Improved Learning Rates for Stochastic Optimization: Two Theoretical Viewpoints

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Abstract

Generalization performance of stochastic optimization stands a central place in learning theory. In this paper, we investigate the excess risk performance and towards improved learning rates for two popular approaches of stochastic optimization: empirical risk minimization (ERM) and stochastic gradient descent (SGD). Although there exists plentiful generalization analysis of ERM and SGD for supervised learning, current theoretical understandings of ERM and SGD either have stronger assumptions in convex learning, e.g., strong convexity, or show slow rates and less studied in nonconvex learning. Motivated by these problems, we aim to provide improved rates under milder assumptions in convex learning and derive faster rates in nonconvex learning. It is notable that our analysis span two popular theoretical viewpoints: stability and uniform convergence. Specifically, in stability regime, we present high probability learning rates of order $O(1/n)$ w.r.t. the sample size $n$ for ERM and SGD with milder assumptions in convex learning and similar high probability rates of order $O(1/n)$ in nonconvex learning, rather than in expectation. Furthermore, this type of learning rate is improved to faster order $O(1/n^2)$ in uniform convergence regime. To our best knowledge, for ERM and SGD, the learning rates presented in this paper are all state-of-the-art.

Keywords: Learning rate, generalization bound, stochastic gradient descent, empirical risk minimization, learning theory

1. Introduction

1.1 Background

Stochastic optimization has played an important role in machine learning as many machine learning problems can be cast into a stochastic optimization problem. In this problem, the goal is to optimize the value of an expected function over some set $W$:

$$\min_{w \in W} F(w) = \mathbb{E}_{z \sim \rho}[f(w; z)],$$

(1)
where \( f(\cdot, z) : W \to \mathbb{R} \) is a random objective function depending on a random variable \( z \in Z \) sampled from a distribution \( \rho \). In statistical learning, this problem is also referred to as risk minimization, in which \( F(w) \) is referred to as population risk, \( z \) can be considered as one sample, \( w \) can be interpreted as a model or hypothesis, and \( f(\cdot, \cdot) \) can be viewed as a loss function. A well-known example is the risk minimization in supervised learning (Vapnik, 1995), defined as follows:

\[
\min_{w \in W} F(w) = \mathbb{E}_{(x,y) \sim \rho}[\ell(w(x), y)],
\]

(2)

where \( z \in Z = X \times Y \subset \mathbb{R}^d \times \mathbb{R} \) and \( f(w; z) = \ell(w(x), y) \) is a loss function measuring the error of a prediction rule \( w(x) \) from the input space \( X \) to the output space \( Y \) in a hypothesis space indexed by \( W \subseteq \mathbb{R}^d \) at a single example \( z \). Nonetheless, we emphasize that the risk minimization problem in (1) is more general than supervised learning in (2) and could be more challenging to be handled (Liu et al., 2018). Since the underlying probability measure \( \rho \) is unknown, so the minimization of the population risk is not accessible. There are two classical approaches for solving the risk minimization problem: sample average approximation (also referred to as empirical risk minimization (ERM)) and stochastic approximation (SA).

In ERM, we minimize the empirical risk defined over a set of \( n \) i.i.d. samples \( S = \{z_1, ..., z_n\} \) drawn from the same distribution \( \rho \):

\[
\min_{w \in W} F_S(w) = \frac{1}{n} \sum_{i=1}^{n} f(w; z_i).
\]

Assume the set \( W \subseteq \mathbb{R}^d \) to be compact and let \( w^* \) be the best parameter within \( W \), which satisfies \( w^* \in \arg \min_{w \in W} F(w) \). Also, let \( \hat{w}^* \in \arg \min_{w \in W} F_S(w) \) be an empirical minima. The generalization performance of ERM is measured by excess risk, defined as

\[
F(\hat{w}^*) - F(w^*).
\]

Most theoretical analysis of ERM are devoted to supervised learning in (2) (Bousquet and Elisseeff, 2002; Koren and Levy, 2015; Zhang et al., 2017; Liu et al., 2020). We just mention some work in the fast rate regime here. For instance, faster rates than \( O(1/\sqrt{n}) \) of ERM have been studied under various conditions (Bartlett et al., 2005; Gonen and Shalev-Shwartz, 2017; Liu et al., 2021), such as low noise (Tsybakov et al., 2004), smoothness (Srebro et al., 2010; Li et al., 2018), strong convexity (Sridharan et al., 2008) and space capacity (Li and Liu, 2021). However, the success of ERM for supervised learning can’t be used directly to stochastic optimization (Shalev-Shwartz et al., 2010). Moreover, as pointed out in (Zhang et al., 2017), literatures about ERM for stochastic optimization are quite limited, and we still lack a full understanding of the theory. We now briefly review the recent work in ERM of stochastic optimization. More introductions of related work are postponed to Section 2. In the work of (Zhang et al., 2017), they derive a series of fast rates of order \( O(1/n^2) \) for ERM of stochastic optimization for convex learning by using pretty strong assumptions: Lipschitz continuous, non-negative, smooth, convex and strongly convex conditions. (Liu et al., 2018) follow their theory and further relax the strongly convex condition to
quadratic growth condition (refer to Assumption 7). Nonetheless, the learning rates presented by (Zhang et al., 2017; Liu et al., 2018) still require pretty strong assumptions and are mostly restricted to convex objective function setting. The above mentioned results (Zhang et al., 2017; Liu et al., 2018) are established via uniform convergence of gradient technique. Other than the uniform convergence tool, stability is also a standard tool for generalization performance analysis. Learning rates established by stability are mostly in expectation (Shalev-Shwartz et al., 2010; Lei and Ying, 2020; Kuzborskij and Lampert, 2018; Zhang et al., 2021; Lei et al., 2021; Zhou et al., 2018) and the existing high probability rates are mostly of order \( O(1/\sqrt{n}) \) (Bousquet et al., 2020; Feldman and Vondrak, 2018; Feldman and Vondrak, 2019). In a breakthrough work, (Klochkov and Zhivotovskiy, 2021) develop the techniques in (Bousquet et al., 2020) and provide the first high probability excess risk bound of order \( O(1/n) \) for ERM via uniform stability, which make it possible to derive \( O(1/n) \)-type rate with high probability by stability theory, rather than in expectation. However, their learning rate for ERM require strong convexity assumption, which are often considered as a pretty strong assumption in stochastic optimization (Karimi et al., 2016; Necoara et al., 2019). Therefore, for ERM of stochastic optimization, how to use stability and uniform convergence theoretical tools to obtain improved rates under milder assumptions in convex learning and derive fast rates in nonconvex learning is still ambiguous.

In SA, we iteratively learn the model from random samples \( z_t \sim \rho, t = 1, ..., n \). In this paper, we focus on one popular stochastic approximation algorithm: stochastic gradient descent (SGD) (Nemirovski et al., 2008), which has found wide application in machine learning due to its simplicity in implementation, low memory requirement and low computational complexity per iteration, as well as good practical behavior (Zhang, 2004; Bach and Moulines, 2013; Rakhlin et al., 2012; Shamir and Zhang, 2013; Bottou et al., 2018; Orabona, 2019; Harvey et al., 2019; Amir et al., 2021). As an iterative algorithm, SGD minimizes empirical errors by moving iterates along the direction of a negative gradient calculated based on a loss function on a single training example or a batch of few examples (Lei and Tang, 2018). This strategy of processing few examples per iteration makes SGD very popular in the big data era (Lei et al., 2019; Bottou, 2010; Mai and Johansson, 2021) and enjoys a better generalization performance than its full-batch gradient descent (GD) counterpart (Amir et al., 2021). Let \( w_1 = 0 \) and \( \{\eta_t\}_{t \in \mathbb{N}} \) be a sequence of positive step sizes. At the \( t \)-th iteration, SGD first draws an index \( j_t \) from the uniform distribution over \( \{1, ..., n\} \), and update the model by

\[
\begin{align*}
    w_{t+1} &= w_t - \eta_t \nabla f(w_t; z_{j_t}), \\
    \end{align*}
\]

where \( \nabla f(w_t; z_{j_t}) \) denotes the gradient of \( f \) w.r.t. the first argument. Note that the randomness of \( w_t \) comes from two sources, one from the sampling of training examples according to \( \rho \) and one from the sampling of the indices \( \{j_t\}_t \) according to the uniform distribution over \( \{1, ..., n\} \). Similarly, the generalization performance of SGD is often measured by excess risk of the last iteration \( w_{T+1} \), defined as

\[
F(w_{T+1}) - F(w^*).
\]

Theoretical guarantees of SGD have been studied broadly and many results are available (Lei et al., 2021). Existing learning rates of SGD are mainly derived in the setting with
only one-pass over the data allowed, that is each example can be used at most once, see (Orabona, 2014; Lin and Zhou, 2017; Orabona, 2019) and references therein. Multi-pass SGD is often adopted in practical applications and the generalization performance analysis of it is more difficult to be handled (Lei et al., 2021), leading to the fact that the generalization analysis of multi-pass SGD is much less than one-pass SGD (Lei et al., 2021). In the related work of multi-pass SGD, (Rosasco and Villa, 2015; Lin and Rosasco, 2017; Dieuleveut et al., 2016; Pillaud-Vivien et al., 2018; Muecke et al., 2019) provide the learning rates for the specific least square loss. In a landmark work of (Bousquet and Bottou, 2007), they develop a framework to study the generalization performance of large scale stochastic optimization algorithms, where three factors influencing the generalization performance are shown as: estimation error, optimization error and approximation error. Under this framework, (Lei et al., 2021; Lei and Tang, 2018, 2021) provide high probability learning rates for multi-pass SGD based on uniform convergence tool. Stability is also used to establish generalization bounds for SGD in expectation (Hardt et al., 2018; Shalev-Shwartz et al., 2010; Lin et al., 2016; Kuzborskij and Lampert, 2018; Lei and Ying, 2020; Lei et al., 2021; Lei and Ying, 2021). However, the learning rates obtained via stability for SGD is at most of order $O(1/n)$ in expectation (Lei and Ying, 2021). Hence, learning rate of order $O(1/n)$ in high probability via stability is still missing no matter what in convex or nonconvex learning. Although uniform convergence can be used to obtain learning rates of order $O(1/n)$ in high probability (Lei et al., 2021; Lei and Tang, 2018, 2021), the results either require strong convexity assumption (Lei and Tang, 2018) or space capacity assumption (Lei et al., 2021) in convex objective function setting. Moreover, the fastest known learning rate in nonconvex learning is of order $O(1/n)$ (Lei and Tang, 2021). Therefore, there naturally raises a question: via uniform convergence theoretical tool, is the faster rate of order $O(1/n^2)$ similar to ERM possible for SGD no matter what in convex or nonconvex learning?

In this paper, we investigate the generalization performance for ERM and multi-pass SGD of stochastic optimization (we omit “multi-pass” in the following statements). We aim to provide improved learning rates by relaxing the existing strong assumptions in convex learning and deriving faster rates in nonconvex learning.

1.2 Main Contributions

In this paper, we provide faster learning rates for ERM and SGD of stochastic optimization. Our theoretical analysis covers two popular theoretical tools: stability and uniform convergence.

Firstly, for ERM, we first provide $O(\frac{1}{n})$-type learning rates via uniform stability and further derive $O(\frac{1}{n^2})$-type rates via uniform convergence of gradients. To be specific, we first consider the convex learning. A learning rate of order $O\left(\frac{\log n \log \left(\frac{1}{\delta}\right)}{n}\right)$ is first derived via uniform stability under nonnegative, Lipschitz continuous and quadratic growth (QG) conditions (Assumptions 1 and 7), where QG condition is a nonconvex curvature condition (Necoara et al., 2019; Karimi et al., 2016) and is much weaker than strong convexity that widely used to derive $O(\frac{1}{n})$-type learning rate. In nonconvex learning, with another projection assumption (Assumption 9), we obtain a learning rate of order $O\left(\frac{\log n \log \left(\frac{1}{\delta}\right)}{n}\right)$, which is the first high probability learning rate studied via stability approach in nonconvex
learning. To our best knowledge, there is only one high probability $O\left(\frac{1}{n}\right)$-type learning rate for ERM obtained by stability technique, which is derived under nonnegative, strong convexity and Lipschitz continuous conditions (Klochkov and Zhivotovskiy, 2021). These $O\left(\frac{1}{n}\right)$-type learning rates are then improved to $O\left(\frac{1}{n^2}\right)$-type rates via uniform convergence of gradients approach. Specifically, in convex learning, under nonnegative and smoothness of the objective function $f$, noise condition and QG condition (Assumptions 2, 4 and 7), when certain sample complexity is satisfied and the minimum population risk $F(w^*) = O\left(\frac{1}{n}\right)$, we present a learning rate of order $O\left(\frac{\log^2(1/\delta)}{n}\right)$, where these assumptions are pretty milder than (Zhang et al., 2017; Liu et al., 2018) by removing the strong convexity and Lipschitz continuous conditions. In nonconvex learning, under nonnegative and smoothness of the objective function, noise condition and PL condition (Assumptions 2, 4 and 8), when certain sample complexity is satisfied and the minimum population risk $F(w^*) = O\left(\frac{1}{n}\right)$, we also provide a learning rate of order $O\left(\frac{\log^2(1/\delta)}{n}\right)$ for ERM, where the PL condition is widely used in nonconvex learning rate analysis (Xu and Zeevi, 2020; Lei and Tang, 2021; Lei and Ying, 2021; Reddi et al., 2016; Zhou et al., 2018; Karimi et al., 2016; Charles and Papailiopoulos, 2018). To our best knowledge, this is the first high probability $O\left(\frac{1}{n}\right)$-type learning rates for ERM in nonconvex learning.

Secondly, for the last iteration produced by (3), SGD, we also first provide $O\left(\frac{1}{n}\right)$-type learning rates via uniform stability and further derive $O\left(\frac{1}{n^2}\right)$-type rates via uniform convergence of gradients. In convex learning, the first learning rate of order $O\left(\frac{1}{n}\right)$ (when hide $\log n$ and $\log(1/\delta)$ terms) is established for non-smooth objectives, followed by a similar order rate for smooth objectives. These results obtained need other nonnegative, Lipschitz continuous, boundness of variance of stochastic gradients and QG (or PL) assumptions (Assumptions 1, 6 and 7 (or 8)), without requiring the strong convexity condition. To our best knowledge, these $O\left(\frac{1}{n}\right)$-type learning rates established for smooth and non-smooth objective functions are the first high probability results for SGD studied via stability approach. With another projection condition (Assumption 9), we derive similar results for SGD in nonconvex learning on smooth and nonsmooth objectives. Furthermore, these learning rates studied by stability tool are also improved to $O\left(\frac{1}{n^2}\right)$-type order rates via uniform convergence of gradients approach. Specifically, in convex learning, under nonnegative and smoothness of the objective function, noise condition, QG, relaxed boundness of gradients and boundness of variance of stochastic gradients conditions (Assumptions 2, 4, 5, 6 and 7), we present a learning rate of order $O\left(\frac{\log n \log^3(1/\delta)}{n^2}\right)$ when certain sample complexity is satisfied and the minimal population risk is of order $O\left(\frac{1}{n}\right)$. To our best knowledge, this is the first high probability $O\left(\frac{1}{n}\right)$-type learning rates for SGD in convex learning, additionally, without the strong convexity assumption. While in nonconvex learning, we first study the general nonconvex objectives and provide decay rates of population gradients. Our theoretical analysis shows that when the PL curvature condition is satisfied, these decay rates of population gradients can be significantly improved from the order $O\left(\frac{1}{\sqrt{n}}\right)$ to $O\left(\frac{1}{n}\right)$. The last learning rate in nonconvex learning is also of order $O\left(\frac{\log n \log^3(1/\delta)}{n^2}\right)$ and further improves the iteration complexity. Moreover, our last learning rate reveals an interesting theoretical finding that over-fitting phenomenon will never happen under PL curvature condition. It should be notable that learning rates in uniform convergence scenario do not require the Lipschitz
continuous condition and projection condition (Assumption 9) any more. To the best of our knowledge, these $O(\frac{1}{\sqrt{r}})$-type learning rates for SGD in nonconvex learning are also the first.

Overall, in this paper, we derive a series of state-of-the-art learning rates for ERM and SGD of stochastic optimization via two theoretical viewpoints: stability and uniform convergence of gradients. Our theoretical analysis span convex learning and nonconvex learning, and the learning rates are all stated in high probability.

**Organization of the paper.** The remainder of this paper is organized as follows. In Section 2, we introduce the related work on our covered theoretical tools: stability and uniform convergence. Assumptions used in this paper are discussed in Section 3. In Section 4, we derive a series of new learning guarantees for ERM and SGD via stability, which is then switched to uniform convergence in Section 5. Section 6 and Section 7 provide the proofs of main theorems. In the last, Sections 8 concludes this paper.

### 2. Related Work

In this section, we first show the related work on stability, followed by the related work on uniform convergence.

#### 2.1 Stability

Algorithmic stability is a fundamental concept in statistical learning theory (Bousquet and Elisseeff, 2002; Bousquet et al., 2020; Klochkov and Zhivotovskiy, 2021), which has a deep connection with learnability (Rakhlin et al., 2005; Shalev-Shwartz and Ben-David, 2014; Shalev-Shwartz et al., 2010). (Shalev-Shwartz et al., 2010) show that there are non-trivial learning problems where uniform convergence does not hold, and yet stability can be identified as the key necessary and sufficient condition for learnability. A training algorithm is stable if small changes in the training set result in small differences in the output predictions of the trained model. The widely used notation of stability is referred to as *uniform stability*. It was introduced in the seminal work of (Bousquet and Elisseeff, 2002), where they use it to study the generalization performance of ERM. This framework was then extended to study randomized learning algorithms (Elisseeff et al., 2005), transfer learning (Kuzborskii and Orabona, 2013), privacy-preserving learning (Bassily et al., 2019; Dwork and Feldman, 2018; Wang et al., 2021), distribution learning (Wu et al., 2020), PAC-Bayesian bounds (London, 2017), pairwise learning (Lei et al., 2020; Shen et al., 2020), non-i.i.d. processes (Mohri and Rostamizadeh, 2009, 2010), stochastic gradient descent (Hardt et al., 2016; Charles and Papailiopoulos, 2018; Yuan et al., 2019; Mai and Johansson, 2021), stochastic gradient Langevin dynamics (Li et al., 2020; Mou et al., 2018), multi-task learning (Liu et al., 2016b), and iterative optimization algorithms (Chen et al., 2018). Some other stability measures include the uniform argument stability (Liu et al., 2017; Yang et al., 2021; Bassily et al., 2020), hypothesis stability (Bousquet and Elisseeff, 2002; Charles and Papailiopoulos, 2018), hypothesis set stability (Foster et al., 2019), on average stability (Shalev-Shwartz et al., 2010; Lei and Ying, 2020; Kuzborskij and Lampert, 2018; Zhang et al., 2021; Lei et al., 2021; Zhou et al., 2018), and locally elastic stability (Deng et al., 2021).
Although stability has been used for generalization performance analysis in large quantities, the learning rates based on stability are usually constructed in expectation (Koren and Levy, 2015; Gonen and Shalev-Shwartz, 2017; Vaskevicius and Zhivotovskiy, 2020; Mourtada et al., 2021; Lei and Ying, 2020, 2021). Recently, uniform stability was developed for providing the sharper high probability guarantees for uniformly stable algorithms in (Bousquet et al., 2020; Feldman and Vondrak, 2018; Feldman and Vondrak, 2019). However, their results still have a term of order $O(1/\sqrt{n})$ in the sampling error, leaving generalization bounds of slow order $O(1/\sqrt{n})$. Fortunately, (Klochkov and Zhivotovskiy, 2021) further develop their results and remove the $O(1/\sqrt{n})$ term, making it possible to build $O(1/n)$ high probability bounds by uniform stability tool (refer to Lemma 35). They provide the learning rates of order $O(1/n)$ for ERM under nonnegative, strongly convex and Lipschitz continuous conditions. However, the strong convexity is a strong condition and the high probability bounds of fast order $O(1/n)$ in SGD and nonconvex ERM are still unknown. Thus, in stability scenario, we focus on the following question: for ERM and SGD of stochastic optimization, can we provide $O(1/n)$-type high probability learning rates under milder conditions in convex learning and derive $O(1/n)$-type high probability learning rates in nonconvex learning?

2.2 Uniform Convergence

Uniform convergence is a central notation for characterizing learnability and also a standard tool for generalization analysis (Shalev-Shwartz and Ben-David, 2014; Shalev-Shwartz et al., 2010). The uniform convergence can be characterized as follows: the empirical risk of hypotheses in the hypothesis class converges to their population risk uniformly (Shalev-Shwartz et al., 2010). In this paper, we are interested in the uniform convergence of gradients, a gap between the gradients of the population risk and the gradients of the empirical risk, which can be studied via the uniform convergence (Mei et al., 2018; Zhang et al., 2017; Foster et al., 2018; Zhang and Zhou, 2019; Lei and Tang, 2021; Zhang and Zhou, 2019; Xu and Zeevi, 2020). Uniform convergence of gradients has been studied for deriving high probability generalization bounds for convex and nonconvex learning and is drawing increasing attention in stochastic optimization. In convex learning, (Zhang et al., 2017) build uniform convergence of gradients based on the covering numbers and the convex property, and derive a series of learning rates of fast order $O(1/n)$ and $O(1/n^2)$ for ERM. Recently, their approach is extended to analyze a variant of the stochastic gradient descent algorithm (Zhang and Zhou, 2019), but their results hold only in expectation rather than in high probability. (Liu et al., 2018) also extend their approach to derive fast rates of ERM and stochastic approximation. However, these results require pretty strong conditions as we discussed in Section 1.1 and are mostly restricted to convex learning. In nonconvex learning, convergence rates of stochastic optimization are first generally stated for the gradients of empirical risks (Reddi et al., 2016; Allen-Zhu and Hazan, 2016; Ghadimi and Lan, 2013; Allen-Zhu, 2018), which not necessarily means that similar convergence rates holds for their population counterparts. Motivated by this, uniform deviation between population and empirical gradients is established based on the covering numbers in (Mei et al., 2018). (Foster et al., 2018) establish the uniform convergence of gradients for the generalized linear models based on a chain rule for vector-valued Rademacher complexity. (Lei and Tang, 2021) establish the uniform convergence of gradients based on Rademacher chaos complexities of order two, which is an extension of
Rademacher complexities to $U$-processes (De la Peña and Giné, 2012; Ying and Campbell, 2010). The above mentioned uniform convergence results in nonconvex learning are established for smooth functions. (Davis and Drusvyatskiy, 2018) study graphical convergence for nonconvex and nonsmooth loss functions. Unfortunately, the fastest known learning rate with high probability for SGD in nonconvex learning is of order $O(1/n)$. A sharp question is that is the faster rates of order $O(1/n^2)$ similar to ERM in (Zhang et al., 2017) possible for SGD in convex and nonconvex learning? In a breakthrough work (Xu and Zeevi, 2020), they establish the uniform convergence of gradients based on “uniform localized convergence” technique (Xu and Zeevi, 2020) in the nonconvex setting. However, how to use this new technique to establish $O(1/n^2)$-type learning rates for ERM and SGD is still unexplored. We will use this uniform convergence of gradients (refer to Lemma 45) combined with other techniques to derive faster learning rates. Therefore, in uniform convergence scenario, we focus on the following question: for ERM and SGD of stochastic optimization, can we provide $O(1/n^2)$-type high probability learning rates under milder conditions in convex learning and derive $O(1/n^2)$-type high probability learning rates in nonconvex learning?

