Self-dual Chern-Simons Vortices on Riemann Surfaces

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Abstract

We study self-dual multi-vortex solutions of Chern-Simons Higgs theory in a background curved spacetime. The existence and decaying property of a solution are demonstrated.

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1 INTRODUCTION

Chern-Simons gauge theories have provide intriguing questions and answers to various subjects of both physics and mathematics. One of interdisciplinary topics attracted attention is so-called self-dual Chern-Simons solitons \[1, 2, 3\]. A natural extension is to include gravity which can be background \[4, 5\] or dynamical \[6, 7\]. Once the Bogomolnyi-type bound is obtained and detailed mathematical properties of those self-dual vortices are studied in the Chern-Simons Higgs model in the presence of background gravity, it would be helpful to address related physics-wise problems involving condensed matter systems, e.g., quantum Hall effects, supergravity, Lorentz-symmetry breaking due to parity-violating term, existence of time-like closed curve around gravitating spinning strings, cosmological implication of cosmic strings, and even cosmological constant problem.

Because it is applicable to diverse fields, mathematical study of self-dual Chern-Simons solitons is going on. The existence of a topological multi-vortex solution of relativistic Chern-Simons Higgs theory in flat \(R^2\) is shown by Wang \[8\]. In the same setting, rotationally-symmetric nontopological solitons and vortices were proven to exist by Spruck and Yang \[9\]. Yang also proved the existence of a topological self-dual multi-vortex solution when the gauge symmetry is extended to non-Abelian \[10\]. When the topological vortices or nontopological solitons are generated in condensed matter systems or in the early universe, they are likely to form a lattice structure or a network. In such sense important works have been done on torus \[11, 12, 13, 14\] or on standard sphere \[4, 15\]. Condensed matter experiments are usually performed by turning on constant external electric or magnetic field. In relation to this, Chae et al. demonstrated the existence of soliton solutions of self-dual Chern-Simons Higgs model coupled to an external background charge density \[16\]. Another study to have cosmological implication was done by Choe with nontopological soliton solutions under decaying metric \[17\].

Now let us take into account curved spacetime geometry of a straight string in the early universe. Then, extremely-small core region of the string is curved by matter fields, and the intermediate region is slightly-curved or locally-flat because of no graviton to the transverse directions. However, the asymptote of the global universe is known to be flat. All of such geometry should be dynamically determined by examining Einstein equations in exact sense, but it is practically too difficult to do with mathematical rigor. A meaningful starting point is to assume a physically-allowable set of background metrics and to study possible string configurations. In this paper, we study Chern-Simons Higgs theory on a uniformly Euclidean metric, which is not necessarily radial. A spatial metric \(\gamma_{ij} = b(x, y)\delta_{ij}\) is called uniformly Euclidean metric if there exist positive constants \(a_1\) and \(a_2\) with \(a_1 \leq b(x, y) \leq a_2\). We show
the existence of a self-dual topological multi-vortex solution and the fast decay property of a solution at infinity. The mathematical conditions we bring up are relevant to the physical situation discussed in the above, e.g., the gravity is not far from that of the flat case at the end of universe.

A brief outline of the paper is in order. In section 2, under the most general static metric, we shall derive the Bogomolnyi type bound of the Chern-Simons Higgs theory in background gravity. In section 3, we present the existence and asymptotic behavior of a solution of the self-dual Chern-Simons vortices. Conclusions with some discussions about our results are presented in section 4.

2 BOGOMOLNYI BOUND OF CHERN-SIMONS HIGGS THEORY IN BACKGROUND GRAVITY

In this section we recapitulate derivation of so-called Bogomolnyi bound of the Chern-Simons Higgs theory coupled to background gravity by assuming the general static metric

$$ds^2 = N^2(x^k)dt^2 - \gamma_{ij}(x^k)dx^idx^j \quad (i, j, k, \ldots = 1, 2),$$

(2.1)

where the metric of two-dimensional spatial hypersurface can always be diagonalized by a conformal gauge $\gamma_{ij} = \delta_{ij}b(x^k)$. Later we shall show that the Bogomolnyi bound is attained only when the lapse function $N(x^i)$ is constant, i.e., $N(x^k) = 1$ after a rescaling of time coordinate $t$.

