Global exponential asymptotic stability of RNNs with mixed asynchronous time-varying delays

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Abstract

The present article addresses the exponential stability of recurrent neural networks (RNNs) with distributive and discrete asynchronous time-varying delays. Some novel algebraic conditions are obtained to ensure that for the model there exists a unique balance point, and it is global exponential asymptotically stable. Meanwhile, it also reveals the difference about the equilibrium point between systems with and without distributed asynchronous delay. One numerical example and its Matlab software simulations are given to illustrate the correctness of the present results.

Keywords: Recurrent neural networks; Equilibrium point; Exponential stability; Mixed asynchronous time-varying delay

1 Introduction

In the last few decades, a number of successful applications of RNNs have been witnessed in many areas, including associative memory, prediction and optimal control, and pattern recognition [1–8]. During the implementation of the operation, time delay is inevitably inherent in the transmission process among neurons on account of limited propagation speed and limited switching of the amplifier [9–13]. In addition, because of the existence of a large number of parallel channels with different coaxial process sizes and lengths, there maybe exist distributions of conduction velocities delays and propagation delays along with these paths. In these cases, we cannot only model the signal propagation with discrete delays due to it not being instantaneous. Thus, it is more suitable to add continuous distribution delays into the neural network model. Moreover, these delays sometimes may produce the desired excellent performance, such as processing moving images between neurons when signals are transmitted, exhibiting chaotic phenomena applied to secure communication. Therefore, it is quite necessary to discuss the dynamical behavior of the neural networks with mixed distributive and discrete delays. And there has been a lot of literature on mixed constant delays [14–19] and time-varying delays [19–24].

Recently, Liu et al. [25] proposed the asynchronous delays, and investigated the exponential stability for complex-valued recurrent neural networks with discrete asynchronous delays. Afterwards, Li et al. [26] presented the stability preservation in discrete
analogue of an impulsive Cohen–Grossberg neural network with discrete asynchronous delays. In the implementation of the operation, time delays are not just discrete asynchronous, but also distributive asynchronous, or even mixed asynchronous. In fact, for a driver, there is not only one kind of delay; his eyes, hands and feet all have delays in responding to the operation. Since the delays are different for different drivers, it needs to be coordinated in the driver’s brain central nervous system. Therefore, the stability analysis of neural networks with distributive and discrete asynchronous delays is a challenge that we should look forward to discussing.

Inspired by the challenge above, we investigated the exponential stability of RNNs with mixed asynchronous time-varying delays. The main contribution was to find some novel sufficient conditions which make the discussed system’s balance point unique and the global exponential asymptotically stable. The rest arrangement of this article are as follows. In the second section, the RNN model with some reasonable assumption is given. The main results are given and proved in the third section. The corollaries and comparisons with the existing literature are given in the fourth section. Section 5 gives a numerical example with comprehensible simulation to illustrate the effectiveness of the main results. In the end of this paper, the conclusion is drawn.

2 Model description
In the present article, we investigate a class of RNNs of \( n \geq 2 \) interconnected neurons as follows:

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij})) \\
&\quad + \sum_{j=1}^{n} d_{ij} \int_{t-h_{ij}(t)}^{t} f_j(x_j(s)) ds + u_i, \\
&\quad t \geq 0, i = 1, 2, \ldots, n, 
\end{align*}
\]

(1)

where \( x_i(t) \) is the state variate at time \( t \) related to the \( i \)th neuron; \( a_i \) is a positive behaved constant; \( f_j(\cdot) \) stands for activation function of the \( j \)th neuron, and it is a globally Lipschitz continuous and differentiable nonlinear function such that

\[
\begin{align*}
|f_j(x) - f_j(y)| &\leq l_i |x - y|, \quad l_i \geq 0, \forall x, y \in \mathbb{R}, \\
|f_j(\cdot)| &\leq M_i, \quad M_i \geq 0;
\end{align*}
\]

(2)

(3)

