Fields and Equations of Classical Mechanics for Quantum Mechanics

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Abstract. A generalized Euler-equation of fluid dynamics is derived for describing many-body states of quantum mechanics for fermion systems. The derived Eulerian equation can be viewed as representing the interaction of two substates, where each substate has its own velocity and pressure fields. These field quantities are given by maps depending on the probability distribution and the phase of the wavefunction. For one-body systems, the Eulerian equation can model either a fluid or particle interpretations for quantum-mechanical states, where the streamlines of the Madelung, or probability, fluid are also the trajectories of the particles. For the fluid interpretation, the mass density is the probability density times the electron mass. The generalized Euler equation is shown to be the gradient of an equation representing the total-energy of the two substates, having two energy fields that are, in general, nonuniform. This total-energy equation is a generalization of the Bernoulli equation of fluid dynamics. The total-energy equation, along with a continuity-equation, is equivalent to the time-dependent Schrödinger equation. An equation is also derived that is equivalent to the main equation of Bohmian mechanics. However, additional identifications are given that are not part of Bohmian mechanics: The quantum potential of Bohmian mechanics is given as a sum of a kinetic energy and pressure fields. Also, the time derivative of the wavefunction phase is replaced by an energy field. In the formalism, field quantities are identified from their placement in equations of classical mechanics. Separately, the field quantities are given by definitions that involve the wavefunction and operators of quantum mechanics. This approach yields, unintended, and unknown energy and pressure fields. These fields, however, are shown to satisfy a continuity equation, an equation that is equivalent to the other equation of Bohmian mechanics. It is also demonstrated that energy conservation holds for both of these energy fields, if the wavefunction is a linear-combination of eigenvectors, where the eigenvectors can be nondegenerate. A detailed investigation is given on the possible behavior, or source, of an electron that has one of the velocity fields. Alternate formulae for this velocity fields are also considered.
1. Introduction

There is a large spectrum of models used to describe physical phenomena. The metric used to evaluate a model, from this spectrum, depends on the academic field where the model is employed. In this paper, quantum-mechanical states are modeled using concepts from classical mechanics. In order to determine the best metric to evaluate this type of model, it is useful to determine where such a model fits into the spectrum of models.

Starting, at one end of the spectrum, is the many-worlds interpretation of quantum mechanics. In this interpretation, there are an infinite number of universes. As far as philosophy is concerned, the impact that such a model has in science is immaterial. Instead, it is evaluated as a deterministic model that does not use a collapse theory to treat the measurement problem. As a useful scientific tool, this model would need to provide verifiable predictions or, at least, provide physical scientists a useful way to think about quantum-mechanical states.

Another model is the De Broglie–Bohm theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], which we call Bohmian mechanics. As a philosophical theory, it is deterministic with hidden variables. In the spectrum of models, it is in the same neighborhood as the many-worlds interpretation. In Bohmian mechanics, in the treatment of an electronic system, each electron, at all times, has assigned to it a definite position and momentum. The trajectory of all particles are determined by an initial configuration, a set of momentums and positions. Since—short of a scientific revolution—it is not possible to determine the particles positions and momentums simultaneously, for all practical purposes, the method is indeterminate. The questions of being deterministic being purely philosophical.

On the other, far end of the spectrum, is the theory of Lewis dot structures used in chemistry. In this model, dots are used to represent valence electrons of atoms and molecules, with rules determining where the dots are placed. The Lewis dot structures contain information about regions of high electron-density, and the electron density is considered a continuum, called a charge cloud. In a typical organic chemistry textbook, there are thousands of Lewis dot structure, but none can be found in physical chemistry or molecular physics textbooks. Rigorous quantum-mechanical methods have not displaced the use of this model. Instead, both models coexists. For example, in crystal field theory [11], lone-pair valence-electrons of ligands split the degeneracy of the d orbitals of the transition-metal cation of a transition metal complex, providing a means to predict if the complex will absorb light in the visible spectrum.

In more rigorous methodology, the charge clouds become the probability densities of electrons from either molecular-orbital, valence-bond theory, or other ab initio approaches [12, 13]. Being a static model, the electron cloud interpretation is, however, incomplete, having only electrostatic fields. An assignment of a kinetic-energy field, where the charge cloud becomes a non-static fluid of charged-mass, could have many applications, and even improve, or supplement, the Lewis dot structure theory. Such a
kinetic-energy field could be used in conjunction with the electrostatic external potential-energy and the electron-electron repulsion-energy, both being classical fields that are widely used in chemistry.

Another method on the spectrum is quantum hydrodynamics [14,15]. This method is closer to the electron-cloud interpretation than the many worlds interpretation, even though it uses equations and ideas from Bohmian mechanics [1,2,3,4,5,6,7,8,9,10].

For one-body states, this method also incorporates ideas from the Madelung fluid [16,17], where Bohmian-mechanics and Madelung-mechanics overlap, sharing the same set of equations and velocity field.

One literal way to view the model of hydrodynamics is as a fluid, where the mass density is the probability density (times the electron mass), and the Bohmian velocity is the velocity field. One way to introduce the thinking behind this identification is to consider the following sequence of conclusions: 1) Since it is not known which Bohmian trajectories to use, all trajectories are used. 2) Since none of the trajectory lines cross, it is useful to identify the set of all such lines as the streamlines of a fluid-velocity field. 3) Using this Madelung fluid, which is called a “probability fluid,” the model can be applied to the study of quantum mechanical states. It this methodology it is immaterial if there is such a fluid. Instead, the information contained in the fluid can be viewed as a way to organize information from the wavefunction. (Note that the Madelung formalism has been generalized to treat many-body systems [10].)

Compared to the information assigned to fluids of classical mechanics, quantum hydrodynamics assigns less information, containing only a velocity field and a mass-density. This suggests that it is incomplete, that there are more classical analogs to be identified. Also, the Hamilton-Jacobi equation of Bohmian mechanics, also used in hydrodynamics, contains the quantum potential $Q$ [1,2,14], which has no classical analog. The dependence of the quantum potential $Q$ on the Laplacian, of the wavefunction, suggests that the velocity field is missing kinetic energy, since kinetic-energy is “stored” in the quantum potential. Also the probability fluid is not assigned a pressure, a field present in all classical fluids.

In hydrodynamics, the velocity field is determined by the phase of the wavefunction. Therefore, the probability fluid of hydrodynamics is not the charge cloud of ab initio theories, since the charge cloud is determined by the probability density. This suggests that these two approaches can be combined into one model.

The assignments of classical-mechanical analogs to functions and equations derived from quantum mechanics have useful applications, as in the many applications of quantum hydrodynamics, discussed below. It is also useful to have core sets of mathematical relationships that are used for different disciplines, as with the Laplace equation. Two mathematical systems, based on completely different axioms, can be equivalent. With this in mind, various models, based on very different identifications, can coexists, with no controversy. Such models require a different metric to evaluate their effectiveness than the metrics used in philosophical theory.

In this field of classical identification, the identification of a classical physical-
property is made by a mathematical analogy. As a generalization of the electron cloud identification, various forms of energy are defined over all space, such that regions of space contain quantitative information about the different energy forms. Satisfactory assignments, of classical mechanical fields to functions from quantum mechanics, should have some correspondence with the quantum states they are applied to. For example, a classical-mechanical velocity field, to be a good match, should have some correspondence with the expectation value of the kinetic energy.

Recently [18, 19], progress has been made in this area of research, with the derivation of an equation that is identified as one from classical mechanics. This equation contains field quantities of classical mechanics, given as formulae of the probability density, applicable to a class of quantum states: The time-independent Schrödinger equation for one-body stationary states with real-valued wavefunctions was shown to be equivalent to a compressible-flow generalization of the Bernoulli equation of fluid dynamics. The kinetic-energy, pressure and mass-density fields are identified from their presence in the Bernoullian equation. The kinetic-energy field naturally yields to the identification of velocity and momentum fields. The derived generalized Bernoulli equation describes compressible, irrotational, steady flow with local variable mass. Over all space, mass is conserved, because the rate of mass creation from the sources are equal to the rate of mass annihilation from the sinks. Also, each fluid element has a constant energy per mass.

This work is continued in this paper, where both the quantum hydrodynamics and the Bernoullian-fluid models are naturally combined into one general model, applicable to all many-body states of quantum mechanics for fermion systems. A total-energy equation is derived. This equation is a generalization of the equations from quantum hydrodynamics and the Bernoullian equation. The total-energy equation can be viewed as a sum of two equations, corresponding to two interacting systems: One of the systems has variable mass from the Bernoullian fluid, and the other one has conserved mass from the Madelung, or probability, fluid. Also, each of the two systems have their own velocity, pressure and energy fields. A generalized Euler equation of fluid dynamics is shown to be implied by the total-energy equation.

Physical properties of classical mechanics are identified as field quantities appearing in equations of classical mechanics, and these fields are maps of the probability distribution and wavefunction phase. The two velocity fields from the Bernoullian fluid and the Madelung fluid are identified in this way. A pressure is also identified as the one from the Bernoullian equation. Furthermore, an energy field is identified as a nonuniform-field generalization of the eigenvalue of the time-independent Schrödinger equation.

A second, independent, method is also presented for the identification of physical properties of classical mechanics for quantum-mechanical systems. These physical properties are also given as fields determined by the wavefunction. For example, two classical momentum-fields, given as an ordered pair, are defined, in a natural way, using the momentum operator and wavefunction of quantum mechanics. (The ordered pair
is the real and imaginary components of the integrand of the expectation-value of the momentum operator, divided by the probability distribution.) This definition yields the same two corresponding velocity fields as the ones mentioned in the previous paragraph, where they were identified by their appearance in classical-mechanical equations. In a similar way, order pairs of field quantities are defined for the energy and pressure. This approach yields the same formulae for the energy and pressure fields mentioned in the previous paragraph.

This approach also gives two unintended consequences: Additional, unknown pressure and energy fields. However, these two field are found to be present in the total-energy equation. They are also shown to appear in an equation that is equivalent to the continuity equation of Bohmian mechanics.

These last two, unintended field quantities, together with the other identified field quantities, gives a classical-mechanical representation, of quantum mechanical systems, that is complete, as far as energy is concerned, where each and every term from the total-energy equation has a classical analog. In the special case of continuity-equation satisfaction, the total-energy equation splits into two equations, that, together with the formulae for the field quantities, are equivalent to the time-dependent Schrödinger equation. Hence, the methodology is also ab initio.

Also, the two nonuniform energy-fields are shown to be conserved in a case where the wavefunction of the time-dependent Schrödinger equation is not necessarily an eigenfunction of the Hamiltonian operator.

While the treatment of energy is complete, the identification of the source of the kinetic energy corresponding to the velocity identified from the Bernoullian equation is an open questions. Progress is made on this issued by exploring of many possibilities. In an attempt to exhaust all possibilities, both particle and fluid descriptions are considered.

Before moving to a description of what is done in the following sections, the next paragraph gives a brief overview of other theoretical results. Applications from quantum hydrodynamics and related fields are also mentioned.

Heifetz and coworkers \cite{20, 21} explores the thermodynamics of Madelung fluids. There are many generalizations of the Madelung equations \cite{22, 23, 24, 25, 26}. The generalization by Broadbridge \cite{23} and Jamali \cite{26} use a complex velocity. Tsekov \cite{27} also uses a complex velocity to derives a complex Navier–Stokes equation. Vadasz \cite{28} derived an extension of the Schrödinger equation from the Navier–Stokes equation. Quantum hydrodynamic theory has been employed to treat systems with single particle wave functions \cite{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44}. The method also also been generalized to treat many particle systems \cite{45, 46}. Application of this formalism include the investigation of spin effects \cite{47, 48}, Bose–Einstein condensates \cite{49}, graphene \cite{50} and plasmas \cite{51, 52, 53}.

Paragraphs that follow indicate the sequence of derivations and results from this paper. Some of the notation used in this overview are introduced. To reduce clutter, this notation is for the special case of one-body expressed in atomic units. Section \ref{13}.
gives a self-contained overview that is easier to follow than the explanations in this introduction. This occurs, because that presentation is not restricted to a sequence that follows the order that the results are obtained. Some readers may prefer to read that material, instead, or in addition to, what follows. Readers can also skip directly to the sections with results, where a brief introduction is given of what is accomplished.

The background needed to fully comprehend the material in this paper involves elementary quantum- and fluid-mechanics. This material is covered in the beginning chapters of many monographs, including Levine [54], for quantum mechanics, and Munson, Young, and Okiishi [55], for fluid mechanics.

Section 2 demonstrates that a many-body generalization of the Bernoulli equation of classical fluid-dynamics is equivalent to the time-independent Schrödinger equation \( \hat{H}\Psi = E\Psi \), in the special case where the wavefunction is real valued. The given derivation is a generalization of a derivation for the special case of one-body [18, 19]. The kinetic-energy \( mu^2/2 \) and pressure \( P \) fields, applicable to quantum mechanical systems, are identified from their presence in the Bernoullian equation: \( mu^2/2 + P/\rho + U = E \), an equation of energy \( E \) conservation, involving the external potential \( U \). The kinetic-energy field \( mu^2/2 \) naturally yields an identification of a velocity field \( u \), and a corresponding particle momentum \( mu \). As far as energy is concerned, there is some flexibility in the choice of the velocity direction, especially the sign. This flexibility in sign is indicated by the notation \( u_\pm \), such that \( u_+ = -u_- \).

Each one of the identified fields of Sec. 2 is defined by a formula involving the probability distribution \( \rho \) and/or derivatives of the probability distribution, e.g., \( P = -\nabla^2\rho/4 \). It is understood that the equivalence of the Schrödinger and Bernoullian equation, mentioned above, requires these mathematical definitions, so, strictly speaking, the equivalence involves an equation set. The derived Bernoullian equation reduces to the well known Bernoulli equation of fluid dynamics for the case of one-body.