3. Assumptions

We denote $\| \cdot \|$ the $\ell_2$ norm, i.e., $\|w\|_2 = \sqrt{\sum_{j=1}^{d} w_j^2}$ for $w = (w_1, ..., w_d)$ and denote $\mathbb{E}$ the expectation. In this paper, we assume that the objective function $f : \mathcal{W} \times \mathcal{Z} \mapsto \mathbb{R}_+$. The following are some assumptions used in this paper. We emphasize here that the following assumptions are scattered in different theorems and many assumptions have close relationship with each other. Also, one of our contributions is to relax the existing strong assumptions in ERM and SGD.

Assumption 1 (Lipschitz Continuous) Assume the objective function $f$ is $L$-Lipschitz. That is, for any $z \in \mathcal{Z}$ and any $w_1, w_2 \in \mathcal{W}$, there has

$$\|f(w_1, z) - f(w_2, z)\| \leq L\|w_1 - w_2\|.$$ 

Remark 1 Lipschitz continuous is a standard assumption and widely used in stability analysis (Bousquet and Elisseeff, 2002; Elisseeff et al., 2005; Shalev-Shwartz et al., 2010; Klochkov and Zhivotovskiy, 2021; Hardt et al., 2016; Charles and Papailiopoulos, 2018), to mention a few. Assumption 1 will only appear in the theorems based on stability.

Assumption 2 (Smoothness) Let $\beta > 0$. For any sample $z \in \mathcal{Z}$ and $w_1, w_2 \in \mathcal{W}$, there has

$$\|\nabla f(w_1, z) - \nabla f(w_2, z)\| \leq \beta\|w_1 - w_2\|.$$ 

Remark 2 The smoothness of objective function, that is the Lipschitz continuous of gradients, is standard and widely used in learning theory (Reddi et al., 2016; Allen-Zhu and Hazan, 2016; Zhang and Zhou, 2019; Mai and Johansson, 2021; Lei and Tang, 2021; Ye et al., 2021). As discussed in Section 2.2, the smoothness condition is often used to establish the uniform convergence of gradients (Mei et al., 2018; Zhang et al., 2017; Foster et al., 2018;
Improved Learning Rates for Stochastic Optimization: Two Theoretical Viewpoints

Lei and Tang, 2021; Zhang and Zhou, 2019). In stability-based stochastic optimization, smoothness condition is also widely used to derive the stability bound of SGD (Hardt et al., 2016; Lei and Ying, 2021; Lei et al., 2021; Zhang et al., 2021; Zhou et al., 2018; Feldman and Vondrak, 2019), to mention a few. In this paper, smoothness is also served as a basic condition in deriving faster rates. Also, we will try to provide faster rates in non-smooth scenario by relaxing the smoothness condition (refer to the following Assumption 3).

Assumption 3 (Hölder Smoothness) Let $P > 0$, $\alpha \in (0, 1]$. We say $f$ has an $\alpha$-Hölder continuous gradient w.r.t. the first argument with parameter $P$ if for all $w_1, w_2 \in W$ and $z \in Z$,

$$\|\nabla f(w_1, z) - \nabla f(w_2, z)\| \leq P\|w_1 - w_2\|^\alpha.$$  

Remark 3 When $\alpha = 0$, then Assumption 3 implies that $f$ is Lipschitz continuous as considered in Assumption 1. If Assumption 3 holds with $\alpha = 1$, then this implies that $f$ is smoothness as considered in Assumption 2. Thus, Assumption 3 characterizes the $\alpha$-Hölder smoothness of $f$ and is a much milder condition than the smoothness of Assumption 2 (Lei and Ying, 2020; Wang et al., 2021). This definition instantiates many non-smooth loss functions. For example, the $q$-norm hinge loss $f(w, z) = (\max(0, 1 - y\langle w, z \rangle))^q$ for classification and the $q$-th power absolute distance loss $f(w, z) = |y - \langle w, z \rangle|^q$ for regression (Lei and Ying, 2020), whose $f$ are $(q - 1)$-Hölder smoothness if $q \in [1, 2]$.

Assumption 4 (Noise Condition) There exists $B_* > 0$ such that for all $2 \leq k \leq n$,

$$\mathbb{E}_{j_t}\left[\|\nabla f(w^*, z)\|^k\right] \leq \frac{1}{2}k!\mathbb{E}\left[\|\nabla f(w^*, z)\|^2\right] B_*^{k-2}.$$  

Remark 4 Assumption 4 is a bernstein condition at the optimal point, which is pretty mild because $B_*$ only depends on gradients at $w^*$. Note that $B_*$ will always exist in the $\mathcal{O}(1/n^2)$ term, it therefore produce little influence to the generalization performance. Assumption 4 is only used in the uniform convergence of gradients regime.

Assumption 5 (Relaxed Boundness of Gradient) Assume the existence of $G > 0$ satisfying

$$\sqrt{\eta_t}\|\nabla f(w_t, z)\| \leq G, \quad \forall t \in \mathbb{N}, z \in Z.$$  

Remark 5 Assumption 5 is introduced in (Lei and Tang, 2021), implying that the product of step size and the norm of gradient is bounded. Compared with the commonly used bounded gradient assumption $\|\nabla f(w_t, z)\| \leq G$ in Assumption 1 (Zhang et al., 2017; Hardt et al., 2016; Kuzborskij and Lampert, 2018; Zhou et al., 2018; Liu et al., 2018; Reddi et al., 2016; Mou et al., 2018; Lian et al., 2015), Assumption 5 is much milder since the step sizes should diminish to zero for the convergence of the algorithm (Lei and Tang, 2021). Assumption 5 is only used in the uniform convergence of gradients regime.

Assumption 6 Assume the existence of $\sigma > 0$ satisfying

$$\mathbb{E}_{j_t}\left[\|\nabla f(w_t, z_{j_t}) - \nabla F_S(w_t)\|^2\right] \leq \sigma^2, \quad \forall t \in \mathbb{N},$$  

where $\mathbb{E}_{j_t}$ denotes the expectation w.r.t. $j_t$. 

Remark 6 Assumption 6 is a standard assumption from the stochastic optimization theory (Bottou et al., 2018; Nemirovski et al., 2008; Ghadimi et al., 2016; Ghadimi and Lan, 2013; Kuzborskij and Lampert, 2018; Lei and Tang, 2021; Zhou et al., 2018), which essentially bounds the variance of the stochastic gradients for dataset $S$ (Zhou et al., 2018).

Assumption 7 (Quadratic Growth) Any function $f : \mathcal{W} \mapsto \mathbb{R}$ satisfies the quadratic growth (QG) condition on $\mathcal{W}$ with parameter $\mu > 0$ if for all $w \in \mathcal{W}$,

$$f(w) - f(w^*) \geq \frac{\mu}{2} \|w - w^*\|^2,$$

where $w^*$ denotes the euclidean projection of $w$ onto the set of global minimizers of $f$ in $\mathcal{W}$ (i.e., $w^*$ is the closest point to $w$ in $\mathcal{W}$).

Remark 7 It is well-known that for smooth convex programming, the optimization error of first order optimization methods, such as projected gradient descent (Nesterov, 2003), SGD (Nemirovski et al., 2008), coordinate gradient descent (Wright, 2015), are converging sublinearly (Necoara et al., 2019; Karimi et al., 2016). Typically, for proving linear convergence of the first order methods one also need to require strong convexity for the objective function (Shalev-Shwartz et al., 2009; Sridharan et al., 2008; Zhang and Zhou, 2019; Hardt et al., 2016). For example, (Zhang et al., 2017) use the strong convexity of the objective function and other conditions to provide $O(1/n^2)$ learning rates for ERM. However, strong convexity of the objective function does not hold for many practical applications. For relaxing the strong convexity assumption, we introduce Assumption 7 and the following Assumption 8. Assumption 7 is one of the weakest curvature condition (Necoara et al., 2019; Karimi et al., 2016) in guaranteeing linear convergence of optimization algorithms (Karimi et al., 2016) as well as fast-rate generalization error (Liu et al., 2018). We provide relationships between the commonly used curvature conditions in learning theory in Lemma 51 in Appendix A. We emphasize that we focus on the fast-rate generalization error under Assumption 7 or 8 in this paper, which are less studied than convergence rate analysis (Lei and Tang, 2021; Karimi et al., 2016). Moreover, the convergence rate analysis, also referred to as optimization error in machine learning, is also involved in our proof, stated in high probability rather than in expectation as in (Karimi et al., 2016; Lei and Ying, 2020, 2021).

Assumption 8 (Polyak-Łojasiewicz) Fix a set $\mathcal{W}$ and let $f^* = \min_{w \in \mathcal{W}} f(w)$. For any function $f : \mathcal{W} \mapsto \mathbb{R}$, we say it satisfies the Polyak-Łojasiewicz (PL) condition with parameter $\mu > 0$ on $\mathcal{W}$ if for all $w \in \mathcal{W}$,

$$f(w) - f^* \leq \frac{1}{2\mu} \|\nabla f(w)\|^2.$$

Remark 8 PL condition is also one of the weakest curvature conditions (refer to Lemma 51), for instance, weaker than “one-point convexity” (Kleinberg et al., 2018; Li and Yuan, 2017), “star convexity” (Zhou et al., 2019), and “quasar convexity” (Hardt et al., 2018; Hinder et al., 2020). PL condition is also referred to as “gradient dominance condition” (Foster et al., 2018) and widely used in the analysis of nonconvex learning (Xu and Zeevi, 2020; Lei and Tang, 2021; Lei and Ying, 2021; Reddi et al., 2016; Zhou et al., 2018; Karimi et al., 2016).
2016; Charles and Papailiopoulos, 2018), to mention a few. Under suitable assumptions on their input, many popular nonconvex objective functions satisfy the PL condition, including: neural networks with one hidden layer (Li and Yuan, 2017), ResNets with linear activations (Hardt and Ma, 2016), linear dynamical systems (Hardt et al., 2018), matrix factorization (Liu et al., 2016a), robust regression (Liu et al., 2016a), phase retrieval (Sun et al., 2018), blind deconvolution (Li et al., 2019), mixture of two Gaussians (Balakrishnan et al., 2017), etc. PL condition indeed implies that every stationary point must be a global optimum. Moreover, according to Lemma 17 in (Charles and Papailiopoulos, 2018) and Appendix A in (Karimi et al., 2016), the PL condition (Assumption 8) implies QG condition (Assumption 7). For this reason and brevity, we use the same parameter $\mu$ in Assumptions 7 and 8. The relationships between PL and other commonly used curvature conditions are also provided in Lemma 51 in Appendix A.

Assumption 9 Let $\pi_S(w)$ denote the closest optimal point of $F_S$ to $w$. Suppose the empirical risk minimizers for $F_S$ and $F_{S'}$, i.e., $\hat{w}^*(S)$ and $\hat{w}^*(S')$, satisfy $\pi_S(\hat{w}^*(S')) = \hat{w}^*(S)$, where $S, S' \in \mathcal{Z}^n$ that differ by at most one example, e.g., $S = \{z_1, ..., z_{i-1}, \hat{z}_i, z_{i+1}, ..., z_n\}$ and $S' = \{z_1, ..., z_{i-1}, \hat{z}'_i, z_{i+1}, ..., z_n\}$.

Remark 9 $\pi_S(w)$ means the projection of $w$ on the set of empirical minimizers of $F_S$. Assumption 9 is introduced in (Charles and Papailiopoulos, 2018) to establish the stability bound for nonconvex learning. This assumption is satisfied if there is a unique minimizer $\hat{w}^*(S)$ for every dataset $S$, but it is relatively strict and can’t apply to the empirical loss with infinitely many global minima. To solve the case with infinitely many global minima, it was discussed in (Charles and Papailiopoulos, 2018) that one could imagine designing $A(S)$, where $A$ is a learning algorithm and $A(S)$ denotes the output of $A$ trained on some dataset $S$, to output a structured empirical risk minimizer, for instance, one such that if algorithm $A$ is operated on dataset $S'$, its projection on the minimizer of $F_S$ would always return $A(S)$. This could be possible if $A(S)$ corresponds to minimizing a regularized, or structured cost function whose set of optimizers only contained a small subset of the global minima of $F_S$. However, for general nonconvex losses, it seems pretty challenging to propose such a structured empirical risk minimizer, and this problem can be still served as an interesting open problem. Assumption 9 will only appear in Section 4.2, which will be successfully removed in the uniform convergence scenario.

4. Stability Scenario

In this section, we define $\mathcal{W} := B_R = \{w \in \mathbb{R}^d : \|w\| \leq R\}$ and denote $A \asymp B$ if there exists universal constants $C_1, C_2 > 0$ such that $C_1A \leq B \leq C_2A$. All theorems established in this section are based on uniform stability tool. Below we give a formal definition of uniform stability.

Definition 10 (Uniform Stability) An algorithm $A$ is uniformly $\epsilon$-stable if for all datasets $S, S' \in \mathcal{Z}^n$ that differ by at most one example and any $z \in \mathcal{Z}$, we have

$$|f(A(S), z) - f(A(S'), z)| \leq \epsilon,$$

where $A : \cup_n \mathcal{Z}^n \mapsto \mathcal{W}$ produce an output model $A(S) \in \mathcal{W}$.
In the following, we use uniform stability to derive improved rates under milder assumptions in convex learning and faster rates in nonconvex learning. In convex learning, we assume the objective function $f$ is convex, while in nonconvex learning $f$ is assumed to be nonconvex.

4.1 Convex Objectives

Our first theorem in convex learning is a probabilistic learning rate for ERM of stochastic optimization.

**Theorem 11** Let $\hat{w}^*(S)$ be the ERM of $F_S$. Suppose Assumption 1 holds and $F_S$ satisfies Assumption 7 with parameter $\mu$. Then, for any $S$ and $\delta > 0$, with probability at least $1 - \delta$, we have

$$F(\hat{w}^*(S)) - F^* = O\left(\frac{\log n}{n} \log \left(\frac{1}{\delta}\right)\right),$$

where $F^* = \inf_{w \in W} F(w)$.

**Remark 12** Since our task is to prove bounds with better rates, that is, which decrease fast with $n \to \infty$, thus for brevity, we leave the explicit results of all Theorems of this paper in proofs. From Theorem 11, one can see that if empirical risk $F_S$ satisfies the QG condition and $f$ are nonnegative convex and Lipschitz continuous, the learning rate of excess risk of ERM of stochastic optimization is of order $O\left(\frac{\log n \log 1/n}{n}\right)$. We now compare Theorem 11 with the most related work (Klochkov and Zhivotovskiy, 2021), which provide the first high probability $O\left(\frac{1}{n}\right)$-type rate for ERM via stability theory and whose learning rate is of order $O\left(\frac{\log n \log 1/n}{n}\right)$ if $f$ are nonnegative convex, Lipschitz continuous and strongly convex. By comparison, our bound is obtained under a much milder QG condition than strong convexity condition (refer to Remark 7).

After the ERM, we begin to analyze the excess risk performance of SGD on two cases: nonsmooth and smooth objective functions. The first is a high probability rate for SGD of stochastic optimization on nonsmooth function case.

**Theorem 13** Suppose Assumptions 1, 3 and 6 holds, and suppose $F_S$ satisfies Assumption 8 with parameter $2\mu$. Let $\{w_t\}$ be the sequence produced by (3) with $\eta_t = \frac{2}{\mu(t + t_0)}$ such that $t_0 \geq \max\left\{\frac{2(2P)^{1/\alpha}}{\mu}, 1\right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$, when $T \asymp n^{2/\alpha}$, we have

$$F(w_{T+1}) - F^* = \begin{cases} O\left(\frac{\log n}{n} \log \left(\frac{1}{\delta}\right)\right) & \text{if } \alpha \in (0, 1), \\
O\left(\frac{\log^{2} n}{n} \log \left(\frac{1}{\delta}\right)\right) & \text{if } \alpha = 1. \end{cases}$$

where $F^* = \inf_{w \in W} F(w)$.

**Remark 14** Theorem 13 shows that if the objective function $f$ is nonnegative convex, Lipshitz continuous and Hölder smooth, $F_S$ satisfies the PL condition and the boundness
if the objective function $f$ is Lipschitz continuous and convex, for any \( \theta \in (12\exp\left\{-\frac{\alpha^2}{2}\right\}, 1) \), with probability at least \( 1 - \theta \), there holds $F(w_T) - F^* = O\left(\frac{\log(n)\log\left(\frac{n}{\delta}\right)}{\sqrt{n}}\right)$ with $T \asymp n^2$ and the constant stepsize $\eta_t = O\left(\frac{1}{n^{1/2}}\right)$, where $w_T = \left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t w_t$. (Yang et al., 2021) follow the theory in (Bassily et al., 2020) and study the pairwise differential privacy problem. It was shown in their paper that if the objective function $f$ is Lipschitz continuous and convex, for any \( \theta \in (0, 1) \), with probability at least \( 1 - \theta \), there holds $F(w_T) - F^* = O\left(\frac{\ln^2 n}{\sqrt{n}}\right)$ with $T \asymp n^2$ and the constant stepsize $\eta_t = O\left(\frac{1}{n^{1/2}}\right)$. High probability learning rates of order $O\left(\frac{1}{\sqrt{n}}\right)$ w.r.t. the sample size $n$ can also be found in Theorem 10 and Theorem 12 in (Wang et al., 2021). Different from the above probabilistic learning rates, (Lei and Ying, 2020) study the expected excess risk. If the objective function $f$ is nonnegative, convex, and $\alpha$-Hölder smooth, and if the stepsize $\eta_t = cT^{-\theta}$ with \( \theta \in [0, 1] \) and $c > 0$, it was shown in (Lei and Ying, 2020) that:

1. $\mathbb{E}F(w_T) - F^* = O\left(n^{-\frac{1}{2}}\right)$ when $\alpha \geq 1/2$, $\theta = 1/2$ and $T \asymp n$;
2. $\mathbb{E}F(w_T) - F^* = O\left(n^{-\frac{1}{2}}\right)$ when $\alpha < 1/2$, $\theta = \frac{3-2\alpha}{2(1-\alpha)}$ and $T \asymp n^{\frac{2-\alpha}{1+\alpha}}$;
3. $\mathbb{E}F(w_T) - F^* = O\left(n^{-\frac{1+\alpha}{2}}\right)$ when $F(w^*) = 0$, $\theta = \frac{3-2\alpha}{2(1-\alpha)}$ and $T \asymp n^{\frac{2-\alpha}{1+\alpha}}$.

To summarize, these learning rates are all of slow order $O\left(\frac{1}{\sqrt{n}}\right)$ w.r.t. the sample size $n$ when stated in high probability (Bassily et al., 2020; Yang et al., 2021; Wang et al., 2021) or at most of order $O\left(\frac{1}{n}\right)$ in expectation (Lei and Ying, 2020). Compared with these results studied on nonsmooth loss, our learning rates have a significant improvement and are studied in high probability.

The second is a probabilistic learning rate for the last iteration produced by (3) in smooth case.

**Theorem 15** Suppose Assumptions 1, 2 and 6 holds, and suppose $F_S$ satisfies Assumption 7 with parameter $2\mu$. Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \frac{2}{\mu(t+t_0)}$ such that $t_0 \geq \max\left\{\frac{4\mu}{\delta}, 1\right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability $1 - \delta$, when $T \asymp n^2$, we have

$$F(w_{T+1}) - F^* = O\left(\frac{\log^2 n \log^2\left(\frac{n}{\delta}\right)}{\sqrt{n} \log^2\left(\frac{1}{\delta}\right)}\right),$$

where $F^* = \inf_{w \in \mathcal{W}} F(w)$. 

13
Remark 16 Theorem 15 can be easily obtained by Theorem 13 since $\alpha = 1$ when $f$ is smooth. A difference is that Theorem 15 replaces the PL condition by the milder QG condition. The learning rate in Theorem 15 is also of fast order $O\left(\frac{\log^2 n}{\eta} \log^2 \left(\frac{1}{\delta}\right)\right)$. We now compare our results in Theorems 13 and 15 with the related work of generalization analysis of SGD in convex learning. In the generalization analysis of multi-pass SGD for the specific least square loss, (Rosasco and Villa, 2015) establish learning rates of order $O\left(n^{-\frac{2}{\eta+1}}\right)$ with $\eta \in (0, 1]$ by integral operator tool, which are of slow order. We then switch to the generalization performance of SGD established via stability. In an seminal work of studying stability bounds of SGD (Hardt et al., 2016), they show expected excess risk bound $\mathbb{E} F(w_T) - F^* = O\left(n^{-\frac{1}{2}}\right)$ for multi-pass SGD if the objective function is smooth, Lipschitz continuous and convex and when $T \asymp n$, where $w_T = \left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t w_t$.

