New proofs of some results on BMO martingales using BSDEs

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Abstract. Using properties of backward stochastic differential equations we give new proofs of some well known results on BMO martingales and improve some estimates of BMO norms.

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1 Introduction

The BMO martingale theory is extensively used to study backward stochastic differential equations (BSDEs). Some properties of BMO martingales was already used by Bismut[3] when he discussed the existence and uniqueness of a solution of some particular backward stochastic Riccati equations, choosing the BMO space for the martingale part of the solution process. In the work of Delbaen et al [5] conditions for the closedness of stochastic integrals with
respect to semimartingales in $L^2$ were established in relation to the problem of hedging contingent claims and linear BSDEs. Most of this conditions deal with BMO martingales and reverse Hölder inequalities. BMO martingales naturally arise in BSDEs with quadratic generators. When the generator of a BSDE has quadratic growth then the martingale part of any bounded solution of the BSDE is a BMO martingale. This fact was proved in [8, 11, 12, 13, 15, 16] under various degrees of generality. Later, the BMO norms were used to prove an existence, uniqueness and stability results for BSDEs, among others in [1, 2, 4, 11, 12, 16, 17].

The aim of this paper is to do the converse: to prove some results on BMO martingales using the BSDE technique.

It is well known that if $M$ is a BMO martingale, then the mapping $\phi : \mathcal{L}(P) \ni X \rightarrow \tilde{X} = \langle X, M \rangle - X \in \mathcal{L}(\tilde{P})$ is an isomorphism of $BMO(P)$ onto $BMO(\tilde{P})$, where $d\tilde{P} = \mathcal{E}_T(M)dP$. E. g., it was proved by Kazamaki [9, 10] that the inequality

$$||\tilde{X}||_{BMO(\tilde{P})} \leq C_K(\tilde{M}) \cdot ||X||_{BMO(P)}$$

is valid for all $X \in BMO(P)$, where the constant $C_K(\tilde{M}) > 0$ is independent of $X$ but depends on the martingale $M$. Using the properties of a suitable BSDE we prove this inequality with a constant $C(M)$ which we express as a linear function of the $BMO(\tilde{P})$ norm of $\tilde{M} = \langle M \rangle - M$ and which is less than $C_K(\tilde{M})$ for all values of this norm.

Using properties of BSDEs we prove also the well known equivalence between BMO property, Muckenhoupt and reverse Hölder conditions (Doleanse-Dade and Meyer [7], Kazamaki [10]) and obtain BMO norm estimates in terms of reverse Hölder and Muckenhoupt constants.

## 2 Reverse Hölder and Muckenhoupt conditions and relations with BSDEs

We start with a probability space $(\Omega, \mathcal{F}, P)$, a finite time horizon $0 < T < \infty$ and a filtration $F = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness.

We recall definitions of BMO martingales, Reverse Hölder and Muckenhoupt conditions (see, e.g., Doleanse-Dade and Meyer [7], or Kazamaki [10]).
Definition 1. A continuous, uniformly integrable martingale \((M_t, \mathcal{F}_t)\) with \(M_0 = 0\) is said to be from the class \(BMO\) if
\[
\|M\|_{BMO} = \sup_{\tau} \left\| E \left[ (\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau \right]^{1/2} \right\|_{\infty} < \infty,
\]
where the supremum is taken over all stopping times \(\tau \in [0, T]\) and \(\langle M \rangle\) is the square characteristic of \(M\).

Denote by \(\mathcal{E}(M)\) the stochastic exponential of a continuous local martingale \(M\):
\[
\mathcal{E}_t(M) = \exp\{M_t - \frac{1}{2} \langle M \rangle_t\}.
\]

Throughout the paper we shall assume that \(M\) is a continuous local martingale with \(\langle M \rangle_T < \infty\) \(P\)-a.s. This implies that \(\mathcal{E}_T(M) > 0\) \(P\)-a.s. and let \(\mathcal{E}_{\tau,T}(M) = \mathcal{E}_T(M)/\mathcal{E}_\tau(M)\).

Definition 2. Let \(1 < p < \infty\). \(\mathcal{E}(M)\) is said to satisfy \((R_p)\) condition if the reverse Hölder inequality
\[
E \left[ \left\{ \mathcal{E}_{\tau,T}(M) \right\}^p \right| \mathcal{F}_\tau \leq C_p
\]
is valid for every stopping time \(\tau\), with a constant \(C_p > 0\) depending only on \(p\).

