Some Explicit Formulas for the Generalized Frobenius-Euler Polynomials of Higher Order

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Abstract. Our aim is to derive some explicit formulas for the generalized Bernoulli and Euler polynomials in terms of Whitney and translated Whitney numbers of the second kind. Also we derive some explicit formulas for the generalized Euler polynomials and Genocchi-like polynomials in terms of generalized Whitney polynomials of the second kind. We provide an algorithm for computing the generalized Frobenius-Euler polynomials of higher order.

1. Introduction

For $v \in \mathbb{C}$ and for $m$ a positive integer, the generalized Pochhammer symbol $(\nu|m)_n$ is defined by

$$(\nu|m)_n := \nu(\nu-m)\cdots(\nu-(n-1)m) \quad \text{with} \quad (\nu|m)_0 = 1,$$

the Pochhammer symbol is $(\nu)_n := (\nu|1)_n$.

Also, the extended binomial polynomial $(\nu|n)$, for a variable $x$, is given by $(\nu|n)!.

Definition 1.1. Let $\alpha$ be a complex number, the generalized Bernoulli polynomials $B_{\alpha}(x,m)_n$ and the generalized Euler polynomials $E_{\alpha}(x,m)_n$ of parameter $m$ are given by the following generating functions, for $|z| < 2\pi$

$$\left(\frac{z}{e^{\nu z} - 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} B_{\alpha}(x,m)_n \frac{z^n}{n!}, \quad \text{and} \quad \left(\frac{2}{e^{\nu z} + 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} E_{\alpha}(x,m)_n \frac{z^n}{n!}. \quad (2)$$

For $m = 1$, we get the generalized Bernoulli polynomials $B_{\alpha}(x,1)_n$ and the generalized Euler polynomials $E_{\alpha}(x,1)_n$, see for instance [12, 16] and references therein.

For $x = 0$, we get the generalized Bernoulli numbers $B_{\alpha}(0,1)_n$ and the generalized Euler numbers $E_{\alpha}(0,1)_n$.

As a second extension to the Frobenius-like polynomials we propose

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Definition 1.2. Let \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), the generalized Frobenius-Euler polynomials of order \( \alpha \), \( H_n^{(m)}(x, \alpha | \lambda) \), \( n \geq 0 \), \( \alpha \in \mathbb{C} \), are defined by the generating function, for \( |z| < 2\pi \)

\[
\left( \frac{1 - \lambda}{e^{\lambda z} - \lambda} \right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(m)}(x, \alpha | \lambda) \frac{z^n}{n!}.
\]

(3)

When \( x = 0 \), we deal with the generalized Frobenius-Euler numbers of order \( \alpha \), \( H_n^{(m)}(0, \alpha | \lambda) = H_n^{(m)}(\alpha | \lambda) \).

We have the following relation

\[
H_n^{(m)}(x, \alpha | \lambda) = \sum_{k=0}^{n} \binom{n}{k} H_n^{(m)}(\alpha | \lambda)x^{n-k}.
\]

For \( m = 1 \), we have the Frobenius-Euler polynomials of order \( \alpha \), \( H_n(x, \alpha | \lambda) \), see for instance [16, 17] and references therein. We have \( E_n^{(m)}(x, \alpha) = H_n^{(m)}(x, \alpha | \lambda) \).

In the special case \( \alpha = m = 1 \), we get the Frobenius-Euler polynomials given by \( H_n^{(1)}(x, 1 | \lambda) = H_n(x, \lambda) \).

For more information we refer to [7, 12, 13].

We will also consider the case where \( \alpha \) be a variable at the end of each of the following sections.

Recently, Boutiche et al. [16] introduce the generalized Stirling polynomials of the second kind as

\[
S^k_n(x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n,
\]

they establish the following explicit formulas for the generalized Bernoulli polynomials and the generalized Euler polynomials in terms of the weighted Stirling polynomials of the second kind \( S^k_n(x) \) respectively,

\[
B_n(x, \alpha) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} (n+\alpha) \binom{n+k-1}{k} S^k_n(x), \quad \text{and} \quad E_n(x, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{k}{k} S^k_n(x).
\]

Also they proved an explicit formula for the Frobenius-Euler polynomials of order \( \alpha \) in terms of the weighted Stirling numbers of the second kind.

