EXPONENTIAL SUMS ON $\mathbb{A}^n$

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Abstract. We discuss exponential sums on affine space from the point of view of Dwork’s $p$-adic cohomology theory.

1. Introduction

Let $p$ be a prime number, $q = p^a$, $\mathbb{F}_q$ the finite field of $q$ elements. Associated to a polynomial $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ and a nontrivial additive character $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ are exponential sums

$$S(\mathbb{A}^n(\mathbb{F}_q), f) = \sum_{x_1, \ldots, x_n \in \mathbb{F}_q} \Psi(\text{Trace}_{\mathbb{F}_q/F_q} f(x_1, \ldots, x_n)) \quad (1.1)$$

and an $L$-function

$$L(\mathbb{A}^n, f; t) = \exp\left(\sum_{i=1}^{\infty} S(\mathbb{A}^n(\mathbb{F}_q), f)\frac{t^i}{i}\right). \quad (1.2)$$

Let $d = \text{degree of } f$ and write

$$f = f^{(d)} + f^{(d-1)} + \cdots + f^{(0)},$$

where $f^{(j)}$ is homogeneous of degree $j$. A by now classical theorem of Deligne\cite{2, Théorème 8.4] says that if $(p, d) = 1$ and $f^{(d)} = 0$ defines a smooth hypersurface in $\mathbb{P}^{n-1}$, then $L(\mathbb{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$, all of whose reciprocal roots have absolute value equal to $q^{n/2}$. This implies the estimate

$$|S(\mathbb{A}^n(\mathbb{F}_q), f)| \leq (d-1)^n q^{n^2/2}. \quad (1.3)$$

In this article, we give a $p$-adic proof of the fact that $L(\mathbb{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$ (equation (2.14) and Theorem 3.8) and give $p$-adic estimates for its reciprocal roots, namely, we find a lower bound for the $p$-adic Newton polygon of $L(\mathbb{A}^n, f; t)^{(-1)^{n+1}}$ (Theorem 4.3). Using general results of Deligne\cite{3}, this information can be used to compute $l$-adic cohomology and hence again obtain the archimedian estimate (1.3) (Theorem 5.3).

For Theorems 3.8 and 4.3, we need to assume only that $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbb{F}_q[x_1, \ldots, x_n]$ (or, equivalently, that $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ have no common zero in $\mathbb{P}^{n-1}$). When $(p, d) = 1$, this is equivalent to Deligne’s hypothesis. When $d$ is divisible by $p$, there are only a few cases satisfying this regular sequence condition. We check them by hand in section 6 to prove the following slight generalization of Deligne’s result.

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Theorem 1.4. Suppose \( \{ \partial f^{(d)}/\partial x_i \}_{i=1}^n \) form a regular sequence in \( F_q[x_1, \ldots, x_n] \). Then \( L(A^n, f; t)^{(-1)^{n+1}} \) is a polynomial of degree \( (d-1)^n \), all of whose reciprocal roots have absolute value equal to \( q^{n/2} \).

In the article [1], we dealt with exponential sums on tori. After a general coordinate change, one can, by using the standard toric decomposition of \( A^n \), deduce most of the results of this article from results in [1]. Our main purpose here is to develop some new methods that will be more widely applicable. For instance, recent results of García [6] on exponential sums on \( A^n \) do not seem to follow from [1].

In contrast with [1], we work systematically with spaces of type \( C(b) \) (convergent series on a closed disk) and avoid spaces of type \( L(b) \) (bounded series on an open disk). This ties together more closely the calculation of \( p \)-adic cohomology and the estimation of the Newton polygon of the characteristic polynomial of Frobenius, eliminating much of section 3 of [1].

Another new feature of this work is the use of the spectral sequence associated to the filtration by \( p \)-divisibility on the complex \( \Omega \cdot C(b) \) (section 3 below). Although the behavior of this spectral sequence is rather simple in the setting of this article (namely, \( E^{r,s}_1 = E^{r,s}_\infty \) for all \( r \) and \( s \)), we believe it will play a significant role in more general situations, such as that of García [6]. We hope the methods developed here will allow us to extend the results of this article to those situations.

2. Preliminaries

In this section, we review the results from Dwork’s \( p \)-adic cohomology theory that will be used in this paper.

Let \( Q_p \) be the field of \( p \)-adic numbers, \( \zeta_p \) a primitive \( p \)-th root of unity, and \( \Omega_1 = Q_p(\zeta_p) \). The field \( \Omega_1 \) is a totally ramified extension of \( Q_p \) of degree \( p - 1 \). Let \( K \) be the unramified extension of \( Q_p \) of degree \( a \). Set \( \Omega_0 = K(\zeta_p) \). The Frobenius automorphism \( x \mapsto x^p \) of \( \text{Gal}(F_q/F_p) \) lifts to a generator \( \tau \) of \( \text{Gal}(\Omega_0/\Omega_1) ( \simeq \text{Gal}(K/Q_p) \) by requiring \( \tau(\zeta_p) = \zeta_p \). Let \( \Omega \) be the completion of an algebraic closure of \( \Omega_0 \). Denote by “ord” the additive valuation on \( \Omega \) normalized by \( \text{ord}_p = 1 \) and by “ord_q” the additive valuation normalized by \( \text{ord}_q = 1 \).