If \(\mathcal{E}(M)\) is a uniformly integrable martingale then by the Jensen inequality we also have that
\[
E \left[ \left\{ \mathcal{E}_{\tau,T}(M) \right\}^p \right| \mathcal{F}_\tau \geq 1.
\]

A condition dual to \((R_p)\) is the Muckenhoupt condition \((A_p)\).

Definition 3. \(\mathcal{E}(M)\) is said to satisfy \((A_p)\) condition for \(1 < p < \infty\) if there is a constant \(D_p > 0\) such that for every stopping time \(\tau \in [0, T]\)
\[
E \left[ \left\{ \mathcal{E}_{\tau,T}(M) \right\}^{-\frac{1}{p-1}} \right| \mathcal{F}_\tau \leq D_p.
\]

Note that, since \(\mathcal{E}(M)\) is a supermartingale, the Jensen inequality implies the converse inequality
\[
E \left[ \left\{ \mathcal{E}_{\tau,T}(M) \right\}^{-\frac{1}{p-1}} \right| \mathcal{F}_\tau \geq \left\{ E \left[ \mathcal{E}_{\tau,T}(M) \right| \mathcal{F}_\tau \right\}^{-\frac{1}{p-1}} \geq 1.
\]

In this paper we shall consider only linear BSDEs of the type
\[
Y_t = Y_0 - \int_0^t [\alpha Y_s + \beta \psi_s] d\langle M \rangle_s + \int_0^t \psi_s dM_s + N_t, \quad Y_T = 1,
\]
where \( \alpha \) and \( \beta \) are constants. A solution of such BSDE we define as a triple \((Y, \psi, N)\), with \( \langle N, M \rangle = 0 \), from the space \( S^\infty \times BMO(P) \times H^2(P) \) equipped with the following norms

\[
|||Y|||_\infty = |||Y_T^*|||_{L^\infty}, \quad \text{where} \quad Y_T^* = \sup_{t \in [0,T]} |Y_t|,
\]

\[
|||\psi \cdot M|||_{BMO(P)} = \sup_{\tau} \left| \frac{1}{E} \int_{\tau}^{T} \psi_s d\langle M \rangle_s |_{\mathcal{F}_\tau} \right|^{1/2},
\]

\[
|||N|||_{H^2} = E^{1/2}[N].
\]

Note that, since the martingale \( M \) is assumed to be continuous, only the latter term of this equation may have the jumps, i.e., \( \Delta Y = \Delta N \). In order to avoid the definition of BMO norms for right-continuous martingales, we are using the \( H^2 \) norms for orthogonal martingale parts. This is sufficient for our goals, since the generators of equations under consideration does not depend on orthogonal martingale parts.

Sometimes we call \( Y \) alone the solution of BSDE, keeping in mind that \( \psi \cdot M + N \) is the martingale part of \( Y \).

**Lemma 1.** Let \( M \) be a continuous local martingale.

a) \( \mathcal{E}(M) \) satisfies (\( R_p \)) if and only if there exists a bounded, positive solution of BSDE

\[
\begin{align*}
Y_t &= Y_0 - \int_0^t \left[ \frac{p(p-1)}{2} Y_s + p\psi_s \right] d\langle M \rangle_s + \int_0^t \psi_s dM_s + N_t, \\
Y_T &= 1.
\end{align*}
\]

(1)

b) \( \mathcal{E}(M) \) satisfies (\( A_p \)) if and only if there exists a bounded, positive solution of equation

\[
\begin{align*}
X_t &= X_0 - \int_0^t \left[ \frac{p}{2(p-1)^2} X_s - \frac{1}{p-1} \varphi_s \right] d\langle M \rangle_s + \int_0^t \varphi_s dM_s + L_t, \\
X_T &= 1.
\end{align*}
\]

(2)

**Proof: a)** Let first show that if \( \mathcal{E}(M) \) satisfies (\( R_p \)) then the process

\[
Y_t = E \left\{ \mathcal{E}_{t,T}(M) \right\}^{p} |_{\mathcal{F}_t}
\]

is a solution of BSDE (1). It is evident that \( Y \) is a bounded positive process and that \( Y \left\{ \mathcal{E}_t(M) \right\}^{p} \) is a uniformly integrable martingale. Therefore, since \( \mathcal{E}_t(M) > 0 \), the process \( Y \) will be a special semimartingale. Let \( Y_t = Y_0 + A_t + m_t \) be the canonical decomposition of \( Y \),
where $m$ is a locally square integrable martingale and $A$ a predictable process of bounded variation. Using the Galtchouk-Kunita-Watanabe decomposition for $m$, we get

$$Y_t = Y_0 + A_t + \int_0^t \psi_s dM_s + N_t,$$

where $N$ is a local martingale orthogonal to $M$.

