Simultaneous Conjugacy Classes of Finite $p$-groups of rank $\leq 5$

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Abstract. For a finite group $G$, we consider the problem of counting simultaneous conjugacy classes of $n$-tuples and simultaneous conjugacy classes of commuting $n$-tuples in $G$. Let $\alpha_{G,n}$ denote the number of simultaneous conjugacy classes of $n$-tuples, and $\beta_{G,n}$ the number of simultaneous conjugacy classes of commuting $n$-tuples in $G$. The generating functions $A_G(t) = \sum_{n \geq 0} \alpha_{G,n} t^n$, and $B_G(t) = \sum_{n \geq 0} \beta_{G,n} t^n$ are rational functions of $t$. This paper concerns the study of normalized functions $A_G(t/|G|)$ and $B_G(t/|G|)$ for finite $p$-groups of rank at most 5.

1. Introduction

Let $G$ be a finite group. Define a group action of $G$ on Cartesian product $G^n$ by simultaneous conjugation:

$$g \cdot (x_1, \ldots, x_n) = (gx_1g^{-1}, \ldots, gx_n g^{-1}).$$

Let $G^{(n)}$ denote the subset of $G^n$ consisting of pairwise commuting tuples:

$$G^{(n)} = \{(x_1, \ldots, x_n) \in G^n \mid [x_i, x_j] = 1 \text{ for all } 1 \leq i, j \leq n\}.$$  

It is clear that the restriction of simultaneous conjugation action on $G^{(n)}$ is also a $G$-action. The enumeration of orbits under the above two group actions is an interesting combinatorial group theory problem. Recently, authors got interested to understand the $G$-orbits in $G^n$ and $G^{(n)}$ for various finite group $G$. If we take $G = GL_n(F_q)$, then $G$ acts on the space $M_n(F_q)^m$ of $m$-tuples of $n \times n$ matrices over $F_q$ and on the set $M_n(F_q)^{(m)}$ of $m$-tuples of commuting matrices from $M_n(F_q)$ by simultaneous conjugation, and in such case, the orbits are called simultaneous similarity classes. Let $a(n, m, q)$ and $c(n, m, q)$ be denote the number of simultaneous similarity classes in $M_n(F_q)^m$ and the number of simultaneous similarity classes in $M_n(F_q)^{(m)}$, respectively. In [16], authors enumerate simultaneous similarity conjugacy classes of tuples of commuting unitary matrices and of commuting symplectic matrices over a finite field $F_q$ of odd size. They studied the number of simultaneous similarity conjugacy classes of commuting elements with the help of branching rules. The orbits of the action of $G$ on $G^n$ by simultaneous conjugation are studied in the context of complete reducibility for algebraic groups (see [2]). In [14], author studied the asymptotic behaviour of $a(n, m, q)$ and $c(n, m, q)$. In [15], author enumerated $c(n, m, q)$ for $n = 2, 3, 4$. With this viewpoint, let $\alpha_{G,n}$ denote the number of $G$-orbits in $G^n$, and $\beta_{G,n}$ the number of $G$-orbits in $G^{(n)}$. Consider the generating functions:

$$A_G(t) = \sum_{n=0}^{\infty} \alpha_{G,n} t^n,$$

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and \[ B_G(t) = \sum_{n=0}^{\infty} \beta_{G,n} t^n. \]

Note that \( G^0 \) and \( G^{(0)} \) are the trivial group, and so \( \alpha_{G,0} = \beta_{G,0} = 1 \). Further, \( \alpha_{G,n} \geq \beta_{G,n} \) with \( \alpha_{G,1} = \beta_{G,1} \) and \( \alpha_{G,1} \) is equal to the number of conjugacy classes of a finite group \( G \). This article is in continuation of [9]. In [9], we have shown that \( A_G(t) \) and \( B_G(t) \) are rational functions of \( t \).

This paper concerns the enumeration of isoclinism \( A \)-equivalent \( G \) and \( \beta \) groups (a group \( G \) where \( G = G/H \) for \( h \in H \)). Theorem 2.1). The normalized invariants \( a_G(g_1Z(G), g_2Z(G)) = [g_1, g_2] \), for \( g_1, g_2 \in G \), and \( a_H(h_1Z(H), h_2Z(H)) = [h_1, h_2] \) for \( h_1, h_2 \in H \).

Hall classified groups into families using the notion of isoclinism in [5]. These families played an important role in the classification of \( p \)-groups. Hall and Senior [6] classified all groups of order \( 2^n \) for \( n \leq 6 \), and James [8] classified all groups of order \( p^n \) for \( n \leq 6 \) for odd primes in terms of isoclinic families. For a family of \( p \)-groups, the smallest \( n \) such that the family has a group of order \( p^n \) is called the rank of the family. In Section 3 we compute the normalized functions \( A_G(t/|G|) \) and \( B_G(t/|G|) \) for isoclinism families of \( p \)-groups of rank up to 5. The results are given in Table 1. In [9], we have shown that the normalized functions \( A_G(t/|G|) \) and \( B_G(t/|G|) \) are invariants of isoclinism families (9 Corollary 4.5 ). Therefore isoclinic groups of the same order are both \( A \)-equivalent and \( B \)-equivalent. Keeping this in mind, we compute \( A_G(t) \) and \( B_G(t) \) for only one group in each isoclinic family. Since our proofs depend heavily on the presentations of the groups of order \( p^6 \) (\( p \) odd) from the paper of James [8] and groups of order \( 2^n \) for \( n \leq 6 \) from the paper Hall and Senior [6], the reader is advised to keep these papers handy.

The normalized invariants \( A_G(t/|G|) \) and \( B_G(t/|G|) \) are the same for families \( \Phi_3 \) and \( \Phi_4 \), and also for families \( \Phi_7 \) and \( \Phi_8 \) of \( p \)-groups for \( p \geq 3 \) (in the classification scheme of James [8]: see Table 1). Thus groups of the same order in these families are \( A \)-equivalent and \( B \)-equivalent.

Our algorithms for computing \( A_G(t) \) and \( B_G(t) \) are easily implemented in sage [12]. Using the GAP interface of sage, it is possible to compute \( A_G(t) \) and \( B_G(t) \) for a large family of groups that are available in GAP. In particular, by locating isoclinism classes of \( p \)-groups in the GAP Small Groups, we are able to verify the results in Table 1. This software is available from the website:

https://www.imsc.res.in/~amri/conjutator/

2. Expression of \( A_G(t) \) and \( B_G(t) \)

For each \( g \in G \), let \( Z_G(g) \) denote its centralizer. The function \( A_G(t) \) can be computed by a simple application of Orbit counting lemma to the action of \( G \) on \( G^n \):

\[
(1) \quad \alpha_{G,n} = \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|^n.
\]
Therefore

\[ A_G(t) = \sum_{n=0}^{\infty} \alpha_{G,n} t^n = \sum_{n=0}^{\infty} \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|^n t^n = \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - |Z_G(g)| t} \]

(2)

Let \( z_m \) denote the number of elements of \( G \) having centralizer of cardinality \( m \) (and therefore conjugacy class of cardinality \( |G|/m \)). Note that \( z_m = 0 \) if \( m > |G| \). In [9], we got the following alternating expression of \( A_G(t) \) (see [9] Section 2):

(3)

\[ A_G(t) = \frac{1}{|G|} \sum_{m=1}^{\infty} \frac{z_m}{1 - mt} \]

We will use equation (3) for the calculation of \( A_G(t) \). Now, let \( c_H \) denote the number of conjugacy classes of \( G \) whose centralizer is isomorphic to a subgroup \( H \) of \( G \) and let \( c_G \) denote the cardinality of the centre of \( G \). Then we have

\[ \beta_{G,n} = \sum_{H} c_H \beta_{H,n-1}, \]

and

(4)

\[ (1 - c_G t) B_G(t) = 1 + \sum_{|H| < |G|} c_H t B_H(t). \]

For the complete detailing of these expression, we suggest reader to go through Section 2 of [9].