\( b_{ij}, c_{ij}, \) and \( d_{ij} \) are the corresponding connection weights associated with the neurons without delays, with discrete delays, and with distributed delays, respectively; \( \tau_{ij}(t) \) corresponds to the discrete asynchronous transmission time-varying delay along with the axon of the unit \( j \) to the unit \( i \) at time \( t \) such that

\[
\tau_{ij}(t) \geq 0, \quad \max_{1 \leq j \leq n} \sup_{t \geq 0} \tau_{ij}(t) \leq \tau_i, \quad 0 \leq \frac{d\tau_{ij}(t)}{dt} \leq \alpha < 1, \quad i = 1, 2, \ldots, n;
\]

(4)

\( h_{ij}(t) \) corresponds to the distributed asynchronous transmission time-varying delay along with the axon of the unit \( j \) to the unit \( i \) at time \( t \), and satisfies

\[
h_{ij}(t) \geq 0, \quad \max_{1 \leq j \leq n} \sup_{t \geq 0} h_{ij}(t) \leq h_i, \quad i = 1, 2, \ldots, n;
\]

(5)

\( u_i \) is a constant, and represents the external input of the \( i \)th neuron.
For model (1), its initial conditions are assumed to be
\[ x_i(s) = \phi_i(s), \quad \forall s \in [-\tau_i, 0], \quad (6) \]
where \( \phi_i(s) \) is real-valued continuous function, and
\[ \tau_i \equiv \max\{\tau_{i, i}, h_{i, i}\}, \quad \forall i \in \{1, 2, \ldots, n\}. \quad (7) \]

Remark 1 \( \tau_{ij}(t) \) and \( h_{ij}(t) \) above receive different information between different nodes at time \( t \), which means that the time-varying delays are asynchronous in system (1). Therefore, model (1) is more general than Refs. [22, 26].

Assume that \( x^* \) is a balance of model (1), and \( x^*_i \) is its \( i \)th component. Then Eq. (1) becomes
\[ a_i x^*_i = \sum_{j=1}^{n} b_{ij} f_j(x^*_j) + \sum_{j=1}^{n} c_{ij} f_j(x^*_j) + \sum_{j=1}^{n} d_{ij} h_{ij}(t) f_j(x^*_j) + u_i. \quad (8) \]

By Ref. [20], we can define the global exponential asymptotic stability of \( x^* \).

**Definition 1** The equilibrium point \( x^* \) in model (1) is said to have global exponential asymptotic stability, if there are \( M \geq 1 \) and \( \gamma > 0 \) such that each solution of Eq. (1) satisfies
\[ \sum_{i=1}^{n} \left| x_i(t) - x^*_i \right| \leq M e^{-\gamma t} \sum_{i=1}^{n} \sup_{s \in \Omega} \left| \phi_i(s) - x^*_i \right|, \quad (9) \]
where \( \phi_i(s) \) is the initial continuous function, and \( \Omega \) is a set of real numbers.

3 **Main results and proofs**

In this section, we will show that there is a unique balance point \( x^* \) in the neural networks (1), and it shows global exponential asymptotic stability.

**Theorem 1** Suppose that (2), (3), and (5) hold. If for each \( i, i \in \{1, 2, \ldots, n\} \), one has
\[ l_i \sum_{j=1}^{n} \left( |c_{ji}| + |b_{ji}| + |d_{ji}| h_{ij} \right) < a_i, \quad (10) \]
then the equilibrium point \( x^* \) exists and is unique in system (1).

**Proof** On account of \( a_i > 0 \), (8) can turn into
\[ \begin{cases} 
    x_i^* = \frac{1}{a_i} \left[ \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + \sum_{j=1}^{n} c_{ij} f_j(x_j^*) + \sum_{j=1}^{n} d_{ij} h_{ij}(t) f_j(x_j^*) + u_i \right], \\
    t \geq 0, i = 1, 2, \ldots, n. 
\end{cases} \quad (11) \]

Let
\[ g_i(x_1, x_2, \ldots, x_n) = \frac{1}{a_i} \left[ \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{j=1}^{n} c_{ij} f_j(x_j) + \sum_{j=1}^{n} d_{ij} h_{ij}(t) f_j(x_j) + u_i \right]. \]
Then we know that $x_i^*$ is a fixed point of the mapping $g_i(x)$ from Eq. (11). Hence the equilibrium point of Eq. (1) can be determined by the fixed points of functions $g_1(x), g_2(x), \ldots, g_n(x)$ within a specific range. Let $x(t)$ be the vector $(x_1(t), x_2(t), \ldots, x_n(t))^T$, and $\Phi$ be a hypercube set defined by

$$\Phi = \left\{ x(t) \left| x_i(t) \leq \frac{1}{a_i} \left( \sum_{j=1}^{n} (|b_{ij}| + |c_{ij}| + |d_{ij}| h_i) M_j + |u_i| \right) , i = 1, 2, \ldots, n \right\}. \quad (12)$$

