NULL BOUNDARY CONTROLLABILITY OF A ONE-DIMENSIONAL HEAT EQUATION WITH INTERNAL POINT MASSES AND VARIABLE COEFFICIENTS

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Abstract. In this paper, we consider a linear hybrid system which is composed of $N + 1$ non-homogeneous thin rods connected by $N$ interior-point masses with a Dirichlet boundary condition on the left end, and Dirichlet control on the right end. Using a detailed spectral analysis and the moment theory, we prove that this system is null controllable at any positive time $T$. To this end, firstly, we implement the Wronskian technique to obtain the characteristic equation for the eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$ associated with this system. Secondly, we provide that the eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$ interlace those of the $N + 1$ decoupled rods with homogeneous Dirichlet boundary conditions, and satisfy the so-called Weyl’s asymptotic formula. Finally, we establish sharp asymptotic estimates of the eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$. As consequence, on one hand, we prove a uniform lower bound for the spectral gap. On another hand, we derive the equivalence between the $H$-norm of the eigenfunctions and their first derivative at the right end. As an application of our spectral analysis, we also present new controllability result for the Schrödinger equation with an internal point mass and Dirichlet control on the left end.

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1. Introduction and main results

The boundary controllability of the so-called "hybrid systems" has been extensively investigated for several decades. This was pioneered in [30] by the moment method for a hybrid system composed of two vibrating strings connected by a point mass. Since then, the controllability of hybrid models for systems of Rayleigh and Euler-Bernoulli beams with interior point masses was considered in [20, 21, 35, 36], see also [17, 22, 34] and references therein. More recently, a variety of other hybrid models for thin rods, quantum boxes and other elastic systems involving point masses have been studied along similar lines. In particular, see [16, 28] for a heat equation with internal point masses, [5, 12, 29] for a Schrödinger equation with internal point masses, and [1 3, 4, 7, 8, 9, 10, 11, 18, 19] for networks of strings with attached masses.

In this paper, we study the boundary null controllability of the temperature of a linear hybrid system consisting of $N + 1$ non-homogeneous rods connected by $N$ point masses. Assume the $N + 1$ non-homogeneous rods occupy the interval $\Omega = (0, L)$, $L > 0$, of the $x$-axis and they are connected by $N$ masses $M_j > 0$ at the points $\ell_j$, $j = 1, ..., N$, where $0 = \ell_0 < \ell_1 < ... < \ell_N < \ell_{N+1} = L$. We partition the domain $\Omega$ as follows:

$$\Omega := \bigcup_{j=0}^{N} \left\{ \Omega_j \cup \{\ell_{j+1}\} \cup \Omega_{j+1} \right\}, \quad \Omega_j = (\ell_j, \ell_{j+1}), \quad j = 0, ..., N.$$  

By means of the scalar functions

$$u_j(t, x), \quad t > 0, \quad x \in \Omega_j, \quad j = 0, ..., N,$$

$$z_j(t), \quad t > 0, \quad j = 1, ..., N,$$

we describe the temperature of the rods $\Omega_j$, and the temperature of the points masses $\ell_j$, respectively. The linear equation modeling heat flow of such a system is as follows:

$$\begin{cases}
(\rho_j(x)\partial_t u_j - \sigma_j(x)\partial_x^2 u_j + q_j(x)u_j)(t, x) = 0, & t > 0, \quad x \in \Omega_j, \quad j = 0, ..., N, \\
u_{j-1}(t, \ell_j) = z_j(t) = u_j(t, \ell_j), & t > 0, \quad j = 1, ..., N, \\
(\sigma_j(\ell_j)\partial_x u_j - \sigma_{j-1}(\ell_j)\partial_x u_{j-1})(t, \ell_j) = M_j \partial_t z_j(t), & t > 0, \quad j = 1, ..., N, \\
u_0(t, \ell_0) = u_0(t, 0) = 0, & t > 0.
\end{cases}$$  

(1.1)
with the control
\[(1.2) \quad u_N(t, \ell_{N+1}) = u_N(t, L) = h(t), \quad t > 0\]
and the initial conditions at \( t = 0 \)
\[(1.3) \quad \begin{cases} u_j(0, x) = u^0_j, \quad x \in \Omega_j, \quad j = 0, \ldots, N, \\ z_j(0) = z^0_j, \quad j = 1, \ldots, N. \end{cases}\]
In System (1.1), for each \( j = 0, \ldots, N \), the coefficients \( \rho_j(x) \) and \( \sigma_j(x) \), represent respectively the density and thermal conductivity of the rods. The potentials are denoted by the functions \( q_j(x) \), \( j = 0, \ldots, N \). Throughout this paper, we assume that the coefficients
\[(1.4) \quad \rho_j, \sigma_j \in H^2(\Omega_j), \quad q_j \in H^1(\Omega_j), \quad j = 0, \ldots, N,\]
and there exist constants \( \rho, \sigma > 0 \), such that
\[(1.5) \quad \rho_j(x) \geq \rho, \quad \sigma_j(x) \geq \sigma, \quad q_j(x) \geq 0, \quad x \in \Omega_j, \quad j = 0, \ldots, N.\]
To state our main null controllability result for system (1.1)-(1.3), we need some definitions and notations. We denote by \( u := \left( u_j \right)_{j=0}^N \) the functions on \( \Omega \) taking their values in \( C \) and let \( u_j(x) \) be the restriction of \( u \) to \( \Omega_j \), \( j = 0, \ldots, N \). Let us define the following Hilbert space
\[(1.6) \quad \mathcal{H} = \prod_{j=0}^N L^2_{\rho_j}(\Omega_j) \times \mathbb{R}^N,\]
which is endowed with the Hilbert structure
\[(1.7) \quad \langle (\hat{u}, \hat{k})^\top, (\tilde{u}, \tilde{k})^\top \rangle_{\mathcal{H}} := \sum_{j=0}^N \int_{\Omega_j} u_j v_j \rho_j(x) dx + \sum_{j=1}^N M_j h_j k_j,\]
where \( \hat{h} = (h_j)_{j=1}^N \), \( \hat{k} = (k_j)_{j=1}^N \in \mathbb{R}^N \), and \( ^\top \) denotes transposition. Hereafter, we use the notation \( f(x)g(x) := fg(x) \). Our first main result is stated as follows:

**Theorem 1.1.** Assume that the coefficients \( \rho_j(x), \sigma_j(x) \) and \( q_j(x) \) satisfy (1.4) and (1.5).
Let \( T > 0 \), then for any initial data \( U^0 := \left( \left( u^0_j \right)_{j=0}^N, \left( z^0_j \right)_{j=1}^N \right)^\top \in \mathcal{H} \) there exists a control \( h \in H^1(0, T) \), given explicitly by the expression (1.9), such that the solution \( U := \left( \left( u_j \right)_{j=0}^N, \left( z_j \right)_{j=1}^N \right)^\top \) of the control system (1.1)-(1.3) satisfies
\[
\begin{cases} 
  u_j(T, x) = 0, & x \in \Omega_j, \quad j = 0, \ldots, N, \\
  z_j(T) = 0, & j = 1, \ldots, N.
\end{cases}
\]
Our approach is mainly based on a precise analysis of the eigenvalue and eigenfunction asymptotics of the corresponding second order eigenvalue problem, and the general moment theory [38, 39]. Firstly, we implement the Wronskian technique (e.g., [26, Chapter 1] and [40, Chapter 1]), to obtain the characteristic equation for the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) associated with System...
for the eigenvalues (\(\mu_n\)) (see Theorem \(\text{5.3}\)). Secondly, we provide the following interlacing property for the eigenvalues (\(\lambda_n\)) (see Theorem \(\text{3.1}\)):

\[
0 < \lambda_1 \leq \mu_1^{N,D} \quad \text{and} \quad \mu_n^{N,D} \leq \lambda_{n+1} \leq \mu_{n+1}^{N,D}, \quad \forall n \in \mathbb{N}^*,
\]

where \(\{\mu_n^{N,D}\}_1^\infty = \bigcup_{j=0}^N \{\mu_n^{j,D}\}_1^\infty\) are the eigenvalues of the \(N+1\) decoupled rods with homogeneous Dirichlet boundary conditions. Then, we establish the Weyl's type asymptotic formula for the eigenvalues (\(\lambda_n\)) (see (1.1)-(1.2)):

\[
\lim_{n \to \infty} \frac{\lambda_n}{n^2 \pi^2} = \gamma = \left( \sum_{j=0}^N \int_{\ell_j}^{\ell_{j+1}} \sqrt{\frac{\rho_j(x)}{\sigma_j(x)}} \, dx \right)^{-2}.
\]

Finally, using the interlacing property (1.7) and the Weyl's formula (1.8), we obtain sharp asymptotic estimates of the eigenvalues (\(\lambda_n\)) (see Theorem \(\text{3.3}\)). Namely, on one hand, we prove a uniform lower bound for the spectral gap (see Theorem \(\text{3.4}\)), namely,

\[
\lambda_{n+1} - \lambda_n \geq 2\gamma \min_{j=0,\ldots,N-1} \left\{ \frac{\rho_j \sigma_j (\ell_{j+1})^{-\frac{1}{2}}}{M_j \omega^2_j}, \frac{(\rho_N \sigma_N (\ell_N))^{-\frac{1}{2}}}{M_N \omega^2_N} \right\}, \quad \text{as } n \to \infty,
\]

and we derive the equivalence between the \(H\)-norm of the eigenfunctions (\(\Phi_n\)) and their first derivative at the right end \(x = L\) (see Proposition \(\text{3.7}\)), that is,

\[
\frac{||\Phi_n||_H}{||\sigma_N(L)\Phi'_n(L)||} \sim \sqrt{\frac{\omega}{2} \frac{\gamma (\rho_N \sigma_N (\ell_N))^{-\frac{1}{2}}}{n \pi}}, \quad \text{as } n \to \infty.
\]

Using these results, we reduce the control problem (1.1)-(1.2) into an equivalent moment problem which will be solved by the general moment theory developed by Fattorini and Russell \(\text{[38, 39]}\).

As an application of our spectral analysis, we also present new controllability result for the following Schrödinger equation with an internal point mass:

\[
i \partial_t u_j (t,x) - \partial_{xx} u_j (t,x) = 0, \quad t > 0, \quad x \in (\ell_j, \ell_{j+1}), \quad j = 0, 1,
\]

\[
u_0 (t, \ell_1) = z(t) = u_1 (t, \ell_1), \quad t > 0,
\]

\[
(\partial_x u_1 - \partial_x u_0) (t, \ell_1) = i \partial_t z(t), \quad t > 0,
\]

\[
u_0 (t, \ell_0) = u_0 (t, 0) = h(t), \quad u_1 (t, \ell_2) = 0 \quad t > 0,
\]

\[
u_0^0 = u_0 (0, x), \quad u_0^0 = u_1 (0, x), \quad z^0 = z(0),
\]

where \(0 = \ell_0 < \ell_1 < \ell_2 = 1\), \(i^2 = -1\) is the imaginary unit, \(h(t)\) is the control. To this end, we assume that

\[
\ell_1 \notin \left\{ \frac{p}{p+1} : p \in \mathbb{N}^* \right\}.
\]
We then prove that the exact controllability of (1.12) can not hold in an asymmetric control space. Namely, we enunciate the following result:

**Theorem 1.2.** Let $T > 0$, and assume that (1.13) holds. Then, for every $(u^0 := (u_0^0, u_1^0), z^0) \top \in H^{-1}(0,1) \times \mathbb{C}$, there exists a control $h(t) \in L^2(0,T)$ such that the solution $U := (u_0(t,x), u_1(t,x), z(t)) \top$ of the problem (1.12) satisfies

$$
u_0(T,x) = u_1(T,x) = z(T) = 0.$$
by Hansen in [29]. In that paper, the author consider System (1.12) with a Dirichlet boundary condition on the left end \( \ell_0 = 0 \), and either Dirichlet or Neumann boundary control on the right end \( \ell_2 = 1 \). In the case of Dirichlet control, the author proves that the exact controllability space is \( H^{-1}(0,1) \times \mathbb{C} \). While, in the case of Neumann control, the exact controllability space is asymmetric with respect to the point mass in the sense that the regularity is one degree higher on the side of the point mass opposite the control. Later on, Avdonin and Edwards [12] studied the Dirichlet boundary controllability of the Schrödinger equation with internal point masses and various homogeneous boundary conditions at one end. Somewhat surprisingly, one of their main results is that System (1.12) is exactly controllable in \( H^{-1}(0,1) \times \mathbb{C} \) if and only if Condition (1.13) is not satisfied. Their proof uses a diophantine approximation argument. As consequence, if (1.13) is fulfilled, the exact controllability space is asymmetric in the sense that the regularity is \( H^{-1}(0,\ell_1) \) on the left side of the point mass and \( H^{-2}(\ell_1,1) \) to the right of the point mass. As we will see in subsection 4.2, the exact controllability space does not depend on the diophantine approximation of \( \ell_1 \). In forthcoming paper [2], we consider the exact controllability of the Schrödinger equation with internal point masses and variable coefficients. In that paper, we assume a Dirichlet boundary condition at one end, and Neumann boundary control on the other end. We prove that this system is exact controllable in asymmetric spaces whose the regularity to the right of each mass exceeds the regularity to the left by one Sobolev order from the controlled end.

This paper is organized as follows: In Section 2 we establish some results which will be used along this work. In subsection 2.1, we show that the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}^*} \) associated with System (1.1)-(1.3) are simple, and we characterize the corresponding eigenfunctions. In subsection 2.2, we investigate the well-posedness of the heat model (1.1)-(1.3). In Section 3, we investigate the main properties of all the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}^*} \): First, in Subsection 3.1, we establish the characteristic equation for the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}^*} \). Subsection 3.2 is devoted to the interlacing property (1.7), and the Weyl’s formula (1.8). In subsection 3.3, we obtain sharp asymptotic estimates of the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}^*} \). The gap condition (1.10), and the equivalence (1.11) are concluded as a consequence. Finally, in Section 4, we prove our main results, namely the null controllability of System (1.1)-(1.3), and then, the exact controllability of the Schrödinger model (1.12).

