Pseudo-effective and nef cones on spherical varieties

Qifeng LI

Abstract

We show that nef cycle classes on smooth complete spherical varieties are effective, and the products of nef cycle classes are also nef. Let \( X \) be a smooth projective spherical variety such that its effective cycle classes of codimension \( k \) are nef, where \( 1 \leq k \leq \dim(X) - 1 \). We study the properties of \( X \). And we show that if \( X \) is a toric variety, then \( X \) is isomorphic to the product of some projective spaces; if \( X \) is toroidal, then \( X \) is isomorphic to a rational homogeneous space; if \( X \) is horospherical, \( \dim(X) \geq 3 \) and \( k = 2 \), then effective divisors on \( X \) are nef; if \( X \) is horospherical and effective divisors on \( X \) are nef, then there is a morphism from \( X \) to a rational homogeneous space such that each fiber is isomorphic to the product of some horospherical varieties of Picard number one.

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1 Introduction

The positivity of divisors and curves occupy an important position in algebraic geometry. And recently, there are some work on the positivity of subvarieties or more generally, on the positivity of cycles, for example [Pet09] [Fulg11] [DELV11] [Ot13] and [Le13]. In the paper [DELV11], they defined pseudo-effective and nef cones and studied the properties of these cones on some Abelian varieties. The aim of this paper is to study the pseudo-effective and nef cones on spherical varieties.

We work on the complex number field \( \mathbb{C} \). Let \( X \) be a smooth complete variety of dimension \( n \). Let \( A^r(X) = \bigoplus_{k=0}^{n} A^k(X) \) be the Chow ring of \( X \). Let \( N^k(X)_{\mathbb{R}} \) be the finite dimensional real vector space of numerical equivalence classes of codimension \( k \) algebraic cycles on
$X$ with real coefficients. Denote by $\text{Eff}^k(X) \subseteq N^k(X)_{\mathbb{R}}$ the cone generated by effective cycles, $\text{Psef}^k(X)$ the closure of $\text{Eff}^k(X)$ in $N^k(X)_{\mathbb{R}}$. Let $\text{Nef}^k(X) = \{ \eta \in N^k(X)_{\mathbb{R}} \mid \eta \cdot \delta \geq 0, \delta \in \text{Psef}^{n-k}(X) \}$, $N_k(X)_{\mathbb{R}} = N^{n-k}(X)_{\mathbb{R}}$, $\text{Eff}_k(X) = \text{Eff}^{n-k}(X)$, $\text{Nef}_k(X) = \text{Nef}^{n-k}(X)$, and $\text{Psef}_k(X) = \text{Psef}^{n-k}(X)$. We call $\text{Eff}^k(X)$, $\text{Psef}^k(X)$, $\text{Nef}^k(X)$ $\text{Psef}^k(X)$ effective cones, pseudo-effective cones, and nef cones respectively, and call their elements effective cycle classes, pseudo-effective cycle classes and nef cycle classes respectively.

Let $G$ be a connected reductive algebraic group, $B$ be a Borel subgroup, and $R_u(B)$ be the unipotent radical of $B$. A normal $G$-variety $X$ is said to be $G$-spherical if there is an open $B$-orbit on $X$.

**Theorem 1.1.** (Theorem 3.4) Let $X$ be a smooth complete $G$-spherical variety of dimension $n$. Then for any integer $k$, $\text{Nef}^k(X) \subseteq \text{Psef}^k(X) = \text{Eff}^k(X)$, and these cones are rational polyhedra. If $\eta_1 \in \text{Nef}^{k_1}(X)$ and $\eta_2 \in \text{Nef}^{k_2}(X)$, then $\eta_1 \cdot \eta_2 \in \text{Nef}^{k_1 + k_2}(X)$.

This theorem answers Problem 6.8 in [DELV11]. And these phenomena are quite different from those on Abelian varieties. Now a natural question arises. What does $\pi$-isomorphic to a rational homogeneous space. The following are our main results.

**Theorem 1.2.** (Theorem 4.12, Theorem 4.17, Theorem 4.28 and Corollary 4.43) Let $X$ be a smooth projective $G$-spherical variety of dimension $n$. Then the following hold.

(i) If $X$ is a toric variety, then $\text{Nef}^k(X) = \text{Psef}^k(X)$ for some $1 \leq k \leq n-1$ if and only if $X$ is isomorphic to the product of some projective spaces.

(ii) If $X$ is toroidal, then $\text{Nef}^k(X) = \text{Psef}^k(X)$ for some $1 \leq k \leq n-1$ if and only if $X$ is isomorphic to a rational homogeneous space.

(iii) If $X$ is $G$-horospherical, $n \geq 3$ and $\text{Nef}^2(X) = \text{Psef}^2(X)$, then $\text{Nef}^1(X) = \text{Psef}^1(X)$.

(iv) If $X$ is $G$-horospherical and $\text{Nef}^1(X) = \text{Psef}^1(X)$, then there is a $G$-equivariant morphism $\pi : X \to G/P$ such that each fiber of $\pi$ is isomorphic to the product of some smooth projective $L$-horospherical varieties of Picard number one, where $P$ is a parabolic subgroup of $G$ and $L$ is a Levi factor of $P$.

This paper is organized as follows. In Section 2 we review some basic notations and results from the Luna-Vust Theory and the Mori Theory on spherical varieties, which are our main tools. In Section 3 we prove Theorem 1.1. In Section 4 we mainly study the smooth projective spherical variety $X$ of dimension $n$ such that $\text{Nef}^k(X) = \text{Psef}^k(X)$ for some $1 \leq k \leq n-1$. In Subsection 4.1 we study the general cases. And we get that if $2 \leq k \leq n-2$ and $\text{Nef}^{1}(X) \neq \text{Psef}^{1}(X)$, then for any birational Mori contraction, the dimension of the exceptional locus is no more than $k-1$. Then we study the $G$-equivariant morphisms from a complete spherical variety to some special rational homogeneous spaces, and these morphisms contribute to the proof of Theorem 1.2 (iv). In Subsection 4.2 and 4.3 we show Theorem 1.2 (i) and (ii) respectively. In Subsection 4.4 we prove Theorem 1.2 (iii) (iv). Since the horospherical cases are complicated, we make some preliminaries in the part 4.4.1. In the part 4.4.2 we show in Proposition 4.27 that the exceptional locus of a birational Mori contraction on a projective $Q$-factorial horospherical variety is irreducible, and then prove Theorem 1.2 (iii). Finally, we prove Theorem 1.2 (iv) in the part 4.4.3 after some reductions in the part 4.4.3.

**Conventions and notations.** Schemes are always assumed to be separated and of finite type over $\mathbb{C}$ and varieties are irreducible and reduced schemes.

Denote by $G$ a connected reductive algebraic group. Let $B$ be a Borel subgroup of $G$, $T$ a maximal torus in $B$. Let $g, b, t$ be the corresponding Lie algebras. Let $S$ be the set of simple
roots. If $I$ is a subset of $S$, then we denote by $P_I$ the corresponding parabolic subgroup of $G$ containing $B$. Denote by $B^-$ the opposite Borel subgroup corresponding to $B$ and by $P^-_I$ the opposite parabolic subgroup corresponding to $P_I$.

For a linear algebraic group $H$, denote by $R_u(H)$ the unipotent radical of $H$. Let $\langle \cdot, \cdot \rangle_H$ be paring between the characters and the coroots on $H$. When there is no confusions, we omit the subscript $H$.

For a group $H$ and an $H$-module $V$, denote by
\[
V^{(H)} = \{v \in V \mid v \neq 0, \text{ and } hv = \chi(h)v \text{ for some character } \chi \in \chi(H)\},
\]
where $\chi(H)$ is the group of characters of $H$. On the other hand, we denote by
\[
V^H = \{v \in V \mid hv = v \text{ for any } h \in H\}.
\]

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2 Spherical varieties

In this section, we will recall some basic notations and results from the Luna-Vust Theory and the Mori Theory on spherical varieties which will be frequently used in this paper.

Let $H$ be an algebraic group, $X$ a scheme (resp. a variety). If there is a morphism $\varphi : H \times X \to X$ such that $\varphi(h_1, \varphi(h_2, x)) = \varphi(h_1 h_2, x)$, $\varphi(e, x) = x$, where $h_1, h_2 \in H$, $x \in X$ and $e$ is the unit of $H$, we say that $H$ acts on $X$ or say that there is an $H$-action on $X$. And $X$ is said to be an $H$-scheme (resp. an $H$-variety). For any $h \in H$ and any $x \in X$, denote by $h \cdot x = \varphi(h, x)$.

Let $X$ be an $H$-scheme. Denote by $S_X, H$ the set of $H$-orbits on $X$, and $S_X^c, H$ the set of closed $H$-orbits on $X$. Let $M \subseteq H$, $Y \subseteq X$ be subsets, then denote by $MY = \{m \cdot y \mid m \in M, y \in Y\}$. If $Y \subseteq X$ is a subscheme such that $HY \subseteq Y$ in the set theory, then we say that $Y$ is $H$-stable. For any point $x \in X$, denote by the isotropy group $H_x = \{h \in H \mid h \cdot x = x\}$.

Definition 2.1. A normal $G$-variety $X$ is $G$-spherical if there is an open $B$-orbit on $X$.

Let $x_0$ be a point in the open $G$-orbit of the $G$-spherical variety $X$, and $H = G_{x_0}$. Then we say that $X$ is a spherical $G/H$-embedding and identify $G/H$ with an open subset of $X$. Note that $G/H$ is itself a $G$-spherical variety, and we call it a homogeneous $G$-spherical variety. Denote by $\partial X = X \setminus (G/H)$, and we call it the boundary of $X$.

Since all Borel subgroups of $G$ are conjugate, a normal $G$-variety $X$ is $G$-spherical if and only if there is an open $B'$-orbit on $X$, where $B'$ is any fixed Borel subgroup of $G$. There are several equivalent definitions of spherical varieties, see for example [Per12] Def. 1.0.1, Thm. 2.1.2, [Br86], and [BLV86].

Proposition 2.2. Let $X$ be a normal $G$-variety. Then the following are equivalent.

(i) $X$ is $G$-spherical.
(ii) $\mathcal{C}(X)^B = \mathcal{C}$;
(iii) for any (or some) Borel subgroup $B'$ of $G$, there is an open $B'$-orbit on $X$;
(iv) $X$ has finitely many $B$-orbits;
(v) every normal $G$-variety containing $G/H$ as the maximal $G$-orbit has only finitely many $G$-orbits, where $G/H \subseteq X$ is an open $G$-orbit;
From now on to the end of Section 2, we assume that $X$ is a spherical $G/H$-embedding. Define a map $f \mapsto \chi_f$ from $C(X)^{(B)}$ to $\chi(B)$, where $\chi_f \in \chi(B)$ satisfies that for all $b \in B$, $b \cdot f = \chi_f(b)$. This is a morphism of abelian groups with kernel $C^*$. Denote by $M_X$ or $M_{G/H}$ the image of this morphism, and identify it with $C(X)^{(B)}/C^*$. Denote by $\text{rank}(X) = \text{rank}(G/H) = \text{rank}(M_X)$ and call it the rank of $X$ or the rank of $G/H$.

Let $N_X = N_{G/H} = \text{Hom}(M_X, \mathbb{Z})$ be the dual of $M_X$. Any valuation $\nu$ on $X$ induces a homomorphism $C(X)^{(B)} \to \mathbb{Q}$ by $f \mapsto \nu(f)$. Hence, $\nu$ induces an element $\rho(\nu) \in \text{Hom}(M_X, \mathbb{Q})$. Thus we can define a morphism $\rho = \rho_{G/H} = \rho_X : \{\text{valuations}\} \to (N_X)_\mathbb{Q}$, where $(N_X)_\mathbb{Q} = N_X \otimes \mathbb{Q}$. Denote by $(M_X)_\mathbb{Q} = M_X \otimes \mathbb{Q}$.

Denote by $\mathcal{V}(G/H)$ the set of $G$-invariant valuations on $X$. By [Kn91] Cor. 1.8, $\rho : \mathcal{V}(G/H) \to (N_{G/H})_{\mathbb{Q}}$ is injective. We regard $\mathcal{V}(G/H)$ as a subset of $(N_X)_\mathbb{Q}$.

A subset $\mathcal{C}$ of a vector space $\mathbb{Q}^n$ is called a cone, if it’s closed under addition and multiplication by $\mathbb{Q}^+ = \{q \in \mathbb{Q} | q \geq 0\}$. For a cone $\mathcal{C}$ in $\mathbb{Q}^n$, we denote by $\mathcal{C}^o$ the interior of it. The cone $\mathcal{C}$ is called strictly convex if $\mathcal{C}$ is strictly convex cone and finitely many elements $v_1, \ldots, v_s \in \mathbb{Q}^n$ such that $\mathcal{C} = \sum_{i=1}^s \mathbb{Q}^+ v_i$.

Proposition 2.3. ([BP87], Cor. 3.2], [Br90]) $\mathcal{V}(G/H)$ is a finitely generated cone in $(N_X)_\mathbb{Q}$. Moreover, there exist linear independent forms $\chi_1, \ldots, \chi_m$ in $(M_{G/H})_{\mathbb{Q}}$ such that $\mathcal{V}(G/H) = \{\nu \in (N_{G/H})_{\mathbb{Q}} | \chi_i(\nu) \geq 0, 1 \leq i \leq m\}$.

It should be noticed that the second assertion of Proposition 2.3 didn’t appear explicitly in [Br90]. Just as what Knop pointed out after Theorem 5.4 in [Kn91], the proof of this assertion in [Br90] is rather involved and rests ultimately on a case-by-case consideration. We call $\mathcal{V}(G/H)$ the valuation cone of $X$ or the valuation cone of $G/H$.

Denote by $\mathcal{B}(X)$ the set of irreducible $B$-stable divisors on $X$. Let $\mathcal{D}(G/H) = \{D \in \mathcal{B}(X) | D \cap (G/H) \neq \emptyset\}$ and $\mathcal{V}_X = \{D \in \mathcal{B}(X) | D \subseteq \partial X\}$. Thus, $\mathcal{B}(X) = \mathcal{V}_X \cup \mathcal{D}(G/H)$, and $\mathcal{V}_X \cap \mathcal{D}(G/H) = \emptyset$. We call the elements in the set $\mathcal{D}(G/H)$ colors of $G/H$, while the elements in the set $\mathcal{V}_X$ are called boundary divisors. For any $Y \in S_{X,G}$, we denote by $\mathcal{D}_Y = \{D \in \mathcal{D}(G/H) | Y \subseteq D\}$, $\mathcal{V}_Y = \{D \in \mathcal{V}_X | Y \subseteq D\}$, and $\mathcal{B}_Y = \mathcal{V}_Y \cup \mathcal{D}_Y$. Let $\mathcal{D}_X = \bigcup_{Y \in S_{X,G}} \mathcal{D}_Y$, and we call its elements colors of $X$.

Definition 2.4. (i) A colored cone is a pair $(\mathcal{C}, \mathcal{D})$ with $\mathcal{C} \subseteq (N_{G/H})_\mathbb{Q}$ and $\mathcal{D} \subseteq \mathcal{D}(G/H)$ having the following properties:

(a) $\mathcal{C}$ is a cone generated by $\rho(\mathcal{D})$ and finitely many elements in $\mathcal{V}(G/H)$;

(b) $\mathcal{C}^o \cap \mathcal{V}(G/H) \neq \emptyset$, i.e. there is a $G$-invariant valuation in the interior of $\mathcal{C}$.

A colored cone $(\mathcal{C}, \mathcal{D})$ is called strictly convex cone if the following holds:

(c) $\mathcal{C}$ is a strictly convex cone and $0 \notin \rho(\mathcal{D})$.

(ii) A pair $(\mathcal{C}_0, \mathcal{D}_0)$ is a colored face of the colored cone $(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}_0$ is a face of $\mathcal{C}$, $\mathcal{C}_0 \cap \mathcal{V}(G/H) \neq \emptyset$ and $\mathcal{D}_0 = \mathcal{D} \cap \rho^{-1}(\mathcal{C}_0)$.

(iii) A colored fan $\mathcal{F}$ is a nonempty finite set of colored cones with the following properties:

(a) Every colored face of $(\mathcal{C}, \mathcal{D}) \in \mathcal{F}$ belongs to $\mathcal{F}$;

(b) For every $\nu \in \mathcal{V}(G/H)$, there is at most one $(\mathcal{C}, \mathcal{D}) \in \mathcal{F}$ such that $\nu \in \mathcal{C}_0$.

A colored fan $\mathcal{F}$ is called strictly convex if $(0, \emptyset) \in \mathcal{F}$, or equivalently, if all elements of $\mathcal{F}$ are strictly convex.

For any $G$-orbit $Y$ on $X$, we denote by $\mathcal{C}_Y = (\mathcal{C}_Y, \mathcal{D}_Y)$, where $\mathcal{C}_Y$ is the cone in $(N_X)_\mathbb{Q}$ generated by all $\rho(\nu_D)$, $D \in \mathcal{B}_Y$. Let $\mathcal{Y}$ be the closure of $Y$ in $X$. For the convenience of discussions, we also denote by $\mathcal{C}_\mathcal{Y} = \mathcal{C}_Y$, $\mathcal{V}_\mathcal{Y} = \mathcal{V}_Y$, $\mathcal{D}_\mathcal{Y} = \mathcal{D}_Y$ and $\mathcal{C}_\mathcal{Y} = \mathcal{C}_\mathcal{Y}$. Denote by $\mathcal{F}_X = \{\mathcal{C}_Y | Y \in S_{X,G}\}$, $\mathcal{C}(X) = \bigcup_{Y \in S_{X,G}} \mathcal{C}_Y$, and $\text{Supp}(\mathcal{F}_X) = \mathcal{C}(X) \cap \mathcal{V}(G/H)$.

A $G$-spherical variety $X$ is said to be simple if it has only one closed $G$-orbit. If $X$ is simple with the unique closed $G$-orbit $Y$, then we denote by $\mathcal{C}(X) = \mathcal{C}_Y$. 

Let $\chi \in \mathfrak{m}$ and $(\cdot , \cdot )$ denote by $l_{\mathfrak{m}} C_{n}$ [Br93, Thm. 1.3(ii)], we can find integers $l_{\nu}$ classes on $X$ that if $\mathfrak{m}$ morphism of colored fans if every element of $l_{\nu}$ embeddings and strictly convex colored fans in $(l_{\nu} C_{n})$. Theorem 2.7. Definition 2.8. Theorem 2.7. (Br93 Thm. 4.1]) Keep notations as above. Let $X$ and $X'$ be a spherical $G/H$-embedding and a spherical $G/H'$-embedding respectively. Then $\phi$ extends to a morphism $X \to X'$ if and only if $F_{X}$ maps to $F_{X}$, if $D$ is dense in $G/H'$. Definition 2.6. Keep notations as above.

(a) Let $(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')$ be colored cones for $G/H, G/H'$ respectively. Then we say that $(\mathcal{C}, \mathcal{D})$ maps to $(\mathcal{C}', \mathcal{D}')$ if $\phi_{*}(\mathcal{C}) \subseteq \mathcal{C}'$ and $\phi_{*}(\mathcal{D}\setminus \mathcal{D}_{\phi}) \subseteq \mathcal{D}'$.

(b) Let $F_{X}$, $F_{X}'$ be colored cones for $G/H, G/H'$ respectively. Then we say that $\phi_{*} : F \to F'$ is a morphism of colored fans if every element of $F$ maps to some element of $F'$.

We call the elements of $PL(X)$ piecewise linear functions on $\mathcal{C}(X)$, while it should be noticed that if $Y_{1}, Y_{2} \in S_{X,G}$, maybe $l_{Y_{1}}|\epsilon_{1} \neq l_{Y_{2}}|\epsilon_{2}$. Suppose that $X$ is moreover complete. Take $l = (l_{Y})_{Y \in S_{X,G}} \in PL(X)$, then for any $Y \in S_{X,G}$, $\dim(\mathcal{C}_{Y}) = \dim(X)_{Q}$ by [Kn91] Thm. 6.3. Thus, $l_{Y}$ can be uniquely extended as a linear function $l_{Y}^{\mathbb{Q}}$ defined on $(N_{X})_{Q}$. If $\delta = \sum_{D \in \mathfrak{B}(X)} n_{D}(\delta)D$ is a Cartier $B$-stable divisor, then by [Br98] Prop. 3.1], for each $Z \in S_{X,G}$, there exists a unique element $\chi_{Z} \in M_{X}$ such that $\nu_{D}(\chi_{Z}) = n_{D}(\delta)$ for all $D \in \mathfrak{B}_{Z}$. These $\chi_{Z}$ determine a unique $l(\delta) = (l_{Y})_{Y \in S_{X,G}}$ such that $l_{Z}|\epsilon_{Z} = \chi_{Z}|\epsilon_{Z}$ for all $Z \in S_{X,G}$. Hence, $l_{Y}(\rho(\nu_{D})) = n_{D}(\delta)$ for all $Y \in S_{X,G}$ and $D \in \mathfrak{B}_{Y}$. By [Br98] Thm. 3.3, if $X$ is moreover projective, then every Cartier $B$-stable divisor $\delta$ is uniquely associated with a well-defined piecewise linear function $l(\delta)$ on $\mathcal{C}(X)$.

Let $X$ be a projective $\mathbb{Q}$-factorial spherical $G/H$-embedding. We will define some 1-cycle classes on $X$ as in the proof of [Br93 Thm. 3.2]. Take $\delta$ to be a Cartier divisor on $X$. Then by [Br93 Thm. 1.3(ii)], we can find integers $n_{D}(\delta)$ such that $\delta = \sum_{D \in \mathfrak{B}(X)} n_{D}(\delta)D$. Let $l(\delta) \in PL(X)$ be the corresponding piecewise linear function on $\mathcal{C}(X)$. In particular, $l(\delta)(\rho(\nu_{D})) = n_{D}(\delta)$ for all $D \in \mathcal{V}_{X} \cup \mathcal{D}_{X}$, and $l_{Y} = l(\delta)|_{\epsilon_{Y}}$ is linear on $\mathcal{C}_{Y}$ for each $Y \in S_{X,G}$. For each $Y \in S_{X,G}$, we denote by $l(\delta|_{Y})$ or $l(\delta|_{Y})$ or $l_{Y}^{\mathbb{Q}}$ the linear extension of $l_{Y}$ to the whole vector space $(N_{X})_{Q}$.

Suppose that $\mu \in F_{X}$ is a wall, i.e. there are two maximal dimensional colored cone $(\mu_{+}, \mathcal{D}_{+})$ and $(\mu_{-}, \mathcal{D}_{-})$ in $F_{X}$ (i.e. $\dim(\mu_{+}) = \dim(\mu_{-}) = \text{rank}(G/H)$) such that $\mu = \mu_{+} \cap \mu_{-}$ and for some subset $\mathcal{D} \subseteq \mathcal{D}_{+} \cap \mathcal{D}_{-}$, $(\mu, \mathcal{D})$ is a colored face of $(\mu_{+}, \mathcal{D}_{+})$ and $(\mu_{-}, \mathcal{D}_{-})$ of codimension one. Let $\chi_{\mu} \in M_{X}$ be the primitive lattice point such that $\chi_{\mu}$ vanishes on $\mu$ and it is positive on $\mu_{+} \setminus \mu$, negative on $\mu_{-} \setminus \mu$. Define $C_{\mu} \in N_{1}(X)_{\mathbb{R}}$ such that

$$\delta \cdot C_{\mu} = (l(\delta, \mu_{+}) - l(\delta, \mu_{-}))/\chi_{\mu}. \quad (1)$$
Remark 2.9. Assume that \( X \) is complete. By [Br93] Prop. 3.3, there is a unique \( B \)-stable smooth rational curve \( C_{YZ} \) such that \( C_{YZ}^B = \{ p_1, p_2 \} \), \( C_\mu \cap Y = \{ p_1 \} \), \( C_\mu \cap Z = \{ p_2 \} \) and \( C_{YZ} = C_\mu \) in \( N_1(X)_R \). We identify \( C_\mu \) with the curve \( C_{YZ} \). Moreover, by the proof of [Br93] Prop. 3.3], \( GC_\mu = \overline{V} \), \( C_\mu = (\overline{V})^{R_\omega(B)} \), \( Gp_1 = Y \) and \( Gp_2 = Z \), where \( V \) is the \( G \)-orbit such that \( \mathcal{E}_V = \mu \).

3 Nef versus effective cycle classes on smooth complete spherical varieties

We will show in Theorem 3.4 that on smooth complete spherical varieties, nef cycle classes are effective and the products of nef cycle classes are also nef.

Let \( \eta = \sum n_i Y_i \) be an algebraic cycle of an \( H \)-scheme \( X \), where \( H \) is an algebraic group, all \( n_i \neq 0 \) and all \( Y_i \) are irreducible closed subschemes. If all \( Y_i \) are \( H \)-stable, then we call \( \eta \) an \( H \)-stable algebraic cycle of \( X \).

If \( X \) is a smooth complete variety, and \( \eta_1, \eta_2, \ldots, \eta_m \in \mathbb{A}^*(X)_R \), then we denote by \( \prod_{i=1}^{m} \eta_i = \eta_1 \cdot \eta_2 \cdot \ldots \cdot \eta_m \). If \( X_1, \ldots, X_m \) are smooth complete varieties and \( \eta_i \in \mathbb{A}^*(X_i)_R \), then we denote by \( \otimes_{i=1}^{m} \eta_i = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_m = \prod_{j=1}^{m} \pi_j^* \eta_i \), where \( \pi_i : \prod_{j=1}^{m} X_j \rightarrow X_i \) is the \( i \)-th projection.

Proposition 3.1. (i) Let \( X \) be a \( \Gamma \)-scheme, where \( \Gamma \) is a connected solvable linear algebraic group. Then any effective algebraic cycle on \( X \) is rationally equivalent to an effective \( \Gamma \)-stable algebraic cycle.

(ii) Let \( X \) be a \( G \)-spherical variety. Then the group of rationally equivalent divisor classes on \( X \) is generated by irreducible \( B \)-stable divisors, while the linearly equivalences are defined by \( \text{div}(f) \) for all \( f \in \mathbb{C}(X)^{(B)} \). More precisely, there is an exact sequence \( M_X \rightarrow \mathbb{Z}(\mathcal{B}(X)) \rightarrow \mathbb{A}^1(X) \rightarrow 0 \). If moreover \( X \) is complete, then the map \( M_X \rightarrow \mathbb{Z}(\mathcal{B}(X)) \) is injective.

