POISSON STRUCTURES COMPATIBLE WITH THE CLUSTER ALGEBRA STRUCTURE IN GRASSMANNIANS

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Abstract. We describe all Poisson brackets compatible with the natural cluster algebra structure in the open Schubert cell of the Grassmannian $G_k(n)$ and show that any such bracket endows $G_k(n)$ with a structure of a Poisson homogeneous space with respect to the natural action of $SL_n$ equipped with an $R$-matrix Poisson-Lie structure. The corresponding $R$-matrices belong to the simplest class in the Belavin-Drinfeld classification. Moreover, every compatible Poisson structure can be obtained this way.

1. Introduction

The notion of a Poisson bracket compatible with a cluster algebra structure was introduced in [GSV1]. It was used to interpret cluster transformations and matrix mutations in cluster algebras from a viewpoint of Poisson geometry. In addition, it was shown in [GSV1] that if a Poisson variety $(M, \{\cdot, \cdot\})$ possesses a coordinate chart that consists of regular functions whose logarithms have pairwise constant Poisson brackets, then one can use this chart to define a cluster algebra $\mathcal{A}_M$, which is closely related (and, under rather mild conditions, isomorphic) to the ring of regular functions on $M$, and such that $\{\cdot, \cdot\}$ is compatible with $\mathcal{A}_M$. This construction was applied to an open cell $G_0^k(n)$ in the Grassmannian $G_k(n)$ viewed as a Poisson homogeneous space under the action of $SL_n$ equipped with the standard Poisson-Lie structure. The resulting cluster algebra $\mathcal{A}_{G_0^k(n)}$ can be viewed as a restriction of the cluster algebra structure in the coordinate ring of $G_k(n)$. This “larger” cluster algebra $\mathcal{A}_{G_k(n)}$ was described in [S] using combinatorial properties of Postnikov’s map from the space of edge weights of a planar directed network into the Grassmannian [P]. Poisson geometric properties of Postnikov’s map are studied in [GSV2]. One of the by-products of this study is the existence of a pencil of Poisson structures on $G_k(n)$ compatible with $\mathcal{A}_{G_0^k(n)}$. It turns out that every bracket in the pencil defines a Poisson homogeneous structure with respect to a Sklyanin Poisson-Lie bracket associated with a solution of the modified classical Yang-Baxter equation (MCYBE) of the form $R_t = R_0 + t\pi_0$, where $t$ is a scalar parameter, $A$ is a certain fixed skew-symmetric $n \times n$ matrix, $R_0 = \pi_+ - \pi_-$, and $\pi_{\pm,0}$ are projections onto strictly upper/strictly lower/diagonal part in $sl_n$ (the standard Poisson-Lie structure corresponds to $t = 0$).

According to the Belavin-Drinfeld classification [BD], skew-symmetric solutions of MCYBE are defined by two types of data: discrete data associated with the Dynkin diagram and called the Belavin-Drinfeld triple and continuous data associated with the Cartan subalgebra. We will say that two $R$-matrices belong to the

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same class if the corresponding Belavin-Drinfeld triples are the same. R-matrices
$R_0$ mentioned above belong to the class associated with the trivial Belavin-Drinfeld
triple. The entire class consists of R-matrices of the form $R_S = R_0 + S\pi_0$, with $S$
arbitrary skew-symmetric. On the other hand, the Poisson pencil described above
does not exhaust all Poisson structures compatible with $\mathcal{A}_{G_k(n)}$. The main goal
of this paper is to prove

**Theorem 1.1.** The Poisson homogeneous structure with respect to the Poisson-Lie
bracket associated with any $R_S$ is compatible with $\mathcal{A}_{G_k(n)}$. Moreover, up to a scalar
multiple, all Poisson brackets compatible with $\mathcal{A}_{G_k(n)}$ are obtained this way.

It should be noted that Poisson brackets compatible with the “larger” cluster
algebra $\mathcal{A}_{G_k(n)}$ are naturally associated with Poisson structures on the Grassmann
cone $C(G_k(n))$ that can be realized as one-dimensional extensions of corresponding
structures on $G_k(n)$. Both the formulation and the proof of Theorem 1.1 can be
modified in a rather straightforward way to the case of the cluster algebra $\mathcal{A}_{G_k(n)}$. A
detailed description of the relationship between $\mathcal{A}_{G_k(n)}$ and $\mathcal{A}_{G_k(n)}$ this modification
relies upon is presented in Chapter 4 of the forthcoming book [GSV3].