Generalization performance analysis was studied separately for smooth and non-smooth loss functions in (Lin et al., 2016). Let $\eta \in (0, 1]$. For smooth loss functions, they show that $\mathbb{E} F(w_T) - F(w^*) = O\left(n^{-\frac{1}{2} \log n}\right)$ with $T \asymp n^{\frac{2}{1+\theta}}$ based on the stability bounds of SGD established in (Hardt et al., 2016). For non-smooth loss functions, it was shown that $\mathbb{E} F(w_T) - F(w^*) = O\left(n^{-\frac{2}{2+\theta}} \log n\right)$ with $T \asymp n^{\frac{2}{3+\theta}}$ by Rademacher complexity tool. Thus the fastest rate in (Lin et al., 2016) is $O\left(\frac{1}{\sqrt{n}}\right)$ in expectation. It was shown in (Feldman and Vondrak, 2019) that for Lipschitz continuous and smooth convex objective function, the learning rate for excess risk is of order $O\left(\frac{\log n}{\sqrt{n}} \log^2 (n/\delta)\right)$ when choose the optimal $T$. The high probability bound was also investigated in (Lei et al., 2021), where they show that $F(w_T) - F(w^*) = O\left(n^{-\frac{2}{2+\theta}} \log^2 (n/\delta)\right)$ when $T \asymp n^{\frac{1}{1+\theta(1-\theta)}}$ with $\theta > 1/2$. There are some work use space capacity assumption to study faster rate for SGD (Lin and Rosasco, 2017; Lei et al., 2021) by Rademacher complexity tool. Very recently, (Bassily et al., 2020; Yang et al., 2021; Wang et al., 2021) extend the discussion in (Hardt et al., 2016) by relaxing the smoothness condition and study the high probability learning rates. However, as discussed in Remark 14, the sharpest rate of their results is of order $O\left(\frac{1}{\sqrt{n}}\right)$ w.r.t. the sample size $n$ when choose the optimal $T$. In a related work of (London, 2017), they study generalization bounds for SGD by using a novel combination of PAC-Bayes and algorithmic stability. Although their results hold for arbitrary posterior distributions on an algorithm’s random hyperparameters, their learning rates are also of slow order $O\left(\frac{1}{\sqrt{n}}\right)$ w.r.t. the sample size $n$. From the comparison, one can see that our learning rates are significantly faster than the related work. Moreover, our learning rates are stated in high probability, which are beneficial to understand the generalization performance of the learned model when restricted to samples as compared to the rates in expectation (Hardt et al., 2016; Lin et al., 2016; Lei and Ying, 2020). Since there are seldom high probability rates of order $O\left(\frac{1}{n}\right)$ for SGD in stability regime, the related work discussed above does not require strong assumptions. In the generalization analysis of SGD based on space complexity technique, strong convexity condition is often required to establish the $O\left(\frac{1}{n}\right)$-type learning rate (Kakade and Tewari, 2008; Kar et al., 2013; Lei and Tang, 2018). While in Theorems 13 and 15, we relax the strong convexity condition by using the milder QG or PL condition.
4.2 Nonconvex Objectives

In this section, we focus on nonconvex objective function $f$ satisfied with PL or QG curvature condition. The first main result in nonconvex learning is a probabilistic learning rate for ERM of stochastic optimization.

**Theorem 17** Assume that $\hat{w}^*(S)$ is the ERM of $F_S$. Suppose Assumptions 1 and 9 hold, and suppose $F_S$ satisfies Assumption 7 with parameter $\mu$. Then, for any $S$ and $\delta > 0$, with probability at least $1 - \delta$, we have

$$F(\hat{w}^*(S)) - F^* = O \left( \frac{\log n}{n} \log \left( \frac{1}{\delta} \right) \right),$$

where $F^* = \inf_{w \in \mathcal{W}} F(w)$.

**Remark 18** Theorem 17 suggests that if the objective function $f$ is nonnegative and Lipschitz continuous, $F_S$ satisfies the QG condition and the projection assumption (Assumption 9) is satisfied, the learning rate of excess risk is of order $O \left( \frac{\log n}{n} \log \left( \frac{1}{\delta} \right) \right)$ for ERM of stochastic optimization. We now compare our results with the most related work (Liu et al., 2018), where they investigate the excess risk of nonconvex ERM of stochastic optimization. Their results don’t require the objective function $f$ to be convex in terms of $w$ or any $z$ and also involve with the QG condition. Specifically, if the objective function $f$ satisfies the Lipschitz continuous condition and the population risk $F$ satisfies the convexity and QG conditions, it was shown in (Liu et al., 2018) that $F(\hat{w}^*(S)) - F^* = O \left( \frac{d \log n + \log(1/\delta)}{n} \right)^{\frac{\alpha}{1-\theta}}$ with $\theta \in (0, 1]$, derived by uniform convergence technique. One can see their learning rate has a dimensionality parameter $d$, thus it cannot be used to guarantee the case when $\mathcal{W}$ is in a separable Hilbert space. On the contrary, our learning rate has no such dilemma.

The second main result in nonconvex learning is a probabilistic learning rate for SGD of stochastic optimization.

**Theorem 19** Suppose Assumptions 1, 3, 6 and 9 hold, and suppose $F_S$ satisfies Assumption 8 with parameter $2\mu$. Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \frac{2}{\mu(t + t_0)}$ such that $t_0 \geq \max \left\{ \frac{2(2P)^{1/\alpha}}{\mu}, 1 \right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$, when $T \asymp n^{2/\alpha}$, we have

$$F(w_{T+1}) - F^* = \begin{cases} 
O \left( \frac{\log n}{n} \log \left( \frac{1}{\delta} \right) \right) & \text{if } \alpha \in (0, 1), \\
O \left( \frac{\log \frac{2}{\alpha}}{n} \log \left( \frac{1}{\delta} \right) \right) & \text{if } \alpha = 1.
\end{cases}$$

where $F^* = \inf_{w \in \mathcal{W}} F(w)$.

**Remark 20** Theorem 19 shows that if the objective function $f$ is nonnegative, Lipshitz continuous and Hölder smooth, the boundness of stochastic gradient variance, PL condition and the projection assumption (Assumption 9) are satisfied, the learning rate of excess
risk of the last iteration of SGD is of order $O\left(\frac{\log n \log \left(\frac{1}{\delta}\right)}{n}\right)$ if $\alpha \in (0, 1)$ and is of order $O\left(\frac{\log^2 n \log^2 \left(\frac{1}{\delta}\right)}{n}\right)$ if $\alpha = 1$ when $T \approx n^{\frac{2}{\alpha}}$. We now compare Theorem 19 with the related work on generalization analysis of SGD in nonconvex learning. Firstly, in (Charles and Papailiopoulos, 2018), based on pointwise hypothesis stability and the optimization error bound in (Karimi et al., 2016), it was shown with probability at least $1 - \delta$ there has $F(w_{T+1}) - F^* = O\left(\frac{1}{\sqrt{n\delta}} + \frac{1}{T^{\frac{1}{4}}\delta^{\frac{1}{2}}}ight)$ for both QG and PL curvature conditions. Combined with the projection assumption (Assumption 9), similar learning rates hold for uniform stability, which are slower than our results. Based on stability bounds established in (Hardt et al., 2016), which are expected uniform stability bounds that are different from the stability bounds this paper derived and are used to derive generalization bounds in expectation, and the optimization error bound in (Karimi et al., 2016), it was shown in (Yuan et al., 2019) that $\mathbb{E}F(w_{T+1}) - F(w^*) = O\left(\frac{1}{\sqrt{n}}\right)$ for PL curvature condition by choosing an optimal $T$. A generalization error bound is provided in (Zhou et al., 2018), that is $F(w_{T+1}) - F_S(w_{T+1}) = O\left(\sqrt{\frac{\log T}{n\delta}}\right)$ with probability at least $1 - \delta$ for PL curvature condition. In a very recent work, (Lei and Ying, 2021) present improved results via on-average stability for SGD learning with PL condition. It was shown that $\mathbb{E}F(w_{T+1}) - F^* = O\left(\frac{1}{n} + \frac{1}{T}\right)$. Although they obtain the $O\left(\frac{1}{n}\right)$-type learning rate, but stated in expectation. To summarize, the generalization performance analysis of nonconvex learning is mainly focusing on PL condition and the mostly stability analysis generally implies a learning rate of slow order $O\left(\frac{1}{\sqrt{n}}\right)$ for SGD in nonconvex learning satisfied with PL condition. Moreover, the above mentioned work mostly use smoothness condition (Yuan et al., 2019; Zhou et al., 2018; Lei and Ying, 2021; Hardt et al., 2016). By comparison, one can see that our learning rate is faster than the related work and stated in high probability, additionally, derived under a relaxed smoothness condition (Assumptions 3). One may consider that the learning rates established in Theorems 17 and 19 have a relative strict assumption (Assumption 9). But as discussed in Remark 9, constructing a structured empirical risk minimizer can be an approach to tackle this challenge. Furthermore, we emphasize that in the following uniform convergence scenario, we provide faster learning rates and do not require this assumption anymore.

**Remark 21** Stability considers the specific model produced by the learning algorithm, thus it can produce dimensionality-independent learning rates, while the dependence on the dimensionality is generally inevitable for the uniform convergence tool (refer to Theorem 28). In stability scenario, the two most important terms that we should consider are high probability stability bound and high probability optimization error. For stability bound of SGD, an popular approach is to use the property of smoothness to establish the nonexpansiveness of gradient mapping (Hardt et al., 2018; Kuzborskii and Lampert, 2018; Feldman and Vondrak, 2019; Le and Ying, 2020; Lei et al., 2021; Lei and Ying, 2021; London, 2017; Bassily et al., 2019; Feldman and Vondrak, 2019; Zhang et al., 2021), however, these stability bounds often studied in expectation. (Feldman and Vondrak, 2019; Bassily et al., 2019; Shen et al., 2020; Wang et al., 2021; Lei et al., 2020) extend this approach and provide high probability stability bounds of SGD. These high probability bounds...
are almost all studied by the Chernoff bounds for Bernoulli variables, which, unfortunately, lead to excess risk bounds of slow order $O\left(\frac{1}{\sqrt{n}}\right)$ w.r.t. the sample size $n$. We provide Theorem 59 in Appendix C to discuss that this method also lead to slow order stability bound in nonconvex learning, and may be vacuous if there has no strong assumptions, since stability bounds in (Feldman and Vondrak, 2019; Bassily et al., 2019; Shen et al., 2020; Wang et al., 2021; Lei et al., 2020) are restricted to convex learning. These problems require us to adopt new perspective to prove faster stability bounds (refer to Section 6.2). Optimization errors have received much attention in the optimization domain (Reddi et al., 2016; Allen-Zhu and Hazan, 2016; Ghadimi and Lan, 2013; Karimi et al., 2016), however, existing results are mainly stated in expectation. Very recently, (Lei and Tang, 2021) provide the high probability optimization error bounds for nonconvex learning, but still requiring the smoothness condition. In this paper, we successfully remove the smoothness condition. Thus, it is notable that our optimization error bounds are established under pretty milder conditions: Assumptions 3, 5 and 6 (refer to Section 6.3).

5. Uniform Convergence Scenario

Let $B(w_0, R) := \{w \in \mathbb{R}^d : \|w - w_0\| \leq R\}$ denote a ball with center $w_0 \in \mathbb{R}^d$ and radius $R$. In this section, we assume the set $W$ satisfy the following condition: $W := B(w^*, R)$, where $w^* \in \arg\min_{w} F(w)$. All theorems established in this section are based on uniform convergence of gradients approach.

5.1 Convex Objectives

In this section, we assume the objective function $f$ is convex. Our first main result in convex learning is a probabilistic learning rate for ERM of stochastic optimization.

**Theorem 22** Suppose Assumptions 2 and 4 hold, and suppose population risk $F$ satisfies Assumption 7 with parameter $\mu$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the sample $S$, when $n \geq \frac{c\beta^2(d + \log(8\log(2nR+2)))}{\mu^2}$, we have

$$F(\hat{w}^*) - F(w^*) = O\left(\frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n}\right),$$

if further assume $F(w^*) = O\left(\frac{1}{n}\right)$, we have

$$F(\hat{w}^*) - F(w^*) = O\left(\frac{\log^2(1/\delta)}{n^2}\right),$$

where $\hat{w}^* \in \arg\min_{w} F_S(w)$ and $c$ is an absolute constant.

**Remark 23** From Theorem 22, one can see that if the objective function $f$ is smooth and nonnegative convex, the bernstein condition at the optimal point is satisfied and $F$ satisfies QG condition, the learning rate of excess risk of ERM of stochastic optimization is of faster order $O\left(\frac{\log^2(1/\delta)}{n^2}\right)$ when certain sample complexity is satisfied and the minimal expected risk is of order $O\left(\frac{1}{n}\right)$. It is notable that the assumption $F(w^*) \leq O\left(\frac{1}{n}\right)$
is common and can be found in (Zhang et al., 2017; Lei et al., 2020; Srebro et al., 2010; Lei et al., 2019; Lei and Ying, 2020; Liu et al., 2018), which is natural since $F(w^*)$ is the minimal population risk. We now compare our result with the related work of ERM in stochastic optimization. It was shown in (Shalev-Shwartz et al., 2009) that if the objective function satisfies the Lipschitz continuous condition, there has the following generalization error bound $F_S(w) - F(w) = O\left(\sqrt{\frac{d \log n \log(d/\delta)}{n}}\right)$ with high probability without requiring the objective function to be convex, implying that the learning rate of ERM is also of order $O\left(\sqrt{\frac{d \log n \log(d/\delta)}{n}}\right)$. A faster learning rate of order $O\left(\frac{d}{n^2}\right)$ of excess risk with probability at least $1 - \delta$ is further established under strongly convex and Lipschitz continuous conditions. The strong convexity condition is relaxed in (Koren and Levy, 2015; Mehta, 2017) by using the exp-concave functions, and they establish a learning rate of order $O\left(\frac{d}{n^2}\right)$ w.r.t. the sample size $n$ and dimensionality $d$ for ERM with high probability when the objective function is exp-concave and Lipschitz continuous. These learning rates are improved in (Zhang et al., 2017), which is the most related work to ours and where they also obtain the $O\left(\frac{1}{n^2}\right)$-type learning rates for ERM by uniform convergence of gradients approach. To be specific, if the objective function is nonnegative, convex and smooth, and the population risk $F$ is Lipschitz continuous, they obtain the learning rate of order $O\left(\frac{d \log n + \log 1/\delta}{n}\right)$ by assuming $F(w^*) = O\left(\frac{1}{n}\right)$; if the objective function is nonnegative, convex and smooth, and the population risk $F$ is Lipschitz continuous and strongly convex, they obtain the learning rate of order $O\left(\frac{\log n + \log 1/\delta}{n^2}\right)$ when certain sample complexity is satisfied and $F(w^*) = O\left(\frac{1}{n}\right)$; if the objective function is nonnegative and smooth, the empirical risk is convex and the population risk is strongly convex, they derive the similar learning rate of order $O\left(\frac{\log n + \log 1/\delta}{n^2}\right)$ when certain sample complexity is satisfied and $F(w^*) = O\left(\frac{1}{n}\right)$. Furthermore, for the generalized linear model, the similar learning rate can be obtained with a better sample complexity. For the generalized linear model, (Lei and Tang, 2021) also obtained improved rates of order $O\left(\frac{d}{n^2}\right)$ with a removed dimensionality $d$ for nonconvex SGD learning with PL curvature condition. However, one can see that the strong convexity and the smoothness conditions are necessary simultaneously in (Zhang et al., 2017). In a recent work (Liu et al., 2018), they use the uniform convergence of gradients in (Zhang et al., 2017) and relax the strong convexity condition. If the objective function $f$ is nonnegative, convex, Lipschitz continuous and smooth, and the population risk $F$ is QG, it was shown in (Liu et al., 2018) that $F(w^*) - F(w^*) = O\left(\frac{d \log n + \log (1/\delta)}{n}\right)^2 + O\left(\frac{d \log n + \log (1/\delta)}{n}\right)$ when certain sample complexity is satisfied. If further assume $F(w^*) = O\left(\frac{1}{n}\right)$, this bound implies a $O\left(\frac{d^2}{n^2}\right)$-type learning rate in high probability. Compared with (Liu et al., 2018), we all assume the QG condition, but we successfully replace the Lipschitz continuous condition by the milder bernstein condition at the optimal point. From the comparison, one can see that our learning rate of order $O\left(\frac{\log^2 (1/\delta)}{n^2}\right)$ is obviously faster and established under milder assumptions.

The second main result in convex learning is a probabilistic learning rate for the last iteration produced by (3).
Theorem 24 Suppose Assumptions 2, 4, 5 and 6 hold, and suppose empirical risk $F_S$ satisfies Assumption 7 with parameter $2\mu$. Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \frac{2}{\mu(t+t_0)}$ such that $t_0 \geq \max \left\{ \frac{4\beta}{\mu}, 1 \right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, when $n \geq c\beta^2(d+\log(\frac{16 \log(2nR+2)}{\mu^2}))$ and $T \asymp n^2$, we have the following result with probability at least $1-\delta$ over the sample $S$,

$$F(w_{T+1}) - F(w^*) = \mathcal{O}\left( \frac{\log n \log^3(1/\delta)}{n^2} + \frac{F(w^*, z) \log(1/\delta)}{n} \right),$$

if further assume $F(w^*) = \mathcal{O}\left( \frac{1}{n} \right)$, we have

$$F(w_{T+1}) - F(w^*) = \mathcal{O}\left( \frac{\log n \log^3(1/\delta)}{n^2} \right).$$

Remark 25 Theorem 24 shows that if the objective function is smooth and nonnegative convex, $F_S$ satisfies the QG condition, the bernstein condition at the optimal point, relaxed boundness of gradient and boundness of variance of stochastic gradients conditions are satisfied, the learning rate of excess risk of the last iteration of SGD is of faster order $\mathcal{O}\left( \frac{\log n \log^3(1/\delta)}{n^2} \right)$ when certain sample complexity is satisfied, $T \asymp n^2$ and $F(w^*) = \mathcal{O}\left( \frac{1}{n} \right)$.

Compared this result with Theorems 13 and 15 that are also high probability learning rates, learning rate in Theorem 24 is improved and obviously in faster order. Therefore, result in Theorem 24 is also faster than rates in (Rosasco and Villa, 2015; Hardt et al., 2016; Lin et al., 2016; Lei and Ying, 2020; Bassily et al., 2020; Feldman and Vondrak, 2019; Lei et al., 2021; Lin and Rosasco, 2017; Yang et al., 2021; Wang et al., 2021; Lei and Ying, 2020), where these work has been discussed in Remark 16. To our best knowledge, this is the first high probability $\mathcal{O}\left( \frac{1}{n^2} \right)$-type learning rate of SGD in convex learning. Moreover, Theorem 24 do not require the strong convexity condition that is a critical assumption in (Zhang et al., 2017) to establish $\mathcal{O}\left( \frac{1}{n^2} \right)$-type learning rate for ERM and the Lipschitz continuous condition in (Liu et al., 2018; Zhang et al., 2017).

### 5.2 Non-Convex Objectives

In this section, we assume the objective function $f$ is nonconvex. The first main result in nonconvex learning is also a probabilistic learning rate for ERM of stochastic optimization.

Theorem 26 Suppose Assumptions 2 and 4 hold, and suppose population risk $F$ satisfies Assumption 8 with parameter $\mu$. Then, for any $\delta > 0$, with probability $1-\delta$ over the sample $S$, when $n \geq \frac{c\beta^2(d+\log(\frac{8 \log(2nR+2)}{\mu^2}))}{\mu^2}$, we have

$$F(\hat{w}^*) - F(w^*) = \mathcal{O}\left( \frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n} \right),$$

if further assume $F(w^*) = \mathcal{O}\left( \frac{1}{n} \right)$, we have

$$F(\hat{w}^*) - F(w^*) = \mathcal{O}\left( \frac{\log^2(1/\delta)}{n^2} \right),$$

where $\hat{w}^* \in \arg\min F_S(w)$ and $c$ is an absolute constant.
Remark 27 Theorem 26 shows that if the objective function $f$ is smooth and nonnegative, the bernstein condition at the optimal point is satisfied and the population risk $F$ satisfies PL condition, the learning rate of ERM of stochastic optimization is of order $O\left(\frac{\log^2(1/\delta)}{n^2}\right)$ when certain sample complexity is satisfied and $F(w^*) = O\left(\frac{1}{n}\right)$. As discussed in Remark 18, if the population risk $F$ satisfies the convexity and QG conditions and the objective function $f$ satisfies the Lipschitz continuous condition, it was shown in (Liu et al., 2018) that $F(\hat{w}^*(S)) - F^* = O\left(\frac{d\log n + \log(1/\delta)}{n}\right)$ with $\theta \in (0, 1]$. By comparison with Theorems 17 and the above nonconvex ERM in (Liu et al., 2018), Theorem 26 obviously provide a faster learning rate. It is notable that Theorem 26 does not require the Lipschitz continuous condition and projection condition (Assumption 9).

We then present the following theorem on the decay rate of population gradients in general nonconvex case.

**Theorem 28** Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \eta_1 t^{-\theta}$, $\theta \in (0, 1)$ and $\eta_1 \leq \frac{1}{2\beta}$. Suppose Assumptions 2, 4, 5 and 6 hold. Then, for any $\delta > 0$, with probability $1 - \delta$, when $T \approx \left(\frac{nd}{\epsilon^2}\right)^{\frac{1}{2\theta - 1}}$, we have

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \begin{cases} 
O\left(\left(\frac{nd}{\epsilon^2}\right)^{\frac{\theta}{2\theta - 1}} \log^3(1/\delta)\right), & \text{if } \theta < 1/2, \\
O\left(\sqrt{\frac{d}{n}} \log(T/\delta) \log^3(1/\delta)\right), & \text{if } \theta = 1/2, \\
O\left(\sqrt{\frac{d}{n}} \log^3(1/\delta)\right), & \text{if } \theta > 1/2.
\end{cases}
$$

Remark 29 Theorem 28 shows that if the objective function is smooth, the bernstein condition at the optimal point, relaxed boundness of gradient and boundness of variance of stochastic gradients conditions are satisfied, the decay rates of the population gradients are shown as above. One can select an appropriate iteration number for early-stopping and an appropriate parameter $\theta$ to balance the iteration complexity and the learning rates. We emphasize that similar results to Theorem 28 have been established in (Lei and Tang, 2021). We provide this theorem here for three reasons: (1) we use different uniform convergence of gradients technique in our proof; (2) to provide the complete analysis for non-convex objectives; (3) to introduce the following Theorem 30 and show that if the empirical risk satisfies the PL curvature condition, the generalization performance of SGD will have a significant improvement.

