DIFFERENTIAL GEOMETRY OF RELATIVE GERBES

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# Table of Contents

1 Introduction 1

2 Relative Homology/Cohomology 5
   2.1 Algebraic Mapping Cone for Chain Complexes 5
   2.2 Algebraic Mapping Cone for Co-chain Complexes 9
   2.3 Kronecker Pairing 10
   2.4 Singular, De Rham, Čech Theory 12
   2.5 Topological Definition of Relative Homology 15
   2.6 An Integrality Criterion 18
   2.7 Bohr-Sommerfeld Condition 18

3 Geometric Interpretation of Integral Relative Cohomology Groups 20
   3.1 Geometric Interpretation of $H^1(\Phi,\mathbb{Z})$ 21
   3.2 Geometric Interpretation of $H^2(\Phi,\mathbb{Z})$ 23
   3.3 Gerbes 26
   3.4 Geometric Interpretation of $H^3(\Phi,\mathbb{Z})$ 31

4 Differential Geometry of Relative Gerbes 34
   4.1 Connections on Line Bundles 34
   4.2 Connections on Gerbes 35
   4.3 Connections on Relative Gerbes 38
   4.4 Cheeger-Simon Differential Characters 40
   4.5 Relative Deligne Cohomology 43
   4.6 Transgression 46

5 Pre-quantization of Group-Valued Moment Maps 49
   5.1 Gerbes over a Compact Lie Group 49
      5.1.1 Some Notations from Lie Groups 49
      5.1.2 Standard Open Cover of $G$ 51
      5.1.3 Construction of the Basic Gerbe 53
      5.1.4 The Basic Gerbe Over SU(n) 55
| Section | Title                                                                 | Page |
|---------|----------------------------------------------------------------------|------|
| 5.2     | The Relative Gerbe for Hol : $\mathcal{A}_G(S^1) \rightarrow G$      | 57   |
| 5.3     | Review of Group-Valued Moment Maps                                   | 60   |
| 5.3.1   | Examples                                                             | 61   |
| 5.3.2   | Products                                                             | 64   |
| 5.3.3   | Reduction                                                            | 64   |
| 5.4     | Pre-quantization of $G$-Valued Moment Maps                           | 65   |
| 5.4.1   | Reduction                                                            | 68   |
| 5.4.2   | A Finite Dimensional Pre-quantum Line Bundle for $\mathcal{M}(\Sigma)$ | 68   |
| 5.4.3   | Pre-quantization of Conjugacy Classes of a Lie Group                | 69   |
| 5.5     | Hamiltonian Loop Group Spaces                                        | 70   |

Bibliography | 73
Chapter 1

Introduction

The motivation of this thesis is to develop pre-quantization of quasi-Hamiltonian spaces with group-valued moment maps by introducing the notion of relative gerbes and addressing their differential geometry.

Giraud [13] first introduced the concept of gerbes in the early 1970s to study non-Abelian second cohomology. Later, Brylinski [7] defined gerbes as sheafs of groupoids with certain axioms, and discussed their differential geometry. He proved that the group of equivalence classes of gerbes gives a geometric realization of integral three cohomology classes on manifolds. Through a more elementary approach, Chatterjee and Hitchin [8, 22] introduced gerbes in terms of transition line bundles for a given cover of the manifold. From this point of view, a gerbe is a one-degree-up generalization of a line bundle, where the line bundle is presented by transition maps. A notable example of a gerbe arises as the obstruction for the existence of a lift of a principal $G$-bundle to a central extension of the Lie group. Another example is the associated gerbe of an oriented codimension three submanifold of an oriented manifold. The third example is what is called “basic gerbe,” which corresponds to the generator of the degree three integral cohomology of a compact, simple and simply
connected Lie group. The basic gerbe over G is closely related to the basic central extension of the loop group, and it was constructed, from this point of view, by Brylinski [7]. Later, Gawedski-Reis [12], for G=SU(n), and Meinrenken [30], in the general case, gave a finite-dimensional construction along with an explicit description of the gerbe connection.

Quasi-Hamiltonian \( G \)-spaces \((M, \omega, \Psi)\) with group-valued moment maps \( \Psi : M \to G \) are introduced in [2]. The 2-form \( \omega \) is not necessarily non-degenerate. However, its kernel is controlled by the minimal degeneracy condition. Conjugacy classes of \( G \), with moment map the inclusion into \( G \), are the main examples of the quasi-Hamiltonian \( G \)-spaces. Another example is the space \( G^{2h} \), with moment map the product of Lie group commutators.

One can define the notion of reduction for group-valued moment maps, and the reduced spaces \( M//G = \Psi^{-1}(e)/G \) are symplectic. The “fusion product” of two quasi-Hamiltonian \( G \)-spaces is a quasi-Hamiltonian \( G \)-space, with moment map the pointwise product of the two moment maps. This product gives a ring structure to the set of quasi-Hamiltonian \( G \)-spaces.

This thesis introduces the notion of relative gerbes for smooth maps of manifolds, and discusses their differential geometry. The equivalence classes of relative gerbes are classified by the relative integral cohomology in degree three. Furthermore, by using the concept of relative gerbes, the pre-quantization of Lie group-valued moment maps is developed, and its equivalence with the pre-quantization of infinite-dimensional Hamiltonian loop group spaces is established.

The organization of this thesis is as follows. In Chapter 2, I discuss the relative (co)homology of a smooth map between two manifolds. When the map is inclusion, the singular relative (co)homology of the map coincides with the singular relative (co)homology of the pair. Also, for a continuous map of topological spaces, the relative (co)homology of the map is isomorphic to the (co)homology of the mapping cone. In Chapter 3, following the Chatterjee-Hitchin perspective on gerbes, I define the notion of relative gerbe for a
Chapter 1. Introduction

smooth map $\Phi \in C^\infty(M, N)$ between two manifolds $M$ and $N$ as a gerbe over the target space together with a quasi-line bundle for the pull-back gerbe. I prove that the group of equivalence classes of relative gerbes can be characterized by the integral degree three relative cohomology of the same map.

Another objective of this thesis is to develop the differential geometry of relative gerbes. More specifically, in Chapter 4, the concepts of relative connection, relative connection curvature, relative Cheeger-Simon differential character, and relative holonomy are introduced. As well, I prove that a given closed relative 3-form arises as a curvature of some relative gerbe with connection if and only if the relative 3-form is integral. I also prove that a relative gerbe with connection for a smooth map $\Phi : M \to N$ generates a relative line bundle with connection for the corresponding map of loop spaces, $L\Phi : LM \to LN$. In addition, I prove that the group of equivalence classes of relative gerbes with connection gives a geometric realization of degree two relative Deligne cohomology.

In Chapter 5, I give an explicit construction of the basic gerbe for $G = SU(n)$, and of suitable multiples of the basic gerbe for the other Lie groups. As well, I show that the construction of the basic gerbe over $SU(n)$ is equivalent to the construction of the basic gerbe in Gawedzki-Reis [12]. In this chapter, I also construct a relative gerbe for the map $\text{Hol} : A_G(S^1) \to G$, where $A_G(S^1)$ is the affine space of connections on the trivial bundle $S^1 \times G$. Inspired by the pre-quantization of Hamiltonian $G$-manifolds, the other objective of this thesis is to construct a method to pre-quantize the quasi-Hamiltonian $G$-spaces with group-valued moment maps. To achieve this, I use the premise of relative gerbes, developed in this thesis. Let $G$ be a compact, simple and simply connected Lie group and $(M, \omega, \Psi)$ be a quasi-Hamiltonian $G$-space. One of the axioms of a group-valued moment map $(M, \omega, \Psi)$ is $d\omega = \Psi^* \eta$, where $\eta$ is the canonical 3-form. This means that the pair $(\omega, \eta)$ is a cocycle for the relative de Rham complex for the moment map $\Psi$. A pre-quantization of a quasi-Hamiltonian $G$-space with $G$-valued moment map is defined in Chapter 5 by a relative gerbe.
with connection with relative curvature $(\omega, \eta) \in \Omega^3(\Psi)$. Based on the results of Chapter 4, $(M, \omega, \Psi)$ is pre-quantizable if and only if the relative 3-form $(\omega, \eta)$ is integral. I prove that, given two pre-quantizable quasi-Hamiltonian $G$-spaces, their fusion product is again pre-quantizable.

In Chapter 5, the pre-quantization conditions for the examples of quasi-Hamiltonian $G$-spaces, described previously, are examined. I show that a given conjugacy class $\mathcal{C} = G \cdot \exp(\xi)$ of $G$ is pre-quantizable when $\xi \in \Lambda^*$, where $\Lambda^*$ is the weight lattice. I also illustrate that $G^{2h}$ has a unique pre-quantization, which enables one to construct a finite dimensional pre-quantum line bundle for the moduli space of flat connections of a closed oriented surface of genus $h$.

Further, recall that there is a one-to-one correspondence between quasi-Hamiltonian $G$-spaces and infinite dimensional loop group spaces [2]. Extending this correspondence, in Chapter 5, I prove that pre-quantization of a quasi-Hamiltonian $G$-spaces with group-valued moment map coincides with the pre-quantization of the corresponding Hamiltonian loop group space.

As a continuation of this thesis, one can consider the “quantization” of quasi-Hamiltonian $G$-spaces. It is expected that this advancement should involve the relative notion of twisted K-theory, to be developed.
Chapter 2

Relative Homology/Cohomology

2.1 Algebraic Mapping Cone for Chain Complexes

**Definition 2.1.1.** Let $f_\bullet : X_\bullet \to Y_\bullet$ be a chain map between chain complexes over $R$ where $R$ is a commutative ring. The algebraic mapping cone of $f$ [10] is defined as a chain complex $\text{Cone}_\bullet(f)$ where

$$\text{Cone}_n(f) = X_{n-1} \oplus Y_n$$

with the differential

$$\partial(\theta, \eta) = (\partial \theta, f(\theta) - \partial \eta).$$

Since $\partial^2 = 0$, we can consider the homology of this chain complex. Define relative homology of $f_\bullet$ to be

$$H_n(f) := H_n(\text{Cone}_\bullet(f)).$$

The short exact sequence of chain complexes

$$0 \to Y_n \xrightarrow{j} \text{Cone}_n(f) \xrightarrow{k} X_{n-1} \to 0$$

where $j(\beta) = (0, \beta)$ and $k(\alpha, \beta) = \alpha$ gives a long exact sequence in homology

$$\cdots \to H_n(Y) \xrightarrow{j} H_n(f) \xrightarrow{k} H_{n-1}(X) \xrightarrow{\delta} H_{n-1}(Y) \to \cdots$$

(2.1.1)
where $\delta$ is the connecting homomorphism.

**Lemma 2.1.1.** The connecting homomorphism $\delta$ is given by $\delta[\gamma] = [f(\gamma)]$ for $\gamma \in X_{n-1}$.

**Proof.** For $\gamma \in X_{n-1}$, we have $k(\gamma,0) = \gamma$. The short exact sequence of chain complexes gives an element $\gamma' \in Y_{n-1}$ such that $j(\gamma') = \partial(\gamma,0) = (\partial\gamma,f(\gamma))$. $\delta$ is defined by $\delta[\gamma] = [\gamma']$. But by definition of $j$, $j(\gamma') = (0,\gamma')$. Therefore $f(\gamma) = \gamma'$. This shows $\delta[\gamma] = [f(\gamma)]$. 

**Definition 2.1.2.** We call a chain map $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ a quasi-isomorphism if it induces isomorphism in cohomology, i.e., $H_{\bullet}(X) \xrightarrow{\cong} H_{\bullet}(Y)$.

**Corollary 2.1.2.** $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ is a quasi-isomorphism if and only if $H_{\bullet}(f) = 0$.

**Proof.** $f$ is a quasi-isomorphism, if and only if the connecting homomorphism in the long exact sequence 2.1.1 is an isomorphism.

**Definition 2.1.3.** A homotopy operator between two chain complexes $f,g : X_{\bullet} \rightarrow Y_{\bullet}$ is a linear map $h : X_{\bullet} \rightarrow Y_{\bullet+1}$ such that

$$h\partial + \partial h = f - g. \quad (\star)$$

In that case, $f$ and $g$ are called chain homotopic and we denote it by $f \simeq g$.

Two chain maps $f : X_{\bullet} \rightarrow Y_{\bullet}$ and $g : Y_{\bullet} \rightarrow X_{\bullet}$ are called homotopy inverse if $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$ are both homotopic to the identity. If $f : X_{\bullet} \rightarrow Y_{\bullet}$ admits a homotopy inverse, it is called a homotopy equivalence. In particular every homotopy equivalence is a quasi-isomorphism.

**Proposition 2.1.3.** Any homotopy between chain maps $f,g : X_{\bullet} \rightarrow Y_{\bullet}$ induces an isomorphism of chain complexes $\text{Cone}(f)_{\bullet}$ and $\text{Cone}(g)_{\bullet}$.

**Proof.** Given a homotopy operator $h$ satisfying $(\star)$, define a map $F : \text{Cone}_{\bullet}(f) \rightarrow \text{Cone}_{\bullet}(g)$ by

$$F(\alpha, \beta) = (\alpha, -h(\alpha) + \beta).$$
Since
\[ \partial F(\alpha, \beta) = (\partial \alpha, g(\alpha) + \partial h(\alpha) + \partial \beta) = (\partial \alpha, f(\alpha) - h \partial(\alpha) + \partial \beta) = F \partial(\alpha, \beta) \]

\( F \) is a chain map and its inverse map is \( F^{-1}(\alpha, \beta) = (\alpha, h(\alpha) + \beta) \).

\[ \square \]

Lemma 2.1.4. \textit{Let}

\[
\begin{array}{cccccc}
0 & \longrightarrow & X_\bullet & \longrightarrow & Y_\bullet & \longrightarrow & Z_\bullet & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{X}_\bullet & \longrightarrow & \tilde{Y}_\bullet & \longrightarrow & \tilde{Z}_\bullet & \longrightarrow & 0
\end{array}
\]

be a commutative diagram of chain maps with exact rows. If two of vertical maps are quasi-isomorphism, then so is the third.

\textit{Proof.} The statement follows from the 5-Lemma applied to the corresponding diagram in homology,

\[
\begin{array}{cccccc}
\cdots & \longrightarrow & H_\bullet(X) & \longrightarrow & H_\bullet(Y) & \longrightarrow & H_\bullet(Z) & \longrightarrow & H_{\bullet-1}(X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_\bullet(\tilde{X}) & \longrightarrow & H_\bullet(\tilde{Y}) & \longrightarrow & H_\bullet(\tilde{Z}) & \longrightarrow & H_{\bullet-1}(\tilde{X}) & \longrightarrow & \cdots
\end{array}
\]

\[ \square \]

Proposition 2.1.5. \textit{Suppose we have the following commutative diagram of chain maps,}

\[
\begin{array}{ccc}
X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\
\Phi_\bullet \downarrow & & \Psi_\bullet \downarrow \\
\tilde{X}_\bullet & \xrightarrow{\tilde{f}_\bullet} & \tilde{Y}_\bullet
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are quasi-isomorphisms. Then the induced map

\[ F : \text{Cone}_\bullet(f) \rightarrow \text{Cone}_\bullet(\tilde{f}), \ (\alpha, \beta) \mapsto (\Phi(\alpha), \Psi(\beta)) \]

is a quasi-isomorphism.
Proof. The map $F$ is a chain map since,

$$\partial F(\alpha, \beta) = \partial (\Phi(\alpha), \Psi(\beta))$$

$$= (\partial \Phi(\alpha), \tilde{f}(\Phi(\alpha)) - \partial \Psi(\beta))$$

$$= (\Phi(\partial(\alpha)), \Psi(f(\alpha) - \partial \beta))$$

$$= F(\partial \alpha, f(\alpha) - \partial \beta)$$

$$= F \partial(\alpha, \beta).$$

The chain map $F$ fits into a commutative diagram,

$$0 \rightarrow Y_\bullet \rightarrow \text{Cone}_\bullet (f) \rightarrow X_{\bullet - 1} \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \rightarrow \tilde{Y}_\bullet \rightarrow \text{Cone}_\bullet (\tilde{f}) \rightarrow \tilde{X}_{\bullet - 1} \rightarrow 0$$

Since $\Phi$ and $\Psi$ are quasi-isomorphisms, so is $F$ by Lemma 2.1.4. \hfill \square

**Proposition 2.1.6.** For any chain map $f_\bullet : X_\bullet \rightarrow Y_\bullet$ there is a long exact sequence

$$\cdots \rightarrow H_{n-1}(\ker f) \overset{j}{\rightarrow} H_n(f) \overset{k}{\rightarrow} H_n(\operatorname{coker} f) \overset{\delta}{\rightarrow} H_{n-2}(\ker f) \rightarrow H_{n-1}(f) \rightarrow \cdots$$

where $j$, $k$ and the connecting homomorphism $\delta$ are defined by

$$j[\theta] = [(\theta, 0)]$$

$$k[(\theta, \eta)] = [\eta \mod f(X)]$$

$$\delta[(\eta \mod f(X))] = [\partial \theta] \in H_{n-2}(\ker f).$$

Here, $\eta \in Y_n$ and $\partial \eta = f(\theta)$ for some $\theta \in X_{n-1}$. In particular, if $f$ is an injection $H_n(f) = H_n(\operatorname{coker} f)$, and if it is onto $H_n(f) = H_{n-1}(\ker f)$.

Proof. Let $\tilde{f}_\bullet : X_\bullet \rightarrow \operatorname{im}(f_\bullet) \subseteq Y_\bullet$ be the chain map $f_\bullet$, viewed as a map into the subcomplex $f_\bullet(X_\bullet) \subseteq Y_\bullet$. We have the following short exact sequence

$$0 \rightarrow \text{Cone}_n(\tilde{f}) \overset{i}{\rightarrow} \text{Cone}_n(f) \overset{k}{\rightarrow} \operatorname{coker}(f_n) \rightarrow 0$$
where $k$ is as above and $i$ is the inclusion map. Therefore we get a long exact sequence

$$\cdots \to H_n(\tilde{f}) \overset{i}{\to} H_n(f) \overset{k}{\to} H_n(\text{coker } f) \to H_{n-1}(\tilde{f}) \to \cdots.$$ (2.1.2)

Let $\tilde{f}'_\bullet : X_\bullet/\text{ker } f_\bullet \to \text{im } f_\bullet$ be the map induced by $f$. Notice that since $\tilde{f}'$ is an isomorphism, therefore $H_\bullet(\tilde{f}') = 0$. By using the long exact sequence corresponding to the short exact sequence

$$0 \to \text{ker } f_\bullet - 1 \overset{j}{\to} \text{Cone } \bullet(\tilde{f}) \overset{\pi}{\to} \text{Cone } \bullet(\tilde{f}') \to 0$$

where $j(\theta) = (\theta, 0)$, and $\pi(\theta, \eta) = (\theta \mod \text{ker } f, \eta)$, we see that $\tilde{j}$ is a quasi-isomorphism. Since $j = i \circ \tilde{j}$, we obtain the long exact sequence

$$\cdots \to H_{n-1}(\text{ker } f) \overset{j}{\to} H_n(f) \overset{k}{\to} H_n(\text{coker } f) \to H_{n-2}(\text{ker } f) \to \cdots.$$ 

To find connecting homomorphism, assume $[\eta \mod f(X)] \in H_n(\text{coker } f)$ for $\eta \in Y_n$. Then $\partial \eta \in f(X)$, i.e., $\partial \eta = f(\theta)$ for some $\theta \in X_{n-1}$. Since

$$f(\partial \theta) = \partial f(\theta) = \partial \partial \eta = 0$$

then $\partial \theta \in \text{ker } f$. Also $k(\theta, \eta) = \eta \mod f(X)$ and $j(\theta) = i \circ \tilde{j}(\partial \theta) = i(\partial \theta, 0) = (\partial \theta, 0) = \partial(\theta, \eta)$. Thus, we have

$$\delta([\eta \mod f(X)]) = [\partial \theta] \in H_{n-2}(\text{ker } f).$$

\[\square\]

### 2.2 Algebraic Mapping Cone for Co-chain Complexes

If $f^\bullet : X^\bullet \to Y^\bullet$ be a co-chain map between co-chain complexes, the algebraic mapping cone of $f$ is defined as a co-chain complex $\text{Cone}^\bullet(f)$ where

$$\text{Cone}^n(f) = Y^{n-1} \oplus X^n$$
with the differential
\[ d(\alpha, \beta) = (f(\beta) - d\alpha, d\beta) \]

Since \( d^2 = 0 \), we can consider the cohomology of this co-chain complex. Define relative cohomology of \( f^\bullet \) to be
\[ H^n(f) := H^n(\text{Cone}^\bullet(f)). \]

**Remark 2.2.1.** Any cochain complex \((X^\bullet, d)\) may be viewed as a chain complex \((\tilde{X}^\bullet, \partial)\) where \( \tilde{X}_n = X^{-n} \) and \( \partial_n = d^{-n} \) \((n \in \mathbb{Z})\). This correspondence takes cochain maps \( f^\bullet : X^\bullet \to Y^\bullet \) into chain maps \( \tilde{f}^\bullet : \tilde{X}^\bullet \to \tilde{Y}^\bullet \) where \( \tilde{f}_n = f^{-n} \) and identifies \( \text{Cone}(\tilde{f}) \) and \( \text{Cone}(f) \) up to a degree shift:
\[
\text{Cone}(\tilde{f})_n = \text{Cone}(f)^{-n} = Y^{-n-1} \oplus X^{-n} \\
\text{Cone}(\tilde{f})_n = \tilde{X}_{n-1} \oplus \tilde{Y}_n = X^{-n+1} \oplus Y^{-n}.
\]

Thus, \( \text{Cone}(f)_n \cong \text{Cone}(\tilde{f})_{n+1} \).

