Optimization reconstruction of biregular term from limited-angle projections

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Abstract. To reduce artifacts of CT image reconstructed from limited-angle projections data, we develop a biregular term optimization algorithm, which introduces the singular value decomposition (SVD) as an additional regularization term on the basis of gradient L0 norm regularization. We combine the alternating direction (ADM) method and the variable splitting method to solve the proposed optimization model. Firstly, the regularization of gradient L0 norm is performed, which takes advantage of the sparsity of the image gradient. Then, the regularization of SVD is carried out, which based on the low rank of the image. Finally, the weighted errors of two regularized image are fed back to the iterative reconstruction process. The experimental results show that, compared with the SART, SART+SVD, SART+L0 algorithms, the maximum peak signal-to-noise ratio (PSNR) of the proposed algorithm is increased by about 28%, 25%, 1%, and the root mean square error (RMSE) is reduced by about 11%, 8%, 0.05%, respectively. The proposed biregular term optimization algorithm can accurately restore image edge, effectively reduce the artifacts reconstructed by limited-angle projections, and significantly improve image quality.

1. Introduction

Computed tomography (CT) can reconstruct the density distribution image of the testing object from certain projections and appropriate algorithm. It has the advantages of non-destructive, high precision, visualization. When the projections is complete, the CT image can be usually reconstructed well by iterative or analytical algorithms. However, only limited-angle projections can be acquired due to the constraints of detection environment, object structure, scanning mode and so on [1].

As far as the problem of limited-angle projections reconstruction is concerned, there will be stripe artifacts and distortion of the image by using analytical algorithm, while Iterative algorithm can effectively eliminate such artifacts by incomplete angle projections. Simulation Algebraic Reconstruction Technique (SART), a typical iterative reconstruction algorithm [2] is widely used because of its good artifact removal ability. However, since the SART algorithm may not converge during the iterative process, the limited-angle artifacts cannot be completely eliminated, thus it is difficult to meet the actual requirements. Compressed Sensing (CS) theory proposed by Candes [3] makes it possible to reconstruct incomplete projections with high quality. In CS theory, the L0 norm of image gradient is taken as a standard measure of sparsity. However, since the L0 norm minimization is an NP-hard problem, it is difficult to traverse the solution in practical applications. Generally, L0 norm is transformed into L1 norm and then convert the non-convex optimization problem into a
convex optimization problem. The reconstructed image is obtained by solving the convex optimization problem with the L1 norm minimization of the sparse image. In 2006, Sidky and Pan [4] combined the Total Variation (TV) minimization with the SART reconstruction algorithm, and obtained the minimum value by the steepest descent method which achieved better optimal reconstruction of images with sparse characteristics. Xu et al. [5] studied a new image smoothing method, which uses the gradient L0 norm minimize of the image with the alternative optimization and half-quadratic splitting. Yu Wei et al. [6-7] proposed a limited-angle projections edge-preserving reconstruction algorithm based on L0 norm regularization optimization, which uses variable splitting [8] and alternating direction method (ADM) [9]. With the maturity of the compressed sensing theory based on sparse representation and the application of related technologies, low rank has gradually attracted people's attention. Cai et al. [10] proposed a kernel norm minimization approximate matrix rank minimization algorithm, which uses singular value shrinkage to approximate kernel norm of image, and achieves low rank of image by SVD and soft threshold constraint [11].

The L0 norm has good performance in artifacts reduction and edge preservation, but may result in excessive edge smoothing or false contours. According to the low rank of the image, SVD is used as the auxiliary regularization term to maintain details and edge, the variable splitting method is adopted to alternately use the L0 norm and the SVD for suppressing the edge over-smoothing. The reconstruction results from simulated projections and actual projections data show that the biregular term optimization algorithm combined with L0 norm and SVD can effectively eliminate limited-angle projections reconstruction artifacts and restore image contours and details.

