On the $A_\alpha$ spectral radius and $A_\alpha$ energy of non-strongly connected digraphs

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Abstract

Let $A_\alpha(G)$ be the $A_\alpha$-matrix of a digraph $G$ and $\lambda_{\alpha 1}, \lambda_{\alpha 2}, \ldots, \lambda_{\alpha n}$ be the eigenvalues of $A_\alpha(G)$. Let $\rho_\alpha(G)$ be the $A_\alpha$ spectral radius of $G$ and $E_\alpha(G) = \sum_{i=1}^{\alpha n} \lambda_{\alpha i}^2$ be the $A_\alpha$ energy of $G$ by using second spectral moment. Let $G^n_\alpha$ be the set of non-strongly connected digraphs with order $n$, which contain a unique strong component with order $m$ and some directed trees which are hung on each vertex of the strong component. In this paper, we characterize the digraph which has the maximal $A_\alpha$ spectral radius and the maximal (minimal) $A_\alpha$ energy in $G^n_\alpha$.

Key Words: $A_\alpha$ spectral radius; $A_\alpha$ energy; non-strongly connected digraphs

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1 Introduction

Let $G = (\mathcal{V}(G), \mathcal{A}(G))$ be a digraph which $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of $G$ and $\mathcal{A}(G)$ is the arc set of $G$. For an arc from vertex $v_i$ to $v_j$, we denote by $(v_i, v_j)$, and $v_i$ is the tail of $(v_i, v_j)$ and $v_j$ is the head of $(v_i, v_j)$. The outdegree $d_i^+ = d_G^+(v_i)$ of $G$ is the number of arcs whose tail is vertex $v_i$ and the indegree $d_i^- = d_G^-(v_i)$ of $G$ is the number of arcs whose head is vertex $v_i$. We denote the maximum outdegree and the maximum indegree of $G$ by $\Delta^+(G)$ and $\Delta^-(G)$, respectively. A walk $\pi$ of length $l$ from vertex $u$ to vertex $v$ is a sequence of vertices $\pi$: $u = v_0, v_1, \ldots, v_l = v$, where $(v_{k-1}, v_k)$ is an arc of $G$ for any $1 \leq k \leq l$. If $u = v$ then $\pi$ is called a closed walk. Let $c_2$ denote the number of all closed walks of length 2. A directed path $P_n$ with $n$ vertices is a digraph which the vertex set is $\{v_i| i = 1, 2, \ldots, n\}$ and the arc set is $\{(v_i, v_{i+1})| i = 1, 2, \ldots, n-1\}$. A directed cycle $C_n$ with $n \geq 2$ vertices is a digraph which the vertex set is $\{v_i| i = 1, 2, \ldots, n\}$ and the arc set is $\{(v_i, v_{i+1})| i = 1, 2, \ldots, n-1\} \cup \{(v_n, v_1)\}$. A digraph $G$ is connected if its underlying graph is connected. A digraph $G$ is strongly connected if for each pair of vertices $v_i, v_j \in \mathcal{V}(G)$, there

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is a directed path from $v_i$ to $v_j$ and one from $v_j$ to $v_i$. A strong component of $G$ is a maximal strongly connected subdigraph of $G$. Throughout this paper, we only consider a connected digraph $G$ containing neither loops nor multiple arcs.

For a digraph $G$ with order $n$, the adjacency matrix $A(G) = (a_{ij})_{n \times n}$ of $G$ is a $(0,1)$-square matrix whose $(i,j)$-entry equals to 1, if $(v_i,v_j)$ is an arc of $G$ and equals to 0, otherwise. The Laplacian matrix $L(G)$ and the signless Laplacian matrix $Q(G)$ of $G$ are $L(G) = D^+(G) - A(G)$ and $Q(G) = D^+(G) + A(G)$, respectively, where $D^+(G) =\text{diag}(d_1^+,d_2^+,...,d_n^+)$ is a diagonal outdegree matrix of $G$. In 2019, Liu et al. [15] defined the $A_\alpha$-matrix of $G$ as

$$A_\alpha(G) = \alpha D^+(G) + (1-\alpha)A(G),$$

where $\alpha \in [0,1]$. It is clear that if $\alpha = 0$, then $A_0(G) = A(G)$; if $\alpha = \frac{1}{2}$, then $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$; if $\alpha = 1$, then $A_1(G) = D^+(G)$. Since $D^+(G)$ is not interesting, we only consider $\alpha \in [0,1)$. The eigenvalue of $A_\alpha(G)$ with largest modulus is called the $A_\alpha$ spectral radius of $G$, denoted by $\rho_\alpha(G)$.

