

TOPOLOGY, CARDINALITY, METRIC SPACES AND THE GENERALIZED CONTINUUM HYPOTHESIS

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Abstract. This is a paper that aims to interpret the cardinality of a set in terms of Baire Category, i.e. how many sets that can be deleted from a set before the set itself becomes negligible. To do this natural tree-theoretic structures such as the Baire topology are introduced, and the Baire Category Theorem is extended to a statement that a \( \mathbb{N} \)-sequentially complete binary tree representation of a Hausdorff topological space that has a clopen base of cardinality \( \mathbb{N} \) and no isolated or discrete points is not the union of \(< \mathbb{N}+1\)-many nowhere dense subsets for cardinal \( \mathbb{N} \geq \mathbb{N}_0 \), where a \( \mathbb{N} \)-sequentially complete topological space is a space where every function \( f : \mathbb{N} \to \{0, 1\} \) is such that \((\forall x)(x \in f \to x \in X) \to (f \in X)\). It is shown that if \( \mathbb{N} < |X| \leq 2^\mathbb{N} \) for \(|X|\) the cardinality of a set \( X \), then it is possible to force \(|X| - \mathbb{N} \times |X| \neq \emptyset\) by deleting a dense sequence of \( \mathbb{N} \) specially selected clopen sets, while if any dense sequence of \( \mathbb{N} + 1 \) clopen sets are deleted then \(|X| - (\mathbb{N} + 1) \times |X| = \emptyset\). This gives rise to an alternative definition of cardinality as the number of basic clopen sets (intervals in fact) needed to be deleted from a set to force an empty remainder. This alternative definition of cardinality is consistent with and follows from the Generalized Continuum Hypothesis, which is shown by exhibiting two models of set theory, one an outer (modal) model, the other an inner, generalized metric model.

1. Introduction

This paper is experimental in the sense that there is very little recent relevant literature in the subject of this paper, and the paper has not been peer reviewed. For these reasons, please email the author if you find any errors or any arguments lack clarity.

In this paper a natural topology is outlined on the natural numbers, real numbers and sets higher up in the von Neumann cumulative hierarchy of pure sets\(^1\) which leads to a change in the definition of cardinality of set in order to support the view that cardinality measures how many topologically negligible sets can be deleted from a set before the set itself becomes negligible. It is then shown that there are models of set theory in which the change of definition of cardinality can be performed (which is exactly when the Generalized Continuum Hypothesis holds).

Before we begin with the development of a natural topology, it is worth noting

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\(^1\) is used as the standard reference for motivating the axioms of set theory (Zermelo-Fraenkel set theory with the Axiom of Choice) and \(^2\) is the standard reference for developments in Zermelo-Fraenkel set theory.

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some assumptions about the universe of sets. Firstly, we identify the set of subsets of a set $X$ of cardinality $\aleph$ with the set of binary sequences of length $\aleph$, called binary $\aleph$-sequences, which are functions $f : \aleph \to \{0, 1\}$, and members of the functions $(\alpha, b)$ are called nodes. This is possible by fixing an enumeration of $X$, $(x_\alpha : \alpha < \aleph)$ (by the Axiom of Choice), and for any subset $Y \subseteq X$ forming the binary $\aleph$-sequence $(b_\alpha : \langle \forall y \in Y \rightarrow b_\alpha = 1 \rangle \lor \langle y \not\in Y \rightarrow b_\alpha = 0 \rangle)$, where the ordinal index of any member $y \in Y$ is taken from the enumeration of $X$ (which includes all members of $Y$). Thus a subset of $X$ can be identified with a binary $\aleph$-sequence, and a set of subsets of $X$ can be identified with a set of binary $\aleph$-sequences. It is natural to think about any set as a tree of binary $\aleph$-sequences for some cardinal $\aleph$, where subtrees may split from a given $\aleph$-sequence at a given node, if we allow a tree to include the degenerate case where all members of the set are subsets of a single branch of the tree (i.e. the tree is a line). It is an obvious but important fact that a tree formed by binary $\aleph$-sequences is a binary tree, i.e. a tree in which every node has at most two successor nodes.

Representation by binary trees also suggests a property of sets that will appear throughout this paper, namely the property of a set $X$ corresponding to every binary $\aleph$-sequence through the tree representing a member of $X$. This property is a kind of completeness, but is in general weaker than compactness (unless $\aleph = \aleph_0$). It is called $\aleph$-sequential completeness. Logically $\aleph$-sequential completeness has the form $(\forall f : \aleph \to \{0, 1\})(\forall x)(x \in f \rightarrow x \in X) \rightarrow (f \in X)$, where $x \in y$ is defined as $\langle \forall z \in y \rightarrow (z \in x) \rangle$. Like completeness in a metric space, $\aleph$-sequential completeness does correspond to a generalized metric condition. $\aleph$-sequential completeness is also a closure condition, but it is stronger than closure because closure depends on which sets are defined to be open. It is in fact a form of absolute closure because the closure does not depend on the embedding space.

We can also note that by the same argument as above any set can be considered as a (possibly infinitely long) binary sequence. An ordinal number, $\alpha$, can be coded (non-uniquely) as a constant sequence of 1s of length $\alpha$, but in order to associate $\alpha < \aleph$ with a unique binary $\aleph$-sequence, $\alpha$ is represented as an initial sequence of $\alpha$ 1s followed by a terminal sequence of 0s. $x \subseteq y$ if $x_\alpha \subseteq y_\alpha$ for all $\alpha < \aleph$ where $x_\alpha$ and $y_\alpha$ are binary representations at position $\alpha$ in a $\aleph$-sequence of sets $x$ and $y$.

It should be apparent that the universe of sets can be regarded as a binary sequence representation of the von Neumann hierarchy of pure sets, $V_0 = \emptyset$, $V_{\alpha+1} = \{x : x \subseteq V_\alpha\}$ and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for $\lambda$ a limit ordinal. Binary $\aleph$-sequences first

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2[7] is a reference for a notion of absolute closure, defined as 'A Hausdorff space $X$ is called absolutely closed if $X$ is closed in every Hausdorff space in which it is imbedded' (see [7] Definition 1.1).

3$\aleph$-sequential completeness is not the same as compactness (because for $\aleph > \aleph_0$ there are infinite covers without finite subcovers) or sequential compactness (because $\aleph$-sequences are longer than $\aleph_0$-sequences for $\aleph > \aleph_0$). In particular $\aleph$-sequential completeness is not the same as the Stone-Cech compactification, because $\aleph$-sequentially complete Baire spaces, unlike Stone spaces over an infinite power set, are not compact for $\aleph > \aleph_0$, and Stone spaces over an infinite power set are larger ($= 2^{2^\aleph}$) and richer than the power set with an $\aleph$-sequentially complete Baire topology (cardinality = $2^{2^\aleph}$), see [1] Theorem 4.3 p. 143 & p. 146. [11] and the online $\pi$-base resource give an excellent view of the landscape of the topological properties of topological spaces.
appear in $V_\alpha$ for some ordinal $\alpha$, and if $\aleph$ is an infinite cardinal then $\alpha > \omega$, where $\omega$ is the least ordinal of cardinality $\aleph_0$.

2. **A natural topology of the natural numbers**

Consider a topology on the natural numbers with closed sets of the form $u_n = \{m \in N : m > n\}$ as well as $\emptyset$ and $N$, where $N$ is the set of natural numbers. These sets are closed sets because $\bigcap_{m < i < n} u_i = u_n$, $\bigcap_{m < i < \omega} u_i = \emptyset$, $u_n \cap N = u_n$, $u_n \cap \emptyset = \emptyset$ and $N \cap \emptyset = \emptyset$ for natural numbers $m$, $n$ and $n > m + 1$ where both appear in the same formula. No new closed sets are introduced by taking finite unions of closed sets, i.e. $\bigcup_{i \in \{m_0, \ldots, m_n\}} u_i = u_{m_0}$ if $m_0 \leq \ldots \leq m_n$ for $n_i \in N$. Rephrasing these statements, it is easy to see that $u_j \subset u_i$ if $j > i$ and for any natural numbers $m$, $n > m + 1$, $\bigcap_{m < i < n} u_i \neq \emptyset$ and $\bigcap_{m < i < \omega} u_i = \emptyset$. Define open sets to be $d_n = N - u_n$ and $\emptyset$ and $N$. Then we see $N - \bigcup_{m < i < n} d_i \neq \emptyset$ and $N - \bigcup_{m < i < \omega} d_i = \emptyset$ for any natural numbers $m$, $n > m + 1$. If we note $|d_{n+1}| - |d_n| = 1$, then we have $|N| \neq \sum_{i=m}^{n} 1$ and $|N| = \sum_{i=m}^{\omega} 1$, or in cardinality terms $\aleph_0 \neq n - m$ and $\aleph_0 = |\omega| \times 1$. These results are not surprising, but it is worth rephrasing: that if we remove any finite set of open sets (not including $N$) from $N$ we have a non-empty remainder, and if we remove any infinite set of open sets from $N$ we have an empty remainder. This shows in this topology that you cannot force$^4 \aleph_0$ to be finite and you cannot force $\aleph_0 \neq |\omega|$. We can state this as:

**Theorem 1.** In a natural topology on the set of natural numbers, $N$, with closed sets of the form $u_n = \{m \in N : m > n\}$ and $\emptyset$ and $N$, you cannot force $\aleph_0$ to be finite and you cannot force $\aleph_0 \neq |\omega|$.

It is possible to reverse the roles of the open and closed sets, but essentially the same topology arises. If, however, (other than closed and open sets $\emptyset$ and $N$) closed sets have the form $u_n = \{m \in N : m \leq n\}$ and open sets have the form $d_n = \{m \in N : m < n\}$, then all open sets are also closed, and all closed sets also open because $m \leq n$ if any only if $m < n + 1$. But then each set of the form $\{n\}$, where $n \in N$, and $\emptyset$ and $N$ are clopen (open and closed). It follows that $X$ has the discrete topology.

3. **A natural topology of the real numbers**

A natural generalization of taking terminal segments of the natural numbers as closed sets is to take closed sets of the real numbers to be the set of real numbers in a set $X$ (expressed as a set of binary sequences) that agree with some $x \in X$ on a particular initial segment of $x$, say $\langle x_m : m < n \rangle$ where $x_m \in \{0, 1\}$, but which do not include $x_n$ and therefore $x$. To be precise, a closed set is a set of the form $u_n(x) = \{y \in X : y_n \neq x_n \land (\forall m < n)(x_m = y_m)\}$ for natural number $n > 0$. This is a variation of the Baire topology, where open sets extend a finite sequence.

