OPEN QUESTIONS IN CLASSICAL GRAVITY

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Abstract

We discuss some outstanding open questions regarding the validity and uniqueness of the standard second order Newton-Einstein classical gravitational theory. On the observational side we discuss the degree to which the realm of validity of Newton's Law of Gravity can actually be extended to distances much larger than the solar system distance scales on which the law was originally established. On the theoretical side we identify some commonly accepted but actually still open to question assumptions which go into the formulating of the standard second order Einstein theory in the first place. In particular, we show that while the familiar second order Poisson gravitational equation (and accordingly its second order covariant Einstein generalization) may be sufficient to yield Newton's Law of Gravity they are not in fact necessary. The standard theory thus still awaits the identification of some principle which would then make it necessary too. We show that current observational information does not exclusively mandate the standard theory, and that the conformal invariant fourth order theory of gravity considered recently by Mannheim and Kazanas is also able to meet the constraints of data, and in fact to do so without the need for any so far unobserved non-luminous or dark matter.

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(1). Introduction

While a great deal of attention is currently being given to the formulating of a quantum theory of gravity, it is remarkable that no such similar attention is being given to the question of what the correct classical theory should be (despite the fact that the correct quantum theory would have to have the correct classical theory as its classical limit); and indeed, not only is the standard Newton-Einstein theory considered to be the correct classical theory, but the general view of the community is that the whole issue has long since been completely settled. Now while there is always some risk in asserting that a theory will be able to accommodate all future data no matter how disparate, there are of course some very good reasons to assign such status to the standard Newton-Einstein gravitational theory. Nonetheless, if a theory is to be the correct theory, then not only must it survive all future testing, but also it must have no open theoretical loose ends. In the present work we shall argue that the standard theory actually still has some unresolved open theoretical loose ends, and, further, that it may even be facing a critical observational moment of truth in regard to galactic rotation curve data with their apparent need for enormous amounts of so far unestablished non-luminous or dark matter. As regards these loose ends, we shall show that the standard theory is only sufficient to meet the demands of data and not in fact necessary; and indeed, we shall demonstrate this in detail by actually presenting an alternate theory of gravity, a fourth order conformal invariant one, which also appears to be able to handle the available data, and to even do so without the need for any dark matter. Now while the interest of the present author is in the fourth order theory per se, as we shall see, its very study enables us to sharpen our evaluation and understanding of the standard theory; and if the standard theory is to continue to be regarded as the correct theory, then it would appear to us that the loose ends which we discuss below not only need to be tidied up, but that they need to be resolved in favor of the standard theory by the identifying of some currently unknown fundamental principle which would then ensure that the standard theory in fact be a necessary and not merely a sufficient theory of gravity.

(2). Observational status of the standard theory

Not only does most of our information regarding gravity derive from a study of the solar system, but also so does most of our intuition. In his study of the solar system Newton established a phenomenological Law of Gravity described by the gravitational potential $V(r) = -MG/r$. This potential exhibits three central
aspects, first that the potential falls like $1/r$ at large distances, second that gravity is universally coupled through a fundamental constant $G$, and third that the gravitational potential is an extensive function of the amount of matter contained in the gravitational source. The law was originally established for weak gravity on distance scales from terrestrial up to solar, but since it possessed a universal aspect, it was intended that it would continue to hold on much larger distance scales where it had not been studied. Moreover, the law actually contains two separate and distinct kinds of universality, a universality in dependence on distance, and a universality in coupling strength. Thus we need to ask whether the potential really is $1/r$ on much larger distance scales, and whether universal coupling actually necessitates the introduction of the fundamental constant $G$ in the first place.

As regards the fitting of the Newtonian potential to the solar system, the theory was challenged in two familiar though differing ways when some deviations were found to the Keplerian $v^2(r) = MG/r$ fall-off expected (Fig. (1)) of the rotational velocities of the planets as a function of their distance from the Sun. Firstly, the motion of the planet Uranus was found to not quite fit the Keplerian expectation, and so it was proposed that there exist yet another, heretofore undetected, outer solar system planet, Neptune, which would perturb Uranus in just the right way. Since it had not been observed at the time it was first proposed, Neptune constitutes an example of what is now called dark matter; and indeed, its subsequent detection was the first successful prediction of the dark matter idea. (For the moment its discovery is actually the only successful prediction of dark matter theory). As regards the inner solar system, it was found there that the motion of the planet Mercury was also not quite Keplerian, and with this planet being so close, it was very easy to ascertain that this time there were no previously unnoticed nearby gravitational sources. Hence the deviation of the motion of the planet Mercury could not be explained by dark matter. Rather, it required something far more radical, an in principle modification, with Einstein explaining the motion of Mercury by providing a complete reformulation of the entire theory of gravity through the introduction of General Relativity. Thus the general wisdom obtained from the solar system is that when there is an observational conflict, either change the matter distribution or change the theory.