The paper is organized as follows. In Section 2 we recall the necessary informa-
tion on cluster algebras and compatible Poisson structures and show how the latter
can be completely described via the use of a toric action. Section 3 provides the
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2. CLUSTER ALGEBRAS AND COMPATIBLE POISSON BRACKETS

We start with the basics on cluster algebras of geometric type. The definition
that we present below is not the most general one, see, e.g., [FZ3, BFZ] for a detailed
exposition. In what follows, we will use notation $[i,j]$ for an interval $\{i,i+1,\ldots,j\}$
in $\mathbb{N}$, and write $[n]$ instead of $[1,n]$.

The **coefficient group** $\mathfrak{P}$ is a free multiplicative abelian group of a finite rank $m$
with generators $g_1,\ldots, g_m$. An **ambient field** is the field $\mathfrak{F}$ of rational functions in $n$
independent variables with coefficients in the field of fractions of the integer group
ring $\mathbb{Z}\mathfrak{P} = \mathbb{Z}[g_1^{\pm 1},\ldots, g_m^{\pm 1}]$ (here we write $x^{\pm 1}$ instead of $x, x^{-1}$).

A seed (of geometric type) in $\mathfrak{F}$ is a pair $\Sigma = (\mathbf{x}, B)$, where $\mathbf{x} = (x_1,\ldots, x_n)$ is a
transcendence basis of $\mathfrak{F}$ over the field of fractions of $\mathbb{Z}\mathfrak{P}$ and $B$ is an $n \times (n+m)$
integer matrix whose principal part $B$ (that is, the $n \times n$ submatrix formed by the
columns $1,\ldots,n$) is skew-symmetrizable. In this paper, we will only deal with the
case when $B$ is skew-symmetric.

The $n$-tuple $\mathbf{x}$ is called a **cluster**, and its elements $x_1,\ldots, x_n$ are called **cluster variables**. Denote $x_{n+i} = g_i$ for $i \in [m]$. We say that $\tilde{\mathbf{x}} = (x_1,\ldots, x_{n+m})$ is an
**extended cluster**, and $x_{n+1},\ldots, x_{n+m}$ are **stable variables**. It is convenient to think
of $\mathfrak{F}$ as of the field of rational functions in $n+m$ independent variables with rational
coefficients.

Given a seed as above, the **adjacent cluster** in direction $k \in [n]$ is defined by

$$
\mathbf{x}_k = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\},
$$

where $\{x_k\}$ is a small subset of $\mathbf{x}$.
where the new cluster variable \( x'_k \) is given by the exchange relation

\[
x_k x'_k = \prod_{1 \leq i \leq n + m} x_i^{b_{ki}^+} + \prod_{1 \leq i \leq n + m} x_i^{-b_{ki}^-};
\]

here, as usual, the product over the empty set is assumed to be equal to 1.

We say that \( \tilde{B}' \) is obtained from \( \tilde{B} \) by a matrix mutation in direction \( k \) and write \( \tilde{B}' = \mu_k(\tilde{B}) \) if

\[
b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k; \\
b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2}, & \text{otherwise.} 
\end{cases}
\]

Given a seed \( \Sigma = (\mathbf{x}, \tilde{B}) \), we say that a seed \( \Sigma' = (\mathbf{x}', \tilde{B}') \) is adjacent to \( \Sigma \) (in direction \( k \)) if \( \mathbf{x}' \) is adjacent to \( \mathbf{x} \) in direction \( k \) and \( \tilde{B}' = \mu_k(\tilde{B}) \). Two seeds are mutation equivalent if they can be connected by a sequence of pairwise adjacent seeds.

The cluster algebra (of geometric type) \( A = A(\tilde{B}) \) associated with \( \Sigma \) is the \( \mathbb{Z}[\mathfrak{g}] \)-subalgebra of \( \mathfrak{g} \) generated by all cluster variables in all seeds mutation equivalent to \( \Sigma \).

Let \( \{\cdot, \cdot\} \) be a Poisson bracket on the ambient field \( \mathfrak{g} \) considered as the field of rational functions in \( n + m \) independent variables with rational coefficients. We say that it is compatible with the cluster algebra \( A \) if, for any extended cluster \( \tilde{\mathbf{x}} = (x_1, \ldots, x_{n+m}) \), one has

\[
\{x_i, x_j\} = \omega_{ij} x_i x_j,
\]

where \( \omega_{ij} \in \mathbb{Z} \) are constants for all \( i, j \in [n + m] \). The matrix \( \Omega^{\tilde{\mathbf{x}}} = (\omega_{ij}) \) is called the coefficient matrix of \( \{\cdot, \cdot\} \) (in the basis \( \tilde{\mathbf{x}} \)); clearly, \( \Omega^{\tilde{\mathbf{x}}} \) is skew-symmetric.

