Chaotic behavior of the $p$-adic Potts–Bethe mapping II

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Abstract. The renormalization group method has been developed to investigate $p$-adic $q$-state Potts models on the Cayley tree of order $k$. This method is closely related to the examination of dynamical behavior of the $p$-adic Potts–Bethe mapping which depends on the parameters $q, k$. In Mukhamedov and Khakimov [Chaotic behavior of the $p$-adic Potts–Bethe mapping. Discrete Contin. Dyn. Syst. 38 (2018), 231–245], we have considered the case when $q$ is not divisible by $p$ and, under some conditions, it was established that the mapping is conjugate to the full shift on $\kappa_p$ symbols (here $\kappa_p$ is the greatest common factor of $k$ and $p - 1$). The present paper is a continuation of the aforementioned paper, but here we investigate the case when $q$ is divisible by $p$ and $k$ is arbitrary. We are able to fully describe the dynamical behavior of the $p$-adic Potts–Bethe mapping by means of a Markov partition. Moreover, the existence of a Julia set is established, over which the mapping exhibits a chaotic behavior. We point out that a similar result is not known in the case of real numbers (with rigorous proofs).

Key words: $p$-adic numbers, Potts–Bethe mapping, chaos, shift
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1. Introduction
The present paper is a continuation of [35], where we have started to investigate the chaotic behavior of the Potts–Bethe mapping over the $p$-adic field (here $p$ is some prime number).
Note that the mapping is governed by

\[ f_{\theta, q, k}(x) = \left( \frac{\theta x + q - 1}{x + \theta + q - 2} \right)^k, \tag{1.1} \]

where \( k, q \in \mathbb{N} \) and \(|\theta - 1|_p < 1\). In [35], we have considered the case when \( q \) is not divisible by \( p \), that is, \(|q|_p = 1\). In that setting, under some conditions, we were able to prove that \( f_{\theta, q, k} \) is conjugate to the full shift on \( \kappa_p \) symbols (here \( \kappa_p \) is the greatest common factor (GCF) of \( k \) and \( p - 1 \)). In the current paper, we are going to study the same Potts–Bethe mapping when \( q \) is divisible by \( p \), that is, \(|q|_p < 1\). It is known that the thermodynamic behavior of the central site of the Potts model with nearest-neighbor interactions on a Cayley tree is reduced to the recursive system which is given by (1.1). The existence of at least two non-trivial \( p \)-adic Gibbs measures indicates that the phase transition may exist. This is closely connected to the chaotic behavior of the associated dynamical system [12, 16, 17, 23, 26, 27]. Therefore, it is important to investigate the chaotic properties of (1.1).

We stress that the Potts–Ising mapping is a particular case of the Potts–Bethe mapping, which can be obtained from (1.1) by putting \( q = 2 \). Recently, in [30, 34] under some condition, a Julia set of the Potts–Ising mapping was described, and it was shown that restricted to its Julia set, the Potts–Ising mapping is conjugate to a full shift. Therefore, it is natural to consider the Potts–Bethe mapping for \( q \geq 3 \) with \(|q|_p < 1\) and \( k \geq 2 \). In [43], all fixed points of \( f_{\theta, q, k} \) were found when \( k = 2 \) and \(|q|_p < 1\). Then, using these fixed points, the dynamics of (1.1) whenever \( k = 2 \) and \(|q|_p < 1\) was investigated in [11, 31, 32]. Recently in [1, 44], the Potts–Bethe mapping was studied for the case \( k = 3 \) and \(|q|_p < 1\).

In the present paper, we are going to consider a more general case, that is, arbitrary \( k \geq 2 \) and \(|q|_p < 1\). To formulate our main result, let us recall some necessary notions.

It is easy to notice that the function (1.1) is defined on \( \mathbb{Q}_p \setminus \{x^{(\infty)}\} \), where \( x^{(\infty)} = 2 - q - \theta \). For the sake of convenience, we write \( \text{Dom}(f_{\theta, q, k}) := \mathbb{Q}_p \setminus \{x^{(\infty)}\} \). Let us denote

\[ \mathcal{P}_{x^{(\infty)}} = \bigcup_{n=1}^{\infty} f_{\theta, q, k}^{-n}(x^{(\infty)}). \]

One can see that the set \( \mathcal{P}_{x^{(\infty)}} \) is at most countable, and could be empty for some \( k, q \) and \( \theta \) (see §3). If it is not empty, then for any \( x_0 \in \mathcal{P}_{x^{(\infty)}} \), there exists an \( n \geq 1 \) such that after \( n \)-times, we will ‘lose’ that point.

For a given mapping \( f \) on \( \mathbb{Q}_p \), we denote by \( \text{Fix}(f) \) the set of all fixed points of \( f \), that is,

\[ \text{Fix}(f) = \{ x \in \mathbb{Q}_p : f(x) = x \}. \]

Let \( f \) be an analytic function and \( x^{(0)} \in \text{Fix}(f) \). We define

\[ \lambda = \frac{d}{dx} f(x^{(0)}). \]

The fixed point \( x^{(0)} \) is called attractive if \( 0 < |\lambda|_p < 1 \), indifferent if \( |\lambda|_p = 1 \), and repelling if \( |\lambda|_p > 1 \).
For an attractive fixed point \( x^{(0)} \) of \( f \), its basin of attraction is defined by
\[
A(x^{(0)}) = \{ x \in \mathbb{Q}_p : \lim_{n \to \infty} f^n(x) = x^{(0)} \},
\]
where \( f^n = f \circ f \circ \cdots \circ f \).

The main result of the present paper is given in the following theorem.

**Theorem 1.1.** Let \( p \geq 3 \), \( k \geq 2 \), \( |q|_p < 1 \), \( |\theta - 1|_p < 1 \), and \( x^*_0 = 1 \). Then the dynamical structure of the system \((\mathbb{Q}_p, f_{\theta,q,k})\) is described as follows.

(A) If \( |k|_p \leq |q + \theta - 1|_p \), then \( \text{Fix}(f_{\theta,q,k}) = \{ x^*_0 \} \) and
\[
A(x^*_0) = \text{Dom}(f_{\theta,q,k}).
\]

(B) Assume that \( |k|_p > |q + \theta - 1|_p \) and \( |\theta - 1|_p < |q|_p^2 \). Then there exists a non-empty set \( J_{f_{\theta,q,k}} \subset \text{Dom}(f_{\theta,q,k}) \backslash \mathcal{P}_x(\infty) \) which is invariant with respect to \( f_{\theta,q,k} \) and
\[
A(x^*_0) = \text{Dom}(f_{\theta,q,k}) \backslash (\mathcal{P}_x(\infty) \cup J_{f_{\theta,q,k}}).
\]
Moreover, if \( \kappa_p \) is the GCF of \( k \) and \( p - 1 \), then the following hold:

(B1) if \( \kappa_p = 1 \), then there exists \( x_0 \in \text{Fix}(f_{\theta,q,k}) \) such that \( x_0 \neq x^*_0 \) and
\[
J_{f_{\theta,q,k}} = \{ x_0 \};
\]

(B2) if \( \kappa_p \geq 2 \), then \( (J_{f_{\theta,q,k}}, f_{\theta,q,k}, | \cdot |_p) \) is topologically conjugate to the full shift dynamics on \( \kappa_p \) symbols.

**Remark 1.2.** It is worth pointing out that, in the present paper, the condition \( |\theta - 1|_p < |q|_p^2 \) is assumed to get essential estimations and calculations to prove the main result. The results of a recent paper [1] show that such a condition could be loosened to \( |\theta - 1|_p < |q|_p \), but only for the case \( k = 3 \) where explicit expressions of the fixed points of the function \( f_{\theta,q,k} \) have essentially been used to get more exact estimations. However, in this paper, we are able to prove the chaoticity of the Potts–Bethe mapping for arbitrary values of \( k \) (under the condition \( |\theta - 1|_p < |q|_p^2 \)) and moreover, we are not even using the existence of the fixed points. Once we have proved that the Potts–Bethe mapping is conjugate to a full shift, then one concludes the existence of the fixed points. Roughly speaking, we are constructing (explicitly) a Markov partition of the mapping (1.1) which allows us to prove the main result of the current paper. However, the results of [1] indicate that the chaoticity of the function (1.1) could be obtained even in the case of \( |q|_p^2 \leq |\theta - 1|_p < |q|_p \), but this will be a topic for another work. Here, it is better to emphasize that the results are valid when \( p \geq 3 \). The case \( p = 2 \) is considered pathological in the \( p \)-adic analysis (see for example [10]). Indeed, in [1], it was established that when \( p = 2 \) and \( k = 3 \), the function (1.1) does not have chaotic behavior. For general values of \( k \), owing to huge calculations and numerous technical issues, this case could be investigated elsewhere.

**Remark 1.3.** In [41, 42], the authors established that the function (1.1) may have at least one fixed point and, moreover, they found a necessary condition (that is \( q \) is divisible by \( p \)) for the existence of more than one fixed point. Therefore, the following conjecture was
formulated: Let \( k \in \mathbb{N}, q \in p\mathbb{N}, \) and \( |\theta - 1|_p < 1, \) then the function (1.1) has at least two fixed points. The formulated Theorem 1.1(A) shows that the mentioned conjecture is not always true.

We stress that, in the \( p \)-adic setting, owing to the lack of a convex structure of the set of \( p \)-adic Gibbs measures, it was quite difficult to constitute a phase transition with some features of the set of \( p \)-adic Gibbs measures. However, Theorem 1.1(B2) yields that the set of \( p \)-adic Gibbs measures is huge which is \textit{a priori} not clear (see [24, 42]). Moreover, the method of the present work allows one to find lots of periodic \( p \)-adic Gibbs measures for the \( p \)-adic Potts model. Furthermore, Theorem 1.1(B) together with the results of [29, 33] will open new perspectives in investigations of generalized \( p \)-adic self-similar sets.