$^4$"Force" is used in its everyday sense, i.e. it is possible with some effort to do something. The powerful mathematical notion of forcing, which amounts to giving a set a property by adding a consistent set of finitely specified conditions, is related to developments later in this paper. (See [this](#) for a clear introduction to set-theoretic forcing).
We see that other closed sets need be added so that closed sets are closed under intersection. We need to add as closed sets \( \{x\} \) for \( x \in X \) because \( \bigcap_{1 \leq i < \omega} u_{n_i}(x_i) = \{x\} \) is possible for some sequence of closed sets \( \langle u_{n_i}(x_i) : 1 < i < \omega \rangle \). The topology can be thought of in terms of trees: if each member of \( X \) is a binary sequence \( \omega \rightarrow \{0, 1\} \), i.e., a binary \( \omega \)-sequence, then every \( x \in X \) is a branch of the tree and \( u_n(x) \) is a subtree that splits from \( x \) at a particular node of the binary sequence \( x \), i.e., \( \langle n, b \rangle \) for natural number \( n \) and \( b \in \{0, 1\} \). It is possible for a point to be represented by two branches in the case where a binary sequence is eventually constant. For example, \( 0111 \ldots \) might be represented by \( 1000 \ldots \) as well. But since there are countably many such double representations, the use of a tree representation is appropriate for studying uncountable sets of real numbers. In the following treatment all isolated members of \( X \), \( x \in X \) which have a highest node \( nd \) such that all other \( y \in X \) split from \( x \) at or below \( nd \), are deleted to simplify the exposition. As there are at most countably many isolated members of \( X \) because there are countably many nodes of type \( nd \), isolated points are well-understood from a cardinality perspective. Moreover, since it is possible for a point to become isolated if other isolated points are removed, by transfinite induction up to a countably infinite ordinal (deleting any \( x \in X \) that is covered only by isolated points of order \( < \alpha \) at limit ordinal \( \alpha \)), we can delete all isolated points and leave either the empty set or a dense-in-itself kernel of the set.

It is also possible to remove closed sets of the form \( u_n(x) \) in a way which is a generalized version of the construction of a Cantor ternary set. Fix an enumeration of countably infinitely many \( x \in X \), say \( \langle x_{\alpha < \omega} \rangle \), treated as branches of binary sequences of length \( \omega \), which are dense in \( X \), i.e., every \( x \in X \) is covered by an \( \omega \)-sequence of \( x_\alpha \).\(^5\) Then for any finite ordinal \( \alpha \) there is a highest node of height \( n(\alpha) \) at which \( x_{\beta < \alpha} \) split from \( x_\alpha \) (where \( n(0) \) is the lowest node from which some branch splits from \( x_0 \)); then proceed along the branch \( x_\alpha \), \( r(\alpha) > 0 \) nodes from which some \( u_{n(\alpha)+r(\alpha)}(x_\alpha) \) splits, and delete any branches that coincide with the terminal segment of \( x_\alpha \) from nodes of height \( r(\alpha) + 1 \) onwards. If \( x_\alpha \) has already been deleted then do nothing. For reference we will call this branch deletion construction from set \( Y \subseteq X \) \( cntr(Y; x_\alpha) \). Finally, at ordinal \( \omega \) take the intersection of all stages of the construction \( \alpha < \omega \). We can write the construction \( X_0 = X \), \( X_{\alpha+1} = \text{cntr}(X_\alpha, x_\alpha) \) for \( \alpha < \omega \) and \( X_\omega = \bigcap_{\alpha < \omega} X_\alpha \).

\(^5\)See \([2]\) s. 2.1, p. 222, for example where the finite (binary for definiteness) sequence \( s \) is the initial segment of \( x \). The case of binary sequences gives rise to the Cantor topology.

\(^6\)An easy way to see that \( \bigcap_{1 \leq i < \omega} u_{n_i}(x_i) = \{x\} \) is possible is to use the tree approach in the main text: at the \( i \)-th split in the binary tree that represents \( X \) choose a subtree which contains a member of \( u_{n_i}(x_i) \), choosing one subtree (using the Axiom of Choice, or choosing the one that starts with \( 0 \) if you want to avoid the Axiom of Choice) if there is a choice. Then the sequences of choices defines a path \( x \) which may be in \( \bigcap_{1 \leq i < \omega} u_{n_i}(x_i) \), and will be if \( x \in X \) because \( x \in u_{n_1}(x_i) \) for each \( 1 < i < \omega \). Conversely, for any \( x \in X \) it is always possible to construct a sequence of \( u_{n_i}(x_i) \) such that \( \bigcap_{1 \leq i < \omega} u_{n_i}(x_i) = \{x\} \), by choosing \( u_{n_i}(x_i) \) such that \( x \in u_{n_i}(x_i) \) at each split.

\(^7\)A set is dense-in-itself if it contains no isolated points. The word “kernel” indicates that the isolated points have been removed and that a non-empty set remains. The construction is from \([2]\) p. 198.

\(^8\)This definition is equivalent to the standard definition of every open neighbourhood of \( x \) having non-empty intersection with the dense set.
Figure 1: Subtrees (clopen intervals) deleted from a binary tree representation of a set. The construction proceeds $r(\alpha)$ nodes that have branches splitting from them (one empty node is skipped in the diagram) them where $x_{\alpha-1}$ splits from $x_\alpha$ and deletes nodes $r(\alpha) + 1$ and higher of $x_\alpha$.

The density of the sequence $\langle x_{\alpha<\omega} \rangle$ in $X$ ensures that $X_{\alpha<\omega} \neq \emptyset$ because each non-empty closed set $u_{n(\alpha)+m}(x_\alpha)$ for $1 \leq m \leq r(\alpha)$ will contain some $x_{\beta>\alpha}$. The resulting Cantor sets, $X_\omega(\langle x_{i<\omega} \rangle)$, are closed and nowhere dense. The reason why the Cantor sets are closed is that $u_{n}(x)$ are closed and open (clopen) since $u_{n}(x)$ contains all of its limits points in $X$, and $X - u_{n}(x)$ contains all of its limit points (so both are clopen), and the Cantor sets constructed at ordinal $\omega$ have the form $X - \bigcup_{\alpha<\omega} u_{n(\alpha)+r(\alpha)}(x_\alpha)$, i.e. the complement of an open set $\bigcup_{\alpha<\omega} u_{n(\alpha)+r(\alpha)}(x_\alpha)$ are known as clopen intervals. To see that a resulting Cantor set is nowhere dense, note that each clopen interval is a maximally dense subset of itself, and since each clopen interval in the tree has a clopen interval deleted from it because the sequence $\langle x_{i<\omega} : x_i \in X_\alpha \rangle$ is dense in $X_\alpha$, no subset of the Cantor set is dense in the tree.

While the tree model of a set of real numbers, $X$, is a strong visual construction, there is a case where the model is not applicable, namely where all $x \in X$ cover a single $\omega$-sequence. In this case, there are no clopen intervals splitting from the single $\omega$-sequence. The members of $X$ will either have a finite length, written $x_n$ for $n \in \omega$, or be an $\omega$-sequence, $x_\omega$. Each $\{x_n\}$ is a closed set because $\{x_n\}$ contains all of its limits points in $X$, and $X - \{x_n\}$ is closed as its closure is $X - \{x_n\}$. Hence $\{x_n\}$ is clopen. $\{x_\omega\}$ is also closed because it contains all of its limits points, and is not open (as $X - \{x_\omega\}$ has $x_\omega$ as a limit). But given that all $\{x_n\}$ are isolated because they are discrete sets, we can remove them, and leave the set $\{x_\omega\}$ or the empty set. If $\{x_\omega\}$ exists, then it too is isolated because it is now a clopen and therefore is a discrete set (as its complement is the empty set). As $|X| \leq \aleph_0$ and $X$ comprises isolated points, this case has been sufficiently characterized from a cardinality perspective.

We can state this as:

$^9$If you proceed along the branch $x_\alpha$ by exactly 1 node, then the intersection will form a single branch, i.e. contain exactly one point.

$^{10}$It is worth noting that $\bigcup u_n(x_\alpha)$ is not in general closed, because all $u_n(x_\alpha)$ could split from a common sequence that is $\notin X$. 
Theorem 2. In the Baire topology on a subset of the set of real numbers, $X$, that comprises a set of binary sequences with a countable basis of clopen intervals and no discrete or isolated points, the Cantor sets $X_\omega$ constructed from $X$ and a dense $\omega$-sequence $(x_{\alpha}\in X)$ by $X_0 = X$, $X_{\alpha+1} = \text{cntr}(X_\alpha; x_\alpha)$ for $\alpha < \omega$ and $X_\omega = \bigcap_{\alpha<\omega} X_\alpha$ are closed and nowhere dense.

In terms of cardinality, note that $X_\omega$ can be empty, but if $X$ is an uncountable sequentially complete set of real numbers (in the sense that every $\omega$-sequence through the tree is a member of the set), then $X$ has cardinality $2^{\aleph_0}$. This is so because every uncountable set of binary sequences must cover an infinite binary tree (because each node is covered by a binary sequence). Remove all isolated binary sequences. Then each binary sequence must split at an arbitrarily high node (i.e., into 0 and 1), since otherwise the binary sequence would be an isolated point; and by sequentially completeness, the tree created is isomorphic to the set of all binary $\omega$-sequences, $2^\omega$, which has cardinality $2^{\aleph_0}$. The subtree generated by the closed nowhere dense set construction, $X_\omega$, also has cardinality $2^{\aleph_0}$, as can be seen by labelling the remaining $u_n(x_\alpha)$ 1.2 et seq and the deleted subtrees 0 and noting that nested $u_n(x_\alpha)$ give rise to sequences that can be labelled using $\omega$-sequences that do not contain 0. We can conclude by sequential completeness that each sequence is a member of $X$. In cardinality terms we have $2^{\aleph_0} - \aleph_0 \times 2^{\aleph_0} = 2^{\aleph_0} \neq \emptyset$.

![Figure 2: A tree representation of a set of binary sequences. The bold line is a path through the tree. A set is sequentially complete if every path through the tree is a member of the set.](image)

We can state this as:

11 Sequential completeness is not a topological notion as the real numbers with the standard open interval topology is homeomorphic to the open interval $(0, 1)$ with the same open interval topology, but the real numbers is sequentially complete and $(0, 1)$ is not because the constant 0 and constant 1 $\omega$-sequences are sequences through the tree but 0 and 1 are not members of $(0, 1)$. Sequential completeness is a metric notion in general, but in the case of the Baire topology it is also set-theoretic (whether any given sequence is a member of a set) and tree-theoretic (whether a sequence is covered by other sequences that split from it at a node implies that the sequence is a member of the set). There is little difference in practice because a Baire space is metrizable with, for example, the metric $d(x, x) = 0$ and $d(x, y) := 2^{-n}$ where $n$ is the height of the lowest node such that $(x)_n \neq (y)_n$.

12 That is, there is a one-to-one mapping of X onto $2^\omega$ that preserves the branch structure.
Theorem 3. In a sequentially complete Baire topology on a set of real numbers, $X$, that comprises a sequentially complete set of binary sequences with a countable basis of clopen intervals and no discrete or isolated points, the Cantor sets have cardinality $2^{\aleph_0}$ and the process of deleting $\omega$ clopen intervals gives rise to the equation $2^{\aleph_0} - \aleph_0 \times 2^{\aleph_0} = 2^{\aleph_0} \neq \emptyset$.

On the other hand, because there are only countably infinitely many clopen intervals (since there are only countably infinitely many nodes from which clopen intervals split from a branch), if we were to delete $\aleph_1$ clopen intervals in a dense way, the empty set would result. This may seem meaningless, but we can say that every $\omega_1$-sequence of Cantor sets, $C_\alpha$, such that $C_\beta \subset C_\gamma$ if $\beta > \gamma$, has a terminal segment of empty sets, i.e. $C_\beta = \emptyset$ for all $\delta > \beta$ for some countable ordinal $\beta$. In cardinality terms we have $2^{\aleph_0} - \aleph_1 \times 2^{\aleph_0} = \emptyset$. Moreover, although we can force $2^{\aleph_0} - \aleph_0 \times 2^{\aleph_0} = \emptyset$ (delete any $\omega$-sequence of all clopen intervals $\subset X$ from $X$), we cannot force $2^{\aleph_0} - \aleph_1 \times 2^{\aleph_0} \neq \emptyset$ as the deletion of any dense uncountable sequence of clopen intervals will result in an empty remainder.