**Theorem 30** Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \eta_1 t^{-\theta}$, $\theta \in (0, 1)$ and $\eta_1 \leq \frac{1}{2\beta}$. Suppose Assumptions 2, 4, 5 and 6 hold, and suppose empirical risk $F_S$ satisfies Assumption 8 with parameter $2\mu$. When $n \geq \frac{c\beta^2(\delta + \log(\frac{16\log(2nR+2)})}{\mu^2}$, for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = O\left(\frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n}\right),
$$

Remark 27 Theorem 26 shows that if the objective function $f$ is smooth and nonnegative, the bernstein condition at the optimal point is satisfied and the population risk $F$ satisfies PL condition, the learning rate of ERM of stochastic optimization is of order $O\left(\frac{\log^2(1/\delta)}{n^2}\right)$ when certain sample complexity is satisfied and $F(w^*) = O\left(\frac{1}{n}\right)$. As discussed in Remark 18, if the population risk $F$ satisfies the convexity and QG conditions and the objective function $f$ satisfies the Lipschitz continuous condition, it was shown in (Liu et al., 2018) that $F(\hat{w}^*(S)) - F^* = O\left(\frac{d\log n + \log(1/\delta)}{n}\right)$ with $\theta \in (0, 1]$. By comparison with Theorems 17 and the above nonconvex ERM in (Liu et al., 2018), Theorem 26 obviously provide a faster learning rate. It is notable that Theorem 26 does not require the Lipschitz continuous condition and projection condition (Assumption 9) in Theorem 17.

We then present the following theorem on the decay rate of population gradients in general nonconvex case.

**Theorem 28** Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \eta_1 t^{-\theta}$, $\theta \in (0, 1)$ and $\eta_1 \leq \frac{1}{2\beta}$. Suppose Assumptions 2, 4, 5 and 6 hold. Then, for any $\delta > 0$, with probability $1 - \delta$, when $T \approx \left(\frac{nd}{\epsilon^2}\right)^{\frac{1}{2\theta - 1}}$, we have

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \begin{cases} 
O\left(\left(\frac{nd}{\epsilon^2}\right)^{\frac{\theta}{2\theta - 1}} \log^3(1/\delta)\right), & \text{if } \theta < 1/2, \\
O\left(\sqrt{\frac{d}{n}} \log(T/\delta) \log^3(1/\delta)\right), & \text{if } \theta = 1/2, \\
O\left(\sqrt{\frac{d}{n}} \log^3(1/\delta)\right), & \text{if } \theta > 1/2.
\end{cases}
$$

Remark 29 Theorem 28 shows that if the objective function is smooth, the bernstein condition at the optimal point, relaxed boundness of gradient and boundness of variance of stochastic gradients conditions are satisfied, the decay rates of the population gradients are shown as above. One can select an appropriate iteration number for early-stopping and an appropriate parameter $\theta$ to balance the iteration complexity and the learning rates. We emphasize that similar results to Theorem 28 have been established in (Lei and Tang, 2021). We provide this theorem here for three reasons: (1) we use different uniform convergence of gradients technique in our proof; (2) to provide the complete analysis for non-convex objectives; (3) to introduce the following Theorem 30 and show that if the empirical risk satisfies the PL curvature condition, the generalization performance of SGD will have a significant improvement.

**Theorem 30** Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \eta_1 t^{-\theta}$, $\theta \in (0, 1)$ and $\eta_1 \leq \frac{1}{2\beta}$. Suppose Assumptions 2, 4, 5 and 6 hold, and suppose empirical risk $F_S$ satisfies Assumption 8 with parameter $2\mu$. When $n \geq \frac{c\beta^2(\delta + \log(\frac{16\log(2nR+2)})}{\mu^2}$, for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = O\left(\frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n}\right),
$$
if further assume $F(w^*) = \mathcal{O}\left(\frac{1}{n}\right)$, we have

$$\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \mathcal{O}\left(\frac{\log^2(1/\delta)}{n^2}\right),$$

when $T \asymp n^{\frac{2}{\theta}}$ if $\theta \in (0, 1/2)$, $T \asymp n^4$ if $\theta = 1/2$ and $T \asymp n^{\frac{2}{1-\theta}}$ if $\theta \in (1/2, 1)$.

Remark 31 Theorem 30 shows that if another PL condition is satisfied, the decay rate of the population gradients can be significantly improved, shown a faster order $\mathcal{O}\left(\frac{\log^2(1/\delta)}{n^2}\right)$. From Theorem 30, one also can see that the corresponding computational complexity varied for different $\theta$ achieves its minimum when taking $\theta$ close to 1/2, while the decay rate of the population gradients is not influenced by $\theta$ when choose the optimal $T$. In the following theorem, we will show that when the stepsize is restricted to $\eta_t = \frac{2}{\mu(t+t_0)}$ such that $t_0 \geq \max\left\{\frac{4\beta}{\mu}, 1\right\}$ for all $t \in \mathbb{N}$, the computation complexity $T$ can further be significantly improved, to the order $\mathcal{O}(n^2)$.

The last main result in nonconvex learning is a probabilistic learning rate for the last iteration of SGD.

Theorem 32 Suppose Assumptions 2, 4, 5 and 6 hold, and suppose empirical risk $F_S$ satisfies Assumption 8 with parameter $2\mu$. Let $\{w_t\}_t$ be the sequence produced by (3) with $\eta_t = \frac{2}{\mu(t+t_0)}$ such that $t_0 \geq \max\left\{\frac{4\beta}{\mu}, 1\right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the sample $S$, when $n \geq \frac{c\delta^3(d+\log(16\log(2nR^2)))}{\mu^2}$ and $T \asymp n^2$, we have

$$F(w_{T+1}) - F(w^*) = \mathcal{O}\left(\frac{\log n \log^3(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n}\right),$$

if further assume $F(w^*) = \mathcal{O}\left(\frac{1}{n}\right)$, we have

$$F(w_{T+1}) - F(w^*) = \mathcal{O}\left(\frac{\log n \log^3(1/\delta)}{n^2}\right).$$

Remark 33 From Theorem 32, one can see that if the objective function is nonnegative smooth, $F_S$ satisfies the PL condition, bernstein condition at the optimal point, relaxed boundedness of gradient and boundness of variance of stochastic gradients are satisfied, the excess risk of the last iteration of SGD shows a fast learning rate of order $\mathcal{O}\left(\frac{\log n \log^3(1/\delta)}{n^2}\right)$ when we run SGD $T \asymp n^2$ times, certain sample complexities is satisfied and $F(w^*) = \mathcal{O}\left(\frac{1}{n}\right)$. To the best of our knowledge, this learning rate is the first $\mathcal{O}(\frac{1}{n})$-type learning rate of SGD in nonconvex learning, in addition, studied in high probability. We now demonstrate an interesting theoretical finding. In the proof of Theorem 32, we use PL property to transform the excess risk of SGD to the norm of gradient of population risk, that is $\|\nabla F(w_{T+1})\|^2$. This term can then be bounded as follows: $2\|\nabla F_S(w_{T+1})\|^2 + 2\|\nabla F(w_{T+1}) - \nabla F_S(w_{T+1})\|^2$, where the first term is optimization error since it is related to the optimization algorithm to minimize the empirical risk $F_S$, and the
second term is statistical error since it is related to the gap between the true gradient and its empirical version based on random samples. By bounding the two terms separately, we finally derive that they can be bounded by $O\left(\frac{\log T \log^3(1/\delta)}{\eta^2}\right)$ and $O\left(\frac{\log T \log^3(1/\delta)}{\eta^2} + \frac{\log^2(1/\delta)}{n^2}\right)$, respectively. One can see that the optimization error decreases as the iteration number increases. It is interesting that the statistical error also decreases as the iteration number increases, which is different from the situation in Theorem 28. In Theorem 28, the statistical errors increase along the learning process because $w_t$ traverses over a ball with an increasing radius as $t$ increases (refer to the proof of Theorem 28). Therefore, Theorem 32 reveals that if the objective function satisfies certain curvature conditions, for instance the PL condition, the overfitting phenomenon will never happen and the generalization would always improve as we increase the training accuracy, which also implies that the classical bias-variance trade-off (that a model should balance under-fitting and over-fitting: rich enough to express underlying structure in data, simple enough to avoid fitting spurious patterns) may be imprecise in some cases. This theoretical finding matches the recent statements in (Belkin et al., 2019), where they provide a “double descent” curve showing how increasing the model capacity beyond the interpolation point results in improved performance. Furthermore, this finding may serve as a perspective for explaining why neural network generalizes well with an over parameterization setting, because there is some empirical evidence suggesting that modern deep neural network might satisfy the PL condition in large neighborhoods of global minimizers (Zhou et al., 2019; Kleinberg et al., 2018).

6. Proofs of Stability Scenario

In this section, we provide the proofs of Theorems in stability scenario.

6.1 Basic Lemma

**Assumption 10 (Generalized Bernstein Condition)** Assume $W^* = \arg\min_{w \in W} F(w)$ is a set of risk minimizers in a closed set $W$. We say that $W$ together with the measure $\rho$ and the function $f$ satisfy the generalized Bernstein assumption if for some $B > 0$ and for any $w \in W$, there is $w^* \in W^*$ such that

$$E(f(w, z) - f(w^*, z))^2 \leq B(F(w) - F(w^*))$$.

**Remark 34** This assumption is a relaxed version of the original Bernstein condition, because it only requires for every $w \in W$ there exists one $w^*$ that could be different for different $w$ such that the above inequality holds, while the original Bernstein condition requires there exists a universal $w^*$ for all $w \in W$ such that the above inequality holds (Liu et al., 2018).

We first introduce a basic lemma used for stability scenario.

**Lemma 35 ((Klochkov and Zhivotovskiy, 2021))** Assume that the objective function $f$ is bounded in $[0, M]$. Let $\mathcal{A}(S)$ be a $\epsilon$-stable algorithm that has the optimization error $\epsilon_{opt}$. Suppose Assumption 10 is satisfied with the parameter $B$. There is an absolute constant $c > 0$ such that the following holds. Fix any $\eta > 0$. Then, with probability at least $1 - \delta$, it
holds that
\[ F(A(S)) - F(w^*) \leq \epsilon_{opt}(A(S)) + \eta \epsilon_{opt}(A(S)) + c(1 + 1/\eta) \left( \epsilon \log n + \frac{M + B}{n} \right) \log \left( \frac{1}{\delta} \right), \]
where \( \epsilon_{opt}(A(S)) = F_S(A(S)) - \min_{w \in W} F_S(w) \).

The following lemma reveals that QG condition (Assumption 1) and Lipschitz continuous condition imply the generalized bernstein condition (Assumption 10).

**Lemma 36** Suppose Assumption 1 holds and the population risk \( F \) satisfies Assumption 7 with parameter \( \mu \), we can derive the generalized Bernstein condition of Assumption 10 with parameter \( \frac{2L^2}{\mu} \).

**Proof** From Assumption 1, the Lipschitz property implies that for any \( w, w^* \in W \), there holds
\[
\mathbb{E}(f(w, z) - f(w^*, z))^2 \leq L^2\|w - w^*\|^2. \tag{4}
\]
From Assumption 7, the QG condition implies that for any \( w \in W \) and the closest optima point \( w^* \) of \( F \) to \( w \),
\[
F(w) - F(w^*) \geq \frac{\mu}{2}\|w - w^*\|^2. \tag{5}
\]
Combined inequality (4) and (5), we have
\[
\mathbb{E}(f(w, z) - f(w^*, z))^2 \leq L^2\|w - w^*\|^2 \leq \frac{2L^2}{\mu}(F(w) - F(w^*)),
\]
which implies for any \( w \in W \), there is \( w^* \in W^* \) such that
\[
\mathbb{E}(f(w, z) - f(w^*, z))^2 \leq \frac{2L^2}{\mu}(F(w) - F(w^*)).
\]
The proof is complete. \( \square \)

### 6.2 Stability Bound

The following lemma shows the uniform stability of ERM of stochastic optimization under Lipschitz continuous and convexity of \( f \) and QG condition.

**Lemma 37** Suppose the objective function \( f \) is convex. Let \( \hat{w}^*(S^i) \) be the ERM of \( F_{S^i}(w) \) that denotes the empirical risk on the samples \( S^i = \{z_1^i, ..., z_n^i\} \) and \( \hat{w}^*(S) \) be the ERM of \( F_S(w) \) on the samples \( S = \{z_1, ..., z_i, ..., z_n\} \). Suppose Assumption 1 holds and \( F_S \) satisfies Assumption 7 with parameter \( \mu \). For any \( S^i \) and \( S \), there holds the following uniform stability bound of ERM:
\[
\forall z \in \mathcal{Z}, \quad |f(\hat{w}^*(S^i), z) - f(\hat{w}^*(S), z)| \leq \frac{4L^2}{n\mu}.
\]
Proof Since $F_S(w) = \frac{1}{n} \left( f(w, z'_i) + \sum_{j \neq i} f(w, z_j) \right)$, we have

\[
F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S))
= \frac{\sum_{j \neq i} (f(\hat{w}^*(S^i), z_j) - f(\hat{w}^*(S), z_j))}{n}
= \frac{f(\hat{w}^*(S^i), z_i) - f(\hat{w}^*(S), z_i)}{n} + \frac{f(\hat{w}^*(S), z'_i) - f(\hat{w}^*(S^i), z'_i)}{n}
+ (F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)))
\leq \frac{2L}{n} \| \hat{w}^*(S^i) - \hat{w}^*(S) \|,
\]

where the first inequality follows from the fact that $\hat{w}^*(S^i)$ is the ERM of $F_S$ and the second inequality follows from the Lipschitz property. Furthermore, for $\hat{w}^*(S^i)$, the convexity of $f$ and the QG property of $F_S$ imply that its closest optima point of $F_S$ is $\hat{w}^*(S)$ (the global minimizer of $F_S$ is unique). Then, there holds that

\[
F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \geq \frac{\mu}{2} \| \hat{w}^*(S^i) - \hat{w}^*(S) \|^2.
\]

Then we get

\[
\frac{\mu}{2} \| \hat{w}^*(S^i) - \hat{w}^*(S) \|^2 \leq F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \leq \frac{2L}{n} \| \hat{w}^*(S^i) - \hat{w}^*(S) \|,
\]

which implies that $\| \hat{w}^*(S^i) - \hat{w}^*(S) \| \leq \frac{4L}{n \mu}$. Combined with the Lipschitz property of $f$ we obtain that for any $S^i$ and $S$

\[
\forall z \in Z, \quad |f(\hat{w}^*(S^i), z) - f(\hat{w}^*(S), z)| \leq \frac{4L^2}{n \mu}.
\]

The proof is complete.

The following lemma shows the uniform stability of SGD of stochastic optimization under Lipschitz continuous and convexity of $f$ and QG condition.

Lemma 38 Suppose the objective function $f$ is convex. Let $w^i_t$ be the output of $F_S(w)$ that denotes the empirical risk on the samples $S^i = \{z_1, \ldots, z'_i, \ldots, z_n\}$ and $w^i_t$ be the output of $F_S(w)$ on $t$-th iteration on the samples $S = \{z_1, \ldots, z_i, \ldots, z_n\}$ in running SGD. Suppose Assumption 1 holds and $F_S$ satisfies Assumption 7 with parameter $\mu$. For any $S^i$ and $S$, there holds the following uniform stability bound of SGD:

\[
\forall z \in Z, \quad |f(w_t, z) - f(w^i_t, z)| \leq 2L \sqrt{\frac{2\epsilon_{opt}(w_t)}{\mu}} + \frac{4L^2}{n \mu},
\]

where $\epsilon_{opt}(w_t) = F_S(w_t) - F_S(\hat{w}^*(S))$ and $\hat{w}^*(S)$ is the ERM of $F_S(w)$.  

24
Thus we have $\|F(w)\|_{\infty} \leq L$. We denote $\hat{w}^*(S)$ be the ERM of $F_S(w)$ and $\hat{w}_S^*$ be the ERM of $F_S(w)$. From Lemma 37, we know that

$$\forall z \in Z, \quad |f(\hat{w}^*(S), z) - f(\hat{w}(S), z)| \leq \frac{4L^2}{n\mu}.$$ 

Also, for $w_t$, the convexity of $f$ and the QG property implies that its closest optima point of $F_S$ is $\hat{w}^*(S)$ (the global minimizer of $F_S$ is unique). Then, there holds that

$$\frac{\mu}{2}||w_t - \hat{w}^*(S)||^2 \leq F_S(w_t) - F_S(\hat{w}^*(S)) = \epsilon_{opt}(w_t).$$

Thus we have $||w_t - \hat{w}^*(S)|| \leq \sqrt{\frac{2\epsilon_{opt}(w_t)}{\mu}}$. A similar relation holds between $\hat{w}^*(S)$ and $w_t$.

Combined with the Lipschitz property of $f$ we obtain that for $\forall z \in Z$, there holds that

$$|f(w_t, z) - f(w_t', z)| \leq |f(w_t, z) - f(\hat{w}^*(S), z)| + |f(\hat{w}^*(S), z) - f(\hat{w}^*(S'), z)| + |f(\hat{w}^*(S'), z) - f(w_t', z)| \leq L||w_t - \hat{w}^*(S)|| + \frac{4L^2}{n\mu} + L||\hat{w}^*(S') - w_t||$$

$$\leq \sqrt{\frac{2\epsilon_{opt}(w_t)}{\mu}} + \sqrt{\frac{4L^2}{n\mu}} + \sqrt{\frac{2\epsilon_{opt}(w_t')}{\mu}}.$$

In the following Lemma 43, for any dataset $S$, the optimization error $\epsilon_{opt}(w_t)$ is uniformly bounded by the same upper bound. Therefore, we write $|f(w_t, z) - f(w_t', z)| \leq 2L\sqrt{\frac{2\epsilon_{opt}(w_t)}{\mu}} + \frac{4L^2}{n\mu}$ here. The proof is complete.

### 6.3 Optimization Error

We now establish optimization error bounds of $\alpha$-Hölder smooth functions. Some lemmas are introduced firstly.

**Lemma 39 ((Ying and Zhou, 2017))** Let $f$ be a differentiable function. Let $\alpha \in (0,1]$ and $P > 0$. If Assumption 3 holds, for any $w_1, w_2 \in W$ and $z \in Z$, then we have

$$f(w_1, z) - f(w_2, z) \leq (w_1 - w_2, \nabla f(w_2, z)) + \frac{P||w_1 - w_2||^{1+\alpha}}{1+\alpha}.$$

**Lemma 40 ((Lei and Tang, 2021))** Let $e$ be the base of the natural logarithm. There holds the following elementary inequalities.

(a) If $\theta \in (0,1)$, then $\sum_{k=1}^{t} k^{-\theta} \leq t^{1-\theta}/(1-\theta)$; 
(b) If $\theta = 1$, then $\sum_{k=1}^{t} k^{-\theta} \leq \log(et)$; 
(c) If $\theta > 1$, then $\sum_{k=1}^{t} k^{-\theta} \leq \frac{\theta}{\theta-1}$. 

25
Lemma 41 Suppose Assumptions 3, 5 and 6 hold. Let \( \{w_t\} \) be the sequence produced by (3) with that \( \eta_t \leq (1/(2P))^{1/\alpha} \) for all \( t \in \mathbb{N} \). Then, for any \( \delta > 0 \), with probability \( 1 - \delta \), we have

\[
\sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2 \leq C \log(2/\delta) + C_t,
\]

where \( C_t = 4 \sup_{z \in \mathcal{Z}} f(0, z) + 4 \max\{PG^2, C_1\} \sum_{k=1}^{t} \eta_k^{2\alpha} + \sum_{k=1}^{t} \eta_k^{1+\alpha} \) and \( C = 32PG^2 + 8 \max\{G^2, 2\sigma^2(2P)^{-1/\alpha}\} \), and where \( C_1 := \frac{1}{1+\alpha}\).\( \frac{1}{2} + (1 + \alpha)\sigma^2 \).

