ASYMPTOTICS OF RESONANCES FOR 1D STARK OPERATORS

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Abstract. We consider the Stark operator perturbed by a compactly supported potentials on the real line. We determine forbidden domain for resonances, asymptotics of resonances at high energy and asymptotics of the resonance counting function for large radius.

1. Introduction and main results

1.1. Introduction. We consider the operator $H = H_0 + V$ acting on $L^2(\mathbb{R})$, where the unperturbed operator $H_0 = -\frac{d^2}{dx^2} + x$ is the Stark operator. Here $x$ is an external electric field and the potential $V = V(x)$, $x \in \mathbb{R}$ is real and satisfies $V \in L^2_{\text{real}}(\mathbb{R})$, $\text{supp} \ V \subset [0, \gamma]$ for some $\gamma > 0$. (1.1)

The operators $H_0$ and $H$ are self-adjoint on the same domain since the operator $V(H_0 - i)^{-1}$ is compact, see e.g. [28, 12, 24, 21]. The spectrum of both $H_0$ and $H$ is purely absolutely continuous and covers the real line $\mathbb{R}$ (see [28, 12, 24]). It is well known that the wave operators $W_{\pm}$ for the pair $H_0, H$ given by

$$W_{\pm} = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$$

exist and are unitary (even for more large class of potentials than considered here, see [28, 12, 24]). Thus the scattering operator $S = W_+ W_-$ is unitary. The operators $H_0$ and $S$ commute and then are simultaneously diagonalizable:

$$L^2(\mathbb{R}) = \bigoplus_{\mathbb{R}} \mathcal{H}_\lambda d\lambda, \quad H_0 = \int_{\mathbb{R}} \lambda I_\lambda d\lambda, \quad S = \int_{\mathbb{R}} S(\lambda) d\lambda;$$

(1.2)

where $I_\lambda$ is the identity in the fiber space $\mathcal{H}_\lambda = \mathbb{C}$ and $S(\lambda)$ is the scattering matrix (which is a scalar function of $\lambda \in \mathbb{R}$ in our case) for the pair $H_0, H$. The function $S(\lambda)$ is continuous in $\lambda \in \mathbb{R}$ and satisfies $S(\lambda) = 1 + o(1)$ as $\lambda \to \pm \infty$. The function $S(\lambda)$ has an analytic extension into the upper half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Im} \lambda > 0\}$ and a meromorphic extension into the lower half-plane $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \text{Im} \lambda < 0\}$, see e.g. [21]. By definition, a zero $\lambda_o \in \mathbb{C}_+$ (or a pole $\lambda_o \in \mathbb{C}_-$) of $S$ is called a resonance. The multiplicity of the resonance is the multiplicity of the corresponding zero (or a pole) of $S$.

Condition C. The potential $V$ satisfies (1.1) and a following asymptotics

$$\int_0^\gamma e^{ikx}V(x)dx = \frac{C_p}{(-i2k)^p} + \frac{O(1)}{k^\nu},$$

(1.3)

as $k \in \mathbb{C}_+, |k| \to \infty$ uniformly in $\arg k \in [0, \pi]$, where $p \in (\frac{1}{2}, 1)$, $p < \nu$ and $(-ik)^p = e^{-ip\frac{\pi}{2}k^p}$.
Remark. 1) Here and below for $\alpha > 0, \lambda \in \mathbb{C}_+$ we define
\[
\lambda^\alpha = |\lambda|^{\alpha} e^{i\alpha \arg \lambda}, \quad \arg \lambda \in [0, \pi] \quad \log \lambda = \log |k| + i \arg \lambda.
\] (1.4)

2) Let $V(x) = \frac{C}{x^4} + V_1(x), x \in [0, \gamma]$ for some $p \in (\frac{1}{2}, 1)$, where $V, V_1, V_1'$ satisfy (1.1). Then $V$ satisfies (1.3) with $C_p = C_1 \Gamma(p)$, see Lemma 1.1.

1.2. Main results. Resonances for the Stark operator perturbed by a compactly supported potential (of a certain class) on the real line were considered in [21]. The following results were proved:

- upper and lower bounds on the number of resonances in complex discs with large radii,
- the trace formula in terms of resonances only,
- it was shown that all resonances determine the potential uniquely.

Our main goal is to obtain global properties of the resonances of $H$. We describe the forbidden domain for resonances.

**Theorem 1.1.** Define the function $\xi(\lambda) = \frac{-i\lambda^{2}}{2\sqrt{\lambda}}, \lambda \in \mathbb{C}_+, \sqrt{\lambda} \in \mathbb{C}_+$ and let $|\lambda| \to \infty$.

i) Let $V$ satisfy (1.1) and $\arg \lambda \in \left[\frac{2\pi}{3}, \pi\right]$. Then $S(\lambda)$ satisfies
\[
S(\lambda) = 1 + \xi(\lambda) O(1),
\] (1.5)
and there are no resonances in the set $\{\varphi \in \left[\frac{2\pi}{3}, \pi\right], |\lambda| \geq \varphi\}$ for some $\varphi > 0$ large enough.

ii) Let $V$ satisfy (1.3) and let $\arg \lambda \in \left[\varepsilon, \frac{2\pi}{3} - \varepsilon\right]$ for any $\varepsilon > 0$. Then $S(\lambda)$ satisfies
\[
S(\lambda) = \xi(\lambda) \frac{C_p + o(1)}{(-i2k)^{p}},
\] (1.6)
and there are no resonances in the set $\{\varphi \in \left[\varepsilon, \frac{2\pi}{3} - \varepsilon\right], |\lambda| \geq \varphi\}$ for some $\varphi > 0$ large enough.