Section 3 gives interpretations, in terms of particles, of the fields of the Bernoullian equation, derived in Sec. 2, for quantum states that satisfy this equation. The kinetic-energy \( mu^2/2 \) and pressure \( P \) fields are then shown to be related to the expectation-value \( \langle \Psi | -\nabla^2/2 | \Psi \rangle \) of the kinetic energy: The kinetic-energy integrand \( \Psi(-\nabla^2/2)\Psi \) is equal to the sum of the kinetic- and pressure-energy fields (per volume), i.e., \( \rho mu^2/2 + P \), where \( (\rho m = m|\Psi|^2) \). The pressure field is also shown to vanish when integrated over all space, implying that the kinetic-energy field \( \rho mu^2/2 \) can replace the kinetic-energy integrand \( \Psi(-\nabla^2)/2\Psi \) in the calculation of the expectation-value of the kinetic energy. This result gives an additional, and independent, classical identification of both the kinetic-energy \( mu^2/2 \) and velocity \( u \) fields that agrees with the assignments given in Sec. 2.

Section 4 works on the main equation from Bohmian mechanics, where the wavefunction is written in polar form \( (\Psi = Re^{iS}) \), defined by functions \( (R^2 = \rho) \) and \( S \). The identification from Sec. 2 involving the kinetic energy, mentioned above, is used to identify the well-known quantum potential \( Q \) from Bohmian mechanics, as a sum
of two fields, involving the kinetic energy and pressure, i.e., \( Q = \frac{mv^2}{2} + \frac{P}{\rho_m} \). When this sum replaces the quantum potential in the Hamilton-Jacobi equation of Bohmian mechanics—that already contains one kinetic-energy field \( \frac{mv^2}{2} \)—the resulting Hamilton-Jacobi equation becomes \( \left( \frac{mv^2}{2} + \frac{mu^2}{2} + \frac{P}{\rho} + U \right) = -\frac{\partial S}{\partial t} \), where \( \partial S \) is the time derivative of the phase, and the equation now has two kinetic energy fields, \( \frac{mv^2}{2} \) and \( \frac{mu^2}{2} \), and one pressure field \( P \). The two kinetic energy fields yield two velocity fields, \( v \) and \( u \), and two corresponding particle momentums, \( mv \) and \( mu \): One momentum \( mv \) is from Bohmian mechanics, and the other one \( mu \) is from the Bernoullian equation developed in Sec. 2.

The resulting Hamilton-Jacobi equation, mentioned in the previous paragraph, together with the continuity equation, is demonstrated to be equivalent to the time-independent Schrödinger equation. For the special case of stationary states, this equation reduces to a generalization of the Bernoullian equation developed in Sec. 2 holding also for complex valued wavefunctions. Also, for stationary states, we have \( (E = -\partial S) \), where \( E \) is the energy eigenvalue \( E \) of the Schrödinger equation, a uniform field.

For the remainder of Sec. 4, it is demonstrated that if the two velocity fields are orthogonal, i.e., \( (v \cdot u = 0) \), then the continuity equation of Bohmian mechanics, \( (\partial \rho = -\nabla \cdot \rho v) \), reduces to a generalization of the Poisson equation, and to a Laplace equation for stationary states. This material in not part of the main logical sequence, and it is not needed for the later results.

In quantum mechanics, momentum, an observable, is defined via axioms involving the momentum operator \( \hat{P} \). In contrast, the above mentioned momentum-particle definitions come from identifications in equations that are implied by Schrödiger equations. It is, therefore, reasonable to investigate if the momentum operator itself can naturally give alternate definitions for the same particle momentums mentioned above. In Sec. 5, this approach is investigated, yielding two particle-moment definitions, as a pair of fields, defined by a complex-valued function \( \Psi^* \hat{P} \Psi / \rho \), involving the momentum operator \( \hat{P} \). One of the particle momentums is taken as the real part of the function, and the other momentum is taken as the imaginary part. The resulting pair of momentums, called momentae, \( mv \) and \( mu \), are the same ones defined in Secs. 2 and 4. Furthermore, a natural definition for a kinetic energy-field is also given, giving the same field as in Secs. 3 and 4. Since the momentums fields, \( mv \) and \( mu \), are irrotational, they are also expressed by their potentials: \( S \) and \( (\theta = -\hbar \ln \rho) \), respectively.

Sec. 6 defines a notational system that reduces clutter in equations for many-body systems, and yields, in most case, many-body equations that are displayed exactly the same way as in the special case of one-body. In order that this system is not confused with sloppy notation, where subscripts are suppressed and understood, the presentation is formal.

After the diversion of Sec. 6, the main logical sequence is continued in Sec. 7 where, as in the particle momentae definitions in Sec. 5, two energy fields are defined as the real and imaginary parts of the function \( \Phi^* \hat{H} \Phi / \rho \), where the \( \hat{H} \) is the Hamiltonian (energy)
operator. Using the Schrödinger Eq. \( i\Phi^*\partial\Phi = \Phi^*\hat{H}\Phi \), the approach yields a pair of energy fields as the real and imaginary parts of \( i\Phi^*\partial\Phi / \rho \): \( E_S = -\partial S \) and \( E_\theta = -\partial \theta \), where \( S \) and \( \theta \) are the momenta potentials, mentioned above.

One of the energy fields \( E_S \), identified by the development of Sec. 7, assigns an energy field to any solution of the time-dependent Schrödinger equation. When \( (-\partial S = E_S) \) is substituted into the Hamilton-Jacobi equation developed in Sec. 4, this equation becomes \( (E_S = mv^2/2 + um^2/2 + P/\rho + U) \), an energy equation. Furthermore, the field \( E_S \) reduces to the energy eigenvalue \( E \) of the Schrödinger equation in the case of time independence.

At this point in Sec. 7, the other energy field \( E_\theta \) is an unknown, unintended consequence of the development. However, later in Sec. 9, this field appears in an energy form of the continuity equation, an equation equivalent to one of the equations of Bohmian mechanics, implied by the Schrödinger equation.

Sec. 8 extends the method of definition for energies and momenta fields to pressures, where, unlike the other cases, the pressure is not an observable of quantum mechanics. A definition is given, such that the two pressures are proportional to the real and imaginary parts of the divergence of the momentum (per volume), \( \nabla \cdot (\Psi^*\hat{P}\Psi) \). This definitions yields the same pressure field \( P \) mentioned above. The second pressure \( P_\nu \), an unknown, appears in the same continuity equation as the energy field \( E_\theta \), mentioned above. (The notation for the other pressure is changed from \( P \) to \( P_u \)).

Sec. 9 uses the continuity equation to derive a number of relations involving the pressures. The energy equation \( (E_\theta = P_\nu/\rho) \), mentioned in the previous paragraph, is obtained in the development, containing both the unknown pressure \( P_\nu \) and energy \( E_\theta \) fields. This energy equation is shown to be equivalent to the continuity equation. Hence, the three defined ordered pairs—\( \{mv, mu\} \), \( \{P_\nu, P_u\} \), and \( \{E_S, E_\theta\} \)—are accounted for, appearing in two equations that, together, are equivalent to the time-dependent Schrödinger equation. Each and every term is identified from the two equations, one term \( U \) being the sum of the classical external-potential and electron-electron repulsion-energy.

Sec. 10 combines the two energy equations, giving a total-energy equation \( (E = 1/2mv^2 + P_\nu\rho^{-1} + 1/2mu^2 + P_u\rho^{-1} + U) \), where \( (E \equiv E_S + E_\theta) \). A many-body generalization of the Euler equation of fluid dynamics is derived from this equation. For one-body states, this Eulerian equation can be viewed as a sum of two equations, corresponding to two interacting systems. One with variable mass with velocity \( u \) and pressure \( P_u \); one with conserved mass with velocity \( v \) and pressure \( P_\nu \). The local time derivative of the velocity fields are shown to satisfy \( (-\nabla E_S = m\partial v) \) and \( (-\nabla E_\theta = m\partial u) \), where, for the case of one spin-free body, values \( \partial v(r, t) \) and \( \partial u(r, t) \) are the time rate of change of the velocity fields at fixed position \( r \in \mathbb{R}^3 \) and time \( t \). (These are not accelerations of fluid elements.)

Sec. 11 investigates the important property of energy conservation for the two energy fields, \( E_\theta \) and \( E_s \), where these fields are not, in general, uniform. The wavefunctions considered are linear combinations of functions, where each function is
an eigenfunction of the same Hamiltonian operator. If such a wavefunction contains nondegenerate eigenfunctions, it does not satisfy the time independent Schrödinger equation. It is demonstrated that, in this case, the two energy quantities are conserved over all space for all times: The two forms of energy can flow in space, but neither energy is created or destroyed.

Sec. 12 considers various issues involving velocity. Sec. 12.1 investigates a velocity compatibility problem for the two velocity fields, where the vector sum $v + u$ does not give the correct total kinetic-energy, if $v \cdot u \neq 0$. Sec. 12.2 considers other possibilities for the velocity $u$ from the Bernoullian equation, especially the velocity direction. From non-relativistic quantum mechanics, a velocity field is derived that has the same direction as a well known one, as an approximation from the Pauli equation. Various issues of local variable mass, angular momentum, and kinetic energy-satisfaction are considered, for both fluid and particle descriptions. While progress is made, the question of the best choice of Bernoullian velocity $u$ remains open.

2. A Classical Mechanics Energy Equation for Stationary States with Real Valued Wavefunctions of Quantum Mechanics

In this section, a many–body generalization of the Bernoulli equation of fluid-dynamics is shown to be equivalent to the time-independent Schrödinger equation, in the case where the wavefunction is real-valued. The kinetic-energy $mu_i^2/2$ and pressure $P_i$ fields, applicable to quantum-mechanical systems, are identified from their presence in the Bernoullian equation, and these fields have formulae dependent on the probability distribution $\Upsilon = |\Psi|^2$. The kinetic-energy fields $mu_i^2/2$ naturally yields the identification of a $n$ velocity fields $u_i$ and particle momentums $mu_i$. The main result of this section is the Bernoullian Eq. (8), that reduces to (11), for the case of one body with probability distribution ($\rho = \Upsilon$).

The $n$-body time-independent Schrödinger equation with a normalized, real-valued eigenfunction $R$, can be written

$$-\frac{\hbar^2}{2m} \sum_{i=1}^{n} R \nabla_i^2 R + \sum_{i=1}^{n} V_i \Upsilon + \frac{1}{2} \sum_{i \neq j}^{n} W_{ij} \Upsilon = E \Upsilon,$$

(1)

where

$$\left[R \nabla_i^2 R\right](x) = R(x) \nabla_{r_i}^2 R(x), \quad x = x_1, \cdots x_n.$$

Also, the $n$-body probability distribution $\Upsilon$ is $\Upsilon = R^2$; the electron coordinate $x_i$ is defined by $x_i = r_i, \omega_i$, where $r_i \in \mathbb{R}^3$ and $\omega_i \in \{-1, 1\}$ are the spatial and spin coordinates, respectively. Furthermore, the functions $V_i$ and $W_{ij}$ are spin-free multiplicative operators. For $n$-electron systems, these are given by the following:

$$[V_i \Upsilon](x) = V(r_i) \Upsilon(x), \quad [W_{ij} \Upsilon](x) = \frac{e^2}{4\pi\varepsilon_0} |r_i - r_j|^{-1} \Upsilon(x),$$

(2)

where $-e$ is the electron charge, $\varepsilon_0$ is the permittivity constant, and the one-body external potential $V$ is a specified real-valued function with domain $\mathbb{R}^3$ such that
\{x | R(x) = 0\} has measure zero. This requirement for V implies that the division of an equation by R or \( \Upsilon \) gives an equation that is defined almost everywhere (a.e.).

Substituting the following equality

\[
-\frac{1}{2} \left[ R \nabla^2 R \right] = \frac{1}{8} \left[ \Upsilon^{-1} \nabla_i \Upsilon \cdot \nabla_i \Upsilon \right] - \frac{1}{4} \nabla_i^2 \Upsilon,
\]

that is proved elsewhere [18, 19], into the Schrödinger equation (1), we obtain

\[
\frac{\hbar^2}{8m} \sum_{i=1}^{n} \Upsilon^{-1} |\nabla_i \Upsilon|^2 - \frac{\hbar^2}{4m} \sum_{i=1}^{n} \nabla_i^2 \Upsilon + \sum_{i=1}^{n} V_i \Upsilon + \frac{1}{2} \sum_{i \neq j}^{n} W_{ij} \Upsilon = E \Upsilon.
\]

Using the mathematical definitions

\[
u_i^{\pm} = \pm \frac{\hbar}{2m} \nabla_i \Upsilon,
\]

\[
P_i = -\frac{\hbar^2}{4m} \nabla_i^2 \Upsilon,
\]

where we call \( \nu_i^{\pm} \) the Bernoullian velocity, and

\[
\frac{1}{2} m \nu_i^2 = \frac{1}{2} m \nu_i^{\pm} \cdot \nu_i^{\pm} = \frac{1}{2} m \left( \frac{\hbar}{2m} \nabla_i \Upsilon \right)^2 = \frac{\hbar^2}{8m} \Upsilon^{-2} |\nabla_i \Upsilon|^2,
\]

we have

\[
\sum_{i=1}^{n} \left( \frac{1}{2} m \nu_i^2 + P_i \Upsilon^{-1} \right) + U = E.
\]

For any point \( x \) such that \( \Upsilon(x) \neq 0 \), this energy equation can also be written

\[
\sum_{i=1}^{n} \left( \frac{1}{2} m \nu_i^2 + P_i \Upsilon^{-1} \right) + U = E.
\]

where

\[
U = \frac{1}{2} \sum_{i \neq j}^{n} W_{ij} + \sum_{i=1}^{n} V_i.
\]

Since the derivation of equation (8) from the Schrödinger equation (1) is reversible, Eqs. (1) and (8) are equivalent, i.e., \( \Upsilon \) is a solution of (8) a.e., if and only if \( \Upsilon \) is a solution of (1) a.e.

