A new universal ratio in Random Matrix Theory and chaotic to integrable transition in Type-I and Type-II hybrid Sachdev-Ye-Kitaev models

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We investigate chaotic to integrable transition in two types of hybrid SYK models which contain both $q = 4$ SYK with interaction $J$ and $q = 2$ SYK with an interaction $K$ in type-I or ($q = 2$)$^2$ SYK with an interaction $\sqrt{K}$ in type-II. These models include hybrid Majorana fermion, complex fermion and bosonic SYK. For the Majorana fermion case, we discuss both $N$ even and $N$ odd case. We make exact symmetry analysis on the possible symmetry class of both types of hybrid SYK in the 10 fold way by Random Matrix Theory (RMT) and also work out the degeneracy of each energy levels. We introduce a new universal ratio which is the ratio of the next nearest neighbour (NNN) energy level spacing to characterize the RMT. We perform exact diagonalization to evaluate both the known NN ratio and the new NNN ratio, then use both ratios to study Chaotic to Integrable transitions (CIT) in both types of hybrid SYK models. Some preliminary results on possible quantum analog of Kolmogorov-Arnold-Moser (KAM) theorem and its dual version in the quantum chaotic side are given. We explore some intrinsic connections between the two complementary approaches to quantum chaos: the RMT and the Lyapunov exponent by the 1/$N$ expansion in the large $N$ limit at a suitable temperature range. We stress the crucial differences between the quantum phase transition (QPT) characterized by renormalization groups at $N = \infty$, 1/$N$ expansions at a finite $N$ and the CIT characterized by the RMT at a finite $N$: the former focus on the ground state and its low energy excitations (edge states in the Fock space), the latter on excited states (bulk states in the Fock space). We also discuss Eigenstate Thermalization hypothesis (ETH)'s power on a quantum chaotic bulk state and its inability to encode the edge states. Comments on some previously related works are given. Some future perspectives, especially the failure of the Zamolodchikov’s c-theorem in 1d CFT RG flow are outlined.

I. INTRODUCTION

In classical chaos, the Lyapunov exponent was used to characterize the exponential growth of two classical trajectories when there are just a tiny difference in the initial conditions. The classical concept of Lyapunov exponent can be extended to its quantum analog to characterize the exponential growth of two initially commuting operators in the early time (up to the Ehrenfest time) under the evolution of a quantum chaotic Hamiltonian. There are recent flurry of research activities to extract the Lyapunov exponent of the Sachdev-Ye-Kitaev (SYK) model and its various variants through evaluating out of time ordered correlation (OTOC) functions. The quantum analog of $\lambda_q$ need to be evaluated away from the thermodynamic limit by a 1/$N$ expansion in the SYK models, also away from the conformal invariant limit due to a leading irrelevant operator and at a finite temperature (ranging from low to infinite temperatures). There are also calculations in the dual bulk quantum black holes in an asymptotic $AdS_5$ geometry to demonstrate the correspondence between the SYK models and Jackiw-Teitelboim gravity. The quantum chaos in the SYK models are due to the quenched disorders and interactions. However, it inspired a new class of clean models called (colored or un-colored) Tensor (Gurau-Witten) model, which share similar quantum chaotic properties as the SYK models at least in the large $N$ limit.

From a completely different perspective and also at a much longer time scale (Heisenberg time), the quantum chaos can also be characterized by the system’s energy level-level correlations encoded in the energy level statistics (ELS) and the spectral form factor (SFF) through random matrix theory (RMT). The ELS and SFF are always evaluated in a finite but sufficiently large system. The ELS of SYK can be described by the Wigner-Dyson (WD) distributions in a $N$ (mod 8) way, $N = 2, 6$ Gaussian unitary ensembles (GUE), $N = 0$ Gaussian orthogonal ensemble (GOE), $N = 4$ Gaussian symplectic ensemble (GSE).

Here we investigate chaotic to integrable transitions (CIT) in two types of hybrid SYK models. The type-I contains both the chaotic $q = 4$ SYK with interaction $J$ and the integrable $q = 2$ SYK with interaction $K$. It violates the particle-hole (PH) symmetry and includes the hybrid Majorana fermion, complex fermion and hardcore bosonic SYK. The type-II includes $q = 4$ SYK of either Majorana or complex fermion with interaction $J$ and...
\( (q = 2)^2 \) SYK with the interaction \( \sqrt{K} \). It preserves the particle-hole (PH) symmetry. In this work, we mainly use the RMT approach, but will also explore some possible connections between the RMT approach and the Lyapunov exponent which can be extracted from the out of time correlation functions (OTOC).

There are direct motivations to study both types of hybrid SYK models. It was also known that the \( q = 4 \) SYK model provides a concrete model to realize Eigen-state Thermalization Hypothesis (ETH)\textsuperscript{43}. Even the entanglement entropy of its ground state satisfies the volume law instead of the more common area law. Here it is important to study how the gapless quantum spin liquid (QSL) responses under the Type I or Type II kinds of integrable perturbations. It would also be interesting to study how the ETH breaks down under the Type I or Type II kind of integrable perturbations. On the dual bulk gravity side, these kind of perturbations are also needed to probe the interior behind the black hole horizon\textsuperscript{44}. Most importantly, in any possible experimental systems to realize SYK models, the Type I and Type II terms are most common perturbations. As shown in Fig[9] in the conclusion section, the two types hybrid SYK models also provide specific examples of RG flow between two CFT\textsubscript{1} fixed points which violate Zamolodchikov’s c-theorem. When comparing with some previously studied hybrid models, one can see there are also many other motivations to study both types of hybrid SYK models. For example, the multi-channel Kondo models show non-Fermi liquid (NFL) behaviours, so it is important to study how the NFL changes under spin or channel anisotropy\textsuperscript{36,38–40}. The Kitaev honeycomb lattice model\textsuperscript{41} hosts a Majorana fermion quantum spin liquid (QSL), so it is important to study how the Majorana fermion QSL changes under adding other interactions such as a Heisenberg interaction or a Dzyaloshinskii-Moriya (DM) term\textsuperscript{42}.

We first introduce a new universal ratio which is the ratio of next nearest neighbor (NNN) energy spacing to characterize the RMT classes and also establish its relations with the well-known ratio of the nearest neighbor (NN) energy spacing\textsuperscript{46}. When a doubly degenerate level is split by a small perturbation, the NN ratio is nearly zero and also rapidly changing, the new NNN ratio can be used to effectively characterize the “hidden” RMT behaviours, especially the CIT. We make exact symmetry analysis to classify the possible symmetry class in the 10-fold way\textsuperscript{39,47,50}, and then perform exact diagonalization (ED) on all these hybrid models. In the Majorana fermion case, we pay special efforts to classify the odd \( N \) case. Both the NN ratio and the NNN ratio are evaluated when a doubly degenerate level is split by a small perturbation. In the type-I hybrid SYK, as \( K/J \) increases, there is always a chaotic to integrable transition (CIT) from GUE to Poisson in all the hybrid Majorana or complex fermionic models, but not the hybrid bosonic model. Starting from the type-I hybrid \( q = 4 \) Majorana or complex fermion SYK side, for the GOE case, there is a GOE to GUE crossover first, then a CIT from the GUE to the Poisson; for the GUE case, there is a direct CIT from the GUE to the Poisson; for the GSE case, any small \( K \) destroys the double degeneracy of the GSE, the NN ratio rises to GUE, then a CIT from the GUE to the Poisson. In this case, the NNN ratio can be most effectively applied to describe the stability regime of GSE near the chaotic side and may also be used to describe the whole crossover until to the integrable side.

In the type-II hybrid SYK, the symmetry analysis alone may not be able to distinguish between the chaotic \( q = 4 \) SYK and integrable \( (q = 2)^2 \) SYK. As \( K/J \) increases, there are always CIT from the corresponding WD distribution dictated by the symmetry of the \( q = 4 \) SYK to the Poisson controlled by the \( (q = 2)^2 \) SYK. When there is a double degeneracy at the \( (q = 2)^2 \) side, the NNN can be most effectively applied to describe the CIT and may also be used to describe the whole crossover until to the quantum chaotic side.

In classical chaos, the classical Kolmogorov-Arnold-Moser (KAM) theorem states that if an integrable Hamiltonian \( H_0 \) is disturbed by a small perturbation \( \Delta H \), which makes the total Hamiltonian \( H = H_0 + \Delta H \) non-integrable. If the two conditions are satisfied: (a) \( \Delta H \) is sufficiently small (b) the frequencies \( \omega_i \) of \( H_0 \) are incommensurate, then the system remains quasi-integrable. Despite some previous efforts, the quantum analogue of the KAM theorem remains elusive. Just like the quantum analog of Lyapunov exponent can be studied in the context of the SYK models\textsuperscript{4–16}, it is important to explore the quantum analog of KAM theorem in both types of hybrid SYK models. We first give a definition of the quantum analogue of the KAM theorem and also its possible dual form which is the stability of quantum chaos in the context of RMT. Then we use both NN and NNN ratio to numerically characterize the KAM regime in the integrable side and the stability of quantum chaos in the chaotic side. We give some preliminary results on possible quantum analog of the KAM theorem, the stability of quantum chaos (can also be called the dual form of the KAM theorem), especially its dependence on the system’s size \( N \) in the two types hybrid SYK models. By pushing the methods developed in\textsuperscript{26,27} further, we leave rigorous mathematical treatments to a future publication\textsuperscript{27}.

We also stress the crucial differences between the onset of quantum phase transitions (QPT) at \( N = \infty \) characterized by renormalization group (RG), \( 1/N \) expansions and the onset of CIT characterized by the RMT at a finite \( N \). The former is focusing on the changes in the ground state and low energy excitations (called edge excitations in the Fock space), there is an associated divergent length scale and quantum critical scalings and a finite size scaling at a finite size. While the latter is focusing on the changes in the correlations in the bulk energy levels in the Fock space, there is no associated divergent length scale, therefore no quantum critical scalings and no finite size scaling at a finite size. We also discuss Eigenstate Thermalization hypothesis (ETH)’s power on a quantum chaotic bulk state and its inability to encode
the ground state and low energy excitations. Some comments on previous works on type-I hybrid SYK models are given. Some perspectives and possible future directions are outlined in the conclusion section. Especially, we point out that the Zamolodchikov’s c-theorem of 2d CFT and its extensions to 2d boundary CFT and higher dimensions break down in 1d CFT RG flow. This fact may be related to $N_{\text{AdS}}/NCFT_1$ may also be dramatically different than its high dimensional counterparts.

As mentioned in the first two paragraphs, there are two complementary approaches to describe the quantum chaos: the OTOC at an early time and the RMT at a later time. However, so far, it seems there is no clear connections between the two different ways to characterize the quantum chaos. In this work, we establish some intrinsic connections between the two approaches: the quantum chaos in the bulk energy levels described by the RMT implies the quantum information scramblings with a non-vanishing Lyapunov exponent $\lambda_L$ by the OTOC in the edge levels (the ground state and low energy levels) in a suitable temperature range.

II. A NEW RATIO TO DESCRIBE RMT: RATIO OF NEXT NEAREST NEIGHBOR ENERGY SPACING

In this section, we first review the known results on the statistics of the NN energy level spacing $(r, \hat{r})$, then introduce a new ratio which is the NN energy level spacing $(r', \hat{r}')$, then establish an approximate, but quite accurate relation between the two.

A. Review on the ELS of NN energy level spacings

Let $\{E_n\}$ be an ordered set of energy levels and $s_n = E_{n+1} - E_n$ are the NN spacings. By considering a $2 \times 2$ matrices system, Wigner derived a simple approximate expression for the distribution function $P(s)$ of the NN spacing,

$$P_{w,\beta}(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

(1)

where $\beta = 1, 2, 4$ is the Dyson index for GOE, GUE and GSE respectively. It is also known that independent random energy levels would yield a Poisson distribution

$$P_p(s) = e^{-s}$$

(2)

However, in order to compare different results from different systems, the energy levels will need an unfolding procedure, which is not convenient when large enough statistics is not available. To get rid of the dependence on the local density of states, it is convenient to look at the distribution of the ratio of two adjacent energy level spacings, $r_n = s_n/s_{n+1}$ which distributes around 1. This quantity has the advantage that it requires no unfolding since ratios of consecutive level spacings are independent of the local density of states.

By considering $3 \times 3$ matrices system, the authors in $^{23,46}$ obtained the Wigner-like surmises of the ratio of consecutive level spacings distribution

$$P_p(r) = \frac{1}{(1+r)^2}, \quad P_w(r) = \frac{1}{Z_\beta (1+r+r^2)^{1+3\beta/2}}$$

(3)

where $\beta = 1, 2, 4$ and $Z_\beta = 8/27, 4\pi/81\sqrt{3}, 4\pi/729\sqrt{3}$ for GOE, GUE and GSE respectively. The distribution $P_w(r)$ has the same level repulsion at small $r$ as $P_w(s)$, namely, $P_w(r) \sim r^\beta$. However, the large $r$ asymptotic behavior $P_w(r) \sim r^{-(2+\beta)}$ is dramatically different than the fast exponential decay of $P_w(s)$.

One may also compute the distribution of the logarithmic ratio $^{23,46}$ $P(\ln r) = P(r) r$. Because $P(\ln r) dr$ is symmetric under $r \leftrightarrow 1/r$, one may confine $0 < r < 1$ and double the probability density $P(\hat{r}) = 2P(r)$. Therefore, the above two distributions have two different sets of expected values of $\hat{r} = \min[r, 1/r]$:

$$\langle \hat{r} \rangle_p = \int_0^1 2r P_p(r) dr = 2 \ln 2 - 1 \approx 0.38629,$$

$$\langle \hat{r} \rangle_w = \int_0^1 2r P_{w,\beta=1,2,4}(r) dr$$

(4)

which is $4 - 2\sqrt{3} \approx 0.53590, 2\sqrt{3}/\pi - 1/2 \approx 0.60266, 32\sqrt{3}/(15\pi) - 1/2 \approx 0.67617$ for GOE, GUE and GSE respectively.

