Computational self-testing of multi-qubit states and measurements

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Abstract
Self-testing is a fundamental technique within quantum information theory that allows a classical verifier to force (untrusted) quantum devices to prepare certain states and perform certain measurements on them. The standard approach assumes at least two spatially separated devices. Recently, Metger and Vidick [MV21] showed that a single EPR pair of a single quantum device can be self-tested under standard computational assumptions. In this work, we generalize their techniques to give the first protocol that self-tests $N$ EPR pairs and measurements in the single-device setting under the same computational assumptions. We show that our protocol can be passed with probability negligibly close to 1 by an honest quantum device using poly($N$) resources. Moreover, we show that any quantum device that fails our protocol with probability at most $\epsilon$ must be poly($N, \epsilon$)-close to being honest in the appropriate sense. In particular, a simplified version of our protocol is the first that can efficiently certify an arbitrary number of qubits of a cloud quantum computer, on which we cannot enforce spatial separation, using only classical communication.

Contents

1 Introduction 2

2 Preliminaries 6
2.1 Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
2.2 Extended noisy trapdoor claw-free function families . . . . . . . . . . . . . . . . . . . . . . . . 6
2.3 Efficient quantum operations and computational indistinguishability . . . . . . . . . . . . . . 8
2.4 Approximation lemmas . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

3 Completeness of self-testing protocol 17

4 Soundness of self-testing protocol 19
4.1 Quantum devices . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
4.2 Reduction to perfect device . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
4.3 Commutation relations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27
4.4 Anti-commutation relations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 29
4.5 Conjugate invariance . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30
4.6 The swap isometry . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35
4.7 Observables under the swap isometry . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
4.8 States under the swap isometry . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37
4.9 Soundness for states and measurements . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37

5 Dimension-testing protocol 53

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1 Introduction

Self-testing is a unique phenomenon of quantum mechanics that allows a classical verifier to force two or more spatially separated quantum provers to prepare certain states and measure them in certain bases up to local isometries. The term “self-test” was first coined by Mayers and Yao [MY04] in 2004 but its concept can be traced back to much earlier works that study the remarkable features of quantum correlations [Bel64, SW87, Tsi87, PR92, BMR92].

To perform a self-test, the verifier asks questions $x$ of the provers and they return answers $a$. The key idea of self-testing is that if $a$ and $x$ obey certain nonlocal correlations, then the verifier can deduce the provers’ behavior assuming they are spatially separated. This assumption, which implies non-communication between the provers, is crucial because otherwise, the provers could reproduce any correlation by using a lookup table.

The literature on this nonlocal type of self-test is vast [SB20]. They address topics such as which correlations can self-test which states, e.g., [CGS17, GKW+18, BKM19]; how efficient and robust a self-test can be, e.g., [MYS12, McK17, NV17, NV18, CRSV18, Fu22]; and how to use self-testing to, e.g., certify a quantum computer’s components [SBWS18], delegate quantum computations [RUV13, CGJV19], and characterize the complexity of quantum correlations [JNV+20].

A limitation of nonlocal self-testing is the assumption of spatial separation. In practice, it is difficult to certify this assumption, especially if the prover’s device is compact or falls outside our physical control. Therefore, it is interesting to ask whether we can replace this assumption with another one so that we can self-test a single quantum device. We illustrate the nonlocal and single-device settings in Fig. 1.

![Figure 1: Self-testing in the nonlocal setting (left) involves (at least) two spatially separated provers that cannot communicate. In the single-device setting (right), there is only one prover.](image)

**Computational self-testing.** Recently, beginning with seminal work by Mahadev [Mah18] on the classical verification of quantum computations, a series of works, e.g., [GV19, BKVV20, CCY20, ACGH20, VZ20, VZ21, KMCVY21, BCM+21, HLG21, LG21, ZKML+21, MV21, MDCAF21, MTH+21], have explored how computational assumptions can be leveraged by a classical verifier to control a single quantum device in certain ways. Typically, the assumption used is that the Learning-With-Errors (LWE) [Reg09] problem is hard to solve efficiently, even for quantum computers, which is a standard assumption. However, except for [GV19, VZ21, MV21, MDCAF21, MTH+21], the level of control established in these works is much weaker than in nonlocal self-testing. For example, if a device passes Mahadev’s verification protocol [Mah18], it only means that, to quote [MV21], “there exists a quantum state such that the distribution over the prover’s answers could have been produced by performing the requested measurements on this state”. We do not know whether the prover actually prepared that state and performed the requested measurements on it.

Metger and Vidick [MV21] are the first to explicitly propose the self-testing of a single device under computational assumptions. We interchangeably refer to this as computational or single-device self-testing. In a sense, their work is a culmination of many previous results because self-testing offers the strongest control. The main limitation of [MV21] and follow-up work [MTH+21] is that they only self-test two and three qubits, respectively. In this work, we introduce a protocol that self-tests a family of $(2N)$-qubit states, including $N$ EPR pairs, and measurements in the computational setting. Our protocol can be passed with probability negligibly close to 1 by an honest quantum device using poly($N$) resources. On the other hand, we show that any quantum device accepted by our protocol with probability $1 - \epsilon$ must be poly($N, \epsilon$)-close\(^1\) to being honest in the appropriate sense.

---

\(^1\)By poly($N, \epsilon$), we mean a real function of $N$ and $\epsilon$ of order $O(N^a \epsilon^b)$ as $N \to \infty$ and $\epsilon \to 0$, where $a, b > 0$ are constants.
Main results. We give a protocol for testing the following states and measurements.

States: \[
\begin{align*}
&\{\ket{\tau^{0,v}} := \ket{v_1} \otimes \cdots \otimes \ket{v_{\theta-1}} \otimes \ket{-}^{v_{\theta+1}} \otimes \cdots \otimes \ket{v_{2N}} \mid \theta \in \{1, \ldots, 2N\}, \ v \in \{0,1\}^{2N}\} \\
&\cup \{\ket{\tau^{0,v}} := \ket{v_1} \otimes \cdots \otimes \ket{v_{2N}} \mid v \in \{0,1\}^{2N}\} \\
&\cup \{\ket{\tau^{0,v}} := \frac{1}{\sqrt{2^N}} \bigotimes_{i=1}^{N} (\sigma^X)^{v_i} \otimes (\sigma^X)^{v_{N+i}} (\ket{0_i}_i \ket{+}_{N+i} + \ket{1_i}_i \ket{-}_{N+i}) \mid v \in \{0,1\}^{2N}\},
\end{align*}
\]
where \(|(-)^a := (\ket{0} + (-1)^a \ket{1})/\sqrt{2}\) for \(a \in \{0,1\}\).

Measurements:

1. \(\Pi_0^u := \ketbra{u_1 \otimes \cdots \otimes u_{2N}}{u_{2N}} \mid u \in \{0,1\}^{2N}\).

2. \(\Pi_1^u := \ketbra{(-)^{u_1} \otimes \cdots \otimes (-)^{u_{2N}} \ket{(-)^{u_{2N}}} \mid u \in \{0,1\}^{2N}\}.

3. \(\Pi_2^u := \ketbra{u_1 \otimes \cdots \otimes u_{2N} \ket{-}^{u_{N+1}} \otimes \cdots \otimes \ket{-}^{u_{2N}} \mid u \in \{0,1\}^{2N}\}.

4. \(\Pi_3^u := \ketbra{(-)^{u_1} \otimes \cdots \otimes (-)^{u_{N+1}} \otimes u_{N+1} \otimes \cdots \otimes u_{2N} \ket{u_{2N}} \mid u \in \{0,1\}^{2N}\}.

Note that we can self-test product states and entangled states together with local measurements in the single-device setting, which is not possible in the nonlocal setting.

Our protocol generalizes the protocols in [GV19, MV21] and uses the Extended Noisy Trapdoor Claw-Free function Families, or ENTCFs, from [Mah18]. An ENTCF consists of two indistinguishable function-pair families, a claw-free family \(\mathcal{F}\) and an injective family \(\mathcal{G}\). It satisfies various properties under the LWE hardness assumption. In our protocol, the classical verifier selects \(\theta \in \{0,1, \ldots, 2N\} \cup \{\diamond\}\) uniformly at random and sends a \(2N\)-tuple of function pairs from \(\mathcal{F} \cup \mathcal{G}\) to the prover such that when \(\theta = 0\), all pairs are from \(\mathcal{G}\), when \(\theta \in \{1, \ldots, 2N\}\), the \(\theta\)th pair is from \(\mathcal{F}\) and the remaining \(2N-1\) pairs are from \(\mathcal{G}\), and when \(\theta = \diamond\), all pairs are from \(\mathcal{F}\). The prover then sends back \(2N\) images, \(y_1, \ldots, y_{2N}\), of these function-pairs – these play the role of a commitment. In the second round, the verifier either (i) checks the commitment by asking for preimages of the \(y_i\) and accepts or rejects accordingly, or (ii) asks for an equation involving the preimages of the \(y_i\). In case (ii), there is a final round where the verifier sends a uniformly random \(q \in \{0,1,2,3\}\) and the prover sends back the result \(u \in \{0,1\}^{2N}\) of performing some measurement \(\{\Pi_q^u\}_u\). The verifier lastly checks that \(u\) is consistent with measuring \(|\tau^{\theta,v}\rangle\), where \(v \in \{0,1\}^{2N}\) is some bitstring that the verifier can work out, using \(\{\Pi_q^u\}_u\) and accepts or rejects accordingly.

Theorem 1.1 (Informal). Let \(\lambda \in \mathbb{N}\) be a security parameter and let \(N = \lambda\). Assuming the LWE problem of size \(\lambda\) cannot be solved in \(\text{poly}(\lambda)\) time, our protocol satisfies the following properties.

Completeness (see Theorem 3.1). Using \(\text{poly}(\lambda)\) qubits and quantum gates, a quantum prover can prepare one of the \(2N\)-qubit states in \(\{|\tau^{\theta,v}\rangle \mid \theta \in \{0,1, \ldots, 2N\} \cup \{\diamond\}, \ v \in \{0,1\}^{2N}\}\) and measure it using one of the measurements in \(\{\{\Pi_q^u\} \mid u \in \{0,1\}^{2N}\} \mid q \in \{0,1,2,3\}\) to pass our protocol with probability \(\geq 1 - \text{negl}(\lambda)\). Moreover, the verifier can be classical and run in \(\text{poly}(\lambda)\) time.

Soundness (see Theorem 4.1). If a quantum prover passes our protocol in \(\text{poly}(\lambda)\) time with probability \(\geq 1 - \epsilon\), then the prover must have prepared a (sub-normalized) state \(\sigma^{\theta,v}\), measured it using \(\{\Pi_q^u\}_u\), and received outcome \(u\), such that

\[
\sum_{v \in \{0,1\}^{2N}} \|V \sigma^{\theta,v} V^\dagger - \ketbra{\tau^{\theta,v}} \otimes \sigma^{\theta,v}\|_1 = O(N^{7/4} \epsilon^{1/32}) \quad \text{and}
\sum_{u,v \in \{0,1\}^{2N}} \|V P_q^u \sigma^{\theta,v} P_q^v V^\dagger - \Pi_q^u \ketbra{\tau^{\theta,v}} \Pi_q^v \otimes \sigma^{\theta,v}\|_1 = O(N^{7/4} \epsilon^{1/32}),
\]
where \(\theta \in \{0,1, \ldots, 2N\} \cup \{\diamond\}, \ q \in \{0,1,2,3\}, \ u,v \in \{0,1\}^{2N}\) are known to the verifier, \(V\) is an efficient isometry independent of \(\{\theta,q,u,v\}\), and the \(\alpha^{\theta,v}\)s are some auxiliary states.
We highlight the \(\text{poly}(N,\epsilon)\) soundness error (or robustness) that we achieve. Good robustness is critical if we want to use our protocol in practice because real quantum devices are imperfect. The more imperfect a device is, the more robust a protocol needs to be to control it.

We also give another protocol for testing the quantum dimension of a single device under computational assumptions. By quantum dimension, we mean the dimension of the device’s quantum memory. Our dimension-testing protocol is a simplified version of our self-testing protocol and we use the soundness part of Theorem 1.1 to prove the following.

**Theorem 1.2** (Informal, see Theorem 5.3). Under the same computational assumptions as in Theorem 1.1, if a quantum prover runs in \(\text{poly}(\lambda)\) time and passes our dimension-testing protocol with probability \(\geq 1 - \epsilon\), then its quantum dimension is at least \((1 - O(N^{7/4}\epsilon^{1/32}))2^N\).

We take the base-2 logarithm of the quantum dimension of a device as a count of its number of qubits. Therefore, the theorem also lower bounds the number of qubits by \(N - O(N^{7/4}\epsilon^{1/32})\). Importantly, \(O(N^{7/4}\epsilon^{1/32}) = \text{poly}(N,\epsilon)\), which means that it suffices to run our protocol \(\text{poly}(N)\) times to estimate \(\epsilon\) to an accuracy that is sufficient to ensure an \(\Omega(N)\) lower bound on the qubit-count. As a single run of our protocol also only takes \(\text{poly}(N)\) time, the total time to implement a dimension test is \(\text{poly}(N)\), which is efficient in the complexity-theoretic sense. Compared to the nonlocal setting, which requires multiple spatially-separated devices, the computational (i.e., single-device) setting is more appropriate for dimension-testing quantum computers. This is because quantum computers are compact devices and typically exist on the cloud so we cannot enforce spatial separation. There is one other setting for dimension tests: the quantum setting where the verifier is itself quantum and the protocols require quantum communication, see, e.g., [CRSV17, CRSV18, CR20]. The computational setting remains advantageous over the quantum setting as long as classical computational resources and classical communication channels remain less expensive than their quantum counterparts. We stress that in the computational setting, the verifier only performs classical computation and only classical communication is required. To the best of our knowledge, our dimension-testing protocol is the first that can test for an arbitrary quantum dimension in the computational setting. In fact, whether this is possible was recently raised as an open question by Vidick in [Vid20, pg. 82].

**Techniques.** The completeness of our self-testing protocol follows straightforwardly from the properties of ENTCFs (see Section 3). The main challenge is to prove its soundness, which we do in Section 4. We start by defining \(4N\) observables of the prover \(\{X_i, Z_i \mid i \in [2N]\}\) using its measurement operators. The strategy is to characterize these observables as the standard \(\sigma_i^X\) and \(\sigma_i^Z\) Pauli observables on \(2N\) qubits where \(i\) indexes those qubits. Then, we characterize the states of the device by their invariance under products of projectors corresponding to these observables; we characterize the measurements of the device as products of these projectors. To characterize \(X_i\) and \(Z_i\), we first generalize the techniques in [MV21] to show that \(X_i\) and \(Z_j\) obey certain state-dependent commutation and anti-commutation relations (Sections 4.3 and 4.4). From Section 4.5 onwards, we introduce new techniques to handle products of projectors (corresponding to these observables). These techniques differ significantly from [MV21] because their techniques are not susceptible to generalization to arbitrary \(N\). These techniques also differ significantly from those used in nonlocal self-testing because we lack the perfect state-independent commutation relations between observables on two spatially-separated provers. More specifically, we introduce a “conjugate invariance” lemma (Proposition 4.29) and a “lifting” lemma (part 5 of Lemma 2.16) that together give us the ability to “commute an observable past a state” by leveraging the cryptographic properties of ENTCFs based on the LWE hardness assumption. We then use this ability to handle products of projectors (see Sections 4.7 to 4.9). This ability is useful because, for example, \(X_1Z_2X_3\psi = Z_2X_1X_3\psi\) does not follow from the commutation relation \(X_1Z_2\psi = Z_2X_1\psi\), where \(\psi\) is some density operator; however, it does follow if we could commute \(X_3\) past \(\psi\) first because \(X_1\) and \(Z_2\) would then be directly next to \(\psi\).

For the dimension test, we use the soundness part of Theorem 1.1 to show that the Hilbert space \(\mathcal{H}\) of the device must be able to accommodate all possible post-measurement states that could result from performing a Hadamard basis measurement of \(N\) qubits in a computational basis state. There are \(2^N\) such post-measurement states and they are all orthogonal which intuitively implies a dimension lower bound of \(2^N\). A formal proof is more challenging because Theorem 1.1 only gives approximate results. We handle the effect of approximation on the dimension bound by proving a technical result, Proposition 5.2, which shows that the rank of a density operator is robust in a specific sense. Another challenge is that we want
to lower bound only the quantum dimension. We do so by decomposing $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_Q$ into its classical ($\mathcal{H}_C$) and quantum ($\mathcal{H}_Q$) parts. We analyze $\dim(\mathcal{H}_Q)$ under the condition that all states and observables of the device are classical on $\mathcal{H}_C$, i.e., block-diagonal in a fixed basis of $\mathcal{H}_C$. To get the polynomially robust bound in Theorem 1.2, we make essential use of the second equation in Eq. (1.1), which requires us to prove approximation lemmas (parts 3 and 4 of Lemma 2.6, and Lemma 2.17) in the presence of projective measurements where we need to control the errors to be independent of the number of outcomes, which is exponential in $N$ in our proof strategy.

**Discussion.** On the technical side, the immediate next step is to expand the number of measurements that we can self-test. Currently, we only self-test 4 measurements. Ideally, we would like to (efficiently) self-test the following set of $2^{2N}$ measurements:

$$\left\{ \left| \Pi_B = \{|B_1(u_1)\rangle\langle B_1(u_1)| \otimes \ldots \otimes |B_{2N}(u_{2N})\rangle\langle B_{2N}(u_{2N})| \right| \right\}_{B \in \{C,H\}^{2N}}, \quad (1.2)$$

where, for $a \in \{0,1\}$, $|B_i(a)\rangle = |a\rangle$ if $B_i = C$ and $|B_i(a)\rangle = |(-)^a\rangle$ if $B_i = H$. If we could also self-test these measurements, then we would be able to use our self-testing protocol in a black-box way for device-independent quantum key distribution [MDCAF21] and oblivious transfer in the bounded storage model [BY21]. [MDCAF21] and [BY21] currently rely on [MV21]. Using multi-qubit self-testing instead of [MV21] allows for the removal of their iid assumptions and improves their soundness errors. One main reason why we only self-test 4 measurements is because our robustness bound scales linearly with the number of measurements (see Proposition 4.9) and we want a polynomial bound.

Besides generalizing to a wider class of measurements, one may wonder if it is possible to further improve the efficiency and robustness of our protocol. When $N = \lambda$, it seems unlikely that the efficiency can be improved because sending (the public keys of) one function pair already requires poly$(\lambda) = \text{poly}(N)$ bits of communication. Turning to robustness, we note that there exists a nonlocal self-test [NV17] which uses poly$(N)$ bits of communication and achieves robustness poly$(\epsilon)$. It might be possible to combine our techniques with those in [NV17] to achieve similar robustness in the computational setting. Another interesting question to ask is what MIP$^*$ protocols can be compiled into computation delegation protocols under computational assumptions. For comparison, it has been shown that classical MIP protocols sound against non-signaling provers can be turned into computation delegation protocols [TKRR13, KRR14]. Lastly, it would be interesting to see if there exists a systematic way to translate nonlocal self-tests into computational ones. We note that [KMCVY21] suggests that the two settings might not be too different at a conceptual level by presenting a test of quantumness in the computational setting that closely resembles the nonlocal CHSH test [CHSH69]. However, it is unknown if this protocol is quantumly sound.

**Organization.** Our paper is organized as follows. In Section 2, we give the preliminaries required. More specifically, we review ENTCFs and prove a variety of approximation lemmas. In Section 3, we describe our protocol and prove that it can be passed with probability negligibly close to 1 by an honest quantum prover using poly$(N)$ resources (Theorem 3.1). In Section 4, we prove that the soundness error of our protocol can be controlled to within poly$(N,\epsilon)$ (Theorem 4.41). In Section 5, we present a simplified version of our self-testing protocol that can efficiently certify an arbitrary quantum dimension (Theorem 5.5).

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2 Preliminaries

2.1 Notation

N is the set of positive integers. For k ∈ N, we write [k] := \{1, 2, \ldots, k\}. Before Section 5, we reserve the letter N for the number of EPR pairs we self-test and so 2N is the number of qubits we self-test. In Section 5, we reserve N for the number of qubits we dimension-test. We do not use special fonts for vectors. Unless otherwise indicated, the (lowercase) letters a, c are reserved for single bits, i.e., a, c ∈ \{0, 1\}; u, v for bitstrings in \{0, 1\}^2N; e, δ for real numbers in (0, 1); and n for a positive integer. When we use these reserved symbols as indices of a sum without specifying the range, the range should be taken as the entire domain of these symbols. For example, \(\sum_a\) always means \(\sum_{a ∈ \{0, 1\}}\). For a set X and a condition C on elements of that set, we use the notation \(\sum_{x ∈ X| C}\) to mean a sum over all x ∈ X that satisfy condition C. The set X can be implicit, so, for example, \(\sum_{v|v_1=a}\) means a sum over all v ∈ \{0, 1\}^2N with v_1 = a. For a finite set X, we use the notation x ← U X to mean that x is sampled from X uniformly at random.

\(H\) denotes an finite-dimensional Hilbert space and \(L(H)\) denotes the set of linear operators on \(H\), \(\text{Pos}(H)\) denotes the positive semi-definite operators on \(H\), \(\text{Pos}(\mathcal{H})\) := \(\{A ∈ L(H) \mid A ≥ 0\}\). We sometimes refer to operators in \(\text{Pos}(H)\) or vectors in \(H\), not necessarily normalized, as (quantum) states. For operators \(A, B ∈ L(H)\), \(A ≥ B\) means \(A − B ≥ 0\), i.e., \(A − B ∈ \text{Pos}(H)\). \(\mathcal{D}(H)\) denotes set of density operators, \(\mathcal{D}(H) := \{A ∈ \text{Pos}(H) \mid Tr[A] = 1\}\). All Hilbert spaces in this work are viewed as \(\mathbb{C}^m\) for some \(m ∈ \mathbb{N}\) under a fixed choice of basis: this is necessary for some notions we use to make sense, for example, quantum gates and the vector-operator correspondence in Section 5.

We write \(λ ∈ \mathbb{N}\) for a security parameter. Most quantities in this work are dependent on \(λ\). Therefore, for convenience, we often make the dependence implicit. A function \(f : \mathbb{N} → \mathbb{R}\) is said to be negligible if for any polynomial \(p\), \(\lim_{λ → ∞} f(λ)p(λ) = 0\). We denote such functions by \(\text{negl}(λ)\).

For an operator \(X ∈ L(H)\), we write \(\|X\|_p := \text{Tr}[|X|^p]^{1/p}\), where \(|X| := \sqrt{X^*X}\), for the Schatten \(p\)-norm. In this work, we mainly work with the norm \(\|X\|_1\), Frobenius norm \(\|X\|_2\) (also written as \(\|X\|_F\)), and operator norm \(\|X\|_∞\). For operators \(A, B \in L(H)\), we use \((A, B) := \text{Tr}[A^*B]\) to denote Hilbert-Schmidt inner product. The commutator and anti-commutator of \(A\) and \(B\) are defined as \([A, B] := AB − BA\) and \([A, B] := AB + BA\) respectively.

The single-qubit Z and X Pauli operators are denoted \(σ_Z := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\) and \(σ_X := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\) which have eigenstates \(\{|0⟩ := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), |1⟩ := \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}\) and \(\{|-⟩ := \frac{1}{√2}(|0⟩ + |1⟩), |+⟩ := \frac{1}{√2}(|0⟩ − |1⟩)\}\) respectively.

An observable on \(H\) refers to a Hermitian operator in \(L(H)\). We say an observable is a binary observable if it has two eigenvalues, −1 and +1. For a binary observable \(O\), we define \(O^{(0)}\) (resp. \(O^{(1)}\)) to be the projector onto the +1 (resp. −1) eigenspace of \(O\). Equivalently, for \(b ∈ \{0, 1\}\), we define \(O^{(b)} := \frac{1}{2}(1 + (−1)^bO)\). Note that \(O = O^{(0)} − O^{(1)}\). We say \(\{P^i \mid i ∈ [n]\}\) is a projective measurement if \(P^i ≥ 0\) for all \(i ∈ [n]\), \(P^i P^j = δ_{i,j} P^i\) for all \(i, j ∈ [n]\), and \(\sum_{i=1}^n P^i = 1\).

2.2 Extended noisy trapdoor claw-free function families

In this subsection, we informally summarize the properties that we employ of Extended Noisy Trapdoor Claw-Free function Families (ENTCFs). Our discussion is based on the arXiv version of [Mah18]. For full details about the properties of ENTCFs, we refer to [Mah18, Definitions 4.1–4.4]. For full details about how to construct ENTCFs under the LWE hardness assumption, we refer to [Mah18, Section 9]: in particular, throughout this work, we make the LWE hardness assumption as described in [Mah18, Definition 3.4] where the LWE parameters are set according to [Mah18, Section 9.1] as functions of the security parameter \(λ\).

Let \(λ ∈ \mathbb{N}\) be a security parameter. Let \(X ⊆ \{0, 1\}^w\) and \(Y\) be finite sets that depend on \(λ\), where \(w = w(λ)\) is some integer that is a polynomially bounded function of \(λ\). An ENTCF consists of two families of function pairs, \(F\) and \(G\). Function pairs from these two families are labeled by public keys. The set of public keys for \(F\) is denoted by \(K_F\), and the set of public keys for \(G\) is denoted by \(K_G\). For \(k ∈ K_F\), a function pair \((f_k, 0, f_k, 1)\) from \(F\) is called a claw-free pair. For \(k ∈ K_G\), a function pair \((f_k, 0, f_k, 1)\) from \(G\) is called an injective pair. For any \(k ∈ K_F ∪ K_G\), the functions \(f_k, 0\) and \(f_k, 1\) map an \(x ∈ X\) to a probability distribution on \(Y\). Note that the keys and function pairs of an ENTCF are functions of \(λ\). We use the terms “efficient” and “negligible” to refer to poly(\(λ\))-time and \(\text{negl}(λ)\) respectively.

An ENTCF satisfies the following properties:
1. **Efficient function generation property** [Mah18, Definitions 4.1 (1), 4.2 (1)]. There exist efficient classical probabilistic algorithms Gen\(_F\) and Gen\(_G\) for \(\mathcal{F}\) and \(\mathcal{G}\) respectively with

\[
\text{Gen}_{\mathcal{F}}(1^k) \rightarrow (k \in \mathcal{K}_{\mathcal{F}}, t_k) \quad \text{and} \quad \text{Gen}_{\mathcal{G}}(1^k) \rightarrow (k \in \mathcal{K}_{\mathcal{G}}, t_k),
\]

where \(t_k\) is known as a trapdoor. We write \(\text{Gen}_{\mathcal{F}}(1^k)_{\text{key}}\) and \(\text{Gen}_{\mathcal{G}}(1^k)_{\text{key}}\) for the marginal distributions of the public key from \(\text{Gen}_{\mathcal{F}}(1^k)\) and \(\text{Gen}_{\mathcal{G}}(1^k)\) respectively.

2. **(Disjoint) trapdoor injective pair property** [Mah18, Definitions 4.1 (2), 4.2 (2)]. For all \(k \in \mathcal{K}_{\mathcal{F}} \cup \mathcal{K}_{\mathcal{G}}\), \(x, x' \in \mathcal{X}\) with \(x \neq x'\), and \(b \in \{0, 1\}\), we have \(\text{Supp}(f_{k,b}(x)) \cap \text{Supp}(f_{k,b}(x')) = \emptyset\).

For all \(k \in \mathcal{K}_{\mathcal{F}}\), there exists a perfect matching \(R_k \subseteq \mathcal{X} \times \mathcal{X}\) of \(\mathcal{X}\) such that \(f_{k,0}(x) = f_{k,1}(x')\) (equal as distributions on \(\mathcal{Y}\)) if and only if \((x, x') \in R_k\). We call any pair \((x, x') \in R_k\) a claw. In particular, for all \(k \in \mathcal{K}_{\mathcal{F}}\), we have \(\bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,0}(x)) = \bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,1}(x))\).

In contrast, for all \(k \in \mathcal{K}_{\mathcal{G}}\), we have \((\bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,0}(x))) \cap (\bigcup_{x' \in \mathcal{X}} \text{Supp}(f_{k,1}(x'))) = \emptyset\).

3. **Efficient range superposition property** [Mah18, Definitions 4.1 (3.c), 4.2 (3.b), 4.3 (1)]. Given \(k \in \mathcal{K}_{\mathcal{F}} \cup \mathcal{K}_{\mathcal{G}}\), there exists an efficient quantum algorithm that prepares a state that is negligibly close to

\[
|\psi\rangle := \frac{1}{\sqrt{2 \cdot |\mathcal{X}|}} \sum_{b \in \{0, 1\}} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \sqrt{(f_{k,b}(x))(y)} |b\rangle |x\rangle |y\rangle,
\]

in trace distance. (In the case \(k \in \mathcal{K}_{\mathcal{F}}\), this property follows from applying [Mah18, Lemma 3.8] to [Mah18, Definition 4.1 (3.c)], as done in [Mah18, start of Section 5.1].)

4. **Adaptive hardcore bit property** [Mah18, Definition 4.1 (4)]. There does not exist an efficient quantum algorithm that, given \(k \leftarrow \text{Gen}_{\mathcal{F}}(1^k)_{\text{key}}\), can compute \(x_b \in \mathcal{X}\) for some \(b \in \{0, 1\}\), \(d \in \{0, 1\}^m \setminus \{0^m\}\), and, with non-negligible advantage, a bit \(d \cdot (x_0 \oplus x_1) \in \{0, 1\}\) such that \((x_0, x_1) \in R_k\). In particular, this means no efficient quantum algorithm can compute a claw \((x_0, x_1) \in R_k\). This is why function pairs from \(\mathcal{F}\) are called claw-free.

5. **Injective invariance property** [Mah18, Definition 4.3 (2)]. There does not exist an efficient quantum algorithm that can distinguish between the distributions \(\text{Gen}_{\mathcal{F}}(1^k)_{\text{key}}\) and \(\text{Gen}_{\mathcal{G}}(1^k)_{\text{key}}\) with non-negligible advantage.

6. **Efficient decoding property** [Mah18, Definitions 4.1 (2, 3.a, 3.b), 4.2 (2, 3.a), 4.3 (1)]. In this paper, we define the following "decoding maps" that decodes the output of functions from an ENTCF. These follow [MV21, Definition 2.1] but we restate them here for completeness.

   (a) Let \(m \in \mathbb{N}\), \(m = \text{poly}(\lambda)\). For \(k \in (\mathcal{K}_{\mathcal{F}} \cup \mathcal{K}_{\mathcal{G}})^m\), \(y \in \mathcal{Y}^m\), \(b \in \{0, 1\}^m\), and \(x \in \mathcal{X}^m\), we define

   \[
   \text{CHK}(k, y, b, x) := \begin{cases} 
   0 & \text{if } y_i \in \text{Supp}(f_{k,b_i}(x_i)) \text{ for all } i \in [m], \\
   1 & \text{otherwise}.
   \end{cases}
   \]  

   (b) For \(k \in \mathcal{K}_{\mathcal{G}}\) and \(y \in \mathcal{Y}\), we define

   \[
   \hat{b}(k, y) := \begin{cases} 
   0 & \text{if } y \in \bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,0}(x)), \\
   1 & \text{if } y \in \bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,1}(x)), \\
   \bot & \text{otherwise}.
   \end{cases}
   \]

   (c) For \(b \in \{0, 1, \bot\}\), \(k \in \mathcal{K}_{\mathcal{F}} \cup \mathcal{K}_{\mathcal{G}}\), and \(y \in \mathcal{Y}\), we define

   \[
   \hat{x}(b, k, y) := \begin{cases} 
   \bot & \text{if } y \notin \bigcup_{x \in \mathcal{X}} \text{Supp}(f_{k,b}(x)) \text{ or } b = \bot, \\
   x & \text{such that } y \in \text{Supp}(f_{k,b}(x)).
   \end{cases}
   \]

   In addition, for \(k \in \mathcal{K}_{\mathcal{G}}\), we use the shorthand \(\hat{x}(k, y) := \hat{x}(\hat{b}(k, y), k, y)\).

---

2Formally, the computed \(d\) needs to be in some smaller set than \(\{0, 1\}^m \setminus \{0^m\}\) but, as that set contains all but a negligible fraction of elements in \(\{0, 1\}^m\) and so behaves like \(\{0, 1\}^m \setminus \{0^m\}\) for our purposes, we equate it to \(\{0, 1\}^m \setminus \{0^m\}\).
In this section, we record \(\text{Definition 2.2}\), which formalizes the notion of efficient quantum operations. We also append a definition for the efficiency of POVMs that is not present in \([\text{MV21}, \text{Definition 2.2}]\).

**Definition 2.1** (Efficient unitaries, isometries, observables, and measurements).

Let \(\{\mathcal{H}_\lambda \mid \lambda \in \mathbb{N}\}\), \(\{\mathcal{H}_{A,\lambda} \mid \lambda \in \mathbb{N}\}\), and \(\{\mathcal{H}_{B,\lambda} \mid \lambda \in \mathbb{N}\}\) be families of finite-dimensional Hilbert spaces where \(\dim(\mathcal{H}_{A,\lambda}) \leq \dim(\mathcal{H}_{B,\lambda})\) for all \(\lambda\).

1. We say a family of unitaries \(\{U_\lambda \in \mathcal{L}(\mathcal{H}_\lambda) \mid \lambda \in \mathbb{N}\}\) is efficient if there exists a classical Turing machine \(M\) that, on input \(1^k\), outputs a description of a quantum circuit with a fixed gate set that implements \(U_\lambda\) in \(\poly(\lambda)\) time.

2. We say a family of isometries \(\{V_\lambda : \mathcal{H}_{A,\lambda} \to \mathcal{H}_{B,\lambda} \mid \lambda \in \mathbb{N}\}\) is efficient if there exists an efficient family of unitaries \(\{U_\lambda \in \mathcal{L}(\mathcal{H}_{B,\lambda}) \mid \lambda \in \mathbb{N}\}\), such that \(V_\lambda = U_\lambda (1_{A,\lambda} \otimes \ket{0_{k(\lambda)}})\), where \(\ket{0_{k(\lambda)}}\) denotes a fiducial state in an ancillary Hilbert space of dimension \(k(\lambda) := \dim(\mathcal{H}_{B,\lambda})/\dim(\mathcal{H}_{A,\lambda})\).

3. We say a family of binary observables \(\{Z_\lambda \in \mathcal{L}(\mathcal{H}) \mid \lambda \in \mathbb{N}\}\) is efficient, if each \(\mathcal{H}_{B,\lambda} \cong (\mathbb{C}^2)^{\poly(\lambda)}\) and there exists a family of efficient unitaries \(\{U_\lambda \in \mathcal{L}(\mathcal{H}_{A,\lambda} \otimes \mathcal{H}_{B,\lambda}) \mid \lambda \in \mathbb{N}\}\) such that for any \(\ket{\psi}_{A,\lambda} \in \mathcal{H}_{A,\lambda}\),
   \[
   U_\lambda^\dagger (\sigma_Z \otimes \id) U_\lambda (\ket{\psi}_{A,\lambda} \otimes \ket{0}_{B,\lambda}) = (Z_\lambda \ket{\psi}_{A,\lambda}) \otimes \ket{0}_{B,\lambda}.
   \]

4. Let \(\{\mathcal{A}_\lambda \subset \mathbb{N} \mid \lambda \in \mathbb{N}\}\) be a family of finite sets. We say a family of projective measurements
   \[
   \{ M_\lambda = \{ M_i^\lambda \in \mathcal{L}(\mathcal{H}_{A,\lambda}) \mid i \in \mathcal{A}_\lambda \} \mid \lambda \in \mathbb{N} \}
   \]
   is efficient if the family of isometries \(\{V_\lambda := \sum_{i \in \mathcal{A}_\lambda} \ket{i} \otimes M_i^\lambda \mid \lambda \in \mathbb{N}\}\) is efficient.

