On the Vapnik-Chervonenkis dimension of products of intervals in $\mathbb{R}^d$

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Abstract

We study combinatorial complexity of certain classes of products of intervals in $\mathbb{R}^d$, from the point of view of Vapnik-Chervonenkis geometry. As a consequence of the obtained results, we conclude that the Vapnik-Chervonenkis dimension of the set of balls in $\ell_d^\infty$ – which denotes $\mathbb{R}^d$ equipped with the sup norm – equals $\lfloor (3d + 1)/2 \rfloor$.

1 Introduction

A classifier $f$ on a measurable space $X$ is a binary function defined on $X$. A class $\mathcal{F}$ of classifiers on $X$ is said to shatter a sample $\sigma \subset X$ of size $n$ if $\mathcal{F}$ can perceive all possible binary labelings of the elements of $\sigma$, that is,

$$\#\{f|_\sigma : f \in \mathcal{F}\} = 2^n.$$  

In other words, $\mathcal{F}$ shatters $\sigma$ when $\#\{C \cap \sigma : \chi_C \in \mathcal{F}\} = 2^n$, where $\chi_C$ denotes the indicator function of $C$. Thus, we can alternatively focus on the sets $\{C : \chi_C \in \mathcal{F}\}$ to study properties of $\mathcal{F}$. More generally, families of classifiers on $X$ are in one-to-one correspondence with families of subsets of $X$. The later are called concept classes, see for example [4]. When a learning algorithm has to efficiently choose a classifier within a family $\mathcal{F}$ that minimizes the learning error, it is often necessary some control on the quantity of different labelings that $\mathcal{F}$ can produce on finite samples of $X$. One of the most important ways to measure the combinatorial complexity of families of classifiers, or equivalently, families of concept classes, is to analyse their Vapnik-Chervonenkis dimension, a concept introduced by Vapnik and Chervonenkis in [5].

The Vapnik-Chervonenkis dimension of a concept class $\mathcal{E}$ on $X$, which we will denote by $\text{VC–dim}(\mathcal{E})$, is defined by

$$\text{VC–dim}(\mathcal{E}) = \sup\{\#\sigma \mid \sigma \subset X \text{ is finite and } \mathcal{E} \text{ shatters } \sigma\}.$$  

We shall also write VC dimension, for short. To illustrate how the information about the VC dimension of a concept class $\mathcal{E}$ guarantees an efficient determination of an appropriate classifier, denote by $\mathcal{N}(\mathcal{E}, n)$ the shattering coefficient of $\mathcal{E}$ on $X$.

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\( \mathcal{E} \) with respect to a sample size \( n \). This is the number of labellings that \( \mathcal{E} \) can produce on a sample of size \( n \), that is,

\[
\mathcal{N}(\mathcal{E}, n) = \max_{\# \sigma = n} \# \{ C \cap \sigma \mid C \in \mathcal{E} \}.
\]

It was proved independently by Sauer in 1972, Shelah in 1972, and Vapnik and Chervonenkis in 1971 that if \( \text{VC} - \dim(\mathcal{E}) = d < \infty \), then \( \mathcal{N}(\mathcal{E}, n) \leq (en/d)^d \), where \( e \) stands for the Euler constant. This implies in particular that the shattering coefficients grow polynomially with respect to the sample size. Furthermore, empirical risk minimization is consistent with respect to \( \mathcal{E} \) if and only if \( \text{VC} - \dim(\mathcal{E}) \) is finite, see [5].

The VC geometry of certain concept classes in \( \mathbb{R}^n \) have received special attention. In his classic paper [2], Dudley showed that the VC dimension of Euclidean balls in \( \mathbb{R}^n \) is \( d + 1 \). Other natural classes that have been studied are the class of products of (possibly degenerate) intervals,

\[
\mathcal{R} := \{ [a_i, b_i] : -\infty \leq a_i < b_i \leq \infty, \; i = 1, 2, \ldots, n \},
\]

and some subclasses of \( \mathcal{R} \). In [7], Dudley and Wenocur proved that \( \text{VC} - \dim(\mathcal{R}) = 2n \). They showed additionally that, under the assumption that each \( a_i \) in [1] equals \(-\infty\), the resulting subclass has VC dimension \( n \). More recently, in [3], Gey determined the VC dimension of the class of axis-parallel cuts in \( \mathbb{R}^n \). More precisely, it was proved that the VC dimension of the concept class

\[
\mathcal{A}_n = \{ \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \leq a \} : i = 1, \ldots, n, \; a \in \mathbb{R} \}
\]

is equal to \( \max \{ m : \binom{m}{\lfloor m/2 \rfloor} \leq n \} \).

