The end of eternal inflation

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Abstract

We propose a new measure for eternal inflation. We require that both conditions, large field fluctuations and smooth homogeneous domains be included in the probability estimate in order to produce successful inflationary pockets. We show that due to the increasing inhomogeneities in the background spacetime fractal, self-reproductions stop within a finite time $t_f$, thus inflation cannot be eternal.

Keywords: inflation, homogeneity, measure

1. Introduction

Inflation is generally believed to be a natural way of solving three classic problems in cosmology: the flatness problem, the homogeneity problem, the horizon problem and the monopole problem. It therefore purports to explain why we live in a state that is well described by a smooth Friedmann–Lemaitre–Robertson–Walker (FLRW) universe. There is a difficulty with this scenario, and that is in the nature of the initial conditions that give rise to such inflation. Suppose that inflation is due to a single scalar field $\varphi$, the inflaton, governed by a potential $V(\varphi)$. In order to account for the observed present homogeneity of the universe, one imposes that at the beginning of inflation the universe must be homogeneous on a scale about $10^4$ times the horizon scale \cite{1}. Thus inflation has merely transformed one homogeneity problem into another one, as has been emphasized by Goldwirth and Piran \cite{2}, Goldwirth \cite{3}, Calzetta and Sakellariadou \cite{4}, and Trodden and Vachaspati \cite{1, 5}. In other words, for inflation to take place successfully, the inflationary universe scenario hypothesizes an incredibly homogeneous initial state. The requirement of such a special initial state is the fine-tuning problem of inflation; or to put it in a slightly different way, this requirement gives rise to the puzzle of why the pre-inflationary state should have such low entropy. The entropy of an initial state is proportional to the volume occupied by it in the phase space of the initial
states. The improbable initial state needed for starting inflation therefore implies that this state occupies a small volume in the phase space of initial conditions.

Once inflation starts, quantum fluctuations of the metric and of the inflaton field $\phi$ generate density perturbations $\delta \rho / \rho$ with a scale invariant spectrum [21]. The dynamics of large wavelength fluctuations is driven not only by the potential term, known as the drift, $\partial \phi / \partial t$, but also by the backreaction of short wavelength modes which provide a diffusion term $f(t, x)$ as described in more detail in the next section. When the dynamics of long wavelength perturbations is dominated by diffusion it leads to the stochastic behavior of these modes. The common lore is that a random large fluctuation will give rise to a newly produced inflating region, a new ‘bubble’ universe. The field undergoes Brownian motion in the diffusion regime, then it enters a regime where large fluctuations are favored and therefore the universe keeps reproducing ad infinitum [6, 7, 10, 13]. The self-reproducing regime is known as eternal inflation and it was first discovered by Linde in [7]. The global geometry of the eternally inflating universe is highly non-trivial, it is a fractal where regions of large field excursions occupy a space volume with dimension $d < 3$ [11, 13].

Unlike scenarios of single shot inflation that are obtained by the extreme fine tuning described above for the initial state, eternal inflation appears to be unavoidable with many regions undergoing inflation. This leads to a picture of a universe filled with many bubbles that are the regions of spacetime that can be considered to be their own universes. It is generic in the sense that all episodes of inflation arise spontaneously, without any fine-tuning of initial conditions and such events continue to occur forever.

The ‘generic’ reproduction of new universes seems to lead to an unbounded growth of phase space; in other words an infinite number of ‘free lunches’ is directly responsible for appearing to violate unitarity on global scales. Eternal inflation then leads to a collection of difficult problems. Spacetime is incomplete in the past, so there must still be some kind of initial singularity [14, 15]. Space has become fractal in nature with non-inflating regions being the fractal set. The measure for the geometry of space has become non-normalizable associated to the apparent loss of unitarity. Tracing back in time this system of unbounded entropy with a fractal space dimension, makes the problem of the improbable initial state an even more exquisitely special choice [17, 18].