In what follows, we denote by \( A(I, J) \) the submatrix of a matrix \( A \) with a row set \( I \) and a column set \( J \). Consider, along with cluster and stable variables \( \tilde{\mathbf{x}} \), another \((n + m)\)-tuple of rational functions denoted \( \tau = (\tau_1, \ldots, \tau_{n+m}) \) and defined by

\[
\tau_j = x_j^{\xi_j} \prod_{k=1}^{n+m} x_k^{b_{jk}},
\]

where \( \tilde{B} = (b_{jk})_{j,k=1}^{n+m} \) is a fixed skew-symmetric matrix such that \( \tilde{B}([n], [n + m]) = \tilde{B} \), \( \xi_j \) is an integer, \( \xi_j = 0 \) for \( 1 \leq j \leq n \). We say that the entries \( \tau_i, i \in [n + m] \), form a \( \tau \)-cluster. It is proved in [GSVI], Lemma 1.1, that \( \xi_j, n+1 \leq j \leq n+m \), can be selected in such a way that the transformation \( \tilde{\mathbf{x}} \mapsto \tau \) is non-degenerate, provided \( \text{rank } \tilde{B} = n \). We denote \( \varsigma = \text{diag}(\xi_i)_{i=1}^{n+m} \) and \( B_{\varsigma} = \tilde{B} + \varsigma \). Nondegeneracy of the transformation \( \tilde{\mathbf{x}} \mapsto \tau \) is equivalent to nondegeneracy of \( B_{\varsigma} \).

Recall that a square matrix \( A \) is reducible if there exists a permutation matrix \( P \) such that \( PAP^T \) is a block-diagonal matrix, and irreducible otherwise. The following result is a particular case of Theorem 1.4 in [GSVI].

**Theorem 2.1.** Assume that \( \text{rank } \tilde{B} = n \) and the principal part of \( \tilde{B} \) is irreducible. Then a Poisson bracket is compatible with \( A(\tilde{B}) \) if and only if its coefficient matrix \( \Omega^\tau \) in the basis \( \tau \) has the following property: the submatrix \( \Omega^\tau([n], [n + m]) \) is proportional to \( \tilde{B} \).
Starting with an arbitrary \{\cdot, \cdot\}^A \n A compatible with \n A, one can suggest an alternative description of all other compatible Poisson brackets via the following construction. Let \(C\) be an invertible \((n + m) \times m\) matrix. Define an action of \((C^*)^m = \{d = (d_1, \ldots, d_m) : d_1 \cdots d_r \neq 0\}\) on \(\bar{x}\) by

\[
\text{d}\bar{x} = \left(x_i \prod_{\alpha=1}^{m} C_{\alpha i}^{\beta} \right)_{i=1}^{n+m}.
\]

We say that \(\text{(2.2)}\) extends to an action of \((C^*)^m\) on \(\n A\) if the action induced by it in any cluster in \(\n A\) has a form \(\text{(2.2)}\) (with possibly different coefficients \(C_{\alpha i}\)). Lemma 2.3 in \(\text{[GSV1]}\) claims that \(\text{(2.2)}\) extends to an action of \((C^*)^m\) on \(\n A\) if and only if \(\bar{B}C = 0\). The same condition guarantees that \(\tau_i(\text{d}\bar{x}) = \tau_i(\bar{x})\) for \(i \in [n]\). Since \(B_{\kappa}\) is invertible, any such \(C\) of full rank has a form \(C = B_{\kappa}^{-1}(n + m), [n + 1, n + m])U\), where \(U\) is any invertible \(m \times m\) matrix.

Next, assume that \((C^*)^m\) is equipped with a Poisson structure given by

\[
\{d_i, d_j\}_V = v_{ij} d_i d_j,
\]

where \(V = (v_{ij})\) is a fixed skew-symmetric matrix.

**Proposition 2.2.** For any \(V\), there exists a Poisson structure \{\cdot, \cdot\}_V^A compatible with \n A such that the map \((C^*)^m \times \n A, \{\cdot, \cdot\}_V^A \times \{\cdot, \cdot\}_0^A) \rightarrow (\n A, \{\cdot, \cdot\}_V^A)\) extended from the action \((\text{d}\bar{x}) \mapsto \text{d}\bar{x}\) is Poisson. Moreover, every compatible Poisson bracket on \(\n A\) is a scalar multiple of \{\cdot, \cdot\}_V^A\) for some \(V\).

