Hybrid Differentially Private Federated Learning on Vertically Partitioned Data

Chang Wang  
Tencent Inc.  
Beijing, China  
coracwang@tencent.com

Jian Liang  
Tencent Inc.  
Beijing, China  
joshualiang@tencent.com

Mingkai Huang  
Tencent Inc.  
Beijing, China  
mingkhuang@tencent.com

Bing Bai  
Tencent Inc.  
Beijing, China  
icebai@tencent.com

Kun Bai  
Tencent Inc.  
Guangzhou, China  
kunbai@tencent.com

Hao Li*  
Tencent Inc.  
Beijing, China  
leehaoli@tencent.com

Abstract

We present HDP-VFL, the first hybrid differentially private (DP) framework for vertical federated learning (VFL) to demonstrate that it is possible to jointly learn a generalized linear model (GLM) from vertically partitioned data with only a negligible cost, w.r.t. training time, accuracy, etc., comparing to idealized non-private VFL. Our work builds on the recent advances in VFL-based collaborative training among different organizations which rely on protocols like Homomorphic Encryption (HE) and Secure Multi-Party Computation (MPC) to secure computation and training. In particular, we analyze how VFL’s intermediate result (IR) can leak private information of the training data during communication and design a DP-based privacy-preserving algorithm to ensure the data confidentiality of VFL participants. We mathematically prove that our algorithm not only provides utility guarantees for VFL, but also offers multi-level privacy, i.e. DP w.r.t. IR and joint differential privacy (JDP) w.r.t. model weights. Experimental results demonstrate that our work, under adequate privacy budgets, is quantitatively and qualitatively similar to GLMs, learned in idealized non-private VFL setting, rather than the increased cost in memory and processing time in most prior works based on HE or MPC. Our codes will be released if this paper is accepted.

1 Introduction

Vertical federated learning (VFL) [20] has been recognized as one of the effective solutions for encouraging enterprise-level data collaborations while respecting data privacy [36], required by the strict government regulations like Europe’s General Data Privacy Regulations (GDPR) [34]. Unlike horizontal federated learning (HFL) [25, 6] setting in which the decentralized datasets share the same feature space but little intersection on the sample space, in VFL setting, the datasets of different organizations share the same or similar sample space but differ in feature space. Therefore the VFL

*Corresponding Author.
participants need to jointly learn a model together [20], rather than independently learn models from their local data like normal HFL participants do.

The difference in data distribution leads to different focus on data protection in HFL and VFL. In HFL, gradients, trained with each participant’s local data and vulnerable to information leakage [29], are sent from each participant to server for a secure aggregation [5]. The numerous participants of HFL are mostly personal smart phones or edge devices with limited computation power and network bandwidth, thus the goal of gradient protection in HFL setting is mainly achieved by differential privacy (DP) [4, 12, 26, 33], secret sharing [5], and so on. In VFL setting, however, few enterprise-level participants jointly learn a machine learning model with their own data by merely exchanging intermediate result (IR), e.g. scalar inner product [20] in generalized linear model (GLM). Since IRs might leak training data information [1, 9], they are mostly protected by additively homomorphic encryption (HE) [20, 29] and secure multi-party computation (MPC) [27] in existing VFL, thanks to the sufficient computation power and network bandwidth of enterprise-level participants. In addition, given by the similar settings and assumptions, most existing VFL algorithms follow traditional privacy-preserving multi-party machine learning methods [30, 4, 18, 3, 8], by applying Taylor approximation to the loss functions, so that HE can be adopted to protect the calculation of polynomial tasks in VFL’s joint training.

We observe several drawbacks of VFL’s data protection using HE. We measure large overhead on memory cost and processing time with HE, similar to the results in [23]. For example, a VFL-based logistic regression task takes few minutes to finish training if IR is not securely computed and transmitted, while an HE version of VFL-based logistic regression takes hours. In addition, as mentioned in [30], most existing VFL methods require a third-party collaborator to ensure data confidentiality during training process. Moreover, It is non-trivial task to approximate certain critical functions, e.g., loss function in machine learning models using only low-degree polynomials before HE, and naive approximation may lead to big errors and makes the solutions intractable [23]. Although many research efforts have been devoted to gradient protection with DP in HFL, surprisingly, we find no prior work on protecting VFL’s data confidentiality using DP. Therefore, we are motivated to propose the first differentially private framework to enforce the data confidentiality of VFL participants with negligible cost, in terms of training time, accuracy, and so on.

The contribution of this paper is threefold. First, to the best of our knowledge, we present HDP-VFL, the first differentially private framework for VFL. By thoroughly analyzing the sensitivity of VFL’s IR and conducting perturbation of IR directly within each training iteration among VFL’s participants, our method doesn’t need to perform Taylor approximation to the loss function, and meanwhile no HE is required, thus HDP-VFL could greatly boost VFL’s performance. Second, we mathematically prove that HDP-VFL not only provides utility guarantees for VFL, but also offers multi-level privacy, i.e. DP w.r.t. IR and JDP w.r.t. model weights, for VFL’s data protection. Third, by not relying on any third-party collaborator to ensure data confidentiality, HDP-VFL is easy to deploy.

2 Related Works

Although the local raw data is not exposed in FL setting, FL on its own still lacks theoretical privacy guarantees [33], and may leak sensitive information about the training data [56]. Therefore, the combination of FL and proper privacy-preserving mechanisms, such as DP [14], HE [30], MPC [17], etc., is a necessity to alleviate FL’s privacy risks.

Privacy-preserving HFL: Most privacy-preserving HFL systems are realized based on DP, MPC, and encryption, due to limited computation power and network bandwidth [5]. For example, Bonawitz et al. [5] proposed a secure aggregation scheme based on MPC to allow server to obtain an aggregation result without learning data information of each participant. Agarwal et al. [2] proposed cpSGD, a communication-efficient DP mechanism using binomial noise to avoid floating point representation issues. McMahan et al. [26] proposed DP-FedAvg, a differentially private version of vanilla FedAvg. Triastcyn and Faltings [33] proposed Bayesian differential privacy, a relaxation of DP for FL with a tighter privacy budget so that FL task over population with similarly distributed data could converge faster than DP-FedAvg. Unlike the existing methods providing gradient-level perturbation, our method focuses on IR perturbation within each multi-party SGD iteration, which is unique in VFL.

Privacy-preserving VFL: Unlike HFL releases summative private information (e.g. averaged gradients) w.r.t. some data instances, VFL releases summative private information (e.g. inner-
products between data and parameters as scalar IR) w.r.t. some dimensions, which requires unique privacy-preserving solutions. With sufficient computation power and network bandwidth, most privacy-preserving VFL systems adopted time-consuming and memory-consuming\cite{20} HE or MPC to protect the IR during joint training\cite{20,10,24} to pursue models with lossless prediction performance, which was assumed to be hard for DP\cite{20} although DP were dominate in traditional research on privacy-preserving machine learning on vertically partitioned data\cite{13,28}. Unlike existing privacy-preserving VFL, our method HDP-VFL proposes using DP to protect the training data of VFL participants. In addition, we mathematically prove HDP-VFL’s multi-level privacy and utility guarantees.

3 Preliminaries

This section reviews key definitions.

**Vertical Federated Learning (VFL).** VFL is applicable to the cases that several datasets, owned by various enterprise-level parties, share the same or similar sample space, i.e., sample IDs, but differ in feature space. Besides, only the party launching a specific joint training task owns the target vector. We define the party with target vector as the “active party” and the others as the “passive party”.

