BOUQUET ALGEBRA OF TORIC IDEALS

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Abstract. To any toric ideal $I_A$, encoded by an integer matrix $A$, we associate a matroid structure called the bouquet graph of $A$ and introduce another toric ideal called the bouquet ideal of $A$. We show how these objects capture the essential combinatorial and algebraic information about $I_A$. Passing from the toric ideal to its bouquet ideal reveals a structure that allows us to classify several cases. For example, on the one end of the spectrum, there are ideals that we call stable, for which bouquets capture the complexity of various generating sets as well as the minimal free resolution. On the other end of the spectrum lie toric ideals whose various bases (e.g. minimal Markov, Gröbner, Graver bases) coincide.

Apart from allowing for classification-type results, bouquets provide a new way to construct families of examples of toric ideals with various interesting properties, e.g., robust, generic, unimodular. The new bouquet framework can be used to provide a characterization of toric ideals whose Graver basis, the universal Gröbner basis, any reduced Gröbner basis and any minimal generating set coincide. We also show that the toric ideal of a general matrix $A$ can be encoded by that of a 0/1 matrix while preserving complexity of its bases. Along the way, we answer two open problems for toric ideals of hypergraphs.

INTRODUCTION

Toric ideals appear prominently in polyhedral geometry, algebraic topology, and algebraic geometry. Naturally, most famous classes of toric ideals come equipped with rich algebraic and homological structure. They also have a common combinatorial feature, namely, equality of various bases. For example, generic toric ideals are minimally generated by indispensable binomials, robust toric by the universal Gröbner basis, those that are Lawrence are minimally generated by the Graver basis, and circuits equal the Graver basis for unimodular toric ideals.

In light of this, the present manuscript offers a combinatorial classification of all toric ideals. This classification simultaneously reveals equality of various distinguished subsets of binomials, provides a unifying framework for studying combinatorial signatures of toric ideals, and introduces a technique to solve several related open problems. Furthermore, it provides a technique to construct infinitely many examples of five important classes of toric ideals, including generic and robust. Before stating our main results, Theorems A-E below, we offer a brief overview of the motivation and some consequences.

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As is customary in the literature, we consider toric ideals to be encoded by a matrix $A$ whose columns are exponents in the monomial parametrization of the corresponding toric variety. Throughout the paper, we also refer to bases of $I_A$ as those of $A$.

**Various bases of toric ideals.** Apart from minimal generating sets, which we may also refer to as minimal Markov bases [23, 3], the well-known distinguished subsets of binomials in a toric ideal $I_A$ are: the set of circuits $C(A)$, the universal Gröbner basis $U(A)$, the set of indispensable binomials $S(A)$, the universal Markov basis $M(A)$, and the Graver basis $Gr(A)$. Briefly, $S(A)$ are binomials appearing in every minimal generating set; $U(A)$ is the union of all of the (finitely many) reduced Gröbner bases; and $Gr(A)$ is a crucial set for integer programming [26, 19] consisting of primitive binomials in the ideal. When $\text{Ker}_Z(A) \cap \mathbb{N}^n = \{0\}$, the generally proper inclusions between them hold as follows.

\[
\begin{align*}
C(A) & \subseteq U(A) \subseteq Gr(A) \\
S(A) & \subseteq M(A)
\end{align*}
\]

In the diagram above, $M$ represents one minimal binomial generating set of $I_A$, or, in the terminology used later in the paper and in some applications, a minimal Markov basis. For a discussion on the inclusions in the general case, see [17].

Equalities between some pairs of these sets were known previously in special cases, perhaps the best known of which is that if $A$ is unimodular then $C(A) = Gr(A)$. One new class we introduce is $\emptyset$-Lawrence ideals, playing a prominent role in our classification; see Theorem C below.

**Five prominent classes of toric ideals.** As mentioned before, apart from classifying toric ideals according to their combinatorial signatures, we provide a technique that constructs infinitely many examples of toric ideals that are of interest in various fields. Let us outline their roles and definitions.

Robust toric ideals are those for which the universal Gröbner basis is a minimal generating set. Unlike the other four classes, robustness makes sense for arbitrary polynomial ideals as well. A beautiful family of robust ideals are ideals of maximal minors of matrices of linear forms that are column-graded [18, Theorem 4.2], which generalize the ideals of maximal minors of a matrix of indeterminates [6, 43], and the ideals of maximal minors of a sparse matrix of indeterminates [9]. Another family of robust ideals is given by the defining ideals $I(\tilde{L})$ of the closure $\tilde{L}$ in $(\mathbb{P}^1)^n$ of linear spaces $L \subset \mathbb{A}^n$, see [1, Theorem 1.3]. Restricting to special cases, of course, reveals more properties. For example, the toric ideal of a simple graph is robust...
if and only if is minimally generated by its Graver basis; this was shown in [11, Theorem 3.2] using the graph-theoretic classification of bases from [39, 45]. In other words, \( S(A) = U(A) = M(A) = Gr(A) \) holds for robust toric ideals of graphs. More generally, \( S(A) = M(A) \) holds for any robust toric ideal, see [11, Theorem 5.10], and thus \( U(A) = M(A) \) also. It is conjectured that all robust toric ideals are minimally generated by their Graver bases, see [11, Question 6.1] and thus robust toric ideals would be precisely the \( \emptyset \)-Lawrence toric ideals.

In the class of generalized robust toric ideals, introduced in [44], the universal Markov basis equals the universal Gröbner basis. One very nice property of this class of ideals is that it properly includes the class of robust toric ideals, see [44, Corollary 5.12] and [44, Example 3.6]. The third class are toric ideals whose circuits equal the Graver basis. A proper subclass is the set of unimodular toric ideals, where equality of these bases has numerous and beautiful consequences in hyperplane arrangements, resolutions, spectral sequences, and algebraic geometry, see [5]. The fourth class are toric ideals whose indispensable binomials form a Graver basis; we name this class \( \emptyset \)-Lawrence and a priori is just a subclass of robust toric ideals. However, if the conjecture of Boocher et al. were true then it would imply that robust toric ideals are exactly \( \emptyset \)-Lawrence. As we will see below, bouquets are exactly what can be used to classify toric ideals that belong to this class. Here, all five sets of binomials from the diagram except \( C(A) \) are equal, expected as second Lawrence liftings are a special case. Finally, the fifth class of toric ideals are those for which the set of indispensable binomials equals the universal Markov basis, that is, they have a unique minimal system of generators. A proper subclass of these are generic toric ideals [37], where all binomials in a minimal generating set have full support.

The fundamental construction: bouquets. Every integer matrix \( A \) comes equipped with a natural oriented matroid structure called the bouquet graph of \( A \), see Definition 1.1. Connected components \( B_1, \ldots, B_s \) of this graph are the bouquets of \( A \) (and, in addition, we distinguish different types of bouquets: free or non-free, and the latter can be mixed or non-mixed; these are technical definitions that are not difficult to check and play a crucial role in the paper). Bouquets are encoded by vectors that, essentially, record the dependencies from the Gale transform of \( A \): the bouquet-index-encoding vectors \( c_B \) and the vectors \( a_B \) whose columns make up the defining matrix \( A_B \) (cf. Definition 1.7) of the bouquet ideal associated to \( I_A \). This new toric ideal encodes the basic properties of \( I_A \) via a structural decomposition of \( A \). The terminology we use comes from the applications to hypergraphs (see Section 5). The graph’s connected components alone were used in [13], under the name coparallelism classes, to provide combinatorial descriptions of self-dual projective toric varieties associated to a non-pyramidal configuration, see [13, Theorem 4.16].

We ask – and answer in various ways – the following question: What does the bouquet structure of \( A \) say about the toric ideal? In particular, we are interested in how bouquets preserve three types of properties of a toric ideal: 1) “all” of its combinatorial properties, that is, the structure of \( \text{Ker}_Z(A) \), Gröbner, Graver, Markov bases, circuits, indispensable binomials, etc.; 2) “essential” combinatorics, that is, \( \text{Ker}_Z(A) \), the Graver basis, and the circuits; and 3) homological information, i.e.,
the minimal free resolution. The following result summarizes how bouquets preserve the essential combinatorics of toric ideals.

**Theorem A** (Theorems 1.9 and 1.11) Let \( A = [a_1, \ldots, a_n] \in \mathbb{Z}^{m \times n} \) and its bouquet matrix \( A_B = [a_{B_1}, \ldots, a_{B_s}] \). There is a bijective correspondence between the elements of \( \text{Ker}_Z(A) \) in general, and \( \text{Gr}(A) \) and \( C(A) \) in particular, and the elements of \( \text{Ker}_Z(A_B) \), and \( \text{Gr}(A_B) \) and \( C(A_B) \), respectively. More precisely, this correspondence is defined as follows: for \( u = (u_1, \ldots, u_s) \in \text{Ker}_Z(A_B) \) then \( B(u) = c_{B_1}u_1 + \cdots + c_{B_s}u_s \in \text{Ker}_Z(A) \).

In particular, Theorem A solves [38, Problem 6.3] for an arbitrary toric ideal; the problem was posed for 0/1 matrices. The basic idea is that the bouquet construction gives a way for recovering all Graver bases elements from the Graver bases elements of a matrix with possibly fewer columns than \( A \).

**Combinatorial classification and some consequences.** As mentioned briefly above, non-free bouquets can be mixed or non-mixed, and of course a toric ideal can have both in its bouquet graph. At one end of the spectrum are the toric ideals with all of the non-free bouquets non-mixed; we call these ideals stable. The notion of stability is quite nice, as it captures the case when passing from \( I_A \) to \( I_{A_B} \) preserves all combinatorial information:

**Theorem B** (Theorem 3.7) Let \( I_A \) be a stable toric ideal. Then the bijective correspondence between the elements of \( \text{Ker}_Z(A) \) and \( \text{Ker}_Z(A_B) \) given by \( u \mapsto B(u) \), is preserved when we restrict to any of the following sets: Graver basis, circuits, indispensable binomials, minimal Markov bases, reduced Gröbner bases (universal Gröbner basis).

Furthermore, in the positively graded case, even the homological information is preserved; see Section 3, which is dedicated to stable toric ideals, and Theorem 3.11 in particular.

At the other end of the spectrum of the classification are toric ideals all of whose bouquets are mixed. In particular, such toric ideals are \( \emptyset \)-Lawrence; however, the converse does not hold: there exist \( \emptyset \)-Lawrence toric ideals that have both mixed and non-mixed bouquets. Thus to capture the correct property we are interested in, we introduce the \( S \)-Lawrence property that, intuitively, “balances” from a trivial condition, namely being \([n]\)-Lawrence, common to all toric ideals in \( K[x_1, \ldots, x_n] \), to a very special class of ideals, those that are \( \emptyset \)-Lawrence. The main result in this direction explains how mixed bouquets capture essential combinatorics of \( A \).

**Theorem C** (Theorem 4.2) Let \( B_1, \ldots, B_s \) be the bouquets of \( A = [a_1, \ldots, a_n] \), and define \( A_B = [a_{B_1}, \ldots, a_{B_s}] \). Let \( S \subset [s] \) be the subset of coordinates corresponding to the mixed bouquets. Then the following are equivalent:

(a) There exists no element in the Graver basis of \( A_B \) which has a proper semiconformal decomposition that is conformal on the coordinates corresponding to \( S \);
(b) The toric ideal of \( A \) is \( \emptyset \)-Lawrence, i.e. \( S(A) = \text{Gr}(A) \);
(c) The toric ideal of \( A_B \) is \( S \)-Lawrence.
In particular, all five sets of $I_A$ from the diagram except $C(A)$ coincide if and only if
the toric ideal of $A_{B}$ is $S$-Lawrence (Definition 4.1), where $S$ is encoded by coordi-
nates of the mixed bouquets of $A$. One beautiful consequence of being $\emptyset$-Lawrence
is that these four bases agree, that is $S(A) = U(A) = M(A) = Gr(A)$, providing
in particular a combinatorial characterization of toric ideals for which $S(A)$ and
$Gr(A)$ are equal. The question about this equality is a long-standing open problem,
although many examples have been discovered [5, 8, 17, 41].

Bouquets of 0/1 matrices: applications and a surprising generality. It is
well known that 0/1 matrices take a special place in the world of toric ideals due
to their applications to biology, algebraic statistics, integer progra-
mation, matroid
theory, combinatorics, see [20, 31, 37, 42, 40]. They are, by definition, incidence
matrices of hypergraphs, and in special cases there are only two nonzero entries in
each column they are incidence matrices of graphs. There is an abundant literature
for toric ideals of graphs, including [33, 39, 45, 46], and the book [47], while toric
ideals of hypergraphs were studied in [27], [38] and [2]. We make several contribu-
tions in this realm, summarized below. They are based on special types of bouquets
for 0/1 matrices, called bouquets with bases, that encode the basic building blocks
of Graver bases elements for hypergraphs. The interested reader will find a more
technical motivation in the introductory paragraphs of Section 5 (Definition 5.1), as
well as solutions to some open problems on hypergraphs.

The following result says, roughly speaking, that the essential combinatorics of
arbitrary toric ideals is encoded by the almost 3-uniform hypergraphs, that is, hy-
pergraphs with edges of cardinality at most 3 and having at least one such edge.
Here, it is important to note that Graver bases of toric ideals of graphs have very
special form and support structure [39, 45], while those of hypergraphs can be quite
complicated ([38]).

**Theorem D** (Theorem 6.2) Given any integer matrix $A$, there exists a hypergraph
$H = (V, E)$ such that $I_H$ is $\emptyset$-Lawrence and there is a bijective correspondence be-
tween the elements of $\text{Ker}\,(Z_A)$, $Gr(A)$, and $C(A)$ and $\text{Ker}\,(Z_H)$, $Gr(H)$, and
$C(H)$, respectively.

Thus toric ideals of almost 3-uniform hypergraphs are “as complicated” as any
arbitrary toric ideals.

Recall that a toric ideal is said to be positively graded if $\text{Ker}\,(Z_A) \cap \mathbb{N}^n = \{0\}$;
this is the case for toric ideals in applications such as algebraic statistics and linear
programming. The final main result shows that if $I_A$ is positively graded, then there
exists a stable toric ideal of a hypergraph $I_H$ of same combinatorial complexity:

**Theorem E** (Theorem 7.2) Let $I_A$ be an arbitrary positively graded nonzero toric
ideal. Then there exists a hypergraph $H$ such that there is a bijective correspondence
between the Graver bases, all minimal Markov bases, all reduced Gröbner bases,
circuits, and indispensable binomials of $I_A$ and $I_H$.

Theorem E can be regarded as a polarization-type operation, in the sense that
one can pass from an arbitrary matrix $A$ to a 0/1 matrix $A_H$ by preserving all the
combinatorial properties of the toric ideal. It in particular implies that any problem
about arbitrary toric ideals involving equality of bases can be reduced to a problem about a toric ideal of a hypergraph defined by a 0/1-matrix. For example, see the conjecture of Boocher discussed at the end of Section 7.

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1. Bouquet decomposition of a toric ideal

Let $K$ be a field and $A = [a_1, \ldots, a_n] \in \mathbb{Z}^{m \times n}$ be an integer matrix with columns $\{a_i\}$. We recall that the toric ideal of $A$ is the ideal $I_A \subset K[x_1, \ldots, x_n]$ given by

$$I_A = (x^{u^+} - x^{u^-} : u \in \text{Ker}(A)),$$

where $u = u^+ - u^-$ is the unique expression of an integral vector $u$ as a difference of two non-negative integral vectors with disjoint support, see [11, Section 4]. As usual, we denote by $x^u$ the monomial $x_1^{u_1} \cdots x_n^{u_n}$, with $u = (u_1, \ldots, u_n)$. Denote by $r$ the $\dim_{\mathbb{Q}} \mathbb{Q}(A)$. Fix a basis $G_1, G_2, \ldots, G_{n-r} \in \mathbb{Z}^n$ for the lattice Ker$(Z_n(A))$, and denote by $G$ the $n \times (n-r)$ matrix with columns $G_1, \ldots, G_{n-r}$. Define $G(A) = \{b_1, \ldots, b_n\}$ to be the set of ordered rows of $G$. The set $G(A)$ is called the Gale transform of $A$, while the vector $G(a_i) := b_i$ is called the Gale transform of $a_i$ for any $i$.

To the columns of $A$ one can associate the oriented matroid $M_A$ (see [11] for details). The support of a vector $u \in \mathbb{Z}^n$ is the set $\text{supp}(u) = \{i|u_i \neq 0\} \subset \{1, \ldots, n\}$. A vector $u \in \text{Ker}(Z_n(A))$ is called a circuit of $A$ if $\text{supp}(u)$ is minimal with respect to inclusion and the coordinates of $u$ are relatively prime. As usual, we denote by $C(A)$ the set of circuits of $A$. A co-vector is any vector of the form $(u \cdot a_1, \ldots, u \cdot a_n)$. A co-circuit of $A$ is any non-zero co-vector of minimal support. A co-circuit with support of cardinality one is called a co-loop. We call the vector $a_i$ free if $\{i\}$ is the support of a co-loop. Since a co-loop is characterized by the property that it belongs to any basis of the matroid, a free vector $a_i$ belongs to any basis of $M_A$. Equivalently, the Gale transform of a free vector, $G(a_i)$ is equal to the zero vector, which means that $i$ is not contained in the support of any minimal generator of the toric ideal $I_A$, or any element in the Graver basis.

Let $E_A$ be the set consisting of elements of the form $\{a_i, a_j\}$ such that there exists a co-vector $c_{ij}$ with support $\{i, j\}$. We denote by $E_A^+$ the subset of $E_A$ where the co-vector is a co-circuit and the signs of the two nonzero coordinates of $c_{ij}$ are distinct, and we denote by $E_A^-$ the subset of $E_A$ where the co-vector is a co-circuit and the signs of the two nonzero coordinates of $c_{ij}$ are the same. Furthermore, we denote by $E_A^0$ the subset of $E_A$ where the co-vector is not a co-circuit. This implies that both $a_i$ and $a_j$ are free vectors. By definition, these three sets $E_A^+, E_A^-, E_A^0$ partition $E_A$.

Definition 1.1. The bouquet graph $G_A$ of $I_A$ is the graph whose vertex set is $\{a_1, \ldots, a_n\}$ and edge set $E_A$. The bouquets of $A$ are the connected components of $G_A$.

It follows from the co-circuit axioms of oriented matroids that each bouquet of $A$ is a clique in $G_A$. If there are free vectors in $A$ they form one bouquet with all edges
in $E_0^A$, which we call the free bouquet of $G_A$. The discussions preceding Definition\[13\] show that the free bouquet of $G_A$ consists of all $a_i$ such that $G(a_i) = 0$. A non-free bouquet is called mixed if it contains at least an edge from $E_A^+$, and non-mixed if it is either an isolated vertex or all of its edges are from $E_A^+$. Moreover, in the following Lemma we give an equivalent description of non-free, mixed and non-mixed bouquets of $A$ in terms of the Gale transform $G(A)$. These descriptions are based on a well-known result about the characterization of co-circuits of cardinality two in terms of Gale transforms, whose proof is included for convenience of the reader. By slight abuse of notation, we identify vertices of $G_A$ with their labels; that is, $a_i$ will be used to denote vectors in the context of $A$ and $M_A$, and vertices in the context of $G_A$. A subbouquet of $G_A$ is an induced subgraph of $G_A$, which is a clique on its set of vertices. Note that a bouquet is a maximal subbouquet. We will say that $A$ has a subbouquet decomposition if there exists a family of subbouquets, say $B_1, \ldots, B_t$, such that they are pairwise vertex disjoint and their union of vertices equals $\{a_1, \ldots, a_n\}$. A subbouquet decomposition always exists if we consider, for example, the subbouquet decomposition induced by all of the bouquets.

**Lemma 1.2.** Suppose that not all of the columns $a_i$ of $A$ are free. Then:

(a) There exists a co-circuit of cardinality two with support $\{i, j\}$ if and only if $a_i, a_j$ are not free and $G(a_i) = \lambda G(a_j)$ for some $\lambda \neq 0$.

(b) $B \subset A$ is a non-free subbouquet if and only if the vector space $< G(a_i) | a_i \in B >$ has dimension one.

(c) The edge $\{a_i, a_j\}$ belongs to $E_A^+$ (respectively $E_A^-$) if and only if $a_i, a_j$ are not free and $G(a_i) = \lambda G(a_j)$ for some $\lambda > 0$ (respectively $\lambda < 0$).

**Proof.** Let $G$ be the $n \times (n - r)$ matrix with the column vectors $G_1, \ldots, G_{n-r}$, and whose set of row vectors is the Gale transform $G(A)$ of $A$. For simplicity set $t = n - r$, $G(a_i) = b_i = (b_{i1}, \ldots, b_{in})$ for all $i \in \{1, \ldots, n\}$, and note that $a_i$ not being free implies $G(a_i) \neq 0$. Definition of $G$ provides:

\[
(1) \quad b_{ik}a_i + \cdots + b_{nk}a_n = 0 \quad \text{for all} \quad k \in \{1, \ldots, t\}.
\]

First we show (a). Assume that there exists a co-circuit $c_{ij}$ of cardinality two, with support $\{i, j\}$: $c_{ij} = (0, \ldots, c_i, \ldots, c_j, \ldots, 0)$ and $c_i, c_j \neq 0$. This implies the existence of a vector $v \in \mathbb{Z}^m$ such that $v \cdot a_k = 0$ for any $k \neq i, j$, $v \cdot a_i = c_i$ and $v \cdot a_j = c_j$. Taking the dot product with $v$ in (1) we obtain $b_{ik}c_i + b_{jk}c_j = 0$ for all $k \in \{1, \ldots, t\}.$ Therefore $G(a_i) = -\sum c_i G(a_j)$, the desired conclusion.

Conversely, let $G(a_i) = \lambda G(a_j)$ for some $\lambda \neq 0$. Since $a_i, a_j$ are not free we also have that $G(a_i), G(a_j) \neq 0$. It is a basic fact in matroid theory, see [36], that the co-circuits of a matroid are the minimal sets having non-empty intersection with every basis of the matroid. Thus, in order to prove the existence of a co-circuit of cardinality two with support $\{i, j\}$, it is enough to prove that 1) any basis of $M_A$ contains either $a_i$ or $a_j$, and 2) there are no co-loops with support $\{i\}$ or $\{j\}$. The latter is automatically satisfied since $a_i, a_j$ are not free. Assume by contradiction that there exists a basis of $\langle a_1, \ldots, a_n \rangle$ which does not contain both $a_i$ and $a_j$. This implies that $a_i = \sum_{k \neq i,j} \beta_k a_k$. Thus, the vector $w$ of this relation, whose $j$-th,
Let $a_i, a_j$ be edges in $E_A \setminus E_A^0$. This implies that (a) holds, and from its proof we obtain $\lambda = -\frac{c_i}{c_j}$. Hence (c) follows at once from the definition of $E_A^+$ and $E_A^-$. \hfill \Box

Due to its importance for later sections, we isolate the following consequence of the proof of Lemma 1.2.