5. Let \(\{\mathcal{A}_\lambda \subset \mathbb{N} \mid \lambda \in \mathbb{N}\}\) be a family of finite sets. We say that a family of POVMs
   \[
   \{ E_\lambda = \{ E_i^\lambda \in \mathcal{L}(\mathcal{H}_{A,\lambda}) \mid i \in \mathcal{A}_\lambda \} \mid \lambda \in \mathbb{N} \}
   \]
   is efficient if there exists an efficient family of isometries \(\{V_\lambda : \mathcal{H}_{A,\lambda} \to \mathcal{H}_{B,\lambda} \mid \lambda \in \mathbb{N}\}\) and an efficient family of projective measurements \(\{ M_\lambda = \{ M_i^\lambda \in \mathcal{L}(\mathcal{H}_{B,\lambda}) \mid i \in \mathcal{A}_\lambda \} \mid \lambda \in \mathbb{N}\}\) such that \(E_i^\lambda = V_\lambda^\dagger M_i^\lambda V_\lambda\) for all \(i \in \mathcal{A}_\lambda\).

We formally define the notion of computational indistinguishability.

**Definition 2.2** (Computational indistinguishability). We say that two families of positive semi-definite operators \(\{\sigma(\lambda)\}_{\lambda \in \mathbb{N}}\) and \(\{\tau(\lambda)\}_{\lambda \in \mathbb{N}}\) are computationally distinguishable with advantage at most \(\delta = \delta(\lambda)\) if the following holds. For all efficient families of POVMs \(\{\{E_\lambda, 1 - E_\lambda\} \mid \lambda \in \mathbb{N}\}\) (with respect to a fixed polynomial in \(\lambda\)), there exists \(\lambda_0 \in \mathbb{N}\), such that for all \(\lambda \geq \lambda_0\), we have
   \[
   |\text{Tr}[E_\lambda \sigma(\lambda)] - \text{Tr}[E_\lambda \tau(\lambda)]| \leq \delta(\lambda).
   \]

In this case, we write \(\sigma \approx_{\delta} \tau\). If instead of \(\delta\), we have \(O(\delta)\) on the right-hand side of Eq. (2.10), we write \(\sigma \approx_{\delta} \tau\). If \(\delta\) can be chosen to be a negligible function of \(\lambda\), then we say that the two families of positive operators are computationally indistinguishable (without qualification).
We sometimes write Eq. (2.10) in terms of an efficient family of algorithms $\mathcal{A}_\lambda$ that output a bit $b \in \{0, 1\}$ corresponding to $\{E_\lambda, 1 - E_\lambda\}$. In this case, $\text{Tr}[E_\lambda \psi(\lambda)]$ is written as $\Pr(\mathcal{A}_\lambda$ outputs 0 on input $\psi(\lambda)$). When working with computational indistinguishability, we often make the dependence on $\lambda$ implicit and abuse language by referring to states or POVMs instead of families of them.

### 2.4 Approximation lemmas

We first recall some facts about operators in $\mathcal{L}(\mathcal{H})$. These will be frequently used without further comment.

1. For $A \in \mathcal{L}(\mathcal{H})$, $|\text{Tr}[A]| \leq \|A\|_1$ and if $A \geq 0$, $\text{Tr}[A] = |\text{Tr}[A]| = \|A\|_1$.

2. For $A, B, C \in \mathcal{L}(\mathcal{H})$ and $p \in [1, \infty]$, $\|ABC\|_p \leq \|A\|_\infty \|B\|_p \|C\|_\infty$.

3. (Hölder’s inequality for Schatten $p$-norms). For $A, B \in \mathcal{L}(\mathcal{H})$ and $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, $\|AB\|_1 \leq \|A\|_p \|B\|_q$. Note: (i) this is stronger than a common form with $|\langle A, B \rangle|$ on the left-hand side, (ii) when $p = q = 2$, this is also known as the Cauchy-Schwarz inequality for Schatten 2-norms.

#### Definition 2.3

In the following, the notation left of “$\iff$” is defined on its right.

1. **Complex vectors.** For $a, b \in \mathbb{C}^n$, we write
   
   \[ a \simeq_\epsilon b \iff \|a - b\|_1 \leq \epsilon \quad \text{and} \quad a \approx_\epsilon b \iff \|a - b\|_1 \leq O(\epsilon). \]  
   (2.11)

2. **State distance.** For $\phi, \psi \in \mathcal{L}(\mathcal{H})$, we write
   
   \[ \phi \simeq_\epsilon \psi \iff \|\phi - \psi\|_1^2 \leq \epsilon \quad \text{and} \quad \phi \approx_\epsilon \psi \iff \|\phi - \psi\|_1^2 \leq O(\epsilon). \]  
   (2.12)

   The use of this notation is usually reserved for when $\phi, \psi$ are quantum states, i.e., elements of $\text{Pos}(\mathcal{H})$, hence the name “state distance”.

3. **State-dependent operator distance.** For $A, B \in \mathcal{L}(\mathcal{H})$ and $\psi \in \text{Pos}(\mathcal{H})$, we write $\|A\|_{\psi}^2 := \text{Tr}[A^\dagger A\psi] = \|A\sqrt{\psi}\|_2^2$ and
   
   \[ A \simeq_{\epsilon, \psi} B \iff \|A - B\|_{\psi}^2 \leq \epsilon \quad \text{and} \quad A \approx_{\epsilon, \psi} B \iff \|A - B\|_{\psi}^2 \leq O(\epsilon). \]  
   (2.13)

   Note that $\simeq$ is “more precise” than $\approx$. For example, if $a, b \in \mathbb{C}$, then $a \simeq b \implies a \approx b$. In these preliminaries, we choose to use $\simeq$ instead of $\approx$ to be more precise (all results still hold under changing $\simeq$ to $\approx$). This extra precision will occasionally be useful when proving our main results. However, we will usually use $\approx$ instead of $\simeq$ as $\approx$ allows us to hide constant factors and is therefore more convenient.

   We use the following lemma to strengthen various bounds in [MV21].

#### Lemma 2.4

Let $\psi \in \text{Pos}(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$. Then

\[ \|A\|_{\psi} \leq \|A\|_\infty \cdot \sqrt{\text{Tr}[\psi]}. \]  
(2.14)

**Proof.** The lemma follows from Hölder’s inequality:

\[ \|A\|_{\psi}^2 = \text{Tr}[A^\dagger A\psi] \leq \|A^\dagger A\psi\|_1 \leq \|A^\dagger A\|_\infty \cdot \|\psi\|_1 = \|A\|_\infty^2 \cdot \text{Tr}[\psi]. \]  
(2.15)

The next lemma is similar to [MV21, Lemma 2.18(ii)] except that we do not require the $n$ to be constant.

#### Lemma 2.5

Let $\psi_i \in \text{Pos}(\mathcal{H})$ for all $i \in [n]$ and $\psi := \sum_{i=1}^n \psi_i$. Let $\epsilon \geq 0$. Let $A, B \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:

1. There exists $\epsilon_1, \ldots, \epsilon_n \geq 0$ with $\sum_{i=1}^n \epsilon_i \leq \epsilon$ such that $A \simeq_{\epsilon_i, \psi_i} B$ for all $i \in [n]$.
2. $A \simeq_{\epsilon, \psi} B$.  

9
Proof. For $1 \implies 2$, consider

$$\text{Tr}[(A - B)^\dagger(A - B)\psi] = \sum_{i=1}^{n} \text{Tr}[(A - B)^\dagger(A - B)\psi_i] \leq \sum_{i=1}^{n} \epsilon_i \leq \epsilon.$$  \hspace{1cm} (2.16)

For $2 \implies 1$, define $\epsilon_i := \|A - B\|^2_{\psi_i} \geq 0$ for $i \in [n]$, so that $A \simeq_{\epsilon_i} \psi_i, B$ by definition. Then

$$\sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} \text{Tr}[(A - B)^\dagger(A - B)\psi_i] = \text{Tr}[(A - B)^\dagger(A - B)\psi] \leq \epsilon.$$  \hspace{1cm} (2.17)

The following replacement lemma will be frequently used in our analysis. Its first and second parts are similar to [MV21, Lemma 2.21] but strengthened to include the trace of the state $\psi$. Its third and fourth parts allow us to replace states and operators in the presence of projective measurements. Importantly, we keep the error independent of the number of projectors constituting the projective measurement. Intuitively, this should be possible because $\sum_i P_i = \mathbb{1}$ for $\{P_i\}_i$ a projective measurement.

Lemma 2.6 (Replacement lemma).

1. Let $\psi \in \text{Pos}(\mathcal{H})$ and $A, B, C \in \mathcal{L}(\mathcal{H})$. If $A \simeq_{\epsilon, \psi} B$ and $\|C\|_\infty = c$ for some constant $c$, then

$$\text{Tr}[CA\psi] \simeq_{c\sqrt{\text{Tr}[\psi]}} \text{Tr}[CB\psi] \quad \text{and} \quad \text{Tr}[AC\psi] \simeq_{c\sqrt{\text{Tr}[\psi]}} \text{Tr}[BC\psi].$$  \hspace{1cm} (2.18)

2. Let $\psi, \psi' \in \mathcal{L}(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$. If $\psi \simeq_{\epsilon} \psi'$ and $\|A\|_\infty = c$ for some constant $c$, then

$$\text{Tr}[A\psi] \simeq_{c\sqrt{\epsilon}} \text{Tr}[A\psi'].$$  \hspace{1cm} (2.19)

3. Let $\psi \in \text{Pos}(\mathcal{H})$, and $A, B \in \mathcal{L}(\mathcal{H})$ be Hermitian. Let $\{X_1, \ldots, X_n\}$ be a set of mutually commuting binary observables and let $\{Y_1, \ldots, Y_n\}$ be another such set. If $A \simeq_{\epsilon, \psi} B$, then, for all $i \in [n]$,

$$\sum_{u \in \{0,1\}^n} |\text{Tr}[(A - B)X^{(u_1)}_1 X^{(u_2)}_2 \cdots X^{(u_n)}_n Y^{(u_{i+1})}_{i+1} \cdots Y^{(u_n)}_n \psi]| \leq \sqrt{\text{Tr}[\psi]} \cdot \epsilon.$$  \hspace{1cm} (2.20)

4. Let $\psi, \psi' \in \mathcal{L}(\mathcal{H})$ be Hermitian, and $\{P_i\}_{i \in [n]}$ be a projective measurement on $\mathcal{H}$. If $\psi \simeq_{\epsilon} \psi'$, then

$$\sum_{i=1}^{n} |\text{Tr}[P_i(\psi - \psi')]| \leq \sum_{i=1}^{n} \|P_i(\psi - \psi')P_i\|_1 \leq \|\psi - \psi'\|_1 \leq \sqrt{\epsilon}.$$  \hspace{1cm} (2.21)

Proof. The first equation of the first part follows from

$$|\text{Tr}[C(A - B)\psi]| = |(C, A - B)\psi| \leq \|C\|_\psi \cdot \|A - B\| \leq c\sqrt{\text{Tr}[\psi]} \sqrt{\epsilon},$$  \hspace{1cm} (2.22)

where, in the last inequality, we use Lemma 2.4 to bound $\|C\|_\psi \leq c\sqrt{\text{Tr}[\psi]}$. The second equation of the first part can be shown analogously.

The second part follows from $|\text{Tr}[A(\psi - \psi')]| \leq \|A\|_\infty \cdot \|\psi - \psi'\|_1$.

Consider the third part. We first write $P^u := X^{(u_1)}_1 X^{(u_2)}_2 \cdots X^{(u_n)}_n, F^u := X^{(u_i)}_i X^{(u_{i+1})}_{i+1} \cdots X^{(u_n)}_n$, and $Q^u := Y^{(u_i)}_i Y^{(u_{i+1})}_{i+1} \cdots Y^{(u_n)}_n$ for convenience. Note that $(P^u)^\dagger = (P^u)^2 = P^u$ by the commutativity of the
Similarly, \((Q^*)^\dagger = (Q^*)^2 = Q^* \) by the commutativity of the \(Y_i\)s. Then, the third part follows from 

\[
\sum_u |\text{Tr}[(A - B)P^u Q^\dagger \psi]| 
\]

\[
= \sum_u |\langle P^u (A - B) \sqrt{\psi}, P^u Q^\dagger \sqrt{\psi} \rangle| 
\]

\[
\leq \sum_u \|P^u (A - B)\|_2 \|P^u Q^\dagger\|_2 
\]

\[
= \sum_u \sqrt{\text{Tr}[P^u (A - B) \psi(A - B)] \sqrt{\text{Tr}[P^u Q^\dagger \psi Q^u]}} 
\]

\[
= \sqrt{\sum_u \text{Tr}[P^u (A - B) \psi(A - B)] \cdot \sum_u \text{Tr}[P^u Q^\dagger \psi Q^u]} 
\]

\[
\leq \sqrt{\epsilon \cdot \sum_{\hat{u}_1, \ldots, \hat{u}_{n-1}} \text{Tr}[Q^\dagger \psi Q^u]} = \sqrt{\text{Tr}[\psi]} \cdot \epsilon 
\]

(2.23)

Finally, consider the fourth part. We write the Hermitian operator \(\sigma := \psi - \psi^\dagger\) in terms of the decomposition \(\sigma = R - S\) where \(R, S\) are positive semi-definite operators with \(RS = 0\), so that \(\|\sigma\| = R + S\). Then, the fourth part follows from 

\[
\sum_{i=1}^n |\text{Tr}[P^i \sigma]| = \sum_{i=1}^n |\text{Tr}[P^i \sigma P^i]| \leq \sum_{i=1}^n \|P^i \sigma P^i\|_1 \leq \sum_{i=1}^n (\|P^i R P^i\|_1 + \|P^i S P^i\|_1) 
\]

\[
= \sum_{i=1}^n (\text{Tr}[P^i R P^i] + \text{Tr}[P^i S P^i]) = \text{Tr}[R + S] = \text{Tr}[\|\sigma\|] = \|\psi - \psi^\dagger\|_1 \leq \sqrt{\epsilon}. 
\]

(2.24)

The next lemma is elementary and states that the trace norm of a sum of orthogonal operators equals the sum of their trace norms. We give a proof for completeness.

**Lemma 2.7.** Suppose \(A_1, A_2, \ldots, A_n \in \mathcal{L}(\mathcal{H})\) are Hermitian and \(A_iA_j = 0\) for all \(i \neq j\), then

\[
\left\| \sum_{i=1}^n A_i \right\|_1 = \sum_{i=1}^n \|A_i\|_1. 
\]

(2.25)

**Proof.** It clearly suffices to prove the lemma for the case \(n = 2\). In this case, let us write \(A := A_1\) and \(B := A_2\). First, note that \(AB = 0 = (AB)^\dagger = B^\dagger A^\dagger = BA\) so \(A\) and \(B\) commute. Therefore, we can simultaneously diagonalize \(A\) and \(B\) and write \(A = \sum_i \lambda_i |i\rangle \langle i|\) and \(B = \sum_j \mu_j |i\rangle \langle i|\) for some \(\lambda_i, \mu_j \in \mathbb{R}\) and \(|i\rangle \in \text{dim}(\mathcal{H})\) an orthonormal basis of \(\mathcal{H}\). Now, \(AB = 0\) implies \(\sum_i \lambda_i \mu_i |i\rangle \langle i| = 0\) and so the sets \(S := \{|i| \lambda_i \neq 0\}\) and \(T := \{|i| \mu_i \neq 0\}\) are disjoint. Therefore, the lemma follows from 

\[
\|A + B\|_1 = \sum_{i \in S} |\lambda_i + \mu_i| = \sum_{i \in S} |\lambda_i| + \sum_{i \in T} |\mu_i| = \|A\|_1 + \|B\|_1. 
\]

(2.26)

\[ \square \]

The next lemma is similar to [MV21, Lemma 2.22] but also strengthened to include the trace of \(\psi\).

**Lemma 2.8.** Let \(A, B, C \in \mathcal{L}(\mathcal{H})\) with \(\|C\|_\infty = c\) for some constant \(c\), and let \(\psi \in \text{Pos}(\mathcal{H})\). Then the following holds:

\[
A \approx_{c, \psi} B \implies A \psi C \approx_{c \text{Tr}[\psi] c} B \psi C \text{ and } C \psi A^\dagger \approx_{c \text{Tr}[\psi] c} C \psi B^\dagger. 
\]

(2.27)

In particular, let \(P, Q \in \mathcal{L}(\mathcal{H})\) be such that \(\|P\|_\infty, \|Q\|_\infty \leq 1\), then

\[
P \approx_{c, \psi} Q \implies P \psi P \approx_{4 \text{Tr}[\psi] c} Q \psi Q. 
\]

(2.28)
Proof. The proof of Eq. (2.27) is the same as that of [MV21, Lemma 2.22] except we use the bound \( \|C\|_\psi^2 \leq c^2 \text{Tr}[\psi] \) (Lemma 2.4). Eq. (2.28) follows from the first part via

\[
P \simeq_{\epsilon,\psi} Q \implies P\psi P \simeq_{\|P\|_\psi^2} P\psi Q \quad \text{and} \quad P\psi Q \simeq_{\|Q\|_\psi^2} P\psi Q \quad \text{(Eq. (2.27))}
\]

\[
\implies P\psi P \simeq_{\epsilon(\|P\|_\psi + \|Q\|_\psi)^2} Q\psi Q \quad \text{(triangle inequality)}
\]

\[
\implies P\psi P \simeq_{\text{4Tr}[\psi] \epsilon} Q\psi Q \quad \text{(||P||_\infty, ||Q||_\infty \leq 1).}
\]

The next lemma is used to bound the distance between post-measurement states.

**Lemma 2.9** (Post-measurement approximation lemma).

1. Let \( \psi_j \in \text{Pos}(\mathcal{H}) \) for \( j \in [n] \) and \( \psi := \sum_{j=1}^n \psi_j \). Let \( \{P^i\}_{i \in [m]} \) and \( \{Q^i\}_{i \in [m]} \) be two projective measurements on \( \mathcal{H} \). If \( \sum_{i,j} \|P^i - Q^j\|_{\psi_j}^2 \leq \epsilon \), then

\[
\sum_{i,j} \|P^i \psi_j^i P^i \otimes |i,j\rangle \langle i,j| \|_{\text{4Tr}[\psi]} \leq \epsilon \sum_{i,j} \|Q^i \psi_j^i P^i \otimes |i,j\rangle \langle i,j| \|_{\text{4Tr}[\psi]}. \quad \text{(2.29)}
\]

2. Let \( \psi_j, \psi_j' \in \text{Pos}(\mathcal{H}) \) for \( j \in [n] \) and \( \psi := \sum_{j=1}^n \psi_j \) and \( \psi' := \sum_{j=1}^n \psi_j' \). Let \( \{P^i\}_{i \in [m]} \) be a projective measurement on \( \mathcal{H} \). If \( \sum_{i=1}^m \|\psi_j - \psi_j'\|_1 \leq \epsilon \), then

\[
\sum_{i,j} \|P^i \psi_j^i P^i \otimes |i,j\rangle \langle i,j| \|_{1} \leq \epsilon \sum_{i,j} \|P^i \psi_j'^i P^i \otimes |i,j\rangle \langle i,j| \|_{1}. \quad \text{(2.30)}
\]

**Proof.** The first part of the lemma follows from:

\[
\left\| \sum_{i,j} P^i \psi_j^i P^i \otimes |i,j\rangle \langle i,j| - \sum_{i,j} Q^i \psi_j^i Q^i \otimes |i,j\rangle \langle i,j| \right\|_1
\]

\[
= \sum_{i,j} \|P^i \psi_j^i P^i - Q^i \psi_j^i Q^i\|_1 \quad \text{(Lemma 2.7)}
\]

\[
\leq \sum_{i,j} \|(P^i - Q^i)\psi_j P^i\|_1 + \|Q^i \psi_j (P^i - Q^i)\|_1 \quad \text{(triangle inequality)}
\]

\[
\leq \sum_{i,j} \|(P^i - Q^i)\sqrt{\psi_j} \|_2 (\sqrt{\psi_j} P^i \|_2 + \sqrt{\psi_j} Q^i \|_2) \quad \text{(Cauchy-Schwarz for Schatten 2-norms)}
\]

\[
= \sum_{i,j} \|(P^i - Q^i)\|_{\psi_j} \left( \sqrt{\text{Tr}[P^i \psi_j]} + \sqrt{\text{Tr}[Q^i \psi_j]}) \right) \quad \text{(definitions)}
\]

\[
\leq \sqrt{\sum_{i,j} \|(P^i - Q^i)\|_{\psi_j}^2} \cdot \left( \sqrt{\sum_{i,j} \text{Tr}[P^i \psi_j]} + \sqrt{\sum_{i,j} \text{Tr}[Q^i \psi_j]} \right) \quad \text{(Cauchy-Schwarz)}
\]

\[
\leq 2\sqrt{\epsilon} \sqrt{\text{Tr}[\psi]} \quad \text{(lemma conditions)}.
\]

The second part of the lemma follows from

\[
\left\| \sum_{i,j} P^i \psi_j^i P^i \otimes |i,j\rangle \langle i,j| - \sum_{i,j} P^i \psi_j'^i P^i \otimes |i,j\rangle \langle i,j| \right\|_1 = \sum_{i,j} \|P^i \psi_j^i P^i - P^i \psi_j'^i P^i\|_1 \leq \sum_j \|\psi_j - \psi_j'\|_1 \leq \epsilon, \quad \text{(2.31)}
\]

where we used Lemma 2.7 for the first equality, part 4 of the replacement lemma (Lemma 2.6) for the first inequality, and the lemma condition for the second inequality.

The next two lemmas are proved in [MV21]. We omit the proofs.
Lemma 2.10 ([MV21, Lemma 2.23]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with $\dim(\mathcal{H}_1) \leq \dim(\mathcal{H}_2)$, $V$ be an isometry: $\mathcal{H}_1 \to \mathcal{H}_2$, and $A$ and $B$ be binary observables on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then, the following holds for all $\psi \in \text{Pos}(\mathcal{H}_1)$:

\[
VAV^\dagger \simeq_{\epsilon,\psi V^\dagger} B \implies A \simeq_{\epsilon,\psi} V^\dagger BV,
\]

\[
A \simeq_{\epsilon,\psi} V^\dagger BV \implies VAV^\dagger \simeq_{\sqrt{\epsilon},\psi V^\dagger} B.
\]

Lemma 2.11 ([MV21, Lemma 2.24]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with $\dim(\mathcal{H}_1) \leq \dim(\mathcal{H}_2)$, $V$ be an isometry: $\mathcal{H}_1 \to \mathcal{H}_2$, and $A$ and $B$ be binary observables on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then, the following holds for all $\psi \in \text{Pos}(\mathcal{H}_1)$ and $b \in \{0,1\}$:

\[
A \simeq_{\epsilon,\psi} V^\dagger BV \implies A^{(b)} \simeq_{\epsilon,\psi} V^{(b)}V,
\]

\[
B \simeq_{\epsilon,\psi V^\dagger} VAV^\dagger \implies B^{(b)} \simeq_{\epsilon,\psi V^\dagger} VA^{(b)}V^\dagger.
\]

The next lemma will be used to characterize the states of the quantum device given a characterization of its observables. The $i \in [n]$ will index qubits.

Lemma 2.12. Let $P_1, \ldots, P_n \in \text{Pos}(\mathcal{H})$ be projectors and $A \in \mathcal{L}(\mathcal{H})$ be such that

\[
\|A - P_i AP_i\|_1 \leq \epsilon_i,
\]

for all $i \in [n]$. Then, writing $P_{1:n} := P_1 P_2 \cdots P_n$, we have

\[
\|A - P_{1:n} P_1 P_{1:n}\|_1 \leq \sum_{i=1}^n \epsilon_i.
\]

Proof. We argue by induction on $n \geq 0$. The base case, $\|A - A\|_1 = 0$, clearly holds.

Now, for $i \in [n+1]$, we write $P_{1:n+1} := P_1 P_2 \cdots P_{n+1}$. Then,

\[
\begin{align*}
\|A - P_{1:n+1} P_1 P_{1:n+1}\|_1 & \leq \|A - P_{2:n+1} P_{2:n+1}\|_1 + \|P_{2:n+1} P_{2:n+1} - P_{1:n+1} P_{1:n+1}\|_1 \\
& \leq \sum_{i=2}^{n+1} \epsilon_i + \|P_{2:n+1} P_{2:n+1} - P_{1:n+1} P_1 P_{2:n+1}\|_1 \\
& \leq \sum_{i=2}^{n+1} \epsilon_i + \|A - P_1 P_1\|_1 \leq \sum_{i=1}^{n+1} \epsilon_i
\end{align*}
\]

which completes the proof.

Unlike the preceding lemmas, all remaining lemmas of this subsection relate to computational indistinguishability. In particular, this means they only hold with respect to efficient operations.

Lemma 2.13 (Triangle inequality for computational indistinguishability). Let $\{\rho_i \mid i \in [n]\} \subseteq \text{Pos}(\mathcal{H})$ be such that $\rho_i \simeq_{\epsilon_i} \rho_{i+1}$ for all $i \in [n-1]$, then

\[
\rho_1 \simeq_{\sum_{i=1}^{n-1} \epsilon_i} \rho_n.
\]

Proof. Let $\{E, 1 - E\}$ be an efficient POVM. The lemma follows from

\[
|\text{Tr}[E \rho_1] - \text{Tr}[E \rho_n]| \leq \sum_{i=1}^{n-1} |\text{Tr}[E \rho_i] - \text{Tr}[E \rho_{i+1}]| \leq \sum_{i=1}^{n-1} \epsilon_i.
\]

\[
\square
\]

13
Lemma 2.14 (Partitioning property of computational indistinguishability). Let $\sigma, \tau \in \text{Pos}(\mathcal{H})$. Let $\{\Pi^i\}_{i \in [n]}$ be an efficient projective measurement. If $\sigma \approx_\delta \tau$, then there exists $\delta_i \geq 0$ such that

$$
\Pi^i \sigma \Pi^i \simeq_{\delta_i} \Pi^i \tau \Pi^i,
$$

(2.38)

for all $i \in [n]$, and $\sum_{i=1}^n \delta_i \leq 2\delta$.

Proof. For $i \in [n]$, let

$$
\alpha_i := \max \text{Tr}[A(\Pi^i \sigma \Pi^i - \Pi^i \tau \Pi^i)] \quad \text{and} \quad \beta_i := \max \text{Tr}[-B(\Pi^i \sigma \Pi^i - \Pi^i \tau \Pi^i)],
$$

(2.39)

where the max is taken over all efficient POVM elements $A$ and $B$. Because $A$ and $B$ can be zero, we see $\alpha_i, \beta_i \geq 0$. Let $A_i$ and $B_i$ be the corresponding maximizers.

By definition, $\delta_i := \max\{\alpha_i, \beta_i\}$ equals how computationally indistinguishable $\Pi^i \sigma \Pi^i$ is from $\Pi^i \tau \Pi^i$. As $\alpha_i, \beta_i \geq 0$, we have $\delta_i \leq \alpha_i + \beta_i$, so it suffices to upper bound $\sum_{i=1}^n \alpha_i + \beta_i$.

Consider the following efficient algorithm for distinguishing $\sigma$ and $\tau$:

1. Measure $\{\Pi^i\}_i$ on the input state and record the outcome.
2. If the outcome is $i$ then measure $\{A_i, 1 - A_i\}$.
3. Output 0 if the result corresponds to $A_i$ and 1 otherwise.

The probability that this algorithm outputs 0 on any input $\psi \in \text{Pos}(\mathcal{H})$ is

$$
\sum_{i=1}^n \text{Tr}[\Pi^i \sigma \Pi^i] \text{Tr}[A_i \cdot \frac{\Pi^i \psi \Pi^i}{\text{Tr}[\Pi^i \psi \Pi^i]}] = \sum_{i=1}^n \text{Tr}[A_i \Pi^i \psi \Pi^i].
$$

(2.40)

Therefore, by the definition of $\sigma \approx_\delta \tau$, we deduce

$$
\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \text{Tr}[A_i (\Pi^i \sigma \Pi^i - \Pi^i \tau \Pi^i)] \leq \delta.
$$

(2.41)

We can similarly show that $\sum_{i=1}^n \beta_i \leq \delta$. Therefore, $\sum_{i=1}^n \delta_i \leq 2\delta$ as required.

We record the following lemma.

Lemma 2.15 ([MV21, Lemma 2.6]). Let $U_1, U_2$ be efficient unitaries on $\mathcal{H}$. Then, $(U_1 + U_2)^\dagger(U_1 + U_2)$ and $(U_1 - U_2)^\dagger(U_1 - U_2)$ are observables and there exists an efficient quantum algorithm that given a state $\psi \in \mathcal{D}(\mathcal{H})$ outputs a bit $b$ with

$$
\text{Pr}[b = 0|\psi] = \frac{1}{4} \text{Tr}[(U_1 + U_2)^\dagger(U_1 + U_2)\psi] \quad \text{and} \quad \text{Pr}[b = 1|\psi] = \frac{1}{4} \text{Tr}[(U_1 - U_2)^\dagger(U_1 - U_2)\psi].
$$

(2.42)

Remark. The proof in [MV21] uses a purification step that we do not know how to make efficient. However, they could have worked directly with mixed states to derive the result (see the proof of Lemma 2.17).

We end this section with two lemmas that allow us to use the computational indistinguishability between two states $\psi$ and $\psi'$ to “lift” a true statement about $\psi$ onto $\psi'$.

Lemma 2.16 (Lifting lemma, see [MV21, Lemma 2.25 (i–v)]). Let $\psi, \psi' \in \mathcal{D}(\mathcal{H})$ be such that $\psi \approx_\delta \psi'$.

1. Let $A$ be an efficient binary observable on $\mathcal{H}$. Then, $\text{Tr}[A\psi] \simeq_{2\delta} \text{Tr}[A\psi']$.
2. Let $A, B$ be efficient binary observables on $\mathcal{H}$. Then,

$$
A \simeq_{\epsilon, \psi} B \implies A \simeq_{\delta + \epsilon, \psi'} B.
$$

(2.43)

3. Let $A, B$ be efficient binary observables on $\mathcal{H}$. Then,

$$
[A, B] \simeq_{\epsilon, \psi} 0 \implies [A, B] \simeq_{\delta + \epsilon, \psi'} 0.
$$

(2.44)

14
4. Let $A, B$ be efficient binary observables on $\mathcal{H}$. Then,
\[ \{A, B\} \simeq_{\epsilon, \psi} 0 \implies \{A, B\} \simeq_{\delta+\epsilon, \psi'} 0. \] (2.45)

5. Let $\mathcal{H}'$ be another Hilbert space with $\dim(\mathcal{H}') \geq \dim(\mathcal{H})$, $U$ an efficient unitary on $\mathcal{H}$, $B$ an efficient binary observable on $\mathcal{H}'$, and $V: \mathcal{H} \to \mathcal{H}'$ an efficient isometry. Then
\[ \Re \text{Tr}[V^\dagger BV U \psi] \simeq_{25} \Re \text{Tr}[V^\dagger BV U \psi'] \quad \text{and} \quad \Re \text{Tr}[U V^\dagger BV \psi] \simeq_{25} \Re \text{Tr}[U V^\dagger BV \psi']. \] (2.46)

**Proof.** The first four parts of the lemma (and their proofs) correspond to [MV21, Lemma 2.25 (i–iv)] verbatim, except that we record the constant factors arising from their proof. The fifth part is implicit in [MV21, Proof of Lemma 2.25(v)]. We present a full proof of the fifth part for completeness.

First, note that the second equation in Eq. (2.46) follows from the first by setting $V$ to $VU^\dagger$, which is efficient. Therefore, it suffices to prove the first equation in Eq. (2.46). Let $W \in \mathcal{L}(\mathcal{H}')$ be an efficient unitary such that $V = W(1 \otimes |0\rangle_{\mathcal{H}''})$ where $\dim(\mathcal{H}'') = \dim(\mathcal{H}')/\dim(\mathcal{H})$. $\mathcal{H}''$ exists because we can add dimension to $\mathcal{H}'$. Since $U \otimes 1_{\mathcal{H}''}$ and $W^\dagger BW$ are efficient unitaries, we can apply Lemma 2.15 to see that there exists an efficient algorithm that outputs a bit $b$ with
\[ \Pr[b = 0|\psi] = \frac{1}{4} \text{Tr}[(U \otimes 1_{\mathcal{H}'}) - W^\dagger BW](U \otimes 1_{\mathcal{H}'}) - W^\dagger BW)(\psi \otimes |0\rangle_{\mathcal{H}'})]. \] (2.47)

Since $\psi \simeq_{\delta} \psi'$, this means
\[ \text{Tr}[(U \otimes 1_{\mathcal{H}'}) - W^\dagger BW](U \otimes 1_{\mathcal{H}'}) - W^\dagger BW)(\psi' \otimes |0\rangle_{\mathcal{H}'})] \approx_{4\delta} \text{Tr}[(U \otimes 1_{\mathcal{H}'}) - W^\dagger BW](U \otimes 1_{\mathcal{H}'}) - W^\dagger BW)(\psi \otimes |0\rangle_{\mathcal{H}'})]. \] (2.48)

On the other hand, we have
\[ \text{Tr}[(U \otimes 1_{\mathcal{H}'}) - W^\dagger BW](U \otimes 1_{\mathcal{H}'}) - W^\dagger BW)(\psi' \otimes |0\rangle_{\mathcal{H}'})] = 2 - 2 \Re \text{Tr}[V^\dagger BVU \psi'], \] (2.49)
\[ \text{Tr}[(U \otimes 1_{\mathcal{H}'}) - W^\dagger BW](U \otimes 1_{\mathcal{H}'}) - W^\dagger BW)(\psi \otimes |0\rangle_{\mathcal{H}'})] = 2 - 2 \Re \text{Tr}[V^\dagger BVU \psi]. \] (2.50)

Therefore,
\[ \Re \text{Tr}[V^\dagger BVU \psi] \approx_{25} \Re \text{Tr}[V^\dagger BVU \psi']. \] (2.51)

In our work, we will need a new type of lifting lemma for handling projective measurements.

**Lemma 2.17** (Lifting-under-projections lemma). Let $\{\Pi^u\}_{u \in \{0, 1\}^n}$ and $\{P^u\}_{u \in \{0, 1\}^n}$ be two efficient projective measurements on $\mathcal{H}$. Let $\psi, \psi' \in \text{Pos}(\mathcal{H})$ be such that $\psi \simeq_{\delta} \psi'$ for some $\delta \geq 0$. Then, there exists $\alpha_u, \beta_u \geq 0$ for $u \in \{0, 1\}^n$ such that
\[ (\Pi^u + P^u)\psi(\Pi^u + P^u) \simeq_{\alpha_u} (\Pi^u + P^u)\psi(\Pi^u + P^u), \]
\[ (\Pi^u - P^u)\psi(\Pi^u - P^u) \simeq_{\beta_u} (\Pi^u - P^u)\psi(\Pi^u - P^u), \] (2.52)
where $\sum_{u \in \{0, 1\}^n} (\alpha_u + \beta_u) \leq 4\delta$.