On one hand, our results shed some light on the question of the investigation of dynamics of rational functions in the \( p \)-adic analysis, because a global dynamical structure of rational maps on \( \mathbb{Q}_p \) remains unclear. Some particular rational functions have been considered in [4, 5, 7, 8, 10, 13–15, 18, 21, 39]. On the other hand, the obtained results may have potential applications in the cryptography to build pseudo-random codes (see [2, 3, 37, 45]). We point out that some \( p \)-adic chaotic dynamical systems have been studied in [9, 45].

2. Preliminaries
2.1. \( p \)-adic numbers. Let \( \mathbb{Q} \) be the field of rational numbers. For a fixed prime number \( p \), every rational number \( x \neq 0 \) can be represented in the form \( x = p^r \frac{n}{m} \), where \( r, n, m \in \mathbb{Z} \), \( m \) is a positive integer, and \( n \) and \( m \) are relatively prime with \( p \): \((p, n) = 1, (p, m) = 1\). The \( p \)-adic norm of \( x \) is given by

\[
|x|_p = \begin{cases} 
    p^{-r} & \text{for } x \neq 0, \\
    0 & \text{for } x = 0. 
\end{cases}
\]

This norm is non-Archimedean and satisfies the so-called strong triangle inequality

\[
|x + y|_p \leq \max\{|x|_p, |y|_p\}.
\]

The completion of \( \mathbb{Q} \) with respect to the \( p \)-adic norm defines the \( p \)-adic field \( \mathbb{Q}_p \). Any \( p \)-adic number \( x \neq 0 \) can be uniquely represented in the canonical form

\[
x = p^{\mathrm{ord}_p(x)}(x_0 + x_1 p + x_2 p^2 + \cdots),
\]

where \( \mathrm{ord}_p(x) \in \mathbb{Z} \) and the integers \( x_j \) satisfy: \( 0 \leq x_j \leq p - 1, \) \( x_0 \neq 0 \). In this case, \( |x|_p = p^{-\mathrm{ord}_p(x)} \).

Recall that \( \mathbb{Q}_p \) is not an ordered field. So, we may compare two \( p \)-adic numbers only with respect to their \( p \)-adic norms.

In what follows, to simplify our calculations, we are going to introduce new symbols ‘\( O \)’ and ‘\( o \)’ (roughly speaking, these symbols replace the notation ‘\( \mod p^k \)’ without noticing the power of \( k \)). Namely, for a given \( p \)-adic number \( x \), by \( O[x] \), we mean a \( p \)-adic number with the norm \( p^{-\mathrm{ord}_p(x)} \), that is, \( |x|_p = |O(x)|_p \). By \( o[x] \), we mean a \( p \)-adic number with a norm strictly less than \( p^{-\mathrm{ord}_p(x)} \), that is, \( |o(x)|_p < |x|_p \). For instance, if \( x = 1 - p + p^2 \), we can write \( x - 1 + p = o[p] \), \( x - 1 = o[1] \), or \( x = O[1] \). The symbols \( O[\cdot] \) and \( o[\cdot] \)
will make our work easier when we need to calculate the \( p \)-adic norm of \( p \)-adic numbers. It is easy to see that \( y = O[x] \) if and only if \( x = O[y] \).

We give some basic properties of \( O[\cdot] \) and \( o[\cdot] \), which will be used later on.

**Lemma 2.1.** Let \( x, y \in \mathbb{Q}_p \). Then the following statements hold.

1. \( O[x]O[y] = O[xy] \).
2. \( xO[y] = O[xy], \ O[y]x = O[xy] \).
3. \( O[x]o[y] = o[xy] \).
4. \( o[x]o[y] = o[xy] \).
5. If \( y \neq 0 \), then \( O[x]/O[y] = O[x/y] \).
6. If \( y \neq 0 \), then \( o[x]/O[y] = o[x/y] \).

For each \( a \in \mathbb{Q}_p, r > 0 \), we denote

\[
B_r(a) = \{ x \in \mathbb{Q}_p : |x - a|_p < r \}.
\]

We recall that \( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \) and \( \mathbb{Z}_p^* = \{ x \in \mathbb{Q}_p : |x|_p = 1 \} \) are the set of all \( p \)-adic integers and \( p \)-adic units, respectively.

The following result is well known as Hensel’s lemma.

**Lemma 2.2.** [6, 22] Let \( F(x) \) be a polynomial whose coefficients are \( p \)-adic integers. Let \( x^* \) be a \( p \)-adic integer such that for some \( i \geq 0 \),

\[
F(x^*) \equiv 0 \pmod{p^{2i+1}}, \quad F'(x^*) \equiv 0 \pmod{p^i}, \quad F'(x^*) \not\equiv 0 \pmod{p^{i+1}}.
\]

Then \( F(x) \) has a \( p \)-adic integer root \( x_\ast \) such that \( x_\ast \equiv x^* \pmod{p^{i+1}} \).

The \( p \)-adic exponential is defined by

\[
\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]

which converges for every \( x \in B_{p^{-1/(p-1)}}(0) \). Denote

\[
\mathcal{E}_p = \{ x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)} \}.
\]

This set is the range of the \( p \)-adic exponential function. The following fact is well known.

**Lemma 2.3.** [40] The set \( \mathcal{E}_p \) has the following properties.

(a) \( \mathcal{E}_p \) is a group under multiplication.
(b) If \( a, b \in \mathcal{E}_p \), then the following are true:

\[
|a - b|_p < \begin{cases} \frac{1}{2}, & p = 2, \\ 1, & p \neq 2 \end{cases} \quad |a + b|_p = \begin{cases} \frac{1}{2}, & p = 2, \\ 1, & p \neq 2. \end{cases}
\]

(c) If \( a \in \mathcal{E}_p \), then there is an element \( h \in B_{p^{-1/(p-1)}}(0) \) such that \( a = \exp_p(h) \).
LEMMA 2.4. Let $k \geq 2$ and $\alpha, \beta \in \mathcal{E}_p$. Then there exists a unique $\gamma \in 1 + p\mathbb{Z}_p$ such that

$$\sum_{j=0}^{k-1} \alpha^{k-j-1} \beta^j = k\gamma. \quad (2.2)$$

Moreover, if $p \neq 2$, then $\gamma \in \mathcal{E}_p$.

Remark 2.5. We notice that Lemma 2.4 has been proved in [35] for $p \neq 2$. The proof of the case $p = 2$ is similar to that one. We notice that this lemma plays a crucial role in our further investigations. Especially, we will often use the fact $\gamma \in \mathcal{E}_p$.

COROLLARY 2.6. Let $k \in \mathbb{N}$. Then

$$\alpha^k - \beta^k = k(\alpha - \beta) + o[k(\alpha - \beta)] \quad \text{for all } \alpha, \beta \in \mathcal{E}_p.$$ 

Proof. Let $\alpha, \beta \in \mathcal{E}_p$. By Lemma 2.4,

$$\sum_{j=0}^{k-1} \alpha^{k-1-j} \beta^j = k + k(\gamma - 1),$$

where $\gamma - 1 = o[1]$.

Hence, Lemma 2.1 implies

$$\alpha^k - \beta^k = (\alpha - \beta) \sum_{j=0}^{k-1} \alpha^{k-j-1} \beta^j$$

$$= k(\alpha - \beta) + k(\alpha - \beta)(\gamma - 1)$$

$$= k(\alpha - \beta) + O[k(\alpha - \beta)]o[1]$$

$$= k(\alpha - \beta) + o[k(\alpha - \beta)],$$

which is the required relation.

Remark 2.7. In our further investigations, we mainly use Corollary 2.6 in the following form. Namely, for $k \in \mathbb{N}$,

$$\alpha^k - 1 = k(\alpha - 1) + o[k(\alpha - 1)] \quad \text{for all } \alpha \in \mathcal{E}_p. \quad (2.3)$$

We notice that a monomial equation $x^k = a$ over $\mathbb{Q}_p$ has been studied in [36, 38]. In our further investigations, we only need the following special case of that equation.

THEOREM 2.8. [36] Let $p \geq 3$ and $a \in \mathcal{E}_p$. Then the following statements hold:

(i) if $|k|_p \leq |a - 1|_p$, then the polynomial $x^k - a$ has no root;

(ii) if $|k|_p > |a - 1|_p$, then for every $\xi \in \{y \in \mathbb{F}_p : y^k \equiv a \pmod{p}\}$, the polynomial $x^k - a$ has a unique root in $B_1(\xi)$.

Here $\mathbb{F}_p$ stands for the ring of integers modulo $p$.

Remark 2.9. Thanks to Theorem 2.8, for every $a \in \mathcal{E}_p$ with $|a - 1|_p < |k|_p$, the equation $x^k = a$ has a single root belonging to $\mathcal{E}_p$, which is called the principal $k$th root and denoted
Chaotic behavior of the p-adic Potts–Bethe mapping II

3439

by \sqrt[k]{a}. In what follows, the symbol \sqrt[k]{a} (for \( a \in \mathbb{E}_p \)) always means the principal kth root of \( a \). Therefore, for \(|a - 1|_p < |k|_p\), all solutions of the monomial equation \( x^k = a \) have the following form: \( x_i = \xi_i \sqrt[k]{a} \), where \( \xi_i^k = 1 \) and \( \sqrt[k]{a} \) is a principal kth root of \( a \).

