Abstract. We show that the theta divisors of general principally polarised abelian varieties can be chosen as smooth irreducible algebraic representatives of the coefficients of the Chern-Dold character in complex cobordisms and describe the action on them of the Landweber-Novikov operations. The link with Milnor-Hirzebruch problem about algebraic representatives in the complex cobordisms is discussed.

1. Introduction

In the complex cobordism theory going back to the foundational works of Milnor and Novikov [15], [16] a crucial role is played by the Chern-Dold character introduced by the first author in [4]. In particular, it appears in the formulation of an analogue of the Riemann-Roch-Grothendieck-Hirzebruch theorem in the theory of complex cobordisms, see [6].

By definition, the Chern-Dold character $\text{ch}_U$ is a natural multiplicative transformation

$$\text{ch}_U : U^*(X) \rightarrow H^*(X; \Omega_U \otimes \mathbb{Q}),$$

where $U^*(X)$ is the complex cobordism ring of a CW-complex $X$ and $\Omega_U = U^*(pt)$ is the cobordism ring of the stably complex manifolds (or, in short, $U$-manifolds). The fundamental Milnor-Novikov result says that the coefficient ring of the theory of $U^*(X)$ is the polynomial ring

$$\Omega_U = \mathbb{Z}[y_1, \ldots, y_n, \ldots], \quad \deg y_n = -2n$$

of infinitely many generators $y_n$, $n \in \mathbb{N}$.

Let $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty)$ be the first Chern classes of the universal line bundle over $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively. The Chern-Dold character is uniquely defined by its action

$$\text{ch}_U : u \rightarrow \beta(z), \quad \beta(z) := z + \sum_{n=1}^{\infty} [B^{2n}] \frac{z^{n+1}}{(n+1)!}$$

where $B^{2n}$ are certain $U$-manifolds (see [4]).

The series $\beta(z)$ is the exponential of the commutative formal group

$$F(u, v) = u + v + \sum_{i,j} a_{i,j} u^i v^j$$

of the geometric complex cobordisms introduced by Novikov in [17], so that

$$F(\beta(z), \beta(w)) = \beta(z + w).$$

Quillen identified this group with Lazard’s universal one-dimensional commutative formal group [18].
The inverse of this series is known as the logarithm of this formal group and can be explicitly given by the Mischenko series [17, 5]:

\[
\beta^{-1}(u) = u + \sum_{n=1}^{\infty} [CP^n] \frac{u^{n+1}}{n+1}.
\]

(2)

The question whether there are natural algebraic representatives of the cobordism classes \([B^{2n}]\) in the exponential of the formal group given by the Chern-Dold character was open for a long time since 1970 work of the first author [4].

In this paper we give the following answer to this question, presenting an explicit form of the series (1) as

\[
\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n] \frac{z^{n+1}}{(n+1)!},
\]

(3)

where \(\Theta^n\) is the theta divisor of a general principally polarised abelian variety \(A^{n+1}\), considered as real manifold of dimension \(2n\). The cobordism class of the theta divisor does not depend on the choice of such variety provided \(\Theta^n\) is smooth, which is true in general case [1].

**Theorem 1.1.** The theta divisor \(\Theta^n\) of a general principally polarised abelian variety \(A^{n+1}\) is a smooth irreducible projective variety, which can be taken as a natural algebraic representative of the coefficient \([B^{2n}]\) in the Chern-Dold character.

As a corollary we have the following representation of the cobordism class of \(U\)-manifold \(M^{2n}\) in terms of theta divisors

\[
[M^{2n}] = \sum_{\omega:|\omega|=n} c^n_{\omega}(M^{2n}) \frac{[\Theta^n]}{(\omega + 1)!},
\]

(4)

where the sum is over all partitions \(\omega = (i_1, \ldots, i_k)\) of \(|\omega| = i_1 + \cdots + i_k = n\),

\[
\Theta^n := \Theta^{i_1} \cdots \Theta^{i_k},
\]

\((\omega + 1)! := (i_1 + 1)! \cdots (i_k + 1)!\) and \(c^n_{\omega}(M^{2n}) \in \mathbb{Z}\) are the Chern numbers of the normal bundle of \(M^{2n}\) (see next section for details).

