Original Article

Time fractional analysis of electro-osmotic flow of Walters’s-B fluid with time-dependent temperature and concentration

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Abstract The present article applies the newly developed concept of Atangana-Baleanu time-fractional derivatives to the electro-osmotic magnetohydrodynamic (MHD) free convectional flow of Walters’-B fluid along with heat and mass distribution. The motion generates in the fluid is due to the oscillations of the vertical plate which is embedded in a porous medium. Employing the idea of Atangana-Baleanu time-fractional order derivatives the conventional model of the Walters’-B liquid is transfigured to the fractional model. With the help of the imposed initial and boundary conditions and by making use of the Laplace transform technique (LTT), analytical solutions are procured for concentration, velocity, and temperature. The impact of various physical parameters on the fluid flow is plotted graphically. It is worth seeing from the graphs of the velocity distribution that the greater values of Walters’-B fluid parameter $C$, cause a decrease in the velocity. Furthermore, it is likewise noted from the graphs that higher values of an electro-osmotic parameter $E_s$ cause a decline in velocity profile. This behavior of electro-osmotic on the velocity profile may also work in the process of separation of the fluids at the atomic level and can play a very important job for medication dischargers.

1. Introduction

Fluids which show non-linear behavior between the stress and rate of strain are labeled as non-Newtonian fluids. Non-Newtonian fluids have physiological and industrial

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1110-0168 © 2019 Faculty of Engineering, Alexandria University. Production and hosting by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).
importance. Many fluids like custards, shampoo, toothpaste, salt solution, starch suspensions, molten polymers paints, ketchup, and blood are non-Newtonian fluids because they show non-linearity in their behavior. Non-Newtonian liquids are vital as a result of their handy uses and applications in our daily life especially in industrial processing like biofluid dynamics, petroleum drilling and many more. The equations produced by the non-Newtonian liquids are naturally non-linear and because of this reason these equations are very enthralling and challenging to solve. Due to the complex and dynamic behavior of the non-Newtonian fluids scientists and analysts have proposed various sorts of mathematical models [1–3]. Non-Newtonian liquids have a subclass called viscoelastic liquids. Those fluids which display both the properties of elasticity and viscosity are named as viscoelastic fluids. Furthermore, viscoelastic properties may also increase or decrease the rate of transfer of heat, contingent on the kinematic attributes of the stream and also on the heading of heat transfer [4]. The model of viscoelastic fluids which has been extensively studied in numerous flow problems was the model developed by Walter’s-B [4]. Walter’s-B fluid is a non-Newtonian differential type of viscoelastic fluid. This model can simulate accurately several difficulties and complex tribological fluids, biotechnological and polymeric fluids. The study of the steady-state flow and transfer of heat rate of Walters-B liquid made on porous rotating disks in parallel plates is analyzed by Rath et al. [5]. They found the numerical solution for laminar free convectional and unsteady Walters'-B liquid using the Crank-Nicolson difference scheme. The authors also have taken in to account the transfer of mass and heat. Furthermore, they examined the profiles of concentration, velocity, and temperature to study the behavior of Schmidt number, Prandtl number, and Viscoelastic parameter. Apart from this they also found out local Sherwood number, Skin-friction, and Nusselt number. The accuracy of the numerical results was checked using graphical analysis. Investigation of the transient rotating hydro-magnetic Walter’s-B fluid regime using LTT is carried out by Nanousis [6]. The authors carried out the study of incompressible, unsteady and electrically-conducting Walters'-B liquid in a rotating medium. Moreover, they found the exact solutions for the velocity profile and for skin-friction using the method of LTT. Using the method of Crank-Nicolson, Sudhakar et al. [7] analyzed Walters-B liquid with the effect of thermophoresis. The authors studied the 2-dimensional incompressible, unsteady and laminar flow of viscoelastic fluid (Walters'-B). The problem is examined under the existence of thermophoresis. The unsteady dimensionless, non-linear and coupled problem is figuring out utilizing the Crank-Nicolson numerical method. The authors found that retardation occurs in concentration and in velocity field by enhancing the value of the thermophoretic parameter, but speed up the temperature profile. Furthermore, the effect of thermophoresis is also studied in case of Newtonian fluid. Khan et al. [8] examined the interaction of magnetic field in Walters-B liquid along with the mass and transfer of heat. The authors studied the phenomenon of mass and heat transfer using the free convection flow of Walter’s-B liquid in the presence of transverse magnetic field. A closed-form solution is found out for concentration, velocity and temperature distribution utilizing the method of Laplace transform. Furthermore, they reduce their work to some well-known results for the Newtonian fluids by taking $\Gamma \to 0$. In addition, the obtained numerical results were justified using the graphical method.