We denote VFL’s datasets as \( D^m = (X^m, y) = \{ (X^1, y), \ldots, (X^m, y) \} \), where \( (X^i, y) \) is the vector of model weights, \( L \) is the objective function, \( \ell \) is the loss function for each data sample, and \( \theta_i \) is the natural parameter for sample \( i \). The \( g(\cdot) \) is a regularization term, such as \( \ell_1 \) or \( \ell_2 \) regularization. To make sure the raw data \( x^A_i \) and \( x^B_i \), and target vector \( y^A_i \), are not exposed to each other, meanwhile gradient and loss calculation are still possible at a single data instance and for any set of outcomes \( S \subseteq R \). To prevent leakage, randomness is introduced into the computation to hide details of individual entries.

**Definition 1** (Differential Privacy \cite{15}). A randomized algorithm \( A : D \rightarrow R \) with domain \( D \) and range \( R \) satisfies \( (\epsilon, \delta) \)-differential privacy if for any two adjacent datasets \( D, D' \in D \) that differ by a single data instance and for any set of outcomes \( S \subseteq R \), the following holds:

\[
\Pr[A(D) \in S] \leq \exp(\epsilon) \Pr[A(D') \in S] + \delta.
\]

The privacy loss pair \( (\epsilon, \delta) \) is referred to as the privacy budget/loss, and it quantifies the privacy risk of algorithm \( A \). The intuition is that it is difficult for a potential attacker to infer whether a certain data point has been changed in, or added into, the input \( D \) based on a change in the output distribution. Consequently, the information of any single data point is protected. In our VFL setting, for active party and passive party, \( x^A_i \) and \( x^B_i \) are treated as a “single entry” by Definition 1 respectively.

**Definition 2** (Joint Differential Privacy \cite{22}). A randomized mechanism \( M : D \rightarrow R \) whose output is an \( n \)-tuple satisfies \( (\epsilon, \delta) \)-joint differential privacy if for any party \( i \in \{ 1, 2, \cdots, m \} \), any two adjacent datasets \( D_i, D'_i \) of party \( i \) that differ by a single data instance, all inputs \( D_{-i} \), from any other parties except for party \( i \), and any set of outcomes \( S \subseteq R^{n-1} \), the following holds:

\[
\Pr[M(D_i ; D_{-i})_{-i} \in S] \leq \exp(\epsilon) \Pr[M(D'_i ; D_{-i})_{-i} \in S] + \delta.
\]
We take a popular machine learning method,\footnote{We assume each passive party only exchanges Sec. We will instantiate \textit{HDP-VFL}. As shown in Algorithm 1, we introduce a differentially private method to calculate the joint training process won’t stop until the model converges or it reaches the maximum iteration.}

\section{HDP-VFL}

This section presents our DP framework for VFL and analyzes its privacy and utility guarantees. Specifically, we present a new analysis of IR perturbation method for VFL-based GLM joint training. Consider a VFL-based GLM joint training algorithm $A$ with $T$ iterations. For iteration $t = 1, \ldots, T$, the IR is exchanged between single active party and passive parties to calculate loss and gradient. The joint training process won’t stop until the model converges or it reaches the maximum iteration.

We assume each passive party only exchanges $\text{Sec}[\text{IR}]$ with active party, and active party exchanges $\text{Sec}[\text{IR}]$ with all passive parties. In such an assumption, “multi-passive-party” setting can be deemed as a simple extension to “single-active-passive-party” setting. Algorithm 4 takes “single-active-passive-party” setting as an example and gives our HDP-VFL algorithm. The $\text{IR}^t_i$ denotes the intermediate result of the GLM in the $t$-th iteration of $i$-th party.

\subsection{Algorithms}

As shown in Algorithm 1, we introduce a differentially private method to calculate $\text{Sec}[\text{IR}]$ to protect the training datasets. Unlike existing HE-based $\text{Sec}[\text{IR}]$ calculation, HDP-VFL doesn’t need to conduct polynomial approximation on loss function before HE can be applied. Instead, we can simply calculate IR’s $\ell_2$ sensitivity and add Gaussian noise correspondingly. In the following sections, we will instantiate HDP-VFL framework by logistic regression and mathematically prove its multi-level privacy and utility guarantees. We will then evaluate our method in Section 6.

\subsection{Examples of HDP-VFL Framework}

We take a popular machine learning method, $\ell_2$-regularized logistic regression with the $\ell_2$ regularization parameter $\lambda$, as an example of our HDP-VFL framework. The objective function is:

\[
\mathcal{L}(w) = \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-y_i x_i \cdot w)) + \frac{\lambda}{2} \|w\|^2, \quad y_i \in \{-1, +1\}, \forall i. \tag{2}
\]

Correspondingly, for $i \in s_t$, each $i$-th entry of $\text{IR}^t_A$ in Algorithm 1 equals

\[
\frac{\partial \ell}{\partial \theta_{i,t}} \bigg|_{\theta_{i,t} = x_i \cdot w^A + \text{Sec}[\text{IR}^t_B]} = \left( \frac{1}{1 + \exp[-y_i (x_i \cdot w^A + \text{Sec}[\text{IR}^t_B])]} - 1 \right) y_i. \tag{3}
\]

Then the update operation with penalty in Algorithm 1 is $\text{Pen}(w_t, g_t^c, \eta, \lambda) = w_t - \eta (g_t + \lambda w_t)$. Other parameters are: $L = 1$, $\beta_0 = 0.25$, $\beta_\eta = 1.1$, $k_y = 1$, which are defined in Section 5.

Examples for other loss functions of GLM and other types of penalties are deferred to Appendix D.
We first show that the differences resulted from adjacent datasets on $y_i$, which can cover a wide range of distributions, including the commonly-encountered Bernoulli, Huber, $\ell_2$-smooth, and $L$-Lipschitz-continuous (defined in Appendix A) w.r.t. the model weights $w$ and $\beta_y$-smooth w.r.t. the natural parameter $\theta_i$, $\forall i$. We also assume $\partial\ell/\partial\theta_i$ is $\beta_y$-Lipschitz-continuous w.r.t. $y_i$, $\forall i$. These properties can cover a wide range of loss functions, including logistic, least square, Huber, $\ell_q$ support-vector-machines loss, losses for Poisson and Gamma regression, etc.

We first show that the differences resulted from adjacent datasets on $w_i$'s are bounded.

Algorithm 1 HDP-VFL

**Input:** Datasets $(X^A, y^A), X^B$. Privacy loss $\epsilon, \delta \geq 0$. Number of epochs $e$. Number of mini-batches $r$. Norm clipping parameter $k > 0$. Number of iterations $T = e \cdot r$. Learning rate $\eta$. Loss function $\ell(\cdot, \cdot)$ with Lipschitz constant $L$ and smooth parameters $\beta_\theta, \beta_y$. Regularization parameter $\lambda$. Target bound $k_y$.