2. Characterization of the eigenelements and Well-posedness

2.1. Characterization of the eigenelements. In this subsection, we establish some spectral results which will be used along this work. First, we prove the existence and uniqueness of
solutions for the initial value problems associated with spectral problem:

\[
(P_N) \begin{cases}
- (\sigma_j(x) \phi_j')' + q_j(x) \phi_j = \lambda \rho_j(x) \phi_j, & x \in \Omega_j, \\
\phi_j-1(\ell_j) = \phi_j(\ell_j), & j = 1, \ldots, N, \\
\sigma_j-1 \phi_j(\ell_j) - \sigma_j \phi_j(\ell_j) = M_j \lambda \phi_j-1(\ell_j), & j = 1, \ldots, N, \\
\phi_1(\ell_1) = \phi_1(0) = 0, & \phi_N(\ell_{N+1}) = \phi_N(L) = 0.
\end{cases}
\]

Then, we study the asymptotic properties of these solutions. As consequence, we show that the eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) of Problem \((P_N)\) \((2.1)\) are simple, and we characterize the associated eigenfunctions. To this end, let us introduce the following Hilbert space

\[
\mathcal{V} = \left\{ \mathbf{u} := (u_j)_{j=0}^N \in \prod_{j=0}^N H^1(\Omega_j) : \begin{cases}
u_0(0) = u_N(L) = 0, \\
u_{j-1}(\ell_j) = u_j(\ell_j), & j = 1, \ldots, N,
\end{cases} \right\},
\]

which endowed with the Hilbert structure

\[
\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}} = \sum_{j=0}^N \int_{\Omega_j} u_j' v_j' \rho_j(x) dx, \quad \mathbf{v} = (v_j)_{j=0}^N.
\]

We consider the following closed subspace of \(\mathcal{V} \times \mathbb{R}^N\),

\[
\mathcal{W} = \left\{ (\hat{\mathbf{u}}, \hat{\mathbf{z}})^{\top} \in \mathcal{V} \times \mathbb{R}^N : \begin{cases}
\hat{\mathbf{z}} := (z_j)_{j=1}^N = (z_1, \ldots, z_N), \\
z_{j-1} = u_{j-1}(\ell_j) = u_j(\ell_j), & j = 1, \ldots, N,
\end{cases} \right\},
\]

which is densely and continuously embedded in the space \(\mathcal{H}\). In the sequel we introduce the operator \(\mathcal{A}\) defined in \(\mathcal{H}\) by setting

\[
\mathcal{A} \mathbf{u} = \left( \frac{1}{\rho_j(x)} (- (\sigma_j(x) u_j')' + q_j(x) u_j) \right)_{j=0}^N, \left( \frac{1}{M_j} (\sigma_{j-1} u_{j-1}'(\ell_j) - \sigma_j u_j'(\ell_j)) \right)_{j=1}^N.
\]

where \(\mathbf{u} = (\hat{\mathbf{u}}, \hat{\mathbf{z}})^{\top}\) on the domain

\[
\mathcal{D}(\mathcal{A}) = \left\{ (\hat{\mathbf{u}}, \hat{\mathbf{z}})^{\top} \in \mathcal{W} : \mathbf{u} = (u_j)_{j=0}^N, \ u_j \in H^2(\Omega_j), \ j = 1, \ldots, N, \right\},
\]

which is dense in \(\mathcal{H}\). Obviously, the spectral problem \((P_N)\) \((2.1)\) is equivalent to the following problem

\[
\mathcal{A} \Phi = \lambda \Phi, \quad \Phi := \left( \phi_j(x, \lambda) \right)_{j=0}^N, \left( \phi_j(\ell_j, \lambda) \right)_{j=1}^N \in \mathcal{D}(\mathcal{A}),
\]

i.e., the eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) of the operator \(\mathcal{A}\) and Problem \((P_N)\) \((2.1)\) coincide together with their multiplicities. Moreover, there is a one-to-one correspondence between the eigenfunctions,

\[
\Phi_n := \left( \phi_j(x, \lambda_n) \right)_{j=0}^N, \left( \phi_j(\ell_j, \lambda_n) \right)_{j=1}^N \quad \leftrightarrow \quad \phi_n := \left( \phi_j(x, \lambda_n) \right)_{j=0}^N, \quad n \in \mathbb{N}^*.
\]

Lemma 2.1. The linear operator \(\mathcal{A}\) is positive and self-adjoint such that \(\mathcal{A}^{-1}\) is compact.
Lemma 2.2. \hspace{1cm}

Let us fix $\ell_j$ respectively. It is known (e.g., [32, Chapter 1] and [40, Chapter 1]), that

\begin{equation}
\sigma_j \phi_j(\ell_j) = \sigma_j \phi_j'(\ell_j) = g_j(\ell_j) \quad \text{for each } j = 1, \ldots, N,
\end{equation}

and the initial conditions

\begin{equation}
\phi_j(0) = \sigma_0 \phi_0'(0) - 1 = 0,
\end{equation}

\begin{equation}
\phi_j(\ell_j) = \phi_{j-1}(\ell_j), \quad j = 1, \ldots, N,
\end{equation}

\begin{equation}
\sigma_j \phi_j'(\ell_j) = \sigma_{j-1} \phi_j'(\ell_j) - M_j \lambda \phi_{j-1}(\ell_j), \quad j = 1, \ldots, N,
\end{equation}

and

\begin{equation}
\phi_N(L) = \sigma_N \phi_N'(L) + 1 = 0,
\end{equation}

\begin{equation}
\phi_{j-1}(\ell_j) = \phi_j(\ell_j), \quad j = 1, \ldots, N,
\end{equation}

\begin{equation}
\sigma_{j-1} \phi_j'(\ell_j) = \sigma_j \phi_j'(\ell_j) + M_j \lambda \phi_j(\ell_j), \quad j = 1, \ldots, N,
\end{equation}

respectively. For each $j = 0, \ldots, N$, let $\hat{\varphi}_j(x, \lambda)$ and $\hat{\psi}_j(x, \lambda)$ are the unique solutions, up to a multiplicative constant, of the subproblems determined by Equation (2.4) in $\Omega_j$, and the initial conditions

\begin{equation}
\hat{\varphi}_j(\ell_j) - 1 = \hat{\varphi}_j'(\ell_j) = 0, \quad j = 0, \ldots, N,
\end{equation}

and

\begin{equation}
\hat{\psi}_j(\ell_j) = \sigma_j \hat{\psi}_j'(\ell_j) - 1 = 0, \quad j = 0, \ldots, N,
\end{equation}

respectively. It is known (e.g., [32 Chapter 1] and [40 Chapter 1]), that $\hat{\varphi}_j(x, \lambda)$ and $\hat{\varphi}_j'(x, \lambda)$ (resp. $\hat{\psi}_j(x, \lambda)$ and $\hat{\psi}_j'(x, \lambda)$) are entire functions of $\lambda$ for each fixed $x \in \Omega_j$, $j = 0, \ldots, N$.

Lemma 2.2. \hspace{1cm}

Let us fix $j \in \{0, \ldots, N\}$, and let $f_j(\lambda)$ and $g_j(\lambda)$ be two analytic functions. Then, the subproblem determined by Equation (2.4) in $\Omega_j$, and the initial conditions

\begin{equation}
\phi_j(\ell_j) = f_j(\lambda), \quad \sigma_j \phi_j'(\ell_j) = g_j(\lambda) \quad \text{or } \phi_j(\ell_{j+1}) = f_j(\lambda), \quad \sigma_j \phi_j'(\ell_{j+1}) = g_j(\lambda)
\end{equation}
has a unique solution \( \phi_j(x, \lambda) \), up to a multiplicative constant,

\[
\phi_j(x, \lambda) = f_j(\lambda) \hat{\varphi}_j(x, \lambda) + g_j(\lambda) \hat{\psi}_j(x, \lambda), \quad x \in \overline{\Omega}_j, \ j = 0, ..., N.
\]

Furthermore, \( \phi_j(x, \lambda) \) and \( \phi'_j(x, \lambda) \) are entire functions of \( \lambda \) for each fixed \( x \in \overline{\Omega}_j, \ j = 0, ..., N \).

**Proof.** By (2.11)-(2.12), the Wronskian

\[
\Delta_j(\lambda) = \hat{\varphi}_j \sigma_j \hat{\psi}'_j(\ell_j) - \hat{\varphi}'_j \sigma_j \hat{\psi}_j(\ell_j) = 1 \neq 0, \ j = 0, ..., N,
\]

and then, \( \hat{\varphi}_j(x, \lambda) \) and \( \hat{\psi}_j(x, \lambda) \) are two linearly independent solutions of Equation (2.4) in \( \Omega_j \).

This implies that any solution \( \phi_j(x, \lambda) \) of the subproblem (2.4), (2.13), can be written in the form

\[
\phi_j(x, \lambda) = C_1 \hat{\varphi}_j(x, \lambda) + C_2 \hat{\psi}_j(x, \lambda), \quad x \in \overline{\Omega}_j, \ j = 0, ..., N,
\]

for some constants \( C_j \neq 0, \ j = 0, 1 \). Using this together with the initial conditions (2.11)-(2.12) and (2.13), we have

\[
\phi_j(x, \lambda) = f_j(\lambda) \hat{\varphi}_j(x, \lambda) + g_j(\lambda) \hat{\psi}_j(x, \lambda), \quad x \in \overline{\Omega}_j, \ j = 0, ..., N,
\]

is a nontrivial solution of the subproblem (2.4), (2.13). The uniqueness of solutions follows from the linearity of the equation (2.4) together with standard theory of differential equations. Since \( f_j(\lambda) \) and \( g_j(\lambda) \) are analytic functions, then from the expression (2.14), \( \phi_j(x, \lambda) \) and \( \phi'_j(x, \lambda) \) are entire functions of \( \lambda \) for each fixed \( x \in \overline{\Omega}_j \).

**Lemma 2.3.** (a) The initial value problem (2.4) - (2.7) has a unique solution, up to a multiplicative constant,

\[
\varphi_N := (\varphi_j(x, \lambda))_{j=0}^N, \quad x \in \overline{\Omega}_j, \ j = 0, ..., N,
\]

where \( \varphi_0(x, \lambda) \) and \( \varphi_j(x, \lambda) \), \( j = 1, ..., N \), are the unique solutions (up to a scalar) of the initial value subproblems determined by Equations (2.4) - (2.5) in \( \Omega_0 \), and Equations (2.4), (2.6) - (2.7) in \( \overline{\Omega}_j \), \( j = 1, ..., N \), respectively. Furthermore, \( \varphi_N(x, \lambda) \) and \( \varphi'_N(x, \lambda) \) are entire functions of \( \lambda \) for each fixed \( x \in \overline{\Omega} \).

(b) The initial value problem (2.4), (2.8) - (2.10) has a unique solution, up to a multiplicative constant,

\[
\psi_N := (\psi_j(x, \lambda))_{j=0}^N, \quad x \in \overline{\Omega}_j, \ j = 0, ..., N,
\]

where \( \psi_N(x, \lambda) \) and \( \psi_j(x, \lambda) \), \( j = 0, ..., N - 1 \), are the unique solutions (up to a scalar) of the initial value subproblems determined by Equations (2.4), (2.8) in \( \Omega_N \), and Equations (2.4), (2.9) - (2.10) in \( \overline{\Omega}_j \), \( j = 0, ..., N - 1 \), respectively. Furthermore, \( \psi_N(x, \lambda) \) and \( \psi'_N(x, \lambda) \) are entire functions of \( \lambda \) for each fixed \( x \in \overline{\Omega} \).
Proof. For $j = 0$, it is known (e.g.,\[32\] Chapter 1 and \[40\] Chapter 1), that the initial value subproblem (2.4)-(2.5) has a unique solution $\varphi_0 := \hat{\psi}_0(x, \lambda)$, $x \in \Omega_0$, up to a multiplicative constant, such that $\psi_0(x, \lambda)$ and $\hat{\varphi}_0(x, \lambda)$ are entire functions of $\lambda$ for each fixed $x \in \Omega_0$. For $j = 1$, let $f_1(\lambda) = \hat{\psi}_0(\ell_1, \lambda)$ and $g_1(\lambda) = \sigma_0(\ell_1)\hat{\psi}_0(\ell_1, \lambda) - M_1\lambda\hat{\psi}_0(\ell_1, \lambda)$. Then by Lemma 2.2 the subproblem determined by Equation (2.4) in $\Omega_1$ and the initial conditions

$$\varphi_1(\ell_1, \lambda) = \hat{\psi}_0(\ell_1, \lambda)$$

and $\sigma_1(\ell_1)\varphi_1(\ell_1, \lambda) = \sigma_0(\ell_1)\hat{\psi}_0(\ell_1, \lambda) - M_1\lambda\hat{\psi}_0(\ell_1, \lambda)$,

has a unique solution $\varphi_1(x, \lambda)$, up to a scalar,

$$\varphi_1(x, \lambda) = \hat{\psi}_0(\ell_1, \lambda)\hat{\varphi}_1(x, \lambda) + \left(\sigma_0(\ell_1)\hat{\psi}_0(\ell_1, \lambda) - M_1\lambda\hat{\psi}_0(\ell_1, \lambda)\right)\hat{\psi}_1(x, \lambda), \quad x \in \Omega_1,$$

where $\hat{\varphi}_1(x, \lambda)$ and $\hat{\psi}_1(x, \lambda)$ are the solutions of the subproblems (2.4), (2.11) and (2.4), (2.12) for $j = 1$, respectively. Furthermore, $\varphi_1(x, \lambda)$ and $\varphi_1'(x, \lambda)$ are analytic functions of $\lambda$ for each fixed $x \in \Omega_1$. For $j = 2$, let $f_2(\lambda) = \varphi_1(\ell_2, \lambda)$ and $g_2(\lambda) = \sigma_1(\ell_2)\varphi_1'(\ell_2, \lambda) - M_2\lambda\varphi_1(\ell_2, \lambda)$. Again by Lemma 2.2 the subproblem determined by Equation (2.4) in $\Omega_2$ and the initial conditions $\varphi_2(\ell_2, \lambda) = f_2(\lambda)$ and $\sigma_2(\ell_2)\varphi_2(\ell_2, \lambda) = g_2(\lambda)$, has a unique solution $\varphi_2(x, \lambda)$, up to a scalar, such that $\varphi_2(x, \lambda)$ and $\varphi_2'(x, \lambda)$ are entire function of $\lambda$ for each fixed $x \in \Omega_2$. Moreover,

$$\varphi_2(x, \lambda) = \varphi_1(\ell_2, \lambda)\hat{\varphi}_2(x, \lambda) + \left(\sigma_1(\ell_2)\varphi_1'(\ell_2, \lambda) - M_2\lambda\varphi_1(\ell_2, \lambda)\right)\hat{\psi}_2(x, \lambda), \quad x \in \Omega_2,$$

where $\hat{\varphi}_2(x, \lambda)$ and $\hat{\psi}_2(x, \lambda)$ are the solutions of the subproblems (2.4), (2.11) and (2.4), (2.12) for $j = 2$, respectively. Now, for each $j = 3, ..., N$, let

$$f_j(\lambda) = \varphi_{j-1}(\ell_j, \lambda), \quad g_j(\lambda) = \sigma_{j-1}(\ell_j)\varphi_{j-1}'(\ell_j, \lambda) - M_j\lambda\varphi_{j-1}(\ell_j, \lambda),$$

and iterating Lemma 2.2. Then for each $j$, the subproblem determined by Equation (2.4) in $\Omega_j$ and the initial conditions

$$\varphi_j(\ell_j, \lambda) = \varphi_{j-1}(\ell_j, \lambda)$$

and $\sigma_j(\ell_j)\varphi_j'(\ell_j, \lambda) = \sigma_{j-1}(\ell_j)\varphi_{j-1}'(\ell_j, \lambda) - M_j\lambda\varphi_{j-1}(\ell_j, \lambda)$,

has a unique solution $\varphi_j(x, \lambda)$, up to a scalar, such that $\varphi_j(x, \lambda)$ and $\varphi_j'(x, \lambda)$ are analytic functions of $\lambda$ for each fixed $x \in \Omega_j$. Consequently, we have the following iteration formula: for each $j = 2, ..., N$, and $x \in \Omega_j$,

