SPACE-TIME ARITHMETIC QUASI-PERIODIC HOMOGENIZATION FOR DAMPED WAVE EQUATIONS
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Abstract. This paper is concerned with space-time homogenization problems for damped wave equations with spatially periodic oscillating elliptic coefficients and temporally (arithmetic) quasi-periodic oscillating viscosity coefficients. Main results consist of a homogenization theorem, qualitative properties of homogenized matrices which appear in homogenized equations and a corrector result for gradients of solutions. In particular, homogenized equations and cell problems will turn out to deeply depend on the quasi-periodicity as well as the log ratio of spatial and temporal periods of the coefficients. Even types of equations will change depending on the log ratio and quasi-periodicity. Proofs of the main results are based on a (very weak) space-time two-scale convergence theory.

1. Introduction and main results

Space-time homogenization problems for hyperbolic equations were first studied by Benoussan, Lions and Papanicolaou. In [4], based on a method of asymptotic expansion, the following wave equation is treated:

$$\frac{\partial^2}{\partial t^2} u_\varepsilon - \text{div} \left( a_{\varepsilon} \nabla u_\varepsilon \right) = f \quad \text{in } \Omega \times (0,T),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $N \geq 1$, $T > 0$, $f = f(x,t)$ is a given data, $a : \mathbb{T}^N \times \mathbb{T} \to \mathbb{R}^{N \times N}$ is an $N \times N$ symmetric matrix field satisfying a uniform ellipticity and 1-periodicity and $a_\varepsilon := a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})$ for $r > 0$ (i.e., $a_\varepsilon$ is $\varepsilon \times \varepsilon^r$-periodic).

The homogenization problem concerns asymptotic behavior as $\varepsilon \to 0_+$ of (weak) solutions $u_\varepsilon = u_\varepsilon(x,t)$. In [4], it is assumed that (weak) solutions $u_\varepsilon = u_\varepsilon(x,t)$ can be expanded as a series:

$$u_\varepsilon(x,t) = \sum_{j=0}^{\infty} \varepsilon^j u_j(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}),$$

where $u_j = u_j(x,t,y,s) : \Omega \times (0,T) \times \mathbb{T}^N \times \mathbb{T} \to \mathbb{R}$ for $j = 0,1,2,\ldots$ are some periodic functions, and then, by substituting (1.2) to (1.1), at a formal level, $u_0 = u_0(x,t)$ turns out to be independent of microscopic variable $(y,s)$ and to solve the following homogenized equation:

$$\frac{\partial^2}{\partial t^2} u_0 - \text{div} \left( a_{\text{hom}} \nabla u_0 \right) = f \quad \text{in } \Omega \times (0,T),$$

where $a_{\text{hom}}$ is the so-called homogenized matrix and represented as

$$a_{\text{hom}} e_k = \int_0^1 \int_{\Delta} a(y,s) \left( \nabla_y \Phi_k(y,s) + e_k \right) dyds \quad \text{for } k = 1,2,\ldots,N.$$
Here $□ := (0, 1)^N$ is a unit cell, $\nabla_y$ stands for the gradient operator with respect to the third variable $y$, \( \{e_k\} = \{[\delta_{jk}]\}_{j=1,2,\ldots,N} \) stands for a canonical basis of $\mathbb{R}^N$ and $\Phi_k : \mathbb{T}^N \times \mathbb{T} \to \mathbb{R}$ (for $k = 1, 2, \ldots, N$) is the *corrector* which will be explained later (see Remark 1.12 below). Moreover, $\Phi_k$ is determined by the so-called *cell problems*. In particular, if the log-ratio of the spatial and temporal periods of the coefficients is the hyperbolic scale ratio (i.e., $r = 1$), then the cell problem is also a wave equation.

$$\partial^2_{ss} \Phi_k - \text{div}_y [a(y, s)(\nabla_y \Phi_k + e_k)] = 0 \quad \text{in } \mathbb{T}^N \times \mathbb{T}$$

(otherwise, cell problems are always elliptic equations, e.g., (1.17) below).

In [4], the following heat equation is also treated:

$$\partial_t u_k - \text{div} (a \nabla u) = f \quad \text{in } \Omega \times (0, T).$$

By substituting (1.2) to (1.4), $u_0 = u_0(x, t)$ is a (weak) solution to the following homogenized equation:

$$\partial_t u_0 - \text{div} (a_{\text{hom}} \nabla u_0) = f \quad \text{in } \Omega \times (0, T),$$

where $a_{\text{hom}}$ is defined by (1.3). Furthermore, if $r = 2$ (i.e., $a_{\varepsilon} = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$), then the unique solution to the following cell problem:

$$\partial_t \Phi_k - \text{div}_y [a(y, s)(\nabla_y \Phi_k + e_k)] = 0 \quad \text{in } \mathbb{T}^N \times \mathbb{T}$$

(as in (1.1), cell problems are always elliptic equations for any $r \neq 2$). Thus the type of the cell problem depends on the log-ratio of the spatial and temporal periods of the coefficients. Moreover, these formal arguments based on the asymptotic expansion for (the Cauchy-Dirichlet problem for) (1.4) are justified via *two-scale convergence theory* by A. Holmbom in [19]. The notion of two-scale convergence was first proposed by G. Nguetseng [24], and then, developed by G. Allaire [2, 3] (see also, e.g., [22, 33, 36]). It enables us to analyze how strong compactness of bounded sequences in Sobolev spaces fails due to their oscillatory behaviors (see (i) and (ii) of Remark 2.5 below). A. Holmbom extended the two-scale convergence theory to space-time homogenization and derived (1.5) and (1.6) rigorously. Moreover, the notion of very weak two-scale convergence is introduced, and then, it plays a crucial role for characterizing homogenized matrices (see Corollary 2.8 below for details). Besides, homogenization problems for various parabolic equations have been studied not only for linear ones but also for nonlinear ones (e.g. [5, 6, 7, 12, 14, 23, 31, 32]). In particular, for p-Laplace type [13, 34] and porous medium type [11], it has been proved that cell problems are given as parabolic equations at the critical scale (i.e., $a_{\varepsilon} = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$ in (1.4)).

On the other hand, the following more general hyperbolic-parabolic equation is treated (e.g. [5, 6, 7, 12, 14, 23, 31, 32]).

$$h_{\varepsilon} \partial^2_{tt} u_{\varepsilon} - \text{div}(a_{\varepsilon} \nabla u_{\varepsilon}) + g_{\varepsilon} \partial_t u_{\varepsilon} = f \quad \text{in } \Omega \times (0, T).$$

Here $h_{\varepsilon}$ and $g_{\varepsilon}$ are $\varepsilon \times \varepsilon^r$-periodic functions rapidly oscillating. Furthermore, [26, 29, 30, 35] deal with nonlinear wave equations, and in particular, in [26, 29], almost periodic settings are studied via $\Sigma$-convergence theory developed in [25]. Here (1.7) is called *damped wave equations* for $h_{\varepsilon} \equiv 1$ and $g_{\varepsilon} > 0$ and it is noteworthy that asymptotic expansions of solutions to damped wave equations are performed with the aid of solutions to diffusion equations (e.g. (1.14)), and moreover, asymptotic behaviors of solutions to damped wave equations are similar to those of diffusion equations as $t \to +\infty$ (see e.g. [18, 28]). Therefore, it is expected that cell problems for (1.7) will change at the critical scale for (1.4) (i.e., $a_{\varepsilon} = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$). However, at least to our knowledge, it does not seem to
occur under the periodic homogenization in the fixed domain except for \( h_\varepsilon = -\varepsilon^2 \) and \( g_\varepsilon \equiv 1 \) (see [4, Chapter 2, Section 4.5] for details).

1.1. Setting of the problem. One of main purposes of the present paper is to find conditions under which the cell problems of (1.7) will be different from elliptic ones (see (1.17) below). As a consequence, we emphasize that the (arithmetic) quasi-periodicity of the time-dependent coefficient \( g_\varepsilon \) in (1.7) is crucial and it is defined as follows.

**Definition 1.1 (Quasi-periodic functions).** The function \( \varphi \in C(\mathbb{R}) \) is said to be (arithmetic) quasi-periodic if it satisfies

\[
\varphi(s+1) = \varphi(s) + C_s \quad \text{for all } s \in [0, 1) \text{ and } C_s \in \mathbb{R}
\]

(i.e., \( \varphi \) is (0, 1)-periodic if \( C_s = 0 \)).

**Remark 1.2.** The notion of quasi-periodicity has been defined in several different ways (see e.g., [9, 11]). We stress that quasi-periodic functions in the sense of Definition 1.1 do not satisfy the almost-periodicity in the sense of Besicovitch, which is known as a generalization of periodicity. Indeed, if \( \varphi \in C(\mathbb{R}) \) is quasi-periodic, there exists a (0, 1)-periodic function \( \varphi_{\text{per}} \in C_{\text{per}}([0,1]) \) such that

\[
\varphi(s) = \varphi_{\text{per}}(s) + C_s s,
\]

which implies that

\[
\left( \limsup_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} |\varphi(s)|^r \, ds \right)^{1/r} = +\infty, \quad \text{for all } r \in [1, +\infty).
\]

Thus \( \varphi \) does not belong to the generalized Besicovitch space \( B^r(\mathbb{R}) \) (see e.g., [8, 21]). Moreover, we shall consider both the effect of the periodic homogenization and the effect of the singular limit due to \( \varphi(\frac{t}{\varepsilon^2}) = \varphi_{\text{per}}(\frac{t}{\varepsilon^2}) + C_s \frac{t}{\varepsilon^2} \) and \( C_s \frac{t}{\varepsilon^2} \to +\infty \) for \( t > 0 \) as \( \varepsilon \to 0_+ \).

In this paper, we shall consider the Cauchy-Dirichlet problem for the following damped wave equation:

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u_\varepsilon - \text{div} [a(t, \frac{x}{\varepsilon}) \nabla u_\varepsilon] + g(t, \frac{\cdot}{\varepsilon}) \partial_t u_\varepsilon &= f_\varepsilon \quad \text{in } \Omega \times (0, T), \\
u_\varepsilon|_{\partial \Omega} &= 0, \quad u_\varepsilon|_{t=0} = v_0^\varepsilon, \quad \partial_t u_\varepsilon|_{t=0} = v_1^\varepsilon.
\end{aligned}
\]

Here we make the following

**Assumption (A).** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( N \geq 1 \).

\( i \) Let \( T > 0, \varepsilon > 0 \) and \( r > 0 \). Let \( v_0^\varepsilon \in H_0^1(\Omega) \) and \( v_1^\varepsilon \in L^2(\Omega) \) be such that

\[
v_0^\varepsilon \to v^0 \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad v_1^\varepsilon \to v^1 \quad \text{weakly in } L^2(\Omega).
\]

Let \( f_\varepsilon, f \in L^2(\Omega \times (0, T)) \) be such that

\[
f_\varepsilon \to f \quad \text{weakly in } L^2(\Omega \times (0, T)).
\]

\( ii \) The \( N \times N \) symmetric matrix \( a \in [C^1(0, T; L^\infty(\mathbb{R}^N))]^{N \times N} \) satisfies a uniform ellipticity, i.e., there exists \( \lambda > 0 \) such that

\[
\lambda |\xi|^2 \leq a(t, y) \xi \cdot \xi \leq |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N \text{ and a.e. } (t, y) \in (0, T) \times \mathbb{R}^N,
\]

and \((0, 1)^N\)-periodicity:

\[
a(t, y + e_j) = a(t, y) \quad \text{a.e. in } (t, y) \in (0, T) \times \mathbb{R}^N.
\]

\[\text{Indeed, setting } \Phi(s) := \varphi(s) - C_s s, \text{ we see that } \Phi(s) \text{ is (0, 1)-periodic.}\]
(iii) Set \( g \in C(\mathbb{R}; \mathbb{R}^+) \) as follows:
\[ g(s) = g_{\text{per}}(s) + C_2 s^2 \quad \text{for all } s \in \mathbb{R}^+. \]
Here \( g_{\text{per}} \) is a \((0,1)\)-periodic function and \( C_2 \geq 0 \) is a constant. In addition, if \( r = 2 \), we further assume \( C_2 \leq \frac{2 \Delta}{C_0} \), where \( C_0 = N/\pi^2 \) is the best constant of the Poincaré inequality on the unit cell, that is,
\[ \|w\|_{L^2(\square)} \leq C_\square \|\nabla w\|_{L^2(\square)} \quad \text{for all } w \in H^1_{\text{per}}(\square). \]

(see Notation below).