The Chern-Simons Higgs theory is described by the action

$$S = \int d^3x \sqrt{g} \left[ \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(|\phi|) \right],$$

(2.2)

where $\phi = e^{i\Theta}|\phi|$ is a complex scalar field, $A_\mu$ a U(1) gauge field, and $D_\mu = \partial_\mu - ieA_\mu$ is gauge-covariant but not covariant under general coordinate transformation. Since the Bogomolnyi limit is our interest, the form of the scalar potential $V(|\phi|)$ is taken to be

$$V(|\phi|) = \frac{e^4}{8\kappa^2} |\phi|^2 (|\phi|^2 - v^2)^2.$$

(2.3)

From here on all the metric components and fields are assumed to be static because the self-dual solitons of our interests are static objects.

Symmetric energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{2}(D_\mu \phi D_\nu \phi + D_\nu \phi D_\mu \phi) - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} D_\rho \phi D_\sigma \phi - V(|\phi|) \right].$$

(2.4)
A physically-meaningful derivation of the Bogomolnyi bound is to investigate vanishing of stress components of the energy-momentum tensor. Since the lapse function \( N(x^i) \) disappears in every stress component by the help of Gauss’ law \( \kappa N B = e^2 A_0 |\phi|^2 \), an appropriate rearrangement of them gives

\[
T^{ij} = \frac{1}{2} \gamma^{ij} \left[ \frac{\kappa^2}{2e^2} \frac{B^2}{|\phi|^2} - V(|\phi|) \right] - \frac{1}{2} (\gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl} - \gamma^{il} \gamma^{jk}) D_k \phi D_l \phi \quad (2.5)
\]

or

\[
T^{ij} = \frac{\kappa^2}{2e^2} \gamma^{ij} \left[ B - \frac{e^3}{2\kappa^2} |\phi|^2(|\phi|^2 - v^2) \right] \left[ B + \frac{e^3}{2\kappa^2} |\phi|^2(|\phi|^2 - v^2) \right]
+ \frac{1}{8} \left\{ \left[ (D^i \phi \mp i \frac{e^{ik}}{\sqrt{\gamma}} \gamma^{kl} D^l \phi) (D^j \phi \pm i \frac{e^{jm}}{\sqrt{\gamma}} \gamma_{mn} D^n \phi) \right.ight.
+ \left. \left. (D^j \phi \pm i \frac{e^{jk}}{\sqrt{\gamma}} \gamma^{kl} D^l \phi) (D^i \phi \mp i \frac{e^{im}}{\sqrt{\gamma}} \gamma_{mn} D^n \phi) \right]\right. 
+ \left. \left. (D^i \phi \pm i \frac{e^{ik}}{\sqrt{\gamma}} \gamma^{kl} D^l \phi) (D^j \phi \mp i \frac{e^{jm}}{\sqrt{\gamma}} \gamma_{mn} D^n \phi) \right]\right. 
+ \left. \left. (D^j \phi \mp i \frac{e^{jk}}{\sqrt{\gamma}} \gamma^{kl} D^l \phi) (D^i \phi \pm i \frac{e^{im}}{\sqrt{\gamma}} \gamma_{mn} D^n \phi) \right]\right. 
\right\}, \quad (2.6)
\]

where \( \gamma^{ij} \) is inverse of the \( \gamma_{ij} \), \( \sqrt{\gamma} = \sqrt{\text{det} \gamma_{ij}} \), and the magnetic field is defined by \( B = - \frac{e^{ij}}{\sqrt{\gamma}} \partial_i A_j \).

