Khovanov homology of graph-links

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Abstract. Graph-links arise as the intersection graphs of turning chord diagrams of links. Speaking informally, graph-links provide a combinatorial description of links up to mutations. Many link invariants can be reformulated in the language of graph-links. Khovanov homology, a well-known and useful knot invariant, is defined for graph-links in this paper (in the case of the ground field of characteristic two).

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§ 1. Introduction

1.1. Graph-links [1]–[3] are combinatorial analogues of virtual links. A ‘diagram’ of a graph-link is a simple unoriented labelled graph, and graph-links themselves are equivalence classes of such graphs modulo formal Reidemeister moves. Graph-links arise in knot theory as the intersection graphs of turning chord diagrams of links. It is known that the intersection graph determines a chord diagram up to mutations [4], [5]. Therefore, any link invariant which is preserved under mutations may be considered as a candidate for the role of an invariant of graph-links. The Jones and Alexander polynomials are examples of such invariants. It may turn out that some link invariants cannot be extended to graph-links, as some links with the same intersection graph may be not transformable into each other by Reidemeister moves and mutations. It is not known whether such links exist. Therefore, it is a nontrivial problem to extend a given link invariant which is preserved under mutations to an invariant of graph-links. In the case of the Jones polynomial this problem was solved by Ilyutko and Manturov [1]. In the current paper we provide a solution for the case of the (odd) Khovanov homology with coefficients in $\mathbb{Z}_2$.

Traldi and Zulli [6], [7] developed a theory parallel to the theory of graph-links. In their approach (called the theory of loop graphs) they use Gauss chord diagrams of knots instead of turning chord diagrams. As was shown by Ilyutko [8], [9], the two theories are closely related. Therefore, the construction described below can also be used to define the Khovanov homology of loop graphs.

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1.2. Khovanov’s discovery of a homological knot invariant in the 1990s [10] has created a new direction in knot theory and led to a number of new invariants, known as the categorifications of polynomial knot invariants. The construction of Khovanov homology for graph-links considered in this article is essentially a combinatorial interpretation of odd Khovanov homology, which was defined in the work of Ozsvath, Rasmussen and Szabó [11]. Bloom showed that the odd Khovanov homology is preserved under mutations [12], so one can expect that this invariant can be extended to graph-links. In fact, Bloom’s construction in the case of characteristic 2 can be transferred directly to graph-links. This construction can also be used as a definition of the ordinary Khovanov homology of graph-links, since the odd and even Khovanov homology theories coincide when considered over a field of characteristic 2. On the other hand, the integral Khovanov homology is not preserved under mutations [13], which eliminates the possibility of defining the appropriate invariant for graph-links. The question of whether the Khovanov homology is preserved under mutations of knots, and whether it is possible to define its analogue for graph-knots, is currently unresolved.

The paper is organized as follows. In §2 we define the Reidemeister moves of labelled graphs and introduce the notion of a graph-link. In §3 we describe the construction of Khovanov homology for graph-links and prove that it is well-defined. In §4 we relate the Khovanov homology to the Jones polynomial of graph-links.

§ 2. Graph-links

Let G be a graph without loops and multiple edges, and let \( \mathcal{V} = \mathcal{V}(G) \) be its vertex set. We assume that the graph G is labelled, that is, each vertex \( v \) of G is labelled by a pair \((a, \alpha)\), where \( a \in \{0, 1\} \) is the framing \( \text{fr}(v) \) of \( v \), and \( \alpha \in \{-1, 1\} \) is the sign \( \text{sgn}(v) \) of \( v \).

We fix an arbitrary ordering of vertices of G and define the incidence matrix \( A(G) = (a_{ij})_{i,j=1,...,n} \) with elements in \( \mathbb{Z}_2 \) as follows: if \( i \neq j \), then \( a_{ij} = 1 \) whenever the vertices \( v_i \) and \( v_j \) are joined by an edge, and \( a_{ij} = 0 \) otherwise. Furthermore, we set \( a_{ii} = \text{fr}(v_i) \).

Let \( v \in \mathcal{V} \). The set of vertices of \( \mathcal{V} \) adjacent to \( v \) is referred to as the neighbourhood of \( v \) and is denoted by \( N_G(v) \) or \( N(v) \).

We define the Reidemeister moves of labelled graphs below. This definition is essentially a reformulation of the definition of the usual Reidemeister moves on link diagrams: we look at how the incidence graphs of turning chord diagrams are transformed when the moves are applied to the original link diagrams. The resulting graph surgeries turn out to be local, which allows us to apply them to arbitrary graphs, including those that do not correspond to any link diagrams.

\( \Omega_1 \). The first Reidemeister move consists in addition/deletion of an isolated vertex with label \((0, \pm 1)\).

\( \Omega_2 \). The second Reidemeister move consists in addition/deletion of a pair of adjacent (respectively, non-adjacent) vertices with labels \((1, -1)\) and \((1, 1)\) (respectively, \((0, -1)\) and \((0, 1)\)) which have identical neighbourhoods.