Moreover, let $\{Q^u\}_{u \in \{0, 1\}^n}$ be another efficient projective measurement. Then, we have
\[ \sum_{u \in \{0, 1\}^n} |\Re \text{Tr}[Q^u \Pi^u P^u \psi] - \Re \text{Tr}[Q^u \Pi^u P^u \psi']| \leq 2\delta. \] (2.53)

**Proof.** Let $\rho_u^\pm := (\Pi^u \pm P^u)\psi(\Pi^u \pm P^u)$ and $\rho_u^\ell \pm := (\Pi^u \pm P^u)\psi'(\Pi^u \pm P^u)$. Let
\[ \alpha_u^+ := \max_{A_u} \text{Tr}[A_u(\rho_u^+ - \rho_u^\ell)] \geq 0 \quad \text{and} \quad \beta_u^+ := \max_{B_u} \text{Tr}[B_u(\rho_u^- - \rho_u^\ell)] \geq 0, \] (2.54)
where the max is taken over efficient POVM elements $A_u$ and $B_u$. Let $A_u^*$ and $B_u^*$ denote the respective maximizers. By the linear combination of unitaries technique [CW12] (also see Lemma 2.15) and the fact that \{\Pi^u\}_{u \in \{0, 1\}^n} and \{P^u\}_{u \in \{0, 1\}^n} are efficient, we see that the following isometry is efficient:

$$V := \frac{1}{2} \left( |0\rangle \otimes \sum_{u \in \{0, 1\}^n} |u\rangle \otimes (\Pi^u + P^u) + |1\rangle \otimes \sum_{u \in \{0, 1\}^n} |u\rangle \otimes (\Pi^u - P^u) \right).$$ (2.55)

In addition, the POVM $\{\Gamma, \mathbb{1} - \Gamma\}$ is efficient, where

$$\Gamma := \sum_{u \in \{0, 1\}^n} |0, u\rangle \langle 0, u| \otimes A_u^* + \sum_{u \in \{0, 1\}^n} |1, u\rangle \langle 1, u| \otimes B_u^*.$$ (2.56)

Therefore, there exists an efficient algorithm that, given any $\sigma \in \text{Pos}(\mathcal{H})$, outputs 0 with probability

$$\text{Tr}[\Gamma \psi \psi^\dagger] = \frac{1}{2} \left( \sum_{u \in \{0, 1\}^n} \text{Tr}[A_u^* (\Pi^u + P^u) \sigma (\Pi^u + P^u)] + \text{Tr}[B_u^* (\Pi^u - P^u) \sigma (\Pi^u - P^u)] \right).$$ (2.57)

Therefore, by the definition of $\psi \simeq_\delta \psi'$, we have $\text{Tr}[\Gamma \psi \psi^\dagger] - \text{Tr}[\Gamma \psi' \psi'^\dagger] \leq \delta$. That is,

$$\frac{1}{2} \sum_{u \in \{0, 1\}^n} (\alpha^+_u + \beta^+_u) \leq \delta.$$ (2.58)

Now, let

$$\alpha_u^- := \max_{A_u} -\text{Tr}[A_u (\rho^+_u - \rho^+_u')] \geq 0 \quad \text{and} \quad \beta_u^- := \max_{B_u} -\text{Tr}[B_u (\rho^-_u - \rho^-_u')] \geq 0,$$ (2.59)

where the maximization is again over efficient POVM elements $A_u$ and $B_u$. We can similarly show that

$$\frac{1}{2} \sum_{u \in \{0, 1\}^n} (\alpha^-_u + \beta^-_u) \leq \delta.$$ (2.60)

But $\alpha_u := \max \{\alpha^+_u, \alpha^-_u\}$ equals how computational indistinguishable $\rho^+_u$ and $\rho^-_u$ are. Likewise $\beta_u := \max \{\beta^+_u, \beta^-_u\}$ equals how computational indistinguishable $\rho^+_u$ and $\rho^-_u$ are. Therefore, we obtain

$$\frac{1}{2} \sum_{u \in \{0, 1\}^n} (\alpha_u + \beta_u) \leq \frac{1}{2} \sum_{u \in \{0, 1\}^n} (\alpha^+_u + \beta^+_u + \alpha^-_u + \beta^-_u) \leq 2\delta.$$ (2.61)

Hence the first part of the lemma.

Now consider the second, “moreover”, part of the lemma. Let $\rho^\pm_{w,u} := Q^w (\Pi^u \pm P^u) \psi (\Pi^u \pm P^u) Q^w$ and $\rho^\pm_{w,u} := Q^w (\Pi^u \pm P^u) \psi (\Pi^u \pm P^u) Q^w$. The partitioning property of computational indistinguishability (Lemma 2.14) shows that there exists $\alpha_{w,u}, \beta_{w,u} \geq 0$ for $u \in \{0, 1\}^n, w \in \{0, 1\}^m$ such that

$$\rho^+_{w,u} \simeq_{\alpha_{w,u}} \rho^+_{w,u} \quad \text{and} \quad \rho^-_{w,u} \simeq_{\beta_{w,u}} \rho^-_{w,u},$$ (2.62)

where $\sum_{w \in \{0, 1\}^m} \alpha_{w,u} \leq 2\alpha_u$ and $\sum_{w \in \{0, 1\}^m} \beta_{w,u} \leq 2\beta_u$.

The second part of the lemma then follows from the first part by

$$\frac{1}{4} \sum_{w \in \{0, 1\}^m, u \in \{0, 1\}^n} |\text{Re} \text{Tr}[Q^w \Pi^u P^u \psi] - \text{Re} \text{Tr}[Q^w \Pi^u P^u \psi']|$$

$$= \frac{1}{4} \sum_{w \in \{0, 1\}^m, u \in \{0, 1\}^n} \left| \frac{1}{4} (\text{Tr}[\rho^+_{w,u}] - \text{Tr}[\rho^-_{w,u}] - (\text{Tr}[\rho^+_{w,u}] - \text{Tr}[\rho^-_{w,u}]) \right|$$

$$\leq \frac{1}{4} \sum_{w \in \{0, 1\}^m, u \in \{0, 1\}^n} (\alpha_{w,u} + \beta_{w,u}) \leq \frac{1}{2} \sum_{u \in \{0, 1\}^n} (\alpha_u + \beta_u) \leq 2\delta,$$ (2.63)

which completes the proof. \qed
3 Completeness of self-testing protocol

In this section, we introduce our self-testing protocol in Fig. 2. We then prove Theorem 3.1 by describing an efficient honest quantum prover that passes our protocol with probability $\geq 1 - \text{neg}(\lambda)$ and noting that the verification can be efficiently performed classically. Note that we set $N = \lambda$ in the protocol.

1. Set $N = \lambda$. Select $\theta \in \{0, 1\}^N \cup \{0, 1\}$ and $\lambda$. Set $2N$ key-trapdoor pairs $(k_1, t_{k_1}), \ldots, (k_{2N}, t_{k_{2N}})$ from an ENTCF according to $\theta$ as follows:

$\theta \in \{0, 1\}^N$: the $\theta$-th key-trapdoor pair is sampled from $\text{Gen}_{\mathcal{F}}(1^\lambda)$ and the remaining $2N - 1$ pairs are all sampled from $\text{Gen}_{\mathcal{G}}(1^\lambda)$.

$\theta = 0$: all the key-trapdoor pairs are sampled from $\text{Gen}_{\mathcal{G}}(1^\lambda)$.

$\theta = \lambda$: all the key-trapdoor pairs are sampled from $\text{Gen}_{\mathcal{F}}(1^\lambda)$.

Send the keys $k = (k_1, \ldots, k_{2N})$ to the prover.

2. Receive $y = (y_1, \ldots, y_{2N}) \in \mathcal{Y}^N$ from the prover.

3. Select round type “preimage” or “Hadamard” uniformly at random and send to the prover.

**Case “preimage”:** receive $(b, v) = (b_1, \ldots, b_{2N}, x_1, \ldots, x_{2N})$ from the prover, where $b \in \{0, 1\}^{2N}$ and $x \in \{0, 1\}^{2N}$. If $\text{CHK}(k, y) = 0$ for all $i \in \{2N\}$, accept, else reject.

**Case “Hadamard”:** receive $d = (d_1, \ldots, d_{2N}) \in \{0, 1\}^{2N}$ from the prover.

Sample $q \leftarrow U \{0, 1, 2, 3\}$ and send to the prover.

Receive $v \in \{0, 1\}^{2N}$ from the prover.

**Case A.** $\theta \in \{0, 1\}^N$, $\theta \leq N$, and $q = 0$: if $b(k_i, y_i) \neq v_i$ for some $i \neq \theta$, reject, else accept.

$q = 1$: if $h(k_\theta, y_\theta, d_\theta) \oplus b(k_{\theta+N}, y_{\theta+N}) \neq v_\theta$, reject, else accept.

$q = 2$: if $b(k_i, y_i) \neq v_i$ for some $i \leq N$ and $i \neq \theta$, reject, else accept.

$q = 3$: if $b(k_i, y_i) \neq v_i$ for some $i > N$ or $b(k_{\theta+N}, y_{\theta+N}) \neq d_\theta$, reject, else accept.

**Case B.** $\theta \in \{0, 1\}^N$, $\theta > N$, and $q = 0$: if $b(k_i, y_i) \neq v_i$ for some $i \neq \theta$, reject, else accept.

$q = 1$: if $h(k_\theta, y_\theta, d_\theta) \oplus b(k_{\theta+N}, y_{\theta+N}) \neq v_\theta$, reject, else accept.

$q = 2$: if $b(k_i, y_i) \neq v_i$ for some $i \leq N$ or $h(k_\theta, y_\theta, d_\theta) \oplus b(k_{\theta+N}, y_{\theta+N}) \neq v_\theta$, reject, else accept.

$q = 3$: if $b(k_i, y_i) \neq v_i$ for some $i > N$ and $i \neq \theta$, reject, else accept.

**Case C.** $\theta = 0$ and $q = 0$: if $b(k_i, y_i) \neq v_i$ for some $i$, reject, else accept.

$q = 1$: accept.

$q = 2$: if $b(k_i, y_i) \neq v_i$ for some $i \leq N$, reject, else accept.

$q = 3$: if $b(k_i, y_i) \neq v_i$ for some $i > N$, reject, else accept.

**Case D.** $\theta = \lambda$ and $q = 0$: accept.

$q = 1$: accept.

$q = 2$: if $v_i \oplus v_{N+i} \neq \tilde{h}(k_{N+i}, y_{N+i}, d_{N+i})$ for some $i \in [N]$, reject, else accept.

$q = 3$: if $v_i \oplus v_{N+i} \neq \tilde{h}(k_i, y_i, d_i)$ for some $i \in [N]$, reject, else accept.

Figure 2: A protocol that self-tests multi-qubit states and measurements of a computationally efficient device.
Theorem 3.1. There exists a quantum prover using \( \poly(\lambda) \) qubits and quantum gates that is accepted by our self-testing protocol with probability \( \geq 1 - \negl(\lambda) \). Moreover, there exists a classical verifier that runs in \( \poly(\lambda) \) time.

Proof. In the first round, for each \( i \in [2N] \), the (honest quantum) prover uses \( k_i \) to prepare a state \( |\psi'_i\rangle \) that is negligibly close to

\[
|\psi_i\rangle := \frac{1}{\sqrt{2^N}} \sum_{b \in \{0,1\}} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \sqrt{(f_{k_i,b}(x))(y)} |b\rangle |x\rangle |y\rangle,
\]

in trace distance, using the efficient range superposition property of ENTCFs; this uses \( \poly(\lambda) \) qubits and quantum gates. Therefore, the prover prepares the tensor product \( \bigotimes_{i=1}^{2N} |\psi'_i\rangle \) using \( O(2N \cdot \poly(\lambda)) = \poly(\lambda) \) qubits and quantum gates. Note that we used \( N = \lambda \) in the last equality.

We have \( \| \bigotimes_{i=1}^{2N} |\psi\rangle - \bigotimes_{i=1}^{2N} |\psi'_i\rangle \| \leq 2N \cdot \negl(\lambda) = \negl(\lambda), \) using \( N = \lambda \). Therefore, the output distributions arising from all subsequent measurements made on \( \bigotimes_{i=1}^{2N} |\psi'_i\rangle \) are the same as those made on \( \bigotimes_{i=1}^{2N} |\psi_i\rangle \) up to \( \negl(\lambda) \) in total variation distance. Hence, for the rest of this proof, we can assume that the prover has actually prepared \( |\psi\rangle := \bigotimes_{i=1}^{2N} |\psi_i\rangle \) because we only need to characterize the probability the prover is accepted up to \( \negl(\lambda) \) additive error to prove the theorem.

Then, the prover measures the \( (\text{image}) \) \( y \) register of \( |\psi_i\rangle \) and sends the outcome to the verifier. By the (disjoint) trapdoor injective pair property of ENTCFs, after the \( y \) measurement, the state \( |\psi_i\rangle \) collapses to \( |\phi_i\rangle |y_i\rangle \), where

\[
|\phi_i\rangle := \begin{cases}
|b(k_i, y_i)\rangle |\hat{x}(k_i, y_i)\rangle & \text{if } k_i \in \mathcal{K}_y, \\
\frac{1}{\sqrt{2}}(|0\rangle |\hat{x}_0(k_i, y_i)\rangle + |1\rangle |\hat{x}_1(k_i, y_i)\rangle) & \text{if } k_i \in \mathcal{K}_x.
\end{cases}
\]

In the following, we use the shorthand \( \hat{b}_i := \hat{b}(k_i, y_i) \in \{0,1\} \) and, for \( a \in \{0,1\}, \hat{x}_{a,i} := \hat{x}(a, k_i, y_i) \in \mathcal{X}. \)

In the second round, there are two cases, “preimage” or “Hadamard”. In the “preimage” case, the prover measures the \( b \) and \( x \) registers of each \( |\phi_i\rangle \) in the computational basis and sends the outcome to the prover. This will always be accepted by the prover using the definition of CHK.

In the “Hadamard” case, the prover measures the \( x \) register of each \( |\phi_i\rangle \) in the Hadamard basis and sends the outcome \( d = (d_1, d_2, \ldots, d_{2N}) \) to the verifier. After this measurement, \( |\phi_i\rangle \) collapses to \( |\alpha_i\rangle |d_i\rangle \), where, if \( \theta \in [2N] \), then

\[
|\alpha_i\rangle = \begin{cases}
|\hat{b}_i\rangle & \text{if } i \neq \theta, \\
(0) + (-1)^{d_x(-\hat{x}_0, e^{\hat{x}_1}, i)} |1\rangle & \text{if } i = \theta.
\end{cases}
\]

if \( \theta = 0 \), then \( |\alpha_i\rangle = |\hat{b}_i\rangle \); and if \( \theta = \lambda \), then \( |\alpha_i\rangle = (0) + (-1)^{d_x(-\hat{x}_0, e^{\hat{x}_1}, i)} |1\rangle / \sqrt{2} \).

In the following, we use the shorthand \( \hat{h}_i := d_i \cdot (\hat{x}_{0,i} \oplus \hat{x}_{1,i}) \in \{0,1\} \) and

\[
\hat{h}' := (\hat{h}_{N+1}, \ldots, \hat{h}_{2N}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N) \in \{0,1\}^{2N}.
\]

For \( v \in \{0,1\}^{2N} \), we also define the state

\[
|\psi^v\rangle := \frac{1}{\sqrt{2^N}} \bigotimes_{i=1}^{2N} (\sigma_X^{v_i} \otimes (\sigma_X^{v_{N+i}} |0\rangle |+\rangle |N_{N+i}\rangle + |1\rangle |-\rangle |N_{N+i}\rangle),
\]

which consists of \( N \) (locally-rotated) EPR pairs.

Then, the prover applies \( N \) controlled-\( \sigma_Z \) gates between the \( i \)-th and \( (N+i) \)-th qubits of \( \bigotimes_{i=1}^{2N} |\alpha_i\rangle \) for all \( i \in [N] \) (note that the controlled-\( \sigma_Z \) gate is independent of which qubit is the control and which qubit is the target). The prover has now prepared the \( 2N \)-qubit state

\[
|\alpha\rangle := \begin{cases}
|\hat{b}_1, \ldots, \hat{b}_{2N} \rangle (|0\rangle |\hat{b}_{2N+\hat{h}_N}^+\rangle |\hat{b}_0, \ldots, \hat{b}_N\rangle) & \text{if } \theta \in [2N], \theta \leq N, \\
|\hat{b}_1, \ldots, \hat{b}_{2N-1} \rangle (|0\rangle |\hat{b}_{2N+\hat{h}_N}^+\rangle |\hat{b}_0, \ldots, \hat{b}_N\rangle) & \text{if } \theta \in [2N], \theta > N, \\
|\hat{b}_1, \ldots, \hat{b}_{2N} \rangle & \text{if } \theta = 0, \\
|\psi^0\rangle & \text{if } \theta = \lambda.
\end{cases}
\]

In the “Hadamard” case, there is a third and final round where the verifier sends a bit \( q \in \{0,1,2,3\} \) to the prover. The prover performs the following \( q \)-dependent measurements.
Case $q = 0$. Measure all $2N$ qubits of $|\alpha\rangle$ in the computational basis.

Case $q = 1$. Measure all $2N$ qubits of $|\alpha\rangle$ in the Hadamard basis.

Case $q = 2$. Measure the first $N$ qubits of $|\alpha\rangle$ in the computational basis and the rest in the Hadamard basis.

Case $q = 3$. Measure the first $N$ qubits of $|\alpha\rangle$ in the Hadamard basis and the rest in the computational basis.

The prover finally sends the outcome $v \in \{0, 1\}^{2N}$ of these measurements to the verifier. The expressions on the right-hand side of Eq. (3.6) imply that the prover passes the last checks made by the verifier with probability $\geq 1 - \text{negl}(\lambda)$. (The only way the prover fails these last checks is if $d_i = 0^w$ for some $i \in [2N]$, which happens with probability at most $O(2N \cdot \text{negl}(\lambda)) = \text{negl}(\lambda)$, using $N = \lambda$.)

The “moreover” part of the theorem follows directly from the efficient function generation and the efficient decoding properties of ENTCFs.

\[ \square \]

4 Soundness of self-testing protocol

In this section, we show that our self-testing protocol achieves $\text{poly}(N, \epsilon)$ soundness error. We start with Section 4.1 where we mathematically model quantum devices. Essentially, a quantum device is a four-tuple $D = (S, M, \Pi, P)$ where $S$ is a set of states, $M$, $\Pi$, and $P$ are, respectively, the measurements it performs to obtain the $d$, $(b, x)$, and $v$ that it sends to the verifier in our protocol. We use $P$ to define observables of the quantum device that we call $X_i$ and $Z_i$. Then, in Section 4.2, we show that it suffices to consider so-called “perfect” quantum devices. We say a quantum device is perfect if it only fails the preimage test with $\text{negl}(\lambda)$ small probability. Sections 4.3 to 4.5 establish three key properties satisfied by states and observables of a quantum device that passes our protocol with high probability. These properties will be used repeatedly when we analyze the swap isometry, $V$. In Section 4.6, we give the formula of $V$ and then describe an efficient quantum circuit that implements it. The next two Sections 4.7 and 4.8 analyze the effect of the swap isometry on observables and states of the device. More specifically, in Section 4.7, we show that $V$ maps the $X_i$ and $Z_i$ observables approximately to the Pauli $\sigma^X$ and $\sigma^Z$ operators acting on the $i$-th qubit of some system. In Section 4.8, we show that $V$ maps the states of the device to states of the form $\tau^{\theta, v} \otimes \alpha^{\theta, v}$ for some positive semi-definite operators $\alpha^{\theta, v}$ that are close to being indistinguishable. Lastly, in Section 4.9, we put everything together to give our main soundness result, Theorem 4.41. In particular, we use results from Section 4.7 to characterize $P_q$ as computational basis measurements when $q = 0$, Hadamard basis measurements when $q = 1$, and a mixture of these when $q \in \{2, 3\}$.

Unlike the proof of completeness in Section 3, we use the adaptive hardcore bit and injective invariance properties of ENTCFs to prove soundness in this section. Therefore, it is necessary for us to make the LWE hardness assumption throughout this section.

4.1 Quantum devices

In this section, we model a general quantum device that can be used by a prover in our protocol specified in Section 3. Our definition is based on the definition given in [MV21, Section 4.1].

Definition 4.1. A device $D = (S, M, \Pi, P)$ is specified by Hilbert spaces named $\mathcal{H}_D$, $\mathcal{H}_Y$, and $\mathcal{H}_R$, with $\dim(\mathcal{H}_Y) = |Y|^{2N}$ and $\dim(\mathcal{H}_R) = 2^{2Nw}$, and the following.

1. A set $S := \{\psi^\theta \mid \theta \in [2N] \cup \{0, \diamond\}\} \subset \mathcal{D}(\mathcal{H}_D \otimes \mathcal{H}_Y)$ of states where each state $\psi^\theta$ is classical on $\mathcal{H}_Y$:

$$\psi^\theta := \sum_{y \in \mathcal{Y}^{2N}} \psi^\theta_y \otimes |y\rangle\langle y|.$$  \hspace{1cm} (4.1)

The state $\psi^\theta_y$ models the prover’s state immediately after returning $y \in \mathcal{Y}^{2N}$ to the verifier if the verifier initially selected $\theta \in [2N] \cup \{0, \diamond\}$. More precisely, $\psi^\theta_y$ (and hence $\psi^\theta$) is a function of the public keys $k \in (K_F \cup K_G)^{2N}$ that the verifier sampled according to $\theta$, as described in the protocol. We choose to make the $k$-dependence implicit for notational convenience.
2. A projective measurement $\Pi$ for the preimage test on $\mathcal{H}_D \otimes \mathcal{H}_Y$:

$$\Pi := \left\{ \Pi^{k,x} := \sum_{y \in \mathcal{Y}^{2N}} \Pi^{k,x}_y \otimes |y\rangle\langle y| \mid b \in \{0, 1\}^{2N}, x \in \mathcal{X}^{2N} \right\}.$$ \hfill (4.2)

The measurement outcome $b, x$ is the prover’s answer for the preimage test.

3. A projective measurement $M$ on $\mathcal{H}_D \otimes \mathcal{H}_Y$ for the prover’s first answer in the Hadamard test:

$$M := \left\{ M^d := \sum_{y \in \mathcal{Y}^{2N}} M^d_y \otimes |y\rangle\langle y| \mid d \in \{0, 1\}^{2Nw} \right\}.$$ \hfill (4.3)

We write $\sigma^\theta(D)$ for the classical-quantum state that results from measuring $M$ on $\psi^\theta$ followed by writing measurement outcome $d$ into another classical register whose Hilbert space is denoted by $\mathcal{H}_R$.

$$\sigma^\theta(D) := \sum_{y \in \mathcal{Y}^{2N}, d \in \{0, 1\}^{2Nw}} \sigma^\theta_{y,d}(D) \otimes |y, d\rangle \langle y, d| \in \mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R,$$ \hfill (4.4)

where $\sigma^\theta_{y,d}(D) := M^d_y \psi^\theta \nu^d_y M^d_y$.

4. Projective measurements $P_0, P_1, P_2, P_3$ on $\mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R$ for the prover’s second answer in the Hadamard test when asked questions $q = 0, 1, 2, 3$ respectively:

$$P_q := \left\{ P^u_q := \sum_{y \in \mathcal{Y}^{2N}, d \in \{0, 1\}^{2Nw}} P^u_{q,y,d} \otimes |y, d\rangle \langle y, d| \mid u \in \{0, 1\}^{2N} \right\}.$$ \hfill (4.5)

The measurement outcome $v$ is the prover’s answer for the question $q$.

**Remark.** We stress that the states $\psi^\theta_y$ depend on the public keys $k \in (\mathcal{K}_G \cup \mathcal{K}_G)^{2N}$ that the verifier sampled according to $\theta$. In particular, all subsequent states of the device also depend on $k$. When we make a statement about such $k$-dependent states labeled by $\theta$, that statement is often understood as holding in expectation over $k$ sampled according to $\theta$. The expectation over $k$ is necessary when the statement is derived using the adaptive hardcore bit or injective invariance properties of ENTCFs, which only hold in expectation over $k$.

Henceforth, unqualified sums over each of the symbols $b, x, y, d$ always refer to sums over $b \in \{0, 1\}^{2N}, x \in \mathcal{X}^{2N}, y \in \mathcal{Y}^{2N},$ and $d \in \{0, 1\}^{2Nw}$ respectively, unless otherwise stated.

In this work, we focus on efficient quantum devices, which are defined below.

**Definition 4.2.** A device $D = (S, \Pi, M, P)$ is efficient if all the states in $S$ can be efficiently prepared and all the measurements $M$, $P$ are efficient.

As in nonlocal self-testing, we want to show that each projector $P^u_q$ behaves like a tensor product of projectors on $2N$ systems. Therefore, we define the marginal observables on each of those systems as done in [MV21, Definition 4.4].

**Definition 4.3** (Marginal observables). Let $D = (S, \Pi, M, P)$ be a device. For $i \in [2N]$, we define the binary observables

$$Z_i(D) := \sum_v (-1)^{v_i} P^w_0 \quad \text{and} \quad X_i(D) := \sum_v (-1)^{v_i} P^w_1.$$ \hfill (4.6)

For $i \in [2N], y \in \mathcal{Y}^{2N},$ and $d \in \{0, 1\}^{2Nw}$, we also define the binary observables

$$Z_{i,y,d}(D) := \sum_v (-1)^{v_i} P^w_{y,0,d} \quad \text{and} \quad X_{i,y,d}(D) := \sum_v (-1)^{v_i} P^w_{1,y,d}.$$ \hfill (4.7)

For $j \in [N]$ and $k \in \{N + 1, \ldots, 2N\}$, we define the binary observables

$$Z_j(D) := \sum_v (-1)^{v_j} P^w_0 \quad \text{and} \quad X_k(D) := \sum_v (-1)^{v_k} P^w_1,$$

$$\bar{Z}_j(D) := \sum_v (-1)^{v_j} P^w_3 \quad \text{and} \quad \bar{X}_k(D) := \sum_v (-1)^{v_k} P^w_3.$$ \hfill (4.8)

Note that $\bar{Z}_j(D)$ commutes with $\bar{X}_k(D)$ and $\bar{X}_j(D)$ commutes with $\bar{Z}_k(D)$ according to these definitions.
Our soundness analysis will characterize the states $\sigma^{\theta,v}$ in the following definition as the states that we are self-testing. In the following definition, for $i \in [2N]$, we write

$$\text{mod}(i + N, 2N) := \begin{cases} i + N & \text{if } i \leq N, \\ i - N & \text{if } i > N. \end{cases} \quad (4.9)$$

**Definition 4.4** (Hadamard round post-$d$-measurement states $\sigma^{\theta,v}$). Let $D$ be a device. For $\theta \in [2N] \cup \{0, \circ\}$ and $v \in \{0, 1\}^{2N}$, we define the state

$$\sigma^{\theta,v}(D) := \sum_{(y, d) \in \Sigma(\theta, v)} \sigma^{\theta}_{y,d}(D) \otimes |y, d\rangle \langle y, d| \in \mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R, \quad (4.10)$$

where,

$$\Sigma(\theta, v) := \begin{cases} \{(y, d) \mid \hat{b}(k_i, y_i) = v_i \text{ for all } i \neq \theta \text{ and } \hat{h}(k_\theta, y_\theta, d_\theta) = v_\theta \oplus v_{\text{mod}(\theta + N, 2N)}\} & \text{if } \theta \in [2N], \\ \{(y, d) \mid \hat{b}(k_i, y_i) = v_i \text{ for all } i\} & \text{if } \theta = 0, \\ \{(y, d) \mid \hat{h}(k_i, y_i, d_i) = v_{\text{mod}(i + N, 2N)} \text{ for all } i\} & \text{if } \theta = \circ. \end{cases} \quad (4.11)$$

In all cases, $(y, d)$ ranges over $\mathcal{Y}^{2N} \times \{0, 1\}^{2N}$, $i$ ranges over $[2N]$, and the states implicitly depend on keys $k \in (\mathcal{K}_X \cup \mathcal{K}_G)^{2N}$ chosen according to $\theta$ as described in the protocol.

Note that $\sum_v \sigma^{\theta,v}(D) \leq \sigma^\theta(D)$ by definition because the $\Sigma(\theta, v)$s partition a subset of $\mathcal{Y}^{2N} \times \{0, 1\}^{2N}$. In particular, taking traces on both sides gives $\sum_v \text{Tr}[\sigma^{\theta,v}(D)] \leq 1$. In the honest case,

$$\text{Try}_R[\sigma^{\theta,v}(D)] = \begin{cases} 2^{-2N}v_1 \ldots v_{\theta-1}(-)^{v_\theta}v_{\theta+1} \ldots v_{2N}\langle v_1 \ldots v_{\theta-1}(-)^{v_\theta}v_{\theta+1} \ldots v_{2N}| & \text{for } \theta \in [2N], \\ 2^{-2N}|v\rangle \langle v| & \text{for } \theta = 0, \\ 2^{-2N}|\psi^v\rangle \langle \psi^v| & \text{for } \theta = \circ, \end{cases} \quad (4.12)$$

where $|v\rangle := |v_1 v_2 \ldots v_{2N}\rangle$ and we recall the definition of $|\psi^v\rangle$ from Eq. (3.5).

In the soundness proof, we will use quantities called $\gamma_p$, $\gamma_T$, and $\gamma_\circ$ to bound how far away the $\sigma^{\theta,v}$ states are from the self-tested states. We define them below.

**Definition 4.5** ($\gamma_p$, $\gamma_T$, and $\gamma_\circ$). Let $D$ be a device. We define the following quantities that all relate to the failure probabilities of $D$.

1. **Preimage test.** For $\theta \in [2N] \cup \{0, \circ\}$, we define

$$t_\theta(D) := \text{Tr} \left[ \sum_{(y, b, x) } \Pi_{b,x}^{\theta} \psi^\theta_{y} \right] = \text{Tr} \left[ \sum_{y, b, x \mid \text{CHK}(k, y, b, x) = 0} \Pi_{b,x}^{\theta} \psi^\theta_{y} \right], \quad (4.13)$$

where an implicit expectation is taken over the keys $k \in (\mathcal{K}_X \cup \mathcal{K}_G)^{2N}$ that are sampled according to $\theta$ as described in the protocol, and

$$\text{INV}(\theta, y) := \{(b, x) \in [0, 1]^{2N} \times \mathcal{X}^{2N} \mid b_i = \hat{b}(k_i, y_i), x_i = \hat{x}(b_i, k_i, y_i) \text{ for all } i\}. \quad (4.14)$$

Note that the set $\text{INV}(\theta, y)$ can be empty due to $\hat{b}$ or $\hat{x}$ returning $\perp$; in that case, $\sum_{(b, x) \in \text{INV}(\theta, y)} \Pi_{b,x}^{\theta} \psi^\theta_{y}$ is taken to mean $0$. We then define

$$\gamma_p(D) := 1 - \min\{t_\theta(D) \mid \theta \in [2N] \cup \{0, \circ\}\}. \quad (4.15)$$

2. **Hadamard non-Bell test** ($\theta \in [2N] \cup \{0\}$). For $\theta \in [2N] \cup \{0\}$ and $i \in [2N]$, we define

$$r_{\theta,i}(D) := \text{Tr} \left[ \sum_v Z_i^{(v)} \sigma^{\theta,v} \right] \quad \text{and} \quad \bar{r}_{\theta,i}(D) := \text{Tr} \left[ \sum_v \bar{Z}_i^{(v)} \sigma^{\theta,v} \right], \quad (4.16)$$

$$s_{\theta,i}(D) := \text{Tr} \left[ \sum_v X_i^{(v)} \sigma^{\theta,v} \right] \quad \text{and} \quad \bar{s}_{\theta,i}(D) := \text{Tr} \left[ \sum_v \bar{X}_i^{(v)} \sigma^{\theta,v} \right]. \quad (4.17)$$
We then define

\[
\gamma_{T,0}(D) := 1 - \min \{ r_{\theta,i}(D) \mid \theta \in [2N] \cup \{0\}, i \in [2N], i \neq \theta \}, \\
\gamma_{T,1}(D) := 1 - \min \{ s_{\theta,0}(D) \mid \theta \in [2N] \}, \\
\tilde{\gamma}_{T,0}(D) := 1 - \min \{ \tilde{r}_{\theta,i}(D) \mid \theta \in [N] \cup \{0\}, i \in [N], i \neq \theta \}, \\
\tilde{\gamma}_{T,0}^{'}(D) := 1 - \min \{ \tilde{r}_{\theta,i}(D) \mid \theta \in \{N + 1, \ldots, 2N\} \cup \{0\}, i > N, i \neq \theta \}, \\
\tilde{\gamma}_{T,1}(D) := 1 - \min \{ \tilde{s}_{\theta,0}(D) \mid \theta \in [2N] \},
\]

and

\[
\gamma_T(D) := \max \{ \gamma_{T,0}(D), \gamma_{T,1}(D), \tilde{\gamma}_{T,0}(D), \tilde{\gamma}_{T,0}^{'}(D), \tilde{\gamma}_{T,1}(D) \}.
\]

3. Hadamard Bell test (\(\theta = \diamond\)). For \(i \in [N]\), we define

\[
\tilde{r}_{\diamond,i}(D) := \sum_v \operatorname{Tr} \left[ (\tilde{Z}_i \tilde{X}_{N+i})^{(v)} \sigma^{\diamond,v} \right],
\]

\[
\tilde{s}_{\diamond,i}(D) := \sum_v \operatorname{Tr} \left[ (\tilde{X}_i \tilde{Z}_{N+i})^{(v)} \sigma^{\diamond,v} \right].
\]

We then define

\[
\tilde{\gamma}_{\diamond,0}(D) := 1 - \min_{i \in [N]} \{ \tilde{r}_{\diamond,i}(D) \}, \quad \tilde{\gamma}_{\diamond,1}(D) := 1 - \min_{i \in [N]} \{ \tilde{s}_{\diamond,i}(D) \}, \quad \text{and} \quad \gamma_{\diamond}(D) := \max \{ \tilde{\gamma}_{\diamond,0}(D), \tilde{\gamma}_{\diamond,1}(D) \}.
\]

We can show \(\sum_v \operatorname{Tr} [\sigma^\diamond,v(D)]\) is \(\gamma_p\) close to 1. We will use the following definition.

**Definition 4.6** (Valid \(y\)). Let \(y \in \mathcal{Y}^{2N}\) and \(k \in (\mathcal{K}_F \cup \mathcal{K}_G)^{2N}\). We say \(y\) is valid (with respect to \(k\)) if \(b(k,i, y_i) \neq \perp\) for all \(i \in [2N]\); equivalently,

\[
y_i \in \bigcup_x (\operatorname{Supp}(f_{k,i,0}(x)) \cup \operatorname{Supp}(f_{k,i,1}(x))), \quad \text{for all } i \in [2N].
\]

Otherwise, we say \(y\) is invalid.

**Lemma 4.7.** Let \(D\) be a device. For all \(\theta \in [2N] \cup \{0, \diamond\}\), \(\| \sigma^\theta(D) - \sum_v \sigma^{\theta,v}(D) \|_1 \leq \gamma_p\).

**Proof.** We have

\[
\left\| \sigma^\theta(D) - \sum_v \sigma^{\theta,v}(D) \right\|_1 = \left\| \sum_{\text{invalid } y} \sum_{d} M_{y}^{d} \psi_{y}^{\theta} M_{y}^{d} \otimes |y,d\rangle \langle y,d| \right\|_1 \\
= \left| \operatorname{Tr} \left[ \sum_{\text{invalid } y} \sum_{d} M_{y}^{d} \psi_{y}^{\theta} M_{y}^{d} \otimes |y,d\rangle \langle y,d| \right] \right| \\
= \sum_{\text{invalid } y} \operatorname{Tr} [\psi_{y}^{\theta}] \leq 1 - t_{\theta} \leq \gamma_p(D),
\]

where the last two inequalities follow from the definitions of \(t_{\theta}\) and \(\gamma_p(D)\) respectively.

We define the following failure probabilities of a device.