**Remark 1.3.** Suppose that $A = \{a_1, \ldots, a_n\}$ has a co-circuit $(0, \ldots, c_i, \ldots, c_j, \ldots, 0)$ with support of cardinality at most two. Then we have $c_i G(a_i) + c_j G(a_j) = 0$.

In particular, it follows from Lemma 1.2(c) that if $\{a_i, a_j\} \in E_A^+$ and $\{a_j, a_s\} \in E_A^+$ then $\{a_i, a_s\} \in E_A^+$, while if $\{a_i, a_j\} \in E_A^-$ and $\{a_j, a_s\} \in E_A^-$ then $\{a_i, a_s\} \in E_A^-$. Combining these considerations provides an algorithm for computing the bouquet graph of a toric ideal through the computation of the Gale transform of $A$, as the following example illustrates.

**Example 1.4.** Let $A$ be the integer matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix},
$$

with columns $\{a_1, \ldots, a_7\}$. A basis for $\text{Ker}_2(A)$ is given by $(1, -1, 0, 0, -1, 1, 0)$ and $(1, 0, -1, -1, 0, 1, 0)$, and thus the Gale transform $G(A)$ of $A$ consists of the following seven vectors: $G(a_1) = (1, 1), G(a_2) = (-1, 0), G(a_3) = (0, -1), G(a_4) = (0, -1), G(a_5) = (-1, 0), G(a_6) = (1, 1)$ and $G(a_7) = (0, 0)$. Hence $a_7$ is a free vector. So, the graph $G_A$ has the vertex set $\{a_1, \ldots, a_7\}$, and applying Lemma 1.2(a) and (b), we see immediately that the edges of $G_A$ are $\{a_1, a_6\}, \{a_2, a_5\}$ and $\{a_3, a_4\}$. Therefore $G_A$ has three bouquets each consisting of a single edge and one additional free bouquet consisting of the isolated vertex $a_7$. Moreover, since $G(a_1) = G(a_6), G(a_2) = G(a_5)$ and $G(a_3) = G(a_4)$, Lemma 1.2(c) provides $\{a_1, a_6\}, \{a_2, a_5\}, \{a_3, a_4\} \in E_A^+$. In this case, all of the non-free bouquets are non-mixed.

In the remainder of this section, we will show that the bouquets of $A$ determine, in a sense, elements of $I_A$. Before stating the main results, the following technical definitions are needed.
First, we define a *bouquet-index-encoding vector* $\mathbf{c}_B$ as follows\footnote{Remark on notation: definition of $\mathbf{c}_B$ does depend on the matrix $A$, but reference to it is suppressed for ease of notation. The underlying matrix $A$ for the bouquet $B$ will always be clear from the context.}. If the bouquet $B$ is free then we set $\mathbf{c}_B \in \mathbb{Z}^n$ to be any nonzero vector such that $\text{supp}(\mathbf{c}_B) = \{ i : \mathbf{a}_i \in B \}$ and with the property that the first nonzero coordinate is positive. For a non-free bouquet $B$ of $A$, consider the Gale transforms of the elements in $B$. All the elements are nonzero and linearly dependent, therefore there exists a nonzero coordinate $j$ in all of them. Let $g_j = \gcd(\lambda G(a_i)_{j} | \mathbf{a}_i \in B)$ and fix the smallest integer $i_0$ such that $\mathbf{a}_{i_0} \in B$. Let $\mathbf{c}_B$ be the vector in $\mathbb{Z}^n$ whose $i$-th coordinate is $0$ if $\mathbf{a}_i \notin B$, and is $\varepsilon_{i_0j} G(a_i)_{j}/g_j$ if $\mathbf{a}_i \in B$, where $\varepsilon_{i_0j}$ represents the sign of the integer $G(a_{i_0})_{j}$.

Thus the $\text{supp}(\mathbf{c}_B) = \{ i : \mathbf{a}_i \in B \}$. Note that the choice of $i_0$ implies that the first nonzero coordinate of $\mathbf{c}_B$ is positive. Since each $\mathbf{a}_i$ belongs to exactly one bouquet the supports of the vectors $\mathbf{c}_B$ are pairwise disjoint. In addition, $\bigcup \text{supp}(\mathbf{c}_B) = [n]$.

**Remark 1.5.** For a non-free bouquet $B$, the definition of vector $\mathbf{c}_B$ does not depend on the nonzero coordinate $j$ chosen. Indeed, let $k \neq j$ be such that $G(a_{i_0})_{j} ≠ 0$ and assume that the bouquet $B$ has $s + 1$ vertices $\mathbf{a}_{i_0}, \ldots, \mathbf{a}_i$. For sake of simplicity denote by $c_l := G(a_{i_0})_{j}$ and $d_l := G(a_i)_{k}$ for all $0 ≤ l ≤ s$. Then we have to prove that $\varepsilon_{i_0j} G(a_{i_0})_{j}/g_j = \varepsilon_{i_0k} d_l/g_k$ for all $l$. The case $s = 0$ is obvious, so we may assume that $s ≥ 1$. Since $B$ is a bouquet then $c_l/d_l = m/n$ for all $l$, for some relatively prime integers $m, n$. On the other hand $g_j = \gcd(c_0, \ldots, c_s)$ implies that $g_j = \lambda_0 c_0 + \cdots + \lambda_s c_s$ for some integers $\lambda_0, \ldots, \lambda_s$. From here we obtain $g_j n/m = \lambda_0 d_0 + \cdots + \lambda_s d_s \in \mathbb{Z}$, and since $g_k = \gcd(d_0, \ldots, d_s)$ then $g_k g_j (n/m)$. Similarly, we obtain that $g_j g_k (m/n)$ and consequently $g_j n = \pm g_k m$, which implies $c_l/d_l = m/n = \pm g_j g_k$. Since $g_j, g_k > 0$, it is easy to see that $(\varepsilon_{i_0j} c_l)/(\varepsilon_{i_0k} d_l) = g_j/g_k$, and therefore we obtain the desired equality.

Whether the bouquet $B$ is mixed or not can be read off from the vector $\mathbf{c}_B$ as follows.

**Lemma 1.6.** Suppose $B$ is a non-free bouquet of $A$. Then $B$ is a mixed bouquet if and only if the vector $\mathbf{c}_B$ has a negative and a positive coordinate.

**Proof.** Let $B$ be a mixed bouquet and assume without loss of generality that $\mathbf{a}_1, \mathbf{a}_2 \in B$ such that $\{ \mathbf{a}_1, \mathbf{a}_2 \} \in E^\circ_A$. Applying Lemma 1.2(c) there exists $\lambda < 0$ such that $G(\mathbf{a}_1) = \lambda G(\mathbf{a}_2)$. Since $\mathbf{a}_1 \in B$ then $\mathbf{a}_1$ is not a free vector and thus there exists an integer $j$ such that $G(\mathbf{a}_1)_{j} ≠ 0$. Then

$$
(c_B)_1(c_B)_2 = \varepsilon_{1j} G(\mathbf{a}_1)_{j}/g_j \varepsilon_{1j} G(\mathbf{a}_2)_{j}/g_j = \lambda \frac{G(\mathbf{a}_2)_{j}^2}{g_j^2} < 0,
$$

and this implies the desired conclusion. Here $(c_B)_1, (c_B)_2$ represent the first two coordinates of the vector $\mathbf{c}_B$. The converse follows immediately with a similar argument. \qed

It follows from Lemma 1.6 and the definition of the vector $\mathbf{c}_B$ that if the non-free bouquet $B$ is non-mixed, then all nonzero coordinates of $\mathbf{c}_B$ are positive.
The following vector now encodes the set of dependencies from the Gale transform, and, therefore, also all of the the essential bouquet information about \( B \).

**Definition 1.7.** Let \( B \) be a bouquet of \( A \) and define

\[
a_B := \sum_{i=1}^{n} (c_B)_i a_i,
\]

where \((c_B)_i\) denotes the \( i \)-th coordinate of the vector \( c_B \). The set of all vectors \( a_B \) corresponding to all bouquets of \( A \) will be denoted by \( A_B \), and thought of as a matrix with columns \( a_B \).

Let us isolate two special cases. First, if \( B \) consists of just an isolated vertex \( a_i \), then, when \( a_i \) is not free \( a_B = a_i \), while otherwise \( a_B \) can be any positive multiple of \( a_i \). Second, if the bouquet graph of \( A \) has no mixed bouquets, then all vectors \( a_B \) corresponding to the non free bouquets \( B \) are just positive linear combinations of the vectors \( a_i \in B \). Finally, note that passing from \( A \) to \( A_B \) might not affect the matrix, since it might happen that \( A = A_B \); a trivial example is \( A_B = (A_B)_B \).

**Remark 1.8.** If \( A \) has a free bouquet, say \( B_s \), then it is easy to see that

\[
\text{Ker}_Z(A_B) = \{(u_1, \ldots, u_{s-1}, 0) : (u_1, \ldots, u_{s-1}) \in \text{Ker}_Z(A_{B'}_B)\},
\]

where \( A_B = [a_{B_1}, \ldots, a_{B_{s-1}}] \). In particular, we notice that for a free bouquet \( B \), even there are infinitely many choices to define \( c_B \) and thus implicitly \( a_B \), \( \text{Ker}_Z(A_B) \) and \( \text{Ker}_Z(A) \) are independent of that choice.

The following construction is the key result of this section, providing a bijection between the kernels of the toric matrix \( A \) and the bouquet matrix \( A_B \) from Definition 1.7.

**Theorem 1.9.** Suppose the bouquets of \( A = [a_1, \ldots, a_n] \) are \( B_1, \ldots, B_s \), and define \( A_B = [a_{B_1}, \ldots, a_{B_s}] \). There is a bijective correspondence between the elements of \( \text{Ker}_Z(A) \) and the elements of \( \text{Ker}_Z(A_B) \) given by the map \( u \mapsto B(u) \), where for \( u = (u_1, \ldots, u_s) \in \text{Ker}_Z(A_B) \)

\[
B(u) := c_{B_1} u_1 + \cdots + c_{B_s} u_s.
\]

**Proof.** We may assume that \( A \) has no free bouquet by Remark 1.8. First we show that for every \( v = (v_1, \ldots, v_n) \in \text{Ker}_Z(A) \) there exists a vector \( u \in \text{Ker}_Z(A_B) \) such that \( v = B(u) \). In order to prove this we may assume, without loss of generality, that there exist integers \( 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n \) such that \( a_1, \ldots, a_{i_{s-1}} \) belong to the bouquet \( B_1 \), and so on, until \( a_{i_{s-1}+1}, \ldots, a_n \) belong to the bouquet \( B_s \). Since \( v \in \text{Ker}_Z(A) \) it follows that there exist integers \( \lambda_1, \ldots, \lambda_{n-r} \) such that \( v = \lambda_1 G_1 + \cdots + \lambda_{n-r} G_{n-r} \). In other words, for each \( k = 1, \ldots, n \),

\[
v_k = \lambda_1 G(a_k)_1 + \cdots + \lambda_{n-r} G(a_k)_{n-r}.
\]

From the formula of the vectors \( c_B \) it follows that \( (c_{B_1})_1 = \varepsilon_{i_1} G(a_1)_1/g_j \) for all nonzero \( g_j = \text{gcd}(G(a_1)_j, \ldots, G(a_n)_j) \). Therefore, \( G(a_1)_j = (c_{B_1})_1 \varepsilon_{i_1} g_j \) for all \( j = 1, \ldots, n-r \), if we put \( g_j = 0 \) when \( G(a_1)_j = 0 \). Then we obtain \( v_1 = \cdots = v_{i_{s-1}} = 0 \),
where \( u_k = \sum_{j=1}^{n-r} \lambda_j G(a_k)_j = \sum_{j=1}^{n-r} \lambda_j \varepsilon_{ij} g_j(c_{B_1})_k \) and

\[
 u_k = \sum_{j=1}^{n-r} \lambda_j G(a_k)_j = \sum_{j=1}^{n-r} \lambda_j \varepsilon_{ij} g_j(c_{B_1})_k = u_1(c_{B_1})_k,
\]

where \( u_1 = \sum_{j=1}^{n-r} \lambda_j \varepsilon_{ij} g_j \). This takes care of the first \( i_1 \) coordinates of \( v \), which correspond to the same bouquet \( B_1 \). The argument for the remaining coordinates follows similarly.

Next we show that \( u \in \text{Ker}_Z(A_B) \) if and only if \( B(u) \in \text{Ker}_Z(A) \). Indeed, \( u = (u_1, \ldots, u_s) \in \text{Ker}_Z(A_B) \) is equivalent to \( \sum_{k=1}^{s} u_k a_{B_k} = 0 \). Replacing every vector \( a_{B_k} \) provides the following equivalent statement: \( u \in \text{Ker}_Z(A_B) \) if and only if

\[
 0 = \sum_{k=1}^{s} u_k a_{B_k}^* = \sum_{k=1}^{s} u_k \left( \sum_{i=1}^{n} (c_{B_k})_i a_i \right) = \sum_{i=1}^{n} \left( \sum_{k=1}^{s} u_k (c_{B_k})_i \right) a_i.
\]

The latter sum being equal to zero is equivalent to

\[
 \sum_{k=1}^{s} (c_{B_k})_1 u_k, \sum_{k=1}^{s} (c_{B_k})_2 u_k, \ldots, \sum_{k=1}^{s} (c_{B_k})_n u_k \in \text{Ker}_Z(A),
\]

or, more concisely,

\[
 \sum_{k=1}^{s} c_{B_k} u_k = B(u) \in \text{Ker}_Z(A),
\]

and the claim follows.

It is an immediate consequence of Theorem 1.9 that the ideals \( I_A \) and \( I_{A_B} \) have the same codimension, as the kernels of the two matrices have the same rank.

**Example 1.10** (Example 1.4 continued). Let \( A \) be the matrix from Example 1.4

It has three bouquets, \( B_1, B_2, B_3 \), with two vertices each: \{\( a_1, a_6 \), \), \{\( a_2, a_5 \) and \{\( a_3, a_1 \)\}, and the isolated vertex \( a_7 \) as the free bouquet \( B_4 \). Let us compute the nonzero vectors \( c_{B_1}, c_{B_2}, c_{B_3}, c_{B_4} \in \mathbb{Z}^7 \). For \( c_{B_1}, j \) can be chosen either 1 or 2, while \( i_0 = 1 \). Fix \( j = 1 \). Thus \( g_1 = \gcd(G(a_1),G(a_6)_1) = 1 \) and the nonzero coordinates of \( c_{B_1} \) are

\[
 (c_{B_1})_1 = \varepsilon_{11} \frac{G(a_1)_1}{g_1} = 1, \quad (c_{B_1})_6 = \varepsilon_{11} \frac{G(a_6)_1}{g_1} = 1.
\]

Hence \( c_{B_1} = (1, 0, 0, 0, 0, 1, 0) \) and \( a_{B_1} = a_1 + a_6 = (1, 1, 1, 1, 0) \). Similarly it can be computed that \( c_{B_2} = (0, 1, 0, 0, 1, 0, 0) \), \( a_{B_2} = a_2 + a_5 = (1, 1, 1, 1, 0) \), \( c_{B_3} = (0, 0, 1, 1, 0, 0, 0) \) and \( a_{B_3} = a_3 + a_4 = (1, 1, 1, 1, 0) \), while by definition \( c_{B_4} = (0, 0, 0, 0, 0, 0, 1) \) and \( a_{B_4} = a_7 = (0, 0, 0, 0, 3) \). Then \( A_B \) consists of four vectors \( a_{B_1}, a_{B_2}, a_{B_3}, a_{B_4} \), and consequently \( \text{Ker}_Z(A_B) = \{ (\alpha + \beta, -\alpha, -\beta, 0) | \alpha, \beta \in \mathbb{Z} \} \).

This encodes the vector

\[
 B((\alpha + \beta, -\alpha, -\beta, 0)) = (\alpha + \beta)c_{B_1} - \alpha c_{B_2} - \beta c_{B_3} = (\alpha + \beta, -\alpha, -\beta, -\beta, -\alpha, \alpha + \beta, 0),
\]

which is a generic element of \( \text{Ker}_Z(A) \).
Let \( \mathbf{u}, \mathbf{w}_1, \mathbf{w}_2 \in \text{Ker}_Z(A) \) be such that \( \mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \). Such a sum is a \textit{conformal decomposition} of \( \mathbf{u} \), written as \( \mathbf{u} = \mathbf{w}_1 + \mathbf{c} \mathbf{w}_2 \), if \( \mathbf{u}^+ = \mathbf{w}_1^+ + \mathbf{w}_2^+ \) and \( \mathbf{u}^- = \mathbf{w}_1^- + \mathbf{w}_2^- \), see [11]. If both \( \mathbf{w}_1, \mathbf{w}_2 \) are nonzero, the decomposition is said to be \textit{proper}. Recall the equivalent definition of the Graver basis \( \mathcal{G}(A) \) as the (finite) set of nonzero vectors in \( \text{Ker}_Z(A) \) for which there is no proper conformal decomposition (see, for example, [Hilbert Algorithm 7.2]).

**Theorem 1.11.** Let \( A \) and \( A_B \) be as in Theorem 1.9. Then there is a bijective correspondence between the Graver basis of \( A_B \) and the Graver basis of \( A \), and a similar bijective correspondence holds between the sets of circuits. Explicitly:

\[
\mathcal{G}(A) = \{ \mathbf{B(u)} | \mathbf{u} \in \mathcal{G}(A_B) \} \quad \text{and} \quad \mathcal{C}(A) = \{ \mathbf{B(u)} | \mathbf{u} \in \mathcal{C}(A_B) \}.
\]

**Proof.** Without loss of generality we may assume, as before, that \( A \) has \( s \) non-free bouquets and there exist integers \( 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n \) such that \( \mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_s} \) belong to the bouquet \( B_{i_1} \), and so on, until \( \mathbf{a}_{i_{s-1}+1}, \ldots, \mathbf{a}_n \) belong to the bouquet \( B_n \). It follows from Theorem 1.9 that the Graver basis elements of \( A \) are of the form \( B(\mathbf{u}) \) with \( \mathbf{u} \in \text{Ker}_Z(A_B) \). Moreover, for any \( \mathbf{u} = (u_1, \ldots, u_s) \in \text{Ker}_Z(A_B) \), Theorem 1.9 implies that

\[
(2) \quad B(\mathbf{u}) = ( (c_{B_1})_{i_1}u_1, \ldots, (c_{B_1})_{i_1}u_1, \ldots, (c_{B_s})_{i_{s-1}+1}u_s, \ldots, (c_{B_s})_{n}u_s).
\]

The bijective correspondence between the Graver basis elements follows at once since for any \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \text{Ker}_Z(A_B) \), we have \( \mathbf{u} = \mathbf{v} + \mathbf{w} \) if and only if \( B(\mathbf{u}) = B(\mathbf{v}) + cB(\mathbf{w}) \). Using Equation (2) again provides that \( \text{Supp}(B(\mathbf{u})) = \cup_{i \in \text{Supp}(\mathbf{u})} \text{Supp}(c_{B_i}) \). Thus, \( \mathbf{u} \) is a circuit if and only if \( B(\mathbf{u}) \) is a circuit.

This correspondence in general does not offer any relation between Markov bases, indispensable binomials and universal Gröbner bases of \( A \) and \( A_B \). On the other hand, Section 3 shows that the correspondence preserves additional sets for the case of stable toric ideals (see Definition 3.1).

**Example 1.12** (Example 1.10 continued). One easily checks that \( \mathcal{G}(A_B) = \mathcal{C}(A_B) = \{(1, -1, 0, 0), (0, 1, -1, 0), (1, 0, -1, 0)\} \), while \( \mathcal{G}(A) = \mathcal{C}(A) \) and consists of the vectors \( (1, -1, 0, 0, -1, 1, 0), (0, 1, -1, -1, 1, 0, 0), (1, 0, -1, -1, 0, 1, 0) \), since

\[
B((u_1, u_2, u_3, u_4)) = u_1c_{B_1} + u_2c_{B_2} + u_3c_{B_3} + u_4c_{B_4} = (u_1, u_2, u_3, u_2, u_1, u_4).
\]

The one-to-one correspondence for the two sets holds in the order indicated. For example,

\[
B((0, 1, -1, 0)) = c_{B_2} - c_{B_3} = (0, 1, -1, -1, 1, 0, 0).
\]

A remark is in order for subbouquets. Theorem 1.9 and Theorem 1.11 are true even if we replace bouquets with proper subbouquets which form a subbouquet decomposition of \( A \). The following sections utilize this when applying the two results.

We conclude this section with a rather surprising application of the bouquet construction. It will turn out that passing from a unimodular matrix \( A \) to its (sub)bouquet matrix \( A_B \) preserves unimodularity. To this end, recall that a matrix \( A \) is called \textit{unimodular} if all of its nonzero maximal minors have the same absolute value, see for example [11] Section 8, page 70. A nice property of a unimodular matrix \( A \) is that for its toric ideal \( I_A \) the set of circuits equals the Graver basis (and
We begin by constructing $i$ that the and $i$ the required conditions. For each $p$ and a positive first coordinate. Define $i$ vectors $c_B, \ldots, c_s$ are encoded by the given vectors. Assume first that $A$ is unimodular. Then it follows from Theorem 2.1 that the set of circuits of $A_B$ equals the Graver basis of $A_B$. It is a known fact that a matrix is unimodular if and only if all initial ideals of its toric ideal are squarefree, see [41, Remark 8.10]. Thus, for proving the unimodularity of $A_B$, it is sufficient to show that the Graver basis of $A_B$, hence also the universal Gröbner basis, consists of vectors with nonzero coordinates only $-1$ or $+1$. Let $u = (u_1, \ldots, u_s)$ be a Graver basis element of $A_B$. Then $B(u) = \sum_{i=1}^s c_B u_i$ is a Graver basis element of $A$ and, since $A$ is unimodular, all coordinates of $B(u)$ are $0, -1, 1$. Because the vectors $c_B$ have the supports pairwise disjoint, the latter condition is fulfilled only if for all $i$ we have $u_i \in \{0, -1, 1\}$ and the nonzero coordinates of $c_B$ are $1$ or $-1$. Thus we obtain the desired conclusion. For the converse, by Theorem 2.1 the Graver basis elements of $A$ are of the form $B(u) = \sum_{i=1}^s c_B u_i$ with $u$ running over all Graver basis elements of $A_B$. Since $A_B$ is unimodular then $u$ has all nonzero coordinates $+1$ or $-1$ and thus, by the hypothesis on $c_B$, it follows that $B(u)$ has all nonzero coordinates $+1$ or $-1$. The unimodularity of $A_B$ implies via Theorem 2.1 the equality of the set of circuits of $A$ and the Graver basis of $A$, and implicitly the equality with the universal Gröbner basis of $A$. Therefore the universal Gröbner basis of $A$ consists also of vectors with nonzero coordinates either $1$ or $-1$, and this implies that $A$ is unimodular.