We propose some explicit formulas of the generalized Frobenius-Euler polynomials of order \( \alpha \) in terms of the generalized Whitney numbers of the second kind and of the generalized translated Whitney numbers of the second kind.

We start by some classical definitions concerning Whitney numbers.

Whitney numbers \( (w_m(n,k))_{k=0}^{\infty} \) of the first kind are given by \( (x-1)^m_n = \sum_{k=0}^{n} w_m(n,k)x^k \), they satisfy the following recursion formula [14],

\[
w_m(n,k) = w_m(n-1,k-1) + (m - 1 - mn) w_m(n-1,k) \quad (1 \leq k \leq n).
\]

Whitney numbers \( (W_m(n,k))_{k=0}^{\infty} \) of the second kind are given by \( x^m = \sum_{k=0}^{\infty} W_m(n,k)(x-1)^k \), they satisfy the following recursion formula [14],

\[
W_m(n,k) = W_m(n-1,k-1) + (mk + 1) W_m(n-1,k) \quad (1 \leq k \leq n).
\]

The exponential generating functions for \( w_m(n,k) \) and \( W_m(n,k) \) are given respectively by, see [3],

\[
\sum_{n=0}^{\infty} w_m(n,k) \frac{z^n}{n!} = (1 + mz)^{-(1/m)} \frac{\ln(1 + mz)}{m^k k!} \quad \text{and} \quad \sum_{n=0}^{\infty} W_m(n,k) \frac{z^n}{n!} = e^{(e^{mx} - 1)^k/k!ln^k}.
\]

For any nonnegative integer \( r \), the \( r \)-Whitney numbers were introduced by Mező [14] as a new class of numbers generalizing the Whitney and \( r \)-Stirling numbers. According to [9, 14], the \( n \)-th power of \( mx + r \)
can be expressed in terms of the falling factorial as follows \((mx + r)^n = \sum_{k=0}^{n} m^k \text{W}_{m,r}(n,k)x^k\), where the coefficients \(\text{W}_{m,r}(n,k)\) are called \(r\)-Whitney numbers of the second kind. Their generating function is

\[
\sum_{n=0}^{\infty} \text{W}_{m,r}(n,k) \frac{z^n}{n!} = e^{z(km - 1)}.
\]

We note that the \(r\)-Whitney numbers of the second kind may be reduced to the \(r\)-Stirling numbers of the second kind by setting \(m = 1\), i.e., \(\text{W}_{1,r}(n,k) = \text{S}(n,k)\) and \(\text{W}_{1,r}(n,k) = \text{S}(n + r,k + r)\).

We introduce now the generalized Whitney polynomials \(\text{W}_{m,r}(x)\) and numbers \(\text{W}_{m,r}^k\) of the second kind:

\[
\text{W}_{m,r}^k(x) = \frac{1}{k!m^k} \sum_{j=0}^{k} (-1)^{j}(\binom{k}{j})(mj + x + 1)^n, \quad \text{we denote } \text{W}_{m,r}^k := \text{W}_{m,r}^k(0).
\]

The corresponding exponential generating function is given by

\[
\sum_{n=0}^{\infty} \text{W}_{m,r}^k(x) \frac{z^n}{n!} = \frac{1}{k!m^k} e^{(x+1)z} e^{zm} - 1
\]

it satisfies the following recurrence relation,

\[
\text{W}_{m,r+1}^k(x) = \text{W}_{m,r}^k(x) + (x + mk + 1)\text{W}_{m,r}(x) \quad (1 \leq k \leq n).
\]

As a consequence, from Identity (6), one can deduce the following results,

\[
\text{W}_{m,r}^k(x) = \sum_{j=0}^{n} \binom{n}{j}\text{W}(j,k)x^{n-j}, \quad \text{where } \text{W}_{m,r}^k(0) = \text{W}(n,k),
\]

\[
\text{W}_{m,r}(r - 1) = \text{W}_{m,r}(n,k) \quad \text{W}_{m,r}^k(nr - 1) = m^{-k}S_n(n + r,k + r), \quad \text{and } \text{W}_{m,r}^k(x) = m^{-k}\text{W}_{m,r}^k\left(\frac{x + 1}{m}\right).
\]

2. Explicit formulas for the generalized Bernoulli polynomials of parameter \(m\).

We begin by establishing an explicit formula of the generalized Bernoulli polynomials of parameter \(m\) in terms of the generalized Whitney polynomials \(\text{W}_{m,r}^k(x)\) of the second kind. According to (1), we have the following result.

