Robust violation of a multipartite Bell inequality from the perspective of a single-system game *

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Recently, Fan et al. [Mod. Phys. Lett. A 36, 2150223 (2021)], presented a generalized Clauser-Horne-Shimony-Holt (CHSH) inequality, to identify N-qubit Greenberger-Horne-Zeilinger (GHZ) states. They showed an interesting phenomenon that the maximal violation of the generalized CHSH inequality is robust under some specific noises. In this work, we map the inequality to the CHSH game, and consequently to the CHSH* game in a single-qubit system. This mapping provides an explanation for the robust violations in N-qubit systems. Namely, the robust violations, resulting from the degeneracy of the generalized CHSH operators correspond to the symmetry of the maximally entangled two-qubit states and the identity transformation in the single-qubit game. This explanation enables us to exactly demonstrate that the degeneracy is $2^{N-2}$.

Keywords: Clauser-Horne-Shimony-Holt game; Bell’s inequality; Greenberger-Horne-Zeilinger states; robust violations

I. INTRODUCTION

Quantum entanglement [1–6], brought about by the superposition principle, puzzled many physicists in the early days of quantum theory. In the original work for Einstein-Podolsky-Rosen (EPR) paradox in entangled states [7], Einstein and his collaborators proposed that quantum mechanics provides probabilistic results because of its incompleteness. In 1964, Bell proposed an inequality [8] to solve the EPR paradox under the assumptions of local reality and hidden variable theory. Such inequality and its generalized versions [9, 10] revealed nonlocality [11–17] in entangled states. The Clauser-Horne-Shimony-Holt inequality [18] is the most widely studied Bell inequality for two-qubit systems, which is written as

$$I_{CHSH}^2 = \langle A_0B_0 \rangle + \langle A_0B_1 \rangle + \langle A_1B_0 \rangle - \langle A_1B_1 \rangle \leq 2,$$ (1)

with $A_a, B_b \ (a, b = 0, 1)$ being measurement settings. It can be violated by all the two-qubit entangled pure states.

Researchers have tried to understand the nonlocality from the perspective of game theory [19–21]. The author of Ref. [19] setted up the so-called CHSH game to show the advantage of quantum strategies. There are two players, Alice and Bob, in the game who cannot communicate with each other. They share a two-qubit system and have the local measurement of $A_a$ and $B_b$ respectively. Here, $a$ and $b = 0, 1$ are two input values. Let $x$ and $y = 0, 1$ represent the outcomes of Alice and Bob. When $x \oplus y = ab$, the players win the game, with $\oplus$ denoting modulo 2 addition. It is directly to find the linear relationship between their success probability

$$\frac{1}{4} \sum_{a, b} \text{Prob} (x \oplus y = ab | a, b)$$ (2)

and the expected value of $\langle I_{CHSH}^2 \rangle$.

Recently, Henaut et al. [21] introduced a single-player CHSH* game with two inputs $a$ and $b$. Any strategy in the CHSH* game can be mapped to a strategy in the CHSH game with two-qubit maximally entangled states. Without loss of generality, let Alice and Bob share one of the Bell states, $|\psi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. The player of the CHSH* game, Carol, has a qubit in state $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. Alice and Bob apply arbitrary local unitary transformations $A_a^T$ and $B_b$ to their qubits, and then measure the Pauli operator $\sigma_x$ on their qubits respectively. This is equivalent to the local measurement of $A_aB_b = A_{a}^{\sigma_x}A_{a}^{T} \otimes B_{b}^{\sigma_x}B_{b}$ on $|\psi_+\rangle$. Carol applies $A_a$ and $B_b$ on the state $|+\rangle$ and measures

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\( \sigma_z \) on her qubit. Similarly, this represents the measurement of \( C_{ab} = A^\dagger_a B^\dagger_b \sigma_x B_a A_b \) on \(|+\rangle \). She wins the game when her outcome \( c = ab \pmod{2} \). The success probabilities of the two games are equal; i.e.,

\[
\frac{1}{4} \sum_{a,b} \text{Prob} \left( x \oplus y = ab | a, b \right) = \frac{1}{4} \sum_{a,b} \text{Prob} \left( c = ab | a, b \right),
\]

which arises from the expected values

\[
\langle I^2_{CHSH} \rangle = \langle I^1_{CHSH} \rangle,
\]

with \( \langle I^2_{CHSH} \rangle = \sum_{a,b} (-1)^{ab} \langle \psi_+ | A^\dagger_a \sigma_x A^\dagger_b \otimes B^\dagger_b \sigma_x B_a | \psi_+ \rangle \) and \( \langle I^1_{CHSH} \rangle = \sum_{a,b} (-1)^{ab} \langle + | A^\dagger B^\dagger \sigma_x B_a A_b | + \rangle \).

