K-THEORY FOR RING C*-ALGEBRAS – THE CASE OF NUMBER FIELDS WITH HIGHER ROOTS OF UNITY

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Abstract. We compute K-theory for ring C*-algebras in the case of higher roots of unity and thereby completely determine the K-theory for ring C*-algebras attached to rings of integers in arbitrary number fields.

1. INTRODUCTION

Recently, a new type of constructions was introduced in the theory of operator algebras, so-called ring C*-algebras. The construction goes as follows: Given a ring $R$, take the Hilbert space $\ell^2(R)$ where $R$ is viewed as a discrete set. Consider the C*-algebra generated by all addition and multiplication operators induced by ring elements. This is the reduced ring C*-algebra of $R$. It is denoted by $\mathfrak{A}(R)$. Such an algebra was first introduced and studied by J. Cuntz in [Cun] in the special case $R = \mathbb{Z}$. As a next step, J. Cuntz and the first named author considered the case of integral domains satisfying a certain finiteness condition in [Cu-Li1]. Motivating examples for such rings are given by rings of integers from algebraic number theory. It turns out that the associated ring C*-algebras carry very interesting structures and admit surprising alternative descriptions (see [Cu-Li1] and [Cu-Li2]). Finally, the most general case of rings without left zero-divisors was treated in [Li].

Of course, whenever new constructions of C*-algebras appear, one of the first problems is to compute their topological K-theory. Usually, this helps a lot in understanding the inner structure of the C*-algebras. In our situation, it even turns out that the ring C*-algebras attached to rings of integers are Kirchberg algebras satisfying the UCT (see [Cu-Li1], §3 and [Li], §5). For such C*-algebras, topological K-theory is a complete invariant. This is why computing K-theory is of particular interest and importance. The first K-theoretic computations were carried out in [Cu-Li2] and [Cu-Li3] for ring C*-algebras attached to rings of integers, but only in the special case where the roots of unity in the number field are given by $+1$ and $-1$. The reason why the general case could not be treated was that a K-theoretic computation for a certain group C*-algebra was missing.

In the present paper, we treat the remaining case of higher roots of unity. The missing ingredient is provided by [La-Lü], where for each number field, the K-theory of the group C*-algebra attached to the semidirect product of the additive group of the ring of integers by the multiplicative group of roots of unity in the number field has been computed. This computation serves as a starting point for our present paper and allows us to follow the strategy from [Cu-Li2] to completely

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determine K-theory for ring C*-algebras associated with rings of integers in number fields.

Let us now formulate our main results. Let $K$ be a number field, i.e., a finite field extension of $\mathbb{Q}$. The ring of integers in $K$, i.e., the integral closure of $\mathbb{Z}$ in $K$, is denoted by $R$. Let $\mathfrak{A}[R]$ be the ring C*-algebra of $R$ defined at the beginning of the introduction (see also §2). Moreover, the multiplicative group $K^\times$ always admits a decomposition of the form $K^\times = \mu \times \Gamma$ where $\mu$ is the group of roots of unity in $K$ and $\Gamma$ is a free abelian subgroup of $K^\times$. Here is our result treating the case of higher roots of unity:

**Theorem 1.1.** Assume that our number field contains higher roots of unity, i.e., $|\mu| > 2$. Then $K_*(\mathfrak{A}[R]) \cong K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)$.

The isomorphism above is meant as an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here $K_*(\mathfrak{A}[R])$ is the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0(\mathfrak{A}[R]) \oplus K_1(\mathfrak{A}[R])$. The exterior $\mathbb{Z}$-algebra $\Lambda^*(\Gamma)$ over $\Gamma$ is endowed with its canonical grading, the group $K_0(C^*(\mu))$ is trivially graded, and we take graded tensor products.

This theorem is the main result of this paper. In combination with the results from [Cu-Li2] and [Cu-Li3], it gives the following complete description of the K-theory of $\mathfrak{A}[R]$ without restrictions on $\mu$:

**Theorem 1.2.** With the same notations as in the previous theorem (but without the assumption $|\mu| > 2$), we have

$$K_*(\mathfrak{A}[R]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \# \{v_R\} \text{ is even,} \\ \Lambda^*(\Gamma) & \text{if } \# \{v_R\} \text{ is odd,} \end{cases}$$

again as $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups.

Here $\# \{v_R\}$ denotes the number of real places of $K$. Note that since we are just identifying $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups, our K-theoretic formulas could be further simplified. But we present the formulas in this way because this description of K-theory naturally comes out of our computations.

Using the classification result for UCT Kirchberg algebras due to E. Kirchberg and C. Phillips (see [Rør Chapter 8]), we obtain from our K-theoretic computations

**Corollary 1.3.** Given two arbitrary rings of integers $R_1$ and $R_2$ in number fields, the corresponding ring C*-algebras $\mathfrak{A}[R_1]$ and $\mathfrak{A}[R_2]$ are always isomorphic.

This result should be contrasted with the observation due to J. Cuntz and C. Deninger that the C*-dynamical system $(\mathfrak{A}[R], \mathbb{R}, \sigma)$ (the higher dimensional analogue of the system introduced in [Cun], it is also the analogue for ring C*-algebras of the system introduced in [C-D-L]) determines the number field in the case of Galois extensions of $\mathbb{Q}$.

This paper is structured as follows: We start with a review on ring C*-algebras attached to rings of integers (§2). In the main part of the paper (§3), we compute the K-theory for ring C*-algebras following the same strategy as in [Cu-Li2]. Here we use results from [La-Lück] on the K-theory of certain group C*-algebras (§3).
the last section, we show how the formalism of the Baum-Connes conjecture can be used to deduce injectivity for certain homomorphisms on the level of K-theory. To do so, we apply tools from algebraic topology. This injectivity result is used in §4 but its proof is given in §5 because it is somewhat independent from the main part.

2. Review

From now on, let $K$ be a number field, i.e., a finite field extension of the rational numbers. Let $R$ be the ring of integers of $K$, i.e., $R$ is the integral closure of $\mathbb{Z}$.

First of all, let us recall the construction of ring C*-algebras: We consider the Hilbert space $\ell^2(R)$ with its canonical orthonormal basis $\{e_r: r \in R\}$. Then we use addition and multiplication in $R$ to define unitaries $U^b$ via $e_r \mapsto e_{br}$ and isometries $S_a$ via $e_r \mapsto e_{ar}$ for all $b$ in $R$, $a$ in $R^\times = R \setminus \{0\}$. The (reduced) ring C*-algebra is given by $\mathfrak{A}[R] := C^*(\{U^b\}, \{S_a\})$, the smallest involutive norm-closed algebra of bounded operators on $\ell^2(R)$ containing the families $\{U^b\}$ and $\{S_a\}$. Note that we write $\{U^b\}$ for $\{U^b: b \in R\}$ and $\{S_a\}$ for $\{S_a: a \in R^\times\}$. We use analogous notations for other families of generators as well. It turns out that $\mathfrak{A}[R]$ is isomorphic to the universal C*-algebra generated by unitaries $\{u^b: b \in R\}$ and isometries $\{s_a: a \in R^\times\}$ satisfying

\begin{align*}
I. \quad & u^b s_a u^d s_c = u^{b+ad} s_{ac} \\
II. \quad & \sum u^b s_a s^*_a u^{-b} = 1
\end{align*}

where we sum over $R/aR = \{b + aR: b \in R\}$ in II. More precisely, Relation I implies that each of the summands $u^b s_a s^*_a u^{-b}$ only depends on the coset $b + aR$, not on the particular representative of the coset. Thus we obtain one summand for each coset in $R/aR$ and we sum up these elements in Relation II. Here $aR$ is the principal ideal of $R$ generated by $a$. We write $e_a$ for the range projection $s_a s^*_a$ of $s_a$.

From the description of $\mathfrak{A}[R]$ as a universal C*-algebra, it follows that $\mathfrak{A}[R] \cong C(\mathfrak{M}) \rtimes^\alpha (R \rtimes R^\times)$. Here $\rtimes^\alpha$ stands for “semigroup crossed product by endomorphisms”. The $R \rtimes R^\times$-action we consider is given by affine transformations as follows: $R$ sits as a subring in its profinite completion $\overline{R}$ and thus acts additively and multiplicatively. Furthermore, it follows from this that $\mathfrak{A}[R]$ is Morita equivalent to the crossed product $C_0(\mathfrak{M}) \rtimes K \rtimes K^\times$, where $\mathfrak{M}$ is the infinite adele space over $K$ and $K \rtimes K^\times$ act on $C_0(\mathfrak{M})$ via affine transformations.

At this point, the duality theorem enters the game. It says that $C_0(\mathfrak{M}) \rtimes K \rtimes K^\times$ is Morita equivalent to $C_0(\mathfrak{M}) \rtimes K \rtimes K^\times$. For the first crossed product, we let $K \rtimes K^\times$ act on $C_0(\mathfrak{M})$ via affine transformations where $\mathfrak{M}$ is the infinite adele space of $K$. So on the whole, we obtain $\mathfrak{A}[R] \sim_M C_0(\mathfrak{M}) \rtimes K \rtimes K^\times$. Actually, the crossed products $C_0(\mathfrak{M}) \rtimes K$ and $C(\overline{R}) \rtimes R$ are Morita equivalent in a $R^\times$-equivariant way. The reason is that the imprimitivity bimodule from [CuL2] §4 carries a canonical $R^\times$-action which is compatible with the $R^\times$-actions on $C_0(\mathfrak{M}) \rtimes K$ and $C(\overline{R}) \rtimes R$. As a consequence, we get that for every (multiplicative) subgroup $\Gamma$ of $K^\times$, the crossed products $C_0(\mathfrak{M}) \rtimes K \rtimes \Gamma$ and $C_0(\Gamma \cdot \overline{R}) \rtimes (\Gamma \cdot R) \rtimes \Gamma$ are Morita equivalent (see [CuL2, Theorem 4.1]). Here $\Gamma \cdot \overline{R}$ is the subring of $\mathfrak{M}$ generated by $\Gamma$ and $\overline{R}$, and $\Gamma \cdot R$ is the subring of $K$ generated by $\Gamma$ and $R$. 
3. K-theory for Certain Group C*-Algebras

We now turn to the case of a number field $K$ and present the proof of Theorem 1.1.

Let $R$ be the ring of integers in $K$. Moreover, let $\mu$ be the group of roots of unity in $K$. This group is always a finite cyclic group generated by a root of unity, say $\zeta$.

The starting point for our K-theoretic computations is the work of M. Langer and the second named author on the K-theory of certain group C*-algebras. More precisely, in [La-Lü], the K-theory of the group C*-algebra of $R \times \mu$ has been computed. Here $R \times \mu$ is the semidirect product obtained from the multiplicative action of $\mu$ on the additive group $(R, +)$. The corresponding group C*-algebra is denoted by $C^*(R \times \mu)$. It is very useful for our purposes that it is even possible to give an almost complete list of generators for the corresponding K-groups. Let us now summarize the results from [La-Lü]:

**Theorem 3.1** (Langer-Lück). With the notations from above, we have

(*) $K_0(C^*(R \times \mu))$ is finitely generated and torsion-free.

(**) Let $\mathcal{M}$ be the set of conjugacy classes of maximal finite subgroups of $R \times \mu$. Then $\sum_{(M) \in \mathcal{M}} \iota_{(M)} : \bigoplus_{(M) \in \mathcal{M}} \bar{R}_C(M) \to K_0(C^*(R \times \mu))$ is injective, i.e., for every $(M) \in \mathcal{M}$, the map $\iota_{(M)}$ is injective and for every $(M_1), (M_2) \in \mathcal{M}$ with $(M_1) \neq (M_2)$ we have $\text{im}((\iota_{(M_1)})_*) \cap \text{im}((\iota_{(M_2)})_* ) = \{0\}$. Moreover, $\text{im}((\iota_*) \cap \left( \sum_{(M) \in \mathcal{M}} \text{im}((\iota_{(M)})_* ) \right) = \{0\}$ and $\text{im}((\iota_* ) \cap \left( \sum_{(M) \in \mathcal{M}} \text{im}((\iota_{(M)})_* ) \right) = \{0\}$ and $\text{im}((\iota_* ) \cap \left( \sum_{(M) \in \mathcal{M}} \text{im}((\iota_{(M)})_* ) \right) = \{0\}$

(***) $K_1(C^*(R \times \mu))$ vanishes.

