INTEGRAL KLEIN BOTTLE SURGERIES AND HEEGAARD FLOER HOMOLOGY

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Abstract. In this paper, we study gluings $X$ of the twisted $I$-bundle over the Klein bottle to knot complements, and investigate which gluings can be realized as integral Dehn surgery along a knot in $S^3$. Closed, orientable 3-manifolds containing a Klein bottle can be presented as such a gluing, and Heegaard Floer homology provides a way to study surgery obstructions and the relevant knot invariants. We prove that if $X$ is 8-surgery along a genus two knot, and can be formed by gluing the twisted $I$-bundle over the Klein bottle to an $S^3$ knot complement, then $X$ is an L-space and $K$ has the same knot Floer homology as $T(2,5)$.

1. Introduction

For a knot $K \subset S^3$, denote the result of $\frac{p}{q}$–Dehn surgery on $K$ as $S^3_{p/q}(K)$. A celebrated theorem of Lickorish and Wallace shows that any closed, orientable 3-manifold may be constructed by integral Dehn surgery on a link in $S^3$ [Lic62, Wal60]. It is then natural to ask which manifolds may be realized as Dehn surgery on a knot in $S^3$, and we will focus on those containing Klein bottles.

Much is known about surgeries $S^3_{p/q}(K)$ containing Klein bottles. Gordon and Luecke [GL95] showed that the surgery slope $p/q$ is integral when $K$ is hyperbolic, and Teragaito [Ter01] extended this condition to $K$ non-cabled and showed that $p$ is divisible by four. In [IT05], Ichihara and Teragaito gave bounds for $|p|$ in terms of the knot genus $g(K)$ when $K$ is non-cabled, and shortly after showed the same bound holds when $K$ is cabled, albeit allowing rational slopes [IT05]. Their combined results prove that if $S^3_{p/q}(K)$ contains a Klein bottle with $K$ non-trivial, then $|p| \leq 4q(K) + 4$ with equality only occurring for specific knots.

The lens spaces containing a Klein bottle are $L(4n, 2n + 1)$ [BW69], and Teragaito proved that genus one knots admitting surgeries containing a Klein bottle are Whitehead doubles [Ter01]. Our focus is then on genus two knots admitting surgeries containing Klein bottles. When $g(K) = 2$, the maximally sloped surgeries containing Klein bottles are $S^3_{\pm 12}(K)$ when either $K = T(2, \pm 5)$ or $K = T(2, \pm 3) \# T(2, \pm 3)$ due to [IT05, Theorem 1]. The next largest slope to consider is then $|r| = 8$, which is our focus.

Suppose $X$ is realizable as 8-surgery on a genus two knot $K$, and contains a Klein bottle. With $N$ denoting the twisted $I$-bundle over the Klein bottle, observe that $X$ contains $N$ as a tubular neighborhood. While $X$ is then viewed as $N$ glued to knot manifold, or a rational homology solid torus, we will focus on pairings with $S^3$ knot complements so that $X = (S^3 \setminus \nu J) \cup_h N$ for some knot $J$ in $S^3$. The gluing $h$ and its effects on $X$ are studied in Subsection 3.1. By utilizing techniques involving knot Floer, HeegaardFloer, and bordered Heegaard Floer homology, we prove

**Theorem 1.1.** Let $X = S^3_{g}(K)$ with $g(K) = 2$ contain a Klein bottle, and suppose $X = (S^3 \setminus \nu J) \cup_h N$. Then $X$ is an L-space, and

- If $J$ is trivial, then $X = (-1; 1/2, 1/2, \frac{2}{5})$ as a Seifert fibered manifold and $K = T(2, 5)$.
- If $J$ is non-trivial, then $J$ is a trefoil and $X$ belongs to an infinite family of such pairings. Further, $K$ satisfies $\widehat{HF}(K) \cong \widehat{HF}(T(2, 5))$ as a genus two L-space knot.

Information given by the large surgery theorem of Osváth-Szabó and Rasmussen [OS04a, Ras03] for $\widehat{HF}(X) \cong \widehat{HF}(S^3_{g}(K))$ together with the immersed curves formulation of bordered Heegaard Floer invariants developed by Hanselman, Rasmussen, and Watson in [HRW16] allows us to determine knots whose complements $S^3 \setminus \nu J$ pair with $N$ to obtain $X$. The versatility...
of this immersed curves package lends itself toward studying Dehn surgery problems, and has already led to fruitful results towards the cosmetic surgery conjecture (see [Han16]).

The techniques used in this paper, and the results in Theorem 1.1, actually hold for complements of knots $J$ in integer homology sphere $L$-spaces $Y$. These are integer homology spheres $Y$ with the simplest Heegaard Floer homology, which is to say $\dim \widehat{HF}(Y) = |H_1(Y, \mathbb{Z})|$. We briefly discuss this generalization in Section 4, and just state that $Y$ is forced to be the Poincaré homology sphere. The example for $J = T(2,3)$ from Theorem 1.1 can be seen in Figure 1, where the count of intersection points corresponds to $\dim \widehat{HF}(X)$. It is worth mentioning that $T(2,5)$ is conjectured to be the only genus two L-space knot.

Figure 1. The pairing of immersed curves for $S^3 \setminus \nu T(2,3)$ in blue and $N$ in red and purple, that computes $\widehat{HF}(X)$.

Ichihara and Teragaito remark that the 2-bridge knot $6_2$ admits an 8-surgery containing a Klein bottle [IT03, Example 5.6]. This knot is not an L-space knot, and so Theorem 1.1 then implies that $S^3_{8/9}(6_2)$ is obtained as $Y \setminus \nu J$ glued to $N$, with $Y$ not an integer homology sphere L-space. This example highlights that gluing along integer homology sphere L-space complements is special. Additionally, there are many examples of genus two cabled knots in $S^3$ admitting rational surgeries $S^3_{8/9}(K)$ that contain Klein bottles. When $K$ is a torus knot the only example arises from the elliptic manifold obtained by Dehn filling $N$ stated in Theorem 1.1. Otherwise $K$ is the $(2,1)$-cable of a genus one knot, and there are many rationally sloped fillings of the associated cable space for $S^3 \setminus \nu K$ that yield Klein bottles [Gor83, Corollary 7.3].

Organization

Throughout this paper we only consider positive surgeries, and remark that the analogous result for negative surgery follows by orientation reversal. Additionally, all manifolds are assumed to be compact, connected, oriented 3-manifolds, unless stated otherwise, and the coefficients in Floer homology are assumed to belong to $\mathbb{F} = \mathbb{F}_2$. We also will typically denote closed manifolds by $X$ or $Y$, and manifolds with (typically torus) boundary by $M$. Also knots $J \subset Y$ bounding a disk are said to be trivial, and figures will have the curve invariant for knot complements in blue, and the curve invariant for the filling manifold in red.

Section 2 summarizes the relevant background from knot Floer and Heegaard Floer homology, and gives an overview of the immersed curves formulation of bordered Floer invariants (for manifolds with torus boundary). We review relevant theorems pertaining to their structure and symmetries.