The transfer of mass and heat in fluids are of much importance in several chemical reactions and industries because they contain mixing mechanisms [9], coaxial mixtures [10], and milk processing [11]. The combined occurrence of mass and heat transfer comes into existence due to the combined effect of thermal bouncy diffusion and diffusion via chemical species, which performs a vital role in chemical engineering, geophysics, and aeronautics. The practical applications of this phenomenon at the industrial level can be seen in food processing, drying food and polymer production [12]. Therefore, the combined phenomena of mass and heat transfer gained a substantial amount of consideration of the researchers and scientists, to work on this numerically or analytically [13–15]. Furthermore, the implementations of magnetohydrodynamic in the field of engineering, astrophysics, and many other industrial areas can be found in [16,17]. Rajput and Kumar [18] explored the applications of MHD flow with the existence of diffusion. Ali et al. [19] studied the flow with natural convection on the inclined plate under the existence of the interfusing effect of heat and mass transfer. The authors found the analytical expressions for velocity, mass concentration and temperature profile using the mathematical tool LTT. They also found expressions for rate of heat transfer, skin friction and rate of mass transfer. The obtained solutions are justified graphically. Ahmad [20] investigated the MHD conducting natural convectional flow along with mass transfer under the existence of thermal radiation and diffusion.

The concept of fractional calculus developed in 1965 from a very interesting question asked by L’Hospital. L’Hospital asked from Leibniz that what would be the possible explanation of $d^{1/2}/dx^{1/2}$. From this question, the fruitful and meaningful full journey of fractional derivatives started. The history of fractional calculus is very rich because many great mathematicians and scientists worked on it and defined different fractional operators with significant properties. Rieman-Liouville, Caputo, Caputo-Kober, Caputo- Hadamard, and Caputo-Fabrizio developed different fractional derivatives models [21–25]. Yang et al. [26] studied the normalized sinc function without a singular kernel utilizing new local fractional derivative. LTT is used by the authors to find the exact solutions for heat diffusion. Furthermore, the authors presented the classical and fractional order results for the sake of comparison. These results are very significant and fruitful in the phenomenon of heat diffusion. The study of the new family of the local systems of PDEs is carried out by Yang et al. [27]. In which semilinear, non-linear local fractional PDEs are considered. Moreover, three different types of local fractional PDEs that are parabolic, hyperbolic and elliptic are discussed. Zhang and Yang [28] developed an analytical technique for solving local fractional, non-linear system of PDEs which arises in mathematical physics. Edeki and Akinlabi [29] reported approximate solutions for the system of time-fractional coupled Burger’s equations by making use of local fractional operators. The cause of producing different time-fractional derivative was the distinct representations of kernel. The different types of time-fractional operators discussed above were having some complications. For example, in Reiman-Liouville case, derivative of the constant is not zero.
This was the limitation of Reiman-Liouville time fraction derivative. This problem was solved by Caputo, but the Caputo time-fractional derivative consists of the kernel singularity. To overcome this problem and to remove the issue which was marked in the Caputo fractional operator for time-fractional derivative, Caputo and Fabrizio developed a new fractional derivative named by Caputo-Fabrizio time-fractional derivatives, consist of special kernel which was based on the exponential function [21,22]. But after few analyses the shortcomings of Caputo-Fabrizio came into light for the researchers. The kernel was found locally in the Caputo-Fabrizio definition for fractional derivative. The disadvantage of local kernel is that does not describe the memory effect. To circumvent this issue, Atangana-Baleanu introduced a new time-fractional derivative which is based on the generalized Mittag-Leffler function namely AB time-fractional model [30] with non-local and non-singular kernel. By employing Caputo-Fabrizio space-fractional model, Atangana and Baleanu developed heat transfer model. Many authors have considered Caputo-Fabrizio or Atangana-Baleanu time-fractional derivatives for different fluids and different geometries. For example, for the very first time Sheikh et al. [31], examined the generalized Casson fluid flow under the effect of free convection. As it was a comparative study, therefore, they used both the fractional approaches that are Caputo-Fabrizio and the newly developed definition of Atangana-Baleanu time-fractional derivatives to fractionalize the model. Because of the comparative study, analytical solutions are brought in for both the cases utilizing the mathematical tool LTT. The obtained outcomes are compared in tabular form and also shown graphically. For time $t = 1$, it is noted that both the obtained velocities are same. Change in velocities is observed for $t < 1$ and $t > 1$. Using the Caputo-Fabrizio time-fractional approach, Al-Mdallal et al. [32], reported the analytical solution of fractional Walter’s-B liquid under the conjugate impact of heat and mass transfer. Use of the fractional-order derivatives can be seen in various fields of science beside of the physics and mathematics. For example, Caputo and Cametti [33], Caputo [34], Jumorie [35], Laffalldano et al. [36], Elsayed [37], Ali et al. [38] worked in biology, economy, demography, geophysics, medicine, and bioengineering respectively. Especially, it has been proved that dealing the viscoelastic fluids [39], non-integer order calculus is a very applicable tool to be used. Using the idea of the Caputo Fabrizio time-derivative fractional, Ali et al. [40] examined the two-phase blood flow of Casson fluid in a vertical cylinder. In continuation of the previous study Ali et al. [41] also examined the Hemodynamic Casson fluid flow in cylindrical coordinates along with temperature distribution by using the CFD. Aamina et al. [42] obtained the exact solutions for free convection flow of Brinkman-type water-based-CNT’s nanofluid using Atangana-Baleanu fractional derivative and highlighted its applications in the solar collector. Utilizing the definition of Caputo-Fabrizio time-fractional derivatives Khan et al. [43] carried out the analysis of fractional Casson fluid. Recently Ali et al. [44] studied the generalized free convection MHD Walters-B liquid model utilizing the idea of the Caputo-Fabrizio time-fractional derivatives. Arqub & Al-Smadi [44], obtained the generalized solutions of Bagley-Torvik and Painleve equations using Atangana-Baleanu fractional approach, considering the Hilbert space. Recently, Arqub & Maayah [45], proposed a numerical reckoning approach based on the RKA for estimating the solutions of a class of Atanagana-Baleanu fractional Volterra integro-differential equations. Besides the above literature review, some important and latest work can be seen in [46,47].