\( \Omega_3 \). The third Reidemeister move is defined as follows. Let \( u, v, w \) be vertices of G with the same label \((0, -1)\), such that \( u \) is adjacent to \( v \) and \( w \) only, and \( v \) and \( w \) are not joined by an edge. Then we change the incidence of \( u \) with all vertices
Any edge of the cube has the form \( s \in N(v) \setminus N(w) \cup N(w) \setminus N(v) \) (the incidence of all other pairs of vertices remains unchanged). Furthermore, we change the signs of the vertices \( v \) and \( w \) to ‘+1’. The inverse operation is also called a third Reidemeister move.

\( \Omega_4 \). The fourth Reidemeister move is defined as follows. Take a pair of adjacent vertices: \( u \) with label \((0, \alpha)\) and \( v \) with label \((0, \beta)\). We change the label of \( u \) to \((0, -\beta)\) and change the label of \( v \) to \((0, -\alpha)\). Furthermore, we change the incidence for each pair of vertices \((t, w)\), where \( t \in N(u) \) and \( w \in N(v) \setminus N(u) \), or \( t \in N(v) \) and \( w \in N(u) \setminus N(v) \).

\( \Omega'_4 \). Another version of the fourth Reidemeister move is obtained if we take a vertex \( v \) with label \((1, \alpha)\). In this case we change the incidence for each pair of vertices \((t, w)\) such that \( t, w \in N(u) \). Furthermore, we change the sign of \( v \) and the framing of all vertices \( w \in N(u) \).

There are no analogues of the fourth Reidemeister moves \( \Omega_4 \) and \( \Omega'_4 \) for ordinary link diagrams. These moves reflect the ambiguity in the choice of a turning go-round on the link diagram: application of a move \( \Omega_4 \) or \( \Omega'_4 \) corresponds to the transition from one turning go-round to another. Note that the original link diagram does not change under this transformation.

**Definition 1.** A graph-link is an equivalence class of a simple labelled graph with respect to the Reidemeister moves \( \Omega_1 – \Omega'_4 \).

### § 3. Khovanov homology of graph-links

Let \( G \) be a simple labelled graph with \( n \) vertices, and \( A = A(G) \) its incidence matrix. We formulate the \( \mathbb{Z}_2 \)-version of Bloom’s construction of the odd Khovanov homology below. The \( \mathbb{Z}_2 \)-coefficients are used because the chain complex in Bloom’s construction is defined by the incidence matrix \( A \), whose elements are in \( \mathbb{Z}_2 \).

We shall refer to subsets of the vertex set \( \mathcal{V} \) as states. Given such a state \( s \subset \mathcal{V} = \mathcal{V}(G) \), we define \( G(s) \) as the full subgraph in \( G \) with the vertex set \( s \), and denote \( A(s) = A(G(s)) \). Consider the vector space

\[
V(s) = \mathbb{Z}_2 \langle x_1, \ldots, x_n \mid r^s_1, \ldots, r^s_n \rangle
\]

generated by \( x_1, \ldots, x_n \) with relations \( r^s_1, \ldots, r^s_n \) given by

\[
r^s_i = \begin{cases} 
  x_i + \sum_{\{j \mid v_j \in s\}} a_{ij} x_j & \text{if } v_i \not\in s, \\
  \sum_{\{j \mid v_j \in s\}} a_{ij} x_j & \text{if } v_i \in s.
\end{cases}
\] (3.1)

The dimension of \( V(s) \) is equal to corank \( A(s) \).

There is a natural bijection between the states \( s \subset \mathcal{V} \) and the vertices of the \( n \)-dimensional cube \( \{0, 1\}^n \): a state \( s \) corresponds to the vector \((\alpha_1, \ldots, \alpha_n)\), where \( \alpha_i = 0 \) if either \( v_i \in s \) and \( \text{sgn}(v_i) = 1 \) or \( v_i \not\in s \) and \( \text{sgn}(v_i) = -1 \); otherwise \( \alpha_i = 1 \). Any edge of the cube has the form \( s \rightarrow s \oplus i \), where the state \( s \oplus i \) is \( s \cup \{v_i\} \) if \( v_i \not\in s \), and is \( s \setminus \{v_i\} \) if \( v_i \in s \). The edge is oriented so that \( v_i \not\in s \) if \( \text{sgn}(v_i) = -1 \), and \( v_i \in s \) if \( \text{sgn}(v_i) = 1 \).
To each edge $s \to s \oplus i$ we assign a linear map of exterior algebras

$$\partial^s_{s \oplus i}: \bigwedge^* V(s) \to \bigwedge^* V(s \oplus i)$$

given by the formula

$$\partial^s_{s \oplus i}(u) = \begin{cases} 
  x_i \wedge u & \text{if } x_i = 0 \in V(s), \\
  u & \text{if } x_i \neq 0 \in V(s).
\end{cases} \quad (3.2)$$

Consider the graded vector space

$$C(G) = \bigoplus_{s \subset V} \bigwedge^* V(s)$$

together with a map $\partial: C(G) \to C(G)$ given by

$$\partial(u) = \sum_{\{s, s' \in V | s \to s'\}} \partial^s_{s'}(u).$$

**Proposition 1.** The map $\partial$ is well-defined and endows $C(G)$ with the structure of a chain complex.

**Proof.** We need to prove that the maps $\partial^s_{s \oplus i}$ are well-defined and that each two-dimensional face of the cube of states is commutative. We shall need several auxiliary lemmata.

