Two-body repulsive bound pairs in multi-body interacting Bose-Hubbard model

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We study the system of multi-body interacting bosons on a two dimensional optical lattice and analyze the formation of bound bosonic pairs in the context of the Bose-Hubbard model. Assuming a repulsive two-body interaction we obtain the signatures of pair formation in the regions between the Mott insulator lobes of the phase diagram for different choices of higher order local interactions. Considering the most general Bose-Hubbard model involving local multi-body interactions we investigate the ground state properties utilizing the cluster mean-field theory approach and further confirm the results by means of sophisticated infinite Projected Entangled Pair States calculations. By using various order parameters we show that this pairing of bosons leads to the pair superfluid phase which lies between two different Mott lobes depending on the choice of the interaction strengths. We also analyze the effect of temperature and establish the stability of the PSF phase against thermal fluctuation.

I. INTRODUCTION

The seminal observation of quantum phase transition between the superfluid(SF) and the Mott insulator(MI) phases in optical lattice [1] and its theoretical prediction [2, 3] have revolutionized the field of strongly correlated quantum matter. The physics which emerges out of the competition between the local two-body interactions and the off-site hopping strengths of the paradigmatic Bose-Hubbard(BH) model is regarded as one of the simplest examples of quantum simulations. The underlying mechanism which drives this interesting phase transition is the high-level of tunability of on-site interactions with respect to the hopping amplitudes using the technique of the Feshbach resonance and/or the lattice strengths. Following this experimental observation, several interesting phenomena have been unveiled at the interface of atomic, molecular, optical and condensed matter physics in recent years considering many variants of the BH model. However, the simple BH model with only on-site interactions itself have revealed a plethora of exotic physics in different contexts [4–6].

Recently, it has been experimentally shown that the ultracold atoms in optical lattice naturally exhibit on-site three and other higher order interactions apart from the conventional two-body one [7, 8]. This finding has further broadened the scope to explore the physics of ultracold matter in presence of multi-body interactions in optical lattices. An immediate usefulness of such multi-body interactions can be understood in the context of the attractive BH model which involves local two-body attractive interactions. It has been shown that for any finite attractive interaction the bosons occupy a single site of an optical lattice leading to collapse [9]. This difficulty can be overcome by including a very strong three-body on-site repulsion which prevents the occupation of a lattice site by more than two atoms and hence the collapse. A recent proposal rigorously shows that an infinitely strong three-body repulsion can arise due to the three-body loss process resulting from the elastic scattering of atoms [10]. This infinite three-body repulsion which is termed as the three-body hardcore constraint, facilitates the formation of attractively bound bosonic pairs. The superfluid of these composite pairs is called the pair superfluid(PSF) phase in optical lattice [10–13] which is an interesting manifestation of competing two and three-body interactions. Several other theoretical proposals have been made recently to control the three-body interactions in various ways in optical lattices [14–16]. Moreover, recent proposal by D. Petrov [17] reveals the possibilities to simultaneously manipulate the higher order multi-body interactions along with the two-body one in atomic systems [17, 18]. This prediction is one step forward in the directions of exploring physics arising due to the on-site interactions in optical lattices. With these types of interactions, the standard BH model gets modified accordingly and one gets a more general BH model with the on-site multi-body interactions given as;

\begin{equation}
H = -t \sum_{\langle i,j \rangle} \left( a_i^\dagger a_j + H.c. \right) + \sum_{p=2}^{M} \left( U_p \sum_{i} \frac{(n_i)!}{p!(n_i-p)!} \right) - \mu \sum_i n_i \tag{1}
\end{equation}

where $a_i^\dagger(a_i)$ is the bosonic creation (annihilation) operator, $n_i$ is the number operator for the $i^{th}$ site, and $\langle i, j \rangle$ denotes the nearest neighbor sites. While $t$ represents the nearest-neighbor hopping amplitude, $U_p$ is the on-site $p$-body interaction strength. Depending on the value of $M$, one gets the corresponding multi-body interacting BH model. $\mu$ is the chemical potential associated to the system in the grand canonical ensemble which decides the number of particles in the system. As mentioned before, this model with only two body interaction $U_2$ exhibits the SF-MI phase transition at integer densities. As a result one gets the well known MI lobes corresponding to different atom densities in the ground state.
phase diagram plotted in the \( \mu \) and \( U_2 \) plane. Hereafter, we denote the MI lobes for different particle densities as MI(\( \rho \)) where \( \rho \) is the ratio between the total number of particles to total number of sites in the systems.

