Green’s function for singular fractional
differential equations and applications

Jinsil Lee\textsuperscript{a} and Yong-Hoon Lee\textsuperscript{b,1}

\textsuperscript{a} Department of Mathematics, University of Georgia, Athens, GA 30606, USA
\textsuperscript{b} Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea
E-mail: jl74942@uga.edu
E-mail: yhlee@pusan.ac.kr

Abstract

In this paper, we study the existence of positive solutions for non-linear fractional differential equation with a singular weight. We derive the Green’s function and corresponding integral operator and then examine compactness of the operator. As an application, we prove an existence result for positive solutions when nonlinear term satisfies either superlinear or sublinear conditions. The proof was mainly employed by Krasnoselski’s classical fixed point theorem.

\textit{MSC (2010):} 34B15, 34B18, 34B27

\textit{Keywords:} fractional differential equation, existence, positive solution, singular weight

1 Introduction

Many problems in physics, control theory, and chemistry can be represented as fractional differential equations (\cite{1,2}). Particularly, many researchers have made contributions to the existence and multiplicity of positive solutions for nonlinear fractional differential equations using Krasnoselski’s fixed point theorem (\cite{7,8,9}). We are interested in applying it to the following equations with a singular weight:

\textsuperscript{1}Corresponding Author
\[
\begin{align*}
D_0^\alpha u(t) + h(t)f(u(t)) &= 0, \quad t \in (0, 1), \\
u(0) &= 0 = u(1),
\end{align*}
\]  
\[\text{(FDE)}\]

where \(D_0^\alpha\) is the Riemann-Liouville fractional derivative of order \(\alpha \in (1, 2]\), \(f \in C([0, \infty), [0, \infty))\) is a given continuous function and \(h \in L^1_{loc}((0, 1), [0, \infty))\) satisfies the following conditions:

\((H_1)\) \[\int_0^1 s^{\alpha-1} h(s) ds < \infty,\]
\((H_2)\) \(h\) is bounded on any compact subinterval in \((0, 1]\).

We notice that coefficient function \(h\) satisfying condition \((H_1)\) may not be integrable near \(t = 0\), as an example, we may consider \(h(t) = t^{-\beta}\) where \(1 < \beta < \alpha\). We see that \(h\) satisfies conditions \((H_1)\) and \((H_2)\) but \(h \notin L^1((0, 1), [0, \infty))\).

Introducing the Green's function for the case that \(h\) is continuous, Bai and Lü [5] consider the following nonlinear problem

\[
\begin{align*}
D_0^\alpha u(t) + f(t, u(t)) &= 0, \quad t \in (0, 1), \\
u(0) &= 0 = u(1),
\end{align*}
\]

where \(f \in C([0, 1] \times [0, \infty), [0, \infty))\). By taking the Riemann-Liouville fractional integral, they set up an equivalent solution operator \(S\) by

\[
Su(t) = \int_0^1 G(t, s) f(s, u(s)) ds
\]

where \(G(t, s)\) defined by

\[
G(t, s) = \begin{cases} 
\frac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\
\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1
\end{cases}
\]

is the Green's function for the fractional differential equation

\[D_0^\alpha u(t) = 0\]

with Dirichlet boundary condition. Analysing this operator, they proved the existence of at least three positive solution of problem (1.1) under some additional conditions on \(f\).

In [7], Jiang and Yuan studied positive solutions of problem (1.1) with the following hypotheses:
(A) There exist \( g \in C([0, \infty], [0, \infty)) \), \( q_1, q_2 \in C((0, 1), (0, \infty)) \) such that
\[
q_1(t)g(u) \leq f(t, t^{\alpha-2}u) \leq q_2(t)g(u),
\]
and \( q_i, \ i = 1, 2 \) also satisfy \( q_i \in L^1(0,1) \).

Under condition (A), they proved that problem (1.1) has at least one positive solution either

(1) \( g_0 = 0, g_\infty = \infty \) or
(2) \( g_\infty = 0, g_\infty = 0, \)

where \( g_0 \) and \( g_\infty \) are defined by \( g_0 = \lim_{u \to 0} \frac{g(u)}{u} \) and \( g_\infty = \lim_{u \to \infty} \frac{g(u)}{u} \). Our concern in this paper is focused on the case that given weight function \( h \) is singular at the boundary which may not be integrable on \((0,1)\), and nonlinear term \( f \) satisfies conditions

(A1) \( f_0 = 0, f_\infty = \infty \), and there exist \( p \) satisfying \( \lim_{u \to \infty} \frac{f(u)}{u^p} = 0 \) or
(A2) \( f_\infty = 0, f_\infty = 0 \).