**Lemma 1.** Let $s$ be a state and let $i$ be an index such that $v_i \not\in s$. We may assume without loss of generality that $A(s \oplus i) = (A a^T \alpha)$, where $A = A(s)$. Then

1) $x_i = 0 \in V(s)$ if and only if $\text{rank } A = \text{rank}(A a^T \alpha)$;
2) $x_i = 0 \in V(s \oplus i)$ if and only if $\text{rank}(A a^T \alpha) + 1 = \text{rank}(A a^T \alpha)$.

**Proof.** The condition $x_i = \sum_{\{j | v_j \in s\}} a_{ij} x_j$ implies that $x_i = 0 \in V(s)$, which means that the vector $a$ is a linear combination of the rows of the incidence matrix $A$. Thus, the first assertion is true.

The identity $x_i = 0 \in V(s \oplus i)$ implies that the vector $(0 \ 1)$ is a linear combination of the rows of the matrix $A(s \oplus i)$. Therefore,

$$\text{rank} \left( \begin{array}{cc} A & a^T \\ a & \alpha \end{array} \right) = \text{rank} \left( \begin{array}{cc} A & a^T \\ a & \alpha \end{array} \right) = \text{rank} \left( \begin{array}{cc} A & 0 \\ a & 0 \end{array} \right) = \text{rank} \left( \begin{array}{c} A \\ a \end{array} \right) + 1.$$

**Lemma 2.** Assume that $A$ is a symmetric matrix satisfying

$$\text{rank} \left( \begin{array}{cc} A & a^T \\ a & \alpha \end{array} \right) = \text{rank} A + 1.$$ 

Then $\text{rank} \left( \begin{array}{c} A \\ a \end{array} \right) = \text{rank} A$. 

Proof. Suppose
\[ \operatorname{rank} \left( \begin{array}{c|c}
A & a \\
\hline
a & \alpha
\end{array} \right) \neq \operatorname{rank} A. \]
Then
\[ \operatorname{rank} \left( \begin{array}{c|c}
A & a^T \\
\hline
a & \alpha
\end{array} \right) = \operatorname{rank} \left( \begin{array}{c}
A \\
a
\end{array} \right), \]
and the vector \( (a^T) \) is a linear combination of the columns of the matrix \( \left( \begin{array}{c|c}
A & a \\
\hline
a & \alpha
\end{array} \right) \).
Therefore, \( a^T \) is a linear combination of the rows of \( A \), and (after application of the transposition) the vector \( a \) is a linear combination of the rows of \( A \). Then \( \operatorname{rank} A = \operatorname{rank} \left( \begin{array}{c|c}
A & a \\
\hline
a & \alpha
\end{array} \right) \), which contradicts the initial assumption.

Lemma 3. For any \( s \) and \( i \) we have:
1) \( \dim V(s \oplus i) = \dim V(s) + 1 \) if and only if \( x_i = 0 \in V(s) \) and \( x_i \neq 0 \in V(s \oplus i) \);
2) \( \dim V(s \oplus i) = \dim V(s) - 1 \) if and only if \( x_i \neq 0 \in V(s) \) and \( x_i = 0 \in V(s \oplus i) \);
3) \( \dim V(s \oplus i) = \dim V(s) \) if and only if \( x_i = 0 \in V(s) \) and \( x_i = 0 \in V(s \oplus i) \).
The case \( x_i \neq 0 \in V(s) \) and \( x_i \neq 0 \in V(s \oplus i) \) does not occur.

Proof. All statements follow from Lemmas 1, 2 and the identities
\[ \dim V(s) = \operatorname{corank} A(s), \quad \dim V(s \oplus i) = \operatorname{corank} A(s \oplus i). \]

We refer to the first two cases of Lemma 3 as even, and to the third case as odd.
It follows from the definition of the differential that \( \partial^s_{s \oplus i} = 0 \) in the odd case. Now we are ready to prove Proposition 1.

Well-definedness of chain maps. Consider the map
\[ \partial^s_{s \oplus i}: \bigwedge^* V(s) \to \bigwedge^* V(s \oplus i). \]
We need to check that relations are mapped to relations, that is, for any element \( u \) and any index \( j \) there exist elements \( u_k \in V(s \oplus i) \) such that
\[ \partial^s_{s \oplus i}(r^s_j \wedge u) = \sum_k r^{s \oplus i}_k \wedge u_k \in V(s \oplus i). \]