In this letter we examine the basics of eternal inflation by addressing the question: what is the probability that random large fluctuations of a scalar field can lead to the self-reproducing inflationary regime? We address this issue by taking into account the two ingredients needed for producing an inflationary bubble, namely the probability that the field will have a large fluctuation and also the probability that this large fluctuation will arise on a highly homogeneous patch of spacetime. As we show next, the probability to get a new bubble universe is

$$P_t = P_n \times P_{st},$$

where $P_n$ is the probability that the field has a large fluctuation and $P_{st}$ is the probability that this fluctuation does not ‘fall’ in an inhomogeneous small domain of the background spacetime. We show that this measure is not infinite, and it does not violate unitarity. Further we show that this measure leads to the conclusion that the probability for producing new inflationary pockets, $P_n$, is even more unlikely than just simply obtaining a random large fluctuation of the scalar field $P_n$, namely $P_{st} \ll 1$ since inhomogeneities from density perturbations grow quite fast thereby making the global spacetime geometry much more inhomogeneous in the future than the geometry at $t = 0$ in the far past when inflation was not eternal. As we show below, generically $P_t \ll 1$. 

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[21]
2. Basics and setup

Let us assume that we have an inflaton field $\phi$ in some slow roll potential, $V(\phi)$ in a FLRW universe with scale factor $a(t)$. In the absence of any spatial dependence of $\phi$, the field evolution is governed by

$$\dot{\phi} + 3H\phi + \frac{dV(\phi)}{d\phi} = 0,$$

where

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right]$$

with $k = 0, \pm 1$ for flat, closed or open universes.

Inflation generates perturbations $\delta \phi$ which are frozen on superhorizon wavelengths with an amplitude $\delta \phi \approx \frac{H}{2\pi}$. Coarse-graining of the subhorizon fluctuations leads to a diffusion source term $f(x,t)$ for the long wavelengths modes [6] such that

$$< f(x_1,t_1) f(x_2,t_2) > \approx \frac{H^4}{4\pi^2} \delta(t_1 - t_2) \frac{\sin z}{z},$$

where $z = aH(t_1 - t_2)$. The coarse-graining of short wavelength modes results in a Langevin equation for the field [6, 7]

$$3H\dot{\phi}(x,t) + \frac{dV(\phi)}{d\phi} = f(x,t),$$

where the second derivative terms have, as usual, been ignored. The probability distribution function (PDF) for the field, $P_\phi$, satisfying equation 5 is given by a Fokker–Planck equation which provides the diffusion equation that describes the Brownian motion of the field

$$\frac{\partial P_\phi}{\partial t} = \frac{1}{3H} \frac{\partial}{\partial \phi} \left[ V'(\phi) P_\phi \right] + \frac{H^3}{8\pi^2} \frac{\partial^3 P_\phi}{\partial \phi^3}$$

$$= \frac{\partial}{\partial \phi} \left[ \frac{1}{3H} P_\phi \frac{dV}{d\phi} + \frac{\partial}{\partial \phi} \left( DP_\phi \right) \right]$$

as was first described by Starobinsky [6]. The diffusion coefficient obtained after coarse-graining is $D = \frac{H^4}{2\pi^2}$. The first term in equation 6 gives the probability current due to the drift from the potential, while the second term gives the current due to the diffusion [6, 7, 10]. If $V(\phi) = \frac{1}{2} m^2 \phi^2$, corresponding to a free field of mass $m$, a stationary solution to equation 6 is the same as the Hartle–Hawking (HH) expression for the probability, $P_\phi \approx e^{-3\phi V(\phi)} \approx e^{-\frac{m^2 \phi^2}{6H^2}}$, while the non-stationary solutions are more complex, as is discussed in detail, for example, in [6, 12, 20]. In what follows, we use proper time denoted by $t$.