Let \( E(t) \) be the Artin-Hasse exponential series:

\[
E(t) = \exp \left( \sum_{i=0}^\infty \frac{t^p^i}{p^i} \right).
\]

Let \( \gamma \in \Omega_1 \) be a solution of \( \sum_{i=0}^\infty t^p^i / p^i = 0 \) satisfying \( \text{ord} \gamma = 1/(p - 1) \) and consider

\[
\theta(t) = E(\gamma t) = \sum_{i=0}^\infty \lambda_i t^i \in \Omega_1[[t]].
\]

The series \( \theta(t) \) is a splitting function in Dwork’s terminology[3]. Furthermore, its coefficients satisfy

\[
\text{ord} \lambda_i \geq i/(p - 1).
\]

We consider the following spaces of \( p \)-adic functions. Let \( b \) be a positive rational number and choose a positive integer \( M \) such that \( Mb/p \) and \( Md/(p - 1) \) are integers. Let \( \pi \) be such that

\[
\pi^{Md} = p
\]
and put \( \tilde{\Omega}_1 = \Omega_1(\pi), \tilde{\Omega}_0 = \Omega_0(\pi) \). The element \( \pi \) is a uniformizing parameter for the rings of integers of \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_0 \). We extend \( \tau \in \text{Gal}(\Omega_0/\Omega_1) \) to a generator of \( \text{Gal}(\tilde{\Omega}_0/\tilde{\Omega}_1) \) by requiring \( \tau(\pi) = \pi \). For \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \), we put \( |u| = u_1 + \cdots + u_n \). Define

\[
C(b) = \left\{ \sum_{u \in \mathbb{N}^n} A_u \pi^{MB[u]} x^u \mid A_u \in \tilde{\Omega}_0 \text{ and } A_u \to 0 \text{ as } u \to \infty \right\}.
\]

For \( \xi = \sum_{u \in \mathbb{N}^n} A_u \pi^{MB[u]} x^u \in C(b) \), define

\[
\text{ord} \xi = \min_{u \in \mathbb{N}^n} \{ \text{ord} A_u \}.
\]

Given \( c \in \mathbb{R} \), we put

\[
C(b, c) = \{ \xi \in C(b) \mid \text{ord} \xi \geq c \}.
\]

Let \( \hat{f} = \sum_u \hat{a}_u x^u \in K[x_1, \ldots, x_n] \) be the Teichmüller lifting of the polynomial \( f \in F_0[x_1, \ldots, x_n] \), i.e., \( (\hat{a}_u)^q = \hat{a}_u \) and the reduction of \( \hat{f} \) modulo \( p \) is \( f \). Set

\[
F(x) = \prod_u \theta(\hat{a}_u x^u),
\]

\[
F_0(x) = \prod_{i=0}^{a-1} \theta((\hat{a}_u x^u)^p).
\]

The estimate (2.2) implies that \( F \in C(b, 0) \) for all \( b < 1/(p-1) \) and \( F_0 \in C(b, 0) \) for all \( b < p/q(p-1) \). Define an operator \( \psi \) on formal power series by

\[
\psi \left( \sum_{u \in \mathbb{N}^n} A_u x^u \right) = \sum_{u \in \mathbb{N}^n} A_{pu} x^u.
\]

It is clear that \( \psi(C(b, c)) \subseteq C(pb, c) \). For \( 0 < b < p/(p-1) \), let \( \alpha = \psi^a \circ F_0 \) be the composition

\[
C(b) \hookrightarrow C(b/q) \xrightarrow{F_0} C(b/q) \xrightarrow{\psi^a} C(b).
\]

Then \( \alpha \) is a completely continuous \( \Omega_0 \)-linear endomorphism of \( C(b) \). We shall also need to consider \( \beta = \tau^{-1} \circ \psi \circ F \), which is a completely continuous \( \tilde{\Omega}_1 \)-linear (or \( \tilde{\Omega}_0 \)-semilinear) endomorphism of \( C(b) \). Note that \( \alpha = \beta^a \).

Set \( \hat{f}_i = \partial \hat{f}/\partial x_i \) and let \( \gamma_l = \sum_{i=0}^l \gamma^{p^i}/p^i \). By the definition of \( \gamma \), we have

\[
\text{ord} \gamma_l \geq \frac{p^{l+1}}{p-1} - l - 1.
\]

For \( i = 1, \ldots, n \), define differential operators \( D_i \) by

\[
D_i = \frac{\partial}{\partial x_i} + H_i,
\]

where

\[
H_i = \sum_{l=0}^{\infty} \gamma_l p^l x_i^{p^l-1} \hat{f}_i^{p^l} (x^p) \in C \left( b, \frac{1}{p-1} - \frac{d-1}{d} \right)
\]

for \( b < p/(p-1) \). Thus \( D_i \) and “multiplication by \( H_i \)” operate on \( C(b) \) for \( b < p/(p-1) \).
To understand the definition of the $D_i$, put

$$\hat{\theta}(t) = \prod_{i=0}^{\infty} \theta(t^i)$$

$$\hat{F}(x) = \prod_{u} \hat{\theta}(\hat{a}_u x^u),$$

so that

$$F(x) = \hat{F}(x)/\hat{F}(x^p),$$

$$F_0(x) = \hat{F}(x)/\hat{F}(x^q).$$

Then formally

$$\alpha = \hat{F}(x)^{-1} \circ \psi^{\alpha} \circ \hat{F}(x)$$

$$\beta = \hat{F}(x)^{-1} \circ \tau^{-1} \circ \psi \circ \hat{F}(x).$$

It is trivial to check that $x_i \partial/\partial x_i$ and $\psi$ commute up to a factor of $p$, hence the differential operators

$$\hat{F}^{-1} \circ x_i \frac{\partial}{\partial x_i} \circ \hat{F} = x_i \frac{\partial \hat{F}}{\partial x_i} / \hat{F}$$

formally commute with $\alpha$ (up to a factor of $q$) and $\beta$ (up to a factor of $p$). From the definitions, one gets

$$\hat{\theta}(t) = \exp \left( \sum_{l=0}^{\infty} \gamma_l t^l \right).$$

It then follows that

$$x_i \partial \hat{F}/\partial x_i / \hat{F} = x_i H_i,$$

which gives

$$x_i \partial \hat{F}/\partial x_i / \hat{F} = x_i H_i,$$

(2.11)

$$\alpha \circ x_i D_i = qx_i D_i \circ \alpha,$$

(2.12)

$$\beta \circ x_i D_i = px_i D_i \circ \beta.$$

Consider the de Rham-type complex $(\Omega_{C(b)}^k, D)$, where

$$\Omega_{C(b)}^k = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} C(b) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and $D : \Omega_{C(b)}^k \to \Omega_{C(b)}^{k+1}$ is defined by

$$D(\xi \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \left( \sum_{i=1}^{n} D_i(\xi) \, dx_i \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$ 