Proof. (*) is [La-Lü] Theorem 0.1, (iii). (**) is [La-Lü] Theorem 0.1, (ii). Note that the maps $\iota_{(M)}$ in our notation are denoted by $\iota_{(M)}$ in [La-Lü], and that $\iota$ in our notation is denoted by $k$ in [La-Lü]. The group $\bar{R}_C(M)$ coincides with the corresponding one in [La-Lü] upon the canonical identification of the representation ring $R_C(M)$ of $M$ with $K_0(C^*(M))$ as abelian groups. Furthermore, (***) is [La-Lü] Theorem 0.1, (iv). \hfill $\square$

Let us now describe $K_0(C^*(R \times \mu))$ in a way which is most convenient for our K-theoretic computations. The idea is to use (*) and (**) from Theorem 3.1 to decompose $K_0(C^*(R \times \mu))$ into direct summands. However, we cannot simply use the subgroups $\text{im}((\iota_*)$ and $\text{im}((\iota_{(M)})_*)$ for $(M) \in \mathcal{M}$ which appear in (**) because
these subgroups might not be direct summands. To solve this problem, we proceed as follows: First of all, we set

\[ K_{inf} := \{ x \in K_0(\mathcal{C}^*(R \rtimes \mu)) : \exists N \in \mathbb{Z}_{>0} \text{ such that } Nx \in \text{im}(\iota_+) \}. \]

Now take a finite subgroup \( M \) of \( R \rtimes \mu \). It has to be a cyclic group. Let \((b, \zeta)\) in \( R \rtimes \mu \) be a generator of \( M \). Note that \( i = m/|M| \) (up to multiples of \( m \)), where \( m = |\mu| \). Let \( \chi \) be a character of \( \mathbb{Z}/|M|\mathbb{Z} \), and denote by \( p_\chi(u^b \zeta) \) the spectral projection \( \frac{1}{|M|} \sum_{j=0}^{M-1} \chi(j + |M|z)(u^b \zeta)^j \). Then \( \text{im}((\iota_M)_+) \) is generated by \( \{ [p_\chi(u^b \zeta)] : 1 \neq \chi \in \mathbb{Z}/|M|\mathbb{Z} \} \). Here \([\cdot] \) denotes the \( K_0 \)-class of the projection in question and \( 1 \in \mathbb{Z}/|M|\mathbb{Z} \) is the trivial character.

It is then clear that the \( K_0 \)-classes \( \{ [p_\chi(u^b \zeta)] \} \) for \((\mu) \neq (M) \in \mathcal{M}, \mathcal{M} = \{(b, \zeta)\} \) and \( 1 \neq \chi \in \mathbb{Z}/|M|\mathbb{Z} \) form a \( \mathbb{Z} \)-basis of \( \sum_{(\mu) \neq (M) \in \mathcal{M}} \text{im}((\iota_M)_+) \). Let us enumerate these \( K_0 \)-classes \( \{ [p_\chi(u^b \zeta)] \} \) by \( y_1, y_2, y_3, \ldots, y_{rk_{fin}} \), where \( rk_{fin} \) is the rank of \( \sum_{(\mu) \neq (M) \in \mathcal{M}} \text{im}((\iota_M)_+) \). As \( K_0(\mathcal{C}^*(R \rtimes \mu)) \) is free abelian, we can recursively find \( \overline{y}_1, \overline{y}_2, \overline{y}_3, \ldots, \overline{y}_{rk_{fin}} \) in \( K_0(\mathcal{C}^*(R \rtimes \mu)) \) such that for every \( 1 \leq j \leq rk_{fin} \),

\[
K_{inf} + \langle \overline{y}_1, \ldots, \overline{y}_j \rangle = \{ x \in K_0(\mathcal{C}^*(R \rtimes \mu)) : \exists N \in \mathbb{Z}_{>0} \text{ such that } Nx \in K_{inf} + \langle y_1, \ldots, y_j \rangle \}. \]

By construction, these elements \( \overline{y}_1, \overline{y}_2, \overline{y}_3, \ldots, \overline{y}_{rk_{fin}} \) are linearly independent. We set \( K_{fin}^\sigma = \langle \overline{y}_1, \overline{y}_2, \overline{y}_3, \ldots, \overline{y}_{rk_{fin}} \rangle \). By construction, \( K_{inf} \cap K_{fin}^\sigma = \{0\} \). Finally, \( \{ [p_\chi(s_\xi)] : 1 \neq \chi \in \mathbb{Z}/m\mathbb{Z} \} \) is a \( \mathbb{Z} \)-basis of \( \text{im}((\iota)_+) \). Enumerate the elements \( \{ p_\chi(s_\xi) \}, 1 \neq \chi \in \mathbb{Z}/m\mathbb{Z} \), by \( z_1, \ldots, z_{m-1} \). Again, there exist \( \overline{z}_1, \ldots, \overline{z}_{m-1} \) in \( K_0(\mathcal{C}^*(R \rtimes \mu)) \) with the property that for every \( 1 \leq l \leq m - 1 \),

\[
K_{inf} + K_{fin}^\sigma + \langle \overline{z}_1, \ldots, \overline{z}_l \rangle = \{ x \in K_0(\mathcal{C}^*(R \rtimes \mu)) : \exists N \in \mathbb{Z}_{>0} \text{ s.t. } Nx \in K_{inf} + K_{fin}^\sigma + \langle z_1, \ldots, z_l \rangle \}. \]

It is again clear that \( \overline{z}_1, \ldots, \overline{z}_{m-1} \) are linearly independent. We set \( K_{fin}^\mu = \langle \overline{z}_1, \ldots, \overline{z}_{m-1} \rangle \). By construction, we have \( (K_{inf} + K_{fin}^\sigma) \cap K_{fin}^\mu = \{0\} \). Thus \( K_0(\mathcal{C}^*(R \rtimes \mu)) = K_{inf} \oplus K_{fin}^\sigma \oplus K_{fin}^\mu \) as \( \text{im}(\iota_+) + \left( \sum_{(M) \in \mathcal{M}} \text{im}((\iota_M)_+) \right) \) is of finite index in \( K_0(\mathcal{C}^*(R \rtimes \mu)) \) by \((**)\) from Theorem 3.1.

4. K-THEORY FOR RING C*-ALGEBRAS

Let us recall the strategy of the previous K-theoretic computations from [Cu-Li 2] for number field without higher roots of unity. We will use the same strategy to treat the case of higher roots of unity.

The first step is to compute K-theory for the sub-C*-algebra \( \mathcal{C}^*(\{u^b\}, s_\zeta, \{e_a\}) \) of \( \mathcal{M}[R] \). Recall that \( e_a \) is the range projection of \( s_a \), i.e., \( e_a = s_a s_a^* \). This sub-C*-algebra can be identified with the inductive limit of the system given by the algebras \( \mathcal{C}^*(\{u^b\}, s_\zeta, e_a) \) for \( a \in R^\times \). Moreover, we can prove that for fixed \( a \) in \( R^\times \), the
algebra \(C^*\{u^b\}, s_\zeta, e_a\) is isomorphic to a matrix algebra over \(C^*(\{u^b\}, s_\zeta) \cong C^*(R \rtimes \mu)\). In this situation, Theorem 3.1 allows us to compute K-theory for \(C^*(\{u^b\}, s_\zeta, \{e_a\})\) using its inductive limit structure.

The next step is to use the duality theorem (§2) to pass over to the infinite adele space. The main point is to prove that the additive action of \(K\) is negligible for K-theory, i.e., that the canonical homomorphism

\[\text{induces an isomorphism on K-theory, at least rationally. This is good enough once}\]

\(\text{we can show that all the K-groups are torsion-free. At this point, we need to know}\)

\(\text{that the canonical homomorphism } C_0(\mathbb{A}_\infty) \rtimes K \to C_0(\mathbb{A}_\infty) \rtimes K \times K^\times\)

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injective on K-theory. The proof of this statement is postponed to §5.

The last step is to compute K-theory for \(C_0(\mathbb{A}_\infty) \rtimes K^\times\) using homotopy arguments and the Pimsner-Voiculescu exact sequence. As we know that \(\mathcal{A}[R]\) is Morita equivalent to \(C_0(\mathbb{A}_\infty) \rtimes K \times K^\times\), we finally obtain the K-theory for the ring \(C^*\)-algebra \(\mathcal{A}[R]\).

### 4.1. Identifying inductive limits

The first step is to compute the K-theory of \(C^*(\{u^b\}, s_\zeta, \{e_a\})\) using its inductive limit structure. Here \(C^*(\{u^b\}, s_\zeta, \{e_a\})\) is the sub-\(C^*\)-algebra of \(\mathcal{A}[R]\) generated by \(\{u^b; b \in R\}, s_\zeta\) and \(\{e_a; a \in R^\times\}\). First of all, note that for all \(a\) and \(c\) in \(R^\times\), we have \(e_a = \sum b + c \in R/\mathbb{Z}/R u^{bc}e_{acm^{-ab}}\). Just conjugate Relation II in §2 (for \(c\) in place of \(a\)), \(1 = \sum u^b e_u^{-b}\); by \(s_a\). Therefore, the \(C^*\)-algebras \(C^*(\{u^b\}, s_\zeta, e_a)\) for \(a\) in \(R^\times\) and the inclusion maps

\(\iota_{a,ac}: C^*(\{u^b\}, s_\zeta, e_a) \to C^*(\{u^b\}, s_\zeta, e_{ac})\)

form an inductive system. Here \(R^\times\) is ordered by divisibility. It is clear that the inductive limit of this system can be identified with \(C^*(\{u^b\}, s_\zeta, \{e_a\})\). Thus our goal is to compute the K-theory of \(C^*(\{u^b\}, s_\zeta, e_a)\) and to determine the structure maps \(\iota_{a,ac}\) on K-theory. Note that \(C^*(\{u^b\}, s_\zeta, e_a)\) is obtained from \(C^*(\{u^b\}, s_\zeta)\) by adding one single projection \(e_a\) and not the whole set of projections \(\{e_a\}\).

Let \(a\) and \(c\) be arbitrary elements in \(R^\times\). Choose a minimal system \(R_a\) of representatives for \(R/\mathbb{A}R\) in \(R\). “Minimal” means that for arbitrary elements \(b_1\) and \(b_2\) in \(R_a\), the difference \(b_1 - b_2\) lies in \(\mathbb{A}R\) (if and) only if \(b_1 = b_2\). We always choose \(R_a\) in such a way that \(0\) is in \(R_a\).

Using the decomposition \(R = \sqcup_{b \in R_a}(b + a\mathbb{A})\) and the inverse of the isomorphism \(\ell^2(R) \cong \ell^2(b + a\mathbb{A}); \varepsilon_r \mapsto \varepsilon_{b + ar}\), we can construct the unitary

\[\ell^2(R) = \bigoplus_{b \in R_a} \ell^2(b + a\mathbb{A}) \cong \bigoplus_{R_a} \ell^2(R)\]

Conjugation with this unitary gives rise to an isomorphism

\[\mathcal{L}(\ell^2(R)) \cong \mathcal{L}(\ell^2(R/\mathbb{A}R)) \otimes \mathcal{L}(\ell^2(R)), T \mapsto \sum_{b, b' \in R_a} e_{b, b'} \otimes (s_a^{b} u^{-b} T u^{b'} s_a)\]

Here \(e_{b, b'}\) is the canonical rank 1 operator in \(\mathcal{L}(\ell^2(R/\mathbb{A}R))\) corresponding to \(b + a\mathbb{A}\) and \(b' + a\mathbb{A}\) sending a vector \(\xi\) in \(\ell^2(R/\mathbb{A}R)\) to \((\xi, \varepsilon_{b' + a\mathbb{A}}) e_{b + a\mathbb{A}}\). In this formula, \(\{e_{b + a\mathbb{A}}; b \in R_a\}\) is the canonical orthonormal basis of \(\ell^2(R/\mathbb{A}R)\).
Let us denote the restriction of this isomorphism to $C^*(\{u^b\}, s_\zeta, e_{ac})$ by $\vartheta_{ac,c}$.

**Lemma 4.1.** For every $a$ and $c$ in $R^\times$, the image of $\vartheta_{ac,c}$ is $\mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c))$. Thus $\vartheta_{ac,c}$ induces an isomorphism
\[
C^*(\{u^b\}, s_\zeta, e_{ac}) \cong \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c)).
\]

Since $R/aR$ is always finite, we know that $\mathcal{L}(\ell^2(R/aR))$ is just a matrix algebra. So it does not matter which tensor product we choose.