By the hypotheses (3) and (5), we can get

$$\left| g_i(x_1^*, x_2^*, \ldots, x_n^*) \right| = \left| \frac{1}{a_i} \left[ \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + \sum_{j=1}^{n} c_{ij} f_j(x_j^*) + \sum_{j=1}^{n} d_{ij} h_i f_j(x_j^*) + u_i \right] \right|$$

$$\leq \frac{1}{a_i} \left[ \sum_{j=1}^{n} |b_{ij}| M_j + \sum_{j=1}^{n} |c_{ij}| M_j + \sum_{j=1}^{n} |d_{ij}| h_i M_j + |u_i| \right]$$

$$\leq \frac{1}{a_i} \left[ \sum_{j=1}^{n} (|b_{ij}| + |c_{ij}| + |d_{ij}| h_i) M_j + |u_i| \right]. \quad (13)$$

Let $g(x)$ be the vector function $(g_1(x), g_2(x), \ldots, g_n(x))^T$. From the continuity of $f_i$, we know that $g(x)$ is a continuous mapping from set $\Phi$ to $\Phi$. By Brouwer’s fixed point theorem, there is at least one $x^* \in \Phi$ such that

$$g_i(x^*) = x_i^* , \quad \forall i \in \{1, 2, \ldots, n\}.$$

It follows that there is at least one equilibrium point in Eq. (1).

Next, we will show the uniqueness of the equilibrium point in Eq. (1).

Let $y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T$ be also an equilibrium point of model (1). From (2), (3), and (8), we can obtain

$$|y_i^* - x_i^*| = \frac{1}{a_i} \left| \sum_{j=1}^{n} (c_{ij} + b_{ij} + d_{ij} h_i(t)) (f_j(y_j^*) - f_j(x_j^*)) \right|$$

$$\leq \frac{1}{a_i} \sum_{j=1}^{n} (|c_{ij}| + |b_{ij}| + |d_{ij}| h_i(t)) |y_j^* - x_j^*|$$

$$\leq \frac{1}{a_i} \sum_{j=1}^{n} l_j (|c_{ij}| + |b_{ij}| + |d_{ij}| h_i) |y_j^* - x_j^*|. \quad (14)$$

Summing over all the neurons that satisfy the inequality (14), we get

$$\sum_{i=1}^{n} |y_i^* - x_i^*| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{a_i} l_j (|c_{ij}| + |b_{ij}| + |d_{ij}| h_i) |y_j^* - x_j^*|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{a_i} l_j (|c_{ij}| + |b_{ij}| + |d_{ij}| h_i) |y_j^* - x_j^*|. \quad (15)$$
It follows that
\[ \sum_{i=1}^{n} |y_i^* - x_i^*| \left[ a_i - l_i \sum_{j=1}^{n} (|c_{ij}| + |b_{ij}| + |d_{ij}|) \right] = 0. \tag{16} \]

According to the condition (10), we can get
\[ x_i^* = y_i^*, \quad i = 1, 2, \ldots, n, \]
implying that there exists a unique equilibrium in model (1).

\[ \square \]

**Theorem 2** Suppose that (2)–(5), and (10) hold, and we have \( \beta \geq 1 \) and \( q > 0 \) such that
\[ \beta = \max_{1 \leq i \leq n} \left\{ 1 + \sum_{j=1}^{n} |c_{ij}| l_j \frac{e^{\alpha t_j}}{1 - \alpha} + \sum_{j=1}^{n} |d_{ij}| l_j \left( \frac{1}{q^2} + \frac{q h_j - 1}{q^2 e^{\alpha t_j}} \right) \right\}. \tag{17} \]

If the equilibrium point \( x^* \) and each solution of Eq. (1) with the initial conditions (6) satisfy
\[ \sum_{i=1}^{n} |x_i(t) - x_i^*| \leq \beta e^{\alpha t} \sum_{i=1}^{n} \sup_{s \in [-\tau_i, 0]} |\varphi_i(s) - x_i^*|, \tag{18} \]
then \( x^* \) is the global exponential asymptotic stability.

