Metric-affine Myrzakulov gravity and its generalizations

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Abstract

Since the discovery of cosmic acceleration, modified gravity theories play an important role in the modern cosmology. In particular, the well-known F(R)-theories reached great popularity motivated by the easier formalism and by the prospect to find a final theories of gravity for the dark scenarios. In the present work, we study some generalizations of F(R), F(T) and F(Q) gravity theories, where \( R, T, Q \) are the Ricci, torsion and nonmetricity scalars. At the beginning, we briefly review the formalism of such theories. Then, we will consider one of their generalizations, the so-called Myrzakulov F(R,T) gravity theory or the MG-I theory. The point-like Lagrangian is explicitly presented. Based on this Lagrangian, the field equations of MG-I are found. For the specific model \( F(R,T) = \mu R + \nu T \), the corresponding exact solutions are derived. Furthermore, we will consider the physical quantities associated to such solutions and we will find how for some values of the parameters the expansion of our universe can be accelerated without introducing any dark component. Finally, some metric-affine Myrzakulov gravity theories with and without boundary term scalars are presented.

Contents

1 Introduction .................................................. 3
2 Preliminaries of \( F(R), F(G) \) and \( F(T) \) gravities ................. 4
2.1 \( F(R) \) gravity ............................................. 4
2.2 \( F(T) \) gravity ............................................. 5
2.3 \( F(G) \) gravity ............................................. 5
3 A naive model of \( F(R,T) \) gravity ................................ 5
4 Reductions, Preliminary classification ............................... 7
4.1 Case: \( F = R \) ............................................ 7
4.2 Case: \( F = T \) ............................................ 8
4.3 Case: \( F = F(T), u = v = 0 \) ............................... 8
4.4 Case: \( F = F(R), u = v = 0 \) ............................... 9
4.5 The \( M_{37A} \) - model ..................................... 9
4.6 The \( M_{37B} \) - model ..................................... 10
4.7 The \( M_{37C} \) - model ..................................... 10
4.8 The \( M_{37D} \) - model ..................................... 10
4.9 The \( M_{37E} \) - model ..................................... 11
4.10 The \( M_{37F} \) - model .................................. 11
4.11 The \( M_{37G} \) - model .................................. 11
5 The particular model: \( F(R,T) = \mu R + \nu T \) .................. 11

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6 Cosmological implications. Dark energy

7 $F(R,T)$ gravity: Bianchi type I model

8 Other generalizations and reductions of some generalized gravity models

9 Other examples of metric-affine Myrzakulov gravity theories

10 Cosmology in MG theories

11 Spherically symmetric and black hole solutions in MG theories

12 MG theories with the boundary term scalars
1 Introduction

Recent observational data imply -against any previous belief- that the current expansion of the universe is accelerating [1]. Since this discovery, the so-called Dark Energy issue has probably become the most ambitious and tantalizing field of research because of its implications in fundamental physics. There exist several descriptions of the cosmic acceleration. Among them, the simplest one is the introduction of small positive Cosmological Constant in the framework of General Relativity (GR), the so-called ΛCDM Model, but it is well accepted the idea according to which this is not the ultimate theory of gravity, but an extremely good approximation valid in the present day range of detection. A generalization of this simple modification of GR consists in considering modified gravitational theories [1-2]. In the last years the interest in modified gravity theories like $F(R)$ and $F(G)$-gravity as alternatives to the ΛCDM Model grew up.

Recently, a new modified gravity theory, namely the $F(T)$-theory, has been proposed. This is a generalized version of the teleparallel gravity originally proposed by Einstein [3]-[16]. In this paper, our aim is to replace these quantities with the other three variables in the form

\[
R_s = g^{\mu\nu} R_{\mu\nu}, \\
G_s = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}, \\
T_s = S_\mu^{\mu\nu} T_{\mu\nu}.
\]

In this paper, our aim is to replace these quantities with the other three variables in the form

\[
R = u + g^{\mu\nu} R_{\mu\nu}, \\
G = w + R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}, \\
T = v + S_\mu^{\mu\nu} T_{\mu\nu},
\]

where $u = u(x; g_{ij}, g_{ij}, g_{ij}, ..., f_j), v = v(x; g_{ij}, g_{ij}, g_{ij}, ..., g_j)$ and $w = w(x; g_{ij}, g_{ij}, g_{ij}, ..., h_j)$ are some functions to be defined. As a result, we obtain some generalizations of the known modified gravity in Friedmann-Robertson-Walker (FRW) universe. The flat FRW space-time is described by the metric

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\]

where $a(t)$ is the scale factor. The orthonormal tetrad components $e_i(x^\mu)$ are related to the metric through

\[
g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu,
\]

where the Latin indices $i, j$ run over 0...3 for the tangent space of the manifold, while the Greek letters $\mu, \nu$ are the coordinate indices on the manifold, also running over 0...3.

$F(R)$ and $F(T)$ modified theories of gravity have been extensively explored and the possibility to construct viable models in their frameworks has been carefully analyzed in several papers (see [17] for a recent review). For such theories, the physical motivations are principally related to the possibility to reach a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields. Recently, it has been registered a renaissance of $F(R)$ gravity and $F(T)$ gravity theories in the attempt to explain the late-time accelerated expansion of the Universe [17]. In the modern cosmology, in order to construct (generalized) gravity theories, three quantities – the curvature scalar, the Gauss–Bonnet scalar and the torsion scalar – are usually used (about our notations see below):
gravity theories. With the FRW metric ansatz the three variables \(R\), \(G\), \(T\) become

\[
\begin{align*}
R_s &= 6(\dot{H} + 2H^2), \\
G_s &= 24H^2(\dot{H} + H^2), \\
T_s &= -6H^2,
\end{align*}
\]

where \(H = (\ln a)_t\). In the contrast, in this paper we will use the following three variables

\[
\begin{align*}
R &= u + 6(\dot{H} + 2H^2), \\
G &= w + 24H^2(\dot{H} + H^2), \\
T &= v - 6H^2.
\end{align*}
\]

This paper is organized as follows. In Sec. 2, we briefly review the formalism of \(F(R)\), \(F(T)\) and \(F(G)\) gravity for FRW metric. In particular, the corresponding Lagrangians are explicitly presented. In Sec. 3, we consider \(F(R, T)\) theory, where \(R\) and \(T\) will be generalized with respect to the usual notions of curvature scalar and torsion scalar. Some reductions of \(F(R, T)\) gravity are presented in Sec. 4. In Sec. 5, the specific model \(F(R, T) = \mu R + \nu T\) is analyzed and in Sec. 6 the exact power-law solution is found; some cosmological implications of the model will be here discussed. The Bianchi type I version of \(F(R, T)\) gravity is considered in Sec. 7. Sec. 8 is devoted to some generalizations of some modified gravity theories. The other examples of metric-affine Myrzakulov gravity theories were presented in Sec. 9. In sections 10, 11 and 12, we briefly mentioned our approach to the problem for the cosmological, spherically symmetric (black hole) solutions in MG including theories with the boundary term scalars. Final conclusions and remarks are provided in Sec. 13.

## 2 Preliminaries of \(F(R)\), \(F(G)\) and \(F(T)\) gravities

At the beginning, we present the basic equations of \(F(R)\), \(F(T)\) and \(F(G)\) modified gravity theories. For simplicity we mainly work in the FRW spacetime.

### 2.1 \(F(R)\) gravity

The action of \(F(R)\) theory is given by

\[
S_R = \int d^4x [F(R) + L_m],
\]

where \(R\) is the curvature scalar. We work with the FRW metric \(10.1\). In this case \(R\) assumes the form

\[
R = R_s = 6(\dot{H} + 2H^2).
\]

The action we rewrite as

\[
S_R = \int dt L_R,
\]

where the Lagrangian is given by

\[
L_R = a^3(F - RF_R) - 6F_R a\dot{a}^2 - 6F_{RR} \dot{R} a^2 - a^3 L_m.
\]

The corresponding field equations of \(F(R)\) gravity read

\[
\begin{align*}
-6\dot{R} F_{RR} - (R - 6H^2)F_R + F &= \rho, \\
-2\dot{R}^2 F_{RRR} + [-4\dot{R} H - 2\dot{R}] F_{RR} + [-2H^2 - 4a^{-1}\ddot{a} + R] F_R - F &= p, \\
\dot{\rho} + 3H(\rho + p) &= 0.
\end{align*}
\]
2.2 \( F(T) \) gravity

In the modified teleparallel gravity, the gravitational action is

\[
S_T = \int d^4 x e [F(T) + L_m],
\]

(2.8)

where \( e = \det (e^i_\mu) = \sqrt{-g} \), and for convenience we use the units \( 16\pi G = \hbar = c = 1 \) throughout.