On the other hand, to distinguish \( N \)-qubit Greenberger-Horne-Zeilinger (GHZ) states, Fan et al. presented a generalized CHSH inequality in the very recent work [22]. It is expressed as

\[
\langle I^N_{CHSH} \rangle = \langle \mathcal{A}_0 \mathcal{B}_0 \rangle + \langle \mathcal{A}_0 \mathcal{B}_1 \rangle + \langle \mathcal{A}_1 \mathcal{B}_0 \rangle - \langle \mathcal{A}_1 \mathcal{B}_1 \rangle \leq 2.
\]

\( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) denote the tensor products of local observables for the first \( N-1 \) qubit. \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) represent two observables of the \( N \)th qubit. The \( N \)-qubit GHZ states can be identified by the maximal violations of the inequality. Besides, they found an interesting quantum phenomenon of robust violations of the generalized CHSH inequality, in which the maximal violation can be robust under some specific noises. Such phenomenon originates from the degeneracy of the largest eigenvalue of Bell-function \( I^N_{CHSH} \).

In this work, we show the mapping between the CHSH* and CHSH games proposed by Henaut et al. [21] can be extended to the generalized CHSH inequality for \( N \)-qubit case. The relations among the inequalities (and the CHSH* game) provide an explanation for the robust violations and give the degeneracy of \( I^N_{CHSH} \). Namely, we map the \( N \)-qubit Bell function \( I^N_{CHSH} \) to \( I^2_{CHSH} \) for the two-qubit case, and consequently to \( I^1_{CHSH} \) for the one-qubit system. For the \( N \)-qubit GHZ states (\(|\psi_+\rangle \) and \(|+\rangle \) for \( N = 2 \) and 1), the expected values satisfy \( |\langle I^1_{CHSH} \rangle| \geq |\langle I^2_{CHSH} \rangle| \geq |\langle I^N_{CHSH} \rangle| \) when \( |\langle I^N_{CHSH} \rangle| \geq 2 \). The equality holds when the generalized CHSH inequality achieves the Tsirelson’s bound, i.e.

\[
\langle I^1_{CHSH} \rangle = \langle I^2_{CHSH} \rangle = \langle I^N_{CHSH} \rangle = \pm 2\sqrt{2}.
\]

This equation is invariable under specific local unitary transformations on the two-qubit and \( N \)-qubit Bell-functions, which corresponds to the identity operation on the single-qubit system. By using this invariance, we exactly demonstrate the degeneracy of \( \langle I^N_{CHSH} \rangle \), which causes the robust violations.

## II. THE GENERALIZED CHSH INEQUALITY AND GAMES

We first introduce the mappings from \( \langle I^N_{CHSH} \rangle \) to \( \langle I^2_{CHSH} \rangle \), where the measured quantum states are the GHZ states

\[
|G\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)
\]

and the Bell state \(|\psi_+\rangle\). Unless explicitly stated otherwise, all the expected values in this paper are of the states \(|G\rangle\), \(|\psi_+\rangle\) and \(|+\rangle\) corresponding to \( N \)-, two- and one-qubit system. The local measurement operators in \( N \)-qubit system can be written as

\[
X_j = \tilde{n}_j \cdot \vec{\sigma}, \quad X'_j = \tilde{n}'_j \cdot \vec{\sigma} \quad (j = 1, 2 \cdots N),
\]

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli matrices, \( \tilde{n}_j = (\sin \alpha_j \cos \varphi_j, \sin \alpha_j \sin \varphi_j, \cos \alpha_j) \) and \( \tilde{n}'_j = (\sin \alpha'_j \cos \varphi'_j, \sin \alpha'_j \sin \varphi'_j, \cos \alpha'_j) \) denote the measurement direction of the \( j \)-th qubit. Fan et al. [22] defined the measurement operators of Alice and Bob in (5) as

\[
\mathcal{A}_0 = \bigotimes_{j=1}^{N-1} X_j, \quad \mathcal{A}_1 = \bigotimes_{j=1}^{N-1} X'_j, \quad \mathcal{B}_0 = X_N, \quad \mathcal{B}_1 = X'_N.
\]

One can derive the first term of \( \langle I^N_{CHSH} \rangle \) as

\[
\langle \mathcal{A}_0 \otimes \mathcal{B}_0 \rangle = \frac{1}{2} [1 + (-1)^N \prod_{j=1}^{N} \cos \alpha_j + \cos(\sum_{j=1}^{N} \varphi_j) \prod_{j=1}^{N} \sin \alpha_j].
\]
Obviously, only the terms $\prod_{j=1}^{N} \cos \alpha_j$, $\cos(\sum_{j=1}^{N} \varphi_j)$ and $\prod_{j=1}^{N} \sin \alpha_j$ are contributed by the projections of the measurement operators in the subspace of $\{|00\ldots0\rangle, |11\ldots1\rangle\}$.

For brevity, we ignore the cases: (i) $(\prod_{j=1}^{N-1} \cos \alpha_j)^2 + (\prod_{j=1}^{N-1} \sin \alpha_j)^2 = 0$; (ii) $(\prod_{j=1}^{N-1} \cos \alpha_j')^2 + (\prod_{j=1}^{N-1} \sin \alpha_j')^2 = 0$; (iii) $\sin \alpha_N = 0$ when $N$ is odd; (iv) $\sin \alpha_N' = 0$ when $N$ is odd. These cases compose a zero measure set, and do not violate the generalized CHSH inequality (5).