2. Generalized Lawrence matrices

This section is dedicated to the construction of a natural inverse procedure of the one given in Section 1. Namely, given an arbitrary set of vectors $a_1, \ldots, a_s$ and vectors $c_1, \ldots, c_s$ that can act as bouquet-index-encoding vectors (cf. definition of $c_B$ in Section 1), the following result constructs a toric ideal $I_A$ whose $s$ subbouquets are encoded by the given vectors.

Recall that an integral vector $a \in \mathbb{Z}^m$ is primitive if the greatest common divisor of all its coordinates is 1.

**Theorem 2.1.** Let $\{a_1, \ldots, a_s\} \subset \mathbb{Z}^m$ be an arbitrary set of vectors. Let $c_1, \ldots, c_s$ be any set of primitive vectors, with $c_i \in \mathbb{Z}^{m_i}$ for some $m_i \geq 1$, each having full support and a positive first coordinate. Define $p = m + \sum_{i=1}^s (m_i - 1)$ and $q = \sum_{i=1}^s m_i$. Then, there exists a matrix $A \in \mathbb{Z}^{p \times q}$ with a subbouquet decomposition, $B_1, \ldots, B_s$, such that the $i^{th}$ subbouquet is encoded by the following vectors: $a_{B_i} = (a_i, 0, \ldots, 0) \in \mathbb{Z}^p$ and $c_{B_i} = (0, \ldots, c_i, \ldots, 0) \in \mathbb{Z}^q$, where the support of $c_{B_i}$ is precisely in the $i^{th}$ block of $\mathbb{Z}^q$ of size $m_i$.

**Proof.** We begin by constructing $A$ explicitly, and then show its subbouquets satisfy the required conditions. For each $i = 1, \ldots, s$, let $c_i = (c_{i1}, \ldots, c_{im_i}) \in \mathbb{Z}^{m_i}$ and
define

$$C_i = \begin{pmatrix} -c_{i2} & c_{i1} \\ -c_{i3} & c_{i1} \\ \vdots & \vdots \\ -c_{im_i} & c_{i1} \end{pmatrix} \in \mathbb{Z}^{(m_i-1) \times m_i}.$$  

Primitivity of each $c_i$ implies that there exist integers $\lambda_{i1}, \ldots, \lambda_{im_i}$ such that $1 = \lambda_{i1}c_{i1} + \cdots + \lambda_{im_i}c_{im_i}$. Fix a choice of $\lambda_{i1}, \ldots, \lambda_{im_i}$, and define the matrices $A_i = [\lambda_{i1}a_i, \ldots, \lambda_{im_i}a_i] \in \mathbb{Z}^{m \times m_i}$. The desired matrix $A$ is then the following block matrix:

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p \times q},$$

where $p = m + (m_1 - 1) + \cdots + (m_s - 1)$ and $q = m_1 + \cdots + m_s$. We will denote the columns of the matrix $A$ by $\beta_1, \ldots, \beta_q$.

In the remainder of the proof, we show that $\beta_1, \ldots, \beta_{m_1}$ belong to the same subbouquet $B_1$. Analogous arguments, with a straightforward shift of the indices, apply to show that $\beta_{m_1+\cdots+m_{i-1}+1}, \ldots, \beta_{m_1+\cdots+m_i}$ belong to the same subbouquet $B_i$ for all $i = 2, \ldots, s$.

Consider the vector $\gamma_{m+i} \in \mathbb{Z}^p$, whose only nonzero coordinate is 1, in position $m + i - 1$, for every $i = 2, \ldots, m_1$. Then the co-vector

$$(\gamma_{m+i} \cdot \beta_1, \ldots, \gamma_{m+i} \cdot \beta_q) = (-c_{1i}, 0, \ldots, c_{1i}, 0, \ldots, 0)$$

has support $\{1, i\}$, and by Remark 1.3 the following relation holds for any $2 \leq i \leq m_1$:

$$-c_{ij}G(\beta_j) + c_{ij}G(\beta_i) = 0.$$  

Since all coordinates of $c_1$ are nonzero, the previous relations imply that $\beta_1, \ldots, \beta_{m_1}$ belong to the same subbouquet, which may or may not be free. Therefore $A$ has $s$ subbouquets.

Finally, we compute $c_{B_1}$. If $B_1$ is free, then by definition we can take $c_{B_1} = (c_1, 0, \ldots, 0)$. Otherwise, if $B_1$ is not free, then there exists a coordinate $j$ such that $G(\beta_j) \neq 0$ for all $i = 1, \ldots, m_1$. Then the relations (4) provide

$$\frac{G(\beta_j)}{c_{11}} = \cdots = \frac{G(\beta_{m_1})}{c_{1m_1}} = \frac{k}{l},$$

with relatively prime integers $k$ and $l$, $l > 0$. Thus $lG(\beta_j) = c_1k$, and consequently $l|c_{1i}$ for all $i$. But the coordinates of $c_1$ being relatively prime implies $l = 1$ and thus $G(\beta_j) = c_1k$ for all $i$. Therefore $g_j := \gcd(G(\beta_1), \ldots, G(\beta_{m_1})) = |k|$. On the other hand, since $G(\beta_j) = c_1k$ and $c_1 > 0$ it follows that $\varepsilon_{1j} = \text{sgn}(k)$, where $\varepsilon_{ij}$ is the sign of $G(\beta_i)$. Hence $G(\beta_i) = c_{1i}k = \varepsilon_{1j}c_{1i}g_j$, and by definition of $c_{B_1}$ we have

$$(c_{B_1})_i = \varepsilon_{1j} \frac{G(\beta_i)_j}{g_j} = \varepsilon_{1j} \varepsilon_{1i}c_{1i} = c_{1i}, \text{ for all } i = 1, \ldots, m_1.$$
Thus $c_{B_i} = (c_1, 0, \ldots, 0)$, and
\[
a_{B_i} = \sum_{i=1}^{m_1} (c_{B_i})_i \beta_i = \sum_{i=1}^{m_1} c_{1i} \beta_i = (a_1, 0, \ldots, 0),
\]
as desired. Note that for the last equality we used that $\lambda_{1i} c_{11} + \cdots + \lambda_{1m_1} c_{1m_1} = 1$.

\[\square\]

**Remark 2.2.** We will call the matrix $A$ defined in Theorem 2.1 the *generalized Lawrence matrix*, since in particular one can recover, after a column permutation, the classical second Lawrence lifting. Indeed, recall from [11, Chapter 7] that the second Lawrence lifting $\Lambda(D)$ of an integer matrix $D = [d_1, \ldots, d_n] \in \mathbb{Z}^{m \times n}$ is defined as
\[
\Lambda(D) = \begin{pmatrix} D & 0 \\ I_n & I_n \end{pmatrix} \in \mathbb{Z}^{(m+n) \times 2n}.
\]
Applying the construction of Theorem 2.1 for the vectors $a_i = d_i$ and $c_i = (1, -1)$ for all $i = 1, \ldots, n$, with $\lambda_{1i} = 1$ and $\lambda_{i2} = 0$, we obtain
\[
A_i = (d_i, 0) \in \mathbb{Z}^{m \times 2}, \quad C_i = (1 \ 1) \in \mathbb{Z}^{1 \times 2},
\]
and thus $A$ is just $\Lambda(D)$ after a column permutation.

As an immediate consequence of Theorem 2.1 we obtain that generalized Lawrence matrices capture the ideal information for any general matrix $A$:

**Corollary 2.3.** For any integer matrix $A$ there exists a generalized Lawrence matrix $A'$ such that $I_A = I_{A'}$, up to permutation of column indices.

**Proof.** Let $A = [a_1, \ldots, a_s] \in \mathbb{Z}^{m \times n}$, and assume that $A$ has $s$ bouquets $B_1, \ldots, B_s$. Then $A_B = [a_{B_1}, \ldots, a_{B_s}] \in \mathbb{Z}^{m \times s}$, and $c_{B_1}, \ldots, c_{B_s} \in \mathbb{Z}^n$ are such that their supports are pairwise disjoint and $\bigcup_{i=1}^s \text{supp}(c_{B_i}) = [n]$. After permuting the columns of $A$ we may assume that $c_{B_i} = (0, \ldots, c_{i1}, \ldots, 0)$ for some $c_i \in \mathbb{Z}^m$ with full support (that is, $m_i = |\text{supp}(c_{B_i})|$) and primitive. Theorem 2.1, applied to the set of vectors $\{a_{B_1}, \ldots, a_{B_s}\}$ and $\{c_{i1}, \ldots, c_{is}\}$, provides the existence of a matrix $A' \in \mathbb{Z}^{p \times n}$ with $s$ subb Bouquets $B'_1, \ldots, B'_s$ and with the property that $a_{B'_i} = (a_{B_i}, 0, \ldots, 0)$ and $c_{B'_i} = (0, \ldots, c_{i1}, \ldots, 0)$ for all $i$. Since $c_{B'_i} = c_{B_i}$ for all $i = 1, \ldots, s$, Theorem 1.9 implies that $\text{Ker}_Z(A) = \text{Ker}_Z(A')$.

\[\square\]

The most important property of Theorem 2.1, often exploited in the next sections, is that it can be used to provide infinite classes of examples. For example, if we want to construct infinitely many unimodular matrices we proceed as follows: let $D = [a_1, \ldots, a_s]$ be any unimodular matrix (for example the incidence matrix of a bipartite graph) and, based on Proposition 1.13 choose arbitrary $c_1, \ldots, c_s$ with entries $-1$ or $1$, and the first nonzero coordinate being $1$. Then Theorem 2.1 yields the generalized Lawrence matrix $A$ such that its subbouquet ideal equals $I_D$, and finally by Proposition 1.13 we obtain that $A$ is also unimodular. Similarly, from an arbitrary unimodular matrix $D$, using any set of vectors $c_1, \ldots, c_s$ satisfying the hypotheses of Theorem 2.1 and such that at least one has one coordinate in absolute value larger than 1, we can construct infinitely many generalized Lawrence matrices.
that are not unimodular, but have the set of circuits equal to the universal Gröbner basis and the Graver basis.

**Remark 2.4.** Theorem 2.1 also solves the following two natural problems. First, given an arbitrary graph $G$ whose connected components are cliques, there exists a matrix $A$ such that the bouquet graph $G_A$ of $A$ is precisely $G$. Second, a stronger statement can be made: given a graph $G$ whose connected components are cliques, along with $+$ and $-$ signs associated to each edge of $G$ according to the sign rules explained following Remark 1.3 there exists a matrix $A$ whose bouquet graph and structure are encoded by $G$.

3. On stable toric ideals

In this section we prove that a certain bouquet structure of $A$, which we call stability, provides many additional properties of the map $u \mapsto B(u)$ between $\text{Ker}_Z(A_B)$ and $\text{Ker}_Z(A)$. We will assume, throughout this section, that $A$ has no free bouquet. If $A$ has free bouquets, then all of the results in this section remain valid, since free bouquets do not affect the kernels $\text{Ker}_Z(A)$ and $\text{Ker}_Z(A_B)$ by Remark 1.8 and Theorem 1.9.

**Definition 3.1.** The toric ideal $I_A$ is called stable if all of the bouquets of $A$ are non-mixed. More generally, the toric ideal $I_A$ is called stable with respect to a subbouquet decomposition of $A$ if there exists a subbouquet decomposition of $A$, such that all of the subbouquets are non-mixed.

Note that there always exists a subbouquet decomposition such that $I_A$ is stable with respect to it: the trivial subbouquet decomposition, i.e. all subbouquets are isolated vertices. In general, there might be several different such subbouquet decompositions. However, there is a canonical (and maximal) subbouquet decomposition such that $I_A$ is stable with respect to it, see Remark 3.13. Still, $I_A$ can be stable with respect to a subbouquet decomposition, without being stable, see Example 3.14. On the other hand, a stable ideal $I_A$ is obviously stable with respect to the subbouquet decomposition given by the family of bouquets of $A$.

In the case of stable toric ideals the map $u \mapsto B(u)$ has the following additional property.

**Remark 3.2.** If all of the bouquets of $A$ are non-mixed, then it follows from Lemma 1.6 that the vectors $c_{B_1}, \ldots, c_{B_s}$ have all nonzero coordinates positive. Then, by Theorem 1.9 it follows that $B(u)^+ = B(u^+)$ and $B(u)^- = B(u^-)$ for every $u \in \text{Ker}_Z(A_B)$.

Stability ensures that several of the properties of $I_A$ are preserved when passing to $I_{A_B}$. We begin with an easy, but crucial, Lemma.

**Lemma 3.3.** Let $I_A$ be a stable toric ideal. Then $I_A$ is positively graded if and only if $I_{A_B}$ is positively graded.

**Proof.** Let $v \in \text{Ker}_Z(A)$. By Theorem 1.9 there exists a vector $u$ such that $v = B(u)$ for some $u \in \text{Ker}_Z(A_B)$. Therefore $v = \sum_{i=1}^{s} c_{B_i} u_i$, and since every $c_{B_i}$ is a nonzero
vector with all nonzero coordinates positive, and their supports are pairwise disjoint, then we obtain that \( \mathbf{0} \neq \mathbf{v} \in \mathbb{N}^n \) if and only if \( \mathbf{0} \neq \mathbf{u} \in \mathbb{N}^n. \)

A Markov basis of \( A \) is a finite subset \( \mathcal{M} \) of \( \text{Ker}_Z(A) \) such that whenever \( \mathbf{w}, \mathbf{v} \in \mathbb{N}^n \) and \( \mathbf{w} - \mathbf{v} \in \text{Ker}_Z(A) \) there exists a subset \( \{ \mathbf{u}_i : i = 1, \ldots, r \} \) of \( \mathcal{M} \) that connects \( \mathbf{w} \) to \( \mathbf{v} \). Here, connectedness means that \( \mathbf{w} - \mathbf{v} = \sum_{i=1}^r \mathbf{u}_i, \) and \( (\mathbf{w} - \sum_{i=1}^p \mathbf{u}_i) \in \mathbb{N}^n \) for all \( 1 \leq p \leq r. \)

A Markov basis \( \mathcal{M} \) is minimal if no proper subset of \( \mathcal{M} \) is a Markov basis. By a fundamental theorem from Markov bases literature (see [22]), a set of vectors is a Markov basis for \( A \) if and only if the corresponding set of binomials (whose exponents are the given vectors) generate the toric ideal \( I_A. \)

**Proposition 3.4.** Let \( I_A \) be a stable toric ideal. Then the map \( \mathbf{u} \mapsto B(\mathbf{u}) \) is a bijective correspondence between the minimal Markov bases of \( A_B \) and the minimal Markov bases of \( A. \)

**Proof.** Let \( \mathcal{M} \) be a Markov basis of \( A_B, \) and \( \mathcal{M}' = \{ B(\mathbf{u}) : \mathbf{u} \in \mathcal{M} \}. \) Let \( \mathbf{w} = B(\mathbf{u}) \in \text{Ker}_Z(A_B), \) for some \( \mathbf{u} \in \text{Ker}_Z(A_B), \) see Theorem 1.9. Then by Remark 3.2 it follows that \( \mathbf{w}^+ = B(\mathbf{u}^+) \) and \( \mathbf{w}^- = B(\mathbf{u}^-). \) Since \( \mathbf{u} \in \text{Ker}_Z(A_B) \) and \( \mathcal{M} \) is a Markov basis for \( A_B \) then there exists a subset \( \{ \mathbf{u}_i : i = 1, \ldots, r \} \) of \( \mathcal{M} \) such that \( \mathbf{u}^+ - \mathbf{u}^- = \sum_{i=1}^r \mathbf{u}_i \) and \( (\mathbf{u}^+ - \sum_{i=1}^p \mathbf{u}_i) \in \mathbb{N}^n \) for all \( 1 \leq p \leq r. \) Thus

\[
\mathbf{w}^+ - \mathbf{w}^- = B(\mathbf{u}) = B\left(\sum_{i=1}^r \mathbf{u}_i\right) = \sum_{i=1}^r B(\mathbf{u}_i),
\]

and

\[
\mathbf{w}^+ - \sum_{i=1}^p B(\mathbf{u}_i) = B(\mathbf{u}^+ - \sum_{i=1}^p \mathbf{u}_i)
\]

belongs to \( \mathbb{N}^n \) by Remark 3.2 since the vector \( \mathbf{u}^+ - \sum_{i=1}^p \mathbf{u}_i \in \mathbb{N}^n \) for all \( 1 \leq p \leq r. \) Therefore \( \mathcal{M}' \) is a Markov basis of \( A. \) Conversely, given any Markov basis \( \mathcal{M}' \) of \( A \) then its elements are of the form \( \{ B(\mathbf{u}) : \mathbf{u} \in \mathcal{M} \cap \text{Ker}_Z(A_B) \}, \) see Theorem 1.9. As before, using Remark 3.2 it can be shown that \( \mathcal{M} \) is a Markov basis of \( A_B. \) This bijective correspondence ensures also that the map \( \mathbf{u} \mapsto B(\mathbf{u}) \) preserves the minimality of the Markov bases. Indeed, if \( \mathcal{M} \) is a minimal Markov basis of \( A_B \) then \( \mathcal{M}' = \{ B(\mathbf{u}) : \mathbf{u} \in \mathcal{M} \} \) is a Markov basis of \( A, \) and if it were not minimal then a proper subset \( \mathcal{M}'_1 \) of it would be minimal, and thus we would obtain that a proper subset of \( \mathcal{M} \) would be a Markov basis of \( A_B, \) a contradiction.

The intersection of all (minimal) Markov bases of \( A \) via the identification of the elements which differ by a sign is called the set of indispensable binomials of \( A, \) and denoted by \( S(A). \) In order to determine the indispensable binomials one has to deal with two cases: \( \text{Ker}_Z(A) \cap \mathbb{N}^n \neq \{ \mathbf{0} \} \) or \( \text{Ker}_Z(A) \cap \mathbb{N}^n = \{ \mathbf{0} \}. \) In the first case, it follows from [15, Theorem 4.18] that \( S(A) = \{ \mathbf{0} \}. \) As a side comment, note that if one restricts to the intersection of all minimal Markov bases of minimal cardinality then it may be at most one binomial in \( S(A). \) In the second case, [16, Proposition 1.1] provides the following useful algebraic characterization, which will be needed in Section 4: the set of indispensable binomials of \( A \) consists of all
binomials $x^u + x^v$ corresponding to the nonzero vectors $u$ in $\text{Ker}_Z(A)$ which have no proper semiconformal decomposition. We recall from [29, Definition 3.9] that for vectors $u, v, w \in \text{Ker}_Z(A)$ such that $u = v + \text{w}$, the sum is said to be a semiconformal decomposition of $u$, written $u = v + \text{w}$, if $v_i > 0$ implies that $w_i \geq 0$, and $v_i < 0$ implies that $v_i \leq 0$ for all $1 \leq i \leq n$. As before we call the decomposition proper if both $v, w$ are nonzero. Note that when writing a semiconformal decomposition of $u$ it is necessary to specify the order of the vectors added.

Proposition 3.5. Let $I_A$ be a stable toric ideal. Then the map $u \mapsto B(u)$ induces a bijective correspondence between the indispensable binomials of $A_B$ and the indispensable binomials of $A$.

Proof. It follows from the above considerations that if $I_A$ is not positively graded then $A$ has no indispensable binomials. Thus by Lemma 3.3 we may assume that $I_A$ is positively graded, otherwise the conclusion holds trivially. Note that if $u = v + \text{w}$ is a proper semiconformal decomposition of $u$ then $B(u) = B(v) + \text{w}$, $B(w)$ is a proper semiconformal decomposition of $B(u)$ and vice versa, since all $c_B, w$ are nonzero and with the nonzero coordinates positive. Thus, by applying [16, Proposition 1.1] we obtain the desired correspondence between the indispensable binomials.

A remark on the proof is in order. Since by definition $S(A)$ is the intersection of all (minimal) Markov bases via the identification of the elements which differ by a sign, then by Proposition 3.3 we could have obtained the desired correspondence between $S(A)$ and $S(A_B)$. However, we prefer to prove it using semiconformal decompositions, as it provides a basis for some constructions in later sections.

Proposition 3.6. Let $I_A$ be a stable toric ideal. Then the map $u \mapsto B(u)$ induces a bijective correspondence between the reduced Gröbner bases of $A_B$ and the reduced Gröbner bases of $A$. In particular, there is a bijective correspondence between the universal Gröbner bases of $A_B$ and $A$.

Proof. As in the proof of Theorem 1.9 we may assume, for ease of notation, that there exist integers $1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n$ such that $a_1, \ldots, a_{i_1}$ belong to the bouquet $B_1$, and so on, until $a_{i_{s-1}+1}, \ldots, a_n$ belong to the bouquet $B_s$. Thus $\text{supp}(c_B) = \{1, \ldots, i_1\}, \ldots, \text{supp}(c_B) = \{i_{s-1} + 1, \ldots, n\}$.