In most works, function \( h \) to be assumed integrable on \((0,1)\) so that the Green’s function can be derived based on the integrability of \( h \) near boundary 0. Under this assumption, the unique solution of (1.1) is represented by the solution of an integral equation (1.2) using the fractional integral. However, if \( h \) is not integrable, we should consider the existence of \( D_{0+}^\alpha u \) and its solution space. Moreover, corresponding Green’s function can not be obtained by obvious modification from the case \( h \in L^1 \). In the paper [8], the researchers considered the existence of the solution for the second order differential equation where the function \( f \) is a given function satisfying Caratheodory’s conditions with singularities at 0 and 1. We are interested in extending the existence results to the fractional case. In [10], Lee and Lee define the solution space and the definition of the solution and derive the Green’s function in this singular situation.

**Lemma 1.1.** ([10]) Assume \( g \) satisfies \((H_1)\) and \((H_2)\), then the following equation
\[
\begin{cases}
D_{0+}^\alpha u(t) + g(t) = 0, & t \in (0,1), \\
u(0) = 0 = u(1),
\end{cases}
\]
is equivalent to the functional integral equation:
\[
u(t) = \int_0^1 G(t,s)g(s)ds,
\]
(1.4)
where \( G(t, s) \) is given in (1.3). Moreover, \( u \) in (1.4) is in \( AC^2[0, 1] \cap E_\alpha \) and \( D_0^\alpha u \) is absolutely continuous in any compact subinterval of \((0, 1)\).

They proved that the solution \( u \) may not be in \( AC^2[0, 1] \) so that we understand a solution \( u \) is in \( E_\alpha \cap AC[0, 1] \) with \( D_0^\alpha u(t) \) which is absolutely continuous in any compact subinterval of \((0, 1)\) and \( u \) satisfies the equation \((FDE_1)\) for almost everywhere \( t \in [0, 1] \) and boundary conditions. In this paper, we introduce some definitions and lemmas related to fractional calculus and Krasnoselski’s classical fixed point theorem in Section 2. In Section 3, we set up a corresponding solution operator of problem \((FDE)\) and prove the existence of a positive solution for the problem.

## 2 Preliminaries

In this section, we introduce some definitions of fractional calculus and some important lemmas, and a theorem that will be used later.

**Definition 2.1.** We first introduce the basic Banach spaces

- \( AC[0, 1] \): the space of absolute continuous functions on \([0, 1]\)
- \( C^1_{\gamma}[0, 1] = \{ u \in C[0, 1] : t^\gamma u'(t) \in C[0, 1], u(0) = 0 = u(1) \} \) with the norm \( \| u \|_{C^1_{\gamma}} = \| u \|_\infty + \| u' \|_{C_{\gamma}} \) where \( 0 < \gamma < 1 \), \( \| u \|_\infty = \max_{t \in [0, 1]} | u(t) | \) and \( \| u \|_{C_{\gamma}} = \max_{t \in [0, 1]} | t^\gamma u(t) | \)
- \( E_\alpha = \{ u \in C[0, 1] : t^{\alpha-1}D_0^{\alpha-1}u(t) \in C[0, 1] \} \) equipped with the norm \( \| u \|_{E_\alpha} = \| u \|_\infty + \| u \|_1 \) where \( \| u \|_1 = \max_{t \in [0, 1]} | t^{\alpha-1}D_0^{\alpha-1}u(t) | \)

**Definition 2.2.** (7) The integral

\[
I_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0
\]

where \( \alpha > 0 \) is called the Riemann-Liouville fractional integral of order \( \alpha \).

**Definition 2.3.** (7) For a function \( u(t) \) given in the interval \([0, \infty)\), the expression

\[
D_0^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_0^t \frac{u(s)}{(t-s)^{n+1}} ds
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of number \( \alpha \), is called the Riemann-Liouville fractional derivative of order \( \alpha \).
Remark 2.4. ([7]) We note for $\lambda > -1$,

$$D^\alpha_{0+} t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda - \alpha}.$$  

giving in particular $D^\alpha_{0+} t^\alpha = 0, m = 1, 2, \ldots, N$, where $N$ is the smallest integer greater than or equal to $\alpha$.