**Proof.** Let \(\Omega^x\) be the coefficient matrix of \{\cdot, \cdot\}_0^A\) in the basis \(\bar{x}\). It easy to see that in the product structure \{\cdot, \cdot\}_V \times \{\cdot, \cdot\}_0^A\) on \((C^*)^m \times \n A\),

\[
\{\text{d}\bar{x}, \text{d}\bar{x}\}_\tau = \left(\Omega^x + CVCT\right)_{\tau i j}(\text{d}\bar{x})_i (\text{d}\bar{x})_j.
\]

Thus, for the action \((\text{d}\bar{x}) \mapsto \text{d}\bar{x}\) to be Poisson, one must have \(\{x_i, x_j\}_V = (\Omega^x + CVCT)_{\tau i j} x_i x_j\) for \(i, j \in [n + m]\). Since \(\tau_i(\text{d}\bar{x}) = \tau_i(\bar{x})\) if \(i \in [n]\), and \(\tau_i(\text{d}\bar{x}) = \tau_i(\bar{x}) m_i(d)\) for some monomials \(m_i(d)\) in \(d\) for \(i \in [n + 1, n + m]\), we see that \(\{\tau_i, \tau_j\}_V = \{\tau_i, \tau_j\}_0^A\) for \(i \in [n]\), \(j \in [n + m]\). Since \{\cdot, \cdot\}_0^A\) is a compatible Poisson bracket, Theorem 2.4 yields that \{\cdot, \cdot\}_V^A\) is compatible as well.

Now, let \(\Omega^\tau\) be the coefficient matrix of \{\cdot, \cdot\}_0^A\) in the basis \(\tau\). Denote \(Z_0 = \Omega^\tau([n + 1, n + m], [n + 1, n + m])\). Consider \(\{\tau_i, \tau_j\}_V^A\) for \(i, j \in [n + 1, n + m]\): \(\{\tau_i, \tau_j\}_V^A = z_{ij} \tau_i \tau_j\). To compute \(Z = (z_{ij})_{n=m+1}^{n+m}\), note that the matrix that describes \{\cdot, \cdot\}_V^A\) in coordinates \(\tau\) is \(\Omega^\tau = B_{\kappa} (\Omega^x + CVCT) B_{\kappa}^T\), and thus

\[
Z = \Omega^\tau([n + 1, n + m], [n + 1, n + m]) = Z_0 + UVU^{-1}.
\]

It is clear that by varying \(V\), one can make \(Z\) to be equal to an arbitrary skew-symmetric \(m \times m\) matrix, and the result follows.

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3. Poisson-Lie groups and Sklyanin brackets

We need to recall some facts about Poisson-Lie groups (see, e.g., [ReST]).

Let \(\mathcal{G}\) be a Lie group equipped with a Poisson bracket \{\cdot, \cdot\}. \(\mathcal{G}\) is called a Poisson-Lie group if the multiplication map

\[
m : \mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto xy \in \mathcal{G}
\]
is Poisson. Perhaps, the most important class of Poisson-Lie groups is the one associated with classical R-matrices.

Let \( \mathfrak{g} \) be a Lie algebra of \( \mathcal{G} \). Assume that \( \mathfrak{g} \) is equipped with a nondegenerate invariant bilinear form \( (\ ,\ ) \). An element \( R \in \text{End}(\mathfrak{g}) \) is a classical R-matrix if it is a skew-symmetric operator that satisfies the modified classical Yang-Baxter equation (MCYBE)

\[
[R(\xi),R(\eta)] - R([R(\xi),\eta] + [\xi,R(\eta)]) = -[\xi,\eta].
\]

(3.1)

Given a classical R-matrix \( R \), \( \mathcal{G} \) can be endowed with a Poisson-Lie structure as follows. Let \( \nabla, \nabla' \) be the right and the left gradients for a function \( f \in C^\infty(\mathcal{G}) \):

\[
(\nabla f(x), \xi) = \frac{d}{dt} f(\exp(t\xi)x)|_{t=0}, \quad (\nabla' f(x), \xi) = \frac{d}{dt} f(x \exp(t\xi))|_{t=0}.
\]

(3.2)

Then the bracket given by

\[
\{f_1, f_2\}_R(x) = \frac{1}{2}(R(\nabla f_1), \nabla' f_2) - \frac{1}{2}(R(\nabla f_2), \nabla f_1)
\]

is a Poisson-Lie bracket on \( \mathcal{G} \) called the Sklyanin bracket.

