Multicomponent bi-superHamiltonian KdV systems

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Abstract

It is shown that a new class of classical multicomponent super KdV equations is bi-superHamiltonian by extending the method for the verification of graded Jacobi identity. The multicomponent extension of super mKdV equations is obtained by using the super Miura transformation.

1 Introduction

The theories of infinite dimensional super integrable systems have drawn a lot of attention in the last two decades, for example, see [1], [2]. The research on the classical multicomponent integrable systems has also become quite active more recently [3], [4], [5], [6], [7], [8], [9]. In this work we construct an extension of classical multicomponent Korteweg de Vries (KdV) system to multicomponent superintegrable systems by employing a bi-superHamiltonian formalism. Such systems are called super because they contain both bosonic and fermionic fields. However there is no a supersymmetry transformation between the fields as it is known from the case of 1-component super KdV equa-
On the other hand there also exist supersymmetric extensions of KdV equation, namely there exist supersymmetry transformations but this bi-superHamiltonian systems have a nonlocal nature. We first introduce skew symmetric super Hamiltonian operators. It is shown that they satisfy the graded Jacobi identity by using the method of prolongation. The set of multicomponent super integrable partial differential equations are derived by introducing associated super Hamiltonians. Furthermore introducing super Miura transformation, a multicomponent super extension modified Korteweg de Vries (mKdV) system is obtained. The paper is organized as follows. In section 2 we investigate the properties of super Hamiltonian operators. It is shown that the second Hamiltonian operator satisfies the graded Jacobi identity by means of constraint between the constant parameters of the system while the first one does trivially. In section 3 the multicomponent super evolution equations are derived from super Hamiltonians. In section 4 we obtain the multicomponent super mKdV equations by using multicomponent super Miura transformation. It is observed that the new systems are reduced to well known systems of one component super evolution equations and in the vanishing fermionic fields limit we get the multicomponent and one component corresponding KdV systems.

2 The super Hamiltonian operators and Jacobi identity

This section is devoted to study of properties of super Hamiltonian operators. We first consider a set of fields $\phi_A$ which contains both commuting and anticommuting fields such as

$$\phi_A = \begin{pmatrix} u_\alpha \\ \xi_a \end{pmatrix} \tag{1}$$

where $u_\alpha(x,t)$ is assumed a commuting (bosonic) field while $\xi_a(x,t)$ is an anticommuting (fermionic) field in 1+1 dimensions, $\alpha = 1, 2, ..., m$ and $a = 1, 2, ..., n$. A $Z_2$ grading is introduced such that $\tilde{p}(\phi)$ equals to zero if $\phi_A$ is commuting or one if it is anticommuting.

The evolution equation of a continuous dynamical Hamiltonian system is
given by

$$
\partial_t \phi_A = \sum_B J_{AB} \frac{\delta H}{\delta \phi_B} = \sum_B J_{AB} E_B(H),
$$

(2)

where $E_B$ is the Euler operator,

$$
E_B = \sum_{k=0}^{\infty} (-\partial_x)^k \frac{\partial}{\partial x^k \phi_B}
$$

(3)

and $J$ is a certain differential operator and $H$ is a suitable functional. Functionals are defined as modulo the integral of total derivative terms as

$$
F = \int F[\phi_B] dx
$$

(4)

where $F[\phi_A]$ is the element of the algebra of functions of $x$, the fields $\phi_A(x)$ and their derivatives. The operator $J$ defines a Poisson bracket as

$$
\{ F, G \} = \sum_{AB} \int [J_{AB} E_B(G)] E_A(F) dx
$$

(5)

Here the ordering of the arbitrary functionals $F$ and $G$ becomes important for the graded systems. The fundamental Poisson bracket is

$$
\{ \phi_A(x), \phi_B(x') \} = J_{AB} \delta(x - x')
$$

(6)

that leads to the following expression for the evolution equation 2

$$
\partial_t \phi_A = \{ \phi_A, H \}.
$$

(7)

$J$ is called a Hamiltonian operator if the Poisson bracket is skew-symmetric as

$$
\{ F, G \} = -(-1)^{\bar{p}(F)} \bar{p}(G) \{ G, F \}
$$

(8)

where the grading $\bar{p}(F)$ equals to zero (one) if an arbitrary functional $F$ is bosonic (fermionic) and satisfies the Jacobi identity which can be given as vanishing the prolongation of an evolutionary vector field, $v_{JH}$, associated to every Hamiltonian $H$ as follows

$$
pr v_{JH}(I) = 0
$$

(9)
where $I$ is the graded cosymplectic functional two-vector given as

$$
I = \frac{1}{2} \sum_{A,B} \int J_{AB} \Theta_B \wedge \Theta_A dx
$$

(10)

here the set $\Theta_A = \{\theta_\alpha, \eta_a\}$ forms a basis of bosonic and fermionic uni-vectors, dual to the one-forms $\{u_\alpha, \xi_a\}$ respectively. Notice that

$$
\Theta_A \wedge \Theta_B = - (-1)^{\tilde{A} \tilde{B}} \Theta_B \wedge \Theta_A
$$

(11)

and $\Theta \wedge \Theta \neq 0$ if $\Theta$ is fermionic.