**Output:** $\hat{w}_A, \hat{w}_B$

1: Conduct entity resolution between parties to obtain common entities and then $r$.
2: Normalize data samples such that for all $i \in \{1, \ldots, n\}$, $||x_i||_2 \leq 1$.
3: Initialize the iteration index $t = 1$.
4: for $a = 1 : e$ do
5: for $j = 1 : r$ do
6: Sample $t$-th mini-batch $X_t$ with the sample indices $s_t \subset \{1, \ldots, n\}$.
7: $IR^A_t = X^A_t w^A_t$,
   Passive party sends $Sec[IR^A_t]$ to active party. // Passive Party
8: $IR^A_t = X^A_t w^A_t + Z^A$, where $Z^A \sim N(0, \sigma_A^2 I)$ is a sample of Gaussian distribution, and
   $\sigma_A = \sqrt{2\log(1.25/\delta)}(\Delta_2(||IR^A||_T^2)/\epsilon)$, where $\Delta_2(||IR^A||_T^2)$ is defined in Lemma 3.
9: $IR^B_t = X^B_t w^B_t$, $Sec[IR^B_t] = IR^B_t$, where $\theta_i,t$ is defined in Eq. (1).
10: Compute gradient $g^A_t = (IR^A_t)^T X^A_t / b$. // Active Party
11: Compute gradient $g^B_t = (Sec[IR^B_t])^T X^B_t / b$. // Passive Party
12: Norm clipping: $w^A_t = Pen(w^A_t, g^A_t, \eta, \lambda)$, $w^B_t = Pen(w^B_t, g^B_t, \eta, \lambda)$.
13: Let $t = t + 1$.
14: end for
15: end for

5 Theoretical Analyses

This section provides privacy guarantees and utility analyses for Algorithm 1. We first define notations and make some assumptions.

**Definition 4 ($\Delta(\cdot)$).** We define $\Delta := ||v - v'||_2$, where $v$ and $v'$ are vectors from two adjacent datasets $D$ and $D'$, respectively, that differ by a single data instance. The changed data instance could be either a pair of $(x^A_i, y^A_i)$ from the active party or a $x^B_j$ from the passive party, $i \in \{1, \ldots, n\}$.

**Variable spaces.** We assume the spaces for model weights and data samples are bounded such that $||w||_2 \leq k$ and $||x||_2 \leq 1, \forall i$, which is natural from the normalization and norm clipping steps of Algorithm 1. For each $i$, we assume that $y_i$ has a sub-exponential distribution with parameters $(\sigma, \nu)$ such that $|y_i| \leq k_y$ with high probability of at least

$$P(|y_i| \leq k_y) \geq \begin{cases} 1 - \exp(-k_y^2/\sigma^2), & 0 \leq k_y \leq \sigma^2/\nu \\ 1 - \exp(-k_y/\nu), & k_y > \sigma^2/\nu, \end{cases}$$

which can cover a wide range of distributions, including the commonly-encountered Bernoulli, Poisson, and Gaussian distributions for logistic, Poisson, and least square regressions, respectively.

**Properties of objective functions.** We assume that the loss function $\ell(\cdot, \cdot)$ in Eq. (1) is $\gamma$-strongly convex, $\beta$-smooth, and $L$-Lipschitz-continuous (defined in Appendix A) w.r.t. the model weights $w$ and $\beta_y$-smooth w.r.t. the natural parameter $\theta_i$, $\forall i$. We also assume $\partial\ell/\partial\theta_i$ is $\beta_y$-Lipschitz-continuous w.r.t. $y_i$, $\forall i$. These properties can cover a wide range of loss functions, including logistic, least square, Huber, $\ell_q$ support-vector-machines loss, losses for Poisson and Gamma regression, etc.
Lemma 2 \((\Delta w_t)^2\) recursion. Assume \(\Delta w_0 = 0\), then we have for any \(\eta \leq \frac{2}{\beta + \gamma}\):

\[
(\Delta w_{t+1})^2 \leq \begin{cases} 
(1 - \frac{2\eta}{b_1})^2 (\Delta w_t)^2 + \frac{4\eta L}{b} \Delta w_t + \frac{4\eta^2 L^2}{b}, & \text{if } t \neq j, j = 0, \ldots, e - 1; \\
(1 - \frac{2\eta}{b_1}) (\Delta w_t)^2, & \text{otherwise}.
\end{cases}
\]

5.1 Privacy Guarantees

This section proves that the \(\text{Sec}[IR^A_t]\)s and \(\text{Sec}[IR^B_t]\)s in HDP-VFL algorithm prevent indirect information leakage from active party’s raw data \((X^A, y^A)\) and passive party’s raw data \(X^B\) respectively. Specifically, we calculate the \(\ell_2\)-sensitivity of \([IR^A_t]_{t=1}^T\) and \([IR^B_t]_{t=1}^T\) and prove that the perturbations make our HDP-VFL algorithm joint differentially private.

Lemma 3 (\(\ell_2\)-sensitivity of \(IR^B_t\)s). Let \(T = e \ast r\) be the number of iterations, the \(\ell_2\)-sensitivity of \(IR^B_t\)s in Algorithm 7 is \(\Delta_2([IR^B_t]_{t=1}^T) = \sqrt{\frac{4L^2 e^2 T^2 y^2}{b}} + 2\frac{\beta_0 k + \beta_y k_y}{b} + 4k^2 e\).

The proofs of both Lemma 2 and 3 are deferred to Appendix B.1 and Appendix B.2.

Lemma 4 (\(\ell_2\)-sensitivity of \(IR^A_t\)s). Let \(T = e \ast r\) be the number of iterations, the \(\ell_2\)-sensitivity of \(IR^A_t\)s in Algorithm 7 is \(\Delta_2([IR^A_t]_{t=1}^T) = \sqrt{\frac{4L^2 e^2 T^2 y^2}{b}} + 2\frac{\beta_0 k + \beta_y k_y}{b} + 4k^2 e\).

The proof of Lemma 4 is deferred to Appendix B.3.

Theorem 1 (DP). Algorithm 7 is \((\epsilon, \delta)\)-differentially private w.r.t \([\text{Sec}[IR^A_t]]_{t=1}^T\) and \([\text{Sec}[IR^B_t]]_{t=1}^T\).

Theorem 2 (JDP). Algorithm 7 is \((\epsilon, \delta)\)-joint differentially private w.r.t \([w^A_t]_{t=1}^T\) and \([w^B_t]_{t=1}^T\).

The proofs of Theorems 1 and 2 are deferred to Appendices B.4 and B.5 respectively.

Theorem 1 shows that through Algorithm 1, first, the perturbations in \(\text{Sec}[IR^B_t]\)s in passive party prevent active party from getting private information about raw data \(x^B_t\), by observing the changes in the sequence of \(\text{Sec}[IR^B_t]\)s; then, the perturbations in \(\text{Sec}[IR^A_t]\)s in active party prevent passive party from getting private information about raw data \(x^A_t\) and \(y^A_t\), by observing the changes in sequence of \(\text{Sec}[IR^A_t]\)s. On the other hand, Theorem 2 further shows that through Algorithm 1, first, the perturbations in \(\text{Sec}[IR^B_t]\)s in passive party prevent active party from getting private information about raw data \(x^B_t\), by observing the changes in the sequence of \(w^B_t\)s; then, the perturbations in \(\text{Sec}[IR^A_t]\)s in active party prevent passive party from getting private information about raw data \(x^A_t\) and \(y^A_t\), by observing the changes in sequence of \(w^A_t\)s.

5.2 Utility Analyses

We build utility analyses for Algorithm 1. Our utility analyses are built upon the error bounds of inexact proximal-gradient descent presented by Schmidt et al. [32].

Let \(w^* = \arg \min w \mathcal{L}(w)\), and \(g(\cdot) = \|\cdot\|_2\). Without loss of generality, we assume that \(\|w_0 - w^*\|_2 = O(k)\). Now, we present guarantees regarding both utility and runtime.