The torsion scalar \( T \) is defined as

\[
T \equiv S_{\rho \mu \nu} T^\rho_{\mu \nu},
\]

(2.9)

where

\[
T^\rho_{\mu \nu} \equiv -e^i_\rho (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu),
\]

(2.10)

\[
K^\rho_{\mu \nu} \equiv -\frac{1}{2} (T^{\rho \nu \mu} - T^{\nu \mu \rho} - T^{\rho \mu \nu}),
\]

(2.11)

\[
S^\rho_{\mu \nu} \equiv \frac{1}{2} (K^\rho_{\mu \nu} + \delta^\rho_\mu T^\theta_{\nu \theta} - \delta^\rho_\nu T^\theta_{\mu \theta}).
\]

(2.12)

For a spatially flat FRW metric \( (10.1) \), as a consequence of equations (2.9) and \( (10.1) \), we have that the torsion scalar assumes the form

\[
T = T_s = -6H^2.
\]

(2.13)

The action (2.8) can be written as

\[
S_T = \int dt L_T,
\]

(2.14)

where the point-like Lagrangian reads

\[
L_T = a^3 (F - F_T T) - 6F_T \dot{a} \dot{a} - a^3 L_m.
\]

(2.15)

The equations of \( F(T) \) gravity look like

\[
12H^2 F_T + F = \rho,
\]

(2.16)

\[
48H^2 F_T \dot{H} - F_T (12H^2 + 4\dot{H}) - F = p,
\]

(2.17)

\[
\dot{\rho} + 3H (\rho + p) = 0.
\]

(2.18)

2.3 \( F(G) \) gravity

The action of \( F(G) \) theory is given by

\[
S_G = \int d^4 x e [F(G) + L_m],
\]

(2.19)

where the Gauss–Bonnet scalar \( G \) for the FRW metric is

\[
G = G_s = 2AH^2 (\dot{H} + H^2).
\]

(2.20)

3 A naive model of \( F(R, T) \) gravity

Our aim in this section is to present a naive version of \( F(R, T) \) gravity. We assume that the relevant action of \( F(R, T) \) theory is given by \( [14] \)

\[
S_{37} = \int d^4 x e [F(R, T) + L_m],
\]

(3.1)

where \( R = u + R_s \) and \( T = v + T_s \) are some dynamical geometrical variables to be defined, and \( R_s \) and \( T_s \) are the usual curvature scalar and the torsion scalar for the FRW spacetime. It is the
so-called $M_{37}$ - model [14]. In this paper we will restrict ourselves to the simple case where for
FRW spacetime $R$ and $T$ are given by

$$
R = u + 6(\dot{H} + 2H^2) = u + R_s, \tag{3.2}
$$

$$
T = v - 6H^2 = v + T_s. \tag{3.3}
$$

As we can see these two variables $(R, T)$ are some analogies (generalizations) of the usual curvature
scalar $(R_s)$ and torsion scalar $(T_s)$ and for obvious reasons we will still continue to call them as the
"curvature" scalar" and the "torsion" scalar. We note that, in general, $u = u(t, a, \dot{a}, \ddot{a}, \ldots; f_i)$
and $v = v(t, a, \dot{a}, \ddot{a}, \ldots; g_i)$ are some real functions, $H = (\ln a)$, while $f_i$ and $g_i$ are some unknown
functions related with the geometry of the spacetime. Finally we can write the $M_{37}$ - model for
the FRW spacetime as

$$
S_{37} = \int d^4x[F(R, T) + L_m], \tag{3.4}
$$

$$
R = u + 6(\dot{H} + 2H^2), \tag{3.5}
$$

$$
T = v - 6H^2. \tag{3.6}
$$

In this paper we restrict ourselves to the case $u = u(a, \dot{a})$ and $v = v(a, \dot{a})$. The scale factor $a(t)$,
the curvature scalar $R$ and the torsion scalar $T(t)$ are taken as independent dynamical variables.
Then, after some algebra the action (3.4) becomes

$$
S_{37} = \int dtL, \tag{3.7}
$$

where the point-like Lagrangian is given by

$$
L_{37} = a^3(F - TF_T - RF_R + vF_T + uF_R) - 6(F_R + F_T)a\dot{a}^2 - 6(F_{RR} + F_{RT}a)a^2\dot{a} - a^3L_m. \tag{3.8}
$$

The corresponding equations of the $M_{37}$ - model assume the form [13]

$$
D_2 F_{RR} + D_1 F_R + JF_{RT} + E_1 F_T + KF = -2a^3\rho, \tag{3.9}
$$

$$
U + B_2 F_{TT} + B_1 F_T + C_2 F_{RRT} + C_1 F_{RT} + C_0 F_R + MF = 6a^2 \rho, \tag{3.10}
$$

$$
\dot{\rho} + 3H(\rho + p) = 0. \tag{3.11}
$$

$$
U = a_3 F_{RRR} + A_2 F_{RR} + A_1 F_R, \tag{3.12}
$$

$$
A_3 = -6R^2a^2, \tag{3.13}
$$

$$
A_2 = -6R^2a^2 - 12\dot{R}a\dot{a} - a^3\dot{R}u_a, \tag{3.14}
$$

$$
A_1 = -6\dot{a}^2 - 12a\ddot{a} + 3a^2\dot{u}_a + a^3\ddot{u}_a - 3a^2(u - R) - a^3u_a, \tag{3.15}
$$

$$
B_2 = -12\ddot{R}\dot{a} + a^3\ddot{T}v_a, \tag{3.16}
$$

$$
B_1 = -6\dot{a}^2 - 12a\ddot{a} + 3a^2\dot{u}_a + a^3\ddot{u}_a - 3a^2(v - T) - a^3v_a, \tag{3.17}
$$

$$
C_2 = -12a^2\ddot{R}\ddot{T}, \tag{3.18}
$$

$$
C_1 = -6a^2\dot{T}^2, \tag{3.19}
$$

$$
C_0 = -12\dot{R}\ddot{a} - 12\ddot{T}\ddot{a} - 6a^2\ddot{T} + a^3\ddot{R}v_a + a^3\ddot{T}u_a, \tag{3.20}
$$

$$
M = -3a^2. \tag{3.21}
$$
We can rewrite the system in terms of $H$ as

\[
DF_{RR} + D_1 F_R + J F_{RT} + E_1 F_T + K F = -2a^3 \rho, \\
U + B_2 F_{TT} + B_1 F_T + C_2 F_{RRT} + C_1 F_{RRT} + C_0 F_{RT} + M F = 6a^2 p, \\
\dot{p} + 3H(\rho + p) = 0,
\]

where

\[
D_2 = -6\dot{\rho}a^2 \dot{a} = -6a^3 H \dot{\bar{R}}, \\
D_1 = -6a^3 H^2 + a^3 u_a \dot{\bar{a}} + 6a^3 (\dot{\bar{H}} + 2H^2) = a^3 u_a \dot{\bar{a}} + 6a^3 (\dot{\bar{H}} + H^2), \\
J = -6a^3 H \dot{\bar{T}}, \\
E_1 = -6a^3 H^2 + a^3 v_a \dot{\bar{a}} - 6a^3 H^2 = -12a^3 H^2 + a^3 v_a \dot{\bar{a}}, \\
K = -a^3.
\]

and

\[
U = A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R, \\
A_4 = -6\dot{\bar{R}}^2 a^2, \\
A_2 = -6\dot{\bar{R}} a^2 - 12\dot{\bar{R}} a \dot{\bar{a}} + a^3 \dot{\bar{R}} u_a = -6\dot{\bar{R}} a^2 - 12\dot{\bar{R}} a \dot{\bar{a}} + a^3 \dot{\bar{R}} u_a, \\
A_1 = -6a^2 - 12a \ddot{\bar{a}} + 3a^2 \dot{\bar{a}} u_a + a^3 \ddot{\bar{a}} - 3a^2 (u - R) - a^3 u_a, \\
B_2 = -12\ddot{\bar{T}} a \dot{\bar{a}} + a^3 \ddot{\bar{T}} v_a, \\
B_1 = -6a^2 - 12a \ddot{\bar{a}} + 3a^2 \dot{\bar{a}} u_a + a^3 \ddot{\bar{a}} - 3a^2 (v - T) - a^3 v_a, \\
C_2 = -12a^2 \dddot{\bar{T}}, \\
C_1 = -6a^2 \dddot{T}, \\
C_0 = -12\dot{\bar{R}} a \dot{\bar{a}} - 12\dot{\bar{T}} a \dot{\bar{a}} - 6a^2 \dddot{T} + a^3 \dot{\bar{R}} v_a + a^3 \dddot{T} u_a, \\
M = -3a^2.
\]

### 4 Reductions. Preliminary classification

Note that the system or admits some important reductions. Let us now present these particular cases.

