Abstract

Recently there are a considerable amount of work devoted to the study of the algorithmic stability and generalization for stochastic gradient descent (SGD). However, the existing stability analysis requires to impose restrictive assumptions on the boundedness of gradients, strong smoothness and convexity of loss functions. In this paper, we provide a fine-grained analysis of stability and generalization for SGD by substantially relaxing these assumptions. Firstly, we establish stability and generalization for SGD by removing the existing bounded gradient assumptions. The key idea is the introduction of a new stability measure called on-average model stability, for which we develop novel bounds controlled by the risks of SGD iterates. This yields generalization bounds depending on the behavior of the best model, and leads to the first-ever-known fast bounds in the low-noise setting using stability approach. Secondly, the smoothness assumption is relaxed by considering loss functions with Hölder continuous gradients for which we show that optimal bounds are still achieved by balancing computation and stability. Finally, we study learning problems with (strongly) convex objectives but non-convex loss functions.

1. Introduction

Stochastic gradient descent (SGD) has become the workhorse behind many machine learning problems. As an iterative algorithm, SGD updates the model sequentially upon receiving a new datum with a cheap per-iteration cost, making it amenable for big data analysis. There is a plethora of theoretical work on its convergence analysis as an optimization algorithm (e.g. Duchi et al., 2011; Lacoste-Julien et al., 2012; Nemirovski et al., 2009; Rakhlin et al., 2012; Shamir & Zhang, 2013; Zhang, 2004).

Concurrently, there are a considerable amount of work with focus on its generalization analysis (Dieuleveut & Bach, 2016; Hardt et al., 2016; Lin et al., 2016; Rosasco & Villa, 2015; Ying & Zhou, 2016). For instance, using the tool of integral operator the work (Dieuleveut & Bach, 2016; Lin & Rosasco, 2017; Rosasco & Villa, 2015; Ying & Pontil, 2008) studied the excess generalization error of SGD with the least squares loss, i.e. the difference between the true risk of SGD iterates and the best possible risk. An advantage of this approach is its ability to capture the regularity of regression functions and the capacity of hypothesis spaces. The results were further extended in Lei & Tang (2018); Lin et al. (2016) based on tools of empirical processes which are able to deal with general convex functions even without a smoothness assumption. The idea is to bound the complexity of SGD iterates in a controllable manner, and apply concentration inequalities in empirical processes to control the uniform deviation between population risks and empirical risks over a ball to which the SGD iterates belong.

Recently, in the seminal work (Hardt et al., 2016) the authors studied the generalization bounds of SGD via algorithmic stability (Bousquet & Elisseeff, 2002; Elisseeff et al., 2005) for convex, strongly convex and non-convex problems. This motivates several appealing work on some weaker stability measures of SGD that still suffice for guaranteeing generalization (Charles & Papailiopoulos, 2018; Kuzborskij & Lampert, 2018; Zhou et al., 2018). An advantage of this stability approach is that it considers only the particular model produced by the algorithm, and can imply generalization bounds independent of the dimensionality.

However, the existing stability analysis of SGD is established under the strong assumptions on the loss function such as the boundedness of the gradient and strong smoothness. Such assumptions are very restrictive which are not satisfied in many standard contexts, e.g. the simple case of least-squares regression, where the model parameter belongs to an unbounded domain. Furthermore, the analysis in the strongly convex case requires strong convexity of each
An advantage of on-average model stability is that the cor-
work in Section 2 and formulate the problem in Section
The paper is structured as follows. We discuss the related
loss function which is not true for many problems such as
the important problem of least squares regression.
In this paper, we provide a fine-grained analysis of sta-

2. Related Work
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1. The stability and generalization for learning without

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- Secondly, we consider loss functions with their gradients
satisfying the Hölder continuity which is a much weaker
condition than the strong smoothness in the literature. Al-
though stability decreases by weakening the smoothness
assumption, optimal generalization bounds can be surpris-

- Thirdly, we study learning problems with (strongly) con-
vex objectives but non-convex individual loss functions. The
nonconvexity of loss functions makes the correspond-

The paper is structured as follows. We discuss the related
work in Section 2 and formulate the problem in Section
3. The stability and generalization for learning without
a bounded gradient assumption is presented in Section 4.
In Sections 5 and 6, we consider problems with relaxed
convexity and relaxed strong convexity, respectively. We
conclude the paper in Section 7.
Generalization Analysis of SGD. A framework to study the generalization performance of large-scale stochastic optimization algorithms was established in Bousquet & Bottou (2008), where three factors influencing generalization behavior were identified as optimization errors, estimation errors and approximation errors. Uniform stability was used to establish generalization bounds $O(1/\sqrt{n})$ in expectation for SGD for convex and strongly smooth cases (Hardt et al., 2016). For convex and nonsmooth learning problems, generalization bounds $O(n^{-\frac{3}{4}})$ were established based on the uniform convergence principle (Lin et al., 2016). An interesting observation is that an implicit regularization can be achieved without an explicit regularizer by tuning either the number of passes or the step sizes (Lin et al., 2016; Rosasco & Villa, 2015). For the specific least squares loss, optimal $Tang, 2018$ or an algorithmic stability approach (Feldman et al., 2016). The above mentioned generalization results are in the form of expectation. High-probability bounds were established based on either a uniform-convergence approach (Lei & Tang, 2018) or an algorithmic stability approach (Feldman & Vondrak, 2019). A novel combination of PAC-Bayes and algorithmic stability was used to study the generalization behavior of SGD, a promising property of which is its applications to all posterior distributions of algorithms’ random hyperparameters (London, 2017).

3. Problem Formulation

Let $S = \{z_1, \ldots, z_n\}$ be a set of training examples independently drawn from a probability measure $\rho$ defined over a sample space $Z = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subseteq \mathbb{R}^d$ is an input space and $\mathcal{Y} \subseteq \mathbb{R}$ is an output space. Our aim is to learn a prediction function parameterized by $w \in \Omega \subseteq \mathbb{R}^d$ to approximate the relationship between an input variable $x$ and an output variable $y$. We quantify the loss of a model $w$ on an example $z = (x, y)$ by $F(w; z)$. The corresponding empirical and population risks are respectively given by

$$F_S(w) = \frac{1}{n} \sum_{i=1}^{n} f(w; z_i) \quad \text{and} \quad F(w) = \mathbb{E}_{z}[f(w; z)].$$

Here we use $\mathbb{E}_{z}[\cdot]$ to denote the expectation with respect to (w.r.t.) $z$. In this paper, we consider stochastic learning algorithms $A$, and denote by $A(S)$ the model produced by running $A$ over the training examples $S$.

We are interested in studying the excess generalization error $F(A(S)) - F(w^*)$, where $w^* \in \arg\min_{w \in \Omega} F(w)$ is the one with the best prediction performance over $\Omega$. It can be decomposed as

$$\mathbb{E}_{S,A}[F(A(S)) - F(w^*)] = \mathbb{E}_{S,A}[F(A(S)) - F_S(A(S))] + \mathbb{E}_{S,A}[F_S(A(S)) - F_S(w^*)]. \quad (3.1)$$

The first term is called the estimation error due to the approximation of the unknown probability measure $\rho$ based on sampling. The second term is called the optimization error induced by running an optimization algorithm to minimize the empirical objective, which can be addressed by tools in optimization theory. A popular approach to control estimation errors is to consider the stability of the algorithm, for which a widely used stability measure is the uniform stability (Elisseeff et al., 2005; Hardt et al., 2016).

**Definition 1** (Uniform Stability). A stochastic algorithm $A$ is $\epsilon$-uniformly stable if for all training examples $S, \tilde{S} \in \mathbb{Z}^n$ that differ by at most one example, we have

$$\sup_z \mathbb{E}_A[f(A(S); z) - f(A(\tilde{S}); z)] \leq \epsilon. \quad (3.2)$$

The celebrated relationship between generalization and uniform stability is established in the following lemma (Hardt et al., 2016; Shalev-Shwartz et al., 2010).

**Lemma 1** (Generalization via uniform stability). Let $A$ be $\epsilon$-uniformly stable. Then

$$|\mathbb{E}_{S,A}[F_S(A(S)) - F(A(S))]| \leq \epsilon.$$

Throughout the paper, we restrict our interest to a specific algorithm called projected stochastic gradient descent.

**Definition 2** (Projected Stochastic Gradient Descent). Let $\Omega \subseteq \mathbb{R}^d$ and $\Pi_{\Omega}$ denote the projection on $\Omega$. Let $w_1 = 0 \in \mathbb{R}^d$ be an initial point and $\{\eta_t\}_{t}$ be a sequence of positive step sizes. Projected SGD updates models by

$$w_{t+1} = \Pi_{\Omega}(w_t - \eta_t \nabla f(w_t; z_i)), \quad (3.3)$$

where $\nabla f(w_t, z_i)$ denotes the gradient of $f$ w.r.t. the first argument and $z_i$ is independently drawn from the uniform distribution over $\{1, \ldots, n\}$.

We now introduce some necessary concepts. We say a differentiable function $g : \mathbb{R}^d \to \mathbb{R}$ is $\sigma$-strongly convex if

$$g(w) \geq g(\bar{w}) + \langle w - \bar{w}, \nabla g(\bar{w}) \rangle + \frac{\sigma}{2} \|w - \bar{w}\|^2 \quad (3.4)$$

for all $w, \bar{w} \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ denotes the inner product and $\|w\|_2$ denotes the $\ell_2$ norm of $w = (w_1, \ldots, w_d)$, i.e., $\|w\|_2 = (\sum_{j=1}^{d} w_j^2)^{\frac{1}{2}}$. If (3.4) holds with $\sigma = 0$, then we say $g$ is convex. We denote $B \preceq \bar{B}$ if there are absolute constants $c_1$ and $c_2$ such that $c_1 B \leq \bar{B} \leq c_2 B$.

4. Stability without Bounded Gradients

An essential assumption to establish the uniform stability of SGD is the uniform Lipschitz continuity (boundedness of gradients) of loss functions as follows (Bousquet & Elisseeff, 2002; Charles & Papailiopoulos, 2018; Hardt et al., 2016; Kuzborskij & Lampert, 2018; Zhou et al., 2018).
Assumption 1. We assume $\|\nabla f(\mathbf{w}; z)\|_2 \leq G$ for all $\mathbf{w} \in \Omega$ and $z \in \mathcal{Z}$.

Unfortunately, the Lipschitz constant $G$ can be very large or even infinite for some learning problems. Consider the simple least squares loss $f(\mathbf{w}; z) = \frac{1}{2}((\mathbf{w}, x) - y)^2$ with the gradient $\nabla f(\mathbf{w}; z) = ((\mathbf{w}, x) - y)x$. In this case the $G$-Lipschitzness of $f$ requires to set $G = \sup_{\mathbf{w} \in \Omega} \sup_{x \in \mathcal{X}} \|((\mathbf{w}, x) - y)x\|_2$, which is infinite if $\Omega$ is unbounded. As another example, the Lipschitz constant of deep neural networks can be prohibitively large. In this case, existing stability bounds fail to yield meaningful generalization bounds. Furthermore, another critical assumption in the literature is the $L$-smoothness on $f$, i.e. for any $z$ and $\mathbf{w}, \tilde{\mathbf{w}} \in \mathbb{R}^d$

$$\|\nabla f(\mathbf{w}, z) - \nabla f(\tilde{\mathbf{w}}, z)\|_2 \leq L \|\mathbf{w} - \tilde{\mathbf{w}}\|_2.$$  \tag{4.1}$$

In this section, we will remove the boundedness assumption on the gradients, and establish stability and generalization only under the assumption where loss functions have H"older continuous gradients—a condition much weaker than the strong smoothness (Lei et al., 2018; Nesterov, 2015; Ying & Zhou, 2017).

Definition 3. Let $L > 0$, $\alpha \in (0, 1]$. We say $f$ is $(\alpha, L)$-H"older continuous if for all $\mathbf{w}, \tilde{\mathbf{w}} \in \mathbb{R}^d$ and $z \in \mathcal{Z}$,

$$\|\nabla f(\mathbf{w}, z) - \nabla f(\tilde{\mathbf{w}}, z)\|_2 \leq L \|\mathbf{w} - \tilde{\mathbf{w}}\|_2^\alpha.$$  \tag{4.2}$$

If (4.2) holds with $\alpha = 1$, then $f$ is strongly smooth as defined by (4.1). Examples of loss functions satisfying Definition 3 include the $q$-norm hinge loss $f(\mathbf{w}; z) = (\max(0, 1 - y(\mathbf{w}, x)))^q$ for classification and the $q$-th power absolute distance loss $f(\mathbf{w}; z) = |y - (\mathbf{w}, x)|^q$ for regression (Steinwart & Christmann, 2008), whose gradients are $(q - 1, C)$-H"older continuous for some $C > 0$ if $q \in (1, 2)$.

4.1. On-average model stability

The key to remove the bounded gradient assumption is the introduction of a novel stability measure which we refer to as the on-average model stability. We use the term “on-average model stability” to differentiate it from on-average stability in Kearns & Ron (1999); Shalev-Shwartz et al. (2010) as we measure stability on model parameters $\mathbf{w}$ instead of function values. Intuitively, on-average model stability measures the on-average sensitivity of models by traversing the perturbation of each single coordinate.

Definition 4 (On-average Model Stability). Let $S = \{z_1, \ldots, z_n\}$ and $\tilde{S} = \{\tilde{z}_1, \ldots, \tilde{z}_n\}$ be drawn independently from $\rho$. For any $i = 1, \ldots, n$, define $S^{(i)} = \{z_1, \ldots, z_{i-1}, \tilde{z}_i, z_{i+1}, \ldots, z_n\}$ as the set formed from $S$ by replacing the $i$-th element with $\tilde{z}_i$. We say a randomized algorithm $A$ is $\ell_1$ on-average model $\epsilon$-stable if

$$\mathbb{E}_{S, \tilde{S}, A}[\frac{1}{n} \sum_{i=1}^n \|A(S) - A(S^{(i)})\|_2] \leq \epsilon,$$

and $\ell_2$ on-average model $\epsilon$-stable if

$$\mathbb{E}_{S, \tilde{S}, A}[\frac{1}{n} \sum_{i=1}^n \|A(S) - A(S^{(i)})\|_2^2] \leq \epsilon^2.$$ 

In the following theorem, we build the connection between generalization in expectation and the on-average model stabilities to be proved in Appendix B. Although the generalization by $\ell_1$ on-average model stability requires Assumption 1, it is removed for $\ell_2$ on-average model stability. We introduce a free parameter $\gamma$ to tune according to the property of problems. Note we require a convexity assumption in Part (c) by considering non-strongly smooth loss functions, and we have $F(A(S))$ on the right-hand side instead of $F_S(A(S))$ as in Part (b). Let $c_{\alpha, 1} = (1 + 1/\alpha)^{1/\alpha} L^{1/\alpha}$.

Theorem 2 (Generalization via Model Stability). Let $S, \tilde{S}$ and $S^{(i)}$ be constructed as Definition 4. Let $\gamma > 0$.

(a) Let $A$ be $\ell_1$ on-average model $\epsilon$-stable and Assumption 1 hold. Then

$$\mathbb{E}_{S, \tilde{S}, A}[F_S(A(S)) - F(A(S))] \leq c_{\alpha, 1} \epsilon \mathbb{E}_{S, \tilde{S}, A}[F(2^{1/\alpha} (A(S)))]$$

(b) If for any $z$, the function $\mathbf{w} \mapsto f(\mathbf{w}; z)$ is nonnegative and $L$-smooth, then

$$\mathbb{E}_{S, \tilde{S}, A}[F(A(S)) - F_S(A(S))] \leq \frac{L}{\gamma} \mathbb{E}_{S, \tilde{S}, A}[F_S(A(S))]$$

$$+ \frac{L + \gamma}{2n} \sum_{i=1}^n \mathbb{E}_{S, \tilde{S}, A}[\|A(S^{(i)}) - A(S)\|_2^2].$$

(c) If for any $z$, the function $\mathbf{w} \mapsto f(\mathbf{w}; z)$ is nonnegative, convex and $\mathbf{w} \mapsto \nabla f(\mathbf{w}; z)$ is $(\alpha, L)$-H"older continuous with $\alpha \in (0, 1]$, then

$$\mathbb{E}_{S, \tilde{S}, A}[F(A(S)) - F_S(A(S))] \leq \frac{c_{\alpha, 1}}{2\gamma} \mathbb{E}_{S, \tilde{S}, A}[F(2^{1/\alpha} (A(S)))]$$

$$+ \frac{\gamma}{2n} \sum_{i=1}^n \mathbb{E}_{S, \tilde{S}, A}[\|A(S^{(i)}) - A(S)\|_2^2].$$

Remark 1. We explain here the benefit of $\ell_2$ on-average model stability. If $A$ is $\ell_2$ on-average model $\epsilon$-stable, then we take $\gamma = \sqrt{2L\mathbb{E}[F_S(A(S))]} / \epsilon$ in Part (b) and derive

$$\mathbb{E}[F(A(S)) - F_S(A(S))] \leq L\epsilon^2 / 2 + \sqrt{2L\mathbb{E}[F_S(A(S))] \epsilon}.$$ 

In particular, if the problem has a low noise in the sense $\mathbb{E}[F_S(A(S))] = O(1/n)$, we derive $\mathbb{E}[F(A(S)) -
we first consider its application to learning with smooth loss which fail to exploit the low-noise condition. Theorem 3 (Stability bounds) (Hardt et al., 2016) for the convex case and the inequality

\[ \text{E}_A \left[ \| w_{t+1} - w_{t+1}^{(i)} \|_2 \right] \leq \sqrt{\frac{2L}{n}} \sum_{j=1}^{t} \eta_j \left( \text{E}_A \left[ \sqrt{F(w_j); z_i} \right] \right) \]

from which and Jensen’s inequality one derives

\[ \frac{1}{n} \text{E}_A \left[ \sum_{i=1}^{n} \| w_{t+1} - w_{t+1}^{(i)} \|_2 \right] \leq \sqrt{\frac{2L}{n}} \sum_{j=1}^{t} \eta_j \left( \left( \frac{1}{n} \sum_{i=1}^{n} f(w_j^{(i)}; z_i) \right)^{\frac{1}{2}} \right). \]

Although the first term in the parentheses can be easily tackled by \( \text{E}_A \left[ \sqrt{F_S(w_j)} \right] \), the second term is difficult to address since \( w_j^{(i)} \) varies for different \( i \) and \( w_j^{(i)} \) depends on \( \tilde{z}_i \). We bypass this obstacle by using an error decomposition to establish a bound analogous to (4.5) but with the involved \( f(w_j^{(i)}; \tilde{z}_i) \) replaced by \( f(w_j; \tilde{z}_i) \), which leads to \( F_S(w_j) \). The \( \ell_2 \) on-average model stability bound (4.4) is proved with a similar idea. As we stated before, an advantage of considering \( \ell_2 \) on-average model stability is that we can fully remove the bounded gradient assumption in the corresponding generalization analysis.