**Proof** By Theorem 1, model (1) exists a unique balance point under the assumptions (2), (3), (5), and (10), and we denote it as \( x^* \). Then from Eq. (1), we have
\[ \frac{d|x_i(t) - x_i^*|}{dt} \leq -a_i |x_i(t) - x_i^*| + \sum_{j=1}^{n} |b_{ij}| l_j |x_j(t) - x_j^*| + \sum_{j=1}^{n} |c_{ij}| l_j |x_j(t - \tau_{ij}(t)) - x_j^*| \]
\[ + \sum_{j=1}^{n} |d_{ij}| \int_{t-h_{ij}(t)}^{t} l_j |x_j(s) - x_j^*| \, ds. \tag{19} \]

Assumed that
\[ y_i(t) = e^{\alpha t} |x_i(t) - x_i^*|, \quad t \geq -\tau_i, i = 1, 2, \ldots, n. \]

Then the derivative along with (19) is
\[ \frac{dy_i(t)}{dt} = q e^{\alpha t} |x_i(t) - x_i^*| + e^{\alpha t} \frac{d|x_i(t) - x_i^*|}{dt} \]
\[ \leq -(a_i - q) y_i(t) + e^{\alpha t} \sum_{j=1}^{n} |b_{ij}| l_j |x_j(t) - x_j^*| + e^{\alpha t} \sum_{j=1}^{n} |c_{ij}| l_j |x_j(t - \tau_{ij}(t)) - x_j^*| \]
\[ + e^{\alpha t} \sum_{j=1}^{n} |d_{ij}| \int_{t-h_{ij}(t)}^{t} l_j |x_j(s) - x_j^*| \, ds. \tag{20} \]
Taking the derivative of (21) into (20), and get

\[
\frac{dy_i(t)}{dt} \leq -(a_i - q) y_i(t) + e^{\alpha t} \sum_{j=1}^{n} |b_{ij}| y_j(t) - x_j^* + e^{\alpha t} \sum_{j=1}^{n} |c_{ij}| l_j |y_j(t - \tau_j(t)) - x_j^*| \\
+ \sum_{j=1}^{n} |d_{ij}| l_j \int_{0}^{h_i} e^{\alpha u} y_j(u + t - h_i) du.
\]

(22)

Consider a Lyapunov function \( V(t) = V(y_1, y_2, \ldots, y_n)(t) \) defined by

\[
V(t) = \sum_{i=1}^{n} \left\{ y_i(t) + \sum_{j=1}^{n} |c_{ij}| l_j e^{\alpha t} \int_{0}^{t} y_j(s) ds \\
+ \sum_{j=1}^{n} |d_{ij}| l_j \int_{0}^{h_i} e^{\alpha u} y_j(u - t - h_i) du \right\}.
\]

(23)

Taking the derivative of \( V(t) \) along with (19), we get

\[
\frac{dV(t)}{dt} = \sum_{i=1}^{n} \left\{ \frac{dy_i(t)}{dt} + \sum_{j=1}^{n} |c_{ij}| l_j e^{\alpha t} \left( y_j(t) - y_j(t - \tau_j(t))(1 - \tau_j(t)) \right) \\
+ \sum_{j=1}^{n} |d_{ij}| l_j \int_{0}^{h_i} e^{\alpha u} y_j(u + t - h_i) du \right\} \\
\leq \sum_{i=1}^{n} \left\{ (a_i - q) y_i(t) - \sum_{j=1}^{n} |b_{ij}| y_j(t) - \sum_{j=1}^{n} |c_{ij}| l_j e^{\alpha t} y_j(t) \\
- \sum_{j=1}^{n} |d_{ij}| l_j \int_{0}^{h_i} e^{\alpha u} y_j(u) du \right\} \\
= \sum_{i=1}^{n} \left\{ (a_i - q) - \sum_{j=1}^{n} |b_{ij}| l_i - \sum_{j=1}^{n} |c_{ij}| l_j e^{\alpha t} y_j(t) \\
- \sum_{j=1}^{n} |d_{ij}| l_i e^{\alpha t} - \frac{1}{q} \right\} y_i(t).
\]