We can state this as:

Theorem 4. In a sequentially complete Baire topology on a set of real numbers, $X$, that comprises a sequentially complete set of binary sequences with a countable basis of clopen intervals and no discrete or isolated points, if $\aleph_1$ clopen intervals are deleted in a dense way, then the empty set results, i.e. $2^{\aleph_0} - \aleph_1 \times 2^{\aleph_0} = \emptyset$.

There is also a connection between this topology and the Baire Category Theorem for compact Hausdorff topological spaces, i.e. that a compact Hausdorff topological space is not the union of countably many closed nowhere dense subsets. A topological space that comprises a sequentially complete set of binary sequences with countably infinitely many clopen basis sets $\mathcal{U}_n(x)$ and no discrete or isolated points,

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13That is, there is no non-empty subset of $X$ that does not have a clopen interval deleted from it. If there were some clopen interval which were not subject to deletion, then the empty set would not result.

14It is always possible to re-order any countably infinite set as a total ordering of order type $\alpha$ for any $\alpha < \omega_1$, but if there were a strictly decreasing nested sequence of Cantor sets of length $\aleph_1$ then as at least one clopen interval is deleted at each step, $X$ would have at least $\aleph_1$ clopen intervals, which is false. Likewise a tree with a path of length $\omega_1$ with $\omega_1$ clopen intervals splitting from it would define a set of clopen intervals of cardinality $\aleph_1$, which does not exist; and thus no tree which has a path of length $\omega_1$ with $\omega_1$ clopen intervals splitting from it represents a set of real numbers.

15If the denseness condition were removed, it would be possible to delete the same clopen interval $\aleph_1$ times, or rather delete it once and then do nothing $\aleph_1$ times.

16A topological space $X$ is compact if for every set of subsets $M \subseteq N$ such that $\bigcup M = X$ there is a finite set of subsets $L \subseteq M$ such that $\bigcup L = X$. A topological space $X$ is locally compact if every $x \in X$ has some open set $U$ and some compact set $C$ such that $x \in U \subseteq C$.

17A topological space is Hausdorff if there are disjoint neighbourhoods around any two distinct points, i.e. Hausdorff $\equiv (\forall x, N)(\forall y, U)(\exists Z \in N)(\exists Z \in N)(x \neq y \implies Y \cap Z = \emptyset)$.

18A space that has a base of countably many open sets is called second-countable.
points, $2^\omega$ for short is compact and Hausdorff in cardinality terms the Baire Category Theorem implies that $2^{\aleph_0}$ is not a retract of the Cantor space $2^{\omega}$ given that each closed nowhere dense set in the compact Hausdorff topological space $2^\omega$ has cardinality $2^{\aleph_0}$.

We can state this as:

**Theorem 5.** In a Hausdorff topological space, $X$, that comprises a sequentially complete set of binary sequences with a countable clopen base and no discrete or isolated points, $X$ is not the union of countably many nowhere dense subsets.

It is also worth noting that we do not need to start with a sequentially complete set $X$ with the Baire topology. If $X$ is not sequentially complete, contains no sequentially complete clopen interval, has a dense-in-itself subset and has cardinality $\aleph_0 < c \leq 2^{\aleph_0}$ such that all clopen sets have cardinality $c$ (removing all clopen intervals of cardinality $< c$ if necessary), then by removing clopen intervals in a dense way following Theorem 4 we see that $c-c \times \aleph_1 = \emptyset$. It is in fact possible using the Cantor construction $X_1 = X$, $X_{\alpha+1} = \text{cnt}(X_\alpha)$ for $\alpha < \omega$ and $X_\omega = \bigcap_{\alpha < \omega} X_\alpha$ to construct an $X_\omega$ which contains any given $\alpha \in X$ by choosing the set $\langle x_\alpha < \omega \rangle$ of $\omega$-sequences to be deleted such that $x_\alpha \in X$, $x_\alpha \neq x$ and $\langle x_\alpha < \omega \rangle$ is dense in $X_\omega$ and by modifying $\text{cnt}$ to increase the value of $r(\alpha)$ so that $x \in X_\alpha$ for all $\alpha < \omega$ and therefore $x \in X_\omega$ by definition.

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19A standard Baire space, $\omega^\omega$, is sequentially complete, but it is not compact nor sequentially compact, in essence because it is too wide: it has unbounded sequences of branches and an infinite cover comprising those branches and subtrees that split from them that does not have a finite subcover. The notation $2^\omega$ reflects the fact that the topological space is actually a Cantor space, i.e. $[0,1]$ with the Baire topology. A Cantor space is compact (as a product of a compact set, namely 2 = \{0,1\}).

20To show compactness from first principles, proceed using a Heine-Borel construction. Assume that a topological space that comprises a sequentially complete set of binary sequences with countably infinitely many clopen basis sets and no discrete or isolated points, $X$, is not compact, i.e. there exists an infinite cover of open sets $\{C_\alpha : \alpha \geq \omega\}$ without a finite open subcover. Then subdivide the set underlying of $X$, $E$ say, into two disjoint clopen intervals $E_1$ and $E_2$ (which exists since $X$ has a clopen basis and the complement of any clopen set is clopen) and iterate the process. At least one clopen interval in each subdivision will not be compact. Because $E$ is a sequentially complete and has a countably infinite basis, if a nested sequence $E_N$ consists of closed non-empty sets, where $N$ is an $\omega$-sequence of finite binary sequences such that if $m \in N$ and $n > m$ and $n \in N$ then $m$ is a subsequence of $n$, the subdivision process $\bigcap_{n \in N} E_N$ will result in a non-empty set. In fact, because $N$ defines a unique point, $\bigcap_{n \in N} E_N$ contains exactly one point in $E_L$. Now every point in $E_L$ will be a member of at least one open set, $C_j$, in the cover, otherwise $E_L$ would not be covered. But $L \in E_n \subseteq C_j$ for some $n \in N$ where $E_n \in E_N$ since an open set $C_j$ such that $L \in C_j$ will include a clopen interval $E_n$ such that $L \in E_n$ because the space has a basis of clopen intervals. By construction any clopen interval will split from $E$ depending only on its first $n$ binary digits for some natural number $n$. Thus if an $E_n$ were a nested sequence of clopen intervals that are not compact, then we would have $L \in E_m \subseteq E_n \subseteq C_j$ for all $m > n$ where $E_m \in E_N$ for any $E_m$ whose members agree with members of $E_j$ on the first $n$ binary digits, which means that $\{C_j\}$ is a single (i.e. finite) cover for $E_{m>\omega}$, contradiction.

To show the space is Hausdorff, note that any two distinct branches $c$ and $d$ will split from one another at a certain node, $n = (n,b)$ where $b \in \{0,1\}$: a $u_n(d_a)$ that includes $c$ and all branches that split from $c$ after node $n$ will have a disjoint union with a $u_n(d_b)$ that includes $d$ and all branches that split from $d$ after node $n$. Hence the space is Hausdorff. 21If $X$ is not compact or sequentially complete, the Baire Category Theorem does not apply. 22There are at least $\aleph_0$ such $\omega$-sequences because if there were a finite number, then some clopen interval would be sequentially complete.
If we consider that each clopen interval is divided into \( r > 1 \) disjoint clopen intervals and one clopen interval is deleted, we can write

\[
U_\alpha = \bigcup_{0 \leq m \leq r(\alpha)} U_{\alpha,m}
\]

and set \( U_{\alpha+1} = U_{\alpha,m} \) for any choice of \( m \) such that \( 1 \leq m \leq r(\alpha) \), where \( U_{\alpha,m} \) are clopen intervals, \( U_{\alpha,0} \) is deleted because \( x_\alpha \) is a branch in \( U_{\alpha,0} \), \( U_1 = X \) and \( U_\omega = \bigcap_{\alpha < \omega} U_\alpha \), for natural numbers \( \alpha, m, r(\alpha) \). Then if \( U_\alpha \) preserves \( y_\alpha \in U_\alpha \), i.e. \( y_\alpha \in U_\omega \), then if \( y_\alpha \in U_{\alpha,m} \) for some \( m > 0 \) there are \( y_{\alpha,s} \in U_{\alpha,s} \neq y_\alpha \) for all \( 1 \leq s \leq r(\alpha) \) and \( s \neq m \). We require that \( U_\alpha \) is constructed to include a clopen interval around branch \( x_\alpha \in U_{\alpha,0} \) (which will be deleted), to preserve \( \bigcup_{1 \leq m < \alpha} \{ y_m \} \) and \( y_\alpha \in U_{\alpha,m} \) for some \( m > 0 \). This requirement can be met by selecting \( y_1 \in U_1 \) such that \( y_1 \neq x_\beta \) for any \( \beta < \omega \) and constructing \( U_{\alpha+1} \) and \( y_{\alpha+1} \) as follows given clopen interval \( U_\alpha \) and \( y_\alpha \in U_\alpha \) which is preserved, i.e. \( y_\alpha \in U_\omega \).

- If \( x_\alpha \) is a branch in \( U_\alpha \): set \( r(\alpha) \) to include the node where \( y_\alpha \) splits from \( x_\alpha \), set \( U_{\alpha,0} \subset U_\alpha \) to be a clopen interval such that \( x_\alpha \) is a branch in \( U_{\alpha,0} \) and \( y_\alpha \notin U_{\alpha,0} \), and set \( U_{\alpha+1} := U_{\alpha,m} \) for any choice of \( 1 \leq m \leq r(\alpha) \) where \( U_\alpha = \bigcup_{0 \leq m \leq r(\alpha)} U_{\alpha,m} \). Set \( y_{\alpha+1} := y_\alpha \) if \( y_\alpha \in U_{\alpha+1} \) and otherwise choose \( y_{\alpha+1} \in U_{\alpha+1} \) such that \( y_{\alpha+1} \neq y_{\leq \alpha \leq \alpha} \) (which is possible since each clopen interval such as \( U_{\alpha+1} \) will have uncountably infinitely many members).

- If \( x_\alpha \) is not a branch in \( U_\alpha \): set \( U_{\alpha+1} := U_\alpha \) and set \( y_{\alpha+1} := y_\alpha \).

Since each \( y_{\alpha+1} \) is preserved by the same construction as was used for \( y_\alpha \), we see that each splitting of a clopen interval into \( r > 1 \) clopen intervals preserves an additional \( r - 1 \) points of \( X \). It follows that it is possible to construct \( X_\omega \) from \( X \), by means of the closed nowhere dense set construction, which contains a dense-in-itself subset of cardinality \( \geq \aleph_0 \). That is, it is possible to force \( c - c \times \aleph_0 \neq \emptyset \).

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**Figure 3:** An example of how a descending sequence of clopen intervals can be forced to contain one point of a set \( X \) per node of a decomposition of \( X \) into clopen

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23The rate of growth of \( r(n) \) depends on the height of the splitting node of \( x_n \) and \( y_n \), which could be set arbitrarily high.
intervals.

We can state this as:

**Theorem 6.** In a Baire topology of an uncountable set of real numbers, $X$, that comprises a set of binary sequences with a countable clopen base and no discrete or isolated points, it is always possible to construct a Cantor set $X_\omega$ from $X$ which contains a dense-in-itself subset of cardinality $\geq \aleph_0$. That is, it is possible to force $\mathcal{C} - \mathcal{C} \times \aleph_0 \neq \emptyset$. But deleting $\aleph_1$ clopen intervals in a dense way results in the empty set, i.e. $\mathcal{C} - \mathcal{C} \times \aleph_1 = \emptyset$.