We extend the mapping $\alpha$ to a mapping $\alpha : \Omega_{C(b)}^k \to \Omega_{C(b)}^k$ defined by linearity and the formula

$$\alpha_k(\xi \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = q^{n-k} \frac{1}{x_{i_1} \cdots x_{i_k}} \alpha(x_{i_1} \cdots x_{i_k} \xi) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$
Equation (2.11) implies that $\alpha$ is a map of complexes. The Dwork trace formula, as formulated by Robba, then gives

$$L(A^n/F_q, f; t) = \prod_{k=0}^{n} \det(I - t\alpha_k | \Omega^k_{C(b)})^{(-1)^{k+1}}.$$  

From results of Serre we then get

$$L(A^n/F_q, f; t) = \prod_{k=0}^{n} \det(I - t\alpha_k | H^k(\Omega^k_{C(b)}, D))^{(-1)^{k+1}},$$

where we denote the induced map on cohomology by $\alpha_k$ also.

3. Filtration by $p$-divisibility

The $p$-adic Banach space $C(b)$ has a decreasing filtration $\{F^r C(b)\}_{r=-\infty}^{\infty}$ defined by setting

$$F^r C(b) = \{ \sum_{u \in \mathbb{N}^n} A_u \pi^{|Mu|} x^u \in C(b) | A_u \in \pi^r \mathcal{O}_{\tilde{\Omega}} \text{ for all } u \},$$

where $\mathcal{O}_{\tilde{\Omega}}$ denotes the ring of integers of $\tilde{\Omega}$. We extend this to a filtration on $\Omega^r_{C(b)}$ by defining

$$F^r \Omega^k_{C(b)} = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} F^r C(b) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$  

This filtration is exhaustive and separated, i.e.,

$$\bigcup_{r \in \mathbb{Z}} F^r \Omega^k_{C(b)} = \Omega^r_{C(b)} \quad \text{and} \quad \bigcap_{r \in \mathbb{Z}} F^r \Omega^k_{C(b)} = (0).$$

We normalize the $D_i$ so that they respect this filtration. Put

$$\epsilon = Mb(d-1) - Md/(p-1),$$

a nonnegative integer. Then

$$\pi^\epsilon D_i(F^r C(b)) \subseteq F^r C(b)$$

and the complexes $(\Omega^r_{C(b)}, D)$, $(\Omega^r_{C(b)}, \pi^\epsilon D)$ have the same cohomology.

Since $(\Omega^r_{C(b)}, \pi^\epsilon D)$ is a filtered complex, there is an associated spectral sequence. Its $E_1$-term is given by

$$E_{1}^{r,s} = H^{r+s}(F^r \Omega^s_{C(b)}/F^{r+1} \Omega^s_{C(b)}).$$

Consider the map $F^0 C(b) \to \mathbf{F}_q[x_1, \ldots, x_n]$ defined by

$$\sum_u A_u \pi^{|Mu|} x^u \mapsto \sum_u \tilde{A}_u x^u,$$

where $\tilde{A}_u$ denotes the reduction of $A_u$ modulo the maximal ideal of $\mathcal{O}_{\tilde{\Omega}_0}$. (Since $A_u \to 0$ as $u \to \infty$, the sum on the right-hand side is finite.) This map induces an isomorphism

$$F^0 \Omega^k_{C(b)}/F^1 \Omega^k_{C(b)} \simeq \Omega^k_{\mathbf{F}_q[x_1, \ldots, x_n]}/\mathbf{F}_q.$$  

In particular,

$$F^0 C(b)/F^1 C(b) \simeq \mathbf{F}_q[x_1, \ldots, x_n].$$
We have clearly
\[ \frac{\partial}{\partial x_i}(F^r C(b)) \subseteq F^{r+1} C(b), \]
and a calculation show that
\[ \pi^r H_i \equiv \pi^{Mb(d-1)} \hat{f}_i \quad (\text{mod } F^1 C(b)) \]
\[ \equiv \pi^{Mb(d-1)} \hat{f}^{(d)}_i \quad (\text{mod } F^1 C(b)), \]
hence under the isomorphism (3.2), the map
\[ \pi^r D_i : F^0 C(b) \to F^0 C(b) \]
induces the map “multiplication by \( \partial f^{(d)}/\partial x_i \)” on \( F_q[x_1, \ldots, x_n] \). More generally, one sees that under the isomorphism (3.1), the map
\[ \pi^r D : F^0 \Omega_C(b) \to F^0 \Omega_C^{k+1}(b) \]
induces the map
\[ \phi_{f^{(d)}} : \Omega_F^n[x_1, \ldots, x_n]/F_q \to \Omega_F^{k+1}[x_1, \ldots, x_n]/F_q \]
defined by
\[ \phi_{f^{(d)}}(\omega) = df^{(d)} \wedge \omega, \]
where \( df^{(d)} \) denotes the exterior derivative of \( f^{(d)} \). We have proved that there is an isomorphism of complexes of \( F_q \)-vector spaces
\[ (F^0 \Omega_C(b)/F^1 \Omega_C(b), \pi^r D) \simeq (\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}). \]

Since multiplication by \( \pi^r \) defines an isomorphism of complexes
\[ (F^0 \Omega_C(b), \pi^r D) \simeq (F^r \Omega_C(b), \pi^r D), \]
we have in fact isomorphisms for all \( r \in \mathbb{Z} \)
\[ (F^r \Omega_C(b)/F^{r+1} \Omega_C(b), \pi^r D) \simeq (\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}). \]