Note that these two conclusions should be well-known, but we fail to find proper references stating them explicitly. For the case when \( X \) is a projective \( \Gamma \)-variety, the statement of (i) has appeared in the proof of [Hi84] Thm. 1], while the case when \( X \) is a projective normal \( \Gamma \)-variety, the statement of (ii) appeared in [Br93] Thm. 1.3(i)]. Recall that the proof of [FMSS95] Thm. 1] was deduced to the case when \( X \) is a projective \( \Gamma \)-variety, and the effectiveness of the cycles are preserved in the proof. Then we can get Proposition 3.1. (i). In fact, when proving [FMSS95] Cor. of Thm. 1], the statement of (i) has been used. For the first assertion of (ii), we only need to notice that \( M_X \cong \mathbb{C}(X)^{(B)}/\mathbb{C}^* \), then it’s just [Br93] Thm. 1.3(ii)], and it’s also a corollary of [FMSS95] Thm. 1]. By [Kn91] Thm. 6.3], when \( X \) is moreover complete, \( \rho(\mathcal{B}(X)) \) generates \((\mathcal{N}X)_R \). Thus, the map \( M_X \rightarrow \mathbb{Z}(\mathcal{B}(X)) \) is injective.

The following proposition is a direct consequence of Proposition 3.1 and [Br89] Prop. 3.1. It also appeared in [Per12] Thm. 3.2.14] with a little different statement, while we can also get the statement here from the proof of [Per12] Thm. 3.2.14].
Proposition 3.2. Let $X$ be a $G$-spherical variety.

(i) The variety $X$ is locally factorial if and only if for any $G$-orbit $Y$, the elements $\rho(v_D)$ form a part of a $\mathbb{Z}$-basis of $N_X$, where $D$ runs over the set $\mathfrak{B}_Y$.

(ii) The variety $X$ is $\mathbb{Q}$-factorial if and only if for any $G$-orbit $Y$, the elements $\rho(v_D)$ are linearly independent in $(N_X)_\mathbb{Q}$, where $D$ runs over the set $\mathfrak{B}_Y$.

Note that by [FMSS95, Thm. 2], if $X$ is a smooth complete spherical variety, then for any integer $k$, $N^k(X) = \mathbb{A}^k(X) = H^{2k}(X)$. In particular, the numerical equivalence and the rational equivalence coincide.

Corollary 3.3. Let $X$ be a $\Gamma$-scheme such that $\Gamma$ has only finitely many orbits, where $\Gamma$ is a connected solvable algebraic group. Let $X'$ be another $\Gamma$-scheme and $\eta \in A^*(X \times X')$ be an effective cycle. Then $\eta = \sum_{i=1}^m c_i \delta_i \otimes \delta'_i$, where $c_i \geq 0$, and $\delta_i, \delta'_i$ can be represented by irreducible $\Gamma$-stable closed subvarieties of $X$ and $X'$ respectively.

Proof. By Proposition 3.1(i), we can assume that $\eta$ is $\Gamma \times \Gamma$-stable and effective. Without loss of generality, we can assume that $\eta$ is represented by an irreducible $\Gamma \times \Gamma$-stable closed subvariety $Z$ of $X \times X'$.

Let $X_0 = X \times X'$ be a variety with a $\Gamma_0$-action such that $\Gamma_0 = \Gamma$ and $g \cdot (x, x') = (g \cdot x, g \cdot e \cdot x') = (g \cdot x, x')$ for all $g \in \Gamma_0$, $x \in X$, and $x' \in X'$. Note that $\Gamma_0$ can be identified with the subgroup $\Gamma \times \{e\}$ of $\Gamma \times \Gamma$. In particular, $Z$ is $\Gamma_0$-stable. By [FMSS95, Lem. 3], $Z \subseteq X_0$ has the form $Z = Y \times Y'$, where $Y \subseteq X$ is a $\Gamma$-stable closed subvariety and $Y' \subseteq X'$ is a closed subvariety. By Proposition 3.1(i) again, $Y'$ is rationally equivalent to an effective $\Gamma$-stable algebraic cycle. The conclusion follows.

Theorem 3.4. Let $X$ be a smooth complete $G$-spherical variety of dimension $n$. Then for any integer $k$, the following hold.

(i) $\text{Eff}^k(X) = P\text{sef}^k(X)$, and it’s a rational polyhedron. Dually, $\text{Nef}^{n-k}(X)$ is also a rational polyhedron.

(ii) $\text{Nef}^k(X) \subseteq P\text{sef}^k(X)$.

(iii) If $\eta \in \mathbb{A}^k(X)$ is a cycle such that $\eta \cdot \mathbb{A}^{n-k}(X) = 0$, then $\eta \cdot \mathbb{A}^*(X) = 0$.

(iv) If $\eta \in \text{Nef}^k(X)$, then for any effective algebraic cycle class $\delta$ in $\mathbb{A}^*(X)$, the intersection $\eta \cdot \delta$ is an effective algebraic cycle class in $\mathbb{A}^*(X)$. In particular, if for all $1 \leq i \leq m$, $\eta_i \in \text{Nef}^k(X)$, then $\prod_{i=1}^m \eta_i \in \text{Nef}^{k_0}(X)$, where $k_0 = \sum_{i=1}^m k_i$.

Proof. The conclusion (i) follows from Proposition 2.2(i)(iv) and Proposition 3.1(i).

Let $\eta \in \mathbb{A}^k(X)$ be any cycle class and $Z \subseteq X$ be a $B$-stable closed subvariety. By Proposition 2.2(i)(iv) and Corollary 3.3 we can assume that $\Delta_*(Z) = \sum_{i=1}^m c_i u_i \otimes v_i$ in $\mathbb{A}^*(X \times X)$, where $\Delta : X \to X \times X$ is the diagonal morphism, $m$ is a positive integer, all $c_i \geq 0$, and all $u_i, v_i$ are algebraic cycle classes represented by irreducible $B$-stable closed subvarieties. Let $\pi_1, \pi_2$ be the two projections from $X \times X$ to the two factors, then $\pi_1 \Delta = id_X$, and $\pi_2 \Delta = id_X$.

Hence, $\eta \cdot Z = \pi_2 \Delta_*(\eta \cdot Z) = \pi_2 \Delta_*(\Delta^* \pi_1^* \eta \cdot Z) = \pi_2 \Delta_*(\sum_{i=1}^m (c_i \eta \cdot u_i) \otimes v_i) = \sum_{\dim(u_i) = k} c_i (\eta \cdot u_i) v_i$. Then the rest of this theorem follows easily from this formula. In particular, for (ii), we can take $\eta \in \text{Nef}^k(X)$ and $Z = X$ to apply this formula. 

Note that the conclusion (i) is indeed a classical result: for the case when $X$ is projective, see [Br93, Thm. 1.3(ii)]; for more general cases, see [FMSS95, Cor. of Thm. 1]. The conclusion (iii) can be deduced from [FMSS95, Cor. of Thm. 2] readily. We state and prove these two conclusions here for the later use and for the discussions of the case when $X$ is a projective $\mathbb{Q}$-factorial toric variety, see Remark 4.12.
Remark 3.5. Let $X$ be a $\mathbb{Q}$-factorial complete $G$-spherical variety of dimension $n$. Define $A^i(X)_\mathbb{Q} \times A^{n-1}(X)_\mathbb{Q} \to \mathbb{Q}$ as the linear extension of the map $Pic(X) \times A^{n-1}(X) \to \mathbb{Z}$, $\delta \times C \mapsto \deg(\delta|_C)$, where $A^{n-1}(X)$ (resp. $A^i(X)$) is the Chow group of curves (resp. Weil divisors), $Pic(X)$ is the Picard group of $X$ and $A^k(X)_\mathbb{Q} = A^k(X) \otimes \mathbb{Q}$ for $k = 1, n - 1$. Thus, $N^k(X)_\mathbb{R}$, $Eff^k(X)$, $Psef^k(X)$, and $Nef^k(X)$ are well-defined for $k = 1, n - 1$. By Proposition 3.3(i), $Eff^k(X) = Psef^k(X)$ for $k = 1, n - 1$. Since the proof of Theorem 3.4 also works when $k = 1$ under our assumption of $X$ here, we know that $Nef^k(X) \subseteq Psef^k(X)$. By the duality, $Nef^{n-1}(X) \subseteq Psef^{n-1}(X)$.

Corollary 3.6. Suppose that for each $1 \leq i \leq m$, $X_i$ is a smooth complete $G_i$-spherical variety, where each $G_i$ is a connected reductive algebraic group. Let $X = \prod_{i=1}^m X_i$ and $r$ be a positive integer. Then $Nef^k(X) = Psef^k(X)$ for all $1 \leq k \leq r$ if and only if $Nef^k(X_i) = Psef^k(X_i)$ for all $1 \leq k \leq r$ and all $1 \leq i \leq m$.

Proof. Note that $X$ is $(\prod_{i=1}^m G_i)$-spherical. By Theorem 3.4(i)(ii), $Nef^k(X) \subseteq Eff^k(X) = Psef^k(X)$ and $Nef^k(X_i) \subseteq Eff^k(X_i) = Psef^k(X_i)$ for all integers $k$ and all $1 \leq i \leq m$.

The “only if” part follows from Theorem 3.4(i)(ii) and the projection formula.

The “if” part: Let $n = \dim(X)$ and $n_i = \dim(X_i)$. For any $1 \leq k \leq r$, take any class $\eta \in Eff^k(X)$, and any class $\delta \in Eff^{n-k}(X)$, then by Corollary 3.3 and the induction on $m$, we get that the class $\eta \in \sum_{i_1 + \ldots + i_m = k} \bigotimes Eff^j(X_{i_j})$ and the class $\delta \in \sum_{i_1 + \ldots + i_m = k} \bigotimes Eff^{n_i - i_j}(X_{i_j})$. By the assumption on $X_i$, $\eta \in \sum_{i_1 + \ldots + i_m = k} \bigotimes Nef^j(X_{i_j})$. Hence, $\eta \cdot \delta \geq 0$, i.e. $Nef^k(X) = Psef^k(X)$ for all $1 \leq k \leq r$.

Remark 3.7. Suppose that for each $1 \leq i \leq m$, $X_i$ is a complete $\mathbb{Q}$-factorial $G_i$-spherical variety, where each $G_i$ is a connected reductive algebraic group. Let $X = \prod_{i=1}^m X_i$. Then $Nef^k(X) = Psef^k(X)$ if and only if $Nef^k(X_i) = Psef^k(X_i)$ for all $1 \leq i \leq m$. By Remark 3.4, $Nef^k(X) \subseteq Eff^k(X) = Psef^k(X)$ and $Nef^k(X_i) \subseteq Eff^k(X) = Psef^k(X_i)$ for all $1 \leq i \leq m$. Note that if we take $r = 1$, then the proof of Corollary 3.7 also works for the case here.

4 Smooth projective spherical varieties whose effective cycle classes of codimension $k$ are nef

In this section, we mainly study the smooth projective spherical variety $X$ of dimension $n$ such that $Nef^k(X) = Psef^k(X)$ for some $1 \leq k \leq n - 1$. We also study some related properties when $X$ may be not smooth, but only $\mathbb{Q}$-factorial. We discuss the general spherical cases, toric cases, toroidal cases and horospherical cases in the corresponding four subsections. The horospherical cases are the most complicated, and we need to make some preparations in the part 4.1.1

4.1 General spherical varieties

There are two main results in this subsection, namely Theorem 4.1 and Theorem 4.5. The former says that if $X$ is a smooth projective variety such that $Nef^k(X) \subseteq Psef^k(X)$ for some $2 \leq k \leq \dim(X) - 2$, then the dimensions of the exceptional loci of the birational Mori contractions on $X$ are less than $k$.

Now let $X$ be a spherical $G/H$-embedding. Denote by $\mathcal{D}_0(G/H) = \{ D \in \mathcal{D}(G/H) \mid \rho(\nu_D) = 0 \}$. In Theorem 4.5 we show that if $X$ is complete, then for any subset $i_1 \subseteq \mathcal{D}_0(G/H)$, we can construct a $G$-equivariant morphism $\pi_0 : X \to G/P_0$, where $P_0$ is a parabolic subgroup of
$G$ related to $D_1$. Moreover, $X$ shares a lot of properties with the fibers. Let $F$ be any fixed fiber of $\pi_0$, then $X$ is smooth (resp. projective, $Q$-factorial, locally factorial) if and only if $F$ is so. If $F$ and $X$ are both projective and $Q$-factorial, then $\text{Nef}^1(X) = \text{Psef}^1(X)$ if and only if $\text{Nef}^1(F) = \text{Psef}^1(F)$.

**Theorem 4.1.** Let $X$ be a smooth projective variety of dimension $n$ such that $\text{Psef}^k(X) \subseteq \text{Nef}^k(X)$ for some $2 \leq k \leq n - 2$. Assume that $R$ is an extremal ray of $\text{NE}(X)$ which can be contracted, i.e. there is a morphism $\pi : X \to Y$ such that $\pi_*O_X = O_Y$ and for any irreducible curve $C$ on $X$, $\pi(C)$ is a point if and only if $C \in R$. Then either $R \subseteq \text{Nef}_1(X)$ or dim$(A) \leq \min\{k-1, n-k-1\}$, where $A \subseteq X$ is the exceptional locus of $\pi$.

**Proof.** Assume that $R \not\subseteq \text{Nef}_1(X)$. Then $\pi$ is birational. Let $A \subseteq X$ be the exceptional locus and $A' = \pi(A)$. By the duality, we can assume that $k \leq \frac{n}{2}$. Now assume that dim$(A) \geq k$. Let $C$ (resp. $E$) be an irreducible curve (resp. a prime divisor) on $X$ such that $C \in R$ and $E \cdot C < 0$.

Case 1. Assume that there exists a point $y \in Y$ such that dim$(\pi^{-1}(y)) \geq k$.

Let $F$ be an irreducible component of $\pi^{-1}(y)$ such that $d = \dim(F) \geq k$. Take $d-1$ general ample divisors $D_1, \ldots, D_{d-1}$ on $X$. Thus, $\prod_{i=1}^{d-1} D_i \cdot F = \lambda C \in N_1(X)_{\mathbb{R}}$ for some $\lambda > 0$. So $\sum_{i=1}^{d-1} D_i \cdot F \cdot E < 0$.

On the other hand, $\prod_{i=1}^{d-k} D_i \cdot F \in \text{Psef}^{n-k}(X)$ and $\prod_{i=d-k+1}^{d-1} D_i \cdot E \in \text{Psef}^k(X) \subseteq \text{Nef}^k(X)$.

Thus, $\sum_{i=1}^{d-1} D_i \cdot F \cdot E \geq 0$. We get a contradiction.

Case 2. Assume that for any $y \in Y$, dim$(\pi^{-1}(y)) \leq k-1$. Let $A_0$ be an irreducible component of $A$ such that $a = \dim(A_0) \geq k$. Let $A'_0 = \pi(A_0)$. Thus, $a' = \dim(A'_0) \geq 1$.

Take $a'$ general very ample Cartier divisors $H_1, \ldots, H_{a'}$ on $Y$. Take a general section $D_i$ of each Cartier divisor $\pi^* O_Y(H_i)$. Take $a - a' - 1$ general ample divisors $D_{a+1}, \ldots, D_{d-1}$ on $X$.

Thus, $\prod_{i=1}^{a-k} D_i \cdot A_0 \in \text{Psef}^{n-k}(X)$ and $\prod_{i=a-k+1}^{a-1} D_i \cdot E \in \text{Psef}^k(X) \subseteq \text{Nef}^k(X)$. So $\sum_{i=1}^{a-1} D_i \cdot A_0 \cdot E \geq 0$.

By the choices of $H_i$ on $Y$, $\prod_{i=1}^{a'} \pi^* H_i \cdot A_0 = \sum_{j=1}^{t} \lambda_j \pi^{-1}(y_j)$, where $t \geq 1$, each $\lambda_j > 0$, and each $\pi^{-1}(y_j)$ is an irreducible closed subvariety of $X$ of dimension $a - a'$. Hence, $\sum_{i=1}^{a-1} D_i \cdot A_0 = \lambda C \in N_1(X)_{\mathbb{R}}$ for some $\lambda > 0$. So $\sum_{i=1}^{a-1} D_i \cdot A_0 \cdot E < 0$. We get a contradiction. \hfill \square

Suppose that $X$ is a smooth projective $G$-spherical variety. By Theorem 3.1(i)(ii), for any integer $k$, $\text{Nef}^k(X) \subseteq \text{Eff}^k(X) = \text{Psef}^k(X)$. By [Br93] Thm. 3.1, every extremal ray of $\text{NE}(X)$ can be contracted. So we have the following

**Corollary 4.2.** Let $X$ be a smooth projective $G$-spherical variety of dimension $n$ such that $\text{Nef}^k(X) = \text{Psef}^k(X)$ for some $2 \leq k \leq n - 2$. If there is an extremal ray $R$ of $\text{NE}(X)$ such that $R \not\subseteq \text{Nef}_1(X)$, then the corresponding contraction $\text{cont}_R : X \to Y$ is birational and the dimension of the exceptional locus $A \subseteq X$ is no more than $\min\{k-1, n-k-1\}$.

For smooth projective Fano varieties, we can get a strong corollary as follows.

**Corollary 4.3.** Let $X$ be a smooth projective Fano variety of dimension $n$. If there is some integer $2 \leq k \leq n - 2$ such that $\text{Psef}^k(X) \subseteq \text{Nef}^k(X)$, then $\text{Psef}^1(X) = \text{Nef}^1(X)$.

**Proof.** By the duality, we can assume that $k \leq \frac{n}{2}$. Now assume that $R \subseteq \text{Psef}_1(X) \setminus \text{Nef}_1(X)$ is an extremal ray. Let $X, Y, \pi, A$ be as in Theorem 4.1. Note that the existence of $\pi = \text{cont}_R$ follows.
from the fact that $X$ is Fano. Then by Theorem 4.1, $\dim(A) \leq k - 1$. By [Io86, Thm. (0.4)], $2 \dim(A) \geq \dim(X) + l(R) - 1$, where $l(R) = \min\{-K_X \cdot C \mid C \text{ is a rational curve and } C \in R\}$. Since $X$ is a Fano variety, $l(R) \geq 1$. Thus, $\dim(A) \geq \frac{1}{2}$. We get a contradiction. 

Let $X$ be a spherical $G/H$-embedding and $Y$ be a $G$-orbit on $X$. By Theorem 2.5, $C_X^G$ is a strictly convex colored cone in $(N_X)_G$. In particular, for each $D \in \mathfrak{D}_Y$, $\rho(\nu_D) \neq 0$. Recall that $\mathfrak{D}_0(G/H) = \{D \in \mathfrak{D}(G/H) \mid \rho(\nu_D) = 0\}$. Hence, $\mathfrak{D}_0(G/H) \subset \mathfrak{D}(G/H) \backslash \mathfrak{D}_X$.

**Lemma 4.4.** Let $X$ be a $\mathbb{Q}$-factorial projective spherical $G/H$-embedding. Take $D \in \mathfrak{D}_0(G/H)$, then

(i) the numerical class $C_{D,Y}$ doesn’t depend on the choice of the closed $G$-orbit $Y$, thus, we can denote it by $C_D$;

(ii) $\mathbb{R}^+C_D$ is an extremal ray of $\text{NE}(X)$ as well as one of $\text{Nef}_1(X)$;

(iii) $\mathbb{R}^+D$ is an extremal ray of $\text{Nef}^1(X)$ as well as one of $\text{Psef}^1(X)$.

Moreover, suppose that $D_1, \ldots, D_m \in \mathfrak{D}_0(G/H)$ are pairwise different, then

(iv) $F = (C_{D_1}, \ldots, C_{D_m})$ is an extremal face of $\text{NE}(X)$ as well as one of $\text{Nef}_1(X)$;

(v) $F^* = (D_1, \ldots, D_m)$ is an extremal face of $\text{Nef}^1(X)$ as well as one of $\text{Psef}^1(X)$.

**Proof.** (i) Take any $\delta \in \text{Pic}(X)$. Then there is an equality $\delta \cdot C_{D,Y} = \delta \cdot C_{D,Y}^* = 0$ for any $Y_1, Y_2 \in S_{X,Y}$ by the formula (2) on page 6. This shows (i).

(ii)(iii) Since $D \in \mathfrak{D}_0(G/H)$, by the formula (2) again, if $D' \in \mathfrak{D}(X) \backslash \{D\}$, then $D' \cdot C_D = 0$. On the other hand, $D \cdot C_D = 1$. By Proposition 3.1(i), $C_D \in \text{Nef}_1(X)$ and $\mathbb{R}^+D$ is an extremal ray of $\text{Psef}^1(X)$.

By the formulas (1) and (2) on page 6 we find that for any wall $\mu \in F_X$, the intersection number $D \cdot C_\mu = 0$ and that for any $Y \in S_{X,G}$ and any $D' \in \mathfrak{D}(G/H) \backslash \{\mathfrak{D}_Y \cup \{D\}\}$, the intersection number $D \cdot C_{D,Y} = 0$. On the other hand, $D \cdot C_D = 1$. By [Br93, Thm. 3.2] and (i) of this lemma, $D \in \text{Nef}^1(X)$ and $\mathbb{R}^+C_D$ is an extremal ray of $\text{NE}(X)$. By Remark 3.3, $\text{Nef}_1(X) \subset \text{Psef}_1(X)$ and $\text{Nef}^1(X) \subset \text{Psef}^1(X)$. Thus, $\mathbb{R}^+C_D$ is an extremal ray of $\text{Nef}_1(X)$ and $\mathbb{R}^+D$ is an extremal ray of $\text{Nef}^1(X)$.

(iv) Now choose $D_1, \ldots, D_m \in \mathfrak{D}_0(G/H)$ which are pairwise different. Let $V_1 \subset N_1(X)_R$ be the subspace generated by $C_{D_1}, \ldots, C_{D_m}$, and $V_2 \subset N_1(X)_R$ be the subspace generated by those $C_{D',Y}$ who are different from $C_{D_1}, \ldots, C_{D_m}$ and those $C_\mu$ where $\mu$ runs over the set of the walls of $F_X$. Take any $C \in V_1 \cap V_2$. If $C \neq 0$, then by the formula (2), the fact $C \in V_1$ implies that we can choose some $\mu$ such that $D_\mu \cdot C \neq 0$. By the formulas (1) and (2), the fact $C \in V_2$ implies that $D_\mu \cdot C = 0$. We get a contradiction. Hence, $V_1 \cap V_2 = 0$. By [Br93, Thm. 3.2], $F$ is an extremal face of $\text{NE}(X)$. By (ii) of this lemma, $F \subset \text{Nef}_1(X)$. By Remark 3.3, $\text{Nef}_1(X) \subset \text{Nef}(X)$. Hence, the extremal face $F$ of $\text{NE}(X)$ is also an extremal face of $\text{Nef}_1(X)$.

(v) Let $W_1 \subset N^1(X)_R$ be the subspace generated by $D_1, \ldots, D_m$, and $W_2 \subset N^1(X)_R$ be the subspace generated by those $D \in \mathfrak{D}(X) \backslash \{D_1, \ldots, D_m\}$. By Proposition 3.1(ii) and [Br93, Cor. 1.3(iv)], $W_1 \cap W_2 = \{0\}$. Thus, by Proposition 3.1(i), $F^*$ is an extremal face of $\text{Eff}^1(X)$. By (iii) of this lemma, $F^* \subset \text{Nef}^1(X)$. By Remark 3.3, $\text{Nef}^1(X) \subset \text{Eff}^1(X) = \text{Psef}^1(X)$. Hence, the extremal face $F^*$ of $\text{Eff}^1(X)$ is also an extremal face of $\text{Nef}^1(X)$. 

**Theorem 4.5.** Let $X$ be a complete spherical $G/H$-embedding, and $\mathfrak{D}_1 \subset \mathfrak{D}_0(G/H)$ be a subset. Denote by $D_0 = \sum_{D \in \mathfrak{D}_1} D$. Then the following hold.

(i) $D_0$ induces a $G$-equivariant morphism $\pi_0 : X \to \bar{X}$, where $\bar{X} = \text{Proj}(R)$, $R = \bigoplus_{m \geq 0} H^0(X, mD_0)$.

Moreover, there is a $G$-equivariant isomorphism $\bar{X} \to G/P_0$, where $P_0$ is a parabolic subgroup of $G$ containing $B$.

(ii) Let $F$ be any fiber of $\pi_0$. Then $X$ is $\mathbb{Q}$-factorial (resp. locally factorial, smooth, projective) if and only if $F$ is $\mathbb{Q}$-factorial (resp. locally factorial, smooth, projective).
(iii) Assume that $X$ is projective and $\mathbb{Q}$-factorial. Then $\text{Nef}^1(X) = \text{Psef}^1(X)$ if and only if $\text{Nef}^1(F) = \text{Psef}^1(F)$.

**Remark 4.6.** The conclusion (iii) in Theorem 4.5 is not true for a general $G$-equivariant morphism from a complete $G$-spherical variety to a rational $G$-homogeneous space. Now we consider such an example. Let $X$ be the blowing-up of $\mathbb{P}^2$ at the point $p = [1, 0, 0]$. Then $X$ is a Hirzebruch surface and it’s a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Denote by $\pi : X \to Y = \mathbb{P}^1$ the fibration. Let $G$ be the set of matrices $(a_{ij})_{1 \leq i, j \leq 3}$ in $\text{GL}_3(\mathbb{C})$ such that the first row is $(1, 0, 0)$ and the first column is $(1, 0, 0)^t$. Thus, $G \cong \text{GL}_2(\mathbb{C})$ is reductive. There is a natural $G$-action on $\mathbb{A}^3_0$. It induces a $G$-action on $\mathbb{P}^2$ fixing the point $p$. Then $X$ is a $G$-spherical variety. Moreover, there is a natural $G$-action on $Y$ such that $\pi$ is $G$-equivariant and $Y$ is $G$-homogeneous. Let $F$ be any fiber of $\pi$. Note that $\text{Nef}^1(F) = \text{Psef}^1(F)$ and $Y$ is a rational $G$-homogeneous space, but $\text{Nef}^1(X) \neq \text{Psef}^1(X)$, since there is a $(-1)$-curve on $X$.

**Lemma 4.7.** Let $X$ be a normal $G$-variety. Then there exists a connected reductive algebraic group $G'$ and a finite surjective morphism $\pi : G' \to G$ of algebraic groups such that under the induced $G'$-action on $X$, every Cartier divisor on $X$ is $G'$-linearizable.