Actually, in 2017, Nikiforov [16] first proposed the $A_\alpha$-matrix of a graph $H$ with order $n$ as

$$A_\alpha(H) = \alpha D(H) + (1-\alpha)A(H),$$

where $D(H) = \text{diag}(d_1,d_2,...,d_n)$ is a diagonal degree matrix of $H$ and $\alpha \in [0,1]$. After that, many scholars began to study the $A_\alpha$-matrices of graphs. Nikiforov et al. [17] gave several results about the $A_\alpha$-matrices of trees and gave the upper and lower bounds for the spectral radius of the $A_\alpha$-matrices of arbitrary graphs. Let $\lambda_1(A_\alpha(H)) \geq \lambda_2(A_\alpha(H)) \geq \cdots \geq \lambda_n(A_\alpha(H))$ be the eigenvalues of $A_\alpha(H)$. Lin et al. [12] characterized the graph $H$ with $\lambda_k(A_\alpha(H)) = an - 1$ for $2 \leq k \leq n$ and showed that $\lambda_n(A_\alpha(H)) \geq 2n - 1$ if $H$ contains no isolated vertices. Liu et al. [13] presented several upper and lower bounds on the $k$-th largest eigenvalue of $A_\alpha$-matrix and characterized the extremal graphs corresponding to some of these bounds. More results about $A_\alpha$-matrix of a graph can be found in [3, 10, 11, 14, 18, 21]. Recently, Liu et al. [14] characterized the digraph which had the maximal $A_\alpha$ spectral radius in $G'\,n$, where $G'\,n$ is the set of digraphs with order $n$ and dichromatic number $r$. Xi et al. [22] determined the digraphs which attained the maximum (or minimum) $A_\alpha$ spectral radius among all strongly connected digraphs with given parameters such as girth, clique number, vertex connectivity or arc connectivity. Xi and Wang [24] established some lower bounds on $\Delta^+ - \rho_\alpha(G)$ for strongly connected irregular digraphs with given maximum outdegree and some other parameters. Ganie and Baghipur [4] obtained some lower bounds for the spectral radius of $A_\alpha(G)$ in terms of the number of vertices, the number of arcs and the number of closed walks of the digraph $G$.

It is well-known that the energy of the adjacency matrix of a graph $H$ first defined by Gutman [3] as $E_A(H) = \sum_{i=1}^{n}\nu_i$, where $\nu_i$ is an eigenvalue of the adjacency matrix of $H$. Peña and Rada [20] defined the energy of the adjacency matrix of a digraph $G$ as $E_A(G) = \sum_{i=1}^{n} |\text{Re}(z_i)|$, where $z_i$ is an eigenvalue of the adjacency matrix of $G$ and $\text{Re}(z_i)$ is the real part of eigenvalue $z_i$. Some results about the energy of the adjacency matrices of graphs and digraphs have been obtained in [2, 3, 4]. Lazić [8] defined the Laplacian energy of a graph $H$ as $LE(H) = \sum_{i=1}^{n} \mu_i^2$ by using second spectral moment, where $\mu_i$ is an eigenvalue of $L(H)$. Perera and Mizoguchi [19] defined the Laplacian energy $LE(G)$ of a digraph $G$ as $LE(G) = \sum_{i=1}^{n} \lambda_i^2$ by using second spectral moment, where $\lambda_i$ is an eigenvalue of $L(G)$. Yang and Wang [25] defined the signless Laplacian energy as $E_{SL}(G) = \sum_{i=1}^{n} q_i^2$ of a digraph $G$ by using second spectral moment, where $q_i$ is an eigenvalue of $Q(G)$. In this paper, we study the
\[ A_\alpha \text{ energy as } E_\alpha(G) = \sum_{i=1}^{n} \lambda_{\alpha i}^2 \text{ of a digraph } G \text{ by using second spectral moment, where } \lambda_{\alpha i} \text{ is an eigenvalue of } A_\alpha(G). \]

Figure 1: An out-star \( \vec{K}_{1,n-1} \), an in-star \( \vec{K}_{1,n-1} \) and a symmetric-star \( \leftrightarrow K_{1,n-1} \)

Next, we will introduce some concepts of digraphs. An arc \((v_i, v_j)\) is said to be simple if \((v_i, v_j)\) is an arc in \( G \) but \((v_j, v_i)\) is not an arc in \( G \). A digraph \( G \) is simple if every arc in \( G \) is simple. An arc \((v_i, v_j)\) is said to be symmetric if both \((v_i, v_j)\) and \((v_j, v_i)\) are arcs in \( G \). A digraph \( G \) is symmetric if every arc in \( G \) is symmetric. Let \( T \) be a directed tree with \( n \) vertices and \( e \) arcs without cycles and \( n = e + 1 \). If \( n = 1 \), then the directed tree is a vertex.

Let \( \vec{K}_{1,n-1} \) be an out-star with \( n \) vertices which has one vertex with outdegree \( n - 1 \) and other vertices with outdegree 0 (see \( \vec{K}_{1,n-1} \) in Figure 1). And the vertex with outdegree \( n - 1 \) is called the centre of \( \vec{K}_{1,n-1} \). Let \( \vec{K}_{1,n-1} \) be an in-star with \( n \) vertices which has one vertex with indegree \( n - 1 \) and other vertices with indegree 0 (see \( \vec{K}_{1,n-1} \) in Figure 1). Let \( \leftrightarrow K_{1,n-1} \) be a symmetric-star with \( n \) vertices which all the arcs are symmetric and have a common vertex (see \( \leftrightarrow K_{1,n-1} \) in Figure 1). Let in-tree be a directed tree with \( n \) vertices which the outdegree of each vertex of the directed tree is at most one. Then the in-tree has exactly one vertex with outdegree 0 and such vertex is called the root of the in-tree (see Figure 2). A \( p \)-spindle with \( n \) vertices is the union of \( p \) internally disjoint \((x, y)\)-directed paths for some vertices \( x \) and \( y \). The vertex \( x \) is said to be the initial vertex of spindle and \( y \) its terminal vertex. A \((p + q)\)-bispindle with \( n \) vertices is the internally disjoint union of a \( p \)-spindle with initial vertex \( x \) and terminal vertex \( y \) and a \( q \)-spindle with initial vertex \( y \) and terminal vertex \( x \). Actually, it is the union of \( p \) \((x, y)\)-directed paths and \( q \) \((y, x)\)-directed paths. We denote the \((p + q)\)-bispindle by \( B[p, q] \) (see \( B[p, q] \) in Figure 3).