$$\varphi_j(x, \lambda) = \varphi_{j-1}(\ell_j, \lambda)\hat{\varphi}_j(x, \lambda) + \left(\sigma_{j-1}(\ell_j)\varphi_{j-1}'(\ell_j, \lambda) - M_j\lambda\varphi_{j-1}(\ell_j, \lambda)\right)\hat{\psi}_j(x, \lambda),$$

where $\hat{\varphi}_j(x, \lambda)$ and $\hat{\psi}_j(x, \lambda)$ are the solutions of the initial value subproblems (2.4), (2.11) and (2.4), (2.12) for $j = 2, ..., N$, respectively. Therefore, the function

$$\varphi_N := (\varphi_j(x, \lambda))^N_{j=0}, \quad x \in \Omega_j, \quad j = 0, ..., N,$$

is a nontrivial solution of Problem (2.4)-(2.7). Since, $\varphi_j(x, \lambda)$ and $\varphi_j'(x, \lambda)$ are analytic functions of $\lambda$ for each fixed $x \in \Omega_j$, then by (2.20), $\varphi_N(x, \lambda)$ and $\varphi_N'(x, \lambda)$ are also entire functions with respect to $\lambda$ for each fixed $x \in \Omega$. We now prove the uniqueness of solutions. Let $\varphi_N^1 :=
(\varphi_j^1(x, \lambda))_{j=0}^N and \varphi_N^2 := (\varphi_j^2(x, \lambda))_{j=0}^N are two linearly independent solutions of Problem (2.4)-(2.7). Then, by the linearity of the equations (2.4)-(2.7), the function
\[ \hat{\varphi}_N := \varphi_N^1 - \varphi_N^2 = (\varphi_j^1(x, \lambda) - \varphi_j^2(x, \lambda))_{j=0}^N, \quad x \in \Omega_j, \quad j = 0, \ldots, N, \]
is a nontrivial solution of the problem determined by Equations (2.4), (2.6)-(2.7), and the initial conditions, \( \varphi_0(0) = \sigma_0 \varphi_0'(0) = 0 \). From this and the uniqueness theorem for the equation (2.4) in \( \Omega_0 \), we get \( \varphi_0^1(x, \lambda) - \varphi_0^2(x, \lambda) \equiv 0 \), and then, by (2.6)-(2.7), one has
\[ \varphi_1(\ell_1) = \sigma_2 \varphi_1'(\ell_1) = 0. \]
Again by the uniqueness theorem for the equation (2.4) in \( \Omega_1 \), \( \varphi_1^1(x, \lambda) - \varphi_1^2(x, \lambda) \equiv 0 \). Iterating this argument, one obtains
\[ \varphi_j^1(x, \lambda) = \varphi_j^2(x, \lambda), \quad x \in \Omega_j, \quad j = 0, \ldots, N, \]
a contradiction. The second statement of the Lemma can be proved in a similar way.

We now prove an asymptotic formula for the solution \( \hat{\varphi}_N(x, \lambda) \) of Problem (2.4)-(2.7). Hereafter, we use these notations
\[ \xi_j(x) = (\rho_j(x)\sigma_j(x))^{-\frac{1}{2}}, \quad \xi_j^* = \xi_j(\ell_j)\xi_j(\ell_j+1), \quad \Upsilon_j = \prod_{k=0}^{j} \xi_k, \]
and
\[ \omega_j(x) = \int_{\ell_j}^{x} \sqrt{\frac{\rho_j(t)}{\sigma_j(t)}} \, dt, \quad \omega_j^* = \omega_j(\ell_j+1), \quad \text{and} \quad \gamma = \sum_{j=0}^{N} \omega_j^*, \quad x \in \Omega_j, \quad j = 0, \ldots, N. \]
One has:

**Proposition 2.4.** Let \( \lambda = \nu^2 \), and let \( \hat{\varphi}_N(x, \lambda) \) be the solution of Problem (2.4)-(2.7) constructed in Lemma 2.3. Then, for each \( j = 2, \ldots, N \), and every \( x \in \Omega_j \),
\[ \frac{(-1)^j \varphi_j(x, \lambda)}{\prod_{k=1}^{j-1} M_k \Upsilon_{j-1} \xi_j(\ell_j) \xi_j(x)} = M_j \nu^{j-1} \prod_{j=0}^{j-1} \sin(\nu \omega_j^*) \sin(\nu \omega_j(x))[1] - \nu^{j-2} \prod_{k=0}^{j-2} \sin(\nu \omega_k^*)[1] \]
\[ \times \left( \frac{\cos(\nu \omega_{j-1}^*) \sin(\nu \omega_j(x))}{\xi_j^1(\ell_j)} + \frac{\sin(\nu \omega_{j-1}^*) \cos(\nu \omega_j(x))}{\xi_j^2(\ell_j)} \right)[1] \]
and
\[ \frac{(-1)^j \xi_j \sigma_j \varphi_j'(x, \lambda)}{\prod_{k=1}^{j-1} M_k \Upsilon_{j-1} \xi_j(\ell_j)} = M_j \nu^{j-1} \prod_{k=0}^{j-1} \sin(\nu \omega_k^*) \cos(\nu \omega_j(x))[1] - \nu^{j-1} \prod_{k=0}^{j-2} \sin(\nu \omega_k^*)[1] \]
\[ \times \left( \frac{\cos(\nu \omega_{j-1}^*) \cos(\nu \omega_j(x))}{\xi_j^1(\ell_j)} - \frac{\sin(\nu \omega_{j-1}^*) \sin(\nu \omega_j(x))}{\xi_j^2(\ell_j)} \right)[1] \]
where \([1] = 1 + O\left(\frac{1}{|\nu|}\right)\), and \( \varphi_j(x, \lambda) \) are given in (2.15).
Proof. It is known (e.g., [32, Chapter 1] and [40, Chapter 1]), that the solutions \( \hat{\varphi}_j(x, \lambda) \) and \( \hat{\psi}_j(x, \lambda) \) of the subproblems determined by Equation (2.4) in \( \Omega_j \), and the initial conditions (2.11) and (2.12), satisfy respectively the asymptotics

\[
\begin{align*}
\left\{ \begin{array}{l}
\hat{\varphi}_j(x, \lambda) = \xi_j(x) \frac{\cos(\sqrt{\lambda} \omega_j(x))}{\xi_j(\ell_j)} [1], \quad j = 0, \ldots, N, \\
\sigma_j(x)\hat{\varphi}_j(x, \lambda) = -\lambda \frac{\xi_j(x)}{\xi_j(\ell_j)\xi_j(x)} [1], \quad j = 0, \ldots, N,
\end{array} \right. \\
\text{and}
\left\{ \begin{array}{l}
\hat{\psi}_j(x, \lambda) = \xi_j(\ell_j)\xi_j(x) \frac{\sin(\sqrt{\lambda} \omega_j(x))}{\sqrt{\lambda}} [1], \quad j = 0, \ldots, N, \\
\sigma_j(x)\hat{\psi}_j(x, \lambda) = \xi_j(\ell_j) \frac{\cos(\sqrt{\lambda} \omega_j(x))}{\xi_j(x)} [1], \quad j = 0, \ldots, N,
\end{array} \right.
\]

as \( |\lambda| \to \infty \), where \([1] = 1 + O \left( \frac{1}{\sqrt{|\lambda|}} \right) \). Let \( j = 1 \), then from the expression (2.17) and the asymptotes (2.20) for \( j = 0 \), we have

\[
\varphi_1(x, \lambda) = -M_1 \xi_0^* \nu \sin(\nu \omega_0^*) \hat{\varphi}_1(x)[1] + \xi_0^* \left( \frac{\cos(\nu \omega_0^*)}{\xi_0^*(\ell_1)} \hat{\varphi}_1(x) + \hat{\varphi}_1(x) \frac{\sin(\nu \omega_0^*)}{\nu} \right) [1], \quad x \in \Omega_1,
\]

as \( |\nu| \to \infty \), where \( \lambda = \nu^2 \), the quantities \( \xi_j^* \) and \( \omega_j^* \) are given by (2.21) and (2.22), respectively. Using this and (2.25), (2.20) for \( j = 1 \), a straightforward calculation gives the following asymptotics

\[
\frac{\varphi_1(x, \lambda)}{\xi_0^* \xi_1(\ell_1) \xi_1(x)} = -M_1 \nu \sin(\nu \omega_0^*) \sin(\nu \omega_1(x)) [1] + \frac{1}{\nu} \left( \frac{\cos(\nu \omega_0^*)}{\xi_0^*(\ell_1)} \hat{\varphi}_1(x) + \frac{\sin(\nu \omega_0^*) \cos(\nu \omega_1(x))}{\xi_1^*(\ell_1)} \right) [1]
\]

and

\[
\frac{\xi_1 \sigma_1 \varphi'_1(x, \lambda)}{\xi_0^* \xi_1(\ell_1)} = -M_1 \nu \sin(\nu \omega_0^*) \cos(\nu \omega_1(x)) [1] + \frac{\cos(\nu \omega_0^*) \cos(\nu \omega_1(x))}{\xi_0^*(\ell_1)} - \frac{\sin(\nu \omega_0^*) \sin(\nu \omega_1(x))}{\xi_1^*(\ell_1)} [1].
\]

In particular, with the convention \( \prod_{j=0}^1 = \prod_{j=0}^{-1} = 1 \), the asymptotes (2.23)–(2.24) hold for \( j = 1 \).

Now, let \( j = 2 \), then by (2.27)–(2.28),

\[
\left\{ \begin{array}{l}
\varphi_1(\ell_2, \lambda) = -M_1 \prod_{j=0}^1 \xi_j^* \sin(\nu \omega_j^*) [1], \\
\sigma_1 \varphi'_1(\ell_2, \lambda) = -M_1 \nu \prod_{j=0}^1 \xi_j^* \sin(\nu \omega_0^*) \frac{\cos(\sqrt{\lambda} \omega_j^*)}{\xi_1^*(\ell_2)} [1],
\end{array} \right.
\]

and hence, from the expression (2.18), one gets

\[
\frac{\varphi_2(x, \lambda)}{M_1 \prod_{j=0}^1 \xi_j^*} = M_2 \nu^2 \prod_{j=0}^1 \sin(\nu \omega_j^*) \hat{\varphi}_2(x)[1] - \left( \nu \sin(\nu \omega_0^*) \frac{\cos(\nu \omega_j^*)}{\xi_1^*(\ell_2)} \hat{\varphi}_2(x) + \hat{\varphi}_2(x) \prod_{j=0}^1 \sin(\nu \Omega_j^*) \right) [1].
\]
From this together with (2.25)-(2.26) for \( j = 2 \), it follows

\[
\frac{\varphi_2(x, \lambda)}{M_1 \prod_{j=0}^{1} \xi_j \ell_2(x)} = M_2 \nu \prod_{j=0}^{1} \sin(\nu \omega_j^*) \sin(\nu \omega_2(x))[1 - \sin(\nu \omega_0^*)[1] \\
\times \left( \frac{\cos(\nu \omega_j^*) \sin(\nu \omega_2(x))}{\xi_j^2(\ell_2)} + \frac{\sin(\nu \omega_j^*) \cos(\nu \omega_2(x))}{\xi_j^2(\ell_2)} \right)[1],
\]

\[
\frac{\xi_2 \sigma_2 \varphi'_2(x, \lambda)}{M_1 \prod_{j=0}^{1} \xi_j^2 \ell_2(x)} = \nu^2 M_2 \prod_{j=0}^{1} \sin(\nu \omega_j^*) \cos(\nu \omega_2(x))[1 - \nu \sin(\nu \omega_0^*)[1] \\
\times \left( \frac{\cos(\nu \omega_j^*) \cos(\nu \omega_2(x))}{\xi_j^2(\ell_2)} - \frac{\sin(\nu \omega_j^*) \sin(\nu \omega_2(x))}{\xi_j^2(\ell_2)} \right)[1],
\]

and this implies that (2.23)-(2.24) hold for \( j = 2 \). For each \( j = 3, ..., N \), following the same argument as above, by using (2.25)-(2.26) and the iteration formula (2.19), we get the asymptotic formulas (2.23)-(2.24). The proof is complete.

□

Theorem 2.5. The eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) of System (\(P_N\)) (2.1) are simple and constitute a sequence of positive real numbers:

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \text{as} \quad n \rightarrow +\infty.
\]

The corresponding eigenfunctions

\[
(\Phi_n)_{n \in \mathbb{N}^*} := \left( (\varphi_j(x, \lambda_n))_{j=0}^{N}, (\varphi_j(\ell_j, \lambda_n))_{j=1}^{N} \right)^{\top}, \quad x \in \Omega_j, \quad j = 0, \ldots, N,
\]

can be chosen to constitute an orthogonal basis of \(H\) with the inner product (1.6), where \(\varphi_j(x, \lambda), \ j = 0, \ldots, N\), are given by (2.15). Moreover, \(\varphi_j(x, \lambda_n), \ j = 2, \ldots, N\), satisfy the asymptotes (2.23)-(2.24) for \(\lambda = \lambda_n\).

Proof. It follows from Lemma 2.4 that the spectrum of the linear operator \(A\) is positive and discrete. Since \(A\) is self-adjoint in \(H\), then by Lemma 2.3 all the eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) of Problem (\(P_N\)) (2.1) are algebraically simple. By the last condition of (2.1) and Equations (2.4)-(2.7), the eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) of Problem (\(P_N\)) (2.1) are solutions of the equation

\[
\varphi_N(\ell_{N+1}, \lambda) = \varphi_N(L, \lambda) = 0,
\]

where \(\varphi_N(x, \lambda)\) is defined in Lemma 2.3. This implies that the corresponding eigenfunctions \((\phi_n)_{n \in \mathbb{N}^*}\) of Problem (\(P_N\)) (2.1) have the unique form, up to a scalar,

\[
\phi_n := (\varphi_j(x, \lambda_n))_{j=0}^{N}, \quad x \in \Omega_j, \quad j = 0, \ldots, N, \ n \in \mathbb{N}^*.
\]

where \(\varphi_j(x, \lambda)\) are given by (2.15). Thus, the expression (2.30) follows from (2.3) and (2.31), which ends the proof of the theorem.
2.2. Well-posedness. In order to study the well-posedness of the heat model (1.1)-(1.3), we apply the semigroup theory. Let us consider the following nonhomogeneous problem with zero boundary conditions

\[
\begin{aligned}
&v_j(t, \ell_j) = z_j(t) = v_j(t, \ell_j), & t > 0, & j = 0, \ldots, N, \\
&(\sigma_j(\ell_j) \partial_x v_j - \sigma_{j-1}(\ell_j) \partial_x v_{j-1}) (t, \ell_j) = M_j \partial_t z_j(t) + g_j(t), & t > 0, & j = 1, \ldots, N, \\
v_0(t, \ell_0) = v_0(t, 0) = 0, & v_N(t, \ell_{N+1}) = v_N(t, L) = 0, & t > 0,
\end{aligned}
\]

and initial conditions at \( t = 0 \)

\[
\begin{aligned}
v_j(0, x) = v_j^0, & x \in \Omega_j, & j = 0, \ldots, N, \\
z_j(0) = z_j^0, & j = 1, \ldots, N.
\end{aligned}
\]

By letting \( V = ((v_j^N)_{j=0}^N, (z_j^N)_{j=1}^N)^\top \) and \( F = ((f_j)_{j=0}^N, (g_j)_{j=1}^N)^\top \), the above problem can be rewritten in the abstract Cauchy problem

\[
\begin{aligned}
&\partial_t V(t) + AV(t) = F(t, x), & t \in (0, \infty), \\
&V(0) = V^0,
\end{aligned}
\]

where \( A \) is defined in (2.2) and \( V^0 = ((v_j^0)_{j=0}^N, (z_j^0)_{j=1}^N)^\top \). By virtue of Lemma 2.1, \( A \) is an infinitesimal generator of a strongly continuous semigroup in \( \mathcal{H} \). Therefore, from the Lumer-Phillips theorem (e.g., [37]), the Cauchy problem (2.32) has a unique mild solution \( V \in C([0, T], \mathcal{H}) \) provided that \( V^0 \in \mathcal{H} \) and \( F \in L^1((0, T); \mathcal{H}) \). Moreover, if \( V^0 \in \mathcal{D}(A) \) and \( F \in C^1([0, T]; \mathcal{H}) \) then (2.32) has a unique classical solution in the space \( C([0, T], \mathcal{D}(A)) \cap W^{1,1}(0, T; \mathcal{H}) \). If we call \( U = ((u_j)_{j=0}^N, (z_j)_{j=1}^N)^\top \) the corresponding solution of (1.1)-(1.3), then the function

\[
V := ((u_j)_{j=0}^{N-1}, u_N - \frac{x-t_N}{t_{N+1}-t_N} h(t), (z_j)_{j=1}^N)^\top
\]

satisfies (2.32) with

\[
\begin{aligned}
V^0 &= \left( u_0^0, \ldots, u_{N-1}^0, u_N^0 - \frac{x-t_N}{t_{N+1}-t_N} h(0) \right), \\
&f_j = 0, j = 0, \ldots, N - 1, \ f_N = - \frac{x-t_N}{t_{N+1}-t_N} (\partial_t h(t) + q_N(x) h(t)), \\
g_j = 0, j = 1, \ldots, N.
\end{aligned}
\]

Consequently, we have the following well-posedness result for the control system (1.1)-(1.3).