Now using the Ito formula we have

$$Y_t \{ E_t(M) \}^p = Y_0 + \int_0^t \left[ \frac{p(p-1)}{2} Y_s + p \psi_s \right] \{ E_s(M) \}^p d\langle M \rangle_s +$$

$$+ \int_0^t \{ E_s(M) \}^p dA_s + \tilde{m}_t,$$

where $\tilde{m}$ is a local martingale.

Because $Y_t \{ E_t(M) \}^p$ is a martingale, equalizing the part of bounded variation to zero, we obtain that

$$A_t = - \int_0^t \left[ \frac{p(p-1)}{2} Y_s + p \psi_s \right] d\langle M \rangle_s,$$

which implies that $Y_t = E \left[ \{ E_{t,T}(M) \}^p | F_t \right]$ is a solution of equation (1).

Now let equation (1) admits a bounded positive solution $Y_t$. Using the Ito formula for the process $Y_t \{ E_t(M) \}^p$ we get that $Y_t \{ E_t(M) \}^p$ is a local martingale. Hence it is a supermartingale, as a positive local martingale. Therefore, from the supermartingale inequality and the boundary condition $Y_T = 1$ we obtain that $E \left[ \{ E_{t,T}(M) \}^p | F_t \right] \leq Y_t$. Because $Y$ is bounded, this implies that $E(M)$ satisfies $(R_p)$ condition.

b) The proof is similar to the proof of the part a), we only need to replace $p$ by $-\frac{1}{p-1}$.

Let $E(M)$ be a uniformly integrable martingale. Denote by $\hat{P}$ a new probability measure defined by $d\hat{P} = E_T(M) dP$ and let $\hat{M} = \langle M \rangle - M$.

Now we shall give a new proof of the well known equivalence (Doleanse-Dade and Meyer [7], Kazamaki [10]) between BMO property, Muckenhoupt and reverse H"older conditions.

**Theorem 1**: Let $E(M)$ be a uniformly integrable martingale. Then the following conditions are equivalent:
i). \( \tilde{M} \in BMO(\tilde{P}) \).

ii). \( \mathcal{E}(M) \) satisfies the \((R_p)\) condition for some \( p > 1 \).

iii). \( M \in BMO(P) \).

iv). \( \mathcal{E}(M) \) satisfies the \((A_p)\) condition for some \( p > 1 \).

Proof: For the sake of simplicity, in all proofs given here, we shall assume without loss of generality that all stochastic integrals are martingales, otherwise one can use the localization arguments.

\( i \implies ii \) Let \( \tilde{M} \in BMO(\tilde{P}) \). According to Lemma 1 it is sufficient to show that equation (1) admits a bounded positive solution for some \( p > 1 \).

Let us rewrite equation (1) in terms of the \( \tilde{P} \)-martingale \( \tilde{M} \):

\[
\begin{cases}
Y_t = Y_0 - \int_0^t \left( \frac{p(p-1)}{2} Y_s + (p-1)\psi_s \right) d\langle M \rangle_s - \int_0^t \psi_s d\tilde{M}_s + N_t, \\
Y_T = 1.
\end{cases}
\]

Since \( \langle N, M \rangle = 0 \), \( N \) is a local \( \tilde{P} \)-martingale orthogonal to \( \tilde{M} \).