3. A critical examination of the probability of eternal inflation

3.1. At the root of the measure problem

The drift term in equation 6 is given by the slope of the potential and it is thus bound by the slow roll parameter. That term describes the rolling down of the classical field. The second term in this equation, the current of the field due to diffusion, dominates the evolution at late
times, since \( <\dot{\phi}^2> \approx D t \) grows without bound at large \( t \). This is the reason why the diffusion term overtakes the drift term in equation 6 as concluded in [6, 7, 12]. At the time this happens, the field enters a regime where large fluctuations are favored. These estimates correspond to comoving volumes of horizon size \( H^{-1} \) with a fixed comoving Planck length. The latter choice is useful in avoiding issues associated with transplanckian physics since a fixed physical Planck length leads to a time-dependent phase space constantly replenished by transplanckian modes.

So far the procedure appears quite straightforward: one implements a coarse-graining scheme by integrating out the environmental degrees of freedom and finds out that the 'bath' leads to a diffusion source term in the evolution equation for the 'system' which sets the field into a random walk. In this case the 'bath' is comprised of subhorizon wavelength modes and the 'system' corresponds to long wavelength modes of the field. The combined dynamics of the drift and diffusion currents in equation 6 determine the PDF for the field \( \phi \) to be found at some sufficiently large value \( \phi_s \approx \phi + \delta \phi \) for a new 'bubble' universe. Applied to a false vacuum decay potential this PDF is nothing more than the nucleation rate of bubbles \( \Gamma \approx e^{-S} \) with a 4-action \( S_4 \), that is the number of bubbles per unit 4-volume, [13]. Applied to new, chaotic slow roll potential, or any other inflationary model, the PDF gives, up to a proportionality constant, the concentration of all the field fluctuations that reach the point \( \phi_s \approx \phi + \delta \phi \), i.e. the number of large fluctuations \( N(\phi) \) that reach the point \( \phi_s \) per unit spacetime volume, that is

\[
P_\phi \approx \frac{N(\phi_s)}{V_4}, \tag{7}
\]

where the time interval \( \delta t = CH^{-1} \) (with \( C < 1 \) a constant depending on the details of the potential), corresponds to intervals during which the field within a Hubble volume remains nearly unchanged. At this stage, the conventional estimate [6, 7, 12] that leads to the conclusion that inflation is eternal is based on the following two assumptions: (i) it assumes that a large fluctuation \( \phi_s \) will automatically guarantee the production of a new inflationary universe; (ii) it assumes that the field PDF should be multiplied by the newly produced inflating volumes in physical coordinates, namely

\[
P_i = P_\phi V_4^{\text{new}} = CH^{-1} P_\phi V_3^{\text{new}}, \tag{8}
\]

where `i` counts the newly produced bubbles.

We believe that assumptions (i) and (ii) above are incorrect and they lead to the paradox of infinite measure. The obvious conclusion obtained by following (i) and (ii) and thus equation 8, which ignores the spacetime homogeneity requirement on the estimates for eternal inflation, is the result that the probability of bubble production, and thus of eternal inflation, can become not only large but in fact infinite due to the 4-volume scaling in equation 7. The main reason for the paradoxical conclusions, such as the infamous infinite measure problem derived from these assumptions, is the fact that a large fluctuation does not necessarily produce a new universe. The unbounded growth of phase space resulting from the conventional estimate of probability becomes apparent since equation 8 introduces an extra source term, \( 3H(\phi)P_\phi \), in the diffusion equation 6 written in physical coordinates, [19]. The probability of the field fluctuation finding an exquisitely homogeneous domain on the global inhomogeneous background spacetime is as crucial to the reproduction process of inflationary pockets as the field fluctuation itself.

