Phantom Dark Energy Models with a Nearly Flat Potential

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We examine phantom dark energy models produced by a field with a negative kinetic term and a potential that satisfies the slow roll conditions: \((1/V)(dV/d\phi) \ll 1\) and \((1/V)(d^2V/d\phi^2) \ll 1\). Such models provide a natural mechanism to produce an equation of state parameter, \(w\), slightly less than \(-1\) at present. Using techniques previously applied to quintessence, we show that in this limit, all such phantom models converge to a single expression for \(w(a)\), which is a function only of the present-day values of \(\Omega_\phi\) and \(w\). This expression is identical to the corresponding behavior of \(w(a)\) for quintessence models in the same limit. At redshifts \(z \lessapprox 1\), this limiting behavior is well fit by the linear parametrization, \(w = w_0 + w_a(1 - a)\), with \(w_a \approx -1.5(1 + w_0)\).

I. INTRODUCTION

Observations [1,2] indicate that approximately 70\% of the energy density in the universe is in the form of an exotic, negative-pressure component, dubbed dark energy. (See Ref. [3] for a recent review). Taking \(w\) to be the ratio of pressure to density for the dark energy:

\[
w = p_{DE}/\rho_{DE}, \tag{1}
\]

recent suggestions suggest that \(w\) is close to \(-1\). For example, if \(w\) is assumed to be constant, then \(-1.1 \lessapprox w \lessapprox -0.9\) [4,5]. If we are interested in dynamical models for dark energy, such as those that arise from a scalar field \(\phi\), then such models can be significantly simplified in the limit that \(w\) is close to \(-1\). This fact was exploited in Ref. [6], which examined quintessence models with a nearly flat potential, defined as a \(V(\phi)\) satisfying the slow-roll conditions:

\[
\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \ll 1, \tag{2}
\]

and

\[
\frac{1}{V}\frac{d^2V}{d\phi^2} \ll 1. \tag{3}
\]

With these assumptions, it is possible to derive a generic expression for \(w\) as a function of the scale factor, \(a\), that provides an excellent approximation to this entire class of potentials. (For other approaches to this problem, see Refs. [7,8,9]).

Here we extend these results to the case of phantom models, i.e., models for which \(w < -1\). It was first noted by Caldwell [10] that observational data do not rule out the possibility that \(w < -1\). These phantom dark energy models have several interesting properties. The density of the dark energy increases with increasing scale factor, and the phantom energy density can become infinite at a finite time, a condition known as the “big rip” [10,11,12]. Further, it has been suggested that the finite lifetime for the universe which is exhibited in these models may provide an explanation for the apparent coincidence between the current values of the matter density and the dark energy density [12]. (See Ref. [14] for a related, but different argument).

A simple way to achieve a phantom model is to take a scalar field Lagrangian with a negative kinetic term. Such models have well-known problems [15,16,17,18], but they nonetheless provide an interesting set of representative phantom models, and they have been widely studied [19,20,21,22,23,24,25,26,27,28]. Here we assume a negative kinetic term, and then use techniques similar to those in Ref. [6] to derive an expression for \(w(a)\) for phantom models satisfying conditions (2) and (3).

We assume a spatially-flat universe containing only nonrelativistic matter and phantom dark energy, since radiation can be neglected in the epoch of interest. Then

\[
H^2 = \rho_T/3, \tag{4}
\]

where \(H = \dot{a}/a\), \(a\) is the scale factor, \(\rho_T\) is the total energy density, and we take \(\hbar = c = 8\pi G = 1\) throughout.