To connect the expected value to the two-qubit system in the state $|\psi_+\rangle$, we define the following two mappings. The first one uniquely leads to a single-qubit observable, for a given $(N-1)$-qubit operator in (9), as

$$\Gamma[\vec{a}] := \vec{u}_a \cdot \vec{\sigma}. \quad (11)$$

Take $\Gamma[\vec{a}_0] := \vec{u}_0 \cdot \vec{\sigma}$ as an example. The Bloch vector $\vec{u}_0 = (\sin \gamma \cos \beta, \sin \gamma \sin \beta, \cos \gamma)$, with $\gamma = \prod_{j=1}^{N-1} \sin \alpha_j / \sqrt{(\prod_{j=1}^{N-1} \cos \alpha_j)^2 + (\prod_{j=1}^{N-1} \sin \alpha_j)^2}$ and $\beta = \sum_{j=1}^{N-1} \varphi_j$. The other vector $\vec{u}_1$ is in the similar form.

The second mapping projects the single-qubit observable in (9) onto the equator of Bloch sphere; i.e.,

$$\Theta[|E_b\rangle] := \vec{r}_b \cdot \vec{\sigma}, \quad (12)$$

with $\vec{r}_0 = (\cos \varphi_N, \sin \varphi_N, 0)$ and $\vec{r}_1 = (\cos \varphi'_N, \sin \varphi'_N, 0)$. When $N$ is even, one can choose

$$A_0 = \Gamma[\vec{a}_0], \quad A_1 = \Gamma[\vec{a}_1], \quad B_0 = B_0, \quad B_1 = B_1,$$

while

$$A_0 = \Gamma[\vec{a}_0], \quad A_1 = \Gamma[\vec{a}_1], \quad B_0 = \Theta[|E_b\rangle], \quad B_1 = \Theta[|E_b\rangle] \quad (14)$$

when $N$ is odd, and obtain the two-qubit Bell function

$$\langle I_{CHSH}^2 \rangle = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle,$$

(15)

corresponding to $\langle I_{CHSH}^N \rangle$ in (5).

The above measurement operators $A_a$ and $B_b$ can always be expressed as $A_a = A^a_x \sigma_x A^T_a$ and $B_b = B^b_x \sigma_x B^T_b$, with $A^T_a$ and $B^T_b$ being two local unitary transformations. That is, any measurement of the Bell function $\langle I_{CHSH}^N \rangle$ in (5) can be mapped to a strategy in the CHSH game, and consequently to the CHSH* game according to the relation $C_{ab} = A^a_x B^b_x \sigma_x B^T_a A^T_b$ given by Henaut et al. [21]. The two maps are constructed based on the fact that, the expected value of $A^a x B^b_x$ on the state $|G\rangle$ is equivalent to the one of $A_a B_b$ on the state $|\psi_+\rangle$. The two single-qubit operators, $A_a$ and $B_b$, are in the form of (8), but the Bloch vector of $A_a$ is in the unit ball in general. Therefore, we introduce a normalization coefficient in (11). Under these mappings, we have the two following theorems.

Theorem 1. \forall $N \geq 3$, under the mappings in (11-15), the violation of the CHSH inequality by the Bell state $|\psi_+\rangle$ is a necessary condition for the violation of the general CHSH inequality by the GHZ state $|G\rangle$. More particularly, when $\langle I_{CHSH}^N \rangle > 2$, $\langle I_{CHSH}^2 \rangle \geq \langle I_{CHSH}^N \rangle$; and when $\langle I_{CHSH}^N \rangle < -2$, $\langle I_{CHSH}^2 \rangle \leq \langle I_{CHSH}^N \rangle$.

Proof. The expected values $\langle I_{CHSH}^N \rangle$ and $\langle I_{CHSH}^2 \rangle$ can be written as

$$\langle I_{CHSH}^N \rangle = \sum_{a,b \in \mathbb{Z}_2} (-1)^{ab} \langle A_a \otimes B_b \rangle,$$

$$\langle I_{CHSH}^2 \rangle = \sum_{a,b \in \mathbb{Z}_2} (-1)^{ab} \langle A_a \otimes B_b \rangle.$$

(16a)

(16b)

The range of $\langle A_a \otimes B_b \rangle$ is $[-1, 1]$. When $\langle I_{CHSH}^N \rangle > 2$, one has

$$\sum_{b \in \mathbb{Z}_2} \langle A_0 \otimes B_b \rangle > 0, \quad \sum_{b \in \mathbb{Z}_2} (-1)^b \langle A_1 \otimes B_b \rangle > 0,$$

$$\sum_{a \in \mathbb{Z}_2} \langle A_a \otimes B_0 \rangle > 0, \quad \sum_{a \in \mathbb{Z}_2} (-1)^a \langle A_a \otimes B_1 \rangle > 0.$$

(17a)

(17b)

