A Critical Dimension in One-dimensional Large-$N$ Reduced Models

Takeshi Morita$^{a,b,*}$, Hiroki Yoshida$^a$

$^a$ Department of Physics, Shizuoka University
836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

$^b$ Graduate School of Science and Technology, Shizuoka University
836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

Abstract

We investigate critical phenomena of the Yang-Mills (YM) type one-dimensional matrix model that is a large-$N$ reduction (or dimensional reduction) of the $D+1$ dimensional $U(N)$ pure YM theory (bosonic BFSS model). This model shows a large-$N$ phase transition at finite temperature, which is analogous to the confinement/deconfinement transition of the original YM theory. We study the matrix model at a three-loop calculation via the “principle of minimum sensitivity” and find that there is a critical dimension $D = 35.5$. At $D \leq 35$, the transition is of first order, while it is of second order at $D \geq 36$. Besides, we evaluate several observables in our method, and they nicely reproduce the existing Monte Carlo results. Through the gauge/gravity correspondence, this transition is expected to be related to a Gregory-Laflamme transition in gravity, and we argue that the existence of the critical dimension is consistent with it.

$^*${E-mail address: morita.takeshi(at)shizuoka.ac.jp}

$^†${E-mail address: yoshida.hiroki.16(at)shizuoka.ac.jp}
Table 1: The critical dimensions of various models. The systems show the first order phase transitions below the critical dimensions and they become of second order above them.

| Model                              | Critical Dimension |
|------------------------------------|--------------------|
| GL (fixed mass)                    | 12.5               |
| GL (fixed temperature)             | 11.5               |
| RP                                 | 11.5               |
| Matrix model (3-loop)              | 35.5               |

1 Introduction

Critical phenomena in physics sometimes show interesting dependences on the numbers of the spatial dimensions. One remarkable example is the Gregory-Laflamme (GL) transition in the \( D+1 \) dimensional gravity with a compact \( S^1 \) circle \[1\]. (See a review \[2\].) By changing the size of the \( S^1 \) from small to large, the stable configuration for a given energy changes from a uniform black string (UBS) to a localized black hole (LBH), and this transition is called the GL transition. A non-uniform black string (NUBS) may appear as an intermediate state in this transition. Surprisingly, the order of this phase transition does depend on \( D \), and it is of first order at \( D \leq 12 \), while is of second order at \( D \geq 13 \) \[3\]. Hence, \( D = 12.5 \) can be regarded as a critical dimension of this transition. Curiously, if we fix the temperature instead of the energy, the critical dimension changes to \( D = 11.5 \) \[4\]. See Table 1.

A related critical dimension appears in the Rayleigh-Plateau (RP) instabilities in liquid too. If we consider a space time \( \mathbb{R}^{D-1,1} \times S^1 \) and set a liquid winding the \( S^1 \) with the same configuration as the UBS. Suppose that the volume of the liquid is fixed and the radius of the \( S^1 \) is increased. Then, above a critical radius, this configuration becomes unstable due to the RP instability, and it tends to be non-uniform. The order of this transition depends on \( D \) similar to the GL transition, and it turned out that the critical dimension is \( D = 11.5 \) \[5, 6\]. The connection between the GL and RP instabilities was also argued in \[5\].

According to the gauge/gravity correspondence \[7, 8\], the GL transition is expected to be qualitatively related to the confinement/deconfinement (CD) transition in the \( D + 1 \) dimensional Yang-Mills (YM) type matrix quantum mechanics, whose action at finite temperature is given by \[9, 10, 11, 12, 13, 14, 15\],

\[
S = \int_0^\beta dt \text{Tr} \left\{ \sum_{I=1}^D \frac{1}{2} (D_t X^I)^2 - \sum_{I,J=1}^D \frac{g^2}{4} [X^I, X^J]^2 \right\}. \tag{1.1}
\]

This model is a large-\( N \) reduction (or dimensional reduction) of the \( D + 1 \) dimensional \( U(N) \)
pure Yang-Mills (YM) theory to one dimension [16]. Here $X^I$ ($I = 1, \cdots, D$) are the $N \times N$ hermitian matrices that are the dimensional reductions of the spatial components of the original $D + 1$ dimensional gauge fields. $D_t := \partial_t - i [A_t, ]$ is the covariant derivative and $A_t$ is the gauge field. $g$ is the coupling constant, and we take the 't Hooft limit $N \to \infty$ and $g \to 0$ with a fixed 't Hooft coupling $\lambda := g^2 N$. Note that this model appears as low energy effective theories of D-branes and membranes in string theories in various situations, and is important in its own right [9, 17, 18, 19, 20, 21, 22].

This model shows a large-$N$ phase transition, which is an analogue of the CD transition of the original YM theory [9, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The order parameter of this transition is the Polyakov loop operators,

$$u_n := \frac{1}{N} \text{Tr} \exp \left( i n \int_0^\beta d t A_t \right).$$

(1.2)

If $\langle u_n \rangle = 0$, ($\forall n$), it indicates a confinement, and, $\langle u_n \rangle \neq 0$, ($\exists n$) shows a deconfinement.