Proof Denote \( \xi_t = \eta_t \langle \nabla F_S(w_t) - \nabla f(w_t, z_{j_t}), \nabla F_S(w_t) \rangle \) and \( \xi'_t = \|\nabla f(w_t, z_{j_t}) - \nabla F_S(w_t)\|^2 - \mathbb{E}_{j_t} \|\nabla f(w_t, z_{j_t}) - \nabla F_S(w_t)\|^2 \). Since the objective function \( f \) satisfies Assumption 3, it is easy to verify that \( F_S \) also satisfies Assumption 3:

\[
\|\nabla F_S(w_1) - \nabla F_S(w_2)\| = \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f(w_1, z_i) - \frac{1}{n} \sum_{i=1}^{n} \nabla f(w_2, z_i) \right\| \\
\leq \frac{1}{n} \sum_{i=1}^{n} \|\nabla f(w_1, z_i) - \nabla f(w_2, z_i)\| \\
\leq P \|w_1 - w_2\|^{\alpha}.
\]

And from Lemma 39, we have

\[
F_S(w_{t+1}) \leq F_S(w_t) + \langle w_{t+1} - w_t, \nabla F_S(w_t) \rangle + \frac{P \|w_{t+1} - w_t\|^{1+\alpha}}{1 + \alpha} \\
= F_S(w_t) + \eta_t \langle \nabla F_S(w_t) - \nabla f(w_t, z_{j_t}), \nabla F_S(w_t) \rangle - \eta_t \|\nabla F_S(w_t)\|^2 \\
+ \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \|\nabla f(w_t, z_{j_t})\|^{1+\alpha} \\
\leq F_S(w_t) + \xi_t - \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \|\nabla f(w_t, z_{j_t})\|^{1+\alpha} \\
\leq F_S(w_t) + \xi_t - \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \left[ \frac{1 - \alpha}{2} + \frac{1 + \alpha}{2} (\|\nabla f(w_t, z_{j_t})\|^{1+\alpha})^{\frac{1}{1+\alpha}} \right] \\
\leq F_S(w_t) + \xi_t - \eta_t \|\nabla F_S(w_t)\|^2 \\
+ \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \left[ \frac{1 - \alpha}{2} + \frac{1 + \alpha}{2} \right] \left[ 2 \xi'_t + 2 \mathbb{E}_{j_t} \|\nabla f(w_t, z_{j_t}) - \nabla F_S(w_t)\|^2 \right] + 2 \|\nabla F_S(w_t)\|^2 \\
\leq F_S(w_t) + \xi_t - \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \left[ \frac{1 - \alpha}{2} + \frac{1 + \alpha}{2} \right] \left[ 2 \xi'_t + 2 \sigma^2 + 2 \|\nabla F_S(w_t)\|^2 \right] \\
\leq F_S(w_t) + \xi_t - 2^{-1} \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P \eta_t^{1+\alpha}}{1 + \alpha} \left[ \frac{1 - \alpha}{2} + (1 + \alpha) \left[ \xi'_t + \sigma^2 \right] \right],
\]

(7)
where the second and the fifth inequality follows from the notation, the third follows from the Young’s inequality: for all $\mu, v \in \mathbb{R}, p^{-1} + q^{-1} = 1, p \geq 0$

$$\mu v \leq p^{-1}|\mu|^p + q^{-1}|v|^q,$$

the sixth inequality follows from Assumption 6, and the last follows from the fact that $P\eta_t^{1+\alpha} \leq P[(1/(2P))^{1/\alpha}\eta_t \leq 2^{-1}\eta_t].$

Denoted by $C_1 := \frac{P}{1+\alpha}(\frac{1}{2} + (1 + \alpha)\sigma^2)$. Taking a summation of the above inequality gives

$$F_S(w_{t+1}) = F_S(w_1) + \sum_{k=1}^{t}(F_S(w_{k+1}) - F_S(w_k)) \leq F_S(w_1) + \sum_{k=1}^{t}\xi_k - 2^{-\alpha-1}\sum_{k=1}^{t}\eta_k\|\nabla F_S(w_k)\|^2 + t\sum_{k=1}^{t}P\eta_k^{1+\alpha}\xi_k' + tC_1\eta_k^{1+\alpha}. \quad (8)$$

Since $\mathbb{E}_{j_k}\xi_k = 0$, thus $\{\xi_k\}$ is a martingale difference sequence. There will hold

$$|\xi_k| \leq \eta_k \left(\|\nabla F_S(w_k)\| + \|\nabla f (w_k, z_{j_k})\|\right) \|\nabla F_S(w_k)\| \leq 2G^2, \quad (9)$$

followed from Assumption 5. Moreover, we have

$$\sum_{k=1}^{t}\mathbb{E}_{j_k}[(\xi_k - \mathbb{E}_{j_k}\xi_k)^2] = \sum_{k=1}^{t}\mathbb{E}_{j_k}(\xi_k)^2 \leq \sum_{k=1}^{t}\eta_k^2\mathbb{E}_{j_k} \left(\|\nabla F_S(w_k) - \nabla f (w_k, z_{j_k})\|^2\right) \|\nabla F_S(w_k)\|^2 \leq \sigma^2 \sum_{k=1}^{t}\eta_k^2\|\nabla F_S(w_k)\|^2 \leq (2P)^{-1/\alpha}\sigma^2 \sum_{k=1}^{t}\eta_k\|\nabla F_S(w_k)\|^2, \quad (10)$$

where the second inequality follows from Assumption 6 and the last inequality follows from the fact that $\eta_t \leq (1/(2P))^{1/\alpha}$ for all $t \in \mathbb{N}$. Substituting (9) and (10) into part (b) of Lemma 53 with $\rho = \min\{1, G^2(2P)^{1/\alpha}(2\sigma^2)^{-1}\}$, we have the following inequality with probability at least $1 - \delta/2$.

$$\sum_{k=1}^{t}\xi_k \leq \frac{\rho(2P)^{-1/\alpha}\sigma^2 \sum_{k=1}^{t}\eta_k\|\nabla F_S(w_k)\|^2}{2G^2} + \frac{2G^2\log(2/\delta)}{\rho} \leq \frac{1}{4} \sum_{k=1}^{t}\eta_k\|\nabla F_S(w_k)\|^2 + 2\log(2/\delta) \max\{G^2, 2\sigma^2(2P)^{-1/\alpha}\}. \quad (11)$$
Similarly, according to Assumption 5, we have

$$|\xi'_k| \leq 2 (\|\nabla F_S(w_k)\|^2 + \|\nabla f(w_k, z_{jk})\|^2) \leq 4\eta_k^{-1}G^2.$$ (12)

Substituting (12) into part (a) of Lemma 53, we have the following inequality with probability at least $1 - \delta/2$

$$\sum_{k=1}^{t} \eta_k^{1+\alpha} \xi'_k \leq 4G^2 \left( 2 \sum_{k=1}^{t} \eta_k^{2\alpha} \log \left( \frac{2}{\delta} \right) \right)^{\frac{1}{2}} \leq 8G^2 \log(2/\delta) + G^2 \sum_{k=1}^{t} \eta_k^{2\alpha},$$ (13)

where the last inequality follows from the Schwartz’s inequality. Substituting (11) and (13) into (8), we have the following inequality with probability at least $1 - \delta$

$$F_S(w_{t+1}) \leq F_S(0) + 2 \log(2/\delta) \max\{G^2, 2\sigma^2(2P)^{-1/\alpha}\} - \frac{1}{4} \sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2$$

$$+ P \left[ 8G^2 \log(2/\delta) + G^2 \sum_{k=1}^{t} \eta_k^{2\alpha} \right] + \sum_{k=1}^{t} C_1 \eta_k^{1+\alpha},$$

which implies that

$$\frac{1}{4} \sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2$$

$$\leq F_S(0) + \log(2/\delta) \left[ 2 \max\{G^2, 2\sigma^2(2P)^{-1/\alpha}\} + 8PG^2 \right] + PG^2 \sum_{k=1}^{t} \eta_k^{2\alpha} + \sum_{k=1}^{t} C_1 \eta_k^{1+\alpha}.$$

Therefore, we have the following inequality with probability at least $1 - \delta$

$$\sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2 \leq C \log(2/\delta) + C_t,$$

where $C_t = 4 \sup_{z \in \mathbb{Z}} f(0, z) + 4 \max\{PG^2, C_1\} (\sum_{k=1}^{t} \eta_k^{2\alpha} + \sum_{k=1}^{t} \eta_k^{1+\alpha})$ and $C = 32PG^2 + 8 \max\{G^2, 2\sigma^2(2P)^{-1/\alpha}\}$. \hfill \blacksquare

**Lemma 42** Suppose Assumptions 3, 5 and 6 hold. Let $\{w_t\}_t$ be the sequence produced by (3) with that $\eta_k \leq (1/(2P))^{1/\alpha}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability $1 - \delta$, we have the following inequality for all $t = 1, \ldots, T$

$$\|w_{t+1}\| \leq C_2 \left( \left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} + 1 + \left( \sum_{k=1}^{t} \eta_k \right)^{\frac{1}{2}} \left( \sum_{k=1}^{t} \eta_k^{2\alpha} + \sum_{k=1}^{t} \eta_k^{1+\alpha} \right)^{\frac{1}{2}} \right) \log \left( \frac{4}{\delta} \right),$$

where $C_2 = \max \left\{ \frac{4G(1/(2P))^{1/2\alpha}}{3}, 2\sigma, 4\sqrt{C} + 4 \sup_{z \in \mathbb{Z}} f(0, z), 4 \sqrt{4 \max\{PG^2, C_1\}} \right\}$. 28
Denoted by $\xi_t = \eta_t (\nabla F_S(w_t) - \nabla f(w_t, z_{j_t}))$. According to (3), we have

$$w_{t+1} = w_t - \eta_t (\nabla f(w_t, z_{j_t}) - \nabla F_S(w_t)) - \eta_t \nabla F_S(w_t).$$

Taking a summation and using $w_1 = 0$, we get

$$w_{t+1} = \sum_{k=1}^{t} \xi_k - \sum_{k=1}^{t} \eta_k \nabla F_S(w_k).$$

There also holds that

$$\|w_{t+1}\| \leq \left\| \sum_{k=1}^{t} \xi_k \right\| + \left\| \sum_{k=1}^{t} \eta_k \nabla F_S(w_k) \right\|. \quad (14)$$

Similarly, $\{\xi_k\}$ is a martingale difference sequence. Firstly,

$$\|\xi_k\| = \eta_k \|\nabla F_S(w_k) - \nabla f(w_k, z_{j_k})\|$$

$$\leq \sqrt{\eta_k} \left( \sqrt{2} \sqrt{\eta_k} \sup_{z \in Z} \|\nabla f(w_k, z)\| \right)$$

$$\leq 2G \sqrt{\eta_k} \leq 2G(2P)^{-1/(2\alpha)}, \quad (15)$$

where the second inequality follows from Assumption 5 and the last inequality follows from the fact that $\eta_t \leq (1/2P)^{1/\alpha}$ for all $t \in \mathbb{N}$. Secondly, according to Assumption 6, we have

$$\sum_{k=1}^{T} E_{j_k}[\|\xi_k\|^2] \leq \sum_{k=1}^{T} \eta_k^2 \sigma^2. \quad (16)$$

Substituting (15) and (16) into Lemma 54, we have the following inequality with probability at least $1 - \delta/2$

$$\max_{1 \leq t \leq T} \left\| \sum_{k=1}^{t} \xi_k \right\| \leq 2 \left( \frac{2G(2P)^{-1/(2\alpha)}}{3} + \sigma \left( \sum_{k=1}^{T} \eta_k^2 \right)^{1/2} \right) \log \frac{4}{\delta}. \quad (17)$$

For the term $\left\| \sum_{k=1}^{t} \eta_k \nabla F_S(w_k) \right\|$, according to Lemma 41 and the Schwartz’s inequality, we have the following inequality with probability at least $1 - \delta/2$,

$$\left\| \sum_{k=1}^{t} \eta_k \nabla F_S(w_k) \right\|^2 \leq \left( \sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\| \right)^2$$

$$\leq \left( \sum_{k=1}^{t} \eta_k \right) \left( \sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2 \right)$$

$$\leq \left( \sum_{k=1}^{t} \eta_k \right) (C \log(4/\delta) + C_t)$$

$$\leq \left( \sum_{k=1}^{t} \eta_k \right) (C \log(4/\delta) + C_T). \quad (18)$$
Substituting (18) and (17) into (14), we have the following inequality with probability at least $1 - \delta$

$$\|w_{t+1}\| \leq 2 \left( \frac{2G(2P)^{-1/(2\alpha)}}{3} + \sigma \left( \sum_{k=1}^{T} \eta_k^2 \right)^{1/2} \right) \log \frac{4}{\delta} + \left( \sum_{k=1}^{T} \eta_k (C \log(4/\delta) + C_T) \right)^{1/2}$$

$$\leq C_2 \left( \left( \sum_{k=1}^{T} \eta_k^2 \right)^{1/2} + 1 + \left( \sum_{k=1}^{T} \eta_k \right)^{1/2} \right) \log(4/\delta),$$

where $C_2 = \max \left\{ \frac{4G(2P)^{-1/(2\alpha)}}{3}, 2\sigma, 4\sqrt{C + 4\sup_{z \in Z} f(0, z)} \right\}$. The proof is complete.

Lemma 43 Suppose Assumptions 3, 5 and 6 hold, and suppose $F_S$ satisfies Assumption 8 with parameter $2\mu$. Let $(w_t)_{t\geq 0}$ be the sequence produced by (3) with $\eta_t = \frac{2(2P)^{1/\alpha}}{\mu}$ such that $t_0 \geq \max \left\{ \frac{2(2P)^{1/\alpha}}{\mu}, 1 \right\}$ for all $t \in \mathbb{N}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$F_S(w_{t+1}) - F_S(\hat{w}^*) = \begin{cases} \mathcal{O} \left( \frac{1}{T} \right) & \text{if } \alpha \in (0, 1), \\ \mathcal{O} \left( \frac{\log(T) \log(1/\delta)}{T} \right) & \text{if } \alpha = 1. \end{cases}$$

Proof When $\eta_t = \frac{2}{\mu(t+t_0)}$ with $t_0 \geq \frac{2(2P)^{1/\alpha}}{\mu}$, we have $\eta_t \leq (2P)^{-1/\alpha}$. Then, from (7), we know that

$$F_S(w_{t+1}) \leq F_S(w_t) + \xi_t - 2^{-1} \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P\eta_t^{1+\alpha}}{1+\alpha} \left[ \frac{1 - \alpha}{2} + (1 + \alpha) (\xi_t' + \sigma^2) \right],$$

where $\xi_t = \eta_t (\nabla F_S(w_t) - \nabla f(w_t, z_{tj}), \nabla F_S(w_t))$ and $\xi_t' = \|\nabla f(w_t, z_{tj}) - \nabla F_S(w_t)\|^2 - \mathbb{E}_{z_t} \|\nabla f(w_t, z_{tj}) - \nabla F_S(w_t)\|^2$. Since $F_S$ satisfies the PL condition with parameter $2\mu$ (which means $F_S(w) - F_S \leq \frac{1}{4\mu} \|\nabla F_S(w)\|^2$ in Assumption 8), we have

$$F_S(w_{t+1}) \leq F_S(w_t) + \xi_t - 4^{-1} \eta_t \|\nabla F_S(w_t)\|^2 + \frac{P\eta_t^{1+\alpha}}{1+\alpha} \xi_t' + C_1 \eta_t^{1+\alpha} + \eta_t \mu (F_S(\hat{w}^*) - F_S(w_t)),$$

where $C_1 = \frac{P}{4\mu} (\frac{1-\alpha}{2} + (1 + \alpha) \sigma^2)$.

There holds that

$$F_S(w_{t+1}) - F_S(\hat{w}^*) + \frac{1}{2\mu(t+t_0)} \|\nabla F_S(w_t)\|^2 \leq \xi_t + \frac{P\eta_t^{1+\alpha}}{1+\alpha} \xi_t' + C_1 \eta_t^{1+\alpha} + \frac{t+t_0-2}{t+t_0} (F(w_t) - F(\hat{w}^*)).$$

Multiply both side by $(t+t_0)(t+t_0-1)$, we have

$$(t+t_0)(t+t_0-1) [F_S(w_{t+1}) - F_S(\hat{w}^*)] + \frac{t+t_0-1}{2\mu} \|\nabla F_S(w_t)\|^2 \leq (t+t_0)(t+t_0-1) \xi_t + P \left( \frac{2}{\mu} \right)^{1+\alpha} \xi_t' (t+t_0)^{-\alpha} (t+t_0-1)$$

$$+ C_1 \left( \frac{2}{\mu} \right)^{1+\alpha} (t+t_0)^{-\alpha} (t+t_0-1) + (t+t_0)(t+t_0-1)(t+t_0-2)(F(w_t) - F(\hat{w}^*)).$$

30
Take a summation from $t = 1$ to $t = T$, we obtain

\[
\sum_{t=1}^{T} \frac{t + t_0 - 1}{2\mu} \|\nabla F_S(w_t)\|^2 + (T + T_0)(T + T_0 - 1)[F_S(w_{T+1}) - F_S(\hat{w}^*)]
\leq \sum_{t=1}^{T} (t + t_0)(t + t_0 - 1)\xi_t + \sum_{t=1}^{T} P \left( \frac{2}{\mu} \right)^{1+\alpha} \xi_t^{1+\alpha}(t + t_0 - 1)
+ \sum_{t=1}^{T} C_1 \left( \frac{2}{\mu} \right)^{1+\alpha} (t + t_0)^{-\alpha}(t + t_0 - 1) + (t_0 - 1)0(F(w_1) - F(\hat{w}^*)).
\]  
(19)

After obtaining this inequality, we switch to bound the norm of $\|w_{t+1}\|$. When $\eta_t = \frac{2}{\mu(t+t_0)}$ and $t_0 \geq 1$, we have

\[
\sum_{t=1}^{T} \eta_t \leq \frac{2}{\mu} \sum_{t=1}^{T} \frac{1}{t + t_0} \leq \frac{2}{\mu} \log(T + 1).
\]

And according to Lemma 42, we have the following inequality for all $t = 1, ..., T$ with probability at least $1 - \delta/2$

\[
\|w_{t+1}\| \leq C_2 \left( \left( \frac{1}{2P} \right)^{\frac{1}{\alpha}} \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} + 1 \right) \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{T} \eta_k^{2\alpha} + \sum_{k=1}^{T} \eta_k^{1+\alpha} \right)^{\frac{1}{2}} \log \left( \frac{8}{\delta} \right)
\leq C_2 \left( \left( \frac{1}{2P} \right)^{\frac{1}{\alpha}} \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} + 1 \right) \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{T} \eta_k^{2\alpha} + \frac{2 + \alpha}{\mu(1+\alpha)} \right)^{\frac{1}{2}} \log \left( \frac{8}{\delta} \right)
\leq 3C_2 \max \left\{ \left( \frac{1}{2P} \right)^{\frac{1}{\alpha}} \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} + 1 \right\} \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} \log \left( \frac{8}{\delta} \right)
\leq C_3 \left( \sqrt{\frac{2}{\mu}} \log \frac{8}{\delta} \right)(T + 1) + \sqrt{\frac{2}{\mu}} \log \frac{8}{\delta} \left( T \right) \left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} \log \left( \frac{8}{\delta} \right)
\leq C_3 \max \left\{ \sqrt{\frac{2}{\mu}}, \left( \sum_{k=1}^{T} \eta_k^{\frac{1}{\alpha}} \right)^{\frac{1}{2}} \right\} \left( \log \frac{8}{\delta} \right)
\leq C_T, \delta,
\]

where the second inequality follows from $\eta_t \leq (2P)^{-1/\alpha}$ and Lemma 40 since $1 + \alpha > 1$ and where in the forth inequality, we denote $C_3 := 3C_2 \max \left\{ \left( \frac{1}{2P} \right)^{\frac{1}{\alpha}} + 1, \left( \frac{4(1+\alpha)}{\mu(1+\alpha)} \right)^{\frac{1}{2}} \right\}$.

The dominated term in the above inequality is $\log \frac{8}{\delta} (T + 1) \left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} \log \left( \frac{8}{\delta} \right)$. According to Lemma 40, if $\alpha \in (0, \frac{1}{2})$, $\left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} = O(T^{(1-2\alpha)/2})$; if $\alpha = \frac{1}{2}$, $\left( \sum_{k=1}^{T} \eta_k^{2\alpha} \right)^{\frac{1}{2}} = O(T^{\frac{1}{2}})$.

\[31\]
\(O(\log^{1/2}(T))\); if \(\alpha \in (\frac{1}{2}, 1]\), \((\sum_{k=1}^{T} \eta_k^{2\alpha})^{\frac{1}{2}} = O(1)\). Hence, we obtain the following result for all \(t = 1, \ldots, T\) with probability at least \(1 - \delta/2\)

\[
||w_{t+1}|| = \begin{cases} 
O \left( \log^{\frac{3}{2}}(T) \right) & \text{if } \alpha \in (0, \frac{1}{2}], \\
O \left( \log(T) \log \left( \frac{1}{\delta} \right) \right) & \text{if } \alpha = \frac{1}{2}, \\
O \left( \log^{\frac{3}{2}}(T) \right) & \text{if } \alpha \in \left( \frac{1}{2}, 1 \right].
\end{cases}
\]

For readability, we denote that \(C_{T,\delta}\) as the upper bound of \(||w_{t+1}||\) for all \(t = 1, \ldots, T\). For example, if \(\alpha = 1/2\), there will hold \(C_{T,\delta} = O(\log(T) \log(1/\delta))\).

We first bound the term \(\sum_{t=1}^{T} (t + t_0)(t + t_0 - 1)\xi_t\) in (19). According to the Hölder smoothness, we have the following inequality for all \(w \in B_R\) and any \(z \in Z\)

\[
||\nabla f(w, z)|| \leq ||\nabla f(0, z)|| + P||w||^\alpha \leq PR^\alpha + \sup_{z \in Z} ||\nabla f(0, z)||. \tag{20}
\]

And, since \(E_{j_t}[\xi_t] = 0\), so \(\{\xi_t\}\) is a martingale difference sequence. The following inequality hold for all \(t = 1, \ldots, T\)

\[
(t + t_0)(t + t_0 - 1)|\xi_t| \leq 2\mu^{-1}(t + t_0 - 1)||\nabla F_S(w_t) - \nabla f(w_t, z_{j_t})||||\nabla F_S(w_t)|| \\
\leq 4\mu^{-1}(T + t_0 - 1) \left( \sup_{z \in Z} ||\nabla f(0, z)|| + P||w_t||^\alpha \right)^2 \\
\leq 4\mu^{-1}(T + t_0 - 1) \left( PC_{T,\delta}^\alpha + \sup_{z \in Z} ||\nabla f(0, z)|| \right)^2,
\]

where the first inequality follows from Schwartz’s inequality and \(\eta_t = \frac{2}{\mu(t+t_0)}\), the third follows from (20), and the last inequality follows from the fact that \(||w_t|| \leq C_{T,\delta}\) in high probability. And we have

\[
E_{j_t}(t + t_0)^2(t + t_0 - 1)^2\xi_t^2 \leq 4\mu^{-2}(t + t_0 - 1)^2||\nabla F_S(w_t)||^2E_{j_t}||\nabla F_S(w_t) - \nabla f(w_t, z_{j_t})||^2 \\
\leq 4\mu^{-2}(t + t_0 - 1)^2\sigma^2||\nabla F_S(w_t)||^2.
\]

32
Denoted by \( b := \sup_{z \in Z} \| \nabla f(0, z) \| \). Applying part (b) of Lemma 53, and setting \( \rho = \min \left\{ 1, (4\sigma^2)^{-1} \left( PC_{T,\delta}^\alpha + b \right)^2 \right\} \), we have with probability at least \( 1 - \delta/4 \),

\[
\sum_{t=1}^{T} (t + t_0)(t + t_0 - 1)\xi_t 
\leq 4\rho \sum_{t=1}^{T} (t + t_0 - 1)^2\sigma^2 \| \nabla F_S(\mathbf{w}_t) \|^2 + \frac{4(T + t_0 - 1) \left( PC_{T,\delta}^\alpha + b \right)^2 \log(\frac{4}{\delta})}{\rho \mu} 
\leq \rho \sum_{t=1}^{T} (t + t_0 - 1)^2\sigma^2 \| \nabla F_S(\mathbf{w}_t) \|^2 + \frac{4(T + t_0 - 1) \left( PC_{T,\delta}^\alpha + b \right)^2 \log(\frac{4}{\delta})}{\rho \mu} 
\leq (4\mu)^{-1} \sum_{t=1}^{T} (t + t_0 - 1)\| \nabla F_S(\mathbf{w}_t) \|^2 + 4\mu^{-1} (T + t_0 - 1) \log(\frac{4}{\delta}) \max \left\{ 4\sigma^2, \left( PC_{T,\delta}^\alpha + b \right)^2 \right\},
\]
conditioned on that \( \| \mathbf{w}_t \| \leq C_{T,\delta} \) with probability at least \( 1 - \delta/2 \).