Although, competing multi-body interactions in the BH model may provide interesting physics, the system with up to the three-body interactions \( U_3 \) has been widely studied in recent years \cite{10, 11, 16, 19–26} revealing various interesting physical phenomena in optical lattices. However, in an interesting proposal in Ref. \cite{16}, it is shown that the strength of the three-body interaction \( U_3 \) can be tuned by coupling it to the Efimov states which leads to a non-trivial form of the interaction \( U_3 = \delta_{n,3} \).

The phase diagram of the BH model in presence of such three-body interaction is obtained by using the simple mean-field theory approach analysis and complemented by the Quantum Monte Carlo (QMC) calculation. This reveals that for attractive \( U_3 \) and repulsive \( U_2 \) the system favors a direct first order transition from the MI(1) to the MI(3) phase by completely suppressing the MI(1) lobe when \( U_3/U_2 > 1 \) even at finite temperature \cite{16}. However, this finding was later found to be inconsistent when compared to the density matrix renormalization group (DMRG) and the cluster mean field theory (CMFT) calculations in one and two dimensional systems respectively by some of us in Ref. \cite{27}. A careful analysis in Ref. \cite{27} showed that there exists no first order transition between the Mott lobes for the parameter choice considered in Ref. \cite{16}. Rather, the competing two and three body interactions lead to the formation of a non-trivial PSF phase in between the MI(1) and MI(3) lobes where the bosons tend to move in pairs even in the presence of the two-body repulsive interactions. This reveals a kind of two-body repulsively bound pairs driven by a mechanism completely different from the one observed in optical lattices by Winkler et al. \cite{28} where the pair formation occurs due to the competition between the two-body interaction \( U_2 \) and the bandwidth.

In this paper we show that to achieve this anomalous pairing of bosons with two-body repulsion, it is not always necessary to consider the specific form of the three-body interaction as discussed in Ref. \cite{16, 27}. The most general BH model given in Eq. 1 with suitable choice of multi-body interactions may stabilize the PSF phase between the Mott lobes which will be discussed in more detail below. The remaining part of the paper is organized as follows. In Sec-II we explain the model considered for this work with a brief information about the methods. In Sec-III we discuss our results in detail and finally we conclude in Sec-IV.

II. METHOD

We numerically investigate the model shown in Eq. 1 by restricting up to four-body interactions for simplicity. The explicit form of the Hamiltonian with all the interactions is given as:

\[
H = -t \sum_{\langle i,j \rangle} (a_i^\dagger a_j + H.c.) + \frac{U_2}{2} \sum_i n_i (n_i - 1) + \frac{U_3}{6} \sum_i n_i (n_i - 1) (n_i - 2) + \frac{U_4}{24} \sum_i n_i (n_i - 1) (n_i - 2) (n_i - 3) - \mu \sum_i n_i \tag{2}
\]

where the terms have their usual meaning as discussed before. In our calculation we set \( t = 1 \) to make all the physical quantities dimensionless. In order to analyze the ground state properties of Eq. 2 we first utilize the self-consistent CMFT approach which is an approximation method based on the simple single site mean-field theory approach \cite{29–31}. In this case, the Hamiltonian is divided into several clusters of finite number of sites and each cluster interacts with the rest of the system in a mean-field way i.e.

\[
H_{CMFT} = \sum_{i,j \in C} H_C - t \sum_{\langle i,j \rangle} (a_i^\dagger \psi_j + H.c.) \tag{3}
\]

Here, \( H_C \) is the cluster Hamiltonian identical to Eq. 2 with index \( i, j \) belonging to the cluster \( C \). The second term which is the mean-field expression for the hopping term from the \( i^{th} \) site at the cluster boundaries(B) to the nearest neighbor \cite{32}. \( \psi \) is the superfluid order parameter which is determined self consistently. In order to obtain the insights about various quantum phases we utilize the average density and the superfluid density of the system \( \rho = 1/L \sum_i n_i \) and \( \rho_s = 1/L \sum_i \psi_i^2 \) respectively computed from the CMFT ground state where \( L \) is number of sites in a cluster. It is well known that the CMFT method is more accurate than the simple mean-field field theory approach and can capture the qualitative picture of the system with less computing effort than the powerful QMC method \cite{12, 27, 29–31, 33}. Note that the accuracy of the method relies on the cluster size. In this case we consider a four site cluster which is sufficient to predict the relevant physics.