4. A NATURAL TOPOLOGY OF SETS OF HIGHER ORDER

The Baire topology can be defined in the case of higher order sets in the same way as real numbers, deleting a dense sets by removing clopen intervals in the same manner as the case of sets of $\mathcal{C}$. If $X$ has no a dense-in-itself kernel then $\mathcal{C} = \mathcal{C} \times \aleph_0$ is not equivalent to a product topology, which is in turn equivalent to allowing only clopen sets that split from a branch of height $n < \omega$ in the case of a finite base for the product.

If $X$ has a dense-in-itself kernel, then it is possible to construct closed nowhere dense sets by removing clopen intervals in the same manner as the case of sets of real numbers, deleting a $\aleph$-sequence $S = \langle x_\beta : \beta < \aleph \rangle$ that is dense in $X$ by means of the construction $\text{cntr}(Y : x_\beta) := Y - u_\beta(x_\beta)$, where $u_\beta(x_\beta) = \{ y : (y)_n(x_\beta) \neq (x_\beta)_n + 1 \}$. $\text{cntr}(Y : x_\beta)$ is the supremum of nodes where $x_\beta$ splits from $x_\beta$ and the offset $r(\beta) > 0$ is any ordinal $r(\beta) < \aleph$ (as in the case of the real numbers, skipping over empty nodes). It follows that we can construct a sequence $X_0 = X$, $X_{\delta + 1} = \text{cntr}(X_\delta : x_\delta)$ for $\delta < \aleph$ and $X_\lambda = \bigcap_{\beta < \lambda} X_\beta$ for limit ordinal $\lambda \leq \aleph$. We claim that $X_\lambda$ is a closed nowhere dense set, which follows because the construction results in sets of the form $X - \bigcup_{\beta < \aleph} u_\alpha(x_\beta)$, i.e. the complement of an open set, and any clopen interval will have a clopen interval deleted from it (since the set of sequences $\{ S : S \in X_\alpha \}$ is dense in $X_\alpha$).

Finally, we note that if $X$ has a linear rather than tree representation in terms of binary $\aleph$-sequences, just as in the case of the real numbers we can remove isolated points by (transfinite) induction, starting at the initial member of the linear order, and proceeding until all members of $X$ have become isolated. In this case $X$ has cardinality $\aleph$.

We can state this as:

**Theorem 7.** In the Baire topology of a set of binary $\aleph$-sequences, $X$, with a basis of clopen intervals of cardinality $\aleph$ and no discrete or isolated points, the Cantor sets $X_\aleph$ constructed from $X$ and a dense $\aleph$-sequence $\langle x_\beta : \beta < \aleph \rangle$ by $X_0 = X$, $X_{\delta + 1} = \text{cntr}(X_\delta : x_\delta)$ for $\delta < \aleph$ and $X_\lambda = \bigcap_{\delta < \lambda} X_\delta$ for limit ordinal $\lambda \leq \aleph$ are closed and nowhere dense.

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24 In this case for a product topology a finite sequence of bounded finite sets (i.e. an initial finite $n$-ary sequence for some natural number $n$) will define the topology.

25 If $X$ does not have a dense-in-itself kernel then $|X| \leq \aleph$. 
In the same way as in the case of the real numbers it is possible to force $|X_\mathcal{R}| \geq \aleph$ by applying the closed nowhere dense set construction to $X$, which has a dense-in-itself subset and has cardinality $\aleph < c \leq 2^{\aleph}$ such that all clopen sets have cardinality $c$ (removing all clopen intervals of cardinality $< c$ if necessary), to construct an $X_\mathcal{R}$ which contains any given $x \in X$ by choosing the set $\langle x_{\alpha<\aleph} \rangle$ of $\aleph$-sequences to be deleted to be such that $x_{\alpha<\aleph} \in X$, $x_{\alpha} \neq x$ and $\langle x_{\alpha<\aleph} \rangle$ is dense in $X_\mathcal{R}$ and by modifying $cntr$ to increase the value of $r(\beta)$ so that $x \in X_\alpha$ for all $\alpha < \aleph$ and therefore $x \in X_\mathcal{R}$ by definition.

If we consider that each clopen interval is divided into $r > 1$ disjoint clopen intervals and one clopen interval is deleted, we can write $U_\alpha = \bigcup_{0 \leq \beta \leq r(\alpha)} U_{\alpha,\beta}$ and $U_{\alpha+1} = U_{\alpha,\beta}$ for any choice of $\beta$ such that $1 \leq \beta \leq r(\alpha)$, where $U_{\alpha,\beta}$ are clopen intervals, $U_{\alpha,0}$ is deleted because $x_{\alpha}$ is a branch in $U_{\alpha,0}$, $U_1 = X$ and $U_\lambda = \bigcap_{\beta < \lambda} U_\beta$, for ordinal numbers $\alpha$, $\beta$, $r(\alpha) < \aleph$ and $\lambda$ a limit ordinal. Then if $U_\alpha$ preserves $y_\alpha \in U_\alpha$, i.e. $y_\alpha \in U_\aleph$, then if $y_\alpha \in U_{\alpha,\beta}$ for some ordinal number $\beta > 0$ there are $y_{\alpha,\gamma} \in U_{\alpha,\gamma} \neq y_\alpha$ for all $1 \leq \gamma \leq r(\alpha)$ and $\gamma \neq \beta$. We require that $U_\alpha$ is constructed to include a clopen set around branch $x_\alpha$ in $U_{\alpha,0}$ (which will be deleted), to preserve $\bigcup_{\gamma < \alpha} \{ y_\gamma \}$ and $y_\alpha \in U_{\alpha,\beta}$ for some $\beta > 0$. This requirement can be met by selecting $y_1 \in U_1$ such that $y_1 \neq x_{\alpha<\aleph}$ and by constructing $U_{\alpha+1}$ and $y_{\alpha+1}$ and $U_\lambda$ and $y_\lambda$ for limit ordinal $\lambda < \aleph$ as follows given clopen interval $U_\alpha$ and $y_\alpha \in U_\alpha$ which is preserved, i.e. $y_\alpha \in U_\aleph$, or clopen intervals $U_{\beta<\lambda}$ and $y_\beta \in U_\beta$ in the case of limit ordinal $\lambda$.

\footnote{There are at least $\aleph$ such $\aleph$-sequences because if there were $< \aleph$, then some clopen interval will be $\aleph$-sequentially complete.}
• If $\alpha < \aleph$ is a successor ordinal:

If $x_\alpha$ is a branch in $U_\alpha$: set $r(\alpha)$ to include the node where $y_\alpha$ splits from $x_\alpha$, set $U_{\alpha,0} \subset U_\alpha$ to be a clopen interval such that $x_\alpha$ is a branch in $U_{\alpha,0}$ and $y_\alpha \notin U_{\alpha,0}$, and set $U_{\alpha+1} := U_{\alpha,\beta}$ for any choice of $1 \leq \beta \leq r(\alpha)$ where $U_\alpha = \bigcup_{0 \leq \beta \leq r(\alpha)} U_{n,m}$. Set $y_{\alpha+1} := y_\alpha$ if $y_\alpha \in U_{\alpha+1}$ and otherwise choose $y_{\alpha+1} \in U_{\alpha+1}$ such that $y_{\alpha+1} \neq y_{1 \leq \beta \leq \alpha}$ (which is possible as there are at least $c > \aleph$ members in any clopen interval such as $U_{\alpha+1}$).

If $x_\alpha$ is a not branch in $U_\alpha$: set $U_{\alpha+1} := U_\alpha$ and set $y_{\alpha+1} := y_\alpha$.

• If $\alpha < \aleph$ is a limit ordinal:

Set $U_\alpha := \bigcap_{1 \leq \beta < \alpha} U_\beta$.

By transfinite induction $U_\alpha$ preserves at least one $y \in U_\alpha$ such that $y \in U_\beta$ is preserved for all $\beta$ such that $1 \leq \beta < \alpha$. But for every such $y$ there is a least ordinal $\alpha \leq \gamma < \aleph$ such that there are no deletions from any subtrees that split from $y$ at or above the $\gamma$-th node of $y$ (as the cardinality of the union of $< \aleph$ ordinals $< \aleph$ is $< \aleph$ using the Axiom of Choice). For ease of reference, the least ordinal $\gamma$ is written as $h(\alpha,y)$. It follows that $U_\alpha$ contains non-empty clopen intervals, $V_\alpha := U_{h(\alpha,y)}$ for all $y \in U_\alpha$.

Set $y_\alpha := y$ for any choice of $y \in X$ such that $y \in U_\beta$ for all $\beta < \alpha$. There is always at least one such $y$ because $y \in V_\alpha \subseteq U_\alpha$.

• If $\alpha = \aleph$:

Set $U_\alpha := \bigcap_{1 \leq \beta < \alpha} U_\beta$. Then each set $U_{h(\aleph,y)} = \{y\}$ and $y_\aleph := y$ for all $y \in U_\aleph$.

The construction also works for $c = \aleph$ as $\alpha + 1 < \aleph$.

Since each $y_\alpha$ can be preserved by the same construction as was used for $y_{\beta < \alpha}$, we see that each splitting of a clopen interval into $r > 1$ clopen intervals for $r < \aleph$ preserves an additional $r - 1$ points of $X$, and all of these points are preserved at limit ordinals (as represented by all possible values of $U_\lambda$ for limit ordinals $\lambda \leq \aleph$). It follows that every $X_\aleph$ generated from $X$ by the closed nowhere dense set construction contains a dense-in-itself subset of cardinality $\geq \aleph$. It follows that if $\aleph < |X| \leq 2^\aleph$ then it is possible to force $|X| - \aleph \times |X| \neq \emptyset$, while if a dense sequence of $\aleph + 1$ clopen intervals are deleted then $|X| - (\aleph + 1) \times |X| = \emptyset$. 
Figure 4: An example of how a descending sequence of clopen intervals has clopen intervals from some limit ordinal onwards, at the point where no clopen intervals have yet been deleted in the construction.

We can state this as:

**Theorem 8.** In a Baire topology of a set of binary $\aleph$-sequences, $X$, such that $\aleph < |X| \leq 2^\aleph$ with a basis of clopen intervals of cardinality $\aleph$ and no discrete or isolated points, it is always possible to construct a Cantor set $X_\aleph$ from $X$, which contains a dense-in-itself subset of cardinality $\geq \aleph$. That is, if $\aleph < |X| \leq 2^\aleph$ then it is possible to force $|X| - \aleph \times |X| \neq \emptyset$, while if a dense sequence of $\aleph + 1$ clopen intervals are deleted then $|X| - (\aleph + 1) \times |X| = \emptyset$.

If $X$ is $\aleph$-sequentially complete and therefore has a base of clopen intervals which are $\aleph$-sequentially complete, i.e. all paths of length $\aleph$ through the interval are members of the interval, then we can claim that it is possible to force $2^\aleph - \aleph \times 2^\aleph = 2^\aleph$ because the same labelling technique can be used on clopen intervals as in the case of the real numbers (all deleted clopen intervals being labelled 0) and we can note that all $\aleph$-sequences of ordinal labels $\aleph > \alpha > 0$ are members of $X$ by $\aleph$-sequential completeness and that the cardinality of $\aleph^{\aleph} = 2^\aleph$. By transfinite induction for $\alpha < \aleph$ with the hypothesis that all clopen intervals $\subseteq X_\alpha$ have cardinality $2^\aleph$, at stage $\alpha + 1$ $X_\alpha$ will be split into $> 1$ and $< \aleph$ clopen intervals with a label $\neq 0$, each of which by the induction hypothesis has cardinality $2^\aleph$, so $X_{\alpha+1}$ as the union of these sets, will also have cardinality $2^\aleph$. For a limit ordinal $\lambda$, all clopen intervals with label 0 can be deleted, and for the clopen intervals remaining $\aleph$-sequential completeness can be applied to the paths between labels formed at stages successor stages $\alpha < \lambda$ to show that $X_\lambda$ has cardinality $2^\aleph$. The latter observation relies on the fact that a strictly descending $\aleph$-sequence of non-empty clopen intervals defines
a single point or branch $x \in X$, and therefore a descending $\alpha < \aleph$-sequence of clopen intervals can be identified with an initial segment of $x$ of length $\alpha$.