We now introduce the super Hamiltonian operators

i) First one:

$$
J_{AB}^{(1)} = \begin{pmatrix}
\delta_{\alpha\beta} \partial_x & 0 \\
0 & \delta_{ab}
\end{pmatrix}
$$

(12)

ii) Second one:

$$
J_{AB}^{(2)} = \begin{pmatrix}
j_{\alpha\beta} & j_{ab} \\
j_{\alpha\beta} & j_{ab}
\end{pmatrix}
$$

(13)

where

$$
j_{\alpha\beta} = b_{\alpha\beta} \partial_x^2 + 2C_{\alpha\beta\gamma} u_\gamma \partial_x + C_{\alpha\beta\gamma} u_{\gamma,x}
$$

(14)

$$
j_{ab} = K_{abcd} \xi_d \partial_x + L_{abcd} \xi_{d,x}
$$

(15)

$$
j_{\alpha\beta} = M_{\alpha\beta\delta} \xi_d \partial_x + N_{\alpha\beta\delta} \xi_{d,x}
$$

(16)

$$
j_{ab} = \Lambda_{ab} \partial_x^2 + \Omega_{ab\gamma} u_{\gamma}
$$

(17)

where $u_{\alpha,x} = \partial_x u_\alpha$ and all coefficients apart from $u(x,t)$ and $\xi(x,t)$ are constants. It is very easy to see that the operator $J_{AB}^{(1)}$ is a Hamiltonian operator because it is skew-symmetric and Jacobi identity is trivially satisfied, there are no variable coefficients in its expression. On the other hand second operator $J_{AB}^{(2)}$, which is skew-symmetric, contains x and t dependent coefficients.
In order to show that it is a Hamiltonian operator the graded Jacobi identity
should be satisfied. The eq. (9) for the second operator becomes

$$\text{pr}_{\nu} J^{\Theta}(I) = \frac{1}{2} \int \text{pr}_{\nu} J^{(2)}_{AB} \Theta_B \wedge \Theta_A dx = 0 \quad (18)$$

Here Einstein sum rule is employed and it will be used from now on. Eq. (18) can be written as

$$\text{pr}_{\nu} J^{\Theta}(I) = \frac{1}{2} \int \left[ \text{pr}_{\nu} J^{(j_{\alpha\beta})} \theta_{\beta} \wedge \theta_{\alpha} + \text{pr}_{\nu} J^{(j_{\alpha\beta})} \eta_{\beta} \wedge \eta_{\alpha} \right] dx = 0 \quad (19)$$

On the other hand, in general,

$$\text{pr}_{\nu} J^{(2)}_{AB} = \sum_{E,F,k} \partial^k_x (J^{(2)}_{EF}) \frac{\partial}{\partial (\partial^k_x \phi_E)} (J^{(2)}_{AB}) \quad (20)$$

Here $k = 0, 1$. Furthermore,

$$\text{pr}_{\nu} J^{(2)}_{AB} = \sum_k \left\{ \partial^k_x (j_{\lambda\rho}) \frac{\partial}{\partial (\partial^k_x \xi_{\rho})} (J^{(2)}_{AB}) + \partial^k_x (j_{\lambda\rho}) \frac{\partial}{\partial (\partial^k_x \xi_{\rho})} (J^{(2)}_{AB}) \right\} \quad (21)$$