Let \( \mathcal{G}_n^m \) be the set of non-strongly connected digraphs with order \( n \), which contain a unique
strong component with order $m$ and some directed trees which are hung on each vertex of
the strong component. For a non-strongly connected digraph $G \in \mathcal{G}_m^m$, we assume that $G^*$
is the unique strong component of $G$ with $m$ vertices and $T^{(i)}$ is the directed tree with $n_i$
vertices which hangs on each vertex of $G^*$, where $n = \sum_{i=1}^m n_i$ and $i = 1, 2, \ldots, m$. Then
the vertex set of $G$ is $\mathcal{V}(G) = \bigcup_{i=1}^m \mathcal{V}(T^{(i)})$, where $\mathcal{V}(T^{(i)}) = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_{n_i}^{(i)}\}$, $\mathcal{V}(G^*) = \{v_1, v_2, \ldots, v_m\}$ and $v_i = u_1^{(i)}$, $i = 1, 2, \ldots, m$. Let $d_{G^*}^{+}(v_i)$ be the outdegree of vertex $u_1^{(i)}$
of $G$ and $d_{G^*}^{+}(v_1) \geq d_{G^*}^{+}(v_2) \geq \cdots \geq d_{G^*}^{+}(v_m)$ be the outdegrees of vertices of $G^*$, where
$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$. We take an example in Figure 4.

Definition 1.1. Let $G \in \mathcal{G}_n^m$ be a non-strongly connected digraph with $n$ vertices.

(i) Let

$$G' = G - \sum_{i=1}^m \sum_{s,t=1}^{n_i} (u_s^{(i)}, u_t^{(i)}) + \sum_{i=1}^m \sum_{j=2}^{n_i} (u_{1}^{(i)}, u_j^{(i)}) ,$$

where $(u_s^{(i)}, u_t^{(i)}) \in \mathcal{A}(G)$, $i = 1,2,\ldots,m$ and $s,t,j = 1,2,\ldots,n_i$. Then $G' \in \mathcal{G}_n^m$ is a non-
strongly connected digraph, which each directed tree $T^{(i)}$ is an out-star $\vec{K}_{1,n_i-1}$ and the centre
of $\vec{K}_{1,n_i-1}$ is $v_i$ of $G^*$, where $i = 1,2,\ldots,m$ (see $G'$ in Figure 5).

(ii) Let

$$G'' = G - \sum_{i=1}^m \sum_{s,t=1}^{n_i} (u_s^{(i)}, u_t^{(i)}) + \sum_{i=1}^m \sum_{j=2}^{n_i} (u_{1}^{(i)}, u_j^{(i)})$$

be the outstar of $G^*$ and the centre
of $\vec{K}_{1,n_i-1}$ is $v_i$. Then $G''$ is a non-strongly connected digraph and the centre
of $\vec{K}_{1,n_i-1}$ is $v_i$ of $G^*$ for each $i = 1,2,\ldots,m$. We take an example in Figure 6.
\[ G' = G' - \sum_{i=2}^{m} \sum_{j=2}^{n_i} (u_s^{(i)}, u_t^{(j)}) + \sum_{i=2}^{m} \sum_{j=2}^{n_i} (u_1^{(i)}, u_j^{(i)}), \]

where \((u_s^{(i)}, u_t^{(i)}) \in A(G), i = 1, 2, \ldots, m\) and \(s, t, j = 1, 2, \ldots, n_i\). Then \(G'' \in \mathcal{G}_n^m\) is a non-strongly connected digraph, which only has an out-star \(\vec{K}_{1,n-m}\) and the centre of \(\vec{K}_{1,n-m}\) is \(v_1\) of \(G^*\), \(v_1\) is the maximal outdegree vertex of \(G^*\) and other directed tree \(T^{(i)}\) is just a vertex \(v_i\) of \(G^*\) for \(i = 2, 3, \ldots, m\) (see \(G''\) in Figure 5).

(iii) Let \(G''\in \mathcal{G}_n^m\) be a non-strongly connected digraph by changing each directed tree in \(G\) to an in-tree which the root of the in-tree is \(v_i\) of \(G^*\), where \(i = 1, 2, \ldots, m\). We take an example in Figure 6.

The arrangement of this paper is as follows. In Section 2, we characterize the digraph which has the maximal \(A_\alpha\) spectral radius in \(\mathcal{G}_n^m\). In Section 3, we characterize the digraph which has the maximal (minimal) \(A_\alpha\) energy in \(\mathcal{G}_n^m\).