**Definition 4.8** (Failure probabilities). Let \(D\) be a device. For \(q \in \{0, 1, 2, 3\}\), we define

\[
\epsilon_p(D) := \Pr(D \text{ fails preimage test } \mid \text{ case: preimage}),
\]

\[
\epsilon_{H,q}(D) := \Pr(D \text{ fails Hadamard test } \mid \text{ case: Hadamard and question } q).
\]

Then,

\[
\epsilon(D) := \epsilon_p(D)/2 + \left( \sum_{q=0}^{3} \epsilon_{H,q}(D) \right)/8
\]

is the overall failure probability.
Henceforth, when $D$ is clear from the context, we mostly drop the $D$-dependence from the quantities

$$\sigma^\theta, \sigma^{\theta,i}, \sigma^{\theta,i,i}, Z_i, X_i, Z_i, x, d, X_i, x, d, \tilde{Z}_i, \tilde{X}_i, t_\theta, r_{\theta,i}, \tilde{r}_{\theta,i}, s_{\theta,i}, \tilde{s}_{\theta,i},$$

$$\tilde{r}_{\theta,i}, \tilde{s}_{\theta,i}, \gamma_P, \gamma_T, 0, \gamma_T, 1, \tilde{\gamma}_T, 0, \tilde{\gamma}_T, 1, \gamma_T, \tilde{\gamma}_o, 0, \tilde{\gamma}_o, 1, \gamma_o, \epsilon_P, \epsilon_H, q, \epsilon.$$

(4.28)

While it is easier to control the soundness error using $\gamma_P$, $\gamma_T$, and $\gamma_o$, they are not immediately observable to the verifier. However, we can bound them by the observable failure probabilities, $\epsilon_P, \epsilon_H, q$, and $\epsilon$, as follows.

**Proposition 4.9** ($\gamma$ bounded by failure probability $\epsilon$). Let $D$ be a device, then we have

$$\gamma_P \leq (2N + 2)\epsilon_P,$$  \hspace{1cm} (4.29)

$$\gamma_T \leq (2N + 2)\epsilon_H, 0,$$  \hspace{1cm} (4.30)

$$\gamma_T, 1 \leq (2N + 2)\epsilon_H, 1,$$  \hspace{1cm} (4.31)

$$\tilde{\gamma}_T, 0 + \tilde{\gamma}_T, 0 + \tilde{\gamma}_T, 1 + \tilde{\gamma}_o, 0 + \tilde{\gamma}_o, 1 \leq (2N + 2)(\epsilon_H, 2 + \epsilon_H, 3).$$  \hspace{1cm} (4.32)

In particular,

$$\gamma_P \leq 2(2N + 2)\epsilon, \quad \gamma_T \leq 8(2N + 2)\epsilon,$$  \hspace{1cm} (4.33)

and $\gamma_o \leq 8(2N + 2)\epsilon$.

**Proof.** As $t_\theta \leq 1$ for all $\theta$, Eq. (4.29) follows from

$$\epsilon_P = 1 - \frac{1}{2N + 2} \left( t_\theta + \sum_{\theta=0}^{2N} t_\theta \right) \geq 1 - \frac{1}{2N + 2} \left( 2N + 1 + \min \{ t_\theta \mid \theta \in [2N] \cup \{0, \diamond \} \} \right) = \frac{\gamma_P}{2N + 2}.$$  \hspace{1cm} (4.34)

As $\Pr(\cap A_i) \leq \min_i \{ \Pr(A_i) \}$ for any events $A_i$ and $r_{\theta,i} \leq 1$ for all $\theta, i$, Eq. (4.30) follows from

$$\epsilon_H, 0 \geq 1 - \frac{1}{2N + 2} \left( 1 + \sum_{\theta=0}^{2N} \min \{ r_{\theta,i} \mid i \in [2N], i \neq \theta \} \right) \hspace{1cm} \geq \frac{\gamma_T, 0}{2N + 2},$$

where the 1 in the brackets in the first line is because our protocol always accepts when $\theta = \diamond$ and $q = 0$.

Eq. (4.31) follows from

$$\epsilon_H, 1 = 1 - \frac{1}{2N + 2} \left( 2 + \sum_{\theta=1}^{2N} s_{\theta,0} \right) \geq 1 - \frac{1}{2N + 2} \left( 2 + 2N - 1 + \min_{\theta \in [2N]} \{ s_{\theta,0} \} \right) = \frac{\gamma_T, 1}{2N + 2}.$$  \hspace{1cm} (4.36)

Now, we also have

$$\epsilon_H, 2 \geq 1 - \frac{1}{2N + 2} \left( \sum_{i=1 \in [N], i \neq \theta}^{N} \min_{i \in [N]} \{ \tilde{r}_{\theta,i} \} + \sum_{\theta=1}^{2N} \tilde{s}_{\theta,0} \geq \min_{\theta \in [N]} \{ \tilde{r}_{\theta,0} \} + \min_{\theta \in [N]} \{ \tilde{s}_{\theta,0} \} \right),$$

$$\epsilon_H, 3 \geq 1 - \frac{1}{2N + 2} \left( \sum_{i=1 \in [N], i \neq \theta}^{N} \min_{i \in [N]} \{ \tilde{r}_{\theta,i} \} + \min_{\theta \in [N]} \{ \tilde{r}_{\theta,0} \} + \min_{\theta \in [N]} \{ \tilde{s}_{\theta,0} \} \right).$$

(4.37)

Therefore,

$$\epsilon_H, 2 + \epsilon_H, 3 \geq 2 - \frac{1}{2N + 2} \left( N + (1 - \tilde{\gamma}_T, 0) + N + (1 - \tilde{\gamma}_T, 0) + 2N - 1 + (1 - \tilde{\gamma}_T, 1) + (1 - \tilde{\gamma}_o, 0) + (1 - \tilde{\gamma}_o, 1) \right) \hspace{1cm} \frac{\gamma_T, 0 + \tilde{\gamma}_T, 0 + \tilde{\gamma}_T, 1 + \tilde{\gamma}_o, 0 + \tilde{\gamma}_o, 1}{2N + 2},$$

(4.38)
where, in the first line, we used
\[
\sum_{\theta=1}^{N} \min_{i \in [N], i \neq \theta} \{ \tilde{r}_{\theta,i} \} + \min_{i \in [N]} \{ \tilde{r}_{0,i} \} \leq N + (1 - \tilde{\gamma}_{T,0}), \quad \sum_{\theta=N+1}^{2N} \min_{i \in [N], i \neq \theta} \{ \tilde{r}_{\theta,i} \} + \min_{i \in [N]} \{ \tilde{r}_{0,i} \} \leq N + (1 - \tilde{\gamma}_{T,0}),
\]
\[
\sum_{\theta=1}^{2N} s_{\theta,0} \leq 2N - 1 + (1 - \tilde{\gamma}_{T,1}), \quad \min_{i \in [N]} \{ \tilde{r}_{0,i} \} = 1 - \tilde{\gamma}_{0,0}, \quad \min_{i \in [N]} \{ \tilde{s}_{0,i} \} = 1 - \tilde{\gamma}_{0,1}.
\]

(4.40)

The last three inequalities of the proposition, in Eq. (4.33), follow immediately from the above by using the definitions \( \gamma_T := \max\{\gamma_{T,0}, \gamma_{T,1}, \tilde{\gamma}_{T,0}, \tilde{\gamma}_{T,1}\} \) and \( \gamma_0 := \max\{\tilde{\gamma}_{0,0}, \tilde{\gamma}_{0,1}\} \) (see Eqs. (4.19) and (4.22) respectively), the expression for the overall failure probability in Eq. (4.27), and the fact that \( \max_i \{a_i\} \leq \sum_i a_i \geq 0 \).

We will also find it useful to define non-negative real numbers labeled by \( \{\zeta, \tilde{\zeta}, \chi, \tilde{\chi}\} \). These numbers can be thought of as being of order \( \gamma_T/2^N \) (see Lemma 4.12). We will later bound the sizes of various quantities by \( \{\zeta, \tilde{\zeta}, \chi, \tilde{\chi}\} \), which helps us bound exponentially large sums involving \( 2^N \) terms.

**Definition 4.10 (\( \zeta, \chi \)).** For \( \theta \in [2N] \cup \{0\}, i \in [2N] \), and \( v \in \{0, 1\}^{2N} \), we define
\[
\zeta(i, \theta, v) := \| Z_i - (-1)^v I \|_{\sigma^{\theta,v}}^2 \quad \text{and} \quad \tilde{\zeta}(i, \theta, v) := \| \tilde{Z}_i - (-1)^v I \|_{\sigma^{\theta,v}}^2.
\]

(4.41)

For \( \theta \in [2N] \), we define
\[
\chi(\theta, v) := \| X_\theta - (-1)^v I \|_{\sigma^{\theta,v}}^2 \quad \text{and} \quad \tilde{\chi}(\theta, v) := \| \tilde{X}_\theta - (-1)^v I \|_{\sigma^{\theta,v}}^2.
\]

(4.42)

For \( i \in [N] \) and \( v \in \{0, 1\}^{2N} \), we define
\[
\tilde{\zeta}_i(i, v) := \| \tilde{Z}_i \tilde{X}_{N+i} - (-1)^v I \|_{\sigma^{\theta,v}}^2 \quad \text{and} \quad \tilde{\chi}_i(i, v) := \| \tilde{X}_i \tilde{X}_{N+i} - (-1)^v I \|_{\sigma^{\theta,v}}^2.
\]

(4.43)

By definition, we have
\[
Z_i \approx_{\zeta(i, \theta, v), \sigma^{\theta,v}} (-1)^v I \quad \text{and} \quad X_\theta \approx_{\chi(\theta, v), \sigma^{\theta,v}} (-1)^v I,
\]
\[
\tilde{Z}_i \approx_{\tilde{\zeta}(i, \theta, v), \sigma^{\theta,v}} (-1)^v I \quad \text{and} \quad \tilde{X}_\theta \approx_{\tilde{\chi}(\theta, v), \sigma^{\theta,v}} (-1)^v I,
\]
\[
\tilde{Z}_i \tilde{X}_{N+i} \approx_{\tilde{\zeta}_i(i, v), \sigma^{\theta,v}} (-1)^v I \quad \text{and} \quad \tilde{X}_i \tilde{X}_{N+i} \approx_{\tilde{\chi}_i(i, v), \sigma^{\theta,v}} (-1)^v I.
\]

(4.44)

By basic algebra, we can express
\[
\zeta(i, \theta, v) / 4 = \text{Tr}[\sigma^{\theta,v}] - \text{Tr}[Z_i^{(v)} \sigma^{\theta,v}] = \| Z_i^{(v)} \|_{\sigma^{\theta,v}}^2,
\]
\[
\chi(\theta, v) / 4 = \text{Tr}[\sigma^{\theta,v}] - \text{Tr}[X_\theta^{(v)} \sigma^{\theta,v}] = \| X_\theta^{(v)} \|_{\sigma^{\theta,v}}^2,
\]
\[
\tilde{\zeta}(i, \theta, v) / 4 = \text{Tr}[\sigma^{\theta,v}] - \text{Tr}[(\tilde{Z}_i \tilde{X}_{N+i})^{(v)} \sigma^{\theta,v}] = \| (\tilde{Z}_i \tilde{X}_{N+i})^{(v)} \|_{\sigma^{\theta,v}}^2,
\]
\[
\tilde{\chi}(i, \theta, v) / 4 = \text{Tr}[\sigma^{\theta,v}] - \text{Tr}[(\tilde{X}_i \tilde{X}_{N+i})^{(v)} \sigma^{\theta,v}] = \| (\tilde{X}_i \tilde{X}_{N+i})^{(v)} \|_{\sigma^{\theta,v}}^2.
\]

(4.45)

The lemma below follows directly from the definitions.

**Lemma 4.11.** For all \( \theta \in [2N] \cup \{0\} \) and \( i \in [2N] \) with \( \theta \neq i \), we have
\[
1 - \gamma_{T,0} \leq r_{\theta,i} = \sum_v \text{Tr}[Z_i^{(v)} \sigma^{\theta,v}] \leq \sum_v \text{Tr}[\sigma^{\theta,v}] \leq \text{Tr}[\sigma^\theta] = 1.
\]

(4.46)

For all \( \theta \in [2N] \), we have
\[
1 - \gamma_{T,1} \leq s_{\theta,0} = \sum_v \text{Tr}[X_\theta^{(v)} \sigma^{\theta,v}] \leq \sum_v \text{Tr}[\sigma^{\theta,v}] \leq \text{Tr}[\sigma^\theta] = 1.
\]

(4.47)
For all $\theta \in [N] \cup \{0\}$ and $i \in [N]$ with $i \neq \theta$, we have
\[
1 - \tilde{\gamma}_{T,0} \leq \tilde{\gamma}_{\theta,i} = \sum_v \text{Tr}[\tilde{Z}_i^{(v)} \sigma^\theta,v] \leq \sum_v \text{Tr}[\sigma^\theta,v] \leq \text{Tr}[\sigma^\theta] = 1.
\] (4.48)

For all $\theta \in \{N+1, \ldots, 2N\} \cup \{0\}$ and $i > N$ with $i \neq \theta$, we have
\[
1 - \tilde{\gamma}_{T,0} \leq \tilde{\gamma}_{\theta,i} = \sum_v \text{Tr}[\tilde{Z}_i^{(v)} \sigma^\theta,v] \leq \sum_v \text{Tr}[\sigma^\theta,v] \leq \text{Tr}[\sigma^\theta] = 1.
\] (4.49)

For all $\theta \in [2N]$, we have
\[
1 - \tilde{\gamma}_{T,1} \leq \tilde{\gamma}_{\theta,0} = \sum_v \text{Tr}[\tilde{X}_\theta^{(v)} \sigma^\theta,v] \leq \sum_v \text{Tr}[\sigma^\theta,v] \leq \text{Tr}[\sigma^\theta] = 1.
\] (4.50)

For all $i \in [N]$, we have
\[
1 - \tilde{\gamma}_{0,0} \leq \tilde{\gamma}_{\theta,i} = \sum_v \text{Tr}[\tilde{Z}_i \tilde{X}_{N+i}^{(v)} \sigma^\theta,v] \leq \sum_v \text{Tr}[\sigma^\theta,v] \leq \text{Tr}[\sigma^\theta] = 1,
\]
\[
1 - \tilde{\gamma}_{0,1} \leq \tilde{\gamma}_{\theta,i} = \sum_v \text{Tr}[\tilde{X}_{N+i} \tilde{Z}_i^{(v)} \tilde{X}_{N+i}^{(v)} \sigma^\theta,v] \leq \sum_v \text{Tr}[\sigma^\theta,v] \leq \text{Tr}[\sigma^\theta] = 1.
\] (4.51)

The following lemma bounds exponential sums of $\zeta(i, \theta, v)$ and $\chi(\theta, v)$ (and their tilde versions) over $v \in \{0, 1\}^{2N}$ using $\gamma_T$. It also bounds exponential sums of $\tilde{\zeta}_\theta(i, v)$ and $\tilde{\chi}_\theta(i, v)$ over $v \in \{0, 1\}^{2N}$ using $\gamma_\theta$.

**Lemma 4.12.** For all $\theta \in [2N] \cup \{0\}$ and $i \in [2N]$ with $i \neq \theta$, we have
\[
\sum_v \zeta(i, \theta, v) \leq 4\gamma_T.
\] (4.52)

For all $\theta \in [2N]$, we have
\[
\sum_v \chi(\theta, v) \leq 4\gamma_T.
\] (4.53)

For all $\theta \in [N] \cup \{0\}$ and $i \in [N]$ with $i \neq \theta$ or $\theta \in \{N+1, \ldots, 2N\} \cup \{0\}$ and $i > N$ with $i \neq \theta$, we have
\[
\sum_v \tilde{\zeta}(i, \theta, v) \leq 4\gamma_T.
\] (4.54)

For all $\theta \in [2N]$, we have
\[
\sum_v \tilde{\chi}(\theta, v) \leq 4\gamma_T.
\] (4.55)

For all $i \in [N]$, we have
\[
\sum_v \tilde{\zeta}_\theta(i, v) \leq 4\gamma_\theta \quad \text{and} \quad \sum_v \tilde{\chi}_\theta(i, v) \leq 4\gamma_\theta.
\] (4.56)

**Proof.** Consider the first inequality. Fix $\theta \in [2N] \cup \{0\}$ and $i \in [2N]$ with $i \neq \theta$. Summing the expression for $\zeta(i, \theta, v)$ in Eq. (4.45) over $v$ and using Lemma 4.11, we obtain
\[
\sum_v \zeta(i, \theta, v) \leq 4 - 4\sum_v \text{Tr}[Z_i^{(v)} \sigma^\theta,v] \leq 4\gamma_{T,0} \leq 4\gamma_T.
\] (4.57)

The argument is analogous for the remaining five inequalities. \qed

Note that the above lemmas control certain parameters of a device using its failure probability. But we have yet to leverage our computational assumptions. The following lemma, which we will frequently use later together with the lifting lemmas (Lemmas 2.16 and 2.17), allows us to use the injective invariance property of ENTCF to further control the device.
Lemma 4.13 (Indistinguishability of \( \{ \psi^\theta \}_\theta \) (resp. \( \{ \sigma^\theta \}_\theta \)). Any pair of states in \( \{ \psi^\theta \}_{\theta \in [2N] \cup \{0, \infty\}} \) (resp. \( \{ \sigma^\theta \}_{\theta \in [2N] \cup \{0, \infty\}} \)) of an efficient device \( D \) are computationally indistinguishable.

Proof. For \( v \in \{0, 1\}^{2N} \), let \( D(v) := (k_1, k_2, \ldots, k_{2N}) \) be the distribution on \( 2N \)-tuples of public keys such that each \( k_i \) is independently distributed according to \( \text{Gen}_C(1^\lambda) \) if \( v_i = 0 \) or \( \text{Gen}_F(1^\lambda) \) if \( v_i = 1 \). For \( \theta \in [2N] \cup \{0, \infty\} \), let

\[
D^\theta := \begin{cases} 
D(\text{str}(\theta)) & \text{if } \theta \in [2N], \\
D(0^{2N}) & \text{if } \theta = 0, \\
D(1^{2N}) & \text{if } \theta = \infty,
\end{cases}
\]

where \( \text{str}(\theta) \in \{0, 1\}^{2N} \) is the 2\( N \)-bit string with a 1 at position \( \theta \) and 0s elsewhere.

The injective invariance property of ENTCFs states that the distributions \( \text{Gen}_C(1^\lambda) \) and \( \text{Gen}_F(1^\lambda) \) are computationally indistinguishable. Therefore, it is clear that for any \( u, v \in \{0, 1\}^{2N} \), with \( u, v \) differing by exactly one bit, \( D(u) \) and \( D(v) \) are computationally indistinguishable. Therefore, any pair of distributions in \( \{ D^\theta \}_{\theta \in [2N] \cup \{0, \infty\}} \) are computationally indistinguishable up to \( 2N \) \( \text{negl}(\lambda) \) by Lemma 2.13 and our setting \( N = \lambda \).

This is because, for any \( \theta, \theta' \in [2N] \cup \{0\} \), \( \text{str}(\theta) \) and \( \text{str}(\theta') \) differ by at most 2 bits while \( \text{str}(\theta) \) and \( 1^{2N} \) differ by at most \( 2N \) bits. As \( D \) can prepare the state \( \psi^\theta \) (resp. \( \sigma^\theta \)) efficiently given keys drawn from \( D^\theta \), all pairs of states in \( \{ \psi^\theta \}_{\theta \in [2N] \cup \{0, \infty\}} \) (resp. \( \{ \sigma^\theta \}_{\theta \in [2N] \cup \{0, \infty\}} \)) must be computationally indistinguishable.

Lemmas 4.12 and 4.13 imply that, on the state \( \sigma^\theta \), \( Z_i \) is close to \( \tilde{Z}_i \) and \( X_i \) is close to \( \tilde{X}_i \). More precisely:

Lemma 4.14. Let \( D \) be an efficient device. For all \( \theta \in [2N] \cup \{0, \infty\} \) and \( i \in [2N] \), we have

\[
\tilde{Z}_i \simeq_{16\gamma_T + 4\gamma_P + \text{negl}(\lambda), \sigma^\theta} Z_i \quad \text{and} \quad \tilde{X}_i \simeq_{16\gamma_T + 4\gamma_P + \text{negl}(\lambda), \sigma^\theta} X_i.
\]

Proof. Consider the first equation of the lemma, that involving \( Z \). It suffices to prove this equation for \( i = 1 \) as the proof for other \( i \) is analogous. In addition, because \( D \) is efficient, \( Z_1 \) and \( \tilde{Z}_1 \) are efficient binary observables. Therefore it suffices to only consider \( \theta = 0 \) (note \( \theta \neq i \)) because of the computational indistinguishability of the \( \sigma^\theta \)'s and part 2 of the lifting lemma (Lemma 2.16).

Now,

\[
\tilde{Z}_1 \simeq_{\zeta(1, 0, v), \sigma^{0, v}} (-1)^{v_1} I \quad \text{and} \quad Z_1 \simeq_{\zeta(1, 0, v), \sigma^{0, v}} (-1)^{v_1} I.
\]

Therefore, by the triangle inequality,

\[
\| \tilde{Z}_1 - Z_1 \|^2_{\sigma^{0, v}} \leq (\zeta(1, 0, v))^{1/2} + \zeta(1, 0, v)^{1/2})^2 \leq 2\zeta(1, 0, v) + 2\zeta(1, 0, v).
\]

Therefore, by Lemma 2.5 and the bounds on \( \zeta \) in Lemma 4.12, which is applicable because \( i \neq \theta \), we have

\[
\tilde{Z}_1 \simeq_{16\gamma_T + \text{negl}(\lambda), \sum_v \sigma^{0, v}} Z_1.
\]

Now, using Lemma 4.7, we have

\[
\| \tilde{Z}_1 - Z_1 \|^2_{\sigma^{0, v}} \leq \text{Tr}[(\tilde{Z}_1 - Z_1)\sigma^0] \leq \|\tilde{Z}_1 - Z_1\|^2 \cdot \gamma_P \leq 4\gamma_P.
\]

Therefore, by Eq. (4.62), we have

\[
\text{Tr}[(\tilde{Z}_1 - Z_1)^2\sigma^0] \leq \text{Tr}[(\tilde{Z}_1 - Z_1)^2\sum_v \sigma^{0, v}] + 4\gamma_P \leq 16\gamma_T + 4\gamma_P + \text{negl}(\lambda),
\]

which completes the proof of the first equation of the lemma.

Now consider the second equation of the lemma, that involving \( X \). It again suffices to prove this equation for \( i = 1 \) as the proof for other \( i \) is analogous. In addition, it suffices to only consider \( \theta = 1 \) (note that \( \theta = i \), unlike before) by the reasoning at the start. The proof is then analogous to the proof of the first equation, except we use the bounds on \( \chi \) in Lemma 4.12, which is applicable because \( \theta = i \). We omit the details. \( \square \)
4.2 Reduction to perfect device

In this section, we follow the strategy of [MV21, Lemma 4.13] by first showing that an arbitrary device is close to a “perfect” device. Then, we restrict attention to perfect devices for the rest of the soundness proof before Section 4.9. A perfect device is one that, for any $\theta \in [2N] \cup \{0, \circ\}$ chosen by the verifier, can always pass the preimage test except with $\text{negl}(\lambda)$ small probability. More formally:

**Definition 4.15** (Perfect device). Let $D = (S, \Pi, M, P)$ be a device. We say $D$ is perfect if $\gamma_D(D) = \text{negl}(\lambda)$.

**Proposition 4.16.** Let $D = (S, \Pi, M, P)$ be an efficient device with $\gamma_D(D) < 1$ and $S = \{\psi^\theta \mid \theta \in [2N] \cup \{0, \circ\}\}$. Then, there exists an efficient perfect device $\tilde{D} = (\tilde{S}, \Pi, M, P)$ which uses the same measurements $\Pi, M, P$ as $D$ and has states $\tilde{S} = \{\tilde{\psi}^\theta \mid \theta \in [2N] \cup \{0, \circ\}\}$ that satisfy: for all $\theta \in [2N] \cup \{0, \circ\}$,

$$\|\psi^\theta - \tilde{\psi}^\theta\|_1 \leq \sqrt{\gamma_D(D)}. \quad (4.65)$$

**Proof.** $\tilde{D}$ can efficiently prepare each state $\tilde{\psi}^\theta$ in $\tilde{S}$ as follows. $\tilde{D}$ first follows $D$ to prepare $\psi^\theta$ using a given set of public keys $k \in (K_X \cup K_d)^{2N}$ sampled according to $\theta$ as described in the protocol. $\tilde{D}$ then applies the efficient unitary $U_\Pi^T$ associated with the efficient measurement $\Pi$, as per Definition 2.1, to create the state

$$\phi^\theta := U_\Pi^T(|0_{2N+2N_w}|0_{2N+2N_w}|\text{anc} \otimes \psi^\theta)U_\Pi^T = \sum_{y,b,x,b',x'} |y,b,x\rangle \langle y,b',x'| \otimes \Pi_y^b \bar{\psi}^\theta \Pi_y^b \bar{x}' . \quad (4.66)$$

On this state, $\tilde{D}$ evaluates the CHK function (which is efficient given only $k$ and not the trapdoor $t_k$, see the efficient decoding property of ENTCFs) on the classical register holding $(y, b, x)$ to create the state

$$\sum_{y,b,x,b',x'} |\text{CHK}(k, y, b, x)\rangle \langle \text{CHK}(k, y', b', x')| \otimes |y, b, x\rangle \langle y, b', x'| \otimes \Pi_y^b \bar{\psi}^\theta \Pi_y^b \bar{x}'. \quad (4.67)$$

$\tilde{D}$ next measures the first (single-bit) register of the state in Eq. (4.67). The probability that the measurement outcome is 0 is equal to $\text{Tr}[\Lambda \phi^\theta]$, where we write $\Lambda$ for the projector

$$\Lambda := \sum_{y,b,x} \sum_{(k,b,x,y) = 0} |y, b, x\rangle \langle y, b, x|. \quad (4.68)$$

Note that $\text{Tr}[\Lambda \phi^\theta] = t_\theta$, where we recall $t_\theta$ from Definition 4.5.

If the measurement outcome is 0, $\tilde{D}$ continues by applying $U_\Pi^T$ to the post-measurement states, and then traces out the anc register to output the state

$$\tilde{\psi}^\theta := \text{Tr}_{\text{anc}} \left[U_\Pi^T \frac{\Lambda \phi^\theta}{\text{Tr}[\Lambda \phi^\theta]} U_\Pi \right] . \quad (4.69)$$

But, using Eq. (4.66), we can write

$$\psi^\theta = \text{Tr}_{\text{anc}} \left[U_\Pi^T \phi^\theta U_\Pi \right] . \quad (4.70)$$

Therefore, since the trace norm cannot increase under partial trace and is unitarily invariant, we obtain

$$\|\tilde{\psi}^\theta - \psi^\theta\|_1 \leq \left\| \frac{\Lambda \phi^\theta}{\text{Tr}[\Lambda \phi^\theta]} - \phi^\theta \right\|_1 \leq 2 \sqrt{1 - \text{Tr}[\Lambda \phi^\theta]} \leq \sqrt{\gamma_D(D)} , \quad (4.71)$$

where we used the gentle measurement lemma in the second inequality, see, e.g., [Wil17, Lemma 9.4.1].

Now, $\tilde{D}$ might not obtain 0 when it measures the first register of the state in Eq. (4.67). In this case, it repeats the above procedure up to $(a fixed) \text{poly}(\lambda)$ times, stopping and outputting $\tilde{\psi}^\theta$ the first time 0 is measured and aborting if 0 is never measured within those poly$(\lambda)$ repeats. But, the probability of aborting is $(1 - t_\theta)^{\text{poly}(\lambda)} \leq \gamma_p^{\text{poly}(\lambda)} \leq \text{negl}(\lambda)$ as required. \hfill \Box

We now prove several lemmas that hold for efficient perfect devices.
**Lemma 4.17.** Let $D$ be a perfect device. For all $\theta \in [2N] \cup \{0, \diamond\}$, we have
\[
\sigma^\theta \simeq_{\negl(\lambda)} \sum_v \sigma^{\theta,v}.
\] (4.72)

**Proof.** The lemma follows from Lemma 4.7 and the definition of a perfect device (Definition 4.15).

**Lemma 4.18 (Collapsing property).** Let $D$ be an efficient perfect device. For all $i \in [2N]$ and $\theta \in [2N] \cup \{0, \diamond\}$, we have
\[
\psi^\theta \simeq_{\negl(\lambda)} \sum_{b,x} \Pi^{b,x} \psi^\theta \Pi^{b,x}.
\]

**Proof.** For $c \in \{0, 1\}$, let $\mathcal{Y}_{i,c} := \{y_i \in \mathcal{Y} \mid \hat{b}(k, y_i) = c\}$. Note that $y \in \mathcal{Y}^{2N}$ is valid (see Definition 4.6) if and only if $y_i \in \mathcal{Y}_{i,0} \cup \mathcal{Y}_{i,1}$ for all $i \in [2N]$.

We first prove the lemma in the case $\theta = 0$, so $k \in k_0^{2N}$. As the device is perfect and $\theta = 0$, with probability $\geq 1 - \negl(\lambda)$, we have
\[
\psi^0_y = \begin{cases} 0 & \text{for all invalid } y \in \mathcal{Y}^{2N}, \\ \Pi_y \psi^0 \Pi_y & \text{for all valid } y \in \mathcal{Y}^{2N}. \end{cases}
\] (4.73)

Therefore, with probability $\geq 1 - \negl(\lambda)$, we have
\[
\sum_{b,x} \Pi^{b,x} \psi^0 \Pi^{b,x} = \sum_{b,x,y} \Pi_y^{b,x} \psi^0 \Psi_y^{b,x} \otimes |y\rangle\langle y| = \sum_{\text{valid } y} \psi^0_y \otimes |y\rangle\langle y| = \psi^0.
\]

When $\theta \neq 0$, the computational indistinguishability of the $\psi^\theta$s (Lemma 4.13) implies
\[
\sum_{b,x} \Pi^{b,x} \psi^\theta \Pi^{b,x} \simeq_{\negl(\lambda)} \sum_{b,x} \Pi^{b,x} \psi^0 \Pi^{b,x} \simeq_{\negl(\lambda)} \psi^0 \simeq_{\negl(\lambda)} \psi^\theta.
\] (4.74)

The conclusion follows by the triangle inequality for computational indistinguishability (Lemma 2.13).

**Lemma 4.19.** Let $D$ be an efficient perfect device. For all $\theta \in [2N]$, we have
\[
\sum_v (-1)^v \Tr[\sigma^{\theta,v}] \simeq_{\negl(\lambda)} 0.
\] (4.75)

**Proof.** The proof is similar to [MV21, Lemma 4.15]. It suffices to prove the lemma for $\theta = 1$ as the proof for other $\theta$ is analogous. Assume for contradiction that Eq. (4.75) is not true. Then, there exists a non-negligible function $\mu(\lambda)$ such that, for infinitely many $\lambda \in \mathbb{N}$, we have
\[
\left| \sum_{v \mid v_1 = 0} \Tr[\sigma^{1,v}] - \sum_{v \mid v_1 = 1} \Tr[\sigma^{1,v}] \right| > \mu(\lambda).
\] (4.76)

We also assume that, for a fixed $\lambda$, the term inside the absolute value sign in Eq. (4.76) is $\geq 0$. The proof is analogous if otherwise.

Consider the following efficient algorithm $\mathcal{A}$ which breaks the adaptive hardcore bit property of ENTCFs:

1. Input: a key $k_1$ sampled from Gen$_x(1^\lambda)_{\text{key}}$.
2. Independently sample $2N - 1$ keys $k_2, \ldots, k_{2N}$, each from Gen$_g(1^\lambda)_{\text{key}}$. (Let $\mathcal{D}^1$ denote the distribution on the keys sampled in the first two steps of $\mathcal{A}$.)
3. Prepare the state \( \psi^1 \), then make measurement obtaining \( y \), then make preimage measurement obtaining \( (b, x) \), and then make Hadamard measurement obtaining \( d \).

4. Output: \( (y_1, b_1, x_1, d_1) \) and print “\( f_{k_1, b_1}(x_1) = y_1 \) and \( \hat{h}(k_1, y_1, d_1) = 0 \)”.

Because \( D \) is perfect, we have, with probability \( \geq 1 - \text{negl}(\lambda) \),

\[
f_{k_1, b_1}(x_1) = y_1. \tag{4.77}
\]

Let \( \mathcal{A}' \) be the same as \( \mathcal{A} \) except that it does not make a preimage measurement and produces no output. By the collapsing property (Lemma 4.18), the state of \( \mathcal{A} \) immediately after the preimage measurement is computationally indistinguishable from its state just before. Therefore, by definition, the state of \( \mathcal{A} \) after the Hadamard measurement is computationally indistinguishable from the state of \( \mathcal{A}' \) after the Hadamard measurement. But note that \( \mathcal{A}' \) behaves exactly like \( D \) up to when the Hadamard measurement is made in the Hadamard case. Therefore, the state of \( \mathcal{A} \) after the Hadamard measurement is computationally indistinguishable from the state of \( D \) after the Hadamard measurement in the Hadamard case.

Now, using the definitions of \( \sigma^1, \sigma \) and \( \Sigma(1, v) \) from Definition 4.4, we see that Eq. (4.76) means

\[
\text{Pr}_{k \sim D_1, D}((y, d) \in \bigcup_{v|_1 = 0}\Sigma(1, v)) > \text{Pr}_{k \sim D_1, D}((y, d) \in \bigcup_{v|_1 = 1}\Sigma(1, v)) + \mu(\lambda). \tag{4.78}
\]

Then, as argued above, we can replace \( D \) by \( \mathcal{A} \) by introducing a negligible term as follows

\[
\text{Pr}_{k \sim D_1, \mathcal{A}}((y, d) \in \bigcup_{v|_1 = 0}\Sigma(1, v)) > \text{Pr}_{k \sim D_1, \mathcal{A}}((y, d) \in \bigcup_{v|_1 = 1}\Sigma(1, v)) + \mu(\lambda) + \text{negl}(\lambda). \tag{4.79}
\]

Now, we can rewrite the probabilities appearing on the left- and right-hand sides to see that

\[
\text{Pr}(\hat{h}(k_1, y_1, d_1) = 0 \text{ and } \bigcap_{i=2}^{2N} \hat{b}(k_i, y_i) \in \{0, 1\}) > \text{Pr}(\hat{h}(k_1, y_1, d_1) = 1 \text{ and } \bigcap_{i=2}^{2N} \hat{b}(k_i, y_i) \in \{0, 1\}) + \mu(\lambda) + \text{negl}(\lambda). \tag{4.80}
\]

But

\[
\text{Pr}(\bigcap_{i=2}^{2N} \hat{b}(k_i, y_i) \in \{0, 1\}) \geq 1 - \text{negl}(\lambda), \tag{4.81}
\]

by the definition of \( \hat{b} \) and the fact \( D \) is perfect. Therefore, as \( \text{Pr}(A \cap B) \geq \text{Pr}(A) + \text{Pr}(B) - 1 \) for any events \( A \) and \( B \), we have

\[
\text{Pr}(\hat{h}(k_1, y_1, d_1) = 0) > \text{Pr}(\hat{h}(k_1, y_1, d_1) = 1) + \mu(\lambda) + \text{negl}(\lambda). \tag{4.82}
\]

Eqs. (4.77) and (4.82) together contradict the adaptive hardcore bit property of ENTCFs.

