SOME OLD AND BASIC FACTS ABOUT RANDOM WALKS ON GROUPS

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Abstract. This note contains old instead of new results about random walks on groups, which may serve as a supplement to the author’s monograph [15]. First, we exhibit a basic exercise on the periodicity classes of random walk. The second topic concerns some basics on ratio limits for random walks, which had been published “only” in German in the 1970ies.

1. Introduction

In all that follows, Γ will be a countable, discrete group written multiplicatively, with elements typically denoted by x, y, etc., unit element e, and μ a probability measure on Γ. The support of μ is

\[ S_\mu = \{ x \in \Gamma : \mu(x) > 0 \} . \]

Recall that the random walk on Γ with law μ is the time-homogeneous Markov chain with state space Γ and transition probabilities \( p_\mu(x, y) = \mu(x^{-1}y) \). The \( n \)-step transition probabilities are then \( p_\mu^{(n)}(x, y) = \mu^{(n)}(x^{-1}y) \), where \( \mu^{(n)} \) is the \( n \)th convolution power of μ. Its support is \( S_\mu^n \), the set of all products \( x_1 \cdots x_n \) of group elements. Thus, the random walk is irreducible in the sense of Markov chains (resp., non-negative matrices) if and only if

\[ \bigcup_{n=1}^\infty S_\mu^n = \Gamma , \]

i.e., the semigroup generated by the support is all of Γ. (In general, for two subsets A, B of Γ, their product is \( AB = \{ xy : x \in A, y \in B \} \), and \( A^{-1} = \{ x^{-1} : x \in A \} \).

• In this note, we always assume irreducibility.

In §2, we recall a basic fact about the period and aperiodicity of the random walk, and in §3, we recall some ratio limit theorems.

2. Aperiodicity

As an irreducible Markov chain, the random walk has a period

\[ d = d(\mu) = \gcd \{ n \in \mathbb{N} : \mu^{(n)}(e) > 0 \} , \]

compare with [15, page 3]. The random walk is called aperiodic, if \( d(\mu) = 1 \). For general \( d \), as for any irreducible Markov chain, there is a partition in subsets

\[ \Gamma = C_0 \cup C_1 \cup \ldots \cup C_{d-1} , \]

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with \( e \in C_0 \), such that the random walk wanders through the set \( C_j \) cyclically: if \( x \in C_j \) and \( y \in C_k \) then \( \mu^{(n)}(x^{-1}y) > 0 \) only when \( n \equiv k - j \pmod{d} \), and then this holds false all but finitely many such \( n \). We now recall a group-theoretical fact which was considered a basic exercise in earlier days but now is sometimes being “discovered” by younger researchers. The following has also has a generalisation to random walks on locally compact groups, which was studied by Woess [14] – which however is written in French and not available online, apart from the author’s webpage.

**Proposition.**

\[
\Gamma_0 = \bigcup_{k=1}^{\infty} S_{\mu}^{-k} S_{\mu}^{k}
\]

is a normal subgroup of \( \Gamma \) with index \( d \). The factor group \( \Gamma / \Gamma_0 \) is cyclic of order \( d \). One has \( C_0 = \Gamma_0 \), and the sets \( C_j \) are the cosets of \( \Gamma_0 \).

The probability measure \( \mu^{(d)} \) is supported by \( \Gamma_0 \) and irreducible on that subgroup, with period 1.

**Proof.** We first show that by (1), one has that \( \Gamma_0 \) is a normal subgroup; this can be found, e.g., in the lecture notes by Mukherjea and Tserpes [7, p. 97].

(a) It is clear that \( e \in \Gamma_0 \) and that \( x^{-1} \in \Gamma_0 \) whenever \( x \in \Gamma_0 \).

(b) If \( x \in \Gamma \) then there is \( n \) such that \( x \in S_{\mu}^n \). Hence \( x^{-1}S_{\mu}^{-k}S_{\mu}^k x \subset S_{\mu}^{-n-k}S_{\mu}^{k+n} \), so that \( x^{-1}\Gamma_0 x \subset \Gamma_0 \).

(c) Finally, by (b),

\[
(S_{\mu}^{-k}S_{\mu}^k)(S_{\mu}^{-n}S_{\mu}^n) \subset (\Gamma_0 S_{\mu}^{-n})S_{\mu}^n = S_{\mu}^{-n}\Gamma_0 S_{\mu}^n \subset \Gamma_0,
\]

so that \( \Gamma_0 \) is closed with respect to the group product.

