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Modeling and optimal control analysis of transmission dynamics of COVID-19: The case of Ethiopia

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Abstract A mathematical model to estimate transmission dynamics of COVID-19 is developed. A real data of confirmed cases for Ethiopia is used for parameter estimation via model fitting. Results showed that, the diseases free and endemic equilibrium points are found to be locally and globally asymptotically stable for \( R_0 < 1 \) and \( R_0 > 1 \) respectively. The basic reproduction number is \( R_0 = 1.5085 \). Optimal control analysis also showed that, combination of optimal preventive strategies such as public health education, personal protective measures and treatment of hospitalized cases are effective to significantly decrease the number of COVID-19 cases in different compartments of the model.

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1. Introduction

It is reported that [1] up to the present, the world has witnessed the happening of seven strains of human coronaviruses including SARS-CoV, MERS-CoV and the current coronavirus 2019-nCoV named COVID-19. COVID-19 was declared a pandemic by the World Health Organization (WHO) on 11 March 2020 [2]. As of July 27th 2020, COVID-19 pandemic has claimed several hundreds of deaths globally and become the greatest threat of its kind in our generation [3]. It also posed a huge risk to public health and economics all over the world [4,5]. The Ethiopian Federal Ministry of Health confirmed its first case of COVID-19 on 13 March 2020, in Addis Ababa [6]. Despite the fact that the number of confirmed cases and deaths reported varies from time to time, the outbreak has taken the lives of several hundreds and affected different livelihoods in Ethiopia.

Since the first case of COVID 19, various mathematicians around the world develop different mathematical models to understand the transmission dynamics of the virus, estimated the basic reproductive number and investigated effects of different intervention strategies via optimal control analysis. For instance, in [7] real data of confirmed cases of COVID-19 in Wuhan, China from January 21, 2020 to January 28, 2020 is used in a model developed to estimate the basic reproductive number \( R_0 = 2.4829 \). The authors developed a fractional model and solved it numerically. A mathematical model for predicting transmission dynamics of the pandemic for Italy is considered in [8]. The result indicated that the combination of restrictive social distancing measures with widespread testing and contact tracing can result in ending
the COVID-19 transmission in Italy. Analysis of Covid-19 infection, based on real data is made in [9] by developing a mathematical model for Wuhan, China.

In [10] fractional order COVID-19 SIDARTHE model is developed to predict the evolution of the pandemic and investigated the impact of different plans to reduce the transmission of the disease with different values of fractional order. There are also many other mathematical models that reported on COVID-19 pandemics, refer [11-14]. Mathematical models developed were mainly used to investigate the effects of different Non-pharmaceutical intervention strategies via simulation using different computing soft-wares [15].

Different scholars used different type of mathematical models in their analysis. For instance, an SEIR model was introduced by Wu et al. [16] for estimating the spread of the pandemic and obtained the basic reproductive number to \( R_0 = 2.68 \). Tang et al. [12] used a deterministic model to estimate the basic reproductive number to be as high as \( R_0 = 6.47 \) and concluded that, contact tracing followed by quarantining and then isolation can mitigate the spread of the pandemic via reducing the basic reproductive number. Kiesha et al. [15] used SEIR model to simulate the outbreak of Coronavirus in Wuhan and indicated that imposing restriction on Wuhan people movement could help in delaying the peak time of the pandemic. Roda et al. [17] claimed that SIR model performs better than SEIR model and used it to predict COVID-19 epidemic in Wuhan and predicted the potential of a second outbreak after the return-to-work in the city. Pang et al. used SEHR mathematical model to investigate the effectiveness of quarantine measures applied in Wuhan city and factors affecting its effectiveness [18].

It is not only for COVID-19 that models were used to study the dynamics but also for several other infectious diseases. In [19] a mathematical model for the deadly disease named Ebola hemorrhage fever is developed and investigated the detail of endemic equilibrium points. SIR model with delay in the context of fractional derivative with Mittag-Leffler is considered in [20]. The author established the global and local stability of disease free and endemic equilibrium points using Lyapunov direct method. Mathematical models that investigated the dynamics of various infectious diseases are numerous in the literature [1,21] and the references there in.

Most of these studies have either restricted their domain of study to a particular country or they used data from a certain particular region. Some of the studies also overlooked the issue of optimal control analysis. Moreover, to the best of authors' knowledge, no study has been conducted on transmission dynamics of COVID-19 for the case of Ethiopia. To this end, the main objective of this work is to develop a mathematical model for the transmission dynamics of COVID-19 followed by mathematical analysis in order to come up with evidences that may help for designing mitigation strategies relevant to the context of Ethiopian.

In this paper a mathematical model for COVID-19 relevant to study the transmission dynamics of coronavirus in Ethiopia is developed. The existence and uniqueness of solution of the model is proved. Local and global stability analysis of the diseases free and endemic equilibrium points are established. Parameter estimation based on real data of confirmed cases is made. The basic reproductive number suggested by the model is calculated. The importance of the basic reproductive number in target setting to control the pandemic is discussed by calculating sensitivity indices of the parameters of the basic reproductive number. Optimal control analysis of the model with three control strategies namely: public health education, personal protective measures and treatment of hospitalized COVID-19 cases were investigated followed by numerical simulation. Discussions, conclusions and limitation of the model are well specified.

2. COVID-19 model formulation

In this work, a mathematical model for COVID-19 transmission dynamics is developed based on compartmental approach of six groups named as \( S \)-susceptible, \( E \)-exposed, \( I \)-symptomatically infected, \( A \)-asymptomatically infected, \( H \)-hospitalized and \( R \)-recovered/ immune compartments. The total population at any time \( t \) is given by \( N(t) = S(t) + E(t) + I(t) + A(t) + H(t) + R(t) \). Humans are recruited into the compartment \( S(t) \) at the constant rate of \( \lambda \) and get infected to coronavirus through contact with either symptomatically or asymptotically infectious individuals with the force of infection \( \alpha(e + A)/N \). The compartment \( E(t) \) gains population from infection induced by coronavirus at the rate of \( \alpha(e + A)/N \). A proportion \( \rho \) of the population in \( E(t) \) progress to compartment of asymptomatic infection \( A(t) \) at the rate of \( \phi \) and the remaining proportion \( 1 - \rho \) progress to symptomatic infection compartment \( I(t) \) with the same rate. Some of the population from compartments \( I(t) \) and \( A(t) \) progress to Isolation/hospitalization compartment \( H(t) \) at the rate of \( \delta \) and \( \lambda \) respectively and the remaining population exit through the compartment of recovery \( R(t) \) as a result of natural immunity and treatment. Some of the population from the compartment \( H(t) \) exit through diseases induced death at the rate of \( \sigma \) and the remaining progress to compartment \( R(t) \). The natural death rate \( \sigma \) is also considered in the model.