#### 4.1 Case: $F = R$

Now we consider the particular case $F = R$. Thus, the system becomes

\[
D_1 + K R = -2a^3 \rho, \\
A_1 + M R = 6a^2 p, \\
\dot{p} + 3H(\rho + p) = 0
\]

or

\[
3H^2 + 0.5(u - \dot{\bar{a}} u_a) = \rho, \\
2\dddot{H} + 3H^2 - 0.5(\dot{\bar{a}} u_a + \frac{1}{3}a \dddot{\bar{a}} - u) = -p, \\
\dot{p} + 3H(\rho + p) = 0.
\]

Let us rewrite this system as

\[
3H^2 = \rho + \rho_c, \\
2\dddot{H} + 3H^2 = -(p + p_c), \\
\dot{p} + 3H(\rho + p) = 0,
\]
where
\[ \rho_c = -0.5(u - \dot{a}v_a), \]
\[ p_c = -0.5(\dot{a}u_a + 3^{-1}a\ddot{a} - u) \] are the corrections to the energy density and pressure. Note that if \( u = 0 \) we obtain the standard equations of GR,
\[ 3H^2 = \rho, \]
\[ 2\dot{H} + 3H^2 = -p, \]
\[ \dot{\rho} + 3H(\rho + p) = 0, \] (4.6)

4.2 Case: \( F = T \)

Let us now consider \( F = T \). Then the system (3.24) leads to
\[ E_1 + KT = -2a^3 \rho, \]
\[ B_1 + MT = 6a^2 p, \]
\[ \dot{\rho} - 3H(\rho + p) = 0, \] (4.7)
or
\[ 3H^2 + 0.5(v - \dot{a}v_a) = \rho, \]
\[ 2\dot{H} + 3H^2 - 0.5(\dot{a}v_a + \frac{1}{3}a\ddot{a} - v) = -p, \]
\[ \dot{\rho} + 3H(\rho + p) = 0. \] (4.8)
The above system can be rewritten as
\[ 3H^2 = \rho + \rho_c, \]
\[ 2\dot{H} + 3H^2 = -(p + p_c), \]
\[ \dot{\rho} + 3H(\rho + p) = 0, \] (4.9)
where
\[ \rho_c = -0.5(v - \dot{a}v_a), \]
\[ p_c = -0.5(\dot{a}v_a + 3^{-1}a\ddot{a} - v) \] are the corrections to the energy density and pressure. Obviously, if \( v = 0 \) we obtain the standard equations of GR (4.6).

4.3 Case: \( F = F(T), \quad u = v = 0 \)

Let us take \( F = F(T), \quad u = v = 0 \). Then, the system (3.24) becomes
\[ E_1 F_T + KF = -2a^3 \rho, \] (4.12)
\[ B_2 F_{TT} + B_1 F_T + MF = 6a^2 p, \] (4.13)
\[ \dot{\rho} + 3H(\rho + p) = 0 \] (4.14)
or
\[ -12a\ddot{a}F_T - a^3 F = -2a^3 \rho, \] (4.15)
\[ -12 \dot{a}aF_{TT} - (36\ddot{a}^2 + 12\dot{a}a)F_T - 3a^2 F = 6a^2 p, \] (4.16)
\[ \dot{\rho} + 3H(\rho + p) = 0. \] (4.17)
This system can be rewritten as

\[ -2TF_T + F = 2\rho, \] \hspace{1cm} (4.18) \\
\[ -8HTF_{TT} + 2(T - 2H)F_T - F = 2p, \] \hspace{1cm} (4.19) \\
\[ \dot{\rho} - 3H(\rho + p) = 0 \] \hspace{1cm} (4.20)

that is the same as (2.16)-(2.18) of \( F(T) \) gravity.

4.4 Case: \( F = F(R), \ u = v = 0 \)

We get the second reduction if we consider the case where \( F = F(R), \ u = v = 0 \). Then the system (4.19) leads to

\[
\begin{align*}
D_2F_{RR} + D_1F_R + KF &= -2a^3\rho, \\
A_3F_{RRR} + A_2F_{RR} + A_1F_R + MF &= 6a^2p, \\
\dot{\rho} + 3H(\rho + p) &= 0 
\end{align*}
\] \hspace{1cm} (4.21)

where

\[
\begin{align*}
A_3 &= -6\dot{\rho}a^2, \\
A_2 &= -6\ddot{\rho}a - 12\dot{\rho}a, \\
A_1 &= -6a^3 - 12\ddot{a} + 3a^2R, \\
D_2 &= -6\dot{\rho}a^2, \\
D_1 &= -6\dot{a}^2 + a^3R, \\
K &= -a^3. 
\end{align*}
\] \hspace{1cm} (4.22)

This system can be written as

\[
\begin{align*}
-6\dot{\rho}a^2F_{RR} + [-6\dot{a}^2 + a^3R]F_R - a^3F &= -2a^3\rho, \\
-6\dot{\rho}^2a^2F_{RRR} + [-12\dot{\rho}a - 6\dot{\rho}a]F_{RR} + [-6\dot{a}^2 - 12\ddot{a} + 3a^2R]F_R - 3a^2F &= 6a^2p, \\
\dot{\rho} + 3H(\rho + p) &= 0. 
\end{align*}
\] \hspace{1cm} (4.30)

As a consequence,

\[
\begin{align*}
6\dot{\rho}HF_{RR} - (R - 6H^2)F_R + F &= 2\rho, \\
-2\dot{\rho}^2F_{RRR} + [-4\dot{\rho}H - 2\ddot{\rho}]F_{RR} + [-2H^2 - 4a^{-1}\ddot{a} + R]F_R - F &= 2p, \\
\dot{\rho} + 3H(\rho + p) &= 0. 
\end{align*}
\] \hspace{1cm} (4.33)

This system corresponds to the one in equations (2.5)-(2.7). We have shown that our model contents \( F(R) \) and \( F(T) \) gravity models as particular cases. In this sense it is the generalizations of these two known modified gravity theories.

4.5 The M\( _{37A} \) - model

For the M\( _{37A} \) - model we have \( u \neq 0, \ v = 0 \) so that

\[
\begin{align*}
S_{37A} &= \int d^4xe[F(R,T) + L_m], \\
R &= u + 6(\dot{H} + 2H^2), \\
T &= -6H^2. 
\end{align*}
\] \hspace{1cm} (4.36)

4.6 The M\( _{37B} \) - model

If we consider the case \( u = 0, \ v \neq 0 \), then we get the M\( _{37B} \) - model with

\[
\begin{align*}
S_{37B} &= \int d^4xe[F(R,T) + L_m], \\
R &= 6(H + 2H^2), \\
T &= v - 6H^2. 
\end{align*}
\] \hspace{1cm} (4.40)
4.7 The M\textsubscript{37C} - model

Now we consider the case \( v = \zeta(u) \). We get the M\textsubscript{37C} - model with

\[
S_{37B} = \int d^4x e[F(R, T) + L_m],
\]

\[
R = u + 6(\dot{H} + 2H^2),
\]

\[
T = \zeta(u) - 6H^2,
\]

where in general \( \zeta \) is a function to be defined e.g. \( \zeta = \zeta(t; a, \dot{a}, \ddot{a}, ..., \zeta; u) \) and \( \zeta \) is an unknown function.