Currently, electro-osmosis achieved the central attention of many researchers and scientists because of its relevance in nano and microscale devices with physical applications in natural chemistry, industrial process, and medicine. Electro-osmosis refers to the movement of liquid in a porous material due to an applied electric field. The concept of making movement by utilizing an outer electric field begins a couple of hundreds of years ago, when Reuss [48], examined this phenomenon experimentally, using clay. In recent years, a bunch of several researchers contributing experimentally, numerically and theoretically for understanding the phenomena of electro-osmosis. The main mechanism of electro-osmosis consists of migration of ions, mean anion moves towards anode and cation moves towards cathode [49]. Schmid’s theory, Buckigham pi theory, Helmholtz-Smudch-Owski theory [50,51], Spiegel fractional model and ion hydration theory [52,53] are the theories used to describe the phenomenon of electro-osmosis. Microfluidics devices play a very important role in many fields, like material synthesis, medical diagnostics, energy harvesting, and chemical analysis. The most basic issue of microfluidics is that how to generate motion in liquid which is traditionally gained through pressure-driven flows. But, these pressure-driven flows become moiling when the channel size reduces to motion scale, and in this process, velocity strongly depends on the size of the channel. Therefore, electro-osmosis phenomena are broadly utilized as an alternative for pumping the liquids in microchannels. The main point which makes electro-osmosis a perfect method of pumping liquids in microchannels is that the velocity of electro-osmotic flows is free of the channel measurements. Electro-osmotic flows of non-Newtonian fluids, for example, blood, protein arrangements, colloidal suspensions, and polymeric arrangements are of great importance. Recently, a number of investigations of electro-osmotic flows of non-Newtonian fluids have been reported with different types of constitutive models, like the Oldroyd-B model [54], power-law model [55–58] Jeffery and generalized Maxwell models [59,60]. Keeping in mind the importance of electro-osmosis Mondal and Shit [61] reported the analysis of transfer of heat in the electro-osmotic flow of a slowly varying symmetric microchannel. Hypothetical investigation of temperature distribution in electro-osmotic flow via a slit microchannel is examined by Hadian et al. [62]. The transfer of heat in non-Newtonian fluids of electro-osmotic flow in a hydrophobic microchannel along with the Navier slip is reported by Misra and Sinha [63]. Ferras et al. [64] found the analytic and semi-analytical solutions of electro-osmotic viscoelastic fluids in microchannels. The electro-osmotic flow of non-Newtonian viscoelastic fluids was studied by many researchers analytically [54,65,66].

Keep in mind the above literature review, it is come to know that yet no work has been investigated to examine the demeanor and significant of the fractional viscoelastic fluid along with the electro-osmotic flow by using the new definition of Atangana-Baleanu fractional derivatives. Furthermore, the consolidated impact of heat and mass transfer with variable temperature and concentration are also considered. Therefore, to bridge this gap, exact solutions for the consolidated impact
of mass and heat transfer under the existence of electro-osmosis of the fractional viscoelastic Walters’-B fluid have been obtained.

2. Problem statement

In the current study, the incompressible unsteady free convection magnetohydrodynamic (MHD) flow of Walter’s-B fluid along with the electro-osmotic effect is considered on a vertical plate. The plate is assumed on the x-axis and the y-axis is normal to the plate as shown in Fig. 1. At first when the time \( t = 0 \) the plate and fluid both are thought to be at rest with ambient temperature \( T_\infty \) and concentration \( C_\infty \). But at \( t = 0^+ \) the plate starts a motion with cosine oscillations. Variable concentration and temperature of the fluid at the time \( t = 0^+ \) are \( C_\infty + (C_v - C_\infty)At \) and \( T_\infty + (T_v - T_\infty)At \) respectively. The problem under consideration can be fully described by continuity equation, momentum, energy and concentration equations.

