Running of Running of the Spectral Index
and
WMAP Three-year data

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ABSTRACT

Three-year data of WMAP implies not only a negative running of the spectral index with large absolute value, but also a large positive running of running of the spectral index with order of the magnitude $10^{-2}$. We calculate the running of running in usual inflation model and noncommutative inflation model. A large tensor-scalar ratio $r \geq 1.23$ is needed in order to fit the WMAP data in the noncommutative inflation model, which roughly saturates the observational upper bound on it.
1 Introduction

An epoch of accelerated expansion in the early universe, inflation [1], dynamically resolves many cosmological puzzles in hot big bang model, such as homogeneity, isotropy and flatness of the universe, and generates superhorizon fluctuations without appealing to fine-tuned initial conditions. These fluctuations become classical after crossing out the Hubble horizon during the period of inflation. During the deceleration phase after inflation they re-enter the horizon, and seed the matter and the radiation fluctuations observed in the universe. The anisotropy in CMB encodes the information for inflation.

However, we still don’t know which of the many versions of inflation model will be picked out by the observations. An important step is to make the observations more and more precise. Now the ΛCDM model remains an excellent fit to the three years WMAP data and other astronomical data [2]. Even though the power-law spectral index model can fit to WMAP three-year data, the running is slightly improved by including the small scale experiments. The spectral index runs from $n_s > 1$ at $k = 0.002\text{Mpc}^{-1}$ to $n_s < 1$ at $k = 0.05\text{Mpc}^{-1}$.

A negative running with large absolute value would be problematic for most inflation models, so that confirmation of this suggestive trend is important for our understanding of early universe physics. We can always reconstruct the potential of inflaton to get such a running, e.g. [4], but the problem is how to interpret it in a fundamental theory. On the other hand, We expect that a running spectral index is related to the physics of the first moments of the big bang and provides some clues into trans-Planckian physics [12]. In the last few years, we found that the noncommutative effects [6–11] always make the power spectrum more blue and the noncommutative effects on the small scale fluctuations can be ignored, which is nicely consistent with observational results. There are also many explanations on the running spectral index [13,14].
Usually, the problem of the running spectral index is how to construct an inflation model in a fundamental theory to provide a negative running with large absolute value. Recently we notice that WMAP three-year results also implies that running of the spectral index also runs significantly from $k = 0.002 \text{Mpc}^{-1}$ to $k = 0.05 \text{Mpc}^{-1}$, which implies a large running of running of the spectral index with order of magnitude $10^{-2}$. This is another problem for the running spectral index.

Our paper is organized as follows. In section 2, we give the definition of the running of running of the spectral index and estimate its value based on the WMAP three-year data. In section 3, we calculate the running of running in usual slow-roll inflation model. We investigate the modification of the running of running in noncommutative inflation model in section 4. Section 5 contains some concluding remarks.

## 2 Running of running of the spectral index and WMAP data

Fit to the WMAP data is described by an eight-parameter model: four parameters for characterizing a Friedmann-Roberton-Walker (FRW) universe (baryonic density, matter density, Hubble constant, optical depth), and four parameters for the primordial power spectra (the amplitude of the power spectra, tensor-scalar ratio, the spectral index and its running). But here we introduce a new parameter, the running of running, for inflation. The reason why we introduce it is that WMAP three-year data implies a large running of running actually.

The definition of the spectral index $n_s$ and its running $\alpha_s$ and its running of running $\beta_s$ are respectively

\begin{align}
  n_s &\equiv 1 + \frac{d \ln \Delta^2_R}{d \ln k}, \\
  \alpha_s &\equiv \frac{dn_s}{d \ln k}, \\
  \beta_s &\equiv \frac{d \alpha_s}{d \ln k},
\end{align}

(2.1) (2.2)
\[ \beta_s \equiv \frac{d\alpha_s}{d \ln k}, \quad (2.3) \]

where \( \Delta_R \) is the primordial amplitude of the power spectrum for the curvature perturbation in comoving gauge and \( k \) is the perturbation mode.