**Theorem 2.1.** The following identity holds true

\[
\text{B}_{m}^{(m)}(x,\alpha) = \sum_{k=0}^{n} (-1)^k \binom{n + k}{k} \binom{n + \alpha}{n - \alpha} \binom{n + \alpha - 1}{k} m^k \text{W}_{m,r}^k(x - 1).
\]

**Proof.** We proceed as in proof of the explicit formula of Srivastava and Todorov [18],

\[
\text{B}_{m}^{(m)}(x,\alpha) = \sum_{k=0}^{n} (-1)^k \binom{n + k}{k} \sum_{j=0}^{n-k} \binom{n}{j} \binom{n}{j} (x - 1)^{n-k-j} km^j \text{W}_{m,r}^{k+j} \sum_{l=0}^{j} \binom{n-k-j}{l} (x - 1)^{l} W_{m,r}^{j-l}.
\]

Since \(\sum_{l=0}^{j} \binom{n-k-j}{l} = \binom{n-k-j}{n-k}\), after some rearrangement, we obtain

\[
\text{B}_{m}^{(m)}(x,\alpha) = \sum_{k=0}^{n} (-1)^k \binom{n + k}{k} \binom{n}{j} \binom{n + k}{j} m^k \sum_{j=0}^{n-k} \binom{n}{j} (x - 1)^{n-k-j} \text{W}_{m,r}^{k+j}.
\]

Employing (8), the result is obtained. \(\square\)
Theorem 3.1. The following relationship holds true

\[
E_n^{(m)}(x, \alpha) = \sum_{k=0}^{n} \left( \frac{-1}{2k!} \right) (\alpha)_k m^k W_{m,n}^{k}(x - 1).
\]  

Proof. We observe that

\[
\sum_{n=0}^{\infty} \left( \frac{-1}{2k!} \right) (\alpha)_k m^k W_{m,n}^{k}(x) \frac{x^m}{m!} = \sum_{k=0}^{\infty} \left( \frac{-1}{2k!} \right) (\alpha)_k m^k \left( \sum_{n=k}^{\infty} W_{m,n}^{k}(x) \frac{x^n}{n!} \right) = e^{x+1} e^{-\frac{x^{m+1}}{m+1} - \frac{x^{m+1}}{m!}} = e^{x+1} e^{-\frac{x^{m+1}}{2} \left( 1 - e^{-\frac{x}{2}} \right)^2},
\]

this concludes the proof. \(\square\)

Using Identity (9), we have the explicit formula

\[
E_n^{(m)}(x, \alpha) = m^n \sum_{k=0}^{n} \left( \frac{-1}{2k!} \right) (\alpha)_k W_{m,n}^{k}(x) \frac{x^m}{m!}.
\]

In particular, for \(m = 1\), we get Relation (3.3) given in [6], and setting \(x = r\) for a fixed \(m\) in (11), we get

\[
E_n^{(m)}(r, \alpha) = \sum_{k=0}^{n} \left( \frac{-1}{2k!} \right) (\alpha)_k W_{m,n}^{k}(n, k).
\]
For $m = 1$ in (12), we have
\[
E_n(r, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k S_r(n + r, k + r).
\]

For $x = \alpha$ in the generating function of Euler polynomials, we get the generating function of $E_n^{(m)}(\alpha, \alpha)$, and get explicitly
\[
E_n^{(m)}(\alpha, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k m^k W_{m,n}^k(\alpha - 1).
\]

For $m = 2$, we get the following result
\[
\left(\frac{1}{\cosh z}\right)^x = \sum_{n=0}^{\infty} E_n^{(2)}(x, x) \frac{z^n}{n!}.
\]

For $x = 1$, we generate the coefficient of the hyperbolic secant number in terms of the generalized Bernoulli number.