With the advent of galactic astronomy it became possible to repeat the solar system type study only on a much larger distance scale. In galaxies it is possible to determine the velocities of the stars themselves as they move in orbit around a galaxy (or, equivalently, the velocities of the $HII$ emission regions which
surround some of the hot stars in the galaxies), as well as the velocities of small amounts of neutral hydrogen gas which are also typically present. Thus one can study the ionized optical spectra directly ($HII$) or the Lyman alpha lines of the neutral gas ($HI$); and, indeed, the original $HII$ optical studies of Rubin, Ford, Thonnard and Burstein$^{15-18}$ yielded rotation curves for spiral galaxies which not only showed no sign of any Keplerian fall-off, but which were even flat in structure out to the largest observed distances. However, eventually, after a concentrated effort to carefully measure the surface brightness (i.e. the stellar luminous matter distribution) of such galaxies, it was gradually realized (see e.g. Kalnajs,$^{19}$ and Kent$^{20}$) that the $HII$ curves could essentially be described by a standard luminous Newtonian prediction after all (even in fact for galaxies such as UGC2885 for which the rotation$^{18}$ and surface brightness$^{20}$ data go out to as much as 80 kpc - see Fig. (2)), simply because the galaxies were extended sources rather than effective pointlike ones such as the Sun, with the observed curves actually being, to a very good approximation, the luminous Newtonian expectation for such extended objects on the observed distance scales. Now, while the Newtonian prediction for such galaxies would then actually show a fall-off at even larger distances, these distances were simply not explorable optically since the signal was too weak; and, generally, the optical data do not (because of their very nature) go out to a large enough distance to be able to explore any substantive deviations from luminous Newtonian behavior. In other words, it is only possible to observe $HII$ spectra in the region where the stars themselves are located, i.e. within the extended luminous matter distribution region itself, a region which turns out to be Newtonian dominated. As we thus see, the flatness observed in the optical data has nothing to do with any asymptotic behaviour of the rotation curves, rather it is merely the Newtonian expectation in the interior optical disk region.

While the $HII$ data do not show any substantive non-canonical behaviour, nonetheless, the pioneering work of Rubin and coworkers brought the whole issue of galactic rotation curves into prominence and stimulated a great deal of study in the field. Now it turns out that the hydrogen gas is distributed in galaxies out to much farther distances than the stars, thus making the $HI$ studies ideal probes of the outer reaches of the rotation curves. One particularly prominent case is the galaxy NGC3198 which was studied by Bosma$^{21,22}$ and then by Begeman$^{23,24}$ (as part of the ongoing University of Groningen Westerbork Synthesis Radio Telescope survey) to interestingly again show a flat rotation curve, but this time in a region way outside the optical disk region charted in Refs. (25) and (26), to thus yield a flatness which was not compatible
with the Newtonian fall-off expected in the outer region. (That HI studies might lead to a conflict with a luminous Newtonian prediction was noted very early by Freeman\textsuperscript{27} from an analysis of NGC300 and M33 and by Roberts and Whitehurst\textsuperscript{28} from an analysis of M31). Thus with the HI studies (there are now about 30 well studied cases) it became clear that there really was a problem with the interpretation of galactic rotation curve data, which the community immediately sought to explain by the introduction of galactic dark matter (the 'Neptune' type solution) since the Newton-Einstein theory was presumed to be universal. Fits to the HI data have been obtained using dark matter (Ref. (26) provides a very complete analysis while Fig. (3) shows a fit to NGC3198 using the data of Refs. (24), (25) and (26)), and while the fits are certainly phenomenologically acceptable, they nonetheless possess certain shortcomings. Far and away their most serious shortcoming is their ad hoc nature, with any found Newtonian shortfall then being retroactively fitted by an appropriately chosen dark matter distribution. In this sense dark matter is not a predictive theory at all but only a parametrization of the difference between observation and the luminous Newtonian expectation. As to possible dark matter distributions, none has convincingly been derived from first principles as a consequence of, say, galactic dynamics or formation theory (for a recent critical review see Ref. (29)), with the most popular being the distribution associated with a modified isothermal gas sphere (a two-parameter spherical matter density distribution $\rho(r) = \rho_o/(r^2 + r_o^2)$ with an overall scale $\rho_o$ and an arbitrarily introduced non-zero core radius $r_o$ which would cause dark matter to predominate in the outer rather than the inner region - even though a true isothermal sphere would have zero core radius).

The appeal of the isothermal gas sphere is that it leads to an asymptotically logarithmic galactic potential and hence to asymptotically flat rotation curves, i.e. it is motivated by the very data that it is trying to explain. However, careful analysis of the explicit dark matter fits is instructive. Recalling that the inner region is already flat for Newtonian reasons, the dark matter parameters are then adjusted so as to join on to the Newtonian piece (hence the ad hoc core radius $r_o$) to give a continuously flat curve in the observed region. This matching of the luminous and dark matter pieces is for the moment completely fortuitous (van Albada and Sancisi\textsuperscript{30} have even referred to it as a conspiracy) and not yet explained by galactic dynamics. Worse, in the fitted region the dark matter contribution is actually still rising, and thus is not yet taking on its asymptotic value. Hence the curves are made flat not by a flat dark matter contribution but rather by an interplay between a rising dark matter piece and a falling Newtonian one, with the asymptotically
flat expectation not yet actually having even been tested. (Prospects for pushing the data out to farther
distances are not good because the \( HI \) surface density typically falls off exponentially fast at the edge of the
explored region). Beyond these fitting questions (two dark matter parameters per galaxy is, however, still
fairly economical), the outcome of the fitting is that galaxies are then 90% or so non-luminous. Thus not
only is the Universe to be dominated by this so far undetected material, the stars in a galaxy are demoted
to being only minor players, an afterthought as it were. Since dark matter only interacts gravitationally it
is extremely difficult to detect (its actual detection with just the right flux would of course be a discovery of
the first magnitude), and since it can be freely reparametrized as galactic data change or as new data come
on line, it hardly qualifies as even being a falsifiable idea, the sine qua non for a physical theory. (While some
possible dark matter candidate particle may eventually be detected, the issue is not whether the particle
exists at all - it may exist for some wholly unrelated reason, but rather whether its associated flux is big
even enough to dominate galaxies). Since the great appeal of Einstein gravity is its elegance and beauty, using
such a band aid solution for it essentially defeats the whole purpose, and is completely foreign to Einstein's
own view of nature.