Next, we observe that \( d(\mu) = \gcd N_{\mu} \), where \( N_{\mu} = \{ n \in \mathbb{N} : e \in S_{\mu}^n \} \), and that the latter set is closed under addition. This implies via elementary number theory that there are \( n_1, n_2 \) such that \( e \in S_{\mu}^{n_1} \cap S_{\mu}^{n_2} \) and \( d(\mu) = n_2 - n_1 \). Therefore

\[
e \in S_{\mu}^{-n_1}S_{\mu}^{n_1}S_{\mu}^{d} \subset \Gamma_0 S_{\mu}^{d}, \quad \text{and} \quad S_{\mu}^{-d} \subset S_{\mu}^{-d}\Gamma_0 S_{\mu}^{d} \subset \Gamma_0.
\]

Therefore also \( S_{\mu}^{nd} \subset \Gamma_0 \) for all \( n \in \mathbb{N} \).

Now suppose that \( S_{\mu}^{k} \cap S_{\mu}^{m} \neq \emptyset \) for \( k \neq m \), and let \( x \) be an element of that set. By (1), there is \( n \) such that \( x^{-1} \in S_{\mu}^n \). Then \( e \in S_{\mu}^{n+k} \cap S_{\mu}^{m+n} \), whence \( k + n \) and \( m + n \) belong to \( N_{\mu} \). We conclude that \( d \) divides \( m - k \).

In particular, let \( x_0 \in S_{\mu} \). If \( x_0^k \in \Gamma_0 \) then \( x_0^k \in S_{\mu}^{-n}S_{\mu}^{-n} \) for some \( n \), so that \( S_{\mu}^{k+n} \cap S_{\mu}^{n} \neq \emptyset \) and \( d \) divides \( k \). Thus, the cosets \( x_0^k\Gamma_0 \), \( k = 0, \ldots, d-1 \) are all distinct, and \( x_0^d\Gamma_0 = \Gamma_0 \). This holds for any \( x_0 \in S_{\mu} \), and it shows that \( \Gamma / \Gamma_0 \) is cyclic of order \( d \).

Turning to the random walk, if \( x \in x_0^k\Gamma_0 \) for \( k \in \{ 0, \ldots, d-1 \} \) and \( p(x, y) = \mu(x^{-1}y) > 0 \) then \( y \in S_{\mu}^{k+1} \) and \( yx_0^d \in S_{\mu}^d \subset \Gamma_0 \). Consequently, \( y \in x_0^{k+1}\Gamma_0 \).

By all that we have proved so far, we have that \( S_{\mu}^n \cap \Gamma_0 = \emptyset \) when \( d \) does not divide \( n \), so that

\[
\bigcup_{n=1}^{\infty} S_{\mu}^{nd} = \Gamma_0,
\]
which means that $\mu^{(d)}$ is irreducible on $\Gamma_0$. Since $N_\mu$ is closed under addition, there is $n_0$ such that $nd \in N_\mu$ for all $n \geq n_0$. Therefore $\mu^{(d)}$ has period 1 on $\Gamma_0$. □

If $\mu$ has period 1, then it is called aperiodic. By the Proposition, we may restrict attention to that case.

3. Spectral radius and convergence

For irreducible $\mu$ as in the preceding section, the number

$$\rho(\mu) = \limsup_{n \to \infty} \mu^{(n)}(x)^{1/n}$$

is independent of $x \in \Gamma$. It is called the **spectral radius** in a variety of references, including [15]. We stick to this terminology, but one should be careful: in general, this is not the spectral radius of an operator; it is the spectral radius (and norm) of $\mu$ as a convolution operator on $\ell^2(\Gamma)$ when $\mu$ is symmetric, i.e., $\mu(x^{-1}) = \mu(x)$ for all $x$, and on a weighted $\ell^2$-space when $\mu$ is reversible, see [15, §10]. The study of this number, for general irreducible Markov chains, goes back to Pruitt [9] and Vere-Jones [13]; see also the monograph by Seneta [11].