Remark 3. Kuzborskij & Lampert (2018) developed an interesting on-average stability bound \( O \left( \frac{\eta}{2} \sum_{j=1}^{t} \eta_j \right) \) under the bounded variance assumption \( \text{E}_{w,z} \left[ \| \nabla f(w_t; z) - \nabla F(w_t; z) \|_2^2 \right] \leq \sigma^2 \) for all \( t \). Although this bound successfully replaces the uniform Lipschitz constant by the milder uniform variance constant \( \tilde{\sigma} \), the corresponding generalization analysis still requires a bounded gradient assumption. A nice property of the stability bound in (Kuzborskij & Lampert, 2018) is that it depends on the quality of the initialization, i.e., the stability increases if we start with a good model. Our stability bound also enjoys this property. As we can see from Theorem 3, the stability increases if we found good models along the optimization process.

Remark 4. The stability bounds in Theorem 3 can be extended to the non-convex case. Specifically, let assumptions of Theorem 3, except the convexity of \( w \mapsto f(w; z) \), hold. Then for any \( p > 0 \) one gets (see Proposition C.3)

\[ \text{E}_A \left[ \sum_{i=1}^{n} \| w_{t+1} - w_{t+1}^{(i)} \|_2 \right] \leq \frac{L}{n} \sum_{j=1}^{t} \eta_j \left( \text{E}_A \left[ F_S(w_j) + F_S(w_j) \right] \right), \]

\[ (1 + p/n)(1 + \eta_t L) \text{E}_A \left[ \sum_{i=1}^{n} \| w_t - w_t^{(i)} \|_2 \right] + \frac{4(1 + p^{-1})L \eta^2}{n} \text{E}_A \left[ F_S(w_t) + F_S(w_t) \right]. \]
Theorem 4 (Generalization bounds). Assume for all \( z \in Z \), the function \( w \mapsto f(w;z) \) is nonnegative, convex and L-smooth. Let \( \{w_i\} \) be produced by (3.3) with nonincreasing step sizes satisfying \( \eta_t \leq 1/(2L) \). Let \( \gamma \geq 1 \). If \( n \geq 4(1 + T/n)(L + \gamma)L\eta_t \sum_{t=1}^T \eta_t^2 \), then

\[
\mathbb{E}_{S,A}[F(w_{T}^{(1)})] - F(w^*) = O\left(\left(\frac{1}{\gamma} + \frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t}\right)F(w^*) + \frac{1}{\sum_{t=1}^T \eta_t} + \frac{\lambda(1 + T/n)}{n}(1 + \frac{T}{t=1} \eta_t^2 F(w^*))\right),
\]

where \( w_{T}^{(1)} = \left(\sum_{t=1}^T \eta_t w_t\right) / \sum_{t=1}^T \eta_t \).

Corollary 5. Assume for all \( z \in Z \), the function \( w \mapsto f(w;z) \) is nonnegative, convex and L-smooth.

(a) Let \( \{w_i\} \) be produced by (3.3) with \( \eta_t = c/\sqrt{T} \leq 1/(2L) \) for a constant \( c > 0 \). If \( n \geq 4Le^2(1 + T/n)(L + \sqrt{n}) \) and \( T \asymp n \), then

\[
\mathbb{E}_{S,A}[F(w_{T}^{(1)})] - F(w^*) = O\left(\frac{F(w^*) + 1}{\sqrt{n}}\right). \tag{4.6}
\]

(b) Let \( \{w_i\} \) be produced by (3.3) with \( \eta_t = \eta_1 \leq 1/(2L) \). If \( F(w^*) = 0 \), \( n \geq 4Le(T + T^2/n)(L + 1)\eta_1^2 \) and \( T \asymp n \), then

\[
\mathbb{E}_{S,A}[F(w_{T}^{(1)})] - F(w^*) = O(1/n). \tag{4.7}
\]

Remark 5. If \( T \asymp n \), then \( (1 + T/n)(L + \sqrt{n}) = O(\sqrt{n}) \). Therefore, the assumption on \( n \) in Part (a) reduces to \( \sqrt{n} \leq \tilde{c} \) for a constant \( \tilde{c} \), which is very mild. If \( T \asymp n \), then \( T + T^2/n = O(n) \). Therefore, our assumption on \( n \) in Part (b) reduces to \( \eta_1^2 = O(\tilde{c}) \) for a constant \( \tilde{c} \).

Remark 6. Based on the stability bound in Hardt et al. (2016), we can show \( \mathbb{E}_{S,A}[F(w_{T}^{(1)})] - F(w^*) \) decays as

\[
2G^2 \sum_{t=1}^T \frac{\eta_t}{n} + O\left(\frac{\sum_{t=1}^T \eta_t^2 F(w^*) + 1}{\sum_{t=1}^T \eta_t}\right), \tag{4.8}
\]

from which one can derive the \( O(1/\sqrt{n}) \) bound at best even if \( F(w^*) = 0 \). The improvement of our bounds over (4.8) is due to the consideration of on-average model stability bounds involving empirical and population risks (we use the same optimization error bounds in these two approaches). Based on the on-average stability bound in Kuzborskij & Lampert (2018), one can derive a generalization bound similar to (4.8) with \( C^2 \) replaced by \( G\sigma \) (\( \sigma \) is the uniform variance constant in Remark 3), which also could not yield a fast bound \( O(1/n) \) if \( F(w^*) = 0 \).

Remark 7. We compare our results with some fast bounds for SGD. Some fast convergence rates of SGD were recently derived for SGD under low noise conditions (Bassily et al., 2018; Ma et al., 2018; Srebro et al., 2010; Zhang & Zhou, 2019) or growth conditions relating stochastic gradients to full gradients (Vaswani et al., 2019). The discussions there mainly focused on optimization errors, which are measured w.r.t. the iteration number \( t \). As a comparison, our fast rates measured by \( n \) are developed for generalization errors of SGD (Part (b) of Corollary 5), for which we need to trade-off optimization errors and estimation errors by stopping at an appropriate iteration number. Fast generalization bounds are also established for the specific least squares based on an integral operator approach (Dieuleveut et al., 2017; Lin & Rosasco, 2018; Mücke et al., 2018; Pillau-Vivien et al., 2018). However, these discussions heavily depend on the structure of the square loss and require capacity assumptions in terms of the decay rate of eigenvalues for the integral operator. As a comparison, we consider general loss functions and do not impose a capacity assumption.

4.3. Non-strongly smooth case

As a further application, we apply our on-average model stability to learning with non-strongly smooth loss functions, which have not been studied in the literature.

Stability bounds. The following theorem to be proved in Appendix D.1 establishes stability bounds. As compared to (4.4), the stability bound below involves an additional term \( O(\sum_{j=1}^t \eta_j^{-2\alpha}) \), which is the cost we pay by relaxing the smoothness condition to a Hölder continuity of gradients.

Theorem 6 (Stability bounds). Assume for all \( z \in Z \), the map \( w \mapsto f(w;z) \) is nonnegative, convex and \( \nabla f(w;z) \) is \((\alpha, L)\)-Hölder continuous with \( \alpha \in (0, 1) \). Let \( S, \tilde{S} \) and \( S^{(i)} \) be constructed in Definition 4. Let \( \{w_i\} \) and \( \{w_t^{(i)}\} \) be...
produced by (3.3) based on $S$ and $S^{(1)}$, respectively. Then

$$
E_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| w_{t+1} - w_{t+1}^{(1)} \|^2_2 \right] = O \left( \sum_{j=1}^{t} \eta_j^2 \right) + 
O \left( n^{-1} + \frac{t}{n} \sum_{j=1}^{t} \eta_j^2 E_A \left[ F_{\hat{S}}^{(2)\infty} (w_j) + F_{\hat{S}}^{(2)\infty} (w_j) \right] \right).
$$

**Generalization bounds.** We now present generalization bounds for learning by loss functions with Hölder continuous gradients, which are specific instantiations of a general result (Proposition D.5) stated and proved in Appendix D.2.

**Theorem 7** (Generalization bounds). Assume for all $z \in Z$, the function $w \mapsto f(w; z)$ is nonnegative, convex, and $\nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1)$. Let $\{w_t\}_t$ be given by (3.3) with $\eta_t = cT^{-\theta}, \theta \in [0, 1], c > 0$.

(a) If $\alpha \geq 1/2$, we can take $\theta = 1/2$ and $T \asymp n$ to derive $E_{S,A}[F(w^{(1)}_T)] - F(w^*) = O(n^{-\frac{1}{2}}).
$

(b) If $\alpha < 1/2$, we can take $T \asymp n^{\frac{2\alpha}{1-2\alpha}}$ and $\theta = \frac{3-3\alpha}{2(2-\alpha)}$ to derive $E_{S,A}[F(w^{(1)}_T)] - F(w^*) = O(n^{-\frac{1}{2}}).
$

(c) If $F(w^*) = 0$, we take $T \asymp n^{\frac{1}{2\alpha}}$ and $\theta = \frac{3-3\alpha - 2\alpha}{4}$ to derive $E_{S,A}[F(w^{(1)}_T)] - F(w^*) = O(n^{-\frac{1}{2} + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots}).$

**Remark 8.** Although relaxing smoothness affects stability by introducing $O(\sum_{j=1}^{T} \eta_j^2)$ in the stability bound, we achieve a generalization bound similar to the smooth case with a similar computation cost if $\alpha \geq 1/2$. For $\alpha < 1/2$, a minimax optimal generalization bound $O(n^{-\frac{1}{2}})$ (Agarwal et al., 2012) can be also achieved with more computation cost as $T \asymp n^{\frac{2\alpha}{1-2\alpha}}$. Analogous to the smooth case, we can derive generalization bounds better than $O(n^{-\frac{1}{2}})$ in the case with low noises. To our best knowledge, this is the first optimistic generalization bound for learning with non-strongly smooth loss functions, which has not been studied yet even in the ERM setting ignoring optimization errors.

**Remark 9.** We can extend our discussion to ERM. If $F_S$ is $\sigma$-strongly convex and $\nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous, we can apply the on-average model stability to show (see Proposition D.7)

$$
E_S \left[ F(A(S)) - F_S(A(S)) \right] = O \left( \frac{E_S \left[ F^{(2)\infty} (A(S)) / (n\sigma) \right]}{/ (n\sigma)} \right),
$$

where $A(S) = \arg \min_{w \in B_d} E_S(w)$. This extends the error bounds developed for ERM with strongly-smooth loss functions (Shalev-Shwartz & Ben-David, 2014, page 143) to the non-strongly smooth case, and removes the $G$-admissibility assumption in Bousquet & Elisseeff (2002). In a low-noise case with a small $E_S[F(A(S))]$, the discussion based on an-on-average stability can imply optimistic generalization bounds for ERM.

### 5. Stability with Relaxed Convexity

We now turn to stability and generalization of SGD for learning problems where the empirical objective $F_S$ is convex but each loss function $f(w; z)$ may be non-convex. Since in an application we may encounter negative $f$, we impose Assumption 1 here and use the arguments based on the uniform stability. The proofs of Theorem 8 and Theorem 9 are given in Appendix E.1.

**Theorem 8.** Let Assumption 1 hold. Assume for all $z \in Z$, the function $w \mapsto f(w; z)$ is $L$-smooth. Let $S = \{z_1, \ldots, z_n\}$ and $\tilde{S} = \{\tilde{z}_1, \ldots, \tilde{z}_n\}$ be two sets of training examples that differ by a single example. Let $\{w_t\}_t$ and $\{\tilde{w}_t\}_t$ be produced by (3.3) based on $S$ and $\tilde{S}$, respectively. If for all $s, \tilde{s}$, $F_s$ is convex, then

$$
\left( E_A \left[ \| w_{t+1} - \tilde{w}_{t+1} \|_2^2 \right] \right)^{\frac{1}{2}} \leq 4G_C \left( \sum_{j=1}^{t} \frac{\eta_j}{n} + 2G \left( C_1 \sum_{j=1}^{t} \frac{\eta_j^2}{n} \right) \right)^{\frac{1}{2}},
$$

where we introduce $C_t = \prod_{j=1}^{t} \left( 1 + L^2 \eta_j^2 \right)$.

**Remark 10.** The derivation of uniform stability bounds in Hardt et al. (2016) is based on the non-expansiveness of the operator $w \mapsto w - \nabla f(w; z)$, which requires the convexity of $w \mapsto f(w; z)$ for all $z$. Theorem 8 relaxes this convexity condition to a milder convexity condition on $F_s$. If $\sum_{j=1}^{\infty} \eta_j^2 < \infty$, the stability bounds in Theorem 8 become $O(n^{-\frac{1}{2}} \sum_{j=1}^{t} \eta_j + n^{-\frac{1}{2}})$ since $C_t < \infty$.

By the proof, Theorem 8 holds if the convexity condition is replaced by $E_A \left[ (w_t - \tilde{w}_t, \nabla F_S(w_t) - \nabla F_S(\tilde{w}_t)) \right] \geq -C\eta_t E_A \left[ \| w_t - \tilde{w}_t \|_2^2 \right]$, for some $C$ and all $t \in N$.

As shown above, minimax optimal generalization bounds can be achieved for step sizes $\eta_t = \eta_1 t^{-\theta}$ for all $\theta \in (1/2, 1)$ as well as the step sizes $\eta_t \asymp 1/\sqrt{T}$ with $T \asymp n$.

**Theorem 9.** Let Assumption 1 hold. Assume for all $z \in Z$, the function $w \mapsto f(w; z)$ is $L$-smooth. Let $\{w_t\}_t$ be produced by (3.3). Suppose for all $S, F_S$ is convex.

(a) If $\eta_t = \eta_1 t^{-\theta}$, $\theta \in (1/2, 1)$, then

$$
E_{S,A}[F(w^{(1)}_T)] - F(w^*) = O \left( n^{-1} T^{1-\theta} + n^{-\frac{1}{2}} + T^{\theta-1} \right).
$$

(b) If $\eta_t = c/\sqrt{T}$ for some $c > 0$ and $T \asymp n$, then

$$
E_{S,A}[F(w^{(1)}_T)] - F(w^*) = O(n^{-\frac{1}{2}}).
$$

**Example: AUC Maximization.** We now consider a specific example of AUC (Area under ROC curve) maximization where the objective function is convex but each loss function may be non-convex. As a widely used method in
imbalanced classification ($Y = \{+1, -1\}$), AUC maximization was often formulated as a pairwise learning problem where the corresponding loss function involves a pair of training examples (Gao et al., 2013; Zhao et al., 2011). Recently, AUC maximization algorithms updating models with a single example per iteration are developed (Liu et al., 2018; Natole et al., 2018; Ying et al., 2016). Specifically, AUC maximization with the least squares loss can be formulated as the minimization of the following objective function

$$
\min_{w \in \Omega} F(w) := p(1-p)\mathbb{E}\left[(1-w^T(x-\hat{x}))^2 \mid y = 1, \tilde{y} = -1\right],
$$

(5.1)

where $p = \Pr\{Y = 1\}$ is the probability of an example being positive. Let $x_+ = \mathbb{E}[X \mid Y = 1]$ and $x_- = \mathbb{E}[X \mid Y = -1]$ be the conditional expectation of $X$ given $Y = 1$ and $Y = -1$, respectively. It was shown that $\mathbb{E}_{i_t}[f(w; z_i)] = F(w)$ for all $w \in \mathbb{R}^d$ (Natole et al., 2018, Theorem 1), where

$$
f(w; z) = (1-p)w^T(x-x_+)^2\mathbb{I}[y_1 = 1] + p(1-p) + 2(1+w^T(x-x_-)w^Tx(x)\mathbb{I}[y_1 = -1] - (1-p)\mathbb{I}[y_1 = 1])
+p(w^T(x-x_-))^2\mathbb{I}[y_1 = -1] - p(1-p)(w^T(x-x_-))^2.
$$

(5.2)

An interesting property is that (5.2) involves only a single example $z$. This observation allows Natole et al. (2018) to develop a stochastic algorithm as (3.3) to solve (5.1). However, for each $z$, the function $z \mapsto f(w; z)$ is non-convex since the associated Hessian matrix may not be positively definite. It is clear that its expectation $F$ is convex.