(24)

Let \( F_i(q_i) \) be an auxiliary continuous function related to index \( i \), defined by

\[
F_i(q_i) = a_i - q_i - l_i \sum_{j=1}^{n} |b_{ij}| - l_i \sum_{j=1}^{n} |c_{ij}| l_j e^{\alpha t} y_j(t) - l_i \sum_{j=1}^{n} |d_{ij}| l_j \left[ h_j + \sum_{k=2}^{\infty} \frac{q_i^{k-1} h_j^k}{k!} \right],
\]

(25)
where $q_i$ is a positive real number, and $i$ is a positive nature number not bigger than $n$. In view of the hypothesis (10), one has

$$F_i(0) = a_i - l_i \sum_{j=1}^{n} |b_{ij}| - l_i \sum_{j=1}^{n} |c_{ij}| - l_i \sum_{j=1}^{n} |d_{ij}| h_j > 0. \quad (26)$$

From the continuity of $F_i$, there exists $q_i^* \in (0, +\infty)$ such that

$$F_i(q_i^*) > 0, \quad i = 1, 2, \ldots, n.$$  

Without loss of generality, let

$$q = \max_{1 \leq i \leq n} \{q_1^*, q_2^*, \ldots, q_n^* \}.$$  

Then

$$F_i(q) = a_i - q - l_i \sum_{j=1}^{n} |b_{ij}| - l_i \sum_{j=1}^{n} |c_{ij}| e^{\tau_i} - l_i \sum_{j=1}^{n} |d_{ij}| h_j \left[ h_j + \sum_{k=2}^{\infty} \frac{q^{k-1} h_j^k}{k!} \right]$$

$$= (a_i - q) - \sum_{j=1}^{n} |b_{ij}| l_i - \sum_{j=1}^{n} |c_{ij}| l_i e^{\tau_i} - \sum_{j=1}^{n} |d_{ij}| l_i e^{h_j} - \frac{1}{q} > 0. \quad (27)$$

Therefore, by (24) and (27), one can see that the derivative of $V(t)$ is smaller than 0 for $t \in [0, +\infty]$. Based on the definition of $V(t)$ and the assumption (4), we obtain

$$\sum_{i=1}^{n} e^{\sigma t} |x_i(t) - x_i^*| \leq V(t) \leq V(0), \quad (28)$$

where

$$V(0) = \sum_{i=1}^{n} \left\{ y_i(0) + \sum_{j=1}^{n} |c_{ij}| l_i \frac{e^{\sigma t_i}}{1 - \alpha} \left[ \int_{-\tau_i(0)}^{0} y_j(s) ds \right] \right.$$  

$$+ \sum_{j=1}^{n} |d_{ij}| l_i \int_{0}^{h_i} e^{\sigma (h_i - u)} \int_{-u}^{0} y_j(w) dw du \right\}$$

$$\leq \sum_{i=1}^{n} \left\{ y_i(0) + \sum_{j=1}^{n} |c_{ij}| l_i \frac{e^{\sigma t_i}}{1 - \alpha} \sup_{s \in [-\tau_i, 0]} y_j(s) \right.$$  

$$+ \sum_{j=1}^{n} |d_{ij}| l_i \int_{0}^{h_i} e^{\sigma (h_i - u)} (h_i - u) du \sup_{s \in [-h_i, 0]} y_j(s) \right\}$$

$$= \sum_{i=1}^{n} \left\{ y_i(0) + \sum_{j=1}^{n} |c_{ij}| l_i \frac{e^{\sigma t_i}}{1 - \alpha} \sup_{s \in [-\tau_i, 0]} y_j(s) \right.$$  

$$+ \sum_{j=1}^{n} |d_{ij}| l_i \left( \frac{1}{q^2} + \frac{q h_i - 1}{q^2} e^{h_i} \right) \sup_{s \in [-h_i, 0]} y_j(s) \right\}$$

$$\leq \sum_{i=1}^{n} \left\{ 1 + \sum_{j=1}^{n} |c_{ij}| l_i \frac{e^{\sigma t_i}}{1 - \alpha} \right.$$  

Combining (17), (28), and (29), one can derive the inequality (18), and thus the equilibrium point $x^*$ of Eq. (1) has the global exponential asymptotic stability on account of Definition 1.