(iv) In addition, if \( C_2 \neq 0 \) and \( 2 < r < +\infty \), then \( a = a(y), v_0, v_1, a(y), g(s) \) and \( f_\varepsilon \) are smooth, \((-\text{div}(a(\varepsilon)\nabla v_\varepsilon)), (v_1^\varepsilon), (f_\varepsilon) \) and \((\partial_t f_\varepsilon)\) are bounded in \( L^2(\Omega), H^1_0(\Omega), L^\infty(0,T;L^2(\Omega)) \) and \( L^2(\Omega \times (0,T)) \), respectively.

In this paper, we shall consider the convergence of solutions \((u_\varepsilon)\) to (1.8) and the homogenized equation as \( \varepsilon \to 0+ \). We also discuss how the homogenized matrix can be represented for each \( r > 0 \).

1.2. Main results. We start with the following definition of weak solutions to (1.8):

**Definition 1.3** (Weak solution of (1.8)). A function \( u_\varepsilon \in L^\infty(0,T;H^1_0(\Omega)) \) is said to be a weak solution to (1.8), if the following (i)-(iii) are all satisfied:

(i) (Regularity) \( u_\varepsilon \in W^{2,2}(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)). \)
(ii) (Initial condition) \( u_\varepsilon(t) \to v_0^\varepsilon \) strongly in \( L^2(\Omega) \) as \( t \to 0_+ \) and \( \partial_t u_\varepsilon(t) \to v_1^\varepsilon \) in \( H^{-1}(\Omega) \) as \( t \to 0_+ \).
(iii) (Weak form) It holds that, for all \( \phi \in H^1_0(\Omega) \),
\[ \langle \partial^2_{\varepsilon t} u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} + A_\varepsilon^t(u_\varepsilon(t), \phi) + \langle g(\frac{1}{\varepsilon}) \partial_t u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} = \langle f_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} \]
for a.e. in \( t \in (0,T) \), where \( A_\varepsilon^t(v,w) \) is a bilinear form in \( H^1_0(\Omega) \) defined by
\[ A_\varepsilon^t(v,w) = \int_\Omega a(t,\frac{x}{\varepsilon}) \nabla v(x) \cdot \nabla w(x) \, dx \quad \text{for } v,w \in H^1_0(\Omega). \]

By Galerkin’s method (cf. [10] Theorem 12.2), we have

**Theorem 1.4** (Existence and uniqueness of weak solutions to (1.8)). Suppose that
\[ a(t,\frac{x}{\varepsilon}) \in [C^4(0,T;L^\infty(\Omega))]^{N \times N}, \quad g(\frac{1}{\varepsilon}) \in C(0,T), \quad f_\varepsilon \in L^2(\Omega \times (0,T)), \]
\[ v_0^\varepsilon \in H^1_0(\Omega), \quad v_1^\varepsilon \in L^2(\Omega). \]
Then for every \( \varepsilon > 0 \) there exists a unique weak solution \( u_\varepsilon \) to (1.8).

Then we first obtain the following homogenization theorem:

**Theorem 1.5** (Homogenization theorem). Suppose that (A) is satisfied. Let \( u_\varepsilon \in L^\infty(0,T;H^1_0(\Omega)) \) be a unique weak solution to (1.8). There exist \( u_0 \in L^\infty(0,T;H^1_0(\Omega)) \) and \( h \in L^2_{\text{loc}}((0,T];H^{-1}(\Omega)) \) such that, for any \( \sigma > 0 \),
\[ u_\varepsilon \to u_0 \quad \text{weakly-* in } L^\infty(0,T;H^1_0(\Omega)), \]
\[ u_\varepsilon \to u_0 \quad \text{strongly in } C([0,T];L^2(\Omega)), \]
\[ g(\frac{1}{\varepsilon}) \partial_t u_\varepsilon \to g_{\text{per}} s \partial_t u_0 + C_\varepsilon h \quad \text{weakly in } \begin{cases} L^2(0,T;H^{-1}(\Omega)) \text{ if } C_\varepsilon = 0, \\ L^2(\sigma,T;H^{-1}(\Omega)) \text{ if } C_\varepsilon \neq 0. \end{cases} \]
Here $\Phi_k$ is a corrector for each $k = 1, \ldots, N$ and it is characterized as follows:

(i) In case $r \in (0, +\infty) \setminus \{2\}$, $\Phi_k \in H^{1}_{\text{per}}(\mathbb{T}^N)/\mathbb{R}$ (see Notation below) is the unique solution to

\begin{equation}
-\text{div}_y [a(t, y)(\nabla_y \Phi_k + e_k)] = 0 \quad \text{in} \quad \mathbb{T}^N \times (0, T),
\end{equation}

where $e_k$ is the $k$-th vector of the canonical basis of $\mathbb{R}^N$.

(ii) In case $r = 2$, $\Phi_k \in L^2(0, T; H^{1}_{\text{per}}(\mathbb{T}^N)/\mathbb{R})$ is the unique solution to

\begin{equation}
C_s \partial_t \Phi_k - \text{div}_y [a(t, y)(\nabla_y \Phi_k + e_k)] = 0 \quad \text{in} \quad \mathbb{T}^N \times (0, T).
\end{equation}

In particular, if either $C_s = 0$ or $a = a(y)$, then $\Phi_k \in H^{1}_{\text{per}}(\mathbb{T}^N)/\mathbb{R}$ is the unique solution to (1.17).

Furthermore, for any $C_s \geq 0$, $u_0$ is the unique weak solution to

\begin{equation}
\begin{cases}
\partial_t^2 u_0 - \text{div} [a_{\text{hom}}(t) \nabla u_0] + g_{\text{per}},_s \partial_t u_0 + C_s h = f & \text{in} \quad \Omega \times (0, T), \\
u_0|_{\partial \Omega} = 0, \quad u_0|_{t=0} = v^0, \quad \partial_t u_0|_{t=0} = \tilde{v}^1.
\end{cases}
\end{equation}

Here $u_0 \equiv v^0$ whenever $C_s \neq 0$, and moreover,

\[\tilde{v}^1 = \begin{cases}
v^1 & \text{if} \quad C_s = 0, \\
0 & \text{if} \quad C_s \neq 0.
\end{cases}\]

Moreover, $a_{\text{hom}}(t)$ is the homogenized matrix given by

\begin{equation}
a_{\text{hom}}(t) e_k = \int_{\mathbb{T}^N} a(t, y)(\nabla_y \Phi_k(t, y) + e_k) \, dy, \quad k = 1, 2, \ldots, N.
\end{equation}

**Remark 1.6.** It is noteworthy that, due to the loss of the time periodicity, the following facts hold:

(i) **(Homogenized equation).** The homogenized equation (1.19) is of the same type as the original equation (1.8) for the periodic case (i.e., $C_s = 0$). On the other hand, for the quasi-periodic case (i.e., $C_s \neq 0$), by the effect of the singular limit of $g$, (1.19) is represented as the following elliptic equation:

\[ -\text{div}(a_{\text{hom}} \nabla u_0) = f - C_s h \quad \text{in} \quad \Omega \times (0, T), \quad u_0 \in H^{1}_0(\Omega).
\]

Furthermore, the limit of the solution to (1.8) coincides with the limit of the initial data $v^0$.

(ii) **(Cell problem).** For the periodic case $C_s = 0$, the corrector $\Phi_k$ is always described as the solution to the elliptic equation (1.17). On the other hand, for the quasi-periodic case, at the critical case $r = 2$, the cell problem (1.18) is different from (1.17) and it is given as the parabolic equation by the effect of the singular limit of $g$. Thus $\Phi_k$ depends on $t \in (0, T)$, and then, qualitative properties of the homogenized matrix $a_{\text{hom}}$ will change due to (1.20) (see Proposition 1.7 below).
Moreover, as for the homogenized matrix, we next have the following

**Proposition 1.7** (Qualitative properties of the homogenized matrix \(a_{\text{hom}}\)). Under the same assumption as in Theorem [1.20], let \(0 < r < +\infty\) and \(a_{\text{hom}}(t)\) be the homogenized matrices defined by \([1.21]\). Then the following (i) and (ii) hold:

(i) (Uniform ellipticity) It holds that

\[
\lambda |\xi|^2 + \lambda \|\nabla_y \Phi_\xi(t)\|_{L^2(\square)}^2 + \frac{C_\ast t}{2} \frac{d}{dt} \|\Phi_\xi(t)\|_{L^2(\square)}^2 \\
\leq a_{\text{hom}}(t) \xi \cdot \xi \\
\leq |\xi|^2 + \|\nabla_y \Phi_\xi(t)\|_{L^2(\square)}^2 + \frac{C_\ast t}{2} \frac{d}{dt} \|\Phi_\xi(t)\|_{L^2(\square)}^2
\]

for any \(\xi \in \mathbb{R}^N\) and a.e. \(t \in (0,T)\), where \(\lambda > 0\) is the ellipticity constant of \(a(t,y)\) defined by \([1.9]\) and \(\Phi_\xi\) is the corrector given by either \([1.17]\) or \([1.18]\) with \(e_k\) replaced by \(\xi \in \mathbb{R}^N\).

(ii) (Symmetry and asymmetry) If \(a(t,y)\) is the symmetric matrix, then \(a_{\text{hom}}(t)\) is the asymmetric matrix for \(r = 2\) and \(C_\ast \neq 0\). Otherwise, \(a_{\text{hom}}(t)\) is also the symmetric matrix.

**Remark 1.8.** We stress that, in the critical case (i.e., \(r = 2\) and \(C_\ast \neq 0\)), even though the elliptic constant of \(a(t,y)\) is independent of \(t \in (0,T)\), that of \(a_{\text{hom}}(t)\) depends on \(t\). Furthermore, the symmetry breaking of \(a_{\text{hom}}(t)\) occurs but it makes no contribution to the divergence (see Remark [5.1] below).

We finally get the following corrector result.

**Theorem 1.9** (Corrector result for time independent coefficients). Suppose that \((\text{A})\) is fulfilled and assume that \(a = a(y)\), \(v^0_\varepsilon\), \(v^1_\varepsilon\), \(a(y)\), \(g(s)\) and \(f_\varepsilon\) are smooth, \((-\text{div}(a(z)\nabla v^0_\varepsilon))\), \((v^1_\varepsilon)\), \((f_\varepsilon)\) and \((\partial_t f_\varepsilon)\) are bounded in \(L^2(\Omega)\), \(H^1(\Omega)\), \(L^\infty(0,T; L^2(\Omega))\) and \(L^2(\Omega \times (0,T))\), respectively. Let \(u_\varepsilon\) and \(u_0\) be the unique solutions to \([1.8]\) and \([1.9]\), respectively. Then it holds that

\[
\lim_{\varepsilon \to 0_+} \int_0^T \int_{\Omega} \left| \nabla u_\varepsilon(x,t) - (\nabla u_0(x,t) + \nabla_y u_1(x,t,\frac{t}{\varepsilon})) \right|^2 \, dx \, dt = 0
\]

for all \(r \in (0, +\infty)\), where \(u_1 = \sum_{k=1}^N \partial_{x_k} u_0 \Phi_k\) and \(\Phi_k \in L^2(0,T; H^1_{\text{per}}(\mathbb{T}^N)/\mathbb{R})\) is the corrector for \(r \in (0, +\infty)\).

As for the time dependent case \(a = a(t,y)\), we have the following corrector result for more specific settings:

**Corollary 1.10** (Corrector result for time dependent coefficients). Suppose that \(C_\ast \neq 0\). In addition, assume that \(a(t,y)\) is smooth and the following \([1.22]-[1.25]\) hold:

\[
\partial_t a(t,y) \xi \cdot \xi \leq 0 \\
\text{for all } \xi \in \mathbb{R}^N \text{ and all } (t,y) \in (0,T) \times \mathbb{R}^N;
\]

\[
- \text{div}(a(0,\xi)\nabla v^0_\varepsilon) \to -\text{div}(a_{\text{hom}}(0)\nabla v^0) \text{ strongly in } H^{-1}(\Omega),
\]

\[
\lim_{\varepsilon \to 0_+} \|v^1_\varepsilon\|_{L^2(\Omega)} = 0,
\]

\[
f_\varepsilon \to f \text{ strongly in } L^2(\Omega \times (0,T)) \text{ or } (f_\varepsilon/\sqrt{t}) \text{ is bounded in } L^2(\Omega \times (0,T)),
\]

In addition, if \(r = 2\) and \(C_\ast \neq 0\), assume that

\[
\partial_t a(t,y) = -a(t,y) \text{ for all } (t,y) \in (0,T) \times \mathbb{R}^N.
\]
Here $a_{\text{hom}}(t)$ is the homogenized matrix defined by (1.20). Let $u_\varepsilon$ and $u_0$ be the unique solutions to (1.8) and (1.19), respectively. Then (1.21) holds.

