On projective varieties with strictly nef tangent bundles

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Abstract. In this paper, we study smooth complex projective varieties $X$ such that some exterior power $\bigwedge^r T_X$ of the tangent bundle is strictly nef. We prove that such varieties are rationally connected. We also classify the following two cases. If $T_X$ is strictly nef, then $X$ isomorphic to the projective space $\mathbb{P}^n$. If $\bigwedge^2 T_X$ is strictly nef and if $X$ has dimension at least 3, then $X$ is either isomorphic to $\mathbb{P}^n$ or a quadric $Q^n$.

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1. Introduction

Throughout this paper, we will study projective varieties defined over $\mathbb{C}$, the field of complex numbers. We recall that a line bundle $L$ over a smooth projective variety $X$ is said to be strictly nef if there is $L \cdot C > 0$ for any curve $C \subset X$, while the Nakai-Moishezon-Kleiman criterion asserts that $L$ is ample if and only if there is $L^{\dim Y} \cdot Y > 0$ for every positive-dimensional subvariety $Y \subset X$. In particular, ampleness implies strict nefness. However, the converse is not true in general, as shown in an example of Mumford (see [Har70, Section 10, Chapter I]). Nevertheless, one might expect more for the canonical bundle $\omega_X$ of $X$. Indeed, on the one hand, since a strictly nef semi-ample line bundle must be ample, the abundance conjecture suggests that if $\omega_X$ is strictly nef, then it should be ample. On the other hand, Campana and Peternell proposed in [CP91, Problem 11.4] the following conjecture.
Conjecture 1.1. Let $X$ be a smooth projective variety. If $\omega_X^{-1}$ is strictly nef, then $\omega_X^{-1}$ is ample, that is, $X$ is a Fano variety.

This conjecture is only verified by Maeda for surfaces (see [Mae93]) and by Serrano for threefolds (see [Ser95], also [Ueh00] and [CCP08]). In this paper, we prove the following theorem, which provides some evidence for this conjecture in all dimensions.

**Theorem 1.2.** Let $X$ be a smooth projective variety of dimension $n$, and let $T_X$ be its tangent bundle. If $\bigwedge^r T_X$ is strictly nef for some $1 \leq r \leq n$, then $X$ is rationally connected. In particular, if $\omega_X^{-1}$ is strictly nef, then $X$ is rationally connected.

We recall that a vector bundle $E$ is said to be strictly nef if the tautological line bundle $O_E(1)$ on the projective bundle $\text{Proj} E$ of hyperplanes is strictly nef. One of the key ingredients for the proof of Theorem 1.2 relies on the recent breakthrough of Cao and Höring on the structure theorems for projective varieties with nef anticanonical bundle (see [Cao16] and [CH17]). Actually, based on their work, we prove the following result, which is essential for Theorem 1.2.

**Theorem 1.3.** Let $X$ be a smooth projective variety with nef anticanonical bundle $\omega_X^{-1}$. Then, up to replacing $X$ with some finite étale cover if necessary, the Albanese morphism $f : X \to A$ has a section $\sigma : A \to X$ such that $\sigma^* \omega_X$ is numerically trivial.

As an application of Theorem 1.2, we prove the following analogue of Mori's characterization of projective spaces (see [Mor79]).

**Theorem 1.4.** Let $X$ be a smooth projective variety of dimension $n$. If $T_X$ is strictly nef, then $X$ is $\mathbb{P}^n$.

In another word, this theorem states that the tangent bundle $T_X$ is strictly nef if and only if it is ample. Along the same lines as Theorem 1.4, we obtain the following characterization as well.

**Theorem 1.5.** Let $X$ be a smooth projective variety of dimension $n \geq 3$. Suppose that $\bigwedge^2 T_X$ is strictly nef, then $X$ is isomorphic to $\mathbb{P}^n$ or a quadric $\mathbb{Q}^n$.

There are also other characterizations of projective spaces or quadrics, see, for example, [KO73], [SY80], [Siu80], [Mok88], [YZ90], [Pet90, Pet91], [CS95], [AW01], [CMSB02], [Miy04], [ADK08], [Hwa13], [DH17], [Yan17], etc.

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2. Basic properties of strictly nef vector bundles

In this section, we collect some basic results on strictly nef vector bundles. The following proposition is analogous to the Barton-Kleiman criterion for nef vector bundles (see for example [Laz04, Proposition 6.1.18]).