2 The maximal \(A_\alpha\) spectral radius of non-strongly connected digraphs

In this section, we will consider the maximal \(A_\alpha\) spectral radius of non-strongly connected digraphs in \(\mathcal{G}_n^m\). Firstly, we list some known results used for later.
Theorem 2.5. Let $G$ be a non-strongly connected digraph with $V(G)$ be a unique strong component of $G$. If $A_{ij} < B_{ij}$ for all $i$ and $j$, then $A < B$. If $A < B$ and $A 
eq B$, then $A < B$. If $A_{ij} < B_{ij}$ for all $i$ and $j$, then $A < B$.

Corollary 2.6. Let $G$ be any digraph with $n$ vertices. Let $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}$ be the eigenvalues of $A_{i}(G)$ and $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of vertices of $G$. For any vertex $v_i$ which is not on the strong components of $G$, we have

$$\lambda_{ai} = \alpha d_i^+.$$
Theorem 2.7. Let $G, G' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 1.1. Then $\rho_\alpha(G') \geq \rho_\alpha(G)$.

Proof. By the definition of $G'$, we know $G' \in \mathcal{G}_n^m$ is a non-strongly connected digraph, which each directed tree $T^{(i)}$ is an out-star $K_{1,n_i-1}$ and the centre of $K_{1,n_i-1}$ is $v_i$ of $G^*$, where $i = 1, 2, \ldots, m$. Then $d_{G^*}^+(v_i) = d_{G^*}^+(u_1^{(i)}) = d_{G^*}^+(v_i) + n_i - 1$, $d_{G^*}^+(u_j^{(i)}) = 0$, where $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n_i$.

First, we consider the $A_\alpha$-eigenvalues of $G'$. From Theorem 2.8, for the vertex $u_j^{(i)}$ which is not on the strong component $G^*$, we have

$$\lambda_{\alpha j}^{(i)}(G') = \alpha d_{G^*}^+(u_j^{(i)}) = 0,$$

where $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n_i$. For the vertex $v_i = u_1^{(i)}$ which is on the strong component $G^*$, the $A_\alpha$-eigenvalues $\lambda_{\alpha 1}^{(i)}(G')$ are equal to the eigenvalues of $A_{11}'$, where

$$A_{11}' = \alpha \text{diag} \{d_{G^*}^+(v_1) + n_1 - 1, d_{G^*}^+(v_2) + n_2 - 1, \ldots, d_{G^*}^+(v_m) + n_m - 1\} + (1 - \alpha)A(G^*).$$

Obviously, $\rho_\alpha(G') = \rho(A_{11}')$.

Next, we consider the $A_\alpha$-eigenvalues of $G$. From Theorem 2.5, for the vertex $u_j^{(i)}$ which is not on the strong component $G^*$, we have

$$\lambda_{\alpha j}^{(i)}(G) = \alpha d_{G^*}^+(u_j^{(i)}),$$

where $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n_i$. For the vertex $v_i = u_1^{(i)}$ which is on the strong component $G^*$, the $A_\alpha$-eigenvalues $\lambda_{\alpha 1}^{(i)}(G)$ are equal to the eigenvalues of $A_{11}$, where

$$A_{11} = \alpha \text{diag} \{d_{G}^+(v_1), d_{G}^+(v_2), \ldots, d_{G}^+(v_m)\} + (1 - \alpha)A(G^*).$$

Hence, $\rho_\alpha(G) = \max_{1 \leq i \leq m, 2 \leq j \leq n_i} \\{\rho(A_{11}), \alpha d_{G}^+(u_j^{(i)})\}$.

Finally, we prove

$$\rho_\alpha(G') = \rho(A_{11}') \geq \rho_\alpha(G) = \max_{1 \leq i \leq m, 2 \leq j \leq n_i} \\{\rho(A_{11}), \alpha d_{G}^+(u_j^{(i)})\}.$$

From Lemma 2.3, since

$$d_{G}^+(v_i) + n_i - 1 \geq d_{G}^+(v_i),$$

we have $A_{11}' \geq A_{11}$. Then $\rho(A_{11}') \geq \rho(A_{11})$. From Lemma 2.4, we have

$$\rho_\alpha(G') \geq \alpha \Delta^+(G') \geq \alpha \Delta^+(G) \geq \alpha d_{G}^+(u_j^{(i)}).$$

Therefore, we have $\rho_\alpha(G') \geq \rho_\alpha(G)$. \hfill \Box

Theorem 2.8. Let $G', G'' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 1.1. If $\alpha \in \left[\frac{d_{G^*}^+(v_1)}{d_{G^*}^+(v_1)+n-m-n_1+1}, 1\right]$, then $\rho_\alpha(G'') \geq \rho_\alpha(G')$; if $\alpha = 0$, then $\rho_\alpha(G'') = \rho_\alpha(G')$. 