2. Combined L0 norm and SVD reconstruction algorithm

2.1 Regularization reconstruction algorithm

The regularization reconstruction algorithm is to attach a prior conditional constraint to the solution of the ill-posed problem, and then transform it into a solution of a well-posed new problem that approximates the original problem. Specifically, the objective function is established and the minimum value is obtained. The process of solving the minimum value is the process of solving the optimal solution of the ill-posed problem. Eq. (1) is the process of solving the optimal solution of the ill-posed problem. Where, $A$ is the system matrix of projections, $X$ is the image to be reconstructed, $P$ denotes the projections data, $\alpha > 0$ is regularization parameter.

$$X = \arg \min \left( \|AX - P\|_2 + \alpha J(x) \right) \quad (1)$$

2.2 L0 norm regularization algorithm

L0 norm of the image gradient reflects the sparsity of the signal and can find a unique sparse solution from Eq. (2). Since the L0 solution problem is an NP-hard problem, L1 is often used for the convex relaxation of L0. In this paper, the gradient of image with L0 norm minimization method proposed by Xu Li et al is used as the L0 norm minimization model. The L0 norm of the image gradient is defined as follows:

$$\|\nabla X\|_0 = \# \{ p \| \partial_x X_p \|_2 + \| \partial_y X_p \|_2 \neq 0 \} \quad (2)$$

In Eq. (2), $\nabla X_p = (\partial_x X_p, \partial_y X_p)^T$ is the gradient of the image at the point $p$, another two auxiliary variables $h_p$ and $v_p$ corresponding to the gradients $\partial_x X_p$ and $\partial_y X_p$ are introduced. $\# \{ \}$ counts the number of $p$ satisfying the condition $\| \partial_x X_p \|_2 + \| \partial_y X_p \|_2 \neq 0$, $\|\nabla X\|_0$ is the number of $p$ that satisfied the condition $\| \partial_x X_p \|_2 + \| \partial_y X_p \|_2 \neq 0$, that is, the L0 norm of the image gradient. The image reconstruction model based on L0 norm is as follows:

2
\[
X^* = \arg\min \left\{ \frac{1}{2} \|AX - P\|^2 + \lambda \|\nabla X\| \right\} \quad (3)
\]

Since the L0 norm is Non-derivative, the global optimal problem is an NP-hard problem, and the traditional gradient descent method is not applicable to Eq. (3). Yu Wei et al. used the variable splitting and the ADM to solve the problem Eq. (3). The method is to relax the problem into two quadratic programming problems, each of which has its closed-form solution. The solution method essentially achieves the L0 norm minimization of image gradient by an approximation to the regularization optimization problem [12]. The specific solution is to introduce the auxiliary variables \(h_p\) and \(v_p\), corresponding to the direction in the \(x\) direction and \(y\) direction, and then decompose into sub-problems, and the decomposition process is as follows:

1. Using augmented Lagrangian function to optimize Eq. (3)

\[
L(X, X_{lo}, d_i) = \frac{1}{2} \|AX - P\|^2 + \lambda \|\nabla X_{lo}\| + \frac{\mu}{2} \|X - X_{lo} - d_i\|^2 \quad (4)
\]

Eq. (4) with each iteration containing the following three steps:

\[
\begin{align*}
X^{k+1} &= \arg\min \left\{ \frac{1}{2} \|AX - P\|^2 + \frac{\mu}{2} \|X - X_{lo}^{(k)} - d_i^{(k)}\|^2 \right\} \\
X_{lo}^{k+1} &= \arg\min \|\nabla X_{lo}^{(k)}\| + \frac{\mu}{2} \|X^{(k+1)} - X_{lo} - d_i^{(k)}\|^2 \\
d_i^{k+1} &= d_i^{(k)} - (X^{(k+1)} - X_{lo}^{(k+1)})
\end{align*}
\]

2. The second item \(\|\nabla X_{lo}^{(k)}\|\) of Eq. (5) decomposed into the following steps:

\[
\|\nabla X_{lo}^{(k)}\| = \arg\min \left\{ \sum_{p=1}^{N} (X_{lo,x} - x_p^{(k+1)} + d_{i,x}^{(k)})^2 + \frac{2}{\mu_i} C(h,v) + \beta \left( \partial_x X_{lo,x} - h_p \right)^2 + \left( \partial_y X_{lo,y} - v_p \right)^2 \right\} \quad (6)
\]

\[
(h^{(r+1)}, v^{(r+1)}) = \arg\min \left\{ \sum_{p=1}^{N} \left( \partial_x X_{lo,x} - h_p \right)^2 + \left( \partial_y X_{lo,y} - v_p \right)^2 \right\} + \frac{2}{\mu_i \beta} C(h,v) \quad (7)
\]

\[
X_{lo}^{(r+1)} = \arg\min \left\{ \sum_{p=1}^{N} (X_{lo,x} - x_p^{(k+1)} + d_{i,x}^{(k)})^2 + \beta \left( \partial_x X_{lo,x} - h_p^{(r+1)} \right)^2 + \left( \partial_y X_{lo,y} - v_p^{(r+1)} \right)^2 \right\} \quad (8)
\]