The above mentioned is the scenario that government of Ethiopia is using in combating the deadly COVID-19 pandemic, in the sense that the data of confirmed cases of COVID-19 in Ethiopia is organized incorporating all the compartments we have used in developing the model of this study. It is these data that we translated into a mathematical model. It seems logical to divide infected cases of COVID-19 [22]. In addition, it is known that coronavirus infected people don’t get infectious immediately after contraction of the virus and this ensures the importance of considering the compartment of exposed cases. As a result, the aforementioned scenario and modification of SEIR [15,21] model are used to develop the present model.

The model developed and the parameters used are indicated in system of differential equation (1) and Table 1.

\[
\begin{align*}
\dot{S} &= \Lambda - \frac{\alpha(t + 1)S}{N} - \sigma S, \\
\dot{E} &= \frac{\alpha(t + 1)S}{N} - (\phi + \sigma)E, \\
\dot{I} &= (1 - \rho)\phi E - (\delta + \theta + \sigma)I, \\
\dot{A} &= \rho \phi E - (\lambda + \beta + \sigma)A, \\
\dot{H} &= \delta I + \lambda A - (\sigma + \mu + \sigma)H, \\
\dot{R} &= \theta I + \beta A + (\sigma + \mu)H - \sigma R.
\end{align*}
\]
In particular, the equations in (1) are positive and hence we can set a lower bound for each of (S, E, I, A, H) and R(t) are all bounded since, \( S(t), E(t), I(t), A(t), H(t), R(t) \leq N(t) \leq \frac{\Lambda}{\mu} \).

Thus, the region
\[
\Omega = \left\{ (S, E, A, I, H, R) \in \mathbb{R}^6 : S(t) + E(t) + A(t) + I(t) + H(t) + R(t) \leq \frac{\Lambda}{\mu} \right\}
\]
is positively invariant region for model (1).

Furthermore if \( N(0) > \frac{\Lambda}{\mu} \) then either the solution of (1) enters \( \Omega \) in a finite time or \( N(t) \) approaches \( \frac{\Lambda}{\mu} \) asymptotically. Hence the region \( \Omega \) attracts all solution of (1) in \( \mathbb{R}^6_+ \).

**Lemma 3: (Existence and Uniqueness)** In model (1) if the initial conditions \( S(0) > 0, E(0) > 0, I(0) > 0, A(0) > 0, H(0) > 0, R(0) > 0, \) and \( t_0 > 0 \), then for all \( t \in \mathbb{R} \) the solutions \( S(t), E(t), I(t), A(t), H(t), R(t) \) exist in \( \mathbb{R}^6_+ \).

**Proof:** Model (1) can be expressed in the form \( \dot{x} = f(x) \) where
\[
\dot{x} = \begin{pmatrix}
S(t) \\
E(t) \\
I(t) \\
A(t) \\
H(t) \\
R(t)
\end{pmatrix},
\phantom{**Proof:**}
f(x) = \begin{pmatrix}
\frac{\Lambda - S(t)S(t) - \mu S(t)}{N(t)} \\
\frac{\alpha}{\mu} S(t) - (\phi + \mu) E(t) \\
\beta S(t) - (\delta + \theta + \mu) I(t) \\
\lambda I(t) - (\lambda + \beta + \mu) A(t) \\
\delta I(t) + \lambda A(t) - (\sigma + \mu + \sigma) H(t) \\
0 + \beta A(t) + (\sigma + \mu) H(t) - \sigma R(t)
\end{pmatrix}.
\]

Since \( f \) has a continuous first derivative in \( \mathbb{R}^6_+ \), it is then locally Lipschitz. As a result, by the well known fundamental existence and uniqueness theorem [23] and Lemma 1 and 2 proved above, there exists a unique, positive and bounded solution for the system of differential equation (1) in \( \mathbb{R}^6_+ \).

**2.2. Equilibrium points**

Solving the dynamic system (1), we obtain two equilibria points:

i. Disease free equilibrium point (DFE): \( N_0 = (S_0, E_0, I_0, A_0, H_0, R_0) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0) \).

ii. Endemic equilibrium point (EEP): \( N^* = (S^*, E^*, I^*, A^*, H^*, R^*) \),

where
\[
S^* = \frac{\alpha}{\mu} E^*; \quad I^* = -\frac{\beta}{\mu} E^*; \quad A^* = -\frac{\alpha}{\mu} E^*; \quad H^* = \frac{\delta}{\mu} E^* + \frac{\delta}{\mu} H^*;
\]
\[
R^* = \frac{1}{\beta} \left( -\frac{\alpha}{\mu} - \frac{\beta}{\mu} E^* + \frac{\alpha}{\mu} H^* + \frac{\delta}{\mu} H^* \right);
\]
and
\[
E^* = \frac{\mu - \mu_1}{\mu} \frac{N - \mu R_0}{\mu_1 R_0} \quad \text{that satisfies the quadratic equation}
\]
\[
m_1 E^2 + m_2 E^2 = 0,
\]
where,
\[
m_1 = \frac{1}{\mu_1} \frac{\mu - \mu_1}{\mu_1} R_0, \quad m_2 = \frac{1}{\mu_1} \frac{N - \mu R_0}{\mu_1 R_0} R_0 = R_0 + R_1 + R_2, \quad R_1 = \frac{\mu}{\mu_1}, \quad R_2 = \frac{\mu}{\mu_1},
\]
\[
l_1 = -\frac{\mu + \mu_1}{\mu_1}, \quad l_2 = (1 - \frac{\mu}{\mu_1}) l_1 = -(\delta + \theta + \mu), \quad l_3 = (\lambda + \beta + \mu),
\]
\[
l_4 = -\frac{\alpha}{\mu}, \quad l_5 = (\sigma + \mu).
\]

As a result a positive endemic equilibrium point exists only if \( R_0 > \frac{\mu_1}{\mu} \).
If we approximate \( \Lambda = \pi N \), then an endemic equilibrium point exists if and only if \( R_0 > 1 \). In the next subsection, it will be made clear that \( R_0 \) is the basic reproductive number.