In this work we continue to investigate the VC complexity of some natural subclasses of \( \mathcal{R} \). As a consequence of our study, we obtain in particular our main theorem, stated below. Let us first establish some notation. \( \ell^d_\infty \) denotes the vector space \( \mathbb{R}^d \) equipped with the norm \( \| \cdot \|_\infty : x \to \max \{|x_1|, \ldots, |x_d|\} \).

Denote by \( \mathcal{C}_d \) the set of all closed balls in \( \ell^d_\infty \), which coincide with the set of closed cubes with sides parallel to the coordinate axes. Our main result reads as follows.

**Main Theorem.** For each \( d \geq 1 \), \( \text{VC} - \dim(\mathcal{C}_d) = \lfloor (3d + 1)/2 \rfloor \).

This result was announced in [1], but there was a flaw in the proof of the inequality \( \text{VC} - \dim(\mathcal{C}_d) \geq \lfloor (3d + 1)/2 \rfloor \). Specifically, in the proof of Lemma 2, it was claimed that the set denoted by \( S' \) is shattered by \( \mathcal{C}_d \), which is false: the subset \( \{ x, y \} \) cannot be carved out of \( S' \) by an element of \( \mathcal{C}_d \) (meaning that there is no \( C \in \mathcal{C}_d \) such that \( C \cap S' = \{ x, y \} \)). This gap does not seem to be easily fixable.

In the present work we prove the announced result approaching it from a different viewpoint. We study the Vapnik-Chervonenkis dimension of certain families of degenerate balls in \( \ell^d_\infty \). A closed (resp. open) degenerate ball in \( \ell^d_\infty \) is any subset of the form \( \prod_{i=1}^d I_i \subseteq \ell^d_\infty \), where each \( I_i \) is a closed (resp. open)
interval, unbounded to at least one side. As the notation suggests, a degenerate ball can be interpreted as a ball with infinite radius. For instance, if we choose a point \( x \in \ell_\infty^d \) and a direction \( v \in \ell_\infty^d \setminus \{0\} \), and let \( B_n \) be the closed ball centered at \( n \cdot v \) and with \( x \) at its boundary, then the pointwise limit on \( n \in \mathbb{N} \) of \( B_n \) is a closed degenerate ball \( D \) with \( x \) in its boundary. One can imagine that \( D \) has its center at infinity, in the direction \( v \). Although the notation might be new, the idea was already used in Vapnik-Chervonenkis theory: in Dudley’s proof in [2] of the fact that the family of euclidean balls in \( \mathbb{R}^d \) has VC dimension \( d + 1 \), an underlying idea is that semi-spaces are pointwise limits of euclidean balls. This motivates a more general and systematic study of combinatorial properties of degenerate balls in finite-dimensional normed spaces.

Throughout this work, we shall use the following additional notation. For any closed subset \( F \) of \( \mathbb{R}^d \), we denote by \( D_F \) the set of all degenerate balls in \( \ell_\infty^d \) containing \( F \). In the case where \( F = \{0\} \), we use the lighter notation \( D_0 \) instead of \( D_{\{0\}} \). In the case where \( F = \emptyset \), we simply write \( D_d \), instead of \( D_{\emptyset} \). Note that \( D_d \) is the set of all degenerate balls in \( \ell_\infty^d \). Our first main result on Vapnik-Cervonenkis complexity of degenerate balls in \( \ell_\infty^d \) is the following.

**Theorem A.** For each nonempty bounded subset \( F \) of \( \mathbb{R}^d \), \( VC \dim(D_F) = \left\lfloor \frac{3d}{2} \right\rfloor \).

We then relate the Vapnik-Cervonenkis geometry of balls and of degenerate balls in \( \ell_\infty^d \) spaces by proving the following.

**Theorem B.** For each \( d \geq 2 \), \( VC \dim(C_d) = VC \dim(D_{d-1}) + 2 \).

Note that Theorems A and B imply the Main Theorem, with the exception of \( VC \dim(C_1) \) - but it is well known, and easily verified, that \( VC \dim(C_1) = 2 \).

The next section is dedicated to proving Theorems A and B.