The velocity field for the flow regime under consideration can be expressed as;

\[
\mathbf{V} = [u(y, t), 0, 0].
\]  \tag{1}

The equation of continuity for the flow is;

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]  \tag{2}

The momentum equation for any fluid by using 2nd law of motion is given as;

\[
\rho \frac{d\mathbf{V}}{dt} = -\nabla P + \mathbf{div} \mathbf{T} + \mathbf{J} \times \mathbf{B} + \overline{\mathbf{R}} + \rho_s \mathbf{E}_c + \rho \mathbf{g}.
\]  \tag{3}

Here \( P \) is the pressure, \( \mathbf{J} \times \mathbf{B} \) represents the cross product of current density and magnetic field, \( \overline{\mathbf{R}} \) is the resistive force due to porosity, \( \rho \mathbf{g} \) are the body forces, \( \rho_s \mathbf{E}_c \) is the electro-osmotic term, in which \( \mathbf{E}_c \) is the applied external electric field and \( \rho \) shows the net electric charge density inside the electrical double layer (EDL), and can be defined as;

\[
\rho_e = -\varepsilon k^2 \psi e^{-ky},
\]  \tag{4}

and

\[
T = -PI + \mu A_1 - K_0 A_2 + K_1 A_1^2,
\]  \tag{5}

where \( A_1 \) and \( A_2 \) are the kinematic tensors, which can be defined as follow [24].

\[
A_1 = \left( \nabla V + (\nabla V)^T \right) \text{ and } A_2 = \frac{\partial A_1}{\partial t} + A_1 (\nabla V + (\nabla V)^T) A_1.
\]  \tag{6}

Using Eq. (1), the above equations take the following form;

\[
\frac{d\mathbf{V}}{dt} = \nabla \varphi(t, y),
\]  \tag{7}

\[
\text{div} \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial y^2} - \frac{\kappa_0}{\rho} \frac{\partial^2 \mathbf{u}}{\partial y^2},
\]  \tag{8}

\[
\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 \mathbf{u},
\]  \tag{9}

\[
\overline{\mathbf{R}} = \phi \left( \frac{\partial}{\partial t} \right) \mathbf{u}.
\]  \tag{10}

By using the above equations from (6)-(10) in Eq. (3), and using Boussinesq approximation, the result obtained is;

\[
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial y^2} - \frac{\kappa_0}{\rho} \frac{\partial^2 u}{\partial y^2} \frac{\phi}{k_p} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right) u - \frac{\sigma B_0^2}{\rho} u
\]  
\[
+ g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty) + \frac{\mathbf{E}_c \mathbf{u}}{\rho},
\]  \tag{11}

The temperature field is defined as;

\[
T = (T(y, t), 0, 0).
\]  \tag{12}

\[
\rho c_p \frac{\partial T}{\partial t} = k_1 \frac{\partial^2 T}{\partial y^2} \quad \text{and} \quad \frac{\partial q}{\partial t} = k_1 \frac{\partial^2 q}{\partial y^2}.
\]  \tag{13}

\[
\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2}.
\]  \tag{14}

The suitable initial and boundary conditions are:

\[
\begin{align*}
  u(y, 0) &= 0, & T(y, 0) &= T_\infty, & C(y, 0) &= C_\infty, \\
  u(0, t) &= UH(t) \cos \omega t, & T(0, t) &= T_\infty + (T_v - T_\infty)At, & C(0, t) &= C_\infty + (C_v - C_\infty)At, \\
  u(\infty, t) &= 0, & T(\infty, t) &= T_\infty, & C(\infty, t) &= C_\infty.
\end{align*}
\]  \tag{15}
In the mathematical problem mentioned above, the \(x\)-component of velocity is represented by \(u\), \(v\) shows kinematic viscosity, Walters’-B fluid parameter is shown by \(k_0\) while, the density of the fluid, the porosity of the porous medium and gravitational acceleration is represented by \(\rho, \phi, g\) respectively. The thermal expansion coefficient is represented by \(\beta_T\), \(T\) shows the temperature of the fluid, the applied outer electric field is represented by \(E_x\), \(\phi\) is used to represent the net electric charge of density. \(c_p\) is the specific heat capacity, \(k_i\) is used to show the thermal conductivity, \(q\), and \(H(t)\) is stand for the radioactive heat flux and the Heaviside step function.

According to the Roseland approximation radiative heat flux is given by:

\[
q_r = \frac{4\sigma_1 \partial T^4}{3k_2},
\]

in which the term \(T^4\) is linearized by employing Taylor expansion with respect to \(T_{\infty}\) up to 2nd terms and the final result is:

\[
T^4 \cong 4TT_{\infty}^3 - 3T_{\infty}^5.
\]

Using Eq. (16) and Eq. (15) in Eq. (12), we get,

\[
\rho c_p \frac{\partial T}{\partial t} = k_1 \left( \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma_1 T_{\infty}^3}{3k_1k_2} \frac{\partial^2 T}{\partial y^2} \right).
\]

