Counting self-conjugate \((s, s+1, s+2)\)-core partitions

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Abstract
We are concerned with counting self-conjugate \((s, s+1, s+2)\)-core partitions. A Motzkin path of length \(n\) is a path from \((0, 0)\) to \((n, 0)\) which stays weakly above the \(x\)-axis and consists of the up \(U = (1, 1)\), down \(D = (1, -1)\), and flat \(F = (1, 0)\) steps. We say that a Motzkin path of length \(n\) is symmetric if its reflection about the line \(x = n/2\) is itself. In this paper, we show that the number of self-conjugate \((s, s+1, s+2)\)-cores is equal to the number of symmetric Motzkin paths of length \(s\), and give a closed formula for this number.

Keywords Core partition · Self-conjugate · Motzkin path

Mathematics Subject Classification 05A15 · 05A17

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1 Introduction

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) be a partition of a positive integer \( n \). The Young diagram of \( \lambda \) is a collection of \( n \) boxes in \( \ell \) rows with \( \lambda_i \) boxes in row \( i \). For example, the Young diagram for \( \lambda = (5, 4, 2) \) is below.

Let the leftmost column be column 1. The box in row \( i \) and column \( j \) is said to be in position \((i, j)\). For the Young diagram of \( \lambda \), the partition \( \lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1}) \) is called the conjugate of \( \lambda \), where \( \lambda'_j \) denotes the number of boxes in column \( j \). A partition whose conjugate is equal to itself is called self-conjugate. For each box in its Young diagram, we define its hook length by counting the number of boxes directly to its right or below, including the box itself. Equivalently, for the box in position \((i, j)\), the hook length of \( \lambda \) is defined by

\[
h(i, j) = \lambda_i + \lambda'_j - i - j + 1.
\]

For a positive integer \( s \), a partition \( \lambda \) is called an \( s \)-core if none of its hook lengths are multiples of \( s \). The study of core partitions arises from the representation theory of the symmetric group \( S_n \) (see [5]). We use the notation of an \((s_1, \ldots, s_k)\)-core if it is simultaneously an \( s_1 \)-core, \ldots, and an \( s_k \)-core.

For a set \( S \) of positive integers, we say that \( a \) is generated by \( S \) if \( a \) can be written as a non-negative linear combination of the elements of \( S \). Let \( P = P_S \) be the set of elements which are not generated by \( S \), and let \((P, <_P)\) be a poset by defining the cover relation so that \( a \) covers \( b \) if and only if \( a - b \in S \). For example, see Fig. 1 for the poset \( P_{\{8,9,10\}} \). For the detailed explanation of poset, we refer the reader to [1,8,11].

For a poset \((P, <_P)\), a set \( I \subset P \) is called a lower ideal of \( P \) if \( a <_P b \) and \( b \in I \) imply \( a \in I \). In [2], Anderson gave a natural bijection between \( s \)-cores and lower ideals of a poset \( P_{\{s\}} \). Moreover, she proved that for relatively prime positive integers \( s \) and \( t \), the number of \((s, t)\)-cores has a nice closed formula by finding a bijection between \((s, t)\)-cores and lattice paths from \((0, 0)\) to \((s, t)\) consisting of north and east steps which stays weakly above the diagonal.

**Fig. 1** The Hasse diagram of \( P_{\{8,9,10\}} \)
Theorem 1 ([2, Theorem 1]) For relatively prime positive integers \( s \) and \( t \), the number of \((s, t)\)-cores is
\[
\frac{1}{s+t} \binom{s+t}{s}.
\]

Since the work of Anderson, the topic counting simultaneous cores has received growing attention (see [1,3,6,9]). In [4], Ford et al. proved the following analog of Anderson’s work.

Theorem 2 ([4, Theorem 1]) For relatively prime positive integers \( s \) and \( t \), the number of self-conjugate \((s, t)\)-cores is
\[
\left( \lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor \right) \left( \lfloor \frac{s}{2} \rfloor \right).
\]

An \((s, k)\)-generalized Dyck path is a path from \((0, 0)\) to \((s, s)\) which stays weakly above the diagonal and consists of the steps \(N_k = (0, k)\), \(E_k = (k, 0)\), and \(D_i = (i, i)\) for \(1 \leq i \leq k - 1\). For example an \((s, 1)\)-generalized Dyck path is a (classical) Dyck path of order \(s\). We say that an \((s, k)\)-generalized Dyck path is symmetric if its reflection about the line \(y = s - x\) is itself. It is often observed that counting the number of simultaneous cores can be described as counting the number of certain paths.