**Remark.** 1) Thus we have $S(\lambda) \to 1$ in (1.5) and $|S(\lambda)| \to \infty$ in (1.6).

2) Consider the Schrödinger operator $hy = -y'' + Vy, y(0) = 0$ with a compactly supported potential $V$ on the half-line. The resolvent $(h - \lambda)^{-1}$ has a meromorphic extension from the first sheet $\Lambda_1 = \mathbb{C} \setminus [0, \infty)$ of the two sheeted simple Riemann surface of the function $\sqrt{\lambda}$ on the second sheet $\Lambda_2 = \mathbb{C} \setminus [0, \infty)$. Each pole defines the resonance and the set of resonances is symmetric with respect to the real line. We consider resonances in $\mathbb{C}_+ \subset \Lambda_2$. Here there are infinitely many resonances (see [31]) and multiplicity of a resonance can be any number (see [15]). Denote by $\mathfrak{N}(r)$ the number of resonances having modulus $\leq r$, each zero being counted according to its multiplicity. Zworski’s result [31] gives
\[
\mathfrak{N}(r) = \frac{2}{\pi} r^\frac{1}{2} (\gamma + o(1)) \quad \text{as} \quad r \to \infty.
\] (1.7)

Moreover, each resonance $\lambda_o$ satisfies $|\lambda_o| \leq C_0 e^{4|\text{Im} \sqrt{\lambda_o}|}$, where $C_0 = \|q\|_1 e^{\|q\|_1}$ and $\|q\|_1 = \int_{\mathbb{R}_+} |q(x)| \, dx$ (see [15]). This gives the forbidden (so-called logarithmic) domain $\{\lambda \in \mathbb{C}_+: |\lambda| > C_0 e^{4|\text{Im} \sqrt{\lambda}|}\}$ for the resonances. Note that due to (1.5) the forbidden domain for $H$ has the form $\{\lambda \in \mathbb{C}_+: |\lambda| > \rho, \arg \lambda \in \left[\frac{2\pi}{3}, \pi\right]\}$ for some $\rho > 0$, see Fig. 1.
1.3. Asymptotics of resonances. Define numbers
\[ z^+_n = \frac{\pi}{2}(p+2) + i \log \frac{3^b}{C_p}, \quad z^-_n = z^+_n - b\pi, \quad b = \frac{p+1}{3}, \quad \rho_r = \pi(2r-1) + \text{Re} z^+_n - \frac{b\pi}{2}. \tag{1.8} \]

**Theorem 1.2.** Let \( V \) satisfy Condition C and let \( b, z_* \), \( \rho \) be given by (1.8) for some integer \( r > 1 \) large enough. Then the function \( S(\lambda) \) in the domain \( D_r = C_+ \setminus \{ |\lambda| < \rho_r^2 \} \) have only simple zeros \( \lambda_{n}^\pm \), \( n \geq r \) labeled by 
\[ |\lambda_{n}^-| < |\lambda_{n+1}^+| < ... \]
with asymptotics
\[ \lambda_{n}^\pm = \left( \pm 3\pi n \right)^2 \left( 1 \pm \frac{i b \log |2\pi n| + z^\pm_n}{3\pi n} + O(1) \right) \] \tag{1.9}
as \( n \to \infty \).

**Remarks.** Note that \( \text{Im} \lambda_{n}^+ = c \log \frac{n}{3^r} + ... \) as \( n \to \infty \). Thus the sequence of the resonances \( \lambda_{n}^+ \in C_+ \) on the second sheet is more and more close to the real line. At the same time we have \( \|Y(\lambda)\| \to 0 \) as \( |\lambda| \to \infty \) on the first sheet \( \lambda \in \overline{C_-} \), see (2.2). It means that perturbed resolvent has the residues at the simple resonances \( \lambda_{n}^+ \), which go zero very fast at large \( n \to \infty \).

Denote by \( \mathcal{N}(r) \) the number of zeros in \( C_+ \) (resonances of \( H \)) of \( S \) having modulus \( \leq r \) and counted according to multiplicity.

**Corollary 1.3.** Let \( V \) satisfy Condition C. Then the counting function \( \mathcal{N}(r) \) satisfies
\[ \mathcal{N}(r) = \frac{4r^2}{3\pi}(1 + o(1)) \quad \text{as} \quad r \to \infty. \tag{1.10} \]

**Remarks.** 1) In the Zworski asymptotics (1.7) the first terms depends on the diameter of support of a potential. In the Stark case (1.10) the first term does not depends on the potential.

2) Roughly speaking, the number of resonances of the perturbed Stark operator \( H \) on the real line corresponds to one for the Schrödinger operator on \( \mathbb{R}^3 \).
1.4. **Brief overview.** A lot of papers are devoted to resonances of the one-dimensional Schrödinger operator, see Froese [5], Hitrik [10], Korotyaev [15], Simon [30], Zworski [31] and references given there. Inverse problems (characterization, recovering, uniqueness) in terms of resonances were solved by Korotyaev for a Schrödinger operator with a compactly supported potential on the real line [17] and the half-line [15], see also Zworski [32], Brown-Knowles-Weikard [2] concerning the uniqueness. The resonances for one-dimensional operators $-\frac{d^2}{dx^2} + V_\pi + V$, where $V_\pi$ is periodic and $V$ is a compactly supported potential were considered by Firsova [4], Korotyaev [18], Korotyaev-Schmidt [20]. Christian森 [C06] considered resonances for steplike potentials. Lieb-Thirring type inequality for the resonances was determined in [19]. The “local resonance” stability problems were considered in [16], [25].