A power-law spectral index model can fit the WMAP three-year data nicely. However the running spectral index provides a slightly better fit to the data [2]. We expect that whether the spectral index runs or not will to be distinguish in the near future. In this paper, we focus on a running spectral index in [2]. The value of the spectral index and its running and the upper bound on the tensor-scalar ration are respectively

\[ n_s = 1.21^{+0.13}_{-0.16}, \quad \alpha_s = -0.102^{+0.050}_{-0.043}, \quad r \leq 1.5, \quad (2.4) \]

at \( k = 0.002\text{Mpc}^{-1} \). But a red power spectrum \( (n_s < 1) \) at \( k = 0.05\text{Mpc}^{-1} \) is favored and the running of the spectral index is not required at more than the 95% confidence level. Thus not only the spectral index runs, but also the running also runs. The value of the running of running of the spectral index is estimated at the linear approximation level as

\[ \beta_s \simeq \frac{\Delta \alpha_s}{\Delta \ln k} = \frac{0 - (-0.102)}{\ln(0.05/0.002)} \simeq 0.0318. \quad (2.5) \]

A running of running of the spectral index with the order of magnitude \( 10^{-2} \) is expected. To run CAMB or CMBfast is needed if one want to get a precise result. So the problems on the running spectral index are not only the negative running with large absolute value, but also the large running of running.

3 Running of running of the spectral index in inflation model

In this paper, we only focus on single field inflation model. The evolution of inflation is governed by the potential of inflaton field. The equations of
motion for an expanding universe containing a homogeneous scalar field in a spatially flat FRW universe are

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_p^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \]  
\[ \ddot{\phi} + 3H \dot{\phi} + V' = 0, \]  
where \( V \) is the potential of inflaton field \( \phi \) and the prime denotes the derivative with respect to \( \phi \). If \( \dot{\phi}^2 \ll V \) and \( |\dddot{\phi}| \ll 3H|\dot{\phi}| \), the inflaton field slowly rolls down its potential and the equations of motion (3.1) become

\[ H^2 = \frac{V}{3M_p^2}, \]  
\[ 3H \dot{\phi} = -V'. \]

To be simple, we define the slow-roll parameters \( \epsilon \) and \( \eta \) as

\[ \epsilon = \frac{M_p^2}{2} \left( \frac{V'}{V} \right)^2, \]  
\[ \eta = M_p^2 \frac{V''}{V}. \]  
The slow-roll conditions become \( \epsilon \ll 1 \) and \( |\eta| \ll 1 \). These slow-roll parameters also characterize the feature of the primordial power spectrum for the perturbations.

The amplitude of the scalar power spectrum for slow-roll inflation can be expressed as (see [3] for a review)

\[ \Delta^2_R = \frac{V/M_p^4}{24\pi^2 \epsilon}, \]  
and the tensor-scalar ratio is

\[ r = 16\epsilon. \]  

From slow-roll conditions, we find

\[ \frac{d}{d\ln k} = -M_p^2 \frac{V'}{V} \frac{d}{d\phi}. \]
and
\[ \frac{d\epsilon}{d\ln k} = 2\epsilon(2\epsilon - \eta), \]  
\[ \frac{d\eta}{d\ln k} = -\xi + 2\epsilon\eta, \]  
\[ \frac{d\xi}{d\ln k} = -\zeta - \eta\xi + 4\epsilon\xi, \]

where
\[ \xi \equiv M_p^2 \frac{V''V'''}{V^2}, \]  
\[ \zeta \equiv M_p^2 \frac{V''^2V'''}{V^3}. \]

Thus the spectral index and its running and its running of running are respectively related to the slow-roll parameters by
\[ s = n_s - 1 = -6\epsilon + 2\eta, \]  
\[ \alpha_s = -24\epsilon^2 + 16\epsilon\eta - 2\xi, \]  
\[ \beta_s = -192\epsilon^3 + 192\epsilon^2\eta - 32\epsilon\eta^2 - 24\epsilon\xi + 2\eta\xi + 2\zeta. \]

For given spectral index \( s = n_s - 1 \) and its running \( \alpha \), solving eq. (3.13) and (3.14), we find the running of running becomes
\[ \beta_s = -\frac{1}{2}s\alpha_s + 9\alpha_s\epsilon - 4\epsilon(s^2 + 15s\epsilon + 30\epsilon^2) + 2\zeta. \]

The first term in R.H.S of (3.16) equals 0.01, since \( s = 0.21 \) and \( \alpha_s = -0.102 \). The second and the third term in R.H.S of (3.16) is negative, since \( \epsilon > 0 \). In order to get large enough running of running, \( \zeta \) should not be smaller than 0.011. As we know, there is no such inflation model with so large running and the running of running (for example, see [3]).

### 4 Running of running of the spectral index in noncommutative inflation model

To make this paper self-consistent, we briefly review the calculation of the primordial power spectrum [9] for noncommutative inflation model. Then
we calculate the running of the spectral index and discuss fitting to WMAP three-year results in this subsection 4.2 and 4.3.

