A CRITERION FOR REFLEXIVITY OF MODULES

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Abstract. Let $M$ be a finitely generated module over a ring $\Lambda$. With certain mild assumptions on $\Lambda$, it is proven that $M$ is a reflexive $\Lambda$-module, once $M \cong M^{**}$ as a $\Lambda$-module.

Let $\Lambda$ be a ring. For each left $\Lambda$-module $X$, let $X^* = \text{Hom}_\Lambda(X, \Lambda)$ denote the $\Lambda$-dual of $X$. This note aims at reporting the following.

**Proposition 1.** Let $\Lambda$ be a ring and let $M$ be a finitely generated left $\Lambda$-module. Assume that one of the following conditions is satisfied.

1. $\Lambda$ is a left Noetherian ring.
2. $\Lambda$ is a semi-local ring, that is $\Lambda/J(\Lambda)$ is semi-simple, where $J(\Lambda)$ denotes the Jacobson radical of $\Lambda$.
3. $\Lambda$ is a module-finite algebra over a commutative ring $R$.

Then, $M$ is a reflexive $\Lambda$-module, that is the canonical map $M \xrightarrow{h} M^{**}$ is an isomorphism if and only if there is at least one isomorphism $M \cong M^{**}$ of $\Lambda$-modules.

To show the above assertion, we need the following. This is well-known, and the proof is standard.

**Lemma 2.** Let $N$ be a right $\Lambda$-module and set $M = N^*$. Then the composite of the homomorphisms

$$M \xrightarrow{h} M^{**} \xrightarrow{(h_N)^*} M$$

equals the identity $1_M$.

**Proof of Proposition 1.** We have only to show the if part. Thanks to Lemma 2, we have a split exact sequence

$$0 \rightarrow M \xrightarrow{h} M^{**} \rightarrow X \rightarrow 0$$

of left $\Lambda$-modules, so that $M \cong M \oplus X$, since $M \cong M^{**}$. Therefore, we get a surjective homomorphism $\varepsilon : M \rightarrow M$ with $\text{Ker} \varepsilon = X$. Hence, if Condition (1) is satisfied, then $X = (0)$, so that $M$ is a reflexive $\Lambda$-module. If Condition (2) is satisfied, then, setting $J = J(\Lambda)$, we get

$$\Lambda/J \otimes_\Lambda M \cong (\Lambda/J \otimes_\Lambda M) \oplus (\Lambda/J \otimes_\Lambda X)$$

whence $\Lambda/J \otimes_\Lambda X = (0)$ by Krull-Schmidt’s theorem, so that $X = (0)$. Suppose that Condition (3) is satisfied. Then, $M \cong M \oplus X$ as an $R$-module, where $M$ is finitely

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generated also as an $R$-module. Consequently, for every $p \in \text{Spec } R$, we get $M_p \cong M_p \oplus X_p$ as an $R_p$-module, whence by the case where Condition (2) is satisfied, $X_p = (0)$ for all $p \in \text{Spec } R$. Thus, $X = (0)$, as claimed. \hfill \Box

**Corollary 3.** Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. If $M \cong M^{**}$ as an $R$-module, then $M$ is a reflexive $R$-module.

**Remark 4.** Let $\Lambda$ be a ring and let $a, b \in \Lambda$ such that $ab = 1$ but $ba \neq 1$. We then have the homomorphism

$\widehat{b} : \Lambda \Lambda \rightarrow \Lambda \Lambda, \quad x \mapsto xb$

is surjective but not an isomorphism. Therefore, setting $X = \text{Ker } \widehat{b}$, we get

$\Lambda \Lambda \cong \Lambda \Lambda \oplus X$.

This example shows that $X$ does not necessarily vanish, even if $M \cong M \oplus X$ and $M$ is a finitely generated module. This example seems also to suggest that Proposition 1 doesn’t hold true without any specific conditions on $\Lambda$.

Let us note one example in order to show how Corollary 3 works at an actual spot. See [1, p.137, the final step of the proof of (4.35) Proposition] also, where one can find a good opportunity of making use of it, from which the motivation for the present research has come.

**Example 5.** Let $k[s, t]$ be the polynomial ring over a field $k$ and set $R = k[s^3, s^2t, st^2, t^3]$. Then $R$ is a normal ring and the graded canonical module $K_R$ of $R$ is given by $K_R = (s^2t, s^3)$. We set $I = (s^2t, s^3)$. Then, since $I$ is a reflexive $R$-module, but not 3-torsionfree in the sense of Auslander-Bridger [1, (2.15) Definition] (because $R$ is not a Gorenstein ring), we must have $\text{Ext}^1_R(R : I, R) \neq (0)$ by [1, (2.17) Theorem]. In what follows, let us check that $\text{Ext}^1_R(R : I, R) \neq (0)$ directly.

