SOME TYPES OF WEAKLY RICCI SYMMETRIC RIEMANNIAN MANIFOLDS
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Abstract
In this paper we discuss when a quasi-conformally flat weakly Ricci symmetric manifold (of dimension greater than 3) becomes a manifold of hyper quasi-constant curvature, a quasi-Einstein manifold and a manifold of quasi-constant curvature. Also we discuss when a pseudo projectively flat weakly Ricci symmetric manifold (of dimension greater than 3) becomes pseudo quasi-constant curvature and a quasi-Einstein manifold and when a $W_2$-flat weakly Ricci symmetric manifold (of dimension greater than 3) becomes a quasi-Einstein manifold.

Keywords: Weakly Ricci symmetric(WRS) manifold, quasi-conformal curvature tensor, hyper quasi-constant curvature tensor, pseudo projective curvature tensor, pseudo quasi-constant curvature, $W_2$-curvature tensor.

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1 Introduction

L.Tamassy and T.Q.Binh[11] introduced weakly symmetric Riemannian manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if the curvature tensor $\bar{R}$ of type $(0,4)$ satisfies the condition

$$\nabla_X \bar{R}(Y, Z, U, V) = A(X)\bar{R}(Y, Z, U, V) + B(Y)\bar{R}(X, Z, U, V) + C(Z)\bar{R}(Y, X, U, V) + D(U)\bar{R}(Y, Z, X, U),$$

(1.1)

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, C, D$ and $E$ are 1-forms (non-zero simultaneously) and $\nabla$ is the operator of covariant differentiation with respect to $g$. The 1-forms are called the associated 1-forms of the manifold and an $n$-dimensional manifold of this kind is denoted by $(WS)_n$. If the associated 1-forms satisfy $B = C$ and $D = E$, the defining condition of a $(WS)_n$ reduces to the following form[4]

$$\nabla_X \bar{R}(Y, Z, U, V) = A(X)\bar{R}(Y, Z, U, V) + B(Y)\bar{R}(X, Z, U, V) + B(Z)\bar{R}(Y, X, U, V) + D(U)\bar{R}(Y, Z, X, U),$$

(1.2)

Let $\{e_i\}$ $i = 1, 2, ..., n$ be an orthonormal basis of the tangent spaces in a neighbourhood of a point of the manifold. Then setting $Y = V = e_i$ in (1.7) and taking summation over $i$, $1 \leq i \leq n$, we get

$$\nabla_X S(Z, U) = A(X)S(Z, U) + B(Z)S(X, U) + D(U)S(X, Z),$$

(1.3)

where $S$ is the Ricci tensor of type $(0,2)$.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci-symmetric if the Ricci tensor $S$ of type $(0,2)$ satisfies the condition

$$\nabla_X S(Z, U) = A(X)S(Z, U) + B(Z)S(X, U) + D(U)S(X, Z).$$

(1.4)

From (1.3) it follows that a $(WS)_n(n > 2)$ is weakly Ricci symmetric (briefly $(WRS)_n(n > 2)$) if [12]

$$B(R(X, Z)U) + D(R(X, U)Z) = 0, \forall X, U, Z \in \chi(M^n).$$

(1.5)
Putting $Z = U = e_i$ in (1.4) we get, 
\[
dr(X) = rA(X) + B(QX) + D(QX),
\]
where $r$ is the scalar curvature of the manifold.

If a $(WRS)_n (n > 2)$ is of zero scalar curvature, then from (1.6),
\[
B(QX) + D(QX) = 0, \quad \forall X.
\]

From (1.6) we have if a $(WRS)_n (n > 2)$ is of non-zero constant scalar curvature, then the 1-form $A$ can be expressed as
\[
A(X) = -\frac{1}{r}[B(QX) + D(QX)], \quad \forall X.
\]

From (1.4) we obtain,
\[
T(QX) = rT(X), \quad \forall X.
\]
where the vector field $\rho$ is defined by
\[
T(X) = g(X, \rho) = B(X) - D(X), \quad \forall X.
\]

Also from (1.6) we obtain,
\[
T(Z)S(X, U) - T(U)S(X, Z) = 0
\]
\[
\forall \text{vector fields } X, Z, U \text{ and } T \text{ is a 1-form.}
\]