| Families | \( A_G(t/|G|) \) | \( B_G(t/|G|) \) |
|---------|----------------|----------------|
| Abelian | \( \frac{1}{1-t} \) | \( \frac{1}{1-t} \) |
| \( \Phi_2, \Gamma_2 \) | \( \frac{1}{1-p^{-1} t} + \frac{p^{-2}}{1-t} \) | \( \frac{-p^{-1}}{1-p^{-1} t} + \frac{1+p^{-1}}{1-p^{-1} t} \) |
| \( \Phi_3, \Phi_4, \Gamma_3, \Gamma_4 \) | \( \frac{1}{1-p^{-1} t} + \frac{p^{-1}-p^{-3}}{1-p^{-1} t} + \frac{p^{-3}}{1-t} \) | \( \frac{-p^{-1}}{1-p^{-1} t} + \frac{1}{1-p^{-2} t} + \frac{p^{-1}}{1-p^{-1} t} \) |
| \( \Phi_5, \Gamma_5 \) | \( \frac{1}{1-p^{-4} t} + \frac{p^{-4}}{1-t} \) | \( \frac{1}{1-p^{-4} t} + \frac{-p^{-1}-p^{-2}-p^{-2}}{1-p^{-2} t} + \frac{p+1+p^{-1}+p^{-2}}{1-p^{-2} t} \) |
| \( \Phi_6 \) | \( \frac{1}{1-p^{-3} t} + \frac{p^{-3}}{1-t} \) | \( \frac{-p^{-1}-p^{-2}}{1-p^{-3} t} + \frac{1+p^{-1}+p^{-2}}{1-p^{-3} t} \) |
| \( \Phi_7, \Phi_8, \Gamma_6, \Gamma_7 \) | \( \frac{1}{1-p^{-2} t} + \frac{p^{-2}-p^{-4}}{1-p^{-1} t} + \frac{p^{-4}}{1-t} \) | \( \frac{-p^{-1}-p^{-2}}{1-p^{-3} t} + \frac{1+p^{-1}+p^{-2}}{1-p^{-3} t} \) |
| \( \Phi_9, \Gamma_8 \) | \( \frac{1}{1-p^{-1} t} + \frac{p^{-1}-p^{-4}}{1-p^{-1} t} + \frac{p^{-4}}{1-t} \) | \( \frac{-p^{-1}}{1-p^{-1} t} + \frac{1}{1-p^{-2} t} + \frac{p^{-1}}{1-p^{-1} t} \) |
| \( \Phi_{10} \) | \( \frac{1}{1-p^{-1} t} + \frac{p^{-1}-p^{-3}}{1-p^{-1} t} + \frac{p^{-3}-p^{-4}}{1-p^{-1} t} + \frac{p^{-4}}{1-t} \) | \( \frac{-p^{-1}}{1-p^{-1} t} + \frac{1-p^{-2}}{1-p^{-1} t} + \frac{p^{-1}+p^{-2}}{1-p^{-1} t} \) |

Table 1. Normalized invariants for isoclinism families of \( p \)-groups of rank up to 5; for the families \( \Gamma_i \) of 2-groups, substitute \( p = 2 \).
3. Preliminary results on $A_G(t)$ and $B_G(t)$ for AC-groups

In this section, we quote some results about $A_G(t)$ and $B_G(t)$ for AC-groups. We start with the following setup. Set

$$X = \{ H \mid 1 \neq H < G \text{ such that } H = Z_G(x) \text{ for some } x \in G \setminus Z(G) \},$$

that is $X$ is a collection of centralizers of non-central elements of the group $G$. Define an equivalence relation $R_1$ on $X$ as follows. We say for $H, K \in X$, $HR_1 K$ if $|H| = |K|$. This implies that there exists integers $n_1, \ldots, n_k$ with $n_i \mid |G|$ for all $i$ and $X = \bigcup_{i=1}^{k} X_{n_i}$, where $X_{n_i} = \{ H \in X \mid |H| = n_i \}$. For each $i$, define an equivalence relation $R_2$ on $X_{n_i}$ as follows. We say for $H, K \in X_{n_i}$, that $HR_2 K$ if $H$ is isomorphic to $K$. Let $Y_{n_i}$ be a set of representatives for the equivalence classes $X_{n_i}/R_2$ for each $i$. Then under the above setup, equation (4) can be written as follows.

$$B_G(t) = \frac{1}{(1 - |Z(G)|/t)} \left( 1 + \sum_{i=1}^{k} \sum_{H \in Y_{n_i}} c_H t B_H(t) \right),$$

where $c_H$ denotes the number of conjugacy classes of $G$ whose centralizer is isomorphic to the subgroup $H$. We will use equation (5) to calculate $B_G(t)$. Now we mention some useful results.

**Lemma 3.1.** Let $G$ be an AC-group. Then

$$B_G(t) = \frac{1}{(1 - |Z(G)|/t)} \left( 1 + \sum_{i=1}^{k} \sum_{H \in Y_{n_i}} c_H t B_H(t) \right).$$

The underlying groups are turn out to be an AC-groups in the following two results.

**Theorem 3.2.** Let $G$ be a $p$-group of order $p^m$ with $|G/Z(G)| = p^2$, where $p$ is a prime number. Then

$$A_G(t) = \frac{1}{p^m} \left( \frac{p^{m-2}}{1 - p^m t} + \frac{p^m - p^{m-2}}{1 - p^{m-1} t} \right)$$

and

$$B_G(t) = \frac{1 - p^{m-3} t}{(1 - p^m t)(1 - p^{m-1} t)}.$$

**Theorem 3.3.** Let $G$ be a $p$-group of order $p^m$ with $|G/Z(G)| = p^3$, where $p$ is a prime number. Then we have the following.

(3.3.1) If $G$ has no abelian maximal subgroup, then

$$A_G(t) = \frac{1}{p^m} \left( \frac{p^{m-3}}{1 - p^m t} + \frac{p^m - p^{m-3}}{1 - p^{m-2} t} \right)$$

and

$$B_G(t) = \frac{1 - p^{m-5} t}{(1 - p^m t)(1 - p^{m-3} t)}.$$

(3.3.2) If $G$ possesses an abelian maximal subgroup, then

$$A_G(t) = \frac{1}{p^m} \left( \frac{p^{m-3}}{1 - p^m t} + \frac{p^{m-1} - p^{m-3}}{1 - p^{m-1} t} + \frac{p^m - p^{m-1}}{1 - p^{m-2} t} \right)$$

and

$$B_G(t) = \frac{1}{(1 - p^{m-3} t)} \left( 1 + \frac{(p^{m-2} - p^{m-4}) t}{1 - p^{m-1} t} + \frac{(p^{m-2} - p^{m-3}) t}{1 - p^{m-2} t} \right).$$

**Remark 3.4.** If $G$ is a non-abelian $p$-group of order $p^3$ then $|G/Z(G)| = p^2$. So, by Theorem 3.3 we get the expression of $A_G(t)$ and $B_G(t)$. Now if $G$ is a $p$-group of order $p^3$, then either $|G/Z(G)| = p^2$ or $|G/Z(G)| = p^3$. Hence again by Theorem 3.2 and Theorem 3.3 we get $A_G(t)$ and $B_G(t)$. 
4. $p$-group of maximal class

A group of order $p^m$ with $m \geq 4$, is of maximal class if it has nilpotency class $m - 1$. Let $G$ be a $p$-group of maximal class with order $p^m$. Following [10] Chapter 3, we define the 2-step centralizer $K_i$ in $G$ to be the centralizer in $G$ of $\gamma_i(G)/\gamma_{i+2}(G)$ (where $\gamma_i(G)$ denotes the $i$th subgroup in the lower central series of $G$) for $2 \leq i \leq m - 2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_1$, $P_i = \gamma_i(G)$ for $2 \leq i \leq m$. Clearly $K_i \geq \gamma_2(G)$ for all $i$. The degree of commutativity $l = l(G)$ of $G$ is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+1}$ for all $i, j \geq 1$ if $P_1$ is not abelian and $l = m - 3$ if $P_1$ is abelian. It is clear that if $G$ is a $p$-group of maximal class which possesses an abelian maximal subgroup, then $P_1$ is abelian.