**Proposition 2.6.** Let \( U^0 = ((u_j^0)_{j=0}^N, (z_j^0)_{j=1}^N)^\top \in \mathcal{H} \) and \( h(t) \in H^1(0, T) \). Then the problem (1.1)-(1.3) has a unique solution

\[
U = ((u_j)_{j=0}^N, (z_j)_{j=1}^N)^\top \in C([0, T], \mathcal{H}).
\]

Moreover, if \( U^0 \in \mathcal{D}(A) \), then \( U \in C([0, T], \mathcal{D}(A)) \cap C^1([0, T], \mathcal{H}) \).

For the Schrödinger model (1.12), we recall the following well-posedness result (see [29]).
Proposition 2.7. Let \( U^0 := (u^0, u_1^0, z^0) \top \in H^{-1}(0, 1) \times \mathbb{C} \) and \( h(t) \in L^2(0, T) \). Then, Problem (1.12) has a unique weak solution (by transposition),

\[
U := (u = (u_0, u_1), z) \top \in C([0, T], H^{-1}(0, 1) \times \mathbb{C})
\]

3. Spectrum

In this section, we investigate the main properties of all the eigenvalues \((\lambda_n)_{n \in \mathbb{N}} \) of Problem (\( P_N \)) (2.1).

3.1. The characteristic equation. In this subsection, we implement the Wronskian technique (e.g., [26, Chapter 1] and [40, Chapter 1]), to obtain the characteristic equation for the eigenvalues \((\lambda_n)_{n \in \mathbb{N}} \) of Problem (\( P_N \)) (2.1). Namely, we enunciate the following result:

Theorem 3.1. \( \lambda \) is an eigenvalue of Problem (\( P_N \)) (2.1), if and only if, the Wronskians

\[
\Delta_j(\lambda) = \varphi_j(x, \lambda)\sigma_j(x)\psi_j'(x, \lambda) - \varphi_j'(x, \lambda)\sigma_j(x)\psi_j(x, \lambda) = 0, \quad \forall x \in \overline{\Omega}_j, \quad \forall j \in \{0, ..., N\},
\]

where \( \varphi_j(x, \lambda) \) and \( \psi_j(x, \lambda) \) are defined by (2.15) and (2.16), respectively.

For the proof of this theorem, we need the following remarkable and useful property of the wronskians \( \Delta_j(\lambda) \).

Lemma 3.2. One has:

\[
(\text{3.1}) \quad \Delta_j(\lambda) = \Delta_k(\lambda), \quad \forall j, k \in \{0, ..., N\} \text{ with } j \neq k.
\]

Proof. By the initial conditions (2.6)-(2.7) and (2.9)-(2.10), one gets

\[
\Delta_{j-1}(\lambda) = \varphi_{j-1}(\lambda)\sigma_j(\lambda)\psi_j'(\lambda) - \varphi_j'(\lambda)\sigma_j(\lambda)\psi_j(\lambda),
\]

\[
= \varphi_j(\ell_j) \left( \sigma_j\psi_j'(\ell_j) + M_j\lambda\psi_j(\ell_j) \right) - \psi_j(\ell_j) \left( \sigma_j\varphi_j'(\ell_j) + M_j\lambda\varphi_j(\ell_j) \right),
\]

\[
= \Delta_j(\lambda), \quad j = 1, ..., N.
\]

By Equations (2.4), \( \partial_x \Delta_j(\lambda) = 0 \), for all \( j \in \{0, ..., N\} \). Thus, from the above, (3.1) follows. \( \square \)

Proof of Theorem 3.1. We first argue by contradiction, so let \( \{\lambda, \phi(x, \lambda)\} \) be an eigenpair of Problem (\( P_N \)) (2.1) and suppose that \( \Delta_{j^*}(\lambda) \neq 0 \) for some \( j^* \in \{0, ..., N\} \). Under this assumption together with Lemma 3.2, it follows

\[
(\text{3.2}) \quad \Delta_j(\lambda) \neq 0, \quad j = 0, ..., N,
\]

and this implies that, \( \varphi_j(x, \lambda) \) and \( \psi_j(x, \lambda) \) are linearly independent solutions of Equation (2.4) in each the subintervals \( \overline{\Omega}_j, \quad j = 0, ..., N \). Consequently, any solution of the problem determined by Equations (2.4), (2.9)-(2.10) may be expressed as a linear combination of \( \varphi_{j^*}(x, \lambda) \) and
Therefore the eigenfunction $\phi(x, \lambda)$ of Problem ($P_N$) (2.1) can be written in the form

$$
\phi := (C_j \varphi_j(x, \lambda) + \hat{C}_j \psi_j(x, \lambda))^N_{j=0}, \quad x \in \Omega_j, \quad j = 0, \ldots, N,
$$

for some constants $C_j$ and $\hat{C}_j$. Substituting this expression into the first two conditions of System ($P_N$) (2.1), we get

$$
C_{j-1} \varphi_{j-1}(\ell_j) + \hat{C}_{j-1} \psi_{j-1}(\ell_j) = C_j \varphi_j(\ell_j) + \hat{C}_j \psi_j(\ell_j), \quad j = 1, \ldots, N,
$$

and

$$
M_j \lambda \left( C_j \varphi_j(\ell_j) + \hat{C}_j \psi_j(\ell_j) \right) = \sigma_{j-1} \left( C_{j-1} \varphi'_{j-1}(\ell_j) + \hat{C}_{j-1} \psi'_{j-1}(\ell_j) \right)
- \sigma_j \left( C_j \varphi'_j(\ell_j) + \hat{C}_j \psi'_j(\ell_j) \right), \quad j = 1, \ldots, N.
$$

Using Conditions (2.6)-(2.7) and (2.9)-(2.10) in (3.4)-(3.5), one has

$$
(\ell_j - \hat{\ell}_j) \varphi_j(\ell_j) + (\hat{C}_j - \hat{\ell}_j) \psi_j(\ell_j) = 0, \quad j = 1, \ldots, N,
$$

and

$$
(\ell_j - \hat{\ell}_j) \sigma_j \varphi'_j(\ell_j) + (\hat{C}_j - \hat{\ell}_j) \sigma_j \psi'_j(\ell_j) = 0, \quad j = 1, \ldots, N.
$$

By multiplying Equations (3.6) and (3.7), respectively, by $\sigma_j \varphi'_j(\ell_j)$ and $\varphi_j(\ell_j)$, a simple calculations yields

$$
(\ell_j - \hat{\ell}_j) (\varphi_j \sigma_j \psi'_j(\ell_j) - \varphi'_j \sigma_j \psi_j(\ell_j)) = (\hat{C}_j - \hat{\ell}_j) \Delta_j(\lambda) = 0, \quad j = 1, \ldots, N.
$$

Similarly,

$$
(\ell_j - \hat{\ell}_j) (\varphi_j \sigma_j \psi'_j(\ell_j) - \varphi'_j \sigma_j \psi_j(\ell_j)) = (\ell_j - \hat{\ell}_j) \Delta_j(\lambda) = 0, \quad j = 1, \ldots, N.
$$

From the above together with (3.2), one has

$$
C_j = C_{j-1} \text{ and } \hat{C}_{j-1} = \hat{C}_j, \quad j = 1, \ldots, N.
$$

From (2.5) and (2.8),

$$
\Delta_1(\lambda) = -\sigma_1(0) \psi_1(0, \lambda) \quad \text{and} \quad \Delta_N(\lambda) = \sigma_N(L) \varphi_N(L, \lambda).
$$

Substituting (3.3) into the last condition of (2.1) and using (3.9), one gets

$$
\hat{C}_1 \psi_1(0, \lambda) = 0 = -\hat{C}_1 \Delta_1(\lambda) \sigma_1(0) \quad \text{and} \quad C_N \varphi_N(L, \lambda) = 0 = C_N \Delta_N(\lambda) \sigma_N(L).
$$

From this together with (1.5), (3.2) and (3.3), we get $C_j = \hat{C}_j = 0$ for all $j \in \{0, \ldots, N\}$. Thus from (3.3), $\phi(x, \lambda) = 0$, a contradiction. Reciprocally, if $\Delta_j(\lambda) = 0$ for all $j \in \{0, \ldots, N\}$, then $\Delta_N(\lambda) = 0$. This implies that,

$$
\varphi_N(x, \lambda) = C \psi_N(x, \lambda), \quad x \in \Omega_N,
$$

for some constant $C \neq 0$, where $\varphi_N(x, \lambda)$ and $\psi_N(x, \lambda)$ are defined by (2.15) and (2.16), respectively. Since $\psi_N(L, \lambda) = 0$, then from (3.10), the solution $\phi_N(x, \lambda)$ of Problem ($P'_N$) (2.4)-(2.7)
satisfies the boundary condition $\phi_N(L, \lambda) = 0$. Thus by Theorem 2.5, \( \{ \lambda, \phi_N(x, \lambda) \} \) is an eigen-
pair of Problem \( (P_N)(2.1) \). The Theorem is proved.

\[ \square \]

3.2. **Interlacing of eigenvalues and Weyl’s formula.** In this subsection, we prove that all
the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) of Problem \( (P_N)(2.1) \) interlace those of the \( N + 1 \) decoupled rods with
homogeneous Dirichlet boundary conditions. As consequence, we establish the Weyl’s formula \( (1.8) \).

Set

\begin{equation}
\Xi_N := \{ \mu_{n,D}^{N,D} \}_{n=1}^{\infty} = \bigcup_{j=0}^{N} \{ \hat{\mu}_{j,D}^{n,J} \}_{n=1}^{\infty},
\end{equation}

where \( (\hat{\mu}_{j,D}^{n,J})_{n \in \mathbb{N}} \), \( j = 1, ..., N \), are the eigenvalues of the \( N + 1 \) Dirichlet subproblems

\begin{equation}
\begin{cases}
-(\sigma_j(x)\phi')' + q_j(x)\phi = \lambda \rho_j(x)\phi, & x \in \Omega_j, \ j = 0, ..., N, \\
\phi(\ell_j) = \phi(\ell_{j+1}) = 0.
\end{cases}
\end{equation}

One has:

**Theorem 3.3.** The eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) and \( (\mu_{n,D}^{N,D})_{n \in \mathbb{N}} \) interlace in the following sense:

\begin{equation}
0 < \lambda_1 \leq \mu_1^{N,D} \text{ and } \mu_1^{N,D} \leq \lambda_{n+1} \leq \mu_{n+1}^{N,D}, \forall n \in \mathbb{N}^*.
\end{equation}

Moreover, the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) satisfy the Weyl’s type asymptotic formula:

\begin{equation}
\lim_{n \to \infty} \frac{\lambda_n}{n^2 \pi^2} = \left( \sum_{j=0}^{N} \int_{\Omega_j} \sqrt{\frac{\rho_j(x)}{\sigma_j(x)}} \ dx \right)^{-2}.
\end{equation}

To this end, let \( (\tilde{\lambda}_n)_{n \in \mathbb{N}} \) denote the eigenvalues of the Problem \( (P_{N-1})(2.1) \)(i.e., Problem \( (2.1) \) with \( N - 1 \) point masses). By virtue of Theorem 2.5 all the eigenvalues \( (\tilde{\lambda}_n)_{n \in \mathbb{N}} \) are
positive and simple:

\begin{equation}
0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \ldots \ldots < \tilde{\lambda}_n < \ldots \ldots \rightarrow \infty.
\end{equation}

Let \( \phi_N(x, \lambda) \) be the solution of the initial problem \( (2.4)-(2.7) \) constructed in Lemma 2.3. Let us introduce the variable complex function

\begin{equation}
F_{N-1}(\lambda) = \frac{\sigma_{N-1}(\ell_N)\varphi_{N-1}(\ell_N, \lambda)}{\varphi_{N-1}(\ell_N, \lambda)}, \lambda \in (-\infty, \tilde{\lambda}_1) \bigcup \bigcup_{n=0}^{\infty} (\tilde{\lambda}_n, \tilde{\lambda}_{n+1}) \bigcup \bigcup_{n=0}^{\infty} (\hat{\lambda}_n, \hat{\lambda}_{n+1}),
\end{equation}

where \( \varphi_{N-1}(x, \lambda) \) is the restriction of \( \phi_N(x, \lambda) \) to the subinterval \( \Pi_{N-1} \). From \( (3.15) \), \( F_{N-1}(\lambda) \) is well-defined on all the intervals \( (-\infty, \tilde{\lambda}_1) \) and \( (\tilde{\lambda}_n, \tilde{\lambda}_{n+1}) \), \( n \in \mathbb{N}^* \). In view of Lemma 2.3 \( F_{N-1}(\lambda) \) is a meromorphic function. Moreover, the poles of the function \( F_{N-1}(\lambda) \) are the
2.3. Let us first prove that

$$\psi_n(x, \lambda)$$ is the restriction of $$\psi_N(x, \lambda)$$ to the subinterval $$\Omega_N$$. Obviously, the poles of the function $$F_N(\lambda)$$ are the eigenvalues $$(\hat{\mu}_n^{N,D})$$ of the Dirichlet problem (3.12) on the subinterval $$\Omega_N$$, while, their zeros are eigenvalues of the problem

$$F_N(\lambda) = \frac{\sigma_N(\ell_N)\psi_N(\ell_N, \lambda)}{\psi_N(\ell_N, \lambda)}, \lambda \in (-\infty, \hat{\lambda}_1) \bigcup \bigcup_{n=0}^{\infty} \left( \hat{\mu}_n^{N,D}, \hat{\mu}_{n+1}^{N,D} \right), n \in \mathbb{N}^*,$$

where $$\psi_N(x, \lambda)$$ is the restriction of $$\psi_N(x, \lambda)$$ to the subinterval $$\Omega_N$$. Obviously, the poles of the function $$F_N(\lambda)$$ are the eigenvalues $$(\hat{\mu}_n^{N,D})$$ of the Dirichlet problem (3.12) on the subinterval $$\Omega_N$$, while, their zeros are eigenvalues of the problem

$$F_N(\lambda) = \frac{\sigma_N(\ell_N)\psi_N(\ell_N, \lambda)}{\psi_N(\ell_N, \lambda)}, \lambda \in (-\infty, \hat{\lambda}_1) \bigcup \bigcup_{n=0}^{\infty} \left( \hat{\mu}_n^{N,D}, \hat{\mu}_{n+1}^{N,D} \right), n \in \mathbb{N}^*.$$

Proposition 3.4. (a) $$F_{N-1}(\lambda)$$ is a decreasing function along each of the intervals $$(-\infty, \hat{\lambda}_1)$$ and $$(\hat{\lambda}_n, \hat{\lambda}_{n+1})$$, $$n \in \mathbb{N}^*$$. Furthermore, it decreases from $$+\infty$$ to $$-\infty$$.