4.8 The M\textsubscript{37D} - model

Now we consider the particular case of \( u = \xi(v) \) and we get the M\textsubscript{37D} - model with

\[
S_{37B} = \int d^4x e[F(R, T) + L_m],
\]

\[
R = \xi(v) + 6(\dot{H} + 2H^2),
\]

\[
T = v - 6H^2,
\]

where in general \( \xi \) is a function to be defined e.g. \( \xi = \xi(t; a, \dot{a}, \ddot{a}, ..., \zeta; v) \) and \( \zeta \) is an unknown function.

4.9 The M\textsubscript{37E} - model

Finally we consider the case \( u = v = 0 \) and we get the M\textsubscript{37E} - model with

\[
S_{37E} = \int d^4x e[F(R, T) + L_m],
\]

\[
R = 6(\dot{H} + 2H^2),
\]

\[
T = -6H^2.
\]

About this model we have some doubt related with the equation

\[
\dot{T} = -2(R + 3T)\sqrt{-\frac{T}{6}}
\]

which follows from (4.49)-(4.50) by avoiding the variable \( H \). This equation tell us that we have only one independent dynamical variable \( R \) or \( T \). It turns out that the model (4.49)-(4.50) is not of the type of \( F(R, T) \) gravity, but is equivalent to \( F(R) \) or \( F(T) \) gravity only. This is why in this paper we introduced some new functions like \( u, v \) and \( w \) with the (temporally?) unknown geometrical nature.

4.10 The M\textsubscript{37F} - model

The M\textsubscript{37F} - model corresponds to the case

\[
R = 0, \quad T \neq 0
\]

that is

\[
u = -6(\dot{H} + 2H^2)
\]

As a consequence the M\textsubscript{37F} - model reads

\[
S_{37F} = \int d^4x e[F(R, T) + L_m],
\]

\[
R = 0,
\]

\[
T = v - 6H^2.
\]

We can see that the M\textsubscript{37F} - model is in fact a generalization of \( F(T) \) gravity.
4.11 The $M_{37G}$ - model

We obtain the $M_{37G}$ - model by assuming

$$R \neq 0, \quad T = 0$$

that is

$$v = 6H^2.$$\hspace{1cm} (4.58)

In this way we write the $M_{37G}$ - model as

$$S_{37J} = \int d^4x\sqrt{F(R, T) + L_m},$$\hspace{1cm} (4.59)

$$R = u + 6(\dot{H} + 2H^2),$$\hspace{1cm} (4.60)

$$T = 0.$$\hspace{1cm} (4.61)

This model is in fact a generalization of $F(R)$ gravity.

5 The particular model: $F(R, T) = \mu R + \nu T$

The equations of $F(R, T)$ gravity are much more complicated with respect to the ones of GR even for FRW metric. For this reason let us consider the following simplest particular model

$$F(R, T) = \nu T + \mu R,$$\hspace{1cm} (5.1)

where $\mu$ and $\nu$ are some real constants. The equations system of $F(R, T)$ gravity becomes

$$\mu D_1 + \nu E_1 + K(\nu T + \mu R) = -2\mu^2, \hspace{1cm} (5.2)$$

$$\mu A_1 + \nu B_1 + M(\nu T + \mu R) = 6a^2p, \hspace{1cm} (5.3)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \hspace{1cm} (5.4)$$

where

$$D_1 = -6a\dot{u}^2 + a^3u_\dot{a} - a^3(u - R) = 6a^2\dot{a} + a^3u_\dot{a} = a^3(6\dot{a} + \dot{u}_a),$$\hspace{1cm} (5.5)

$$E_1 = -6a\dot{u}^2 + a^3v_\dot{a} - a^3(v - T) = -12a\dot{a}^2 + a^3v_\dot{a} = a^3(-12\dot{a}^2 + \dot{v}_a),$$\hspace{1cm} (5.6)

$$K = -a^3, \hspace{1cm} (5.7)$$

$$A_1 = 12\dot{a}^2 + 6a\dot{u} + 3a^2u_\dot{a} + a^3u_\dot{a} - a^3u_a, \hspace{1cm} (5.8)$$

$$B_1 = -24a\dot{a}^2 - 12a\dot{u} + 3a^2v_\dot{a} + a^3v_\dot{a} - a^3v_a, \hspace{1cm} (5.9)$$

$$M = -3a^3, \hspace{1cm} (5.10)$$

$$R = u + 6\frac{\dot{a}^2}{a^2} = u + 6(\dot{H} + 2H^2), \hspace{1cm} (5.11)$$

$$T = v - 6\frac{\dot{a}^2}{a^2} = v - 6H^2.$$\hspace{1cm} (5.12)

We get

$$-6(\mu + \nu)\frac{\dot{a}^2}{a^2} + \mu a_\dot{u} + \nu v_\dot{a} - \mu \dot{u} - \nu \dot{v} = -2\rho, \hspace{1cm} (5.13)$$

$$-2(\mu + \nu)(\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}}{a}) + \mu a_\dot{u} + \nu v_\dot{a} - \mu \dot{u} + \nu \dot{v} + \frac{\mu}{3}(\dot{u}_a - u_a) + \frac{\nu}{3}(\dot{v}_a - v_a) = 2p, \hspace{1cm} (5.14)$$

$$\dot{\rho} + 3H(\rho + p) = 0. \hspace{1cm} (5.15)$$

May rewrite it as

$$3(\mu + \nu)\frac{\dot{a}^2}{a^2} - 0.5(\mu a_\dot{u} + \nu v_\dot{a} - \mu \dot{u} - \nu \dot{v}) = \rho, \hspace{1cm} (5.16)$$

$$(\mu + \nu)(\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}}{a}) - 0.5(\mu a_\dot{u} + \nu v_\dot{a} - \mu \dot{u} - \nu \dot{v}) - \frac{\mu}{6}(\dot{u}_a - u_a) - \frac{\nu}{6}(\dot{v}_a - v_a) = -p, \hspace{1cm} (5.17)$$

$$\dot{\rho} + 3H(\rho + p) = 0. \hspace{1cm} (5.18)$$
or

\[
3(\mu + \nu)H^2 - 0.5(\mu \dot{a}u_a + \nu \dot{a}v_a - \mu u - \nu v) = \rho. (5.19)
\]

\[
(\mu + \nu)(2\dot{H} + 3H^2) - 0.5(\mu \dot{a}u_a + \nu \dot{a}v_a - \mu u - \nu v) - \frac{\mu}{6}a(\dot{u}_a - u_a) - \frac{\nu}{6}a(\dot{v}_a - v_a) = -\dot{\rho}. (5.20)
\]

\[
\dot{\rho} - 3H(\rho + p) = 0. (5.21)
\]

This system contains 2 equations and 5 unknown functions \((a, \rho, p, u, v)\). Note that the EoS parameter is given by

\[
\omega = \frac{p}{\rho} = -1 - \frac{2(\mu + \nu)\dot{H} - \frac{\mu}{6}a(\dot{u}_a - u_a) - \frac{\nu}{6}a(\dot{v}_a - v_a)}{3(\mu + \nu)H^2 - 0.5(\mu \dot{a}u_a + \nu \dot{a}v_a - \mu u - \nu v)}. (5.22)
\]

Now we assume

\[u = \alpha a^n, \quad v = \beta a^m,\] (5.23)

where \(n, m, \alpha, \beta\) are some real constants so that we have

\[
u = \alpha \left(\frac{\nu}{\beta}\right)^{\frac{\alpha}{\beta}}, \quad v = \beta \left(\frac{\nu}{\alpha}\right)^{\frac{\alpha}{\beta}}, (5.24)
\]