For dimensional analysis non-dimensional variables are:

\[
u^* = \frac{u}{U_0}, \quad y^* = \frac{U_0}{v} y, \quad t^* = \frac{U_0}{v} t, \quad \theta = \frac{T - T_{\infty}}{T_u - T_{\infty}},
\]

\[
\phi = \frac{C - C_\infty}{C_v - C_\infty}
\]

into Eqs. (11)-(14) and Eq. (17), dropping * for the simplicity, yielding the following result;

\[
\Gamma \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right) - k_{\text{eff}} u - E_x e^{\gamma v} + G m e + G n \theta,
\]

\[
\frac{\partial \theta}{\partial t} = Pr_{\text{eff}} \frac{\partial^2 \theta}{\partial y^2}.
\]
\[
\frac{\partial \phi}{\partial t} = \frac{1}{Sc} \frac{\partial^2 \phi}{\partial y^2},
\]

\[
\begin{align*}
\phi(y, 0) &= 0, & \theta(y, 0) &= 0, & \phi(y, 0) &= 0, \\
u(0, t) &= H(t) \cos(\alpha t), & \theta(0, t) &= t, & \theta(0, t) &= t, \\
u(\infty, t) &= 0, & \theta(\infty, t) &= 0, & \phi(\infty, t) &= 0,
\end{align*}
\]

(21)

where

\[
\Gamma = \frac{k_{ef} \lambda^2}{\eta^2},
\]

is Walter’s-B fluid term, \( \frac{1}{k} \) is permeability parameter, \( Gr = \frac{g \beta \Delta T \sqrt{U_0}}{\alpha} \) is the thermal Grashof number, \( Gm = \frac{\mu \alpha \sqrt{U_0}}{\eta^2} \) is the mass Grashop number, \( Sc = \frac{\nu}{\alpha} \) represents the Schmidt number, \( Pr = \frac{\nu}{\lambda} \) is the Prandtl number, \( M = \frac{\mu \alpha}{\eta^2} \) is a magnetic parameter, \( R = \frac{\lambda \sqrt{U_0}}{\alpha \gamma} \) is radiation parameter, \( Pr_{eff} = \frac{Pr}{Sc} \) is the effective Prandtl number, \( Es = \frac{\mu \alpha \sqrt{U_0}}{\eta^2} \) dimensionless electro-osmosis parameter, and \( \Gamma_0 = 1 + \frac{1}{k}, k_{ef} = k + M, k_3 = \frac{4 \eta \alpha}{\eta^2} \) are the constants.

To transform the classical model of Walter’s-B into the time-fractional model we replace \( \frac{\partial}{\partial t} \) by AB fractional operator \( ^{AB}D_t^\alpha \), Eqs. (18)–(20) adopt the following shape:

\[
\Gamma_0 ^{AB}D_t^\alpha \phi(y, t) = \frac{\partial^2 \phi}{\partial y^2} - \Gamma AB^\alpha \frac{\partial^2 \phi}{\partial y^2} - k_{ef} \phi - Es \phi + Gr \theta,
\]

(23)

\[
^{AB}D_t^\alpha \theta(y, t) = \frac{1}{Pr_{eff}} \frac{\partial^2 \theta}{\partial y^2},
\]

(24)

\[
^{AB}D_t^\alpha \phi(y, t) = \frac{1}{Sc} \frac{\partial^2 \phi}{\partial y^2},
\]

(25)

where \( ^{AB}D_t^\alpha (\cdot) \) stands for the Atangana-Baleanu time-fractional operator of order \( \alpha \), which is defined as \([31]\):

\[
^{AB}D_t^\alpha f(t) = \frac{1}{1 - \frac{\alpha}{2}} \int_0^t E_\alpha \left( \frac{-2(\alpha - t)^\alpha}{1 - \alpha} \right) f(\tau) d\tau, \quad \text{for} \ 0 < \alpha < 1.
\]
The generalized Mittag-Leffler function can be defined as
\[ E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \]

3. Solutions of the problem

Upon taking the Laplace transform on Eqs. (22)–(24) and using relevant initial conditions from equation (22), yields to:
\[ L(ABt)\theta(y, t) = L\left(\frac{1}{Pr_{eff}} \frac{\partial^2 \theta}{\partial y^2}\right) \]

\[ pr_{eff}\left(\frac{q^2}{q^2 + \frac{1}{1 - \alpha}}\right) \theta(y, q) = \frac{d^2 \theta(y, q)}{dy^2} \]

\[ \frac{d^2 \theta(y, q)}{dy^2} = \frac{Pr_{eff}q^4a_0}{q^4 + a_1} \theta(y, q) \]

\[ \frac{d^2 \theta(y, q)}{dy^2} - \frac{q^2 a_2}{q^4 + a_1} = 0 \] (29)

\[ m^2 - \frac{q^2 a_2}{q^4 + a_1} = 0 \] (30)

\[ \bar{\theta}(y, q) = c_1e^{-\sqrt{\frac{m^2 q^4}{q^4 + a_1}}} + c_2e^{\sqrt{\frac{m^2 q^4}{q^4 + a_1}}} \] (31)