### 2.3. The basic reproductive number

The basic reproductive number, denoted by \( R_0 \), is defined as the number of secondary infections appearing from one infected individual. It provides a threshold condition for the stability of the system. Finding \( R_0 \) through Jacobian approach using linearization of the dynamic system often doesn’t work for complex systems [24]. Thus, we obtained \( R_0 \) by establishing the next generation matrix, \( TV^{-1} \), and calculating a spectral radius of the matrix \( TV^{-1} \) at \( N_0 \) [25]. The matrices \( T \) and \( V^{-1} \) are obtained by linearizing the mathematical model (1) about DFE that resulted in the Jacobian matrix \( J_{DE} \) given as:

\[
J_{DE} = \begin{bmatrix}
-\sigma & 0 & -\sigma
-\sigma & 0 & 0
0 & (1-\rho)\varphi & -\delta + \beta + \sigma
0 & 0 & -\delta
\end{bmatrix}
\]

After constructing a matrix (\( M \)) from infectious components of the \( J_{DE} \) and partitioning it into transition (\( V \)) and transition (\( T \)) matrices, we have:

\[
M = T + V = \begin{bmatrix}
-\sigma & 0 & -\sigma
-\sigma & 0 & 0
0 & (1-\rho)\varphi & -\delta + \beta + \sigma
0 & 0 & -\delta
\end{bmatrix}
\]

where,

\[
T = \begin{bmatrix}
0 & \sigma & \sigma & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
V^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

The spectral radius of the next generation matrix \( \rho(TV^{-1}) \) is the basic reproductive number given by:

\[
R_0 = \frac{\rho(1-\rho)\varphi}{(\sigma + \sigma)(\delta + \beta + \sigma) - \sigma \rho \varphi}.
\]

Note that, \( R_0 \) can also be written as \( R_0 = R_1 + R_2, R_1 = \frac{\rho(1-\rho)\varphi}{\sigma(\delta + \beta + \sigma)}, R_2 = \frac{\sigma \rho \varphi}{\sigma(\delta + \beta + \sigma)} \) where \( l_i, i = 1, \ldots, 6 \) are defined earlier.

### 3. Stability analysis

This section deals with the local stability analysis of the diseases free and the endemic equilibrium points.

#### 3.1. Local stability analysis of DFE and EEP

**i. Local Stability analysis of DFE**

**Theorem 1:** The diseases free equilibrium point, \( N_0 = (S_0, E_0, I_0, A_0, H_0, R_0) = (\frac{\sigma}{\beta}, 0.0, 0.0, 0.0) \) of (1) is locally asymptotically stable if and only if \( R_0 < 1 \).

**Proof:** The three negative eigenvalues of the Jacobian matrix (2) are given by \( e_1 = e_2 = -\sigma \) and \( e_3 = -(\sigma + \mu + \sigma) \). The remaining three eigenvalues are determined if they have negative real part or not, from the following characteristic equation by Routh-Hurwitz stability criteria:

\[
\phi(e) = e^3 + a_2e^2 + a_1e + a_0 = 0
\]

where

\[
a_2 = -(l_1 + l_3 + l_4),
a_1 = l_1l_3 + l_1l_4 + l_3l_4 - a_0p - a_0l_2,
a_0 = -l_1l_3l_4 + a_0p l_3 + a_0l_3l_4.
\]

**Necessary Condition:** Since both \( R_1 \) and \( R_2 \) are positive and \( R_1 + R_2 = R_0 \), then from the characteristic equation (4), we have \( a_2 > 0 \) and \( a_1 = l_1l_3(1-R_1) + l_1l_4(1-R_2) + l_3l_4 > 0 \) only if \( R_0 < 1 \). Moreover, since \( l_1l_3l_4 < 0 \), then \( a_0 = -l_1l_3l_4 + a_0p l_3 + a_0l_3l_4 < 0 \), only if \( R_0<1 \). The sufficient condition:

\[
a_0 - a_1a_2 = 2l_1l_3l_4(l_1 + l_3)(1 - R_1) + (l_1 + l_3)(l_1l_4(1 - R_2) + (l_1 + l_3)(l_3l_4) < 0, \text{ for } R_1 < 1, R_2 < 1, R_0 < 1, \text{ and by Routh-Hurwitz stability criteria all the eigenvalues of the characteristic equation } \phi(e) = e^3 + a_2e^2 + a_1e + a_0 = 0, \text{ have a negative real part. Hence, from the two conditions above, the diseases free equilibrium point } N_0 \text{ is locally asymptotically stable if and only if } R_0 < 1.
\]

**ii. Local Stability analysis of EEP**

**Theorem 2:** The Endemic equilibrium points of (1), \( N^* = (S^*, E^*, I^*, A^*, H^*, R^*) \) is locally asymptotically stable if and only if \( R_0 > 1 \).