**Proposition 2.1.** Let $E$ be a vector bundle on a smooth projective variety $X$. Then the following conditions are equivalent.

1. $E$ is strictly nef.
2. For any smooth projective curve $C$ with a finite morphism $\nu : C \to X$, and for any line bundle quotient $\nu^*(E) \to L$, one has $\deg L > 0$.

**Proof.** For a fixed smooth projective curve $C$, we know that any non-constant morphism $\mu : C \to \text{Proj } E$ whose image is horizontal over $X$ corresponds one-to-one to a finite morphism $\nu : C \to X$ with a line bundle quotient $\nu^*E \to L$. Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\mu} & \text{Proj } E \\
\downarrow{\nu} & & \downarrow{f} \\
X & & 
\end{array}
\]

such that $L = \mu^*O_E(1)$, where $O_E(1)$ is the tautological line bundle of $\text{Proj } E$ (see for example [Har77, Proposition II.7.12]).

1. $\Rightarrow$ (2). Let $\nu : C \to X$ be a finite morphism, where $C$ is a smooth projective curve. Assume that there is a line bundle quotient $\nu^*E \to L$. Let $\mu : C \to \text{Proj } E$ be the induced morphism as in diagram (2.1). Then we have

$$
\deg L = \deg \mu^*O_E(1) = O_E(1) \cdot \mu(C).
$$

Since $O_E(1)$ is strictly nef, we obtain that $\deg L > 0$.

2. $\Rightarrow$ (1). Let $B$ be a curve in $\text{Proj } E$. Let $f : \tilde{B} \to \text{Proj } E$ be its normalization. If $B$ is vertical over $X$, then we have $O_E(1) \cdot B > 0$. If $B$ is horizontal over $X$, then the natural morphism $g : \tilde{B} \to X$ is finite. In this case, there is an induced line bundle quotient $g^*E \to Q$ such that $Q \cong f^*O_E(1)$. We note that $O_E(1) \cdot B = \deg Q$, which is positive by hypothesis. Thus $E$ is strictly nef.

We also have the following list of properties of strictly nef vector bundles.

**Proposition 2.2.** Let $E$ and $F$ be two vector bundles on a smooth projective variety $X$. Then we have the following assertions.

1. $E$ is a strictly nef if and only if for every smooth projective curve $C$ and for any non-constant morphism $f : C \to X$, $f^*E$ is strictly nef.
(2) If $E$ is strictly nef, then any non-zero quotient bundle $Q$ of $E$ is strictly nef.
(3) If $E \oplus F$ is strictly nef, then both $E$ and $F$ are strictly nef.
(4) If the symmetric power $\text{Sym}^k E$ is strictly nef for some $k \geq 1$, then $E$ is strictly nef.
(5) Let $f : Y \to X$ be a finite morphism such that $Y$ is a smooth projective variety. If $E$ is strictly nef, then so is $f^*E$.
(6) Let $f : Y \to X$ be a surjective morphism such that $Y$ is a smooth projective variety. If $f^*E$ is strictly nef, then $E$ is strictly nef.
(7) If $E$ is strictly nef, then $h^0(X, E^* \otimes L) = 0$ for any numerically trivial line bundle $L$.

Proof. (1) follows directly from Proposition 2.1. For (2), we note that there is a natural embedding $\iota : \text{Proj} Q \to \text{Proj} E$ such that $\iota^*O_E(1) \cong O_Q(1)$. Hence if $E$ is strictly nef, then so is $Q$. (3) follows from (2). For (4), we note that there is a Veronese embedding $v : \text{Proj} E \to \text{Proj} (\text{Sym}^k E)$ such that $v^*O_{\text{Sym}^k E}(1) \cong O_E(k)$. This implies (4).

For (5), we notice that for any smooth projective curve $C$ with a finite morphism $\nu : C \to X$, the composition $f \circ \nu : C \to X$ is also finite. The assertion then follows from Proposition 2.1.

Now we consider (6). We note that for every curve in $X$, there is a curve in $Y$ which maps onto it. Hence by (1), we only need to prove the case when $X$ and $Y$ are smooth curves. In this case, there is a natural finite surjective morphism $g : \text{Proj} (f^*E) \to \text{Proj} E$ induced by $f$. Moreover, we have $O_{f^*E}(1) \cong g^*O_E(1)$. By assumption, $O_{f^*E}(1)$ is a strictly nef line bundle. Since $g$ is finite surjective, this implies that $O_E(1)$ is also strictly nef. Therefore, $E$ is strictly nef.