Lemma 2.5. ([5]) Assume that $u \in C(0,1) \cap L(0,1)$. For $\alpha > 0$, $D^\alpha_{0+} u(t) = 0$ has a unique solution

$$u(t) = c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \cdots + c_n t^{\alpha - n}, \quad c_i \in \mathbb{R}, i = 1, 2, \ldots, n$$

where $n$ is the smallest integer greater than or equal to $\alpha$.

As $D^\alpha_{0+} I^\alpha_{0+} u(t) = u(t)$ for all $u \in C(0,1) \cap L(0,1)$. From Lemma 2.5, we deduce the following statement.

Lemma 2.6. ([5], [9]) Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$I^\alpha_{0+} D^\alpha_{0+} u(t) = u(t).$$

Moreover, if $0 < \alpha < 1$ and $u(t) \in C[0,1]$, then $D^\alpha_{0+} u(t) \in C(0,1) \cap L(0,1)$ and

$$I^\alpha_{0+} D^\alpha_{0+} u(t) = u(t).$$

In the paper ([5], [7]), the writers introduce useful properties of the Green’s function $G(t,s)$ defined by (1.3) as follows:

Lemma 2.7. ([5], [7]) The Green function $G(t,s)$ satisfies the following conditions:

(1) $G(t,s) \in C([0,1] \times [0,1]),$ and $G(t,s) > 0$ for $t, s \in (0,1),$

(2) $\max_{0 \leq t \leq 1} G(t,s) = G(s,s), \quad s \in (0,1)$

(3) $G(t,s) = G(1-s,1-t)$, for $t, s \in (0,1),$

(4) $\frac{\alpha - 1}{\Gamma(\alpha)} (1-t)(1-s)^{\alpha-1} s \leq G(t,s) \leq \frac{1}{\Gamma(\alpha)} (1-t)(1-s)^{\alpha-2}.$

(5) $G(t,s) \leq \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} t^{\alpha-2}$.

The property (1), (2) are proved in Lemma 2.4 by monotonicity of $G(t,s)$ in the paper [5]. And the authors in [7] deduced properties (3) and (4). From the properties (3) and (4), we can have

$$G(t,s) = G(1-s,1-t) \leq \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} (t)^{\alpha-2}.$$  

(2.1)
Theorem 2.8. (Fixed point theorem of cone expansion/compression type)

Let $E$ be a Banach space and let $K$ be a cone in $E$. Assume that $\Omega_1$ and $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$. Assume that $T : K \cap (\overline{\Omega_2 \setminus \Omega_1}) \to K$ is completely continuous such that either

1. $\|Tu\| \leq \|u\|$, for $u \in K \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$, for $u \in K \cap \partial \Omega_2$, or
2. $\|Tu\| \geq \|u\|$, for $u \in K \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$, for $u \in K \cap \partial \Omega_2$.

Then $T$ has a fixed point in $K \cap (\overline{\Omega_2 \setminus \Omega_1})$.

3 An Application to Nonlinear Problems

Our goal in this section is to prove an existence result for following nonlinear problem

\[
\begin{aligned}
&{D}_{0+}^{\alpha}u(t) + h(t)f(u(t)) = 0, \quad t \in (0, 1), \\
&u(0) = u(1) = 0 = u(1),
\end{aligned}
\]  

$(FD)$

where ${D}_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (1, 2]$, $f \in C([0, \infty), [0, \infty))$ and $h$ satisfies $(H_1)$ and $(H_2)$. In [10], the researchers proved that a solution of

\[ u(t) = \int_0^1 G(t, s)h(s)f(u(s))ds := Su(t), \]

where $G(t, s)$ is the Green’s function given by (1.3) satisfies the equation $(FD)$. In this paper, we use the fixed point method to find a solution of the integral problem and this fixed point satisfies our equation $(FD)$. We now state our main theorem in this section. For $u \in C[0, 1]$, we define a nonlinear operator $S : C[0, 1] \to C[0, 1]$ by

\[ Su(t) = \int_0^1 G(t, s)h(s)f(u(s))ds. \]