2.2. p-adic subshift. Let \( f : X \rightarrow \mathbb{Q}_p \) be a map from a compact open set \( X \) of \( \mathbb{Q}_p \) into \( \mathbb{Q}_p \). We assume that (i) \( f^{-1}(X) \subset X \); (ii) \( X = \bigcup_{j \in I} B_r(a_j) \) can be written as a finite disjoint union of balls of centers \( a_j \) and of the same radius \( r \) such that for each \( j \in I \), there is an integer \( \tau_j \in \mathbb{Z} \) such that

\[
|f(x) - f(y)|_p = p^{\tau_j}|x - y|_p, \quad x, y \in B_r(a_j).
\]  

(2.4)

For such a map \( f \), define its Julia set by

\[
J_f = \bigcap_{n=0}^{\infty} f^{-n}(X).
\]  

(2.5)

It is clear that \( f^{-1}(J_f) = J_f \) and then \( f(J_f) \subset J_f \). The triple \((X, J_f, f)\) is called a p-adic weak repeller if all \( \tau_j \) in (2.4) are non-negative, but at least one is positive. We call it a p-adic repeller if all \( \tau_j \) in (2.4) are positive. For any \( i \in I \), we let

\[
I_i := \{ j \in I : B_r(a_j) \cap f(B_r(a_i)) \neq \emptyset \} = \{ j \in I : B_r(a_j) \subset f(B_r(a_i)) \}
\]

(the second equality holds because of the expansiveness of the ultrametric property). Then define a matrix \( A = (a_{ij})_{I \times I} \), called incidence matrix, as follows:

\[
a_{ij} = \begin{cases} 
1 & \text{if } j \in I_i, \\
0 & \text{if } j \notin I_i.
\end{cases}
\]

If \( A \) is irreducible, we say that \((X, J_f, f)\) is transitive. Here the irreducibility of \( A \) means that for any pair \((i, j) \in I \times I \), there is a positive integer \( m \) such that \( a_{ij}^{(m)} > 0 \), where \( a_{ij}^{(m)} \) is the entry of the matrix \( A^m \).

Given \( I \) and the irreducible incidence matrix \( A \) as above, we denote

\[
\Sigma_A = \{(x_k)_{k \geq 0} : x_k \in I, \ A_{x_k,x_{k+1}} = 1, \ k \geq 0\},
\]

which is the corresponding subshift space, and let \( \sigma \) be the shift transformation on \( \Sigma_A \). We equip \( \Sigma_A \) with a metric \( d_f \) depending on the dynamics, which is defined as follows. First, for \( i, j \in I, \ i \neq j \), let \( \kappa(i, j) \) be the integer such that \(|a_i - a_j|_p = p^{-\kappa(i,j)}\). It is clear that \( \kappa(i, j) < \log_p(r) \). By the ultrametric inequality,

\[
|x - y|_p = |a_i - a_j|_p, \quad i \neq j \quad \text{for all } x \in B_r(a_i), \text{ for all } y \in B_r(a_j).
\]

For \( x = (x_0, x_1, \ldots, x_n, \ldots) \in \Sigma_A \) and \( y = (y_0, y_1, \ldots, y_n, \ldots) \in \Sigma_A \), define

\[
d_f(x, y) = \begin{cases} 
p^{-\tau_0 - \tau_1 - \cdots - \tau_{n-1} - \kappa(x_n,y_n)} & \text{if } n \neq 0, \\
p^{-\kappa(x_0,y_0)} & \text{if } n = 0
\end{cases}
\]

where \( n = n(x, y) = \min\{i \geq 0 : x_i \neq y_i\} \). It is clear that \( d_f \) defines the same topology as the classical metric which is defined by \( d(x, y) = p^{-n(x,y)} \).
Theorem 2.10. [9] Let \((X, J_f, f)\) be a transitive \(p\)-adic weak repeller with incidence matrix \(A\). Then the dynamics \((J_f, f, |\cdot|_p)\) is isometrically conjugate to the shift dynamics \((\Sigma_A, \sigma, d_f)\).

3. Proof of Theorem 1.1: part (A)

In what follows, we always assume that \(p \geq 3\) and \(|q|_p < 1\). To prove Theorem 1.1(A), we need the following auxiliary lemma.

Lemma 3.1. Let \(p \geq 3\) and \(k \in \mathbb{N}\). If \(a \in \mathcal{E}_p\) and \(|a - 1|_p \geq |k|_p\), then \(|x^k - a|_p \geq |a - 1|_p\) for any \(x \in \mathbb{Q}_p\).

Proof. Take an arbitrary \(a \in \mathcal{E}_p\) such that \(|a - 1|_p \geq |k|_p\). We distinguish three cases.

Case \(x \notin \mathbb{Z}^*_p\). Then we immediately get \(|x^k - 1|_p \geq 1\). From \(|a - 1|_p < 1\), using the strong triangle inequality, one has \(|x^k - a|_p \geq 1\). This yields that \(|x^k - a|_p > |a - 1|_p\).

Case \(x \in \mathcal{E}_p\). Then noting \(|x - 1|_p < 1\), owing to Corollary 2.6, we obtain \(|x^k - 1|_p < |k|_p\). The last one together with \(|a - 1|_p \geq |k|_p\) implies \(|x^k - a|_p = |a - 1|_p\).

Case \(x \in \mathcal{Z}^*_p \setminus \mathcal{E}_p\). In this case, \(x\) has the following canonical form:

\[
x = x_0 + x_1 \cdot p + x_2 \cdot p^2 + \cdots ,
\]

where \(2 \leq x_0 \leq p - 1\) and \(0 \leq x_i \leq p - 1, i \geq 1\). Then \((x/x_0) \in \mathcal{E}_p\). According to Remark 2.7,

\[
(x/x_0)^k - 1 = O[k(x - x_0)] = o[k].
\]

Consequently, \(|x^k - x_0^k|_p < |k|_p\), which yields \(|x^k - 1|_p = |x_0^k - 1|_p\). Now we need to check two subcases, \(|x_0^k - 1|_p = 1\) and \(|x_0^k - 1|_p < 1\), separately.

Suppose that \(|x_0^k - 1|_p = 1\). Then, owing to \(|a - 1|_p < 1\), one has \(|x^k - a|_p = 1\).

Hence, \(|x^k - a|_p > |a - 1|_p\).

Let us assume that \(|x_0^k - 1|_p < 1\). For convenience, let us write \(k = mp^s\), where \(s \geq 1\) and \((m, p) = 1\). Then noting \(x_0^p \equiv x_0(\mod\, p)\), from \(x^{mp^s} \equiv 1(\mod\, p)\), we obtain \(|x_0^m - 1|_p < 1\). Thanks to Remark 2.7, one finds

\[
x_0^{mp^s} - 1 = p^s(x_0^m - 1) + o[p^s(x_0^m - 1)],
\]

which yields \(|x_0^k - 1|_p < |k|_p\). Hence, from \(|a - 1|_p \geq |k|_p\), it follows that \(|x^k - a|_p = |a - 1|_p\). Consequently, \(|x^k - a|_p = |a - 1|_p\). This completes the proof.

Remark 3.2. We notice that the set \(P_{x^{(\infty)}}\) is empty if \(|k|_p \geq |q + \theta - 1|_p\). Indeed, from \(x^{(\infty)} \in \mathcal{E}_p\), where \(x^{(\infty)} = 2 - q - \theta\) and \(|x^{(\infty)} - 1|_p \geq |k|_p\), owing to Lemma 3.1, we infer that

\[
|f^n_{\theta,q,k}(x) - x^{(\infty)}|_p \geq |x^{(\infty)} - 1|_p\quad \text{for all } n \in \mathbb{N},\text{ for all } x \in \text{Dom}(f_{\theta,q,k}).
\]

Hence, noting \(x^{(\infty)} \neq 1\), we can conclude that \(P_{x^{(\infty)}} = \emptyset\).
Let us define
\[ g_{\theta,q}(x) = \frac{\theta x + q - 1}{x + \theta + q - 2}. \] (3.1)

In our further investigations, we use the following simple property of the function \( g_{\theta,q} \):
\[ g_{\theta,q}(x) - 1 = \frac{(\theta - 1)(x - 1)}{x + \theta + q - 2}. \] (3.2)

We notice that \( f_{\theta,q,k}(x) = (g_{\theta,q}(x))^k \) for any \( x \in \text{Dom}(f_{\theta,q,k}) \). It is clear that the function \( f_{\theta,q,k} \) has a fixed point \( x_0 = 1 \).

**Proof of Theorem 1.1:** (A). Let \( |k|_p \leq |q + \theta - 1|_p \) and denote
\[ K_1 = \{ x \in \mathbb{Q}_p : |x - 1|_p < |q + \theta - 1|_p \}, \]
\[ K_2 = \{ x \in \mathbb{Q}_p : |x - 1|_p = |x - 2 + q + \theta|_p \}. \]

First, we show that \( f_{\theta,q,k}(x) \in K_1 \cup K_2 \) for any \( x \notin K_1 \cup K_2 \). Then we prove that \( f_{\theta,q,k}(x) \in K_1 \) for any \( x \in K_2 \). Finally, we show that \( f_{\theta,q,k}^n(x) \to 1 \) for any \( x \in K_1 \).

Indeed, let \( x \notin K_1 \cup K_2 \). From \( |q + \theta - 1|_p < 1 \), owing to Lemma 3.1,
\[ |f_{\theta,q,k}(x) - 2 + q + \theta|_p \geq |q + \theta - 1|_p, \]
which is equivalent to either \( |f_{\theta,q,k}(x) - 1|_p < |q + \theta - 1|_p \) or \( |f_{\theta,q,k}(x) - 1|_p = |f_{\theta,q,k}(x) - 2 + q + \theta|_p \). This yields that \( f_{\theta,q,k}(x) \in K_1 \cup K_2 \).

Now assume that \( x \in K_2 \). Then by (3.2),
\[ g_{\theta,q}(x) - 1 = (\theta - 1)O[1] = O[\theta - 1] = o[1] \]
which means \( g_{\theta,q}(x) \in \mathcal{E}_p \). Then thanks to Remark 2.7,
\[ |f_{\theta,q,k}(x) - 1|_p < |k|_p. \]
The last one together with \( |k|_p \leq |q + \theta - 1|_p \) implies \( |f_{\theta,q,k}(x) - 1|_p < |q + \theta - 1|_p \) and hence \( f_{\theta,q,k}(x) \in K_1 \).

