What kind of “complexity” is dual to holographic complexity?

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**ABSTRACT:** It is assumed that the holographic complexities such as the complexity-volume (CV) conjecture and the complexity-action (CA) conjecture are dual to complexity in field theory. However, because the definition of the complexity in field theory is still not complete, the confirmation of the holographic duality of the complexity is ambiguous. To improve this situation, we approach the problem from a different angle. We first identify minimal and genuine properties that the field theory dual of the holographic complexity should satisfy without assuming anything from the circuit complexity or the information theory. Based on these properties, we propose a field theory formula dual to the holographic complexity. Our field theory formula implies that the complexity between certain states in two dimensional CFTs is given by the Liouville action, which is compatible with the path-integral complexity. It also gives natural interpretations for both the CV and CA holographic conjectures and identify what the reference states are in both cases. When applied to the thermo-field double states, it also gives consistent results with the holographic results in the CA conjecture: both the divergent term and finite term. We also make a comment on the difference between our proposal and the Fubini-Studi distance.
1 Introduction

Recently, the concepts in quantum information theory have been applied to investigate the theory of gravity and black holes. In particular, a concept named “complexity”, which comes from the quantum circuit complexity in quantum information theory, was introduced for the study of the black hole interior. The complexity in quantum circuits can be defined for both operators and states. Roughly speaking, the complexity of an operator is the minimal number of required gates\footnote{The gates are basic building blocks to construct the quantum circuit.} when we use quantum circuits to simulate it. The complexity of state is the minimal number of required gates when we use quantum circuits to transform a reference state to a target state.

The motivation to introduce the complexity into the black holes physics was to understand about the fire-wall model of the black hole \cite{1} and the growth rate of the Einstein-Rosen bridge for the AdS black holes \cite{2-4}. Refs. \cite{2} and \cite{5} proposed two holographic conjectures to compute the complexity for some particular quantum states which are dual...
to boundary time slices of an eternal asymptotic AdS black hole. They are called the
complexity-volume (CV) conjecture and the complexity-action (CA) conjecture.

The CV conjecture states that the complexity is proportional to the maximum volume
of space-like hypersurfaces. Suppose \( t_L \) and \( t_R \) are two time slices at the left and right
boundaries of an external asymptotic AdS black hole. Then the CV conjecture is given by

\[
C = \max_{\partial \Sigma = t_L \cup t_R} \frac{\text{Vol}(\Sigma)}{G_N \ell},
\]

where \( \Sigma \) is a spacelike surface which connects the time slices \( t_L \) and \( t_R \) of two boundaries,
\( G_N \) is the Newton’s gravity constant and \( \ell \) is a length scale. The CA conjecture states that
the complexity associated to two boundary time slices is given by the on-shell action in the
Wheeler-DeWitt (WdW) patch

\[
C = \frac{S_{\text{WdW,on-shell}}}{\pi \hbar}.
\]

The WdW patch is the closure of all spacelike surfaces which connect \( t_L \) and \( t_R \).

Many works have been done to study the properties of the conjectures (1.1) and (1.2):
the time-evolution of the holographic complexity in the CV or CA conjectures [6–8], the
action growth rate and the Lloyd’s bound in various gravity systems [9–17], the UV diver-
gent structures of the holographic complexity [18, 19], the quench effects in the holographic
complexity [20–22] and so on. Besides these two conjectures, other conjectures for the
complexity were also proposed in holography for different systems and purposes (see, for
example, Refs [23–29]).

Compared with much progress on the complexity in gravity side, the precise meaning
and a proper definition of the complexity in quantum field theory is still incomplete. One
natural idea to define the complexity in field theory is as follows.

1. Start with various well-established models of circuit complexity and generalize them
   for field theory in certain ways.

2. Analyze the consequences of those generalizations and figure out which one is com-
   patible with the holographic complexity.

This idea is based on the assumption that the holographic complexities such as the CV or
CA conjectures are indeed dual to a kind of “circuit complexity”. See Fig. 1(a). Following
this idea, there have been many attempts to generalize the concept of complexity of dis-
crete quantum circuits to continuous systems: “complexity geometry” [30–32], Fubini-study
metric [33], and path-integral optimization [26, 27, 34, 35]. See also [36–41].

In particular, the complexity geometry is the most studied. The basic idea was first
proposed by Nielsen et al. [42–44], in which the authors constructed a continuum approxi-
mation of the circuit complexity which involves the geodesic distance in a certain geometry
called “complexity geometry”. See for examples [7, 45–55, 55–61]. Along this road, there
have been many works showing positive supports in identifying the field theory complexity
in the sense of the agreement with the holographic complexity. However, a few fundamental
questions still remain.
Figure 1. Two different methods to understand the holographic complexity. (a) Assume that the holographic complexity is a precise dual of a specific generalization of the circuit complexity. We investigate different possible generalizations of the circuit complexity in field theory and then seek for the best one compatible with the holographic complexity. (b) Do not make any assumption on what the field theory dual of the holographic complexity is. First, just try to find out various possible candidates of the field dual of the holographic complexity. Next, we ask if they have any relationship to the circuit complexity or something else.

The most important one is what the reference states in the CV and CA conjectures are. Both the CV and CA conjectures are expected to describe the complexity between states, which will be meaningful only if both the reference state and the target state are identified clearly. The target state is dual to the thermofield double (TFD) state associated with time slices at the boundary [62]. However, the reference state is unclear in the statements of both the CV and CA conjectures.

The other question is how to understand different behaviors of the time-evolution in the CV and CA conjectures. Though both the CV and CA conjectures show that the complexity grows linearly at late time limit, they show different behaviors at early time. In the CV conjecture, the complexity grows as $t^2$ at early time while, in the CA conjecture, it first keeps constant and then suddenly obtain a negative infinite growth rate after a certain time. See Refs. [6, 7] for more detailed discussions about the time evolutions of the complexity in the CV and CA conjectures. This difference may imply that two conjectures describe two different complexities (or something else) in field theory.

There is another important issue to consider when we identify the field theory dual of the holographic complexity: the properties of the proposed complexity in field theory may differ for different models. Different from the geometrization method of Nielsen’s, Refs. [26, 27, 34] proposed the “path-integral complexity” to describe the complexity between the field operator eigenstate and the ground state of a 2-dimensional conformal field theory (CFT). It states that the complexity can be given by the on-shell Liouville action. This is based on the tensor network renormalizations [63] in constructing the ground state. Ref. [64] also proved the Einstein’s equation in 2+1 dimensional case could be obtained by minimizing such a complexity. Recently, Ref. [65] offers a viewpoint to connect the path integral complexity and circuit complexity and tries to fill up the gap between these two different proposals in field theory. The “path-integral complexity” has an important
Figure 2. A schematic explanations for possible relationships between the circuit complexity and the field-theory duals of the holographic complexity. (a) The field theory dual of the holographic complexity belongs to the continuous version of the circuit complexity. (b) The field theory dual of the holographic complexity and a continuous version of the circuit complexity share some common properties, but, there are still differences too.

property: it is bases-independent and unitary invariant

Though a large amount of efforts have been made along the road shown in Fig. 1(a), there is still no proposal which can be completely compatible with any holographic complexity. It motivated us to step back and ask a question: is the quantity so-called the “holographic complexity” really a “complexity” in field theory? Even if it is the case, it is possible that that the field theory dual of the holographic complexity may belong to a different type of complexity, which is not necessarily the same as a continuous version of the circuit complexity. In this case, the field theory dual of the holographic complexity and the continuous version of the circuit complexity may share some common properties but there may be differences too. See Fig. 2 for a schematic explanation. For now, it seems that there is no evidence to rule out any one in Fig. 2. Thus, let us keep this possibilities open.

The way in Fig. 1(a) will work only for the case in Fig. 2(a). If the relationship shown in Fig. 2(b) is valid, then the way in Fig. 1(a) is not suitable. Instead, a different method shown in Fig. 1(b) is more promising:

1. First try to find all possible field theory duals of the holographic complexity, by considering the genuine properties of the holographic complexity, without assuming anything from field theory.

2. Next, we check if there is any candidate which can match with the basic requirements of the circuit complexity (non-negativity, right-invariance, triangle inequality, et.)

This method indeed can cover both possibilities in Fig. 2. The main goal of this paper is to make a step towards this new road.

We assume that the holographic complexity ($C_V$ or $C_A$) has a field theory dual denoted by $\tilde{C}$

$$C_{V(A)} = \tilde{C}(|\psi\rangle, |R\rangle).$$

which describes an unknown relationship between a target state $|\psi\rangle$ and a reference state $|R\rangle$. In this paper, we use the notation $\tilde{C}$ instead of $C$, to denote it is the specific ‘complexity’ related with the holographic complexity. In principle, it may be different from the usual
circuit complexity or any other quantum computational concept. Our first task is to find if there is any special properties of this function \( \bar{C} \). We emphasize again that we do not assume anything from field theory. i.e. at this stage \( \bar{C}(|\psi\rangle, |R\rangle) \) may not correspond to a specific kind of “circuit complexity”. We just try to investigate the properties of the holographic complexity and find out what the possible candidates of the function \( \bar{C} \) are.