We read the first-order Bogomolnyi equations from Eq. (2.6)

\[
B = \mp \frac{e^3}{2\kappa^2} |\phi|^2(|\phi|^2 - v^2), \quad (2.7)
\]

\[
D_i \phi \mp i \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} D_k \phi = 0. \quad (2.8)
\]

The second equation (2.8) expresses the spatial components of the gauge field \( A_i \) in terms of the scalar field, i.e., \( eA_i = \partial_i \Theta \mp \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_k \ln |\phi| \). Substituting it into the first Bogomolnyi equation (2.7) together with the conformal gauge, we have

\[
\partial^2 \ln |\phi| = \frac{e^4}{2\kappa^2} b |\phi|^2(|\phi|^2 - v^2) \mp \epsilon^{ij} \partial_i \partial_j \Theta, \quad (2.9)
\]

where Dirac-delta function like contribution of the scalar phase \( \Theta \) comes from multi-valued function such as \( \Theta = \sum_{k=1}^n \tan^{-1} \frac{x^2 - x_k^2}{x^2 - x^2_k} \).

Let us check a consistency condition that whether or not the Bogomolnyi equations (2.7) \( \sim \) (2.8) reproduce second-order Euler-Lagrange equations. Since we used the Gauss’ law, let us consider scalar field equation;

\[
\frac{1}{\sqrt{g}} D_\mu (\sqrt{g} g^{\mu \nu} D_\nu \phi) = - \frac{\phi}{|\phi|} \frac{dV}{d|\phi|}. \quad (2.10)
\]
For a static configuration, insertion of the Bogomolnyi equations (2.7), (2.8), (2.8) into the scalar equation (2.10) leads to
\[
\frac{1}{N} \gamma^{ij} \partial_i N \partial_j |\phi| = 0. \tag{2.11}
\]
As it is well-known, for every configuration of the self-dual solitons, derivative of the scalar amplitude vanishes nowhere, and both derivatives, \(\partial_i N\) and \(\partial_j |\phi|\), are not perpendicular each other. Then, the lapse function \(N\) should be a constant which we set to be one by a rescaling of time variable, i.e., \(dt \rightarrow dt/N\). Note that the spatial components of the gauge-field equation are automatically reproduced for \(N = 1\) without giving any additional constraint.

Now that we have the condition \(N = 1\), the derivation of the Bogomolnyi bound reduces to the original one by Schiff \([4]\). The energy is exactly proportional to the magnetic flux \(\Phi = \int d^2 x \sqrt{\gamma} B\) as follows
\[
E = \int d^2 x \sqrt{\gamma} \left[ \frac{\kappa^2 B^2}{2e^2 |\phi|^2} + \frac{1}{2} \gamma^{ij} D_i \phi D_j \phi + V(|\phi|) \right]
\]
\[= \int d^2 x \sqrt{\gamma} \left\{ \frac{\kappa^2}{2e^2 |\phi|^2} \left[ B \pm \frac{e^3}{2\kappa^2} |\phi|^2 (|\phi|^2 - v^2) \right]^2 \right. \]
\[+ \frac{1}{4} \gamma^{ij} D_i \phi \mp i \sqrt{\gamma} \gamma^{ik} D_k \phi \right] (D_j \phi \pm i \sqrt{\gamma} \gamma^{jm} D_m \phi)
\[\geq \left| \frac{ev^2}{2} \Phi \right|. \tag{2.12}
\]
The first and second lines of Eq. (2.12) vanish by substituting the Bogomolnyi equations (2.7)~(2.8), and the last total-divergence term in the third line of Eq. (2.12) does not contribute to the energy since U(1) current decays rapidly at spatial asymptote.

We read possible boundary conditions of the scalar amplitude at spatial infinity from the scalar potential (2.3), that is, \(\lim_{|x| \rightarrow \infty} |\phi| \rightarrow 0\) or \(v\). The former is a nontopological soliton or vortex, and the latter a topological vortex. All of them carry the magnetic flux \(\Phi\) (or equivalently U(1) charge \(Q = e \int d^2 x \sqrt{\gamma} A_0 |\phi|^2\) related by the Gauss’ law), and spin
\[
J \equiv \int d^2 x \sqrt{\gamma} \epsilon_{ij} x^i T^j_0 \tag{2.14}
\]
\[= \int d^2 x \sqrt{\gamma} \frac{1}{2} \sqrt{\gamma} \epsilon_{ij} x^i (D^2 \phi D_0 \phi + D_0 \phi D^i \phi) \tag{2.15}
\]
\[= \frac{e^2}{8\kappa} \int d^2 x \sqrt{\gamma} x^i \partial_i (|\phi|^2 - v^2)^2, \tag{2.16}
\]
which distinguishes the Chern-Simons solitons from the solitons in Abelian Higgs model.
3 EXISTENCE OF A SOLUTION