## 2 Proof of main results

Let us start by establishing some notation. For each \( S \subset \mathbb{R}^d \), \( \text{co}(S) \) denotes the convex hull of \( S \), as usual. We define the **rectangular hull of** \( S \) as being the smallest product of (possibly degenerate) intervals containing \( S \), and denote it by \( \square \text{-hull}(S) \).

To prove Theorems A and B, the following Propositions [1] and [2] will be of use.

**Proposition 1.** Let \( F \) be a nonempty bounded closed subset of \( \mathbb{R}^d \). Then, \( VC \dim(D_F) = VC \dim(D_{\square}(F)) \).

**Proof.** Note that \( D_F = D_{\square}(F) \), where \( R = \square \text{-hull}(F) \). We can assume then, without loss of generality, that \( F \) is rectangular: \( F = [a_1, b_1] \times \cdots \times [a_d, b_d] \). For
each \( i \in \{1, \ldots, d\} \), define

\[
p_i(x) = \begin{cases} 
  x - a_i, & \text{if } x < a_i, \\
  0, & \text{if } a_i \leq x \leq b_i, \\
  x - b_i, & \text{if } b_i < x.
\end{cases}
\]

Define \( P : \mathbb{R}^d \to \mathbb{R}^d \) by \( P(p_1(x_1), \ldots, p_d(x_d)) \). It is readily verified that \( P \) is surjective and satisfies the following properties:

1. for each \( D \in \mathcal{D}_d^F \), \( P(D) \in \mathcal{D}_d^0 \), and
2. for each \( D \in \mathcal{D}_d^0 \), \( P^{-1}(D) \in \mathcal{D}_d^F \).

Let us verify that \( P \) also satisfies the following additional property:

3. \( P \) is injective when restricted to any \( \mathcal{D}_d^F \)-shattered subset of \( \mathbb{R}^d \).

Indeed, let \( x \in \mathbb{R}^d \), and let \( D \in \mathcal{D}_d^F \) be such that \( x \in D \). Then

\[
D \supset \square\text{-hull}(F \cup \{x\}) = \text{co}([a_1, b_1] \cup \{x_1\}) \times \cdots \times \text{co}([a_d, b_d] \cup \{x_d\}).
\]

Suppose now that \( y \in \mathbb{R}^d \) is such that \( P(y) = P(x) \). Then for each \( i \in \{1, \ldots, d\} \),

\[
p_i(y) = p_i(x) \Rightarrow \begin{cases} 
  y_i = x_i, & \text{if } x_i < a_i \text{ or } x_i > b_i \\
  y_i \in [a_i, b_i], & \text{if } x_i \in [a_i, b_i]
\end{cases}
\]

\[
\Rightarrow y_i \in \text{co}([a_i, b_i] \cup \{x_i\}).
\]

It follows that \( y \in \square\text{-hull}(F \cup \{x\}) \subset D \). This proves that \( x \) and \( y \) cannot be separated by an element of \( \mathcal{D}_d^F \), from which Property 3 follows.

Let \( S \subset \mathbb{R}^d \) be a \( \mathcal{D}_d^F \)-shattered set. By Property 3, \( \#P(S) = \#S \). Let us verify that \( P(S) \) is \( \mathcal{D}_d^0 \)-shattered. Indeed, let \( P(S') \) be an arbitrary subset of \( P(S) \). Since \( S \) is \( \mathcal{D}_d^F \)-shattered, there exists \( D \in \mathcal{D}_d^F \) with \( D \cap S = S' \). By Property 1, \( P(S) \in \mathcal{D}_d^0 \), and on the other hand, \( P(D) \cap P(S) = P(S') \). This shows that \( P(S) \) is \( \mathcal{D}_d^0 \)-shattered. We conclude that \( \dim(\mathcal{D}_d^0) \geq \dim(\mathcal{D}_d^F) \).

Suppose now that \( S \subset \mathbb{R}^d \) is \( \mathcal{D}_d^0 \)-shattered. For each \( s \in S \), choose any \( t_s \in P^{-1}(s) \), and define \( T = \{t_s : s \in S\} \). We claim that \( T \) is \( \mathcal{D}_d^F \)-shattered. Indeed, let \( T' = \{t_s : s \in S'\} \) be any subset of \( T \). Since \( S \) is \( \mathcal{D}_d^0 \)-shattered, there exists \( D \in \mathcal{D}_d^0 \) such that \( S \cap D = S' \). By Property 2, \( P^{-1}(S) \in \mathcal{D}_d^F \), and it clearly satisfies \( P^{-1}(S) \cap T = T' \). This shows that \( \dim(\mathcal{D}_d^F) \geq \dim(\mathcal{D}_d^0) \), and we are done.