Remark 1 Let \(s\) be a positive integer.
1. The number of \((s, s+1)\)-cores is the \(s\)th Catalan number \(C_s = \frac{1}{s+1} \binom{2s}{s}\) which counts the number of Dyck paths of order \(s\).
2. The number of self-conjugate \((s, s+1)\)-cores is \(\left( \lfloor \frac{s}{2} \rfloor \right)\) which counts the number of symmetric Dyck paths of order \(s\).

In [1], Amdeberhan and Leven expanded Anderson’s results to \((s, s+1, \ldots, s+k)\)-cores.

Theorem 3 ([1, Theorem 4.2]) The following are equal:
(a) The number of \((s, s+1, \ldots, s+k)\)-cores.
(b) The number of \((s, k)\)-generalized Dyck paths.
(c) The number of lower ideals in \(P_{\{s,s+1,\ldots,s+k\}}\).

We note that \((s, 2)\)-generalized Dyck paths are equivalent to Motzkin paths of length \(s\). From Theorem 3, one can obtain the following corollary.

Corollary 1 For a positive integer \(s\), the number of \((s, s+1, s+2)\)-cores is
\[
M_s = \sum_{i=0}^{s} \frac{1}{i+1} \binom{s}{2i} \binom{2i}{i},
\]
the \(s\)th Motzkin number which counts the number of Motzkin paths of length \(s\).
We note that Yang et al. [11] proved Corollary 1 independently.
It is natural to ask whether the number of self-conjugate \((s, s + 1, s + 2)\)-cores and the number of symmetric Motzkin paths of length \(s\) are equal from Remark 1 and Corollary 1. In this paper, we prove that these two quantities are equal by showing that they satisfy the same recurrence relation. Furthermore, we give a closed formula for these numbers.

2 Poset structure for self-conjugate \((s, s + 1, s + 2)\)-cores

In this section, we construct a poset whose lower ideals with some restrictions are corresponding to self-conjugate \((s, s + 1, s + 2)\)-cores, and then give a simple diagram to visualize that poset.

For a partition \(\lambda\), let \(MD(\lambda)\) denote the set of main diagonal hook lengths. Therefore, \(MD(\lambda)\) is a set of distinct odd integers when \(\lambda\) is self-conjugate. In [4], Ford et al. gave a useful result for determining self-conjugate \(s\)-cores.

Proposition 1 [4, Proposition 3] Let \(\lambda\) be a self-conjugate partition. Then \(\lambda\) is an \(s\)-core partition if and only if both of the following hold:

(a) If \(h \in MD(\lambda)\) with \(h > 2s\), then \(h - 2s \in MD(\lambda)\).
(b) If \(h_1, h_2 \in MD(\lambda)\), then \(h_1 + h_2 \not\equiv 0 \pmod{2s}\).

Note that for a self-conjugate \((s, s + 1, s + 2)\)-core partition \(\lambda\), \(h \in MD(\lambda)\) and \(h > 2s + 4\) imply that \(h - 2s, h - (2s + 2), h - (2s + 4) \in MD(\lambda)\) by Proposition 1 (a).

To construct a poset with the partial order \(\preceq\) if and only if the set \(MD(\lambda)\) is a set of distinct odd integers when \(\lambda\) is self-conjugate. In [4], Ford et al. gave a useful result for determining self-conjugate \(s\)-cores.

Proposition 2 Let \(\lambda\) be a self-conjugate partition. Then \(\lambda\) is an \((s, s + 1, s + 2)\)-core partition if and only if the set \(MD(\lambda)\) is a lower ideal of \(\hat{P}_{\{s, s+1, s+2\}}\) with no elements \(h_1, h_2\) such that \(h_1 + h_2 \in \{2s, 2s + 2, 2s + 4\}\).
For a positive integer \( s \), the number of self-conjugate \((s, s + 1, s + 2)\)-cores is equal to the number of self-conjugate \((2s, 2s + 2, 2s + 3)\)-cores.