We can state this as:

**Theorem 9.** In a $\aleph$-sequentially complete Baire topology of a set of binary $\aleph$-sequences, $X$, with a basis of clopen intervals of cardinality $\aleph$ and no discrete or isolated points, the Cantor sets have cardinality $2^\aleph$ and the process of deleting $\aleph$ clopen intervals gives rise to the equation $2^\aleph - \aleph \times 2^\aleph = 2^\aleph$.

The Baire Category Theorem can also be generalized to the statement that in a $\aleph$-sequentially complete Hausdorff topological space, $X$, that comprises a $\aleph$-sequentially complete set of binary $\aleph$-sequences with a clopen base of cardinality $\aleph$ and with no discrete or isolated points, $X$ is not the union of $< \aleph + 1$-many nowhere dense subsets for $\aleph \geq \aleph_0$. It is worth noting that a $\aleph$-sequentially complete Hausdorff space $X$ that comprises a $\aleph$-sequentially complete set of binary $\aleph$-sequences with a clopen base of cardinality $\aleph$ is neither compact nor metrizable for $\aleph > \aleph_0$, but there is a generalized metric function that can be used.

In [3] R. Kopperman showed that it possible to replace the set of real numbers in the definition of a metric space with a commutative semi-group and for every topology to find a suitable commutative semi-group for which a metric can be introduced to the topological space (which may not be symmetric or separate distinct members of the topological space). Let $(2^\aleph, \oplus)$ be a structure defined as follows. If $2^\aleph$ is the set of functions $\aleph \rightarrow 2$ and $a, b \in 2^\aleph$, i.e. are binary $\aleph$-sequences, then treat $a$ and $b$ as $\aleph$-sequences of real numbers in the range $[0, \infty)$, $\langle a_1, ..., a_{<\alpha}, \ldots \rangle$ and $\langle b_1, ..., b_{<\alpha}, \ldots \rangle$ for real numbers $a_\alpha, b_\alpha \in [0, \infty)$, and define $a \oplus b$ as the $\aleph$-sequence $\langle a_1 + b_1, ..., a_{<\alpha} + b_{<\alpha}, \ldots \rangle$.

Let us denote a clopen interval comprising binary $\square$-sequences from $a \leq b$ to $b$ by $([a, b])_\square$ and the half-open interval from $a < b$ to $b$ by $[a, b)[\aleph]$. Let us

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27See for example [6] Proposition 3.8 p. 213 for the case $\aleph = \aleph_0$.

28Construct a cover $Z$ of $X$ as follows. Fix a branch $x \in X$ and add to $Z$ all disjoint clopen intervals that split from $x$. Then add to $Z$ a clopen interval that splits from $y \in X$ such that $y \neq x$ at a node of index $> \aleph_0$. $Z$ has no finite open subcover if the base of $X$ has cardinality $\aleph > \aleph_0$ because there are at least $\aleph_0$ disjoint clopen intervals in the cover such that removal of any one such set would not result in a cover of $X$. A corollary is that there is a descending $\aleph$-sequence of clopen sets $\langle x_\alpha, \alpha < \aleph \rangle$ (complements of clopen intervals) such that all finite intersections of $x_\alpha$ are non-empty while $\bigcap_{\beta < \aleph} X_\beta = \emptyset$. The failure of compactness means that the topological space cannot be characterized by convergent ultrafilters, but it possible nevertheless to characterize $X$ by the set of strictly descending $\aleph$-sequences of clopen intervals converging to a point $x$, and in fact a generalized local compactness condition does hold for any $\aleph$-sequentially complete Hausdorff topological space that has a clopen base of cardinality $\aleph$: if for every $\beta < \aleph \bigcap_{\alpha < \beta} F_\alpha \neq \emptyset$ then $\bigcap_{\alpha < \beta} F_\alpha \neq \emptyset$ for any strictly descending $\aleph$-sequence of non-empty clopen intervals $F_\alpha$, i.e. $F_\beta \subset F_\gamma$ if ordinal $\gamma < \beta$. This follows by following the branch from which successive nested clopen intervals split.

29By the Nagata-Smirnov metrization theorem $2^\aleph$ for $\aleph > \aleph_0$ is not metrizable as it is Hausdorff and regular (since any two points can be separated by clopen neighbourhood), but does not have a countable locally finite base (since there are uncountably many clopen intervals and if every member of $2^\aleph$ is only a member of finitely many clopen intervals, the family of clopen intervals in the base is uncountable).

30A semi-group is defined like a group but may lack an inverse operation to the group operation.
now define \(d(x, y)\) for binary \(\mathbb{N}\)-sequences of the form \(\langle x, \ldots, x_{\alpha < \omega}, \ldots \rangle\) where real number \(x_\alpha \in ([0, 1])^{[\omega]}\), by \(d(x, x) := 0\) and \(d(x, y) := 1_{\alpha(x, y)}\), i.e. where there is a 1 only in the \(\alpha\)-th digit of an \(\mathbb{N}\)-sequence of binary \(\omega\)-sequences with a binary point (a real number) and 0 for all other digits, and \(\alpha\) is a successor ordinal that is the height of the lowest node where \((x)_\alpha \neq (y)_\alpha\). This is an unambiguous definition because each \(x_\alpha \in ([0, 1])\) can be represented as a real number with 0 in front of the binary point (because 1.000... can also be written 0.111...). We can thus skip the 0 before the binary point in the real number representation uniquely identifying the height of the lowest node where \((x)_\alpha \neq (y)_\alpha\). In practice we will leave the binary point in place for clarity. Surprisingly we have \(d(x, y) \leq \frac{1}{2} = \langle 0.1, 0, 0, 0, \ldots \rangle\) because the first node after 0 is the first node that \(x\) and \(y\) can differ. On the other hand \(d(x, y) + d(y, z) \leq 1\), and a sum of natural number \(n\) such distances is bounded by \(n/2\).

We can show that if \(x\) and \(y\) are binary \(\mathbb{N}\)-sequences in the clopen interval \(([0, 1])^{[\mathbb{N}]\rangle\) then \(([0, 1])^{[\mathbb{N}]}\) forms a metric space.\(^{31}\) We have \(d(x, y) = d(y, x)\), \(d(x, y) \geq 0\) and \(d(x, y) = 0 \rightarrow x = y\) immediately from the definition of \(d\) and the fact that all real numbers in \(x\) and \(y\) start with 0. We also have \(d(x, y) + d(y, z) \geq d(x, z)\) because:

\[a) \text{If } \alpha(x, z) > \alpha(x, y) \text{ then } \alpha(y, z) = \alpha(x, y), \text{ and } d(x, y) + d(y, z) = 1_{\alpha(x, y)} + 1_{\alpha(y, z)} = 1_{\alpha(x, y)} > 1_{\alpha(x, z)}.\]

\[b) \text{If } \alpha(x, z) < \alpha(x, y) \text{ then } \alpha(y, z) = \alpha(x, z), \text{ and } d(x, y) + d(y, z) = 1_{\alpha(x, y)} + 1_{\alpha(y, z)} > 1_{\alpha(x, z)}.\]

\[c) \text{If } \alpha(x, z) = \alpha(x, y) \text{ then we have } d(x, y) + d(y, z) = 1_{\alpha(x, y)} + 1_{\alpha(y, z)} > 1_{\alpha(x, z)}.\]

\[d) \text{If } x = y \text{ then } \alpha(x, z) = \alpha(y, z), \text{ and } d(x, y) + d(y, z) = 1_{\alpha(x, z)}; \text{ if } y = z \text{ then } \alpha(x, y) = \alpha(x, z), \text{ and } d(x, y) + d(y, z) = 1_{\alpha(x, y)} = 1_{\alpha(x, z)}; \text{ and if } x = z \text{ then } \alpha(x, y) = \alpha(y, z), \text{ and } d(x, y) + d(y, z) = 1_{\alpha(x, y)} + 1_{\alpha(y, z)} > 0.\]

We can define clopen intervals in the Baire topology as \(\{y : d(x, y) = 1_{\alpha(x, y)}\}\).

\[\begin{align*}
\text{Figure 5: Diagrams showing the different cases in the generalized metric of } ([0, 1])^{[\mathbb{N}]}\text{.}
\end{align*}\]

The clopen interval \(([0, 1])^{[\mathbb{N}]\rangle\) was chosen for simplicity, and it has the closure

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\(^{31}\)In fact \(([0, 1])^{[\mathbb{N}]\rangle\) is also an ultrametric space as \(\max(d(x, y), d(y, z)) \geq d(x, z)\), see Figure 5.
property \( d(x, y) \oplus d(y, z) \in ([0, 1])[\aleph] \) if \( x, y, z \in ([0, 1])[\aleph] \); but exactly the same generalized metric works on the interval \([0, \infty])[\aleph] \). Any binary real number can be padded with 0s in front of the binary point if necessary to have a prefix of the same length as any other binary real number, and all binary digits in the prefix are treated as negative whole number offsets from the binary point. For example, to calculate the \( d(x, y) \) where \( x = 11.000 \ldots \) and \( y = 100.000 \ldots \) the prefix of \( x \) can be padded to 011 and \( d(x, y) = 100.000 \ldots \), which is at position -3 with respect to the binary point. It is true that \( d(x, y) \oplus d(y, z) \in [0, \infty][\aleph] \) if \( x, y, z \in [0, \infty][\aleph] \), but of course \([0, \infty])[\aleph] \) is not closed under upward limits, i.e., \( d(x, y) \to \infty \) if \( x \) is fixed and \( y \to \infty \) or vice versa, and \( \infty / [0, \infty] \). The clopen interval \(([0, 1])[\aleph] \) is therefore a better representation of the set of all binary \( \aleph \)-sequences.

We should be clear that for \( \aleph > \aleph_0 \) the generalized metric space is not compact. The reason is that, as we have seen, it is possible to have an \( \omega \)-sequence of non-empty clopen and totally bounded intervals \( \langle X_\alpha, \alpha < \omega \rangle \) such that \( X_\alpha \subseteq X_\beta \) if \( \beta \leq \alpha < \omega \) and \( \bigcap_{\beta < \omega} X_\beta = \emptyset \). But the following statements are true in a generalized metric space. If \( y \) is a limit point of non-empty \( \bigcap_{\alpha < \aleph} X_\beta \) where \( \langle X_\alpha, \alpha < \aleph \rangle \) is an \( \aleph \)-sequence of non-empty clopen intervals such that \( X_\beta \subseteq X_\gamma \) if \( \gamma < \beta \), then \( y \in \bigcap_{\alpha < \aleph} X_\beta \). Moreover, as noted in Footnote 28 and which can be seen from the proof of the generalized Baire Category Theorem below, in a \( \aleph \)-sequentially complete Hausdorff space every strictly descending nested \( \aleph \)-sequence of non-empty clopen intervals converges to exactly one point. Furthermore, the compactness condition can be replaced by a generalized compactness condition in a \( \aleph \)-sequentially complete Hausdorff topological space called \( \aleph \)-compactness: if for every \( \beta < \aleph \) \( \bigcap_{\alpha < \beta} X_\alpha \neq \emptyset \) then \( \bigcap_{\alpha < \aleph} X_\alpha \neq \emptyset \) for any strictly descending \( \aleph \)-sequence of non-empty clopen intervals \( \langle X_\alpha, \alpha < \aleph \rangle \), i.e., \( X_\beta \subseteq X_\gamma \) if ordinal \( \gamma < \beta \). It is therefore true that a \( \aleph \)-sequentially complete \( \langle 2^\aleph, \oplus \rangle \)-generalized metric space is a \( \aleph \)-compact topological space (i.e., a topological space such that each closed set satisfies the \( \aleph \)-compactness condition).