Throughout this section, we denote that \((M, \gamma)\) is a two-dimensional complete Riemann surface which is diffeomorphic to \(R^2\) with the metric \(\gamma_{ij} = b(x, y)\delta_{ij}\). We assume that there exist positive constants \(a_1\) and \(a_2\) with \(a_1 \leq b(x, y) \leq a_2\) for all \(z = (x, y) \in R^2\) \((x^1 = x\) and \(x^2 = y\) from here on). Let \(\Delta = \frac{1}{\det(\gamma_{ij})} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\) \((\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\), \(|\nabla u|\)\(|\nabla u|_E\) and \(\delta (\delta_E)\) is the Laplacian, the norm of the gradient and Dirac-delta function with respect to the metric \(\gamma_{ij}\) (Euclidean metric). We denote \(dz = dx dy\), \(dV_\gamma = b(x, y) dz\) and \(H^2_1\) be the Sobolev space, which is the completion of \(C^\infty_c(M)\) with respect to the norm \(||w|| = (\int_M |\nabla w|^2 + w^2 \ dV_\gamma)^{\frac{1}{2}}\).

In this section, we show the following Theorem.

**Theorem 1:** There exists a solution for the following self-dual Chern-Simons vortex equation on \((M, \gamma)\)

\[
\Delta w = e^w(e^w - 1) + 4\pi \sum_{k=1}^n \delta(z - z_k) \tag{3.1}
\]

with the boundary condition \(\lim_{|z| \to \infty} w = 0\). Moreover, \(w\) satisfies \(-ae^{-b|x|} \leq w(x) < 0\) at infinity for some positive constants \(a\) and \(b\).

The above equation (3.1) comes from Eq. (2.9) by rescaling the scalar field \(|\phi| = ve^w\) and the spatial coordinates \(x^i \to \frac{e^\sigma}{\epsilon^{1/2}} x^i\).

When there is no vortex, there is no \(H^2_1\) solution for the following equation (3.2),

\[
\Delta w = e^w(e^w - 1), \tag{3.2}
\]

because

\[
\int_{\Omega} e^w(e^w - 1)w dV_\gamma = -\int_{\Omega} |\nabla w|^2 dV_\gamma + \int_{\partial \Omega} w \frac{\partial w}{\partial \eta} dS \tag{3.3}
\]

for a sufficiently large smooth domain \(\Omega\) (see Ref. [3]). Applying the same method to the domain outside of vortices, we see that any \(H^2_1\) solution \(w\) of Eq. (3.1) satisfies \(w \leq 0\).

**Proof of Theorem 1:** To show the existence of a solution, we follow [3]. Take \(u_0\) be

\[
u_0 = -\sum_{k=1}^n \ln(1 + \mu |z - z_k|^{-2}), \tag{3.4}\]

then

\[
\Delta_0 u_0 = -4 \sum_{k=1}^n \frac{\mu}{(\mu + |z - z_k|^2)^2} + 4\pi \sum_{k=1}^n \delta_E(z - z_k). \tag{3.5}\]
Note that for any given smooth function $f(z)$,

\[
\int_M \Delta \ln |z - z_k|^2 f(z) dV_\gamma = \int_M \frac{1}{b(x, y)} \Delta_0 \ln |z - z_k|^2 f(z) b(x, y) dxdy = \int_{R^2} \Delta_0 \ln |z - z_k|^2 f(z) dxdy = 4\pi f(z_k). \tag{3.6}
\]

Therefore, $\Delta \ln |z - z_k|^2 = 4\pi \delta(z - z_k)$ and $\Delta E \ln |z - z_k|^2 = 4\pi \delta_E(z - z_k)$. Define $h_0$, $h$ and $B$ as the followings,

\[
h_0 = 4 \sum_{k=1}^{n} \frac{\mu}{(\mu + |z - z_k|^2)^2}, \quad h = h_0 / b,
\]

and

\[
B = e^{u_0} = \prod_{k=1}^{n} \frac{|z - z_k|^2}{\mu + |z - z_k|^2}. \tag{3.7}
\]