**Proof:** The Jacobian matrix \( J_{ep} \) of (1) at the EEP is given by:

\[
J_{ep} = \begin{bmatrix}
-l_7 - \sigma & -e_3 & -e_1 & 0 & 0 & 0 \\
l_1 & l_1 & l_3 & l_6 & 0 & 0 \\
0 & l_2 & l_5 & 0 & 0 & 0 \\
0 & 0 & \delta & \lambda & l_5 & 0 \\
0 & 0 & 0 & \beta & l_6 & -\sigma
\end{bmatrix}
\]

where

\[
l_1 = -(\sigma + \sigma), l_2 = (1-\rho)\varphi, l_3 = -(\sigma + \beta + \sigma), l_4 = -(\lambda + \beta + \sigma), l_5 = -(\sigma + \mu + \sigma), l_6 = (\sigma + \mu), l_7 = \frac{\sigma \rho \varphi}{\sigma(\delta + \beta + \sigma)}, l_8 = \frac{\sigma}{\beta}.
\]

The two negative eigenvalues of the Jacobian matrix \( J_{ep} \) are \( e_1 = -\sigma, e_2 = l_3 = -(\sigma + \mu + \sigma) \). The sign of the real part of the remaining four eigenvalues are determined from a characteristic equation (5) by Routh-Hurwitz stability criteria:

\[
f(e) = e^3 + b_3e^2 + b_2e + b_1e + b_0,
\]

where,
\[ b_1 = \sigma - I - I - I + I; \]
\[ b_2 = I + I + I + I - I - I - I + I; \]
\[ b_3 = I + I - I + I - I + I + I + I; \]
\[ b_4 = I + I - I + I + I + I - I + I; \]

**Necessary Condition:** The coefficient \( b_i \) is positive and \( b_2, b_1, b_0 \) can be shown to be positive as follows:

\[ b_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\lambda_3 \lambda_2} + I + I - I - I + I + I + I > 0, \]
\[ b_1 = \sigma I + I + I + I - I - I - I + I > 0, \]
\[ b_0 = \sigma I + I + I + I - I - I - I + I > 0. \]

**Sufficient Condition:** Furthermore, by Routh-Hurwitz stability criteria all the eigenvalues of the characteristic equation (5) have negative real part since it can be shown that \( b_0 b_1^2 - b_2 b_1 b_0 < 0 \).

Hence, EEP \( N^* = (S^*, E^*, I^*, A^*, H^*, R^*) \) is locally asymptotically stable if and only if \( R_0 > 1 \). Note that the EEP is positive if and only if \( R_0 > 1 \) as it is discussed in the previous sections.

### 3.2. Global stability analysis DFE and EEP

In this subsection we showed global asymptotical stability of the DFE and EEP using Lyapunov function method.

#### i. Global Stability Analysis of DFE

The locally asymptotically stability of the DFE proved earlier indicates that COVID-19 can be eliminated from Ethiopia when \( R_0 \) is smaller than the initial size of the cases in each of the compartments or in the society is very close to the DFE; initial size is in the basin of attraction of DFE, \( N_0 = (S_0, R_0, I_0, A_0, H_0, R_0) = \left( \frac{\lambda_3}{\lambda_2}, 0, 0, 0, 0, 0 \right) \). In order to show that the elimination of COVID-19 is independent of the initial size of the number of cases in the society, it is necessary to show that the DFE is globally asymptotically stable if \( R_0 < 1 \). This case is explored by following the following theorem.

**Theorem 3:** If \( R_0 < 1 \), then the DFE given by \( N_0 = (S^*, E^*, I^*, A^*, H^*, R^*) \) defined in section 2.1 is globally asymptotically stable in the positively invariant region \( \Omega \) defined in section 2.

**Proof:** Consider a Lyapunov function candidate

\[ V(S, E, I, A, H, R) = \left( S - S_0 - S_0 \ln \frac{S}{S_0} \right) + E + I + A + H + R \]

Differentiating \( V(S, E, I, A, H, R) \) with respect to time in the direction of the solution of (1), and then substituting the appropriate values from (1) and \( S_0 = \frac{\lambda_3}{\lambda_2} \) leads to,

\[ \dot{V} = \Lambda \left( 2 - \frac{\lambda_3}{S} - \frac{\lambda_2}{S_0} \right) - \sigma (E + I + A + H + R), \]

where \( \delta_n = \frac{\lambda_1 \lambda_2 \lambda_0}{\lambda_3} \).

Since \( \delta_n \left( \frac{\lambda_3}{\lambda_2} \right) \) is non-negative we have,

\[ \dot{V} \leq \Lambda \left( 2 - \frac{\lambda_3}{S} - \frac{\lambda_2}{S_0} \right) - \sigma (E + I + A + H + R) \]
\[ = \Lambda \left( \frac{2SS_0 - (S^* + S)}{SS_0} \right) - \sigma (E + I + A + H + R) \]
\[ = - \Lambda \left( \frac{(S - S^*)^2}{SS_0} \right) - \sigma (E + I + A + H + R) \]

Note that, by the inequality of arithmetic and geometric means we have

\[ \frac{2SS_0 - (S^* + S)}{SS_0} \leq 0 \]

Thus we have proved that \( V \) is a Lyapunov function and \( \dot{V} \leq 0 \) and \( \dot{V} = 0 \) if and only if \( S_0 = S, E = I = A = H = R = 0 \). Therefore, it follows that the largest invariant set in \( \{(S, E, I, A, H, R) \in \Omega : \dot{V} = 0 \} \) is \( N_0 = \left( \frac{\lambda_3}{\lambda_2}, 0, 0, 0, 0, 0 \right) \). Thus, by LaSalle’s invariance principle the DFE, is globally asymptotically stable.

Note that, since the proof is general without considering \( R_0 < 1 \) as necessary condition, it holds true if \( R_0 < 1 \) in particular. This completes proof of the theorem.

#### ii. Global Stability Analysis of EEP

The global stability of EEP is explored by proving Theorem 4.

**Theorem 4:** If \( R_0 > 1 \), then the EEP given by \( N^* = (S^*, E^*, I^*, A^*, H^*, R^*) \) defined in section 2.1 is globally asymptotically stable in the region \( \Omega \).

