The impact of section properties on thin walled beam sections with restrained torsion

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Abstract: In this paper the influence of section properties on thin walled beam sections with restrained torsion is studied by considering the angle of twisting for non-uniform torsion. The study is based on the governing equation for non-uniform torsion of thin-walled beam of open and closed cross-sections, including the restrained behavior. The hyperbolic shape functions is used to generate the general solution for homogenous equation as it gives the exact results in accordance with the analytical solution. This study is focused on a bar subjected to torsion with warping of wide-ranging open and closed cross-sections with different value of characteristics number for torsion. The impact of the characteristics number for torsion on different section type is studied and evaluated. The obtained results are compared based on the influence of section properties on the components of the total applied torsion and presented graphical for different value of characteristics number for torsion.

Keywords: thin walled structures, open and closed sections, non-uniform torsion, restrained torsion, characteristic number for torsion, twisting angle.

1. Introduction

Nowadays most structural components of transportation facilities are designed using thin-walled components due to their advantages of high strength, light in weight, sustainable to unbalanced loadings and structural efficiency. Regardless of their applications, the methods of analysis include the effects of cross-section type, non-uniform torsion and restraints of thin-walled members should be considered for the proper design and identifying their behaviour. The importance of restrained torsion of thin walled section has grown significantly as the deformations and stresses caused by torsion affects the behaviour of the structures with open as well as closed section. Generally, thin-walled sections do not behave according to the law of the plane sections employed by Euler-Bernoulli-Navier however, the general theory of thin-walled section is developed by Vlasov [1–2]. In addition, thin-walled structures like plates and shells associated with finite element formulations are the most common construction elements in nature and technology [3-4].

When a bar is subjected to a non-uniform torsion and its angle of twist per unit length is not constant along its axis. Several researchers have dealt with restrained torsion of thin walled beams of different cross sectional type [5–9]. Saint-Venant driven models have been widely studied as a groundworks and the new torsion element of thin-walled beams including shear deformation which accounts for the warping deformation and shear deformation due to restrained torsion are developed [10-19]. Neglecting these warping stresses may generate significant errors specially for open profile torsion or shear-bending of short beams, and the situation may be even more critical for composite beams [20-21]. Warping-based stresses and deformations in closed sections are assumed to be insignificant and have been therefore...
neglected [6]. Restraining the warping deformation causes the twist rate to be zero at the point of restraint and this causes a local effective torsional stiffening that affects the global torsional response of the beam [13]. In this article, the function of the twisting angle is implemented from the solution of the hyperbolic equation of restrained torsion of thin-walled bars as it gave the exact result like the analytical solution.

The main objective of this paper is to show the consequence of thin-walled open and closed sections depending on the geometrical properties of the section. The twisting angle is expressed as a function of the characteristic number for torsion, which depends on the ratio of the pure torsional rigidity (GJ), in the sense of the Saint-Venant theory, to the sectional rigidity (EC_{90}). The distribution of these stresses through the thin walled section depends to a large extent on the cross sectional geometry and specifically whether it is open or closed section type [12, 20, 21]. Thin-walled sections, for the reason that of their specific properties, have a unique internal force factor - bimoment, which in certain cases can cause large normal stresses in the cross-section. In order to determine the stresses with good accuracy mixed FEM can be used [21].

The behavior of a bar with restrained torsion using the governing equation for non-uniform torsion of thin-walled sections is considered. This study is applied generally to bars of closed and/or open sections of thin walled steel cross section subjected to different torsional loading and to the most general torsional boundary conditions. The variation of primary torsional moment (M_{TP}), secondary torsional moments (M_{TS}) and warping moments (M_{\omega}) for different value of characteristic number for torsion (\theta) are presented graphically. The twisting angle is expressed as a function of the characteristic number of non-uniform torsion with or without consideration of warping effects as theories and as analysis using commercial FEM codes. The variation of the warping and sectional rigidity of the cross section along the beam results in the variation of the characteristic number which defining the non-uniform behaviour of the torsional loading in the beam section.