Lemma 5. For all \(t \in \{1, \ldots, T\}\), denote the gradient error caused by noise by \(e^t = \frac{1}{b} \sum_{i \in s_t} \nabla l(x^A_t, w^A_t, y^A_t) - g^A_t, g^B_t\), where \(g^A_t\) and \(g^B_t\) are defined in Algorithm 7. It holds that \(\|e^t\| = O\left(\sqrt{\frac{\sum_{i \in s_t} T^2}{b}} \sqrt{\frac{\beta_0 k + \beta_y k_y}{b}} + \frac{2(\beta_0 k + \beta_y k_y)}{b} + \frac{4k^2 e}{\epsilon}\right)\).

Theorem 3. For \(E = \mathcal{L}(\frac{1}{b} \sum_{t=1}^T w_t) - \mathcal{L}(w^*)\), we have, with high probability,

\[
E = O\left(\left[\frac{k \sqrt{\beta}}{2} + \frac{T}{\sqrt{b}} \frac{\log(1.25/\delta)}{\epsilon} \sqrt{\frac{\sum_{i \in s_t} T^2}{b}} \sqrt{\frac{2(\beta_0 k + \beta_y k_y)}{b}} + \frac{2(\beta_0 k + \beta_y k_y)}{b} + \frac{4k^2 e}{\epsilon}\right] \right).
\]

The proof of lemma 5 and theorem 3 are deferred to Appendix B.6 and Appendix B.7.
6 Experiments

This section evaluates the proposed HDP-VFL method instantiated by a VFL-based regularized logistic regression task. We address three questions: (Q1) How is HDP-VFL’s privacy-accuracy tradeoff? (Q2) How does HDP-VFL’s hyper-parameters affect HDP-VFL’s accuracy under certain privacy requirement? (Q3) How is HDP-VFL’s runtime overhead?

6.1 Methods for Comparison

For the regularized logistic regression task, we evaluate five types of methods: 1) single-party method, which trains a logistic regression model by active party and its dataset alone; 2) traditional centralized non-FL method, which trains a regularized logistic regression model with all datasets located at a single party; 3) idealized non-private VFL method, which jointly trains a logistic regression model, with datasets partitioned at two parties, by exchanging intermediate result IR directly; 4) HE-VFL method, which jointly trains a logistic regression model, with datasets partitioned at two parties, by a) approximating loss and gradient to low-degree polynomial representations, and b) exchanging HE-based polynomial Sec[IR] between parties; 5) our HDP-VFL method, which is similar to idealized non-private VFL method except that differentially private Sec[IR] is exchanged between parties.

We implement three VFL-based methods in FATE-1.3 [35], an open source platform for VFL research. For single-party and centralized non-FL methods, we leverage the logistic regression classifier from sklearn. We use three real-world datasets from UCI Machine Learning Repository [12] for our evaluation, detailed in Table 1. We split the datasets vertically into two sub-datasets with comparable amount of attributes and distribute them to active party and passive party respectively. We use test accuracy as our evaluation metric. All experimental data is average of 10 runs.

Table 1: Datasets for Active Party and Passive Party

| Datasets | Task                  | # of Samples | # of Attributes (Active) | # of Attributes (Passive) |
|----------|-----------------------|--------------|-------------------------|--------------------------|
| Breast   | Binary Classification | 569          | 11                      | 20                       |
| Credit   | Binary Classification | 30000        | 14                      | 10                       |
| Adult    | Binary Classification | 32561        | 7                       | 8                        |

6.2 Implementation Details

We set $\lambda = 0.001$ as default for all our datasets. The epoch number $e$ and weight constraint $k$ are HDP-VFL’s two important hyper-parameters which will affect IR’s $\ell_2$ sensitivity. Normally the larger the sensitivity value, the larger the noise needed to maintain differentially private, and the lower the accuracy. We tune these hyper-parameters for the best privacy-accuracy tradeoffs. Specifically, we tune $e$ in $[5, 15]$ and $k$ in $[0.1, 1]$ using 5-fold cross-validation method on the training datasets. We set $\delta = 0.01$ according to the work of Boyd et al. [7].

6.3 Privacy-Accuracy Tradeoff

First we study HDP-VFL’s tradeoff between the privacy requirement in specific range and the accuracy of a binary classification task. By adjusting the parameters mentioned in Section 6.2, Figure 1 reports the HDP-VFL’s results on privacy and accuracy tradeoff.

From the results we can see that the best accuracy result HDP-VFL could achieve within the given privacy $\epsilon$ range in $[0.001, 10]$ is comparable to single-party method, centralized method, idealized non-private VFL method, and HE-VFL method which is deemed as lossless. This indicates that HDP-VFL could achieve high accuracy when privacy budget is sufficient, e.g. above 10, but low accuracy, only half of the lossless accuracy, when privacy budget is very tight, e.g. below 0.1. In practice using our HDP-VFL method, we set $\epsilon = 1$ which achieves acceptable accuracy-privacy tradeoffs. The privacy-accuracy tradeoff evaluation result on the full range of $\epsilon$ are shown in the supplementary material (Appendix C).
Figure 1: \textit{HDP-VFL}'s privacy-accuracy tradeoff results using public datasets. We set mini-batch size $b = 3200$, $\lambda = 0.001$, epoch number $e = 10$, and weight constraint $k = 1$.

### 6.4 Effects of Hyper-parameters

Then we study how the hyper-parameters affect its accuracy under certain privacy requirement. For each hyper-parameter under a given range, e.g., range in Section 6.2, we choose three values, e.g., lower bound, upper bound, and a middle value, to study the privacy-accuracy tradeoffs. Figure 2 shows the results of tuning hyper-parameters epoch number $e$ and weight constraint $k$.

Figure 2: Privacy-accuracy tradeoff when tuning hyper-parameters of \textit{HDP-VFL} on Breast dataset.

From the results we can see that under different privacy budget, \textit{HDP-VFL} could achieve different accuracy results with different hyper-parameter value. For example, by changing $e$ from 5 to 15 under $\epsilon = 1$, \textit{HDP-VFL}'s accuracy drops from 0.9 to around 0.6. Similarly, by changing $k$ from 0.1 to 0.5, the accuracy increases from 0.5 to around 0.9. The reason behind this is that \textit{HDP-VFL}'s hyper-parameters affect $\text{IR}$'s $\ell_2$ sensitivity, as analyzed in Lemma 3 and Lemma 4. This also indicates that under certain privacy budget, the hyper-parameter tuning should be targeting at minimizing $\text{IR}$'s $\ell_2$ sensitivity.

### 6.5 Runtime Overhead

Figure 3: The runtime overhead of idealized non-private VFL, HE-VFL, and \textit{HDP-VFL}. In (a), we change the number of epochs from 1 to 10 and set the size of dataset to 10000. In (b), we vary the dataset size from 10,000 to 100,000 and set the number of epochs to 1.
Finally we study \textit{HDP-VFL}'s runtime overhead. We mainly compare runtime overhead of idealized non-private VFL, HE-VFL, and \textit{HDP-VFL}. From the result in Figure 3 we can see that \textit{HDP-VFL} achieve the similar runtime overhead as the idealized non-private VFL, whereas HE-VFL has the largest runtime overhead, roughly 2 ~ 3 times slower than both non-private VFL and JDP-VFL. More importantly, we can see the runtime overheads of three VFL methods are proportional to the number of epochs and samples. This result strongly indicates that \textit{HDP-VFL} could significantly save joint training time under VFL setting where both parties have large amount of data samples.

