Coherence and Rydberg Blockade of Atomic Ensemble Qubits

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Coherence and Rydberg blockade of atomic ensemble qubits

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We demonstrate $|W\rangle$ state encoding of multi-atom ensemble qubits. Using optically trapped Rb atoms the $T_2$ coherence time is 2.6(3) ms for $N = 7.6$ atoms and scales approximately inversely with the number of atoms. Strong Rydberg blockade between two ensemble qubits is demonstrated with a fidelity of 0.89(1) and a fidelity of $\sim 1.0$ when postselected on control ensemble excitation. These results are a significant step towards deterministic entanglement of atomic ensembles.

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Qubits encoded in hyperfine states of neutral atoms are a promising approach for scalable implementation of quantum information processing[1]. While a qubit can be encoded in a pair of ground states of a single atom, it is also possible to encode a qubit, or even multiple qubits, in an $N$ atom ensemble by using Rydberg blockade to enforce single excitation of one of the qubit states[2, 3]. Ensemble qubits have several interesting features in comparison to single atom qubits. Using an array of traps it is simpler to prepare many ensemble qubits with $N \geq 1$ for each ensemble, than it is to prepare an array with exactly one atom in each trap which remains an outstanding challenge[4–6]. In addition, a $|W\rangle$ state ensemble qubit encoding is maximally robust against loss of a single atom[7], which can be remedied with error correction protocols[8], while atom loss is a critical error for single atom qubits. Furthermore an ensemble encoding facilitates strong coupling between atoms and light, an essential ingredient for quantum networking protocols[9] and atomic control of photonic interactions in Rydberg blocked ensembles[10]. As the atom-light coupling strength grows with the number of atoms, recent experiments[10],[11] and theory proposals[12] are based on ensembles with $N > 100$. We are focused here on studying the physics of ensembles for computational qubits and therefore work with smaller ensembles with up to $N \sim 10$ atoms.

In this letter we demonstrate and study the coherence and interactions of atomic ensemble qubits. We measure the $T_2$ coherence time of ensemble qubits achieving a ratio of coherence time to single qubit $\pi$ rotation time of $\sim 2600$. We furthermore proceed to demonstrate strong Rydberg blockade between two, spatially separated ensemble qubits. With the recent demonstration of entanglement between a Rydberg excited ensemble and a propagating photon[13] these results establish a path towards both local and remote entanglement of arrays of ensemble qubits, which will enable enhanced quantum repeater architectures[14].

The computational basis states of the ensemble qubits are

$$\begin{align*}
|0\rangle &= |0_1...0_N\rangle, \\
|I\rangle &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |0_10_2...1_j...0_N\rangle,
\end{align*}$$

where $|0_j\rangle$ and $|1_j\rangle$ are two ground states of the $j^{th}$ atom in an $N$ atom sample[15]. The state $|I\rangle$, which is a symmetric superposition of one of the $N$ atoms being excited, is commonly referred to as a $|W\rangle$ state.

Gate protocols for ensemble qubits differ from the single atom qubit case [2, 16] as all operations must use blockade to prohibit multi-atom excitation. Gate operations are performed via the collective, singly excited Rydberg state

$$|\bar{r}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |0_10_2...r_j...0_N\rangle,$$

where $|r_j\rangle$ is the Rydberg state of the $j^{th}$ atom. A single qubit rotation $R(\theta, \phi)$ with angle $\theta$ and phase $\phi$ between ensemble states $|0\rangle, |I\rangle$ is implemented as the three pulse sequence $|I\rangle \xrightarrow{\Omega_{\pi}} |\bar{r}\rangle, |\bar{r}\rangle \xrightarrow{\Omega_{R(\theta,\phi)}} |0\rangle, |\bar{r}\rangle \xrightarrow{\Omega_{\pi}} |I\rangle$. Note that the coupling strength between states $|I\rangle, |\bar{r}\rangle$ is the single atom Rabi frequency $\Omega$ while the coupling between $|0\rangle, |\bar{r}\rangle$ is at the collective Rabi frequency $\Omega_N = \sqrt{N}\Omega$. Since $\Omega_N$ depends on $N$, the one-qubit gate pulse lengths depend on the number of atoms. A $C_Z$ gate between control and target ensembles $c, t$ is implemented as the three pulse sequence $|I\rangle_c \xrightarrow{\Omega_{\pi}} |\bar{r}\rangle_c, |\bar{r}\rangle_c \xrightarrow{\Omega_{R(\theta,\phi)}} |\bar{r}\rangle_t, |\bar{r}\rangle_c \xrightarrow{\Omega_{\pi}} |I\rangle_c$. The $C_Z$ gate pulses do not depend on the number of atoms. The $N$ dependence of the one-qubit gates can be strongly suppressed using adiabatic pulse sequences so that high fidelity gate operations are possible with small, but unknown values of $N$[17].