**Example 1** For a self-conjugate \((8, 9, 10)\)-core partition \( \lambda = (6, 3, 3, 1, 1, 1) \), the set \( MD(\lambda) = \{11, 3, 1\} \) of main diagonal hook lengths is a lower ideal of \( \tilde{P}_{8,9,10} \) with no elements \( h_1, h_2 \) such that \( h_1 + h_2 \in \{16, 18, 20\} \).

To clearly show the conditions in Proposition 1(b), we construct the *modified diagram* for \( \tilde{P}_{8,9,10,11} \) as follows. We rotate \( R_5 \) 180 degrees and then locate it below \( Q_5 \). After that we add a dotted edge to connect a lower element from \( Q_5 \) to a lower element from \( R_5 \) whenever their sum equals \( 2s, 2s + 2, \) or \( 2s + 4 \) so that at most one end point of each dotted edge can be selected for the lower ideal corresponding to an \((s, s + 1, s + 2)\)-core. See Fig. 4 for example.

We note that the sets of minimal elements in \( Q_{2s} \) and \( Q_{2s+1} \) are the same as \( \{1, 3, \ldots, 2s - 1\} \), and the sets of minimal elements in \( R_{2s} \) and \( R_{2s+1} \) are \( \{2s + 3, 2s + 5, \ldots, 4s - 1\} \) and \( \{2s + 5, 2s + 7, \ldots, 4s + 1\} \), respectively. Hence, both \( \tilde{P}_{2s,2s+1,2s+2} \) and \( \tilde{P}_{2s+1,2s+2,2s+3} \) are equivalent to the disjoint union of two posets \( P_{s+1,s+2,s+3} \) and \( P_{s+1,s+2,s+3} \). We also note that \( h_1 + h_2 \in \{4s, 4s + 2, 4s + 4\} \) if and only if \( h_1 + (h_2 + 2) \in \{4s + 2, 4s + 4, 4s + 6\} \) for \( h_1 \in \{1, 3, \ldots, 2s - 1\} \) and \( h_2 \in \{2s + 3, 2s + 5, \ldots, 4s - 1\} \). Hence, the modified diagrams of \( \tilde{P}_{2s,2s+1,2s+2} \) and \( \tilde{P}_{2s+1,2s+2,2s+3} \) are also equivalent.

Thus, we have the following proposition.

**Proposition 3** For a positive integer \( s \), the number of self-conjugate \((s, s + 1, s + 2)\)-cores is equal to the number of self-conjugate \((2s + 1, 2s + 2, 2s + 3)\)-cores.
Fig. 4 The modified diagrams of $\tilde{P}_{[8,9,10]}$ and $\tilde{P}_{[9,10,11]}$

Fig. 5 Symmetric Motzkin paths of length 4

3 Counting self-conjugate simultaneous core partitions

In this section, we give a formula for the number of symmetric Motzkin paths, and then show that the number of self-conjugate $(s, s+1, s+2)$-cores and the number of symmetric Motzkin paths of length $s$ have the same recurrence relation.

3.1 Counting symmetric Motzkin paths

For a fixed $i$, there are $C_i \binom{n}{2i}$ Motzkin paths with exactly $i$ up steps since there are $C_i$ Dyck paths and there are $\binom{n}{2i}$ ways to insert $n-2i$ flat steps into a Dyck path with $i$ up steps. We say that a Motzkin path of length $n$ is symmetric if its reflection about the line $x = n/2$ is itself. Let $S_n$ denote the number of symmetric Motzkin paths of length $n$. For example, $S_0 = 1$, $S_1 = 1$, $S_2 = 2$, $S_3 = 2$, and $S_4 = 5$. Figure 5 shows five symmetric Motzkin paths of length 4.

We note that the $(n+1)$st step of any symmetric Motzkin path of length $2n+1$ must be a flat step, and therefore, there is a natural bijection between symmetric Motzkin paths of length $2n+1$ and that of length $2n$ so that $S_{2n+1} = S_{2n}$. Now, we count the number of symmetric Motzkin paths.