7 Conclusions

Privacy-preserving vertical federated learning (VFL) is one of the effective solutions for enterprise-level data collaborations while respecting data privacy. However, the commonly used HE-based VFL suffers from the increased cost in memory and processing time when the number of training samples is huge. This paper studies this issue and presents \textit{HDP-VFL}, the first differentially private framework for VFL. By analyzing the sensitivity of VFL's intermediate result (IR) and conducting perturbation of IR directly within each training iteration, \textit{HDP-VFL} doesn’t need the Taylor approximation step and the third-party collaborator of HE-VFL, thus \textit{HDP-VFL} is easy to deploy. We mathematically prove that \textit{HDP-VFL} provides multi-level privacy and utility guarantees. Experimental results show the effectiveness of \textit{HDP-VFL}. 
Broader Impact

As any federated learning related research which trades communication efficiency and training time for data privacy and thus has an impact on energy consumption, our work, which focuses on acceleration of the vertical federated training process without compromising privacy guarantees, is no exception. Specifically, this work has a positive impact on society to respect data privacy, by complying with government regulations like GDPR\cite{GDPR}. when conducting collaborative machine learning tasks on personal data or enterprise data. At the same time, this work may have some negative consequences: 1) our work uses differentially private method and mathematical proofs to replace the time-consuming and memory-consuming homomorphic encryption based privacy-preserving federated training process, thus it may be difficult, when privacy budget is abnormally tight, to gain a lossless joint model as homomorphic encryption based vertical federated learning (VFL); 2) the low performance joint model, under abnormally tight privacy budget, may fail to deliver the expected outcomes for data collaboration between organizations; 3) our method inherits the same limitation of existing VFL, which requires datasets of organizations have to share the same or similar sample space but differ in feature space. Furthermore, we should be cautious of the fact that our method only protects enterprise data, and how enterprise data is collect from personal data is beyond the scope of this work. Finally, this work does leverage biases in the data, which is the primary task of this work.

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A Definitions of Properties for Loss Functions

**Definition 5.** Let \( f : \mathcal{W} \rightarrow \mathbb{R} \) be a function, where \( \mathcal{W} \) is a hypothesis space equipped with the standard inner product and \( \ell_2 \) norm \( \| \cdot \| : 

1) \( f \) is \( L \)-Lipschitz if for any \( u, v \in \mathcal{W} 

\| f(u) - f(v) \| \leq L \| u - v \| ; \)

2) \( f \) is \( \beta \)-smooth if

\[ \| \nabla f(u) - \nabla f(v) \| \leq \beta \| u - v \| ; \]

3) \( f \) is \( \gamma \)-strongly convex if

\[ f(u) \geq f(v) + \langle \nabla f(v), u - v \rangle + \frac{\gamma}{2} \| u - v \|^2 . \]

**Post-Processing immunity.** This property helps us use the output of a differentially private algorithm without additional information leaking, as long as we do not touch the dataset \( D \) again.

**Property 1** (Post-Processing immunity. Proposition 2.1 in Dwork et al. [15]). Let algorithm \( A_1(B_1) : \)

\( D \rightarrow I_1 \in \mathcal{R} \) be an \( (\epsilon, \delta) \) - differential privacy algorithm, and let \( f : \mathcal{R} \rightarrow \mathcal{R}' \) be an arbitrary mapping. Then, algorithm \( A_2(B_2) : D \rightarrow I_2 \in \mathcal{R}' \) is still \((\epsilon, \delta)\) - differentially private, i.e., for any \( S \subseteq \mathcal{R}, \)

\[ \mathbb{P}(I_2 \in S \mid B_2 = D) \leq e^\epsilon \mathbb{P}(I_2 \in S \mid B_2 = D') + \delta. \]

B Proof of Results In The Main Text

B.1 Proof of Lemma 2((\( \Delta w_t \))\(^2 \) recursion)

**Proof.** Let \( S \) denote the mini-batch of data with the sample indices \( s \in \{1, \ldots, n\} \) and \( |s| = b, \) and let \( w_t \) denote the model weights in the \( t \)-th step of HDP-VFL's joint training described in Algorithm 1. Let \( \mathcal{F}(w_t, S) = \frac{1}{b} \sum_{i \in s} \ell(w_t, x_i) \) denote the average loss function for \( S \). Let \( S' \) be the “neighboring data” of \( S \), and let \( w'_t \) be the model weights trained from \( S' \). To calculate the recursion of \((\Delta w_t)\)^2, consider two cases of \( S \) and \( S' \): 1) \( S \) is not changed in the \( t \)-th step of HDP-VFL, thus \( S = S' \); 2) \( S \) and \( S' \) are neighboring data differing in just one element \( x_i \rightarrow x'_i \) or \( (x_i, y_i) \rightarrow (x'_i, y'_i) \). We omit \( y'_i \) when \( (x_i, y_i) \) change to \( (x'_i, y'_i) \) for short. Following the proof of Lemma 3.7.3 of Hardt et al. [19], we have:

**Case 1:** no data instance in \( S \) is changed, we have

\[
(\Delta w_{t+1})^2 = \|w_{t+1} - w'_{t+1}\|^2 \\
= \|w_t - \eta \nabla \mathcal{F}(w_t, S) - w'_t + \eta \nabla \mathcal{F}(w'_t, S)\|^2 \\
= \|w_t - w'_t\|^2 + \eta^2 \|\nabla \mathcal{F}(w'_t, S) - \nabla \mathcal{F}(w_t, S)\|^2 \\
- 2\eta \langle w_t - w'_t, \nabla \mathcal{F}(w'_t, S) - \nabla \mathcal{F}(w_t, S) \rangle \\
= \|w_t - w'_t\|^2 + \eta^2 \left( \frac{1}{b} \sum_{i=1}^{b} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i)) \right)^2 \\
- 2\eta \langle w_t - w'_t, \frac{1}{b} \sum_{i=1}^{b} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i)) \rangle \\
\leq (1 - \frac{2\eta \beta \gamma}{\beta + \gamma}) (\Delta w_t)^2 - \left( \frac{2\eta \beta}{\beta + \gamma} \right) \eta^2 \frac{1}{b} \sum_{i=1}^{b} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i))^2 \\
\leq (1 - \frac{2\eta \beta \gamma}{\beta + \gamma}) (\Delta w_t)^2,
\]

where the first inequality, using the following inequality:

\[ \langle w_t - w'_t, \nabla \ell(w_t, x) - \nabla \ell(w'_t, x) \rangle \geq \frac{\beta \gamma}{\beta + \gamma} \|w_t - w'_t\|^2 + \frac{1}{\beta + \gamma} \|\nabla \ell(w_t, x) - \nabla \ell(w'_t, x)\|^2 .\]
Case 2): one data instance in $S$ is changed, we have

\[
(\Delta w_{t+1})^2 = \|w_{t+1} - w'_{t+1}\|^2 \\
= \|w_t - \eta(\nabla F(w_t, S) - w' + \eta \nabla F(w'_t, S'))\|^2 \\
= \|w_t - w'_t\|^2 + \eta^2 \|\nabla F(w'_t, S') - \nabla F(w_t, S)\|^2 - 2\eta \langle w_t - w'_t, \nabla F(w'_t, S') - \nabla F(w_t, S) \rangle \\
\leq \|w_t - w'_t\|^2 + \eta^2 \|\frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i))\|^2 \\
- \nabla \ell(w'_t, x_i))\|^2 - 2\eta \langle w_t - w'_t, \frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i)) \rangle \\
+ \eta^2 \|\frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i))\|^2 - 2\eta \langle w_t - w'_t, \frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i)) \rangle \\
\leq (1 - \frac{2\eta(b-1)b\gamma}{b(b+\gamma)})(\Delta w_t)^2 - \frac{2\eta}{b+\gamma} - \eta^2 \|\frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i))\|^2 \\
+ \eta^2 \|\frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i))\|^2 - 2\eta \langle w_t - w'_t, \frac{1}{b} \sum_{i=1}^{b-1} (\nabla \ell(w_t, x_i) - \nabla \ell(w'_t, x_i)) \rangle \\
\leq (1 - \frac{2\eta(b-1)b\gamma}{b(b+\gamma)})(\Delta w_t)^2 + \frac{4\eta L}{b} \Delta w_t + \frac{4\eta^2 L^2}{b^2}.
\]