Since the original proposal for eternal inflation in [7], there have been many measures put forth, all of which try to avoid infinities by introducing some ad hoc cutoff surface and by reparametrizing time. All of them are pathological in the sense that infinities persist. Besides,
they also produce a new set of paradoxes such as the ‘youthness paradox’, the ‘oldness paradox’ and the ‘Boltzmann Brain’ paradox [9, 16]. Most of the popular measures can be written in the following form \( P = P_\beta \times V_\beta \) where \( V_\beta = e^{3Ht_\beta} \), with \( \beta \) the time parameter depending on the definition of the timeslicing \( t_\beta \), [9].

We propose a new physical measure for the probability of eternal inflation \( P_i \) by replacing \((i)\) with the requirement of homogeneity on the initial state of any newly produced bubble for any potential inflationary domain and any fluctuation. We propose here that the PDF of the field \( \phi \) should be multiplied with the probability \( P_{st} \) of finding a homogeneous domain in a highly inhomogeneous background in physical coordinates, produced by inflation, instead of multiplying it by the volume as previously done in equation 7. The reason for the multiplication of the two probabilities is based on the fact that both conditions that must be met for the successful creation of an inflationary pocket, namely the constraint of large field fluctuations and the constraint of smooth Hubble sized domains on the global spacetime, are random and thus independent of one another.

As is well known, having a large field fluctuation is a necessary but not sufficient condition for starting inflation since the spacetime region where this fluctuation arises should be exquisitely fine tuned to be smooth on scales larger than the Hubble volume. This is the unpopular fine tuning problem of inflation. Therefore, it does not matter how many large fluctuations we have, they will not give rise to inflationary pockets if they arise on inhomogeneous domains on the spacetime background. The two key issues relevant for the existence of any inflationary pocket that we propose must be demanded, are: \((a)\) the probability of finding a large fluctuation \textit{and}, \((b)\) the probability that this large field excursion finds an homogeneous region in a highly inhomogeneous global background. Previously, the commonly advocated approach relied on taking advantage of the exponential growth of the volume in physical coordinates by multiplying the probability of large field fluctuations \( P_\phi \) with this newly created physical volume \((ii)\) and \((iii)\). As we describe next, volume multiplication of \( P_\phi \) changes a measure into a number count of fluctuations. Volume multiplication does not remove the two constraints needed for inflation, and in fact this volume multiplication is directly responsible for the problem of infinities in all previously suggested measures.

3.2. Our proposal: the probability of eternal inflation

Demanding \((a)\) large field fluctuations and \((b)\) initial homogeneous spacetime regions in estimating the probability of eternal inflation, leads to the expression of equation 1 for the probability of inflation \( P_i \), namely \( P = P_\beta \times P_{st} \).

The proposed probability in equation 1 is our main point. Equation 1 provides the correct estimate for the probability of producing new inflationary regions, i.e. the measure of eternal inflation, since it takes into account both conditions that are needed for producing an inflationary universe, namely large field fluctuations on smooth homogeneous spacetime domains. The reason why previous estimates of measures based on criterion \((ii)\) led to false results and infinite measures is because the estimated quantity from equation 7 is in fact the number of fluctuations passing through \( \phi \) anywhere on the background space, and not the PDF for bubble production. Let us recall that the PDF for the field \( P_\phi \) simply provides the concentrations of fluctuations with strength \( \delta \phi \) that go through some space time point \((x, t)\),