Next consider a state of a one-body system, such that \( R(r, \pm 1) = \phi(r) \alpha(\pm 1) \), where \( \alpha \) is the spin function that satisfies \( \alpha(1) = 1 \) and \( \alpha(-1) = 0 \). Hence, \( \Upsilon(r, 1) = \phi^2(r) = \rho(r) \), and the last equality defines the one-body probability density \( \rho \). In this special case, with \( U = V \), (4), (5) and (8) can be written

\[
u_i^{\pm} = \pm \frac{\hbar}{2m} \nabla_i \rho, \quad P = -\frac{\hbar^2}{4m} \nabla_i^2 \rho, \]

\[
\frac{1}{2} m \nu_i^2 + P \rho^{-1} + U = E.
\]

These equations have been used to treat one-body stationary-states of quantum-mechanical systems with real-valued wave-functions as flows of a fluid, where Eq. (11) is identified as a compressible-flow generalization of the Bernoulli equation with body
force $U$, pressure $p$, velocity field $u_\pm$, and mass density $\rho_m = m \rho$ [18, 19]. Also, $E\rho$ is the energy per volume, and $E/m$ is the energy per mass, of the fluid elements. Furthermore, $m$ is total the mass of the fluid, equal to the electron mass. The fields $mu_\pm$ and $\rho_m u_\pm$ are called particle- and fluid-momentums, respectively. 

The velocity choice $u_\pm$ of Eq. (10) for one-body systems, appears in other investigations. The function $\varrho_m |u_\pm|^2/2$ is a term of the Hamiltonian functional of the generalized fluid-dynamics formalism by Broer [56], where the Hamiltonian functional is derived from the time-dependent Schrödinger equation, and $\varrho_m$ satisfies $\varrho_m = 2\rho_m$. Salesi [57] obtains a Lagrangian function that is equivalent to the Madelung equations, and this function contains the term $\rho_m |u_\pm|^2/2$. He shows that a variational approach of the Lagrangian has $\rho_m |u_\pm|^2/2$ determining the quantum potential $Q$ of Bohmian mechanics. He interprets $|u_\pm|^2/2$ as the internal energy of the relative motion in the center of mass coordinate frame from the Zitterbewegung (ZWB) model of spin. The velocity magnitude $|u_\pm|$ follows as a non-relativistic approximation of a velocity expression of Hestenes [58] of Schrödinger–Pauli theories. Furthermore, Tsekov [27] obtains the same velocity choice $u_\pm$ of Eq. (10) as the imaginary component of a complex velocity, where the formalism involves diffusion.

The velocity choice $u_-$, called downhill flow, has the fluid particles move in the direction of lower density; similarly, choice $u_+$ is called uphill flow. For later use, we note that the Bernoullian velocity definition (4) can be generalized to

$$u_{i\pm} = u_{i\pm} \hat{s}_i, \quad u_{i\pm} = \pm \frac{\hbar}{2m} \frac{\nabla_i \Upsilon}{\Upsilon}, \quad (12)$$

where $\hat{s}_i = \hat{s}_i(r)$ is a unit vector at our disposal. This generalizations yields kinetic energies $mu_i^2$ that are the same as (3), and the energy equation (8) does not depend on the unit vector $\hat{s}_i$. 

3. Interpretations Based on Classical Mechanics

As in Bohmian Mechanics [12], in this section, we present a particle interpretation for one-body states that satisfy the energy energy (11), and then the interpretation is generalized to many-body states that satisfy the Bernoullian Eq. (8). The kinetic-energy integrand of quantum mechanics is shown to be equal to the sum of the kinetic- and pressure-energy fields (per volume), as indicated in Eq. (14). The pressure fields are shown to vanish when integrated over all space, implying that the kinetic-energy field $\rho_m(u_1^2 + \cdots u_n^2)/2$ can replace the kinetic-energy integrand in the calculation of the expectation-value of the kinetic energy. This result gives an additional, and independent, classical identification of both the kinetic-energy $mu_i^2/2$ and velocity $u_i$ fields that agrees with the assignments given in Sec. 2.

For one-body states, when the particle is at position $r \in \mathbb{R}^3$, we interpret $|u_\pm|(r)$ and $P(r)$ as the speed and pressure, respectively, defined by Eqs. (10). Also, for the energy equation (14), we take $[mu^2/2]$, $[pp^{-1}]$, $U$, and $E$ to be the kinetic-, “compression”-, potential-, and total-energy of the particle, respectively, where these
scalar fields have domain $\mathbb{R}^3$, and $E$ is uniform. Since the kinetic energy depends on the particle’s mass and speed only, we only require the velocity to satisfy Eq. (12). Thus, we do not require the velocity direction to be either of the directions given by irrotational vector field (10). Further consideration of the velocity choice is given in Sec. (12.2), including a well known velocity field that is derived from relativistic quantum-mechanics.

The interpretations for one-body states have a natural generalization to many-body states, with energy equation (8). For example, as in Bohmian mechanics, the configuration $\mathbf{x} = x_1, \cdots x_n$ is taken to mean that one electron is located at $r_1$ with spin-variable value $\omega_1$, another one is at $r_2$ with $\omega_2$, and so on. The function
\begin{equation}
\frac{1}{2}m[u_i(x_1, x_2, \cdots x_i, \cdots x_n)]^2,
\end{equation}
is interpreted as the kinetic energy of the $i$th particle from configuration $\mathbf{x}$, that is, it is the kinetic energy of a particle located at $x_i$, where the $i$th particle speed is $|u_{i\pm}(x)|$. Similar interpretations are given to the other energy terms from (8), and $P_i(x)$ can be interpreted as the pressure subjected to the $i$th particle from configuration $\mathbf{x}$. With these interpretations, except for the spin dependence and the probability distribution $\Upsilon$, equation (8) is interpreted as a classical energy equation $H = E$, with a Hamiltonian function $H$ that depends on the probability distribution $\Upsilon$, partial derivatives of the probability distribution, and the potential energy function $U$.

By comparing (11) and (7), we obtain an equality satisfied by the integrand of the expectation-value of the kinetic energy from quantum mechanics:
\begin{equation}
-\frac{\hbar^2}{2m}\sum_{i=1}^{n}[R\nabla_i^2 R] = \Upsilon \sum_{i} \left( \frac{1}{2}mu_i^2 + P_i \Upsilon^{-1}\right).
\end{equation}
From this equation and the above particle interpretations, it follows that the value at the point $\mathbf{x}$ of the integrand, of the expectation value of the kinetic energy, is the sum of the $n$ particle kinetic- and compression-energies for configuration $\mathbf{x}$, multiplied by the weight $\Upsilon$, and $\Upsilon$ is required to be normalized, i.e.,
\begin{equation}
\sum_{\omega_1, \cdots, \omega_n} \int_{\mathbb{R}^3n} \Upsilon = 1.
\end{equation}
Note that the compression energy $P_i \Upsilon^{-1}$ times the probability distribution $\Upsilon$ is the pressure $P_i$ for the $i$th particle.

For each Cartesian coordinate $\alpha_i \in \{x_i, y_i, z_i\}$, ($i = 1, \cdots n$), we require the wavefunction to satisfy
\begin{equation}
\lim_{\alpha_i \to \pm \infty} R(\mathbf{x}) = \lim_{\alpha_i \to \pm \infty} \frac{\partial R}{\partial \alpha_i} = 0.
\end{equation}
Hence
\begin{equation}
\int_{-\infty}^{\infty} \frac{\partial^2 \Upsilon}{\partial \alpha_i^2} d\alpha_i = \frac{\partial \Upsilon}{\partial \alpha_i} \bigg|_{-\infty}^{\infty} = 2 R \frac{\partial R}{\partial \alpha_i} \bigg|_{-\infty}^{\infty} = 0,
\end{equation}
and therefore
\begin{equation}
\int_{\mathbb{R}^3} \nabla_i^2 \Upsilon \, d\mathbf{r}_i = 0.
\end{equation}
This result combined with (5) gives
\[
\int_{\mathbb{R}^3} P_i(x) \, dr_i = 0.
\] (15)
Therefore, it follows from Eq. (15), that for a fixed “partial” configuration
\[x'_i = x_1 \cdots x_{i-1}, x_{i+1} \cdots x_n,
\]
where \(x_i\) is excluded, that the function
\[P_{x'_i}(x_i) = P_i(x_1 \cdots x_m), \quad x_i \in \mathbb{R}^3,
\]
where \(x'_i\) is considered a parameter, must have both positive and negative values on subspaces with nonzero measures, or be the zero function a.e. Also, it follows from Eqs. (14) and (15) that the pressure \(P_i\) does not contribute to the expectation value of the kinetic energy.

Using equality (15) for \(P_i\), and integrating (14) over the \(3n\) spatial coordinates and summing over the \(n\) spin coordinates from the set \(\Omega\), we have
\[
\langle T \rangle[R] = -\frac{\hbar^2}{2m} \sum_{\Omega} \int_{\mathbb{R}^{3n}} \sum_{i=1}^n R \nabla_i^2 R = \sum_{\Omega} \int_{\mathbb{R}^{3n}} \sum_{i=1}^n \frac{1}{2} \Upsilon \mu_i^2.
\] (16)
Hence
\[
\langle T \rangle[\Upsilon] = \left\langle \sum_{\Omega} \sum_{i=1}^n \frac{1}{2} \mu_i^2 \right\rangle,
\] (17)
where \(\langle T \rangle[\Upsilon]\) is the expectation value of the kinetic energy. This result supports the interpretations given in Sec. 2 for the kinetic energy and speed. For a one-body system represented by the spin orbital \(\phi_\alpha\), as is done in the derivation of Eqs. (10) and (11), Eq. (17) reduces to
\[
\langle T \rangle = \int \frac{1}{2} \rho_m u^2 \, dr,
\]
where \(\rho_m = m \rho\), and for a fluid interpretation, \(\rho_m\) is the mass density.

4. Extended Bohmian Mechanics

In this section, the kinetic-energy integrand result (14) from the previous section is used to identify the quantum potential \(Q\) from Bohmian mechanics. This relation is substituted into the Hamilton-Jacobi equation of Bohmian mechanics. The gives result (26), containing the pressures \(P_i\) and velocity \(u_i\) fields from the Bernoullian equation (8), and also the original velocity \(v_i\), defined by Eq. (22) from Bohminan Mechanics. Also, \(S\) is the wavefunction phase of the polar form \((\Psi = Re^{iS/\hbar})\) of the wavefunction. The resulting equation (26), together with the continuity equation (20) of Bohminan mechanics, along with the given definitions for the fields, are shown to be equivalent to the time-dependent Schrödinger equation (18). For the special case of stationary states, we have \((-\partial S = E\)) where \(E\) is the eigenvalue of the time independent Schrödinger equation (18), and the resulting equation (27) is a generalization of the Bernoullian Eq. (8), holding also for complex valued wavefunction.
The time-dependent Schrödinger equation is

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \sum_{i} \nabla_i^2 \Psi + U \Psi = \hat{H} \Psi, \tag{18} \]

where \( \partial \Psi(t) = \partial \Psi/\partial t, \) \( U = U(x, t) \) is given by \( \Psi = \Psi(x, t) \) is the \( n \)-body time-dependent wavefunction, and we use the same notation as in the previous sections, e.g., \( x_i = r_i, \omega_i \). Let the spin coordinates \( \omega_i, \cdots \omega_n \) be specified parameters. Hence, \( \Psi = \Psi(r, t) \), where \( r = r_1, \cdots r_n \), permitting us to consider \( \Psi \) to be a function of the time and the spatial coordinates only. For Bohmian mechanics \([1, 2, 3, 4, 5, 14, 6, 7, 8, 9, 10]\), the wavefunction is represented in polar form is

\[ \Psi = R e^{iS/\hbar}, \tag{19} \]

where \( R \) and \( S \) are time-dependent real-valued functions. When this ansatz is substituted into the time-dependent Schrödinger equation \( \text{(18)} \), after significant manipulations, the following two equations can be obtained \([10]\):

\[ \partial \Upsilon + \sum_{i=1}^{n} \nabla_i \cdot (\Upsilon v_i) = 0, \tag{20} \]

\[ -R\partial S = \sum_{i=1}^{n} \left( \frac{1}{2m} R \nabla_i S \cdot \nabla_i S - \frac{\hbar^2}{2m} \nabla_i^2 R \right) + UR, \tag{21} \]

where the probability distribution is \( \Upsilon = \Psi \Psi^* = R^2, \) \( \partial S(x, t) = \partial S(x, t)/\partial t, \) and the velocity \( v_i \) of the \( i \)th particle is defined below. Equation \( \text{(20)} \) is called the continuity equation. In the special case of a one-body system, with \( \Upsilon = \rho \), this equation has the same form as the continuity equation from fluid dynamics \([55, 60]\), a statement of the conservation of mass, where the mass density is \( m\rho \). The above two equations \( \text{(20)} \) and \( \text{(21)} \) are identical to Eqs. \( \text{(12)} \) and \( \text{(6)} \) in the manuscript by Renziehausen and Barth \([10]\), for the special case considered here where there is only one kind of particle, e.g., only electrons. Also, by examining the mathematics used in the derivation of \( \text{(20)} \) and \( \text{(21)} \), it is easy to demonstrate that these two equations, taken together, are equivalent to the Schrödinger Eq. \( \text{(18)} \). The quantum potential \( Q \), presented below, is defined by Eq. \( \text{(18)} \) in the same manuscript.

Bohmian mechanics also defines the following two functions:

\[ v_i = \frac{\nabla_i S}{m} = \text{Im} \left( \frac{\hbar}{m} \frac{\nabla_i \Psi}{\Psi} \right), \tag{22} \]

\[ Q = -\frac{1}{R} \frac{\hbar^2}{2m} \nabla_i^2 R = \Upsilon^{-1} \left( -R \frac{\hbar^2}{2m} \nabla_i^2 R \right), \tag{23} \]

where \( Q \) is known as the Bohm quantum potential \([1, 2, 14]\), and we call \( v_i \) the de Broglie velocity. The function value \( v_i(x_1, \cdots x_i, \cdots x_n) \) is interpreted as the velocity of the \( i \)th particle, i.e., the velocity of the particle located at \( x_i \) for the configuration \( x = x_1, \cdots x_n \). The identity given in \( \text{(22)} \) is included because sometimes the second definition is used when discussing Bohmian mechanics. This identity is proved in Appendix \[\text{Appendix B}\] where a relation between the two velocities, \( v_i \) and \( u_i \), is also given.
Substituting the above two definitions into (21), and dividing by \( R \), we get
\[
-\partial S = \sum_i \frac{1}{2} m v_i^2 + Q + U,
\]  
where \( v_i^2 = |v_i|^2 \). This is a Hamilton–Jocobi equation [14] with the addition of the quantum potential \( Q \). By direct substitution, it is easily seen that Eq. (24) along with definitions (22) and (23) is equivalent to Eq. (21). Hence, Eq. (24), together with definitions (22) and (23), and the continuity equation (20), are equivalent to the Schrödinger Eq. (18).