### 6. Stability with Relaxed Strong Convexity

#### 6.1. Stability and generalization errors

Finally, we consider learning problems with strongly convex empirical objectives but possibly non-convex loss functions. Theorem 10 provides stability bounds, while the minimax optimal generalization bounds $O(1/(n \sigma))$ are presented in Theorem 11. The proofs are given in Appendix F.

**Theorem 10.** Let Assumptions in Theorem 8 hold. Suppose for all $S \subseteq \mathbb{Z}$, $F_S$ is $\delta_S$-strongly convex. Then, there exists a constant $t_0$ such that for SGD with $\eta_t = 2/((t + t_0)\sigma_S)$ we have

$$
\left(\mathbb{E}_{\omega} [\|w_{t+1} - \omega_{t+1}\|^2]\right)^{\frac{1}{2}} \leq \frac{4G}{\sigma_S} \left(\frac{1}{\sqrt{n(t+t_0)}} + \frac{1}{n}\right).
$$

**Remark 11.** Under the assumption $w \mapsto f(w; z)$ is $\sigma$-strongly convex and smooth for all $z$, it was shown that $\mathbb{E}_{\omega} [\|w_{t+1} - \omega_{t+1}\|^2] = O(1/(n \sigma))$ for $\eta_t = O(1/\sigma(t))$ (Hardt et al., 2016). Indeed, this strong convexity condition is used to show that the operator $w \mapsto w - \nabla f(w; z)$ is contractive. We relax the strong convexity of $f(w; z)$ to the strong convexity of $F_S$. Our stability bound holds even if $w \mapsto f(w; z)$ is non-convex. If $t = n$, then our stability bound coincides with the one in Hardt et al. (2016) up to a constant factor.

**Theorem 11.** Let Assumption 1 hold. Assume for all $z \in \mathbb{Z}$, the function $w \mapsto f(w; z)$ is $L$-smooth. Suppose for all $S \subseteq \mathbb{Z}$, $F_S$ is $\sigma_S$-strongly convex. Then, there exists some $t_0$ such that for SGD with $\eta_t = 2/((t + t_0)\sigma_S)$ and $T \gg n$ we have

$$
\mathbb{E}_{S, \omega} [F(w^{(2)}_T) - F(w^*)] = O(\mathbb{E}[1/(n \sigma_S)]),
$$

where $w^{(2)}_T = (\sum_{t=1}^T (t + t_0 - 1)w_t) / \sum_{t=1}^T (t + t_0 - 1).

#### 6.2. Application: least squares regression

We now consider an application to learning with the least squares loss, where $f(w; z) = \frac{1}{2}(w, x - y)^2$. Let $\Omega = \{w \in \mathbb{R}^d : \|w\|_2 \leq R\}$. In this case, (3.3) becomes

$$
w_{t+1} = \Pi_{\Omega} (w_t - \eta \langle w_t, x_i \rangle x_i),
$$

(6.1)

where $\Pi_{\Omega}(w) = \min_{R} \|R\|_1 \min \omega \mid \|\omega\|_2 = 1$. Note that each individual loss function $f(w; z_i)$ is non-strongly convex. However, as we will show below, the empirical objective satisfies a strong convexity on a subspace containing the iterates $\{w_t\}$. For any $S = \{z_1, \ldots, z_n\}$ let $C_S = \frac{1}{n} \sum_{i=1}^n x_i x_i^T$ be the empirical covariance matrix and $\sigma_S$ be the minimal positive eigenvalue of $C_S$. Then it is clear from (6.1) that $S$ belongs to the range of $C_S$. Then we consider $S \subseteq \mathbb{Z}$ differ from $S$ by a single example. For simplicity, we assume $S$ and $S$ differ by the first example and denote $S = \{z_1, z_2, \ldots, z_n\}$. We construct a set $S = \{0, z_2, \ldots, z_n\}$. Let $\{w_t\}, \{\omega_t\}$ and $\{\tilde{w}_t\}$ be the sequence by (6.1) based on $S, S$ and $S$, respectively. Then our previous discussion implies that $w_t - \omega_t \in \text{Range}(C_S), \tilde{w}_t - \omega_t \in \text{Range}(C_S)$ for all $t \in \mathbb{N}$ (Range($C_S$) $\subseteq$ Range($C_S$)).

Therefore, we can apply Theorem 10 with $S = \omega_t, \tilde{w}_t$ and $\sigma_S$ to derive (note $\nabla F_S(w) = C_S w - \frac{1}{n} \sum_{i=1}^n y_i x_i$)

$$
\mathbb{E}_{\omega} [\|w_{t+1} - \omega_{t+1}\|^2] \leq \frac{4G}{\sigma_S} \left(\frac{1}{\sqrt{n(t+t_0)}} + \frac{1}{n}\right).
$$

(6.2)
A similar inequality also holds for \( \mathbb{E}_A[\| \tilde{w}_{t+1} - \tilde{w}_{t+1}\|_2] \), which together with the subadditivity of \( \| \cdot \|_2 \) immediately gives the following stability bound on \( \mathbb{E}_A[\| w_{t+1} - \tilde{w}_{t+1}\|_2] \).

**Corollary 12.** Let \( f(w; z) = \frac{1}{2}((w, x) - y)^2 \) and \( \Omega = \{ w \in \mathbb{R}^d : \| w \|_2 \leq R \} \) for some \( R > 0 \). There exists an \( t_0 \) such that for (6.1) with \( \eta_j = 2/(\sigma_j^2(j + t_0)) \) we have

\[
\mathbb{E}_A[\| w_t - \tilde{w}_t\|_2] = O\left( \frac{1}{\sqrt{n(t + t_0)}\sigma_S} + \frac{1}{n\sigma_S} \right).
\]

**7. Conclusions**

In this paper, we study stability and generalization of SGD by removing the bounded gradient assumptions, and relaxing the smoothness assumption and the convexity requirement of each loss function in the existing analysis. We introduce a novel on-average model stability able to capture the risks of SGD iterates, which implies fast generalization bounds in the low-noise case. For all considered problems, we show that our stability bounds can imply minimax optimal generalization bounds by balancing optimization and estimation errors. We apply our results to practical learning problems to justify the superiority of our approach over the existing stability analysis. Our results can be extended to stochastic proximal gradient descent, high-probability bounds and SGD without replacement (details are given in Appendix G). In the future, it would be interesting to study stability bounds for other stochastic optimization algorithms, e.g., Nesterov’s accelerated variants of SGD (Nesterov, 2013).

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A. Optimization Error Bounds

For a full picture of generalization errors, we need to address the optimization errors. This is achieved by the following lemma. Parts (a) and (b) consider the convex and strongly convex empirical objectives, respectively (we make no assumptions on the convexity of each \( f(\cdot, z) \)). Note in Parts (c) and (d), we do not make a bounded gradient assumption. As an alternative, we require convexity of \( f(\cdot, z) \) for all \( z \). An appealing property of Parts (c) and (d) is that it involves \( O(\sum_{j=1}^{t} \eta_{j}^{2} F_{S}(w)) \) instead of \( O(\sum_{j=1}^{t} \eta_{j}^{2}) \), which is a requirement for developing fast rates in the case with low noises.

Our discussion on optimization errors requires to use a self-bounding property for functions with H"{o}lder continuous gradients, which means that gradients can be controlled by function values. The case \( \alpha = 1 \) was established in Srebro et al. (2010).

Lemma A.1 (Ying & Zhou 2017). Assume for all \( z \in Z \), the map \( w \mapsto f(w; z) \) is nonnegative, and \( w \mapsto \nabla f(w; z) \) is \((\alpha, L)\)-H"{o}lder continuous with \( \alpha \in (0, 1] \). Then for \( c_{\alpha,1} = (1 + 1/\alpha)^{-\frac{1}{\alpha}} L^{-\frac{1}{\alpha}} \), we have

\[
\|\nabla f(w, z)\|_{2} \leq c_{\alpha,1} f_{\infty}^{\frac{1}{\alpha}} (w, z), \quad \forall w \in \mathbb{R}^{d}, z \in Z.
\]

Lemma A.2. (a) Let \( \{w_{t}\}_{t} \) be produced by (3.3) and Assumption 1 hold. If \( F_{S} \) is convex, then for all \( t \in \mathbb{N} \) and \( w \in \Omega \)

\[
E_{A}[F_{S}(w_{t}^{(1)})] - F_{S}(w) \leq \frac{G^{2} \sum_{j=1}^{t} \eta_{j}^{2} + \|w\|_{2}^{2}}{2 \sum_{j=1}^{t} \eta_{j}},
\]

where \( w_{t}^{(1)} = \sum_{j=1}^{t} \eta_{j} w_{j} / \sum_{j=1}^{t} \eta_{j} \).

(b) Let \( F_{S} \) be \( \sigma_{S} \)-strongly convex and Assumption 1 hold. Let \( t_{0} \geq 0 \) and \( \{w_{t}\}_{t} \) be produced by (3.3) with \( \eta_{t} = 2/(\sigma_{S}(t + t_{0})) \). Then for all \( t \in \mathbb{N} \) and \( w \in \Omega \)

\[
E_{A}[F_{S}(w_{t}^{(2)})] - F_{S}(w) = O(1/(t\sigma_{S}) + \|w\|_{2}^{2}/t^{2}),
\]

where \( w_{t}^{(2)} = \sum_{j=1}^{t} (j + t_{0} - 1) w_{j} / \sum_{j=1}^{t} (j + t_{0} - 1) \).

(c) Assume for all \( z \in Z \), the function \( w \mapsto f(w; z) \) is nonnegative, convex and \( L \)-smooth. Let \( \{w_{t}\}_{t} \) be produced by (3.3) with \( \eta_{t} \leq 1/(2L) \). If the step size is nonincreasing, then for all \( t \in \mathbb{N} \) and \( w \in \Omega \) independent of the SGD algorithm \( A \)

\[
\sum_{j=1}^{t} \eta_{j} E_{A}[F_{S}(w_{j}) - F_{S}(w)] \leq (1/2 + L\eta_{1})\|w\|_{2}^{2} + 2L \sum_{j=1}^{t} \eta_{j}^{2} F_{S}(w).
\]

(d) Assume for all \( z \in Z \), the function \( w \mapsto f(w; z) \) is nonnegative, convex, and \( \nabla f(w; z) \) is \((\alpha, L)\)-H"{o}lder continuous with \( \alpha \in (0, 1] \). Let \( \{w_{t}\}_{t} \) be produced by (3.3) with nonincreasing step sizes. Then for all \( t \in \mathbb{N} \) and \( w \in \Omega \) independent of the SGD algorithm \( A \) we have

\[
2 \sum_{j=1}^{t} \eta_{j} E_{A}[F_{S}(w_{j}) - F_{S}(w)] \leq \|w\|_{2}^{2} + c_{\alpha,1} \left( \sum_{j=1}^{t} \eta_{j}^{2} \eta_{j}^{-\frac{1}{\alpha}} \left( \eta_{1}\|w\|_{2}^{2} + 2 \sum_{j=1}^{t} \eta_{j}^{2} F_{S}(w) + \sum_{j=1}^{t} \eta_{j}^{2} \right) \right)^{1+\frac{1}{\alpha}}
\]

where \( c_{\alpha,2} = \frac{1-\alpha}{1+\alpha} (2\alpha/(1+\alpha))^{\frac{2\alpha}{1+\alpha}} \).

Proof. Parts (a) and (b) can be found in the literature (Lacoste-Julien et al., 2012; Nemirovski et al., 2009). We only prove Parts (c) and (d). We first prove Part (c). The projection operator \( \Pi_{\Omega} \) is non-expansive, i.e.,

\[
\|\Pi_{\Omega}(w) - \Pi_{\Omega}(\bar{w})\|_{2} \leq \|w - \bar{w}\|_{2}.
\]

By the SGD update (3.3), (A.1), convexity and Lemma A.1, we know

\[
\|w_{t+1} - w\|_{2}^{2} \leq \|w_{t} - \eta_{t} \nabla f(w_{t}; z_{i}) - w\|_{2}^{2}
\]

\[
= \|w_{t} - w\|_{2}^{2} + \eta_{t}^{2} \|\nabla f(w_{t}; z_{i})\|_{2}^{2} + 2\eta_{t} (w - w_{t}; \nabla f(w_{t}; z_{i})
\]

\[
\leq \|w_{t} - w\|_{2}^{2} + 2\eta_{t}^{2} Lf(w; z_{i}) + 2\eta_{t} (f(w; z_{i}) - f(w_{t}; z_{i}))
\]

\[
\leq \|w_{t} - w\|_{2}^{2} + 2\eta_{t} f(w; z_{i}) - \eta_{t} f(w_{t}; z_{i}).
\]

(A.2)
where the last inequality is due to $\eta_t \leq 1/(2L)$. It then follows that
\[
\eta_t f(w_t; z_{i_t}) \leq \|w_t - w\|_2^2 - \|w_{t+1} - w\|_2^2 + 2\eta_t f(w; z_{i_t}).
\]

Multiplying both sides by $\eta_t$ and using the assumption $\eta_{t+1} \leq \eta_t$, we know
\[
\eta_t^2 f(w_t; z_{i_t}) \leq \eta_t \|w_t - w\|_2^2 - \eta_t \|w_{t+1} - w\|_2^2 + 2\eta_t^2 f(w; z_{i_t})
\leq \eta_t \|w_t - w\|_2^2 - \eta_{t+1} \|w_{t+1} - w\|_2^2 + 2\eta_t^2 f(w; z_{i_t}).
\]

Taking a summation of the above inequality gives ($w_1 = 0$)
\[
\sum_{j=1}^{t} \eta_j^2 f(w_j; z_{i_j}) \leq \eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 f(w; z_{i_j}).
\]

Taking an expectation w.r.t. $A$ gives (note $w_j$ is independent of $i_j$)
\[
\sum_{j=1}^{t} \eta_j^2 E_A[F_S(w_j)] = \sum_{j=1}^{t} \eta_j^2 E_A[f(w_j; z_{i_j})] \leq \eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 E_A[F_S(w)].
\] (A.3)

On the other hand, taking an expectation w.r.t. $i_t$ over both sides of (A.2) shows
\[
2\eta_t [F_S(w_{t+1}) - F_S(w)] \leq \|w_t - w\|_2^2 - E_{i_t} [\|w_{t+1} - w\|_2^2] + 2\eta_t^2 L F_S(w_t).
\]

Taking an expectation on both sides followed with a summation, we get
\[
2 \sum_{j=1}^{t} \eta_j E_A[F_S(w_j)] - F_S(w)] \leq \|w\|_2^2 + 2L \sum_{j=1}^{t} \eta_j^2 E_A[F_S(w_j)]
\leq (1 + 2L\eta_1)\|w\|_2^2 + 4L \sum_{j=1}^{t} \eta_j^2 E_A[F_S(w)],
\]

where the last step is due to (A.3). The proof is complete since $w$ is independent of $A$.