2. Method of research
The governing equation for non-uniform torsion is derived for bars of thin-walled sections with a local coordinate system defined by y_1, y_2, y_3 axes as shown in figure 1.

\[ \frac{d}{dy_1} \left( I \frac{d\theta}{dy_1} \right) = M_{TP} + M_{TS} + M_{\omega} \]

\[ M_{TP} = M_{TP}(\theta) \]
\[ M_{TS} = M_{TS}(\theta) \]
\[ M_{\omega} = M_{\omega}(\theta) \]

If a bar undergoes non-uniform torsion, the total applied twisting moment M_T is resisted by an internal twisting moment that consists of two components the primary and secondary torsional moments. The M_{TP} is the twisting moment due to uniform torsion (St Venant torsion) and the M_{TS} is the twisting moment due to warping restraint. The variation of internal twisting moments are expressed as a function of the characteristic number of non-uniform torsion. The Governing Equation for non-uniform torsion is given in equation 1, as the rate of change of the total twisting moment in a bar is in equilibrium with the applied load as given below.
\[ EC_n \frac{d^4 \beta_i}{dy_i^4} - GJ \frac{d^2 \beta_i}{dy_i^2} = 0, \quad \frac{d^4 \beta_i}{dy_i^4} - \left( \frac{\theta}{a} \right)^2 \frac{d^2 \beta_i}{dy_i^2} = 0 \quad (1) \]

\[ \theta := a \sqrt{\frac{GJ}{EC_n}} \]

\[ \beta_i \text{ angle of twist,} \quad J \text{ torsion constant} \]

\[ E, G \text{ elastic constants,} \quad m_T \text{ twisting load per unit length of bar} \]

\[ C \text{ warping constant,} \quad a \text{ length of the bar} \]

\[ \text{characteristic number for torsion} \]

The value of \( \theta \) depends on the geometrical properties (area, second moment of area, warping constant, torsion constant) which showed that the torsional behavior of the section can also be described similarly by geometrical properties. Restraint of warping produces longitudinal stresses, shear stresses however the longitudinal warping stresses are greatest at the flange tips. For the verification of combined bending and torsion, it is more convenient to use the value of the warping moment in the flange, rather than the longitudinal warping stress. For an element with constant \( M_T \) the warping torsion generates a generalized force called Bimoment \( (M_w) \) given as follows:

\[ M_w = - EC_n \frac{d^4 \beta_i}{dy_i^4} \quad (2) \]

The general solution for the homogeneous equation is satisfied by the following assumed twisting angle \( (\beta_i(y_i)) \) which satisfies the equation (3). The equation yields to the exact solutions for angle of twists, its derivatives, twisting moments, and bimoments at a node.

\[ \beta_i = c_1 \sinh \frac{\theta y_i}{a} + c_2 \cosh \frac{\theta y_i}{a} + c_3 \frac{y_i}{a} + c_4 \quad (3) \]

The free coefficients \( c_1 \) to \( c_4 \) are determined so that the boundary conditions at the ends of the bar are satisfied. Based on figure 1, if the vertex \( y_i = 0 \) is fixed and a twisting moment \( M_T \) is applied at vertex \( y_i = a \), which is free to warp. Therefore, solving equation (3) by considering the boundary conditions the free coefficients are given below:

\[ c_3 = \frac{M_T a}{GJ} \quad c_1 = -\frac{c_3}{\theta} \quad c_2 = \frac{c_3}{\theta} \tanh \theta \quad & \quad c_4 = -\frac{c_3}{\theta} \tanh \theta \]

The above coefficients are substituted into expression (3) and the angle of twist is given as follow:

\[ \beta_i = \frac{M_T a}{GJ} \left( \tanh \theta \left( \frac{\theta y_i}{a} - 1 \right) - \left( \frac{\theta y_i}{a} - \frac{y_i}{a} \right) \right) \quad (4) \]

Based on the equations (1), (2) and (4) the \( M_{Tp}, M_{Ts} \) and \( M_w \) for a node are given below in the equations (5) to (7) respectively.

\[ M_{Tp} = M_T + M_T \tanh \theta \frac{\theta y_i}{a} - \frac{\theta y_i}{a} \quad (5) \]

\[ M_{Ts} = -M_T \tanh \theta \frac{\theta y_i}{a} - \frac{\theta y_i}{a} \quad (6) \]

\[ M_w = -M_T \left( \tanh \theta \left( \frac{\theta y_i}{a} - \frac{\theta y_i}{a} \right) \right) \quad (7) \]
Tables 1 to 3 show comparatively for different value of $\theta$ along the span of the beam, the initial value of $\theta$ is 1 and the maximum is considered to be 60. The variation of $M_{TP}$, $M_{TS}$ and $M_w$ for different value of characteristic number for torsion are presented. The results are tabulated below to bars of closed and/or open sections of thin walled steel cross section subjected to different torsional loading and to the most general torsional boundary conditions. In table 1, the values are obtained based on equation (5) as it is expressed in a dimensionless form as shown below for diverse values of $\theta$.