Finally, we suppose that \( x \in K_1 \). It then follows from (3.2) that
\[ g_{\theta,q}(x) - 1 = \frac{(\theta - 1)(x - 1)}{O[q + \theta - 1]} = (\theta - 1)o[1] = o[\theta - 1] = o[1]. \]
This again means \( g_{\theta,q}(x) \in \mathcal{E}_p \). By Remark 2.7,
\[ f_{\theta,q,k}(x) - 1 = O\left[ \frac{k(\theta - 1)(x - 1)}{q + \theta - 1} \right]. \]
Noting \( |q + \theta - 1|_p > |k(\theta - 1)|_p \), from the last one,
\[ |f_{\theta,q,k}(x) - 1|_p < |x - 1|_p. \]
Hence,
\[ |f_{\theta,q,k}^n(x) - 1|_p \leq \frac{1}{p^n}|x - 1|_p, \]
which yields \( f_{\theta,q,k}^n(x) \to 1 \) as \( n \to \infty \). This completes the proof. \( \square \)
4. Proof of Theorem 1.1: the first part of (B)
In this section, we are going to study the dynamics of \( f_{θ,q,k} \) when \( |θ - 1|_p < |q^2|_p \) and \( |q|_p < |k|_p \). In what follows, the following auxiliary fact is needed.

**Proposition 4.1.** Let \( p \geq 3 \) and \( |θ - 1|_p < |q|_p < |k|_p \). If \( x \in \text{Dom}(f_{θ,q,k}) \) with \( |x - 2 + q + θ|_p > |θ - 1|_p \), then \( f^n_{θ,q,k}(x) \to 1 \) as \( n \to ∞ \).

**Proof.** First, we notice that \( |x - 2 + q + θ|_p > |θ - 1|_p \) implies \( |x - 1 + q|_p > |θ - 1|_p \). Owing to \( |θ - 1|_p < |q|_p \), we are going to consider two cases: (i) \( |x - 1 + q|_p \geq |q|_p \) and (ii) \( |θ - 1|_p < |x - 1 + q|_p < |q|_p \).

**Case (i).** Let \( |x - 1 + q|_p \geq |q|_p \). This means that either \( x \in B_{|q|_p}(1) \) or \( |x - 1 + q|_p = |x - 1|_p \). First, we show that the condition \( |x - 1 + q|_p = |x - 1|_p \) yields \( f^n_{θ,q,k}(x) \in B_{|q|_p}(1) \). Furthermore, we establish that \( f^n_{θ,q,k}(x) \to 1 \) for any \( x \in B_{|q|_p}(1) \).

Let us pick \( x \in \mathbb{Q}_p \) such that \( |x - 1 + q|_p = |x - 1|_p \). Then \( |q|_p \leq |x - 1|_p \). Keeping in mind \( θ - 1 = o[q] \), one finds \( θ - 1 = o[x - 1 + q] \) and
\[
x - θ + q - 2 = O[x - 1 + q] = O[x - 1].
\]

So, by (3.2),
\[
g_{θ,q}(x) - 1 = \frac{(θ - 1)(x - 1)}{O[x - 1]} = o[q]O[1] = o[q].
\]

Because \( |k|_p \leq 1 \), owing to Remark 2.7, we obtain \( |f^n_{θ,q,k}(x) - 1|_p < |q|_p \), which implies \( f^n_{θ,q,k}(x) \in B_{|q|_p}(1) \).

Now let us suppose that \( x \in B_{|q|_p}(1) \). Then by (3.2),
\[
g_{θ,q}(x) - 1 = \frac{O[q](x - 1)}{q + o[q]} = \frac{o[q](x - 1)}{O[q]} = o[1](x - 1) = o[x - 1].
\]

Hence, again thanks to Remark 2.7, one has \( |f^n_{θ,q,k}(x) - 1|_p < |x - 1|_p \), which yields
\[
|f^n_{θ,q,k}(x) - 1|_p \leq \frac{1}{p^n}|x - 1|_p \quad \text{for all } n \in \mathbb{N}.
\]

So, \( f^n_{θ,q,k}(x) \to 1 \) as \( n \to ∞ \).

**Case (ii).** Let \( |θ - 1|_p < |x - 1 + q|_p < |q|_p \). Then
\[
g_{θ,q}(x) - 1 = \frac{o[x - 1 + q]O[q]}{O[x - 1 + q]} = o[1]O[q] = o[q].
\]

Again, Remark 2.7 yields \( |f^n_{θ,q,k}(x) - 1|_p < |q|_p \). Hence, by (i), we have \( f^n_{θ,q,k}(x) \to 1 \) as \( n \to ∞ \). This completes the proof.

**Corollary 4.2.** Let \( p \geq 3 \) and \( |θ - 1|_p < |q|_p < |k|_p \). If \( |x - 1 + q|_p \geq |q|_p \), then \( f^n_{θ,q,k}(x) \to 1 \) as \( n \to ∞ \).

**Proof.** Let \( |x - 1 + q|_p \geq |q|_p \). By \( |θ - 1|_p < |q|_p \) and the strong triangle inequality, one finds \( |x - 2 + q + θ|_p > |θ - 1|_p \). Hence, the last one owing to Proposition 4.1 yields \( f^n_{θ,q,k}(x) \to 1 \).
LEMMA 4.3. Let $p \geq 3$ and $|\theta - 1|_p < |q|_p < |k|_p$. If $|x - 2 + q + \theta|_p < |q(\theta - 1)|_p$, then $f^n_{\theta,q,k}(x) \to 1$ as $n \to \infty$.

**Proof.** Take arbitrary $x \in \text{Dom}(f_{\theta,q,k})$ such that $|x - 2 + q + \theta|_p < |q(\theta - 1)|_p$. Then

$$\frac{\theta x + q - 1}{x + q + \theta - 2} = \theta - \frac{(q + \theta - 1)(\theta - 1)}{x + q + \theta - 2} = \theta + \frac{O[q(\theta - 1)]}{o[q(\theta - 1)]]} = \frac{O[q(\theta - 1)]}{o[q(\theta - 1)]},$$

which yields $|f^n_{\theta,q,k}(x)|_p > 1$. Hence, $|f^n_{\theta,q,k}(x) - 2 + q + \theta|_p > |\theta - 1|_p$. Then by Proposition 4.1, we obtain the desired assertion.

Our aim is to construct a set $X \subset \text{Dom}(f_{\theta,q,k})$ for which a triple $(X, J_{f_{\theta,q,k}}, f_{\theta,q,k})$ is a transitive $p$-adic repeller. Thanks to Proposition 4.1 and Lemma 4.3, the required set $X$ should be a subset of the following set:

$$Y = \left\{ x \in U : |q(\theta - 1)|_p \leq |f_{\theta,q,k}(x) - 2 + q + \theta|_p \leq |\theta - 1|_p \right\},$$

where

$$U := \bigcup_{n \in \mathbb{Q}_p: |n|_p \leq |n|_p \leq 1} B_{|q(\theta - 1)|_p}(2 - q - \theta + \eta(\theta - 1)).$$

One can see that for $|q|_p \leq |\eta|_p \leq 1$, we have $x_\eta \in Y$ if and only if

$$\begin{cases} 
  x_\eta = 2 - q - \theta + \eta(\theta - 1) + o[q(\theta - 1)], \\
  |q(\theta - 1)|_p \leq |f_{\theta,q,k}(x_\eta) - 2 + q + \theta|_p \leq |\theta - 1|_p.
\end{cases} \tag{4.1}$$

**Remark 4.4.** We notice that if for $|q|_p \leq |\eta|_p \leq 1$ one of the assumptions of (4.1) does not hold, then $f^n_{\theta,q,k}(x_\eta) \to 1$ as $n \to \infty$.

Now we are going to find a necessary condition for $\eta \in \mathbb{Q}_p$ which yields (4.1).

**PROPOSITION 4.5.** Let $p \geq 3$, $|k|_p > |q|_p$ and $|\theta - 1|_p < |q^2|_p$. Assume that for $\eta \in \mathbb{Q}_p$ with $|q|_p \leq |\eta|_p \leq 1$, (4.1) holds. Then the following statements are true:

1. If $|\eta|_p = |q|_p$, then $((\eta - q)/\eta)^k = 1 + o[1]$;
2. If $|\eta|_p > |q|_p$, then $\eta = k - ((k - 1)q)/2) + (((k - 1)(k - 2)q^2)/6k) + o[q]$.

**Proof.** Assume that (4.1) holds. Then

$$f_{\theta,q,k}(x_\eta) = 2 - q + \theta + O[\theta - 1] = 1 - q + o[q] = 1 + o[1]. \tag{4.2}$$

(1) Let $|\eta|_p = |q|_p$. By (4.2), one finds

$$\left(1 - \frac{q}{\eta} + \frac{(\eta - 1)(\theta - 1)}{\eta}\right)^k = 1 + o[1]. \tag{4.3}$$

Noting $|\theta - 1|_p < |q|_p$, we obtain $((\eta - 1)(\theta - 1))/\eta = o[1]$. Plugging the last one into (4.3),

$$\left(\frac{\eta - q}{\eta} + o[1]\right)^k = 1 + o[1]. \tag{4.4}$$

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Finally, keeping in mind $|k|_p \leq 1$, from (4.4),
\[
\left( \frac{\eta - q}{\eta} \right)^k = 1 + o[1].
\]

(2) Let $|\eta|_p > |q|_p$. First, let us assume that $|k|_p \leq |\eta - k|_p$. Then, using the strong triangle inequality, we can easily check
\[
\frac{k}{\eta} \neq 1 + o[1]. \tag{4.5}
\]
From $|\theta - 1|_p < |q^2|_p$ and $|q|_p < |\eta|_p$,
\[
g_{\theta,q}(x_\eta) = 1 - \frac{q}{\eta} + o[q].
\]
Keeping in mind $f_{\theta,q,k}(x_\eta) = (g_{\theta,q}(x_\eta))^k$, by (2.3),
\[
f_{\theta,q,k}(x_\eta) = 1 - \frac{kq}{\eta} + o\left[ \frac{kq}{\eta} \right]. \tag{4.6}
\]
Plugging (4.5) into (4.6) yields
\[
f_{\theta,q,k}(x_\eta) - 1 \neq -q + o[q],
\]
but it contracts to $f_{\theta,q,k}(x_\eta) - 1 + q = o[q]$. This means that $|\eta|_p > |q|_p$ and (4.1) hold only for $|\eta - k|_p < |k|_p$.