We now prove Part (d). Analogous to (A.2), one can show for loss functions with Hölder continuous gradients
\[
\|w_{t+1} - w\|_2^2 \leq \|w_t - w\|_2^2 + c_{\alpha, 1}\eta_t^2 \|f(w_t; z_{i_t})\|^{\frac{p}{p+q}} + 2\eta_t (f(w; z_{i_t}) - f(w_t; z_{i_t})).
\] (A.4)

By the Young’s inequality
\[
ab \leq p^{-1}|a|^p + q^{-1}|b|^q, \quad a, b \in \mathbb{R}, p, q > 0 \text{ with } p^{-1} + q^{-1} = 1,
\] (A.5)

we know
\[
\eta_t c_{\alpha, 1}^2 = \frac{\eta_t^2}{\eta_t^2} f(w_t; z_{i_t}) = \left(\frac{1 + \alpha}{2\alpha} f(w_t; z_{i_t})\right)^{\frac{p}{p+q}} \left(\frac{2\alpha}{1 + \alpha}\right)^{\frac{p}{p+q}} c_{\alpha, 1}^2 \eta_t
\leq \left(\frac{2\alpha}{1 + \alpha}\right)^{\frac{p}{p+q}} c_{\alpha, 1}^2 \eta_t + \left(\frac{2\alpha}{1 + \alpha}\right)^{\frac{p}{p+q}} c_{\alpha, 1}^2 \eta_t
\leq f(w_t; z_{i_t}) + c_{\alpha, 2}\eta_t^2.
\]

Combining the above two inequalities together, we get
\[
\eta_t f(w_t; z_{i_t}) \leq \|w_t - w\|_2^2 - \|w_{t+1} - w\|_2^2 + 2\eta_t f(w; z_{i_t}) + c_{\alpha, 2}\eta_t^2.
\]

Multiplying both sides by $\eta_t$ and using $\eta_{t+1} \leq \eta_t$, we derive
\[
\eta_t^2 f(w_t; z_{i_t}) \leq \eta_t \|w_t - w\|_2^2 - \eta_{t+1} \|w_{t+1} - w\|_2^2 + 2\eta_t^2 f(w; z_{i_t}) + c_{\alpha, 2}\eta_t^2.
\]
We now prove Part (b). According to (B.1) due to the $\mathcal{L}$-smoothness, we have

$$\sum_{j=1}^{t} \eta_j^2 f(w_j; z_{i_j}) \leq \eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 f(w; z_{i_j}) + c_{\alpha,2} \sum_{j=1}^{t} \eta_j^\frac{2-\alpha}{\alpha}, \quad (A.6)$$

Taking a summation of the above inequality gives

$$\sum_{j=1}^{t} \eta_j^2 f(w_j; z_{i_j}) \leq \eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 f(w; z_{i_j}) + c_{\alpha,2} \sum_{j=1}^{t} \eta_j^\frac{2-\alpha}{\alpha}, \quad (A.7)$$

where in the last step we have used (A.6). Taking an expectation on both sides of (A.4), we know

$$2\eta_t \mathbb{E}_A [F_S(w_t) - F_S(w)] \leq \mathbb{E}_A [\|w_t - w\|_2^2] - \mathbb{E}_A [\|w_t+1 - w\|_2^2] + c_{\alpha,1} \eta_t^2 \mathbb{E}_A [f^{\frac{2-\alpha}{\alpha}}(w_t; z_{i_t})].$$

Taking a summation of the above inequality gives

$$\sum_{j=1}^{t} \eta_j \mathbb{E}_A [F_S(w_j) - F_S(w)] \leq \|w\|_2^2 + c_{\alpha,1} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_A [f^{\frac{2-\alpha}{\alpha}}(w_j; z_{i_j})]$$

$$\leq \|w\|_2^2 + c_{\alpha,1} \left( \sum_{j=1}^{t} \eta_j^2 \right)^{\frac{1-\alpha}{2}} (\eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_A [F_S(w)] + c_{\alpha,2} \sum_{j=1}^{t} \eta_j^\frac{2-\alpha}{\alpha})^{\frac{2-\alpha}{\alpha}},$$

where we have used (A.7) and the concavity of $x \mapsto x^{\frac{2-\alpha}{\alpha}}$ in the last step. The proof is complete by noting the independence between $w$ and $A$.

\[\square\]

### B. Proofs on Generalization by On-average Model Stability

To prove Theorem 2, we introduce an useful inequality for $L$-smooth functions $w \mapsto f(w; z)$ (Nesterov, 2013)

$$f(w; z) \leq f(\tilde{w}; z) + \langle w - \tilde{w}, \nabla f(\tilde{w}; z) \rangle + \frac{L\|w - \tilde{w}\|_2^2}{2}. \quad (B.1)$$

**Proof of Theorem 2.** Due to the symmetry, we know

$$\mathbb{E}_{S, A} [F(A(S)) - F_S(A(S))] = \mathbb{E}_{S, \tilde{S}, A} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( F(A(S^{(i)})) - F_S(A(S)) \right) \right]$$

$$= \mathbb{E}_{S, \tilde{S}, A} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( f(A(S^{(i)}); z_i) - f(A(S); z_i) \right) \right], \quad (B.2)$$

where the last identity holds since $A(S^{(i)})$ is independent of $z_i$. Under Assumption 1, it is then clear that

$$\mathbb{E}_{S, A} [F(A(S)) - F_S(A(S))] \leq \mathbb{E}_{S, \tilde{S}, A} \left[ \frac{G}{n} \sum_{i=1}^{n} \| A(S^{(i)}) - A(S^{(i)}) \|_2 \right].$$

This proves Part (a).

We now prove Part (b). According to (B.1) due to the $L$-smoothness of $f$ and (B.2), we know

$$\mathbb{E}_{S, A} [F(A(S)) - F_S(A(S))] \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S, \tilde{S}, A} \left[ \langle A(S^{(i)}) - A(S), \nabla f(A(S); z_i) \rangle + \frac{L}{2} \| A(S^{(i)}) - A(S) \|_2^2 \right].$$
According to the Schwartz’s inequality we know
\[
\langle A(S^{(i)}) - A(S), \nabla f(A(S); z_i) \rangle \leq \|A(S^{(i)}) - A(S)\|_2 \|\nabla f(A(S); z_i)\|_2
\]
\[
\leq \frac{\gamma}{2} \|A(S^{(i)}) - A(S)\|_2^2 + \frac{1}{2 \gamma} \|\nabla f(A(S); z_i)\|_2^2
\]
\[
\leq \frac{\gamma}{2} \|A(S^{(i)}) - A(S)\|_2^2 + \frac{L}{\gamma} f(A(S); z_i),
\]
where the last inequality is due to the self-bounding property of smooth functions (Lemma A.1). Combining the above two inequalities together, we derive
\[
\mathbb{E}_{S,A}[F(A(S)) - F_S(A(S))] \leq \frac{L + \gamma}{2n} \sum_{i=1}^{n} \mathbb{E}_{S,S,A}[\|A(S^{(i)}) - A(S)\|_2^2] + \frac{L}{n \gamma} \sum_{i=1}^{n} \mathbb{E}_{S,A}[f(A(S); z_i)].
\]
The stated inequality in Part (b) then follows directly by noting \(\frac{1}{n} \sum_{i=1}^{n} f(A(S); z_i) = F_S(A(S))\).

Finally, we consider Part (c). By (B.2) and the convexity of \(f\), we know
\[
\mathbb{E}_{S,A}[F(A(S)) - F_S(A(S))] \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S,S,A}\left[ (A(S^{(i)}) - A(S), \nabla f(A(S^{(i)}); z_i)) \right].
\]

By the Schwartz’s inequality and Lemma A.1 we know
\[
\langle A(S^{(i)}) - A(S), \nabla f(A(S^{(i)}); z_i) \rangle \leq \frac{\gamma}{2} \|A(S^{(i)}) - A(S)\|_2^2 + \frac{1}{2 \gamma} \|\nabla f(A(S^{(i)}); z_i)\|_2^2
\]
\[
\leq \frac{\gamma}{2} \|A(S^{(i)}) - A(S)\|_2^2 + \frac{c_{\alpha,1}}{2 \gamma} \frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n} (A(S^{(i)}); z_i).
\]
Combining the above two inequalities together, we get
\[
\mathbb{E}_{S,A}[F(A(S)) - F_S(A(S))] \leq \frac{\gamma}{2n} \sum_{i=1}^{n} \mathbb{E}_{S,S,A}[\|A(S^{(i)}) - A(S)\|_2^2] + \frac{c_{\alpha,1}}{2 \gamma} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S,S,A}[f(\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}) (A(S^{(i)}); z_i)].
\]
Since \(x \mapsto x^\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}\) is concave and \(z_i\) is independent of \(A(S^{(i)})\), we know
\[
\mathbb{E}_{S,S,A}[f(\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}) (A(S^{(i)}); z_i)] \leq \mathbb{E}_{S,S,A} \left[ \left( \mathbb{E}_{z_i} \left[ f(A(S^{(i)}); z_i) \right] \right)^{\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}} \right] = \mathbb{E}_{S,S,A} \left[ F^{\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}} (A(S^{(i)})) \right] = \mathbb{E}_{S,A} \left[ F^{\frac{2^{\frac{\sqrt{\alpha}}{\sqrt{\gamma}}}}{n}} (A(S)) \right].
\]

A combination of the above two inequalities then gives the stated bound in Part (c). The proof is complete. \(\square\)

C. Proof on Learning without Bounded Gradients: Strongly Smooth Case

C.1. Stability bounds

A key property on establishing the stability of SGD is the non-expansiveness of the gradient-update operator established in the following lemma.

**Lemma C.1 (Hardt et al. 2016).** Assume for all \(z \in \mathcal{Z}\), the function \(w \mapsto f(w; z)\) is convex and \(L\)-smooth. Then for \(\eta \leq 2/L\) we know
\[
\|w - \eta \nabla f(w; z) - \tilde{w} + \eta \nabla f(\tilde{w}; z)\|_2 \leq \|w - \tilde{w}\|_2.
\]

Based on Lemma C.1, we establish stability bounds of models for SGD applied to two sets differing by a single example.

**Lemma C.2.** Assume for all \(z \in \mathcal{Z}\), the function \(w \mapsto f(w; z)\) is nonnegative, convex and \(L\)-smooth. Let \(S, \tilde{S}\) and \(S^{(i)}\) be constructed as Definition 4. Let \(\{w_i\}\) and \(\{w_i^{(i)}\}\) be produced by (3.3) with \(\eta_i \leq 2/L\) based on \(S\) and \(S^{(i)}\), respectively. Then for any \(p > 0\) we have
\[
\mathbb{E}_A \left[ \|w_{t+1}^{(i)} - w_{t+1}^{(i)}\|_2 \right] \leq \sqrt{\frac{2L}{n}} \sum_{j=1}^{t} \eta_j \mathbb{E}_A \left[ \sqrt{f(w_j; \tilde{z}_i)} + \sqrt{f(w_j; z_i)} \right] \tag{C.1}
\]
We first prove Eq. (C.1). Since the last inequality is due to Lemma A.1. Combining (C.3), the above inequality together and noticing the distribution of $i$, we derive

$$\nabla f(w_t; \tilde{z}_i) = \nabla f(w_t; \tilde{z}_i) - (\nabla f(w_t; \tilde{z}_i) - \nabla f(w_t; z_i))$$

to derive

$$\|w_{t+1} - w_{t+1}^{(i)}\|_2 \leq \|w_t - \eta_t \nabla f(w_t; z_i) - w_t^{(i)} + \eta_t \nabla f(w_t^{(i)}, \tilde{z}_i)\|_2$$

$$\leq \|w_t - \eta_t \nabla f(w_t; \tilde{z}_i) - w_t^{(i)} + \eta_t \nabla f(w_t^{(i)}, \tilde{z}_i)\|_2 + \eta_t \|\nabla f(w_t^{(i)}) - \nabla f(w_t; z_i)\|_2$$

where the second inequality follows from the sub-additivity of $\| \cdot \|_2$, the third inequality follows from Lemma C.1 and the last inequality is due to Lemma A.1 on the self-bounding property of smooth functions.

We first prove Eq. (C.1). Since $i$ is drawn from the uniform distribution over $\{1, \ldots, n\}$, we can combine Eqs. (3.3) and (C.5) to derive

$$\mathbb{E}_A[\|w_{t+1} - w_{t+1}^{(i)}\|_2] \leq \mathbb{E}_A[\|w_t - w_t^{(i)}\|_2] + \frac{\sqrt{2L} \eta_t}{n} \mathbb{E}_A[f(w_t; \tilde{z}_i) + \sqrt{f(w_t; z_i)}]$$

Taking a summation of the above inequality and using $w_1 = w_1^{(i)}$ then give (C.1).

We now turn to (C.2). For the case $i_t = i$, it follows from (C.4) and the standard inequality $(a+b)^2 \leq (1+p)a^2 + (1+1/p)b^2$ that

$$\|w_{t+1} - w_{t+1}^{(i)}\|_2^2 \leq (1+p)\|w_t - w_t^{(i)}\|_2^2 + (1+1/p)\eta_t^2 \|\nabla f(w_t; \tilde{z}_i) - \nabla f(w_t; z_i)\|_2^2$$

where the last inequality is due to Lemma A.1. Combining (C.3), the above inequality together and noticing the distribution of $i$, we derive

$$\mathbb{E}_A[\|w_{t+1} - w_{t+1}^{(i)}\|_2^2] \leq (1+p/n)\mathbb{E}_A[\|w_t - w_t^{(i)}\|_2^2] + \frac{4(1+p-1)L\eta_t^2}{n} \mathbb{E}_A[f(w_t; \tilde{z}_i) + f(w_t; z_i)]$$

Multiplying both sides by $(1+p/n)^{-(t+1)}$ yields that

$$(1+p/n)^{-(t+1)}\mathbb{E}_A[\|w_{t+1} - w_{t+1}^{(i)}\|_2^2] \leq (1+p/n)^{-t}\mathbb{E}_A[\|w_t - w_t^{(i)}\|_2^2] + \frac{4(1+p-1)L(1+p/n)^{-(t+1)}\eta_t^2}{n} \mathbb{E}_A[f(w_t; \tilde{z}_i) + f(w_t; z_i)].$$
We now turn to (4.4). It follows from Lemma C.2 (Eq. (C.2)) that the stated bound then follows.

The proof is complete.

Proof of Theorem 3. We first prove (4.3). According to Lemma C.2 (Eq. (C.1)), we know

\[
\mathbb{E}_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \right] \leq 2L \frac{n}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{t} \eta_j \mathbb{E}_A \left[ \sqrt{f(\mathbf{w}_j; \hat{z}_i)} + \sqrt{f(\mathbf{w}_j; z_i)} \right].
\]

It then follows from the concavity of the square-root function and the Jensen’s inequality that

\[
\mathbb{E}_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \right] \\
\leq \frac{2L}{n} \sum_{j=1}^{t} \eta_j \mathbb{E}_A \left[ \sqrt{n-1} \sum_{i=1}^{n} f(\mathbf{w}_j; \hat{z}_i) \right] + \frac{2L}{n} \sum_{j=1}^{t} \eta_j \mathbb{E}_A \left[ \sqrt{n-1} \sum_{i=1}^{n} f(\mathbf{w}_j; z_i) \right] \\
= \frac{2L}{n} \sum_{j=1}^{t} \eta_j \mathbb{E}_A \left[ \sqrt{F_S(\mathbf{w}_j)} + \sqrt{F_S(\mathbf{w}_j)} \right].
\]

This proves (4.3).

We now turn to (4.4). It follows from Lemma C.2 (Eq. (C.2)) that

\[
\mathbb{E}_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \right] \leq \frac{4(1 + p^{-1})L(1 + p/n)^{t-j}\eta_j^2}{n^2} \mathbb{E}_A \left[ f(\mathbf{w}_j; \hat{z}_i) + f(\mathbf{w}_j; z_i) \right] \\
= \frac{4(1 + p^{-1})L(1 + p/n)^{t-j}\eta_j^2}{n} \mathbb{E}_A \left[ F_S(\mathbf{w}_j) + F_S(\mathbf{w}_j) \right].
\]

The proof is complete.

Proposition C.3 (Stability bounds for non-convex learning). Let Assumptions of Theorem 3 hold except that we do not require the convexity of \( \mathbf{w} \mapsto f(\mathbf{w}; z) \). Then for any \( p > 0 \) we have

\[
\mathbb{E}_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \right] \\
\leq (1 + p/n)(1 + \eta t) L \mathbb{E}_A \left[ \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{w}_t - \mathbf{w}_t^{(i)} \|_2 \right] + \frac{4(1 + p^{-1})L\eta_t^2}{n} \mathbb{E}_A \left[ F_S(\mathbf{w}_t) + F_S(\mathbf{w}_t) \right].
\]

Proof. If \( i_t \neq i \), then by the \( L \)-smoothness of \( f \) we know

\[
\| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \leq \| \mathbf{w}_t - \mathbf{w}_t^{(i)} \|_2 + \eta t \| \nabla f(\mathbf{w}_t; z_i) - \nabla f(\mathbf{w}_t^{(i)}, z_i) \|_2 \leq (1 + \eta t) \| \mathbf{w}_t - \mathbf{w}_t^{(i)} \|_2. \tag{C.7}
\]

If \( i_t = i \), then analogous to (C.4) but using the \((1 + \eta t)\)-expansiveness of the operator \( \mathbf{w} \mapsto \mathbf{w} - \eta_t \nabla f(\mathbf{w}; \hat{z}_i) \) (as (C.7)) instead of Lemma C.1, one gets

\[
\| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \leq (1 + \eta t) \| \mathbf{w}_t - \mathbf{w}_t^{(i)} \|_2 + \eta t \| \nabla f(\mathbf{w}_t; \hat{z}_i) - \nabla f(\mathbf{w}_t; z_i) \|_2.
\]

Then, analogous to the deduction of (C.6), one can get

\[
\| \mathbf{w}_{t+1} - \mathbf{w}_t^{(i)} \|_2 \leq (1 + p)(1 + \eta t)^2 \| \mathbf{w}_t - \mathbf{w}_t^{(i)} \|_2 + 4(1 + p^{-1})L\eta_t^2 f(\mathbf{w}_t; \hat{z}_i) + 4(1 + p^{-1})L\eta_t^2 f(\mathbf{w}_t; z_i).
\]
By the uniform distribution of \( i_t \in \{1, 2, \ldots, n\} \) we can combine the above inequality and \((C.7)\) to derive

\[
E_A[\|w_{t+1} - w^{(i)}_{t+1}\|_2^2] \leq (1 + p/n)(1 + \eta_t)L^2E_A[\|w_t - w^{(i)}_t\|_2^2] + \frac{4(1 + p^{-1})Ln\eta_t^2}{n}E_A[f(w_t; \tilde{z}_i) + f(w_t; z_i)].
\]

It then follows that

\[
E_A\left[\frac{1}{n} \sum_{i=1}^n \|w_{t+1} - w^{(i)}_{t+1}\|_2^2\right]
\leq (1 + p/n)(1 + \eta_t)L^2E_A\left[\frac{1}{n} \sum_{i=1}^n \|w_t - w^{(i)}_t\|_2^2\right] + \frac{4(1 + p^{-1})Ln\eta_t^2}{n}E_A\sum_{i=1}^n [f(w_t; \tilde{z}_i) + f(w_t; z_i)]
\leq (1 + p/n)(1 + \eta_t)L^2E_A\left[\frac{1}{n} \sum_{i=1}^n \|w_t - w^{(i)}_t\|_2^2\right] + \frac{4(1 + p^{-1})Ln\eta_t^2}{n}E_A[F_{\tilde{S}}(w_t) + F_S(w_t)].
\]

The proof is complete.