Section 3 introduces $X$ as $N$ glued to a knot complement along $h$, and establishes the necessary lemmas to prove Theorem 1.1. When $J$ is trivial, $X$ is an elliptic manifold that falls under Doig’s classification of finite, non-cyclic surgeries for $p \leq 9$ [Doi15]. We then consider such pairings with $J$ non-trivial, using Floer homology techniques to conclude $J$ is a trefoil.
Section 4 mentions some generalizations for pairings involving knot complements where \( Y \) is not an integer homology sphere L-space, and provides a few homological constraints that narrow the problem.

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2. Background Material

We will assume the reader is familiar with the constructions of knot Floer homology \( \text{OS04a, Ras03} \) and Heegaard Floer homology \( \text{OS04c} \), and just include relevant theorems before introducing bordered Heegaard Floer homology for manifolds with torus boundary \( \text{LOT18} \) in its immersed curves formulation \( \text{HRW16, HRW18} \).

2.1. Knot Floer and Heegaard Floer homology. The Heegaard Floer homology theory of Ozsváth and Szabó, and knot Floer homology theory of Ozsváth and Szabó, and independently Rasmussen have proven to be very powerful invariants of 3-manifolds and knots. To an oriented knot \( K \) in an integer homology sphere \( Y \), Ozsváth, Szabó, and Rasmussen associate a bi-graded, finitely generated vector space that decomposes as

\[
\widehat{HF}(Y, K) = \bigoplus_{i,s} \widehat{HF}_i(Y, K, s).
\]

The integers \( i \) and \( s \) denote the Maslov (or homological) and Alexander gradings, respectively. We may often suppress the Maslov grading from notation when not needed, and the ambient 3-manifold from notation when \( Y = S^3 \).

For knots \( K \subset S^3 \), knot Floer homology categorifies the Alexander polynomial, and leads to knot genus and fiberedness detection results. Ozsváth and Szabó showed in \( \text{OS04a} \) that knot Floer homology detects the knot genus exactly via

\[
g(K) = \max\{ s \geq 0 \mid \widehat{HF}(K, s) \neq 0 \}.
\]

Together with work of Ghiggini and Ni, it also detects precisely if a knot is fibered \( \text{OS04d, Ghi08, Ni07} \):

\[
K \subset S^3 \text{ fibered} \iff \widehat{HF}(K, g(K)) \cong \mathbb{F}.
\]

Further, knot Floer homology determines whether a knot in \( S^3 \) is a trefoil or figure-eight knot following from genus and fiberedness detection \( \text{Ghi08} \). These detection results will allow us to detect which \( J \) provide complements that pair with \( N \) to yield \( X \) as 8-surgery on a genus two knot.

For a closed, oriented 3-manifold \( Y \), Ozsváth and Szabó associate to it a finitely-generated vector space that decomposes as

\[
\widehat{HF}(Y) = \bigoplus_{s \in \text{Spin}^c(Y)} \widehat{HF}(Y, s),
\]

over \( \text{Spin}^c \) structures \( \text{OS04c} \). For a rational homology sphere \( Y \), we have that \( \dim \widehat{HF}(Y, s) \geq 1 \) for any \( s \in \text{Spin}^c(Y) \). A rational homology sphere \( Y \) satisfying \( \dim \widehat{HF}(Y, s) = 1 \) for all \( s \in \text{Spin}^c(Y) \) is called a Heegaard Floer L-space, generalizing the situation for lens spaces.

For a null-homologous knot \( K \) in a 3-manifold \( Y \), the full knot Floer complex \( CFK^\infty(Y, K) \) is a \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain complex. For our purposes, we may take \( Y = S^3 \) and simply write \( CFK^\infty(K) \) in this setting. For \( X \) a subset of \( \mathbb{Z} \oplus \mathbb{Z} \), let \( CX \) be the subgroup of \( CFK^\infty(K) \) generated by those elements with filtration level \( (i, j) \in X \). Further, for \( s \in \mathbb{Z} \) define the subcomplexes

\[
\widehat{A}_s = C\{ \max\{i, j - s\} = 0 \},
\]

and the respective homology groups \( \widehat{A}_s = H_*(\widehat{A}_s) \). This notation is suggestive of a connection to the hat-flavor of Heegaard Floer homology we will see in Theorem 2.1 below.
Let us identify Spin$^c(S^3_p(K))$ with $\mathbb{Z}/p\mathbb{Z}$ as in [OS04 Subsection 2.4], and denote the correspondence using $[s] \in \text{Spin}^c(S^3_p(K))$ for $[s] \in \mathbb{Z}/p\mathbb{Z}$. The following theorem is considered the large surgery theorem, and shows that $\widehat{HF}(S^3_p(K), [s])$ and $\hat{A}_s$ are isomorphic under certain conditions.

**Theorem 2.1** ([OS04a, Ras03]). For $p \gg 0$ and any $s \in \mathbb{Z}$ with $|s| \leq p/2$, there is an isomorphism

\[ \widehat{HF}(S^3_p(K), [s]) \cong \hat{A}_s. \]

Here, $[s] \in \mathbb{Z}/p\mathbb{Z}$ is the corresponding Spin$^c$ structure on $S^3_p(K)$.

**Remark.** The large surgery theorem holds for all Spin$^c$ structures when $p \geq 2g(K) - 1$.

We are particularly interested in using Theorem 2.1 to study 8-surgery on knots $K$ with $g(K) = 2$. In this case, we have

**Proposition 2.2.** Let $K \subset S^3$ have $g(K) = 2$. Then $\dim \widehat{HF}(S^3_p(K), [s]) = 1$ for at least five of the eight $[s] \in \text{Spin}^c(S^3_p(K))$.

**Proof.** We have that $\hat{A}_{-s}$ and $\hat{A}_s$ are isomorphic due to Lemma 2.3 of [HLW15], following from the fact that $CFK^\infty(K)$ is filtered chain homotopy equivalent under reversing the roles of $i$ and $j$. Further, $\hat{A}_s \cong \widehat{CF}(S^3)$ for $s \geq g(K)$ by definition, and so $\hat{A}_s \cong \mathbb{F}$ for $|s| \geq 2$. Since $S^3_h(K)$ is large surgery, Theorem 2.1 implies $\widehat{HF}(S^3_h(K), [s]) \cong \mathbb{F}$ for $s \in \mathbb{Z}$ satisfying $|s| \neq 0, \pm 1 \in \mathbb{Z}/8\mathbb{Z}$. □

This simple structure of $\widehat{HF}(S^3_h(K))$ will be very useful towards establishing Theorem 1.1, where we appeal to counting the number of $t \in \text{Spin}^c(X)$ supporting $\dim \widehat{HF}(X, t) > 1$. This large surgery is also particularly special because of the following proposition.