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Proof. By the definition of $G''$, we know $G'' \in \mathcal{G}_n^m$ is a non-strongly connected digraph, which only has an out-star $\overrightarrow{K}_{1,n-m}$ and the centre of $\overrightarrow{K}_{1,n-m}$ is $v_1$ of $G^*$, $v_1$ is the maximal outdegree vertex of $G^*$ and each other directed tree $T^{(i)}$ is just a vertex $v_i$ of $G^*$ for $i = 2, 3, \ldots, m$. The vertex set $\mathcal{V}(G'') = \mathcal{V}(\overrightarrow{K}_{1,n-m}) \cup (\mathcal{V}(G^*) - v_1)$, where $\mathcal{V}(\overrightarrow{K}_{1,n-m}) = \{u^{(1)}_1, u^{(1)}_2, \ldots, u^{(1)}_{n-m+1}\}$, $\mathcal{V}(G^*) = \{v_1, v_2, \ldots, v_m\}$ and $v_1^{(1)} = v_1$. Then $d^+_G(v_1) = d^+_G(v_1) + n - m$, $d^+_G(v_j^{(1)}) = 0$ and $d^+_G(v_i) = d^+_G(v_i)$, where $i = 2, 3, \ldots, m$ and $j = 2, 3, \ldots, n - m + 1$. Since $d^+_G(v_1) \geq d^+_G(v_2) \geq \cdots \geq d^+_G(v_m)$, by Lemma 2.4, we have
\[
\rho_\alpha(G'') \geq \alpha \Delta^+(G'') = \alpha \left(d^+_G(v_1) + n - m\right).
\]
From the proof of Theorem 2.7, we have $\rho_\alpha(G') = \rho(A'_{11})$. By Lemma 2.1, we have
\[
\min_{1 \leq i \leq m} R_i(A'_{11}) \leq \rho(A'_{11}) \leq \max_{1 \leq i \leq m} R_i(A'_{11}).
\]
Then
\[
\rho_\alpha(G') \leq \max_{1 \leq i \leq m} \{1 - \alpha \} d^+_G(v_i) + \alpha \left(d^+_G(v_i) + n_i - 1\right) = \max_{1 \leq i \leq m} \left\{d^+_G(v_i) + \alpha(n_i - 1)\right\}.
\]
Without loss of generality, let $\max \\left\{d^+_G(v_i) + \alpha(n_i - 1)\right\} = d^+_G(v_t) + \alpha(n_t - 1)$. That is $d^+_G(v_t) + \alpha(n_t - 1) \geq d^+_G(v_i) + \alpha(n_i - 1)$.

If $\alpha \neq 0$, we have $n_t \geq \frac{d^+_G(v_t) - d^+_G(v_i) + \alpha n_i}{\alpha}$. Next, we prove
\[
\rho_\alpha(G'') \geq \alpha \left(d^+_G(v_t) + n - m\right) \geq d^+_G(v_t) + \alpha(n_t - 1) \geq \rho_\alpha(G').
\]
We only need to prove
\[
\alpha \left(d^+_G(v_t) + n - m\right) \geq d^+_G(v_t) + \alpha \left(d^+_G(v_1) - d^+_G(v_t) + \alpha n_1\right) - 1.
\]
That is, $1 > \alpha \geq \frac{d^+_G(v_1)}{d^+_G(v_1) + n - m - n_1 + 1}$.

If $\alpha = 0$, we have $\rho_0(G') = \rho(A(G^*))$ and $\rho_0(G'') = \rho(A(G^*))$, then $\rho_\alpha(G'') = \rho_\alpha(G')$.

Therefore, if $\alpha \in \left[\frac{d^+_G(v_t)}{d^+_G(v_1) + n - m - n_1 + 1}, 1\right]$, then $\rho_\alpha(G'') \geq \rho(G')$; if $\alpha = 0$, then $\rho_\alpha(G'') = \rho_\alpha(G')$.

From Theorems 2.7 and 2.8 we have the following theorem.

**Theorem 2.9.** Among all digraphs in $\mathcal{G}_n^m$, if $\alpha \in \left[\frac{d^+_G(v_1)}{d^+_G(v_1) + n - m - n_1 + 1}, 1\right]$ or $\alpha = 0$, then $G''$ is a digraph which has the maximal $A_\alpha$ spectral radius.

We only obtain $G''$ is a digraph having the maximal $A_\alpha$ spectral radius in $\mathcal{G}_n^m$ when $\alpha \in \left[\frac{d^+_G(v_1)}{d^+_G(v_1) + n - m - n_1 + 1}, 1\right]$ or $\alpha = 0$. However, we know $\frac{d^+_G(v_1)}{d^+_G(v_1) + n - m - n_1 + 1} = \frac{d^+_G(v_1)}{d^+_G(v_1) + \sum_{i=2}^m(n_i - 1)}$.

The bigger $\sum_{i=2}^m(n_i - 1)$ is, the smaller $\frac{d^+_G(v_1)}{d^+_G(v_1) + n - m - n_1 + 1}$ is. And if $\sum_{i=2}^m(n_i - 1) \rightarrow 0$, then $n_1 - 1 \rightarrow n - m$ and $G' \rightarrow G''$. So we think the result is true for all $\alpha \in [0, 1)$. Therefore, we give the following conjecture.

**Conjecture 2.10.** Among all digraphs in $\mathcal{G}_n^m$, for any $\alpha \in [0, 1)$, $G''$ is a digraph which has the maximal $A_\alpha$ spectral radius.
3 The maximal $A_\alpha$ energy of non-strongly connected digraphs

In this section, we will consider the maximal $A_\alpha$ energy of non-strongly connected digraphs in $G^m_n$. Firstly, we will introduce some basic concepts of $A_\alpha$ energy of digraphs.