**Proof.** We take a similar proof with [Per, Prop. 2.3.1]. By [Per, Cor. 2.2.7], there exists a point $x_0 \in X$ such that there is a connected reductive algebraic group $G'$ and a finite surjective morphism $\pi : G' \to G$ of algebraic groups such that $\text{Pic}(G') = 0$. Since $G$ is reductive, $G'$ is also reductive. Denote by $\varphi : G' \times X \to X, (g', x) \mapsto \pi(g') x$, i.e. $\varphi$ defines the $G'$-action on $X$ induced by the $G$-action. Let $p_{G'}$ and $p_X$ be the projections from $G' \times X$ to $G'$ and $X$ respectively. Take any Cartier divisor $\delta$ on $X$. By [Per, Lem. 2.2.3], there exists a point $x_0 \in X$ such that $\varphi^* \delta = p_{G'}^*(\varphi^* \delta_{|G' \times \{x_0\}}) \otimes p_X^* (\varphi^* \delta_{|e \times X})$. Since Pic($G'$) = 0, $\varphi^* \delta_{|G' \times \{x_0\}} = \mathcal{O}_{G'}$. Since $e$ has a trivial action on $X$, $\varphi^* \delta_{|e \times X} = \delta$ in Pic($X$). Thus, $\varphi^* \delta = p_X^* \delta$ in Pic($G' \times X$). By [Per, Cor. 2.1.7], $\delta$ is $G'$-linearizable.

**Proof of Theorem 4.4.** By [Br89, Prop. 3.1], $\mathcal{O}_X(D_0) = \mathcal{O}(m_0D_0)$ is a Cartier divisor. By [Per, Thm. 1.6], there exists a positive integer $m_0$ such that $\mathcal{O}_X(m_0D_0)$ is $G$-linearizable. Thus, by [Per, Lem. 2.3.2], $H^0(X, mm_0D_0)$ is a $G$-module for each $m \geq 0$. For any $m \geq 1$, let $s \in H^0(X, mm_0D_0)^{(B)}$ be the global section corresponding to $mm_0D_0$. Take any $s' \in H^0(X, mm_0D_0)^{(B)}$, then there is some element $f \in \mathbb{C}(X)^{(B)}$ such that $s' = fs$. Thus, div($f$) + $mm_0D_0$ is a $B$-stable effective divisor. In particular, the only possible poles of $f$ are in the support of $D_0$. By Proposition 3.1(ii), $\mathcal{V}(f) = 0$ for any $D \in \mathcal{D}_1$. Hence, $f$ is a regular function on $X$. Since $X$ is complete, we know that $f$ is a constant function, i.e. $\dim H^0(X, mm_0D_0)^{(B)} = 1$. Thus, $H^0(X, mm_0D_0)$ is an irreducible $G$-module. Hence, $R_0 = \bigoplus_{m \geq 0} H^0(X, mm_0D_0)$ is generated by $H^0(X, m_0D_0)$. Therefore, by taking a similar argument with the last paragraph, we can know that $R = \bigoplus_{m \geq 0} H^0(X, mD_0)$ is generated by $H^0(X, D_0)$, and $D_0$ induces a natural $G'$-equivariant morphism $\pi' : X \to X'$, where $X' = \text{Proj}(R)$. Note that the inclusion $R_0 \subseteq R$ induces a natural isomorphism $\text{Proj}(R) \cong \text{Proj}(R_0)$. We identify them. Hence, $\pi_0$ coincides with $\pi'$. Therefore, $D_0$ induces a natural $G$-equivariant morphism $\pi_0 : X \to X$, where $X = \text{Proj}(R)$, and $R = \bigoplus_{m \geq 0} H^0(X, mD_0)$. 

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We claim that $\tilde{X}$ is $G$-equivariantly isomorphic to a rational homogeneous space $G/P_0$, where $P_0$ is a parabolic subgroup of $G$ containing $B$. Otherwise, let $Z$ be a closed $G$-orbit on $\tilde{X}$ and $O_{\tilde{X}}(A)$ be the Cartier divisor on $\tilde{X}$ such that $H^0(X, A) = H^0(X, m_0 D_0)$. Thus, $A$ is very ample and $G$-linearizable. There exists some positive integer $m$ such that the sheaf $O_{\tilde{X}}(m A) \otimes I_Z$ is globally generated, where $I_Z$ is the ideal sheaf corresponding to $Z$. On the other hand, $H^0(\tilde{X}, O_{\tilde{X}}(m A) \otimes I_Z) \subseteq H^0(\tilde{X}, O_{\tilde{X}}(m A))$ is a nonzero $G$-submodule. However, $H^0(\tilde{X}, O_{\tilde{X}}(m A)) = H^0(X, O_X(m m_0 D_0))$ is an irreducible $G$-module and $O_{\tilde{X}}(m A)$ is globally generated. Hence, $O_{\tilde{X}}(m A) \otimes I_Z = O_{\tilde{X}}(m A)$, which is contradicted with the assumption that $Z \neq X$. The conclusion follows.

Before proving Theorem 4.5(ii)(iii), we make some remarks. Firstly, note that it suffices to prove these two conclusions for a special fiber. Our approach is as follows. We take a special fiber $X_0$ and show that $X_0$ is a spherical variety. Roughly speaking, $F_X \subseteq F_{X_0}$ and their difference $F_{X_0} \setminus F_X$ can be described clearly. In this case, we say that $F_X$ is immersed into $F_{X_0}$, for the precise meaning, see Definition 4.8. The study of these two colored fans helps us to complete the proof of Theorem 4.5.

Definition 4.8. Let $X$ be a spherical $G/H$-embedding and $X'$ be a spherical $G'/H'$-embedding, where $G, G'$ are connected reductive algebraic groups. We say that there is an immersion from $F_X$ to $F_{X'}$, if the following conditions are satisfied.

(i) There is an isomorphism of abelian groups $\phi : N_{G/H} \to N_{G'/H'}$. We also use $\phi$ to denote the linear extension to $(N_{G/H}, \mathbb{Q}) \to (N_{G'/H'}, \mathbb{Q})$.

(ii) $\phi(V(G/H)) \subseteq V(G'/H')$.

(iii) There is a bijective map $\varphi : S(X, X') \to V_X \cup D_X$, such that $V_X \cup D_X \subseteq S(X, X') \subseteq B(X)$ and for any $D \in S(X, X')$, there is an equality $\rho_{G'/H'}(\nu_D(\varphi(D))) = \phi(\rho_{G/H}(\nu_D))$.

(iv) There is an injective map $\psi : S_{X,G} \to S_{X',G'}$ such that $\psi(S_{X,G}) = \{Y' \in S_{X',G'} | \mathcal{E}_{X', \varphi}(\psi(\nu_D(\varphi(D)))) \neq 0\}$, and for any $Y \in S_{X,G}$ and any $D \in S(X, X')$, $D \supseteq Y$ if and only if $\varphi(D) \supseteq \psi(Y)$.

Remark 4.9. Keep notations as in Definition 4.8 and assume that all the conditions (i) - (iv) in it are satisfied.

(i) We can define an injective map $\Phi : F_X \to F_{X'}$, such that for any $Y \in S_{X,G}$, $\Phi(\mathcal{E}_{X,Y}) = \mathcal{E}_{X', \varphi(Y)}$. We say that $\Phi$ is induced by $\varphi$. We identify $M_X$ with $M_{X'}$, as groups and identify $S(X, X')$ with $V_X \cup D_X$, in the set theory. Thus, $\Phi$ is a natural inclusion, i.e. $F_X \subseteq F_{X'}$. This is the reason why we use the word “immersion” in Definition 4.8. Moreover, we know that the set $F_{X'} \setminus F_X = \{\mathcal{E}_{X', \varphi}(Y) | \mathcal{E}_{X,Y}(\nu_D(\varphi(D))) = 0\}$. When there is an immersion $\Phi : F_X \to F_{X'}$, we will always identify $M_X$ with $M_{X'}$, $S(X, X')$ with $V_{X'} \cup D_{X'}$, and $\Phi$ with the natural inclusion.

(ii) Assume that $S(X, X') \subseteq \Omega \subseteq B(X)$. If there is an injective map $\Psi : \Omega \to B(X)$ such that $\Psi_S(X, X') = \varphi$, and for any $D \in \Omega$, there is an equality $\rho_{G'/H'}(\nu_D(\varphi(D))) = \phi(\rho_{G/H}(\nu_D))$, then we say that the immersion $\Phi : F_X \to F_{X'}$ is compatible with $\Psi$. We also say $\phi : N_{G/H} \to N_{G'/H'}$ is compatible with $\Psi$, or say that $\phi^* : M_{G/H} \to M_{G/H}$ is compatible with $\Psi$, where $\phi^*$ is the dual map of $\phi$.

(iii) If moreover $V(G/H) = V(G'/H')$, then $F_X = F_{X'}$. We say that $F_X$ is isomorphic to $F_{X'}$. When there is no confusion, we also say that $F_X$ is identified with $F_{X'}$.

(iv) Assume that $G = G'$ and $H \subseteq H'$. Then there is a natural inclusion $M_{G/H} \to M_{G/H}$. By Theorem 4.5(ii)(iii), there is a corresponding $G$-equivariant morphism $X \to X'$. If moreover $H = H'$, then $F_X = F_{X'}$, and $X$ is $G$-equivariantly isomorphic to $X'$ by Theorem 4.5(ii). This is the reason why we use the word “isomorphic” in (iii).

The following is the key lemma of Theorem 4.5 and the lemma itself is meaningful.
Lemma 4.10. Keep notations as in Theorem 4.5. Let $L_0 = P_0 \cap P^-_0$ be the standard Levi factor of $P_0$ and $P^-_0$. Let $\bar{x}_0 \in \bar{X}$ be the point such that $G \bar{x}_0 = P^-_0$. Denote by $X_0 = \pi^{-1}_0(\bar{x}_0)$. Identify $\bar{X}$ with $G/P^-_0$. Then the following hold.

(a) $X_0$ is a complete spherical $L_0/H_0$-embedding, where $H_0 = L_0 \cap G_{x_0}$ and $x_0$ is any point in the open $B$-orbit of $X$ such that $x_0 \in X_0$. Moreover, $B_0 x_0$ is an open subset of $X_0$, where $B_0 = B \cap L_0$.

(b) There is a $P_0$-equivariant isomorphism $\tau : R_u(P_0) \times X_0 \to \pi^{-1}_0(\bar{x}_0)$.

(c) The natural morphism $\phi : M_X \to M_{X_0}$ induced by the restriction of rational functions is an isomorphism of free abelian groups.

(d) The map $\Psi : \mathcal{B}(X) \setminus \mathcal{D}_1 \to \mathcal{B}(X_0), D \to D \cap X_0$ is bijective and it’s compatible with the morphism $\phi$. And $\pi_0^D : \mathcal{D}(G/P^-_0) \to \mathcal{D}_1, D' \to \pi^{-1}_0(D')$ is a bijective map.

(e) The isomorphism $\phi^{-1} : M_{X_0} \to M_X$ induces an immersion $\Phi : \overline{\mathcal{F}}_X \to \overline{\mathcal{F}}_{X_0}$ of colored fans, where $\Phi(\mathcal{C}_Y) = \mathcal{C}_{Y \cap x_0}$ for all closed $G$-orbit on $X$.

(f) If $\mathcal{D}_1 = \mathcal{D}_0(G/H)$, then $\mathcal{D}_0(L_0/H_0) = \emptyset$.

Proof. (a) Firstly, note that $B \bar{x}_0$ is the open $B$-orbit on $\bar{X}$. Since $\bar{X}$ is $G$-homogeneous and $\pi_0$ is $G$-equivariant, for any $Y \in S_{X,G}$, $\pi_0(Y) = \bar{X}$ and $\pi_0(Y^\circ) = B \bar{x}_0$, where $Y^\circ$ is the open $B$-orbit on $Y$, which exists by [Kn91, Cor. 2.2]. In particular, we can choose a point $x_0$ on the open $B$-orbit of $X$ such that $\pi_0(x_0) = \bar{x}_0$. Without loss of generality, we can assume that $H = G_{x_0}$.

In particular, $P^-_0 = \{ g \in G | g x_0 = X_0 \}$ and $H \subseteq P^-_0$.

Let $B_0 = B \cap L_0 = B \cap P^-_0$ and $H_0 = H \cap L_0$. Then $B x_0 \cap X_0 = \{ b \cdot x_0 | b \in B, \pi_0(b \cdot x_0) = \bar{x}_0 \} = (B \cap P^-_0) x_0 = B_0 x_0$. Since $B x_0$ is an open subset of $X$, $B_0 x_0$ is an open subset of $X_0$. Moreover, $(L_0)_{x_0} = G_{x_0} \cap L_0 = H \cap L_0 = H_0$. Since $X$ is complete and $X_0$ is a closed subscheme of $X$, we know that $X_0$ is complete. Hence, to show that $X_0$ is a complete spherical $L_0/H_0$-embedding, we only need to show that $X_0$ is irreducible and normal.

Now we consider the following diagram (\ast):

\[ \begin{array}{ccc}
G \times X_0 & \xrightarrow{\theta} & X \\
\downarrow p & & \downarrow \pi_0 \\
G & \xrightarrow{\pi} & G/P^-_0.
\end{array} \]

In the diagram (\ast), $\theta(g,x) = g \cdot x$, $p(g,x) = g$, and $\pi(g) = g \cdot \bar{x}_0$ for all $g \in G$, and $x \in X_0$. It’s easy to see that the diagram (\ast) is cartesian, i.e. $G \times X_0$ is the fiber product of $G$ and $X$ over $G/P^-_0$. Since $\pi$ is a smooth morphism, $\theta$ is a smooth morphism to a normal variety. Thus, $G \times X_0$ is normal, which implies that $X_0$ is normal. On the other hand, since $\pi_0$ is $G$-equivariant and $P^-_0/H$ is irreducible, the fibers of $\pi_0$ are irreducible. In particular, $X_0$ is irreducible. Hence, $X_0$ is a complete spherical $L_0/H_0$-embedding and $B_0 \cdot x_0 \subseteq X_0$ is an open subset.

(b) Since $R_u(P_0) \cap G_{\bar{x}_0} = R_u(P_0) \cap P^-_0 = \{ e \}$, we know that $R_u(P_0) \bar{x}_0 \cong R_u(P_0)$. There is a natural morphism $\tau : R_u(P_0) \times X_0 \to R_u(P_0) X_0, (v,x) \mapsto v \cdot x$. Note that $B P^-_0 = B \subset P^-_0$ in $G$, which implies that $B \bar{x}_0 = P \bar{x}_0 = R_u(P_0) \bar{x}_0$ in $X$. Hence, $R_u(P_0) X_0 = \pi^{-1}_0(R_u(P_0) \bar{x}_0) = \pi^{-1}_0(B \bar{x}_0) = \pi^{-1}_0(P_0 \bar{x}_0)$ is a $P_0$-stable open subset of $X$. Consider the following commutative diagram:

\[ \begin{array}{ccc}
R_u(P_0) \times X_0 & \xrightarrow{\tau} & R_u(P_0) X_0 \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
R_u(P_0) \bar{x}_0 & \xrightarrow{\phi_1^{-1}} & R_u(P_0) \bar{x}_0.
\end{array} \]

For any $v \in R_u(P_0)$, the morphism $\phi_1^{-1}(v \cdot \bar{x}_0) \xrightarrow{\tau} \phi_2^{-1}(v \cdot \bar{x}_0)$ is an isomorphism. Thus, $\tau : R_u(P_0) \times X_0 \to R_u(P_0) X_0$ is an isomorphism of varieties.
Now we define a $P_0$-action on $R_u(P_0) \times X_0$ as follows. Since $P_0$ is a semi-direct product of $R_u(P_0)$ and $L_0$, for every element $p \in P_0$, there is a unique pair $(l, v) \in L_0 \times R_u(P_0)$ such that $p = lv$. Then for any point $(v_0, x) \in R_u(P_0) \times X_0$, define the action $p \cdot (v_0, x) = (lv_0^{-1}l, x)$. Thus, the morphism $P_0 \times R_u(P_0) \times X_0 \to R_u(P_0) \times X_0, (p, v_0, x) \mapsto p \cdot (v_0, x)$ defines a $P_0$-action on $R_u(P_0) \times X_0$, and $\tau$ is a $P_0$-equivariant isomorphism.

(c) Firstly, we want to use the isomorphism $\tau$ in (b) to show that there is an isomorphism $\mathbb{C}(R_u(P_0) \times X_0)(B) \cong \mathbb{C}(X_0)(B_0)$, which is similar with the proof of [Per] Lem. 10.1.11.

Now we define the following two morphisms:

$$\varphi_1 : \mathbb{C}(R_u(P_0) \times X_0)(B) \to \mathbb{C}(X_0)(B_0)$$

$$f \mapsto f|_{x_0} \times X_0,$$

and

$$\varphi_2 : \mathbb{C}(X_0)(B_0) \to \mathbb{C}(R_u(P_0) \times X_0)(B)$$

$$f_0 \mapsto (R_u(P_0) \times X_0 \to C, (v_0, x) \mapsto f_0(x)).$$

We need to show that $\varphi_2$ is well-defined. Take $f_0 \in \mathbb{C}(X_0)(B_0), (v_0, x) \in R_u(P_0) \times X_0$ and $b \in B$. Since $b \in P_0$, there is a unique pair $(l, v) \in L_0 \times R_u(P_0)$ such that $b^{-1} = lv$. Thus, $v \in B$ and $l \in L_0 \cap B = B_0$. Let $f = \varphi_2(f_0)$. We get that $(b \cdot f)(v_0, x) = f(b^{-1} \cdot (v_0, x)) = f(lv_0^{-1}l, x) = f_0(l \cdot x) = (l^{-1} \cdot f_0)(x) = x \cdot f_0(l^{-1} \cdot f_0(x)) = x \cdot f_0(l^{-1}) = \chi_0(b, l^{-1}) = \chi_0(b, l^{-1})$, i.e. $b \cdot f = \chi_0(b, l^{-1})$. Hence, $f \in \mathbb{C}(C(R_u(P_0) \times X_0)(B))$, and $\varphi_2$ is well-defined.

It’s clear that $\varphi_1$ is well-defined and $\varphi_1$, $\varphi_2$ are the inverse of each other. Therefore, $\mathbb{C}(R_u(P_0) \times X_0)(B) \cong \mathbb{C}(X_0)(B_0)$.

By the isomorphism $\tau$ in (b), $\mathbb{C}(X)(B) = \mathbb{C}(\pi_0^{-1}(B\tilde{x}_0))(B) \cong \mathbb{C}(R_u(P_0) \times X_0)(B) \cong \mathbb{C}(X_0)(B_0)$. Thus, there is an isomorphism of groups $\phi : M_X \to M_{X_0}$. Denote by $\phi^*$ the dual map $N_{X_0} \to N_X$.

(d) Take any $D' \in \mathfrak{D}(G/P_0^-)$. Since $\pi_0$ is $G$-equivariant and the fibers are irreducible, $\pi_0^{-1}(D')$ is an irreducible $B$-stable divisor, i.e. $\pi_0^{-1}(D') \in \mathfrak{D}(G/H)$. On the other hand, by the definition of $\pi_0$, for any $D'' \in \mathfrak{D}_1, \pi_0(D'') \in \mathfrak{D}(G/P_0^-)$. Moreover, $\sum_{D' \in \mathfrak{D}_1} \pi_0(D)$ is an ample $B$-stable divisor on $G/P_0^-$. By the intersection theory on rational homogeneous spaces (see for example [Br05 Prop. 1.3.6]), this implies that $D' \subseteq \bigcup_{D' \in \mathfrak{D}_1} \pi_0(D)$. So $\pi_0^{-1}(D') \in \mathfrak{D}_1, \pi_0^{-1}(D') = D''$.

Let $S_0 = \{D \in \mathfrak{B}(X) \mid D \cap \pi_0^{-1}(B\tilde{x}_0) \neq \emptyset\}$. By (b), there is a bijective map $S_0 \to \mathfrak{B}(X_0), D \mapsto D \cap X_0$. We claim that $S_0 = \mathfrak{B}(X) \setminus \mathfrak{D}_1$. Note that the claim is equivalent to the assertion that for any $D \in \mathfrak{B}(X), \pi_0(D) = G/P_0^-$ if and only if $D \in S_0$.

Consider the following commutative diagram (**):

\[
\begin{array}{ccc}
R_u(P_0) \times X_0 & \xrightarrow{\tau} & X \\
\phi_1 \downarrow & & \downarrow \pi_0 \\
R_u(P_0)\tilde{x}_0 & \xrightarrow{i} & G/P_0^-.
\end{array}
\]

In the diagram (**), the morphism $i$ is the inclusion and $R_u(P_0) \times X_0 \to X$ is the composition $R_u(P_0) \times X_0 \to \pi_0^{-1}(B\tilde{x}_0) \subseteq X$, and $\tau, \phi_1$ are as those in the commutative diagram in the proof of (b).

For any $D \in S_0, \pi_0(D) = B\pi_0(D) \supseteq B\tilde{x}_0$, i.e. $\pi_0(D) = G/P_0^-$. For any $D \in \mathfrak{B}(X) \setminus S_0, D \subseteq X \setminus \pi_0^{-1}(B\tilde{x}_0)$. So $\pi_0(D) \neq G/P_0^-$. Hence, the claim holds, i.e. $\Psi : \mathfrak{B}(X) \setminus \mathfrak{D}_1 \to \mathfrak{B}(X_0), D \mapsto D \cap X_0$ is a bijective map.
Now we consider the inverse of the map $\Psi$. By (b), for any $D \in \mathfrak{B}(X_0)$, $\Psi^{-1}(D) = R_{\nu}(F_0)D = BBD$. Hence, for any $f \in \mathbb{C}(X)^{(B)}$, we have $\nu_D(f|_{X_0}) = \nu_{\Psi^{-1}(D)}(f)$, i.e. $\rho_{X_0}(D) = \phi(\rho_X(\Psi^{-1}(D)))$. So $\Psi$ is compatible with $\phi$. Thus, (d) holds.

(e) Remark that by (e), the condition (i) in Definition 4.8 is satisfied and we can identify $(N_X)_Q$ with $(N_{X_0})_Q$ as vector spaces. By (d), the condition (iii) in Definition 4.8 is satisfied. We identify $\mathfrak{B}(X)\backslash D_1$ with $\mathfrak{B}(X_0)$ in the set theory, when there is no confusion.

Now we check the condition (ii) in Definition 4.8. Let $p_{X_0} : R_u(F_0) \times X_0 \rightarrow X_0$ be the projection. Define

$$
p_{X_0} : \mathbb{C}(X_0) \rightarrow \mathbb{C}(R_u(F_0) \times X_0) \cong \mathbb{C}(X),
\quad f_0 \mapsto f_0 \circ p_{X_0},
$$

and

$$(p_{X_0})_* : \{\text{valuations on } X\} \rightarrow \{\text{valuations on } X_0\},
\mu \mapsto (\mathbb{C}(X_0)\backslash \{0\} \rightarrow \mathbb{Q}, f_0 \mapsto \mu(p_{X_0}^*f_0)).$$

Note that $\mathbb{C}(X_0)^{(B_0)} \xrightarrow{p_{X_0}^*} \mathbb{C}(X)^{(B)}$ induces a morphism $M_{X_0} \rightarrow M_X$, which is exactly the inverse of the isomorphism $\phi : M_X \rightarrow M_{X_0}$ in (c).

On the other hand, take any $\mu \in \mathcal{V}(G/H)$, i.e. $\mu$ is a $G$-invariant valuation on $X$. Take any $l \in L_0$, then $(p_{X_0})_*\mu(l \cdot f_0) = \mu(p_{X_0}(l \cdot f_0)) = \mu(l \cdot p_{X_0}^*(f_0)) = \mu(p_{X_0}^*(f_0)) = (p_{X_0})_*\mu(f_0)$, i.e. $(p_{X_0})_*\mu$ is a $L_0$-invariant valuation on $X_0$. Therefore, the condition (ii) in Definition 4.8 is also satisfied.

Now we define a map $\psi : S_{X,G,Y_0} \rightarrow S_{X,G,Y_0}$, for any $D \in \mathfrak{B}_{GY_0}$, $\Psi(D) = D \cap X_0 \supseteq GY_0 \cap X_0 \supseteq Y_0$, i.e. $\Psi(D) \in \mathfrak{B}_{Y_0}$, which implies that $\mathcal{E}_{GY_0} \subseteq \mathcal{E}_{Y_0}$. Since $X = GX_0$ and $X_0 = \bigcup_{Y_0 \in S_{X,G,Y_0}} Y_0$, we know that $X = \bigcup_{Y_0 \in S_{X,G,Y_0}} Y_0$, which implies that $\psi$ is surjective.

For any $Y \in S_{X,G}^c$, take any $Y_1, Y_2 \in S_Y \cap X_0$, i.e. $Y \cap X_0 \neq \emptyset$. Since $Y$ is $G$-homogeneous, $\psi(Y_1) = \psi(Y_2) = Y$. Thus, $\mathcal{E}_Y \subseteq \mathcal{E}_{Y_1} \cap \mathcal{E}_{Y_2}$. Since $X$ is a complete variety, $Y$ is $G$-equivariantly isomorphic to a rational homogeneous space. In particular, as a $G$-spherical variety, $\text{rank}(Y) = 0$. Thus, by [Ku91] Thm. 6.3, $\text{dim}(\mathcal{E}_Y) = \text{rank}(M_X)$, which implies that $\mathcal{E}_Y \subseteq \mathcal{E}_{Y_1} \cap \mathcal{E}_{Y_2}$. By Theorem 2.5, $\mathcal{E}_{Y_1} \cap \mathcal{E}_{Y_2} \cap \mathcal{V}(L_0/\mathfrak{H}_0) \supseteq \mathcal{E}_Y \cap \mathcal{V}(G/H)$ $\neq \emptyset$. By the definition of colored fans, $\mathcal{E}_{Y_1} = \mathcal{E}_{Y_2}$, i.e. $Y_1 = Y_2$. Hence, the $L_0$-stable set $Y \cap X_0$ is indeed a $L_0$-orbit. So the map $\eta : S_{X,G} \rightarrow S_{X,G,Y_0}$ is well-defined and $\psi \circ \eta = \text{id}_{S_{X,G}}$.

For any $Y \in S_{X,G}$ and any $D \in \mathfrak{B}(X_0)$, if $D \supseteq \eta(Y)$, then $\Psi^{-1}(D) = BBD \supseteq B(Y^0 \cap X_0) = Y^0$, where $Y^0$ is the open $B$-orbit on $Y$, which exists by [Ku91] Cor. 2.2. It should also be noticed that the fact $Y^0 \cap X_0 \neq \emptyset$ follows from the facts that $\pi_0(Y) = G/P_0$ and $\pi_0(Y^0)$ is the open $B$-orbit on $G/P_0$. Thus, $\mathcal{E}_Y \subseteq \mathcal{E}_{Y^0}$. Hence, $\mathcal{E}_Y(Y) = \mathcal{E}_{Y^0} \cap \mathcal{V}(L_0/\mathfrak{H}_0) \cap \mathcal{V}(G/H)$ for all $Y \in S_{X,G}$. By the fact $\mathcal{V}(G/H) \subseteq \mathcal{V}(L_0/\mathfrak{H}_0)$ and Theorem 2.5, $\eta$ can be extended uniquely to $S_{X,G} \rightarrow S_{X,G,Y_0}$ satisfying the condition (iv) in Definition 4.8.