Remark 2 The constant $\beta \geq 1$, which plays a significant role in the index of convergence of model (1), relies on the distributive delay $h_j$ and delay $\tau_j$ for $j = 1, 2, \ldots, n$. If either the discrete delay $\tau_j$ or the distribution delay $h_j$ of (17) is sufficiently large, namely, the discrete asynchronous delays $\tau_{ij}(t)$ of (4) and distributed asynchronous delays $h_{ij}(t)$ of (5) are sufficiently large, then $\beta$ will be large enough, and thus the convergence time towards the equilibrium point will be longer. Therefore, the convergence time of model (1) can be shortened only if the two delays are reduced appropriately in the process of operation coordination.

4 Corollaries and comparisons

By Theorem 1 and Theorem 2, we will have the following corollaries. Meanwhile, we also will compare the conclusions of this paper with the existing literature.

When $h_{ij}(t) = 0$ for $i, j \in \{1, 2, \ldots, n\}$, Eq. (1) changes into the following neural networks:

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij}(t))) + u_i, \quad t \geq 0,$$

and its initial conditions are

$$x_i(s) = \varphi_i(s), \quad \forall s \in [-\tau_i, 0],$$

where $i$ is a positive integer not bigger than $n$.

**Corollary 1** Assume that (2) and (3) are true. If for each $i, i \in \{1, 2, \ldots, n\}$, one has

$$l_i \sum_{j=1}^{n} (|c_{ji}| + |b_{ji}|) < a_i,$$

then the equilibrium point $x^*$ exists and is unique in the system (30).

**Corollary 2** Suppose that (2)–(4), and (32) hold. If there exist two constants $\beta \geq 1$ and $q > 0$ such that

$$\beta = \max_{i \leq j \leq n} \left\{ 1 + \sum_{j=1}^{n} |c_{ji}| l_j \frac{e^{q\tau_j}}{1 - \alpha} \right\},$$

then the equilibrium point $x^*$ exists and is unique in the system (30).
and the equilibrium point \( x^* \) and each solution of Eq. (30) with the initial conditions (31) satisfy

\[
\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq \beta e^{-\alpha t} \sum_{i=1}^{n} \sup_{s \in [-\tau_i, 0]} |\phi_i(s) - x_i^*|,
\]

(34)

then \( x^* \) is the global exponential asymptotic stability.

**Remark** 3 By Ref. [26], the equilibrium point of model (30) with discrete asynchronous time-varying delay is the same to that without delays. Meanwhile \( h_i(t) \neq 0 \), by Theorem 1, the equilibrium point of model (1) will be affected by \( h_i(t), t > 0 \).

When \( \tau_{ij}(t) = 0 \) for \( i, j \in \{1, 2, \ldots, n\} \), Eq. (1) turns into the following neural networks:

\[
\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij} \int_{t-h_i(t)}^{t} f_j(x_j(s)) ds + u_i, \quad t \geq 0,
\]

(35)

and its initial conditions are

\[
x_i(s) = \psi_i(s), \quad \forall s \in [-h_i, 0],
\]

(36)

where \( i \) is a natural number, belonging to the set \( \{1, 2, \ldots, n\} \).

**Corollary 3** Suppose that (2), (3), and (5) hold. If for each \( i, \in \{1, 2, \ldots, n\} \), we have

\[
l_i \sum_{j=1}^{n} (|b_{ji}| + |d_{ji}|h_j) < a_i,
\]

(37)

then the equilibrium point \( x^* \) exists and is unique in the system (35).