Proposition 4 The number of symmetric Motzkin paths of length $n$ is

$$S_n = \sum_{i \geq 0} \binom{\frac{n}{2} - \frac{i}{2}}{i} \binom{i}{\frac{i}{2}}.$$ 

Proof It is sufficient to enumerate symmetric Motzkin paths of length $2n$. Suppose we are given a symmetric Dyck path with $i$ up steps, so that it has $2n - 2i$ flat steps. To
obtain a symmetric Motzkin path of length \(2n\) with \(i\) up steps, we consider inserting \(n-i\) flat steps into the first half of the given symmetric Dyck path. Since there are \(\binom{n}{i/2}\) symmetric Dyck paths with \(i\) up steps as in Remark 1, and there are \(\binom{n}{i}\) ways to insert flats, the number of symmetric Motzkin paths of length \(2n\) with \(i\) up steps is \(\binom{n}{i}(\binom{n}{i/2})\). Therefore, \(S_{2n} = S_{2n+1} = \sum_{i \geq 0} \binom{n}{i}\binom{n}{i/2}\). □

Now, we consider a recurrence relation of \(S_{2n}\) involving \(M_n\). For a symmetric Motzkin path \(P = P_1 P_2 \cdots P_{2n}\) of length \(2n\), where \(P_i\) denotes the \(i\)th step, let \(k \leq n\) be the largest number such that \(P\) meets the \(x\)-axis at \((k, 0)\). We note that if \(k = n\), then both of \(P_1 P_2 \cdots P_n\) and \(P_{n+1} P_{n+2} \cdots P_{2n}\) are Motzkin paths of length \(n\) which are symmetric to each other. On the other hand, if \(k < n\), then \(P_{k+1} = U, P_{2n-k} = D\), the subpath \(P_{k+2} P_{k+3} \cdots P_{2n-k-1}\) is a symmetric Motzkin path of length \(2n-2k-2\), and both of two subpaths \(P_1 P_2 \cdots P_k\) and \(P_{2n-k+1} P_{2n-k+2} \cdots P_{2n}\) are Motzkin paths of length \(k\) which are symmetric to each other. Hence, we have a relation between \(S_{2n}\) and \(M_n\):

\[
S_{2n} = M_n + \sum_{k=0}^{n-1} S_{2n-2k-2} M_k.
\] (1)

Equation (1) can also be found in the OEIS as A005773 [7].

### 3.2 Counting self-conjugate \((2s, 2s+1, 2s+2)\)-cores

The following lemma plays an important role to obtain a relation for the number of self-conjugate \((2s, 2s+1, 2s+2)\)-cores.

**Lemma 1** For a positive integer \(s\), the number of self-conjugate \((2s, 2s+1, 2s+2)\)-cores \(\lambda\) satisfying that \(2s - 1 \in MD(\lambda)\) is equal to the number of self-conjugate \((2s-2, 2s-1, 2s)\)-cores.

**Proof** We consider Diagrams I, II, III, and IV in Fig. 6. Diagram I shows the modified diagram of \(\tilde{P}_{[2s,2s+1,2s+2]}\). (We use gray color for the elements in the set \(A = \{a \mid 2s + 3 < p a\} \cup \{2s + 5 < p a\}\) and for the lines with an end point belonging to \(A\).) Diagram II is obtained from Diagram I by removing all gray terms. Diagram III is obtained from Diagram II by changing a solid line to a dotted line whenever a line has an end point \(\leq 2s - 3\) and vice versa. Diagram IV shows the modified diagram of \(\tilde{P}_{[2s-2,2s-1,2s]}\).

By Proposition 2, there is a bijection between self-conjugate \((2s, 2s+1, 2s+2)\)-cores \(\lambda\) with \(2s - 1 \in MD(\lambda)\) and lower ideals \(I\) of \(\tilde{P}_{[2s,2s+1,2s+2]}\) containing \(2s - 1\) and no elements \(h_1, h_2\) such that \(h_1 + h_2 \in \{4s, 4s+2, 4s+4\}\). Since \(2s - 1 \in I\) implies that \(2s + 3, 2s + 5 \not\in I\), it is sufficient to consider lower ideals of the induced subposet of \(\tilde{P}_{[2s,2s+1,2s+2]}\) which is obtained by excluding three elements \(2s - 1, 2s + 3\), and \(2s + 5\) from \(\tilde{P}_{[2s,2s+1,2s+2]}\). Therefore, we consider Diagram II instead of Diagram I.

