ON THE NUMBER OF TOPOLOGIES ON A FINITE SET

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Abstract. We denote the number of distinct topologies which can be defined on the set $X$ with $n$ elements by $T(n)$. Similarly, $T_0(n)$ denotes the number of distinct $T_0$ topologies on the set $X$. In the present paper, we prove that for any prime $p$, $T(p^k) \equiv k + 1 \mod p$, and that for each non-negative integer $n$ there exists a unique $k$ such that $T(p+n) \equiv k$. We calculate $k$ for $n = 1, 2, 3, 4$. We give an elementary proof for a result of Z.I. Borevich to the effect that $T_0(p+n) \equiv T_0(n+1) \mod p$.

1. INTRODUCTION

Given a finite set $X$ with $n$ elements, let $\mathcal{T}(X)$ and $\mathcal{T}_0(X)$ be the family of all topologies on $X$ and the family of all $T_0$ topologies on $X$, respectively. We denote the cardinality of $\mathcal{T}(X)$ by $T(n)$ and the cardinality of $\mathcal{T}_0(X)$ by $T_0(n)$. There is no simple formula giving $T(n)$ and $T_0(n)$.

Calculation of these sequences by hand becomes very hard for $n \geq 4$. The online encyclopedia of N. J. A. Sloane [1] gives the values of $T(n)$ and $T_0(n)$ for $n \leq 18$. For a more detailed discussion of results in literature we refer to the article by Borevich [2].

In the present paper, we prove that for any prime $p$, $T(p^k) \equiv k + 1 \mod p$, and that for each non-negative integer $n$ there exists a unique $k$ such that $T(p+n) \equiv k$. We calculate $k$ for $n = 0, 1, 2, 3, 4$. We give an elementary proof for a result of Borevich [2] to the effect that $T_0(p+n) \equiv T_0(n+1) \mod p$.

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2. **Main Results**

Let $G$ be a group acting on the finite set $X$. Then the action of $G$ on $X$ can be extended to the action of $G$ on $\mathcal{F}(X)$ by setting $g\tau = \{gU|\text{for } U \in \tau\}$ where $\tau \in \mathcal{F}(X)$ and $gU = \{gu|u \in U\}$. Now, set $Fix(\mathcal{F}(X)) = \{\tau \in \mathcal{F}(X) | g\tau = \tau \text{ for all } g \in G\}$. Notice that if $G$ is a $p$-group for a prime $p$, $|\mathcal{F}(X)| \equiv |Fix(\mathcal{F}(X))| \mod p$ as every non-fixed element of $\mathcal{F}(X)$ has orbit with cardinality a positive power of $p$. For a given topology $\tau \in \mathcal{F}(X)$ and $x \in X$, we denote the intersection of all open sets of $\tau$ including $x$ by $O_x$. Note that $O_x \in \tau$ as we are in the finite case.

**Definition 2.1.** A base $\mathcal{B}$ of a topology $\tau$ is called a minimal base if any base of the topology contains $\mathcal{B}$.

**Proposition 2.2.** Let $\tau \in \mathcal{F}(X)$ and $M_\tau$ be the set of all distinct $O_x$ for $x \in X$. A base $\mathcal{B}$ of $\tau$ is a minimal base if and only if $\mathcal{B} = M_\tau$.

By the proposition, we extend the definition: a base $\mathcal{B}$ on $X$ is minimal if $\mathcal{B} = M_\tau$ where $\tau$ is the topology generated by $\mathcal{B}$.