Where \(h_p\) and \(v_p\) is auxiliary variables. The gradient is controlled by automatic \(\partial_x X_p\) and \(\partial_y X_p\) tuning parameter \(\beta\).

\[
C(h,v) = \sum_{p=1}^{N} H(|h_p| + |v_p|) \quad (9)
\]

\[
H(|h_p| + |v_p|) = \begin{cases} 
1, & |h_p| + |v_p| \neq 0 \\
0, & |h_p| + |v_p| = 0
\end{cases} \quad (10)
\]

To solve the optimization problem Eq. (8), we use a fast method which incorporates the convolution theorem of Fourier transform and the diagonalized derivative operator expressed [7], the optimization is as follows:
The optimization objective function proposed in this paper is as follows:

\[ X = \arg\min \left( \frac{1}{2} \| AX - P \|^2 + \lambda_2 \| X \|^2 \right) \]  

(16)

Using variable splitting alternate direction optimization method, the problem is decomposed into three sub-problems. At first, the variable splitting algorithm is used to separate the part from \( \| VX \| \) and \( \| X \| \), and then the augmented Lagrangian can be given as Eq. (17):

\[ L(X, X_{\text{Lo}}, X_{\text{SVT}}, d_1, d_2) = \frac{1}{2} \| AX - P \|^2 + \lambda_2 \| VX_{\text{Lo}} \|^2 + \lambda_2 \| X \|^2 + \frac{\lambda_2}{2} \| X - X_{\text{SVT}} - d_2 \|^2 + \frac{\lambda_1}{2} \| X - X_{\text{Lo}} - d_1 \|^2 \]

(17)
Where, $X_{L0}$ is the image optimized for the L0 norm, $X_{SVD}$ is the image optimized of the SVD, $\mu$ is the Lagrangian multiplier introduced for the L0 norm term, and $\mu_2$ is the Lagrangian multiplier introduced for the SVD term, $d_1$ is the error of the optimized image from L0 norm optimization, $d_2$ is the error of the optimized image from SVD. The iteration formulas for sub-problems are as follows:

$$
\begin{align}
X^{k+1} &= \arg \min_{X} \frac{1}{2} \|AX - P\|^2 + \frac{\mu}{2} \|X - X_{L0}^{(k)} - d_1^{(k)}\|^2 + \\
&\quad \frac{\mu_2}{2} \|X - X_{SVD}^{(k)} - d_2^{(k)}\|^2 \\
X_{L0}^{k+1} &= \arg \min_{X} \|AX_{L0}^{(k)}\|^2 + \frac{\mu}{2} \|X_{L0}^{(k+1)} - X_{L0} - d_1^{(k+1)}\|^2 \\
X_{SVD}^{k+1} &= \arg \min_{X} \|AX_{SVD}^{(k)}\|^2 + \frac{\mu_2}{2} \|X_{SVD}^{(k+1)} - X_{SVD} - d_2^{(k+1)}\|^2 \\
d_1^{k+1} &= d_1^{(k)} - (X^{(k+1)} - X_{L0}^{(k+1)}) \\
d_2^{k+1} &= d_2^{(k)} - (X^{(k+1)} - X_{SVD}^{(k+1)})
\end{align}
$$

(18)

The specific steps of this method are as follows:

① Initialize the reconstruction parameters. $X = 0$, $d_1 = 0$, $d_2 = 0$, $X_{L0} = 0$, $X_{SVD} = 0$, and $\lambda$ is iterative relaxation factor, $\lambda_1$ and $\lambda_2$ are regularization parameter, $u_1 = \frac{2}{\mu_1}$ is weight of L0 smoothing, $\beta$ is L0 smoothing rate, $u_2 = \frac{2}{\mu_2}$ is singular value threshold, $\sigma = 0.00001$;

② The first item in Eq. (18) is solved by the SART [2] to obtain the reconstructed image, and the image $X$ is corrected by feedback errors from feedback errors from SVD and L0 norm, respectively;

③ Using the L0 norm to minimize the second term in Eq. (18), the solution $X - d_1$ is minimized $X_{L0}$, and the error $d_1$ in Eq. (19) is updated;

④ Using SVD and soft threshold constraint to solve the third term in Eq. (17). In SVD process, the optimized image $X_{L0}$ is replaced by $X$ in step 3 to obtain the optimal solution, and the error $d_1$ in Eq. (19) is updated;

⑤ If the convergence condition $\|X - X_{SVD}\| \leq \sigma$ is not met, return to step ② and continue with the next iteration, otherwise ends reconstruction.