Next, we mention some results for one-dimensional perturbed Stark operators:

- the scattering theory was considered by Rejto-Sinha [28], Jensen [12], Liu [24];
- the inverse scattering problem are studied by Calogero-Degasperis [3], Kachalov-Kurylev [14], Kristensson [22], Lin-Qian-Zhang [23];
- there are a lot of results about the resonances, where the dilation analyticity techniques are used, see e.g., Herbst [6], Jensen [13] and references therein. Note that compactly supported potentials are not treated in these papers.
- There are interesting results about resonances for one-dimensional Stark-Wannier operators $-\frac{d^2}{dx^2} + \varepsilon x + V_\pi$, where the constant $\varepsilon > 0$ is the electric field strength and $V_\pi$ is the real periodic potential: Agler-Froese [1], Herbst-Howland [7], Jensen [11].

Finally we note the that Herbst and Mavi [8] considered resonances of the Stark operator perturbed by delta-potentials.

1.5. **Plan of the paper.** In Section 2 we recall well known results on basic estimates for the Stark operator in a form useful for our approach. In Section 3 we prove the main results. The Appendix contains technical estimates needed to obtain sharp asymptotics [19].

### 2. Properties of S-matrix

2.1. **The well-known facts.** We denote by $C$ various possibly different constants whose values are immaterial in our constructions. We introduce resolvents $R(\lambda) = (H - \lambda)^{-1}$ and $R_0(\lambda) = (H_0 - \lambda)^{-1}$ and operators $Y, Y_0$ by

$$
Y(\lambda) = |V|^{\frac{1}{2}} R(\lambda) V^{\frac{1}{2}}, \quad Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}, \quad \lambda \in \mathbb{C}_\pm,
$$

$$
V = |V|^{\frac{1}{2}} V^{\frac{1}{2}}, \quad V^{\frac{1}{2}} = |V|^{\frac{1}{2}} \text{sign } V. \quad (2.1)
$$

Let $B_1$ be the trace class equipped with the norm $\| \cdot \|_{B_1}$. We recall results from [21].

**Lemma 2.1.** Let the potential $V$ satisfy (1.1) and let $\alpha < 1$. Then and the operator-valued functions $Y_0(\lambda)$ and $Y(\lambda)$ are uniformly Hölder on $\mathbb{C}_\pm$ in the $B_1$–norm and satisfy

$$
\sup_{\lambda \in \mathbb{C}_\pm} (1 + |\lambda|)^{\frac{1}{\alpha}} (\|Y_0(\lambda)\|_{B_1} + \|Y(\lambda)\|_{B_1}) < \infty. \quad (2.2)
$$

Moreover, they have meromorphic extensions into the whole complex plane.
2.2. The spectral representation for $H_0$. We will need some facts concerning the spectral decomposition of the Stark operator $H_0$. Let $\mathcal{U} : f \mapsto \tilde{f}$ be the unitary transformation on $L^2(\mathbb{R})$, which can be defined on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by the explicit formula

$$
\tilde{f}(x) = (\mathcal{U}f)(x) = \int_{\mathbb{R}} \text{Ai}(y-x)f(y)dy,
$$

(2.3)

see e.g. [24], where $\text{Ai}(\cdot)$ is the Airy function:

$$
\text{Ai}(z) = \frac{1}{\pi} \int_{0}^{\infty} \cos\left(\frac{t^3}{3} + tz\right) dt, \quad \forall \ z \in \mathbb{R}.
$$

(2.4)

The unitary transformation (2.3) carries $H_0$ over into multiplication by $x$ in $L^2(\mathbb{R}, dx)$:

$$(\mathcal{U}H_0\mathcal{U}^*)f(x) = x\tilde{f}(x), \quad \tilde{f} \in \mathcal{D}(x).
$$

(2.5)

The Airy function $\text{Ai}(z), z \in \mathbb{C}$ is entire, satisfies the equation $\text{Ai}''(z) = z\text{Ai}(z)$ and the following asymptotics

$$
\text{Ai}(z) = \frac{1}{2z^{\frac{2}{3}}\sqrt{\pi}} e^{-\frac{2}{3}z^\frac{3}{2}} \left(1 + O(z^{-\frac{3}{2}})\right), \quad \text{if} \quad |\arg z| < \pi - \varepsilon,
$$

$$
\text{Ai}(-z) = \frac{1}{2z^{\frac{2}{3}}\sqrt{\pi}} \left[\sin \vartheta + O(z^{-\frac{3}{2}}e^{i\text{Im} \vartheta})\right], \quad \text{if} \quad |\arg z| \leq \varepsilon, \quad \vartheta = \frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4},
$$

(2.6)

as $|z| \to \infty$ uniformly in $\arg z$ for any fixed $\varepsilon > 0$ (see (4.01)-(4.05) from [26]).

Introduce the space $L^p(\mathbb{R})$ equipped by the norm $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^\frac{1}{p} \geq 0$ and let $\|f\|^2 = \|f\|_2^2$. Recall results from [21].