Note that Eq. (14) is an equality holding for two times differentiable real-valued functions, where \( u_i^2 = |u_i|^2 \) and \( P_i \) are given by Eq. (4) and (5), respectively. Next we extend the interpretations of the functions \( u_{i\pm} \) and \( P_i \) from Eqs. (4) and (5) to the case where \( R \) is the real-valued factor of the time-dependent wavefunctions \( \Psi \), given by ansatz (19), and note that these functions also appear in (14). Making these interpretations and substituting Eq. (14) into (23), we discover
\[
Q = \sum_i \frac{1}{2} m u_i^2 + \sum_i P_i \Upsilon^{-1},
\]  
where \( Q \) is a sum of the \( n \)-particle kinetic energy \( \sum_i \frac{1}{2} m u_i^2 \) and the \( n \)-particle compression-energy \( \sum_i P_i \Upsilon^{-1} \). The kinetic energy portion of this quantum-potential \( Q \) expression agrees with the one from Salesi [57], where no interpretation is given for the other term of \( Q \).

Substituting (25) into (24) gives the desired result:
\[
-\partial S = \sum_i \left( \frac{1}{2} m v_i^2 + \frac{1}{2} m u_i^2 + P_i \Upsilon^{-1} \right) + U.
\]  
This equation is a further development of (24), containing two kinetic energy terms, a compression energy term \( \sum_i P_i \Upsilon^{-1} \), and the external potential \( U \), given by (9). It seems reasonable at this point to assume that the total velocity of the \( i \)th particle is \( u_{i\pm} + v_i \). Eq. (26) is a variant of the Hamilton–Jocobi equation. The right-hand-side of (26) can be interpreted as the time-dependent energy, i.e., a Hamiltonian function. For the left-hand side, from Eq. (22), \( S \) can be interpreted as the momentum potential for each of the \( n \) particles, but only including the de Broglie velocity \( v_i \) portion of the total velocity.

By direct substitution, Eqs. (24) and (25) are equivalent to Eq. (26). Hence, Eq. (26) together with Eq. (20) along with the above definitions for the fields \( v_i, u_i, \) and \( P_i \), are equivalent to the Schrödinger Eq. (18).

If \( \Psi \) is a stationary state then \( \Psi(x) = R(x)e^{-iEt/\hbar} \), giving \( S(t) = -Et \). Hence, from (26), we have
\[
\sum_i \left( \frac{1}{2} m v_i^2 + \frac{1}{2} m u_i^2 + P_i \Upsilon^{-1} \right) + U = E.
\]  
This equation is a generalization of Eq. (15), holding for complex valued wavefunctions. Multiplying this equation by \( \Upsilon \), using the time-independent Schrödinger equation,
$(\Psi^* \bar{H} \Psi = E \Upsilon)$ with Hamiltonian definition (18), integrating the result over the $3n$ spatial coordinates, summing over the $n$ spin coordinates, and using also Eq. (15), we find that

$$\langle T \rangle [\Upsilon] = \left\langle \sum_{\Omega} \sum_{i=1}^{n} \left( \frac{1}{2} m u_i^2 + \frac{1}{2} m v_i^2 \right) \right\rangle,$$

a generalization of Eq. (17). This result supports the interpretation of speed and kinetic energy involving the velocity vector $v_i$.

Note that the Bernoullian velocity $u_{i\pm}$ does not solve a continuity equation like (20), instead, from (4) and (5), we have

$$\nabla_i \cdot (u_{i\pm} \Upsilon) = \pm \frac{\hbar}{2m} \nabla^2 \Upsilon = \mp \frac{\hbar}{2m} P_i.$$ (28)

This equation should not be confused with the continuity equation implied by the time-dependent Schrödinger equation that implies probability conservation. That equation remains satisfied, since our probability distributions under consideration have wavefunctions that satisfy the time dependent Schrödinger equation.

Consider a one-body state where the velocity field is (10), i.e., $\rho_m u_{\pm} = \pm \hbar \nabla \rho_m / 2m$. This equation is Fick’s law of diffusion [61] with diffusion coefficient ($D_{\pm} = \mp \hbar / 2m$), but since (28) is not in the proper form of a continuity equation, the well known diffusion equation ($\partial \rho_m = D \nabla^2 \rho_m$) [61] is not satisfied.

For the remainder of this section, the continuity equation (20) is shown to reduce the the Poisson equation, and to the Laplace equation for stationary states, if the velocity fields are orthogonal $v \cdot u = 0$. This material is not needed for the later results.

Next we examine the continuity equation (20), the one implied by the time-dependent Schrödinger equation. Let a sum of the subscript $i$ over $\{1, \cdots, n\}$ be understood, and we suppress the $i$ subscript on the del operators $\nabla$.

The process of expanding out the divergence term from (20) and then using (22) is

$$0 = \partial \Upsilon + \nabla \cdot (\Upsilon v_i) = \partial \Upsilon + \nabla \Upsilon \cdot \nabla \bar{S} + \Upsilon \nabla^2 \bar{S},$$

where the de Broglie velocity is $\nabla_i \bar{S} = \nabla_i \bar{S} / m = v_i$. Hence,

$$-\partial \Upsilon = \nabla \Upsilon \cdot \nabla \bar{S} + \Upsilon \nabla^2 \bar{S}, \quad \bar{S} \dot{=} \bar{S} / m.$$ (29)

We take this opportunity to obtain the continuity equation under two special conditions:

If $\nabla \Upsilon \cdot \nabla \bar{S} = 0$, then $\nabla^2 \bar{S} = -\Upsilon^{-1} \partial \Upsilon = -\partial \ln \Upsilon$, (30)

If $\nabla \Upsilon \cdot \nabla \bar{S} = 0$ and $\partial \Upsilon = 0$, then $\nabla^2 \bar{S} = 0$.

These equations are the Poisson and Laplace equations, respectively. A rule for taking the logarithm of a dimensioned quantity, as in $\ln \Upsilon$, is given in the second paragraph of the next section. Note that the equation $\nabla \Upsilon \cdot \nabla \bar{S} = 0$ is equivalent to the two velocity fields being orthogonal, i.e., $v_i \cdot u_i = 0$, and this can be verified by examining the definitions of the two velocity fields, Eq. (4) and (22).
5. Momentum Potentials and the Kinetic Energy Fields

Definition. The particle momenta set is ordered pair \( \{ m v, m u_\pm \} \), where the momenta members are defined by Eqs. (22) and (4).

In previous sections, definitions of the two particle-momenta, \( m u_\pm \) and \( m v \), are made when these fields make an appearance in classical-mechanical equations, where the classical-mechanical equations are implied by Schrödinger equations. In quantum mechanics, momentum, an observable, is defined via axioms involving the momentum operator. Therefore, it seems reasonable to investigate if rules can be found that yield the particle momenta using the momentum operator, giving independent, and alternative, definitions of these fields that do not involve Schrödinger equations. This approach is investigated in this section, yielding two particle-moment definitions, as a pair of fields, defined by a complex-valued function involving the momentum operator: One particle momentum is taken as the real part of the function; one is taken as the imaginary part. The resulting pair of momenta are the momenta, \( m u \) and \( m v \), defined above. Furthermore, using the momentum operator, a natural definition for a kinetic energy-field is also given, giving the same field as in Sec. 2. To reduce clutter in the derivation, we begin with a one-body state and use the down-hill velocity choice \( u_- \).

Let \( \zeta \) be a constant with the same units as the probability density \( \rho \), i.e., per volume. Let the natural logarithmic function \( \ln \) be refined by \( \ln \rho = \log_e(\rho/\zeta) \), taking care of the requirement that the function \( \ln \) is only defined on dimensionless quantities, and this definition can be used for other dimensioned quantities with the modification understood. This \( \ln \) definition is useful when only derivatives of \( \ln \) are assigned meaning. Let

\[
\theta_\pm = \pm \frac{\hbar}{2} \ln \rho. \tag{31}
\]

Using this definition, and the velocity definitions (22) and (10), the irrotational particle-momenta can be expressed using their potentials:

\[
m v = \nabla S, \quad m u_\pm = \nabla \theta_\pm. \tag{32}
\]

Let \( \hat{P} \) be the momentum operator for quantum mechanical states of one-body systems. Let the two particle-momentums of a state with wavefunction \( \Psi \) be the real and imaginary parts of \( (\Psi^* \hat{P} \Psi)\rho^{-1} \), where \( \rho = \Psi^* \Psi \) is the probability distribution. Similarly, the fluid momentums per volume are defined as the real and imaginary parts of \( \Psi^* \hat{P} \Psi \). With the definition \( i\hbar^{-1} \hat{P} = \nabla \) in mind, we obtain the following equation sequence:

\[
\Psi^* \nabla \Psi = i\hbar^{-1} \Psi^* \hat{P} \Psi = Re^{-iS/\hbar} \nabla (Re^{iS/\hbar}) = i\hbar^{-1} \rho \nabla S + R \nabla R,
\]

\[
i\hbar^{-1} \Psi^* \hat{P} \Psi = i\hbar^{-1} \rho \nabla S + \frac{1}{2} \nabla \rho,
\]

\[
\Psi^* \hat{P} \Psi = \rho \nabla S - \frac{\hbar}{2} \nabla \rho.
\]
Using the momente potential definitions \((32)\), we obtain the objective:

\[
\Psi^* \hat{P} \Psi = \rho \nabla S + i \rho \nabla \theta = \rho_m v + i \rho_m u, \tag{33}
\]

\[
\frac{\Psi^* \hat{P} \Psi}{\Psi^* \Psi} = \nabla S + i \nabla \theta = mv + imu, \tag{34}
\]

where \(\rho_m = m \rho\), \(u = u_\downarrow\), and we will continue using the downhill velocity choice \(u_\downarrow\). Since any complex function can be written in polar form, the above formulae for momentums \(m u\) and \(m v\) are determined by the momentum operator \(\hat{P}\). Since \(u_\downarrow\) has the same sign as \(v\) in the above equations, we use this as justification for choosing \(u_\downarrow\) over \(u_\uparrow\). However, the corresponding \(u_\uparrow\) equations are obtained simply by replacing \(u_\downarrow\) with \(-u_\downarrow\). Note that the definition \(v_\pm = \pm \nabla S/m\) corresponds to two linearly independent wavefunctions \(\text{Re}^\pm i S/\hbar\), with the same \(R\) function, if \(S \neq 0\). Also note that the particle-momentum definition \((33)\) for the Bernoullian velocity \(u\) fixes the direction of this velocity field to within a sign, removing the flexibility in the direction of \(u\) for its corresponding kinetic energy, as pointed out in the end of Sec. 2.

Next we give a natural definition for the kinetic energy \(K\), and obtain its formula:

\[
K = K_v + K_u = \frac{1}{2m} \frac{\Psi^* \hat{P} \Psi}{\Psi^* \Psi} = \frac{1}{2m} \left( m v^2 + m u^2 \right) = \frac{1}{2m} |\nabla S|^2 + \frac{1}{2m} |\nabla \theta_\pm|^2, \tag{35}
\]

and this equation also defines two kinetic energy formulae, \(K_v\) and \(K_u\). These definitions yield the same fields as identified in previous sections, where both fields appear in Eq. \((26)\).

Since the generalization to the \(n\)-body case is trivial, and discussed below, the above labeled three equations gives support for the interpretations for the velocities functions present in the energy equation Eq. \((26)\). It also suggests that the function \(P_i\) from Eq. \((26)\) is not a kinetic energy term. The function \(P_i\) in Eq. \((26)\) is also probably not one associated with a body force, since such potentials are usually universal, and, therefore, would not have a formula that depends on the probability density, as in definition \((5)\).

The \(n\)-body case is easily obtained by the replacements \(X \to X_i\) and \(\nabla Y \to \nabla_i Y\) where

\[
X \in \{ \hat{P}, v, u, K, K_v, K_u \}, \quad \nabla Y \in \{ \nabla S, \nabla \theta \},
\]

and where the equations still hold by summation over the dummy index \(i\). For example, Eq. \((35)\) becomes

\[
\sum_{i=1}^n K_i = \sum_{i=1}^n K_{vi} + K_{ui} = \frac{1}{2m} \sum_{i=1}^n \left( \frac{\Psi^* \hat{P} \Psi}{\Psi^* \Psi} \right) = \frac{1}{2m} \sum_{i=1}^n \left( |\nabla_i S|^2 + \frac{1}{2m} |\nabla_i \theta_i|^2 \right). \tag{36}
\]

This equation also holds without the summations.
6. Notation Involving Sets with the Same Cardinality

An examination of Eq. (7) indicates that there are three shortcomings in its notation: 1) clutter caused by the summation symbols \( \Sigma \), 2) eyestrain from the subscript symbols \( i \) and \( j \), and 3) over accent from too many capital letters. In this section, we introduce new notation to remove these shortcomings and put the \( n \)-body case on the same footing as the one-body case, applicable almost all of the equations encountered in this work.