It remains to prove (7). We remark that $E \otimes L^{-1}$ is still strictly nef for $L^{-1}$ is numerically trivial. Thus, by replacing $E$ with $E \otimes L^{-1}$, we may assume that $L$ is trivial. Assume by contradiction that $h^0(X, E^*) > 0$. By [DPS94, Proposition 1.16], there exists a nowhere vanishing section $\sigma \in H^0(X, E^*)$. Then $\sigma$ induces a subbundle $O_X \to E^*$ as well as a quotient bundle $E \to O_X$. This contradicts Proposition 2.1. \qed

3. Strictly nef bundles on curves

In this section, we will look at strictly nef vector bundle $E$ on a smooth projective curve $C$. If $C$ is rational, then $E$ is a direct sum of line bundles. Hence $E$ is strictly nef if and only if $E$ is ample in this case. However, on a smooth curve $C$ of genus at least 2, there exists a strictly nef vector bundle $E$ which is also a Hermitian flat stable vector bundle (see [Har70, Section 10, Chapter I]). In particular, this bundle $E$ is not ample. Now it remains to look at the case when $C$ is elliptic. We observe the following fact.

Theorem 3.1. Let $E$ be a vector bundle on an elliptic curve $C$. If $E$ is strictly nef, then $E$ is ample.
For the proof of this theorem, we will first prove the following lemma. Recall that a vector bundle \( E \) is called numerically flat if both \( E \) and \( E^* \) are nef, or equivalently, if both \( E \) and \( \det(E^*) \) are nef (see [DPS94, Definition 1.17]).

**Lemma 3.2.** Let \( E \) be a strictly nef vector bundle on smooth projective variety \( X \) such that \( \omega_X \) is numerically trivial. Then \( \det E \) is not numerically trivial.

**Proof.** By Beauville’s decomposition theorem (see [Bea83, Théorème 1]), there is a finite étale cover \( X' \to X \) such that \( X' \) is the product of an abelian variety and a simply connected variety. We note that the pullback of \( E \) on \( X' \) is still strictly nef by Proposition 2.2. Hence by replacing \( X \) with \( X' \) if necessary, we may assume that \( X \) is the product of an abelian variety and a simply connected variety. In particular, the fundamental group \( \pi_1(X) \) is commutative.

Assume by contradiction that \( \det E \) is numerically trivial. Then \( E \) is numerically flat, and so is \( E^* \). By [DPS94, Theorem 1.18], \( E^* \) admits a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_p = E^*
\]

of subbundles such that the quotients \( E_k/E_{k-1} \) are Hermitian flat. In particular, \( E_1 \) is Hermitian flat, and is defined by a unitary representation of the fundamental group \( \pi_1(X) \). Since \( \pi_1(X) \) is an abelian group, such a representation is the direct sum of one-dimensional representations. Hence \( E_1 \) is a direct sum of flat line bundles. Let \( Q \) be one of them. Then there is a line bundle quotient \( E \to L \) with \( L = Q^{-1} \). Moreover, since \( L \) is also flat, it is numerically trivial. This contradicts Proposition 2.1.

Now we can conclude Theorem 3.1.

**Proof of Theorem 3.1.** The vector bundle \( E \) can be decomposed as \( E = \bigoplus E_i \) so that each \( E_i \) is an indecomposable bundle. By Proposition 2.2, each \( E_i \) is strictly nef. Hence we have \( \deg(E_i) > 0 \) by Lemma 3.2. This implies that \( E \) is ample (see [Har71, Theorem 1.3] or [Gie71, Theorem 2.3]).

**Remark 3.3.** From Proposition 2.2 and Theorem 3.1, we can obtain that if a projective variety \( X \) contains the image of an elliptic curve which is not a point, then the determinant of every strictly nef vector bundle \( E \) on \( X \) is not numerically trivial. In particular, if \( E \) is strictly nef over a projective variety \( X \) with pseudo-effective \( \omega_X^{-1} \), then by [BDPP13, Corollary 0.3] and Lemma 3.2, \( \det E \) is not numerically trivial.

4. Sections of Albanese morphisms

In this section, we shall prove Theorem 1.3. We will divide this section into two parts. In the first one, we will prove a theorem on periodic points for group actions on projective schemes. By using this theorem, we will conclude Theorem 1.3.
4.1. Periodic points for linear actions of abelian groups. The goal of this subsection is to prove the following theorem.

