Algebraic characterization of the isometries of the hyperbolic 5-space

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Abstract Let \( GL(2, \mathbb{H}) \) be the group of invertible \( 2 \times 2 \) matrices over the division algebra \( \mathbb{H} \) of quaternions. \( GL(2, \mathbb{H}) \) acts on the hyperbolic 5-space as the group of orientation-preserving isometries. Using this action we give an algebraic characterization of the orientation-preserving isometries of the hyperbolic 5-space. Along the way we also determine the conjugacy classes and the conjugacy classes of centralizers or the \( z \)-classes in \( GL(2, \mathbb{H}) \).

Keywords Hyperbolic 5-space · Isometries · Quaternions · \( z \)-Classes

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1 Introduction

Let \( \mathbb{H}^{n+1} \) denote the \( (n + 1) \)-dimensional hyperbolic space. The conformal boundary of the hyperbolic space is the \( n \)-dimensional sphere \( S^n \). Let \( \mathbb{E}^n \) denote the \( n \)-dimensional euclidean space. We identify \( S^n \) with the extended euclidean space \( \hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{ \infty \} \). Let \( Io(n+1) \) denote the group of orientation-preserving isometries of \( \mathbb{H}^{n+1} \). Classically, one uses the ball model of \( \mathbb{H}^{n+1} \) to define the dynamical type of an isometry. In this model an isometry is elliptic if it has a fixed point on the disk. An isometry is parabolic, resp. hyperbolic, if it is not elliptic and has one, resp. two fixed points on the conformal boundary of the hyperbolic space. If in addition to the fixed points one also consider the “rotation-angles” of an isometry, the above classification of the dynamical types can be made finer.

Let \( S \) be an orthogonal transformation of \( \mathbb{E}^n \). The rotation angles of \( S \) correspond to each pair of complex conjugate eigenvalues of \( S \). For each pair of complex conjugate eigenvalues \( \{ e^{i\theta}, e^{-i\theta} \} \), \( -\pi \leq \theta \leq \pi, \theta \neq 0 \), we assign a rotation angle to \( S \). If \( S \) has \( k \) rotation angles, it is called a \( k \)-rotation. Now suppose \( T \) is an isometry of \( \mathbb{H}^{n+1} \). Then it follows from the description of the conjugacy class of \( T \) that one can associate to \( T \) an orthogonal transfor-
motion $A_T$ of $\mathbb{E}^n$. For our purpose it is enough to consider the case when $n$ is even, that is the dimension of $\mathbb{H}^{n+1}$ is odd. In this case, by Lefschetz fixed-point theorem, every isometry has a fixed point on $\mathbb{S}^n$. Hence the restriction of $T$ to $\mathbb{S}^n$ can be conjugated to a similarity $f_T$ of $\mathbb{E}^n$. The orthogonal transformation $A_T$ is associated to this similarity $f_T$ of $\mathbb{E}^n$. With respect to a suitable coordinate system, $f_T$ is of the form $rA_Tx + b$, $r > 0$.

We call $T$ a $k$-rotatory elliptic (resp. $k$-rotatory parabolic, resp $k$-rotatory hyperbolic) if it is elliptic (resp. parabolic, resp. hyperbolic) and $A_T$ is a $k$-rotation. A 0-rotatory parabolic (resp. a 0-rotatory hyperbolic) is called a translation (resp. a stretch). For more details of this classification cf. [9].

Recall that in dimension 3, the group $GL(2, \mathbb{C})$, or equivalently, $\mathbb{S}L(2, \mathbb{C})$, acts as the linear fractional transformations of the boundary sphere $\mathbb{S}^2$ and the dynamical types are characterized by the trace [3,18] and trace $^2 \det$ [9], respectively. In [5] Parker et al have given an algebraic characterization of the dynamical types of the orientation-preserving isometries of the hyperbolic 4-space. Parker et al have offered the characterization after identifying the group of isometries with a proper subgroup of $GL(2, \mathbb{H})$ which preserves the unit disk on $\mathbb{H}$.

Our interest in this paper are the orientation-preserving isometries of the hyperbolic 5-space. The conformal boundary $\mathbb{S}^4$ of the hyperbolic 5-space is identified with the extended quaternionic plane $\mathbb{H} = \mathbb{H} \cup \{\infty\}$. The group $GL(2, \mathbb{H})$ acts on $\mathbb{H}$ as the linear fractional transformations:

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} : Z \mapsto (aZ + b)(cZ + d)^{-1}.
$$

Under this action $GL(2, \mathbb{H})$ can be identified with the identity component of the full group of Möbius transformations of $\mathbb{S}^4$. We have proved this fact in Sect. 3. Another proof can be found in [20]. Comparable versions are available in [1,6,12,13,19]. The proof we give here is different from the existing proofs and is more geometric.

The group $GL(2, \mathbb{H})$ can be embedded in $GL(4, \mathbb{C})$ as a subgroup. Using this embedding and the representation of the isometries of $\mathbb{H}^5$ as $2 \times 2$ matrices over the quaternions, we offer an algebraic characterization of the dynamical types. Our main theorem is the following.

**Theorem 1.1** Let $f$ be an orientation-preserving isometry of $\mathbb{H}^5$. Let $f$ be induced by $A$ in $GL(2, \mathbb{H})$. Let $A_C$ be the corresponding element in $GL(4, \mathbb{C})$. Let the characteristic polynomial of $A_C$ be

$$
\chi(A_C) = x^4 - 2a_3x^3 + a_2x^2 - 2a_1x + a_0.
$$

Then $a_0 > 0$. Define,

$$
c_1 = \frac{a_1^2}{a_0 \sqrt{a_0}}, \quad c_2 = \frac{a_2}{\sqrt{a_0}}, \quad c_3 = \frac{a_3^2}{\sqrt{a_0}}.
$$

Then we have the following.

(i) $A$ acts as a 2-rotatory hyperbolic if and only if $c_1 \neq c_3$.
(ii) $A$ acts as a 2-rotatory elliptic if and only if $c_1 = c_3$, $c_2 < c_1 + 2$.
(iii) $A$ acts as an 1-rotatory hyperbolic if and only if $c_1 = c_3$, $c_2 > c_1 + 2$.