**Corollary 4** Suppose that (2), (3), (5), and (36) hold. If we have \( \beta \geq 1 \) and \( q > 0 \) such that

\[
\beta = \max_{1 \leq i \leq n} \left\{ 1 + \sum_{j=1}^{n} |d_{ji}|l_i \left( \frac{1}{q^2} + \frac{qh_j - 1}{q^2} e^{qh_j} \right) \right\},
\]

(38)

and the equilibrium point \( x^* \) and each solution of Eq. (35) with the initial conditions (36) satisfy

\[
\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq \beta e^{-\alpha t} \sum_{i=1}^{n} \sup_{s \in [-h_i, 0]} |\phi_i(s) - x_i^*|,
\]

(39)

then \( x^* \) has the global exponential asymptotic stability.
When $\tau_{ij}(t) = \tau(t)$ and $h_{ij}(t) = h(t)$ for $i, j \in \{1, 2, \ldots, n\}$, let $\sup_{t \geq 0} \tau(t) \leq \tau$, and $\sup_{t \geq 0} h(t) \leq h$. Hence Eq. (1) changes into

$$\begin{cases}
\frac{dx_i(t)}{dt} = -a_{ix_i(t)} + \sum_{j=1}^{n} b_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}f_j(x_j(t - \tau(t))) \\
+ \sum_{j=1}^{n} d_{ij} \int_{t-h(t)}^{t} f_j(x_j(s)) \, ds + u_i,
\end{cases}
$$

$i = 1, 2, \ldots, n, t \geq 0,$

(40)

and its initial conditions are

$$\begin{cases}
x_i(s) = \phi_i(s), \quad \forall s \in [-\tau, 0], \\
i = 1, 2, \ldots, n, t \geq 0,
\end{cases}
$$

(41)

where $\phi_i(s)$ is real-valued continuous functions, and

$$\tau = \max_{t \geq 0} \{\tau, h\}.
$$

(42)

**Corollary 5** Suppose that (2), (3), and (5) hold. If for each $i, i \in \{1, 2, \ldots, n\}$, we have

$$l_i \sum_{j=1}^{n} (|b_{ij}| + |c_{ij}| + |d_{ij}| h) < a_i,
$$

(43)

then the equilibrium point $x^*$ exists and is unique in the system (40).

**Corollary 6** Suppose that (2)–(5), and (43) hold, and we have $\beta \geq 1$ and $q > 0$ such that

$$\beta = \max_{1 \leq i \leq n} \left[ 1 + \sum_{j=1}^{n} |c_{ij}| l_i \frac{e^{\beta \tau}}{1 - \alpha} + \sum_{j=1}^{n} |d_{ij}| l_i \left( \frac{1}{q^2} + \frac{q - 1}{q^2} e^{\beta h} \right) \right].
$$

(44)

If the equilibrium point $x^*$ and each solution of Eq. (40) with the initial conditions (41) satisfy

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq \beta e^{-\beta t} \sum_{i=1}^{n} \sup_{s \in [-\tau, 0]} |\phi_i(s) - x_i^*|,
$$

(45)

then $x^*$ has the global exponential asymptotic stability.

**Remark 4** The system (40) is one of the RNNs with distributive and discrete delays, which occur to the literature [22]. In this paper, we show the balance $x^*$ of Eq. (40) is the global exponential asymptotic stability via the Lyapunov function method, which is different from the method used in the literature [22].
5 Simulation example

Example We consider the following two-dimensional neural network model:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -5x_1(t) + 0.2f_1(x_1(t)) + 0.3f_2(x_2(t)) + 1.7f_1(x_1(t - \tau_{11}(t))) \\
&\quad + 0.8f_2(x_2(t - \tau_{12}(t))) + 0.3 \int_{t-\tau_{11}(t)}^{t} f_1(x_1(s)) \, ds \\
&\quad + 0.2 \int_{t-\tau_{12}(t)}^{t} f_2(x_2(s)) \, ds + 2.5, \\
\frac{dx_2(t)}{dt} &= -7x_2(t) + 0.1f_1(x_1(t)) + 0.4f_2(x_2(t)) + 0.8f_1(x_1(t - \tau_{21}(t))) \\
&\quad + 0.9f_2(x_2(t - \tau_{22}(t))) - 0.2 \int_{t-\tau_{21}(t)}^{t} f_1(x_1(s)) \, ds \\
&\quad + 0.4 \int_{t-\tau_{22}(t)}^{t} f_2(x_2(s)) \, ds + 1.5,
\end{align*}
\]

(46)

where \(f_i(x_i(t)) = \tanh(x_i(t))\), \(\tau_{11}(t) = \tau_{21}(t) = 0.49 + 0.49 \sin(0.02t)\), \(\tau_{12}(t) = \tau_{22}(t) = 0.48 + 0.48 \cos(0.03t)\), \(h_{11}(t) = h_{21}(t) = 1.5 + e^{-t}\), and \(h_{12}(t) = h_{22}(t) = 3 + e^{-t}\). For the initial condition is considered that \(x_1(s) = \ln(s + 3.9)\), \(x_2(s) = 0.4 e^{-0.7}\), \(s \in [-4, 0]\).