**Lemma.** For every $x \in \Gamma$, one has convergence:

$$\lim_{n \to \infty} \mu^{(n)}(x)^{1/n} = \rho(\mu),$$

and the sequence $\mu^{(n)}(e)^{1/n}$ converges to its limit from below. Furthermore, there is $k_0$ such that

$$\mu^{(n)}(e) < \rho(\mu)^n \quad \text{strictly for all } n \geq k_0.$$

**Proof.** The convergence result is the multiplicative version of Fekete’s Lemma [2]. Let $a_n = \mu^{(n)}(e)$. Then $a_n > 0$ for all $n \geq n_0$ and $a_m a_n \leq a_{m+n} \leq 1$. For fixed $m \in n$, let $n = qm + r$ with $n_0 \leq r = r_n < n_0 + m$. Then

$$a_n \geq a_r a_m^q \geq c_m a_m^q,$$

where $c_m = \min\{a_r : n_0 \leq r < n_0 + m\} > 0$.

Therefore $a_n^{1/n} \geq c_m^{1/n} a_m^{q/n}$, and letting $n \to \infty$,

$$\rho(\mu) \geq \liminf_{n \to \infty} a_n^{1/n} \geq a_m^{1/m}.$$

This holds for every $m$, and letting $m \to \infty$, we get that $a_m^{1/m}$ tends to $\rho(\mu)$ from below. For general $x$, there is $k$ such that $\mu^{(k)}(x) > 0$, and $\mu^{(n)}(x) \geq \mu^{(k)}(x) \mu^{(n-k)}(e)$. Taking $n^{th}$ roots on both sides, we get that also $\mu^{(n)}(x)^{1/n}$ converges.

The last observation is due to Gerl [4]: fix $x_0 \in S_{\mu} \setminus \{e\}$. Then $x_0^{-1} \in S_{\mu}^{r_0}$ for some $r_0 \in \mathbb{N}$, and by the above,

$$\mu^{(n)}(x_0) \mu^{(n)}(x_0^{-1}) > 0 \quad \text{for all } n \geq n_0 + r_0 =: k_0$$

Now suppose that for some $n \geq k_0$, $\mu^{(n)}(e) = \rho(\mu)^n$. Then

$$\rho(\mu)^{2n} \geq \mu^{(2n)}(e) = \sum_{x \in \Gamma} \mu^{(n)}(x) \mu^{(n)}(x^{-1}) = \rho(\mu)^{2n} + \sum_{x \neq e} \mu^{(n)}(x) \mu^{(n)}(x^{-1}).$$
But then $\mu^{(n)}(x_0)\mu^{(n)}(x_0^{-1}) = 0$, a contradiction. \hfill \square

The following result of Gerl [4] is less straightforward, and was the primary motivation for writing this little note. It is based on an argument in [3], compare also with Guivarc’h [5, pp. 18-19]. According to Le Page [6], the argument can be traced back to Orey and Kingman [8].

**Theorem 1.** If $\mu$ is irreducible and aperiodic on $\Gamma$, then

$$\lim_{n \to \infty} \frac{\mu^{(n+1)}(x)}{\mu^{(n)}(x)} = \rho(\mu) \quad \text{for every } x \in \Gamma.$$  

**Proof.** Write $\rho = \rho(\mu)$. Recall the transition probabilities $p(x, y) = \mu(x^{-1}y)$ of the random walk. Irreducibility yields that there is a positive $\rho$-subharmonic function $h : \Gamma \to (0, \infty)$, that is,

$$\sum_{y \in \Gamma} \mu(x^{-1}y)h(y) \leq \rho h(x) \quad \text{for every } x \in \Gamma.$$  

See [9], [11] or [15, Lemma 7.2]. In many cases, there even is such a function which is $\rho$-harmonic, i.e., equality holds at every $x$.

We now define a new Markov chain on $\Gamma$, the $h$-process, with transition probabilities

$$p_{h}(x, y) = \frac{\mu(x^{-1}y)h(y)}{\rho h(x)}.$$  

This is in general not a group-invariant random walk. The transition matrix (denoted $Q$ in [4])

$$P_{h} = (p_{h}(x, y))_{x, y \in \Gamma}$$  

is substochastic, i.e., $\sum_{y} p_{h}(x, y) \leq 1$ for all $x$, so that there may be a positive probability that the Markov chain “dies” at $x$. Furthermore, along with $\mu$ it is irreducible and aperiodic: for all $x, y$, there is $n_{x,y}$ such that $p_{h}^{(n)}(x, y) > 0$ for all $n \geq n_{x,y}$, where $p_{h}^{(n)}(x, y)$ is the $(x, y)$-entry of the matrix power $P_{h}^{n}$. We have by Lemma 3

$$p_{h}^{(n)}(x, y) = \frac{\mu^{(n)}(x^{-1}y)h(y)}{\rho^{n} h(x)} \quad \text{and} \quad \lim_{n \to \infty} p_{h}^{(n)}(x, y)^{1/n} = 1 \quad \text{for all } x, y \in \Gamma.$$  