**Proof.** A direct computation yields
\[
\begin{align*}
\vartheta_{ac,c}(u^b e_a u^{-b'}) &= e_{b',b} \otimes 1 \quad \text{for all } b, b' \in R_a; \\
\vartheta_{ac,c}(u^{ab}) &= 1 \otimes u^b \quad \text{for all } b \in R; \\
\vartheta_{ac,c}(\sum_{b \in R_a} u^b e_a u^{-b} s_\zeta) &= 1 \otimes s_\zeta; \\
\vartheta_{ac,c}(\sum_{b \in R_a} u^b e_{ac} u^{-b}) &= 1 \otimes e_c.
\end{align*}
\]

Our claim follows from the observation that $C^*(\{u^b\}, s_\zeta, e_{ac})$ is generated by $u^b e_a u^{-b'} (b, b' \in R_a); u^{ab} (b \in R); \sum_{b \in R_a} u^b e_a u^{-b} s_\zeta$ and $\sum_{b \in R_a} u^b e_{ac} u^{-b}$.

\[\square\]

Let us now fix minimal systems of representatives $\mathcal{R}_a$ for every $a$ in $R^\times$ as explained before the previous lemma (we will always choose $0 \in \mathcal{R}_a$). As $R/aR$ is finite for every $a$ in $R^\times$, we know that $\mathcal{L}(\ell^2(R/aR))$ is simply a matrix algebra of finite dimension. Thus we can use the previous lemma to identify $K_0(C^*(\{u^b\}, s_\zeta, e_a))$ and $K_0(C^*(\{u^b\}, s_\zeta))$ via $(\rho_{1,a})_*(\vartheta_{ac,1})_*$. Here $\rho_{c,a}$ (for $a$ and $c$ in $R^\times$) is the canonical homomorphism
\[
C^*(\{u^b\}, s_\zeta, e_c) \to \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c)); x \mapsto e_{0,0} \otimes x.
\]

**Lemma 4.2.** We have
\[
(\rho_{1,ac})_*^{-1}(\vartheta_{ac,1})_* (\iota_{a,ac})_* (\vartheta_{ac,1})_*^{-1}(\rho_{1,a})* = (\rho_{1,ac})_*^{-1}(\vartheta_{ac,1})_* (\iota_{1,c})*.
\]

In other words, under the K-theoretic identifications above, the map $(\iota_{a,ac})_*$ corresponds to $(\iota_{1,c})*$. This observation is helpful because it says that we only have to determine the homomorphisms $\iota_{1,c}$ on K-theory.

**Proof.** It is immediate that
\[
\rho_{c,a} = \vartheta_{ac,c} \circ \text{Ad}(s_a)
\]
as homomorphisms $C^*(\{u^b\}, s_\zeta, e_c) \to \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c))$. Here we mean by $\text{Ad}(s_a)$ the homomorphism $C^*(\{u^b\}, s_\zeta, e_c) \to C^*(\{u^b\}, s_\zeta, e_{ac})$; $x \mapsto s_axs_a^*$. It would be more precise to write $\text{Ad}(s_a)|_{C^*(\{u^b\}, s_\zeta, e_{ac})}$, but it will become clear from the context on which domain $\text{Ad}(s_a)$ is defined.

We know by Relation I in §2 that
\[
\text{Ad}(s_a) \circ \text{Ad}(s_c) = \text{Ad}(s_{ac}).
\]
So, using (2), we can deduce from (3) that

\[ (4) \quad \vartheta_{ac,c}^{-1} \circ \rho_{c,a} \circ \vartheta_{a,c}^{-1} \circ \rho_{1,c} = \vartheta_{ac,1}^{-1} \circ \rho_{1,ac}. \]

Moreover, \( \text{Ad} \left( s_a \right) \circ \iota_{1,c} = \iota_{a,ac} \circ \text{Ad} \left( s_a \right) \) and (2) imply

\[ (5) \quad \vartheta_{ac,c}^{-1} \circ \rho_{c,a} \circ \iota_{1,c} = \iota_{a,ac} \circ \vartheta_{a,1}^{-1} \circ \rho_{1,a}. \]

Finally, we compute

\[
\begin{align*}
(\rho_{1,ac})_*^{-1}(\vartheta_{ac,1})_* (u_{ac})_* (\vartheta_{a,c})_* (\rho_{1,a})_* & \\
(\rho_{1,c})_*^{-1}(\vartheta_{c,1})_* (\rho_{c,a})_* (\vartheta_{ac,c})_* (\iota_{a,ac})_* (\vartheta_{a,c})_* (\rho_{1,a})_* & \\
(\rho_{1,c})_*^{-1}(\vartheta_{c,1})_* (\rho_{c,a})_* (\vartheta_{ac,c})_* (\rho_{1,ac})_* (\rho_{1,c})_* (\iota_{1,c})_* & \\
= (\rho_{1,c})_*^{-1}(\vartheta_{c,1})_* (\iota_{1,c}).
\end{align*}
\]

Therefore it remains to determine

\[ (\rho_{1,c})_*^{-1}(\vartheta_{c,1})_* (\iota_{1,c})_* : K_0(C^*(\{u^b\}, \zeta)) \to K_0(C^*(\{u^b\}, \zeta)). \]

Let us denote this map by \( \eta_c \), i.e., \( \eta_c = (\rho_{1,c})_*^{-1}(\vartheta_{c,1})_* (\iota_{1,c})_* \). In conclusion, we have identified the K-theory of \( C^*(\{u^b\}, \zeta) \) with \( C^*(\{u^b\}, \zeta) \) carry over to \( C^*(\{u^b\}, \zeta) \). In the sequel, we use the same notations as in \( \S 3 \) but everything should be understood modulo this canonical isomorphism \( C^*(\{u^b\}, \zeta) \cong C^*(\{u^b\}, \zeta) \).

### 4.2. The structure maps

First of all, we can canonically identify \( C^*(\{u^b\}, \zeta) \) with \( C^*(\mathbb{R} \times \mu) \) because \( \mathbb{R} \times \mu \) is amenable. Therefore, all the results from \( \S 3 \) carry over to \( C^*(\{u^b\}, \zeta) \). In conclusion, we have identified the K-theory of \( C^*(\{u^b\}, \zeta) \) with \( C^*(\mathbb{R} \times \mu) \).

To determine \( \eta_c \), we use the decomposition \( K_0(C^*(\{u^b\}, \zeta)) = K_{inf} \oplus K_{fin} \oplus K_{fin}^{\mu} \) with the particular \( \mathbb{Z} \)-basis \( \langle \overline{g}_1, \ldots, \overline{g}_{rk_{fin}} \rangle \) and \( \{ \overline{x}_1, \ldots, \overline{x}_{m-1} \} \) of \( K_{fin}^{\mu} \) and \( K_{fin}^{\mu} \), respectively (see \( \S K(\text{group-C}) \)). Moreover, let \( [1] \in K_0(C^*(\{u^b\}, \zeta) be the K_0-class of the unit in \( C^*(\{u^b\}, \zeta) \) and denote by \( \infty \{1\} \) the subgroup of \( K_0(C^*(\{u^b\}, \zeta) \) generated by \( [1] \). As the canonical inclusion \( \mathbb{C} \cdot 1 \hookrightarrow C^*(\{u^b\}, \zeta) \) splits (a split is given by \( C^*(\{u^b\}, \zeta) \cong C^*(\mathbb{C}) \to C^*(\mathbb{C} \cdot 1) \equiv \mathbb{C} \cdot 1 \)), it is clear that \( \infty \{1\} \) is a direct summand of \( K_0(C^*(\{u^b\}, \zeta)) \), hence of \( K_{inf} \).

Furthermore, note that it suffices to determine the structure maps \( \eta_c \) for \( c \in \mathbb{Z}_{\geq 1} \) with the property that \( \prod_{i=1}^{m-1}(1 - \zeta^i) \) divides \( c \) because these elements form a cofinal set in \( \mathbb{R}^\times \) with respect to divisibility.

Our goal is to prove

**Proposition 4.3.** There exists a subgroup \( K_{inf}^\mathbb{Z} \) of \( K_{inf} \) together with a \( \mathbb{Z} \)-basis \( \{ \overline{x}_1, \ldots, \overline{x}_{rk_{inf}} \} \) of \( K_{inf}^\mathbb{Z} \) such that \( K_{inf} = \infty \{1\} \oplus K_{inf}^\mathbb{Z} \) and that with respect to the \( \mathbb{Z} \)-basis \( \{ [1], \overline{x}_1, \ldots, \overline{x}_{rk_{inf}}, \overline{g}_1, \ldots, \overline{g}_{rk_{fin}}, \overline{x}_1, \ldots, \overline{x}_{m-1} \} \) of \( K_0(C^*(\{u^b\}, \zeta)) \),
This matrix is subdivided according to the decomposition $K(K(C^*(\{u^b\})), s_\zeta)) = \langle [1] \rangle \oplus K^{\zeta}_{inj} \oplus K^{\mu}_{fin}$. Moreover, the diagonal of the box
\[
\begin{pmatrix}
\cdots & * & * \\
0 & \cdots & * \\
0 & 0 & c^n \\
0 & 0 & 1 \\
\end{pmatrix}
\]
describing the $K^{\zeta}_{inj}$-$K^{\zeta}_{inj}$-part of this matrix consists of powers of $c$ with decreasing exponents. The least exponent $\zeta$ can be $0$ only if $n$ is even, and in that case, the $0$-th power $c^0$ can appear only once on the diagonal.

The proof of this proposition consists of two parts which are treated in the following two paragraphs.

4.2.1. The infinite part.

**Lemma 4.4.** For $c \in \mathbb{Z}_{>1}$, we have $\eta_c(K_{inj}) \subseteq K_{inj}$. Moreover, there is a subgroup $K^\zeta_{inj}$ of $K_{inj}$ and a $\mathbb{Z}$-basis $\{\varpi_1, \ldots, \varpi_{K^\zeta_{inj}}\}$ of $K^\zeta_{inj}$ such that $K_{inj} = \langle [1] \rangle \oplus K^\zeta_{inj}$ and, for every $c \in \mathbb{Z}_{>1}$, $\eta_c|_{K_{inj}}$, as a map $K_{inj} \to K_{inj}$, is of the form
\[
\begin{pmatrix}
\cdots & * \\
0 & \cdots \\
0 & 0 \\
\end{pmatrix}
\]
with respect to the decomposition $K_{inj} = \langle [1] \rangle \oplus K^\zeta_{inj}$ and the chosen $\mathbb{Z}$-basis of $K^\zeta_{inj}$. Here, as in the proposition, $\zeta$ can be $0$ only if $n$ is even, and in that case, the $0$-th power $c^0$ can only appear at most once on the diagonal.

**Proof.** Let us choose a suitable $\mathbb{Z}$-basis for $K_{inj}$ and determine $\eta_c|_{K_{inj}}$. First of all, under the canonical identification $C^*(\{u^b\}, s_\zeta) \cong C^*(R \times \mu)$, the sub-$C^*$-algebra $C^*(\{u^b\})$ corresponds to $C^*(R)$. So the inclusion map $\iota : C^*(R) \hookrightarrow C^*(R \times \mu)$ corresponds to the canonical inclusion $C^*(\{u^b\}) \hookrightarrow C^*(\{u^b\}, s_\zeta)$ which we denote by $\iota$ as well. Let $\omega_1, \ldots, \omega_n$ be a $\mathbb{Z}$-basis for $R$ and let $u(i) := u^{\omega_i}$. Since $C^*(\{u^b\})$ is isomorphic to $C^*(R) \cong C^*(\mathbb{Z}^n)$ ($R$ is viewed as an additive group), a $\mathbb{Z}$-basis for $K_0(C^*(\{u^b\}))$ is given by
\[
\{[u(i_1)] \times \cdots \times [u(i_k)] : i_1 < \cdots < i_k, k \text{ even} \}.
\]
Here $\times$ is the exterior product in K-theory as described in [Hig-Roe]. Moreover, $[\cdot]$ denotes the $K_1$-class of the unitary in question.