Assume that $f$ satisfies $(A1)$. Then, there exists a constant $p > 1$, $r$, $R$, $\epsilon$ and $M_1$ such that

\[ |f(u)| < \epsilon |u| \text{ for } |u| < r, \]
\[ |f(u)| < \epsilon |u|^p \text{ for } |u| > R, \]

and

\[ |f(u)| < M_1 \text{ for } r \leq |u| \leq R \]
and then we have

\[|(Su)(t)| \leq \int_0^1 G(t, s)h(s)f(u(s))ds \]

\[\leq \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}h(s)\frac{f(u(s))}{\Gamma(\alpha)}ds\]

\[< \int_{|u|<\tau} s^{\alpha-1}(1-s)^{\alpha-1}h(s)\frac{\epsilon|u(s)|}{\Gamma(\alpha)}ds + \int_{r<|u|<R} s^{\alpha-1}(1-s)^{\alpha-1}h(s)\frac{M_1}{\Gamma(\alpha)}ds\]

\[+ \int_{|u|>R} s^{\alpha-1}(1-s)^{\alpha-1}h(s)\frac{\epsilon|u(s)|^p}{\Gamma(\alpha)}ds\]

\[< \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}h(s)ds\hat{M} < \infty\]

where \(\hat{M} = \epsilon\|u\|_\infty + M_1 + \epsilon\|u\|_\infty^p\). Moreover, by the similar calculation in the paper \[10\], we have

\[D_{0+}^{\alpha-1}Su(t) = \int_0^t (1-\tau)^{\alpha-1} - 1)h(\tau)f(u(\tau))d\tau - \int_t^1 (1-\tau)^{\alpha-1}h(\tau)f(u(\tau))d\tau\]

and then

\[|t^{\alpha-1}D_{0+}^{\alpha-1}Su(t)|\]

\[= t^{\alpha-1} \int_0^t (1-\tau)^{\alpha-1}h(\tau)f(u(\tau))d\tau + t^{\alpha-1} \int_0^1 (1-\tau)^{\alpha-1}h(\tau)f(u(\tau))d\tau\]

\[= t^{\alpha-1} \int_0^t \int_0^1 (\alpha - 1)s^{\alpha-2}ds h(\tau)f(u(\tau))d\tau + t^{\alpha-1} \int_0^1 \frac{\tau^{\alpha-1}}{\tau^{\alpha-1}}(1-\tau)^{\alpha-1}h(\tau)f(u(\tau))d\tau\]

\[\leq t^{\alpha-1} \int_0^t (\alpha - 1)(1-\tau)^{\alpha-2}(1-\tau^{\alpha-1})h(\tau)d\tau \hat{M} + t^{\alpha-1} \int_0^1 \frac{\tau^{\alpha-1}}{\tau^{\alpha-1}}(1-\tau)^{\alpha-1}h(\tau)d\tau \hat{M}\]

\[\leq [(\alpha - 1) \int_0^t (1-\tau)^{\alpha-2}\tau h(\tau)d\tau + \int_0^1 \tau^{\alpha-1}(1-\tau)^{\alpha-1}h(\tau)d\tau] \hat{M}\]

Since \((1-\tau)^{\alpha-1} - 1)h(\tau), \tau h(\tau) \in L^1(0,1)\), we conclude that \(Su : C[0, 1] \rightarrow E_\alpha[0, 1]\) is well defined and bounded when \(f\) satisfies (A1). With the similar technique in the proof of Lemma 3.2 in \[10\], it follows that \(Su \in AC[0, 1] \cap E_\alpha[0, 1]\) and therefore \(u = Su \in E_\alpha[0, 1]\). Similarly, we get the same result if \(f\) satisfies (A2). As a results, if we can find the fixed point of \(u = Su\) in \(C[0, 1]\), it can be a solution of our equation (\(FD\)) and then \(u \in E_\alpha[0, 1]\). So, we can get the following Theorem

**Theorem 3.1.** Assume that the given function \(h \in L^1_{\text{loc}}((0, 1), [0, \infty))\) satisfies (\(H_1\), \(H_2\)) and \(f\) satisfies either (A1) or (A2). Then problem (\(FD\)) has at least one positive solution in \(E_\alpha[0, 1]\).
For this, let $E_\alpha$ be endowed with the ordering $u \leq v$ if $u(t) \leq v(t)$ for all $t \in [0, 1]$. We define a cone $\mathcal{K} \subseteq E_\alpha$ by

$$\mathcal{K} = \{ u \in C[0, 1] \mid u(t) \geq 0, u(t) \geq (\alpha - 1)t(1 - t)\|u\|_\infty \}.$$ 

We first check the compactness of operator $S$.

