).
[5] B. Gledzer, Dokl. Akad. Nauk SSSR 209, 1046 (1973) [Soviet Physics Dokl. 18, 216 (1973)]; Yamada and K. Okhitani, J. Phys. Soc. Jpn. 56, 4210 (1987).
[6] V. S. L’vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, Phys. Rev. E 58, 1811 (1998).
[7] Pissarenko, L. Biferale, D. Courvoisier, U. Frisch, and M. Vergassola, Phys. Fluids A 5, 2533 (1993).
[8] D. H. Wacks and C. F. Barenghi, Phys. Rev. B 84, 184505 (2011).
[9] L. Boué, V. L’vo, A. Pomyalov, and I. Procaccia, Phys. Rev. B 85 104502 (2012).
[10] R. J. Donnelly and C. F. Barenghi, J. Phys. Chem. Ref. Data 27, 6, (1998).
[11] Notice that we observe the energy accumulation for scales, larger then intervortex distance $\ell$, while in [12] it happens at smaller scales.
[12] A. W. Baggaley and C. F. Barenghi, Phys. Rev. E 84, 067301 (2011).
[13] V.S. L’vo, S.V. Nazarenko and L. Skrbek, J. Low Temp. Phys. 145, 125 (2006).
[14] N.E. L. Haugen and A. Brandenburg, Phys. Rev. E 70, 026405 (2004); U. Frisch, S. Kurien, R. Pandit, W. Pauls, S.S. Ray, A. Wirth, J-Z. Zhu, Phys. Rev. Lett. 101, 145401 (2008).
[15] V. S. L’vo, A. Pomyalov and I. Procaccia, Phys. Rev. E 63, 056118 (2001).