**Theorem 4.1.** Let $G$ be a finitely generated abelian group. Assume that $G$ acts on a projective scheme $Z$ such that there is a $G$-equivariant ample line bundle $L$ on $Z$. Then the action of $G$ on $Z$ has a periodic point.

We recall that if $G$ acts on a projective scheme $Z$, then a line bundle $L$ on $Z$ is said to be $G$-equivariant if there is an isomorphism $g^*L \cong L$ for any element $g \in G$ which is compatible with the group structure of $G$. A (closed) point $z \in Z$ is said to be a periodic point for an element $g \in G$ if $g^k.z = z$ for some positive integer $k$. A point $z$ is called a periodic point for the action of $G$ if there is a positive integer $k$ such that $g^k.z = z$ for all elements $g \in G$.

**Proof of Theorem 4.1.** By replacing $L$ with some positive power of it if necessary, we can assume that $L$ is very ample. Then there is a nature linear action of $G$ on $W = H^0(Z, L)$, which induces a natural action of $G$ on Proj$W$. Moreover, there is a $G$-equivariant embedding $Z \to$ Proj$W$. The theorem is then equivalent to the following proposition.

**Proposition 4.2.** Let $G$ be a finitely generated abelian group. Let $V$ be a linear representation of $G$. Assume that $Z \subseteq \mathbb{P}(V)$ is a closed subscheme which is stable under the induced action of $G$ on $\mathbb{P}(V)$, where $\mathbb{P}(V)$ is the projective space of lines in $V$. Then the action of $G$ on $Z$ has a periodic point.

We will first prove Proposition 4.2 in the case when $G$ is generated by one element.

**Lemma 4.3.** With the notations in Proposition 4.2, if we assume that $G$ is generated by one element, then the action of $G$ on $Z$ has a periodic point.

**Proof.** Assume that $G$ is generated by an element $g$. Then it is enough to prove that some positive power of $g$ has a periodic point in $Z$. Hence during the proof, we will replace $g$ with some positive power of it if necessary. Let $(x_0, \ldots, x_m)$ be a coordinates system of $V$ such that the vector with coordinates $(1, 0, \ldots, 0)$ in $V$ is an eigenvector for $g$. Let $[x_0 : \cdots : x_m]$ be the induced homogeneous coordinates system of $\mathbb{P}(V)$.

We will prove the lemma by induction on the dimension $m$ of $\mathbb{P}(V)$. If $m = 1$, then $Z$ is either a finite set or the whole $\mathbb{P}(V)$. If $Z$ is a finite set, then all of its points are periodic. If $Z = \mathbb{P}(V)$, then the point $[1 : 0]$ belongs to $Z$ and is fixed by $g$. Hence the lemma is true in this case.

Assume that the lemma is true in dimensions smaller than $m \geq 2$. If the point $y$ with coordinates $[1 : 0 : \cdots : 0]$ is in $Z$, then it is a fixed-point for $g$ and we are done. Assume that $y$ does not belong to $Z$. Then the rational projection $[x_0 : \cdots : x_m] \mapsto [x_1 : \cdots : x_m]$ is a well-defined morphism on $Z$. We denote this projection by $\varphi : Z \to \mathbb{P}^{m-1}$. We note that $\varphi$ is proper, and hence the image $Z' = \varphi(Z)$ is a closed subscheme of
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$\mathbb{P}^{m-1}$. Since $(1,0,...,0) \in V$ is an eigenvector for $g$, the action of $G$ on $\mathbb{P}(V)$ descends naturally to an action of $G$ on $\mathbb{P}^{m-1}$ and the projection $\varphi$ is $G$-equivariant. Thus $Z'$ is stable under the action of $G$ on $\mathbb{P}^{m-1}$. By induction hypothesis, there is a point $z' \in Z'$ which is a periodic point for the action of $G$. By replacing $g$ with some positive power if necessary, we may assume that $z'$ is a fixed-point. Let $L$ be the line joining $z'$ and $y$ in $\mathbb{P}(V)$. Then $L \cap Z$ is non-empty and stable under the action of $G$. Since we have assumed that $y \notin Z$, the intersection $L \cap Z$ is a proper subscheme of $L$. Thus it is a finite set. In particular, every point in $Z \cap L$ is a periodic point for $g$. This completes the proof.

□

Now we can conclude Proposition 4.2.