Since according to [1] the Todd genus \(Td(B^{2n}) = (-1)^n\) we have the following formula for Todd genus for any \(U\)-manifold \(M^{2n}\)

\[
Td(M^{2n}) = \sum_{\omega:|\omega|=n} c^n_{\omega}(M^{2n}) \frac{(-1)^n}{(\omega + 1)!}.
\]

(5)

This implies the divisibility condition on the Chern numbers \(c^n_{\omega}(M^{2n})\) of \(U\)-manifolds. More divisibility conditions we can get applying to formula (4) the Landweber-Novikov operations and taking the Todd genus (see the last section).

The action of the Landweber-Novikov operations on the theta divisors can be described explicitly.

Let \(\omega = (i_1, \ldots, i_k)\) be a partition of \(|\omega| := i_1 + \cdots + i_k\) and \(S_\omega[M]\) be the result of the action of the Landweber-Novikov operation \(S_\omega\) on \(U\)-manifold \(M\) defined in terms of its stable normal bundle (see [13,17]).
Let \((k) = (k, 0, \ldots, 0)\) be a one-part partition and consider the smooth complete intersection
\[
\Theta^{n-k}_k = \Theta^n \cap \Theta^n(a_1) \cap \ldots \Theta^n(a_k)
\]
of \(\Theta^n\) with \(k\) general translates \(\Theta^n(a_i), a_i \in A^{n+1}\) of the theta divisor \(\Theta^n\). Let \(D = c_1(\mathcal{L}) \in H^2(A^{n+1}, \mathbb{Z})\) be Poincare dual cohomology class of \(\Theta^n\), then \(\Theta^{n-k}_k\) is the Poincare dual to \(D^{k+1} \in H^{2k+2}(A^{n+1}, \mathbb{Z})\). Note that, due to a theorem of Mattuck, for a general abelian variety \(D^m\) generates the corresponding Hodge group \(H^{2m}_{\text{Hodge}}(A^{n+1})\) for all \(m = 1, \ldots, n + 1\) (see section 17.4 in \[3\]).

**Theorem 1.2.** For every partition \(\omega\) with \(|\omega| < n\) the Landweber-Novikov cobordism class \(S_{\omega}[\mathbb{B}^{2n}]\) has a smooth irreducible algebraic representative.

More precisely, if \(\omega \neq (k)\) for some \(k \in \mathbb{N}\) then \(S_{\omega}[\mathbb{B}^{2n}] = S_{\omega}[\Theta^n] = 0\), while for \(\omega = (k)\) we have
\[
S_{(k)}[\mathbb{B}^{2n}] = S_{(k)}[\Theta^n] = [\Theta^{n-k}_k].
\]

We consider also the most degenerate case of abelian variety \(A^{n+1} = \mathcal{E}^{n+1}\), where \(\mathcal{E}\) is an elliptic curve. In that case the theta-divisor is singular, but we show that there is a real-analytic representative of the same cohomology class given in terms of the classical elliptic functions.

In the last section we discuss the link with Milnor-Hirzebruch problem about characteristic numbers of smooth irreducible algebraic varieties.

2. Complex bordisms and cobordisms

Let \(M^m\) be a smooth closed real manifold. By *stable complex structure* (or, simply *U-structure*) on \(M^m\) we mean an isomorphism of real vector bundles
\[
i_M : TM^m \oplus (2N - m)_{\mathbb{R}} \to \nu \xi,
\]
where \(TM^m\) is the tangent bundle of \(M^m\), \((2N - m)_{\mathbb{R}}\) is trivial real \((2N - m)\)-dimensional bundle over \(M^m\) and \(\nu \xi\) is a complex vector bundle over \(M^m\) considered over reals. A manifold \(M^m\) with a chosen U-structure is called U-manifold.

Note that complex structure in the stable tangent bundle \(TM^m\) determines complex structure in the stable normal bundle \(\nu M^m\).