B. Introduction and calculation of the ELS of NNN energy level spacings

When there are nearly double degenerated energy levels, we find that it is convenient to introduce next nearest-neighbor (NNN) spacings, $s'_n = E_{2n+1} - E_{2n-1}$ and ratios $r'_n = s'_n/s'_{n-1}$. Here we study the distribution of the ratio of the two NNN spacings by exploring the exact calculation for $5 \times 5$ matrices.

For a Poisson ensemble, it is more convenient to work with $P(s_1, s_2, s_3, s_4)$ and the ratio of consecutive NNN level spacings is $r' = (s_3 + s_4)/(s_1 + s_2)$. The NNN ratio distribution can be calculated from

$$P_p^{(2)}(r') = \int \prod_{i=1}^4 ds_i P(s_1, s_2, s_3, s_4) \delta \left( r' - \frac{s_3 + s_4}{s_1 + s_2} \right).$$

(5)

Since energy levels are not correlated in Poisson ensemble, one can rewrite $P(s_1, s_2, s_3, s_4) = \prod_{i=1}^4 P_p(s_i)$ and evaluate the integral to obtain a simple result

$$P_p^{(2)}(r') = \frac{6r'}{(1+r')^3}.$$ 

(6)

It is easy to see that $P_p^{(2)}(r') \sim r'$ when $r'$ is small, and $P_p^{(2)}(r') \sim r'^{-3}$ when $r'$ is large. Interestingly, there is a NNN level repulsion in Poisson ensemble which intuitively can be understood due to the interruption of the intermediate level.
For various Gaussian ensembles, it is good to start from the joint probability distribution \( \rho(E_1, E_2, E_3, E_4, E_5) \) and the ratio of consecutive NNN level spacings is \( r' = (E_5 - E_3)/(E_3 - E_1) \). The NNN ratio distribution can be calculated from

\[
P^{(2)}_{w,\beta}(r') = \int \prod_{i=1}^{5} dE_i \rho(E_1, E_2, E_3, E_4, E_5) \delta \left( r' - \frac{E_5 - E_3}{E_3 - E_1} \right)
\]

where the joint probability distribution takes the form

\[
\rho(E_1, E_2, E_3, E_4, E_5) = C_\beta \prod_{1 \leq i < j \leq 5} |E_i - E_j|^\beta \prod_{i=1}^{5} e^{-E_i^2/2}.
\]

The integral can be simplified as

\[
P^{(2)}_{w,\beta}(r') \sim \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz f(x, y, z, r'), \quad f(x, y, z, r') = e^{-\frac{1}{2}[(x+y)^2+(x+z)^2+(y+z)^2+(1-r')x(y-z)]}
\]

\[
\times x[r'(1+r')x^3(r'x-y)(r'x+z)(y+x)(x+y)(x-z)(y+z)]^{\beta/2},
\]

where \( \sim \) means the normalization constant is ignored. The integrals can be evaluated analytically, but its expression is lengthy. Here, we just show analytically its asymptotic behaviour: \( P^{(2)}_{w,\beta}(r') \sim r'3\beta+1 \) when \( r' \) is small, and \( P^{(2)}_{w,\beta}(r') \sim r'^{-3\beta+1} \) when \( r' \) is large. If comparing them with those of \( P_w(r) \), one can see that the asymptotic behaviours of the NNN level statistics with index \( \beta \) are the same as those of the NN with index \( 3\beta + 1 \) (See Eq. 10) below for all the ranges). Similar to the Poisson case discussed above, the intermediate energy level induces much stronger level repulsions between NNN. In Fig. 1, we compare \( P_{w,\beta} \) obtained from Eq. 8 against the numerical simulations of the corresponding WD ensembles. We find nearly perfect agreement in all ranges of \( r' \).

It is easy to check that \( P^{(2)}(r') \) have the same symmetry as \( P(r) \), namely, \( P^{(2)}(r') = \frac{1}{r'} P^{(2)}(\frac{1}{r'}) \), thus we can still define \( \tilde{r} \). From the NNN ratio distribution given in Eq. 2 for Poisson and Eq. 8 for WD, one can calculate expectation value for \( r' \) and \( \tilde{r}' \) as

\[
\langle r' \rangle = \int_0^\infty dr' r' P(r'), \quad \langle \tilde{r}' \rangle = \int_0^1 dr' 2r' P(r')
\]

which are listed in Table I.

### C. An approximate but accurate relation between the ELS of NNN and those of NN

In fact, instead of lengthy results from exact evaluation of integral, we find an approximate relation between \( P^{(3)}(r') \) and \( P(r) \): \( P^{(2)}_{w,\beta}(r') \approx P_{w,3\beta+1}(r) \) (10)

The difference of the two was shown in Fig 2. The very small deviation shows that the approximation in Eq. 10 is quite accurate. In the last two lines in the TABLE I, we also list the numerical values of \( \langle r' \rangle, \langle \tilde{r}' \rangle \) and \( \langle \tilde{r}' \rangle \) using \( P_{w,3\beta+1}(r) \). All these values are very close to those using \( P^{(2)}(r') \) in Eq. 8. For example, just take \( \langle \tilde{r}' \rangle \) for the GSE case, one can see the relative difference \( \frac{0.7672 - 0.7910}{0.7910} = \sim 3\% \) is very small.

In fact, one can see Eq. 10 can also work well when putting \( \beta = 0 \). Namely, when the NN satisfies Poisson, the NNN seems fit GOE approximately. For example, from the TABLE I, \( \langle r' \rangle \) and \( \langle \tilde{r}' \rangle \) for Poisson are 2 and 0.5 which are not too much away from \( \langle r \rangle \) and \( \langle \tilde{r} \rangle \) for GOE (1.75 and 0.53590). At least, one can use GOE for an quick eye guides to judge the NNN for the Poisson as we did in all the following figures.

In the following sections, we will apply both NN and NNN ELS to study the CIT in the two types of hybrid Sachdev-Ye-Kitaev models.
In the $K = 0$ limit, the $q = 4$ SYK Hamiltonian $H_{M4}$ is non-integrable at any finite $N$. In the following, we discuss when $N$ is even or odd respectively. For even number of sites $N = 2N_c$, one can split the sites to even and odd, then introduce $N_c$ complex fermions\textsuperscript{10,23,25} by $c_i = (\chi_{2i} - i\chi_{2i-1})/\sqrt{2}, c_i^\dagger = (\chi_{2i} + i\chi_{2i-1})/\sqrt{2}$ and define the PH symmetry operator to be\textsuperscript{52}

$$P = K \prod_{i=1}^{N_c} (c_i^\dagger + c_i) \quad (12)$$

It is easy to show $P^2 = (-1)^{\left\lfloor \frac{N_c}{2} \right\rfloor} P c_i P = \eta c_i, P c_i^\dagger P = \eta c_i^\dagger$ where $\eta = (-1)^{\left\lfloor \frac{N_c-1}{2} \right\rfloor}$. The total number of fermions $Q_c = \sum_{i=1}^{N_c} c_i^\dagger c_i$. It is not a conserved quantity, but the parity $(-1)^Q_c$ is in $H_{M4}$. Then $PQ_cP^{-1} = N_c - Q_c$ which justifies $P$ as an anti-unitary PH transformation. $P$ also commutes with the Hamiltonian\textsuperscript{51} $[P, H_{M4}] = 0$. Depending on $N$ (mod 8), the ELS satisfies GUE when $N$ (mod 8) = 2, 6, GOE when $N$ (mod 8) = 0, GSE when $N$ (mod 8) = 4, 8, 23, 25. The ELS, the degeneracy at a given parity sector, and the total parity sector are listed in the Table IIa. Its low energy excitation level spacing is $e^{-s_0 N}$ which leads to extensive $T = 0$ entropy $s_0 = 0.232424...$ (in the $N \to \infty$ limit before $T \to 0$ limit\textsuperscript{5,7,9}). The system’s many body DOS (different from the single particle DOS) has been worked out in\textsuperscript{52} to be similar, but different than the semi-circle DOS. The quasi-particle picture completely breaks down. This is similar to the $U(1)/Z_2$ Dicke model in the $Z_2$ limit (inside the superradiant phase) where the ELS satisfies the GOE distribution\textsuperscript{67,68} (see Sec V-C). its two lowest energy splitting between two opposite parities is $e^{-s_0 N}$ which is due to the instanton quantum tunneling (QT) process. Of course, the ground state of $q = 4$ SYK is a gapless quantum spin liquid. While that of $U(1)/Z_2$ Dicke model is a gapped $Z_2$ symmetry broken super-radiant state at $N = \infty$.

However, when $N$ is odd, the above procedures for even $N$ needs to be modified. In fact, one can still take the advantage of the above representation with $N$ even case by adding $\chi_{N+1} = \chi_{\infty}$ to make the parity conservation explicit, but also enlarge the Hilbert space twice. Similar strategy was used before to study the symmetry protected topological phase of odd number of Majorana chains\textsuperscript{56} and the ELS of the SYK model with $N$ odd\textsuperscript{23,25}. Then one can still define $P$ with $N_c = \frac{N-1}{2}$ as before. All the commutation relations still apply.

So when $N$ (mod 8) = 1, 5, $N_c$ is even, then $P^2 = -1, +1$ respectively. Under $P$, $Q_c$ maps to the same parity sector. So it is in GSE and GOE with degeneracy $d = 2$ and $d = 1$ respectively at a given parity sector. When using the $Z$ operator (see Eq. (13) below) which maps $Q_c$ to $Q_c + 1$, so it establishes the connection between the two parity sectors. So it has the degeneracy $d_t = 2 + 2$ and $d_t = 1 + 1$ when considering both parities.

When $N$ (mod 8) = 1, 5, $N_c$ is odd, then $P^2 = +1, -1$ respectively. However, under $P$, $Q_c$ maps to the opposite

III. TYPE-I HYBRID SACHDEV-YE-KITAEV MODELS

By Type-I, we mean the integrable side is given by $q = 2$ SYK which breaks the PH symmetry Eq. (12) of the $q = 4$ SYK. We will discuss Majorana and complex type-I hybrid SYK respectively.

A. The hybrid of $q = 2$ and $q = 4$ Majorana fermion SYK

The CIT may be first investigated in the hybrid of $q = 2$ and $q = 4$ Majorana fermion SYK, which also known as type-I hybrid Majorana SYK model:

$$H_{M4} = \sum_{i<j<k<l} N \ J_{ijkl} \chi_i \chi_j \chi_k \chi_l + i \sum_{i<j} N K_{ij} \chi_i \chi_j \quad (11)$$

where $J_{ijkl}, K_{ij}$ are real and satisfy the Gaussian distributions with $(J_{ijkl}) = 0, (J_{ijkl}^2) = 3! J^2 / N^3$ and $(K_{ij}) = 0, (K_{ij}^2) = K^2 / N$ respectively. In the following, we denote the first term of $H_{M4}, q = 4$ SYK model, as $H_{M4}$, and the second term, $q = 2$ SYK model, as $H_{M2}$.

In the $J = 0$ limit, the $q = 2$ SYK $H_{M2}$ breaks the PH symmetry Eq. (12) It is non-interacting, so must be integrable. Its single particle density of state (DOS) satisfies the semi-circle law\textsuperscript{11}, while its many body ELS satisfies the Poisson distribution\textsuperscript{11,25}. Its many body low or high energy excitation level spacing is $\sim 1/N$, the low energy quasi-particle picture holds. Its $T = 0$ entropy density $s_0$ vanishes. This is very similar to the $U(1)/Z_2$ Dicke model\textsuperscript{54,65} in the $U(1)$ limit (inside the superradiant phase) where the many body ELS also satisfies the Poisson distribution, and the low energy excitation level spacing is also $\sim D \sim 1/N$ where $D$ is the phase diffusion constant.

![FIG. 2: Difference of $\delta P^{(2)} = P^{(2)}_{W,0}(r) - P^{(2)}_{W,0+1}(r)$. The solid line is from Eq.8. The numerical data in $\delta P^{(2)} = P^{(2)}_{W,0}(r) - P^{(2)}_{W,0+1}(r)$ are taken from Fig. 4. Small $|\delta P^{(2)}|$ justifies validation of Eq. (10).](image)
connection between the two opposite parity sectors. This

FIG. 3: Even $N$: the evolution of the ELS for type-I hybrid
Majorana SYK model with even $N$. (a) and (c) are GUE, (b)
is GOE, (d) is GSE on the $q = 4$ side. (r) (black curve), and
($\tilde{r}'$) (orange curve) for the GSE case in (d) evaluated
for $N^c = 14, 16, 18, 20$ and are averaged over 100, 80, 60,
40 samplings respectively. Notably, in the GSE case in (d),
the NN ratio ($\tilde{r}$) is rapidly changing near the $q = 4$ side,
so it is quite difficult to determine the stability regime of the
quantum chaos. Fortunately, the NNN ratio ($\tilde{r}'$) shows
a nice plateau regime near the $q = 4$ side, the quantum chaos
stability regime can be easily identified and listed in the Ta ble
III-VI. This dramatic advantage of the new NNN ratio over
the known NN ratio when there is a double degeneracy was
further demonstrated in all the following relevant figures when
there are double degeneracy on the chaotic side or integrable
side.

sector $Q_c+1$. So one may only use $P$ to establish
the connection between the two opposite parity sectors. This
forced us to look for a new operator which may map $Q_c$ into
the same sector and still commutes with the Hamiltonian.
This new operator is found to be:

\[ Z = P \chi_\infty = K \prod_{i=1}^{N_c-1} (c_i^\dagger + c_i) \quad (13) \]

which can be written down by just changing $N_c$ in $P$ to
$N_c-1$. So it will play a complementary role as $P$ which
will be analyzed in the following.