Let $E_\alpha(G)$ be $A_\alpha$ energy of a digraph $G$. By using second spectral moment, Xi [22] defined the $A_\alpha$ energy as $E_\alpha(G) = \sum_{i=1}^{n} \lambda_{\alpha i}^2$, where $\lambda_{\alpha i}$ is an eigenvalue of $A_\alpha(G)$. She also obtained the following result.

**Lemma 3.1.** (22) Let $G$ be a connected digraph with $n$ vertices and $c_2$ be the number of all closed walks of length 2. Let $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of vertices of $G$. Then

$$E_\alpha(G) = \sum_{i=1}^{n} \lambda_{\alpha i}^2 = \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + (1-\alpha)^2 c_2.$$ 

Obviously, we can get the following results.

**Theorem 3.2.** Let $G$ be a connected digraph with $n$ vertices and $e$ arcs. Let $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of vertices of $G$.

(i) If $G$ is a simple digraph, then

$$E_\alpha(G) = \alpha^2 \sum_{i=1}^{n} (d_i^+)^2.$$ 

(ii) If $G$ is a symmetric digraph, then

$$E_\alpha(G) = \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + (1-\alpha)^2 e.$$ 

**Proof.** From Lemma 3.1, the conclusion is obvious. \qed

Let $c_2$ be the number of all closed walks of length 2 of a digraph. From Theorem 3.2, we have the following results.

**Example 3.3.** We give some $A_\alpha$ energies of special digraphs as follows:

1. $E_\alpha(P_n) = \alpha^2(n-1)$;
2. $E_\alpha(C_n) = \begin{cases} \alpha^2 n, & \text{if } n > 2; \\ 2(\alpha^2 - 2\alpha + 1), & \text{if } n = 2; \end{cases}$
3. $E_\alpha(K_{1,n-1}^+) = \alpha^2(n-1)^2$;
4. $E_\alpha(K_{1,n-1}^-) = \alpha^2(n-1)$;
5. $E_\alpha(K_{1,n-1}) = \alpha^2 n(n-1) + 2(1-\alpha)^2 (n-1)$;
6. $E_\alpha(\infty[m_1, m_2, \ldots, m_t]) = \alpha^2 (t^2 + n - 1) + (1-\alpha)^2 c_2$;
7. $E_\alpha(B[p,q]) = \alpha^2 (p^2 + q^2 + n - 2) + (1-\alpha)^2 c_2$.

**Lemma 3.4.** (22) Let $T$ be a directed tree with $n$ vertices. Then

$$\alpha^2 (n-1) \leq E_\alpha(T) \leq \alpha^2(n-1)^2.$$ 

Moreover, $E_\alpha(T) = \alpha^2(n-1)$, if and only if $T$ is an in-tree with $n$ vertices; $E_\alpha(T) = \alpha^2(n-1)^2$ if and only if $T$ is an out-star $K_{1,n-1}^-$. 


Next, we give our main results.

**Theorem 3.5.** Let \( G, G' \in \mathcal{G}_n^m \) be two non-strongly connected digraphs as defined in Definition 1.1. Then \( E_\alpha(G') \geq E_\alpha(G) \) with equality holds if and only if \( G \cong G' \).

**Proof.** By the definition of \( G \), we know \( G \in \mathcal{G}_n^m \) is a non-strongly connected digraph with order \( n \), which contains a unique strong component with order \( m \) and some directed trees which are hung on each vertex of the strong component. From Lemma 3.4, we know the maximal \( A_\alpha \) energy of \( T(i) \) is

\[
\left( E_\alpha(T(i)) \right)_{\max} = \alpha^2(n_i - 1)^2,
\]

where \( i = 1, 2, \ldots, m \). Then we have

\[
E_\alpha(G) = \alpha^2 \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left( d_{G}^+(u_j^{(i)}) \right)^2 + (1 - \alpha)^2 c_2
\]

\[
= \alpha^2 \sum_{i=1}^{m} \left( d_{G}^+(u_1^{(i)}) + d_{T(i)}^+(u_1^{(i)}) \right)^2 + \alpha^2 \sum_{i=1}^{m} \sum_{j=2}^{n_i} \left( d_{G}^+(u_j^{(i)}) \right)^2 + (1 - \alpha)^2 c_2
\]

\[
= \alpha^2 \sum_{i=1}^{m} \left( d_{G}^+(v_i) \right)^2 + \left( d_{T(i)}^+(u_1^{(i)}) \right)^2 + 2d_{G}^+(v_i)d_{T(i)}^+(u_1^{(i)})
\]

\[
+ \alpha^2 \sum_{i=1}^{m} \sum_{j=2}^{n_i} \left( d_{T(i)}^+(u_j^{(i)}) \right)^2 + (1 - \alpha)^2 c_2
\]

\[
= \alpha^2 \sum_{i=1}^{m} \left( d_{G}^+(v_i) \right)^2 + 2\alpha^2 \sum_{i=1}^{m} \left( d_{T(i)}^+(u_j^{(i)}) \right)^2 + 2\alpha^2 \sum_{i=1}^{m} d_{G}^+(v_i)d_{T(i)}^+(v_i) + (1 - \alpha)^2 c_2
\]

\[
\leq \alpha^2 \sum_{i=1}^{m} \left( d_{G}^+(v_i) \right)^2 + \alpha^2 \sum_{i=1}^{m} (n_i - 1)^2 + 2\alpha^2 \sum_{i=1}^{m} d_{G}^+(v_i)(n_i - 1) + (1 - \alpha)^2 c_2
\]

\[
= \alpha^2 \sum_{i=1}^{m} \left( d_{G}^+(v_i) + (n_i - 1) \right)^2 + (1 - \alpha)^2 c_2
\]

\[
= E_\alpha(G').
\]