A product compatible collection \( N \cdot \) is a family containing sets \( A_1, A_2, \ldots \) with the same cardinality. Also, the following operations are defined:

\[
\begin{align*}
\sum_{i \in I} x_i, & \quad x_i \in A \in N \cdot, \\
\sum_{i \in I} x_i y_i, & \quad x_i \in A \in N \cdot, \quad y_i \in B \in N \cdot,
\end{align*}
\]

where \( I \) is an index set, and each of the members of \( N \cdot \) are indexed by \( I \). Since \( x \) and \( y \) are not defined elements of \( A \), \( B \), we define these symbols, in equations, to be

\[
x \doteq \sum_{i \in I} x_i, \quad xy \doteq \sum_{i \in I} x_i y_i.
\]

We also give another meaning to the symbols \( x \) and \( y \) when they are not in an equation: \( x \) is the set \( A \); \( y \) is the set \( B \), e.g., \( x = \{ x_i \mid i \in I \} \). Hence, it is not necessary to use the symbols \( A \) and \( B \). Also, if the cardinality is one, then the single element \( x_1 \) from the set \( x \) is also denoted by \( x \). This convention is also used for \( f(C) \), the image of the set \( C \) under the function \( f \) by writing \( f(x) \), if \( C = \{ x \} \), instead of the notation \( f(\{ x \}) \). It can also be convenient to be able to append \( N \cdot \) with elements. So we define \( N \cdot \doteq N \cdot + z \) to mean that \( N \cdot \cup \{ z \} \) is now the new definition of \( N \cdot \), and this can be understood to have been done, when there is no misunderstanding.

To apply this notation to Eq. (36), we define a member of \( N \cdot \) for each variable with an \( i \) subscript, and these variables are indexed by \( (I = \{ 1, \cdots n \}) \). Examples being \( (K = \{ K_i \mid i \in I \}), \ (u^2 = \{ u^2_i \mid i \in I \}), \ (\nabla S = \{ \nabla_i S \mid i \in I \}) \), and we also use

\[
|\nabla S|^2 = \nabla S \cdot \nabla S \doteq \sum_i \nabla_i S \cdot \nabla S_i.
\]

The result of this notation change is that Eqs. (36) and (35) represent the same equation. Also the application of notation to (8) gives (11), and (26) becomes

\[
-\partial S = \frac{1}{2}mv^2 + \frac{1}{2}mu^2 + P\rho^{-1} + U,
\]

where \( U \) does not use this notation, being still defined by Eq. (9). In this equation, since the left-hand side cannot be expressed as a sum over \( n \) terms, only the right-hand side is summed over the index set \( I \), except for \( U \). In order that a summation can be given explicitly, if it is useful to do so, let the operator \( \Sigma \) be defined by

\[
\sum x \doteq \sum_{i \in I} x_i, \quad \sum xy \doteq \sum_{i \in I} x_i y_i, \quad \text{and so on},
\]

giving

\[
-\partial S = U + \sum \frac{1}{2}mv^2 + \frac{1}{2}mu^2 + P\rho^{-1},
\]
where $\sum$ acts on all terms on its right side, until either an equal sign or the equation end.

In the above equation and Eq. (37), the over accented $\Upsilon$ is now $\rho$, and $P$ could be changed to $p$. I find that subscripts get in the way of the process of the evaluation of an equation. For subscripted equations, in the process of weeding out the clutter, I find myself staring at the little subscripts, instead of taking in the whole equation all at once. I also find that two many capital symbols, especially ones in Greek, make it difficult to distinguish and classify the factors of terms in equations. This may be the reason that functional analysis monographs usually use symbols like $x$ and $y$ for maps and reserve capitals for sets.

7. Energies

In this section, as in the particle momentae definitions in Sec. 5, two energy fields are defined as the real and imaginary parts of the function $\Psi^* \hat{H} \Psi / \rho$, where $\hat{H}$ is the Hamiltonian (energy) operator. Using the Schrödinger Eq. ($i\Psi^* \partial \Psi = \Psi^* \hat{H} \Psi$), the approach yields a pair of energy fields as the real and imaginary parts of $i\Psi^* \partial \Psi / \rho$: $(E_S = -\partial S)$ and $(E_\theta = -\partial \theta)$, where $S$ and $\theta$ are the momentae potentials, defined by Eq. (32). The energy field $(E_S = -\partial S)$ is substituted into the Hamilton-Jacobi Eq. (37) of Sec. 4 giving an energy equation, Eq. (44). In the special case of time independence, the energy field $E_S$ reduces to the energy eigenvalue $E$ of the Schrödinger equation. Later in Sec. 9, the other energy field $E_\theta$, an unknown, makes an appearance in an energy form of the continuity equation, an equation equivalent to the Bohmian-mechanics continuity Eq. (20).

Let the energy functions $E_S$ and $E_\theta$ be the real and imaginary parts of the function $\rho^{-1} \Psi \hat{H} \Psi$, where the Schrödinger Eq. (18), can be written

$$i\hbar \rho^{-1} \Psi^* \partial \Psi = \rho^{-1} \Psi \hat{H} \Psi. \quad (38)$$

Setting $\Psi = \phi e^{iS/\hbar}$, and working on the left-hand side of (38), we obtain the following equation sequence:

$$e^{-iS/\hbar} \phi \partial (\phi e^{iS/\hbar}) = \phi \partial \phi + i\hbar^{-1} \partial S,$$

$$e^{-iS/\hbar} \phi \partial (\phi e^{iS/\hbar}) = \frac{1}{2} \partial \rho + i\hbar^{-1} \partial S,$$

$$i\hbar e^{-iS/\hbar} \phi \partial (\phi e^{iS/\hbar}) = -\rho \partial S + \frac{\hbar}{2} \partial \rho,$$

$$i\hbar \rho^{-1} \Psi^* \partial \Psi = -\partial S + \frac{\hbar}{2} \partial \ln \rho. \quad (39)$$

Using definition (31) with $\theta = \theta_-$ and Eq. (38), we discover that

$$\rho^{-1} \Psi \hat{H} \Psi = i\hbar \rho^{-1} \Psi^* \partial \Psi = -\partial S - i\partial \theta,$$

and the above definitions give the desired result:

$$E_S = -\partial S, \quad E_\theta = -\partial \theta = \frac{\hbar}{2} \partial \ln \rho, \quad (41)$$
such that

\[
\frac{i\hbar \Psi^* \partial \Psi}{\Psi^* \Psi} = \frac{\Psi^* \hat{H} \Psi}{\Psi^* \Psi} = E_S + iE_\theta,
\]  
(42)

and \( \partial \ln \rho = \rho^{-1} \partial \rho \) is used in Eq. (39). It is worth noting that, since \( \partial \ln \rho = \rho^{-1} \partial \rho \), the right-hand side equation of (41), can be written

\[
E_\theta \rho = -\rho \partial \theta = \frac{\hbar}{2} \partial \rho.
\]  
(43)

Equation (41) gives a further support for the choice \( \theta = \theta_- \), since the corresponding function, \( -\partial S \), has a minus sign. Henceforth, in part to reduce clutter, we use the choice \( \theta = \theta_- \), giving the downhill choice \( u_- \). Also, substituting (41) into (37) with the \( P \) notation changed to \( P_u \), we have

\[
E_S = \frac{1}{2}mv^2 + \frac{1}{2}mu^2 + P_u \rho^{-1} + U,
\]  
(44)

and here we use the notation from Sec. 6 for \( n \)-bodies. Since Eq. (37), together with the equation (20), are equivalent to the Schrödinger’ (18), the above equation defines an energy field \( E_S \), not necessarily uniform and constant in time, that is assigned to time-dependent systems that satisfy the time-dependent Schrödinger equation (18).

It is worth noting that, to return an equation like Eq. (44) to the subscripted notation, a subscript, say \( j \), is appended to the pressure and velocity (or speed) fields—\( u^2, v^2 \), and \( P_u \) in Eq. (44)—followed by a summation over the dummy index \( j \). Using this rule makes the derivations of Sec. (10) easier to follow.

8. Pressures

This section extends the method of definition for energies and momenta fields to pressures. This physical property has not been defined as an observable for quantum mechanics, and, therefore, it does not have a linear-operator assignment. Definition (47), involving the divergence of momentum density, yields the same pressure \( P \) as in the previous sections, defined in Eqs. (10) or (5), where the notation is changed from \( P \) to \( P_u \). The second pressure \( P_v \), an unknown, appears in an energy form of the continuity equation (52) from Sec. 9, along with the energy field \( E_\theta \).

We begin with the notation from Sec. 6 and then switch to the subscripted notation in the next section. We also continue to use the choice \( u = u_- \). Let the the pressures, \( P_u \) and \( P_v \), be defined, in the following manner, where they are proportional to the divergence of a momentum density:

\[
P_u \doteq \frac{\hbar}{2m} \nabla \cdot \rho_m u_-,
\]

\[
P_v \doteq \frac{\hbar}{2m} \nabla \cdot \rho_m (-v).
\]  
(45)

Using definition (10) for the velocity \( u_- \), we have \( \rho_m u_- = -(\hbar/2) \nabla \rho \), giving the previous definition (10) for the pressure with \( P_u = P \):

\[
P_u \doteq \frac{\hbar}{2m} \nabla \cdot \rho_m u_- = \frac{\hbar}{2m} \nabla \cdot \left( -\frac{\hbar}{2} \nabla \rho \right) = -\frac{\hbar^2}{4m} \nabla^2 \rho.
\]  
(46)
Using definitions (45) and (33), \( \Psi^* \hat{P} \Psi = \rho_m v + i \rho_m u \), we obtain an alternate definition for the pressures:

\[
\frac{\hbar}{2m} \nabla \cdot (\Psi^* \hat{P} \Psi) = -P_v + iP_u. \tag{47}
\]

The sign difference between the two pressures suggests that one of these pressures would be better with a sign change, and the resulting function could be considered a tension. However, since the pressures, especially \( P_u \), are not thermodynamic pressures, i.e., they can be negative valued, such a sign change in the equations seems to make little or no difference in the given interpretations. The other possibility is to use the uphill velocity choice \( u_+ \).

9. Equalities involving Time Derivatives and Pressures

This section uses the continuity equation to derive a number of relations involving the pressures, most having time derivatives. An energy equation (52) is obtained, containing both the unknown pressure \( P_v \) and energy \( E_\theta \) fields, and this equation is shown to be equivalent to the continuity equation (20). At the conclusion of this section, we have a total of three defined ordered pairs—\( \{mv, mu\} \), \( \{P_v, Pu\} \), and \( \{E_S, E_\theta\} \)—that are present in two main equations, (14) and (52). These two equations, along with the definitions of the three ordered pairs of fields, are equivalent to the time-dependent Schrödinger equation (18), indicating that the method is ab initio. Each and every term is identified from the two equations, one term \( U \) being the classical electrostatic terms (16), indicating that the approach is complete in its energy description.

In what follows in this section, because the continuity equation is used, it is necessary to use the subscripted notation. For example, the above definitions are

\[
P_{ui} \triangleq \frac{\hbar}{2m} \nabla_i \cdot \rho_m u_i, \quad P_{vi} \triangleq \frac{\hbar}{2m} \nabla_i \cdot \rho_m (-v_i). \tag{48}
\]

For the pressure \( P_{vi} \), we multiply the continuity Eq. (20), using the notation \( \rho = \Upsilon \), by \( \hbar/2 \) and \( m/m \), giving

\[
\frac{\hbar}{2} \partial \rho = -\frac{\hbar}{2m} \sum_i \nabla_i \cdot (\rho_m v_i),
\]

and it follows from the above definition (18) for \( P_{vi} \) that

\[
\frac{\hbar}{2} \partial \rho = \sum_i P_{vi}, \tag{49}
\]

a form of the continuity equation. Taking a gradient of this result and switching the order, we have

\[
\sum_i \nabla_j P_{vi} = \frac{\hbar}{2} \partial (\nabla_j \rho).
\]

Using the equality \( \nabla \rho_j = -(2/\hbar) u_j \rho_m \) from Eq. (10), and interchanging the dummy indices, we obtain the additional result:

\[
\sum_j \nabla_i P_{vj} = -\partial (\rho_m u_i), \quad i \in \{1, \cdots n\}. \tag{50}
\]
This equation is used in the next section. For the special case of one body, the resulting equation,
\[-\nabla P_v = \partial(\rho_m u),\]
is not Newton’s second law, since, as discussed in the next section after Eq. (55), the partial time-derivative is a local derivative, assigned to fixed points in the field, instead of a time derivative from an equation of motion that is assigned to a particle, as it moves through space.

Applying the operator \((\hbar/2m)\nabla\iota\) to Eq. (50),
\[
\frac{\hbar}{2m} \sum_j \nabla^2_i P_{vj} = -\frac{\hbar}{2m} \partial[\nabla_i \cdot (\rho_m u_i)],
\]
and using the definition (48), \(P_{ui} = (\hbar/2m)\nabla \cdot \rho_m u_i\), and switching the order, we discover that the two types of pressures are related by
\[
\partial P_{ui} = -\frac{\hbar}{2m} \sum_j \nabla_i^2 P_{vj},
\]
the Poisson equation for the one-body case.

For an energy equation for \(E_\theta\), we substitute (41), written \(((\hbar/2)\partial\rho = E_\theta \rho)\), into (49), \(((\hbar/2)\partial\rho = P_{v1} + \cdots P_{vn})\), giving
\[
E_\theta = \rho^{-1} \sum_i P_{vi}.
\]
Hence, \(E_\theta \rho\) is a pressure with corresponding compression energy \(E_\theta\).

Since the derivation of (52) from (20) is reversible, Eq. (52) is mathematically equivalent to the continuity equation (20). The reversibility is proven by 1) doing the derivation in reverse, where \(P_{vi}\) is nothing more then a mathematical field, defined by the right-hand side of Eq. (48), and the field \(v_i\), present in that equation, is defined by (4). By 2) noting that the satisfaction of the field \(E_\theta\), in Eq. (41), follows from its mathematical definition (41).

Equation (52) is used in the next section.

10. The Total Energy and Euler Equation

This section combines the two energy equations, Eqs. (44) and (52), giving a total-energy equation (53), containing all three ordered pairs of fields. An \(n\)-body generalization of the Euler equation of fluid dynamics is derived from this equation, Eq. (62), and this derivation is much longer than previous ones. This Eulerian equation is viewed as a sum of two equations corresponding to two interacting systems. The local time derivative of the velocity fields are shown to satisfy Eqs. (55), where, for the case of spin-free one-body states, \(\partial v(r, t)\) and \(\partial u(r, t)\) are the time rate of change of the velocity fields at time \(t\) and fixed point \(r \in \mathbb{R}\).
Combining Eq. (44) and (52), for the one-body case, we obtain a definition of the total energy $E$

$$E \doteq E_S + E_\theta = \frac{1}{2} m v^2 + P \rho^{-1} + \frac{1}{2} m u^2 + P_u \rho^{-1} + U,$$  \hspace{1cm} (53)

and an alternative worth mention is $E \doteq |E_S + i E_\theta|$. The above equation holds for $n$-bodies using the notation from Sec. 6. Using the subscripted notation, it can also be derived using equations (41), (26), and (52).