Take $w = u_0 + u$, then Eq. (3.1) turns out to be

\[
\Delta u = Be^u(Be^u - 1) + h. \tag{3.8}
\]

A critical point of functional $E$ defined on $H^1_\gamma$ is a solution of Eq. (3.8) where

\[
E(u) = \int_M |\nabla u|^2 + (Be^u - 1)^2 + 2hu dV_\gamma. \tag{3.9}
\]

By the basic inequality,

\[
(e^u - 1)^2 \geq \frac{|u|^2}{(1 + |u|)^2}, \tag{3.10}
\]

and

\[
\int_M 2hu dV_\gamma = \int_{R^2} 2h_0 u dz \leq 2 \left( \int_{R^2} h_0^2 dz \right)^{\frac{1}{2}} \left( \int_{R^2} u^2 dz \right)^{\frac{1}{2}}. \tag{3.11}
\]

Note that there exist constants $c_1$ and $c_2$ such that

\[
2 \left( \int_{R^2} h_0^2 dz \right)^{\frac{1}{2}} \leq \frac{c_1}{\sqrt{\mu}} \tag{3.12}
\]

and

\[
\int_M (B - 1)^2 dV_\gamma \leq c_2. \tag{3.13}
\]
Note that Eq. (3.13) holds when $dV_\gamma < cr^3\epsilon dr$ for any positive constant $c$ and any positive small constant $\epsilon$. The second term of Eq. (3.9) can be estimated as

\[
\int_M (Be^u - 1)^2 dV_\gamma \geq \frac{1}{2} \int_M B^2(e^u - 1)^2 - (B - 1)^2 dV_\gamma. \tag{3.14}
\]

Let us define $\Omega_1 = \{ x \in M | B^2(x) \leq 1/2 \}$ and $|\Omega_1|$ be the area of $\Omega_1$. The finiteness of $|\Omega_1|$ implies

\[
\int_{\Omega_1} \left( B^2 - \frac{1}{2} \right) \frac{|u|^2}{(1 + |u|)^2} dV_\gamma \geq \int_{\Omega_1} \frac{1}{2} \frac{|u|^2}{(1 + |u|)^2} dV_\gamma \geq -\frac{1}{2} |\Omega_1|. \tag{3.15}
\]

From Eqs. (3.10) and (3.15), there is a constant $c_3$ that

\[
\int_M B^2(e^u - 1)^2 dV_\gamma \geq \int_M B^2|u|^2 (1 + |u|)^2 dV_\gamma \\
= \int_{M - \Omega_1} B^2|u|^2 dV_\gamma + \int_{\Omega_1} B^2|u|^2 dV_\gamma > \frac{1}{2} \int_M |u|^2 (1 + |u|)^2 dV_\gamma - c_3 \tag{3.16}
\]

For $f \in H^1_1(R^2)$, $\int_{R^2} f^2 dz \leq \frac{1}{4} \left( \int_{R^2} |\nabla f|^2 dz \right)^2$. Set $f = u^2$ and we have

\[
\int_{R^2} u^4 dz \leq \left( \int_{R^2} |u\nabla u|dV_\gamma \right)^2 \leq \int_{R^2} u^2 dz \int_{R^2} |\nabla u|^2 dz, \tag{3.17}
\]

and

\[
\left( \int_{R^2} u^2 dz \right)^2 \leq \left[ \int_{R^2} \frac{|u|^2}{1 + |u|} (1 + |u|)|u|dz \right]^2 \\
\leq \int_{R^2} \left( \frac{|u|^2}{1 + |u|} \right)^2 dz \int_{R^2} u^2(1 + |u|)^2 dz \leq 2 \int_{R^2} \left( \frac{|u|^2}{1 + |u|} \right)^2 dz \int_{R^2} u^2 + u^4 dz. \tag{3.18}
\]