**Proposition 2.3.** $S^3_h(K)$ is irreducible for any knot $K$ with $g(K) = 2$.

**Proof.** To generate a contradiction, suppose that $S^3_h(K)$ is reducible. From [MS03], we see that $S^3_h(K)$ reducible implies $1 < |p| \leq 2g(K) - 1$ for $K$ non-cabled. So it must be the case that $K$ is the $(r, s)$-cable of some knot $K'$, where $r$ and $s$ are coprime and positive with $s > 1$. The cabling conjecture holds for cable knots, and so the slope $p = rs$ provided by the cabling annulus is the only reducing slope for $S^3_h(K)$. In this case we have $S^3_{hs}(K) \cong L(s, r)\# S^3_{3/8}(K')$, and so

\[ S^3_h(K) \cong L(8, 1)\# S^3_{3/8}(K'). \]

Let $[s_i] \in \text{Spin}^c(S^3_h(K))$ restrict to $[s'_i] \times [s_0]$, where $[s'_i] \in \text{Spin}^c(L(8, 1))$ and $[s_0] \in \text{Spin}^c(S^3_{3/8}(K'))$. The Künneth formula for the hat-flavor of Heegaard Floer homology [OS04b, Theorem 1.5] implies

\[ \widehat{HF}(S^3_h(K), [s_i]) = H_*(\widehat{CF}(L(8, 1), [s'_i])) \otimes_{\mathbb{F}} \widehat{CF}(S^3_{3/8}(K'), [s_0])), \]

since $L(8, 1)$ is a Lens space. Theorem 2.1 forces $\dim \widehat{HF}(S^3_h(K), [s_i]) = \dim \hat{A}_i$, and since $\dim \hat{A}_i = 1$ for $|s| \geq g(K)$, we see that $S^3_h(K)$ is an L-space and $K'$ is an L-space knot. From [OS11, Proposition 9.5], the $\nu$ invariant for $K'$ must be trivial, which implies $K'$ is the unknot by [OS11, Proposition 9.6]. Therefore $K$ is trivial as the $(1, 8)$-cable of the unknot, and so $S^3_h(K) \cong L(8, 1)$, yielding a contradiction. □

We also have the following immediate corollary, which is useful for the case when $Y \neq S^3$ discussed in Section 1.

**Corollary 2.4.** Let $M = Y \setminus \nu J$ be a knot manifold. If $X = M \cup_h N$ is realizable as $S^3_h(K)$ for $g(K) = 2$, then $M$ is irreducible.
2.2. The pairing theorem and immersed curves. Bordered Heegaard Floer homology, introduced by Lipshitz, Ozsváth, and Thurston, provides a way of computing $\widehat{HF}(X)$ by decomposing $X$ along an essential surface, and then recovering its Floer homology by a suitable means of pairing the relative Floer invariants for the decomposed pieces \cite{LOT18}. While defined for general manifolds with boundary, we are interested in manifolds with torus boundary.

For $M_1$ a compact, oriented 3-manifold with torus boundary, they associate to it a differential module called a type D structure $\overline{CFD}(M_1, \alpha_1, \beta_1)$, depending on a parameterization of $\partial M_1$. Together with a suitably dual object $\overline{CFA}(M_2, \alpha_2, \beta_2)$, they prove that the hat-flavor of the Heegaard Floer homology of their gluing is given by the box tensor product of these modules. That is, they show in \cite{HRW18} that

$$\widehat{HF}(M_1 \cup_h M_2) = H_*(\overline{CFA}(M_2, \alpha_2, \beta_2) \boxtimes \overline{CFD}(M_1, \alpha_1, \beta_1)).$$

Computations with the box tensor product are decently involved, and so the geometric interpretation of these invariants due to Hanselman, Rasmussen, and Watson in \cite{HRW16} offers a more tractable approach. Their geometric construction lives in the punctured torus, which we now discuss.

**Definition 2.5.** Let the punctured torus $T_M$ be defined as $(H_1(\partial M; \mathbb{R})/H_1(\partial M; \mathbb{Z})) \setminus \{z\}$, where $z = (1 - \epsilon, 1 - \epsilon)$ for $\epsilon$ small. We refer to $z$ as the marked point, and orient $T_M$ so that the $y$-axis projects to $\alpha$ and the $x$-axis projects to $\beta$, with $\alpha, \beta$ specifying the handle decomposition of $\partial M \setminus z$.

The bordered invariant $\overline{CFD}(M)$ is captured as a collection $\widehat{HF}(M)$ of immersed, closed curves (each possibly decorated with local systems) in the punctured torus $T_M$. This formulation has the advantage of casting the pairing theorem as the Lagrangian intersection Floer homology of the immersed curves of the paired manifolds. Before proceeding, we make the following cautionary remark.

**Remark.** The manifolds encountered in this paper are all loop type, which means their immersed curve invariants have one-dimensional local systems. One suspects that application of these techniques to general knot manifolds would require working with non-trivial local systems. Further, we do not require Maslov grading information, so any grading/phantom arrows connecting curve components are ignored. We refer the reader to \cite{HRW18} for more information on these facets of the theory.

**Theorem 2.6** (\cite[Theorem 2]{HRW18}). Consider the pairing $X = M_1 \cup_h M_2$, where the $M_i$ are compact, oriented 3-manifolds with torus boundary and $h : \partial M_2 \to \partial M_1$ is an orientation reversing homeomorphism for which $h(z_2) = z_1$. Then

$$\widehat{HF}(X) \cong HF(\widehat{HF}(M_1), h(\widehat{HF}(M_2))),$$

where intersection Floer homology is computed in $T_{M_1}$ and the isomorphism is one of relatively graded vector spaces that respects the Spin$^c$ decomposition. Restriction on Spin$^c$ structures gives a surjective map $\pi : \text{Spin}^c(X) \to \text{Spin}^c(M_1) \times \text{Spin}^c(M_2)$.

We will almost always find it more convenient to carry out the intersection Floer homology in specific covers of $T_{M_1}$.

**Definition 2.7** (\cite{HRW18}). Let $\overline{T}_M$ be the cover of $T_M$ associated with the kernel of the composition $\pi_1(T_M) \to \pi_1(\partial M) \to H_1(\partial M) \to H_1(M)$, and $p : \overline{T}_M \to T_M$ the projection. Further, let $\overline{T}$ denote the cover $\mathbb{R}^2/(\frac{1}{2} + \mathbb{Z})^2$.

The lifts of $\widehat{HF}(M)$ to these covers are useful for identifying properties that the invariant encodes. Additionally, it is almost always more convenient to lift to $\overline{T}$ for homology computations. The immersed curve invariant decomposes over Spin$^c$ structures as

$$\widehat{HF}(M) \cong \bigoplus_{\mathfrak{s} \in \text{Spin}^c(M)} p(\overline{\widehat{HF}(M, \mathfrak{s})}),$$

where $\overline{\widehat{HF}(M, \mathfrak{s})}$ denotes the lift to $\overline{T}_{M, \mathfrak{s}}$ of the part of $\overline{\widehat{HF}(M)}$ associated to $\mathfrak{s}$ \cite[Theorem 7]{HRW16}. This lift $\overline{\widehat{HF}(M, \mathfrak{s})}$ is well-defined up to action by the deck group of $p$, and will typically
be taken to be centered about the origin in $T_{M,b}$. When $H_1(M) \cong \mathbb{Z}$, which is encountered in the case of $S^3$ knot complements, the cover $T_M$ may be identified with the infinite cylinder $S^1 \times (\mathbb{R} \setminus (\frac{1}{2} + \mathbb{Z}))$ with lifts $\tau_i$ of the basepoint at coordinates $(0, i + \frac{1}{2})$.