Similarly, when $\langle I_{CHSH}^N \rangle < -2$, the greater-than signs in the four inequalities (17) become the less-than signs. Let us denote the normalization constants of $\Gamma[\vec{a}_0]$ and $\Gamma[\vec{a}_1]$ as $\varepsilon = \frac{1}{\sqrt{(\prod_{j=1}^{N-1} \cos \alpha_j)^2 + (\prod_{j=1}^{N-1} \sin \alpha_j)^2}}$ and $\varepsilon' = \frac{1}{\sqrt{(\prod_{j=1}^{N-1} \cos \alpha_j')^2 + (\prod_{j=1}^{N-1} \sin \alpha_j')^2}}$. They satisfy $\varepsilon \geq 1$ and $\varepsilon' \geq 1$. 
When \( N \) is even,

\[
\begin{align*}
\langle A_0 \otimes B_0 \rangle &= \frac{\varepsilon}{\sin \alpha_N} \langle A_0 \otimes E_0 \rangle \\
\langle A_0 \otimes B_1 \rangle &= \frac{\varepsilon}{\sin \alpha_N^*} \langle A_0 \otimes E_1 \rangle \\
\langle A_1 \otimes B_0 \rangle &= \varepsilon' \langle A_1 \otimes U_0 \rangle \\
\langle A_1 \otimes B_1 \rangle &= \varepsilon' \langle A_1 \otimes U_1 \rangle.
\end{align*}
\] (18)

Multiplying by weighting coefficients \((-1)^{a+b}\) and summing up them, according to the inequalities (17a) one can find \( \langle \hat{I}_{\text{CHSH}}^2 \rangle \geq \langle \hat{I}_{\text{CHSH}}^N \rangle \), when \( \langle \hat{I}_{\text{CHSH}}^N \rangle > 2 \). Evidenced by the same token, when \( \langle \hat{I}_{\text{CHSH}}^N \rangle < -2 \), \( \langle \hat{I}_{\text{CHSH}}^2 \rangle \leq \langle \hat{I}_{\text{CHSH}}^N \rangle \).

When \( N \) is odd, one has

\[
\begin{align*}
\langle A_0 \otimes B_0 \rangle &= \frac{\varepsilon}{\sin \alpha_N} \langle A_0 \otimes E_0 \rangle \\
\langle A_0 \otimes B_1 \rangle &= \frac{\varepsilon}{\sin \alpha_N^*} \langle A_0 \otimes E_1 \rangle \\
\langle A_1 \otimes B_0 \rangle &= \varphi' \langle A_1 \otimes \sin \alpha_N \rangle \\
\langle A_1 \otimes B_1 \rangle &= \varphi' \langle A_1 \otimes \sin \alpha_N^* \rangle.
\end{align*}
\] (19)

When \( \langle \hat{I}_{\text{CHSH}}^N \rangle > 2 \), according to the inequalities (17b), it is direct to obtain

\[
\frac{1}{\sin \alpha_N} \sum_{a \in \mathbb{Z}_2} (\langle A_0 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle) + \frac{1}{\sin \alpha_N^*} \sum_{a \in \mathbb{Z}_2} (-1)^a (\langle A_0 \otimes B_1 \rangle) > \langle \hat{I}_{\text{CHSH}}^N \rangle.
\] (20)

The form of (10) leads to \( \frac{\langle A_0 \otimes B_0 \rangle}{\sin \alpha_N} \) and \( \frac{\langle A_0 \otimes B_1 \rangle}{\sin \alpha_N^*} \in [-1, 1] \). Consequently,

\[
\frac{1}{\sin \alpha_N} (\langle A_0 \otimes E_0 \rangle + \langle A_0 \otimes E_1 \rangle) > 0,
\]

\[
\frac{1}{\sin \alpha_N^*} (\langle A_1 \otimes E_0 \rangle - \langle A_1 \otimes E_1 \rangle) > 0.
\] (21)

Multiplying by weighting coefficients \((-1)^{a+b}\) and summing up the terms in (19), according to the relations (20) and (21) one can find \( \langle \hat{I}_{\text{CHSH}}^2 \rangle \geq \langle \hat{I}_{\text{CHSH}}^N \rangle \), when \( \langle \hat{I}_{\text{CHSH}}^N \rangle > 2 \). Similarly, when \( \langle \hat{I}_{\text{CHSH}}^N \rangle < -2 \), \( \langle \hat{I}_{\text{CHSH}}^2 \rangle \leq \langle \hat{I}_{\text{CHSH}}^N \rangle \). This ends the proof.

\[ \square \]

There are two corollaries of Theorem 1 as follows. (i) The maximal violations of the GHZ state cannot exceed the Tsirelson’s bound \( \pm 2\sqrt{2} \), which has been found in Ref. [22]. (ii) When the expected value for the GHZ state \( \langle \hat{I}_{\text{CHSH}}^N \rangle = \pm 2\sqrt{2} \), the corresponding \( \langle \hat{I}_{\text{CHSH}}^2 \rangle = \langle \hat{I}_{\text{CHSH}}^N \rangle \).

**Theorem 2.** When the expected value for the GHZ state saturates the Tsirelson’s bound, \( \langle \hat{I}_{\text{CHSH}}^N \rangle = \pm 2\sqrt{2} \), the Bloch vectors of the operators in \( \hat{A}_0 \) and \( \hat{A}_1 \) in (9) are restricted as follows three cases

\[
\begin{align*}
\text{(i)} & \quad \vec{n}_j = (0, 0, \pm 1), & \vec{n}'_j = (\cos \phi'_j, \sin \phi'_j, 0) \\
\text{(ii)} & \quad \vec{n}_j = (\cos \phi, \sin \phi, 0), & \vec{n}'_j = (0, 0, \pm 1) \\
\text{(iii)} & \quad \vec{n}_j = (\cos \phi, \sin \phi, 0), & \vec{n}'_j = (\cos \phi'_j, \sin \phi'_j, 0).
\end{align*}
\] (22)

Only the case (iii) exists in the system with an odd \( N \).