First, we follow a usual way in theoretical physics: symmetry is important. We will argue that, the diffeomorphic invariance of the holographic complexity implies that, at least for a large class of infinite dimensional unitary group (strictly speaking, faithful unitary representation of an infinite dimensional Lie group) \( \mathcal{G} \), the function \( \bar{C} \) is invariant under a transformation of \( \mathcal{G} \):

\[
\forall \hat{U} \in \mathcal{G}, \quad \bar{C}(|\psi\rangle, |R\rangle) = \bar{C}(\hat{U}|\psi\rangle, \hat{U}|R\rangle) .
\]

(1.4)

This property of the holographic complexity, however, cannot be read from the circuit complexity. This shows that the field theory dual of the holographic complexity has infinitely many symmetries so infinite constraints.

The second important property of \( \bar{C} \) is that it is an extensive quantity in thermodynamic limit

\[
\bar{C}(|\psi_1\rangle \otimes |\psi_2\rangle, |R_1\rangle \otimes |R_2\rangle) = \bar{C}(|\psi_1\rangle, |R_1\rangle) + \bar{C}(|\psi_2\rangle, |R_2\rangle) .
\]

(1.5)

This property comes from a fact that the holographic complexity is proportional to the volume of the boundary slice. This is also a special property of the holographic complexity and shows an essential difference from the entanglement entropy.

By combining these two basic properties, we propose a class of possible simple candidates for the function \( \bar{C} \) and choose particular forms as examples to make detailed discussion. We will show many interesting consequences. It proves that the state complexity in 2D CFTs is given by the Liouville action, which is consistent with the path-integral complexity. It also gives natural explanations for both the CV and CA conjectures. In particular, it clarifies what the target and reference states are in the CV and CA conjectures. In other words, our proposal answers two aforementioned unsolved problems.

The paper is organized as follows: In Sec. 2, we explain why the holographic complexity implies the equations (1.4) and (1.5). In Sec. 3 we give a class of simple candidates for the function \( \bar{C} \) and show that they can exhibit very rich contents including the new interpretation of the CV and CA conjecture. In Sec. 4 we apply our proposal to the TFD states and show further consistency checks with holographic complexities. In Sec. 5 we make a comment on a difference between our proposal and the Fubini-Study distance. We conclude in Sec. 6.

2 Basic properties of holographic complexity

2.1 Holographic complexity is diffeomorphic invariant

One standard way to investigate physical systems of which structures are not well known is to start with symmetry. Along this line, we may ask if there is any universal symmetry in the CV and CA conjectures? We think the answer is positive. In both CV and CA conjectures, the complexity is given as a geometric quantity of the bulk spacetime. This
implies that there is an important and universal symmetry in the holographic complexity: it is invariant under a bulk diffeomorphic transformations. If we make a bulk diffeomorphic transformation, the boundary theory will be also transformed by the induced transformation.\(^2\)

Let us consider a simple case in a pure AdS\(_{d+1}\) spacetime. There are two kinds of transformations: a conformal group in a boundary theory and an isometric group in AdS spacetime. The two groups are both isomorphic to SO(2,\(d\)). Every transformation of SO(2,\(d\)) in the bulk corresponds to a conformal transformation at the boundary CFT and versa vice. Though CFT quantities such as correlation functions, generating functional and partition function are invariant, the operators and quantum states will obtain a SO(2,\(d\)) transformation and, in general, this transformation is not an identical transformation.\(^3\)

Suppose that the states \(|\psi\rangle\) and \(|R\rangle\) are a target state and a reference state in a CFT, of which the ‘complexity’ is given by \(\tilde{C}(|\psi\rangle, |R\rangle)\). By the holographic duality, in the corresponding bulk theory, there are bulk metric \(g_{\mu\nu}\) and matter fields \(A_i\) (\(i\) stands for different matter fields). By using the CV or CA conjecture \(\tilde{C}(|\psi\rangle, |R\rangle)\) can be computed as

\[
\tilde{C}(|\psi\rangle, |R\rangle) = C_{V(A)}(g_{\mu\nu}, A_i),
\]

where \(C_{V(A)}(g_{\mu\nu}, A_i)\) stands for the holographic complexity computed by the metric \(g_{\mu\nu}\) in the CV or CA conjecture.

Suppose now that \(\hat{U}_\phi\) is an SO(2,\(d\)) transformation in the CFT Hilbert space, which transforms \(|\psi\rangle\) and \(|R\rangle\) to \(\hat{U}_\phi|\psi\rangle\) and \(\hat{U}_\phi|R\rangle\). Accordingly, there is a corresponding bulk diffeomorphism \(\phi\), which induces a pull-back transformation \(\phi^*\) for the the bulk metric \(g_{\mu\nu} \mapsto \phi^*(g_{\mu\nu})\) and for matter fields \(A_i \rightarrow \phi^*(A_i)\). By symmetry, we have

\[
\tilde{C}(\hat{U}_\phi|\psi\rangle, \hat{U}_\phi|R\rangle) = C_{V(A)}(\phi^*(g_{\mu\nu}), \phi^*(A_i)).
\]

In the pure AdS case, the diffeomorphism \(\phi\) is an isometry \(\phi^*(g_{\mu\nu}) = g_{\mu\nu}\). Thus, we have the following important result on the field theory dual of the holographic complexity

\[
\forall \hat{U} \in \text{SO}(2, d), \quad \tilde{C}(|\psi\rangle, |R\rangle) = \tilde{C}(\hat{U}|\psi\rangle, \hat{U}|R\rangle).
\]

The same result holds even if the diffeomorphism is not an isometry. Suppose that \(\mathcal{M}_{d+1}\) is an arbitrary asymptotically AdS spacetime and \(\phi: \mathcal{M}_{d+1} \rightarrow \mathcal{M}_{d+1}\) is an arbitrary diffeomorphic transformation, which transforms the boundary time slices and bulk metric \(\{t_L, t_R, g_{\mu\nu}\}\) into \(\{\phi(t_L), \phi(t_R), \phi^*(g_{\mu\nu})\}\). The holographic complexity computed by them are the same, i.e.,

\[
C_{V(A)}(t_L, t_R, g_{\mu\nu}) = C_{V(A)}(\phi(t_L), \phi(t_R), \phi^*(g_{\mu\nu})).
\]

\(^2\)In this paper, when we talk about spacetime transformation, we always use “active viewpoint”: the coordinates is fixed but metric is transformed. We will also use the “active viewpoint” to consider transformations of states in the Hilbert space and generators in Lie algebra: the bases of Hilbert space and Lie algebra are unchanged but the quantum states and generators are changed.

\(^3\)For example, the gauge transformation of a U(1) gauge theory does not change the partition function, correlation functions but it will induce a unitary transformation in the Hilbert space. The gauge transformation: \(\{A, \phi\} \rightarrow \{A + \nabla A, \phi - \partial A\}\) will induce a unitary transformation in the Hamiltonian and quantum states: \(H \mapsto \hat{U} H \hat{U}^\dagger\) and \(|\psi\rangle \mapsto \hat{U}|\psi\rangle\) with \(\hat{U} = \exp(iq A)\). Here \(q\) is the charge.
This transformation will also induce a transformation on the Hilbert space in the boundary theory, i.e., \( \hat{U}_\phi : \mathcal{H} \rightarrow \mathcal{H} \). This transformation must be unitary as no information will be lost by the diffeomorphic transformation. If \( \{ |\psi\rangle, |R\rangle \} \subset \mathcal{H} \) is a pair of a target state and a reference state, the diffeomorphic transformation \( \phi \) will induce a new pair of target state and reference state \( \{ \hat{U}_\phi |\psi\rangle, \hat{U}_\phi |R\rangle \} \subset \mathcal{H} \). Then, we have the following equations for the function \( \bar{C} \)

\[
\bar{C}(|\psi\rangle, |R\rangle) = \mathcal{C}_{V(A)}(t_L, t_R, g_{\mu\nu}), \quad \bar{C}(\hat{U}_\phi |\psi\rangle, \hat{U}_\phi |R\rangle) = \mathcal{C}_{V(A)}(\phi(t_L), \phi(t_R), \phi^*(g_{\mu\nu})) , \quad (2.5)
\]
which imply

\[
\bar{C}(|\psi\rangle, |R\rangle) = \bar{C}(\hat{U}_\phi |\psi\rangle, \hat{U}_\phi |R\rangle) . \quad (2.6)
\]

All bulk diffeomorphic transformations form an infinite dimensional Lie group, which induces an infinite dimensional unitary group (strictly speaking, a faithful unitary representation) \( \mathcal{G} \) on the boundary Hilbert space. Thus, we have the following symmetry for the field theory dual of the holographic complexity

\[
\forall \hat{U} \in \mathcal{G}, \quad \bar{C}(|\psi\rangle, |R\rangle) = \bar{C}(\hat{U}_\phi |\psi\rangle, \hat{U}_\phi |R\rangle) . \quad (2.7)
\]

Note that the holographic complexity gives a strong condition to its field theory dual: it must have infinitely many constraints from the symmetries, which may imply the following. If the field theory dual of the holographic complexity is a kind of continuous version of the usual circuit complexity, these constraints may not allow us to choose the gates and penalties in the circuit complexity artificially. From the perspective of the usual circuit complexity, it seems too strong because, in the usual circuit, the gates and penalties can be arbitrarily chosen and, in general, the complexity is not invariant under unitary transformations. However, again, our strategy here is not to have any prejudice from the field theoretic or quantum computational concept. We want to figure out where the holographic complexity leads us, wherever it is.