For any index \( j \) we have \( r^s_j = r^{s \oplus i}_j + \alpha x_i \) for some \( \alpha \in \mathbb{Z}_2 \). If \( x_i = 0 \in V(s \oplus i) \), then
\[ \partial^s_{s \oplus i}(r^s_j \wedge u) = r^s_j \wedge u = r^{s \oplus i}_j \wedge u + \alpha x_i \wedge u = r^{s \oplus i}_j \wedge u', \]
in the space \( V(s \oplus i) \), where \( u' = u \) or \( u' = 0 \) (in the odd case). If \( x_i \neq 0 \in V(s \oplus i) \), then \( x_i = 0 \in V(s) \) and
\[ \partial^s_{s \oplus i}(r^s_j \wedge u) = x_i \wedge r^s_j \wedge u = x_i \wedge r^{s \oplus i}_j \wedge u + \alpha x_i \wedge x_i \wedge u = r^{s \oplus i}_j \wedge (x_i \wedge u). \]
In either case, the map \( \partial^s_{s \oplus i} \) is well-defined.
Commutativity of two-dimensional faces. A two-dimensional face of the cube of states has the form

\[ \bigwedge^* V(s \oplus i \oplus j) \xrightarrow{\partial_{s \oplus i \oplus j}} \bigwedge^* V(s \oplus i \oplus j) \]

The diagrams without odd edges can be split into five types according to how they change the dimension of the spaces

\[ V(s'), \quad s' = s, s \oplus i, s \oplus j, s \oplus i \oplus j, \]

see Fig. 1. Here the number at the vertex corresponding to a state \( s' \) is equal to the difference \( \dim V(s') - \dim V(s) = \text{corank} A(s') - \text{corank} A(s) \), and the edge corresponding to a map \( \partial'_{s'} \) is labelled by \( z = 1, x_i \) or \( x_j \) if \( \partial'_{s'}(u) = z \wedge u \).

Two-dimensional faces of types 1, 2, 3 are obviously commutative. Any face of type 4 is also commutative since \( x_i = x_j = 0 \in V(s \oplus i \oplus j) \). For a face of type 5, we need to show that \( x_i = x_j \in V(s \oplus i \oplus j) \). We have the following three possible cases, up to reordering of variables:

1. \( \text{sgn}(v_i) = \text{sgn}(v_j) = -1 \). Then \( v_i, v_j \in s \oplus i \oplus j \). We can assume without loss of generality that \( v_i \) and \( v_j \) are the last vertices of the state \( s \oplus i \oplus j \). The incidence matrix \( A(s \oplus i \oplus j) \) has the form

\[
\begin{pmatrix}
A & a^T & b^T \\
\alpha & 0 & 0 \\
\beta & 0 & 0
\end{pmatrix}
\]

where \( A = A(s) \).
The identity \( \text{corank } A(s \oplus i) = \text{corank } A(s) - 1 \) implies that \( x_i = 0 \in V(s \oplus i) \). Then the vector \((0 1)\) is a linear combination of the rows of the matrix \( A(s \oplus i) = (A_{\alpha} a_\alpha) \). The same linear combination of the rows of the matrix \( A(s \oplus i \oplus j) \) is equal to \((0 1 \delta)\), for some \( \delta \in \mathbb{Z}_2 \). If \( \delta = 0 \), then \( x_i = 0 \in V(s \oplus i \oplus j) \), which is impossible. Thus, \( \delta = 1 \), and we obtain the relation \( x_i + x_j = 0 \) in the space \( V(s \oplus i \oplus j) \).

2. \( \text{sgn}(v_i) = -1, \text{sgn}(v_j) = 1 \). Then \( v_i \in s \oplus i \oplus j \) and \( v_j \notin s \oplus i \oplus j \). We may assume that the incidence matrix \( A(s \oplus i) \) has the form (3.3), where \( A = A(s \oplus j) \).

We have \( x_i = 0 \in A(s \oplus i) \). This implies that the vector \((0 1 0)\) is a linear combination of the rows of the incidence matrix \( A(s \oplus i) \). If the coefficient of the row \((b \gamma \beta)\) in this linear combination is zero, then the same linear combination of the rows of \( A(s \oplus i \oplus j) = (A_{\alpha} a_\alpha) \) gives the vector \((0 1)\). Hence \( x_i = 0 \in V(s \oplus i \oplus j) \), which is a contradiction. Therefore, the coefficient of \((b \gamma \beta)\) is 1. Then the vector \((b \gamma + 1)\) is a linear combination of the rows of \( A(s \oplus i \oplus j) \), which implies that \( x_i + x_j = 0 \in V(s \oplus i \oplus j) \).

3. \( \text{sgn}(v_i) = \text{sgn}(v_j) = -1 \). Then the matrix \( A(s) \) has the form (3.3), where \( A = A(s \oplus i \oplus j) \). The identity \( x_i + x_j = 0 \in V(s \oplus i \oplus j) \) corresponds to the vector-relation \( a + b \). Since

\[
\text{rank } A(s) = \text{rank } A(s \oplus j) = \text{rank } \begin{pmatrix} A & a^T \\ a & \alpha \end{pmatrix} = \text{rank } \begin{pmatrix} A & a^T \\ a & \alpha & b^T \end{pmatrix},
\]

the last row of the matrix \( A(s) \) is a linear combination of the other rows. If the coefficient of the row \((a \alpha \gamma)\) in this linear combination is zero, then the vector \((b \gamma \beta)\) is a linear combination of the rows of \( (A \ a^T \ b^T) \), and the vector \( b \) is a linear combination of the rows of \( A \). Hence \( x_j = 0 \in V(s \oplus i \oplus j) \), which is a contradiction. Therefore, the coefficient of \((a \alpha \gamma)\) is nonzero, and the vector \( a + b \) is a linear combination of the rows of \( A \). This implies that \( x_i + x_j = 0 \in V(s \oplus i \oplus j) \).