Using this correspondence, results for the mapping cone of chain maps directly carry over to cochain maps.

### 2.3 Kronecker Pairing

For a chain complex \( X^\bullet \), the dual co-chain complex \((X')^\bullet\) is defined by \((X')^n = \text{Hom}(X_n, R)\) with the dual differential.

**Proposition 2.3.1.** Let \( f^\bullet : X^\bullet \to Y^\bullet \) be a map between chain complexes and \((f')^\bullet : (Y')^\bullet \to (X')^\bullet\) be its dual cochain map. Then the bilinear pairing
\[ \text{Cone}^n(f') \times \text{Cone}_n(f) \to R \]
given by the formula
\[ \langle (\alpha, \beta), (\theta, \eta) \rangle = \langle \alpha, \theta \rangle - \langle \beta, \eta \rangle \]
for \((\alpha, \beta) \in \text{Cone}^n(f')\) and \((\theta, \eta) \in \text{Cone}_n(f)\), induces a pairing in cohomology/homology

\[ H^n(f') \times H_n(f) \to \mathbb{R}. \]

**Proof.** It is enough to show that a cocycle paired with a boundary is zero and a coboundary paired with a cycle is zero. Let \((\alpha, \beta) = \partial(\alpha', \beta')\) and \(\partial(\theta, \eta) = 0\). Therefore by definition

\[
\alpha = f' \beta' - d\alpha', \quad \beta = d\beta'
\]

and

\[
\partial \eta = f(\theta), \quad \partial \theta = 0.
\]

\[
\langle (\alpha, \beta), (\theta, \eta) \rangle = \langle \alpha, \theta \rangle - \langle \beta, \eta \rangle
\]

\[
= \langle f' \beta', \theta \rangle - \langle d\alpha', \theta \rangle - \langle d\beta', \eta \rangle
\]

\[
= \langle f' \beta', \theta \rangle - \langle \alpha', \partial \theta \rangle - \langle \beta', \partial \eta \rangle
\]

\[
= \langle f' \beta', \theta \rangle - \langle \beta', f(\theta) \rangle
\]

\[
= 0.
\]

Similarly we can prove that a co-boundary paired with a cycle is zero.

**Lemma 2.3.2.** If \(f : X_\bullet \to Y_\bullet\) is a chain map, and \((f')^\bullet : (Y')^\bullet \to (X')^\bullet\) be its dual cochain map, then \(\text{Cone}^\bullet(f') = (\text{Cone}_\bullet(f))'\).

**Proof.** Notice that \(\text{Cone}^n(f') = (\text{Cone}_n(f))' = (X^{n-1})' \oplus (Y^n)'\). It follows from definitions that

\[
\langle d(\alpha, \beta), (\theta, \eta) \rangle = \langle (\alpha, \beta), \partial(\theta, \eta) \rangle.
\]

Therefore differential of \(\text{Cone}^n(f')\) is dual of differential of \(\text{Cone}_n(f)\).
2.4 Singular, De Rham, Čech Theory

In this Section, we fix two manifolds $M$ and $N$ and a map $\Phi \in C^\infty(M, N)$.

**Singular relative homology**: Consider the push-forward map $\Phi_* : S_q(M, R) \to S_q(N, R)$, where $R$ is a commutative ring and $S_q(M, R), S_q(N, R)$ are the singular chain complexes of $M$ and $N$ respectively. Singular relative homology is the homology of the chain complex $\text{Cone}_* (\Phi_*)$, and is denoted $H_* (\Phi, R)$.

**Singular relative cohomology**: Consider the pull-back map $\Phi^* : S^q(N, R) \to S^q(M, R)$, where $R$ is a commutative ring and $S^q(M, R), S^q(N, R)$ are the singular co-chain complex of $M$ and $N$ respectively. Singular relative cohomology is the cohomology of the co-chain complex $\text{Cone}^* (\Phi^*)$, and is denoted $H^*(\Phi, R)$.

**De Rham relative cohomology**: For $\Phi \in C^\infty(M, N)$, consider the pull-back map

$$\Phi^* : \Omega^q(N) \to \Omega^q(M)$$

between differential co-chain complexes. We denote the cohomology of $\text{Cone}^*(\Phi^*)$ by $H^*_{\text{dR}}(\Phi)$ and will call it de Rham relative cohomology.

**Čech relative cohomology**: Let $A$ be a $R$-module, and $U = \{U_\alpha\}$ be a good cover of a manifold $M$, i.e., all the finite intersections are contractible. For any collection of indices $\alpha_0, \cdots, \alpha_p$ such that $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset$, let

$$U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$

A Čech-p-cochain $f \in \tilde{C}^p(U, A)$ is a function

$$f = \prod_{\alpha_0 \cdots \alpha_p} f_{\alpha_0 \cdots \alpha_p} : \prod_{\alpha_0 \cdots \alpha_p} U_{\alpha_0 \cdots \alpha_p} \to A$$

where $f_{\alpha_0 \cdots \alpha_p}$ is locally constant and anti-symmetric in indices. The differential is defined by

$$(df)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{\alpha_0 \cdots \tilde{\alpha}_i \cdots \alpha_{p+1}}$$
where the hat means that the index has been omitted. Since \( d \circ d = 0 \), we can define Čech cohomology groups with coefficients in \( A \) by
\[
\hat{H}^p(M, A) := H^p(\check{C}(U, A)).
\]

Let \( U = \{U_i\}_{i \in I}, \ V = \{V_j\}_{j \in J} \) be good covers of \( M \) and \( N \) respectively such that there exists a map \( r : I \to J \) with \( \Phi(U_i) \subseteq V_{r(i)} \). Let \( \check{C}^\bullet(M, A), \check{C}^\bullet(N, A) \) be the Čech complexes for given covers, where \( A \) is an \( R \)-module. Using the pull-back map \( \Phi^* : \check{C}^\bullet(N, A) \to \check{C}^\bullet(M, A) \), we define the relative Čech cohomology to be the cohomology of \( \text{Cone}^\bullet(\Phi^*) \). Denote this cohomology by \( \hat{H}^\bullet(\Phi, A) \).

Suppose that \( A \) is one of the sheaves \( \mathbb{Z}, \mathbb{R}, \mathbb{U}(1), \mathcal{O}^{\mathbb{R}} \). Denote the space of \( k \)-cochains of the sheaf \( A \) on \( M \) and \( N \) respectively by \( C^k(M, A) \) and \( C^k(N, A) \). Here the differential is defined as above. Again, we have an induced map
\[
\Phi^* : C^k(N, A) \to C^k(M, A)
\]

Denote the cohomology of \( \text{Cone}^\bullet(\Phi^*) \) by \( H^\bullet(\Phi, A) \).

**Theorem 2.4.1.** There is a canonical isomorphism \( H^n_{dR}(\Phi) \cong H^n(\Phi, \mathbb{R}) \).

**Proof.** Let \( S_{sm}^\bullet(M, \mathbb{R}) \) and \( S_{sm}^\bullet(N, \mathbb{R}) \) be the smooth singular cochain complex of \( M \) and \( N \) respectively [4]. Consider the following diagram :

\[
\begin{array}{ccc}
\Omega^n(N) & \xrightarrow{\Phi^*} & \Omega^n(M) \\
g^n \downarrow & & f^n \downarrow \\
S_{sm}^n(N, \mathbb{R}) & \xrightarrow{\Phi^*} & S_{sm}^n(M, \mathbb{R})
\end{array}
\]

where \( f^n \) is defined by \( f^n(\omega) : \sigma \mapsto \int_{\Delta_n} \sigma^* \omega \), for \( \omega \in \Omega^n(M) \) and \( \sigma \in S_{sm}^n(M) \) is a smooth singular \( n \)-simplex. \( g^* \) is defined in similar fashion. From these definitions, it is clear that the diagram commutes. \( f^* \) and \( g^* \) are quasi-isomorphisms by de Rham Theorem [4]. Define
Chapter 2. Relative Homology/Cohomology

$k^\bullet : \Omega^\bullet(\Phi, \mathbb{R}) \to S^\bullet_{\text{sm}}(\Phi, \mathbb{R})$ by $k^\bullet(\alpha, \beta) = (f^{*-1}(\alpha), g^\bullet(\beta))$. We can use Proposition 2.1.5 and deduce that $k^\bullet$ is a quasi-isomorphism. There is a co-chain map

$l^\bullet : S^\bullet(M, \mathbb{R}) \to S^\bullet_{\text{sm}}(M, \mathbb{R})$

given by the dual of the inclusion map in chain level. In ( [33], p.196) it is shown that $l^\bullet$ is a quasi-isomorphism. Therefore if we use Proposition 2.1.5 again, we get

$H^n(\Phi, \mathbb{R}) \cong H^n_{\text{sm}}(\Phi, \mathbb{R})$.

Together we have $H^\bullet(\Phi, \mathbb{R}) \cong H^\bullet_{dR}(\Phi, \mathbb{R})$.

\[\square\]

**Theorem 2.4.2.** For $\Phi \in C^\infty(M, N)$, there is an isomorphism $H_{dR}^q(\Phi) \cong \tilde{H}^q(\Phi, \mathbb{R})$.

**Proof.** Let $\mathcal{U} = \{U_i\}_{i \in I}$, $\mathcal{V} = \{V_j\}_{j \in J}$ be good covers of $M$ and $N$ together with a map $r : I \to J$ such that $\Phi(U_i) \subseteq V_{r(i)}$. Define the double complex $E^{p,q}(M) = \check{C}^p(M, \Omega^q)$ where $\check{C}^p(\mathcal{U}, \Omega^q)$ is the set of q-forms $\omega_{\alpha_0 \cdots \alpha_p} \in \Omega^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p})$ anti-symmetric in indices with the differential $d$ defined as before. Let $E^n(M) = \bigoplus_{p+q=n} E^{p,q}(M)$ be the associated total complex. The map $\Phi : M \to N$ induces chain maps $\Phi^* : E^n(N) \to E^n(M)$. We denote the corresponding algebraic mapping cone by $E^n(\Phi)$. The inclusion $\check{C}^n(M, \mathcal{U}) \to E^n(M)$ is a quasi-isomorphism ([4], p. 97). We have a similar quasi-isomorphism for $N$ and since inclusion maps commute with pull-back of $\Phi$, we get a quasi-isomorphism $\check{C}^n(\Phi) \to E^n(\Phi)$. Thus we get isomorphism

$\check{H}^n(\Phi, \mathbb{R}) \cong H^n(\Phi)$.

(2.4.1)

The map $\Omega^n(M) \to E^{0,n}(M) \subset E^n(M)$ given by restrictions of forms $\alpha \mapsto \alpha|_{U_i}$ is a quasi-isomorphism ([4], p. 96). Again these maps commute with pull back, and hence define a quasi-isomorphism $\Omega^n(\Phi) \to E^n(\Phi)$ which means

$H^n_{dR}(\Phi) \cong H^n(\Phi)$.

(2.4.2)

By combining Equation 2.4.1 and Equation 2.4.2, we get $\check{H}^\bullet(\Phi, \mathbb{R}) \cong H^\bullet_{dR}(\Phi)$ \[\square\]
Remark 2.4.1. A modification of this argument, working instead with the double complex $C^p(M, S^q)$ given by collection of $S^q(U_{\alpha_0, \ldots, \alpha_p})$ gives isomorphism between Čech relative cohomology and singular relative cohomology with integer coefficients, hence

$$\check{H}^q(\Phi, \mathbb{Z}) \cong H_S^q(\Phi, \mathbb{Z}).$$

2.5 Topological Definition of Relative Homology

Let $\Phi : M \to N$ be an inclusion map, then the push-forward map $\Phi_* : S_\ast(M, R) \to S_\ast(N, R)$ is injection. Proposition 2.1.6 shows that $H_\ast(\Phi) \cong H_\ast(S(N)/S(M)) = H_\ast(N, M; R)$. $H_\ast(N, M; R)$ is known as relative homology. Obviously this is a special case of what we defined to be a relative singular homology of an arbitrary map $\Phi : M \to N$.

Given a continuous map $f : X \to Y$ of topological spaces, define mapping cylinder

$$\text{Cyl}_f = \frac{(X \times I) \sqcup Y}{(x, 1) \sim f(x)}$$

and mapping cone

$$\text{Cone}_f = \frac{\text{Cyl}_f}{X \times \{0\}}$$

Let $\text{Cone}(X) := X \times I/X \times \{0\}$.
There are natural maps

\[ i : Y \hookrightarrow \text{Cone}_f, \quad j : \text{Cone}(X) \to \text{Cone}_f. \]

Note that \( j \) is an inclusion only if \( f \) is an inclusion. There is a canonical map,

\[ h : \tilde{S}_{n-1}(X) \to \tilde{S}_n(\text{Cone}(X)) \]

with the property, \( h \circ \partial + \partial \circ h = k \), where \( h \) is defined by replacing a singular n-simplex with its cone and \( k : X \hookrightarrow \text{Cone}(X) \) is the inclusion map. Define a map

\[ l_n : \text{Cone}_n(f \ast) \to S_n(\text{Cone}_f), \quad (x, y) \mapsto j_\ast(h(x)) - i_\ast(y). \]
Theorem 2.5.1. $l_*$ is a chain map and is a quasi-isomorphism. Thus,

$$H_n(f) \cong H_n(\text{Cone}_f).$$

Proof. Recall that $\partial(x, y) = (\partial x, f_*(x) - \partial y)$. Since

$$l(\partial(0, y)) + \partial l(0, y) = l((0, -\partial y)) - \partial i_* y$$

$$= i_*(\partial y) - \partial i_* y \quad (2.5.1)$$

and

$$l(\partial(x, 0)) + \partial l(x, 0) = l((\partial x, f(x))) + \partial j_* h(x)$$

$$= j_* h(\partial x) - i_* f(x) + \partial j_* h(x) \quad (2.5.2)$$

$$= j_* k_*(x) - i_* f(x)$$

$$= 0$$

therefore $\partial l + l \partial = 0$. Consider diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & S_n(\text{Cyl}_f) & \longrightarrow & S_n(\text{Cone}_f) & \longrightarrow & S_n(\text{Cone}_f, \text{Cyl}_f) & \longrightarrow & 0 \\
& & \uparrow l & & \uparrow l & & \uparrow l & & \\
0 & \longrightarrow & S_n(Y) & \longrightarrow & \text{Cone}_n(f_*) & \longrightarrow & S_{n-1}(X) & \longrightarrow & 0
\end{array}
$$

where the first row corresponds to the pair $(\text{Cone}_f, \text{Cyl}_f)$ and the right vertical arrow comes from

$$S_{n-1}(X) \rightarrow S_n(\text{Cone}(X), X) \cong_{\text{exision}} S_n(\text{Cone}_f, \text{Cyl}_f).$$

The diagram commutes, and the rows are exact. Since the right and left vertical maps are quasi-isomorphisms, hence so is the middle map. \qed
2.6 An Integrality Criterion

If A and B are R-modules, then any homomorphism $\kappa : A \to B$ induces homomorphisms $\kappa : H^n(\Phi, A) \to H^n(\Phi, B)$ and $\kappa : H_n(\Phi, A) \to H_n(\Phi, B)$. In particular the injection $\iota : \mathbb{Z} \to \mathbb{R}$ induces a homomorphism $\iota : H^n(\Phi, \mathbb{Z}) \to H^n(\Phi, \mathbb{R})$.

A class $[\gamma] \in H^n(\Phi, \mathbb{R})$ is called integral in case $[\gamma]$ lies in the image of the map $\iota$.

**Proposition 2.6.1.** A class $[(\alpha, \beta)] \in H^n(\Phi, \mathbb{R})$ is integral if and only if $\int_\theta \alpha - \int_\eta \beta \in \mathbb{Z}$ for all cycles $(\theta, \eta) \in \text{Cone}_n(\Phi, \mathbb{Z})$.

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H^n(\Phi, \mathbb{R}) & \longrightarrow & \text{Hom}(H_n(\Phi, \mathbb{R}), \mathbb{R}) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \iota & & \tau & & \uparrow \\
0 & \longrightarrow & \text{Ext}(H_n(\Phi, \mathbb{Z})) & \longrightarrow & H^n(\Phi, \mathbb{Z}) & \longrightarrow & \text{Hom}(H_n(\Phi, \mathbb{Z}), \mathbb{Z}) & \longrightarrow & 0
\end{array}
\]

where $H^n(\Phi, \mathbb{R}) \to \text{Hom}(H_n(\Phi, \mathbb{R}), \mathbb{R})$ and $\tau$ are pairing given by integral. The map $\tilde{\iota}$ is inclusion map, considering the fact that

$\text{Hom}(H_n(\Phi, \mathbb{R}), \mathbb{R}) = \text{Hom}(H_n(\Phi, \mathbb{Z}), \mathbb{R})$.

Thus $[(\alpha, \beta)] \in H^n(\Phi, \mathbb{R})$ is integral if $\int_\theta \alpha - \int_\eta \beta \in \mathbb{Z}$ for all cycles $(\theta, \eta) \in \text{Cone}_n(\Phi, \mathbb{Z})$.

2.7 Bohr-Sommerfeld Condition

Let $(N, \omega)$ be a symplectic manifold. Recall that an immersion $\Phi : M \to N$ is isotropic if $\Phi^* \omega = 0$. It is called Lagrangian if furthermore $\dim M = \frac{1}{2} \dim N$. Suppose $H_1(N, \mathbb{Z}) = 0$
and $\omega$ is integral. A Lagrangian immersion $\Phi : M \to N$ is said to satisfy the Bohr-Sommerfeld ([16], [25]) condition if for all 1-cycles $\gamma \in S_1(M)$

$$\int_D \omega \in \mathbb{Z} \quad \text{where} \quad \partial D = \Phi(\gamma).$$

Note that since $\omega$ is integral, this does not depend on the choice of $D$. Also, if $\omega = d\theta$ is exact (for example for the cotangent bundles), the condition means that

$$\int_{\gamma} \Phi^* \theta \in \mathbb{Z} \quad \text{for all 1-cycles } \gamma.$$

In terms of relative cohomology, the condition means that $(0, \omega) \in \Omega^2(\Phi)$ defines an integral class in $H^2_{dR}(\Phi)$. The interesting feature of this situation is that the forms on $M, N$ are fixed, and defines a condition on the map $\Phi$.

Example 2.7.1. Let $N = \mathbb{R}^2$, $M = S^1$, $\omega = dx \wedge dy$, $\Phi =$ inclusion map. Then the immersion $\Phi : S^1 \hookrightarrow \mathbb{R}^2$ satisfies the Bohr-Sommerfeld condition.
Chapter 3

Geometric Interpretation of Integral
Relative Cohomology Groups

Let $\Phi \in C^\infty(M, N)$ where $M$ and $N$ are manifolds. Let $U = \{U_i\}_{i \in I}$, $V = \{V_j\}_{j \in J}$ be good covers of $M$ and $N$ respectively such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$.