**Proof:** Suppose the basic reproductive number \( R_0 > 1 \) so that the EEP exists. Consider a Lyapunov function candidate \( L \) defined by,

\[ L(S, E, I, A, H, R) = \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + (E - E^* + E^* \ln \frac{E}{E^*}) + (I - I^* + I^* \ln \frac{I}{I^*}) + (A - A^* + A^* \ln \frac{A}{A^*}) + (H - H^* + H^* \ln \frac{H}{H^*}) + (R - R^* + R^* \ln \frac{R}{R^*}) \]

Differentiating \( L \) in the direction of the solution of (1) leads to,

\[ \frac{dL}{dt} = \left( S - S^* \right) \dot{S} + \left( E - E^* \right) \dot{E} + \left( I - I^* \right) \dot{I} + \left( A - A^* \right) \dot{A} + \left( H - H^* \right) \dot{H} + \left( R - R^* \right) \dot{R} \]

Replacing \( S, E, I, A, H, \) \( R \) by their respective expression from model (1) we get,

\[ \dot{S} = \left( \frac{\lambda_3}{\lambda_2} \right) \Lambda \left( \frac{\lambda_1 \lambda_2 \lambda_0}{\lambda_3} \right) - \sigma (E + I + A + H + R) \]
\[ + \Lambda \left( \frac{2SS_0 - (S^* + S)}{SS_0} \right) - \sigma (E + I + A + H + R) \]
\[ - \Lambda \left( \frac{(S - S^*)^2}{SS_0} \right) - \sigma (E + I + A + H + R) \]

Replacing \( S, E, I, A, H, \) \( R \) by their respective expression from model (1) we get,
\[
\begin{align*}
\frac{dS}{dt} &= -\left(\frac{S - S'}{S} + \frac{(x(i + A) + \sigma)}{N}\right) + \Lambda + \left(\frac{x(i + A)}{N} + \sigma\right) S' - \frac{E}{E^2} (\sigma + \sigma) \frac{S'}{S} \\
&= \left(-\frac{S'}{S} \Lambda + \frac{S'(x(i + A) + \sigma)S'}{N} + \frac{(x(i + A) + \sigma)S'}{N}\right) + \left(\frac{E}{E^2} \left(\sigma + \sigma\right) + \frac{(x(i + A) + \sigma)S'}{N}\right) \\
&= \frac{E}{E^2} \left(\sigma + \sigma\right) + \frac{(x(i + A) + \sigma)S'}{N} - \frac{E}{E^2} \frac{x(i + A) + \sigma}{N} S'.
\end{align*}
\]

Note the implication of Theorem 4 is that, COVID-19 will be asymptotically stable. Theorem 1 suggests that a reproductive number of 0.83 is calculated on the relation \(\Lambda/\sigma = N(t)\), and hence \(\Delta = \sigma N(t) = 4576/day\). Since 17,530 people were infected and 274 of them confirmed dead in 140 days from March to 31 July 2020, COVID-19 induced death rate is calculated to be \(\sigma = 0.00001/day\). It is also learnt from the data that, in the last 140 days of the pandemic 5700 hospitalized or isolated cases were recovered from the diseases. Hence, the recovery rate \(\mu = \frac{6700}{17530} = 0.00273/day\).

Some of the remaining parameters are estimated from the mathematical model (1) via model fit using a computing software MatLab2018a by optimal estimation of parameters; least square curve fit along with Trust-Region-Reflective to find bounded estimated parameters. The result of model fit versus confirmed cases of COVID-19 is portrayed in Fig. 1. Thus, values of the parameters used in this work are obtained from assumption, calculation, model fitting and from the literature and are indicated in Table 2.

Based on the parameter values in Table 2, the basic reproductive number for Ethiopia is calculated as \(R_0 = 1.5085\), \(R_1 = 0.6788\), \(R_2 = 0.8297\). Here, \(R_0 = 0.6788\) is the contribution of symptomatic cases to susceptible transmission and \(R_2 = 0.8297\) is the contribution from asymptomatic to susceptible transmission. The mathematical model used in this work suggested a reproductive number of \(R_0 = 1.5085\); that is in the range of what WHO suggested; from 1.4 to 2.5 [29]. However, \(R_0 = 1.5085\) is less than several other estimations of the basic reproductive number such as in [7,12,16,30].

The EEP is calculated based on the parameter values of Table 1 given by \(N_0 = (79290000, 10654, 1034.5, 1264.5, 24468, 37042000)\). The eigenvalues of the Jacobian matrix of (1) at the EEP is given by \(-1.0183, -0.2719, -0.1032, -0.0001\) which are all negative and hence the endemic equilibrium point is locally asymptotically stable, and is in agreement with the analytical proof made in subsection 3.2.

In order to determine the best control measures using the basic reproductive number, the normalized sensitivity index, which measures the the relative change of \(R_0\) with respect to \(\omega\) denoted by \(Y_\omega\), and defined as \(Y_{\omega} = (\omega/R_0)\partial R_0/\partial \omega\) is useful [22,25]. For this model, the the normalized sensitivity index of the basic reproductive number (3) are given as: \(Y_{\sigma} = 1\), \(Y_{\rho} = -0.45\), \(Y_{\Lambda} = -1.6470\), \(Y_{\beta} = -0.1225\), \(Y_{\varphi} = -0.0002873\), \(Y_{\nu} = -0.3275\), \(Y_{\mu} = -0.5500\), \(Y_{\omega} = -0.00038381\), \(Y_{\nu} = -0.000054999\).
The magnitude of sensitivity analysis of the reproductive number (3) shows that the most sensitive parameters in descending order are $\rho$, $\beta$, $e$, $\theta$, $\delta$, ... and so on. Increasing the parameter values of $\rho$, $\beta$, $\theta$, $\delta$ will reduce the value of $R_0$ and then suppress COVID-19 transmission. Moreover, decreasing the parameter values of $\alpha$ and $e$ also suppresses COVID-19 transmission by reducing $R_0$.