**Lemma 3.2.** Let $S : \mathcal{K} \to E_\alpha$ be the operator defined by

$$Su(t) = \int_0^1 G(t, s)h(s)f(u(s))ds$$

where $f \in C([0, \infty), [0, \infty))$ satisfying either (A1) or (A2) and $h \in L^1_{\text{loc}}((0, 1), (0, \infty))$ satisfies (H1) and (H2). Then $S : \mathcal{K} \to \mathcal{K}$ is completely continuous.

**Proof.** Since the Green’s function, $h$ and $f$ are nonnegative,

$$Su(t) = \int_0^1 G(t, s)h(s)f(u(s))ds \geq 0.$$ 

By Lemma 2.7 we obtain

$$Su(t) = \int_0^1 G(t, s)h(s)f(u(s))ds \geq \frac{\alpha - 1}{\Gamma(\alpha)}t^{\alpha - 1}(1 - t)\int_0^1 s(1 - s)^{\alpha - 1}h(s)f(u(s))ds.$$ 

$$\|Su\|_\infty \leq \frac{1}{\Gamma(\alpha)}t^{\alpha - 2}\int_0^1 s(1 - s)^{\alpha - 1}h(s)f(u(s))ds.$$ 

Hence,

$$(\alpha - 1)t(1 - t)\|Su\|_\infty \leq Su(t).$$ 

This implies that $Su \in \mathcal{K}$. Let $\{u_k\} \subseteq E_\alpha$ be a convergent sequence to $u \in E_\alpha$. For any given $\epsilon > 0$, we let

$$\epsilon_1 = \frac{\epsilon}{\int_0^1 G(s, s)h(s)ds}.$$ 

By the continuity of a function $f$, there exists $r > 0$ such that for any $u_k$ with $|u - u_k| < r$, we have

$$|Su(t) - Su_k(t)| \leq \int_0^1 |G(t, s)h(s)[f(u(s)) - f(u_k(s))]|ds$$

$$\leq \int_0^1 G(s, s)h(s)|f(u(s)) - f(u_k(s))|ds$$

$$\leq \int_0^1 G(s, s)h(s)ds\epsilon_1 < \epsilon.$$
By the continuity of $f$ and the condition $(H1), (H2)$, we can conclude that the operator $S$ is continuous. Let $\mathcal{M}$ be a bounded subset in $\mathcal{K}$. Then we get

$$|Su(t)| \leq \int_0^1 G(s, s) h(s) f(u(s)) ds \leq M \int_0^1 G(s, s) h(s) ds,$$

where $M = \sup_{u \in \mathcal{M}} \| f \circ u \|_{\infty}$. Hence, $S(\mathcal{M})$ is bounded by conditions $(H1)$ and $(H2)$.

Now, we prove that $(Su)(\mathcal{M})$ is relatively compact subset of $C[0, 1]$. Let $\{u_k\} \subseteq \mathcal{M}$. It is proved that $Su' \in L^1(0, 1)$ in Section 3 in [10]. From the fact that $f(u)$ is bounded, it follows that for any $0 \leq t_1 < t_2 \leq 1$,

$$|Su(t_2) - Su(t_1)| \leq \int_{t_1}^{t_2} |Su'(s)| ds \to 0$$

as $|t_1 - t_2| \to 0$ for all $u \in \mathcal{M}$. Consequently, by Arzela-Ascoli theorem, $\{Su(t)\}_{u \in \mathcal{M}}$ is relatively compact. Therefore $S : \mathcal{K} \to \mathcal{K}$ is completely continuous and the proof is done.

We now prove Theorem 3.1

**Proof of Theorem 3.1** Case 1. $f_0 = 0, f_{\infty} = \infty$.

By condition $f_0 = 0$, we may choose $r$ satisfying for $0 < u \leq r$,

$$f(u) \leq \varepsilon u,$$

where $\frac{1}{2} > \varepsilon > 0$ satisfies

$$\frac{\varepsilon}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1} h(s) ds \leq 1.$$