**Lemma 2.3.** $\tau \in Fix(\mathcal{F}(X))$ if and only if $M_\tau$ is $G$-invariant.

**Proof.** Let $\tau \in Fix(\mathcal{F}(X))$ and $O_x \in M_\tau$. We need to show that $gO_x \in M_\tau$. Let $gx = y$ for $g \in G$. As $g\tau = \tau$, $gO_x \in \tau$. Thus, $gO_x$ is an open set containing the element $gx = y$. Hence, $O_y \subseteq gO_x$. Since $g^{-1}y = x$, we can show that $O_x \subseteq g^{-1}O_y$ which force $gO_x = O_y$. So, $M_\tau$ is $G$-invariant. If $M_\tau$ is $G$-invariant then clearly the topology $\tau$ generated by this base is $G$-invariant. Hence, $\tau \in Fix(\mathcal{F}(X))$. This proof also shows that $G$ acts on $M_\tau$ if $\tau \in Fix(\mathcal{F}(X))$. \(\square\)

**Theorem 2.4.** $T(p^k) \equiv k+1 \pmod{p}$ where $k$ is a non-negative integer and $p$ is a prime number.
Proof. Without loss of generality, let $X$ be the cyclic group of order $p^k$, that is, $X = C_{p^k}$. Clearly $X$ acts on $X$ by left multiplication. By extending this action, $X$ acts on $\mathfrak{T}(X)$. Notice that $|\mathfrak{T}(X)| \equiv |\text{Fix}(\mathfrak{T}(X))| \mod p$ as $X$ is a $p$-group. It is left to show that $\text{Fix}(\mathfrak{T}(X))$ has $k + 1$ elements. Let $\tau \in \text{Fix}(\mathfrak{T}(X))$ and let $O_x, O_y \in M_\tau$ for $x, y \in X$. Then $(yx^{-1})O_x$ is an open set including $y$. Hence, $O_y \subseteq (yx^{-1})O_x$ which means $|O_y| \leq |O_x|$. The other inclusion can be done similarly so $|O_x| = |O_y|$ for all $x, y \in X$. Now, if $m \in O_x \cap O_y$, then $O_m \subseteq O_x \cap O_y$. As their orders are equal, we must have $O_x = O_y$ or $O_x \cap O_y = \emptyset$. Thus, $X$ is a disjoint union of elements of $M_\tau$. $X$ also acts on $M_\tau$ by Lemma 2.3. and this action is transitive. Let $e$ be the identity element of $X$ and $\text{Stab}(O_e)$ be the stabilizer of $O_e$ in $X$. We have $\text{Stab}(O_e), O_e = O_e$. Since $e \in O_e$, $\text{Stab}(O_e) \subseteq O_e$. As $|X : \text{Stab}(O_e)| = |M_\tau| = \frac{|X|}{|O_e|}$, we have $|\text{Stab}(O_e)| = |O_e|$ which implies that $\text{Stab}(O_e) = O_e$. Hence, the set $M_\tau$ is the set of left cosets of a subgroup of $X$. Since the chosen topology $\tau$ from $\text{Fix}(\mathfrak{T}(X))$ uniquely determines $M_\tau$ and $M_\tau$ uniquely determines a subgroup $O_e$ of $X$, we have an injection from $\text{Fix}(\mathfrak{T}(X))$ to the set of all subgroups of $X$. Conversely, for a subgroup $H$ of $X$, the set of left cosets of $H$ form a minimal base for a topology. The topology $\tau$ generated by this base is an element of $\text{Fix}(\mathfrak{T}(X))$ by Lemma 2.3. Hence, the cardinality of $\text{Fix}(\mathfrak{T}(X))$ is equal to the number of the subgroups of $X$, which is $k + 1$.

By applying same method in the above proof, we can also show that $T_0(p^k) \equiv 1 \mod p$. Actually, Z. I. Borevich proved more general result about $T_0(n)$. Now, we will establish elementary proof for the theorem of Z. I. Borevich.
Theorem 2.5 (Z. I. Borevich). For a given nonnegative integer $n$ and prime $p$, $T_0(n + p) \equiv T_0(n + 1) \mod p$.