**Proposition 3.0.1.** $H^q(\Phi, \mathbb{Z}) \cong H^{q-1}(\Phi, U(1))$ for $q \geq 1$.

**Proof.** We have the following long exact sequence

$$
\cdots \to H^{q-1}(M, \mathbb{R}) \to H^q(\Phi, \mathbb{R}) \to H^q(N, \mathbb{R}) \to H^q(M, \mathbb{R}) \to \cdots.
$$

Since $H^\ast(M, \mathbb{R}) = 0$ and $H^\ast(N, \mathbb{R}) = 0$, we see that $H^q(\Phi, \mathbb{R}) = 0$ for $q > 0$. By using the long exact sequence associated to exponential sequence

$$
0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp} U(1) \to 0, \quad (\star)
$$

we can deduce that $H^q(\Phi, \mathbb{Z}) \cong H^{q-1}(\Phi, U(1))$ for $q \geq 1$. \qed
3.1 Geometric Interpretation of $H^1(\Phi, \mathbb{Z})$

Let $X$ be a manifold. We say that a function $f \in C^\infty(X, U(1))$ has global logarithm if there exists a function $k \in C^\infty(X, \mathbb{R})$ such that $f = \exp((2\pi \sqrt{-1})k)$.

**Definition 3.1.1.** Two maps $f, g : X \to U(1)$ are equivalent if $f/g$ has a global logarithm.

The short exact sequence of sheaves ($\star$) gives an exact sequence of Abelian groups

$$0 \to H^0(X, \mathbb{Z}) \to C^\infty(X, \mathbb{R}) \xrightarrow{\exp} C^\infty(X, U(1)) \to H^1(X, \mathbb{Z}) \to 0.$$ 

This shows there is a one-to-one correspondence between equivalence classes and elements of $H^1(X, \mathbb{Z})$. We are looking for a geometric realization of $H^1(\Phi, \mathbb{Z})$ for a smooth map $\Phi : M \to N$. Let

$$\mathcal{L} := \{(k, f) | \Phi^* f = \exp((2\pi \sqrt{-1})k)\} \subset C^\infty(M, \mathbb{R}) \times C^\infty(N, U(1)).$$

$\mathcal{L}$ has a natural group structure. There is a natural group homomorphism,

$$\tau : C^\infty(N, \mathbb{R}) \to \mathcal{L}$$

where $\tau$ is defined for $l \in C^\infty(N, \mathbb{R})$ by

$$\tau(l) = (\Phi^* l, \exp((2\pi \sqrt{-1})l)).$$

**Definition 3.1.2.** We say $(k, f), (k', f') \in \mathcal{L}$ are equivalent if $f/f' = \exp((2\pi \sqrt{-1})h)$ for some function $h \in C^\infty(N, \mathbb{R})$ such that

$$\Phi^* h = k - k'.$$

The set of equivalence classes is a group $\mathcal{L}/\tau(C^\infty(N, \mathbb{R}))$.

**Theorem 3.1.1.** There exists an exact sequence of groups

$$C^\infty(N, \mathbb{R}) \xrightarrow{\tau} \mathcal{L} \to H^1(\Phi, \mathbb{Z}) \to 0.$$

Thus $H^1(\Phi, \mathbb{Z})$ parameterizes equivalence classes of pairs $(k, f)$. 
Proof. The first step will be constructing a group homomorphism

\[ \chi : \mathcal{L} \to H^1(\Phi, \mathbb{Z}). \]

Given \((k, f)\), let \(l_j \in C^\infty(V_j, \mathbb{R})\) be local logarithms for \(f|_{V_j}\), that is \(f|_{V_j} = \exp((2\pi \sqrt{-1})l_j)\).

On overlaps, \(a_{jj'} := l_j' - l_j : V_{jj'} \to \mathbb{Z}\) defines a Čech cocycle in \(\check{\mathcal{C}}^1(N, \mathbb{Z})\). Let

\[ b_i := \Phi^*l_{r(i)} - k|_{U_i} : U_i \to \mathbb{Z}. \]

Since \(b_i' - b_i = \Phi^*a_{r(i)r(i')}\), so that \((b, a)\) defines a Čech cocycle in \(\check{\mathcal{C}}^1(\Phi, \mathbb{Z})\). Given another choice of local logarithms \(\tilde{l}_j\), the Čech cocycle changes to

\[ \tilde{b}_i = b_i + \Phi^*c_{r(i)}, \quad \tilde{a}_{jj'} = a_{jj'} + c_{j'} - c_j \]

where \(c_j = \tilde{l}_j - l_j : V_j \to \mathbb{Z}\). Thus \((\tilde{b}, \tilde{a}) = (b, a) + d(0, c)\) and \(\chi(k, f) := [(b, a)] \in H^1(\Phi, \mathbb{Z})\) is well-defined. Similarly, if \((b, a) = d(0, c)\) then the new local logarithms \(\tilde{l}_j = l_j - c_j\) satisfy \(\tilde{a}_{jj'} = 0\) which means that the \(\tilde{l}_j\) patch to a global logarithm \(\tilde{L}\). \(b_i = \Phi^*c_{r(i)}\) implies that \(k|_{U_i} = \Phi^*\tilde{l}_{r(i)}\), which means \(k = \Phi^*\tilde{L}\). This shows that the kernel of \(\chi\) consists of \((k, f)\) such that there exists \(l \in C^\infty(N, \mathbb{R})\) with \(f = \exp((2\pi \sqrt{-1})l)\) and \(k = \Phi^*l\), i.e., \(ker(\chi) = im(\tau)\).

Finally, we are going to show that \(\chi\) is surjective. Suppose that \((b, a) \in \check{\mathcal{C}}^1(\Phi, \mathbb{Z})\) is a cocycle. Then

\[ a_{jj''} - a_{jj'} + a_{jj'} = 0 \quad (3.1.1) \]

\[ \Phi^*a_{r(i)r(i')} = b_{i'} - b_i \quad (3.1.2) \]

Choose a partition of unity \(\sum_{j \in J} h_j = 1\) subordinate to the open cover \(V = \{V_j\}_{j \in J}\). Define \(f_j \in C^\infty(V_j, U(1))\) by

\[ f_j = \exp(2\pi \sqrt{-1} \sum_{p \in J} a_{jp}h_p). \]
By applying (2.1.1) on $V_j \cap V_{j'}$ we have

$$f_j f_{j'}^{-1} = \exp(2\pi \sqrt{-1} \sum_{p \in J} a_{jp} h_p) \exp(-2\pi \sqrt{-1} \sum_{p \in J} a_{jp'} h_p)$$

$$= \exp(2\pi \sqrt{-1} \sum_{p \in J} a_{jj'} h_p)$$

$$= 1.$$ 

Hence the $f_i$ define a map $f \in C^\infty(N, U(1))$ such that $f|_{V_j} = f_j$. Define $k_i \in C^\infty(U_i, \mathbb{R})$ by

$$k_i = \sum_{p \in J} (\Phi^* a_{r(i)p} + b_i) \Phi^* h_p.$$ 

(3.1.3)

Since $b_i \in \mathbb{Z}$, $\exp((2\pi \sqrt{-1}) k_i) = \Phi^* f|_{U_i}$. We check that on overlaps $U_i \cap U_{i'}$, $k_i - k_{i'} = 0$, so that the $\{k_i\}$ defines a global function $k \in C^\infty(M, \mathbb{R})$ with $\Phi^* f = \exp((2\pi \sqrt{-1}) k)$. Indeed by applying (2.1.1) and (2.1.2) on $U_i \cap U_{i'}$ we have

$$\sum_{p \in J} (\Phi^* a_{r(i)p} + \Phi^* a_{pr(i')}) + b_i - b_{i'} \Phi^* h_p = \sum_{p \in J} (\Phi^* a_{r(i)j} + b_i - b_{i'}) \Phi^* h_p = 0.$$

By construction $\chi(b, a) = [(k, f)]$ which shows $\chi$ is surjective. 

Remark 3.1.1. Any $(k, f) \in \mathcal{L}$ defines a $U(1)$-valued function on the mapping cone, $\text{Cone}_\Phi = N \cup_\Phi \text{Cone}(M)$, given by $f$ on $N$ and by $\exp((2\pi \sqrt{-1}) t k)$ on $\text{Cone}(M)$. Here $t \in I$ is the cone parameter. Hence, one obtains a map

$$\mathcal{L} \to H^1(\text{Cone}_\Phi, \mathbb{Z}) \cong H^1(\Phi, \mathbb{Z}).$$

This gives an alternative way of proving the Theorem 2.1.1.

### 3.2 Geometric Interpretation of $H^2(\Phi, \mathbb{Z})$

Denote the group of Hermitian line bundles over $M$ with $\text{Pic}(M)$ and the subgroup of Hermitian line bundles over $M$ which admits a unitary section with $\text{Pic}_0(M)$. Recall that
there is an exact sequence of Abelian groups

\[ 0 \to Pic_0(M) \hookrightarrow Pic(M) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \to 0 \]

defined as follows. For the line bundle \( L \) over \( M \) with transition maps \( c_{ii'} \in C^\infty(U_{ii'}, U(1)) \) over good cover \( \{U_i\}_{i \in I} \) for \( M \), \( \delta(L) \) is the cohomology class of the 2-cocycle \( \{a_{ii'i''} : U_{ii'i''} \to \mathbb{Z}\} \) given as

\[ a_{ii'i''} := \left( \frac{1}{2\pi \sqrt{-1}} (\log c_{ii'i} - \log c_{ii'i''} + \log c_{ii''}) \right) \in \mathbb{Z}. \]

Thus we can say two Hermitian line bundles \( L_1 \) and \( L_2 \) over \( M \) are equivalent if and only if \( L_1L_2^{-1} \) admits a unitary section. The exact sequence shows \( H^2(M, \mathbb{Z}) \) parameterizes the equivalence classes of line bundles \([27]\). The class \( \delta(L) := c_1(L) \) is called the first Chern class of \( L \). Similarly, for a smooth map \( \Phi : M \to N \) we are looking for a geometric realization of \( H^2(\Phi, \mathbb{Z}) \).

**Definition 3.2.1.** Suppose \( \Phi \in C^\infty(M, N) \) and \( L_1, L_2 \) are two Hermitian line bundles over \( N \), and \( \sigma_1, \sigma_2 \) are unitary sections of \( \Phi^*L_1, \Phi^*L_2 \). Then we say \((L_1, \sigma_1)\) is equivalent to \((L_2, \sigma_2)\) if \( L_1L_2^{-1} \) admits a unitary section \( \tau \) and there is a map \( f \in C^\infty(M, \mathbb{R}) \) such that \((\Phi^*\tau)/\sigma_1\sigma_2^{-1} = \exp((2\pi \sqrt{-1})f))\).

This defines an equivalence relation among \((\sigma, L)\), where \( L \) is a Hermitian line bundle over \( N \), and \( \sigma \) is a unitary section of \( \Phi^*L \).

**Definition 3.2.2.** A relative line bundle for \( \Phi \in C^\infty(M, N) \) is a pair \((\sigma, L)\), where \( L \) is a Hermitian line bundle over \( N \) and \( \sigma \) is an unitary section for \( \Phi^*L \). Define the group of relative line bundles

\[ Pic(\Phi) = \{(\sigma, L) | L \in Pic(N), \sigma \text{ a unitary section of } \Phi^*L \} \]

and a subgroup of it

\[ Pic_0(\Phi) = \{(\sigma, L) \in Pic(\Phi) | \exists \text{ a unitary section } \tau \text{ of } L \text{ and } k \in C^\infty(M, \mathbb{R}) \text{ with } \Phi^*\tau/\sigma = \exp((2\pi \sqrt{-1})k) \}. \]
Example 3.2.1. Let \((N, \omega)\) be a compact symplectic manifold of dimension \(2n\), and let \(L \to N\) be a line bundle with connection \(\nabla\) whose curvature is \(\omega\), i.e., \(L\) is a pre-quantum line bundle with connection. A Lagrangian submanifold \(N\) satisfies the Bohr-Sommerfeld condition if there exists a global non-vanishing covariant constant \((=\text{flat})\) section \(\sigma_M\) of \(\Phi^*L\), where \(\Phi : M \to N\) is inclusion map (1.6, [25]). For any Lagrangian submanifold \(M\), \((\sigma_M, L) \in \text{Pic}(\Phi)\).

Theorem 3.2.1. There is a short exact sequence of Abelian groups

\[
0 \to \text{Pic}_0(\Phi) \to \text{Pic}(\Phi) \to H^2(\Phi, \mathbb{Z}) \to 0.
\]

Thus \(H^2(\Phi, \mathbb{Z})\) parameterizes the set of equivalence classes of pairs \((\sigma, L)\).

Proof. We can identify \(H^2(\Phi, \mathbb{Z})\) with \(H^1(\Phi, U(1))\) by Proposition 2.0.1. Let \((\sigma, L) \in \text{Pic}(\Phi)\). Let \(\{V_j\}_{j \in J}\) be a good cover of \(N\) and \(\{U_i\}_{i \in I}\) be good cover of \(M\) such that there exist a map \(r : I \to J\) with \(\Phi(U_i) \subseteq V_r(i)\). Choose unitary sections \(\sigma_j\) of \(L|_{V_j}\). The corresponding transition functions for \(L\) are

\[
g_{jj'} \in C^\infty(V_{jj'}, U(1)) \quad j, j' \in J, \quad g_{jj'} \sigma_j = \sigma_{j'} \quad \text{on} \quad V_{jj'}
\]

Define \(f_i = \Phi^*(\sigma_{r(i)})/\sigma\) on \(U_i\). Then

\[
f_i f_{i'}^{-1} = (\Phi^*(\sigma_{r(i)})/\sigma). (\Phi^*(\sigma_{r(i')})/\sigma)^{-1} = \Phi^* g_{r(i)r(i')}
\]

(3.2.1)

Since

\[
(\delta g)_{r(i)r(i')r(i'')} = 1,
\]

then \((f_i, g_{r(i)r(i')})\) is a cocycle in \(\check{\mathcal{C}}^1(\Phi, \mathbb{Z})\). If we change local sections \(\sigma_j, j \in J\), then \((f_i, g_{r(i)r(i')})\) will shift by a co-boundary. Define

\[
\chi : \text{Pic}(\Phi) \to H^1(\Phi, U(1)), \quad (\sigma, L) \mapsto [(f, g)].
\]
To find the kernel of $\chi$, suppose that $(f, g) = \delta(t, c)$. Thus $g = \delta c$ and $f = \phi^*(c) \exp(2\pi i h)^{-1}$, where $h$ is the global logarithm of $t$. Define local section $\tau_j := \sigma_j/c_j$ on $V_j$. Since on $V_{jj'}$

$$\sigma_j/c_j = \sigma_{j'}/c_{j'},$$

then we obtain a global section $\tau$. On the other hand,

$$\Phi^*\sigma_{r(i)}/\sigma = f_i = \phi^* c_{r(i)} \exp((2\pi \sqrt{-1}) h)^{-1}.$$ 

Therefore $\phi^* \tau/\sigma = \exp((2\pi \sqrt{-1}) h)^{-1}$. This exactly shows that the kernel of $\chi$ is $Pic_0(\Phi)$.

Next we are going to show $\chi$ is onto. Let $(f_i, g_{jj'}) \in C^1(\Phi, U(1))$ be a cocycle. Pick a line bundle $L$ over $N$ with $g_{jj'}$ corresponding to local sections $\sigma_j$. $\Phi^*\sigma_{r(i)}/f_i$ defines local sections for $\Phi^*L$ over $U_i$. On $U_i \cap U_j$

$$\Phi^*\sigma_{r(i)}/f_i = \Phi^*\sigma_{r(i')}/f_j$$

which defines a global section $\sigma$ for $\Phi^*L$. By construction, $\chi(\sigma, L) = [(f, g)]$. This shows $\chi$ is onto.

**Remark 3.2.1.** A relative line bundle $(L, \sigma)$ for the map $\Phi : M \to N$, defines a line bundle over the mapping cone, $\text{Cone}_\Phi = N \cup_{\Phi} \text{Cone}(M)$. This line bundle is given by $L$ on $N \subset \text{Cone}_\Phi$ and by the trivial line bundle on $\text{Cone}(M)$. The section $\sigma$ is used to glue these two bundles. Hence, one obtain a map

$$Pic(\Phi) \to H^2(\text{Cone}_\Phi, \mathbb{Z}) \cong H^2(\Phi, \mathbb{Z}).$$

### 3.3 Gerbes

Our main references for this Section are [22], [8] and [21].

Let $U = \{U_i\}_{i \in I}$ be an open cover for a manifold $M$. It will be convenient to introduce
the following notations. Suppose there is a collection of line bundles $L_{i(0),\ldots,i(n)}$ on $U_{i(0),\ldots,i(n)}$.
Consider the inclusion maps,
$$
\delta_k : U_{i(0),\ldots,i(n+1)} \to U_{i(0),\ldots,i(k),\ldots,i(n+1)} \quad (k = 0, \ldots, n+1)
$$
and define Hermitian line bundles $(\delta L)_{i(0),\ldots,i(n+1)}$ over $U_{i(0),\ldots,i(n+1)}$ by
$$
\delta L := \bigotimes_{k=0}^{n+1} (\delta_k^* L)^{(-1)^k}.
$$
Notice that $\delta(\delta L)$ is canonically trivial. If we have a unitary section $\lambda_{i(0),\ldots,i(n)}$ of $L_{i(0),\ldots,i(n)}$ for each $U_{i(0),\ldots,i(n)} \neq \emptyset$, then we can define $\delta \lambda$ in a similar fashion. Note that $\delta(\delta \lambda) = 1$ as a section of trivial line bundle.

**Definition 3.3.1.** A gerbe on a manifold $M$ on an open cover $U = \{U_i\}_{i \in I}$ of $M$, is defined by Hermitian line bundles $L_{i'i''}$ on each $U_{i'i''}$ such that $L_{i'i''} \cong L_{i'i''}^{-1}$ and a unitary section $\theta_{i'i''}$ of $\delta L$ on $U_{i'i''}$ such that $\delta \theta = 1$ on $U_{i'i''',i''''}$ Denote this data by $\mathcal{G} = (U, L, \theta)$.

We denote the set of all gerbes on $M$ on the open cover $U = \{U_i\}_{i \in I}$ by $\text{Ger}(M, U)$. Recall that an open cover $V = \{V_j\}_{j \in J}$ is a refinement of open cover $U = \{U_i\}_{i \in I}$ if there is a map $r : J \to I$ with $V_j \subset U_i$. In that case we get a map
$$
\text{Ger}(M, U) \hookrightarrow \text{Ger}(M, V).
$$
Define the group of gerbes on $M$ by
$$
\text{Ger}(M) = \lim_{\longrightarrow} \text{Ger}(M, U).
$$
We can define the product of two gerbes $\mathcal{G}$ and $\mathcal{G}'$ to be the gerbe $\mathcal{G} \otimes \mathcal{G}'$ consisting of an open cover of $M$, $V = \{V_j\}_{j \in J}$ (common refinement of open covers of $\mathcal{G}$ and $\mathcal{G}'$), line bundles $L_{i'i''} \otimes L'_{i'i''}$ on $V_{i'i''}$ and unitary sections $\theta_{i'i''} \otimes \theta'_{i'i''}$ of $\delta(L \otimes L')$ on $V_{i'i''}
$
$\mathcal{G}^{-1}$ the dual of a gerbe $\mathcal{G}$, is defined by dual bundles $L_{i'i''}^{-1}$ on $U_{i'i''}$ and sections $\theta^{-1}$ of $\delta(L^{-1})$ over $U_{i'i''}$. Therefore we have a group structure on $\text{Ger}(M)$. If $\Phi : M \to N$ be a
smooth map between two manifolds and \( \mathcal{G} \) be a gerbe on \( N \) with open cover \( \mathcal{V} = \{ V_j \}_{j \in J} \), the pull-back gerbe \( \Phi^* \mathcal{G} \) is simply defined on \( \mathcal{U} = \{ U_i \}_{i \in I} \) where \( \Phi(U_i) \subset V_{r(i)} \) for a map \( r : I \rightarrow J \), line bundles \( \Phi^* L_{r(i)r(i')} \) on \( U_{ii'} \) and unitary sections \( \theta \) of \( \delta(\Phi^* L) \) on \( U_{ii'i'} \).

**Definition 3.3.2.** (quasi-line bundle): A quasi-line bundle for the gerbe \( \mathcal{G} \) on a manifold \( M \) on the open cover \( \mathcal{U} = \{ U_i \}_{i \in I} \) is given by:

1. A Hermitian line bundle \( E_i \) over each \( U_i \)

2. Unitary sections \( \psi_{ii'} \) of

\[
(\delta E^{-1})_{ii'} \otimes L_{ii'}
\]

such that \( \delta\psi = \theta \).

Denote this quasi-line bundle with \( \mathcal{L} = (E, \psi) \).