Two closed smooth real \(m\)-dimensional manifolds \(M_1\) and \(M_2\) are called bordant if there exists real \((m + 1)\)-dimensional U-manifold \(W\) such that the boundary \(\partial W\) is a disjoint union of \(M_1^m\) and \(M_2^m\) and the restriction of the stable tangent bundle \(TW\) to \(M_i\) coincides with \(TM_i, i = 1, 2\).

Similarly, Two closed smooth real \(m\)-dimensional manifolds \(M_1\) and \(M_2\) are called cobordant if there exists real \((m + 1)\)-dimensional U-manifold \(W\) such that the boundary \(\partial W\) is a disjoint union of \(M_1^m\) and \(M_2^m\) and the restriction of the stable normal bundle \(\nu W\) to \(M_i\) coincides with \(\nu M_i, i = 1, 2\).

Define the following operations in corresponding equivalence classes of U-manifolds.

Disjoint union \([M_1^m] \cup [M_2^m]\) of two closed \(m\)-dimensional U-manifolds is U-manifolds. Define the sum of the corresponding bordism classes as
\[
[M_1^m] + [M_2^m] = [M_1^m \cup M_2^m].
\]
Similarly define the product of bordism classes by
\[ [M_1^{m_1}] [M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}], \]
where \( M_1^{m_1} \times M_2^{m_2} \) is the direct product of \( M_1^{m_1} \) and \( M_2^{m_2} \). This defines the commutative graded ring
\[ \Omega^U = \sum_{m \geq 0} \Omega^U_m, \]
where \( \Omega^U_m \) is the group of bordism classes of \( m \)-dimensional \( U \)-manifolds.

Similarly, we have the graded ring \( \Omega^*_U = \sum_{m \geq 0} \Omega^*_U m \), where \( \Omega^*_U m \) is the group of cobordism classes of \( m \)-dimensional \( U \)-manifolds.

From the correspondence between stably complex structures in tangent and normal bundles it follows that the groups \( \Omega^U_m \) and \( \Omega^*_U m \) and the rings \( \Omega^U * \) and \( \Omega^*_U * \) are isomorphic. This isomorphism can be extended to Poincare duality between complex bordisms and cobordisms.

The following fundamental result is due to Milnor and Novikov.

**Theorem 2.1.** (Milnor [15], Novikov [16]) The graded complex bordism ring \( \Omega^U * \) is isomorphic to the graded polynomial ring \( \mathbb{Z}[a_2, a_4, \ldots, a_{2n}, \ldots] \) of infinitely many variables \( a_{2n}, n \in \mathbb{N} \), where \( \deg a_{2n} = 2n \). In particular, \( \Omega^U_{2n-1} = 0 \).

Let \( \omega = (i_1, \ldots, i_k), i_1 \geq i_2 \geq \cdots \geq i_k \) be a partition of \( n \): \( i_1 + \cdots + i_k = n \). Using the standard splitting principle one can define the Chern classes \( c_\omega(TM) \in H^{2n}(M, \mathbb{Z}) \) of a \( U \)-manifold \( M \) corresponding to the monomial symmetric functions \( m_\omega(t) = t_1^{i_1} \cdots t_k^{i_k} + \ldots \).

The Chern numbers \( c_\omega(M^{2n}), |\omega| = n \) of \( U \)-manifold \( M^{2n} \) are defined as the value of the cohomology class \( c_\omega(TM^{2n}) \) on the fundamental cycle \( < M^{2n} > \):
\[ c_\omega(M^{2n}) := (c_\omega(TM^{2n}), < M^{2n} >). \] (7)

We have \( \pi(n) \) Chern numbers \( c_\omega(M^{2n}) \), where \( \pi(n) \) is the number of partitions of \( n \), which depend only on the bordism class of \( M^{2n} \).

**Corollary 2.2.** Two closed \( 2n \)-dimensional \( U \)-manifolds \( M_1 \) and \( M_2 \) are \( U \)-bordant if and only if all the corresponding Chern numbers are the same.

It will be more convenient for us to use the Chern numbers \( c'_\omega(M^{2n}) \) defined using the stable normal bundle \( \nu M^{2n} \):
\[ c'_\omega(M^{2n}) := (c_\omega(\nu M^{2n}), < M^{2n} >). \] (8)

They can be expressed through the usual Chern numbers \( c_\omega(M^{2n}) \) and contain the same information about \( U \)-manifold \( M^{2n} \).