The experimental setting is as described in [18]. In brief we prepare a cold sample of $^{87}$Rb atoms in a magneto-optical trap (MOT) and then load a variable number of atoms into optical dipole traps. The dipole traps shown in Fig. 1 are formed by focusing 1064 nm light to waists ($1/e^2$ intensity radii) of 3.0 $\mu$m. The atoms are cooled to a temperature of $\sim 150 \mu$K in 1-1.5 mK.
deep optical potentials. This gives approximately Gaussian shaped density distributions with typical standard deviations $\sigma_{\perp} = 0.7 \, \mu m$ perpendicular to the long trap axis and $\sigma_z = 7 \, \mu m$ parallel to the long axis. The estimated density at trap center is $n/N = 5 \times 10^{16} \text{ m}^{-3}$. We apply a bias magnetic field along the trap axis of $B_z = 0.24 \, \text{mT}$ and optically pump into $|0\rangle \equiv |5s_{1/2}, f = 2, m_f = 0\rangle$ using $\pi$ polarized 795 nm light resonant with $|5s_{1/2}, f = 2\rangle \to |5p_{1/2}, f = 2\rangle$ and 780 nm repump light resonant with $|5s_{1/2}, f = 1\rangle \to |5p_{3/2}, f = 2\rangle$.

Rydberg excitation coupling $|0\rangle, |\vec{r}\rangle$ is performed by off-resonant two-photon transitions via $5p_{3/2}$[19] using counter-propagating 780 nm and 480 nm light. With $\sigma_+$ polarization for both beams we couple to the Rydberg state $|\vec{r}\rangle = |n d_{5/2}, m_j = 5/2\rangle$ which is selected with a $B_z = 0.37 \, \text{mT}$ bias field. The other qubit ground state is $|1\rangle \equiv |5s_{1/2}, f = 1, m_f = 0\rangle$. Coupling between $|1\rangle, |\vec{r}\rangle$ is performed with 780 and 480 nm light where 780 nm have the same propagation vector and polarization but a frequency difference of 6.8 GHz corresponding to the $^{87}\text{Rb} f = 1 \leftrightarrow f = 2$ clock frequency. In the experiments reported below we used Rydberg levels $7d_{5/2}$ and $111d_{5/2}$. In both cases strong blockade was observed in individual ensembles with no evidence for double excitation of the logical $|\bar{1}\rangle$ state[18]. While we do not observe double excitation of $|\bar{1}\rangle$, experiments with two ensembles do show evidence for double excitation of the Rydberg state $|\vec{r}\rangle$, which plays a role in limiting the fidelity with which we can prepare the $|\bar{1}\rangle$ state.

We proceed to demonstrate the coherence of the ensemble states of Eq. (1) using Ramsey interferometry. The amplitude of the Ramsey signal is used to quantify the presence of $N$ atom entanglement in the ensemble, as has been observed in other recent experiments[20, 21]. Details of the analysis showing that 82±6% of the atoms participate in the entangled $|W\rangle$ state are presented in the supplemental material[22]. We load 3 < $\bar{N}$ < 10 atoms into one of the optical traps. The number of atoms loaded for each measurement follows a Poisson distribution with mean $\bar{N}$. Each measurement starts with optical pumping into $|0\rangle$ followed by the pulse sequence

$$|\psi\rangle = R_1(\pi)R(\pi/2)R_1(\pi)G(t)R_1(\pi)R_0(\pi/2)|0\rangle. \tag{2}$$