\(\square\)

### C.2. Generalization bounds

To prove Theorem 4 on generalization errors, we introduce a general result on controlling the population risk of \( w_{t+1} \) by empirical risks of SGD iterates.

**Proposition C.4.** Assume for all \( z \in Z \), the function \( w \mapsto f(w; z) \) is nonnegative, convex and \( L \)-smooth. Let \( w_t \) be produced by (3.3) with \( \eta_t \leq 2/L \). Let \( \gamma > 0, p > 0 \). If \( n \geq 4(1 + p^{-1})(L + \gamma)L(1 + p/n)^{-1} \sum_{j=1}^t \eta_j^2 \), then the following inequality holds for all \( w \in \Omega \) independent of \( S \)

\[
E_{S,A}[F(w_{t+1})] \leq (1 + L/\gamma)E_{S,A}[F_S(w_{t+1})] + \frac{4(1 + p^{-1})(L + \gamma)L(2 + L\gamma^{-1})(1 + p/n)^{-1}}{n}(\eta_t \|w\|_2^2 + 2 \sum_{j=1}^t \eta_j^2 F(w)).
\]

**Proof.** Let \( A(S) \) be the \((t+1)\)-th iterate of SGD applied to the dataset \( S \). Taking expectation on both sides of (4.4) and noticing \( E_{\tilde{S}}[F_{\tilde{S}}(w_j)] = F(w_j) \) since \( w_j \) is independent of \( \tilde{S} \), we derive

\[
E_{S,\tilde{S},A}\left[\frac{1}{n} \sum_{i=1}^n \|w_{t+1} - w^{(i)}_{t+1}\|_2^2\right] \leq \sum_{j=1}^t \frac{4(1 + p^{-1})L(1 + p/n)^{t-j} \eta_j^2}{n}E_{S,A}[F(w_j) + F_S(w_j)].
\]

We plug the above inequality into Part (b) of Theorem 2, and derive

\[
E_{S,A}\left[F(w_{t+1}) - (1 + L\gamma^{-1})F_S(w_{t+1})\right] \leq \frac{2(1 + p^{-1})(L + \gamma)L}{n} \sum_{j=1}^t (1 + p/n)^{t-j} \eta_j^2 E_{S,A}[F(w_j) - (1 + L/\gamma)F_S(w_j) + (2 + L\gamma^{-1})F_S(w_j)].
\]

Let \( \delta_j = \max \{E_{S,A}[F(w_j) - (1 + L/\gamma)F_S(w_j)], 0\} \) for all \( j \in \mathbb{N} \). Then, it is clear that

\[
\delta_{t+1} - \frac{2(1 + p^{-1})(L + \gamma)L(2 + L\gamma^{-1})(1 + p/n)^{t-1}}{n} \sum_{j=1}^t \eta_j^2 E_{S,A}[F_S(w_j)]
\leq \frac{2(1 + p^{-1})(L + \gamma)L(1 + p/n)^{t-1}}{n} \sum_{j=1}^t \eta_j^2 \max_{1 \leq j \leq t+1} \delta_j.
\]
Since the above inequality holds for all \( t \) and the above upper bound of \( \{ \delta_t \} \) is an increasing function of \( t \), we know
\[
\max_{1 \leq j \leq t+1} \delta_j \leq \frac{2(1+p^{-1})(L+\gamma)L(1+p/n)^{t-1}}{n} \sum_{j=1}^{t} \eta_j^2 \max_{1 \leq j \leq t+1} \delta_j + \frac{2(1+p^{-1})(L+\gamma)L(2+L\gamma^{-1})(1+p/n)^{t-1}}{n} \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}[F_S(w_t)] \\
\leq \frac{1}{2} \max_{1 \leq j \leq t+1} \delta_j + \frac{2(1+p^{-1})(L+\gamma)L(2+L\gamma^{-1})(1+p/n)^{t-1}}{n} (\eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}[F_S(w)])
\]
where the last inequality is due to the assumption \( n \geq 4(1+p^{-1})(L+\gamma)L(1+p/n)^{t-1} \sum_{j=1}^{t} \eta_j^2 \) and (A.3). It then follows that
\[
\max_{1 \leq j \leq t+1} \delta_j \leq \frac{4(1+p^{-1})(L+\gamma)L(2+L\gamma^{-1})(1+p/n)^{t-1}}{n} (\eta_1 \|w\|_2^2 + 2 \sum_{j=1}^{t} \eta_j^2 \mathbb{E}_{S,A}[F_S(w)])
\]
The stated bound then follows from the definition of \( \delta_j \), and the independence between \( w \) and \( S \). The proof is complete.

**Proof of Theorem 4.** According to Part (c) of Lemma A.2 with \( w = w^* \), we know the following inequality for all \( T \)
\[
\sum_{t=1}^{T} \eta_t \mathbb{E}_{A}[F_S(w_t) - F_S(w^*)] \leq \left( 1/2 + L\eta_1 \right) \|w^*\|_2^2 + 2L \sum_{t=1}^{T} \eta_t^2 F_S(w^*). 
\]  
(C.8)

We choose \( p = n/T \), then \( (1+p/n)^{T-1} = (1+1/T)^{T-1} \leq e \). Therefore,
\[
n \geq 4(1+T/n)(L+\gamma)L(1+p/n)^{T-1} \sum_{t=1}^{T} \eta_t^2 \geq 4(1+p^{-1})(L+\gamma)L(1+1/T)^{T-1} \sum_{t=1}^{T} \eta_t^2
\]
and Proposition C.4 holds with \( t = T \). According to Proposition C.4 with \( w = w^*, t = T \) and \( p = n/T \), we know the following inequality for all \( t \leq T \)
\[
\mathbb{E}_{S,A}[F(w_t)] \leq \left( 1 + L/\gamma \right) \mathbb{E}_{S,A}[F_S(w_t)] + \frac{4(1+p^{-1})(L+\gamma)L(2+L\gamma^{-1})(1+1/T)^{t-2}}{n} (\eta_t \|w^*\|_2^2 + 2 \sum_{j=1}^{t-1} \eta_j^2 F(w^*)). 
\]
Multiplying both sides by \( \eta_t \) followed with a summation gives
\[
\sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t)] \leq \left( 1 + L/\gamma \right) \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t)] + \frac{4(1+T/n)(L+\gamma)L(2+L\gamma^{-1})e}{n} \sum_{t=1}^{T} \eta_t \|w^*\|_2^2 + 2 \sum_{j=1}^{T-1} \eta_j^2 F(w^*)
\]
Putting (C.8) into the above inequality then gives
\[
\sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t)] \leq \left( 1 + L/\gamma \right) \left( \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w^*)] + (1/2 + L\eta_1) \|w^*\|_2^2 + 2L \sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w^*)] \right) \\
+ \frac{4(1+T/n)(L+\gamma)L(2+L\gamma^{-1})e}{n} \sum_{t=1}^{T} \eta_t \|w^*\|_2^2 + 2 \sum_{j=1}^{T-1} \eta_j^2 F(w^*)
\]
Since $E_S[F_S(w^*)] = F(w^*)$, it follows that
\[
\sum_{t=1}^{T} \eta_t E_{S,A}[F(w_t) - F(w^*)] \leq \frac{L}{\gamma} \sum_{t=1}^{T} \eta_t F(w^*) + \left(1 + L/\gamma\right) \left((1/2 + L\eta_t)\|w^*\|_2^2 + 2L \sum_{t=1}^{T} \eta_t^2 F(w^*)\right) \\
+ \frac{4(1 + T/n)(L + \gamma)(2 + L\gamma^{-1})c}{n} \sum_{t=1}^{T} \eta_t \|w^*\|_2^2 + 2 \sum_{j=1}^{t-1} \eta_j^2 F(w^*)
\]

The stated inequality then follows from Jensen’s inequality. The proof is complete.

**Proof of Corollary 5.** We first prove Part (a). For the chosen step size, we know
\[
\sum_{t=1}^{T} \eta_t^2 = c^2 \sum_{t=1}^{T} \frac{1}{T} = c^2 \quad \text{and} \quad \sum_{t=1}^{T} \eta_t = c\sqrt{T}.
\]
We choose $\gamma = \sqrt{n}$, then
\[
n \geq 4Le c^2 (1 + T/n)(L + \sqrt{n}) = 4(1 + T/n)(L + \gamma)Le \sum_{t=1}^{T} \eta_t^2
\]
and therefore Theorem 4 holds. The stated bound (4.6) then follows from Theorem 4, $\gamma = \sqrt{n}$ and (C.9).

We now prove Part (b). We choose $\gamma = 1$, then
\[
n \geq 4Le (T + T^2/n)(L + 1)\eta_t^2 = 4(1 + T/n)(L + \gamma)Le \sum_{t=1}^{T} \eta_t^2
\]
and therefore Theorem 4 holds. The stated bound (4.7) then follows from Theorem 4, $F(w^*) = 0$ and $\gamma = 1$. The proof is complete.

**D. Proof on Learning without Bounded Gradients: Non-strongly Smooth Case**

**D.1. Stability bounds**

Theorem 6 is a direct application of the following general stability bounds with $p = n/t$. Therefore, it suffices to prove Theorem D.1.

**Theorem D.1.** Assume for all $z \in \mathcal{Z}$, the function $w \mapsto f(w; z)$ is nonnegative, convex and $\nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1)$. Let $S$, $\tilde{S}$ and $S^{(i)}$ be constructed as Definition 4 and $c_{\alpha,3} = \frac{\sqrt{1 - \alpha}}{\sqrt{1 + \alpha}} (2^{-\alpha} L)^{-1/\alpha}$. Let $w_t$ and $w_t^{(i)}$ be the $t$-th iterate produced by (3.3) based on $S$ and $S^{(i)}$, respectively. Then for any $p > 0$ we have
\[
E_A \left[ \frac{1}{n} \sum_{i=1}^{n} \|w_{t+1} - w_{t+1}^{(i)}\|_2^{2} \right] \leq c_{\alpha,3}^2 \sum_{j=1}^{t} \left(1 + p/n\right)^{t-j} \eta_j^{2/\alpha} \\
+ 2(1 + p^{-1})c_{\alpha,3}^2 \sum_{j=1}^{t} \left(1 + p/n\right)^{t-j} \eta_j^{2/\alpha} E_A \left[ \tilde{F}_{S}^{2\alpha} (w_j) + \tilde{F}_{S}^{2\alpha} (w_j) \right].
\]

We require several lemmas to prove Theorem D.1. The following lemma establishes the co-coercivity of gradients for convex functions with Hölder continuous gradients. The case $\alpha = 1$ can be found in Nesterov (2013).

**Lemma D.2 (Ying & Zhou 2017).** Assume for all $z \in \mathcal{Z}$, the map $w \mapsto f(w; z)$ is convex, and $w \mapsto \nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1]$. Then for all $w, \tilde{w}$ we have
\[
\langle w - \tilde{w}, \nabla f(w; z) - \nabla f(\tilde{w}; z) \rangle \geq \frac{2L^{-\alpha}}{1 + \alpha} \| \nabla f(w; z) - \nabla f(\tilde{w}; z) \|_2^{2 + \alpha}. \]
The following lemma controls the expansive behavior of the operator $w \mapsto w - \eta \nabla f(w; z)$ for convex $f$ with Hölder continuous gradients.

**Lemma D.3.** Assume for all $z \in \mathcal{Z}$, the map $w \mapsto f(w; z)$ is convex, and $w \mapsto \nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1)$. Then for all $w \in \mathbb{R}^d$ and $\eta > 0$ there holds

$$
\|w - \eta \nabla f(w; z) - \tilde{w} + \eta \nabla f(\tilde{w}; z)\|^2_2 \leq \|w - \tilde{w}\|^2_2 + c^2_{\alpha, \eta} \eta^{\frac{2}{1-\alpha}}.
$$

**Proof.** The following equality holds

$$
\|w - \eta \nabla f(w; z) - \tilde{w} + \eta \nabla f(\tilde{w}; z)\|^2_2 = \|w - \tilde{w}\|^2_2 + \eta^2 \|\nabla f(w; z) - \nabla f(\tilde{w}; z)\|^2_2 - 2\eta \langle w - \tilde{w}, \nabla f(w; z) - \nabla f(\tilde{w}; z) \rangle.
$$

According to Lemma D.2, we know

$$
\|\nabla f(w; z) - \nabla f(\tilde{w}; z)\|^2_2 \leq \left( \frac{L}{\alpha} (1 + \alpha) \right)^\frac{2\alpha}{2-\alpha} \|w - \tilde{w}, \nabla f(w; z) - \nabla f(\tilde{w}; z)\|^2_2 + \|w - \tilde{w}\|^2_2 + \eta^2 \|\nabla f(w; z) - \nabla f(\tilde{w}; z)\|^2_2.
$$

where we have used Young’s inequality (A.5). Plugging the above inequality back into (D.3), we derive

$$
\|w - \eta \nabla f(w; z) - \tilde{w} + \eta \nabla f(\tilde{w}; z)\|^2_2 \leq \|w - \tilde{w}\|^2_2 + \frac{1 - \alpha}{1 + \alpha} \eta^{2+\frac{2\alpha}{1-\alpha}} (2^{-\alpha} L)^{\frac{1}{2-\alpha}},
$$

The proof is complete with the definition of $c_{\alpha, \eta}$ given in Theorem D.1. \qed

Based on Lemma D.3, we establish stability bounds of models for SGD applied to two sets differing by a single example.

**Lemma D.4.** Assume for all $z \in \mathcal{Z}$, the function $w \mapsto f(w; z)$ is nonnegative and convex, and $\nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1)$. Let $S, \tilde{S}$ and $S^{(i)}$ be constructed as Definition 4. Let $\{w_t\}$ and $\{w_t^{(i)}\}$ be produced by (3.3) based on $S$ and $S^{(i)}$, respectively. Then for any $p > 0$ we have

$$
E_A \left[ \|w_{t+1} - w_{t+1}^{(i)}\|_2^2 \right] \leq c^2_{\alpha, \eta} \sum_{j=1}^t (1 + p/n)^{t+1-j} \eta^\frac{2}{1-\alpha} + \frac{2(1+p-1)c^2_{\alpha, \eta}}{n} \sum_{j=1}^t (1 + p/n)^{t+1-j} \eta^\frac{2}{1-\alpha} E_A \left[ f^{\frac{2\alpha}{1-\alpha}}(w_j; z_i) + f^{\frac{2\alpha}{1-\alpha}}(w_j; z_i) \right].
$$

**Proof.** For the case $i_t \neq i$, it follows from Lemma D.3 that

$$
\|w_{t+1} - w_{t+1}^{(i)}\|^2_2 \leq \|w_t - \eta_t \nabla f(w_t; z_t) - w_t^{(i)} + \eta_t \nabla f(w_t^{(i)}; z_t)\|^2_2 \leq \|w_t - w_t^{(i)}\|^2_2 + c^2_{\alpha, \eta} \eta_t^{\frac{2}{1-\alpha}}.
$$

If $i_t = i$, analogous to the derivation of (C.4) and using the standard inequality $(a + b)^2 \leq (1 + p)a^2 + (1 + 1/p)b^2$, we get

$$
\|w_{t+1} - w_{t+1}^{(i)}\|^2_2 \leq (1 + p)\|w_t - \eta_t \nabla f(w_t; z_t) - w_t^{(i)} + \eta_t \nabla f(w_t^{(i)}; z_t)\|^2_2 + (1 + p - 1)\eta_t^2 \|\nabla f(w_t; z_t) - \nabla f(w_t^{(i)}; z_t)\|^2_2,
$$

where the last step is due to Lemma D.3. Combining the above two inequalities together, using the self-bounding property (Lemma A.1) and noticing the distribution of $i_t$, we derive

$$
E_A \left[ \|w_{t+1} - w_{t+1}^{(i)}\|^2_2 \right] \leq (1+p/n) \left( E_A \left[ \|w_t - w_t^{(i)}\|^2_2 \right] + c^2_{\alpha, \eta} \eta_t^{\frac{2}{1-\alpha}} \right) + \frac{2(1+p-1)c^2_{\alpha, \eta}}{n} E_A \left[ f^{\frac{2\alpha}{1-\alpha}}(w_t; z_i) + f^{\frac{2\alpha}{1-\alpha}}(w_t; z_i) \right].
$$
Then for all

Theorem 7 can be considered as an instantiation of the following proposition on generalization error bounds with specific

Taking a summation of the above inequality and using \( w_1 = w_1^{(i)} \), we derive

The stated bound then follows. The proof is complete.

**Proof of Theorem D.1.** According to Lemma D.4, we know

It then follows from the concavity of the function \( x \mapsto x^{2n/n} \) and the Jensen’s inequality that

The stated inequality then follows from the definition of \( F_S \) and \( F_S' \). The proof is complete.