4.1 The primordial power spectrum and the running of running in noncommutative inflation

Noncommutative spacetime naturally emerges in string theory [5], which implies a new uncertainty relation

$$\Delta t_p \Delta x_p \geq l_s^2,$$  \hspace{1cm} (4.1)

where $t_p$ and $x_p$ are the physical time and space, $l_s$ is uncertainty length scale or string scale in string theory. Here we follow the toy model in [6]. The spacetime noncommutative effects are encoded in a new product among functions, namely the star product, replacing the usual algebra product. The evolution of the background is homogeneous and the standard cosmological equations of the inflation does not change.

In order to make the uncertainty relationship in (4.1) more clear in FRW background, we introduce another time coordinate $\tau$ in the noncommutative spacetime such that the metric takes the form

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{\tau}^2.$$  \hspace{1cm} (4.2)

Now the uncertainty relationship (4.1) becomes

$$\Delta \tau \Delta x \geq l_s^2.$$  \hspace{1cm} (4.3)

The star product can be explicitly defined as

$$f(\tau, x) \star g(\tau, x) = e^{-\frac{i}{4}l_s^2(\partial_{\tau} \delta_{\tau'} - \partial_{\tau'} \delta_{\tau})} f(\tau, x)g(\tau', x')|_{\tau' = \tau, x' = x}.$$  \hspace{1cm} (4.4)

Since the comoving curvature perturbation $R$ depends on the space and time, the equation of motion for $R$ is modified by the noncommutative effects

$$u''_k + \left(k^2 - \zeta_k''/\zeta_k\right)u_k = 0,$$  \hspace{1cm} (4.5)
where

\[ z_k^2(\tilde{\eta}) = z^2y_k^2(\tilde{\eta}), \quad y_k^2 = (\beta_k^+ \beta_k^-)^{\frac{1}{2}}, \]  
(4.6)

\[ \frac{d\tilde{\eta}}{d\tau} = \left( \frac{\beta_k^-}{\beta_k^+} \right)^{\frac{1}{2}}, \quad \beta_k^\pm = \frac{1}{2}(a^{\pm 2}(\tau + l_s^2 k) + a^{\pm 2}(\tau - l_s^2 k)), \]

here \( \mathcal{R}_k(\tilde{\eta}) = u_k(\tilde{\eta})/z_k(\tilde{\eta}) \) is the Fourier modes of \( \mathcal{R} \) in momentum space and the prime denotes derivative with respect to the modified conformal time \( \tilde{\eta} \). The deviation from the commutative case encodes in \( \beta_k^\pm \) and the corrections from the noncommutative effects can be parameterized by \( \frac{Hk}{aM_s^2} \).

After a lengthy but straightforward calculation, we get

\[ \frac{z_k''}{z_k} = 2(aH)^2 \left( 1 + \frac{5}{2}\epsilon - \frac{3}{2}\eta - 2\mu \right), \]
(4.7)

\[ aH \simeq -\frac{1}{\tilde{\eta}}(1 + \epsilon + \mu), \]

where

\[ \mu = \frac{H^2k^2}{(a^2M_s^4)} \]
(4.8)

is the noncommutative parameter and \( M_s = l_s^{-1} \) is the noncommutative mass scale or string mass scale. Solving eq. \( (4.6) \) yields the amplitude of the scalar comoving curvature fluctuations in noncommutative spacetime

\[ \Delta_R^2 \simeq \frac{k^3}{2\pi^2} |\mathcal{R}_k(\tilde{\eta})|^2 = \frac{V}{2FM_s^4}(1 + \mu)^{-4-6\epsilon+2\eta}, \]
(4.9)

where \( H \) and \( V \) take the values when the fluctuation mode \( k \) crosses the Hubble radius \( (z_k''/z_k = k^2) \), \( k \) is the comoving Fourier mode. Using \( (4.8) \) and \( (3.7) \), we obtain

\[ \frac{d\mu}{d\ln k} \simeq -4\epsilon\mu. \]
(4.10)

Thus the spectral index and its running and its running of running are respectively

\[ s = n_s - 1 = -6\epsilon + 2\eta + 16\epsilon\mu, \]
(4.11)
\[ \alpha_s = -24\epsilon^2 + 16\epsilon\eta - 2\xi - 32\epsilon\eta\mu, \]  
(4.12)  
\[ \beta_s = -192\epsilon^3 + 192\epsilon^2\eta - 32\epsilon^2\eta^2 - 24\epsilon\xi + 2\eta\xi + 2\zeta \]  
+ \[ 64\epsilon^2\mu - 64\epsilon^2\eta\mu + 32\epsilon\xi\mu. \]  
(4.13)  

The tensor-scalar ratio is still related to \( \epsilon \) by \( r = 16\epsilon \). Here we only sketch out the brief derivation of the primordial power spectrum for fluctuations. When \( \mu = 0 \), the results in noncommutative inflation are the same as those in commutative inflation exactly.