More particularly, following the works in [21,24,31] and targeting the second most sensitive parameter $\alpha$, the target reproduction number has entry $S = \{(1,1)\}$; the term representing $K_s$ of the entry $(1,1)$ of the next generation matrix $TV^{-1}$ discussed in section 2.3. Since it can easily be shown that the spectral radius of $(TV^{-1} - K_s) < 1$, we have the spectral radius of the target matrix $K_s(I - TV^{-1} + K_s)^{-1} = R_1/(1 - R_1)$, where $Id$ is the $4 \times 4$ identity matrix and, $K_s = \begin{pmatrix} R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It then follows that, if the transmission from asymptomatic cases to suspected cases is reduced by a fraction of at least $1 - (1 - R_1)/R_0$, (i.e. $\alpha \leq 0.47$), then the transmission of COVID-19 will die out from Ethiopia. Of course for $\alpha = 0.47$, we have new basic reproductive number given as $R'_1 = 0.3190$, $R'_2 = 0.3900$, $R'_3 = R'_1 + R'_2 = 0.7090 < 1$.

### Table 2

Values of the model parameters corresponding to the COVID-19 case in Ethiopia.

| No. | Parameters | Value       | Source     |
|-----|------------|-------------|------------|
| 1   | $\Lambda$  | 4576/day    | Calculated |
| 2   | $\alpha$   | 1           | Fitted     |
| 3   | $\varepsilon$ | 1           | Fitted     |
| 4   | $\rho$     | 0.8300      | Assumed    |
| 5   | $\varphi$  | 1/7 = 0.143/day | [28]      |
| 6   | $\delta$   | 0.06813/day | Fitted     |
| 7   | $\theta$   | 0.18219/day | Fitted     |
| 8   | $\beta$    | 1/day       | Assumed    |
| 9   | $\sigma$   | 4.11x10^{-5}/day | Calculated |
| 10  | $\mu$      | 0.00273/day | Calculated |
| 11  | $\lambda$  | 0.00001/day | Fitted     |
| 12  | $\sigma$   | 0.00011/day | Calculated |

### 5. Optimal control analysis

Control measures play significant role in mitigating the transmission of COVID-19. In this study, three intervention strategies to contain the transmission of the pandemic that are time dependent control variables namely: $u_1$ represents a control strategy through public health education about COVID-19, $u_2$ represents a control strategy of using personal protective measures such as wearing face mask, regular hand washing, social distancing which are considered operative approaches in Ethiopian context; and $u_3$ represents treatment of COVID-19 patients in hospitals or isolation centers to minimize suffering from the diseases.

By incorporating these control variables in (1), we obtain the state equation:

\[
S = \Lambda - \frac{\mu}{C0}(u_1 + \sigma)S, \\
E = \frac{\mu}{C0} - (\alpha + \theta + \mu + u_3)E, \\
I = (1 - \varphi)E - (\delta + \theta + \mu + u_3)I, \\
A = \varphi E - (\lambda + \beta + \mu + u_3)A, \\
H = \delta I + \lambda A - (\sigma + \mu + \sigma + u_3 + u_1)H, \\
R = (\varphi + u_3)I + (\beta + u_3)A + \mu_0 S + (\theta + \mu + u_3 + u_3)H - \sigma R 
\]

Now, the objective is to find an optimal control for the preventive strategies/measures $u_1$, $u_2$, and $u_3$ for the relatively reduced costs of the prevention strategies. The necessary and sufficient conditions for the existence of optimal control are established by Pontryagin maximum principle [32].

The function that minimizes the number of Exposed cases $E$, number of symptomatically infected cases $I$, and number of asymptotically infected cases $A$ and number of hospitalized cases $H$ over a time interval of $[0, T]$ can be defined as,

\[
J_{opt}(u, \Omega) = \int_0^T \left( c_0 E(t) + c_1 I(t) + c_2 A(t) + c_3 H(t) \\
+ \frac{b_1}{2} u_1^2 + \frac{b_2}{2} u_2^2 + \frac{b_3}{2} u_3^2 \right) dt, 
\]

where $\Omega$ denotes biological feasible region defined in Section 2, $c_0, c_1, c_2, c_3$ are positive weights to balance the factors and $b_1, b_2, b_3$ measures the relative cost of intervention strategies over the time range of $[0, T]$. Minimizing equation (7) provides an optimal control $u^*_i$, $i = 1, 2, 3$ such that

\[
J_{opt}(u^*_1, u^*_2, u^*_3) = \min_{w \in W} J_{opt}(w_1(t), w_2(t), w_3(t)), 
\]
where the control set is given by $U = \{u_i(t) : 0 \leq u_i(t) \leq 1, 0 \leq t \leq T, i = 1, 2, 3\}$ subjected to the constraint given by system of differential equation (6).

The necessary conditions that need to be satisfied by optimal control called Pontryagin’s Maximum Principle converts equation (6) and (7) into a problem of minimizing point-wise a Hamiltonian $Hs$ with respect to $u_i(t)$, where $Hs$ is defined as,

$$Hs(y, u_1(t), u_2(t), u_3(t), \lambda, t) = (c_0 E(t) + c_1 I(t) + c_2 A(t) + c_3 H(t) + \frac{b_1}{2} + b_2 + \frac{b_3}{2} u_2^2(t))$$

$$+ \lambda_1 h_1(t) + \lambda_2 h_2(t) + \lambda_3 h_3(t) + \lambda_4 h_4(t),$$

where,

$$h_1 = A - \left(\delta + \sigma + \mu + u_2\right),$$

$$h_2 = \frac{A - \delta}{\phi} - \left(\rho + \sigma\right),$$

$$h_3 = (1 - \rho)\phi E - (\delta + \sigma + \mu + u_2)I,$$

$$h_4 = p\rho E - (\lambda + \beta + \sigma + u_2)A,$$

$$h_5 = 2\theta + A - (\sigma + \mu + \sigma + u_2 + u_3)H,$$

$$h_6 = (\theta + u_2)I + (\beta + u_2)A + u_4 + S + (\sigma + \mu + u_2 + u_3)H - \sigma R.$$