Here $R_0(\theta)$ is a pulse of area $\theta$ between states $|0\rangle, |\vec{r}\rangle$ and $R_1(\theta)$ is a pulse of area $\theta$ between states $|\bar{1}\rangle, |\vec{r}\rangle$. The first $R_0(\pi/2)$ pulse creates an equal superposition $\frac{|0\rangle + |\vec{r}\rangle}{\sqrt{2}}$. This is then mapped to $\frac{|0\rangle + |\bar{1}\rangle}{2}$ with a $R_1(\pi)$ pulse, we wait a gap time $t$ described by an operator $G(t)$, map $|\bar{1}\rangle \to |\vec{r}\rangle$ with a $R_1(\pi)$ pulse, and then perform another $\pi/2$ pulse between $|0\rangle, |\vec{r}\rangle$. Finally, any population left in $|\vec{r}\rangle$ is mapped back to $|\bar{1}\rangle$ with another $R_1(\pi)$ pulse. Atoms in state $|0\rangle$ are then pushed out of the trap using unbalanced radiation pressure from a beam resonant with $|5s_{1/2}, f = 2\rangle \to |5p_{3/2}, f = 3\rangle$ while the dipole trap light is chopped on and off. For the push out step a bias field is applied along $x$ the narrow axis of the dipole traps, and the circularly polarized push out beam propagates along $x$. This is followed by a measurement of the number of atoms remaining in the dipole trap giving the data in Fig. 2. The amplitude of the Ramsey interference at short gap times is limited by the $|W\rangle$ state preparation fidelity of about 50% for the atom number used in the figure. The fidelities of the $R_0(\pi)$ and $R_1(\pi)$ pulses used to prepare $|W\rangle$ are estimated to each be at least 90% on the basis of previous experiments[18] and the strong inter-ensemble blockade effect we report below. We attribute the limited $|W\rangle$ state preparation fidelity to Rydberg dephasing. Periodic fluorescence measurements of the mean atom number (described in the supplemental

![FIG. 1. (color online) Experimental geometry a) and transitions used for qubit control b). The Raman light is only used for preparation of product states, as discussed in connection with Fig. 3.](image1)

![FIG. 2. (color online) Ramsey interference measurement of qubit coherence for $\bar{N} = 7.6$. The peak-peak amplitude of the oscillation as a function of the gap time gives $T_2 = 2.6(3) \, \text{ms}$. The circles are data points with ±$\sigma$ error bars and the dashed and solid lines are fits to the functions $v(t)$, $v(t)$ defined in the text. The gap time is the time $t$ between the $R_1(\pi)$ pulses in Eq. (2). All data have been corrected for ≈ 1.5% probability per atom of the blow away giving an unwanted transition from $|0\rangle \to |1\rangle$. The inset shows the Ramsey oscillations for gap times of 0 (solid line), 0.5 ms (dashed line), and 2.5 ms (dashed-dotted line).](image2)
The principal sources of decoherence in this experiment are expected to be magnetic noise, motional dephasing, and atomic collisions [23]. For small atom numbers and low collision rates we fit the Ramsey signal to the expression [24] \( v_\text{R}(t, T_2) = v_0/\sqrt{1 + (e^{2/3} - 1) \left(\frac{T_2}{T}\right)^{2/3}} \)

and in the collision dominated regime we use a Gaussian form \( v_\text{G}(t) = v_0 e^{-(t/T_2)^2} \) where \( v_0 \) is the amplitude at \( t = 0 \). Both functional forms give the same \( T_2 \) time within our experimental error bars of \( T_2 = 2.6 \pm 0.3 \text{ ms} \).

The expression [26] bound drifts to \( 6.7 < \bar{N} < 9 \), during the 12 hour measurement of this data set.