The complex \( (\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}) \) is isomorphic to the Koszul complex on \( F_q[x_1, \ldots, x_n] \) defined by \( \{ \partial f^{(d)}/\partial x_i \}_{i=1}^n \). If we assume \( \{ \partial f^{(d)}/\partial x_i \}_{i=1}^n \) form a regular sequence in \( F_q[x_1, \ldots, x_n] \), we get
\[ H^i(\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}) = 0 \quad \text{for } i \neq n, \]
\[ \dim_{F_q} H^n(\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}) = (d-1)^n. \]

It follows from these equations that
\[ E_1^{r,s} = 0 \quad \text{if } r + s \neq n \]
\[ \dim_{F_q} E_1^{r,s} = (d-1)^n \quad \text{if } r + s = n. \]

The first of these equalities implies that all the coboundary maps \( d_1^{r,s} \) are zero, hence the spectral sequence converges weakly, i. e.,
\[ E_1^{r,s} \simeq F^r H^{r+s}(\Omega_C(b), \pi^r D)/F^{r+1} H^{r+s}(\Omega_C(b), \pi^r D). \]

This spectral sequence actually converges. First observe the following. Let \( x^i, i = 1, \ldots, (d-1)^n, \) be monomials in \( x_1, \ldots, x_n \) such that the cohomology classes \( \{ [x^i \ dx_1 \wedge \cdots \wedge dx_n] \}_{i=1}^{(d-1)^n} \) form a basis for \( H^n(\Omega_F^n[x_1, \ldots, x_n]/F_q, \phi_{f^{(d)}}) \). Then the
images of the cohomology classes \(\{[\pi^r x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n]\}_{i=1}^{(d-1)n}\) in \(E^{r,s}_1\) form a basis for \(E^{r,s}_1\) when \(r + s = n\).

**Theorem 3.8.** Suppose \(\{\partial f^{(d)}/\partial x_i\}_{i=1}^n\) form a regular sequence in \(F_q[x_1, \ldots, x_n]\). Then

1. \(H^1(\Omega^i_{C(b)}, \pi^s D) = 0\) if \(i \neq n\),
2. the cohomology classes \([x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n], i = 1, \ldots, (d - 1)n\), form a basis for \(H^n(\Omega^i_{C(b)}, \pi^s D)\).

**Proof.** Suppose \(i \neq n\) and let \(\eta \in \Omega^i_{C(b)}\) with \(\pi^s D(\eta) = 0\). For some \(r\) we have \(\eta \in F^r \Omega^i_{C(b)}\). Equations (3.3) and (3.4) then imply that

\[\eta = \pi \eta_1 + \pi^s D(\zeta_1)\]

with \(\eta_1 \in F^r \Omega^i_{C(b)}\) and \(\zeta_1 \in F^r \Omega^{i-1}_{C(b)}\). Suppose that for some \(t \geq 1\) we have found \(\eta_t \in F^r \Omega^i_{C(b)}\) and \(\zeta_t \in F^r \Omega^{i-1}_{C(b)}\) such that

\[\eta = \pi^t \eta_t + \pi^s D(\zeta_t)\]

and such that

\[\zeta_t - \zeta_{t-1} \in F^{r+t-1} \Omega^{i-1}_{C(b)}\].

Applying \(\pi^s D\) to both sides of (3.9) gives

\[\pi^{t+1} D(\eta_t) = 0\],

hence \(\pi^s D(\eta_t) = 0\) since multiplication by \(\pi\) is injective on \(\Omega^i_{C(b)}\). Equations (3.3) and (3.4) give

\[\eta_t = \pi \eta_{t+1} + \pi^s D(\zeta_{t+1})\],

with \(\eta_{t+1} \in F^r \Omega^{i}_{C(b)}\) and \(\zeta_{t+1} \in F^r \Omega^{i-1}_{C(b)}\). If we put \(\zeta_{t+1} = \zeta_t + \pi^s \zeta_{t+1}\), then substitution into (3.9) gives

\[\eta = \pi^{t+1} \eta_{t+1} + \pi^s D(\zeta_{t+1})\]

with

\[\zeta_{t+1} - \zeta_t \in F^{r+t} \Omega^{i-1}_{C(b)}\].

It is now clear that the sequence \(\{\zeta_t\}_{t=1}^{\infty}\) converges to an element \(\zeta \in F^r \Omega^{i-1}_{C(b)}\) such that \(\eta = \pi^s D(\zeta)\). This proves the first assertion.

It follows easily from (3.3) that the \(\{[x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n]\}_{i=1}^{(d-1)n}\) are linearly independent, hence it suffices to show that they span \(H^n(\Omega^i_{C(b)}, \pi^s D)\). Let \(\eta \in F^r \Omega^n_{C(b)}\). From (3.3) we have

\[\eta = \sum_{i=1}^{(d-1)n} c_i^{(1)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi \eta_1 + \pi^s D(\zeta_1),\]

where \(c_i^{(1)} \in \tilde{\Omega}_0\), \(c_i^{(1)} x^{\mu_i} \in F^r C(b), \eta_1 \in F^r \Omega^i_{C(b)}, \zeta_1 \in F^r \Omega^{i-1}_{C(b)}\). Suppose we can write

\[\eta = \sum_{i=1}^{(d-1)n} c_i^{(1)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^s \eta_t + \pi^s D(\zeta_t)\]
with $c_i^{(t)} \in \tilde{\Omega}_0$, $c_i^{(t)} x^{\mu_i} \in F^r C(b)$, $\eta_t \in F^r \Omega_{C(b)}^n$, and $\zeta_t \in F^r \Omega_{C(b)}^{n-1}$ such that

$$(c_i^{(t)} - c_i^{(t-1)}) x^{\mu_i} \in F^{r+1} C(b)$$

$$\zeta_t - \zeta_{t-1} \in F^{r+1} \Omega_{C(b)}^{n-1}.$$  

By (3.3) we have

$$\eta_t = \sum_{i=1}^{(d-1)^n} c_i^{(t)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi \eta_{t+1} + \pi^x D(\zeta_t'),$$