\[
P_\phi \simeq \frac{\mathcal{N}(\phi_0 + \delta \phi)}{V_\phi}.
\]
Where $N$ is the number of all field excursions that start at $\phi_0$ and fluctuate to the point $\phi_0 + \delta\phi$ and $V_4$ is the 4-volume. If we were to multiply this expression with the 3-volume of space in physical coordinates, as was done in previous estimates of the measure $(i)$ and $(ii)$, then what we are really calculating is not a probability but a number flux, namely the number of all field fluctuations that go through the point $\phi_0 + \delta\phi$ at some time $t$ anywhere in the physical space 3-volume $V_3$. Certainly, the number of fluctuations should be infinite, thus it is no surprise that the measure always ends up being infinite. The previous estimate of probability for eternal inflation based on assumptions $(i)$ and $(ii)$ rather paradoxically leads to a growing phase space. These two assumptions taken together lead to a picture where one estimates the total number of fluctuations anywhere in the physical space time volume (which of course is infinite) and equates that with a probability distribution of inflating pockets, therefore a growing phase space of inflation. To illustrate this point further we can use the example of false vacuum decay: if we were interested in the total number of instanton bubbles anywhere in the spacetime volume, then the number of instantons would obviously be infinite even for the dilute gas case, since in physical coordinates the space volume is $V_i \approx a^3 \approx e^{3Ht}$ and therefore according to equation 7 we should have that the probability of eternal inflation is $P_i \approx H^{-1}V_i \approx H^{-2}e^{3Ht} \rightarrow 0$.

Next, we would like to estimate the probability of large fluctuations finding homogeneous domains on the background spacetime, $P_{st}$, in order to investigate the results of the new measure equation 1. Generally in a highly inhomogeneous spacetime such as this, homogeneous regions are the inflating domains. Therefore, $P_{st}$ is given by the ratio of volumes occupied by inflating regions over the global volume of spacetime

$$P_{st} = \frac{V_{st}}{V_{tot}}.$$  \hspace{1cm} (10)

3.3. The probability of homogeneous domains $P_{st}$

A consistent treatment of eternal inflation should incorporate the relation of diffusive perturbations in equation 6 with the global spacetime metric via Einstein equations. Long wavelength inhomogeneities produce density perturbations $\delta \rho$ which grow at large scales. Thus the metric of spacetime on global scales is quite complex and highly inhomogeneous. Part of this structure can be captured by the following metric with a space and time dependent scale factor $a(x, t)$

$$ds^2 \approx dr^2 - a(x, t)^2 dx^2.$$  \hspace{1cm} (11)

Let us emphasize the complex inhomogeneous structure of spacetime resulting from the dissipative field dynamics by reviewing some previous results obtained in [12]. These authors [12] used a conformally Newtonian coordinate system in order to relate the field to the metric

$$ds^2 \approx (1 + 2\Phi(x, \tau)) dr^2 - (1 - 2\Psi(x, \tau)) dx^2$$  \hspace{1cm} (12)

(with $\tau$ the usual conformal time), and show that the relation obtained by Einstein equations and the perturbation equation, $\frac{d\Phi}{d\tau} \approx -2\Phi$ and $\Psi \approx \Phi$ holds for at least around an inflating region. In this gauge, the relation between the background potential $\Phi$ and the field fluctuations leads in a straightforward manner to a diffusion equation for $\Phi$ which is analogous to equation 6 for the field diffusion. That is, the metric of spacetime responds directly to field excursions due to diffusion, as it should based on Einstein equations. The background potential $\Phi$ has a dispersion $\Delta \Phi$ which also grows with time [12].
\[ \Delta \phi^2 \approx <\phi^2> \approx \frac{2V^2}{3(3\pi VM)^{1/2}} \tau. \] (13)

The growing dispersion on the metric is not surprising since we expect \( \rho \gg \rho_0 \) at large scales. Since \( \frac{\rho}{\rho_0} \approx -2\Phi \) during inflation then naturally density perturbations grow with time with the same dispersion as \( \Delta \phi^2 \).