(b) $$F_N(\lambda)$$ is an increasing function from $$-\infty$$ to $$+\infty$$ along each of the intervals $$(-\infty, \hat{\mu}_1^{N,D})$$ and $$(\hat{\mu}_n^{N,D}, \hat{\mu}_{n+1}^{N,D})$$, $$n \in \mathbb{N}^*$$.

Proof. Let $$\varphi_j(x, \lambda)$$ be the solution of the initial problem $$(\mathcal{P}_N^0)$$ (2.4), (2.7) constructed in Lemma 2.3. Let us first prove that

$$\partial_\lambda F_{N-1}(\lambda) = \frac{-1}{\varphi_{N-1}'(\ell_N, \lambda)} \left( \sum_{j=0}^{N-1} \int_{\Omega_j} \varphi_j^2(x, \lambda) dx + \sum_{j=1}^{N-1} M_j \varphi_j^2(\ell_j, \lambda) \right),$$

where $$\varphi_j(x, \lambda)$$ are given by (2.13). To this end, let $$\lambda, \mu \in (-\infty, \hat{\lambda}_1)$$ or $$(\hat{\lambda}_n, \hat{\lambda}_{n+1})$$, and let us denote by

$$\tilde{\Delta}_j(x) = \varphi_j(x, \lambda) \sigma_j(x) \varphi_j'(x, \mu) - \varphi_j'(x, \lambda) \sigma_j(x) \varphi_j(x, \mu), x \in \overline{\Omega}_j, j = 0, \ldots, N - 1,$$

where $$\lambda \neq \mu$$. By (2.4),

$$\tilde{\Delta}_j(x) = \varphi_j(x, \lambda) (\sigma_j \varphi_j)'(x, \mu) - (\sigma_j \varphi_j)'(x, \lambda) \varphi_j(x, \mu)$$

$$= (q_j(x) - \mu \rho_j(x)) \varphi_j(x, \lambda) \varphi_j(x, \mu) - (q_j(x) - \lambda \rho_j(x)) \varphi_j(x, \lambda) \varphi_j(x, \mu)$$

$$= (\lambda - \mu) \rho_j(x) \varphi_j(x, \lambda) \varphi_j(x, \mu), x \in \overline{\Omega}_j, j = 0, \ldots, N - 1,$$

and this implies that

$$\sum_{j=0}^{N-1} \left( \tilde{\Delta}_j(\ell_j + 1) - \tilde{\Delta}_j(\ell_j) \right) = (\lambda - \mu) \sum_{j=0}^{N-1} \int_{\Omega_j} \varphi_j(x, \lambda) \varphi_j(x, \mu) dx.$$
From (2.15), \( \hat{\Delta}_0(\ell_0) = 0 \), and then, by (3.19), it follows

\[
(3.20) \quad \hat{\Delta}_{N-1}(\ell_N) = \sum_{j=1}^{N-1} (\hat{\Delta}_j(\ell_j) - \hat{\Delta}_{j-1}(\ell_j)) + (\lambda - \mu) \int_{\Omega_j} \varphi_j(x, \lambda) \varphi_j(x, \mu) dx.
\]

Using the initial conditions (2.6)–(2.7), one gets

\[
\hat{\Delta}_{j-1}(\ell_j) = \varphi_{j-1}(\ell_j, \lambda) \sigma_{j-1}(\ell_j) \varphi'_{j-1}(\ell_j, \mu) - \sigma_{j-1}(\ell_j) \varphi'_{j-1}(\ell_j, \lambda) \varphi_{j-1}(\ell_j, \mu)
\]

\[
= \varphi_j(\ell_j, \lambda) \left( \sigma_j \varphi'_j + M_j \mu \varphi_j \right)(\ell_j, \mu) - \left( \sigma_j \varphi'_j + M_j \lambda \varphi_j \right)(\ell_j, \lambda) \varphi_j(\ell_j, \mu)
\]

\[
= \hat{\Delta}_j(\ell_j) + M_j(\mu - \lambda) \varphi_j(\ell_j, \lambda) \varphi_j(\ell_j, \mu), \quad j = 1, \ldots, N - 1,
\]

and this implies that,

\[
\sum_{j=1}^{N-1} (\hat{\Delta}_j(\ell_j) - \hat{\Delta}_{j-1}(\ell_j)) = (\lambda - \mu) \sum_{j=1}^{N-1} M_j \varphi_j(\ell_j, \lambda) \varphi_j(\ell_j, \mu).
\]

From this together with (3.20), one has

\[
(3.21) \quad \frac{\hat{\Delta}_{N-1}(\ell_N)}{\sigma_{N-1}(\ell_N)} = \varphi_{N-1}(\ell_N, \lambda) \frac{\varphi'_{N-1}(\ell_N, \mu) - \varphi'_{N-1}(\ell_N, \lambda)}{\lambda - \mu} \frac{\varphi_{N-1}(\ell_N, \mu) - \varphi_{N-1}(\ell_N, \lambda)}{\lambda - \mu}
\]

\[
= \frac{1}{\sigma_{N-1}(\ell_N)} \left( \sum_{j=0}^{N-1} \int_{\Omega_j} \varphi_j(x, \lambda) \varphi_j(x, \mu) dx + \sum_{j=1}^{N-1} M_j \varphi_j(\ell_j, \lambda) \varphi_j(\ell_j, \mu) \right).
\]

Thus, passing to the limit as \( \mu \to \lambda \) in (3.21) and dividing both sides by \( \varphi_{N-1}^2(\ell_N, \lambda) \), we get (3.18). We now prove that

\[
(3.22) \quad \lim_{\lambda \to +\infty} F_{N-1}(\lambda) = +\infty.
\]

Let \( \lambda = |\nu|^2 \), where \( \nu \in \mathbb{R}^* \). By Lemma 2.4, one has

\[
(3.23) \quad F_{N-1}(\lambda) \sim \frac{i |\nu| \cos(i |\nu| \omega_{N-1}^*) [1]}{2 \xi_{N-1}^2(\ell_N) \sin(i |\nu| \omega_{N-1}^*) [1]}, \quad \text{as} \ |\nu| \to \infty,
\]

where \([1] = 1 + \mathcal{O}\left( \frac{1}{|\nu|} \right)\), \( \sqrt{i} = -1 \) is the imaginary unit, \( \xi_{N-1} \) and \( \omega_{N-1}^* \) are defined by (2.21) and (2.22), respectively. Since, \( \sin(i |\nu|) = i \sinh(|\nu|) \) and \( \cos(i |\nu|) = \cosh(|\nu|) \), then by (3.23), one gets

\[
F_{N-1}(\lambda) \sim \frac{|\nu|}{\xi_{N-1}^2(\ell_N)}, \quad \text{as} \ |\nu| \to \infty,
\]

and this proves (3.22). On the other hand, the poles \( \left( \hat{\lambda}_n \right)_{n \in \mathbb{N}^*} \) and the zeros of function \( F_{N-1}(\lambda) \) do not coincide, since otherwise, \( \hat{\lambda}_n \) would be an eigenvalue of Problem (\( P_{N-1} \)) (2.1) for which \( \varphi_{N-1}(\ell_N, \hat{\lambda}_n) = \varphi_{N-1}(\ell_N, \hat{\lambda}_n) = 0 \), a contradiction. From this together with (3.18), it follows

\[
\lim_{\lambda \to \hat{\lambda}_n} F_{N-1}(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \hat{\lambda}_n} F_{N-1}(\lambda) = +\infty, \quad n \in \mathbb{N}^*.
\]

and this ends the proof of the statement (a) of Proposition 3.4. The second statement of the proposition can be proved in a same way. The proof is complete. \( \square \)
We are now ready to prove Theorem 3.3.

**Proof.** Let \( \{\lambda_n\}_{n \in \mathbb{N}^*} \) are the eigenvalues the problem (\( \mathcal{P}_{N-1} \)) (2.1) (i.e., Problem (2.1) with \( N - 1 \) point masses). Set \( \Gamma := \{\gamma_n\}_1^\infty = \{\mu_n^{1,D}\}_1^\infty \cup \{\lambda_n\}_1^\infty \), where \( \{\mu_n^{1,D}\}_{n \in \mathbb{N}^*} \) are the eigenvalues of the Dirichlet subproblem (3.12) for \( x \in \Omega_N \). Since \( \{\lambda_n\}_{n \in \mathbb{N}^*} \) and \( \{\mu_n^{1,D}\}_{n \in \mathbb{N}^*} \) are simple, then \( \Gamma \) has a decomposition \( \Gamma = \Gamma^* \cup \Gamma^+ \), where

\[
\Gamma^* := \{\gamma_n^*, \text{ for some } n \in \mathbb{N}^*\} = \{\gamma_n \in \Gamma : \lambda_j = \mu_k^{1,D} \text{ for some } j, k \in \mathbb{N}^*\}
\]

and

\[
\Gamma^+ := \{\gamma_n^+\}_1^\infty = \Gamma \setminus \Gamma^* : 0 < \gamma_1^+ < \gamma_2^+ < \ldots < \gamma_n^+ < \ldots \xrightarrow{n \to \infty} \infty.
\]

We prove (3.13)- (3.14) by induction. It is known [16, Corollary 3.4], that the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}^*} \) of Problem (\( \mathcal{P}_1 \)) (2.1) satisfy:

\[
0 < \lambda_1 \leq \mu_1^{1,D}, \mu_{n+1}^{1,D} \leq \lambda_n \leq \mu_n^{1,D}, \forall n \in \mathbb{N}^*,
\]

and

\[
\lim_{n \to \infty} \frac{\lambda_n}{n^2 \pi^2} = \left( \frac{1}{\sqrt{\pi}} \int_{\Omega_1} \sqrt{\frac{\rho(x)}{\sigma(x)}} \, dx \right)^{-2},
\]

where \( \mu_2^{1,D} \in \Xi_1 \), and \( \Xi_1 \) is defined by (3.11). This means that (3.13)- (3.14) hold in the case \( N = 1 \). Assume that (3.26)- (3.27) hold for \( j \leq N - 1 \), then by induction hypothesis, the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}^*} \) of Problem (\( \mathcal{P}_{N-1} \)) (2.1) satisfy:

\[
0 < \lambda_1 \leq \mu_1^{N-1,D}, \mu_{n+1}^{N-1,D} \leq \lambda_n \leq \mu_n^{N-1,D}, \forall n \in \mathbb{N}^*,
\]

and

\[
\lim_{n \to \infty} \frac{\lambda_n}{n^2 \pi^2} = \left( \frac{1}{\sqrt{\pi}} \int_{\Omega_1} \sqrt{\frac{\rho(x)}{\sigma(x)}} \, dx \right)^{-2},
\]

where \( \mu^{N-1,D}_n \in \Xi_{N-1} \), and \( \Xi_{N-1} \) is defined by (3.11). First, we prove the interlacing formula (3.13). In view of Theorem 3.1

\[
\varphi_N(\ell, \lambda_n) \sigma_N(\ell, \lambda_n) \psi_N(\ell, \lambda_n) - \varphi'_N(\ell, \lambda_n) \sigma_N(\ell, \lambda_n) \psi_N(\ell, \lambda_n) = 0, \quad n \in \mathbb{N}^*,
\]

where \( \varphi_N(x, \lambda) \) and \( \psi_N(x, \lambda) \) are respectively given by (2.15) and (2.16). Then, we have only examine the following cases:

**Case 1.** If \( \varphi_N(\ell, \lambda_n) \neq 0, n \in \mathbb{N}^* \), then by (3.30), we get \( \psi_N(\ell, \lambda_n) \neq 0 \). This means that

\[
\lambda_n \in \Pi := \left( -\infty, \gamma_1^+ \right) \cup \left\{ \bigcup_{n=0}^\infty (\gamma_n^+, \gamma_{n+1}^+) \right\}, \quad \gamma_n^+ \in \Gamma^+, \quad n \in \mathbb{N}^*,
\]

where \( \Gamma^+ \) is defined by (3.26). From (2.6)- (2.7) and (3.30), one obtains

\[
\sigma_N(\ell, \lambda_n) \varphi_{N-1} \psi_N(\ell, \lambda_n) - \sigma_{N-1}(\ell, \lambda_n) \varphi_N \psi_{N-1}(\ell, \lambda_n) + M_N \lambda_n \psi_N \varphi_{N-1}(\ell, \lambda_n) = 0,
\]
which can be rewritten in the form

\[ F_N(\lambda_n) - F_{N-1}(\lambda_n) = -M_N \lambda_n, \quad \lambda_n \in \Pi, \]

where \( F_{N-1}(\lambda) \) and \( F_N(\lambda) \) are respectively given by (3.10) and (3.14). By (3.31), we have \( \gamma_n^+ \neq \gamma_{n+1}^+, n \in \mathbb{N}^* \). By Proposition 3.4, \( F_N(\lambda) - F_{N-1}(\lambda) \) is an increasing function from \(-\infty\) to \(+\infty\) along each of the intervals \((-\infty, \gamma_1^+)\) and \((\gamma_j^+, \gamma_{j+1}^+)\), \( n \in \mathbb{N}^* \). Clearly, that the solution of the equation (3.33) are the eigenvalues \( \lambda_{n} \in N^* \) of Problem (PN) (2.1). Moreover, if \( F_N(\lambda') - F_{N-1}(\lambda') \) for some \( \lambda' \), then \( \lambda' \) is an eigenvalue of the problem (PN) (2.1) for \( M_N = 0 \). Consequently, from the curves of the functions \( F_N(\lambda) - F_{N-1}(\lambda) \) and \(-M_N \lambda \), one has

\[ 0 < \lambda_1 < \lambda_1' < \gamma_1^+ \text{ and } \gamma_n^+ < \lambda_{n+1} < \lambda_{n+1}' < \gamma_{n+1}^+, \quad n \in \mathbb{N}^*, \]

where \((\lambda_n')_{n \in \mathbb{N}^*}\) are the eigenvalues of Problem (PN) (2.1) for \( M_N = 0 \). Since \( \gamma_n^+ \neq \gamma_{n+1}^+ \), then by (3.28), we have

\[ \gamma_n^+ = \hat{\lambda}_j \text{ and } \gamma_{n+1}^+ = \hat{\mu}_k^{N,D} \text{ or } \gamma_n^+ = \hat{\mu}_k^{N,D} \text{ and } \gamma_{n+1}^+ = \hat{\lambda}_j, \quad j, k \in \mathbb{N}^*. \]

We may assume without loss of generality that in (3.34), \( \gamma_n^+ = \hat{\lambda}_j \) and \( \gamma_{n+1}^+ = \hat{\mu}_k^{N,D} \). Then, by (3.33), (3.28) and (3.31), one gets

\[ 0 < \lambda_1 < \lambda_1' < \mu_k^{N,D} \text{ and } \mu_k^{N,D} < \lambda_{n+1} < \lambda_{n+1}' < \mu_k^{N,D}, \quad n \in \mathbb{N}^*, \]

with \( \mu_k^{N,D} = \mu_k^{N-1,D} \) and \( \mu_k^{N,D} = \mu_k^{N-1,D} \). This ends the proof of (3.13) in this case.

