EXTENSIONS OF THE UNIVERSAL THETA DIVISOR

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Abstract. The Jacobian varieties of smooth curves fit together to form a family, the universal Jacobian, over the moduli space of smooth marked curves, and the theta divisors of these curves form a divisor in the universal Jacobian. In this paper we describe how to extend these families over the moduli space of stable marked curves (or rather an open subset thereof) using a stability parameter. We then prove a wall-crossing formula describing how the theta divisor varies with the stability parameter.

We use that result to analyze a divisor on the moduli space of smooth marked curves that has recently been studied by Grushevsky–Zakharov, Hain and Müller. In particular, we compute the pullback of the theta divisor studied in Alexeev’s work on stable abelian varieties and in Caporaso’s work on theta divisors of compactified Jacobians.

1. Introduction

In this paper we describe how the theta divisor of a compactified universal Jacobian varies with a stability parameter and then use this result to analyze a divisor on the moduli space of smooth marked curves recently studied by Samuel Grushevsky, Richard Hain, Fabian Müller, and Dmitry Zakharov.

Let us begin by recalling that earlier work. Given a sequence \( \vec{d} = (d_1, \ldots, d_n) \) of integers with \( \sum d_j = g - 1 \) and at least one \( d_j \) negative, the subset

\[
\mathcal{D}_{\vec{d}} := \{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} : h^0(C, \mathcal{O}(d_1 p_1 + \cdots + d_n p_n)) \neq 0 \}
\]

is a proper closed subset of \( \mathcal{M}_{g,n} \), so it has an associated fundamental class \( [\mathcal{D}_{\vec{d}}] \in \mathbb{A}^1(\mathcal{M}_{g,n}) \), and we can consider the problem of extending \( [\mathcal{D}_{\vec{d}}] \) to a Chow class \( [\overline{\mathcal{D}}_{\vec{d}}] \in \mathbb{A}^1(\overline{\mathcal{M}}_{g,n}) \) on the Deligne–Mumford compactification, or an open subscheme thereof, and then describing \( [\mathcal{D}_{\vec{d}}] \) in terms of standard generators. F. Müller extended \( \mathcal{D}_{\vec{d}} \) to its Zariski closure \( \overline{\mathcal{D}}_{\vec{d}}(\overline{\mathcal{M}}) \) and proved [Müller13 Theorem 5.6]:

\[
(1) \quad [\overline{\mathcal{D}}_{\vec{d}}(\overline{\mathcal{M}})] = -\lambda + \sum_{j=1}^n \binom{d_j + 1}{2} \cdot \psi_j - \sum_{i,S} \left( \frac{|d_S - i| + 1}{2} \cdot \delta_{i,S} - \sum_{S \subseteq S^+} \left( \frac{|d_S - i| + 1}{2} \cdot \delta_{i,S} \right) \cdot \delta_{irr} \right).
\]

Here \( d_S := \sum_{j \in S} d_j \) and \( S^+ := \{ j \in \{1, \ldots, n\} : d_j > 0 \} \). Müller’s result answers a question Hain attributes to Joe Harris [Hain13 page 561].

R. Hain extended \( [\mathcal{D}_{\vec{d}}] \) to a rational Chow class \( [\overline{\mathcal{D}}_{\vec{d}}(\mathcal{H})] \) using the formalism of theta functions and then proved [Hain13 Theorem 11.7]:

\[
(2) \quad [\overline{\mathcal{D}}_{\vec{d}}(\mathcal{H})] = -\lambda + \sum_{j=1}^n \binom{d_j + 1}{2} \cdot \psi_j - \sum_{i,S} \left( \frac{d_S - i + 1}{2} \cdot \delta_{i,S} \right) + \frac{\delta_{irr}}{8}.
\]
Using different methods, both results were reproved by S. Grushevsky and D. Zakharov [GZ14a, Theorem 2, Theorem 6], who further developed these ideas in [GZ14b].

A third way of extending \([D_\vec{d}]\) was suggested by Hain [Hai13, Section 11.2, page 561]. If \(J_{g,n} \to \mathcal{M}_{g,n}\) is the family of degree \(g-1\) Jacobians associated to the universal curve over \(\mathcal{M}_{g,n}\) (so the fiber of \(J_{g,n} \to \mathcal{M}_{g,n}\) over \((C, p_1, \ldots, p_n)\) is the moduli scheme of degree \(g-1\) line bundles on \(C\)), then the rule \((C, p_1, \ldots, p_n) \mapsto \mathcal{O}(d_1 p_1 + \cdots + d_n p_n)\) defines a morphism

\[
D_{\vec{d}}: \mathcal{M}_{g,n} \to J_{g,n}
\]

with the property that \(D_{\vec{d}}\) is the preimage of the theta divisor

\[
\Theta := \{(C, p_1, \ldots, p_n; F) : h^0(C, F) \neq 0\}.
\]

Thus one way to extend \(D_{\vec{d}}\) is to extend (3) to a morphism

\[
s_{\vec{d}}: \mathcal{M}_{g,n} \to \mathcal{J}_{g,n},
\]

into an extension \(\mathcal{J}_{g,n}\) of \(J_{g,n}\), to extend the theta divisor to a divisor \(\overline{\Theta}\) on \(\mathcal{J}_{g,n}\), and then to take the preimage \(s_{\vec{d}}^{-1}(\overline{\Theta})\). The difficulty in carrying out this idea is that the obvious extension of \(J_{g,n}\) is badly behaved. The family \(\mathcal{F} \to \mathcal{M}_{g,n}\) of moduli spaces of degree \(g-1\) line bundles on stable marked curves exists, but it fails to be separated. In particular, \(s_{\vec{d}}\) does extend to a morphism into \(\mathcal{F}\), but there is not a unique extension, an issue already observed by Hain, who remarks that this is a “subtle problem” [Hai13, Section 11.2, page 561].

One way to extend \(J_{g,n}\) is to use the theory of degenerate principally polarized abelian varieties. The family \((J_{g,n}/\mathcal{M}_{g,n}, \Theta)\) is a family of principally polarized torsors for abelian varieties, and this family uniquely extends to a family \((\mathcal{J}_{g,n}/\mathcal{M}_{g,n}, \overline{\Theta})\) of stable semiabelic pairs, or stable principally polarized degenerate abelian varieties. The morphism \(s_{\vec{d}}\) is a rational map into \(\mathcal{J}_{g,n}\), and we can use it extend \([D_{\vec{d}}]\) as

\[
[D_{\vec{d}}(\text{SP})] := s_{\vec{d}}^{-1}(\overline{\Theta}).
\]

An alternative approach, the focus of the present paper, is to extend \(J_{g,n}\) as moduli space of sheaves. The failure for \(\mathcal{F}_{g,n}\) to be separated is intimately related to an invariant of a line bundle on a reducible curve: the multidegree. The multidegree \(\deg(F)\) of a line bundle \(F\) is defined to be the vector whose components are the degrees of the restrictions of \(F\) to the irreducible components of \(C\). To have a well-behaved moduli space of line bundles, one typically imposes a numerical condition on the multidegree of a line bundle, i.e. a stability condition. There is now a large body of literature on how to construct a moduli space associated to a stability condition, and we build upon that literature to construct a collection of extensions of \(J_{g,n}(\phi)\) indexed by a linear algebra parameter \(\phi\).

We construct these extensions only when \(3g - 3 + n \geq 0\) and \(n > 0\), so in particular we do not construct families over \(\overline{\mathcal{M}}_{g,0}\). We use the assumption \(n > 0\) at several places, e.g. in Proposition 11 where the assumption is used to construct a certain line bundle.

We also do not construct families over all of \(\overline{\mathcal{M}}_{g,n}\) and instead construct families over the open substack \(\mathcal{M}_{g,n}^{(0)}\) parameterizing curves with loop-free circuit rank 0. The techniques of this paper can be applied to larger open substacks of \(\overline{\mathcal{M}}_{g,n}\) and to \(\overline{\mathcal{M}}_{g,0}\), but...
we restrict our attention to $\mathcal{M}^{(0)}$ because this stack is natural to consider when studying the problem of extending a divisor on $\mathcal{J}_{g,n}$ such as $\Theta$, as we explain in Section 3.4.

We construct the extensions in Section 3. There we construct an affine space $V_{g,n}$, the stability space, and for every nondegenerate element $\phi \in V_{g,n}$ a family of moduli spaces $\mathcal{J}_{g,n}(\phi) \rightarrow \mathcal{M}_{g,n}^{(0)}$ of Jacobians.

The affine space $V_{g,n}$ decomposes into a stability polytope decomposition, a decomposition into polytopes such that $\mathcal{J}_{g,n}(\phi_1) = \mathcal{J}_{g,n}(\phi_2)$ if and only if $\phi_1$ and $\phi_2$ lie in a common polytope. For every nondegenerate $\phi \in V_{g,n}$, we construct a divisor $\overline{\mathcal{O}}(\phi)$ extending $\Theta$ and then we describe the dependence of the associated Chow class $\overline{u}(\phi)$ on $\phi$ as follows. For any two nondegenerate stability parameters $\phi_1$ and $\phi_2$, there is a distinguished isomorphism between the Chow groups $\Lambda^1(\mathcal{J}_{g,n}(\phi_1)) \cong \Lambda^1(\mathcal{J}_{g,n}(\phi_2))$, and using this isomorphism, we can form the difference in the Chow group.

Our main result describes this difference. By Lemmas 2 and 6 in Section 3, crossing a wall in the stability space $V_{g,n}$ corresponds to changing the stable bidegree of a line bundle on a general element of $\Delta_{i,S} \subset \overline{\mathcal{M}}_{g,n}$ from $(d-1,g-d)$ to $(d,g-1-d)$, and leaving the stable bidegree on the general element of $\Delta_{i',S'}$ unchanged for all other $(i',S') \neq (i,S)$. Let $\phi_1$ be a stability parameter in $V_{g,n}$ corresponding to the first stability condition, and let $\phi_2$ be a stability parameter corresponding to the second one. Our main result is Theorem 17 in Section 4: the wall-crossing formula

**Theorem.**

\[ \overline{u}(\phi_2) - \overline{u}(\phi_1) = (d - i) \cdot \delta_{i,S} \tag{5} \]

We use this formula in Section 5 to study different extensions of $D_{\overline{\mathcal{O}}}$: Specifically, for every nondegenerate stability parameter $\phi$ the morphism $s_{\overline{\mathcal{O}}}$ extends uniquely to a morphism

\[ s_{\overline{\mathcal{O}}} : \mathcal{M}^{(0)}_{g,n} \rightarrow \mathcal{J}_{g,n}(\phi), \tag{6} \]

so we can form the preimage

\[ \overline{D}_{\overline{\mathcal{O}}}(\phi) := s_{\overline{\mathcal{O}}}^{-1}(\overline{\mathcal{O}}(\phi)). \]

We compute the class of these divisors:

**Theorem.** For a nondegenerate stability parameter $\phi$, we have

\[ [D_{\overline{\mathcal{O}}}(\phi)] = -\lambda + \sum_{j=1}^{n} \left( d_{j} + 1 \right) \cdot \psi_{j} + \sum_{i,S} \left( \left( \frac{d(i,S) - i + 1}{2} \right) - \left( \frac{d_{S} - i + 1}{2} \right) \right) \cdot \delta_{i,S}, \tag{7} \]

where $d(i,S)$ is the unique integer such that $(d(i,S), g-1-d(i,S))$ is the bidegree of a $\phi$-stable line bundle on a general element of $\Delta_{i,S} \subset \overline{\mathcal{M}}_{g,n}$.

This is Theorem 22. We describe the relation between the divisors $[D_{\overline{\mathcal{O}}}(\phi)]$, $[D_{\overline{\mathcal{O}}}(\text{Ha})]$, $[D_{\overline{\mathcal{O}}}(\text{SP})]$, $[D_{\overline{\mathcal{O}}}(\text{Miu})]$ in Sections 5.1, 5.2, 5.3 respectively. In particular, we prove:
Corollary. The pullback of the theta divisor of the family of stable semiabelic pairs extending $(\mathcal{J}_{g,n}, \Theta)$ satisfies

\[
[D_d^{j}(SP)] = -\lambda + \sum_{j=1}^{n} \left( \frac{d_j + 1}{2} \right) \cdot \psi_j - \sum_{i,S} \left( \frac{d_s - i + 1}{2} \right) \cdot \delta_{i,S} \\
= [D_d^{j}(Ha)] - \frac{\delta_{uv}}{8} \\
= [D_d^{j}(\phi)] \text{ for any } \phi \text{ satisfying Lemma 17}
\]

This is Corollary 24. As is explained in Section 5.2 this is also the pullback of the theta divisor studied in Caporaso’s works [Cap08a, Cap09].

After this paper was first posted to the arXiv, the authors were made aware of related work of Bashar Dudin. In [Dud15], Dudin computes the pullback of the theta divisor of the family of stable semiabelic pairs studied in Caporaso’s works [Cap08a, Cap09].

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After this paper was first posted to the arXiv, the authors were made aware of related work of Bashar Dudin. In [Dud15], Dudin computes the pullback of the theta divisor of certain compactified universal Jacobians that are constructed by Melo in an upcoming paper. He computes the pullback of such a theta divisor to be the class in Equation (8), and the authors expect that the restriction of Melo’s family to $\mathcal{M}_{g,n}^{(0)}$ is $\mathcal{J}_{g,n}(\phi)$ for a $\phi$ satisfying the conditions of Lemma 17. The authors first become aware of Dudin’s work on July 14, 2015 when Dudin emailed the authors. The authors first posted their preprint to the arXiv on July 13, 2015 and first publicly presented their work in a seminar on March 10, 2015. Dudin posted his paper to the arXiv on May 12, 2015.

2. Conventions

A curve over a field $\text{Spec}(F)$ is a $\text{Spec}(F)$-scheme $C/\text{Spec}(F)$ that is proper over $\text{Spec}(F)$, geometrically connected, and pure of dimension 1. A curve $C/\text{Spec}(F)$ is a nodal curve if $C$ is geometrically reduced and the completed local ring of $C \otimes \overline{F}$ at a non-regular point is isomorphic to $\overline{F}[x,y]/(xy)$. Here $\overline{F}$ is an algebraic closure of $F$.

A family of curves over a $k$-scheme $T$ is a proper, flat morphism $C \to T$ whose fibers are curves. A family of curves $C \to T$ is a family of nodal curves if the fibers are nodal curves.

If $F$ is a rank 1, torsion-free sheaf on a nodal curve $C$ with irreducible components $\{C_i\}$, then we define the multidegree by $\deg(F) := (\deg(F_{C_i})$. Here $F_{C_i}$ is the maximal torsion-free quotient of $F \otimes O_{C_i}$.

Given a ring $R$ and a set $S$, we write $R^S$ for the $R$-module of functions $S \to R$, a free $R$-module with basis indexed by $S$.