Since the maps corresponding to odd edges of the cube of states are zero, we need only to verify the commutativity of the two types of two-dimensional faces containing odd edges shown in Fig. 2.

A diagram of type 6 is commutative since

\[
\dim V(s \oplus i) = \dim V(s \oplus i \oplus j),
\]

so that \( x_j = 0 \in V(s \oplus i \oplus j) \).

In order to establish the commutativity of a diagram of type 7 we need to show that \( x_i = 0 \in V(s \oplus i \oplus j) \). Suppose this is not true.

Let

\[
\text{sgn}(v_i) = \text{sgn}(v_j) = -1.
\]
Then the incidence matrix has the form (3.3). The identity $x_j = 0 \in V(s \oplus i)$ implies that
\[
\text{rank } \begin{pmatrix} A & a^T \\ a & \alpha \\ b & \gamma \end{pmatrix} = \text{rank } \begin{pmatrix} A & a^T \\ a & \alpha \end{pmatrix}.
\]

Then the vector $b$ is a linear combination of the rows of the matrix $(A_a)$. Since $\text{rank}(A_a) = \text{rank } A$, the vector $b$ is also a linear combination of the rows of $A$. On the other hand, the identity $x_j \neq 0 \in V(s)$ implies that $\text{rank}(A_b) \neq \text{rank } A$. This is a contradiction.

The cases when the vertices have other signs are analogous.

This finishes the proof of Proposition 1.

**Definition 2.** We refer to the homology $\overline{Kh}(G)$ of the complex $(C(G), \partial)$ as the *reduced (odd) Khovanov homology* of a labelled graph $G$.

The main result of this section is that the Khovanov homology is a well-defined invariant of graph-links.

**Theorem 1.** The Khovanov homology $\overline{Kh}(G)$ is invariant under the Reidemeister moves $\Omega_1-\Omega_4$.

**Proof.** Let $G$ be a labelled graph and let $\tilde{G}$ be the graph obtained from $G$ by applying one of the Reidemeister moves $\Omega_1-\Omega_4$.

**Invariance under $\Omega_1$.** Let $\tilde{G}$ be obtained from $G$ by adding an isolated vertex $v$. The complex $C(\tilde{G})$ is isomorphic to a product of complexes $C(G) \otimes C(v)$, where $C(v)$ is the complex
\[
\mathbb{Z}_2 \xrightarrow{x^\wedge} \bigwedge^* \mathbb{Z}_2 \langle x \rangle
\]
if $\text{sgn}(v) = -1$, and is the complex
\[
\bigwedge^* \mathbb{Z}_2 \langle x \rangle \xrightarrow{x=0} \mathbb{Z}_2
\]
if $\text{sgn}(v) = 1$. In either case, $H_*(C(v)) = \mathbb{Z}_2 \cdot 1$, where $1 \in H_0(C(v))$ for $\text{sgn}(v) = 1$ and $1 \in H_1(C(v))$ for $\text{sgn}(v) = -1$. We therefore obtain
\[
\overline{Kh}(\tilde{G}) = \overline{Kh}(G) \otimes \overline{Kh}(v) \cong \overline{Kh}(G).
\]

**Invariance under $\Omega_2$.** Suppose the graph $\tilde{G}$ is obtained by adding two vertices $v$ and $w$ with $\text{sgn}(v) = 1$ and $\text{sgn}(w) = -1$. We may assume without loss of generality that the incidence matrix $A(\tilde{G})$ has one of the following two forms:
\[
\begin{pmatrix} 0 & 0 & a^T \\ 0 & 0 & a^T \\ a & a & A(G) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & a^T \\ 1 & 1 & a^T \\ a & a & A(G) \end{pmatrix}.
\]

In either case, for any vertex $s \in V(G)$, we have
\[
\text{corank } A(G(s)) = \text{corank } A(\tilde{G}(s)),
\]
\[
\text{corank } A(\tilde{G}(s \cup \{v\})) = \text{corank } A(\tilde{G}(s \cup \{w\})) = \text{corank } A(\tilde{G}(s \cup \{v, w\})) - 1.
\]
These relations determine the type of the upper and the left maps in the complex $C(\tilde{G})$ written in the form

$$
\begin{array}{c}
C_{vw} \xrightarrow{1} C_w \\
\uparrow x_2 \\
C_v \xrightarrow{\partial} C
\end{array}
$$

Here the space $C_v$ is generated by chains whose state contains the vertex $v$ and does not contain the vertex $w$. The spaces $C_v$, $C_w$ and $C_{vw}$ are defined similarly.