3. Experiments and results

In order to objectively evaluate the reconstruction results of the proposed method, the peak signal-to-noise ratio (PSNR) and root mean square error (RMSE) are used for quantitative evaluation [14], which are defined as follows.

$$
\text{PSNR} = 10 \log_{10} \left( \frac{\text{MAX}^2(I_{mm})}{\frac{1}{hw} \sum_{i=1}^{h} \sum_{j=1}^{w} (I(i,j) - I_{mm}(i,j))^2} \right)
$$

(20)

$$
\text{RMSE} = \sqrt{\frac{1}{hw} \sum_{i=1}^{h} \sum_{j=1}^{w} (I(i,j) - I_{mm}(i,j))^2}
$$

(21)
Where $I_{true}$ is the ideal image (or original image), $I$ is the reconstructed image, $\text{MAX}(I_{true})$ is the maximum value of the pixel values in the ideal image, $h$ is the pixel length and $w$ is width in the ideal image.

### 3.1 Digital Shepp-Logan phantom experiment

In practical applications, the projections data inevitably contains noise. In order to verify the feasibility of our reconstruction algorithm, in the simulation experiment, The Gaussian noise with zero average value and 0.4% of the maximum value of the projections data as the standard deviation is added to the projections data. The proposed biregular term optimization algorithm (Proposed algorithm) are compared with the iterative algorithm (SART), the iterative algorithm based on SVD (SART+SVD), and the iterative algorithm based on based on L0 norm regularization (SART+L0) respectively. Shepp-Logan phantom is used as simulate studies, the image size is 256×256 pixels, the number of detectors is 512, the distance between source and rotation center (SOD) is 250mm, and the distance from the source to the detector (SDD) is 500mm. The pixel size of the object reconstructed is 0.3×0.3mm², and the scanning angular range is [0°, 90°] with 90 projections.

**Figure 1.** Shepp-Logan phantom

The reconstruction parameters for four algorithms are shown in Table1, and the reconstruction results are shown in Figure2. Figure3 is the local enlargement of Figure2.

**Table 1. Reconstruction parameters of Shepp-Logan phantom for four algorithms**

| Parameter         | $\lambda$ | $\lambda_1$ | $\lambda_2$ | $\lambda'$ | $\beta$ | $\nu$ |
|-------------------|-----------|--------------|--------------|------------|---------|-------|
| SART              | 0.8       |              |              |            |         |       |
| SART+SVD          | 0.8       | 0.8          |              |            |         | 0.055 |
| SART+L0           | 0.8       | 0.8          | 0.00085      | 5          |         |       |
| Proposed algorithm| 0.8       | 0.8          | 0.3          | 0.00085    | 5       | 0.00046 |

**Figure 2.** Images of the Shepp-Logan phantom reconstructed by four algorithms, and the display window is [0 1].
Furthermore, the best images reconstructed by four methods are compared by PSNR and RMSE, respectively. In the 285th iteration of SART, the image signal-to-noise ratio is the highest and the root mean square error is the smallest, while the SART+SVD, SART+L0 and the algorithm are in the 1000th iteration with the highest signal-to-noise ratio and the smallest root mean square error. So the SART algorithm selects the 285th image, and the SART+SVD, SART+L0 and the proposed algorithm selects the image of the 1000th iteration. The comparison results are shown in Table 2.

| Quantitative assessment | SART     | SART+SVD | SART+L0   | Proposed algorithm |
|-------------------------|----------|----------|-----------|--------------------|
| PSNR                    | 67.1695  | 69.2644  | 93.8407   | 94.7639            |
| RMSE                    | 0.1117   | 0.0878   | 0.0052    | 0.0047             |

It can be concluded from Figure 2 and Figure 3 that the SART+SVD has sharper contours, fewer artifacts and smoother details than SART when the number of iterations is the same; SART+L0 has sharper details, fewer artifacts, better details than SART, SART+SVD; the proposed algorithm has more complete details and fewer artifacts than SART, SART+SVD and SART+L0.