Then, the previous system (5.16)-(5.18) leads to

\[
3(\mu + \nu)\ddot{a} - \frac{\mu}{6}a(\dot{u}_a) - \frac{\nu}{6}a(\dot{v}_a) = \rho, (5.25)
\]

\[
(\mu + \nu)(\ddot{a}^2 + 2\dot{a}) + \frac{\mu}{6}(n + 3)\dot{a}^n + \frac{\nu}{6}(\beta + 3)\dot{a}^m = -p, (5.26)
\]

\[
\dot{\rho} + 3H(\rho + p) = 0. (5.27)
\]

or

\[
3(\mu + \nu)H^2 - 0.5(\mu \dot{a}u_a + \nu \dot{a}v_a - \mu u - \nu v) = \rho, (5.28)
\]

\[
(\mu + \nu)(2\dot{H} + 3H^2) + \frac{\mu}{6}(n + 3)\dot{a}^n + \frac{\nu}{6}(\beta + 3)\dot{a}^m = -p, (5.29)
\]

\[
\dot{\rho} + 3H(\rho + p) = 0. (5.30)
\]

6 Cosmological implications. Dark energy

Here we are interested in the cosmological implications of the model relating to the dark energy problem. In order to satisfy our interest, let us consider the power-law solution in the form

\[
a = a_0 t^\eta, (6.1)
\]

where \(a_0\) and \(\eta\) are constants. Thus,

\[
\rho = 3(\mu + \nu)\eta^2 t^{-2} + 0.5(\mu a_0^n t^m + \nu a_0^m t^m), (6.2)
\]

\[
p = -[(\mu + \nu)(-2\eta + 3\eta^2)t^{-2} + \frac{\mu}{6}(n + 3)a_0^n t^m + \frac{\nu}{6}(\beta + 3)a_0^m t^m]. (6.3)
\]

The EoS parameter reads

\[
\omega = \frac{\rho}{\rho} = -1 - \frac{-2\eta(\mu + \nu) + \frac{\mu}{6}a_0^n t^m + \frac{\nu}{6}(\beta + 3)a_0^m t^m}{3(\mu + \nu)\eta^2 t^{-2} + 0.5(\mu a_0^n t^m + \nu a_0^m t^m)}. (6.4)
\]

These expressions still content some unknown constant parameters. We assume that these parameters have the following values, namely \(\mu = \nu = 1 = m = n = \alpha = \beta = a_0\), \(\eta = 2/3\). Thus in this case one has

\[
\rho = \frac{8}{3} t^{-2} + t^{2/3}, (6.5)
\]

\[
p = -\frac{4}{3} t^{2/3}, (6.6)
\]
so that the EoS takes the form
\[ \rho = \frac{512}{81} p + \frac{3}{4} p^3. \]  
(6.7)

Furthermore, the EoS parameter becomes
\[ \omega(t) = \frac{p}{\rho} = -\frac{4}{3} + \frac{8}{3} t - \frac{8}{3} t^2 = -\frac{4}{3} + \frac{8}{3} t. \]  
(6.8)

Hence, we see that \( \omega(0) = 0, \omega(1) = -4/11 = -0.36 \) and \( \omega(\infty) = -4/3 \approx -1.33 \), so that our particular case admits the phantom crossing for \( \omega = -1 \) as \( t_0 = \frac{8}{3} \). In Fig.1 we plot the evolution of the EoS parameter with respect to the cosmic time \( t \). It is interesting to compare this result with the torsionless case with \( \nu = \alpha = \beta = 0 \), by taking the same values for all the other parameters, namely \( \mu = 1 \) and \( \eta = 2/3 \), which is the case of GR. As a consequence \( p = 0 \) and \( \rho = \frac{8}{3} t^2 \), which describe the dust matter.

### 7 \( F(R, T) \) gravity: Bianchi type I model

The results of the section 3 can be extendent to the other metric. As an example, let us consider the \( M_{37} \) - model for the Bianchi type spacetime. The corresponding metric is given by
\[ ds^2 = -dr^2 + A^2 dx_1^2 + B^2 dx_2^2 + C^2 dx_3^2, \]  
(7.1)

In this case the \( M_{37} \) - model reads as
\[ S_{39} = \int d^4x \left[ F(R, T) + L_m \right], \]  
(7.2)
\[ R = u + 2 \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{A B}{AB} + \frac{A C}{AC} + \frac{B C}{BC} \right), \]  
(7.3)
\[ T = v - 2 \left( \frac{A B}{AB} + \frac{A C}{AC} + \frac{B C}{BC} \right). \]  
(7.4)

Here \( u = u(t, A, B, C, \dot{A}, \dot{B}, \dot{C}, \ddot{A}, \ddot{B}, \ddot{C}, \ldots : f_i) \) and \( v = v(t, A, B, C, \dot{A}, \dot{B}, \dot{C}, \ddot{A}, \ddot{B}, \ddot{C}, \ldots ; g_i) \).

### 8 Other generalizations and reductions of some generalized gravity models

#### 8.1 The \( F(G) \) with \( w \) field

Now we consider the \( M_{39} \) - model which looks like
\[ S_{39} = \int d^4x \left[ F(G) + L_m \right], \]  
(8.1)
\[ G = w + 24 H^2 (\dot{H} + \dot{H}^2), \]  
(8.2)
\[ w = w(t, a, \dot{a}, \ddot{a}, \ldots ; h_i) \]  
(8.3)

where, again, \( w = w(t, a, \dot{a}, \ddot{a}, \ldots ; h_i) \) is a real function and \( h_i \) is an unknown function related to the geometry of the spacetime. If \( w = 0 \) the \( M_{39} \) - model reduces to the usual \( F(G) \) gravity with \( G = G_s = 24 H^2 (\dot{H} + \dot{H}^2) \).

#### 8.2 The \( M_{40} \) - model

Now we consider the \( M_{40} \) - model which reads
\[ S_{40} = \int d^4x \left[ F(R, G) + L_m \right], \]  
(8.4)
where
\[ R = u + 6(\dot{H} + 2H^2), \quad (8.5) \]
\[ G = w + 2A H^2(\dot{H} + H^2), \quad (8.6) \]
\[ u = u(t, a, \dot{a}, \ddot{a}, \ldots; f_i), \quad (8.7) \]
\[ w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i). \quad (8.8) \]

Here, \( u = u(t, a, \dot{a}, \ddot{a}, \ldots; f_i) \) and \( w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i) \) are some real functions and \( f_i, h_i, g_i \) are some unknown functions related to the geometry of the spacetime. Note that if we put \( u = w = 0 \), the \( M_{40} \) - model reduces to the usual \( F(R, G) \) gravity.

### 8.3 The \( M_{38} \) - model

Let us consider the following action of the \( M_{38} \) - model
\[
S_{38} = \int d^4x [F(G, T) + L_m], \quad (8.9)
\]
where
\[
G = w + 2A H^2(\dot{H} + H^2), \quad (8.10)
\]
\[
T = v - 6H^2, \quad (8.11)
\]
\[
w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i), \quad (8.12)
\]
\[
v = v(t, a, \dot{a}, \ddot{a}, \ldots; g_i). \quad (8.13)
\]

Here in general \( w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i) \) and \( v = v(t, a, \dot{a}, \ddot{a}, \ldots; g_i) \) are some real functions and \( h_i \) and \( g_i \) are some unknown functions related with the geometry of the spacetime.

### 8.4 The \( M_{41} \) - model

Now we consider the \( M_{41} \) - model with the following action
\[
S_{41} = \int d^4x [F(R, G, T) + L_m], \quad (8.14)
\]
where
\[
R = u + 6(\dot{H} + 2H^2), \quad (8.15)
\]
\[
G = w + 2A H^2(\dot{H} + H^2), \quad (8.16)
\]
\[
T = v - 6H^2, \quad (8.17)
\]
\[
u = u(t, a, \dot{a}, \ddot{a}, \ldots; f_i), \quad (8.18)
\]
\[
w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i), \quad (8.19)
\]
\[
v = v(t, a, \dot{a}, \ddot{a}, \ldots; g_i). \quad (8.20)
\]

Here, again, \( u = u(t, a, \dot{a}, \ddot{a}, \ldots; f_i) \), \( w = w(t, a, \dot{a}, \ddot{a}, \ldots; h_i) \) and \( v = v(t, a, \dot{a}, \ddot{a}, \ldots; g_i) \) are some real functions and \( f_i, h_i, g_i \) are some unknown functions related to the geometry of the spacetime.