Let $C = C_p$ be the cyclic group of order $p$ and $N$ be a set with $n$ elements. Without loss of generality, let $X$ be a disjoint union of $C$ and $N$ so that the cardinality of $X$ is $p + n$. We define the action of $C$ on $X$ in a following way: for $c \in C$ and for $x \in X$, $c \ast x = cx$ if $x \in C$ and $c \ast x = x$ if $x \in N$. Then the action of $C$ on $X$ can be extended to the action of $C$ on $\mathcal{T}_\sigma(X)$. As $|\mathcal{T}_\sigma(X)| \equiv |\text{Fix}(\mathcal{T}_\sigma(X))| \mod p$, it is left to show that $|\text{Fix}(\mathcal{T}_\sigma(X))| = T_0(n + 1)$. Let $\tau \in \text{Fix}(\mathcal{T}_\sigma(X))$ and let $x, y \in C$ with $x \neq y$. We know that $yx^{-1}O_x = O_y$. Then we can observe that $O_x \cap N = O_y \cap N$ as $N$ is fixed by $C$. Similarly, $O_x \cap C$ and $O_y \cap C$ are disjoint or equal. But we can not have $O_x = O_y$ as it is a $T_0$ topology. Hence $O_x \cap C \neq O_y \cap C$. Then we must have $O_x \cap C = \{x\}$ for all $x \in X$. For $a \in N$, $x(C \cap O_a) = C \cap O_a$ which force that $C \cap O_a$ is $C$ or $\emptyset$. Set $\tilde{X} = X/ \sim$ where $x \sim y$ if $x, y \in C$. Notice that the minimal base $M_\tau$ of $X$ induce a minimal base on $\tilde{X}$ by setting $\bar{O}_x = \bar{O}_x$. It is easy to see that this map is one to one on the minimal bases of $\tau \in \text{Fix}(\mathcal{T}_\sigma)$. Hence, $|\text{Fix}(\mathcal{T}_\sigma(X))| \leq T_0(n + 1)$. Conversely, Let $\bar{\tau} \in \mathcal{T}_\sigma(\tilde{X})$, then set $O_x = \{x\} \cup (O_x \setminus \{\bar{x}\})$. Note that for $a \in N$, $\bar{a} = a$. It is easy to see that induced minimal base is $C$-invariant. Then the induced topology $\tau \in \text{Fix}(\mathcal{T}_\sigma(X))$. Having other inclusion concludes the proof.

Corollary 2.6. $T_0(p^k) \equiv 1 \mod p$ where $k$ is a non-negative integer and $p$ is a prime number.

The proof of next theorem is similar with the proof of Theorem 2.5. For clarity, we repeat some arguments.

Theorem 2.7. For a given nonnegative integer $n$, there exists a uniqe integer $k$ such that $T(p + n) \equiv k \mod p$ for all primes $p$. 
Proof. If the given integer \( n = 0 \), \( T(p) \equiv 2 \mod p \) by Theorem 1. Hence we can assume that \( n > 0 \). Let \( C = C_p \) be the cyclic group of order \( p \) and \( N \) be a set with \( n \) elements. We set \( X \) and define the action of \( C \) on \( X \) as in the proof the previous theorem. Then the action of \( C \) on \( X \) can be extended to the action of \( C \) on \( \mathcal{T}(X) \). As \(|\mathcal{T}(X)| \equiv |\text{Fix} (\mathcal{T}(X))| \mod p\), it is left to show that \(|\text{Fix} (\mathcal{T}(X))|\) does not depend on the choice of prime \( p \). By the Lemma 2.3 and Lemma 2.4, \(|\text{Fix} (\mathcal{T}(X))|\) is equal to the number of \( C \)-invariant minimal basis on \( X \). Let \( x, y \in C \). Notice that \( O_x \) completely determine \( O_y \) as \((yx^{-1})O_x = O_y \). Then \(|O_x \cap C| = |O_y \cap C|\) as \(yx^{-1}(O_x \cap C) = O_y \cap C \). Now, it is easy to see that \( O_x \cap C \) and \( O_y \cap C \) is either disjoint or equal. As cardinality of \( C \) is prime, \( O_x \cap C \) is equal to \( \{x\} \) or \( C \). If \( a \in N \) then we must have \( gO_a = O_a \) for all \( g \in C \) as \( ga = a \). Hence, \( O_a \cap C = \emptyset \) or \( O_a \cap C = C \). Thus, number of the elements of the \( C \) has no contribution to the the number of the possible minimal bases. Now we know that such \( k \) exist. If \( k' \) is also such integer than \( k \equiv k' \mod p \) for all prime \( p \). Hence \( k - k' \) is divisible by all primes which force \( k - k' = 0 \). 