Given the results obtained from the solar system study, one can also consider a galactic 'Mercury' type
solution; and indeed, in the absence of revealed wisdom, it would appear that the community should actually
have no choice but to explore both viewpoints without preconceived prejudice to see what may emerge, and
only then to make a judgement. Nonetheless, this is not how the subject has been treated, with only a
few brave souls being prepared to contemplate the possibility that the Newton-Einstein theory may actually
require modification (how in principle it is possible to do this by exploiting the theory’s loose ends while still
retaining its tested features will be discussed below in Sec. (3)). From the point of view of non-relativistic
theory alone, the motion of a test particle is given by inserting the law of force into Newton’s second law of
motion. There thus appear to be three possible empirical options. One is to retain Newton’s gravitational
force law but then increase the number of sources (i.e. dark matter); the second is to change the law of force
(e.g. Sanders\textsuperscript{29} with his proposed additional exponential potential which is motivated by ideas based on a
possible grand unification of the fundamental interactions); and the third is to change Newtonian dynamics
itself (Milgrom\textsuperscript{31}). The Milgrom option in particular has been extensively explored in the literature and is
currently phenomenologically viable (Ref. (32) provides a recent analysis). For our purposes in this paper
however, we do not want to address the non-relativistic galactic rotation curve question per se but rather to
explore the freedom still available at the relativistic level, to see what alternatives relativistic theory might
yield, and to only then go to the non-relativistic limit and attempt to fit data.

Motivated by a desire to have a theory of gravity which obeys the general covariance principle and
also the additional demand of local conformal invariance (the invariance under arbitrary local conformal
stretchings $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$ which is now thought to be enjoyed by the other three fundamental
interactions, the strong, electromagnetic and weak), Mannheim and Kazanas$^{1-14}$ have suggested that gravity
be based not on the Einstein-Hilbert action

$$I_{EH} = -\int d^4x (-g)^{1/2} \frac{1}{16\pi G} R_{\alpha}^\alpha$$ (1)

but rather on the conformal invariant fourth order action

$$I_W = -\alpha \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$$ (2)

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and $\alpha$ is a purely dimensionless coefficient. They explored the
structure of this theory to obtain the complete, exact, non-perturbative exterior vacuum solution associated
with a static, spherically symmetric gravitational source such as a star in the theory, viz. (up to an
unobservable overall conformal factor)

$$-g_{00} = 1/g_{rr} = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \beta r - kr^2$$ (3)

where $\beta$, $\gamma$, and $k$ are three appropriate integration constants. As can be seen, for small enough values of
the linear and quadratic terms (i.e. on small enough distance scales) the solution reduces to the familiar
Schwarzschild solution of Einstein gravity, with the conformal theory then enjoying the same successes as
the Einstein theory on those distance scales. On larger distance scales the theory begins to differ from the
Einstein theory through the linear potential term, and (with the quadratic term only possibly being impor-
tant cosmologically, and with the $\beta\gamma$ product terms being numerically negligible) yields a non-relativistic
gravitational potential

$$V(r) = -\beta/r + \gamma r/2$$ (4)

which can then be fitted to galactic data in the weak gravity limit.
In order to apply the potential of Eq. (4) to a galaxy, we note that spiral galaxies typically consist of an axisymmetric disk of stars with surface density \(N \exp(-R/R_o)/2\pi R_o^2\) where \(N\) is the number of stars and \(R_o = 1/\alpha\) is the scale length of the luminous matter distribution. Integrating Eq. (4) over such a distribution yields\(^7\) for the rotational velocities the extremely compact, exact expression

\[ rV'(r) = (N\beta\alpha^3 r^2/2)[I_0(\alpha r/2)K_0(\alpha r/2) - I_1(\alpha r/2)K_1(\alpha r/2)] + (N\gamma\alpha r^2/2)I_1(\alpha r/2)K_1(\alpha r/2) \]  

(5)

an expression which behaves asymptotically as \(N\beta/r + N\gamma r/2 - 3N\gamma R_o^2/4r\) as would be expected, and which is now ready for fitting.