B.Das and A.Bhattacharyya studied some types of weakly symmetric Riemannian manifolds in 2011[3]. In 2000, U.C.De et al. discussed about weakly symmetric and weakly Ricci symmetric K-contact manifolds[5] and in 2012, A.A.Shaikh and H.Kundu discussed about weakly symmetric and weakly Ricci symmetric warped product manifolds[10]. Various types of works on weakly Ricci symmetric manifolds is done by S.Jana, A.Shaikh[7] and U.C.De and G.C.Ghosh[6]. Motivated from their work we have established some new results on weakly Ricci symmetric manifolds. In this paper we study quasi-conformally flat, pseudo projectively flat and $W_2$-flat weakly Ricci symmetric manifolds. Here we prove a quasi-conformally flat $(WRS)_n (n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature and this manifold of non-vanishing scalar curvature is a quasi-Einstein manifold and a manifold of quasi-constant curvature with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0 \quad \forall X$, where $B, D$ are 1-forms(non-zero simultaneously). Also we obtain that a pseudo projectively flat $(WRS)_n (n > 3)$ with non-zero constant scalar curvature is a manifold of pseudo quasi-constant curvature and with non-vanishing scalar curvature is a quasi-Einstein manifold with respect to $T$. In the last section we prove that a $W_2$-flat $(WRS)_n (n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to $T$.

A quasi-conformal curvature tensor $C^*$ is defined by[14]
\[
C^*(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\]
\[
- \frac{a}{2}[\gamma - 1 + 2b][g(Y, Z)X - g(X, Z)Y],
\]
where $a, b$ are constants, $g$ is the Riemannian metric and $R, Q, \gamma$ are the Riemannian curvature tensor of type $(1,3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature respectively.

Chen and Yano[2] introduced a Riemannian manifold $(M^n, g)(n > 3)$ of quasi-constant curvature which is conformally flat and its curvature tensor $\bar{R}$ of type $(0,4)$ is of the form
\[
\bar{R}(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W)
\]
\[
+ g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)],
\]
where $a, b$ are non-zero scalars.

A Riemannian manifold $(M^n, g)(n > 3)$ is said to be of hyper quasi-constant curvature if it is conformally flat and its curvature tensor $\bar{R}$ of type $(0,4)$ satisfies the condition[9]
\[
\bar{R}(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + g(X, W)P(Y, Z) - g(X, Z)P(Y, W)
\]
\[
+ g(Y, Z)P(X, W) - g(Y, W)P(X, Z),
\]
where \(a\) is a non-zero scalar and \(P\) is a tensor of type \((0,2)\).

A pseudo projective curvature tensor \(P\) is defined by\[^{13}\]
\[
P(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{a}{n-1} + b[g(Y, Z)X - g(X, Z)Y],
\](1.15)
where \(a, b\) are constants such that \(a, b \neq 0\); \(R, S, r\) are the Riemannian curvature tensor, the Ricci tensor and scalar curvature respectively.

A Riemannian manifold \((M^n, g)(n > 3)\) is said to be of pseudo quasi-constant curvature if it is pseudo projectively flat and its curvature tensor \(\hat{R}\) of type \((0,4)\) satisfies\[^{3}\]
\[
\hat{R}(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + P(Y, Z)g(X, W) - P(X, Z)g(Y, W),
\](1.16)
where \(a\) is a constant and \(P\) is a tensor of type \((0,2)\).

Pokhariyal and Mishra have defined the \(W_2\)-curvature tensor on a differential manifold of dimension \(n\) is given by\[^{8}\]
\[
W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX].
\](1.17)

All these notions will be required in the next sections.

## 2 Quasi-conformally flat \((WRS)_n(n > 3)\)

In this section we prove that a quasi-conformally flat weakly Ricci symmetric manifold \((WRS)_n(n > 3)\) of non-zero constant scalar curvature is a manifold of hyper quasi constant curvature and this manifold of non-vanishing scalar curvature is a quasi-Einstein manifold and a manifold of quasi-constant curvature with respect to the 1-form defined by \(T(X) = B(X) - D(X) \neq 0\ \forall X\), where \(B, D\) are 1-forms (non-zero simultaneously).