The key issue in these estimates is the fact that globally the universe becomes highly inhomogeneous. Further work showed that in fact the global geometry of the universe is that of a fractal whereby homogeneous regions with \( \Delta \phi \ll O(1) \) occupy a very small volume of dimension less than three [11]. Since the spacetime metric tracks the field then the probability of finding homogeneous regions at large times, estimated in [11] is

\[ P \approx \int_{\phi_0}^{\phi_0} P(\phi) d\phi \approx \frac{4}{\pi} e^{\frac{2\Delta \phi^2}{4\rho_0^2}} \]

where \( \phi_0 \) is the boundary value of the field’s excursion. This estimate leads to inhomogeneous spatial regions with fractal dimension \( d \) having a volume

\[ V_{3h} \approx e^{d \phi_0}, \quad d = 3 - \frac{D\pi^2}{4\rho_0^2} \] (14)

For the case of an effective potential \( V = V_0 - m^2\phi^2 / 2 \) near the beginning of an inflationary epoch, with \( m^2 \ll H^2 \), the fractal dimension is \( d = 3 - \frac{m^2}{H^2} \) and the probability of not having inhomogeneous regions is \( P \approx e^{-\frac{2\Delta \phi^2}{m^2 H^2}} \). Similarly for \( V = V_0 - \lambda\phi^4 / 4 \), \( d = 3 - \frac{4\lambda}{3H^2} \). Starobinsky and Yokoyama [20] studied the stochastic behaviour of the field on a deSitter (dS) background and estimated the two point correlation functions in space and time, and showed that the spatial correlation size does not depend on time, i.e. the physical size of domains with growing and decreasing field values remains constant, independent of the dS background expansion.

The implications of these results are that since the number of homogeneous and inhomogeneous domains, even on a dS background, scales with the expansion then we can use the scaling properties of fractals to estimate the probability that a large field fluctuation finds a homogeneous domain on this fractal background. The evolution of each h-domain with \( \Delta \phi \leq H \) is equivalent to the evolution of a single large volume \( V_{3h}^{tot} \) with size \( R \) which started inflating at the initial time \( t = 0 \) [11], since the stochastic field excursions and therefore the evolution of each h-domain is independent from each other. Therefore, the total spatial volume for all homogeneous domains of size \( x \ll R \) in this background of size \( R \) occupies a volume \( V^h \approx x^d \) due to the self similar properties of the fractal geometry. It follows that the probability of finding a homogeneous region of size \( x \) at any time ‘t’ in this inhomogeneous background of size \( R \) is

\[ P_t = \frac{V^h}{V_{3h}^{tot}} = \left( \frac{x}{R} \right)^{3-d}. \] (15)

For example, for chaotic or new inflation type potentials \( P_t \approx (\frac{x}{R})^{\frac{2\rho_0}{m^2}} \). Note that \( R = H^{-1}a(t) \) where \( a(t) \) is some averaged value of the background expansion scale factor. Then

\[ P_t \approx (Hx)^{\frac{m^2}{3H^2}} e^{\left( \frac{m^2 H^2}{3H^2} \right) t}. \] (16)

This relation holds for any potential \( V \) provided that the deviation \( \alpha \) of the fractal dimension from the background space of dimension three, \( \alpha = [3 - d] \) replaces \( \alpha = m^2 / 3H^2 \) in the above example. It is clear from equation 16 that inhomogeneities grow too much within a time \( t_f \approx \frac{3H}{m^2} \approx (aH)^{-1} \). The critical timescale for the growth of inhomogeneities implies that
the probability of a fluctuation finding a homogeneous region in order to produce an inflationary pocket, is incredibly small for all times \( t \geq t_f \), according to our proposed measure. This result should be expected of any fractal geometry since the volume of homogeneous regions has a dimension less than three, and thus cannot fill the 3-volume of space and this homogeneous volume is increasingly diminishing with time relative to the background volume, as we proved in equations 15 and 16. The resulting spacetime becomes highly inhomogeneous within a time interval, \( 0 < t < t_f \), and the probability of finding smooth domains becomes smaller and smaller. Large fluctuations cannot give rise to inflationary pockets without arising on homogeneous domains. In short, inflation cannot self reproduce beyond the point \( t > t_f \).