A convenient application of the pairing theorem for immersed curves applies to Dehn surgery, so that
\[
\widehat{HF}(S^3/K) \cong \widehat{HF}(S^3/\nu J, h(\widehat{HF}(D^2 \times S^1))),
\]
where $h$ is the surgery map. Figure 2 provides the computation of $\widehat{HF}(S^3(T(2, 5)))$, where the four Spin$^c$ structures are in correspondence with the four lifts of $h(\widehat{HF}(D^2 \times S^1))$ required to lift all intersections to $T_M$. It is immediate that this manifold is an L-space when lifting to $T_M$, showing the advantage of working in the cover.

Figure 2. The pairing of $\widehat{HF}(S^3/T(2, 5))$ and $h(\widehat{HF}(D^2 \times S^1))$ that computes $\widehat{HF}(S^3(T(2, 5)))$.

We now introduce the immersed curve invariant for the twisted $I$-bundle over the Klein bottle, denoted $N$. Supporting two different Seifert structures, $N$ has two slopes $\phi_0$ and $\phi_1$ on $\partial N$ that correspond to the fiber slope of the structure with base orbifold a Möbius band and $D^2(2, 2)$, respectively. These slopes are dual, and so they form a parameterization $\{\phi_0, \phi_1\}$ for $\partial N$. The slope $\phi_0$ is the rational longitude of $N$, or the unique slope in $H_1(N)$ with finite order [Wat12]. The curve $\phi_1$ includes in $H_1(N)$ as twice a primitive curve, and we will assume this parameterization for $\partial N$ is taken throughout the rest of the paper.

The bordered invariant $\widehat{CFD}(N, \phi_1, \phi_0)$, computed in [BGW13], yields $\widehat{HF}(N)$ shown in Figure 3 after applying the algorithm in [HRW18 Subsection 1.2]. We caution the reader that the notation for the idempotent decomposition of the torus algebra $A$ is different between these two references. The bordered invariant has two distinct components corresponding to the two torsion Spin$^c$ structures $s_i \in \text{Spin}^c(N)$. We will refer to the component that pulls tight around the basepoint as the component corresponding to $s_1$, and the loose component as the one corresponding to $s_0$.

If $J$ is a knot in $S^3$ (and more generally an integer homology sphere L-space), then there is a simple algorithm to construct $\widehat{HF}(S^3 \setminus J)$ from $\text{CFK}^-(S^3, K)$.
The immersed curve invariant \( \hat{HF}(M) \) encodes many algebraic properties as geometric symmetries. For example, orientation reversal yields \( \hat{HF}(-M) = \hat{HF}(M) \), but the boundary parameterization changes. Seen in \( T_M \), the invariant \( \hat{HF}(-M) \) is obtained by reflecting about the rational longitude. The invariant also carries information about Turaev torsion and the Thurston norm for knot complements \([HRW18]\).

A structural property of immersed curves is their invariance (as unlabelled curves) under action by the elliptic involution, with \( z \) fixed, of \( \partial M \setminus z \) \([HRW18]\) Theorem 7]. More concretely, Hanselman, Rasmussen, and Watson show that \( \hat{CFD}(M, c(s)) \cong E \boxtimes \hat{CFD}(M, s) \), meaning that Spin\(^c\) conjugation on the level of bordered invariants achieves the same resulting curve invariant as what would arise from the box tensor product with a particular type DA structure associated with elliptic involution. This feature of \( \hat{HF}(M) \) can be seen when a rotation of \( \pi \) about the origin is applied to our chosen lift \( \tilde{HF}(M, s) \) in \( T_M \).

There is also a convenient way to arrange the curves \( \hat{HF}(M) \) to ensure that the intersection Floer homology of pairings is minimal.

**Definition 2.8.** Fix a metric on the torus \( T_M \). We say \( \hat{HF}(M) \) is in pegboard form if the immersed curves are homotoped to have minimal length in \( T_M \), where the curve remains outside an \( \varepsilon \)-ball of \( z \).

The resulting curve is a pegboard representative for \( \hat{HF}(M) \), and we may lift these to both \( \tilde{T}_M \) and \( \tilde{T} \), where each lift of \( z \) has an \( \varepsilon \)-ball disjoint from the lift of \( \hat{HF}(M) \). Pegboard forms are invaluable for pairing, since pulling curves tight homotopes away pseudo-holomorphic disks that do not contribute to the intersection Floer homology of a pairing. This ensures that the resulting Floer homology is minimal \([HRW16]\) Lemma 47].

When \( M = S^3 \setminus \nu J \), we may enumerate the lifts of the meridian \( \mu \) to \( \tilde{T}_M \) so that the lift between basepoints \( \pi_s \) and \( \pi_{s+1} \) is \( \tilde{\mu}_{[s]} \). To introduce more properties, it will be useful to reference specific regions of the covers \( \tilde{T}_M \) and \( \tilde{T} \cong \mathbb{R}^2 \setminus \left( \frac{1}{2} + \mathbb{Z} \right)^2 \), so we make the following notational conventions.

**Definition 2.9.** Let \( R \) denote the neighborhood(s) of \( \tilde{\mu} \) in \( T_M \), and let \( \tilde{R}_j \) denote its lifts to the cover \( \tilde{T}_M \). If \( H_1(M) \cong \mathbb{Z} \) so that \( T_M \cong S^1 \times (\mathbb{R} \setminus \frac{1}{2} \mathbb{Z}) \), define \( n_i \) to be the number of curve segments of the pegboard representative of \( \hat{HF}(M, s) \) that are homotopic to \( \tilde{\mu}_i \) within \( \tilde{R}_j \).
For example, the curves in Figure 4 both have \( n_{\pm 1} = 1 \), but \( T(2, 3) \# T(2, 3) \) has \( n_0 = 3 \) while \( T(2, 5) \) has \( n_0 = 1 \). They both satisfy \( n_i = 0 \) for \( |i| > 1 \).

When \( M = S^3 \setminus \nu J \), the lifts \( \hat{H}F(M, s) \) satisfy a conveniently simple form. Each \( \hat{H}F(M, s) \) consists of inessential curves, or curves that are null-homotopic after allowing homotopies through the basepoints, and a single essential curve \( \tau_s \) that is homotopic to the homological longitude when allowing homotopies through the basepoints \([HRW18, \text{Corollary 63}]\). The pegboard representative of \( \hat{H}F(M, s) \) encodes many numerical invariants of \( J \). For example, the genus is given by the maximum height \( \lceil \text{height} \rceil \) of \( \tau_s \), around which the pegboard representative of \( \hat{H}F(M, s) \) wraps in \( \overline{T}_M \), yielding \( g(J) = [s] \). Additionally, the height \( \text{height} \) of the first \( \tau_s \) around which \( \tau \) wraps is precisely the Oszváth-Szabó invariant \( \tau(J) \). Hom’s \( \epsilon \) invariant may also be determined from \( \hat{H}F(M, s) \) by observing \( \tau \) after it wraps around this basepoint: the curve turns upwards, downwards, or continues straight corresponding to \( \epsilon(J) \) being 1, -1, and 0, respectively. Notice that the essential curve \( \tau \) only continues straight if \( \tau(J) = 0 \). Together, these invariants determine the slope of the curve \( \tau \) in \( \overline{T}_M \setminus R \), given by \( 2\tau(J) - \epsilon(J) \).