Let $\nu_c$ be the endomorphism on $C^*(\{u^b\})$ defined by $\nu_c(u^b) = u^cb$. We have
\begin{equation}
(\theta_{c,1} \circ \iota_{1,c} \circ \iota \circ \nu_c)(u^b) = \theta_{c,1}(u^b) = 1 \otimes u^b
\end{equation}
for all $b$ in $R$. Thus $\theta_{c,1} \circ \iota_{1,c} \circ \iota \circ \nu_c = (1 \otimes \text{id}) \circ \iota$. We conclude that
\begin{align*}
\eta_c(\iota_*(\text{[u}(i_1)\text{]}_1 \times \cdots \times [u(i_k)\text{]}_1)) &= \langle \rho_1, c \rangle \iota_*(\text{[u}(i_1)\text{]}_1 \times \cdots \times [u(i_k)\text{]}_1) \\
&= c^{-k}(\rho_1, c \rangle \iota_*(\text{[u}(i_1)\text{]}_1 \times \cdots \times [u(i_k)\text{]}_1)) \\
&= c^{-k}(\rho_1, c \rangle \iota_*(\text{[u}(i_1)\text{]}_1 \times \cdots \times [u(i_k)\text{]}_1)).
\end{align*}

Now, let $H_k$ be the subgroup of $K_0(C^*(\{u^b\}))$ generated by the $K_0$-classes $[u(i_1)\text{]}_1 \times \cdots \times [u(i_k)\text{]}_1]$ for $i_1 < \cdots < i_k$ where $k$ is fixed. We have
\[ K_0(C^*(\{u^b\})) = \bigoplus_{k \geq 0 \text{ even}} H_k. \]

We claim that $\ker(\iota_*)$ is compatible with this decomposition, i.e.,
\[ \ker(\iota_*) = \bigoplus_{k \geq 0 \text{ even}} (H_k \cap \ker(\iota_*)). \]

Proof of the claim:

Let $h$ be in $\ker(\iota_*)$. We can write
\begin{equation}
h = \sum_{k \geq 0 \text{ even}} h_k
\end{equation}
with $h_k \in H_k$. We have to show that for every $k$, the summand $h_k$ lies in $\ker(\iota_*)$. Let us assume that there are at least two non-zero summands in (8), because otherwise, there is nothing to show. Now, equation (7) tells us that
\begin{equation}
\eta_c \circ \iota_* = \iota_* \circ \bigoplus_k (c^{-k} \cdot \text{id}_{H_k})
\end{equation}
on $K_0(C^*(\{u^b\})) = \bigoplus_k H_k$. Thus $\bigoplus_k (c^{-k} \cdot \text{id}_{H_k})(h) = \sum_k c^{-k} \cdot h_k$ lies in $\ker(\iota_*)$ as well. This implies
\[ \ker(\iota_*) \ni c^n h - \bigoplus_k (c^{-k} \cdot \text{id}_{H_k})(h) = \sum_{k \geq 2 \text{ even}} (c^n - c^{-k}) \cdot h_k. \]

Proceeding inductively, we obtain that for every even number $j \geq 2$,
\begin{equation}
\sum_{k \geq j \text{ even}} (c^n - c^{-k})(c^{n-2} - c^{-k}) \cdots (c^{n-j+2} - c^{-k}) \cdot h_k
\end{equation}
lies in $\ker(\iota_*)$. Taking $j$ to be the highest index for which the summand $h_j$ in (8) is not zero, the term in (10) will be a non-zero multiple of the highest term in (8). As both $K_0(C^*(\{u^b\}))$ and $K_0(C^*(\{u^b\}, s_c))$ are free abelian, we conclude that the highest term itself must lie in $\ker(\iota_*)$. Working backwards, we obtain that for every $k$, the summand $h_k$ lies in $\ker(\iota_*)$. This proves our claim.

Now, for every $k$, $H_k \cap \ker(\iota_*) = \ker(\iota_*/H_k)$ is a direct summand of $H_k$ because $K_0(C^*(\{u^b\}, s_c))$ is free abelian. Thus we can choose subgroups $I_k$ of $H_k$ so that
\[ H_k = I_k \oplus (H_k \cap \ker(\iota_*)). \]
As \( \ker(\iota_\nu) = \bigoplus_k (H_k \cap \ker(\iota_\nu)) \), we have \( K_0(C^*(\{u^b\})) = (\bigoplus_k I_k) \oplus \ker(\iota_\nu) \).

We can choose a \( \mathbb{Z} \)-basis for \( \bigoplus_k I_k \) in \( \bigcup_k H_k \). As \( H_0 \cap \ker(\iota_\nu) = \{0\} \), we have \( I_0 = H_0 = \{1\} \) so that we can let \([1]\) be a basis element. Moreover, \( H_n \) is non-trivial only if \( n \) is even, and in that case \( \text{rk}(H_n) = 1 \) so that there is at most one basis element in \( H_n \).

Now \( \iota_\nu \) maps \( \bigoplus_k I_k \) isomorphically into \( \text{im}(\iota_\nu) \subseteq K_0(C^*(\{u^b\}, s_\zeta)) \), so that the \( \mathbb{Z} \)-basis of \( \bigoplus_k I_k \) chosen above is mapped to a \( \mathbb{Z} \)-basis of \( \text{im}(\iota_\nu) \). By (10), we know that if we order this \( \mathbb{Z} \)-basis in the right way (corresponding to the index \( k \)), we obtain that \( \eta_{\nu}|_{\text{im}(\iota_\nu)} \) as an endomorphism of \( \text{im}(\iota_\nu) \) is given by

\[
\begin{pmatrix}
  c^n & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & c^j
\end{pmatrix}
\]

where the exponents of \( c \) on the diagonal are monotonously decreasing. The entry \( c^n \) corresponds to the basis element \([1]\), and \( c^0 \) can only appear at most once on the diagonal (if it appears, it has to be in the lower right corner according to our ordering). We enumerate this \( \mathbb{Z} \)-basis of \( \text{im}(\iota_\nu) \) by \( x_0, \ldots, x_{\text{rk}_{in,f}} \) according to our ordering, so \( x_0 = [1] \).

By construction (see (11)), \( \langle x_0, \ldots, x_{\text{rk}_{in,f}} \rangle = \text{im}(\iota_\nu) \) is of finite index in \( K_{in,f} \), so that we can choose a \( \mathbb{Z} \)-basis \( \{\tau_0, \ldots, \tau_{\text{rk}_{in,f}}\} \) of \( K_{in,f} \) with the property that

\[
\langle \tau_0, \ldots, \tau_j \rangle = \{x \in K_{in,f} \mid \exists N \in \mathbb{Z}_{>0} \text{ with } Nx \in \langle \tau_0, \ldots, \tau_j \rangle \}
\]

for every \( 0 \leq j \leq \text{rk}(K_{in,f}) - 1 = \text{rk}_{in,f} \). In particular, we have \( \tau_0 = [1] \). It then follows that \( \eta_{\nu}(K_{in,f}) \subseteq K_{in,f} \) and that with respect to the \( \mathbb{Z} \)-basis \( \{\tau_j\} \), the matrix describing \( \eta_{\nu}|_{K_{in,f}} \) as an endomorphism of \( K_{in,f} \) is of the form

\[
\begin{pmatrix}
  c^n & * & \cdots & 0 \\
  * & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & c^j
\end{pmatrix}
\]

Recall that \( \zeta \) can be 0 only if \( n \) is even, and in that case, the 0-th power \( c^0 \) can only appear at most once on the diagonal. Now set \( K_{in,f}^\zeta := \langle \tau_1, \tau_2, \ldots, \tau_{\text{rk}_{in,f}}^\zeta \rangle \). Then we have \( K_{in,f} = \langle [1] \rangle \oplus K_{in,f}^\zeta \) by construction, and \( \tau_1, \tau_2, \ldots, \tau_{\text{rk}_{in,f}}^\zeta \) is a \( \mathbb{Z} \)-basis of \( K_{in,f}^\zeta \) with the desired properties.

This first lemma settles the \( K_{in,f} \)-part.

Up to now, we have only used that \( c \) is an integer bigger than 1, the extra condition that \( \prod_{i \mid \text{im}, 1 \leq i \leq m} (1 - c^i) \) divides \( c \) was not used in our arguments up to this point.

But for the finite part, this condition plays a crucial role.

\[4.2.2. \text{ The finite part.}\]
Lemma 4.5. Assume that $c$ is an integer bigger than 1 and that $\prod_{i|m, 1 \leq i < m} (1 - \zeta^i)$ divides $c$. Then
\[
\eta_c([p_\chi(s_\xi)]) \in [p_\chi(s_\xi)] + \sum_{(\mu) \neq (M) \in M} \text{im}((t_{M})_*) \text{ for all } \chi \in \mathbb{Z}/m\mathbb{Z}
\]
and
\[
\eta_c \left( \sum_{(\mu) \neq (M) \in M} \text{im}((t_{M})_*) \right) \in ([1]).
\]

Proof. Let $M$ be a maximal finite subgroup of $R \rtimes \mu$, and choose a generator $(b, \zeta') \in R \rtimes \mu$ of $M$. Our aim is to compute $\eta_c([p_\chi(u_b^s s_\xi)])$ (with $\chi \in \mathbb{Z}/m\mathbb{Z}$). By definition, $\eta_c = (\rho_{1,c})^{-1}(\vartheta_{c,1})_*(t_{1,c})_*$. So we have to examine $(\vartheta_{c,1} \circ t_{1,c})(u_b^s s_\xi)$. Take two elements $d, d'$ in the system $\mathcal{R}_c$ of representatives for $R/cR$. The $(d, d')$-th entry of $(\vartheta_{c,1} \circ t_{1,c})(u_b^s s_\xi)$ is given by
\[
s_{c}^{*} u^{-d} u_b^s s_{c}^{*} u^{d'} s_{c}^{*} = \begin{cases} s_{c}^{*} u^{-d + b + \zeta' d'} s_{c}^{*}, & \text{if } -d + b + \zeta' d' \notin cR, \\ 0, & \text{if } -d + b + \zeta' d' \in cR. \end{cases}
\]
Therefore, the matrix $(\vartheta_{c,1} \circ t_{1,c})(u_b^s s_\xi)$ has exactly one non-zero entry in each row and column. In other words, for fixed $d$ in $\mathcal{R}_c$, there exists exactly one $d' \in \mathcal{R}_c$ with $-d + b + \zeta' d' \in cR$, namely the element in $\mathcal{R}_c$ which represents the coset $\zeta^{-1}(d - b) + cR$. There is only one such element because $\mathcal{R}_c$ is minimal.

Moreover, $u_b^s s_\xi$ is a cyclic element: Its $\frac{m}{\zeta'}$-th power is 1.

These two observations imply that $\frac{m}{\zeta'} \sum_{j=0}^{m-1} \chi(j + \frac{m}{\zeta'} \mathbb{Z})(u_b^s s_\xi)^j = p_\chi(u_b^s s_\xi)$ can be decomposed into irreducible summands, and each of these summands has to be a projection. "Irreducible" means that once we apply $(\vartheta_{c,1} \circ t_{1,c})$, we obtain an irreducible matrix. Thus up to conjugation by a permutation matrix, $(\vartheta_{c,1} \circ t_{1,c})(p_\chi(u_b^s s_\xi))$ is of the form
\[
\begin{pmatrix} p_1 & 0 \\ p_2 & \ddots \\ 0 & \ddots \end{pmatrix}
\]
where the $p_i$ are projections of certain sizes. Of course, conjugation by a permutation matrix does not have any effect in K-theory. This means that we obtain
\[
\eta_c([p_\chi(u_b^s s_\xi)]) = (\rho_{1,c})^{-1}([p_1] + [p_2] + \cdots)
\]
where $\rho_{1,c}$ is the homomorphism
\[
C^*(\{u_b^s\}, s_c) \rightarrow \mathcal{L}(\ell^2(R/cR)) \otimes C^*(\{u_b^s\}, s_c); \; x \mapsto e_{0,0} \otimes x.
\]
Here $\mathcal{L}(\ell^2(R/cR)) \cong M_{c^\infty}(\mathbb{C})$ and $e_{0,0}$ is a minimal projection. So it remains to find out what these irreducible summands $p_i$ give in K-theory.

First of all, we look at the case $b = 0, i = 1$, i.e., we consider $p_\chi(s_\xi)$. Irreducible summands of size 1 must be of the form $p_\chi(u_b^s s_\xi)$. What we want to show now is that there is only one 1-dimensional summand which gives the class of $p_\chi(s_\xi)$. To do so, we take a 1-dimensional summand corresponding to the position $d$ (for some $d$ in $\mathcal{R}_c$). The $(d, d)$-th entry of $(\vartheta_{c,1} \circ t_{1,c})(s_\xi)$ is given by $u_c^{\xi^{-1}(\zeta d - d)} s_\xi$. By
Theorem 4.3] the corresponding projection (i.e., $p_\chi(u^{e-1}(\zeta d - d)s_\zeta)$) gives a $K_0$-class in $\text{im}(\iota_M)_*$ if and only if the subgroups $\langle c^{-1}(\zeta d - d), \zeta \rangle$ and $\langle (0, \zeta) \rangle$ of $R \rtimes \mu$ are conjugate. This is equivalent to $c^{-1}(\zeta d - d) \in (\zeta - 1) \Leftrightarrow d \in cR$. But as $R_c$ is minimal, this happens for exactly one element in $R_c$ (by our convention, this element has to be 0, but this is not important at this point). Moreover, if $d$ lies in $cR$, then $[p_\chi(u^{e-1}(\zeta d - d)s_\zeta)] = [p_\chi(s_\zeta)]$ in $K_0$. So from the 1-dimensional summands we obtain in $K_0$ exactly once the class $[p_\chi(s_\zeta)]$ and some other classes in $\sum_{(\mu) \notin (M) \in M} \text{im}((\iota_M)_*)$.