Let $B_r = \{u \in C[0, 1]| \| u \|_{\infty} < r\}$, then for $u \in \mathcal{K} \cap \partial B_r$, we obtain

$$Su(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1} h(s) f(u(s)) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1} h(s) \varepsilon u(s) ds$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1} h(s) ds \| u \|_{\infty}$$

$$\leq \| u \|_{\infty}.$$

Therefore, $\| Su \|_{\infty} \leq \| u \|_{\infty}$. Since $f_{\infty} = \infty$, there exists $M^* > 0$ such that $f(u) \geq \rho u$, for $u > M^*$, where $\rho > 0$ is chosen so that

$$\frac{\rho(\alpha - 1)}{16} \int_{\frac{3}{4}}^\frac{1}{2} G\left(\frac{3}{4}, s\right) h(s) ds \geq 1.$$
Let us take $R > \max\{\frac{16}{\alpha - 1}M^*, r\}$ and $B_R = \{u \in C[0, 1] \|u\|_\infty < R\}$, then for $u \in K \cap \partial B_R$, 
$$u(t) \geq \frac{\alpha - 1}{16} \|u\|_\infty > M^*,$$
for $t \in [\frac{1}{4}, \frac{3}{4}]$ and

$$\|Su\|_\infty \geq Su\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, s\right) h(s) f(u(s)) ds \geq \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) f(u(s)) ds \geq \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) \rho u(s) ds \geq \rho \frac{\alpha - 1}{16} \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) ds \geq \|u\|_\infty.$$

Therefore, by Theorem 2.8, $S$ has a fixed point $u$ in $u \in K \cap (\overline{B_R} \setminus B_r)$.

Case 2. $f_0 = \infty$, $f_\infty = 0$.
By condition $f_0 = \infty$, we may choose $r_1$ so that $f(u) \geq Lu$, for $0 < u \leq r_1$ where $L > 0$ satisfies

$$\frac{L(\alpha - 1)}{16} \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) ds \geq 1.$$

Let $B_{r_1} = \{u \in C[0, 1] \|u\|_\infty < r_1\}$, then for $u \in K \cap \partial B_{r_1}$, we have

$$\|Su\|_\infty \geq Su\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, s\right) h(s) f(u(s)) ds \geq \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) f(u(s)) ds \geq \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) Lu(s) ds \geq \frac{L(\alpha - 1)}{16} \int_{\frac{3}{4}}^1 G\left(\frac{1}{2}, s\right) h(s) ds \|u\|_\infty \geq \|u\|_\infty.$$
Since \( f_\infty = 0 \), there exists \( L_2 > 0 \) such that \( f(u) \leq \zeta u \) for \( u > L_2 \) where \( \zeta > 0 \) satisfies

\[
\frac{\zeta}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1}h(s)ds < 1.
\]

And choose \( R_2 \) satisfying

\[
R_2 > \max\{L_2, \frac{\max_{0 \leq u \leq L_2} |f(u)| \int_0^1 (s(1-s))^{\alpha-1}h(s)ds}{\Gamma(\alpha) - \zeta \int_0^1 (s(1-s))^{\alpha-1}h(s)ds}\}.
\]

Then we obtain

\[
Su(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-1}h(s)f(u(s))ds \\
\leq \frac{1}{\Gamma(\alpha)} \left[ \int_{0 \leq u \leq L_2} (s(1-s))^{\alpha-1}h(s)f(u(s))ds + \int_{L_2 \leq u \leq R_2} (s(1-s))^{\alpha-1}h(s)f(u(s))ds \right] \\
\leq \frac{1}{\Gamma(\alpha)} \max_{0 \leq u \leq L_2} |f(u)| \int_{0 \leq u \leq L_2} (s(1-s))^{\alpha-1}h(s)ds \\
+ \int_{L_2 \leq u \leq R_2} (s(1-s))^{\alpha-1}h(s)\zeta u(s)ds \\
\leq \frac{1}{\Gamma(\alpha)} \left( \max_{0 \leq u \leq L_2} |f(u)| + \zeta \|u\|_{\infty} \right) \int_0^1 (s(1-s))^{\alpha-1}h(s)ds \\
\leq R_2 = \|u\|_{\infty}.
\]

This implies that \( \|Su\|_{E\alpha} \leq \|u\|_{\infty} \) for \( u \in \mathcal{K} \cap \partial B_{R_2} \), and thus \( S \) has a fixed point \( u \) in \( \mathcal{K} \cap (B_{R_2} \setminus B_{r_1}) \).