One can show that $Z \chi_i Z = (-1)^N \eta \chi_i$, where $i = \infty$
is excluded and the number $\eta = (-1)^{[\frac{N_c-1}{2}]}$ is defined
below Eq. (12). It is also easy to see that $Z^2 = \eta$. Using the
fact that the Hamiltonian does not contain the fermions
added at infinity, one can show that it still commutes with the Hamiltonian $[Z, H_{M4}] = 0$. It also leads to
$ZQ_cZ^{-1} = N_c - 1 - Q_c + 2n_\infty$ where $n_\infty = c_\infty^\dagger c_\infty = \frac{1}{2} - i \chi_\infty \chi N$.

So when $N$ (mod 8) = 1, 5, $N_c$ is odd. Under $Z$, $Q_c$
maps to the same sector. $Z^2 = 1$ and $Z^2 = -1$
respectively, so it is in GOE and GSE with degeneracy
d = 1 and d = 2 respectively at a given parity sector.

When using the $P$ operator (see above) which maps $Q_c$
to $Q_c + 1$, so it establishes the connection between the
two parity sectors. Then it has the degeneracy $d_1 = 1 + 1$
and $d_2 = 2 + 2$ when considering both parities.

Now we apply the PH transformation to the hybrid
SYK model Eq. (11). The parity $(-1)^Q_c$ remains to be
conserved. However, $P$ ( when $N$ (mod 8) = 1, 5, one
use $Z$, in all the other cases, one use $P$ ) is not con-
sewed anymore due to $\{ P, H_2 \} = 0$ ( or $\{ Z, H_2 \} = 0$).
So the hybrid SYK does not have the PH symmetry any-
more. Just from symmetry point of view (the 10-fold way
classification scheme), the hybrid Majorana SYK Eq. (11)
belongs to the class A, so may satisfy GUE for any ratio
of $K/J$. Our ED studies at a given parity $(-1)^Q_c$
sector were shown in Fig. 3 for even $N$ and Fig. 4 for odd $N$.
In the following, we will discuss them respectively.

For even $N$, there are 3 cases: (a) For $N$ (mod 8) = 0,
the $q = 4$ Majorana fermion SYK at $K = 0$ is in GOE,
the hybrid is in the GUE around $K/J = e^{-4.5} \sim 0.03$ to
1, there is a crossover from GOE to GUE first, then a
CIT from GUE to the Poisson as $K/J$ increases.

(b) For $N$ (mod 8) = 2, 6, the $q = 4$ Majorana fermion
SYK at $K = 0$ is in GUE, it stays in the GUE until
$K/J = 1$, then there is a CIT from GUE to the Poisson
as $K/J$ increases.

(c) For $N$ (mod 8) = 4, the $q = 4$ Majorana fermion
SYK at $K = 0$ is in GSE at $K = 0$. Because $P^2 = -1$,
any energy level is doubly degenerate at a given parity
sector, so when doing ELS, we only pick up one of the
doubly degenerate levels to demonstrate the GSE. Any
small $K$ breaks the degeneracy, then we may consider
both sets of energy levels, a small $K$ makes ($\tilde{r}$) small, so
TABLE II: The ELS and degeneracy of the Majorana fermion $q = 4$ SYK model at $N$ even in (a) or odd in (b). The degeneracy $d_t$ is at both parity sectors. (a) When $N$ is even, there is only one anti-unitary operator $P$. When $N$ (mod 8) = 2, 6, $P$ maps $Q_e$ into $Q_e + 1$. When $N$ (mod 8) = 0, 4, $Q_e$ and $Q_e + 1$ are completely dependent, no operator connects between the two opposite parities. (b) When $N$ is odd, after adding one Majorana fermion at $N + 1 = \infty$, one doubles the Hilbert space, also introduces one more conserved quantity (the parity), there are two anti-unitary operators $P$ and $Z$. When $N$ (mod 8) = 3, 7, $P$ maps $Q_e$ to itself, $Z$ maps $Q_e$ into $Q_e + 1$. When $N$ (mod 8) = 1, 5, $Z$ maps $Q_e$ to itself, $P$ maps $Q_e$ into $Q_e + 1$. So $P$ and $Z$ exchange their roles in the two cases. So the $d_t$ in the odd $N$ case is the degeneracy in the enlarged Hilbert space which is the twice of the original one.

| $N$ (mod 8) | 0 | 2 | 4 | 6 |
|-------------|---|---|---|---|
| ELS         | GOE | GUE | GSE | GUE |
| $\beta$     | 1 | 2 | 4 | 2 |
| $Q$         | $d = 1$ | $d = 1$ | $d = 2$ | $d = 1$ |
| $Q_t$       | $d_t = 1$ | $d_t = 1 + 1$ | $d_t = 2$ | $d_t = 1 + 1$ |
| $N$ (mod 8) | 1 | 3 | 5 | 7 |
| ELS         | GOE | GSE | GSE | GOE |
| $\beta$     | 1 | 4 | 4 | 1 |
| $Q$         | $d = 1$ | $d = 2$ | $d = 2$ | $d = 1$ |
| $Q_t$       | $d_t = 1 + 1$ | $d_t = 2 + 2$ | $d_t = 2 + 2$ | $d_t = 1 + 1$ |

$\langle \hat{r} \rangle$ starts from zero and increases as $K/J$ increases, then reaches the GUE in the range as $e^{-4}$ to $e^{0.5}$. There is a CIT from GUE to the Poisson as $K/J$ increases. So in this case, using the NN ELS is not enough. One may start to use the combination of NN and the NNN ELS presented in Sec.2. The $\langle \hat{r} \rangle$ was also shown in Fig.3d. It is convenient to combine both $\langle \hat{r} \rangle$ and $\langle \hat{r} \rangle$ into the same Fig.3d. When $\langle \hat{r} \rangle$ in (d) is close to be zero, the NN $\langle \hat{r} \rangle$ in (d) still shows GSE until $K/J \sim e^{-5}$. When $\langle \hat{r} \rangle$ in (d) reaches the plateau value $\sim 0.60266$ of GUE with $\beta = 2$, then according to Eq.11, $\langle \hat{r} \rangle$ in (d) reaches the corresponding plateau value listed in Table I as $\sim 0.7344$ with a RMT index $3\beta + 1 = 7$. When $\langle \hat{r} \rangle$ in (d) reaches the plateau value $\sim 0.38629$ of Poisson, according to Eq.11, $\langle \hat{r} \rangle$ in (d) reaches a corresponding plateau value listed in Table I as $\sim 0.5$. As pointed out in Sec.2, it is only slightly below the GOE value $0.53590$ with the RMT index $3\beta + 1 = 1$.

For odd $N$, due to the absence of the GUE in Table IIb, at the $q = 4$ side, there are only the GSE case in (a) and (d), the GOE case in (b) and (c). They show similar evolution patterns as the corresponding GSE and GOE cases at the $q = 4$ side for even $N$ cases shown in Fig.2d.

Obviously, at any given disorder realization of $K_{ij}$, the eigen-energies of $q = 2$ SYK is in-commensurate (so the disorder in SYK may play a similar role as the Berry phase in the $J - U(1)/Z_2$ Dicke model), a quantum version of Kolmogorov-Arnold-Moser (KAM) theorem (See Sec V-C ) should apply when $J/K$ is sufficiently small (See Sec V-C). So the ELS changes from the Poisson to GUE around some critical $(J/K)_c$ values. The mean value $\langle \hat{r} \rangle$ also changes from its Poisson value to the corresponding GUE value. It is similar to the $\beta_3$ in the $U(1)/Z_2$ Dicke model where the ELS changes from the Poisson to GOE when inside the superradiant phase $g/g_c \geq 1$ (see Fig.2a in58). Of course, in contrast to the Dicke model, due to the quenched disorders, there is no regular regime (see Fig.2b in58) at any values of $J/K$.

The procedures used for the Majorana fermion in the last subsection (of course, no odd $N$ case anymore) can also be applied to study the $q = 2$ and $q = 4$ type-I hybrid complex SYK:

$$q_c = \sum_i (c_i^\dagger c_i - 1/2)$$

The $q_c$ can be applied to study the $q = 2$ and $q = 4$ type-I hybrid complex SYK:

$$H_C = \sum_{i \leq j < k < l} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \sum_{i < j} K_{ij} c_i^\dagger c_j - \mu q_c$$

where $\langle J_{ijkl} \rangle = 0$, $\langle (J_{ijkl})^2 \rangle = 3J^2/N^3$ drawing from the Gaussian distribution $P[J_{ijkl}] \sim e^{-A|J_{ijkl}|^2/2J^2}$ where $A = N^3/3!$. In general, $J_{ijkl} = -J_{ij;kl}$, $J_{ijkl} = -J_{ij;kl}$, $J_{ijkl}$ = $J_{ij;kl}$. We also take the four site indices $i, j, k, l$ are all different to keep the PH symmetry explicit at $\mu = 0$. $K_{ij}^* = K_{ij}$ is a Hermitian matrix satisfying $\langle K_{ij} \rangle = 0$, $\langle (K_{ij})^2 \rangle = K^2/N$.

In the $K/J = 0$ limit, the $q = 4$ complex SYK is non-integrable at any finite $N$. Under the PH transformation, $q_c \rightarrow -q_c$. So $q_c = 0$ is the PH symmetric and only happens when $N$ is even. At the half-filling $q_c = 0$, the
system also has the maximum zero temperature entropy $s_{\beta=0}$.
Away from the half-filling $\mu \neq 0$, it breaks the $P$ symmetry. 
$-N/2 < q_c \neq 0 < N/2$ corresponds to a non-vanishing electric field in a charged black hole in an asymptotic $AdS_2$ bulk.
It was shown that when $N \equiv 0 \mod 4$, the ELS is GOE and GSE respectively. But when $N \equiv 1 \mod 4$ and $q_c = \pm 1/2$, it is GUE. In fact, as long as $q_c \neq 0$, there is no PH symmetry anymore, so it is in GUE regardless of the value of $N \equiv 0 \mod 4$.

Now we apply the PH transformation to the hybrid SYK model Eq.(15) when $K/J \neq 0$. The fermion number remains to be conserved. However, $P$ is not conserved anymore due to $\{P, H_2\} = 0$ (here and thereafter, $H_2$ excludes $\mu$ in Eq.(15)). So the hybrid SYK does not have the PH symmetry anymore. Just from symmetry point of view, the hybrid complex SYK belongs to the class A in the 10 fold way classification, so may satisfy GUE for any ratio of $K/J$. Our ED studies at a given $q_c$ in Fig.5 shows that this is true only in the intermediate regimes of $K/J$. The KAM theorem and its dual form applies in the two ends regimes $K/J \ll 1$ where the ELS remains Poission and $K/J \gg 1$ where the ELS becomes the WD determined by the $q = 4$ SYK respectively.

In our ED studies, we first study $q = 4$ SYK to reproduce the known results[3], then look at the $q = 2$ SYK to show that it indeed satisfies the Poisson distribution. Then we study the hybrid model Eq.(15) at a given $q_c$ at any $K/J$. The ED result is shown in Fig.6 where there are also 3 cases:

(a) For $N_c \equiv 0 \mod 4$, at the half filling $q_c = 0$, the hybrid complex fermion SYK is in the GUE in a wide range near $K/J = 1$, there is a crossover from GOE to GUE, then a CIT from GUE to the Poisson as $K/J$ increases.

(b) Away from the half filling $q_c \neq 0$, regardless of $N_c \equiv 0 \mod 4$, the $q = 4$ SYK is in GUE. The hybrid complex fermion SYK stays in the GUE across $K/J = 1$ until to $K/J \sim e^{1.5}$, then there is a CIT from GUE to the Poisson as $K/J$ increases further.

(c) For $N_c \equiv 2 \mod 4$, at the half filling $q_c = 0$, the complex $q = 4$ fermion SYK is in GSE at $K = 0$. Because $P^2 = -1$, any energy level is doubly degenerate, so when doing ELS, we only pick up one of the doubly degenerate levels to demonstrate the GSE at $K/J = 0$. Any small $K$ breaks the degeneracy, then we may need to consider both sets of energy levels, then it is easy to see that any small $K$ makes $\langle \tilde{r} \rangle$ small, so $\langle \tilde{r} \rangle$ starts from zero and increases as $K/J$ increases. The hybrid complex fermion SYK is in the GUE in a wide range near $K/J = 1$. There is a CIT from GUE to the Poisson as $K/J$ increases further.

So in this case, using the NN ELS is not enough. One must to use the combination of NN and the NNN ELS presented in Sec.2. The $\langle \tilde{r}' \rangle$ was also shown in Fig.5. It is complete to combine both $\langle \tilde{r} \rangle$ and $\langle \tilde{r}' \rangle$ into the same figure. When $\langle \tilde{r} \rangle$ is close to be zero, the NNN $\langle \tilde{r}' \rangle$ still shows GSE until $K/J \sim e^{-5}$. When $\langle \tilde{r} \rangle$ reaches the plateau value $\sim 0.60266$ of GUE with $\beta = 1$, then according to Eq.10 $\langle \tilde{r}' \rangle$ reaches a corresponding plateau value listed in Table I as $\sim 0.7344$ with a RMT index $3\beta+1 = 4$. When $\langle \tilde{r} \rangle$ reaches the plateau value $\sim 0.38629$ of the Poisson, according to Eq.10 $\langle \tilde{r}' \rangle$ reaches a corresponding plateau value listed in Table I as $\sim 0.5$. It is only slightly below the GOE value 0.53590 with the RMT index $3\beta+1 = 1$.