2.2 Holographic complexity is extensive

Another important property we learn from the holographic complexity is that the complexity between a boundary state and an unknown reference state is proportional to the volume at the boundary time slices when the volume is large enough and the boundary state is uniform, i.e.,

\[
\bar{C}(|\psi\rangle, |R\rangle) \propto V_{\text{bd}}, \quad \text{if} \quad V_{\text{bd}} \rightarrow \infty . \quad (2.8)
\]

Here, \( V_{\text{bd}} \) is the volume of boundary slices, not the volume in any bulk region.

In order to see what we can obtain from this holographic property, let us consider the complexity between two states \(|\psi\rangle\) and \(|R\rangle\) which contain two independent sub-systems \(A\) and \(B\), e.g. see the Fig. 3. The systems \(A\) and \(B\) are locally the same and have the volume \(V_A\) and \(V_B\), respectively. When two sub-systems are separated far enough, the target state \((|\psi\rangle)\) and the reference state \((|R\rangle)\) of \(A \cup B\) can be written in terms of the direct product of two independent sub-systems approximately

\[
|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B , \quad |R\rangle = |R\rangle_A \otimes |R\rangle_B . \quad (2.9)
\]
By the holographic result (2.8), we have the following relationships

\[ \bar{C}(|\psi\rangle_A, |R\rangle_A) = cV_A, \quad \bar{C}(|\psi\rangle_B, |R\rangle_B) = cV_B, \quad \text{and} \quad \bar{C}(|\psi\rangle, |R\rangle) = c(V_A + V_B), \quad (2.10) \]

with a real number \( c \). From Eqs. (2.10) and (2.9) we obtain the following property.

**Extensive property**: the complexity of the product states of continuous systems in thermodynamic limit is extensive i.e.,

\[ \bar{C}(|\psi\rangle_A \otimes |\psi\rangle_B, |R\rangle_A \otimes |R\rangle_B) = \bar{C}(|\psi\rangle_A, |R\rangle_A) + \bar{C}(|\psi\rangle_B, |R\rangle_B), \quad (2.11) \]

if the states \( |\psi\rangle_A, |\psi\rangle_B, |R\rangle_A \) and \( |R\rangle_B \) have infinite volume \( V \), infinite degrees of freedom \( N \) with a finite \( N/V \).

Thermodynamic limit is needed because the holographic result (2.8) is valid in that limit. This property shows a very important difference compared with the entanglement entropy, as the latter in general is proportional to the area. This property is also not easy to be found if we think only from the perspective of circuits complexity. If the field theory dual of the holographic complexity is a kind of continuous version of the circuit complexity, it should have very special properties which do not usually (easily) appear in the studies of the circuit complexity.

3 Field theory dual of the holographic complexity and applications

3.1 Proposal and infinitesimal triangle inequality

We have clarified two basic properties of a field theory dual of the holographic complexity: Eq. (2.7) and Eq. (2.11). The next task is to find possible mathematical formulas for \( \bar{C} \) satisfying those properties. Since Eq. (2.7) and Eq. (2.11) do not lead to a unique formula for \( \bar{C} \) we want to propose one possibility which looks minimal and most relevant to our purpose.

Note that the unitary group \( \mathcal{G} \), which is a faithful unitary representation of diffeomorphism group of \( \mathcal{M}_{d+1} \), is a very large group. If we assume that a field theory dual of the
holographic complexity should have a natural mathematical form, then a simple choice for $\bar{C}$ is
\[ \bar{C}(|\psi\rangle, |R\rangle) = f(\langle \psi | R \rangle), \quad (3.1) \]
where $f$ is an unknown function. If we combine it with the property (2.11), we conclude that for a complex number $x$, the simplest $f(x)$ is
\[ f(x) = \alpha_1 \text{Re} \ln x + \alpha_2 |\text{Im} \ln x|, \quad (3.2) \]
for any constant real numbers $\alpha_1$ and $\alpha_2$. By choosing $\alpha_1 = -1, \alpha_2 = 1$ we have
\[ \bar{C}(|\psi\rangle, |R\rangle) = -\text{Re} \ln x + |\text{Im} \ln x| = -\ln |\langle \psi | R \rangle| + |\text{Im} \ln \langle \psi | R \rangle|. \quad (3.3) \]
As the complexity has the freedom of an overall factor, only the ratio $\alpha_1/\alpha_2$ is relevant. We assume $\alpha_1/\alpha_2 = -1$ in this paper.

Usually, “complexity” stands for a kind of “distance” so is expected to satisfy the triangle inequality. Indeed, Eq. (3.3) satisfies the triangle inequality under certain conditions. Let us first prove the “infinitesimal version” of the triangle inequality. Let us consider arbitrary three infinitesimally close quantum states $|R\rangle, |T\rangle$ and $|\psi\rangle$ as shown in Fig. 4. There may be three hermitian Hamiltonians (may not be unique) $\{H_1, H_2, H_3\}$ and an infinitesimal parameter $\delta > 0$, which satisfy
\[ |\psi\rangle = e^{-i\delta H_1}|R\rangle, \quad |T\rangle = e^{-i\delta H_2}|\psi\rangle = e^{-i\delta H_3}|R\rangle. \quad (3.4) \]
The complexities between these three quantum states are labeled by $\bar{C}_1, \bar{C}_2$ and $\bar{C}_3$. See Fig. 4.

Because
\[ \langle \psi | R \rangle = \langle R | e^{i\delta H_1} | R \rangle = 1 + i\delta \langle R | H_1 | R \rangle + O(\delta^2), \quad (3.5) \]
our proposal (3.3) gives
\[ \bar{C}_1 = \delta |\langle R | H_1 | R \rangle| + O(\delta^2). \quad (3.6) \]
Similarly, we have
\[ \bar{C}_2 = \delta |\langle \psi | H_2 | \psi \rangle| + O(\delta^2), \quad \bar{C}_3 = \delta |\langle R | H_3 | R \rangle| + O(\delta^2). \quad (3.7) \]
$\bar{C}_2$ can be also expressed as
\[ \bar{C}_2 = \delta |\langle R | H_2 | R \rangle| + O(\delta^2), \quad (3.8) \]
because
\[
\langle \psi | H_2 | \psi \rangle = \langle R | e^{i \delta H_1} H_2 e^{-i \delta H_1} | R \rangle = \langle R | H_2 | R \rangle + \mathcal{O}(\delta) .
\] (3.9)

Furthermore, \( \tilde{C}_3 \) can be expressed as
\[
\tilde{C}_3 = \delta |\langle R | H_2 + H_1 | R \rangle| + \mathcal{O}(\delta^2) .
\] (3.10)

because, up to order \( \mathcal{O}(\delta) \),
\[
\langle R | T \rangle = 1 - i \delta \langle R | H_3 | R \rangle = \langle R | e^{-i \delta H_2} e^{-i \delta H_1} | R \rangle = 1 - i \delta \langle R | H_2 + H_1 | R \rangle ,
\] (3.11)

which yields \( \langle R | H_3 | R \rangle = \langle R | H_2 + H_1 | R \rangle \).

Using the fact
\[
|\langle R | H_1 | R \rangle| + |\langle R | H_2 | R \rangle| \geq |\langle R | H_2 | R \rangle + \langle R | H_2 | R \rangle| = |\langle R | H_1 + H_2 | R \rangle| ,
\] (3.12)

we have
\[
\tilde{C}_1 + \tilde{C}_2 \geq \tilde{C}_3 .
\] (3.13)

This shows that our proposal satisfies the triangle inequality for infinitesimally close states.

Note that this “infinitesimal triangle inequality” does not imply the triangle inequality for arbitrary states. The proof of general triangle inequality needs more preparations and we will come back to this point at end of Sec. 3.2.

**A few comments** Let us make a few comments on Eq. (3.3). From the perspective of the usual circuit complexity, Eq. (3.3) is too simple and may lose many interesting properties of the circuit complexity. This is true. However, we recall that the motivation of this paper: we do not study how to use the properties of the usual circuit complexity to recover the holographic results. Instead, we try to understand what a possible field theory dual of the holographic complexity is, whatever it is. Even if it turns out to be a kind of circuit complexity, it must be a very “special” circuit complexity because it contains properties such as (2.7) and (2.11) which do not appear in the usual circuit complexity.