Proof of Proposition 4.2. We first note that it is enough to prove that some subgroup of $G$ of finite index has a periodic point. Hence during the proof we may replace $G$ by some subgroup of finite index of it if necessary. In particular, we may assume that $G$ is torsion-free. Moreover, without loss of generality, we may assume that the representation $V$ of $G$ is faithful. We will prove by induction on the rank of $G$. If the rank is one, then the theorem follows from Lemma 4.3.

Assume that the theorem holds for ranks smaller than $k \geq 2$. Assume that $G$ has rank $k$. Let $\{g_1,...,g_k\}$ be a set of generators of $G$. Let $F$ be the subgroup generated by $g_1$ and let $H$ be the subgroup generated by $\{g_2,...,g_k\}$. Then by Lemma 4.3, the action of $F$ on $Z$ has a periodic point. By replacing $g_1$ with some positive power of it and $G$ by some subgroup of finite index if necessary, we may assume that the action of $F$ on $Z$ has a fixed-point. In particular, the set $Z^F$ is not empty. We note that $Z^F$ is also a closed subscheme of $\mathbb{P}^m$. Moreover, it is stable under the actions of $H$ since $G$ is an abelian group. By induction hypothesis, the action of $H$ on $Z^F$ has a periodic point $z$. Then $z$ is a periodic point for the action of $G$ on $Z$. This completes the proof.

□

4.2. Proof of Theorem 1.3. We will finish the proof of Theorem 1.3 in this subsection. We will need the following two lemmas.

Lemma 4.4. Let $B$ be a smooth projective variety and let $\pi : \tilde{B} \to B$ be the universal cover with Galois group $G = \pi_1(B)$. Let $V$ be a linear representation of $G$ and let $E$ be the corresponding flat vector bundle over $B$. Then there is a one-to-one correspondence between the set of $G$-fixed-points $y \in \text{Proj} V$ and the set of codimension one flat subbundles $F \to E$.

Proof. There is a one-to-one correspondence between the set of $G$-fixed-points $y$ and the set of codimension one subrepresentations $W \subseteq V$. Moreover, the set of codimension one subrepresentations $W \subseteq V$ is one-to-one correspondent to the set of codimension one flat subbundles $F \to E$. □

In the next lemma, we consider the following situation. Let $Y$ be a projective variety and let $H$ be a very ample line bundle on $Y$. Let $B$ be a smooth projective variety
with fundamental group $G = \pi_1(B)$. Let $\hat{B} \to B$ be the universal cover. Assume that there is an action of $G$ on $Y$ such that $H$ is $G$-equivariant. Let $G$ act on $Y \times \hat{B}$ diagonally and let $X$ be the quotient $(Y \times \hat{B})/G$. Then there is a natural fibration $f : X \to B$, and $H$ descends to a line bundle $L$ on $X$. Moreover, the natural linear action of $G$ on $V = H^0(Y, H)$ induces a flat vector bundle structure on $E = f_*L$.

**Lemma 4.5.** With the notation in the paragraph above, we assume that there is a $G$-fixed-point $y \in Y$. Then on the one hand, $y$ induces a section $\sigma : B \to X$ of $f$. On the other hand, $y$ also induces a short exact sequence of flat vector bundles

$$0 \to F \to E \to Q \to 0$$

such that $Q \cong \sigma^*L$.

**Proof.** Let $P = \text{Proj} E$ and let $p : P \to B$ be the natural projection. We note that there is a $G$-equivariant embedding $Y \to \text{Proj} V$, which induces a closed embedding $X \to P$. Moreover, we have $\mathcal{O}_E(1)|_X \cong L$.

Since $y \in Y \subseteq \text{Proj} V$ is a $G$-fixed-point, it corresponds to a codimension one flat subbundle $F \to E$ by Lemma 4.4. Let $Q$ be the quotient $E/F$. Then the quotient map $E \to Q$ induces a section $\mu : B \to P$ of $p$ such that $\mu^*\mathcal{O}_E(1) \cong Q$. Moreover, for every $b \in B$, $\mu(b) \in P_b$ corresponds exactly to $y \in \text{Proj} V$. Since $y \in Y$, the section $\mu$ factors through a section $\sigma : B \to X$ of $f$. Since $\mathcal{O}_E(1)|_X = L$, we have $Q \cong \sigma^*L$. This completes the proof of the lemma.