where $c_i' \in \tilde{\Omega}_0$, $c_i' x^{\mu_i} \in F^r C(b)$, $\eta_{t+1} \in F^r \Omega_{C(b)}^n$, $\zeta_t' \in F^r \Omega_{C(b)}^{n-1}$. If we put $c_i^{(t+1)} = c_i^{(t)} + \pi^x c_i'$ and $\zeta_{t+1} = \zeta_t + \pi^x \zeta_t'$, then

$$\eta = \sum_{i=1}^{(d-1)^n} c_i^{(t+1)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^{x+1} \eta_{t+1} + \pi^x D(\zeta_{t+1})$$

with

$$(c_i^{(t+1)} - c_i^{(t)}) x^{\mu_i} \in F^{r+1} C(b)$$

$$\zeta_{t+1} - \zeta_t \in F^{r+1} \Omega_{C(b)}^{n-1}.$$  

It follows that the sequences \( \{c_i^{(t)}\}_{t=1}^{\infty} \subseteq \tilde{\Omega}_0 \) and \( \{\zeta_t\}_{t=1}^{\infty} \) converge, say, \( c_i^{(t)} \to c_i \in \tilde{\Omega}_0, \zeta_t \to \zeta \in F^r \Omega_{C(b)}^{n-1} \), and that these limits satisfy

$$\eta = \sum_{i=1}^{(d-1)^n} c_i x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^x D(\zeta)$$

with $c_i x^{\mu_i} \in F^r C(b)$. This completes the proof of the second assertion.

The following result is a consequence of the proof of Theorem 3.8.

**Proposition 3.11.** Under the hypothesis of Theorem 3.8, if $\eta \in F^r \Omega_{C(b)}^n$, then there exist \( \{c_i^{(d-1)^n}\}_{i=1}^{\infty} \subseteq \tilde{\Omega}_0 \) such that in \( H^n(\Omega_{C(b)}, \pi^x D) \) we have

$$[\eta] = \sum_{i=1}^{(d-1)^n} [c_i x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n],$$

where $c_i x^{\mu_i} \in F^r C(b)$ for $i = 1, \ldots, (d-1)^n$.

4. p-adic estimates

It follows from (2.14) and Theorem 3.8 that

$$L(A^n, f; t)^{(-1)^{n+1}} = \text{det}(1 - t\alpha_n \mid H^n(\Omega_{C(b)}, D))$$

is a polynomial of degree $(d-1)^n$ (by \( \mathbb{F}_q \), zero is not an eigenvalue of $\alpha_n$). We estimate its $p$-adic Newton polygon. Note that

$$H^n(\Omega_{F_q[x_1, \ldots, x_n]/F_q, \phi_f(a)}) \simeq F_q[x_1, \ldots, x_n]/(\partial f^{(d)}/\partial x_1, \ldots, \partial f^{(d)}/\partial x_n)$$

is a graded $F_q[x_1, \ldots, x_n]$-module. Let $H^n(\Omega_{F_q[x_1, \ldots, x_n]/F_q, \phi_f(a)})^{(m)}$ denote its homogeneous component of degree $m$. It follows from (3.4) that its Hilbert-Poincare
series is \((1 + t + \cdots + t^{d-2})^n\). Write

\[(4.2) \quad (1 + t + \cdots + t^{d-2})^n = \sum_{m=0}^{n(d-2)} U_m t^m,\]

so that

\[U_m = \dim_{F_q} H^n(F_q[x_1, \ldots, x_n]/F_q[\phi_{f(a)}]^{(m)}).\]

Equivalently,

\[U_m = \text{card}\{x^{\mu} \mid |\mu_i| = m\}.\]

Let \(\beta_n \) be the endomorphism of \(H^n(\Omega_{C(b)}, D)\) constructed from \(\beta\) as \(\alpha_n\) was constructed from \(\alpha\), i.e.,

\[\beta_n(\xi dx_1 \wedge \cdots \wedge dx_n) = \frac{1}{x_1 \cdots x_n} \beta(x_1 \cdots x_n \xi) dx_1 \wedge \cdots \wedge dx_n.\]

Then \(\beta_n\) is an \(\tilde{\Omega}_1\)-linear endomorphism of \(H^n(\Omega_{C(b)}, D)\) and \(\alpha_n = (\beta_n)^a\).

**Theorem 4.3.** Suppose \(\{\partial f(d)/\partial x_i\}_{i=1}^n\) form a regular sequence in \(F_q[x_1, \ldots, x_n]\). Then the Newton polygon of \(L(A^n, f; t)(-1)^{n+1}\) with respect to the valuation “ord\(_q\)” lies on or above the Newton polygon with respect to the valuation “ord\(_q\)” of the polynomial

\[n(d-2) \prod_{m=0}^{n(d-2)} (1 - q^{(m+n)/d}) U_m.\]

We begin with a reduction step. Let \(\gamma_j\) be an \(\tilde{\Omega}_0\)-integral basis for \(\tilde{\Omega}_0\) over \(\tilde{\Omega}_1\). Then under the hypothesis of Theorem 3.8, the cohomology classes

\[[\gamma_j x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n], \quad i = 1, \ldots, (d-1)^n, \quad j = 1, \ldots, a,\]

form a basis for \(H^n(\Omega_{C(b)}, D)\) as \(\tilde{\Omega}_1\)-vector space. We estimate \(p\)-adically the entries of the matrix of \(\beta_n\) with respect to a certain normalization of this basis, namely, we set

\[\xi(i,j) = (\pi^{Mb/p})^{|\mu_i|+n} \gamma_j x^{\mu_i}\]

and use the cohomology classes \([\xi(i,j) dx_1 \wedge \cdots \wedge dx_n]\). This normalization is chosen so that

\[x_1 \cdots x_n \xi(i,j) \in C(b/p, 0),\]

and

\[\beta(x_1 \cdots x_n \xi(i,j)) \in C(b, 0)\]

hence

\[\beta(x_1 \cdots x_n \xi(i,j)) \in C(b, 0)\]