In Eq. (3.3) there is an imaginary part of a complex number. If the inner product \( \langle \psi | R \rangle \) is not a real number, there may be ambiguities in two aspects. The first one is due to the multiple valued function “\( \ln(\cdot) \)”. For any complex number \( x = \rho e^{i \theta} \), we have \( \ln x = \ln \rho + i \theta + 2n\pi i \) with \( n = 0, \pm 1, \pm 2, \cdots \). The second one is due to the fact that two state vectors \( |\psi\rangle \) and \( e^{i \theta} |\psi\rangle \) describe the same physical state. These two aspects imply that our formula (3.3) has an ambiguity of adding arbitrary constants.

Interestingly, this ambiguity may correspond to a fact that, in the CA conjecture, the action of theory has a freedom of adding arbitrary constants. The CA conjecture connects the on-shell action to the complexity and the action has a freedom of adding a constant term. Thus, any complexity theory, if it is claimed to be dual to the CA conjecture, must have a freedom of adding arbitrary constant. In the Nielsen’s complexity geometry, we has a freedom to choose an overall factor but do not have a freedom to add a constant. From this perspective, it seems difficult to use the Nielsen’s “complexity geometry” to find a dual of the CA conjecture.
If we consider a continuous “time” dependent state $|\psi(t)\rangle$ with $|\psi(0)\rangle = |R\rangle$, then the complexity $\mathcal{C}(t)$ will depend on $t$, too. It is natural to require $\mathcal{C}(t)$ is also the continuous function and $\mathcal{C}(0) = 0$. In this case, the ambiguity of a phase factor disappears.

In the following sections, we will show that Eq. (3.3) indeed can be understood as a kind of “complexity”, i.e., a kind of minimal “cost”. However, in general it will not be a usual circuit complexity. We will investigate some implications of the proposal (3.3), which support that the proposal (3.3) may be a correct field theory dual of the holographic complexity.

### 3.2 Path-integral formula and triangle inequality

In this subsection, we compute the complexity for pure states by the path integral formulation. Suppose that $|\psi_0\rangle$ is a normalized initial state, $|\psi(t)\rangle$ is a target state and the time evolution is given by a time evolution operator $\hat{U}(t)$. Without loss of generality, we may consider a quantum mechanics case and assume that the configuration space is one dimensional. The Feynman propagator reads

$$K(x_2, t_2; x_1, 0) := \langle x_2 | \hat{U}(t) | x_1 \rangle = \frac{1}{\mathcal{N}} \int_{x(0)=x_1}^{x(t_2)=x_2} D[x] \exp \left\{ \frac{i}{\hbar} S[x(t)] \right\}, \quad (3.14)$$

where $S[x(t)]$ is the classical action functional and $\mathcal{N}$ is a normalized factor. The transition amplitude between $|\psi_0\rangle$ and $|\psi(t)\rangle$ is

$$Z := \langle \psi(t) | \psi_0 \rangle = \int dx_2 dx_2 \psi^*_0(x_2) K(x_2, t_2; x_1, 0) \psi_0(x_1), \quad (3.15)$$

where $\psi_0(x) := \langle x | \psi_0 \rangle$ is the wave function of the initial state.

The complexity between $|\psi_0\rangle$ and $|\psi(t)\rangle$ is given by Eq. (3.3):

$$\mathcal{C}(t_2) = -\ln |Z| + \text{Im} \ln Z, \quad (3.16)$$

so the time evolution of the complexity depends on the initial state and the action of the system.

A similar expression can be obtained in quantum field theory. The complexity between the state $|\Psi\rangle$ and $|\Phi\rangle$ in field theory can be expressed as a functional integration,

$$\mathcal{C} = -\ln |Z| + \text{Im} \ln Z, \quad (3.17)$$

and

$$Z = \langle \Phi | \Psi \rangle = \int D[\varphi(x)] \Psi^*[\varphi(x)] \Psi[\varphi(x)] \quad (3.18)$$

where $\Psi[\varphi(x)] = \langle \varphi | \Psi \rangle$ is the wave functional of the state $|\Psi\rangle$. For a time evolution case, the complexity between $|\Psi(t)\rangle$ and $|\Psi_0\rangle = |\Psi(0)\rangle$ is given by Eq. (3.17) with

$$Z = \langle \Psi(t) | \Psi_0 \rangle = \int D[\varphi_1(x)] D[\varphi_2(x)] \Psi^*_0[\varphi_2(x)] \Psi_0[\varphi_1(x)] K[\varphi_2(x), t_2; \varphi_1(x), 0], \quad (3.19)$$

where

$$K[\varphi_2(x), t_2; \varphi_1(x), t_1] = \frac{1}{\mathcal{N}[\varphi_1, \varphi_2]} \int_{\varphi(x,t_1)=\varphi_1(x)}^{\varphi(x,t_2)=\varphi_2(x)} D[\varphi(x)] \exp \left\{ \frac{i}{\hbar} S[\varphi] \right\}. \quad (3.20)$$
Here, \( N[\varphi_1, \varphi_2] \) is the normalization factor and satisfies \( N[\varphi_1, \varphi_1] = 1 \). If we consider the classical limit \( \hbar \to 0 \) and assume the target state and reference state are eigenstates of the field operator, we have the following approximation

\[
Z = K \approx \frac{1}{N[\varphi_1, \varphi_2]} \exp \left\{ \frac{i}{\hbar} S_{\text{cl}}[\varphi] \right\} .
\]  

(3.21)

Here \( S_{\text{cl}}[\varphi] \) is the classical on-shell action. Up to the leading order of \( \hbar \), we have

\[
\bar{C} \approx \frac{1}{\hbar} \min \{|S_{\text{cl}}[\varphi]| \mid \forall \varphi(x, t), \text{ s.t. } \varphi(x, t_1) = \varphi_1(x), \varphi(x, t_2) = \varphi_2(x)\} ,
\]  

(3.22)

Here we assume \( \text{Re}(\ln N[\varphi_1, \varphi_2]) \ll S_{\text{cl}}/\hbar \) and \( \text{Im}(\ln N[\varphi_1, \varphi_2]) \ll S_{\text{cl}}/\hbar \). Here the minimization means that we choose the minimal on-shell action if the classical paths are not unique.

In particular, we are interested in the complexity between the field operator eigenstate \(|\varphi_0\rangle\) and the ground state \(|\Omega\rangle\) for a given Hamiltonian. The field operator eigenstate is the continuous limit of the product state in the configuration space, which is assumed as the reference state in the path-integral complexity. The inner product between these two states can be obtained by the Euclidean path integral:

\[
Z_E := \langle \varphi_0(x)|\Omega \rangle = \frac{1}{N} \int_{\varphi(x,0) = \varphi_0(x)} D[\varphi(x)] \exp \left\{ -\frac{1}{\hbar} S_E[\varphi] \right\} ,
\]  

(3.23)

where \( \varphi_0(x) = \langle x|\varphi_0 \rangle \), and \( S_E[\varphi] \) is the Euclidean action, and the normalization factor \( N \) is defined as

\[
N := \int_{\varphi(x,0) = \Omega(x)} D[\varphi(x)] \exp \left\{ -\frac{1}{\hbar} S_E[\varphi] \right\} .
\]  

(3.24)

so that \(|\langle \Omega|\Omega \rangle| = 1\).

The absolute symbol in the right-hand of Eq. (3.23) has been dropped as the function in the integration is positive definite. The upper bound of integration is omitted since, in the Euclidean case, the \( \varphi(x, \infty) \) is the ground state \( \Omega(x) = \langle x|\Omega \rangle \) and we do not need to specialize it. In the classical limit \( \hbar \to 0 \), the complexity between the ground state and a given eigenstate of the field operator is approximately \(^4\)

\[
\bar{C} = -\ln |\langle \varphi_0(x)|\Omega \rangle| = -\ln Z_E \\
\approx \frac{1}{\hbar} \min \{S_{\text{on-shell}}[\varphi] - S_0\} ,
\]  

(3.25)

where \( S_{\text{on-shell}}[\varphi] \) is the Euclidean on-shell action and \( S_0 = \ln N \approx S_E[\Omega] \) is the Euclidean on-shell action for the ground state (\( |\phi_0 \rangle = |\Omega \rangle \)). Here the minimization means that we choose the minimal on-shell action if the classical paths are not unique.

The equations (3.22) and (3.25) show that our proposal (3.3) indeed define a kind of “complexity” if we use the action (or Euclidean action) to define the cost. However, this complexity has many essential differences compared with the circuit complexity: (1) it does

\(^4\)In the Euclidean path integral, there is no imaginary part.
not allow people to choose the “gates” and penalties artificially; (2) it may be negative; (3) it is unitary invariant.