So, suppose $|\eta - k|_p < |k|_p$, which implies $|\eta|_p = |k|_p$. Now we prove our assertion by contradiction. Suppose in contrary,
\[
\left| \eta - k + \frac{(k - 1)q}{2} - \frac{(k - 1)(k - 2)q^2}{6k} \right|_p \geq |q|_p. \tag{4.7}
\]
Noting $|q|_p < |k|_p \leq 1$, we then can easily check the following:
\[
\left| \frac{(k - 1)q}{2} \right|_p \leq |q|_p,
\]
\[
\left| \frac{(k - 1)(k - 2)q^2}{6k} \right|_p \leq |q|_p.
\]
These inequalities together with (4.7) yield
\[
\left| \eta - k + \frac{(k - 1)q}{2} - \frac{(k - 1)(k - 2)q^2}{6k} \right|_p = \max\{|\eta - k|_p, |q|_p\}. \tag{4.8}
\]
Owing to $|\theta - 1|_p < |q^2|_p$ and $|\eta|_p = |k|_p$, we have
\[
f_{\theta,q,k}(x_\eta) = \left( 1 - \frac{q}{\eta} + \frac{(\eta - 1)(\theta - 1)}{\eta} \right)^k
= \left( 1 - \frac{q}{\eta} + o\left[ \frac{q^2}{k} \right] \right)^k.
\]
\[
= \left( 1 - \frac{q}{\eta} \right)^k + o\left[ \frac{q^2}{k} \right]
\]

\[
= \left( 1 - \frac{q}{k} \sum_{n=0}^{\infty} \left( \frac{k - \eta}{k} \right)^n \right)^k + o\left[ \frac{q^2}{k} \right]
\]

\[
= 1 - q \sum_{n=0}^{\infty} \left( \frac{k - \eta}{k} \right)^n + \frac{(k - 1)q^2}{2k} - \frac{(k - 1)(k - 2)q^2}{6k^2} + o\left[ \frac{q^2}{k} \right]
\]

\[
= 1 - q + \frac{q}{k} \left( \eta - k + \frac{(k - 1)q^2}{2} - \frac{(k - 1)(k - 2)q^2}{6k} \right) - q \sum_{n=2}^{\infty} \left( \frac{k - \eta}{k} \right)^n + o\left[ \frac{q^2}{k} \right]. \quad (4.9)
\]

We calculate the norm of \( q \sum_{n=2}^{\infty} ((k - \eta)/k)^n \). Keeping in mind \( |\eta - k|_p < |k|_p \), by the strong triangle inequality,

\[
\left| q \sum_{n=2}^{\infty} \left( \frac{k - \eta}{k} \right)^n \right|_p = \left| \frac{q(k - \eta)^2}{k^2} \right|_p . \quad (4.10)
\]

So, we need to calculate the norm of \( (q(k - \eta)^2)/k^2 \). One can see that

\[
\left| \frac{(k - \eta)^2}{k} \right|_p < |\eta - k|_p \leq \max\{|\eta - k|_p, |q|_p\}.
\]

The last inequality together with (4.10) yields

\[
\left| q \sum_{n=2}^{\infty} \left( \frac{k - \eta}{k} \right)^n \right|_p < \left| \frac{q}{k} \right|_p \cdot \max\{|\eta - k|_p, |q|_p\}. \quad (4.11)
\]

Then by (4.8) and (4.11), using the strong triangle inequality, one finds

\[
\left| \frac{q}{k} \left( \eta - k + \frac{(k - 1)q^2}{2} - \frac{(k - 1)(k - 2)q^2}{6k} \right) - q \sum_{n=2}^{\infty} \left( \frac{k - \eta}{k} \right)^n \right|_p \geq \left| \frac{q^2}{k} \right|_p . \quad (4.12)
\]

Hence, plugging (4.12) into (4.9) and noting \( |k|_p \leq 1 \), one finds

\[
|f_{\theta, q, k}(x_\eta) - 1 + q|_p \geq |q^2|_p .
\]
which together with $|\theta - 1|_p < |q^2|_p$ implies $|f_{\theta,q,k}(x_\eta) - 2 + q + \theta|_p > |\theta - 1|_p$, which contradicts (4.1). This means that if $|\eta|_p > |q|_p$, (4.1) holds, then

$$
\eta = k - \frac{(k - 1)q}{2} + \frac{(k - 1)(k - 2)q^2}{6k} + o[1].
$$

Remark 4.6. One can see that if $|\eta|_p = |q|_p$ and $((\eta - q)/\eta)^k \in \mathcal{E}_p$, then $((\eta - q)/\eta) \in \mathbb{Z}_p^* \setminus \mathcal{E}_p$. This means that there exists a root of unity $\xi \neq 1$ such that $(\eta - q)/\eta = \xi + o[1]$, which yields $\eta = q/(1 - \xi) + o[1]$. Without loss of generality for $\xi = 1$, we put $\eta = k - (((k - 1)q)/2) + (((k - 1)(k - 2)q^2)/6k) + o[q]$. Consequently, we have found a relation between all roots of unity and all $\eta \in \mathbb{Q}_p$ for which (4.1) holds.

Let us denote

$$
\text{Sol}_p(x^k - 1) = \{\xi \in \mathbb{Z}_p^* : \xi^k = 1\}, \quad \kappa_p = \text{card}(\text{Sol}_p(x^k - 1)),
$$

where $\text{card}(A)$ is the cardinality of a set $A$.

We point out that $\kappa_p$ is the number of solutions of the equation $x^k = 1$ in $\mathbb{Q}_p$. From the results of [38], we infer that $\kappa_p$ is the GCF of $k$ and $p - 1$. Therefore, it is clear that $1 \leq \kappa_p \leq k$.

For a given $\xi_i \in \text{Sol}_p(x^k - 1)$, $i \in \{1, \ldots, \kappa_p\}$, we define

$$
x_{\xi_i} = \begin{cases} 
1 - q + (k - 1)\left(1 - \frac{q}{2} + \frac{(k - 2)q^2}{6k}\right)(\theta - 1) & \text{if } \xi_i = 1, \\
2 - q - \theta + \frac{q}{1 - \xi_i}(\theta - 1) & \text{if } \xi_i \neq 1,
\end{cases}
$$

and

$$
X = \bigcup_{i=1}^{\kappa_p} B_r(x_{\xi_i}), \quad r = |q(\theta - 1)|_p.
$$

(4.13)

(4.14)

By construction, the set $X$ is a subset of $\mathcal{E}_p \setminus \{1\}$.

Thanks to Remark 4.4, as a corollary of Proposition 4.5, we can formulate the following result which describes the basin of attraction of $x^*_0 = 1$.

PROPOSITION 4.7. Let $p \geq 3$ and $|k|_p > |q|_p$. If $|\theta - 1|_p < |q^2|_p$, then

$$
\lim_{n \to \infty} f_{\theta,q,k}^n(x) = 1 \quad \text{for all } x \in \text{Dom}(f_{\theta,q,k}) \setminus X.
$$

The next result shows that the set $X$ (given by (4.14)) consists of disjoint balls.

LEMMA 4.8. Let $p \geq 3$ and $|\theta - 1|_p < |q^2|_p < |k^2|_p$. If $x_{\xi_i}$ is given by (4.13) and $r = |q(\theta - 1)|_p$, then $B_r(x_{\xi_i}) \cap B_r(x_{\xi_j}) = \emptyset$ if $i \neq j$. 

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Proof. Let $x_{\xi_i}$ and $x_{\xi_j}$ be given by (4.13), where $i \neq j$. We consider two cases.

Case $\xi_i = 1$ and $\xi_j \neq 1$. Then from (4.13),
\[
x_{\xi_i} - x_{\xi_j} = \left( k - \frac{(k-1)q}{2} + \frac{(k-1)(k-2)q^2}{6k} - \frac{q}{1 - \xi_j} \right)(\theta - 1)
\]
\[
= (k + o[k])(\theta - 1)
\]
\[
= O(k(\theta - 1)],
\]
which implies that $|x_{\xi_i} - x_{\xi_j}|_p > |q(\theta - 1)|_p$. Hence, $B_r(x_{\xi_i}) \cap B_r(x_{\xi_j}) = \emptyset$.

Case $\xi_i \neq 1$ and $\xi_j \neq 1$. In this case,
\[
x_{\xi_i} - x_{\xi_j} = \frac{(\xi_i - \xi_j)q(\theta - 1)}{(1 - \xi_i)(1 - \xi_j)} = \frac{O[1]q(\theta - 1)}{O[1]} = O[q(\theta - 1)],
\]
which means $|x_{\xi_i} - x_{\xi_j}|_p = r$. Hence, we infer that $B_r(x_{\xi_i}) \cap B_r(x_{\xi_j}) = \emptyset$.