Define the mapping \( H : S^\infty \times BMO(\tilde{P}) \times H^2(\tilde{P}) \rightarrow S^\infty \times BMO(\tilde{P}) \times H^2(\tilde{P}) \), which maps \( (y, \psi, n) \in S^\infty \times BMO(\tilde{P}) \times H^2(\tilde{P}) \) onto the solution \( (Y, \Psi, N) \) of the BSDE (1), i.e.,

\[
Y_t = E^{\tilde{P}} \left[ 1 + \int_t^T \left( \frac{p(p-1)}{2} y_s + (p-1)\psi_s \right) d\langle M \rangle_s \Bigg| \mathcal{F}_t \right]
\]

and

\[
- \int_0^t \Psi_s d\tilde{M}_s + N_t = E^{\tilde{P}} \left[ 1 + \int_t^T \left( \frac{p(p-1)}{2} y_s + (p-1)\psi_s \right) d\langle M \rangle_s \Bigg| \mathcal{F}_t \right].
\]

We shall show that there exists \( p > 1 \) such that this mapping is a contraction. Let

\[
\delta Y = Y^1 - Y^2, \quad \delta y = y^1 - y^2, \quad \delta \Psi = \Psi^1 - \Psi^2, \quad \delta \psi = \psi^1 - \psi^2, \quad \delta N = N^1 - N^2.
\]

It is evident that \( \delta Y_T = 0 \) and

\[
\delta Y_t = \delta Y_0 - \int_0^t \left( \frac{p(p-1)}{2} \delta y_s + (p-1)\delta \psi_s \right) d\langle M \rangle_s - \int_0^t \delta \Psi_s d\tilde{M}_s + \delta N_t.
\]

According to the Ito formula, applied for \( (\delta Y_t)^2 - (\delta Y_T)^2 \) and taking conditional expectations we have

\[
(\delta Y_t)^2 + E^{\tilde{P}} \left[ \int_\tau^T (\delta \Psi_s)^2 d\langle M \rangle_s \Bigg| \mathcal{F}_\tau \right] + E^{\tilde{P}} \left[ (\delta N)_T - [\delta N]_\tau \Bigg| \mathcal{F}_\tau \right] =
\]
\[
E \tilde{P} \left[ \int_T \tau \delta Y_s \delta y_s d(M)_s \bigg| F_\tau \right] + E \tilde{P} \left[ \int_T 2(p-1) \delta Y_s \delta \psi_s d(M)_s \bigg| F_\tau \right]
\]
and using elementary inequalities we obtain
\[
(\delta Y_\tau)^2 + E \tilde{P} \left[ \int_T (\delta \Psi_s)^2 d(M)_s \bigg| F_\tau \right] + E \tilde{P} \left[ [\delta N]_T - [\delta N]_\tau \bigg| F_\tau \right] \leq
\]
\[
\leq \frac{p(p-1)}{2} \left\| M \right\|^2_{BMO(\tilde{P})} \cdot ||\delta Y\|_\infty^2 + \frac{p(p-1)}{2} \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} \cdot ||\delta y\|_\infty^2 +
\]
\[
+ (p-1) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} \cdot ||\delta Y\|_\infty^2 + (p-1) \left\| \int \delta \psi d\tilde{M} \right\|^2_{BMO(\tilde{P})}.
\]

Because the right hand side of the inequality does not depend on \( \tau \), we will have
\[
\left( 1 - p(p-1) \right) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} - 2(p-1) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} \|\delta Y\|_\infty^2 +
\]
\[
+ \left\| \int \delta \psi d\tilde{M} \right\|^2_{BMO(\tilde{P})} + ||\delta N||^2_{L^2(\tilde{P})} \leq
\]
\[
\leq p(p-1) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} \|\delta y\|_\infty^2 + 2(p-1) \left\| \int \delta \psi d\tilde{M} \right\|^2_{BMO(\tilde{P})}.
\]

Since
\[
1 - (p-1)(p+2) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})} < 1
\]
for \( p \) sufficiently close to 1, one can make the constant of \( ||\delta Y||^2_\infty \) in the left-hand side of (5) positive and we finally obtain the inequality
\[
\|\delta Y\|_\infty^2 + \left\| \int \delta \psi d\tilde{M} \right\|^2_{BMO(\tilde{P})} + ||\delta N||^2_{H^2(\tilde{P})} \leq
\]
\[
\leq \alpha(p) \cdot ||\delta y\|_\infty^2 + \beta(p) \cdot \left\| \int \delta \psi d\tilde{M} \right\|^2_{BMO(\tilde{P})},
\]
where
\[
\alpha(p) = \frac{p(p-1) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})}}{1 - (p-1)(p+2) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})}},
\]
\[
\beta(p) = \frac{2(p-1)}{1 - (p-1)(p+2) \left\| \tilde{M} \right\|^2_{BMO(\tilde{P})}}.
\]
It is easy to see that \( \lim_{p \downarrow 1} \alpha(p) = \lim_{p \downarrow 1} \beta(p) = 0 \). So, if we take \( p^* \) such that \( \alpha(p^*) < 1 \) and \( \beta(p^*) < 1 \) we obtain that the mapping \( H \) is a contraction and there exists a unique solution \( (Y, \Psi, N) \) of (1) in \( S^\infty \times BMO(\tilde{P}) \times H^2(\tilde{P}) \).