Recall that \( n_{-i} = n_i \) due to the invariance of \( \hat{H}F(M) \) under the action of the hyperelliptic involution. It will be particularly useful to characterize those knots whose complements have curve invariants with minimal \( n_i \) for all \( i \in \mathbb{Z} \). The following lemma is surely known to experts of the field, but is included here for clarity. It also serves as an example of the computations with immersed curves to come.

**Lemma 2.10.** Let \( M = S^3 \setminus \nu J \) with \( J \) non-trivial. Then \( J \) is an L-space knot if and only if \( \hat{H}F(M, s) \) pulls tight to a curve with \( n_i = 1 \) for \( |i| < g(J) \) and \( n_i = 0 \) for \( |i| \geq g(J) \).

**Proof.** When \( J \) is a genus \( g \) knot with an L-space surgery \( S^3_p(J) \), bymirroring if necessary we may take \( p \) to be positive. A surgery exact triangle argument shows that \( S^3_{p+1}(J) \) is an L-space surgery, and likewise for \( S^3_k(J) \) with integral \( k > p \). For some \( k > 2g - 1 \), Theorem 2.1 then additionally provides that

\[
\hat{H}F(S^3_k(J), [s]) \cong \hat{A}_s
\]

for all \( s \in \mathbb{Z} \). Then each \( \hat{A}_s \cong \mathbb{F} \) since \( S^3_k(J) \) is an L-space. We can view \( S^3_k(J) \) as the \( +k \)-sloped pairing of \( D^2 \times S^1 \) to \( M \), so that Theorem 2.6 guarantees

\[
\hat{H}F(S^3_k(J)) \cong HF(\hat{H}F(M), h(\hat{H}F(D^2 \times S^1)))
\]

where \( h \) is the surgery map.

Analogous to the discussion following the pairing theorem involving the \( S^3_2(T(2,5)) \) example, precisely \( k \) lifts of the \( +k \)-sloped curve \( \hat{H}F(D^2 \times S^1) \) are required to lift all intersections in \( T_M \).
to $\overline{T}_M$, and each lift is in correspondence to precisely one Spin$^c$ structure of Spin$^c(S^3(J))$. These differ in height by one in $\overline{T}_M$, and each lift must intersect $\sigma_{s \delta}$ at least once. Recall that $\sigma_{s \delta}$ is homotopic to the homological longitude of $M$, which for an $S^3$ knot complement is the curve $([-\frac{1}{2}, \frac{1}{2}] \times \{0\})/\sim$ in $\overline{T}_M$. Then if $\tilde{HF}(M)$ contains any inessential curve segments homotopic to some lift $\pi_i$, we would have $\dim \tilde{HF}(S^3(J), [s]) > 1$ for some $s$. As $S^3(J)$ is an L-space, we must have $n_i = 1$ for $|i| < g(J)$ and $n_i = 0$ otherwise.

If $\tilde{HF}(M, s)$ has a pegboard representative satisfying $n_i = 1$ for $|i| < g(J)$ and $n_i = 0$ for $|i| \geq g(J)$, then by mirroring if necessary we may suppose $\tau(J) > 0$. The invariant $\tilde{HF}(M, s)$ consists of just $\sigma_{s \delta}$, and has slope $2\tau(J) + \epsilon(J)$ outside of $R$ and pulls tight to vertical segments between adjacent basepoints $\sigma_{s \delta}$ for $|s| \leq \frac{2g(J)-1}{2}$.

When pairing with $\tilde{HF}(D^2 \times S^1)$ having slope $k > 2\tau(J) - \epsilon(J)$, each lift intersects $\tilde{HF}(M, s)$ at most once. Therefore $S^3(J)$ is an L-space for $k > 2\tau(J) - \epsilon(J)$, and so $J$ is an L-space knot. □

Remark. For an L-space knot $J$, we have $|\tau(J)| = g(J)$ and so the pegboard representative of $\tilde{HF}(M)$ takes on one of two mirrored forms depending on the sign of $\tau(J)$. These are illustrated in Figure 6 for a genus two knot.

3. Pairings and immersed curves

As alluded to in the introduction, if a closed, oriented 3-manifold $X$ contains a Klein bottle, then we may view $X$ as the $\nu$-gluing of $Y = M \cup_h N$ of a rational homology solid torus $M$ and the twisted I-bundle over the Klein bottle $N$. Alternatively, we can view $M = Y \setminus \nu J$ as a knot manifold, that is the complement of a knot $J$ in $Y$ some rational homology sphere. We specialize to gluings of $N$ to $S^3$ knot complements, and determine which knot complements pair to obtain manifolds $X$ that are realizable as 8-surgery on a genus two knot. We first establish conventions for the gluing map $h$, and then approach the lemmas required to prove the main theorem.

3.1. Pairings. We will typically order the summands of homology with the summand generated by the rational longitude first, such as in $H_1(\partial N) \cong \mathbb{Z}[\phi_0] \oplus \mathbb{Z}[\phi_1]$. To study pairings in depth, let us introduce notational conventions.

Definition 3.1. Let $X = (S^3 \setminus \nu J) \cup_h N$ be the gluing of $N$ to the complement $S^3 \setminus \nu J$, where the orientation-reversing gluing induces $h_*$ on homology given by

$$[h_*] = \begin{pmatrix} q & r \\ p & s \end{pmatrix}.$$ 

We also say $h$ is a slope $p/q$ gluing/pairing, corresponding to the slope of $h_*(\phi_0)$.

We are interested in gluings that yield $H_1(X) \cong \mathbb{Z}/8\mathbb{Z}$, so consider the following proposition.

Proposition 3.2. Let $h$ be defined as above. Then $|H_1(X)| = 8$ if only if $|p| = 2$, and we have

$$H_1(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} & s \equiv 0 \pmod{2} \\ \mathbb{Z}/8\mathbb{Z} & s \not\equiv 0 \pmod{2} \end{cases}$$

Proof. Recall the parameterization on $\partial N$ by the rational longitude $\phi_0$ and our chosen dual curve $\phi_1$, so that $H_1(\partial N) \cong \mathbb{Z}[\phi_0] \oplus \mathbb{Z}[\phi_1]$. As a slight abuse of notation, let $\phi_0$ and $\phi_1$ also denote the inclusion of these slopes in $H_1(N)$. Since the inclusion of $\phi_1$ to $H_1(N)$ is not primitive but twice some primitive curve $x$, we have $H_1(N) \cong \mathbb{Z}/2\mathbb{Z}[\phi_0] \oplus \mathbb{Z}[x]$ [Wat12 Subsection 3.1]. For
the knot complement, \( H_1(\partial(S^3 \setminus \nu J)) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{2\mu} \) and \( H_1(S^3 \setminus \nu J) \cong \mathbb{Z}_{2\mu} \), with the inclusions \([\mu]\) primitive and \([\lambda]\) trivial.