The proof of the generalized Baire Category Theorem proceeds as follows (broadly following 3.83Ac 213 for the case \( \aleph = \aleph_0 \)). Let us suppose for contradiction that \( X = \bigcup_{\alpha < \aleph} C_\alpha \) for closed nowhere dense sets \( C_\alpha \). We claim we can find a \( \aleph \)-sequence of non-empty closed sets \( \langle D_\alpha : \alpha < \aleph \rangle \) such that \( D_0 \subseteq X \), \( D_\beta \subseteq D_\alpha \) if \( \alpha < \beta \) and \( C_\alpha \cap D_\alpha \subseteq D_{\alpha+1} = \emptyset \). This is possible because for every non-empty open set \( O \), \( O - C_\alpha \) is a non-empty open set as \( O \) has a non-empty interior and \( C_\alpha \) has an empty interior. We choose \( D_0 \subseteq X \) to be a clopen interval (since the space has a clopen base), \( D_{\alpha+1} \subseteq D_\alpha - C_\alpha \) to be a clopen interval (as a clopen subset of the non-empty interior of \( D_\alpha \)) and \( D_\lambda := \bigcap_{\alpha < \lambda} D_\alpha \) for limit ordinals \( \lambda \). We can see that \( D_{\alpha < \aleph} \neq \emptyset \) because at limit ordinals, \( \lambda \), the node from which the clopen interval \( D_\lambda \) splits from some branch \( x \in X \) has an ordinal which is the limit of an \( \alpha \)-sequence for \( \alpha < \aleph \) of ordinals \( < \aleph \) (because the branches are of length \( \aleph \)), and thus the node has an ordinal \( < \aleph \) (by the Axiom of Choice). As \( D_{\alpha < \aleph} \) can be

\[ \text{if } \bigcap_{\beta < \aleph} X_\beta \neq \emptyset \text{ and } y \in \bigcap_{\beta < \aleph} X_\beta - \bigcap_{\beta < \aleph} X_\beta \text{ then } y \notin X_\alpha \text{ for some } \alpha < \aleph, \text{ and since } X_\alpha \text{ is clopen, } y \notin X_\alpha \text{ and hence } y \notin \bigcap_{\beta < \aleph} X_\beta. \text{ Since } \bigcap_{\beta < \aleph} X_\beta \subseteq \bigcap_{\beta < \aleph} X_\beta \text{ by definition of closure, it follows that } y \notin \bigcap_{\beta < \aleph} X_\beta, \text{ contradiction.} \]

\[ \text{In fact the } \aleph \text{-sequence of initial segments from which nested clopen intervals split defines a branch that is in } D_\aleph. \]
viewed as a clopen interval splitting from a branch, and that clopen interval is then split at some branch in the interval at higher ordinals, it follows by $\aleph$-sequential completeness that $D_\alpha$ can be identified with a set containing an $\aleph$-sequence of $2^\aleph$, i.e. a set containing a single point. Using this observation we have $\bigcap_{\alpha<\aleph} D_\alpha = \{x\}$ for $x \in 2^\aleph$, and since $D_0 \subseteq X$, $x \in X$. However, as $C_\alpha \cap D_{\alpha+1} = \emptyset$, $x \notin \bigcup_{\alpha<\aleph} C_\alpha$. Hence $X \neq \bigcup_{\alpha<\aleph} C_\alpha$, as was to be proved.

We can state these results as:

**Theorem 10.** (Generalized Baire Category Theorem) In a $\aleph$-sequentially complete Hausdorff topological space, $X$, that comprises a $\aleph$-sequentially complete set of binary $\aleph$-sequences with a clopen base of cardinality $\aleph$ and with no discrete or isolated points, $X$ is not the union of $< \aleph + 1$-many nowhere dense subsets for $\aleph \geq \aleph_0$.

**Theorem 11.** A $\aleph$-sequentially complete Hausdorff topological space that comprises a $\aleph$-sequentially complete set of binary $\aleph$-sequences is not compact and not metrizable for $\aleph > \aleph_0$, but it is possible to use a generalized metric and every strictly descending $\aleph$-sequence of non-empty clopen intervals converges to exactly one point.

5. A Modal Model of Set Theory

We have seen from Theorem 10 that in a $\aleph$-sequentially complete Hausdorff topological space, $X$, that has a clopen base of cardinality $\aleph$ and with no discrete or isolated points, $X$ is not the union of $< \aleph + 1$-many nowhere dense subsets for $\aleph \geq \aleph_0$. But is it the case that $X$ is the union of $\aleph + 1$ closed nowhere dense sets if the cardinality of $X > \aleph$? The answer is that this result is possible because it can be forced if the $\aleph + 1$ closed nowhere dense sets are dense in $X$, but the forcing is quite natural. [5] provides a clear explanation of set theoretic forcing. It will be seen that the result is independent of Zermelo Fraenkel set theory with the Axiom of Choice (ZFC). The result can be seen by means of the following construction.

If we represent members of a $\aleph$-sequentially complete Hausdorff topological space, $X$, as $< \aleph + 1$-sequences, we can define $X((x_1, \ldots, x_\alpha<\aleph+1); (y_1, \ldots, y_\beta<\aleph+1))$ as the generalized Cantor (i.e. closed nowhere dense) set that results from the construction in Theorem 8 that preserves members of $(x_1, \ldots, x_\alpha<\aleph+1)$ and deletes members of $(y_1, \ldots, y_\beta<\aleph+1)$, where each $x_\gamma \leq \alpha$, $y_\gamma \leq \beta \in X$ and $x_\gamma \neq y_\delta$ for all $\gamma \leq \alpha, \beta \leq \beta$. Note that the choice of $x_\gamma$ depends on $y_\delta \leq y_\gamma$. Consider a $\aleph+1$-sequence $X_{\alpha<\aleph+1}((x_1, \ldots, x_\alpha); (y_1, \ldots, y_\alpha))$ of $\aleph$-sequentially complete closed nowhere dense sets, which is possible because it is always possible to cover a branch of $\aleph$ nodes with disjoint sets of branches with $\aleph$ members. Then we can see that:

$$\bigcup_{\alpha<\aleph+1} X_\alpha((x_1, \ldots, x_\alpha); (y_1, \ldots, y_\alpha)) \cup \bigcup_{\alpha<\aleph+1} X_{\alpha<\aleph+1}((y_1, \ldots, y_\alpha); (x_1, \ldots, x_\alpha))$$

is dense in $X$ (because $(x_1, \ldots, x_\aleph) \cup (y_1, \ldots, y_\aleph)$ is dense in $X$) and it is possible for $(x_1, \ldots, x_{\aleph+1})$ and $(y_1, \ldots, y_{\aleph+1})$ to each have $\aleph+1$ members if the cardinality of $X > \aleph$. While it is not in general true that the union of $\aleph + 1$ closed nowhere dense sets, $X_{\alpha<\aleph+1}$, that are dense in $X$ is $X$ (because a $\aleph$-sequence may exist which is covered by $\aleph$-sequences that is in $X$ but is not in the union), it is true in a natural model of $X$. That model is a transitive outer model model of cardinality $\aleph + 1$ (see for example [3] in the case of countable transitive outer models), in which the
forcing partially ordered functions are \( f : \aleph + 1 \to \{ Y : Y \subseteq X \} \), where \( f_\beta = X_\beta \), where as above each \( X_\beta \) is a function of \( \langle x_1, \ldots, x_\beta \rangle \), \( \langle y_1, \ldots, y_\beta \rangle \) and of the function \( r : \beta \to \beta \) used to control the preservation of \( \langle x_1, \ldots, x_\alpha \rangle \) and the deletion of \( \langle y_1, \ldots, y_\alpha \rangle \). Now in the following keep \( r \) fixed. To see that \( f \) defines a partial ordering, note that \( f_\alpha \subseteq f_\beta \) for \( \aleph + 1 > \alpha > \gamma \) since \( X_\gamma \subseteq X_\alpha \) for \( \langle x_1, \ldots, x_\alpha \rangle \) extending \( \langle x_1, \ldots, x_\gamma \rangle \)\(^3\)  

\[
F = \bigcup_{\beta < \aleph + 1} f_\beta \quad \text{is a function because if:} \\
F(\langle x_1, \ldots, x_{\aleph + 1} \rangle; \langle y_1, \ldots, y_\aleph \rangle) \neq F(\langle w_1, \ldots, w_{\aleph + 1} \rangle; \langle z_1, \ldots, z_\aleph \rangle)
\]

then it follows that:

\[
f_\beta(\langle x_1, \ldots, x_{\gamma \beta} \rangle; \langle y_1, \ldots, y_\aleph \rangle) \neq f_\beta(\langle w_1, \ldots, w_{\gamma \beta} \rangle; \langle z_1, \ldots, z_\aleph \rangle)
\]

for some \( \beta < \aleph + 1 \) by definition of union, and there is a correspondence (possibly many to one) between \( \langle x_1, \ldots, x_\beta \rangle \) and \( X_\beta \). This implies that:

\[
\langle x_1, \ldots, x_{\gamma \beta}; y_1, \ldots, y_\aleph \rangle \neq \langle w_1, \ldots, w_{\gamma \beta}; z_1, \ldots, z_\aleph \rangle
\]

since \( f_\beta \) is a function and hence:

\[
\langle x_1, \ldots, x_{\gamma \beta}; y_1, \ldots, y_\aleph \rangle \neq \langle w_1, \ldots, w_{\gamma \beta}; z_1, \ldots, z_\aleph \rangle.
\]

\( F \) is onto \( X - \bigcup_{\beta < \aleph} \{ y_\beta \} \) because if \( x \neq y_\beta < \aleph \) and \( x \notin \text{ran}(F) \) for \( x \in X \) then for some \( \gamma < \aleph + 1 \) we can add \( x \) to be preserved by \( X_\gamma \) and all \( X_\alpha > \gamma \) for \( \alpha < \aleph + 1 \). Since the same argument works for \( \langle y_1, \ldots, y_\aleph \rangle; \langle x_1, \ldots, x_{\alpha < \aleph + 1} \rangle \) with \( G = \bigcup_{\beta < \aleph + 1} g_\alpha \), showing \( G \) is onto \( X - \bigcup_{\beta < \aleph} \{ x_\alpha \} \), we see that \( F \cup G \) is a function onto \( X \). This model is natural because it is completely described by binary \( \aleph + 1 \)-sequences that can be instantiated and that control membership of the closed nowhere dense sets.

To see that this result is independent of ZFC, we note that the function \( F \cup G \) is a function from a set of cardinality \( \aleph + 1 \) onto a set of cardinality \( 2^\aleph \) (i.e. from \( \aleph + 1 \) onto \( X \)). Hence \( \aleph + 1 \geq 2^\aleph \). Since \( 2^\aleph \geq 2^{\aleph + 1} \) by Cantor’s theorem, GCH follows. Conversely if GCH is true, any union of closed nowhere dense sets, such as \( \{ x \} \) for \( x \in X \), has cardinality \( \aleph + 1 = 2^\aleph \), and hence \( X \) is the union of \( \aleph + 1 \) closed nowhere dense sets.