and

\[\frac{1}{x_1 \cdots x_n} \beta(x_1 \cdots x_n \xi(i,j)) \in \pi^{Mb_n} C(b, 0).\]
This says that
\[ \beta_n(\xi(i, j) \, dx_1 \wedge \cdots \wedge dx_n) \in F^{Mn} \Omega^n_{C(b)}, \]
hence by Proposition 3.11 and the properties of an integral basis we have
\[ [\beta_n(\xi(i, j) \, dx_1 \wedge \cdots \wedge dx_n)] = \sum_{i', j'} A(i', j'; i, j) [\gamma_{j'} x_{i'} dx_1 \wedge \cdots \wedge dx_n] \]
with
\[ A(i', j'; i, j) \in \pi^{Mb(\mu | i + n)} \Omega_{\bar{b}_0}. \]
This may be rewritten as
\[ [\beta_n(\xi(i, j) \, dx_1 \wedge \cdots \wedge dx_n)] = \sum_{i', j'} B(i', j'; i, j) [\xi(i', j') \, dx_1 \wedge \cdots \wedge dx_n] \]
with
\[ B(i', j'; i, j) \in \pi^{Mb(\mu | i + n)(1-1/p)} \Omega_{\bar{b}_0}. \]
i. e., the \((i', j')\)-row of the matrix \(B(i', j'; i, j)\) of \(\beta_n\) with respect to the basis \(\{[\xi(i, j) \, dx_1 \wedge \cdots \wedge dx_n]\}_{i,j}\) is divisible by
\[ \pi^{Mb(\mu | i + n)(1-1/p)}. \]
This implies that \(\det_{\bar{b}_1}(I - \beta_n | H^n(\Omega^n_{C(b)}, D))\) has Newton polygon (with respect to the valuation “ord”) lying on or above the Newton polygon (with respect to the valuation “ord”) of the polynomial
\[ n(d-2) \prod_{m=0}^{n} (1 - \pi^{Mb(m+n)(1-1/p)} t^a U_m). \]
But \(\det_{\bar{b}_1}(I - \beta_n | H^n(\Omega^n_{C(b)}, D))\) is independent of \(b\), so we may take the limit as \(b \to p/(p-1)\) to conclude that its Newton polygon lies on or above the Newton polygon of
\[ n(d-2) \prod_{m=0}^{n} (1 - p^{(m+n)/d} t^a U_m). \]
Theorem 4.3 now follows from Lemma 4.4.

Let \(\{\rho_i\}_{i=1}^{(d-1)n}\) be the reciprocal roots of \(L(A^n, f; t)^{(-1)n+1}\) and put
\[ \Lambda(f) = \prod_{i=1}^{(d-1)n} \rho_i \in \mathbb{Q}(\zeta_p). \]
Theorem 4.3 implies that
\[ \text{ord}_q \Lambda(f) \geq \frac{1}{d} \sum_{m=0}^{n(d-2)} (m + n)U_m. \]
But it follows from (4.2) evaluated at \(t = 1\) that
\[ \sum_{m=0}^{n(d-2)} U_m = (d-1)^n \]
and from the derivative of (4.2) evaluated at \(t = 1\) that
\[ \sum_{m=0}^{n(d-2)} mU_m = n(d-1)^n(d-2)/2. \]
We thus get the following.

**Corollary 4.5.** Under the hypothesis of Theorem 4.3,

\[ \text{ord}_q \Lambda(f) \geq \frac{n(d - 1)^n}{2}. \]

It can be proved directly by $p$-adic methods that equality holds in Corollary 4.5. We shall derive this equality in the next section by $l$-adic methods.

5. $l$-adic Cohomology

Let $l$ be a prime, $l \neq p$. There exists a lisse, rank-one, $l$-adic étale sheaf $\mathcal{L}_q(f)$ on $\mathbb{A}^n$ with the property that

(5.1) \[ L(\mathbb{A}^n, f; t) = L(\mathbb{A}^n, \mathcal{L}_q(f); t), \]

where the right-hand side is a Grothendieck $L$-function. By Grothendieck’s Lefschetz trace formula,

(5.2) \[ L(\mathbb{A}^n, f; t) = \prod_{i=0}^{2n} \det(I - tF | H^i_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f)))^{(-1)^{i+1}}, \]

where $H^i_c$ denotes $l$-adic cohomology with proper supports and $F$ is the Frobenius endomorphism. The $H^i_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f))$ are finite-dimensional vector spaces over a finite extension $K_i$ of $\mathbb{Q}_l$ containing the $p$-th roots of unity. We combine Theorem 3.8 and Corollary 4.5 with general results of Deligne\[2\] to prove the following theorem of Deligne\[2\] Théorème 8.4.

**Theorem 5.3.** Suppose $(p, d) = 1$ and $f^{(d)} = 0$ defines a smooth hypersurface in $\mathbb{P}^{n-1}$. Then

1. $H^i_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f)) = 0$ if $i \neq n$,
2. $\dim_{K_i} H^i_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f)) = (d - 1)^n$,
3. $H^i_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f))$ is pure of weight $n$.

**Proof:** We consider the theorem to be known for $n = 1$ and prove it for general $n \geq 2$ by induction. For $\lambda \in \mathbb{F}_q$, set

\[ f_\lambda(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \lambda) \in \mathbb{F}_q(\lambda)[x_1, \ldots, x_{n-1}]. \]

Since the generic hyperplane section of a smooth variety is smooth, we may assume, after a coordinate change if necessary, that the hyperplane $x_n = 0$ intersects the variety $f^{(d)} = 0$ transversally in $\mathbb{P}^{n-1}$. Thus $f^{(d)}(x_1, \ldots, x_{n-1}, 0) = 0$ defines a smooth hypersurface in $\mathbb{P}^{n-2}$. But

\[ f^{(d)}_\lambda = f^{(d)}(x_1, \ldots, x_{n-1}, 0), \]

so by the induction hypothesis the conclusions of the theorem are true for all $f_\lambda$.