Corollary 3.2. We have the following identity
\[
\text{sech } z = \sum_{n=0}^{\infty} E_n^{(2)}(1, 1) \frac{z^n}{n!}.
\]

4. Explicit formulas for the generalized Frobenius-Euler Polynomials

Now, we deal with the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, \alpha|\lambda)$ of order $\alpha \in \mathbb{C}$, with $\lambda \neq 1$. We express it in terms of the generalized Whitney polynomials of the second kind as follows

Theorem 4.1. The following relationship holds true
\[
H_n^{(m)}(x, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_{m,n}^k(x - 1). \tag{13}
\]

Proof. From (6), we have
\[
\sum_{k=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_{m,n}^k(x) \right) \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k \left( \sum_{n=0}^{\infty} W_{m,n}^k(x) \frac{z^n}{n!} \right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k \frac{e^{(x+1)z} - e^{(x-1)z}}{m^k e^{xz}} = e^{(x+1)z} \sum_{k=0}^{\infty} (\alpha)_k \frac{k!}{(\lambda - 1)^k} \left( \frac{e^{(x+1)z} - e^{(x-1)z}}{m^k e^{xz}} \right) = \sum_{n=0}^{\infty} \frac{H_n^{(m)}(x + 1, \alpha|\lambda) \frac{z^n}{n!}}{m^n}
\]

which gives the desired result. \(\square\)

Setting $m = 1$, we have
\[
H_n(x, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} e^k(x), \text{ which is due to [6, Eq. (7)].}
\]

For $x = 1$ and $\alpha = s$ ($s$ being a positive integer) in (13), we have
\[
H_n^{(m)}(1, s|\lambda) = \sum_{k=0}^{n} \frac{(s)_k}{(\lambda - 1)^k} m^k W_{m,n}(k).
\]

Substituting $s = 1$ in (13), we have
\[
H_n^{(m)}(x + 1|\lambda) = H_n^{(m)}(x, \lambda) = \sum_{k=0}^{n} \frac{k!}{(\lambda - 1)^k} m^k W_{m,n}^k(x - 1).
\]

For $x = r$ in (13), we have
\[
H_n^{(m)}(r, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_{m,n}(n, k).
\]

In particular, for $m = 1$, we have
\[
H_n(r, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} S_r(n + r, k + r), \text{ which is given in [16, Remark 3].}
\]

Setting $\lambda = -1$, we obtain Formula (11) for the generalized Euler polynomials $E_n^{(m)}(x, \alpha)$.

In particular, for $m = 1$, we have
\[
E_n(x, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k S^k_n(x), \text{ which is given in [6, Eq. 3.3].}
\]

Setting $x = \alpha$ in the generating function (3), we get the generating function $H_n^{(m)}(x, \alpha|\lambda)$, and get explicitly
\[
H_n^{(m)}(x, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_{m,n}^k(x - 1).
\]

According to (13) and (5), we have, for $m > 0$. 

Proposition 4.2. We have $H_n^{(m)}(x,\alpha|\lambda) = \sum_{i=0}^{n} \binom{n}{i} m^{n-i} \left( \sum_{0 \leq j \leq k \leq n} \frac{(\alpha)_k}{(\lambda - 1)^k} \frac{(-1)^{k-j}}{k!} (j)_i \right) x^i$.

Proof. In fact, $H_n^{(m)}(x,\alpha|\lambda) = \sum_{i=0}^{n} \binom{n}{i} m^{n-i} \left( \sum_{0 \leq j \leq k \leq n} \frac{(\alpha)_k}{(\lambda - 1)^k} \frac{(-1)^{k-j}}{k!} (j)_i \right) x^i = \sum_{i=0}^{n} \binom{n}{i} m^{n-i} \left( \sum_{0 \leq j \leq k \leq n} \frac{(\alpha)_k}{(\lambda - 1)^k} \frac{(-1)^{k-j}}{k!} (j)_i \right) x^i$.