After comparison and simple calculation, we know that Eq. (46) represents a two-dimensional RNN with discrete and distributed asynchronous time-varying delays, and satisfies all the assumptions of Theorem 1 and Theorem 2. In Fig. 1, we illustrate state trajectories of \(x_1\) and \(x_2\) of model (46), which show that model (46) has a unique and exponential asymptotically stable equilibrium point.

In Fig. 2, we illustrate state trajectories of \(x_1\) and \(x_2\) of model (46) under three other different initial conditions: \(x_1(s) = 0.4, x_2(s) = 0.7; x_1(s) = -0.5 + e^{s}, x_2(s) = 0.4e^{s}\) and \(x_1(s) = -0.9, x_2(s) = -0.3\), which demonstrate that exponential convergence of model (46) is global. Therefore, Figs. 1 and 2 show fully the effectiveness of our results in this paper.

Taking out the distributed delay, model (46) turns into a two-dimensional RNN with discrete asynchronous time-varying delays. Figure 3 illustrates its state trajectories, which are marked as \(x1\)-discrete and \(x2\)-discrete, respectively. Meanwhile, Fig. 3 also shows the state trajectories of \(x_1\) and \(x_2\) of model (46) without delays. From Fig. 3, we see that the dynamical behaviors of Eq. (46) without distributed delay and Eq. (46) without delay are converging toward the same equilibrium point.

![Figure 1: State trajectories of model (46)](image-url)
Removing the discrete delay, model (46) becomes a two-dimensional RNN with distributive asynchronous time-varying delays. Figure 4 illustrates its state trajectories, which are marked as $x_1$-distributed and $x_2$-distributed, respectively. Meanwhile, Fig. 4 also demonstrate the state trajectories of $x_1$ and $x_2$ of model (46) without delays. From Fig. 4, we see that the trajectories of Eq. (46) with distributive asynchronous time-varying delays and Eq. (46) without delay are different, which implies that the dynamical behavior of model (46) is affected by the distributed asynchronous time-varying delay.

In Fig. 5, we illustrate the state trajectories of model (46) with delays (i): $\tau_{11}(t) = \tau_{21}(t) = 0.49 + 0.49 \sin(0.02t)$, $\tau_{12}(t) = \tau_{22}(t) = 0.48 + 0.48 \cos(0.03t)$, $h_{11}(t) = h_{21}(t) = 1.5 + e^{-t}$, $h_{12}(t) = h_{22}(t) = 3 + e^{-t}$, and model (46) with delays (ii): $\tau_{11}(t) = \tau_{21}(t) = 0.11 + 0.11 \sin(0.02t)$, $\tau_{12}(t) = \tau_{22}(t) = 0.09 + 0.09 \cos(0.03t)$, $h_{11}(t) = h_{21}(t) = 0.1 + e^{-t}$, $h_{12}(t) = h_{22}(t) = 0.2 + e^{-t}$, which are marked as $x_1$-big and $x_2$-big, and $x_1$-small and $x_2$-small, respectively. Obviously, the maximum of all delays in (i) is bigger than that in (ii). From Fig. 5,
we see that the stability convergence time of neural networks with big upper bound delay is longer than that of neural networks with small upper bound delay.

6 Conclusion

In the present paper, we discuss the RNNs with mixed asynchronous time-varying delays. By the Lyapunov function method, some algebra conditions are given to make the investigated model have a unique and global exponential asymptotically stable balance point. Meanwhile, we also show that the balance point of the neural networks with distributive asynchronous time-varying delays is different from that without distributed delays. Finally, one numerical example and its simulation are given to demonstrate the effectiveness of our results. The considered neural networks in this paper can be further discussed as regards their discrete-time analogue, and also can be investigated as regards their dynamical characteristics by adding pulses.
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