The connection between the CD transition and the GL transition can be intuitively understood as follows. The diagonal components of $X^I$ can be regarded as the positions of $N$ particles (or D-branes). If we take the static diagonal gauge $(A_t)_{ij} = \alpha_i \delta_{ij}$ ($i, j = 1, \cdots, N$), $\alpha_i$ also describe the positions of the particles. (Here the configuration space of the gauge field is regarded as a real space.) Particularly, $\alpha_i$ show the periodicity $\alpha_i = \alpha_i + 2\pi/\beta$, and this space is actually an $S^1$. At large-$N$, these particles may behave as a static fluid in the $D + 1$ dimension, and their distribution would be uniform, non-uniform or localized along the $S^1$ as schematically shown in FIG. 1. Now the connection to the GL transition in the

\footnote{Generally, it is non-trivial to compute such a D-brane distribution in matrix models [22, 27, 34].}
gravity would be clear. These configurations would correspond to an UBS, NUBS and LBH, respectively. Note that the temporal component $\alpha_i$ of the gauge theory corresponds to the spatial $S^1$ direction in the gravity. As we have mentioned, the UBS is stable when the size of the $S^1$ is small. Correspondingly, the uniform distribution in FIG 1 is stable at a small $2\pi/\beta$, which means a low temperature. We can easily see that $\langle u_n \rangle = 0$ in the uniform distribution, and this is consistent with the confinement at low temperatures.

Since the critical dimensions appear in the GL and RP transitions, the existence of the critical dimension in the CD transition of the matrix model is expected. Indeed, several evidences for this conjecture have been found [29]. For small $D$, Monte Carlo (MC) simulations show that the order of the CD transition up to $D = 25$ would be of first order [29, 32]. On the other hand, at large-$D$, we can analyze the model analytically through the $1/D$ expansion, and find the second order CD transition [28]. Hence, a critical dimension would exist in the matrix model too. In this letter, we analyze the matrix model by using so called “principle of minimum sensitivity” [35], and we will see that the critical dimension is $D = 35.5$ at a three-loop calculation.

2 Our Analysis

2.1 Analysis via the Principle of Minimum Sensitivity

To investigate the phase structure of the model (1.1), we employ the principle of minimum sensitivity. Such an analysis was first done by Kabat and Lifschytz [23], but we use a different approach in order to study the details of the phase transition.

We deform the model (1.1) as

$$S = S_0 + \kappa S_{\text{int}},$$

$S_0 = \int_0^\beta dt \text{Tr} \left\{ \sum_{i=1}^D \frac{1}{2} \left( D_t X^I \right)^2 + \frac{M^2}{2} \left( X^I \right)^2 \right\},$

$S_{\text{int}} = \int_0^\beta dt \text{Tr} \left\{ \sum_{i=1}^D -\frac{M^2}{2} \left( X^I \right)^2 - \sum_{i,j=1}^D \frac{g^2}{4} [X^I, X^J]^2 \right\}.$

More precisely, the temporal direction in the gravity is also periodic in order to make the system to be at a finite temperature. This temporal direction corresponds to the T-dual of one of $X^I$, say $X^1$, in the matrix model. If the temperature in the gravity is sufficiently high, the size of the $S^1$ after the T-dual is large, and we can ignore the periodicity of $X^1$ in the matrix model [9].

There are several studies, which apply the principle of minimum sensitivity to YM type matrix models [36, 37, 38, 39, 40].
Here we have introduced the deformation parameter \( \kappa \) and \( M \). If we take \( \kappa = 1 \), this model is equivalent to the original model (1.1).

We integrate out \( X^I \) through the perturbative calculations with respect to \( \kappa \), and derive the effective action of the Polyakov loop \( \{u_n\} \). The relevant terms at low temperatures, where all \( u_n \) are small, is given by [20, 41]

\[
S_{\text{eff}}(\{u_n\}, M) = N^2 \left( \beta f_0 + f_1|u_1|^2 + f_2|u_1|^4 + f_3|u_2|^2 + f_4(u_2u_{-1}^2 + u_{-2}u_1^2) + \cdots \right). \tag{2.2}
\]

Here \( f_0 \) is a function of \( M \) while \( f_i \) (\( i = 1, 2, 3, 4 \)) are functions of \( M \) and \( T := 1/\beta \). The explicit expressions of \( f_0 \) and \( f_i \) at three-loop order are shown in (A.14) - (A.18). (If we are interested in the two-loop results \footnote{We need at least two-loop to apply the principle of minimum sensitivity.}, we remove the terms proportional to \( \kappa^2 \) in these equations.)