Consider the morphism of $\mathbb{F}_q$-schemes $\sigma : \mathbb{A}^n \to \mathbb{A}^1$ which is projection onto the $n$-th coordinate. The Leray spectral sequence for the composition of $\sigma$ with the structural morphism $\mathbb{A}^1 \to \text{Spec}(\mathbb{F}_q)$ is

(5.4) \[ H^i_c(\mathbb{A}^1 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, R^j \sigma_! \mathcal{L}_q(f)) \Rightarrow H^{i+j}_c(\mathbb{A}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{L}_q(f)). \]

Proper base change implies that for $\lambda \in \mathbb{F}_q$ and $\lambda^\flat$ a geometric point over $\lambda$

(5.5) \[ (R^j \sigma_! \mathcal{L}_q(f))^{\flat} = H^j_c(\sigma^{-1}(\lambda)) \times_{\mathbb{F}_q(\lambda)} \overline{\mathbb{F}}_q, \mathcal{L}_q(f^{\flat})). \]
Applying the induction hypothesis to $f_\lambda$ shows that the right-hand side of (5.5) vanishes for all $\lambda \in F_q$ if $j \neq n - 1$. It follows that the Leray spectral sequence collapses and we get

\[ H^i_c(A^1 \times F_q, \bar{F}_q, R^{n-1}_i \sigma L_\Phi(f)) = H^{i+n-1}_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)). \]

Since $\dim A^1 = 1$, the left-hand side of (5.6) can be nonzero only for $i = 0, 1, 2$. However, $H^{n-1}_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)) = 0$ because $A^n$ is smooth, affine, of dimension $n$, and $L_\Phi(f)$ is lisse on $A^n$. This proves that $H^i_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)) = 0$ except possibly for $i = n, n + 1$.

By (5.2) we then have

\[ L(A^n, f; t)^{(1)^{n+1}} = \frac{\det(I - tF | H^n_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)))}{\det(I - tF | H^{n+1}_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)))}. \]

Since $L_\Phi(f)$ is pure of weight 0, Deligne’s fundamental theorem tells us that $H^n_c(A^n \times F_q, \bar{F}_q, L_\Phi(f))$ is mixed of weights $\leq n$. Equation (5.5) and the induction hypothesis applied to $f_\lambda$ tell us that $R^{n-1}_i \sigma L_\Phi(f)$ is pure of weight $n - 1$ and that all fibers of $R^{n-1}_i \sigma L_\Phi(f)$ have the same rank, namely, $(d - 1)^{n-1}$. It follows from Katz, Corollary 6.7.2, that $R^{n-1}_i \sigma L_\Phi(f)$ is lisse on $A^1$. Equation (5.6) with $i = 2$ now implies, by Deligne, Corollaire 1.4.3, that $H^{n+1}_c(A^n \times F_q, \bar{F}_q, L_\Phi(f))$ is pure of weight $n + 1$, hence there can be no cancellation on the right-hand side of (5.7). However, Theorem 3.8 implies that $L(A^n, f; t)^{(1)^{n+1}}$ is a polynomial of degree $(d - 1)^n$, so we must have

\[ H^{n+1}_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)) = 0 \]

and

\[ \dim K_r H^n_c(A^n \times F_q, \bar{F}_q, L_\Phi(f)) = (d - 1)^n. \]

This establishes the first two assertions of the theorem.

To prove the last assertion of the theorem, note that $|\rho_i| \leq q^{n/2}$ for every $i$ and every archimedean absolute value since $H^n_c(A^n \times F_q, \bar{F}_q, L_\Phi(f))$ is mixed of weights $\leq n$. Thus we have

\[ |\Lambda(f)| \leq q^{n(d-1)^n/2} \]

for every archimedean absolute value on $Q(\zeta_p)$. By Corollary 4.5, we have

\[ |\Lambda(f)|_p \leq q^{-n(d-1)^n/2} \]

for every normalized archimedean absolute value on $Q(\zeta_p)$ lying over $p$, and it is well-known that $|\rho_i|_p = 1$ for every nonarchimedean absolute value lying over any prime $p' \neq p$. It then follows from the product formula for $Q(\zeta_p)$ that equality holds in (5.8) (and also in (5.9)), which implies the last assertion of the theorem.

6. Proof of Theorem 1.4

It remains to consider the case where $p$ divides $d$. The Euler relation becomes

\[ \sum_{i=1}^{n} x_i \frac{\partial f^{(d)}}{\partial x_i} = 0. \]

The regular sequence hypothesis then implies that

\[ x_i \in \left( \frac{\partial f^{(d)}}{\partial x_1}, \ldots, \frac{\partial f^{(d)}}{\partial x_i}, \ldots, \frac{\partial f^{(d)}}{\partial x_n} \right), \]
hence there is an equality of ideals of $F_q[x_1, \ldots, x_n]$

\[(x_1, \ldots, x_n) = \left( \frac{\partial f^{(d)}}{\partial x_1}, \ldots, \frac{\partial f^{(d)}}{\partial x_n} \right). \tag{6.1} \]

Conversely, if (6.1) holds, then $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ is a regular sequence. Equation (6.1) implies that $d = 2$, hence $p = 2$ as well, thus $f$ is a quadratic polynomial in characteristic 2. We may assume $f$ contains no terms of the form $x_i^2$ by the following elementary lemma.