Note that if we flip Diagram III, then it is equivalent to Diagram IV. To prove the lemma, we show that Diagrams II and III have the same number of lower ideals by constructing a bijection \(\phi\) between the set of lower ideals \(I\) of Diagram II and the set of lower ideals \(J\) of Diagram III.

If \(I\) is a lower ideal of Diagram II, then \(I\) satisfies the following:
Fig. 6 The modified diagrams for Lemma 1
Counting self-conjugate \((s, s + 1, s + 2)\)-core partitions

Diagram III in Fig. 7.

It is easy to check that \(\varphi\) is the bijection defined in the proof of Lemma 1.

**Example 2** For self-conjugate \((8, 9, 10)\)-core partitions \(\lambda\) such that \(7 \in MD(\lambda)\), let \(I_1, \ldots, I_{13}\) be their corresponding lower ideals of Diagram II associated with \(\tilde{P}_{[8,9,10]}\). Now, we list \(I_i\) and \(J_i = \varphi(I_i)\) for \(i = 1, \ldots, 13\), where \(\varphi\) is the bijection defined in the proof of Lemma 1.

\[
\begin{align*}
I_1 &= \emptyset & J_1 &= \emptyset & I_2 &= \{1\} & J_2 &= \{1\} \\
I_3 &= \{3\} & J_3 &= \{3\} & I_4 &= \{5\} & J_4 &= \{5\} \\
I_5 &= \{1, 3\} & J_5 &= \{1, 3\} & I_6 &= \{1, 5\} & J_6 &= \{1, 5\} \\
I_7 &= \{3, 5\} & J_7 &= \{3, 5\} & I_8 &= \{1, 3, 5\} & J_8 &= \{1, 3, 5\} \\
I_9 &= \{1, 3, 5, 21\} & J_9 &= \{21\} & I_{10} &= \{1, 3, 5, 21, 23\} & J_{10} &= \{21, 23\} \\
I_{11} &= \{3, 5, 23\} & J_{11} &= \{23\} & I_{12} &= \{1, 3, 5, 23\} & J_{12} &= \{1, 23\} \\
I_{13} &= \{15\} & J_{13} &= \{1, 3, 5, 15\}
\end{align*}
\]

We note that for each \(i\), \(I_i\) is a lower ideal of Diagram II and \(J_i\) is a lower ideal of Diagram III in Fig. 7.

The following proposition is a generalization of Lemma 1.

**Proposition 5** Let \(s\) and \(k\) be positive integers such that \(k \leq s\).

(a) The number of self-conjugate \((2s, 2s + 1, 2s + 2)\)-cores \(\lambda\) with \(1, 3, \ldots, 2s - 1 \notin MD(\lambda)\) is \(M_s\).

(b) The number of self-conjugate \((2s, 2s + 1, 2s + 2)\)-cores \(\lambda\) satisfies that

\[
2k - 1 \in MD(\lambda) \quad \text{and} \quad 2k + 1, 2k + 3, \ldots, 2s - 1 \notin MD(\lambda)
\]

is the number of self-conjugate \((2k - 2, 2k - 1, 2k)\)-cores multiplied by \(M_{s-k}\).
Proof (a) The number of self-conjugate \((2s, 2s + 1, 2s + 2)\)-core partitions \(\lambda\) with \(1, 3, \ldots, 2s - 1 \notin MD(\lambda)\) is equal to the number of lower ideals \(I\) of \(\tilde{P}_{(2s, 2s+1, 2s+2)}\) with \(1, 3, \ldots, 2s - 1 \notin I\). Note that if \(1, 3, \ldots, 2s - 1 \notin I\), then \(I\) is a lower ideal of \(R_{2s}\) which is equivalent to the poset \(P_{(s,s+1,s+2)}\). It follows from Corollary 1 that the number of lower ideals \(I\) with \(1, 3, \ldots, 2s - 1 \notin I\) is \(M_s\).