Let $G = \{y^{u_1^+} - y^{u_1^-}, \ldots, y^{u_s^+} - y^{u_s^-}\}$ be a reduced Gröbner basis of $I_{A_B}$ with respect to a monomial order $<_1$ on $K[y_1, \ldots, y_s]$. By [41, Proposition 1.11] there exists a weight vector $\omega = (\omega_1, \ldots, \omega_s) \in \mathbb{N}^s$ such that the monomial order is given by $\omega$, that is $\text{in}_<(I_{A_B}) = \omega(I_{A_B})$. Without loss of generality we may assume that $y^{u_s^+} > y^{u_s^-}$, i.e. the dot product $\omega \cdot u_i > 0$, for any $i = 1, \ldots, t$. We will define a monomial order $<_{1}$ on $K[x_1, \ldots, x_n]$ such that the set $G' = \{x^{B(u_1^+)} - x^{B(u_1^-)}, \ldots, x^{B(u_s^+)} - x^{B(u_s^-)}\}$ is a reduced Gröbner basis of $I_A$ with respect to $<_1$. For this, we first define the vector

$$
\omega' = \left(\frac{\omega_1}{(c_B)_{i_1}, \ldots, (c_B)_{i_1}}, \frac{\omega_1}{(c_B)_{i_1}}, \frac{\omega_2}{(c_B)_{i_1+1}(i_2 - i_1)}, \ldots, \frac{\omega_s}{(c_B)_{i_1}(n - i_{s-1})}\right),
$$

and let $<_{\omega'}$ be an arbitrary monomial order on $K[x_1, \ldots, x_n]$. We define $<_1$ to be the monomial order $<_{\omega'}$ on $K[x_1, \ldots, x_n]$ induced by $<$ and $\omega'$, see [41, Chapter 1] for
definition. Next we prove that $y^{u^+} > y^{u^-}$ implies that $x^{B(u^+)} > x^{B(u^-)}$. Indeed, since $y^{u^+} > y^{u^-}$ then $\omega \cdot u > 0$, where $u = (u_1, \ldots, u_s)$. Thus

$$
\omega' \cdot B(u) = \frac{\omega_1}{(c_B)_1 t_1}(c_B)_1 u_1 + \frac{\omega_1}{(c_B)_2 t_1}(c_B)_2 u_1 + \cdots = \omega \cdot u > 0,
$$

which implies that $x^{B(u^+)} > x^{B(u^-)}$, as desired. In particular, we obtain that $x^{B(u^+)} > x^{B(u^-)}$ for all $i$, and consequently $(x^{B(u^+)}), \ldots, x^{B(u^+)}) \subset \text{in}_{<1}(I_A)$. For the converse inclusion let $x^{B(u^+)} - x^{B(u^-)} \in I_A$ be an arbitrary element, see Theorem 1.10 and say that $x^{B(u^+)} > x^{B(u^-)}$, the other case being similar. It follows from the previous considerations that $y^{u^+} > y^{u^-}$ and thus there exists an integer $i$ such that $y^{u^+} | y^{u^-}$. Since $u \mapsto B(u)$ is a linear map and all $c_B$ are nonnegative then $x^{B(u^+)} | x^{B(u^-)}$. Since divisibility is compatible to any monomial order, then via Remark 3.2 we get that $x^{B(u^+)} \in (x^{B(u^+)}, \ldots, x^{B(u^+)})$ and thus $G'$ is a Gröbner basis of $I_A$ with respect to $<_1$. Finally, to prove that $G'$ is reduced we argue by contradiction. This implies that there exists an integer $i$ such that $x^{B(u_i)} \in \text{in}_{<1}(I_A)$, so $x^{B(u_i)}$ is divisible by some $x^{B(u)}$. This in turn implies that $y^{u^+} | y^{u^-}$, a contradiction since $G$ is reduced. Therefore we obtain $G'$ is reduced Gröbner basis with respect to $<_1$, as desired.

Conversely, let $G' = \{x^{B(u_1^+)} - x^{B(u_1^-)}, \ldots, x^{B(u_s^+)} - x^{B(u_s^-)}\}$ be a reduced Gröbner basis of $I_A$ with respect to a monomial order $<_1$ on $K[x_1, \ldots, x_n]$. Using similar arguments as above one can prove that the set $G = \{y^{u_1^+} - y^{u_1^-}, \ldots, y^{u_s^+} - y^{u_s^-}\}$ is a reduced Gröbner basis of $A_B$ with respect to the monomial order $<_1$ on $K[y_1, \ldots, y_s]$ defined as follows

$$
y^u < y^v \text{ if and only if } x^{B(u)} < x^{B(v)}.
$$

That $<_1$ is a monomial order follows easily once we note that $<_1$ is well defined, since all $c_B$'s are nonnegative. Therefore the proof is complete. □

In summary, Theorem 1.11 can be combined with Propositions 3.3, 3.5, 3.6 to provide, in particular, justification for the terminology ‘stable’ toric ideals:

**Theorem 3.7.** Let $I_A$ be a stable toric ideal. Then the bijective correspondence between the elements of Ker$_Z(A)$ and Ker$_Z(A_B)$ given by $u \mapsto B(u)$, is preserved when we restrict to any of the following sets: Graver basis, circuits, indispensable binomials, minimal Markov bases, reduced Gröbner bases (universal Gröbner basis).

**Example 3.8.** Based on Theorem 2.1 there are infinitely many stable toric ideals. In fact, to construct them is enough to consider matrices $A$ obtained as in Theorem 2.1, starting from arbitrary vectors $a_i$’s, but considering only vectors $c_i$’s with positive coordinates. Then via Remark 3.2 the corresponding subbouquets of $A$ are either free or non-mixed, and thus $I_A$ is a stable toric ideal.

We conclude this section with an application of the stable toric ideals to the construction of generic toric ideals, an open problem posed by Miller and Sturmfels [32, Section 9.4]. Recall from [37] that a toric ideal $I_A$ is called generic if it is minimally generated by binomials with full support, that is $I_A = (x^{u_1} - x^{v_1}, \ldots, x^{u_s} - x^{v_s})$, and none of the vectors $u_i - v_i$ has a zero coordinate. The following result
states that in the case of stable toric ideals genericity is preserved when passing from $A$ to $A_B$ and conversely.

**Theorem 3.9.** Let $I_A$ be a stable toric ideal. Then $I_A$ is a generic toric ideal if and only if $I_{A_B}$ is a generic toric ideal.

**Proof.** Applying Theorem 3.7 we know that the map $u \mapsto B(u)$ induces a bijective correspondence between the minimal Markov bases. Thus, since $B((u_1, \ldots , u_s)) = \sum_{i=1}^{s} c_B u_i$ and $\cup_i \text{Supp}(c_{B_i}) = [u]$, then $\mathcal{M}$ is a minimal Markov basis of $A_B$, with each vector having full support if and only if $\{B(u) : u \in \mathcal{M}\}$ is a minimal Markov basis of $A$ with each vector having full support. \hfill \Box

**Remark 3.10.** In particular, Section 2 provides a way to construct an infinite class of generic toric ideals starting from an arbitrary example of a generic toric ideal. More precisely, one can use the examples of generic toric ideals, see [35, Example 2.3, Theorem 3.11. Let $I_A \subset S = K[x_1, \ldots , x_n]$ be a stable positively graded toric ideal and $A_B = [a_{B_1}, \ldots , a_{B_s}]$ with $I_{A_B} \subset R = K[y_1, \ldots , y_s]$. If we denote by $\mathbb{F}_*$ the minimal $\mathcal{N}A_B$-graded free resolution of $R/I_{A_B}$, then $\mathbb{F}_* \otimes_R S$ is a minimal $\mathcal{N}A$-graded free resolution of $I_A$.

**Proof.** Since $I_A$ is a stable toric ideal then all the non-free bouquets are non-mixed. Without loss of generality, we may assume, as in the proof of Theorem 1.9 that $A$ has no free bouquet and there exist integers $1 \leq i_1 < i_2 < \cdots < i_s-1 \leq n$ such that $a_1, \ldots , a_{i_1}$ belong to the bouquet $B_1$, and so on, until $a_{i_{s-1}+1}, \ldots , a_n$ belong to the bouquet $B_s$. In particular, by Remark 3.2 all nonzero coordinates of $c_{B_1}, \ldots , c_{B_s}$ are positive. We define the following $K$-algebra homomorphism

\[ \phi : R \rightarrow S \]

\[ y_i \mapsto x_i^{(c_{B_1})_i} \cdots x_i^{(c_{B_s})_i} \]

\[ y_s \mapsto x_{i_{s-1}+1}^{(c_{B_{s-1}})_{i_{s-1}+1}} \cdots x_n^{(c_{B_s})_n} \].

The homomorphism $\phi$ is well-defined since all nonzero coordinates of $c_{B_j}$ are positive, and is a graded homomorphism with respect to the gradings induced by 1) $\mathcal{N}A_B$ on $R$, that is $\deg(y_i) = a_{B_i}$ for all $i = 1, \ldots , s$, and 2) $\mathcal{N}A$ on $S$, that is $\deg(x_j) = a_j$ for all $j = 1, \ldots , n$. Indeed,

\[ \deg(y_j) = a_{B_j} = \sum_{(c_{B_j})_k \neq 0} (c_{B_j})_k a_k = \sum_{k=i_{j-1}+1}^{i_j} (c_{B_j})_k a_k = \deg(x_{i_{j-1}+1}^{(c_{B_{j-1}})_{i_{j-1}+1}} \cdots x_n^{(c_{B_s})_n}) \]

the last one being the degree $\deg(\phi(y_j))$ for all $j$. In addition,

\[ I_A = (x^{B(u)^+} - x^{B(u)^-} : u \in \text{Ker}_Z(A_B)) = (x^{B(u)^+} - x^{B(u)^-} : u \in \text{Ker}_Z(A_B)), \]

\[ 20 \]
where the first equality follows from Theorem 1.9 and the second equality from Remark 3.2, which implies \( \phi(I_{AB}) = I_A \), since by definition \( \phi(y^u - y^{-u}) = x^B(u^+) - x^B(u^-) \) for all \( u \in \text{Ker}_Z(A_B) \).

Applying now [24, Theorem 18.16] we obtain that \( \phi \) is flat. This implies that the natural map \( I_{AB} \otimes_R S \to I_A \) is an isomorphism of graded \( S \)-modules; see [24, Proposition 6.1]. Finally, flatness of \( \phi \) ensures that tensoring the minimal graded free resolution of \( I_{AB} \) as \( R \)-module with \( S \) we obtain the minimal graded free resolution of \( I_A \) as \( S \)-module, as desired. \( \square \)

**Remark 3.12.** Let \( I_A \) be a stable toric ideal. It follows from Theorem 3.7 that \( I_A \) is a robust toric ideal if and only if \( I_{AB} \) is a robust toric ideal. In particular, using the same strategy described above for generic lattice ideals, we can construct robust toric ideals that are different from the ones considered in [10] [11] which, in fact, correspond to toric ideals of graphs and toric ideals generated by degree two binomials. Using again Theorem 3.7 we also have that \( I_A \) is a generalized robust toric ideal if and only if \( I_{AB} \) is a generalized robust toric ideal (see [44] for the definition of generalized robust toric ideal).

**Remark 3.13.** As a concluding remark of this section, given that stability preserves a lot of information, we discuss the case when a given ideal is not stable but we still wish to preserve all the combinatorial and algebraic information. To that end, suppose that \( I_A \) is stable with respect to a certain subbouquet decomposition, say \( B_1, \ldots, B_t \). Then, the subbouquet ideal \( I_A' \) associated to this decomposition preserves all “combinatorial data” of \( I_A \), and its minimal graded free resolution can be read off from the one of \( I_A \). Indeed, the reader will have noted that, throughout this section, the only part of the stability hypothesis on \( I_A \) that is relevant for the proofs is the fact that the bouquets are non-mixed. Maximality of the bouquets was not assumed. Hence, the results hold true for any subbouquet decomposition. Therefore, in case of a non-stable toric ideal, we are motivated to look for a “maximal” subbouquet decomposition such that toric ideal is stable with respect to it. There is indeed such a subbouquet decomposition and a canonical way to obtain it. Since \( I_A \) is not stable, there exists at least one mixed bouquet. Each mixed bouquet has a subbouquet decomposition into two maximal non-mixed subbouquets. To see this let us consider a mixed bouquet \( B \), which by definition contains an edge \( \{a_i, a_j\} \in E^{-}\). By Lemma 1.2 we have \( G(a_i) = \lambda G(a_j) \) for some \( \lambda < 0 \). We define \( B_1 \) to be the clique induced on the subset of vertices of \( B \) for which the Gale transform is a positive multiple of \( G(a_i) \), while \( B_2 \) is the clique induced on the subset of vertices of \( B \) for which the Gale transform is a positive multiple of \( G(a_j) \). It is straightforward to see that \( B_1 \) and \( B_2 \) are the desired maximal non-mixed subbouquets. Thus considering the non-mixed bouquets of \( I_A \) and taking all maximal non-mixed subbouquets of all mixed bouquets we obtain the desired canonical subbouquet decomposition. Note also that this subbouquet decomposition is non-trivial only if at least one of the non-mixed subbouquets is not an isolated vertex. This canonical subbouquet decomposition provides the subbouquet ideal \( I_{A'_B} \). Finally, we can pass from \( I_{A'_B} \) to \( I_{AB} \) through a bouquet graph whose non-mixed bouquets are isolated vertices, and mixed bouquets (if any) are singleton edges.
The following example explains the general strategy of passing from a toric ideal to its bouquet ideal through a subbouquet ideal, where the subbouquet is chosen such that the toric ideal is stable with respect to the underlying subbouquet decomposition.

**Example 3.14.** Let $A = [a_1, \ldots, a_9]$ be the integer matrix

$$
\begin{pmatrix}
3 & 0 & 0 & 0 & 4 & 5 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
\end{pmatrix}.
$$

It has three non-free bouquets, $B_1, B_2, B_3$, with $B_1, B_3$ mixed and $B_2$ non-mixed, hence $I_A$ is not stable. The bouquet $B_1$ is on the set of vertices $\{a_1, a_2, a_3, a_4\}$ with $G(a_1) = G(a_2) = -G(a_3) = -G(a_4)$, $B_2$ is the isolated vertex $a_5$, and $B_3$ has the set of vertices $\{a_6, a_7, a_8, a_9\}$ with $G(a_6) = G(a_7) = -G(a_8) = -G(a_9)$. Each of the mixed bouquets $B_1, B_3$ has two maximal non-mixed subbouquets $B'_1, B''_1$ and $B'_3, B''_3$, respectively. Moreover, the encoding vectors are $c_{B'_1} = (1, 1, 0, 0, 0, 0, 0, 0, 0)$, $c_{B''_1} = (0, 0, 1, 1, 0, 0, 0, 0, 0)$, $c_{B_2} = (0, 0, 0, 0, 1, 0, 0, 0, 0)$, $c_{B'_3} = (0, 0, 0, 0, 1, 1, 0, 0, 0)$ and $c_{B''_3} = (0, 0, 0, 0, 0, 0, 1, 1, 1)$. Therefore, the associated subbouquet matrix $A_B = [a_{B'_1}, a_{B''_1}, a_{B_2}, a_{B'_3}, a_{B''_3}] \in \mathbb{Z}^{7 \times 5}$ is

$$
\begin{pmatrix}
3 & 0 & 4 & 5 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

where the last row of bold zeros stands for the zero block matrix from $\mathbb{Z}^{4 \times 5}$. Since $I_A$ is stable with respect to the subbouquet decomposition given by the subbouquets $B'_1, B''_1, B_2, B'_3, B''_3$, this implies that the conclusions of Theorem 3.7 and Theorem 3.11 do apply, when restricted to $I_A$ and $I_{A_B}$. In other words, when passing from $I_A$ to $I_{A_B}$ we preserve all the combinatorial and algebraic data. Finally, the matrix $A'_B$ has three non-free bouquets, two mixed and one non-mixed, with the two mixed consisting of a single edge $\{a_{B'_1}, a_{B''_1}\}$ and $\{a_{B'_3}, a_{B''_3}\}$, respectively, while the non-mixed one is the isolated vertex $a_{B_2}$. Computing the bouquet matrix of $A'_B$ we obtain

$$
A_B = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}^{7 \times 3},
$$

and the toric ideal of $A_B$ equals the toric ideal of the monomial curve $(3 \ 4 \ 5)$. Comparing now $I_A$ (and thus $I_{A_B}$) with $I_{A_B}$ we note, for example, that a minimal Markov basis of $I_A$ has six elements, while a minimal Markov basis of $I_{A_B}$ has only three.
4. A combinatorial characterization of toric ideals whose Graver bases are minimal Markov bases

It is well known from [41, Proposition 7.1] that the Graver basis, the universal Gröbner basis, any reduced Gröbner basis and any minimal Markov basis are equal for the toric ideal of the second Lawrence lifting of an arbitrary integer matrix. It is also well known that Lawrence liftings are not the only matrices with this property. For example, it was shown to hold for 2-regular uniform hypergraphs by Gross and Petrović in [27], and for robust toric ideals of graphs by Boocher et al. in [11]. Furthermore, such examples can have both mixed and non-mixed bouquets, as in Example 4.3(b). Therefore, it is clear that bouquets alone do not capture equality of bases.

The main result of this section, Theorem 4.2, is a characterization of toric ideals whose bases are equal. It relies on two additional ingredients. The first one is the familiar notion of a semiconformal decomposition (cf. Section 3), which provides an algebraic characterization of equality of bases. The second one is the new concept of \( S \)-Lawrence ideals that generalizes the classical second Lawrence lifting.

Before we proceed, recall that if \( \ker Z(A) \cap \mathbb{N}^n \neq \{0\} \), the four sets of bases can never be simultaneously equal by [15, Theorem 4.18]. Thus we may assume for the rest of this section that \( \ker Z(A) \cap \mathbb{N}^n = \{0\} \), which is equivalent to saying that \( I_A \) is positively graded. Recall also that the fiber \( F_u \) of a monomial \( x^u \) is the set \( \{ t \in \mathbb{N}^n : u - t \in \ker Z(A) \} \). When \( I_A \) is positively graded, \( F_u \) is a finite set.

**Definition 4.1.** Fix a subset \( S \) of \([n]\). The toric ideal \( I_A \) is said to be \( S \)-Lawrence if and only if for every element \( u \in Gr(A) \), there exists no element \( w \) in the fiber of \( u^+ \), different from \( u^+ \) and \( u^- \), such that the \( S \)-part of \( w \) belongs to \( P_S(u) \).

The \( S \)-Lawrence property is a natural one. Namely, by (the conformal) definition of the Graver basis, every toric ideal is \( S \)-Lawrence in the case when \( S = [n] \). On the other hand, if \( S = \emptyset \) and \( I_A \) is \( \emptyset \)-Lawrence, then for every \( u \in Gr(A) \) the fiber of \( u^+ \) consists of just two elements \( u^+, u^- \). In particular, the fiber of \( u^+ \) being finite implies that all fibers of \( I_A \) are finite (see for example [15, Proposition 2.3]), and thus \( I_A \) is positively graded. Since the fiber of each Graver basis element \( u \) consists of the two elements \( u^+, u^- \), whose supports are disjoint, every Graver basis element is in a minimal Markov basis, and thus indispensable via [14, Corollary 2.10]. Hence the two bases are equal: \( Gr(A) = S(A) \). In summary, \( I_A \) is \( \emptyset \)-Lawrence if and only if \( I_A \) is positively graded and \( Gr(A) = S(A) \). Note that when \( S(A) = Gr(A) \) then the Graver basis, the universal Gröbner basis, any reduced Gröbner basis and any minimal Markov basis of \( A \) are equal.

We are now ready to state the desired combinatorial characterization.
Theorem 4.2. Let $B_1, \ldots, B_s$ be the bouquets of $A = [a_1, \ldots, a_n]$, and define $A_B = [a_{B_1}, \ldots, a_{B_s}]$. Let $S \subset [s]$ be the subset of coordinates corresponding to the mixed bouquets. Then the following are equivalent:

(a) There exists no element in the Graver basis of $A_B$ which has a proper semiconformal decomposition that is conformal on the coordinates corresponding to $S$.

(b) The following sets coincide:

- the Graver basis of $A$,
- the universal Gröbner basis of $A$,
- any reduced Gröbner basis of $A$,
- any minimal Markov basis of $A$.

(c) The toric ideal of $A_B$ is S-Lawrence.

Proof. We may assume that the $s$ bouquets of $A$ are not free and partition the set \{a_1, \ldots, a_n\} into $s$ subsets such that the first $i_1$ vectors belong to the bouquet $B_1$, the next $i_2$ vectors belong to the bouquet $B_2$, and so on, the last $i_s$ vectors belong to the bouquet $B_s$.

In order to prove the theorem we analyze two cases: 1) $S = \emptyset$ and 2) $S \neq \emptyset$. If $S = \emptyset$, then the hypothesis of Theorem 4.2 translates to $I_A$ being a stable, positively graded toric ideal. It follows from the remarks after Definition 4.1 that condition (c) is equivalent to $I_{AB}$ being positively graded and $Gr(A_B) = S(A_B)$. On the other hand, since $I_A$ is a stable, positively graded toric ideal, Lemma 3.3 implies that $I_{AB}$ is also positively graded. Thus (a) is equivalent, via [16, Proposition 1.1], to $I_{AB}$ being positively graded and $Gr(A_B) = S(A_B)$. Therefore, (a) and (c) are equivalent. Note that equivalence of (a) with (b) follows from Theorem 3.7.

If $S \neq \emptyset$, we assume that $B_1, \ldots, B_t$ are the mixed bouquets of $A$ for some $t$ with $t \leq s$. We will show the equivalence of (a) and (b), and then of (a) and (c).

(b) $\Rightarrow$ (a): Since the four sets coincide and $I_A$ is positively graded, $Gr(A) = S(A)$. Assume by contradiction that there exists a Graver basis element $u$ of $A_B$ which has a proper semiconformal decomposition that is conformal on the components corresponding to $S$. This implies that there exist nonzero vectors $v, w \in Ker_Z(A_B)$ such that $u = v + sc \ w$ and $(u_1, \ldots, u_t) = (v_1, \ldots, v_t) + sc (w_1, \ldots, w_t)$. Since $u \in Gr(A_B)$ it follows from Theorem 3.11 that $B(u) \in Gr(A)$. We will prove that $B(u) = B(v) + sc \ B(w)$, a contradiction to our assumption that $Gr(A) = S(A)$. Note that conformality implies that $v_i$ and $w_i$ have the same sign for all $i = 1, \ldots, t$ thus

\begin{equation}
(c_{B_1}u_1, \ldots, c_{B_t}u_t) = (c_{B_1}v_1, \ldots, c_{B_t}v_t) + sc (c_{B_1}w_1, \ldots, c_{B_t}w_t).
\end{equation}

Note that $t < s$, for otherwise we obtain $B(u) = B(v) + sc B(w)$ and thus $B(u) \notin Gr(A)$, a contradiction. Since the bouquets $B_{t+1}, \ldots, B_s$ are not mixed, it follows that the vectors $c_{B_{t+1}}, \ldots, c_{B_s}$ have all coordinates positive and thus

\begin{equation}
(c_{B_{t+1}}u_{t+1}, \ldots, c_{B_s}u_s) = (c_{B_{t+1}}v_{t+1}, \ldots, c_{B_s}v_s) + sc (c_{B_{t+1}}w_{t+1}, \ldots, c_{B_s}w_s).
\end{equation}

Combining (3) and (4) we obtain $B(u) = B(v) + sc B(w)$, and the claim follows.