At this stage, we take \( \kappa = 1 \). Although the initial model (2.1) at \( \kappa = 1 \) is independent of the deformation parameter \( M \), the obtained effective action does depend on \( M \). Here, we fix \( M \) so that the \( M \) dependence of the effective action becomes a minimum. This prescription is so called “the principle of minimum sensitivity” [35]. Although the validity of such a prescription is generally not ensured, it works very well for many models. We will compare our results with the existing studies in order to test our analysis.

At low temperatures, \( f_1, f_2, f_3 > 0 \) from (A.15), (A.16) and (A.17). Then, the stable configuration in the effective action (2.2) is \( u_1 = u_2 = 0 \), and we can approximate \( S_{\text{eff}} = N^2 \beta f_0 \). Thus, \( M \) at low temperatures is fixed so that the \( M \) dependence of \( f_0 \) is minimized, hence

\[
|\partial_M f_0(M = M_0)| = \min |\partial_M f_0(M)|, \tag{2.3}
\]

where \( M_0 \) denotes the value of \( M \) that minimizes \( |\partial_M f_0| \). In the two-loop effective action, \( f_0 \) has a single extremum \( \partial_M f_0 = 0 \) from (A.14), and this point gives \( M_0 \) as

\[
M_0 = \lambda^{1/3}(D - 1)^{1/3}; \quad (2\text{-loop}). \tag{2.4}
\]

In the three-loop effective action, \( f_0 \) does not have any extremum. However, it has an inflection point \( \partial^2_M f_0 = 0 \), which minimizes (2.3), and we obtain

\[
M_0 = \frac{15^{1/3}\lambda^{1/3}}{2}(D - 3/4)^{1/3}, \quad (3\text{-loop}). \tag{2.5}
\]
2.2 The Confinement/Deconfinement phase transition

As temperature increasing, \( f_1(M_0, T) \) becomes negative, and \( u_1 \) and \( u_2 \) may obtain non-zero vevs, indicating a deconfinement. This is the CD transition in this model. Near the critical temperature, \( u_1 \) and \( u_2 \) would be small and we can perturbatively treat them in the effective action \((2.2)\). Correspondingly, \( M \) can be expanded as

\[
M = M_0 + M_1 |u_1|^2 + M_2 |u_1|^4 + M_3 |u_2|^2 + M_4 (u_2 u_{-1} + u_{-2} u_1^2) + \cdots .
\] (2.6)

Here, in the two-loop theory, \( M_0 \) is given by \((2.4)\) and \( M_i (i = 1, \cdots , 4) \) are fixed through the condition \( \partial_M S_{\text{eff}} = 0 \) in \((2.2)\). In the three-loop theory, \( M_0 \) is given by \((2.5)\) and the condition \( \partial_{M_i}^2 S_{\text{eff}} = 0 \) determines \( M_i \). (See the details in Appendix A.2)

Then, by substituting \((2.6)\) into the effective action \((2.2)\) and using the small \( \{u_n\} \) expansion, we obtain

\[
S_{\text{eff}}(\{u_n\}) = N^2 \left( \beta f_0 + \tilde{f}_1 |u_1|^2 + \tilde{f}_2 |u_1|^4 + \tilde{f}_3 |u_2|^2 + \tilde{f}_4 (u_2 u_{-1} + u_{-2} u_1^2) + \cdots \right).
\] (2.7)
Here
\[
\bar{f}_i = f_i + (\partial_M f_0) M_i, \quad (i = 1, 3, 4),
\]
\[
\bar{f}_2 = f_2 + (\partial_M f_0) M_2 + \frac{1}{2} \left( \partial_M^2 f_0 \right) M_1^2 + (\partial_M f_1) M_1, \quad (2.8)
\]
where \( f_i \) are evaluated at \( M = M_0 \). Finally, by integrating out \( u_2 \), we reach a Landau-Ginzburg type effective action for \( u_1 \),
\[
S_{\text{eff}}(u_1) = N^2 \left( \beta f_0 + a(T)|u_1|^2 + b(T)|u_1|^4 + \cdots \right),
\]
\[
a(T) = \bar{f}_1, \quad b(T) = \bar{f}_2 - \frac{\bar{f}_1^2}{\bar{f}_3}. \quad (2.9)
\]

Now we can easily see the phase structure \([20, 24, 41]\). If \( a > 0 \), \( u_1 = 0 \) may be stable and the system is confined. If \( a < 0 \), \( u_1 = 0 \) is unstable and \( u_1 \) has to develop a non-zero vev, and it is deconfinement. Thus, we can derive the critical temperature \( T_c \) by solving \( a(T_c) = 0 \).

The results are shown in FIG. 2.