First, consider the exact sequence

$0 \rightarrow R \rightarrow R : I \rightarrow \text{Ext}^1_R(R/I, R) \rightarrow 0$

induced from the sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$. Taking the $R$-dual of it again, we get the exact sequence

$0 \rightarrow R : (R : I) \rightarrow R \rightarrow \text{Ext}^1_R(\text{Ext}^1_R(R/I, R), R) \rightarrow \text{Ext}^1_R(R : I, R) \rightarrow 0$,

that is

$0 \rightarrow R/I \rightarrow \text{Ext}^1_R(\text{Ext}^1_R(R/I, R), R) \rightarrow \text{Ext}^1_R(R : I, R) \rightarrow 0$.

Therefore, the homomorphism

$\sigma : R/I \rightarrow \text{Ext}^1_R(\text{Ext}^1_R(R/I, R), R)$

should not be an isomorphism. Because

$\text{Hom}_{R/(f)}(\text{Hom}_{R/(f)}(R/I, R/(f)), R/(f)) \cong \text{Ext}^1_R\text{Ext}^1_R(R/I, R), R)$

for every $0 \neq f \in I$, thanks to Corollary 3, the assertion that $\sigma$ is not an isomorphism is equivalent to saying that $R/I$ is not a reflexive $R/(f)$-module for some $0 \neq f \in I$. In the following, we shall confirm that $R/I$ is not a reflexive $R/(s^3)$-module. Before starting
work, we would like to note here and emphasize that if we do not make use of Corollary 3, we must certify the above homomorphism $\sigma$ to be induced from the canonical map

$$\frac{R}{I} \xrightarrow{h_{R/I}} \text{Hom}_{R/(s^3)}(\text{Hom}_{R/(s^3)}(\frac{R}{I}, \frac{R}{(s^3)}), \frac{R}{(s^3)}),$$

which provably makes a tedious calculation necessary.

We set $T = \frac{R}{(s^3)}$ and $J = (\overline{s^2t}, \overline{st^2})$ in $T$, where $\overline{\cdot}$ denotes the image in $T$. Notice that $\text{Hom}_T(\frac{R}{I}, T) \cong (0) :_T I = J$ and $\text{Hom}_T(T/J, T) \cong (0) :_T J = (\overline{s^2t})$. Therefore, from the exact sequence

$$0 \to J \to T \to T/J \to 0,$$

we get the exact sequence

$$0 \to (\overline{s^2t}) \to T \to \text{Hom}_T(J, T) \to \text{Ext}^1_T(T/J, T) \to 0,$$

that is the exact sequence

$$(E) \quad 0 \to \frac{R}{I} \to \text{Hom}_T(J, T) \to \text{Ext}^1_T(T/J, T) \to 0,$$

which guarantees it suffices to show $\text{Ext}^1_T(T/J, T) \neq (0)$, since $\text{Hom}_T(J, T) = \text{Hom}_T(\text{Hom}_T(\frac{R}{I}, T), T)$. We now identify

$$R = k[X, Y, Z, W]/I_2(\frac{X}{Y} \frac{Z}{W}),$$

where $k[X, Y, Z, W]$ denotes the polynomial ring over $k$, $I_2(\mathbb{M})$ stands for the ideal of $k[X, Y, Z, W]$ generated by the $2 \times 2$ minors of a matrix $\mathbb{M}$, and $X, Y, Z, W$ correspond to $s^3, s^2t, st^2, t^3$, respectively. We denote by $x, y, z, w$ the images of $X, Y, Z, W$ in $T$. Then, $T/J$ has a $T$-free resolution

$$\ldots \to T^\oplus 6 \begin{pmatrix} y & z & 0 & 0 & 0 & 0 \\ -x & 0 & w & 0 & 0 & 0 \\ 0 & -x & -z & 0 & 0 & 0 \\ 0 & y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to T^3 \xrightarrow{(x \ y \ z)} T \to T/J \to 0.$$

Taking the $T$-dual of the resolution, we have $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \text{Ker} [T^\oplus 3 \xrightarrow{T^3} T^\oplus 6]$, but $\begin{pmatrix} y \\ 0 \end{pmatrix} \neq \alpha \begin{pmatrix} x \\ y \end{pmatrix}$ for any $\alpha \in T$. Thus, $\text{Ext}^1_T(T/J, T) \neq (0)$, so that the exact sequence (E) shows $\frac{R}{I}$ is not a reflexive $T$-module. Hence, by Corollary 3 the homomorphism

$$\sigma : \frac{R}{I} \to \text{Ext}^1_R(\text{Ext}^1_R(\frac{R}{I}, R), R)$$

is not an isomorphism. Thus, $\text{Ext}^1_R(K^*_R, R) \neq (0)$, and $K_R$ is not 3-torsionfree.

References

[1] M. Auslander and M. Bridger, Stable module theory, Amer. Math. Soc., Memoirs, 94, 1969.