2.1. Graphs. A graph $\Gamma$ is a pair $(\text{Vert}(\Gamma), \text{Edge}(\Gamma))$ consisting of a set $\text{Vert}(\Gamma)$ (the vertex set) and a subset $\text{Edge}(\Gamma)$ (the edge set) of the quotient of $\text{Vert}(\Gamma) \times \text{Vert}(\Gamma)$ by the equivalence relation $(v_1, v_2) \equiv (v_2, v_1)$. If $(v_1, v_2)$ represents an edge $e \in \text{Edge}(\Gamma)$, then we define $v_1$ and $v_2$ to be the endpoints of $e$. A loop based at $v$ is an edge whose endpoints both equal $v$.

A $n$-marked graph is a graph $\Gamma$ together with a (genus) map $g: \text{Vert}(\Gamma) \to \mathbb{N}$ and a (marking) map $p: \{1, \ldots, n\} \to \text{Vert}(\Gamma)$. We call $g(v)$ the genus of $v \in \text{Vert}(\Gamma)$. If $v = p(j)$, then we say that the marking $j$ lies on the vertex $v$. We say that an $n$-marked graph is stable if every vertex $v$ of genus 0 has the property the number of edges with $v$ as an endpoint plus the number of markings $j \in \{1, \ldots, n\}$ mapping to $v$ is at least 3.
(when counting edges, count a loop based at \(v\) twice). The (arithmetic) genus of the graph is \(g(\Gamma) := \sum_{v \in \text{Vert}(\Gamma)} g(v) - \# \text{Vert}(\Gamma) + \# \text{Edge}(\Gamma) + 1\).

If \(\Gamma\) is a \(n\)-marked graph and \(e \in \text{Edge}(\Gamma)\) is an edge, the contraction of \(e\) in \(\Gamma\) is the graph \(\Gamma'\) where the edge \(e\) is removed, the two endpoints \(w_1\) and \(w_2\) of \(e\) are replaced by a unique vertex \(w'\), and the genus and marking functions are extended to \(w'\) by \(p'(j) := w'\) whenever \(p(j)\) equals \(w_1\) or \(w_2\), and

\[
g'(w') := \begin{cases} 
g(w_1) + g(w_2) & \text{when } e \text{ is not a loop,} \\
g(w_1) + g(w_2) + 1 & \text{when } e \text{ is a loop.}
\end{cases}
\]

In this paper, a subgraph is always assumed to be proper and complete. A subgraph is given the induced genus and marking functions.

2.2. Moduli of curves. Throughout the paper, we fix integers \(g \geq 0\) and \(n \geq 1\) (if \(g = 0\), then \(n \geq 3\)).

**Definition 1.** If \((C, p_1, \ldots, p_n)\) is a stable marked curve, we define the dual graph \(\Gamma_C\) to be the \(n\)-marked graph whose vertices are the irreducible components of \(C\) decorated by their geometric genera and whose edges are the nodes of \(C\). The loop-free dual graph \(\Gamma_C\) is the graph obtained from \(\Gamma_C\) by contracting all loops. We say that \(\Gamma_C\) (or alternatively \(C\)) has loop-free circuit rank 0, written \(b_1(\Gamma_C) = 0\), if \(\Gamma_C\) is a tree. If \((p_1, \ldots, p_n)\) are markings of \(C\), then we define the corresponding markings of \(\Gamma_C\) to be the assignment \(\{1, \ldots, n\} \rightarrow \text{Vert}(\Gamma_C)\) that sends \(j\) to the irreducible component containing \(p_j\).

Given a stable marked graph \(\Gamma\), we define \(\mathcal{M}_{g,n}(\Gamma)\) to be the locally closed substack of \(\overline{\mathcal{M}}_{g,n}\) parameterizing curves with dual graph \(\Gamma\). We define \(\mathcal{M}^{(0)}_{g,n} \subset \overline{\mathcal{M}}_{g,n}\) to be the open substack parameterizing curves of loop-free circuit rank 0.

In this paper we will work with several divisors and their classes in \(\overline{\mathcal{M}}_{g,n}\). Because every such divisor is completely determined by its restriction to \(\mathcal{M}^{(0)}_{g,n}\), we will sometimes abuse the notation and denote a divisor on \(\overline{\mathcal{M}}_{g,n}\) and on \(\mathcal{M}^{(0)}_{g,n}\) with the same symbol.

**Definition 2.** For a given pair \((i, S)\) with \(i \in \{0, \ldots, g\}\) and \(S \subset \{1, \ldots, n\}\) such that \(1 \in S\) and

\[
\# S \leq n - 2 \text{ if } i = g, \\
\# S \geq 2 \text{ if } i = 0;
\]

we define \(\Gamma(i, S)\) to be the graph with two vertices \(v_1\) and \(v_2\) and one edge connecting them, and with genera \(g(v_1) = i\) and \(g(v_2) = g - i\), and markings

\[
p(j) = \begin{cases} 
v_1 & \text{if } j \in S; \\
v_2 & \text{otherwise.}
\end{cases}
\]

The boundary divisor \(\Delta_{i,S}\) is the closure of \(\mathcal{M}_{g,n}(\Gamma(i, S))\) in \(\overline{\mathcal{M}}_{g,n}\). The boundary divisor \(\Delta_{irr}\) is the closure of the locus of irreducible, singular curves.

The restriction of the universal curve \(\pi: \mathcal{C}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}\) to \(\Delta_{i,S}\) has two irreducible components, and we write \(\mathcal{C}^*_{i,S}\) for the irreducible component that contains the markings \(S\) and \(\mathcal{C}_{i,S}\) for the other irreducible component.
We require that \((i,S)\) satisfy \((i)\) so that pairs \((i,S)\) are in bijection with the boundary divisors distinct from \(\Delta_{irr}\).

3. Stability Conditions

In this section we define families \(\mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}\) that extend the universal Jacobian \(\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}\) and effective divisors \(\Theta(\phi) \subset \mathcal{J}_{g,n}(\phi)\) that extend the family of theta divisors \(\Theta \subset \mathcal{J}_{g,n}\). The families are indexed by stability parameters \(\phi \in V_{g,n}\) lying in an affine space \(V_{g,n}\). We describe the dependence of \(\mathcal{J}_{g,n}(\phi)\) on \(\phi\) by constructing a polytope decomposition of \(V_{g,n}\), called the stability polytope decomposition, with the property that \(\mathcal{J}_{g,n}(\phi_1) = \mathcal{J}_{g,n}(\phi_2)\) if and only if \(\phi_1\) and \(\phi_2\) lie in a common polytope.

Our construction of the \(\mathcal{J}_{g,n}(\phi)\)'s is perhaps not the first construction that one might try. A natural first approach is to define \(V_{g,n}\) to be the relative ample cone \(\text{Amp}\) inside the relative Néron–Severi space \(\text{Pic}(C_{g,n})_R/\pi^* \text{Pic}(\mathcal{M}_{g,n}^{(0)})_R\) and then for \(\phi \in \text{Amp}\) to set \(\mathcal{J}_{g,n}(\phi)\) equal to the moduli space of degree \(g-1\) rank 1, torsion-free sheaves that are slope semistable with respect to \(\phi\). For our purposes, this does not lead to a satisfactory theory because, as was observed in [Ale04, 1.7], the condition of slope stability with respect to \(\phi\) on degree \(g-1\) sheaves is independent of \(\phi\), so this approach produces only one family \(\mathcal{J}_{g,n}(\phi)\). Furthermore, this family is a stack with points that have positive dimensional stabilizers (or is a highly singular coarse moduli scheme depending on how one tries to construct \(\mathcal{J}_{g,n}(\phi)\)) because there are sheaves that are strictly semistable, and the presence of positive dimensional stabilizers complicates the intersection theory of \(\mathcal{J}_{g,n}(\phi)\) (see e.g. [EGS13]). Below we modify this (unsuccessful) approach to construct a large collection of families \(\mathcal{J}_{g,n}(\phi)\) that are smooth Deligne–Mumford stacks.

This section is organized as follows. In Section 3.1 we define \(\phi\)-stability and related concepts, in Section 3.2 we define the stability polytope decomposition, and then in Section 3.3 we construct the family \(\mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}\) of compactified Jacobians and the family of theta divisors \(\Theta(\phi) \subset \mathcal{J}_{g,n}(\phi)\) associated to a nondegenerate stability parameter \(\phi\). Finally in Section 3.4 we make some remarks about the definition of \(V_{g,n}\), \(\mathcal{J}_{g,n}(\phi)\) and their relations to constructions from the literature.

3.1. Stability Conditions: The Stability Space. The stability condition we study is the following.

**Definition 3.** Given a stable marked graph \(\Gamma\) of genus \(g\), define \(V(\Gamma) \subset \mathbb{R}^{\text{Vert}(\Gamma)}\) to be the affine subspace of vectors \(\phi\) satisfying

\[
\sum_{v \in \text{Vert}(\Gamma)} \phi(v) = g - 1.
\]

If \(C\) is a stable marked curve with dual graph \(\Gamma\) and \(C_0 \subset C\) is a subcurve with dual graph \(\Gamma_0 \subset \Gamma\), then we write \(\deg_C F\) or \(\deg_{\Gamma_0} F\) for the degree of the maximal torsion-free quotient of \(F \otimes \mathcal{O}_{C_0}\) and \(C_0 \cap C_0'\) or \(\Gamma_0 \cap \Gamma_0'\) for the set of edges \(e \in \text{Edge}(\Gamma)\) that join a vertex of \(\Gamma_0\) to a vertex of its complement \(\Gamma_0'\).
Given \( \phi \in V(\Gamma) \) we define a degree \( g - 1 \) rank 1, torsion-free sheaf \( F \) on a curve \( C/k \) defined over an algebraically closed field to be \( \phi \)-semistable (resp. \( \phi \)-stable) if

\[
\deg_{\Gamma_0}(F) \geq \sum_{v \in \text{Vert}(\Gamma_0)} \phi(v) - \frac{\#(\Gamma_0 \cap \Gamma_0')}{2} \quad \text{(resp. >)}
\]

for all proper subgraphs \( \Gamma_0 \subset \Gamma \). We say that \( \phi \in V(\Gamma) \) is nondegenerate if every \( \phi \)-semistable sheaf is \( \phi \)-stable.

**Remark 1.** Nondegenerate \( \phi \)'s exist since e.g. any \( \phi \) with irrational coefficients must be general.

**Remark 2.** We have defined a sheaf \( F \) to be \( \phi \)-semistable if and only if it is slope semistable (resp. stable) if and only if \( \phi(A,M) \in V(\Gamma_C) \) is defined by setting for \( v \in \text{Vert}(\Gamma_C) \)

\[
(12) \quad \phi(A,M)(v) := \frac{\deg_s(A)}{\deg(A)} \deg(M) + \frac{\deg_s(\omega_C)}{2} - \deg_s(M),
\]

then a degree \( g - 1 \) rank 1, torsion-free sheaf \( F \) is \( \phi(A,M) \)-semistable (resp. \( \phi(A,M) \)-stable) if and only if \( F \otimes M \) is slope semistable (resp. stable) with respect to \( A \).

**Proof.** By elementary algebra, this is a consequence of the explicit computation of semistability in [Ale04, pages 1245–1246] or [CMKV12].

Motivated by the lemma, we make the following definition.

**Definition 4.** We define the canonical parameter \( \phi_{\text{can}} \in V(\Gamma) \) of a stable marked curve \( C \) with dual graph \( \Gamma \) by setting \( \phi_{\text{can}}(v) = \frac{\deg_s(\omega_C)}{2} \) for \( v \in \text{Vert}(\Gamma) \).

Concretely \( \phi_{\text{can}}(v) = 2g(v) - 2 + \#\text{Edge}(N_v) \) where \( N_v \subset \Gamma \) is the neighbourhood of \( v \). A sheaf \( F \) is \( \phi_{\text{can}} \)-semistable if and only if it is slope semistable with respect to an ample line bundle, i.e. \( \phi_{\text{can}} = \phi(A,O_C) \) for some (equivalently all) ample \( A \).

Next we define a stability space that controls families \( \mathcal{J}_{g,n}(\phi) \) over \( \mathcal{M}_{g,n}^{(0)} \).
Definition 5. Suppose that $c: \Gamma_1 \to \Gamma_2$ is a contraction of stable marked graphs. We say that $\phi_1 \in V(\Gamma_1)$ is compatible with $\phi_2 \in V(\Gamma_2)$ with respect to $c$ if
\begin{equation}
\phi_2(v_2) = \sum_{c(v_1)=v_2} \phi_1(v_1)
\end{equation}
for all vertices $v_2 \in \text{Vert}(\Gamma_2)$.

Define the stability space to be the subset of
\begin{equation}
V_{g,n} \subset \prod_{b_1(\Gamma)=0} V(\Gamma)
\end{equation}
that consists of vectors $\phi = (\phi(\Gamma))$ such that $\phi(\Gamma_1)$ is compatible with $\phi(\Gamma_2)$ with respect to every contraction $c: \Gamma_1 \to \Gamma_2$.

The canonical parameter $\phi_{\text{can}} \in V_{g,n}$ is defined to be $\phi_{\text{can}} := (\phi_{\text{can}}(\Gamma))$.

Given $\phi \in V_{g,n}$ we say that a degree $g-1$ rank 1, torsion-free sheaf $F$ on a stable marked curve $(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}^{(0)}$ is $\phi$-semistable (resp. $\phi$-stable) if $F$ is $\phi(\Gamma)$-semistable (resp. $\phi(\Gamma)$-stable) for $\Gamma$ the dual graph of $C$. We say that $\phi \in V_{g,n}$ is nondegenerate if $\phi(\Gamma)$ is nondegenerate for all $\Gamma$.

Remark 3. In addition to compatibility with contractions, a natural condition to impose on a stability parameter is that it is invariant under automorphisms, i.e. $\phi(\Gamma)(v) = \phi(\Gamma)(\alpha(v))$ for all graphs $\Gamma$ and all graph automorphisms $\alpha: \Gamma \to \Gamma$. We believe this condition follows from compatibility with contractions, although this becomes false if $n$ is allowed to be 0. We do not pursue these issues here because they are not needed.

Remark 4. In Definition 5 we defined $V_{g,n}$ to be the subset of $\phi$’s that are compatible with contractions in order to ensure that there is a well-behaved moduli stack $\mathcal{F}_{g,n}(\phi)$ associated to a nondegenerate stability parameter $\phi$ (the existence of $\mathcal{F}_{g,n}(\phi)$ is Proposition 11 below). Without the compatibility condition, a suitable moduli stack may not exist. The essential point is this: Suppose $F_\eta$ is a line bundle on a stable marked curve $C_\eta$ that specializes to $F_s$ on $C_s$ within some 1-parameter family. If $C_{0,s}$ is an irreducible component of $C_\eta$, then that irreducible component specializes to a subcurve $C_{0,s}$, and the degrees are related by
\begin{equation}
\text{deg}_{C_{0,s}}(F_\eta) = \text{deg}_{C_{0,s}}(F_s)
\end{equation}
(by continuity of the Euler characteristic).