For each state $s$ in $C_{vw}$ we define a linear function $f: V(s) \to \mathbb{Z}_2$ by the formula $f(P_i \lambda_i x_i) = \lambda_1 + \lambda_2$. The function $f$ is well-defined because it vanishes on any relation: $f(r_i^s) = a_i1 + a_i2 = 0$ since $a_i1 = a_i2$. Then

$$
\wedge^* V(s) = \wedge^* \ker f \oplus x_2 \wedge^* V(s), \quad C_{vw} = X \oplus x_2 C_{vw}.
$$

The subcomplex $X \to C_w$ is contractible. Therefore, the homology of the complex $C(\tilde{G})$ is equal to the homology of the quotient complex

$$
\begin{array}{c}
x_2 C_{vw} \\
\uparrow x_2 \\
C_v \xrightarrow{\partial} C
\end{array}
$$

This quotient complex also becomes contractible after factorization by the subcomplex $C$. Therefore, the complex $C(\tilde{G})$ has the same homology as $C = C(G)$.

**Invariance under $\Omega_3$.** We may assume without loss of generality that the vertices $u$, $v$, $w$ involved in the third Reidemeister move have numbers 1, 2, 3 in $\mathcal{V}(G) = \mathcal{V}(\tilde{G})$. Then the incidence matrices of the graphs $G$ and $\tilde{G}$ have the following form:

$$
A(G) = \begin{pmatrix}
0 & 1 & 1 & 0^T \\
1 & 0 & 0 & a^T \\
1 & 0 & 0 & b^T \\
0 & a & b & B
\end{pmatrix}, \quad A(\tilde{G}) = \begin{pmatrix}
0 & 0 & 0 & (a + b)^T \\
0 & 0 & 0 & b^T \\
0 & 0 & 0 & a^T \\
(a + b) & a & b & B
\end{pmatrix}.
$$

Denote $\tilde{V}(s) = V(\tilde{G}(s))$. Then, for any state $s \subset \mathcal{V}(G) \setminus \{u, v, w\}$, we have

$$
V(s) \cong \tilde{V}(s), \quad V(s \oplus v) \cong \tilde{V}(s \oplus v), \quad V(s \oplus w) \cong \tilde{V}(s \oplus w),
$$

$$
V(s \oplus v \oplus w) \cong \tilde{V}(s \oplus u \oplus v) \cong \tilde{V}(s \oplus u \oplus w),
$$

$$
V(s \oplus u \oplus v \oplus w) \cong \tilde{V}(s \oplus u), \quad V(s \oplus u \oplus v) \cong V(s \oplus u \oplus w) \cong V(s),
$$

and the corresponding isomorphisms of exterior algebras commute with the differential.
We write the complexes $C(G)$ and $C(\tilde{G})$ in the form of cubes:

\[
\begin{array}{c}
\begin{array}{c}
C_u \\ x_1
\end{array}
\xrightarrow{1}
\begin{array}{c}
C_{uw} \rightarrow C_{uvw} \\
C_w \\
C_v
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\tilde{C}_{uw} \\
x_1
\end{array}
\xrightarrow{1}
\begin{array}{c}
\tilde{C}_{uvw} \rightarrow \tilde{C}_u \\
\tilde{C}_{uw} \\
\tilde{C}_v \\
\tilde{C}_w
\end{array}
\end{array}
\end{array}
\]

For each state $s$ in $C_u$, we consider the linear function $f : V(s) \rightarrow \mathbb{Z}_2$ given by the formula $f(\sum_i \lambda_i x_i) = \lambda_1$. This function is well-defined, and it induces a decomposition

\[
\bigwedge^* V(s) = \bigwedge^* \ker f \oplus x_1 \bigwedge^* V(s), \quad C_u = X \oplus x_1 C_u.
\]

Consider the subcomplex

\[
\begin{array}{c}
\begin{array}{c}
X
\end{array}
\xrightarrow{1}
\begin{array}{c}
C_{uw} \rightarrow C_{uvw} \\
C_w \\
C_v
\end{array}
\end{array}
\end{array}
\]

The quotient complex $C \rightarrow x_1 C_u$ is contractible, so that the homology of $C(G)$ is isomorphic to the homology of the subcomplex above. This subcomplex contains a contractible part $X \rightarrow \partial(X)$. The maps $X \rightarrow C_{uw}$ and $X \rightarrow C_{uw}$ are isomorphisms, so we obtain the following complex after taking quotients:

\[
\begin{array}{c}
\begin{array}{c}
C_{uw} \rightarrow C_{uvw} \\
C_{uw} \\
C_w \\
C_v
\end{array}
\end{array}
\]

where the spaces $C_v$ and $C_w$ are identified.

Similar arguments reduce the complex $C(\tilde{G})$ to the complex

\[
\begin{array}{c}
\begin{array}{c}
\tilde{C}_{uw} \rightarrow \tilde{C}_u \\
\tilde{C}_{uw} \\
\tilde{C}_v \\
\tilde{C}_w
\end{array}
\end{array}
\]
(for states $s$ in $\tilde{C}_{uvw}$ we need to define the function $f : \tilde{V}(s) \to \mathbb{Z}_2$ by the formula $f(\sum_i \lambda_i x_i) = \lambda_1 + \lambda_2 + \lambda_3$). Both resulting complexes are isomorphic to the complex

$$C_v \oplus C_w \to C \oplus C_{vw} \to C_{uvw}.$$ 

Therefore, the homology of complexes $C(G)$ and $C(\tilde{G})$ coincide.