It remains to examine higher dimensional summands. We want to show that all the higher dimensional summands give rise to $K_0$-classes in $\sum_{(\mu) \notin (M) \in M} \text{im}((\iota_M)_*)$. Let us take a summand of size $j$ with $j > 1$. This means that the $j$-th power of $(\vartheta_{e,1} \circ \iota_{1,e})(s_\zeta)$ has a non-zero diagonal entry, say at the $(d,d)$-th position. This entry is $u^{e-1}(\zeta d - d)s_\zeta^{ij}$. Now we prove a result in a bit more generality than actually needed at this point. But later on, we will come back to it.

**Lemma 4.6.** Assume that for an irreducible summand of $(\vartheta_{e,1} \circ \iota_{1,e})(p_\chi(u^b s_\zeta^{ij}))$, $j \in \mathbb{Z}_{>1}$ is the smallest number such that the $j$-th power of this summand has non-zero diagonal entries. Let one of these non-zero diagonal entries be $u^b s_\zeta^{ij}$ for some $b$ in $R$. Then the $K_0$-class of this summand coincides with $[p_\chi(u^b s_\zeta^{ij})]$ where $\tilde{\chi}$ is the restriction of $\chi \in \langle \zeta \rangle$ to $\langle \zeta^j \rangle$.

**Proof of Lemma 4.6** Up to conjugation by a permutation matrix, the irreducible summand of $u^b s_\zeta$ we are considering is of the form

\[
\begin{pmatrix}
0 & x_j \\
x_1 & \ddots & \ddots \\
0 & \cdots & x_{j-1} & 0
\end{pmatrix}
\]

All the entries lie in $C^*(\{u^b\}, s_\zeta)$.

The $j$-th power is given by

\[
\begin{pmatrix}
x_jx_{j-1} \cdots x_2x_1 & 0 \\
0 & x_1x_jx_{j-1} \cdots x_2 & 0 \\
0 & \cdots & x_{j-1}x_{j-2} \cdots x_1x_j
\end{pmatrix}
\]

By assumption, $x_jx_{j-1} \cdots x_2x_1 = u^b s_\zeta^{ij}$. Then the irreducible summand of

\[
p_\chi(u^b s_\zeta) = \frac{1}{m!} \sum_{k=0}^{m-1} \chi(j)(u^b s_\zeta)^k
\]

is given by

\[
\begin{pmatrix}
p_\chi(u^b s_\zeta^{ij}) & p_\chi(u^b s_\zeta^{ij}) \cdot x_1^* & \cdots \\
x_1 \cdot p_\chi(u^b s_\zeta^{ij}) & x_1 \cdot p_\chi(u^b s_\zeta^{ij}) \cdot x_1^* & \cdots \\
\vdots & \vdots & \ddots \\
x_{j-1} \cdots x_1 \cdot p_\chi(u^b s_\zeta^{ij}) & x_{j-1} \cdots x_1 \cdot p_\chi(u^b s_\zeta^{ij}) \cdot x_1^* & \cdots
\end{pmatrix}
\]
The $k$-th column is given by the product of the first column with $(x_{k-1} \cdots x_1)^*$ from the right.

But then,

$$
\begin{pmatrix}
p_{\hat{\xi}}(u^b s_{\xi^j}) & 0 & \ldots \\
\vdots & \ddots & 0 \\
x_{j-1} \cdots x_1 \cdot p_{\hat{\xi}}(u^b s_{\xi^j}) & 0 & \ldots
\end{pmatrix}
$$

is a partial isometry with entries in $C^*(\{u^b\}, s_{\xi})$ whose range projection is precisely the irreducible summand from above and whose support projection is

$$
\begin{pmatrix}
p_{\hat{\xi}}(u^b s_{\xi^j}) & 0 & \ldots \\
\vdots & \ddots & 0 \\
0 & \ldots & 0
\end{pmatrix}.
$$

This proves Lemma 4.6.

**Corollary 4.7.** If in the situation of Lemma 4.6, we have $j = m$, i.e., $u^b s_{\xi^j} = 1$, then the corresponding irreducible summand of $(\vartheta_{c,1} \circ t_{1,c})(p_b (u^b s_{\xi^j}))$ gives the $K_0$-class [1].

Now let us continue the proof of Lemma 4.6. We go back to the higher dimensional summands of $(\vartheta_{c,1} \circ t_{1,c})(s_{\xi})$. We were considering the $(d, d)$-th position with entry $u^{c^{-1}(\zeta' d - d)}s_{\xi^j}$. Lemma 4.6 tells us that this irreducible summand gives $[p_{\hat{\xi}}(u^{c^{-1}(\zeta' d - d)}s_{\xi^j})]$. By Theorem 3.1, this $K_0$-class lies in $	ext{im} \left( \langle t_{1,\mu} \rangle \right)$ if and only if the corresponding subgroup $\langle (c^{-1}(\zeta' d - d), \zeta') \rangle$ is conjugate to a subgroup of $\langle \{0, \zeta'\} \rangle$. In case $\zeta' \neq 1$, this happens if and only if $c^{-1}(\zeta' d - d) \in (\zeta' - 1) \Leftrightarrow d \in cR \Leftrightarrow d = 0$ by our choice of $R_c$. But for $d = 0$, the summand we get is of size 1 (see above). This contradicts $j > 1$. If $\zeta' = 1$, then we obtain a projection whose $K_0$-class is [1] by Corollary 4.7.

This proves $\eta_c([p_{\hat{\xi}}(s_{\xi})]) = [p_{\hat{\xi}}(s_{\xi})] + \sum_{(\mu) \neq (M) \in M_c} \text{im} \left( \langle t_{1,\mu} \rangle \right)$ for all $\xi \in \mathbb{Z}/m\mathbb{Z}$.

It remains to prove $\eta_c \left( \sum_{(\mu) \neq (M) \in M_c} \text{im} \left( \langle t_{1,\mu} \rangle \right) \right) \in \langle [1] \rangle$. Take a maximal finite subgroup $M$ with $(M) \neq (\mu)$. Let $(b, \zeta')$ be a generator of $M$, and consider the element $[p_b (u^b s_{\xi^j})]$. How do the irreducible summands of $(\vartheta_{c,1} \circ t_{1,c})(p_b (u^b s_{\xi^j}))$ look like? We claim that each of these summands must have size $m$. To show this, let $j$ be the size of such a summand. This means that there exists $d \in R_c$ such that

$$
\begin{align*}
s_c^* u^{-d} (u^b s_{\xi^j})^j u^d s_c &= s_c^* u^{-d} u^b \xi_{ij}^{c^{-1}(\zeta' d - d)} s_{\xi^j} u^d s_c \\
&= s_c^* u^{-d} \xi_{ij}^{\frac{1}{1-\zeta'} b + \zeta' d} s_{\xi^j} \neq 0.
\end{align*}
$$

This happens if and only if $-d + \frac{1}{1-\zeta'} b + \zeta' d \in cR$. Now, if $\zeta'^j \neq 1$, then we can proceed as follows: As $\prod_{i=1}^{j} (1 - \zeta'^i)$ divides $c$ by assumption, we know that $1 - \zeta'^i$ divides $cR$, so that $-d + \frac{1}{1-\zeta'} b + \zeta' d \in cR$ implies that $b$ lies in $(1 - \zeta'^i)R$. But this is a contradiction to $(M) \neq (\mu)$. Thus we must have $\zeta'^j = 1$, which by minimality of $j$ implies $j = \frac{m}{d}$, as claimed.
Therefore all these irreducible summands give the \( K_0 \)-class \([1]\) (see Corollary 4.7). This proves that

\[
\eta_c \left( \sum_{(\mu) \neq (M) \in \mathcal{M}} \text{im} \left( (\iota_M)_* \right) \right) \in ([1]).
\]

\[\square\]

**Corollary 4.8.** With respect to the \( \mathbb{Z} \)-basis \( \overline{y}_1, \ldots, \overline{y}_{rk}_{fin} \) and \( \overline{x}_1, \ldots, \overline{x}_{m-1} \) of \( K_{fin}^c \) and \( K_{fin}^\mu \), respectively, and with respect to the \( \mathbb{Z} \)-basis \( \left\{ [1], \overline{x}_1, \ldots, \overline{x}_{rk}_{fin}, \overline{y}_1, \ldots, \overline{y}_{rk}_{fin}, \overline{x}_1, \ldots, \overline{x}_{m-1} \right\} \) of \( K_0(C^*\{u^b\}, s_\zeta) \), for every \( c \in \mathbb{Z}_{>1} \) with the property that \( \prod_{1 \leq i < m} (1 - \zeta_i) \) divides \( c \), we have that \( \eta_c \mid_{K_{fin}^c \oplus K_{fin}^\mu} : K_{fin}^c \oplus K_{fin}^\mu \to K_0(C^*\{u^b\}, s_\zeta) \) is of the form

\[
\begin{pmatrix}
* & * \\
* & * \\
0 & 1 \\
0 & \ddots \\
0 & 1
\end{pmatrix}.
\]

**Proof.** This follows from Lemma 4.5 and the way the basis elements \( \overline{y}_1, \ldots, \overline{y}_{rk}_{fin} \) and \( \overline{x}_1, \ldots, \overline{x}_{m-1} \) were chosen (see § 3). \[\square\]

With this corollary, together with Lemma 4.4 we have completed the proof of Proposition 4.3.

### 4.3. K-theory for a sub-C*-algebra

Now we can compute the K-theory of \( C^*\{u^b\}, s_\zeta, \{e_a\} \). We can also determine \( \text{Ad} (s_c) = s_c \cup s_c^* \) on K-theory.

**Proposition 4.9.** We have

\[
K_0(C^*\{u^b\}, s_\zeta, \{e_a\}) \cong \mathbb{Q}^{k(K_{inf}) - \delta} \oplus \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}.
\]

Here \( \delta = 1 \) if for all \( c \in \mathbb{Z}_{>1} \) divisible by \( \prod_{1 \leq i < m} (1 - \zeta_i) \), the least exponent of \( c \) in the matrix describing \( \eta_c \) (see Proposition 4.3) is 0. Otherwise \( \delta = 1 \).

\( K_1(C^*\{u^b\}, s_\zeta, \{e_a\}) \) vanishes.

Moreover, there exists a \( \mathbb{Q} \)-basis of \( \mathbb{Q}^{k(K_{inf}) - \delta} \) and a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1} \) such that, for all \( c \in \mathbb{Z}_{>1} \) with \( \prod_{1 \leq i < m} (1 - \zeta_i) \) dividing \( c \), the homomorphism \( \text{Ad} (s_c) \)
is of the form
\[
\begin{pmatrix}
\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c^{-n} & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
\]
if $\delta = 1$

and
\[
\begin{pmatrix}
\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c^{-n} & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
\]
if $\delta = 0$.

Note that in the first box on the diagonal of these matrices, all the diagonal entries are always strictly less than 1.

\textbf{Proof.} We know that $C^*(\{u_b\}, s_\zeta, \{e_a\})$ can be identified with the inductive limit obtained from the C*-algebras $C^*(\{u_b\}, s_\zeta, e_a)$, for $a \in R^\times$ and the inclusion maps $\iota_{a, ac}$: $C^*(\{u_b\}, s_\zeta, e_a) \to C^*(\{u_b\}, s_\zeta, e_{ac})$. Using continuity of K-theory, together with Lemma 4.1 and Lemma 4.2, we obtain for $i = 0, 1$:

\[K_i(C^*(\{u_b\}, s_\zeta, \{e_a\})) \cong \lim_{\rightarrow c} K_i(C^*(\{u_b\}, s_\zeta), \eta_c).\]

From this, it is immediate that $K_1(C^*(\{u_b\}, s_\zeta, \{e_a\}))$ vanishes by $(***)$ in Theorem 3.1. Moreover, the description for $K_0(C^*(\{u_b\}, s_\zeta, \{e_a\}))$ can be deduced from the description of $\eta_c$ in Proposition 4.3.