Let \((M^n, g)(n > 3)\) be a quasi-conformally flat \((WRS)_n\). Then from (1.12) we obtain,
\[
\hat{R}(X, Y, Z, U) = \frac{a}{2}S(Y, Z)g(X, U) - S(X, Z)g(Y, U) + g(Y, Z)S(X, U) - g(X, Z)S(Y, U)
+ \frac{a}{n-1} \left[\frac{1}{n} + \frac{2a}{n}\right][g(Y, Z)g(X, U) - g(X, Z)g(Y, U)],
\](2.1)
where \(g(\hat{R}(X, Y)Z, U) = \hat{R}(X, Y)Z, U\).

Putting \(X = U = e_i\) in (2.1) where \(\{e_i\}\) is an orthonormal basis of the tangent space at each point of the manifold and taking the summation over \(i\), where \(1 \leq i \leq n\), we get
\[
S(Y, Z) = \alpha g(Y, Z),
\](2.2)
where \(\alpha = 2\frac{\frac{2}{n-2}}{n-1} \left[\frac{1}{n} + \frac{1}{n} \left(1 + \frac{2(n-1)}{n}\right)\right]\).

From (2.2) we get
\[
(\nabla_X S)(Y, Z) = \alpha' d\gamma(X)g(Y, Z),
\](2.3)
where \(\alpha' = 2\frac{\frac{2}{n-2}}{n-1} \left[\frac{1}{n} + \frac{1}{n} \left(1 + \frac{2(n-1)}{n}\right)\right]\).

Using (2.3) and (1.4) we get,
\[
(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Y) = S(Y, Z)[A(X) - D(X)] - S(X, Y)\{A(Z) - D(Z)\}
= \alpha'[d\gamma(X)g(Y, Z) - d\gamma(Z)g(Y, X)].
\](2.4)

Let \(\rho_1, \rho_2, \rho_3\) be the associated vector fields corresponding to the 1-forms \(A, B, D\) respectively, i.e., \(g(X, \rho_1) = A(X), g(X, \rho_2) = B(X), g(X, \rho_3) = D(X)\).

Putting \(Z = \rho_2\) in (2.4) we get,
\[
[A(X) - D(X)]B(Y) - S(X, Y)[A(\rho_2) - D(\rho_2)]
= \alpha'[d\gamma(X)B(Y) - d\gamma(\rho_2)g(Y, X)]
\]
\[
\gamma \left( B(Y) \{ \gamma A(X) + B(QX) + D(QX) \} - g(X, Y) \{ \gamma A(p_2) + B(Qp_2) + D(Qp_2) \} \right). \tag{2.5}
\]

If the manifold has non-zero constant scalar curvature, then by virtue of (1.8), (2.5) yields that
\[
[A(X) - D(X)]B(QY) - S(X, Y)[A(p_2) - D(p_2)] = 0 \tag{2.6}
\]
where \( \bar{B}(Y) = B(QY) \) and \( \alpha_1 = \frac{1}{\overline{A(p_2) - D(p_2)}}, \alpha_2 = -\frac{1}{\overline{A(p_2) - D(p_2)}}. \)

This leads to the following theorem:

**Theorem 2.1:** In a quasi-conformally flat \((WRS)_n(n > 3)\) of non-zero constant scalar curvature, the Ricci tensor \( S \) has the form (2.7).

Again using (1.8) in (2.7) we get,
\[
S(X, Z) = \alpha_1 \bar{B}(Z) - \frac{1}{\bar{\alpha}}[\bar{B}(X) + \bar{D}(X)] + \alpha_2 D(X) \bar{B}(Z), \tag{2.8}
\]
where \( \bar{D}(X) = D(QX) \).

Using (2.8) in (2.1) we obtain,
\[
\bar{R}(X, Y, Z, W) = \alpha \left[ g(Y, Z)[g(X, W) - g(X, Z)g(Y, W)] + g(X, W)p(Y, Z) - g(X, Z)p(Y, W) \right. \\
\left. + g(Y, Z)p(X, W) - g(Y, W)p(X, Z) \right] + \gamma \left[ \bar{B}(Y) \bar{B}(Z) + \alpha_2 D(Y) \bar{B}(Z) - \frac{\bar{\alpha}}{\gamma} \bar{B}(Z) \bar{D}(Y) \right]. \tag{2.9}
\]

Hence comparing (2.9) with (1.14) we can state the following theorem:

**Theorem 2.2:** A quasi-conformally flat \((WRS)_n(n > 3)\) of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

Now putting \( U = \rho \) in (1.11) and then using (1.19) we get,
\[
T(Z)S(X, \rho) - T(\rho)S(X, Z) = 0 \\
\Rightarrow T(Z)T(QX) - T(\rho)S(X, Z) = 0 \\
\Rightarrow \gamma T(Z)T(X) - T(\rho)S(X, Z) = 0. \tag{2.10}
\]

Let us suppose that a \((WRS)_n(n > 3)\) is quasi-conformally flat and of non-zero scalar curvature.