(a) $\Rightarrow$ (b): Assume that there exists no element in the Graver basis of $A_B$ which has a proper semiconformal decomposition that is conformal on the components
corresponding to $S$. We argue by contradiction and suppose that $S(A)$ is properly contained in $\mathcal{G}r(A)$. Then, by Theorem 1.11 there exists an element $B(u) \in \mathcal{G}r(A)$ for some $u \in \mathcal{G}r(A_2)$ such that $B(u) \notin S(A)$. Applying again [16] Proposition 1.1 we obtain that there exist nonzero vectors $v, w \in \text{Ker}_Z(B)_{sc}$ such that $B(u) = B(v) + sc B(w)$. We will prove that $u = v + sc w$ and $(u_1, \ldots, u_t) = (v_1, \ldots, v_t) + c (w_1, \ldots, w_t)$, a contradiction to our hypothesis. Since $B_1, \ldots, B_t$ are mixed each one of the vectors $c_{B_1}, \ldots, c_{B_t}$ has one negative and one positive coordinate by Lemma 1.6. Since $B(u) = B(v) + sc B(w)$, it follows that

$$(c_{B_1} u_1, \ldots, c_{B_t} u_t) = (c_{B_1} v_1, \ldots, c_{B_t} v_t) + sc (c_{B_1} w_1, \ldots, c_{B_t} w_t),$$

and thus $u_i$ and $w_i$ have the same sign for all $1 \leq i \leq t$. Therefore we obtain that $(u_1, \ldots, u_t) = (v_1, \ldots, v_t) + c (w_1, \ldots, w_t)$. If $t = s$ then the proof of this implication is complete. If $t < s$, then bouquets $B_{t+1}, \ldots, B_s$ are not mixed, and consequently all coordinates of the vectors $c_{B_{t+1}}, \ldots, c_{B_s}$ are positive. Since

$$(c_{B_{t+1}} u_{t+1}, \ldots, c_{B_s} u_s) = (c_{B_{t+1}} v_{t+1}, \ldots, c_{B_s} v_s) + sc (c_{B_{t+1}} w_{t+1}, \ldots, c_{B_s} w_s),$$

it follows that $(u_{t+1}, \ldots, u_s) = (v_{t+1}, \ldots, v_s) + sc (w_{t+1}, \ldots, w_s)$. This implies that $u = v + sc w$ and the proof is complete.

$(a) \Rightarrow (c)$: Assume by contradiction that $I_{AB}$ is not $S$-Lawrence. This implies that there exists a vector $u \in \mathcal{G}r(A_B)$ and a positive vector $w \in \mathcal{F}_{u^+} \setminus \{u^+, u^-\}$ such that the $S$-part of $w$ belongs to $P_S(u)$. We claim that $(\ast): u = (u^+ - w) + sc (w - u^-)$ and the sum is conformal on the coordinates corresponding to $S$, which leads us to a contradiction and the claim follows. We will prove first that the sum is conformal on the coordinates of $S$. Let $i$ be an integer of $S$. If $u_i > 0$ then $u_i = u_i^+, u_i^- = 0$, and by the hypothesis $w_i \leq u_i^+$. Thus $u_i^+ - w_i \geq 0$, $w_i - u_i^- = w_i \geq 0$. If $u_i = 0$ then $u_i^+ = u_i^- = 0$ and by hypothesis also $w_i = 0$, so $u_i^+ - w_i = w_i - u_i^- = 0$. Finally, if $u_i < 0$ then $u_i = u_i^-, u_i^+ = 0$, $w_i \leq u_i^-$, $u_i^+ - w_i = -w_i \leq 0$, $-w_i - u_i^- \leq 0$. Hence it follows that the sum $(\ast)$ is conformal on the coordinates corresponding to $S$, and implicitly the sum $(\ast)$ is also semiconformal on the coordinates corresponding to $S$. It remains to check semiconformality only on the coordinates not in $S$. There are such coordinates, since otherwise we obtain that the sum is conformal, a contradiction to $u \in \mathcal{G}r(A_B)$. Let $i \not\in S$ be an integer. If $u_i^+ - w_i > 0$ then $u_i^+ > 0$, $u_i^- = 0$, and thus $w_i - u_i^- = w_i \geq 0$. On the other hand if $w_i - u_i^- < 0$ then $u_i^- > 0$, $u_i^+ = 0$ and we obtain $u_i^+ - w_i = -w_i \leq 0$. Therefore we have proved that the sum $(\ast)$ is semiconformal, and the claim follows.

$(c) \Rightarrow (a)$: Assume by contradiction that there exists an element $u \in \mathcal{G}r(A_B)$ which has a proper semiconformal decomposition, that is conformal on the coordinates corresponding to $S$. This implies that there exist nonzero vectors $v, w \in \text{Ker}_Z(A_2)$ such that $u = v + sc w$ and $(u_1, \ldots, u_t) = (v_1, \ldots, v_t) + c (w_1, \ldots, w_t)$. By definition, semiconformality implies that $u^+ \geq v^+$ and $u^- \geq w^-$. We claim that the vector $z = u^- - v = u^- + w$ is positive, and thus $z \in \mathcal{F}_{u^+}$, $z$ is different from $u^+$ and $u^-$ and the $S$-part of $z$ belongs to $P_S(u)$. This yields a contradiction to the hypothesis that $I_{AB}$ is $S$-Lawrence, and the proof is complete. Since $z = u^+ - v = u^- - v^+ + v^-$, then $z \geq 0$ follows from $u^+ \geq v^+$. Moreover, $v, w \neq 0$, provides that $z \neq u^+, u^-$. Finally, that the $S$-part of $z$ belongs to $P_S(u)$ holds.
immediately after analyzing the cases 1) $i \in S$ with $u_i > 0$, 2) $i \in S$ with $u_i = 0$, and 3) $i \in S$ with $u_i < 0$.

Theorem 4.2 can be used to easily provide examples of toric ideals, different from the second Lawrence liftings, such that the four bases are equal; this is further developed in Section 6. In the following example we show how to use the equivalence of (a) and (c) from Theorem 4.2.

**Example 4.3.** a) Let $A = [a_1, \ldots, a_{13}] \in \mathbb{Z}^{15 \times 13}$, where $a_1, \ldots, a_{12}$ are the support vectors of the edges $E_1, \ldots, E_{12}$ from Example 5.2(a) on the set of vertices $\{x, v_1, \ldots, v_{14}\}$, and $a_{13} = (5, 0, \ldots, 0)$. With respect to the basis $G_1, G_2$ of $\ker\mathbb{Z}(A)$, where

$$G_1 = (0, 0, 0, 0, 0, 5, 5, 5, -5, -5, -5, -4),$$

and

$$G_2 = (1, 1, 1, -1, -1, -2, -2, -2, -2, 2, 2, 2, 1),$$

we see that the Gale transforms are: $G(a_1) = G(a_2) = G(a_3) = -G(a_4) = -G(a_5) = (0, 1), G(a_6) = \cdots = G(a_9) = -G(a_{10}) = \cdots = -G(a_{12}) = (5, -2)$, and $G(a_{13}) = (-4, 1)$. Thus the bouquet graph $G_A$ has three non-free bouquets $B_1, B_2, B_3$: The first two, $B_1$ and $B_2$, are mixed bouquets corresponding to the sets of vertices $\{a_1, \ldots, a_5\}$ and $\{a_6, \ldots, a_{12}\}$, respectively. The third, $B_3$, is an isolated vertex $a_{13}$. Hence $A_B = [a_{B_1}, a_{B_2}, a_{B_3}]$, where $a_{B_1} = (3, 0, \ldots, 0), a_{B_2} = (4, 0, \ldots, 0), a_{B_3} = (5, 0, \ldots, 0), s = 3, S = \{1, 2\}$. The Graver basis elements of $A_B$ are $(4, -3, 0), (1, -2, 1), (3, -1, -1), (2, 1, -2), (5, 0, -3), (1, 3, -3), (0, 5, -4)$. Since $(4, -3, 0) = (3, -1, -1) + sc (1, -2, 1)$, and the sum is conformal on the first two coordinates, i.e. on $S$, it follows that the condition (a) of Theorem 4.2 is not satisfied and therefore the four sets are not simultaneously equal. Note that in this case the toric ideal of $A_B$ is positively graded.

b) Consider the graph from Figure 1 and denote by $A$ its incidence matrix whose column vectors $e_1, \ldots, e_{15}$ are given by the support vectors of the edges $e_1, \ldots, e_{15}$.

![Figure 1](image-url)
Then the bouquet graph associated to $A$ has nine non-free bouquets $B_1, \ldots, B_9$: the first six are mixed bouquets corresponding to the edges $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\{e_6, e_7\}$, $\{e_8, e_9\}$, $\{e_{11}, e_{12}\}$, $\{e_{13}, e_{14}\}$, and the last three are the isolated vertices $e_{10}, e_{15}$. Therefore not all bouquets of $G_A$ are mixed, $A_B = \{a_{B_1}, \ldots, a_{B_9}\}$, where $a_{B_i} = e_1 - e_2$, $a_{B_2} = e_3 - e_4$, $a_{B_3} = e_6 - e_7$, $a_{B_4} = e_8 - e_9$, $a_{B_5} = e_{11} - e_{12}$, $a_{B_6} = e_{13} - e_{14}$, $a_{B_7} = e_5$, $a_{B_8} = e_{10}$, $a_{B_9} = e_{15}$ and $S = \{1, \ldots, 6\}$. The toric ideal of $A_B$ is not positively graded, and thus all the fibers of $I_{A_B}$ are infinite. We will check that $I_{A_B}$ is $S$-Lawrence in order to use Theorem 4.2 to conclude that the four bases are equal. The Graver basis of $A_B$ has fifteen elements, one of which is $u = (0, 1, 1, 0, 0, 0, -1, 1, 0)$. The degree of the fiber of $u^+$, $u^-$ is equal to $a_{B_2} + a_{B_3} + a_{B_8} = a_{B_7} = (1, 0, 1, 0, 0, 0, 0, 0, 0)$. Since the fiber of $u^+$ is found by computing all nonnegative solutions of the equation $\sum_{i=1}^9 n_i a_{B_i} = (1, 0, 1, 0, 0, 0, 0, 0, 0)$, any element in the fiber is of one of the following three types of nonnegative vectors

$$(\alpha, \alpha, \beta, \beta, \gamma, \gamma, 1, 0, 0), \ (\alpha, \alpha + 1, \beta, \beta - 1, \gamma, \gamma, 1, 0, 0), \ (\alpha, \alpha + 1, \beta, \beta, \gamma, \gamma - 1, 0, 0, 1).$$

Let $w$ be in the fiber of $u^+$ such that the $S$-part of $w$ belongs to the $S$-parallelepiped of $u$, that is $w_1 = w_4 = w_8 = 0$ and $w_2, w_3 \leq 1$. It follows immediately that $w$ is either of the first type with $\alpha = 0, \beta = 0, \gamma = 0$, which implies $w = u^-$, or of the second type with $\alpha = 0, \beta = 1, \gamma = 0$, and then $w = u^+$. Hence there exists no vector $w$ different from $u^+$ and $u^-$ which belongs to $P_S(u)$. Similarly one can check for the rest of the elements of $\mathcal{G}(A_B)$ and conclude that $I_{A_B}$ is $S$-Lawrence. Therefore we obtain that the four sets are equal. The same conclusion could have been drawn for the toric ideal $I_A$ as a consequence of the graph-theoretical description of the Graver, universal Gröbner, and any minimal Markov basis of $A$ given in [39, 45].

As an immediate consequence of Theorem 4.2 we obtain the following.

**Corollary 4.4.** Suppose that every non-free bouquet of $A$ is mixed. Then the following sets of binomials coincide:

1. the Graver basis of $A$,
2. the universal Gröbner basis of $A$,
3. any reduced Gröbner basis of $A$,
4. any minimal Markov basis of $A$.

**Proof.** Note first that if every non-free bouquet is mixed then $\ker_z(A) \cap \mathbb{N}^n = \{0\}$. As before, we may assume that $A$ has no free bouquet and since all $s$ bouquets are mixed then $S = \{1, \ldots, s\}$, and thus condition (a) of Theorem 4.2 is trivially satisfied. An application of Theorem 4.2 leads to the desired conclusion. \qed

Note that the condition that each non-free bouquet of $A$ is mixed is not a sufficient condition, as Example 1.3(b) shows. We close this section by showing how one recovers [11, Theorem 7.1] from Corollary 4.4.

**Remark 4.5.** Let $D \in \mathbb{Z}^{m \times n}$ be an integer matrix and $\Lambda(D)$ its second Lawrence lifting. We denote by $\alpha_1, \ldots, \alpha_n$ the columns of $D$. By Remark 2.2, $\Lambda(D)$ is after a column permutation just the matrix $A$ constructed in Theorem 2.1 for the vectors $a_i = \alpha_i$ and $c_i = (1, -1)$ for all $i = 1, \ldots, n$. Applying Theorem 2.1 we obtain
that $A$ has $n$ subbouquets, which are either free or mixed by Lemma 1.6. Thus by Corollary 4.4 we obtain the desired conclusion.

Interestingly, all previously known examples of equality $Gr(A) = S(A)$, that is, of $\emptyset$-Lawrence toric ideals in the literature not arising from graphs, as those from [11], and Example 4.3a), are such that matrices $A$ have only mixed bouquets. Our results provide many additional such easy-to-construct examples, e.g. Corollary 4.4 recovering, in particular, [11 Proposition 7.1] as a special case. On the other hand, Example 4.3b) constructs an $\emptyset$-Lawrence toric ideal that has both mixed and non-mixed bouquets. However, we do not have an example of an $\emptyset$-Lawrence matrix $A$ that has no mixed bouquet, raising the natural question: Is it true that if $I_A$ is $\emptyset$-Lawrence then $A$ has at least one mixed bouquet? For readers interested in this question, we gather a few remarks. Note that if $A$ has $s$ bouquets then, since $I_{AB}$ is always $[s]$-Lawrence the consequence of Theorem 4.2 is that: in the case when $A$ has all of the bouquets mixed then the converse is also true, and thus $I_A$ is $\emptyset$-Lawrence. In addition, when $A$ has no mixed bouquets, Theorem 4.2 implies that $I_A$ is $\emptyset$-Lawrence if and only if $I_{AB}$ is $\emptyset$-Lawrence; also a consequence of Theorem 3.7. Moreover, at the moment, we do not know whether $S$-Lawrence toric ideals have special properties when $\emptyset \subset S \subset [n]$.

5. Hypergraphs and bouquets with bases

As this section consists of three parts, we begin with a ‘roadmap’ to point out the main definitions and results. As described in the Introduction, this section is concerned with toric ideals of 0/1 matrices, which are, by definition, incidence matrices of hypergraphs. To understand this class of toric ideals, we need two main definitions: a bouquet with basis in Definition 5.1 lacking in the previous literature and a monomial walk in Definition 5.7 recovering the meaning of monomial walks on graphs from [39], [47]. The first two subsections are motivated by [38] Problem 6.2], and they, essentially, partially solve its generalization. Theorem 5.3 classifies bouquets with bases, which are either free or mixed subbouquets, while the running example shows its implications on identifying and constructing elements in the Graver basis of a hypergraph. Proposition 5.10 shows that certain hypergraphs based on a sunflower are bouquets with bases and, in particular, recovers the main structural result of [38]. As another application of bouquets with bases, Proposition 5.13 generalizes a result by [21] and [45] for graphs to uniform hypergraphs and showing that the universal Gröbner and Graver bases differ for complete uniform hypergraphs with enough vertices. However, in general, there exist hypergraphs that do not admit such a subbouquet structure. In this case, we provide general constructions in Sections 6 and 7.

Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph on the set of vertices $V = \{x_1, \ldots, x_m\}$ with edge set $\mathcal{E} = \{E_1, \ldots, E_n\}$, where each $E_i$ is a subset of $\{x_1, \ldots, x_m\}$. When $|E_i| = 2$ for all $i$ and $E_i \neq E_j$ for all $i \neq j$, we have a finite simple graph, and we will specialize to this case to recall results about graphs. We denote by $\alpha_E$ the support (column) vector of an edge $E$, and thus the toric ideal $I_\mathcal{H}$ is the toric ideal of the matrix $[\alpha_{E_1}, \ldots, \alpha_{E_n}]$. Hereafter, for ease of notation, various bases of $I_\mathcal{H}$.
will be referred to as bases of \( \mathcal{H} \); the reader may simply keep in mind that the underlying toric matrix is the incidence matrix of the hypergraph \( \mathcal{H} \); for example, \( Gr(\mathcal{H}) := Gr(A_{\mathcal{H}}) \) where \( A_{\mathcal{H}} \) is the vertex-edge incidence matrix of \( \mathcal{H} \).

5.1. **Bouquets with bases.** Let \( \mathcal{H} = (V, \mathcal{E}) \) be a hypergraph and \( U \subset V \). We define the multiset \( U_{\mathcal{E}} = \{ E \cap U | E \in \mathcal{E}, E \cap U \neq \emptyset \} \) and the set \( \mathcal{E}_U = \{ E \in \mathcal{E} | E \cap U \neq \emptyset \} \). Note that the multiset \( U_{\mathcal{E}} \) and the set \( \mathcal{E}_U \) have the same number of elements: \( \mathcal{E}_U \) is the set of edges of \( \mathcal{H} \) that intersect the vertex set \( U \), while \( U_{\mathcal{E}} \) consists of their restrictions to \( U \).

**Definition 5.1.** The set of edges \( \mathcal{E}_U \) of \( \mathcal{H} \) is called a *bouquet with basis \( U \subset V \) if the toric ideal of the multi-hypergraph \( (U, U_{\mathcal{E}}) \) is principal, generated by an element \( e^c - e^{c^-} \) with \( \text{supp}(c) = \mathcal{E}_U \), and such that the first nonzero coordinate of \( c \) is \( \alpha^c > 0 \). Here the toric ideal of \( (U, U_{\mathcal{E}}) \) is contained in the polynomial ring with variables indexed by the non-empty subsets \( e_i := E_i \cap U \).

Moreover, for a bouquet with basis \( \mathcal{E}_U \), we define the vector \( c_{\mathcal{E}_U} \in \mathbb{Z}^n \) such that \( (c_{\mathcal{E}_U})_E = c_E \) for any \( E \in \mathcal{E}_U \) and 0 otherwise, and the vector \( a_{\mathcal{E}_U} = \sum_{E \in \mathcal{E}_U} c_{\mathcal{E}_U} \alpha^E \in \mathbb{Z}^m \). Here \( c_E \) denotes the coordinate of \( c \) corresponding to the edge \( E \), and note from the definition of \( c \) that \( c_E \neq 0 \) for all \( E \in \mathcal{E}_U \).

Given the apparent similarity with Definition 1.1, one would expect a certain relationship between a bouquet with basis of a hypergraph \( \mathcal{H} \) and a bouquet of its incidence matrix. Even more, one may ask whether \( c_{\mathcal{E}_U} \) and \( a_{\mathcal{E}_U} \), the encoding vectors of a bouquet with basis, are analogous to \( c_B \) and \( a_B \) as defined in Section 1. The following examples show that a bouquet with basis may correspond to a non-free bouquet or a free bouquet of the incidence matrix. This correspondence is the natural one, i.e., a bouquet with basis \( \mathcal{E}_U \) corresponds to the set of vectors \( \{ \alpha_E : E \in \mathcal{E}_U \} \).

**Example 5.2.** a) Consider the hypergraph \( \mathcal{H} = (V, \mathcal{E}) \) from Figure 2 with the set of vertices \( V = \{x, v_1, \ldots, v_{22}\} \) and whose set of edges \( \mathcal{E} \) consists of the following 20 edges: \( E_1 = \{x, v_1, v_2\}, E_2 = \{x, v_3, v_4\}, E_3 = \{x, v_5, v_6\}, E_4 = \{v_1, v_3, v_5\}, E_5 = \{v_2, v_4, v_6\}, E_6 = \{v_7, v_8\}, E_7 = \{x, v_9, v_{10}\}, E_8 = \{v_1, v_{11}, v_{12}\}, E_9 = \{v_{13}, v_{14}\}, E_{10} = \{v_{15}, v_{16}\}, E_{11} = \{v_{17}, v_{18}\}, E_{12} = \{v_{19}, v_{20}\}, E_{13} = \{v_{21}, v_{22}\}, E_{14} = \{x, v_1, v_3, v_5\}, E_{15} = \{x, v_7, v_8\}, E_{16} = \{x, v_{11}, v_{12}\}, E_{17} = \{x, v_{13}, v_{14}\}, E_{18} = \{x, v_{15}, v_{16}\}, E_{19} = \{x, v_{17}, v_{18}\}, E_{20} = \{x, v_{19}, v_{20}\}, E_{21} = \{x, v_{21}, v_{22}\} \).

The first bouquet with basis \( \mathcal{E}_U_1 \) has five edges, the basis \( U_1 \) is the set \( \{v_1, \ldots, v_6\} \), the vector \( c_1 \) corresponds to the binomial generator \( e_1 e_2 e_3 - e_4 e_5 \) of the toric ideal of \( (U_1, U_{\mathcal{E}}) \), and thus \( c_{\mathcal{E}_U_1} = (1, 1, 1, -1, -1, 0, \ldots, 0) \in \mathbb{Z}^{20} \) and

\[
a_{\mathcal{E}_U_1} = \alpha_{E_1} + \alpha_{E_2} + \alpha_{E_3} - \alpha_{E_4} - \alpha_{E_5} = (3, 0, \ldots, 0) \in \mathbb{Z}^{23}.
\]

The second bouquet with basis \( \mathcal{E}_U_2 \) has seven edges, the basis \( U_2 \) is the set \( \{v_7, \ldots, v_{14}\} \), the vector \( c_2 \) corresponds to the binomial generator \( e_6 e_7 e_8 e_9 - e_{10} e_{11} e_{12} \) of the toric ideal of \( (U_2, U_{\mathcal{E}}) \), and thus the encoding vectors are

\[
c_{\mathcal{E}_U_2} = (0, 0, 0, 0, 1, 1, 1, -1, -1, 0, \ldots, 0) \in \mathbb{Z}^{20}.
\]
and
\[ a_{E_{v_2}} = \sum_{i=6}^{9} \alpha_{E_i} - \sum_{j=10}^{12} \alpha_{E_j} = (4,0,\ldots,0) \in \mathbb{Z}^{23}. \]

The third bouquet with basis \( E_{U_3} \) has eight edges, the basis \( U_3 = \{v_{15},\ldots,v_{22}\} \), the vector \( c_3 \) corresponds to the binomial generator \( e_{13}e_{14}c_2^2e_{16} - e_{17}c_{18}e_{19}e_{20} \) of the toric ideal of \( (U_3, U_{3e}) \), and thus \( c_{E_{U_3}} = (0,\ldots,0,1,1,2,1,-1,-1,-1,-1) \in \mathbb{Z}^{20} \) and
\[ a_{E_{U_3}} = \alpha_{E_{13}} + \alpha_{E_{14}} + 2\alpha_{E_{15}} + \alpha_{E_{16}} - \sum_{j=17}^{20} \alpha_{E_j} = (5,0,\ldots,0) \in \mathbb{Z}^{23}. \]

On the other hand, if \( A = A_H \) is the incidence matrix of the hypergraph \( H \) with columns \( \alpha_{E_1},\ldots,\alpha_{E_{20}} \), then \( G_A \) has three mixed non-free bouquets \( B_1, B_2, B_3 \). More precisely, the first bouquet \( B_1 \) corresponds to the vectors \( \alpha_{E_1},\ldots,\alpha_{E_5} \), the second bouquet \( B_2 \) to the vectors \( \alpha_{E_6},\ldots,\alpha_{E_{12}} \), and the third bouquet \( B_3 \) to the vectors \( \alpha_{E_{13}},\ldots,\alpha_{E_{20}} \). Moreover, it turns out that \( c_{B_i} = c_{E_{U_i}} \) and \( a_{B_i} = a_{E_{U_i}} \) for all \( i \).