\textbf{Proposition 2.} Let \( B \) be \( \mathcal{D}_d^F \) or \( C_d \). Suppose that there exists a \( B \)-shattered set \( A \subset \mathbb{R}^d \) with \( \#A = n \). Then, there exists a \( B \)-shattered set \( A' \subset \mathbb{R} \) with \( \#A' = n \) and such that, for each coordinate projection \( \pi_j : \mathbb{R}^d \to \mathbb{R} \), \( \#\pi_j(A') = n \).

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Proof. We shall prove the statement for $B = C_d$. The proof for $D^0_d$ can be easily adapted. Suppose that $A \subset \mathbb{R}^d$ is $C_d$-shattered and that $\# A = n$. Let $x$ be some point in $A$. Consider $C_1, \ldots , C_k \subset C_d$ that shatter $A$. For each $j = 1, \ldots , k$ such that $x \in C_j$, note that $d(A \setminus \{x\}, C_j) > 0$, since $C_j$ is closed. This implies that we can substitute $C_j$ by another cube $C'_j$ with same center but slightly bigger, so that $C'_j \cap A = C_j \cap A$, but now we guarantee that $x$ is an interior point of $C'_j$. Choose some open set $V_j$ with $x \in V_j \subset C'_j$. For the $j = 1, \ldots , k$, such that $x \notin C_j$, put $C_j' = C_j$ and choose an open neighborhood $V_j$ of $x$ such that $V_j \cap C'_j = \emptyset$.

Let $V = \cap_{j=1}^k V_j$. Note that $C'_1, \ldots , C'_k$ shatter $A$, but also shatter $(A \setminus \{x\}) \cup \{x\}$ for any choice of $x' \in V$. Since the set

$$\cup_{j=1}^d \pi_j^{-1}(\pi_j(A \setminus \{x\}))$$

is nowhere dense in $\mathbb{R}^d$, we can choose some $x' \in V$ such that, for each $j = 1, \ldots , d$, $\pi_j(x') \notin \pi_j(A \setminus \{x\})$.

Repeating this process recursively to each point of $A$ at a time, we obtain $A'$, which clearly satisfies the desired properties. \hfill \Box

Remark. The same works for for the set of non-degenerate rectangles in $\mathbb{R}^d$, or the set of closed balls with respect to any norm in $\mathbb{R}^d$. These cases will not be used in what follows, though.

We are in position to prove the main results.

Proof of Theorem A. Let us start by showing that $\text{VC} - \dim(D^0_d) \geq \lfloor 3d/2 \rfloor$. Suppose that $d$ is even. In this case, we can write $3d/2$ instead of $\lfloor 3d/2 \rfloor$. The proof in this case will follow by a two-step induction on $d$. Note that $D^0_2 \geq 3$, since for instance it is readily verified that $\{(-1,1), (1,-1), (2,1)\}$ is $D^0_2$-shattered. Suppose now that $\text{VC} - \dim(D^0_d) \geq 3d/2$. This means that there exists a $D^0_d$-shattered set $S \subset \mathbb{R}^d$ with $\# S = 3d/2$. We shall show that there is a $D^0_{d+2}$-shattered subset of $\mathbb{R}^{d+2}$ with $3(d+2)/2 = \# S + 3$ elements, implying that $\text{VC} - \dim(D^0_{d+2}) \geq 3(d+2)/2$. Indeed, let $X$ be a $D^0_d$-shattered subset of $\mathbb{R}^d$ with $\# X = 3$. We claim that the set

$$T = \{(s, 0) : s \in S\} \cup \{(0, x) : x \in X\} \subset \mathbb{R}^d \times \mathbb{R}^2 = \mathbb{R}^{d+2}$$

is $D^0_{d+2}$-shattered. In effect, let $T'$ be any subset of $T$. Then there are $S' \subset S$ and $X' \subset X$ with $T' = \{(s, 0) : s \in S'\} \cup \{(0, x) : x \in X'\}$. Since $S$ is $D^0_d$-shattered, there exists $I = \prod_{i=1}^d I_i \in D^0_d$ with $S \cap I = S'$. Since $X$ is $D^0_2$-shattered, there exists $J = \prod_{i=d+1}^{d+2} I_i \in D^0_d$ with $X \cap J = X'$. Then, $I \times J = \prod_{i=1}^{d+2} I_j \in D^0_{d+2}$ satisfies $T \cap I \times J = T'$. Since $\# T = 3(d+2)/2$, we are done for $d$ even.