The equality holds if and only if

\[
\sum_{j=1}^{n_i} \left( d_{T(i)}^+(u_j^{(i)}) \right)^2 + 2d_{G}^+(v_i)d_{T(i)}^+(v_i) = (n_i - 1)^2 + 2d_{G}^+(v_i)(n_i - 1),
\]

for all \( i = 1, 2, \ldots, m \). Anyway, the strong component \( G^* \) does not change, so \( d_{G}^+(v_i) \) does not change. That is, \( d_{G}^+(u_1^{(i)}) = d_{T(i)}^+(v_i) = n_i - 1 \), and \( d_{G}^+(u_j^{(i)}) = 0 \), where \( i = 1, 2, \ldots, m \) and \( j = 2, 3, \ldots, n_i \). Then each directed tree \( T(i) \) is an out-star \( \overrightarrow{K}_{1,n_i-1} \).

Hence, we have \( E_\alpha(G') \geq E_\alpha(G) \) with equality holds if and only if \( G \cong G' \). \( \square \)

**Theorem 3.6.** Let \( G', G'' \in \mathcal{G}_n^m \) be two non-strongly connected digraphs as defined in Definition 1.1. Then \( E_\alpha(G'') \geq E_\alpha(G') \) with equality holds if and only if \( G' \cong G'' \).
Proof. By the definition of $G''$, we know $G'' \in \mathcal{G}^m_n$ is a non-strongly connected digraph which only has an out-star $K_{1,n-m}$ and the centre of $K_{1,n-m}$ is $v_1$ of $G^*$, $v_1$ is the maximal outdegree vertex of $G^*$ and other directed tree $T^{(i)}$ is just a vertex $v_i$ of $G^*$ for $i = 2, 3, \ldots, m$. Then we have

$$E_\alpha(G'') = \alpha^2 (d^+_{G^*}(v_1) + n - m)^2 + \alpha^2 \sum_{i=2}^{m} (d^+_{G^*}(v_i))^2 + (1 - \alpha)^2 c_2.$$  

Since

$$E_\alpha(G') = \alpha^2 \sum_{i=1}^{m} (d^+_{G^*}(v_i) + (n_i - 1))^2 + (1 - \alpha)^2 c_2$$

$$= \alpha^2 \left( \sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + \sum_{i=1}^{m} (n_i - 1)^2 + 2 \sum_{i=1}^{m} d^+_{G^*}(v_i)(n_i - 1) \right) + (1 - \alpha)^2 c_2$$

$$\leq \alpha^2 \left( \sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + \left( \sum_{i=1}^{m} (n_i - 1) \right)^2 + 2 \sum_{i=1}^{m} d^+_{G^*}(v_i)(n_i - 1) \right) + (1 - \alpha)^2 c_2$$

$$= \alpha^2 \left( \sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + (n - m)^2 + 2d^+_{G^*}(v_1)(n - m) \right) + (1 - \alpha)^2 c_2$$

$$= \alpha^2 (d^+_{G^*}(v_1) + n - m)^2 + \alpha^2 \sum_{i=2}^{m} (d^+_{G^*}(v_i))^2 + (1 - \alpha)^2 c_2$$

$$= E_\alpha(G').$$

The equality holds if and only if

$$\sum_{i=1}^{m} (n_i - 1)^2 + 2 \sum_{i=1}^{m} d^+_{G^*}(v_i)(n_i - 1) = \left( \sum_{i=1}^{m} (n_i - 1) \right)^2 + 2 \sum_{i=1}^{m} d^+_{G^*}(v_1)(n_i - 1).$$

Anyway, the strong component $G^*$ does not change, so $d^+_{G^*}(v_i)$ does not change. That is, $n_i - 1 = 0$ for all $i = 2, 3, \ldots, m$ and $n_1 = n - m + 1$. Then the directed tree $T^{(1)}$ is an out-star $K_{1,n-m}$, and each other directed tree is a vertex $v_i$, where $i = 2, 3, \ldots, m$. Hence, we have $E_\alpha(G'') \geq E_\alpha(G')$ with equality holds if and only if $G' \cong G''$. \hfill \Box

Theorem 3.7. Let $G, G'' \in \mathcal{G}^m_n$ be two non-strongly connected digraphs as defined in Definition \ref{definition:non-strongly-connected}. Then $E_\alpha(G) \geq E_\alpha(G'')$ with equality holds if and only if $G \cong G''$.