IV. TYPE II HYBRID SYK MODELS

By Type-II, we mean the integrable side is given by $(q = 2)^2$ SYK which keeps the PH symmetry Eq.(12) of the $q = 4$ SYK. We will discuss Majorana and complex type-II hybrid SYK respectively. The results can be contrasted with the corresponding Type-I hybrid SYK models.

A. The particle-hole $P$ or Z conserving hybrid $q = 4$ Majorana fermion SYK

It may also be interesting to study the CIT in the type-II $q = 4$ hybrid Majorana fermion SYK which keeps the PH symmetry at any ratio of $K/J$.

$$H_{M-II, \pm} = \sum_{i<j<k<l}^N J_{ijkl} \chi_i \chi_j \chi_k \chi_l \pm [\sum_{i<j}^N K_{ij} \chi_i \chi_j]^2$$

where $J_{ijkl}$, $K_{ij}$ are real and satisfy the Gaussian distributions with $\langle J_{ijkl} \rangle = 0$, $\langle J_{ijkl}^2 \rangle = 3J^2/N^3$ and $\langle K_{ij} \rangle = 0$, $\langle K_{ij}^2 \rangle = 3K^2/N^3$. 

FIG. 5: The evolution of the ELS for type-I hybrid complex SYK model. (a) is GOE (b) and (d) are GUE, (c) is GSE on the $q = 4$ side. $\langle \tilde{r} \rangle$ (black curve), and $\langle \tilde{r}' \rangle$ (orange curve) for GSE case in (c) evaluated at $N_c = 12, 13, 14, 15$ and are averaged over 100, 50, 50, 50 samplings respectively. Note the advantages of using $\langle \tilde{r}' \rangle$ over $\langle \tilde{r} \rangle$ in (c), especially in the quantum chaos side.
\langle K_{ij} \rangle = 0, \langle K_{ij}^2 \rangle = K/N \text{ respectively. Note that here we use } \langle K_{ij}^2 \rangle \sim K \text{ to make } K/J \text{ dimensionless.}

Obviously, the second term of \( H_{M,II,+} \) can be written as \( H_{M,II,+}^2 \). In contrast to Eq. (14) it still keeps the \( P \) symmetry. So symmetry analysis alone cannot distinguish between \( H_{M,II} \) and \( H_{M,II}^2 \) despite the former is chaotic, the latter is integrable. Because \( H_{M,II} \) is integrable, so is \( H_{M,II}^2 \). This can be most conveniently seen from the NN ratio \( r_n = s_n/s_{n+1} \) of the \( H_{M,II} \). Then \( r_n \) of the \( H_{M,II}^2 \) can be written as:

\[
R_n = \frac{E^2_n - E^2_{n-1}}{E^2_{n+1} - E^2_n} \sim \frac{E_n - E_{n-1}}{E_{n+1} - E_n} = r_n
\]

where, similar to the cancelation of the density of states in \( r_n \), the center of two NN energies just cancels in the ratio. Very similarly, one can show that the ratio of the NNN spacing \( r'_n \sim r''_n \). So the ELS of \( H_{M,II}^2 \) remains Poisson.

This model with both \( \pm \) sign was studied before in [23] by \( 1/N \) expansion and by the RG analysis at \( N = \infty \). By performing the RG on the coefficient of the \( H_{M,II}^2 \) term around the \( q = 4 \) SYK conformally invariant fixed point, it was found that the + sign is marginally irrelevant (Fig. 2), so the \( q = 4 \) non-integrable SYK NFL fixed point is stable in the IR against the \( +H_{M,II}^2 \) perturbation. However, the – is marginally relevant (Fig. 1), it flows to the integrable FL fixed point controlled by \( -H_{M,II}^2 \). However, our ED results show that there is very little differences between the two signs in ELS. This fact could be explained as follows:

\[
-H_{M,II,-}[J, K] = \sum_{i<j<k<l} J'_{ijkl} \chi_i \chi_j \chi_k \chi_l + \left[ \sum_{i<j} J_{ijkl} \chi_i \chi_j \right]^2 = H_{M,II,+}[J', K],
\]

where \( J'_{ijkl} = -J_{ijkl} \). So \( J' \) and \( J \) satisfy the same distribution.

Let \( E^+_n \) be an ordered set of energy levels of \( H_{M,II,+}[J, K] \), then \( s^+_n = E^+_n - E^+_{n-1} > 0 \) are the NN spacings. Then \( E^n_+ = E^+_n \) is the corresponding ordered set of energy levels of \( H_{M,II,-}[J', K] \) and \( s_n = E^-_n - E^-_{n-1} = E^+_{n+1} - E^+_n = s^+_n \) are the corresponding NN spacings. Similarly, the NNN spacings \( s'_n = E^+_{2n-1} - E^+_{2n-2} > 0 \). Because \( (J', K) \) satisfy the same distribution, we conclude \( H_{M,II,\pm} \) satisfy the same ELSs. This is confirmed by our ED. So we just show our + sign results in Fig. 6 and 7. However, there is an exchange between the ground state and the highest energy state in the \( \pm \) sign, so the \( H_{M,II,\pm} \) will have completely different ground states. This can also be seen by the RG analysis in Fig. 2b, 1b, 12. This fact may show that the CIT characterized by the RMT at a finite \( N \) may be complementary to the QPT characterized by the RG at \( N = \infty \).

Further elucidations on the intricate relation between RG and RMT will be given in Sec. V.

Because \( \{ P, H_{M,II,\pm} \} = 0 \) or \( \{ Z, H_{M,II,\pm} \} = 0 \) (when \( N \) (mod 8) = 1, 5, one use \( Z \), in all the other cases, one use \( P \)), so symmetry analysis alone cannot distinguish between \( H_{M,II} \) and \( H_{M,II}^2 \). So the symmetry classification in Sec. 3 still applies to this Type-II hybrid Majorana fermion SYK at any \( J/K \). So the Table II still holds. Our ED studies in Fig. 8 and 7 shows that this is true up to \( K/J \sim e^{-2} \). Then the KAM theorem applies at small \( K/J \ll 1 \) where the ELS becomes Poissonian. There is a CIT from the corresponding WD to the Poisson as \( K/J \) increases. However, the degeneracy in Table I remains true at any \( K/J \). In the following, we discuss \( N \) even and odd case respectively.

For \( N \) even shown in Fig. 6 there is always a CIT from the corresponding WD to the Poisson. Several salient features need to be stressed. For \( N \) (mod 8) = 0 in (b), one may also look at the ELS from the \( H_{M,II}^2 \) side when \( K/J \ll 1 \). Because \( \{ P, H_{M,II} \} = 0 \), then \( \psi, \bar{\psi} \) are still in the same parity sector, but have two different eigenvalues \( \pm \lambda \), so are orthogonal. But \( \{ P, H_{M,II}^2 \} = 0 \), a pair of orthogonal eigenstates \( (\psi, \bar{\psi}) \) have the same eigenvalue \( \lambda^2 \). So \( H_{M,II}^2 \) is doubly degenerate at \( J = 0 \). However, the double degeneracy is broken by any \( J > 0 \). Then using the NN ELS is not enough. One must use the combination of NN and the NNN ELS presented in Sec. 2. In contrast to all the previous cases with \( P \) or \( Z \) violating...
For $N$ (mod $8) = 0$, the energy levels are doubly degenerate at any $J/K$. For $N$ (mod $8) = 4$ in (d), there is a CIT from GSE to Poisson as $K/J$ increases. As shown in Table II, there is always a double degeneracy at any $J/K$. Although $\langle \tilde{r}' \rangle$ in (b) is quite similar to $\langle \tilde{r}' \rangle$ in (d), both of which looks like to show a CIT from GSE to Poisson, they have very different physical meanings. The former is actually a CIT from GOE to Poisson. While the latter is a true CIT from GSE to Poisson with a double degeneracy.

### B. The particle-hole $P$ conserving hybrid $q = 4$ complex fermion SYK

Now we study the CIT in the following $q = 4$ type-II hybrid complex fermion SYK model which also keeps the $P$ symmetry at any ratio of $K/J$:

$$H_{C-II} = \sum_{i<j,k<l}^{N} J_{ij,kl} c_{ij}^\dagger c_{kl}^\dagger c_{kl} c_{ij} + \sum_{i<j}^{N} K_{ij} c_{ij}^\dagger c_{ij}^\dagger - \mu q_c,$$

where $K_{ij}^* = K_{ji}$ is a Hermitian matrix satisfying $\langle K_{ij} \rangle = 0, |\langle K_{ij} \rangle|^2 = K/N$. Note that here we use $\langle K_{ij}^2 \rangle \sim K$ to make $K/J$ dimensionless.

Type-I hybrid SYK models where the doubly degeneracy comes from the $q = 4$ non-integrable side, here the doubly degeneracy comes from the integrable side. It is best to combine $\langle \tilde{r} \rangle$ and $\langle \tilde{r}' \rangle$ in Fig.7, and read them from the $(q = 2)^2$ integrable side. When $\langle \tilde{r} \rangle$ is close to be zero, the NNN $\langle \tilde{r}' \rangle$ still shows Poisson until $J/K \sim e^{-\beta}$. When $\langle \tilde{r} \rangle$ reaches the plateau value $\sim 0.53590$ of the GOE with $\beta = 1$, then according to Eq.10 $\langle \tilde{r}' \rangle$ reaches an corresponding plateau value listed in Table I as $\sim 0.6709$ which is quite close to GSE with a RMT index $3\beta + 1 = 4$.

For $N$ (mod $8) = 4$ in (d), there is a CIT from GSE to Poisson as $K/J$ increases. As shown in Table II, there is always a double degeneracy at any $J/K$. Although $\langle \tilde{r}' \rangle$ in (b) is quite similar to $\langle \tilde{r}' \rangle$ in (d), both of which looks like to show a CIT from GSE to Poisson, they have very different physical meanings. As said above, $\langle \tilde{r}' \rangle$ with $N$ (mod $8) = 0$ in (b) represents NNN ELS, so it stands for a CIT from GOE to Poisson. While $\langle \tilde{r} \rangle$ with $N$ (mod $8) = 4$ in (d) represents NN ELS, so it is a true CIT from GSE to Poisson.

For $N$ (mod $8) = 2, 6$ in (c) and (a), there is no degeneracy at any $J/K$. There is a CIT from GUE to Poisson as $K/J$ increases.

The odd $N$ case is shown in Fig.7. As shown in Table I, one continue to use $P$ symmetry when $N$ (mod $8) = 3, 7$, but must use $Z$ symmetry when $N$ (mod $8) = 1, 5$. Due to the absence of GUE, there are only two cases: the CIT from GSE to Poisson in (a) and (d) and the CIT from GOE to Poisson in (b) and (c). They show similar evolution patterns as the corresponding CITs for even $N$ case shown in Fig.6. Again, although $\langle \tilde{r}' \rangle$ in (b) and (c) are quite similar to $\langle \tilde{r} \rangle$ in (a) and (d), both of which looks like to show a CIT from GSE to Poisson, they have very different physical meanings. The former is actually a CIT from GOE to Poisson. While the latter is a true CIT from GSE to Poisson with a double degeneracy.
regardless of the value of $N \pmod{4}$). Our ED studies\textsuperscript{27} in Fig.8 shows that this is true until $K/J \sim e^{-1}$. Then the KAM theorem applies al small $J/K \ll 1$ where the ELS becomes Poisson. However, the degeneracy remains true at any $J/K$. There is a CIT from the corresponding WD to the Poisson as $K/J$ increases.

The rest of discussions are quite similar to the $P$ or $Z$ conserving Type-II hybrid Majorana fermion SYK discussed in the last subsection. Similar to Eq.\textsuperscript{19} it is easy to show that because $H_{C2}$ is integrable, so is $H'_{C2}$, so its ELS remains Poisson. Furthermore, the eigenvalue of $H'_{C2}$ is always positive. Both $\pm$ sign in the second term need to be considered. In the RG sense, we expect the $+$ $\sim$ $WD$ to the Poisson as in Fig.8. There is always a CIT from the corresponding $NN$ and the NNN ELS presented in Sec.2. In contrast to the previous $P$ violating type-I hybrid complex SYK model where the doubly degeneracy comes from the $q = 4$ chaotic side, here the doubly degeneracy comes from the integrable side. It is complete to combine both $\langle \tilde{r} \rangle$ and $\langle \tilde{r}' \rangle$ in Fig.\textsuperscript{a} and read them from the $q = 2$ integrable side. When $\langle \tilde{r} \rangle$ is close to be zero, the NNN $\langle \tilde{r}' \rangle$ shows Poisson until $J/K \sim e^{-6}$. When $\langle \tilde{r} \rangle$ reaches the plateau value $\sim 0.53590$ for the GOE with $\beta = 1$, then according to Eq.\textsuperscript{10} $\langle \tilde{r}' \rangle$ reaches a corresponding plateau value listed in Table I as $\sim 0.6709$ which is quite close to the GSE with a RMT index $3\beta + 1 = 4$. There is always a double degeneracy at any $J/K$ in (c). Although $\langle \tilde{r}' \rangle$ in (a) is quite similar to $\langle \tilde{r} \rangle$ in (c), both of which looks like to show a CIT from GSE to Poisson, they have very different physical meanings. As said above, $\langle \tilde{r}' \rangle$ in (a) when $N_c \pmod{4} = 0, q_c = 0$ represents NNN ELS, so it stands for a CIT from GOE to Poisson. While $\langle \tilde{r} \rangle$ in (c) when $N_c \pmod{4} = 2, q_c = 0$ represents NN ELS, so it stands for a CIT from GSE to Poisson and each energy level is doubly degenerate at any $K/J$.