By introducing

$$C_{\alpha\beta\lambda} = C_{\beta\alpha\lambda}$$
$$\Omega_{ab\lambda} = \Omega_{ba\lambda}$$
$$\Omega_{ab\lambda} K_{\lambda cd} = \Omega_{ac\lambda} K_{\lambda bd} \quad (22)$$
and by using eq. (21) in the eq. (20), we finally obtain

\[ \rho v_J(I) = \frac{1}{2} \int \left\{ C_{\alpha\beta\lambda} b_{\lambda\rho} (\theta_{\rho,xxx} \wedge \theta_{\beta} \wedge \theta_{\alpha} - 2 \theta_{\rho,xx} \wedge \theta_{\beta,xx} \wedge \theta_{\alpha} - 2 \theta_{\rho,xx} \wedge \theta_{\beta,x} \wedge \theta_{\alpha,x}) \\
+ C_{\alpha\beta\lambda} C_{\lambda\rho\gamma} (u_{\gamma} \theta_{\rho,x} \wedge \theta_{\beta} \wedge \theta_{\alpha} + u_{\gamma,x} \theta_{\rho} \wedge \theta_{\beta} \wedge \theta_{\alpha} + 4 u_{\gamma} \theta_{\rho,x} \wedge \theta_{\beta,x} \wedge \theta_{\alpha} + 2 u_{\gamma,x} \theta_{\rho} \wedge \theta_{\beta,x} \wedge \theta_{\alpha}) \\
+ M_{\alpha\beta\lambda} (L_{\alpha\beta} + M_{\alpha\beta} - N_{\alpha\beta}) \xi_{\alpha} \eta_{\beta} \wedge \theta_{\beta,x} \wedge \theta_{\alpha,x} \\
- N_{\alpha\beta}(K_{\alpha\beta} - L_{\alpha\beta} + N_{\alpha\beta}) \xi_{\beta,x} \eta_{\alpha,x} \wedge \theta_{\beta} \wedge \theta_{\alpha} \\
+ [2 C_{\alpha\beta\gamma} K_{\gamma\beta a} - M_{\alpha\beta}(K_{\alpha\beta c} - L_{\alpha\beta c} + N_{\alpha\beta c})] \xi_{\alpha} \eta_{\beta} \wedge \theta_{\beta,x} \wedge \theta_{\alpha} \\
+ [2 C_{\alpha\beta\gamma} L_{\gamma\beta a} - N_{\alpha\beta}(M_{\beta\gamma c} - N_{\beta\gamma c} + L_{\beta\gamma c})] \xi_{\alpha} \eta_{\beta} \wedge \theta_{\beta,x} \wedge \theta_{\alpha} \\
+ [\Lambda_{\alpha}(K_{\alpha\beta c} + M_{\alpha\beta c}) - 6 \Omega_{\alpha\beta\lambda} b_{\lambda\alpha}] \eta_{\alpha,xx} \wedge \eta_{xx} \wedge \theta_{\alpha} \\
+ [\Lambda_{\alpha}(M_{\alpha\beta c} - N_{\alpha\beta c} + L_{\alpha\beta c}) - 2 \Omega_{\alpha\beta\lambda} b_{\lambda\alpha}] \eta_{\alpha,xxx} \wedge \eta_{\alpha} \wedge \theta_{\alpha} \\
+ [2 \Omega_{\alpha\beta\gamma} C_{\lambda\alpha\beta} + \frac{1}{2} \Omega_{\alpha\beta\gamma} (N_{\alpha\beta c} - 4 L_{\alpha\beta c})] u_{\beta,x} \eta_{\alpha} \wedge \eta_{\beta} \wedge \theta_{\alpha} \\
- \Omega_{\alpha\beta}(M_{\alpha\beta c} - N_{\alpha\beta c} + L_{\alpha\beta c}) u_{\beta,x} \eta_{\alpha} \wedge \eta_{\beta} \wedge \theta_{\alpha} \\
+ \frac{1}{3} \Omega_{\alpha\beta}[3 L_{\alpha\beta} - K_{\alpha\beta}] \xi_{\alpha} \eta_{\alpha} \wedge \eta_{\beta} \wedge \eta_{\gamma} \right\} dx = 0 \] (23)

As it can easily be seen there is a trivial solution for eq. (23) in which all constant coefficients vanish. There exists a non-trivial solution as

\[
\begin{align*}
 b_{\alpha\lambda} C_{\lambda\beta\gamma} &= b_{\beta\lambda} C_{\lambda\alpha\gamma} \\
 C_{\alpha\beta\lambda} C_{\lambda\gamma\rho} &= C_{\alpha\gamma\lambda} C_{\lambda\beta\rho} \\
 K_{\lambda\alpha b} &= M_{\alpha\lambda b} \\
 K_{\lambda\alpha b} &= 3 L_{\alpha\lambda b} \\
 2 M_{\lambda\alpha b} &= 3 N_{\alpha\lambda b} \\
 \Lambda_{\alpha c} M_{\alpha\beta c} &= 3 b_{\alpha\beta} \Omega_{\alpha\beta} \\
 M_{\alpha\beta c} K_{\beta\alpha c} &= M_{\alpha\beta c} K_{\alpha\beta c} \\
 C_{\alpha\beta\gamma} K_{\gamma\beta c} &= K_{\alpha\beta c} K_{\beta\alpha c}
\end{align*}
\] (24)