### 8.5 The \( M_{42} \) - model

Let us consider the \( M_{42} \) - model with the action
\[
S_{42} = \int d^4x [F(R, T) + L_m], \quad (8.21)
\]
where
\[
R = T\phi + 6(\dot{H} + 2H^2), \quad (8.22)
\]
\[
T = R\varphi - 6H^2. \quad (8.23)
\]
Here \( u = T \phi, \quad v = R \varphi \), where \( \phi = \phi(t, a, \dot{a}, \ddot{a}, \ldots; \phi_i) \) and \( \varphi = \varphi(t, a, \dot{a}, \ddot{a}, \ldots; \varphi_i) \) are some unknown functions. This model admits (at least) two important particular cases.

\section*{9 Other examples of metric-affine Myrzakulov gravity theories}

Consider the metric-affine spacetime with the affine connection \( \tilde{\Gamma}_{\lambda \mu \nu} \). Then the torsion and non-metricity tensors are given by

\[ T^\lambda_{\mu \nu} = 2 \tilde{\Gamma}_{\lambda [\mu \nu]}, \quad (9.1) \]
\[ Q_{\lambda \mu \nu} = \tilde{\nabla}_{\lambda} g_{\mu \nu}. \quad (9.2) \]

The corresponding covariant derivative of an arbitrary vector \( v^\lambda \) can be split into a Riemannian contribution and a distortion tensor

\[ \tilde{\nabla}_\mu v^\lambda = \nabla_\mu v^\lambda + N^\lambda_{\rho \mu} v^\rho, \quad (9.3) \]

where

\[ N^\lambda_{\rho \mu} = K^\lambda_{\rho \mu} + L^\lambda_{\rho \mu}. \quad (9.4) \]

Here the contortion and disformation tensors read as

\[ K^\lambda_{\rho \mu} = \frac{1}{2}(T^\lambda_{\rho \mu} - T^\lambda_{\mu \rho} - T^\lambda_{\mu \rho}), \quad (9.5) \]
\[ L^\lambda_{\rho \mu} = \frac{1}{2}(Q^\lambda_{\rho \mu} - Q^\lambda_{\mu \rho} - Q^\lambda_{\mu \rho}), \quad (9.6) \]

respectively. The commutation of the covariant derivatives takes the form

\[ [\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] v^\lambda = \tilde{R}^\lambda_{\rho \mu \nu} v^\rho + T^\rho_{\mu \nu} \tilde{\nabla}_{\rho} v^\lambda, \quad (9.7) \]

where

\[ \tilde{R}^\lambda_{\rho \mu \nu} = \partial_\rho \tilde{\Gamma}^\lambda_{\mu \nu} - \partial_\nu \tilde{\Gamma}^\lambda_{\rho \mu} + \tilde{\Gamma}^\sigma_{\rho \mu} \tilde{\Gamma}^\lambda_{\sigma \nu} - \tilde{\Gamma}^\lambda_{\rho \sigma} \tilde{\Gamma}^\sigma_{\mu \nu}. \quad (9.8) \]

Note that the geometric structure of the metric-affine spacetime is determined by three tensors: the metric tensor \( g_{\mu \nu} \), the torsion tensor \( T^\lambda_{\rho \mu} \) and the nonmetricity tensor \( Q_{\lambda \mu \nu} \). The torsion tensor is the antisymmetric part of the connection and the nonmetricity tensor measures the failure of the connection to be metric compatible. Note that these three tensors can be computed once an affine connection \( \tilde{\Gamma}_{\alpha \beta \lambda} \) is given. In this metric-affine spacetime, let us introduce five scalars - \( R, T, Q, G, B \), where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar. Below \( T \) is the trace of the energy-momentum tensor. In the previous sections, we have considered the Myrzakulov gravity-I (MG-I) which has the following action

\[ S = \int \sqrt{-g} d^4x [F(R, T) + L_m], \quad (9.9) \]

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( L_m \) is the matter Lagrangian. This MG-I is some kind generalization (unification) of the well-known \( F(R) \) and \( F(T) \) gravity theories. We now going to present some other examples of metric-affine Myrzakulov gravity theories, also abbreviated below as MG-N, where N=I, II, III, IV, ... (see, also, Table 1 and Table 2).
9.1 MG-I

The action of the Myrzakulov gravity - I (MG-I) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T) + 2\kappa^2 L_m \right], \]  

(9.10)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( L_m \) is the matter Lagrangian. This MG-I is some kind generalizations of the well-known \( F(R) \) and \( F(T) \) gravity theories. If exactly, the MG-I is the unification of the \( F(R) \) and \( F(T) \) theories.

9.2 MG-II

The action of the Myrzakulov gravity - II (MG-II) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, Q) + 2\kappa^2 L_m \right], \]  

(9.11)

where \( R \) is the curvature scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

9.3 MG-III

The action of the Myrzakulov gravity - III (MG-III) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(T, Q) + 2\kappa^2 L_m \right], \]  

(9.12)

where \( T \) is the torsion scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

9.4 MG-IV

The action of the Myrzakulov gravity - IV (MG-IV) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T, T) + 2\kappa^2 L_m \right], \]  

(9.13)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( T \) is the trace of the energy-momentum tensor.

9.5 MG-V

The action of the Myrzakulov gravity - V (MG-V) is given by

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T, Q) + 2\kappa^2 L_m \right], \]  

(9.14)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

9.6 MG-VI

The action of the Myrzakulov gravity - VI (MG-VI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, Q, T) + 2\kappa^2 L_m \right], \]  

(9.15)

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor of our generalized spacetime.
9.7 **MG-VII**
The action of the Myrzakulov gravity - VII (MG-VII) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, T) + 2\kappa^2 L_m],
\]  
(9.16)

and \(T\) is the torsion scalar, \(Q\) is the nonmetricity scalar and \(T\) is the trace of the energy-momentum tensor of the metric-affine spacetime.

9.8 **MG-VIII**
The action of the Myrzakulov gravity - VIII (MG-VIII) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, T) + 2\kappa^2 L_m],
\]  
(9.17)

where \(R\) is the curvature scalar, \(T\) is the torsion scalar, \(Q\) is the nonmetricity scalar and \(T\) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor) of the metric-affine spacetime.

9.9 **MG-IX**
The action of the Myrzakulov gravity - IX (MG-IX) has the following form

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G) + 2\kappa^2 L_m],
\]  
(9.18)

where \(R\) is the curvature scalar, \(T\) is the torsion scalar, \(G\) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.10 **MG-X**
The action of the Myrzakulov gravity - X (MG-X) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G) + 2\kappa^2 L_m],
\]  
(9.19)

where \(R\) is the curvature scalar, \(Q\) is the nonmetricity scalar, \(G\) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.11 **MG-XI**
The action of the Myrzakulov gravity - XI (MG-XI) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G) + 2\kappa^2 L_m],
\]  
(9.20)

where \(T\) is the metric-affine torsion scalar, \(Q\) is the metric-affine nonmetricity scalar and \(G\) is the metric-affine Gauss-Bonnet scalar of our metric-affine spacetime.

9.12 **MG-XII**
The action of the Myrzakulov gravity - XII (MG-XII) has the following form

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, T) + 2\kappa^2 L_m],
\]  
(9.21)

where \(R\) is the metric-affine curvature scalar, \(T\) is the metric-affine torsion scalar, \(G\) is the metric-affine Gauss-Bonnet scalar and \(T\) is the trace of the energy-momentum tensor.
9.13 MG-XIII
The action of the Myrzakulov gravity - XIII (MG-XIII) is given by
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, G) + 2\kappa^2 L_m], \] (9.22)
where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.14 MG-XIV
The action of the Myrzakulov gravity - XIV (MG-XIV) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G, T) + 2\kappa^2 L_m], \] (9.23)
where \( R \) is the metric-affine curvature scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

9.15 MG-XV
The action of the Myrzakulov gravity - XV (MG-XV) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, T) + 2\kappa^2 L_m], \] (9.24)
and \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of our metric-affine spacetime.