We may state this result as:

**Theorem 12.** *(Not provable in ZFC, equivalent to GCH)* In a \( \aleph \)-sequentially complete Hausdorff topological space, \( X \), that comprises a \( \aleph \)-sequentially complete set of binary \( \aleph \)-sequences with a clopen base of cardinality \( \aleph \) and with no discrete or isolated points, \( X \) is the union of \( \aleph + 1 \)-many nowhere dense subsets for \( \aleph \geq \aleph_0 \).

The construction showing that \( X \) is the union of \( \leq \aleph + 1 \) closed nowhere dense sets is naturally constructed in \( V_\alpha \) for some \( \alpha > \omega(\aleph_0) = \omega \), and uses the following argument: if a counterexample could be produced, the construction could be applied to the counterexample, showing that the counterexample would not be an actual counterexample. It natural to think of these constructions taking place in a modal model of ZFC (such as the S4 modal model of [10]). As a reminder, an S4-modal

\(^3\)The \( \aleph + 1 \)-sequence of sets \( \langle X_\alpha < \aleph + 1 \rangle \) must be eventually constant \( C X \in < \aleph + 1 \) steps; otherwise the preservation of \( \aleph + 1 \) members of \( X \) will result in \( X \).
model of set theory is a 4-tuple \((G, R, D^G, F)\), where \(G\) is a set of forcing conditions, \(R\) is a reflexive and transitive relation on \(G\), \(D^G\) is the domain of sets corresponding to \(G\) and \(F\) is a mapping from forcing conditions to quantifier-free sentences in set theory with constants in \(D^G\) such that \(V\) can be extended to all sentences in set theory with \(D^G\) by means of the forcing relation \(\Vdash\). We have \(p \Vdash A\) if \(A \in F(p)\), \(p \Vdash \neg X\) if \(p \not\Vdash X\), \(p \Vdash X \land Y\) if \(p \Vdash X\) and \(p \Vdash Y\), \(p \Vdash X \lor Y\) if \(p \Vdash X\) or \(p \Vdash Y\), \(p \Vdash (\exists x) P(x)\) if \(p \Vdash P(d)\) for some \(d \in D^G\), \(p \Vdash (\forall x) P(x)\) if \(p \Vdash P(d)\) for all \(d \in D^G\), \(p \Vdash \neg X\) if \(q \Vdash X\) for every \(q \in G\) such that \(R(p, q)\), and \(p \Vdash \exists X\) if \(q \Vdash X\) for some \(q \in G\) such that \(R(p, q)\). In a modal model a sentence of set theory \(X\) is translated to a sentence of modal set theory written \([X]\), by induction: \([A] = \Box A\) for atomic \(A\), \([\neg X]\) = \(\Box \neg [X]\), \([X \land Y]\) = \(\Box [X] \land [Y]\), \([X \lor Y]\) = \(\Box [X] \lor [Y]\), \([\exists x] P(x)\) = \(\Box (\exists x)[P(x)]\), \([\forall x] P(x)\) = \(\Box (\forall x)[P(x)]\), and it is proven that the translation of every instance of an axiom of \(ZFC\) is true for each forcing condition of the model. The model that we have constructed is then \(\langle \{x_\alpha, y_\alpha, X_\alpha: \alpha < \aleph + 1\}, \subseteq, 2^\aleph, F : \{x_\alpha y_\alpha, X_\alpha\} \rightarrow \{x_\alpha \in \aleph \in X_{\beta \geq \alpha}, y_\alpha \notin X_{\beta \geq \alpha}\}\rangle\) where \(\{x_\alpha y_\alpha, X_\alpha\}\) are as described in Theorem 8.

6. A Generalized Metric Model of Set Theory

We can also use the fact (see Theorem 11) that any initial segment of \(V, V_\alpha = 2^\aleph\) for some cardinal \(\aleph\), can be considered as a \((2^\aleph, \oplus)\)-generalized metric space with the Baire topology on any set \(X \subseteq 2^\aleph\) comprising binary \(\aleph\)-sequences (or equivalently \(\aleph\)-sequences of real numbers). That is to say, that for infinite \(\alpha\) and \(\aleph V_\alpha\) can be represented as a clopen interval \(([0, 1])[\aleph]\) for binary \(\aleph\)-sequences for some length \(\aleph\). As we have seen, if all real numbers in the \(\aleph\)-sequence start with the same number \((0, 0\text{ in the case of }([0, 1])[\aleph])\) then all binary \(\aleph\)-sequences can still be represented (by ignoring the constant number before the binary point). It is therefore reasonable to represent \(V_\alpha\) as a clopen interval \(([0, 1])[\aleph]\) for binary \(\aleph\)-sequences for some length \(\aleph\). That \(V_\alpha\) is a generalized metric space does not alter what sets exist, as those sets will be sets of binary \(\aleph\)-sequences (for example most will not be \(\aleph\)-sequentially complete); the constraint of being a generalized metric space only determines how far apart points in the space are.

It is then possible to decide the membership of \(X\) in \(\aleph \in \aleph + 1\) steps by enumeration as follows. Consider a clopen interval, \(([0, 1])[\aleph]\), which is linearly ordered lexicographically, \(i.e.\ z < y\) if \((\exists \alpha < \aleph)(z_\alpha < y_\alpha) \land (\forall \beta < \alpha)(z_\beta = y_\beta)\) for \(v_\alpha\) the \(\alpha\)-th binary member of the \(\aleph\)-sequence \(w\), and assume that each binary \(\aleph\)-sequence \(z\) in \(2^\aleph\) is marked with 1 or 0 depending whether \(z \in \aleph\) or not, which is decidable only if you find the location of \(z\) in the interval. The latter assumption reflects the fact that when you search for \(x\) in a linearly ordered set it is either present in its place in the order (when \(x \in \aleph\)) or it is not (when \(x \notin \aleph\)). Before we begin the construction, we will need the ability to divide a binary \(\aleph\)-sequence (of a \(\aleph\)-sequence of real numbers) by 2. This is just standard binary division by 2 with carries to the right if necessary.

To start the construction, bisect the interval to give a point \(m = (0.1, 0.0, 0.0, \ldots)\). Now set \(r \colonequals m\). If the midpoint \(r = x\) then we can decide whether \(x \in \aleph\) or \(x \notin \aleph\) and stop. Otherwise test whether \(x < r\). If \(x < r\) then consider the
clopen interval \(((0, r)][\aleph] \); and if \(x > r\) consider the clopen interval \(([r, 1)][\aleph]\). Iterate the bisection construction as follows\(^{35}\): \(\text{cl}_1 = ([0, 1)][\aleph] \), \(\text{cl}_{\alpha+1} = \text{Bi}(\text{cl}_\alpha; x)\) and \(\text{cl}_\lambda = \bigcap_{\alpha < \lambda} \text{cl}_\alpha\) for limit ordinal \(\lambda\) (which is the unique maximal clopen interval \(\subseteq ([0, 1)][\aleph]\) such that for all \(z \in \text{cl}_\lambda\) the initial \(\lambda\)-sequence of \(z\) is \(x[\lambda] := \langle x_\alpha : \alpha < \lambda \rangle\), i.e. \((x[\lambda] || \langle 0, 0, 0, \ldots \rangle, x[\lambda] || \langle 1, 1, 1, \ldots \rangle)\)), where \(||\) is concatenation, \((0, 0, 0, \ldots)\) and \((1, 1, 1, \ldots)\) are \(\mathbb{N}\)-sequences that stand for \(\mathbb{N}\) concatenated \(\omega\)-sequences \((0, 0, 0, \ldots)\) and \((0, 1, 1, \ldots)\) respectively, \(\text{Bi}(([a,b])[\aleph]; x) = ([a,r])[\aleph]\) and \(x_\alpha = 0\) if \(([a,b])[\aleph] = d_\alpha\) and \(x < r\) for the midpoint \(r = (b-a)/2\), \(\text{Bi}(([a,b])[\aleph]; x) = ([r,b])[\aleph]\) and \(x_\alpha = 1\) if \(([a,b])[\aleph] = d_\alpha\) and \(x > r\), and the iteration stops if \(x = r\) (and one can decide whether \(r \in X\)). It is clear that the construction will terminate in \(\leq \aleph\) steps, as a nested sequence of clopen intervals can only comprise \(\aleph\) members, as that is how many bits there are in the single binary \(\aleph\)-sequence in any non-empty intersection of a nested sequence of clopen intervals. If \(x \in X\) has not been confirmed in \(< \aleph\) steps, then at the \(\aleph\) step \(\text{cl}_\aleph = ([x,x]) = \{ x \}\), and at ordinal step \(\aleph + 1\) (i.e. 1 after \(\aleph\)) we can then decide whether \(x \in X\) given that \(x\) has been located.

But the condition that \(x \in X\) can be decided by enumeration in \(\leq \aleph\) steps is equivalent to GCH as can be shown as follows.

**Theorem 13.** (Not provable in ZFC) GCH is equivalent to\(^{36}\) the assertion that the amount of information needed to decide the relation \(x \in X\) by an interleaved enumeration of \(X\) or \(2^\aleph - X\) is \(\leq \aleph + 1\), for any given binary \(\aleph\)-sequence \(x\) of length at most cardinal \(\aleph \geq \aleph_0\) and \(X\) has cardinality \(\leq 2^\aleph\).

**Proof.** Assume that:

a) \(\emptyset \subseteq X \subseteq 2^\aleph\),

b) \(X\) has cardinality \(\aleph < c < 2^\aleph\),

C) Any \(x \in X\) is expressed as a binary sequence of length at most cardinal \(\aleph \geq \aleph_0\), and

\(^{35}\)The clopen intervals are not subsets of \(X\) in general but are subsets of \(2^\aleph\).

\(^{36}\)Strictly the inference from the information limitation principle to GCH is probabilistic (true almost always) in cardinality terms rather than logically necessary.
d) The amount of information needed to decide the relation \( x \in X \) by an interleaved enumeration of \( X \) or \( 2^\aleph_0 - X \) is \( < \aleph_0 + 1 \).