**Proof.** According to Theorem 1, \( \langle \hat{I}_{\text{CHSH}}^N \rangle = \langle \hat{I}_{\text{CHSH}}^2 \rangle \) requires \( \varepsilon = \varepsilon' = 1 \). That is, \( \prod_{j=1}^{N-1} \cos \alpha_j^2 + \prod_{j=1}^{N-1} \sin \alpha_j^2 = (\prod_{j=1}^{N-1} \cos \alpha_j^2)^2 + (\prod_{j=1}^{N-1} \sin \alpha_j^2)^2 = 1 \). Either all of the Bloch vectors \( \vec{n}_j \), with \( j = 1...N-1 \), are parallel to \( z \) axis, or perpendicular to \( z \) axis. This holds true for \( \vec{n}'_j \) with \( j = 1...N-1 \).

When \( N \) is odd, \( \langle \hat{I}_{\text{CHSH}}^N \rangle = \langle \hat{I}_{\text{CHSH}}^2 \rangle \) also requires \( \sin \alpha_N = \sin \alpha_N^* = 1 \). Hence, the Bloch vectors of \( B_0 \) and \( B_1 \) in (14) are perpendicular to \( z \) axis. The ones of \( A_0 \) and \( A_1 \) should also be perpendicular to \( z \) axis, to enable \( \langle \hat{I}_{\text{CHSH}}^2 \rangle \) to achieve \( \pm 2\sqrt{2} \). The correspondence in (14) leads to that only the case (iii) is allowed.
When \( N \) is even, \( \vec{n}_j \) and \( \vec{n}_j' \) cannot be simultaneously parallel to \( z \) axis. This is naturally derived from the fact that the measurement directions of \( A_0 \) and \( A_1 \) in (14) are perpendicular when \( \langle I_{CHSH}^2 \rangle = \pm 2\sqrt{2} \). In brief, the three cases of the Bloch vectors, (i), (ii) and (iii), are possible when the GHZ state achieves the maximal violations of the generalized CHSH inequality, with \( N \) being even. This ends the proof.

\[
\square
\]

III. ROBUST VIOLATIONS OF THE GENERALIZED CHSH INEQUALITY

A. Framework

In this section, we show that the above mappings provide an explanation for the quantum phenomenon of robust violations of the generalized CHSH inequality [22]. Based on the explanation, one can exactly demonstrate the degeneracy of the Bell function \( I_{CHSH}^N \), which corresponds to the dimension of noises for robust violations.

According to the results in section II and Ref. [22], when the \( N \)-qubit Bell function \( \langle I_{CHSH}^N \rangle = \pm 2\sqrt{2} \), the corresponding \( \langle I_{CHSH}^2 \rangle \), and consequently \( \langle I_{CHSH}^1 \rangle \), reach \( \pm 2\sqrt{2} \). In addition, the corresponding terms in the three Bell functions are equal; i.e.,

\[
\langle C_{ab} \rangle = \langle A_a B_b \rangle = \langle \mathbb{A}_a \mathbb{B}_b \rangle, \tag{23}
\]

with

\[
\langle C_{ab} \rangle = \langle +| A_a^b B_b^j \sigma_x B_a^u A_u^j |+ \rangle, \tag{24a}
\]

\[
\langle A_a B_b \rangle = \langle \psi_+| A_a^u \sigma_x A_a^T \otimes B_b^j \sigma_x B_b^j |\psi_+ \rangle. \tag{24b}
\]

One always can insert the \( 2 \times 2 \) unit operator, \( \mathbb{1} = u^* u^T \) with \( u \) being unitary, between the two unitary operators in \( \langle C_{ab} \rangle \), as \( \langle +| A_a^b B_b^j \sigma_x B_a^u A_u^j |+ \rangle = \langle +| A_a^b u^* u^T B_b^j \sigma_x B_b^j u^* u^T A_u^j |+ \rangle \). It is equivalent to a local unitary transformation on the two qubit system as

\[
\langle \psi_+| A_a^u \sigma_x A_a^T \otimes B_b^j \sigma_x B_b^j |\psi_+ \rangle = \langle \psi_+| (u^\dagger \otimes u^T) (A_a^u \sigma_x A_a^T \otimes B_b^j \sigma_x B_b^j) (u \otimes u^*) |\psi_+ \rangle \tag{25}
\]

\[
= \langle \psi_+| (u^\dagger A_a u \otimes (u^T B_b u^*) |\psi_+ \rangle. \tag{26}
\]

This actually gives the symmetry operations of the Bell state \( |\psi_+ \rangle \), that

\[
(u \otimes u^*) |\psi_+ \rangle = |\psi_+ \rangle, \tag{26}
\]

and the relation between different choices of observers achieving the maximal violation.