**Lemma 4.20.** Let \( D \) be an efficient perfect device. For all \( i, \theta \in [2N] \), we have

\[
\text{Tr}[X_i\sigma^\theta] \simeq 2^{-\gamma_T + \text{negl}(\lambda)} 0. \tag{4.83}
\]

**Proof.** Because \( D \) is efficient, \( X_i \) is an efficient binary observable. Therefore, it suffices to consider \( \theta = i \) by using the computational indistinguishability of the \( \sigma^\theta \)'s (Lemma 4.13) and part 1 of the lifting lemma (Lemma 2.16). Moreover, we only consider \( \theta = i = 1 \) as the argument is analogous otherwise. Then, the lemma follows from

\[
\text{Tr}[X_1\sigma^1] = \sum_v \text{Tr}[-(-1)^{v_i}(2X_1^{(v_i)} - 1)]\sigma^1,v] = \sum_v (-1)^{v_i} \text{Tr}[\sigma^1,v] - \frac{1}{2} \sum_v (-1)^{v_i} \chi(1, v) \simeq 2^{-\gamma_T} \text{negl}(\lambda), \tag{4.84}
\]

where the middle equality uses Eq. (4.45) and the last approximation uses Lemmas 4.12 and 4.19.

### 4.3 Commutation relations

**Proposition 4.21.** Let \( D \) be an efficient perfect device. For all \( i, j \in [2N] \) with \( i \neq j \), we have

\[
[Z_i, Z_j] = 0, \quad [X_i, X_j] = 0, \quad \text{and} \quad [Z_i, X_j] \simeq 2^{-\gamma_T + \text{negl}(\lambda), \sigma^\theta} 0. \tag{4.85}
\]
Proof. The first two equations follow directly from Definition 4.3. Consider the last equation. Because \( D \) is efficient, \( Z_i \) and \( X_i \) are efficient binary observables. Therefore, by the computational indistinguishability of the \( \sigma^a \)'s (Lemma 4.13) and part 3 of the lifting lemma (Lemma 2.16), it suffices to prove the last equation for \( \theta = j \). It also suffices to consider \( i = 1 \) and \( j = \theta = 2 \) as the proof is analogous for other \( i, j, \theta \) with \( i \neq j \).

For the purposes of this proof, we write \( \zeta \) for \( \zeta(1, 2, v) \) and \( \chi \) for \( \chi(2, v) \) for convenience. By definitions, we have \( Z_1 \approx_{\zeta, \sigma^2, v} (-1)^{v_1} I \) and \( X_2 \approx_{\chi, \sigma^2, v} (-1)^{v_2} I \). Therefore,

\[
Z_1 X_2 \approx_{\chi, \sigma^2, v} (-1)^{v_2} Z_1 \approx_{\zeta, \sigma^2, v} (-1)^{v_2 + v_1} I \approx_{\chi, \sigma^2, v} X_2 (-1)^{v_1} \approx_{\zeta, \sigma^2, v} X_2 Z_1. \tag{4.86}
\]

Therefore, by the triangle inequality, we have

\[
Z_1 X_2 \approx_{\chi + \zeta, \sigma^2, v} X_2 Z_1. \tag{4.87}
\]

By linearity of the trace we can sum the above equation up over \( v \in \{0, 1\}^{2N} \) to deduce

\[
Z_1 X_2 \approx \sum_v \chi_v \sum_v \zeta_v \sigma^2_v X_2 Z_1. \tag{4.88}
\]

Using \( \sum_v \chi_v \leq 4 \gamma_T \) and \( \sum_v \zeta_v \leq 4 \gamma_T \) from Lemma 4.12, and using \( \sigma^2 \approx_{\text{negl}(\lambda)} \sum_v \sigma^{2_v} \) from Lemma 4.17, which uses the perfectness of \( D \), we obtain

\[
Z_1 X_2 \approx_{\gamma_T + \text{negl}(\lambda), \sigma^2} X_2 Z_1, \tag{4.89}
\]

which completes the proof. \( \square \)

4.4 Anti-commutation relations

We prove the following proposition in this subsection.

**Proposition 4.22.** Let \( D \) be an efficient perfect device. For all \( i, \theta \in [2N] \), we have

\[
\{Z_i, X_i\} \approx_{\sqrt{\gamma_T} + \text{negl}(\lambda), \sigma^\theta} 0. \tag{4.90}
\]

We need a few lemmas first.

**Lemma 4.23.** Let \( D \) be an efficient device. For all \( i, \theta \in [2N] \), we have

\[
\sum_a \text{Tr}[X_i Z_i^{(a)} \sigma^\theta Z_i^{(a)}] \approx_{\sqrt{\gamma_T} + \text{negl}(\lambda)} 0. \tag{4.91}
\]

**Proof.** We prove the lemma for \( i = 1 \) as the proof is analogous for other \( i \). Now, the states in \( \{\sum_a Z_i^{(a)} \sigma^\theta Z_i^{(a)} | \theta \in [2N]\} \) are computationally indistinguishable because they can be efficiently prepared from \( \sigma^\theta \) by measuring \( Z_1 \) (\( Z_1 \) is efficient because \( D \) is efficient) and the \( \sigma^\theta \)'s are computationally indistinguishable (Lemma 4.13). In addition, because \( D \) is efficient, \( X_1 \) is an efficient binary observable. Therefore, by part 1 of the lifting lemma (Lemma 2.16), it suffices to prove the current lemma for \( \theta = 2 \).

For \( a \in \{0, 1\} \), we define

\[
\sigma(a) := \sum_{v: v_1 = a} \sigma^{2_v} \quad \text{and} \quad \zeta(a) := \sum_{v: v_1 = a} \zeta(1, 2, v). \tag{4.92}
\]

Note that \( \sigma^2 = \sigma(0) + \sigma(1) \) by definition and \( \zeta(0) + \zeta(1) \leq 4 \gamma_T \) by Lemma 4.12.

By Eq. (4.45), we have

\[
\zeta(a)/4 = \text{Tr}[\sigma(a)] - \text{Tr}[Z_1^{(a)} \sigma(a)], \tag{4.93}
\]

which can be re-expressed as

\[
Z_1^{(a)} \approx_{\zeta(a), \sigma(a)} I, \quad Z_1^{(a)} \approx_{\zeta(a), \sigma(a)} 0. \tag{4.94}
\]

Therefore, by Lemma 2.8, we have

\[
Z_1^{(a)} \sigma(a) Z_1^{(a)} \approx_{\zeta(a), \sigma(a)} 0, \quad Z_1^{(a)} \sigma(a) Z_1^{(a)} \approx_{\zeta(a)} 0. \tag{4.95}
\]
Lemma 4.12, the above equation means we obtain
\[ \sum_{a,v} Z_{1}^{(a)} \sigma^{2,v} Z_{1}^{(a)} = \sum_{a} Z_{i}^{(a)} (\sigma(0) + \sigma(1)) Z_{i}^{(a)} \approx (\sqrt{\zeta(0)} + \sqrt{\zeta(1)})^2 \sigma^2. \]  

Recalling \( \zeta(0) + \zeta(1) \leq 4\gamma_T \) by Lemma 4.12, the above equation means
\[ \sum_{a,v} Z_{1}^{(a)} \sigma^{2,v} Z_{1}^{(a)} \approx_{\gamma_T} \sigma^2. \]  

Finally, using the above equation and Lemma 4.20, we obtain
\[ \sum_{a} \text{Tr}[X_{1} Z_{1}^{(a)} \sigma^{2} Z_{1}^{(a)}] = \text{Tr}\left[X_{1} \sum_{a,v} Z_{1}^{(a)} \sigma^{2,v} Z_{1}^{(a)}\right] \approx_{\gamma_T} \text{Tr}[X_{1} \sigma^2] \approx_{\gamma_T} 0, \]  

which completes the proof.

To state and prove the next lemma, we need the following definition.

**Definition 4.24.** Let \( D \) be a device. For keys \( k \in (K_X \cup K_G)^{2N}, b \in \{0,1\}^{2N}, y \in \mathcal{Y}^{2N} \), we define \( \hat{x}(b,k,y) \in X^{2N} \) component-wise by \( \hat{x}(b,k,y)_i := \hat{x}(b_i,k_i,y_i) \) for all \( i \in [2N] \). Then, we define
\[ \hat{\Pi}_y^{b} := \Pi^{b,k,y}_y \quad \text{and} \quad \hat{\Pi}^{b} := \sum_y \hat{\Pi}_y^{b} \otimes |y\rangle\langle y|, \]  

which can be thought of as projectors onto the correct preimage answers. Note that if \( \hat{x}(b_i,k_i,y_i) = \perp \) for some \( i \in [2N] \), then \( \hat{\Pi}_y^{b} = 0 \). Moreover, for \( a \in \{0,1\} \) and \( i \in [2N] \), we define
\[ \hat{\Pi}_{y}^{i,a} := \sum_{b|b_i=a} \hat{\Pi}_y^{b}, \quad \hat{\Pi}^{i,a} := \sum_y \hat{\Pi}_{y}^{i,a} \otimes |y\rangle\langle y|, \quad \text{and} \quad Z_{i,d}^{(a)} := \sum_y Z_{i,y,d}^{(a)} \otimes |y\rangle\langle y|. \]  

We also note that, by previous definitions, we have
\[ Z_{i,y,d}^{(a)} = \sum_{v|v_i=a} P_{0,y,d}^{v} \quad \text{and} \quad Z_{i,d}^{(a)} = \sum_{d} Z_{i,d}^{(a)} \otimes |d\rangle\langle d|. \]  

**Lemma 4.25.** Let \( D \) be a device. Let \( \theta \in [2N] \cup \{0\} \) and \( b, b' \in \{0,1\}^{2N} \) be such that \( b_j \neq b'_j \) for some \( j \in [2N] \) with \( j \neq \theta \). Then, \( \hat{\Pi}_y^{b} \psi_y^{b} \hat{\Pi}_y^{b'} = 0 \), for all \( y \in \mathcal{Y}^{2N} \).

**Proof.** We consider the following three cases, which cover all possibilities:

1. \( \hat{b}(k_j,y_j) = b_j \). In this case, we have \( y_j \in \cup_{x \in X} \text{Supp}(f_{k_j,b_j}(x)) \). Now, \( k_j \in K_G \) as \( j \neq \theta \). Therefore, \( b'_j \neq b_j \) implies \( y_j \notin \cup_{x \in X} \text{Supp}(f_{k_j,b'_j}(x)) \). So \( \hat{x}(b'_j,k_j,y_j) = \perp \). So \( \hat{\Pi}_y^{b'} = 0 \).

2. \( \hat{b}(k_j,y_j) = b'_j \). In this case, by an analogous argument, we obtain \( \hat{\Pi}_y^{b} = 0 \).

3. \( \hat{b}(k_j,y_j) = \perp \). In this case, we have \( \hat{x}(b_j,k_j,y_j) = \perp \). So \( \hat{\Pi}_y^{b} = \hat{\Pi}_y^{b'} = 0 \).

In all cases, we see \( \hat{\Pi}_y^{b} \psi_y^{b} \hat{\Pi}_y^{b'} = 0 \). Hence the lemma.

**Lemma 4.26.** Let \( D \) be an efficient perfect device. For all \( i, \theta \in [2N] \), we have
\[ \sum_{a,y,d} ||M^{d}_{y} \hat{\Pi}_{y}^{i,a} - Z_{i,y,d}^{(a)} ||_{\psi_y^{b}}^{2} \approx_{2\gamma_T + \text{negl}(\lambda)} 0. \]  

(4.102)
Proof. We prove the lemma for \( i = 1 \) as the proof for other \( i \) is analogous. Expanding the left-hand side of Eq. (4.102), we obtain
\[
\sum_{a,y,d} \| M_y \hat{1}_{y,a} - Z_{1,y,d} M_y \|_{\psi_y}^2
= \sum_{a,y,d} \text{Tr}[(\hat{1}_{y,a} M_y - M_y Z_{1,y,d}) (M_y \hat{1}_{y,a} - Z_{1,y,d} M_y) \psi_y]
= \sum_{a,y,d} \text{Tr}[(\hat{1}_{y,a} M_y \hat{1}_{y,a} \psi_y) + \sum_{a,y,d} \text{Tr}[M_y Z_{1,y,d} M_y \psi_y] - \sum_{a,y,d} \text{Tr}[M_y Z_{1,y,d} M_y (\hat{1}_{y,a} \psi_y + \psi_y \hat{1}_{y,a})]].
\]

(4.103)

Since \( \psi^\theta = \sum_y \psi_y \otimes |y \rangle \langle y |, \ M^d = \sum_y M^d_y \otimes |y \rangle \langle y |, \ \hat{1}_{y,a} = \sum_y \hat{1}_{y,a} \otimes |y \rangle \langle y | \) and \( Z_{1,y,d} = \sum_y Z_{1,y,d} \otimes |y \rangle \langle y | \), we can simplify the left-hand side of Eq. (4.102) as
\[
\sum_{a,y,d} \text{Tr}[\hat{1}_{1,a} M^d \hat{1}_{1,a} \psi^\theta] + \sum_{a,y,d} \text{Tr}[M^d Z_{1,a,d} M^d \psi^\theta] - \sum_{a,y,d} \text{Tr}[Z_{1,a,d} M^d (\hat{1}_{1,a} \psi^\theta + \psi^\theta \hat{1}_{1,a}) M^d].
\]

(4.104)

Because the device is perfect, according to Definition 4.8, the first term is
\[
\sum_{a,y,d} \text{Tr}[\hat{1}_{1,a} M^d \hat{1}_{1,a} \psi^\theta] = \sum_a \text{Tr}[\hat{1}_{1,a} \left( \sum_d Z_{1,d}^a \right) M^d \psi^\theta] = \sum_a \text{Tr}[\hat{1}_{1,a} \psi^\theta] = 1 - \text{negl}(\lambda).
\]

(4.105)

The second term is
\[
\sum_{a,y,d} \text{Tr}[Z_{1,a,d} M^d \hat{1}_{1,a} \psi^\theta] = \sum_{a,y,d} \text{Tr}[Z_{1,a,d} M^d \hat{1}_{1,a} \psi^\theta] = 1.
\]

(4.106)

For the third term, we first notice that
\[
\| \hat{1}_{1,y} - \hat{1}_{1,y} \|^2_{\psi^\theta} = 1 - \text{Tr}[(\hat{1}_{1,y} + \hat{1}_{1,y}) \psi^\theta] \leq \text{negl}(\lambda),
\]

(4.107)

that is
\[
\hat{1}_{1,y} + \hat{1}_{1,y} \approx_{\text{negl}(\lambda), \psi^\theta} 1.
\]

(4.108)

Then, we have
\[
\sum_{a} \text{Tr}\left[ \left( \sum_{d} Z_{1,d}^a M^d \right) (\hat{1}_{1,a} \psi^\theta + \psi^\theta \hat{1}_{1,a}) \right]
\approx_{\text{negl}(\lambda)} \sum_{a,y,d} \text{Tr}[M^d Z_{1,a,d} \hat{1}_{1,a} \psi^\theta (\hat{1}_{1,a} + \hat{1}_{1,a}) M^d] - \sum_{a,y,d} \text{Tr}[Z_{1,a,d} M^d (\hat{1}_{1,a} \psi^\theta + \psi^\theta \hat{1}_{1,a}) M^d],
\]

(4.109)

where the first approximation follows from the replacement lemma (Lemma 2.6) with Eq. (4.108), \( \| \hat{1}_{1,a} \|_\infty \leq 1 \), and \( \| \sum_{d} M^d Z_{1,d}^a M^d \|_\infty \leq 1 \). The last bound follows from \( 0 \leq \sum_{d} M^d Z_{1,d}^a M^d \leq \sum_{d} M^d = 1 \) which is implied by \( 0 \leq Z_{1,d}^a \leq 1 \).

Now, the second term appearing on the last line Eq. (4.109) is zero, which follows from: \( \hat{1}_{1,a} \psi^\theta \hat{1}_{1,1-a} + \hat{1}_{1,1-a} \psi^\theta \hat{1}_{1,a} \) is independent of \( a \). \( \sum_{a} Z_{1,d}^a = 1 \), \( \sum_{d} M^d = 1 \), and \( \hat{1}_{1,0} \hat{1}_{1,0} = 0 \).

We argue that the first term appearing on the last line Eq. (4.109) is close to 2. For \( \theta = 0 \), we have
\[
\sum_{d} M^d \hat{1}_{1,a} \psi^\theta \hat{1}_{1,a} M^d \otimes |d \rangle \langle d | = \sum_{y \cdot d} \sum_{b,b' | b_1 = b_1'} M_y \hat{1}_{y} \psi_y \hat{1}_{y} M_y \otimes |y \rangle \langle y |, d \rangle \langle d |
\]

(4.110)

\[
\sum_{y \cdot d} \sum_{b,b' | b_1 = b_1'} M_y \hat{1}_{y} \psi_y \hat{1}_{y} M_y \otimes |y \rangle \langle y |, d \rangle \langle d |
\]

(4.111)

\[
\sum_{\text{valid}} M_y \hat{1}_{y} \psi_y M_y \otimes |y \rangle \langle y |, d \rangle \langle d |
\]

(4.112)

\[
\sum_{\text{valid}} \sigma^0, a \cdot d \]

(4.113)
where we used Lemma 4.25 in the second equality, and \( \theta = 0 \) (so \( k \in \mathcal{K}_Q^{2N} \)) and the perfectness of \( D \) in the third (approximate) equality.

Therefore, we have

\[
\sum_{a,d} \text{Tr}[Z_{1,d}^{(a)} M^d \hat{\Pi}^{1,a} \psi^0 \hat{\Pi}^{1,a} M^d] = \sum_a \text{Tr}[Z_1^{(a)} \sum_d M^d \hat{\Pi}^{1,a} \psi^0 \hat{\Pi}^{1,a} M^d \otimes |d\rangle \langle d|] \\
\simeq_{\text{negl}(\lambda)} \sum_v \text{Tr}[Z_1^{(v)}] \sigma^0,v \geq 1 - \gamma_T. 
\] (4.114)

But given any state \( \psi^\theta \) of \( D \), \( \sum_{a,d} \text{Tr}[Z_{1,d}^{(a)} M^d \hat{\Pi}^{1,a} \psi^\theta \hat{\Pi}^{1,a} M^d] \) can be estimated by the following efficient algorithm that is independent of \( \theta \):

1. Measure \( \psi^\theta \) using the efficient measurement \( \{ \sum_{b_2,..b_{2N}, x} \Pi_{b_2,..b_{2N}, x} \}_{b_1} \) to obtain outcome \( b_1 \in \{0,1\} \).
2. Measure \( M^d \) to obtain outcome \( d \in \{0,1\}^{2Nw} \).
3. Measure \( Z_{1,d} \) to obtain bit \( b'_1 \).
4. Output 1 if \( b_1 = b'_1 \) and 0 otherwise.

Because \( D \) is perfect, the state of the system after the first step is negligibly close to \( \sum_a \hat{\Pi}^{1,a} \psi^\theta \hat{\Pi}^{1,a} \) in trace distance. After the second step, the state becomes \( \sum_{a,d} M^d \hat{\Pi}^{1,a} \psi^\theta \hat{\Pi}^{1,a} M^d \). Therefore, the probability the algorithm outputs 1 is negligibly close to \( \sum_{a,d} \text{Tr}[Z_{1,d}^{(a)} M^d \hat{\Pi}^{1,a} \psi^\theta \hat{\Pi}^{1,a} M^d] \).

Therefore, Eq. (4.114) and the computational indistinguishability of the \( \psi^\theta \)'s (Lemma 4.13) gives

\[
\sum_{a,d} \text{Tr}[Z_{1,d}^{(a)} M^d \hat{\Pi}^{1,a} \psi^\theta \hat{\Pi}^{1,a} M^d] \geq 1 - \gamma_T + \text{negl}(\lambda). 
\] (4.115)

Overall, putting together the bounds we have derived for each of the three terms in Eq. (4.104), we obtain

\[
\sum_{a,y,d} \|M^d \hat{\Pi}^{a}_{1,y} - Z_{1,y,d}^{(a)} M^d \|_2^2 \leq 2\gamma_T + \text{negl}(\lambda), 
\] (4.116)

which completes the proof. \( \square \)

**Lemma 4.27.** Let \( D \) be an efficient perfect device. For all \( \theta \in [2N] \) and \( c \in \{0,1\} \), we have

\[
\sum_{a,c} \text{Tr}[X_{\theta} Z_{\theta}^{(a)} \sigma^0,c Z_{\theta}^{(a)}] \approx_{\sqrt{\gamma_T} + \text{negl}(\lambda)} 0. 
\] (4.117)

**Proof.** It suffices to prove the lemma for \( \theta = 1 \) as the argument is analogous for other \( \theta \). Define

\[
E_c := \sum_{a,v|v_1=c} \text{Tr}[X_1 Z_{1}^{(a)} \sigma^1,v Z_{1}^{(a)}]. 
\] (4.118)

By Lemma 4.23, we have \( E_0 + E_1 \approx_{\sqrt{\gamma_T} + \text{negl}(\lambda)} 0 \). Therefore, it suffices to show that \( E_0 \approx_{\sqrt{\gamma_T} + \text{negl}(\lambda)} E_1 \).

Using the definitions of \( \sigma^1,v \) and \( \Sigma(1,v) \) from Definition 4.4, and using Definition 4.3, we have

\[
E_c = \sum_{a,v|v_1=c} \sum_{y,d \in \Sigma_{1,v}} \text{Tr}[X_1 Z_{1}^{(a)} M_y^d \psi^1_y M_y^d \otimes |y,d\rangle \langle y,d| Z_{1}^{(a)}] \\
= \sum_{a,v|v_1=c} \sum_{y,d \in \Sigma_{1,v}} \text{Tr}[X_{1,y,d} Z_{1,y,d}^{(a)} M_y^d \psi^1_y M_y^d Z_{1,y,d}^{(a)}] \\
= \sum_{a,y} \sum_{d \mid k_{1,y,d} = c} \text{Tr}[X_{1,y,d} Z_{1,y,d}^{(a)} M_y^d \psi^1_y M_y^d Z_{1,y,d}^{(a)}]. 
\] (4.119)
Now, also define

\[
F_c := \sum_{a,y} \sum_{d} \text{Tr}[X_{1,y,d} M_y^d \hat{\Pi}_y^1 \psi_y^1 \hat{\Pi}_y^1 M_y^d].
\] (4.120)

By the technical Claim 4.28, deferred to after this proof, we have \(E_c \approx \sqrt{\tau} F_c\). Therefore, to prove the lemma, it suffices to prove \(F_0 \approx_{\text{negl}} F_1\). To this end, we first substitute the definition \(\hat{\Pi}_y^1 := \sum_{b|b_1 = a} \hat{\Pi}_y^b\) into the expression for \(F_c\):

\[
F_c = \sum_{b,y} \sum_{d} \text{Tr}[X_{1,y,d} M_y^d \hat{\Pi}_y^b \psi_y^b \hat{\Pi}_y^b M_y^d] + \sum_{a,y} \sum_{d} \sum_{b_1 \neq b} \text{Tr}[X_{1,y,d} M_y^d \hat{\Pi}_y^b \psi_y^b \hat{\Pi}_y^b M_y^d].
\] (4.121)

But by Lemma 4.25, the second term is 0. Therefore,

\[
F_c = \sum_{b,y} \sum_{d} \text{Tr}[X_{1,y,d} M_y^d \hat{\Pi}_y^b \psi_y^b \hat{\Pi}_y^b M_y^d].
\] (4.122)

With the above expression in hand, we prove \(F_0 \approx_{\text{negl}} F_1\) by contradiction. Assume that there exists a non-negligible function \(\mu\) such that \(F_0 - F_1 > \mu(\lambda) > 0\) for some \(\lambda\) (as in the proof of Lemma 4.19, the case \(F_1 - F_0 > \mu(\lambda) > 0\) can be handled analogously). We can then construct the following efficient algorithm \(A\) which breaks the adaptive hardcore bit property of ENTCFs.

1. Input: a key \(k_1\) sampled from \(\text{Gen}_X(1^\lambda)_{\text{key}}\).
2. Independently sample \(2N - 1\) keys \(k_2, \ldots, k_{2N-1}\), each from \(\text{Gen}_E(1^\lambda)_{\text{key}}\).
3. Prepare the state \(\psi^1\), then make measurement obtaining \(y\), then make preimage measurement obtaining \((b, x)\), then make Hadamard measurement obtaining \(d\), and then measure \(X_1\) obtaining (single-bit) \(h\).
4. Output: \((y_1, b_1, x_1, d_1, h)\) and print “\(f_{k_1, b_1}(x_1) = y_1\) and \(\hat{h}(k_1, y_1, d_1) = h\)”.

Because \(D\) is perfect, we have, with probability \(\leq 1 - \text{negl}(\lambda)\),

\[
f_{k_1, b_1}(x_1) = y_1.
\] (4.123)

Write \(\hat{h}\) as shorthand for \(\hat{h}(k_1, y_1, d_1)\). Using \(F_c = \Pr_{A}(h = 0, \hat{h} = c) - \Pr_{A}(h = 1, \hat{h} = c)\), we find

\[
\Pr_{A}(\hat{h} = h) = \Pr_{A}(\hat{h} = h = 0) + \Pr_{A}(\hat{h} = h = 1) = \Pr_{A}(\hat{h} = 0, h = 1) + \Pr_{A}(\hat{h} = 1, h = 0) + F_0 - F_1
\] (4.124)

\[
> \Pr_{A}(\hat{h} \neq h) + \mu(\lambda).
\]

Eqs. (4.123) and (4.124) contradict the adaptive hardcore bit property of ENTCFs as required.

Claim 4.28. Under the same setup as in the proof of Lemma 4.27, for any \(c \in \{0, 1\}\), we have

\[
E_c \approx \sqrt{\tau} F_c.
\] (4.125)

Proof. By simple algebra, we have

\[
E_c - F_c = \sum_{a,y} \sum_{d} \text{Tr}[X_{1,y,d} Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^1 \psi_y^1] + \sum_{a,y} \sum_{d} \text{Tr}[X_{1,y,d} M_y^d - M_y^d \hat{\Pi}_y^1 \psi_y^1]
\] (4.126)

Therefore, by Cauchy-Schwarz, we have

\[
|E_c - F_c| \leq \sum_{a,y,d} \|X_{1,y,d} Z_{1,y,d}^a M_y^d\| \psi_y^1 \cdot \|Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^1 \psi_y^1\| + \|Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^1 \psi_y^1\| \cdot \|X_{1,y,d} M_y^d \hat{\Pi}_y^1 \psi_y^1\|.
\] (4.127)
where we dropped the restriction on $d$, which only makes the upper bound looser.

Again by Cauchy-Schwarz, we have

$$
|E_c - F_c| \leq \sqrt{\sum_{a,y,d} \|X_{1,y,d} Z_{1,y,d}^a M_y^d d\|_2^2} \cdot \sqrt{\sum_{a,y,d} \|Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^a\|_2^2} + \sqrt{\sum_{a,y,d} \|Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^a\|_2^2} \cdot \sqrt{\sum_{a,y,d} \|X_{1,y,d} M_y^d \hat{\Pi}_y^a\|_2^2}.
$$

(4.128)

Now, we have $\sum_{a,y,d} \|X_{1,y,d} Z_{1,y,d}^a M_y^d\|_2^2 = 1$ and $\sum_{a,y,d} \|X_{1,y,d} M_y^d \hat{\Pi}_y^a\|_2^2 \approx \text{negl}(\lambda)$, where the first equality directly follows from definitions and the second equality follows as $D$ is perfect.

Hence,

$$|E_c - F_c| \leq 2 \sum_{a,y,d} \|Z_{1,y,d}^a M_y^d - M_y^d \hat{\Pi}_y^a\|_2 \leq 2\sqrt{T},
$$

(4.129)

which completes the proof.

Using the last lemma, Lemma 4.27, we can now prove Proposition 4.22 as follows.

**Proof of Proposition 4.22.** Because $D$ is efficient, $X_i$ and $Z_i$ are efficient binary observables. Therefore, it suffices to prove the proposition for $\theta = i$ by the computational indistinguishability of the $\sigma^\theta$s (Lemma 4.13) and part 4 of the lifting lemma (Lemma 2.16). Moreover, it suffices to only prove the proposition for $\theta = 1$ as the argument is analogous otherwise.

$$
\text{Tr}[\{Z_1, X_1\}^2] = 4 \sum_v \text{Tr}[(X_1 Z_1^{(0)} X_1 Z_1^{(0)}) + Z_1^{(1)} X_1 Z_1^{(1)}] \sigma^{1,v})
$$

$$
= 4 \sum_v (-1)^{v_1} \text{Tr}[(Z_1^{(0)} X_1 Z_1^{(0)}) + Z_1^{(1)} X_1 Z_1^{(1)}] \sigma^{1,v}) + O(\sqrt{\chi(1,v)} \cdot \sqrt{\text{Tr}[\sigma^{1,v})]}
$$

(4.130)

where the first equality is by simple algebra, the second equality is by Definition 4.10 and the replacement lemma (note that $\|Z_1^{(a)} X_1 Z_1^{(a)}\|_\infty \leq 1$ for $a \in \{0, 1\}$), and the last approximation is by Lemma 4.27 with $\theta = 1$ (which uses the perfectness of $D$). But, by Cauchy-Schwarz, we have

$$
\sum_v \sqrt{\chi(1,v)} \cdot \sqrt{\text{Tr}[\sigma^{1,v})}] \leq \sqrt{\left(\sum_v \chi(1,v)\right)\left(\sum_v \text{Tr}[\sigma^{1,v})]\right)} \leq \sqrt{T}.
$$

(4.131)

Therefore, Eq. (4.130) implies that

$$
\text{Tr}[\{Z_1, X_1\}^2] \approx \sqrt{T + \text{negl}(\lambda)} 0,
$$

(4.132)

which completes the proof.

### 4.5 Conjugate invariance

In this section, we prove that the $\sigma^\theta$ states remain close to invariant under conjugation by $Z$ and $X$. This means that we can commute $Z$ and $X$ past $\sigma^\theta$, a property that will be crucial in our analysis of the effect of the swap isometry in subsequent subsections.

**Proposition 4.29** (Conjugate invariance). For all $i, \theta \in [2N]$ with $i \neq \theta$, we have

$$
Z_i \sigma^\theta Z_i \approx \gamma_T + \text{negl}(\lambda) \quad \text{and} \quad X_\theta \sigma^\theta X_\theta \approx \gamma_T + \text{negl}(\lambda),
$$

(4.133)

In particular, $Z_i \sigma^\theta \approx \gamma_T + \text{negl}(\lambda) \quad \text{and} \quad X_\theta \sigma^\theta \approx \gamma_T + \text{negl}(\lambda)$.

$\sigma^\theta$ by the unitary invariance of the trace norm.
Proof. The two equations involving tilde observables in Eq. (4.133) follow from the two involving non-tilde observables by Lemma 4.14 and Lemma 2.8. The proofs of the two equations involving non-tilde observables in Eq. (4.133) are analogous. We prove the first in detail and comment on the minor changes required to prove the second.

To prove the first equation, it suffices to prove it for \( i = 1 \) and \( \theta = 2 \) as the proof for other \( i \neq \theta \) is analogous. By the triangle inequality and Lemma 4.17, we have

\[
\| Z_1 \sigma_2 Z_1 - \sigma_2 \|_1 \leq \sum_v \| Z_1 \sigma_2 Z_1 - \sigma_2 \|_1 + \text{negl}(\lambda). \tag{4.134}
\]

We bound each term in the sum. Recall the definition \( \zeta(1, 2, v) := \| Z_1 - (-1)^v \mathbb{1} \|_{\sigma_2,v} \) from Definition 4.10. Therefore, by Lemma 2.8, we have

\[
\| Z_1 \sigma_2 Z_1 - \sigma_2 \|_1 = \| Z_1 \sigma_2 Z_1 - (-1)^v \mathbb{1} \sigma_2 Z_1 - (-1)^v \mathbb{1} \|_1 \leq 2 \sqrt{\zeta(1, 2, v) \cdot \text{Tr}[\sigma_2,v]} + \text{negl}(\lambda). \tag{4.135}
\]

Therefore, resuming Eq. (4.134):

\[
\| Z_1 \sigma_2 Z_1 - \sigma_2 \|_1 \leq 2 \sum_v \sqrt{\zeta(1, 2, v) \cdot \text{Tr}[\sigma_2,v]} + \text{negl}(\lambda) \leq 4 \sqrt{\gamma_{T,1}} + \text{negl}(\lambda), \tag{4.136}
\]

where the last inequality follows from Cauchy-Schwarz and bounds on \( \zeta \) (and \( \chi \)) in Lemma 4.12. Hence we have proved the first equation.

To prove the second equation, it suffices to prove it for \( \theta = 1 \). Then, we can exactly reuse the above proof after changing the symbols \( \{ Z_1, \sigma_2, \sigma_2, v, \zeta(1, 2, v) \} \) to \( \{ X_1, \sigma_1, \sigma_1, v, \chi(1, v) \} \), respectively.

4.6 The swap isometry

Definition 4.30. Let \( D \) be a device and let \( \mathcal{H} := \mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R \). The swap isometry is the map \( \mathcal{V} : \mathcal{H} \to \mathbb{C}^{2^N} \otimes \mathcal{H} \) defined by

\[
\mathcal{V} = \sum_{u \in \{0,1\}^{2N}} |u\rangle \otimes \prod_{i \in [2N]} X_i^{u_i} \prod_{j \in [2N]} Z_j^{(u_j)}. \tag{4.137}
\]

Note that by the commutation relations (Proposition 4.21) the ordering within each \( \prod_{i \in [2N]} \) does not matter.

Lemma 4.31. Let \( D \) be an efficient device. Then, \( \mathcal{V} \) is efficient.

Proof. \( \mathcal{V} \) can be implemented efficiently by the following circuit:

1. Initialize \( 2N \) ancilla qubits to \( |0\rangle \).
2. Apply \( H^{2N} \) on the ancilla.
3. Apply \( 2N \) c-Z gates where the control is by the \( i \)-th ancilla qubit for \( i = 1, \ldots, 2N \).
4. Apply \( H^{2N} \) on the ancilla.
5. Apply \( 2N \) c-X gates where the control is by the \( i \)-th ancilla qubit for \( i = 1, \ldots, 2N \).

That the above circuit indeed gives \( \mathcal{V} \) can be verified by a direct calculation. We illustrate \( \mathcal{V} \) when \( 2N = 4 \) in Fig. 3 below.
Lemma 4.14. \( V \) is perfect). The first equation follows by direct calculation using the fact that, for all \( j \) and trivially on the other 2\( N - 1 \) qubits.

\[ \sigma_k^Z \] and \( \sigma_k^X \) to be the Pauli operator on 2\( N \) qubits that acts as \( \sigma^Z \) (resp. \( \sigma^X \)) on the \( k \)th qubit and trivially on the other 2\( N - 1 \) qubits.

Lemma 4.32. Let \( D \) be an efficient perfect device. For all \( k \in [2N] \) and \( \theta \in [2N] \cup \{0, \infty\} \), we have
\[
\begin{align*}
\mathcal{V}^\dagger(\sigma_k^Z \otimes \mathbb{1})\mathcal{V} &= Z_k, \\
\mathcal{V}^\dagger(\sigma_k^X \otimes \mathbb{1})\mathcal{V} &\approx_{N, \sqrt{\gamma_T}, \sigma^\#} X_k, \\
\mathcal{V}^\dagger(\sigma_k^Z \otimes \mathbb{1})\mathcal{V} &\approx_{\gamma_T + \text{negl}(\lambda), \sigma^\#} \tilde{Z}_k, \\
\mathcal{V}^\dagger(\sigma_k^X \otimes \mathbb{1})\mathcal{V} &\approx_{N, \sqrt{\gamma_T}, \sigma^\#} \tilde{X}_k.
\end{align*}
\] (4.138)

Proof. The last two equations follow immediately from the first two and Lemma 4.14 (note \( \gamma_T = \text{negl}(\lambda) \) as \( D \) is perfect). The first equation follows by direct calculation using the fact that, for all \( i \in [2N] \), we have \( X_i^2 = \mathbb{1} \) and \( \sum_{\theta \in \{0, 1\}} Z_i^{(\theta)} = \mathbb{1} \).