We then focus on the term \( \sum_{t=1}^{T} C_1 \rho \left( \frac{2}{\mu} \right)^{1+\alpha} (t + t_0)^{-\alpha}(t + t_0 - 1) \) in (19). It is clear that

\[
\sum_{t=1}^{T} (t + t_0)^{-\alpha}(t + t_0 - 1) \leq \sum_{t=1}^{T} (t + t_0)^{1-\alpha} \leq \int_1^{T} (t + t_0)^{1-\alpha} dt \leq \frac{(T + t_0)^{2-\alpha}}{2 - \alpha} = O(T^{2-\alpha}).
\]

For the term \( \rho \sum_{t=1}^{T} \xi'_t (t + t_0)^{-\alpha}(t + t_0 - 1) \) in (19), we have

\[
|\xi'_t| \leq \| \nabla f(\mathbf{w}_t, z_{j_t}) - \nabla F_S(\mathbf{w}_t) \|^2 \leq 2\| \nabla f(\mathbf{w}_t, z_{j_t}) \|^2 + 2\| \nabla F_S(\mathbf{w}_t) \|^2 \leq 4 \left( PC_{T,\delta}^\alpha + b \right)^2,
\]
where the last inequality follows from (20). Applying part (a) of Lemma 53, we have the following inequality with probability at least \( 1 - \delta/4 \)

\[
\sum_{t=1}^{T} (t + t_0)^{-\alpha}(t + t_0 - 1)\xi'_t \leq 4 \left( PC_{T,\delta}^\alpha + b \right)^2 \left( 2 \sum_{t=1}^{T} (t + t_0)^{-2\alpha}(t + t_0 - 1)^2 \log(\frac{4}{\delta}) \right)^{1/2},
\]
conditioned on that \( \| \mathbf{w}_t \| \leq C_{T,\delta} \) with probability at least \( 1 - \delta/2 \). Moreover, it is clear that

\[
\sum_{t=1}^{T} (t + t_0)^{-2\alpha}(t + t_0 - 1)^2 \leq \sum_{t=1}^{T} (t + t_0)^{2-2\alpha} \leq \int_1^{T} (t + t_0)^{2-2\alpha} dt \leq \frac{(T + t_0)^{3-2\alpha}}{3 - 2\alpha} = O(T^{3-2\alpha}).
\]
Therefore, we have the following result with probability at least \( 1 - \delta/4 \)

\[
\sum_{t=1}^{T} (t + t_0)^{-\alpha}(t + t_0 - 1)\xi'_t = O \left( \left( C_{T,\delta}^{2\alpha} T^{\frac{4}{3} - \alpha}, \log(\frac{4}{\delta}) \right) \right),
\]

33
conditioned on that \( \|w_t\| \leq C_{T,\delta} \) with probability at least \( 1 - \delta/2 \).

Substituting these bounds into (19), we finally have the following inequality with probability at least \( 1 - \delta \):

\[
T(T-1)[F_S(w_{T+1}) - F_S(\hat{w}^*)] 
\leq - \sum_{t=1}^{T} \frac{t + t_0 - 1}{4\mu} \|\nabla F_S(w_t)\|^2 + 4\mu^{-1}(T + t_0 - 1) \log \left( \frac{4}{\delta} \right) \max \left\{ 4\sigma^2, (PC_{T,\delta} + b)^2 \right\} 
+ \mathcal{O} \left( C_{T,\delta}^2T^{3/2-\alpha} \log \frac{1}{\delta} \right) + \mathcal{O} \left( T^{2-\alpha} \right) + (t_0 - 1)t_0(F(w_1) - F(\hat{w}^*))
\]

which implies that

\[
F_S(w_{T+1}) - F_S(\hat{w}^*) = \begin{cases} 
\mathcal{O} \left( \frac{1}{T} \right) & \text{if } \alpha < 1, \\
\mathcal{O} \left( \frac{\log(T) \log^2(1/\delta)}{T} \right) & \text{if } \alpha = 1.
\end{cases}
\]

The proof is complete.

6.4 Proof of Theorem 11

**Proof** Since \( \hat{w}^*(S) \) is ERM, thus the optimization error in Lemma 35 has \( \epsilon_{opt}(\hat{w}^*(S)) = 0 \). Since Assumption 1 holds and \( F_S \) satisfies Assumption 7 with parameter \( \mu \), we know that the uniform stability of \( \hat{w}^*(S) \) is \( \frac{4L^2}{n\mu} \) from Lemma 37 and that \( B = \frac{2L^2}{\mu} \) from Lemma 36. Moreover, since \( \|w\| \leq R \), there holds that \( |f(w) - f(0)| \leq \|w\| \leq R \), which implies that

\[
f(w) \leq R + \sup_{z \in \mathcal{Z}} f(0, z) \tag{21}
\]

Substituting these results into Lemma 35 and fixing any \( \eta > 0 \), we have the following inequality with probability at least \( 1 - \delta \):

\[
F(\hat{w}^*(S)) - F(w^*) \leq c(1 + 1/\eta) \left( \frac{4L^2}{n\mu} \log n + \frac{R + \sup_{z \in \mathcal{Z}} f(0, z) + 2L^2/\mu}{n} \right) \log \left( \frac{1}{\delta} \right),
\]

where \( c \) is an absolute constant. The proof is complete.

6.5 Proof of Theorem 13

**Proof** According to Lemma 17 in (Charles and Papailiopoulos, 2018) and Appendix A in (Karimi et al., 2016), the PL condition (Assumption 8) implies the QG condition (Assumption 7). Thus, under Lipschitz continuous condition (Assumption 1) and Assumption 8 with parameter \( 2\mu \), the uniform stability bound of SGD is

\[
\forall z \in \mathcal{Z}, \quad |f(w_t, z) - f(w^*_t, z)| \leq 2L \sqrt{\frac{\epsilon_{opt}(w_t)}{\mu}} + \frac{2L^2}{n\mu},
\]

\[
34
\]
from Lemma 38.

When the step size $\eta_t = \frac{2}{\mu(t+t_0)}$ with $t_0 \geq \max \left\{ \frac{2(2P)^{1/\alpha}}{\mu}, \frac{\mu}{2} \right\}$ for all $t \in \mathbb{N}$, we know that $\eta_t \leq \min \left\{ (1/(2P))^{1/\alpha}, \frac{\mu}{2} \right\}$ and $\eta_t$ decreases fast, thus Assumption 5 is much milder than the Lipschitz continuous condition (Assumption 1) as discussed in Remark 5. For brevity, we consider the case that $\sqrt{\eta_t} \|\nabla f(w_t, z)\| \leq G \leq \sqrt{\eta_t} L$, which means Lipschitz continuous (Assumption 1) implies Assumption 5.

Hence, under Assumptions 1, 3, 6 and PL condition (Assumption 8) of empirical risk $F_S$, from Lemma 43, the optimization error of $w_{T+1}$ is

$$
\epsilon_{opt}(w_{T+1}) := F_S(w_{T+1}) - F_S(\bar{w}) = \begin{cases} O\left(\frac{1}{\alpha} \log \frac{1}{\delta}\right) & \text{if } \alpha \in (0, 1), \\
O\left(\frac{\log(T) \log^2(1/\delta)}{T}\right) & \text{if } \alpha = 1,
\end{cases}
$$

with probability at least $1 - \delta$.

Substituting these bounds into Lemma 35 combined with the boundness of $f$ in (21) and $B = L^2/\mu$ from Lemma 36, we finally obtain that with probability at least $1 - \delta$,

$$
F(w_{T+1}) - F(w^*) = O \left( \sqrt{\epsilon_{opt}(w_{T+1})} \log n \log \left( \frac{1}{\delta} \right) + \frac{1}{n} \log n \log \left( \frac{1}{\delta} \right) + \epsilon_{opt}(w_{T+1}) \right).
$$

If $\alpha \in (0, 1)$, take $T \asymp n^{2\alpha}$, we have the following excess risk bound with probability at least $1 - \delta$,

$$
F(w_{T+1}) - F(w^*) = O \left( \frac{1}{n} \log n \log \left( \frac{1}{\delta} \right) \right).
$$

If $\alpha = 1$, take $T \asymp n^2$, we have the following excess risk bound with probability at least $1 - \delta$,

$$
F(w_{T+1}) - F(w^*) = O \left( \frac{1}{n} \log^2 n \log^2 \left( \frac{1}{\delta} \right) \right).
$$

The proof is complete.

6.6 Proof of Theorem 15

**Proof** In Remark 3, we have already known that when $\alpha = 1$, the $\alpha$-Hölder smoothness condition becomes the smoothness condition (Assumption 2). From Lemma 51 in Appendix A, one can see that smooth convex function with QG condition (Assumption 7) implies the PL condition (Assumption 8). The assumptions therefore are satisfied to prove the bound by following the proof in Section 6.5. From (22), when Assumptions 1, 2 and 6 holds and $F_S$ satisfies Assumption 7 with parameter $2\mu$, taking $T \asymp n^2$, we have the following excess risk with probability at least $1 - \delta$,

$$
F(w_{T+1}) - F(w^*) = O \left( \frac{1}{n} \log^2 n \log^2 \left( \frac{1}{\delta} \right) \right).
$$

The proof is complete.
6.7 Proof of Theorem 17
Proof Since \( \hat{w}^*(S) \) is ERM, thus the optimization error \( \epsilon_{opt}(\hat{w}^*(S)) = 0 \). From (6), under Assumption 1, we know that
\[
F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S^i)) \leq \frac{2L}{n} ||\hat{w}^*(S^i) - \hat{w}^*(S)||.
\]
Furthermore, for empirical risk minimizers \( \hat{w}^*(S^i) \) and \( \hat{w}^*(S) \), there holds that \( \pi_S(\hat{w}^*(S^i)) = \hat{w}^*(S) \) according to Assumption 9. Therefore, combined with the QG property (Assumption 7) of \( F_S \), we have
\[
F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \geq \frac{\mu}{2} ||\hat{w}^*(S^i) - \hat{w}^*(S)||^2.
\]
Then we get
\[
\frac{\mu}{2} ||\hat{w}^*(S^i) - \hat{w}^*(S)||^2 \leq F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \leq \frac{2L}{n} ||\hat{w}^*(S^i) - \hat{w}^*(S)||,
\]
which implies that \( ||\hat{w}^*(S^i) - \hat{w}^*(S)|| \leq \frac{4\mu}{n\mu} \). Combined with the Lipschitz continuous condition of \( f \), we know the stability bound of \( \hat{w}^*(S) \) is \( \frac{4L^2}{n\mu} \).

Substituting the above results, \( B = 2L^2/\mu \) from Lemma 36 and the boundness of \( f \) in (21) into Lemma 35 and fixing any \( \eta > 0 \), we have the following excess risk with probability at least \( 1 - \delta \),
\[
F(\hat{w}^*(S)) - F(w^*) \leq c(1 + 1/\eta) \left( \frac{4L^2}{n\mu} \log n + \frac{R + \sup_{z \in \mathcal{Z}} f(0, z) + 2L^2/\mu}{n} \right) \log \left( \frac{1}{\delta} \right),
\]
where \( c \) is an absolute constant. The proof is complete.

6.8 Proof of Theorem 19
Proof We first consider the optimization error of \( w_{T+1} \). When the step size \( \eta_t = \frac{2}{\mu(t+t_0)} \) with \( t_0 \geq \max \left\{ \frac{2(2P)^{1/\alpha}}{\mu}, 1 \right\} \) for all \( t \in \mathbb{N} \), we know that \( \eta_t \) decreases fast and Assumption 5 is much milder than the Lipschitz continuous condition (Assumption 1). For brevity, we also consider the case that \( \sqrt{\eta_t} \|\nabla f(w_t, z)\| \leq G \leq \sqrt{\eta_t}L \), which means Lipschitz continuous (Assumption 1) implies Assumption 5. Hence, under Assumptions 1, 3, 6 and PL condition (Assumption 8) of empirical risk \( F_S \), from Lemma 43, the optimization error of \( w_{T+1} \) is
\[
\epsilon_{opt}(w_{T+1}) := F_S(w_{T+1}) - F_S(w^*) = \begin{cases} O \left( \frac{1}{\mu(t+t_0)} \right) & \text{if } \alpha \in (0, 1), \\ O \left( \frac{\log(T) \log(1/\delta)}{T} \right) & \text{if } \alpha = 1, \end{cases}
\]
with probability at least \( 1 - \delta \).

We then establish the uniform stability bound of SGD. Let the closest empirical risk minimizer of \( F_S \) to \( w_t \) be \( \hat{w}^*(S) \) and the closest empirical risk minimizer of \( F_{S_t} \) to \( w^*_t \) be \( \hat{w}^*(S^i) \). Firstly, from (6), we know that
\[
F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \leq \frac{2L}{n} ||\hat{w}^*(S^i) - \hat{w}^*(S)||.
\]
Then, for empirical risk minimizers $\hat{w}^*(S^i)$ and $\tilde{w}^*(S)$, there holds that $\pi_S(\hat{w}^*(S^i)) = \tilde{w}^*(S)$ according to Assumption 9. And we know that the PL condition (Assumption 8) implies the QG condition (Assumption 7) (Charles and Papailiopoulos, 2018; Karimi et al., 2016). Thus, according to the QG condition of $F_S$ with parameter $2\mu$ we have

$$F_S(\hat{w}^*(S^i)) - F_S(\tilde{w}^*(S)) \geq \mu \|\tilde{w}^*(S^i) - \hat{w}^*(S)\|^2.$$  

Then we get

$$\mu \|\tilde{w}^*(S^i) - \hat{w}^*(S)\|^2 \leq F_S(\hat{w}^*(S^i)) - F_S(\hat{w}^*(S)) \leq \frac{2L}{n} \|\tilde{w}^*(S^i) - \hat{w}^*(S)\|,$$

which implies that $\|\tilde{w}^*(S^i) - \hat{w}^*(S)\| \leq \frac{2L}{n\mu}$. Combined with the Lipschitz property of $f$ we obtain that

$$\forall z \in Z, \quad |f(\tilde{w}^*(S^i), z) - f(\hat{w}^*(S), z)| \leq \frac{2L^2}{n\mu}.$$  

We further can obtain that for $\forall z \in Z$,

$$|f(w_t, z) - f(w^*_t, z)|$$

$$\leq |f(w_t, z) - f(\tilde{w}^*(S), z)| + |f(\tilde{w}^*(S), z) - f(\hat{w}^*(S^i), z)|$$

$$+ |f(\hat{w}^*(S^i), z) - f(w^*_t, z)|$$

$$\leq L\|w_t - \tilde{w}^*(S)\|_i + \frac{2L^2}{n\mu} + L\|\tilde{w}^*(S^i) - w^*_t\|$$

$$\leq 2L \sqrt{\frac{F_S(w_t) - F_S(\tilde{w}^*(S))}{\mu}} + \frac{2L^2}{n\mu}$$

$$\leq 2L \sqrt{\frac{\epsilon_{opt}(w_t)}{\mu}} + \frac{2L^2}{n\mu}, \quad (23)$$

where the second inequality follows from Lipschitz property of $f$ and third inequality follows from the QG condition since the closest empirical risk minimizer of $F_S$ to $w_t$ is $\tilde{w}^*(S)$.

Substituting the optimization error bound $\epsilon_{opt}(w_{T+1})$ and the uniform stability bound (23) into Lemma 35 combined with the boundness of $f$ in (21) and $B = L^2/\mu$ from Lemma 36, we have the following result with probability at least $1 - \delta$

$$F(w_{T+1}) - F(w^*) = O \left( \sqrt{\epsilon_{opt}(w_{T+1}) \log n \log \left( \frac{1}{\delta} \right)} + \frac{1}{n} \log n \log \left( \frac{1}{\delta} \right) + \epsilon_{opt}(w_{T+1}) \right).$$

If $\alpha \in (0, 1)$, take $T \asymp n^{\frac{2}{\alpha}}$, we have the following excess risk bound with probability at least $1 - \delta$,

$$F(w_{T+1}) - F(w^*) = O \left( \frac{1}{n} \log n \log \left( \frac{1}{\delta} \right) \right).$$
If $\alpha = 1$, take $T \asymp n^2$, we have the following excess risk bound with probability at least $1 - \delta$,

$$F(w_{T+1}) - F(w^*) = O\left(\frac{1}{n} \log^{3/2} n \log^2 \left(\frac{1}{\delta}\right)\right).$$

The proof is complete.

7. Proofs of Uniform Convergence Scenario

In this section, we provide the proofs of Theorems in uniform convergence scenario.

7.1 Basic Lemma

Definition 44 For every $\alpha > 0$, we define the Orlicz -- $\alpha$ norm of a random $v$:

$$\|v\|_{\text{Orlicz} - \alpha} = \inf\{K > 0 : \mathbb{E} \exp((|v|/K)^\alpha) \leq 2\}.$$

A random variable (or vector) $X \in \mathbb{R}^d$ is $K$-sub-exponential if $\forall \lambda \in \mathbb{R}^d$, we have

$$\|\lambda^T X\|_{\text{Orlicz} - 1} \leq K\|\lambda\|^2.$$

A random variable (or vector) $X \in \mathbb{R}^d$ is $K$-sub-Gaussian if $\forall \lambda \in \mathbb{R}^d$, we have

$$\|\lambda^T X\|_{\text{Orlicz} - 2} \leq K\|\lambda\|^2.$$

The following result is a lemma on uniform convergence of gradients.

Lemma 45 ((Xu and Zeevi, 2020)) For all $w_1, w_2 \in W$, we assume that $\frac{\nabla f(w_1, z) - \nabla f(w_2, z)}{\|w_1 - w_2\|}$ is a $\gamma$-sub-exponential random vector. Formally there exists $\gamma > 0$ such that for any unit vector $u \in B(0, 1)$ and $w_1, w_2 \in W$,

$$\mathbb{E}\left\{\exp\left(\frac{|u^T (\nabla f(w_1, z) - \nabla f(w_2, z))|}{\gamma \|w_1 - w_2\|}\right)\right\} \leq 2.$$

Then $\forall \delta \in (0, 1)$, with probability $1 - \delta$, for all $w \in W$, there holds that

$$\|\nabla F(w) - \nabla F_S(w) - (\nabla F(w^*) - \nabla F_S(w^*))\| \leq c\gamma \max\left\{\|w - w^*\|, \frac{1}{n}\right\}\left(\sqrt{\frac{d + \log 4\log_2(2nR+2)}{n}} + \frac{d + \log 4\log_2(2nR+2)}{n}\right),$$

where $c$ is an absolute constant and $w^* \in \arg\min_{W} F(w)$.

The following Lemma 46 is a basic Lemma for proving Theorems in Section 5.
**Lemma 46** Suppose Assumptions 2 and 4 hold. For all \( w \in W \) and any \( \delta > 0 \), with probability at least \( 1 - \delta \),

\[
\| \nabla F(w) - \nabla F_S(w) \| \\
\leq c' \beta \max \left\{ \| w - w^* \|, \frac{1}{n} \right\} \eta + \frac{B_* \log(4/\delta)}{n} + \sqrt{\frac{2E[\| \nabla f(w^*, z) \|^2 \log(4/\delta)]}{n}},
\]

where \( c' \) is an absolute constant and \( \eta = \sqrt{\frac{d + \log(16 \log^3(2nR + 2))}{n}} + d + \log(16 \log^3(2nR + 2)). \)

If further population risk \( F \) satisfies Assumption 8 with parameter \( \mu \), then for any \( \delta > 0 \), when \( n \geq \frac{c^2(d + \log(16 \log^3(2nR + 2)))}{\mu^2} \), with probability at least \( 1 - \delta \),

\[
\| \nabla F(w) - \nabla F_S(w) \| \\
\leq \| \nabla F_S(w) \| + \frac{\mu}{n} + \frac{2B_* \log(4/\delta)}{n} + 2\sqrt{\frac{2E[\| \nabla f(w^*, z) \|^2 \log(4/\delta)]}{n}},
\]

and

\[
\| \nabla F(w) \| \leq 2 \| \nabla F_S(w) \| + \frac{\mu}{n} + \frac{2B_* \log(4/\delta)}{n} + 2\sqrt{\frac{2E[\| \nabla f(w^*, z) \|^2 \log(4/\delta)]}{n}}.
\]

**Proof** According to smoothness (Assumptions 2), for any sample \( z \in Z \) and \( w_1, w_2 \in W \), there has

\[
\| \nabla f(w_1, z) - \nabla f(w_2, z) \| \leq \beta \| w_1 - w_2 \|.
\]

Assume any unit vector \( u \in B(0, 1) \), we have

\[
|u^T(\nabla f(w_1, z) - \nabla f(w_2, z))| \leq \| u \| \| \nabla f(w_1, z) - \nabla f(w_2, z) \| \\
\leq \beta \| w_1 - w_2 \|.
\]

Then we have

\[
\frac{|u^T(\nabla f(w_1, z) - \nabla f(w_2, z))|}{\beta \| w_1 - w_2 \|} \leq 1.
\]

That is

\[
\mathbb{E} \left\{ \exp \left( \frac{\ln 2|u^T(\nabla f(w_1, z) - \nabla f(w_2, z))|}{\beta \| w_1 - w_2 \|} \right) \right\} \leq 2.
\]

So we obtain that for all \( w_1, w_2 \in W \), \( \frac{\nabla f(w_1, z) - \nabla f(w_2, z)}{\| w_1 - w_2 \|} \) is a \( \frac{\beta}{\ln 2} \)-sub-exponential random vector.

From Lemma 45, under the smoothness assumption, for \( \forall \delta \in (0, 1) \) and all \( w \in W \), we have the following inequality with probability at least \( 1 - \delta \)

\[
\| (\nabla F(w) - \nabla F_S(w) ) - (\nabla F(w^*) - \nabla F_S(w^*) ) \| \\
\leq c \frac{\beta}{\ln 2} \max \left\{ \| w - w^* \|, \frac{1}{n} \right\} \left( \sqrt{\frac{d + \log^4(2nR + 2)}{n}} + \frac{d + \log^4(2nR + 2)}{n} \right),
\]

(24)
where $c$ is an absolute constant.