In a recent comprehensive analysis of the \(HI\) rotation curves of spiral galaxies Casertano and van Gorkom\(^33\) have found that the data fall into essentially four general groups characterized by specific correlations between the maximum rotation velocity and the luminosity. In order of increasing luminosity the four groups are dwarf, intermediate, compact bright, and large bright galaxies. Thus as a first attempt at data fitting we have chosen to study one representative galaxy from each group, respectively the galaxies DDO154 (a gas dominated rather than star dominated galaxy), NGC3198, NGC2903, and NGC5907. This will immediately enable us to test the flexibility of our theory, as well as confront the systematics apparent in dark matter fits to the same four groups where it is typically found that the more luminous the galaxy the proportionately less dark matter seems to be needed. For NGC3198 we use the rotation curve of Ref. (24) and the surface brightness data of Refs. (25) and (26), for NGC 2903 the data are taken from Refs. (23) and (25), for NGC 5907 from Refs. (30) and (34), and for DDO154 from Refs. (35) and (36). The fits we obtain are shown in Figs. (4-7) (the details are given in Ref. (7) which also assesses the quality level required of fitting - typically up to 5% - or even 10% for gas dominated galaxies); and, as can be seen, the model does quite well with the data, being able to even reproduce the luminosity trend found in the dark matter fits with the more luminous galaxies being proportionately more Newtonian. The mass to light ratios we find in the fits (viz. \(M/L(154)=1.4, M/L(3198)=4.2, M/L(2903)=3.5, M/L(5907)=6.1\)) are similar to those found in the dark matter fits, with the mass to light ratio essentially increasing with luminosity to thus make the more luminous galaxies more Newtonian dominated. For the linear terms we obtain a galactic \(1/\gamma = 1/N\gamma_{\text{star}}\) of order the Hubble length (viz. \(1/\gamma(154)=4.0 \times 10^{29}\) cm., \(1/\gamma(3198)=2.9 \times 10^{29}\) cm., \(1/\gamma(2903)=1.3 \times 10^{29}\) cm.,
1/\gamma(5907)=1.7 \times 10^{29}\text{cm.}, an intriguing fact which suggests a possible cosmological origin for \gamma. (Such a value for a galactic \gamma would then make \gamma_{\text{star}} = \gamma/N completely insignificant on solar system distance scales, to thus permit the Newtonian potential to nicely dominate there as required). As regards our fitting, we see that the linear term is essentially replacing the dark matter contribution (c.f. Figs. (3) and (5)), so that our fits appear to be able to handle galactic rotation curves without the need for any dark matter at all. That our model is able to fit the data sample at all is perhaps initially surprising, since, given its linear potential, the model predicts that rotational velocities will ultimately rise asymptotically. However, because the galaxies are extended sources and not pointlike ones, we see that the falling Newtonian piece and the rising linear piece are able to compensate each other to produce effectively flat rotation curves in the observed region (the right hand side of Eq. (5) is simply a very slowly varying function in this region), while also satisfying the luminosity correlation at the same time. Thus it would appear to us that at the present time one cannot categorically assert that the sole gravitational potential on all distance scales is the Newtonian one, and that, in the linear potential, the standard 1/r potential would not only appear to have a companion but to have one which would even dominate over it asymptotically. Indeed, the very need for dark matter in the standard theory may simply be due to trying to apply just the simple Newtonian potential in a domain for which there is no prior (or even current for that matter) justification. As regards our fitting, we see that is not in fact necessary to demand flatness in the asymptotic region in order to obtain flat rotation curves in the explored intermediate region. Thus, unlike the dark matter fits, we do not need to know the structure of the data prior to the fitting, or need to adapt the model to a presupposed asymptotic flatness. Further, not only is our linear potential theory more motivated than the dark matter models (Eq. (4) arises in a fundamental, uniquely specified theory), it possesses one fewer free parameter per galaxy (\gamma instead of \rho_o and r_o). Consequently, according to the usual criteria for evaluating rival theories, it is to be preferred. However, in order to fully merit this status, the theory has to confront Einstein gravity head on, a task to which we now turn.

(3). The loose ends of the standard theory

In his monumental development of General Relativity, Einstein set out to accomplish three basic goals; first, to extend to accelerating observers the relativistic invariance properties of physical systems he had
established for observers moving with uniform velocity; second, to find a natural explanation for the observed equality of inertial and gravitational masses; and third, to construct a field theory of gravity, a theory which at the time did not even obey Special Relativity. While these issues are separate and distinct, the distinctions between them are easily blurred, leaving the impression that the only prescription that works is the one based on the use of the Einstein-Hilbert action of Eq. (1), with any possible departure from this prescription guaranteed to prove fatal - to both the candidate alternative and to its proponents. Nonetheless, it is worthwhile to go over the entire package piece by piece, to see whether there is still any place where it is possible to make changes.

As regards the issue of accelerating observers, we note first that this issue has nothing to do with gravity per se, since observers can accelerate in flat spacetime; and, anyway, gravity is not the only force that can produce accelerations, electromagnetism can do so too and that certainly is not thought to necessitate curvature. Thus the issue for accelerating observers in flat spacetime is how to construct equations of motion on whose physics they all can agree. Consider, for instance, a standard free particle of kinematic mass \( m \) (the discussion here follows Ref. (6)) moving in flat spacetime according to the special relativistic generalization of Newton’s second law of motion, viz.

\[
\frac{d^2 \xi^\alpha}{d\tau^2} = 0 \quad , \quad R_{\mu\nu\sigma\tau} = 0
\]  

(6)

where \( d\tau = (-\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta)^{1/2} \) is the proper time and \( \eta_{\alpha\beta} \) is the flat spacetime metric, and where we have indicated explicitly that the Riemann tensor is (for the moment) zero. Transforming to an arbitrary coordinate system \( x^\mu \), and using the definitions

\[
\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \quad , \quad g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}
\]

(7)

enables us to write Eq. (6) in the form

\[
m \left( \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = 0 \quad , \quad R_{\mu\nu\sigma\tau} = 0
\]  

(8)

with the invariant proper time then taking the form \( d\tau = (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2} \). As derived, Eq. (8) so far only holds in a strictly flat spacetime with zero Riemann curvature tensor, and indeed Eq. (8) is only a covariant rewriting of the special relativistic Newtonian second law of motion, i.e. it covariantly describes what an
observer with a non-uniform velocity in flat spacetime sees. (While the four velocity \(dx^\lambda/d\tau\) is a general covariant vector, its ordinary derivative \(d^2x^\lambda/d\tau^2\) (which samples adjacent points and not merely the point where the four velocity itself is calculated) is not, and it is only the entire left-hand side of Eq. (8) which transforms as a general covariant four acceleration). Thus we see that in general it is Eq. (8) which should be taken as Newton’s second law of motion (in flat spacetime) and not Eq. (6), to thus show that general covariance is not in principle related to curvature, and that it is a principle which should be imposed on all physical theories independent of whether or not the spacetime in which we live is curved.