Near \( T = T_c \), \( a(T) \) can be expanded as \( a(T) = -c(T - T_c) + \cdots \) \((c := \partial a/\partial T > 0)\) and we obtain the classical solution of \( u_1 \) in (2.9) as
\[
\begin{align*}
\left. u_1 \right|_{T_c} &\simeq \sqrt{\frac{c(T - T_c)}{2b}}. \quad (2.10)
\end{align*}
\]

Therefore, if \( b(T_c) \) is positive, it indicates a non-trivial solution in \( T \geq T_c \), which implies a continuous second order phase transition. If \( b(T_c) \) is negative, an unstable solution exists in \( T \leq T_c \), and a first order phase transition occurs at \( T_0 \), which should be slightly below \( T_c \). See FIG. 3.

### 2.3 Critical dimension

We plot \( b(T_c) \) with respect to \( D \) in FIG. 4. At two-loop order, \( b \) is always negative and it predicts the first order phase transition. At three-loop order, \( b \) becomes positive at \( D = 36 \), and it changes to the second order one. Thus, the critical dimension of this model is \( D = 35.5 \) at three-loop.

### 3 Discussions

We have shown that the critical dimension of the matrix model (1.1) is 35.5 at three-loop. The existence of a critical dimension has been predicted through the MC \([29]\) and the \( 1/D \)
Figure 3: Schematic plots of the free energy vs. temperature. The rigid lines depict stable and meta-stable phases, while the dashed lines depict unstable phases. The $u_1 = 0$ phase lines are horizontal and do not depend on $T$ due to the large-$N$ volume independence. In the first order phase transition case (the left panel), the unstable phase with $u_1 \neq 0$ merges to the $u_1 = 0$ branch at $T = T_c$. Thus, the $u_1 \neq 0$ solution (2.10) near $T_c$ exists in $T \leq T_c$. Note that the phase transition occurs not at $T_c$ but at $T_0$ shown in the figure. In the second order phase transition case (the right panel), the stable $u_1 \neq 0$ solution (2.10) appears in $T \geq T_c$. Therefore, through (2.10), the signature of $b$ at $T = T_c$ determines the order of the phase transition.

Figure 4: The value of $b$ at the critical temperature. The negative and positive $b$ indicate the first and second order phase transitions, respectively. At three-loop, $b$ becomes positive at $D = 36$, and hence the critical dimension is 35.5. (See the right panel.) The $1/D$ result is from (4.29) in [28]. The MC simulations show first order transitions at least up to $D = 25$ [29, 32].
expansion \cite{28}, and our result is consistent with them. Besides, our result establishes the strong similarity between the GL, RP and the CD in the matrix model \cite{11}. This similarity may arise because the matrix model may describe a kind of fluid as depicted in FIG. 1. (The obtained critical dimension is different from the gravity, but it would be not a problem because we cannot expect any quantitative agreement in this correspondence \cite{9, 12}.)

However, our analysis relies on the perturbative calculation and the principle of the minimum sensitivity, and $D = 35.5$ is not conclusive. We need the higher order loop calculations to ensure it. (At large-$D$, these corrections may make our results approach to those of the $1/D$ expansion \cite{28}.) Also, there are several varieties of the principle of the minimum sensitivity \cite{40}, and we need to check whether our results depend on these schemes.

Another remaining problem is understanding the properties of the first order phase transition at $T_0$ shown in FIG. 3 in $D \leq 35$. Above $T_0$, the stable configuration would be non-uniform distribution or localized one depending on $D$. If the stable configuration is a non-uniform distribution, another phase transition to a localized distribution must occur at a higher temperature. Indeed, these transitions have been found in the GL and RP transitions \cite{2, 6, 43, 44}.

In order to investigate this issue, we need to evaluate the effective action at finite $\{u_n\}$, and thus we cannot use the expansion (2.6). Besides, we need to calculate higher order couplings of the Polyakov loops such as $|u_1|^6$ in the effective action (2.2). We leave this problem for future work.

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\footnote{In the second order phase transition case ($D \geq 36$), the stable configuration just above $T_c$ is the non-uniform distribution \cite{20, 24, 28, 31}.}
A The details of the analysis

A.1 The derivation of the effective action (2.2)

We compute the effective action of the Polyakov loop \( \{u_n\} \) by integrating out \( X^I \) through the three-loop perturbative calculation in (2.1) with respect to \( \kappa \), and will obtain the expansion,

\[
S_{\text{eff}}(\{u_n\}, M) = \sum_{m=1}^{3} \kappa^{m-1} S_{m\text{-loop}}.
\]

The analysis mainly follows that of the massive BFSS model [20]. In order to compute this expansion, we use the propagator of \( X^I \) in the static diagonal gauge \( (A_t)_{ij} = \alpha_i \delta_{ij} \) [28],

\[
\langle X^I_{ij}(t) X^J_{kl}(0) \rangle = \delta_{ik} \delta_{jl} \frac{1}{2M} e^{i(\alpha_i - \alpha_j)||t||} \left[ e^{-M||t||} \sum_{n=0}^{\infty} x^n u^n_i u^n_j + e^{M||t||} \sum_{n=1}^{\infty} x^n u^n_{-n} u^n_j \right],
\]

where \( x = e^{-\beta M} \) and \( u^i_n = e^{i\beta \alpha_i} \) which satisfies \( \sum_{i=1}^{N} u^i_n = Nu_n \) from (1.2). \( ||t|| \) denotes \( ||t + n\beta|| = t \) for \( 0 \leq t < \beta \).