To make the proof readable, we denote \( \eta = \sqrt{d + \log \frac{8 \log 2 (2nR+2)}{n}} + \frac{d+\log \frac{8 \log 2 (2nR+2)}{n}}{4} \). From (24), there exists a constant \( c' \) such that \( \forall \delta > 0 \), with probability at least \( 1 - \frac{\delta}{2} \), there holds

\[
\| \nabla F (w) - \nabla F_S (w) \| - \| \nabla F (w^*) - \nabla F_S (w^*) \| \leq c' \beta \max \left\{ \| w - w^* \| , \frac{1}{n} \right\} \eta. \tag{25}
\]

From Lemma 55 (vector Bernstein inequality), under Assumption 4, we have the following inequality with probability at least \( 1 - \frac{\delta}{2} \),

\[
\| \nabla F (w^*) - \nabla F_S (w^*) \| \leq \frac{B_* \log (4/\delta)}{n} + \sqrt{\frac{2 \mathbb{E}[\| \nabla f (w^*, z) \|^2 \log (4/\delta)]}{n}}, \tag{26}
\]

where the above inequality follows with the optimality property \( \nabla F (w^*) = 0 \). Combined (26) and (25), we obtain that with probability at least \( 1 - \delta \), there holds

\[
\| \nabla F (w) - \nabla F_S (w) \| \\
\leq c' \beta \max \left\{ \| w - w^* \| , \frac{1}{n} \right\} \eta + \frac{B_* \log (4/\delta)}{n} + \sqrt{\frac{2 \mathbb{E}[\| \nabla f (w^*, z) \|^2 \log (4/\delta)]}{n}}. \tag{27}
\]

From (27), we get the following inequality with probability at least \( 1 - \delta \),

\[
\| \nabla F (w) \| - \| \nabla F_S (w) \| \\
\leq \| \nabla F (w) - \nabla F_S (w) \| \\
\leq c' \beta \left( \| w - w^* \| + \frac{1}{n} \right) \eta + \frac{B_* \log (4/\delta)}{n} + \sqrt{\frac{2 \mathbb{E}[\| \nabla f (w^*, z) \|^2 \log (4/\delta)]}{n}}. \tag{28}
\]

From Lemma 51 in Appendix A, we know that smoothness (Assumptions 2) and PL condition (Assumption 8) of \( F \) imply that for all \( w \in \mathcal{W} \), there has

\[
\| \nabla F (w) \| \geq \mu \| w - w^* \|, \tag{29}
\]

where \( w^* \) is the closest optima point of \( F \) to \( w \). Therefore, combined (28) and (29), we obtain that for all \( w \in \mathcal{W} \), there exists a minimizer \( w^* \), there holds the following inequality with probability at least \( 1 - \delta \)

\[
\mu \| w - w^* \| \leq \| \nabla F (w) \| \leq \| \nabla F_S (w) \| + c' \beta \left( \| w - w^* \| + \frac{1}{n} \right) \eta \\
+ \frac{B_* \log (4/\delta)}{n} + \sqrt{\frac{2 \mathbb{E}[\| \nabla f (w^*, z) \|^2 \log (4/\delta)]}{n}}.
\]

Let \( c = \max \{ 4c'^2, 1 \} \). When

\[
n \geq \frac{c \beta^2 (d + \log (\frac{8 \log (2nR+2)}{\delta}))}{\mu^2},
\]

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we have \( c' \beta \eta \leq \mu/2 \), followed from the fact that \( \frac{\mu}{c} \leq 1 \) (Nesterov, 2003). So we have

\[
\|w - w^*\| \leq \frac{2}{\mu} \left( \|\nabla F_S(w)\| + \frac{B_s \log(4/\delta)}{n} + \sqrt{\frac{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)}{n}} + \frac{\mu}{2n} \right). \tag{30}
\]

Substituting (30) into (27), we obtain that for all \( w \in W \), when

\[
n \geq \frac{c \beta}{2} \left( d + \log \left( \frac{8 \log(2nR + 2)}{\delta} \right) \right) \frac{\mu^2}{n},
\]

with probability at least \( 1 - \delta \),

\[
\|\nabla F(w) - \nabla F_S(w)\| \\
\leq \|\nabla F_S(w)\| + \frac{\mu}{n} + 2 \frac{B_s \log(4/\delta)}{n} + 2 \sqrt{\frac{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)}{n}}.
\]

Substituting (30) into (28), we obtain that for all \( w \in W \), when

\[
n \geq \frac{c \beta}{2} \left( d + \log \left( \frac{8 \log(2nR + 2)}{\delta} \right) \right) \frac{\mu^2}{n},
\]

with probability at least \( 1 - \delta \),

\[
\|\nabla F(w)\| \leq 2 \|\nabla F_S(w)\| + \frac{\mu}{n} + 2 \frac{B_s \log(4/\delta)}{n} + 2 \sqrt{\frac{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)}{n}}.
\]

The proof is complete. \( \blacksquare \)

### 7.2 Optimization Error

We now introduce the optimization error bound, which actually has been proved in Section 6.3. Here we only need the results of \( \alpha = 1 \) corresponding to the smoothness condition. For brevity, we just sketch the important proof.

**Lemma 47** Let \( \{w_t\} \) be the sequence produced by (3) with \( \eta_t \leq \frac{1}{2c} \) for all \( t \in \mathbb{N} \). Suppose Assumptions 2, 5 and 6 hold. Then, for any \( \delta \in (0, 1) \), the following inequality holds with probability \( 1 - \delta \)

\[
\sum_{k=1}^{t} \eta_k \|\nabla F_S(w_k)\|^2 \leq C \log(2/\delta) + C_t,
\]

where \( C_t = 4 \sup_{z \in \mathbb{Z}} f(0, z) + 8\beta \max\{G^2, \sigma^2\} \sum_{k=1}^{t} \eta_k^2 \) and \( C = 32\beta G^2 + 8 \max\{G^2, \sigma^2(\beta)^{-1}\} \).

**Lemma 48** Let \( \{w_t\} \) be the sequence produced by (3) with \( \eta_t \leq \frac{1}{2c} \). Suppose Assumptions 2, 5 and 6 hold. Then, for any \( \delta \in (0, 1) \), with probability \( 1 - \delta \) we have uniformly for all \( t = 1, ..., T \)

\[
\|w_{t+1} - w^*\| \leq C_2' \left( \left( \sum_{k=1}^{t} \eta_k^2 \right)^{1/2} + 1 \right) \left( \left( \sum_{k=1}^{t} \eta_k \right)^{1/2} + 1 \right) \log(4/\delta),
\]

where \( C_2' = \max \left\{ \left( \frac{4\sqrt{2G(1/2\beta)^{1/2}}}{3}, 2\sqrt{2\sigma}, 4\sqrt{2C + 8 \sup_{z \in \mathbb{Z}} f(0, z)} + 4\sqrt{8 \max\{\beta G^2, C_1\}}, \|w^*\| \right\} \),

which is independent of \( T \) and \( \delta \).
Proof. In this proof, we assume \( \|w^*\| < \infty \) as did in (Lei and Tang, 2018). From Lemma 42, it is easy to verify that with probability \( 1 - \delta \)

\[
\|w_{t+1} - w^*\| \\
\leq \|w_{t+1}\| + \|w^*\|
\]

\[
\leq C_2' \left( \left( \sum_{k=1}^{T} \eta_k^2 \right)^{1/2} + 1 \right) \left( \left( \sum_{k=1}^{T} \eta_k \right)^{1/2} + 1 \right) \log \left( \frac{4}{\delta} \right),
\]

where \( C_2' = \max \left\{ \frac{4\sqrt{2G(1/2\beta)^{1/2}}}{3}, 2\sqrt{2} \sigma, 4\sqrt{2C + 8 \max_{z \in Z} f(0, z)} \right\} \).

Remark 49: Since we assume \( W := B(w^*, R) \) in the uniform convergence scenario, where \( B(w_0, R) := \{ w \in \mathbb{R}^d : \|w - w_0\| \leq R \} \), thus we focus on \( \|w_{t+1} - w^*\| \) in this section.

Lemma 50: Suppose Assumptions 2, 5 and 6 hold, and suppose \( F_S \) satisfies Assumption 8 with parameter \( 2\mu \). Let \( \{w_t\}_t \) be the sequence produced by (3) with \( \eta_t = \frac{2}{\mu(t + t_0)} \) such that \( t_0 \geq \max \left\{ \frac{4 \beta}{\mu}, 1 \right\} \) for all \( t \in \mathbb{N} \). Then, for any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
F_S(w_{T+1}) - F_S(w^*) = \mathcal{O} \left( \frac{\log(T) \log^3(1/\delta)}{T} \right).
\]

Proof. Follow the proof of Lemma 43, it is easy to obtain that for all \( t = 1, ..., T \) with probability at least \( 1 - \delta/2 \)

\[
\|w_{t+1} - w^*\| = \begin{cases} 
\mathcal{O} \left( \log^2(T) T^{(1 - 2\alpha)/2} \log(1/\delta) \right) & \text{if } \alpha \in (0, \frac{1}{2}), \\
\mathcal{O} \left( \log(T) \log(1/\delta) \right) & \text{if } \alpha = 1/2, \\
\mathcal{O} \left( \log^\frac{1}{2}(T) \log(1/\delta) \right) & \text{if } \alpha \in (\frac{1}{2}, 1].
\end{cases}
\]

Then reach the inequality (20), we have the following inequality for all \( w \in B(w^*, R) \) and any \( z \in Z \)

\[
\|\nabla f(w, z)\| \leq \|\nabla f(0, z)\| + \beta \|w\| \\
\leq \beta \|w - w^* + w^*\| + \sup_{z \in Z} \|\nabla f(0, z)\| \\
\leq \beta \|w - w^*\| + \beta \|w^*\| + \sup_{z \in Z} \|\nabla f(0, z)\|.
\]

Denoted by \( b = \beta \|w^*\| + \sup_{z \in Z} \|\nabla f(0, z)\| \) and follow the proof of Lemma 43, the conclusion can be obtained.

\[ \square \]
And according to (Nesterov, 2003), there holds the following property for smooth functions:

\[
\frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - \inf_{w \in W} f(w).
\]

Combined with Lemma 50, it can be derived that

\[
\|\nabla F_S(w_{T+1})\|^2 = O\left(\frac{\log(T)\log(1/\delta)}{T}\right).
\]  \hfill (31)

### 7.3 Proof of Theorem 22

**Proof** Since the objective function \(f\) is smooth and convex, thus \(F(w)\) is a smooth convex function. From Lemma 51 in Appendix A, if \(F\) further satisfies Assumption 7, then \(F(w)\) satisfies the PL property:

\[
F(w) - F(w^*) \leq \frac{\|\nabla F(w)\|^2}{2\mu}, \quad \forall w \in W.
\]  \hfill (32)

Therefore, to bound the excess risk \(F(\hat{w}^*) - F(w^*)\), we need to bound the term \(\|\nabla F(\hat{w}^*)\|\).

Then, from Lemma 46, under smoothness, convexity, QG of \(F\) and noise assumptions, we know that for any \(\delta > 0\), when \(n \geq \frac{c^2\beta^2(d+\log(\beta\log(2\beta+1)))}{\mu^2}\), with probability at least \(1 - \delta\),

\[
\|\nabla F(w)\| \leq 2\|\nabla F_S(w)\| + \frac{\mu}{n} + 2\frac{B_*\log(4/\delta)}{n} + 2\sqrt{2\mathbb{E}[\|\nabla f(w^*, z)\|^2]\log(4/\delta)}.
\]

Since \(f\) is convex, we have \(\nabla F_S(\hat{w}^*) = 0\). Thus with probability at least \(1 - \delta\), we obtain

\[
\|\nabla F(\hat{w}^*)\| \leq \frac{2B_*\log(4/\delta)}{n} + 2\sqrt{2\mathbb{E}[\|\nabla f(w^*, z)\|^2]\log(4/\delta)} + \frac{\mu}{n}.
\]

Then, it can be derived that

\[
F(\hat{w}^*) - F(w^*) \leq \frac{\|\nabla F(\hat{w}^*)\|^2}{2\mu}
\]

\[
\leq \frac{1}{2\mu} \left(\frac{2B_*\log(4/\delta)}{n} + 2\sqrt{2\mathbb{E}[\|\nabla f(w^*, z)\|^2]\log(4/\delta)} + \frac{\mu}{n}\right)^2
\]  \hfill (33)

\[
\leq \frac{2B_*^2\log^2(4/\delta)}{\mu n^2} + \frac{4\mathbb{E}[\|\nabla f(w^*, z)\|^2]\log(4/\delta)}{\mu n} + \frac{2\mu}{n^2}.
\]

When \(f\) is nonnegative and \(\beta\) smooth, from Lemma 4.1 of (Srebro et al., 2010), we have

\[
\|\nabla f(w^*, z)\|^2 \leq 4\beta f(w^*, z),
\]

thus we have

\[
\mathbb{E}[\|\nabla f(w^*, z)\|^2] \leq 4\beta \mathbb{E}f(w^*, z) = 4\beta F(w^*).
\]  \hfill (34)
Substituting (34) to (33), we finally obtain the following inequality with probability at least \(1 - \delta\),

\[
F(\hat{w}^*) - F(w^*) \leq \frac{2B^2_* \log^2(4/\delta)}{\mu n^2} + \frac{16\beta F(w^*) \log(4/\delta)}{\mu n} + 2\mu n^2.
\]

The proof is complete. 

### 7.4 Proof of Theorem 24

**Proof** Since the objective function \(f\) is smooth convex and \(F_S\) satisfies Assumption 7 with parameter \(2\mu\), thus \(F(w)\) is smooth convex and satisfies Assumption 7 with parameter \(2\mu\).

Then, from Lemma 51 in Appendix A, we have

\[
F(w) - F(w^*) \leq \frac{\|\nabla F(w)\|^2}{4\mu}, \quad \forall w \in W.
\]

(35)

From Lemma 46, under Assumptions 2 and 4 and PL condition of \(F_S\), for any \(\delta > 0\), when \(n \geq \frac{c\beta^2(d + \log(16 \log(2nR + 2)))}{\mu^2}\), we have the following inequality with probability at least \(1 - \delta\),

\[
\|\nabla F(w)\| \leq 2 \|\nabla F_S(w)\| + \frac{2\mu}{n} + 2 \frac{B_* \log(4/\delta)}{n} + 2 \sqrt{\frac{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)}{n}},
\]

That is with probability at least \(1 - \delta/2\), when \(n \geq \frac{c\beta^2(d + \log(16 \log(2nR + 2)))}{\mu^2}\),

\[
\|\nabla F(w_{T+1})\| \leq 2 \|\nabla F_S(w_{T+1})\| + \frac{2\mu}{n} + 2 \frac{B_* \log(8/\delta)}{n} + 2 \sqrt{\frac{2E[\|\nabla f(w^*, z)\|^2] \log(8/\delta)}{n}}.
\]

(36)

Therefore, to obtain the bound of \(\|\nabla F(w_{T+1})\|\), the next key step is to bound the term \(\|\nabla F_S(w_{T+1})\|\).

From (31), we know that under Assumptions 2, 5, 6 and the PL condition of \(F_S\), with probability at least \(1 - \delta/2\), there holds

\[
\|\nabla F_S(w_{T+1})\|^2 = O \left( \frac{\log(T) \log^3(1/\delta)}{T} \right).
\]

(37)

Combined (37) and (36), we obtain the following result with probability at least \(1 - \delta\)

\[
\|\nabla F(w_{T+1})\|^2 = O \left( \frac{\log T \log^3(1/\delta)}{T} \right) + O \left( \frac{E[\|\nabla f(w^*, z)\|^2] \log(1/\delta)}{n} + \frac{\log^2(1/\delta)}{n^2} \right)
\]

\[
= O \left( \frac{\log T \log^3(1/\delta)}{T} \right) + O \left( \frac{F(w^*, z) \log(1/\delta)}{n} + \frac{\log^2(1/\delta)}{n^2} \right),
\]

where the last inequality follows from (34). If we consider the case \(F(w^*, z) = O(\frac{1}{n})\) as did in (Zhang et al., 2017), and take \(T \asymp n^2\), we finally derive the following result with probability at least \(1 - \delta\)

\[
\|\nabla F(w_{T+1})\|^2 = O \left( \frac{\log n \log^3(1/\delta)}{n^2} \right).
\]

(38)
Substituting (38) into (35), we have that, for any $\delta > 0$, when $n \geq \frac{c\beta^2(d + \log(\frac{16\log(2nR + 2)}{\delta})))}{\mu^2}$, with probability at least $1 - \delta$
\[
F(w_{T+1}) - F(w^*) \leq \frac{1}{4\mu} \|\nabla F(w_{T+1})\|^2 = o\left(\frac{\log n \log^2(1/\delta)}{n^2}\right).
\]
The proof is complete.

7.5 Proof of Theorem 26

**Proof** Since $F(w)$ satisfies the PL property (Assumption 8),
\[
F(w) - F(w^*) \leq \frac{\|\nabla F(w)\|^2}{2\mu}, \quad \forall w \in \mathcal{W}.
\]
Therefore, to bound the excess risk $F(\hat{w}^*_T) - F(w^*)$, we need to bound the term $\|\nabla F(\hat{w}^*_T)\|$. From Lemma 46, under Assumptions 2 and 4 and PL condition of population risk $F$, for any $\delta > 0$, when $n \geq \frac{c\beta^2(d + \log(\frac{16\log(2nR + 2)}{\delta})))}{\mu^2}$, we have the following inequality with probability at least $1 - \delta$,
\[
\|\nabla F(w)\| \leq 2 \|\nabla F_S(w)\| + \frac{\mu}{n} + 2B_s \log(4/\delta) + 2\sqrt{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)}.
\]
Since $\nabla F_S(\hat{w}^*_T) = 0$, thus, with probability at least $1 - \delta$, we obtain
\[
\|\nabla F(\hat{w}^*_T)\| \leq 2B_s \log(4/\delta) + 2\sqrt{2E[\|\nabla f(w^*, z)\|^2] \log(4/\delta)} + \frac{\mu}{n}.
\]
Following the proof of Theorem 22, we finally obtain the following inequality with probability at least $1 - \delta$,
\[
F(\hat{w}^*_T) - F(w^*) \leq \frac{2B_s^2 \log^2(4/\delta)}{\mu n^2} + \frac{16\beta F(w^*) \log(4/\delta)}{\mu n} + \frac{2\mu}{n^2}.
\]
The proof is complete.

7.6 Proof of Theorem 28

**Proof** When $\eta_t = \eta_1t^{-\theta}, \theta \in (0, 1)$ with $\eta_1 \leq \frac{1}{2\beta}$ and Assumptions 2, 5 and 6 hold, according to Lemma 47, for any $\delta \in (0, 1)$, we obtain that, with probability at least $1 - \delta/3$,
\[
\sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \sum_{t=1}^{T} \eta_t \|\nabla F(w_t) - \nabla F_S(w_t) + \nabla F_S(w_t)\|^2
\]
\[
\leq 2 \sum_{t=1}^{T} \eta_t \|\nabla F(w_t) - \nabla F_S(w_t)\|^2 + 2 \sum_{t=1}^{T} \eta_t \|\nabla F_S(w_t)\|^2
\]
\[
\leq 2 \sum_{t=1}^{T} \eta_t \max_{t=1,...,T} \|\nabla F(w_t) - \nabla F_S(w_t)\|^2 + o\left(\sum_{t=1}^{T} \eta_t^2 + \log \left(\frac{1}{\delta}\right)\right).
\]
Then we can get
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 
\leq 2 \max_{t=1,\ldots,T} \| \nabla F(w_t) - \nabla F_S(w_t) \|^2 + \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{t=1}^{T} \eta_t^2 + \log \left( \frac{1}{\delta} \right) \right).
\]
Combined with Lemma 46, under Assumptions 2, 4, 5 and 6, we have the following inequality with probability at least 1 - \frac{2\delta}{3},
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 
\leq 2 \max_{t=1,\ldots,T} \left[ c' \beta \max \left\{ \| w_t - w^* \|, \frac{1}{n} \right\} \eta + B_2 \log \left( \frac{12}{\delta} \right) \right] + \sqrt{2 \mathbb{E} \left[ \| \nabla f(w^*, z) \|^2 \right] \log \left( \frac{12}{\delta} \right)} \Bigg) \right]
+ \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{t=1}^{T} \eta_t^2 + \log \left( \frac{1}{\delta} \right) \right),
\] (39)
where \( \eta = \sqrt{\frac{d + \log \frac{24 \log_2 (2nR^2 + 2)}{n}}{n}} + \frac{d + \log \frac{24 \log_2 (2nR^2 + 2)}{n}}{n}. \)

From Lemma 40 and Lemma 48, we have the following inequality with probability 1 - \delta/3 uniformly for all \( t = 1, \ldots, T \)
\[
\| w_{t+1} - w^* \| \leq C'_2 \left( \left( \sum_{k=1}^{T} \eta_k^2 \right)^{1/2} + 1 \left( \sum_{k=1}^{T} \eta_k \right)^{1/2} \right) \log(12/\delta)
\leq C_{T, \delta} := \begin{cases} 
\mathcal{O}(\log(1/\delta))T^{2-\theta}, & \text{if } \theta < 1/2 \\
\mathcal{O}(\log(1/\delta))T^{1+ \frac{1}{2}} \log^{1/2} T, & \text{if } \theta = 1/2 \\
\mathcal{O}(\log(1/\delta))T^{1-\theta}, & \text{if } \theta > 1/2
\end{cases}
\] (40)
Back to (39), if we focus on the dominated term, we can derive the following result with probability at least 1 - \frac{2\delta}{3},
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 
= \mathcal{O} \left( \max_{t=1,\ldots,T} \| w_t - w^* \| \frac{d + \log \frac{24 \log_2 (2nR^2 + 2)}{n}}{n} \right) + \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{t=1}^{T} \eta_t^2 + \log \left( \frac{1}{\delta} \right) \right)
= \mathcal{O} \left( C^2_{T, \delta} \frac{d + \log \frac{1}{\delta}}{n} \right) + \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{k=1}^{T} \eta_k^2 + \log \left( \frac{1}{\delta} \right) \right),
\] (41)
Substituting (40) to (41) and together with Lemma 40, we finally obtain the following inequality with probability at least $1 - \delta$

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \begin{cases} \\
\mathcal{O}(\xi_{n,d,\delta}) T^{2-3\theta} + \mathcal{O}(\log(1/\delta) T^{-\theta}), & \text{if } \theta < 1/2 \\
\mathcal{O}(\xi_{n,d,\delta}) T^{1/2} \log T + \mathcal{O}(\log(T/\delta) T^{-\theta}), & \text{if } \theta = 1/2 \\
\mathcal{O}(\xi_{n,d,\delta}) T^{1-\theta} + \mathcal{O}(\log(1/\delta) T^{\theta-1}), & \text{if } \theta > 1/2 
\end{cases}
$$

where $\xi_{n,d,\delta} = \frac{d + \log \frac{1}{\delta}}{n} \log^2(1/\delta)$.