To preserve the equations (23) and the value of \( \langle I_{CHSH}^N \rangle \), the local unitary operator \( u \) can only have some special forms, which we will list in the following part. For a given \( u \), the corresponding transformation of the \( N \)-qubit system can be written as

\[
\langle G| \mathbb{A}_a \otimes \mathbb{B}_b |G \rangle = \langle G| (\bigotimes_{j=1}^N u_j^j) (\mathbb{A}_a \otimes \mathbb{B}_b) (\bigotimes_{j=1}^N u_j^j) |G \rangle, \tag{27}
\]

where \( u_N = u^* \). At this point, note that, \( \mathbb{B}_b = B_b \) even if \( N \) is odd. In addition, the set of \( u_j \) \((j = 1...N - 1)\) is not unique. This is because the mapping from \( \langle I_{CHSH}^N \rangle \) to \( \langle I_{CHSH}^2 \rangle \) is many-to-one. Similarly, the degree of freedom of \( u \) in (25) also comes from the many-to-one relationship between \( A_a B_b \) and \( C_{ab} \). Generally, \( (\bigotimes_{j=1}^N u_j^j) |G \rangle \) is different with \( |G \rangle \), which indicates the largest eigenvalue of \( I_{CHSH}^N \) is degenerate. Mixing or superposing \( |G \rangle \) with the states in the subspace does not affect \( \langle I_{CHSH}^N \rangle \). This is the quantum phenomenon of robust violations of Bell’s inequality for the GHZ state presented by Fan et al. [22].

B. Degeneracy

Then, we give the details of the local unitary transformations and degeneracy. According to their relationship between the Bloch vectors of and the \( z \) axis, there are six cases of the observers in \( I_{CHSH}^2 \), as shown in Fig. 1. Only the measurement directions (up to rotations about the \( z \) axis) of \( A_a \) are plotted, since \( B_b \) can be uniquely determined
Fig. 1. Relationships between z axis and the measurement directions of \(A_0\) and \(A_1\) in \(I_{CHSH}^2\), when the corresponding \(\langle I_{CHSH}^N \rangle\) reaches the maximal quantum violation.

by \(A_0\) when \(\langle I_{CHSH}^2 \rangle\) reaches the maximal violation. These six cases have a two-to-one correspondences with the three possible choices of the \(N\)-qubit operators in Theorem 2 which are (i): 1, 2; (ii): 3, 4; and (iii): 5, 6.

According to the equivalence relations under the local unitary transformations on the \(N\)-qubit system and the exchange between \(\mathcal{A}_0\) and \(\mathcal{A}_1\), it is sufficient to consider only the degeneracy of \(I_{CHSH}^N\) corresponding to cases (1) and (5). For the case (1), there are six types of the unitary operator \(u\) to consider, corresponding to the six cases in Fig. 1 as the final states of \(A_0\). However, only two types of \(u\) for the case (5) need to be considered, corresponding to the final states in (5) and (6). This is because, it is equivalent to the one in case (1), if the \(N\)-qubit operators \(I_{CHSH}^N\) can be transformed by \(\bigotimes_{j=1}^{N} u_j\) into the cases (i) or (ii) in Theorem 2. As show in Fig. 2, for fixed \(\mathcal{A}_0\) and \(u\), one can derive the unitary operators \(u_j\) by requiring \(\Gamma(\bigotimes_{j=1}^{N-1} u_j^\dagger \mathcal{A}_0 \bigotimes_{j=1}^{N-1} u_j) = u^\dagger \Gamma(\mathcal{A}_0)u\). We remark that, the initial and final directions of two Bloch vectors can uniquely determine a \(2 \times 2\) unitary operator, up to a phase factor which does not affect value of \(\langle I_{CHSH}^N \rangle\) in (27).

Fig. 2. Procedure to determine the requirements for \(u_j\), \(X_j\) and \(X'_j\).

Case (1).— The case (1) exists only in the system with an even \(N\). The angles \(\varphi_j'\) in \(\mathcal{A}_1\) can always be adjusted to zero by local rotations about the z axis, which transforms \(A_1\) into \(\sigma_z\) simultaneously. In addition, the single-qubit operators in \(\mathcal{A}_0\), \(X_j = \pm \sigma_z\) have even minus signs. These minus signs can be removed by qubit flips without affecting the corresponding \(A_0\). Therefore, one can choose the initial observables as

\[
X_1 = \ldots = X_{N-1} = \sigma_z, \quad X'_1 = \ldots = X'_{N-1} = \sigma_x; \quad A_0 = \sigma_z, \quad A_1 = \sigma_x.
\]  

(28)

To construct the local unitary transformations \(u\) and \(u_j\), we introduce three sets of unitary operators as

\[
u^{(1)} = 1, \quad u^{(2)} = \sigma_x, \quad u^{(3)} = \exp(i \frac{\sigma_y \pi}{2}), \quad u^{(4)} = \sigma_x u^{(3)}, \quad u^{(5)} = \exp(-i \frac{\sigma_x \pi}{2}), \quad u^{(6)} = \sigma_x u^{(5)}; 
\]  

(29a)

\[
u^{(1)} = u^{(1)}, \quad v^{(2)} = u^{(2)}, \quad v^{(3)} = u^{(3)}, \quad v^{(4)} = u^{(4)}, \quad v^{(5)} = \begin{cases} u^{(5,6)} & N/2 \text{ is even,} \\ u^{(6,5)} & N/2 \text{ is odd;} \end{cases}
\]