**Lemma 4.1.** Let $G$ be a $p$-group of maximal class and order $p^m$ with positive degree of commutativity. Suppose $s \in G \setminus P_1$, $s_i \in P_1 \setminus P_2$ and $s_i = [s_{i-1}, s]$ for $1 \leq i \leq m - 1$. Then

\begin{align*}
(4.1.1) & G = \langle s, s_1 \rangle, P_i = \langle s_i, \ldots, s_{m-1} \rangle, |P_i| = p^{m-i} \text{ for } 1 \leq i \leq m - 1 \text{ and } P_{m-1} = Z(G) \text{ of order } p. \\
(4.1.2) & Z_G(s) = \langle s \rangle P_{m-1}, \text{ and } |Z_G(s)| = p^2. 
\end{align*}

**Proof.** (4.1.1) follows from [10] Lemma 3.2.4. (4.1.2) Since $G$ has positive degree of commutativity, the 2-step centralizers of $G$ are all equal [10] Corollary 3.2.7. Now (4.1.2) follows by [7] Hilfssatz III 14.13.$\square$

**Theorem 4.2.** Let $G$ be a $p$-group of maximal class and order $p^m$ with positive degree of commutativity.

\begin{align*}
(4.2.1) & \text{If } G \text{ possesses an abelian maximal subgroup, then } \\
A_G(t) &= \frac{1}{p^m} \left( \frac{p}{1 - pt} + \frac{p^{m-1}}{1 - pt} + \frac{p^{m-1} - p}{1 - pt^2} + \frac{p^{m-3} - p}{1 - pt^4} \right). \\
(4.2.2) & \text{If } [P_1, P_3] = 1 \text{ and } G \text{ possesses no abelian maximal subgroup, then } \\
A_G(t) &= \frac{1}{p^m} \left( \frac{p}{1 - pt} + \frac{p^{m-1}}{1 - pt} + \frac{p^{m-3} - p}{1 - pt^2} + \frac{p^{m-3} - p}{1 - pt^4} \right). 
\end{align*}

**Proof of (4.2.1).** Since $P_1$ is abelian, $Z_G(g) = P_1$ for any $g \in P_1 \setminus Z(G)$. Now by Lemma 4.1 we have $|Z_G(g)| = p^2$ for $g \in G \setminus P_1$. Thus $X_{p^{m-1}} = P_1 \setminus Z(G)$ and $X_{p^2} = G \setminus P_1$ and so the result follows from (3).$\square$

**Proof of (4.2.2).** Since $P_1$ is a maximal subgroup of $G$, $P_1$ is non-abelian. As $[P_1, P_3] = 1$, we have $P_3 \leq Z(P_1)$ and therefore $|P_1/Z(P_3)| = p^2$. This shows that, for $x \in P_1 \setminus Z(P_1)$, $C_{P_1}(x)$ is abelian and therefore $P_1$ is an AC-group. This yields that $|C_{P_1}(x)| = p^{m-2}$ for each $x \in P_1 \setminus Z(P_1)$ and $|C_{P_1}(x)| = |C_G(x)| = p^{m-1}$ for each $x \in Z(P_1) \setminus Z(G)$.

**Claim:** $Z_{P_1}(x) = Z_G(x)$ for each $x \in P_1 \setminus Z(P_1)$.

On the contrary, suppose $g \in Z_G(x) \setminus C_{P_1}(x)$. Then $g \in G \setminus P_3$ and $x \in Z_G(g)$. Therefore $|Z_G(g)| \geq p^3$, which is a contradiction by Lemma 4.1. This proves the claim. Therefore by the above observations and Lemma 4.1 we have $X_{p^2} = G \setminus P_1$, $X_{p^{m-2}} = P_1 \setminus Z(P_1)$ and $X_{p^{m-1}} = Z(P_1) \setminus Z(G)$. Hence the result follows from (3).$\square$

**Theorem 4.3.** Let $G$ be a $p$-group of maximal class and order $p^m$ with positive degree of commutativity.

\begin{align*}
(4.3.1) & \text{If } G \text{ possesses an abelian maximal subgroup, then } \\
B_G(t) &= \frac{1}{(1 - pt)} \left( 1 + \frac{(p^{m-2} - 1)t}{(1 - p^{m-1})t} + \frac{(p^2 - p)t}{1 - p^2} \right). \\
(4.3.2) & \text{If } [P_1, P_3] = 1 \text{ and } G \text{ possesses no abelian maximal subgroup, then } \\
B_G(t) &= \frac{1}{(1 - pt)} \left( 1 + \frac{(p^{m-4} - 1)t(1 - p^{m-4})t}{(1 - p^{m-2})t(1 - p^{m-3})t} + \frac{(p^{m-3} - p^{m-5})t}{1 - p^{m-2}t} + \frac{(p^2 - p)t}{1 - p^2} \right). 
\end{align*}
Proof of (4.3.1). In view of the proof of (1.2.1), we have $P_1$ is an abelian subgroup of order $p^{m-1}$, $Z_G(x) = P_1$ for all $x \in P_1 \setminus Z(G)$ and $|Z_G(x)| = p^2$ for all $x \in G \setminus P_1$. This implies that the total number of conjugacy classes is equal to $k(G) = |Z(G)| + \frac{|P_1||Z(G)|}{p} + \frac{|G||P_1|}{p^{m-2}} = p^{m-2} + p^2 - 1$. Suppose $Y_{p^2} = \{H_1, \ldots, H_n\}$. Then $c_{H_1} + \cdots + c_{H_n} = k(G) - (c_{P_1} + |Z(G)|) = p^{m-2} + p^2 - 1 - (p^{m-2} - 1 + p) = p^2 - p$. Therefore by Lemma 3.1 we get

$$B_G(t) = \frac{1}{1 - pt} \left(1 + \frac{c_{P_1} t}{1 - p^{m-1} t} + \sum_{H \in Y_{p^2}} \frac{c_H t}{1 - |H| t}\right).$$

Proof of (4.3.2). In view of the proof of (1.2.2), we have the following observations.

1. $P_1$ is a non-abelian AC-group of order $p^{m-1}$ with $|P_1/Z(P_1)| = p^2$.
2. For all $x \in P_1 \setminus Z(P_1)$, $Z_G(x) = Z_{P_1}(x)$ is an abelian subgroup of order $p^{m-2}$.
3. For all $x \in Z(P_1) \setminus Z(G)$, $Z_G(x) = P_1$ is of order $p^{m-1}$.
4. $|Z_G(x)| = p^2$ for all $x \in G \setminus P_1$.

By using the above observations, we have that the total number of conjugacy classes is equal to $k(G) = |Z(G)| + \frac{|P_1||Z(G)|}{p} + \frac{|G||P_1|}{p^{m-2}} = p^{m-4} + p^{m-3} + p^{m-5} - 1$. Therefore by equation (5), we get

$$B_G(t) = \frac{1}{1 - pt} \left(1 + \frac{c_{P_1} tB_{P_1}(t)}{1 - p^{m-2} t} + \sum_{H \in Y_{p^2}} \frac{c_H t}{1 - |H| t}\right).$$

By Theorem 3.2 we get

$$B_{P_1}(t) = \frac{(1 - p^{m-4} t)}{(1 - p^{m-2} t)(1 - p^{m-3} t)}.$$ 

Now use the value of $c_{P_1}$, $c_K$ and $B_{P_1}(t)$ in equation (6), we get

$$B_G(t) = \frac{1}{1 - pt} \left(1 + \frac{(p^{m-4} - 1)t(1 - p^{m-4} t)}{(1 - p^{m-2} t)(1 - p^{m-3} t)} + \frac{(p^{m-3} - p^{m-5})t}{(1 - p^{m-2} t)} + \frac{(p^2 - p)t}{(1 - p^2 t)}\right).$$

This proves the theorem. 

5. Isoclinism families of rank at most 5

Let $p$ be an odd prime number. The approach of P. Hall to the classification of $p$-groups was based on the concept isoclinism, which was introduced by P. Hall himself. Since isoclinism is an equivalence relation on the class of all groups, one can consider equivalence class widely known as isoclinism family. It follows from [5] p. 135 that there exists at least one finite $p$-group $H$ in each isoclinism family such that $Z(H) \leq H'$. Such a group $H$ is called a stem group in the isoclinism family. In other words, every isoclinism family contains a stem group. Two stem groups in the same family are necessarily of the same order, and if $G$ is any group, then the order of the stem groups in the isoclinism family of $G$ is given by $|G/Z(G)|/|Z(G) \cap G'|$. The stem groups of an isoclinic family are all the groups in the family with the minimal order (within the family) [5] Section 3. In the case of isoclinism families of $p$-groups, stem groups will themselves be $p$-groups. If the stem groups of an isoclinic family of group $G$ has order $p^r$, then $r$ is called the rank of family [5] Section 4. [18, Section 3].
The $p$-groups of rank at most $5$ are classified in 10 isoclinism families respectively by P. Hall \[8\]. In this section, we use the notation of the paper \[8\]. We will use not only the results of the previous sections but, more crucially, also use the classification of $p$-groups of rank $\leq 5$ by R. James (\[8\] Section 4.5). We pick one stem group $G$ from each isoclinism family $\Phi_k$ of $p$-groups of rank at most $5$ and compute the generating functions $A_G(t)$ and $B_G(t)$. Then we use \[9\] Theorem 4.4 to get generating functions for all groups in the isoclinic family $\Phi_k$. We note that this gives us generating functions for all $p$-groups of order at most $p^5$.