Therefore, we have shown that there is an immersion of colored fans $\Phi : \mathcal{F}_X \rightarrow \mathcal{F}_{X_0}$ induced by the map $\phi^{-1} : M_{X_0} \rightarrow M_X$ in the sense of Definition 4.8 and $\Phi(\mathcal{E}_{Y_0}) = \mathcal{E}_{Y_0 \cap X_0}$ for all $Y \in S_{X,G}^c$.

(f) It’s a direct consequence of the compatibility of $\Psi$ and $\phi$. More precisely, if $\mathcal{D}_1 = \mathcal{D}_0(G/H)$, then by (d), for any $D \in \mathcal{D}(L_0/\mathfrak{H}_0)$, $\Psi^{-1}(D) = BBD \supseteq B(Y^0 \cap X_0) = Y^0$, where $Y^0$ is the open $B$-orbit on $Y$, which exists by [Ku91] Cor. 2.2. It should also be noticed that the fact $Y^0 \cap X_0 \neq \emptyset$ follows from the facts that $\pi_0(Y) = G/P_0$ and $\pi_0(Y^0)$ is the open $B$-orbit on $G/P_0$. Thus, $\mathcal{E}_Y \subseteq \mathcal{E}_{Y^0}$. Hence, $\mathcal{E}_Y(Y) = \mathcal{E}_{Y^0} \cap \mathcal{V}(L_0/\mathfrak{H}_0) \cap \mathcal{V}(G/H)$ for all $Y \in S_{X,G}$. By the fact $\mathcal{V}(G/H) \subseteq \mathcal{V}(L_0/\mathfrak{H}_0)$ and Theorem 2.5, $\eta$ can be extended uniquely to $S_{X,G} \rightarrow S_{X,G,Y_0}$ satisfying the condition (iv) in Definition 4.8.

Now we can complete the proof of Theorem 4.8.

Proof of Theorem 4.8 (ii), (iii). Keep notations as in Lemma 4.10. (ii) Note that by Lemma 4.10 (d)(c) and Remark 4.9 (i), we get the following fact

$$
\mathcal{F}_X \subseteq \mathcal{F}_{X_0}, \quad \text{and} \quad \mathcal{F}_{X_0} \cap \mathcal{V}(G/H) = \emptyset.
$$

(3)
By Proposition 3.2, the fact (3) implies that if $X_0$ is locally factorial (resp. $\mathbb{Q}$-factorial), then $X$ is locally factorial (resp. $\mathbb{Q}$-factorial).

Conversely, if $X$ is locally factorial (resp. $\mathbb{Q}$-factorial, smooth), then by Lemma 4.10(b), $X_0$ is also locally factorial (resp. $\mathbb{Q}$-factorial, smooth).

If $X_0$ is smooth, then the morphism $\pi_0$ is a smooth morphism from $X$ to a smooth variety $G/P_0^-$. Hence, $X$ is a smooth variety.

If $X$ is projective, then its closed subvariety $X_0$ is also projective.

Now we assume that $X_0$ is projective. Take any Cartier $B_0$-stable divisor $d_0 = \sum_{D \in \mathcal{B}(X_0)} n_D D$. Then by the fact (3), and [Br89 Prop. 3.1], the $B$-stable divisor $d = \sum_{D \in \mathcal{B}(X_0)} n_D \Psi^{-1}(D)$ is Cartier. Note that by [Br89 Prop. 3.1], the $B$-stable divisor $D_0$ is also Cartier.

If $d_0$ is moreover ample, then by the fact (3), Lemma 4.10(d), Definition 2.4 and the formulas (1)(2) on page 6, we can get the following two conclusions.

- If $\mu$ is a wall in $F_X$, then $\mu_0$ is a wall in $F_{X_0}$, where $\mu = \phi^*(\mu_0)$ and $\phi^*: (N_{X_0})_Q \to (N_X)_Q$ is the dual map induced by the isomorphism $\phi$ in Lemma 4.10(c). Moreover, $(D_0 + \delta) \cdot C_\mu = \delta \cdot C_\mu = \delta_0 \cdot C_\mu_0 > 0$.
- If $Y \in S^X_{X,G}$ and $D \in \mathcal{D}(G/H) \setminus (\mathcal{D}_1 \cup \mathcal{D}_Y)$, then $Y \cap X_0 \in S^X_{X_0,L_0}, \Psi(D) \in \mathcal{D}(L_0/H_0) \setminus \mathcal{D}_Y \cap X_0$ and $(D_0 + \delta) \cdot C_{D,Y} = \delta \cdot C_{D,Y} = \delta_0 \cdot C_{\Psi(D),Y \cap X_0} > 0$.

Note that if $Y \in S^X_{X,G}$ and $D \in \mathcal{D}_1$, then by the formula (2) on page 6, $(D_0 + \delta) \cdot C_{D,Y} = D \cdot C_{D,Y} = 1 > 0$. Hence, by [Br89 Thm. 3.2(ii)], the Cartier divisor $D_0 + \delta$ is ample, and $X$ is projective.

(iii) The “only if” part: Denote by $l: X_0 \to X$ the natural inclusion. Then by Proposition 3.1(i) and Lemma 4.10(d), $\operatorname{Psef}^1(X_0) \subseteq l^*(\operatorname{Psef}^1(X))$. By the assumption that $\operatorname{Psef}^1(X) = \operatorname{Nef}^1(X)$ and the projection formula, $\operatorname{Psef}^1(X_0) \subseteq \operatorname{Nef}^1(X_0)$. Hence, $\operatorname{Nef}^1(X_0) = \operatorname{Psef}^1(X_0)$.

For the “if” part, we want to apply [Br93 Thm. 3.2(ii)] and the formulas (1)(2) on page 6. Let $D_1 \in V(X) \setminus \mathcal{D}_1$, $\mu$ is a wall in $F_X$, $Y \in S^X_{X,G}$ and $D \in \mathcal{D}(G/H) \setminus (\mathcal{D}_Y \cup \mathcal{D}_1)$. Then by Lemma 4.10(d), the fact (3), and the formulas (1)(2) on page 6, we know that $D_1 \cdot C_\mu = \Psi(D_1) \cdot C_\mu_0 > 0$ and $D_1 \cdot C_{D,Y} = \Psi(D_1) \cdot C_{\Psi(D),Y \cap X_0} > 0$, where $\mu_0 = (\phi^*)^{-1}(\mu)$ is the corresponding wall of $F_{X_0}$.

On the other hand, by Lemma 3.1(i)(iii), for any $D \in \mathcal{D}_1 \subseteq \mathcal{D}_0(G/H)$, $D \in \operatorname{Nef}^1(X)$ and $C_D \in \operatorname{Nef}^1(X)$. Hence, by Proposition 3.1(ii) and [Br93 Thm. 3.2(ii)], $\operatorname{Nef}^1(X) = \operatorname{Psef}^1(X)$.

\[\square\]

\textbf{Remark 4.11.} \textit{Keep notations as in Theorem 4.5 and Lemma 4.10. Define $G \times^{P_0^-} X_0$ as the set of the quotient of $G \times X_0$ by equivalences $(gp,x) \sim (g,px)$ for all $g \in G, p \in P_0^-$ and $x \in X_0$. Then $G \times^{P_0^-} X_0$ is bijective to $X$ as sets. Thus, $X$ is the geometric quotient of $G \times X_0$ by $P_0^-$.}

In general, $X$ is not isomorphic to the product $G/P_0^- \times X_0$. In Subsection 4.4, when discussing the pseudo-effective and nef cones of smooth projective horospherical varieties, we will give a counterexample (see Example 4.3). The parabolic subgroup $P_0^-$ has a close relationship with the set $\mathcal{D}_1$, especially in the horospherical cases (see Theorem 4.5). Moreover, we will prove the converse of Theorem 4.5 in the horospherical cases (see Theorem 4.5).

\section{4.2 Toric varieties}

By the discussions in [Fult93 Chapter 5], we can define an intersection theory on $\mathbb{Q}$-factorial complete toric varieties with the intersection numbers in $\mathbb{Q}$. The main result in this subsection is Theorem 4.13 who says that if $X$ is a projective $\mathbb{Q}$-factorial variety such that $\operatorname{Nef}^k(X) = \operatorname{Psef}^k(X)$ for some $1 \leq k \leq \dim(X) - 1$, then $X$ is a quotient of the product of some projective spaces by a finite group.

Let $X$ be a $\mathbb{Q}$-factorial complete toric variety, and $F_X$ be the fan of $X$. Take a cone $\sigma \in F_X$, and assume $v_1, \ldots, v_k$ to be the primitive lattice points along rays of $\sigma$. Let $N_\sigma = N_X \cap \text{span}(\sigma)$,
where \( \text{span}(\sigma) \subseteq (N_X)_\mathbb{Q} \) is the subspace generated by \( \sigma \). Define \( \text{mult}(\sigma) = [N_\sigma : \sum_{i=1}^k \mathbb{Z}v_i] \). Let \( \tau \in \mathbb{P}_X \) be a cone such that \( \sigma \cap \tau = \{0\} \), then define

\[
V(\sigma) \cdot V(\tau) = \begin{cases} 
\frac{\text{mult}(\sigma) \cdot \text{mult}(\tau)}{\text{mult}(\gamma)} V(\gamma), & \text{if } \gamma = \langle \sigma, \tau \rangle \in \mathbb{P}_X, \\
0, & \text{otherwise},
\end{cases}
\]

where \( V(\sigma), V(\tau), \) and \( V(\gamma) \) are the orbit closures on \( X \) of the corresponding cones. When \( X \) is a smooth complete toric variety, this intersection theory coincides with the usual one.

**Remark 4.12.** Let \( X \) be a \( \mathbb{Q} \)-factorial complete toric variety. By the intersection theory on it, the cones \( \text{Eff}^k(X), \text{Psef}^k(X), \) and \( \text{Nef}^k(X) \) are well-defined.

(i) Let \( Y \) be another \( \mathbb{Q} \)-factorial complete toric variety and \( f : Y \to X \) be a toric morphism. We want to show the projection formula for \( f \). More precisely, we claim that the equality \( f_*(f^*\eta \cdot \xi) = \eta \cdot f_*\xi \) holds for all \( \eta \in A^k(X)_\mathbb{Q} \) and \( \xi \in A^m(Y)_\mathbb{Q} \), where \( 0 \leq k \leq \dim(X) \) and \( 0 \leq m \leq \dim(Y) \).

By Proposition 3.1(i) and the linearity of the formulas at the two sides, to show the claim, we can assume that \( \eta \) is represented by an orbit closure. Since \( X \) is \( \mathbb{Q} \)-factorial, there exist Cartier divisors \( \eta_1, \ldots, \eta_k \) and a positive rational number \( \lambda \) such that \( \eta = \lambda \prod_{i=1}^k \eta_i \). Thus, by the induction on \( k \) and the projection formula for Cartier divisors, \( f_*(f^*\eta \cdot \xi) = f_*(f^*(\lambda \prod_{i=1}^k \eta_i) \cdot \xi) = \lambda f_*(\prod_{i=1}^m f^*\eta_i \cdot \xi) = \lambda \prod_{i=1}^k \eta_i \cdot f_*\xi = \eta \cdot f_*\xi \).

(ii) By (i), we can see that the proof of Theorem 3.4 holds for \( X \) without any change. In particular, for any integer \( k \), \( \text{Nef}^k(X) \subseteq \text{Eff}^k(X) = \text{Psef}^k(X) \) and the products of nef cycle classes are nef.

**Theorem 4.13.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric variety of dimension \( n \). Then the following are equivalent.

(i) There exists some \( 1 \leq k \leq n-1 \) such that \( \text{Psef}^k(X) = \text{Nef}^k(X) \).

(ii) For all \( 1 \leq k \leq n-1 \), \( \text{Psef}^k(X) = \text{Nef}^k(X) \).

(iii) There is a finite surjective toric morphism \( f : \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_v} \to X \), where \( \rho \) is the Picard number of \( X \) and \( d_1, \ldots, d_v \) are some positive integers such that their sum is just \( n \).

If \( X \) is moreover smooth, then the conditions above are also equivalent to

(iv) \( X \cong \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_v} \), where \( \rho, d_1, \ldots, d_v \) are as in (iii).

**Proof.** Note that by the projection formula, (iii) implies (ii). Hence, by [FS09] Prop. 5.3, to complete the proof, we only need to show the following two claims for the case when \( X \) is a \( \mathbb{Q} \)-factorial projective toric variety of dimension \( n \).

Claim 1: If \( \text{Psef}^1(X) = \text{Nef}^1(X) \), then for all \( 2 \leq k \leq n-1 \), \( \text{Psef}^k(X) = \text{Nef}^k(X) \).

Claim 2: If \( \text{Psef}^k(X) = \text{Nef}^k(X) \) for some \( 2 \leq k \leq n-1 \), then \( \text{Psef}^1(X) = \text{Nef}^1(X) \).

By Proposition 3.1(i), for both claims, we only need to analysis the intersections of orbit closures.

The claim 1 is a direct consequence of the fact that the products of nef cycles on \( X \) are nef and the fact that each orbit closure on \( X \) is the intersection of some prime divisors up to a multiplicity.

To show the claim 2, it suffices to show that for any \( (n-1) \)-dimensional cone \( \sigma \in \mathbb{P}_X \) and any primitive lattice point \( v \) which generates a ray of \( \mathbb{P}_X \), the intersection number \( V(\sigma) \cdot V(v) \) is nonnegative.

Assume that \( \sigma = \langle v_1, \ldots, v_{n-1} \rangle \), where \( v_i \) is the primitive lattice point on a ray \( R_i \) of \( \sigma \). By reordering \( v_i \), we can assume that \( v_i \neq v \) for all \( i = 1, \ldots, k-1 \). Since \( X \) is \( \mathbb{Q} \)-factorial, \( \mathbb{Q}^+v \cap \langle v_1, \cdots, v_{k-1} \rangle = \{0\} \).
If \( v, v_1, \ldots, v_{k-1} \) don’t generate any \( k \)-dimensional cone in \( F_X \), then \( V(v) \cdot \prod_{i=1}^{k-1} V(v_i) = 0. \)

If \( v, v_1, \ldots, v_{k-1} \) generate a cone \( \gamma \) in \( F_X \), then \( V(v) \cdot \prod_{i=1}^{k-1} V(v_i) = \frac{1}{\text{mult}(\gamma)} V(\gamma). \)

All in all, \( V(v) \cdot \prod_{i=1}^{k-1} V(v_i) \in \text{Psef}^k(X) = \text{Nef}^k(X). \) Denote by \( \tau \) the face of \( \sigma \) generated by \( v, v_1, \ldots, v_{n-1}, \) then \( \prod_{i=k}^{n-1} V(v_i) = \frac{1}{\text{mult}(\tau)} V(\tau) \in \text{Psef}^{n-k}(X). \) Hence, \( V(\sigma) \cdot V(v) = \text{mult}(\sigma) V(v) \cdot \prod_{i=1}^{k-1} V(v_i) \cdot V(\tau) \geq 0. \) Then the claim 2 holds. The conclusion follows.

\[ \Box \]

4.3 Toroidal varieties

In this subsection, we show in Theorem 4.16 that if \( X \) is a smooth projective \( G \)-spherical variety such that \( \mathcal{O}_X = \emptyset \) and \( \text{Nef}^k(X) = \text{Psef}^k(X) \) for some \( 1 \leq k \leq \dim(X) - 1 \), then \( X \) must be isomorphic to a rational homogeneous space. Note that this theorem is a generalization of Theorem 4.13.

**Definition 4.14.**

(i) A spherical \( G/H \)-embedding \( X \) is toroidal, if \( \mathcal{O}_X = \emptyset \).

(ii) Let \( \Gamma \) be a connected linear algebraic group. A \( \Gamma \)-variety \( X \) is said to be regular if the following conditions are satisfied:

(a) there is an open \( \Gamma \)-orbit on \( X \);

(b) the closure of every \( \Gamma \)-orbit is smooth;

(c) for any \( Y \in S_{X, \Gamma} \), if its closure \( \overline{Y} \neq X \), then \( \overline{Y} \) is the transversal intersection of the orbit closures of codimension one containing \( \overline{Y} \);

(d) for any point \( x \in X \), the isotropy group \( \Gamma_x \) has a dense orbit in the normal space of the orbit \( \Gamma \cdot x \) in \( X \).

(iii) Let \( \Gamma \) be a connected algebraic group, \( X \) be a smooth \( \Gamma \)-variety, and \( D \subseteq X \) be a \( \Gamma \)-stable divisor with normal crossings. We say that \( X \) is a log homogeneous \( \Gamma \)-variety with boundary \( D \), if the natural morphism of \( G \)-linearized sheaves \( \mathcal{O}_X \otimes \mathfrak{h} \to \mathcal{T}_X(-\log D) \) is surjective, where \( \mathfrak{h} \) is the Lie algebra of \( \Gamma \) and \( \mathcal{T}_X(-\log D) \) is the logarithmic tangent sheaf corresponding to \( D \).

**Proposition 4.15.** ([BB96, Prop. 2.2.1], [Br04, Cor. 2.14, Cor. 3.2.2]) Let \( X \) be a smooth complete \( G \)-variety. Then \( X \) is a toroidal spherical variety if and only if it’s a regular variety if and only if it’s a log homogeneous variety.

**Theorem 4.16.** Let \( X \) be a smooth projective toroidal \( G/H \)-embedding of dimension \( n \). Then the following are equivalent:

(i) there exists some \( 1 \leq k \leq n - 1 \) such that \( \text{Nef}^k(X) = \text{Psef}^k(X) \);

(ii) for all \( 1 \leq k \leq n - 1 \), \( \text{Nef}^k(X) = \text{Psef}^k(X) \);

(iii) \( X \) is isomorphic to a rational homogeneous space.

Note that in the condition (iii) of the theorem, \( X \) may be not \( G \)-homogeneous, but it’s homogeneous under the action of a larger group.

**Proof.** We depart the condition (i) into the following two conditions:

(a) there is an equality \( \text{Nef}^1(X) = \text{Psef}^1(X) \);

(b) there exists some \( 2 \leq k \leq n - 2 \) such that \( \text{Nef}^k(X) = \text{Psef}^k(X) \).

(iii) \( \Rightarrow (ii) \) Since \( X \) is a rational homogeneous space, it’s well-known that there are two dual bases of \( H^*(X) \), see for example [Br05, Prop. 1.3.6]. Note that for the rational homogeneous
space $X$, the natural cycle map $A^*(X) \to H^*(X)$ is an isomorphism. Thus, the conclusion (ii) follows.

(ii) $\Rightarrow$ (ib) is trivial.

(ib) $\Rightarrow$ (ia) Assume that $\text{Nef}^1(X) \neq \text{Psef}^1(X)$, then by the duality, we can choose an extremal ray $R$ of $NE(X)$ such that $R \notin \text{Nef}_1(X)$. Assume that $E$ is an irreducible $B$-stable divisor such that $E : R < 0$. Choose a curve $C \in R$ and let $F = \overline{EC}$. Since $E \cdot C < 0$ and $D_X = \emptyset$, we know that $E \in V_X$ and $E \supseteq F$.

By [Br93 Thm. 3.1], there exists a corresponding contraction $\pi = \text{cont}_R : X \to X'$. Thus, $F$ is a component of the exceptional locus. Assume that $a = \text{dim}(F)$ and $b = \text{dim}(\pi(F))$. Note that $F = \overline{GC}$ is a $G$-orbit closure. Let $D_1 = E, D_2, \ldots, D_t$ be all the boundary divisors containing $F$. By Proposition 4.15 and the definition of regular varieties, $t$ is exactly the codimension of $F$ in $X$, and $t \geq D_i = F$ in $A^*(X)$.

Take $b$ general very ample divisors $H'_{t+1}, \ldots, H'_{t+b}$ on $X'$. Let $D_i$ be a general global section of $\pi^*H'_j$ for $t + 1 \leq j \leq t + b$. Take $n - t - b - 1$ general ample divisor $D_{t+b+1}, \ldots, D_{n-1}$ on $X$.

Thus, $\prod_{i=1}^{n-1} D_i = \lambda C$ in $N_1(X)_R$, where $\lambda > 0$. Hence, $E \cdot \prod_{i=1}^{n-1} D_i < 0$.

On the other hand, $\prod_{i=1}^{k} D_i \in \text{Psef}^k(X) = \text{Nef}^k(X)$, and $E \cdot \prod_{i=k+1}^{n-1} D_i \in \text{Psef}^{n-k}(X)$. Thus, $E \cdot \prod_{i=1}^{n-1} D_i = (E \cdot \prod_{i=k+1}^{n-1} D_i) \cdot \prod_{i=1}^{k} D_i \geq 0$. We get a contradiction. Thus, the conclusion (ia) holds.

(ia) $\Rightarrow$ (iii) Denote by $X_1, \ldots, X_m$ all the boundary divisors and identify $\partial X$ with the divisor $\sum_{i=1}^{m} X_i$. By Proposition 4.15, $X$ is a regular variety. Hence, each $X_i$ is a smooth variety.

Denote by $l_i : X_i \to X$ the natural inclusion and $N_{X_i/X}$ the normal bundle of $X_i$ in $X$. Then by [BB96 Prop. 2.3.2, Prop. 4.1.1], there are two exact short sequences

$$0 \to \mathbb{T}_X(-\log \partial X) \to \mathbb{T}_X \to \bigoplus_{i=1}^{m} (l_i)_* N_{X_i/X} \to 0,$$

and

$$0 \to H^0(X, \mathbb{T}_X(-\log \partial X)) \to H^0(X, \mathbb{T}_X) \to \bigoplus_{i=1}^{m} H^0(X, (l_i)_* N_{X_i/X}) \to 0.$$

Note that the second exact sequence appeared in the proof of [BB96 Prop. 4.11].

Now we consider the following commutative diagram:

$$\newcommand{\map}{\phi} \begin{array}{ccc}
H^0(X, \mathbb{T}_X(-\log \partial X)) \otimes \mathcal{O}_X & \xrightarrow{\map_1} & H^0(X, \mathbb{T}_X) \otimes \mathcal{O}_X \\
\map_2 & & \map_3 \\
\mathbb{T}_X(-\log \partial X) & \xrightarrow{\map_3} & \mathbb{T}_X \\
\end{array}
$$

By Proposition 4.15 and the definition of log homogeneous varieties, $\mathbb{T}_X(-\log \partial X)$ is globally generated, i.e. $\map_1$ is surjective. By (ia), each $X_i$ is a nef Cartier divisor on $X$. By [Per12 Cor. 3.2.11], $\mathcal{O}_X(X_i)$ is globally generated, which implies that $N_{X_i/X} = (\mathcal{O}_X(X_i))|_{X_i}$ is globally generated. Thus, $\map_3$ is surjective. Then by the Short Five Lemma, $\map_2$ is surjective, i.e. the tangent bundle $\mathbb{T}_X$ is globally generated, which implies that $X$ is a homogeneous variety. Since spherical varieties are rational (see for example [Per12 Cor. 2.1.3]), $X$ is isomorphic to a rational homogeneous space.

$\square$
4.4 Horospherical varieties

The main results in this subsection are as follows. Let $X$ be a smooth projective horospherical $G/H$-embedding such that $H \supseteq R_u(B)$ and $N_G(H) = P_I$. Theorem 4.28 says that if $\text{Nef}^2(X) = \text{Psef}^2(X)$ and $\dim(X) \geq 3$, then $\text{Nef}^1(X) = \text{Psef}^1(X)$. Corollary 4.43 says that if $\text{Nef}^1(X) = \text{Psef}^1(X)$, then there is a $G$-equivariant morphism $\pi : X \to G/P_{\mathfrak{S}_{\mathfrak{D}_0}(G/H)}$ and each fiber is isomorphic to the product of some smooth projective $L$-horospherical varieties of Picard number one, where $\mathfrak{D}_0(G/H)$ is identified with a subset of $S \setminus I$ by Remark 4.18 in the following, and $L$ is a Levi factor of $P_{\mathfrak{S}_{\mathfrak{D}_0}(G/H)}$. Corollary 4.43 is a direct consequence of Theorem 4.29 and Theorem 4.41.

This subsection is organized as follows. In the part 4.4.1, we study some properties on horospherical varieties. In the part 4.4.2, we prove Theorem 4.28 which has been described as above. In the part 4.4.3, we study the $G$-equivariant morphism $\pi$ mentioned above. In the part 4.4.4, we describe of the fiber of $\pi$.

4.4.1 Preliminaries

In order to study the smooth projective horospherical varieties whose effective cycles of codimension $k$ are nef for $k = 1$ or $2$, we need to study some related properties on horospherical varieties. These results are easy to prove, but we fail to find references to include them.

Now we describe the contents in this part. Let $X$ be a horospherical $G/H$-embedding such that $H \supseteq R_u(B)$. Firstly, we make a remark on the isotropy group $G_x$ and its normalizer $N_G(G_x)$ for a point $x \in X$. Then we show in Corollary 4.21 that if $X$ is $\mathbb{Q}$-factorial, then there is a one to one correspondence between the rays of $\mathbb{R} X$ and the elements in $V_X \cup \mathfrak{D}_X$. We continue to recall the description of the Picard number of $X$ (assuming $X$ to be complete) and the descriptions of the $G$-orbits on $X$. Finally, we study the relationship between the images of $G$-orbits and the images of colored cones in Proposition 4.26.

Definition 4.17. A spherical $G/H$-embedding $X$ is $G$-horospherical if there is a point $x \in G/H$ such that $G_x \supseteq R_u(B)$. We also say that $X$ is a horospherical $G/H$-embedding.

Firstly, we will summarize some well-known results on horospherical varieties, which will be used frequently in the following.

Remark 4.18. Let $X$ be a horospherical $G/H$-embedding such that $H \supseteq R_u(B)$. Then the following hold.

(i) $N_G(H)$ is a parabolic subgroup of $G$ containing $B$, and $TH = HT = N_G(H)$.

(ii) Let $I$ be the subset of $S$ such that $P_I = N_G(H)$. Then $P_I/H$ is isomorphic to a torus, $M_{G/H} \cong \chi(P_I/H)$ and $H = \text{Ker}_{P_I} M_{G/H}$. We identify $M_{G/H}$ and $\chi(P_I/H)$. Moreover, $P_I$ is the unique parabolic subgroup $P$ of $G$ containing $B$ such that $H$ is the intersection of some characters of $P$.

(iii) There is a bijective map $S \setminus I \to \mathfrak{D}(G/H), \alpha \mapsto D_\alpha$. When there is no confusions, we will identify them. Moreover, this identity only depends on the triple $(B, T, U)$, where $U = G/H$ is the open $G$-orbit.