To prove the first part of (B) of Theorem 1.1, we define the following set:
\[
J_{f_{a,q,k}} = \bigcap_{n=1}^{\infty} f_{\theta, a,q,k}^{-n}(X). \tag{4.15}
\]

Remark 4.9. In [36], we have considered the following function over $\mathbb{Q}_p$ ($p \geq 3$):
\[
f_{b,c,d}(x) = \left( \frac{bx - c}{x - d} \right)^k, \quad b, c, d \in \mathbb{E}_p, \ c \neq bd.
\]
It was proved that the mapping $f_{b,c,d}$ has exactly $\kappa_p + 1$ fixed points belonging to $\mathbb{E}_p$ if $d = 1 - b + c$ and $|b - 1|_p < |c - 1|_p^2 < |k|_p^2$ (see [36, Theorem 4.5]). One can see that if one takes $b = \theta$, $c = 1 - q$, and $d = 2 - q - \theta$, then the function $f_{b,c,d}$ reduces to $f_{\theta, a,q,k}$. So, as a corollary of the mentioned result and noting that $\text{Fix}(f_{\theta, a,q,k}) \cap (\mathbb{Q}_p \setminus X) = \{1\}$, we conclude that if $|\theta - 1|_p < |q|_p^2 < |k|_p^2$, then $f_{\theta, a,q,k}$ has exactly $\kappa_p$ fixed points belonging to $X$. This yields $J_{f_{\theta, a,q,k}} \neq \emptyset$ for $|\theta - 1|_p < |q|_p^2 < |k|_p^2$. Moreover, we may check that for every $i \in \{1, 2, \ldots, \kappa_p\}$, there exists a unique fixed point of $f_{\theta, a,q,k}$ in $B_r(x_{\xi_i})$ (see Proposition 5.5).

Proof of Theorem 1.1: (B). By Proposition 4.7, the set $\mathcal{P}_x(\infty)$ can not belong to $\text{Dom}(f_{\theta, a,q,k}) \setminus X$. Then $\mathcal{P}_x(\infty) \subset X$. According to the construction of $J_{f_{\theta, a,q,k}}$ (see (4.15)), we conclude that $J_{f_{\theta, a,q,k}} \cap \mathcal{P}_x(\infty) = \emptyset$. However, owing to Remark 4.9, the set $J_{f_{\theta, a,q,k}}$ is not empty and by the construction, it is invariant with respect to $f_{\theta, a,q,k}$. Then for any $x \notin J_{f_{\theta, a,q,k}} \cup \mathcal{P}_x(\infty)$, there exists a number $m \geq 1$ such that $f_{\theta, a,q,k}^m(x) \notin X$. Hence, owing to Proposition 4.7, we infer that $f_{\theta, a,q,k}^n(x) \to 1$ as $n \to \infty$. The proof is complete.

5. Proof of Theorem 1.1: parts (B1) and (B2)
In the following, we need some auxiliary facts.
Lemma 5.1. Let $p \geq 3$ and $|k|_p > |q|_p$. Then for any $a \in B_{|q^2|_p}(1 - q)$, the equation $x^k = a$ has a unique solution $x_*$ on $\mathcal{E}_p$. Moreover, this solution satisfies

$$x_* - 1 + \frac{q}{k} + \frac{(k - 1)q^2}{2k^2} - \frac{(k - 1)(k - 2)q^3}{6k^3} = o\left[\frac{q^2}{k}\right].$$

(5.1)

Proof. Let $|k|_p > |q|_p$ and $a \in B_{|q^2|_p}(1 - q)$. For convenience, we use the canonical form of $a$:

$$a = 1 + a_t p^t + a_{t+1} p^{t+1} + \cdots$$

We note that $|k|_p > p^{-t}$. Let us put $x_t = 1$ and define a sequence $\{x_{n+t}\}_{n \geq 1}$ as follows:

$$x_{n+t} = x_{n+t-1} + \frac{a - x^k_{n+t-1}}{k}.$$  

(5.2)

First, by induction, let us show that $x_{n+t+1} \in \mathcal{E}_p$ for any $n \geq 1$. It is clear that $x_t \in \mathcal{E}_p$ and, therefore, we assume that $x_{n+t+1} \in \mathcal{E}_p$ for some $n \geq 1$. Then, owing to Remark 2.7, we obtain

$$x^k_{n+t+1} - 1 = k(x_{n+t+1} - 1) + o(k(x_{n+t+1} - 1)),$$

which is equivalent to

$$|x^k_{n+t+1} - 1|_p < |k|_p.$$  

The last inequality together with $|a - 1|_p < |k|_p$ implies that $|x_{n+t+1} - x_{n+t}|_p < 1$. Consequently, from $x_{n+t+1} \in \mathcal{E}_p$, we find $x_{n+t} \in \mathcal{E}_p$. Hence, $x_{n+t} \in \mathcal{E}_p$ for any $n \geq 1$.

Owing to Corollary 2.6, by (5.2), we have

$$x^k_{n+t} - x^k_{n+t-1} = k(x_{n+t} - x_{n+t-1}) + o[k(x_{n+t} - x_{n+t-1})]$$

$$= a - x^k_{n+t-1} + o[a - x_{n+t-1}],$$

which means

$$|x^k_{n+t} - a|_p < |x^k_{n+t-1} - a|_p.$$  

Hence, there exists a number $n_0 \geq 1$ such that

$$|x^k_{n_0+t} - a|_p \leq |(a - 1)^2|_p.$$  

Now, let us consider a polynomial $F(x) = x^k - a$. It is easy to check that

$$|F'(x_{n_0+t-1})|_p = |k|_p, \quad \text{and} \quad |F(x_{n_0+t-1})|_p \leq |(a - 1)^2|_p.$$  

So by $|k^2|_p > |(a - 1)^2|_p$ and Hensel’s lemma, we conclude that $F$ has a root $x_*$ such that

$$|x_* - x_{n_0+t-1}|_p \leq |(a - 1)^2|_p.$$  

From $x_{n_0+t-1} \in \mathcal{E}_p$, we infer that $x_* \in \mathcal{E}_p$. The uniqueness of the solution on $\mathcal{E}_p$ immediately follows from Remark 2.7.
Suppose that $x_* \in \mathcal{E}_p$ is a solution of $x^k - a = 0$. Let us show that it can be represented by (5.1). It can be checked that $x_*$ has the following form:

$$x_* = 1 - \frac{q}{k} + \alpha_*,$$

where $\alpha_* = o[q/k]$. Indeed, because $x_* \in \mathcal{E}_p$, there exists $y_* \in p\mathbb{Z}_p$ such that $x_*$ can be represented as follows: $x_* = 1 + y_* + o[y_*]$. Then by Remark 2.7, we have $x_*^k = 1 + ky_* + o[ky_*]$. By assumption, $a = 1 - q + o[q^2]$. Hence, we obtain the following implications:

$$x_*^k - a = 0 \implies ky_* + q = o[q] \implies y_* = -\frac{q}{k} + o\left[\frac{q}{k}\right] \implies x_* = 1 - \frac{q}{k} + o\left[\frac{q}{k}\right].$$

Furthermore, from (5.3), one finds

$$a = x_*^k = 1 + k\left(-\frac{q}{k} + \alpha_*\right) + k(k-1)\left(-\frac{q}{k} + \alpha_*\right)^2$$

$$+ \frac{k(k-1)(k-2)}{6} \left(-\frac{q}{k} + \alpha_*\right)^3 + o\left[\frac{q^2}{k}\right],$$

$$= 1 - q + k\alpha_* + \frac{(k-1)q^2}{2k} - \frac{(k-1)(k-2)q^3}{6k^2} + o\left[\frac{q^2}{k}\right].$$

Plugging $a = 1 - q + o[q^2]$ into the last equality,

$$k\alpha_* + \frac{(k-1)q^2}{2k} - \frac{(k-1)(k-2)q^3}{6k^2} = o\left[\frac{q^2}{k}\right].$$

Hence,

$$\alpha_* = -\frac{(k-1)q^2}{2k^2} + \frac{(k-1)(k-2)q^3}{6k^3} + o\left[\frac{q^2}{k^2}\right].$$

Putting the last one into (5.3) yields (5.1), which completes the proof. \qed

**Remark 5.2.** We point out that in [38], the existence of solutions of the equation $x^k = a$ on $\mathbb{Z}_p^*$ has been obtained, but an advantage of Lemma 5.1 is that it provides the uniqueness of solution in $\mathcal{E}_p$ with an explicit expression which is essential in our investigation.

On the set $X$ (see (4.14)), the mapping $f_{\theta,q,k}$ has exactly $\kappa_p$ inverse branches:

$$h_i(x) = \frac{(q + \theta - 2)\xi_i \sqrt[p]{x} - q + 1}{\theta - \xi_i \sqrt[p]{x}},$$

where $\xi_i^k = 1$, $i \in \{1, \ldots, \kappa_p\}$ and $\sqrt[p]{x}$, as before, is a principal root of $x \in X$ (see Remark 2.9).
PROPOSITION 5.3. Let $p \geq 3$ and $|k|_p > |q|_p$. If $|\theta - 1|_p < |q^2|_p$, then

$$h_i(X) \subset B_r(x_{\xi_i}).$$

(5.4)

Proof. Let $x \in X$. We consider two cases: $\xi_i = 1$ and $\xi_i \neq 1$.

Case $\xi_i = 1$. In this case, we have

$$h_i(x) - x_{\xi_i} = \left(\frac{q + \theta - 2}{\theta - \sqrt[k]{x}} - q + 1\right) - \left(1 - q + (k - 1)\left(1 - \frac{q}{2} + \frac{(k - 2)q^2}{6k}\right)(\theta - 1)\right)$$

$$= (\theta - 1)(q + \theta - 1 + (k - ((k - 1)q)/2) + ((k - 1)(k - 2)q^2)/6k)(\sqrt[k]{x} - \theta))$$

$$= (\theta - 1)(q + (k - ((k - 1)q)/2) + ((k - 1)(k - 2)q^2)/6k)(\sqrt[k]{x} - 1) + o[q^2].$$

(5.5)

However, owing to Lemma 5.1,

$$\sqrt[k]{x} - 1 = -\frac{q}{k} - \frac{(k - 1)q^2}{2k^2} + \frac{(k - 1)(k - 2)q^3}{6k^3} + o\left[\frac{q^2}{k^2}\right].$$

(5.6)

Furthermore, keeping in mind $|q|_p < |k|_p$, we can easily check the following:

$$q + \left(k - \frac{(k - 1)q}{2} + \frac{(k - 1)(k - 2)q^2}{6k}\right)(\sqrt[k]{x} - 1) = o\left[\frac{q^2}{k}\right].$$

(5.7)

Plugging (5.6), (5.7) into (5.5), one has

$$h_i(x) - x_{\xi_i} = \frac{(\theta - 1)o[q^2/k]}{O[q/k]} = (\theta - 1)o[\frac{q}{k}] = o[q(\theta - 1)].$$

This means $h_i(x) \in B_r(x_{\xi_i})$. The arbitrariness of $x \in X$ yields (5.4).