In particular, for all $x$,

$$0 < p_{h}^{(n)}(x, x) = \frac{\mu^{(n)}(e)}{\rho^{n} h(x)} < 1 \quad \text{for all } n \geq k_0.$$  

We now fix $m \geq k_0$ and set $a = a_{m} = 1 - p_{h}^{(m)}(x, x)$, so that $0 < a < 1$, as well as

$$Q = \frac{1}{a}(P_{h}^{m} - (1 - a)I),$$  

where $I$ is the identity matrix over $\Gamma$. (The matrix $Q$ is denoted $R$ in [4].) We shall also need the matrix $E$ over $\Gamma$ with all entries $= 1$. For the next lines of the proof, we
just write $P$ for $P^n_h$. Then $Q$ is also non-negative, substochastic and irreducible, and $P = aQ + (1 - a)I$. Note that $Q$ commutes with $P_h$, and that $P_hE \leq E$. Then

$$P^n = \sum_{k=0}^{n} p_a(n, k) Q^k,$$

where $p_a(n, k) = \binom{n}{k} a^k (1 - a)^{n-k}$.

The latter is the probability that the sum $S_n = X_1 + \cdots + X_n$ of i.i.d. Bernoulli random variables with $\Pr[X_k = 1] = a$ has value $k$. For $\varepsilon > 0$, consider the set

$$C_n(\varepsilon) = \{ k \in \{0, \ldots, n\} : p_a(n, k) \leq (1 + \varepsilon) p_a(n + 1, k + 1) \}$$

and its complement $C_n(\varepsilon)^c$ in $\{0, \ldots, n\}$. Then

$$\sum_{k \in C_n} p_a(n, k) = \Pr\left[ \frac{S_n + 1}{n + 1} - a > \varepsilon a \right].$$

This is a large deviation probability, which is well known to decay exponentially, i.e., there is $\delta > 0$ such that it is $\leq e^{-n\delta}$. In our specific case, this can also be verified by combinatorial computations. See e.g. Rényi [10, p. 324]. Then, using matrix products and elementwise inquality between matrices,

$$\frac{1}{a} P^{n+1} - \frac{1-a}{a} P^n = QP^n = \sum_{k=0}^{n} p_a(n, k) Q^{k+1} \leq e^{-\delta n} E + (1 + \varepsilon) \sum_{k \in C_n} p_a(n + 1, k + 1) Q^{k+1} \leq e^{-\delta n} E + (1 + \varepsilon) P^{n+1}.$$

Reassembling the terms,

$$\left( 1 - \frac{a}{1-a} \varepsilon \right) P^{n+1} \leq \frac{a}{1-a} e^{-\delta n} E + P^n.$$

We multiply from the left with $P^r_h$, where $r \in \mathbb{N}_0$ is arbitrary, and get for the matrix elements

$$\left( 1 - \frac{a}{1-a} \varepsilon \right) \frac{p_h^{(mn+m+r)}(x, y)}{p_h^{(mn+r)}(x, y)} \leq \frac{a}{1-a} \frac{e^{-\delta n}}{p_h^{(mn+r)}(x, y)} + 1.$$

We are not dividing by 0 if $n$ is sufficiently large, and since $p_h^{(mn+r)}(x, y)^{1/n} \rightarrow 1$, the right hand side tends to 1 as $n \rightarrow \infty$. Since we can choose $\varepsilon$ arbitrarily small, we get

$$\limsup_{n \rightarrow \infty} \frac{p_h^{(mn+m+r)}(x, y)}{p_h^{(mn+r)}(x, y)} \leq 1,$$

and this holds for every $m \geq k_0$ and every $r \geq 0$.