**Example 3.3.** In [6], Kong and Wang examined the existence of solution for the following equation for the following equation

\[
\begin{aligned}
D_{0+}^\alpha u(t) + \omega(t)u^\theta = 0, & \quad 0 < \theta < 1 < \alpha < 2 \\
u(0) = 0 = u(1),
\end{aligned}
\]

(3.1)

where \( \omega \in \mathcal{C}[0,1] \). They showed that (3.1) has at least one positive solution.

Furthermore, if we take \( f(t, u) = \omega(t)u^\theta \), then \( f(t, f_\infty u) = \omega(t)t^{(\alpha-2)\theta}u^\theta \).

By choosing \( g(u) = u^\theta \) and \( q_1(t) = q_2(t) = \omega(t)t^{(\alpha-2)\theta} \), we get \( q_1, q_2 \in L(0, 1) \) and \( g_0 = \infty, g_\infty = 0 \), thus \( f \) satisfies condition (A) in [7] so that we get the same result by applying Theorem 1.5 in [7] to problem (3.1). When \( \omega \) is
given as \( \omega(t) = t^{-\beta}, 1 < \beta < \alpha \), then problem (3.1) becomes

\[
\begin{aligned}
D_0^\alpha u(t) + t^{-\beta} u^\theta &= 0, \\
0 < \theta < \alpha - 1, 1 < \beta < \alpha, 1 < \alpha < 2 \\
u(0) = 0 = u(1).
\end{aligned}
\]  

(3.2)

and the above results are not applicable since \( \omega \notin C[0,1] \) and condition (A) is not valid. Indeed, if we take \( f(t,u) = t^{-\beta} u^\theta \), then

\[
f(t, t^{\alpha-2} u) = t^{-\beta + (\alpha-2)\theta} u^\theta
\]

in which \( t \)-term is not integrable by the fact that \( -\beta - \theta < -\beta + (\alpha - 2)\theta < -\beta < -1 \). Therefore it is impossible to find \( q_1, q_2 \) satisfying condition (A).

But \( \omega(t) = t^{-\beta}, 1 < \beta < \alpha \) satisfies \( (H_1), (H_2) \) and \( f(u) = u^\theta \) satisfies conditions \( (A2) \). Therefore, we can apply Theorem 3.1 to guarantee the existence of at least one positive solution in \( E_\alpha[0,1] \) for (3.2).

**Acknowledgment**

The authors express their gratitude to anonymous referees for their helpful suggestions which improved final version of this paper.

**Funding**

This work was supported by the National Research Foundation of Korea, Grant funded by the Korea Government (MEST) (NRF2016R1D1A1B04931741).

**Availability of data and materials**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**Competing interests**

The authors declare that there is no competing interests for this paper.

**Author’s contributions**

All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.
Author’s information

Jinsil Lee, Department of Mathematics, University of Georgia, Athens, GA 30606, USA. E-mail: jl74942@uga.edu
Yong-Hoon Lee, Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea. E-mail: yhlee@pusan.ac.kr

Publisher’s Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Igor Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering V198, Academic Press 1999.

[2] Tomas Kisela, *Fractional Differential Equations and Their Applications*, Fakulta strojnih inzenyrstvi. 2008

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and application of fractional differential equation, North-Holl and Mathematics Studies, 2006, 1-523.

[4] Zhongli Wei, *Positive solution of singular Dirichlet boundary value problems for second order differential equation system*, J. Math. Anal. Appl. 328 (2007) 1255-1267.

[5] Zhanbing Bai and Haishen Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. 311 (2005) 495-505.

[6] Qingkai Kong and Min Wang, *Positive solutions of nonlinear fractional boundary value problems with Dirichlet boundary conditions*, Electron. J. Qual. Theory Differ. Equ. 17 (2012) 1-13.

[7] Daqing Jiang, Chengjun Yuan The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, Nonlinear Analysis: Theory, Methods & Applications Volume 72, Issue 2, 15 January 2010, Pages 710-719
[8] Ahmed M.A.El-Sayed, Fatma M.Gaafar, *Existence of Solutions for Singular Second-Order Ordinary Differential Equations with Periodic and Deviated Nonlocal Multipoint Boundary Conditions*, Hindawi, Journal of Function Spaces, Volume 2018, Article ID 9726475, 11 pp.

[9] Xinwei Su *Boundary value problem for a coupled system of nonlinear fractional differential equations* Applied Mathematics Letters 22 (2009) 64-69

[10] Jinsil Lee, Yong-Hoon Lee *Regularity of solutions for singular fractional differential equation*, preprint