(iv) For any $\alpha \in S \setminus I$, $\rho(\nu_{D_\alpha}) = \alpha^\vee |_{M_{G/H}}$, where $\alpha^\vee$ is the corresponding coroot.

(v) Let $B'$ be any Borel subgroup of $G$, then by [Ha73, Cor. 21.3A], there exists some element $g \in G$ such that $R_u(B') = gR_u(B)g^{-1}$. Let $x = H/H \in G/H \subseteq X$. Then $G_{g^{-1}}x = gHg^{-1} \supseteq R_u(B')$. So $X$ is also a horospherical $G/H'$-embedding such that $H' \supseteq R_u(B')$ for any given Borel subgroup $B'$ and some corresponding $H'$. Therefore, when saying that $X$ is a horospherical $G/H$-embedding, we can either assume $H \supseteq R_u(B)$ or assume $H \supseteq R_u(B')$ for any fixed Borel group $B'$ of $G$. This doesn’t matter. In particular, we can assume $H \supseteq R_u(B^-)$ and $N_G(H) = P_I$ if necessary.
Note that by [Pa06, Prop. 1.3], \(N_G(H)\) is a parabolic subgroup \(P\) of \(G\), and it’s the unique parabolic subgroup \(P\) of \(G\) containing \(B\) such that \(H\) is the intersection of the kernels of some characters of \(P\). Then we can easily know the conclusions \((i)\) and \((ii)\). The conclusion \((iii)\) follows from the Bruhat decomposition of a rational homogeneous space and the one to one correspondence between the \(B\)-orbits on \(G/H\) and the \(B\)-orbits on \(G/P\) induced by the natural morphism \(\pi: G/H \rightarrow G/P\). The conclusion \((iv)\) follows from the isomorphism in \((ii)\).

For a spherical \(G/H\)-embedding \(X\), we also use \(\rho\) to denote the composition of the maps \(\mathfrak{B}(X) \rightarrow \{\text{valuations}\} \rightarrow \langle N_X \rangle_q \rightarrow \nu_D \rightarrow \rho(\nu_D)\).

**Proposition 4.19.** Let \(X\) be a spherical \(G/H\)-embedding and \(Y\) be a \(G\)-orbit on \(X\). Then \(Y' = Y \cap \rho^{-1}(\mathfrak{C}_Y)\).

**Proposition 4.20.** Let \(X\) be a spherical \(G/H\)-embedding and \(Y\) be a \(G\)-orbit. Then \(\mathcal{D}_Y \cap \rho^{-1}(\mathcal{V}(G/H)) = \{D \in \mathcal{D}_X | \rho(\nu_D) \in \mathfrak{C}_Y \cap \mathcal{V}(G/H)\}\). If \(X\) is moreover G-horospherical, then \(\mathcal{D}_Y = \{D \in \mathcal{D}_X | \rho(\nu_D) \in \mathfrak{C}_Y\}\), and \(\mathcal{B}_Y = \{D \in \mathcal{V}_X \cup \mathcal{D}_X | \rho(\nu_D) \in \mathfrak{C}_Y\}\).

**Corollary 4.21.** Let \(X\) be a \(\mathbb{Q}\)-factorial horospherical \(G/H\)-embedding. Then there is a bijective map \(\psi: \mathcal{V}_X \cup \mathcal{D}_X \rightarrow \{\text{rays of } \mathbb{F}_X\}, D \mapsto \mathbb{Q}^+ \rho(\nu_D)\).

**Corollary 4.22.** Let \(X\) be a complete \(\mathbb{Q}\)-factorial \(G/H\)-horospherical variety. Then the Picard number of \(X\) is \(\rho(X) = m - r + d\), where \(m\) is the number of rays in \(\mathbb{F}_X\), \(r = \text{rank}(G/H)\), and \(d\) is the number of elements in the set \(\mathcal{D}(G/H) \setminus \mathcal{D}_X\).
Remark 2.9. Then there is a wall \( \mu \) parabolic subgroup of \((\mathbf{C} \rightarrow \mathbf{P}^n)\) corresponding morphism. Let 4.23(\( \pi \)): 

Lemma 4.24. Let 

Corollary 4.25. Proposition 4.23. 

Denote by \( G \rightarrow H \rightarrow G \) homogeneous, there exists an element \( g \in G \) such that \( g \cdot x_0 = x \). Thus, \( G_x = gPg^{-1} \) is a parabolic subgroup. By [G07] Prop. 11.1], there is a maximal torus \( T' \subseteq G_x \cap P \). Thus, \( G_x \supseteq R_u(B)T' = B \). Then by [Hil75] Cor. 23.1], \( G_x = P \). In particular, \( g \in N_G(P) = P \). Hence, \( x = g \cdot x_0 = x_0 \).

Corollary 4.25. Keep notations as in Proposition 4.23. Then the following hold. 

(i) \( K \supseteq H \), \( M_{\text{ev}} = M_{G/K} \) and \( N_G(K) = P_{\text{ev}, Y} \).

(ii) Denote by \( x = H/H \in G/H \). If there is a \( G \)-equivariant morphism \( \phi : G/H \rightarrow Y \), then \( N_G(G_{\phi(x)}) = P_{\text{ev}, Y} \).

Proof. (i) By Proposition 4.23 and Remark 4.13(i), \( K \supseteq H \), \( M_{\text{ev}} \subseteq M_{G/K} \) and \( N_G(K) \subseteq P_{\text{ev}, Y} \).

Note that \( G/K \) is homogeneous horospherical variety. Thus, by Remark 4.13(i), \( \dim(G/K) = \dim(M_{G/K} + \dim(N_G(K)) \). By the facts \( M_{\text{ev}} \subseteq M_{G/K} \), \( N_G(K) \subseteq P_{\text{ev}, Y} \) and Proposition 4.23, \( \dim(M_{G/K} + \dim(N_G(K)) \). Thus, \( M_{\text{ev}} \) is a subgroup of \( M_{G/K} \) with a finite index. Since \( N_G(K) \) and \( P_{\text{ev}, Y} \) are parabolic subgroups with the same dimension and \( N_G(K) \subseteq P_{\text{ev}, Y} \), we know that \( N_G(K) = P_{\text{ev}, Y} \).

By Remark 4.13(ii), \( M_{G/K} = \chi(P_{\text{ev}, Y}/K) \subseteq \chi(P_I/H) = M_{G/H} \). Since \( M_{\text{ev}} \) is a subgroup of \( M_{G/K} \) with a finite index, \( (M_{\text{ev}})_Q = (M_{G/K})_Q \) as subspaces of \( (M_{G/H})_Q \). In particular, \( M_{G/K} \subseteq M_{G/H} \cap (M_{\text{ev}})_Q \). By Proposition 4.23(i), \( (M_{\text{ev}})_Q \subseteq (M_{G/H})_Q \cap \xi_Y^\perp \). Thus, \( M_{G/K} \subseteq M_{G/H} \cap (M_{\text{ev}})_Q \subseteq \xi_Y^\perp = M_{G/H} \cap \xi_Y^\perp = M_{\text{ev}} \). Hence, \( M_{G/K} = M_{\text{ev}} \).

(ii) Identify \( Y \) with \( G/K \). Denote by \( y = K/K \in G/K \). Let \( y' = \phi(x), K' = G_{y'}, P = P_{\text{ev}, Y} \) and \( z = P/P \in G/P \). Denote by \( \pi_1 : G/H \rightarrow Y \) and \( \pi_2 : Y \rightarrow G/P \) the natural morphisms induced by the inclusions \( H \subseteq K \subseteq P \). By Lemma 4.21, \( \pi_2(y') = z \). Thus, \( y' = (\pi_2)^{-1}(z) = P/K \subseteq Y \). In particular, there is some element \( g \in P \) such that \( g \cdot y = y' \). Thus, \( K' = gKg^{-1} \) and \( N_G(K') = gN_G(K)g^{-1} = gPg^{-1} = P \).

The following properties are direct consequences of the corresponding propositions in Section but we will use them frequently later. For the convenience of discussions, we state them explicitly.

Proposition 4.26. (i) Let \( X \) be a horospherical \( G/H \)-embedding, \( Y \) be a \( G \)-orbit on \( X \) and \( \sigma \) be a face of \( \xi_Y \). Then there exists a unique \( G \)-orbit \( V(\sigma) \) on \( X \) such that \( \xi_{V(\sigma)} = \sigma \). Moreover, \( Y \subseteq V(\sigma) \) and \( \xi_{V(\sigma)} \) is a colored face of \( \xi_Y \).

(ii) Let \( X \) be a spherical \( G/H \)-embedding and \( X' \) be spherical \( G/H' \)-embedding. Assume that \( H \subseteq H' \) and there is an \( G \)-equivariant morphism \( \pi : X \rightarrow X' \). Extending the natural morphism \( G/H \rightarrow G/H' \). Denote by \( \phi : (N_X)_Q \rightarrow (N_X')_Q \) the corresponding morphism. Let \( Y \subseteq X \) and \( Y' \subseteq X' \) be \( G \)-orbits such that \( \phi(\xi_Y)^\circ \subseteq \xi_Y^\perp \). Then \( \pi(Y) = Y' \).

(iii) Let \( X \) be a horospherical \( G/H \)-embedding and \( X' \) be a horospherical \( G/H' \)-embedding. Assume that \( X \) is projective and \( \mathbb{Q} \)-factorial, \( H \subseteq H' \) and there is a \( G \)-equivariant morphism \( \pi : X \rightarrow X' \). Extending the natural morphism \( G/H \rightarrow G/H' \). Denote by \( \phi : (N_X)_Q \rightarrow (N_X')_Q \) the corresponding morphism. Let \( Y \subseteq X \) and \( Y' \subseteq X' \) be two closed \( G \)-orbits such that \( \phi(\xi_Y) \subseteq \xi_Y \). Then there is a wall \( \mu \) in \( \mathbb{F}_X \) such that \( \pi(C_\mu) \) is a point, where \( C_\mu \) is the curve described in Remark 2.9.
Proof. (i) It's a direct consequence of Theorem 2.3 and [Kn91] Cor. 6.2.

(ii) By Theorem 2.7 and the fact \( \phi(C_Y)^* \cap C'_Y \neq \varnothing \), we know that \( \phi(C_Y)^o \subseteq C'_Y \). Then by Theorem 2.7 and Theorem 2.5, \( \pi \) maps \( X_Y \) into \( X'_Y \), where \( X_Y \) (resp. \( X'_Y \)) is the simple spherical open subvariety of \( X \) (resp. \( X' \)) with the unique closed \( G \)-orbit \( Y \) (resp. \( Y' \)). In particular, the \( G \)-orbit \( Z = \pi(Y) \) on \( X' \) satisfies that \( \overline{Z} \subseteq Y' \). By Theorem 2.7, \( \phi(C_Y) \subseteq C_Z \) and \( C_Z \) is a face of \( C'_Y \). Since \( C_Y \) is a strictly convex cone and \( C_Z \cap C'_Y \supseteq \phi(C_Y) \cap C'_Y \neq \varnothing \), we know that \( C_Z = C'_Y \). By (i), \( Z = Y' \), i.e. \( \pi(Y) = Y' \).

(iii) By [Kn91] Thm. 6.3, \( \dim(C_Y) = \dim(N_X)_Q \). Note that \( \phi : (N_X)_Q \to (N_X)_Q \) is a surjective morphism of vector spaces. Hence, \( \dim(\phi(C_Y)) = \dim(N_X)_Q \). By Theorem 2.5, \( C_Y \) and \( C_Y \) are strictly convex cones. Then \( \phi(C_Y) \) is also a convex cone. Note that \( \phi(C_Y)^o \subseteq \phi(C_Y)^o \), i.e. \( \phi(\partial C_Y) \supseteq \partial \phi(C_Y) \). Note that \( \partial \phi(C_Y) \cap C'_Y \neq \varnothing \). Thus, there exists a face \( \mu \subseteq C_Y \) of codimension one such that \( \phi(\mu) \cap C'_Y \neq \varnothing \). Thus, \( \phi(\mu)^o \subseteq C'_Y \). By (i), \( \mu \) is a wall of \( \mathbb{F}_X \) and there exists a \( G \)-orbit \( V \) on \( X \) such that \( C_Y = \mu \). By (ii), \( \pi(V) = Y' \).

By Remark 2.19, \( C_\mu = (\mathbb{F}_X)^{\mu}R_{a}(B) \). Thus, \( \pi(C_\mu) \subseteq (Y')^{R_{a}(B)} \). Note that \( Y' \) is a complete \( G \)-orbit, then by Lemma 4.24 there is only one \( R_a(B) \)-fixed point on \( Y' \). The conclusion follows.

4.4.2 Smooth projective horospherical varieties whose effective cycle classes of codimension two are nef

In this part, we firstly show in Proposition 4.27 that the exceptional locus of a birational Mori contraction of a projective \( \mathbb{Q} \)-factorial G-horospherical variety is irreducible and we can even describe the corresponding colored cone. Then we show that if \( X \) is a smooth projective horospherical \( G/H \)-embedding of dimension \( n \geq 3 \) such that \( \text{Nef}^2(X) = \text{Psef}^2(X) \), then \( \text{Nef}^1(X) = \text{Psef}^1(X) \).

Let \( X \) be a projective \( \mathbb{Q} \)-factorial spherical \( G/H \)-embedding of rank \( r \) and \( \mu \) be wall of \( \mathbb{F}_X \). Choose primitive lattice points \( e_1, \ldots, e_{r+1} \) in \( N_X \) such that \( \mu = \langle e_1, \ldots, e_{r-1} \rangle, \mu_+ = \langle e_1, \ldots, e_r \rangle \) and \( \mu_- = \langle e_1, \ldots, e_{r-1}, e_{r+1} \rangle \), where \( \mu_+, \mu_- \subseteq \mathbb{D}(G/H) \) and \( \mu \subseteq \mathbb{D}_+ \cap \mathbb{D}_- \). Then there is a unique sequence \( a_1, \ldots, a_{r+1} \in \mathbb{Q} \) such that \( \sum_{i=1}^{r+1} a_i e_i = 0 \) and \( a_{r+1} = 1 \). Note that \( a_r > 0 \).

By reordering \( e_i \), we can assume that

\[
\begin{cases}
  a_i = 0, & 1 \leq i \leq \alpha,
  a_i = 0, & \alpha + 1 \leq i \leq \beta,
  a_i > 0, & \beta + 1 \leq i \leq r + 1.
\end{cases}
\]

If all \( a_i \) are nonnegative, then let \( \alpha = 0 \). Note that \( 0 \leq \alpha \leq \beta \leq r - 1 \).

**Proposition 4.27.** Let \( X \) be a projective \( \mathbb{Q} \)-factorial horospherical \( G/H \)-embedding, and \( R \) be an extremal ray of \( NE(X) \). Denote by \( \pi = \text{cont}_R : X \to X' \) the corresponding contraction. Assume that \( \pi \) is a birational morphism. Let \( A \) be the exceptional locus of \( \pi \) and \( A' = \pi(A) \). Then \( A \) is an \( \overline{G} \)-orbit closure.

(i) Suppose that \( R = \mathbb{R}^+ C_\mu \), where \( \mu \) is a wall in \( \mathbb{F}_X \). Keep notations as the discussions above. Then \( A \) is the \( \overline{G} \)-orbit closure on \( X \) such that \( C_A = \langle e_1, \ldots, e_\alpha \rangle \), and \( C_A' = \langle e_1, \ldots, e_\beta, e_{\beta+1}, \ldots, e_{r+1} \rangle \).

(ii) Suppose that \( R \) doesn’t contain any \( C_\mu \) for walls \( \mu \) in \( \mathbb{F}_X \) and that \( R = \mathbb{R}^+ C_{D,Y} \), where \( Y \) is a closed \( G \)-orbit on \( X \) and \( D \in \mathbb{D}(G/H) \setminus \mathbb{D}_Y \). Then \( \rho(\nu_D) \in C_Y, A \) is the \( \overline{G} \)-orbit closure on \( X \) such that \( C_A = \langle e_1, \ldots, e_s \rangle \), and \( C_A' = \langle C_A, \mathbb{D}_A \cup \{D\} \rangle \), where \( C_Y = \langle e_1, \ldots, e_r \rangle \), \( \rho(\nu_D) = \sum_{i=1}^s a_i e_i \), and \( a_i > 0 \) for all \( 1 \leq i \leq s \).
Proof. By [Br93] Thm. 3.2(ii), either $R = \mathbb{Q}^+C_\mu$ for some wall $\mu$ in $F_X$ or $R = \mathbb{Q}^+C_{D,Y}$ for some closed $G$-orbit $Y$ and some $D \in \mathcal{D}(G/H)\setminus \mathcal{D}_Y$. Hence, if the conclusions (i)(ii) hold, then $A$ is a $G$-orbit closure. So we can turn to show the conclusions (i) and (ii).

(i) Denote by $\varphi : F_X \to F_{X'}$ the morphism of colored fans corresponding to $\pi$. Let $\sigma = (e_1, \ldots, e_s)$, and $\tau = (e_1, \ldots, e_{\alpha+1}, \ldots, e_{r+1})$. Denote by $Z$ the closed $G$-orbit such that $C_\sigma \subseteq \langle e_1, \ldots, e_r \rangle$. By [Br93] Prop. 4.6, $\mathcal{C}_\sigma(Z) = \langle e_1, \ldots, e_{r+1} \rangle$, and $\tau$ is a face of $C_\sigma(Z)$. Thus, by Proposition 4.26(ii), there exists a $G$-orbit $V(\sigma)$ on $X$ such that $\mathcal{C}_V(\sigma) = \sigma$ and a $G$-orbit $V'(\tau)$ on $X'$ such that $\mathcal{C}_{V'}(\tau) = \tau$.

Denote by $X_{-\sigma} = X' \setminus V(\sigma)$ and $X'_{-\tau} = X' \setminus V'(\tau)$. By Theorem 2.5 $F_X \setminus F_{X_{-\sigma}} = \{ (\mathcal{C}, D) \mid \mathcal{C} \supseteq \sigma \}$, and $F_X \setminus F_{X'_{-\tau}} = \{ (\mathcal{C}', \mathcal{D}') \mid \mathcal{C}' \supseteq \tau \}$. By [Br93] Prop. 4.6, $F_{X_{-\sigma}}$ is identified with $F_{X'_{-\tau}}$ by $\varphi$. By Theorem 2.5 and Theorem 2.7 $X_{-\sigma}$ is $G$-equivariantly isomorphic to $X'_{-\tau}$ by the morphism $\pi$. In particular, $A \subseteq V(\sigma)$.

Denote by $X_V(\sigma)$ (resp. $X_V'(\tau)$) the simple spherical open subvariety of $X$ (resp. $X'$) with the unique closed $G$-orbit $V(\sigma)$ (resp. $V'(\tau)$). Since $X_{V(\sigma)} \setminus V(\sigma) \subseteq X_{-\sigma}$, $\pi$ maps $X_{V(\sigma)} \setminus V(\sigma)$ isomorphically to its image.

On the other hand, $\sigma \subseteq \tau$, i.e. $\varphi$ maps $\sigma$ into $\tau$. By Theorem 2.7, $\pi$ maps $X_V(\sigma)$ into $X_{V'(\tau)}$. In particular, $\pi(V(\sigma)) \supseteq V'(\tau)$. Denote by $Z = \pi(V(\sigma))$. Then $\mathcal{C}_Z$ is a colored face of $\mathcal{C}_V(\tau)$. The fact that $a_i < 0$ for all $1 \leq i \leq \alpha$ and $a_i > 0$ for all $\beta + 1 \leq i \leq r + 1$ implies $\sigma^\alpha \subseteq \tau^\beta$. By Proposition 4.26(ii), $\pi(V(\sigma)) = Z = V(\tau)$. By Corollary 4.25(i), $\operatorname{rank}(V(\sigma)) = \operatorname{rank}(V'(\tau)) = \operatorname{rank}^\beta + r - \beta$. In particular, $\operatorname{rank}(V(\sigma)) \neq \operatorname{rank}(V'(\tau))$. Thus, $\pi|_{V(\sigma)} : V(\sigma) \to V'(\tau)$ is not an isomorphism. Hence, $A \cap V(\sigma) \neq \emptyset$.

Since $A$ is closed and $G$-stable, $\sigma \supseteq \mathcal{G}(A \cap V(\sigma)) = V(\sigma)$, hence $A = V(\sigma)$. Since $A$ is complete, $A' = \pi(V(\sigma)) = V'(\tau)$.

(ii) Denote by $\varphi : F_X \to F_{X'}$ the morphism of colored fans corresponding to $\pi$.

Assume that $\rho(\nu_D) \notin \mathcal{C}_Y$. By [Br93] Prop. 3.4, $\langle \langle \mathcal{C}_Y, \rho(\nu_D) \rangle, D_Y \cup \{ D \} \rangle$ is a colored cone in $F_{X'}$. By the fact $\mathcal{C}_Y \not\subseteq \langle \langle \mathcal{C}_Y, \rho(\nu_D) \rangle \rangle$ and Proposition 4.26(iii), there is a wall $\mu$ in $F_X$ such that $C_\mu$ is contorted by $\pi$, i.e. $C_\mu \subset R$. It’s contradicted with our assumption. Hence, $\rho(\nu_D) \in \mathcal{C}_Y$.

Assume that $\mathcal{C}_Y = (e_1, \ldots, e_r)$. Note that $\sigma = (\sigma, \rho(\nu_D))$ is a face of $\mathcal{C}_Y = (\mathcal{C}_Y, \rho(\nu_D))$. By [Br93] Prop. 3.4 and Theorem 2.5 there is a $G$-orbit $Y$ on $X'$ such that $\mathcal{C}_Y = (\mathcal{C}_Y, D_Y \cup \{ D \})$. By Proposition 4.26(iii), $\pi(\nu_D) = \nu_Y$. By Proposition 4.26(i), there is a $G$-orbit $V$ on $X$ and a $G$-orbit $V'$ on $X'$ such that $\mathcal{C}_V = (\mathcal{C}_V, \nu_Y) = \sigma$. Denote by $F_{-\sigma}$ the subset of $F_X$ such that $F_X \setminus F_{-\sigma} = \{ (\mathcal{C}, \mathcal{D}) \mid \mathcal{C} \supseteq \sigma \}$, and $F'_{-\sigma}$ the subset of $F_X'$ such that $F_X' \setminus F'_{-\sigma} = \{ (\mathcal{C}', \mathcal{D}') \mid \mathcal{C}' \supseteq \sigma \}$. By Theorem 2.5 the spherical $G/H$-embeddings $X_{-\sigma}$ and $X'_{-\tau}$ satisfy that $F_{X_{-\sigma}} \setminus F_{-\sigma} = \pi(F_{X_{-\sigma}} \setminus F_{-\sigma})$, where $X_{-\sigma} = X \setminus V$ and $X'_{-\tau} = X' \setminus V'$. By [Br93] Prop. 3.4, $F_{-\sigma}$ is identified with $F'_{-\sigma}$ by $\varphi$. By Theorem 2.5 and Theorem 2.7 $X_{-\sigma}$ is $G$-equivariantly isomorphic to $X'_{-\tau}$ by the morphism $\pi$. In particular, $A \subseteq V$.

On the other hand, by Proposition 4.26(ii), $\pi(V) = V'$. By Theorem 2.5 $\mathcal{C}_Y$ (resp. $\mathcal{C}_V$) is a colored face of $\mathcal{C}_V$ (resp. $\mathcal{C}_V'$. Since $D_Y = D_Y \cup \{ D \}$, $D'_{V'} = D_Y \cup \{ D \}$. Then by Proposition 4.26(iii), $\pi(V) \neq \pi(V')$. In particular, $A \cap V \neq \emptyset$.

Since $A$ is closed and $G$-stable, $\sigma \supseteq \mathcal{G}(A \cap V) = V$. Hence, $A = V$, and $\mathcal{C}_A = \mathcal{C}_V = \sigma$. Since $A$ is complete, $A' = \pi(V) = V'$, and $\mathcal{C}_{A'} = \mathcal{C}_V = (\mathcal{C}_V, D_Y \cup \{ D \}) = (\mathcal{C}_A, D_A \cup \{ D \})$.

Theorem 4.28. Let $X$ be a smooth projective horospherical $G/H$-embedding of dimension $n \geq 3$ such that $\operatorname{Nef}^2(X) = \operatorname{Psef}^2(X)$. Then $\operatorname{Nef}^2(X) = \operatorname{Psef}^2(X)$.

Proof. When $n = 3$, the conclusion follows from the duality. So we can assume $n \geq 4$. Now assume that $H \supseteq R_u(B)$, $N_G(H) = F_i$ and there is an extremal ray $R$ of $N E(X)$ such that $R \not\subseteq \operatorname{Nef}^2(X)$.
By [Br93] Thm. 3.1, there exists a corresponding contraction \( \pi = \cont_R : X \to Y \). Let \( A \subseteq X \) be the exceptional locus, and \( A' = \pi(A) \). By Corollary 1.2, \( \dim(A) = 1 \) and \( \dim(A') = 0 \).

By Proposition 1.27, \( A \) is a \( G \)-orbit closure. Thus, \( A \) is an irreducible \( G \)-stable curve on \( X \) and \( A' \) is a \( G \)-stable point on \( Y \). Now we will get contradictions case by case as follows.

Case 1. Assume that there exist irreducible divisors \( D \) and \( E \) such that

(i) \( D \) is \( B \)-stable and \( D \cdot A < 0 \),

(ii) \( E \) is \( G \)-stable and \( A \subseteq E \), and

(iii) \( D \neq E \).

By Proposition 3.2(ii), \( \mathbb{Q}^+ \rho(v_E) \) is an extremal ray of \( \mathcal{C}_A \), and there is a unique proper face \( \mathcal{C} \) of \( \mathcal{C}_A \) of codimension one not containing \( \rho(v_E) \). By Proposition 1.20(i), there is a unique \( G \)-orbit closure \( W \) such that \( \mathcal{C}_W = \mathcal{C} \), and \( A \subseteq W \). Note that \( \mathcal{D}_W \subseteq \mathcal{D}_A \subseteq \mathcal{D}_W \cup \{ E \} \) by Proposition 4.20 and Proposition 3.2. Since \( E \in \mathcal{V}_X \), \( \mathcal{D}_A = \mathcal{D}_W \).

Since \( E \) and \( W \) are both \( G \)-stable, \( E \cap W \) is the union of some \( G \)-orbits. Take any \( G \)-orbit \( V \) contained in \( E \cap W \). By Theorem 2.3, \( \mathcal{C}_V \supseteq \langle \mathcal{C}_W, \rho(\nu_V) \rangle = \mathcal{C}_A \). Thus, \( V \subseteq A \). Hence, \( E \cap W \subseteq A \), i.e. \( A = E \cap W \). Therefore, \( \dim(W) = 2 \), and \( E \cdot W = \lambda A \) in \( N_1(X)_{R} \) for some \( \lambda > 0 \). Thus, \( D \cdot E \cdot W < 0 \).