(b) The tetrahedron is an example of a bouquet with basis, and the basis can be chosen to be any facet. Let \( V = \{v_1,v_2,v_3,v_4\} \) and \( E = \{E_1 = \{v_2,v_3,v_4\}, E_2 = \{v_1,v_2,v_3\}, E_3 = \{v_1,v_2,v_4\}, E_4 = \{v_1,v_3,v_4\}\} \). If we consider as basis the set \( U_1 = \{v_1,v_2,v_3\} \) then \( U_{U_1} = \{e_1 = \{v_2,v_3\}, e_2 = \{v_1,v_3\}, e_3 = \{v_1,v_2\}, e_4 = \{v_1,v_2,v_3\}\} \) and the toric ideal of \( (U_1, U_{U_1}) \) is generated by the element \( e^{e^+} - e^{e^-} = e_1e_2e_3 - e_4^2 \), with \( c = (1,1,1,-2) \). Therefore, \( c_{E_{U_1}} = (1,1,1,-2) \) and \( a_{E_{U_1}} = \alpha_{E_1} + \alpha_{E_2} + \alpha_{E_3} - 2\alpha_{E_4} = (0,0,0,3) \). The same example has four different representations as a bouquet with basis \( U \), each one of them having pairwise distinct vectors \( c_{E_{U}} \) and \( a_{E_{U}} \), respectively. More precisely, if \( U_2 = \{v_1,v_2,v_4\} \) then \( c_{E_{U_2}} = (1,-2,1,1) \) and \( a_{E_{U_2}} = (0,0,3,0) \); if \( U_3 = \{v_1,v_3,v_4\} \) then \( c_{E_{U_3}} = (1,-2,1,1) \) and \( a_{E_{U_3}} = (0,3,0,0) \), and if \( U_4 = \{v_2,v_3,v_4\} \) then \( c_{E_{U_4}} = (2,-1,-1,-1) \) and \( a_{E_{U_4}} = (-3,0,0,0) \). Note that \( I_H \) is the zero ideal and the Gale transforms \( G(\alpha_{E_i}) \) are zero vectors for all \( i \). Therefore, the bouquet graph of \( I_H \) has only the free bouquet \( B \) consisting of all vertices \( \alpha_{E_1},\ldots,\alpha_{E_4} \). In particular, by definition of \( c_{B} \) and \( a_{B} \) we observe that we can choose these vectors to be any of the 4 pairs of vectors obtained before as \( c_{E_{U}} \) and \( a_{E_{U}} \).

In light of the two situations discussed in Example 5.2, the following result clarifies the ‘bouquet with basis’ terminology. As we will see, a bouquet with basis corresponds to a subbouquet of the incidence matrix of the hypergraph, where the
correspondence is the natural one associating to a set of edges of a hypergraph the set of corresponding support column vectors of its incidence matrix. Thus, from now on, by abuse of notation, we will identify the bouquet with basis $\mathcal{E}_U$ with the corresponding subbouquet of the incidence matrix. Furthermore, we also note that $c_{\mathcal{E}_U}$ matches the definition of $c_B$ from Section 4 and the same holds for $a_{\mathcal{E}_U}$.

**Theorem 5.3.** A bouquet with basis of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ is either a free subbouquet or a mixed subbouquet of the incidence matrix of $\mathcal{H}$.

*Proof.* Let $\mathcal{E}_U$ be a bouquet with basis for some $U \subset V$, and set $\mathcal{E}_U = \{E_1, \ldots, E_s\}$. Furthermore, denote by $M \in \mathbb{Z}^{m \times n}$ the incidence matrix of the hypergraph $(V, \mathcal{E})$. We may arrange $M$ so that its rows are indexed first by vertices in $U$ and then vertices in $V \setminus U$, and its columns are indexed first by the edges $E_1, \ldots, E_s$ and next by the remaining edges, if any. Note that the submatrix of $M$ corresponding to the rows indexed by $U$ and the first $s$ columns, denoted by $M_U$, is the incidence matrix of the multi-hypergraph $(U, \mathcal{U}_E)$, while the submatrix of $M$ corresponding to the rows indexed by $U$ and the rest of the columns is $0$. Finally, denote by $G = (g_{ij}) \in \mathbb{Z}^{n \times r}$ the Gale transform of $M$, and according to the labeling of the columns of $M$ its first $s$ rows are $G(\alpha_{E_1}), \ldots, G(\alpha_{E_s})$. By definition of the Gale transform, column $j$ of $G$ is in the kernel of $M$. Therefore $(g_{1j}, \ldots, g_{sj}) \in \text{Ker } M_U$ and since $\mathcal{E}_U$ is a bouquet with basis then $(g_{1j}, \ldots, g_{sj})$ is a multiple of $c = (c_{E_1}, \ldots, c_{E_s})$. Thus $g_{ij} = \lambda_j c_{E_i}$ for all $i, j$ and implicitly we obtain $G(\alpha_{E_i}) = c_{E_i}(\lambda_1, \lambda_2, \ldots, \lambda_r)$ for all $i \in \{1, \ldots, s\}$. Then it follows at once that $\alpha_{E_1}, \ldots, \alpha_{E_s}$ belong to the same subbouquet $B$ of $M$. In addition, since $\text{supp}(c) = \mathcal{E}_U$ then $c_{E_i} \neq 0$ for all $i \in \{1, \ldots, s\}$. Thus there are two possibilities: $G(\alpha_{E_i}) = 0$ for all $i$ or $G(\alpha_{E_i}) \neq 0$ for all $i$. In the first case we obtain that $B$ is a free subbouquet, while in the second $B$ is a non-free subbouquet. Moreover, the toric ideal of $(U, \mathcal{U}_E)$ being positively graded implies that the vector $c_{\mathcal{E}_U}$ has at least one positive and one negative coordinate, and by Lemma 4.6 we obtain that $B$ is mixed. $\Box$

Let us point out that a bouquet with basis can be a proper subbouquet of a bouquet. Indeed, let $A'$ be the submatrix of the incidence matrix of the hypergraph $\mathcal{H}$ from Example 5.2(a) corresponding to the first twelve columns. As it was already noticed the first five edges form a bouquet with basis, and the other seven edges form another bouquet with basis. In contrast, one can easily see that the bouquet graph of $A'$ has one mixed bouquet - consisting of all column vectors of $A'$, and thus the two bouquets with bases are proper subbouquets.

The previous Theorem shows that bouquets with bases are always subbouquets. The natural converse question arises: Can there exist (sub)bouquets of the incidence matrix of a hypergraph which are not bouquets with bases? The answer is yes, and as an example consider the complete graph $K_4$ on four vertices, whose incidence matrix is the submatrix corresponding to the first 6 columns of $A$ from Example 1.3. As we have seen in Section 1, $K_4$ has three non mixed bouquets, each one being an edge. On the other hand, $K_4$ does not have any bouquets with basis, since by Theorem 5.3 this would imply the existence either of a mixed non-free subbouquet or a free subbouquet.
The main consequence of Theorem 5.3 is that if the edge set of a hypergraph $H$ can be partitioned into bouquets with bases, then the toric ideal $I_H$ is easier to describe, via Theorems 1.9 and 1.11. The following example captures this remark.

Example 5.4 (Example 5.2(a), continued). The hypergraph $H$ has three bouquets with bases $E_{U_1}, E_{U_2}, E_{U_3}$ which partition $E$. Therefore, the subbouquet ideal of $A_H$ is given by the toric ideal of the matrix whose columns are $a_{E_{U_1}} = (3, 0, \ldots, 0), a_{E_{U_2}} = (4, 0, \ldots, 0), a_{E_{U_3}} = (5, 0, \ldots, 0) \in \mathbb{Z}^{23}$, which is the same as the toric ideal of the monomial curve $A = (3 4 5)$. Computing with $[1]$ we obtain that the Graver basis of $I_A$ consists of seven elements $(4, -3, 0), (1, -2, 1), (3, -1, -1), (2, 1, -2), (5, 0, -3), (1, 3, -3), (0, 5, -4)$. Therefore, by Theorem 1.11, the Graver basis of the toric ideal of the hypergraph consists of seven elements. For example, $(1, -2, 1)$ corresponds to the following Graver basis element of $I_H$:

$$c_{E_{U_1}} - 2c_{E_{U_2}} + c_{E_{U_3}} = (1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1)$$

and it encodes the binomial

$$E_1 E_2 E_3 E_4 E_5 E_6 E_7 E_8 E_9 E_{10} E_{11} E_{12} E_{13} E_{14} E_{15} E_{16} - E_4 E_5 E_6 E_7 E_8 E_9 E_{10} E_{11} E_{12} E_{13} E_{14} E_{15} E_{16} E_{17} E_{18} E_{19} E_{20}.$$ 

This binomial corresponds to the primitive monomial walk (see Definition 5.7) depicted below, where the three copies of the vertex $x$ should be identified, but are shown separately for better visibility.

![Figure 3. A primitive monomial walk of $H$](image)

In particular, combining Theorem 5.3 with Corollary 4.4 we obtain the following:

Corollary 5.5. If the set of edges of a hypergraph $H$ can be partitioned into bouquets with bases then the following sets of binomials coincide:

1. the Graver basis of $H$,
2. the universal Gröbner basis of $H$,
3. any reduced Gröbner basis of $H$,
4. any minimal Markov basis of $H$.

Remark 5.6. As a second application of Corollary 4.4 we give a new class of hypergraphs which satisfy the conclusion of Corollary 5.5, and whose building blocks are not necessarily bouquets with bases. Let $H = (V, E)$ be a hypergraph such that there exists $U \subset V$ with the property that $U \cap E \neq \emptyset$ for all $E \in E$ and every vertex of $U$ belongs to exactly two edges. Denote by $U = \{v_1, \ldots, v_t\} \subset \{v_1, \ldots, v_m\} = V$, $E = \{E_1, \ldots, E_n\}$ and let $B$ be a non-free bouquet of $A_H$ (if such a $B$ does not exist then $I_H = 0$ and we are done). This implies that there exists an $i \in [n]$ such that
\( \alpha_{E_i} \in B \). By definition of \( \mathcal{H} \) there exists \( v_j \in E_i \cap E_k \) with \( j \leq m \) and \( k \neq i \). Then the vector \( u \in \mathbb{Z}^m \), whose only nonzero coordinate is 1 on the \( j \)-th position satisfies the following equalities

\[
\begin{align*}
    u \cdot \alpha_{E_i} &= 1, \\
    u \cdot \alpha_{E_k} &= 1, \\
    u \cdot \alpha_{E_l} &= 0 
\end{align*}
\]

for all \( l \neq i, k \).

Therefore the vector \( c_{ik} := (u \cdot \alpha_{E_1}, \ldots, u \cdot \alpha_{E_n}) \) has support \( \{i, k\} \) and \( G(\alpha_{E_i}) + G(\alpha_{E_k}) = 0 \), see Remark \( 1.3 \). Since \( B \) is non-free, \( G(\alpha_{E_i}) = -G(\alpha_{E_k}) \neq 0 \), and thus \( B \) is mixed. Therefore by Corollary \( 4.4 \) we obtain the desired conclusion, and in particular we recover [27, Proposition 4.5]. Imposing in the definition of \( \mathcal{H} \) that \( U = V \), thus making \( \mathcal{H} \) a 2-regular hypergraph, we also recover [27, Proposition 4.2].

Of course, general hypergraphs do not admit a partition of their edge sets into bouquets with bases, let alone mixed subbouquets, as it was shown earlier for \( K_4 \). Infinitely many such examples can be constructed; see Section \( 7 \) for details.

5.2. **Sunflowers.** In this subsection we identify some interesting examples of bouquets with bases, namely, the so-called ‘sparse bouquets’ from [38] (of which there was no formal definition!) These hypergraphs are built on sunflowers. The sunflower is highly structured and useful in the hypergraph literature; for example, it is guaranteed to occur in hypergraphs with large enough edge sets, independently of the size of the vertex set (see e.g. [30].)

General properties of bouquets with bases studied in the previous subsection allow us to not only recover the theorems about existence of Graver basis elements, but also to: 1) describe completely their Graver basis elements by identifying the bouquet ideal, thus the \( a_{E_i} \)'s, 2) show that these sunflowers are actually \( \emptyset \)-Lawrence, and 3) identify Graver basis elements of any hypergraphs which have sunflowers as subhypergraphs. In addition, unlike [38], we do not specialize to uniform hypergraphs.

In Section \( 1 \) it was shown that the bouquet graph of \( A \) encodes the Graver basis of \( I_A \). On the other hand, [38] showed that the Graver basis of \( I_{\mathcal{H}} \) is encoded by primitive monomial walks on the hypergraph. The two concepts are consolidated in the following definition.

**Definition 5.7.** Let \( (\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}) \) be a multiset collection of edges of \( \mathcal{H} = (V, \mathcal{E}) \). We denote by \( \deg_{\mathcal{E}_{\text{blue}}} (v) \) and \( \deg_{\mathcal{E}_{\text{red}}} (v) \) the number of edges of \( \mathcal{E}_{\text{blue}} \) and \( \mathcal{E}_{\text{red}} \) containing the vertex \( v \), respectively. We say that \( (\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}) \) are balanced on \( U \subset V \) if \( \deg_{\mathcal{E}_{\text{blue}}} (v) = \deg_{\mathcal{E}_{\text{red}}} (v) \) for each vertex \( v \in U \).

The vector \( a_{\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}} = (\deg_{\mathcal{E}_{\text{blue}}} (v) - \deg_{\mathcal{E}_{\text{red}}} (v))_{v \in V} \) is called the vector of imbalances of \( (\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}) \) and its support is contained in the complement of \( U \) in \( V \). If \( a_{\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}} = 0 \) then we say that \( (\mathcal{E}_{\text{blue}}, \mathcal{E}_{\text{red}}) \) is a monomial walk.\(^2\)

\(^2\)For completeness, note that the support of a monomial walk, considered as a multi-hypergraph, was called a monomial hypergraph in [38], but this definition does not make an appearance in our results. Instead, we focus on bouquets with bases and the corresponding \( U_\mathcal{E} \) and \( \mathcal{E}_U \).
Every monomial walk encodes a binomial $f_{E_B, E_R} = \prod_{E \in E_B} E - \prod_{E \in E_R} E$ in $I_H$. A monomial walk $(E_B, E_R)$ is said to be primitive if there do not exist proper sub-multisets $E_B' \subset E_B$ and $E_R' \subset E_R$ such that $(E_B', E_R')$ is also a monomial walk. The toric ideal $I_H$ is generated by binomials corresponding to primitive monomial walks, see [38, Theorem 2.8].

**Remark 5.8.** Let $H = (V, E)$ be a hypergraph, $U \subset V$ such that $E_U$ is a bouquet with basis. Since the toric ideal of the (multi)hypergraph $(U, E_U)$ is principal generated by $e^{c^+} - e^{c^-}$ with $\text{supp}(c) = E_U$, the primitive monomial walk $(E_{blue}, E_{red})$ encoded by $(U, E_U)$ is the following: $E_{blue}$ is the multiset consisting of the edges $e_1, e_2, \ldots, e_r$ with multiplicities $c_1, c_2, \ldots, c_r$, respectively, while $E_{red}$ is the multiset consisting of the edges $e_{r+1}, \ldots, e_i$ with multiplicities $-c_{r+1}, \ldots, -c_i$, respectively. Here $c_1, c_2, \ldots, c_l$ are the positive coordinates of $c$, while $c_{r+1}, \ldots, c_l$ are the remaining coordinates of $c$, all negative. If we consider $E_{blue}$ to be the multiset collection of edges $E_1, E_2, \ldots, E_r$ with multiplicities $c_1, c_2, \ldots, c_r$, respectively, and $E_{red}$ the multiset collection of edges $E_{r+1}, \ldots, E_i$ with multiplicities $-c_{r+1}, \ldots, -c_i$, respectively then $(E_{blue}, E_{red})$ are balanced on $U$. Moreover, notice that the vector of imbalances $\omega_{E_{blue}, E_{red}}$ equals $\omega_{E_U}$, and as explained before $c_{E_U}$ determines $(E_{blue}, E_{red})$. For example, considering the bouquet with basis $E_U$ from Example 5.2(a), the toric ideal of $(U, E_U)$ was generated by $e_{13}e_{14}e_{15}e_{16} - e_{17}e_{18}e_{19}e_{20}$, and thus the corresponding $E_{blue}$ and $E_{red}$ are depicted with the corresponding colors in the rightmost part of Figure 3.

Recall that a matching on a hypergraph $H = (V, E)$ is a subset $M \subset E$ of pairwise disjoint edges. A matching is called perfect if it covers all the vertices of the hypergraph. A hypergraph is said to be connected if its primal graph is connected, where the primal graph has the same vertex set as the hypergraph and an edge between any two vertices contained in the same hyperedge.

A hypergraph $H = (V, E)$ is a sunflower if, for some vertex set $C$, $E_i \cap E_j = C$ for all edges $E_i, E_j \in E, i \neq j$ and $C \subseteq E$ for all edges $E \in E$. The set of vertices of $C$ is called the core of the sunflower, and each $E_i$ is called a petal. A matched-petal sunflower is a hypergraph consisting of a sunflower and a perfect matching on the non-core vertices. Note that the set of edges of a matched-petal sunflower partitions into the edges of sunflower, i.e., petals, and the edges of the matching, while its set of vertices is just the set of the vertices of the sunflower. A matched-petal sunflower $H$ is called connected if the (multi)hypergraph $H - C$ is connected, where $C$ represents the core vertices. Here, $H - C$ is the (multi)hypergraph consisting of the restricted sunflower: $(V \setminus C, E')$ where $E' = \{ E \setminus C : E \in E \}$, and the edges of the perfect matching of $H$.

A matched-petal partitioned-core sunflower is a hypergraph $H$ consisting of a collection of vertex-disjoint sunflowers $S_1, S_2, \ldots, S_l$ and a perfect matching on the

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3 Let us also relate this definition to [38], where the authors defined a monomial sunflower: the multihypergraph with $\omega_{E_{blue}, E_{red}} = 0$ whose support is a matched-petal sunflower. However, there is an important distinction: not every matched-petal sunflower is the support of a monomial sunflower. (As a monomial sunflower is an example of a monomial hypergraph, this definition also isn’t used in this manuscript.)
union of non-core vertices, that is $\cup_i (S_i \setminus C_i)$. A matched-petal partitioned-core sunflower is called connected if $\mathcal{H} - C$ is connected, where $C = \cup_i C_i$. A matched-petal relaxed-core sunflower is a hypergraph $\mathcal{H}$ consisting of a collection of sunflowers $S_1, S_2, \ldots, S_l$ with cores $C_1, \ldots, C_l$ respectively, which may only intersect at their cores, and a perfect matching on the union of the non-core vertices of the sunflowers. A matched-petal relaxed-core sunflower is called connected if the (multi)hypergraph $\mathcal{H} - C$ is connected, where $C = \cup_i C_i$.

**Remark 5.9.** The definitions above resemble various monomial walks (based on various types of sunflowers) introduced in [38]. In contrast, here we allow hypergraphs to be non-uniform and consider the supporting sunflowers as sets instead of multisets.

As far as the standard terminology is concerned, note that a matched-petal sunflower is a particular case of matched-petal partitioned-core sunflower, which in turn is a particular case of matched-petal relaxed-core sunflower. Each of the first two subhypergraphs from Figure 2 are connected matched-petal sunflowers, while the third one is not. If we identify the vertices $x_1, x_2$ and $x_3$ from Figure 4 then the hypergraph consisting of the three depicted connected matched-petal sunflowers is a non-connected matched-petal sunflower.

**Proposition 5.10.** Every connected matched-petal relaxed-core sunflower is a bouquet with basis, where the basis consists of the non-core vertices.

**Proof.** Let $\mathcal{H}$ be a connected matched-petal relaxed-core sunflower consisting of $l$ sunflowers $S_1, \ldots, S_l$ with $C_i$ being the core vertices of sunflower $S_i$ for all $i$, and let $U = \cup_i S_i \setminus C_i$. Note that by definition the set of edges $E$ of $\mathcal{H}$ partitions into the set of petals of the sunflowers $S_1, \ldots, S_l$, labeled $E_1, \ldots, E_l$, and the edges of the matching, labeled $E_{l+1}, \ldots, E_k$. In order to prove that $E_U$ is a bouquet with basis, we note first that the vector $c = (1, 1, \ldots, 1, -1, \ldots, -1) \in \mathbb{Z}^k$ corresponds to a binomial in the toric ideal of $(U, \mathcal{U}_E)$, where the number of 1’s equals the number of petals, while the number of −1’s equals the number of edges of the matching. Assume that $u = (u_1, \ldots, u_t, u_{t+1}, \ldots, u_k)$ is an arbitrary vector corresponding to a binomial in the toric ideal of $(U, \mathcal{U}_E)$. It remains to prove that $u_1 = \cdots = u_t$ and $u_{t+1} = \cdots = u_k = -u_1$. Let $E, E' \in \mathcal{U}_E$ be two different edges, restrictions of two petals and thus non-empty, and let $v \in E, v' \in E'$.