Now suppose that $d$ is odd. It is a general observation that $\text{VC} - \dim(D^0_d) > \text{VC} - \dim(D^0_{d-1})$, since whenever $S \subset \mathbb{R}^{d-1}$ is shattered by $D^0_{d-1}$, then the set $\{(x, 0) : x \in S\} \cup \{(0, 1)\} \subset \mathbb{R}^d$ is clearly shattered by $D^0_d$. It follows that, also for odd $d$,

$$\text{VC} - \dim(D^0_d) \geq \text{VC} - \dim(D^0_{d-1}) + 1 = \frac{3(d-1)}{2} + 1 = \frac{3d}{2} - \frac{1}{2} = \lfloor \frac{3d}{2} \rfloor.$$
It remains to show that $\text{VC-dim}(D^n_d) \leq \lfloor 3d/2 \rfloor$, for each $d \geq 1$. This part of the proof departs from the main idea used in [11 Theorem 1]. Suppose that $S$ is a $D^n_d$-shattered subset of $\mathbb{R}^d$, with $\# S = d + n$. For each $i \in \{1, \ldots, d\}$, choose $l_i$ and $u_i$ with minimal, and respectively maximal, $i$th coordinate amongst $S$. Note that each $s \in S$ should appear on the list

$$l_1, u_1, l_2, u_2, \ldots, l_d, u_d.$$  

(2)

Indeed, it would be otherwise impossible to carve out $S \setminus \{s\}$ from $S$ with an element of $D^n_d$, since $s$ would be in the rectangular envelope of $S \setminus \{s\}$.

Let $k$ be the number of elements of $S$ which appear on the list (2) exactly once. Assume that $k \geq d + 1$. By the pigeonhole principle, there exists an $i$ such that both $l_i$ and $u_i$ appear on the list exactly once. Without loss of generality, assume that $i = d$. Let $[a_1, b_1] \times \cdots \times [a_d, b_d]$ the the rectangular envelope of $S \setminus \{l_d, u_d\}$. Since $l_d$ and $u_d$ appear on (2) only once, we have that

$$a_i \leq \pi_i(l_d) \leq b_i, \quad \text{and} \quad a_i \leq \pi_i(u_d) \leq b_i, \quad \text{for each} \quad i \in \{1, \ldots, d - 1\}. \quad (3)$$

Suppose that a product of intervals $I = \prod_{i=1}^d I_i$ carves out $S \setminus \{l_d, u_d\}$ from $S$. $I$ must contain $[a_1, b_1] \times \cdots \times [a_d, b_d]$. It follows from (3) and the fact that $l_d$ has minimal $d$th coordinate among $S$, that $l_d$ is bounded from below by $\pi_d(l_d)$. Analogously, $u_d$ is bounded from above by $\pi_d(u_d)$. $I_d$ is therefore a bounded interval, and it follows that $I$ is not an element from $D^n_d$. This shows in particular that $S \setminus \{l_d, u_d\}$ cannot be carved out from $S$ by an element from $D^n_d$, a contradiction. Then, $k \leq d$.

Once the $k$ points of $S$ that appear only once in (2), $2d - k$ slots remain to be filled with the $d + n - k$ points of $S$, which appear on (2) at least twice. It follows that $2(d + n - k) \leq 2d - k$, thus $2n \leq k$. We conclude that

$$\# S = d + n \leq d + \frac{k}{2} \leq \frac{3d}{2}.$$  

\[\square\]

\textbf{Proof of Theorem B.} Let $A \subset \mathbb{R}^d$ be a $C_d$-shattered set with $\# A = \text{VC-dim}(C_d)$. By Proposition 1 we can assume that $\# \pi_j(A) = \# A$, for each $j = 1, \ldots, d$. Let $I_1 \times \cdots \times I_d$ be the rectangular envelope of $A$. Without loss of generality, assume that $|I_d| = \max\{|I_1|, \ldots, |I_d|\}$, where $|I_j|$ denotes the length of the interval $I_j$.