In summary, the recursion about $\Delta w_t$ is:

\[
(\Delta w_{t+1})^2 \leq \begin{cases} 
(1 - \frac{2\eta(b-1)b\gamma}{b(b+\gamma)})(\Delta w_t)^2 + \frac{4\eta L}{b} \Delta w_t + \frac{4\eta^2 L^2}{b^2}, & \text{if } t = j * b, j = 0, \ldots, e - 1; \\
(1 - \frac{2\eta(b-1)b\gamma}{b(b+\gamma)})(\Delta w_t)^2, & \text{otherwise}.
\end{cases}
\]

From the above recursion we can know that, $(\Delta w_{t+1})^2 \leq (\Delta w_t)^2$ for Case 1) and $(\Delta w_{t+1})^2 \leq (\Delta w_t + \frac{2\eta L}{b})^2$ in Case 2). Consider the assumption $\Delta w_0 = 0$, then we have $(\Delta w_T)^2 \leq \frac{2\eta L}{b}.
\]

\]

\[\Box\]

B.2 Proof of Lemma 5 [\ell_2-sensitivity of IR_t^B]

\textbf{Proof.} Let $b$ be the mini-batch size, $r$ be the number of mini-batches, $e$ be the number of epochs, $T = r * e$, $\|x\| \leq 1$, $\|w^B\| \leq k$. $S$ and $S'$ are neighboring data differing in just one element $x_i^B \rightarrow x_i'^B$. Similar to B.1, consider two cases of $S$ and $S'$: 1) $S$ is not changed in the $t$-th step of HDP-VFL, thus $S = S'$; 2) $S$ and $S'$ are neighboring data differing in just one element $x_i^B \rightarrow x_i'^B$.

First, consider the $\Delta_2(\text{IR}_t^B)$ in passive party’s single step:

Case 1): $x_i^B$ is unchanged:

\[
\Delta(\text{IR}_t^B) = \left\| \sum_{i=1}^{b} (w_i^B x_i^B - w_i'^B x_i'^B) \right\| \\
\leq \sum_{i=1}^{b} (\|x_i^B\| \|w_i^B - w_i'^B\|)^2 \\
\leq \sqrt{b(\Delta w_t)^2}.
\]
Case 2): \( x_i^B \rightarrow x'_i^B \):

\[
\Delta(\mathbf{IR}_t^B) = \sqrt{\sum_{t' \neq i} (w_t^B x_{i}^B - w_{t'}^B x_{i}^B)^2 + (w_t^B x_i - w_{t'}^B x_i')^2}
\]

\[
\leq \sqrt{\sum_{t' \neq i} (w_t^B x_{i}^B - w_{t'}^B x_{i}^B)^2 + (|w_t^B x_i - w_{t'}^B x_i'| + |w_t^B x_i - w_{t'}^B x_i'|)^2}
\]

\[
\leq \sqrt{(b - 1)(\Delta w_t)^2 + (2k + \Delta w_t)^2}
\]

\[
= \sqrt{b(\Delta w_t)^2 + 4k\Delta w_t + 4k^2}.
\]

Then, consider multiple steps for the passive party:

\[
\Delta([\mathbf{IR}_t^B]_t^{T}) = \sqrt{\sum_{t=1}^{T} (\Delta(\mathbf{IR}_t^B))^2}
\]

\[
\leq \sqrt{(T - e)b(\Delta w_T)^2 + e(b(\Delta w_T)^2 + 4k\Delta w_T + 4k^2)}.
\]

Combining the proof of lemma 2 we have:

\[
\Delta([\mathbf{IR}_t^B]_t^{T}) \leq \sqrt{\frac{T(2enL)^2}{b} + \frac{8ke^2nL}{b} + 4ck^2}
\]

\[
= \sqrt{\frac{4L^2e^2n^2T^2}{b} + \frac{8kLe^2n}{b} + 4k^2e}
\]

\[
= \Delta(\mathbf{IR}_t^B). \quad \Box
\]

**B.3 Proof of Lemma 4 [\( \ell_2 \)-sensitivity of \( \mathbf{IR}_t^A \)'s]**

**Proof.** Consider \( \mathbf{IR}_t^A = [h(w_t, x, y_i)]_{i \in s_t} \), where \( s_t \) is the indices of the mini-batch for the \( t \)-th step, and

\[
h(w_t, x_i, y_i) = \frac{\partial \ell}{\partial \theta; t \mid \theta; t = x; w^i + sec(\mathbf{IR}_t^B)}.
\]

Let \( h(\cdot) \) denote \( h \) w.r.t. the enclosed variable.

Assume there exist constants \( \beta_y, \beta_x, \beta_w > 0 \), such that for all \( y, y', x, x', w, w' \)

\[
|h(y) - h(y')| \leq \beta_y|y - y'|
\]

\[
\|h(x) - h(x')\| \leq \beta_x\|x - x'\|
\]

\[
\|h(w) - h(w')\| \leq \beta_w\|w - w'\|
\]

For generalized linear model, \( \theta = xw \). Because \( \ell(\cdot, \cdot) \) is \( \beta_\theta \)-smooth w.r.t. \( \theta \), then

\[
\frac{\partial \ell}{\partial \theta \partial x} = \frac{\partial \ell}{\partial \theta \partial \theta \partial x} \leq \beta_\theta\|w\| \leq \beta_\theta k,
\]

\[
\frac{\partial \ell}{\partial \theta \partial w} = \frac{\partial \ell}{\partial \theta \partial \theta \partial w} \leq \beta_\theta\|x\| \leq \beta_\theta.
\]

Therefore, we have \( \beta_x = \beta_\theta k, \beta_w = \beta_\theta \).

First, consider one step for the active party. We consider two cases:

**Case 1**: no instance is changed, we have:

\[
\Delta([h(w_t, x_i, y_i)]_{i \in s_t}) = \sqrt{\sum_{i \in s_t} (h(w_t, y_i) - h(w'_t, y_i))^2}
\]

\[
\leq \sqrt{b(\Delta w_t)^2}.
\]

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Algorithm 1 satisfies:

\[
|\Delta([h(x_i, w_t, y_i)]_{t=1}^T)|^2 \\
= \sum_{i \neq i'} (h(w_t, x_{i'}, y_{i'}) - h(w_t, x_i, y_{i'}))^2 + \sum_{i \in S} (h(w_t, x_i, y_i) - h(w_t', x_i, y_i))^2 \\
\leq (b - 1)(\beta w \Delta w_t)^2 + (h(w_t, x_i, y_i) - h(w_t', x_i, y_i)) \\
\leq b(\beta w \Delta w_t)^2 + 2(2\beta x + 2\beta y k_y)\beta w \Delta w_t + (2\beta x + 2\beta y k_y)^2.
\]