Returning to Eq. (53), in the special case where $v \cdot u = 0$, we can write

$$\frac{1}{2} m w^2 + P \rho^{-1} + U = E,$$  \hspace{1cm} (54)

where

$$w^2 = w \cdot w, \quad w = u + v, \quad P = P_v + P_u.$$  \hspace{1cm} For a one-body system, Eq. (53) has the same form as the Bernoullian’ (11). While one objective of this work is to avoid complex-valued physical-properties, it is still worth mentioning that the definition

$$w^2 = |v + i u|^2,$$

gives (54) from (53) for the general case, where $v \cdot u \neq 0$ is permitted. The consequences of $v \cdot u = 0$ for the continuity equation is given at the end of section 4.

Next we derived a pair of equations involving $\partial v$ and $\partial u$, needed for the derivation of an Eulerian equation that follows. This is accomplished by taking the gradient of $E_S = - \partial S$ and $E_\theta = - \partial \theta$, i.e., Eq. (11), and then using the particle momentum definitions (32), giving

$$- \nabla E_S = m \partial v, \quad - \nabla E_\theta = m \partial u.$$  \hspace{1cm} (55)

Despite the forms of these equations, they are not to be confused with Newton’s second law for particles with conservative forces $- \nabla E_S$ and $- \nabla E_\theta$, respectively. In classical fluid dynamics [55], $\partial v$ and $\partial u$ are local accelerations: For a fixed position in space $r \in \mathbb{R}$ and velocity field $u$, $\partial u(r, t)$ is the time rate of change of the velocity fields at point $r$ and time $t$. For steady flow, these partial derivatives vanish, and the acceleration of a fluid particle is obtained by the time derivative of the composite, $u = u(r)$, such that $r = r(t)$. For the case of nonsteady flow, we have $u = u(r, t)$, with $r = r(t)$, where the nonpartial, total derivative of $u$ is called the material derivative [55, 60, 62].

Next we derive an Euler equation for one-body systems, and, separately, obtain the $n$-body equation. We begin by taking the gradient of (44),

$$\nabla E_S = \frac{1}{2} m \nabla v^2 + \frac{1}{2} m \nabla u^2 + \nabla \left( P_u \rho^{-1} \right) + \nabla U.$$  \hspace{1cm} (56)

Substituting the left equation of (55) into Eq. (56), and then adding the resulting equation to Eq. (50), in the form $\rho^{-1} \partial (\rho m u) + \rho^{-1} \nabla P_v = 0$, we obtain the desired result for one-body systems:

$$m \partial v + \rho^{-1} \partial (\rho m u) + \frac{1}{2} m \nabla \left( v^2 + u^2 \right) + \rho^{-1} \nabla P_v + \nabla \left( P_u \rho^{-1} \right) + \nabla U = 0.$$  \hspace{1cm} (57)
For the $n$-body case, we begin with Eq. (44), return to the subscripted notation with summation index $j$, take a gradient $\nabla_i$ of that equation, and then follow the procedure above for the one-body case with $\nabla$ replaced by $\nabla_i$, giving the following equation sequence:

$$\nabla_i E_S = \frac{1}{2} m \nabla_i v_j^2 + \frac{1}{2} m \nabla_i u_j^2 + \nabla_i \left( P_{uj} \rho^{-1} \right) + \nabla_i U,$$

$$-\nabla_i E_S = m \partial_i v, \quad -\nabla_i E_\theta = m \partial_i u,$$

$$\text{(50)}: \quad \rho^{-1} \partial (\rho_m u_i) + \rho^{-1} \sum_j \nabla_i P_{vj} = 0,$$

$$m \partial_i v + \rho^{-1} \partial (\rho_m u_i) + \frac{1}{2} m \nabla_i (v_j^2 + u_j^2) + \rho^{-1} \nabla_i P_{vj} + \nabla_i \left( P_{uj} \rho^{-1} \right) = -\nabla_i U. \quad (58)$$

where there is an understood sum over index $j$ and $i \in \{1, \cdots n\}$.

Next, we put the one-body equation (57) in a form so that it can be compared to the variable mass Euler equation (A.7),

$$\frac{\partial}{\partial t} (\rho_m u) + \frac{1}{2} \rho_m \nabla u^2 + \nabla \cdot (\rho_m u) u + \nabla p + q \rho \nabla \Phi = 0,$$

derived in Appendix A, with pressure $p$, force per charge $(-\nabla \Phi)$, mass density $\rho_m$, and charge density $q \rho$. First note the following identity used below:

$$\rho \nabla \left( P \rho \right) = -P \frac{\nabla \rho}{\rho} + \nabla P. \quad (59)$$

Multiplying Eqs. (45) and (10), given by

$$\nabla \cdot \rho_m u = \frac{2m}{h} P_u, \quad u = -\frac{h}{2m} \nabla \rho, \quad u = u_-, \quad$$

we obtain

$$\nabla \cdot (\rho_m u) u = -\frac{\nabla \rho}{\rho}. \quad$$

Substituting this one into identity (59) gives the desired equality for $P_u$:

$$\rho \nabla \left( \frac{P_u}{\rho} \right) = \nabla \cdot (\rho_m u) u + \nabla P_u. \quad (60)$$

Substituting this equation into the Eq. (57), after multiplying the equation by $\rho$, gives the final result:

$$\rho_m \partial \mathbf{v} + \partial (\rho_m \mathbf{u}) + \frac{1}{2} \rho_m \nabla (v^2 + u^2) + \nabla (P_v + P_u) + \nabla \cdot (\rho_m \mathbf{u}) \mathbf{u} + \rho \nabla U = 0. \quad (61)$$

In order to assign a meaning to portions of Eq. (61), we write the equation as a sum of two equations: $(\text{EQ}_u + \text{EQ}_v)$, where

$$\text{EQ}_u \equiv \partial (\rho_m \mathbf{u}) + \frac{1}{2} \rho_m \nabla u^2 + \nabla \cdot (\rho_m \mathbf{u}) \mathbf{u} + \nabla P_u + \rho \nabla U = 0,$$

$$\text{EQ}_v \equiv \rho_m \partial \mathbf{v} + \frac{1}{2} \rho_m \nabla v^2 + \nabla P_v = 0.$$

For one-body systems, these are Euler equations of fluid dynamics: Equation $\text{EQ}_u$ is the Euler Equation (A.7) with variable mass; Equation $\text{EQ}_u$ is the Euler Equation (A.8).
with conserved mass, and there is flexibility in the placement of the body forces (per volume) $\rho \nabla U$. Both equations are for irrotational flow.

For the $n$-body case, the derivation of the generalization of Eq. (60) is

$$
\rho \nabla \left( \frac{P_j}{\rho} \right) = -P_j \frac{\nabla \rho}{\rho} + \nabla P_j,
$$

(48)

$$
\nabla_j \cdot (\rho_m u_j) = \frac{2m}{\hbar} P_{uj}, \quad u_i = -\frac{\hbar}{2m} \frac{\nabla \rho}{\rho}, \quad u = u_-, \nabla_j \cdot (\rho_m u_j) u_i = -P_{uj} \frac{\nabla \rho}{\rho},
$$

$$
\rho \nabla_i \left( \frac{P_{uj}}{\rho} \right) = \nabla \cdot (\rho_m u_j) u_i + \nabla P_{uj},
$$

giving from (58),

$$
\rho_m \partial \mathbf{v}_i + \partial (\rho_m \mathbf{u}_i) + \frac{1}{2} \rho_m \nabla_i \left( v_j^2 + u_j^2 \right) + \nabla_i (P_{vj} + P_{uj}) + \nabla_j \cdot (\rho_m u_j) u_i = -\rho \nabla_i U,
$$

where there is an understood summation over index $j$ and $i \in \{1, \cdots n\}$. Note that we can still use the notation system for Sec 6 for the $j$ dummy index:

$$
\rho_m \partial \mathbf{v}_\Delta + \partial (\rho_m \mathbf{u}_\Delta) + \frac{1}{2} \rho_m \nabla_i |w|_2^2 + \nabla_i (P_\Delta + P_\mathbf{u}) + \nabla_j \cdot (\rho_m u_j) u_\Delta = -\rho \nabla_i U,
$$

(62)

where $\Delta \in \{1, \cdots n\}$, $w = v + iu$ and $P = P_\mathbf{v} + P_\mathbf{u}$. This equation lacks symmetry, with respect to the interchange of $\mathbf{v}$ and $\mathbf{u}$, because one of these velocities satisfies a continuity equation and one does not. The symmetry is recovered by adding the continuity equation $(\partial \rho + \nabla \cdot \rho \mathbf{v})\mathbf{v}_i = 0$ satisfied by $\mathbf{v}_i$.

11. A Linear Combination of Eigenfunctions

In this section, we investigate the energy fields $E_s$ and $E_\theta$, given by Eqs. (41) and (44), for energy conservation. Wavefunctions are considered that are linear combinations of basis functions, where each function from the linear combination is, separately, an eigenfunction of the same Hamiltonian operator with a time-independent external potential. If members of such a linear combination are not degenerate, the linear combination still satisfies the time dependent Schrödinger equation (18), but the linear combination does not satisfy the time-independent Schrödinger equation. In other words, the linear combination is not an eigenfunction of the Hamiltonian operator. We show that these quantities have the important property of being conserved over all space for each moment of time. In other words, while energy flows from one region of space to another, the total energy, over all space, is fixed in time.

Using atomic units, let $\phi_1$ and $\phi_2$ be real-valued orthonormal functions that are eigenfunctions of a Hamiltonian operator $\hat{H}$. Let

$$
\psi_i(t) = e^{-i\epsilon_i t}, \quad |C_1|^2 + |C_2|^2 = 1, \quad C_i \in \mathbb{R}, \quad \rho_i \doteq \phi_i \phi_i, \quad i \in \{1, 2\}, \quad \beta \doteq C_1 C_2.
$$
In the following equation sequence, we start with the linear combination wavefunction \( \Psi \), and calculate the left-hand side of Eq. (42), multiplied by \( \rho \). In the following equation sequence, we start with the linear combination wavefunction

\[
\Psi(r, t) = C_1 \phi_1(r) \psi_1(t) + C_2 \phi_2(r) \psi_2(t),
\]

\[
i \partial \Psi = C_1 \epsilon_1 \phi_1 \psi_1 + C_2 \epsilon_2 \phi_2 \psi_2,
\]

\[
\Psi^* i \partial \Psi = (C_1 (\phi_1 \psi_1)^* + C_2 (\phi_2 \psi_2)^*) (C_1 \epsilon_1 \phi_1 \psi_1 + C_2 \epsilon_2 \phi_2 \psi_2),
\]

\[
\Psi^* i \partial \Psi = C_1^2 \epsilon_1 \rho_1 + C_2^2 \epsilon_2 \rho_2 + \beta (\epsilon_1 \phi_1 \phi_2^* \psi_1 + \epsilon_2 \phi_2 \phi_1^* \psi_2),
\]

\[
\Psi^* i \partial \Psi = C_1^2 \epsilon_1 \rho_1 + C_2^2 \epsilon_2 \rho_2 + \beta \phi_1 \phi_2 \left( \epsilon_1 e^{i(\epsilon_2 - \epsilon_1)t} + \epsilon_2 e^{-i(\epsilon_2 - \epsilon_1)t} \right).
\]

Using (42), we obtain the desired energy fields per volume:

\[
E^*_\rho = \beta \phi_1 \phi_2 (\epsilon_2 - \epsilon_1) \sin [(\epsilon_1 - \epsilon_2)t], \quad (63)
\]

\[
E^*_S = C_1^2 \epsilon_1 \rho_1 + C_2^2 \epsilon_2 \rho_2 + \beta \phi_1 \phi_2 (\epsilon_1 + \epsilon_2) \cos [(\epsilon_2 - \epsilon_1)t]. \quad (64)
\]

Since the functions \( \phi_1 \) and \( \phi_2 \) are orthonormal, the spacial energy averages, obtained by integrating (63) and (64) over all space \( \mathbb{R}^{3n} \) and summing over all spin variables, are

\[
E^*_\rho (\text{avg}) = 0, \quad \text{and} \quad E^*_S (\text{avg}) = C_1^2 \epsilon_1 + C_2^2 \epsilon_2.
\]

Since these are constants, independent of time, both of these energies are conserved over all space.

If we repeat the derivation above with the following change \( C_2 \rightarrow -C_2 \), then the sign of the \( \beta \) terms in Eq. (63) and (64) change. If, instead, we make the following change \( C_2 \rightarrow iC_2 \), nothing dramatic happens: The \( \beta \) terms in Eq. (64), for \( E_S \), and Eq. (63), for \( E_\rho \), switch places.

For the general case, using a set of \( m \) orthonormal spatial wavefunctions \( \{ \phi_1, \cdots \phi_m \} \) that are eigenfunctions of a single Hamiltonian operator, the same result is obtained: \( E_S \) and \( E_\rho \) are conserved over space. This result is easily seen to follow because, with no loss of generality, the members of the set \( \{ \phi_1, \cdots \phi_m \} \) are mutually orthogonal. The same result is obtained for a wavefunction defined by an infinite sequence of orthonormal functions, since each term of the corresponding sequences, for both \( E_S \) and \( E_\rho \), are conserved.