The proof is summarized in the tables below, where a \( \checkmark \) means that the option is possible and \( \times \) means that the option is impossible.

|                  | Enumerate \( X \) | Enumerate \( 2^\aleph_0 - X \) |
|------------------|-------------------|-------------------------------|
| \( x \in X \)    | \( < c \checkmark \) | \( 2^\aleph_0 \times \)       |
| \( x \notin X \) | \( c \times \)     | \( < 2^\aleph_0 \checkmark \) |

*Table 1: The number of steps to decide \( x \in X \) by enumeration*

| \( < c \) | Proof Ref. | \( c \) | Proof Ref. |
|-----------|------------|--------|------------|
| \( \aleph_0 + 1 < c \times \) | 1 | \( \aleph_0 + 1 < c \times \) | 4 |
| \( \aleph_0 + 1 = c \checkmark \) | 2 | \( \aleph_0 + 1 = c \times \) | 5 |
| \( \aleph_0 + 1 > c \times \) | 3 | \( \aleph_0 + 1 > c \times \) | 3 |

| \( < 2^\aleph_0 \) | Proof Ref. | \( 2^\aleph_0 \) | Proof Ref. |
|-------------------|------------|----------------|------------|
| \( \aleph_0 + 1 < 2^\aleph_0 \times \) | 1 | \( c < 2^\aleph_0 \times \) | 8 |
| \( \aleph_0 + 1 = 2^\aleph_0 \checkmark \) | 6 | \( c < 2^\aleph_0 \times \) | 8 |
| \( \aleph_0 + 1 > 2^\aleph_0 \times \) | 7 | \( c < 2^\aleph_0 \times \) | 8 |

*Table 2: The possible cardinal relationships for the number of steps in Table 1 and proof references*

Proof references:

1. \( x \in X \) would almost always be decided in \( \geq \aleph_0 + 1 \) bits for a given enumeration of \( X \), contradicting assumption d).
2. \( \aleph_0 + 1 = c \) is consistent with assumption d), as \( x \in X \) would be decided in \( < c = \aleph_0 + 1 \) steps by enumeration.
3. \( \aleph_0 + 1 > c \) contradicts assumption b) \( \aleph_0 < c \), as there would be a cardinal strictly between \( \aleph_0 \) and \( \aleph_0 + 1 \).
4. \( x \in X \) would almost always be decided in \( > \aleph_0 + 1 \) bits for a given enumeration of \( X \), contradicting assumption d).
5. \( \aleph_0 + 1 = c \) implies that \( \aleph_0 + 1 \) bits are needed to decide \( x \in X \) by enumerating all of \( X \), which contradicts assumption d).
6. \( \aleph_0 + 1 = 2^\aleph_0 \) is consistent with assumption d), as \( x \in X \) would be decided in \( < 2^\aleph_0 = \aleph_0 + 1 \) steps by enumeration.
7. \( \aleph_0 + 1 > 2^\aleph_0 \) contradicts Cantor’s theorem that \( \aleph_0 + 1 \leq 2^\aleph_0 \).
8. \( c < |2^\aleph_0 - X| = 2^\aleph_0 \) and therefore \( x \in X \) could always be decided in \( < 2^\aleph_0 \) steps by enumeration of \( X \).

We can conclude that if \( x \in X \) then \( c = \aleph_0 + 1 \) and if \( x \notin X \) then \( \aleph_0 + 1 = 2^\aleph_0 \). Using predicate logic\(^{37}\) we can conclude \((\exists x)(x \in X) \rightarrow c = \aleph_0 + 1 \) and \((\forall x)(x \in 2^\aleph_0 - X) \rightarrow \aleph_0 + 1 = 2^\aleph_0 \). Since both \( X \) and \( 2^\aleph_0 - X \) are not empty we can conclude that \( c = \aleph_0 + 1 = 2^\aleph_0 \), which contradicts the assumption that \( c < 2^\aleph_0 \). GCH then

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\(^{37}\)Existential elimination: for example, assume \((\exists x)(x \in X)\) and \((\forall x)(x \in X \rightarrow c = \aleph_0 + 1)\), then if \( c \neq \aleph_0 + 1 \) then by contraposition \((\forall x)(x \notin X)\) and hence \( \neg(\exists x)(x \in X) \), contradiction; hence \( c = \aleph_0 + 1 \).
Conversely, assume GCH. Then if \( x \in X \) then by GCH \( x \) will be enumerated in \( < |X| \leq 2^\aleph_0 = \aleph_0 + 1 \) steps. While if \( x \notin X \) then \( x \) will be enumerated in \( < |2^\aleph_0 - X| = 2^\aleph_0 = \aleph_0 + 1 \) steps. In either case then \( x \in X \) can be decided by enumeration in \( < \aleph_0 + 1 \) steps, i.e. in \( < \aleph_0 + 1 \) bits. \( \square \)

Remark 14. What this result shows that if the class of all pure sets \( V \) is considered to be a hierarchy of \( \langle 2^\aleph_0, \oplus \rangle \)-generalized metric spaces, then GCH holds. It is of course not true that the class of all pure sets in \( V \) needs to be a hierarchy of \( \langle 2^\aleph_0, \oplus \rangle \)-generalized metric spaces, but it is a natural construction of \( V \) based on a natural topology of sets.

7. Alternative Definition of Cardinality

Having shown that there are models of ZFC in which Theorems 10 and 12 are true, we can now redefine cardinality to reflect cardinality in this model (in which GCH is true). In terms of the normal definition of cardinality, \( 2^\aleph_0 - (\aleph_0 + 1) \times 2^\aleph_0 = \emptyset \) is not surprising; it simply says that \( 2^\aleph_0 - \max(\aleph_0 + 1, 2^\aleph_0) = 2^\aleph_0 - 2^\aleph_0 = \emptyset \). However, the fact that \( \aleph_0 + 1 \) is the least cardinal number with the property that \( 2^\aleph_0 - (\aleph_0 + 1) \times 2^\aleph_0 = \emptyset \) is forced in the Baire topology suggests a modification to the definition of cardinal number. Intuitively, the idea is that iterating the closed nowhere dense set construction in a dense way on a nowhere dense set will produce ever sparser nowhere dense sets, but the deletion of \( \aleph_0 + 1 \) such nowhere dense sets in a dense way results in the empty set. Cardinality in these terms then measures how sparse a set can be before it ceases to exist. Or, in the spirit of the Baire Category Theorem, cardinality measures how many negligible sets you need to add together before a non-negligible set is formed.

This argument suggests a change of definition of cardinality, namely a set \( X \) (represented as a tree \( T \)) has cardinality \( \aleph_0 \) if \( \aleph_0 \) is the largest cardinal such that every dense (in the sense that every non-empty open set of the tree has non-empty intersection with the sequence), non-repeating sequence of (clopen) splitting subtrees of \( T \) of length \( \aleph_0 \) has an empty remainder after removal of \( \aleph_0 \) subtrees in this sequence from \( T \), while it is possible to construct a non-empty remainder after removal of any subsequence of length \( < \aleph_0 \). This construction is always possible because the number of splitting subtrees is the same as the number of nodes and the branch length, and it is always possible to delete \( < \aleph_0 \)-many paths (i.e. \( \aleph_0 \)-sequences that are not branches) or branches in a way that leaves a dense sequence of splitting subtrees.\(^{38} \) This is so by the result in Section 4 that any set \( X \) with a dense-in-itself subset and of cardinality \( \aleph_0 < c \leq 2^\aleph_0 \) has, under the Baire topology, the property that \( c - c \times (\aleph_0 + 1) = \emptyset \) and it is possible to force \( c - c \times \aleph_0 \neq \emptyset \) using the closed nowhere dense set construction.

In logical terms the change in definition of cardinality can be stated as follows:

- \( |T| = \alpha \leftrightarrow (P(T, \alpha) \land (\forall \gamma : Card(\gamma))(\gamma \geq \alpha \rightarrow \neg P(T, \gamma))) \), where 
- \( \alpha \) is a cardinal, \( Card(\alpha) \)

\(^{38} \)The condition to delete \( < \aleph_0 \)-many branches ensures that a tree of cardinality \( \aleph_0 \) does not have cardinality \( \aleph_0 + 1 \)
T is a binary tree with a root

- $P(T, \beta) := \forall u_{\eta<\beta} : S(T, \{u_{\eta<\beta}\})(\bigcap_{\eta<\beta} u_\eta = \emptyset) \land (\forall \gamma < \beta \exists u_\delta < \gamma : S(T, \{u_{\delta<\gamma}\})(\bigcap_{\delta<\gamma} u_\delta \neq \emptyset))$

- $S(T, \{u_{\eta<\beta}\}) := \forall u_\theta : \forall \alpha < \beta : \forall \gamma < \beta (\forall u_\eta \in \{u_{\eta<\beta}\})(\forall u_\lambda \in \{u_{\eta<\beta}\})(u_\theta = u_\lambda \rightarrow \theta = \lambda)$ [non-repeating sequence]

- $D(T, \{u_{\eta<\beta}\}) := (\exists \theta \leq \alpha)(\forall u_\beta \in \{u_{\eta<\beta}\}) : B(T, \{u_{\eta<\beta}\})(\exists u_\kappa \in \{u_{\eta<\beta}\})(w_\delta \cap u_\kappa \neq \emptyset)$ [dense sequence]

- $C(T, \{u_{\eta<\beta}\}) := (\forall \rho < \xi)(u_\rho \neq \emptyset \land \mathrm{ClopenSplit}(T, u_\rho))$ [sequence of non-empty clopen splitting subtrees]

- $\mathrm{ClopenSplit}(T, u) := (\exists x(\exists \beta : \mathrm{Ord}(\beta))(u = \{y \in T : y \neq x \land (\forall \gamma < \beta)(x_\gamma = y_\gamma)\}))$

- $\mathrm{Ord}(\alpha) := (\alpha = 0 \lor (\exists \beta < \alpha)(\mathrm{Ord}(\beta) \land \alpha = \beta \lor \{\beta\}) \lor (\alpha = \bigcup_{\beta < \alpha} \mathrm{Ord}(\beta)) \lor (\exists f : \beta \rightarrow \alpha) \mathrm{Sur}(f)\}$

- $\mathrm{Sur}(f) := F_{n}(f : X \rightarrow Y) \land (\forall y)(\exists x(y = f(x)) \land (\forall x \in X)(\forall y \in Y)(x = y \rightarrow f(x) = f(y))$

In the case of sets of natural numbers, a number $n$ can be represented by a sequence $\langle 1, \ldots, 1, 0, \ldots \rangle$, i.e. $n$ 1s and then a terminal 0-sequence of 0s. Removal of a dense sequence of clopen subtrees is visually the removal of all of the terminal $\omega$-sequences of 0s (and of course the initial sequence of 1s) because in the discrete topology each set $\{n\}$, i.e. $<1$. Thus $\aleph_0 - \aleph_0 \times \aleph_0 = \emptyset$ has solution $\aleph_0 = \aleph_0$; and $\aleph_0$ has cardinality $\aleph_0$ and a finite set with $n$ members has cardinality $n$, as before.

It can be seen that according to the modified definition of cardinality $2^{\aleph_0} = \aleph_1$ for all cardinals $\aleph_k > \aleph_0$, and there are no cardinals $\aleph_k + 1 > \aleph_k < 2^{\aleph_0}$, i.e. that the Generalized Continuum Hypothesis is true in the sense of the new definition of cardinality.

8. Conclusions

There is a natural way to measure size of sets, which is given by how many clopen subtrees need to be deleted in a dense way from a binary tree of binary $\aleph$-sequences before the empty set results (or a countable set of isolated points). This definition of cardinality works well for a set universe which satisfies the Axiom of Choice, when all sets of size $\leq 2^{\aleph_0}$ are sets of binary $\aleph$-sequences. The price to be paid for the use of a Baire topology (in which clopen subtrees exist) is that the Baire topology is pathological in several respects: clopen sets are totally disconnected by definition, and Baire topological spaces, such as clopen interval $([0,1])[\aleph_0]$, comprise a set of binary $\aleph$-sequences that are not compact or metrizable (for $\aleph > \aleph_0$). That said, since all sets can be regarded as $\aleph$-tuples of real numbers, there is a natural generalized metric and in a $\aleph$-sequentially complete topological space, every strictly nested decreasing $\aleph$-sequence of clopen intervals is a set with a single element. The class of sets is then quite well behaved under the assumption of the Axiom of Choice,

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39A topological space $X$ is **totally disconnected** if for every two points $x, y \in X$ such that $x \neq y$ there are disjoint open sets $O_1$ and $O_2$ such that $x \in O_1, y \in O_2$ and $O_1 \cup O_2 = X$. 
although this good behaviour does not extend to the properties of sets that can be created using the Axiom of Choice (see [8] for a selection of such sets).

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