For an analogous lower bound, we use the set

$$D_n(\varepsilon) = \{ k \in \{0, \ldots, n\} : p_a(n + 1, k + 1) \leq (1 + \varepsilon) p_a(n, k) \}$$
and observe that
\[
\sum_{k \in C_n} p_a(n+1, k+1) = \Pr \left[ \frac{S_{n+1}+1}{n+1} - a < -\frac{\varepsilon}{1+\varepsilon} a \right].
\]
also decays exponentially, and is bounded by \(e^{-\delta n}\) for some \(\delta > 0\). Then
\[
P^{n+1} \leq e^{-\delta n} E + (1 + \varepsilon) \sum_{k \in D_n(\varepsilon)} p_a(n, k) Q^{k+1}
\]
\[
\leq e^{-\delta n} E + (1 + \varepsilon) P^n Q = e^{-\delta n} E + (1 + \varepsilon) \left( \frac{1}{a} P^{n+1} - \frac{1-a}{a} P^n \right).
\]
Reassembling the terms,
\[
(1 + \varepsilon) P^n \leq \frac{a}{1-a} e^{-\delta n} E + \left( 1 + \frac{\varepsilon}{1-a} \right) P^{n+1}.
\]
Proceeding as above, we get for all \(m \geq k_0\) and all \(r \geq 0\)
\[
\lim_{n \to \infty} \inf \frac{p_{h^{(mn+m+r)}}(x,y)}{p_{h^{(mn+r)}}(x,y)} \geq 1,
\]
for every \(m \geq k_0\) and every \(r \geq 0\). Since these two numbers can be chosen arbitrarily, it is an easy exercise to deduce from (5) and (6) that
\[
\lim_{n \to \infty} \frac{p_{h^{(n+1)}}(x,y)}{p_{h^{(n)}}(x,y)} = 1,
\]
and the stated result follows. \(\square\)

**Remark.** The last theorem is not restricted to random walks on groups. If \(P\) is the transition matrix of an irreducible Markov chain on a countable set – say – \(\Gamma\), and it is *strongly aperiodic*, that is,
\[
\gcd \{ n \in \mathbb{N} : \inf_x p^{(n)}(x, x) > 0 \} = 1,
\]
then the same proof applies to show that
\[
\lim_{n \to \infty} \frac{p^{(n+1)}(x,y)}{p^{(n)}(x,y)} = \rho(P) \quad \text{for all } x, y \in \Gamma.
\]

In the situation of Theorem 1, assume for simplicity that \(S_{\mu}\) is finite. Then it is well known and easy to deduce that the sequence of measures
\[
\left( \frac{\mu^{(n)}}{\mu^{(n)}(e)} \right)_{n \in \mathbb{N}}
\]
is relatively compact in the topology of pointwise convergence, and every limit measure \(\nu\) satisfies the convolution equation
\[
\mu \ast \nu = \rho(\mu) \cdot \nu,
\]
or in other words, the function $h(x) = \nu(x^{-1})$ is $\rho(\mu)$-harmonic, that is
\[
\sum_y \mu(x^{-1}y) h(y) = \rho(\mu) h(x) \quad \text{for all } x \in \Gamma.
\]

[4] and (under slightly different conditions, where the state space is not necessarily discrete) [5] prove the following ratio limit theorem.

**Theorem 2.** Assume that $\mu$ is irreducible and aperiodic, and that $S_\mu$ is finite. Suppose that (P) is a certain property of positive measures $\nu$ on $\Gamma$ such that

- every limit along a pointwise convergent subsequence

\[
\left( \frac{\mu^{(n_k)}}{\mu^{(n_k)}(e)} \right)_{k \in \mathbb{N}}
\]

must have property (P), and

- there is a unique positive measure $\nu$ with $\nu(e) = 1$ that satisfies (8).

Then
\[
\lim_{n \to \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(e)} = \nu(x) \quad \text{for all } x \in \Gamma.
\]

The proof is clear in view of the observations preceding the statement of Theorem 2, i.e., relative compactness and (8), which are left to the reader as exercises or to be looked up in old references. Finiteness of $S_\mu$ can be relaxed. Typical examples of application are (virtually) Abelian groups, where property (P) is empty, because there is a unique solution $\nu$ of (8): there is a good amount of literature from the 1960ies on this. The most significant reference for Abelian groups is Stone [12].

Other typical examples are isotropic random walks on free groups, and property (P) is that $\nu$ is also isotropic; see [4]. It should be noted that in those cases, one also has stronger results, namely local limit theorems; see [15, Chapter III].

**Author’s final, personal remarks.** When I wrote the monograph [15] in the 2nd half of the 1990ies, ratio limit theorems were not an active topic, but replaced by the study of the asymptotic type of random walk transition probabilities and the sharper local limit theorems. Therefore, having to keep the book size under control, I had “sacrificed” the material of ratio limit theorems. In the meantime, the subject has “woken up” again, e.g. in a recent paper of Dor-On [1] and current work of Dougall and Sharp.

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