**Proposition 3.3.1.** Any two quasi-line bundle over a given gerbe differ by a line bundle.

**Proof.** Consider two quasi-line bundles \( \mathcal{L} = (E, \psi) \) and \( \tilde{\mathcal{L}} = (\tilde{E}, \tilde{\psi}) \) for the gerbe \( \mathcal{G} = (\mathcal{U}, L, \theta) \). \( \psi_{ii'} \otimes \tilde{\psi}_{ii'}^{-1} \) is a unitary section for

\[
E_{ii'} \otimes E_i^{-1} \otimes L_{ii'}^{-1} \otimes \tilde{E}_{ii'}^{-1} \otimes \tilde{E}_i \otimes L_{ii'} \cong E_{ii'} \otimes E_i^{-1} \otimes \tilde{E}_{ii'}^{-1} \otimes \tilde{E}_i \cong E_{ii'} \otimes \tilde{E}_{ii'}^{-1} \otimes E_i^{-1} \otimes \tilde{E}_i.
\]

Therefore \( E \otimes \tilde{E}^{-1} \) defines a line bundle over \( M \).

Denote the group of all gerbes on \( M \) related to the open cover \( \mathcal{U} = \{ U_i \}_{i \in I} \) which admits a quasi-line bundle by \( \operatorname{Ger}_0(M, \mathcal{U}) \). Define

\[
\operatorname{Ger}_0(M) = \varinjlim \operatorname{Ger}_0(M, \mathcal{U}).
\]
Proposition 3.3.2. There exists a short exact sequence of groups

\[ 0 \to \text{Ger}_0(M) \to \text{Ger}(M) \xrightarrow{\chi} H^3(M, \mathbb{Z}) \to 0. \]

Proof. Identify \( H^3(M, \mathbb{Z}) \) with \( H^2(M, U(1)) \). Consider the gerbe \( G \) on \( M \). Refine the cover such that all \( L_{ii'} \) admit unitary sections \( \sigma_{ii'} \). Define

\[ t := (\delta \sigma)\theta^{-1}. \]

Thus, \( \delta t = 1 \) which means \( t \) is a cocycle. Define

\[ \chi(G) := [t]. \]

Different sections shift the cocycle by \( \delta \check{C}^1(M, U(1)) \) which shows \( \chi \) is well-defined. Also \( \chi(G \otimes G') = [tt'] = \chi(G)\chi(G') \), which proves \( \chi \) is a group homomorphism. Next, we will show that the kernel of \( \chi \) is \( \text{Ger}_0(M) \). For \( G \in \text{Ger}_0(M) \), choose a quasi-line bundle \( \mathcal{L} = (E, \psi) \).

Thus, \( t = \delta(\sigma\psi^{-1}) \). Hence, \( \chi(G) = [t] = 1 \). Conversely, if \( [t] = 1 \), then

\[ t = \delta t' \]

and by defining the new sections \( \sigma' = t'\sigma \) we see that \( \delta \sigma' = t\delta \sigma = \theta \) which shows that \( G \) admits a quasi-line bundle.

Finally, we show that \( \chi \) is onto. If \( \mathcal{U} = \{U_i\}_{i \in I} \) be an open cover of \( M \) and \( t_{ii'i''} \) is a cocycle \( \check{C}^2(M, U(1)) \), then define a gerbe \( G \) on \( M \) by trivial line bundle \( L_{ii'} \) on \( U_{ii'} \) and unitary sections \( \sigma_{ii'} \) on \( U_{ii'} \). Define \( \theta = t\delta \sigma \). Since \( \delta t = 1 \), then \( \delta \theta = 1 \). By construction \( \chi(G) = [t] \).

Definition 3.3.3. Let \( G \in \text{Ger}(M). \) \( \chi(G) \in H^2(M, U(1)) \cong H^3(M, \mathbb{Z}) \) is called Dixmier-Douady class of the gerbe \( G \) which we denote it by \( \text{D.D.} G \).

A gerbe admits a quasi-line bundle if and only if its Dixmier-Douady class is zero by Proposition 3.3.2.
Example 3.3.1. Let $G$ be a Lie group and $1 \to U(1) \to \hat{G} \to G \to 1$ a central extension. Suppose $\pi : P \to M$ is a principal $G$-bundle. A lift of $\pi : P \to M$ is a principal $\hat{G}$-bundle $\hat{\pi} : \hat{P} \to M$ together with a map $q : \hat{P} \to P$ such that $\hat{\pi} = \pi \circ q$ and the following diagram commutes:

$$\begin{diagram}
G \times \hat{P} & \rightarrow & \hat{P} \\
\downarrow & & \downarrow q \\
G \times P & \rightarrow & P
\end{diagram}$$

Suppose that $\{U_i\}_{i \in I}$ be an open cover of $M$ such that $P|_{U_i} := P_i$ has a lift $\hat{P}_i$. Define $\hat{G}$-equivariant Hermitian line bundles

$$E_i = \hat{P}_i \times_{U(1)} \mathbb{C} \rightarrow P|_{U_i}.$$ 

Since $U(1)$ acts by weight 1 on $E_i$, it acts by weight 0 on $E_i \otimes E_i^{-1} := E_{i\iota}$ on $U_{i\iota}$. Therefore $G$ acts on $E_{i\iota}$ and $E_{i\iota}/G$ is a well-defined Hermitian line bundle namely $L_{i\iota}$. By construction $\delta L$ is trivial on $U_{i\iota \iota}$, therefore we can pick trivial section $\theta$ which obviously satisfy the relation $\delta \theta = 1$. This shows the obstruction to lifting $P$ to $\hat{P}$ defines a gerbe $G$.

If $E_i \to U_i$ define a quasi-line bundle $L$ for $G$, then the line bundles $\hat{E}_i := E_i \otimes \pi^* L_i^{-1}$ patch together to a global $\hat{G}$-equivariant line bundle $\hat{E} \to P$ and the unit circle bundle defines a global lift $\hat{P} \to P$. Conversely, if $P$ admits a global lift $\hat{P}$ and $\hat{P}_i := \hat{P}|_{U_i}$, then $L_{i\iota}$ is trivial which shows the resulting gerbe is a trivial one.

Example 3.3.2. Take $N \subset M$ to be an oriented codimension 3 submanifold of an n-oriented manifold $M$. The tubular neighborhood $U_0$ of $N$ has the form $P \times_{SO(3)} \mathbb{R}^3$ where $P \to N$ is the frame bundle. Let $U_1 = M - N$. Then $U_0 \cap U_1 \cong P \times_{SO(3)} (\mathbb{R}^3 - 0)$. Over $(\mathbb{R}^3 - 0) \cong S^2 \times (0, \infty)$, we have degree 2 line bundle $E$ which is $SO(3)$ equivariant. Thus, $L_{01} := P \times_{SO(3)} E$.
Chapter 3. Geometric Interpretation of Integral Relative Cohomology Groups

is a line bundle over $U_0 \cap U_1$ which defines the only transition line bundle. Since there is no triple intersection, this data defines a gerbe over $M$.

3.4 Geometric Interpretation of $H^3(\Phi, \mathbb{Z})$

Definition 3.4.1. A relative gerbe for $\Phi \in C^\infty(M, N)$ is a pair $(\mathcal{L}, \mathcal{G})$, where $\mathcal{G}$ is a gerbe over $N$ and $\mathcal{L}$ is a quasi-line bundle for $\Phi^* \mathcal{G}$.

Notation: Let $\Phi \in C^\infty(M, N)$. Then

$$Ger(\Phi) = \{(\mathcal{L}, \mathcal{G})|(\mathcal{L}, \mathcal{G})\text{is a relative gerbe for }\Phi \in C^\infty(M, N).\}$$

$$Ger_0(\Phi) = \{(\mathcal{L}, \mathcal{G}) \in Ger(\Phi)|\mathcal{G}\text{ admits a quasi-line bundle }\mathcal{L}'\text{ s.th the line bundle }\mathcal{L} \otimes \Phi^* \mathcal{L}'^{-1}$$

admits a unitary section\}$$

Example 3.4.1. Consider a smooth map $\Phi : M \to N$ with dim $M \leq 2$. Let $\mathcal{G}$ be a gerbe on $N$. Since $\Phi^* \mathcal{G}$ admits a quasi-line bundle say $\mathcal{L}$, $(\mathcal{L}, \mathcal{G})$ is a relative gerbe.

Theorem 3.4.1. There exists a short exact sequence of Abelian groups

$$0 \to Ger_0(\Phi) \hookrightarrow Ger(\Phi) \xrightarrow{\kappa} H^3(\Phi, \mathbb{Z}) \longrightarrow 0$$

Proof. We can identify $H^3(\Phi, \mathbb{Z}) \cong H^2(\Phi, U(1))$. Let $\{V_j\}_{j \in J}$ be a good cover of $N$ and $\{U_i\}_{i \in I}$ be good cover of $M$ such that there exist a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$. Let $(\mathcal{L}, \mathcal{G}) \in Ger(\Phi)$. Refine the gerbe $\mathcal{G} = (\mathcal{U}, L, \theta)$ sufficiently such that all $L_{jj'}$ admit unitary sections $\sigma_{jj'}$. Then, define $t_{jj'j''} \in \check{C}^2(N, U(1))$ by

$$t := (\delta \sigma)(\theta)^{-1}.$$ 

Since $\delta \theta = 1$ and $\delta(\delta \sigma) = 1$, we have $\delta t = 1$. Let $\mathcal{L} = (E, \psi)$ be a quasi-line bundle for $\Phi^* \mathcal{G}$ with unitary sections $\psi_{ii'}$ for line bundles $((\delta E)_{ii'})^{-1} \otimes \Phi^* L_{r(i)r(i')}. Define s_{ii'} \in \check{C}^1(M, U(1)) by

$$s_{ii'} := (\psi_{ii'})^{-1}((\delta \lambda)^{-1}_{ii'} \otimes \Phi^* \sigma_{r(i)r(i')}$$
where $\lambda_i$ is a unitary section for $E_i$. Now
\[
(\Phi^*t^{-1})\delta(\Phi^*\sigma) = \Phi^*\theta
= \delta\psi
= (\delta s)^{-1}(\delta\delta\lambda^{-1} \otimes \delta\Phi^*\sigma)
= (\delta s)^{-1}\delta\Phi^*\sigma.
\] (3.4.1)
This proves that $\delta s = \Phi^*t$. Define the map
\[
k: \text{Ger}(\Phi) \to H^2(\Phi, U(1)), \quad \kappa(\mathcal{L}, \mathcal{G}) = [(s, t)].
\]
It is straightforward to check that this map is well-defined, i.e., it is independent of the choice of $\sigma_{ji'}$ and $\lambda_i$. Conversely, given $[(s, t)] \in H^2(\Phi, U(1))$, we can pick $\mathcal{G}$ such that $\theta = t^{-1}(\delta\sigma)$ and define
\[
\psi_{ii'} = s^{-1}_{ii'}((\delta\lambda^{-1})_{ii'} \otimes \Phi^*\sigma_{r(i)r(i')}).
\]
Since $\delta s = \Phi^*t$, then $\mathcal{L} = (E, \psi)$ defines a quasi-line bundle for $\Phi^*\mathcal{G}$. The construction shows $\kappa(\mathcal{L}, \mathcal{G}) = [(s, t)]$. Therefore $\kappa$ is onto.

We now show that $\ker(\kappa) = \text{Ger}_0(\Phi)$. Assume $\kappa(\mathcal{L}, \mathcal{G}) = [(s, t)]$ is a trivial class. Therefore there exists $(\rho, \tau) \in \tilde{C}^1(\Phi, U(1))$ such that $(s, t) = \delta(\rho, \tau) = (\Phi^*\tau(\delta\rho)^{-1}, \delta\tau)$. $t = \delta\tau$ shows that $\mathcal{G}$ admits a quasi-line bundle $\mathcal{L}'$. Thus, $\mathcal{L} \otimes \Phi^*\mathcal{L}'^{-1}$ defines a line bundle over $M$. The first Chern class of this line bundle is given by the cocycle $s(\Phi^*\tau)^{-1}$. The condition $s = (\Phi^*\tau)\delta\rho^{-1}$ shows that this cocycle is exact, i.e., the line bundle $\mathcal{L} \otimes \Phi^*\mathcal{L}'^{-1}$ admits a unitary section. Thus, $\ker(\kappa) \subseteq \text{Ger}_0(\Phi)$. Conversely, if $(\mathcal{L}, \mathcal{G}) \in \text{Ger}_0(\Phi)$ then the above argument, read in reverse, shows that $(s, t)$ is exact. Hence, $\text{Ger}_0(\Phi) \subseteq \ker(\kappa)$. \qed

Remark 3.4.1. A relative (topological) gerbe $(\mathcal{L}, \mathcal{G}) \in \text{Ger}(\Phi)$ defines a (topological)gerbe over the mapping cone by “gluing” the trivial gerbe over $\text{Cone}(M)$ with the gerbe $\mathcal{G}$ over $N \subset \text{Cone}_\Phi$. Here, the line bundles $E_i$ which define $\mathcal{L}$ play the role of transition line bundles. For gluing of gerbes see [32].
Example 3.4.2. Let $1 \to U(1) \to \hat{G} \to G \to 1$ be a central extension of a Lie group $G$. Suppose $\Phi \in C^\infty(M,N)$ and $Q \to N$ is a principal $G$-bundle. If $P = \Phi^*Q \to M$ admit a lift $\hat{P}$, then we get an element of $H^3(\Phi, \mathbb{Z})$.

Example 3.4.3. Suppose $G$ is a compact Lie group. Recall that the universal bundle $EG \to BG$ is a (topological) principal $G$-bundle with the property that any principal $G$-bundle $P \to B$ is obtained as the pull-back by some classifying map $\Phi : B \to BG$. While the classifying bundle is infinite dimensional, it can be written as a limit of finite dimensional bundles $E_nG \to B_nG$. For instance, if $G = U(k)$, one can take $E_nG$ the Stiefel manifold of unitary $k$-frames over the Grassmanian $Gr_{\mathbb{C}}(k,n)$. Furthermore, if $B$ is given, any $G$-bundle $P \to B$ is given by a classifying map $\Phi : B \to B_nG$ for some fixed $n$, sufficiently large $n$ depending only on $\dim B$ [24].

It maybe shown that $H^3(BG, \mathbb{Z})$ classifies central extension $1 \to U(1) \to \hat{G} \to G \to 1$ [6]. For $n$ sufficiently large, $H^3(B_nG, \mathbb{Z}) = H^3(BG, \mathbb{Z})$. Hence, we find that $H^3(\Phi, \mathbb{Z})$ classifies pairs $(\hat{G}, \hat{P})$, where $\hat{G}$ is a central extension of $G$ by $U(1)$ and $\hat{P}$ is a lift of $\Phi^*EG$ to $\hat{G}$.
Chapter 4

Differential Geometry of Relative Gerbes

4.1 Connections on Line Bundles

Let $L$ be a Hermitian line bundle with Hermitian connection $\nabla$ over a manifold $M$. In terms of local unitary sections $\sigma_i$ of $L|_{U_i}$ and the corresponding transition maps $g_{ii'}: U_{i'} \to U_i$, connection 1-forms $A_i$ on $U_i$ are defined by $\nabla \sigma_i = (2\pi \sqrt{-1}) A_i \sigma_i$. On $U_{i'}$, we have

$$(2\pi \sqrt{-1})(A_{i'} - A_i) = g_{ii'}^{-1} dg_{ii'}.$$

Hence, the differentials $dA_i$ agree on overlaps. The curvature 2-form $F$ is defined by $F|_{U_i} = : dA_i$. The cohomology class of $F$ is independent of the chosen connection. The cohomology class of $F$ is the image of the Chern class $c_1(L) \in H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. A given closed 2-form $F \in \Omega^2(M, \mathbb{R})$ arises as a curvature of some line bundle with connection if and only if $F$ is integral [7].

The line bundle with connection $(L, \nabla)$ is called flat if $F = 0$. In this case, we define
the holonomy of \((L, \nabla)\) as follows: We assume that the open cover \(\{U_i\}_{i \in I}\) is a good cover of \(M\). Therefore \(A_i = df_i\) on \(U_i\), where \(f_i : U_i \to \mathbb{R}\) is a smooth map on \(U_i\). Then,
\[
d(2\pi \sqrt{-1}(f_{i'} - f_i) - \log g_{ii'}) = 0.
\]
Thus,
\[
c_{ii'} := (2\pi \sqrt{-1}(f_{i'} - f_i) - \log g_{ii'})
\]
are constants. Since \(\log g\) is only defined modulo \(2\pi \sqrt{-1}\mathbb{Z}\), so what we have is a collection of constants \(\tilde{c}_{ii'} := c_{ii'} \mod \mathbb{Z}\). Different choices of \(f_i\), shift this cocyle with a coboundary. The 1-cocycle \(\tilde{c}_{ii'}\) represents a Čech class in \(\check{H}^1(M, U(1))\) which is called the holonomy of the flat line bundle \(L\) with connection \(\nabla\).

\[
\text{Let } L \to M \text{ be a line bundle with connection } \nabla, \text{ and } \gamma : S^1 \to M \text{ a smooth curve. The holonomy of } \nabla \text{ around } \gamma \text{ is defined as the holonomy of the line bundle } \gamma^*L \text{ with flat connection } \gamma^*\nabla.
\]

### 4.2 Connections on Gerbes

**Definition 4.2.1.** Let \(G = (U, L, \theta)\) be a gerbe on a manifold \(M\). A gerbe connection on \(G\) consist of connections \(\nabla_{ii'}\) on line bundles \(L_{ii'}\) such that \((\delta \nabla)_{ii'ii''} \theta_{ii'i''} := (\nabla_{ii''} \otimes \nabla_{ii'}^{-1} \otimes \nabla_{ii'}) \theta_{ii'i''} = 0\) together with 2-forms \(\varpi_i \in \Omega^2(U_i)\) such that on \(U_{ii'}, (\delta \varpi)_{ii'} = F_{ii'}\) the curvature of \(\nabla_{ii'}\). We denote this connection gerbe by a pair \((\nabla, \varpi)\).

Since \(F_{ii'}\) is a closed 2-form, the de Rham differential \(\kappa |_{U_i} := d\varpi_i\) defines a global 3-form \(\kappa\) which is called the curvature of the gerbe connection. \([\kappa] \in H^3(M, \mathbb{R})\) is the image of the Dixmier-Douady class of the gerbe under the induced map by inclusion
\[
\iota : H^3(M, \mathbb{Z}) \to H^3(M, \mathbb{R}).
\]
A given closed 3-form \(\kappa \in \Omega^2(M, \mathbb{R})\) arises as a curvature of some gerbe with connection if and only if \(i\kappa\) is integral [22].
Example 4.2.1. Suppose \( \pi : P \to B \) is a principal \( G \)-bundle, and
\[
1 \to U(1) \to \hat{G} \to G \to 1
\]
a central extension. In Example 3.3.1, we described a gerbe \( \mathcal{G} \), whose Dixmier-Douady class is the obstruction to the existence of a lift \( \tilde{\pi} : \hat{P} \to B \). We will now explain (following Brylinski [7], see also [14]) how to define a connection on this gerbe. We will need two ingredients:

(i) A principal connection \( \theta \in \Omega^1(P, \mathfrak{g}) \),

(ii) A splitting \( \tau : P \times_G \hat{\mathfrak{g}} \to B \times \mathbb{R} \) of the sequence of vector bundles
\[
0 \to B \times \mathbb{R} \to P \times_G \hat{\mathfrak{g}} \to P \times_G \mathfrak{g} \to 0
\]
associated to the sequence of Lie algebras \( 0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0 \).