4. Discussion

In estimating the likelihood of homogeneous domains \( P_{st} \), we treated the field distribution and therefore the geometry of Hubble patches as an independent random walk. All h-volumes are considered independent from each other since each domain containing some field value is a stochastic variable, with roughly half of the domains having large field fluctuations and the other half having small fluctuations.

The global structure of spacetime is a crucial element in determining the probability of new pockets being produced. Pockets of eternal inflation from random field fluctuations do not occur inside a single h-volume but rather throughout the global spacetime. If fluctuations are assumed to arise only inside a single h-domain, that would render the rest of the spacetime volume irrelevant due to the absence of correlations among h-domains. In that case the theory becomes unfalsifiable. The correct treatment requires the global approach where the field is distributed randomly everywhere on the spacetime volume. Some domains in that global volume can be smooth and homogeneous, and some other domains are highly inhomogeneous. In fact the fractal nature of spacetime globally is a sufficient indicator that the overwhelming majority of the volume will be contained in the highly inhomogeneous domains. The volume occupied by ‘smooth’ domains is miniscule compared to the volume of the inhomogeneous part, \( P_{st} \ll 1 \), whereas field excursions due to diffusion arise randomly everywhere in the global background. If a random large field excursion happens to arise in a smooth domain of this volume, then inflation occurs and a pocket universe is produced. But, if it arise on inhomogeneous domains, then inflation will not occur despite the fact that the field fluctuation may be large.

Since large field excursions arise randomly, then the probability that large field excursions will scan smooth domains in the global spacetime depends on how many smooth domains can be found on the global spacetime relative to the total volume. The volume of smooth domains occupies a space with dimension less than 3, leading to the conclusion that the inhomogeneous part of the volume is the overwhelming dominant volume of spacetime. Letting the field go into diffusion excursions on such background spacetime results in the chance that the field ‘visits’ a smooth domain to be incredibly small. By randomness, the field will be diffusing into inhomogeneous domains in the large majority of excursions. In the large time limit the whole volume of spacetime is contained almost entirely in the inhomogeneous volume. In the limit of vanishing volume of homogeneous domains, which roughly occurs during time \( t_f \), the magnitude of field excursions does not matter because inflating domains can not be produced no matter what fluctuation value the field may take.

The condition for producing an inflating pocket is a large field fluctuation arising in a smooth domain of a size of at least the Hubble volume. Density fluctuations grow on superhorizon scales and can overclose the universe if they are too large, e.g. if \( \delta \rho / \rho \sim O(1) \) at
distances $10^9 H^{-1}$. Normally homogeneity is required not only inside bubble-volume domains but also in a range of superhubble distances since all fluctuations contribute to the anisotropies field. For example in the case of our universe, density fluctuation should be less than $10^{-5}$ up to $10^4 H^{-1}$ distances. Therefore our estimate of $P_{st}$ is generous since we did not impose a condition on the size of the domain needed to produce an inflating pocket in estimating measure. Instead we counted all the volumes occupied by homogeneous domains independent of their size when estimating the likelihood of the field finding roughly a Hubble volume in the fractal, $P_{st}$. If we were to count only smooth domains above a certain size in $P_{st}$ and discount those smooth regions that may be smaller than a h-volume, then the probability of homogeneous domains $P_{st}$ would be even smaller.

The crucial point accounted for in the new measure we propose is that finding smooth domains in a fractal spacetime should have been a real concern in estimating the existence of eternal inflation and thus should be included in the measure.

The full measure we proposed here, $P = P_B \times P_d$ for example in the case of chaotic type inflation where the field has to fluctuate upwards in the potential, yields the following
density

\[ P = P_B \times P_d \approx B e^{-\frac{(\phi - \phi_0)^2}{2\sigma^2} (\mathcal{H}x)^d} e^{\frac{\sigma^2}{m^2}} \]  

(17)

with $B$ a normalization constant. The case of false vacuum decay potential [24] leads to exactly the same result by replacing in equation 17 the fluctuation PDF with the bubble nucleation rate $P_B = \Gamma t$ and the fractal dimension $\frac{d}{m} = [3 - d]$ with the result relevant to the false vacuum decay fractal $[3 - d] = -\frac{D}{D_t} \Gamma$ obtained in [15].