V. CONTRAST KAM THEOREM OF INTEGRABILITY WITH STABILITY OF QUANTUM CHAOS

In classical chaos, the classical Kolmogorov-Arnold-Moser (KAM) theorem\textsuperscript{28} states that if an integrable Hamiltonian $H_0$ is disturbed by a small perturbation $\Delta H$, which makes the total Hamiltonian $H = H_0 + \Delta H$, non-integrable. If the two conditions are satisfied: (a) $\Delta H$ is sufficiently small (b) the frequencies $\omega_i$ of $H_0$ are in-commensurate, then the system remains quasi-integrable. It plays important roles to study the stability of solar systems. It remains an outstanding problem to find an quantum analogue of the KAM theorem. Here, we define the quantum analog of the classical KAM theorem as the range of a chaotic perturbation which still keeps the ELS in the Poissonian. The quantum KAM theorem in the context of RMT: For any integrable system subject to an chaotic perturbation, the KAM theorem holds when the chaotic perturbation is below the average many body energy level spacing of the integrable system.

This theorem should hold to any quantum chaotic systems such as hybrid SYK models or Dicke models.\textsuperscript{29} Instead of providing a rigorous mathematical proof of this quantum KAM theorem which is left to a future publication,\textsuperscript{21} we will first give some instructive, but naive estimates which lead to an exponential scaling law, then point out it should be replaced by a large power law. Its precise form may be determined by using more rigorous mathematical treatments\textsuperscript{29} in a future publication.

A. Scaling forms of the KAM theorem in the Type-I and Type-II hybrid Majorana or complex SYK

For the hybrid Majorana fermion SYK. The size of the Hilbert space is $2^N/2$. On the integrable side, in the $N \to \infty$ limit, although the single body density of states (DOS) satisfies the semi-circle law, the many body DOS satisfies something similar to a Gaussian which has a variance $\sigma = \sqrt{N/8}$ and also an exponentially decay in the two tails (see appendix B), so average bulk many body energy level spacing is $\sim K\sqrt{N/8} \times 2 \times 3/2 \times 2^{-N/2} \sim K\sqrt{N}2^{-N/2}$ where $2/3$ takes care of only $2/3$ states are

\[ \begin{array}{c}
\text{K/J} \\
q=4 \rightarrow q=2
\end{array} \]

\[ \begin{array}{c}
\text{K/J}' \\
(q=2)^2 \rightarrow -(q=2)^2
\end{array} \]

\[ \begin{array}{c}
\text{K/J} \\
q=4 \rightarrow (q=2)^2
\end{array} \]

\[ \begin{array}{c}
\text{K/J}' \rightarrow \end{array} \]

FIG. 9: RG flow of type-I and type-II hybrid SYK models between two different CFT fixed points. (a) The RG flow of the type-I hybrid SYK models. The $K/J', J' = \sqrt{J/K} + K$ is relevant to the $q = 4$ SYK fixed point. So the ground state is always a Fermi liquid (FL) with well defined low energy quasi-particle excitations. (b) Upper, The RG flow of the type-II hybrid SYK models with + sign. The $K/J'$ is marginally relevant to the $q = 4$ SYK fixed point. So the ground state is always a Non-Fermi liquid (NFL) without the low energy quasi-particle excitations.
within the width ±σ = \sqrt{\frac{N}{8}}. Note that the low or high lying spacing is \sim 1/N (which is much larger than the bulk energy level spacing) indicating the existence of the quasi-particles. When doing the ED, one can simply throw away these low and high energy levels. Because all these many body energy levels are un-correlated and satisfy the Possion distribution, so naively we expect that when the chaotic perturbation J is smaller than this bulk average spacing, the KAM theorem holds:

\[ (J/K)_M \sim \sqrt{N}2^{-N/2} = \sqrt{N}e^{-N\ln 2/2} \]  

(20)

This scaling seems matche well with the ED results listed in Table III for Type I and Table V for Type II hybrid Majorana SYK. Note that the prefactor \sqrt{N} may not be important in the large N limit, but is important when comparing with the ED data at the sizes we can perform the ED. For example, taking N = Nc = 14, we can see J/K = 4 \times 2^{-5} \sim 2^{-5} \sim 0.03 which is quite close to that listed in the Table III. The KAM regime in the other sizes seem also fit the scaling law well. However, as argued below Eq\[21\] the naive Eq\[20\] should be replaced by a large power law \sim \frac{1}{N^{\alpha_M}}, \alpha_M \gg 1. Unfortunately, the ED at the available sizes may not be able to distinguish the two quite different scaling forms.

For the hybrid complex fermion SYK. Due to the conserved total charge to be \frac{N}{2} at the half filling (other filling can be similar discussed). The size of the Hilbert space is \( C_N^{-\alpha M} = \frac{N!}{(\frac{N}{2})!^2}. \) By using Sterling formula \ln N! = N\ln N - N which holds for a large N, one can still show \( C_N^{-\alpha M} \sim 2^N/\sqrt{N} \) for a large N. On the integrable side, in the N \to \infty limit, although the single body density of states (DOS) satisfies the semi-circle law, the many body DOS satisfies something similar to a Gaussian which has a variance \( \sigma = \sqrt{N/4} \) and also an exponentially decay in the two tails (see appendix B), so average bulk many body energy level spacing is \( \sim K \sqrt{N/4} \times 2 \times 2^{-2N/\sqrt{N}} \sim 3/2K N^{-N}. \) Again, because all these many body energy levels are un-correlated and satisfy the Possion distribution, so naively we expect that when the chaotic perturbation J is smaller than this average spacing, the KAM theorem holds:

\[ (J/K)_C \sim 3/2N2^{-N} = 3/2N e^{-N\ln 2} \]  

(21)

In practice, when comparing the data in Table IV for Type I and Table VI for Type II, it is more accurate to use the actual \( C_N^{-\alpha M} \) instead of its Sterling form for the sizes we can perform the ED. For example, taking N = Nc = 12, then \( C_{12}^{-\alpha M} \sim 10^{-3}, \) we can see \( J/K \sim 6 \times 10^{-3} \) which seems close to that listed in the Table IV and VI. The KAM regime in the other sizes in Table IV and VI seem also fit the scaling law well.

However, as already alerted below Eq\[20\] the naive Eq\[21\] should be replaced by a large power law \sim \frac{1}{N^{\alpha_C}}, \alpha_C \gg 1. Unfortunately, the ED at the available sizes may not be able to distinguish the two quite different scaling forms. The arguments go as follow: in the integrable side, taking two nearest neighbouring (NN) bulk states \( |B1\rangle = |n_1, n_2, \ldots, n_N\rangle_{B1} \) with the eigen-energy \( E_{B1} = \sum n_i^{B1}\epsilon_i \) and \( |B2\rangle = |n_1, n_2, \ldots, n_N\rangle_{B2} \) with the eigen-energy \( E_{B2} = \sum n_i^{B2}\epsilon_i. \) The NN means the eigen-energy difference \( E_{B1} - E_{B2} \) is of the many-body origin Eq\[21\]. The single particle spacing in the integral side \( h_2 = \epsilon_2 - \epsilon_1 \sim K/N. \) So \( |B1\rangle, |B2\rangle \) must differ by \sim N/2 particle occupations \( n_i \) to get a such a small spacing listed in Eq\[21\]. However, every chaotic perturbation \( H_4 \) involves only 4 particle moves in \( n_i, \) so typically one need \sim N/2 \times 1/4 = N/8 steps to connect the two NN bulk states. The energy of intermediate states involve only 4 single particles, so the change of energy could at most be 4K/N, then by N/8 steps of chaotic perturbation which connects \( |B1\rangle \) to \( |B2\rangle, \) one can estimate the off-diagonal matrix element \( \langle B1|H_4|B2\rangle \sim (J/(4K/N))^{N/8} \sim (N/J/K)^{N/8} \) which is also exponentially small when \( N/J/K \ll 1. \) Due to the PH symmetry at the half filling and the self-average in the quenched disorders in the large N limit, one also expect that the diagonal energy shift is tiny \( \langle B1|H_4|B1\rangle - \langle B2|H_4|B2\rangle \sim 0. \) As argued in\[20\], the diagonal energy shift does not lead to the change of ED anyway. The energy level repulsion is solely due to the off-diagonal matrix element \( \langle B1|H_4|B2\rangle. \) So we expect the naive Eq\[21\] should be replaced by a large power law \sim \frac{1}{N^{\alpha_C}}. The large power \( \alpha_C \gg 1 \) will be determined by a rigorous mathematical derivation in\[20\]. A similar analysis can be applied to the Majorana fermion case to determine the large power \( \alpha_M \) in Eq\[20\].

**B. The lower bound of the dual form of the KAM theorem in the Type-I hybrid Majorana or complex SYK**

As mentioned in Sec.III-A, one may also study the quantum analog of the KAM theorem from a dual point of view, namely, the stability of quantum chaos of the quantum chaotic \( q = 4 \) SYK against the integrable perturbation as one turns on the quantum chaotic perturbation \( q = 4 \) SYK does not fit the semi-circle law well, but still close to its form near the band edge. The 1/N expansion can only get the low end of the DOS which scales as \( \sqrt{E}. \) Its whole form can only be achieved by ED. So we use the results in the ED achieved in\[21\], the bandwidth is \( 2\sqrt{M} N J \) where \( M = 0.05 \) is the ground state energy per site in the unit of N.J. Then the average many body energy level spacing is \( \sim 2^{M} \sqrt{N} N J^{-2/N^2} \) which defines the Heisenberg time \( t_H \sim \frac{1}{M} e^N. \) Note that the low or high lying spacing is \( \sim e^{-\sigma N} \) (which is still much larger than the bulk energy level spacing) indicating the absence of the low energy quasi-particles. In sharp contrast to the KAM theorem in the integral side, all these many body energy levels are correlated and satisfy various WD distributions, so the arguments used to establish the KAM
Eq.\[24\] on the integral sides break down. We expect that when the integrable perturbation \( K \) is smaller than this average spacing, it remains the same ELS. Naively, the lower bound of the dual form of the KAM takes

\[
(K/J)^M > 2\epsilon_0^M N 2^{-N/2} \sim 2\epsilon_0^M N e^{-N \ln 2/2} \quad (22)
\]

This lower bound of the scaling seems match well with the ED results for GOE and GSE cases listed in Table III for Type I. For example, taking \( N = 16 \), we can see \((K/J)^M \) is quite similar to that in the Majorana fermion case, namely the RMT behaviour \( \sqrt{E - E_0} \). However, its whole form at the half-filling can only be obtained by the ED which is shown in the appendix C Fig[12] where \( \epsilon_0^C \sim 0.15 \) is the ground state energy per site in the unit of \( N J \). Then the average many body energy level spacing is \( 2\epsilon_0^C N J \sqrt{N} e^{-N/2} \). Then Eq.\[22\] can be simply replaced by:

\[
(K/J)_C > 2\epsilon_0^C N^{\frac{3}{2}} 2^{-N} \sim 2\epsilon_0^C N^{\frac{3}{2}} e^{-N \ln 2} \quad (23)
\]

This lower bound of the scaling seems match well with the ED results for GOE and GSE cases listed in Table IV for Type I. For example, taking \( N = 12 \), we can see \((K/J)_C \) is quite close to be semi-circle with the bandwidth \( 2\epsilon_0^C N J \) which need to be evaluated by some rigorous mathematical treatments\[22\]. Again, this lower bound of the scaling seems match well with the ED results for GOE and GSE cases listed in Table III.

Due to the strong energy level correlations among the energy levels on the \( q = 4 \) chaotic side, we expect that the lower bounds Eq.\[22\] can be improved significantly. In fact, by using the ETH of the two NN bulk states in the chaotic side in Sec.VI-C, one can show the diagonal energy shift is tiny shown in Eq.\[27\]. Again, as argued in\[78\], the diagonal energy shift does not lead to the change of ELS anyway. The energy level repulsion is solely due to the off-diagonal matrix element \( (B1|H_2|B2) \) in Eq.\[27\] which need to be evaluated by some rigorous mathematical treatments\[22\]. A similar analysis can be applied to the dual Majorana fermion case.

Of course, Eq.\[22\] do not apply to the GUE case for Type I and any case in Type II. In all these cases, there are no changes in ELS from \( q = 4 \) side to the bulk, there is a direct CIT from the corresponding WD in the \( q = 4 \) side to the Possion at the \( q = 2 \) side. The stability of the quantum chaos does not vanish in the thermodynamic limit.

C. Contrast numerical data with the scaling form of the quantum KAM theorem and its dual form

Now from Fig.3-8 we summarize the stability of quantum chaos near \( q = 4 \) SYK with the KAM theorem near the integrable \( q = 2 \) SYK in type-I and \( q = 2 \) SYK in type-II in the following 4 Tables from which one can conclude the data support the KAM theorem and the lower bound of its dual form presented in the last subsection.

(1) The validity regime of KAM theorem seem comparable in type-I and type-II. In all the cases, its validity regime gets smaller in exponentially fast as the system size gets larger. These facts are consistent with the KAM theorem in the hybrid SYK models conjectured above.

(2) In type-I, the stability of GOE is comparable to that of GSE. They are also consistent with the KAM in the sizes we studied, also approaches zero as the system’s size increases. This is consist with the lower bound of the dual form of the KAM theorem conjectured above. The GOE or GSE will turn into GUE when \( K/J \) increases further.

However, the stability of the GUE is much more robust than the two. It is also much more robust than the KAM. This is due to the fact that GUE is dictated by the symmetry classification anyway, so it exists as an intermediate regime in all the three cases, then the stability regime of the GUE at the \( q = 4 \) side is greatly enhanced. Obviously, it remains a finite value in the thermodynamic limit.