From (2.10) we have,
\[
S(X, Z) = \bar{\alpha}T(X)T(Z), \tag{2.11}
\]
where \( \bar{\alpha} = \frac{\gamma}{T(\rho)} \).

Again, a Riemannian manifold is called quasi-Einstein if its Ricci tensor is of the form\[1\]−
\[
S = pg + q\omega \otimes \omega, \tag{2.12}
\]
where \( p, q \) are scalars of which \( q \neq 0 \) and \( \omega \) is a 1-form.

Comparing (2.11) and (2.12) we can state the following theorem:

**Theorem 2.3:** A quasi-conformally flat \((WRS)_n(n > 3)\) of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form \( T \) defined by \( T(X) = B(X) - D(X) \neq 0 \) \( \forall X \).

Again using (2.11) in (2.1) we obtain,
\[
\bar{R}(X, Y, Z, W) = l[\gamma g(Y, Z)[g(X, W) - g(X, Z)g(Y, W)] + \delta g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)], \tag{2.13}
\]
where \( l = \frac{\bar{\alpha}}{\gamma^2} \) and \( \delta = -\frac{\bar{\alpha}}{\gamma} \).

Hence comparing (2.13) with (1.13) we have the following theorem:
\textbf{Theorem 2.4:} A quasi-conformally flat $(WRS)_n(n > 3)$ of non-vanishing scalar curvature is a manifold of quasi-constant curvature with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0 \ \forall X.$

Using the expression of $T$ in (2.13) we have,
\[
\begin{align*}
\hat{R}(X, Y, Z, W) &= l[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + g(X, Z)\{\delta BD\} \{Y, Z\} - g(X, Z)\{\delta BD\} \{Y, W\} \\
&+ g(Y, Z)\{\delta BD\} \{X, W\} - g(Y, W)\{\delta BD\} \{X, Z\},
\end{align*}
\]
where $\{\delta BD\} = \delta(BB - BD - DB + DD)$.

Comparing the above relation with (1.14) we can state:

\textbf{Corollary 2.1:} A quasi-conformally flat $(WRS)_n(n > 3)$ of non-zero scalar curvature is a manifold of hyper quasi-constant curvature.

\section{Pseudo-projectively flat $(WRS)_n(n > 3)$}

In this section we obtain that a pseudo projectively flat $(WRS)_n(n > 3)$ with non-zero constant scalar curvature is a manifold of pseudo-quasi constant curvature and with non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0 \ \forall X$, where $B, D$ are 1-forms(non-zero simultaneously).

Let $(M^n, g)(n > 3)$ be a pseudo-projectively flat $(WRS)_n$. Then from (1.15) we obtain,
\[
\begin{align*}
\hat{R}(X, Y, Z, W) &= -\frac{\alpha}{n} [S(Y, Z)g(X, U) - S(X, Z)g(Y, U)] + \frac{\alpha}{n} [\frac{\partial}{\partial Y} + b][g(Y, Z)g(X, U) \\
&- g(X, Z)g(Y, U)].
\end{align*}
\]

(3.1)

Putting $X = U = e_i$ in (3.1) where \{e_i\} is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $1 \leq i \leq n$, we get,
\[
S(Y, Z) = \alpha g(Y, Z),
\]
where $\alpha = \frac{1}{n}$.

(3.2) indicates that a pseudo-projectively flat manifold is an Einstein manifold.

Now from (3.2) we get,
\[
(\nabla_X S)(Y, Z) = \alpha_1 dr(X)g(Y, Z),
\]
where $\alpha_1 = \frac{1}{n}$.

(3.3)

Similarly $(\nabla_Z S)(Y, X) = \alpha_1 dr(Z)g(Y, X)$.