Through the one-loop integral, we obtain

\[
S_{1\text{-loop}}/N^2 = \frac{D\beta M}{2} + \sum_{n=1}^{\infty} \frac{1 - D x^n}{n} |u_n|^2.
\]

At two-loop, we obtain

\[
S_{2\text{-loop}} = \left\langle \int_0^{\beta} dt \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 - \frac{M^2}{2} (X^I)^2 \right) \right\rangle,
\]

where

\[
\left\langle \int_0^{\beta} dt \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 \right) \right\rangle = \frac{\beta N^2 \lambda}{8M^2} D(D-1) + \frac{\beta N^2 \lambda}{4M^2} D(D-1) \sum_{n=1}^{\infty} (x^{2n} + 2x^n)|u_n|^2
\]

\[
+ \frac{\beta N^2 \lambda}{8M^2} D(D-1)(x^2 + 2x^3)(u_1^2 u_{-2} + u_{-1}^2 u_2) + \cdots,
\]

\[
\left\langle \int_0^{\beta} dt \text{Tr} \left( -\frac{M^2}{2} (X^I)^2 \right) \right\rangle = -M^2 \frac{\partial}{\partial (M^2)} S_{1\text{-loop}} = -\frac{DN^2 \beta M}{4} - \frac{DN^2 \beta M}{2} \sum_{n=1}^{\infty} x^n |u_n|^2.
\]

Here (A.5) has been computed via the planar diagram depicted in FIG. 5, and \( \cdots \) denotes the irrelevant terms at low temperatures. On the other hand, (A.6) can be generated from the one-loop result (A.3).
Figure 5: Planar diagrams to compute the effective action. At three-loop order, the diagrams A, B and C correspond to $S_{3\text{-loop}}^A$, $S_{3\text{-loop}}^B$ and $S_{3\text{-loop}}^C$ in (A.7), respectively.

In order to compute the three-loop corrections, we need to evaluate

$$-rac{1}{2} \left< \int_0^\beta dt \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 \right) \int_0^\beta dt' \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 \right) \right>_{\text{connected}} = S_{3\text{-loop}}^A + S_{3\text{-loop}}^B + S_{3\text{-loop}}^C.$$  \hspace{1cm} (A.7)

Here the last three terms are from the three diagrams depicted in FIG. 5 and we obtain

$$S_{3\text{-loop}}^A/N^2 = -\beta \frac{3 \lambda^2}{128 M^5} D(D-1)$$

$$- \frac{3 \beta \lambda^2}{32 M^5} D(D-1)(2\beta M x^2 - x^3 + x^2 + 3x) |u_1|^2$$

$$- \frac{3 \beta \lambda^2}{64 M^5} D(D-1)(2\beta M x^2 - 2x^4 + 5x^2) |u_1|^4$$

$$+ \frac{3 \beta \lambda^2}{32 M^5} D(D-1) \left( 4\beta M x^4 - x^6 + x^4 + 3x^2 \right) |u_2|^2$$

$$- \frac{3 \beta \lambda^2}{32 M^5} D(D-1) \left( 4\beta M x^3 - x^5 - x^4 + 3x^3 + 2x^2 \right) (u_2 u_{-1} + u_{-2} u_1^2)$$

$$+ \cdots, \hspace{1cm} (A.8)$$

$$S_{3\text{-loop}}^B/N^2 = -\beta \frac{\lambda^2}{32 M^5} D(D-1)^2$$

$$- \frac{\beta \lambda^2}{16 M^5} D(D-1)^2 \left( \beta M (2x^2 + x) + 3x^2 + 3x \right) |u_1|^2$$

$$- \frac{3 \beta \lambda^2}{16 M^5} D(D-1)^2 (\beta M x^3 + x^3) |u_1|^4$$

$$- \frac{\beta \lambda^2}{16 M^5} D(D-1)^2 \left( 2\beta M \left( 2x^4 + x^2 \right) + 3x^4 + 3x^2 \right) |u_2|^2$$

$$- \frac{\beta \lambda^2}{32 M^5} D(D-1)^2 \left( 2\beta M \left( 2x^4 + 3x^3 + x^2 \right) + 3x^4 + 6x^3 + 3x^2 \right) (u_2 u_{-1} + u_{-2} u_1^2)$$