Let $V$ satisfy (1.4). Then the functionals $\Psi(\lambda) : L^2(\mathbb{R}) \rightarrow \mathbb{C}$ given by

$$
\Psi(\lambda)f = \int_{\mathbb{R}} \text{Ai}(x-\lambda)|V(x)|^\frac{1}{2} f(x) dx \quad \forall \ \lambda \in \mathbb{R},
$$

(2.7)

and the mapping $\Psi_1(\lambda) = \Psi(\lambda)^*$, for all $\lambda \in \mathbb{R}$ have analytic extensions from the real line into the whole complex plane and satisfy

$$
\|\Psi(\lambda)\|^2 = \|\Psi_1(\lambda)\|^2 = \int_{\mathbb{R}} |\text{Ai}(x-\lambda)|^2 |V(x)| dx \quad \forall \ \lambda \in \mathbb{C}.
$$

(2.8)

2.3. The scattering matrix. Recall that the S-matrix $S(\lambda)$ is a scalar function of $\lambda \in \mathbb{R}$, acting as multiplication in the fiber spaces $\mathbb{C} = \mathcal{H}_\lambda$. The stationary representation for the scattering matrix has the form (see e.g. [28] [12] [24]):

$$
S(\lambda) = I + \mathcal{A}(\lambda), \quad \lambda \in \mathbb{R}; \quad \mathcal{A} = \mathcal{A}_0 - \mathcal{A}_1,
$$

$$
\mathcal{A}_0(\lambda) = -2\pi i \Psi(\lambda)V_\lambda \Psi_1(\lambda), \quad \mathcal{A}_1(\lambda) = 2\pi i \Psi(\lambda)V_\lambda Y(\lambda + i0) \Psi_1(\lambda),
$$

(2.9)

where $V_\lambda = \text{sign} V$, $\Psi_1(\lambda) = \Psi(\lambda)^*$. Note that due to Lemma 2.1 the operator $Y(\lambda \pm i0)$ is continuous in $\lambda \in \mathbb{R}$. The function $S(\lambda)$ is continuous in $\lambda \in \mathbb{R}$ and satisfies $S(\lambda) - 1 = O(\lambda^{-\frac{3}{2}})$ as $\lambda \to \pm \infty$. In order to study S-matrix we define the function $X$ by

$$
X(\lambda) = 2\pi \int_{\mathbb{R}} |\text{Ai}(x-\lambda)|^2 |V(x)| dx, \quad \lambda \in \mathbb{C}.
$$

(2.10)
Fredholm determinants.

2.4. Resonances for the operator $H$ were discussed in [21], where a central role was played by the Fredholm determinant. We recall some results from [21]. Under condition (1.1) each operator $Y_0(\lambda), \text{Im} \lambda \neq 0$, is trace class and thus we can define the determinant:

$$D_\pm(\lambda) = \det(I + Y_0(\lambda)), \quad \lambda \in \mathbb{C}_\pm.$$  \hfill (2.13)

Here the function $D_\pm(\lambda), \lambda \in \mathbb{C}_\pm$ is analytic in $\mathbb{C}_\pm$ and satisfies

$$\overline{D_+(\lambda)} = D_-(\overline{\lambda}) \quad \forall \ \lambda \in \mathbb{C}_+, \hfill (2.14)$$

$$D_\pm(\lambda) = 1 + O(\lambda^{-\frac{a}{2}}) \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{C}_\pm, \hfill (2.15)$$

for any fixed $a \in (0,1)$, uniformly with respect to $\arg \lambda \in [0, \pm \pi]$. Furthermore, the determinant $D_\pm(\lambda), \lambda \in \mathbb{C}_\pm$ has an analytic continuation into the entire complex plane. Moreover, for each $\lambda \in \mathbb{R}$ the S-matrix $S(\lambda)$ for the operators $H_0, H$ has the form:

$$S(\lambda) = \frac{D_-(\lambda - i0)}{D_+(\lambda + i0)} \quad \forall \ \lambda \in \mathbb{R}. \hfill (2.16)$$

Furthermore, by (2.16), the S-matrix $S(\lambda), \lambda \in \mathbb{R}$ has an analytic extension into the whole upper half plane $\mathbb{C}_+$ and a meromorphic extension into the whole lower half plane $\mathbb{C}_-$. The zeros of $S(\lambda), \lambda \in \mathbb{C}_+$ coincide with the zeros of $D_-$ and the poles of $S(\lambda), \lambda \in \mathbb{C}_-$ are precisely the zeros of $D_+$. In [21] we defined the resonances of the perturbed Stark operator $H$ as the zeros of the analytic continuation of the determinant $D_+(\lambda), \lambda \in \mathbb{C}_+$ in the lower half-plane $\mathbb{C}_-$. Due to (2.14) the sets of zeros of $D_\pm$ in $\mathbb{C}_\mp$ are symmetric with respect to the real line. Thus in order to study resonances it is enough to consider $D_+$ or $D_-$. By the identity (2.16), the resonances equivalently can be characterized as zeros of the scattering matrix in the lower half plane $\mathbb{C}_+$ or the poles of the scattering matrix in the lower half plane $\mathbb{C}_-$. Note, however, that the Riemann surface of $S(\lambda)$ is the complex plane $\mathbb{C}$, while for the determinant the natural domain of analyticity consists of two disconnected copies of $\mathbb{C}$, corresponding to analytic continuation from $\mathbb{C}_+$ to $\mathbb{C}_-$ and vice versa.