Proof. From Lemma \ref{lemma:minimal-energy} we know the minimal $A_\alpha$ energy of $T^{(i)}$ is

$$\left( E_\alpha(T^{(i)}) \right)_{\text{min}} = \alpha^2 (n_i - 1),$$

where $i = 1, 2, \ldots, m$. Similar to the proof of Theorem \ref{theorem:energy-comparison} we can get the result easily. And

$$E_\alpha(G'') = \alpha^2 \sum_{i=1}^{n} (d^+_{G^*}(v_i))^2 + \alpha^2 (n - m) + (1 - \alpha)^2 c_2.$$  

\hfill \Box
Moreover, the first equality holds if and only if each directed tree is in-tree which the root of the in-tree is \(v_i\) of \(G^*\), where \(i = 1, 2, \ldots, m\); the second equality holds if and only if \(G \in G_n^m\) only has an out-star \(K_{1,n-m}\) and the centre of \(K_{1,n-m}\) is \(v_1\) of \(G^*\), \(v_1\) is the maximal outdegree vertex of \(G^*\) and each other directed tree \(T^{(j)}\) is just a vertex \(v_i\) of \(G^*\) for \(i = 2, 3, \ldots, m\).

From Corollary 3.9 we can get some bounds of \(A_\alpha\) energies of special non-strongly connected digraphs.

**Corollary 3.10.** The bounds of \(A_\alpha\) energies of special non-strongly connected digraphs \(\tilde{U}_n^m\), \(\infty[m_1, m_2, \ldots, m_t]\) and \(B[p, q]\).

(i) Let \(\tilde{U}_n^m \in G_n^m\) be a unicyclic digraph with order \(n\) which contains a unique directed cycle \(C_m\) and some directed trees which are hung on each vertex of \(C_m\), where \(m \geq 2\). Then

\[
2\alpha^2 + \alpha^2(n - 2) + 2(1 - \alpha)^2 \leq E_\alpha(\tilde{U}_n^m) \leq \alpha^2(n - 1)^2 + \alpha^2 + 2(1 - \alpha)^2,
\]

and

\[
\alpha^2 m + \alpha^2(n - m) \leq E_\alpha(\tilde{U}_n^m) \leq \alpha^2(n - m + 1)^2 + \alpha^2(m - 1) (m \geq 3).
\]

Moreover, the first equality holds if and only if each directed tree is in-tree which the root of the in-tree is connected with \(C_m\); the second equality holds if and only if \(\tilde{U}_n^m \in G_n^m\) only has an out-star \(K_{1,n-m}\) and the centre of \(K_{1,n-m}\) is an any vertex of \(C_m\).

(ii) Let \(\infty[m_1, m_2, \ldots, m_t] \in G_n^m\) be a generalized \(\infty\)-digraph with order \(n\) which contains \(\infty[m_1, m_2, \ldots, m_t]\) and some directed trees which are hung on each vertex of \(\infty[m_1, m_2, \ldots, m_t]\), where \(2 = m_1 = \cdots = m_s < m_{s+1} \leq \cdots \leq m_t\), \(m = \sum_{i=1}^{t} m_i - t + 1\) and the common vertex of \(t\) directed cycles \(C_{m_i}\) is \(v\). Then

\[
\alpha^2(m - 1 + t^2) + \alpha^2(n - m) + 2s(1 - \alpha)^2 \leq E_\alpha(\infty[m_1, m_2, \ldots, m_t])
\]

\[
\leq \alpha^2(n - m + t^2) + \alpha^2(m - 1) + 2s(1 - \alpha)^2.
\]

Moreover, the first equality holds if and only if each directed tree is in-tree which the root of the in-tree is connected with \(\infty[m_1, m_2, \ldots, m_t]\); the second equality holds if and only if \(\infty[m_1, m_2, \ldots, m_t] \in G_n^m\) only has an out-star \(K_{1,n-m}\) and the centre of \(K_{1,n-m}\) is \(v\).

(iii) Let \(B[p, q] \in G_n^m\) be a digraph with order \(n\) vertices which contains \(B[p, q]\) and some directed trees which are hung on each vertex of \(B[p, q]\), where \(V(B[p, q]) = m\) and \(p \geq q\). If both \((x, y)\) and \((y, x)\) are arcs in \(B[p, q]\), then

\[
\alpha^2(m - 2 + p^2 + q^2) + \alpha^2(n - m) + 2(1 - \alpha)^2 \leq E_\alpha(\tilde{B}[p, q])
\]
\[
\leq \alpha^2 (n - m + p)^2 + \alpha^2 (m - 2 + q^2) + 2(1 - \alpha)^2.
\]

Otherwise,
\[
\alpha^2 (m - 2 + p^2 + q^2) + \alpha^2 (n - m) \leq E_\alpha(\overline{B}[p, q]) \leq \alpha^2 (n - m + p)^2 + \alpha^2 (m - 2 + q^2).
\]

Moreover, the first equality holds if and only if each directed tree is in-tree which the root of the in-tree is connected with \(B[p, q]\); the second equality holds if and only if \(\overline{B}[p, q] \in \mathcal{G}_{\alpha}^m\) only has an out-star \(\overline{K}_{1,n-m}\) and the centre of \(\overline{K}_{1,n-m}\) is \(x\).

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