Equation (16) is exactly the condition that the degree vector $\text{deg}(F)$ is compatible with the contraction $c: \Gamma_{C_s} \to \Gamma_{C_\eta}$. Thus when defining stability conditions on line bundles, it is natural to require that the degree vectors of stable line bundles are compatible with contractions, and this holds when the line bundles are the $\phi$-stable line bundles for a stability parameter $\phi$ that is compatible with contractions.

We conclude the section by proving that a nondegenerate stability parameter $\phi \in V_{g,n}$ is determined by its components $\phi(\Gamma)$ for $\Gamma$ a 2-vertex graph.

Lemma 2. The restriction of the natural projection
\begin{equation}
\prod_{b_1(\Gamma)=0} V(\Gamma) \to \prod_{b_1(\Gamma)=2} V(\Gamma)
\end{equation}
to $V_{g,n}$ is a bijection.

Proof. To begin, we examine the condition that

$$
\phi \in \prod_{b_1(\Gamma) = 0} V(\Gamma)
$$

is compatible with contractions. Let $\Gamma$ be a stable graph with $b_1(\Gamma) = 0$. If $e \in \text{Edge}(\Gamma)$ is an edge that is not a loop, then $\Gamma - e$ has two connected components, say $\Gamma^+$ and $\Gamma^-$. If $c$ is the contraction that contracts all edges of $\Gamma$ except for $e$, then $\Gamma^+$ and $\Gamma^-$ are contracted to two distinct vertices, say $v^+$ and $v^-$ respectively. The vector $\phi$ is compatible with $c$ if and only if the following equalities are satisfied:

$$
\sum_{v \in \text{Vert}(\Gamma^+)} \phi(\Gamma)(v) - \sum_{v \in \text{Vert}(\Gamma^-)} \phi(\Gamma)(v) = \phi(c(\Gamma))(v^+) - \phi(c(\Gamma))(v^-),
$$

$$
\sum_{v \in \text{Vert}(\Gamma)} \phi(\Gamma)(v) = g - 1.
$$

(18)

Varying over all nonloops $e \in \text{Edge}(\Gamma)$, the equations in (18) form a system of $\# \text{Edge}(\Gamma) + 1 = \# \text{Vert}(\Gamma)$ inhomogeneous equations in $\# \text{Vert}(\Gamma)$ variables. Furthermore, the associated system of homogeneous equations is nondegenerate (induct on $\# \text{Edge}(\Gamma)$ to show that if $v_0 \in \text{Vert}(\Gamma)$ is a leaf, then the determinant of the system associated to $\Gamma$ equals twice the determinant associated to the graph obtained by contracting the unique non-loop containing $v_0$). In particular, $\phi(\Gamma) \in V(\Gamma)$ is the unique vector satisfying (18).

It immediately follows that the projection (17) is injective.

We establish surjectivity as follows. Given

$$
\phi \in \prod_{\# \text{Vert}(\Gamma) = 2} V(\Gamma),
$$

define, for $\Gamma$ a stable marked graph with $b_1(\Gamma) = 0$, $\phi(\Gamma)$ to be the unique solution to (18) and then define

$$
\phi := (\phi(\Gamma)) \in \prod_{b_1(\Gamma) = 0} V(\Gamma).
$$

This vector is compatible with all contractions. Indeed, given a contraction $c: \Gamma_1 \to \Gamma_2$, define $\phi'(\Gamma_2) \in V(\Gamma_2)$ by setting

$$
\phi'(\Gamma_2)(v_2) = \sum_{c(v_1) = v_2} \phi(\Gamma_1)(v_2).
$$

Then both $\phi'(\Gamma_2)$ and $\phi(\Gamma_2)$ satisfy (18), so $\phi'(\Gamma) = \phi(\Gamma)$, proving that

$$
\phi := (\phi(\Gamma)) \in \prod_{b_1(\Gamma) = 0} V(\Gamma)
$$

is compatible with contractions and thus the surjectivity of (17).
3.2. Stability Conditions: The stability polytope decomposition. Here we define the polytope decompositions of $V(\Gamma)$ and $V_{g,n}$ that describe how $\phi$-stability depends on $\phi$.

We will use the following definition and lemma to construct the decomposition.

**Definition 6.** A subgraph $\Gamma_0 \subset \Gamma$ is said to be elementary if both $\Gamma_0$ and its complement $\Gamma_0^c$ are connected.

**Remark 5.** The vertex set of an elementary subgraph is an elementary cut in the sense of [OS79, page 31].

**Remark 6.** When $b_1(\Gamma) = 0$ (the case of present interest), a subgraph $\Gamma_0 \subset \Gamma$ is elementary if and only if $\Gamma_0 \cap \Gamma_0^c$ consists of a single edge.

**Lemma 3.** Let $(C, p_1, \ldots, p_n)$ be a stable marked curve and $\phi \in V(\Gamma_C)$. A rank 1, torsion-free sheaf $F$ is $\phi$-semistable (resp. $\phi$-stable) if and only if Inequality (10) holds for all elementary subgraphs $\Gamma_0$ of $\Gamma_C$.

**Proof.** Set $\Gamma := \Gamma_C$. It is enough to show that if $F$ satisfies Inequality (12) for all elementary subgraphs $\Gamma_0 \subset \Gamma$, then it satisfies the inequality for all subgraphs. First, consider the case where $\Gamma_0^c$ is connected. Let $\Gamma_1, \ldots, \Gamma_n$ be the connected components of $\Gamma$. Each of the connected components $\Gamma_1, \ldots, \Gamma_n$ is an elementary subgraph of $\Gamma$. Indeed, the complement of $\Gamma_i$ in $\Gamma$ is

$$\Gamma_i^c = \Gamma_0^c \cup \Gamma_1 \cup \Gamma_2 \cdots \cup \Gamma_{i-1} \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_n,$$

and we can connect each $\Gamma_j$ for $j \neq i$ to $\Gamma_0^c$ as follows. Since $\Gamma$ is connected, for $j \neq i$, we can connect any vertex in $\Gamma_j$ to any vertex in $\Gamma_0^c$ by a path in $\Gamma$. Pick one such path $v_0, v_1, \ldots, v_n$ that has minimal length. There is no consecutive pair $v_i, v_{i+1}$ of vertices with $v_i \in \text{Vert}(\Gamma_{k_1})$ and $v_{i+1} \in \text{Vert}(\Gamma_{k_2})$ for distinct $k_1, k_2$ because no edge joins $\Gamma_{k_1}$ to $\Gamma_{k_2}$. Furthermore, the first vertex that lies in $\Gamma_0^c$ must be $v_n$ by minimality, so the vertices $v_1, \ldots, v_{n-1}$ must all lie in $\Gamma_j$. This proves that $\Gamma_i$ is elementary.

By hypothesis, Inequality (12) holds for the subgraphs $\Gamma_1, \ldots, \Gamma_n$ and combining those inequalities with the triangle inequality, we get Inequality (12) for $\Gamma_0$. This proves the lemma under the assumption that $\Gamma_0^c$ is connected.

For arbitrary $\Gamma$ we argue as follows. Inequality (12) is symmetric with respect to replacing $\Gamma_0$ with $\Gamma_0^c$, so the result follows immediately when $\Gamma_0$ is connected and from the case that $\Gamma_0$ is connected, the general case then follows by expressing $\Gamma_0$ as a union of connected components and applying the triangle inequality. □

**Definition 7.** Let $\Gamma$ be a stable marked graph of genus $g$. To a subgraph $\Gamma_0 \subset \Gamma$ and an integer $d \in \mathbb{Z}$ we associate the affine linear function $\ell(\Gamma_0, d) : V(\Gamma) \to \mathbb{R}$ defined by

$$\ell(\Gamma_0, d)(\phi) := d - \sum_{v \in \Gamma_0} \phi(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2}.$$

We call

$$H(\Gamma_0, d) := \{ \phi \in V(\Gamma) : \ell(\Gamma_0, d)(\phi) = 0 \}.$$
a stability hyperplane if \( \Gamma_0 \subset \Gamma \) is an elementary subgraph. A connected component of the complement of all stability hyperplanes in \( V(\Gamma) \)

\[
V(\Gamma) - \bigcup_{\Gamma_0 \subset \Gamma \text{ elementary}} \{ \phi \in V(\Gamma) : \ell(\Gamma_0, d)(\phi) = 0 \}
\]

is defined to be a stability polytope, and the set of all stability polytopes is defined to be the stability polytope decomposition of \( V(\Gamma) \).

By definition if \( \phi_0 \) is a nondegenerate stability parameter, the stability polytope \( P \) containing \( \phi_0 \) can be written as:

\[
P = \{ \phi \in V(\Gamma) : \ell(\Gamma_0, d)(\phi) > 0 \text{ for all } \ell(\Gamma_0, d) \text{ s.t. } \ell(\Gamma_0, d)(\phi_0) > 0 \}. \tag{19}
\]

The stability polytope \( P \) is a rational bounded convex polytope because in Equation \( \text{(19)} \) only finitely many \( \ell(\Gamma_0, d) \)'s are needed to define \( P \).

**Example 1.** When \( \Gamma_C \) has a single vertex, \( V(\Gamma) \) is a 0-dimensional affine space. There are no elementary subgraphs of \( \Gamma \), so there is only one stability polytope: \( V(\Gamma) \) itself.

**Example 2.** Suppose that \( \Gamma \) is the graph depicted in Figure 1. The associated stability polytopes are depicted in Figure 2. The affine space \( V(\Gamma) \) is 1-dimensional, and a stability polytope is an open line segment with endpoints at two consecutive half-integer points. More precisely, if \( \bar{d} = (d(v_1), d(v_2)) \in V_Z(\Gamma) \) is an integral vector, then the set of solutions to

\[
d(v_1) - \phi(v_1) + 1/2 > 0, \\
d(v_2) - \phi(v_2) + 1/2 > 0
\]

is a stability polytope \( P(\bar{d}) \) that can be described as the relative interior of the convex hull of \( (d(v_1) - 1/2, d(v_2) + 1/2) \) and \( (d(v_1) + 1/2, d(v_2) - 1/2) \), and every stability polytope can be written as \( P(\bar{d}) \) for a unique \( \bar{d} \).

One property of the graph depicted in Figure 1 is that, for every nondegenerate stability parameter \( \phi \), there is a unique \( \phi \)-stable multidegree. This is more generally true for loop-free circuit rank 0 graphs:

**Lemma 4.** Let \( \Gamma \) be a graph with loop-free circuit rank 0 and \( \phi \in V(\Gamma) \) a nondegenerate stability parameter. Then there exists a \( \phi \)-stable line bundle \( F \). Furthermore, any two \( \phi \)-stable line bundles have the same multidegree.

**Proof.** We first prove that any two \( \phi \)-stable line bundles have the same multidegree. Endow \( \Gamma \) with the structure of a rooted tree by arbitrarily picking a vertex \( r_0 \in \text{Vert}(\Gamma) \) as the root. We prove the lemma by working one vertex at a time, starting with the leaves and ending with the root. Suppose \( v_0 \in \text{Vert}(\Gamma) \) is a vertex that is not the root \( r_0 \).
Define $\Gamma_0 \subset \Gamma$ to be the induced subgraph on the set of vertices consisting of $v_0$ and its descendants. For a line bundle $F$, the $\phi$-stability inequality (12) for $\Gamma_0$ takes the form

$$ \frac{1}{2} \left| \text{deg}_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi(v) \right| < 1/2. $$

This proves that the partial degree $\text{deg}_{\Gamma_0}(F)$ of a $\phi$-stable line bundle is uniquely determined (and equal to the integer nearest to $\sum_{v \in \Gamma_0} \phi(v)$). Since $\text{deg}_{\Gamma_0}(F) = \text{deg}_{\Gamma_2}(F)$ for $\Gamma_1 := \Gamma_0 - \{v_0\}$, it follows by reverse induction on the depth of $v_0$ that $\text{deg}_{\Gamma_0}(F)$ is also uniquely determined. Given that $\text{deg}_{\Gamma_0}(F)$ is uniquely determined for all $v_0 \neq r_0$, we can conclude that $\text{deg}_{\Gamma_0}(F)$ is also uniquely determined as $\text{deg}_{r_0}(F) = g-1-\text{deg}_{\Gamma_2}(F)$ for $\Gamma_2 := \Gamma - \{r_0\}$. This proves any two $\phi$-stable line bundles have the same multidegree. For existence, observe that we can inductively construct a $\phi$-semistable line bundle $F$ by requiring that $\text{deg}_{\Gamma_0}(F)$ equals the nearest integer to $\sum_{v \in \Gamma_0} \phi(v)$ (i.e. equals the unique solution to Inequality (20)).

**Lemma 5.** Let $(C, p_1, \ldots, p_n)$ be a stable marked curve such that the dual graph $\Gamma_C$ has loop-free circuit rank 0. Given two nondegenerate stability parameters $\phi_1, \phi_2 \in V(\Gamma_C)$, $\phi_1$-stability coincides with $\phi_2$-stability if and only if there exists a stability polytope containing both $\phi_1$ and $\phi_2$.

**Proof.** If $\phi_1, \phi_2$ both lie in a stability polytope, then $\phi_1$-stability coincides with $\phi_2$-stability by Lemma 3. Conversely suppose that $\phi_1$ and $\phi_2$ lie in distinct stability polytopes. By definition there exists $d \in \mathbb{Z}$ and $\Gamma_0 \subset \Gamma$ an elementary subgraph such that $\ell(\Gamma_0, d)(\phi_1) > 0$ but $\ell(\Gamma_0, d)(\phi_2) < 0$. There exists (by Lemma 4) a $\phi_1$-stable line bundle $F$, and its multidegree satisfies

$$ \left| \text{deg}_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi_1(v) \right| < 1/2, $$

**Figure 2.** The stability polytopes of a two-vertex graph $\Gamma$
i.e. \( \deg_{\Gamma_0}(F) \) is the least integer greater than \( \sum_{v \in \Gamma_0} \phi_1(v_0) - 1/2 \). In particular, \( \deg_{\Gamma_0}(F) \leq d \), so

\[
\deg_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi_2(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2} = (\deg_{\Gamma_0}(F) - d) + (d - \sum_{v \in \Gamma_0} \phi_2(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2})
\]

\[
= (\deg_{\Gamma_0}(F) - d) + \ell(\Gamma_0, d)(\phi_2)
\]

\[
< 0,
\]

and \( F \) is not \( \phi_2 \)-stable. \( \square \)

**Remark 7.** Lemma 5 becomes false if the stability hyperplanes are defined to be the subsets \( \{ \ell(\Gamma_0, d)(\phi) = 0 \} \) with \( \Gamma_0 \subset \Gamma \) a possibly non-elementary subgraph. For example, if \( \Gamma \) is the graph depicted in Figure 3, then the decomposition by black rectangles depicted in Figure 4 is the stability polytope decomposition of \( V(\Gamma) \) (or more precisely its isomorphic image under the projection \( V(\Gamma) \to \mathbb{R}^2, \phi \mapsto (\phi(v_1), \phi(v_2)) \)). The subdivision of the polytope decomposition given by the dotted and solid lines is the decomposition by the hyperplanes \( \{ \ell(\Gamma_0, d)(\phi) = 0 \} \) with \( \Gamma_0 \subset \Gamma \) a possibly non-elementary subgraph.