The groups are given by polycyclic presentation, in which all the relations of the form $[x, y] = x^{-1}y^{-1}xy = 1$ between the generators have been omitted from the list.

**Lemma 5.1.** Let $G$ be a $p$-group of order $p^m$ of rank $3$. Then

\begin{align*}
A_G(t/|G|) &= \frac{1 - p^{-2}}{1 - p^{-1}t} + \frac{p^{-2}}{1 - t} \\
B_G(t/|G|) &= \frac{1 - p^{-3}}{1 - p^{-2}t} + \frac{1 + p^{-1}}{1 - p^{-1}t}.
\end{align*}

**Proof.** All $p$-groups of rank $3$ are isoclinic and belong to the family $\Phi_2$. If $G \in \Phi_2$, then $|G/\Phi(G)| = p^2$ \[5\] Section 4. Now we get $A_G(t)$ and $B_G(t)$ using Theorem 3.2. We normalize this expression to obtain the result.

**Lemma 5.2.** Let $G$ be a $p$-group of order $p^m$ of rank $4$. Then

\begin{align*}
A_G(t/|G|) &= \frac{1 - p^{-1}}{1 - p^{-2}t} + \frac{p^{-1} - p^{-3}}{1 - p^{-1}t} + \frac{p^{-3}}{1 - t} \\
B_G(t/|G|) &= \frac{-p^{-1}}{1 - p^{-3}t} + \frac{1}{1 - p^{-2}t} + \frac{p^{-1}}{1 - p^{-1}t}.
\end{align*}

**Proof.** All $p$-groups of rank $4$ are isoclinic and belong to the family $\Phi_3$ \[5\] Section 4. We consider the group $G = \Phi_3(1^4)$. The group $G$ is a stem group in isoclinic family $\Phi_3$ and $|G/\Phi(G)| = p^3$ \[8\] Section 4.5. Groups of order $p^4$ have a maximal subgroup which is abelian \[13\] 6.5.1. Using (5.3.2), we get

\[A_G(t) = \frac{1}{p^4} \left( \frac{p}{1 - p^4t} + \frac{p^3 - p}{1 - p^3t} + \frac{p^4 - p^3}{1 - p^2t} \right).\]

and

\[B_G(t) = \frac{1}{1 - pt} \left( 1 + \frac{(p^2 - 1)t}{1 - p^3t} + \frac{(p^2 - p)t}{1 - p^2t} \right).\]

We normalize the expression to obtain the result.

**Lemma 5.3.** Let $G$ be a $p$-group of order $p^m$ of rank $5$ and let $G \in \Phi_4$. Then

\begin{align*}
A_G(t/|G|) &= \frac{1 - p^{-1}}{1 - p^{-2}t} + \frac{p^{-1} - p^{-3}}{1 - p^{-1}t} + \frac{p^{-3}}{1 - t} \\
B_G(t/|G|) &= \frac{p^{-1}}{1 - p^{-3}t} + \frac{1}{1 - p^{-2}t} + \frac{p^{-1}}{1 - p^{-1}t}.
\end{align*}

**Proof.** Let $G = \Phi_4(1^5)$ be a stem group in the family $\Phi_4$. The group $G$ has a polycyclic presentation

\[G = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 | [\alpha_i, \alpha] = \beta_i, \alpha^p = \alpha_i^p = \beta_i^p = 1 (i = 1, 2) \rangle.\]

Here $Z(G) = \langle \beta_1, \beta_2 \rangle$, $|G/Z(G)| = p^3$ and $H = \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \rangle$ is a maximal subgroup of $G$ which is abelian. Using (5.3.2), we get

\[A_G(t) = \frac{1}{p^5} \left( \frac{p^2}{1 - p^5t} + \frac{p^4 - p^2}{1 - p^4t} + \frac{p^5 - p^4}{1 - p^3t} \right).\]
and
\[ B_G(t) = \frac{1}{1 - p^2t} \left( 1 + \frac{(p^3 - p)t}{1 - p^4t} + \frac{(p^3 - p^2)t}{1 - p^3t} \right). \]

We normalize the expression to obtain the result.

**Lemma 5.4.** Let \( G \) be a \( p \)-group of order \( p^m \) of rank 5 and let \( G \in \Phi_5 \). Then
\[
(5.4.1) \quad A_G(t)/|G| = \frac{1 - p^{-4}}{1 - p^{-3}t} + \frac{p^{-4}}{1 - t},
\]
\[
(5.4.2) \quad B_G(t)/|G| = \frac{1}{1 - p^{-4}t} + \frac{-p - 1 - p^{-1} - p^{-2}}{1 - p^{-3}t} + \frac{p + 1 + p^{-1} + p^{-2}}{1 - p^{-2}t}.
\]

**Proof.** Let \( G = \Phi_5(1^5) \) be a stem group in the family \( \Phi_5 \). The group \( G \) has a polycyclic presentation
\[
G = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle.
\]
We compute that \( Z(G) = \langle \beta \rangle \) is group of order \( p \) and \( G/Z(G) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is an elementary \( p \)-group of order \( p^4 \). This implies that \( G \) is an extraspecial \( p \)-group of order \( p^5 \). Using [9] Theorem 7.4 and [9] Theorem 7.5, we get
\[
A_G(t) = \frac{1}{p^5} \left( \frac{p}{1 - p^3t} + \frac{p^5 - p}{1 - p^4t} \right)
\]
and
\[
B_G(t) = \frac{1 - t}{(1 - pt)(1 - p^4t)}
\]
respectively. We normalize the expression to obtain the result.

**Lemma 5.5.** Let \( G \) be a \( p \)-group of order \( p^m \) of rank 5 and let \( G \in \Phi_6 \). Then
\[
(5.5.1) \quad A_G(t)/|G| = \frac{1 - p^{-3}}{1 - p^{-2}t} + \frac{p^{-3}}{1 - t},
\]
\[
(5.5.2) \quad B_G(t)/|G| = \frac{-p^{-1} - p^{-2}}{1 - p^{-3}t} + 1 + \frac{p^{-1} + p^{-2}}{1 - p^{-2}t}.
\]

**Proof.** Let \( G = \Phi_6(1^5) \) be a stem group in the family \( \Phi_6 \). The group \( G \) has a polycyclic presentation
\[
G = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta_i^p = 1 \rangle.
\]
Here \( Z(G) = \langle \beta_1, \beta_2 \rangle \), \( |G/Z(G)| = p^3 \). Note that \( G \) has no maximal subgroup which is abelian. Using [3.3.1], we get
\[
A_G(t) = \frac{1}{p^3} \left( \frac{p^2}{1 - p^3t} + \frac{p^5 - p}{1 - p^4t} \right)
\]
and
\[
B_G(t) = \frac{1 - t}{(1 - p^2t)(1 - p^4t)}.
\]
We normalize the expression to obtain the result.

**Lemma 5.6.** Let \( G \) be a \( p \)-group of order \( p^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Phi_7 \). Then
\[
(5.6.1) \quad A_G(t)/|G| = \frac{1 - p^{-2}}{1 - p^{-2}t} + \frac{p^{-2} - p^{-4}}{1 - p^{-4}t} + \frac{p^{-4}}{1 - t},
\]
\[
(5.6.2) \quad B_G(t)/|G| = \frac{-p^{-1} - p^{-2}}{1 - p^{-3}t} + 1 + \frac{p^{-1} + p^{-2}}{1 - p^{-2}t}.
\]
Proof. Let $G = \Phi_2(1^5)$ be a stem group in the family $\Phi_2$. The group $G$ has a polycyclic presentation

$$G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^{p} = \alpha_1^{p} = \beta^{p} = 1 \ (i = 1, 2) \rangle,$$

where $\alpha_1^{p}$ denotes $\alpha_1^{p} \alpha_2^{p} \alpha_3^{p}$. If $p > 3$, the relation $\alpha_1^{p} = 1$ together with other power relations, imply $\alpha_1^p = 1$. If $p = 3$, the relation $\alpha_1^{p} = 1$, together with other power relations, implies $\alpha_1 \alpha_3 = 1$. We compute for all prime numbers $p$, the group $G$ has centre $Z(G) = \langle \alpha_3 \rangle$ and $G' = \langle \alpha_2, \alpha_3 \rangle$.