**D.2. Generalization errors**

Theorem 7 can be considered as an instantiation of the following proposition on generalization error bounds with specific choices of \( \gamma, T \) and \( \theta \). In this subsection, we first give the proof of Theorem 7 based on Proposition D.5, and then turn to the proof of Proposition D.5.

**Proposition D.5.** Assume for all \( z \in \mathbb{Z} \), the function \( w \mapsto f(w; z) \) is nonnegative, convex, and \( \nabla f(w; z) \) is \((\alpha, L)\)-Hölder continuous with \( \alpha \in (0, 1) \). Let \( \{w_t\} \) be produced by (3.3) with step sizes \( \eta_t = cT^{-\theta}, \theta \in [0, 1] \) satisfying \( \theta \geq (1 - \alpha)/2 \). Then for all \( T \) satisfying \( n = O(T) \) and any \( \gamma > 0 \) we have

**Proof of Theorem 7.** It can be checked that \( \theta \) considered in Parts (a)-(c) satisfy \( \theta \geq (1 - \alpha)/2 \). Therefore Proposition D.5 holds. If \( \gamma = \sqrt{n} \), then by Proposition D.5 we know

\[
E_{S,A}[F(w_T^{(1)})] - F(w^*) = O\left(n^{-1/2}T^{1-\theta} + O(T^{1-\theta}) + \theta - 1\right) + O\left(T^{1-\theta} + O(T^{1-\theta}) + O(T^{1-\theta})\right).
\]
We first prove Part (a). Since $\alpha \geq 1/2$, $\theta = 1/2$ and $T \asymp n$, it follows from (D.5) that
\[ \mathbb{E}_{S,A}[F(w^{(1)}_T)] - F(w^*) = O(n^{-\frac{1}{2}}) + O(n^{\frac{3}{4} + 1 + \frac{1}{\alpha}}) + O(n^{-\frac{3}{2}(\frac{3}{2} - \frac{1}{\alpha})}) = O(n^{-\frac{1}{2}}), \]
where we have used $\frac{3}{2} - \frac{1}{1-\alpha} \leq -\frac{1}{2}$ due to $\alpha \geq 1/2$. This shows Part (a).

We now prove Part (b). Since $\alpha < 1/2$, $T \asymp n^{\frac{2-\alpha}{\alpha}}$ and $\theta = \frac{3-3\alpha}{2(2-\alpha)} \geq \frac{3}{2}$ in (D.5), the following inequalities hold
\[ \sqrt{n}T^{1-\frac{2\theta}{1+\theta}} \asymp \sqrt{n}T^{1-\frac{3-3\alpha}{(2-\alpha)(1+\alpha)}} \asymp \sqrt{n}T^{-\frac{1+\alpha}{2(2-\alpha)(1+\alpha)}} \asymp n^{-\frac{1}{2}}, \]
\[ n^{-\frac{3}{2}} T^{2-2\theta} \asymp n^{-\frac{3}{2}} T^{\frac{4-3\alpha+3\alpha}{2-\alpha}} \asymp n^{-\frac{3}{2}} T^{\frac{1+\alpha}{2-\alpha}} \asymp n^{-\frac{1}{2}}, \]
\[ T^{\theta-1} \asymp T^{\frac{3-3\alpha+4\alpha}{2(2-\alpha)}} \asymp T^{-\frac{1+\alpha}{2(2-\alpha)}} \asymp n^{-\frac{1}{2}}, \]
\[ T^{-\theta} \asymp n^{\frac{2-\alpha}{\alpha}} \asymp n^{\frac{3(3-2\alpha)}{2(2-\alpha)(1+\alpha)}} = O(n^{-\frac{1}{2}}), \]
where we have used $(3\alpha - 3)/(1+\alpha) \leq -1$ due to $\alpha < 1/2$. Furthermore, since $\theta \geq 1/2$ and $\alpha < 1/2$ we know $\frac{3\alpha - 3}{1+\alpha} \leq -\frac{1}{2}$ and $T \asymp n^{\frac{2-\alpha}{\alpha}} \geq n$. Therefore $T^{\frac{a\theta - a\alpha - 2\alpha}{1+\alpha}} = O(T^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}})$. Plugging the above inequalities into (D.5) gives the stated bound in Part (b).

We now turn to Part (c). Since $F(w^*) = 0$, Proposition D.5 reduces to
\[ \mathbb{E}_{S,A}[F(w^{(1)}_T)] - F(w^*) = O\left( n^{-\frac{1}{2}} T^{\frac{2\theta}{1+\theta}} + n^{-\frac{3}{2}} T^{\frac{2-2\theta}{1+\theta}} \right) + O\left( T^{-\frac{1}{2}} \right) + O\left( T^{\frac{a\theta - a\alpha - 2\alpha}{1+\alpha}} \right). \]

With $\gamma = nT^{\theta-1}$, we further get
\[ \mathbb{E}_{S,A}[F(w^{(1)}_T)] - F(w^*) = O\left( n^{-1} T^{\frac{1}{1+\alpha}} \right) + O\left( nT^{-\frac{(1+\alpha)\theta}{1+\alpha}} \right) + O\left( n^{-1} T^{\frac{(\theta-1)(\alpha-1)}{1+\alpha}} \right) + O\left( T^{\frac{a\theta - a\alpha - 2\alpha}{1+\alpha}} \right). \]

For the choice $T = n^{\frac{1}{2+2\theta}}$ and $\theta = \frac{3-3\alpha}{4}$, we know $1 - \theta = (1+\alpha)^2/4$ and therefore
\[ \left( n^{-1} T^{\frac{1}{1+\alpha}} \right)^{\frac{1+\alpha}{1+\alpha}} \asymp \left( n^{-1} n^{\frac{3}{4} + \frac{1+\alpha}{4}} \right)^{\frac{1+\alpha}{1+\alpha}} = n^{-\frac{1+\alpha}{2}}, \]
\[ nT^{-\frac{(1+\alpha)\theta}{1+\alpha}} \asymp n^{-\frac{2\theta}{1+\alpha}} = n^{\frac{2-\alpha+2\alpha}{4(1+\alpha)}} = n^{\frac{2-\alpha}{2(1+\alpha)}} = n^{-\frac{1+\alpha}{2}}, \]
\[ n^{-1} T^{\frac{(\theta-1)(\alpha-1)}{1+\alpha}} \asymp n^{-1} T^{\frac{(1-\alpha)(1+\alpha)^2}{4(1+\alpha)}} \asymp n^{-1} T^{\frac{(1-\alpha)(1+\alpha)^2}{4}} \asymp n^{-\frac{1+\alpha}{2}}, \]
\[ T^{\theta-1} \asymp n^{-\frac{2\alpha}{1+\alpha}} + n^{-\frac{1+\alpha}{2}}. \]

Furthermore,
\[ (\theta - 1)(1+\alpha) - (a\theta - \theta - 2\alpha) = 2\theta + \alpha - 1 = 2^{-1}(3 - \alpha^2 - 2\alpha + 2\alpha - 2) \geq 0 \]
and therefore
\[ T^{\theta-1} \geq n^{-\frac{a\theta - a\alpha - 2\alpha}{1+\alpha}}. \]
Plugging the above inequalities into (D.6) gives the stated bound in Part (c). The proof is complete.

To prove Proposition D.5, we first introduce an useful lemma to address some involved series.

**Lemma D.6.** Assume for all $z \in \mathbb{R}$, the function $w \mapsto f(w; z)$ is nonnegative, convex, and $\nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1)$. Let $\{w_t\}$ be produced by (3.3) with step sizes $\eta_t = cT^{-\theta}$, $\theta \in [0, 1]$ satisfying $\theta \geq \frac{1-\alpha}{2}$. Then
\[ \left( \sum_{t=1}^{T} \eta_t^2 \right)^{\frac{1}{1+\alpha}} \left( \eta_t \|w^*\|_2^2 + 2 \sum_{t=1}^{T} \eta_t^2 F(w^*) + c_{\alpha, 2} \sum_{t=1}^{T} \eta_t^{\frac{3}{2}} \right)^{\frac{2\alpha}{1+\alpha}} = O(T^{\frac{1-\alpha - 2\theta}{1+\alpha}}) + O(T^{1-2\theta} F^{\frac{2\alpha}{1+\alpha}}(w^*)), \]
\[ \sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)]^{\frac{2\alpha}{1+\alpha}} = O(T^{\frac{1-\alpha - 2\theta}{1+\alpha}}) + O(T^{1-2\theta} F^{\frac{2\alpha}{1+\alpha}}(w^*)), \]
\[ \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t)]^{\frac{2\alpha}{1+\alpha}} = O(T^{\frac{(1-\alpha)(1-\theta)}{1+\alpha}}) + O(T^{1-\theta} F^{\frac{2\alpha}{1+\alpha}}(w^*)). \]
We now consider (D.8). Taking an expectation over both sides of (A.6) with where we have used (D.7) in the last step. This shows (D.8).

According to the Jensen’s inequality and the concavity of \( x \), we get

\[
\sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)] \leq \eta_1 \|w^*\|_2^2 + 2 \sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S}[F^*(w)] + c_{\alpha,2} \sum_{t=1}^{T} \eta_t^{\frac{3-\alpha}{\alpha}}.
\]

According to the Jensen’s inequality and the concavity of \( x \), we know

\[
\sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)] \leq \sum_{t=1}^{T} \eta_t^2 \left( \sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)] \right)^{\frac{2\alpha}{1+\alpha}} \\
\leq \left( \sum_{t=1}^{T} \eta_t^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left( \eta_1 \|w^*\|_2^2 + 2 \sum_{t=1}^{T} \eta_t^2 F(w) + c_{\alpha,2} \sum_{t=1}^{T} \eta_t^{\frac{3-\alpha}{\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \\
= O(T^{\frac{1-\alpha-2\theta}{1+\alpha} \log T}) + O(T^{1-\theta} F^{\frac{2\alpha}{1+\alpha}}(w^*)),
\]

where we have used (D.7) in the last step. This shows (D.8).

Finally, we show (D.9). Since we consider step sizes \( \eta_t = cT^{-\theta} \), it follows from (D.8) that

\[
\sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t)] \leq \left( cT^{-\theta} \right)^{-1} \sum_{t=1}^{T} \eta_t^2 \mathbb{E}_{S,A}[F_S(w_t)] \leq O(T^{\frac{1-\alpha-2\theta}{1+\alpha}} + O(T^{1-\theta} F^{\frac{2\alpha}{1+\alpha}}(w^*)�
\]

This proves (D.9) and finishes the proof.

**Proof of Proposition D.5.** Since \( \mathbb{E}_{S}[F_S(w^*)] = F(w^*) \), we can decompose the excess generalization error into an estimation error and an optimization error as follows

\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t (\mathbb{E}_{S,A}[F(w_t)] - F(w^*)) = \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t) - F_S(w_t)] \\
+ \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t) - F_S(w^*)]. \tag{D.10}
\]

Our idea is to address separately the above estimation error and optimization error.
We first address **estimation errors**. Taking expectations on both sides of (D.1) and noticing \( \mathbb{E}_S[F_S^{\frac{2\alpha}{T}}(w_j)] \leq F_S^{\frac{2\alpha}{T}}(w_j) \), we get

\[
\mathbb{E}_{S,A} \left[ \frac{1}{n} \sum_{t=1}^{n} \|w_{t+1} - w_{t+1}^{(i)}\|^2 \right] \leq c_{\alpha,3}^2 \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} + 2(1 + p^{-1})c_{\alpha,1} \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} \mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) + F_S^{\frac{2\alpha}{T}}(w_j) \right].
\]

Plugging the above inequality back into Theorem 2 (Part (c)) with \( A(S) = w_{t+1} \), we derive

\[
\mathbb{E}_{S,A} \left[ F(w_{t+1}) - F_S(w_{t+1}) \right] \leq \frac{c_{\alpha,1}}{2\gamma} \mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) \right] + 2^{-1} \gamma c_{\alpha,3}^2 \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} + 2(1 + p^{-1}) \gamma c_{\alpha,1} \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} \mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) + F_S^{\frac{2\alpha}{T}}(w_j) \right].
\]

By the concavity and sub-additivity of \( x \mapsto x^{\frac{2\alpha}{T}} \), we know

\[
\mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) \right] \leq (\mathbb{E}_{S,A}[F(w_j)] - \mathbb{E}_{S,A}[F_S(w_j)]) + \mathbb{E}_{S,A}[F_S(w_j)] \frac{2\alpha}{T} \leq \delta_j \frac{2\alpha}{T} + (\mathbb{E}_{S,A}[F_S(w_j)]) \frac{2\alpha}{T},
\]

where we denote \( \delta_j = \max\{\mathbb{E}_{S,A}[F(w_j)] - \mathbb{E}_{S,A}[F_S(w_j)], 0\} \) for all \( j \in \mathbb{N} \). It then follows from \( p = n/T \) that

\[
\delta_{t+1} \leq \frac{c_{\alpha,1}}{2\gamma} \left( \frac{\delta_{t+1}}{n} + (\mathbb{E}_{S,A}[F_S(w_{t+1})]) \frac{2\alpha}{T} \right) + 2^{-1} \gamma c_{\alpha,3}^2 \sum_{j=1}^{t} \frac{(1 + T/n)^{t+1-j} \eta_j^{2}}{n} + \frac{2(1 + p^{-1}) \gamma c_{\alpha,1} \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} \mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) + F_S^{\frac{2\alpha}{T}}(w_j) \right]}{n}.
\]

For all \( t \leq T \), from which we get

\[
\delta_{t+1} \leq \tilde{C}_{t,1} \max_{1 \leq j \leq t+1} \delta_j \frac{2\alpha}{T} + \tilde{C}_{t,2},
\]

where \( \tilde{C}_{t,1} = c_{\alpha,1} \left( \frac{c_1(1+T/n)}{n} \sum_{j=1}^{t} \eta_j^{2} + \frac{1}{2\gamma} \right) \) and

\[
\tilde{C}_{t,2} = 2^{-1} \gamma c_{\alpha,3}^2 \sum_{j=1}^{t} \frac{(1 + T/n)^{t+1-j} \eta_j^{2}}{n} + 2\gamma c_{\alpha,1} \left( 1 + \frac{T}{n} \right) \sum_{j=1}^{t} \frac{(1 + p/n)^{t+1-j} \eta_j^{2}}{n} \mathbb{E}_{S,A} \left[ F_S^{\frac{2\alpha}{T}}(w_j) \right] \frac{2\alpha}{T} + \frac{c_{\alpha,1}}{2\gamma} \left( \mathbb{E}_{S,A}[F_S(w_{t+1})]) \frac{2\alpha}{T} \right).
\]

Since Eq. (D.11) holds for all \( t \) and the right-hand side is an increasing function of \( t \), we know

\[
\max_{1 \leq j \leq t+1} \delta_j \leq \tilde{C}_{t,1} \max_{1 \leq j \leq t+1} \delta_j \frac{2\alpha}{T} + \tilde{C}_{t,2} \leq 2 \max \left\{ \tilde{C}_{t,1}, \max_{1 \leq j \leq t+1} \delta_j \frac{2\alpha}{T}, \tilde{C}_{t,2} \right\},
\]

from which we know

\[
\max_{1 \leq j \leq t+1} \delta_j \leq \max \left\{ \left( 2\tilde{C}_{t,1} \right) \frac{1+\alpha}{T}, 2\tilde{C}_{t,2} \right\}.
\]

It then follows from the definition of \( \delta_j \) that

\[
\left( \sum_{i=1}^{T} \frac{1}{\eta_i} \right)^{-1} \sum_{i=1}^{T} \eta_i \left( \mathbb{E}_{S,A}[F(w_i)] - \mathbb{E}_{S,A}[F_S(w_i)] \right) \leq \left( \sum_{i=1}^{T} \frac{1}{\eta_i} \right)^{-1} \sum_{i=1}^{T} \eta_i \left( 2\tilde{C}_{t,1} \right) \frac{1+\alpha}{T} + 2\tilde{C}_{t-1,2}. \tag{D.12}
\]

For \( \eta_i = cT^{-\theta} \) we know \( \tilde{C}_{t,1} \) is increasing w.r.t. \( t \) and \( T = O(n) \)

\[
\left( \sum_{i=1}^{T} \frac{1}{\eta_i} \right)^{-1} \sum_{i=1}^{T} \eta_i \left( 2\tilde{C}_{t,1} \right) \frac{1+\alpha}{T} \leq \left( 2\tilde{C}_{T,1} \right) \frac{1+\alpha}{T} = O \left( \left( n^{-2}T^{-1-2\theta} + \gamma^{-1} \right) \frac{1+\alpha}{T} \right). \tag{D.13}
\]
and (notice the first two terms of $\tilde{C}_{t,2}$ are increasing w.r.t. $t$)

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \tilde{C}_{t,1,2}
= O(\gamma) \left(\sum_{t=1}^{T} \eta_t \frac{1}{n^2} + n^{-2} T \sum_{t=1}^{T} \eta_t^2 (\mathbb{E}_{S,A}[F_S(w_t)])^\frac{2\alpha}{1+n} + O(\frac{1}{\gamma}) \sum_{t=1}^{T} \eta_t \left(\mathbb{E}_{S,A}[F_S(w_t)]\right)^\frac{2\alpha}{1+n}\right)
= O(\gamma) \left(T^{1-\frac{2\alpha}{1+n}} + n^{-2} T \sum_{t=1}^{T} \eta_t^2 (\mathbb{E}_{S,A}[F_S(w_t)])^\frac{2\alpha}{1+n} + O\left(T^{-\gamma}\frac{1}{\gamma - n\gamma} + T^{1-\theta} F^{\frac{2\alpha}{1+n}} (w^*)\right)\right)
= O(\gamma) \left(T^{1-\frac{2\alpha}{1+n}} + n^{-2} T \sum_{t=1}^{T} \eta_t^2 (\mathbb{E}_{S,A}[F_S(w_t)])^\frac{2\alpha}{1+n} + O\left(T^{-\gamma}\frac{1}{\gamma - n\gamma} + T^{1-\theta} F^{\frac{2\alpha}{1+n}} (w^*)\right)\right).
$$

We now consider optimization errors. By Lemma A.2 (Part (d) with $w = w^*$) and the concavity of $x \mapsto x^\frac{2\alpha}{1+n}$, we know

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F_S(w_t) - F_S(w^*)]
\leq \left(\sum_{t=1}^{T} \eta_t\right)^{-1} \mathbb{E}_{S,A}[F_S(w_t) - F_S(w^*)]
= O\left(T^{1-\gamma} \frac{2\alpha}{1+n} + n^{-2} T \sum_{t=1}^{T} \eta_t^2 F(w^*) + c_{\alpha,2} \sum_{t=1}^{T} \eta_t^2 \right)^\frac{2\alpha}{1+n}
= O(T^{\gamma - \theta} + O(T^{\frac{1-\gamma}{\gamma - n\gamma} + \theta}) + O(T^{-\gamma}\frac{1}{\gamma - n\gamma} + T^{1-\theta} F^{\frac{2\alpha}{1+n}} (w^*)�
$$

where we have used (D.7) in the last step.