Case $\xi_i \neq 1$. Then,

$$h_i(x) - x_{\xi_i} = \frac{(q + \theta - 2)\xi_i\sqrt[k]{x} - q + 1}{\theta - \xi_i\sqrt[k]{x}} - \left(2 - q - \theta + \frac{q}{1 - \xi_i}(\theta - 1)\right)$$

$$= \frac{(\theta - 1)(q - (\theta q/(1 - \xi_i)) + (\xi_i\sqrt[k]{x}q/(1 - \xi_i)) + \theta - 1)}{\theta - 1 + 1 - \xi_i\sqrt[k]{x}}$$

$$= \frac{(\theta - 1)(q - (q/(1 - \xi_i)) + (\xi_i q/(1 - \xi_i)) + o[q])}{O[1]}$$

$$= \frac{(\theta - 1)o[q]}{O[1]} = o[q(\theta - 1)].$$

The last one implies $h_i(x) \in B_r(x_{\xi_i})$. Again, owing to the arbitrariness of $x \in X$, we obtain (5.4).
COROLLARY 5.4. Let \( p \geq 3 \) and \( |k|_p > |q|_p \). If \( |	heta - 1|_p < |q^2|_p \) and \( X \) is the set given by (4.14), then the following statements hold:

(i) \( f^{-1}_{\theta,q,k}(X) \subset X \);
(ii) \( B_r(x_{\xi_i}) \subset f_{\theta,q,k}(B_r(x_{\xi_j})) \) for any \( i, j \in \{1, \ldots, \kappa_p\} \).

PROPOSITION 5.5. Let \( p \geq 3 \), \( |k|_p > |q|_p \). If \( |	heta - 1|_p < |q^2|_p \) and \( X \) is the set given by (4.14), then the following statements hold:

(a) if \( \xi_i = 1 \), then

\[
|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = \frac{|q(x - y)|_p}{|k(\theta - 1)|_p} \quad \text{for any} \ x, y \in B_r(x_{\xi_i}); \tag{5.8}
\]

(b) if \( \xi_i \neq 1 \), then

\[
|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = \frac{|k(x - y)|_p}{|q(\theta - 1)|_p} \quad \text{for any} \ x, y \in B_r(x_{\xi_i}). \tag{5.9}
\]

Proof. (a) Recall that for \( \xi_i = 1 \),

\[
x_{\xi_i} = 1 - q + (k - 1)\left(1 - \frac{q}{2} + \frac{(k - 2)q^2}{6k}\right)(\theta - 1).
\]

Thus for \( x \in B_r(x_{\xi_i}) \), by (3.2), we have

\[
g_{\theta,q}(x) - 1 = \frac{(\theta - 1)(-q + o[q])}{k(\theta - 1) + o[k(\theta - 1)]} = O\left[\frac{q}{k}\right].
\]

This means \( g_{\theta,q}(x) \in \mathcal{E}_p \). Then, owing to Corollary 2.6, for any \( x, y \in B_r(x_{\xi_i}) \),

\[
|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = |k(g_{\theta,q}(x) - g_{\theta,q}(y))|_p. \tag{5.10}
\]

However,

\[
g_{\theta,q}(x) - g_{\theta,q}(y) = \frac{(\theta - 1)(q + \theta - 1)(x - y)}{(x - 2 + q + \theta)(y - 2 + q + \theta)}
\]

\[
= \frac{O[q(\theta - 1)](x - y)}{(k(\theta - 1) + o[k(\theta - 1)])^2}
\]

\[
= \frac{O[q](x - y)}{O[k^2(\theta - 1)]}.
\]

Plugging the last one into (5.10) implies (5.8).

(b) Recall that for \( \xi_i \neq 1 \),

\[
x_{\xi_i} = 2 - q - \theta + \frac{q(\theta - 1)}{1 - \xi_i}.
\]

Then for \( x \in B_r(x_{\xi_i}) \),

\[
g_{\theta,q}(x) = 1 + \frac{(\theta - 1)(x - 1)}{x - 2 + q + \theta} = 1 + \frac{(\theta - 1)(-q + o[q])}{(q(\theta - 1))/(1 - \xi_i) + o[q(\theta - 1)]} = \xi_i + o[1].
\]

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So, \(|g_{\theta,q}(x)|_p = 1\). Moreover, \((g_{\theta,q}(x)/g_{\theta,q}(y)) \in \mathcal{E}_p\) for any \(x, y \in B_r(x_\xi)\). Then, owing to Corollary 2.6,

\[ |f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = |k(g_{\theta,q}(x) - g_{\theta,q}(y))|_p. \quad (5.11) \]

However, from

\[
g_{\theta,q}(x) - g_{\theta,q}(y) = (q + \theta - 1)(x - y)/(x - 2 + q + \theta)(y - 2 + q + \theta)
\]

\[
= O[q(\theta - 1)](x - y)/(1 - \xi) + o[q(\theta - 1)]^2
\]

\[
= (x - y)/O[q(\theta - 1)]
\]

and (5.11), we arrive at (5.9).

\[ \Box \]

**Proof of Theorem 1.1.** (B1) Assume that \(x^k = 1\) has only one solution. Then the set \(X\) given by (4.14) consists of only one ball \(B_r(x_1)\), where

\[
x_1 = 1 - q + (k - 1)(1 - q/2)(\theta - 1), \quad r = |q(\theta - 1)|_p.
\]

By the proof for the case (B),

\[
A(x_0^*) = \text{Dom}(f_{\theta,q,k}) \setminus (J_{f_{\theta,q,k}} \cup \mathcal{P}_{x(\infty)}),
\]

where \(x_0^* = 1\), and \(J_{f_{\theta,q,k}}\) is given by (4.15). By Proposition 5.5, for any \(x, y \in B_r(x_1)\), we have

\[ |f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p > p^2|x - y|_p, \]

which implies that \(|J_{f_{\theta,q,k}}| \leq 1\). Thanks to Remark 4.9, we have \(J_{f_{\theta,q,k}} \neq \emptyset\). Because \(|J_{f_{\theta,q,k}}| \leq 1\), one has \(J_{f_{\theta,q,k}} = \{x_*\}\), where \(x_* \in \text{Fix}(f_{\theta,q,k}) \cap (\mathcal{E}_p \setminus \{1\})\).

(B2) Assume that \(x^k = 1\) has \(\kappa_p\) solutions. Consider the set \(X\) defined by (4.14). Then according to Corollary 5.4(i) and Proposition 5.5, the triple \((X, J_{f_{\theta,q,k}}, f_{\theta,q,k})\) is a \(p\)-adic repeller. Owing to Corollary 5.4(ii), the corresponding incidence matrix \(A\) has dimension \(\kappa_p \times \kappa_p\) and can be written as follows:

\[
A = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{pmatrix}.
\]

This means that the triple \((X, J_{f_{\theta,q,k}}, f_{\theta,q,k})\) is transitive. Hence, Theorem 2.10 implies that the dynamics \((J_{f_{\theta,q,k}}, f_{\theta,q,k}, \cdot|_p)\) is topologically conjugate to the full shift dynamics on \(\kappa_p\) symbols. \[ \Box \]
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A. Appendix

A.1. p-adic measure. Let \((X, B)\) be a measurable space, where \(B\) is an algebra of subsets \(X\). A function \(\mu : B \rightarrow \mathbb{Q}_p\) is said to be a p-adic measure if for any \(A_1, \ldots, A_n \subset B\) such that \(A_i \cap A_j = \emptyset (i \neq j)\), the equality holds

\[
\mu \left( \bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu(A_j).
\]

A p-adic measure is called a probability measure if \(\mu(X) = 1\). A p-adic probability measure \(\mu\) is called bounded if \(\sup\{|\mu(A)|_p : A \in B\} < \infty\). For more detailed information about p-adic measures, we refer to \[19–21\].

A.2. Cayley tree. Let \(\Gamma^k_+ = (V, L)\) be a semi-infinite Cayley tree of order \(k \geq 1\) with the root \(x^0\) (whose each vertex has exactly \(k + 1\) edges, except for the root \(x^0\), which has \(k\) edges). Here, \(V\) is the set of vertices and \(L\) is the set of edges. The vertices \(x\) and \(y\) are called nearest neighbors and they are denoted by \(l = \langle x, y \rangle\) if there exists an edge connecting them. A collection of the pairs \(\langle x, x_1 \rangle, \ldots, \langle x_{d-1}, y \rangle\) is called a path from the point \(x\) to the point \(y\). The distance \(d(x, y), x, y \in V\), on the Cayley tree, is the length of the shortest path from \(x\) to \(y\).

\[
W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.
\]

The set of direct successors of \(x\) is defined by

\[
S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n.
\]

Observe that any vertex \(x\) has \(k\) direct successors.

A.3. p-adic quasi-Gibbs measure. In this section, we recall the definition of p-adic quasi-Gibbs measure (see \[25\]).