**Remark 1.11.** Initial data $v^0_\varepsilon \in H^1_0(\Omega)$ satisfying (1.23) can actually be constructed (see e.g. [10] pp. 236)).

**Remark 1.12.** From Theorem 1.9 it holds that

$$u_\varepsilon \not\to u_0 \quad \text{strongly in } L^2(0, T; H^1_0(\Omega))$$

in general due to the oscillation of the third term $u_1(x, t, \cdot)$ as $\varepsilon \to 0_+$. Thus $u_1(x, t, \cdot)$ plays a role as the corrector term recovering the strong compactness in this topology. For this reason, $\Phi_k$ is often called a corrector.

### 1.3. Plan of the paper and notation.

This paper is organized as follows. In the next section, we summarize relevant material on space-time two-scale convergence. Section 3 is devoted to proving uniform estimates for solutions $u_\varepsilon$ to (1.8) as $\varepsilon \to 0_+$. Furthermore, we shall prove their weak(-*) and strong convergences. In Section 4, we shall prove Theorem 1.5. To prove Proposition 1.7, we shall discuss qualitative properties of the homogenized matrix $a_{\text{hom}}(t)$ in Section 5. The final section is devoted to proofs of Theorem 1.9 and Corollary 1.10.

**Notation.** Throughout this paper, $C > 0$ denotes a non-negative constant which may vary from line to line. In addition, the subscript $A$ of $C_A$ means dependence of $C_A$ on $A$. Let $\delta_{ij}$ be the Kronecker delta, $e_i = (\delta_{ij})_{1 \leq j \leq N}$ be the $i$-th vector of the basis of $\mathbb{R}^N$, $\| \cdot \|_{H^1_0(A)}$ be defined by $\| \cdot \|_{H^1_0(A)} := \| \nabla \cdot \|_{L^2(A)}$ for domains $A \subset \mathbb{R}^N$, $\nabla$ and $\nabla_y$ denote gradient operators with respect to $x$ and $y$, respectively, and $\text{div}$ and $\text{div}_y$ denote divergence operators with respect to $x$ and $y$, respectively. Furthermore, we shall use the following notation:

- $\Box = (0, 1)^N$, $I = (0, T)$, $J = (0, 1)$, $dZ = dydsdt$.
- Define the set of smooth $\Box$-periodic functions by

$$C^{\infty}_{\text{per}}(\Box) = \{ w \in C^{\infty}(\Box) : w(\cdot + \epsilon_k) = w(\cdot) \text{ in } \mathbb{R}^N \text{ for } 1 \leq k \leq N \}.$$

- We also define $W^{1,q}_{\text{per}}(\Box)$ and $L^q_{\text{per}}(\Box)$ as closed subspaces of $W^{1,q}(\Box)$ and $L^q(\Box)$ by

$$W^{1,q}_{\text{per}}(\Box) = \overline{C^{\infty}_{\text{per}}(\Box)}^{\| \cdot \|_{W^{1,q}(\Box)}}, \quad L^q_{\text{per}}(\Box) = \overline{C^{\infty}_{\text{per}}(\Box)}^{\| \cdot \|_{L^q(\Box)}},$$

respectively, for $1 \leq q < +\infty$. In particular, set $H^1_{\text{per}}(\Box) := W^{1,2}_{\text{per}}(\Box)$. We shall simply write $L^q(\Box)$ instead of $L^q_{\text{per}}(\Box)$, unless any confusion may arise.

- We often write $L^q(\Omega \times \Box)$ instead by $L^q(\Omega; L^q_{\text{per}}(\Box))$ since $L^q_{\text{per}}(\Box)$ is reflexive Banach space for $1 < q < +\infty$.

- Define the mean $\langle w \rangle_y := \int_{\Box} w(y) dy$ in $y$ of $w \in L^1(\Box)$.
- We set $W^{1,q}_{\text{per}}(\Box)/\mathbb{R} = \{ w \in W^{1,q}_{\text{per}}(\Box) : \langle w \rangle_y = \int_{\Box} w(y) dy = 0 \}$.

- Furthermore, let $X$ be a normed space with a norm $\| \cdot \|_X$ and a duality pairing $\langle \cdot, \cdot \rangle_X$ between $X$ and its dual space $X^*$. Moreover, we write $X^N = X \times X \times \cdots \times X$ ($N$-product space), e.g., $[L^2(\Omega)]^N = L^2(\Omega; \mathbb{R}^N)$.

In order to clarify variables of integration, we shall often write, e.g., $\| u(\cdot) \|_{L^q(\Omega)}$ and $\| u(x, \cdot) \|_{L^q(\Omega)}$ instead of $\| u(\cdot) \|_{L^q(\Omega)}$ and $\| u(x, \cdot) \|_{L^q(\Omega)}$, respectively. We often write $u(t)$ instead of $u(\cdot, t)$ for each $t \in I$ and $u : \Omega \times I \to \mathbb{R}$. 


2. Space-time two-scale convergence theory

In this section, we introduce the notion of space-time two-scale convergence and briefly summarize its crucial properties (see, e.g., [24], [23], [22], [36] and [19] for more details). Throughout this section, let $q \in [1, +\infty]$ and $q'$ denote the Hölder conjugate of $q$ (i.e., $1/q + 1/q' = 1$ if $1 < q < +\infty$; $q' = 1$ if $q = +\infty$; $q' = +\infty$ if $q = 1$) unless any confusion may arise. Furthermore, let $I = (0, T)$, $\square = (0, 1)^N$ and $J = (0, 1)$.

We first define a class of test functions for the space-time two-scale convergence, called admissible test functions.

**Definition 2.1** (Admissible test function). Let $q' \in [1, +\infty]$ and let $X \subset L^{q'}(\Omega \times I \times \square \times J)$ be a separable normed space equipped with norm $\| \cdot \|_X$. Then $(X, \| \cdot \|_X)$ is called an admissible test function space (for the weak space-time two-scale convergence in $L^q(\Omega \times I \times \square \times J)$), if it holds that, for all $\Psi \in X$, $(x, t) \mapsto \Psi(x, t, \frac{r}{\varepsilon}, \frac{y}{\varepsilon})$ is Lebesgue measurable in $\Omega \times I$ for $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0^+} \| \Psi(x, t, \frac{r}{\varepsilon}, \frac{y}{\varepsilon}) \|_{L^{q'}(\Omega \times I)} = \| \Psi(x, t, y, s) \|_{L^{q'}(\Omega \times I \times \square \times J)},$$

$$\| \Psi(x, t, \frac{r}{\varepsilon}, \frac{y}{\varepsilon}) \|_{L^{q'}(\Omega \times I)} \leq C \| \Psi(x, t, y, s) \|_X \quad \text{for } \varepsilon > 0.$$  

Moreover, $\Psi \in X$ is called an admissible test function (for the weak space-time two-scale convergence in $L^q(\Omega \times I \times \square \times J)$).

The following fact is well known and often used, in particular, to discuss weak convergence of periodic test functions.

**Proposition 2.2** (Mean-value property). Let $w \in L^q(\square \times J)$ and set $w_\varepsilon(x, t) = w(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ for $\varepsilon > 0$ and $0 < r < +\infty$. For any bounded domain $\Omega \subset \mathbb{R}^N$ and any bounded interval $I \subset \mathbb{R}$, it holds that

$$\begin{cases} w_\varepsilon \to \langle w \rangle_{y,s} \text{ weakly in } L^q(\Omega \times I) & \text{if } q \in [1, +\infty), \\
w_\varepsilon \to \langle w \rangle_{y,s} \text{ weakly-* in } L^\infty(\Omega \times I) & \text{if } q = +\infty \end{cases}$$

as $\varepsilon \to 0^+$. Here $\langle w \rangle_{y,s}$ denotes the mean of $w$, i.e.,

$$\langle w \rangle_{y,s} = \int_0^1 \int_{\square} w(y, s) dy ds.$$

**Proof.** See [10], Theorem 2.6. \hfill \square

Now, we are in a position to define the notion of space-time two-scale convergence in the following.

**Definition 2.3** (Weak space-time two-scale convergence and very weak two-scale convergence).

(i) A bounded sequence $(v_\varepsilon)$ in $L^q(\Omega \times I)$ is said to weakly space-time two-scale converge to a limit $v \in L^q(\Omega \times I \times \square \times J)$ if it holds that

$$\lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega v_\varepsilon(x, t) \Psi(x, t, \frac{r}{\varepsilon}, \frac{y}{\varepsilon}) dx dt = \int_0^T \int_\Omega \int_0^1 \int_{\square} v(x, t, y, s) \Psi(x, t, y, s) dZ$$

for any admissible function $\Psi \in X \subset L^q(\Omega \times I \times \square \times J)$ and it is denoted by $v_\varepsilon \overset{\text{wast}}{\to} v$ in $L^q(\Omega \times I \times \square \times J)$.  

Proof. See \[19, Corollary 3.3\].

As a limit (and let \(v\) be a bounded sequence in \(L^q(\Omega \times I \times \square \times J)\)) we choose \(\Psi(x,y,t,s) = \phi(x)b(y)\psi(t)c(s)\) for any \(\phi \in C_c^\infty(\Omega), b \in C^\infty_{\text{per}}(\square) / \mathbb{R}, \psi \in C^\infty_c(I)\) and \(c \in C^\infty_{\text{per}}(J)\), and then, it is written by

\[
v_k \xrightarrow{\text{w} \text{w}} v \quad \text{in} \quad L^q(\Omega \times I \times \square \times J).
\]

Remark 2.4. Due to the extension of the original definition in \[24, 2\] of the test function \(\Psi \in L^q(\Omega \times I; C^\infty_{\text{per}}(\square \times J))\) the boundedness of \(v_k\) is essential. Indeed, some counterexamples that the (weak space-time) two-scale limit does not coincide with the weak limit are known in \[22, \text{Examples 11 and 12}\].

Remark 2.5. As for the relation between weak or strong convergence and weak space-time two-scale convergence, the following holds:

(i) If \(v_k \xrightarrow{\text{w} \text{w}} v \) in \(L^q(\Omega \times I \times \square \times J)\), then \(v_k \rightharpoonup \langle v \rangle_{y,s}\) weakly in \(L^q(\Omega \times I)\).

(ii) If \(v_k \rightarrow \hat{v}\) strongly in \(L^1(\Omega \times I)\), then \(v_k \xrightarrow{\text{w} \text{w}} \hat{v}\) in \(L^q(\Omega \times I \times \square \times J)\).

The following theorem is concerned with weak space-time two-scale compactness of bounded sequences in \(L^q(\Omega \times I)\).

**Theorem 2.6** (Weak space-time two-scale compactness). Let \(q \in (1, \infty)\). Then, for any bounded sequence \((v_k)\) in \(L^q(\Omega \times I)\), there exist a subsequence \((\varepsilon_k)\) of \((\varepsilon)\) such that \(\varepsilon_k \rightarrow 0_+\) and a limit \(v \in L^q(\Omega \times I \times \square \times J)\) such that

\[
v_{\varepsilon_k} \xrightarrow{\text{w} \text{w}} v \quad \text{in} \quad L^q(\Omega \times I \times \square \times J).
\]

**Proof.** See \[19, Theorem 2.3\].

As for weak space-time two-scale compactness of gradients, we obtain

**Theorem 2.7** (Weak space-time two-scale compactness for gradients). Let \(q \in (1, +\infty)\) and let \((v_k)\) be a bounded sequence in \(W^{1,q}(\Omega \times I)\). Then there exist a subsequence \(\varepsilon_k \rightarrow 0_+\), a limit \(v \in L^q(\Omega \times I)\) and a function \(v_1 \in L^q(\Omega \times I; W^{1,q}_{\text{per}}(\square \times J) / \mathbb{R})\) such that

\[
\nabla_{t,x} v_{\varepsilon_k} \xrightarrow{\text{w} \text{w}} \nabla_{t,x} v + \nabla_{s,y} v_1 \quad \text{in} \quad [L^q(\Omega \times I \times \square \times J)]^{N+1}.
\]

Here and henceforth, \(\nabla_{t,x} = (\partial_{t}, \partial_{x_1}, \ldots, \partial_{x_N})\) and \(\nabla_{s,y} = (\partial_{s}, \partial_{y_1}, \ldots, \partial_{y_N})\).