Incidentally, we note in passing that Eq. (8) provides an answer to Mach’s criticism of Newton’s second law of motion. Mach sought to have some dynamical interplay between local and global physics which would select out (the non-relativistic limit) of Eq. (6) as special. Since Eq. (6) is not generally covariant, there is no need to have any such dynamics for it, rather it is Eq. (8) which is physically meaningful in general, and for a single particle Eq. (8) will reduce to Eq. (6) in flat spacetime in Cartesian coordinates simply because it is covariantly allowed to do so. However, for a system of two particles in mutual rotation around each other in flat spacetime it is impossible to find a coordinate transformation which will bring Eq. (8) to the form of Eq. (6) for both of them at once, thus making Coriolis forces physical, again without the need for any mystical interaction with the rest of the Universe. (That does not mean that dynamically there is no such interaction, only that it is not needed in order to meet Mach’s concerns). Thus covariance itself appears to be the answer to Mach and not Einstein gravity per se. (Moreover, even when there is curvature, our ability to remove the Christoffel symbols over some local region (they can always be removed at one point because they do not transform as tensors) would, if anything, be enhanced by having little or no curvature, i.e. by having a weak coupling with the rest of the Universe rather than a strong one).

Now, of course, we still do need to generalize Eq. (8) to include gravity since gravity is also a physical force, and the great insight of Einstein was to then realize that in a non-flat spacetime if the gravitational field emerged as being in the Christoffel symbol, then since the two terms on the left hand side of Eq. (8) have the same coefficient, the equality of gravitational and inertial masses would then be assured, with Eq. (8) being replaced by

\[
m \left( \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = 0 \quad , \quad R_{\mu\nu\sigma\tau} \neq 0,
\]

Equation (9) achieves two things, one is it establishes the metric as the gravitational field in the first place,
and the other is it specifies how a test particle is to couple to gravity. Since geodesic motion follows from the
covariant conservation of the energy-momentum tensor of test particles, and since this covariant conservation
occurs in any theory which is coordinate invariant, the validity of Eq. (9) for test particles in no way entails
the need to use the Einstein-Hilbert action. Thus the question of what (covariant) equation of motion is
to fix the Christoffel symbols which are to go into Eq. (9) is in principle an independently explorable issue
which is decoupled from the validity or otherwise of the equivalence principle, and constitutes the only piece
of the entire Einstein package which would still appear to admit of further analysis.

As to the question of what specific equation of motion is in fact to be obeyed by the gravitational field,
it is here that we run into the theoretical loose ends we alluded to in Sec. (1). Thus while it can safely be
said that gravity is indeed a metric based field theory, covariance alone is simply not sufficient to specify the
gravitational action with more input being needed, since any covariant scalar action whatsoever will meet
that requirement. Now in a sense it is quite remarkable that Einstein did not appeal to something like a
fundamental principle which would then unambiguously specify the appropriate action, since that would be
more in keeping with the viewpoint which motivated his explanation of the equality of gravitational and
inertial masses. Rather, he chose the action of Eq. (1) simply because it would nicely recover the standard
non-relativistic Newtonian phenomenology while also yielding calculable and testable covariant corrections
to it, and it is to this aspect of the theory which we now turn.

The variation of the action of Eq. (1) with respect to the metric yields, in the presence of a matter
source with energy-momentum tensor $T_{\mu\nu}$, the equation of motion

$$R_{\mu\nu} - g_{\mu\nu}R^\alpha_\alpha/2 = -8\pi G T_{\mu\nu}$$

The great appeal of Eq. (10) is that, first, in the weak gravity limit it reduces to the second order Poisson
equation

$$\nabla^2 \phi(r) = g(r)$$

with its familiar exterior Newtonian potential solution, viz.

$$\phi(r > R) = -\frac{1}{r} \int_0^R dr' g(r') r'^2$$

for a spherically symmetric, static source with radius $R$; while, second it yields relativistic corrections to
this non-relativistic theory. The observational confirmation of these corrections on terrestrial to solar system
distance scales not only established the validity of the Einstein theory on those scales but seems to have
established it on all others too, even though many other theories could potentially have the same leading
perturbative structure on a given distance scale and yet differ radically elsewhere.

While Eq. (10) was chosen so as to yield Eq. (11), it is a curious but not well appreciated fact\(^8\) that
there is actually not yet any evidence for the independent validity of Eq. (11) in the interior of a gravitational
source - in the exterior region alone one can only test the validity of the (Newtonian) solution in that region,
this not being sufficient to infer the validity of the Poisson equation itself in that and all other regions as
well. Specifically, suppose a theory has a Newtonian potential as its exterior solution. Then for a set of such
Newtonian sources we may determine the potential of the system via the summation

\[
\phi(r) = \sum \frac{1}{|r - r'|}
\]

(13)
a summation which of course reduces to Eq. (12) for an appropriately chosen \(g(r)\) which at this point would
only need to be identified as the number density of such sources. Thus, when derived from the non-relativistic
theory alone the source needed for the Poisson equation is the number density, whereas, when derived from
the weak gravity limit of the Einstein theory, the source would be the energy density. While these two
densities are of course proportional in the weak binding limit, we note that in general in order to recover
Newton’s original result that the gravitational potential be an extensive function of the amount of matter
in the gravitating source, we only need establish proportionality to the number density, and not necessarily
to the energy density, a point on which we will capitalize below when we study the fourth order theory. Via
Eqs. (12) and/or (13) we thus determine the Newtonian potential of a set of sources outside of the region
where the sources themselves are located.