12. Discussion of issues of velocity

This section considers issues involving velocity. Sec. 12.1 investigates a velocity compatibility problem for the two velocity fields as a vector sum \( \mathbf{v} + \mathbf{u} \). Sec. 12.2 considers other possibilities for the Bernoullian velocity \( \mathbf{u} \). Various issues of local variable mass, angular momentum, spin, and kinetic-energy satisfaction are considered, for both fluid and particle descriptions. The \( ns \) states of hydrogen atom are considered for the investigation. The objective is to determine what velocity is the best match for the Bernoullian velocity \( \mathbf{u} \), and to summarize the information available. For the most part, only one-body systems are considered.
12.1. A velocity compatibility problem

For the case of one-body, with a time-independent complex-valued wavefunction, let the velocity be \( \dot{q} = u + v \). Since

\[
\frac{1}{2} m \dot{q} \cdot \dot{q} \neq \frac{1}{2} m u \cdot u + \frac{1}{2} m v \cdot v ,
\]

if \( u \cdot v \neq 0 \), the single particle does not have the correct kinetic energy. Hence, the two kinetic energies seem to represent different forms of energy. Another justification, for this posit, is that the Eulerian Eq. (61) is consistent with two interacting “substates,” where each one can have their own energy forms. One possibility for the kinetic-energy identification, involving the Bernoullian velocity, is that the vector \( |u| \) represents an average speed of Brownian-like motion, which, given the direction of the velocity vector \( u \) from Eq. (10), might be restricted to one-dimension. In other words, the electron oscillates back and forth. However, if the motion is random, the electron in the hydrogen atom would drift away from the nucleus.

It is well known that, given an ensemble of a quantum state with a distribution of initial configurations, distributed according to the Born rule, that the Born rule is preserved over time: The distribution of members of the ensemble satisfy the Born rule at all times [63]. If we insist that this rule must also hold for the velocity sum \( \dot{v} + \dot{u} \), this greatly restricts the possibilities for the direction of velocity \( u \), given that the corresponding speed \( |u| \) should agree with the kinetic-energy expectation-value. This problem does not occur if the velocity \( u \) is associated with some sort of motion that does not cause electron translation. In that case, except for “local motion,” an electron, in a state described by a real-valued wavefunction, can be considered static, and the issue of the static electron, from Bohmian mechanics in such cases, is resolved.

This general idea of two sources of kinetic energy agrees with Salesi’s arguments [57], obtained from approximations of relativistic quantum-mechanics. He assigns the kinetic energy from the Madelung velocity \( v \) to the center of mass motion, and the Bernoullian velocity \( u \) to internal energy of the electron. The electron is considered as having a body occupying space of non-zero measure, undergoing motion relative to its center of mass. Salesi also presents an argument that the internal energy of the electron is due to Zitterbewegung, from rapid oscillation of the electron [64], a model that might be able to describe electron spin.

However, there is another possibility: The two electron velocities can be considered perpendicular. Given, as presented below, that the direction of the velocity vector \( u \) is completely changed when including relativistic approximations into the picture, and that the meaning of the velocity formulae derived from relativistic theory are open to interpretation, perpendicular velocities is not out of the question. For that reason, other possibilities are considered in the subsections that follow.

It is worth mentioning that Eq. (35) indicates that perpendicular velocities, \( u \cdot v = 0 \), implies the satisfaction of the Poisson Equation \( \nabla^2 \tilde{S} = -\partial \ln \rho \). It is easily proven that that Poisson-equation satisfaction, implies \( u \cdot v = 0 \), if the wavefunction defined by \( R = \sqrt{\rho} \) and \( S \), satisfies the Schrödinger equation.
12.2. Bernoullian fluid-velocity field direction and spin

In this subsection, for hydrogen ns states and a fluid description, we consider two Bernoullian velocity-fields: One with zero angular momentum from non-relativistic quantum mechanics, Eq. (10), and a well known one from relativistic quantum mechanics, depending on electron spin, for both fluid are particle descriptions. Ignoring spin, the velocity choice (10) seems to be the best match for nonrelativistic quantum mechanics, given that it appears in a natural way in the momentum definition (33). Also, the magnitude of this velocity choice appears in the kinetic energy expression of the Bernoullian Eq. (11), and in the expression for the kinetic-energy expectation value (14). As for the velocity from relativistic quantum mechanics, we show that the direction of this velocity choice can be obtained from non-relativistic quantum mechanics by requiring a continuity-equation satisfaction for the hydrogen ns states. This continuity-equation involving velocity \( u \) differs, and should not be confused with, the one from time-dependent quantum-mechanics—involving velocity \( v \) in Bohmian mechanics—that implies probability conservation. Since all stationary states have time independent probability densities, this time-dependent continuity equation remains satisfied for states considered.

For the one-electron systems that satisfy the energy equation (8), both \( |u_\pm| \) and \( \rho_m \) are fixed, and, therefore, only the unit vector \( \hat{s}_i \) of Eq. (12) is at our disposal. Also, given \( \hat{s}_i \), we find no grounds for choosing between the two velocity directions that satisfy \( u_+ = -u_- \).

12.2.1. Bernoullian velocity choice from a fluid or internal-energy description

Zero angular momentum with variable mass. Since the ns wavefunctions of hydrogen are eigenfunctions of \( \hat{L}^2 \)—the operator for the square of the magnitude of the angular momentum—and the eigenvalues are zero, it is reasonable to choose a radially directed momentum, so that the angular momentum vector-field is the zero function. For the ns states of hydrogen, this requires the velocity field given by Eq. (10). However, since the flow is also steady, the flow must have local variable mass, since a steady radial flow requires infinite mass, if mass is conserved. A generalization of the continuity equation of fluid mechanics—but not related to a Schrödinger equation—involving this velocity \( u_\pm \) choice, is obtained from (10):

\[
\nabla \cdot (u_\pm \rho) = \pm \frac{\hbar}{2m} \nabla^2 \rho = \pm \frac{2}{\hbar} P,
\]

where \( P \) is the pressure. Hence, the flows do not locally conserve mass, since \( \{ r \in \mathbb{R}^3 | P(r) = 0 \} \) has measure zero. These steady flows with variable mass are stabilized by a continuous creation and annihilation of matter. Over all space, the flows conserve mass, because the sources cancel the sinks, and this follows because Eq. (65) integrated over all space vanishes. Fluid flows of the local variable-mass velocity-choice (10) is explored elsewhere, including the investigation of the vector fields of the hydrogen atom 1s and 2s states [18 19].
Mass conservation.

Since local variable mass is a non-classical element for a fluid, and we wish to have the minimum number of such elements, it is reasonable to investigate the consequences of choosing the unit vector $\hat{s}$ of the velocity field $u$ from Eq. (12), or some modification of this definition, so that the resultant classical continuity-equation yields local conservation of mass, where we consider steady flow. For use below, possibilities for the Bernoullian velocity $u$ are considered, where $u$ is not necessarily given by $u_\pm$ from Eq. (10). Also, sometimes $u$, like $u_\pm$, will be unsigned, having both choices, but in such cases the “±” symbol is not used.

First consider incompressible flow along each streamline, but where the mass density $\rho_m$ is not uniform. In other words, each streamlines has a constant mass density $\rho_m$. However, since we have $\nabla \rho_m(\mathbf{r}) \neq 0$ for points $\mathbf{r} \in \mathbb{R}^3$ almost everywhere [18, 19], $\rho_m$ is not, and cannot be uniform. Since such streamlines are on a level surface of $\rho_m$, they have directions $u/|u|$ that are perpendicular to $\nabla \rho_m$, giving

$$\nabla \cdot (u\rho_m) = \nabla \rho_m \cdot u + \rho_m \nabla \cdot u = \rho_m \nabla \cdot u = 0,$$

(66)

if mass is conserved. In other words, local conservation of mass for an electron flow is satisfied if the volumetric dilatation rate $\nabla \cdot u$ vanish, i.e., each fluid elements has a constant volume. This is the same requirement for the conservation of mass of a classical incompressible fluid [55, 60, 62], but in this case, the fluid is incompressible only on a streamline, with the mass density changing from one streamline to another.

For the hydrogen $n\ell s$ states, the directions of $-\nabla \rho_m$ is in the radial direction $\hat{r}$. Therefore, for the streamline not to cross and also be perpendicular to $\nabla \rho$ to conserve mass, all the streamline must have a common rotational axis. Let this rotational axis be the $z$-axis. Hence, for mass conservation, the streamlines must always be in the $\hat{\phi}$ direction, where $\phi$ the azimuthal angle of spherical coordinates, since otherwise they would not always be perpendicular to $\nabla \rho$, giving local variable mass. Hence, for both conservation of mass and the proper treatment of kinetic energy, from the consideration of the $n\ell s$ states of hydrogen, we obtain the following velocity choice:

$$u = \pm |u_\pm| \hat{\phi}, \quad \phi = \frac{u_\pm \times \hat{z}}{|u_\pm \times \hat{z}|}.$$

(67)

where $u_\pm$ is given by Eq. (10). Also, definition (67) can be generalized from hydrogen $n\ell s$ states to all quantum states, and this expression is a special case of definitions (12).

The above expression (67), derived from non-relativistic quantum mechanics, can be compared with the velocity field formula obtained for spin one-half particles from approximations of the Pauli equations [65, 66, 67, 57]:

$$u = u_\pm \times \hat{z} = \frac{\nabla \rho}{m\rho} \times s_\pm = |u_\pm| \sin \theta \hat{\phi}, \quad s_\pm = \pm \frac{\hbar}{2} \hat{z},$$

(68)

also giving streamlines for the $n\ell s$ states of hydrogen with mass conservation, and it is easily verified that $\nabla \cdot u = 0$ is satisfied using spherical coordinates, giving incompressible flow. Also, note that this fluid-velocity choice from relativistic quantum-mechanics has the same direction as the non-relativistic one (67) above. However, definition (68) does
not satisfy \(|u| = |u_\pm|\), except on the \(x-y\) plane where \(u\) and \(\hat{z}\) are perpendicular, and, therefore, velocity choice (68) does not not satisfy the Bernoullian equation (11), an equation implied by a Schrödinger equation.

In turns out, however, that the velocity choice (68) and the non-relativistic one (67) with conserved mass are related. Salesi also demonstrated in his treatment, by using equalities from relativistic quantum mechanics, that the velocity magnitude, or speed, from (68) reduces to \(|u_\pm|\), so choice (68) reduced to (67). In addition, he demonstrated that such a velocity choice satisfies a continuity equation, also from relativistic quantum mechanics. Therefore, the velocity choice (67) seems reasonable, and there are three identifications of its possible source: a fluid-velocity field, internal energy of a static electron, or as a particle velocity, considered below.

Summary with a fluid model. The requirements of zero-angular momentum and the agreement with the kinetic-energy expectation-value is satisfied by the local variable-mass choice (10). This choice also satisfies (non local) mass conservation. A zero angular momentum for a fluid flow over all space is incompatible with local mass conservation with a finite total mass; mass must be created in some regions and destroyed in others. The requirement of local mass-conservation satisfaction and the agreement with the kinetic-energy expectation-value is satisfied by choice (67), which includes information from relativistic quantum mechanics.

12.2.2. Velocity Choice with a dynamic particle description For one-body systems, since the fluid streamlines are the possible paths for the particle, it can be useful to think about the fluid velocity field \(u_\pm\) when considering the electron as a particle. If we choose the velocity to be the minus choice of \(u_\pm\) given by (10), then an \(ns\) electron moves steadily away from the nucleus, suggesting an unstable atom, and a similar argument is applicable for the plus choice of \(u_\pm\), where the electron heads towards the nucleus. Therefore, some modification, involving addition elements, is needed to make the system stable. For example, let \(|u_\pm|\) be the average speed of one-dimensional radially-directed motion, discussed above. Another interpretation is that an electron in the hydrogen atom with zero-spin is unstable, suggesting the need for a relativistic correction involving spin, as discussed above for the fluid description, which we consider below for a particle description.

The derivation of the velocity choices (67) and (68) for a fluid description is easily modified for a particle description, by replacing mass-conservation satisfaction with continuity-equation satisfaction. For these choices, the \(ns\) electron is in a stable circular orbit centered on the \(z\) (vertical) axis with angle \(\theta\) off this axis. Because of the cross product in (68), the speed, a scalar field, is proportional to \(\sin \theta\). With the requirement that the electron motion yields the correct kinetic energy, in order to satisfy the energy equation (8), the electron must be restricted to the \(x-y\) plane, giving \((\theta = \pi/2)\), or else, it will not have sufficient kinetic energy. It is demonstrated elsewhere \[18, 19\] that the \(1s\) electron speed calculated by Eq. (10), which is equal to the speed from choice from (67), or (68) restricted to the the \(x-y\) plane, is a constant, and it is the same speed as
the one for the first Bohr orbit of hydrogen. Colijn and Vrscay use velocity choice (68) to examine the Bohmian trajectories of electronic states of the hydrogen atom [66].

**Summary for ns states of hydrogen for a dynamic particle description.** The velocity choice (10) requires additional elements for stability. One-dimensional motion with the average speed given by choice (10) is one possibility. The requirements of continuity-equation and kinetic-energy expectation-value satisfaction with velocity choice (68) is given by a spin one-half electron with a circular orbit around the nucleus, on the \( x-y \) plane, and perpendicular to the spin vector \( s \), giving two possible direction, one for each spin state. The Bohr model of atomic hydrogen is a special case with a known, and fixed, orbit radius. Velocity choice (67) satisfies both the continuity equation and kinetic energy requirements, with no modifications.

The restriction of the hydrogen \( ns \) states to the \( x-y \) plane is not satisfactory, since one would want the electron to be able to visit any location, in a manner that has some sort of association with the probability density \( \rho \), required for applications with energy fields spread over all space. The velocity choice (67) is not incompatible with this requirement.

**Overall conclusion.** Velocity choice (10) describes spin-zero particles as fluids with local, variable mass. For a given stationary state, velocity choice (67) describes a spin one-half particle, either as a static particle with some sort of, unknown, internal energy, or as a particle orbiting, that is not necessarily on a plane containing the nucleus, having orbital-angular momentum. Since the curl of the velocity field (67) is, in general, nonzero, if such a particle is endowed a body, it spins while it orbits, and, therefore, has spin-angular momentum. For a corresponding fluid identification, the flow is rotational with a forced vortex.

13. Summary

To reduce clutter, except for two equations near the end, this summary gives the equations for the special case of one-body. These become \( n \)-body equations, using the notational system of Sec. 6.