To further clarify the sensitivity to collisional dephasing, Fig. 3 shows the measured \( T_2 \) for different \( \bar{N} \), including the case of \( \bar{N} = 1 \) Fock states which are selected using an additional fluorescence measurement before the Ramsey sequence [18]. We see that \( T_2 \sim 1/\bar{N} \), in contrast to the \( 1/\bar{N}^2 \) scaling observed for GHZ states [25]. The observed \( 1/\bar{N} \) scaling for \( |W\rangle \) states is expected for decoherence dominated by collisions since the collision rate per atom is proportional to \( \bar{N} \). For comparison, the \( T_2 \) time was also measured for product states \( |\psi\rangle \sim (|0\rangle - i|1\rangle)^{\otimes N} \). These states were prepared using a two-frequency Raman laser coupling \( |0\rangle \) and \( |1\rangle \) via the \( 5D_{3/2} \) level [26] as shown in Fig. 1. Comparison of the \( |1\rangle \) (\( |W\rangle \) state) and product state coherence data suggests that for \( N > 5 \) the coherence time is limited by collisions. For \( N < 5 \) as well as for the \( \bar{N} = 1 \) Fock state data the product states show longer coherence time. The coherence of the \( |W\rangle \) states is measured by comparison with a phase reference defined by the beatnote of the 780\(_0\) and 780\(_1\) Rydberg lasers which have a measured beatnote linewidth of 100 Hz FWHM.

This linewidth is consistent with the observed shorter coherence time of the \( |W\rangle \) states compared to the product states which are referenced to the Raman laser beatnote which is in turn locked to a stable 6.8 GHz microwave oscillator. We anticipate that compensated optical traps and dynamical decoupling methods together with an optical lattice to reduce collisional effects can be used to greatly extend these coherence times [27].

To demonstrate ensemble-ensemble blockade we load atoms into control (c) and target (t) dipole traps, optically pump into \( |0\rangle_c |0\rangle_t \) and apply one of two sequences. Preparation of a superposition of \( |0\rangle \) and \( |1\rangle \) in the target qubit is effected by the sequence \( U_a |0\rangle_c |0\rangle_t = R_{\phi\pi}(\pi)R_{0\pi\pi}(\theta)|0\rangle_c |0\rangle_t \). This should ideally leave the qubits in the joint state \( |0\rangle_c \cos(\theta/2) |0\rangle_t - \sin(\theta/2) |1\rangle_t \) with the probability of preparing \( |1\rangle_t \) proportional to \( \sin^2(\theta/2) \), as is shown in Fig. 4a). We see the expected time dependence with a peak probability of \( P_{|1\rangle_t} \sim 0.52 \), consistent with our earlier study of Fock state preparation [18].

Rydberg blockade between two ensembles is observed with the sequence \( U_b |0\rangle_c |0\rangle_t = R_{1\pi\pi}(\pi)R_{1\pi\pi}(\pi)R_{0\pi\pi}(\pi)R_{0\pi\pi}(\pi) |0\rangle_c |0\rangle_t \). Here we have used state \( |0\rangle \) of the control ensemble to block the target transfer with the final \( R_{1\pi\pi}(\pi) \) pulse ideally leaving the qubits in the joint state \( |1\rangle_c |0\rangle_t \). The data in Fig. 4a) show a ratio of \( P_{|1\rangle_t}(U_b)/P_{|1\rangle_t}(U_a) = 0.11(1) \), i.e. a blockade fidelity of 0.89. This implies that the success probability of the transition \( R_{0\pi\pi}(\pi) |0\rangle_c \rightarrow |1\rangle_c \) is bounded below by the \( |1\rangle_t \) population ratio for the two sequences. We infer that at least one atom is excited to the Rydberg state \( |r\rangle_c \) with probability \( \geq 0.89(1) \).

As a further check on the inter-site blockade fidelity,
events where the control site ends in state $|1\rangle_c$ after sequence $U_b$, and the target population is shown in Figure 4b), along with the expected blow-away leakage rate of the control and target sites which is measured to be 0.2%/atom. From the data it can be seen that the post-selected results are consistent with perfect inter-site blockade.

The observed high blockade fidelity exceeds that originally achieved in experiments with single atom qubits[28, 29], and is certainly sufficient to create entanglement between ensemble qubits. What has so far limited a demonstration of deterministic entanglement is the relatively low probability of up to 62% [18] with which the ensemble state $|1\rangle$ can be prepared. In order to gain insight into what is limiting the state preparation fidelity we looked for signatures of Rydberg-Rydberg interactions concurrently with strong blockade. Ideally the probability of preparing $|1\rangle_c$ with sequence $U_b$, should be independent of the pulse area $\theta$ applied to the target ensemble. However a clear dependence on $\theta$ can be seen in Fig. 5a).