**Invariance under $\Omega_4$.** Suppose the vertices $u$ and $v$ involved in the Reidemeister move $\Omega_4$ have numbers $p$ and $q$ in $V(G) = V(\tilde{G})$. The coefficients of the incidence matrices $A(G) = (a_{ij})$ and $A(\tilde{G}) = (\tilde{a}_{ij})$ are related by the identity

$$\tilde{a}_{ij} = \begin{cases} 
  a_{ij} + a_{ip}a_{jq} + a_{iq}a_{jp}, & \{i, j\} \cap \{p, q\} = \emptyset, \\
  a_{ij}, & \{i, j\} \cap \{p, q\} \neq \emptyset.
\end{cases}$$

Consider the map $\varphi : C(G) \to C(\tilde{G})$ whose value on the states is given by

$$\varphi(s) = \begin{cases} 
  s \cup \{u, v\}, & \{u, v\} \cap s = \emptyset, \\
  s \setminus \{u, v\}, & \{u, v\} \cap s = \{u, v\}, \\
  s, & \{u, v\} \cap s \neq \emptyset, \{u, v\},
\end{cases}$$

and also define linear maps $\Phi : V(s) \to \tilde{V}(\varphi(s))$ given by

$$\Phi(x_i) = \begin{cases} 
  x_i, & i \neq p, q, \\
  x_q, & i = p, \\
  x_p, & i = q.
\end{cases}$$

The map $\Phi$ is an isomorphism of linear spaces, and being extended to an isomorphism of exterior algebras

$$\bigwedge^* V(s) \to \bigwedge^* \tilde{V}(\varphi(s))$$

it defines an isomorphism of graded linear spaces $\Phi : C(G) \to C(\tilde{G})$. This $\Phi$ is also a chain map. Thus, the complexes $C(G)$ and $C(\tilde{G})$ are isomorphic, together with their homology.

**Invariance under $\Omega'_4$.** Suppose the vertex $v$ involved in the Reidemeister move $\Omega'_4$ has number $p$. The coefficients of the incidence matrices $A(G) = (a_{ij})$ and $A(\tilde{G}) = (\tilde{a}_{ij})$ are related by the identity

$$\tilde{a}_{ij} = \begin{cases} 
  a_{ij} + a_{ip}a_{jp}, & i, j \neq p, \\
  a_{ip}, & j = p, \\
  a_{pj}, & i = p.
\end{cases}$$

We define the map $\varphi : C(G) \to C(\tilde{G})$ whose value on the states is given by

$$\varphi(s) = s \oplus p,$$

and also define linear maps $\Phi : V(s) \to \tilde{V}(\varphi(s))$ by $\Phi(x_i) = x_i, i = 1, \ldots, n$. The maps $\Phi$ give rise to an isomorphism of complexes $C(G)$ and $C(\tilde{G})$, so that their homology coincide.

**Corollary 1.** The Khovanov homology $\overline{Kh}(G)$ is an invariant of graph-links.
§ 4. Jones polynomial of graph-links and Khovanov homology

Let \( G \) be a simple labelled graph. The complex \( C(G) \) has two gradings: the homological grading \( M_0 \) and the algebraic grading \( \text{deg} \) in the graded algebras \( \wedge^* V(s) \). The differential is not homogeneous with respect to \( \text{deg} \), but it respects the grading \( Q_0 \) whose value on the element \( u \in \wedge^r V(s) \) is given by

\[
Q_0(u) = \dim \mathbb{Z}_2 V(s) - 2r + M_0(s).
\]

The differential increases the homological grading \( M_0 \) by one, and does not change the grading \( Q_0 \). These two gradings define the decomposition of the Khovanov homology into a direct sum

\[
\overline{Kh}(G) = \bigoplus_{m,q \in \mathbb{Z}} \overline{Kh}(G)_{(m,q)}.
\]

The following definition of the Kauffman bracket for graph-links was given by Ilyutko and Manturov [1]:

**Definition 3.** Let \( G \) be a simple labelled graph with \( n \) vertices. The Kauffman polynomial of \( G \) is defined by

\[
\langle G \rangle(a) = \sum_{s \subset \mathcal{V}(G)} a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\text{corank} A(s)},
\]

where \( \alpha(s) = \# \{ v \in s \mid \text{sgn}(v) = -1 \} + \# \{ v \not\in s \mid \text{sgn}(v) = 1 \} \) and \( \beta(s) = n - \alpha(s) \).