We are interested in the case \( \mathcal{G} = SL_n \) and \( \mathfrak{g} = sl_n \) equipped with the trace-form

\[
(\xi, \eta) = \text{Tr}(\xi \eta).
\]

In this case, the right and left gradients are

\[
\nabla f(x) = x \, \text{grad} f(x), \quad \nabla' f(x) = \text{grad} f(x) \, x,
\]

where

\[
\text{grad} f(x) = \left( \frac{\partial f}{\partial x_{ij}} \right)_{i,j=1}^n,
\]

and the Sklyanin bracket becomes

\[
\{f_1, f_2\}_R(x) = \frac{1}{2}(R(\text{grad} f_1(x) \, x), \text{grad} f_2(x) \, x) - \frac{1}{2}(R(x \, \text{grad} f_1(x)), x \, \text{grad} f_2(x)).
\]

(3.4)

Every \( \xi \in \mathfrak{g} \) can be uniquely decomposed as

\[
\xi = \pi_-(\xi) + \pi_0(\xi) + \pi_+(\xi),
\]

where \( \pi_+ (\xi) \) and \( \pi_- (\xi) \) are strictly upper and lower triangular and \( \pi_0(\xi) \) is diagonal. The simplest classical R-matrix on \( sl_n \) is given by

\[
R_0(\xi) = \pi_+(\xi) - \pi_-(\xi) = (\text{sign}(j-i)\xi_{ij})_{i,j=1}^n.
\]

(3.5)

Substituting \( R = R_0 \) into (3.4), we find the bracket for a pair of matrix entries:

\[
\{x_{ij}, x_{i'j'}\}_{R_0} = \frac{1}{2} (\text{sign}(i'-i) + \text{sign}(j'-j)) x_{ij} x_{i'j'}.
\]

(3.6)

It is known (see [ReST]) that if \( R_0 \) is the standard R-matrix, \( S \) is any linear operator on the space of traceless diagonal matrices that is skew-symmetric with respect to the trace-form, and \( \pi_0 \) is the natural projection onto the subspace of diagonal matrices, then

\[
R_S = R_0 + S\pi_0
\]

(3.7)

satisfies MCYBE (3.1), and thus gives rise to a Sklyanin Poisson-Lie bracket. The operator \( S \) can be identified with an \( n \times n \) skew-symmetric matrix whose kernel contains the vector \((1, \ldots, 1)\) and thus is uniquely determined by its \((n-1) \times (n-1)\)
submatrix \((s_{ij})_{i,j=1}^{n-1}\), which we will also denote by \(S\). Slightly abusing notation, we denote the remaining elements of the above \(n \times n\) skew-symmetric matrix by

\[
s_{in} = -\sum_{j=1}^{n-1} s_{ij}, \quad s_{nj} = -\sum_{i=1}^{n-1} s_{ij}.
\]

The Sklyanin bracket \((3.3)\) that corresponds to \((3.7)\) can be written in terms of matrix entries as

\[
\{x_{ij}, x_{i'j'}\}_R = \{x_{ij}, x_{i'j'}\}_{R_0} + \frac{1}{2} (s_{ii'} - s_{jj'}) \cdot x_{ij} x_{i'j'}.
\]

Let \(\mathcal{H}\) denote the subgroup of diagonal matrices in \(SL_n\):

\[
\mathcal{H} = \{\text{diag}(d_1, \ldots, d_n) : d_1 \cdots d_n = 1\}.
\]

For any skew-symmetric matrix \(V = (v_{ij})_{i,j=1}^{n-1}\), define a Poisson bracket \(\{\cdot, \cdot\}_V^\mathcal{H}\) on \(\mathcal{H}\) by

\[
\{d_i, d_j\}_V^\mathcal{H} = v_{ij} d_i d_j;
\]

here \(v_{in}\) and \(v_{nj}\) have the same meaning as \(s_{in}\) and \(s_{nj}\) above.

In what follows, we denote the Poisson manifolds \((\mathcal{H}, \{\cdot, \cdot\}_V^\mathcal{H})\) and \((SL_n, \{\cdot, \cdot\}_{R_0})\) by \(\mathcal{H}(V)\) and \(SL_n^{\{\cdot, \cdot\}_R}\), respectively.

Next, for \(S\) defined as in \((3.7)\), consider the direct product of Poisson manifolds \(\mathcal{H}(1/2S) \times SL_n^{\{0\}} \times H(1-1/2S)\); the product structure we denote below simply by \(\{\cdot, \cdot\}\).