5. Generalized Frobenius-Genocchi polynomials of higher order

Here, we give an explicit formula for Frobenius-Genocchi polynomials introduced by Yaşar and Özarslan [19], by the mean of the following generating function (1

\[ \frac{(1 - \lambda) z}{e^z - \lambda} \left( e^{\alpha x} - 1 \right) = \sum_{n=0}^{\infty} G_n^{(m)}(x,\alpha|\lambda) \frac{z^n}{n!}. \]

When $x = 0$, $G_n^{(m)}(0,\alpha|\lambda) = G_n^{(m)}(\alpha|\lambda)$ denote the generalized Frobenius-Genocchi numbers of order $\alpha$. For a unified extension of Euler and Genocchi polynomials, see [2]

The generalized Genocchi polynomials $G_n^{(m)}(x,\alpha)$, given by $G_n^{(m)}(x,\alpha) = G_n^{(m)}(x,\alpha - 1)$ are defined by the following generating function

\[ \left( \frac{2z}{e^z + 1} \right)^\alpha = \sum_{n=0}^{\infty} G_n^{(m)}(x,\alpha) \frac{z^n}{n!}. \]

By setting $\alpha = m = 1$ in (15), we obtain the classical Genocchi polynomials $G_n(x) = G_n^{(1)}(x,1)$.

According to (3), (14) and (13), we have the corollary

Corollary 5.1. The following explicit formula holds true

\[ G_n^{(m)}(x,l|\lambda) = \frac{n!}{(n-1)!} H_n^{(m)}(x,l|\lambda) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} \frac{(l)_k}{(\lambda - 1)^k} m^k W_{m,n-k}(x - 1). \]

By setting $m = 1$, Corollary 5.1 is reduced to the [6, Eq. 12].

By substituting $x = 1$ in (16), we obtain the following explicit formula in terms of the Whitney numbers of the second kind $G_n^{(m)}(1,l|\lambda) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} \frac{(l)_k}{(\lambda - 1)^k} m^k W_{m,n-k}(n - l, k)$.

If we set $x = r$, we obtain $G_n^{(m)}(r,l|\lambda) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} \frac{(l)_k}{(\lambda - 1)^k} m^k W_{m,r}(n - l, k)$.

By setting $\lambda = -1$ in (16), we obtain an explicit formula for the generalized Genocchi polynomials $G_n^{(m)}(x,l)$ of order $l$ as follows, $G_n^{(m)}(x,l) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{2^k} (l)_k m^k W_{m,n-k}(x - 1)$.

Setting $x = l$ in the generating function (14), we have \[ \frac{(1 - \lambda) z}{e^{\alpha n-l - \lambda e^z}} \left( e^{\alpha l} - 1 \right) = \sum_{n=0}^{\infty} G_n^{(m)}(l|\lambda, l) \frac{z^n}{n!}, \]

which have the explicit formula $G_n^{(m)}(l,l|\lambda) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} \frac{(l)_k}{(\lambda - 1)^k} m^k W_{m,n-k}(l - 1)$.
6. What about translated Whitney numbers

We introduce the generalized translated Whitney polynomials of the second kind by

\[ W_{n}^{k,m}(x) = \frac{1}{k!m!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x + mj)^n. \]

The exponential generating function of \( W_{n}^{k,m}(x) \) is given by

\[
\sum_{n=0}^{\infty} W_{n}^{k,m}(x) \frac{s^n}{n!} = \frac{1}{k!m!} e^{s(x + ms) - 1},
\]

and they satisfy the following recurrence relation

\[ W_{n+1}^{k,m}(x) = -W_{n}^{k-1,m}(x) + (x + mk)W_{n}^{k,m}(x), \quad (1 \leq k \leq n). \]

From (17), we have \( W_{n}^{k,m}(x + 1) = W_{n}^{k,m}(x) \quad (1 \leq k \leq n). \)

For \( m = 1 \), we recover the classical generalized Stirling numbers of the second kind \( S_{\alpha}^{n}(x) \).

For \( x = 0 \), we have the translated Whitney numbers of the second kind \( W_{n}^{m}(n,k) \), see [1].