Then, according to the definition 1, Algorithm 1 is differentially private w.r.t $S = [Sec[x_i, y_i]]_{t=1}^T$. Because we add perturbation to each element of the sequence by the $\ell_2$ sensitivity of $[IR^A_t]_{t=1}^T$, i.e., $\Delta_2([IR^A_t]_{t=1}^T)$ defined in Lemma 3 using the Gaussian Mechanism introduced in Lemma 1 with the standard deviation of $\sqrt{2\log(1.25/\delta)}\Delta_2([IR^A_t]_{t=1}^T)$, then by Lemma 1 we have for all adjacent databases $D^B$, $D'\bar{B}$ that differ in a single data instance $x_i^B \rightarrow x'_i^B$, and for any set $S \in \mathbb{R}^{T \times b}$, the Algorithm satisfies:

\[
P([Sec[IR^B_t]]_{t=1}^T \in S|D^B) \leq \exp(\epsilon)P([Sec[IR^B_t]]_{t=1}^T \in S|D') + \delta.
\]

Similarly, for the active party, the view of the passive party includes $[Sec[IR^B_t]]_{t=1}^T$. Because we add perturbation to each element of the sequence by the $\ell_2$ sensitivity of $[IR^A_t]_{t=1}^T$, i.e., $\Delta_2([IR^A_t]_{t=1}^T)$ defined in Lemma 4 using the Gaussian Mechanism introduced in Lemma 1 with the standard deviation of $\sqrt{2\log(1.25/\delta)}\Delta_2([IR^A_t]_{t=1}^T)$, then by Lemma 1 we have for all adjacent databases $D^A$, $D'\bar{A}$ that differ in a single data instance $(x_i^A, y_i^A) \rightarrow (x'_i^A, y'_i^A)$, and for any set $S \in \mathbb{R}^{T \times b}$, the Algorithm satisfies:

\[
P([Sec[IR^A_t]]_{t=1}^T \in S|D^A) \leq \exp(\epsilon)P([Sec[IR^A_t]]_{t=1}^T \in S|D') + \delta.
\]

Such properties can be easily demonstrated to hold for multiple passive parties.

Then, according to the definition 3, Algorithm 1 is $(\epsilon, \delta)$-differentially private w.r.t $[Sec[IR^A_t]]_{t=1}^T$ and $[Sec[IR^B_t]]_{t=1}^T$. 

B.5 Proof of Theorem 2 [HDP-VFL’s Joint Differential Privacy Guarantees]

Proof. For the passive party, the view of the active party includes \([w_t^A]^T_{t=1}\). Since the mapping \(\text{Sec}[IR_t^R] \rightarrow w_t^A\) does not touch any unperturbed sensitive information of \(DB\), the Post-Processing immunity property (property 1) can be applied such that combining the proof of Theorem 1, we have for all adjacent databases \(DB, DR\) that differ in a single data instance \(x_t^B \rightarrow x_t^B\), and for any set \(S \in \mathbb{R}^{T \times d}\), the Algorithm \(A\) satisfies:

\[
P([w_t^A]^T_{t=1} \in S | DB, DA) \leq \exp(\epsilon)P([w_t^A]^T_{t=1} \in S | DR, DA) + \delta.
\]

Similarly, for the active party, the view of the passive party includes \([w_t^B]^T_{t=1}\). Since the mapping \(\text{Sec}[IR_t^R] \rightarrow w_t^B\) does not touch any unperturbed sensitive information of \(DA\), the Post-Processing immunity property (property 1) can be applied such that combining the proof of Theorem 1, we have for all adjacent databases \(DA, DR\) that differ in a single data instance \(x_t^A \rightarrow x_t^A\), and for any set \(S \in \mathbb{R}^{T \times d}\), the Algorithm \(A\) satisfies:

\[
P([w_t^B]^T_{t=1} \in S | DA, DB) \leq \exp(\epsilon)P([w_t^B]^T_{t=1} \in S | DA, DB) + \delta.
\]

Such properties can be easily demonstrated to hold for multiple passive parties.

Then, according to the definition 2, Algorithm \(A\) is \((\epsilon, \delta)\)-joint differentially private w.r.t \([w_t^A]^T_{t=1}\) and \([w_t^B]^T_{t=1}\).

B.6 Proof of Lemma 5 [Utility Analyses]

Proof. In the \(t\)-th step of HDP-VFL, the gradient error caused by the noisy data (IR) is:

\[
\|e_t\|_2 = \frac{1}{b} \sum_{i \in S_t} \nabla \ell(x_t, w_t, y_i) - (g_t^A, g_t^B)
\]

\[
\leq \frac{1}{b} \sum_{i \in S_t} (h(x_t, w_t, y_i) - \sum_{i \in S_t} (h(x_t, w_t + z_t^B, y_i) + z_t^A)) x_t
\]

\[
= \frac{1}{b} \sum_{i \in S_t} x_t \left[ h(x_t, w_t, y_i) - h(x_t, w_t + z_t^B, y_i) - z_t^A \right]
\]

\[
\leq \frac{1}{b} \sum_{i \in s_t} |h(x_t, w_t, y_i) - h(x_t, w_t^i + z_t^B, y_i) - z_t^A|
\]

\[
\leq \frac{1}{b} \sum_{i \in s_t} |h(x_t, w_t, y_i) - h(x_t, w_t^i + z_t^B, y_i)| + |z_t^A|
\]

\[
\leq \frac{1}{b} \sum_{i \in s_t} \beta z_t \left( |x_t - x_t| + |z_t^A| \right)
\]

\[
\leq \frac{1}{b} \sum_{i \in s_t} \beta \left( |x_t| + |z_t^B| + |z_t^A| \right)
\]

\[
\leq \frac{1}{b} \sum_{i \in s_t} \beta (2k + |z_t^B| + |z_t^A|).
\]

Because \(z_t^A \sim N(0, \sigma_A^2)\), \(z_t^B \sim N(0, \sigma_B^2)\), \(\sigma_A = \sqrt{2 \log(1.25/\delta)} \frac{\Delta_2([IR_t^A]^T_{t=1})}{\epsilon}\), and \(\sigma_B = \sqrt{2 \log(1.25/\delta)} \frac{\Delta_2([IR_t^B]^T_{t=1})}{\epsilon}\). According to tail inequality of Gaussian variable \(z \sim N(0, \sigma^2)\) such that \(P(|z| \leq v) \geq 1 - \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-z^2/2}\) for \(z > 0\). Then for a constant \(C > 0\), with high probability
of at least $1 - \frac{\sqrt{\pi}}{\sqrt{e}} e^{-\frac{c^2}{2}}$ we have: $|z| \leq C\sigma = O(\sigma)$, then we have:

$$
\|e^t\| = O\left(\frac{\sqrt{2 \log(1.25/\delta)}}{\epsilon} \left[ \beta_0 \left( 2k + \sqrt{\frac{4L^2e^2T\eta^2}{b} + \frac{8kLe^2\eta}{b} + 4k^2e} \right) 
+ \sqrt{\frac{4\beta_0^2L^2e^2T\eta^2}{b} + 8(\beta_0k + \beta_yk_y)\beta_0L e^2\eta}{b} + 4(\beta_0k + \beta_yk_y)^2e} \right) \right]
$$

$$
= O\left(\frac{\sqrt{2 \log(1.25/\delta)}}{\epsilon} \sqrt{\frac{4\beta_0^2L^2e^2T\eta^2}{b} + 8(\beta_0k + \beta_yk_y)\beta_0L e^2\eta}{b} + 4(\beta_0k + \beta_yk_y)^2e} \right)
$$

$$
= O\left(\frac{\sqrt{2 \log(1.25/\delta)}}{\epsilon} \sqrt{\frac{4\beta_0^2L^2e^2T\eta^2}{b} + 2(\beta_0k + \beta_yk_y)\beta_0L e^2\eta}{b} + (\beta_0k + \beta_yk_y)^2e} \right).
$$