For a given lift \( \tilde{\pi} : \hat{P} \to B \), with corresponding projection \( q : \hat{P} \to P \), we say that a principal connection \( \tilde{\theta} \in \Omega^1(\hat{P}, \hat{\mathfrak{g}}) \) lifts \( \theta \) if its image under \( \Omega^1(\hat{P}, \hat{\mathfrak{g}}) \to \Omega^1(P, \mathfrak{g}) \) coincides with \( q^* \theta \). Given such a lift with curvature
\[
F^{\tilde{\theta}} = d\tilde{\theta} + \frac{1}{2} [\tilde{\theta}, \tilde{\theta}] \in \Omega^2(\hat{P}, \hat{\mathfrak{g}})_{\text{basic}} = \Omega^2(B, P \times_G \mathfrak{g}),
\]
let \( K^{\tilde{\theta}} := \tau(F^{\tilde{\theta}}) \in \Omega^2(B, \mathbb{R}) \) be its “scalar part”. Any two lifts \( (\hat{P}, \hat{\theta}) \) of \( (P, \theta) \) differs by a line bundle with connection \( (L, \nabla^L) \) on \( B \). Twisting a given lift \( (\hat{P}, \hat{\theta}) \) by such a line bundle, the scalar part changes by the curvature of the line bundle
\[
K^{\tilde{\theta}} + \frac{1}{2\pi\sqrt{-1}} \text{curv}(\nabla^L) \ [7]. \tag{4.2.1}
\]
In particular, the exact 3-form \( dK^{\tilde{\theta}} \in \Omega^3(B) \) only depend on the choice of splitting and the connection \( \theta \). (It does not depend on choice of lift.) In general, a global lift \( \hat{P} \) of \( P \) does not exist. However, let us choose local lifts \( (\hat{P}_i, \hat{\theta}_i) \) of \( (P |_{U_i}, \theta) \). Denote the scalar part of \( F^{\tilde{\theta}_i} \) with \( \omega_i \in \Omega^2(U_i) \), and let \( L_{i|i'} \to U_{i|i'} \) be the line bundle with connection \( \nabla^{L_{i|i'}} \) defined by two lifts \( (\hat{P}_i |_{U_{i|i'}}, \hat{\theta}_i) \) and \( (\hat{P}_{i'} |_{U_{i|i'}}, \hat{\theta}_{i'}) \). By Equation \( 4.2.1 \),
\[
(\delta \omega)_{ii'} = \frac{1}{2\pi\sqrt{-1}} \text{curv}(\nabla^{L_{i|i'}}).
\]
On the other hand, the connection $\delta \nabla^L$ on $\left( \delta L \right)_{ii'i''} = L_{i''i'} L_{ii'}^{-1} L_{ii'}$ is just the trivial connection on the trivial line bundle. Hence, we have defined a gerbe connection.

A quasi-line bundle $(E, \psi)$ with connection $\nabla^E$ for this gerbe with connection gives rise to a global lift $(\hat{P}, \hat{\theta})$ of $(P, \theta)$, where $\hat{P} |_{U_i}$ is obtained by twisting $\hat{P}_i$ by the line bundle with connection $(E_i, \nabla^{E_i})$. The error 2-form is the scalar part of $F^{\hat{\theta}}$.

**Definition 4.2.2.** Let $G$ be a gerbe with connection with a quasi-line bundle $L = (E, \psi)$. A connection on a quasi-line bundle consists of connections $\nabla^E_i$ on line bundles $E_i$ with curvature $F^{E_i}$ such that

$$(\delta \nabla^E)_{ii'i'} := \nabla^E_i \otimes (\nabla^E_i)^{-1} \cong \nabla_{ii'i'}.$$ 

Also, the 2-curvatures obey $(\delta F^{E})_{ii'i'} = F^{E}_{ii'i'}$. We denote this quasi-line bundle with connection by $(L, \nabla^E)$. Locally defined 2-forms $\omega |_{U_i} = \omega_i - F^{E_i}$ patch together to define a global 2-form $\omega$ which is called the error 2-form [21].

**Remark 4.2.1.** The difference between two quasi-line bundles with connections is a line bundle with connection, with the curvature equal to the difference of the error 2-forms.

Let $G = (U, L, \theta)$ be a gerbe with connection on $M$. Again, assume that $U$ is a good cover. Let $t \in \check{C}^2(\check{M}, U(1))$ be a representative for the Dixmier-Douady class of $G$. Then, we have a collection of 1-forms $A_{ii'} \in \Omega^1(U_{ii'})$ and 2-forms $\omega_i \in \Omega^2(U_i)$ such that

$$\kappa |_{U_i} = d\omega_i$$

$$\delta \omega = dA$$

$$(2\pi \sqrt{-1}) \delta A = t^{-1} dt.$$ 

If $\kappa = 0$, we say the gerbe is flat. In this case by using Poincaré Lemma, $\omega_i = d\mu_i$ on $U_i$ and on $U_{ii'}$,

$$(\delta \omega)_{ii'} = d\delta (\mu)_{ii'} = dA_{ii'}.$$
Thus, again by Poincaré Lemma

\[ A_{ii'} - (\delta\mu)_{ii'} = dh_{ii'}. \]

By using \((2\pi\sqrt{-1})\delta A = t^{-1}dt\), we have

\[ d((2\pi\sqrt{-1})\delta h - \log t) = 0. \]

Therefore, what we have is the collection of constants \(c_{ii'i''} \in \check{C}^2(M, \mathbb{R})\). Since \(\log\) is defined modulo \(2\pi\sqrt{-1}\mathbb{Z}\), we define

\[ \tilde{c}_{ii'i''} := c_{ii'i''} \mod \mathbb{Z}. \]

The 2-cocycle \(\tilde{c}_{ii'i''}\) represents a Čech class in \(\check{H}^2(M, U(1))\), which we call it the holonomy of the flat gerbe with connection. Let \(\sigma : \Sigma \to M\) be a smooth map, where \(\Sigma\) is a closed surface. The holonomy of \(\mathcal{G}\) around \(\Sigma\) is defined as the holonomy of the \(\sigma^*\mathcal{G}\) of the flat connection gerbe \(\sigma^*(\nabla, \varpi)( [22], [28])\).

### 4.3 Connections on Relative Gerbes

Let \(\Phi \in C^\infty(M, N)\) and \(U = \{U_i\}_{i \in I}, V = \{V_j\}_{j \in J}\) are good covers of \(M\) and \(N\) respectively such that there exists a map \(r : I \to J\) with \(\Phi(U_i) \subseteq V_{r(i)}\).

**Definition 4.3.1.** A relative connection on a relative gerbe \((\mathcal{L}, \mathcal{G})\) consist of gerbe connection \((\nabla, \varpi)\) on \(\mathcal{G}\) and a connection \(\nabla^E\) on the quasi-line bundle \(\mathcal{L} = (E, \psi)\) for the \(\Phi^*\mathcal{G}\).

Consider a relative connection on a relative gerbe \((\mathcal{L}, \mathcal{G})\). Define the 2-form \(\tau\) on \(M\) by

\[ \tau |_{U_i} := \Phi^* \varpi_{r(i)} - F_i^E. \]

Thus, \((\tau, \kappa) \in \Omega^3(\Phi)\) is a relative closed 3-form which we call it the curvature of the relative connection.
Theorem 4.3.1. A given closed relative 3-form \((\tau, \kappa) \in \Omega^3(\Phi)\) arises as a curvature of some relative gerbe with connection if and only if \((\tau, \kappa)\) is integral.

Proof. Let \((\tau, \kappa) \in \Omega^3(\Phi)\) be an integral relative 3-form. By Proposition 2.6.1,

\[ \int_\alpha \kappa - \int_\beta \tau \in \mathbb{Z}, \quad (4.3.1) \]

where \(\alpha \subset N\) is a smooth 3-chain and \(\Phi(\beta) = \partial \alpha\), i.e., \((\beta, \alpha) \in \text{Cone}_3(\Phi, \mathbb{Z})\) is a cycle. If \(\alpha\) be a cycle then \((0, \alpha) \in \text{Cone}_3(\Phi, \mathbb{Z})\) is a cycle. In this case equation 4.3.1 shows that for all cycles \(\alpha \in S_3(N, \mathbb{Z})\),

\[ \int_\alpha \kappa \in \mathbb{Z}. \]

Therefore we can pick a gerbe \(\mathcal{G} = (\mathcal{V}, L, \theta)\) with connection \((\nabla, \varpi)\) over \(N\) with curvature 3-form \(\kappa\). Denote \(\tau_i := \tau |_{U_i}\). Define \(F^E_i \in \Omega^2(U_i)\) by

\[ F^E_i = \Phi^* \varpi_{r(i)} - \tau_i. \]

Let \((\alpha, \beta) \in \text{Cone}_3(\Phi, \mathbb{Z})\) be a cycle. Then

\[ \int_\beta F^E_i = \int_\beta (\Phi^* \varpi_{r(i)} - \tau_i) \]

\[ = \int_{\Phi(\beta)} \varpi - \int_\beta \tau \]

\[ = \int_\alpha d\varpi - \int_\beta \tau \]

\[ = \int_\alpha \kappa - \int_\beta \tau \in \mathbb{Z}. \]

Therefore we can find a line bundle \(E_i\) with connection over \(U_i\) whose curvature is equal to \(F^E_i\). Over \(U_{ii'}\), the curvature of two line bundles \(\Phi^* L_{ii'}\) and \(E_{ii'} \otimes E_{ii}^{-1}\) agrees. We can assume that the open cover \(\mathcal{U} = \{U_i\}_{i \in I}\) is a good cover of \(M\). Thus, there is a unitary section \(\psi_{ii'}\) for the line bundle \(E_i \otimes E_{ii}^{-1} \otimes \Phi^* L_{ii'}\) such that \(\delta \psi = \Phi^* \theta\). Therefore we get a quasi-line bundle \(\mathcal{L} = (E, \psi)\) with connection for \(\Phi^* \mathcal{G}\). By construction the curvature of the relative gerbe \((\mathcal{L}, \mathcal{G})\) is \((\tau, \kappa)\). Conversely, for a given relative gerbe with connection \((\mathcal{L}, \mathcal{G})\) we have \(\int_\beta F^E_i \in \mathbb{Z}\) where \(\beta \subset M\) is a 2-cycle which gives us 4.3.1. \(\square\)
Suppose that $G$ is a gerbe with a flat connection gerbe $(∇, ω)$ on $N$ and $L$ a quasi-line bundle with connection for $Φ^∗G$. Since $κ = 0$, as it explained in previous Section, we get 2-cocyles $\tilde{c}_{ii'}$ which represents a cohomology class in $\tilde{H}^2(M, U(1))$. Since $Φ^∗G$ is trivializable, we can find a collection of maps $f_{ii'}$ on $U_{ii'}$ such that $δf = Φ^∗t$ where $j = r(i)$ and $j' = r(i')$. Define $k_{ii'} ∈ \mathbb{R}$ by the formula

$$k_{ii'} =: (2π\sqrt{-1})Φ^∗h_{ii'} − \log f_{ii'},$$

and

$$\tilde{k}_{ii'} := k_{ii'} \mod \mathbb{Z}.$$ 

Thus,

$$Φ^∗\tilde{c} = δ\tilde{k}.$$ 

Define the relative holonomy of the pair $(G, L)$ by the relative class $[(\tilde{k}, \tilde{c})] ∈ H^2(Φ, U(1))$.

**Definition 4.3.2.** Let the following diagram be commutative:

$$S^1 \xrightarrow{i} \Sigma \xrightarrow{\psi} \tilde{\psi} \xrightarrow{\phi} M \xrightarrow{Φ} N$$

where $Σ$ is a closed surface, $i$ is inclusion map and all other maps are smooth. Suppose $G$ is a gerbe with connection on $N$ and $Φ^∗G$ admits a quasi-line bundle $L$ with connection. Clearly $\tilde{ψ}^∗G$ is a flat gerbe and since $i^∗\tilde{ψ}^∗G = ψ^∗Φ^∗G$, $i^∗\tilde{ψ}^∗G$ admits a quasi line bundle with connection which is equal to $ψ^∗L$. We can define the holonomy of the relative gerbe around the commutative diagram to be the same as holonomy of the pair $(ψ^∗L, ψ^∗G)$.

### 4.4 Cheeger-Simon Differential Characters

In this section, I develop a relative version of Cheeger-Simon differential characters. Denote the smooth singular chain complex on a manifold $M$ by $S^*_{sm}(M)$. Let $Z^*_{sm}(M) ⊆ S^*_{sm}(M)$
be the sub-complex of smooth cycles. Recall that a differential character of degree $k$ on a manifold $M$ is a homomorphism

$$ j : Z_{k-1}^s(M) \rightarrow U(1), $$

such that there is a closed form $\alpha \in \Omega^k(M)$ with

$$ j(\partial x) = \exp\left(2\pi \sqrt{-1} \int_x \alpha \right) $$

for any $x \in S^s_k(M)$ [9].

A connection on a line bundle defines a differential character of degree 2, with $j$ the holonomy map. Similarly, a connection on a gerbe defines a differential character of degree 3. In more details:

Any smooth $k$-chain $x \in S^s_k(M)$ is realized as a piecewise smooth map

$$ \varphi_x : K_x \rightarrow M, $$

where $K_x$ is a $k$-dimensional simplicial complex [20]. Then, by definition

$$ \int_{K_x} \alpha = \int_x \alpha, \quad \alpha \in \Omega^k(M). $$

Suppose that $y \in Z^s_2(M)$, $y = \Sigma \epsilon_i \sigma_i$ with $\epsilon_i = \pm 1$. Assume $\mathcal{G}$ is a gerbe with connection over $M$. Since $H^3(K_y, \mathbb{Z}) = 0$, $\varphi_y^* \mathcal{G}$ admits a piecewise smooth quasi-line bundle $\mathcal{L}$ with connection. That is, a quasi-line bundles $\mathcal{L}_i$ for all $\varphi_y^* \mathcal{G} \mid_{\Delta^k_{\sigma_i}}$, such that they agree on the matching boundary faces. Let $\omega \in \Omega^2(K_y)$ be the error 2-form and define

$$ j(y) := \exp\left(2\pi \sqrt{-1} \int_{K_y} \omega \right). $$

Any two quasi-line bundles differ by a line bundle and hence different choices for $\mathcal{L}$, changes $\omega$ by an integral 2-form. This shows that $j$ is well-defined. Assume that $y = \partial x$. Since the components of $K_x$ with empty boundary will not contribute, we can assume that each
component of $K_x$ has non-empty boundary. Since $H^3(K_x, \mathbb{Z}) = 0$, we can choose a quasi-line bundle with connection for $\varphi_x^*G$ with error 2-form $\omega$. Let $k$ be the curvature of $G$, since $\varphi_x^*k = d\omega$, by stokes’ Theorem we have

$$\int_{K_x} k = \int_{K_x} d\omega = \int_{\partial K_x} \omega = \int_{K_y} \omega.$$ 

This shows that $j$ is a differential character of degree 3.

**Definition 4.4.1.** Let $\Phi \in C^\infty(M, N)$ be a smooth map between manifolds. A relative differential character of degree $k$ for the map $\Phi$ is a homomorphism

$$j : Z_k^{sm}(\Phi) \to U(1),$$

such that there is a closed relative form $(\beta, \alpha) \in \Omega^k(\Phi)$ with

$$j(\partial(y, x)) = \exp(2\pi \sqrt{-1}(\int_y \beta - \int_x \alpha))$$

for any $(y, x) \in S_k^{sm}(\Phi)$.

**Theorem 4.4.1.** A relative connection on a relative gerbe defines a relative differential character of degree 3.

**Proof.** Let $\Phi \in C^\infty(M, N)$ be a smooth map between manifolds and consider a relative gerbe $(\mathcal{L}, \mathcal{G})$ with connection. Let $(y, x) \in S_1^{sm}(\Phi)$ be a smooth relative singular cycle, i.e.,

$$\partial y = 0$$

and

$$\Phi_*(y) = \partial x.$$ 

Let $K_y$ and $K_x$ be the corresponding simplicial complex, and

$$\Phi : K_y \to K_x.$$
be the induced map. Given a relative connection, choose a quasi-line bundle $\mathcal{L}'$ for $\varphi^*_x \mathcal{G}$ and a unitary section $\sigma$ of the line bundle $H := \varphi^*_y \mathcal{L} \otimes (\Phi^* \mathcal{L}')^{-1}$. Let $\tilde{\omega} \in \Omega^2(N)$ be the error 2-form for $\mathcal{L}'$ and $A \in \Omega^1(M)$ be the connection 1-form for $H$ with respect to $\sigma$. Define a map $j$ by

$$j(y, x) := \exp \left( 2\pi \sqrt{-1} \left( \int_{K_x} \tilde{\omega} - \int_{K_y} A \right) \right).$$

If we choose another quasi-line bundle for $\varphi^*_x \mathcal{G}$, the difference of error 2-forms will be an integral 2-form and changing the section $\sigma$ will shift connection 1-form $A$ to $A + A'$, where $A'$ is an integral 1-form. This proves that

$$j : Z^m_2(\Phi) \to U(1)$$

is well-defined. Let $k$ be the curvature 3-form for $\mathcal{G}$ and $\omega$ be the error 2-form for $\mathcal{L}$. Then $(\omega, k) \in \Omega^3(\Phi)$ and

$$j(\partial(y, x)) = j(\partial y, \Phi_*(y) - \partial x)$$

$$= \exp \left( 2\pi \sqrt{-1} \left( \int_{K_{\Phi_*(y) - \partial x}} \tilde{\omega} - \int_{K_{\partial y}} A \right) \right)$$

$$= \exp \left( 2\pi \sqrt{-1} \left( \int_{K_{\Phi_*(y)}} \tilde{\omega} - \int_{K_{\partial x}} \tilde{\omega} - \int_{K_{\partial y}} A \right) \right)$$

$$= \exp \left( 2\pi \sqrt{-1} \left( \int_{K_y} (\Phi_* \tilde{\omega} - dA) - \int_{K_x} d\tilde{\omega} \right) \right)$$

$$= \exp \left( 2\pi \sqrt{-1} \left( \int_{K_y} \omega - \int_{K_x} k \right) \right).$$

Thus, $j$ is a relative differential character in degree 3.

\[\Box\]

### 4.5 Relative Deligne Cohomology

Suppose we have a co-chain complex of sheaves $\underline{A}^*$ over a manifold $M$. For an open cover $\mathcal{U} = \{U_i\}_{i \in I}$, define $A^p_{\mathcal{U}} := \check{C}^p(\mathcal{U}, \underline{A}^q)$ to be the Čech $p$-cochains with values in the sheaf $\underline{A}^q$. On $A^p_{\mathcal{U}}$ we have two differentials, $d^q : A^p_{\mathcal{U}} \to A^{p+q}_{\mathcal{U}}$ which is induced from the differential of...
co-chain complex of sheaves $A^\bullet$ and Čech coboundary map $\delta : A^{p,q}_U \to A^{p+1,q}_U$. Since $\delta d = d\delta$, $d^2 = 0$ and $\delta^2 = 0$, $A^\bullet_U$ is a double complex. We denote the total complex of this double complex by $T^\bullet_{(U,A)}$. A refinement $V < U$ will induce mappings $T^\bullet_{(U,A)} \to T^\bullet_{(V,A)}$. The sheaf hypercohomology groups ([7], [15]) of $M$ with values in the cochain complex of sheaves $A^\bullet$ are defined by

$$H^\bullet(M, A) := \lim_{\to} H^\bullet(T^\bullet_{(U,A)}).$$

For any $f \in C^\infty(U_i, U(1))$ define

$$d \log f := \frac{1}{2\pi \sqrt{-1}} f^{-1} df \in \Omega^1(U_i).$$

It is known that classes in the first Deligne hypercohomology group

$$H^1(M; U(1) \xrightarrow{d \log} \Omega^1)$$

are in a one-to-one correspondence with isomorphism classes of line bundle with connection on $M$ and classes in the second Deligne hypercohomology group

$$H^2(M; U(1) \xrightarrow{d \log} \Omega^1 \xrightarrow{d} \Omega^2)$$

are in a one-to-one correspondence with isomorphism classes of gerbes with connection on $M$ [7]. Suppose $M$ and $N$ are two manifolds. Fix a map $\Phi \in C^\infty(M, N)$. Let $U = \{U_i\}_{i \in I}$, $V = \{V_j\}_{j \in J}$ be good covers of $M$ and $N$ respectively such that there exists a map $r : I \to J$ with $\Phi(U_i) \subseteq V_{r(i)}$. Also assume $A^\bullet$ and $B^\bullet$ are two cochain complexes of sheaves over $M$ and $N$ respectively and one is given a homomorphism of sheaf complexes $\Phi^* : B^\bullet \to A^\bullet$. Then $\Phi$ induces mappings map

$$(\Phi^*)^{p,q} : B^{p,q}_V \to A^{p,q}_U.$$

$\Phi^*$ will induce mappings

$$\Phi^* : T^\bullet_{(V,B)} \to T^\bullet_{(U,A)}.$$
Denote the algebraic mapping cone of \((\tilde{\Phi}^\ast)\) by \(\text{Cone}_{(U, V)}(\tilde{\Phi}^\ast)\) and its corresponding cohomology by \(H^\bullet_{(U, V)}(\tilde{\Phi})\). If \((U', V')\) is a double-refinement of \((U, V)\), i.e., \(U'\) is a refinement of \(U\), \(V'\) is a refinement of \(V\) and there are maps \(r : I \to J\) with \(\Phi(U_i) \subseteq V_{r(i)}\), \(r' : I' \to J'\) with \(\Phi(U'_i) \subseteq V'_{r'(i)}\), then we will have induced mappings
\[
\text{Cone}_{(U', V')}(\tilde{\Phi}^\ast) \to \text{Cone}_{(U, V)}(\tilde{\Phi}^\ast).
\]

Define the relative sheaf hypercohomology groups \(H^\bullet(\Phi; A, B)\) by

\[
H^\bullet(\Phi; A, B) := \lim_{\to} H^\bullet_{(U, V)}(\Phi).
\]

**Definition 4.5.1.** We define relative Deligne cohomology of \(\Phi\) by

\[
H^\bullet(\Phi; U(1) \xrightarrow{\text{dlog}} \Omega^1; U(1) \xrightarrow{\text{dlog}} \Omega^1 \xrightarrow{d} \Omega^2)
\]

and we will denote it by \(H^\bullet_D(\Phi)\).