(b) Diagram A in Fig. 8 shows the modified diagram of \(\tilde{P}_{(2s, 2s+1, 2s+2)}\). (We use gray color for the elements in the set \(C = \{a | 2k + 2i - 1 < \rho \ a \text{ for } i = 1, 2, \ldots s - k\} \cup \{2k - 1\} \) and for the lines with an end point belonging to \(C\).) Diagram B in Fig. 8 is obtained from Diagram A by removing all gray terms. More precisely, Diagram B is the disjoint union of two posets, say \(B_1\) and \(B_2\), where \(B_1\) is the maximal induced subposet of \(\tilde{P}_{(2s, 2s+1, 2s+2)}\) whose minimal elements are \(1, 3, \ldots, 2k - 1, 4s - 2k + 7, \ldots, 4s - 1\) and \(B_2\) is the maximal induced subposet of \(\tilde{P}_{(2s, 2s+1, 2s+2)}\) whose minimal elements are \(2s + 3, 2s + 5, \ldots, 4s - 2k - 1\). To enumerate lower ideals \(I\) of \(\tilde{P}_{(2s, 2s+1, 2s+2)}\) with \(2k - 1 \in I\) and \(2k + 1, 2k + 3, \ldots, 2s - 1 \notin I\), it is sufficient to consider Diagram B instead of Diagram A. Similar to the proof of Lemma 1, it can be proven that the sets of lower ideals in \(B_1\) and \(\tilde{P}_{(2k-2,2k-1,2k)}\) are equinumerous. Since \(B_2\) is equivalent to the poset \(P_{(2s-2k-2,2s-2k-1,2s-2k)}\), the number of lower ideals \(I\) satisfying \(2k - 1 \in I\) and \(2k + 1, 2k + 3, \ldots, 2s - 1 \notin I\) is \(M_{s-k}\) times the number of self-conjugate \((2k - 2, 2k - 1, 2k)\)-cores by Corollary 1. \(\square\)
Now, we are ready to prove our main theorem.

**Theorem 4** For a positive integer \( s \), the number of self-conjugate \((s, s+1, s+2)\)-cores is

\[
\sum_{i \geq 0} \left( \left\lfloor \frac{s}{2} \right\rfloor \binom{i}{\left\lfloor \frac{i}{2} \right\rfloor} \right),
\]

which counts the number of symmetric Motzkin paths of length \( s \).

**Proof** Let \( a_s \) denote the number of self-conjugate \((s, s+1, s+2)\)-cores. From Proposition 3, we have \( a_{2s+1} = a_{2s} \). Hence, it is sufficient to show that \( a_{2s} = S_{2s} \).

For \( 1 \leq k \leq s \), let \( A_k \) be the set of self-conjugate \((2s, 2s+1, 2s+2)\)-cores \( \lambda \) that satisfies

\[
2k - 1 \in MD(\lambda) \quad \text{and} \quad 2k + 1, 2k + 3, \ldots, 2s - 1 \notin MD(\lambda),
\]

and let \( A_0 \) be the set of self-conjugate \((2s, 2s+1, 2s+2)\)-cores \( \lambda \) with \( 2i - 1 \notin MD(\lambda) \) for \( 1 \leq i \leq s \). Then, \( A_0 \cup A_1 \cup \cdots \cup A_s \) is the set of self-conjugate \((2s, 2s+1, 2s+1)\)-cores and

\[
a_{2s} = |A_0| + |A_1| + \cdots + |A_s|.
\]

From Proposition 5, we have \( |A_0| = M_s \) and \( |A_k| = a_{2k-2}M_{s-k} \) for \( 1 \leq k \leq s \), and therefore,

\[
a_{2s} = M_s + \sum_{k=1}^{s} a_{2k-2}M_{s-k} = M_s + \sum_{k=0}^{s-1} a_{2s-2k-2}M_k.
\]

Since the relation between \( a_{2s} \) and \( M_s \) is equivalent to (1) and \( a_0 = S_0 = 1 \), we have come to a conclusion that \( a_{2s} = S_{2s} = \sum \binom{s}{i} \left\lfloor \frac{i}{\left\lfloor i/2 \right\rfloor} \right\rfloor \) by Proposition 4. \( \square \)

Encouraged by this success, we offer the following generalized conjecture.

**Conjecture 1** For positive integer \( s \) and \( k \), the number of self-conjugate \((s, s+1, \ldots, s+k)\)-cores is equal to the number of symmetric \((s, k)\)-generalized Dyck paths.

**Remark 2** Yan et al. [10] recently proved Conjecture 1.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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