Concerning the description of $(\text{Ad } (s_\zeta))_*$, let us explain the case $\delta = 1$. The case $\delta = 0$ is similar. We observe that $(\text{Ad } (s_\zeta))_*$ is given by the inverse of the homomorphism on

\[K_0(C^*(\{u_b\}, s_\zeta, \{e_a\})) \cong \lim_{\rightarrow c} K_0(C^*(\{u_b\}, s_\zeta), \eta_c).\]

induced by $\eta_c$. This follows from (2). We obtain that there exists a $Q$-basis of $Q^{rk(K_{1, i})-\delta}$ and a $Z$-basis of $Z^\delta \oplus Z^{m-1}$ such that $\text{Ad } (s_\zeta)$ is of the form

\[
\begin{pmatrix}
\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c^{-n} & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
\]
Modifying the $\mathbb{Z}$-basis if necessary, we can find a new $\mathbb{Z}$-basis for $\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}$ such that the two boxes in the upper right corner of (13) vanish. □

4.4. Passing over to the infinite adele space. We can now compute K-theory for certain crossed products involving the profinite completion of $R$. Using the duality theorem, we are then able to pass over to the infinite adele space.

**Corollary 4.10.** We have

\begin{align*}
K_0(C(\mathbb{R}) \rtimes R \rtimes \mu) &\cong Q^{rk(K_{nf})-\delta} \oplus \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1} \\
K_1(C(\mathbb{R}) \rtimes R \rtimes \mu) &\cong \{0\}
\end{align*}

Proof. These results follow from the duality theorem (see §2 and Proposition 4.9). □

For the next result, we need Proposition 5.1.

**Corollary 4.11.** With respect to the same bases as in Proposition 4.9, we must have that $\text{Ad}(s_c)$ is of the form

\[
\begin{pmatrix}
    c^{-n} & & & * \\
    & \ddots & & \\
    0 & \ddots & \ddots & \\
    0 & & 0 & \text{id}_{\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}}
\end{pmatrix}
\]

for all $c \in \mathbb{Z}_{>1}$ divisible by $\prod_{1 \leq i < m}(1 - \zeta^i)$. Moreover, if $n$ is even, $\delta$ must be 1.

Proof. If $n = [K : \mathbb{Q}]$ is odd, $\delta$ must vanish and $m$ must be 2 as $K$ admits an embedding into $\mathbb{R}$ so that $\mu$ must be $\{\pm 1\}$. So in that case, there is nothing to prove.

Now let us consider the case that $n = [K : \mathbb{Q}]$ is even. First of all, we know that under the canonical isomorphism $C(\mathbb{R}) \rtimes R \rtimes \mu \cong C^*\left(\{u^b\}, \{s_c\}, \{e_a\}\right)$, the endomorphism $\beta_c^{(fm)}$ of $C(\mathbb{R}) \rtimes R \rtimes \mu$ induced by multiplication with $c$ corresponds to $\text{Ad}(s_c)$. From Proposition 4.9 we see that $\text{rk}(\ker(\text{id} - (\text{Ad}(s_c)_*))) \leq m$ and that we can only have equality if $\delta = 1$ and $\text{Ad}(s_c)$ is the identity on $\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}$ in Proposition 4.9. Thus the same holds for $\text{rk}(\ker(\text{id} - (\beta_c^{(fm)}))_*)$.

As against that, we know that $C_0(\mathbb{A}_\infty) \rtimes K \sim M C(\mathbb{R}) \rtimes R$ in a $R^\times$-equivariant way (see §B). Therefore, we can identify $K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu)$ with $K_*(C(\mathbb{R}) \rtimes R \rtimes \mu)$ so that $(\beta_c)_*$ is the automorphism of $C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ induced by multiplication with $c$ corresponds to $(\beta_c^{(fm)})_*$. Thus $\text{rk}(\ker(\text{id} - (\beta_c^{(fm)})_*)) = \text{rk}(\ker(\text{id} - (\beta_c)_*))$.

But the canonical homomorphism $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ maps into $\ker(\text{id} - (\beta_c^{(fm)}))$,
(βc)∗ in K0 since βc is homotopic to the identity on C0(ℍ∞) × μ (recall that n is even). Moreover, by Proposition 4.11 we know that this canonical homomorphism is injective on K0. As K0(C0(ℍ∞) × μ) ≤ K0(C∗(μ)) ≤ Zm by equivariant Bott periodicity (see [Bla, Theorem 20.3.2], n is even), we conclude that rk (ker (id − (βc)∗)) ≥ m. So rk (ker (id − (βc)∗)) must be m, and our assertion follows. □

Corollary 4.12. If K has higher roots of unity (|μ| > 2), then

\[ K_0(C_0(ℍ∞) × K × μ) \cong \mathbb{Q}^{rk(K_{n-1})−1} ⊕ \mathbb{Z}^m \quad \text{and} \quad K_1(C_0(ℍ∞) × K × μ) \cong \{0\}. \]

Moreover, there exists a Q-basis for \( \mathbb{Q}^{rk(K_{n-1})−1} \) and a Z-basis for \( \mathbb{Z}^m \) such that for \( c \) in \( \mathbb{Z}_{≥1} \) divisible by \( ∏_{i=|m|,1≤i<m}(1−ζ^i) \), \((βc)∗ : K_0(C_0(ℍ∞) × K × μ) → K_0(C_0(ℍ∞) × K × μ)\) is of the form

\[
\begin{pmatrix}
    c^{-n} & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 0
\end{pmatrix}
\]

(18)

Proof. \(|μ| > 2\) implies that n is even. So the second statement in Corollary 4.11 tells us that \( δ = 1 \). Thus our first two statements about \( K_0 \) and \( K_1 \) follow from (10) and (17). Furthermore, we can identify \( K_∗(C_0(ℍ∞) × K × μ) \) with \( K_∗(C(\overline{R}) × R × μ) \) so that \((βc)∗\) corresponds to \((βc_{λ_{fin}})∗\). We can also identify \( C(\overline{R}) × R × μ \) with \( C∗(\{a^b, s_e, \{e_a\}\}) \) so that \( βc_{λ_{fin}} \) corresponds to \( Ad(s_e) \). Thus our second statement about \((βc)∗\) follows from Corollary 4.11. □

Corollary 4.13. If K has higher roots of unity (|μ| > 2), then we have for all \( c \) in \( \mathbb{Z}_{≥1} \) divisible by \( ∏_{i=|m|,1≤i<m}(1−ζ^i) \) that \( K_i(C_0(ℍ∞) × K × (μ × (c))) \cong \mathbb{Z}^m \) (i = 0, 1).

Proof. Just plug in the results from the last corollary into the Pimsner-Voiculescu sequence for \( C_0(ℍ∞) × K × (μ × (c)) \cong (C_0(ℍ∞) × K × μ) × βc Z. \) □

4.5. End of proof. We are now ready to prove our main result. First of all, let us fix one integer \( c > 1 \) with the property that \( ∏_{i=|m|,1≤i<m}(1−ζ^i) \) divides \( c \). In addition, we can choose \( c_1, c_2, \ldots \) in \( K^x \) such that \( c, c_1, c_2, \ldots \), are free generators of a free abelian subgroup \( Γ \) of \( K^x \) with \( K^x = μ × Γ \). Let \( Γ_j := \langle c, c_1, \ldots, c_j \rangle \) \( (Γ_0 = \langle c \rangle) \).

Proposition 4.14. The canonical homomorphism

\[ C_0(ℍ∞) × (μ × Γ_j) → C_0(ℍ∞) × K × (μ × Γ_j) \]

is a rational isomorphism for all \( i ≥ 0 \). Moreover, \( β_{c_{i+1}} \), the automorphism induced by multiplication with \( c_{i+1} \), is the identity on \( K_∗(C_0(ℍ∞) × K × (μ × Γ_j)) \).

Here \( K_∗ \) stands for the \( \mathbb{Z}/2\mathbb{Z} \)-graded abelian group \( K_0 \oplus K_1 \).
Corollary 4.15. settles the case

\[ β \in \text{such a way that the map on K-theory induced by the canonical homomorphism} \]

\[ (\beta) \text{ follows that} \]

\[ C \xrightarrow{C \times (\mu \times \Gamma_0)} K \]

Now we know that \( \beta \) implies \( (\beta) \xrightarrow{\text{isomorphism when we restrict its image to the copy of}} \]

\[ K \]

By Proposition 5.1, we know that \( C \xrightarrow{\text{injective map on}} \]

\[ C \]

and \( C \xrightarrow{\text{is homotopic to the identity on}} \]

\[ C \xrightarrow{\text{on}} \]

\[ \text{Comparing the ranks, we deduce that this injective map must be a rational isomorphism when we restrict its image to the copy of} \]

\[ K \]

Since this copy is isomorphic to \( K_1(C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_0)) \) (for \( i = 0, 1 \)) via the maps in the second exact sequence \( \text{(19)} \), we obtain that the canonical homomorphism \( C_0(\mathbb{A}_\infty) \times (\mu \times \Gamma_0) \xrightarrow{\text{induces a rational isomorphism on}} \]

\[ K \]

Now we know that \( \beta_{c_i} \) is homotopic to the identity on \( C_0(\mathbb{A}_\infty) \times (\mu \times \Gamma_0) \). This implies \( (\beta_{c_i})_* = \text{id on} \)

\[ K \]

As the canonical homomorphism \( C_0(\mathbb{A}_\infty) \times (\mu \times \Gamma_0) \xrightarrow{\text{induces a rational isomorphism on}} \]

\[ K \]

we know that \( (\beta_{c_i})_* \otimes \text{id} \text{ on} \)

\[ K \]

But as \( K_*(C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_0)) \) is free abelian (see Corollary 4.13, \( \Gamma_0 = (\gamma) \)), it follows that \( (\beta_{c_i})_* \) must be the identity on \( K_*(C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_0)) \). This settles the case \( j = 0 \).

The remaining induction step is proven in a similar way. We just have to use that \( \beta_{c_{j+1}} \) is homotopic to the identity on \( C_0(\mathbb{A}_\infty) \times (\mu \times \Gamma_j) \). \( \square \)

Corollary 4.15. For every \( i \in \mathbb{Z}_{\geq 0}, \) we can identify

\[ K_*(C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_j)) \text{ with} \]

\[ K_0(C^*\{(\mu)\}) \otimes \mathbb{Z} \Lambda^*(\Gamma_j) \]

in such a way that the map on K-theory induced by the canonical homomorphism

\[ C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_j) \rightarrow C_0(\mathbb{A}_\infty) \times K \times (\mu \times \Gamma_{j+1}) \]

corresponds to the canonical map

\[ K_0(C^*\{(\mu)\}) \otimes \mathbb{Z} \Lambda^*(\Gamma_j) \rightarrow K_0(C^*\{(\mu)\}) \otimes \mathbb{Z} \Lambda^*(\Gamma_{j+1}) \]
induced by the inclusion $\Gamma_j \hookrightarrow \Gamma_{j+1}$ for all $j \in \mathbb{Z}_{\geq 0}$.

Proof. This follows inductively on $j$ using $(\beta_{j+1})_* = \id$ on $K_\ast(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_j))$ (see Proposition 4.13) and the Pimsner-Voiculescu exact sequence. The induction starts with Corollary 4.13.

Finally, we arrive at

$$K_\ast(C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times \cong \varinjlim K_\ast(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_j))$$

$$\cong \varinjlim K_0(C^\ast(\mu)) \otimes_\mathbb{Z} \Lambda^\ast(\Gamma_j) \cong K_0(C^\ast(\mu)) \otimes_\mathbb{Z} \Lambda^\ast(\Gamma).$$

This completes the proof of our main result, Theorem 1.1.

Proof of Corollary 5.1. By [Cu-Li1, Theorem 3.6], the ring $C^\ast$-algebras of rings of integers are simple and purely infinite, and by Corollaries 3 and 4 in [Li], these ring $C^\ast$-algebras are nuclear and satisfy the UCT. Moreover, these ring $C^\ast$-algebras are obviously unital and separable, and it is easy to see that for these algebras, the class of the unit in $K_0$ vanishes. Thus [Ror] Theorem 8.4.1 (iv) tells us that two such ring $C^\ast$-algebras are isomorphic if and only if their $K$-groups are isomorphic.

Now our corollary follows from Theorem 1.1.

□

Remark 4.16. In [Cu-Li2, Remark 6.6], it was observed that the same ideas which lead to the duality theorem also yield

$$C_0(\mathbb{A}) \rtimes K \rtimes K^\times \sim_M C^\ast(K \rtimes K^\times).$$

With the same strategy as in the proof of Theorem 1.1, we can now compute $K_\ast(C^\ast(K \rtimes K^\times))$. We start with computing $K_\ast(C^\ast(K \rtimes \mu))$. To this end, we write $C^\ast(K \rtimes \mu)$ as an inductive limit where all the $C^\ast$-algebras are given by $C^\ast(R \rtimes \mu)$ and the connecting maps are induced by multiplication with elements from $R^\times$.