Using Eqs. (3.17) and (3.18), we obtain

\[
\int_{R^2} u^2 dz \leq 2 \int_{R^2} \left( \frac{|u|^2}{1 + |u|} \right)^2 dz \left( 1 + \int_{R^2} |\nabla u|^2 dz \right). \tag{3.19}
\]
By the Hölder inequality and Eq. (3.19), \( u \in H^2_1(R^2) \) satisfies
\[
\left( \int_{R^2} u^2 dz \right)^{\frac{1}{2}} \leq \int_{R^2} \frac{|u|^2}{(1+|u|)^2} dz + 2 \int_{R^2} |\nabla u|_E^2 dz + 2. \tag{3.20}
\]

From the above, there exists a constant \( c_4 \) such that
\[
E(u) \geq \int_M |\nabla u|^2 dV_\gamma + \frac{1}{2} \int_M \frac{|u|^2}{(1+|u|)^2} dV_\gamma - \frac{c_1}{\sqrt{\mu}} \left[ \int_{R^2} \frac{|u|^2}{(1+|u|)^2} dz + 2 \int_{R^2} |\nabla u|_E^2 dz + 2 \right] - c_3 - c_2
\]
\[
\geq \left( 1 - \frac{2c_1}{\sqrt{\mu}} \right) \int_M |\nabla u|^2 dV_\gamma + \left( \frac{1}{2} - \frac{c_1}{a_1 \sqrt{\mu}} \right) \int_M \frac{|u|^2}{(1+|u|)^2} dV_\gamma - c_4, \tag{3.21}
\]
where we used \( \int_M |\nabla u|^2 dV_\gamma = \int_{R^2} |\nabla u|_E^2 dz \). From Eq. (3.20) and by taking large \( \mu \), there exist constants \( c_5 \) and \( c_6 \) such that
\[
E(u) \geq c_5 \left( \int_M |\nabla u|^2 + u^2 dV_\gamma \right)^{1/2} - c_6. \tag{3.23}
\]

Therefore \( E(u) \) is coercive on \( H^2_1 \) and \( \inf_{u \in H^2_1} E(u) \) is finite. Moreover, \( E(u) \) is weakly lower semi-continuous functional on \( H^2_1 \). We take a minimizing sequence \( \{u_n\} \) for \( \inf_{u \in H^2_1} E(u) \). Then \( \{u_n\} \) is bounded on \( H^2_1 \), which has a subsequence \( \{u_{n_k}\} \) converging to \( u \in H^2_1 \), a minimizer for \( \inf_{u \in H^2_1} E(u) \). By the elliptic regularity, \( u \) is smooth. Finally, \( u \) satisfies Eq. (3.8).

Next we study the behavior of solution of Eq. (3.1). Since \( \Delta w \leq 0 \) and \( w \leq 0 \), we have \( -c ||w||_{L^2(B(x,1))} \leq w(x) < 0 \) at infinity for some positive constant \( c \) (see Ref. [18]). Therefore \( w \) decays to zero uniformly at infinity. For a sufficiently small positive constant \( \delta \), \( -\delta < w < 0 \) implies
\[
w \Delta_0 w = b e^w (e^w - 1) w \geq \frac{w^2}{2}. \tag{3.24}
\]

By Jaffe and Taubes [19] methods (page 83), \( -ae^{-b|x|} \leq w(x) < 0 \) for some positive constants \( a \) and \( b \) at infinity.

Remarks: Since \( a_2 \) does not appear in Eq. (3.22), we can generalize Theorem 1 if the integral value of Eq. (3.13) is bounded. For example, Theorem 1 holds if \( c'dr \leq dV_\gamma = b(x, y) dx \leq cr^{3-\epsilon} dr \) at infinity for any positive constants \( c' \), \( c \) and any small positive constant \( \epsilon \).
4 CONCLUDING REMARKS

We extend the existence and decay property of topological multi-vortex solutions of Chern-Simons Higgs theory in a general background curved spacetime, which have been studied on flat space or on special background metric. Finding the borderline of growth or decaying condition of the given metric, which gives Theorem 1, is an interesting question. Related issues, e.g., the existence of nontopological solitons and vortices, self-dual topological vortices in a suitably decaying metric and Chern-Simons solitons under a dynamical gravity, need further study.

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