**B.7 Proof of Theorem 3 [Utility Analyses]**

**Proof.** Use Proposition 1 of Schmidt et al. [32], we have:

$$
L\left(\frac{1}{T} \sum_{t=1}^{T} w_t\right) - L(w^*) \leq \frac{\beta}{2T} \left( \|w_0 - w^*\| + 2 \sum_{t=1}^{T} \|e^t\| \right)^2.
$$

Then we replace the gradient error in Proposition 1 with the $\|e^t\|$ calculated in lemma 5 and we get:

$$
L\left(\frac{1}{T} \sum_{t=1}^{T} w_t\right) - L(w^*) = O\left( k \sqrt{\frac{\beta}{T}} 
+ 2 \left( \sqrt{\frac{\log(1.25/\delta)}}{\epsilon} \sqrt{\frac{4\beta_0^2L^2e^2T\eta^2}{b} + 2(\beta_0k + \beta_yk_y)\beta_0L e^2\eta}{b} + (\beta_0k + \beta_yk_y)^2e} \right)^2 \right).
$$

**C Privacy-Accuracy Tradeoff**

In this section, we report the test-accuracy results on the full range of $\epsilon$ in $[0.001, 1000]$, as mentioned in Section 6.2 in Figure 4. From the results we can see that HDP-VFL’s accuracy is comparable to other evaluated methods if the privacy budget is sufficient, e.g., above 1.

Figure 4: HDP-VFL’s privacy-accuracy tradeoff results using public dataset. We set mini-batch size $b = 3200$, $\lambda = 0.001$, epoch number $e = 10$, weight constraint $k = 1.$
D Extensions To Other Loss Functions and Penalties

This section introduces additional loss functions and penalties which support the mainstream machine learning tasks. We show that HDP-VFL in Algorithm 1 can cover these commonly-encountered objective functions by merely changing some parameters, and the theoretical results still hold.

D.1 Extensions To Other Loss Functions

This section first introduces two additional losses for linear regression and classification, respectively, and then introduces losses for general applications, including Poisson regression and Gamma regression.

D.1.1 Least Square Loss

The least square loss is often used for linear regression which is widely applied for continuous-variable prediction. The loss function is as follows.

\[ \ell(x_i, w, y_i) = (y_i - x_i w)^2, \quad y_i \in \mathbb{R}. \quad (5) \]

Correspondingly, for \( i \in s_t \), each \( i \)-th entry of \( \text{IR}_t^A \) in Algorithm 1 equals

\[ \frac{\partial \ell}{\partial \theta_{i,t}} \bigg|_{\theta_{i,t} = x_i w^A + \text{Sec}[\text{IR}_t^B]}, \quad (6) \]

We can normalize the targets by subtracting the mean and dividing the standard deviation to approximate standard normal variables, then with high probability, other parameters are: \( L = 6, \beta_0 = 2, \beta_y = 2, k_y = 3 \).

D.1.2 \( \ell_2 \)-loss Support Vector Machine

The support vector machine is widely applied for classification. Enjoying smooth properties, the \( \ell_2 \)-loss support vector machine is popular. The loss function is as follows.

\[ \ell(x, w, y_i) = (\max(0, 1 - y_i x w))^2, \quad y_i \in \{-1, +1\}. \quad (7) \]

Correspondingly, for \( i \in s_t \), each \( i \)-th entry of \( \text{IR}_t^A \) in Algorithm 1 equals

\[ \frac{\partial \ell}{\partial \theta_{i,t}} \bigg|_{\theta_{i,t} = x_i w^A + \text{Sec}[\text{IR}_t^B]}, \quad (8) \]

Other parameters are: \( L = 2, \beta_0 = 2, \beta_y = 2, k_y = 1 \).

D.1.3 Losses for The Exponential Dispersion Family

For general applications, this section introduce a type of loss function that follow a distribution from the exponential dispersion family [21]:

\[ \ell(y_i; \theta_i, \phi) = \frac{y_i \theta_i - b(\theta_i)}{a(\phi)} + c(y_i; \phi), \quad (9) \]

where \( \theta_i = x_i w \) is the natural parameter, \( \phi \) is the dispersion parameter, and \( a(\cdot), b(\cdot), c(\cdot) \) are known functions determined by the specific distribution, with some abuse of notation. This type of loss function covers a wide range of distribution, including Bernoulli, Normal, Poisson, and Gamma distributions for logistic regression, least square regression, Poisson regression, and Gamma regression, respectively. The specific forms of \( a(\cdot), b(\cdot), c(\cdot) \) for these distributions are listed in Table 2.

Correspondingly, for \( i \in s_t \), each \( i \)-th entry of \( \text{IR}_t^A \) in Algorithm 1 equals

\[ \frac{\partial \ell}{\partial \theta_{i,t}} \bigg|_{\theta_{i,t} = x_i w^A + \text{Sec}[\text{IR}_t^B]}, \quad (10) \]

Other parameters are: \( L = k_y / a(\phi), \beta_0 = \sup |b''(\cdot)|, \beta_y = 1/a(\phi) \).
Table 2: Some common distributions in the exponential dispersion family.

| Distribution     | $\theta$ | $\phi$ | $a(\phi)$ | $b(\theta)$ | $c(y; \phi)$     |
|------------------|----------|--------|------------|--------------|------------------|
| Bernoulli($p$)   | $\log\{p(1 - p)^{-1}\}$ | 1      | 1          | $\log(1 + e^\theta)$ | 0                |
| Normal($\mu$, $\sigma^2$) | $\mu$    | $\sigma^2$ | $\phi$    | $\theta^2/2$ | $-(y^2 \phi^{-1} + \log 2\pi)/2$ |
| Poisson($\lambda$) | $\log \lambda$ | 1      | 1          | $e^\theta$   | $-\log y!$       |
| Gamma($\alpha$, $\beta$) | $-\beta/\alpha$ | $1/\alpha$ | $\phi$    | $-\log(-\theta)$ | $\log(\alpha \gamma^{\alpha - 1}/\Gamma(\alpha))$ |

D.2 Extensions To Other Penalties

This section introduces two popular penalties. Since these penalties result in element-wise operations which do not involve data instances, no additional privacy concern is required to address. Therefore, the privacy and utility bounds still hold.

D.2.1 $\ell_1$ Norm Penalty

$\ell_1$ norm penalty is popular to introduce sparseness into model weights for interpretation or information compression. For a $\ell_1$ norm penalty $\lambda \|w\|_1$, one can update by proximal operators:

$$
\text{Pen}(w_i, g_i, \eta, \lambda) = \text{sign}(\tilde{w}_i) \max\{0, |\tilde{w}_i| - \eta \lambda\}
$$

$$
\tilde{w}_i = w_i - \eta g_i.
$$

D.2.2 Elastic Net Penalty

Elastic net penalty is effective to achieve both sparseness and accurate estimation, which is a compromise between $\ell_1$ and $\ell_2$ norm regularization.

For an elastic net penalty norm penalty $\lambda[\|w\|_1 + (\mu/2)\|w\|_2^2]$, one can also update by proximal operators:

$$
\text{Pen}(w_i, g_i, \eta, \lambda) = \frac{1}{1 + \eta \lambda \mu} \text{sign}(\tilde{w}_i) \max\{0, |\tilde{w}_i| - \eta \lambda\}
$$

$$
\tilde{w}_i = w_i - \eta g_i.
$$