The Mayer-Vietoris sequence for \( X = (S^3 \setminus \nu J) \cup_h N \) then implies
\[
H_1(X) \cong \frac{(H_1(S^3 \setminus \nu J) \oplus H_1(N))}{f_*(H_1(\partial N))},
\]
where \( f_* \) maps \( H_1(\partial N) \) into \( H_1(N) \) by inclusion and into \( H_1(S^3 \setminus \nu J) \) through \( h_* \) and inclusion. The quotient identifies \( \phi_0 \sim q\lambda + p\mu \) and \( \phi_1 = 2x \sim r\lambda + s\mu \). Since \( \phi_0 \) has order two and \( \lambda \) has order one, \( H_1(X) \) has the following presentation:
\[
H_1(X) \cong \{(\lambda, \mu, \phi_0, x) \mid \lambda = 0, 2\phi_0 = 0, \phi_0 = q\lambda + p\mu, 2x = r\lambda + s\mu\}
\cong \{(\mu, x) \mid 2p\mu = 0, 2x = s\mu\}.
\]
From this we see that \(|H_1(X)| = 8\) if and only if \(|p| = 2\), and that \( H_1(X) \) is cyclic when \( s \neq 0 \) (mod 2).

3.2. Dehn twisting invariance. The type D structure \( \tilde{CFD}(N) \) enjoys a form of Dehn twisting invariance that leaves the homotopy class of the associated immersed curve \( \tilde{HF}(N) \) unchanged. A Heegaard Floer homology solid torus \( M \) is a rational homology solid torus satisfying
\[
\tilde{CFD}(M, \mu_M, \lambda_M) \cong \tilde{CFD}(M, \mu_M + \lambda_M, \lambda_M),
\]
with \( \lambda_M \) the rational longitude of \( M \) and \( \mu_M \) any slope dual to \( \lambda \). The twisted \( I \)-bundle over the Klein bottle is shown to be a Heegaard Floer homology solid torus in [BGW, Proposition 7], and inspection of \( \tilde{HF}(N) \) in Figure 3 reveals that the curve invariant can be homotoped (without crossing the basepoint) to lie within a neighborhood of the rational longitude.

Performing \( n \) Dehn twists along \( \phi_0 \), and then gluing \( N \) is equivalent to post-composing \([h_*]\) with
\[
[T_n] = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \text{yielding} \quad [h_* \circ T_n] = \begin{pmatrix} q & r+np \\ p & s+np \end{pmatrix}.
\]
Gluing \( S^3 \setminus \nu J \) and \( N \) by either map yields manifolds with the same hat-flavor of Heegaard Floer homology [HRW2, Corollary 27], and also preserves the mod \( p \) residue class of \([s]\). So we restrict attention to the maps \( h \) with \( 0 \leq s < p \).

The manifolds of interest are those with \( H_1(X) \cong \mathbb{Z}/8\mathbb{Z} \), and so we need only consider \( s \equiv 1 \) (mod 2) due to Proposition 3.2. We will see that the pairing slope \( \pm 2/q \) must in fact be integral when \( J \) is non-trivial, so we will take \( q = \pm 1 \). The Dehn twisting invariance of \( \tilde{CFD}(N) \) allows us to choose \( s = \mp 1 \), and the prototypical gluings \( X \) will then be achieved by gluing \( N \) along
\[
[h^{\pm 2}] = \begin{pmatrix} \pm 1 & 0 \\ 2 & \mp 1 \end{pmatrix}.
\]

3.3. Proving Theorem 1.1. Suppose \( X = (S^3 \setminus \nu J) \cup_h N \) is realizable as \( S^3(K) \) with \( g(K) = 2 \). The primary strategy is to use Theorem 2.6 to interpret \( \dim \tilde{HF}(X, t) \) as the minimal intersection count of \( \tilde{HF}(S^3 \setminus \nu J, s) \) and some lift of \( h(\tilde{HF}(N, s_k)) \), where \( t \in \text{Spin}^c(X) \). Together with Proposition 2.2 requiring five of these eight Spin\(^c\) structures \( t \) to support trivial \( \tilde{HF}(X, t) \), we can constrain the form of \( \tilde{HF}(S^3 \setminus \nu J) \). This will ultimately determine \( J \), and determine \( K \) up to knot Floer homology.

We first investigate pairings with \( J \) trivial, and then turn to the case where \( J \) is non-trivial. Continuing as before, let \( M \) will denote the knot complement \( M = S^3 \setminus \nu J \).

Lemma 3.3. Suppose \( X = (Y \setminus \nu J) \cup_h N \) contains a Klein bottle and is realized as \( S^3(K) \) with \( g(K) = 2 \). If \( J \) is trivial, then \( Y = S^3 \), \( K = T(2, 5) \), and \( X = (-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) as a Seifert fibered manifold.

Proof. We know from Corollary 2.4 that \( M \) is irreducible, and so \( Y = S^3 \) and \( M = D^2 \times S^1 \). Such a pairing \( X \) is a Dehn filling \( N(\alpha) \), where \( \alpha \) is a slope on \( \partial N \).

The twisted \( I \)-bundle over the Klein bottle \( N \) has a Seifert structure with base orbifold \( D^2(2, 2) \) [LW14], and we may parametrize \( \partial N \) using \{\( \phi_0, \phi_1 \)\} as before, where \( \phi_0 = \lambda_N \) is the rational longitude of \( N \) and \( \phi_1 \) is our preferred choice of curve dual to \( \phi_0 \). Recall that \( N \) is a
Heegaard Floer homology solid torus, which implies that the Dehn filling $N(\alpha)$ is an L-space for all $\alpha \neq \phi_0$ [HRW18, Theorem 26]. Thus, $\widehat{HF}(K) \cong \widehat{HF}(T(2, 5))$ since $K$ is a genus two L-space knot.

We have $N(\phi_1) = \mathbb{RP}^3 \# \mathbb{RP}^3$ and $I_2(N(\phi_0))$ non-finite, and so we can consider $\alpha \neq \phi_0, \phi_1$. Any Dehn filling $N(\alpha)$ for which $\alpha \neq \phi_0, \phi_1$ admits a pair of Seifert structures with base orbifolds $\mathbb{RP}^2(\Delta(\alpha, \phi_0))$ and $S^3(2, 2, \Delta(\alpha, \phi_1))$ [LW14]. The filling $N(\alpha)$ is a lens space when $\Delta(\alpha, \phi_1) = 1$, and otherwise has finite, non-cyclic $\pi_1(X)$ since $\partial N$ compresses in $S^3 \setminus \nu J$. We handle first the possibility that $X \cong L(8, q)$ using the $d$-invariants, also known as the Heegaard Floer correction terms, from [OS03].