Now suppose we want to determine the potential inside a star. In the non-relativistic limit we would
then take the individual atoms in the star to be the gravitating sources needed for the Poisson equation and
calculate the potential exterior to them but interior to the star again via Eqs. (12) and (13), i.e. again exterior
to the (microscopic this time) gravitating sources, to again give us an extensive potential in the weak gravity
and weak binding limits. Thus even interior to a star we are still only using the exterior Newtonian potential
in this limit. The only place where the Poisson equation would have explicit independent consequences would
be in the interior of the atoms, with a truly interior solution taking the form

\[ \phi(r < R) = -\frac{1}{r} \int_0^r dr' \frac{g(r')}{r'^2} - \int_R^r dr' \frac{g(r')}{r'} \] (14)

However, interior to atoms Eq. (14) has never been tested, simply because gravity is not the predominant force there. Thus as of today the dynamical consequences of the second order gravitational Poisson equation which go beyond those of Newton’s Law have yet to be tested, thus making it unnecessary to find a covariant generalization of the second order Poisson equation; rather one only needs to recover its exterior solution in some limit.

As regards recovering this exterior solution, we recall of course that the exterior solution to the Einstein equation of Eq. (10) is simply the familiar Schwarzschild solution to the Ricci flat condition \( R_{\mu\nu} = 0 \), a solution which nicely contains both the Newtonian potential and its relativistic corrections; and as we have just seen that is all we need for the standard non-relativistic phenomenology which would just sum and count the number of such sources. Now it was noted by Eddington in the very early days of Relativity that this same exterior non-relativistic and relativistic phenomenology would follow if some other function of the Ricci tensor vanished, either a higher power of it or some derivative of it instead, since such a situation would still admit of the vanishing of the Ricci tensor itself. With this being precisely the case which occurs in fourth order theories of gravity (theories which are just as covariant as the second order one), Eddington then threw out a challenge to the community to tell him which one the correct theory of gravity was. This question has generally been ignored by the community and never satisfactorily resolved, even though it goes to the very heart of the question of the uniqueness of gravitational theory.

While the vanishing of a derivative or a higher power of the Ricci tensor implies the vanishing of the Ricci tensor, the reverse is not true, since the vanishing of a function of the Ricci tensor could have other, possibly less desirable solutions as well, and thus in order to answer Eddington’s question it is necessary to find such other solutions and identify their observational implications. This was then the brief that Mannheim and Kazanas followed. The variation of the conformal invariant action of Eq. (2) with respect to the metric yields the equation of motion (see e.g. Ref. (1))

\[ W_{\mu\nu} = \frac{1}{2} g_{\mu\nu} (R^\alpha_{\alpha} \, ;^\beta)_{;^\beta} + R_{\mu\nu} \, ;^\beta_{;^\beta} - R_{\mu} \, ;^\beta_{;^\nu;^\beta} - R_{\nu} \, ;^\beta_{;^\mu;^\beta} - 2 R_{\mu\beta} R_{\nu} \, ;^\beta + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \]
\[
-\frac{2}{3}g_{\mu\nu}(R^\alpha_{\beta})_{;\beta} + \frac{2}{3}(R^\alpha_{\beta\nu})_{;\mu} + \frac{2}{3}R^\alpha_{\alpha\nu} - \frac{1}{6}g_{\mu\nu}(R^\alpha_{\alpha})^2 = \left(\frac{1}{4\alpha}\right)T_{\mu\nu}
\]

with \(W_{\mu\nu}\) thus replacing the Einstein tensor in Eq. (10). Despite its somewhat forbidding appearance, Mannheim and Kazanas\(^1\) were able to solve the exterior static, spherically symmetric vacuum problem associated with Eq. (15) exactly to find the metric of Eq. (3) as its complete and exact solution. Thus the Schwarzschild solution is nicely recovered as Eddington had already noted, together with the new linear term whose potential role in galaxies was discussed above. Thus not only have we found the extra non-Schwarzschild solutions, we have seen that rather than be undesirable, they may even enjoy some observational support.

In the presence of a source, Mannheim and Kazanas went further to find\(^8\) that for the associated interior problem Eq. (15) reduces without approximation to the remarkably compact equation

\[
\nabla^4 B(r) = \frac{(rB''')'}{r} = f(r)
\]

where \(B(r) = -g_{00}(r)\) and where the source is given by

\[
f(r) = 3(T^0_0 - T^r_r)/4\alpha B(r)
\]

Thus the metric of Eq. (3) is immediately recognized as the (exact) solution to a fourth order Laplace equation. In the presence of the source the fourth order Poisson equation of Eq. (16) is also readily integratable to yield the exact exterior solution (we ignore an uninteresting \(w - kr^2\) particular integral)

\[
\begin{align*}
B(r > R) &= -\frac{1}{6r} \int_0^R dr' f(r') r'^4 - \frac{r}{2} \int_0^R dr' f(r') r'^2
\end{align*}
\]

to thus both recover the linear and Newtonian potential terms of Eq. (3) and enable us to express their coefficients as appropriate moments of the source function \(f(r)\), viz.