On the other hand, the fact \( D \neq E \) implies that \( D \cdot E \in \text{Psef}^2(X) = \text{Nef}^2(X) \). Hence, \( D \cdot E \cdot W \geq 0 \). We get a contradiction.

Case 2. Assume that every irreducible \( G \)-stable divisor \( E \) has property \( E \cdot A \geq 0 \).

By [Per12] Thm. 3.3.9, \( -K_X \cdot A \geq 0 \). Thus, the length \( l(R) = -K_X \cdot A \) of the extremal ray \( R \) is nonnegative. By [Br93] Thm. (0.4)], \( \dim(X) \leq 2 \dim(A) - l(R) + 1 \leq 3 \). It’s contradicted with our assumption on the dimension of \( X \).

Case 3. Assume that there is an irreducible \( G \)-stable divisor \( E \) such that \( E \cdot A < 0 \), and \( E \) is the unique irreducible \( B \)-stable divisor having a negative degree on the curve \( A \).

By [Br93] Thm. 3.2(ii)], either \( A = \lambda C_\mu \) in \( N_1(X)_{R} \) for some wall \( \mu \) of \( F_X \) and \( \lambda > 0 \), or \( C = nC_{D,Z} \) in \( N_1(X)_{R} \) for some closed \( G \)-orbit \( Z \), \( D \in \mathcal{D}(G/H) \backslash \mathcal{D}_Z \) and \( \lambda > 0 \).

Case 3.1. Assume that \( A = \lambda C_\mu \) in \( N_1(X)_{R} \), where \( \mu \) is a wall in \( F_X \) and \( \lambda > 0 \).

Let \( \mu, \mu_+, \mu_-, e_i, a_i, \alpha, \beta, r \) be as in the discussions before Proposition 1.27. Let \( D_\alpha \) be the irreducible \( B \)-stable divisor on \( X \) such that \( \rho(v_{D_\alpha}) \in \mathbb{Q}^+ e \), which is unique by Corollary 1.21. By Proposition 1.27, \( \mathcal{C}_A = \langle e_1, \ldots, e_r \rangle \), \( \mathcal{C}_A' = \langle e_1, \ldots, e_\alpha, e_{\beta+1}, \ldots, e_{r+1} \rangle \). By Proposition 1.23, \( 1 = \dim(A) = r - \alpha + \dim(G/P_{\alpha,D}) \). Since \( \alpha \leq \beta \leq r - 1 \), we know that \( \alpha = \beta = r - 1 \) and \( S = I \cup \mathcal{D}_A \). Thus, \( \mathcal{D}(G/H) = S/I \cup \mathcal{D}_A \subseteq \{ D_{e_1}, \ldots, D_{e_r} \} \).

On the other hand, by the formula (1) on page 6, the intersection number \( D_{e_i} \cdot C_\mu \) has the same sign as \( a_i \). In particular, \( D_{e_i} \cdot C_\mu < 0 \) for all \( 1 \leq i \leq \alpha \). By the uniqueness of \( E \) in the assumption of Case 3, \( \alpha = 1 \) and \( D_{e_1} = E \). Hence, \( \text{rank}(G/H) = r = \alpha + 1 = 2 \) and \( \mathcal{D}(G/H) \subseteq \{ E \} \). However, \( E \) is \( G \)-stable. So \( \mathcal{D}(G/H) = \emptyset \), i.e. \( P_1 = G \). Thus, \( \dim(X) = \dim(G/H) = \text{rank}(G/H) + \dim(G/P_1) = 2 \), which is contradicted with our assumption on the dimension of \( X \).

Case 3.2. Assume that there doesn’t exist any wall \( \mu \) in \( F_X \) such that \( A = \lambda C_\mu \) in \( N_1(X)_{R} \) for some \( \lambda > 0 \). By [Br93] Thm. 3.2(ii)], we can assume that \( C = \lambda C_{D,Z} \) in \( N_1(X)_{R} \), where \( \lambda > 0 \), \( Z \) is a closed \( G \)-orbit on \( X \) and \( D \in \mathcal{D}(G/H) \backslash \mathcal{D}_Z \). By the assumption, we can apply [Br93] Prop. 3.4 to get \( F_Y \).

Denote by \( \mathcal{E}_Z = \langle e_1, \ldots, e_r \rangle \), and \( \rho(v_D) = \sum a_i e_i \). Thus, by the formula (2) on page 6, \( D \cdot C_{D,Z} = 1 \), \( D_{e_i} \cdot C_{D,Z} = -a_i \) for all \( 1 \leq i \leq r \), and for any \( D' \in \mathcal{B}(X) \backslash \{ D_{e_1}, \ldots, D_{e_r} \} \), the intersection number \( D' \cdot C_{D,Z} \geq 0 \).

By Proposition 1.27, \( \rho(v_D) \in \mathcal{E}_Z \), i.e. \( a_i \geq 0 \) for all \( 1 \leq i \leq r \). Then by the uniqueness of \( E \) in the assumption of Case 3, we can assume that \( E = D_{e_1} \), \( a_1 > 0 \) and \( a_i = 0 \) for all \( 2 \leq i \leq r \), i.e. \( \rho(v_D) = a_1 e_1 = a_1 \rho(v_E) \). By Proposition 1.27(ii), \( E \) is the exceptional locus of.
\( \pi \), i.e. \( E = A \). Since \( A \) is a curve and \( E \) is a prime divisor on \( X \), we know that \( \dim(X) = 2 \). It is contradicted with our assumption on the dimension of \( X \). Hence, the conclusion follows. \( \square \)

4.4.3 Reduction morphisms on horospherical varieties

In this part, we analyse the horospherical cases of Theorem 4.3 and show its converse in the horospherical cases.

Let \( X \) be a complete horospherical \( G/H \)-embedding such that \( H \supseteq R_u(B^-) \) and \( N_G(H) = P_1^{-} \). By Remark 4.18, \( \mathcal{D}(G/H) \) is identified with \( S \setminus I \). Let \( \mathcal{D}_1 \) be a subset of \( \mathcal{D}_0(G/H) \). By Theorem 4.35(i), \( D_0 \) induces a \( G \)-equivariant morphism \( \pi_0 : X \to G/P_0^{-} \), where \( D_0 = \sum_{D \in \mathcal{D}_1} D \), and \( P_0^{-} \) is some parabolic subgroup of \( G \) containing \( B^- \). Let \( x_0 = H/H \subseteq G/H, \pi_0(x_0) \) and \( X_0 = \pi_0^{-1}(x_0) \). Note that \( Bx_0 \) is the open \( B \)-orbit on \( X \) and \( G_{\pi_0}(x_0) = P_0^{-} \). Then by Lemma 4.10(a)-(e), \( X_0 \) is a complete spherical \( L_0/H_0 \)-embedding and there is an immersion \( \Phi : \mathbb{F}_X \to \mathbb{F}_{X_0} \), where \( L_0 = P_0 \cap P_0^{-} \) and \( H_0 = H \cap L_0 \).

**Theorem 4.29.** Keep notations as in the discussions above. Then the following hold.

(i) \( P_0 = P_S \cap \mathcal{D}_1, H_0 \supseteq R_u(B_0^-) \) and \( X_0 \) is a complete horospherical \( L_0/H_0 \)-embedding.

(ii) For any \( \alpha \in I \cup \mathcal{D}_1, \langle M_{L_0/H_0}, \alpha^\vee \rangle_G = 0 \).

(iii) \( \alpha \)-embedding such that \( \pi_0 \circ \rho_{\alpha(\cdot)} \) acts trivially on \( X_0 \).

(iv) The map \( \Phi : \mathbb{F}_X \to \mathbb{F}_{X_0} \) is an isomorphism of colored fans.

**Proof.** (i) Consider the restricted morphism \( \pi_0|_{G/H} : G/H \to G/P_0^{-} \). By Lemma 4.24, this morphism is induced by the inclusion \( H \subseteq P_0^{-} \). By Lemma 4.10(d), \( \pi^2 : \mathcal{D}(G/P_0^{-}) \to \mathcal{D}_1, D' \mapsto \pi_0^{-1}(D') \) is a bijective map. Apply Remark 4.18(iii) to \( G/H \) and \( G/P_0^{-} \) respectively, we get that \( P_0 = P_S \cap \mathcal{D}_1 \). As a closed subscheme of \( X \), \( X_0 \) is complete. Note that \( H_0 = H \cap L_0 \supseteq R_u(B^-) \).

Hence, the spherical \( L_0/H_0 \)-embedding is \( L_0 \)-horospherical.

(ii) By Lemma 4.10(c) and Remark 4.18(ii), \( M_{L_0/H_0} = M_{G/H} \subseteq \chi(P_1^{-}) \). Thus, for all \( \alpha \in I \), \( \langle M_{L_0/H_0}, \alpha^\vee \rangle_G = 0 \).

By Remark 4.18(iii)(iv) and the fact \( \mathcal{D}_1 \subseteq \mathcal{D}_0(G/H) \), \( \beta^\vee|_{M_{G/H}} = \rho(\nu_{\alpha_D}) = 0 \) for all \( \beta \in \mathcal{D}_1 \).

Hence, \( \langle M_{L_0/H_0}, \beta^\vee \rangle_G = 0 \).

(iii) Note that \( (B \cdot x_0) \cap X_0 = \{ b \cdot x_0 \mid b \in B, \pi(b \cdot x_0) = \pi_0(x_0) \} = \{ b \cdot x_0 \mid b \in G_{\pi_0}(x_0) \cap B \} = B_0 \cdot x_0 \).

Thus, \( P_0^{-} \cdot x_0 \subseteq B_0 \cdot x_0 \) is an open subset of \( X_0 \).

Note that \( (P_0^{-})_{x_0} = H \cap P_0^{-} = H \supseteq R_u(B^-) \supseteq R_u(P_0^{-}) \).

Take any point \( x \in P_0^{-} \cdot x_0 \). There exists some element \( p \in P_0^{-} \) such that \( p \cdot x_0 = x \). Thus, \( (P_0^{-})_x = p(P_0^{-})_{x_0}p^{-1} = pR_u(P_0^{-})p^{-1} = R_u(P_0^{-}) \).

Hence, \( R_u(P_0^{-}) \subseteq \bigcap_{x_0} (P_0^{-})_x \), i.e. \( R_u(P_0^{-}) \) act trivially on the open subset \( P_0^{-} \cdot x_0 \) of the irreducible variety \( X_0 \). Therefore, \( R_u(P_0^{-}) \) act trivially on \( X_0 \).

(iv) By Lemma 4.10(e), \( \mathbb{F}_X \subseteq \mathbb{F}_{X_0} \) and \( \mathbb{F}_{X_0} = \{ (\mathcal{C}, \mathcal{D}) \in \mathbb{F}_{X_0} \mid \mathcal{C} \cap \mathcal{D} = \emptyset \} \).

However, by [Kn91, Cor. 6.2], \( \mathcal{V}(G/H) = (N_X)_{\mathbb{Q}} \). Hence, \( \mathbb{F}_X = \mathbb{F}_{X_0} \).

**Theorem 4.30.** Let \( \mathcal{D}_1 \subseteq S \) be a subset. Denote by \( P_0 = P_S \setminus \mathcal{D}_1, L_0 = P_0 \cap P_0^{-} \) the standard Levi factor and \( B_0 = B \cap L_0 \). Suppose that the following three conditions hold.

(a) There is a \( P_0^{-} \)-action on a normal variety \( X_0 \) such that \( R_u(P_0^{-}) \) acts trivially on \( X_0 \), and \( X_0 \) is a horospherical \( L_0/H_0 \)-embedding, where \( H_0 \supseteq R_u(B_0^-) \).

(b) For any \( \alpha \in \mathcal{D}_1, \langle M_{L_0/H_0}, \alpha^\vee \rangle_G = 0 \).

(c) \( N_{L_0}(H_0) = P_{1,L_0} \) is the parabolic subgroup of \( L_0 \) containing \( B_0 \) corresponding to the subset \( I \setminus \{ \mathcal{D}_1 \} \) of simple roots of \( L_0 \). And for any \( \alpha \in I, \langle M_{L_0/H_0}, \alpha^\vee \rangle_G = 0 \).

Let \( H = K \cap P_i \cdot M_{L_0/H_0} \), where \( P_i \) is the parabolic subgroup of \( G \) containing \( B \) corresponding to the subset \( I \setminus \{ \mathcal{D}_1 \} \) of simple roots of \( G \). Then the following hold.

(i) The variety \( X = G \times^{P_i} X_0 \) is a horospherical \( G/H \)-embedding, and \( \mathbb{F}_X = \mathbb{F}_{X_0} \).
(ii) There is an inclusion \( \mathcal{D}_1 \subseteq \mathcal{D}_0(G/H) \).

(iii) Assume that \( X_0 \) is complete. Let \( \pi_0 : X \to \widetilde{X} \) be the morphism corresponding to \( \mathcal{D}_1 \) as in Theorem 4.29(i). Then \( \widetilde{X} \) is \( G \)-equivariantly isomorphic to \( G/P_- \) and \( \pi_0^{-1}(x_0) \) is \( P_- \)-equivariantly isomorphic to \( X_0 \).

Note that \( M_{L_0/H_0} \subseteq \chi(B_0) = \chi(B) \). Hence, the pairing \((-,-)_G\) in the conditions (a) and (b) are well-defined. By Remark \( \text{[4.11] (ii)} \), \( N_{L_0}(H_0) \) is a parabolic subgroup of \( L_0 \) containing \( B_0^- \). Since \( L_0 = P_{S \setminus \mathcal{D}_1} \cap P_{S \setminus \mathcal{D}_1}^- \), the set of the simple roots of \( L_0 \) corresponding to \((B_0,T)\) is \( S \setminus \mathcal{D}_1 \). Hence, the condition (c) is well-defined. Remark that the conditions (a)\((b)\)(c) are corresponding to the conclusions (i)\((ii)\)(iii) in Theorem \( \text{[4.29]} \). So Theorem \( \text{[4.30]} \) is indeed the converse of Theorem \( \text{[4.5]} \) in the horospherical cases.

**Proof.** The condition (c) implies that \( M_{L_0/H_0} \subseteq \chi(P_I) \). So \( H = \text{Ker}_{P_I}M_{L_0/H_0} \) is well-defined, and \( H \supseteq R_a(B) \). In particular, \( G/H \) is a homogeneous horospherical variety. We will show this theorem in two steps.

**Step 1.** We will construct a horospherical \( G/H \)-embedding \( X' \) with the colored fan isomorphic to \( \mathbb{F}X_0 \).

Note that \( P^-_I \cap L_0 \) is a parabolic subgroup of \( L_0 \) containing \( B_0^- \). Then by the one to one correspondence between parabolic subgroups of \( L_0 \) containing \( B_0^- \) and subsets of \( S \setminus \mathcal{D}_1 \), \( P^-_I \cap L_0 \). Hence, \( H \cap L_0 = \text{Ker}_{P^-_I \cap L_0}M_{L_0/H_0} = \text{Ker}_{P^-_I \cap L_0}M_{L_0/H_0} = H_0 \), where the last equality follows from Remark \( \text{[4.18] (ii)} \).

Note that \( M_{L_0/H_0} \subseteq \chi(P^-_I/H) \subseteq \chi(P^-_I/H_0) \). By Remark \( \text{[4.18] (ii)} \), \( M_{G/H} = \chi(P^-_I/H) \) and \( M_{L_0/H_0} = \chi(P^-_I/H_0) \). Hence, \( M_{G/H} = M_{L_0/H_0} \), and \( N_{G/H} = N_{L_0/H_0} \). Remark that (ii) of this theorem follows from the condition (b) and the identity \( M_{G/H} = M_{L_0/H_0} \).

By considering the natural morphisms \( L_0 \to P^+_0 \to P^-_0/H \), we get an inclusion \( L_0/H_0 \subseteq P^-_0/H \). On the other hand, \( P^-_0 = R_a(P^-_0)_0 \) and \( R_a(P^-_0)_0 \subseteq H \). Hence, \( L_0/H_0 = P^+_0/H \), and \( G/H \cong G \times P^-_0 \to P^-_0/H \cong G \times P^-_0 \). This implies that there is an injective map \( \Psi : \mathcal{D}(L_0/H_0) \to \mathcal{D}(G/H), D \mapsto B \times D \). From the inclusion \( L_0/H_0 = P^+_0/H \subseteq G/H \), we can get a natural morphism \( \phi : \mathcal{C}(G/H)^{(B)} \to \mathcal{C}(L_0/H_0)^{(B_0)}, f \mapsto f|_{L_0/H_0} \). Since \( M_{G/H} = M_{L_0/H_0}, \phi \) is an isomorphism. Moreover, \( \rho_{L_0/H_0}(\nu_D) = \rho_{G/H}(\nu_D|_{L_0/H_0}) \) for all \( D \in \mathcal{D}(L_0/H_0) \). Thus, \( \mathcal{D}(L_0/H_0) \) can be identified with a subset of \( \mathcal{D}(G/H) \). By \( \text{[Kno91], Cor. 6.2]}, \( V(G/H) \cong (N_{G/H})_0/Q \) and \( \mathcal{Y}_0 = (N_{L_0/H_0})_0/Q \). Therefore, \( \mathbb{F}X_0 \) is a colored fan in \( (N_{G/H})_0 = (N_{L_0/H_0})_0 \). By Theorem \( \text{[2.30]} \), there exists a horospherical \( G/H \)-embedding \( X' \) such that \( \mathbb{F}X' = \mathbb{F}X_0 \).

**Step 2.** We claim that \( X' \) is \( G \)-equivariantly isomorphic to \( G \times P^-_0 \). 

For the moment, we assume that \( X_0 \) is complete. Then by \( \text{[Kno91], Thm. 4.2, Cor. 6.2]}, \( X' \) is also complete. By Theorem \( \text{[4.29] (iii)} \) and the condition (a), the conclusion (iii) holds if we replace \( X \) by \( X' \). By Remark \( \text{[4.11]} \) \( X' \) is \( G \)-equivariantly isomorphic to \( G \times P^-_0 \). Hence, when \( X_0 \) is complete, the conclusions (i) and (iii) hold.

Now we deal with the general case when \( X_0 \) may be not complete. Let \( \overline{X_0} \) be a normal \( G \)-equivariant completion of \( X_0 \). Let \( \overline{X_0} \) be a horospherical \( G/H \)-embedding such that \( \mathbb{F}X' = \mathbb{F}X_0 \) as in Step 1. By the discussions above, \( \overline{X'} = G \times P^-_0 \overline{X_0} \). By Theorem \( \text{[2.30]} \) \( X' \) is the fiber product of \( G \times P^-_0 \overline{X_0} \) over \( G \times P^-_0 \overline{X_0} \), i.e. \( X' = G \times P^-_0 X_0 \). Hence, (i) holds.

**4.4.4 Smooth projective horospherical varieties whose effective divisors are nef**

Our main result in this part is Corollary \( \text{[4.38]} \) which has been described in the introduction of Subsection 4.4. It’s a direct consequence of Theorem \( \text{[4.29]} \) and Theorem \( \text{[4.31]} \). So our main job in this part is to show Theorem \( \text{[4.31]} \), who says that if \( X \) is a smooth projective horospherical \( G/H \)-embedding such that \( \mathcal{D}_0(G/H) = \emptyset \) and \( \text{Nef}^1(X) = \text{Psef}^1(X) \), then \( X \) is isomorphic to the product of some smooth projective \( G \)-horospherical varieties of Picard number one.
Our approach to show Theorem 1.4.1 is as follows. Firstly, by studying the toric variety associated with \( X \), we show that \( F_X \) is a product \( \prod_i F_i \), (for the precise meaning, see Definition 1.3.7). Then we show that each \( F_i \) can be identified with the colored fan of a \( G \)-horospherical variety \( X_i \). Finally, we show that \( \prod_i X_i \) is a horospherical \( G/H \)-embedding such that \( F_{\prod_i X_i} = F_X \), which implies that \( X \) is \( G \)-equivariantly isomorphic to \( \prod_i X_i \).

Guided by the approach above, we organize this part as follows. We firstly study the properties of the toric variety associated with \( X \). Then we study the conditions when the product of several \( G \)-horospherical varieties \( X_i \) is itself \( G \)-horospherical, and study the colored fan of \( \prod_i X_i \). After these discussions, we turn to the proofs of Theorem 1.4.1 and Corollary 1.4.3. In the end, we give an example to explain Corollary 1.4.3 and to show that the fibration \( \pi_0 \) in Theorem 1.5 may be nontrivial.

Let \( X \) be a horospherical \( G/H \)-embedding such that \( H \supseteq R_u(B) \) and \( N_G(H) = P_I \). We will define a toroidal horospherical variety \( T(X) = G \times P_I \) as follows. The maximal torus \( T \subseteq B \) satisfies that \( P_I = TH \). Thus the torus \( T_H = P_I/H \) is a quotient of \( T \). Let \( F = \{ \mathcal{C} \mid (\mathcal{C}, D) \in F_X \} \). Then, \( F \) is a fan in \( (N_X)_\mathbb{Q} \). By Theorem 2.2, there is a unique toric variety \( Y \) with the fan \( F_Y = F \) and the maximal torus orbit \( T_H \). Since \( P_I/H = T_H \subseteq Y \), we get that \( H \) has a trivial action on \( Y \), and \( P_I \) has a natural action on \( Y \). In the set theory, we define \( T(X) = G \times P_I \) to be the quotient of \( G \times Y \) by equivalent relations that \( (g, p) \sim (gp, y) \) for all \( g \in G, p \in P_I \), and \( y \in Y \). Thus, \( T(X) \) is a toroidal horospherical \( G/H \)-embedding such that \( N_{T(X)} = N_X \) and \( F_{T(X)} = \{ (\mathcal{C}, \emptyset) \mid (\mathcal{C}, D) \in F_X \} \).

Every \( G \)-orbit on \( T(X) \) has the form \( G \times P_I Z \), where \( Z \) is a \( T_H \)-orbit on \( Y \). Note that for each cone \( (\mathcal{C}, \emptyset) \in F_{T(X)} \), there is a unique subset \( I(\mathcal{C}) \subseteq S \setminus I \) such that \( (\mathcal{C}, I(\mathcal{C})) \in F_X \). By Theorem 2.4 there is a unique birational \( G \)-equivariant morphism \( T_X : T(X) \to X \) extending the identity \( G/H = G/H \) and inducing the morphism \( F_{T(X)} \to F_X, (\mathcal{C}, \emptyset) \to (\mathcal{C}, I(\mathcal{C})) \).

**Remark 4.31.** Let \( X \) be a toroidal \( G \)-horospherical variety. By Theorem 2.2, \( T(X) = X \). Hence, there is a parabolic subgroup \( P \) of \( G \) and a toric variety \( Y \) such that \( Y \) is a \( P \)-variety and \( X \) is \( G \)-equivariantly isomorphic to \( G \times P \). This can also be seen from [Pa0a, Example 1.13(2), Thm. 1.17, Example 1.19(3)]].

Keep notations as in the discussions above. In fact, we can read a lot of information of \( X \) from \( Y \) or inversely. Remark that \( Y \) is a \( T \)-horospherical variety which implies that theories and results on horospherical varieties can be applied to \( Y \). If \( \mu \) is a wall of \( F_X \), then it is also a wall of \( F_Y \). Denote by \( C^Y_{\mu} \) the \( T \)-stable irreducible curve on \( Y \) corresponding to \( \mu \). For any ray \( R = \mathbb{R}^+.e \) of \( F_X \), denote by \( D^Y_{\mu} \) or \( D^Y_{e} \) the corresponding \( T \)-stable prime divisor on \( Y \).

**Proposition 4.32.** Let \( X \) be a projective \( \mathbb{Q} \)-factorial horospherical \( G/H \)-embedding such that \( H \supseteq R_u(B), N_G(H) = P_I \), and \( T(X) = G \times P_I \). Then the following hold.

(i) \( T(X) \) is a projective \( \mathbb{Q} \)-factorial toroidal horospherical \( G/H \)-embedding and \( Y \) is a projective \( \mathbb{Q} \)-factorial toric variety.

(ii) The variety \( Y \) is locally factorial if and only if \( Y \) is smooth and only if \( T(X) \) is smooth.

(iii) If moreover \( Nef^T(X) = Psef^T(X) \), then \( Nef^T(Y) = Psef^T(Y) \).

**Proof.** (i) The conclusion that \( T(X) \) and \( Y \) are \( \mathbb{Q} \)-factorial follows from Proposition 3.2(ii).

Take any ray \( R \) in \( F_X \). By Corollary 4.21, there exists a unique prime divisor \( D_R \) such that \( \rho_X(\nu_{D_R}) \subseteq R \). Since \( \rho_Y(\nu_{D^Y_R}) \) is the primitive lattice point in \( R \), there is a positive integer \( d_R \) such that \( \rho_X(\nu_{D_R}) = d_R \rho_Y(\nu_{D^Y_R}) \).

Let \( \delta = \sum_{R \in \mathbb{D}(G/H) \setminus \mathbb{D}_X} n_{D}(\delta)D \) be an ample Cartier divisor on \( X \), where \( R \) runs over all the rays in \( F_X \). Denote by \( \delta^Y = \sum_{R \in \mathbb{D}(G/H) \setminus \mathbb{D}_X} n_{D}(\delta)D^Y_R \), where \( R \) runs over all the rays in \( F_X \).
By [Br98], Prop. 3.1, $\delta^Y$ is a Cartier divisor on $Y$. By the formula (1) on page 5 for any wall $\mu$ in $\mathcal{F}_Y$ and any $D \in \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$, $D \cdot C_\mu = 0$ and $\delta^Y \cdot C^Y_\mu = \delta \cdot C_\mu > 0$. Thus, $\delta^Y$ is an ample Cartier divisor on $Y$, i.e. $Y$ is projective.

Let $\delta'$ be an ample Cartier divisor on $G/P_1$. Then by [Br98], Prop. 3.1, Thm. 3.3, $\delta^Y + \pi^*\delta'$ is an ample Cartier divisor on $\mathcal{T}(X)$. Hence, $\mathcal{T}(X)$ is projective.

(iii) follows from Proposition 3.2(i) and [Pa09] Thm. 2.6.

(iii) By the formula (1) on page 5 for each wall $\mu$ of $\mathcal{F}_X$, $C^Y_\mu \in \text{Nef}_1(Y)$. Thus, $\text{Nef}_1(Y) = \text{Psef}_1(Y)$ and $\text{Nef}^{-1}(Y) = \text{Psef}^{-1}(Y)$.