The $d$-invariants $d(L(p, q, [s])$ for lens spaces are well known and may be computed recursively using [OS03, Proposition 4.8], together with $d(-Y, s) = -d(Y, s)$ and $d(L(1, 1), 0) = 0$. These invariants for 8-surgeries are determined in [NW15, Theorem 26]. Thus, $\widehat{HF}(K) \cong \widehat{HF}(T(2, 5))$ since $K$ is an L-space.

We have $d(L(8, \pm 1), [s]) = \pm \frac{7}{4}$ for some $s \in \mathbb{Z}$ and $d(L(8, \pm 3), [t]) = \pm \frac{5}{8}$ for some $t \in \mathbb{Z}$. Both of these differ from any $d(S^3_8(K), [s])$, and so $X$ is not an L-space.

In [Doi15, Theorem 2], Doig classifies finite, non-cyclic surgeries for $|r| \leq 9$, with $r$ the surgery slope. Among these, the manifolds with $|I_2(X)| = 8$ are the dihedral manifolds $\pm S^3_8(T(2, 3)) = (-1; \frac{1}{2}, \frac{3}{2}, \frac{7}{2})$, and $\pm S^3_8(T(2, 5)) = (-1; \frac{1}{2}, \frac{1}{2}, \frac{7}{2})$ since eight is a characterizing slope for $T(2, 5)$ due to [NZ18]. For $T(2, 3)$, we have $V_0(T(2, 3)) = 0$ and so $d(\pm S^3_8(T(2, 3), [1]) = -d(S^3_8(T(2, 3), [1]) = -7/8$, which differs from any $d(S^3_8(K), [s])$ when $K$ has the same knot Floer homology as $T(2, 5)$. Therefore, we must have that $K = T(2, 5)$.

Now focusing on non-trivial knots $J$, we will show that the pairing slope must be integral before handling the crucial lemma for proving the main theorem.

There is a convenient visual way to track Floer homology associated to a given Spin$^c$ structure of $X$, that we have already encountered in the example of Figure 2, and in proving Lemma 2.10. By Theorem 2.6, we have $\widehat{HF}(X, t) \cong \widehat{HF}(\widehat{HF}(M, s), h(\widehat{HF}(N), s_k))$, where restriction gives $\pi(t) = s \times s_k$. Intersection points $x$ and $y$ from pairing contribute homology associated to the same Spin$^c$ structure $t$ with $\pi(t) = s \times s_k$ if and only if there exist paths $p_0$ from $x$ to $y$ in $\widehat{HF}(M, s)$ and $p_1$ from $x$ to $y$ in $h(\widehat{HF}(N, s_k))$ such that the concatenation of $p_0$ with $-p_1$ lifts to a closed, piecewise smooth path in $\tilde{T}$ [HRW18, Section 2]. When $h$ has slope $\pm 2/q$, single lifts $h(\widehat{HF}(N, s_k))$ of the component of $h(\widehat{HF}(N))$ corresponding to $s_k$ will fail to lift any intersection in $T_M$ generated by this component. We require two lifts of the components of $h(\widehat{HF}(N))$, totaling eight curves visible in $\tilde{T}_M$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The 2/3-sloped curves of $N$ associated to $s_1$ in $T_M$.}
\end{figure}
This situation is depicted in Figure 7 for the curves of \( N \) associated to \( s_1 \) with filling slope \( 2/3 \), and Figure 8 provides an example of the four required curves of \( h(\bar{HF}(N, s_1)) \) in the \( +2 \)-sloped pairing of the twisted \( I \)-bundle over the Klein bottle with the right-handed trefoil complement. Notice that the lifted curves cannot share the same Spin\(^c\) grading as there is no path between the curves in \( h(\bar{HF}(N, s_k)) \). For this reason, we can associate the eight \( t \in \text{Spin}^c(X) \) with these lifted curves of \( h(\bar{HF}(N, s_1)) \) and \( h(\bar{HF}(N, s_1)) \) where convenient.

**Proposition 3.4.** If \( X = (S^3 \setminus \nu J) \cup_h N \) contains a Klein bottle and is realized as \( S_0^2(K) \) with \( g(K) = 2 \) and \( J \) non-trivial, then the filling slope of \( h \) is \( \pm 2 \).

**Proof.** Suppose for the sake of contradiction that \( X \) contains a Klein bottle and is realized as \( S_0^2(K) \) with \( g(K) = 2 \), where the filling slope of \( h \) is \( 2/q \) for \( |q| > 1 \) due to Proposition 3.2. Further, assume the immersed curves \( \bar{HF}(M, s) \) and \( \bar{HF}(N, s_k) \) are pulled tight to pegboard form, thereby ensuring that their intersection Floer homology is minimal. Following the discussion above, we think of the eight Spin\(^c\) structures of \( X \) in correspondence with the eight curves needed to lift all intersections in \( T_M \).

If \( |q| > 1 \), then the filling slope of \( h \) satisfies \( |2/q| < 1 \). It is then immediate that all four lifts of the loose curves \( h(\bar{HF}(N, s_0)) \) intersect each \( \bar{p}_i \) more than once. This would imply \( \bar{HF}(X, t) \geq 2n_i \) for every \( t \in \text{Spin}^c(X) \), and so Proposition 2.2 forces \( n_i = 0 \) for all \( |i| < g(J) \). This condition is satisfied only by the unknot, which is the contradiction we sought. \( \square \)

Now that the pairing slope is required to be integral, we prove the final remaining lemma that will enable us to establish Theorem 1.1. Recall \( h^{\pm 2} \) defined towards the end of Subsection 3.2.

**Lemma 3.5.** Let \( J \) be non-trivial and consider the pairings \( X^{\pm 2} = (S^3 \setminus \nu J) \cup_{h^{\pm 2}} N \). If \( X^{\pm 2} = S_0^2(K) \) with \( g(K) = 2 \), then \( J = T(2, 0, \pm 3) \).

**Proof.** Let \( M = S^3 \setminus \nu J \) and pull \( \bar{HF}(M, s) \) tight to pegboard form. Further, recall the discussion about distinguishing the curves that generate Floer homology associated to a given \( t \in \text{Spin}^c(X) \) in the proof of Lemma 3.4. From Theorem 2.6 we have
\[
\bar{HF}(X^{\pm 2}, t) \cong HF(\bar{HF}(M, s), h^{\pm 2}(\bar{HF}(N, s_k))),
\]
where \( t \in \pi^{-1}(s \times s_k) \). When \( \bar{HF}(M, s) \) is pulled tight, all intersections of components of \( h^{\pm 2}(\bar{HF}(N, s_k)) \) with \( \bar{HF}(M, s) \) are with either its essential curve \( \bar{\gamma}_s \) outside of the \( \bar{R}_j \)'s, or within the neighborhoods \( \bar{R}_j \) containing the lifts \( \bar{\mu}_i \). Any curve of \( h^{\pm 2}(\bar{HF}(N, s_k)) \) intersects precisely one lift \( \bar{\mu}_i \) each time it passes through a region \( \bar{R}_j \). Each such occurrence of intersection with the \( n_i \) segments of \( \bar{HF}(M, s) \), we have a contribution of \( n_i \) to \( \text{dim} \bar{HF}(X) \) in the associated Spin\(^c\) structure to that curve of \( \bar{HF}(N, s_k) \).