Since \( \alpha(p) \) and \( \beta(p) \) are decreasing functions of \( p \in (1, \infty) \), the norms \( ||Y||_\infty \) and \( ||\Psi \cdot \tilde{M}||_{BMO(\tilde{P})} \) are uniformly bounded, as functions of \( p \) for \( p \in [1, p^*] \). Therefore, for any \( p \in [1, p^*] \) we have

\[
Y_t = E^\tilde{P} \left[ 1 + \int_t^T \left( \frac{p(p-1)}{2} Y_s + (p-1) \Psi_s \right) d\langle M \rangle_s \bigg| \mathcal{F}_t \right] \tag{7}
\]

and

\[
Y_t \geq 1 - \frac{p(p-1)}{2} ||Y||_\infty ||\tilde{M}||_{BMO(\tilde{P})} - \frac{p-1}{2} ||\tilde{M}||_{BMO(\tilde{P})} - \frac{p-1}{2} ||\Psi \cdot \tilde{M}||_{BMO(\tilde{P})} \geq 0
\]

for some \( p \) sufficiently close to 1. Hence, there exists a bounded positive solution of equation (1) for some \( p > 1 \), which implies that \( \mathcal{E}(M) \) satisfies the \( R_p \) condition, according to Lemma 1.

\( ii) \implies iii) \) Let \( \mathcal{E}(M) \) be a uniformly integrable martingale and satisfies the \( (R_p) \) condition for some \( p > 1 \). Then the process \( Y_t = E \left[ \{ \mathcal{E}_{t,T}(M) \}^p \big| \mathcal{F}_t \right] \) is a solution of equation (1) and satisfies the two-sided inequality

\[
1 \leq Y_t \leq C_p.
\]

Using the Ito formula for \( e^{-\beta Y_t} - e^{-\beta Y_t} \) and taking conditional expectations we have

\[
e^{-\beta} - e^{-\beta Y_t} = \beta \frac{p(p-1)}{2} E \left[ \int_T^T \kappa e^{-\kappa_s} d\langle M \rangle_s \bigg| \mathcal{F}_t \right] + \frac{\beta^2}{2} E \left[ \int_T^T e^{-\beta Y_s} d\langle N^c \rangle_s \bigg| \mathcal{F}_t \right] + E \left[ \sum_{\tau < s \leq T} (e^{-\beta Y_s} - e^{-\beta Y_{s-}} + \beta e^{-\beta Y_{s-}} \Delta Y_s) \bigg| \mathcal{F}_t \right].
\]

Since \( \frac{\beta^2}{2} \psi_s^2 + \beta \psi_s \geq -\frac{\beta^2}{2}, \quad e^{-\beta Y_s} - e^{-\beta Y_{s-}} + \beta e^{-\beta Y_{s-}} \Delta Y_s \geq 0 \) and \( Y_t \geq 1 \), taking \( \beta > \frac{1}{p-1} \) we obtain the inequality

\[
\frac{p}{2} (\beta(p-1) - p) e^{-\beta C_p} E \left[ \langle M \rangle_T - \langle M \rangle_T \bigg| \mathcal{F}_t \right] \leq e^{-\beta} - e^{-\beta C_p},
\]

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which implies that

$$||M||_{BMO(P)}^2 \leq \frac{2(e^{\beta(C_p-1)} - 1)}{p(\beta(p-1) - p)}$$

for any $\beta > \frac{p}{p-1}$.

$iii) \implies iv)$ If $M$ is a $BMO(P)$ martingale, then according to Lemma 1 it is sufficient to show that equation (2) admits bounded positive solution for some $p > 1$, which can be proved similarly to the implication $i) \implies ii)$. By the same way one can show that for the mapping $H$

$$X_t = E \left[ 1 + \int_t^T \left[ \frac{p}{2(p-1)} x_s - \frac{1}{p-1} \varphi_s \right] d\langle M \rangle_s \bigg| \mathcal{F}_t \right],$$

where $-\int_0^t \Phi_s dM_s + L_t$ is the martingale part of $X$, the inequality (3) holds with

$$\alpha(p) = \frac{p||M||_{BMO(P)}^2}{(p-1)^2 - (3p-2)||M||_{BMO(P)}^2},$$

$$\beta(p) = \frac{2(p-1)}{(p-1)^2 - (3p-2)||M||_{BMO(P)}^2},$$

where $\lim_{p \to \infty} \alpha(p) = \lim_{p \to \infty} \beta(p) = 0$. So if we take $p$ large enough we obtain that the mapping $H$ is a contraction.