Now by Lemma 4.5, $(G, Z(G))$ is a Camina pair and hence for every element $g \in G \setminus Z(G)$, $gZ(G) \subseteq cl_G(g)$. We use the fact if $p \mid X(t)$ and $|X(t)| = p^3$, then subgroups $gG, G, G'$ are isomorphic. We note that $c_{K_1} + c_{K_2} = p^2 - 1$. Now we compute $B_{K_1}(t)$ and $B_{K_2}(t)$. Both the subgroups $K_1$ and $K_2$ are of order $p^4$ with centre of order $p^2$. We use Theorem 5.2 to obtain

$$B_{K_1}(t) = B_{K_2}(t) = \frac{1 - pt}{(1 - pt)(1 - p^3t)}.$$

Next if $x \in G \setminus H$, then $|Z_G(x)| = p^3$ and if $x \in H \setminus Z(G)$, then $|Z_G(x)| = p^4$ to obtain the total number of conjugacy classes of $G$ is equal to $r = |Z(G)| + \frac{|G| - |H|}{p} + \frac{|G| - |H|}{p} = p + (p^2 - 1) + (p^3 - p) = p^3 + p^2 - 1$.

Now we consider the case when $x \in H \setminus Z(G)$. If $x \in G' \setminus Z(G)$ then we compute that $Z_G(x)$ is isomorphic to subgroup $K_1 = \langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle$ of order $p^4$ with centre $\langle \alpha_2, \alpha_3 \rangle$ of order $p^2$. If $x \in H \setminus G'$ then we compute that $Z_G(x)$ is isomorphic to subgroup $K_2 = \langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle$ of order $p^3$ with centre $\langle \alpha_3, \beta \rangle$ of order $p^2$. Moreover if $p > 3$ then subgroups $K_1$ and $K_2$ are isomorphic. We note that $c_{K_1} + c_{K_2} = p^2 - 1$. Now we compute $B_{K_1}(t)$ and $B_{K_2}(t)$. Both the subgroups $K_1$ and $K_2$ are of order $p^4$ with centre of order $p^2$. We use Theorem 5.2 to obtain

$$B_{K_1}(t) = B_{K_2}(t) = \frac{1 - pt}{(1 - pt)(1 - p^3t)}.$$

We normalize the expression to obtain the result.\qed

Lemma 5.7. Let $G$ be a p-group of order $p^m$ of rank 5 and $G$ belongs to isoclinic family $\Phi_8$. Then

$$A_G(t/G) = \frac{1 - p^2}{1 - p^3 t} + \frac{p - p^4}{1 - p - 1} + \frac{1 + p^2}{1 - p^2 t} - \frac{1 - p^2}{1 - p^3 t},$$

$$B_G(t/G) = \frac{1 - p^2}{1 - p^3 t} + \frac{1 + p}{1 - p^2 t} - \frac{1 - p}{1 - p^3 t}.$$
\[ \alpha_2^{-1}\alpha_1\alpha_2 = \alpha_1\beta, \beta, \alpha_2] = \beta^p, [\beta, \alpha_2^p] = 1 \text{ and } [\alpha_1, \alpha_2^p] = (\beta\alpha_2^p)^p\alpha_2^p. \]  

With these relations, it is easy to see that \( X_{p^3} = \Phi(G) \setminus Z(G) \) and \( X_{p^3} = G \setminus \Phi(G). \) Using equation (3), we get

\[
A_G(t) = \frac{1}{p^3} \left( \frac{p}{1 - p^3t} + \frac{p^3 - p}{1 - p^3t} + \frac{p^5 - p^3}{1 - p^3t} \right). 
\]

Now we compute \( B_G(t). \) We use the fact if \( x \in G \setminus \Phi(G), \) then \( |Z_G(x)| = p^3 \) and if \( x \in \Phi(G) \setminus Z(G), \) then \( |Z_G(x)| = p^4 \) to obtain the total number of conjugacy classes of \( G \) is equal to \( r = |Z(G)| + \frac{|\Phi(G)| - |Z(G)|}{p} + \frac{|G| - |\Phi(G)|}{p^2} = p + (p^2 - 1) + (p^3 - p) = p^3 + p^2 - 1. \)

Now we consider the case when \( x \in \Phi(G)/Z(G). \) Here we have two subcases. If \( x \in \{(\alpha_2^p)^i(\beta^p)^i \mid 1 \leq j \leq p - 1, 0 \leq i \leq p - 1\}, \) then \( Z_G(x) \) is isomorphic to group \( K_1 = \langle \alpha_2, \beta \mid [\alpha_2, \beta] = \beta^p \rangle \) of order \( p^4 \).

If \( x \in \{(\alpha_2^p)^i(\beta^p)^i \mid 1 \leq j \leq p - 1, 1 \leq i \leq p^2 - 1, i \) is not multiple of \( p\}, \) then \( Z_G(x) \) is isomorphic to group \( K_2 = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = (\alpha_1^p\alpha_2^{-1})^p\alpha_2^p \rangle \) of order \( p^4 \). We note that \( c_{K_1} = p - 1 \) and \( c_{K_2} = p^2 - p. \)

Next if \( x \in G/\Phi(G), \) then using relations of group, we compute that \( Z_G(x) = \langle x, Z(G) \rangle \) for \( x \in G/\Phi(G), \) which is abelian. Suppose \( Y_{p^3}(G) = \{H_1, \ldots, H_t\}. \) Then \( c_{H_1} + \cdots + c_{H_t} = p^3 - p. \) Therefore by (4), we get

\[
B_G(t) = \frac{1}{1 - pt} \left( e_{K_1}(t)B_{K_1}(t) + e_{K_2}(t)B_{K_2}(t) + \sum_{H \in Y_{p^3}(G)} \frac{e_H t}{1 - |H|t} \right). 
\]

Now we compute \( B_{K_1}(t) \) and \( B_{K_2}(t). \) It is easy to observe that \( |Z(K_1)| = |Z(K_2)| = p^2. \) Both the groups \( K_1 \) and \( K_2 \) are of order \( p^4 \), using Theorem 3.2 we get

\[
B_{K_1}(t) = B_{K_2}(t) = \frac{1 - pt}{1 - p^2t(1 - p^3t)}. 
\]

Use the value of \( c_{K_1}, c_{K_2}, B_{K_1}(t) \) and \( B_{K_2}(t) \) in (7), we get

\[
B_G(t) = \frac{1}{1 - pt} \left( 1 + \frac{(p - 1)t(1 - pt)}{(1 - p^2t)(1 - p^3t)} + \frac{(p^2 - p)t(1 - pt)}{(1 - p^2t)(1 - p^3t)} + \frac{(p^3 - p)t}{1 - p^3t} \right)
\]

\[
= \frac{1}{1 - pt^3(1 - p^2t)}. 
\]

We normalize the expression to obtain the result. \qedhere

**Lemma 5.8.** Let \( G \) be a \( p \)-group of order \( p^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Phi_9. \) Then

\[
(5.8.1) \ A_G(t/|G|) = \frac{1}{p - 1} + \frac{1}{p - 1 - p^3t} + \frac{1}{p - 1 - p^4t} + \frac{p^4}{1 - t} + \frac{p^3}{1 - p^3t} + \frac{p}{1 - p^4t}.
\]

\[
(5.8.2) \ B_G(t/|G|) = \frac{1}{p - 1} + \frac{1}{p - 1 - p^3t} + \frac{1}{p - 1 - p^4t} + \frac{p^4}{1 - t} + \frac{p^3}{1 - p^3t} + \frac{p}{1 - p^4t}.
\]

**Proof.** Let \( G = \Phi_9(1^5) \) be a stem group in the isoclinic family \( \Phi_9. \) The group \( G \) has a polycyclic presentation

\[
G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^{p^i} = \alpha^{(p)}_{i+1} = 1 \ (i = 1, 2, 3) \rangle,
\]

where \( \alpha^{(p)}_{i+1} \) denotes \( \alpha^{p^i+1}_1 \alpha^{p^i+2}_2 \cdots \alpha^{p^i+p-1}_i. \) If \( p > 3, \) the relation \( \alpha^{(p)}_1 = 1, \) together with other power relations, imply \( \alpha^p = 1 \) and this forces \( \alpha_{i+1} = 1 \) for \( i = 1, 2, 3. \) If \( p = 3, \) then the relations \( \alpha^{(p)} = 1 \) for \( i = 1, 2, 3 \) imply \( \alpha_2^3 = \alpha_3^3 = 1 \) and the relation \( \alpha^{(p)}_1 = 1, \) together with other relations, imply \( \alpha_1^3 \alpha_3 = 1. \) For all prime numbers \( p, \) nilpotency class of \( G \) is 4 and \( M = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is a maximal subgroup which is abelian and degree of commutativity is positive. Using Theorem 4.2 we get

\[
A_G(t) = \frac{1}{p^3} \left( \frac{p}{1 - p^3t} + \frac{p^3 - p^4}{1 - p^3t} + \frac{p^4 - p}{1 - p^4t} \right).
\]
We normalize the expression to obtain the result.

\[ B_G(t) = \frac{1}{1 - pt} \left( 1 + \frac{(p^3 - 1)t}{1 - p^4t} + \frac{(p^2 - 1)p^t}{1 - p^2t} \right). \]

We normalize the expression to obtain the result.