### Table 1. Variation of $M_{TP}$ for different value of characteristic number for torsion

| $y_1/a$ | $\theta = 1$ | $\theta = 2.5$ | $\theta = 5$ | $\theta = 7.5$ | $\theta = 10$ | $\theta = 20$ | $\theta = 40$ | $\theta = 60$ |
|---------|--------------|----------------|--------------|----------------|--------------|--------------|--------------|--------------|
| 0       | 1.00         | 1.00           | 1.00         | 1.00           | 1.00         | 1.00         | 1.00         | 1.00         |
| 0.2     | 0.87         | 0.61           | 0.37         | 0.22           | 0.14         | 0.02         | 0.00         | 0.00         |
| 0.4     | 0.77         | 0.38           | 0.14         | 0.05           | 0.02         | 0.00         | 0.00         | 0.00         |
| 0.6     | 0.70         | 0.25           | 0.05         | 0.01           | 0.00         | 0.00         | 0.00         | 0.00         |
| 0.8     | 0.66         | 0.18           | 0.02         | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |
| 1       | 0.65         | 0.16           | 0.00         | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |

Similarly, in tables 2 and 3, the values are obtained based on equations (5) and (6) respectively and they are expressed in a dimensionless form as shown below for different values of $\theta$.

### Table 2. Variation of $M_{TS}$ for different value of characteristic number for torsion

| $y_1/a$ | $\theta = 1$ | $\theta = 2.5$ | $\theta = 5$ | $\theta = 7.5$ | $\theta = 10$ | $\theta = 20$ | $\theta = 40$ | $\theta = 60$ |
|---------|--------------|----------------|--------------|----------------|--------------|--------------|--------------|--------------|
| 0       | 0.00         | 0.00           | 0.00         | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |
| 0.2     | 0.13         | 0.39           | 0.63         | 0.78           | 0.86         | 0.98         | 1.00         | 1.00         |
| 0.4     | 0.23         | 0.62           | 0.86         | 0.95           | 0.98         | 1.00         | 1.00         | 1.00         |
| 0.6     | 0.30         | 0.75           | 0.95         | 0.99           | 1.00         | 1.00         | 1.00         | 1.00         |
| 0.8     | 0.34         | 0.82           | 0.98         | 1.00           | 1.00         | 1.00         | 1.00         | 1.00         |
| 1       | 0.35         | 0.84           | 1.00         | 1.00           | 1.00         | 1.00         | 1.00         | 1.00         |

### Table 3. Variation of $M_w$ for different value of characteristic number for torsion with $\theta/a = 4$

| $y_1/a$ | $\theta = 1$ | $\theta = 2.5$ | $\theta = 5$ | $\theta = 7.5$ | $\theta = 10$ | $\theta = 20$ | $\theta = 40$ | $\theta = 60$ |
|---------|--------------|----------------|--------------|----------------|--------------|--------------|--------------|--------------|
| 0       | -0.25        | -0.25          | -0.25        | -0.25          | -0.25        | -0.25        | -0.25        | -0.25        |
| 0.2     | -0.14        | -0.15          | -0.09        | -0.06          | -0.03        | 0.00         | 0.00         | 0.00         |
| 0.4     | -0.10        | -0.09          | -0.03        | -0.01          | 0.00         | 0.00         | 0.00         | 0.00         |
| 0.6     | -0.07        | -0.05          | -0.01        | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |
| 0.8     | -0.03        | -0.02          | 0.00         | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |
| 1       | 0.00         | 0.00           | 0.00         | 0.00           | 0.00         | 0.00         | 0.00         | 0.00         |
3. Results and discussion