Write $I_d = [a_d, b_d]$, and let $x, y \in A$ be such that $\pi_d(x) = a_d$ and $\pi_d(y) = b_d$. Let $A'$ be some subset of $A \setminus \{x, y\}$. Since $A$ is $C_d$-shattered, there must be some $C_{A'} \in C_d$ with $C_{A'} \cap A = A' \cup \{x, y\}$. Consider $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ the projection onto the first $d - 1$ coordinates, and note that $\pi(C_{A'})$ is a $(d - 1)$-dimensional cube such that $\pi(C_{A'}) \cap \pi(A) = \pi(A') \cup \{x, y\}$. Since $C_{A'}$ contains $x$ and $y$, the side of the cube $\pi(C_{A'})$ is greater than the diameter of $\pi(A)$ in $\mathbb{R}^{d-1}$. It follows that there exists $D_{A'} \in D^{(x,y)}_{d-1}$ such that $D_{A'} \cap \pi(A) = \pi(A' \cup \{x, y\})$. It follows that $\{D_{A'} : A' \subset A \setminus \{x, y\}\} \subset D^{(x,y)}_{d-1}$ shatters the set $\pi(A \setminus \{x, y\})$ in $\mathbb{R}^{d-1}$. Since $\# \pi(A \setminus \{x, y\}) = \# A - 2 = \text{VC-dim}(C_d) - 2$, it follows that
VC − dim(D^{(x,y)}_{d-1}) ≥ VC − dim(C_d) − 2. From Proposition 1 it follows that VC − dim(D^{(x,y)}_{d-1}) ≥ VC − dim(C_d) − 2.

Now let A ⊂ ℝ^{d-1} be finite, D^0_{d-1}-shattered set, and let L > 0 be such that A is contained in the closed ℓ^d_∞ ball centered at 0 and with radius L/2. Consider in ℝ^d the points x = (0, ..., 0, L) and y = (0, ..., 0, −L), and let A' = \{a, 0 : a ∈ A\} ∪ \{x, y\}. To show that VC − dim(C_d) ≥ VC − dim(D^0_{d-1}) + 2, it suffices to verify that A' is C_d-shattered. In effect, let B ⊂ A'. Since π(B) \ {0} ⊂ A, π(B) \ {0} is carved out by some D ∈ D^0_{d-1}. Note that, for each M ≥ L, π(B) \ {0} can also be carved out by some C ∈ C_d which contains 0 and has ∥ · ∥_∞-diameter M. Let us define an element C' ∈ C_d depending on B ∩ \{x, y\}, as follows:

1. if B ∩ \{x, y\} = \{x, y\}, consider some C ∈ C_{d-1} which contains 0 and has ∥ · ∥_∞-diameter 2L. Define C' = C × [−L, L];
2. if B ∩ \{x, y\} = \{x\}, consider some C ∈ C_{d-1} which contains 0 and has ∥ · ∥_∞-diameter L. Define C' = C × [0, L];
3. analogously, if B ∩ \{x, y\} = \{y\}, consider some C ∈ C_{d-1} which contains 0 and has ∥ · ∥_∞-diameter L. Define C' = C × [−L, 0];
4. if B ∩ \{x, y\} = \emptyset, consider some C ∈ C_{d-1} which contains 0 and has ∥ · ∥_∞-diameter L. Define C' = C × [−L/2, L/2];

In each case, C' carves B out of A', which concludes the proof.

3 Final remarks

The natural follow up to this work would be to determine the exact Vapnik-Chervonenkis dimension of the set D_d of all degenerate balls in ℓ^d_∞. Note that one can easily obtain the comparison

\[\left\lfloor \frac{3d}{2} \right\rfloor = VC − dim(D^0_d) \leq VC − dim(D_d) \leq VC − dim(C_d) = \left\lfloor \frac{3(d+1)}{2} \right\rfloor.\]

Indeed, the first inequality is clear since D^0_d ⊂ D_d. The second inequality follows from the following fact: if we can carve out a subset S' from a set S with a degenerate ball, then we can carve out S' from S with an appropriate ball with large enough radius. Computing \(\lfloor 3d/2 \rfloor\) and \(\lfloor (3d + 1)/2 \rfloor\) in the cases where \(d\) is odd and even separately gives us the following result, which is a direct consequence of Theorem A and the main theorem.

**Proposition 3.** Let \(d \geq 1\). If \(d\) is odd, \(VC − dim(D_d) = (3d + 1)/2\). If \(d\) is even, \(\frac{3d}{2} \leq VC − dim(D_d) \leq \frac{3d}{2} + 1\).
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