Suppose that \( \Gamma = \Gamma_C \) is the dual graph of \( C \). When \( \phi \in V(\Gamma) \) crosses a dotted line, the set of vectors \( \tilde{d} \in \mathbb{Z}^{\text{Vert}(\Gamma)} \) satisfying \( |\tilde{d}(\Gamma_0) - \sum_{v \in \Gamma_0} \phi(v_0)| \leq \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2} \) changes, but the subset of vectors of the form \( \tilde{d} = \deg(F) \) for \( F \) a \( \phi \)-semistable sheaf does not change.

\[ g_2 \]
\[ g_1 \]
\[ g_2 \]
\[ g_1 \]

**Figure 3.** A tree \( \Gamma \) with three vertices, \( g_1 + g_2 + g_3 = g \)

\[ g_2 \]
\[ g_1 \]
\[ g_2 \]
\[ g_1 \]

**Figure 4.** The stability polytopes of \( V(\Gamma) \)
We now define the combinatorial objects that control the stability conditions over $\mathcal{M}_{g,n}^{(0)}$.

**Definition 8.** For $\Gamma$ a stable marked graph of genus $g$ with $b_1(\Gamma) = 0$, $\Gamma_0 \subset \Gamma$ an elementary subgraph, and $d \in \mathbb{Z}$ an integer, we define
\[
\{ \phi \in V_{g,n} : \ell(\Gamma_0, d)(\phi(\Gamma)) = 0 \}
\]
to be a **stability hyperplane**. A **stability polytope** for $V_{g,n}$ is defined to be a connected component of
\[
V_{g,n} - \bigcup_{\Gamma_0 \subset \Gamma \text{ elementary}} \{ \phi \in V_{g,n} : \ell(\Gamma_0, d)(\phi(\Gamma)) = 0 \}.
\]
The collection of all stability polytopes is defined to be the **stability polytope decomposition** of $V_{g,n}$.

As before, the stability polytopes in $V_{g,n}$ are rational bounded convex polytopes.

Having shown in Lemma 2 that a stability parameter $\phi \in V_{g,n}$ is determined by 2-vertex graphs, we now prove analogous statements about stability hyperplanes and polytopes.

**Lemma 6.** If $H \subset V_{g,n}$ is a stability hyperplane, then
\[
H = H(\Gamma, d) := H(i, S, d)
\]
for $\Gamma = \Gamma(i, S)$ a stable marked graph with two vertices and one edge (see Definition 2).

**Proof.** Let
\[
H = \{ \ell(\Gamma_1, d)(\phi(\Gamma_2)) = 0 \}
\]
be a given stability hyperplane (so $\Gamma_2$ is a stable marked graph of loop-free circuit rank 0, $\Gamma_1 \subset \Gamma_2$ an elementary subgraph, and $d \in \mathbb{Z}$ an integer).

Define $c : \Gamma_2 \to \Gamma$ to be the contraction that contracts $\Gamma_1$ to a vertex $w_1$ and $\Gamma_0$ to a vertex $w_2$, so $\Gamma$ is a stable marked graph with two vertices. By compatibility we have $\phi(\Gamma)(w_1) = \sum_{v \in \Gamma_1} \phi(\Gamma_2)(v)$, so
\[
\{ \ell(w_1, d)(\phi(\Gamma)) = 0 \} = \{ \ell(\Gamma_1, d)(\phi(\Gamma_2)) = 0 \}
\]
\[= H. \]

Lemma 8 implies that when $\phi \in V_{g,n}$ varies in such a way that $\phi$-semistability changes, that variation is already witnessed over a graph with two vertices. As a corollary, we obtain the following description of stability polytopes:

**Corollary 7.** Given a stability polytope $\mathcal{P}(\Gamma)$ for every stable marked graph $\Gamma$ with $b_1(\Gamma) = 0$ and $\# \text{Vert}(\Gamma) = 2$, there exists a unique stability polytope $\mathcal{P} \subset V_{g,n}$ such that the projection onto $V(\Gamma)$ is $\mathcal{P}(\Gamma)$ for all 2-vertex graphs $\Gamma$.

**Proof.** This follows from Lemma 2 together with Lemma 6. Uniqueness is Lemma 2. To prove existence, by the same lemma there exists $\phi_0 \in V_{g,n}$ satisfying $\phi_0(\Gamma) \in \mathcal{P}(\Gamma)$ for all 2-vertex graphs $\Gamma$. By Lemma 6, $\phi_0$ is not contained in a stability hyperplane, so it is contained in a unique stability polytope that satisfies the desired condition by the same lemma.
To study the set of stability polytopes, we introduce the following group action.

**Definition 9.** Define $W_\mathbb{Z}(\Gamma) \subset \mathbb{Z}^{\text{Vert}(\Gamma)}$ to be the subgroup of sum-zero vectors. The **natural action** of $W_\mathbb{Z}(\Gamma)$ on $V(\Gamma)$ is the translation action, $(\psi + \phi)(v) = \psi(v) + \phi(v)$.

**Lemma 8.** The natural action of $W_\mathbb{Z}(\Gamma)$ on $V(\Gamma)$ maps stability polytopes to stability polytopes.

**Proof.** This follows from the identity

$\ell(\Gamma_0,d)(\psi + \phi) = \ell(\Gamma_0,d - \psi(\Gamma_0))(\phi).$

**Definition 10.** We define $(W_{g,n})_\mathbb{Z}$ to be the additive subgroup

$$(W_{g,n})_\mathbb{Z} \subset \prod_{b_1(\Gamma) = 0} W_\mathbb{Z}(\Gamma)$$

that consists of vectors satisfying the contraction compatibility condition:

$$\psi(\Gamma_2)(v_2) = \sum_{c(v_1) = v_2} \psi(\Gamma_1)(v_1)$$

for all contractions $c: \Gamma_1 \to \Gamma_2$ and all vertices $v_2 \in \text{Vert}(\Gamma_2)$. The **natural action** of $(W_{g,n})_\mathbb{Z}$ on $V_{g,n}$ is defined by $(\psi + \phi)(\Gamma) = \psi(\Gamma) + \phi(\Gamma)$.

**Lemma 9.** The natural action of $(W_{g,n})_\mathbb{Z}$ on $V_{g,n}$ maps stability polytopes to stability polytopes.

**Proof.** This follows from Lemma 8.

We define $\text{Pic}^0(C_{g,n}/\mathcal{M}_{g,n}^{(0)})$ to be the subgroup of the group $\text{Pic}(C_{g,n})/\pi^*(\text{Pic}(\mathcal{M}_{g,n}^{(0)}))$ generated by the images of line bundles $F$ with the property that the restriction to any fiber of $\pi: C_{g,n} \to \mathcal{M}_{g,n}^{(0)}$ has degree 0. The multidegree defines a homomorphism $\text{deg}: \text{Pic}^0(C_{g,n}/\mathcal{M}_{g,n}^{(0)}) \to (W_{g,n})_\mathbb{Z}$. The **natural action** of $\text{Pic}^0(C_{g,n}/\mathcal{M}_{g,n}^{(0)})$ on $V_{g,n}$ is the action induced by the action of $(W_{g,n})_\mathbb{Z}$ via the homomorphism $\text{deg}$. The subschemes $C_{i,S}^+ \subset C_{g,n}$ associated with each component over $\Delta_{i,S}$ in the universal curve (see Definition 2) are effective Cartier divisors, so their associated line bundles $\mathcal{O}(C_{i,S}^+)$ are defined, and we use them to prove the following lemma.

**Lemma 10.** The subgroup of $\text{Pic}^0(C_{g,n}/\mathcal{M}_{g,n}^{(0)})$ generated by the line bundles $\mathcal{O}(C_{i,S}^+)$ acts freely and transitively on the set of stability polytopes in $V_{g,n}$.

**Proof.** Consider the line bundle $F := \mathcal{O}(C_{i,S}^+)$ associated to a pair $(i,S)$ as in Definition 2. Its degree vector $\text{deg}(F) \in (W_{g,n})_\mathbb{Z}$ satisfies

$$(21) \quad \text{deg}(F)(\Gamma) = \begin{cases} (-1,+1) & \text{if } \Gamma = \Gamma(i,S); \\ (0,0) & \text{if } \Gamma \neq \Gamma(i,S), \# \text{Vert}(\Gamma) = 2. \end{cases}$$

Using the description of stability polytopes associated to a graph with two vertices in Example 2 we conclude that the subgroup generated by the $\mathcal{O}(C_{i,S}^+)$’s acts transitively.
on the image of $V_{g,n}$ in
\[
\prod_{\# \text{Vert}(\Gamma) = 2 \atop b_1(\Gamma) = 0} V(\Gamma),
\]
and from Corollary [7] we deduce that the same is true for $V_{g,n}$. To see that the action is free, observe that an element of $\text{Pic}^0(C_{g,n}/\mathcal{M}^{(0)}_{g,n})$ acts as translation by its multidegree, so an element acting trivially must have multidegree 0 and the only such element is the identity. \qed

### 3.3. Stability Conditions: Representability

We now restrict our attention to a stability parameter $\phi \in V_{g,n}$ that is nondegenerate. Given such a $\phi$, we construct a family $\mathcal{J}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$ of compactified Jacobians and a family of theta divisors $\overline{\Theta}(\phi) \subset \mathcal{J}_{g,n}(\phi)$, and then we describe some properties of $\overline{\Theta}(\phi)$.

**Definition 11.** Given $T$ a $k$-scheme and $(C/T, p_1, \ldots, p_n) \in \mathcal{M}^{(0)}_{g,n}(T)$ a family of stable marked curves, a family of degree $g - 1$ rank 1, torsion-free sheaves on $C/T$ is a locally finitely presented $\mathcal{O}_C$-module $F$ that is $\mathcal{O}_T$-flat and has rank 1, torsion-free fibers. Given $\phi \in V_{g,n}$, we say $F$ is a family of $\phi$-semistable sheaves if the fibers are $\phi$-semistable.

**Definition 12.** Given $\phi \in V_{g,n}$, define $\mathcal{J}^{\text{pre}}_{g,n}(\phi)$ to be the category fibered in groupoids whose objects are tuples $(C, p_1, \ldots, p_n; F)$ consisting of a family of stable genus $g$, $n$-marked curves $(C/T, p_1, \ldots, p_n)$ and a family of $\phi$-semistable rank 1, torsion-free sheaves $F$ on $C/T$. The morphisms of $\mathcal{J}^{\text{pre}}_{g,n}(\phi)$ are defined by defining a morphism from $(C, p_1, \ldots, p_n; F) \in \mathcal{J}^{\text{pre}}_{g,n}(\phi)(T)$ to $(C', p_1', \ldots, p_n'; F') \in \mathcal{J}^{\text{pre}}_{g,n}(\phi)(T')$ lying over a $k$-morphism $t: T \to T'$ to be a tuple consisting of an isomorphism of marked curves $\tilde{t}: (C, p_1, \ldots, p_n) \cong (C', p_1', \ldots, p_n')$ and an isomorphism of $\mathcal{O}_C$-modules $F \cong \tilde{t}^*(F')$.

With this definition, for every object $(C, p_1, \ldots, p_n; F)$ of $\mathcal{J}^{\text{pre}}_{g,n}(\phi)(T)$ the rule that sends $g \in G_m(T)$ to the automorphism of $F$ defined by multiplication by $g$ defines an embedding $G_m(T) \to \text{Aut}(C, p_1, \ldots, p_n; F)$ that is compatible with pullbacks. The image of this embedding is contained in the center of the automorphism group, so the rigidification stack in the sense of [ACV03, Section 5.1.5] is defined, and we call this stack the universal family of $\phi$-compactified Jacobians $\mathcal{J}_{g,n}(\phi)$.

**Proposition 11.** Assume $\phi$ is nondegenerate. Then the forgetful morphism $\mathcal{J}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$ is representable, proper, and flat. In particular, $\mathcal{J}_{g,n}(\phi)$ is a separated Deligne–Mumford stack. Furthermore, $\mathcal{J}_{g,n}(\phi)$ is $k$-smooth.

**Proof.** To begin, we prove the statement for one specific $\phi$. Pick an odd number $B > 2g - 2 + n - 1$ and then set,
\[
b := B - (2g - 2 + n - 1),
\]
\[
A := \omega(bp_1 + p_2 + \cdots + p_n),
\]
\[
M := \mathcal{O}(p_1).
\]
The authors claim that $\phi_0 := \phi(A, M)$ is nondegenerate. If the claim failed, then the expression in Equation [13] would be an integer for some proper subcurve $Y \subset X$ of a
stable marked curve or equivalently the slope \( \deg_{\Gamma_{k}}(A)/\deg(A) \) would be a half-integer for some proper subgraph \( \Gamma_{0} \subset \Gamma_{C} \), but this is impossible because the slope is of the form \( k/B \) for \( k \in \mathbb{Z}, 0 < k < B \). This proves the claim.

Lemma \[1\] identifies \( \phi_{0} \)-stability with slope stability, so the representability result \[\text{Sim94}, \text{Theorem 1.21}\] implies that \( \mathcal{J}_{g,n}(\phi_{0}) \to \mathcal{M}_{g,n}^{(0)} \) is representable and proper (the conclusion in loc. cit. that étale locally a universal family of sheaves exists is equivalent to the representability of \( \mathcal{J}_{g,n}(\phi_{0}) \)). The flatness of \( \mathcal{J}_{g,n}(\phi_{0}) \to \mathcal{M}_{g,n}^{(0)} \) and the \( k \)-smoothness of \( \mathcal{J}_{g,n}(\phi_{0}) \) follow from a modification of the deformation theory argument in \[\text{CMKV12}\].

For an arbitrary nondegenerate \( \phi \), we argue as follows. By Lemma \[10\] for a given nondegenerate \( \phi \in V_{g,n} \), there is a line bundle \( L \in \text{Pic}^{0}(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)}) \) so that \( \phi \) and \( \deg(L) + \phi_{0} \) lie in the same stability polytope and then the rule \( F \mapsto F \otimes L \) identifies \( \mathcal{J}_{g,n}(\phi_{0}) \) with \( \mathcal{J}_{g,n}(\phi) \).

**Remark 8.** When \( \phi \) is degenerate, the authors expect that \( \mathcal{J}_{g,n}(\phi) \) is still an algebraic stack, but then the forgetful morphism \( \mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)} \) is not representable, and \( \mathcal{J}_{g,n}(\phi) \) is not Deligne–Mumford. We do not pursue this issue here because we have no use for these more general families in this paper.

**Proposition 12.** For \( \phi \in V_{g,n} \) nondegenerate, the fibers of the forgetful morphism \( \mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)} \) are irreducible.