In addition to the CMFT approach, we have employed the infinite Projected Entangled Pair States (iPEPS) algorithm which are two dimensional tensor network techniques \cite{34, 35}. Such techniques are built upon genuine quantum correlations and hence, goes beyond mean field calculations. Besides, we can directly target the thermodynamic limit by assuming translational invariance over some sites. Another advantage is that unlike QMC techniques, it does not suffer from the infamous sign problem for fermionic and frustrated systems \cite{36}. For these reasons PEPS techniques have been used in the past to study hard condensed matter problems such as frustrated kagome antiferromagnets \cite{37–40} and real materials \cite{41–43}. It has been able to beat state-of-the-art
QMC calculations in finding the ground state energy of the doped Hubbard model [44], and helped settle controversies that would have otherwise been difficult such as the magnetization plateaus of the Shastry-Sutherland model[45], phase diagrams of steady states of dissipative spin models[46], etc. The technique has now been extended to finite temperature calculations [47–50] and the difficult problem of time evolution in 2D [51–53].

For the purpose of this work, we use an iPEPS with a two-site unit cell in the thermodynamic limit. We approximate the ground state of the Hamiltonian given in Eq. 2 using the so-called simple update [54] with bond dimension \( D = 2 \) and \( D = 4 \) which proves sufficient for our purpose. After analyzing the zero temperature phase diagram of the system we investigate the effect of thermal fluctuation in the system. The finite temperature calculations are done with an annealing algorithm [50] with infinite Projected Entangled Pair Operators (iPEPO) which are mixed state version of iPEPS [46, 55]. The expectation values are computed using the Corner Transfer Matrix Renormalization (CTMRG) method [56, 57].

III. RESULTS AND DISCUSSION

In this section we move on to discuss our results in detail which are obtained by using the CMFT and iPEPS approach for \( M = 4 \) of Eq. 1. Keeping terms up to \( M = 4 \) in the model(1) we have three different interactions in the system such as \( U_2, U_3 \) and \( U_4 \). Note that in our analysis the focus is to analyze the two-body repulsive bound pairs. Therefore, our obvious choice is to keep \( U_2 > 0 \). In this case, we consider attractive(repulsive) three-(four-)body interactions i.e. \( U_3 < 0 \) and \( U_4 > 0 \). For simplicity, we define two ratios such as \( U_4/U_3 \) and \( U_3/U_2 \) and analyze the ground state phase diagram of the system. In the case of the BH model shown in Eq. 1, it is well known that the presence of interaction up to \( U_3 \) largely affects the SF-MI phase transitions with modified Mott lobes at higher densities. While the MI lobes corresponding to \( \rho \geq 2 \) get enlarged by the three-body repulsion \( U_3 \) [19, 21], an attractive \( U_3 \) results in shrinking up of the higher MI lobes [21]. However, in this case we show that a large four-body repulsive interaction \( U_4 \) leads to interesting phenomena. In this case, the MI(3) lobe expands by simultaneously shrinking the MI(2) lobe which eventually disappears for some specific ratio of interactions defined above.