**Lemma 3.1.** The map \(\mathcal{H}(1/2S) \times SL_n^{\{0\}} \times H(1-1/2S) \to SL_n^{\{S\}}\) given by \((D_1, X, D_2) \mapsto D_1 X D_2\) is Poisson.

\[\text{Proof.}\] Denote \(D_1 X D_2\) by \(\tilde{X} = (\tilde{x}_{ij})_{i,j=1}^{n}\). Let \(D_k = \text{diag}(d_{kl})_{l=1}^{n}\) for \(k = 1, 2\). Then \(\{\tilde{x}_{ij}, \tilde{x}_{i'j'}\} = \{\tilde{x}_{ij}, \tilde{x}_{i'j'}\}_{R_0} + x_{ij} x_{i'j'} \cdot d_1 d_2 d_1 d_2\). The second term is equal to

\[
\frac{1}{2} (s_{ii'} - s_{jj'}) x_{ij} x_{i'j'} d_1 d_2 d_1 d_2 = \frac{1}{2} (s_{ii'} - s_{jj'}) \tilde{x}_{ij} \tilde{x}_{i'j'},
\]

and the claim follows by \((3.8)\). \(\square\)

4. **Grassmannians**

4.1. Let \(\mathcal{P}\) be a Lie subgroup of a Poisson-Lie group \(\mathcal{G}\). A Poisson structure on the homogeneous space \(\mathcal{P} \setminus \mathcal{G}\) is called **Poisson homogeneous** (with respect to the Poisson-Lie structure on \(\mathcal{G}\)) if the action map \(\mathcal{P} \setminus \mathcal{G} \times \mathcal{G} \to \mathcal{P} \setminus \mathcal{G}\) is Poisson. In particular, if \(\mathcal{P}\) is a parabolic subgroup of a simple Lie group \(\mathcal{G}\) equipped with the standard Poisson-Lie structure, then \(\mathcal{P} \setminus \mathcal{G}\) is a Poisson homogeneous space. We will be interested in the case when \(\mathcal{G} = SL_n\) equipped with the bracket \((3.3)\) and

\[
\mathcal{P} = \mathcal{P}_k = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : A \in GL_k, C \in GL_{n-k} \right\}.
\]

The resulting homogeneous space is the Grassmannian \(G_k(n)\) equipped with what we will call the **standard Poisson homogeneous structure** \(\{\cdot, \cdot\}_0^{Gr}\). We will recall an explicit expression of this Poisson structure on the open Schubert cell \(G_k^n = \{X \in G_k(n) : x_{\{j\}} \neq 0\}\). Here we use the same notation for an element of the Grassmannian and its matrix representative \(X\), and \(x_I\) denotes the Plücker coordinate that corresponds to a \(k\)-element subset \(I \subset [n]\). Elements of \(G_k^n\) can be represented by matrices of the form \([1_k Y]\), and the entries of the
$k \times (n-k)$ matrix $Y$ serve as coordinates on $G_k^0(n)$. In terms of matrix elements $y_{ij}$ of $Y$, the Poisson homogeneous bracket looks as follows \cite{GSV1}:

\begin{equation}
\{y_{ij}, y_{\alpha \beta}\}^G_{10} = \frac{\text{sign}(\alpha - i) - \text{sign}(\beta - j)}{2} y_{i\beta} y_{\alpha j}.
\end{equation}

We denote $G_k(n)$ equipped with the Poisson bracket \eqref{eq:Gk(n)} by $G_k(n)^{(0)}$.

\textbf{Proposition 4.1.} (i) For an arbitrary skew-symmetric operator $S$, there exists a Poisson bracket $\{\cdot, \cdot\}^S_G$ on $G_k(n)$, unique up to a scalar multiple, such that

the natural action $(X, D) \mapsto XD$ is a Poisson map from $G_k(n)^{(0)} \times H(\mathfrak{sl}_n)$ to $G_k(n)^{(S)} := \left( G_k(n), \{\cdot, \cdot\}^G_{10} \right)$.

(ii) The bracket $\{\cdot, \cdot\}^G_{10}$ is a Poisson homogeneous structure on $G_k(n)$ with respect to the bracket $\{\cdot, \cdot\}_{R_S}$ on $SL_n$ defined by \eqref{eq:R_S}.