$$+ \cdots, \hspace{1cm} (A.9)$$
\[ S_{3\text{-loop}}^{C}/N^2 = -\beta \frac{\lambda^2}{16M^5} D(D - 1)^2 \]
\[ \quad - \frac{\beta \lambda^2}{8M^5} D(D - 1)^2 (\beta M(x + 1)^2 + x^3 + 2x^2 + 3x) |u_1|^2 \]
\[ \quad - \frac{\beta \lambda^2}{8M^5} D(D - 1)^2 (2\beta M x^4 + x^4 + 2x^2) |u_1|^4 \]
\[ \quad - \frac{\beta \lambda^2}{8M^5} D(D - 1)^2 (2\beta M (x^6 + 2x^4 + x^2) + x^6 + 2x^4 + 3x^2) |u_2|^2 \]
\[ \quad - \frac{\beta \lambda^2}{8M^5} D(D - 1)^2 (\beta M (2x^5 + x^3 + 4x^3 + x^2) + x^5 + 4x^3 + 3x^2) (u_2 u_{-1} + u_{-2} u_1^2) + \cdots. \] (A.10)

In addition, we need to compute
\[ -\left\langle \int_0^\beta dt \text{Tr} \left( -\frac{M^2}{2} (X^I)^2 \right) \int_0^\beta dt' \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 \right) \right\rangle_{\text{connected}} \]
\[ = -M^2 \frac{\partial}{\partial M^2} \left\langle \int_0^\beta dt \text{Tr} \left( -\frac{g^2}{4} [X^I, X^J]^2 \right) \right\rangle \]
\[ = \frac{\beta N^2 \lambda}{8M^2} D(D - 1) \]
\[ + \frac{\beta N^2 \lambda}{4M^2} D(D - 1) \sum_{n=1}^\infty ((n \beta M + 1)x^{2n} + (2 + n \beta M)x^n) |u_n|^2 \]
\[ + \frac{\beta N^2 \lambda}{8M^2} D(D - 1)((\beta M + 1)x^2 + (3\beta M + 2)x^3)(u_1^2 u_{-2} + u_{-1}^2 u_2) + \cdots, \quad (A.11) \]

and
\[ -\frac{1}{2} \left\langle \int_0^\beta dt \text{Tr} \left( -\frac{M^2}{2} (X^I)^2 \right) \int_0^\beta dt' \text{Tr} \left( -\frac{M^2}{2} (X^I)^2 \right) \right\rangle_{\text{connected}} \]
\[ = \frac{1}{2} M^4 \frac{\partial^2}{\partial (M^2)^2} S_{1\text{-loop}} \]
\[ = -\beta \frac{DN^2 M}{16} - \frac{DN^2}{8} \sum_{n=1}^\infty (n (\beta M)^2 + \beta M) x^n |u_n|^2 \] (A.12)

The three-loop correction \( S_{3\text{-loop}} \) in (A.1) is given as the sum of (A.7), (A.11) and (A.12).

By using these results, we can read off the effective action (2.2) at three-loop order,
\[ S_{\text{eff}}(\{u_n\}, M) = N^2 \left( \beta f_0 + f_1 |u_1|^2 + f_2 |u_1|^4 + f_3 |u_2|^2 + f_4(u_2 u_{-1} + u_{-2} u_1^2) + \cdots \right). \] (A.13)