For \( x = r \), we have the translated \( r \)-Whitney numbers of the second kind \( W_{n+r}^{m}(n+r,k+r) \), see also [1].

This section will be presented without proofs (they are similar to precedent ones).

6.1. Generalized Bernoulli polynomials

Employing the generalized translated Whitney polynomials \( W_{n}^{k,m}(x) \) of the second kind, we derive the following result concerning the generalized Bernoulli polynomials of parameter \( m \),

**Theorem 6.1.** The following relationship holds true

\[ B_{n}^{(m)}(x, \alpha) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} \binom{n+k+1}{k} (x+\alpha-k-1) m^{k} W_{n+k}^{k,m}(x). \]

Setting \( x = \alpha \), we get \( B_{n}^{(m)}(\alpha, \alpha) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} \binom{n+\alpha+k-1}{k} m^{k} W_{n+k}^{k,m}(\alpha). \)

For \( m = 1 \), we get the explicit formula for the generalized Bernoulli polynomials \( B_{n}(x, \alpha) \), see [6, Eq. 2.1].

Setting \( x = 0 \) in Theorem 6.1, we get an explicit formula for the generalized Bernoulli numbers \( B_{n}^{(m)} \).

For \( x = r \) and \( \alpha = -k \) (in \( \mathbb{N} \)), we get

\[ B_{n}^{(m)}(x, -k) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n+j}{j} \binom{n-k}{j} \binom{j-k-1}{k} m^{k} W_{r}^{m,n+j}(r+j). \]

And setting \( x = r \) and \( \alpha = 1 \), we get

\[ B_{n}^{(m)}(r, 1) = \sum_{k=0}^{n} (-1)^{n} \binom{n+k}{k} \binom{n+1}{k+1} m^{k} W_{n+k}^{m}(n+k). \]

6.2. Generalized Euler polynomials

An explicit formula for the generalized Euler polynomials in terms of the translated Whitney numbers:

**Theorem 6.2.** The following relationship holds true

\[ E_{n}^{(m)}(x, \alpha) = \sum_{k=0}^{n} \frac{(-1)^{k}}{2^{k}} \binom{n+k}{k} m^{k} W_{n+k}^{k,m}(x). \]

In particular, for \( m = 1 \), we have an explicit formula given in [6, Eq. (3.3)].

For \( x = 0 \), we get an explicit formula for the generalized Euler numbers \( E_{n}^{(m)}(0, \alpha) = \sum_{k=0}^{n} \frac{(-1)^{k}}{2^{k}} \binom{n+1}{k} m^{k} W_{n}^{m}(n,k). \)

Setting \( x = r \), we get

\[ E_{n}^{(m)}(r, \alpha) = \sum_{k=0}^{n} \frac{(-1)^{k}}{2^{k}} \binom{n+k}{k} m^{k} W_{r}^{m,n+r}(n+r,k+r). \]

Setting \( x = \alpha \), we have

\[ E_{n}^{(m)}(\alpha, \alpha) = \sum_{k=0}^{n} \frac{(-1)^{k}}{2^{k}} \binom{n+k}{k} m^{k} W_{n+k}^{k,m}(\alpha). \]
6.3. Generalized Frobenius-Euler polynomials

In the previous section, we proposed an explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, \alpha|\lambda)$: what about the generalized Frobenius-Euler numbers $H_n^{(m)}(\alpha|\lambda)$? So let’s setting $H_n^{(m)}(x, \alpha|\lambda)$ in Formula (13), we obtain an explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, \alpha|\lambda)$ in terms of the generalized translated Whitney numbers.

An explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, \alpha|\lambda)$ is given by the following theorem.

**Theorem 6.3.** The following relationship holds true

$$H_n^{(m)}(x, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_n^{k,m}(x).$$ (18)

For $x = \alpha$, we have $H_n^{(m)}(\alpha, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_n^{k,m}(\alpha)$.

For $m = 1$, we obtain $H_n^{(1)}(x, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} W_n^{k,m}(x)$, which is done by Boutiche et al. [6, Eq. (7)].