We can then use Theorem 1.1 to determine the corresponding inductive limit in $K$-theory. To complete our computation, we proceed in an analogous manner as in Paragraph 4.3. We choose a free abelian subgroup $\Gamma$ of $K^\times$ such that $K^\times = \mu \rtimes \Gamma$ and then use the Pimsner-Voiculescu sequence iteratively. As a final result, we obtain that the canonical homomorphism $C^\ast(K^\times) \to C^\ast(K \rtimes K^\times)$ induces an isomorphism on $K$-theory. Thus, for every number field $K$, we obtain

$$K_\ast(C_0(\mathbb{A}) \rtimes K \rtimes K^\times) \cong K_\ast(C^\ast(K \rtimes K^\times)) \cong K_\ast(C^\ast(K^\times)) \cong K_0(C^\ast(\mu)) \otimes_\mathbb{Z} \Lambda^\ast(\Gamma).$$

5. Injectivity of certain inclusions on $K$-theory

We want to prove

Proposition 5.1. For every number field $K$, the homomorphism

$$K_0(C_0(\mathbb{A}_\infty) \rtimes \mu) \to K_0(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu)$$

induced by the canonical map $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ is injective.

Recall that $K \rtimes \mu$ acts on $C_0(\mathbb{A}_\infty)$ via affine transformations as in § 2. This proposition is needed in the proof of Theorem 1.1 more precisely, it is needed in
the proofs of Corollary 4.11 and Proposition 4.14 We have postponed the proof of this proposition until now because it is independent from the previous sections.

5.1. Induction and restriction. In this section let G be a discrete group and let A be a G-C*-algebra, i.e., a C*-algebra A with left G-action. We write g • a for this action. We write elements in $C_0(G, A)$ as finite sums of the form $\sum g_i \cdot a_i$. Let us present some elementary facts about induction and restriction which hold for reduced as well as full crossed products. But as we will consider amenable groups anyway later on, we only treat the case of reduced crossed products and remark that full crossed products can be studied in a similar way.

Let $\iota : H \to G$ be an injective group homomorphism. The homomorphism $\text{id}_A \rtimes_r \iota : A \rtimes_r H \to A \rtimes_r G$ induces the map called induction with $\iota$,

$$\iota_* = \text{ind} : K_i(A \rtimes_r H) \to K_i(A \rtimes_r G).$$

(21)

Now suppose that the index of the image of $\iota$ in G is finite. We want to construct maps in the “wrong” direction, i.e., a map $K_i(A \rtimes_r G) \to K_i(A \rtimes_r H)$. To simplify notations, we think of H as a subgroup of G via $\iota$. On $K_0$, we proceed as follows:

We obtain an isomorphism of (left) $A \rtimes_r H$-modules

$$\bigoplus_{\gamma H \in G/H} A \rtimes_r \gamma H \overset{\cong}{\to} \text{res}^{A \rtimes_r G}_{A \rtimes_r H}(A \rtimes_r G)$$

(22)

sending $(x_{\gamma H})_{\gamma H}$ to $\sum_{\gamma H \in G/H} x_{\gamma H} \cdot \gamma^{-1}$ after choices of representatives $\gamma \in \gamma H$ for every $\gamma H \in G/H$. Hence $A \rtimes_r H$ is a finitely generated free $A \rtimes_r H$-module. This implies that the restriction of every finite generated projective $A \rtimes_r G$-module to $A \rtimes_r H$ is again a finitely generated projective $A \rtimes_r H$-module. Hence we obtain a homomorphism $\iota^* = \text{res} : K_0(A \rtimes_r G) \to K_0(A \rtimes_r H)$ which is called restriction with $\iota$.

Here is an alternative construction which has the advantage that it works for $K_1$ as well: First of all, we represent $A$ faithfully on a Hilbert space $H$. Then $A \rtimes_r H$ is faithfully represented on $H \otimes \ell^2(G)$ via $(a \cdot g)(\xi \otimes \varepsilon) = ((g^{-1} \cdot a)\xi) \otimes \varepsilon_{g\gamma}$. We identify $A \rtimes_r G$ with concrete operators on $H \otimes \ell^2(G)$ via this representation. Now fix representatives $\gamma \in \gamma H$ for every $\gamma H \in G/H$. From the (set-theoretical) bijection $G = \cup_{\gamma H} \gamma H \cong \cup_{\gamma H} H$ we obtain a unitary

$$H \otimes \ell^2(G) \cong \bigoplus_{G/H} H \otimes \ell^2(H)$$

$$\sum_{\gamma} \sum_{h} \lambda_{\gamma h} \xi_{\gamma h} \otimes \varepsilon_{\gamma h} \mapsto \left(\sum_{h} \lambda_{\gamma h} \xi_{\gamma h} \otimes \varepsilon_{h}\right).$$

Conjugation by this unitary yields the identification

$$\mathcal{L}(H \otimes \ell^2(G)) \cong M_{[G:H]}(\mathcal{L}(H \otimes \ell^2(H))), \quad T \mapsto (P_{\ell^2(H)}\gamma^{-1}TP_{\ell^2(H)})_{\gamma,\gamma'}$$

where $P_{\ell^2(H)}$ is the orthogonal projection onto the subspace $\ell^2(H)$ of $\ell^2(G)$.

A straightforward computation shows that this isomorphism sends the operator $a \cdot g$ to the matrix whose $(\gamma, \gamma')$-th entry is $(\gamma^{-1} \cdot a) \cdot (\gamma^{-1} g_{\gamma'})$ if $\gamma^{-1} g_{\gamma'}$ lies in $H$ and 0 if $\gamma^{-1} g_{\gamma'} \notin H$. In particular, $A \rtimes_r G$ is mapped to $M_{[G:H]}(A \rtimes_r H)$. This
homomorphism induces the desired map \( i^* : K_1(A \rtimes_r G) \to K_1(M_{G[H]}(A \rtimes_r H)) \cong K_1(A \rtimes_r H) \).

Moreover, these restriction maps do not depend on the choices of the representatives \( \gamma \) in \( \gamma H \in G/H \). The reason is that for two different choices, the constructed homomorphisms \( A \rtimes_r G \to M_{G[H]}(A \rtimes_r H) \) turn out to be unitarily equivalent, hence they induce the same map in K-theory.

If \( i : H \to G \) is an inclusion of subgroups, one often writes \( \text{res}_i = \text{res}_i^G \) and \( \text{ind}_i = \text{ind}_i^G \). For \( g \in G \) conjugation defines an isomorphism of C*-algebras \( c(g) : A \rtimes G \to A \rtimes G \) sending \( x \in A \rtimes G \) to \( gxg^{-1} \). The two endomorphisms \( \text{ind}_{c(g)} \) and \( \text{res}_{c(g)} \) of \( K_0(A \rtimes G) \) are the identity. Hence in the next lemma the choice of representatives \( \gamma \in H\gamma K \) for an element \( H\gamma K \in H \backslash G/K \) does not matter. It is a variation of the classical Double Coset Formula.

**Lemma 5.2.** Let \( H, K \subseteq A \) be two subgroups of \( G \). Suppose that \( H \) has finite index in \( G \). Then, for \( i = 0, 1 \), we get the following equality of homomorphisms \( \text{res}_{i}^H \circ \text{ind}_{i}^G \): \[ \text{res}_{i}^H \circ \text{ind}_{i}^G = \sum_{H\gamma K \in H \backslash G/K} \text{ind}_{c(\gamma)}:K_{\gamma^{-1}H\gamma} \to H \circ \text{res}_{K}^{K_{\gamma^{-1}H\gamma}}, \]
where \( c(\gamma) \) is conjugation with \( \gamma \), i.e., \( c(\gamma)(k) = \gamma k \gamma^{-1} \).

**Proof.** Since \( H \) has finite index in \( G \), \( K \cap \gamma^{-1}H \gamma \) has finite index in \( K \) and \( H \backslash G/K \) is finite. Hence the expression appearing in Lemma 5.2 makes sense.

On \( K_0 \), we can proceed as follows: Let \( P \) be a finitely generated projective \( A \rtimes_r K \)-module. Fix choices of representatives \( \gamma \in H\gamma K \) for every \( H\gamma K \in H \backslash G/K \). Next we claim that the following homomorphism \[ \bigoplus_{H\gamma K \in H \backslash G/K} \text{ind}_{c(\gamma)}(\text{res}_{K}^{K_{\gamma^{-1}H\gamma}} P) = \bigoplus_{H\gamma K \in H \backslash G/K} A \rtimes_r H \otimes_A \text{ind}_{c(\gamma)}(K_{\gamma^{-1}H\gamma}) P \]
\[ \cong \text{res}_{i}^H \circ \text{ind}_{i}^G P = A \rtimes_r G \otimes_A A \rtimes_r K P. \]

is an isomorphism of \( A \rtimes_r K \)-modules. Its restriction to the summand for \( H\gamma K \in H \backslash G/K \) sends \( x \otimes p \) for \( x \in A \rtimes_r K \) and \( p \in P \) to \( x\gamma \otimes p \). Since it is natural and compatible with direct sums, it suffices to show bijectivity for \( P = A \rtimes_r H \) what is straightforward.

Again, we present an alternative proof which works in general (i.e., for \( i = 1 \) as well). Choose representatives \( \gamma \in H\gamma K \) for every \( H\gamma K \in H \backslash G/K \). For every such \( \gamma \), choose representatives \( \kappa_\gamma \in K_{\gamma^{-1}H\gamma} \) for every \( \kappa_\gamma(K \cap \gamma^{-1}H\gamma) \in K/(K \cap \gamma^{-1}H\gamma) \). The first observation is that the products \( \kappa_\gamma \gamma^{-1} \) form a full set of representatives for \( G/H \), i.e., we can write \( G \) as a disjoint union as follows: \( G = \bigcup_{\gamma \in \kappa_{\cdot}^{-1}}(\kappa_{\cdot} \gamma^{-1}H) \).

Now we use the representatives \( \{ \kappa_\gamma \gamma^{-1} \} \) of \( G/H \) to construct as above the homomorphism \( A \rtimes_r G \to M_{G[H]}(A \rtimes_r H) \) which induces \( \text{res}^G \) on K-theory. The composition of this map with the canonical map \( A \rtimes_r K \to A \rtimes_r A \) is given by \( C_\epsilon(K, A) \ni a \cdot k \mapsto (x_{\kappa_{\cdot} \gamma^{-1}, \kappa_{\cdot}^{-1}} \gamma^{-1}) \in M_{G[H]}(A \rtimes_r H) \).
with
\[
x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}} = \begin{cases} ((\gamma\kappa_{\gamma^{-1}}^{-1} \cdot a) \cdot (\gamma\kappa_{\gamma'^{-1}}^{-1}k\kappa_{\gamma'}^{-1}) & \text{if } \gamma\kappa_{\gamma^{-1}}^{-1}k\kappa_{\gamma'}^{-1} \in H, \\ 0 & \text{else.} \end{cases}
\]

The second observation is that the matrix \((x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}})\) can be decomposed into smaller matrix blocks since for \(\gamma \neq \gamma'\), \(\gamma\kappa_{\gamma^{-1}}^{-1}k\kappa_{\gamma'}^{-1} \) does not lie in \(H\) no matter which \(k \in K\) we take. This holds because \(\gamma \neq \gamma'\) implies \((H\gamma K) \cap (H\gamma'K) = \emptyset\) by our choice of the \(\gamma_s\). Hence in K-theory, we obtain that the class of \((x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}})\) is the sum over \(\gamma\) of the classes of \((x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}})_{K,\kappa,\kappa'}\).

Now the third observation is that
\[
x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}} = \begin{cases} ((\gamma\kappa_{\gamma^{-1}}^{-1} \cdot a) \cdot (\gamma\kappa_{\gamma'^{-1}}^{-1}k\kappa_{\gamma'}^{-1}) & \text{if } \gamma\kappa_{\gamma^{-1}}^{-1}k\kappa_{\gamma'}^{-1} \in K \cap \gamma^{-1}H\gamma, \\ 0 & \text{else.} \end{cases}
\]
This means that the map \(a \cdot k \mapsto (x_{\kappa,\gamma^{-1},\kappa',\gamma'^{-1}})_{K,\kappa,\kappa'}\) is precisely the composition with \(c(\gamma)\) (or rather the extension of \(c(\gamma)\) to matrices) of one of the maps \(A \rtimes_r K \to M[M[K;K]\cap\gamma^{-1}H\gamma](A \rtimes_r (K \cap \gamma^{-1}H\gamma))\) which induce \(\text{res}_K^{K\cap\gamma^{-1}H\gamma}\). This proves the Double Coset Formula.