**Lemma 4.33.** Let $X$ be a projective $\mathbb{Q}$-factorial horospherical $G/H$-embedding such that $\text{Nef}^1(X) = \text{Psef}^1(X)$. Then $\mathfrak{D}(G/H)\setminus \mathfrak{D}_X = \mathfrak{D}_0(G/H)$, i.e. for all $D \in \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$, $\rho(D) = 0$.

**Proof.** By Theorem 2.5 for each $G$-orbit $Y$ on $X$, $\mathcal{E}_Y$ is a strictly convex colored cone. Hence, for all $D \in \mathcal{Y}_X$, $\rho(D) \neq 0$. Therefore, $\mathfrak{D}_0(G/H) \subseteq \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$. Now assume that the inclusion is strict, then there is a divisor $D \in \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$ such that $\rho(D) \neq 0$.

Note that $\text{Supp}(\mathcal{F}_X) = \mathcal{V}(G/H) = (N_X)_Q$ by [Kn91] Thm. 4.2, Cor. 6.2. Thus, there is a closed $G$-orbit $Y$ such that $\rho(D) \in \mathcal{E}_Y$. The fact $D \in \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$ implies that $D \in \mathfrak{D}(G/H)\setminus \mathfrak{D}_Y$. Then there is a $B$-stable 1-cycle class $C_{D,Y}$ in $N_1(X)_\mathbb{R}$ defined as in the formula (2) on page 6.

Assume $\mathcal{E}_Y = (e_1, \ldots, e_r)$. Then $\rho(D) = \sum_{i=1}^r a_i e_i$, where all $a_i \geq 0$ and there is an $i_0$ such that $a_{i_0} > 0$.

By the definition of $\mathcal{E}_Y$, there is an irreducible $B$-stable divisor $D'$ such that $\rho(D') = b_{i_0} e_{i_0}$, where $b_{i_0} > 0$. Then by the formula (2) on page 5 the intersection number $D' \cdot C_{D,Y} = -\frac{a_{i_0}}{b_{i_0}} < 0$. Hence, $D' \in \text{Psef}^1(X)\setminus \text{Nef}^1(X)$. We get a contradiction.

Remark that the condition $\text{Nef}^1(X) = \text{Psef}^1(X)$ in Lemma 4.33 is necessary to guarantee $\mathfrak{D}(G/H)\setminus \mathfrak{D}_X = \mathfrak{D}_0(G/H)$. In general, the inclusion $\mathfrak{D}_0(G/H) \subseteq \mathfrak{D}(G/H)\setminus \mathfrak{D}_X$ is strict. For example, we have the following Proposition 4.34.

**Proposition 4.34.** Let $X$ be a smooth projective horospherical $G/H$-embedding such that $\text{rank}(X) = 1$ and $\mathfrak{D}_X \neq \emptyset$. Choose any $D \in \mathfrak{D}_X$, then $\rho(D) \neq 0$. There is a closed $G$-orbit $Z$ such that $\mathcal{E}_Z = Q^+ \rho(D) \in \mathcal{F}_X$. Let $\tilde{X} = Bl_Z(X)$ be the blowing-up of $X$ along $Z$. Then $\tilde{X}$ is a smooth projective horospherical $G/H$-embedding, and $\pi^{-1}_*(-1) \in \mathfrak{D}(G/H)\setminus (\mathfrak{D}_X \cup \mathfrak{D}_0(G/H))$, where $\pi : \tilde{X} \to X$ is the natural morphism, and $\pi^{-1}_*D$ is the strict transform of $D$.

**Proof.** By [Kn91] Thm. 4.2, Cor. 6.2, and Proposition 4.26(i), there are exactly three $G$-orbits $G/H, Z, Y$ on $X$, the two $G$-orbits $Z$ and $Y$ are complete, and $-\mathcal{E}_Z = \mathcal{E}_Z = Q^+ \rho(D)$. Hence, $Y$ and $Z$ are isomorphic to rational $G$-homogeneous spaces. In particular, $Z$ is smooth. Hence, $\tilde{X}$ is smooth. Since $Z$ is a closed $G$-stable subvariety on $X$, there is a natural $G$-action on $\tilde{X}$ such that $\pi$ is $G$-equivariant. Since $G^{-1}(G/H) = G/H$ is open in $\tilde{X}$, $\tilde{X}$ is a complete smooth horospherical $G/H$-embedding. Since $X$ is projective and $\pi$ is a projective morphism, $\tilde{X}$ is also projective.

Since $\pi$ is $G$-equivariant, $\pi^{-1}_*D$ is a prime $B$-stable divisor and it is not $G$-stable. Thus, $\pi^{-1}_*D \in \mathfrak{D}(G/H)$. Since $\pi$ is birational and $\pi$ maps $(\pi^{-1}_*D) \cap (G/H)$ isomorphically to $D \cap (G/H)$, we know that $\mathcal{E}(X)^{G/H} = \mathcal{E}(\tilde{X})^{G/H}$ and $\rho(\tilde{X}(\pi^{-1}_*D)) = \rho_X(D) \neq 0$, i.e. $\pi^{-1}_*D \notin \mathfrak{D}_0(G/H)$.

Moreover, since $\tilde{X}$ is the blowing-up of $X$ along $Z$, there are exactly three $G$-orbits $G/H, \pi^{-1}(Z)$ and $\pi^{-1}(Y)$ on $\tilde{X}$, where $\pi^{-1}(Z)$ and $\pi^{-1}(Y)$ are the inverse images of $Z$ and $Y$ respectively. By considering the morphism $\pi$, we know that $\pi^{-1}_*D$ contains neither $\pi^{-1}(Z)$ nor $\pi^{-1}(Y)$. Hence, $\pi^{-1}_*D \notin \mathfrak{D}_X$.
Proposition 4.35. Assume that for each $1 \leq i \leq m$, $X_i$ is a horospherical $G/H_i$-embedding such that $H_i \supseteq R_\alpha(B)$ and $N_G(H_i) = P_{I_i}$. Denote by $X = \prod_{i=1}^m X_i$, $H = \prod_{i=1}^m H_i$, and $N_G(H) = P_I$.

Then the following are equivalent:

(i) $X$ is a $G$-horospherical variety;
(ii) $X$ is a horospherical $G/H$-embedding;
(iii) the natural injective morphism $\pi: G/H \to \prod_{i=1}^m G/H_i$ is an open immersion;
(iv) $\pi$ is an isomorphism;
(v) $M_{G/H} = \bigoplus_{i=1}^m M_{G/H_i}$, and if $i \neq j$, then for any root $\alpha \in S\backslash I_i$ and any root $\beta \in S\backslash I_j$, $\alpha$ and $\beta$ lie in different connected components of the Dynkin diagram of $G$.

Before proving this proposition, we give a lemma to explain the conditions in it.

Lemma 4.36. Keep notations as in Proposition 4.35. Then the following hold.

(i) $I = \bigcap_{i=1}^m I_i$ and $P_I = \bigcap_{i=1}^m G/P_{I_i}$.

(ii) Let $\psi: G/P_I \to \prod_{i=1}^m G/P_{I_i}$ be the natural morphism. Then $\psi$ is an isomorphism if and only if for any $i \neq j$, any root $\alpha \in S\backslash I_i$ and any root $\beta \in S\backslash I_j$, $\alpha$ and $\beta$ lie in different connected components of the Dynkin diagram of $G$.

Proof. (i) Denote by $P' = \bigcap_{i=1}^m P_i$. Then $P_{I_i} = \bigcap_{i=1}^m P_{I_i}$. Let $M' = \sum_{i=1}^m M_{G/H_i} \subseteq \chi(P_{I'})$, and $H' = \ker P_{I'}, M'$. By Remark 4.18(iii)(ii) $P_{I'} = \bigcap_{i=1}^m TH_i \supseteq TH = P_I$, and $M_{G/H_i} = \chi(P_{I_i}/H_i) \subseteq \chi(P_{I}/H) = M_{G/H}$. Hence, $M' \subseteq M_{G/H}$, $H' = \ker P_{I_i}, M' \supseteq \ker P_{I_i}, M = H$ and $H' = \ker P_{I_i}, M' \subseteq \ker P_{I_i}, M_{G/H_i} = H_i$. The later inclusion implies that $H' \subseteq \bigcap_{i=1}^m H_i = H$. Thus, $H' = H$, and $P_{I'} = TH' = TH = P_I$, i.e. $I = I' = \bigcap_{i=1}^m I_i$ and $P_I = \bigcap_{i=1}^m G/P_{I_i}$.

(ii) By (i), $\psi$ is always injective. Since $G/P_I$ is complete, $\psi$ is an isomorphism if and only if the natural morphism $\Psi: G \to \prod_{i=1}^m G/P_{I_i}$ is dominant. Let $R^- \cap R_{-\alpha}^-$ be the set of negative roots of $G$, and $R_{-\alpha}$ be the set of negative roots generated by $-\alpha$ where $\alpha \in I$. Note that the tangent space $T_{\alpha}$ of $G/P_{I_i}$ at the origin $P_{I_i}/P_{I_i}$ is naturally isomorphic to $\bigoplus_{\alpha \in R^- \cap R_{-\alpha}^-} g_\alpha$, and the tangent space $T_{\alpha}G$ of $G$ at the origin is naturally isomorphic to its Lie algebra $g$. Hence, $\Psi$ is dominant if and only if the map between the tangent spaces at the origins $T_{\Psi}: T_G \to \bigoplus_{i=1}^m T_{I_i}$ is surjective. The later holds if and only if for any $i \neq j$, $(R^- \cap R_{-\alpha}) \cup (R^- \cap R_{-\alpha}) = \emptyset$, i.e. $R^- = R_{-\alpha}^- \cup R_{-\alpha}^-$. By checking the Dynkin diagrams of simple groups case by case, we know that this is equivalent to the fact that for any $i \neq j$, any root $\alpha \in S\backslash I_i$ and any root $\beta \in S\backslash I_j$, $\alpha$ and $\beta$ lie in different connected components of the Dynkin diagram of $G$.

Proof of Proposition 4.35. (ii) $\Rightarrow$ (iii) By the definition of spherical varieties, $G/H$ is the unique open $G$-orbit on $X$. On the other hand, $\prod_{i=1}^m G/H_i$ is a $G$-stable open subset on $X$. Hence, $\pi$ is an open immersion.
(iii) ⇒ (iv) Consider the following commutative diagram (*):

\[
\begin{array}{ccc}
G/H & \overset{\pi}{\longrightarrow} & \prod_{i=1}^{m} G/H_i \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
G/P_I & \overset{\psi}{\longrightarrow} & \prod_{i=1}^{m} G/P_{I_i}.
\end{array}
\]

Since \( \pi \) is dominant, \( \phi_2 \) is surjective and \( \psi \circ \phi_1 = \phi_2 \circ \pi \), we know that \( \psi \) is dominant, which implies that \( \psi \) is surjective, since \( G/P_I \) is complete. On the other hand, \( \psi \) is injective, since \( P_I = \bigcap_{i=1}^{m} P_{I_i} \). Hence, \( \psi \) is an isomorphism. We identify \( G/P_I \) with \( \prod_{i=1}^{m} G/P_{I_i} \) in the following.

Now we consider the fibers of \( \phi_1 \) and \( \phi_2 \). Denote by \( x = P_I/P_I \in G/P_I = \prod_{i=1}^{m} G/P_{I_i} \).

Since \( \pi \) is an open immersion, the morphism \( \phi_1^{-1}(x) \xrightarrow{\sim} \phi_2^{-1}(x) \) is an open immersion. Thus, the torus \( P_I/H \) is a subgroup of the torus \( \prod_{i=1}^{m} P_{I_i}/H_i \) with finite index, which implies that \( P_I/H = \prod_{i=1}^{m} P_{I_i}/H_i \). Thus, for any point \( x' \in G/P_I \), \( \phi_1^{-1}(x') \xrightarrow{\sim} \phi_2^{-1}(x') \) is an isomorphism.

Hence, \( \pi \) is an isomorphism.

(iv) ⇒ (ii) Since \( B^{-}H \) is an open subset of \( G \), \( \prod_{i=1}^{m} G/H_i \) has an open \( B^{-} \)-orbit. By Proposition 4.2(i)(iii), \( X \) is a horospherical \( G/H \)-embedding.

(ii) ⇒ (i) is trivial.

(i) ⇒ (iv) Let \( \hat{x} \in X \) be a point such that \( G_x \supseteq R_a(B) \). Denote by \( p_i : X \to X_i \) the \( i \)-th projection, \( \hat{H} = G_{\hat{x}} \), \( \hat{x}_i = p_i(\hat{x}) \), and \( H_i = G_{\hat{x}_i} \). Since \( G \cdot \hat{x} \) is an open \( G \)-orbit on \( X \), \( p_i(G \cdot \hat{x}) = G/H_i \). In particular, \( \hat{x}_i \in G/H_i \). Now we consider the composition \( G \cdot \hat{x} \xrightarrow{\phi_1} G/H_i \xrightarrow{\phi_2} G/P_I \). By Lemma 4.13, \( \hat{x}_i \in \phi_1^{-1}(P_{I_i}/P_{I_i}) \). Thus, \( \hat{H}_i \subseteq P_{I_i} \), and there is an element \( g_i \in P_{I_i} \) such that \( g_i x_i = \hat{x}_i \), where \( x_i = H_i/H \in G/H_i \). Hence, \( \hat{H}_i = g_i H_i g_i^{-1} \) and \( N_G(H_i) = g_i N_G(H) g_i^{-1} = P_{I_i} \).

Denote by \( P_{I_i} = N_G(H_i) \). Note that \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_m) \) in \( X = \prod_{i=1}^{m} X_i \). Thus, \( \hat{H} = \bigcap_{i=1}^{m} H_i \).

Apply the equivalence of the conclusions (ii) and (iv) in this theorem to \( \hat{H} \) and \( \hat{H}_i \) instead of \( H \) and \( H_i \), then we get that \( \hat{\pi} : G/\hat{H} = G \cdot \hat{x} \to \prod_{i=1}^{m} G/\hat{H}_i \) is an isomorphism. Note that \( G/\hat{H}_i = G \cdot \hat{x}_i = G/H_i \), and \( G/H \subseteq \prod_{i=1}^{m} G/H_i \). Thus, \( G/H \subseteq G \cdot \hat{x} \). Since \( G \cdot \hat{x} \) is \( G \)-homogeneous, \( G/H = G \cdot \hat{x} \). Hence, the natural morphism \( \pi \) is an isomorphism.

(iv) ⇒ (v) Recall the first part of the proof of (iii) ⇒ (iv), we know that \( \psi \) is an isomorphism. By Lemma 4.36(ii), the second assertion of (v) holds. Since \( \pi \) is an isomorphism, the corresponding fibers \( P_I/H \) and \( \prod_{i=1}^{m} P_{I_i}/H_i \) are isomorphic to each other. By Remark 4.13(ii),

\[
M_{G/H} = \chi(P_I/H) \cong \bigoplus_{i=1}^{m} \chi(P_{I_i}/H_i) = \bigoplus_{i=1}^{m} M_{G/H_i}.
\]

(v) ⇒ (iv) Consider the commutative diagram (*) in the proof of (iii) ⇒ (iv). By Lemma 4.36(ii), \( \psi : G/P_I \to \prod_{i=1}^{m} G/P_{I_i} \) is an isomorphism. We identify them and still denote by \( x = P_I/P_I \in G/P_I \). Then \( \phi_1^{-1}(x) = P_I/H \) and \( \phi_2^{-1}(x) = \prod_{i=1}^{m} P_{I_i}/H_i \) are two tori. Since
\[
\chi(P_i/H) = M_{G/H} = \bigoplus_{i=1}^m M_{G/H_i} = \bigoplus_{i=1}^m \chi(P_i/H_i), \quad \text{we get that } P_i/H \cong \prod_{i=1}^m P_i/H_i. \text{ Thus, for any point } x' \in G/P_i, \phi_1^{-1}(x') \rightarrow \phi_2^{-1}(x') \text{ is an isomorphism. Hence, } \pi \text{ is an isomorphism.} \]

**Definition 4.37.** Let \( G/H \) be a homogeneous spherical variety, and \( F \) be a colored fan in \( (N_{G/H})_Q \). We say that \( F \) is a product of \( F_1, \ldots, F_m \) and denote by \( F_X = \prod_{i=1}^m F_i \), if there is a decomposition \( N_{G/H} = \bigoplus_{i=1}^m N_i \) of abelian groups such that

(i) \( F_i = \{(C, \mathcal{D}) \in F \mid C \subseteq (N_i)_Q \} \);

(ii) for any \( C^i = (C_i, \mathcal{D}_i) \in F_i, \prod_{i=1}^m C^i = \bigcup F \), where \( \prod_{i=1}^m C^i = (\prod C_i) \cup \mathcal{D}_i \), and every colored cone in \( F \) is of this form.

**Proposition 4.38.** Assume that for each \( 1 \leq i \leq m, X_i \) is a horospherical \( G/H_i \)-embedding such that \( H_i \supseteq R_\alpha(B) \). If \( X = \prod_{i=1}^m X_i \) is \( G \)-horospherical, then \( F_X = \prod_{i=1}^m F_{X_i} \).

**Proof.** Let \( H = \bigcap_{i=1}^m H_i \). Denote by \( P_i = N_{G/H_i} \) and \( P_i = N_{G/H} \). By Proposition 4.35(i)(iv)(v),

\[
M_{G/H} = \bigoplus_{i=1}^m M_{G/H_i}, \quad N_{G/H} = \bigoplus_{i=1}^m N_{G/H_i} \quad \text{and} \quad G/H = \prod_{i=1}^m G/H_i. \]

Thus, for any \( D \in \mathcal{D}(G/H_i) \),

\[
\pi^{-1}(D) \in \mathcal{D}(G/H) \quad \text{and} \quad \rho_{G/H}(\nu_{D}) = \rho_{G/H_i}(\nu_{D}) \in (N_{G/H_i})_Q \subseteq (N_{G/H})_Q, \]

where \( \pi_i : X \rightarrow X_i \) is the \( i \)-th projection. Note that by [Kn91 Cor. 6.2], \( \mathcal{V}(G/H) = (N_{G/H})_Q \). So \( \prod_{i=1}^m F_{X_i} \) is a colored fan in \( (N_{G/H})_Q \).

Let \( X' \) be a horospherical \( G/H \)-embedding such that \( F_{X'} = \prod_{i=1}^m F_{X_i} \). By Theorem 2.7, there is a \( G \)-equivariant morphism \( \Phi_j : X' \rightarrow X_j \) extending the natural morphism \( G/H \rightarrow G/H_i \), and inducing the morphism of colored fans \( \phi_j : F_{X'} \rightarrow \prod_{i=1}^m F_{X_i} \). Define \( \Phi : X' \rightarrow X = \prod_{i=1}^m X_i, x \mapsto (\Phi_1(x), \ldots, \Phi_m(x)) \). Denote by \( \phi = \prod_{i=1}^m \phi_i : F_{X'} \rightarrow \prod_{i=1}^m F_{X_i} \rightarrow F_X \) the morphism of colored fans corresponding to \( \Phi \).

Step 2. For the moment, we assume that all \( X_i \) are complete. By [Kn91 Thm. 4.2, Cor. 6.2], \( X' \) is also complete, which implies that the birational morphism \( \Phi \) is surjective. By Theorem 2.7, \( \pi_i \) corresponds to a morphism of colored fans \( \psi_i : F_X \rightarrow F_{X_i} \). Consider the morphism of colored fans \( \psi = \prod_{i=1}^m \psi_i : F_X \rightarrow \prod_{i=1}^m F_{X_i} = F_{X'}, C_Y \rightarrow \prod_{i=1}^m \psi_i(C_Y) \) and the identity \( G/H = \prod_{i=1}^m G/H_i \). By Theorem 2.7, they correspond to a \( G \)-equivariant morphism \( \Psi : X \rightarrow X' \).

Take any \( Y' \in S_{X'} \). Let \( Y = \Phi(Y') \) and \( Y_i = \Phi_i(Y') = \pi_i \circ \Phi(Y') = \pi_i(Y) \). By the construction of \( F_{X'}, C_{Y'} = \prod_{i=1}^m C_{Y_i} \). By Theorem 2.7, \( \phi(C_{Y'}) \subseteq C_{Y'} \) and \( \psi_i(C_{Y'}) \subseteq C_{Y_i} \). Thus, \( \psi \circ \phi(C_{Y'}) \subseteq \prod_{i=1}^m C_{Y_i} = C_{Y} \). Since \( \phi \) and \( \psi \) are compatible with the isomorphisms of vector spaces \( (N_{X'})_Q \rightarrow (N_X)_Q \) and \( (N_X)_Q \rightarrow (N_{X'})_Q \) respectively, we get that \( \psi \circ \phi(C_{Y'}) = C_{Y'} \). By Proposition 4.20, \( \rho_X((\psi \circ \phi)^{-1}(\rho_Y(C_{Y'}))) = \rho_Y(C_{Y'}) \subseteq \rho_X((\psi \circ \phi)^{-1}(\rho_Y(C_{Y'}))) = D_{X'} \cap \rho_X((\psi \circ \phi)^{-1}(\rho_Y(C_{Y'}))) = D_{Y'} \).

Thus, \( \psi \circ \phi(C_{Y'}) = C_{Y'} \). Hence, \( \psi \circ \phi = i_{Y'} \). By Theorem 2.7 and Theorem 2.3, \( \Psi \circ \Phi = i_{X'} \). In particular, \( \Phi \) is an injective morphism. Thus, the birational surjective \( G \)-equivariant morphism \( \Phi \) is an isomorphism and for each \( Y_i \in S_{X_i} \), \( \prod_{i=1}^m Y_i \) is itself a \( G \)-orbit and \( C_{Y_i} = \prod_{i=1}^m C_{Y_i} = \prod_{i=1}^m C_{Y_i} \).

Then the general case when \( X_i \) may not be complete follows from this fact and Theorem 2.5 by taking a \( G \)-equivariant normal completion for each \( X_i \). \( \Box \)
Remark 4.39. If \( X = \prod_{i=1}^{m} X_i \) is only assumed to be \( G \)-spherical in Proposition 4.38, then maybe \( \mathbb{F}_X \neq \prod_{i=1}^{m} \mathbb{F}_{X_i} \). We can consider an example as follows. Let \( X = X_1 \times X_2 \), where \( X_1 = G/P_{S \setminus \{\alpha_1\}} \). Assume for the moment that \( X \) is \( G \)-spherical. Then \( \mathbb{F}_X = \mathbb{F}_{X_1} \times \mathbb{F}_{X_2} \) if and only if \( \text{rank}(X) = 0 \) if and only if \( X \) is \( G \)-homogeneous is and only if \( X \) is \( G \)-equivariantly isomorphic to \( G/P_{S \setminus \{\alpha_1, \alpha_2\}} \). By Lemma 4.40(ii), the last assertion is equivalent to the fact that \( \alpha_1 \) and \( \alpha_2 \) lie in different connected component of the Dynkin diagram of \( G \). In particular, if \( G \) is a simple group, then \( \mathbb{F}_X \neq \mathbb{F}_{X_1} \times \mathbb{F}_{X_2} \). However, by [L95], there exist such an example that \( G \) is simple and simply connected, and \( G/P_{S \setminus \{\alpha_1\}} \times G/P_{S \setminus \{\alpha_2\}} \) is \( G \)-spherical, (see Table I there).

Proposition 4.40. Assume that for each \( 1 \leq i \leq m \), \( X_i \) is a complete \( \mathbb{Q} \)-factorial horospherical \( G/H_i \)-embedding. If \( X = \prod_{i=1}^{m} X_i \) is \( G \)-horospherical, then the following hold.

(i) \( \rho(X) = \sum_{i=1}^{m} \rho(X_i) \), where \( \rho(X) \) (resp. \( \rho(X_i) \)) is the Picard number of \( X \) (resp. \( X_i \)).

(ii) \( \text{Nef}^1(X) = \text{Nef}^1(X_i) \) if and only if \( \text{Nef}^1(X) = \text{Nef}^1(X_i) \) for all \( 1 \leq i \leq m \).

Proof. (i) By Proposition 4.38 \( \mathbb{F}_X = \prod_{i=1}^{m} \mathbb{F}_{X_i} \), which implies that \( S_{X,G} = \big\{ \prod_{i=1}^{m} Y_i \mid Y_i \in S_{X_i,G} \big\} \).

Hence, there is a bijection between \( \mathcal{V}_X \) and the disjoint union \( \bigcup_{i=1}^{m} \mathcal{V}_{X_i} \). By Remark 4.15(i)(ii), we can assume that \( H_i \supseteq R_u(B) \) and \( N_G(H_i) = P_{H_i} \). Denote by \( H = \bigcap_{i=1}^{m} H_i \) and \( P = N_G(H) \).

By Lemma 4.36(i), \( S \setminus I = \bigcup_{i=1}^{m} S \setminus I_i \). By Proposition 4.35(i)(v), \( M_{G/H} = \bigoplus_{i=1}^{m} M_{G/H_i} \), and if \( i \neq j \), then \( (S \setminus I_i) \cap (S \setminus I_j) = \emptyset \). Hence, by Proposition 3.1(i) and Remark 4.15(iii), \( \rho(X) = \sum_{i=1}^{m} \rho(X_i) \).

(ii) follows from Remark 4.41.

Now we are able to show the main theorem in this part, which is a part of the characterization of the smooth projective horospherical varieties whose effective divisors are nef.

Theorem 4.41. Let \( X \) be a smooth projective horospherical \( G/H \)-embedding such that \( H \supseteq R_u(B) \), \( \mathcal{D}_0(G/H) = \emptyset \) and \( \text{Nef}^1(X) = \text{Nef}^1(X) \). Then there is a \( G \)-equivariant isomorphism \( \Phi : X \to \prod_{i=1}^{m} X_i \), where \( \rho(X) \) is the Picard number of \( X \), and each \( X_i \) is a smooth projective horospherical \( G/H_i \)-embedding of Picard number one. Moreover, the natural morphism \( \pi : G/H \to \prod_{i=1}^{m} G/H_i \) is an isomorphism, \( \Phi \) extends \( \pi \), and \( H = \bigcap_{i=1}^{m} H_i \).

Proof. Denote by \( T(X) = G \times \mathbb{P}^{r_1} Y \) the corresponding toroidal horospherical variety. Then by Proposition 4.32 \( Y \) is a smooth projective toric variety such that \( \text{Nef}^1(Y) = \text{Nef}^1(Y) \). By Theorem 4.13 \( Y \cong \mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_{\rho(Y)}} \), where \( \rho(Y) \) is the Picard number of \( Y \) and \( \sum_{i=1}^{r_i} r_i \) equals to the rank \( r \) of \( G/H \).