**Theorem 4.5.1.** There is a one-to-one correspondence between classes of \(H^2_D(\Phi)\) and isomorphism classes of relative gerbe with connection.

**Proof.** Let \(G = (V, L, \theta)\) be a gerbe with connection on \(N\). Assume that \(V\) is a good cover. Let \(t \in \check{C}^2(N, U(1))\) be a representative for the Dixmier-Douady class of \(G\). Then, we have a collection of 1-forms \(\varpi_{jj'} \in \Omega^1(V_{jj'})\), 2-forms \(\omega_j \in \Omega^2(V_j)\) and 3-form \(\kappa \in \Omega^3(N)\) such that

\[
\kappa|_{V_j} = d\varpi_j, \quad \delta\varpi = dA, \quad (2\pi\sqrt{-1})\delta A = t^{-1} dt.
\]

This defines a class \((\varpi, A, t) \in H^2(N; U(1) \xrightarrow{\text{dlog}} \Omega^1 \xrightarrow{d} \Omega^2)\). Let \(\mathcal{L}\) be a quasi-line bundle with connection for \(\Phi^\ast G\). We can use \(\Psi\) to get \(h \in \check{C}^1(M, U(1))\) such that

\[
\Phi^\ast t = \delta h.
\]
Also by using the identity $(\delta F)_{i'i'} = (\Phi^* F)_{r(i)r(i')}$, we can find $l \in \Omega^1(U_{i'i'})$ such that

$$\Phi^* A = \delta l$$

Therefore, a relative gerbe $(\mathcal{L}, \mathcal{G})$ gives a class $[((l,h),(\varpi,A,t))] \in H^2_D(\Phi)$. \qed

### 4.6 Transgression

For a manifold $M$, we denote its loop space by $LM$. In this Section first we will construct a line bundle with connection over $LM$ by transgressing a gerbe with connection over $M$. A map $\Phi \in C^\infty(M,N)$ induces a map $L\Phi \in C^\infty(LM,LN)$. We will prove that a relative gerbe with connection on $\Phi$ produce a relative line bundle with connection on $L\Phi$ by transgression. In cohomology language it means that there is a map

$$T : H^*_D(\Phi) \to H^{*-1}_D(L\Phi).$$

**Proposition 4.6.1.** (Parallel transportation) Suppose that $\mathcal{G}$ is a gerbe with connection on $M \times [0,1]$ and $\mathcal{G}_0 = \mathcal{G}|_{(M \times \{0\})}$. There is a natural quasi-line bundle with connection for the gerbe $\pi^* \mathcal{G}_0 \otimes \mathcal{G}^{-1}$, where $\pi$ is the projection map

$$\pi : M \times [0,1] \to M \times \{0\}.$$ 

**Proof.** It is obvious that we can get a quasi-line bundle with connection for the gerbe $\pi^* \mathcal{G}_0 \otimes \mathcal{G}^{-1}$. We will specify a quasi-line bundle $\mathcal{L}_\mathcal{G}$ for the gerbe $\pi^* \mathcal{G}_0 \otimes \mathcal{G}^{-1}$ by the following requirements:

1. The pull-back $\iota^* \mathcal{L}_\mathcal{G}$ is trivial, while $\iota$ is inclusion map

$$\iota : M \times \{0\} \hookrightarrow M \times [0,1].$$

2. Let $\eta \in \Omega^3(M \times [0,1])$ be the connection 3-form for $\pi^* \mathcal{G}_0 \otimes \mathcal{G}^{-1}$. We have $\iota^* \eta = 0$. Let $\chi \in \Omega^2(M \times [0,1])$ be canonical primitive of $\eta$ given by transgression. Then we choose the
connection on $\mathcal{L}_G$ such that its error 2-form is $\chi$. Any two such quasi-line bundles differ by a flat line bundle over $M \times [0,1]$ and which is trivial over $M \times \{0\}$.\hfill\qed

**Theorem 4.6.2.** A gerbe $\mathcal{G}$ with connection on $M \times S^1$, induces a line bundle $E_{\mathcal{G}}$ with connection on $M$. Also, a quasi-line bundle with connection for $\mathcal{G}$ induces a unitary section of $E_{\mathcal{G}}$.

**Proof.** $M \times S^1 = M \times [0,1]/\sim$, where the equivalence relation is defined by $(m,0) \sim (m,1)$ for $m \in M$. Therefore $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}|_{M \times \{1\}/\sim}$ is a trivial gerbe and $\mathcal{L}_G|_{M \times \{1\}/\sim}$ is a quasi-line bundle with connection for this trivial gerbe, i.e., a line bundle with connection $E_{\mathcal{G}}$ for $M$. If we change $\mathcal{L}_G$ to another natural quasi-line bundle with connection, the difference between two line bundles over $M \times S^1$ is a trivial line bundle. This shows the assignment $\mathcal{G} \rightarrow E_{\mathcal{G}}$ is well-defined.

If the gerbe $\mathcal{G}$ admits a quasi-line bundle $\mathcal{L}$, then $(\pi^*\mathcal{L}_0) \otimes (\mathcal{L}^{-1})$ and $\mathcal{L}_G$ are two quasi-line bundles for the gerbe $\pi^*\mathcal{G}_0 \otimes \mathcal{G}^{-1}$, where $\mathcal{L}_0 = \mathcal{L}|_{M \times \{0\}}$. Thus $\pi^*\mathcal{L}_0 \otimes \mathcal{L}^{-1} \otimes (\mathcal{L}_G)^{-1}$ defines a line bundle over $M \times S^1 = M \times [0,1]/\sim$. This line bundle over $M$ defines a map $s : M \rightarrow U(1)$. $(\pi^*\mathcal{L}_0) \otimes (\mathcal{L}^{-1})|_{M \times \{0\}/\sim}$ is the trivial line bundle $E$. Since $E_{\mathcal{G}} \otimes E^{-1} = s$, therefore $E_{\mathcal{G}}$ admits a unitary section.\hfill\qed

**Remark 4.6.1.** Let $X = LM$ and $\mathcal{G}$ is a gerbe with connection $M$. Consider the evaluation map

$$e : LM \times S^1 \rightarrow M.$$  

Then $e^*\mathcal{G}$ will induce a line bundle with connection on $LM$.

**Theorem 4.6.3.** For a given map $\Phi \in C^\infty(M,N)$, a relative gerbe with connection $\mathcal{G}_\Phi$ will induce a relative line bundle with connection $E_{L\Phi}$.

**Proof.** The relative gerbe $\mathcal{G}_\Phi$ is a gerbe $\mathcal{G}$ on $N$ together with a quasi-line bundle with connection $\mathcal{L}$ for the pull-back gerbe $\Phi^*\mathcal{G}$. The gerbe $\mathcal{G}$ induces a line bundle with connection
$E_G$ and the quasi-line bundle with connection $\mathcal{L}$ for $\Phi^*G$ induces a unitary section $s$ for the line bundle with connection $(L\Phi)^*E_G$ by Theorem 4.6.2. Thus, the pair $(s, E_G)$ defines a relative line bundle with connection $E_{L\Phi}$.

\qed
Chapter 5

Pre-quantization of Group-Valued Moment Maps

5.1 Gerbes over a Compact Lie Group

It is well-known fact that for a compact, simple, simply connected Lie group the integral cohomology $H^\bullet(G, \mathbb{Z})$ is trivial in degree less than three, while $H^3(G, \mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. The gerbe corresponding to the generator of $H^3(G, \mathbb{Z})$ is called the basic gerbe over $G$. In this section, I give an explicit construction of the basic gerbe for $G = SU(n)$, and of suitable multiples of the basic gerbe for the other Lie groups. This gerbe plays an important role in pre-quantization of the quasi-Hamiltonian $G$-spaces.

5.1.1 Some Notations from Lie Groups

Let $G$ be a compact, simple simply connected Lie group and with a maximal torus $T$. Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of $G$ and $T$ respectively. Denote by $\Lambda \subset \mathfrak{t}$ the integral lattice,
given as the kernel of 
\[ \exp : t \to T. \]

Let \( \Lambda^* = Hom(\Lambda, \mathbb{Z}) \subset t^* \) be its dual weight lattice. Recall that any \( \mu \in \Lambda^* \) defines a homomorphism 
\[ h_\mu : T \to U(1), \exp \xi \mapsto e^{2\pi \sqrt{-1}\langle \mu, \xi \rangle}. \]

This identifies \( \Lambda^* = Hom(T, U(1)) \). Let \( \mathcal{R} \subset \Lambda^* \) be the set of roots, i.e., the non-zero weights for the adjoint representation. Define 
\[ t^{reg} := t \setminus \bigcup_{\alpha \in \mathcal{R}} \ker \alpha. \]

The closures of the connected components of \( t^{reg} \) are called Weyl chambers. Fix a Weyl chamber \( t_+ \). Let \( \mathcal{R}_+ \subset \Lambda^* \) be the set of the positive roots, i.e., roots that are non-negative on \( t_+ \). Then \( \mathcal{R} = \mathcal{R}_+ \cup -\mathcal{R}_+ \). A positive root is called simple, if it cannot be written as the sum of positive roots. We denote the set of simple roots by \( S \). The set of simple roots \( S \subset \mathcal{R}_+ \) forms a basis of \( t \), and

\[ t_+ = \{ \xi \in t \mid \langle \alpha, \xi \rangle \geq 0, \forall \alpha \in S \}. \]

Any root \( \alpha \in \mathcal{R} \) can be uniquely written as
\[ \alpha = \Sigma k_i \alpha_i, \quad k_i \in \mathbb{Z}, \alpha_i \in S. \]

The height of \( \alpha \) is defined by \( ht(\alpha) = \Sigma k_i \). Since \( g \) is simple, there is a unique root \( \alpha_0 \) with \( ht(\alpha) \geq ht(\alpha_0) \) for all \( \alpha \in \mathcal{R} \) which is called the lowest root. The fundamental alcove is defined as
\[ \mathfrak{A} = \{ \xi \in t_+ \mid \langle \alpha_0, \xi \rangle \geq -1 \}. \]

The basic inner product on \( g \) is the unique invariant inner product such that \( \alpha.\alpha = 2 \) for all long roots \( \alpha \) which we use it to identify \( g^* \cong g \). The mapping \( \xi \to Ad G(\exp \xi) \) is a homeomorphism from \( \mathfrak{A} \) onto \( G/Ad G \), the space of the conjugacy classes in \( G \). Therefore the
fundamental alcove parameterizes conjugacy classes in $G$ [11]. We will denote the quotient map by $q : G \to \mathfrak{a}$.

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left and right invariant Maurer Cartan forms. If $L_g$ and $R_g$ denote left and right multiplication by $g \in G$, then the values of $\theta^L_g$ and $\theta^R_g$ at $g$ are given by

$$
\theta^L_g = dL_{g^{-1}} : TG_g \to TG_e \cong \mathfrak{g}, \quad \theta^R_g = dR_{g^{-1}} : TG_g \to TG_e \cong \mathfrak{g}.
$$

For any $g \in G$,

$$
\theta^L_g = Ad_g(\theta^R_g).
$$

For any invariant inner product $B$ on $\mathfrak{g}$, the form

$$
\eta := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)
$$

is bi-invariant since the inner product is invariant. Any bi-invariant form on a Lie group is closed, therefore $\eta$ is a closed 3-form. Its cohomology class represents the generator of $H^3(G, \mathbb{R}) = \mathbb{R}$ if we assume that $G$ is compact and simple. If in addition $G$ is simply connected, then $H^3(G, \mathbb{Z}) = \mathbb{Z}$ and one can normalize the inner product such that $[\eta]$ represents an integral generator [18]. We will say that $B$ is the inner product at level $\lambda > 0$ if $B(\xi, \xi) = 2\lambda$ for all short lattice vectors $\xi \in \Lambda$. The inner product at level $\lambda = 1$ is called the basic inner product. (It is related to the Killing form by a factor $2c_\mathfrak{g}$, where $c_\mathfrak{g}$ is the dual Coxeter number of $\mathfrak{g}$.) Suppose $G$ is simply connected and simple. It is known that the 3-form defined by Equation 5.1.1 is integral if and only if its level $\lambda$ of $B$ is an integer.

### 5.1.2 Standard Open Cover of G

Let $\mu_0, \cdots, \mu_d$ be the vertices of $\mathfrak{a}$, with $\mu_0 = 0$. Let $\mathfrak{a}_j$ be the complement of the closed face opposite to the vertex $\mu_j$. The standard open cover of $G$ is defined by the pre-images
$V_j = q^{-1}(\mathfrak{A}_j)$. Denote the centralizer of $\exp \mu_j$ by $G_j$. Then the flow-out $S_j = G_j, \exp(\mathfrak{A}_j)$ is an open subset of $G_j$, and is a slice for the conjugation action of $G$. Therefore

$$G \times_{G_j} S_j = V_j.$$ 

More generally let $\mathfrak{A}_I = \cap_{j \in I} \mathfrak{A}_j$, and $V_I = q^{-1}(\mathfrak{A}_I)$. Then $S_I = G_I, \exp(\mathfrak{A}_I)$ is a slice for the conjugation action of $G$ and therefore

$$G \times_{G_I} S_I = V_I.$$ 

We denote the projection to the base by

$$\pi_I : V_I \to G/G_I.$$ 

**Lemma 5.1.1.** $\eta_G$ is exact over each of the open subsets $V_j$.

*Proof.* $S'_j := G_j \cdot (\mathfrak{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$ and is $G_j$-equivariantly diffeomorphic with $S_j$. We can extend this retraction from $S_j$ onto $\exp(\mu_j)$ to a $G$-equivariant retraction from $V_j$ onto $C_j = q^{-1}(\mu_j)$. But since $d_G \omega_{C_j} + \iota_{C_j}^* \eta_G = 0$, then $\eta_G$ is exact over $V_j$. 

Let $\iota_j : C_j \to V_j$ and $\pi_j : V_j \to G/G_j = C_j$ denote the inclusion and the projection respectively. The retraction from $V_j$ onto $C_j$ defines a $G$-equivariant homotopy operator

$$h_j : \Omega^p(V_j) \to \Omega^{p-1}(V_j).$$

Thus,

$$d_G h_j + h_j d_G = Id - \pi_j^* \iota_j^*.$$ 

Define the equivariant 2-form $\varpi_j$ on $V_j$ by $(\varpi_j)_G = h_j \eta_G - \pi_j^* \omega_{C_j}$. Write $(\varpi_j)_G = \varpi_j - \theta_j$ where $\varpi_j \in \Omega^2(V_j)$ and $\theta_j \in \Omega^0(V_j, \mathfrak{g})$. For any conjugacy class $C \subset V_j$, $\iota^*(\varpi_j)_G + \omega_C$ is an equivariantly closed 2-form with $\theta_j$ as its moment map. Therefore $\iota^*(\varpi_j)_G + \omega_C = \theta_j^*(\omega_O)_G$, where $(\omega_O)_G$ is the symplectic form on the (co)-adjoint orbit $O = \theta_j(C)$. 


Proposition 5.1.2. Over $V_{ij} = V_i \cap V_j$, $\theta_i - \theta_j$ takes values in the adjoint orbit $O_{ij}$ through $\mu_i - \mu_j$. Furthermore,

$$(\varpi_i)_G - (\varpi_j)_G = \theta_{ij}^* (\omega_{O_{ij}})_G$$

where $\theta_{ij} := \theta_i - \theta_j : V_{ij} \to O_{ij}$, and $(\omega_{O_{ij}})_G$ is the equivariant symplectic form on the orbit.

Proof. Let $\nu : \mathfrak{a}_j \to \mathfrak{t}$ be the inclusion map. Then

$$\tilde{h}_j \circ (\exp |_{\mathfrak{a}_j}) \frac{1}{2} (\theta^L + \theta^R) = \tilde{h}_j \circ d\nu = \nu - \mu_j$$

where $\tilde{h}_j$ is the homotopy operator for the linear retraction of $\mathfrak{t}$ onto $\mu_j$. This proves that $(\exp |_{\mathfrak{a}_j})^* \theta_j = \nu - \mu_j$. Therefore, for $\xi \in \mathfrak{a}_{ij}$ we have,

$$\theta_{ij}(\exp \xi) = (\xi - \mu_i) - (\xi - \mu_j) = \mu_j - \mu_i.$$ 

Therefore $\theta_{ij}$ takes values in the adjoint orbit through $\mu_j - \mu_i$ by equivariance. The difference $\varpi_i - \varpi_j$ vanishes on $T$ and is therefore determined by its contractions with generating vector fields. But $\theta_{ij}$ is a moment map for $\varpi_i - \varpi_j$, hence $\varpi_i - \varpi_j$ equals to the pull-back of the symplectic form on $O_{ij}$. \qed

5.1.3 Construction of the Basic Gerbe

Let $G$ be a compact, simple, simply connected Lie group, and $B$ is an invariant inner product at integral level $k > 0$. Use $B$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{t} \cong \mathfrak{t}^*$. We assume that under this identification, all vertices of the alcove are contained in the weight lattice $\Lambda^* \subset \mathfrak{t}$. This is automatic if $G$ is the special unitary group $A_d = SU(d+1)$ or the compact symplectic group $C_d = Sp(2d)$. In general, the following table lists the smallest integer $k$ with this property [5]:
For constructing the basic gerbe over $G$, we pick the standard open cover of $G$, $\mathcal{V} = \{V_i, i = 0, \cdots, d\}$. For any $\mu \in \Lambda^*$, with stabilizer $G_\mu$, define a line bundle

$$L_\mu = G \times_{G_\mu} \mathbb{C}_\mu$$

with the unique left invariant connection $\nabla$, where $\mathbb{C}_\mu$ is the 1-dimensional $G_\mu$-representation with infinitesimal character $\mu$. $L_\mu$ is a $G$-equivariant pre-quantum line bundle for the orbit $\mathcal{O} = G \cdot \mu$. Therefore

$$\frac{i}{2\pi} \text{curve}_G(\nabla) = (\omega_\mathcal{O})_G := \omega_\mathcal{O} - \Phi_\mathcal{O}$$

where $\omega_\mathcal{O}$ is a symplectic form for the inclusion map $\Phi_\mathcal{O} : \mathcal{O} \to \mathfrak{g}^*$. Define line bundles

$$L_{ij} := \theta_{ij}^*(L_{\mu_j - \mu_i})$$

equipped with the pull-back connection. In three fold intersection $V_{ijk}$, the tensor product $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$ is the pull-back of the line bundle over $G/G_{ijk}$ which is defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of $G_{ijk}$. Therefore it is canonically trivial with trivial connection. The trivial sections $t_{ijk} = 1$, satisfy $\delta t = 1$ and $(\delta \nabla)t = 0$. Define $(F_j)_G = (\varpi_j)_G$. Since

$$\delta(F)_G = \theta_{ij}^*(\omega_\mathcal{O}_{ij})_G = \frac{1}{(2\pi \sqrt{-1}) \text{curve}_G(\nabla^{ij})},$$

then $\mathcal{G} = (\mathcal{V}, L, t)$ is a gerbe with connection $(\nabla, \varpi)$. The construction of the basic gerbe is discussed in more general cases in [30, 3].
Chapter 5. Pre-quantization of Group-Valued Moment Maps

5.1.4 The Basic Gerbe Over SU(n)

In this Section, I show that the our construction of the basic gerbe over $SU(n)$ is equivalent to the construction of the basic gerbe in Gawedzki-Reis [12].