In Fig. 1, we depict the phase diagram corresponding to the ground state of Eq. 1 in the \( \mu/U_3 \) and \( t/U_2 \) plane for \( U_4/U_3 = -3 \) and \( U_3/U_2 = -2 \). The MI lobes are denoted by the continuous lines and the dashed line separates the empty state. The SF to MI transitions are characterized by examining the behavior of change in the total density of the system \( \rho \) and the superfluid density \( \rho_s \) with respect to increase in chemical potential \( \mu \). In the SF phase \( \rho \) increases continuously with increase in \( \mu \). However, in the MI phase \( \rho \) remains constant for a range of \( \mu \) and at the same time \( \rho_s \) vanishes. In Fig. 2(a) we show the \( \mu - \frac{\rho}{\rho_s} \) plot for various values of \( U_2 = 18, \ 24 \) and 60 which cut through different regions of the phase diagram of Fig. 1 indicating the MI plateaus and the SF regions. The end points of the plateaus correspond to two different chemical potentials \( \mu^+ \) and \( \mu^- \) of the system defined as

\[
\mu^+ = E_L(N + 1) - E_L(N); \quad \mu^- = E_L(N) - E_L(N - 1).
\]

Here, \( E(N) \) denotes the ground state energy of the system with \( N \) particles. The difference \( G = \mu^+ - \mu^- \) quan-

FIG. 1: (color online)CMFT phase diagram for \( U_3/U_2 = -2.0 \) and \( U_4/U_3 = -3.0 \). Solid lines show the boundaries of MI phases and dashed line separates the empty state. Inset shows the PSF-SF boundary marked by green line with circles.

FIG. 2: (color online) (a) \( \mu \) vs. \( \mu/U_2 \) and (b) \( \rho_s \) vs. \( \mu/U_2 \) plots for several cuts through the phase diagram of Fig. 1 corresponding to \( U_2 = 18 \) (black dot-dashed), 24 (red dashed), 60 (green solid line).
identifies the gap in the MI phase which vanishes in the SF phase. The signatures of the MI and SF phases are also confirmed from the \( \frac{\mu}{U} - \rho_s \) plot in Fig. 2(b) which shows finite(zero) superfluid density in the SF(MI) phase.

Interesting thing happens in the regime of large interactions. It can be seen from Fig. 3 that for large \( U_2 = 50 \), there are discrete jumps in \( \rho \) (blue down triangles) with respect to increase in \( \frac{\mu}{U} \) in the region between two plateaus corresponding to the MI(1) and MI(3) phases. This indicates a change in the particle number \( \Delta N = 2 \) in the region which is a signature of pair formation. We can identify this phase as the PSF phase which can be confirmed from the pair correlation functions \([12, 27, 58]\). To this end we compute the \( n \)-particle correlation functions defined as

\[
\Gamma_n(i, j) = \langle (a_i^n) \rangle.
\]

In Fig. 3 we also plot the correlation functions \( \Gamma_n(i, j) \) for \( n = 1, 2 \) and 3 corresponding to the single-(black circles), two-(red squares) and three-particles(green up triangles) respectively for different values of \( \mu/U_2 \) at a fixed \( t/U_2 = 0.02 \) of the phase diagram given in Fig. 1. It can be seen that at the plateau regions corresponding to the MI(1) and MI(3) phases, all the correlation functions are vanishingly small. However, for the values of \( \rho \) away from the plateau regions i.e. \( 1 < \rho < 3 \), \( \Gamma_2 \) clearly dominates over \( \Gamma_1 \) and \( \Gamma_3 \). This is a clear indication of the existence of the PSF phase which is sandwiched between the MI(1) and MI(3) lobes in the large \( U_2 \) regime as shown in Fig. 1. There exists a SF-PSF phase transition at these densities indicated by the green circles.

As mentioned before, the CMFT approach can predict the quantum phases qualitatively and efficiently. However, to concretely establish the existence of the PSF phase of these two-body repulsively bound pairs we use the iPEPS method discussed before. In our simulation we use various physical quantities to identify different quantum phases. The gapped MI phases are identified by looking at the behavior of the chemical potential \( \mu \) with respect to the average density \( \rho \). The SF and the PSF phases are characterized by their respective order parameters defined as:

\[
O_{SF} = |\langle a_i \rangle|^2
\]

and

\[
O_{PSF} = |\langle a_i^2 \rangle|^2.
\]