(3.4)

Subtracting (3.4) from (3.3) we have,
\[
(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = \alpha_1 [dr(X)g(Y, Z) - dr(Z)g(Y, X)].
\]

(3.5)

Interchanging $X, U$ in (1.4) and then subtracting the resultant from (1.4) we obtain by virtue of (3.5) we obtain,
\[
[A(X) - D(X)]S(U, Z) - [A(U) - D(U)]S(X, Z) = \alpha_1 [dr(X)g(Z, U) - dr(U)g(Z, X)].
\]

(3.6)

Let $\rho_1, \rho_2, \rho_3$ be the associated vector fields corresponding to the 1-forms $A, B, D$ respectively, i.e., $g(X, \rho_1) = A(X), g(X, \rho_2) = B(X), g(X, \rho_3) = D(X)$.

Substituting $U$ by $\rho_2$ in (3.6) and then using (1.6) we get,
\[
[A(X) - D(X)]B(QZ) - [A(\rho_2) - D(\rho_2)]S(X, Z) = \alpha_1 [B(Z)\{rA(X) + B(QX) + D(QX)\}
\]

\]

...
Theorem 3.3: A pseudo-projectively flat manifold of pseudo quasi-constant curvature. If the manifold has non-zero constant scalar curvature, then by virtue of (1.8), (3.7) yields that
\[ [A(\rho_2) - D(\rho_2)]S(X, Z) - [A(X) - D(X)]B(QZ) = 0 \] (3.8)
\[ \Rightarrow S(X, Z) = \alpha_2 A(X)B(Z) + \alpha_3 D(X)B(Z), \] where \( \bar{B}(Z) = B(QZ) \), \( \alpha_2 = \frac{1}{A(\rho_2) - D(\rho_2)} \), \( \alpha_3 = -\frac{1}{A(\rho_2) - D(\rho_2)} \),

This leads to the following:

**Theorem 3.1:** In a pseudo-projectively flat \((WRS)_n(n > 3)\) of non-zero constant curvature, the Ricci tensor \( S \) has the form (3.9).

\[ \gamma S(X, Z) = -\frac{\alpha}{\alpha n - 1} B(Z)(B(X) + \bar{B}(X)) + \alpha_3 D(X)B(Z). \] (3.10)

Using (3.10) in (3.1) we have,
\[ \bar{R}(X, Y, Z, W) = \alpha [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + g(X, W)P(Y, Z) - g(Y, W)P(X, Z), \] (3.11)
where \( \alpha = \frac{a}{an - 1} + b \) and \( P(Y, Z) = -\frac{2}{\alpha n - 1} B(Z)(B(Y) + \bar{B}(Y)) + \alpha_3 D(Y)B(Z) \).

Comparing (3.11) with (1.16) we can state the following theorem:

**Theorem 3.2:** A pseudo-projectively flat \((WRS)_n(n > 3)\) of non-zero constant scalar curvature is a manifold of pseudo quasi-constant curvature.

Now putting \( U = \rho \) in (1.11) and then using (1.9) we get,
\[ rT(X)T(Z) - T(\rho)S(X, Z) = 0. \] (3.12)

From (3.12) we have,
\[ S(X, Z) = \frac{1}{T(\rho)}T(X)T(Z). \] (3.13)

Now \( T(\rho) \neq 0 \) for if \( T(\rho) = 0 \), then (3.13) implies that \( rT(X)T(Z) = 0 \) which implies that \( r = 0 \) (as \( T(X) \neq 0 \) \( \forall X \)), which is a contradiction.

Hence \( S(X, Z) = \alpha_1 T(X)T(Z) \), where \( \alpha_1 \) is a non-zero scalar.

Hence comparing (3.14) with (2.12) we can state that:

**Theorem 3.3:** A pseudo-projectively flat \((WRS)_n(n > 3)\) of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form \( T \) defined by \( T(X) = B(X) - D(X) \neq 0 \) \( \forall X \).

Again using (3.14) in (3.1) we obtain,
\[ \bar{R}(X, Y, Z, W) = \gamma_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \delta_1 [T(Y)T(Z)g(X, W) - T(X)T(Z)g(Y, W)], \] where \( \gamma_1 = \frac{\alpha}{\alpha n - 1} + b \) and \( \delta_1 = -\frac{b}{\alpha_1} \).

Comparing (3.15) with (1.16) we have the following theorem:

**Theorem 3.4:** A pseudo-projectively flat \((WRS)_n(n > 3)\) of non-vanishing scalar curvature
is a manifold of pseudo quasi-constant curvature with respect to 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0$ $\forall X$.