**Proof.** This follows from Lemma \[4\] In a fiber of \( \mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)} \), the locus of line bundles of fixed multidegree is \( k \)-smooth and connected (as it is a torsor for the generalized Jacobian, a semiabelian variety). Since there is a unique \( \phi \)-stable multidegree of a line bundle, we conclude that the line bundle locus in a fiber is connected. But the line bundle locus is also dense in its fiber (since e.g. a tangent space computation shows that this locus is the \( k \)-smooth locus), so we conclude that the fiber is irreducible. \( \square \)

We now turn our attention to the theta divisor and its associated Chow class. The theta divisor \( \Theta(\phi) \subset \mathcal{J}_{g,n}(\phi) \) is an effective divisor supported on the locus of sheaves that admit a nonzero global section, but it is not uniquely determined by its support because \( \Theta(\phi) \) can be nonreduced (see Corollary \[9\]). We define \( \Theta(\phi) \) using the formalism of the determinant of cohomology, a formalism we use in Section \[4\] to compute intersection numbers. More precisely, the theta divisor is defined in terms of the cohomology of the following sheaf:

**Definition 13.** The universal family of sheaves \( F_{\text{uni}} \) on \( \mathcal{J}_{g,n}^{\text{pre}}(\phi) \times \mathcal{M}_{g,n}^{(0)} \mathcal{C}_{g,n} \) is defined to be the family of \( \phi \)-stable sheaves that corresponds to the identity under the 2-Yoneda Lemma. A sheaf \( F_{\text{can}} \) on \( \mathcal{J}_{g,n}(\phi) \times \mathcal{M}_{g,n}^{(0)} \mathcal{C}_{g,n} \) is defined to be a **tautological family of sheaves** if \( F_{\text{can}} \) is the pullback \( (\sigma \times 1)^{*} F_{\text{uni}} \) of the universal sheaf for some section \( \sigma \) of the natural morphism \( \mathcal{J}_{g,n}^{\text{pre}}(\phi) \to \mathcal{J}_{g,n}(\phi) \).

Concretely \( F_{\text{can}} \) is a \( \mathcal{J}_{g,n}(\phi) \)-flat family of rank 1, torsion-free sheaves on \( \mathcal{J}_{g,n}(\phi) \times \mathcal{M}_{g,n}^{(0)} \mathcal{C}_{g,n} \) such that the restriction to the fiber of \( \mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)} \) over a point \( (C, p_{1}, \ldots, p_{n}; F) \in \mathcal{M}_{g,n}^{(0)} \) is isomorphic to \( F \).
Lemma 13. The rigidification morphism $\mathcal{J}_{g,n}^\text{pre}(\phi) \to \mathcal{J}_{g,n}(\phi)$ admits a section. In particular, $\mathcal{J}_{g,n}(\phi)$ admits a tautological family $F_{\text{taut}}$.

Proof. A section is defined by rigidifying sheaves along the marking $p_1$. More formally, consider the morphism $\mathcal{J}_{g,n}^\text{pre}(\phi) \to \mathcal{J}_{g,n}(\phi)$ that sends a tuple $(C/T, p_1, \ldots, p_n; F)$ to $(C/T, p_1, \ldots, p_n; F \otimes (\pi_T)^*(p_1^*(F)^{-1}))$. Here $\pi_T: C \to T$ is the structure morphism. To see this morphism is well-defined, observe $F \otimes \pi_T^*(p_1^*(F)^{-1})$ is a flat family $\phi$-stable sheaves because this sheaf is Zariski locally isomorphic to $F$ over $T$ (as $p_1^*(F)$ is a line bundle). Furthermore, $\mathcal{J}_{g,n}^\text{pre}(\phi) \to \mathcal{J}_{g,n}(\phi)$ has the property that the image of $\mathcal{G}_m(T) \subset \text{Aut}(C/p_1, \ldots, p_n; F)$ is mapped to the identity in $\text{Aut}(C/p_1, \ldots, p_n; F \otimes \pi_T^*(F)^{-1})$ (a scalar $g \in \mathcal{G}_m(T)$ acts by $g$ on $F$, by $g^{-1}$ on $\pi_T^*(F)^{-1}$, so by $gg^{-1} = 1$ on the tensor product). By the universal property of rigidification the morphism $\mathcal{J}_{g,n}^\text{pre}(\phi) \to \mathcal{J}_{g,n}(\phi)$ factors as $\mathcal{J}_{g,n}^\text{pre}(\phi) \to \mathcal{J}_{g,n}(\phi) \to \mathcal{J}_{g,n}(\phi)$, and $\mathcal{J}_{g,n}(\phi) \to \mathcal{J}_{g,n}^\text{pre}(\phi)$ defines the desired section. \hfill $\square$

Remark 9. The tautological family $F_{\text{taut}}$ is not uniquely determined. Given a tautological family $F_{\text{taut}}$ and a line bundle $L$ on $\mathcal{J}_{g,n}(\phi)$, $F_{\text{taut}} \otimes \pi^*(L)$ is also a tautological family. However, every tautological family is of the form $F_{\text{taut}} \otimes \pi^*L$ for some line bundle $L$ on $\mathcal{J}_{g,n}(\phi)$ by the Seesaw theorem.

We now construct the theta divisor as the determinant of the cohomology of $F_{\text{taut}}$. Recall the more general construction of the determinant of an element of the derived category. Generalizing earlier work with Mumford, Knudsen proved that the rule that assigns to a bounded complex $E$ of vector bundles on $\mathcal{J}_{g,n}(\phi)$ the line bundle $\det(E) := \bigotimes (\wedge^\text{max} E)^{(-1)^i}$ extends to a rule that assigns an isomorphism of line bundles to a quasi-isomorphism of perfect complexes [Knu02 Theorem 2.3], so the determinant of an object in the bounded derived category is defined. (See also [Est01 Section 6.1] for a more explicit approach in the special case of a family of curves, the case of current interest.) The derived pushforward $\mathbb{R}\pi_*F_{\text{taut}}$ of a tautological family is an element of the bounded derived category by the finiteness theorem [Ill05 Theorem 8.3.8], so in particular, its determinant $\det(\mathbb{R}\pi_*F_{\text{taut}})$ is defined.

The inverse line bundle $\det(\mathbb{R}\pi_*F_{\text{taut}})^{-1}$ admits a distinguished nonzero global section that is constructed as follows. The morphism $\pi: \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}^{(0)}$ has relative cohomological dimension 1, so $\mathbb{R}\pi_*F_{\text{taut}}$ can be represented by a 2-term complex of vector bundles $E^0 \to E^1$. The generic fiber of this complex computes the cohomology of a degree $g-1$ sheaf, so it has Euler characteristic zero (by the Riemann–Roch formula). We deduce that $\text{rank } E^0 = \text{rank } E^1$, and so the top exterior power $\det(d) := \wedge^\text{max}(d)$ is a global section of

$$\mathcal{H}om\left(\det E^0, \det E^1\right) = \left(\det E^0\right)^{-1} \otimes E^1 = \det(\mathbb{R}\pi_*F_{\text{taut}})^{-1}.$$ 

A direct computation shows that $\det(d) \in H^0(\mathcal{M}_{g,n}^{(0)}, \det(\mathbb{R}\pi_*F_{\text{taut}})^{-1})$ is independent of the choice of complex $E$ (i.e. that $\det(d)$ is preserved by isomorphisms induced by quasi-isomorphisms; see [Est01 Observation 43]).

The line bundle $\det(\mathbb{R}\pi_*F_{\text{taut}})$ is uniquely determined even though $F_{\text{taut}}$ is not:
Lemma 14. If $F_{\text{tau}}$ and $G_{\text{tau}}$ are two tautological families on $\mathcal{J}_{g,n}(\phi)$, then
\[
\det(\mathbb{R}\pi_* F_{\text{tau}}) = \det(\mathbb{R}\pi_* G_{\text{tau}}),
\]
and this identification identifies
\[
\det(d) \in H^0(\mathcal{M}_{g,n}^{(0)}, \det(\mathbb{R}\pi_* F_{\text{tau}})^{-1})
\]
with
\[
\det(e) \in H^0(\mathcal{M}_{g,n}^{(0)}, \det(\mathbb{R}\pi_* G_{\text{tau}})^{-1}).
\]

Proof. By Remark 3, $G_{\text{tau}} = F_{\text{tau}} \otimes \pi^*(M)$ for some line bundle $M$ on $\mathcal{J}_{g,n}(\phi)$, so the result follows from the projection property of the determinant [Est01, Proposition 44(3)]. □

Definition 14. The theta divisor $\Theta(\phi) \subset \mathcal{J}_{g,n}(\phi)$ is the effective Cartier divisor defined by $(\det(\mathbb{R}\pi_* F_{\text{tau}})^{-1}, \det(d))$ for $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1$ a 2-term complex of vector bundles that represents $\mathbb{R}\pi_* F_{\text{tau}}$. The theta divisor Chow class $\bar{\Theta}(\phi) \in \mathcal{A}^1(\mathcal{J}_{g,n}(\phi))$ is the fundamental class of $\Theta(\phi)$.

We conclude this section by describing some of the properties of $\bar{\Theta}(\phi)$.

Lemma 15. The theta divisor $\bar{\Theta}(\phi)$ is supported on the locus of points $(C, p_1, \ldots, p_n; F) \in \mathcal{J}_{g,n}(\phi)$ with $H^0(C, F) \neq 0$.

Proof. Fix a 2-term complex of vector bundles $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1$ that represents $\mathbb{R}\pi_* F_{\text{tau}}$, so that $\Theta(\phi) = \{\det(d) = 0\}$. Given a point $(C, p_1, \ldots, p_n; F)$, write
\[
\mathcal{E} \otimes k(\text{point}) := \mathcal{E}^0 \otimes k(\text{point}) \xrightarrow{d} \mathcal{E}^1 \otimes k(\text{point})
\]
for the fiber of $\mathcal{E}^0 \to \mathcal{E}^1$ at $(C, p_1, \ldots, p_n; F)$. The point $(C, p_1, \ldots, p_n; F)$ lies in $\bar{\Theta}(\phi)$ if and only if the complex $\mathcal{E} \otimes k(\text{point})$ has nonzero cohomology, and because the formation of $\mathbb{R}\pi_* F_{\text{tau}}$ commutes with base change [Illo5 Theorem 8.3.2], the cohomology groups of $\mathcal{E} \otimes k(\text{point})$ are $H^0(C, F)$ and $H^1(C, F)$. □

Next we characterize when $\bar{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat.

Lemma 16. If $\mathcal{P} \subset \mathcal{V}_{g,n}$ is a stability polytope, then for $\phi \in \mathcal{P}$, the natural projection $\bar{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat if and only if $\phi_{\text{can}} \in \mathcal{P}$.

Proof. Since $\bar{\Theta}(\phi)$ is an effective divisor on $\mathcal{J}_{g,n}(\phi)$ and $\mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat, $\bar{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat if and only if $\bar{\Theta}(\phi)$ does not contain a fiber of $\mathcal{J}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$. The lemma thus follows from Lemma 15 and [Bea77 Lemma (2.1)]. □

3.4 Stability Conditions: Concluding Remarks. We conclude with some remarks, beginning with remarks about the families $\mathcal{J}_{g,n}(\phi)$ and their relation to families already existing in the literature. By definition $\mathcal{J}_{g,n}(\phi)$ is the moduli space of $\phi$-semistable rank 1, torsion-free sheaves. Our definition of $\phi$-semistability, Definition 3, is an extension of the definition given by Oda–Seshadri [OS79] (for degree 0 rank 1, torsion-free sheaves on a single nodal curve), and our proof of Proposition 11 shows that $\phi$-semistability can be (non-canonically) identified with slope semistability in the sense of [Sim94]. The
authors expect that \( \overline{J}_{g,n}(\phi) \) can alternatively be constructed as a family of quasi-stable compactified Jacobians in the sense of Esteves \cite{Est01}.

Earlier Melo constructed a compactified universal Jacobian over \( \overline{M}_{g,n} \) in \cite{Me11}. Her compactification is different from the ones studied in this paper as e.g. it is not always a Deligne–Mumford stack (as the hypothesis to \cite{Me11} Proposition 8.3] fails; her compactification is also not a moduli stack of torsion-free sheaves on stable curves, but the authors expect one can identify it with such a stack by an argument similar to \cite[Theorem 10.3.1]{Pan96}). Melo’s paper builds upon a large body of work on constructing compactifications over \( \overline{M}_{g,0} \) \cite{Cap94, Pan96, Jar00, Cap08b, Mel09}.

The difference between the different compactifications over \( \overline{M}_{g,n} \) is somewhat subtle. To describe the difference, fix a stable marked curve \((C, p_1, \ldots, p_n) \in \Delta_{i,S}\) that has two \( k \)-smooth irreducible components and assume \( i, g - i > 0 \) (so \( C \) does not have a rational tail) and then examine the corresponding fiber \( \overline{J}_C \) of \( \overline{J} \to \overline{M}_{g,n}^{(0)} \) for \( \overline{J} \to \overline{M}_{g,n}^{(0)} \), a family extending the universal Jacobian. For most extensions, \( \overline{J}_C \) is isomorphic to the product \( J^+ \times J^- \) of the Jacobians of the irreducible components \( C^+, C^- \) of \( C \). There are, however, many ways of interpreting this scheme as a moduli space of sheaves: For any pair of integers \((d_+, d_-) \) with \( d_+ + d_- = g - 1 \), we can extend \( \overline{J}_{g,n} \) by taking \( \overline{J}_C \) to be the moduli space of bidegree \((d_+, d_-) \) line bundles on \( C \) and then restriction defines an isomorphism \( \overline{J}_C \cong J^{d_+} \times J^{d_-} \) with the product of the moduli space of degree \( d_+ \) line bundles on \( C^+ \) and the moduli space of degree \( d_- \) line bundles on \( C^- \). (This moduli space is, for example, the fiber of \( \overline{J}_{g,n}(\phi) \to \overline{M}_{g,n}^{(0)} \) for a suitably chosen \( \phi \).

We do not study the moduli space \( \overline{J}_C(\phi_{can}) \) associated to the canonical parameter in this paper, but this moduli space has frequently been studied in the literature: by \cite[Theorem 10.3.1]{Pan96} this is the moduli space constructed in \cite{Cap94, Cap08b}, and it plays a distinguished role in the study of degenerations of abelian varieties (see Section 5.2).

The moduli space \( \overline{J}_C(\phi_{can}) \) can be described as follows. There are three types of \( \phi_{can} \)-semistable sheaves, all of which are strictly semistable: line bundles of bidegree \((i - 1, g - i) \), line bundles of bidegree \((i, g - i - 1) \), and sheaves that are the direct image of a line bundle of bidegree \((i - 1, g - i - 1) \) on the normalization. Restriction again defines an isomorphism between the coarse moduli space \( \overline{J}_C(\phi_{can}) \) (in the sense of \cite{Sim94}) and \( J^{i-1} \times J^{g-i-1} \), so \( \overline{J}_C(\phi_{can}) \) is (non-canonically) isomorphic to \( \overline{J}_C(\phi) \) for any general \( \phi \), but this isomorphism cannot be chosen in a way that identifies moduli functors. Furthermore, while the scheme \( \overline{J}_C(\phi) \) is a fine moduli space, \( \overline{J}_C(\phi_{can}) \) is naturally the coarse space of an algebraic stack with \( G_m \) stabilizers.