We now prove that our proposal (3.3) satisfies the triangle inequality not only for the infinitesimally close states but also for general states which have classical correspondences. Here, we assume that the classical trajectory is stable so the on-shell action is locally minimal. Consider three states \( \{|\varphi_i\rangle\} \) \((i=1,2,3)\), which are the eigenstates of field operator \( \varphi \) and correspond to the classical field configurations \( \varphi_i|_x = \varphi_i(x) \). Then according to our formula (3.22), up to the leading order we have

\[
\tilde{C}(|\varphi_i\rangle, |\varphi_j\rangle) = \frac{1}{\hbar} \min \{|S_{cl}[\varphi]| \mid \forall \varphi(x,t), \ s.t. \ \varphi(x,t_1) = \varphi_i(x), \ \varphi(x,t_2) = \varphi_j(x)\}, \tag{3.26}
\]

with \( i, j = 1, 2, 3 \). By this formula, we find

\[
\tilde{C}(|\varphi_i\rangle, |\varphi_j\rangle) + \tilde{C}(|\varphi_j\rangle, |\varphi_k\rangle) \geq \tilde{C}(|\varphi_i\rangle, |\varphi_k\rangle), \quad i, j, k = 1, 2, 3. \tag{3.27}
\]

Thus, for the quantum states which have classical correspondences, our formula gives us a kind of “distance”.

Let us make a few comments. Our proof of the triangle inequality Eq. (3.27) here is valid only for the quantum states which have classical correspondences. Otherwise, it is out of our scope. For example, we do not mean to apply our proposal (3.3) to some states which appear in quantum information processes, quantum circuits or quantum computations which do not have classical correspondences and even may not have Lagrangian formalism. In these cases, there are well-developed complexity theories in that context.

As we have repeatedly emphasized, the purpose of this paper is not to prove the holographic complexity is really equivalent to “circuit complexity” in every sense. It is still an open question and it is not something obvious a priori. We try to understand, without any prejudice, what the possible field theory dual of holographic complexity should be and then try to find certain relationships to the “complexity” of quantum circuits or quantum computations, if any.

Note that the boundary theory of an asymptotically AdS spacetime is not an arbitrary tunable quantum theory as in quantum circuits or quantum computations. Instead, it is a field theory which has classical correspondence. Our formula (3.3) is designed for such cases and, for such cases, we find our proposal satisfies the triangle inequality.

### 3.3 Proof for path-integral complexity

From Eq. (3.23), we can prove the conjecture about the “path-integral complexity” proposed by Refs. [26, 27] as follows. Let us consider a 2-dimensional conformal field theory embedded in a higher D-dimensional flat space \((2 < D < 25)\), which contains arbitrary matter fields coupling with string worldsheet. The classical action reads,

\[
S := (2\pi\alpha)^{-1}S_X + S_m[\varphi;g_{ab}], \tag{3.28}
\]

where \( S_X := \int d^2x g^{\mu\nu}\partial_\mu X^\mu \partial_\nu X^\nu \eta_{\mu\nu} \) and \( S_m \) is the string worldsheet action and a conformal matter fields action respectively. \( \alpha \) is the string coupling constant and is proportional to
string length square, $\eta_{\mu\nu}$ is the Minkowski metric at the D-dimensional background space. $g_{ab}$ is the induced metric in the worldsheet.

The Euclidean action can be written as \[ S_E = S_m[\varphi, \delta_{ab}] + \frac{1}{2\pi\alpha} (S_X[X^\mu, \delta_{ab}] + S_L[\phi, \delta_{ab}] + S_{gh}[b^{ab}, c_a, \delta_{ab}]), \] (3.29)

where $S_L[\phi, \delta_{ab}] := c/(24\pi) \int d^2x \left[ T^{ab} \delta_{ab} \phi \partial^2 \phi + \mu \partial \phi \right]$ is the Liouville action with the central charge $c$ and $S_{gh}[b^{ab}, c_a, \delta_{ab}]$ is the action for ghost fields. Assume $|\varphi_0\rangle$ is one common eigenstate when $\varphi = \varphi_0$, $X^\mu = X_0^\mu$ and $g_{ab}^{(E)} = \delta_{ab}$; and $|\Omega_\phi\rangle$ is the ground state satisfying $g_{ab}^{(E)}|z=z_0=0\rangle = e^{2\phi(z)}\delta_{ab}$, where $z$ is the Euclidean time and $z_0 = \epsilon \ll 1$ is a UV cut-off. Then we have

\[ \langle \varphi_0|\Omega_\phi\rangle = \left[\int D[\phi] D[D] D[X] D[b] D[c] \exp \left\{ -\frac{1}{\hbar} S_E \right\} \right] \langle \varphi_0|\Omega_0\rangle, \] (3.30)

where $|\Omega_0\rangle$ is the ground state when $\phi = 0$. Thus, the complexity between $|\varphi_0\rangle$ and $|\Omega_\phi\rangle$ reads

\[ \bar{C}[\phi] = -\ln \int_{\phi(x,z = \epsilon) = \phi(x)} D[\phi] \exp \left( -\frac{S_L}{2\pi\alpha\hbar} \right) - \ln \langle \varphi_0|\Omega_0\rangle, \] (3.31)

It is interesting to compare our result Eq. (3.31) with the proposal of the path integral complexity in Refs. [26, 27]. Refs. [26, 27] conjectured that the complexity between ground state $|\Omega\rangle$ and the field operator eigenstate $|\varphi_0\rangle$ was given by the on-shell Liouville action. In the small $\hbar\alpha$ limit, the saddle point approximation of Eq. (3.31) yields

\[ \bar{C} = \bar{C}(0) + \frac{S_L^{(c)}[\phi]}{2\pi\hbar\alpha}[1 + O(\hbar\alpha)], \] (3.32)

where $S_L^{(c)}[\phi]$ is the classical on-shell action of the Liouville action with the boundary condition $\phi(x, \epsilon) = \phi(x)$. Thus, we see that the conjecture of Refs. [26, 27] only includes the leading order term in classical limit. $\bar{C}(0)$ corresponds to $S_0$ in Eq (3.25).

Ref. [64] also gave a more exact diagrammatic argument about why $\bar{C}$ should be proportional to $S_L$ by the relationship between discretized path integrals and tensor network renormalization algorithm [63]. Our result is purely algebraic and the starting point has no relationship with the tensor network renormalization. This agreement, notwithstanding the different method, is an evidence supporting our proposal.

### 3.4 Relation to holographic conjectures

In this section, we will show that both the CV and CA conjectures can be understood from our proposal, which serves as other supporting evidences for our proposal. We also clarify what the reference states are in these two conjectures and why two conjectures have different behaviors at early time [6, 7].

---

5There are two different limits that we can recover the proposal for the Liouville action: $\hbar \to 0$ and $\alpha \to 0$. The former corresponds to the usual classical limit while the later corresponds to the weak coupling limit between the matter and string/gravity.
3.4.1 CV conjecture

Let us consider the ground state $|\Omega\rangle$ for a given CFT Hamiltonian $H_0$. Let us make a perturbation on this Hamiltonian $H_\delta = H_0 + H_I\delta$ with an infinitesimal parameter $\delta$ and obtain the perturbed ground state $|\Omega_\delta\rangle$. Suppose that $\mathcal{L}_0$ and $\mathcal{L}_\delta$ are the Euclidean Lagrangians of two Hamiltonians, then we have [69, 70]

$$\langle \Omega_\delta | \Omega_0 \rangle = \frac{1}{\sqrt{Z_0 Z_\delta}} \int \mathcal{D}\phi \exp \left[ -d^d x \left( \int_{-\infty}^0 d\tau \mathcal{L}_0 + \int_0^\infty d\tau \mathcal{L}_\delta \right) \right], \quad (3.33)$$

where $\phi$ stands for the field variable. It has been show in Refs. [69, 70] that Eq. (3.33) has the following expansion with respective to $\delta$

$$\langle \Omega_\delta | \Omega_0 \rangle = 1 - G_{\delta\delta} \delta^2 + \mathcal{O}(\delta^4), \quad (3.34)$$

where $G_{\delta\delta} \in \mathbb{R}$ is called the information metric [69] or fidelity of susceptibility [71]. Neglecting the higher order of $\delta$ we find that the complexity between $|\Omega\rangle$ and $|\Omega_\delta\rangle$ is

$$\mathcal{C}(|\Omega\rangle, |\Omega_\delta\rangle) = -\ln |\langle \Omega_\delta | \Omega_0 \rangle| = G_{\delta\delta} \delta^2. \quad (3.35)$$

Thus, we find a simple relationship between the complexity of the perturbed ground state and the information metric

$$\mathcal{C}(|\Omega\rangle, |\Omega_\delta\rangle) \propto G_{\delta\delta}. \quad (3.36)$$

Refs. [69, 70] have given some nontrivial evidence to show that, in conformal field theories with a small perturbation by a primary operator, the information metric is approximately given by a volume of the maximal time slice in the AdS spacetime, i.e.,

$$G_{\delta\delta} \propto \max_{\partial \Sigma = t_L \cup t_R} \text{Vol}(\Sigma). \quad (3.37)$$

Thus, with the holographic duality (3.37), we obtain, by Eq. (3.36),

$$\mathcal{C}(|\Omega\rangle, |\Omega_\delta\rangle) \propto \max_{\partial \Sigma = t_L \cup t_R} \text{Vol}(\Sigma). \quad (3.38)$$

This is nothing but the CV conjecture! The ground state of a CFT in holography is the TFD state dual to the double-sided black hole geometry. Thus, according to our proposal, the complexity in the CV conjecture can be interpreted as the complexity between the TFD state and its perturbed TFD state under a marginal operator, not the complexity between a TFD state and an unknown “simple” reference state, which is usually assumed in most literatures.