Setting $x = 0$ and $\alpha = s$ (s being a positive integer) in (18), we have an explicit formula for the generalized Frobenius-Euler numbers $H_n^{(m)}(s|\lambda), H_n^{(m)}(s|\lambda) = \sum_{k=0}^{n} \frac{(s)_k}{(\lambda - 1)^k} W_n^{k,m}(n, k)$.

By substituting $\alpha = 1$, we have $H_n^{(m)}(x, 1|\lambda) = \sum_{k=0}^{n} \frac{k!}{(\lambda - 1)^k} m^k W_n^{k,m}(x)$.

For $x = r$, we have $H_n^{(m)}(r, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_n^{k,m}(n + r, k + r)$.

In particular, for $m = 1$, we obtain $H_n^{(1)}(r, \alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda - 1)^k} S_r(n + r, k + r)$, which is in [16, Remark 3].

Setting $\lambda = -1$, we obtain the following explicit formula for the generalized Euler polynomials $E_n^{(m)}(x, \alpha)$,

$$E_n^{(m)}(x, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k m^k W_n^{k,m}(x).$$

In particular, for $m = 1$, we obtain the following explicit formula for the Euler polynomials $E_n^{(1)}(x, \alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k S_r(x)$, which was given by Boutiche et al. [6, Eq. (3.3)].

**Corollary 6.4.** The explicit formula for the generalized Frobenius-Genocchi polynomials of order $l$, $l \in \mathbb{N}$ is

$$G_n^{(m)}(x, l|\lambda) = \frac{n!}{(n - l)!} H_n^{(m)}(x, l|\lambda) = \frac{n!}{(n - l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda - 1)^k} m^k W_n^{k,m}(x).$$

By setting $x = 0$, we obtain the following explicit formula for the generalized Frobenius-Genocchi numbers of order $\alpha$,

$$G_n^{(m)}(l|\lambda) = \frac{n!}{(n - l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda - 1)^k} m^k W_n^{k,m}(n - l, k).$$

For $m = 1$, we obtain the explicit formula $G_n^{(1)}(l|\lambda) = \frac{n!}{(n - l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda - 1)^k} S(n - l, k)$, see [16, Eq. (12)].
By setting $x = r$, we obtain

$$G_n^{(m)}(r, l|\lambda) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \left(\frac{l}{\lambda - 1}\right)^k m^k W_n^{(m)}(n - l + r, k + r).$$

In particular, for $m = 1$, we obtain the explicit formula

$$G_n^{(1)}(r, l|\lambda) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \left(\frac{l}{\lambda - 1}\right)^k S_r(n - l + r, k + r).$$

By setting $\lambda = -1$, we have an explicit formula for the generalized Genocchi polynomials $G_n^{(m)}(x) of order l$,

$$G_n^{(m)}(x, l) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \left(-1\right)^k \left(\frac{l}{\lambda - 1}\right)^k m^k W_n^{(m)}(n - l - k).$$

For $m = 1$, we get an explicit formula for the Genocchi polynomials $G_n^{(1)}(x, l)$ of order $l$, see [16, Eq. (12)].

For $x = 0$, we get an explicit formula for the generalized Genocchi numbers $G_n^{(m)}(l)$ of order $l$,

$$G_n^{(m)}(l) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \left(-1\right)^k \left(\frac{l}{\lambda - 1}\right)^k m^k S_n(n - l, k).$$

7. Recurrence relations for the generalized Frobenius-Euler polynomials of higher order

In this section, we propose an algorithm based on a three-term recurrence relation for calculating the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, a|\lambda)$ of order $a$.

First, by setting $x = 0$, we obtain the following explicit formula for the generalized Frobenius-Euler numbers $H_n^{(m)}(a|\lambda) = \sum_0^n \frac{(\alpha)_k}{(\lambda - 1)^k} m^k W_n^{(m)}(n, k)$.