The special unitary group is the classical group:

$$SU(n) = \{ A \in U(n) \mid \det A = 1 \},$$

which is a compact connected Lie group of dimension equal to $n^2 - 1$ with Lie algebra equal to the space:

$$su(n) = \{ A \in L_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) \mid A^* + A = 0 \text{ and } trA = 0 \}.$$

Any matrix $A \in SU(n)$ is conjugate to a diagonal matrix with entries

$$\text{diag} \left( \exp((2\pi\sqrt{-1})\lambda_1(A)), \cdots, \exp((2\pi\sqrt{-1})\lambda_n(A)) \right)$$

where $\lambda_1(A), \cdots, \lambda_n(A) \in \mathbb{R}$ are normalized by the identity $\Sigma_{i=1}^n \lambda_i(A) = 0$ and

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \geq \lambda_1(A) - 1. \quad (5.1.2)$$

Consider the following maximal torus of $SU(n)$,

$$T = \{ A \in SU(n) \mid A \text{ is diagonal} \}.$$

Let $t$ be the Lie algebra of $T$. Thus, $t \cong \{ \lambda \in \mathbb{R}^n \mid \Sigma_{i=1}^n \lambda_i = 0 \}$. The roots $\alpha \in \mathcal{R} \subset t^*$ are the linear maps:

$$\alpha_{ij} : t \to \mathbb{R}, (\lambda_1, \cdots, \lambda_n) \mapsto \lambda_i - \lambda_j, \ i \neq j,$$

and the set of simple roots is

$$\mathcal{S} = \{ \alpha_{1,2}, \alpha_{2,3}, \cdots, \alpha_{n-1,n} \}.$$

The lowest root is $\alpha_{n,1}$ ( [26], Appendix C). Choose the following Weyl chamber

$$t_+ = \{ (\lambda_1, \cdots, \lambda_n) \in t \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}.$$
In that case the fundamental alcove is

\[ \mathfrak{A} = \{ (\lambda_1, \ldots, \lambda_n) \in t \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_1 - 1 \} \].

The basic inner product on \( t \) is induced from the standard basic inner product on \( \mathbb{R}^n \). We can use this inner product to identify \( t \cong t^* \). Under this identification \( \alpha_{i,j} = e_i - e_j \), where \( \{e_i\}_{i=1}^n \) is the standard basis for \( \mathbb{R}^n \). The fundamental weights are given by

\[ \mu_i = \{ \lambda \in \mathfrak{A} \mid \lambda_1 = \lambda_2 = \cdots = \lambda_i > \lambda_{i+1} = \cdots = \lambda_n = \lambda_1 - 1 \} \].

\( SU(n)_{reg} = \{ A \in SU(n) \mid \text{all eigenvalues of } A \text{ have multiplicity one} \} \cong G \times T \). int \( \mathfrak{A} \cong G/T \).

For \( i \in \{1, \ldots, n\} \), define

\[ \mathfrak{A}_i := \{ \lambda \in \mathfrak{A} \mid \lambda_1 \geq \cdots \geq \lambda_i > \lambda_{i+1} \geq \cdots \geq \lambda_n \geq \lambda_1 - 1 \} \].

Thus, the standard open cover for \( SU(n) \) is \( V = \{ V_i \}_{i=1}^n \), where \( V_i = q^{-1}(\mathfrak{A}_i) \). Each \( SU(n)_{ij} \) is isomorphic to \( U(n-1) \) with the center isomorphic to \( U(1) \). Over the set of regular elements all the inequalities are strict and we have \( n \) equivariant line bundles \( E_1, \ldots, E_n \) defined by the eigenlines for the eigenvalues \( \exp((2\pi \sqrt{-1})\lambda_i(A)) \). For \( i < j \), the tensor product \( E_{i+1} \otimes \cdots \otimes E_j \rightarrow SU(n)_{reg} \) extends to a line bundle \( E_{ij} \rightarrow V_{ij} \). For \( i < j < k \), we have a canonical isomorphism \( E_{ij} \otimes E_{jk} \cong E_{ik} \) over \( V_{ijk} \). These line bundles together with corresponding isomorphisms define a gerbe over \( SU(n) \), in Gawedzki-Reis sense, which represents the generator of \( H^3(SU(n), \mathbb{Z}) \). Each \( E_i = G \times_T \mathbb{C}\nu_i \) for some \( \nu_i \in \Lambda^* \). In fact, by using the standard action of \( T \subset SU(n) \) on \( \mathbb{C}^n \), one can see that

\[ \nu_i = e_i - \frac{1}{n}(1, \ldots, 1) \].

Since \( \mu_i = \Sigma_{k=1}^i e_k - \frac{i}{n} \Sigma_{k=1}^n e_k \), therefore, \( \mu_i = \Sigma_{k=1}^i \nu_k \). Recall that in Section 5.1.3, to construct the basic gerbe, we defined \( L_{ij} := \theta_{ij}^*(L_{\mu_j - \mu_i}) \) on \( V_{ij} \). Thus, for \( i < j \) we have

\[
L_{ij} = \theta_{ij}^*(L_{\mu_j - \mu_i}) \\
= \theta_{ij}^*(L_{\Sigma_{k=1}^j \nu_k - \Sigma_{k=1}^i \nu_k}) \\
= \theta_{ij}^*(L_{\Sigma_{k=1}^{i+1} \nu_k}) = E_{ij}
\]
5.2 The Relative Gerbe for $\text{Hol} : \mathcal{A}_G(S^1) \to G$

Denote the affine space of connection on the trivial bundle $S^1 \times G$ by $\mathcal{A}_G(S^1)$. Thus, $\mathcal{A}_G(S^1) = \Omega^1(S^1, \mathfrak{g})$. The loop group $LG = \text{Map}(S^1, G)$ acts on $\mathcal{A}_G(S^1)$ by gauge transformations:

$$g \cdot A = \text{Ad}_g(A) - g^* \theta_R.$$  

(5.2.1)

Taking the holonomy of a connection defines a smooth map

$$\text{Hol} : \mathcal{A}_G(S^1) \to G$$

with equivariance property $\text{Hol}(g \cdot A) = \text{Ad}_{g(0)} \text{Hol}(A)$. If $\mathfrak{g}$ carries an invariant inner product $B$, we write $Lg^*$ instead of $\mathcal{A}_G(S^1)$ using the natural pairing between $\Omega^1(S^1, \mathfrak{g})$ and $Lg = \Omega^0(S^1, \mathfrak{g})$. We will refer to this action as the coadjoint action. However, notice that the action 5.2.1 is not the point wise action. Recall that we have constructed in Section 5.1.3,

- a) An open cover $\mathcal{V} = \{V_0, \cdots, V_d\}$ of $G$ such that $V_j/G = \mathfrak{A}_j$, where $d = \text{rank}G$.

- b) For each $V_j$, a unique $G$-equivariant deformation retraction on to a conjugacy class $C_j = G \cdot \exp(\mu_j)$, where $\mu_j$ is the vertex of $\mathfrak{A}_j$. This deformation retraction descends to the linear retraction of $\mathfrak{A}_j$ to $\mu_j$.

- c) 2-forms $\omega_j \in \Omega^2(V_j)$, with $d\omega_j = \eta|_{V_j}$, such that the pull-back onto $C_j$ is the invariant 2-form for the conjugacy class $C_j$.

Lemma 5.2.1. There exists a unique $LG$-equivariant retraction from $\tilde{V}_j := \text{Hol}^{-1}(V_j)$ onto the coadjoint orbit $O_j = LG \cdot \mu_j$, descending to the linear retraction of $\mathfrak{A}_j$ onto $\mu_j$.

Proof. The holonomy map sets up a one-to-one correspondence between the sets of $G$-conjugacy classes and coadjoint $LG$-orbits, hence both are parameterized by points in the alcove. The evaluation map $LG \to G, g \mapsto g(1)$ restricts to an isomorphism $(LG)_j \cong G_j$.
Hence,

\[ LG \times_{(LG)} \tilde{S}_j = \tilde{V}_j \]

where \( \tilde{S}_j = \text{Hol}^{-1}(S_j) \) and \( S_j = G_j \cdot \exp(\mathfrak{a}_j) \). Therefore the unique equivariant retraction from \( V_j \) onto \( C_j \), which descends to the linear retraction of \( \mathfrak{a}_j \) onto the vertex \( \mu_j \), lifts to the desired retraction.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{V}_j & \xrightarrow{\tilde{\pi}_j} & \mathcal{O}_j \\
\downarrow \text{Hol} & & \downarrow \text{Hol} \\
V_j & \xrightarrow{\pi_j} & C_j
\end{array}
\]

where \( \tilde{\pi}_j : \tilde{V}_j \to \mathcal{O}_j \) is the projection that we get from retraction. Let \( \sigma_j \in \Omega^2(\tilde{V}_j) \) denote the pull-backs under \( \tilde{\pi}_j \) of the symplectic forms on \( \mathcal{O}_j \).

**Lemma 5.2.2.** On overlaps \( \tilde{V}_j \cap \tilde{V}_{j'} \), \( \sigma_j - \sigma_{j'} = \text{Hol}^*(\omega_j - \omega_{j'}) \).

**Proof.** Both sides are closed \( LG \)-invariant forms, for which the pull-back to \( t \subset L\mathfrak{g}^* \) vanishes. Hence, it suffices to check that at any point \( \mu \in \mathcal{O}_j \cap \mathcal{O}_{j'} \subset t \subset L\mathfrak{g}^* \), the contraction with \( \zeta_{L\mathfrak{g}^*} \) are equal for \( \zeta \in L\mathfrak{g} \). We have

\[
\iota(\zeta)\sigma_j = \tilde{\pi}_j^* dB(\Phi_j, \zeta)
\]

\[
= dB(\tilde{\pi}_j^* \Phi_j, \zeta)
\]

where \( \Phi_j : \mathcal{O}_j \hookrightarrow L\mathfrak{g}^* \) is inclusion. But

\[
(\tilde{\pi}_j^* \Phi_j - \tilde{\pi}_{j'}^* \Phi_{j'}) |_{g \mu} = g \cdot \mu_j - g \cdot \mu_{j'}
\]

\[
= (\text{Ad}_g(\mu_j) - g^* \theta^R) - (\text{Ad}_g(\mu_{j'}) - g^* \theta^R)
\]

\[
= \text{Ad}_g(\mu_j - \mu_{j'}).
\]
This, however, is another moment map for \( \text{Hol}^*(\varpi_j - \varpi'_j) \).

We conclude that the locally defined forms

\[
\varpi |_{\tilde{V}_j} = \text{Hol}^*(\varpi_j) - \sigma_j
\]

patch together to define a global 2-form \( \varpi \in \Omega^2(\mathfrak{g}^*) \). Form the properties of \( \varpi_j \) and \( \sigma_j \) we read off:

(i) \( d\varpi = \text{Hol}^* \eta \),

(ii) \( \iota(\zeta \mathfrak{Lg}^*) \varpi = \frac{1}{2} B(\text{Hol}^*(\theta^L + \theta^R), \zeta(0)) - dB(\mu, \zeta) \).

Here \( \mu : \mathfrak{Lg}^* \to \mathfrak{g}^* \) is the identity map. Such a 2-form was constructed in [2] using a different method.

Consider the case \( G = SU(n) \) or \( G = Sp(n) \), which are the two cases where the vertices of the alcove lie in the weight lattice \( \Lambda^* \), where we identify \( \mathfrak{g}^* \cong \mathfrak{g} \) using the basic inner product. Let

\[
U(1) \to \hat{\mathcal{L}G} \to \mathcal{L}G
\]

denote the \( k \)-th power of the basic central extension of the loop group [31]. That is, on the Lie algebra level the central extension

\[
\mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g}
\]

is defined by the cocycle,

\[
(\xi_1, \xi_2) \mapsto \int_{S^1} B(\xi_1, d\xi_2) \quad \xi \in \mathfrak{g} = \Omega^0(S^1, \mathfrak{g}))
\]

where \( B \) is the inner product at level \( k \). The coadjoint action of \( \hat{\mathcal{L}G} \) on \( \hat{\mathfrak{Lg}}^* = \mathfrak{Lg}^* \times \mathbb{R} \) preserves the level sets \( \mathfrak{Lg}^* \times \{t\} \), and the action for \( t = 1 \) is exactly the gauge action of \( \mathcal{L}G \) considered above. Since \( \mu_j \in \Lambda^* \), the orbits \( \mathcal{L}G \cdot \mu_j = \mathcal{O}_j \) carry \( \hat{\mathcal{L}G} \)-equivariant pre-quantum line bundles \( \mathcal{L}_\mathcal{O}_j \to \mathcal{O}_j \), given explicitly as

\[
\mathcal{L}_\mathcal{O}_j = \hat{\mathcal{L}G} \times_{(\hat{\mathcal{L}G})} \mathbb{C}_{(\mu_j, 1)}.
\]
Here $(\hat{L}G)_j$ is the restriction of $\hat{L}G$ to the stabilizer $(LG)_j$ of $\mu_j \in t \subset Lg^*$, and $C_{(\mu_j, 1)}$ denotes the 1-dimensional representation of $(\hat{L}G)_j$ with weight $(\mu_j, 1) \in \Lambda^* \times \mathbb{Z}$. Let $E_j \to \tilde{V}_j$ be the pull-back $\tilde{\pi}_j^* L_{\mathcal{O}_j}$. On overlaps, $\tilde{V}_j \cap \tilde{V}_{j'}$, $E_j \otimes E_{j'}^{-1}$ is an associated bundle for the weight $(\mu_j, 1) - (\mu_{j'}, 1) = (\mu_j - \mu_{j'}, 0)$. Therefore, $E_j \otimes E_{j'}^{-1}$ is an $LG$-equivariant bundle. It is clear by construction that $E_j \otimes E_{j'}^{-1}$ is the pull-back of the pre-quantum line bundle over $\mathcal{O}_{jj'} \subset g^*$. Taking all this information together, we have constructed an explicit quasi-line bundle for the pull-back of the $k$-th power of the basic gerbe under holonomy map $\text{Hol} : Lg^* \to G$ with error 2-form equal to $\varpi \in \Omega^2(Lg^*)$.

### 5.3 Review of Group-Valued Moment Maps

Suppose $(M, \omega)$ is a symplectic manifold together with a symplectic action of a Lie group $G$. This action called Hamiltonian if there exists a smooth equivariant map

$$\Phi : M \to g^*$$

such that

$$\iota(\xi_M)\omega + d\langle \Phi, \xi \rangle = 0$$

for all $\xi \in g$, where $\xi_M$ is the vector field on $M$ generated by $\xi \in g$, i.e.,

$$\xi_M(m) = \frac{d}{dt} |_{t=0} \exp(t\xi) \cdot m.$$  

The map $\Phi$ and the triple $(M, \omega, \Phi)$ are known as moment map and Hamiltonian $G$-manifold respectively [17]. Let $G$ be a compact Lie group. Fix an invariant inner product $B$ on $g$, which we use to identify $g \cong g^*$. Since the exponential map $\exp : g \to G$ is a diffeomorphism in a neighborhood of the origin, the composition map

$$\Psi := \exp \circ \Phi : M \to G$$

inherits the properties of the moment map $\Phi$ and vice versa.
Definition 5.3.1. A quasi-Hamiltonian $G$-space with group-valued moment map is a triple $(M, \omega, \Psi)$ consisting of a $G$-manifold $M$, an invariant 2-form $\omega \in \Omega^2(M)$, and an equivariant smooth map $\Psi : M \to G$ such that

1. $d\omega = \Psi^*\eta$ where $\eta \in \Omega^3(G)$ is the 3-form defined by $B$. This condition is called the relative cocycle condition.

2. $\iota(\xi_M)\omega = \frac{1}{2} B(\Psi^*(\theta^L + \theta^R), \xi)$. This condition is called the moment map condition.

3. The $\ker(\omega_m) \in T_m(M)$ for $m \in M$ consists of all $\xi_M(m)$ such that

   $$(Ad_{\Psi(m)} + 1)\xi = 0.$$ 

This is called the minimal degeneracy condition.

5.3.1 Examples

Example 5.3.1. Consider a Hamiltonian $G$-manifold $(M, \omega, \Phi)$ such that the image of $\Phi$ is a subset of the set of regular values for the exponential map. Then $(M, \Upsilon, \Psi)$ is a quasi-Hamiltonian $G$-space with group-valued moment map, where

$$\Psi = \exp \circ \Phi$$

and

$$\Upsilon := \omega + \Phi^*\varpi$$

where $\varpi \in \Omega^2(\mathfrak{g})$ is the primitive for $\exp^*\eta$ given by the de Rham homotopy operator for the vector space $\mathfrak{g}$. The converse is also true, provided that $\Psi(M)$ lies in a neighborhood of the origin on which the exponential map is a diffeomorphism.
Example 5.3.2. Let $\mathcal{C} \subset G$ be a conjugacy class of $G$. The triple $(\mathcal{C}, \omega, \Phi)$ is a quasi-Hamiltonian $G$-space with group-valued moment map where $\Phi : \mathcal{C} \hookrightarrow G$ is inclusion and $\omega_g(\xi_C(g), \zeta_C(g)) = \frac{1}{2}B((\text{Ad}_g - \text{Ad}_{g^{-1}})\xi, \zeta)$ \[19\].

Example 5.3.3. Given an involutive Lie group automorphism $\rho \in \text{Aut}(G)$, i.e., $\rho^2 = 1$, one defines twisted conjugacy classes to be the orbits of the action $h \cdot g = \rho(h)gh^{-1}$. $G$ is a symmetric space

$$G = G \times G/(G \times G)^\rho$$

where $\rho(g_1, g_2) = (g_2, g_1)$. The map $G \times G \to \mathbb{Z}_2 \ltimes G \times G, (g_1, g_2) \mapsto (\rho^{-1}, g_1, g_2)$ takes the twisted conjugacy classes of $G \times G$ to conjugacy classes of the disconnected group $\mathbb{Z}_2 \ltimes G \times G$. Thus by using example 5.3.2 the group $G$ itself becomes a group-valued Hamiltonian $\mathbb{Z}_2 \ltimes G \times G$, with 2-form $\omega = 0$, moment map $g \mapsto (\rho, g, g^{-1})$ and action $(g_1, g_2) \cdot g = g_2gg_1^{-1}, \rho \cdot g = g^{-1}$.

Example 5.3.4. Let $D(G)$ be a product of two copies of $G$. On $D(G)$, we can define a $G \times G$ action by

$$(g_1, g_2).(a, b) = (g_1ag_2^{-1}, g_2ag_1^{-1}).$$

Define a map

$$\Psi : D(G) \to G \times G, \quad \Psi(a, b) = (ab, a^{-1}b^{-1})$$

and let the 2-form $\omega$ be defined by

$$\omega = \frac{1}{2}((\text{Pr}_1^* \theta^L, \text{Pr}_2^* \theta^R) + (\text{Pr}_1^* \theta^R, \text{Pr}_2^* \theta^L))$$

where $\text{Pr}_1$ and $\text{Pr}_2$ are projections to the first and second factor. Then the triple $(D(G), \omega, \Psi)$ is a Hamiltonian $G \times G$-manifold with group-valued moment map.
Example 5.3.5. Let $G = SU(2)$ and $M = S^4$ the unit sphere in $\mathbb{R}^5 \cong \mathbb{C}^2 \times \mathbb{R}$, with SU(2)-action induced from the action on $\mathbb{C}^2$. $M$ carries the structure of a group-valued Hamiltonian $SU(2)$-manifold, with the moment map $\Psi : M \to SU(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \to S^2$. For details, see [1]. This example is generalized by Hurtubise-Jeffrey-Sjamaar in [23] to $G = SU(n)$ acting on $M = S^{2n}$ (viewed as unit sphere in $\mathbb{C}^n \times \mathbb{R}$).