$iv) \implies i)$ The proof is similar to the proof of the implication $ii) \implies iii)$. In particular, for the BMO norm of $\tilde{M}$ the following inequality holds

$$||\tilde{M}||_{BMO(P)}^2 \leq \frac{2(p-1)^2}{p(\beta - p)} (e^{\beta(D_p-1)} - 1)$$

for any $\beta > p$, where $D_p$ is a constant from Definition 3.

$\square$

### 3 Girsanov’s transformation of BMO martingales and BSDEs

Let $M$ be a continuous local $P$-martingale such that $\mathcal{E}(M)$ is a uniformly integrable martingale and let $d\tilde{P} = \mathcal{E}_T(M)dP$. To each continuous local
martingale $X$ we associate the process $\tilde{X} = \langle X, M \rangle - X$, which is a local $\tilde{P}$-martingale according to Girsanov's theorem. We denote this map by $\varphi : \mathcal{L}(P) \to \mathcal{L}(\tilde{P})$, where $\mathcal{L}(P)$ and $\mathcal{L}(\tilde{P})$ are classes of $P$ and $\tilde{P}$ local martingales.

Let consider the process

$$ Y_t = E^{\tilde{P}}[\langle X \rangle_T - \langle X \rangle_t | \mathcal{F}_t] = E[\mathcal{E}_{t,T}(M)(\langle X \rangle_T - \langle X \rangle_t) | \mathcal{F}_t]. \quad (8) $$

Since $\langle \tilde{X} \rangle = \langle X \rangle$ under either probability measure, it is evident that $||Y||_{\infty} = ||\tilde{X}||^2_{BMO(\tilde{P})}$.

Let $M \in BMO(P)$. According to Theorem 1 condition $(R_p)$ is satisfied for some $p > 1$. The $(R_p)$ condition and conditional energy inequality (Kazamaki [10], page 29) imply that for any $X \in BMO(P)$ the process $Y$ is bounded, i.e., $\varphi$ maps $BMO(P)$ into $BMO(\tilde{P})$. Moreover, as proved by Kazamaki [9, 10], $BMO(P)$ and $BMO(\tilde{P})$ are isomorphic under the mapping $\phi$ and for all $X \in BMO(P)$ the inequality

$$ ||\tilde{X}||^2_{BMO(\tilde{P})} \leq C^2_K(\tilde{M}) \cdot ||X||^2_{BMO(P)} \quad (9) $$

is valid, where

$$ C^2_K(\tilde{M}) = 2p \cdot 2^{1/p} \sup_{\tau} \left\| E^{\tilde{P}}\left[ \mathcal{E}_{\tau,T}(\tilde{M}) \right]^{(p-1)/p} | \mathcal{F}_{\tau} \right\|_{\infty}^{(p-1)/p}, \quad (10) $$

and $p$ is such that

$$ ||\tilde{M}||_{BMO(\tilde{P})} < \sqrt{2}(\sqrt{p} - 1). \quad (11) $$

Note that the similar inequality holds for the inverse mapping $\phi^{-1}$.

Now we give an alternative proof of this assertion, which improves also the constant in the inequality [2].

**Theorem 2.** If $M \in BMO(P)$, then $\tilde{X} = (X) \rightarrow \tilde{X}$ is an isomorphism of $BMO(P)$ onto $BMO(\tilde{P})$. In particular, the inequality

$$ \frac{1}{1 + \frac{\sqrt{2}}{2} ||M||_{BMO(P)}} ||X||_{BMO(P)} \leq ||\tilde{X}||_{BMO(\tilde{P})} \leq \left( 1 + \frac{\sqrt{2}}{2} ||M||_{BMO(\tilde{P})} \right) ||X||_{BMO(P)}. \quad (12) $$
is valid for any $X \in BMO(P)$.