(3) In type-II, the stability of quantum chaos is even among GOE, GUE and GSE. This is because they are dictated by the symmetry classification at the corresponding \( N \) values anyway. They are also much more robust than the KAM theorem.

| \( N \) | 14 | 16 | 18 | 20 |
|---|---|---|---|---|
| ELS at \( K = 0 \) | GUE | GOE | GUE | GSE |
| QCS: \( K/J \) | \( e^0 = 1 \) | \( e^{-5.5} \approx 0.004 \) | \( e^0 \) | \( e^{-5.5} \) |
| KAM: \( J/K \) | \( e^{-3} \approx 0.05 \) | \( e^{-3.5} \approx 0.04 \) | \( e^{-4} \approx 0.02 \) | \( e^{-5} \approx 0.01 \) |
| \( N \) | 13 | 15 | 17 | 19 |
| ELS at \( K = 0 \) | GSE | GOE | GUE | GSE |
| QCS: \( K/J \) | \( e^{-4.5} \approx 0.01 \) | \( e^{-5} \approx 0.007 \) | \( e^{-5.5} \approx 0.004 \) | \( e^{-5.5} \) |
| KAM: \( J/K \) | \( e^{-3} \approx 0.05 \) | \( e^{-3.5} \approx 0.04 \) | \( e^{-4} \approx 0.02 \) | \( e^{-5} \approx 0.01 \) |

It remains interesting to test the KAM theorem applied to the hybrid SYK models and its dual form in much large size systems. Unfortunately, the ED can not get to much larger size than \( N = 20 \) achieved in this paper. In Type-I Majorana fermion hybrid SYK, in the
GOE and GSE cases such as Fig.3b,d for even $N$, Fig.4 for odd $N$, Type-I complex fermion hybrid SYK, Fig.5a,c, we only identified the KAM theorem from the integrable side and the low bound of its dual form from the quantum chaos side. However, there is no physical understanding on the two ends of the intermediate GUE plateau. The two crossover regimes near the two ends may not satisfy ETH. As alerted at the end of Sec.V-B, in the GUE case (on the $q = 4$ SYK side) of the type-I SYK and all the type-II hybrid SYK models, there is no change from the ELS from the $q = 4$ SYK side to that in the bulk of the hybrid SYK models, so the quantum chaos is always much more robust against an integrable perturbation than the other cases. There is no analytic estimate on the stability regime of this robust quantum chaos. Of course, in contrast to the KAM and its dual form, this robust quantum chaos regime stays robust in the thermodynamic limit.

### VI. BULK (OR INTERMEDIATE) STATES CHARACTERIZED BY THE RMT VERSUS EDGE (NAMELY, LOW AND HIGH ENERGY) STATES CAPTURED BY RG OR $1/N$ EXPANSION

It was well established that in terms of symmetries and the space dimension, the RG (including the DMRG, MPS and tensor network) can be used to classify many body quantum phases and quantum phase transitions at $T = 0$ and classical phase transitions at finite temperatures. The RG focus on infra-red (IR) behaviours of the system which are determined by the ground state and low energy excitations. The RG is also intimately connected to general relativity through the holographic principle. RG is usually applied to a system in the thermodynamic limit to characterize the quantum phase transitions (QPT). When it gets to a finite system, finite size scalings near the QCP can be used to extract the information at a finite $N$ from the knowledge at $N = \infty$. For a zero dimensional system such as SYK models or $U(1)/Z_2$ Dicke models with infinite-range interaction, one can use a $1/N$ expansion to study the similar properties.

While the 10-fold way RMT classification scheme to describe quantum chaos in a many body system only depends on two anti-unitary or one unitary chiral operators, seems independent of space dimension. It covers all the energy levels of the system, so can be used to characterize the CIT. RMT is usually applied to a system with a finite size $N$. In a completely different context, the 10-fold way can also be used to classify topological equivalent class for non-interacting (single particle) topological insulators and topological superconductors. In this case, it also depends on the space dimension and has the Bott periodicity $d \to d + 8$.

#### A. QPT versus CIT

The differences and connections between RG at $N = \infty$, then a $1/N$ expansion at a finite $N$ and the RMT at a finite $N$ are explicitly demonstrated in the two types of SYK models in this work. For example, in the thermodynamic limit $N = \infty$, the fermions in the type-I hybrid SYK models have scaling dimension $1/2$ and $1/4$ at the $q = 2$ and $q = 4$ SYK respectively. So the $q = 2$ ($q = 4$) SYK is a stable (unstable) conformably invariant fixed point (Fig[3a]). Any $J (K)$ is irrelevant (relevant) to the $q = 2$ ($q = 4$) SYK. At any $K/J \neq 0$, the ground state is controlled by the $q = 2$ SYK fixed point, so it is always a non-chaotic Fermi liquid with well defined low energy quasi-particle excitations. There is no quantum phase transition (QPT) in the type-I hybrid SYK models. There is no finite temperature transitions either. However, as shown in Fig.3-8, there is always a chaotic to integrable transition (CIT) from the GUE to Poisson. Of course, the name of CIT need to be interpreted correctly, because it is dramatically different than...
a QPT. The former is characterized by RMT, the latter is by RG or \( 1/N \) expansion. As shown in these figures, if there is a symmetry change from the \( q = 4 \) SYK to the hybrid SYK, there are also crossovers between different WD ensembles as \( K/J \) increases.

Quantum or topological phase transitions always start from the zero temperature \( T = 0 \), then raise up to a low temperature \( T \ll \beta J < N \). Especially one can find scaling functions for various physical quantities near the QPT. If putting the system at a finite size, one may also do finite size scaling to identify the QPT by numerical simulations. It is a bottom/up approach. On the other hand, the Quantum chaos can only happen at non-zero temperature. In fact, it could even start from the infinite temperature \( T = \infty \), then lower down to the low temperature \( T \ll \beta J < N \). So it is a top/down approach. Therefore the two approaches are complementary to each other.

QPT only involves the change of the ground states and the low energy excitations, while the high energy states are irrelevant. There is a divergent length scale (the time scale is related to the length scale by a dynamic exponent \( z \)) and associated scalings near a QPT in the thermodynamic limit \( N = \infty \). While the ELS involves the bulk energy levels at a finite but large enough \( N \), the low or high energy levels are irrelevant (see appendix D). So the QPT characterized by RG and the CIT characterized by the RMT are dramatically different, but complementary to each other. Despite the absence of QPT in the two types of the hybrid SYK models presented in Sec.III and IV respectively, there could still be a CIT which is dramatically different than QPT. There is no divergent length (or time) scales, no associated scalings, therefore no finite size scalings near a CIT, but there is a KAM in the integrable side and its dual form in the chaotic side if there is a change of level statistics as presented in Sec.V. However, independent of the \(+/-\) sign, there is always a CIT from the corresponding WD of \( q = 4 \) SYK to the Poisson as shown in Fig.67 and 8. In fact, as explained in Sec.3.3.3.4, despite the \(+/-\) sign leads to dramatically different ground states, the ELS is identical. So here we provide an interesting example where the ELS are the same, but the ground states are dramatically different.

This fact demonstrates explicitly the dramatic differences between the two classification schemes which are complementary to each other, the RG focus on the ground state and low or high energy excitations, while the RMT focus on the bulk high energy levels (see appendix D).

For another example, as shown in appendix A, the ground states of the hybrid bosonic SYK is a quantum spin glass which breaks ergodicity. There is a finite temperature phase transition at \( T = T_{QSG} \) from the QSG to a paramagnet where the ergodicity is restored. However, as shown in Fig.10 in the appendix A, the ELS stays as GOE or GUE. There is no CIT. In order to probe the quantum spin glass ground state at \( T < T_{QSG} \), one need to focus on the low energy excitations which, as alerted above, can not be described by the RMT, need to be investigated by RG or \( 1/N \) expansion.

In any cases, the RMT of the bulk states are quite insensitive to these edge (low or high energy) states. Namely, when evaluating the ELS, incorporating or throwing away these edge states will not affect the ELS. To see these edge states, RG or \( 1/N \) expansion may be used to study the ground state and its low or high energy excitations. In fact, as outlined in the introduction, using \( 1/N \) expansion, many previous works studied the conformally invariant QSL and also its extensive low energy excitations with the energy spacing \( \sim e^{-\gamma_0 N} \) which leads to the zero temperature entropy \( s_0 \) (taking \( T \to 0 \) limit after taking \( N \to \infty \) limit which belongs to the regime (b) in the following subsection).

### B. Remarks on Lyapunov exponents from OTOC and SFF in the hybrid SYK models

As stressed in the conclusion of Ref.\textsuperscript{28}, there are at least two completely ways to characterize the quantum chaos. One way is through evaluating the regulated out-of-time-ordered correlation function (OTOC). Another way is to use the RMT to characterize the quantum chaos at a very late time. As demonstrated in all the previous sections, when collecting the ELS of the bulk energy levels, low and high energy levels can be simply thrown away without affecting the bulk ELS (see appendix D). So the two ways are complementary to each other.

The OTOC of the two bosonic or fermionic Hermitian operators \( V^\dagger = V, W^\dagger = W \) is defined by:

\[
F(t) = Tr[yW(t)y^V(t)yV(t)yW(t)yV(0)]
\]

where \( y = e^{-\beta H/4}/Z^{1/4} \) is one quarter of the density matrix and \( Z(\beta) = Tr e^{-\beta H} \) is the partition function. In the SYK models, \( V = W = \psi \) for Majorana or complex SYK respectively. Its early time exponential growth can be characterized by a quantum Lyapunov exponent, while its late time approaches a constant, so become featureless. The early time behaviour is mainly determined by the ground state and the low energy excitations, while the bulk energy levels are irrelevant (see appendix D). The OTOC reflects the ground state and low energy excitations, so it is directly related to the RG description and can be addressed by \( 1/N \) expansion. It seems quite in-sensitive to the 10-fold way global discrete symmetry classification, therefore independent of \( N \mod 8 \) or \( N_c \mod 4 \).
For the $q = 4$ Majorana SYK model, as shown in\textsuperscript{33}, at any finite temperature, one must consider the effects of the finite temperature $\beta J$ versus the finite size $N$. (a) At very low temperatures $\beta J > N$, the OTOC takes a power law $\sim t^{-6}$ in the long time limit $t_H > t > \beta > N/J$, so the Lyapunov exponent $\lambda_L$ cannot even be defined at such a low temperature. (b) At intermediate temperatures $1 < \beta J < N$, then the OTOC takes the exponential form in the early time up to the Ehrenfest (or the scrambling time $t_s \sim \beta \log N$) which defines the Lyapunov exponent $\lambda_L$, but still decays as $\sim t^{-6}$ in the long time limit $t_H > t > N/J > \beta$. (c) In the high temperature range $\beta J < 1 \ll N$, the physics seems dominated by the microscopic energy scale $J$, the Lyapunov exponent $\lambda_L \sim J$.

It remains interesting to study how the $K$ term changes the behaviours of the OTOC and the associated Lyapunov exponent $\lambda_L$ at all the three temperature regimes (a),(b) and (c). In both types of hybrid SYK models, the three temperature regimes are still determined by the competition between the finite size $N$ and the finite temperatures, so adding a $K$ term will not change such a division. (a) is still not the regime to even define a Lyapunov exponent, so we only need to focus on (b) and (c). In the regime (b) $1 \ll \beta J < N$. We expect that for both type I and type II hybrid SYK models, the Lyapunov exponent $\lambda_L > 0$ can be also computed in all the quantum chaotic regimes defined in the RMT sense. We expect $\lambda_L = 0$ in the KAM regime. In the regime (c), we expect that in the quantum chaotic side, any small $K$ will reduce the Lyapunov exponent to $\lambda_L \sim J - K + \cdots$. It will vanish in the KAM regime. Because at such a high temperature, all the energy levels are involved, so we expect there may exist a one to one correspondence between $\lambda_L$ and the KAM theorem from the RMT in the regime (c).

As said in the introduction, in addition to the ELS in the RMT classifications, the CIT may also be dynamically diagnosed from the spectral form factor (SFF)\textsuperscript{25,35}:

$$g(t, \beta) = \frac{\langle Z(\beta, t) Z^*(\beta, t) \rangle}{\langle Z(\beta) \rangle^2}$$

(25)

where $Z(\beta, t) = Tr(e^{-(\beta + i\beta)t}H)$ and the disorder average was taken separately in the enumerator and denominator.

The SFF at $1 < \beta J < N$ may also be used to measure the dynamic (time-dependent) chaotic behaviours of the two types of hybrid SYK models. A slope-dip-ramp-plateau structure in the time evolution was considered to be evidence for the chaotic behaviours. This feature should disappear in the KAM regime. It remains interesting to study its evolution in the quantum chaotic regime in both regimes (b) and (c).

The constraints of the symmetries $P$ in Eq.(12) and $Z$ in Eq.(13) put on the OTOC Eq.(24) and the SFF will also need to be explored.

C. Remarks on Eigenstate Thermalization hypothesis (ETH): its power on bulk states and inability to encode the edge states

It is interesting to note that the RMT was originally proposed to study statistically the many body energy level correlations of a nuclei with a large atomic number to hold large number of electrons\textsuperscript{26,27}. Then it was used to classify the quantum chaos of non-interacting electrons moving in a random potential which may show metal to Anderson insulator transition\textsuperscript{28}. There is a corresponding Chaotic to Integrable transition (CIT) where the single particle ELS satisfies WD in the metal, while Poisson in the Anderson insulator.