**Proof.** See \[22, Theorem 20\].

As a corollary of Theorem 2.7, the following is obtained.

**Corollary 2.8** (cf. \[15, 16, 19\]). Under the same assumptions as in Theorem 2.7, it holds that

\[
\frac{v_{\varepsilon_k}}{\varepsilon_k} \xrightarrow{\text{w} \text{w}} v \quad \text{in} \quad L^q(\Omega \times I \times \square \times J).
\]

**Proof.** See \[19, Corollary 3.3\].
To discuss convergence of solutions \( u_\varepsilon \) for (1.3), we shall first verify their uniform boundedness, and then, we shall prove their weak or strong convergences.

**Lemma 3.1 (Uniform estimates).** Let \( u_\varepsilon \in L^\infty(I; H^1_0(\Omega)) \) be the unique weak solution of (1.3) under the same assumptions as in Theorem 3.2 and let \( I_\sigma = (\sigma,T) \) for any \( \sigma > 0 \). Then the following (i)-(vi) hold:

(i) \((u_\varepsilon)\) is bounded in \( L^\infty(I; H^1_0(\Omega)) \),
(ii) \((\partial_t u_\varepsilon)\) is bounded in \( L^\infty(I; L^2(\Omega)) \),
(iii) \((\sqrt{t}\varepsilon^{-2}\partial_t u_\varepsilon)\) is bounded in \( L^2(\Omega \times I) \), provided that \( C_\varepsilon \neq 0 \),
(iv) \((\partial^2_{tt} u_\varepsilon + g(\frac{\varepsilon}{\sigma})\partial_t u_\varepsilon)\) is bounded in \( L^2(I; H^{-1}(\Omega)) \),
(v) \((\partial^2_{tt} u_\varepsilon)\) is bounded in \( L^2(I; H^{-1}(\Omega)) \) if \( C_\varepsilon = 0 \),
(vi) \((t\varepsilon^{-2}\partial_t u_\varepsilon)\) is bounded in \( L^2(I_\sigma; H^{-1}(\Omega)) \), provided that \( C_\varepsilon \neq 0 \).

**Proof.** Recall (1.10), i.e.,
\[
\langle \partial^2_{tt} u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} + A_\varepsilon(u_\varepsilon(t), \phi) + \langle g(\frac{\varepsilon}{\sigma}) \partial_t u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} = \langle f_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)}
\]
for all \( \phi \in H^1_0(\Omega) \). Testing it by \( \partial_t u_\varepsilon \) (see Remark 3.2 below), we deduce by the symmetry of \( a(t,y) \) that
\[
(3.1) \int_\Omega a(t, \frac{\varepsilon}{\sigma}) \nabla u_\varepsilon(x,t) \cdot \nabla \partial_t u_\varepsilon(x,t) \, dx
= \frac{1}{2} \frac{d}{dt} \int_\Omega a(t, \frac{\varepsilon}{\sigma}) \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \, dx - \frac{1}{2} \int_\Omega \partial_t a(t, \frac{\varepsilon}{\sigma}) \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \, dx
\]
a.e. in \( I \). Thus we have
\[
(3.2) \frac{1}{2} \int_0^s \frac{d}{dt} \left[ \int_\Omega \left[ |\partial_t u_\varepsilon(x,t)|^2 + a(t, \frac{\varepsilon}{\sigma}) \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \right] \right] \, dx \, dt
\]
for all \( s \in I \). Then we observe from the uniform ellipticity (1.1), (3.2) and (A) that
\[
\|
\partial_t u_\varepsilon(s)
\|_{L^2(\Omega)}^2 + \lambda \| u_\varepsilon(s) \|_{H^1_0(\Omega)}^2
\leq
\| v_\varepsilon \|_{L^2(\Omega)}^2 + \| \nabla \partial_t u_\varepsilon(t) \|_{H^1_0(\Omega)}^2 + \| \nabla u_\varepsilon(t) \|_{H^1_0(\Omega)}^2 + \int_0^s \frac{d}{dt} \left( \| \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 + \int_\Omega a(t, \frac{\varepsilon}{\sigma}) \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \, dx \right) \, dt
\]

\[
+ 2 \int_0^s \int_\Omega f_\varepsilon(x,t) \partial_t u_\varepsilon(x,t) \, dx \, dt - \int_0^s \left( g_{\text{per}}(\frac{\varepsilon}{\sigma}) + C_\varepsilon \frac{\varepsilon}{\sigma} \right) \| \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 \, dt
\]

\[
+ 2 \int_0^s \left[ \| f_\varepsilon(t) \|_{L^2(\Omega)} \| \partial_t u_\varepsilon(t) \|_{L^2(\Omega)} + \left( \beta - C_\varepsilon \frac{\varepsilon}{\sigma} \right) \| \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 \right] \, dt
\]
we get (v) if $C$ for all $0 < \sigma < T$.

Here $\beta = \max_{s \in [0,1]} |g_{\text{per}}(s)|$. From the boundedness of $(f_\varepsilon)$ in $L^2(\Omega \times I)$, we get

$$
\int_0^T \left( \| \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 + \| u_\varepsilon(t) \|_{H^1_0(\Omega)}^2 \right) dt - C_s\int_0^T \| \sqrt{\varepsilon t^{-r}} \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 dt.
$$

which together with Gronwall’s inequality yields (i) and (ii). Moreover, (iii) also follows from (i), (ii) and (3.3). We next prove (iv). For any $\phi \in H^1_0(\Omega)$, the weak form (1.10) yields

$$
\langle \partial^2_{tt} u_\varepsilon(t) + g(\frac{1}{\varepsilon^2}) \partial_t u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} \leq \| \phi \|_{H^1_0(\Omega)} \left( \| f_\varepsilon(t) \|_{H^{-1}(\Omega)} + \| u_\varepsilon(t) \|_{H^1_0(\Omega)} \right).
$$

Here we used the fact that

$$
|a(t, y)\xi \cdot \zeta| \leq |\xi||\zeta|
$$

for all $\xi, \zeta \in \mathbb{R}^N$ and a.e. $(t, y) \in I \times \mathbb{R}^N$, which follows from the Rayleigh-Ritz variational principle. By the boundedness of $(f_\varepsilon)$ in $L^2(I; H^{-1}(\Omega))$ together with (i) and (3.3), we deduce that

$$
\int_0^T \| \partial^2_{tt} u_\varepsilon(t) + g(\frac{1}{\varepsilon^2}) \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 dt 
\overset{\mathbf{1.3}}{\leq} \int_0^T \left( \| f_\varepsilon(t) \|_{L^2(\Omega)}^2 + \| u_\varepsilon(t) \|_{H^1_0(\Omega)}^2 \right) dt 
\leq 2 \left( \| f_\varepsilon \|_{L^2(I; H^{-1}(\Omega))}^2 + \| u_\varepsilon \|_{L^2(I; H^1_0(\Omega))}^2 \right),
$$

which implies that (iv) holds true. Here noting that

$$
\| \partial^2_{tt} u_\varepsilon + C_s t \varepsilon^{-r} \partial_t u_\varepsilon \|_{L^2(I; H^{-1}(\Omega))} 
\leq \| \partial^2_{tt} u_\varepsilon + g(\frac{1}{\varepsilon^2}) \partial_t u_\varepsilon \|_{L^2(I; H^{-1}(\Omega))} + \beta \| \partial_t u_\varepsilon \|_{L^2(I; H^{-1}(\Omega))} < C,
$$

we get (v) if $C_s = 0$. As for $C_s \neq 0$, we infer that

$$
\int_0^T t \left( \| \partial^2_{tt} u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 + \| C_s t \varepsilon^{-r} \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 \right) dt 
\overset{\mathbf{1.5}}{=} \int_0^T t \left( \| \partial^2_{tt} u_\varepsilon(t) + C_s t \varepsilon^{-r} \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 - 2 \langle \partial^2_{tt} u_\varepsilon(t), C_s \varepsilon^{-r} \partial_t u_\varepsilon(t) \rangle_{H^{-1}(\Omega)} \right) dt 
\leq CT - \int_0^T C_s t^2 \left( \| \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 dt 
\overset{\mathbf{3.6}}{=} CT - C_s T^2 \| \sqrt{\varepsilon^{-r}} \partial_t u_\varepsilon(T) \|_{H^{-1}(\Omega)}^2 + 2C_s \int_0^T \| \sqrt{t \varepsilon^{-r}} \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 dt \leq C.
$$

Hence we have

$$
\int_0^T \left( \| \partial^2_{tt} u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 + \| \varepsilon^{-r} C_s \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 \right) dt 
\leq \frac{1}{\sigma} \int_0^T t \left( \| \partial^2_{tt} u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 + \| \varepsilon^{-r} C_s \partial_t u_\varepsilon(t) \|_{H^{-1}(\Omega)}^2 \right) dt \leq \frac{C}{\sigma}
$$

for all $0 < \sigma < T$, which implies (v) and (vi).

□
The argument mentioned above is not fully rigorous in view of the regularity of weak solutions. However, an approximate solution constructed by the Galerkin method satisfies all the estimates and (weak) lower semicontinuity of norms assures the assertions.

Moreover, for smooth data, we have

**Lemma 3.3 (Uniform estimates with smooth data).** Suppose that \( a = a(y), v^0, v^1, a(y), g(s) \) and \( f_\varepsilon \) are smooth, \((-\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon)), (v^1_\varepsilon), (f_\varepsilon) \) and \((\partial_t f_\varepsilon)\) are bounded in \(L^2(\Omega), H^1_0(\Omega), L^\infty(I; L^2(\Omega)) \) and \(L^2(\Omega \times I)\), respectively. Let \( u_\varepsilon \in L^\infty(I; H^1_0(\Omega)) \) be the unique weak solution to (1.8). Assume that (A) holds. Let \( I_\sigma = (\sigma, T) \) for any \( \sigma > 0 \). Then the following (i)-(v) hold:

(i) \((-\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon)) \) is bounded in \(L^\infty(I; L^2(\Omega))\),

(ii) \((\partial_t u_\varepsilon)\) is bounded in \(L^\infty(I; H^1_0(\Omega))\),

(iii) \((\partial^2_{tt} u_\varepsilon + g(\frac{x}{\varepsilon})\partial_t u_\varepsilon)\) is bounded in \(L^\infty(I; L^2(\Omega))\),

(iv) \((\partial^2_{tt} u_\varepsilon)\) is bounded in \(L^2(\Omega \times I_\sigma)\),

(v) \((\varepsilon^{-1}\partial_t u_\varepsilon)\) is bounded in \(L^2(\Omega \times I_\sigma)\), provided that \( C_\varepsilon \neq 0 \).