Consider the second equation. For \( i \in [2N] \), \( k \in [2N] \), and \( u = (u_1, \ldots, u_{2N}) \in \{0, 1\}^{2N-i+1} \), we write

1. \( Z_i(u) \) for the product of the 2\( N \)th (from the left) operator down to the \( i \)th operator in the 2\( N \)-tuple \( (Z_1^{(u_1)}, Z_2^{(u_2)}, \ldots, Z_k^{(u_k)}, \ldots, Z_{2N}^{(u_{2N})}) \). More precisely
\[
Z_i(u) := Z_{2N}^{(u_{2N})} Z_{2N-1}^{(u_{2N-1})} \cdots Z_1^{(u_i)}.
\] (4.139)

2. \( Z_{k,i}(u) \) for the product of the 2\( N \)th operator down to the \( i \)th operator in the 2\( N \)-tuple \( (Z_1^{(u_1)}, Z_2^{(u_2)}, \ldots, Z_k^{(u_k)}, Z_k^{(u_k)} Z_{k+1}^{(u_{k+1})}, \ldots, Z_{2N}^{(u_{2N})}) \). More precisely
\[
Z_{k,i}(u) := \begin{cases} 
Z_{2N}^{(u_{2N})} Z_{2N-1}^{(u_{2N-1})} \cdots Z_i^{(u_i)} & \text{if } k < i, \\
Z_{2N}^{(u_{2N})} Z_{2N-1}^{(u_{2N-1})} \cdots Z_k^{(u_k)} Z_{k+1}^{(u_{k+1})} \cdots Z_{2N}^{(u_{2N})} & \text{if } k \geq i.
\end{cases}
\] (4.140)

We also write \( Z_{2N+1}(u) := \mathbb{1} \) and \( Z_{k,2N+1}(u) := \mathbb{1} \) for all \( k \in [2N] \).

For \( i \in [2N] \) and \( k \in [2N] \), we define the isometry \( \mathcal{V}_{k,i} \) according to whether \( k < i \) or \( k \geq i \). If \( k \geq i \), we define \( \mathcal{V}_{k,i} \) by the following circuit:

1. Initialize 2\( N \) ancilla qubits to \( |0\rangle \).
2. Apply \( H^{\otimes (2N-i+1)} \) on the \( i \)-to-2\( N \)th ancilla qubits.
3. Apply \( (2N - i + 1) \) \( c-Z_j \) gates where the control is by the \( j \)-th ancilla qubit and \( j = i, \ldots, 2N \).
4. Apply $H^\otimes(2N-i+1)$ on the $i$-to-$2N$th ancilla qubits.
5. apply a single c-$X_k$ gate where the control is by the $k$-th ancilla qubit.

If $k < i$, we define $V_{k,i}$ by the same circuit except, in the second step, we additionally apply an $H$ gate on the $k$-th ancilla qubit. We also define $V_{k,2N+1} := 1$ for all $k \in [2N]$.

We illustrate $V_{k,i}$ when $2N = 4$, $k = 1$, and $i = 3$ below.

![Figure 4](image-url)

Figure 4: Illustration of $V_{k,i}$ when $2N = 4$, $k = 1$, and $i = 3$.

By direct calculation, it can be seen that, for all $i \in [2N+1]$ and $k \in [2N]$, we have

$$V_{k,i}^\dagger(\sigma^X_k \otimes \mathbb{I})V_{k,i} = \sum_{u=(u_1,...,u_{2N}) \in \{0,1\}^{2N-i+1}} Z_{k,i}(u)X_k Z_i(u).$$  \hspace{1cm} (4.141)

For $i \in [2N+1], k \in [2N]$, we define

$$\hat{X}_{k,i} := V_{k,i}^\dagger(\sigma^X_k \otimes \mathbb{I})V_{k,i}.$$  \hspace{1cm} (4.142)

The reason for this notation is that the right-hand side increasingly “looks like” $X_k$ as $i$ increases. Indeed,

$$\hat{X}_{k,2N+1} = X_k.$$  \hspace{1cm} (4.143)

Now, since $V_{k,i}$ is an isometry, Eq. (4.142) implies that

$$\|\hat{X}_{k,i}\|_\infty \leq 1,$$  \hspace{1cm} (4.144)

which will be crucial in the following main argument when we use the replacement lemma.

We proceed with our main argument. By direct calculation, we have

$$V^\dagger(\sigma^X_k \otimes \mathbb{I})V = \sum_{u \in \{0,1\}^{2N}} Z_{k,i}(u)X_k Z_i(u).$$  \hspace{1cm} (4.145)

Note that $V^\dagger(\sigma^X_k \otimes \mathbb{I})V = V_{k,1}^\dagger(\sigma^X_k \otimes \mathbb{I})V_{k,1} = \hat{X}_{k,1}$.

By direct expansion, we have

$$\|V^\dagger(\sigma^X_k \otimes \mathbb{I})V - X_k\|_{\sigma^\theta}^2 = \text{Tr}[(V^\dagger(\sigma^X_k \otimes \mathbb{I})V - X_k)^2\sigma^\theta]$$

$$= 1 + \text{Tr}[V^\dagger(\sigma^X_k \otimes \mathbb{I})V^\dagger(\sigma^X_k \otimes \mathbb{I})V\sigma^\theta] - 2\text{Re Tr}[V^\dagger(\sigma^X_k \otimes \mathbb{I})V X_k \sigma^\theta]$$

$$= 1 + \text{Tr}[(V^\dagger(\sigma^X_k \otimes \mathbb{I})V^\dagger(\sigma^X_k \otimes \mathbb{I})V\sigma^\theta - 2\text{Re Tr}[V^\dagger(\sigma^X_k \otimes \mathbb{I})V X_k \sigma^\theta]]$$

$$\leq 2 - 2\text{Re Tr}[V^\dagger(\sigma^X_k \otimes \mathbb{I})V X_k \sigma^\theta],$$  \hspace{1cm} (4.146)

where, in the last inequality, we used the fact that $V^\dagger$ is a projector (Hermitian and idempotent) and $(\sigma^X_k \otimes \mathbb{I})V\sigma^\theta V^\dagger(\sigma^X_k \otimes \mathbb{I})$ is a positive semi-definite operator.
We now show that the last term, \( \Re \text{Tr}[\mathcal{V}^\dagger(\sigma_k^X \otimes I)\mathcal{V} X_k \sigma_k^\theta] \), is close to 1. By the lifting lemma (Lemma 2.16, part 5) and the efficiency of \( \mathcal{V} \) (Lemma 4.31), it suffices to consider the case when \( \theta = k \).

In the following, the terms \{commute, replace, conjugate\} are shorthand for \{Proposition 4.21, Lemma 2.6, Proposition 4.29\} respectively. When using the replacement lemma, we crucially use \( \|\hat{X}_{k,i}\|_\infty \leq 1 \) (Eq. (4.144)).

\[
\begin{align*}
\text{Tr}[\mathcal{V}^\dagger(\sigma_k^X \otimes I)\mathcal{V} X_k \sigma_k^k] &= \text{Tr}[\hat{X}_{k,1} X_k \sigma_k^k] \\
&= \sum_a \text{Tr}[Z_{k,i}(u) X_k Z_1(u) X_k \sigma_k^k] \\
&= \sum_a \text{Tr}[Z_{1}^{(a)} \left( \sum_{u_2, \ldots, u_N} Z_{k,2}(u) X_k Z_2(u) \right) X_k \sigma_k^k] \\
&= \sum_a \text{Tr}[Z_{1}^{(a)} \hat{X}_{k,2} Z_{1}^{(a)} X_k \sigma_k^k] \\
&\approx \sqrt{\tau N} \sum_a \text{Tr}[Z_{1}^{(a)} \hat{X}_{k,2} Z_{1}^{(a)} X_k \sigma_k^k] \\
&= \sum_a \text{Tr}[Z_{1}^{(a)} \hat{X}_{k,2} X_k \sigma_k^k] \\
&= \text{Tr}[\hat{X}_{k,2} X_k \sigma_k^k]
\end{align*}
\]

Therefore, we have \( \text{Tr}[\hat{X}_{k,1} X_k \sigma_k^k] \approx \sqrt{\tau N} \text{Tr}[\hat{X}_{k,2} X_k \sigma_k^k] \). Using the above reasoning another \( k - 2 \) times and the triangle inequality gives \( \text{Tr}[\hat{X}_{k,2} X_k \sigma_k^k] \approx \sqrt{\tau N} \text{Tr}[\hat{X}_{k} X_k \sigma_k^k] \). Then by the definitions of \( \{\hat{X}_{k,2}, \hat{X}_{k,k+1}, \mathcal{V}_{k+1}\} \) and direct calculation, \( \text{Tr}[\hat{X}_{k,2} X_k \sigma_k^k] = \sum_a \text{Tr}[Z_{k}^{(a)} \hat{X}_{k,k+1} Z_{k}^{(a)} X_k \sigma_k^k] \). Using \( Z_{k}^{(a)} = (1 + (-1)^a Z_k)/2 \) to perform the sum over \( a \) gives

\[
\text{Tr}[\hat{X}_{k,2} X_k \sigma_k^k] = \frac{1}{2} (\text{Tr}[\hat{X}_{k,k+1} X_k \sigma_k^k] - \text{Tr}[Z_{k} \hat{X}_{k,k+1} Z_{k} X_k \sigma_k^k])
\]

(4.147)

We approximate both terms using the reasoning at the beginning another \( 2N - k \) times and the triangle inequality (when handling the second term, we additionally use the commutativity of the \( Z_{is} \)):

\[
\frac{1}{2} \text{Tr}[\hat{X}_{k,k+1} X_k \sigma_k^k] - \frac{1}{2} \text{Tr}[Z_{k} \hat{X}_{k,k+1} Z_{k} X_k \sigma_k^k] \approx (2N-k)\sqrt{\tau N} \frac{1}{2} \text{Tr}[\hat{X}_{k,2N+1} X_k \sigma_k^k] - \frac{1}{2} \text{Tr}[Z_{k} \hat{X}_{k,2N+1} Z_{k} X_k \sigma_k^k]
\]

\[
= \frac{1}{2} \text{Tr}[X_k X_k \sigma_k^k] - \frac{1}{2} \text{Tr}[Z_{k} X_k Z_{k} X_k \sigma_k^k]
\]

\[
= 1 - \frac{1}{2} \text{Tr}[Z_{k} X_k Z_{k} X_k \sigma_k^k],
\]

(4.148)

where the second equation is by Eq. (4.143) and the last equation is by \( X_k^2 = I \).

Applying the triangle inequality to all of the above approximations and then taking the real part gives

\[
\Re \text{Tr}[\mathcal{V}^\dagger(\sigma_k^X \otimes I)\mathcal{V} X_k \sigma_k^\theta] \approx N\sqrt{\tau N} \frac{1}{2} - \frac{1}{4} (\text{Tr}[Z_{k} X_k Z_{k} X_k \sigma_k^k] + \text{Tr}[X_k Z_{k} X_k Z_{k} \sigma_k^k])
\]

(4.149)

By directly unpacking the definition of the anti-commutation relation \( \{X_k, Z_k\} \approx \sqrt{\tau_+} \sigma_k^0 \) (Proposition 4.22), we see that \( \text{Tr}[Z_{k} X_k Z_{k} X_k \sigma_k^k] + \text{Tr}[X_k Z_{k} X_k Z_{k} \sigma_k^k] \approx \sqrt{\tau_+ \text{negl}(\lambda)} - 2 \) (no replacement lemma used). Hence

\[
\Re \text{Tr}[\mathcal{V}^\dagger(\sigma_k^X \otimes I)\mathcal{V} X_k \sigma_k^\theta] \approx N\sqrt{\tau N} 1,
\]

(4.150)

and hence the lemma. □
Lemma 4.33. Let $D$ be an efficient perfect device. For $k \in [N]$ and $\theta \in [2N] \cup \{0, \circ\}$, we have

$$
\mathcal{V}(\sigma^X_k \otimes \sigma^Z_{N+k} \otimes \mathbb{1})\mathcal{V} \approx_{N^{1/4}\gamma_2^{1/8}, \sigma^\theta} \tilde{X}_k \tilde{Z}_{N+k};
$$

$$
\mathcal{V}(\sigma^Z_k \otimes \sigma^X_{N+k} \otimes \mathbb{1})\mathcal{V} \approx_{N^{1/4}\gamma_2^{1/8}, \sigma^\theta} \tilde{Z}_k \tilde{X}_{N+k}.
$$

(4.151)

Proof. By Lemma 2.10 and Lemma 4.32, for all $k \in [2N]$ and $\theta \in [2N] \cup \{0, \circ\}$, we have

$$
\mathcal{V} \tilde{Z}_k \mathcal{V}^\dagger \approx_{\sqrt{\gamma_T^{1/4}, \sigma^\theta}} \sigma^Z_k \otimes \mathbb{1},
$$

$$
\mathcal{V} \tilde{X}_k \mathcal{V}^\dagger \approx_{\sqrt{\gamma_T^{1/4}, \sigma^\theta}} \sigma^Z_k \otimes \mathbb{1}.
$$

(4.152)

It suffices to prove the first equation for $k = 1$ as the proof for other $k$ is analogous. In the following, the terms {indist., lift, conjugate, replace} are shorthand for {Lemma 4.13, Lemma 2.16 (part 5), Proposition 4.29, Lemma 2.6} respectively.

$$
\text{Tr}[(\mathcal{V}(\sigma^X_1 \otimes \sigma^Z_{N+1} \otimes \mathbb{1})\mathcal{V} - \tilde{X}_1 \tilde{Z}_{N+1})^\dagger(\mathcal{V}(\sigma^X_1 \otimes \sigma^Z_{N+1} \otimes \mathbb{1})\mathcal{V} - \tilde{X}_1 \tilde{Z}_{N+1})\sigma^\theta] = 2 - 2 \text{Re } \text{Tr}[\mathcal{V}(\sigma^1_X \otimes \sigma^Z_{N+1} \otimes \mathbb{1})\mathcal{V}\tilde{X}_1 \tilde{Z}_{N+1}\sigma^\theta]
$$

(rearrange)

$$
\approx_{\gamma_T^{1/4}+\text{negl}(\lambda)} 2 - 2 \text{Re } \text{Tr}[\mathcal{V}(\sigma^1_X \otimes \sigma^Z_{N+1} \otimes \mathbb{1})\mathcal{V}\tilde{X}_1 \tilde{Z}_{N+1}^\dagger\mathcal{V}^\dagger]
$$

(indist. and lift)

$$
\approx_{\sqrt{\gamma_T^{1/4}+\text{negl}(\lambda)}} 2 - 2 \text{Re } \text{Tr}[\mathcal{V}(\sigma^1_X \otimes \sigma^Z_{N+1} \otimes \mathbb{1})\mathcal{V}\tilde{X}_1 \sigma^\theta]
$$

(conjugate and replace)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \sigma^Z_{N+1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(Eq. (4.152) and replace)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \mathbb{1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(trace is cyclic and $\mathcal{V}^\dagger \mathcal{V} = \mathbb{1}$)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \mathbb{1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(trace is cyclic and $\mathcal{V}^\dagger \mathcal{V} = \mathbb{1}$)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \mathbb{1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(conjugate and replace)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \mathbb{1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(trace is cyclic and $\mathcal{V}^\dagger \mathcal{V} = \mathbb{1}$)

$$
= 2 - 2 \text{Re } \text{Tr}[(\mathcal{V}\tilde{Z}_{N+1}\mathcal{V}^\dagger)(\sigma^1_X \otimes \mathbb{1} \otimes \mathbb{1})(\mathcal{V}\tilde{X}_1 \mathcal{V}^\dagger)(\sigma^1 \mathcal{V}^\dagger)]
$$

(Eq. (4.152) and replace)

$$
= 0.
$$

Therefore, by the triangle inequality, we have

$$
\mathcal{V}(\sigma^X_k \otimes \sigma^Z_{N+k} \otimes \mathbb{1})\mathcal{V} \approx_{N^{1/4}\gamma_2^{1/8}, \sigma^\theta} \tilde{X}_k \tilde{Z}_{N+k},
$$

(4.153)

and hence the first equation of the lemma.

The proof for the second equation is similar. We may again only consider the case $k = 1$. Then, the second equation follows from the same steps as above except we lift to $\theta = N + 1$ at the lifting step. \qed

4.8 States under the swap isometry

In this subsection, we show that the states $\sigma^\theta,v$ of a device that passes our protocol with high probability must be close to states that we call $\tau^\theta,v$ under the swap isometry. The states $\tau^\theta,v$ are defined in

Definition 4.34 (density operators $\tau^\theta,v$). Let $v \in \{0, 1\}^{2N}$. For $\theta \in [2N] \cup \{0, \circ\}$, we define the 2N-qubit density operator $\tau^\theta,v := |\tau^\theta,v\rangle \langle \tau^\theta,v|$, according to the following three cases.

$$
|\tau^\theta,v\rangle := \begin{cases} 
|v_1\rangle \otimes \cdots \otimes |v_{\theta-1}\rangle \otimes \langle (-)^{\theta}v| \otimes |v_{\theta+1}\rangle \otimes \cdots \otimes |v_{2N}\rangle & \text{if } \theta \in [2N], \\
|v\rangle := |v_1\rangle \otimes \cdots \otimes |v_{2N}\rangle & \text{if } \theta = 0, \\
|\psi^{\circ}\rangle & \text{if } \theta = \circ,
\end{cases}
$$

(4.154)

where $|\psi^{\circ}\rangle$ is as defined in Eq. (3.5)
Lemma 4.32.

Let $a \in \{0, 1\}$ and $k \in [2N]$, we define

$$
|a\rangle \langle a|_k := \mathbb{1}_2 \otimes (k-1) \otimes |a\rangle \langle a| \otimes \mathbb{1}_2 \otimes (2N-k).
$$

\[ (4.155) \]

For $v \in \{0, 1\}^2N$ and $j \in [2N]$, we define

$$
|v\rangle_{2k} |v\rangle_{2N} := \mathbb{1}_2 \otimes (j-1) \otimes |v\rangle_j \otimes \ldots \otimes |v\rangle_{2N} |v\rangle_{2N}.
$$

\[ (4.156) \]

For $i, \theta \in [2N]$ and $v \in \{0, 1\}^2N$, we define

$$
\chi'(i, \theta, v) := ||X_i - \mathcal{V}^l(\sigma^X_i \otimes 1\mathbb{I})\mathcal{V}^l||_{\sigma^{\theta,v}},
$$

\[ (4.157) \]

so that, by definition,

$$
X_i \approx \chi'(i, \theta, v), \sigma^\theta, v \mathcal{V}^l(\sigma^X_i \otimes 1\mathbb{I})\mathcal{V}^l,
$$

\[ (4.158) \]

By Lemma 4.32 and Lemma 4.33, we see that, for all $i, \theta \in [2N]$,

$$
\sum_v \chi'(i, \theta, v) = O(N \sqrt{T}), \quad \sum_v \chi'(i, \theta, v) = O(N^{1/4} \gamma_T^{1/8}), \quad \text{and} \quad \sum_v \chi'(i, \theta, v) = O(N^{1/4} \gamma_T^{1/8}).
$$

(4.159)

With the above definitions in place, we can prove the following lemma.

**Lemma 4.35.** Let $D$ be an efficient perfect device. For all $\theta \in [2N] \cup \{0\}$ and $v \in \{0, 1\}^2N$, there exists a positive semi-definite operator $\sigma^{\theta,v}$ such that

$$
\sum_v \|\mathcal{V}^{\theta,v} - \tau^{\theta,v} \otimes \sigma^{\theta,v}\|_1 \leq O(N^{3/2} \gamma_T^{1/4}).
$$

(4.160)

**Proof.** It suffices to prove the lemma for $\theta = 1$ as the proof for other $\theta$ is analogous (the proof for $\theta = 0$ is also simpler). First, we prove the following claim using Lemma 4.32.

**Claim 4.36.**

1. There exists a positive semi-definite operator $\beta^v$, such that

$$
\|\mathcal{V}^{\theta,v} - (\beta^{\theta,v})\|_1 \leq \epsilon(v),
$$

(4.161)

where $\epsilon(v) := O(\chi(1, v) + \sqrt{\text{Tr}[\sigma^{\theta,v}] \cdot \chi'(1, 1, v)})^{1/2} \sqrt{\text{Tr}[\sigma^{\theta,v}]}$.

2. For all $k \in [2N]$ with $k \geq 2$, we have $|||v_k\rangle\langle v_k| \otimes 1 - \mathbb{1}||^2_{\mathcal{V}^{\theta,v} \mathcal{V}^l} = \zeta(k, 1, v)$.

**Proof of Claim 4.36.** To prove the first part, consider

$$
\|\mathcal{V}^{\theta,v} - (\beta^{\theta,v})\|_1 \leq \text{Tr}[\sigma^{\theta,v} - \beta^{\theta,v}] = \text{Tr}[\mathcal{V}^{\theta,v} - \mathcal{V}^{l}((\beta^{\theta,v}) - (\beta^{\theta,v})\] 1 - \mathbb{1}])\mathcal{V}^{\theta,v}\mathcal{V}^l
$$

(4.162)

where the inequality uses the replacement lemma (Lemma 2.6) with the definition of $\chi'(1, 1, v)$ in Eq. (4.157) and Lemma 2.11, and the last equality is by Eq. (4.45).
Using Lemma 2.8 with the above equation, we obtain
\[ \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \|_1 \leq \epsilon(v), \] (4.163)
where
\[ \epsilon(v) := O([\chi(1, v) + \sqrt{\text{Tr}[\sigma^{1,v}] \cdot \chi'(1, v)}]^{1/2} \sqrt{\text{Tr}[\sigma^{1,v}]}) \] (4.164)
But \( \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \| \) is of the form \(|\langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta^v\) for
\[ \beta^v := \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \| \). (4.165)

Hence the first part of the claim.

The second part is simpler. Using the first equation of Lemma 4.32 and Eq. (4.45) \((k \geq 2)\), we see
\[ \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \|_1 \leq \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger\|_1 - \|\langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \|_1 \]
using the first part of Claim 4.36 to bound the first term and the second part of Claim 4.36 with Lemma 2.8 to bound the second term:
\[ \leq 2\epsilon(v) + 2\sqrt{\chi(k, 1, v)} \cdot \text{Tr}[\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger] = 2\epsilon(v) + 2\sqrt{\chi(k, 1, v)} \cdot \text{Tr}[\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger]. \] (4.166)

That is, for all \( k \in [2N] \) with \( k \geq 2 \), we have
\[ \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \|_1 \leq 2\epsilon(v) + 2\sqrt{\chi(k, 1, v)} \cdot \text{Tr}[\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger]. \] (4.167)

We then have
\[ \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \mathbb{1} \rangle \sigma^{1,v}\mathcal{V}^\dagger \|_1 \]
using the triangle inequality:
\[ \leq \|\mathcal{V}\sigma^{1,v}\mathcal{V}^\dagger - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta^v \rangle \|_1 \]
\[ + \|\langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta^v \rangle - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta \rangle \|_1 \]
using Claim 4.36:
\[ \leq \epsilon(v) + \|\langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta^v \rangle - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta \rangle \|_1 \]
\[ - \langle \langle (-)^{v_i} \rangle \langle (-)^{v_i} \rangle_1 \otimes \beta \rangle \|_1 \]
using Lemma 2.12 with Eq. (4.167):
\[ \leq (2N - 1) \epsilon(v) + 2 \sum_{k=2}^N \sqrt{\text{Tr}[\sigma^{1,v}]} \cdot \chi(k, 1, v). \]

Finally, using the properties of \( \{\chi', \chi, \zeta\} \) given in Eq. (4.159) and Lemma 4.12, the definition of \( \epsilon(v) \) in
Eq. (4.164), and the Cauchy-Schwarz inequality, we obtain
\[
\sum_{v \in \{0,1\}^{2N}} \|V^{1,v}V^\dagger - (\langle v_{2:2N}\rangle \langle v_{2:2N} | \otimes 1 \rangle) \langle (-)^{v_1}_1 \otimes \beta^v \rangle \langle v_{2:2N}\rangle \langle v_{2:2N} | \otimes 1 \rangle\|_1 \\
\leq O\left(N \sum_v \epsilon(v) + \sum_v \sum_{k=2}^{2N} \sqrt{\text{Tr}[\sigma^{1,v} \cdot \zeta(k,1,v)]}\right) \\
\leq O\left(N \sum_v \left(\chi(1,v) + \sqrt{\text{Tr}[\sigma^{1,v} \cdot \chi'(1,1,v)]}^{1/2} \sqrt{\text{Tr}[\sigma^{1,v}]} \right) + N \sqrt{\gamma_T}\right) \\
\leq O\left(N \left(\sum_v \chi(1,v) + \sqrt{\text{Tr}[\sigma^{1,v} \cdot \chi'(1,1,v)]}^{1/2} \right) + N \sqrt{\gamma_T}\right) \\
\leq O(N(1/2) + \text{negl}(\lambda))^{1/2} + N \sqrt{\gamma_T} \leq O(N^{3/2} \gamma_T^{1/2}).
\]
Noting that \( (\langle v_{2:2N}\rangle \langle v_{2:2N} | \otimes 1 \rangle) \langle (-)^{v_1}_1 \otimes \beta^v \rangle \langle v_{2:2N}\rangle \langle v_{2:2N} | \otimes 1 \rangle \) is of the form
\[
\tau^{1,v} \otimes \alpha^{1,v} = |(-)^{v_1}_1 \otimes |v_2\rangle \langle v_2| \otimes \cdots \otimes |v_{2N}\rangle \langle v_{2N}| \otimes \alpha^{1,v},
\]
for some positive semi-definite operator \( \alpha^{1,v} \), establishes the lemma.

\( \square \) Proof of Lemma 4.35.

**Lemma 4.37.** Under the same setup as Lemma 4.35, the following holds.

1. Let \( \theta \in [2N] \cup \{0\} \). For all \( v \in \{0,1\}^{2N} \), we have \( \alpha^{\theta,v} \leq O(1/2) \).

2. Let \( \theta, \varphi \in [2N] \cup \{0\} \). For all \( v \in \{0,1\}^{2N} \), there exists \( \delta_{\theta,\varphi}(v) \geq 0 \) with \( \sum_v \delta_{\theta,\varphi}(v) = O(N^{3/4} \gamma_T^{1/8}) \) such that \( \alpha^{\theta,v} \leq \delta_{\theta,\varphi}(v) \alpha^{\varphi,v} \).

3. All pairs of states in the set \( \{\alpha^{\theta,v} \, | \, \theta \in [2N] \cup \{0\}, \, v \in \{0,1\}^{2N}\} \) are computationally indistinguishable up to \( O(N^{3/4} \gamma_T^{1/8}) \).

4. Let \( \alpha := \alpha^{1,0,N} \). For all \( \theta \in [2N] \cup \{0\} \), we have \( V^{0} \approx \alpha^{1,0,N} \).

**Proof.** We restrict our attention to \( \theta \in [2N] \) since analogous but simpler proofs suffice to handle the case \( \theta = 0 \). We will use the following restatement of Lemma 4.35 in the proof of first two parts: there exists \( \epsilon = O(N^{3/2} \gamma_T^{1/4}) \) such that, for all \( \theta \in [2N] \), we have
\[
\sum_v \|V^{\theta,v}V^\dagger - \tau^{\theta,v} \otimes \alpha^{\theta,v}\|_1 \leq \epsilon.
\]

(4.169)

In this proof, we will occasionally use the notation \( \alpha(\theta,v) \) for \( \alpha^{\theta,v} \) and \( \tau(\theta,v) \) for \( \tau^{\theta,v} \) for clarity. We now consider the first part of the lemma. It suffices to consider \( \theta = 1 \) as the proof for other \( \theta \) is analogous.

For \( j \in [2N] \) and \( w \in \{0,1\}^{2N-1} \), let
\[
w(j,0) := w_1 w_2 \cdots w_{j-1} 0 w_j \cdots w_{2N-1} \in \{0,1\}^{2N}, \\
w(j,1) := w_1 w_2 \cdots w_{j-1} 1 w_j \cdots w_{2N-1} \in \{0,1\}^{2N}.
\]

(4.170)

We prove the first part of the lemma via Claim 4.38. Claim 4.38 places bounds on sums of the computational distinguishing advantages between certain pairs of states in \( \{\alpha^{\theta,v} \, | \, v \in \{0,1\}^{2N}\} \) for each fixed \( \theta \). The bounds in Claim 4.38 are more refined than the bounds stated in the first part of the lemma.

**Claim 4.38.** For all \( j \in [2N] \), there exists a function \( \delta_j : \{0,1\}^{2N-1} \rightarrow \mathbb{R}_{\geq 0} \), with \( \sum_{w \in \{0,1\}^{2N-1}} \delta_j(w) \leq O(N^{3/4} \gamma_T^{1/8}) \), such that, for all \( w \in \{0,1\}^{2N-1} \), we have \( \alpha^{1,w(0)} \approx \delta_j(w) \alpha^{1,w(j,1)} \).
To see how the first part of Lemma 4.37 follows from the claim, observe that for any $v \in \{0,1\}^{2N}$,
\[
\alpha^{1,v_1 \ldots v_{2N-1}v_{2N}} \approx \delta_{2N(v_1 \ldots v_{2N-1})} \alpha^{1,v_1 \ldots v_{2N-2}v_{2N-1}0} \approx \delta_{2N-1(v_1 \ldots v_{2N-2})} \alpha^{1,v_1 \ldots v_{2N-2}00}.
\]
By the triangle inequality for computational indistinguishability (Lemma 2.13), we see that $\alpha^{1,v_1 \ldots v_{2N-1}v_{2N}}$ is computationally indistinguishable from $\alpha^{1,02N}$ up to order
\[
\sum_{j=1}^{2N} \delta_j(v_1 \ldots v_{j-1}1^{2N-j}) \leq \sum_{j=1}^{2N} \sum_{w \in \{0,1\}^{2N-1}} \delta_j(w) \leq O(N^{7/4} \gamma^{1/8}),
\]
where the second inequality uses the claim.

**Proof of Claim 4.38.** Suppose that for each $j \in [2N]$ and $w \in \{0,1\}^{2N-1}$, there exists an efficient POVM $\{E_{j,w}^{(0)}, E_{j,w}^{(1)}\}$ that distinguishes $\alpha^{1,w(j,0)}$ from $\alpha^{1,w(j,1)}$ with advantage $\delta_j(w) \geq 0$, i.e.,
\[
\Tr[E_{j,w}^{(0)}(\alpha^{1,w(j,0)} - \alpha^{1,w(j,1)})] = \delta_j(w).
\]
We use $\{E_{j,w}^{(0)}, E_{j,w}^{(1)}\}$ to define the following POVM elements:
\[
\Gamma_j := \mathcal{V}\left( \sum_{w \in \{0,1\}^{2N-1}} \tau^{1,w(j,0)} \otimes E_{j,w}^{(0)} + \sum_{w \in \{0,1\}^{2N-1}} \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)} \right) \mathcal{V},
\]
\[
\Gamma_{j,0} := \mathcal{V}\left( \sum_{w \in \{0,1\}^{2N-1}} \tau^{1,w(j,1)} \otimes E_{j,w}^{(0)} \right) \mathcal{V},
\]
\[
\Gamma_{j,1} := \mathcal{V}\left( \sum_{w \in \{0,1\}^{2N-1}} \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)} \right) \mathcal{V}.
\]
The three POVMs $\{\Gamma_j, I - \Gamma_j\}$, $\{\Gamma_{j,0}, I - \Gamma_{j,0}\}$, and $\{\Gamma_{j,1}, I - \Gamma_{j,1}\}$ can all be efficiently implemented as the POVM $\{E_{j,w}^{(0)}, E_{j,w}^{(1)}\}$ is assumed to be efficient, $\mathcal{V}$ is efficient, and $\tau^{1,v}$ is an efficient projector for all $v \in \{0,1\}^{2N}$. To clarify, $\{\Gamma_j, I - \Gamma_j\}$, for example, can be implemented efficiently as follows:

1. Apply $\mathcal{V}$.
2. Measure the first qubit in the Hadamard basis and the remaining $2N - 1$ qubits in the computational basis to obtain $u \in \{0,1\}^{2N}$.
3. (a) If $u$ is consistent with $\tau^{1,w(j,0)}$ for some $w \in \{0,1\}^{2N-1}$, measure $E_{j,w}^{(0)}$ and output $a \in \{0,1\}$;

(b) If $u$ is consistent with $\tau^{1,w(j,1)}$ for some $w \in \{0,1\}^{2N-1}$, measure $E_{j,w}^{(1)}$ and output $a \in \{0,1\}$.

Using Lemma 4.13 and the efficiency of the POVMs above, we have
\[
\Tr[\Gamma_j \sigma^1] - \Tr[\Gamma_j \sigma^k] = \text{negl}(\lambda),
\]
\[
\Tr[\Gamma_{j,0} \sigma^1] - \Tr[\Gamma_{j,0} \sigma^k] = \text{negl}(\lambda),
\]
\[
\Tr[\Gamma_{j,1} \sigma^1] - \Tr[\Gamma_{j,1} \sigma^k] = \text{negl}(\lambda).
\]
On the other hand, consider the following manipulations of the left-hand side of the first equation in Eq. (4.175). In the following, the sum over the symbol $w$ is always over $w \in \{0,1\}^{2N-1}$ and $j, k \in [2N]$.
\[
\Tr[\Gamma_j \sigma^1] - \Tr[\Gamma_j \sigma^k] = \Tr[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w}^{(0)} ) \mathcal{V} \sigma^1 \mathcal{V}^\dagger] + \Tr[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)} ) \mathcal{V} \sigma^1 \mathcal{V}^\dagger] - \Tr[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w}^{(0)} ) \mathcal{V} \sigma^k \mathcal{V}^\dagger] - \Tr[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)} ) \mathcal{V} \sigma^k \mathcal{V}^\dagger].
\]
Adding and subtracting $\text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^1 \nu^\dagger] + \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^k \nu^\dagger]$, we rearrange the above equation as

$$
\text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] = \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w})\sigma^1 \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^1 \nu^\dagger] \\
+ \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^k \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w})\sigma^k \nu^\dagger] \\
+ \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)})\sigma^1 \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(1)})\sigma^k \nu^\dagger].
$$

(4.177)

Using the definitions of $\Gamma_{j,0}$ and $\Gamma_{j,1}$, the above can be rewritten as

$$
\text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] = \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w})\sigma^1 \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\sigma^1 \nu^\dagger] \\
+ \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^k \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\sigma^k \nu^\dagger] \\
+ \text{Tr}[\Gamma_{j,0} \sigma^1] - \text{Tr}[\Gamma_{j,0} \sigma^k] \\
+ \text{Tr}[\Gamma_{j,1} \sigma^1] - \text{Tr}[\Gamma_{j,1} \sigma^k].
$$

(4.178)

Note that the expressions on the last two lines are negligible using the last two equations in Eq. (4.175) and $\sigma^6 \approx \text{negl}(\lambda) \sum_{w \in \{0,1\}^{2N}} \sigma^{0,u}$ (Lemma 4.17). Therefore, we have

$$
\text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] = \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w})\sigma^1 \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\sigma^1 \nu^\dagger] \\
+ \text{Tr}[(\sum_w \tau^{1,w(j,1)} \otimes E_{j,w})\sigma^k \nu^\dagger] - \text{Tr}[(\sum_w \tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\sigma^k \nu^\dagger] + \text{negl}(\lambda)
$$

(4.179)

Now, we use the replacement lemma (Lemma 2.6) together with the following observations to replace $\nu^\dagger$ by $\tau^{\theta,u} \otimes \alpha^{\theta,u}$ for $\theta \in \{1, k\}$:

1. $\|\sum_w (\tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\|_\infty \leq 1$ because the $\tau^{1,w(j,0)}$s are orthonormal and $0 \leq E_{j,w}^{(0)} \leq 1$, likewise $\|\sum_w (\tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\|_\infty \leq 1$; and

2. $\sum_u \sqrt{\text{Tr}[\sigma^{\theta,v}] \cdot \|\sigma^{\theta,v}\|_1} - \tau^{\theta,v} \otimes \alpha^{\theta,v}\|_1 \leq \sqrt{\tau}$ by Eq. (4.169) and Cauchy-Schwarz.