In a phantom model with negative kinetic term and potential \(V(\phi)\), the energy density and pressure of the phantom are given by

\[
\rho_\phi = -(1/2)\dot{\phi}^2 + V(\phi), \tag{5}
\]

and

\[
p_\phi = -(1/2)\dot{\phi}^2 - V(\phi), \tag{6}
\]

so that the equation of state parameter is

\[
w = (1/2)\dot{\phi}^2 + V(\phi)/(1/2)\dot{\phi}^2 - V(\phi). \tag{7}
\]

The evolution of \(\phi\) is given by

\[
\ddot{\phi} + 3H\dot{\phi} - V'(\phi) = 0. \tag{8}
\]

where the prime denotes the derivative with respect to \(\phi\). A field evolving according to equation (8) rolls uphill in the potential.
This equation of motion can be rewritten as \[27, 28\]

\[\pm \frac{V'}{V} = \sqrt{-\frac{3(1 + w)}{\Omega_\phi} \left[ 1 + \frac{1}{6} \frac{d \ln(x)}{d \ln(a)} \right]}, \tag{9}\]

where \(\Omega_\phi\) is the density of the phantom field in units of the critical density (note that \(\Omega_\phi\) evolves with time). In equation (9), \(x = \phi^2 / 2V\), so that \(x\) and \(w\) are related via

\[x = -\frac{1 + w}{1 - w}. \tag{10} \]

(Note that in equation (9) we use the sign convention in \[28\], rather than the one used in \[27\]). Equation (9) is the phantom version of the quintessence equation of motion first derived in Ref. \[29\]; it differs from the quintessence equation only in the sign of \(1 + w\) on the right-hand side.

We are interested in the limit where \(w\) is near \(-1\), so following Ref. \[2\], we define \(\beta = -(1 + w)\), where we take \(\beta\) to be small and positive. This will allow us to drop terms of higher order in \(\beta\). Then equation (9) becomes

\[\beta' = -3\beta(2 + \beta) + \lambda(2 + \beta)\sqrt{3\beta \Omega_\phi}, \tag{11}\]

where we have defined \(\lambda = V'/V\). The fractional density in phantom dark energy, \(\Omega_\phi\), evolves as

\[\Omega_\phi' = 3(1 + \beta)\Omega_\phi - (1 - \Omega_\phi). \tag{12}\]

Combining equations (11) and (12) yields

\[\frac{d\beta}{d\Omega_\phi} = -3\beta(2 + \beta) + \lambda(2 + \beta)\sqrt{3\beta \Omega_\phi} \tag{13}\]

This can be compared to the corresponding equation for \(d\gamma / d\Omega\) for quintessence, where \(\gamma = 1 + w\) \[3\]:

\[\frac{d\gamma}{d\Omega_\phi} = -3\gamma(2 - \gamma) + 3\gamma \sqrt{3\gamma \Omega_\phi} \tag{14}\]

Clearly, equations (13) and (14) predict that \(\beta(\Omega_\phi)\) for a phantom field and \(\gamma(\Omega_\phi)\) for a quintessence field will evolve quite differently. Now, however, we make our slow-roll assumptions. As in Ref. \[3\], we take \(\beta \ll 1\) and we assume \(\lambda\) to be a constant, \(\lambda_0\), given by the initial value of \(V'/V\). Both of these results are a consequence of equations (2) and (3); see Ref. \[3\] for the details. Then equation (13) becomes

\[\frac{d\beta}{d\Omega_\phi} = -\frac{2\beta}{\Omega_\phi(1 - \Omega_\phi)} + \frac{2}{3} \lambda_0 \left(1 - \frac{\sqrt{3\beta}}{\Omega_\phi} \right). \tag{15}\]

This is identical to the equation one obtains, in the corresponding limit, for \(\gamma(\Omega_\phi)\) for quintessence \[3\]. Thus, in this limit, \(1 + w(\Omega_\phi)\) for quintessence and \(-1 - w(\Omega_\phi)\) for a phantom field evolve in exactly the same way. Using the exact solution for equation (13) from Ref. \[3\], we obtain

\[1 + w = \frac{\lambda_0^2}{3} \left[ \frac{1}{\Omega_\phi} - 2 \left(1 - \frac{1}{\Omega_\phi} \right) \ln \left(1 + \sqrt{\Omega_\phi} \right) \right]^2. \tag{16}\]