**Proof.** Test (1.10) by \(-\text{div}(a(\frac{x}{\varepsilon})\nabla \partial_t u_\varepsilon)\). Then we observe that

\[
\int_0^s \int_\Omega \partial^2_{tt} u_\varepsilon(x, t) \left(-\text{div}(a(\frac{x}{\varepsilon})\nabla \partial_t u_\varepsilon(x, t)) \right) \, dx \, dt
\]

\[
= \int_0^s \int_\Omega \partial_t \left( \nabla \partial_t u_\varepsilon(x, t) \right) \cdot a(\frac{x}{\varepsilon}) \nabla \partial_t u_\varepsilon(x, t) \, dx \, dt
\]

\[
= \frac{1}{2} \int_0^s \frac{d}{dt} \left( \int_\Omega a(\frac{x}{\varepsilon}) \nabla \partial_t u_\varepsilon(x, t) \cdot \nabla \partial_t u_\varepsilon(x, t) \, dx \right) \, dt
\]

\[
\geq \frac{\lambda}{2} \|\partial_t u_\varepsilon(s)\|^2_{H^1_0(\Omega)} - \frac{1}{2} \|v^1_\varepsilon\|^2_{H^1_0(\Omega)}
\]

and

\[
\int_0^s \int_\Omega (-\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x, t))) \left(-\text{div}(a(\frac{x}{\varepsilon})\nabla \partial_t u_\varepsilon(x, t)) \right) \, dx \, dt
\]

\[
= \frac{1}{2} \|\partial_t u_\varepsilon(s)\|^2_{L^2(\Omega)} - \frac{1}{2} \|\partial_t u_\varepsilon(s)\|^2_{L^2(\Omega)}
\]

for all \( s \in I \). Hence we derive that

\[
\frac{\lambda}{2} \|\partial_t u_\varepsilon(s)\|^2_{H^1_0(\Omega)} + \frac{1}{2} \| -\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon(s))\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2} \|v^1_\varepsilon\|^2_{H^1_0(\Omega)} + \frac{1}{2} \| -\text{div}(a(\frac{x}{\varepsilon})\nabla v^0_\varepsilon)\|^2_{L^2(\Omega)}
\]

\[
+ \int_\Omega f_\varepsilon(x, s) \left(-\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x, s)) \right) \, dx - \int_\Omega f_\varepsilon(x, 0) \left(-\text{div}(a(\frac{x}{\varepsilon})\nabla v^0_\varepsilon(x)) \right) \, dx
\]

\[
- \int_0^s \int_\Omega \partial_t f_\varepsilon(x, t) \left(-\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x, t)) \right) \, dx \, dt
\]

\[
= \frac{1}{2} \|v^1_\varepsilon\|^2_{H^1_0(\Omega)} + \| -\text{div}(a(\frac{x}{\varepsilon})\nabla v^0_\varepsilon)\|^2_{L^2(\Omega)}
\]

\[
+ \|f_\varepsilon(s)\|^2_{L^2(\Omega)} + \frac{1}{4} \| -\text{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon(s))\|^2_{L^2(\Omega)} + \frac{1}{2} \|f_\varepsilon(0)\|^2_{L^2(\Omega)}
\]
\[ + \frac{1}{2} \| \partial_t u \|_{L^2(\Omega \times I)}^2 + \frac{1}{2} \int_0^s \| - \text{div}(a(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon(t)) \|_{L^2(\Omega)}^2 \, dt - \int_0^s \lambda g(\frac{\cdot}{\varepsilon}) \| \nabla \partial_t u_\varepsilon(t) \|_{L^2(\Omega)}^2 \, dt \geq 0 \]

which together with Gronwall’s inequality yields (i) and (ii). Hence one can derive that

\[ ||\partial^2_t u_\varepsilon + g(\frac{\cdot}{\varepsilon}) \partial_t u_\varepsilon||_{L^\infty(I;L^2(\Omega))} \leq ||f||_{L^\infty(I;L^2(\Omega))} + || - \text{div}(a(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon)||_{L^\infty(I;L^2(\Omega))}, \]

which implies (iii). As in the proofs of (3.5) and (3.6), (iv) and (v) follow from (3.7).

\[ \square \]

Applying Lemma 3.1 we next get the following

**Lemma 3.4** (Weak(-star) and strong convergences). Let \( u_\varepsilon \in L^\infty(I;H^1_0(\Omega)) \) be the unique weak solution of (1.8) under the same assumption as in Lemma 3.1. Then there exist a subsequence (\( \varepsilon_n \)) of (\( \varepsilon \)), \( u_0 \in L^\infty(I;H^1_0(\Omega)) \), \( w \in L^2(I;H^{-1}(\Omega)) \) and \( h \in L^2_{\text{loc}}((0,T];H^{-1}(\Omega)) \) such that, for any \( \sigma \in I \),

\[ u_\varepsilon \rightharpoonup u_0 \quad \text{weakly-}^* \quad \text{in} \quad L^\infty(I;H^1_0(\Omega)), \]

\[ \partial_t u_\varepsilon \rightharpoonup \partial_t u_0 \quad \text{weakly-}^* \quad \text{in} \quad L^\infty(I;L^2(\Omega)), \]

\[ \partial^2_{tt} u_\varepsilon + g(\frac{\cdot}{\varepsilon}) \partial_t u_\varepsilon \rightharpoonup w \quad \text{weakly in} \quad L^2(I;H^{-1}(\Omega)), \]

\[ t\varepsilon_n^{-r} \partial_t u_\varepsilon \rightharpoonup h \quad \text{weakly in} \quad L^2(I;H^{-1}(\Omega)) \quad \text{if} \quad C_* \neq 0, \]

\[ u_\varepsilon \rightharpoonup u_0 \quad \text{strongly in} \quad C(\overline{T};L^2(\Omega)), \]

\[ \text{weakly in} \quad C(\overline{T};H^{-1}(\Omega)) \quad \text{if} \quad C_* = 0, \]

\[ \text{strongly in} \quad C(\overline{T};H^{-1}(\Omega)) \quad \text{if} \quad C_* 
eq 0, \]

\[ \sqrt{\varepsilon} \partial_t u_\varepsilon \rightarrow 0 \quad \text{strongly in} \quad L^2(\Omega \times I) \quad \text{if} \quad C_* = 0. \]

In particular, if \( C_* \neq 0 \), then \( \partial_t u_0(\cdot,t) \equiv 0 \) for a.e. \( t \in I \), and hence, \( u_0 \) is independent of \( t \in I \), i.e., \( u_0 = u_0(x) \). Furthermore, there exists \( w_1 \in L^2(\Omega \times I;H^1_{\text{per}}(\Box \times J)/\mathbb{R}) \) such that

\[ \partial_t u_\varepsilon \rightharpoonup \partial_t w_1 \quad \text{in} \quad L^2(\Omega \times I \times \Box \times J), \]

\[ a(t,\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon \rightharpoonup a(t,y)(\nabla u_0 + \nabla y w_1) \quad \text{in} \quad [L^2(\Omega \times I \times \Box \times J)]^N. \]

Thus it holds that

\[ a(t,\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon \rightarrow \langle a(t,\cdot)(\nabla u_0 + \nabla y w_1) \rangle_{y,s} \quad \text{weakly in} \quad [L^2(\Omega \times I)]^N, \]

where

\[ \langle a(t,\cdot)(\nabla u_0(x,t) + \nabla y w_1(x,t,\cdot)) \rangle_{y,s} = \int_{\Box} a(t,y)(\nabla u_0(x,t) + \nabla y w_1(x,t,y,\cdot)) \, dy. \]

**Proof.** Thanks to Lemma 3.1 we readily obtain (3.8)-(3.12). Furthermore, from (i) and (ii) of Lemma 3.1 the Ascoli-Arzelà theorem yields (3.13). In the same way, (3.14) also
holds true by (ii) and (v) of Lemma 3.1 As for (3.15), noting by (iii) of Lemma 3.1 that
$$\lim_{\varepsilon \to 0^+} \| \sqrt{\varepsilon} \partial_t u_{\varepsilon} \|_{L^2(\Omega \times I)}^2 \leq \lim_{\varepsilon \to 0^+} C \varepsilon r = 0,$$
we obtain (3.15). Thus \( u_0 = u_0(x) \), provided that \( C_* \neq 0 \). We finally show (3.16), (3.17) and (3.18). All the assumptions of Theorem 2.7 can be checked by (i) and (ii) of Lemma 3.1 Hence (3.16) holds true. Moreover, note that, for any \( \Psi \in [L^2(\Omega^2)]^N \), \( \Psi \) and \( a(t, y) \Psi \) are admissible test functions in \([L^2(\Omega^2)]^N \) (see 22 Theorems 2 and 4 for details) and define \( \Xi \in [L^2(\Omega^2)]^N \) by
$$a(t, \frac{\varepsilon}{\varepsilon_n}) \nabla u_{\varepsilon_n} \frac{2}{\varepsilon_n} \Xi \quad \text{in} \quad [L^2(\Omega^2)]^N.$$ 
Then, Theorem 2.7 yields that
$$\int_0^T \int_\Omega \int_0^1 \Xi(x, t, y, s) \cdot \Psi(x, t, y, s) \, dZ = \lim_{\varepsilon_n \to 0^+} \int_0^T \int_\Omega \int_0^1 \nabla u_{\varepsilon_n}(x, t) \cdot \left[ u(t, \frac{\varepsilon_n}{\varepsilon_n}) \Psi(x, t, \frac{\varepsilon_n}{\varepsilon_n}) \right] \, dx \, dt$$
$$= \int_0^T \int_\Omega \int_0^1 \left( \nabla u_0(x, t) + \nabla_y w_1(x, t, y, s) \right) \cdot \left[ u(t, \frac{\varepsilon_n}{\varepsilon_n}) \Psi(x, t, \frac{\varepsilon_n}{\varepsilon_n}) \right] \, dx \, dt,$$
which implies (3.17), and hence, (i) of Remark 2.5 yields (3.18). This completes the proof.

4. PROOF OF THEOREM 1.5

We first derive the homogenized equation by setting
$$j_{\text{hom}}(x, t) := \left\langle a(t, \cdot) \left( \nabla u_0(x, t) + \nabla_y w_1(x, t, \cdot, \cdot) \right) \right\rangle_{y, s}.$$ 
Recalling (3.10) and (3.18), we observe that, for all \( \phi \in H^1_0(\Omega) \) and \( \psi \in C_c^\infty(I) \),
$$\int_0^T \langle f(t), \phi \rangle_{H^1_0(\Omega)} \psi(t) \, dt$$
$$= \lim_{\varepsilon_n \to 0^+} \int_0^T \langle f_{\varepsilon_n}(t), \phi \rangle_{H^1_0(\Omega)} \psi(t) \, dt$$
$$= \lim_{\varepsilon_n \to 0^+} \int_0^T \left[ \langle \partial_t^2 \varepsilon_n(t) + g(\frac{\varepsilon_n}{\varepsilon_n}) \partial_t u_{\varepsilon_n}(t), \phi \rangle_{H^1_0(\Omega)} + \left( a(t, \frac{\varepsilon_n}{\varepsilon_n}) \nabla u_{\varepsilon_n}(t), \nabla \phi \rangle_{L^2(\Omega)} \right) \psi(t) \, dt$$
$$= \lim_{\varepsilon_n \to 0^+} \int_0^T \left[ \langle w, \phi \rangle_{H^1_0(\Omega)} + \langle j_{\text{hom}}(t), \nabla \phi \rangle_{L^2(\Omega)} \right) \psi(t) \, dt.$$ 
Here \( w \) can be regarded as
$$w = \partial_t^2 u_0 + \langle g_{\text{per}} \rangle_{s} \partial_t u_0 + C_* h.$$ 
Actually, due to \( \psi \in C_c^\infty(\Omega) \), this follows from (3.12), (3.14) and Proposition 2.2. Hence, by the arbitrariness of \( \psi \in C_c^\infty(I) \), \( u_0 \) turns out to be a weak solution to
$$\begin{cases} \partial_t^2 u_0 - \text{div} j_{\text{hom}} + \langle g_{\text{per}} \rangle_{s} \partial_t u_0 + C_* h = f & \text{in} \quad \Omega \times I, \\ u_0 |_{\partial \Omega} = 0, \quad u_0 |_{t=0} = v_0, \quad \partial_t u_0 |_{t=0} = \hat{v}_1. \end{cases}$$
where

\[ \tilde{v}^1 = \begin{cases} v^1 & \text{if } C_* = 0, \\ 0 & \text{if } C_* \neq 0. \end{cases} \]

Indeed, noting that

\[ \|u_0(0) - v^0\|_{L^2(\Omega)} \leq \|u_0(0) - u_{\epsilon_n}(0)\|_{L^2(\Omega)} + \|u_{\epsilon_n}(0) - v^0\|_{L^2(\Omega)} \]

\[ \leq \|u_0 - u_{\epsilon_n}\|_{C(\overline{\Omega}, L^2(\Omega))} + \|v^0 - v^0\|_{L^2(\Omega)}, \]

we see by (3.13) and (A) that

\[ \|u_0(0) - v^0\|_{L^2(\Omega)} \leq \limsup_{\epsilon_n \to 0^+} \|u_0 - u_{\epsilon_n}\|_{C(\overline{\Omega}, L^2(\Omega))} + \limsup_{\epsilon_n \to 0^+} \|v_{\epsilon_n}^0 - v^0\|_{L^2(\Omega)} = 0, \]

which implies that \( u_0(x, 0) = v^0 \). Thus \( u_0 \equiv v^0 \) by \( \partial_t u_0 \equiv 0 \), provided that for \( C_* \neq 0 \).