It turns out that the Kauffman bracket coincides up to shift of grading with the Euler characteristic of the bigraded Khovanov homology:

**Theorem 2.** The following identity holds for any labelled graph \( G \):

\[
\sum_{m,q \in \mathbb{Z}} (-1)^m \dim \mathbb{Z}_2 \overline{Kh}(G)_{(m,q)} \cdot t^q = (-it^{1/2})^n \langle G \rangle(it^{-1/2}).
\]

**Proof.** The left hand side of the identity coincides with the Euler characteristic of the complex \( C(G) \). For each state \( s \subset \mathcal{V}(G) \) of homological grading \( \beta(s) \), the corresponding space of chains \( \wedge^* V(s) \) contributes

\[
(-1)^{\beta(s)}(t + t^{-1})^{\dim V(s)} t^{\beta(s)}
\]

to the Euler characteristic. Since

\[
\dim V(s) = \text{corank} A(s) \quad \text{and} \quad \beta(s) = -\frac{1}{2}(\alpha(s) - \beta(s) - n),
\]

the sum over all states is equal to the right hand side of the identity.

**Example 1.** Let \( W_5 \) be the Bouchet graph [14] (see Fig. 3) with all vertices labelled by \((0,+)\). The Khovanov homology of this labelled graph \( G \) was calculated by Bloom:

\[
\overline{Kh}_{(1,0)}(G) = \overline{Kh}_{(2,0)}(G) = \overline{Kh}_{(3,2)}(G) = \overline{Kh}_{(3,4)}(G) = \overline{Kh}_{(4,6)}(G) = \overline{Kh}_{(5,6)}(G) = \mathbb{Z}_2,
\]
and the other homology groups are zero. The Kauffman bracket is given by

\[ \langle G \rangle(a) = -a^{-2} - a^2. \]

This implies that the graph-link \( G \) is nontrivial. Furthermore, it turns out that \( G \) is a non-realizable graph-link (see [3]). This means that any labelled graph which is obtained from \( G \) by Reidemeister moves cannot be realized as the intersection graph of a chord diagram.

![Figure 3. Bouchet graph \( W_5 \).](image)

Note that the graph \( G \) is not a graph-knot in the sense of Definition 4 below.

The gradings \( M_0 \) and \( Q_0 \) of the Khovanov complex are not preserved by Reidemeister moves, as is shown in Table 1.

Here \( \Omega_1^\pm \) denotes the addition of an isolated vertex with sign \( \pm 1 \), and \( \Omega_2 \) denotes a move adding two vertices. The shift in the grading caused by application of a Reidemeister move is given in the corresponding cell of the table.

It follows that the groups \( \overline{Kh}(G)_{(m,q)} \) are not invariants of graph-links. Nevertheless, they can be normalized to produce invariants of graph-knots.

**Definition 4** (see [1]). A graph-link \( G \) is called a graph-knot if \( \text{corank}(A(G)+E) = 0 \), where \( E \) denotes the unit matrix.

The relation \( \text{corank}(A(G) + E) = 0 \) is preserved under Reidemeister moves, so the notion of a graph-knot is well-defined.

|       | \( M_0 \) | \( Q_0 \) |
|-------|----------|----------|
| \( \Omega^-_1 \) | 0        | -1       |
| \( \Omega^+_1 \) | 1        | 2        |
| \( \Omega_2 \)   | 1        | 1        |
| \( \Omega_3 \)   | 0        | 0        |
| \( \Omega_4 \)   | 0        | 0        |
| \( \Omega'_4 \)  | 0        | 0        |
Definition 5 (see [1]). The twisting number of a graph $G$ with $n$ vertices is defined as the sum
\[ w(G) = \sum_{i=1}^{n} (-1)^{\text{corank} B_i(G)} \, \text{sgn} \, v_i, \]
where $B_i(G) = A(G) + E + E_{ii}$, and the matrix $E_{ii}$ has only one nonzero element, which lies in the $i$th row and the $i$th column.

The twisting number is invariant under the moves $\Omega_2$–$\Omega_4$. The move $\Omega_{1} \pm 1$ changes the twisting number by $\mp 1$. Using this fact and Table 1 we construct two gradings which are preserved under Reidemeister moves:

\[ M = M_0 + \frac{w(G) - n(G)}{2}, \quad Q = Q_0 + \frac{3w(G) - n(G)}{2}, \]

where $n(G)$ is the number of vertices in the graph $G$.

We denote by $\overline{\text{Kh}}_{m,q}(G)$ the homogeneous part of degree $M = m$ and $Q = q$ in the homology $\overline{\text{Kh}}(G)$. The groups $\overline{\text{Kh}}_{m,q}(G)$ are invariants of graph-knots.

The twisting number allows us to define another invariant of graph-knots:

Definition 6 (see [1]). The Jones polynomial of a graph-knot $G$ is defined by
\[ X(G)(a) = (-a)^{-3w(G)} \langle G \rangle(a). \]

Theorem 2 implies

Theorem 3. The bigraded Khovanov homology $\overline{\text{Kh}}_{m,q}(G)$ of graph-knots is the categorification of the Jones polynomial, that is,
\[ \sum_{m,q \in \mathbb{Z}} (-1)^m \dim_{\mathbb{Z}} \overline{\text{Kh}}_{m,q}(G) \cdot t^q = X(G)(it^{-1/2}). \]

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