**Case 2.** If \( \varphi_N(\ell_N, \lambda_n) = 0 \), for some \( n \in \mathbb{N}^* \), then by (3.29), we have \( \psi_N(\ell_N, \lambda_n) = 0 \). This implies \( \lambda_n \in \Gamma^* \), where \( \Gamma^* \) is defined by (3.24). Consequently, \( \lambda_n, n \in \mathbb{N}^* \), is simultaneously an eigenvalue of the Dirichlet subproblem (3.12) for \( x \in \Omega_N \) and Problem \( (P_{N-1}) \) (2.1). Thus from (3.24) and (3.28), one has

\[ \lambda_{n+1} = \gamma_n^+ = \hat{\mu}_k^{N,D} = \lambda_{n+1}', \quad \text{for some } n \in \mathbb{N}^*, \]

and then (3.13) follows in this case. This completes the proof of interlacing formula (3.13). By (3.29), the eigenvalues \((\lambda_n')_{n \in \mathbb{N}^*}\) of the problem (PN) (2.1) for \( M_N = 0 \) satisfy the asymptote

\[ \left( \frac{\lambda_n'}{n^2 \pi^2} \right) = \lim_{\ell_N \to L} \left( \frac{\hat{\lambda}_n(\ell_N)}{n^2 \pi^2} \right) = \left( \sum_{j=0}^{N-1} \int_{\Omega_N} \frac{\rho_j(x)}{\sigma_j(x)} dx + \int_{\ell_{N-1}}^{L} \sqrt{\frac{\rho(x)}{\sigma(x)}} dx \right)^{-2}, \]

where \((\hat{\lambda}_n)_{n \in \mathbb{N}^*}\) are the eigenvalues of Problem (PN) (2.1).

\[ \rho(x) = \begin{cases} 
\rho_{N-1}(x), & x \in \Omega_{N-1}, \\
\rho_N(x), & x \in \Omega_N,
\end{cases} \quad \text{and } \sigma(x) := \begin{cases} 
\sigma_{N-1}(x), & x \in \Omega_{N-1}, \\
\sigma_N(x), & x \in \Omega_N.
\end{cases} \]

From (3.35) - (3.36), we have

\[ \mu_k^{N,D} \leq \lambda_n \leq \lambda_n' \leq \mu_k^{N,D} \leq \lambda_{n+1} \leq \lambda_{n+1}' \leq \mu_k^{N,D}, \quad n \in \mathbb{N}^*, \]

and then, by (3.37) we get the Weyl's asymptotic formula (3.14). The proof is complete. \( \square \)
3.3. Sharp asymptotics of the eigenvalues and spectral gap. In this subsection, we establish sharp asymptotic estimates for eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) of Problem \((\mathcal{P}_N)^{(2.1)}\). As consequence, we prove that the spectral gap \(|\lambda_{n+1} - \lambda_n|\) is uniformly positive. Namely, we enunciate the following result:

**Theorem 3.5.** Set \(\Lambda^* = \{n+1 : \mu_n^{N,D} = \lambda_{n+1} = \mu_{n+1}^{N,D} \text{ for some } n \in \mathbb{N}^*\}\), and let

\[
Q_j^* := \frac{(\ell_{j+1} - \ell_j)^2}{\gamma_j^2} \int_{\Omega_j} \left( \rho_j^{-1} q_j(x) - \{\rho_j^{-3} \sigma_j(x)\}^{\frac{1}{2}} \left[ \sigma_j(x) \left( \{\sigma_j \rho_j(x)\}^{-\frac{1}{2}} \right) \right]^{\prime} \right)^{\prime} dx, \quad j = 0, \ldots, N.
\]

Then, the set of eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) of Problem \((\mathcal{P}_N)^{(2.1)}\) is asymptotically splits into \(N + 1\) branches \(\{\lambda_n^j\}_{n \in \mathbb{N}^*}, \quad j = 0, \ldots, N\), such that:

(a) for large \(n + 1 \in \Lambda^*\),

\[
\lambda_{n+1}^j = \left( n \pi \frac{\sqrt{\lambda_{n+1}^j}}{\omega_j^*} + \frac{Q_j^*}{2(n+1)} + O\left( \frac{1}{n^2} \right) \right), \quad j = 0, \ldots, N,
\]

(b) for large \(n + 1 \in \mathbb{N}^* \setminus \Lambda^*\), one has the asymptotes:

\[
\begin{align*}
\sqrt{\lambda_{n+1}^*} &= \frac{\pi}{\omega_j^*} + \frac{Q_j^*}{2n} + \frac{\gamma_j^2 (\ell_{j+1})}{M_{j+1} \omega_j^* \pi} + O\left( \frac{1}{n^2} \right), \quad j = 0, \ldots, N - 1, \\
\sqrt{\lambda_{n+1}^N} &= \frac{\pi}{\omega_N^*} + \frac{Q_N^*}{2n} + \frac{\gamma_N^2 (\ell_N)}{M_N \omega_N^* \pi} + O\left( \frac{1}{n^2} \right),
\end{align*}
\]

where the quantities \(\xi_j, \gamma\) and \(\omega_j^*\) are respectively given in \((2.21)\) and \((2.22)\).

Moreover,

\[
\lambda_{n+1} - \lambda_n \geq 2\gamma \min_{j=0, \ldots, N-1} \left\{ \frac{\xi_j^2 (\ell_{j+1})}{M_{j+1} \omega_j^{*2}}, \frac{\xi_N^2 (\ell_N)}{M_N \omega_N^{*2}} \right\}, \quad n \to \infty.
\]

**Remark 3.6.** It should be noted that if \(n \in \Lambda^*\), then at most \(N + 1\) of the eigenvalues \((\tilde{\mu}_n^{i,D})_{n \in \mathbb{N}^*}\) of the \(N + 1\) Dirichlet subproblems \((3.12)\) can coincide. This follows from the simplicity of the eigenvalues \((\tilde{\mu}_n^{i,D})_{n \in \mathbb{N}^*}\), \(j = 0, \ldots, N\). For example, in the case of constant coefficients \(\rho_j = \sigma_j = 1, q_j = 0, \ell_0^* = \ell_j + 1 - \ell_j, \) and \(\ell_0^* = \ell_{j+1}^*\) for all \(j \in \{0, \ldots, N\}\). In particular, in this case \(\Lambda^* = \mathbb{N}^*\). Conversely, if \(\ell_j^* \in \mathbb{R}_+ \setminus \mathbb{Q}\) for all \(j, k \in \{0, \ldots, N\}\) with \(j \neq k\), then \(\Lambda^* \equiv \emptyset\).

**Proof.** From the interlacing theorem \((3.9)\) we deduce that between two consecutive eigenvalues \(\mu_n\) and \(\mu_{n+1}\) there is only one eigenvalue \(\lambda_n\) of Problem \((\mathcal{P}_N)^{(2.1)}\). Consequently by \((3.11)\), the set of eigenvalues \((\lambda_n)_{n \in \mathbb{N}^*}\) may be decomposed as:

\[
\{\lambda_n\}_{n \in \mathbb{N}^*} = \bigcup_{j=0}^N \{\lambda_n^j\}_{n \in \mathbb{N}^*} := \sqrt{\lambda_{n+1}^j} = \sqrt{\tilde{\mu}_n^{j,D}} + \kappa_n^j, \quad j = 0, \ldots, N
\]

for some sequences \((\kappa_n^j)_{n \in \mathbb{N}^*} \geq 0\). Let \(n+1 \in \Lambda^*\), i.e., \(\mu_n^{N,D} = \lambda_{n+1} = \mu_{n+1}^{N,D}\), then by \((3.12)\), we get

\[
\sqrt{\lambda_{n+1}^j} = \sqrt{\tilde{\mu}_n^{j,D}}, \quad j = 0, \ldots, N.
\]
Using the modified Liouville transformation (e.g., [32] Chapter 1),
\[
t = \frac{\ell_{j+1} - \ell_j}{\gamma_j} \int_{x_j}^x \sqrt{\frac{\rho_j(t)}{\sigma_j(t)}} dt + \ell_j \quad \text{and} \quad \hat{\phi}(t) = (\rho_j \sigma_j(x))^{\frac{j}{4}} \phi(x), \; j = 0, ..., N.
\]

Problem (3.12) can be written in the following form
\[
\begin{align*}
-\hat{\phi}'' + Q_j(t) \hat{\phi} &= \frac{\gamma_j^2}{(\ell_{j+1} - \ell_j)^2} \lambda \hat{\phi}, \; t \in \Omega_j, \; j = 0, ..., N, \\
\hat{\phi}(\ell_j) &= \hat{\phi}(\ell_{j+1}) = 0,
\end{align*}
\]
where
\[
Q_j := \frac{(\ell_{j+1} - \ell_j)^2}{\gamma_j^2} \left( \rho_j^{-1} q_j - \rho_j^{-3} \sigma_j \left[ \left( \sigma_j \rho_j^{-\frac{3}{2}} \right)' \right]' \right), \quad j = 0, ..., N.
\]

It is known (e.g., [32, Chapter 1] and [40, Chapter 1]), that the eigenvalues \((\hat{\mu}_n^{j,D})_{n \in \mathbb{N}}\) of the \(N + 1\) Dirichlet subproblems (3.44) satisfy the asymptotics
\[
\sqrt{\hat{\mu}_n^{j,D}} = \frac{n \pi}{\omega_j^*} + \frac{1}{2n} \int_{x_j}^x Q_j(x)dx + O \left( \frac{1}{n^2} \right), \; j = 0, ..., N.
\]

Therefore (3.39) is a simple deduction from (3.43) and (3.45). Now, let \(n + 1 \in \mathbb{N}^* \setminus \Lambda^*\), i.e., \(\mu_n^{j,D} \neq \mu_{n+1}^{j,D}\). Then by Theorem 3.3, (3.39) and (3.42), one has
\[
\sqrt{\lambda_{n+1}} = \frac{n \pi}{\omega_j^*} + \frac{Q_j}{2n} + O \left( \frac{1}{n^2} \right) + n_{\sigma}^j, \quad \text{with} \quad (n_{\sigma}^j)_{n \in \mathbb{N}} > 0, \; j = 0, ..., N.
\]

By Proposition 2.4 (2.26) (for \(j = 0\)) and (2.27), we have
\[
\begin{align*}
\frac{\varphi_j(\ell_{j+1}, \lambda)}{\sigma_j \varphi_j'(\ell_{j+1}, \lambda)} &= \frac{\xi_j^2(\ell_{j+1}) \sin(\sqrt{\lambda} \omega_j^*)}{\sqrt{\lambda} \cos(\sqrt{\lambda} \omega_j^*)} [1], \\
\frac{\varphi_N(\ell_N, \lambda)}{\sigma_N \varphi_N'(\ell_N, \lambda)} &= \frac{1}{M_N \lambda} [1], \quad j = 0, ..., N - 1,
\end{align*}
\]

where \(\varphi_j(x, \lambda)\) are given in (2.13). Similarly, let \(\psi_{-j}(x, \lambda)\) be the solution of the initial problem (2.24), (2.28) - (2.10) constructed in Lemma 2.25. Then by (2.28),
\[
\begin{align*}
\frac{\psi_N(\ell_N, \lambda)}{\xi_N(\ell_N) c_1(\ell_N, \lambda)} &= \sin(\sqrt{\lambda} \omega_N) [1], \\
\frac{\xi_N(\ell_N) c_1(\ell_N, \lambda)}{\sigma_N \psi_N'(\ell_N, \lambda)} &= \frac{\xi_N(\ell_N)}{\cos(\sqrt{\lambda} \omega_N)} [1],
\end{align*}
\]

and by (2.29) - (2.10), one has
\[
\frac{\psi_j(\ell_{j+1}, \lambda)}{\sigma_j \psi_j'(\ell_{j+1}, \lambda)} = \frac{1}{M_{j+1} \lambda} [1], \quad j = 0, ..., N - 1,
\]

where \(\psi_j(x, \lambda)\) are given by (2.10). From Theorem 3.1 and (3.42), it follows
\[
\begin{align*}
\frac{\varphi_j(\ell_{j+1}, \lambda)}{\sigma_j \varphi_j'(\ell_{j+1}, \lambda)} &= \frac{\psi_j(\ell_{j+1}, \lambda)}{\psi_j'(\ell_{j+1}, \lambda)}, \quad j = 0, ..., N - 1,
\end{align*}
\]
where \( \varphi_N(x, \lambda) \) and \( \psi_N(x, \lambda) \) are respectively given by (2.15) and (2.16). Therefore, from (3.47) - (3.49) and (3.50), we get

\[
\left\{
\begin{array}{l}
\sin(\sqrt{\lambda_n} \omega_n) [1] = \frac{\xi_j^2(\ell_j+1)}{M_{j+1} \sqrt{\lambda_n}}, \quad j = 0, \ldots, N - 1, \\
\cos(\sqrt{\lambda_n} \omega_n) \end{array}
\right.
\]

(3.51)

where the quantities \( \xi_j, \gamma \) and \( \omega_j^* \) are respectively given in (2.21) and (2.22). Hence by (3.46) and (3.51),

\[
\left\{
\begin{array}{l}
\sin(\kappa_n^j \omega_j^*) [1] = \frac{\xi_j^2(\ell_j+1)}{M_{j+1} \sqrt{\lambda_n}}, \quad j = 0, \ldots, N - 1, \\
\cos(\kappa_n^j \omega_j^*) \end{array}
\right.
\]

(3.52)

It is easy to see that \( \kappa_n^j \to 0 \), as \( n \to \infty \), \( j = 0, \ldots, N \). Therefore, by the Weyl's formula (3.44), (3.46) and (3.52), we get the asymptotes (3.40). Hence by (3.46) and (3.53),

\[
\sqrt{\lambda_{n+1}^j + \sqrt{\mu_{n+1}^j D}} \sim \frac{2n \pi}{\omega_j^*} \quad \text{and} \quad \left\{
\begin{array}{l}
\sqrt{\lambda_{n+1}^j - \sqrt{\mu_{n+1}^j D}} \sim \frac{\gamma \xi_j^2(\ell_j+1)}{M_{j+1} \omega_j^* n \pi}, \quad j = 0, \ldots, N - 1, \\
\sqrt{\lambda_{n+1}^j - \sqrt{\mu_{n+1}^j D}} \sim \frac{\gamma \xi_j^2(\ell_j+1)}{M_{j+1} \omega_j^* n \pi}, \quad j = 0, \ldots, N - 1.
\end{array}
\right.
\]