Notice that each \( \bar{p}_i \) has a lift in \( \bar{T} \) that intersects two of the four lifted curves of \( h^2(\bar{HF}(N, s_1)) \), which is showcased in Figure 8. Further, each \( \bar{p}_i \) has a lift in \( \bar{T} \) that intersects two of the four lifted curves of the loose component \( h^2(\bar{HF}(N, s_0)) \). For all \( i \in \mathbb{Z} \), we have \( n_i \) contributing to \( \text{dim} \bar{HF}(X, t) \) in at least four different Spin\(^c\) structures. Proposition 2.2 then forces \( n_i \leq 1 \) for all \( i \in \mathbb{Z} \). The presence of an inessential curve component of \( \bar{HF}(M, s) \) would force some \( n_i > 1 \), and so \( \bar{HF}(M, s) \) must not have any such components. Therefore, \( J \) is an \( L \)-space knot by Lemma 2.10.

In the remark following that lemma we see that

**Figure 8.** Intersections of the lifts of \( h^2(\bar{HF}(N, s_1)) \) with the lifts of \( \bar{p}_i \).
the pegboard representative of $\widehat{HF}(M) = \tau_s$ for an L-space knot complement is completely determined by $\tau(J)$.

If $|\tau(J)| > 1$, four lifted components of $h^2(\widehat{HF}(N))$ intersect $\tau_s$ more than once. Two of the four lifted curves of $h^2(\widehat{HF}(N, s_1))$ pairing to yield excess Floer homology are shown in Figure 9 when $\tau(J) = 2$. For this case, the other two lifted curves that pair to yield excess Floer homology are lifts of $h^2(\widehat{HF}(N, s_0))$ that intersect lifts of $\tau_{\pm 1}$. When $|\tau(J)| > 2$, all eight lifts of $h^2(\widehat{HF}(N, s_1))$ intersect $\tau_s$ multiple times. Once again, Proposition 2.2 forces $|\tau(J)| = 1$ for $X$ to be realizable as 8-surgery on a genus two knot.

Then as an L-space knot with $|\tau(J)| = g(J) = 1$, we have $J$ is fibered and so $J$ is a trefoil. Among the four possible pairings, $X^{+2}$ requires $J = T(2, 3)$ and $X^{-2}$ requires $J = T(2, -3)$ in order to adhere to Proposition 2.2. In both cases the situation for $X$ holds for integer homology sphere L-spaces different from $Y$. Therefore $K$ is a genus two L-space knot, and so $\widehat{HFK}(K) \cong \widehat{HFK}(T(2, 5))$. □

We now prove Theorem 1.1.

Proof. Lemma 3.3 using Doig’s classification handles the case when $J$ is trivial, implying $X = (-1; \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ as a Seifert fibered manifold. Such an $X$ is an L-space and self is a characterizing slope for $T(2, 5)$ due to [NZ18], and so $K = T(2, 5)$. If $J$ is non-trivial, then Proposition 3.4 requires $X$ to be obtained by pairing $N$ with $M$ along $h^{\pm 2}$. Lemma 3.5 implies $J$ is a trefoil and that $X$ is an L-space, and so $\widehat{HFK}(K) \cong \widehat{HFK}(T(2, 5))$ as well. In fact, the latter case yields an infinite family of manifolds with equivalent Floer homology by Dehn twisting $N$ along $\phi_0$, as mentioned in Subsection 3.2.

Determining the $d$-invariants associated to these pairings should provide a way to further refine the possibilities for surgeries. While Dehn twisting invariance allows for an infinite family of pairings with $N$ to have equivalent relatively-graded $\widehat{HF}(X)$, we should suspect that the $d$-invariants change.

4. The situation for $Y \neq S^3$

As alluded to in the introduction, the results in this paper are stated for knots $J$ in $S^3$ but hold more generally for knots in integer homology sphere L-spaces. When $Y$ is an integer homology sphere L-space different from $S^3$, we can use Corollary 2.4 together with a corollary of Baldwin and Vela-Vick. If $J \subset Y$ is a nullhomologous knot with irreducible complement and $\dim \widehat{HFK}(Y, K) = 3$, their work implies that $Y \setminus J \cong S^3 \setminus \nu T(2, \pm 3)$ [BVV18]. So when $Y \neq S^3$, we must have that $Y$ is the Poincaré homology sphere. We used Floer homology considerations to constrain the form of $\widehat{HF}(Y \setminus J)$, which together with known properties coming from the algorithm from [HRW18] Proposition 47 allowed us to determine the possible knots $J$. This algorithm is the immersed curves form of a theorem of Lipshitz, Ozsváth, and Thurston that computes the type D structure for a knot complement from $\text{CFK}^-(Y, K)$, which holds for integer homology sphere L-spaces $Y$ [LOT18]. In the absence of a full analog of this algorithm, we could potentially handle the remaining cases for $Y$ just by knowing what form the essential curve components must take in $T_M$. In particular, in [KWZ20] Section 5 it is suggested that the essential curve component potentially picks up a non-trivial local system.
Still, we can narrow down some properties of $Y$ through homological means. Consider the pairing $X = M \bigcup \mathcal{N}$, where $M$ is a rational homology solid torus and $h$ is a slope $p$ orientation-reversing homeomorphism on $\partial \mathcal{N}$, defined as in Definition \[3.1\] Let $\lambda_M$ denote the rational longitude of $M$, and $\mu_M$ the respective dual curve. Then we have

$$|H_1(X)| = 4d|H| \Delta(\lambda_M, \mu_M),$$

where $d = o(\lambda_M)$ in $H_1(M)$ and $H$ is the torsion subgroup of $H_1(M)$ \[BGW13\] Section 3. Since $h$ has gluing slope $p$, we have $\Delta(\lambda_M, \mu_M) = p$, and so $|H_1(X)| = 4pd|H| = 8$.

Recall that $\lambda_M$ generates $\text{Ker}(H_1(\partial M) \to H_1(M)) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. The rational longitude includes $i_*(\lambda_M) \in H \subset H_1(M)$ with finite order (and is unique among slopes in $H_1(\partial M)$ with this property \[Wat13\]). Then we must have $d = 1$, as otherwise $d > 1$ implies $|H| \geq d > 1$. Thus, $\lambda_M$ includes as a null-homologous curve.

Suppose $|H| = 2$, so that $p = 1$. A similar computation to that in the proof of Proposition \[3.2\] shows that the induced map $h_*$ in the Mayer-Vietoris sequence for $X = M \bigcup N$ does not interact with $H$. This implies that $H_1(X)$ carries $H$ as a free summand. However $|H| = 2$ is not relatively prime to the orders of the other summands of $H_1(X)$, and so we must have $|H| = 1$ to have a pairing with cyclic $H_1(X)$. We are left with $|H| = 1$ and $p = 2$, which has $M$ as the exterior of a knot in an integer homology sphere and the curve invariant $\overline{HF}(M)$ lifts to the familiar infinite cylindrical cover $\mathcal{T}_M$.

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