**Remark 4.6.** We note that the section $\sigma : B \to X$ in the lemma above depends only on the fixed-point $y$, and is independent of the choice of the line bundle $H$. Indeed, $\sigma(B)$ is the quotient $\langle \{y\} \times \hat{B} \rangle/G$.

Now we can prove Theorem 1.3.

**Proof of Theorem 1.3.** As in [Cao16, Corollary 4.16], by replacing $X$ with some finite étale cover, we may assume that the fibers of $f$ are simply connected. If $A$ is a point or if $f$ is an isomorphism, then there is nothing to prove. Assume that $A$ has positive dimension and that $f$ is not an isomorphism. Let $L$ be an $f$-relatively very ample divisor and let $E = f_*L$. Assume that rank $E = m+1$ with $m \geq 0$. There is an isogeny $p : A' \to A$ such that $p^*(\text{det} E)$ is divisible by $m+1$. Since we have assumed that the fibers of $f$ are simply connected, the natural morphism $X \times_A A' \to A'$ is still the Albanese morphism. Hence, by replacing $X$ with $X \times_A A'$, we may assume that $\text{det} E = N^{m+1}$ for some line bundle $N$. By replacing $L$ with $L - f^*N$, we may then assume that $\text{det} E$ is trivial.

Let $\pi : \tilde{A} \to A$ be the universal cover with Galois group $G = \pi_1(A)$. Let $\tilde{X}$ be the fiber product $X \times_A \tilde{A}$ and let $p : \tilde{X} \to X$ be the natural morphism. By [Cao16, Lemma 4.15], $f_* (kL)$ is a numerically flat vector bundle on $A$ for any positive integer $k$. In particular, $E$ is numerically flat and hence is a flat vector bundle (see [Ver04, Lemma 6.5 and Corollary 6.6]). Let $(V, \rho)$ be the corresponding representation of $G$. 

Then there is a $G$-equivariant isomorphism $(\text{Proj} E) \times_A \tilde{A} \cong (\text{Proj} V) \times \tilde{A}$, where the action of $G$ on $\text{Proj} V$ is the one induced by $\rho$. By [Cao16, Proposition 2.8], there is a $G$-stable subvariety $Y \subseteq \text{Proj} V$ such that there is a $G$-equivariant isomorphism $	ilde{X} \cong Y \times \tilde{A}$ which makes the following diagram commute

$$
\begin{array}{ccc}
Y \times \tilde{A} & \xrightarrow{pr_1} & (\text{Proj} V) \times \tilde{A} \\
\downarrow & & \downarrow \\
X & \rightarrow & \text{Proj} E
\end{array}
$$

Let $H = \mathcal{O}_{\text{Proj} V}(1)|_Y$. Then, on $	ilde{X}$, we have $p^*L \cong pr_1^*H$, where $pr_1 : \tilde{X} \rightarrow Y$ is the natural projection induced by the isomorphism $\tilde{X} \cong Y \times \tilde{A}$. After all, we have $X \cong (Y \times \tilde{A})/G$, and we are in the same situation as in Lemma 4.5.

We note that $G$ is an abelian group. By Theorem 4.1, the action of $G$ on $Y$ has a periodic point for $H$ is a $G$-equivariant ample line bundle. Hence there is a subgroup $G'$ of $G$ of finite index such that the action of $G'$ on $Y$ has a fixed-point $y \in Y$. The quotient $\tilde{X}/G'$ is a finite étale cover of $X$, and the natural morphism $\tilde{X}/G' \rightarrow \tilde{A}/G'$ is the Albanese morphism for we have assumed that the fibers of $f$ are simply connected. Hence, by replacing $X$ with $\tilde{X}/G'$ if necessary, we may assume that the action of $G$ on $Y$ has a fixed-point $y \in Y$. This fixed-point induces a section $\sigma : \tilde{A} \rightarrow X$ of the Albanese morphism $f$ by Lemma 4.5.