Consequently,

\[
\lambda_{n+1}^j - \mu_{n+1}^j D \sim \frac{2n \pi}{\omega_j^*} \quad \text{and} \quad \lambda_{n+1}^j - \mu_{n+1}^j D \sim \frac{2n \pi}{\omega_j^*} \quad \text{as} \quad n \to \infty, \quad j = 0, \ldots, N - 1.
\]

Case 2. If \( n \in \mathbb{N} \setminus \Lambda^* \), then \( n + 1 \in \mathbb{N} \setminus \Lambda^* \) or \( n + 1 \in \Lambda^* \). Clearly, if \( n + 1 \in \mathbb{N} \setminus \Lambda^* \), then (3.53) is satisfied. Now, let \( n + 1 \in \Lambda^* \). By (3.39),

\[
\sqrt{\lambda_{n+1}^j} = \sqrt{\mu_{n+1}^j D} = \frac{2n \pi}{\omega_j^*} + \frac{Q_j^*}{2n} + O\left(\frac{1}{n^2}\right), \quad j = 0, \ldots, N.
\]

(3.54)

Since \( n \in \mathbb{N} \setminus \Lambda^* \), then there exist sequences \( \{\lambda_n^j\}_{n \in \mathbb{N}^*} \) such that \( \sqrt{\lambda_n^j} = \sqrt{\mu_{n+1}^j - \kappa_n^j} \), \( j = 0, \ldots, N \). As above, one has

\[
\left\{
\begin{array}{l}
\lambda_n^j = \frac{n \pi}{\omega_j^*} + \frac{Q_j^*}{2n} - \frac{\gamma \xi_j^2(\ell_j+1)}{M_{j+1} \omega_j^* n \pi} + O\left(\frac{1}{n^2}\right), \quad j = 0, \ldots, N - 1, \\
\lambda_n^j = \frac{n \pi}{\omega_j^*} + \frac{Q_j^*}{2n} - \frac{\gamma \xi_j^2(\ell_j+1)}{M_{j+1} \omega_j^* n \pi} + O\left(\frac{1}{n^2}\right).
\end{array}
\right.
\]

(3.55)

From this and (3.54), it follows

\[
\lambda_{n+1}^j - \lambda_n^j = \frac{2n \pi}{\omega_j^*} \quad \text{and} \quad \lambda_{n+1}^j - \lambda_n^j = \frac{2n \pi}{\omega_j^*} \quad \text{as} \quad n \to \infty, \quad j = 0, \ldots, N - 1.
\]
By the interlacing theorem 3.3 we have $\lambda^j_{n+1} - \lambda^j_n \geq \lambda^j_{n+1} - \tilde{\mu}^j_{n+1}$, $j = 0, ..., N$. Thus, from (3.53) and (3.55), one gets

$$\lambda^j_{n+1} - \lambda^j_n \geq 2\gamma \min_{j=0, ..., N-1} \left\{ \frac{\xi^j_j(\ell_j^+)}{M^j_j + 1 \omega^{j^2}} \right\} \xi^j_N(\ell_N) \frac{2}{M^j_N \omega^{j^2}} \sqrt{\omega^j} \gamma \xi_N(L), \quad \text{as } n \to \infty.$$

The proof is complete. \(\square\)

In the next result, we establish the equivalence between the $H$-norm of the eigenfunctions $(\Phi^j_n)_{n \in \mathbb{N}}$, and their first derivative at the right end $x = L$.

**Proposition 3.7.** Let $(\Phi^j_n)_{n \in \mathbb{N}}$ be the sequence of eigenfunctions of Problem $(P_N)$ (2.1) constructed in Theorem 2.4. One has:

$$\frac{||\Phi^j_n||_H}{|\sigma_N(L)\phi^j_n(L,\lambda)|} \sim \sqrt{\frac{\omega^j_N \gamma \xi_N(L)}{2\pi}}, \quad \text{as } n \to \infty,$$

where the quantities $\xi_N$, $\gamma$ and $\omega^j_N$ are respectively given in (2.24) and (2.26).

**Proof.** By the change of variables $X = \omega_j(x)$, one has

$$\int_{\Omega_j} \xi^j_j(x) \sin^2(\sqrt{\omega_j}(x)) \rho_j(x) \, dx = \int_{\Omega_j} \xi^j_j(x) \cos^2(\sqrt{\omega_j}(x)) \rho_j(x) \, dx = \frac{\omega^j}{2}[1],$$

$$\int_{\Omega_j} \xi^j_j(x) \sin(\sqrt{\omega_j}(x)) \cos(\sqrt{\omega_j}(x)) \rho_j(x) \, dx = \frac{\sin^2(\sqrt{\omega^j})}{2\sqrt{\lambda}}[1], \quad j = 0, ..., N,$$

where $[1] = 1 + O(\frac{1}{\sqrt{\lambda}})$, the quantities $\xi_j$, $\gamma$ and $\omega_j$ are respectively given in (2.21) and (2.26). From this together with (2.20) (for $j = 0$) and (2.27),

$$||\varphi_0||^2_{L^2_\rho_0} = \xi^j_0(\ell_0) \frac{\omega^j_0}{2\lambda}[1] \quad \text{and} \quad ||\varphi_1||^2_{L^2_{\rho_1}} = (M_1 \xi_0^j \xi_1(\ell_1) \sin(\nu \omega_0^j))^2 \frac{\omega^j_1}{2}[1].$$

Similarly by (2.26), we get

$$||\varphi_j||^2_{L^2_{\rho_j}} = \left(\sqrt{\lambda^{j-1}} \Upsilon_j \xi_j(\ell_j) \prod_{k=1}^j \prod_{i=0}^{j-1} \sin(\sqrt{\lambda} \omega^j_i) \right)^2 \frac{\omega^j}{2}[1], \quad j = 2, ..., N,$$

where $\Upsilon_j$ are defined by (2.21). Thus, by the above asymptotes,

$$\sum_{j=0}^N ||\varphi_j||^2_{L^2_{\rho_j}} = ||\varphi_N||^2_{L^2_{\rho_N}}[1] = \left(\sqrt{\lambda^{N-1}} \Upsilon_N \xi_N(\ell_N) \prod_{k=1}^N \prod_{i=0}^{N-1} \sin(\sqrt{\lambda} \omega^N_i) \right)^2 \frac{\omega^N}{2}[1].$$

Again by (2.23) + (2.24) and (2.20) + (2.27), it follows

$$\sum_{j=1}^N M_j \phi^2_j(\ell_j) = \phi^2_N(\ell_N)[1] = \left(\sqrt{\lambda^{N-2}} \Upsilon_{N-1} \prod_{k=1}^N \prod_{i=0}^{N-1} \sin(\sqrt{\lambda} \omega^N_i) \right)^2 \frac{\omega^N}{2}[1],$$

and

$$(-1)^N \sigma_N \phi'_N(L) = \left(\sqrt{\lambda} \Upsilon_{N-1} \xi_N(\ell_N) \prod_{k=1}^N \prod_{i=0}^{N-1} \sin(\sqrt{\lambda} \omega^N_k) \right) \frac{\cos(\sqrt{\lambda} \omega^N_N)}{\xi_N(L)}[1].$$
Thus, by (2.30), (3.57)-(3.59), one gets
$$\frac{||\Phi_n||_{H}^2}{|\sigma_N(L)\varphi_N'(L,\lambda_n)|^2} = \frac{||\varphi_N(x,\lambda_n)||_{L^2_N}}{|\sigma_N(L)\varphi_N'(L,\lambda_n)|^2}[1] = \frac{\omega_N^2(L)}{2\lambda_n \cos^2(\sqrt{\lambda_n} \omega_N^*)}[1].$$
Or equivalantly (by Theorem 3.31),
$$\frac{||\Phi_n||_{H}^2}{|\sigma_N(L)\varphi_N'(L,\lambda_n)|^2} = \frac{\omega_N^2(L)}{2\lambda_n \left(1 - \sin^2(\sqrt{\lambda_n} \omega_N^*)\right)}[1], \quad j = 0, ..., N.$$  
It is easy to see from (3.39)-(3.40) that
$$\left|\sin(\sqrt{\lambda_n} \omega_N^*)\right| \leq C_j, \quad j = 0, ..., N,$$
for some constants $C_j > 0$. Therefore, from this and (3.60) together with the Weyl’s formula (3.14), we get the equivalence (3.56). The proof is complete. □

4. Controllability

In this section, we prove our main results, namely the null controllability of System (1.11)-(1.12), and then, the exact controllability of the Schrödinger model (1.12).

4.1. Null controllability of the heat model (1.11)-(1.12). In this subsection, we prove Theorem 1.11. We do it by reducing the control problem to problem of moments. Then, we will solve this problem of moments using the theory developed in [38, 39]. To this end, let us consider the so-called adjoint problem, that is,

$$\begin{align*}
(p_j(x)\partial_t \hat{u}_j + \partial_x (\sigma_j(x)\partial_x \hat{u}_j) - q_j(x)\hat{u}_j)(t,x) = 0, & \quad t > 0, \ x \in \Omega_j, \ j = 0, ..., N, \\
\hat{u}_{j-1}(t,\ell_j) = \hat{z}_j(t) = \hat{u}_{j}(t,\ell_j), & \quad t > 0, \ j = 1, ..., N, \\
(\sigma_{j-1}(\ell_j)\partial_x \hat{u}_{j-1} - \sigma_j(\ell_j)\partial_x \hat{u}_j)(t,\ell_j) = M_j \partial_t \hat{z}_j(t), & \quad t > 0, \ j = 1, ..., N, \\
\hat{u}_0(t,0) = 0, \ \hat{u}_N(t,L) = 0 & \quad t > 0,
\end{align*}$$

with final data at $t = T > 0$ given by
$$\begin{align*}
\hat{u}_j(T,x) = \hat{u}_j^T, \ x \in \Omega_j, & \quad j = 0, ..., N, \\
\hat{z}_j(T) = \hat{z}_j^T, & \quad j = 1, ..., N.
\end{align*}$$

By letting $\hat{U} = \left((\hat{u}_j)^N_{j=0},(\hat{z}_j)^N_{j=1}\right)^T$, the above problem can be written as
$$\partial_t \hat{U}(t) = A\hat{U}(t), \quad \hat{U}(T) = \hat{U}^T, \quad t \in (0, \infty),$$
where $A$ is defined in (2.2) and $\hat{U}^T = \left((\hat{u}_j)^N_{j=0},(\hat{z}_j)^N_{j=1}\right)^T$. Then, we have the following characterization of the null-controllability property.
Lemma 4.1. System (1.1)-(1.3) is null-controllable in time $T > 0$, if and only if, for any initial data $U^0 = \left( (u^0_j)_{j=0}^N, (z^0_j)_{j=1}^N \right)^T \in \mathcal{H}$, there exists a control function $h(t) \in H^1(0,T)$, such that, for any $\hat{U}^T = \left( (\hat{u}^T_j)_{j=0}^N, (\hat{z}^T_j)_{j=1}^N \right)^T \in \mathcal{H}$
\begin{equation}
\left\langle U^0, (\hat{u}_j(0,x))_{j=0}^N, (\hat{z}_j(0))_{j=1}^N \right\rangle^T_{\mathcal{H}} = \sigma_N(L) \int_0^T h(t) \partial_x \hat{u}_N(t,L) dt
\end{equation}
where $\hat{U} = \left( (\hat{u}_j)_{j=0}^N, (\hat{z}_j)_{j=1}^N \right)^T$ is the solution of the adjoint problem (4.1)-(4.2).

Proof. We proceed as in the classical duality approach. We first multiply the $N+1$ equations in (1.1) by $(\hat{u}_j)_{j=0}^N$, to obtain
\begin{equation}
\sum_{j=0}^N \int_{\ell_j}^{\ell_{j+1}} \partial_t u_j \hat{u}_j dt \rho_j(x) dx = \int_0^T \sum_{j=0}^N \int_{\ell_j}^{\ell_{j+1}} (\partial_x (\sigma_j(x) \partial_x u_j) - q_j(x) u_j) \hat{u}_j dt dx,
\end{equation}
where $\hat{U} = \left( (\hat{u}_j)_{j=0}^N, (\hat{z}_j)_{j=1}^N \right)^T$ is the solution of Problem (4.1)-(4.2). Integration by parts leads to
\begin{equation}
\sum_{j=0}^N \int_{\ell_j}^{\ell_{j+1}} u_j \hat{u}_j |_{t=0}^{t=T} \rho_j(x) dx = \int_0^T \sum_{j=0}^N \left( \partial_j(x) \partial_x u_j \hat{u}_j |_{x=\ell_j}^{x=\ell_{j+1}} - \sigma_j(x) \partial_x \hat{u}_j |_{x=\ell_j}^{x=\ell_{j+1}} \right) dt.
\end{equation}
Since
\begin{equation}
\sum_{j=0}^N \sigma_j(x) \partial_x u_j \hat{u}_j |_{x=\ell_{j+1}}^{x=\ell_j} = - \sum_{j=1}^N M_j \partial_t \hat{z}_j(t),
\end{equation}
and
\begin{equation}
\sum_{j=0}^N \sigma_j(x) \partial_x u_j \hat{u}_j |_{x=\ell_{j+1}}^{x=\ell_j} = \sum_{j=1}^N M_j \partial_t \hat{z}_j(t) + \sigma_N(L) h(t) \partial_x \hat{u}_j(t,L),
\end{equation}
then by (4.3), one gets
\begin{equation}
\sum_{j=0}^N \int_{\ell_j}^{\ell_{j+1}} u_j \hat{u}_j |_{t=0}^{t=T} \rho_j(x) dx + \sum_{j=1}^N M_j \hat{z}_j(t) |_{t=0}^{t=T} = - \int_0^T \sigma_N(L) \partial_x \hat{u}_N u_N(t,L) dt.
\end{equation}
Equivalently,
\begin{equation}
\left\langle U(T), \hat{U}^T \right\rangle_{\mathcal{H}} = \left\langle U^0, \hat{U}(0) \right\rangle_{\mathcal{H}} - \int_0^T \sigma_N(L) \partial_x \hat{u}_N u_N(t,L) dt,
\end{equation}
where
\begin{equation}
U(T) := \left( (u_j(T,x))_{j=0}^N, (z_j(T))_{j=1}^N \right)^T \text{ and } \hat{U}(0) := \left( (\hat{u}_j(0,x))_{j=0}^N, (\hat{z}_j(0))_{j=1}^N \right)^T.
\end{equation}
Now, we assume that (4.3) holds. Then by (4.5), one has
\begin{equation}
\left\{ \begin{array}{l}
u_j(T,x) = 0, \quad \forall x \in \Omega_j, \; \forall j \in \{0, \ldots, N\}, \\
z_j(T) = 0, \quad \forall j \in \{1, \ldots, N\}.
\end{array} \right.
\end{equation}
Thus, the solution $U$ is controllable to zero and $h(t)$ is a control of Problem (1.1)-(1.3). Conversely, if $h(t)$ is a control of Problem (1.1)-(1.3) for which (4.6) holds. Thus by (4.5), we get (4.3). The Lemma is proved. \qed
We are now ready to reduce the control problem (1.1)-(1.3) to a moment problem. Let \((\Phi_n)_{n \in \mathbb{N}^*}\) be the sequence of eigenfunctions of Problem \((\mathcal{P}_N)\) constructed in Theorem 2.5, then any terminal data \(\hat{U}^T := \left((\hat{u}_T^j)_{j=0}^N, (\hat{z}_T^j)_{j=1}^N\right) \in \mathcal{H}\) for the adjoint problem (4.1)-(4.2) can be written as

\[
\hat{U}^T = \sum_{n \in \mathbb{N}^*} \frac{(\hat{U}_n, \Phi_n)_{\mathcal{H}}}{\|\Phi_n\|^2} \Phi_n(x),
\]

where the Fourier coefficients \(\hat{U}_n^T = \frac{(\hat{U}_n, \Phi_n)_{\mathcal{H}}}{\|\Phi_n\|^2}, n \in \mathbb{N}^*\), belong to \(\ell^2(\mathbb{N})\). Hence, the solution \(\hat{U}(t, x) = \left((\hat{u}_j(t, x))_{j=0}^N, (\hat{z}_j(t, x))_{j=1}^N\right)\) of (4.1)-(4.2) is given by

\[
\hat{U}(t, x) = \sum_{n \in \mathbb{N}^*} \hat{U}_n^T e^{-\lambda_n(T-t)} \Phi_n(x),
\]

and we have

\[
\partial_t \hat{U}(t, L) = \partial_x \hat{u}_N(t, L) = \sum_{n \in \mathbb{N}^*} \hat{U}_n^T e^{-\lambda_n(T-t)} \varphi'_N(L, \lambda_n),
\]

where \(\varphi_N(x, \lambda)\) is defined by (2.14). Using this fact in (1.3), on gets the following lemma.