2.5. Estimates on Airy functions. In order to study S-matrix we need the asymptotics of the Airy function from [2.6]. Furthermore we have

$$\begin{cases} (x - \lambda)^{-\frac{1}{2}} = (-\lambda)^{-\frac{1}{2}}(1 + O(|\lambda|^{-1})), \\ (x - \lambda)^{\frac{3}{2}} = (-\lambda)^{\frac{3}{2}} + \frac{3}{2}x(-\lambda)^{\frac{1}{2}} + O(|\lambda|^{-\frac{1}{2}}) \end{cases}, \quad |\arg \lambda| \geq \varepsilon, \hfill (2.17)$$
and
\[
\begin{cases}
\lambda - x \sim (\lambda - x)^{-\frac{1}{4}} (1 + O(|\lambda|^{-1})), \\
\lambda - x \sim (\lambda - x)^{-\frac{3}{4}} x + O(|\lambda|^{-\frac{5}{4}})
\end{cases}
\]
\quad \text{as } |\lambda| \to \infty, \quad |\arg \lambda| \leq \varepsilon,
\tag{2.18}
\]
locally uniformly in \(x \in \mathbb{R}\), as \(|\lambda| \to \infty\). Here and bellow we use the following definitions
\[
\lambda = |\lambda| e^{i\varphi} \in \mathbb{C}_+, \quad \varphi \in [0, \pi] \quad k = \sqrt{\lambda} \in \mathbb{C}_+,
\]
\[
-\frac{i}{2} \lambda^\frac{3}{2} = -i |\lambda|^\frac{3}{2} (c + is) = |\lambda|^\frac{3}{2} s - i |\lambda|^\frac{3}{2} c,
\tag{2.19}
\]
\[
c = \cos \frac{3\varphi}{2}, \quad s = \sin \frac{3\varphi}{2} \in \begin{cases}
(0, 1) \quad \varphi \in \left[0, \frac{2\pi}{3}\right] \\
(-1, 0) \quad \varphi \in \left[\frac{2\pi}{3}, \pi\right].
\end{cases}
\]
These estimates give as \(|\lambda| \to \infty\) and let \(k = \sqrt{\lambda} \in \mathbb{C}_+\):

• Let \(\varphi \in [\varepsilon, \pi]\) for some \(\varepsilon > 0\). Then one has
\[
|\text{Ai}(x - \lambda)|^2 = \frac{i}{4k\pi} e^{-i\Phi} \left(1 + \frac{O(1)}{k}\right), \quad \Phi = \frac{4}{3} k^3 - 2xk,
\tag{2.20}
\]
and, in particular,
\[
|\text{Ai}(x - \lambda)|^2 = \frac{1}{4\pi |k|} e^{\text{Im} \Phi} \left(1 + \frac{O(1)}{|k|}\right).
\tag{2.21}
\]

• Let \(|\varphi| \leq \varepsilon\) for some \(\varepsilon > 0\). Then the following holds true:
\[
|\text{Ai}^2(x - \lambda) = \frac{1 + \sin \Phi}{2\pi k} + \frac{O(e^{\text{Im} \Phi})}{\lambda}.
\tag{2.22}
\]
All estimates \((2.20)-(2.22)\) are locally uniform in \(x\) on bounded intervals.

**Lemma 2.3.** Let \(\lambda = |\lambda| e^{i\varphi} \in \mathbb{C}_+,\) and \(k = \sqrt{\lambda} \in \mathbb{C}_+\) and \(\varepsilon > 0\). Recall that \(\Phi = \frac{4}{3} k^3 - 2xk\).

i) Define the function \(\xi(\lambda) = \frac{e^{-\frac{4}{3} k^\frac{3}{2}}}{2\sqrt{\lambda}}, \lambda \in \mathbb{C}_+.\) Let \(\varphi \in [\varepsilon, \pi].\) Then
\[
\mathcal{A}_0(\lambda) = \xi(\lambda) J_0(\lambda),
\]
\[
J_0(\lambda) = \int_0^\gamma e^{2ikV(x)} \left(1 + \frac{O(1)}{k}\right) dx,
\tag{2.23}
\]
\[
X(\lambda) = |\xi(\lambda)| J_1(\lambda),
\]
\[
J_1(\lambda) = \int_0^\gamma e^{-2ikV(x)} |V(x)| \left(1 + \frac{O(1)}{k}\right) dx,
\tag{2.24}
\]

ii) Let \(|\varphi| \leq \varepsilon\) and let \(V_0 = \int_0^\infty V(x) dx\). Then
\[
\mathcal{A}_0(\lambda) + \frac{iV_0}{k} = -\frac{i}{k} \int_0^\gamma V(x) \left[\sin \Phi + \frac{O(e^{\text{Im} \Phi})}{k}\right] dx.
\tag{2.25}
\]

**Proof.** Substituting asymptotics \((2.20)-(2.22)\) into \((2.11)\) we obtain \((2.23)-(2.25)\).
3. PROOF OF MAIN THEOREMS

We describe the Forbidden domain for resonances.