To check \( \partial_t u_0(x, 0) = v^1(x) \) a.e. in \( \Omega \) for \( C_* = 0 \), let \( \psi \in C^\infty(I) \) be such that \( \psi(T) = 0 \) and \( \psi(0) = 1 \). Then we infer that, for all \( \phi \in C^\infty_c(\Omega) \),

\[ \int_\Omega v^1(x) \phi(x) \, dx \overset{(A)}{=} \lim_{\epsilon_n \to 0^+} \int_\Omega v_{\epsilon_n}^1(x) \phi(x) \, dx \]

\[ + \lim_{\epsilon_n \to 0^+} \int_0^T \left\langle \frac{\partial^2 u_{\epsilon_n}}{\partial x^2} (t) + g\left(\frac{\partial u_{\epsilon_n}}{\partial x}\right) \partial_t u_{\epsilon_n} (t), \phi \right\rangle_{H^1_0(\Omega)} \psi(t) \, dt \]

\[ + \int_0^T \int_\Omega \left[ a(t, \frac{\partial u_{\epsilon_n}}{\partial x}) \nabla u_{\epsilon_n} (x, t) \cdot \nabla \phi (x) \psi(t) - f_{\epsilon_n} (x, t) \phi (x) \psi(t) \right] \, dx \, dt \]

\[ = \int_0^T \int_\Omega \left[ -\partial_t u_0 (x, t) \phi (x) \partial_t \psi (t) + g(\frac{\partial u_0}{\partial x}) \partial_t u_0 (x, t) \phi (x) \psi(t) \right. \]

\[ + a(t, \frac{\partial u_0}{\partial x}) \nabla u_0 (x, t) \cdot \nabla \phi (x) \psi(t) - f_0 (x, t) \phi (x) \psi(t) \] \[ + j_{\text{hom}} (x, t) \cdot \nabla \phi (x) \psi(t) - f(x, t) \phi (x) \psi(t) \] \[ \left. \int_\Omega \partial_t u_0 (x, 0) \phi (x) \, dx, \right] \]

which together with the arbitrariness of \( \phi \in C^\infty_c(\Omega) \) yields that \( \partial_t u_0 (x, 0) = v^1(x) \) a.e. in \( \Omega \) for \( C_* = 0 \).

The rest of the proof is to show that

\[ j_{\text{hom}} = a_{\text{hom}}(t) \nabla u_0 (x, t). \]

Here \( a_{\text{hom}}(t) \) is the homogenized matrix defined by (3.19). Thus it suffices to prove (1.16), that is,

\[ \langle w_1 \rangle_s = u_1 := \sum_{k=1}^{N} \partial_{x_k} u_0(x, t) \Phi_k(t, y), \]

where \( \Phi_k \) is the corrector defined by either (1.17) or (1.18). Indeed, if (4.1) holds, then we derive that

\[ j_{\text{hom}} (x, t) = \langle a(t, \cdot) (\nabla u_0 (x, t) + \nabla y w_1 (x, t, \cdot)) \rangle_{y,s} \]
\[ \int_0^T a(t, y) \left( \nabla u_0(x, t) + \sum_{k=1}^N \partial_{x_k} u_0(x, t) \nabla \Phi_k(t, y) \right) dy = \sum_{k=1}^N \left( \int_0^T a(t, y) (\nabla \Phi_k(t, y) + \varepsilon_k) dy \right) \partial_{x_k} u_0(x, t) = a_{hom}(t) \nabla u_0(x, t), \]

which implies (4.3). Hence \( u_0 \) turns out to be a unique weak solution to (1.19). Indeed, this follows from the uniqueness of the corrector \( \Phi_k \) and the similar argument as in Theorem 1.4 if \( C_* = 0 \) and \( u_0 \equiv \psi^0 \) whenever \( C_* \neq 0 \). Thus we have

\[ u_* \to u_0 \text{ as } \varepsilon \to 0_+ \]

without taking any subsequence \((\varepsilon_n)\). Therefore, (1.11)-(1.15) hold by Lemma 3.4 and (1.1). Thus we get all the assertions.

In the rest of this section, we shall prove (1.4) for all \( 0 < r < +\infty \). To this end, we show the following

**Lemma 4.1.** Under the same assumption as in Theorem 1.5, it holds that

\[
\lim_{\varepsilon \to 0_+} \varepsilon^{1-r} \int_0^T \int_\Omega \left[ -\partial_t u_{\varepsilon_n}(x, t) \partial_x c \left( \frac{1}{\varepsilon_n} \right) + C_* t \partial_t u_{\varepsilon_n}(x, t) c \left( \frac{1}{\varepsilon_n} \right) \right] \phi(x) b \left( \frac{x}{\varepsilon_n} \right) \psi(t) \, dx \, dt
+C_* \cdot \lim_{\varepsilon \to 0_+} \varepsilon^{1-r} \int_0^T \int_\Omega a(t, \frac{x}{\varepsilon_n}) \nabla u_{\varepsilon_n}(x, t) \cdot \phi(x) \nabla \Phi \left( \frac{x}{\varepsilon_n} \right) \psi(t) \, dx \, dt = 0
\]

for all \( \phi \in C^\infty_c(\Omega) \), \( b \in C^\infty_c(\square) \), \( \psi \in C^\infty_c(I) \) and \( c \in C^\infty_c(J) \).

**Proof.** Taking a difference of the weak forms for (1.8) and (1.2) and recalling \( w \) in (1.1), we observe that

\[
0 = \int_0^T \left\langle \partial^2_{tt} u_{\varepsilon_n}(t) + g \left( \frac{x}{\varepsilon_n} \right) \partial_t u_{\varepsilon_n}(t) - w(t), \phi \right\rangle_{H^1_0(\Omega)} \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dt
+C_* \cdot \int_0^T \int_\Omega \left( a(t, \frac{x}{\varepsilon_n}) \nabla u_{\varepsilon_n}(x, t) - j_{hom}(x, t) \right) \cdot \nabla \Phi \left( \frac{x}{\varepsilon_n} \right) \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dx \, dt
-
\int_0^T \int_\Omega (f_\varepsilon - f)(x, t) \phi(x) b \left( \frac{x}{\varepsilon_n} \right) \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dx \, dt
= \int_0^T \int_\Omega \partial_t u_{\varepsilon_n}(x, t) \phi(x) b \left( \frac{x}{\varepsilon_n} \right) \left( \partial_x \psi(t) c \left( \frac{1}{\varepsilon_n} \right) + \psi(t) \varepsilon^{-r} \partial_x c \left( \frac{1}{\varepsilon_n} \right) \right) \, dx \, dt
+C_* \cdot \int_0^T \int_\Omega \left( g_{per} \left( \frac{x}{\varepsilon_n} \right) + C_* \frac{1}{\varepsilon_n} \right) \partial_t u_{\varepsilon_n}(x, t) \phi(x) b \left( \frac{x}{\varepsilon_n} \right) \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dx \, dt
-
\int_0^T \left\langle w(t), \phi \right\rangle_{H^1_0(\Omega)} \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dt
+C_* \cdot \int_0^T \int_\Omega (a(t, \frac{x}{\varepsilon_n}) \nabla u_{\varepsilon_n}(x, t) - j_{hom}(x, t)) \cdot \nabla \phi \left( \frac{x}{\varepsilon_n} \right) b \left( \frac{x}{\varepsilon_n} \right) \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dx \, dt
-
\int_0^T \int_\Omega (f_\varepsilon - f)(x, t) \phi(x) b \left( \frac{x}{\varepsilon_n} \right) \psi(t) c \left( \frac{1}{\varepsilon_n} \right) \, dx \, dt.
\]
Multiplying both sides by $\varepsilon$, we conclude that

\begin{equation}
\varepsilon_n (A)
\end{equation}

Lemma 4.2. For any $\phi \in C_c^\infty(\Omega)$, $\psi \in C_c^\infty(I)$ and $c \in C_0^\infty(I)$ to show (4.4).

Employing Lemma 3.4 and Corollary 2.8 we shall apply (4.5) for any $\phi \in C_c^\infty(\Omega)$, $b \in C_0^\infty(\Box)/\mathbb{R}$, $\psi \in C_c^\infty(I)$ and $c \in C_0^\infty(I)$ holds.

Lemma 4.2. For any $0 < r \leq 1$, (4.4) holds.

Proof. Set $c(s) \equiv 1$ in (1.17). By (3.16), the first term in (4.5) is zero. Thanks to (3.17), we deduce by (4.3) that

\begin{equation}
\int_0^T \int_0^1 \int_0^\Box a(t,y) \left( \nabla u_0(x,t) + \nabla_y w_1(x,t,y,s) \right) \cdot \phi(x) \nabla_y b(y) \psi(t)(s) dZ = 0.
\end{equation}

From the arbitrariness of $\phi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(I)$, we get

\begin{equation}
\int_\Box a(t,y) \left( \nabla u_0(x,t) + \nabla_y w_1(x,t,y) \right) \cdot \nabla_y b(y) dy = 0 \quad \text{a.e. in } \Omega \times I.
\end{equation}

Recalling that

\begin{equation}
u_1 = \sum_{k=1}^N \partial_{x_k} u_0(x,t) \Phi_k(y),
\end{equation}

where $\Phi_k$ is the solution to (1.17), we check that

\begin{equation}
\int_\Box a(t,y) \left( \nabla u_0(x,t) + \nabla_y u_1(x,t,y) \right) \cdot \nabla_y b(y) dy
\end{equation}

\begin{equation}
= \sum_{k=1}^N \partial_{x_k} u_0(x,t) \int_\Box a(t,y) \left( \nabla_y \Phi_k(y) + e_k \right) \cdot \nabla_y b(y) dy \equiv 0.
\end{equation}
Hence (4.7) with \( \langle w_1 \rangle_s \) replaced by \( u_1(x,t,y) \) holds. Setting \( b = (\langle w_1 \rangle_s - u_1)(x,t,\cdot) \) and subtracting (4.7) for \( \langle w_1 \rangle_y \) and \( u_1 \), we deduce by the Poincaré-Wirtinger inequality that

\[
0 = \int_\square a(t,y) \nabla_y (\langle w_1 \rangle_s - u_1)(x,t,y) \cdot \nabla_y ((\langle w_1 \rangle_s - u_1)(x,t,y)) \, dy
\]

\[
\geq \lambda \| \nabla_y ((\langle w_1 \rangle_s - u_1)(x,t)) \|_{L^2(\square)}^2 \geq \frac{\lambda}{C_1} \| (\langle w_1 \rangle_s - u_1)(x,t) \|_{L^2(\square)}^2,
\]

which implies that

\[
\langle \partial_t \frac{1}{s} \rangle \quad \text{(4.8)}
\]

which implies that

\[
\langle \partial_t \frac{1}{s} \rangle \quad \text{(4.8)}
\]

Thus (3.17) and (4.5) yield (4.7) for the case \( 1 < r < \infty \). This completes the proof.

Before discussing the case \( r > 1 \), we claim that

\[
w_1 = w_1(x,t,y) \quad \text{for all } r \in (1, +\infty).
\]

Indeed, multiplying both sides by \( \varepsilon_n^{-2(1-r)} \) in (4.6), we see that the third term in (4.6) is zero as \( \varepsilon_n \to 0_+ \) due to (3.15), and then, one can derive by Lemma 3.1 and Corollary 2.8 that

\[
0 = - \lim_{\varepsilon_n \to 0_+} \varepsilon_n^{r-1} \int_0^T \int_\Omega \partial_t u_{\varepsilon_n}(x,t) \phi(x)b(\varepsilon_n) \psi(t) \partial_s c(\varepsilon_n) \, dx \, dt
\]

\[
= \lim_{\varepsilon_n \to 0_+} \varepsilon_n^{r-1} \int_0^T \int_\Omega u_{\varepsilon_n}(x,t) \phi(x)b(\varepsilon_n) \partial_t \psi(t) \partial_s c(\varepsilon_n) \, dx \, dt
\]

\[
+ \lim_{\varepsilon_n \to 0_+} \int_0^T \int_\Omega \frac{u_{\varepsilon_n}}{\varepsilon_n}(x,t) \phi(x)b(\varepsilon_n) \psi(t) \partial_{ss} c(\varepsilon_n) \, dx \, dt
\]

\[
= \int_0^T \int_\Omega \int_0^1 \int_\square w_1(x,t,y,s) \phi(x)b(y) \psi(t) \partial_{ss} c(s) \, dZ,
\]

which implies that \( \partial_s w_1 \) is independent of \( s \in J \) and so is \( w_1 \) by \( J \)-periodicity. Thus \( w_1 \in L^2(\Omega \times I; H^1_{\text{per}}(\square)/\mathbb{R}) \) for all \( r > 1 \).