The equivariant de Rham complex is defined as

$$\Omega^k_G(M) = \bigoplus_{2i+j=k} (\Omega^j(M) \otimes S^i(g^*))^G$$

where $S(g^*)$ is the symmetric algebra over the dual of the Lie algebra of $G$. Elements in this complex can be viewed as equivariant polynomial maps from $g$ into the space of differential forms. $\Omega_G(M)$ carries an equivariant differential $d_G$ of degree 1,

$$(d_G \alpha)(\xi) := d\alpha(\xi) + \iota(\xi_M)\alpha(\xi).$$

Since $(d + \iota(\xi_M))^2 = L(\xi_M)$ and we are restricting on the equivariant maps, $d_G^2 = 0$. The equivariant cohomology is the cohomology of this co-chain complex [18]. The canonical 3-form $\eta$ has a closed equivariant extension $\eta_G \in \Omega^3_G(G)$ given by

$$\eta_G(\xi) := \eta + \frac{1}{2}B(\theta^L + \theta^R, \xi).$$

We can combine the first two conditions of the definition of a group-valued moment map and get the condition

$$d_G \omega = \Psi^* \eta_G.$$
5.3.2 Products

Suppose \((M, \omega, (\Psi_1, \Psi_2))\) is a group-valued Hamiltonian \(G \times G\)-manifold. Then \(\tilde{M} = M\) with diagonal action, moment map \(\tilde{\Psi} = \Psi_1 \Psi_2\) and 2-form

\[
\tilde{\omega} = \omega - \frac{1}{2} B(\Psi_1^* \theta^L, \Psi_2^* \theta^R)
\]

is a group-valued quasi-Hamiltonian \(G\)-space. If \(\tilde{M} = M_1 \times M_2\) is a direct product of two group-valued quasi-Hamiltonian \(G\)-spaces, we call \(\tilde{M}\) the fusion product of \(M_1\) and \(M_2\). This product is denoted by \(M_1 \circledast M_2\). If we apply fusion to the double \(D(G)\), we obtain a group-valued quasi-Hamiltonian \(G\)-space with \(G\)-action

\[
g \cdot (a, b) = (Ad_g a, Ad_g b),
\]

moment map

\[
\Psi(a, b) = aba^{-1}b^{-1} \equiv [a, b],
\]

and 2-form

\[
\omega = \frac{1}{2}(B(Pr_1^* \theta^L, Pr_2^* \theta^R) + B(Pr_1^* \theta^R, Pr_2^* \theta^L) - B((ab)^* \theta^L, (a^{-1}b^{-1})^* \theta^R)).
\]

Fusion of \(h\) copies of \(D(G)\) and conjugacy classes \(C_1, \cdots, C_r\) gives a new quasi-Hamiltonian space with the moment map

\[
\Psi(a_1, b_1, \cdots, a_h, b_h, d_1, \cdots, d_r) = \prod_{j=1}^h [a_j, b_j] \prod_{k=1}^r d_k.
\]

5.3.3 Reduction

The symplectic reduction works as usual:

If \((M, \omega, \Psi)\) be a Hamiltonian \(G\)-space with group-valued moment map and the identity element \(e \in G\) be a regular value of \(\Psi\), then \(G\) acts locally freely on \(\Psi^{-1}(e)\) and therefore \(\Psi^{-1}(e)/G\) is smooth. Furthermore, the pull- back of \(\omega\) to identity level set descends to a
symplectic form on \( M//G := \Psi^{-1}(e)/G \). For instance, the moduli space of flat \( G \)-bundles on a closed oriented surface of genus \( h \) with \( r \) boundary components, can be written

\[
\mathcal{M}(\Sigma; C_1, \cdots, C_r) = G^{2h} \otimes C_1 \otimes \cdots \otimes C_r//G = \Psi^{-1}(e)/G
\]

where the \( j \)-th boundary component is the bundle corresponding to the conjugacy class \( C_j \).

More details can be found in [2], [1].

### 5.4 Pre-quantization of \( G \)-Valued Moment Maps

We know that a symplectic manifold \( (M, \omega) \) is pre-quantizable (admits a line bundle \( L \) over \( M \) with curvature 2-form \( \omega \)) if the 2-form \( \omega \) is integral. In this Section, we will first introduce a notion of a pre-quantization of a space with \( G \)-valued moment map and then give a similar criterion for being pre-quantizable.

**Definition 5.4.1.** Let \( G \) be a compact connected Lie group with canonical 3-form \( \eta \). Fix a gerbe \( \mathcal{G} \) on \( G \) with connection \( (\nabla, \varpi) \) such that \( \text{curv}(\mathcal{G}) = \eta \). A pre-quantization of \( (M, \omega, \Psi) \) is a relative gerbe with connection \( (\mathcal{L}, \mathcal{G}) \) corresponding to the map \( \Psi \) with relative curvature \( (\omega, \eta) \).

Since \( \eta \) is closed 3-form and \( \Psi^* \eta = d \omega \), \( (\omega, \eta) \) defines a relative cocycle. Recall from chapter 1 that a class \([ (\omega, \eta) ] \in H^3(\Psi, \mathbb{R})\) is integral if and only if \( \int_\beta \eta - \int_\Sigma \omega \in \mathbb{Z} \) for all relative cycles \( (\beta, \Sigma) \in C_3(\Psi, \mathbb{R}) \).

**Remark 5.4.1.** \((M, \omega, \Psi)\) is pre-quantizable if and only if \([ (\omega, \eta) ] \) is integral by Theorem 4.3.1.

**Theorem 5.4.1.** Suppose \( M_i, \ i = 1, 2 \) are two quasi-Hamiltonian \( G \)-spaces. The fusion product \( M_1 \circledast M_2 \) is pre-quantizable if both \( M_1 \) and \( M_2 \) are pre-quantizable.
Proof. Let Mult : \( G \times G \to G \) be group multiplication and \( \text{Pr}_i : G \times G \to G \), \( i = 1, 2 \) projections to the first and second factors. Since

\[
\text{Mult}^* \eta = \text{Pr}_1^* \eta + \text{Pr}_2^* \eta + \frac{1}{2} B(\text{Pr}_1^* \theta_L, \text{Pr}_1^* \theta_R),
\]

we get a quasi-line bundle with connection for the gerbe \( \text{Mult}^* G \otimes (\text{Pr}_1^* G)^{-1} \otimes (\text{Pr}_2^* G)^{-1} \) such that the error 2-form is equal to \( \frac{1}{2} B(\text{Pr}_1^* \theta_L, \text{Pr}_1^* \theta_R) \). Any two such quasi-line bundles differ by a flat line bundle with connection. Let \( \Psi_i, i = 1, 2 \) be moment maps for \( M_i \), \( i = 1, 2 \) respectively and \( \Psi = \Psi_1 \Psi_2 \) be the moment map for their fusion product \( M_1 \otimes M_2 \). Thus,

\[
\Psi^* G = (\Psi_1 \times \Psi_2)^* \text{Mult}^* G = (\Psi_1^* \times \Psi_2^*) ((\text{Pr}_1^* G) \otimes (\text{Pr}_2^* G)) = \Psi_1^* G \otimes \Psi_2^* G.
\]

Therefore \( M_1 \otimes M_2 \) is pre-quantizable if and only if both \( M_1 \) and \( M_2 \) are pre-quantizable. \( \square \)

Proposition 5.4.2. Suppose \( G \) is simple and simply connected. Let \( k \in \mathbb{Z} \) be the level of \( (M, \omega, \Psi) \). Suppose \( H^2(M, \mathbb{Z}) = 0 \). Then there exists a pre-quantization of \( (M, \omega, \Psi) \) if and only if the image of

\[
\Psi^* : H^3(G, \mathbb{Z}) \to H^3(M, \mathbb{Z})
\]

is \( k \)-torsion.

Proof. By assumption, \([\eta]\) represents \( k \) times the generator of \( H^3(G, \mathbb{Z}) \). If \( H^2(M, \mathbb{Z}) = 0 \), the long exact sequence:

\[
\cdots \to H^2(G, \mathbb{Z}) \to H^2(M, \mathbb{Z}) \to H^3(\Psi, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \xrightarrow{\Psi^*} H^3(M, \mathbb{Z}) \to \cdots
\]
shows that the map $H^3(\Psi, \mathbb{Z}) \to H^3(G, \mathbb{Z})$ is injective. In particular, $H^3(\Psi, \mathbb{Z})$ has no torsion, and $(M, \omega, \Psi)$ is pre-quantizable if and only if $[\eta]$ is in the image of the map $H^3(\Psi, \mathbb{Z}) \to H^3(G, \mathbb{Z})$, i.e., in the kernel of $H^3(G, \mathbb{Z}) \to H^3(M, \mathbb{Z})$. This exactly means that the image of this map is $k$-torsion.

**Proposition 5.4.3.** If $H_2(M, \mathbb{Z}) = 0$ a pre-quantization exists. More generally, if $H_2(M, \mathbb{Z})$ is $r$-torsion, a level $k$ pre-quantization exists, where $k$ is a multiple of $r$.

**Proof.** If $rH_2(M, \mathbb{Z}) = 0$, for any cycle $S \in C_2(M)$, there is a 3-chain $T \in C_3(M)$ with $\partial T = r \cdot S$. If $\Psi(S) = \partial B$, $\Psi(T) - rB$ is a cycle and

$$
\int_S k\omega - \int_B k\eta = \frac{k}{r}(\int_T d\omega - \int_{rB} \eta) = \frac{k}{r}(\int_T \Psi^*\eta - \int_{rB} \eta) = \frac{k}{r}(\int_{\Psi(T)} \eta - \int_{rB} \eta) = \frac{k}{r}(\int_{\Psi(T) - rB} \eta) \in \mathbb{Z}.
$$

(5.4.1)

By Remark 5.4.1 $(M, \omega, \Psi)$ is pre-quantizable.

**Example 5.4.1.** $M = S^4$ carries the structure of a group-valued Hamiltonian $SU(2)$-manifold, with the moment map $\Psi : M \to SU(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \to S^2$.

By Proposition 5.4.3 this $SU(2)$-valued moment map is pre-quantizable.
5.4.1 Reduction

Let $G$ be a simply connected Lie group. Fix a pre-quantization $\mathcal{L}$ for a space with $G$-valued moment map $(M, \omega, \Psi)$.

\[
\begin{array}{ccc}
\mathcal{L} & \rightarrow & G \\
\downarrow & & \downarrow \\
M & \xrightarrow{\Psi} & G \\
\uparrow \iota & & \uparrow \iota \\
\Psi^{-1}(e) & \xrightarrow{\Psi} & \{e\}
\end{array}
\]

Since $\mathcal{G}|_{\Psi^{-1}(e)}$ is equal to trivial gerbe, $\mathcal{L}|_{\Psi^{-1}(e)}$ is a line bundle with connection with curvature $(\iota_{\Psi^{-1}(e)})^*\omega$. Since $G$ is simply connected and the 2-form $(\iota_{\Psi^{-1}(e)})^*\omega$ is $G$-basic, there exists a unique lift of the $G$-action to $\mathcal{L}|_{\Psi^{-1}(e)}$ in such a way that the generating vector fields on $\mathcal{L}|_{\Psi^{-1}(e)}$ are horizontal. This is a special case of Kostant’s construction [27]. In conclusion, we get a pre-quantum line bundle over $\Psi^{-1}(e)/G$.

5.4.2 A Finite Dimensional Pre-quantum Line Bundle for $\mathcal{M}(\Sigma)$

Let $M = G^{2h}$ where $G$ is a simply connected Lie group and consider the map

$\Psi : M \rightarrow G$

with the rule

$\Psi(a_1, \cdots, a_h) = \prod_{i=1}^{h} [a_i, b_i]$.

Let $\mathcal{G}$ be the basic gerbe with the connection on $G$ and $\text{curv}(\mathcal{G}) = \eta$. The moduli space of flat $G$-bundles on a closed oriented surface $\Sigma$ of genus $h$ is equal to

$\mathcal{M}(\Sigma) = G^{2h} / G = \Psi^{-1}(e) / G$.

Since $G$ is simply connected, $H^2(G, \mathbb{Z}) = H^2(G^{2h}, \mathbb{Z}) = 0$. $H^3(G^{2h}, \mathbb{Z}) \cong \mathbb{Z}$ is torsion free therefore by Proposition 4.3.3 there exists a unique quasi-line bundle $\mathcal{L}$ for the gerbe $\Psi^* \mathcal{G}$. 
Pick a connection for this quasi-line bundle and call the error 2-form $\nu$. Therefore $d(\nu - \omega) = 0$. This together with the fact that $H^2(M, \mathbb{Z}) = 0$ allow us to modify quasi-line bundle with connection such that triple $(M, \omega, \Psi)$ is pre-quantizable. By reduction we get a pre-quantum line bundle over $\Psi^{-1}(e)/G = \mathcal{M}(\Sigma)$.

### 5.4.3 Pre-quantization of Conjugacy Classes of a Lie Group

Let $G$ be a simple, simply connected compact Lie group. Fix an inner product $B$ at level $k$. The map

$$\exp : \mathfrak{g} \rightarrow G$$

takes (co)adjoint orbits $\mathcal{O}_\xi$ to conjugacy classes $\mathcal{C} = G \cdot \exp(\xi)$.

Any conjugacy class $\mathcal{C} \subseteq G$ is uniquely a $G$-valued quasi-Hamiltonian $G$-space $(\mathcal{C}, \omega, \Psi)$, where $\Psi : \mathcal{C} \hookrightarrow G$ is inclusion map, as it explained in example 5.3.2. Suppose $(\beta, \Sigma) \in \text{Cone}_n(\Psi, \mathbb{Z})$ is a cycle. We want to see under which conditions $\mathcal{C}$ is pre-quantizable at level $k$. Equivalently, we are looking for conditions which implies

$$k(\int_\beta \eta - \int_\Sigma \omega) \in \mathbb{Z}$$

where $\eta = \frac{1}{12} B(\theta_L^L, [\theta_L^L, \theta_L^L])$ is canonical 3-form. Consider the basic gerbe $\mathcal{G} = (\mathcal{V}, L, \theta)$ with connection on $G$ with curvature $\eta$. For all $\mathcal{C} \subseteq G$ there exists a unique $\xi \in \mathfrak{a}$ such that $\exp(\xi) \in \mathcal{C}$. Let

$$\iota_\mathcal{C} : \mathcal{C} \rightarrow G$$

be inclusion map assume that $\varpi_0$ is the primitive of $\eta$ on $V_0$, i.e.,

$$\eta |_{V_0} = d\varpi_0$$

where $V_0$ contains $\mathcal{C}$. Recall from Section 5.1.3 that

$$\omega_\mathcal{C} = \theta_0^* (\omega_{\mathcal{O}_\xi})_G - \iota_\mathcal{C}^* (\varpi_0)_G$$
and pull-back of the $\theta$ to $\mathcal{C}$ is zero. Thus,

$$k(\int_{\Sigma} \omega_{\mathcal{C}} \beta - \int_{\beta} i_\xi^* \eta) = k(\int_{\Sigma} \theta_0^*(\omega_{\mathcal{O}_\xi}) - \int_{\beta} i_\xi^*(\omega_0) - \int_{\beta} i_\xi^*(\eta))$$

$$= k(\int_{\Sigma} \theta_0^*(\omega_{\mathcal{O}_\xi})).$$

$(\omega_{\mathcal{C}}, i_\xi^* \eta)$ is integral if and only if the symplectic 2-form $k\omega_{\mathcal{O}_\xi}$ is integral. It is a well-known fact from symplectic geometry that $k\omega_{\mathcal{O}_\xi}$ is integral if and only if $B(\xi) \in \Lambda_k^* := \Lambda^* \cap k\mathfrak{a}$, by viewing $B$ as a linear map $t \to t^*$.

### 5.5 Hamiltonian Loop Group Spaces

Fix an invariant inner product $B$ on $\mathfrak{g}$. Assume that $G$ is simple and simply connected. Recall that a Hamiltonian loop group manifold is a triple $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$ where $\tilde{M}$ is an (infinite-dimensional) $LG$-manifold, $\tilde{\omega}$ is an invariant symplectic form on $\tilde{M}$, and $\tilde{\Psi} : \tilde{M} \to L\mathfrak{g}^*$ an equivariant map satisfying the usual moment map condition,

$$\iota(\xi_{\tilde{M}})\tilde{\omega} + dB(\tilde{\Psi}, \xi) = 0 \quad \xi \in \Omega^0(S^1, \mathfrak{g}).$$

**Example 5.5.1.** Let $\mathcal{O} \subset L\mathfrak{g}^*$ be an orbit of the loop group action. Then $\mathcal{O}$ carries a unique structure for a Hamiltonian $LG$-manifold when the moment map is inclusion and the 2-form is

$$\tilde{\omega}_{\mathcal{O}}(\xi_{\mathcal{O}}(\mu), \eta_{\mathcal{O}}(\mu)) = \langle d_\mu \xi, \eta \rangle = \int_{S^1} B((d_\mu \xi), \eta).$$

The based loop group $\Omega G \subset LG$ consisting of loops that are trivial at the origin of $S^1$, acts freely on $L\mathfrak{g}^*$ and the quotient is just the holonomy map. There is a one-to-one correspondence between quasi-Hamiltonian $G$-spaces $(M, \omega, \Psi)$ and Hamiltonian $LG$-spaces
with proper moment maps $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$, where

\[
M = \tilde{M}/\Omega G, \\
\text{Hol} \circ \tilde{\Psi} = \Psi \circ \text{Hol}, \\
\tilde{\omega} = \text{Hol}^* \omega - \tilde{\Psi} \varpi.
\]

This is called \textit{Equivalence Theorem} in [2]. We thus, have a commutative diagram:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{\Psi}} & Lg^* \\
\text{Hol} \downarrow & & \text{Hol} \downarrow \\
M & \xrightarrow{\Psi} & G
\end{array}
\]

\textbf{Theorem 5.5.1. (Equivalence Theorem for Pre-quantization)} There is a one-to-one correspondence between pre-quantizations of quasi-Hamiltonian $G$-spaces with group valued moment maps and pre-quantizations of the corresponding Hamiltonian $LG$-spaces with proper moment maps.

\textit{Proof.} Assume that we have constructed a pre-quantization of a quasi-Hamiltonian $G$-space with group-valued moment map $(M, \omega, \Psi)$ with the corresponding Hamiltonian $LG$-space $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$. Thus, we have a relative gerbe mapping to the basic gerbe over $G$. Pullback of this quasi-line bundle under $\text{Hol} : \tilde{M} \to M$, gives a quasi-line bundle of the gerbe $\text{Hol}^* \Psi^* \mathcal{G} = \tilde{\Psi}^* \text{Hol}^* \mathcal{G}$ over $\tilde{M}$. But recall that, we have a quasi-line bundle for $\text{Hol}^* \mathcal{G}$ as it explained in Section 5.2. Therefore the difference between these two quasi-line bundles with connection is a line bundle with connection $\tilde{L} \to \tilde{M}$ with the curvature 2-form $\tilde{\omega} = \text{Hol}^* \omega - \tilde{\Psi} \varpi$ by Remark 4.2.1. Note also that if the quasi-line bundle for $\Psi : M \to G$ is $G$-equivariant, then since the quasi-line bundle for $Lg^* \to G$ is $\hat{L}G$-equivariant, the line
bundle \( \tilde{L} \) will be \( \widehat{L}G \) equivariant. Conversely, suppose that we are given a \( \widehat{L}G \)-equivariant line bundle over \( \tilde{M} \), where \( U(1) \subset \widehat{L}G \) acts with weight 1. The difference of this \( \widehat{L}G \)-equivariant line bundle and the \( \widehat{L}G \)-equivariant quasi-line bundle for \( \text{Hol}^* \Psi^* \mathcal{G} \) (constructed in Section 5.2), is a quasi-line bundle with error 2-form \( \text{Hol}^* \omega \). By descending of this quasi-line bundle to \( M \), we can get the desired quasi-line bundle for \( \Psi^* \mathcal{G} \).

The argument, given here applies in greater generality:

For any \( \widehat{L}G \)-equivariant line bundle \( \tilde{L} \to \tilde{M} \), where the central extension \( U(1) \subset \widehat{L}G \) acts with weight \( k \in \mathbb{Z} \), there is a corresponding relative gerbe at level \( k \) with respect to the map \( \Psi : M \to G \). Indeed, the given quasi-line bundle for \( \tilde{\Psi}^* \text{Hol}^* \mathcal{G}^k \) is given by \( \widehat{L}G \) equivariant line bundles over \( \text{Hol}^{-1} \Psi^{-1}(V_j) \) at level \( k \). Twisting by \( \tilde{L} \), we get new quasi-line bundle where \( U(1) \subset \widehat{L}G \) acts trivially. The quotient therefore descends to a quasi-line bundle over \( M \). For instance, Meinrenken and Woodward construct for any Hamiltonian loop group space a so-called “canonical line bundle” in [29], which is \( \widehat{L}G \)-equivariant at level \( 2c \), where \( c \) is the dual Coxeter number. Therefore this line bundle gives rise to a distinguished element of \( H^3(\Phi, \mathbb{Z}) \) at level \( 2c \). Notice that \( M \) and \( \tilde{M} \) are not pre-quantizable necessarily.
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