3.4.2 CA conjecture

Let us turn to the CA conjecture. We first consider Euclidean cases. From Eq. (3.25) we see that the complexity between the field operator eigenstate and the ground state of a given Hamiltonian is given by the partition function of the boundary field theory

$$\mathcal{C} = -\ln Z_{bd}[\phi(x)]. \quad (3.39)$$
On the other hand, the partition function of the boundary field theory in AdS/CFT correspondence is given by the partition function of a bulk gravity theory in asymptotic AdS spacetime

\[ Z_{\text{bd}}[\phi(x)] = Z_{\text{bulk}}[g_{\mu\nu}, \phi(x, z)], \tag{3.40} \]

with matter fields which satisfy the boundary condition \( \phi(x, z)|_{z=0} = \phi(x) \). Then we can find that Eq. (3.39) reads

\[ \bar{C} = -\ln \int D[g_{\mu\nu}] D[\phi] \exp \left\{ -\frac{1}{\hbar} S_E[g_{\mu\nu}, \phi(x, z)] \right\}. \tag{3.41} \]

where \( S_E \) is the Euclidian action of the bulk gravity with matters. In the weak gravity limit, we have the following leading order approximation

\[ \bar{C} \approx \frac{1}{\hbar} S_{E, \text{on-shell}}[g_{\mu\nu}, \phi(x, z)] = \frac{1}{\hbar} \left[ \int_{-\infty}^{\infty} dt \int_{V(t)} H_E(g_{\mu\nu}, \phi) d^d x + S_{\text{bd}} \right], \tag{3.42} \]

where \( H_E(g_{\mu\nu}, \phi) \) is the Euclidean Hamiltonian density, \( V(t) \) is a time slice in the bulk at time \( t \), and \( S_{\text{bd}} \) is a suitable boundary term. We assume that the bulk spacetime is \((d + 1)\)-dimensional.

In the Lorentzian case, we consider the complexity between two "field operators eigenstates". The complexity is given by \( \bar{C} = -\ln |Z| + \text{Im} \ln Z \), where \( Z \) is the inner product of two "field operators eigenstates" and is given by the path-integral

\[ Z = \int D[g_{\mu\nu}] D[\phi] \exp \left\{ \frac{i}{\hbar} S[g_{\mu\nu}, \phi(x, z)] \right\}. \tag{3.43} \]

In the classical limit \( \hbar \to 0 \), \( \bar{C} \) is dominated by the imaginary part of \( Z \) and so we have

\[ \bar{C} \approx \frac{1}{\hbar} |S_{\text{on-shell}}[g_{\mu\nu}, \phi(x, z)]| = \frac{1}{\hbar} \left[ \int_{-\infty}^{\infty} dt \int_{V(t)} L(g_{\mu\nu}, \phi) d^d x + S_{\text{bd}} \right], \tag{3.44} \]

where \( L(g_{\mu\nu}, \phi) \) is the Lagrangian density of the gravity theory with bulk matters. In order to compute the integration (3.44), we need to carefully define the integration region. The time slices \( V(t) \) should satisfy a suitable boundary condition so that the bulk region \( \cup_{t \in \mathbb{R}} V(t) \) can correspond to the boundary states given by \( t_L \) and \( t_R \). Thus, we require the time slices \( V(t) \) satisfies the following boundary condition

\[ \forall t, \quad \partial V(t) = t_L \cup t_R. \tag{3.45} \]

In the Lorentzian case, the set of \( V(t) \) is just the WdW-patch (see Fig. 5) and so we find

\[ \bar{C} \approx \frac{1}{\hbar} \left[ \int_{\text{WdW}} L(g_{\mu\nu}, \phi) d^{d+1} x + S_{\text{bd}} \right]. \tag{3.46} \]

The absolute sign disappears because it have been shown the on-shell action of WdW-patch is always positive [72]. Eq. (3.46) is nothing but the CA conjecture! There is another way to understand the physical meaning of such integration domain. When the boundary slice
In the Lorentzian case, after we fix the boundary condition for the time slice $\partial V(t) = t_L \cup t_R$, the bulk region to compute the on-shell action is just the WdW patch. The WdW-patch is also the entanglement wedge of boundary slice $t_L \cup t_R$.

$t_L \cup t_R$ has infinite volume (in 2-D boundary, it means the lengths of $t_L$ and $t_R$ are infinite), the entanglement surface then connect two slices and the entanglement wedge of $t_L \cup t_R$ is just the WdW-patch. Thus, we just choose the entanglement wedge of $t_L \cup t_R$ as the integration domain.

We see that the complexity in the CA conjecture describes the complexity between a ground state and the field operator eigenstate of a boundary field theory. In the holographic duality, the ground state of the boundary field theory is the TFD state corresponding to the double-sided AdS black hole. Thus, the CA conjecture gives the complexity between a TFD state and the eigenstate of the field operator of the boundary field theory. Note that it is not the complexity between two TFD states. This explains why the CV and CA conjecture show very different time-evolution behaviors at early time, which was reported in [6, 7].

4 Applications to the TFD states

In this section we will use our proposal to study the complexity of the TFD states. We will show our proposal can reproduce some results of the CA conjecture: both the divergent term and the finite term.

4.1 Complexity of time-independent TFD states

To compare with the CA conjecture, we consider a general TFD state

$$|\text{TFD}\rangle := \frac{1}{\sqrt{Z(\beta)}} \sum_{E_n} e^{-\beta E_n / 2} |E_n\rangle_R |E_n\rangle_L, \quad (4.1)$$

with the inverse temperature $\beta$, eigen-energy $E_n$ and partition function

$$Z(\beta) := \sum_{E_n} e^{-\beta E_n}. \quad (4.2)$$

As we have argued, in the CA conjecture, the reference state should be the eigenstate of the field operator. To write down such a state, in general we need the detailed action of a theory. Thus, we cannot directly construct such a reference state only based on the general
TFD state (4.1). Nevertheless, this reference state is a kind of “simple” state which has no spatial entanglement and there is another type of “no-spatial-entanglement” state. We know that the strong thermal fluctuation will destroy any quantum correlation, so the TFD state at infinite temperature has no spatial entanglement. Based on this property, we use the following state as a reference state

$$|\tilde{R}_\alpha^{(\varepsilon)}\rangle := \frac{1}{\sqrt{Z(\varepsilon\alpha)}} \sum_{E_n} e^{-\varepsilon\alpha E_n/2} |E_n\rangle R |E_n\rangle_L,$$

(4.3)

where \(\varepsilon \to 0\) stands for the UV cut-off and \(\alpha\) is an arbitrary positive constant. Though this state may not be an eigenstate of a field operator, it is still a kind of classical state with no spatial entanglement and we will show that it can be used as a reference state to reproduce basic properties of the CA conjecture.

As \(\langle \tilde{R}_\alpha^{(\varepsilon)} | \text{TFD} \rangle\) is a real number, the complexity between \(|\tilde{R}_\alpha^{(\varepsilon)}\rangle\) and \(|\text{TFD}\rangle\) in our proposal reads

$$\bar{C} = -\ln\langle \tilde{R}_\alpha^{(\varepsilon)} | \text{TFD} \rangle.$$

(4.4)

It is easy to find

$$\langle \tilde{R}_\alpha^{(\varepsilon)} | \text{TFD} \rangle = \frac{1}{\sqrt{Z(\beta)Z(\alpha\varepsilon)}} \sum_{E_n} e^{-(\beta+\alpha\varepsilon)E_n/2} = \frac{Z(\beta/2)}{\sqrt{Z(\beta)Z(\alpha\varepsilon)}},$$

(4.5)

where we neglected the \(\varepsilon\) term in \(\beta + \alpha\varepsilon\). Then our proposal (4.4) gives

$$\bar{C} = -\ln\frac{Z(\beta/2)}{\sqrt{Z(\beta)Z(\alpha\varepsilon)}}.$$  

(4.6)

For a CFT theory in \(d\)-dimensional spacetime, the partition function of a thermal field has the universal form

$$Z(\beta) \propto \exp(b_d V \beta^{1-d}) = \exp(b_d VT^{d-1}),$$

(4.7)

where \(b_d\) is a constant and proportional to the central charge of the CFT. Plugging Eq. (4.7) in Eq. (4.6) we obtain

$$\bar{C} = \frac{1}{2} \alpha^{1-d} b_d V [\varepsilon^{1-d} + \alpha^{d-1}(1 - 2^{2-d}) T^{d-1}].$$

(4.8)

Note that the overall factor of complexity is irrelevant. It has been shown that the complexity of the CA conjecture in the AdS-Schwarzschild black hole [18] is given by

$$C = \frac{\ln d}{4\pi} V \varepsilon^{1-d} + \frac{d-2}{4\pi d \cot(\pi/d)} \left(\frac{4\pi}{d}\right)^{d-1} VT^{d-1},$$

(4.9)

where we set \(G_N = 1\). By choosing the parameter \(\alpha\) suitably in (4.8), we can recover the holographic result (4.9) exactly.