Proof of Theorem 1.1. Let \( \lambda = |\lambda| e^{i\varphi}, \varphi \in [0, \pi] \) and \( |\lambda| \to \infty \). Consider the case \( \varphi \in [\varepsilon, \pi] \) for some \( \varepsilon > 0 \) and let \( a < 1 \). Using (2.12), (2.23), (2.24) we obtain

\[
\mathcal{A}(\lambda) = \mathcal{A}_0(\lambda) + \mathcal{A}_1(\lambda) = \xi(\lambda)(J_0(\lambda) + J_1(\lambda)O(|\lambda|^{-\frac{2}{p}})).
\]

(3.1)

i) Let \( \varphi \in [\frac{2\pi}{3}, \pi] \). Then (3.1) and the identity \( |\xi(\lambda)| = \frac{e^{-\frac{4i\lambda^\frac{3}{2}}{2\lambda^\frac{3}{2}}}}{2\lambda^\frac{3}{2}} \) yield (1.5), since we have \( |J_0(\lambda)| = O(||V||_1) \) and \( |J_1(\lambda)| = O(||V||_1) \) and \( s = \sin \frac{3\varphi}{2} \in (-1,0) \).

ii) Let \( V \) satisfy (1.3) and let \( \varphi \in [\frac{2\pi}{3} - \varepsilon, \varepsilon] \). Then \( s = \sin \frac{3\varphi}{2} \in (0,1) \) and

\[
\text{Im} \lambda^\frac{3}{2} = \frac{3}{2} \sin \frac{\varphi}{2}, \quad \sin \frac{3\varphi}{2} \geq c_\varepsilon > 0.
\]

(3.2)

From Lemma 2.3 and (1.3) we obtain

\[
J_0(\lambda) = \int_0^\gamma e^{ik\lambda x} V(x) \left(1 + O\left(\frac{1}{k}\right)\right) dx = \frac{C_p + o(1)}{(-i2k)^p},
\]

(3.3)

and

\[
J_1(\lambda) \leq C \int_0^\gamma e^{-2i\text{Im} k} |V(x)| dx \leq C ||V||_1.
\]

(3.4)

Substituting (3.3), (3.4) into (3.1) we obtain

\[
\mathcal{A}(\lambda) = \xi(\lambda) \left(\frac{C_p + o(1)}{(-i2k)^p} + o(|k|^{-a})\right) = \xi(\lambda) \frac{C_p + o(1)}{(-i2k)^p}.
\]

which yields (1.6), since we can take \( a > p \). □

Now we are ready to determine asymptotics of resonances.

Proof of Theorem 1.2. i) Let \( \arg \lambda = \varphi \in [\frac{2\pi}{3} - \varepsilon, \frac{2\pi}{3}] \) for some small \( \varepsilon > 0 \) and \( |\lambda| \to \infty \). Let for shortness \( k = \sqrt{\lambda} \in \mathbb{C}^+ \). From (2.12), (2.23), (2.24) we have

\[
\mathcal{A}(\lambda) = \mathcal{A}_0(\lambda) + \mathcal{A}_1(\lambda) = \xi(\lambda)(J_0(\lambda) + O(|\lambda|^{-\frac{2}{p}})),
\]

(3.5)

\[
J_0(\lambda) = \int_0^\gamma e^{ik\lambda x} V(x) \left(1 + O\left(\frac{1}{k}\right)\right) dx = \frac{C_p}{(-i2k)^p} + w_0(\lambda), \quad w_0(\lambda) = O(1)^{1/\nu},
\]

for any fix \( a < 1 \), where \( C_p \) is defined by (1.3). This yields

\[
\mathcal{A}(\lambda) = -\frac{iC_pe^{-\frac{4i\lambda^3}{3}}}{(-i2k)^p+1}(1 + w(\lambda)), \quad w(\lambda) = O(\lambda^{-\frac{2}{p}}), \quad s = \min\{\nu, a\} - p > 0,
\]

for some function \( w \) analytic in \( \mathbb{C}^+ \). We have the equation for zeros of \( S \):

\[
1 - \frac{iC_pe^{-\frac{4i\lambda^3}{3}}}{(-i2k)^p+1}(1 + w(\lambda)) = 0.
\]

(3.6)

We rewrite this equation in terms of the new variable \( z = \frac{4}{3} \lambda^\frac{3}{2} \in \mathbb{C}^+ \). From (3.6) we obtain that the corresponding zeros satisfy the following equation:

\[
1 + g(z) - \frac{e^{-iz+iz^b}}{z^b} = 0,
\]

(3.7)
where \( g(z) := \frac{w(\lambda(z))}{w(\lambda(z))} = O(z^{-\frac{3}{4}}) \) as \( |z| \to \infty \) and
\[
b = \frac{p + 1}{3}, \quad 2k = (3z)^{\frac{1}{3}} \in \mathbb{C}_+, \quad z_s = \frac{\pi}{2} (p + 2) + i \log \frac{3^b}{C_p}, \quad (3z)^{\frac{1}{3}} e^{-i\frac{\pi}{2}(p+1)} = z^b e^{-i\frac{\pi}{2}(p+1)+b \log 3}.
\]

(3.8)

All zeros of the equation (3.7) were determined in Lemma 1.2 and from this lemma we have
\[
z_n = z_n^0 + z_s - b \pi + O(n^{-s/3}) \quad \text{as} \quad n \to -\infty,
\]
where \( z_n^0 \) is defined by (1.8). Then these asymptotics for give \( \lambda_n^- = (\frac{4}{3} z_n^-)^{\frac{1}{3}} \):
\[
\lambda_n^- = \mu_n \left( 1 - \frac{i y_n^0 + z_s - \pi b}{x_n^0} + O(1)\right)^{\frac{4}{3}} = \mu_n \left( 1 - \frac{2 i y_n^0 + z_s - \pi b}{3 x_n^0} + O(1)\right)
\]
where \( \mu_n = (\frac{4}{3} z_n^-)^{\frac{1}{3}} = |x_n^0|^{\frac{1}{3}} e^{i \frac{2}{3} \pi}, \) which yields (1.9) for \( \lambda_n^-, n \to \infty \)