Differentiating the Hamiltonian function with respect to the compartment variables gives the adjoint variables $\lambda_i, j \in \{S, E, I, A, H, R\}$ corresponding to the system given as follows:

$$\lambda_1 = \lambda_1 (u_1 + \sigma + \rho (A + E))/N - \lambda_1 \mu_1 - \left(\lambda_2 \phi (A + E)/N, \right.$$

$$\lambda_2 = \lambda_2 (\rho + \sigma) - \lambda_1 \rho - \lambda_1 + \lambda_3 \phi,$$

$$\lambda_3 = \lambda_3 (\theta + \mu + \rho) - \lambda_2 \theta + \lambda_2 c_1 - (\lambda_2 \theta + \lambda_2 c_1 - (\mu \lambda_3 S)/N + (\sigma \lambda_3 A)/N, \right.$$ $$\lambda_4 = \lambda_4 (\lambda + \beta + \sigma + u_2 - \lambda_3 \beta + \lambda_3 - \lambda_3 (\lambda_3 S)/N + (\lambda_3 S)/N, \right.$$ $$\lambda_5 = \lambda_5 (\sigma + \mu + u_2 + u_3 + \mu) - \lambda_1 (\sigma + \mu + u_2 + u_3),$$

$$\lambda_6 = \lambda_6 E,$$

and $\lambda_S, \lambda_E, \lambda_I, \lambda_A, \lambda_H, \lambda_R$ are the adjoint variables, $\lambda_j = (\lambda_S, \lambda_E, \lambda_I, \lambda_A, \lambda_H, \lambda_R), j = \{S, E, I, A, H, R\}.$

Now, setting the transversality condition

$$\lambda_j (T) = 0, j \in \{S, E, I, A, H, R\},$$

we obtain the optimal controls and the optimality conditions respectively as

$$u_1(t) = \left(\lambda_S - \lambda_1\right) / \lambda_1 E,$$

$$u_2(t) = \left(\lambda_E + \lambda_2\lambda_1 - \lambda_1 \lambda_3 A + \lambda_3\lambda_1 (A + E)\right) / \lambda_1 E,$$

$$u_3(t) = \left(\lambda_A + \lambda_4\lambda_1 - \lambda_1\lambda_3 H + \lambda_3\lambda_1 (A + E)\right) / \lambda_1 E,$$

and

$$u_4(t) = \min \left[\max \left(0, \left(\frac{\lambda_4}{\lambda_1} - \lambda_1\right) S, \right) 1\right],$$

$$u_5(t) = \min \left[\max \left(0, \left(\frac{\lambda_5\lambda_1 - \lambda_1 \lambda_3 A \lambda_1 - \lambda_1 \lambda_3 A + \lambda_1 \lambda_3 H}{\lambda_1 E}, \right) 1\right].$$

Note that, state equation (6), the adjoint equation (9) together with the characterization of the optimal control (13) and the transversality condition (11) are said to be Optimality system. The sections and subsection treated above describe analytical behaviors of the optimal control. The corresponding numerical simulation of the optimal control is treated in the next section.

### 6. Numerical simulation

In this section numerical solution for the optimality system is presented. The initial conditions used in the simulation are $S(0) = 110000000, E(0) = 200, I(0) = 1, A(0) = 0, H(0) = 0, R(0) = 0, N(0) = 110000201$, while the parameter values are indicated in Table 2.

The simulation to state equation (6) constrained by adjoint system of equations (10) together with the transversality condition (11) and the characterization equation (13) of the optimal control $u_i^*$ are accomplished by forward-backward sweep scheme and Matlab 2018a. The Matlab code used for the scheme is adapted from Marcheva [33].

The model explored the effect of intervention practices in mitigating the transmission of COVID-19 in Ethiopia using the three intervention strategies as a control measures. To compare numerical results from their corresponding simulation, we considered different cases namely: combination of the three intervention strategies, one intervention strategy at a time and two of the strategies at a time. Accordingly, the following simulation scenarios were considered.

**Cases I:** This is the case when mathematical model (6) is simulated without any of the control measures $(u_i(t) = u_2(t) = u_3(t) = 0)$ i.e. none of the intervention strategies were optimally practiced and the simulation result is shown in Fig. 2.

As can be seen from Fig. 2 above, the peak for the spread of the virus is observed around 238th day of the pandemic. The result indicated that there are about 5,095,000 exposed, 603,500 asymptomatic, 479,100 symptomatic and 1,154,000 hospitalized cases at the peak.

**Case II:** In this case, all the three intervention strategies namely: educating the public $(u_1(t)\neq0)$, using of personal protective measures $(u_2(t)\neq0)$, and treating the hospitalized cases $(u_3(t)\neq0)$ applied. The simulation results of this case are shown in Figs. 3 and 4 for optimal control functions and number of cases respectively.

Fig. 3 reveals that, educating the public $(u_1(t))$ and using personal protective measures $(u_2(t))$ need to be maintained at the maximum level for the first 5 and 8 days respectively. Using personal protective measures need to continue up to the end of the pandemic with the intensity indicated in the same figure whereas treating the hospitalized cases $(u_3(t))$ is required to be continued throughout the period of the pandemic at the maximum level.

If the three intervention strategies were applied optimally at the same time as in Fig. 3 from the onset of the pandemic, then their reducing effect of the number of COVID-19 cases would have been as indicated in Fig. 4. That is, applying the three control strategies at the same time from the onset of the pandemic optimally would reduce the number of cases in the compartments significantly to the extent that there are no infected and hospitalized cases.

**Case III:** This case considered the combination of optimal personal protective measure and optimal treatment of hospitalized cases with no any public health education intervention
The simulation result of the effect of these two intervention strategies is almost identically the same as in Fig. 4.

Fig. 2 Number of cases without intervention.

Fig. 3 Optimal functions.

(u_1(t) = 0, u_2(t) ≠ 0, u_3(t) ≠ 0). The simulation result of the effect of these two intervention strategies is almost identically the same as in Fig. 4.

Fig. 4 Number of cases with all the intervention strategies.

Fig. 5 reveals the optimal profile of the combination of intervention strategies u_2(t) and u_3(t). As can be seen from figure above, applying the combination of control strategies...
Case IV: This case analyzes the effect of optimal treatment of hospitalized cases without optimal practice of personal protective measures and public health education \( (u_1(t) = 0, \ u_2(t) = 0, \ u_3(t) = 0) \). The simulation result for the effect of optimally maintaining this intervention strategy is shown in the Fig. 6.