Let \(\iota : H \to G\) be the inclusion of a normal subgroup of finite index. Denote by \(N_{G/H} \in \mathbb{Z}[G/H]\) the norm element, i.e., \(N_{G/H} = \sum_{H} \gamma H\). If \(M\) is any \(\mathbb{Z}[G/H]\)-module, then multiplication with \(N_{G/H}\) induces a map \(\mathbb{Z} \otimes \mathbb{Z}[G/H] M \to M^{G/H}\) whose kernel and whose cokernel are annihilated by multiplication with \([G : H]\). Denote by \(c(g) : A \rtimes_r H \to A \rtimes_r H\) and by \(c(g) : A \rtimes_r G \to A \rtimes_r G\) the ring homomorphisms obtained by conjugation with \(g\), i.e., they send \(x\) to \(gxg^{-1}\). The induction homomorphism \(\text{ind}_{(g)} : K_i(A \rtimes_r G) \to K_i(A \rtimes_r H)\) is the identity. The induction homomorphism \(\text{ind}_{(g)} : K_i(A \rtimes_r H) \to K_i(A \rtimes_r G)\) is the identity provided that \(g \in H\). Since \(c(g_1) \circ c(g_2) = c(g_1g_2)\) holds for \(g_1, g_2 \in G\) and \(c(1) = \text{id}\), we obtain a \(G/H\)-action on \(K_i(A \rtimes_r H)\). The group homomorphisms \(c(g) \circ \iota\) and \(\iota \circ c(g)\) agree. Hence the map \(\iota_* = \text{ind}_{\iota} : K_i(A \rtimes_r H) \to K_i(A \rtimes_r G)\) factors over the canonical projection \(K_i(A \rtimes_r H) \to \mathbb{Z} \otimes \mathbb{Z}[G/H] K_i(A \rtimes_r H)\) to a homomorphism \(\iota^* : \mathbb{Z} \otimes \mathbb{Z}[G/H] K_i(A \rtimes_r H) \to K_i(A \rtimes_r H)\). In addition, the homomorphism \(A \rtimes_r H \to M[M[K;K]\cap\gamma^{-1}H\gamma](A \rtimes_r H)\) which induces res factors with \(c(g)\) (the extended map on matrices) up to unitary equivalence, and therefore the map \(\iota^* : K_i(A \rtimes_r G) \to K_i(A \rtimes_r H)^{G/H}\) factors over the inclusion \(K_i(A \rtimes_r H)^{G/H} \to K_i(A \rtimes_r H)\) to a map \(\iota^* : K_i(A \rtimes_r G) \to K_i(A \rtimes_r H)^{G/H}\). These considerations and Lemma 5.3 imply

**Lemma 5.3.** Let \(\iota : H \to G\) be the inclusion of a normal subgroup of finite index. Then we obtain a commutative diagram such that the kernel and cokernel of the lower horizontal map are annihilated by \([G : H]\).
Moreover, we have:

**Corollary 5.4.** Let $F$ be a finite group and $A$ be a $F$-$C^*$-algebra. Then the inclusion induces a map $K_i(A) \to K_i(A \rtimes F)$ that factors over the canonical projection $K_i(A) \to \mathbb{Z} \otimes_{ZF} K_i(A)$ to a map $\mathbb{Z} \otimes_{ZF} K_i(A) \to K_i(A \rtimes F)$ whose kernel is annihilated by multiplication with $|F|$.

**Proof.** This follows directly from Lemma 5.3 applied to $H = \{1\}$ and $G = F$. Of course, since $F$ is finite, we do not need to distinguish between reduced and full crossed products. $\square$

### 5.2. Injectivity after inverting orders.

Now we consider the case $G = R \rtimes \mu$. Actually, we can treat a slightly more general situation, namely that $G = \mathbb{Z}^n \rtimes F$ for a finite cyclic group $F$ where the conjugation action of $F$ on $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$. Let $A$ be a $G$-$C^*$-algebra. Since $G$ is amenable, we do not have to distinguish between full and reduced crossed products. It is clear that $A \rtimes G \cong (A \rtimes \mathbb{Z}^n) \rtimes F$ where $\mathbb{Z}^n$ acts on $A$ via the restricted action and $F$ acts on $A \rtimes \mathbb{Z}^n$ via $c(f)$, i.e., $f \cdot (a \cdot g) = (f \cdot a) \cdot (fgf^{-1})$ for $a \in A$ and $g \in \mathbb{Z}^n$.

**Theorem 5.5.** Suppose that the map $K_i(A) \to K_i(A \rtimes \mathbb{Z}^n)$ is injective after inverting $|F|$. Then the map $K_i(A \rtimes F) \to K_i(A \rtimes G)$ induced by the canonical homomorphism $A \rtimes F \to A \rtimes G$ is injective after inverting $|F|$.

The proof of this theorem needs some preparation.

**Lemma 5.6.** Let $\mathcal{M}$ be a complete system of representatives of conjugacy class of maximal finite subgroups of $G$. Let $m$ be the least common multiple of the orders of the subgroups $M \in \mathcal{M}$.

Then for every $M \in \mathcal{M}$ the canonical homomorphisms $A \to A \rtimes M \to A \rtimes G$ induce a map

$$
\ker(\mathbb{Z} \otimes_{ZF} K_i(A) \to K_i(A \rtimes G)) \to \ker(K_i(A \rtimes M) \to K_i(A \rtimes G))
$$

which is bijective after inverting $m$.

**Proof.** We have the following $G$-pushout (compare [La-Lü](4.5))

$$
\begin{array}{ccc}
\prod_{M \in \mathcal{M}} G \times_M EM & \longrightarrow & EG \\
\prod_{M \in \mathcal{M}} \text{id}_G \times_M f_M & \downarrow & f \\
\prod_{M \in \mathcal{M}} G \times_M \{\bullet\} & \longrightarrow & EG
\end{array}
$$

Applying equivariant K-homology with coefficients in $A$, we obtain a long exact sequence (the Mayer-Vietoris sequence associated with the pushout)

$$
\cdots \to \bigoplus_{M \in \mathcal{M}} K^G_i(G \times_M EM; A) \to K^G_i(EG; A) \oplus \bigoplus_{M \in \mathcal{M}} K^G_i(G \times_M \{\bullet\}; A) \\
\to K^G_i(EG; A) \to \bigoplus_{M \in \mathcal{M}} K^G_{i-1}(G \times_M EM; A) \\
\to K^G_{i-1}(EG; A) \oplus \bigoplus_{M \in \mathcal{M}} K^G_{i-1}(G \times_M \{\bullet\}; A) \to K^G_{i-1}(EG; A) \to \cdots
$$
There is a spectral sequence converging to \( K_{*}^{\text{G}}(G \times EM; A) \) whose \( 2 \)-term is \( H_{p,q}^{\text{G}}(G \times EM; K_{0}(A)) \). Since \( M \) is finite, we know that \( H_{p,q}^{\text{G}}(G \times EM; K_{0}(A)) \cong H_{p}^{\text{EM}}(EM; K_{0}(A)) \) is annihilated by multiplication with \( |M| \) for \( p \geq 1 \). Since \( H_{0}^{\text{G}}(G \times EM; K_{0}(A)) \) can be identified with \( \mathbb{Z} \otimes_{\mathbb{Z}M} K_{0}(A) \), the edge homomorphism

\[
\mathbb{Z} \otimes_{\mathbb{Z}M} K_{0}(A) \to K_{*}^{\text{G}}(G \times EM; A)
\]

is bijective after inverting \( |M| \). Its composite with

\[
K_{*}^{\text{G}}(\text{id} \times_{M} f_{M}) : K_{*}^{\text{G}}(G \times EM; A) \to K_{*}^{\text{G}}(G \times M \{ \bullet \}; A) \cong K_{1}(A \times M)
\]

is the map \( \mathbb{Z} \otimes_{\mathbb{Z}M} K_{1}(A) \to K_{1}(A \times M) \) induced by the canonical homomorphism \( A \to A \times M \). The kernel of this map \( \mathbb{Z} \otimes_{\mathbb{Z}M} K_{1}(A) \to K_{1}(A \times M) \) is annihilated by multiplication with \( |M| \) by Corollary 5.4. This already implies injectivity of the map in (23) after inverting \( m \). But this also implies that the long exact sequence (24) yields after inverting \( m \) the short exact sequence

\[
0 \to \bigoplus_{n} \mathbb{Z} \otimes_{\mathbb{Z}M} K_{n}(A)[\frac{1}{m}] \to K_{n}^{\text{G}}(EG; A)[\frac{1}{m}] \oplus \bigoplus_{m \in M} K_{n}(A \times M)[\frac{1}{m}] \to K_{n}(A \times G)[\frac{1}{m}] \to 0
\]

where we used that \( G \) is amenable, hence satisfies the Baum-Connes conjecture with coefficients, i.e., assumes \( K_{*}^{\text{G}}(EG; A) \to K_{1}(A \times G) \) is an isomorphism (see [Hig-Kas]). Exactness of (25) immediately yields surjectivity of the map in (23) because \( \bigoplus_{n} \mathbb{Z} \otimes_{\mathbb{Z}M} K_{n}(A)[\frac{1}{m}] \to \bigoplus_{m \in M} K_{n}(A \times M)[\frac{1}{m}] \) is induced by the canonical homomorphisms \( A \to A \times M \) as explained above and \( \bigoplus_{m \in M} K_{n}(A \times M)[\frac{1}{m}] \to K_{n}(A \times G)[\frac{1}{m}] \) is induced by the canonical homomorphisms \( A \times M \to A \times G \). This proves our claim. \( \square \)

**Lemma 5.7.** For \( i = 0 \) or \( 1 \), suppose that the map \( K_{i}(A) \to K_{i}(A \times \mathbb{Z}^{n}) \) is injective after inverting \( |F| \). Then the map \( \mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A) \to \mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A \times \mathbb{Z}^{n}) \) coming from the canonical homomorphism \( A \to A \times \mathbb{Z}^{n} \) is injective after inverting \( |F| \).

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A) & \to & \mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A \times \mathbb{Z}^{n}) \\
\downarrow N_{F} & & \downarrow N_{F} \\
K_{i}(A) & \to & K_{i}(A \times \mathbb{Z}^{n})
\end{array}
\]

The vertical maps denoted by \( N_{F} \) are given by multiplication with the norm element \( N_{F} \) and are isomorphisms after inverting \( |F| \). The two lower vertical arrows are the canonical inclusions. The horizontal arrows are induced by the canonical homomorphism \( A \to A \times \mathbb{Z}^{n} \). Since the lower horizontal arrow is injective after inverting \( |F| \) by assumption, the same is true for the upper horizontal arrow. \( \square \)

**Proof of Theorem 5.3** By assumption, the canonical homomorphism \( A \to A \times \mathbb{Z}^{n} \) is injective in K-theory once we invert \( |F| \). Hence Lemma 5.7 implies that \( \mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A) \to \mathbb{Z} \otimes_{\mathbb{Z}F} K_{i}(A \times \mathbb{Z}^{n}) \) is injective after inverting \( |F| \). Now Corollary 5.1 tells
us that the map $\mathbb{Z} \otimes_{\mathbb{Z} F} K_i(A \rtimes \mathbb{Z}^n) \to K_i((A \rtimes \mathbb{Z}^n) \rtimes F) \cong K_i(A \rtimes G)$ is injective after inverting $|F|$. Hence $\mathbb{Z} \otimes_{\mathbb{Z} F} K_i(A) \to K_i(A \rtimes G)$ is injective after inverting $|F|$. Finally apply Lemma 5.6 in the case $M = F$.

\[\square\]

5.3. Injectivity. Finally, we are ready for the

**Proof of Proposition 5.7.** Let $c$ be some element in $R^\times$. Since the additive action of $c^{-1}R$ is homotopic to the trivial action, an iterative application of the Pimsner-Voiculescu sequence implies that the canonical homomorphism $C_0(\mathbb{A}_\infty) \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R)$ is injective on $K_0$. Thus Theorem 5.5 yields that $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu$ is injective on $K_0$ after inverting $|\mu|$. By equivariant Bott periodicity (see [Bla] Theorem 20.3.2, $n$ is even), we know that $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu) \cong K_0(C^*(\mu))$ and the latter group is free abelian. Thus the canonical homomorphism $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu$ itself must be injective on $K_0$. But then, since

$$C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu = \bigcup_{c \in R^\times} C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu,$$

the canonical homomorphism $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ must be injective on $K_0$ as well by continuity of $K_0$.

\[\square\]

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