By the previous therem we see that \( k \) is uniquely determined by \( n \). Hence we can use \( k(n) \) to denote \( k \) for a given \( n \). The proof the theorem also gives algorithm to calculate \( k(n) \) for a given \( n \) but when \( n \) is larger, calculation of congruence becomes difficult. Here, we calculate for \( n = 1 \).

**Corollary 2.8.** \( T(p + 1) \equiv 7 \mod p \) for all primes \( p \).

**Proof.** We follow the proof the Theorem 2.6. We need to show that it has exactly seven \( C \)-invariant minimal bases. Let \( x \in C \) then \( O_x \cap C = C \) or \( O_x \cap C = \{x\} \).
Case 1: Let \( O_x \cap C = C \) then \( O_x = C \) or \( O_x = C \cup \{a\} \). For both cases \( O_a = \{a\} \) or \( O_a = C \cup \{a\} \). We count 4 different possibilities.

Case 2: Let \( O_x \cap C = \{x\} \) then \( O_x = \{x\} \) or \( O_x = \{x,a\} \) where \( a \) is the unique element of \( N \). If \( O_x = x \) then \( O_a = \{a\} \) or \( O_a = C \cap N \). If \( O_x = \{x,a\} \) then \( O_a \subseteq O_x \) which force \( O_a = \{a\} \). We count three possible sub-cases. All together we have 7 possible cases.

We develop new method to calculate \( k(n) \) for larger \( n \). But the method requires to know some values of \( T(s) \) for some \( s \).

**Theorem 2.9.** The sequence \( k(n) \) satisfies the following inequality: 
\[ T(n+1) < k(n) < 2T(n+1). \]

**Proof.** We again the follow the proof of the Theorem 2.6. To count \( k(n) \), we have two main case,

**Case 1:** Let \( O_x \cap C = C \) for \( x \in C \). Then \( O_y \cap C = C \) for all \( y \in C \) and \( O_x = O_y \) for all \( x,y \in C \). If \( a \in N \) then \( O_a \cap C \) equals to \( C \) or \( \emptyset \). Hence we can see whole \( C \) as a one element. Then we have \( T(n+1) \) possible case.

**Case 2:** Let \( O_x \cap C = \{x\} \) for \( x \in C \). Again we have \( O_x = O_y \) for all \( x,y \in C \). Hence we have at most \( T(n+1) \) possible case. But we can not have exactly \( T(n+1) \) possible cases. To see this set \( O_x = \{x,a\} \) for \( a \in N \). Then \( O_a \) can not be \( \{x,a\} \). Then the result follows.

\( \square \)

**Theorem 2.10.** The sequence \( k(n) := 7, 51, 634, 12623 \) for \( n = 1, 2, 3, 4 \) respectively.

We only show the calculation of \( k(2) \). The others calculation are similar. By previous theorem \( T(3) < k(2) < 2 \times T(3) \) so \( 29 < k(2) < 58 \).

Since

\[ T(4) \equiv k(2) \mod 2 \]
\[ T(5) \equiv k(2) \mod 3 \]
By solving the above congruence relation, we get $k(2) \equiv 21 \mod 30$.

By the inequality we have $k(2) = 51$.

For $n = 3, 4$, we have same procedure. For $n \geq 5$, we do not have unique solution satisfying the inequality.

Closed form of $k(n)$ seems to be another open problem. Hence, calculation of $k(n)$ for specific $n$ or some better lower and upper bounds can be seen as new problem arising from this article.

References

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