### 4.2 Model-independent analysis

In this subsection, we use the slow-roll parameters to make model-independent analysis on fitting to WMAP three-year results.

First, we ignore \( \xi \) and \( \zeta \), there are only three parameters \( \epsilon, \eta \) and \( \mu \). For given spectral index \( n_s = 1.21 \), the slow-roll parameter \( \eta \) is related to \( \eta \) and \( \mu \) by (4.11). Now the running and running of running are showed in Fig. 1.

![Figure 1: Ignoring \( \xi \) and \( \zeta \), we shows the running and the running of running which depend on the value of \( \epsilon \) and \( \mu \), where \( \eta \) is related to \( \epsilon \) and \( \mu \) by the spectral index \( n_s = 1.21 \).](image)

If we take \( n_s = 1.21 \), \( \alpha_s = -0.102 \) and \( \beta_s = 0.0318 \) as input, there is no free parameter. Now we find \( \epsilon = 0.082, \eta = 0.133 \) and \( \mu = 0.33 \). The tensor-scalar ratio equals \( r = 1.31 \). The values of parameters are reasonable and the noncommutative inflation can fit to the data.
Next we only ignore \( \zeta \). Now a new parameter \( \xi \) appears and the constraint on \( \epsilon, \eta \) and \( \mu \) is looser than previous case. We solve \( \eta \) and \( \xi \) in (4.11) and (4.12). We find

\[
\xi = \frac{1}{2}(-\alpha_s + 8s - 16s\mu\epsilon + 8(1 - 8\mu)(3 - 4\mu)\epsilon^2). \tag{4.14}
\]

Now the running of running becomes

\[
\beta_s = -\frac{1}{2}s\alpha_s + 9\alpha_s\epsilon - 4\epsilon(s^2 + 15s\epsilon + 30\epsilon^2) \\
+ 8(-\alpha_s + s^2 + 64\epsilon + 324\epsilon^2)\epsilon\mu \\
- 512(s + 17\epsilon)\epsilon^2\mu^2 + 6144\epsilon^3\mu^3. \tag{4.15}
\]

For given spectral index \( n_s = 1.21 \) and its running \( \alpha_s = -0.102 \), the running of running is showed in Fig. 2.

![Figure 2: Ignoring \( \zeta \), we show the running of running which depends on the value of \( \epsilon \) and \( \mu \), where \( \eta \) and \( \xi \) are related to \( \epsilon \) and \( \mu \) by the spectral index \( n_s = 1.21 \) and the running of the spectral index \( \alpha_s = -0.102 \).](image)

For ignoring \( \zeta \), there are four free parameters, \( \epsilon, \eta, \xi \) and \( \mu \). However there are only three constraints. A degree of freedom is left. We show the constraint on the \( \epsilon, \eta \) and \( \mu \) in Fig. 3. According to this figure, we find a large tensor-scalar ratio \( r \geq 1.23 \) is needed. On the other hand, the amplitude of the power spectrum is \( \Delta_R = 2.09 \times 10^{-9} \) [2]. Using (4.9), we find the Hubble parameter during the period of inflation is roughly \( \sqrt{\frac{\pi^2}{2}}\Delta R M_p \approx 10^{-4} M_p \approx \).
2.4 \times 10^{14} \text{Gev}. Since \mu \sim H^4/M_s^4 \simeq 0.3, the noncommutative scale or string scale is roughly 3.2 \times 10^{14} \text{Gev}, which is lower than GUT scale.

The constraints for WMAP+SDSS is similar to WMAP+2dFGRS. For the case with tensor perturbations, WMAP+SDSS gives a more stringent constraint on the tensor-scalar ratio as

$$r \leq 0.67 \quad \text{WMAP+SDSS}, \quad r \leq 1.0 \quad \text{WMAP+2df},$$

(4.16)

at 95\% CL. If so, the noncommutative inflation cannot provide a running with large enough absolute value within reasonable parameters space. On the other hand, if we ignore the tensor perturbations, the spectral index and its running are respectively [15]

$$n_s = 0.895^{+0.041}_{-0.042}, \quad \alpha_s = -0.040^{+0.027}_{-0.027},$$

(4.17)

for WMAP+SDSS. Now the value of running of running is roughly \beta_s \simeq 0.0124. Setting \xi = \zeta = 0, we find \epsilon = 0.081, \eta = 0.117, \mu = 0.113 and the tensor-scalar ratio is \( r = 1.3 \). Taking \xi into account, we show the fit to the running spectral index in fig. 4 and we find \( r \geq 1.25 \) is needed. Considering the upper bound on the tensor-scalar ration given by WMAP
Figure 4: Ignoring $\zeta$, we show the constraint on $r = 16\epsilon$, $\eta$ and $\mu$ for $n_s = 0.895$, $\alpha_s = -0.04$ and $\beta_s = 0.0124$.