\[
\beta(2 - 3\beta) = \frac{1}{6} \int_0^R dr' f(r') r'^4, \quad \gamma = -\frac{1}{2} \int_0^R dr' f(r') r'^2
\]

We thus see that while a second order Poisson equation is sufficient to yield Newton’s Law of Gravity, it is not in fact necessary, with a fourth order Poisson equation being just as capable. While both the second and fourth order Poisson equations contain the Newtonian potential as the dominant solution on shorter
distance scales, our ability to use the second order Poisson equation on larger distance scales comes through
the assertion that the potential is still strictly Newtonian on those distance scales also, something which
while widely believed has yet to be established. Thus the case for the unique use of the second order Poisson
equation (and accordingly of its second order Einstein generalization) on large distance scales has yet to be
made; and of course, should such a case eventually actually be made, then after all that, dark matter would
still have to be found.

While the expressing of the Newtonian coefficient as a fourth moment rather than a second moment
integral of the energy-momentum tensor is unfamiliar, we should note that it violates no known observational
information, since it is only required to hold within fundamental microscopic sources (or within macroscopic
compact sources for which nothing at all is currently observationally known). As long as such microscopic
sources put out $1/r$ potentials then the macroscopic weak gravity potential (the only one which has so far
been explored observationally) still scales as their number in the standard way, this being all that is required
for the standard Newtonian phenomenology, as we noted above. (In the Einstein theory the coefficient of
the $1/r$ potential is identified as the second moment integral of the energy density of the source and called
its mass. However, despite its commonplace familiarity, we are not aware that such integrals have ever been
performed for either microscopic or macroscopic sources for that matter, and it is simply not known whether
the mass of the Sun as determined by the gravitational motions of the planets is in fact equal to the integral
of its energy density, since the energy density function of the Sun is unknown). Finally, we also note that the
integrals in Eq. (19) are all proportional to the inverse of the fundamental coupling constant $\alpha$ introduced
in the conformal action of Eq. (2). Thus our theory meets Newton’s final demand, namely that gravity be
universally coupled; and interestingly we see that the theory is able to do so without the need to introduce
$G$ at all. Since in the standard theory $G$ is not independently measurable anyway as it only ever appears in
the product $MG$ (its canonically quoted value just represents the choice of units in which to measure mass),
we see, that despite its longevity, Newton’s constant may not actually play any fundamental role in physics
at all. This of course would be a severe problem for the Einstein-Hilbert action since it gives fundamental
status to $G$. (It would also be a severe problem for string theory since the string tension is taken to be the
Planck length - of course string theories with their intrinsic fundamental string tension scale are anyway
already excluded by the underlying conformal invariance assumption on which the action of Eq. (2) is based.
We leave it up to the reader to decide whether the incompatibility of conformal gravity (a four-dimensional field theory which is even renormalizable in its own right) and string theory is virtue or a vice).

As the above argument shows, the standard second order theory is only sufficient to yield the Newtonian phenomenology and not necessary, with it being somewhat dangerous to identify the complete covariant theory simply from a knowledge of its first few perturbative terms in a restricted kinematic regime. We believe this to be a major loophole in the standard theory which still needs to be addressed. Beyond the phenomenological issue of what relativistic theory is mandated by non-relativistic observation, the standard theory comes with a few other loose ends. Specifically, since no principle was offered which would uniquely select out the Einstein-Hilbert action (to thus make the above discussion necessary as well as sufficient), no principle was offered which would exclude a possible cosmological constant term, and indeed since its inception Einstein gravity has been plagued by the cosmological constant problem. The conformal theory makes three specific contributions to this problem. First, the starting conformal symmetry not only unambiguously specifies the action (Eq. (2) is the unique conformally invariant action) but by doing so it thereby automatically excludes any cosmological constant term at all at the level of the input Lagrangian. Second, the conformal theory possesses no Planck scale, so that any induced cosmological term would not be required to be of Planck density. And third, if any cosmological term is induced in $T_{\mu\nu}$ as a consequence of the scale breaking Higgs mechanism which is needed to induce masses into the conformal invariant theory in the first place, then since the conformal $T_{\mu\nu}$ is traceless (and remains so even after the symmetry breaking), any induced cosmological term would have to be of the same order of magnitude as all the other terms in $T_{\mu\nu}$ which for cosmology would mean the energy density of the particles in the Universe. (This feature is exhibited explicitly in the model presented in Ref. (5) where it is also shown that conformal cosmology has no flatness problem, another major problem for the standard theory). Moreover, while it still needs to be explored in detail, a cosmological constant term of order of magnitude the cosmological energy density might provide a solution to the current age problem of the Universe, with recent astrophysical measurements indicating an age for the Universe which may be less than that of some of its constituents. Thus it appears to us that the origin of the cosmological constant problem in the standard theory may be traced back to an insufficiently motivated choice of starting action, with the Einstein theory simply lacking an additional invariance principle which would remove all ambiguity.
One final loose end in the standard theory is in our understanding not of the gravitational side of the equation of motion but rather in our understanding of the energy-momentum side. Unlike the situation that existed in Einstein’s time, we no longer regard mass as being kinematical, but rather it is now thought of as being dynamically induced in theories in which there are no fundamental scales at the level of the action at all, with all scales being induced dynamically in the vacuum; and indeed this motivated Mannheim and Kazanas to study conformal gravity in the first place. In conformal invariant theories the energy-momentum tensor is kinematically traceless. Thus, if we believe that the microscopic strong, electromagnetic and weak interactions are responsible for the masses of elementary particles, and hence for the masses of macroscopic systems such as stars, then we should naturally only consider traceless conformal invariant energy-momentum tensors from the beginning. Consequently it is somewhat difficult to understand why any such energy-momentum tensor should be set equal to the non-conformal invariant, non-traceless Einstein tensor as would be required by Eq. (10). Moreover, in the conformal case $T_{\mu\nu}$ could not be the familiar non-traceless kinematical test particle $T_{\mu\nu}$, since a conformal, dynamical particle would have to be accompanied by the Higgs field which gave it its mass in the first place. Nonetheless, it turns out that such dynamical particles still move on geodesics with the Higgs field energy and momentum actually being found to simply decouple from their motion. Thus even the standard test particle Bianchi identity argument (another cornerstone of the standard theory) is irrelevant to geodesic motion if particles have dynamical masses. As regards mass, we make one final comment. The equivalence principle relates gravitational and inertial masses. Since we now believe that inertial masses are dynamical through the Higgs mechanism, we are thus led to consider gravitational masses as being dynamical too. However, one prominent gravitational mass, the Planck mass, is not dynamical. We are thus led naturally to consider removing it from physics; and thus the apparent lack of compatibility between the Einstein theory and current ideas on mass and the structure of the other fundamental interactions represents another loose end for the Einstein theory, one which could even hinder a possible unification of gravity with the other interactions.