\textbf{Proof.} (i) Let $X = [1_k Y] \in G_k^0(n)$, $D = \text{diag}(d_1, \ldots, d_n) \in H$ and let $[1_k \tilde{Y}]$ be the matrix that represents the element $XD \in G_k^0(n)$. Then $\tilde{y}_{ij} = y_{ij} d_{j+k}/d_i$, and the Poisson bracket of any two Plücker coordinates $\tilde{y}_{ij}$ and $\tilde{y}_{\alpha \beta}$ in the product structure $\{\cdot, \cdot\}_{0}^G \times \{\cdot, \cdot\}_{H}^S$ is equal to

$$\frac{\text{sign}(\alpha - i) - \text{sign}(\beta - j)}{2} \tilde{y}_{i\beta} \tilde{y}_{\alpha j} + \frac{s_{i, \beta + k} + s_{j+k, \alpha} - s_{i, \alpha} - s_{j+k, \beta + k}}{2} \tilde{y}_{ij} \tilde{y}_{\alpha \beta}.$$

Thus, the bracket defined on $G_k^0(n)$ by the formula

$$\{y_{ij}, y_{\alpha \beta}\}_{0}^G = \{y_{ij}, y_{\alpha \beta}\}_{10}^G + \frac{s_{i, \beta + k} + s_{j+k, \alpha} - s_{i, \alpha} - s_{j+k, \beta + k}}{2} y_{ij} y_{\alpha \beta}$$

is the unique, up to a scalar multiple, Poisson bracket that makes the map $(X, D) \mapsto XD$ Poisson. Since $G_k^0(n)$ is an open dense subset in $G_k(n)$ the claim follows.

(ii) To see that $\{\cdot, \cdot\}_{S}^G$ is Poisson homogeneous with respect to $\{\cdot, \cdot\}_{R_S}$, we need to check that the natural action of $SL_n$ on $G_k(n)$ defines a Poisson map from $G_k(n)^{(S)} \times SL_n^{(S)}$ to $G_k(n)^{(S)}$. Instead of a straightforward calculation, we can use the fact that this is true for $S = 0$ and Lemma 3.1. Indeed, it is easy to check that both Poisson maps $G_k(n)^{(0)} \times H(\mathfrak{sl}_n) \to G_k(n)^{(S)}$ given by $(X, D_1) \mapsto XD_1$ and $H(\mathfrak{sl}_n) \times SL_n^{(0)} \to SL_n^{(S)}$ given by $(D_1, X, D_2) \mapsto D_1 XD_2$ are surjective. Therefore, we can replace the map $G_k(n)^{(S)} \times SL_n^{(S)} \to G_k(n)^{(S)}$ by the map

$$G_k(n)^{(0)} \times H(\mathfrak{sl}_n) \times H(\mathfrak{sl}_n) \times SL_n^{(0)} \times H(\mathfrak{sl}_n) \to G_k(n)^{(S)}$$

given by $(X, D_1, D_2, A, D_3) \mapsto XD_1 D_2 A D_3$. It is easy to check that $(D_1, D_2) \mapsto D_1 D_2$ is a Poisson map from $H(\mathfrak{sl}_n) \times H(\mathfrak{sl}_n)$ onto $H(\mathfrak{sl}_n)$, which, in turn, is a Poisson-Lie subgroup of $SL_n^{(0)}$. We thus arrive to the Poisson map $G_k(n)^{(0)} \times SL_n^{(0)} \times H(\mathfrak{sl}_n) \to G_k(n)^{(S)}$ given by $(X, \tilde{A}, D_3) \mapsto X \tilde{A} D_3$ with $\tilde{A} = D_1 D_2 A$. The standard Poisson homogeneous structure ensures that the map $G_k(n)^{(0)} \times SL_n^{(0)} \to G_k(n)^{(0)}$ is Poisson, and it remains to use part (i) of Proposition 4.1 to complete the proof. \hfill \Box

4.2. Now, we recall the construction of the cluster algebra $\mathcal{A}_{G_k^0(n)}$ associated with the open cell $G_k^0(n)$ as described in \cite{GSV1, GSV2}. Denote $m = n - k$. For every
\[ i \in [k], j \in [m] \text{ put} \]
\[ I_{ij} = \begin{cases} [i + 1, k] \cup [j + k, i + j + k - 1], & \text{if } i \leq m - j + 1 \\ ([k] \setminus [i + j - m, i]) \cup [j + k, n], & \text{if } i > m - j + 1. \end{cases} \]

Denote the Plücker coordinate \( x_{I_{ij}} \) by \( x(i, j) \).