**Lemma 5.9.** Let \( G \) be a \( p \)-group of order \( p^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Phi_{10} \). Then

\[
\begin{align*}
A_G(t/|G|) &= \frac{1}{1 - p^{-1}} \frac{p^{-1} - p^{-3}}{1 - p^{-2}t} + \frac{p^{-3} - p^{-4}}{1 - p^{-4}t} + \frac{p^{-4}}{1 - t} \quad \text{(5.10.1)} \\
B_G(t/|G|) &= \frac{1}{1 - p^{-1}} \frac{1 - p^{-3}}{1 - p^{-2}t} + \frac{p^{-3} - p^{-4}}{1 - p^{-4}t} + \frac{p^{-4}}{1 - t} \quad \text{(5.10.2)}
\end{align*}
\]

**Proof.** Let \( G = \Phi_{10}(1^5) \) be a stem group of isoclinic family \( \Phi_{10} \). The group \( G \) has a polycyclic presentation

\[ G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_1^{(p)} = 1 \ (i = 1, 2, 3) \rangle, \]

where \( \alpha_1^{(p)} \) denotes \( \alpha_i^{(p)} \). If \( p > 3 \), the relation \( \alpha_1^{(p)} = 1 \), together with other power relations, imply \( \alpha_i^{(p)} = 1 \) and this forces \( \alpha_i^{(p)} = 1 \) for \( i = 1, 2, 3 \). If \( p = 3 \), then the relations \( \alpha_1^{(p)} = 1 \) for \( i = 1, 2, 3 \) imply \( \alpha_3^3 \alpha_4 = \alpha_3^3 = \alpha_3^3 = 1 \) and the relation \( \alpha_1^{(p)} = 1 \), together with other relations, imply \( \alpha_1^3 \alpha_2^3 = 1 \). For all prime numbers \( p \), the group \( G \) has centre \( Z(G) = \langle \alpha_4 \rangle, \ G' = \gamma_2(G) = \langle \alpha_2, \alpha_3, \alpha_4 \rangle, \gamma_3(G) = \langle \alpha_3, \alpha_4 \rangle, \) and \( \gamma_4 = \gamma_5(G) \) The nilpotency class of \( G \) is 4 and degree of commutativity is positive. Observe that \( P_1 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is non-abelian, \( [P_1, P_3] = 1 \) and \( G \) has no maximal subgroup which is abelian. Using Theorem 4.3, we get

\[ A_G(t) = \frac{1}{p^3} \left( \frac{p}{1 - p^4t} + \frac{p^5 - p^4}{1 - p^2t} + \frac{p^4 - p^2}{1 - p^2t} + \frac{p^2 - p}{1 - p^2t} \right). \]

We further use Theorem 4.3 to obtain

\[ B_G(t) = \frac{1}{1 - pt} \left( 1 + \frac{(p - 1)t(1 - pt)}{(1 - p^2t)(1 - p^2t)} + \frac{(p^2 - 1)t}{1 - p^2t} + \frac{(p^2 - p)t}{1 - p^2t} \right). \]

We normalize the expression to obtain the result.

**2-groups of rank at most 5.** Since the classification of \( p \)-groups of rank \( \leq 5 \) in [8] are for odd primes, we consider the 2-groups separately in this subsection. The following lemmas are useful in computations of the expression of \( A_G(t) \) and \( B_G(t) \) for 2-groups of rank \( \leq 5 \). For sake of completion, in the first lemma of this subsection, we consider all the dihedral groups \( D_{2n} := \{ \alpha, \beta \mid \alpha^n = \beta^2 = 1, \alpha^2 = \alpha^{-1} \} \), when \( n \) is an even number.

**Lemma 5.10.** Let \( D_{2n} := \{ \alpha, \beta \mid \alpha^n = \beta^2 = 1, \alpha^2 = \alpha^{-1} \} \) be dihedral group of order \( 2n \). If \( n \) is even then

\[
\begin{align*}
A_{D_{2n}}(t) &= \frac{1}{2n} \left( \frac{2}{1 - 2nt} + \frac{n}{1 - 4t} + \frac{n - 2}{1 - nt} \right) \quad \text{(5.10.1)} \\
B_{D_{2n}}(t) &= \frac{1}{(1 - 2t)} \left( 1 + \frac{n - 2}{2(1 - nt)} + \frac{2t}{1 - 4t} \right) \quad \text{(5.10.2)}
\end{align*}
\]

**Proof.**

(5.10.1) The group \( D_{2n} \) has centre \( \langle \alpha \rangle \) of size two, when \( n \) is an even number. A non-central element \( x \) in subgroup \( \langle \alpha \rangle \) has \( Z_{D_{2n}}(x) = \langle \alpha \rangle \). Let \( x \in D_{2n} \setminus \langle \alpha \rangle \), then \( Z_{D_{2n}}(x) = \langle Z(D_{2n}), x \rangle \) is a group of order 4. Therefore \( X_n = n - 2 \) and \( X_4 = n \). We get \( A_{D_{2n}}(t) \) using (8).

(5.10.2) Note that \( D_{2n} \) has \( 4 + \frac{4 - 2}{2} \) conjugacy classes, when \( n \) is even number. There are \( \frac{4 - 2}{2} \) conjugacy classes whose representative has centralizer \( \langle \alpha \rangle \). Thus \( c_{\langle \alpha \rangle} = \frac{4 - 2}{2} \). The centralizer of representative of remaining two non-central conjugacy class is isomorphic to
the Klein’s four group. Using (4), we compute
\[ B_G(t) = \frac{1}{1 - 2t} \left( 1 + \frac{(n - 2)t}{2(1 - nt)} + \frac{2t}{1 - 4t} \right). \]

This completes the proof.

**Lemma 5.11.** Let \( G \) be a maximal class group of order \( 2^n; n \geq 4 \). Then

\[
(5.11.1) \quad A_G(t) = \frac{1}{2^n} \left( \frac{2}{1 - 2^n} + \frac{2^{n-1}}{1 - 4t} + \frac{2^{n-1} - 2}{1 - 2^n1} \right). \\
(5.11.2) \quad B_G(t) = \frac{1}{(1 - 2t)} \left( 1 + \frac{(2^{n-2} - 1)t}{2(1 - nt)} + \frac{2t}{1 - 4t} \right). 
\]

**Proof.** All groups of order \( 2^n \) with nilpotency class \( n - 1 \) are isoclinic (see [1]). Further if \( n \geq 4 \), then this class contains the following three groups [3, Theorem 4.5];

(5.11.1) \( D_{2^n} = \langle \alpha, \beta \mid \alpha^{2^{n-1}} = \beta^2 = 1, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \)

(5.11.2) \( SD_{2^n} = \langle \alpha, \beta \mid \alpha^{2^{n-1}} = \beta^2 = 1, \beta\alpha\beta^{-1} = \alpha^{n-2} \rangle \)

(5.11.3) \( Q_{2^n} = \langle \alpha, \beta \mid \beta^2 = \alpha^{2^{n-2}}, \alpha^{2^{n-1}} = 1, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \)

The isoclinic groups of same order have same \( A \) and \( B \) functions (see [9, Corollary 4.5]). Now the result follows from Lemma 5.10.