The measure in equation 17 is finite, normalizable, and it preserves unitarity. The latter can be seen by the fact that the diffusion PDF for the field is directly converted to a diffusion PDF for the entropy of fluctuations, $S \approx \log [P_B]$. Since individually large fluctuations spread in phase space due to diffusion and dissipation, then the volumes their trajectories occupy in the phase space grows. But the whole closed system that includes all the excursions does not change. The closed system occupies a volume in phase space bound by a finite entropy $S$ given by the background horizon $H^{-2}$. The trajectory of each individual fluctuation can only spread and grow within the finite phase space volume occupied by the closed system. In short, the phase space volume of the whole system and its finite entropy are larger than that of individual bubbles and fluctuations. Although the entropy of these subsystems can grow, it is finite and can only grow inside the bounds provided by the whole background system. The entropy of the background system is that of the field at the hypersurface of initial conditions, i.e. the entropy at the initial moment $t = 0$ when inflation first switches on, which does not change.

The new measure we propose removes some of the problems and paradoxes related to previous measures of eternal inflation. Previous measures were unbounded because they resulted in large fluctuations being favored and equated that process with the production of new bubble universes. The probability of decaying to false vacuum or jumping higher up the potential were favored over the decay to lower energy states. Further, damping was missing from the Einstein–Smoluchowski (E–S) type relation of dissipation to damping, due to the infinities produced. It is easy to see that with the new measure the E–S relation $D/t = k T$, where $\mu$ is the damping coefficient, is satisfied since $T \approx H^{-1}$, therefore $\mu \approx DH$ is related to $\mu \approx \left(\frac{2\rho}{2m}\right)^{\frac{1}{2}}$, [22].

Before the measure proposed here, we were in a situation where either probability was not conserved or, it was not finite, or both. Besides, the width of the wavepacket $<\delta \phi^2> \approx Dt \gg 1$ loses quantum coherence which makes the exponential expansion of the
newly produced \( h \)-domains nearly impossible \([20]\). Even fluctuations with \( \langle \delta p^2 \rangle \sim \ll 1 \), run the danger of \( \langle h \rangle \gg 1 \) by the uncertainty principle in which case slow roll inflation can not be obtained despite large field fluctuations. Taking into account the homogeneity requirement precludes the existence of coherence for the wavepacket, which ensures that we recover the correspondence with the usual semiclassical equations of inflation where coherent quantum fluctuations behave as classical states \([23]\).

5. Conclusion

We have shown that inflation cannot sustain an eternal replenishment of ‘free lunches’ of new universes after a time \( t > t_f \approx \frac{1}{d - 3/2} \), where \( d \) is the fractal dimension of spacetime, since the background space grows highly inhomogeneous. Therefore, one of the two conditions for producing inflationary regions, namely: (a) a large fluctuation and (b) an exquisitely homogeneous domain, is not fulfilled after time \( t > t_f \).

The new measure we advocate here takes both requirements of producing inflationary pockets into account, in the probability expression. In contrast to our measure, the homogeneity condition (b) into the probability measure was not included in the previous proposals for the measure of eternal inflation. The new measure is finite, normalizable, and it preserves unitarity.

We have shown here that, despite the diffusive behaviour of field fluctuations, the probability of subsequent episodes of inflation is highly suppressed relative to even the probability of single shot inflation, which is already incredibly small. The reason for this behaviour arises from the fact that the growing diffusion in the field triggers a growing dispersion of the spacetime metric as given by the gravitational potential related to density perturbations. Ultimately the growing dispersion in metric fluctuations results in a highly inhomogeneous spacetime, a fractal spacetime. Thus inflation cannot be eternal.

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