We thus believe that we have identified some open questions for classical gravity. And even if the conformal theory advanced here were to fall by the wayside (this seems to be a somewhat widespread hope), nonetheless, that would not suddenly make the standard theory right, and the questions we have asked of it would still require answering. Moreover, if the Einstein theory is in fact correct, then these questions must
not only be answered, but they must be resolved in its favor, and, instead, a theory such as the Einstein theory deserves that they be answered. In conclusion we would like to quote from Binney and Tremaine\textsuperscript{37} who remarked in their book (p. 637): "If a new theory of gravity is required, it will ultimately be accepted because of its beauty and unifying properties rather than because it eliminates the need for dark matter". We believe that the conformal theory presented here meets all of these requirements (since it possesses an additional (conformal) symmetry beyond that of the Einstein theory, the theory not only enjoys all the beauty and elegance of General Relativity, it even has some more). Whether Binney and Tremaine are correct in saying that the theory will ultimately be accepted remains however to be seen.

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Figure Captions

Figure (1). The Keplerian expectation for the velocities of the solar system planets as they orbit the Sun, with distance expressed in Astronomical Units (the mean Earth - Sun distance).

Figure (2). The luminous Newtonian expectation for the HI velocity profile of the galaxy UGC2885 obtained by integrating the Newtonian potential over the stellar surface luminosity distribution (Ref. (20)) whose logarithm is shown in the upper diagram. The bars show the rotational velocity data of Ref. (18) with their quoted errors. Note that all the data are confined to the 4 scale length stellar distribution region (a distribution which is parametrized by a leading exponential with $R_o = 21.7$ kpc).

Figure (3). Phenomenological dark matter fit to the HI rotational velocity data of Ref. (24) for the galaxy NGC3198. The luminous Newtonian contribution is obtained by integrating the Newtonian potential over the stellar (Ref. (26)) and gaseous (Ref. (25)) surface distributions shown in the upper two diagrams (the stellar is represented via its logarithm). The dark matter contribution is obtained by integrating the
Newtonian potential over an isothermal sphere. The full curve shows the overall prediction. Note that the HI data extend to 11 stellar scale lengths, a distance which is way beyond the stellar region.

Figure (4). The calculated rotational velocity curve associated with the conformal gravity potential $V(r) = -\beta/r + \gamma r/2$ for the representative gas dominated dwarf irregular galaxy DDO154. The bars show the HI rotational velocity data points of Ref. (36) with their quoted errors, the full curve shows the overall theoretical velocity prediction as a function of distance from the galactic center, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials would produce when integrated over the luminous matter distribution of the galaxy given in Ref. (36). No dark matter is assumed.

Figure (5). The calculated rotational velocity curve associated with the conformal gravity potential $V(r) = -\beta/r + \gamma r/2$ for the representative intermediate sized galaxy NCC3198. The bars show the HI rotational velocity data points of Ref. (24) with their quoted errors, the full curve shows the overall theoretical velocity prediction as a function of distance from the galactic center, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials would produce when integrated over the luminous matter distribution of the galaxy given in Refs. (25) and (26). No dark matter is assumed.

Figure (6). The calculated rotational velocity curve associated with the conformal gravity potential $V(r) = -\beta/r + \gamma r/2$ for the representative compact bright galaxy NGC2903. The bars show the HI rotational velocity data points of Ref. (23) with their quoted errors, the full curve shows the overall theoretical velocity prediction as a function of distance from the galactic center, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials would produce when integrated over the luminous matter distribution of the galaxy given in Ref. (25). No dark matter is assumed.

Figure (7). The calculated rotational velocity curve associated with the conformal gravity potential $V(r) = -\beta/r + \gamma r/2$ for the representative large bright galaxy NGC5907. The bars show the HI rotational velocity data points of Ref. (30) with their quoted errors, the full curve shows the overall theoretical velocity prediction as a function of distance from the galactic center, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials would produce when integrated over the luminous matter distribution of the galaxy given in Ref. (34). No dark matter is assumed.