**Proof:** Similarly to Lemma 1 one can show that for any $X \in BMO(P)$ the process $Y$ (defined by (13)) is a positive bounded solution of the BSDE

$$
\begin{align*}
Y_t &= Y_0 - \langle X \rangle_t - \int_0^t \varphi_s d\langle M \rangle_s + \int_0^t \varphi_s dM_s + L_t, \\
Y_T &= 0.
\end{align*}
$$

(13)

Applying the Ito formula for $(Y_\tau + \varepsilon)^p - (Y_T + \varepsilon)^p$ where $0 < p < 1$, $\varepsilon > 0$ and taking conditional expectations we obtain

$$(Y_\tau + \varepsilon)^p - \varepsilon^p = E\left[\int_\tau^T p(Y_s + \varepsilon)^{p-1}d\langle X \rangle_s \big| \mathcal{F}_\tau\right] + \frac{p(1-p)}{2} E\left[\int_\tau^T (Y_s + \varepsilon)^{p-2}d\langle L^c \rangle_s \big| \mathcal{F}_\tau\right] +
\frac{p}{2(1-p)} (Y_s + \varepsilon)^{p-2} \varphi_s^2 + p(Y_s + \varepsilon)^{p-1} \varphi_s + \frac{p}{2(1-p)} (Y_s + \varepsilon)^p \\ - E[\Sigma_{\tau<s\leq T}((Y_s + \varepsilon)^p - (Y_s^- + \varepsilon)^p - p(Y_s^- + \varepsilon)^{p-1}\Delta Y_s) \big| \mathcal{F}_\tau].
$$

(14)

Because $f(x) = x^p$ is concave for $p \in (0,1)$, the last term in (14) is positive. Therefore, using the inequality

$$
\frac{p}{2(1-p)} (Y_s + \varepsilon)^{p-2} \varphi_s^2 + p(Y_s + \varepsilon)^{p-1} \varphi_s + \frac{p}{2(1-p)} (Y_s + \varepsilon)^p \geq 0
$$

from (14) we obtain

$$(Y_\tau + \varepsilon)^p - \varepsilon^p \geq E\left[\int_\tau^T p(Y_s + \varepsilon)^{p-1}d\langle X \rangle_s \big| \mathcal{F}_\tau\right] -
\frac{p}{2(1-p)} E\left[\int_\tau^T (Y_s + \varepsilon)^p d\langle M \rangle_s \big| \mathcal{F}_\tau\right].
$$

(15)

Since $0 < p < 1$

$$
p(||Y||_\infty + \varepsilon)^{p-1} E[\langle X \rangle_T - \langle X \rangle_\tau \big| \mathcal{F}_\tau] \leq E\left[\int_\tau^T p(Y_s + \varepsilon)^{p-1}d\langle X \rangle_s \big| \mathcal{F}_\tau\right],
$$

from (15) we have

$$
p(||Y||_\infty + \varepsilon)^{p-1} E[\langle X \rangle_T - \langle X \rangle_\tau \big| \mathcal{F}_\tau] \leq (Y_\tau + \varepsilon)^p - \varepsilon^p + \frac{p}{2(1-p)} E\left[\int_\tau^T (Y_s + \varepsilon)^p d\langle M \rangle_s \big| \mathcal{F}_\tau\right]
$$

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and taking norms in the both sides of the latter inequality we obtain
\[ p(||Y||_{\infty}+\epsilon)^{p-1}||X||_{BMO(P)}^2 \leq (||Y||_{\infty}+\epsilon)^p - \epsilon^p + \frac{p}{2(1-p)}(||Y||_{\infty}+\epsilon)^p ||M||_{BMO(P)}^2. \]

Taking the limit when \( \epsilon \to 0 \) we will have that for all \( p \in (0, 1) \)
\[ ||X||_{BMO(P)}^2 \leq \left( \frac{1}{p} + \frac{1}{2(1-p)}||M||_{BMO(P)}^2 \right) \cdot ||Y||_{\infty}. \]