We compute these parameters for several values of \( U_2 \) and find signatures of different phases and phase transitions similar to that obtained using the CMFT method. In Fig. 4(a), we plot \( \rho \) (black circles), \( O_{SF} \) (red squares) and \( O_{PSF} \) (green triangles) against \( \mu/U_2 \) for fixed \( t/U_2 = 0.015 \). Note that the choice of \( U_4 \) restricts the local Hilbert space to a maximum of three particles per site and simplifies our iPEPS calculation while retaining the underlying physics of the system. It can be clearly seen from Fig. 4(a) that there exists two Mott plateaus at \( \rho = 1 \) and 3 corresponding to the gapped MI(1) and MI(3) phases. Inside these plateau regions both the superfluid order parameters vanish. However, in the region between the two MI phases, the value

![FIG. 3: (color online) Correlation functions \( \Gamma_n(i, j) \) for a cut along \( 1/U_2 = 0.02 \) in the phase diagram Fig. 1. The corresponding values of \( \rho \) (blue down triangles) are shown for comparison.](image-url)

![FIG. 4: (color online) (a) iPEPS data for \( \rho, O_{SF} \) and \( O_{PSF} \) for \( t/U_2 = 0.015 \) showing the MI(1), MI(3) and the PSF phases. (b) Similar calculations for a cut passing through the normal Superfluid (SF) region at \( t/U_2 = 0.04 \). This phase is characterized by a non-vanishing value of both \( O_{SF} \) as well as \( O_{PSF} \) while the PSF phase is characterized by a vanishing \( O_{SF} \) and non-zero \( O_{PSF} \).](image-url)
of $O_{SF}$ remains vanishingly small, where as $O_{PSF}$ becomes finite indicating the existence of the PSF phase. We have also performed the same calculation for a different cut that passes through the region of normal superfluid (SF) as shown in Fig. 4 (b) for $t/U_2 = 0.04$. We find that both the $O_{SF}$ as well as the $O_{PSF}$ are non-zero in this region which defines our SF phase.

It can be noted that the physics obtained using the CMFT approach and the iPEPS method are similar to the one reported in Ref. [27]. The important difference is the choice of the interactions. We explicitly show that in the presence of two-body repulsion the bosons prefer to move in pairs due to the large three-body attraction and four-body repulsion. The physics of the pair formation and the PSF phase on top of the MI(1) phase can be understood from the energy consideration as discussed in Ref. [27]. Due to the large three-body attraction the system will tend to acquire two particles at a time to reach the energy minimum by forming a trimer. However, because of the presence of uniform two body repulsion from the MI(1) background the added particles tend to move in pairs without affecting the system energy. This leads to the PSF phase in the system. This is indeed an interesting manifestation of the multi-body interactions in the Bose-Hubbard model. We would like to mention that this pair formation is not limited to the region between the MI(1) and MI(3) lobes. One can in principle create the PSF phase between higher Mott lobes such as between the MI(2) and MI(4) lobes. To achieve this one needs to consider a five-body interaction term by keeping terms up to $M = 5$ in the model given in Eq. 1. Using the CMFT calculation we verify that the PSF in this case can be obtained for suitable choice of repulsive $U_2$, $U_3$, $U_5$ and attractive $U_4$ terms in Eq. 1. Because of the attractive nature of $U_4$ in this case, the value of $U_5$ has to be very strong and repulsive to prevent the collapse.

After discussing the zero temperature phase diagram of the model shown in Eq. 2 we embark on to analyze the effect of temperature on the system. As it is well known that the temperature is an unavoidable parameter in the real cold gas experiment [11, 59, 60] and the quantum phases are fragile in presence of thermal fluctuation, it is pertinent to examine the stability of the PSF phase. At this stage, we perform finite temperature calculations using iPEPS to check the survival of the PSF phase by gradually increasing the system temperature $T$. We compute the different order parameters for these thermal states i.e. $O_{SF}$ and $O_{PSF}$ along with $\rho$ for the same choice of parameters considered in Fig. 4(a) for the zero temperature calculation and plot them in Fig. 5. We show two different values of temperature such as $T = 0.1$ and $T = 0.2$ at which the PSF phase clearly survives which can be seen from the finite values of $O_{PSF}$. Above this temperature the PSF phase slowly disappear. This confirms that the PSF phase is stable against the thermal fluctuation.