Lemma 4.2. Problem (1.1)-(1.3) is null-controllable in time \(T > 0\) if and only if for any \(U^0 = \sum_{n \in \mathbb{N}^*} \frac{(U^0, \Phi_n)_{\mathcal{H}}}{\|\Phi_n\|^2} \Phi_n \in \mathcal{H}\), there exists a function \(h(t) \in H^1(0, T)\) such that

\[
e^{-\lambda_n T} (U^0, \Phi_n)_{\mathcal{H}} = \sigma_N(L) \varphi'_N(L, \lambda_n) \int_0^T h(T-t)e^{-\lambda_n t} dt, \forall n \in \mathbb{N}^*.
\]

We are now in a position to prove Theorem 1.1.

Proof. From the Weyl's formula (3.14),

\[
\sum_{n \in \mathbb{N}^*} \frac{1}{\lambda_n} < \infty,
\]

and then by Theorem 3.5, we deduce that there exists a biorthogonal sequence \((\Theta_n(t))_{n \in \mathbb{N}^*}\) to the family of exponential functions \(e^{-\lambda_n t})_{n \in \mathbb{N}^*}\) (see [38, 39]) such that

\[
\int_0^T \Theta_n(t)e^{-\lambda_n t} dt = \delta_{nm} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}
\]

Again by (3.14) together with the general theory developed in [38], it follows that there exists constants \(C_j(T) > 0\) (depending on \(T\)) and \(\tilde{C}_j > 0\) such that for any \(j \in \mathbb{N}\),

\[
\|\Theta_n(t)\|_{H^j(0, T)} \leq C_j(T)e^{\tilde{C}_j n}, \quad n \in \mathbb{N}^*.
\]

Let \((\Phi_n)_{n \in \mathbb{N}^*}\) be the sequence of eigenfunctions of Problem \((\mathcal{P}_N)\) constructed in Theorem 2.5, then

\[
\Phi_n(L) = \varphi'_N(L, \lambda_n) \neq 0, \forall n \in \mathbb{N}^*.
\]
where \( \varphi_N(x, \lambda) \) is defined by (2.13). Indeed, if \( \Phi'_n(L) = 0 \), then the restriction \( \varphi_N(x, \lambda) \) of \( \Phi_n \) to \( \Omega_N \) satisfies, \( \varphi_N(L, \lambda_n) = \varphi'_N(L, \lambda_n) = 0 \). Thus, \( \varphi_N(x, \lambda_n) = 0 \), a contradiction. Therefore from the above, we infer that an explicit formal solution of the moment problem (4.7) is given by

\[
\Phi_n(0,0,T) = \frac{\lambda_n^T}{\sigma N \varphi'_N(L, \lambda_n)} e^{-\lambda_n T} \Theta_n(t).
\]

As a consequence, the task consists in showing that the series \( k(t) \) convergence in \( H^1(0,T) \). From (4.8) and (4.9), we obtain by Cauchy-Schwarz inequality that

\[
\|h\|_{H^1(0,T)} \leq C_1(T) \left( \sum_{n \in \mathbb{N}^*} \|\Phi_n\|_H^2 e^{-\lambda_n T} \right)^{1/2} \|U^0\|_H.
\]

Therefore from Proposition 4.7, the Weyl’s formula (3.14) and (4.10), it follows

\[
\|h\|_{H^1(0,T)} \leq C(T) \sqrt{\frac{\omega_j}{2}} \xi_j \sum_{n \in \mathbb{N}^*} \gamma e^{-\frac{\omega_j}{n \pi}} \|U^0\|_H,
\]

for some new constant \( C(T) > 0 \), where the quantities \( \xi_j, \gamma \) and \( \omega_j \) are respectively given in (2.21) and (2.22). This proves the convergence of the series \( h(t) \) and finishes the proof of Theorem 1.1. \( \square \)

4.2. Exact controllability of the Schrödinger model (1.12). In this subsection, we prove Theorem 1.2.

Proof. By means of Lions HUM method (see [31]), controllability properties of the Schrödinger model (1.12) can be reduced to suitable observability inequalities for the adjoint system. As (1.12) is reversible in time, we are reduced to the same system without control. Let \( \tilde{U} := (\tilde{u}_0(t, x), \tilde{u}_1(t, x), \tilde{x}(t))^\top \) be the unique solution of Problem (1.12) with \( h(t) \equiv 0 \). It is easy to show that

\[
\tilde{U}(t, x) := \sum_{n \in \mathbb{N}^*} c_n e^{i\lambda_n t} \hat{\Phi}_n(x) \in C([0, T], H^1_0(0, 1) \times \mathbb{C}), c_n \in \ell^2(\mathbb{N}^*),
\]

where \( (\lambda_n)_{n \in \mathbb{N}^*} \) are the eigenvalues of the spectral problem

\[
\begin{align*}
-\phi''_j &= \lambda \phi_j, \quad x \in (\ell_j, \ell_{j+1}), \quad j = 0, 1, \\
\phi_0(\ell_1) &= \phi_1(\ell_1), \quad (\phi'_0 - \phi'_1)(\ell_1) = \lambda \phi_0(\ell_1), \\
\phi_0(\ell_0) &= \phi_0(0) = 0, \quad \phi_1(\ell_2) = \phi_1(1) = 0,
\end{align*}
\]

and \( \left( \hat{\Phi}_n \right)_{n \in \mathbb{N}^*} \) are the associated eigenfunctions, which are normalized in the Hilbert space

\( \mathcal{H} = \prod_{j=0}^1 L^2(\ell_j, \ell_{j+1}) \times \mathbb{C} \) so that \( \lim_{n \to \infty} \|\hat{\Phi}_n\|_\mathcal{H} = 1 \). Consequently, the task now is to prove following observability inequality:

\[
\int_0^T |\partial_x \tilde{u}_0(t, 0)|^2 \, dt \preceq \|\tilde{U}(0, x)\|^2_{H^1_0(0, 1) \times \mathbb{C}}, \quad \forall \ T > 0.
\]
To this end, following Lemma 2.3 and Proposition 2.4, it easy to see that the problem determined by Equations (4.12)-(4.13), and the initial conditions \( \phi_0(0) = \phi_0'(0) - 1 = 0 \), has a unique solution

\[
\varphi(x, \lambda) := \begin{cases} 
\varphi_0(x, \lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}, & x \in [0, \ell_1], \\
\varphi_1(x, \lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} - \sin(\sqrt{\lambda}) \sin(\sqrt{\lambda}(x - \ell_1)), & x \in [\ell_1, 1].
\end{cases}
\]

Similarly, the problem determined by Equations (4.12)-(4.13), and the initial conditions \( \phi_0(1) = \phi_0'(1) + 1 = 0 \), has a unique solution

\[
\psi(x, \lambda) := \begin{cases} 
\psi_0(x, \lambda) = \frac{\sin(\sqrt{\lambda}(1-x))}{\sqrt{\lambda}} - \sin(\sqrt{\lambda}(1 - \ell_1)) \sin(\sqrt{\lambda}(l_1 - x)), & x \in [0, \ell_1], \\
\psi_1(x, \lambda) = \frac{\sin(\sqrt{\lambda}(1-x))}{\sqrt{\lambda}}, & x \in [\ell_1, 1].
\end{cases}
\]

Following an argument similar to that in the proof of Theorem 2.5, we deduce that the eigenfunctions \( (\Phi_n(x))_{n \in \mathbb{N}} \), associated with Problem (4.12)-(4.14) taken the form

\[
(\mathbf{\Phi}_n(x))_{n \in \mathbb{N}^*} := \left( (\varphi_j(x, \lambda_n))_{j=0}^1 \right)_{n \in \mathbb{N}^*}, \quad x \in [\ell_j, \ell_{j+1}], \quad j = 0, 1,
\]

where \( \varphi_j(x, \lambda) \) are given by (4.16). Consequently, the eigenfunctions

\[
(\mathbf{\Phi}_n(x)) := \mathbf{\Phi}_n(x) / \| \mathbf{\Phi}_n \|_H, \quad \forall n \in \mathbb{N}^*,
\]

can be chosen to constitute an orthonormal basis of \( \mathcal{H} \), and then, the space \( H_0^1(0, 1) \times \mathbb{C} \) can be characterized as

\[
H_0^1(0, 1) \times \mathbb{C} = \left\{ u(x) = \sum_{n \in \mathbb{N}^*} c_n \mathbf{\Phi}_n(x) : \| u \|^2_{H_0^1(0, 1) \times \mathbb{C}} = \sum_{n \in \mathbb{N}^*} \lambda_n |c_n|^2 < \infty \right\}.
\]

By (4.16) and (4.18), a simple calculation yields

\[
| \mathbf{\Phi}_n(0) | := \| \mathbf{\Phi}_n(0) \|_H = \frac{1}{\| \mathbf{\Phi}_n \|_H} \sqrt{\frac{2}{1 - \ell_1} \frac{1}{\sin(\sqrt{\lambda_n \ell_1})} [1]},
\]

where \( [1] = 1 + O \left( \frac{1}{\sqrt{\lambda_n}} \right) \). From Theorem 3.3, \( \varphi_j(\ell_1, \lambda) = \psi_j(\ell_1, \lambda) \), \( j = 0, 1 \), and then by (4.16) and (4.17), one has

\[
\frac{\sin(\sqrt{\lambda_n \ell_1})}{\cos(\sqrt{\lambda_n \ell_1})} = \frac{1}{\sqrt{\lambda_n}} [1] \quad \text{and} \quad \frac{\sin(\sqrt{\lambda_n (1 - \ell_1)})}{\cos(\sqrt{\lambda_n (1 - \ell_1)})} = \frac{1}{\sqrt{\lambda_n}} [1].
\]

Let \( \{ \mu_n \}_{n=1}^\infty = \left\{ \left( \frac{\pi}{\ell_1} \right)^2 \right\}_{n=1} \cup \left\{ \left( \frac{\pi}{1 - \ell_1} \right)^2 \right\}_{n=1} \). Then, under Condition (4.13) together with Theorem 3.3

\[
0 < \lambda_1 < \inf \left\{ \left( \frac{\pi}{\ell_1} \right)^2, \left( \frac{\pi}{1 - \ell_1} \right)^2 \right\}, \quad \mu_n < \lambda_{n+1} < \mu_{n+1}, \quad \forall n \in \mathbb{N}^*.
\]

and

\[
\lambda_n \sim n^2 \pi^2,
\]
Following an argument similar to that in the proof of Theorem 2.5, using (4.21)-(4.22) and (4.23), we deduce that the set of eigenvalues \( \{ \lambda_n \}_{n \in \mathbb{N}^*} \) is asymptotically splits into two branches \( \{ \lambda_j^2 \}_{n \in \mathbb{N}^*}, j = 0, 1 \), such that:

\[
(4.24) \quad \sqrt{\lambda_j^2} = \frac{n \pi}{\ell_1} + \mathcal{O}\left( \frac{1}{n^2} \right) \quad \text{and} \quad \sqrt{\lambda_j^2} = \frac{n \pi}{1 - \ell_1} + \mathcal{O}\left( \frac{1}{n^2} \right).
\]

Consequently,

\[
(4.25) \quad \lambda_{n+1} - \lambda_n \geq 2 \min\left\{ \frac{1}{\ell_1^2}, \frac{1}{(1 - \ell_1)^2} \right\}, \text{ as } n \to \infty,
\]

and since \( 1 = 1 - \ell_1 + \ell_1 \), by (4.20), one gets the equivalence

\[
(4.26) \quad |\hat{\Phi}_n'(0)| \asymp \frac{1}{n} \sim \left| \frac{\sin \left( \sqrt{\lambda_j^2} \ell_1 \right)}{\sqrt{\lambda_j^2} \ell_1} \right|, j = 0, 1.
\]

From (4.22)-(4.23), we find that the Beurling upper density of the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}^*} \),

\[
D^+(\lambda_n) := \lim_{r \to \infty} \frac{n^+(r, \lambda_n)}{r} = \lim_{n \to \infty} \frac{1}{n \pi} = 0,
\]

where \( n^+(r, \lambda_n) \) denotes the maximum number of terms of the sequence \( (\lambda_n)_{n \in \mathbb{N}^*} \) contained in an interval of length \( r \). Therefore Beurling’s Theorem (e.g., [23]) states that for any \( T > 0 \), the family \( \{ e^{i \lambda_n t} \}_{n \in \mathbb{N}^*} \) forms a Riesz basis in \( L^2(0, T) \). Furthermore, for every \( T > 0 \),

\[
\int_0^T \left| \sum_{n \in \mathbb{N}^*} \chi_n e^{i \lambda_n t} \right|^2 dt \asymp \sum_{n \in \mathbb{N}^*} |\chi_n|^2,
\]

for all sequences of complex numbers \( (\chi_n)_{n \in \mathbb{N}^*} \). Let \( \chi_n = c_n \hat{\Phi}_n(0) \), then by (4.11) and (4.15),

\[
\int_0^T |\partial_x \hat{u}_0(t, 0)|^2 dt \asymp \sum_{n \in \mathbb{N}^*} \left| c_n \hat{\Phi}_n(0) \right|^2, \quad \forall \ T > 0.
\]

Therefore, from this, (4.19) and (4.26), we get the observability inequality (4.15). The proof is complete.

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