Using initial and boundary conditions to find \( c_1 \) and \( c_2 \) we get the following value;

\[ c_1 = \frac{1}{q^2} \text{ and } c_2 = 0 \] (32)

By putting the value of \( c_1 \) in Eq. (31), we get Eq. (33)

\[ \bar{\theta}(y, q) = \frac{1}{q^2} \exp\left(-\sqrt{\frac{a_2 q^4}{q^4 + a_1}}\right) \] (33)

To avoid repetition, the following equations can be produced by using the same procedure mentioned above:

Fig. 4  Profile of velocity for various values of \( \alpha \) and \( Es \)
\[
\ddot{u}(y, q) = \frac{1}{q^2} \exp \left( -y \sqrt{\frac{bq^2}{q^2 + a_1}} \right),
\]

where \(a_1 = \frac{q^2}{L_y}, \quad a_2 = Sca_0, \quad b = Sca_0.\)

Now applying Laplace transform upon Eq. (23) and also using Eqs. (33) and (34) into Eq. (23), yields to:

\[
\ddot{U}(y, q) = c_3e^{-\sqrt{\frac{L_y}{q^2}}(q^2 + a_1)} + c_4e^{\sqrt{\frac{L_y}{q^2}}(q^2 + a_1)} + \frac{E_0(q^2 + a_1)\exp(-y)}{q(q^2 + a_1) - qE_0}\]

\[
-\frac{Gr_1(q^2 + a_1)^2e^{-\sqrt{\frac{L_y}{q^2}}(q^2 + a_1)}}{q^2(a_2q^2 + a_2q^2 + a_1)} - \frac{E_0(q^2 + a_1)^2e^{-\sqrt{\frac{L_y}{q^2}}(q^2 + a_1)}}{q^2(a_2q^2 + a_2q^2 + a_1)}
\]

Using the boundary conditions for finding the values of \(c_3\) and \(c_4\) putting in Eq. (35), we get Eq. (36)

\[
\ddot{u}(y, q) = \frac{a}{q^2 + a} \exp \left( -y \sqrt{\frac{bq^2}{q^2 + a}} \right)
\]

\[
+ \frac{Gr_2(q^2 + a_1)^2}{q(q^2 + a_1) - qE_0} \exp \left( -y \sqrt{\frac{L_y}{q^2 + a}} \right)
\]

\[
+ \frac{Gr_1(q^2 + a_1)^2}{q(q^2 + a_1) - qE_0} \exp \left( -y \sqrt{\frac{L_y}{q^2 + a}} \right)
\]

\[
- \frac{E_0(q^2 + a_1)}{q(q^2 + a_1) - qE_0} \exp(-ky)
\]

\[
- \frac{Gr_1(q^2 + a_1)^2}{q(q^2 + a_1)} \exp \left( -y \sqrt{\frac{bq^2}{q^2 + a}} \right)
\]

\[
- \frac{Gr_1(q^2 + a_1)^2}{q(q^2 + a_1) - qE_0} \exp \left( -y \sqrt{\frac{L_y}{q^2 + a}} \right)
\]

where
\( a_1 = k_{eff} a_1, \quad \Gamma_1 = \Gamma_0 a_0, \quad \Gamma_2 = \Gamma a_0, \)
\( \Gamma_3 = 1 - \Gamma_2, \quad \Gamma_4 = k_{eff} + \Gamma_1, \)
\( \Gamma_5 = \frac{\alpha_1}{\Gamma_3}, \quad \Gamma_6 = \frac{\alpha_1}{\Gamma_5}, \quad \Gamma_7 = \frac{\alpha_1}{\Gamma_3}, \quad \Gamma_8 = \frac{\alpha_1}{\Gamma_5}, \)
\( \Gamma_9 = 1 - \Gamma_8, \quad \Gamma_{10} = \Gamma_7 + \Gamma_8, \)
\( \Gamma_{11} = \frac{a_1}{\Gamma_{10}}, \quad \Gamma_{12} = \frac{a_1}{\Gamma_8}, \quad \delta_1 = a_2 - \Gamma_6, \)
\( \delta_2 = a_2 \Gamma_7 - a_1 \Gamma_6 - \Gamma_2 \Gamma_6, \quad \delta_3 = a_1 \Gamma_6, \quad \delta_4 = b - \Gamma_6, \)
\( \delta_5 = b \Gamma_7 - a_1 \Gamma_6 - \Gamma_3 \Gamma_6, \quad E_{31} = \frac{\alpha_1}{\Gamma_5}, \quad G_{31} = \frac{\alpha_1}{\Gamma_5}, \)
\( G_{m1} = \frac{\alpha_1}{\Gamma_5}, \quad G_{m2} = \frac{\alpha_1}{\Gamma_5}, \)
\( l_1 = \frac{\alpha_1}{\Gamma_3}, \quad l_2 = -\frac{\alpha_1}{\Gamma_5}, \quad l_3 = \frac{\alpha_1}{\Gamma_5}, \quad l_4 = -\frac{\alpha_1}{\Gamma_5}, \)
\( L_1 = \frac{b}{2} + \frac{\sqrt{b^2 - 4a_1}}{2}, \quad L_2 = \frac{b}{2} - \frac{\sqrt{b^2 - 4a_1}}{2}, \)
\( L_3 = \frac{b}{2} + \frac{\sqrt{b^2 - 4a_1}}{2}, \quad L_4 = \frac{b}{2} - \frac{\sqrt{b^2 - 4a_1}}{2}. \)