The \( 2 \)-groups of rank at most 5 are classified in 8 isoclinic families by Hall and Senior [6]. We compute generating functions \( A_G(t) \) and \( B_G(t) \) taking one stem group from each isoclinic family of rank at most 5. Then we use Theorem [9, Theorem 4.4] to get the generating functions for all the groups in the isoclinic family. We use the notation from the book by Hall and Senior [6, Chapter 5]

**Lemma 5.12.** Let \( G \) be a group of order \( 2^m \) of rank 3. Then

\[
(5.12.1) \quad A_G(t/|G|) = \frac{1 - 2^2}{1 - 2^2t} + \frac{2^2}{1 - t} \\
(5.12.2) \quad B_G(t/|G|) = \frac{1 - 2^-1}{1 - 2^-2t} + \frac{1 + 2^-1}{1 - 2^-1t} 
\]

**Proof.** All \( 2 \)-groups of rank 3 are isoclinic and belong to the family \( \Gamma_2 \). If \( G \in \Gamma_2 \), then \(|G/Z(G)| = 2^2 \) [3, Section 4]. Now \( A_G(t) \) and \( B_G(t) \) is given by Lemma 5.2. We normalize these expressions to get the result.

**Lemma 5.13.** Let \( G \) be a group of order \( 2^m \) of rank 4. Then

\[
(5.13.1) \quad A_G(t/|G|) = \frac{1 - 2^-1}{1 - 2^-2t} + \frac{2^-1 - 2^-3}{1 - 2^-2t} + \frac{2^-3}{1 - t} \\
(5.13.2) \quad B_G(t/|G|) = \frac{1 - 2^-1}{1 - 2^-2t} + \frac{1}{1 - 2^-2t} + \frac{2^-1}{1 - 2^-1t} 
\]

**Proof.** All \( 2 \)-groups of rank 4 are isoclinic and belong to the family \( \Gamma_3 \) [5, Section 4]. Let \( G = \Gamma_3a_1 \) be a stem group of the isoclinic family \( \Gamma_3 \). The group \( G \) has a polycyclic presentation:

\[ \langle \alpha, \beta \mid \alpha^8 = \beta^2 = 1, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle. \]

The group \( G \) is the dihedral group of order 16. Now using Lemma 5.10 we get

\[ A_G(t) = \frac{1}{16} \left( \frac{2}{1 - 16t} + \frac{8}{1 - 4t} + \frac{6}{1 - 8t} \right) \]

and

\[ B_G(t) = \frac{1}{1 - 2t} \left( 1 + \frac{3t}{1 - 8t} + \frac{2t}{1 - 4t} \right). \]

We normalize these expressions to get the result.

**Lemma 5.14.** Let \( G \) be a group of order \( 2^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Gamma_4 \). Then
(5.14.1) \( A_G(t/|G|) = \frac{1 - 2^{-1}}{1 - 2^{-2}t} + \frac{2^{-1} - 2^{-3}}{1 - 2^{-1}t} + \frac{2^{-3}}{1 - t}. \)

(5.14.2) \( B_G(t/|G|) = \frac{-2^{-1}}{1 - 2^{-3}t} + \frac{1}{1 - 2^{-2}t} + \frac{2^{-1}}{1 - 2^{-1}t}. \)

**Proof.** Let \( G = \Gamma_4a_2 \) be a stem group of isoclinic family \( \Gamma_4 \). The group \( G \) has a polycyclic presentation:

\[ \langle \alpha_1, \alpha_2, \beta \mid \alpha_1^4 = \alpha_2^3 = \beta^2 = 1, \beta \alpha_1 \beta^{-1} = \alpha_1^{-1}, \beta \alpha_2 \beta^{-1} = \alpha_2^{-1} \rangle. \]

Here, \( Z(G) = \langle \alpha_1^2, \alpha_2^2 \rangle \). Again \( |G/Z(G)| = 8 = 2^3 \) and \( G \) has maximal subgroup \( H = \langle \alpha, \alpha_2 \rangle \) of order 16 which is abelian. Using (3.3.2), we get

\[ A_G(t) = \frac{1}{32} \left( \frac{4}{1 - 32t} + \frac{12}{1 - 16t} + \frac{16}{1 - 8t} \right), \]

and

\[ B_G(t) = \frac{1}{1 - 4t} \left( \frac{6t}{1 - 16t} + \frac{4t}{1 - 8t} \right). \]

We normalize these expressions to get the result.

\[ \square \]

**Lemma 5.15.** Let \( G \) be a group of order \( 2^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Gamma_5 \). Then

(5.15.1) \( A_G(t/|G|) = \frac{1 - 2^{-4}}{1 - 2^{-2}t} + \frac{2^{-4}}{1 - t}. \)

(5.15.2) \( B_G(t/|G|) = \frac{1}{1 - 2^{-3}t} + \frac{-3 - 2^{-1} - 2^{-2}}{1 - 2^{-2}t} + \frac{3 + 2^{-1} + 2^{-2}}{1 - 2^{-1}t}. \)

**Proof.** Let \( G = \Gamma_5a_1 \) be a stem group of isoclinic family \( \Gamma_5 \). The group \( G \) has a polycyclic presentation:

\[ \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid \alpha_1^2 = \alpha_2^3 = \alpha_3^3 = \alpha_4^3 = \beta^2 = 1, \alpha_1 \alpha_2 \alpha_1^{-1} = \beta \alpha_3 \beta^{-1}, \alpha_1 \alpha_4 \alpha_1^{-1} = \beta \alpha_4 \beta^{-1}, \alpha_2 \alpha_3 \alpha_2^{-1} = \beta \alpha_3^{-1} \rangle. \]

It is easy to verify that \( Z(G) = \langle \beta \rangle \) is group of order 2 and \( G/Z(G) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is an elementary abelian 2-group of order \( 2^4 \). Therefore \( G \) is an extraspecial 2-group. Now using [9] Theorem 7.4], we get

\[ A_G(t) = \frac{1}{32} \left( \frac{2}{1 - 32t} + \frac{30}{1 - 16t} \right), \]

and using [9] Theorem 7.5, we get

\[ B_G(t) = \frac{1 - t}{(1 - 2t)(1 - 16t)}. \]

We normalize them to get the result.

\[ \square \]

**Lemma 5.16.** Let \( G \) be a group of order \( 2^m \) of rank 5 and \( G \) belongs to isoclinic family \( \Gamma_6 \). Then

(5.16.1) \( A_G(t/|G|) = \frac{1 - 2^{-2}}{1 - 2^{-2}t} + \frac{2^{-2} - 2^{-4}}{1 - 2^{-1}t} + \frac{2^{-4}}{1 - t}. \)

(5.16.2) \( B_G(t/|G|) = \frac{-2^{-1} - 2^{-2}}{1 - 2^{-3}t} + \frac{1 + 2^{-1} + 2^{-2}}{1 - 2^{-2}t}. \)

**Proof.** Let \( G = \Gamma_6a_1 \) be a stem group of isoclinic family \( \Gamma_6 \). The group \( G \) has a polycyclic presentation:

\[ \langle \alpha, \beta_1, \beta_2 \mid \alpha^8 = \beta_1^2 = \beta_2^2 = 1, \beta_1 \alpha_1 \beta_1^{-1} = \alpha_1^{-1}, \beta_2 \alpha_2 \beta_2^{-1} = \alpha_2^{-1} \rangle. \]

Observe that \( Z(G) = \langle \alpha_1^4 \rangle \) is cyclic group of order 2 and the derived subgroup \( G' = \langle \alpha_2^4 \rangle \) is cyclic group of order 4. Using the fact that for \( x \in G \setminus G' \), the conjugacy class of \( x \) is contained in coset \( xG' \) for all \( x \in G \setminus G' \), we get \( |Z_G(x)| = 8 \) or 16 for all \( x \in G \setminus Z(G) \). Consider the normal abelian subgroup \( H = \langle \alpha_2, \beta_2 \rangle \) of order 8. By relations among generators, it is routine check that \( X_{16} = H \setminus Z(G) \) and \( X_8 = G \setminus H \). Using [9], we get

\[ A_G(t) = \frac{1}{32} \left( \frac{2}{1 - 32t} + \frac{6}{1 - 16t} + \frac{24}{1 - 8t} \right). \]
Using these observations, we get that the total number conjugacy classes of $G$ is equal to $r = |Z(G)| + \frac{G-H}{p} + \frac{H-Z(G)}{p} = 2 + 3 + 6 = 11.$

If $x \in H \setminus G'$, we compute that $Z_G(x)$ is either isomorphic to subgroup $K_1 = \langle \alpha^2, \beta_1, \beta_2 \rangle$ or to subgroup $K_2 = \langle \alpha^2, \alpha \beta_1, \beta_2 \rangle$ of order 16 with centre $\langle \alpha^4, \beta_1 \rangle$ of order 4. If $x \in G' \setminus Z(G)$, Now observe that $Z_G(x)$ is isomorphic to subgroup $K_3 = \langle \alpha, \beta_2 \rangle$ of order 16 with centre $\langle \alpha^2 \rangle$ of order 4. Thus we have $Y_{16}(G) = \{K_1, K_2, K_3\}$. We notice that $c_{K_1} + c_{K_2} + c_{K_3} = 3$.