Using the expression of $T$ in (3.15), it can be easily seen that
$$R(X, Y, Z, W) = \gamma_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \{\delta_1 BD\}(Y, Z)g(X, W)$$
$$- \{\delta_1 BD\}(X, Z)g(Y, W),$$
where $\{\delta_1 BD\} = \delta_1(BB - BD - DB + DD)$.

Comparing the above relation with (1.16) we can state:

**Corollary 3.1:** A pseudo-projectively flat $(WRS)_n(n > 3)$ of non-zero scalar curvature is a manifold of pseudo quasi-constant curvature.

### 4 $W_2$-flat $(WRS)_n(n > 3)$

In this section we prove that a $W_2$-flat $(WRS)_n(n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0$ $\forall X$, where $B, D$ are I-forms (non-zero simultaneously).

Let $(M^n, g)(n > 3)$ be a $W_2$-flat $(WRS)_n$. Then from (1.17) we have,
$$R(X, Y, Z, W) = \frac{1}{n!} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W)].$$

Putting $X = W = e_i$ in (4.1), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, where $1 \leq i \leq n$, we obtain,
$$S(Y, Z) = \alpha g(Y, Z),$$
where $\alpha = \frac{1}{n}$.

From (4.2) we get,
$$(\nabla_X S)(Y, Z) = \alpha' dr(X)g(Y, Z),$$
where $\alpha' = \frac{1}{n}$.

Using (4.3) and (1.4) we get,
$$(\nabla_Y S)(Y, Z) - (\nabla_Z S)(Y, X)$$
$$= S(Y, Z)[A(X) - D(X)] - S(X, Y)[A(Z) - D(Z)]$$
$$= \alpha'[dr(Y)g(Z, g(Y, X)]$$
$$= \alpha'[dr(Y)\{rA(X) + B(QX) + D(QX)\} - g(X, Y)\{rA(\rho_2) + B(Q\rho_2) + D(Q\rho_2)\}] - g(X, Y)\{rA(\rho_2) + B(Q\rho_2) + D(Q\rho_2)\}].$$

Let $\rho_1, \rho_2, \rho_3$ be the associated vector fields corresponding to the 1-forms $A, B, D$ respectively, i.e., $g(X, \rho_1) = A(X), g(X, \rho_2) = B(X), g(X, \rho_3) = D(X)$.

Putting $Z = \rho_2$ in (4.4) we get,
$$[A(X) - D(X)]B(Y) - [A(\rho_2) - D(\rho_2)]S(X, Y)$$
$$= \alpha'[dr(Y)\{rA(X) + B(QX) + D(QX)\} - g(X, Y)\{rA(\rho_2) + B(Q\rho_2) + D(Q\rho_2)\}].$$

If the manifold has non-zero constant scalar curvature, then by virtue of (1.8), (4.5) yields that
$$[A(X) - D(X)]B(Y) - [A(\rho_2) - D(\rho_2)]S(X, Y) = 0$$
$$\Rightarrow S(X, Y) = \alpha_1 A(X)B(Y) + \alpha_2 D(X)B(Y),$$
where $B(Y) = B(QY), \alpha_1 = \frac{1}{A(\rho_2) - D(\rho_2)}, \alpha_2 = -\frac{1}{A(\rho_2) - D(\rho_2)}$.
This leads to the following:

**Theorem 4.1:** In a $W_2$-flat $(WRS)_n(n > 3)$ of non-zero constant scalar curvature, the Ricci tensor $S$ has the form (4.6).

Putting $U = \rho$ in (1.11) and then using (1.9) we get,

$$rT(X)T(Z) = T(\rho)S(X, Z).$$

(4.7)

Let us now suppose that a $(WRS)_n(n > 3)$ is $W_2$-flat and of non-zero scalar curvature.

From (4.7) we have,

$$S(X, Z) = T(\rho)T(X)T(Z).$$

(4.8)

Now $T(\rho) \neq 0$ as if $T(\rho) = 0$ then (4.8) implies that $rT(X)T(Z) = 0 \Rightarrow r = 0$ (as $T(X) \neq 0 \forall X$), which is a contradiction.

Hence $S(X, Z) = \tilde{\alpha}_1 T(X)T(Z)$,

(4.9)

where $\tilde{\alpha}_1$ is a non-zero scalar.

Hence comparing (4.9) with (2.12) we can state that:

**Theorem 4.2:** A $W_2$-flat $(WRS)_n(n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form $T$ defined by $T(X) = B(X) - D(X) \neq 0 \forall X$.

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