Here
\[ f_0 = \frac{D M}{2} + \kappa \left( \frac{\lambda}{8M^2} D(D - 1) - \frac{D M}{4} \right) \]
\[ + \kappa^2 \left[ -\frac{3\lambda^2}{128M^5} D(D - 1) - \frac{3\lambda^2}{32M^5} D(D - 1)^2 + \frac{\lambda}{8M^2} D(D - 1) - \frac{D M}{16} \right], \] (A.14)
\[ f_1 = 1 - D x + \kappa \left( \frac{\beta \lambda}{4M^2} D(D-1)(x^2 + 2x) - \frac{1}{2} \beta D M x \right) \]
\[ + \kappa^2 \left[ -\frac{3 \beta \lambda^2}{32 M^5} D(D-1)(2 \beta M x^2 - x^3 + x^2 + 3x) - \frac{\beta \lambda^2}{16 M^5} D(D-1)^2(\beta M(2x^2 + x) + 3x^2 + 3x) \right. \]
\[ - \frac{\beta \lambda^2}{8 M^5} D(D-1)^2(\beta M(x^2 + 1)^2 + x^3 + 2x^2 + 3x) + \frac{\beta \lambda}{4 M^2} D(D-1)(x^2(\beta M + 1) + x(\beta M + 2)) \]
\[ \left. - \frac{1}{8} D x ((\beta M)^2 + \beta M) \right], \quad (A.15) \]
\[ f_2 = \kappa^2 \left[ -\frac{3 \beta \lambda^2}{64 M^5} D(D-1)(2 \beta M x^2 - 2x^4 + 5x^2) - \frac{3 \beta \lambda^2}{16 M^5} D(D-1)^2(\beta M x^3 + x^3) \right. \]
\[ - \frac{\beta \lambda^2}{8 M^5} D(D-1)^2(2 \beta M x^4 + x^4 + 2x^2) \left. \right], \quad (A.16) \]
\[ f_3 = \frac{1}{2} (1 - D x^2) + \kappa \left( \frac{\beta \lambda}{4 M^2} D(D-1)(x^4 + 2x^2) - \frac{1}{2} \beta D M x^2 \right) \]
\[ + \kappa^2 \left[ -\frac{3 \beta \lambda^2}{32 M^5} D(D-1)(4 \beta M x^4 - x^6 + x^4 + 3x^2) \right. \]
\[ - \frac{\beta \lambda^2}{16 M^5} D(D-1)^2(2 \beta M(2x^4 + x^2) + 3x^4 + 3x^2) \]
\[ - \frac{\beta \lambda^2}{8 M^5} D(D-1)^2(2 \beta M(x^6 + 2x^4 + x^2) + x^6 + 2x^4 + 3x^2) \]
\[ + \frac{\beta \lambda}{4 M^2} D(D-1)((2 \beta M + 1)x^4 + (2 + 2 \beta M)x^2) - \frac{D}{8}(2(\beta M)^2 + \beta M)x^2 \right], \quad (A.17) \]
\[ f_4 = \kappa \frac{\beta \lambda}{8 M^2} D(D-1)(x^2 + 2x^3) + \kappa^2 \left[ -\frac{3 \beta \lambda^2}{32 M^5} D(D-1)(4 \beta M x^3 - x^5 - x^4 + 3x^3 + 2x^2) \right. \]
\[ - \frac{\beta \lambda^2}{32 M^5} D(D-1)^2(2 \beta M(2x^4 + 3x^3 + x^2) + 3x^4 + 6x^3 + 3x^2) \]
\[ - \frac{\beta \lambda^2}{8 M^5} D(D-1)^2(\beta M(2x^5 + 4x^3 + x^2) + x^5 + x^4 + 3x^3 + 2x^2) \]
\[ + \frac{\beta \lambda}{8 M^2} D(D-1)((\beta M + 1)x^2 + (3 \beta M + 2)x^2) \right]. \quad (A.18) \]

We have used \( x = e^{-\beta M} \).

**A.2 The details of the principle of minimum sensitivity analysis**

We analyze the effective action \( (A.13) \) and discuss the phase structure of the model. We will mainly show the analysis at two-loop, since the three-loop analysis is almost parallel.
(Recall that we remove $O(\kappa^2)$ terms in (A.13) when we consider the two-loop theory.) We set $\kappa = 1$ hereafter.

First, we consider a low temperature regime. There, $x = e^{-\beta M}$ would be small, and $f_1$ and $f_3$ would be positive. Then, to make the effective action (A.13) small, $\langle u_1 \rangle = \langle u_2 \rangle = 0$ would be favored. Thus, the effective action (A.13) would become $S_{\text{eff}} = \beta N^2 f_0(M)$.

Here, we need to determine $M$. As we have discussed, we fix $M$ such that the $M$ dependence of the effective action is minimized. At low temperatures, it implies that we need to find $M$ that minimizes $|\partial_M f_0(M)|$ as (2.3). From (A.14), we find that at $M_0 = \lambda^{1/3}(D-1)^{1/3}$,

$$\partial_M f_0$$

becomes 0 and is minimized. Then we obtain the free energy at low temperatures as

$$F = S_{\text{eff}}/\beta = N^2 f_0(M_0) = \frac{3}{8} N^2 D M_0 = \frac{3}{8} N^2 D \lambda^{1/3}(D-1)^{1/3}. \quad \text{(A.20)}$$

This result is shown in FIG. 2 and Table 2. We find good agreement with the MC results even at two-loop order.

Next, in order to investigate the phase transition, we compute $M_i$ in (2.6) and $\bar{f}_i$ in (2.8) near the critical temperature. However, since $\partial_M f_0 = 0$ at $M = M_0$, we obtain $\bar{f}_i = f_i$ for $i = 1, 3, 4$ and we need to evaluate only $M_1$ and $\bar{f}_2$. By substituting the expansion (2.6) into the equation $\partial_M S_{\text{eff}} = 0$, we find

$$M_1 = -\frac{\partial_M f_1}{\partial^2_M f_0}, \quad \bar{f}_2 = -\frac{1}{2} \frac{(\partial_M f_1)^2}{\partial^2_M f_0}. \quad \text{(A.21)}$$

Now, we are ready to discuss the critical phenomena. As we have argued below (2.9), the critical temperature can be found through

$$0 = f_1 |_{M=M_0} = 1 - D e^{-\beta M_0} + \frac{D}{4} \beta M_0 e^{-2\beta M_0}. \quad \text{(A.22)}$$

This equation can be solved numerically and the result is summarized in FIG. 2 and Table 3. Again our results seem to be consistent with the MC results.