Eq. (36) can be expressed in an equivalent but more simplified form as:

\[
\tilde{u}(y, \tau) = \frac{1}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) + \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) \\
+ \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) + \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) \\
+ \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) - \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) \\
- \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) - \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) \\
- \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) - \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) \\
+ \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right) + \frac{\pi}{\sqrt{\pi \tau}} \exp \left(-y \sqrt{\Gamma_6 \sqrt{\frac{\tau}{\Gamma_5}}} \right).
\]

(37)

where
\[\Psi_1 = \frac{(a_1b_1)^2}{L_1L_2}, \quad \Psi_2 = \frac{(a_2b_2)^2}{L_1L_2}, \quad \Psi_3 = \frac{(a_3b_3)^2}{L_1L_2}, \quad \Psi_4 = \frac{(a_4b_4)^2}{L_1L_2}, \quad \Psi_5 = \frac{(a_5b_5)^2}{L_1L_2}, \quad \Psi_6 = \frac{(a_6b_6)^2}{L_1L_2}, \quad \Psi_7 = \frac{(a_7b_7)^2}{L_1L_2}, \quad \Psi_8 = 1 - \Gamma_1, \quad b_0 = \Psi_1Gr_2, \]

\[b_1 = \Psi_2Gr_2, \quad b_2 = \Psi_3Gr_2, \quad b_3 = \Psi_4Gr_2, \quad b_4 = \Psi_5Gr_2, \quad b_5 = \Psi_6Gr_2, \quad b_7 = \Psi_7Es_1, \quad b_7 = \Psi_8Es_1.\]

Eqs. (33), (34) and (37) may be precised in equivalent but in more applicable shape like:

\[\tilde{\phi}(y, q) = \tilde{\phi}(y, q, 0, a_2, 0, a_1),\]

\[\tilde{\phi}(y, q) = \tilde{\phi}(y, q, 0, 0, a_1),\]

\[u(y, t) = \cos t \phi_y (y, \sqrt{\Gamma_0} t, 0, 1, 0, a_1) + h(t) + b_0 \phi_y (y, \sqrt{\Gamma_0} t, 0, 1, 0, a_1) + h(t) + b_1 \phi_y (y, \sqrt{\Gamma_0} t, 1, 1, 0, a_1) + h(t) + b_2 \phi_y (y, \sqrt{\Gamma_0} t, 1, 1, 0, a_1) + \ldots\]

where \(*\) denotes the convolution product and

\[L^{-1} \left( \frac{1}{q^2 + r} \right) = h(t) = \frac{1}{1 - 2r} t^{1-r}.\]

\[L^{-1} \left( \frac{1}{q^2 + r} \right) = h_1(t) = \frac{1}{1 - 2r} t^1.\]

\[L^{-1} \left( \frac{1}{q^2 + r} \right) = R_{t0}(0, t).\]

\[L^{-1} \left( \frac{1}{q^4 + \Gamma_1} \right) = R_{t0}(0, t).\]

\[\phi(y, t; s_1, s_2, s_3, s_4) = \frac{1}{\pi} \int_0^\infty \chi(y, t; s_1, s_2, s_3, s_4) \times \exp(-tr - ur^2 \cos r \sin \phi) dr du,\]

\[\phi_1(y, t; s_2, s_3, s_4) = \frac{1}{\pi} \int_0^\infty \psi(y, t; s_2, s_3, s_4) \times \exp(-tr - ur^2 \cos r \sin \phi) dr du,\]

\[\chi(y, t; s_1, s_2, s_3, s_4) = e^{-s_1t - \sqrt{s_1} y} - \frac{\sqrt{s_1}}{2 \sqrt{\pi}} \int_0^\infty t e^{-t^2} \exp(s_1 t) \, dt + \frac{y^2}{4u^2} - s_2 u I_1 (2 \sqrt{(s_3 - s_4) ur}) \, dr du,\]

\[\psi(y, t; s_2, s_3, s_4) = \frac{s_1}{\pi} e^{-\sqrt{s_1} y} - \frac{\sqrt{s_1}}{2 \sqrt{\pi}} \int_0^\infty \frac{1}{u} \times \exp(-\frac{y^2}{4u^2} - s_2 u I_1 (2 \sqrt{(s_3 - s_4) ur}) du.\]

applying the inverse Laplace transform technique, upon Eqs. (38)-(40), obtaining the following solution:

\[\theta(y, t) = \phi(y, t, 0, a_2, 0, a_1),\]