The similarities and differences of $M_{TP}$ and $M_{TS}$ for different value of $\theta$ are presented graphically in figure 2 referring the results in tables 1 to 3. Considering a point nearly to the support at $y_1 = 0$ of figure 1, the total torsional moment is largely carried by the primary twisting moment, whereas the secondary twisting moment is small. At other points within the span, the total torsion is carried partly as St Venant torsion (i.e. by the St Venant shear stresses) and partly as warping torsion (i.e. by the shear stresses caused by the restraint of warping). The value of $\theta$ for the closed and open sections differ largely, accordingly for closed section the shear stress is constant while for open section varies its direction and magnitude across the thickness. The section property for open and closes sections vary with respect to torsional and warping constants thus the value of $\theta$ differs. For small value of $\theta$ both torsion mechanisms contribute to $M_T$ throughout the beam in both cases but with the increasing of the value of $\theta$ the influence of the twisting moment varies as shown in the figure 2. In view of figure 2 with the increasing value of $\theta$, $M_{TP}$ reduces fast and St. Venant in the major part of the beam dominates the torsional moment. Similarly, with the increasing value of $\theta$, the St. Venant moment is growing and the torsional moment $M_T$ is dominated by St. Venant in the major part of the beam in both cases. For value of $\theta$ greater than 20, the total torsional moment is restricted to small length (approximately 0.2a) near to the support as shown in figure 2 and its magnitude changes rapidly in both cases.

![Figure 2. Variation of $M_{TP}$ and $M_{TS}$ for different value of $\theta$.](image)

Likewise, the behavior and variation of $M_{\theta\alpha}$ for different value of $\theta$ are presented graphically in figure 3. The characteristics number for torsion is an indicator of how quickly the effect of warping restraint dissipates and the warping moment varies based on the restrained conditions of the beam. The variation of $M_{\theta\alpha}$ shows greater values at the fixed support as the warping prevented and it induces the largest normal stresses as shown in figures 3. At the free end, the beam can warp freely therefore the normal stresses and $M_{\theta\alpha}(a)$ are zero. The graphs shown below are plotted for different value of ratio of $\theta$ to a ($\theta/a = 2$ and 4) as it is a general case. The distribution of $M_{\theta\alpha}$ varies differently in both cases with altering values of $\theta$ and accordingly the normal warping stress differs along the beam based on the position of restrained torsion and section type.
Figure 3. Variation of $M_a$ for different value of $\theta$ and for $\theta/a = 2$ and 4.

A combined graph for $M_{TP}$, $M_{TS}$ and $M_{a}$ for value of $\theta = 1$ and $\theta = 10$ is shown in the figure 4. Referring figure 4, the values of $M_{TP}$, $M_{TS}$ and $M_{a}$ differ for $\theta = 1$ and for $\theta = 10$. For $\theta = 1$, the total torsional moment components are extent throughout the span of the beam as shown in figure 4 and its magnitude changes gradually as the value of $\theta$ is small. If the value of $\theta$ is small, it is most common for open sections. For $\theta = 10$, the total torsional moment is restricted to small length near to the support and its magnitude changes rapidly. If value of $\theta$ is large, it is most common for closed sections.

Figure 4. Combined $M_{TP}$, $M_{TS}$ and $M_{a}$ for value of $\theta = 1$ and $\theta = 10$

4. **Conclusion**

The variation of the warping and sectional rigidity of the cross section along the beam results in the variation of the characteristic number. In this study, we used the hyperbolic shape function than polynomial shape functions as it gives the exact results as per the analytical solution. The discrepancy in the characteristic number may describe the non-uniform behaviour of the torsion. With the increasing value of $\theta$, $M_{TP}$ reduces fast and St. Venant moment dominates the major part of the beam torsional
moment in both cases. For value of $\theta$ greater than 20, the total torsional moment is restricted to small length (roughly 0.2a) nearer to the support and its magnitude changes rapidly in both cases. The distribution of $M_a$ varies differently in both cases with altering values of $\theta$ and accordingly the normal warping stress differs along the beam based on the position of restrained torsion and section type. A combined graph is given in figure 4 for $M_{TP}, M_{TS}$ and $M_a$ for $\theta = 1$, the total torsional moment components are extent throughout the span and changes steadily as the value of $\theta$ is small. For $\theta = 10$, the total torsional moment is restricted to small interval near to the support and change their magnitude rapidly.

Acknowledgment
This paper is financially supported by the Ministry of Education and Science of the Russian Federation on the program to improve the competitiveness of Peoples’ Friendship University of Russia (RUDN University) among the world’s leading research and education centers in the 2016-2020.

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