It is interesting to compare our result with other proposals, which start from the complexity of quantum circuits: for example, the proposal of Ref. [73] based on the cMERA or the proposal of Ref. [33] based the Fubini-Study metric. Both methods can reproduce the
leading divergent term of the holographic complexity. However, in Ref. [73], $L^1$ norm is assumed artificially and, in Ref. [33] the generators set is restrict to be SU(1,1) and $L^1$ norm is also assumed artificially. In addition, though the leading divergent term was reproduced in their work, the temperature-dependent term of Eq. (4.9) cannot be reproduced.

In our framework, Eq. (4.8) naturally appears without artificial assumptions such as $L^1$ norm or specific generators set. We stress that we obtained the correct temperature-dependent term as well as the divergence structure. To our knowledge, this is the only case that a field theory proposal reproduce the complete form of the holographic result (4.9).

4.2 Time evolution and compatibility with holographic results

We also would like to make a short comment regarding the inner product between a TFD state and its time-evolution state at large time limit. A time-dependent TFD state is given by

$$|\text{TFD}(t)\rangle := \frac{1}{\sqrt{Z(\beta)}} \sum_{E_n} e^{-(\beta+2it)E_n/2} |E_n\rangle_R |E_n\rangle_L.$$  \hspace{1cm} (4.10)

The complexity between $|\text{TFD}(t)\rangle$ and $|\text{TFD}(0)\rangle$ in our proposal reads

$$\bar{C}(t) = -\ln |F(t)| + \text{Im} \ln F(t),$$ \hspace{1cm} (4.11)

where

$$F(t) := \langle \text{TFD}(t)|\text{TFD}(0)\rangle.$$ \hspace{1cm} (4.12)

In the CA conjecture, the reference state is conjectured to be a kind of “simple” state. In this paper, we have argued that this “simple” state should be an eigenstate of the field operator and we denote it by $|R\rangle$. Though, in the previous subsection, we used $|\tilde{R}(\epsilon)\rangle$ (4.3) as a reference state and reproduced a few basic properties of complexity in the CA conjecture, here we do not assume $|R\rangle = |\tilde{R}(\epsilon)\rangle$.

Let us introduce three different complexities: i) $\bar{C}_R(0)$ is the complexity between $|R\rangle$ and the initial TFD state $|\text{TFD}(0)\rangle$, ii) $\bar{C}_R(t)$ is the complexity between $|R\rangle$ and the time-dependent TFD state $|\text{TFD}(t)\rangle$, iii) $\bar{C}(t)$ is the complexity between $|\text{TFD}(0)\rangle$ and $|\text{TFD}(t)\rangle$. If the complexity in the CA conjecture stands for a kind of distance [30–32], the triangle inequality implies

$$\bar{C}_R(t) - \bar{C}_R(0) \leq \bar{C}(t).$$ \hspace{1cm} (4.13)

In the CA conjecture, it has been discovered that the complexity between $|R\rangle$ and $|\text{TFD}(t)\rangle$ will grow linearly at late-time limit. Thus, if we accept the result of the CA conjecture as $\bar{C}_R(t)$, we may conclude that $\bar{C}(t)$ should increase forever as time goes on: $\bar{C}(\infty) \rightarrow \infty$, i.e. the complexity between between $|\text{TFD}(0)\rangle$ and $|\text{TFD}(t)\rangle$ should grow to infinity as $t \rightarrow \infty$. Let us check it concretely.

Plugging Eq. (4.10) into Eq. (4.12) we obtain

$$F(t) = \frac{1}{Z(\beta)} \sum_{E_n} e^{-(\beta+it)E_n} = \frac{Z(\beta + it)}{Z(\beta)}.$$ \hspace{1cm} (4.14)
In Eq. (4.14) we have
\[ Z(\beta + it) = \sum_n e^{-\beta E_n} e^{iE_n t}. \] (4.15)

In the continuum limit, we may replace the sum with the integral:
\[ Z(\beta + it) = \int_0^\infty N(E) e^{-\beta E} e^{iEt} dE, \] (4.16)

where the density of state \( N(E) \) is introduced and \( N(E) dE \) is the state number when energy is in \( E \sim E + dE \). It is clear that
\[ \int_0^\infty |N(E) e^{-\beta E}| dE = \int_0^\infty N(E) e^{-\beta E} dE = Z(\beta), \] (4.17)

which is finite (here we assume that volume \( V \) is large but finite). Then the Riemann-Lebesgue lemma says that
\[ \lim_{t \to \infty} Z(\beta + it) = \lim_{t \to \infty} \int_0^\infty N(E) e^{-\beta E} e^{-itE} dE = 0, \] (4.18)

so
\[ \lim_{t \to \infty} F(t) = 0. \] (4.19)

It implies that the complexity between \( |\text{TFD}(t)\rangle \) and \( |\text{TFD}(0)\rangle \) will grow forever.\(^6\) This is another nice consistency check of our proposal with the CV conjecture. As we do not know the reference state exactly just from the general ‘formal definition’ of the TFD state (4.1), we cannot directly compute \( \bar{C}_R(t) \). Thus, the consistency check we just showed is the best we can do.

5 A comment on Fubini-Study distance

In this section, we discuss how our proposal can resolve the problem in the Fubini-Study distance. In Ref. [32] it was argued that that the function of the Fubini-Study distance could not be the complexity because it cannot distinguish one-flip from multi-flips. However, this should not be considered as an objection to using the inner product. Our purpose here is to show that this problem (distinguishing one-flip from multi-flips) may be solved by our proposal (3.3).

For example, let us consider the following two states in a \( n \)-qubit system
\[ |\psi_1\rangle = |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_n\rangle, \] (5.1)
and
\[ |\psi_2\rangle = |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle, \] (5.2)

with \( a_i = 0, 1 \) and \( b_i = 0, 1 \). Let us first consider the Fubini-Study distance:
\[ D_{FS} = \arccos \left| \prod_{i=1}^n \langle a_i | b_i \rangle \right|. \] (5.3)

\(^6\)See Ref. [37] for another way of computation of \( F(t) \) by an analytic continuation of Eq. (4.7). In our opinion, it seems that a simple analytical continuation may be misleading.
If we start with $|\psi_1\rangle = |\psi_2\rangle$ we can flip only one qubit to change the complexity from zero into $\pi/2$. This does not reflect the property of the complexity as we expect changing just one qubit should not change the complexity too much if the system is large enough. If we start with $\langle \psi_1 | \psi_2 \rangle = 0$, flipping some of the qubit will not change the complexity at all. This again does not reflect the property of the complexity: the state is changing but the complexity is not.

Next, let us consider our proposal:

$$\bar{C} = \bar{C}_r + \bar{C}_{\text{Im}}, \quad (5.4)$$

where

$$\bar{C}_r = - \sum_{i=1}^{n} \ln |\langle a_i | b_i \rangle|, \quad \bar{C}_{\text{Im}} = \left| \text{Im} \sum_{i=1}^{n} \ln |\langle a_i | b_i \rangle| \right|. \quad (5.5)$$

Thanks to “log”, naively every flip will have one-unit of the complexity, which is a desired property for the complexity. However, in this case, one-unit of the complexity seems infinite. This infinity problem can be resolved if we note that our formula (3.3) are proposed for the system in the continuum and thermodynamics limit rather than finite discrete systems.

Let us focus on the real part $\bar{C}_r$. For a continuous 1 dimensional system

$$\bar{C}_r = - \ln |\langle \psi_1 | \psi_2 \rangle| = -2L \int \ln |\langle a(k) | b(k) \rangle| dk, \quad (5.6)$$

where the discrete index $i$ is replaced by the continuous label $k$ and $L$ has the dimension of $|k|^{-1}$. Let us assume the states are ‘regular’ states, which means that the inner product

$$\langle a(k) | b(k) \rangle \text{ is the analytical function of } k. \quad (5.7)$$

If we change the states of $k \in (k_0 - \delta, k_0 + \delta)$ so that $\langle a(k_0) | b(k_0) \rangle \neq 0 \rightarrow \langle \tilde{a}(k_0) | \tilde{b}(k_0) \rangle = 0$, the change of the complexity is

$$\delta \bar{C}_r = -L \int_{-\delta}^{\delta} [\ln |\langle \tilde{a}(k_0 + x) | \tilde{b}(k_0 + x) \rangle| - \ln |\langle a(k_0 + x) | b(k_0 + x) \rangle|] dx. \quad (5.8)$$

This integration is finite again thanks to ‘log’, although $\langle \tilde{a}(k_0 + x) | \tilde{b}(k_0 + x) \rangle$ is zero at $x = 0$. In the limit of $\delta \to 0$, which corresponds to “flipping exactly one qubit”, $\delta \bar{C}_r = 0$ as expected. The same is also true for $\bar{C}_{\text{Im}}$.