9.16 MG-XVI
The action of the Myrzakulov gravity - XVI (MG-XVI) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, G, T) + 2\kappa^2 L_m], \] (9.25)
where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

9.17 MG-XVII
The action of the Myrzakulov gravity - XVII (MG-XVII) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(Q, G) + 2\kappa^2 L_m], \] (9.26)
where \( Q \) is the metric-affine nonmetricity scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.18 MG-XVIII
The action of the Myrzakulov gravity - XVIII (MG-XVIII) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G) + 2\kappa^2 L_m], \] (9.27)
where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.
9.19 MG-XIX

The action of the Myrzakulov gravity - XIX (MG-XIX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, G, T) + 2\kappa^2 L_m], \]  
(9.28)

where \( T \) is the metric-affine torsion scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

9.20 MG-XX

The action of the Myrzakulov gravity - XX (MG-XX) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, B) + 2\kappa^2 L_m], \]  
(9.29)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar and \( L_m \) is the matter Lagrangian. This MG-I is some kind generalizations of the well-known \( F(R) \) and \( F(T) \) gravity theories. If exactly, the MG-I is the unification of the \( F(R) \) and \( F(T) \) theories.

9.21 MG-XXI

The action of the Myrzakulov gravity - XXI (MG-XXI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, B) + 2\kappa^2 L_m], \]  
(9.30)

where \( R \) is the curvature scalar, \( B \) is the boundary term scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

9.22 MG-XXII

The action of the Myrzakulov gravity - XXII (MG-XXII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, B) + 2\kappa^2 L_m], \]  
(9.31)

where \( T \) is the torsion scalar, \( B \) is the boundary term scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

9.23 MG-XXIII

The action of the Myrzakulov gravity - XXIII (MG-XXIII) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, B, T) + 2\kappa^2 L_m], \]  
(9.32)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor.

9.24 MG-XXIV

The action of the Myrzakulov gravity - XXIV (MG-XXIV) is given by

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, B) + 2\kappa^2 L_m], \]  
(9.33)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.
9.25 MG-XXV
The action of the Myrzakulov gravity - XXV (MG-XXV) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, B, T) + 2\kappa^2 L_m], \] (9.34)

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of our generalized spacetime.

9.26 MG-XXVI
The action of the Myrzakulov gravity - XXVI (MG-XXVI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, B, T) + 2\kappa^2 L_m], \] (9.35)

and \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

9.27 MG-XXVII
The action of the Myrzakulov gravity - XXVII (MG-XXVII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, B) + 2\kappa^2 L_m], \] (9.36)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor) of the metric-affine spacetime.

9.28 MG-XXVIII
The action of the Myrzakulov gravity - XXVIII (MG-XXVIII) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \] (9.37)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar, \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.29 MG-XXIX
The action of the Myrzakulov gravity - XXIX (MG-XXIX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G, B) + 2\kappa^2 L_m], \] (9.38)

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar, \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.30 MG-XXX
The action of the Myrzakulov gravity - XXX (MG-XXX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, B) + 2\kappa^2 L_m], \] (9.39)

where \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of our metric-affine spacetime.
The action of the Myrzakulov gravity - XXXI (MG-XXXI) has the following form

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B, \mathcal{T}) + 2\kappa^2 L_m],$$

(9.40)

where $R$ is the metric-affine curvature scalar, $T$ is the metric-affine torsion scalar, $G$ is the metric-affine Gauss-Bonnet scalar, $B$ is the boundary term scalar and $\mathcal{T}$ is the trace of the energy-momentum tensor.

The action of the Myrzakulov gravity - XXXII (MG-XXXII) is given by

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, G, B) + 2\kappa^2 L_m],$$

(9.41)

where $R$ is the curvature scalar, $T$ is the torsion scalar, $Q$ is the nonmetricity scalar, $B$ is the boundary term scalar and $G$ is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

The action of the Myrzakulov gravity - XXXIII (MG-XXXIII) reads as

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G, B, T) + 2\kappa^2 L_m],$$

(9.42)

where $R$ is the metric-affine curvature scalar, $Q$ is the metric-affine nonmetricity scalar, $G$ is the metric-affine Gauss-Bonnet scalar, $B$ is the boundary term scalar and $T$ is the trace of the energy-momentum tensor of the metric-affine spacetime.

The action of the Myrzakulov gravity - XXXIV (MG-XXXIV) reads as

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, B, T) + 2\kappa^2 L_m],$$

(9.43)

and $T$ is the metric-affine torsion scalar, $Q$ is the metric-affine nonmetricity scalar, $G$ is the metric-affine Gauss-Bonnet scalar, $B$ is the boundary term scalar and $\mathcal{T}$ is the trace of the energy-momentum tensor of our metric-affine spacetime.

The action of the Myrzakulov gravity - XXXV (MG-XXXV) reads as

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, G, B, T) + 2\kappa^2 L_m],$$

(9.44)

where $R$ is the metric-affine curvature scalar, $T$ is the metric-affine torsion scalar, $Q$ is the metric-affine nonmetricity scalar, $G$ is the metric-affine Gauss-Bonnet scalar, $B$ is the boundary term scalar and $\mathcal{T}$ is the trace of the energy-momentum tensor of the metric-affine spacetime.

The action of the Myrzakulov gravity - XXXVI (MG-XXXVI) reads as

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(Q, G, B) + 2\kappa^2 L_m],$$

(9.45)

where $Q$ is the metric-affine nonmetricity scalar, $B$ is the boundary term scalar and $G$ is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.
9.37 MG-XXXVII

The action of the Myrzakulov gravity - XXXVII (MG-XXXVII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m]. \]  
(9.46)

where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

9.38 MG-XXXVIII

The action of the Myrzakulov gravity - XXXVIII (MG-XXXVIII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, G, B, T) + 2\kappa^2 L_m]. \]  
(9.47)

where \( T \) is the metric-affine torsion scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.
Table 1: Metric-affine Myrzakulov gravity theories

| N  | Name                      | Action                                         |
|----|----------------------------|------------------------------------------------|
| 1  | Myrzakulov Gravity - I (MG-I) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T) + 2k^2 L_m \right]$ |
| 2  | Myrzakulov Gravity - II (MG-II) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, G) + 2k^2 L_m \right]$ |
| 3  | Myrzakulov Gravity - III (MG-III) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, Q) + 2k^2 L_m \right]$ |
| 4  | Myrzakulov Gravity - IV (MG-IV) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T) + 2k^2 L_m \right]$ |
| 5  | Myrzakulov Gravity - V (MG-V) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, G) + 2k^2 L_m \right]$ |
| 6  | Myrzakulov Gravity - VI (MG-VI) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, Q) + 2k^2 L_m \right]$ |
| 7  | Myrzakulov Gravity - VII (MG-VII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T) + 2k^2 L_m \right]$ |
| 8  | Myrzakulov Gravity - VIII (MG-VIII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, G, T) + 2k^2 L_m \right]$ |
| 9  | Myrzakulov Gravity - IX (MG-IX) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, G, G) + 2k^2 L_m \right]$ |
| 10 | Myrzakulov Gravity - X (MG-X) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, Q, T) + 2k^2 L_m \right]$ |
| 11 | Myrzakulov Gravity - XI (MG-XI) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, Q, G) + 2k^2 L_m \right]$ |
| 12 | Myrzakulov Gravity - XII (MG-XII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, Q, Q) + 2k^2 L_m \right]$ |
| 13 | Myrzakulov Gravity - XIII (MG-XIII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, G) + 2k^2 L_m \right]$ |
| 14 | Myrzakulov Gravity - XIV (MG-XIV) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, Q) + 2k^2 L_m \right]$ |
| 15 | Myrzakulov Gravity - XV (MG-XV) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, T) + 2k^2 L_m \right]$ |
| 16 | Myrzakulov Gravity - XVI (MG-XVI) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, T, G) + 2k^2 L_m \right]$ |
| 17 | Myrzakulov Gravity - XVII (MG-XVII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, T, Q) + 2k^2 L_m \right]$ |
| 18 | Myrzakulov Gravity - XVIII (MG-XVIII) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, T, T) + 2k^2 L_m \right]$ |
| 19 | Myrzakulov Gravity - XIX (MG-XIX) | $S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R(T, T, T, T, T, G) + 2k^2 L_m \right]$ |

10 Cosmology in MG theories

Consider the FRW universe. The flat FRW spacetime is described by the metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$  \hspace{1cm} (10.1)

where $a = a(t)$ is the scale factor. Let $R, T, Q$ are the Ricci, torsion, nonmetricity scalars. For the FRW metric they have the forms: i) $R = R_0$, where $T = Q = 0$; ii) $T = T_0$, where $R = Q = 0$; iii) $Q = Q_0$, where $R = T = 0$. For the FRW metric, they have the forms:

$$R_0 = 6(\dot{H} + 2H^2),$$  \hspace{1cm} (10.2)
$$T_0 = -6H^2,$$  \hspace{1cm} (10.3)
$$Q_0 = 6H^2,$$  \hspace{1cm} (10.4)

where $H = \ln a_t$ is the Hubble parameter. In the metric-affine spacetime case, we assume that the Ricci, torsion and nonmetricity scalars take the forms

$$R = 6(\dot{H} + 2H^2) + u,$$  \hspace{1cm} (10.5)
$$T = -6H^2 + v,$$  \hspace{1cm} (10.6)
$$Q = 6H^2 + w.$$  \hspace{1cm} (10.7)

Similarly, we can write the boundary term scalar $(B)$ and the GB scalar as

$$G = G_0 + p,$$  \hspace{1cm} (10.8)
$$B = B_0 + f,$$  \hspace{1cm} (10.9)

where $u, v, w, p, f$ are some real functions of $t, a, \dot{a}, \ddot{a}$. 