We will use the following convenient class of \( U \)-manifolds from [7].

Let \( M^{2n} \) be a smooth real manifold of dimension \( 2n \). A complex framing of \( M^{2n} \) is a choice of complex line bundle \( \mathcal{L} \) on \( M^{2n} \), such that the direct sum \( TM^{2n} \oplus \mathcal{L} \) admits a structure of trivial complex vector bundle. Thus complex framing is a \( U \)-structure of very special type. The examples of such structures is given by the following construction.

Let \( X \) be a complex manifold of complex dimension \( n + 1 \) with holomorphically trivial tangent bundle and \( L \) be a complex line bundle over \( X \). Let \( S \) be a real-analytic section \( S : X \to L \), transversal to the zero section and...
consider \( M^{2n} = \{ x \in X : S(x) = 0 \} \subset X \), which is a smooth real-analytic submanifold of \( X \). Then the line bundle \( \mathcal{L} = i^*(L) \), where \( i : M^{2n} \to X \) is the embedding, determines the complex framing on \( M \).

An explicit example of such submanifold for \( X \) being a product of \( n + 1 \) elliptic curves will be discussed in section 5.

3. Theta divisors of abelian varieties

Consider now our main example, when \( X = A^{n+1} \) is principally polarised abelian variety \( A^{n+1} = \mathbb{C}^{n+1}/\Gamma \) with lattice \( \Gamma \) generated by the columns of the \( (n+1) \times 2(n+1) \) matrix \((I \, \tau)\) with complex symmetric \((n+1) \times (n+1)\) matrix \( \tau \) having positive imaginary part [8]. It has a canonical line bundle \( L \) with one-dimensional space of sections generated by the classical Riemann \( \theta \)-function

\[
\theta(z, \tau) = \sum_{l \in \mathbb{Z}^{n+1}} \exp\{\pi i(l, \tau l) + 2\pi i(l, z)\}, \quad z \in \mathbb{C}^{n+1}.
\]

(9)

The corresponding theta divisor \( \Theta^n \subset A^{n+1} \) given by \( \theta(z, \tau) = 0 \) is known (after Andreotti and Mayer [1]) to be smooth for general principally polarised abelian variety \( A^{n+1} \).

In particular, for \( n = 1 \) a generic abelian surface is Jacobi variety of a smooth genus 2 curve \( C \) with theta divisor \( \Theta^1 \sim C \), for \( n = 2 \) this is Jacobi variety of genus 3 curve \( C \) with \( \Theta^2 \cong S^2(C) \) being smooth for all non-hyperelliptic \( C \). For \( n \geq 3 \) the general case is not Jacobian with theta divisor smooth outside complex codimension 1 locus in the moduli space of the abelian varieties. For more detail on geometry of theta divisors we refer to the survey [9] by Grushevsky and Hulek.

The line bundle \( L \) is ample with \( L^3 \) known (after Lefschetz [3]) to be very ample in the sense that the sections of \( L^3 \) determine the embedding of \( A^{n+1} \) into corresponding projective space \( \mathbb{P}^N, \quad N = 3^{n+1} - 1 \). The corresponding quadratic and cubic equations, defining the image in \( \mathbb{P}^N \), were described by Birkenhake and Lange [2] (see also Ch. 7 in [3]). For the elliptic curves this reduces to the Hasse cubic equation

\[ x^3 + y^3 + z^3 = 3\lambda xyz. \]

Note that the line bundle \( L^2 \) is very ample only on the quotient \( X/\mathbb{Z}_2 \) by involution \( z \to -z \), which is known as \textit{Kummer variety}. It has the sections given by \( \theta_{\epsilon, \delta}^2(z) \), where the theta-functions with characteristics \((\epsilon, \delta) \in \mathbb{Z}_2^{2n}\) are defined by

\[
\theta_{\epsilon, \delta}(z) = \sum_{l \in \mathbb{Z}^{n+1}} \exp \pi i(l + \epsilon/2, \tau l + \epsilon/2) + 2\pi i(l + \epsilon/2, z + \delta/2), \quad z \in \mathbb{C}^{n+1}.
\]

In that case its image in \( \mathbb{P}^N, \quad N = 2^{n+1} - 1 \) is determined by certain quadratic relations, in dimension 1 described by Jacobi.