IV. CONCLUSIONS

In this paper we analyze a multi-body interacting Bose-Hubbard model and show the possibility of creating two-body repulsive bound bosonic pairs in a two dimensional optical lattice due to the combined effects of the multi-body interactions. We establish that for a very strong four-body repulsion a suitable ratio between the three-body attraction and two-body repulsion leads to the pair formation and hence the PSF phase between the MI(1) and MI(3) lobes. This fact is concretely demonstrated by analyzing the ground state properties of the BH model using the CMFT approach as well as the iPEPS method. Moreover, we show that this pair formation is is stable against the thermal fluctuation for some finite values of temperature. Due to the recent development in the field of ultracold quantum gas experiments, if it can be made possible to engineer the multi-body interactions among the bosons then it will be possible to create the repulsively bound pairs in an alternate way as opposed to the already observed one [28]. Moreover, this finding may provide scope to create and manipulate the number of pairs in a controlled manner. As mentioned before, the pair formation is not limited to the two-body repulsion. One can in principle create them due to the two and three-body repulsion by suitable choice of higher order interactions.

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[1] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature 415, 39 (2002).
[2] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).
[3] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. 81, 3108 (1998).
[4] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[5] M. P. Kennett, ISRN Condensed Matter Physics 2013, 1 (2013), ISSN 2090-7400.
[6] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, Advances in Physics 56, 243 (2007).
[7] S. Will, T. Best, U. Schneider, L. Hackermüller, D.-S. Lühmann, and I. Bloch, Nature 465, 197 (2010).
[8] M. J. Mark, E. Haller, K. Lauber, J. G. Danzl, A. J. Daley, and H.-C. Nägerl, Phys. Rev. Lett. 107, 175301 (2011).
[9] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[10] A. J. Daley, J. M. Taylor, S. Diehl, M. Baranov, and P. Zoller, Phys. Rev. Lett. 102, 040402 (2009).
[11] L. Bonnes and S. Wessel, Phys. Rev. Lett. 106, 185302 (2011).
[12] M. Singh, T. Mishra, R. V. Pai, and B. P. Das, Phys. Rev. A 90, 013625 (2014).
[13] Y.-C. Chen, K.-K. Ng, and M.-F. Yang, Phys. Rev. B 84, 092503 (2011).
[14] P. R. Johnson, E. Tiesinga, J. V. Porto, and C. J. Williams, New Journal of Physics 11, 093022 (2009).
[15] A. J. Daley and J. Simon, Phys. Rev. A 89, 053619 (2014).
[16] A. Safavi-Naini, J. von Stecher, B. Capogrosso-Sansone, and S. T. Rittenhouse, Phys. Rev. Lett. 109, 135302 (2012).
[17] D. S. Petrov, Phys. Rev. A 90, 021601 (2014).
[18] D. S. Petrov, Phys. Rev. Lett. 112, 103201 (2014).
[19] M. Singh, A. Dhar, T. Mishra, R. V. Pai, and B. P. Das, Phys. Rev. A 85, 051604 (2012).
[20] B.-I. Chen, X.-b. Huang, S.-p. Kou, and Y. Zhang, Phys. Rev. A 78, 043603 (2008).
[21] T. Sowiński, Phys. Rev. A 85, 065601 (2012).
[22] T. Sowiński, R. W. Chhaijlany, O. Dutta, L. Tagliacozzo, and M. Lewenstein, Phys. Rev. A 92, 043615 (2015).
[23] S. Ejima, F. Lange, H. Fehske, F. Gebhard, and K. z. Münster, Phys. Rev. A 88, 063625 (2013).