Since $\mathcal{H}$ is connected, there exists a path in the primal graph of $(U, \mathcal{U}_E)$ from $v$ to $v'$, that is $v = v_0, v_1, \ldots, v_r = v'$. We construct inductively a sequence of edges of $\mathcal{U}_E$: let $s_1$ be the largest number such that $v_{s_1} \in E := E_1$. Since $E, E'$ are restrictions of petals then $E \cap E' = \emptyset$, and thus $s_1 < r$. By definition of $s_1$, there exists an edge of $\mathcal{U}_E$, say $E_2$, such that $v_{s_1} \in E_1 \cap E_2$. Let $s_2$ be the largest number such that $v_{s_2} \in E_2$. If $s_2 = r$ then $E' = E_2$ and we stop, otherwise we continue. In this way we obtain a sequence of edges $E = E_1, \ldots, E_p = E'$ of $\mathcal{U}_E$ for some $p \geq 2$.

Denote by $u_{ij}$ the coordinate of $u$ corresponding to $E_j$ for all $j = 1, \ldots, p$ (that is $|u_{ij}|$ is the exponent of $E_j$ in the binomial). Since the binomial corresponding to $u$ is in the toric ideal of $(U, \mathcal{U}_E)$, every vertex of $U$ is balanced. Moreover, since every vertex of $U$ belongs to exactly two edges, $v_{s_j}$ being balanced implies that $u_{ij+1} = -u_{ij}$.
for all $j < p$. Thus we obtain $u_{ij} = (-1)^{i+j+1}u_1$ for all $j = 1, \ldots, p$, and in particular, if $E_j$ is the restriction of a petal then $E_{j+1}$ is an edge of a matching. Therefore, we obtain that for any distinct edges $E, E'$ of $\mathcal{U}_E$ the corresponding coordinates of $\mathbf{u}$ are either equal or negatives of each other, with equality holding if and only if $E, E'$ are simultaneously either restrictions of petals or edges of the matching $M$. Hence we get that $\mathbf{u} = (u_1, \ldots, u_1, -u_1, \ldots, -u_1)$, which implies that the toric ideal of $(\mathcal{U}, \mathcal{U}_E)$ is principal, and generated by $e^{c^+} - e^{c^-}$. Thus $\mathcal{E}_U$ is a bouquet with basis, as desired.

Since the connected components of a matched-petal relaxed-core sunflower are connected matched-petal relaxed-core sunflowers, then by Proposition 5.10 and Corollary 5.3 we obtain the following:

**Proposition 5.11.** Let $\mathcal{H}$ be a matched-petal relaxed-core sunflower. Then $I_{\mathcal{H}}$ is $\emptyset$-Lawrence.

In particular, one recovers [38, Theorem 4.12] and implicitly [38, Proposition 4.5] and [38, Proposition 4.9]. To see this, we first identify the subbouquets of a matched-petal relaxed-core sunflower $\mathcal{H}$ and their corresponding $\mathbf{a}$-vectors. If $\mathcal{H}_1, \ldots, \mathcal{H}_t$ are the connected components of $\mathcal{H}$, that is matched-petal relaxed-core sunflowers with the sets of core vertices $C_1, \ldots, C_t$, then denote by $C = \{v_1, \ldots, v_s\}$ the union $\cup_i C_i$ of core vertices of $\mathcal{H}$. Moreover, we label the edges such that petals are labeled first, while the edges of the matching are labeled last. By Proposition 5.10 $\mathcal{H}_1, \ldots, \mathcal{H}_t$ are bouquets with bases, the bases are the sets of non-core vertices, and the vectors $\mathbf{a}_{\mathcal{H}_i}$ can be computed as vectors of imbalances induced by $c_{\mathcal{H}_i}$, as explained in Remark 5.8. Therefore, if we label the rows of the incidence matrix of $\mathcal{H}$ first according to the vertices from $C$, then it follows from Remark 5.8 that for each $j$ the vector $\mathbf{a}_{\mathcal{H}_j}$ has at most the first $s$ components nonzero and for each $i = 1, \ldots, s$ the $i$-th coordinate of $\mathbf{a}_{\mathcal{H}_j}$ is equal to $d_{ij}$, where $d_{ij}$ is the number of petals of $\mathcal{H}_j$ containing the vertex $v_i$. Now it is obvious via Theorem 1.11 that $I_{\mathcal{H}} \neq 0$ if and only if $I_D \neq 0$, where $D$ is the matrix $(d_{ij}) \in \mathbb{Z}^{s \times t}$, and their Graver basis are in bijective correspondence, and this is essentially the content of [38, Theorem 4.12].

**Example 5.12.** Consider $\mathcal{H}$ to be the matched-petal relaxed-core sunflower whose three connected components $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are depicted below, and with cores $C_i = \{x_i\}$ for $i = 1, 2, 3$. Note that each $\mathcal{H}_i$ is a connected matched-petal sunflower, and thus a bouquet with basis by Proposition 5.10. If $x_i \neq x_j$ for all $i \neq j$ then the union of core vertices is $C = \{x_1, x_2, x_3\}$ and the matrix $(d_{ij}) \in \mathbb{Z}^{3 \times 3}$ is the displayed matrix $D_1$. We recall from the previous paragraph that $d_{ij}$ is the number of petals of $\mathcal{H}_j$ containing $x_i$. Thus in this case $I_\mathcal{H} = 0$.

$$
D_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 3 & 4 & 5 \end{pmatrix}
$$

If $x_1 = x_2 \neq x_3$ then the union of core vertices is $C = \{x_1, x_3\}$ and we obtain the corresponding matrix $D_2 \in \mathbb{Z}^{2 \times 3}$. Therefore in this case $I_\mathcal{H} \neq 0$ is principal. Finally, if $x_1 = x_2 = x_3$ then $C = \{x_1\}$ and the matrix $D_3$ is given above.
Specializing the previous discussions to a matched-petal sunflower $H$ with $t \geq 1$ connected components one obtains the following classification. The toric ideal $I_H = 0$ if and only if $t = 1$, that is $H$ is a connected matched-petal sunflower. The toric ideal $I_H \neq 0$ if and only if $t > 1$, in which case the subbouquet ideal is just the toric ideal of the monomial curve $(d_1 \ldots d_t)$, where $d_i$ represents the number of petals of the $i$-th connected component, containing the core.

5.3. Graver basis elements of hypergraphs. In cases of interest, it may happen that all (or almost all) bouquets are singletons, seemingly implying that the bouquet construction does not offer anything. However, this is not the case since by [41, Proposition 4.13] if we restrict to a submatrix $C$ of $A$ obtained by deleting some of the columns of $A$ then the Graver basis, universal Gröbner basis and circuits of $C$ are included in the corresponding ones of $A$. Based on this and the bouquet techniques we construct in Proposition 5.13 an element in the Graver basis of the uniform complete hypergraph with large enough number of vertices, which is not in the universal Gröbner basis.

In [21, 45] it was shown that the universal Gröbner basis and the Graver basis of the toric ideal of the complete graph $K_n$ are identical for $n \leq 8$ and differ for $n \geq 9$. As one application of the bouquet technique, we show that the universal Gröbner basis and the Graver basis of the toric ideal of the complete $d$-uniform hypergraph $K_n^d$ differ for $n \geq (d+1)^2$, by giving a non trivial example of an element in the Graver basis which does not belong to the universal Gröbner basis. First note that all of the bouquets of the complete $d$-uniform hypergraph $K_n^d$, for large $n$, are singletons. Then, restricting to a subhypergraph $H$ of $K_n^d$, as explained before, we can apply the bouquet techniques to find an element from the Graver basis of $H$ that belongs also to the Graver basis of $K_n^d$.

We consider a hypergraph $H_{d+1} = (V, E)$ with $(d+1)^2$ vertices $V = \{v_{ij} | 1 \leq i, j \leq d+1\}$ and $(d+2)(d+1)$ edges: let $E_j = \{v_{ki} | 1 \leq k \leq d+1, k \neq j\}$ for $1 \leq j \leq d+1$ and $E_{ij} = \{v_{ik} | 1 \leq k \leq d+1, k \neq j\}$ for $1 \leq i, j \leq d+1$. The hypergraph $H_4$, for $d = 3$, is depicted in Figure 5.

Note that $H_{d+1}$ is a subhypergraph of $K_n^d$ for $n \geq (d+1)^2$. In addition the set of edges of $H_{d+1}$ partitions into $d+1$ bouquets with bases $E_{U_1}, \ldots, E_{U_{d+1}}$ and $d+1$ single edges $E_1, \ldots, E_{d+1}$. Here $E_{U_i} = \{E_{ij} | 1 \leq j \leq d+1\}$ and $U_i = \{v_{ij} | 2 \leq j \leq d+1\}$ for all $i = 1, \ldots, d+1$, while the principal generator of the toric ideal of the hypergraph $(U_i, U_{e^i})$ is the binomial $\prod_{j \neq 1} e_{ij}^j - e_{i1}^{d-1}$, see Definition 5.1. Moreover, the incidence matrix of $H_{d+1}$ has the rows indexed by the vertices in the following
way: \( v_{11}, v_{21}, \ldots, v_{d+1,1}, v_{12}, v_{22}, \ldots, v_{d+1,d+1} \) and the columns indexed by the edges in the following way \( E_{11}, E_{12}, \ldots, E_{1,d+1}, E_{21}, E_{22}, \ldots, E_{d+1,d+1}, E_{1}, \ldots, E_{d+1} \). With respect to this labeling of the edges and similarly to the computations from Example 5.2(b) we obtain that \( c_{E_{i1}} = (d - 1, -1, \ldots, -1, 0, \ldots, 0) \), \( c_{E_{i2}} = (0, d - 1, -1, \ldots, -1, 0, \ldots, 0) \), and so on \( c_{E_{i,d+1}} = (0, \ldots, 0, d - 1, -1, \ldots, -1, 0) \) are vectors of \( \mathbb{Z}^{(d+2)(d+1)} \), where \( 0 \in \mathbb{Z}^{d+1} \) represents the vector with all coordinates zero. By Remark 5.8 the vector \( a_{E_{i1}} \in \mathbb{Z}^{(d+1)^2} \) has all coordinates zero except the one corresponding to the vertex \( v_{i1} \), which equals \(-d\). For the singleton subbouquets \( E_{1}, \ldots, E_{d+1} \), the encoding vectors are: \( c_{E_{1}} = (0, \ldots, 0, 1, 0, \ldots, 0) \), \( \ldots \), \( c_{E_{d+1}} = (0, \ldots, 0, 0, \ldots, 0, 1) \), while \( a_{E_{i}} = \alpha_{E_{i}} \) for all \( i = 1, \ldots, d + 1 \).

![Figure 5. \( \mathcal{H}_4 \)](image)

For the rest of this subsection we use for simplicity the binomial representation for an element in the toric ideal of a hypergraph instead of the vector representation. It is easy to see that the following three types of binomials belong to the toric ideal of the complete \( d \)-uniform hypergraph \( K_{n}^{d} \):

a) \( d + 1 \) binomials of the form

\[
g_{i} := E_{i}^{d-1} \prod_{j \neq 1} E_{ij} - E_{i1}^{d-1} \prod_{j \neq i} E_{j},
\]

b) \( d + 1 \) binomials of the form

\[
h_{i} := \prod_{i \neq l} \prod_{j \neq 1} E_{ij} - E_{l}^{d} \prod_{i \neq l} E_{i1}^{d-1},
\]

and c) one binomial of the form

\[
\prod_{i \neq l} \prod_{j \neq 1} E_{ij} - \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1}.
\]

Furthermore, it can be shown that they are all elements of the Graver basis of \( K_{n}^{d} \), although we will prove next just for the last binomial.

**Proposition 5.13.** The universal Gröbner basis of the toric ideal of the complete \( d \)-uniform hypergraph \( K_{n}^{d} \) differs from the Graver basis for \( n \geq (d + 1)^2 \).
\begin{proof}
First we prove that the binomial
\[ f = \prod_{i} \prod_{j \neq 1} E_{ij} - \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1} \]
does not belong to the universal Gröbner basis of the toric ideal of the complete \( d \)-uniform hypergraph \( K_n^d \). We argue by contradiction. Assume that there exists a monomial order \( \succ \) such that the binomial \( f \) belongs to the reduced Gröbner basis with respect to the order \( \succ \). Since \( E_{i1}^{d-1} \prod_{j \neq i} E_{j} \) divides properly \( \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1} \), it follows that
\[ E_{i}^{d-1} \prod_{j \neq 1} E_{ij} > E_{i1}^{d-1} \prod_{j \neq i} E_{j}. \]
Indeed, if this is not the case then \( \text{in}_{\prec}(g_{i}) = E_{i1}^{d-1} \prod_{j \neq i} E_{j} \). From the previous divisibility it follows that: 1) if \( \text{in}_{\prec}(f) = \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1} \) then we contradict that \( f \) is in the Gröbner basis, or 2) if \( \text{in}_{\prec}(f) = \prod_{i} E_{i} \prod_{j \neq 1} E_{ij} \) then we contradict that \( f \) is reduced. Thus we obtain the inequality (5).

Also \( \prod_{i \neq i} \prod_{j \neq 1} E_{ij} \) divides \( \prod_{i} \prod_{j \neq 1} E_{ij} \), and similarly to the proof of (5) it can be shown that
\[ \prod_{i \neq 1} \prod_{j \neq 1} E_{ij} < \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1}. \]
Taking the product of inequalities (5) when \( i \) runs from 1 to \( d + 1 \) and canceling common terms we obtain
\[ \prod_{i} \prod_{j \neq 1} E_{ij} > \prod_{i} E_{i} \prod_{i} E_{i1}^{d-1}. \]
Similarly, taking the product of inequalities (6) we obtain
\[ (\prod_{i} \prod_{j \neq 1} E_{ij})^{d} < (\prod_{i} E_{i} \prod_{i} E_{i1}^{d-1})^{d}, \]
a contradiction.

In order to prove that the binomial \( f \) belongs to the Graver basis of \( K_n^d \) it is enough to prove via [41, Proposition 4.13] that \( f \) belongs to the Graver basis of its subhypergraph \( \mathcal{H}_{d+1} \). The partition of the edges of \( \mathcal{H}_{d+1} \) into \( d + 1 \) bouquets with bases \( E_{U_{1}}, \ldots, E_{U_{d+1}} \) and \( d + 1 \) single edges \( E_{1}, \ldots, E_{d+1} \) induces the matrix \( A_B \in \mathbb{Z}^{(d+1)^2 \times 2(d+1)} \)
\[ A_B = [a_{E_{U_{1}}}, \ldots, a_{E_{U_{d+1}}}, a_{E_{1}}, \ldots, a_{E_{d+1}}] = \begin{pmatrix}
-d & 0 & \ldots & 0 & 0 & 1 & \ldots & 1 \\
0 & -d & \ldots & 0 & 1 & 0 & \ldots & 1 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & -d & 1 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}, \]
where \( a_{E_{i}} = \alpha_{E_{i}} \) for all \( i \), the first \( d + 1 \) rows are indexed after the vertices \( v_{11}, \ldots, v_{d+1,1} \) and the last row represents block matrix \( 0 \in \mathbb{Z}^{d(d+1) \times 2(d+1)} \). As it
was noticed in the comments prior to this proposition the binomial $f$ corresponds
to the vector $v = (1 - d, 1, \ldots, 1 - d, 1, \ldots, 1 - d, 1, \ldots, 1, -1, \ldots, -1) \in \text{Ker}(A_{d+1})$.

Applying Theorem 1.9 we have a bijective correspondence between \( \text{Ker}(A_B) \) and \( \text{Ker}(A_{d+1}) \) and given by

\[
B(u_1, \ldots, u_{2d+2}) = \sum_{i=1}^{d+1} c_{E_i} u_i + \sum_{i=1}^{d+1} c_{E_i} u_{d+1+i}.
\]

Therefore replacing the formulas obtained before for $c_{E_i}$ and $c_{E_i}$, for all $i = 1, \ldots, d + 1$, we obtain that $v = B(u)$ where $u = (1, \ldots, 1, -1, \ldots, -1) \in \mathbb{Z}^{2d+2}$ with equally many 1’s and −1’s. By Theorem 1.11 it is enough to prove that $u$ belongs to the Graver basis of $A_B$ to conclude that $v$ (and thus $f$) is in the Graver basis of $H_{d+1}$. Assume by contradiction $u = u^+ - u^-$ is not in the Graver basis of $A_B$. Then there exists $0 \neq w \in \text{Ker}(A_B)$ such that $w^+ \preceq u^+$ and $w^- \preceq u^-$, with at least one inequality strict, see Section 1. In terms of coordinates this can be restated: $w_i \in \{0, 1\}$ for all $i = 1, \ldots, d + 1$ and $w_i \in \{-1, 0\}$ for all $i = d + 2, \ldots, 2d + 2$, and with at least one coordinate zero. If $w^+ = u^+$ then since $w \in \text{Ker}(A_B)$ we obtain $w^- = u^-$, a contradiction. Otherwise there exists $i \in \{1, \ldots, d + 1\}$ such that $w_i = 0$, and without loss of generality we may assume that $w_1 = 0$. Since $w \in \text{Ker}(A_B)$ then $w_{d+3} + \cdots + w_{2d+2} = 0$, and thus $w_{d+3} = \ldots = w_{2d+2} = 0$. If $w_{d+2} = 0$ then $w = 0$, a contradiction, so we necessarily have $w_{d+2} = -1$. This leads to $-dw_i = 1$ for all $i = 1, \ldots, d + 1$ a contradiction to $w \in \mathbb{Z}^{2d+2}$. Therefore $u$ belongs to the Graver basis of $A_B$, and we are done.

6. Complexity of hypergraphs

It is well-known that the elements of the Graver basis of a toric ideal of a graph are of a rather special form: the exponents of each variable in an element of the Graver basis is either one or two, see, for example, [39] and references therein, and are completely determined by their support. In other words, one can not find two elements in the Graver basis of a toric ideal of a graph with the same support. In [38] it was shown that exponents in the elements of the Graver basis of a toric ideal of a hypergraph can be arbitrarily high, and are not uniquely determined by their supports, see Example 5.4. Thus the natural question is: how complicated are the Graver basis, the universal Gröbner basis or a Markov basis of a hypergraph?

The bouquet ideal technique can be used to prove that toric ideals of hypergraphs are as complicated as toric ideals in general. Namely, for any toric ideal, there exists a toric ideal of a hypergraph with “worse” Graver basis, universal Gröbner basis and Markov bases, as well as circuits. “Worse” means that the corresponding set for the toric ideal of a hypergraph has at least the same cardinality and elements of higher degrees than the corresponding elements of the toric ideal.

We start first with an example, which shows the details of the general construction.
Example 6.1. Let
\[ A = \begin{pmatrix} -1 & -1 & 2 & 2 \\ -2 & 2 & -1 & 0 \end{pmatrix}, \]
be the matrix whose columns are denoted by \( a_1, \ldots, a_4 \). We construct a hypergraph \( H = (V, E) \) such that the bouquet graph of \( H \) has four bouquets with basis \( E_{U_1}, \ldots, E_{U_4} \), and with the property that the corresponding subbouquet ideal (associated to the vectors \( a_{E_{U_1}}, \ldots, a_{E_{U_4}} \)) is \( I_A \). The latter will follow since the vectors \( a_{E_{U_1}}, \ldots, a_{E_{U_4}} \) are “essentially” the vectors \( a_1, \ldots, a_4 \), in the sense that each \( a_{E_{U_i}} \in \mathbb{Z}^{|V|} \) is just the natural embedding of \( a_i \) in \( \mathbb{Z}^{|V|} \) with the other coordinates 0. The construction of \( H \) is carried out in three steps.

**Step 1.** Every non-zero entry of the matrix \( A \) is used to construct a sunflower, which will be the building blocks of the desired hypergraph. Precisely, for a positive entry \( \lambda \) of the matrix \( A \) we consider a 3-uniform sunflower with one core vertex and \( |\lambda| \) petals, while for a negative entry \( \lambda \) we consider the sunflower with one core vertex and \( |\lambda| + 1 \) petals, which is almost 3-uniform, meaning that only one petal has two vertices, the other have three. In the particular case of our example the matrix \( A \) has three different nonzero entries \(-1, -2, 2\), and thus we have two almost 3-uniform sunflowers and one 3-uniform sunflower pictured below, see Figure 6.

![Figure 6](image)

**Step 2.** Next we construct for each column \( a_j \) a connected matched-petal partitioned-core sunflower \( H_j \). For this we use the sunflowers constructed in **Step 1** in the following way: if \( a_{ij} > 0 \) we use the previously constructed sunflower, otherwise the sunflower obtained by deleting the petal with two vertices from the corresponding sunflower. Finally, \( H_j \) consists of these disjoint 3-uniform sunflowers, and a perfect matching on the union of non-core vertices. Then we add to \( H_j \) the deleted petals with two vertices to obtain \( H'_j \). For example, since \( a_1 = (-1, -2) \) we take the sunflowers corresponding to \(-1\) and \(-2\) from **Step 1** (see the leftmost picture of Figure 7), then delete the petals with two vertices to construct the connected matched-petal partition-core sunflower \( H_1 \) (see the middle picture of Figure 7), and finally, add the two petals of cardinality 2 to obtain \( H'_1 \) (the rightmost picture of Figure 7).

For sake of completeness, we have labeled the vertices of sunflowers from Figure 7 according to the general construction of Theorem 6.2 and we denote their edges by \[ e_0(11) = \{u(11), v_1\}, e_1(11) = \{u(11), v_1(11), v_2(11)\}, e_0(21) = \{u(21), v_2\}, e_1(21) = \{u(21), v_1(21), v_2(21)\}, e_2(21) = \{u(21), v_3(21), v_4(21)\}, e_1 = \{v_2(11), v_1(21)\}, e_2 = \{v_2(21), v_3(21)\}, e_3 = \{v_4(21), v_1(11)\}. \] One should note that there is not a unique
The column vectors of the incidence matrix of H correspond, in this order, to the edges e1, e0, e4, e2, and e3, which is a bouquet with basis \( H'_1 \) constructed in Figure 7, the others being computed similarly. We assume that the column vectors of the incidence matrix of H are indexed such that the first eight correspond, in this order, to the edges \( e_1(11), e_0(11), e_4(21), e_2(21), e_0(21), e_1, e_2, e_3 \), while the first two rows are indexed by the vertices \( v_1 \) and \( v_2 \).