group, we conclude that the noncommutative cannot provide a good fit to WMAP+SDSS data.

4.3 Check some typical inflation models

In this subsection we check two inflation models: power-law inflation model and chaotic inflation model. The small field inflation models predict a small amplitude of tensor perturbations and the corrections from the spacetime noncommutative effects can be neglected.

Power-law inflation is governed by the potential of inflaton with

$$
V = V_0 \exp \left(-\sqrt{\frac{2}{p} \phi M_p} \right).
$$

The slow-roll parameters are

$$
\epsilon = 1/p, \quad \eta = 2/p, \quad \xi = 4/p^2, \quad \zeta = 8/p^3.
$$

The spectral index and its running and its running of running are respectively

$$
s = n_s - 1 = -\frac{2}{p} + \frac{16}{p^2} \mu,
$$
αs = −\frac{64}{p^2} \mu, \quad (4.21) \\
βs = \frac{256}{p^3} \mu. \quad (4.22)

In commutative spacetime, \( \mu = 0 \), the spectral index does not run at all. In noncommutative case, there are only two parameters, \( p \) and \( \mu \). Requiring \( n_s = 1.21 \) and \( \alpha_s = -0.102 \) yields \( p = 13.9 \) and \( \mu = 0.307 \) and thus \( \beta_s = 0.0293 \), which is slightly smaller than that from data.

Chaotic inflation is dominated by the inflaton potential with

\[ V = \frac{\lambda}{p} \phi^p. \quad (4.23) \]

The value of inflaton at the time corresponding to number of e-folds \( N \) before the end of inflation is given by

\[ \phi_N = \sqrt{2pNM_p}. \quad (4.24) \]

The slow-roll parameters are expressed as

\[ \epsilon = \frac{p}{4N}, \quad \eta = \frac{p-1}{2N}, \quad \xi = \frac{(p-1)(p-2)}{4N^2}, \quad \zeta = \frac{(p-1)(p-2)(p-3)}{8N^3}. \quad (4.25) \]

The spectral index and its running and its running of running are

\[ s = -\frac{p+2}{2N} + \frac{4p}{N} \mu, \quad (4.27) \]
\[ \alpha_s = -\frac{p+2}{2N^2} - \frac{4p(p-1)}{N^2} \mu, \quad (4.28) \]
\[ \beta_s = -\frac{p+2}{N^3} + \frac{4p(p-1)(p-2)}{N^3} \mu. \quad (4.29) \]

For \( N=50 \), requiring \( n_s = 1.21 \) and \( \alpha_s = -0.102 \) yields \( p = 13.9 \) and \( \mu = 0.233 \) and thus \( \beta_s = 0.0158 \). For \( n_s = 1.21 \) and \( \alpha_s = -0.102 \), we scan the range with \( N \in [47, 61] \) and show the running of running in Fig. 4.

However, we can not find a reasonable solution for \( n_s = 1.21, \alpha_s = -0.102 \) and \( \beta_s = 0.0318 \). The value of running of running is roughly 0.0245 which is not large enough to fit the data.
Figure 5: The line corresponds to $n_s = 1.21$, $\alpha_s = -0.102$ for $N \in [47, 61]$.

5 Conclusion

In this paper we point out that there is another problem for the running spectral index, i.e. the running of the spectral index also runs. It is a new challenge for inflation model building. However noncommutative inflation model still provide a nice explanation on the running of running. But a large amplitude of the tensor perturbations is needed. Noncommutative power-law inflation is still nice, but the noncommutative chaotic inflation can not provide a large enough running of running.

If we take the running of running into account, the tensor-scalar ratio roughly saturates the upper bound on it in WMAP three-year data and the noncommutative scale or string scale is roughly $3.2 \times 10^{14}$Gev. However, noncommutative inflation model cannot provide a nice fit to WMAP+SDSS data. We expect that the observations, e.g. more years data of WMAP and Planck, can confirm or rule it out in near future at high level of confidence.

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