Similarly, from Table 2, the PSNR and RMSE of the proposed algorithm are better than those of SART, SART+SVD and SART+L0.

3.2 Physical phantom experiment

Further, two sets of actual projections data are selected to analyse the reconstruction quality of each algorithm. In the first set of experiments, the projections data of the cheese mode of the sculptured CT are selected [15]. The scanning method is equal distance fan beam, the scanning angular range is [90°, 210°] with 120 projections, SOD is 404.3mm, SDD is 547.8mm, the pixel size is 0.05×0.05mm², and the number of detectors is 2240.

![Cheese CT image reconstructed by FBP from complete projections](image)

Figure 4. Cheese CT image reconstructed by FBP from complete projections

The reconstruction parameters of each algorithm are shown in Table 3. The reconstruction results are shown in Figure 5. Figure 6 is the local enlargement of Figure 5.
Table 3. Reconstruction parameters of cheese model CT image for four algorithms

| Parameter       | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda^*$ | $\beta$ | $\mu$   |
|-----------------|------------|------------|------------|----------|--------|------|
| SART            | 0.8        |            |            |          |        |      |
| SART+SVD        | 0.8        | 0.8        |            |          |        | 0.0020|
| SART+L0         | 0.8        | 0.98       |            | 0.0000125| 8.4    |      |
| Proposed algorithm | 0.8     | 0.98       | 0.4        | 0.0000125| 8.4    | 0.0015|

(a) SART       (b) SART+SVD (c) SART+L0 (d) Proposed algorithm

Figure 5. Cheese model CT images reconstructed by four algorithms, and the display window is [0 1].

(a) SART       (b) SART+SVD (c) SART+L0 (d) Proposed algorithm

Figure 6. The local enlargement of Figure 5, and the display window is [0 0.046].

Since the lack of ideal images, PSNR and RMSE cannot be used for evaluation. Therefore, the reconstructed image and the local enlargement image are used for qualitative evaluation. It can be seen from Figure 5 and Figure 6 that SART+SVD has sharper edges, fewer artifacts and smoother details than the SART; SART+L0 has sharper edges, fewer artifacts and better details than SART, SART+SVD and SART+L0; the edge contours of proposed algorithm are more complete and fewer artifacts than SART, SART+SVD and SART+L0.

In the second set of experiment, the projections data of walnut are selected[16], scanning method is equal distance fan beam, scanning angular range is [120°, 280°], with 160 projections, SOD is 110.0mm, SDD is 300mm, the pixel size is 0.05×0.05mm² and the number of detectors is 2296.
Figure 7. Walnut CT Image reconstructed by FBP from complete projections

The reconstruction parameters of each algorithm are shown in Table 4, the reconstruction results are shown in Figure 8. Figure 9 is the local enlargement of Figure 8.

Table 4. Reconstruction parameters of walnut CT image for different algorithms

| Parameter       | \( \lambda \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda' \) | \( \beta \) | \( \nu \) |
|-----------------|---------------|-----------------|-----------------|---------------|-----------|---------|
| SART            | 0.8           |                 |                 |               |           |         |
| SART+SVD        | 0.8           | 0.8             |                 |               |           | 0.015   |
| SART+L0         | 0.8           | 0.8             | 0.00002         | 3             |           |         |
| Proposed algorithm | 0.8           | 0.8             | 0.35            | 0.00002       | 3         | 0.0165  |

Figure 8. Walnut CT images reconstructed by different algorithms, and the display window is [0 1].

From Figure 8 and Figure 9, it is also concluded that SART+SVD has smoother details and fewer artifacts than SART; SART+L0 has sharper edges, fewer artifacts and better details than SART, SART+SVD and SART+L0; the proposed algorithm has better edge-preserving and fewer artifacts than SART, SART+SVD and SART+L0.
4. Conclusion
Aimed to overcome the problem of image reconstruction with limited-angle projections, we analysed the disadvantage of the two optimization methods (L0 norm and SVD), and proposed biregular term optimization algorithm combining L0 norm and SVD. Reconstruction results from simulation data and actual scanning projections data demonstrate that the proposed algorithm can effectively eliminate artifacts of limited-angle projections reconstructed image and well preserves image edges and image structural details comparing to the SART method, the SART with L0 regularization, SART with SVD regularization. In order to extensively apply CT detection equipment and effectively solve practical application problems, subsequent research work will further explore how to simplify reconstruction parameters of the optimization.

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