ii) Let \( \varphi \in (0, \varepsilon) \) and let \( k = \sqrt{\lambda} \in \mathbb{C}_+ \) and \( \Phi = \frac{1}{3} k^3 - 2 k x. \) Then from (2.25) we obtain
\[
\mathcal{A}_0(\lambda) = -i \frac{V_0}{k} + \mathcal{A}_{01}(\lambda) + \mathcal{A}_{02}(\lambda), \quad \mathcal{A}_{02}(\lambda) = \frac{1}{|\lambda|} \int_0^\gamma |V(x)| O(e^{i \text{Im} \Phi}) dx,
\]
where
\[
\mathcal{A}_{01}(\lambda) = -i \frac{i}{k} \int_0^\gamma V(x) \sin \Phi dx = \frac{1}{2k} \int_0^\gamma V(x) (e^{-i \Phi} - e^{i \Phi}) dx = \frac{1}{2k} \int_0^\gamma V(x) (e^{-i \Phi} + O(1)) dx
\]
since for \( |k| \to \infty \) we have
\[
\text{Im} \Phi(\lambda, x) = 2|k| \left( 2|k|^2 \frac{3\phi}{3} - x \sin \phi \right) \sim 2|k| \phi (2|k|^2 - x) > 4\phi|k|^3.
\]

Similar arguments and (2.12) yield for any \( \text{fix} a < 1: \)
\[
A_1(\lambda) = \xi(\lambda) \frac{O(1)}{k^a}.
\]
Thus collecting asymptotics of \( A_0 \) and \( A_1 \) we obtain
\[
A(\lambda) = \xi(\lambda) \left( \int_0^\gamma V(x) e^{i 2 x k} dx + \frac{O(1)}{k} + \frac{O(1)}{k^a} \right) = \frac{e^{-i \frac{4}{3} k^3} C_p}{2k(-i2k)^a} \left( 1 + \frac{O(1)}{k^a} \right).
\]

(3.10)

This yields
\[
A(\lambda) = -i C_p e^{-i \frac{4}{3} k^3} \frac{1 + g(\lambda)}{(-i2k)^{p+1}} (1 + g(\lambda)), \quad g(\lambda) = O(k^{-a}), \quad s = \min\{\nu, a\} - p.
\]

We have the equation for zeros of \( S: \)
\[
1 - \frac{i C_p e^{-i \frac{4}{3} k^3}}{(-i2k)^{p+1}} (1 + g(\lambda)) = 0,
\]

(3.11)

We rewrite this equation in terms of the new variable \( z = \frac{4}{3} \lambda^{\frac{1}{3}} \in \mathbb{C}_+. \) From (3.11) we obtain that the corresponding zeros satisfy the following equation:
\[
1 - \frac{e^{-i z + iz_s}}{z^b} (1 + g(\lambda(z))) = 0, \quad \iff \quad \frac{1}{(1 + g(\lambda(z)))} - \frac{e^{-i z + iz_s}}{z^b} = 0
\]

(3.12)
where 
\[ 2k = (3z)^{\frac{1}{3}} \in \mathbb{C}+, \quad (-i2k)^{p+1} = (3z)^{b}e^{-i\frac{\pi}{3}(p+1)} = z^{b}e^{-i\frac{\pi}{3}(p+1)+b\log 3}. \]
From Lemma 4.2(ii) we deduce that the zeros \( z_n \) of the equation (3.12) have the form \( z_n = z_n^0 + z_\ast + O(n^{-\frac{1}{3}}) \), which yields asymptotics of \( \lambda_n^+ = (3z_n/4)^{\frac{2}{3}} \) in (1.10) as \( n \to \infty \). 

**Proof of Corollary 1.3.** From Theorem 1.2 we obtain
\[
\# \{ n \in \mathbb{Z} : |\lambda_n| < r \} = \# \left\{ n \in \mathbb{Z} : \frac{4}{3}|\lambda_n|^{\frac{2}{3}} < \frac{4}{3}r^{\frac{2}{3}} \right\} = \frac{4r^{\frac{2}{3}}}{3\pi}(1 + o(1)) \quad \text{as} \quad r \to \infty
\]
which yields (1.10). 

**4. Model equations**

We discuss Condition C.

**Lemma 4.1.** Let \( \gamma > 0 \) and \( p \in (0, 1) \). Let \( k \in \overline{\mathbb{C}_+} \) and \( |k| \to \infty \). Then
\[
\int_{0}^{\gamma} e^{ikx} \frac{dx}{x^{1-p}} = \frac{\Gamma(p)}{(-ik)^p} + O(1) / k,
\]
uniformly in \( \arg k \in [0, \pi] \), where \( \Gamma \) is the Gamma function.

**Proof.** Let \( k = r\omega, \omega = e^{i\phi}, \phi \in [0, \pi] \). We have
\[
J = \int_{0}^{\gamma} e^{ikx} \frac{dx}{x^{1-p}} = \int_{0}^{\infty} e^{ikx} \frac{dx}{x^{1-p}} - J,
\]
where
\[
J = \int_{r\gamma}^{\infty} e^{iky} \frac{dy}{y^{1-p}} = \frac{1}{i\omega r^p} \int_{r\gamma}^{\infty} e^{i\omega y} \frac{dy}{y^{1-p}} = \frac{1}{i\omega r^p} \left( -e^{i\omega r^\gamma} \right) + O(1) / r^p.
\]
which yields \( |J| \leq \frac{C}{r^p} \). Moreover, using the identity 3.381 from Gradshteyn-Ryzhyk
\[
\int_{0}^{\infty} e^{ikx} \frac{dx}{x^{1-p}} = \frac{\Gamma(p)}{(-ik)^p} = e^{ip\frac{\pi}{2} - p\log k} \Gamma(p), \quad (p, k) \in (0, 1) \times \overline{\mathbb{C}_+},
\]
we obtain (4.1). 