There is one important family \( \overline{J} \) with the property that \( \overline{J}_C \) is not isomorphic to \( J^+ \times J^- \): the family constructed by Caporaso in \cite{Cap94} (over the moduli space of unmarked curves \( \overline{M}_{g,0} \), a space we do not consider here). That family has the property that \( \overline{J}_C \) is not \( J^+ \times J^- \) but rather the quotient \( J^+ \times J^- / \text{Aut}(C) \). The appearance of this quotient is related to stack-theoretic issues; in loc. cit. \( \overline{J} \) is a family over the coarse moduli scheme of the moduli stack \( \overline{M}_{g,0} \) rather than over the stack itself.

For the families we construct in this paper, the fiber \( \overline{J}_C \) is isomorphic to a product of Jacobians, and different families only differ on the level of moduli functors. The authors expect this is an artifact of the fact that we study extensions of \( \overline{J}_{g,n} \to \overline{M}_{g,n} \) to a family over \( \overline{M}_{g,n}^{(0)} \) rather than all of \( \overline{M}_{g,n} \), for examples in \cite{MRVJ2} suggest that there are many...
different schemes that extend the universal Jacobian to a family of moduli spaces over all of $\mathcal{M}_{g,n}$.

This brings us to the second topic of discussion: the stability space $V_{g,n}$. We have defined $V_{g,n}$ so that it controls families over $\mathcal{M}_{g,n}$. A consequence of Corollary 19 in Section 3 is that the decomposition of $V_{g,n}$ defined by the variation of the theta divisor essentially coincides with the stability polytope decomposition, the only difference being that the theta divisor is constant on all the (finitely many) polytopes that contain $\phi_{\text{can}}$ in their closures (a consequence of Lemma 16).

The authors believe that $\mathcal{M}_{g,n}^{(0)}$ is the largest open substack $W \subset \mathcal{M}_{g,n}$ that is a union of topological strata, with the property that the different theta divisors are essentially in bijection with the different extensions of $\mathcal{J}_{g,n}$ to a family over $W$.

4. Wall-Crossing formula for the Theta divisor

In this section we restrict our attention to nondegenerate stability parameters $\phi \in V_{g,n}$, and study how the theta divisor class $\overline{\theta}(\phi)$ varies with $\phi$ by proving a wall-crossing formula. The main result of this section is Equation (22), which we prove by applying the Grothendieck-Riemann-Roch theorem to a test curve.

In the previous section we defined a stability space $V_{g,n}$ (Definition 5); for any $\phi \in V_{g,n}$ we then defined a universal $\phi$-compactified Jacobian $\mathcal{J}_{g,n}(\phi)$ over $\mathcal{M}_{g,n}^{(0)}$ (Definition 12), and a theta divisor $\overline{\theta}(\phi) \subset \mathcal{J}_{g,n}(\phi)$ (Definition 13). In Definition 8 we endowed the stability space $V_{g,n}$ with a stability polytope decomposition. In the paragraph below we prove that the Picard groups of $\mathcal{J}_{g,n}(\phi_1)$ and of $\mathcal{J}_{g,n}(\phi_2)$ are isomorphic for nondegenerate $\phi_1, \phi_2 \in V_{g,n}$. In fact, the isomorphism is induced by an isomorphism between the moduli stacks $\mathcal{J}_{g,n}(\phi_1)$ and $\mathcal{J}_{g,n}(\phi_2)$, and Lemma 10 prescribes a distinguished choice of such an isomorphism.

Indeed, for every pair of stability polytopes $P_1, P_2 \subset V_{g,n}$, by Lemma 10 there exists a unique element $\mathcal{L}(P_1, P_2)$ in the subgroup of Pic$^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ generated by the components $\mathcal{O}(\mathcal{C}_{g,n}^i)$ such that the rule $F \mapsto F \otimes \mathcal{L}(P_1, P_2)$ defines an isomorphism $\mathcal{J}_{g,n}(\phi_1) \rightarrow \mathcal{J}_{g,n}(\phi_2)$ for all $\phi_1 \in P_1$ and $\phi_2 \in P_2$. In this section we will always identify $\mathcal{J}_{g,n}(\phi_1)$ and $\mathcal{J}_{g,n}(\phi_2)$ using the isomorphism $\mathcal{L}(P_1, P_2)$. Our main result is a formula for the difference $\overline{\theta}(\phi_2) - \overline{\theta}(\phi_1)$.

We describe this difference between theta divisors by fixing a stability wall (or facet) $H$, and then describing the difference between the theta classes associated to two stability polytopes $P_1$ and $P_2$ that have $H$ as a common facet. By Lemma 6 a wall $H = H(i, S, d)$ in the stability space $V_{g,n}$ is determined by a stable graph with two vertices and one edge $\Gamma(i, S)$, and by an integer $d$. The polytope $P_2$ is a translate of $P_1$, and we fix the convention that

$$P_2(i, S) = P_1(i, S) + (1, -1).$$

In more concrete terms, for a general fiber over $\Delta_{i, S}$, the $\phi$-stable sheaves are the line bundles of bidegree

$$(d - 1, g - d) \text{ when } \phi \in P_1, \text{ and } (d, g - 1 - d) \text{ when } \phi \in P_2.$$
Theorem 17. Let $\phi_1, \phi_2$ be nondegenerate stability parameters that belong to stability polytopes $\mathcal{P}_1, \mathcal{P}_2$ of $V_{g,n}$ whose common facet is the wall $H(\Gamma(i, S), d) = H(i, S, d)$. Then

$$\overline{\theta}(\phi_2) - \overline{\theta}(\phi_1) = \left( \phi_2^\tau(i, S) + \frac{1}{2} - i \right) \cdot \delta_{i, S}$$

(22)

(As is customary, we have written $\delta_{i, S}$ for the pullback of the boundary divisor class along $J_{g,n} \to M_{g,n}^{(0)}$).

Proof. Choosing tautological bundles $F_{\tau\text{a}}(\phi_1)$ and $F_{\tau\text{a}}(\phi_2)$ as in Lemma 13 we have $F_{\tau\text{a}}(\phi_2) \cong F_{\tau\text{a}}(\phi_1) \otimes \mathcal{O}(C_{i, S}^-)$.

By Definition 14 the left-hand side of (22) is the first Chern class of the line bundle

$$L := \left( \det(\mathbb{R}^*F_{\tau\text{a}}(\phi_1) \otimes \mathcal{O}(C_{i, S}^-)) \right)^{-1} \otimes \det(\mathbb{R}^*F_{\tau\text{a}}(\phi_1)).$$

We claim that the line bundle $L$ is the pullback of $\mathcal{O}(\Delta_{i, S})^c$ for some $c \in \mathbb{Z}$. Indeed, over the complement of $\Delta_{i, S}$ the restriction of $\mathcal{O}(C_{i, S}^-)$ is trivial. Since the formation of the determinant of cohomology commutes with base change, the restriction of $L$ to $J_{g,n} - \Delta_{i, S}$ is also trivial, which implies our claim.

The integer $c$ is determined by computing the other two integers in the equality

$$c \cdot \deg(\mathcal{O}(\Delta_{i, S})|_T) = \deg L|_T$$

(24)

Let $(\pi_T: C \to T, p_1, \ldots, p_n)$ be a test curve for $M_{g,n}^{(0)}$: the pullback to a $k$-smooth curve $T \to M_{g,n}^{(0)}$ of the universal curve $\pi:C_{g,n} \to M_{g,n}^{(0)}$ and of the universal sections. Every such $T \to M_{g,n}^{(0)}$ lifts to a morphism $T \to J_{g,n}(\phi_1)$; equivalently there exists a family of $\pi_T$-fiberwise $\phi_1$-stable sheaf $F$ on $C$. This is a consequence of the existence of a section of the forgetful morphism $J_{g,n}(\phi_1) \to M_{g,n}^{(0)}$, and we will construct several such sections in the beginning of Section 5, see (39) and (40).

From Proposition 20 below (a Grothendieck-Riemann-Roch calculation) we deduce that the right-hand side of (24) equals

$$\deg L|_T = - \deg \left( \pi_T^* \left( (\text{ch}(C_{i, S}^-) - \text{ch}(F)) \cap \text{td} C \right) \right).$$

(25)

In Construction 1 we produce an explicit test curve for $M_{g,n}^{(0)}$ whose intersection with the divisor $\Delta_{i, S}$ is the class of one point:

$$\deg \left( \mathcal{O}(\Delta_{i, S})|_T \right) = 1.$$  

(26)

We then prove in Lemma 21 that this test curve, the right-hand side of (25) equals

$$\deg L|_T = - \deg \left( F|\mathcal{C}_{i, S}^- \right) + (g - i) = -(g - 1 - d + 1) + (g - i) = (d - i).$$

(27)

Combining Equation (24) with (26) and (27) gives $c = d - i$, which concludes the proof of Theorem 17. □

Remark 10. Using the classical Riemann-Roch formula, we can express the coefficient of $\delta_{i, S}$ in Formula (22) as the Euler characteristic of a line bundle.

Writing $C_{i, S} = C^+ \cup C^-$ for a general fiber of $\pi^{-1}(\Delta_{i, S}) \to \Delta_{i, S}$, we have the equalities

$$d - i = \deg(F_{\tau\text{a}}(\phi_1)|_{C^+}) + 1 - i = \chi(C^+, F_{\tau\text{a}}(\phi_1)).$$
Formula (22) can therefore be written in the form
\[(28) \quad \overline{c}(\phi_2) - \overline{c}(\phi_1) = \chi(C^+, F_{\text{tau}}(\phi_1)) \cdot \delta_{i,S} - \chi(C^-, F_{\text{tau}}(\phi_2)) \cdot \delta_{i,S}.
\]

Let us now present some easy corollaries of formula (22). As a consequence of Lemma 16, \(\overline{c}(\phi_2)\) equals \(\overline{c}(\phi_1)\) when the canonical parameter \(\phi_{\text{can}}\) belongs to the closures of the stability polytopes \(P_1\) and \(P_2\). Theorem 17 provides the converse implication.

**Corollary 18.** If \(\overline{c}(\phi_1)\) equals \(\overline{c}(\phi_2)\), then \(\phi_{\text{can}}\) belongs to the closures of both \(P_1\) and \(P_2\).

When \(\phi\) satisfies the conditions of Corollary 18, \(\overline{c}(\phi)\) is reduced. In the following corollary we determine all the stability parameters whose associated theta divisor is reduced.

**Corollary 19.** If \(\phi \in V_{g,n}\) belongs to the stability polytope \(P\), then \(\overline{c}(\phi)\) is reduced if and only if there exists a stability polytope \(Q\) such that \(P \cap Q = \emptyset\) and \(\phi_{\text{can}} \in Q\).

We conclude this section by providing the proof of the auxiliary results that we used to prove Theorem 17.

**Proposition 20.** Let \((\pi_T: C \to T, p_1, \ldots, p_n, F)\) be a test curve for \(J_{g,n}(\phi)\). Then the following equality
\[\left(c_1(\mathbb{R}\pi_* F(C^-_{i,S})) - c_1(\mathbb{R}\pi_* F)) \cap [T] = c_{\tau T}\left((\text{ch} F(C^-_{i,S}) - \text{ch}(F)) \cap \text{td } C\right)\right]
holds in the Chow group of 0-cycles on \(T\).

**Proof.** The 0-th and 1-st higher pushforwards of \(F\) and \(F(C^-_{i,S})\) under \(\pi_T\) are sheaves of the same rank. Indeed, taking higher pushforwards commutes with base change, and the 0-th and 1-st cohomology of \(F\) and \(F(C^-_{i,S})\) on the fiber of a geometric point in \(T\) have the same dimension by the Riemann-Roch formula for curves, since the fiberwise degree is \(g - 1\). Therefore we have that both the degree-0 Chern characters
\[\text{ch}_0(\mathbb{R}\pi_* F), \quad \text{ch}_0(\mathbb{R}\pi_* F(C^-_{i,S}))\]
vanish, so we deduce the following equality in the Chow group of 0-cycles on \(T\):
\[(29) \quad c_1(\mathbb{R}\pi_* F(C^-_{i,S})) - c_1(\mathbb{R}\pi_* F)) \cap [T] = c_{\tau T}\left((\text{ch} \mathbb{R}\pi_* F(C^-_{i,S}) - \text{ch} \mathbb{R}\pi_* F)) \cap \text{td } C\right),\]
and the statement follows by combining Equations (29) and (30).

**Construction 1.** For each pair \((i,S)\) as in Definition 2 we construct a test curve \((\pi_T: C \to T, p_1, \ldots, p_n)\) whose intersection with \(\Delta_{i,S}\) is the class of one point, and whose general fiber is in \(\Delta_{i,S,\{1\}}\), as shown in Figure 5 (The special cases \(i = 0\) and \(|S| = 2\) are left to the scrupulous reader).

Fix a general genus \(g - i\) marked curve \((T, (S^c \cup \{1\}))\) and a general genus \(i\) marked curve \((T', S \cup \{\bullet\} \setminus \{1\})\). In \(T \times T\) the diagonal intersects the locus
\[T \times \{p_k: k \in S^c \cup \{1\}\}\]
at the points \(\{(p_k, p_k): k \in S^c \cup \{1\}\}\), and we define the blow-up of these points to be \(\overline{C}_1 \to T \times T\). We then define \(\overline{C}_2\) to be \(T \times T'\).
Figure 5. The general fiber and the special fiber of the test curve

The diagonal map $\Delta: T \to T \times T$ induces a morphism $s_1: T \to \tilde{C}_1$, and we define the morphism $s_2: T \to \tilde{C}_2$ as the constant $\bullet$ section of the first projection map. We then define $C$ to be the following pushout (which exists by [Fer03, Theorem 5.4]):

$$T \cup T \xrightarrow{s_1 \cup s_2} \tilde{C}_1 \cup \tilde{C}_2 \xrightarrow{\nu} j: C.$$ 

Projection onto the first component defines a morphism $\tilde{C}_1 \cup \tilde{C}_2 \to T$ that induces a morphism $\pi_T: C \to T$ by the universal property of pushouts. The morphism $\pi_T$ inherits pairwise disjoint sections $p_1, \ldots, p_n$, whose images lie in the fiberwise $k$-smooth locus.

The family $\pi_T$ then defines a morphism $T \to \mathcal{M}_{0}^{g,n}$ whose intersection with $\Delta_i,S$ is the class of one point (by construction the total space $C$ of the family is $k$-smooth at the unique node of type $\Delta_{i,S}$).