Recently, there is a renewed interest on Eigenstate Thermalization hypothesis (ETH) in many body interacting systems\textsuperscript{30}. It states that for any excited (also called bulk) state $|\psi\rangle$ with eigen-energy $E$ which is above the ground state energy by a finite amount in the thermodynamic limit: $\lim_{V \rightarrow \infty} \frac{|E - E_0|}{V} \neq 0$, one may define a temperature $\beta$ corresponding to the state: $\langle \psi|H|\psi\rangle = Tr He^{-\beta H}/Z$ where $Z = Tr e^{-\beta H}$ is the partition function, then ETH implies than for any local operator $O$:

$$\langle \psi|O|\psi\rangle = Tr O e^{-\beta H}/Z$$

(26)

The entanglement entropy of the excited state $|\psi\rangle$ satisfies the volume law, while that of the ground state or all the low energy states satisfies the more common area law.

Now one can use Eq.(26) to evaluate the diagonal energy level shift on the chaotic side due to the integrable perturbation $O = H_2$. Taking the complex SYK model at half-filling for an example, assuming the temperatures corresponding to the two NN bulk state $|\psi_1\rangle, |\psi_2\rangle$ in the chaotic side are $\beta_1, \beta_2$, then one can see:

$$\langle \psi_1|H_4|\psi_1\rangle - \langle \psi_2|H_4|\psi_2\rangle = \frac{1}{Z} Tr H_4 e^{-\beta_1 H_4 \delta \beta}$$

(27)

Equating Eq.(27) to Eq.(28) leads to $\delta \beta = \beta_1 - \beta_2 \sim N^{3/2}2^{-N}$. Then one can immediately see the diagonal energy level shift:

$$\langle \psi_1|H_2|\psi_1\rangle - \langle \psi_2|H_2|\psi_2\rangle = \frac{1}{Z} Tr H_2 H_4 e^{-\beta_1 H_4 \delta \beta} \sim N^{3/2}2^{-N}$$

(28)

which is very tiny. Then one may focus on the off-diagonal matrix element $\langle \psi_1|H_2|\psi_2\rangle$. By using the Cauchy inequality $|\langle \psi_1|O|\psi_2\rangle|^2 < \langle \psi_1|O|\psi_1\rangle \langle \psi_2|O|\psi_2\rangle$, one can establish the bound:

$$|\langle \psi_1|H_2|\psi_2\rangle| < Tr H_2 e^{-\beta_1 H_4}/Z$$

(29)

which maybe useful to evaluate the dual form Eq.(23) in Sec. V.

Because ETH focus on excited states, so it should be closely related to RMT. Indeed, for interacting many body systems, the quantum chaos imply the ETH or vice versa\textsuperscript{31}. The results achieved in this paper show that...
the ETH of \( q = 4 \) SYK should be preserved in all the plateau regimes in Fig.3-8 satisfying a WD class in the hybrid SYK models. While the KAM theorem implies the violation of ETH or broken ergodicity. The transition regimes between different plateaus may not satisfy ETH either.

The ETH only applies to the bulk states, so it has a very serious limitation: it has no saying on ground state and the low energy excitations which can be defined as \( \lim_{N \to \infty} \frac{E_N}{N^2} = 0 \). They are nothing but the edge states. So the conjecture that a single bulk eigenstate encodes the information of the full Hamiltonian clearly fails on the edge states. A complete understanding of the system needs not only the knowledge of the bulk states in a statistics way by a RMT, but also the edge states by more quantitative approach such as \( 1/N \) expansion or RG.

D. Comments on some early works on type-I hybrid Majorana SYK models

In this section, we comment on previous works\(^{73,74} \) on the type-I hybrid SYK model and also point out their main differences from our work.

The special GUE case (a) and (c) in Fig.8 (but not the other two cases of GOE in (b) and GSE in (d) ) in type-I Majorana fermion hybrid SYK Eq.(11) at even \( N \) was studied in\(^{73} \). However, the most interesting case of the interruption of GUE in the intermediate ranges of \( K/J \) in the other two cases (b) and (d) is absent in this special GUE case (a) and (c). Furthermore, these authors in-correctly interpret the CIT as a true QPT at \( T = 0 \) or a classical phase transition at \( T > 0 \). So it does not correspond to the Hawking-Page transition in its bulk gravity dual as claimed in this work. It was known that the Hawking-Page transition in the bulk may be dual to a true quark-gluon confinement to deconfinement transition at \( T > 0 \) in the boundary. The dramatic differences between the CIT characterized by RMT at a finite \( N \) and a true quantum or a classical phase transitions characterized by RG at \( N = \infty \) was stressed in the last subsection. The finite size scalings to locate a quantum critical point (QCP) only apply to the later, not to the former. Unfortunately, the authors in\(^{73} \) still tried to fit their data to a finite size scaling without success.

After submitting the first version of our work to arXiv, we got to know the authors in\(^{54} \) have also studied the other two cases of (b) and (d). Furthermore, they also evaluated the Thouless energy scale \( E_{th} = \frac{N^2}{16} \Delta \) where \( \Delta \) is the average many body energy level spacing beyond which the RMT breaks down. However, this reference did not (1) introduce the new NNN ratio \( r' \) (2) do the \textit{odd} \( N \) case (3) study the type II cases (4) address the possible deep relations between the RG at \( N = \infty \), \( 1/N \) expansion at a finite \( N \) and the RMT at a finite \( N \). As shown in Sec.III-A,Sec.IV-A, the odd number of sites are in different classes in both classifications and ED\(^{23} \). In the GSE case in Fig.3-5, which has the double degeneracy at the \( q = 4 \) side, the new NNN ratio \( r' \) must be used to describe the stability regime of the quantum chaos in the GSE side and also describe the whole crossover from GSE to the GUE, then the CIT from GUE to the KAM side with the Poisson distribution. The new NNN ratio must be used to describe the KAM theorem in the integrable side when there is a double degeneracy in the integrable side as are the cases in Fig.6b, Fig.7b,c and Fig.8a in the type-II hybrid SYK models.

VII. CONCLUSIONS AND PERSPECTIVES

RG is a semi-group which may not have an inverse. In the coarse graining process, some information gets lost, the self similar phenomena start to emerge only in the low energy limit. What kind of information is lost ? We believe what is lost is the RMT information. It is worth to point out that in relativistic quantum field theories, the low energy and high energy levels are closely related due to the Lorentz invariance. So the RG can be equivalently performed by removing the IR or UV divergencies. This procedure is well established by the dimensional regularization in relativistic quantum field theories. It has also been applied to non-relativistic quantum field theories to describe the superfluid to Mott transition\(^{22} \) with the long range Coulomb interaction and also the quantum Hall plateau-plateau transition in a periodic potential with the long range Coulomb interaction\(^{30,31} \). We expect the bulk energy levels of these systems are also described by the RMT. Similarly, the QCD is described by the asymptotical freedoms in the high energy ( or short distance ) limit, but its bulk energy levels are described by the RMT\(^{32,34} \).

Here we introduced a new universal ratio which is the ratio of the next nearest neighbour (NNN) energy level spacing to characterize the random matrix behaviours. It must be used when there is a double ( \( d = 2 \) ) degeneracy near the chaotic side or near the integrable side. This new universal ratio is particular useful when numerically characterizing the KAM regime in the integrable side and the stability of quantum chaos in the chaotic regime. In Sec.V, we only present some very preliminary results on the analytical scaling forms of the KAM and its dual form. Their complete and rigorous forms will be given in a separate publication. If there are higher order such as \( d = 3, 4 \) degeneracy near the chaotic side or near the integrable side, then one may need to introduce more universal ratios such as the NN, NNN, NNNN ratios or the whole series whose physical meanings remain to be explored. This is similar to Tensor category, upto higher order tensor category are needed to characterize the topological phases\(^{34} \).

In a recent work\(^{35} \), we studied quantum chaos in 2- or 4-colored SYK models and also CIT in 2- or 4-colored hybrid SYK models. These colored SYK models provide concrete examples of classifying quantum chaos in a sys-
tem with multiple conserved quantities which show richer RMT classes than the conventional SYK models. The NN ratio should also find its applications in numerically studying KAM and its dual form in the 2- or 4-colored hybrid SYK models. It may also be interesting to study the ELS of the two indices SYK model with the two large numbers: \( N \) (the number of sites) and the \( M \) (the group of \( O(M) \) or \( SU(M) \)). Depending on the relations between \( N \) and \( M \), it may show either chaotic QSL or symmetry-broken QSG ground state.

As shown in, the stability of quantum chaos of black holes against a constant \( q = 2 \) SYK terms may be used to explore the interior behind the black hole horizon. Implications of the results achieved here on the bulk gravity side or on quantum error corrections to AdS/CFT need to be explored.

Fig.9b1 and Fig.9b2 show that the \( q = 4 \) chaotic SYK CFT fixed point and the \( (q = 2)^2 \) integral CFT flow towards each other with \( \pm \) sign respectively. In 2d CFT, Zamolodchikov’s \( \c \)-theorem states that RG flows only from a CFT with central charge \( c_1 \) to another one with \( c_2 < c_1 \). One can construct a \( c \)-function which monotonically decrease from \( c_1 \) to \( c_2 \). The 2d \( \c \)-theorem can also be extended to higher dimensions through F-theorem in odd dimension or a-theorem in even dimension. In the 2d boundary CFT (BCFT), there is a also a \( g \)-theorem which states that there is only one way RG flow from a BCFT with the boundary degeneracy \( g_1 \) to another one with \( g_2 < g_1 \). However, there is no such a \( c \)-theorem in a 1d CFT. This maybe special to the 1d c-theorem in a 1d CFT. This may also be related to the distinction of \( N \) AdS\( 2/NCFT \) from its higher dimension counterparts. It remains important to explore further the possible deep mechanism on the two ways RG flow between the FL and NFL fixed points in 1d CFT.

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![FIG. 10: ELS of the hybrid bosonic SYK.](image)

(a) For \( N_b \) even, at the half filling \( q_0 = 0 \), it is always in GOE for any ratio of \( K/J \). (b) Away from the half filling \( q_0 \neq 0 \), it is always in GUE for any ratio of \( K/J \), regardless of \( N_b \) is even or odd. In contrast to the fermion cases presented in the main text, there is no CIT in the bosonic hybrid SYK. However, it is not known if there is a QPT between the bosonic QSG\(_2\) and QSG\(_4\) in the ground state.

Appendix A: The hybrid of \( q = 2 \) and \( q = 4 \) bosonic SYK

This appendix was cited in Sec.VI-A. The procedures for fermions presented in the main text can also be applied to study the \( q = 2 \) and \( q = 4 \) hybrid bosonic SYK:

\[
\mathcal{H}_B = \sum_{i<j<k<l}^N J_{ijkl} b_i^\dagger b_j^\dagger b_k b_l + \sum_{i<j}^N K_{ij} b_i^\dagger b_j - \mu q_b \quad (A1)
\]

where, in general, \( J_{ijkl} = J_{jkl,i} = J_{ijkl} = J_{ikj,l} = J_{kl,i}j \) and \( \langle J_{ijkl} \rangle^2 = 3 J^2 / N^3 \). \( q_b = \sum_{i,j} (b_i^\dagger b_i - 1/2) \) is just the boson analog of Eq. Following, we take the four site indices \( i, j ; k, l \) are all different to keep the PH symmetry explicit at \( \mu = 0 \). \( K_{ij}^* = K_{ij}^* \) is a Hermitian matrix satisfying \( \langle K_{ij} \rangle = 0 \), \( \langle K_{ij}^2 \rangle = K^2 / N \).

In the \( K/J \leq 0 \) limit, the \( q = 4 \) bosonic SYK was studied by the ED in, a Quantum spin glass (QSG) ground state was expected in the thermodynamic limit \( N \rightarrow \infty \). One can define the particle-hole symmetry operator to be \( P = K \prod_{i=1}^N (b_i^\dagger + b_i) \). The boson charge operator is \( Q_b = \prod_{i=1}^N (b_i^\dagger + b_i) \). It is easy to show \( P^2 = 1, P b_i P = b_i^\dagger, P b_i^\dagger P = b_i, Pb_i P = N - Q_b \). \( [P, H_{M14}] = 0 \). For \( N \) even, at half filling \( q_b = 0 \), it is in GOE. However, as long as \( q_b = 0 \), there is no PH symmetry anymore, it is in GUE regardless of the value of \( N \) is even or odd.

Now we apply the PH transformation to the bosonic hybrid SYK model Eq. In contrast to the fermionic hybrid SYK models, \( [P, H_2] = 0 \), so the PH symmetry is preserved in the hybrid bosonic SYK model. In the \( J/K = 0 \) limit, we expect that the ELS for \( q = 2 \) bosonic SYK is the same as \( q = 4 \) bosonic SYK: when \( q_b = 0 \), it is in GOE, when \( q_b \neq 0 \), it is in GUE. This is in sharp contrast to the \( q = 2 \) fermionic SYK which is non-interacting, so integrable. While the \( q = 2 \) bosonic SYK is interacting (the bosons on the same site behaves as fermions, but different sites as bosons), so non-integrable and is already a quantum chaotic system.
So we expect the ELS of the hybrid bosonic SYK stays the same from $q = 4$, all the way down to $q = 2$. This is indeed confirmed by our ED results shown in Fig. 10 there are only two cases here (a) For $N_b$ even, at the half filling $q_b = 0$, it is always in GOE for any ratio of $K/\theta$. (b) Away from the half filling $q_b \neq 0$, it is always in GUE for any ratio of $K/\theta$, regardless of $N_b$ is even or odd.

In short, in contrast to all the hybrid fermionic models discussed in the main text, the KAM theorem does not apply in the bosonic hybrid model where there is no CIT. Here we only focused on the ELS of bulk spectrum. It supports the claim made in Sec.VI-A on the relation between the edge versus bulk energy levels and belong to the class A in the BSCFS.