We choose \( c(s) \equiv 1 \) in (4.3) below. Then one can get the following

**Lemma 4.3.** For any \( 1 < r \leq 2 \), (4.3) holds.

**Proof.** As for the periodic case, (4.7) follows from (4.5) and (3.17) with \( c(s) \equiv 1 \) and \( C_* = 0 \). Thus the assertion is obtained as in the proof of Lemma 4.2. We next consider the quasi-periodic case. Applying Corollary 2.8 to (4.5) with \( c(s) \equiv 1 \), we deduce that

\[
\lim_{\varepsilon_n \to 0_+} \varepsilon_n^{1-r} \int_0^T \int_\Omega C_* \partial_t u_{\varepsilon_n}(x,t) \phi(x)b(\varepsilon_n) \psi(t) \, dx \, dt
\]

\[
= \lim_{\varepsilon_n \to 0_+} \varepsilon_n^{r-2} \int_0^T \int_\Omega C_* \frac{u_{\varepsilon_n}}{\varepsilon_n}(x,t) \phi(x)b(\varepsilon_n) \partial_t \psi(t) \, dx \, dt
\]

\[
= \begin{cases} 
0 & \text{if } 1 < r < 2, \\
- \int_0^T \int_\Omega \int_\square C_* w_1(x,t,y) \phi(x)b(y) \partial_t (\psi(t)) \, dy \, dx \, dt & \text{if } r = 2.
\end{cases}
\]

Thus (3.17) and (4.5) yield (4.7) for the case \( 1 < r < 2 \).

On the other hand, for \( r = 2 \), we find by (3.17), (4.5) and (4.8) that

\[
\int_0^T \int_\Omega \langle C_* \partial_t w_1(x,t,\cdot), b \rangle_{H^1_{\text{per}}(\square)/\mathbb{R}} \phi(x) \psi(t) \, dx \, dt
\]
Here we used the fact that
\[ \psi \quad \text{is } \mathcal{C}^2 \quad \text{in } \Omega \quad \text{and} \quad \mathcal{H}^1_{\text{per}}(\Omega) \]
and that
\[ \mathcal{H}^1_{\text{per}}(\Omega) = (\mathcal{H}^1_{\text{per}}(\Omega))^* \]
for a.e. \( (x, t) \in \Omega \times I \). Thus (4.9) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
(4.10) follows, and then the arbitrariness of \( \phi \in \mathcal{C}^\infty(\Omega) \) yields that
\[ \mathcal{C}_s T \int_0^T \int_\Omega \phi(x) \, d\nu \, dt = \frac{C_* T}{2} \int_0^T \int_{\partial \Omega} \left| \nabla \cdot \left( -\nabla \psi + \psi \nabla \mathcal{H}_\mathcal{C} \right) \right| \, d\nu \, dt \]
which implies \( w_1 = u_1 \). Furthermore, (4.11) yields the uniqueness of (1.18), and moreover, if \( a = a(y) \), then \( \Phi_k \) is the solution to (1.17) due to the uniqueness of (1.18). This completes the proof. \( \square \)

We finally discuss the case where \( 2 < r < +\infty \).

**Lemma 4.4.** For any \( 2 < r < +\infty \), (4.4) holds.

**Proof.** Due to (3.17) and (4.5), it suffices to check

\[
\lim_{\varepsilon_n \to 0^+} \varepsilon_n^{1-r} \int_0^T \int_\Omega C_s t \partial_t u_{\varepsilon_n}(x,t) \phi(x) b(\frac{x}{\varepsilon_n}) \psi(t) c(\frac{t}{\varepsilon_n}) \, dx \, dt = 0
\]

with \( c(s) \equiv 1 \). It is clear if \( C_s = 0 \). If \( C_s \neq 0 \), since \( (t \varepsilon^{-r} \partial_t u_{\varepsilon}) \) is bounded in \( L^2(\Omega \times I) \) by (v) of Lemma 3.3, we find by \( \psi \in C^\infty_c(I) \) that

\[
\varepsilon_n^{1-r} \left| \int_0^T \int_\Omega C_s t \partial_t u_{\varepsilon_n}(x,t) \phi(x) b(\frac{x}{\varepsilon_n}) \psi(t) \, dx \, dt \right| \leq C \varepsilon_n \| t \varepsilon^{-r} \partial_t u_{\varepsilon_n} \psi \|_{L^1(\Omega \times I)} \to 0 \text{ as } \varepsilon_n \to 0^+,
\]

which completes the proof. \( \square \)

By Lemmas 4.2, 4.3 and 4.4 we obtain (4.4) for all \( r \in (0, +\infty) \). Therefore, Theorem 1.7 is proved.

5. PROOF OF PROPOSITION 1.7

We consider the case of \( C_s \neq 0 \) and \( r = 2 \) only (see [11, Proposition 1.8] for the proof of the other case). We first prove (i). For each \( \xi \in \mathbb{R}^N \), there exists a unique solution \( \Phi_\xi \in L^2(I; H^1_{\text{per}}(\square)/\mathbb{R}) \) to

\[
C_s t \partial_t \Phi_\xi - \text{div}_y \left[ a(t,y)(\nabla_y \Phi_\xi + \xi) \right] = 0 \quad \text{in } I \times \square.
\]

Using (1.20) and (1.9), we derive that, for a.e. \( t \in I \),

\[
a_{\text{hom}}(t) \xi \cdot \xi = \int_{\square} \left[ a(t,y)(\nabla_y \Phi_\xi(t,y) + \xi) \cdot \xi \right] \, dy
\]

\[
\geq \int_{\square} a(t,y)(\nabla_y \Phi_\xi(t,y) + \xi) \cdot (\nabla_y \Phi_\xi(t,y) + \xi) \, dy + C_s t \frac{d}{dt} \| \Phi_\xi(t) \|^2_{L^2(\square)}
\]

\[
\geq \lambda \int_{\square} |\xi + \nabla_y \Phi_\xi(t,y)|^2 \, dy + C_s t \frac{d}{dt} \| \Phi_\xi(t) \|^2_{L^2(\square)}
\]

\[
\geq \lambda \left( |\xi|^2 + \| \nabla_y \Phi_\xi(t) \|^2_{L^2(\square)} \right) + C_s t \frac{d}{dt} \| \Phi_\xi(t) \|^2_{L^2(\square)}.
\]

Here we used the fact that \( \langle \nabla_y \Phi_\xi(t, \cdot) \rangle_y = 0 \). In an analogous way, we get

\[
a_{\text{hom}}(t) \xi \cdot \xi \leq \left( |\xi|^2 + \| \nabla_y \Phi_\xi(t) \|^2_{L^2(\square)} \right) + C_s t \frac{d}{dt} \| \Phi_\xi(t) \|^2_{L^2(\square)}.
\]

We next prove (ii). Let \( \Phi_j \) be the unique solution to (5.1) with \( \xi \) replaced by \( e_j \). Then we observe from the symmetry of \( a(t,y) \) that, for a.e. \( t \in I \),

\[
\int_{\square} a(t,y)(\nabla_y \Phi_k(t,y) + e_k) \cdot e_j \, dy
\]

\[
= \int_{\square} a(t,y)(\nabla_y \Phi_k(t,y) + e_k) \cdot e_j \, dy
\]

\[
= \int_{\square} a(t,y)(\nabla_y \Phi_k(t,y) + e_k) \cdot e_j \, dy
\]

\[
= \int_{\square} a(t,y)(\nabla_y \Phi_k(t,y) + e_k) \cdot e_j \, dy
\]

\[
= \int_{\square} a(t,y)(\nabla_y \Phi_k(t,y) + e_k) \cdot e_j \, dy
\]
the symmetry of the Hessian that, for a.e. \( t \)

\[
\text{However, for the second term in the last line, it follows that}
\]

\[
\int \nabla \Phi_k(t, y) e_j \cdot \nabla \Phi_j(t, y) \, dy + \left\langle C_s t \partial_t \Phi_k(t), \Phi_j(t) \right\rangle_{H^1_{\text{per}}(\mathbb{R}^d)} \neq 0 \quad \text{for } j \neq k,
\]

which completes the proof.

**Remark 5.1.** The skew symmetric part of \( a_{\text{hom}}(t) \) is defined by

\[
a_{\text{hom}}^\text{skew}(t) e_j \cdot e_k = \left( \frac{a_{\text{hom}}(t) - a_{\text{hom}}(t)}{2} \right) e_j \cdot e_k
\]

\[
= -\frac{1}{2} C_s t \left( -\left\langle \partial_t \Phi_j(t), \Phi_k(t) \right\rangle_{H^1_{\text{per}}(\mathbb{R}^d)} + \left\langle \partial_t \Phi_k(t), \Phi_j(t) \right\rangle_{H^1_{\text{per}}(\mathbb{R}^d)} \right).
\]

Then we note that the skew-symmetric part of \( a_{\text{hom}}(t) \) makes no contribution to the divergence for a.e. \( t \in I \). Assume that \( u_0 \) and \( \Phi_k \) are smooth enough. Then we find by the symmetry of the Hessian that, for a.e. \( t \in I \),

\[
\text{div}(a_{\text{hom}}^\text{skew}(t) \nabla u_0) = \frac{1}{2} \text{div}(a_{\text{hom}}^\text{skew}(t) \nabla u_0)
\]

\[
- \frac{1}{4} \sum_{j,k=1}^N \left[ C_s t \int \Phi_j(t, y) \Phi_k(t, y) \, dy \right] \partial^2_{y_j y_k} u_0
\]

\[
= \frac{1}{2} \text{div}(a_{\text{hom}}^\text{skew}(t) \nabla u_0)
\]

\[
+ \frac{1}{4} \sum_{j,k=1}^N \left[ C_s t \int \Phi_j(t, y) \Phi_k(t, y) \, dy \right] \partial^2_{y_j y_k} u_0
\]

\[
= \frac{1}{2} \text{div}(a_{\text{hom}}^\text{skew}(t) \nabla u_0) - \frac{1}{2} \text{div}(a_{\text{hom}}^\text{skew}(t) \nabla u_0) = 0,
\]

which yields the assertion.

## 6. Proof of a Corrector Result

This section is devoted to proving Theorem 1.9 and Corollary 1.10.

### 6.1. Proof of Theorem 1.9

Let \( a_\varepsilon = a(\varepsilon \cdot) \) for the sake of simplicity. To show Theorem 1.9 we observe from (1.9) that

\[
\lambda \int_0^T \int_\Omega \left| \nabla u_\varepsilon - (\nabla u_0 + \nabla_y u_1(x, t, \varepsilon)) \right|^2 \, dx \, dt
\]
In what follows, we shall estimate these three terms, $I_1^\varepsilon$, $I_2^\varepsilon$, and $I_3^\varepsilon$ for all $r \in (0, +\infty)$.

We first estimate $I_1^\varepsilon$.

**Lemma 6.1.** Under the same assumption as in Theorem 1.5, it holds that

$$
\lim_{\varepsilon \to 0_+} \int_0^T \int_\Omega a_\varepsilon \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \, dx dt
= \int_0^T \int_\Omega \int_\square a(y)(\nabla u_0(x,t) + \nabla y_1(x,t,y)) \cdot (\nabla u_0(x,t) + \nabla y_1(x,t,y)) \, dy dx dt.
$$

**Proof.** From (A), (3.13) and (iii) of Lemma 3.3 it follows that

$$
\int_0^T \int_\Omega a_\varepsilon \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) \, dx dt
= \int_0^T \int_\Omega \int_\square a(y)(\nabla u_0(x,t) + \nabla y_1(x,t,y)) \cdot (\nabla u_0(x,t) + \nabla y_1(x,t,y)) \, dy dx dt
$$

as $\varepsilon \to 0_+$. Here we used the fact that $\Phi_k$ is the unique solution to (1.17) due to $a = a(y)$. This completes the proof. \hfill \Box

Before discussing the limit of $I_2^\varepsilon$, recall that $u_1$ is written by $u_1 = \sum_{k=1}^N \partial_{x_k} u_0 \Phi_k$. Due to the smoothness of $a(y)$, noting that $a(y)(\nabla u_0 + \nabla y_1)$ belongs to $[L^2(\Omega \times I; C_{\text{per}}(\square))]^N$, we see by [22, Theorem 4] that it is an admissible test function in $[L^2(\Omega \times I \times \square)]^N$. Hence (3.17) yields that

$$
I_2^\varepsilon \to \int_0^T \int_\Omega \int_\square a(y)(\nabla u_0(x,t) + \nabla y_1(x,t,y)) \cdot (\nabla u_0(x,t) + \nabla y_1(x,t,y)) \, dy dx dt
$$

as $\varepsilon \to 0_+$.