We note that the anticanonical bundle $\omega_Y^{-1}$ of $Y$ is nef for $\omega_X^{-1}$ is nef. Moreover, $\omega_Y^{-1}$ is canonically $G$-equivariant. Hence $\omega_X^{-a} \otimes H$ is a $G$-equivariant ample line bundle for any positive integer $a$. On $\tilde{X}$, we have

$$
p^*(\omega_X^{-a} \otimes L) \cong \omega_X^{-a} \otimes p^*L \cong pr_1^*(\omega_Y^{-a} \otimes H).
$$

Let $r_a$ be some large enough positive integer such that $(\omega_Y^{-a} \otimes H)^{r_a}$ is very ample. Then there is a natural linear action of $G$ on $H^0(Y, (\omega_Y^{-a} \otimes H)^{r_a})$, which induces a flat vector bundle structure on $E_a = f_*((\omega_X^{-a} \otimes L)^{r_a})$. By Lemma 4.5, the $G$-fixed-point $y$ induces a short exact sequence of flat vector bundles

$$
0 \rightarrow F_a \rightarrow E_a \rightarrow Q_a \rightarrow 0,
$$

such that $Q_a \cong \sigma^*(-\omega_X^{-a} \otimes L)^{r_a}$. Since $Q_a$ is flat, it is numerically trivial. Thus $\sigma^*(-\omega_X^{-a} \otimes L)$ is numerically trivial. Since this is true for all positive integer $a$, we obtain that $\sigma^*\omega_X$ is numerically trivial. This completes the proof of the theorem for $\omega_A$ is trivial.

\[\square\]

5. Projective manifolds with strictly nef tangent bundles

In this section, we will prove Theorem 1.2, Theorem 1.4 and Theorem 1.5. At first, we observe the following fact.
Lemma 5.1. Let $X = Y \times Z$ be a variety of dimension $n$ which is a product of two smooth projective varieties of positive dimensions. If $\wedge^r T_X$ is strictly nef for some $1 \leq r \leq n$, then both $Y$ and $Z$ are uniruled.

Proof. We have $T_X \cong E \oplus F$, where $E$ and $F$ are the pullbacks of the tangent bundles of $Y$ and $Z$ respectively. Then we have

$$\wedge^s T_X \cong \bigoplus_{a+b=r} \wedge^a E \otimes \wedge^b F.$$

Let $s$ be the dimension of $Y$. We will first show that $r > s$. Suppose by contradiction that $r \leq s$. On the one hand, $\wedge^r E$ is a direct summand of $\wedge^r T_X$. On the other hand, for any $y \in Y$, the restriction of $\wedge^r E$ on the fiber $X_y$ is trivial. Hence, $(\wedge^r T_X)|_{X_y}$ cannot be strictly nef by Proposition 2.2. This is a contradiction.

As a consequence, we obtain that $N = \wedge^s E \otimes \wedge^{r-s} F$ is a direct summand of $\wedge^r T_X$. In particular, $N$ is strictly nef by Proposition 2.2. Let $z$ be a point in $Z$. Since $(\wedge^{r-s} F)|_{X_z}$ is trivial and since $N|_{X_z}$ is still strictly nef, we conclude that $(\wedge^s E)|_{X_z}$ is strictly nef. This implies that $\omega_Y^{-1}$ is strictly nef for $X_z \cong Y$. Hence $Y$ is uniruled by [MM86, Corollary 2]. By symmetry, we can also show that $Z$ is uniruled.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We note that $\omega_X^{-1}$ is nef. We will first show that the augmented irregularity $\tilde{q}(X)$ is zero. Assume the opposite. By replacing $X$ with some finite étale cover if necessary, we may assume that the irregularity $q(X)$ is equal to $\tilde{q}(X) > 0$. Let $f : X \to A$ be the Albanese morphism. Then $\dim A = q(X) > 0$. Thanks to Theorem 1.3, by replacing $X$ with some finite étale cover if necessary, we may assume that there is a section $\sigma : A \to X$ such that $\sigma^* \omega_X$ is numerically trivial. On the one hand, we remark that $\det (\sigma^* \wedge^r T_X) \cong \sigma^* \omega_X^{-t}$ is numerically trivial, where $t$ is the binomial number $\binom{n}{r-1}$. One the other hand, since $\sigma^* \wedge^r T_X$ is strictly nef, $\det (\sigma^* \wedge^r T_X)$ cannot be numerically trivial by Lemma 3.2. We obtain a contradiction.

Therefore, we have $\tilde{q}(X) = 0$. In particular, $X$ has finite fundamental group. Let $\tilde{X} \to X$ be the universal cover. Then by [CH17, Theorem 1.2], we have $\tilde{X} \cong Y \times F$ such that $\omega_{\tilde{X}}$ is trivial and that $F$ is rationally connected. By Lemma 5.1, $Y$ must be a point for $\wedge^r T_{\tilde{X}}$ is strictly nef. Hence $\tilde{X}$ is rationally connected and so is $X$. 

The following corollary is a direct consequence of Theorem 1.2.