Let $\zeta_p$ be a primitive $p$-th root of unity. Since $\Psi$ is a nontrivial additive character of $F_q$, there exists a nonzero $b \in F_q$ such that $\Psi(x) = \zeta_p \text{Tr}_{F_q/F_p}(bx)$.

\[(6.2) \]

**Lemma 6.3.** Let $a \in F_q$, $a \neq 0$, and choose $c \in F_q$ such that $c^p = (ab)^{-1}$. Then

\[\sum_{x_1, \ldots, x_n \in F_q} \Psi(f(x_1, \ldots, x_n) + ax_n^p) = \sum_{x_1, \ldots, x_n \in F_q} \Psi(f(x_1, \ldots, x_n) + ac^{p-1}x_n).\]

**Proof.** Making the change of variable $x_n \mapsto cx_n$, the sum becomes

\[\sum_{x_1, \ldots, x_n \in F_q} \Psi(f(x_1, \ldots, x_{n-1}, cx_n) + b^{-1}x_n^p) = \sum_{x_1, \ldots, x_n \in F_q} \Psi(f(x_1, \ldots, x_{n-1}, cx_n) + b^{-1}x_n)\Psi(b^{-1}(x_n^p - x_n)).\]

But by (6.2),

\[\Psi(b^{-1}(x_n^p - x_n)) = \zeta_p^{\text{Tr}_{F_q/F_p}(x_n^p - x_n)} = 1\]

since $\text{Tr}_{F_q/F_p}(x_n^p - x_n) = 0$ for all $x_n \in F_q$. Making the change of variable $x_n \mapsto c^{-1}x_n$ now gives the lemma.

By the lemma, we may assume our quadratic polynomial $f$ has the form

\[f = \sum_{1 \leq i < j \leq n} a_{ij}x_ix_j + \sum_{k=1}^n b_kx_k + c,\]

where $a_{ij}, b_k, c \in F_q$. This gives

\[f^{(2)} = \sum_{1 \leq i < j \leq n} a_{ij}x_ix_j.\]

Let $A = (A_{ij})$ be the $n \times n$ matrix defined by

\[A_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ a_{ji} & \text{if } i > j. \end{cases}\]

Thus $A$ is a symmetric matrix with zeros on the diagonal. One checks that

\[\begin{bmatrix} \frac{\partial f^{(2)}}{\partial x_1} \\ \vdots \\ \frac{\partial f^{(2)}}{\partial x_n} \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\]

therefore (6.1) holds if and only if $\det A \neq 0$. 

We now evaluate the exponential sum

\[(6.4) \sum_{x_1, \ldots, x_n \in F_q} \Psi \left( \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{k=1}^{n} b_k x_k + c \right). \]

**Proposition 6.5.** If \( n \) is odd, then (6.1) cannot hold. If \( n \) is even and (6.1) holds, then the sum (6.4) equals \( \zeta q^{n/2} \), where \( \zeta \) is a root of unity.

**Proof.** If \( n = 1 \), then \( \det A = 0 \), so (6.1) cannot hold. If \( n = 2 \), then \( \det A \neq 0 \) if and only if \( a_{12} \neq 0 \). It is then easy to check that the sum (6.4) equals

\[ \Psi \left( \frac{b_1 b_2 + c}{a_{12}} \right) q. \]

Thus the proposition holds for \( n = 1, 2 \). Suppose \( n \geq 3 \). The sum (6.4) can be rewritten as

\[ \sum_{x_1, \ldots, x_n \in F_q} \Psi \left( \sum_{1 \leq i < j \leq n-1} a_{ij} x_i x_j + \sum_{k=1}^{n-1} b_k x_k + c \right) \sum_{x_n \in F_q} \Psi \left( \left( \sum_{i=1}^{n-1} a_{in} x_i + b_n \right) x_n \right). \]

But

\[ \sum_{x_n \in F_q} \Psi \left( \left( \sum_{i=1}^{n-1} a_{in} x_i + b_n \right) x_n \right) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n-1} a_{in} x_i + b_n \neq 0 \\ q & \text{if } \sum_{i=1}^{n-1} a_{in} x_i + b_n = 0, \end{cases} \]

hence (6.4) equals

\[(6.6) q \sum_{x_1, \ldots, x_{n-1} \in F_q} \Psi \left( \sum_{1 \leq i < j \leq n-1} a_{ij} x_i x_j + \sum_{k=1}^{n-1} b_k x_k + c \right). \]

Since we are assuming \( A \) is invertible, some \( a_{in} \) must be nonzero, say, \( a_{n-1,n} \neq 0 \). By making the change of variable \( x_{n-1} \mapsto (a_{n-1,n})^{-1} x_{n-1} \), we may assume \( a_{n-1,n} = 1 \). Solving \( a_{1n} x_1 + \cdots + a_{n-1,n} x_{n-1} + b_n = 0 \) for \( x_{n-1} \) and substituting into the expression in the additive character, we see that (6.6) equals

\[(6.7) q \sum_{x_1, \ldots, x_{n-2} \in F_q} \Psi \left( \sum_{1 \leq i < j \leq n-2} a'_{ij} x_i x_j + \sum_{k=1}^{n-2} b'_k x_k + c \right), \]

where

\[ a'_{ij} = a_{ij} + a_{i,n-1} a_{jn} + a_{j,n-1} a_{in}. \]

Let \( A' = (A'_{ij}) \) be the \((n-2) \times (n-2)\) matrix constructed from the \( a'_{ij} \) as \( A \) was constructed from the \( a_{ij} \). We explain the connection between \( A \) and \( A' \). Let \( \tilde{A} \) be the \( n \times n \) matrix obtained from \( A \) by replacing row \( i \) by

\[ \text{row } i + a_{in} \text{ (row } n-1) + a_{i,n-1} \text{ (row } n) \]

for \( i = 1, \ldots, n-2 \). Keeping in mind that \( a_{n-1,n} = 1 \), we see that

\[ \tilde{A} = \begin{bmatrix} A' & 0 & 0 \\ a_{1,n-1} & \cdots & a_{n-2,n-1} \\ a_{1n} & \cdots & a_{n-2,n} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}. \]
In particular, $\det A' = \det A$. We can repeat this procedure starting with the sum (6.7) and continue until we are reduced to the one or two variable case, according to whether $n$ is odd or even. If $n$ is odd, this implies $\det A = 0$, a contradiction. Thus there does not exist a quadratic polynomial $f$ satisfying (6.1) in this case. If $n$ is even, this shows that (6.4) equals $q^{n/2}$ times a root of unity, which is the desired result.

A straightforward calculation using Proposition 6.5 then shows that the corresponding $L$-function has the form asserted in Theorem 1.4.

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