Finally, consider the hypergraph \( H = (V, E) \) with \(|V| = 28 \) and \(|E| = 26\), obtained from \( H'_1, \ldots, H'_4 \) by taking the vertex set to be the union of the set of vertices of \( H'_1, \ldots, H'_4 \), and the set of edges is the (disjoint) union of the set of edges of \( H'_1, \ldots, H'_4 \). Furthermore, we compute the subbouquet vectors of \( H'_1, \ldots, H'_4 \). For this, we analyze in detail only the first hypergraph \( H'_1 \) which is a bouquet with basis \( U_1 \), constructed in Figure 7, the others being computed similarly. We assume that the column vectors of the incidence matrix of H are indexed such that the first eight correspond, in this order, to the edges \( e_1(11), e_0(11), e_4(21), e_2(21), e_0(21), e_1, e_2, e_3 \), while the first two rows are indexed by the vertices \( v_1 \) and \( v_2 \).

Then \( c_{H'_1} = (1, -1, 1, 1, -2, -1, -1, -1, 0) \in \mathbb{Z}^{26} \) and

\[
a_{H'_1} = e_1(11) - e_0(11) + e_4(21) + e_2(21) - 2e_0(21) - e_1 - e_2 - e_3 = (-1, -2, 0) \in \mathbb{Z}^{28}.
\]

Analogously, with respect to a similar order for the rest of the edges of the other three bouquets, we obtain the following \( c_{H'_2} = (0, 1, -1, 1, 1, -1, -1, -1, 0, 0) \), \( c_{H'_3} = \)
Theorem 6.2. Given any integer matrix $A$ without any zero row or zero column, there exists a hypergraph $\mathcal{H} = (V, E)$ such that:

1. There is a bijective correspondence, $u \mapsto B(u)$, between $\text{Ker}_Z(A)$ and $\text{Ker}_Z(\mathcal{H})$, and between $\text{Gr}(A)$ and $\text{Gr}(\mathcal{H})$.

2. For every $u \in \text{Gr}(A)$ we have $\deg(x^{u^+} - x^{u^-}) \leq \deg(x^{B(u^+)} - x^{B(u^-)})$.

3. The toric ideal $I_\mathcal{H}$ is $\emptyset$-Lawrence, i.e. the following four sets of $I_\mathcal{H}$ coincide:
   - the Graver basis of $\mathcal{H}$,
   - the universal Gröbner basis of $\mathcal{H}$,
   - any reduced Gröbner basis of $\mathcal{H}$,
   - any minimal Markov basis of $\mathcal{H}$.

Proof. Let $A = [a_1, \ldots, a_n] \in \mathbb{Z}^{m \times n}$. We will construct a hypergraph $\mathcal{H}$ whose subbouquet ideal is $I_A$, and with the property that all its non-free subbouquets are mixed. This will imply at once conditions (1) and (3), by Theorems 1.1.1 and Corollary 5.5. To this end, let $\{v_1, \ldots, v_m\}$ be a set of vertices, and for each nonzero entry of the matrix $a_{ij}$ we introduce the following new vertices:

1. if $a_{ij} > 0$ the set $\{v_1(ij), v_2(ij), \ldots, v_{2a_{ij}}(ij)\}$ of $2a_{ij}$ vertices,
2. if $a_{ij} < 0$ the set $\{u(ij), v_1(ij), v_2(ij), \ldots, v_{-2a_{ij}}(ij)\}$ of $-2a_{ij} + 1$ vertices,

as well as the following edges:

3. if $a_{ij} > 0$ the $a_{ij}$ edges $e_s(ij) = \{v_i, v_{2s-1}(ij), v_{2s}(ij)\}$, where $1 \leq s \leq a_{ij}$,
4. if $a_{ij} < 0$ the $-a_{ij} + 1$ edges $e_s(ij) = \{u(ij), v_{2s-1}(ij), v_{2s}(ij)\}$, where $1 \leq s \leq -a_{ij}$ and $e_0(ij) = \{v_i, u(ij)\}$.

Note that for each $a_{ij} \neq 0$ the hypergraph $S_{ij}$, on the set of vertices defined item (1) (item 2) with the set of edges defined in item (3) (item 4, respectively) is a sunflower with the core $v_i$ (respectively $u(ij)$). Furthermore, each sunflower $S_{ij}$ is either a 3-uniform sunflower if $a_{ij} > 0$ or an almost 3-uniform sunflower if $a_{ij} < 0$ (by almost we mean that all the edges have three vertices except $e_0(ij)$, which has only two vertices). In addition, for any fixed $j$ the sunflowers $S_{1j}, \ldots, S_{mj}$ are vertex disjoint by definition. For each nonzero column $a_j = (a_{1j}, \ldots, a_{mj})$ we construct
Remark 6.3. It follows from the proof of Theorem 6.2 that the hypergraph $H_j = (V_j, \mathcal{E}_j)$ on the set of vertices

$$V_j = \bigcup_{i : a_{ij} > 0} \{v_1(ij), v_2(ij), \ldots, v_{2a_{ij}}(ij), v_i\} \cup \bigcup_{i : a_{ij} < 0} \{v_1(ij), v_2(ij), \ldots, v_{-2a_{ij}}(ij), u(ij)\},$$

with the core vertices

$$C_j = \bigcup_{i : a_{ij} > 0} \{v_i\} \cup \bigcup_{i : a_{ij} < 0} \{u(ij)\}.$$

For this is enough to describe the construction of $H_1 = (V_1, \mathcal{E}_1)$, the others being similar. Consider all nonzero entries of the column $a_1 = (a_{11}, \ldots, a_{m1})$. To each nonzero $a_{11}$ we associate the sunflower $S_{11}$ if $a_{11} > 0$, or the sunflower denoted by $S_{11} \setminus \{e_0(i1)\}$ obtained from $S_{11}$ by removing the edge $e_0(i1)$, if $a_{11} < 0$. Note that $S_{11} \setminus \{e_0(i1)\}$ is a 3-uniform sunflower with core the set $\{u(i1)\}$, and its vertex set is obtained from the vertex set of $S_{11}$ by removing $v_i$. The matched-petal partitioned-core hypergraph $H_1$ with vertex set $V_1$, consists of the vertex-disjoint sunflowers $S_i$ (for those $i$ with $a_{1i} > 0$) and $S_i \setminus \{e_0(i1)\}$ (for those $i$ with $a_{1i} < 0$), and with the following perfect matching on the set $V_j \setminus C_j$, of non-core vertices:

$$\{v_{2(11)}, v_{3(11)}, \ldots, v_{2|a_{11}| - 2(11)}, v_{2|a_{11}|-1(11)}\}, \{v_{2|a_{11}|(11)}, v_{1(21)}\},$$

$$\{v_{2(21)}, v_{3(21)}, \ldots, v_{2|a_{m1}| - 2(m1)}, v_{2|a_{m1}|-1(m1)}\}, \{v_{2|a_{m1}|(m1)}, v_{1(11)}\},$$

where for convenience of notation we assumed that all $a_{1i} \neq 0$ (see for an example the blue edges from Figures 7 and 8). This perfect matching on the non-core vertices ensures that $H_j = (V_j, \mathcal{E}_j)$ is a connected matched-petal partitioned-core sunflower.

Then we “extend” the hypergraph $H_j$ to the hypergraph $H'_j = (V'_j, \mathcal{E}'_j)$, where $V'_j = V_j \cup \{v_i : a_{ij} < 0\}$ and $\mathcal{E}'_j = \mathcal{E}_j \cup \{e_0(ij) : a_{ij} < 0\}$, which turns out to be a bouquet with basis the set $U_j = V'_j \setminus \{v_i : a_{ij} \neq 0\}$. Alternatively, we can write $H'_j = \mathcal{E}_{U_j}$ for all $j = 1, \ldots, n$. Finally, the hypergraph $H = (V, \mathcal{E})$ we are looking for is obtained from $H'_1, \ldots, H'_n$ as follows $V = \bigcup_{j=1}^n V'_j$, and $\mathcal{E} = \bigcup_{j=1}^n \mathcal{E}'_j$. Note that since $A$ has no zero row, then $\{v_1, \ldots, v_m\} \subset V$. Since each $H'_j$ is a bouquet with basis, then it follows immediately from construction that $H$ has $n$ bouquets with bases. Moreover, the resulting bouquet vector of $H'_j = \mathcal{E}_{U_j}$ is $a_{\mathcal{E}_{U_j}} = (a_j, 0) \in \mathbb{Z}^{|V|}$ and thus we obtain that the subbouquet ideal of $I_H$ is $I_A$, as desired. \hfill \square

Remark 6.3. It follows from the proof of Theorem 6.2 that the hypergraph $H$ has edges of size at most 3, which implies that almost 3-uniform hypergraphs are good enough to capture any strange behavior.

7. Hypergraphs encode all positively graded toric ideals

This section presents a correspondence between positively graded (general) toric ideals and stable toric ideals of hypergraphs. Namely, given a positively graded $I_A$, Theorem 7.2 constructs a stable toric ideal of a hypergraph $H = H(A)$ that has the same combinatorial complexity. In particular, the Graver bases of the ideals $I_A$ and $I_H$ have the same number of elements, and the same holds for their universal Gröbner and Markov bases, as well as indispensable binomials and circuits.
For the remainder of this section, fix the following notation:

\[
\Sigma_n := \begin{pmatrix}
0 & 1 & \ldots & 1 \\
1 & 0 & 1 & \ldots & 1 \\
& \ddots & \ddots & \ddots & \ddots \\
1 & \ldots & 1 & 0 & 1 \\
1 & \ldots & 1 & 0 & 1
\end{pmatrix} \in \mathbb{Z}^{n \times n}
\]

and

\[
\varepsilon_{k,n} := \underbrace{(1 \cdots 1 0 \cdots 0)}_{\text{k times}} \in \mathbb{Z}^n,
\]

where \(1 \leq k < n\).

The following example illustrates a general construction which is the basis for the proof of Theorem 7.2.

**Example 7.1.** Given a non-negative integer matrix \(D\), we seek a procedure to create a 0/1 matrix \(A\) such that there is a bijective correspondence between distinguished sets of binomials of \(I_D\) and \(I_A\), namely, each of: Graver basis, circuits, indispensable binomials, minimal Markov bases, reduced Gröbner bases (universal Gröbner basis).

Let \(D = (d_{ij}) = \begin{pmatrix}
1 & 3 & 2 & 0 & 1 \\
3 & 2 & 1 & 3 & 2 \\
3 & 0 & 2 & 2 & 1
\end{pmatrix} \in \mathbb{N}^{3 \times 5}\).

Let \(\delta_i := \max_j \{d_{ij}\}\) be the maximum entry in column \(i\) of \(D\), and \(j_i := \min\{l : d_{li} = \delta_i\}\) be the index of the first row where \(\delta_i\) appears. For the given matrix \(D\), these values are as follows: \(\delta_1 = 3, \delta_2 = 3, \delta_3 = 2, \delta_4 = 3\) and \(\delta_5 = 2\); and \(j_1 = 2, j_2 = 1, j_3 = 1, j_4 = 2\) and \(j_5 = 2\). Define \(\delta := \sum_{i=1}^{5}(\delta_i + 1) = 18\) and \(l = 3 - |\{j_1, \ldots, j_5\}| = 1\), the number of rows that do not contain any column-maximum entry \(\delta_i\). We construct a 0-1 matrix \(A = M_H\) of size \(19 \times 18\), the incidence matrix of a hypergraph \(H\), such that its subbouquet ideal will be \(I_D\). Here, \(19 = \delta + l\) and \(18 = \delta\). The matrix \(A\) is constructed from \(D\) in two steps.

**Step 1.** Every column-maximum entry \(\delta_i\) defined above will determine a set of horizontal blocks of the matrix \(A\) as follows. For each row index \(k \in \{j_i\}_{i=1}^{5}\), consider the set of all \(\delta_i\)'s appearing on the \(k\)-th row of \(D\). For each such \(\delta_i\), construct a \((\delta_i + 1) \times \delta\) block matrix by concatenating (horizontally) the following 5 sub-blocks: block \(i\) shall consist of the matrix \(\Sigma_{\delta_i+1}\), while for each \(l \neq i\), block \(l\) shall consist of \(\delta_i + 1\) copies of \(\varepsilon_{d_{li},\delta_i+1}\).

For example, the first row of \(D\) contains two column-maximum entries, \(\delta_2\) and \(\delta_3\). The first entry will generate a row-block of size \((3 + 1) \times 18\) inside \(A\), while the second entry will generate a row-block of size \((2 + 1) \times 18\). First, the entry \(\delta_2 = 3\) requires concatenating five sub-blocks: the first one consisting of \(4 = \delta_2 + 1\) copies of \(\varepsilon_{d_{11},4} = (1 0 0 0)\); the second one, \(\Sigma_4\); the third one consisting of 4 copies of \(\varepsilon_{d_{13},4} = (1 1 0 0)\); the fourth one, 4 copies of \(\varepsilon_{d_{14},4} = (0 0 0 0)\); and the fifth, 4 copies of \(\varepsilon_{d_{15},4} = (1 0 0 0)\). This resulting row-block of \(A\) is displayed below, with the distinguished sub-block corresponding to \(\Sigma_4\) colored in blue.
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}.
\]

Similarly, the second entry, \( \delta_3 = 2 \), will generate the following \( 3 \times 18 \) row-block:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The remaining row indices are \( j_4 = j_5 = 2 \), thus we consider the second row of \( D \). It contains three \( \delta_i \) values: \( \delta_1, \delta_4, \delta_5 \), and thus gives rise to three row-blocks of similar structure. These are the third, fourth and fifth row-blocks of the matrix \( A \) in Equation (7).

**Step 2.** For each row \( k \) of \( D \) containing none of the column-maxima \( \delta_i \)'s, that is, for all row indices \( k \in \{1,2,3\} \setminus \{j_i\}_{i=1}^5 \), we will create one additional row of \( A \). This row will be obtained by concatenating (in this order) the following vectors \( \varepsilon_{d_{k1}+1}, \varepsilon_{d_{k2}+1}, \varepsilon_{d_{k3}+1}, \varepsilon_{d_{k4}+1}, \varepsilon_{d_{k5}+1} \). For example, the third row of \( D \) does not contain any \( \delta_i \) and thus the row of \( A \) obtained from concatenating the vectors

\[
\varepsilon_{d_{31}} = (1 1 1 0), \ \varepsilon_{d_{32}} = (0 0 0 0), \ \varepsilon_{d_{33}} = (1 1 1 0), \ \varepsilon_{d_{34}} = (1 1 0 0) \quad \text{and} \quad \varepsilon_{d_{35}} = (1 0 0)
\]

is

\[
(1 1 1 0 | 0 0 0 0 | 1 1 1 0 1 0 0 0 | 1 0 0).
\]

Finally, the matrix \( A = [a_1, \ldots, a_{18}] \) is obtained by concatenating (vertically) the two row-blocks generated by the first row of \( D \), the three row-blocks generated by the second row of \( D \), and the row generated by the third row of \( D \); see Equation (7).

Note that the submatrix of \( A \) generated by the first row of \( D \) has \( 4+3 = 7 \) rows, the one generated by the second row of \( D \) has \( 4+4+3 = 11 \) rows, and the one generated by the third row of \( D \) has 1 row. The matrix \( A \) has 5 non-mixed bouquets such that \( a_1, a_2, a_3, a_4 \) belong to the first bouquet \( B_1 \), \( a_5, a_6, a_7, a_8 \) to \( B_2 \), \( a_9, a_{10}, a_{11} \) to \( B_3 \), \( a_{12}, a_{13}, a_{14}, a_{15} \) to \( B_4 \), and \( a_{16}, a_{17}, a_{18} \) to \( B_5 \). Moreover all nonzero components of \( c_{B_1, \ldots, c_{B_5}} \in \mathbb{Z}^{18} \) are 1, and thus \( a_{B_1} = a_1 + \cdots + a_4 \in \mathbb{Z}^{19} \) is the transposed vector of

\[
(1 1 1 1 1 1 1 1 1 3 3 3 3 3 3 3 3 3 3 3 3),
\]

where the first block has 7 coordinates, the second one 11 coordinates, and the last one 1 coordinate. Similarly, \( a_{B_2} = a_5 + \cdots + a_8 \) is the transposed vector of

\[
(3 3 3 3 3 3 3 3 3 2 2 2 2 2 2 2 2 2 0),
\]

and so on. Thus \( I_A = I_D \), and since \( I_A \) is a stable toric ideal the desired bijective correspondence follows from Theorem 3.7.
\[ (7) \ A = M_\mathcal{H} = \]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

**Theorem 7.2.** Let \( I_D \) be an arbitrary positively graded nonzero toric ideal. Then there exists a hypergraph \( \mathcal{H} \) such that there is a bijective correspondence between the Graver bases, all minimal Markov bases, all reduced Gröbner bases, circuits, and indispensable binomials of \( I_D \) and \( I_\mathcal{H} \).

**Proof.** Since \( I_D \) is a positively graded toric ideal, we may assume by [32, Corollary 7.23] that \( D = (d_{ij}) \in \mathbb{N}^{m \times n} \). Furthermore, every column of \( D \) is nonzero, since, otherwise, if the \( j \)-th column is 0, then \( x_j - 1 \in I_D \), a contradiction to \( I_D \) being positively graded. We will construct the incidence matrix \( A \) of a hypergraph \( \mathcal{H} \), such that its subbouquet ideal is equal to \( I_D \). For this define column-maximum entries \( \delta_i := \max\{d_{ji} : j = 1, \ldots, m\} \) for all \( i = 1, \ldots, n \) and set \( \delta = n + \sum_{i=1}^n \delta_i \). Note that \( \delta_i > 0 \) for every \( i = 1, \ldots, m \). Moreover, denote by \( j_i := \min\{k : d_{ki} = \delta_i\} \) for all \( i = 1, \ldots, n \) and denote by \( l = m - \#\{j_i : 1 \leq i \leq n\} \).

Then the 0-1 matrix \( A \in \mathbb{N}^{(\delta+1) \times \delta} \) consists of several blocks, concatenated vertically, of the following two types: 1) For each \( k \in \{j_1, \ldots, j_n\} \), and for each column-maximum entry \( \delta_i \) located on the \( k \)-th row of \( D \), construct \( \delta_i + 1 \) rows of \( A \) by concatenating (horizontally) \( n \) block matrices. Here, the \( i \)-th block is \( \Sigma_{\delta_i+1} \), while for each \( l \neq i \) the \( l \)-th block consists of \( \delta_i + 1 \) copies of the row sub-vector \( \varepsilon_{d_{ki}, \delta_i+1} \). 2) For each row index \( k \in [m] \setminus \{j_1, \ldots, j_n\} \) of \( D \), that is, each row not containing any column-maximum entry \( \delta_i \), construct a row of \( A \) by by concatenating (in this order) the following vectors

\[ (\varepsilon_{d_{k1}, \delta_1+1}\varepsilon_{d_{k2}, \delta_2+1} \cdots \varepsilon_{d_{kn}, \delta_n+1}). \]
Assume now that the columns of $A$ are labeled $a_1, \ldots, a_5$. We prove that the first $\delta_1 + 1$ column vectors of $A$ belong to the same subbouquet $B_1$, the next $\delta_2 + 1$ belong to the same subbouquet $B_2$, and so on, until the last $\delta_n + 1$ column vectors belong to the same subbouquet $B_n$. We will prove this only for the first $\delta_1 + 1$ columns of $A$, the other cases being similar. By the definition of $A$ we have that in the submatrix of $A$ determined by the columns $a_1, \ldots, a_{\delta_1 + 1}$ there exist integers $i_1, \ldots, i_1 + \delta_1 = i_2$ such that the matrix corresponding to these rows and the columns $a_1, \ldots, a_{\delta_1 + 1}$ is $\Sigma_{\delta_1 + 1}$. We consider the vectors $c_{i,i+1} \in \mathbb{Z}^{i+1}$ for every $i = i_1, \ldots, i_1 + \delta_1 - 1$, whose only nonzero coordinates are $1$ on position $i$, and $-1$ on position $i + 1$. Then the co-vector $(c_{i_1,i_1+1} \cdot a_1, \ldots, c_{i_1,i_1+1} \cdot a_5) = (-1, 1, 0, \ldots, 0)$ has support of cardinality two and $G(a_1) = G(a_2)$. Here, we used the fact that in the horizontal block containing $\Sigma_{\delta_1 + 1}$ (and corresponding to the rows $i_1, \ldots, i_2$) all the entries corresponding to the rows $i_1, i_1 + 1$ and columns $3, \ldots, 5$ are identical, by definition. Similarly, using all the vectors $c_{i,i+1}$ for $i = i_1, \ldots, i_2 - 1$, we obtain $G(a_1) = \cdots = G(a_{\delta_1 + 1})$, and thus they all belong to the same subbouquet $B_1$, which is either free or non-mixed. In any case, the vector $c_{B_1} = (1, 0, \ldots, 0) \in \mathbb{Z}^\delta$ with $1 = (1, \ldots, 1) \in \mathbb{Z}^{\delta_1 + 1}$ and $a_{B_1} = a_1 + \cdots + a_{\delta_1 + 1}$. Analogously, we have $c_{B_2} = (0, 1, \ldots, 0)$ with $1 = (1, \ldots, 1) \in \mathbb{Z}^{\delta_2 + 1}$ and $a_{B_2} = a_{\delta_1 + 2} + \cdots + a_{\delta_1 + \delta_2 + 2}$, and so on, until $c_{B_2} = (0, 0, \ldots, 1)$ with $1 = (1, \ldots, 1) \in \mathbb{Z}^{\delta_n + 1}$ and $a_{B_n} = a_{\delta_n} + \cdots + a_5$. One can easily see from the construction of $A$ that $I_{A_B} = I_D$. In addition, since $I_D \neq 0$ then at least one subbouquet is non-free, and therefore $I_H = I_A$ is a stable toric ideal. Theorem 3.7 now provides the desired conclusion. 

In conclusion, certain problems about arbitrary positively graded toric ideals can be reduced to problems about toric ideals of hypergraphs. For example, in [11] Boocher et. al proved that, for robust toric ideals of graphs, the Graver basis is a minimal Markov basis. Equivalently, robust toric ideals of graphs are $\emptyset$-Lawrence. They ask if this property is true in general for any robust ideal. To prove such a statement, it is enough to prove it only for toric ideals of hypergraphs. Indeed, if $I_A$ is any robust toric ideal then Theorem 7.2 shows that $I_H$ is robust. Then if one can prove that robust ideals of hypergraphs have the property that the Graver basis is a minimal Markov basis then it follows, again from Theorem 7.2 that the ideal $I_A$ has this property.

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