Next, we consider the case when $x \in G \setminus H$. In this case, $|Z_G(x)| = 8$. We further observe that $Z_G(x)$ is abelian in this case, as $x \in Z(Z_G(x))$ and $x \notin Z(G)$ implies $Z(G) \subseteq Z(Z_G(x))$. Therefore $|Z(Z_G(x))| > 2$ and $Z_G(x)$ is an abelian group. Suppose $Y_{8}(G) = \{H_1, \ldots, H_t\}$. Then $c_{H_1} + \ldots + c_{H_t} = 6$. Therefore by [5], we get

\begin{equation}
B_G(t) = \frac{1}{1 - 2t} \left( 1 + \frac{\sum_{K \in Y_{8}(G)} c_K B_K(t) + \sum_{H \in Y_{8}(G)} c_H t}{1 - |H|/t} \right).
\end{equation}

Now to find out $B_{G_i}(t)$, we compute $B_{K_i}(t)$ for $i = 1, 2, 3$. We know that $K_i/Z(K_i) = 4 = 2^2$ for $i = 1, 2, 3$. We use Theorem 3.2, we get

$$B_{K_i}(t) = \frac{1 - 2t}{(1 - 4t)(1 - 8t)} \quad \text{for} \quad i = 1, 2, 3.$$  

Therefore by (5), we get

\begin{equation}
B_G(t) = \frac{1}{1 - 2t} \left( 1 + \frac{3t(1 - 2t)}{(1 - 4t)(1 - 8t)} + \frac{6t}{(1 - 8t)} \right) = \frac{(1 - t)}{1 - 8t}(1 - 4t).
\end{equation}

We normalize the expressions of $A_G(t)$ and $B_G(t)$ to get the result. \hfill \square

**Lemma 5.17.** Let $G$ be a group of order $2^m$ of rank 5 and $G$ belongs to isoclinic family $\Gamma_7$. Then

\begin{align}
(5.17.1) \quad A_G(t/|G|) &= \frac{1 - 2^{-2}}{1 - 2^{-2}t} + \frac{2^{-2} - 2^{-4}}{1 - 2^{-4}t} + \frac{1}{1 - t} \\
(5.17.2) \quad B_G(t/|G|) &= \frac{1 - 2^{-2} - 2^{-2}}{1 - 2^{-3}t} + \frac{1}{1 - 2^{-2}t}.
\end{align}

**Proof.** Let $G = \Gamma_{7a_1}$ be a stem group of isoclinic family $\Gamma_7$. The group $G$ has a polycyclic presentation:

$$\langle \alpha, \beta_1, \beta_2, \beta_3 \mid \beta_1^2 = \beta_2^2 = \beta_3^2 = \alpha^4 = 1, \alpha \beta_1 \alpha^{-1} = \beta_1, \beta_2, \alpha \beta_3 \alpha^{-1} = \beta_2 \beta_3 \rangle.$$  

Observe that $Z(G) = \langle \beta_1 \rangle$ is cyclic group of order 2 and the derived subgroup $G' = \langle \beta_1, \beta_2 \rangle$ is cyclic group of order 4. Using the fact that for $x \in G \setminus G'$, the conjugacy class of $x$ is contained in coset $xG'$ for all $x \in G \setminus G'$, we get $|Z_G(x)| = 8$ or 16 for all $x \in G \setminus Z(G)$. Consider the normal abelian subgroup $H = \langle \alpha^2, \beta_1, \beta_2 \rangle$ of order 8. By relations among generators, it is routine check that $X_{16} = H \setminus Z(G)$ and $X_8 = G \setminus H$. Now using (4), we get

$$A_G(t) = \frac{1}{32} \left( \frac{2}{1 - 32t} + \frac{6}{1 - 16t} + \frac{24}{1 - 8t} \right).$$

Using these observations, we get that the total number conjugacy classes of $G$ is equal to $r = |Z(G)| + \frac{G-H}{p} + \frac{H-Z(G)}{p} = 2 + 3 + 6 = 11.$

If $x \in H \setminus G'$, we compute that $Z_G(x)$ is either isomorphic to subgroup $K_1 = \langle \alpha, \beta_1, \beta_2 \rangle$ or to subgroup $K_2 = \langle \beta_1, \beta_2, \beta_3 \alpha \rangle$ of order 16 with centre $\langle \alpha^2 \rangle$ of order 4. If $x \in G' \setminus Z(G)$. We compute that $Z_G(x)$ is isomorphic to subgroup $K_3 = \langle \alpha^2, \beta_1, \beta_2, \beta_3 \rangle$ of order 16 with centre $\langle \alpha^2 \rangle$ of order 4. Thus we have $Y_{16}(G) = \{K_1, K_2, K_3\}$. We notice that $c_{K_1} + c_{K_2} + c_{K_3} = 3$.
Next, we consider the case when $x \in G \setminus H$. In this case $|Z_G(x)| = 8$. We further observe that $Z_G(x)$ is abelian in this case, as $x \in Z(Z_G(x))$ and $x \notin Z(G)$ implies $Z(G) \subseteq Z(Z_G(x))$. Therefore $|Z(Z_G(x))| > 2$ and $Z_G(x)$ is an abelian group. Suppose $Y_k(G) = \{H_1, \ldots, H_t\}$. Then $c_{H_1} + \cdots + c_{H_t} = 6$. Therefore by [4], we get

$$B_G(t) = \frac{1}{(1 - 2t)} \left(1 + \sum_{K \in Y_k(G)} c_k t B_K(t) + \sum_{H \in Y_k(G)} \frac{c_H t}{(1 - |H| t)} \right).$$

Next, to find out $B_G(t)$, we compute $B_{K_i}(t)$ for $i = 1, 2, 3$. We know that $K_i / Z(K_i) = 4 = 2^2$ for $i = 1, 2, 3$. By using Theorem 5.2 we get

$$B_{K_i}(t) = \frac{1 - 2t}{(1 - 4t)(1 - 8t)} \quad \text{for } i = 1, 2, 3.$$ 

Therefore by (9), we get

$$B_G(t) = \frac{1}{(1 - 2t)} \left(1 + \frac{3t(1 - 2t)}{(1 - 4t)(1 - 8t)} + \frac{6t}{(1 - 8t)} \right) \quad \frac{(1 - t)}{(1 - 8t)(1 - 4t)}.$$ 

We normalize them to get the result.

**Lemma 5.18.** Let $G$ be a group of order $2^n$ of rank 5 and $G$ belongs to isoclinic family $\Gamma_8$. Then

$$A_G(t/|G|) = \frac{1 - 2^{-1}}{1 - 2^{-3} t} + \frac{2^1 - 2^{-4}}{1 - 2^{-1} t} + \frac{2^{-4}}{1 - 7 t}.$$ 

$$B_G(t/|G|) = \frac{1 - 2^{-1}}{1 - 2^{-4} t} + \frac{1}{1 - 2^{-3} t} + \frac{2^{-1}}{1 - 2^{-1} t}.$$

**Proof.** Let $G = \Gamma_{8a1}$ be a stem group $G$ of isoclinic family $\Gamma_8$. The group $G$ is dihedral group of order 32. Therefore from Lemma 5.10, we get

$$A_G(t) = \frac{1}{32} \left(\frac{2}{1 - 32t} + \frac{16}{1 - 4t} + \frac{14}{1 - 16t}\right)$$

and

$$B_G(t) = \frac{2 - 22t + 8t^2}{2(1 - 2t)(1 - 16t)(1 - 4t)}.$$ 

We normalize these expressions to get the result.

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