As can be seen from Fig. 6, using the intervention strategy ‘treating hospitalized COVID-19 cases’ alone with the maximum effort couldn’t reduce the number of exposed, symptomatic and asymptomatic cases as the number of cases in Fig. 6 is the same as number of cases in Fig. 2 for the compartments \( E, I \) and \( A \). Applying this control strategy alone reduced the number of hospitalized cases from 1,154,000 (Fig. 2) to 27,130 (Fig. 6) at the peak. Further, the control profile of this case is the same as case II for \( u_1(t) \) of Fig. 3.

Case V: In this case, optimal practice of public health education is considered without the intervention of personal protective measures and treatment of hospitalized cases \( (u_2(t) = 0, \ u_2(t) = 0, \ u_3(t) = 0) \). The simulation result for the effect of optimally practicing this intervention strategy is presented in Fig. 7 for the optimal profile of \( u_1(t) \) and in Fig. 8 for the number of cases in the compartments \( E, I, A \) and \( H \).

As can be seen from Fig. 7, the control profile that the optimal time range for public education is ranging from 176th –186th only for 10 days. Further, Fig. 8 reveals that the peak of the pandemic for this case is around 176th day but it is around 238th day in Fig. 2.

One can also analyze from Figs. 2 and 8 that using optimal public education alone could have reduced the duration of the pandemic, the peak period and number of cases in each of the compartments.

Case VI: This case considered the optimal practice of personal protective measure without intervention of the other two control strategies: public health education and treatment of hospitalized cases \( (u_1(t) = 0, \ u_2(t) = 0, \ u_3(t) = 0) \). The sim-
ulation result of the control profile is depicted in Fig. 9 where as its effect on the number of cases in the compartments $E$, $I$, $A$ and $H$ identically the same as in case II.

It can be observed from Fig. 9 above, the usage of personal protective measures is optimal for the first 120 days of the duration of the pandemic. Comparing cases II and VI, one can learn that both cases lead to the required result of mitigating the transmission of the pandemic independently. However, in case VI the practices of personal protective measures need to be applied optimally for the first 129 days as compared to 8 days for case II.

Despite the effort made by the Gvernment to teach the public and many Ethiopians effort to maintain personal protective measures and treatments of hospitalized cases being underway, the transmission of COVID-19 is increasing from time to time (see Fig. 1). Consequently, it seems reasonable to make
numerical investigation of nonoptimal intensity of practicing personal protective strategies to appreciate their effect in decreasing the number of cases in the compartments of model (6). Accordingly, we assumed $u_1(t) = 0 = u_2(t)$ and simulated the model for different percentage values of personal protective measure $u_2(t) = 1\%, 5\%, 9\%$ and the result is shown in Fig. 10 below.

From Fig. 10 it can be inferred that, if at least 1% from each of the symptomatic, asymptomatic and hospitalized cases maintained practicing personal protective measures at an optimal level from the onset of the pandemic, then the number of cases in the compartments $E$, $I$, $A$ and $H$ would have reduced (black) as compared to the graph with no personal protective (red) measures ($u_2(t) = 0$). The figure further reveals that a pretty improved result would have been obtained for $u_2 = 5\%$ and 9% in the different compartments.

7. Discussion

In this study, a mathematical model for transmission dynamics of COVID-19 for the case of Ethiopia is developed and its different properties including local stability analysis of the diseases free and endemic equilibrium points have been checked. Some of the parameter estimates were taken from Spencer et al., [12] and the remaining parameters were computed via model fitting based on real daily data of COVID-19 confirmed cases of Ethiopia from 13 March to 31 July 2020. It is then calculated that the basic reproductive is $R_0 = 1.5085$.

An optimal analysis of the model for the purpose of assessing the effect of public health education, effect of personal protective measures and effect of treating hospitalized or isolated cases in mitigating transmission of COVID-19 was conducted. The result showed that the optimal practice of combination of all the three intervention strategies significantly reduces the number of exposed, symptomatic, asymptomatic and hospitalized cases (see Figs. 3 and 4).

Likewise, optimal usage of personal protective measures alone led to the required decreases in the number of cases in the compartments except that the optimal application of the control measure needs to be maintained relatively for a longer period of time (see Fig. 9). It is also found that combining control strategies personal protective measures and treatment of hospitalized cases (Case III) is as good as combining the three strategies (case I) in combating the deadly COVID-19 pandemic in Ethiopia.

8. Conclusion

In this work, the parameters that need to be targeted for suppression of COVID-19 pandemic by decreasing the basic reproductive number, are identified. For instance, it is shown that if the target is to decreasing the rate of transmission from asymptotically infected to suspected individuals, $a$, then reducing this parameter to at least $a = 0.47$ can suppress coronavirus from Ethiopia. The same type of work can be conducted in different countries by adapting to their situation. Moreover, optimal combination of the three intervention strategies proposed in this work or optimal combination of any two of the strategies or optimal practice of personal protective measures alone led to the required decreases in the number of COVID-19 cases in the compartments as shown in the simulation results.

Hence, the result of this study can be used as a policy input for the government of Ethiopia and other countries. The government of Ethiopia has to take necessary measures to make personal protective measures a mandatory practice throughout the period of the pandemic. Personal protective measures such as wearing facemask, regular hand washing and social distancing if practiced with optimum effort, can significantly decrease the disturbing effect of COVID-19 and safeguarded the nation.
The government can also target its control strategy on reducing the basic reproductive number, taking into account the sensitivity analysis results of this work while considering the limitations of this work given below.

9. Limitations of the model

It is ethical to make clear that mathematical models are in general approximations of real phenomena and as such they are naturally inaccurate. Moreover, the parameters used in any mathematical models are determined based on observations and experimentations using different numerical methods of computing software and hence are uncertain. Therefore, readers should take into account the limitations of the models while interpreting the findings.

Declaration of Competing Interest

The authors declared that there is no conflict of interest.

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