Through the discussions around (2.10), the order of the phase transition is determined by the signature of $b$ defined in (2.9) at the critical temperature. We numerically see that it is always negative as shown in FIG. 4 and indicates the first order phase transition for any $D$. 

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Note that, at large-$D$, we can solve (A.22) analytically and obtain several quantities as

\[ f_0 = \frac{3}{8}D(\lambda D)^{1/3} \left( 1 - \frac{1}{3D} + O(1/D^2) \right), \]

\[ \beta_c = \frac{\log D}{(\lambda D)^{1/3}} \left( 1 + \frac{1}{12D} + O(1/D^2) \right), \]

\[ b|_{T=T_c} = -\frac{1}{6D} \log D \left( 1 + O(1/D) \right). \] \hspace{1cm} (A.23)

Thus, $b$ is negative, which does not agree with the $1/D$ expansion [28]. Somehow the leading order of $f_0$ and $T_c$ in our results are precisely coincident with those of the $1/D$ expansion [28], although the $1/D$ corrections differ. (See Eq.(4.27) and (4.30) in [28].) Since the results of the $1/D$ expansion [28] would be reliable at large-$D$, our $f_0$ and $T_c$ at two-loop order are accidentally very good at large-$D$.

The three-loop calculation is almost parallel to the two-loop analysis. One significant difference is the minimum of $|\partial_M f_0|$ in (2.3) is not zero. Hence we need to find the minimum via $\partial^2_M f_0 = 0$, and obtain

\[ M_0 = \frac{15^{1/3} \lambda^{1/3}}{2} (D - 3/4)^{1/3}. \] \hspace{1cm} (A.24)

The rest of the calculations are straightforward. We obtain the free energy in the confinement phase as

\[ F = N^2 f_0(M_0) = N^2 \lambda^{1/3} \frac{D(1412D - 1187)}{160(30(4D - 3))^{2/3}}. \] \hspace{1cm} (A.25)

This result is shown in FIG. 2 and Table 2.

We fix $M_i$ via $\partial^2_M S_{\text{eff}} = 0$ near the critical temperature, and obtain

\[ M_i = -\frac{\partial^2_M f_i}{\partial^3_M f_0}, \quad (i = 1, 3, 4), \]

\[ M_2 = -\frac{1}{\partial^3_M f_0} \left( \partial^2_M f_2 - \partial^3_M f_1 \frac{\partial^2_M f_1}{\partial^3_M f_0} + \frac{1}{2} \partial^4_M f_0 \left( \frac{\partial^2_M f_1}{\partial^3_M f_0} \right)^2 \right), \] \hspace{1cm} (A.26)

where $f_i$ are evaluated at $M = M_0$. Then $\tilde{f}_i$, $a$ and $b$ are derived through (2.8) and (2.9). Finally, by solving $a(T_c) = 0$ and evaluating $b(T_c)$, we obtain the results shown in FIG. 2 and 4.

Different from the two-loop case, we cannot solve $T_c$ in the three-loop case analytically even at large-$D$. 

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Table 2: Free energy $F/N^2$ in the confinement phase. We have used the unit $\lambda = 1$. The two-loop and three-loop results are from (A.20) and (A.25), respectively. The $1/D$ expansion results are from (4.27) in [28]. The MC results are from the unpublished data in [29].

| $D$ | two-loop | three-loop | $1/D$ expansion | MC ($T = 0.50$) |
|-----|----------|------------|----------------|----------------|
| 2   | 0.75     | 0.72       | 0.76           | 0.70 ($N = 60$) |
| 3   | 1.42     | 1.37       | 1.41           | 1.42 ($N = 32$) |
| 4   | 2.16     | 2.09       | 2.15           | 2.11 ($N = 32$) |
| 5   | 2.98     | 2.88       | 2.95           | 2.93 ($N = 24$) |
| 6   | 3.85     | 3.71       | 3.82           | 3.81 ($N = 32$) |
| 9   | 6.75     | 6.52       | 6.71           | 6.66 ($N = 32$) |
| 13  | 11.2     | 10.8       | 11.1           | 11.0 ($N = 32$) |

Table 3: Critical temperature $T_c$. We have used the unit $\lambda = 1$. The $1/D$ expansion results are from (4.30) in [28]. The MC results are from [29, 32]. In the MC results, we show $T_0$ defined in FIG.3, which should be slightly below $T_c$.

| $D$ | two-loop | three-loop | $1/D$ expansion | MC |
|-----|----------|------------|----------------|----|
| 2   | 1.65     | 1.61       | 1.34           | 1.32 |
| 3   | 1.26     | 1.20       | 1.08           | 1.10 |
| 9   | 0.938    | 0.889      | 0.892          | 0.901 |
| 15  | 0.906    | 0.867      | 0.879          | 0.884 |
| 20  | 0.903    | 0.869      | 0.882          | 0.884 |
| 25  | 0.906    | 0.877      | 0.889          | 0.89  |
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