Appendix B: The many-body density of states of $q = 2$ Majorana and $q = 2$ complex SYK models

In this appendix, we provide the many body energy distributions of the $q = 2$ Majorana and $q = 2$ complex SYK models which are needed to derive the scaling forms of the KAM theorems for the corresponding hybrid SYK models in Sec. V. Surprisingly, there is no previous works to discuss the many body energy level distributions of the $q = 2$ Majorana or complex SYK models.

![Graph of N=36 Majorana SYK and N=24 complex SYK many-body DOS](image)

FIG. 11: The Many-body density of states of $q = 2$ Majorana and complex SYK models: (a) Majorana fermion case at a given parity (b) complex fermion case at half-filling. Both many body DOS are close to be a Gaussian distribution with a small deviation in the center. However, it is difficult to resolve the precise nature of the band edge numerically.

1. $q = 2$ Majorana SYK model.

The $q = 2$ Majorana SYK model in Eq.11 is defined as $H_{\chi} = \sum_{1 \leq i < j \leq N} K_{ij} \chi_i \chi_j$, where $K_{ij}$ is real random number and drawn from the Gaussian distribution with zero mean and $K^2/N$ variance. So its single particle energy levels fit rigorously the RMT description with the matrix size $L = N$.

One can calculate its all many-body energy levels $\{E_{\chi}\}$ at a given parity sector by diagonalizing a $2^N/2 \times 2^N/2$ sparse matrix (assuming $N$ is even). Obviously, $\langle E_{\chi}\rangle = 0$. The data $\langle E_{\chi}^2\rangle$ is listed in Table VII. We find the many-body ELS of Majorana SYK model is a Poisson and the many-body energy density of states (DOS) $\rho(E_{\chi})$ satisfies a Gaussian distribution with zero mean and variance $\sigma^2 = 0.127 N - 0.157 \approx N/8$. This value matches the analytic estimate of the second moment $T_i H^k/T_i = (\frac{q^2}{2})^k/2^{2k}$. Putting $k = 2, q = 2$ and $\langle K^2 \rangle = K^2/N$, we find it is $K^2 N/8$. The fitted Gaussian distribution and $\rho(E_{\chi})$ are shown in Fig.11.

2. $q = 2$ Complex SYK model.

The $q = 2$ complex SYK model in Eq.14 is defined as $H_c = \sum_{1 \leq i < j \leq N} K_{ij} c_i^* c_j$, where $K_{ij} = K_{ji}^{*}$ is random complex number drawn from the Gaussian distribution with $\langle K_{ij}\rangle = 0$ and $\langle |K_{ij}|^2\rangle = K^2/N$. The fermion number $Q_c = \sum_i c_i^* c_i = q$ is conserved. So its single particle energy levels fit rigorously the RMT description with the matrix size $L = N$.

One can calculate its all many-body energy levels $\{E_c\}$ in the half-filling sector ($q = N/2$ for $N$ even) by diagonalizing a $C_N^{N/2} \times C_N^{N/2}$ sparse matrix (assuming $N$ is even). Obviously, $\langle E_c\rangle = 0$. The data of $\langle E_c^2\rangle$ is listed in Table VIII. We find the many-body ELS is a Poisson and many-body energy DOS $\rho(E_c)$ satisfies a Gaussian distribution with zero mean and $\sigma^2 = 0.250 N + 0.002 \approx N/4$ variance. The fitted Gaussian distribution and $\rho(E_c)$ are shown in Fig.11.

Appendix C: The many-body density of states of $q = 4$ complex SYK models

The many body DOS for the $q = 4$ Majorana SYK was studied by $1/N$ in[20] and ED in[22]. It was found that globally it is like Gaussian with the variance $\sqrt{2J^2 N/64}$, locally look like the semi-circle DOS with $\sqrt{E - E_0}$ behaviour near the band edge (either low or high). The many body DOS for the $q = 4$ complex SYK is shown in Fig.12.

At fixed $q = 4$, as $N$ gets large, the many body DOS of Majorana or complex SYK globally approaches the
Gaussian with a width $\sigma \sim \sqrt{N}$. However, near the band edge $E_0 \sim N$, it shows the $\sqrt{E - E_0}$ behaviour. So near the band edge, it locally behaves as a semi-circle. The $1/N$ expansion at a fixed $q$ at the temperatures $1 < \beta J < N$ can only resolve the local $\sqrt{E - E_0}$ behaviour near the band edge, but not the global behaviour. In the double scaling limit $N \to \infty$ and $q \to \infty$, but keep $\lambda = q^2/N$ fixed, the action can be mapped to a 2d Liouville CFT in the kinetic space with the central charge $c \sim N/q^2 = 1/\lambda$ which is solvable at all energy scales. The DOS was shown to be a Gaussian when $c \gg 1$, a semi-circle when $c \ll 1$, a more complicated form when $c \sim 1$ (see also the footnote [46] in Ref.54). However, only in the triple scaling limit $\lambda \to 0$ (namely, the central charge $c \to \infty$) and the low energy limit $E - E_0 \to 0$, at a fixed $(E - E_0)/J\lambda = z$, it reduces to the 1d Schwarzian which is the low energy limit of the SYK model.

Appendix D: The edge (low and high energy) states versus the bulk (intermediate energy) states in $q = 2$ and $q = 4$ complex SYK

In this appendix, we aim to demonstrate the connections and differences between the edge and bulk states in the Fock space. They can be studied by studied by $1/N$ expansion or RG and RMT respectively. This appendix is heavily cited in Sec.VI.

1. $q = 2$ Complex SYK model.

The $q = 2$ complex SYK was written in Eq.15

$$H_2 = \sum_{i<j}^N K_{ij}c_i^\dagger c_j - \mu Q_c$$

(D1)

Instead of using the grand-canonical ensemble, we choose canonical one with a fixed fermion number $Q_c = \sum_{i=1}^N c_i^\dagger c_i$. Then, the total Hilbert space can be decomposed into $N$ blocks as $2^N = \sum_{Q=0}^N C_N^Q$. For $Q_c = 1$, it is nothing but the single-particle sector with the dimension $C_N^1 = N$. Under the PH transformation $P$, it is mapped to its particle-hole conjugate sector $C_N^{N-1} = N$ which is nothing but the single-hole sector. Surprisingly and interestingly, only the single-particle and the single-hole sector are chaotic in the GUE, all the other multi-particle sectors $q \geq 2$ are integrable in the Possionian (Fig.13).

As shown in Sec.V, the low energy spacing $\sim 1/N$ which stands for the low energy quasi-particle excitations, due to the exact mirror symmetry, the high energy spacing is also $\sim 1/N$ which stands for the high energy quasi-particle excitations. The bulk energy spacing is much smaller $\sim 1/C_N^{N/2} \sim \sqrt{N}2^{-N}$ at a large $N$. The KAM theorem is determined by the bulk energy spacing. All the bulk energy levels are described by the RMT in the Possion statistics. The bulk energy level statistics is quite in-sensitive to the edge energy levels. Namely, incorporating or throwing away these edge (low or high energy) levels will not affect the bulk Possion statistics.

2. $q = 4$ Complex SYK model.

The $q = 4$ complex SYK was also written in Eq.15

$$H_4 = \sum_{i<j,k<l}^N J_{ij,k,l} c_{i,j}^\dagger c_{k,l}^\dagger c_l c_j - \mu Q_c$$

(D2)

As shown in Sec.VI, $[P, H_2] = 0$, so in contrast to $H_2$, $H_4$ does not have a mirror symmetry for any given random realization of $J_{ij,k,l}$. However, it still have an approximate mirror symmetry for any given realization.
of $J$ due to the following argument similar to Eq.\[18\]

$$- H_4 = - \sum_{i<j,k<l}^N J_{ij;kl} c_i^\dagger c_j c_k c_l = \sum_{i<j,k<l}^N J'_{ij;kl} c_i^\dagger c_j c_k c_l$$

where $J'_{ij;kl} = -J_{ij;kl}$. Note that we are still confining to the $Q_c = \sum_{i=1}^N c_i^\dagger c_i$ sector. Obviously, $J'$ and $J$ satisfy the same distribution. At a large enough $N_c$, it is self-averaging, so there is still an approximate mirror symmetry at any filling for a given random realization of $J_{ij;kl}$.

This is indeed the case shown in Fig.\[14\] As shown in Sec.VI, the low energy spacing $\sim e^{-s_0 N}$ due to the approximate mirror symmetry, the high energy spacing is also $\sim e^{-s_0 N}$. The bulk energy spacing is much smaller $\sim 1/C_N^{1/2} \sim \sqrt{N}e^{-s_0 N}$. The KAM theorem is determined by the bulk energy spacing. All the bulk energy levels are described by the RMT in the GOE. The bulk energy level statistics is quite in-sensitive to the edge energy levels. Namely, incorporating or throwing away these edge (low or high energy) levels will not affect the bulk GOE statistics.

It remains interesting to examine how the edge and bulk states evolve from the $q = 2$ side in Fig.\[13\] to that in the $q = 4$ side in Fig.\[14\] in the Type-I and Type-II hybrid complex SYK models.

FIG. 14: The edge and bulk energy levels of $q = 4$ complex SYK and $N_c = 12$. The energy levels are labeled by conserved quantity, number of fermion $Q_c$. Only the $Q_c = 6$ sectors satisfy GOE, others are GUE. Note: For $Q_c = 1$, $N_c = 1$ stand for a single particle or hole, it is an exact zero modes with the degeneracy $N_c$. $Q_c = 0, 12$ are two trivial sectors. The many body DOS at the half-filling is given in Fig.\[12\]
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In fact, equivalently, one can also define the PH symmetry operator to be \( R = (P - 1)^{Q_2} = ( - 1)^{Nc} ( - 1)^{Q_2} P = K \prod_{i=1}^{Nc} (c_i^+ - c_i) \). It is easy to show \( R^2 = ( - 1)^{\sum_{i<j}^{Nc} s_{i,j}} R \) which justifies \( R \) as an anti-unitary PH transformation. Here \( R \) does not provide any new information than \( P \), so is redundant. But they become important and in-dependent anti-unitary operators in the \( N = 1 \) SUSY SYK for a unified and compact classification scheme on both SYK and \( N = 1 \) SUSY SYK, see 40.

Here we choose even site \( \chi_{2i} \) to be real, odd site \( \chi_{2i-1} \) to be pure imaginary. Then as shown in our recent work 40, by choosing this way, when \( N \) is odd, one must use the extended scheme (namely adding a Majorana fermion at \( \infty \) to enlarge the Hilbert space twice), also introduce one more conserved quantity: the parity. However, if one choose the other way around, namely, even site \( \chi_{2i} \) to be pure imaginary, odd site \( \chi_{2i-1} \) to be real, then when \( N \) is odd, one can use either the minimum scheme without adding a Majorana fermion at \( \infty \) or the extended scheme.

It seems that so far, Ref. 40 and this work are the only two papers which have discussed the classifications and presented EDs in the odd \( N \) case. But the two papers still use different anti-unitary operators to do the RMT classifications. For example, Ref. 40 constructed an anti-unitary Time-reversal operator in the many body Hilbert space, while we did not have such a kind of operator. It is still not known to the authors what are the relations between the two different sets of anti-unitary operators. For a unified minimum classification scheme, see Ref. 40.

In fact, here we discuss a canonical ensemble instead of a grand canonical ensemble. So it has a fixed \( N \) instead of a fixed \( \mu \).
we add a Majorana fermion at $N + 1 = \infty$. (3) using the Clifford algebra for both $N = 2k$ even and $N = 2k + 1$ odd case. We achieved the same ED results in all the three ways.

58 To do the ED, following Ref.\textsuperscript{2}, one puts the Qb hard core bosons on a chain with $N$ sites $i = 1, 2, \cdots, N$. The hard core boson case is simpler than its fermionic counterpart, because one do not need to attach a Jordan-Wigner string of $\sigma_z$, namely, $b_i = \sigma_i^z$, $b_i^\dagger = \sigma_i^+$ to map the hard core boson model to a quantum spin model.

59 It was known that the quantum ferromagnetic and anti-ferromagnetic Heisenberg model differs by only a sign. Despite they have dramatically different ground states and low energy excitations, using the same argument here, one can see they share the same ELS.

60 This choice is important when doing ED at a finite size $N$. However, choosing the four site indices all different or not makes no difference in the $1/N$ expansion.

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88 In the $U(1)$ Dicke model, at any finite $N$, the gapless Goldstone mode at $N = \infty$ was lifted to a pseudo-Goldstone one with a finite gap $\Delta \sim D/N$ where $D$ is the diffusion constant subject to an oscilliation due to the Berry phase. Then when adding a small CRW term $\beta = g'/g = \beta_{U(1)} \sim 1/N^2$ smaller than this gap opened by the finite size effect, then the pseudo-Goldstone mode remains robust. This is more similar to the $q = 2$ integrable side where the quasi-particle exists and has a gap $\sim 1/N$, but dramatically different than the $q = 4$ quantum chaotic side where the quasi-particle breaks down and has a much smaller gap $\sim e^{-c_0 N}$ and all the low energy levels are also strongly repellled.

89 The edge states here mean the many body energy levels near the ground state, so they are ground state plus low energy excitations. The edge exponents in the RMT for the 7 classes with a mirror symmetry mean the asymptotic behaviour of the lowest eigenvalue $\lambda_1$ near the origin $E = 0$ which, in the present case, is one of the bulk states. In some literatures, the former is called soft edge, the latter is hard edge.

90 There should be an exact proof on this Gaussian distribution in the many body DOS, see\textsuperscript{3}.

91 The two ways flow between the two CFT\textsubscript{1} fixed points in Fig.9b1 and 9b2 violates the Zamolodchikov’s c-theorem
on RG flow between two $\text{CFT}_2$ fixed points. See the conclusion section for more discussions.