The initial cluster of \( \mathcal{A}_\Gamma \) is given by
\[ x = x(k, n) = \left\{ \frac{x(i, j)}{x[k]} : i \in [k], j \in [m] \right\}. \]

Stable variables are \( \frac{x(1, 1)}{x[k]}, \ldots, \frac{x(k, 1)}{x[k]}, \frac{x(k, 2)}{x[k]}, \ldots, \frac{x(k, m)}{x[k]} \). The entries of \( \tilde{B} \) that correspond to \( x \) are all 0 or \( \pm 1 \)s. Thus it is convenient to describe \( \tilde{B} \) by a directed graph \( \Gamma(\tilde{B}) \).

\[ \tau_{ij} = \frac{x(i + 1, j + 1)x(i, j + 1)x(i + 1, j)}{x(i + 1, j + 1)x(i, j + 1)x(i + 1, j)}, \quad i \in [k - 1], j \in [2, m], \]

where \( x(0, j) = x(i, m + 1) = 1 \).

**Lemma 4.2.** Functions \( f, \alpha \) are invariant under the natural action of \( \mathcal{H} \) on \( \Gamma_k(n) \).

**Proof.** Let \( X \in \Gamma_k(n), D = \text{diag}(d_1, \ldots, d_n) \in \mathcal{H} \) and \( \tilde{X} = XD \). For any subset \( I = \{i_1, \ldots, i_l\} \subset [n] \) denote \( d^I = \prod_{i=1}^l d_{i_i} \). Then, using (4.2), we obtain
\[ \tilde{x}(i, j) = x(i, j)d^{i,j} = \begin{cases} x(i, j) \frac{d[k]d[i+j+k-1]}{d[i]d[k+i+j-k-1]}, & \text{if } i \leq m - j + 1, \\ x(i, j) \frac{d[k]d[i+j-m-1]}{d[i]d[k+i+j-k-1]}, & \text{if } i > m - j + 1, \end{cases} \]
and the equality \( \tau_{ij}(\tilde{X}) = \tau_{ij}(X) \) follows from (4.1) by trivial cancellation. \( \square \)
Now we are ready to prove

**Theorem 4.3.** A Poisson structure $\{\cdot, \cdot\}$ on $G_k(n)$ is compatible with $A_{G_k^0(n)}$ if and only if a scalar multiple of $\{\cdot, \cdot\}$ defines a Poisson homogeneous structure with respect to $\{\cdot, \cdot\}_{RS}$ for some skew-symmetric operator $S$.

**Proof.** It follows from Theorem 5.4 in [GSV2] that $\{\cdot, \cdot\}^{Gr}_{G_k^0(n)}$ is compatible with $A_{G_k^0(n)}$. The number of stable variables for $A_{G_k^0(n)}$ is $n - 1$. Since $\mathcal{H}$ is isomorphic to $(\mathbb{C}^*)^{n-1}$, Lemma 4.2 guarantees that the map $(X, D) \mapsto XD$ translates into an action of $(\mathbb{C}^*)^{n-1}$ on $A_{G_k^0(n)}$ as described in Section 2. Then Proposition 2.2 and Proposition 4.1 imply that every compatible Poisson bracket on $A_{G_k^0(n)}$ is a scalar multiple of $\{\cdot, \cdot\}^{Gr}_{S}$ for some skew-symmetric operator $S$ on the space of traceless diagonal $n \times n$ matrices. Since $\{\cdot, \cdot\}^{Gr}_{S}$ is a unique Poisson homogeneous with respect to $\{\cdot, \cdot\}_{RS}$ (see, e.g. [D]), the claim follows. \qed

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**References**

[BD] A. A. Belavin, A. A., V. G. Drinfeld, *Triangle equations and simple Lie algebras*, Soviet Sci. Rev. Sect. C Math. Phys. Rev. 4 (1984), 93–165.

[BFZ] A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*. Duke Math. J. 126 (2005), 1–52.

[D] V. G. Drinfeld, *On Poisson homogeneous spaces of Poisson-Lie groups*, Theoret. and Math. Phys. 95 (1993), no. 2, 524–525.

[FZ1] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. 12 (1999), 335–380.

[FZ2] S. Fomin and A. Zelevinsky, *Total Positivity: tests and parametrizations*, Math. Intelligencer. 22 (2000), 23–33.

[FZ3] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), 497–529.

[GSV1] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Poisson geometry*. Mosc. Math. J. 3 (2003), 899–934.

[GSV2] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Poisson geometry of directed networks in a disk*, Selecta Mathematica 15, no. 1, 61-103.

[GSV3] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Poisson geometry*, book in preparation.

[P] A. Postnikov, *Total positivity, Grassmannians and networks*, arXiv: math/0609704.

[ReST] A. Reyman and M. Semenov-Tian-Shansky *Group-theoretical methods in the theory of finite-dimensional integrable systems*. Encyclopaedia of Mathematical Sciences, vol.16, Springer–Verlag, Berlin, 1994 pp. 116–225.

[S] J. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. 92 (2006), 345–380.