\[\phi(y, t) = \phi(y, t, 0, b, 0, a_1).\]
4. Results and discussion

In the current article, the classical model of Walter’s-B fluid is generalized using the new definition of the Atangana-Baleanu fractional model. To make non-dimensional the governing equations, the dimensionless variables are used. Closed-form expressions are brought in for velocity, concentration and temperature profiles by applying the technique of Laplace transform. The obtained solutions are expressed in the form of an integral function. The influence of various physical embedded parameters like fractional order parameter $\alpha$, electro-osmosis parameter $E_s$, Prandtl number $Pr$, Schmidt number $Sc$, mass Grashof number $G_m$, thermal Grashof number $Gr$, Walter’s-B fluid parameter $C$, and the consolidate impact of permeability and magnetic parameter $K_{eff}$ are examined. Figs. 2–8 show the impact of $\alpha$ on velocity. Furthermore, the graphs show that greater values of $\alpha$ produce an enhancement in the velocity profile, in simple words it can be stated that velocity is an increasing function of $\alpha$.

In Fig. 2, the influence $\Gamma$ has been shown from various graphs. It is can be observed from these graphs that increasing values of $\Gamma$ produces a decrease in the velocity profile. As $\Gamma$ has a direct relation to the viscoelasticity of the fluid, so as $\Gamma$ increases the viscosity and elasticity of the fluid increase and in result the motion of the fluid retards. It is also worth noting that the Newtonian fluid ($\Gamma = 0$) has greater velocity as compare to Walters’s-B fluid, the fact is that viscosity of Walters-B liquid is higher than Newtonian liquid.

Fig. 3, sketches the influence of $K_{eff}$ on velocity distribution of Walters-B liquid. It is observed from the graphs that due to rising values of $K_{eff}$ a retard in the velocity is noted, and it is true because greater values of $K_{eff}$ strengthening the resistive forces in flow and consequently velocity retards.

The behavior for distinct values of $E_s$ on velocity profile in Fig. 4. A fall in the velocity profile of generalized Walters’B fluid is noted. This is happened due to the greater values of $E_s$. The reason behind this retardation is that, when an outer electric field is applied to the fluid flow, (EDL) electric double
layer gains many charges which produce resistive forces in the fluid which in result retards the fluid motion. It is worth mentioning that this retardation can play a worthy role in medical research, in the process of chemical separation techniques, and furthermore in some other mechanical branches.

The effect of $Gr$ and $Gm$ on fluid flow is illustrated in Figs. 5 and 6. It has been observed that the velocity profile enhanced in the response to rising in the values of $Gr$ and $Gm$. Physically it is true, because of greater values of $Gr$ and $Gm$ gives rise to the buoyancy forces which weaken the viscous forces and accelerate the velocity.

The influence of Prandtl number $Pr$, on velocity distribution, is sketched in Fig. 7. As $Pr$ is the ratio between kinematic viscosity “$\nu$” and thermal conduction “$k$”. Therefore, greater values of $Pr$ giving strength to the viscosity and as a result layers thicker with respect to thermal boundary layers and fall in velocity profile occur.

Fig. 8, shows the behavior of $Sc$ on the velocity profile. It is interesting to see that for greater values of $Sc$ the velocity profile decreases. The fact behind this is that $Sc$ is the ratio between the viscous forces and mass diffusion, therefore greater values of $Sc$ providing strength to the viscous forces and weakness to the mass diffusion which in result produces retardation in the velocity profile.

5. Conclusion

In the present article, the principle motivation behind this study is to explore the electro-osmotic flow influence of fractional Walter’s-B fluid under the amalgamated effect of heat and mass transfer along with variable concentration and temperature. Moreover, understanding the significance of non-integer order calculus in the real-world phenomena ordinary Walters-B liquid model has been transmuted to the generalized
model utilizing the definition of AB time-fractional derivative. Making use of the technique of Laplace transform is to obtain the exact expressions for velocity, concentration and temperature distributions. To show the effects of various embedded parameters on the velocity field, various graphs have been plotted. It is worth mentioning that all the graphs satisfy the imposed boundary conditions, which show the coherence of our obtained solutions. It is additionally worth seeing from the graphs in Fig. 4 that by enhancing the domain of the electro-osmotic parameter $E_s$, the profile of the velocity falls. This phenomenon is exceptionally significant in the process of electronically controlling the flow of fluid via the junction. The electro-osmotic system also helps in controlling the damp rising in the walls of the building [64,67], especially in very thick walls. Furthermore, this phenomenon may also work in the process of separation of the fluids at the atomic level and can play a very important job for medication dischargers [65,68]. Apart from this, it is worth observed from the graphs that giving rise to the values of $Sc$, $Gr$, $Pr$ and $K_{aw}$ brings retardation in the velocity distribution and greater values of $x$, $Gr$ and $Gm$ produce increment in the velocity profile.

Furthermore, the present problem can be extended in the future by taking different nanoparticles, different geometry, ramped wall temperature, Newtonian heating and thermodiffusion.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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