Plugging the above optimization error bound and the estimation error bound (D.12), (D.13), (D.14) back into the error decomposition (D.10), we finally derive the following generalization error bounds

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \mathbb{E}_{S,A}[F(w_t) - F(w^*)] = O\left((n^{-2}\gamma T^{2-2\theta} + \gamma^{-1})^{1+n}\right)
+ O\left(T^{1-\frac{2\alpha}{1+n}} + n^{-2} T \sum_{t=1}^{T} \eta_t^2 F(w^*) + c_{\alpha,2} \sum_{t=1}^{T} \eta_t^2 \right)^\frac{2\alpha}{1+n}.
$$

The stated inequality then follows from the convexity of $F$. The proof is complete.

D.3. Empirical Risk Minimization with Strongly Convex Objectives

In this section, we present an optimistic bound for ERM with strongly convex objectives based on the $\ell_2$ on-average model stability. We consider nonnegative and convex loss functions with Hölder continuous gradients.

**Proposition D.7.** Assume for any $z$, the function $w \mapsto f(w; z)$ is nonnegative, convex and $w \mapsto \nabla f(w; z)$ is $(\alpha, L)$-Hölder continuous with $\alpha \in (0, 1]$. Let $A$ be the ERM algorithm, i.e., $A(S) = \arg\min_{w \in \mathbb{R}^d} F_S(w)$. If for all $S$, $F_S$ is $\sigma$-strongly convex, then

$$
\mathbb{E}_{S} \left[F(A(S)) - F_S(A(S))\right] \leq \frac{2\alpha_2}{n\sigma} \mathbb{E}_{S} \left[F^{\frac{2\alpha}{1+n}} (A(S))\right].
$$

**Proof.** Let $\tilde{S}$ and $S^{(i)}, i = 1, \ldots, n$, be constructed as Definition 4. Due to the $\sigma$-strong convexity of $F_{S^{(i)}}$ and $\nabla F_{S^{(i)}}(A(S^{(i)})) = 0$ (necessity condition for the optimality of $A(S^{(i)})$), we know

$$
F_{S^{(i)}}(A(S)) - F_{S^{(i)}}(A(S^{(i)})) \geq 2^{-1} \sigma \|A(S) - A(S^{(i)})\|_2^2.
$$

Taking a summation of the above inequality yields

$$
\frac{1}{n} \sum_{i=1}^{n} \left(F_{S^{(i)}}(A(S)) - F_{S^{(i)}}(A(S^{(i)}))\right) \geq \frac{\sigma}{2n} \sum_{i=1}^{n} \|A(S) - A(S^{(i)})\|_2^2.
$$

(D.15)
According to the definition of $S^{(i)}$, we know
\[
n \sum_{i=1}^{n} F_{S^{(i)}}(A(S)) = \sum_{i=1}^{n} \left( \sum_{j \neq i} f(A(S); z_j) + f(A(S); \tilde{z}_i) \right) = (n - 1) \sum_{j=1}^{n} f(A(S); z_j) + \sum_{i=1}^{n} f(A(S); \tilde{z}_i) = (n - 1)nF_S(A(S)) + nF_S(A(S)).
\]
Taking an expectation and dividing both sides by $n^2$ give ($A(S)$ is independent of $\tilde{S}$)
\[
\frac{1}{n} \mathbb{E}_{S, \tilde{S}} \left[ \sum_{i=1}^{n} F_{S^{(i)}}(A(S)) \right] = \frac{n - 1}{n} \mathbb{E}_S [F_S(A(S))] + \frac{1}{n} \mathbb{E}_S [F(A(S))].
\]

Furthermore, by symmetry we know
\[
\frac{1}{n} \mathbb{E}_{S, \tilde{S}} \left[ \sum_{i=1}^{n} F_{S^{(i)}}(A(S)) \right] = \mathbb{E}_S [F_S(A(S))].
\]
Plugging the above identity and (D.16) back into (D.15) gives
\[
\frac{\sigma}{2n} \sum_{i=1}^{n} \mathbb{E}_{S, \tilde{S}} \left[ \|A(S^{(i)}) - A(S)\|_2^2 \right] \leq \frac{1}{n} \mathbb{E}_{S, \tilde{S}} \left[ (n - 1)F_S(A(S)) + nF_S(A(S)) - F_S(A(S)) \right].
\]

We can now apply Part (c) of Theorem 2 to show the following inequality for all $\gamma > 0$ (notice $A$ is a deterministic algorithm)
\[
\mathbb{E}_S \left[ F(A(S)) - F_S(A(S)) \right] \leq \frac{c_{\alpha, 1}}{2\gamma} \mathbb{E}_S \left[ F^{\frac{1}{1+\alpha}}(A(S)) \right] + \frac{\gamma}{n\sigma} \mathbb{E}_S \left[ F(A(S)) - F_S(A(S)) \right].
\]
Taking $\gamma = n\sigma/2$, we derive
\[
\mathbb{E}_S \left[ F(A(S)) - F_S(A(S)) \right] \leq \frac{c_{\alpha, 1}}{n\sigma} \mathbb{E}_S \left[ F^{\frac{1}{1+\alpha}}(A(S)) \right] + \frac{1}{2} \mathbb{E}_S \left[ F(A(S)) - F_S(A(S)) \right],
\]
from which we can derive the stated inequality. The proof is complete.

\[
\square
\]

**E. Proofs on Stability with Relaxed Convexity**

**E.1. Stability and generalization errors**

For any convex $g$, we have (Nesterov, 2013)
\[
\langle w - \tilde{w}, \nabla g(w) - \nabla g(\tilde{w}) \rangle \geq 0, \quad w, \tilde{w} \in \mathbb{R}^d.
\]

**Proof of Theorem 8.** Without loss of generality, we can assume that $S$ and $\tilde{S}$ differ by the first example, i.e., $z_1 \neq \tilde{z}_1$ and $z_i = \tilde{z}_i, i \neq 1$. According to the update rule (3.3) and (A.1), we know
\[
\|w_{t+1} - \tilde{w}_{t+1}\|_2^2 \leq \|w_t - \eta_t \nabla f(w_t; z_i) - \tilde{w}_t + \eta_t \nabla f(\tilde{w}_t; \tilde{z}_i)\|_2^2
\]
\[
= \|w_t - \tilde{w}_t\|_2^2 + \eta_t^2 \|\nabla f(w_t; z_i) - \nabla f(\tilde{w}_t; \tilde{z}_i)\|_2^2 + 2\eta_t (w_t - \tilde{w}_t, \nabla f(\tilde{w}_t; \tilde{z}_i) - \nabla f(w_t; z_i)).
\]

We first study the term $\|\nabla f(w_t; z_i) - \nabla f(\tilde{w}_t; \tilde{z}_i)\|_2^2$. The event $i_t \neq 1$ happens with probability $1 - 1/n$, and in this case it follows from the smoothness of $f$ that ($z_{i_t} = \tilde{z}_{i_t}$)
\[
\|\nabla f(w_t; z_i) - \nabla f(\tilde{w}_t; \tilde{z}_i)\|_2 \leq L\|w_t - \tilde{w}_t\|_2.
\]
The event $i_t = 1$ happens with probability $1/n$, and in this case
\[
\|\nabla f(w_t; z_i) - \nabla f(\tilde{w}_t; \tilde{z}_i)\|_2 \leq \|\nabla f(w_t; z_i)\|_2 + \|\nabla f(\tilde{w}_t; \tilde{z}_i)\|_2 \leq 2G.
\]
Therefore, we get
\[ \mathbb{E}_{t_i} \left[ \| \nabla f(w_t; z_{t_i}) - \nabla f(\bar{w}_t; \bar{z}_{t_i}) \|_2^2 \right] \leq \frac{ (n-1) L^2 }{ n } \| w_t - \bar{w}_t \|_2^2 + \frac{4G^2}{ n }. \]  
(E.3)

It is clear
\[ \mathbb{E}_{t_i} \left[ f(w_t; z_{t_i}) \right] = F_S(w_t) \quad \text{and} \quad \mathbb{E}_{t_i} \left[ f(\bar{w}_t; \bar{z}_{t_i}) \right] = F_S(\bar{w}_t). \]

Therefore, by (E.1) we derive
\[ \mathbb{E}_{t_i} \left[ (w_t - \bar{w}_t, \nabla f(\bar{w}_t; \bar{z}_{t_i}) - \nabla f(w_t; z_{t_i})) \right] = (w_t - \bar{w}_t, \nabla F_S(\bar{w}_t) - \nabla F_S(w_t)) \]
\[ = \frac{1}{n} (w_t - \bar{w}_t, \nabla f(\bar{w}_t; \bar{z}_{1}) - \nabla f(w_t; z_{1})) + (w_t - \bar{w}_t, \nabla F_S(\bar{w}_t) - \nabla F_S(w_t)) \]
\[ \leq \frac{2G \| w_t - \bar{w}_t \|_2}{n}. \]  
(E.4)

Plugging (E.3) and the above inequality back into (E.2), we derive
\[ \mathbb{E}_{t_i} \left[ \| w_{t+1} - \bar{w}_{t+1} \|_2^2 \right] \leq \| w_t - \bar{w}_t \|_2^2 + \frac{4G\eta_t \| w_t - \bar{w}_t \|_2}{n} + \eta_t^2 \left( \frac{ (n-1) L^2 }{ n } \| w_t - \bar{w}_t \|_2^2 + \frac{4G^2}{ n } \right) \]

and therefore
\[ \mathbb{E}_A \left[ \| w_{t+1} - \bar{w}_{t+1} \|_2^2 \right] \leq (1 + L^2 \eta_t^2) \mathbb{E}_A \left[ \| w_t - \bar{w}_t \|_2^2 \right] + 4G \left( \frac{ \eta_t \mathbb{E}_A \| w_t - \bar{w}_t \|_2 }{ n } + \frac{G \eta_t^2}{ n } \right). \]

(E.5)

By the above recurrence relationship and \( w_1 = \bar{w}_1 \), we derive
\[ \mathbb{E}_A \left[ \| w_{t+1} - \bar{w}_{t+1} \|_2^2 \right] \leq 4G \sum_{j=1}^{t} \prod_{j=j+1}^{t} \left( 1 + L^2 \eta_j^2 \right) \left( \frac{ \eta_j \mathbb{E}_A \| w_j - \bar{w}_j \|_2 }{ n } + \frac{G \eta_j^2}{ n } \right) \]
\[ \leq 4G \prod_{j=1}^{t} \left( 1 + L^2 \eta_j^2 \right) \sum_{j=1}^{t} \left( \frac{ \eta_j \max_{1 \leq j \leq t} \mathbb{E}_A \| w_j - \bar{w}_j \|_2 }{ n } + \frac{G \eta_j^2}{ n } \right). \]

Since the above inequality holds for all \( t \in \mathbb{N} \) and the right-hand side is an increasing function of \( t \), we get
\[ \max_{1 \leq j \leq t+1} \mathbb{E}_A \left[ \| w_j - \bar{w}_j \|_2^2 \right] \leq 4GC_t \sum_{j=1}^{t} \left( \frac{ \eta_j \max_{1 \leq j \leq t+1} \mathbb{E}_A \| w_j - \bar{w}_j \|_2 }{ n } + \frac{G \eta_j^2}{ n } \right). \]

It then follows that (note \( \mathbb{E}_A \| w_j - \bar{w}_j \|_2 \leq (\mathbb{E}_A \| w_j - \bar{w}_j \|_2^2)^{\frac{1}{2}} \))
\[ \max_{1 \leq j \leq t+1} \mathbb{E}_A \left[ \| w_j - \bar{w}_j \|_2^2 \right] \leq 4GC_t \sum_{j=1}^{t} \frac{ \eta_j }{ n } \max_{1 \leq j \leq t+1} \left( \mathbb{E}_A \| w_j - \bar{w}_j \|_2^2 \right)^{\frac{1}{2}} + 4G^2C_t \sum_{j=1}^{t} \frac{ \eta_j^2 }{ n }. \]

Solving the above quadratic function of \( \max_{1 \leq j \leq t+1} \left( \mathbb{E}_A \| w_j - \bar{w}_j \|_2^2 \right)^{\frac{1}{2}} \) then shows
\[ \max_{1 \leq j \leq t+1} \left( \mathbb{E}_A \| w_{j+1} - \bar{w}_{j+1} \|_2^2 \right)^{\frac{1}{2}} \leq 4GC_t \sum_{j=1}^{t} \frac{ \eta_j }{ n } + 2G \left( C_t \sum_{j=1}^{t} \frac{ \eta_j^2 }{ n } \right)^{\frac{1}{2}}. \]

The proof is complete. \( \square \)

To prove Theorem 9, we require a basic result on series.

**Lemma E.1.** We have the following elementary inequalities.
We denote by \( \epsilon_{\text{stab}}(A, n) \) the infimum over all \( \epsilon \) for which (3.2) holds, and omit the tuple \((A, n)\) when it is clear from the context.

**Proof of Theorem 9.** For the step sizes considered in both Part (a) and Part (b), one can check that \( \sum_{t=1}^T \eta_t^2 \) can be upper bounded by a constant independent of \( T \). Therefore, \( C_t < C \) for all \( t = 1, \ldots, T \) and a universal constant \( C \). We can apply Lemma A.2 (Part (a)) on optimization errors to get

\[
\mathbb{E}_A[F_S(w_T^{(1)})] - F_S(w^*) = O\left( \frac{\sum_{t=1}^T \eta_t^2 + \|w^*\|^2_2}{\sum_{t=1}^T \eta_t} \right). \tag{E.6}
\]

By the convexity of norm, we know

\[
\mathbb{E}_A[\|w_T^{(1)} - w_T^{(1)}\|_2] \leq \left( \sum_{t=1}^T \eta_t \right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_A[\|w_t - \bar{w}_t\|_2] = O\left( \sum_{t=1}^T \eta_t + n^{-\frac{1}{2}} \left( \sum_{t=1}^T \eta_t^2 \right)^{\frac{1}{2}} \right),
\]

where we have applied Theorem 8 (the upper bound in Theorem 8 is an increasing function of \( t \)). It then follows from the Lipschitz continuity that \( \epsilon_{\text{stab}} = O\left( \sum_{t=1}^T \frac{n}{\eta_t^2} + n^{-\frac{1}{2}} \left( \sum_{t=1}^T \eta_t^2 \right)^{\frac{1}{2}} \right) \). This together with the error decomposition (3.1), Lemma 1 and the optimization error bound (E.6) shows

\[
\mathbb{E}_{S,A}[F(w_T^{(1)})] - F(w^*) = O\left( \sum_{t=1}^T \frac{\eta_t}{n} + n^{-\frac{1}{2}} \left( \sum_{t=1}^T \eta_t^2 \right)^{\frac{1}{2}} \right) + O\left( \sum_{t=1}^T \frac{\eta_t^2 + \|w^*\|^2_2}{\sum_{t=1}^T \eta_t} \right). \tag{E.7}
\]

For the step sizes \( \eta_t = \eta_t t^{-\theta} \) with \( \theta \in (1/2, 1) \), we can apply Lemma E.1 to show

\[
\mathbb{E}_{S,A}[F(w_T^{(1)})] - F(w^*) = O\left( \sum_{t=1}^T \frac{t^{-\theta}}{n} + n^{-\frac{1}{2}} \left( \sum_{t=1}^T t^{-(2\theta)} \right)^{\frac{1}{2}} \right) + O\left( \frac{\sum_{t=1}^T t^{-2\theta} + \|w^*\|^2_2}{\sum_{t=1}^T t^{-\theta}} \right)
\]

\[
= O\left( n^{-1} T^{1-\theta} + n^{-\frac{1}{2}} + T^{\theta-1} \right).
\]

This proves the first part.