Therefore, continuing the above:

$$
\text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] \geq -4\sqrt{\tau} + \text{negl}(\lambda) + \sum_{u,w} \text{Tr}[(\tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\tau^{1,u} \otimes \alpha^{1,u}] - \sum_{u,w} \text{Tr}[(\tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\tau^{1,u} \otimes \alpha^{1,u}] \\
+ \sum_{u,w} \text{Tr}[(\tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\tau^{k,u} \otimes \alpha^{k,u}] - \sum_{u,w} \text{Tr}[(\tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\tau^{k,u} \otimes \alpha^{k,u}].
$$

(4.180)

We simplify the first line using the fact that $\text{Tr}[\tau^{1,v} \tau^{1,u}] = \delta_{u,v}$ for all $u, v \in \{0,1\}^{2N}$, which follows from the definition of $\tau^{1,u}$ given in Eq. (4.154), and obtain

$$
\text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] \geq -4\sqrt{\tau} + \text{negl}(\lambda) + \sum_{w} \text{Tr}(E_{j,w}^{(0)} (\alpha^{1,w(j,0)} - \alpha^{1,w(j,1)})) \\
+ \sum_{u,w} \left(\sum_{u} \text{Tr}[(\tau^{1,w(j,1)} \otimes E_{j,w}^{(0)})\tau^{k,u} \otimes \alpha^{k,u}] - \sum_{u} \text{Tr}[(\tau^{1,w(j,0)} \otimes E_{j,w}^{(0)})\tau^{k,u} \otimes \alpha^{k,u}] \right).
$$

(4.181)
As \( \text{Tr}[\Gamma_j \sigma^1] - \text{Tr}[\Gamma_j \sigma^k] = \text{negl}(\lambda) \) from Eq. (4.175), we obtain

\[
\text{Tr} \left[ \sum_w E^{(0)}_{j,w} \left( \alpha^{1,w(j,0)} - \alpha^{1,w(j,1)} \right) \right] \\
\leq 4\sqrt{\epsilon} + \text{negl}(\lambda) + \sum_w (\sum_u \text{Tr}[(\tau^{1,w(j,0)} \otimes E^{(0)}_{j,w}) \tau^{k,u} \otimes \alpha^{k,u}] - \sum_u \text{Tr}[(\tau^{1,w(j,1)} \otimes E^{(0)}_{j,w}) \tau^{k,u} \otimes \alpha^{k,u}]).
\]

Taking

\[
k := \begin{cases} 
2 & \text{if } j = 1, \\
2 & \text{if } j > 1,
\end{cases}
\]

and crucially using the fact that \( w(j,0) \) and \( w(j,1) \) differ only on bit \( j \), we find that the last term of Eq. (4.182) is exactly zero. To be clear, for all \( w \in \{0,1\}^{2N-1} \), we can use \( \text{Tr}[\cdot|\cdot] = 1/2 \) and \( \text{Tr}[\cdot|\cdot] = \text{negl}(\lambda) \) for all \( a, b \in \{0,1\} \) to directly calculate

\[
\sum_u \text{Tr}[(\tau^{1,w(j,0)} \otimes E^{(0)}_{j,w}) \tau^{k,u} \otimes \alpha^{k,u}] = \sum_u \text{Tr}[(\tau^{1,w(j,1)} \otimes E^{(0)}_{j,w}) \tau^{k,u} \otimes \alpha^{k,u}]
\]

\[
\begin{cases} 
\frac{1}{2} \sum_{(a,b) \in \{0,1\}^2} \text{Tr}[E^{(0)}_{1,u} \alpha(2, abz_2 \ldots z_{2N-1})] & \text{if } j = 1, \\
\frac{1}{2} \sum_{(a,b) \in \{0,1\}^2} \text{Tr}[E^{(0)}_{j,w} \alpha(j, aw_2 \ldots w_{j-1} b z_1 \ldots z_{2N-j})] & \text{if } j > 1.
\end{cases}
\]

Therefore, we have

\[
\text{Tr} \left[ \sum_w E^{(0)}_{j,w} \left( \alpha^{1,w(j,0)} - \alpha^{1,w(j,1)} \right) \right] \leq 4\sqrt{\epsilon} + \text{negl}(\lambda).
\]

Using \( \epsilon = O(N^{3/2} \gamma^{1/4}) \) from Eq. (4.169) and the definition \( \delta_j(w) := \text{Tr}[E^{(0)}_{j,w} \left( \alpha^{1,w(j,0)} - \alpha^{1,w(j,1)} \right)] \) from Eq. (4.173), this means

\[
\sum_w \delta_j(w) \leq O(N^{3/4} \gamma^{1/8}).
\]

Recalling that \( j \) can be any integer in \([2N]\) establishes the claim. \( \square \) Proof of Claim 4.38.

Hence, as explained below the statement of Claim 4.38, the first part of the lemma follows.

We now consider the second part of the lemma. Assume without-loss-of-generality that \( \theta < \varphi \) and then fix \( \theta, \varphi \). Suppose that for each \( v \in \{0,1\}^{2N} \), there exists an efficient POVM \( \{E^{(0)}_v, E^{(1)}_v\} \) that distinguishes \( \alpha^{\theta,v} \) from \( \alpha^{\varphi,v} \) with advantage \( \delta(v) \geq 0 \), i.e.,

\[
\text{Tr}[E^{(0)}_v (\alpha^{\theta,v} - \alpha^{\varphi,v})] = \delta(v).
\]

We use \( \{E^{(0)}_v, E^{(1)}_v\} \) to define the following POVM elements:

\[
\Lambda_\theta := \mathbb{V}^\dagger \left( \sum_{v \in \{0,1\}^{2N}} \tau^{\theta,v} \otimes F^{(0)}_v \right) \mathbb{V}.
\]

\( \{\Lambda_\theta, \mathbb{I} - \Lambda_\theta\} \) is an efficient POVM by an argument similar to that below Eq. (4.174). Therefore, we have

\[
\text{Tr}[\Lambda_\theta(\sigma^\theta - \sigma^\varphi)] = \text{negl}(\lambda).
\]

On the other hand, consider

\[
\text{Tr}[\Lambda_\theta (\sigma^\theta - \sigma^\varphi)]
\]

using definitions and \( \sigma^\theta \sim \text{negl}(\lambda) \sum_{u \in \{0,1\}^{2N}} \sigma^{\theta,u} \) (Lemma 4.17):

\[
= \text{negl}(\lambda) + \sum_u \text{Tr}[\sum_v (\tau^{\theta,v} \otimes F^{(0)}_v) \mathbb{V} \sigma^{\theta,u} \mathbb{V}^\dagger] - \sum_u \text{Tr}[\sum_v (\tau^{\theta,v} \otimes F^{(0)}_v) \mathbb{V} \sigma^{\varphi,u} \mathbb{V}^\dagger].
\]

46
Lemma 2.6 similarly to how it is used in the proof of Claim 4.38:

\[ \geq -2\sqrt{\epsilon} + \sum_{u,v} \text{Tr}[\tau^{\theta,u} \otimes F_v^{(0)}] \alpha^{\theta,u} \otimes \alpha^{\tau,v} - \sum_{u,v} \text{Tr}[\tau^{\theta,v} \otimes F_v^{(0)}] \tau^{\tau,u} \otimes \alpha^{\tau,v} \]

we simplify the first term using the fact that \( \text{Tr}[\tau^{\theta,u} \tau^{\theta,v}] = \delta_{u,v} \) for all \( u, v \in \{0,1\}^{2N} \), which follows from the definition of \( \tau^{\theta,v} \) given in Eq. (4.154):

\[ = -2\sqrt{\epsilon} + \sum_v \text{Tr}[F_v^{(0)} \alpha^{\theta,v}] - \sum_{u,v} \text{Tr}[\tau^{\theta,v} \otimes F_v^{(0)}] \tau^{\tau,u} \otimes \alpha^{\tau,v} \]

we simplify the second term using the definition of \( \tau^{\theta,v} \) given in Eq. (4.154), \( \text{Tr}[\tau^{\theta,v} \otimes b] = \delta_{u,b} \) for all \( u, b \in \{0,1\} \), and \( \theta < \varphi \):

\[ = -2\sqrt{\epsilon} + \sum_v \text{Tr}[F_v^{(0)} \alpha^{\theta,v}] - \frac{1}{4} \sum_{(a,b) \in \{0,1\}^2} \text{Tr}[F_v^{(0)} \alpha(\varphi, u, v_{\varphi-1} b v_{\varphi+1} \ldots v_{2N})]. \]

By Claim 4.38, for all \( v \in \{0,1\}^{2N} \) and \( a, b \in \{0,1\}^2 \), the state

\[ \alpha(\varphi, v_{\varphi-1} b v_{\varphi+1} \ldots v_{2N}) \]

is computationally indistinguishable from \( \alpha^{\varphi,v} \) up to (using the notation of Claim 4.38):

\[ \delta_\theta(v_{\varphi-1} b v_{\varphi+1} \ldots v_{2N}) + \delta_\varphi(v_{\varphi-1} b v_{\varphi+1} \ldots v_{2N}). \]

Therefore, continuing the above and using Claim 4.38:

\[ \text{Tr}[\Lambda_\theta(\sigma^\theta - \sigma^\varphi)] \geq -2\sqrt{\epsilon} + \sum_v \text{Tr}[F_v^{(0)} (\alpha^{\theta,v} - \alpha^{\varphi,v})] - O(N^{3/4} \gamma_T^{1/8}) - \text{negl}(\lambda). \] (4.192)

Using \( \text{Tr}[\Lambda_\theta(\sigma^\theta - \sigma^\varphi)] = \text{negl}(\lambda) \) from Eq. (4.189), \( \epsilon = O(N^{3/2} \gamma_T^{1/4}) \) from Eq. (4.169), and the definition \( \delta(v) := \text{Tr}[F_v^{(0)} (\alpha^{\theta,v} - \alpha^{\varphi,v})] \) from Eq. (4.187), this means

\[ \sum_v \delta(v) \leq O(N^{3/4} \gamma_T^{1/8}), \] (4.193)

which establishes the second part of the lemma.

The third part of the lemma follows from its first and second parts and the triangle inequality for computational indistinguishability (Lemma 2.13).

Consider the fourth part of the lemma. By Lemma 4.35 and the third part of the lemma, we have

\[ \forall \sigma^{\theta,v} \bar{\sigma} \approx_{(N^{3/4} \gamma_T^{1/4})} \sum_v \tau^{\theta,v} \otimes \alpha^{\theta,v} \approx_{2^{2N \cdot (N^{7/4} \gamma_T^{1/8})}} \sum_v \tau^{\theta,v} \otimes \alpha^{1,2^{2N}} = 1^2 \otimes 2^{2N} \otimes \alpha, \] (4.194)

where \( \alpha := \alpha^{1,2^{2N}} \). Note that the first approximation in Eq. (4.194) also holds in the sense of computational indistinguishability up to \( O(N^{3/4} \gamma_T^{1/4}) \) (see [GV19, Lemma 2.4]). Therefore, by the triangle inequality for computational indistinguishability (Lemma 2.13) and dropping lower order terms, we obtain

\[ \forall \sigma^{\theta,v} \bar{\sigma} \approx_{2^{2N \cdot (N^{7/4} \gamma_T^{1/8})}} 1^2 \otimes 2^{2N} \otimes \alpha. \] (4.195)

Hence the fourth part of the lemma. \[ \square \] Proof of Lemma 4.37.

Lemma 4.35 and Lemma 4.37 characterize the states \( \sigma^{\theta,v} \) for \( \theta \in [2N] \). For \( \theta = \odot \), we have

**Lemma 4.39.** Let \( D \) be an efficient perfect device. For all \( v \in \{0,1\}^{2N} \), there exists a positive semi-definite operator \( \alpha^{\odot,v} \) such that

\[ \sum_v \| \forall \sigma^{\odot,v} \bar{\sigma} \|_1 \leq O(N^{1/2} + N^{11/16} \gamma_T^{1/128}). \] (4.196)

In addition, let \( \alpha \) be as defined in Lemma 4.37. Then, we have \( \alpha^{\odot,v} \approx_{\delta(v)} \alpha \), for some \( \delta(v) \geq 0 \) such that

\[ \sum \delta(v) \leq O(2^{2N} N^{7/4} \gamma_T^{1/32} + N \gamma_T^{1/2}). \] (4.197)
Proof. The proof is similar to the proof of Lemma 4.35. We provide the details for completeness.

We will use the fact that, for all \( v \in \{0, 1\}^{2N} \), we have

\[
|\psi^v\rangle \langle \psi^v| \otimes 1 = \prod_{i=1}^{N} (\sigma^Z_i \otimes \sigma^X_{N+i} \otimes 1)^{v(i)} (\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_N+i)}. \tag{4.198}
\]

For all \( i \in [N] \), we have

\[
\|((\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_N+i)} \otimes 1 - 1\|_{\psi^o,v,\psi^v}^2 = \text{Tr}[\mathcal{V}^{\psi^o,v\psi^v}] - \text{Tr}[(\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_N+i)} \mathcal{V}^{\psi^o,v\psi^v}] = \text{Tr}[\sigma^o,v] - \text{Tr}[\mathcal{V}^{\psi^o,v\psi^v}] \leq \text{Tr}[\sigma^o,v] - \text{Tr}[(\tilde{x}_o, z_{N+i})^{(v_N+i)} \sigma^o,v] + O(\sqrt{\text{Tr}[\sigma^o,v] \cdot \tilde{c}_o(i, v)}) = \tilde{x}_o(i, v) + O(\sqrt{\text{Tr}[\sigma^o,v] \cdot \tilde{c}_o(i, v)}), \tag{4.199}
\]

where the inequality uses the replacement lemma (Lemma 2.6) with the definition of \( \tilde{x}_o \) in Eq. (4.157) and Lemma 2.11, and the last equality is by Eq. (4.45). Similarly, we have

\[
\|((\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_i)} \otimes 1 - 1\|_{\psi^o,v,\psi^v}^2 \leq \tilde{c}_o(i, v) + O(\sqrt{\text{Tr}[\sigma^o,v] \cdot \tilde{c}_o(i, v)}). \tag{4.200}
\]

Using Lemma 2.8 with Eqs. (4.199) and (4.200), we obtain

\[
\|
\mathcal{V}^{\psi^o,v\psi^v} - (\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_N+i)} \mathcal{V}^{\psi^o,v\psi^v} (\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_N+i)}\|_1 \leq \epsilon_1(i, v),
\]

\[
\|
\mathcal{V}^{\psi^o,v\psi^v} - (\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_i)} \mathcal{V}^{\psi^o,v\psi^v} (\sigma^X_i \otimes \sigma^Z_{N+i} \otimes 1)^{(v_i)}\|_1 \leq \epsilon_2(i, v), \tag{4.201}
\]

where

\[
\epsilon_1(i, v) := O\left(\left(\tilde{x}_o(i, v) + \sqrt{\text{Tr}[\sigma^o,v] \cdot \tilde{c}_o(i, v)}\right)^{1/2} \sqrt{\text{Tr}[\sigma^o,v]}\right),
\]

\[
\epsilon_2(i, v) := O\left(\left(\tilde{c}_o(i, v) + \sqrt{\text{Tr}[\sigma^o,v] \cdot \tilde{c}_o(i, v)}\right)^{1/2} \sqrt{\text{Tr}[\sigma^o,v]}\right). \tag{4.202}
\]

Using Eq. (4.198) and Lemma 2.12 with Eq. (4.201) gives

\[
\|
\mathcal{V}^{\psi^o,v\psi^v} - |\psi^v\rangle \langle \psi^v| \mathcal{V}^{\psi^o,v\psi^v} |\psi^v\rangle \langle \psi^v| \|_1 \leq \sum_{i=1}^{N} \epsilon_1(i, v) + \epsilon_2(i, v). \tag{4.203}
\]

Therefore, taking a sum over \( v \) gives

\[
\sum_v \|
\mathcal{V}^{\psi^o,v\psi^v} - |\psi^v\rangle \langle \psi^v| \mathcal{V}^{\psi^o,v\psi^v} |\psi^v\rangle \langle \psi^v| \|_1 \leq \sum_{i=1}^{N} \sum_v \epsilon_1(i, v) + \epsilon_1(i, v) \leq O(N\gamma_o^{1/2} + N^{17/16} \gamma^{1/32}). \tag{4.204}
\]

But \( |\psi^v\rangle \langle \psi^v| \mathcal{V}^{\psi^o,v\psi^v} |\psi^v\rangle \langle \psi^v| \) is of the form \( \tau^{o,v} \otimes \alpha^{o,v} \) for some positive semi-definite operator \( \alpha^{o,v} \). Hence the first part of the lemma.

For the “in addition” part, consider

\[
\sum_v \tau^{o,v} \otimes \alpha^{o,v} \approx (N_o^{1/2} + N^{17/16} \gamma^{1/32}) \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \approx \text{negl}(\lambda) \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \approx \tilde{c}^{2N N^{\gamma/4} \gamma^{1/32} + N \gamma^{1/2}} \mathcal{V}^{\psi^o,v\psi^v}. \tag{4.205}
\]

where the last approximation is from Lemma 4.37. Therefore, the triangle inequality for computational indistinguishability (Lemma 2.13) gives

\[
\sum_v \tau^{o,v} \otimes \alpha^{o,v} \approx \tilde{c}^{2N N^{\gamma/4} \gamma^{1/32} + N \gamma^{1/2}} \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \approx \text{negl}(\lambda) \mathcal{V}^{\psi^o,v\psi^v} \mathcal{V}^{\psi^o,v\psi^v} \approx \tilde{c}^{2N N^{\gamma/4} \gamma^{1/32} + N \gamma^{1/2}} \mathcal{V}^{\psi^o,v\psi^v}. \tag{4.206}
\]
Now, suppose $\alpha^{\otimes v}$ is computationally distinguishable from $\alpha$ with advantage $\delta(v) \geq 0$. This means that there exists an efficient POVM $\{E_v, 1 - E_v\}$ such that $\text{Tr}[E_v(\alpha^{\otimes v} - \alpha)] = \delta(v)$. We define $\Gamma := \sum_v \tau^{\otimes v} \otimes E_v$. \{\Gamma, 1 - \Gamma\} is an efficient POVM by an argument similar to that below Eq. (4.174). We have
\[
\text{Tr} \left[ \Gamma \left( \sum_v \tau^{\otimes v} \otimes \alpha^{\otimes v} - \frac{1}{2} \otimes 2N \otimes \alpha \right) \right] = \text{Tr} \left[ \sum_v \tau^{\otimes v} \otimes E_v (\alpha^{\otimes v} - \alpha) \right] = \sum_v \delta(v).
\] (4.207)

Therefore, Eq. (4.206) implies
\[
\sum_v \delta(v) \leq O(2^{2N} N^{7/4} \gamma_T^{1/32} + N \gamma_0^{1/2}).
\] (4.208)

\section{4.9 Soundness for states and measurements}

We now put everything together to give our main theorem. First, in most of the preceding analysis, we have considered the prover as an efficient \textit{perfect} device in the knowledge that the states $\psi^0$ of the actual device are equal to those of a perfect device up to error $\sqrt{\gamma_T}$ in trace distance, see Proposition 4.16. In fact, $\gamma_P$ also controls how close the states $\sigma^{\otimes v}$ of the actual device are to those of a perfect device:

**Lemma 4.40.** Let $D$ be an efficient device and $\tilde{D}$ the perfect device associated with $D$ according to Proposition 4.16. Respectively, let $\sigma^{\otimes v}, \tilde{\sigma}^{\otimes v} \in \text{Pos}(\mathcal{H})$ be the states prepared by $D$ and $\tilde{D}$ in the Hadamard round as defined in Definition 4.4. Then, for all $\theta \in [2N] \cup \{0, 1\}$, we have
\[
\sum_{v \in \{0, 1\}^{2N}} \|\sigma^{\otimes v} - \tilde{\sigma}^{\otimes v}\|_1 \leq \sqrt{\gamma_P} + 2\gamma_P.
\] (4.209)

**Proof.** Since $\sigma^0$ is obtained from $\psi^0$ by a quantum channel, which cannot increase trace distance, Proposition 4.16 implies $\|\sigma^0 - \tilde{\sigma}^0\|_1 \leq \sqrt{\gamma_P}$. As the operators in $\{\sigma^{\otimes v} - \tilde{\sigma}^{\otimes v}\}_{v \in \{0, 1\}^{2N}}$ are Hermitian and pairwise multiply to 0 by Definition 4.4, we have
\[
\sum_{v \in \{0, 1\}^{2N}} \|\sigma^{\otimes v} - \tilde{\sigma}^{\otimes v}\|_1 = \left\| \sum_{v \in \{0, 1\}^{2N}} (\sigma^{\otimes v} - \tilde{\sigma}^{\otimes v}) \right\|_1 \leq \|\sigma^0 - \tilde{\sigma}^0\|_1 + 2\gamma_P \leq \sqrt{\gamma_P} + 2\gamma_P,
\] (4.210)

where the first equality uses Lemma 2.7 and the first inequality uses Lemma 4.7. Hence the lemma. \qed

To state our theorem, we recall the definition of $\tau^{\otimes v}$ from Definition 4.34: $\tau^{\otimes v} := |\tau^{\otimes v}\rangle \langle \tau^{\otimes v}|$, where
\[
|\tau^{\otimes v}\rangle := \begin{cases} |v_0\rangle \otimes \cdots \otimes |v_{2N}\rangle & \text{if } \theta \in [2N], \\ |v\rangle := |v_1\rangle \otimes \cdots \otimes |v_{2N}\rangle & \text{if } \theta = 0, \\ |\psi\rangle & \text{if } \theta = 1. \end{cases}
\] (4.211)

We also write, for all $u \in \{0, 1\}^{2N}$, and $q \in \{0, 1, 2, 3\}$, $|u(q)\rangle$ for the $2N$-qubit state
\[
|u(q)\rangle := \begin{cases} |u_1, u_2, \ldots, u_{2N}\rangle & \text{if } q = 0, \\ H^{\otimes 2N} |u_1, u_2, \ldots, u_{2N}\rangle & \text{if } q = 1, \\ |u_1, u_2, \ldots, u_{2N}\rangle H^{\otimes N} |u_{N+1}, \ldots, u_{2N}\rangle & \text{if } q = 2, \\ (H^{\otimes N} |u_1, u_2, \ldots, u_{2N}\rangle) |u_{N+1}, \ldots, u_{2N}\rangle & \text{if } q = 3. \end{cases}
\] (4.212)

Note that $\Pi^u_q = |u(q)\rangle \langle u(q)|$ for $q \in \{0, 1, 2, 3\}$ and $u \in \{0, 1\}^{2N}$, where $\Pi^u_q$ is as defined in the introduction. We finally stress that the theorem holds under the LWE hardness assumption made throughout this work.

**Theorem 4.41.** Let $D$ be an efficient device. Let $\mathcal{H} := \mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R$ be the Hilbert space of $D$. Let $\mathcal{Y} : \mathcal{H} \rightarrow \mathbb{C}^{2^{2N}} \otimes \mathcal{H}$ be the swap isometry defined in Definition 4.30. For $\theta \in [2N] \cup \{0, 1\}$ and $v \in \{0, 1\}^{2N}$, let $\sigma^{\otimes v} \in \text{Pos}(\mathcal{H})$ be the states that $D$ prepares after returning the first answer in the Hadamard round, as defined in Definition 4.4. Let $\{E^u_q\}_{u \in \{0, 1\}^{2N}}$ be the measurements $D$ performs to return the second answer in the Hadamard round when asked question $q$.
Suppose that \( D \) fails the protocol in Fig. 2 with probability at most \( \epsilon \). Then, there exist states 
\( \alpha^{\theta,v} \mid \theta \in [2N] \cup \{0,1\}, v \in \{0,1\}^{2N} \) that are computationally indistinguishable in the way specified in Lemma 4.37 (if \( \theta \in [2N] \cup \{0\} \)) and Lemma 4.39 (if \( \theta = \infty \)) such that

\[
\sum_{v \in \{0,1\}^{2N}} \| \mathcal{V} \sigma^{\theta,v} \|^1 \otimes \| v \rangle\langle v \| \approx N^{7/2} \epsilon^{1/16}, \tag{4.213}
\]

\[
\sum_{u,v \in \{0,1\}^{2N}} \| \mathcal{V} P^u_q \sigma^{\theta,v} \|^1 P^u_q \otimes \| u \rangle\langle u \| \approx N^{7/2} \epsilon^{1/16}, \tag{4.214}
\]

**Proof.** Let \( \bar{D} \) be the perfect device associated with \( D \) according to Proposition 4.16. Let \( \tilde{\sigma}^{\theta,v} \) be the states of \( \bar{D} \) corresponding to \( \sigma^{\theta,v} \) of \( D \).

We first prove Eq. (4.213), which characterizes the pre-measurement states \( \sigma^{\theta,v} \). We have

\[
\sum_{v} \| \mathcal{V} \sigma^{\theta,v} \|^1 - \| \tilde{\sigma}^{\theta,v} \|^1 \| \otimes \| v \rangle\langle v \| \leq \sum_{v} \| \mathcal{V} \sigma^{\theta,v} \|^1 - \| \mathcal{V} \tilde{\sigma}^{\theta,v} \|^1 \|_1 + \sum_{v} \| \mathcal{V} \tilde{\sigma}^{\theta,v} \|^1 - \| \sigma^{\theta,v} \|^1 \|_1 \
\leq \sqrt{7} \gamma P + O(N^{3/2} \gamma T^{1/4} + N^{1/2} \gamma T^{1/4} + N^{17/16} \gamma T^{1/32}),
\]

where, in the last inequality, we used Lemma 4.40 to bound the first term and added the bounds in Lemmas 4.35 and 4.39 to bound the second term in a way that is independent of \( \theta \).

To get a bound in terms of \( N \) and the failure probability, \( \epsilon \), we can use Proposition 4.9, which says \( \gamma P, \gamma T, \gamma_0 = O(\sqrt{N} \epsilon) \). Therefore, we obtain

\[
\sum_{v} \| \mathcal{V} \sigma^{\theta,v} \|^1 - \| \tilde{\sigma}^{\theta,v} \|^1 \| \leq O(N^{7/4} \epsilon^{1/32}). \tag{4.216}
\]

Hence Eq. (4.213). (The approximation error is squared due to the definition of \( \approx \) for states in Definition 2.3.)

We now prove Eq. (4.214), which characterizes the post-measurement states \( P^u_q \sigma^{\theta,v} P^u_q \|^1 \). We have

\[
\sum_{u,v} \| \mathcal{V} P^u_q \sigma^{\theta,v} P^u_q \|^1 - \| \tilde{\sigma}^{\theta,v} \|^1 \| \otimes \| u \rangle\langle u \| \leq \sum_{u,v} \| \mathcal{V} P^u_q \sigma^{\theta,v} P^u_q \|^1 - \| \mathcal{V} \tilde{\sigma}^{\theta,v} \|^1 \|_1 + \sum_{u,v} \| \mathcal{V} \tilde{\sigma}^{\theta,v} \|^1 - \| \sigma^{\theta,v} \|^1 \|_1 \
\leq \sqrt{7} \gamma P + 2 \gamma P + O(N^{3/2} \gamma T^{1/4} + N^{1/2} \gamma T^{1/4} + N^{17/16} \gamma T^{1/32}),
\]

where the second inequality uses part 4 of the replacement lemma (Lemma 2.6) and the third inequality uses Lemma 4.40.

Equation (4.217) allows us to assume that \( D \) is already perfect and add on \( \sqrt{7} \gamma P + 2 \gamma P \) to the bound we derive on the top expression in Eq. (4.217) only at the end. Henceforth, we assume \( D \) is perfect until indicated otherwise.

We only provide a proof of the second part of the theorem for the case \( q = 1 \) because the proof for the case \( q \in \{0,2,3\} \) is analogous and will give no worse bound (up to constants). The bound would be no worse for \( q \in \{0,2,3\} \) because, in Lemma 4.32, the characterization of \( Z_k, \bar{Z}_k, \bar{X}_k \) (which corresponds to \( q \in \{0,2,3\} \)) is no worse than the characterization of \( X_k \) (which corresponds to \( q = 1 \)).

For \( u \in \{0,1\}^{2N} \), we have

\[
P^u_1 = \prod_{i \in [2N]} X_{i}(u_i). \tag{4.218}
\]

Now, Lemma 4.32 and Lemma 2.10 gives: for all \( i \in [2N] \) and \( \theta \in [2N] \cup \{0,1\} \), \( \mathcal{V} X_i \|^1 \approx_{\delta, \mathcal{V} \sigma^{\theta,v} \|^1} \sigma_i^{X_i} \otimes 1 \), where \( \delta := (N \sqrt{T \gamma})^{1/2} \). Therefore, by Lemma 2.11, we have

\[
\mathcal{V} X^{(u_i)}_i \|^1 \approx_{\delta, \mathcal{V} \sigma^{\theta,v} \|^1} |(-u_i)\rangle\langle(-u_i)| \otimes 1. \tag{4.219}
\]
We write
\[ X_j[u] := X_{2N}^{(u_2N)} X_{2N-1}^{(u_{2N-1})} \cdots X_1^{(u_2N)} = X_j^{(u_j)} X_{j+1}^{(u_{j+1})} \cdots X_{2N}^{(u_{2N})}, \quad \text{(4.220)} \]
and recall \(|(-)_{2N}^{(u_2N)} (\cdots)_{2N-1}^{(u_{2N-1})} \cdots (\cdots)_{2N}^{(u_{2N})} = \sigma^2(j-1)|(-)_{2N}^{(u_2N)} (\cdots)_{2N-1}^{(u_{2N-1})} \cdots (\cdots)_{2N}^{(u_{2N})} \rangle \langle |(-)_{2N}^{(u_2N)} (\cdots)_{2N-1}^{(u_{2N-1})} \cdots (\cdots)_{2N}^{(u_{2N})}| \rangle \text{ from Eq. (4.156)).}

Then, by direct calculation, we have
\[
\| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \| \quad \text{and Eq. (4.221)} \quad \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \|.
\]
\[ \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \| \quad \text{and Eq. (4.221)} \quad \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \|.
\]
\[ \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \| \quad \text{and Eq. (4.221)} \quad \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \|.
\]
\[ \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \| \quad \text{and Eq. (4.221)} \quad \| V X_1[u] \| \leq \sum_{j=1}^{\infty} \left( |(-)_{2N}^{(u_2N)} \langle \cdots \langle u \cdots \rangle \rangle \right)^{1/2} \sigma^2 \| \| V \|. \]

We proceed to show that Eq. (4.221) summed over \( u \) is close to zero. We do this by showing that the vector
\[
\sum_u \text{Re} \text{Tr}[[(-)^{u}] ((-)^{u} \otimes 1) V X_1[u] \sigma^\theta V^\dagger] | u \rangle
\]
is approximately equal to both
\[
\sum_u \text{Tr}[[X_1[u] \sigma^\theta] | u \rangle \quad \text{and} \quad \sum_u \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) V X_1[u] \sigma^\theta V^\dagger] | u \rangle
\]
in \( \ell_1 \)-norm distance.

In the following, we recall that, for vectors \( x, y \in \mathbb{C}^n \), we defined \( x \approx_y \) \( y \) to mean \( \| x - y \|_1 \leq O(\epsilon) \), where \( \| \cdot \|_1 \) denotes the \( \ell_1 \)-norm. We give justifications for the steps below that do not involve an exact equality afterwards. The exact equalities follow from \( V^\dagger V = \mathbb{1} \), \( X_1[u] = X_2[u] X_1^{(u_1)} \), \( ((-)^{u_1}) (-)^{u_1} | 1 \rangle = ((-)^{u_1}) (-)^{u_1} | 1 \rangle = ((-)^{u_1}) (-)^{u_1} | 1 \rangle \), and the cyclicity of the trace.

\[
\sum_u \text{Re} \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) V X_1[u] \sigma^\theta V^\dagger] | u \rangle
\]
\[ = \sum_u \text{Re} \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) (V X_2[u] X_1^{(u_1)}) V^\dagger (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ \approx \text{neglect(\lambda)} \sum_u \text{Re} \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) (V X_2[u] X_1^{(u_1)}) V^\dagger (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ \approx \sqrt{\frac{\lambda}{n}} \sum_u \text{Re} \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) (V X_2[u] X_1^{(u_1)}) V^\dagger (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ = \sum_u \text{Re} \text{Tr}[[V X_1^{(u_1)} V^\dagger] ((-)^{u} ((-)^{u} \otimes 1) (V X_2[u] V^\dagger) (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ \approx \sqrt{\frac{\lambda}{n}} \sum_u \text{Re} \text{Tr}[[((-)^{u_1}) ((-)^{u_1} \otimes 1) ((-)^{u_1} \otimes 1) (V X_2[u] V^\dagger) (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ = \sum_u \text{Re} \text{Tr}[[((-)^{u_1}) ((-)^{u_1} \otimes 1) ((-)^{u_2} \otimes 1) ((-)^{u_2} \otimes 1) (V X_2[u] V^\dagger) (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ \approx \text{neglect(\lambda)} \sum_u \text{Re} \text{Tr}[[((-)^{u_1}) ((-)^{u_1} \otimes 1) ((-)^{u_2} \otimes 1) ((-)^{u_2} \otimes 1) (V X_2[u] V^\dagger) (V \sigma^\theta V^\dagger) | u \rangle
\]
\[ = \sum_u \text{Re} \text{Tr}[[((-)^{u}) ((-)^{u} \otimes 1) (V X_2[u] \sigma^\theta V^\dagger) | u \rangle
\]

We justify the four steps above that do not involve an exact equality as follows:

1. In the third line, we use the “moreover” part of the lifting-under-projections lemma (Lemma 2.17) with \( Q^\omega = \mathbb{1}, P^\omega = ((-)^{u}) ((-)^{u} \otimes 1), \psi = V X_1[u] V^\dagger, \psi' = V \sigma^\theta V^\dagger, \) and \( \psi'' = V \sigma^\theta V^\dagger. \)

2. In the fourth line, we use part 4 of the replacement lemma (Lemma 2.6) with \( P^\omega = ((-)^{u}) ((-)^{u} \otimes 1), \)
\[ \psi = \frac{1}{2} V X_2[u] X_1^{(u_1)} V^\dagger, \quad \text{and} \quad \psi' = \frac{1}{2} V X_2[u] \sigma^\theta X_1^{(u_1)} V^\dagger, \]
\[ \psi'' = \frac{1}{2} V X_2[u] \sigma^\theta X_1^{(u_1)} V^\dagger. \]

Note that \( \| \psi - \psi' \|_1 \leq O(\sqrt{\lambda}) \) follows from \( \| V \|_\infty = \| X_2[u] \|_\infty = 1 \) and Proposition 4.29.
3. In the sixth line, we use part 3 of the replacement lemma (Lemma 2.6) with $A = \mathcal{V}X_i^{(u_i)}\psi^\dagger$, $B = \langle(-u_i)|(-u_i)_1 \otimes 1, X_i^{(u_i)} = \langle(-u_i)|(-u_i)_1 \otimes 1$ for $i \in [2N]$, $Y_i = \mathcal{V}X_i^{(u_i)}\psi^\dagger$ for $i \in \{2, \ldots, 2N\}$, and $\psi = \mathcal{V}_{\sigma^1}\psi$. Importantly, we use $A \approx_{\delta', \psi} B$ from Eq. (4.219).