Thus \( J^{(2)}_{AB} \) becomes a Hamilton operator with the set of equations (24). It describes the second Poisson structure. For KdV equation one can easily show that sum of two Hamilton operators of bi-Hamiltonian structure is also a
Hamilton operator because one of the Hamilton operator \((J = \partial_x)\) trivially satisfies Jacobi Identity [10]. In our case \(J^{(1)}_{AB} + J^{(2)}_{AB}\) satisfies the graded Jacobi identity with the condition

\[
\Omega_{ab\beta} - M_{a\beta b} - \frac{1}{2}(K_{\beta ab} + L_{\beta ab} - N_{a\beta b}) = 0 \tag{25}
\]

and using our solution (24) this equation becomes

\[
\Omega_{ab\beta} + N_{a\beta b} = 2K_{\beta ab}, \tag{26}
\]

and furthermore, we obtain

\[
\Omega_{ab\beta} = 4L_{\beta ab} \tag{27}
\]

Thus the Hamilton operators \(J^{(1)}_{AB}\) and \(J^{(2)}_{AB}\) constitute a super Hamiltonian pair. We can now rewrite the second operator in terms of \(L_{\lambda ab}\) as

\[
J^{(2)}_{AB} = \begin{pmatrix}
    b_{\alpha\beta \gamma} \partial_{\gamma} + c_{\alpha\beta \gamma} (u_{\alpha} \partial_{x} + \partial_{x} u_{\gamma}) & L_{abc} (2\xi_{c} \partial_{x} + \partial_{x} \xi_{c}) \\
    L_{\alpha \beta \gamma} (\xi_{c} \partial_{x} + 2 \partial_{x} \xi_{c}) & \Lambda_{ab} \partial_{x}^{2} + 4L_{\lambda ab} u_{\lambda}
\end{pmatrix} \tag{28}
\]

However the equations (24) and (25) provides information about an algebra related to the evolution equations. In the next section we shall derive the corresponding evolution equations that are coupled multicomponent super KdV equations.

3 The multicomponent super KdV systems

Bi-Hamiltonian formalism suggests the existence of infinitely many conserved quantities \(\{H_k\}\) satisfying the recursion relation

\[
\sum_{B} J^{(2)}_{AB} E_{B}(H_{k-1}) = \sum_{B} J^{(1)}_{AB} E_{B}(H_k) \tag{29}
\]

where \(k = 1, 2, 3, \ldots\). These infinitely many conserved quantities provide an extension of super KdV hierarchy to multicomponent super KdV hierarchy. We now introduce the first two conserved quantities to obtain first member of evolution equations as

\[
H_0 = \frac{1}{2} \int [-\delta_{\alpha\beta} u_{\alpha} u_{\beta} + \delta_{ab} \xi_{a} \xi_{b, x}] dx \tag{30}
\]
and
\[ H_1 = \frac{1}{2} \int \left[ -b_{\alpha\beta} u_{\alpha,x} u_{\beta,x} + C_{\alpha\beta\gamma} u_{\alpha} u_{\beta} u_{\gamma} 
- \Lambda_{\alpha\beta} \xi_{\alpha,x} \xi_{\beta,xx} + 2K_{\alpha\beta\gamma} u_{\alpha} \xi_{\beta} \xi_{\beta,x} \right] dx. \] (31)

Then one can easily derive integrable super coupled integrable evolution equations, which admit infinitely many conserved quantities due to the recursion relations (29), by using
\[ \partial_t \phi_A = \sum_B J_{AB}^{(1)} E_B (H_1) = \sum_B J_{AB}^{(2)} E_B (H_0). \] (32)

In this way we get the new class of integrable multicomponent super KdV equations by using the equations (24) and (26) as follows
\[ u_{\alpha,t} = b_{\alpha\beta} u_{\beta,xxx} + 3C_{\alpha\beta\gamma} u_{\beta,x} u_{\gamma} + K_{\alpha\beta\lambda} \xi_{\lambda,xx} \] (33)
\[ \xi_{\alpha,t} = \Lambda_{\alpha\beta} \xi_{\alpha} \xi_{\beta} + K_{\alpha\beta\lambda} (\xi_{\lambda} u_{\lambda,x} + 2u_{\lambda} \xi_{\beta,x}) \] (34)