Corollary 5.2. Let $f : Y \to X$ be a smooth surjective morphism between projective manifolds. If $\omega_Y^{-1}$ is strictly nef, then $X$ is rationally connected.

In the following, we will present two applications of Theorem 1.2 on characterizations of projective spaces and quadrics. We will prove Theorem 1.4 and Theorem 1.5 successively.
Proof of Theorem 1.4. By Theorem 1.2 and the structure theorem for smooth projective varieties with nef tangent bundles (see [DPS94, Main Theorem]), we deduce that $X$ is a Fano variety. For any rational curve $f : \mathbb{P}^1 \to X$, the bundle $f^*T_X$ is ample by Proposition 2.1. Moreover, since there is a non-zero morphism from $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ to $f^*T_X$, we obtain that $\deg f^*\omega_X^{-1} \geq n + 1$. Hence $X \cong \mathbb{P}^n$ by [CMSB02, Corollary 0.3].

Proof of Theorem 1.5. We know that $X$ is rationally connected from Theorem 1.2. For a rational curve $f : \mathbb{P}^1 \to X$, we can write

$$f^*T_X \cong \left( \bigoplus_{a_i > 0} \mathcal{O}_{\mathbb{P}^1}(a_i) \right) \bigoplus \left( \bigoplus_{b_j \leq 0} \mathcal{O}_{\mathbb{P}^1}(b_j) \right).$$

Since $\wedge^2 T_X$ is strictly nef, so is $f^*(\wedge^2 T_X)$ by Proposition 2.1. Hence $\sharp \{b_j\} \leq 1$. If $\sharp \{b_j\}$ is 0, then $f^*T_X$ is ample, and $\deg f^*\omega_X^{-1} \geq n + 1$ by the same argument as in the proof of Theorem 1.4.

If $\sharp \{b_j\}$ is 1, then we can assume that

$$f^*T_X \cong \left( \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i) \right) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-c)$$

with $0 < a_1 \leq a_2 \leq \cdots \leq a_{n-1}$ and $c \geq 0$. Since $f^*(\wedge^2 T_X)$ is strictly nef, we must have $a_1 - c > 0$. Moreover, since there is a natural non-zero morphism from $T_{\mathbb{P}^1}$ to $f^*T_X$, there exists some $i$ such that $a_i \geq 2$. If $c > 0$, then $a_1$ is at least 2 and we have

$$\deg f^*\omega_X^{-1} = (a_1 - c) + a_2 + \cdots + a_{n-1} \geq 1 + 2(n - 2) = 2n - 3 \geq n.$$\n
If $c = 0$, we also have

$$\deg f^*\omega_X^{-1} = a_1 + a_2 + \cdots + a_{n-1} \geq (n - 2) + 2 = n.$$\n
After all, we always have $\deg f^*\omega_X^{-1} \geq n$. By [DH17, Corollary D], $X$ is isomorphic to $\mathbb{P}^n$, or a quadric $Q^n$, or a projective bundle over some smooth curve. It remains to rule out the case of projective bundles. Assume by contradiction that $X \cong \text{Proj} V$, where $V$ is a vector bundle over a smooth projective curve $B$. We note that $B$ is isomorphic to $\mathbb{P}^1$ for $X$ is rationally connected. We may assume that $V = \bigoplus_{i=1}^n \mathcal{O}_B(d_i)$ with $0 = d_1 \leq d_2 \leq \cdots \leq d_n$. If $\pi : X \to B$ is the natural projection, then we have

$$\omega_X^{-1} = (\pi^*\omega_B^{-1}) \otimes \omega_X^{-1}$$

\begin{align*}
&\cong (\pi^*\mathcal{O}_B(2)) \otimes (\mathcal{O}_B(n) \otimes \pi^*(\det V)^{-1}) \\
&= \mathcal{O}_V(n) \otimes \left( \pi^*\mathcal{O}_B\left(2 - \sum_{i=1}^n d_i\right) \right).
\end{align*}
We note that the quotient morphism \( V \to O_B(d_1) \) induces a section \( \sigma : B \to X \) of \( \pi \) such that \( \sigma^*O_V(1) \cong O_B(d_1) = O_B \). Thus we have

\[
\sigma^*\omega_X^{-1} \cong O_B \left( 2 - \sum_{i=1}^n d_i \right).
\]

This shows that \( \deg \sigma^*\omega_X^{-1} \leq 2 < n \), which is a contradiction.  

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