(29b)

\[
u^{(1)} = \sigma_x v^{(1)}, \quad w^{(2)} = \sigma_x v^{(2)}, \quad w^{(3)} = \sigma_x v^{(3)}, \quad w^{(4)} = \sigma_x v^{(4)}, \quad w^{(5)} = \sigma_x v^{(5)}, \quad w^{(6)} = \sigma_x v^{(6)}.
\]

(29c)

For an arbitrary superscript \(\nu = 1, \ldots, 6\), one can check the \(N\)-qubit GHZ state has a symmetry as

\[
u^{(\nu)} \otimes_{N-1} u^{(\nu)^*} |G\rangle = \exp(i \phi) |G\rangle,
\]

(30)
where $v^{(u)} \otimes^{N-1} v^{(u)}$ denotes the direct product of $N - 1$ $v^{(u)}$ on the first $N - 1$ qubits and $\phi \in [0, 2\pi]$ is a phase factor. This can be regarded as an extension of the symmetry of the Bell state in (26).

The operator $u$, transforming the initial $A_u$ into the case (26), can be universally written as

$$u = u^{(\nu)} \exp(i \frac{\sigma_x^u \delta}{2})$$  (31)

with $\delta \in [0, 2\pi]$. According to the correspondences between the six cases and the three possible choices in Theorem 2, there are two alternative forms of $u_j$ as

$$u_j = v^{(u)} \exp(i \frac{\sigma_x^u \delta_j}{2}) \quad \text{or} \quad w^{(u)} \exp(i \frac{\sigma_x^u \delta_j}{2}),$$  (32)

with $j = 1, ..., N - 1$ and $\delta_j \in [0, 2\pi]$. Applying them onto $A_u$ and $A_u$ and requiring $\Gamma(\otimes_{j=1}^{N-1} u_j A_u \otimes_{j=1}^{N-1} u_j) = u \Gamma(A_u)u$, one can easily obtain the two conditions on $u_j$, as

$$\sum_{j=1}^{N-1} \delta_j = \delta \mod 2\pi,$$  (33)

and the number of $w^{(u)}$ in $\otimes_{j=1}^{N-1} u_j$ being even.

Then, $\otimes_{j=1}^{N} u_j |G\rangle$ are eigenstates of the initial $I^N_{CHSH}$, with the same eigenvalue as $|G\rangle$. By utilizing the forms of $u_j$ in (32) and $u_N = u^*$ in (31), one can derive these states in three steps: (1) rotations about the z axis with $\exp(i \frac{\sigma_x^u \delta_j}{2})$ and $\exp(-i \frac{\sigma_x^u \delta}{2})$; (2) $v^{(u)}$, including the ones in $w^{(u)}$, and $w^{(u)^*}$; (3) the even number of $\sigma_x$ or $\sigma_z$ factored out from $v^{(u)}$. The state $|G\rangle$ is invariant under the operations in the first two steps, because of the condition (33) and the symmetry (30). Consequently, $\otimes_{j=1}^{N} u_j |G\rangle$ are equivalent to the results of $|G\rangle$ multiplied by even number of $\sigma_x$ or $\sigma_z$. Since $|G\rangle$ is invariant under even number of $\sigma_z$, the degenerate states are given by qubit flips in pairs (i.e., application of even number of $\sigma_x$) on $|G\rangle$. The degeneracy can be directly derived as

$$C_N^0 + C_N^2 + \ldots + C_N^{N-2} = 2^{N-2}.  \quad (34)$$

**Case (5).** – One can always adjust the angles $\phi_j$ in $A_u$ to zero by using local rotations about the z axis, which transform $A_0$ into $\sigma_x$ and $A_1$ into $\sigma_y$ simultaneously. Then, the initial observables can be choose as

$$X_j = \sigma_x, \quad X'_j = \cos \phi_j \sigma_x + \sin \phi_j \sigma_y; \quad A_0 = \sigma_x, \quad A_1 = \sigma_y,$$  (35)

with $j = 1, ..., N - 1$ and $\sum_{j=1}^{N-1} \phi_j = \pi/2 \mod 2\pi$.

In order to express in a similar way as the case (1), we define

$$u^{(5)} = 1, \quad u^{(5)} = u^{(5)}, \quad u^{(5)} = \sigma_x u^{(5)}; \quad (36a)$$  

$$u^{(6)} = \sigma_x, \quad u^{(6)} = u^{(6)}, \quad u^{(6)} = \sigma_x u^{(6)}.$$  (36b)

Similarly, the $N$-qubit GHZ state has a symmetry as

$$v^{(\mu)} \otimes^{N-1} v^{(\mu)^*} |G\rangle = \exp(i\theta) |G\rangle,$$  (37)

with $\mu = 5, 6$ and $\theta \in [0, 2\pi]$ being a phase factor. The operators transforming the initial $A_u$ and $A_u$ into the case (26), can be written as

$$u = u^{(\mu)} \exp(i \frac{\sigma_x^u \delta}{2}),$$  (38)

$$u_j = v^{(\mu)} \exp(i \frac{\sigma_x^u \delta_j}{2}) \quad \text{or} \quad w^{(\mu)} \exp(i \frac{\sigma_x^u \delta_j}{2}),$$  (39)

with $\delta, \delta_j \in [0, 2\pi]$ and $j = 1, ..., N - 1$.