A generalized Euler equation of fluid dynamics, displayed below, is derived. This equation is implied by the \( n \)-body time-dependent Schrödinger equation, where the fields from the equation have the same domain as the wavefunction. These fields are maps of the probability distribution \( (\rho = |\Psi|^2) \) and the phase factor \( S \) of the wavefunction in polar form \( (\Psi = Re^{iS}/\hbar, R^2 = \rho) \). For example, one of the velocity fields is \( (u = -(\hbar/2)\nabla \ln \rho) \). The statements in this summary of the equivalence of equations, and an implication, require these field definitions as auxiliary constraints. The field definitions are given near the end of this summary.

The derived Eulerian equation, mentioned above, for the special case of one-body quantum-states, can be viewed as a sum of two equations (62), \( (EQ_u + EQ_v) \), corresponding to two interacting systems: One equation \( EQ_v \) with conserved mass, involving velocity \( v \) and pressure \( P_v \); one equation \( EQ_u \) with variable mass, involving
velocity $u$, pressure $P$, and the external potential $U$ from the Schrödinger equation. Explicitly, these equations are the following:

\[
\text{EQ}_v = \rho_m \partial v + \frac{1}{2} \rho_m \nabla v^2 + \nabla P_v, \quad \rho_m = m \rho, \quad v = |v|,
\]

\[
\text{EQ}_u = \partial (\rho_m u) + \frac{1}{2} \rho_m \nabla u^2 + \nabla \cdot (\rho_m u) u + \nabla P_u + \rho \nabla U,
\]

where $\rho_m$ is the mass density for the corresponding classical fluid, and $\partial$ is the partial time derivative.

An $n$-body generalization of the Euler equation ($\text{EQ}_u + \text{EQ}_v$) is also derived. In turn, this Eulerian equation is derived from the gradient of a total-energy equation:

\[
E = \frac{1}{2} m (u^2 + v^2) + \rho^{-1} (P_u + P_v) + U,
\]

that is also a generalization of the Bernoulli equation of fluid dynamics, where $U$ is the sum of the external potential and the two-body electron-electron repulsion-energy operator. In turn, this Bernoullian-equation generalization is a sum of two, energy-field equations ($E = E_S + E_\theta$):

\[
E_S = \frac{1}{2} m u^2 + \frac{1}{2} m v^2 + P_u \rho^{-1} + U, \quad E_\theta = \rho^{-1} P_v.
\]

Together, these two equations are demonstrated to be equivalent to the time-dependent Schrödinger equations. In the special case of time-independence, $E_S$ is the energy eigenvalue of the Schrödinger equation. If, in addition, the wavefunction is real valued, we have $E_\theta = 0$, $v^2 = 0$, and the surviving equation for the energy field $E_S$ reduces to the derived, Bernoullian Eq. (11): ($E_S = \frac{1}{2} m u^2 + P_u \rho^{-1} + U$).

The above two displayed energy-equations, for $E_S$ and $E_\theta$, are also equivalent to the two equations of Bohmian-mechanics:

\[
-\partial S = \frac{1}{2} m v^2 + Q + U, \quad \partial \rho + \nabla \cdot (\rho v).
\]

These two equations are Hamilton-Jacobi and continuity equations, respectively. The quantum potential $Q$ is demonstrated to given by Equation (25): ($Q = \frac{1}{2} m v^2 + P_u \rho^{-1}$).

The three pairs of fields of momentum ($v, u$), pressure ($P_v, P_u$), and energy ($E_S, E_\theta$) in the energy equations above, are defined by the real and imaginary parts of functions. A kinetic-energy field is also defined. These functions involve the wavefunction and either the momentum $\hat{P}$ or energy $\hat{H}$ operators of quantum mechanics. The pair of particle momentums, $m v$ and $m u$, called momentae, are defined by

\[
\rho^{-1} \Psi \hat{P} \Psi \doteq m v + i m u,
\]

giving the formulae

\[
m v = \nabla S, \quad m u = -(\hbar/2) \nabla \ln \rho \doteq \nabla \theta,
\]

where $S$ and $\theta$ are the momentae potentials, and where $\theta$ is defined by the above equation (to within an additive constant). The kinetic-energy field is defined to be $|\Psi \hat{P} \Psi|^2/2m$,
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\[ (\mu^2 + m v^2)/2. \]
Similarly, the energy fields are defined by

\[ \rho^{-1} \Psi^* \hat{H} \Psi = E_S + i E_\theta, \]

where the time-dependent Schrödinger equation \((i \hbar \Psi^* \partial / \rho = \rho^{-1} \Psi^* \hat{H} \Psi)\) is substituted to obtain formulae involving time derivatives:

\[ E_S = -\partial S, \quad E_\theta = -\partial \theta. \]

Also, the pressures are defined by

\[ \psi \rho \frac{1}{\rho} \psi^* \hat{P} \psi \dot{\psi} = -P + i P_u. \]

giving the formulae:

\[ P_u = (\hbar/2m) \nabla \cdot \rho \nabla u, \quad P_v = -(\hbar/2m) \nabla \cdot \rho \nabla v. \]

Other relations are derived from the continuity equations involving the pressures and time derivatives:

\[ P_v = (\hbar/2) \partial \rho, \]

\[ \nabla_i P_v = -\partial (\rho u_i), \quad i \in \{1, \cdots n\}. \]

\[ -(\hbar/2m) \nabla_i^2 P_v = \partial P_{u_i}, \quad i \in \{1, \cdots n\}, \]

where the last two equations reduce to the special case of one-body by suppressing the \(i\) and \(j\) subscripts.

Sec. \((11)\) demonstrates energy conservation for the nonuniform energy fields, \(E_S\) and \(E_S\), in cases where the wavefunction is a nondegenerate linear-combination of eigenfunctions of a time-independent Hamiltonian operator \(\hat{H}\), where the nondegenerate linear-combination is not an eigenfunction of the Hamiltonian operator. Sec. \((12)\) gives a thorough discussions on the compatibility of the two velocity fields for vector addition, and the consideration of many possibilities for modifications of the Bernoullian velocity \(u\), to obtain the best match from classical mechanics.

14. Discussion

Given the wave-particle duality of quantum mechanics, it is not surprising that the Eulerian equation \((61)\) seem to describe two interaction substates, since both of the concepts indicate a doubling of the degrees of freedom. The presence of two substates, justifies the use of more complicated physical models to describe the electron, even though there are numerous constraints. However, this produces a new problem of uniqueness, at least for the identification of a model for the kinetic energy coming from the Bernoullian velocity \(u\): There seems to be too many possibilities, especially considering the possibilities explored in Sec. \((12)\), and the observation that some of the various properties are compatible. For an over the top example, the electron has a compressible body that stores energy, and rotates, translates, and/or “zitters,” while being in a medium under pressure or tension, gaining and losing mass as it goes, and,
all the while, diffusing through the medium. To identify the best model, comparisons with experiment would be useful.

Declarations

The author did not receive support from any organization for the submitted work. The author has no relevant financial or non-financial interests to disclose.

Data availability statement

All data that support the findings of this study are included within the article.

Appendix A. The variable mass Euler equation for irrotational flows

In the derivation of the Euler equation, the differential form of the momentum-balance equation is used that does not hold for systems that do not conserve mass. Since our systems under consideration have variable mass, we need a form of the Euler equation that does not use the continuity equation. It is trivial to derive variable mass momentum-balance equations, one simply uses a standard derivation [68], but refrain from utilizing the continuity equation, leading to some extra terms in the working equation. Here we derive the variable-mass Euler equation for irrotational flows. I follow the vector-calculus approach and notation of Kelly [62].

We start with the momentum balance equation for a fluid subject to a Coulombic body-force with force per charge \((-\nabla \Phi)\) with mass density \(\rho_m\), charge density \(q\rho\) and \(\rho = \rho_m/m\):

\[
\frac{d}{dt} \int_V \rho_m u \, dV = \int_S \sigma \mathbf{n} \, dS + \int_V q \rho (-\nabla \Phi) \, dV, \tag{A.1}
\]

where \(\sigma\) is the stress tensor and \(\mathbf{n}\) is the normal unit vector to the surface \(S\), the border of the subspace \(V\). First we work on the left-hand side. Using the Reynolds’ transport theorem, and the definition,

\[
\int_V g \frac{d(dV)}{dt} \doteq \int_V g \nabla \cdot u \, dV,
\]

where \(g\) is an arbitrary function, we obtain

\[
\frac{d}{dt} \int_{V(t)} \rho_m u \, dV = \int_V \left( \rho_m \frac{d u}{d t} + u \frac{d \rho_m}{d t} \right) \, dV + \int_V \rho_m u \frac{d(dV)}{dt} \\
- = \int_V \left( \rho_m \frac{\partial u}{\partial t} + \rho_m (\text{grad} u) u + \left( \frac{\partial \rho_m}{\partial t} + \nabla \rho_m \cdot u \right) u + (\nabla \cdot u) \rho_m u \right) \, dV,
\]

where \([\text{grad} u]_{ij} = \partial u_i / \partial x_j\) and \([\text{grad} u u]_i = (\partial u_i / \partial x_j) u_j\), summed over \(j\). Hence

\[
\frac{d}{dt} \int_V \rho_m u \, dV = \int_V \left( \rho_m \frac{\partial u}{\partial t} + \rho_m (\text{grad} u) u + \mathbf{M} u \right) \, dV, \tag{A.2}
\]
where
\[ M = \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = \dot{\rho}_m + (\nabla \cdot \mathbf{u})\rho_m, \]  
(A.3)
and the continuity equation for systems that conserve mass is \( M = 0 \). Note that the definition above permits the use of a product rule of differentiation on the factors of the integrand: \( \rho_m \times \mathbf{u} \times dV \), where \( dV \) is “considered” a factor, even though the symbol is excluded in some notations for integrals.

For the surface integral of Eq. (A.1), we apply the divergence theorem:
\[ \int_{S} \sigma \mathbf{n} dS = \int_{V} \text{div} \sigma dV. \]  
(A.4)
Substituting Eq. (A.2) and (A.4) into (A.1) gives the differential momentum-balance equation with variable mass:
\[ \rho_m \frac{\partial \mathbf{u}}{\partial t} + \rho_m (\text{grad} \mathbf{u}) \mathbf{u} + M \mathbf{u} = \text{div} \sigma + q\rho (-\nabla \Phi), \]  
(A.5)
where we removed the integrations and obtained a true statement, since the equation with the integrations holds for all subspaces \( V \).

Next we consider only inviscid flows. By definition, these satisfy \( \sigma = -p \mathbf{I} \), where \( p \) is the pressure. Using this equality and a vector identity, the first term on the right-hand side (A.5) for inviscid fluids becomes
\[ \text{div} \sigma = -\text{div} (p \mathbf{I}) = -\nabla p. \]

Next we require \( \mathbf{u} \) to be irrotational, i.e., \( \nabla \times \mathbf{u} = 0 \). This permits the use of the following equality:
\[ (\text{grad} \mathbf{u}) \mathbf{u} = \frac{1}{2} \nabla u^2, \quad \text{if} \quad \nabla \times \mathbf{u} = 0. \]  
(A.6)
Substituting the above two equations into (A.5) we obtained the desired equation:
\[ \rho_m \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \rho_m \nabla u^2 + \left( \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) \right) \mathbf{u} = -\nabla p - q\rho \nabla \Phi, \]
and we also used (A.3). This equation can also be written
\[ \frac{\partial}{\partial t} (\rho_m \mathbf{u}) + \frac{1}{2} \rho_m \nabla u^2 + \nabla \cdot (\rho_m \mathbf{u}) \mathbf{u} + \nabla p + q\rho \nabla \Phi = 0. \]  
(A.7)
This equation is the variable-mass Euler equation for the special case of irrotational flow. In other words, Eq. (A.7) is applicable to flows that are irrotational, compressible, inviscid, and variable-mass. We also require the body force to be Coulombic, but, obviously, \( q\rho \nabla \Phi \) can be replaced by \( \rho_m F \), where \( F \) is a force per mass. In the special case where the flow is steady, incompressible, and mass is conserved, the division of (A.7) by \( \rho_m \), followed by integration, yields the Bernoulli equation (11).

In the special case where mass is conserved, explicitly given by
\[ \frac{\partial}{\partial t} \rho_m + \nabla \cdot (\rho_m \mathbf{u}) \mathbf{u} = 0, \]
Eq. (A.7) reduces to the familiar form:
\[ \rho_m \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \rho_m \nabla u^2 + \nabla p + q\rho \nabla \Phi = 0. \]  
(A.8)
Appendix B. Equalities for the Velocities

Next we show that

\[ u_i^\pm = \pm \text{Re} \left( \frac{\hbar}{m} \nabla_i \Psi \right), \quad v_i = \text{Im} \left( \frac{\hbar}{m} \nabla_i \Psi \right), \]

where the second one is well known. To reduce clutter we suppress the \( i \) subscripts. We require \( u^\pm \) to be defined by (4) and we use \( v = \nabla S/m \), from (22), to define \( v \). Starting with the ansatz \( \Psi = \text{Re}(i S/\hbar) \) we have

\[
\nabla \Psi = (\nabla R)e^{iS/\hbar} + i\hbar^{-1}\text{Re}(iS/\hbar)\nabla s,
\]

\[
\frac{\hbar}{m} \nabla \Psi = \frac{\hbar}{m}(\nabla R)R^{-1} + i \frac{\nabla S}{m}.
\]

Taking the imaginary part of this equation, and using \( v = \nabla S/m \), gives the second one from (B.1). The real part of the above equation is

\[
\pm \text{Re} \left( \frac{\hbar}{m} \nabla \Psi \right) = \pm \frac{\hbar}{m} \nabla R.
\]

Starting with (4) for \( u^\pm \) we have

\[
u^\pm = \pm \frac{\hbar}{2m} \frac{\nabla Y}{Y} = \pm \frac{\hbar}{2m} \frac{\nabla R^2}{R^2} = \pm \frac{\hbar}{m} \frac{\nabla R}{R}.
\]

So the first one from (B.1) is also true.

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