Denote by \( Y_i = \mathbb{P}^{r_i} \), and \( N_i = 	ext{supp}(\mathbb{F}_{Y_i}) \cap N_X \). Then \( N_X = \bigoplus_{i=1}^{m} N_i \). For any cone \( \mathcal{C} \in \mathcal{F}_Y \), define \( \Phi(\mathcal{C}) = \{ D \in \mathcal{D}_X \mid \rho(D) \in \mathcal{C} \} \). By Proposition 4.20 if \( \mathcal{C} \in \mathcal{F}_{Y_i} \), then \( (\mathcal{C}, \Phi(\mathcal{C})) \in \mathcal{F}_X \).

Let \( \mathcal{F}_i = \{ (\mathcal{C}, \Phi(\mathcal{C})) \mid \mathcal{C} \in \mathcal{F}_{Y_i} \} \). The fact \( \mathcal{F}_Y = \prod_{i=1}^{\rho(Y)} \mathcal{F}_i \) and Proposition 4.20 imply that \( \mathbb{F}_X = \prod_{i=1}^{\rho(Y)} \mathbb{F}_i \).
By Lemma 4.33, \( \mathcal{D}(G/H) = \mathcal{D}_X \). Hence, by Lemma 4.42 in the following, there exists a smooth projective horospherical \( G/H_i \)-embedding \( X_i \) for each \( 1 \leq i \leq \rho(Y) \) such that \( F_{X_i} = F_i \), \( \mathcal{D}(G/H_i) = \mathcal{D}_{X_i} \), \( H = \prod_{i=1}^{\rho(Y)} H_i \), \( X \) is \( G \)-equivariantly isomorphic to \( \prod_{i=1}^{\rho(Y)} X_i \), and this isomorphism extends the isomorphism \( G/H \to \prod_{i=1}^{\rho(Y)} G/H_i \). By Proposition 4.22 and Corollary 4.21 the Picard numbers \( \rho(X) = \rho(Y) \) and \( \rho(X_i) = \rho(Y_i) = 1 \). The conclusion follows.

**Lemma 4.42.** Let \( X \) be a smooth horospherical \( G/H \)-embedding such that \( H \supseteq R_n(B) \), \( N_G(H) = P_I \), \( \mathcal{D}(G/H) = \mathcal{D}_X \) and \( F_X = \prod_{i=1}^{m} F_i \). Then there exists a smooth horospherical \( G/H_i \)-embedding \( X_i \) for each \( 1 \leq i \leq m \) such that \( H_i \supseteq H \), \( \mathcal{D}(G/H_i) = \mathcal{D}_{X_i} \), \( F_{X_i} = F_i \), \( H = \prod_{i=1}^{m} H_i \), the natural morphism \( G/H \to \prod_{i=1}^{m} G/H_i \) is an isomorphism, and this isomorphism can be extended to a \( G \)-equivariant isomorphism \( X \to \prod_{i=1}^{m} X_i \).

**Proof.** Step 1. In this step, we will construct a horospherical \( G/H_i \)-embedding \( X_i \) for each \( i \) such that \( F_{X_i} = F_i \).

Let \( \mathcal{D}_i = \{ D \in \mathcal{D}_X \mid \rho(D) \in (N_i)_Q \} \) and \( I_i = S \setminus \mathcal{D}_i \). The fact \( F_X = \prod_{i=1}^{m} F_i \) implies that \( \mathcal{D}_X \) is the disjoint union of those \( \mathcal{D}_i \), where \( 1 \leq i \leq m \). Thus, \( \bigcap_{i=1}^{m} I_i = I \), \( \bigcap_{i=1}^{m} P_{I_i} = P_I \) and if \( i_1 \neq i_2 \), then \( I_{i_1} \cap I_{i_2} = S \).

Let \( M_i = \text{Hom}(N_i, \mathbb{Z}) \). Then \( M_{G/H} = \bigoplus_{i=1}^{m} M_i \). Recall that \( \chi(P_i) = \{ \chi \in \chi(B) \mid \langle \chi, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I \} \) and \( \chi_{(P_i)} = \{ \chi \in \chi(B) \mid \langle \chi, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I_i \} \). Take any \( \chi_1 \in M_i \), then by Remark 4.18(ii), \( \chi_1 \in M_{G/H} \subseteq \chi(P_i) \). Moreover, for any \( \alpha_1 \in I_i \setminus I \), there exists some \( j \neq i \) such that \( \alpha_1 \in S \setminus I_j \). Thus, \( \rho(\nu_{\alpha_1}) \in (N_j)_Q \subseteq M_i \cap (N_X)_Q \). In particular, \( \langle \chi_1, \alpha_1^\vee \rangle = 0 \) i.e. \( \chi_1 \in \chi(P_i) \). Hence, \( M_i \subseteq \chi(P_i) \cap M_{G/H} \). Let \( H_i = \text{Ker} P_i \). Then \( H_i \supseteq \text{Ker} P_i M_{G/H} = H \supseteq R_n(B) \). By Remark 4.18(ii) and the definition of \( H_i \), we get that \( N_G(H_i) = P_{I_i} \), \( M_{G/H_i} = \chi(P_i) \) and \( H_i = \text{Ker} P_i M_{G/H_i} \). Hence, \( M_i \) is a subgroup of \( M_{G/H} \) with a finite index. So \( (M_i)_Q = (M_{G/H})_Q \) as subspaces of \( (M_{G/H})_Q \). By the decomposition \( M_{G/H} = \bigoplus_{i=1}^{m} M_i \), we can know that \( M_{G/H_i} \subseteq (M_i)_Q \cap M_{G/H} = M_i \). Hence, \( M_{G/H_i} = M_i \) and \( N_{G/H_i} = N_i \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
G/H & \xrightarrow{\phi_i} & G/H_i \\
\downarrow & & \downarrow \\
G/P_I & \xrightarrow{\phi_I} & G/P_{I_i}.
\end{array}
\]

By Remark 4.18(iii), \( \mathcal{D}(G/H_i) \) can be identified with \( S \setminus I_i \). Denote by \( D_{\alpha,i} \) the \( B \)-stable divisor on \( G/H_i \) corresponding to the simple root \( \alpha \). Thus, \( \phi_i^{-1}(D_{\alpha,i}) = D_{\alpha} \in \mathcal{D}(G/H) = S \setminus I \) and \( \rho_G/H(\nu_{\alpha,i}) = \chi_{(P_i)}(\nu_{\alpha,i}) \in (N_G(H_i))_Q \). Thus, \( \phi_i^{-1} : \mathcal{D}(G/H_i) \to \mathcal{D}_{X_i}, D \mapsto \phi_i^{-1}(D) \) is a well-defined bijection. Hence, \( F_i \) is a colored fan in \( (N_i)_Q = (N_{G/H_i})_Q \). By Theorem 2.5, there exists a horospherical \( G/H_i \)-embedding \( X_i \) such that \( F_{X_i} = F_i \).

Step 2. Claim: \( H = \bigcap_{i=1}^{m} H_i \) and \( \prod_{i=1}^{m} X_i \) is a horospherical \( G/H \)-embedding.
We know from Remark 4.18(ii) that \( \prod_{i=1}^{m} H_i = \prod_{i=1}^{m} (\text{Ker} \, p_i \, M_{G/H_i}) = \text{Ker} \, \prod_{i=1}^{m} (\bigcup_{i=1}^{m} M_{G/H_i}) = H. \)

In fact, on each connected component \( \Gamma \) of the Dynkin diagram of \( G \), there is at most one \( 1 \leq i \leq m \) such that \( S \setminus I_i \cap \Gamma \neq \emptyset \). Otherwise, assume that \( \alpha_1, \alpha_2 \in \Gamma \) are two simple roots such that \( \alpha_1 \in S \setminus I_i, \alpha_2 \in S \setminus I_i, i \neq i_2 \) and there is no simple roots in \( S \setminus I_i \) between \( \alpha_1 \) and \( \alpha_2 \) on \( \Gamma \). Since \( S \setminus I_i \) is bijectively corresponding to \( \mathcal{D}_X \), for each \( i \), there are maximal dimensional colored cones \((\mathcal{C}_1, \mathcal{D}_1) \subset \mathbb{F}_{X_1} \), \((\mathcal{C}_2, \mathcal{D}_2) \subset \mathbb{F}_{X_2} \) such that \( D_{\alpha_1} \in \mathcal{D}_1, D_{\alpha_2} \in \mathcal{D}_2 \). The fact \( \mathbb{F}_X = \prod_{i} \mathbb{F}_{X_i} \) implies that there is a maximal dimensional colored cone \((\mathcal{C}, \mathcal{D}) \subset \mathbb{F}_X \) such that \((\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_2, \mathcal{D}_2) \) are colored faces of \((\mathcal{C}, \mathcal{D}) \subset \mathbb{F}_X \) under the natural inclusions \( \mathbb{F}_{X_i} \subset \mathbb{F}_X \). This is contradicted with [Pa06 Thm. 2.6]. Then by Proposition 4.35(ii)(iv)(v), the claim holds, and the natural morphism \( \pi : G/H \to \prod_{i=1}^{m} G/H_i \) is an isomorphism.

By Proposition 4.38 and Theorem 2.5 \( X \) is \( G \)-equivariantly isomorphic to \( \prod_{i=1}^{m} X_i \), and the isomorphism extends \( \pi \).

**Corollary 4.43.** Let \( X \) be a smooth projective horospherical \( G/H \)-embedding such that \( \text{Nef}^1(X) = \text{Psef}^1(X) \). Then there exists a \( G \)-equivariant morphism \( \pi : X \to G/P_{S \setminus \mathcal{D}_0(G/H)} \) such that each fiber is isomorphic to the product of some smooth projective \( L_0 \)-horospherical varieties of Picard number one, where \( L_0 = P_{S \setminus \mathcal{D}_0(G/H)} \cap P_{S \setminus \mathcal{D}_0(G/H)} \).

**Proof.** By Remark 4.18(v), we can assume that \( H \supseteq R_0(B^-) \). Let \( D_0 = \sum_{D \in \mathcal{D}_0(G/H)} D \). Then by Theorem 4.35(i) and Theorem 4.29(i), \( D_0 \) induces a \( G \)-equivariant morphism \( \pi_0 : X \to G/P_{S \setminus \mathcal{D}_0(G/H)} \) such that \( G \pi_0 = P_{S \setminus \mathcal{D}_0(G/H)} \). Let \( \pi_0^{-1}(x_0), L_0 = P_{S \setminus \mathcal{D}_0(G/H)} \cap P_{S \setminus \mathcal{D}_0(G/H)} \) and \( X_0 = \pi_0^{-1}(x_0) \). Then by Lemma 4.10(f) and Theorem 4.29(i), \( X_0 \) is a complete horospherical \( L_0/H_0 \)-embedding such that \( \mathcal{D}_0(L_0/H_0) = \emptyset \) and \( H_0 \supseteq R_0(B^-) \), where \( B_0 = B \cap L_0 \). By Theorem 4.35(ii)(iii), \( X_0 \) is a smooth projective variety such that \( \text{Nef}^1(X_0) = \text{Psef}^1(X_0) \). By Theorem 4.41 \( X_0 \) is isomorphic to the product of some smooth projective \( L_0 \)-horospherical varieties of Picard number one. Note that different fibers of \( \pi_0 \) are isomorphic to each other. The conclusion follows.

**Remark 4.44.** The smooth projective \( G \)-horospherical varieties of Picard number one have been classified in [Pa09]. Thus, by Theorem 4.37 Lemma 4.10 Theorem 4.29 Theorem 4.30 Proposition 4.30 and Theorem 3.14 we can give a complete characterization of the smooth projective \( G \)-horospherical varieties whose effective divisors are nef.

**Example 4.45.** Let \( M \) be a complex vector space of dimension \( m \geq 5 \), and \( L, E \) be subspaces such that \( \dim_{\mathbb{C}}(L) = 1, \dim_{\mathbb{C}}(E) = m + 1 \) and \( M = L \oplus E \). Let \( X \) be the set of pairs \((W, V) \in \text{Gr}(k-1, E) \times \text{Gr}(k+1, M) \) such that \( W \subseteq E \cap V \), where \( 2 \leq k \leq m - 1 \). Associate \( X \) with the reduced closed subvariety structure. Take \( e_0 \in L \setminus \{0\} \) and \( \{e_1, e_2, \ldots, e_{m+1}\} \) to be a basis of \( E \). Let \( G = SL_{m+1}(\mathbb{C}) \), consider the natural \( G \)-action on \( E \) and let \( G \) acts trivially on \( L \). Thus, \( X \) is naturally a \( G \)-variety. This variety will help us to understand the characterization of smooth projective horospherical varieties whose effective divisors are nef. More precisely, we have the following Proposition 4.46.

**Proposition 4.46.** Keep notations as in Example 4.45. Then the following hold.

(i) \( X \) is a smooth projective \( G \)-horospherical variety of Picard number two such that \( \text{Nef}^1(X) = \text{Psef}^1(X) \).
(ii) $X$ is not homogeneous under the action of $Aut^0(X)$, where $Aut^0(X)$ is the connected component of the automorphism group of $X$ containing the identity.

(iii) $X$ is not isomorphic to a nontrivial product of two varieties.

(iv) Let $D_0 = \sum_{D \in \Delta_0(G/H)} D$, where $H = G_x \supseteq R_u(B^-)$ for some point $x$ in the open $B$-orbit. Then $D_0$ induces a $G$-equivariant morphism $\pi_0 : X \rightarrow G/P_0^-$. Let $F$ be a fiber of $\pi_0$, then $X$ is not isomorphic to $G/P_0^- \times F$.

Keep notations as above. Take $x_i = (W_i, V_i)$ for $i = 0, 1, 2$, where $w_0 = e_1 \wedge \cdots \wedge e_k \wedge (e_0 + e_{k+1})$, $v_0 = e_1 \wedge \cdots \wedge e_k \wedge e_0$, and $v_2 = e_1 \wedge \cdots \wedge e_k \wedge e_{k+1}$ are the corresponding representatives under the Plücker coordinates. Let $X_i = G \cdot x_i$ and $H_i = G_{x_i}$. Let $S = \{\alpha_1, \ldots, \alpha_m\}$, where $\alpha_i$ is the $i$-th simple root of $G$ by the standard notations. Let $\omega_i$ be the $i$-th fundamental dominant weight of $G$.

Lemma 4.47. Keep the notations as above, then

(i) $X$ is the disjoint union of $X_0, X_1$ and $X_2$;

(ii) the isotropy groups $H_0 = Ker_{PS_{\{\alpha_k\}}(\omega_k - \omega_{k+1})}$, $H_1 = PS_{\{\alpha_{k-1}, \alpha_k\}}$, and $H_2 = PS_{\{\alpha_{k-1}, \alpha_{k+1}\}}$.

(iii) $X$ is a projective horospherical $G/H_0$-embedding.

Proof. Take any point $x = (W, V) \in X$, where $w = e_1' \wedge \cdots \wedge e_{k-1}'$ and $v = e_1' \wedge \cdots \wedge e_{k-1}' \wedge e_k' \wedge e_{k+1}'$ are the corresponding Plücker coordinates. If $V \subseteq E$, then there is an element $g \in G$ such that $ge_i = \lambda_i e'_i$, $\lambda_i \in C^*$, $i = 1, \ldots, k$, hence $x \in X_0$. If $L \subseteq V$, then we can assume that $e'_i \in E$ and $e_{k+1}' = e_0$. There is an element $g \in G$ such that $ge_i = \lambda_i e'_i$, $\lambda_i \in C^*$, $i = 1, \ldots, k$, hence $x \in X_1$.

If $L \nsubseteq V$ and $V \nsubseteq E$, then we can assume that $e'_k \in E$ and $e_{k+1}' = e_0 + t \tilde{e}_{k+1}$, where $t \in C^*$ and $\tilde{e}_{k+1} \in E$. There is an element $g' \in G$ such that $g'e_i = \lambda_i e'_i$, $\lambda_i \in C^*$, $i = 1, \ldots, k$. This implies that there is a $g \in G$ such that $ge_i = \lambda_i e'_i$, $\lambda_i \in C^*$, $i = 1, \ldots, k - 1$. There is an element $g' \in G$ such that $g'e_i = \lambda_i e'_i$, $\lambda_i \in C^*$, $i = 1, \ldots, k$. This implies $x \in X_0$. This shows that $X = X_0 \cup X_1 \cup X_2$.

Take any element $h \in G$. Then $h \in H_1$ if and only if $h v_0 = \mu v_0$ for some $\mu \in C^*$ and $h v_2 = \mu v_2$ for some $\mu \in C^*$ and $h v_0 = \mu v_0$ for some $\mu \in C^*$ if and only if $h \in P_{S_{\{\alpha_{k-1}\}}}$. On the other hand, $v_0 = v_1 + v_2$. And $h v_1 = \mu v_1$ for some $\mu \in C^*$ if and only if $h \in P_{S_{\{\alpha_k\}}}$. What's more, in this case, $\mu_1 = \omega_k(h)$. Similarly, $h v_2 = \mu v_2$ for some $\mu_2 \in C^*$ if and only if $h \in P_{S_{\{\alpha_{k+1}\}}}$. And in this case, $\mu_2 = \omega_k(h)$. Thus, $h v_0 = \mu v_0$ for some $\mu \in C^*$ if and only if $h v_1 = \mu v_1$, $h v_2 = \mu v_2$ for some $\mu_1 = \mu_2 \in C^*$, i.e. if and only if $h \in Ker_{P_{S_{\{\alpha_k\}}}}(\omega_k - \omega_{k+1})$. This implies (ii) and the fact $X_i \cap X_j = \emptyset$ if $i \neq j$, i.e. (i) holds too.

Since $H_0 \supseteq R_u(B)$, $X_0 = G/H_0$ is a homogeneous $G$-horospherical variety. The fact $\dim(X_i) \leq \dim(X) - 2$ for $i = 1, 2$ implies that $X$ is a normal variety. Thus, $X$ is a horospherical $G/H_0$-embedding. Since $X$ is a closed subvariety of $Gr(k - 1, E) \times Gr(k + 1, M)$, $X$ is projective. Thus, (iii) holds.

Proof of Proposition 4.46. Let $X \xrightarrow{\pi_1} Y_1 \subseteq Gr(k - 1, E)$ and $X \xrightarrow{\pi_2} Y_2 \subseteq Gr(k + 1, M)$ be the restrictions of the two projections from $Gr(k - 1, E) \times Gr(k + 1, M)$ to its factors, where $Y_1, Y_2$ are the images. Then $Y_1 = Gr(k - 1, E)$ and each fiber $\pi_1^{-1}(x)$ is isomorphic to $Gr(2, M/V)$, where $x = (W, V)$. Thus, $\pi_1$ is a smooth morphism, and $X$ is a smooth variety.

The construction of $X$ implies $Y_2 = Gr(k + 1, M)$. For any $x = (W, V) \in X_0 \cup X_1$, the fiber $\pi_2^{-1}(x) \cong Gr(k - 1, E \cap V) \cong P^{k-1}$. For any $x = (W, V) \in X_2$, the fiber $\pi_2^{-1}(x) \cong Gr(k - 1, E \cap V) \cong Gr(2, V)$. By [Br1] Prop. 2.1, $X_2$ is stable under the action of $Aut^0(X)$, which implies the conclusion (ii).

By our previous discussions, we know that $G/H_0$ is the $G$-orbit on $X$, $M_{G/H} = Z(\omega_k - \omega_{k+1})$, $V_X = \emptyset$, $D(G/H_0) = \{D_{\alpha_{k-1}}, D_{\alpha_k}, D_{\alpha_{k+1}}\}$, $D_0(G/H_0) = \{D_{\alpha_{k-1}}\}$, $\rho(\nu_{D_{\alpha_k}}) = \cdots$
\[-\rho(\nu_{D_{\alpha+k+1}}), \mathcal{E}_{X_1} = (\mathbb{Q}^+ \rho(\nu_{D_{\alpha+k+1}}), \{D_{\alpha+k+1}\}) \text{ and } \mathcal{E}_{X_2} = (\mathbb{Q}^+ \rho(\nu_{D_{\alpha}}), \{D_{\alpha}\}). \] Moreover, \(X\) is a smooth projective horospherical \(G/H_0\)-embedding.

We know from Theorem 4.29(i) that both \(G/P^0\) and \(F\) have positive dimensions. Hence, the conclusion (iv) follows from the conclusion (iii).

By Proposition 5.11(ii) and [Br93, Cor. 1.3(iv)], \(D_{\alpha}\) and \(D_{\alpha+k+1}\) are numerically equivalent in \(N^1(\mathbb{X})\), the Picard number \(\rho(X) = 2\), and \(\mathbb{R}^+ D_{\alpha+k}, \mathbb{R}^+ D_{\alpha\,k}\) are exact the two extremal rays of \(\text{Psef}^1(X)\).

By Theorem 1.5(i) and Theorem 4.29(i), \(D_{\alpha+k-1}\) induces a \(G\)-equivariant morphism \(\pi_0 : X \to G/P^0(\mathbb{X}) = Y_1\). By Lemma 4.24, \(\pi_1|_{X_0} = \pi_0|_{X_0}\). Note that the morphisms of colored fans corresponding to \(\pi_1\) and \(\pi_0\) are both \(\mathbb{F}_X \to \{(pt, 0)\}\). By Theorem 2.7, \(\pi_1 = \pi_0\). By Theorem 4.3(iii), \(\text{Nef}^1(X) = \text{Psef}^1(X)\). Thus, \(\pi_1\) is a Mori contraction of \(X\) and the conclusion (i) holds.

On the other hand, \(\pi_2^* \mathcal{O}_{Y_2}(1)\) is not ample and \(\pi_2\) doesn’t factor through \(\pi_1\). This implies that \(\pi_2\) factors through the Mori contraction different from \(\pi_1\). The facts \(\pi_2\) is smooth, \(\pi_2\) has connected fibers and the Picard number \(\rho(X) = 2\) imply that \(\pi_2\) is indeed the other Mori contraction of \(X\).

Suppose that \(X\) is isomorphic to a nontrivial product \(X_1 \times X_2\), then the fact \(\rho(X) = 2\) implies that \(\pi_1^* : X \to X_1\) and \(\pi_2^* : X \to X_2\) are the two Mori contractions. Thus, by reordering \(X_1\) and \(X_2\) if necessary, \(X_1 = Y_1\) and \(X_2 = Y_2\). It’s contradicted with the fact that \(\dim(X) < \dim(Y_1) + \dim(Y_2)\). Hence, the conclusions (iii) and (iv) hold. \(\square\)

References

[BM13] V. Batyrev, A. Moreau, The arc space of horospherical varieties and motivic integration, Compositio Math. 149 (2013), 1327-1352.

[BB96] F. Bien, M. Brion, Automorphisms and local rigidity of regular varieties, Compositio Math. 104 (1996), no. 1, 1-26.

[Br86] M. Brion, Quelques propriétés des espaces homogènes sphériques, Manuscripta Math. 55 (1986), no. 2, 191-198.

[Br89] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphériques, Duke Math. J. 58 (1989), no. 2, 397-424.

[Br90] M. Brion, Vers une généralisation des espaces symétriques, J. Algebra 134 (1990), no. 1, 115-143.

[Br93] M. Brion, Variétés sphériques et théorie de Mori, Duke Math. J. 72 (1993), no. 2, 369-404.

[Br05] M. Brion, Lectures on the geometry of flag varieties, Topics in cohomological studies of algebraic varieties, 33-85, Trends Math., Birkhäuser, Basel, 2005.

[Br07] M. Brion, Log homogeneous varieties, Actas del XVI Coloquio Latinoamericano de Álgebra, 1-39, Biblioteca de la Revista Matemática Iberoamericana, Madrid, 2007.

[Br11] M. Brion, On automorphism groups of fiber bundles, Publ. Mat. Urug. 12 (2011), 39-66.

[BLV86] M. Brion, D. Luna, Th. Vust, Espaces homogènes sphériques, Invent. Math. 84 (1986), no. 3, 617-632.

[BP87] M. Brion, F. Pauer, Valuations of spaces homogènes sphériques, Comment. Math. Helv. 62 (1987), no. 2, 265-285.

[DELV11] O. Debarre, L. Ein, R. Lazarsfeld, C. Voisin, Pseudoeffective and nef classes on abelian varieties, Compos. Math. 147 (2011), no. 6, 1793-1818.
[FS09] O. Fujino, H. Sato, Smooth projective toric varieties whose nontrivial nef line bundles are big, Proc. Japan Acad. Ser. A Math. Sci. 85 (2009), no. 7, 89-94.

[Fulg11] M. Fulger, The cones of effective cycles on projective bundles over curves, Math. Z. 269 (2011), no. 1-2, 449-459.

[Fult93] W. Fulton, Introduction to toric varieties, Annals of Math. Studies, 131, Princeton University Press, Princeton, NJ, 1993.

[FMSS95] W. Fulton, R. MacPherson, F. Sottile, B. Sturmfels, Intersection theory on spherical varieties, J. Algebraic Geom. 4 (1995), no. 1, 181-193.

[Gi07] P. Gille, Rationality properties of linear algebraic group and Galois cohomology, http://www.math.ens.fr/~gille/prenotes/mcm.pdf, 2007.

[Hi84] A. Hirschowitz, Le groupe de Chow équivariant, C. R. Acad. Sci. Paris. Sér. I Math. 298 (1984), no. 5, 87-89.

[Hu75] J. E. Humphreys, Linear algebraic groups, Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York-Heidelberg, 1975.

[Io86] P. Ionescu, Generalized adjunction and applications, Math. Proc. Cambridge Philos. Soc. 99 (1986), no. 3, 457-472.

[Kn91] F. Knop, The Luna-Vust theory of Spherical Embeddings, Proceedings of the Hyderabad Conference on Algebraic Groups, (Hyderabad, 1989), 225-249, Manoj-Prakashan, Madras, 1991.

[Le13] B. Lehmann, Geometric characterizations of big cycles, arXiv: 13090880, 2013.

[Li94] P. Littelmann, On spherical double cones, J. Algebra 166 (1994), no. 1, 142-157.

[Ot13] J. C. Ottem, On subvarieties with ample normal bundle, arXiv: 13092263, 2013.

[Pa06] B. Pasquier, Variétés horosphériques de Fano, PhD thesis, http://tel.archives-ouvertes.fr/docs/00/11/19/12/PDF/versionfinale.pdf, 2006.

[Pa09] B. Pasquier, On some smooth projective two-orbit varieties with Picard number 1, Math. Ann. 344 (2009), no. 4, 963-987.

[Per] N. Perrin, Introduction to spherical varieties, http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/spherical.pdf.

[Per12] N. Perrin, On the geometry of spherical varieties, arXiv: 1211.1277, 2012.

[Pet09] T. Peternell, Submanifolds with ample normal bundles and a conjecture of Hartshorne, Interactions of classical and numerical algebraic geometry, 317-330, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[Su75] H. Sumihiro, Equivariant completion. II. J. Math. Kyoto Univ. 15 (1975), no.3, 573-605.