**Lemma 4.2.** i) Let \( (b, z_{\ast}) \in (0, 1) \times \mathbb{C} \) and let \( \mathcal{D}_r = \overline{\mathbb{C}_+} \setminus \{|z| < \rho_r\} \), where the radius \( \rho_r = \pi(2r - 1) + \text{Re} z_{\ast} - \frac{b\pi}{2} \) for some integer \( r > 1 \) large enough. Consider a function \( F(z) = e^{-i\gamma(z-z_\ast)}, z \in \overline{\mathbb{C}_+} \). Then the function \( F - 1 \) in the domain \( \mathcal{D}_r \) have only simple zeros \( z_n, n \geq r \) given by
\[
z_n = z_n^0 + z_\ast + bu_n + O\left(\frac{\log^2 n}{n^2}\right),
\]
where
\[
u_n = \frac{iyn_0 + z_\ast}{x_n^0}, \quad \begin{cases} z_n^+ = z_n^0 + z_\ast - b\pi, \quad z_n^0 = x_n^0 + iy_n^0, \quad x_n^0 = 2\pi n, \quad n \in \mathbb{Z}, \\ y_n^0 = b \log |2\pi n| \in \mathbb{R}_+ \end{cases}
\]
ii) Let in addition \( g \) be an analytic function in \( \overline{\mathbb{C}_+} \) and satisfies \( g(z) = O(z^{-\beta}) \) as \( |z| \to \infty, z \in \overline{\mathbb{C}_+} \) uniformly in \( \arg z \in [0, \pi] \) for some \( \beta \in (0, 1) \). Then the function \( F - 1 - g \) in the domain \( \mathcal{D}_r \) for some \( r > 0 \) large enough have only simple zeros \( z_n \in \mathcal{D}_r, n \geq r \) given by
\[
z_n^\pm = z_n^0 + z_\ast + O(n^{-\beta}) \quad \text{as} \quad n \to \infty.
\]
Proof. i) Let $z \in \mathcal{D}_r$ be a zero of $F - 1$. Then letting $z_0 = x_0 + iy_0 \in \mathbb{C}$, we have

$$1 = |F(z)| = \frac{e^{(y-y_0)}}{|z|^b} \Rightarrow y - y_0 = \log|z|^b = b\log|x| + O(1/x),$$

as $|z| \to \infty$, since $|x| \to \infty$ in this case. Let

$$z = z_0^o + z_0 + t, \quad t \in \mathbb{C}, \quad |t| < \pi, \quad u = \frac{iy_0^o + z_0 + t}{x_0^o} = u_n + \frac{t}{x_0^o},$$

Consider the first case, let $z = x + iy \in \mathcal{D}_r$ be a zero of $F - 1$ and $x \to \infty$. The proof for the second case $x \to -\infty$ is similar. We have

$$F(z_0^o + z_0 + t) = \frac{e^{-i(z_0^o + t)}}{(z_0^o + z_0 + t)^b} = \frac{e^{y_0^o}}{(x_0^o)^b} \cdot \frac{e^{-it}}{(1 + u)^b} = \frac{e^{-it}}{(1 + u)^b}.$$

Thus asymptotics $u \to 0, u_n \to 0$ yield $e^{-it} \to 1$ and $t \to 0$. Then from the equation $e^{-it} = (1 + u)^b$ and $u = u_n + \frac{t}{x_0^o} \to 0, t \to 0$ we obtain

$$1 - it(1 + O(t)) = 1 + bu(1 + O(u)) \Rightarrow -it(1 + O(t)) = bu(1 + O(u)),$$

and we get (1.3), since

$$t = -\frac{bu_n}{i + \frac{b}{x_0^o}}(1 + O(u_n)) = ibu_n(1 + O(u_n)).$$

ii) As above the zeros of $F - 1 - g$ have the form $z = z_0^o + z_0 + \tau, \quad \tau \in \mathbb{C}, \quad |	au| < \pi$. Substituting this into the equation $F = 1 + g$ we obtain

$$F(z_0^o + z_0 + \tau) = \frac{e^{y_0^o}}{(x_0^o)^b} \cdot \frac{e^{-i\tau}}{(1 + u)^b} = \frac{e^{-i\tau}}{(1 + u)^b} = 1 + g(z_0^o + z_0 + \tau), \quad (4.5)$$

where

$$u = \frac{iy_0^o + z_0 + \tau}{x_0^o} = u_n + \frac{\tau}{x_0^o} \to 0, \quad u_n \to 0 \quad \text{and} \quad g(z_0^o + z_0 + \tau) = O(n^{-\beta}).$$

This yields $e^{-i\tau} \to 1$ and $\tau \to 0$. Thus from (1.5) we obtain

$$1 - i\tau(1 + O(\tau)) = (1 + bu(1 + O(u)))(1 + O(n^{-\beta}) \Rightarrow -i\tau(1 + O(\tau)) = O(n^{-\beta})$$

Then we get $\tau = O(n^{-\beta})$ which yields (1.4). ■

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