4. In the second-to-last line, we again use the “moreover” part of the lifting-under-projections lemma (Lemma 2.17), this time with $Q^u = \langle(-u)\rangle(-u)_1 \otimes 1, \Pi^u = \langle(-u_2)_2 \otimes 1, P^u = \mathcal{V}X_2[u]V^\dagger, \psi = \mathcal{V}_{\sigma^1}\psi, \psi' = \mathcal{V}_{\sigma^\theta}\psi$.

Comparing the expressions in the first and last lines, we can see that we can change in the same way another $2N - 1$ times, such that $\sigma^\theta$ is replaced by $\sigma^k$ by the lifting-under-projections lemma (Lemma 2.17) at time $k$, to obtain

$$\sum_u \text{Re} \text{Tr}([\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}X_1[u]\sigma^\theta V^\dagger |u), (4.242)$$

where

$$\delta' := N(\sqrt{\delta} + \sqrt{\gamma_T} + \text{negl}(\lambda)) \leq O(N^{5/4}\gamma_T^{1/8}). \quad (4.225)$$

But $\text{Re} \text{Tr}([\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}_{\sigma^\theta} V^\dagger = \text{Tr}([\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}_{\sigma^\theta} V^\dagger |\geq 0$. Therefore,

$$\sum_u \text{Re} \text{Tr}([\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}X_1[u]\sigma^\theta V^\dagger |u) \approx_{\delta'} \sum_u \text{Tr}[\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}_{\sigma^\theta} V^\dagger |u). \quad (4.226)$$

By an analogous argument, we can “merge” $\langle(-u)\rangle(-u)$ into $\mathcal{V}X_1[u]V^\dagger$ (instead of the other way round as done above) to obtain

$$\sum_u \text{Re} \text{Tr}([\langle(-u)\rangle(-u) \otimes 1] \mathcal{V}X_1[u]\sigma^\theta V^\dagger |u) \approx_{\delta'} \sum_u \text{Tr}[\mathcal{V}X_1[u]\sigma^\theta |u). \quad (4.227)$$

Substituting Eqs. (4.226) and (4.227) into Eq. (4.221) gives

$$\sum_u \|\mathcal{V}X_1[u]V^\dagger - \langle(-u)\rangle(-u) \otimes 1\|_{\mathcal{V}_{\sigma^\theta} V^\dagger}^2 \leq O(\delta'). \quad (4.228)$$

Then, applying Lemma 2.5 to Eq. (4.228) gives

$$\sum_{u,v}^2 \|\mathcal{V}X_1[u]V^\dagger - \langle(-u)\rangle(-u) \otimes 1\|_{\mathcal{V}_{\sigma^\theta} V^\dagger}^2 \leq O(\delta'). \quad (4.229)$$

Therefore, we have

$$\sum_{u,v} \mathcal{V}P_1^u\sigma^\theta,v P_1^v \mathcal{V}^\dagger |u, v\rangle\langle u, v| \leq \sum_{u,v} (\mathcal{V}X_1[u]V^\dagger)(\mathcal{V}_{\sigma^\theta,v} V^\dagger)(\mathcal{V}X_1[u]V^\dagger) \otimes |u, v\rangle\langle u, v| \quad (\text{Eq. (4.218) and } \mathcal{V}^\dagger \mathcal{V} = 1)$$

$$\approx_{\delta'} \sum_{u,v} \langle(-u)\rangle(-u) \otimes 1) \mathcal{V}_{\sigma^\theta,v} V^\dagger (\mathcal{V}X_1[u]V^\dagger) \otimes |u, v\rangle\langle u, v| \quad \left(\text{Part 1 of Lemma 2.9 with Eq. (4.229)}\right)$$

$$\approx_{\delta'^2} \sum_{u,v} \langle(-u)\rangle(-u) \otimes 1) \tau^\theta,v \otimes \alpha^\theta,v (-u) \otimes 1) \otimes |u, v\rangle\langle u, v| \quad \left(\text{Part 2 of Lemma 2.9 with Lemmas 4.35 and 4.39}\right),$$

where

$$\delta'^2 := N^{3/2} \gamma_T^{1/4} + N^{5/2} \gamma_T^{1/2} + N^{17/16} \gamma_T^{1/32}. \quad (4.230)$$

Note that the above approximations are for operators and we recall that for two operators $A$ and $B$, we defined $A \approx_{\epsilon} B$ to mean $\|A - B\|_1^2 \leq O(\epsilon)$, where we stress the square on the Schatten 1-norm. Therefore, also recalling $|u^{(1)}\rangle := (-u)^{\dagger}$, we can use the triangle inequality to obtain

$$\sum_{u,v} \|\mathcal{V}P_1^u\sigma^\theta,v P_1^v \mathcal{V}^\dagger - |u^{(1)}\rangle\langle u^{(1)}| |u^{(1)}\rangle\langle u^{(1)}| \otimes \alpha^\theta,v \|_1 \leq O(\sqrt{\delta'} + \delta''). \quad (4.231)$$
At this point, we drop the assumption that $D$ is perfect. In this case, as explained at the start of proof of Eq. (4.214), we obtain the bound

$$\sum_{u,v} \| u P_1^{\delta,v} P_1^{\delta,u} v \|_1 - \langle u^{(1)} | \tau^{\delta,u} v^{(1)} \rangle = \langle u^{(1)} | (u^{(1)} \otimes \sigma^{\delta,u}) \rangle \leq \sqrt{\gamma P} + 2 \gamma P + O(\sqrt{\delta'} + \delta'')$$

(4.232)

where we substituted the definitions of $\delta'$ and $\delta''$, given in Eqs. (4.225) and (4.230) respectively, in the second inequality and used Proposition 4.9 in the last inequality. Hence the theorem.

5 Dimension-testing protocol

In this section, we present a protocol in Fig. 5 for testing the quantum dimension of an efficient device. This protocol is a simplified version of the self-testing protocol in Fig. 2.

1. Set $N = \lambda$. Select $\theta \leftarrow U\{0, 1, \ldots, N\}$, sample $N$ key-trapdoor pairs $(k_1, t_{k_1}), \ldots, (k_N, t_{k_N})$ from an ENTCF such that, if $\theta = 0$, all pairs are sampled from $Gen_q(1^\lambda)$, otherwise the $\theta$-th key-trapdoor pair is sampled from $Gen_x(1^\lambda)$ and the remaining $N - 1$ pairs are all sampled from $Gen_q(1^\lambda)$.

2. Send the keys $k = (k_1, \ldots, k_N)$ to the prover.

3. Receive $y = (y_1, \ldots, y_N) \in Y^N$ from the prover.

4. Select round type “preimage” or “Hadamard” uniformly at random and send to the prover.

- **Case “preimage”:** receive
  
  $$(b, x) = (b_1, \ldots, b_N, x_1, \ldots, x_N)$$
  
  from the prover, where $b \in \{0, 1\}^N$ and $x \in \{0, 1\}^{Nw}$.

  If $CHK(k_i, y_i, b_i, x_i) = 0$ for all $i \in [N]$, **accept**, else **reject**.

- **Case “Hadamard”:** receive
  
  $$d = (d_1, \ldots, d_N) \in \{0, 1\}^{Nw}$$
  
  from the prover.

  Sample $q \leftarrow U\{0, 1\}$ and send $q$ to the prover.

  Receive $v = \{0, 1\}^N$ from the prover.

  - **Case A.** $\theta = 0$ and
    
    $q = 0$: if $b(k_i, y_i) \neq v_i$ for some $i \in [N]$, **reject**, else **accept**.
    
    $q = 1$: **accept**.

  - **Case B.** $\theta \in [N]$ and
    
    $q = 0$: if $b(k_i, y_i) \neq v_i$ for some $i \neq \theta$, **reject**, else **accept**.
    
    $q = 1$: if $b(k_\theta, y_\theta, d_\theta) \neq v_\theta$, **reject**, else **accept**.

Figure 5: A protocol that tests the quantum dimension of a computationally efficient device.

There exists an efficient honest quantum prover that is accepted by the dimension-testing protocol with probability $1 - \negl(\lambda)$. The strategy of this prover follows that described in the proof of Theorem 3.1, except it is simpler because there is no need to apply controlled-$Z$ gates. In particular, when $q = 0$, the honest prover measures $N$ qubits in the computational basis, and when $q = 1$, it measures $N$ qubits in the Hadamard basis. Moreover, there exists an efficient classical verifier, which again follows from the efficient function generation and efficient decoding properties of ENTCFs. We omit a formal completeness proof.
The intuition behind the soundness of this protocol is that, when it is passed with high probability, Proposition 4.41 guarantees the existence of a quantum state $\rho^*$ on the quantum part of the prover’s memory that is close to the maximally mixed state up to some isometry. More specifically, $\rho^*$ comes from using Theorem 4.41 to force the prover to perform a Hadamard basis measurement on $N$ qubits that are in a computational basis state. Then, the main proposition of this section, Proposition 5.2, shows that the guarantee on $\rho^*$ is strong enough for us to lower bound the rank of $\rho^*$, which is also a lower bound on the quantum dimension of the prover’s memory. We proceed to give a formal proof of soundness and stress that we are making the LWE hardness assumption throughout the rest of this section.

To prove Proposition 5.2, we use the vector-operator correspondence mapping, vec, as defined in [Wat18, Chapter 1.1.2]. Let $A \in \mathcal{L}(\mathcal{H})$. Informally, vec$(A) \in \mathcal{H} \otimes \mathcal{H}$ is the column vector formed by concatenating the rows of $A$ vertically. We will use the following properties of vec without further comment:

1. Suppose $A$ is of the form $\sum_i \lambda_i |v_i\rangle\langle u_i|$ with $\lambda_i \geq 0$. Then, vec$(\sqrt{A}) = \sum_i \sqrt{\lambda_i} |v_i\rangle\langle u_i|$, where the overline denotes element-wise complex conjugation.
2. The $\ell_2$-norm of vec$(A)$ equals the Frobenius norm of $A$. In symbols, $\|\text{vec}(A)\| = \|A\|_F$.
3. If $B, C \in \mathcal{L}(\mathcal{H})$, then $(B \otimes C)\text{vec}(A) = \text{vec}(BAC^\dagger)$, where $^\dagger$ denotes the transpose. In particular, if $U \in \mathcal{L}(\mathcal{H})$ is unitary, then $(U \otimes \overline{U})\text{vec}(A) = \text{vec}(UA\overline{U}^\dagger)$.

We also need the following technical lemma.

**Lemma 5.1 ([Wat18, Lemma 3.34]).** Let $P_1, P_2 \in \text{Pos}(\mathcal{H})$ be positive semi-definite operators. Then,

$$\|\sqrt{P_1} - \sqrt{P_2}\|_F \leq \sqrt{\|P_1 - P_2\|_1}. \quad (5.1)$$

**Proposition 5.2.** Let $\rho, \alpha \in D(\mathcal{H})$ be density operators. If there exists a unitary $U \in \mathcal{L}(\mathbb{C}^{2^n} \otimes \mathcal{H})$ such that

$$\|U(|0\rangle\langle 0| ^\otimes n \otimes \rho)U^\dagger - 2^{-n}1 \otimes \alpha\|_1 \leq \epsilon, \quad (5.2)$$

then $\text{Rank}(\rho) \geq (1-\epsilon)2^n$.

**Proof.** Let us write the eigen-decompositions of $\rho$ and $\alpha$ as $\rho = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|$ and $\alpha = \sum_{k=1}^{N'} \mu_k |u_k\rangle\langle u_k|$, where $\lambda_i, \mu_k > 0$. Note that Rank$(\rho) = N$; $\sum_i \lambda_i = 1$ because $\rho$ is normalized; $\mu_k \leq 1$ for all $k \in [N']$ because $\alpha$ is normalized. We have

$$\text{vec}(\sqrt{0\rangle\langle 0| ^\otimes n \otimes \rho}) = \sum_{i=1}^{N} \sqrt{\lambda_i} |0\rangle ^\otimes n |v_i\rangle \otimes |0\rangle ^\otimes n |v_i\rangle, \quad (5.3)$$

$$\text{vec}(\sqrt{2^{-n}1 \otimes \alpha}) = \sum_{k=1}^{N'} \left( \sum_{x \in \{0, 1\}^n} \sqrt{\mu_k/2^n} |x\rangle |u_k\rangle \otimes |x\rangle |u_k\rangle \right).$$

Therefore, the condition of the proposition, Eq. (5.2), implies that

$$\left\| U \otimes \overline{U} \left( \sum_{i=1}^{N} \sqrt{\lambda_i} |0\rangle ^\otimes n |v_i\rangle \otimes |0\rangle ^\otimes n |v_i\rangle \right) - \sum_{k=1}^{N'} \left( \sum_{x \in \{0, 1\}^n} \sqrt{\mu_k/2^n} |x\rangle |u_k\rangle \otimes |x\rangle |u_k\rangle \right) \right\|_F \leq \epsilon.$$  

$$\left\| U |0\rangle ^\otimes n \otimes \sqrt{\rho} |0\rangle ^\otimes n |0\rangle ^\otimes n \otimes \rho |0\rangle ^\otimes n \otimes \sqrt{\rho} |0\rangle ^\otimes n |0\rangle ^\otimes n \otimes \rho |0\rangle ^\otimes n \right\|_F \\ = \left\| U |0\rangle ^\otimes n \otimes \sqrt{\rho} |0\rangle ^\otimes n |0\rangle ^\otimes n \otimes \rho |0\rangle ^\otimes n \right\|_F \leq \|U|0\rangle ^\otimes n \otimes \sqrt{\rho} |0\rangle ^\otimes n \right\|_F \leq \epsilon. \quad (5.4)$$

The key observation of this proof is the following. Let $|\alpha\rangle, |\beta\rangle$ be bipartite states on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $|\alpha\rangle$ is normalized and has Schmidt rank $R$, and the Schmidt coefficients of $|\beta\rangle$ are each at most $\delta$. Then,

$$|\langle \alpha | \beta \rangle|^2 \leq R\delta^2. \quad (5.5)$$

54
To see this, write $|\alpha\rangle = \sum_{i=1}^{R} a_i |e_i\rangle \otimes |e'_i\rangle$ and $|\beta\rangle = \sum_{j=1}^{d} b_j |f_j\rangle \otimes |f'_j\rangle$ in terms of their Schmidt decompositions, where $d := \min\{\dim \mathcal{H}, \dim \mathcal{H}'\}$. Then, by using the Cauchy-Schwarz inequality twice,

$$|\langle \alpha | \beta \rangle| \leq \|\sum_{i=1}^{R} a_i |e_i\rangle \otimes |e'_i\rangle\| \|\sum_{j=1}^{d} b_j |f_j\rangle \otimes |f'_j\rangle\|$$

$$\leq b \left( \sum_{i=1}^{R} a_i^2 \right)^{1/2} \left( \sum_{j=1}^{d} (|f_j\rangle \langle f_j|) |e'_i\rangle^2 \right)^{1/2} \leq b \sqrt{R}.$$  \hspace{1cm} (5.6)

To conclude the proof, we use the above observation as follows. Set

$$|\alpha\rangle := U \otimes U \left( \sum_{i=1}^{N} \lambda_i \frac{1}{2^n} |0\rangle \otimes |v_i\rangle \right) \quad \text{and} \quad |\beta\rangle := \sum_{k=1}^{N'} \sqrt{\mu_k/2^n} \langle u_k | \otimes |x\rangle$$

then,

$$|\langle \alpha | \beta \rangle| \geq \Re \langle \alpha | \beta \rangle = \frac{1}{2} (\|\alpha\|^2 + \|\beta\|^2 - \|\alpha\| - \|\beta\|)^2 \geq 1 - \epsilon/2.$$ \hspace{1cm} (5.8)

Now, $|\alpha\rangle$ is normalized (because $\sum \lambda_i = 1$) and has Schmidt rank $N$, and the Schmidt coefficients of $|\beta\rangle$ are each at most $\sqrt{\mu_k/2^n} \leq 2^{-n/2}$. Therefore, by the above observation,

$$N \geq (1 - \epsilon/2)^2 / (2^{-n/2}) \geq (1 - \epsilon) 2^n,$$ \hspace{1cm} (5.9)

which completes the proof. \qed

**Remark.** Some ideas behind our proof of Proposition 5.2 come from the proof of [JNV+20, Theorem 8.3].

We now use Proposition 5.2 to prove the main theorem of this section. Much of the proof is devoted to bookkeeping to ensure that the (normalized) density operator condition in Proposition 5.2 is satisfied and that we are bounding the quantum dimension.

**Theorem 5.3.** Let $D$ be an efficient device with Hilbert space $\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_Y \otimes \mathcal{H}_R$. Let the classical-quantum decomposition of $\mathcal{H}$ be $\mathcal{H}_C \otimes \mathcal{H}_Q$, so that all states and observables of $D$ on $\mathcal{H}$ are classical on $\mathcal{H}_C$, i.e., block-diagonal in a fixed basis $\{|c\rangle | c \in \{|1\rangle, |0\rangle\}^{\dim(\mathcal{H}_C)}\}$ of $\mathcal{H}_C$. If $D$ can pass the dimension test protocol of Fig. 5 with probability $\geq 1 - \epsilon$, then the quantum dimension of $D$, $\dim(\mathcal{H}_Q)$, is at least $(1 - O(N^{7/4} \epsilon^{1/32}))2^N$.

**Proof.** It suffices to assume $\epsilon = O(1/N^{56})$, else the bound on $\dim(\mathcal{H}_Q)$ holds vacuously as $N \to \infty$. In this proof, we use $\text{poly}(N, \epsilon)$ to mean a polynomial of order $O(N^{7/4} \epsilon^{1/32})$ for convenience.

We model $D$ as in Section 4.1. By following the arguments in Section 4, we can prove the following analogue of Theorem 4.41:

$$\sum_{v \in \{0,1\}^N} \sum_{u \in \{0,1\}^N} \mathcal{V} P_v^{u^0} P_v^{u^1} \otimes |u, v\rangle \langle u, v| \approx_{\text{poly}(N, \epsilon)} \sum_{v \in \{0,1\}^N} \sum_{u \in \{0,1\}^N} 2^{-N} |u^{(1)}\rangle \langle u^{(1)}| \otimes \alpha^0 \otimes |u, v\rangle \langle u, v|.$$ \hspace{1cm} (5.10)

For convenience, let us write $\sigma^v := \sigma^{0,v}$, $\tau^v := \tau^{0,v}$, and $\alpha^v := \alpha^{0,v}$. Then, Eq. (5.10) implies

$$\sum_{v \in \{0,1\}^N} \sum_{u \in \{0,1\}^N} \|\mathcal{V} P_v^{u^0} \sigma^v P_v^{u^1} \otimes 2^{-N} |u^{(1)}\rangle \langle u^{(1)}| \otimes \alpha^v\|_1 \leq \text{poly}(N, \epsilon).$$ \hspace{1cm} (5.11)

Let $S := \{v \in \{0,1\}^N \mid \text{Tr}[\sigma^v] > 0\}$. For $v \in S$, we write $\hat{\sigma}^v := \sigma^v / \text{Tr}[\sigma^v]$ and $\hat{\alpha}^v := \alpha^v / \text{Tr}[\sigma^v]$. Then,

$$\sum_{v \in S} \text{Tr}[\sigma^v] \|\mathcal{V} \left( \sum_{u \in \{0,1\}^N} P_v^{u^0} \hat{\sigma}^v P_v^{u^1} \right) \otimes 2^{-N} \sum_{u \in \{0,1\}^N} |u^{(1)}\rangle \langle u^{(1)}| \otimes \hat{\alpha}^v\|_1 \leq \text{poly}(N, \epsilon).$$ \hspace{1cm} (5.12)
Note that $\sum_{u \in \{0,1\}^N} |u^{(1)}⟩⟨u^{(1)}| = \mathbb{1}$.

Let $v_{\min}$ be the $v \in S$ that minimizes $\|\mathcal{V}(\sum_{u \in \{0,1\}^N} P_\mu^u \sigma^u P_\mu^u)\mathcal{V}^\dagger - 2^{-N} \mathbb{1} \otimes \tilde{\alpha}^v\|_1$. Then, Lemma 4.7 gives

$$\left\|\mathcal{V}\left(\sum_{u \in \{0,1\}^N} P_\mu^u \sigma^u P_\mu^u\right)\mathcal{V}^\dagger - 2^{-N} \mathbb{1} \otimes \tilde{\alpha}^v\right\|_1 \leq \text{poly}(N,\epsilon)/(1 - \gamma_P) \leq \text{poly}(N,\epsilon), \quad (5.13)$$

where the last inequality holds by Proposition 4.9 and $\epsilon = O(1/N)$.

Note that $\tilde{\alpha}^{v_{\min}}$ is not necessarily normalized, but we can show $\text{Tr}[\tilde{\alpha}^{v_{\min}}]$ is close to 1 as follows. To simplify notation, let $\sigma := \tilde{\alpha}^{v_{\min}}$ and $\tilde{\alpha} := \tilde{\alpha}^{v_{\min}}$. Then,

$$|1 - \text{Tr}[\tilde{\alpha}]| = \left\|\mathcal{V}\left(\sum_{u \in \{0,1\}^N} P_\mu^u \sigma^u P_\mu^u\right)\mathcal{V}^\dagger - 2^{-N} \mathbb{1} \otimes \tilde{\alpha}\right\|_1 \leq \left\|\mathcal{V}\left(\sum_{u \in \{0,1\}^N} P_\mu^u \sigma^u P_\mu^u\right)\mathcal{V}^\dagger - 2^{-N} \mathbb{1} \otimes \tilde{\alpha}\right\|_1 \leq \text{poly}(N,\epsilon). \quad (5.14)$$

Let $\rho := \sum_{u \in \{0,1\}^N} P_\mu^u \sigma^u P_\mu^u$. From the definition of $\mathcal{V}$ in Fig. 3, we see that there exists a unitary $U \in \mathcal{L}(\mathbb{C}^{2^n} \otimes \mathcal{H})$ of the form

$$U = \sum_{c=1}^{\dim \mathcal{H}_C} |c⟩⟨c| \otimes U_c, \quad (5.15)$$

where each $U_c \in \mathcal{L}(\mathbb{C}^{2^n} \otimes \mathcal{H}_Q)$ is unitary, such that

$$\|U(|0⟩⟨0|^{\otimes N} \otimes \rho)U^\dagger - 2^{-N} \mathbb{1} \otimes \alpha\|_1 \leq \text{poly}(N,\epsilon). \quad (5.16)$$

Here, we used the fact that the controlled-Z, and controlled-X, gates appearing in $\mathcal{V}$ are both block-diagonal in the $\{|c⟩⟨c|\}_c$ basis because $Z_i$ and $X_i$ are observables of $D$ on $\mathcal{H}$.

We can also write $\rho$ and $\alpha$ as

$$\rho = \sum_{c=1}^{\dim \mathcal{H}_C} |c⟩⟨c| \otimes \rho_c \quad \text{and} \quad \alpha = \sum_{c=1}^{\dim \mathcal{H}_C} |c⟩⟨c| \otimes \alpha_c, \quad (5.17)$$

where $\rho_c, \alpha_c \in \text{Pos}(\mathcal{H}_Q)$ are such that $\sum_c \text{Tr}[\rho_c] = \sum_c \text{Tr}[\alpha_c] = 1$.

Then, Eq. (5.16) implies

$$\sum_{c=1}^{\dim \mathcal{H}_C} \|U_c|0⟩⟨0|^{\otimes N} \otimes \rho_c U_c^\dagger - 2^{-N} \mathbb{1} \otimes \alpha_c\|_1 \leq \text{poly}(N,\epsilon). \quad (5.18)$$

Analogously to how we handled the sum over $v \in \{0,1\}^N$ in Eq. (5.11), we can use Eq. (5.18) to show that there exists some $c^* \in [\dim \mathcal{H}_C]$ and (normalized) density operators $\rho^*$ and $\alpha^*$ on $\mathcal{H}_Q$ such that

$$\|U_{c^*}|0⟩⟨0|^{\otimes N} \otimes \rho^* U_{c^*}^\dagger - 2^{-N} \mathbb{1} \otimes \alpha^*\|_1 \leq \text{poly}(N,\epsilon). \quad (5.19)$$

Finally, we apply Proposition 5.2 to obtain

$$\dim \mathcal{H}_Q \geq \text{Rank}(\rho^*) \geq (1 - \text{poly}(N,\epsilon))2^N, \quad (5.20)$$

which completes the proof. □
References

[ACGH20] Gorjan Alagic, Andrew M. Childs, Alex B. Grilo, and Shih-Han Hung. Non-interactive Classical Verification of Quantum Computation. In Theory of Cryptography, pages 153–180. Springer International Publishing, 2020. [p. 2]

[BCM+21] Zvika Brakerski, Paul Christiano, Urmila Mahadev, Umesh Vazirani, and Thomas Vidick. A cryptographic test of quantumness and certifiable randomness from a single quantum device. Journal of the ACM, 68(5), 2021. doi:10.1145/3441309. [p. 2]

[Bel64] J. S. Bell. On the Einstein Podolsky Rosen paradox. Physics Physique Fizika, 1:195–200, 1964. doi:10.1103/PhysicsPhysiqueFizika.1.195. [p. 2]

[BKM19] Spencer Breiner, Amir Kalev, and Carl A. Miller. Parallel Self-Testing of the GHZ State with a Proof by Diagrams. Electronic Proceedings in Theoretical Computer Science, 287:43–66, 2019. doi:10.4204/eptcs.287.3. [p. 2]

[BKVV20] Zvika Brakerski, Venkata Koppula, Umesh V. Vazirani, and Thomas Vidick. Simpler Proofs of Quantumness. In Proceedings of the 15th Conference on the Theory of Quantum Computation, Communication, and Cryptography (TQC), 2020. [p. 2]

[BMR92] Samuel L. Braunstein, A. Mann, and M. Revzen. Maximal violation of Bell inequalities for mixed states. Physical Review Letters, 68:3259–3261, 1992. doi:10.1103/PhysRevLett.68.3259. [p. 2]

[BY21] Anne Broadbent and Peter Yuen. Device-independent oblivious transfer from the bounded-quantum-storage-model and computational assumptions, 2021. arXiv:2111.08595 [p. 5]

[CCY20] Nai-Hui Chia, Kai-Min Chung, and Takashi Yamakawa. Classical Verification of Quantum Computations with Efficient Verifier. In Theory of Cryptography, pages 181–206. Springer International Publishing, 2020. [p. 2]

[CGJV19] Andrea Coladangelo, Alex B. Grilo, Stacey Jeffery, and Thomas Vidick. Verifier-on-a-Leash: New Schemes for Verifiable Delegated Quantum Computation, with Quasilinear Resources. In Advances in Cryptology – EUROCRYPT 2019, pages 247–277, 2019. [p. 2]

[CGS17] Andrea Coladangelo, Koon Tong Goh, and Valerio Scarani. All pure bipartite entangled states can be self-tested. Nature Communications, 8(1):15485, 2017. doi:10.1038/ncomms15485. [p. 2]

[CHSH69] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. Proposed experiment to test local hidden-variable theories. Physical Review Letters, 23(15):880, 1969. [p. 5]

[CR20] Rui Chao and Ben W. Reichardt. Quantum dimension test using the uncertainty principle, 2020. arXiv:2002.12432 [p. 4]

[CRSV17] Rui Chao, Ben W. Reichardt, Chris Sutherland, and Thomas Vidick. Overlapping Qubits. In Christos H. Papadimitriou, editor, 8th Innovations in Theoretical Computer Science Conference (ITCS 2017), volume 67 of Leibniz International Proceedings in Informatics (LIPIcs), pages 48:1–48:21, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. arXiv:1701.01062 doi:10.4230/LIPIcs.ITCS.2017.48. [p. 4]

[CRSV18] Rui Chao, Ben W. Reichardt, Chris Sutherland, and Thomas Vidick. Test for a large amount of entanglement, using few measurements. Quantum, 2:92, 2018. doi:10.22331/q-2018-09-03-92. [pp. 2, 4]

[CW12] Andrew M. Childs and Nathan Wiebe. Hamiltonian simulation using linear combinations of unitary operations. Quantum Info. Comput., 12(11–12):901–924, 2012. [p. 16]
[Fu22] Honghao Fu. Constant-sized correlations are sufficient to self-test maximally entangled states with unbounded dimension. *Quantum*, 6:614, 2022. doi:10.22331/q-2022-01-03-614. [p. 2]

[GKW+18] Koon Tong Goh, Jedrzej Kaniewski, Elie Wolfe, Tamás Vértesi, Xingyao Wu, Yu Cai, Yeong-Cherng Liang, and Valerio Scarani. Geometry of the set of quantum correlations. *Physical Review A*, 97:022104, 2018. doi:10.1103/PhysRevA.97.022104. [p. 2]

[GMP22] Alexandru Gheorghiu, Tony Metger, and Alexander Poremba. Quantum cryptography with classical communication: Parallel remote state preparation for copy-protection, verification, and more, 2022. Manuscript forthcoming. [p. 5]

[GV19] Alexandru Gheorghiu and Thomas Vidick. Computationally-Secure and Composable Remote State Preparation. In *Proceedings of the 60th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 1024–1033, 2019. doi:10.1109/FOCS.2019.00066. [pp. 2, 3, 47]

[HLG21] Shuichi Hirahara and François Le Gall. Test of Quantumness with Small-Depth Quantum Circuits. In *Proceedings of the 46th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 59:1–59:15, 2021. doi:10.4230/LIPIcs.MFCS.2021.59. [p. 2]

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP*=RE, 2020. arXiv:2001.04383. [pp. 2, 55]

[KMCVY21] Gregory D. Kahanamoku-Meyer, Soonwon Choi, Umesh V. Vazirani, and Norman Y. Yao. Classically-Verifiable Quantum Advantage from a Computational Bell Test, 2021. arXiv:2104.00687 [pp. 2, 5]

[KRR14] Yael Tauman Kalai, Ran Raz, and Ron D. Rothblum. How to delegate computations: The power of no-signaling proofs. In *Proceedings of the 46th ACM Symposium on the Theory of Computing (STOC)*, page 485–494, New York, NY, USA, 2014. Association for Computing Machinery. doi:10.1145/2591796.2591809. [p. 5]

[LG21] Zhenning Liu and Alexandru Gheorghiu. Depth-efficient proofs of quantumness, 2021. arXiv:2107.02163 [p. 2]

[Mah18] U. Mahadev. Classical Verification of Quantum Computations. In *Proceedings of the 59th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 259–267, 2018. arXiv:1804.01082 doi:10.1109/FOCS.2018.00033. [pp. 2, 3, 6, 7]

[McK17] Matthew McKague. Self-testing in parallel with CHSH. *Quantum*, 1:1, 2017. doi:10.22331/q-2017-04-25-1. [p. 2]

[MDCAF21] Tony Metger, Yfke Dulek, Andrea Wei Coladangelo, and Rotem Arnon-Friedman. Device-independent quantum key distribution from computational assumptions. *New Journal of Physics*, 2021. [pp. 2, 5]

[MTH+21] Akihiro Mizutani, Yuki Takeuchi, Ryo Hiromasa, Yusuke Aikawa, and Seiichiro Tani. Computational self-testing for entangled magic states, 2021. arXiv:2111.02700 [p. 2]

[MV21] Tony Metger and Thomas Vidick. Self-testing of a single quantum device under computational assumptions. *Quantum*, 5:544, 2021. doi:10.22331/q-2021-09-16-544. [pp. 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 27, 28]

[MY04] Dominic Mayers and Andrew Yao. Self Testing Quantum Apparatus. *Quantum Info. Comput.*, 4(4):273–286, 2004. [p. 2]

[MYS12] M McKague, T H Yang, and V Scarani. Robust self-testing of the singlet. *Journal of Physics A: Mathematical and Theoretical*, 45(45):455304, 2012. doi:10.1088/1751-8113/45/45/455304. [p. 2]
[NV17] Anand Natarajan and Thomas Vidick. A Quantum Linearity Test for Robustly Verifying Entanglement. In Proceedings of the 49th ACM Symposium on the Theory of Computing (STOC), page 1003–1015, New York, NY, USA, 2017. Association for Computing Machinery. doi:10.1145/3055399.3055468. [pp. 2, 5]

[NV18] Anand Natarajan and Thomas Vidick. Low-Degree Testing for Quantum States, and a Quantum Entangled Games PCP for QMA. In Proceedings of the 59th IEEE Symposium on Foundations of Computer Science (FOCS), pages 731–742, 2018. doi:10.1109/FOCS.2018.00075. [p. 2]

[PR92] Sandu Popescu and Daniel Rohrlich. Which states violate Bell’s inequality maximally? Physics Letters A, 169(6):411–414, 1992. doi:10.1016/0375-9601(92)90819-8. [p. 2]

[Reg09] Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. Journal of the ACM, 56(6), 2009. doi:10.1145/1568318.1568324. [p. 2]

[RUV13] Ben W. Reichardt, Falk Unger, and Umesh Vazirani. Classical command of quantum systems. Nature, 496(7446):456–460, 2013. doi:10.1038/nature12035. [p. 2]

[SB20] Ivan Šupić and Joseph Bowles. Self-testing of quantum systems: a review. Quantum, 4:337, 2020. doi:10.22331/q-2020-09-30-337. [p. 2]

[SBWS18] Pavel Sekatski, Jean-Daniel Bancal, Sebastian Wagner, and Nicolas Sangouard. Certifying the Building Blocks of Quantum Computers from Bell’s Theorem. Physical Review Letters, 121:180505, 2018. doi:10.1103/PhysRevLett.121.180505. [p. 2]

[SW87] Stephen J. Summers and Reinhard Werner. Maximal violation of Bell’s inequalities is generic in quantum field theory. Communications in Mathematical Physics, 110(2):247–259, 1987. doi:10.1007/BF01207366. [p. 2]

[TKR13] Yael Tauman Kalai, Ran Raz, and Ron D. Rothblum. Delegation for bounded space. In Proceedings of the 45th ACM Symposium on the Theory of Computing (STOC), page 565–574, New York, NY, USA, 2013. Association for Computing Machinery. doi:10.1145/2488608.2488679. [p. 5]

[Tsi87] B. S. Tsirel’son. Quantum analogues of the Bell inequalities. The case of two spatially separated domains. Journal of Soviet Mathematics, 36(4):557–570, 1987. doi:10.1007/BF01663472. [p. 2]

[Vid20] Thomas Vidick. Course FSMP, Fall’20: Interactions with Quantum Devices, 2020. Lecture notes available at: http://users.cms.caltech.edu/~vidick/teaching/fsmp/fsmp.pdf. Date accessed: 25th January 2022. [p. 2]

[VZ20] Thomas Vidick and Tina Zhang. Classical zero-knowledge arguments for quantum computations. Quantum, 4:266, 2020. doi:10.22331/q-2020-05-14-266. [p. 2]

[VZ21] Thomas Vidick and Tina Zhang. Classical Proofs of Quantum Knowledge. In Advances in Cryptology – EUROCRYPT 2021, pages 630–660, 2021. doi:10.1007/978-3-030-77886-6_22. [p. 2]

[Wat18] John Watrous. The Theory of Quantum Information. Cambridge University Press, 2018. doi:10.1017/9781316848142. [p. 54]

[Wil17] Mark M. Wilde. Quantum Information Theory. Cambridge University Press, 2017. arXiv:1106.1445v8 doi:10.1017/9781316809976.001. [p. 27]

[ZKML+21] Daiwei Zhu, Gregory D. Kahanamoku-Meyer, Laura Lewis, Crystal Noel, Or Katz, Bahaa Harraz, Qingfeng Wang, Andrew Risinger, Lei Feng, Debopriyo Biswas, Laird Egan, Alexandru Gheorghiu, Yunseong Nam, Thomas Vidick, Umesh Vazirani, Norman Y. Yao, Marko Cetina, and Christopher Monroe. Interactive Protocols for Classically-Verifiable Quantum Advantage, 2021. arXiv:2112.05156 [p. 2]