When $\theta < 1/2$, we set $T \asymp (nd^{-1})^{\frac{1}{2(1-\theta)}}$, then we obtain the following result with probability at least $1 - \delta$

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \mathcal{O}\left((nd^{-1})^{\frac{\theta}{2(1-\theta)}} \log^3(1/\delta)\right).
$$

When $\theta = 1/2$, we set $T \asymp nd^{-1}$, then we obtain the following result with probability at least $1 - \delta$

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \mathcal{O}\left(\sqrt{\frac{d}{n}} \log(T/\delta) \log^3(1/\delta)\right).
$$

When $\theta > 1/2$, we set $T \asymp (nd^{-1})^{\frac{1}{2(1-\theta)}}$, then we obtain the following result with probability at least $1 - \delta$

$$
\left(\sum_{t=1}^{T} \eta_t\right)^{-1} \sum_{t=1}^{T} \eta_t \|\nabla F(w_t)\|^2 = \mathcal{O}\left(\sqrt{\frac{d}{n}} \log^3(1/\delta)\right).
$$

The proof is complete.

7.7 Proof of Theorem 30

**Proof** If Assumptions 2 and 4 hold and $F_S$ satisfies Assumption 8 with parameter $2\mu$, from Lemma 46, for all $w \in \mathcal{W}$ and any $\delta > 0$, when $n \geq \frac{c\beta^2(d + \log(\frac{n\log(2n\beta+2)}{\mu^2}))}{\mu^2}$, with probability at least $1 - \delta$,

$$
\|\nabla F(w)\| \leq 2 \|\nabla F_S(w)\| + \frac{2\mu}{n} + 2\frac{B_s \log(4/\delta)}{n} + 2\sqrt{\frac{2\mathbb{E}[\|\nabla F(w^*, z)\|^2\log(4/\delta)]}{n}}.
$$

47
Then we can derive that, when \( n \geq \frac{c^2\beta^2(d+\log(16\log(2nR)))}{\mu^2} \), with probability at least \( 1 - \delta/2 \), there holds
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 \leq 8 \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F_S(w_t) \|^2 + \frac{8\mu^2}{n^2} + \frac{8B_2^2 \log^2(8/\delta)}{n^2} + \frac{16E[\| \nabla f(w^*, z) \|^2 \log(8/\delta)]}{n}.
\]
(42)

When \( \eta_t = \eta_t t^{-\theta}, \theta \in (0, 1) \) with \( \eta_t \leq \frac{1}{T^2} \) and Assumptions 2, 5 and 6 hold, according to Lemma 47, we obtain the following inequality with probability at least \( 1 - \delta/2 \),
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F_S(w_t) \|^2 \leq \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{t=1}^{T} \eta_t^2 + \log \left( \frac{1}{\delta} \right) \right).
\]
(43)

Combined (42) and (43), we derive that, if \( n \geq \frac{c^2\beta^2(d+\log(16\log(2nR))}{\mu^2} \), with probability at least \( 1 - \delta \)
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 \leq \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \mathcal{O} \left( \sum_{k=1}^{T} \eta_k^2 + \log \left( \frac{1}{\delta} \right) \right) + \mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} + \frac{E[\| \nabla f(w^*, z) \|^2 \log(1/\delta)]}{n} \right).
\]

Based on Lemma 40 and (34), we finally obtain the following inequality with probability at least \( 1 - \delta \),
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 = \begin{cases} 
\mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n} \right) + \mathcal{O}(\log(1/\delta)T^{-\theta}), & \text{if } \theta < 1/2, \\
\mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n} \right) + \mathcal{O}(\log(T/\delta)T^{-1/2}), & \text{if } \theta = 1/2, \\
\mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} + \frac{F(w^*) \log(1/\delta)}{n} \right) + \mathcal{O}(\log(1/\delta)T^{\theta-1}), & \text{if } \theta > 1/2.
\end{cases}
\]

When \( \theta < 1/2 \), we set \( T \asymp n^{2\theta} \) and assume \( F(w^*) = \mathcal{O}(\frac{1}{n}) \) as did in (Zhang et al., 2017), then we obtain the following result with probability at least \( 1 - \delta \)
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 = \mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} \right).
\]

When \( \theta = 1/2 \), we set \( T \asymp n^4 \) and assume \( F(w^*) = \mathcal{O}(\frac{1}{n}) \), then we obtain the following result with probability at least \( 1 - \delta \)
\[
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 = \mathcal{O} \left( \frac{\log^2(1/\delta)}{n^2} \right).
\]
When $\theta > 1/2$, we set $T \asymp n^{1-2\theta}$ and assume $F(w^*) = O(\frac{1}{n})$, then we obtain the following result with probability at least $1 - \delta$

$$
\left( \sum_{t=1}^{T} \eta_t \right)^{-1} \sum_{t=1}^{T} \eta_t \| \nabla F(w_t) \|^2 = O \left( \frac{\log^2 (1/\delta)}{n^2} \right).
$$

The proof is complete. 

7.8 Proof of Theorem 32

Proof Since $F_S$ satisfies the PL assumption (Assumption 8) with parameter $2\mu$, we have

$$
F(w) - F(w^*) \leq \frac{\| \nabla F(w) \|^2}{4\mu}, \forall w \in W. \quad (44)
$$

So to bound $F(w_{T+1}) - F(w^*)$, we need to bound the term $\| \nabla F(w_{T+1}) \|^2$. And there holds

$$
\| \nabla F(w_{T+1}) \|^2 = 2 \| \nabla F(w_{T+1}) - \nabla F_S(w_{T+1}) \|^2 + 2 \| \nabla F_S(w_{T+1}) \|^2. \quad (45)
$$

From (31), under assumptions 2, 5, 6 and the PL property of empirical risk $F_S$, we know that with probability at least $1 - \delta/2$

$$
\| \nabla F_S(w_{T+1}) \|^2 = O \left( \frac{\log T \log^2 (1/\delta)}{T} \right). \quad (46)
$$

We now bound the first term $\| \nabla F(w_{T+1}) - \nabla F_S(w_{T+1}) \|^2$. From Lemma 46, if Assumptions 2 and 4 hold and $F_S$ satisfies Assumption 8, for all $w \in W$ and any $\delta > 0$, when $n \geq \frac{C_\beta (d + \log (16 \log (2nR + 2)))}{\mu^2}$, with probability at least $1 - \delta/2$, there holds

$$
\| \nabla F_S(w_{T+1}) \|^2 \leq 2 \| \nabla F(w_{T+1}) \|^2 + 2 \| \nabla F_S(w_{T+1}) \|^2 \leq \| \nabla F_S(w_{T+1}) \|^2 + 2 \| \nabla F_S(w_{T+1}) \|^2 + 2 \| \nabla f(w^*, z) \|^2 \log (8/\delta)
$$

where the last inequality follows from (34). Then we can derive that

$$
\| \nabla F(w_{T+1}) - \nabla F_S(w_{T+1}) \|^2 = O \left( \frac{\log T \log^3 (1/\delta)}{T} \right) + O \left( \frac{\log^2 (1/\delta)}{n^2} + \frac{F(w^*) \log (1/\delta)}{n} \right)
$$

$$
= O \left( \frac{\log T \log^3 (1/\delta)}{T} \right) + O \left( \frac{\log^2 (1/\delta)}{n^2} \right), \quad (47)
$$

where the second equality follows from the fact that assuming $F(w^*) = O(\frac{1}{n})$. Substituting (47) and (46) into (45), we have

$$
\| \nabla F(w_{T+1}) \|^2 = O \left( \frac{\log T \log^3 (1/\delta)}{T} \right) + O \left( \frac{\log^2 (1/\delta)}{n^2} \right). \quad (48)
$$
Then substituting (48) into (44). When we choose $T \asymp n^2$, we obtain

$$F(w_{T+1}) - F(w^*) = O\left(\frac{\log n \log^3(1/\delta)}{n^2}\right).$$

Hence, when assumptions are satisfied, $n \geq \frac{c \beta^2 (d+\log(16 \log(2nR+2)))}{\mu^2}$ and $\eta_t = \frac{2}{\mu(t+t_0)}$ such that $t_0 \geq \max \left\{ \frac{A \beta}{\mu}, 1 \right\}$ for all $t \in \mathbb{N}$, for all $w \in \mathcal{W}$ and any $\delta > 0$, with probability at least $1 - \delta$, we have $F(w_{T+1}) - F(w^*) = O\left(\frac{\log n \log^3(1/\delta)}{n^2}\right)$ when we choose $T \asymp n^2$ and assume $F(w^*) = O(\frac{1}{n})$. The proof is complete.

8. Conclusions

This paper presents improved learning rates for two important approaches of stochastic optimization: ERM and SGD. We develop theoretical analysis via two viewpoints: uniform stability and uniform convergence of gradients. In uniform stability regime, we relax the strong assumptions in convex learning and provide faster learning rates in nonconvex learning. These faster learning rates are further be improved in uniform convergence of gradients regime. Overall, we obtain a series of state-of-the-art learning rates under milder assumptions in convex learning and state-of-the-art rates in nonconvex learning. It would be interesting to extend our analysis to distribution learning (Lin and Zhou, 2018) and other stochastic optimization methods, such as stochastic variance-reduced optimization (Reddi et al., 2016; Allen-Zhu and Hazan, 2016), stochastic proximal gradient descent (Karimi et al., 2016), stochastic coordinate gradient descent (Wright, 2015).

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Appendix A: Curvature Conditions

Lemma 51 Let \( F(w) \) be \( \beta \) smooth, and assume that \( W_* \) be a non-empty solution set of \( \arg \min_{w \in W} F(w) \). For any \( w \in W \), let \( w^* = \arg \min_{u \in W_*} \| u - w \| \) denote an optimal solution closest to \( w \).

1. **Strong convexity (SC.)** (Nesterov, 2003): for all \( w_1, w_2 \in W \) we have
   \[
   F(w_1) - F(w_2) \geq \nabla F(w_2)^T (w_1 - w_2) + \frac{\mu}{2} \|w_1 - w_2\|^2.
   \]

2. **Weak strong convexity (WSC.)** (Necoara et al., 2019): for all \( w \in W \) we have
   \[
   F(w^*) - F(w) \geq \nabla F(w)^T (w^* - w) + \frac{\mu}{2} \|w^* - w\|^2.
   \]

3. **Restricted secant inequality (RSI.)** (Zhang and Yin, 2015): for all \( w \in W \) we have
   \[
   \nabla F(w)^T (w - w^*) \geq \mu \|w^* - w\|^2.
   \]

4. **Polyak-Lojasiewicz (PL.)** (Polyak, 1963; Lojasiewicz, 1963): for all \( w \in W \) we have
   \[
   F(w) - F(w^*) \leq \frac{1}{2\mu} \|\nabla F(w)\|^2.
   \]

5. **Error Bound (EB.)** (Luo and Tseng, 1995): for all \( w \in W \) we have
   \[
   \|\nabla F(w)\| \geq \mu \|w - w^*\|^2.
   \]

6. **Quadratic Growth (QG.)** (Anitescu, 2006): for all \( w \in W \) we have
   \[
   F(w) - F(w^*) \geq \frac{\mu}{2} \|w - w^*\|^2.
   \]

There holds that:

\[
(SC) \rightarrow (WSC) \rightarrow (RSI) \rightarrow (EB) \equiv (PL) \rightarrow (QG).
\]

If we further assume that \( F(w) \) is convex then we have

\[
(RSI) \equiv (EB) \equiv (PL) \equiv (QG).
\]

Remark 52 As we mentioned in Section 5, strong convexity of the objective function do not hold for many practical applications, such as least squares and logistic regression. This situation has motivated a variety of alternatives to strong convexity in the literature, to guarantee the linear convergence rates. Curvature conditions (2-6) are the relaxation of strong convexity. The proof of Lemma 51 can be adapted from (Karimi et al., 2016) and the relationships among many of those conditions for non-smooth functions can be found in (Drusvyatskiy and Lewis, 2018). From Lemma 51, one can see that QG is the weakest assumption among those considered. In this paper, we are focusing on the two conditions: PL and QG. The PL condition implies that all critical points are global minima without requiring convexity. The QG condition further relax this, allowing for critical points that are not global minima, while still enforcing that locally, the function grows quadratically away from global minima. (Charles and Papailiopoulos, 2018) illustrate two examples \( x^2 + \sin^2(x) \) and \((x - 2)^2(x + 2)^2\) for PL and QG, respectively, readers interested can have a reference.
Appendix B: Concentration Inequalities

**Lemma 53 ((Lei et al., 2021))** Let $z_1, \ldots, z_n$ be a sequence of random variables such that $z_k$ may depend the previous variables $z_1, \ldots, z_{k-1}$ for all $k = 1, \ldots, n$. Consider a sequence of functionals $\xi_k(z_1, \ldots, z_k)$, $k = 1, \ldots, n$. Let $\sigma_n^2 = \sum_{k=1}^n E_{z_k}[(\xi_k - E_{z_k}[\xi_k])^2]$ be the conditional variance.

(a) Assume $|\xi_k - E_{z_k}[\xi_k]| \leq b_k$ for each $k$. Let $\delta \in (0, 1)$. With probability at least $1 - \delta$

$$\sum_{k=1}^n \xi_k - \sum_{k=1}^n E_{z_k}[\xi_k] \leq \left(2 \sum_{k=1}^n b_k^2 \log \frac{1}{\delta} \right)^{\frac{1}{2}}.$$

(b) Assume $|\xi_k - E_{z_k}[\xi_k]| \leq b$ for each $k$. Let $\rho \in (0, 1]$ and $\delta \in (0, 1)$. With probability at least $1 - \delta$ we have

$$\sum_{k=1}^n \xi_k - \sum_{k=1}^n E_{z_k}[\xi_k] \leq \frac{\rho \sigma_n^2}{b} + \frac{b \log \frac{1}{\delta}}{\rho}.$$

**Lemma 54 ((Tarres and Yao, 2014))** Let $\{\xi_k\}_{k \in \mathbb{N}}$ be a martingale difference sequence in $\mathbb{R}^d$. Suppose that almost surely $\|\xi_k\| \leq D$ and $\sum_{k=1}^t E[\|\xi_k\|^2 | \xi_1, \ldots, \xi_{k-1}] \leq \sigma_t^2$. Then, for any $0 < \delta < 1$, the following inequality holds with probability at least $1 - \delta$

$$\max_{1 \leq j \leq t} \left\| \sum_{k=1}^j \xi_k \right\| \leq 2 \left( \frac{D}{3} + \sigma_t \right) \log \frac{2}{\delta}.$$

**Lemma 55 ((Pinelis, 1994; Pinelis et al., 1999))** Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables taking values in a real separable Hilbert space. Assume that $E[X_i] = \mu$, $E[\|X_i - \mu\|^2] = \sigma_i^2$, $\forall 1 \leq i \leq n$. If for all $1 \leq i \leq n$, vector $X_i$ satisfying the following Bernstein condition with parameter $B$

$$E \left[ \|X_i - \mu\|^k \right] \leq \frac{1}{2} k! \sigma_i^2 B^{k-2}, \forall 2 \leq k \leq n.$$

Then for all $\delta \in (0, 1)$, with probability $1 - \delta$, there holds that

$$\left\| \frac{1}{n} \sum_{i=1}^n X_i - \mu \right\| \leq \frac{B \log(2/\delta)}{n} + \sqrt{\frac{2 \sigma_i^2 \log(2/\delta)}{n}}.$$

**Remark 56** Part (a) of Lemma 53 (Boucheron et al., 2013) and Part (b) of Lemma 53 (Zhang, 2005) are concentration inequalities on martingales. Lemma 54 is a pinelis-bernstein inequality for martingale difference sequences in a Hilbert space (Lei and Tang, 2021). Lemma 55 is a vector Bernstein concentration inequality.

Appendix C: Stability Bound

In this section, we provide an uniform stability bound with high probability for nonconvex objectives.
Lemma 57 ((RaghavanPrabhakar, 1988)) Let \( s_t \in (0, 1] \) and let \( X_t \) be a Bernoulli variable with parameter \( p_t \) independently, i.e., \( X_t \sim p_t \). Denote \( \mu = \sum_{t=1}^{T} s_t p_t \). Then, for all \( k > 0 \),
\[
\mathbb{P}\left( \sum_{t=1}^{T} s_t X_t \geq (1 + k)\mu \right) \leq \left[ \frac{e^k}{(1 + k)(1 + k)} \right]^\mu.
\]

Lemma 58 ((Hardt et al., 2016)) Assume that \( S \) and \( S' \) differ by the \( i \)-th datum, i.e., \( z_i \neq z'_i \). Let \( \{w_t\}_{t=1}^{T} \) and \( \{w'_t\}_{t=1}^{T} \) be a sequence produced by SGD based on \( S \) and \( S' \), respectively. And let \( G_1, ..., G_t \) and \( G'_1, ..., G'_t \) be the gradient update rules, i.e., \( G_t(w_t) = w_t - \nabla f(w_t) \). If the objective function \( f \) satisfies Assumptions 1 and 2, then
\[
\|w_{t+1} - w'_{t+1}\| \leq \begin{cases} (1 + \eta_t \beta)\|w_t - w'_t\|, & \text{if } G_t = G'_t, \\ \|w_t - w'_t\| + 2\eta_t L, & \text{otherwise.} \end{cases}
\]

Theorem 59 Suppose the objective function \( f \) satisfies Assumptions 1 and 2. Let \( \eta_t = \frac{c}{t+1} \) such that \( \eta_t \in (0, 1] \) for all \( t \in \mathbb{N} \) and \( w_t \) be the \( t \)-th iteration produced by (3). Then, for any \( \delta \in (0, 1) \), we have the following uniform stability bound with probability at least \( 1 - \delta \)
\[
\epsilon(w_{t+1}) = \mathcal{O}\left(t^{c\beta} \sqrt{\frac{\log(n/\delta)}{n}}\right).
\]

Proof Assume that any datasets \( S, S' \in \mathbb{Z}^n \) differ by the \( i \)-th datum, i.e., \( z_i \neq z'_i \). Let \( \{w_t\}_{t=1}^{T} \) and \( \{w'_t\}_{t=1}^{T} \) be a sequence produced by (3) based on \( S \) and \( S' \), respectively. For any \( t \in [T] \), we consider the following two cases.
Case 1: If \( i_t \neq i \), Lemma 58 implies that
\[
\|w_{t+1} - w'_{t+1}\| \leq (1 + \eta_t \beta)\|w_t - w'_t\|.
\]

Case 2: If \( i_t = i \), Lemma 58 implies that
\[
\|w_{t+1} - w'_{t+1}\| \leq \|w_t - w'_t\| + 2\eta_t L.
\]

Combined with the two cases, we obtain that
\[
\|w_{t+1} - w'_{t+1}\| \leq (1 + \eta_t \beta)\|w_t - w'_t\| + 2\eta_t L_{i_t = i} = \eta_t,
\]
where \( L_{i_t = i} \) is the indicator function, i.e., \( L_{i_t = i} = 1 \) if \( i_t = i \) and 0 otherwise. Applying the above inequality recursively, we get
\[
\|w_{t+1} - w'_{t+1}\| \leq \prod_{k=1}^{t} (1 + \eta_k \beta)\|w_1 - w'_1\| + \left(2L \sum_{k=1}^{t} L_{i_k = i} \eta_k\right) \prod_{j=k+1}^{t} (1 + \eta_j \beta).
\]
Since $w_1 = w'_1$, we get
\[ \|w_{t+1} - w'_{t+1}\| \leq \left( 2L \sum_{k=1}^{t} \mathbb{I}_{i_k=i} \eta_k \right) \prod_{j=k+1}^{t} \left( 1 + \eta_j \beta \right) \]
\[ \leq \left( 2L \sum_{k=1}^{t} \mathbb{I}_{i_k=i} \eta_k \right) \exp \left( \sum_{j=k+1}^{t} \eta_j \beta \right) \]
\[ \leq \left( 2L \sum_{k=1}^{t} \mathbb{I}_{i_k=i} \eta_k \right) \exp \left( c \beta \log \left( \frac{t+1}{k+1} \right) \right) \]
\[ \leq (t+1)^c \beta \left( 2L \sum_{k=1}^{t} \mathbb{I}_{i_k=i} \eta_k \frac{1}{(k+1)^c \beta} \right) \]
\[ = 2Lc(t+1)^c \beta \left( \sum_{k=1}^{t} \mathbb{I}_{i_k=i} \frac{1}{(k+1)^c \beta} \right), \]
where the second inequality follows from that $1 + x \leq \exp(x)$ for all $x$. Denoted by $s_k = \frac{1}{(k+1)^c \beta}$. Let $X = \sum_{k=1}^{t} \mathbb{I}_{i_k=i} s_k$ and $a = \sum_{k=1}^{t} s_k$, we have $\mu = \frac{a}{n}$. Lemma 57 implies that with probability at least $1 - \exp\left\{ -\frac{a^2}{3n} \right\}$ for any $\delta \in (0, 1)$, there holds that $X \leq (1 + \delta) \frac{a}{n}$. Thus, with probability at least $1 - \delta$, we have
\[ X \leq \left( 1 + \sqrt{\frac{3n \log(1/\delta)}{a}} \right) \frac{a}{n}. \] (49)

By taking a union bound of probabilities over $i = 1, ..., n$, with probability at least $1 - \delta$, there holds
\[ \|w_{t+1} - w'_{t+1}\| \leq 2Lc(t+1)^c \beta \left( \frac{\sum_{k=1}^{t} s_k}{n} + \sqrt{\frac{3 \log(n/\delta)}{n} \sum_{k=1}^{t} s_k} \right) \]
\[ \leq (t+1)^c \beta \mathcal{O} \left( \sqrt{\frac{\log(n/\delta)}{n}} \right), \]
where the second inequality follows from Lemma 40.

Combined with the Lipschitz property of $f$, we can obtain the uniform stability bound for SGD in nonconvex learning. The proof is complete.

**Remark 60** We just provide the result corresponding to step-size sequence $\eta_t = \mathcal{O}(1/t)$, while stability bounds of other step-size sequences can be easily derived followed from the above proof. From Theorem 59, one can see that, in nonconvex learning, the stability bound also includes a term of slow order $\mathcal{O} \left( \frac{1}{\sqrt{n}} \right)$ (see (49)) when studied by Chernoff bounds for Bernoulli variables. Moreover, this bound of order $\mathcal{O} \left( t^c \beta \sqrt{\frac{\log(n/\delta)}{n}} \right)$ requires $c$ to be extremely small to prevent the stability bound from being vacuous. These problems require us to adopt new perspective to prove faster stability bounds.
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