Therefore,
\[ ||X||_{BMO(P)}^2 \leq \min_{p \in (0, 1)} \left( \frac{1}{p} + \frac{1}{2(1-p)}||M||_{BMO(P)}^2 \right) \cdot ||Y||_{\infty} = \]
\[ = \left( 1 + \frac{\sqrt{2}}{2} ||M||_{BMO(\tilde{P})} \right)^2 \cdot ||Y||_{\infty}, \quad (16) \]

since the minimum of the function \( f(p) = \frac{1}{p} + \frac{1}{2(1-p)}||M||_{BMO(\tilde{P})}^2 \) is attained for \( p^* = \frac{\sqrt{2}}{\sqrt{2} + ||\tilde{M}||_{BMO(\tilde{P})}} \) and \( f(p^*) = \left( 1 + \frac{\sqrt{2}}{2} ||M||_{BMO(\tilde{P})} \right)^2. \)
Thus, from (16)
\[ \frac{1}{\left( 1 + \frac{\sqrt{2}}{2} ||M||_{BMO(\tilde{P})} \right)} ||X||_{BMO(P)} \leq ||\tilde{X}||_{BMO(\tilde{P})}. \]

Now we can use inequality (16) for the Girsanov transform of \( \tilde{X} \).
Since \( dP/d\tilde{P} = \mathcal{E}_T^{-1}(M) = \mathcal{E}_T(\tilde{M})dP, \tilde{M}, \tilde{X} \in BMO(\tilde{P}) \) and
\[ \varphi(\tilde{X}) = \tilde{X} - \langle \tilde{X}, \tilde{M} \rangle = X, \]
from (16) we get the inverse inequality:
\[ ||\tilde{X}||_{BMO(\tilde{P})} \leq \left( 1 + \frac{\sqrt{2}}{2} ||\tilde{M}||_{BMO(\tilde{P})} \right) ||X||_{BMO(P)}. \quad (17) \]

Let us compare the constant
\[ C(\tilde{M}) = 1 + \frac{\sqrt{2}}{2} ||\tilde{M}||_{BMO(\tilde{P})} \]
from (12) with the corresponding constant \( C_K(\tilde{M}) \) from (9) (Kazamaki [10]).
Since
\[ E^\tilde{P} \left[ \{ \mathcal{E}_{\tau,T}(\tilde{M}) \}^{-\frac{1}{\tau - T}}|\mathcal{F}_\tau \right] \geq 1, \]
the constant \( C_K(\tilde{M}) \) is more than \( \sqrt{2^p} \), where \( p \) is such that \( \|\tilde{M}\|_{BMO(\tilde{P})} < \sqrt{2}(\sqrt{p} - 1) \). Since the last inequality is equivalent to the inequality \( p > (1 + \frac{\sqrt{2}}{2}\|\tilde{M}\|_{BMO(\tilde{P})})^2 \), we obtain that at least
\[ C^2(\tilde{M}) \leq \frac{1}{2} C^2_K(\tilde{M}). \]

From inequality (12) it follows the following simple corollary, which cannot be deduced from inequality (11).

**Corollary.** Let \((M^n, n \geq 1)\) be a sequence of \( BMO(P) \) martingales such that \( \lim_{n \to \infty} \|M^n\|_{BMO(P)} = 0 \). Let \( dP^n = \mathcal{E}_T(M^n)dP \) and \( \tilde{X}^n = X - \langle X, M^n \rangle \). Then for any \( X \in BMO(P) \)
\[ \lim_{n \to \infty} \|\tilde{X}^n\|_{BMO(P^n)} = \|X\|_{BMO(P)}. \]

**Proof.** The second inequality of (12), applied for \( X = M^n \) and \( M = M^n \) gives
\[ \|\tilde{M}^n\|_{BMO(P^n)} \leq \left( 1 + \frac{\sqrt{2}}{2}\|\tilde{M}^n\|_{BMO(P^n)} \right)\|M^n\|_{BMO(P)}. \]
Therefore,
\[ \frac{\sqrt{2}}{2} + 1/\|\tilde{M}^n\|_{BMO(P^n)} \leq \|M^n\|_{BMO(P)}, \]
which implies that \( \lim_{n \to \infty} \|M^n\|_{BMO(P^n)} = 0 \). Now, passing to the limit in the two-sided inequality (12) we obtain
\[ \|X\|_{BMO(P)} \leq \lim_{n \to \infty} \|\tilde{X}^n\|_{BMO(P^n)} \leq \|X\|_{BMO(P)}. \]

**Remark.** Note that the converse of Theorem 2 is also true. I.e., if \( M \) is a continuous local martingale and \( \mathcal{E}(M) \) is a uniformly integrable martingale, Schachermayer [18] proved that if \( M \notin BMO(P) \) then the map \( \varphi \) is not an isomorphism from \( BMO(P) \) into \( BMO(\tilde{P}) \).
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