We believe this effect is due to long range interactions, where the amplitude for Rydberg atom excitation in the target site is sufficiently blocked to prevent it from making the transfer to $|1\rangle_t$ with any significant probability, yet the target ensemble Rydberg excitation still interacts with the control ensemble strongly enough to disrupt the control ensemble state transfer. A similar situation of partial blockade together with decoherence of multi-atom ground-Rydberg Rabi oscillations was reported earlier in [19].

A two-atom Rydberg interaction effect should scale with the Rydberg double excitation probability, i.e. $P_{R} \propto \Omega_{N}^{2}/B^{2}$, where $B$ is the ensemble mean blockade shift[30]. To check this, we extract the slopes from linear fits to the $P_{|1\rangle_t}(\theta)$ data for small $\theta$ and compare to the scaling parameter

$$F = \Omega_{N}^{2} \left[ \frac{(n/n_{0})^{12}}{(R/R_{0})^{6}} \right]^{-2} \propto P_{\text{double}}. \quad (3)$$

Here $n$ is the Rydberg principal quantum number and $R$ is the site-site separation. The larger $F$ is for a given set of parameters, the stronger the Rydberg-Rydberg interaction, and thus the larger the slope of $dP_{|1\rangle_t}(\theta)/d\theta$. Indeed, this is the behavior we observe, as shown in Fig. 5b), for a range of $N$, $R$, and $n$.

This interaction effect hints at the possible mechanism responsible for the observed reduction in the probability $P_{|1\rangle}$ of preparing the collective qubit state in a single ensemble. The spatial extent of one ensemble is $\sim 2\sigma_{z} = 14 \ \mu m$ giving a length scale in between the lower two data sets in Fig. 5a). The intra-ensemble Rydberg interactions are significantly stronger than between atoms located in different ensembles at the same separation because the dipole-dipole interaction angular factors favor atom pairs separated along $z[30]$. These considerations imply that lack of perfect blockade leading to long range Rydberg-Rydberg interactions in a single ensemble only partially explains the observed maximum of $P_{|1\rangle} = 0.62$ [18]. Another candidate explanation is very strong interactions at short range in a single ensemble which mix levels together and open anti-blockade resonance channels[31]. The doubly excited molecular energy structure becomes difficult to calculate with confidence at short range, with many molecular potentials near resonant[32]. For our typical Rydberg state $97d_{5/2}$ this characteristic separation is $\sim 5 \ \mu m$, and for a 6 atom sample with our ensemble spatial distributions an average of 7 atom pairs out of 15 have $R < 5 \ \mu m$. We conjecture that the strong, short range interactions give an amplitude for double excitation, resulting in Rydberg-Rydberg interactions which dephase the ground-Rydberg rotations needed for state preparation, thereby limiting the probability of preparing the ensemble $|1\rangle$ state. A related reduction of the fidelity of Rydberg mediated atom-photon coupling in dense ensembles due to Rydberg-ground state interactions has also been observed[11].

In conclusion, we have demonstrated the coherence of ensemble qubit basis states. The coherence time scales approximately inversely with the number of atoms, but is still several ms and 2600 times longer than our characteristic gate time for $N \sim 10$. Additionally we have demonstrated inter-ensemble blockade with a fidelity of 0.89 and $\sim 1.0$ when post-selecting on control ensemble excitation. We identified Rydberg-Rydberg interactions from weak double excitations, either at long or short range, as a possible mechanism limiting the fidelity of ensemble state preparation. Future work towards ensemble entanglement and quantum computation will explore the use of a background optical lattice to better localize the ensembles while limiting uncontrolled short range interactions.

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15. Equation (1) is an approximation to \(|\tilde{I}| = \sum_{j=1}^{N} \frac{1}{\Omega_j} |0_10_2...0_j...0_N\rangle \) with \(\Omega_j\) the Rabi frequency of the \(j\)th atom. For our experiments the Rabi frequency is approximately constant across the ensemble so \(|\tilde{I}| = N^{-1/2} \sum_{j=1}^{N} |0_10_2...0_j...0_N\rangle\) and Eq. (1) can be written as

\(|\tilde{I}| = N^{-1/2} \sum_{j=1}^{N} |0_10_2...0_j...0_N\rangle\).