We finally estimate $I_3^\varepsilon$. Thanks to $\nabla y \Phi_k \in C_{\text{per}}(\square)$, one can derive by Proposition 2.2 that

$$
I_3^\varepsilon = \int_0^T \int_\Omega \left[ a_\varepsilon \nabla u_0 \cdot \nabla u_0 + 2a_\varepsilon \nabla y_1(x,t,\frac{x}{\varepsilon}) \cdot \nabla u_0 + a_\varepsilon \nabla y_1 \cdot \nabla y_1(x,t,\frac{x}{\varepsilon}) \right] \, dx dt
$$

as $\varepsilon \to 0_+$.\hfill \Box
Consequently, with the aid of Lemma 6.1, 6.1 and 6.2, we obtain
\[ \lim_{\varepsilon \to 0} (I_1^\varepsilon - 2I_2^\varepsilon + I_3^\varepsilon) = 0, \]
which completes the proof.

6.2. **Proof of Corollary 1.10.** The strategy of the proof of Corollary 1.10 is the same as Theorem 1.9 and it suffices to show the following

**Lemma 6.2.** Under the same assumption as in Corollary 1.10, it holds that
\[
\limsup_{\varepsilon \to 0} \int_0^T \int_{\Omega} a(t, \xi) \nabla u_\varepsilon(x, t) \cdot \nabla u_\varepsilon(x, t) \, dx \, dt
\leq \int_0^T \int_{\Omega} a(t, y) (\nabla u_0(x) + \nabla_y u_1(x, t, y)) \cdot (\nabla u_0(x) + \nabla_y u_1(x, t, y)) \, dy \, dx \, dt.
\]

**Proof.** Let \( a_\varepsilon = a(t, \frac{\xi}{\varepsilon}) \) for simplicity. Define \( E^\varepsilon(u_\varepsilon(t)) \) by
\[
E^\varepsilon(u_\varepsilon(t)) = \frac{1}{2} \| \partial_\rho u_\varepsilon(t) \|^2_{L^2(\Omega)} + \frac{1}{2} \int_{\Omega} a_\varepsilon \nabla u_\varepsilon(x, t) \cdot \nabla u_\varepsilon(x, t) \, dx + \int_0^t \int_{\Omega} g(\frac{\xi}{\varepsilon}) \| \partial_\rho u_\varepsilon(\rho) \|^2_{L^2(\Omega)} \, d\rho - \frac{1}{2} \int_0^t \int_{\Omega} \partial_\rho a(\rho, \frac{\xi}{\varepsilon}) \nabla u_\varepsilon(x, \rho) \cdot \nabla u_\varepsilon(x, \rho) \, dx \, d\rho
\]
for \( t \in I \). From (i), (ii) and (iii) of Lemma 3.1 and (A), we have \( |E^\varepsilon(u_\varepsilon(t))| \leq C \) for all \( t \in \overline{I} \). Moreover, we derive by Remark 3.2 that, for any \( t \in I \) and \( h \in (0, T - t) \),
\[
|E^\varepsilon(u_\varepsilon(t + h)) - E^\varepsilon(u_\varepsilon(t))|
\leq \int_t^{t+h} \left| \langle \partial_{\rho^2} u_\varepsilon(\rho), \partial_\rho u_\varepsilon(\rho) \rangle_{H^2(\Omega)} + \left( a_\varepsilon \nabla u_\varepsilon(\rho), \nabla \partial_\rho u_\varepsilon(\rho) \right)_{L^2(\Omega)} + g(\frac{\xi}{\varepsilon}) \| \partial_\rho u_\varepsilon(\rho) \|^2_{L^2(\Omega)} \right| \, d\rho
\leq \int_t^{t+h} \| f_\varepsilon(\rho) \|_{L^2(\Omega)} \| \partial_\rho u_\varepsilon(\rho) \|_{L^2(\Omega)} \, d\rho \leq \| f_\varepsilon \|_{L^2(\Omega \times I)} \| \partial_\rho u_\varepsilon \|_{L^2(\Omega \times I)} \| \partial_\rho u_\varepsilon \|_{L^2(\Omega \times I)} \| \partial_\rho u_\varepsilon \|_{L^2(\Omega)},
\]
which along with (ii) of Lemma 3.1 and the boundedness of \((f_\varepsilon)\) in \( L^2(\Omega \times I) \) yields the equicontinuous of \( t \mapsto E^\varepsilon(u_\varepsilon(t)) \) on \( \overline{I} \). Then Ascoli-Arzelà’s theorem ensures that there exists \( \xi \in C(\overline{I}) \) such that
\[
E^\varepsilon(u_\varepsilon) \to \xi \quad \text{strongly in } C(\overline{I}).
\]
Furthermore, it holds that, for any \( t \in I \),
\[
E^\varepsilon(u_\varepsilon(t))
= \int_0^t \left( \langle \partial_{\rho^2} u_\varepsilon(\rho), \partial_\rho u_\varepsilon(\rho) \rangle_{H^2(\Omega)} + \left( a_\varepsilon \nabla u_\varepsilon(\rho), \nabla \partial_\rho u_\varepsilon(\rho) \right)_{L^2(\Omega)} + g(\frac{\xi}{\varepsilon}) \| \partial_\rho u_\varepsilon(\rho) \|^2_{L^2(\Omega)} \right) \, d\rho
+ \frac{1}{2} \| v_\varepsilon^0 \|^2_{L^2(\Omega)} + \frac{1}{2} \int_{\Omega} a(0, \frac{\xi}{\varepsilon}) \nabla v_\varepsilon^0(x) \cdot \nabla v_\varepsilon^0(x) \, dx
\]
and, by (1.23), (1.24) and (1.25), \( \xi(t) \) is identified with
\[
\xi(t) = \frac{1}{2} \int_{\Omega} a_{\text{hom}}(0) \nabla v^0(x) \cdot \nabla v^0(x) \, dx.
\]
Hence defining \( J(t) \) by

\[
J(t) := \int_0^t \int_\Omega a(\rho, y) \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) \cdot \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) dxd\rho
\]

and noting by [1,22] and [3, Example 1 in Section 7] that

\[
\lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega a_\varepsilon \nabla u_\varepsilon(x, t) \cdot \nabla u_\varepsilon(x, t) \, dxdt \\
\leq \lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega (2E^\varepsilon(u_\varepsilon)(t)) \, dt + \lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega \int_\Omega \rho \partial a(\rho, \frac{x}{\varepsilon}) \nabla u_\varepsilon(x, \rho) \cdot \nabla u_\varepsilon(x, \rho) \, dxd\rho dt
\]

\[
\leq T \int_\Omega a(0, y) \left( \nabla u_0(x) + \nabla y u_1(x, 0, y) \right) \cdot \nabla u_0(x) \, dydx + \int_0^T J(t) \, dt.
\]

We estimate the second term below. Since \( u_1(x, \cdot, \cdot) \) is smooth in \( \square \times (\eta, T) \) for all \( \eta \in I \) due to the smoothness of \( a(t, y) \), it holds that

\[
\int_\eta^T \int_\eta^T \int_\square \partial a(\rho, y) \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) \cdot \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) dxd\rho dt
\]

\[
= \int_\eta^T \int_\Omega a(t, y) \left( \nabla u_0(x) + \nabla y u_1(x, t, y) \right) \cdot \left( \nabla u_0(x) + \nabla y u_1(x, t, y) \right) dxdy dt
\]

\[
- (T - \eta) \int_\Omega \int_\Omega a(\eta, y) \left( \nabla u_0(x) + \nabla y u_1(x, \eta, y) \right) \cdot \left( \nabla u_0(x) + \nabla y u_1(x, \eta, y) \right) dydx
\]

\[
- 2 \int_\eta^T \int_\Omega \int_\Omega \rho \partial a(\rho, y) \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) \cdot \nabla y \partial \rho u_1(x, \rho, y) \, dxd\rho dt
\]

\[
=: J_1^\eta + J_2^\eta + J_3^\eta.
\]

Then one can verify that

\[
J_1^\eta \to \int_\Omega \int_\square \int_\eta^T a(t, y) \left( \nabla u_0(x) + \nabla y u_1(x, t, y) \right) \cdot \left( \nabla u_0(x) + \nabla y u_1(x, t, y) \right) dxdy dt
\]

and

\[
J_2^\eta \to -T \int_\Omega \int_\square \int_\eta^\Omega a(0, y) \left( \nabla u_0(x) + \nabla y u_1(x, 0, y) \right) \cdot \nabla u_0(x) \, dydx
\]

as \( \eta \to 0^+ \). Here we used the fact [1,18] at \( t = 0 \) in (6.4). Furthermore, if \( r \neq 2 \), \( J_3^\eta = 0 \) readily follows due to \( u_1 = u_1(x, y) \). On the other hand, if \( r = 2 \), [1,18] and [1,26] yield that

\[
J_3^\eta = 2 \int_\eta^T \int_\Omega \int_\square \partial a(\rho, y) \left( \nabla u_0(x) + \nabla y u_1(x, \rho, y) \right) \cdot \nabla y \partial \rho u_1(x, \rho, y) \, dxd\rho dt
\]

\[
\leq 2 \int_\eta^T \int_\Omega \int_\square a(t, y) \left( \nabla u_0(x) + \nabla y u_1(x, t, y) \right) \cdot \nabla y \partial \rho u_1(x, t, y) \, dxdy dt
\]
\[-2(T - \eta) \int_\Omega \int_\Omega a(\eta, y) \left( \nabla u_0(x) + \nabla_y u_1(x, \eta, y) \right) \cdot \nabla_y \partial_\rho u_1(x, \eta, y) \, dydx \]
\[-2 \int_\eta^T \int_\eta^t \int_\Omega \int_\Omega a(\rho, y) \left( \nabla u_0(x) + \nabla_y u_1(x, \rho, y) \right) \cdot \nabla_y \partial_{\rho \rho}^2 u_1(x, \rho, y) \, dydxd\rho dt \]
\[\leq -2 \int_\eta^T C_\ast t \| \partial_\rho u_1(t) \|^2_{L^2(\Omega \times \Box)} \, dt + 2(T - \eta) C_\ast \eta \| \partial_\rho u_1(\eta) \|^2_{L^2(\Omega \times \Box)} \]
\[+ \int_\eta^T \int_\eta^t C_\ast \frac{d}{d\rho} \| \partial_\rho u_1(\rho) \|^2_{L^2(\Omega \times \Box)} \, d\rho dt \]
\[\leq - \int_\eta^T C_\ast t \| \partial_\rho u_1(t) \|^2_{L^2(\Omega \times \Box)} \, dt + 2(T - \eta) C_\ast \eta \| \partial_\rho u_1(\eta) \|^2_{L^2(\Omega \times \Box)} \leq 0 \]
as $\eta \to 0_+$. Hence we conclude that
\[\int_0^T J(t) \, dt = \lim_{\eta \to 0_+} J_1^\eta + \lim_{\eta \to 0_+} J_2^\eta + \lim_{\eta \to 0_+} J_3^\eta \]
\[\leq \int_0^T \int_\Omega \int_\Omega a(t, y) \left( \nabla u_0(x) + \nabla_y u_1(x, t, y) \right) \cdot \left( \nabla u_0(x) + \nabla_y u_1(x, t, y) \right) \, dydxt \]
\[\quad - T \int_\Omega \int_\Omega a(0, y) \left( \nabla u_0(x) + \nabla_y u_1(x, 0, y) \right) \cdot \nabla u_0(x) \, dydx, \]
which completes the proof. \qed

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