Lemma 21. On the test curve $(\pi_T: C \to T, p_1, \ldots, p_n)$ defined in Construction 1, we have

$$\deg \left( \left( \text{deg}(F(C_{i,S}) - \text{deg}(F)) \cap \text{td}(C) \right) \right) = \deg \left( L^1_{i,S} \right) - (g - i)$$

Proof. Since we are after the calculation of the degree of a 0-cycle, from now on it will be enough to compute all classes modulo numerical equivalence. In the calculations we will follow standard notation: we omit writing fundamental classes, we write $(a,b)$ to denote the ruling on a product, and we write $[\text{pt}]$ for the class of a point.

We claim that the Todd class of $C$ equals

$$\text{td}(C) = \nu_*(\text{td}(\tilde{C}_1 \times \tilde{C}_2)) - j_*\text{td}(T)).$$

Indeed, the Todd class of the singular variety $C$ is defined in [Fiu98, Chapter 18] as $\tau_C(O_C)$, for $\tau_C$ a group homomorphism from the $K$-theory of coherent sheaves on $C$ to the rational Chow group of $C$. Formula (32) follows by applying $\tau_C$ to the short sequence of sheaves

$$0 \to O_C \to \nu_*(O_{\tilde{C}_1 \times \tilde{C}_2}) \to j_*O_T \to 0,$$

which is exact by the pushout construction.

We denote by $\nu_1$ and $\nu_2$ the restrictions of $\nu$ to the two components of the normalization $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$. Applying the formulas for the Todd class of a product and of a
blow-up of a $k$-smooth surface at a point, we compute

\[
\begin{align*}
\text{td}(T) &= 1 + (1 - g + i)[pt], \\
\text{td}(\tilde{C}_1) &= 1 + (1 - g + i, 1 - g + i) - \frac{1}{2} \sum_{k \in S \cup \{1\}} E_k + (1 - g + i)^2[pt], \\
\text{td}(\tilde{C}_2) &= 1 + (1 - g + i, 1 - i) + (1 - g + i)(1 - i)[pt].
\end{align*}
\]

(Here $E_k$ denotes the exceptional fibers of the surface $\tilde{C}_1$.)

The relevant Chern characters on $\tilde{C}_1$ are

\[
\begin{align*}
\text{ch} \nu_1^* \mathcal{O}(C_{i,s}^+) &= 1 + E_1 - \frac{1}{2}[pt], \\
\text{ch} \nu_1^* \mathcal{O}(C_{i,s}^-) &= 1 + (1, 0) - E_1 - \frac{1}{2}[pt];
\end{align*}
\]

and the relevant Chern characters on $\tilde{C}_2$ are

\[
\begin{align*}
\text{ch} \nu_2^* \mathcal{O}(C_{i,s}^+) &= 1 + (1, 0), \\
\text{ch} \nu_2^* \mathcal{O}(C_{i,s}^-) &= 1.
\end{align*}
\]

To calculate the degree on the left-hand side of (31), we compute

\[
\begin{align*}
\deg \left( (\text{ch} \nu_1^* F(C_{i,s}^-) - \text{ch} \nu_1^* F) \cap \text{td} \tilde{C}_1 \right) &= (1 - g + i) - \frac{1}{2} + \deg (F_{C_{i,s}^-}) - \frac{1}{2} \\
&= \deg (F_{C_{i,s}^-}) - (g - i), \\
\deg \left( (\text{ch} \nu_2^* F(C_{i,s}^-) - \text{ch} \nu_2^* F) \cap \text{td} \tilde{C}_2 \right) &= 0,
\end{align*}
\]

where the last two expressions vanish because the curve $C_{i,s}^-$ has empty intersection with $\nu_2(\tilde{C}_2)$ and with $j(T)$.

Altogether, taking $(36) + (37) - (38)$, we find that the degree on the left-hand side of (31) equals $\deg (F_{C_{i,s}^-}) - (g - i)$. This concludes the proof of Lemma 21.
the family of line bundles $\mathcal{O}_C(D)$ is fiberwise $\phi_d$-stable and the rule \eqref{eq:theta-dual} defines a section $s_d$ of $\overline{\mathcal{J}}_{g,n}(\phi_d) \to \mathcal{M}^{(0)}_{g,n}$.

More generally, for any nondegenerate stability parameter $\phi \in V_{g,n}$, we define the divisor
\begin{equation}
D(\phi) := d_1 p_1 + \ldots + d_n p_n + \sum_{i,S} \left( d_S - \left( \phi^+ (i, S) + \frac{1}{2} \right) \right) \cdot C_{i,S}^+.
\end{equation}

The family of line bundles $\mathcal{O}_C(D(\phi))$ is fiberwise $\phi$-stable by construction, and the rule \eqref{eq:theta-dual} defines a section $s_d$ of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$. We then define $D_d(\phi) := s_d^{-1}(\theta(\phi))$.

In the following we compute the pullback of the theta class via $s_d$. Observe that the pullback along $s_d$ induces a well-defined group homomorphism $\text{Pic}(\overline{\mathcal{M}}_{g,n}) \to \text{Pic}(\mathcal{M}^{(0)}_{g,n})$, and the latter is isomorphic to $\text{Pic}(\mathcal{M}_{g,n})$ because the complement of $\mathcal{M}^{(0)}_{g,n}$ in $\mathcal{M}_{g,n}$ has codimension 2. Recall that the integral Picard group of $\overline{\mathcal{M}}_{g,n}$ is generated (freely when $g \geq 3$) by the first Chern class of the Hodge bundle $\lambda$, the first Chern classes of the cotangent line bundles to the $j$-th marking $\psi_j$, the boundary strata classes $\delta_{i,S}$ and $\delta_{irr}$.

**Theorem 22.** The pullback of $\overline{\theta}(\phi_d)$ from $\overline{\mathcal{J}}_{g,n}(\phi_d)$ to $\overline{\mathcal{M}}_{g,n}$ is given by
\begin{equation}
s_d^* \overline{\theta}(\phi_d) = -\lambda + \sum_{j=1}^n \frac{(d_j + 1)}{2} \cdot \psi_j.
\end{equation}

More generally, for any nondegenerate $\phi \in V_{g,n}$, we obtain the equality
\begin{equation}
s_d^* \overline{\theta}(\phi) = -\lambda + \sum_{j=1}^n \frac{(d_j + 1)}{2} \cdot \psi_j + \sum_{i,S} \left( \left( \phi^+ (i, S) + \frac{1}{2} \right) - \left( \frac{d_S - i + 1}{2} \right) \right) \cdot \delta_{i,S}.
\end{equation}

**Proof.** Assuming \eqref{eq:theta-dual} holds, Formula \eqref{eq:theta-dual-general} follows by the wall-crossing Formula \eqref{eq:wall-crossing}.

We prove equality \eqref{eq:theta-dual}. We define $D$ to the effective divisor $\sum_{j=1}^n d_j p_j$ in $\mathcal{C}_{g,n}$. As we observed earlier, the line bundle $\mathcal{O}(D)$ is fiberwise $\phi_d$-stable. We have
\begin{equation}
s_d^* \overline{\theta}(\phi_d) = -s_d^* c_1(\pi_* (F_{\text{ram}})) = -c_1(\pi_* (\mathcal{O}(D)))
= -\left[ \text{ch}(\pi_* (\mathcal{O}(D)) \cap \text{td} (\mathcal{M}^{(0)}_{g,n}))_{\text{codim}=1} \right.
= -\pi_* \left[ \text{ch}(\mathcal{O}(D) \cap \text{td} (\mathcal{C}_{g,n}))_{\text{codim}=2} \right]
= \pi_* \left[ -\frac{D^2}{2} + D \cdot \frac{K_{g,n}}{2} - \text{td}_2 (\mathcal{C}_{g,n}) \right],
\end{equation}

where we applied the definition of theta divisor, the fact that $\text{ch}_0(\pi_* (\mathcal{O}(D)))$ equals zero, and then the Grothendieck-Riemann-Roch formula for stacks (see e.g. \cite[Theorem 3.5]{edidin}).

The first term in \eqref{eq:theta-dual} equals
\begin{equation}
- \pi_* \left( \frac{D^2}{2} \right) = \frac{1}{2} \sum_{j=1}^n d_j^2 \psi_j,
\end{equation}

because two different sections $p_j$ and $p_k$ are by definition disjoint, and by the very definition of the $\psi$-classes:
\begin{equation}
\psi_j := -\pi_* (p_j^2).
\end{equation}
To compute the second and third terms in (43), we identify the universal curve $C_{g,n}$ with $\mathcal{M}_{g,n} + 1$. The canonical class equals

$$K = K_{\mathcal{M}_{g,n}} = 13\lambda + \psi - 2\delta,$$

where $\delta := \delta_{\text{irr}} + \sum \delta_{i,S}$ and $\psi := \sum_{j=1}^n \psi_j$.

Using the pushforward formulas

$$\pi_*(p_j \cdot \lambda) = \lambda,$$

$$\pi_*(p_j \cdot \psi_k) = \begin{cases} 0 & \text{when } j = k, \\ \psi_k & \text{when } j \neq k, \end{cases}$$

$$\pi_*(p_j \cdot \delta_{i,\text{irr}}) = \delta_{i,\text{irr}},$$

$$\pi_*(p_j \cdot \delta_{i,S}) = \begin{cases} \delta_{i,S} & \text{when } \{p_j, p_{n+1}\} \subseteq S \text{ or } \{p_j, p_{n+1}\} \subseteq S^c, \\ 0 & \text{otherwise}, \end{cases}$$

(where $\delta_{0,j}$ is interpreted as $-\psi_j$ in the last formula), the second term in (43) becomes

$$\pi_*\left(\left[\mathcal{D}\right] \cdot \frac{K_{C_{g,n}}}{2}\right) = \frac{13}{2} (g - 1) \cdot \lambda + \sum_{j=1}^n \frac{g - 1 + d_j}{2} \cdot \psi_j - (g - 1) \cdot \delta.$$

Finally, the third term equals

$$-\pi_* (\text{td}_2(C_{g,n})) = -\left(\frac{g - 1}{2} \cdot (13\lambda + \psi - 2\delta) + \lambda\right).$$

Indeed $\text{td}_2 = K^2 / 12$, and we read the formula for $c_2$ in [Bin05, page 765]. (Note that the formula for the second Chern class appears with an error in the coefficient of $\kappa_2$, which should be $-\frac{1}{4}$.) This can be quickly checked by applying the Grothendieck-Riemann-Roch formula to the sheaf $\omega_{\mathcal{M}}^\times(p_1 + \ldots + p_n)$ along the universal curve $\pi: C_{g,n} \to \mathcal{M}_{g,n}$. The pushforward (46) can then be computed with the aid of the pushforward formulas

$$\pi_*(K^2) = \pi_*(\pi^* K + \omega_{\pi}) = 2 \cdot \pi_*(\omega_{\pi}) \cdot K + \pi_*(\omega_{\pi}^2) = 2 \cdot (2g - 2) \cdot (13\lambda + \psi - 2\delta) + 12\lambda - \delta,$$

$$\pi_*(\kappa_2) = 12\lambda + \psi - \delta,$$

$$\pi_*(\xi_{\text{irr}}(\psi + \psi)) = 2 \cdot \delta_{\text{irr}},$$

$$\pi_*(\xi_{i,S}(1 \otimes \psi + \psi \otimes 1)) = \delta_{i,S},$$

$$\pi_*(\xi_{0,(j,n+1)}(1 \otimes \psi + \psi \otimes 1)) = \psi_j.$$

(Following the notation from [Bin05], here $\xi_{\text{irr}}$ and $\xi_{i,S}$ are the gluing maps, and $\kappa_2$ is the Arbarello-Cornalba kappa class.

Plugging the three terms (44) (45) and (46) in equation (43), we deduce (41). $\square$

We now compare our result with pullbacks of the theta divisor that have recently been studied by different authors.
5.1. The class of Hain. Hain studied a problem similar to the problem of computing the Abel map for stable curves. In [Hai13] he computed the pullback to $\overline{M}_{g,n}$ of a theta divisor $\theta_\alpha$ on a stack $\mathcal{J}_g$ that he calls the universal Jacobian. The stack $\mathcal{J}_g$ is not the universal Jacobian studied in this paper (which, on smooth marked curves, is the moduli stack of degree $g-1$ line bundles), but rather it is the moduli stack of multidegree $0$ line bundles on unmarked stable curves. However, the construction of the theta divisor involves a certain choice of a degree $g-1$ line bundle $\alpha$, a theta characteristic, on a certain cover of $\overline{M}_{g,n}$, and translation by $\alpha$ identifies the moduli space of degree $0$ line bundles with the moduli space of degree $g-1$ line bundles.

Hain’s theta divisor is also different from the theta divisors studied in this paper and this difference is more significant. The theta divisors studied in this paper are constructed using the description of the universal Jacobian $\mathcal{J}_{g,n}(\phi)$ as a moduli space of sheaves, i.e. by forming the determinant of the cohomology of a tautological sheaf. By contrast, Hain’s theta divisor is constructed using the description of $\mathcal{J}_g$ as family of abelian varieties, i.e. by using the formalism of theta functions [Hai13, Section 11.2, page 561]. (Alternatively, in the proof of [Hai13, Theorem 11.7], Hain proves that $\theta_\alpha$ can be described using Hodge theory, as the divisor class $\lambda/2 + \tilde{\phi}_{H(\cdot)}$.) Not only is Hain’s theta divisor constructed in a different manner, but as we now explain, $[\mathcal{D}_d(\phi)]$ is not equal to any $[\mathcal{D}_d(\phi)]$ because the second divisor class is integral but the first is not.

In [Hai13, Theorem 11.7], Hain computes the pullback of $\theta_\alpha$ by $s_d$ as:

\[
[\mathcal{D}_d(\phi)] = -\lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \cdot \psi_j - \sum_{i,S} \binom{d_S - i + 1}{2} \cdot \delta_{i,S} + \delta_{irr}/8.
\]

Grushevsky-Zakharov gave an alternative proof of this result in [GZ14a, Theorem 2, Equation (3.4)]. (A caution to the reader: the definition of the theta divisor in [GZ14a] is different from the definition in [Hai13]. Over the locus of compact type curves, the theta divisor is defined on [GZ14a, page 4053, second paragraph] to be the image of an Abel map out of a symmetric power. It is significant that this is taken as the definition over the locus of compact type curves and not over all of $\overline{M}_{g,n}$. While the image of the Abel map is a divisor class defined over all of $\overline{M}_{g,n}$, it is not a divisor class whose pullback is $[\mathcal{D}_d(\phi)]$ because the image of the Abel map, being the image of a rational morphism between Deligne-Mumford stacks that are representable over $\overline{M}_{g,n}$, is an integral Chow class, and as such, its pullback by $s_d$ cannot equal a nonintegral class such as $[\mathcal{D}_d(\phi)]$.)

The divisor class $[\mathcal{D}_d(\phi)]$ is nonintegral (and hence so is $\theta_\alpha$) because of the term $\delta_{irr}/8$ in (47). However, as was observed in [Hai13], $[\mathcal{D}_d(\phi)] - \delta_{irr}/8$ is an integral divisor class on $\overline{M}_{g,n}$, and this last divisor is one of the divisors constructed in this paper:

\[
[\mathcal{D}_d(\phi)] = [\mathcal{D}_d(\phi_0)] + \delta_{irr}/8
\]

for any stability parameter $\phi_0$ satisfying the condition in Lemma 16.