23
11 Spherically symmetric and black hole solutions in MG theories

Let us now present our idea to study, for example, the black hole solutions of MG theories. For this aim, we consider the following static and spherically symmetric metric

\[ ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2 \theta d\phi^2), \]  \hspace{1cm} (11.1)

where \( A(r) \), \( B(r) \) and \( C(r) \) are real functions of the radial coordinate \( r \). Consider two connections: the Levi-Civita connection and the Weitzenböck connection. First, let us consider the Levi-Civita connection case. In this case, the nonmetricity and torsion scalar \( S \) are equal to zero that is \( S = 0 \). Then the corresponding Ricci scalar has the form

\[ R_0 = \frac{A''}{AB} + 2\frac{C'}{ABC} + \frac{A'C'}{ABC} - \frac{A'^2}{2AB^2} - \frac{C'^2}{2BC^2} - \frac{A'B'}{2AB^2} - \frac{B'C'}{2B^2C} - \frac{2}{C}. \]  \hspace{1cm} (11.2)

Here and below primes denote differentiation with respect to the radial coordinate \( r \). Let us now consider the Weitzenböck connection case. In this case, the Ricci scalar and nonmetricity scalar are equal to zero that is \( R_0 = Q_0 = 0 \) and the torsion scalar is given by

\[ T_0 = \frac{C'(2AC + AC')}{2ABC^2}. \]  \hspace{1cm} (11.3)

Similarly, we can calculate the nonmetricity scalar \( Q_0 \). For the metric (11.1), it has the form

\[ Q_0 = -\frac{C'(2AC + AC')}{2ABC^2}. \]  \hspace{1cm} (11.4)

where \( R_0 = T_0 = 0 \). The geometry of the MG theories is the metric-affine spacetime. For that reason, now let us consider the more general case, namely, the metric-affine spacetime. For this metric-affine spacetime, we have the metric-affine connection. In this metric-affine case, the Ricci...

| N  | Name                        | Action                                                                 |
|----|-----------------------------|------------------------------------------------------------------------|
| 1  | Myrzakulov Gravity - XX (MG-XX) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, B) + 2k^2L_m \right) \] |
| 2  | Myrzakulov Gravity - XXI (MG-XXI) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, Q, B) + 2k^2L_m \right) \] |
| 3  | Myrzakulov Gravity - XXII (MG-XXII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B) + 2k^2L_m \right) \] |
| 4  | Myrzakulov Gravity - XXIII (MG-XXIII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, B, T) + 2k^2L_m \right) \] |
| 5  | Myrzakulov Gravity - XXIV (MG-XXIV) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, Q, B) + 2k^2L_m \right) \] |
| 6  | Myrzakulov Gravity - XXV (MG-XXV) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B) + 2k^2L_m \right) \] |
| 7  | Myrzakulov Gravity - XXVI (MG-XXVI) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B, T) + 2k^2L_m \right) \] |
| 8  | Myrzakulov Gravity - XXVII (MG-XXVII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B, T) + 2k^2L_m \right) \] |
| 9  | Myrzakulov Gravity - XXVIII (MG-XXVIII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, Q, B) + 2k^2L_m \right) \] |
| 10 | Myrzakulov Gravity - XXIX (MG-XXIX) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, Q, B) + 2k^2L_m \right) \] |
| 11 | Myrzakulov Gravity - XXX (MG-XXX) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B) + 2k^2L_m \right) \] |
| 12 | Myrzakulov Gravity - XXXI (MG-XXXI) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, B, T) + 2k^2L_m \right) \] |
| 13 | Myrzakulov Gravity - XXXII (MG-XXXII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, Q, B, T) + 2k^2L_m \right) \] |
| 14 | Myrzakulov Gravity - XXXIII (MG-XXXIII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, Q, G, B) + 2k^2L_m \right) \] |
| 15 | Myrzakulov Gravity - XXXIV (MG-XXXIV) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, G, B) + 2k^2L_m \right) \] |
| 16 | Myrzakulov Gravity - XXXV (MG-XXXV) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, Q, G, B) + 2k^2L_m \right) \] |
| 17 | Myrzakulov Gravity - XXXVI (MG-XXXVI) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(Q, G, B) + 2k^2L_m \right) \] |
| 18 | Myrzakulov Gravity - XXXVII (MG-XXXVII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(R, T, G, B) + 2k^2L_m \right) \] |
| 19 | Myrzakulov Gravity - XXXVIII (MG-XXXVIII) | \[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( F(T, G, B, T) + 2k^2L_m \right) \] |
scalar, the torsion scalar and the nonmetricity scalar take the forms
\[ R = R_0 + u, \]  
\[ T = T_0 + v, \]  
\[ Q = Q_0 + w. \]

Here the metric-affine contributions are given by the following functions
\[ u = u(A, B, C, A', B', C', A'', B'', C''), \]  
\[ v = v(A, B, C, A', B', C', A'', B'', C''), \]  
\[ w = w(A, B, C, A', B', C', A'', B'', C''). \]

They are some real functions of the metric tensor components \( g_{ij} \) (11.1).

### 12 MG theories with the boundary term scalars

Next, we very briefly mention the main moments of MG theories with the boundary term scalars. According our idea, we assume that the boundary term scalar has the form
\[ B = B_0 + f. \]

Similarly, we can write the GB scalar for the metric-affine spacetime as
\[ G = G_0 + p. \]

In the last two equations, \( p \) and \( f \) are metric-affine contributions and some functions of \( A, B, C \) and their derivatives.

### 13 Conclusion

As it is well known, modified gravity theories play an important role in modern cosmology. In particular, the well-known \( F(R) \) and \( F(T) \) theories are useful tools to study dark energy phenomena motivated at a fundamental level. In the present work, we have considered the more general theory, namely the \( F(R, T) \)-models.

At first, we have written the equations of the model and we have found their several reductions. In particular, the Lagrangian has been explicitly constructed. The corresponding exact solutions are found for the specific model \( F(R, T) = \mu R + \nu T \) theory, for which the universe expands as \( a(t) = a_0 t^\eta \). Furthermore, we have considered the physical quantities corresponding to the exact solution, and we have found that it can describe the expansion of our universe in an accelerated way without introducing the dark energy.

Some remarks are in order. Of course many aspects of \( F(R, T) \) theory are actually unexplored. For example, we do not have any realistic model which fits the cosmological data, unlike \( F(R) \) or \( F(T) \) theory. We do not know viability conditions of the models, what forms of \( F(R, T) \) can be derived from fundamental theories and so on (it may be extremely important to reconstruct a \( F(R, T) \)-theory by starting from some basalical principles). On the other hand, we have here shown that the \( F(R, T) \) models can be serious candidates as modified gravity models for the dark energy. Also we note that the behaviour and the results of \( F(R, T) \)-gravity may be extremely different with respect to the ones of GR, \( F(R) \) and \( F(T) \) gravity theories, so that only the observation of our universe may discriminate between the various gravity theories. We not want here discuss merits and demerits of the models above, since we think that it requires some more accurate investigations related to cosmological applications.

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