Part (b) follows by plugging (C.9) into (E.7). The proof is complete.

\[\square\]

**F. Proofs on Stability with Relaxed Strong Convexity**

**Proof of Theorem 10.** Due to the \( \sigma_S \)-strong convexity of \( F_S \), we can analyze analogously to (E.4) to derive

\[
\mathbb{E}_t\left[ (w_t - \bar{w}_t, \nabla f(\bar{w}_t; z_i) - \nabla f(w_t; z_i)) \right] \leq \frac{2G\|w_t - \bar{w}_t\|_2}{n} - \sigma_S\|w_t - \bar{w}_t\|_2^2.
\]

Therefore, analogous to the derivation of (E.5) we can derive

\[
\mathbb{E}_A[\|w_{t+1} - \bar{w}_{t+1}\|_2^2] \leq (1 + L^2 \eta_t^2 - 2\sigma_S \eta_t)\mathbb{E}_A[\|w_t - \bar{w}_t\|_2^2] + 4G\left( \frac{G\eta_t^2}{n} + \eta_t \mathbb{E}_A[\|w_t - \bar{w}_t\|_2] \right)
\]

\[
\leq (1 + L^2 \eta_t^2 - \frac{3}{2} \sigma_S \eta_t)\mathbb{E}_A[\|w_t - \bar{w}_t\|_2^2] + \frac{4G^2 \eta_t^2}{n} + \frac{8G^2 \eta_t}{n^2 \sigma_S},
\]

where we have used

\[
\frac{4G}{n} \mathbb{E}_A[\|w_t - \bar{w}_t\|_2] \leq \frac{8G^2}{n^2 \sigma_S} + \frac{\sigma_S \mathbb{E}_A[\|w_t - \bar{w}_t\|_2]}{2}.
\]
We find \( t_0 \geq 4L^2/\sigma_S^2 \). Then \( \eta_t \leq \sigma_S/(2L^2) \) and it follows that
\[
\mathbb{E}_A[\|w_{t+1} - \tilde{w}_{t+1}\|^2] \leq (1 - \sigma_S\eta_t)\mathbb{E}_A[\|w_t - \tilde{w}_t\|^2] + \frac{4G^2\eta_t^2}{n} + \frac{8G^2\eta_t}{n^2\sigma_S}.
\]
Multiplying both sides by \((t + t_0)(t + t_0 - 1)\) yields
\[
(t + t_0)(t + t_0 - 1)\mathbb{E}_A[\|w_{t+1} - \tilde{w}_{t+1}\|^2] \leq (t + t_0 - 1)(t + t_0 - 2)\mathbb{E}_A[\|w_t - \tilde{w}_t\|^2] + \frac{8G^2(t + t_0 - 1)}{n\sigma_S}(\eta_t + \frac{2}{n\sigma_S}).
\]
Taking a summation of the above inequality and using \( w_1 = \tilde{w}_1 \) then give
\[
(t + t_0)(t + t_0 - 1)\mathbb{E}_A[\|w_{t+1} - \tilde{w}_{t+1}\|^2] \leq \frac{8G^2}{n\sigma_S} \sum_{j=1}^t (j + t_0 - 1)(\eta_j + \frac{2}{n\sigma_S})
\]
\[
= \frac{8G^2}{n\sigma_S} \left( \sum_{j=1}^t (j + t_0 - 1)\eta_j + \frac{2}{n\sigma_S} \sum_{j=1}^t (j + t_0 - 1) \right)
\]
\[
\leq \frac{8G^2}{n\sigma_S} \left( \frac{2t}{\sigma_S} + \frac{t(t + 2t_0 - 1)}{n\sigma_S} \right).
\]
It then follows
\[
\mathbb{E}_A[\|w_{t+1} - \tilde{w}_{t+1}\|^2] \leq \frac{16G^2}{n\sigma_S^2} \left( \frac{1}{t + t_0} + \frac{1}{n} \right).
\]
The stated bound then follows from the elementary inequality \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \). The proof is complete.

Proof of Theorem 11. By the convexity of norm, we know
\[
\mathbb{E}_A[\|w_{T}^{(2)} - \tilde{w}_{T}^{(2)}\|_2] \leq \left( \sum_{t=1}^T (t + t_0 - 1) \right)^{-1} \sum_{t=1}^T (t + t_0 - 1) \mathbb{E}_A[\|w_t - \tilde{w}_t\|_2]
\]
\[
\leq \frac{4G}{\sigma_S} \left( \sum_{t=1}^T (t + t_0 - 1) \right)^{-1} \sum_{t=1}^T (t + t_0 - 1) \left( \frac{1}{\sqrt{n(t + t_0)}} + \frac{1}{n} \right)
\]
\[
= O(\sigma_S^{-1}(nT)^{-\frac{1}{2}} + n^{-1}),
\]
where we have used Lemma E.1 in the last step. Since the above bound holds for all \( S, \tilde{S} \) differing by a single example, it follows that \( \ell_1 \) on-average model stability is bounded by \( O(\mathbb{E}_S[\sigma_S^{-1}((nT)^{-\frac{1}{2}} + n^{-1})) \). By Part (b) of Lemma A.2 we know
\[
\mathbb{E}_A[F_S(w_{T}^{(2)})] - F_S(w^*) = O(1/(T\sigma_S) + \|w^*\|^2/T^2).
\]
It then follows from (3.1) and Part (a) of Theorem 2 that
\[
\mathbb{E}_{S,A}[F(w_{T}^{(2)})] - F(w^*) = O(\mathbb{E}_S[\sigma_S^{-1}((nT)^{-\frac{1}{2}} + n^{-1})]] + O(\mathbb{E}_S[1/(T\sigma_S)] + 1/T^2).
\]
The stated bound holds since \( T \asymp n \). The proof is complete.

Proposition F.1. Let \( S = \{z_1, \ldots, z_n\} \) and \( C_S = \frac{1}{n} \sum_{i=1}^n x_ix_i^T \). Then the range of \( C_S \) is the linear span of \( \{x_1, \ldots, x_n\} \).

Proof. It suffices to show that the kernel of \( C_S \) is the orthogonal complement of \( V = \text{span}\{x_1, \ldots, x_n\} \) (we denote \( \text{span}\{x_1, \ldots, x_n\} \) the linear span of \( x_1, \ldots, x_n \). Indeed, for any \( x \) in the kernel of \( C_S \), we know \( C_Sx = 0 \) and therefore \( x^TC_Sx = \frac{1}{n} \sum_{i=1}^n (x_i^T x)^2 = 0 \), from which we know that \( x \) must be orthogonal to \( V \). Furthermore, for any \( x \) orthogonal to \( V \), it is clear that \( C_Sx = 0 \), i.e., \( x \) belongs to the kernel of \( C_S \). The proof is complete.
G. Extensions

In this section, we present some extensions of our analyses. We consider three extensions: extension to stochastic proximal gradient descent, extension to high probability analysis and extension to SGD without replacement.

G.1. Stochastic proximal gradient descent

Our discussions can be directly extended to study the performance of stochastic proximal gradient descent (SPGD). Let \( r : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) be a convex regularizer. SPGD updates the models by

\[
\mathbf{w}_{t+1} = \text{Prox}_{\eta_t r}(\mathbf{w}_t - \eta_t \nabla f(\mathbf{w}_t; z_{i_t})),
\]

where \( \text{Prox}_{g}(\mathbf{w}) = \arg \min_{\mathbf{w} \in \mathbb{R}^d} [g(\mathbf{w}) + \frac{1}{2} \| \mathbf{w} - \mathbf{w}\|_2^2] \) is the proximal operator. SPGD has found wide applications in solving optimization problems with a composite structure (Parikh & Boyd, 2014). It recovers the projected SGD as a specific case by taking an appropriate \( r \). Our stability bounds for SGD can be trivially extend to SPGD due to the non-expansiveness of proximal operators:

\[
\|\text{Prox}_{g}(\mathbf{w}) - \text{Prox}_{g}(\tilde{\mathbf{w}})\|_2 \leq \|\mathbf{w} - \tilde{\mathbf{w}}\|_2, \forall \mathbf{w}, \tilde{\mathbf{w}} \text{ if } g \text{ is convex}.
\]

G.2. Stability bounds with high probabilities

We can also extend our stability bounds stated in expectation to high-probability bounds, which would be helpful to understand the fluctuation of SGD w.r.t. different realization of random indices.

**Proposition G.1.** Let Assumption 1 hold. Assume for all \( z \in \mathcal{Z} \), the function \( \mathbf{w} \mapsto f(\mathbf{w}; z) \) is convex and \( \mathbf{w} \mapsto \nabla f(\mathbf{w}; z) \) is \((\alpha, L)\)-Hölder continuous with \( \alpha \in (0, 1] \). Let \( S = \{z_1, \ldots, z_n\} \) and \( \tilde{S} = \{\tilde{z}_1, \ldots, \tilde{z}_n\} \) be two sets of training examples that differ by a single example. Let \( \{\mathbf{w}_t\}_t \) and \( \{\tilde{\mathbf{w}}_t\}_t \) be produced by (3.3) based on \( S \) and \( \tilde{S} \), respectively, and \( \delta \in (0, 1) \).

If we take step size \( \eta_j = ct^{-\theta} \) for \( j = 1, \ldots, t \) and \( c > 0 \), then with probability at least \( 1 - \delta \)

\[
\|\mathbf{w}_{t+1} - \tilde{\mathbf{w}}_{t+1}\|_2 = O\left(t^{1-\frac{\theta}{2+\theta}} + n^{-1} t^{1-\theta} \left(1 + \sqrt{nt^{-1} \log(1/\delta)}\right)\right).
\]

High-probability generalization bounds can be derived by combining the above stability bounds and the recent result on relating generalization and stability in a high-probability analysis (Bousquet et al., 2019; Feldman & Vondrak, 2019).

To prove Proposition G.1, we need to introduce a special concentration inequality called Chernoff’s bound for a summation of independent Bernoulli random variables (Boucheron et al., 2013).

**Lemma G.2** (Chernoff’s Bound). Let \( X_1, \ldots, X_t \) be independent random variables taking values in \( \{0, 1\} \). Let \( X = \sum_{t=1}^t X_t \) and \( \mu = \mathbb{E}[X] \). Then for any \( \delta \in (0, 1) \) with probability at least \( 1 - \exp \left(-\mu \delta^2 / 3\right) \) we have \( X \leq (1 + \delta)\mu \).

**Proof of Proposition G.1.** Without loss of generality, we can assume that \( S \) and \( \tilde{S} \) differ by the first example, i.e., \( z_1 \neq \tilde{z}_1 \) and \( z_i = \tilde{z}_i \) for \( i \neq 1 \). If \( i_t \neq 1 \), we can apply Lemma D.3 and (A.1) to derive

\[
\|\mathbf{w}_{t+1} - \tilde{\mathbf{w}}_{t+1}\|_2 \leq \|\mathbf{w}_t - \eta_t \nabla f(\mathbf{w}_t; z_{i_t}) - \tilde{\mathbf{w}}_t + \eta_t \nabla f(\tilde{\mathbf{w}}_t; z_{i_t})\|_2 \\
\leq \|\mathbf{w}_t - \tilde{\mathbf{w}}_t\|_2 + c_{\alpha, \beta} \eta_t^{\frac{1}{\alpha-1}}.
\]

If \( i_t = 1 \), we know

\[
\|\mathbf{w}_{t+1} - \tilde{\mathbf{w}}_{t+1}\|_2 \leq \|\mathbf{w}_t - \eta_1 \nabla f(\mathbf{w}_t; z_1) - \tilde{\mathbf{w}}_t + \eta_1 \nabla f(\tilde{\mathbf{w}}_t; \tilde{z}_1)\|_2 \\
\leq \|\mathbf{w}_t - \tilde{\mathbf{w}}_t\|_2 + 2\eta_1 G.
\]

Combining the above two cases together, we derive

\[
\|\mathbf{w}_{t+1} - \tilde{\mathbf{w}}_{t+1}\|_2 \leq \|\mathbf{w}_t - \tilde{\mathbf{w}}_t\|_2 + c_{\alpha, \beta} \eta_t^{\frac{1}{\alpha-1}} + 2\eta_t G_{[i_t = 1]}.
\]

Taking a summation of the above inequality then yields

\[
\|\mathbf{w}_{t+1} - \tilde{\mathbf{w}}_{t+1}\|_2 \leq c_{\alpha, \beta} \sum_{j=1}^t \eta_j^{\frac{1}{\alpha-1}} + 2G \sum_{j=1}^t \eta_j G_{[i_j = 1]}.
\]
Applying Lemma G.2 with $X_j = 1_{[i_j=1]}$ and $\mu = t/n$ (note $E_A[X_j] = 1/n$), with probability $1 - \delta$ there holds

$$\sum_{j=1}^{t} 1_{[i_j=1]} \leq \frac{t}{n} \left( 1 + \sqrt{3nt^{-1} \log(1/\delta)} \right).$$

Therefore, for the step size $\eta_j = ct^{-\theta}$, $j = 1, \ldots, t$, we know

$$\|w_{t+1} - \tilde{w}_{t+1}\|_2 \leq c_{\alpha,3} t^{-1-\frac{3}{4\theta}} + 2G_{cn}^{-1} \left( 1 + \sqrt{3nt^{-1} \log(1/\delta)} \right) t^{1-\theta}.$$

The proof is complete. \qed

### G.3. SGD without replacement

Our stability bounds can be further extended to SGD without replacement. In this case, we run SGD in epochs. For the $k$-th epoch, we start with a model $w^k_1 \in \mathbb{R}^d$, and draw an index sequence $(i^k_1, \ldots, i^k_n)$ from the uniform distribution over all permutations of $\{1, \ldots, n\}$. Then we update the model by

$$w^k_{t+1} = w^k_t - \eta^k_t \nabla f(w^k_t; z^k_t), \quad t = 1, \ldots, n,$$  \hspace{1cm} (G.1)

where $\{\eta^k_t\}$ is the step size sequence. We set $w^{k+1}_1 = w^{k}_{n+1}$, i.e., each epoch starts with the last iterate of the previous epoch. The following proposition establishes stability bounds for SGD without replacement when applied to loss functions with Hölder continuous gradients.

**Proposition G.3.** Suppose assumptions of Proposition G.1 hold. Let $\{w_t\}_t$ and $\{\tilde{w}_t\}_t$ be produced by (G.1) based on $S$ and $\tilde{S}$, respectively. Then

$$E_A[\|w^{K+1}_1 - \tilde{w}^{K+1}_1\|_2] \leq \frac{2G}{n} \sum_{k=1}^{K} \sum_{t=1}^{n} \eta^k_t + c_{\alpha,3} \sum_{k=1}^{K} \sum_{t=1}^{n} (\eta^k_t)^{\frac{1}{1-\theta}}.$$

**Proof.** Without loss of generality, we can assume that $S$ and $\tilde{S}$ differ by the first example, i.e., $z_1 \neq \tilde{z}_1$ and $z_i = \tilde{z}_i$ for $i \neq 1$. Analogous to the proof of Proposition G.1, we derive the following inequality for all $k \in \mathbb{N}$ and $t = 1, \ldots, n$

$$\|w^k_{t+1} - \tilde{w}^k_{t+1}\|_2 \leq \|w^k_t - \tilde{w}^k_t\|_2 + c_{\alpha,3} (\eta^k_t)^{\frac{1}{1-\theta}} \mathbb{I}_{[i^k_t \neq 1]} + 2\eta^k_t G e^{1_{[i^k_t=1]}}.$$

Taking a summation of the above inequality from $t = 1$ to $n$ gives

$$\|w^k_{n+1} - \tilde{w}^k_{n+1}\|_2 \leq \|w^k_1 - \tilde{w}^k_1\|_2 + c_{\alpha,3} \sum_{t=1}^{n} (\eta^k_t)^{\frac{1}{1-\theta}} \mathbb{I}_{[i^k_t \neq 1]} + 2G \sum_{t=1}^{n} \eta^k_t 1_{[i^k_t=1]}.$$

Let $i^k$ be the unique $t \in \{1, \ldots, n\}$ such that $i^k_t = 1$. Since $w^{k+1}_1 = w^{k}_{n+1}$, we derive

$$\|w^k_{1} - \tilde{w}^{k+1}_1\|_2 \leq \|w^k_1 - \tilde{w}^k_1\|_2 + c_{\alpha,3} \sum_{t=1}^{n} (\eta^k_t)^{\frac{1}{1-\theta}} + 2G \eta^k_k.$$

Since we draw $(i^k_1, \ldots, i^k_n)$ from the uniform distribution of all permutations, $i^k$ takes an equal probability to each $1, \ldots, n$. Therefore, we can take expectations over $A$ to derive

$$E_A[\|w^{k+1}_1 - \tilde{w}^{k+1}_1\|_2] \leq E_A[\|w^k_1 - \tilde{w}^k_1\|_2] + c_{\alpha,3} \sum_{t=1}^{n} (\eta^k_t)^{\frac{1}{1-\theta}} + \frac{2G \sum_{t=1}^{n} \eta^k_t}{n}.$$

We can take a summation of the above inequality from $k = 1$ to $K$ to derive the stated bound. The proof is complete. \qed
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