Moreover, the formalism of stability parameters developed in this paper illuminates some of the structure of (47). The term $\lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \cdot \psi_j$ is $[\mathcal{D}_d(\phi_d)]$, while the term $\sum \binom{d_S - i + 1}{2} \cdot \delta_{i,S}$ is a wall-crossing term, the difference between $[\mathcal{D}_d(\phi_d)]$ and $[\mathcal{D}_d(\phi_0)]$ described by Theorem 17.
5.2. The stable pairs class. In the introduction we introduced the divisor \([\overline{\mathcal{D}}_g(\text{SP})]\) that is the pullback of the theta divisor of the unique family of stable semiabelic (or quasiabelian) pairs extending the principally polarized universal Jacobian. Here we describe this extension in greater detail.

Recall that a stable semiabelic pair is a pair \((\overline{\mathcal{P}}, D)\) consisting of a (possibly reducible) seminormal projective variety \(\overline{\mathcal{P}}\) with a suitable action of a seminormal variety \(G\) together with an ample effective divisor \(D \subset \overline{\mathcal{P}}\) that does not contain a \(G\)-orbit [Ale02 Definition 1.1.9]. Stable semiabelic pairs satisfy a stable reduction theorem [Ale02 Theorem 5.7.1] that implies there is, up to isomorphism of pairs, at most one extension of the family of principally polarized Jacobians \((\overline{\mathcal{J}}_{g,n}/\mathcal{M}_{g,n}, \Theta)\) to a family of stable semiabelic pairs \((\overline{\mathcal{J}}_{g,n}/\mathcal{M}_{g,n}^{(0)}, \overline{\Theta})\).

For \(n = 0\) (a case not studied here), Alexeev has proven that this unique extension exists and is realized by the Caporaso–Pandharipande family, the family of compactified Jacobians associated to the degenerate parameter \(\phi\) [Ale02 Definition 5.3, Corollary 5.4]. For \(n > 0\), the unique extension \((\overline{\mathcal{J}}_{g,n}/\mathcal{M}_{g,n}, \overline{\Theta})\) of \(\mathcal{J}_{g,n}\) is the pullback of \((\overline{\mathcal{J}}_{g,0}/\mathcal{M}_{g,0}, \overline{\Theta})\) by the forgetful morphism \(\mathcal{M}_{g,n} \to \mathcal{M}_{g,0}\).

An alternative description of this extension is provided by the following lemma:

**Lemma 23.** For \(\phi\) satisfying the condition from Lemma [16], the restriction of the pair \((\overline{\mathcal{J}}_{g,n}(\phi)/\mathcal{M}_{g,n}^{(0)}, \overline{\Theta}(\phi))\) to the open substack \(\mathcal{U} \subset \mathcal{M}_{g,n}^{(0)}\) of stable curves with at most 1 node is a stable semiabelic pair.

**Proof.** The main point to prove is that a fiber of \(\overline{\Theta}(\phi)\)|\(\mathcal{U} \to \mathcal{U}\) is ample and does not contain an orbit of the action of the multidegree 0 Jacobian, and we prove this by directly computing the theta divisor, which has a particularly simple structure. To begin, observe that both \(\mathcal{J}_{g,n}(\phi)|\mathcal{U}\to \mathcal{U}\) and \(\overline{\Theta}(\phi)|\mathcal{U} \to \mathcal{U}\) are flat by Proposition [11] and Lemma [16] so it is enough to fix a marked curve \((C, p_1, \ldots, p_n) \in \mathcal{U}\) and prove that the fiber \(\mathcal{J}_C\) with the effective divisor \(\overline{\Theta}_C\) is a stable semiabelic variety.

Alexeev has proved quite generally that the compactified Jacobian of a nodal curve is a stable semiabelic variety [Ale04 Theorem 5.1], so to prove the specific pair \((\mathcal{J}_C, \overline{\Theta}_C)\) is a stable pair, we need to prove that \(\overline{\Theta}_C\) is ample and does not contain an orbit of the action of the moduli space \(\mathcal{J}_C\) of multidegree 0 line bundles. There are two cases to consider: when \(C\) is irreducible and when \(C\) is reducible.

When \(C\) is irreducible, \(\overline{\Theta}_C\) is ample by [Sou94, Corollary 14] and does not contain a group orbit by the proof of [Sou94, Proposition 7]. When \(C\) is reducible, \(C\) must have two irreducible components, \(C^+\) and \(C^-\), and the computation from Example 2 shows that the \(\phi\)-stable sheaves are either the line bundles of bidegree \((g^+, g^-)\) or the line bundles of bidegree \((g^+, g^- - 1)\). In the first case, restricting to components defines an isomorphism \(\mathcal{J}_C(\phi) \cong \mathcal{J}_{C^+}^{g^+} \times \mathcal{J}_{C^-}^{g^-}\) that identifies \(\overline{\Theta}(\phi)\) with \(p_1^*(\text{node } + \overline{\Theta}_{C^+}) + p_2^*(\overline{\Theta}_{C^-})\).

(Here \(p_1, p_2\) are the projection morphisms. This identifies \((\mathcal{J}_C(\phi), \overline{\Theta}_C)\) as the product of principally polarized varieties, and such a product satisfies the desired conditions. The case of bidegree \((g^+, g^- - 1)\) is entirely analogous, with the roles of \(C^+\) and \(C^-\) being switched.)

**Remark 11.** Observe that Lemma 23 implies that the unique extension of \((\overline{\mathcal{J}}_{g,n}, \overline{\Theta})\) to a family of stable pairs over \(\mathcal{U}\) admits multiple descriptions as a moduli space. The
authors expect this remains true over $\mathcal{M}^{(0)}_{g,n}$ but, as our goal is to establish (8), we do not pursue this issue here.

An immediate consequence is

**Corollary 24.** Equation (8) holds.

**Proof.** By Lemma [23] $[\overline{D}_d(S\mathrm{P})] = [\overline{D}_d(\phi)]$ for any $\phi$ satisfying the conditions from Lemma [16]. The other equalities follow from Hain’s result (47) and Theorem [22].

5.3. **The class of Müller.** Müller studied a different extension of $[D_d]$ in [Müll13]. Under the assumption that some $d_j$ is negative, he defined $\overline{D}_d(\mathrm{Mi}) \subset \overline{\mathcal{M}}_{g,n}$ to be the Zariski closure of $D_d$ and then computed

$$\overline{D}_d(\mathrm{Mi}) = -\lambda + \sum_{j=1}^n \left( \frac{d_j + 1}{2} \right) \cdot \psi_j - \sum_{i,S} \left( \frac{|d_S - i| + 1}{2} \right) \cdot \delta_{i,S} - \sum_{i,S} \left( \frac{d_S - i + 1}{2} \right) \cdot \delta_{i,S}$$

in [Müll13] Theorem 5.6. (Here $S^+ := \{ j \in \{1, \ldots, n\}: d_j > 0 \}$). Grushevsky-Zakharov gave an alternative proof of this in [GZ14a, Theorem 2].

Comparing (42) with (48), we see that if $\phi_0$ is as in Lemma [16], then

$$[\overline{D}_d(\phi_0)] = [\overline{D}_d(\mathrm{Mi})] + \sum_{(i,S) \in T_d} (i - d_S) \cdot \delta_{i,S},$$

where $T_d$ is defined by

$$T_d := \{(i,S): d_j > 0 \text{ for all } j \in S, \text{ and } d_S < i\}.$$

Inspecting Equation (49), we see that the divisor classes $[\overline{D}_d(\phi_0)]$ and $[\overline{D}_d(\mathrm{Mi})]$ are equal if and only if $T_d = \emptyset$. In that case, not only are the divisor classes equal, but the subschemes $\overline{D}_d(\phi)$ and $\overline{D}_d(\mathrm{Mi})$ coincide:

**Corollary 25.** The inclusion of the closed substack $\overline{D}_d(\mathrm{Mi})$ in $\overline{D}_d(\phi)$ is an isomorphism if and only if $\phi = \phi_0$ and $T_d = \emptyset$.

**Proof.** The content of the claim is that if the divisor classes $[\overline{D}_d(\mathrm{Mi})]$ and $[\overline{D}_d(\phi)]$ are equal, then the inclusion is an isomorphism. First reduce to the setting of divisors on a projective scheme by picking an étale cover $\overline{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ with $\overline{\mathcal{M}}$ projective. Setting $D_1$ and $D_2$ respectively equal to the the pullbacks of $\overline{D}_d(\mathrm{Mi})$ and $\overline{D}_d(\phi)$, we have an exact sequence of coherent sheaves

$$0 \to \mathcal{O}_{D_2}(-D_1) \to \mathcal{O}_{D_2} \to \mathcal{O}_{D_1} \to 0.$$

Now pick an ample line bundle $A$ on $\overline{\mathcal{M}}$ and consider the Hilbert polynomials $p_1(t)$ and $p_2(t)$ respectively associated to $D_1$ and $D_2$. The polynomial $p_1$ has degree $\dim(\overline{\mathcal{M}}) - 1$ and leading term the degree of $D_1$ (computed with respect to $A$). Since $D_1$ is linearly equivalent to $D_2$ by hypothesis, $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$ have the same degree, and so $p_2 - p_1$ has degree strictly less than $\dim(\overline{\mathcal{M}}) - 1$. By additivity $p_2 - p_1$ equals the Hilbert polynomial of $\mathcal{O}_{D_2}(-D_1)$, and we conclude that this last Hilbert polynomial has degree strictly less than $\dim(\overline{\mathcal{M}}) - 1$. 

This is only possible if \( O_{D_2}(-D_1) \) equals zero. Indeed, \( O_{D_2}(-D_1) \) is locally principal, so if \( O_{D_2}(-D_1) \) was nonzero, then its support would have dimension \( \dim(M) - 1 \). Since \( O_{D_2}(-D_1) = 0 \), the inclusion \( D_1 \to D_2 \) is an isomorphism. \( \square \)

While \( [\overrightarrow{D}\{M\}] \) is not always equal to some \( [\overrightarrow{D}(\phi)] \), it is always equal to a divisor class on \( \mathcal{J}_{g,n}(\phi) \): the difference \( [\overrightarrow{D}(\phi_0)] - \sum (i - d_S) \cdot \delta_{i,S} \). However, this divisor class can be chosen to be an effective divisor only when the hypotheses of Corollary \( \ref{corollary} \) hold.

**Proposition 26.** The class \( [\overrightarrow{D}\{M\}] \) equals the pullback via \( s_d \) of an effective divisor on \( \mathcal{J}_{g,n}(\phi) \) if and only if it equals \( [\overrightarrow{D}(\phi_0)] \).

**Proof.** The pullback homomorphism \( s_d^* \text{Pic}(\mathcal{J}_{g,n}(\phi)) \to \text{Pic}(\mathcal{M}^{(0)}_{g,n}) \) is injective because the image of each boundary class is a boundary class, and \( \overline{\theta}(\phi) \) is not the linear combination of boundary classes. Moreover, because the Picard groups of \( \mathcal{J}_{g,n}(\phi) \) are all isomorphic, it is enough to consider the case \( \phi = \phi_0 \). Therefore, if this effective divisor exists it has to be linearly equivalent to the difference

\[
\overline{\theta}(\phi_0) - \sum_{(i,S) \in T_d} (i - d_S) \cdot \delta_{i,S}.
\]

The proof is then concluded by applying Lemma \( \ref{lemma} \). \( \square \)

**Lemma 27.** The only effective divisor linearly equivalent to \( \overline{\theta}(\phi) \) is \( \overline{\theta}(\phi) \) itself.

**Proof.** It is enough to prove that the space of global sections of \( O(\overline{\theta}(\phi)) \) is 1-dimensional, and we prove this by computing the direct image of \( O(\overline{\theta}(\phi)) \) under the natural morphism \( \mathcal{J}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n} \). We compute this direct image by using the theorem on cohomology and base change together with results about the theta divisor of a Jacobian variety.

Let \( \mathcal{U} \subset \mathcal{M}^{(0)}_{g,n} \) be the open substack parameterizing marked curves with at most one node. Given \( (C, p_1, \ldots, p_n) \in \mathcal{U} \), consider the restriction \( O(\overline{\theta}_{C}(\phi)) \) of \( O(\overline{\theta}(\phi)) \) to the fiber \( J_C \) over \( C \). The authors claim that the space of global sections of this restriction is 1-dimensional. In the proof of Lemma \( \ref{lemma} \) we observe that when \( C \) is of compact type \( (\mathcal{J}_{C}, \overline{\theta}_{C}(\phi)) \) is a product of principally polarized Jacobians, so the result follows from the Riemann–Roch formula for abelian varieties \( \text{Mum}08 \) page 140]. When \( C \) is not of compact type, \( (\mathcal{J}_{C}, \overline{\theta}_{C}(\phi)) \) is a principally polarized rank 1 degeneration (in the sense of \( \text{Mum}83 \)), and the desired computation is done on \( \text{Mum}83 \) page 4] (by reducing the claim to a result about the normalization of \( \mathcal{J}_{C} \), which is a \( \mathbb{P}^1 \)-bundle over an abelian variety).

Now consider the global section of \( O(\overline{\theta}(\phi))|\mathcal{U} \) that is the image of 1 under the natural inclusion \( O_{\mathcal{U}} \to O(\overline{\theta}(\phi))|\mathcal{U} \). This section has nonzero restriction to every fiber of \( \mathcal{J}_{g,n}(\phi)|\mathcal{U} \to \mathcal{U} \) (because \( \overline{\theta}(\phi) \) does not contain a fiber), so by dimension considerations, this restriction is a generator. We can conclude that the hypothesis of the cohomology and base change theorem \( \text{Ill05} \) Theorem 8.3.2] is satisfied, so the formation of the direct image of \( O(\overline{\theta}(\phi))|\mathcal{U} \) under \( \mathcal{J}_{g,n}(\phi)|\mathcal{U} \to \mathcal{U} \) commutes with base change. In particular, the natural inclusion from \( O_{\mathcal{U}} \) to the direct image has the property that the induced map on stalks is always surjective. We conclude that this inclusion is an isomorphism,
so
\[ h^0(\mathcal{J}_{g,n}(\phi), \mathcal{O}(\overline{\mathcal{S}}(\phi))\mathcal{U}) = h^0(\mathcal{U}, \mathcal{O}_\mathcal{U}) = 1. \]

The last equality follows because \( \mathcal{O}_\mathcal{U} \) satisfies condition S2 and the complement of \( \mathcal{U} \) has codimension 2. For similar reasons,
\[ h^0(\mathcal{J}_{g,n}(\phi), \mathcal{O}(\overline{\mathcal{S}}(\phi))) = h^0(\mathcal{J}_{g,n}(\phi), \mathcal{O}(\overline{\mathcal{S}}(\phi))\mathcal{U}). \]

\[ \square \]

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TO BE ADDED AFTER REFEREEING PROCESS.

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