[24] J. Silva-Valencia and A. Souza, Eur. Phys. J. B 85, 161 (2012).
[25] C. Avila, R. Franco, A. Souza, M. Figueira, and J. Silva-Valencia, Physics Letters A 378, 3233 (2014), ISSN 0375-9601.
[26] A. F. Hincapie-F, R. Franco, and J. Silva-Valencia, Phys. Rev. A 94, 033623 (2016).
[27] M. Singh, S. Greschner, and T. Mishra, Phys. Rev. A 98, 023615 (2018).
[28] K. Winkler, G. Thalhammer, F. Lang, R. Grimm, J. Hecker Denschlag, A. J. Daley, A. Kuantian, H. P. Büchler, and P. Zoller, Nature 441, 853 (2006), ISSN 1476-4687.
[29] T. McIntosh, P. Pisarski, R. J. Gooding, and E. Zaremba, Phys. Rev. A 86, 013623 (2012).
[30] D. Yamamoto, A. Masaki, and I. Danshita, Phys. Rev. B 86, 054516 (2012).
[31] S. R. Hassan and L. de’ Medici, Phys. Rev. B 81, 035106 (2010).
[32] A. Tomadin, V. Giovannetti, R. Fazio, D. Gerace, I. Carusotto, H. E. Türeci, and A. Imamoglu, Phys. Rev. A 81, 061801 (2010).
[33] D.-S. Lühmann, Phys. Rev. A 87, 043619 (2013).
[34] J. Jordan, R. Orús, G. Vidal, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. 101, 250602 (2008).
[35] F. Verstraete and J. I. Cirac, arXiv e-prints cond-mat/0407066 (2004), cond-mat/0407066.
[36] P. Corboz, R. Orús, B. Bauer, and G. Vidal, Phys. Rev. B 81, 165104 (2010).
[37] H. J. Liao, Z. Y. Xie, J. Chen, Z. Y. Liu, H. D. Xie, R. Z. Huang, B. Normand, and T. Xiang, Phys. Rev. Lett. 118, 137202 (2017).
[38] T. Picot and D. Poilblanc, Phys. Rev. B 91, 064415 (2015).
[39] T. Picot, M. Ziegler, R. Orús, and D. Poilblanc, Phys. Rev. B 93, 064007 (2016).
[40] A. Kshetrimayum, T. Picot, R. Orús, and D. Poilblanc, Phys. Rev. B 94, 235146 (2016).
[41] A. Kshetrimayum, C. Balz, B. Lake, and J. Eisert, arXiv e-prints arXiv:1904.00028 (2019), 1904.00028.
[42] C. Boos, S. P. G. Crone, I. A. Niesen, P. Corboz, K. P. Schmidt, and F. Mila, Phys. Rev. B 100, 140413 (2019).
[43] C. Boos, S. P. G. Crone, I. A. Niesen, P. Corboz, K. P. Schmidt, and F. Mila, arXiv e-prints arXiv:1903.07887 (2019), 1903.07887.
[44] P. Corboz, Phys. Rev. B 93, 045116 (2016).
[45] P. Corboz, T. M. Rice, and M. Troyer, Phys. Rev. Lett. 113, 046402 (2014).
[46] A. Kshetrimayum, H. Weimer, and R. Orús, Nat. Commun. 8, 1291 (2017).
[47] P. Czarnik, L. Cincio, and J. Dziarmaga, Phys. Rev. B 86, 245101 (2012).
[48] P. Czarnik and J. Dziarmaga, Phys. Rev. B 92, 035152 (2015).
[49] P. Czarnik, M. M. Rams, and J. Dziarmaga, Phys. Rev. B 94, 235142 (2016).
[50] A. Kshetrimayum, M. Rizzi, J. Eisert, and R. Orús, Phys. Rev. Lett. 122, 070502 (2019).
[51] C. Hubig and J. I. Cirac, SciPost Phys. 6, 31 (2019).
[52] P. Czarnik, J. Dziarmaga, and P. Corboz, Phys. Rev. B 99, 035115 (2019).
[53] A. Kshetrimayum, M. Gohl, and J. Eisert, arXiv e-prints arXiv:1910.11359 (2019), 1910.11359.
[54] H. C. Jiang, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. 101, 090603 (2008).
[55] H. Weimer, A. Kshetrimayum, and R. Orús, arXiv e-prints arXiv:1907.07079 (2019), 1907.07079.
[56] R. Orús and G. Vidal, Phys. Rev. B 80, 094403 (2009).
[57] R. Orús, Phys. Rev. B 85, 205117 (2012).
[58] M. Singh, S. Mondal, B. K. Sahoo, and T. Mishra, Phys. Rev. A 96, 053604 (2017).
[59] F. Gerbier, Phys. Rev. Lett. 99, 120405 (2007).
[60] T. K. Kopec and M. W. Szymanski, Physics Letters A 378, 3402 (2014), ISSN 0375-9601.