If we try to use our proposal for the discrete system, such as the $n$-qubit system, we have to make a suitable regularization in the argument of “$\ln$” and a suitable discretization on the integration measure “$\int dk$”. For a $n$-qubit system, a convenient method is that

$$L \int dk \to \sum, \quad (5.9)$$

for a discretization and

$$\ln |\langle \cdot | \cdot \rangle| \to \ln(|\langle \cdot | \cdot \rangle| + \varepsilon), \quad (5.10)$$
with $\bar{\varepsilon} \ll 1$ for a regularization. Then, for two states Eq. (5.1) and (5.2) in the $n$-qubit system, the discrete version of Eq. (5.6) is

$$\bar{C}_r = -\sum_{i=1}^{n} \ln \left( \frac{|\langle a_i | b_i \rangle| + \bar{\varepsilon}}{1 + \bar{\varepsilon}} \right), \quad (5.11)$$

where the denominator $1 + \bar{\varepsilon}$ is introduced to ensure that the complexity between the same states is zero. Suppose that we start with the case $|\psi_1 \rangle = |\psi_2 \rangle$, so $\bar{C}(|\psi_2 \rangle, |\psi_1 \rangle) = 0$. If we flip only one qubit in $|\psi_2 \rangle$, then the complexity will change by $\bar{C}_0 = \bar{C}_r = -2 \ln(\bar{\varepsilon}/(1 + \bar{\varepsilon})) \sim -2 \ln \bar{\varepsilon}$. If we flip $n$ qubits in $|\psi_2 \rangle$ the complexity is changed by $n\bar{C}_0$, which is a desired property of the complexity.

At first glance, the cut-off term $\bar{\varepsilon}$ in Eq. (5.10) looks artificial. However, this may be understood as follows. Suppose that we want to create two qubits $|a_{th} \rangle$ and $|b_{th} \rangle$. Mathematically, it is easy to write down but physically we have to use some physical systems to realize them. Due to unavoidable quantum and thermal fluctuations, what we really observe are two states $|a_{ob} \rangle$ and $|b_{ob} \rangle$ and their inner product is

$$|\langle a_{ob} | b_{ob} \rangle| = |\langle a_{th} | b_{th} \rangle| + \varepsilon, \quad (5.12)$$

Thus, we find

$$\bar{\varepsilon} = |\langle a_{ob} | b_{ob} \rangle| - |\langle a_{th} | b_{th} \rangle|, \quad (5.13)$$

where the “$\bar{X}$” stands for the average observations of the variable $X$ and $\bar{\varepsilon}$ is an “error” which is due to the intrinsic quantum and thermal fluctuations. When $|\langle a_{th} | b_{th} \rangle| > 0$, the $|\langle a_{ob} | b_{ob} \rangle|$ can be larger or less than $|\langle a_{th} | b_{th} \rangle|$, i.e. $\varepsilon$ can be positive or negative, so $\bar{\varepsilon} \rightarrow 0$. However, if $|\langle a_{th} | b_{th} \rangle| = 0$, we will always have $|\langle a_{ob} | b_{ob} \rangle| \geq 0$ so we find $\bar{\varepsilon} > 0$. The value of $\bar{\varepsilon} > 0$ is determined by the physical limit of observation. We, in principle, could reduce it by using complicated systems, which will increase the complexity as expected. It is reflected in our formula $\bar{C}_0 \sim -\ln \bar{\varepsilon}$ in the previous paragraph.

6 Conclusions

Complexity is a quantum informational quantity, which essentially depends on the choices of the basic operations (gates) and their costs (penalties). There are proposals for the holographic duals of the complexity. In the complexity-volume and the complexity-action conjectures, the complexity is conjectured to be the volume of a maximal spatial hyper-surface or the on-shell action of the bulk theory in a special spacetime region. In both conjectures, they neither tell us what the fundamental operators (gates) are nor tell us how to choose the costs (penalties). Thus, there are two fundamental questions: if the holographic complexity is indeed a kind of complexity, in their field theory duals, what are the basic gates and their costs?

Towards the answer to these fundamental questions, we choose a different strategy. We do not assume anything from usual concepts from the complexity in quantum information theory. In particular, we do not assume that the field theory dual of holographic complexity is a continuum version of the circuit complexity. Without any prejudice, we start with
the inherent properties of the holographic complexity and try to understand the essential features that the field theory duals should have. In principle, the dual may not be any kind of complexity.

We argue that any field theory dual of the holographic complexity should have two basic properties: (1) it is invariant under infinitely many independent unitary transformations and (2) it is extensive in thermodynamic limit. These two basic properties are inferred from the holographic complexity itself but cannot be obtained by the general analyses of quantum informational setups including “circuit complexity” or “operators complexity”. Guided by these two properties, we proposed a possible candidate for the field theory dual of the holographic complexity:

$$\mathcal{C}(|\psi_1\rangle, |\psi_2\rangle) = \alpha_1 \ln |\langle\psi_1|\psi_2\rangle| + \alpha_2 |\text{Im} \ln \langle\psi_1|\psi_2\rangle|.$$  (6.1)

This formula looks very simple but its contents are indeed rich.

First, we have shown that the complexity between the ground state of a given Hamiltonian and the eigenstate of the field operator is given by the partition function. In the classical limit, the partition function is given by the on-shell Euclidian action, by which we gave a proof for the relationship between the “path-integral complexity” and the Liouville action.

Second, we also used our proposal to give natural explanations for both the CV and CA conjectures and clarified the reference and target states of them. The CV conjecture is dual to the complexity between the TFD state and its perturbed TFD state under a marginal operator. The CA conjecture is dual to the complexity between a TFD state and the eigenstate of the field operator of the boundary field theory. This difference between the CV and CA conjecture from a field theory perspective naturally explains why the holographic CV and CA conjecture show different time evolution.

Finally, if our proposal is applied to the TFD states we find that it is compatible with the holographic complexities at late time. Furthermore, our proposal can reproduce the holographic results of the CA conjecture well. It yields both the correct divergent structure and the temperature-dependent term. To our knowledge, this is the only field theory proposal that reproduces both terms in the result from the CA conjecture. Our proposal also can resolve the problems in the Fubini-Study distance.

One basic property of our proposal (6.1) is that it is unitary invariant. It is often claimed [32, 74] that the complexity must be non-unitary invariant because a unitary-invariant complexity cannot reproduce the “expected” time evolution of the complexity: for a chaotic system with $N$ degrees of freedom, the complexity evolves as time goes, in three stages: linear growth until $t \sim e^N$, saturation and small fluctuations after then, and quantum recurrence at $t \sim e^{e^N}$. However, the counter example of this claim is shown in Ref. [75], where the unitary-invariant or bi-invariant complexity can indeed realize the expected time evolution. The example in [75] is a supporting evidence for our claim that the field theory dual of the holographic complexity may be unitary-invariant. What is more, the unitary invariance also matches with the fact that holographic complexity is diffeomorphic invariant. Every bulk diffeomorphic transformation will induce a unitary transformation on the boundary states, so the boundary complexity should be invariant under infinitely
many independent unitary transformations. To the best of our knowledge, there is no non-unitary invariant complexity which is invariant under these infinitely many independent unitary transformations.

We want to emphasize that there is nothing wrong with the “non-unitary-invariance” of the complexity in real quantum circuits. The essential question we are asking in this paper is “what is the boundary field theory dual corresponding to the holographic complexity?”. Therefore, the properties of the complexity of the real circuits are never requirements. They must be consequences, if possible. If we assume that the holographic complexity is dual to a kind of continuum version of the discrete circuit complexity, in our opinion, some of the properties of the real circuit complexity need to be modified to satisfy two properties we proposed of this paper.

One may argue that, our proposal is basically a certain function of an inner product, which describes some properties of “overlap” between two states. Regarding this viewpoint, we want to make two comments. First, we have shown in the end of Secs. 3.1 and 3.2 that our proposal satisfies the triangle inequality so stands for a kind of “distance”. Second, if a simple function of the inner product can reproduce most of basic properties of holographic complexity, there is no reason to naively abandon the possibility that holographic complexity in fact describes some properties of “overlap” of boundary states. Holographic complexity may or may not be related to kind of complexity, which is still an open question, to our understanding. We believe that looking at the problem from a different angle will be a meaningful starting point to understand holographic complexity better.

Acknowledgments

The work of K.-Y. Kim was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning(NRF- 2017R1A2B4004810) and GIST Research Institute(GRI) grant funded by the GIST in 2020. C. Niu is supported by the Natural Science Foundation of China under Grant No. 11805083. C.Y. Zhang is supported by Project funded by China Postdoctoral Science Foundation. R.-Q. Yang is supported by the Natural Science Foundation of China under Grant No. 12005155. We also would like to thank the APCTP(Asia-Pacific Center for Theoretical Physics) focus program,”Holography and geometry of quantum entanglement” in Seoul, Korea for the hospitality during our visit, where part of this work was done.

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