Let \(q \geq 2\) and \(\Phi = \{1, 2, \ldots, q\}\). Here, \(\Phi\) is called a state space and is assigned to the vertices of the tree \(\Gamma^k_+ = (V, \Lambda)\). A configuration \(\sigma\) on \(V\) is then defined as a function \(x \in V \rightarrow \sigma(x) \in \Phi\); in a similar manner, one defines configurations \(\sigma_n\) and \(\omega\) on \(V_n\) and \(W_n\). The set of all configurations on \(V\) (respectively \(V_n, W_n\)) coincides with \(\Omega = \Phi^V\) (respectively \(\Omega_{V_n} = \Phi^{V_n}, \quad \Omega_{W_n} = \Phi^{W_n}\)). One can see that \(\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}\). Using this, for given configurations \(\sigma_{n-1} \in \Omega_{V_{n-1}}\) and \(\omega \in \Omega_{W_n}\), we define their concatenation by

\[
(\sigma_{n-1} \vee \omega)(x) = \begin{cases} 
\sigma_{n-1}(x) & \text{if } x \in V_{n-1}, \\
\omega(x) & \text{if } x \in W_n.
\end{cases}
\]

It is clear that \(\sigma_{n-1} \vee \omega \in \Omega_{V_n}\).
The Hamiltonian of the $p$-adic Potts model on $\Omega V_n$ is

$$H_n(\sigma) = J \sum_{(x,y) \in L_n} \delta_{\sigma(x)\sigma(y)},$$

(A.1)

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant and $\delta_{ij}$ is the Kroneker’s symbol.

A construction of a generalized $p$-adic quasi-Gibbs measure corresponding to the model is given below.

Assume that $h : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p / \Phi_1$ is a mapping, that is, $h_x = (h_{1,x}, h_{2,x}, \ldots, h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ ($i \in \Phi$) and $x \in V \setminus \{x^{(0)}\}$. Given $n \in \mathbb{N}$, we consider a $p$-adic probability measure $\mu^{(n)}_{h,\rho}$ on $\Omega V_n$ defined by

$$\mu^{(n)}_{h,\rho}(\sigma) = \frac{1}{Z_n^{(h)}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}.$$  

(A.2)

Here, $\sigma \in \Omega V_n$, and $Z_n^{(h)}$ is the corresponding normalizing factor

$$Z_n^{(h)} = \sum_{\sigma \in \Omega V_n} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}.$$  

(A.3)

We are interested in the construction of an infinite volume distribution with given finite-dimensional distributions. More exactly, we would like to find a $p$-adic probability measure $\mu_h$ on $\Omega V_n$ which is compatible with the given ones $\mu^{(n)}_{h,\rho}$, that is,

$$\mu_h(\{\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n\}) = \mu^{(n)}_{h,\rho}(\sigma_n) \quad \text{for all } \sigma_n \in \Omega V_n, \ n \in \mathbb{N}. \quad (A.4)$$

We say that the $p$-adic probability distributions (A.2) are compatible if for all $n \geq 1$ and $\sigma \in \Phi^{V_n - 1}$:

$$\sum_{\omega \in \Omega W_n} \mu^{(n)}_{h,\rho}(\sigma_{n-1} \lor \omega) = \mu^{(n-1)}_{h,\rho}(\sigma_{n-1}). \quad (A.5)$$

This condition, according to the Kolmogorov extension theorem (see [20]), implies the existence of a unique $p$-adic measure $\mu_h$ defined on $\Omega$ with a required condition (A.4). Such a measure $\mu_h$ is said to be a generalized $p$-adic Gibbs measure corresponding to the model [25, 26]. If one has $h_x \in E_p$ for all $x \in V \setminus \{x^{(0)}\}$, then the corresponding measure $\mu_h$ is called a $p$-adic Gibbs measure (see [41]).

By $\mathcal{QG}(H)$, we denote the set of all generalized $p$-adic Gibbs measures associated with functions $h = \{h_x, \ x \in V\}$. If there are at least two distinct generalized $p$-adic Gibbs measures such that at least one of them is unbounded, then we say that a phase transition occurs.

The following statement describes conditions on $h_x$ guaranteeing compatibility of $\mu^{(n)}_{h,\rho}(\sigma)$.

**Theorem A.1.** [25] The measures $\mu^{(n)}_{h,\rho}, n = 1, 2, \ldots$ (see (A.2)) associated with the $q$-state Potts model (A.1) satisfy the compatibility condition (A.5) if and only if for any
n ∈ \mathbb{N}, the following equation holds:

\[ \hat{h}_x = \prod_{y \in S(x)} F(\hat{h}_y, \theta). \]  

(A.6)

Here and below, a vector \( \hat{h} = (\hat{h}_1, \ldots, \hat{h}_{q-1}) \in \mathbb{Q}_p^{q-1} \) is defined by a vector \( h = (h_1, h_2, \ldots, h_q) \in \mathbb{Q}_p^q \) as follows:

\[ \hat{h}_i = \frac{h_i}{h_q}, \quad i = 1, 2, \ldots, q - 1 \]  

(A.7)

and the mapping \( F : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \rightarrow \mathbb{Q}_p^{q-1} \) is defined by \( F(x; \theta) = (F_1(x; \theta), \ldots, F_{q-1}(x; \theta)) \) with

\[ F_i(x; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta}, \quad x = \{x_i\} \in \mathbb{Q}_p^{q-1}, \quad i = 1, 2, \ldots, q - 1. \]  

(A.8)

Let us first observe that the set \((1, \ldots, 1, h, 1, \ldots, 1) (m = 1, \ldots, q - 1) \) is invariant for the equation (A.6). Therefore, in what follows, we restrict ourselves to one of such vectors, let us say \((h, 1, \ldots, 1) \).

In [32], to establish the phase transition, we considered translation-invariant (that is, \( h = (h_x)_{x \in V \setminus \{x_0\}} \) such that \( h_x = h_y \) for all \( x, y \)) solutions of (A.6). Then the equation (A.6) is reduced to the following one:

\[ h = f_{\theta, q, k}(h), \]  

(A.9)

where \( f_{\theta, q, k} \) is given by (1.1).

Hence, to establish the existence of the phase transition, when \( k = 2 \), we showed in [41] that (A.9) has three non-trivial solutions if \( q \) is divisible by \( p \). Note that the full description of all solutions of the last equation has been carried out in [43] when \( k = 2 \). Certain periodic points of \( f_{\theta, q, k} \) have been carried out in [1, 28, 31].

REFERENCES

[1] M. A. Kh. Ahmad, L. M. Liao and M. Saburov. Periodic \( p \)-adic Gibbs measures of \( q \)-state Potts model on Cayley tree: the chaos implies the vastness of \( p \)-adic Gibbs measures. J. Stat. Phys. 171(2018), 1000–1034.
[2] V. Anashin. Non-Archimedean ergodic theory and pseudorandom generators. Comput. J. 53(2010), 370–392.
[3] V. Anashin, A. Khrennikov and E. Yurova. \( T \)-functions revisited: new criteria for bijectivity/transitivity. Des. Codes Cryptogr. 71(2014), 383–407.
[4] R. Benedetto. Reduction, dynamics, and Julia sets of rational functions. J. Number Theory 86 (2001), 175–195.
[5] R. Benedetto. Hyperbolic maps in \( p \)-adic dynamics. Ergod. Th. & Dynam. Sys. 21(2001), 1–11.
[6] Z. I. Borevich and I. R. Shafarevich. Number Theory, Academic Press, New York, 1966.
[7] H. Diao and C. E. Silva. Digraph representations of rational functions over the \( p \)-adic numbers. p-Adic Numbers Ultrametric Anal. Appl. 3 (2011), 23–38.
O. N. Khakimov. On a generalized Potts–Behtee mapping. Discrete Contin. Dyn. Syst. 39 (2019), no. 4, 2317–2330.

O. N. Khakimov and F. Mukhamedov. On a class of rational maps with good reduction. Theoret. and Math. Phys. 194 (2018), no. 2, 1171–1190.

O. N. Khakimov and F. Mukhamedov. On metric properties of unconventional limit sets of contractive p-adic dynamical systems. Dyn. Syst. 31 (2016), 506–524.

F. Mukhamedov and O. Khakimov. On Julia set and chaos in p-adic Ising model on the Cayley tree. Math. Phys. Anal. Geom. 20 (2017), Article no. 23, 14 pp.

F. Mukhamedov and O. Khakimov. Chaotic behavior of the p-adic Potts–Behtee mapping. Discrete Contin. Dyn. Syst. 38 (2018), 231–245.

F. Mukhamedov and O. Khakimov. P-adic monomial equations and their perturbations. Izv. Math. 84 (2020), 348–360.

F. Mukhamedov, B. Omirov and M. Saburov. On cubic equations over p-adic fields. Int. J. Number Theory 10 (2014), 1171–1190.

F. Mukhamedov and M. Saburov. On equation $x^q = a$ over $\mathbb{Q}_p$. J. Number Theory 133 (2013), 55–58.

F. Mukhamedov, M. Saburov and O. Khakimov. p-adic Ising–Vannimenus model on an arbitrary order Cayley tree. J. Stat. Mech. Theory Exp. 2015 (2015), P05032.
[40] F. Mukhamedov, M. Saburov and O. Khakimov. Translation-invariant p-adic quasi Gibbs measures for the Ising–Vannimenus model on a Cayley tree. *Theoret. and Math. Phys.* **187**(1) (2016), 583–602.

[41] F. M. Mukhamedov and U. A. Rozikov. On Gibbs measures of p-adic Potts model on the Cayley tree. *Indag. Math. (N.S.)* **15** (2004), 85–100.

[42] F. M. Mukhamedov and U. A. Rozikov. On inhomogeneous p-adic Potts model on a Cayley tree. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8** (2005), 277–290.

[43] U. A. Rozikov and O. N. Khakimov. Description of all translation-invariant p-dic Gibbs measures for the Potts model on a Cayley tree. *Markov Process. Related Fields* **21** (2015), 177–204.

[44] M. Saburov and M. A. Kh. Ahmad. On descriptions of all translation invariant p-adic Gibbs measures for the Potts model on the Cayley tree of order three. *Math. Phys. Anal. Geom.* **18** (2015), Article no. 26, 33 pp.

[45] C. F. Woodcock and N. P. Smart. p-adic chaos and random number generation. *Exp. Math.* **7** (1998), 333–342.