We define the sets $J = \{1, 2, 3, \ldots, N - 1\}$, $K$ and $L$, with $K \subseteq J$ and $L$ being its complementary set. The elements of $K$ are the subscripts of $u_j$ with $w^{(\mu)}$, and ones of $L$ are for $v^{(\mu)}$. Applying the operators (38) onto $A_u$ and $A_u$ and requiring $\Gamma(\otimes_{j=1}^{N-1} u_j A_u \otimes_{j=1}^{N-1} u_j) = u \Gamma(A_u)u$, one can easily obtain

$$\sum_{j=1}^{N-1} \delta_j = \delta \mod 2\pi,$$  (40)
and

$$\sum_{k \in K} \phi'_j = 0 \mod \pi. \tag{41}$$

The latter condition is on the initial observables $X'_j$, which is a difference with the case 1. The states $\bigotimes_{j=1}^N u_j|G\rangle$ can be derived by following the same three steps in the case 1, which lead to

$$\bigotimes_{j=1}^N u_j|G\rangle = \bigotimes_{k \in K} \sigma^k_x|G\rangle, \tag{42}$$

with $\sigma^k_x$ being the Pauli operator $\sigma_x$ of the $k$-th qubit.

For a fixed $K$, $\bigotimes_{k \in K} \sigma^k_x|G\rangle$ reaches the maximal violations, only when the initial $I_{CHSH}^N$ satisfies the condition (41). Therefore, the number of $K$, with which the condition (41) is satisfied, gives the degeneracy of the largest eigenvalue of $I_{CHSH}^N$. Then, the maximum degeneracy in the case 5 is $2^{N-2}$, which can be obtained based on the following facts. The condition (41) cannot be fulfilled simultaneously by a subset of $J$ and its complementary set, which sets $2^{N-2}$ as the upper limit on the degeneracy. A simple construction to reach the upper limit is that, $\phi'_1 = ... = \phi'_{N-2} = 0$ and $\phi'_{N-1} = \pi/2$.

**Example.**—An arbitrary choice of the operators $A_a$ and $B_a$ reaching the maximal violation of $|G\rangle$ can always be transformed into the above two cases by local unitary operations. The degenerate subspace can also be derived by the same local unitary operations on the above results. We show these by using the example with $N = 4$ provided in Ref. [22], which belongs to the case 5.

The parameters of the observables $X_j$ and $X'_j$ are given by $\varphi_1 = \varphi_2 = \varphi'_4 = 0$, $\varphi'_1 = \varphi'_2 = \varphi_4 = \frac{\pi}{4}$, $\varphi_3 = -\frac{\pi}{4}$ and $\varphi'_3 = \frac{\pi}{4}$. Then, $\langle I_{CHSH}^N \rangle = \langle I_{CHSH}^N \rangle = 2\sqrt{2}$. The operators can be adjusted into the simple form (35) by using $\tau_3 = \exp(i\frac{\pi}{4} \frac{\pi}{4})$ and $\tau_4 = \exp(-i\frac{\pi}{4} \frac{\pi}{4})$ on the third and forth qubits. These lead to $\varphi_3 \to 0$, $\varphi_4 \to \pi/4$, $\varphi'_3 \to \pi/2$ and $\varphi'_4 \to -\pi/4$. Then, the subsets of $J$, with which the condition (41) are fulfilled, are given by

$$K = \emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}. \tag{43}$$

Applying $\tau_3$ and $\tau_4$ onto $\bigotimes_{k \in K} \sigma^k_x|G\rangle$, one obtains the four degenerate states as

$$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \quad \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$\frac{1}{\sqrt{2}}(e^{-i\frac{\pi}{4}}|1010\rangle + e^{i\frac{\pi}{4}}|0101\rangle), \quad \frac{1}{\sqrt{2}}(e^{-i\frac{\pi}{4}}|0110\rangle + e^{i\frac{\pi}{4}}|1001\rangle), \tag{44}$$

which are the same as the results in Ref. [22].

**IV. SUMMARY**

In summary, we relate the two recent topics in the area of Bell-nonlocality, which are the robust violations of Bell inequality of the GHZ states [22] and the single-qubit quantum game [21]. Namely, we present the mapping from the generalized CHSH inequality, to distinguish the GHZ states constructed by Fan et al. [22] to the CHSH game, and consequently to the single-qubit CHSH* game [21]. These relationships provide an explanation for the robust violations of the generalized CHSH inequality in $N$-qubit systems. The identity transformation in the CHSH* game, corresponds to the symmetry of the two-qubit Bell state, and further leads to the local unitary transformations generating the degenerate subspace of the $N$-qubit Bell function. An arbitrary superposition or mixture in the subspace leads to the same expected value of the Bell function, which is the quantum phenomenon of robust violations. Based on the explanation, we exactly prove that the maximal degeneracy is $2^{N-2}$. It would be interesting to extend the mapping among the systems with different numbers of subsystems to explore more topics in the area of Bell-nonlocality and entanglement, such as the identification of W states.
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