By means of the Stirling transform, we obtain

$$\frac{(\alpha)_n}{(\lambda - 1)^n} m^n = \sum_{k=0}^n H_n^{(m)}(a|\lambda) = \sum_{k=0}^n w^m(n, k) H_n^{(m)}(a|\lambda).$$

Now, we introduce the sequence $(A_n^{(m)}(\alpha))$ with two indices:

$$A_n^{(m)}(\alpha) = A_n^{(m)}(\alpha) = \frac{(\lambda - 1)^n}{m^n(\lambda - 1)} \sum_{k=0}^n \frac{m^k W_n^{(m)}(n, k)}{H_n^{(m)}(a|\lambda)}.$$ (19)

with $A_{0|\lambda} = 1$, $A_{0|\lambda} = H_n^{(m)}(a|\lambda)$, where $w^m(n, k)$ are the translated Whitney numbers of the first kind, given by $(x|m)_n = \sum_{k=0}^n w^m(n, k)x^k$, and satisfying the recurrence relation (see for details [1]),

$$w^m(n, k) = w^m(n - 1, k - 1) + m(n - 1)w^m(n - 1, k).$$

Theorem 7.1. The sequence $A_n^{(m)}(\alpha)$ satisfies the following three-term recurrence relation,

$$A_{n+1}^{(m)}(\alpha) = \frac{m(l + \alpha)}{\lambda - 1} A_{n+1}^{(m)}(\alpha) - m l A_n^{(m)}(\alpha), \quad n \geq 0, \quad with A_{0|\lambda} = 1.$$(20)

Proof. From (7) and (19), we have

$$A_{n+1}^{(m)} = \frac{(\lambda - 1)^{n+1}}{m^n(\lambda - 1)} \sum_{k=0}^{n+1} \frac{w^m(l, k - 1) H_n^{(m)}(a|\lambda) + m l w^m(l, k) H_n^{(m)}(a|\lambda)}{H_n^{(m)}(a|\lambda)} = \sum_{k=0}^{n+1} \frac{w^m(l, k) H_n^{(m)}(a|\lambda)}{H_n^{(m)}(a|\lambda)} = \frac{(\lambda - 1)^{n+1}}{m^n(\lambda - 1)} A_{n+1}^{(m)} + \frac{(\lambda - 1)^{n+1}}{m^n(\lambda - 1)} A_{n+1}^{(m)}.$$

Finally, we consider the polynomials $A_n^{(m)}(x, \alpha)$ defined by $A_{0|\lambda}^{(m)}(x, \alpha) = 1$, $A_{0|\lambda}^{(m)}(x, \alpha) = H_n^{(m)}(x, a|\lambda)$ and

$$A_n^{(m)}(x, \alpha) = \sum_{k=0}^n \binom{n}{k} A_k^{(m)}(\alpha) x^{n-k}.$$ (21)
Theorem 7.2. The polynomials $A_{n,j}^{(l,m)}(x,\alpha)$ satisfy the following three-term recurrence relation:

$$A_{n+1,j}(x) = (x - ml)A_{n,j}(x) + \frac{m(x + l)}{\lambda - 1}A_{n,j+1}(x),$$

with an initial sequence given by $A_{0,j}(x) = 1$.

Proof. From (7) and (21), we have

$$A_{n+1,j}(x) = \sum_{k=0}^{n} \binom{n}{k} A_{k,j}^{(l,m)}(\alpha)x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{\lambda^{k+1}} A_{k+1,j}^{(l,m)}(\alpha) + \frac{(\lambda-1)}{\alpha + 1} A_{k,j}^{(l,m)}(\alpha) \right)x^{n-k} =$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{\lambda^{k+1}} A_{k,j}^{(l,m)}(\alpha)x^{n-k+1} - \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\alpha + 1} A_{k+1,j}^{(l,m)}(\alpha)x^{n-k+1} + \frac{(\lambda-1)}{\alpha + 1} A_{n+1,j}(x) =$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{\lambda^{k+1}} A_{k,j}^{(l,m)}(\alpha)x^{n-k+1} + \frac{(\lambda-1)}{\alpha + 1} A_{n+1,j}(x) =$$

$$\frac{1}{\lambda^{n+1}} A_{n+1,j}(x) + (ml - x)A_{n,j}(x).$$

Then $A_{n+1,j}(x) = (x - ml)A_{n,j}(x) + \frac{m(x + l)}{\lambda - 1}A_{n,j+1}(x)$. □

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