Further, when \(1 + w\) is close to \(-1\), we can approximate the solution to equation (12) as \[6, 7\]

\[\Omega_\phi = \left[ 1 + \left(\Omega_{\phi 0}^{-1} - 1\right) a^{-3} \right]^{-1}, \tag{17}\]

where \(\Omega_{\phi 0}\) is the present-day value of \(\Omega_\phi\), and we take \(a = 1\) at the present. Combining equations (16) and (17), and normalizing to \(w = w_0\) at present, we obtain:

\[1 + w = (1 + w_0) \left[ \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}} \right] + \frac{1}{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}} \tag{18}\]

\[\times \left[ \frac{1}{\sqrt{\Omega_{\phi 0}}} - \left(1 - \Omega_{\phi 0}^{-1} \right) \frac{\tanh^{-1} \sqrt{\Omega_{\phi 0}}}{\Omega_{\phi 0}^{1/2}} \right]^{-2}. \]

This equation is identical to the corresponding result for \(1 + w\) for the case of quintessence \[3\]. The fact that these two different models (phantom and quintessence) yield an identical form for \((1 + w)/(1 + w_0)\) is not a \textit{a priori} obvious; indeed, this result is valid only in the “slow roll” limit considered here.

We now compare the approximation given in equation (15) to exact numerical results. Kujat et al. \[28\] showed that phantom potentials with a negative kinetic term can be divided into three broad classes, depending on the late-time asymptotic behavior of \(V'(\phi)/V(\phi)\). When \(V'/V \to \pm \infty\) at late times, \(w_\phi \to -\infty\). These models include, for instance, negative power laws. For models in which \(V'/V\) asymptotically approaches a constant, \(w\) also approaches a constant, \(w_0\), where \(w_0 < -1\). This class of models includes exponential potentials. Finally, when \(V'/V \to 0\) at late times, \(w_\phi \to -1\). This class of models includes, for example, positive power law potentials. This final class of models, for which \(w \to -1\), can be further subdivided into models which yield a future singularity (a “big rip”), and those which do not. The exact conditions necessary to avoid a big rip involve an integral function of \(V(\phi)\) \[28\], but for power-law potentials, the condition is simpler: for \(V(\phi) = \phi^\alpha\), \(\alpha > 4\) yields a big rip, while \(\alpha \leq 4\) yields no future singularity \[25, 28\].

We have therefore chosen to compare our analytic approximation for \(w(a)\) with a model from each of these four classes. The results are displayed in Fig. 1. As in the case of quintessence, the agreement between the actual evolution and our approximation in equation (15) is remarkably good, with the error in \(w\) less than 0.5%. Our analytic approximation is well-fit by a linear relation, \[30\]

\[w(a) = w_0 + w_\alpha (1 - a). \tag{19}\]

in the regime \(\alpha > 0.5\), which corresponds to a redshift \(z < 1\). Further, for the value of \(\Omega_{\phi 0}\) adopted here (\(\Omega_{\phi 0} = 0.7\)), we have \(w_\alpha \approx -1.5(1 + w_0)\). This is the same relation between \(w_\alpha\) and \(w_0\) noted empirically for linear
FIG. 1: The evolution of \( w \) for the phantom field evolving in a nearly flat potential as a function of the scale factor, \( a \), normalized to \( a = 1 \) at the present, with \( \Omega_{\phi 0} = 0.7 \) and \( w_0 = -1.1 \). Solid curve is our analytic result for the behavior of phantom models with a nearly flat potential and \( w \) near \(-1\). Other curves give the true evolution for the potentials \( V(\phi) = \phi^6 \) (dotted), \( V(\phi) = \phi^2 \) (short dash), \( V(\phi) = \phi^{-2} \) (long dash), and \( V(\phi) = \exp(\lambda\phi) \) (dot-dash).

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