In the bosonic limit when the fermionic variables vanish the system reduces to multicomponent KdV systems, known as degenerate Svinolupov system in which $b_{\alpha\beta}$ is nondiagonalizable \[4\], \[3\]. In this case, 1-component limit is the KdV equation. Furthermore if we choose the coefficients $b_{11} = -1$, $\Lambda_{11} = -4$, $C_{111} = 2$, $K_{111} = 3$ satisfying the constraint equations (24) and variables $u_1 = u$ and $\xi_1 = \xi$ eqs.(33-34) becomes
\[ u_t = -u_{xxx} + 6u u_x + 3\xi_{xx} \] (35)
\[ \xi_t = -4\xi_{xxx} + 6u \xi_x + 3u_{x} \xi. \] (36)

These are super KdV equations given in references \[10\], \[11\]. In other words, our equations reduce to one of the known 1-component super KdV equations which consists of one bosonic and one fermionic variables.

4 The multicomponent super mKdV systems

In this section we first introduce a super extension of Miura transformation using the notation of previous sections. The multicomponent super Miura transformation is
\[ u_{\alpha} = v_{\alpha,x} + \frac{1}{2} C_{\alpha\beta\gamma} v_{\beta} v_{\gamma} + K_{\alpha\beta\gamma} \bar{\xi}_{\beta} \bar{\xi}_{\gamma} \] (37)
\[ \xi_{\alpha} = \bar{\varepsilon}_{\alpha,x} + \frac{1}{3} M_{\alpha\beta\gamma} v_{\lambda} \bar{\xi}_{\beta} \] (38)
where \( v(x, t) \) and \( \varepsilon(x, t) \) are new bosonic and fermionic variables, respectively. The multicomponent super mKdV equations can be obtained from the multicomponent super KdV equations by the multicomponent super Miura transformations. This implies that any solution of the multicomponent super mKdV equations gives a solution of the multicomponent super KdV equations through the multicomponent super Miura transformations. When we substitute the transformation (37) into the eqs. (33) we get the multicomponent super mKdV equations as

\[
v_{\alpha,t} = -v_{\alpha,xxx} + \frac{3}{2} C_{\alpha\beta\gamma} C_{\beta\lambda\rho} v_{\lambda} v_{\rho} v_{\gamma,x} \\
+ \frac{1}{8} K_{\beta mn} C_{\alpha\beta\gamma} (2 v_{\gamma,x} \varepsilon_{m,n,x} + v_{\gamma} \varepsilon_{m} \varepsilon_{n,x}) \\
+ \frac{1}{4} K_{\alpha mn} \varepsilon_{n,x} \varepsilon_{m,xx} \tag{39}
\]

\[
\varepsilon_{a,t} = -4 \varepsilon_{a,xxx} - K_{\beta ab} (v_{\beta,x} \varepsilon_{b} + 2 v_{\beta,x} \varepsilon_{b,x}) \\
+ K_{\beta ab} C_{\beta\lambda\rho} (v_{\lambda} v_{\rho} \varepsilon_{b,x} + v_{\lambda,x} v_{\rho} \varepsilon_{b}) \tag{40}
\]

by employing the constraints (24) on the coefficients. As in the case of multicomponent super KdV equations, the eqs. (39-40) reduces to

\[
v_t = \partial_x (2v^3 - v_{xx} + \frac{3}{4} \varepsilon \varepsilon_{xx} + \frac{3}{2} v \varepsilon \varepsilon_x) \tag{41}
\]

\[
\varepsilon_t = -4 \varepsilon_{xxx} + (6v v_x - 3v_{xx}) \varepsilon + 6(v^2 - v_x) \varepsilon_x \tag{42}
\]

in the 1-component limit, by taking the coefficients as \( C_{111} = 2, K_{111} = 3 \) and the variables as \( v_1 = v \) and \( \varepsilon_1 = \varepsilon \). This is the super extension of the mKdV equation given by Kuperschmidt [10].

### 5 Conclusion

In this work we have found the new class of integrable multicomponent super KdV equations. It is shown that they are bi-superHamiltonian. The graded Jacobi identity associated to the Poisson structure defined by super Hamiltonian operators is satisfied by imposing constraints (24) on the coefficients introduced in the super Hamiltonian operators. These constraints could be important to describe the structure associated to our evolution equations. It is natural to expect that such relations would also imply the existence of
generalized symmetries. Furthermore by introducing a super Miura transformation a super extension of multicomponent mKdV equations is obtained. This system also possesses the structure described by the constraints (24). We have shown that our equations are reduced to well known 1-component super equations and multicomponent and 1-component bosonic equations.

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