Let \( D = c_1(\mathcal{L}) \in H^2(\Theta^n, \mathbb{Z}) \) be the first Chern class of line bundle \( \mathcal{L} \). Then the total Chern class \( c(\Theta^n) = 1 + c_1(\Theta^n) + \cdots + c_n(\Theta^n) \) of \( \Theta^n \) satisfies

\[ c(\Theta^n)(1 + D) = 1 \]

(10)
since the tangent bundle of an abelian variety is trivial. This means that the Euler characteristic
\[ \chi(\Theta^n) = c_n(\Theta^n) = (-1)^n \mathcal{D}^n = (-1)^n (n + 1)!, \]
(11) since \( \mathcal{D}^n = (n + 1)! \) (see next section).

The Betti numbers of the theta divisors \( \Theta^n \) are not difficult to compute, see e.g. [13]. Indeed, by Lefschetz theorem the embedding \( i : \Theta^n \to A^{n+1} \) induces the isomorphisms
\[ i^* : H^k(A^{n+1}, \mathbb{Z}) \to H^k(\Theta^n, \mathbb{Z}) \]
for \( k < n \). This means that for \( k < n \) the Betti numbers are
\[ b_k(\Theta^n) = \binom{2n + 2}{k} = b_{2n-k}(\Theta^n). \]
The remaining middle Betti number can be found then using the formula (11) for the Euler characteristic:
\[ b_n(\Theta^n) = (n + 1)! + \frac{n}{n + 2} \binom{2n + 2}{n + 1}. \]
Since the cohomology groups of \( \Theta^n \) have no torsion [13], this defines them uniquely, but multiplication structure seems still to be understood. Note that the signature \( \tau(\Theta^n) \) for even \( n \) has been computed in [7]:
\[ \tau(\Theta^n) = 2^{n+2}(2^{n+2} - 1) \left( \frac{n}{n + 2} \right) B_{n+2}, \]
(12)
where \( B_n \) are Bernoulli numbers.

In the simplest case \( n = 1 \) the general abelian variety is Jacobi variety of some genus 2 curve \( \Gamma \), so by Riemann theorem in this case the theta divisor \( \Theta^1 = \Gamma \) is genus two curve.

For all \( n \) the smooth theta divisor \( \Theta^n \) is a projective variety of general type. Indeed, by the adjunction formula the canonical class \( K_{\Theta^n} = \mathcal{L} := i^*(L) \) is ample. In particular, \( \mathcal{L} \) is known to have \( n \)-dimensional space of sections generated by the partial derivatives \( \partial_{\xi} \theta(z, \tau) \) of the theta function. By Bertini theorem the system of equations
\[ \theta(z, \tau) = 0, \partial_{\xi_1} \theta(z, \tau) = 0, \ldots, \partial_{\xi_k} \theta(z, \tau) = 0, \quad z \in A^{n+1} \]
(13)
with generic \( \xi_1, \ldots, \xi_k \in \mathbb{C}^{n+1} \) determine smooth complete intersections \( \Theta_k^{-k} \subset \Theta^n \subset A^{n+1} \) mentioned above.

The canonical class of \( S = \Theta_k^{-k} \subset A^{n+1} \) is \( K_S = \mathcal{O}((k + 1)D)|_S \), where \( D \) is the principal polarisation divisor of \( A^{n+1} \). It is ample, so \( S \) is of general type as well. In particular, \( \Theta_n^0 \) consists of \( (n + 1)! \) points and \( \Theta_n^{1-n-1} \) is a curve with Euler characteristic \( \chi = n(n + 1)! \).

Note that the varieties \( \Theta_k^{-k} \) with \( k \neq n \) are irreducible, since they are smooth and (by Lefschetz theorem) their Betti number \( b_0 = 1 \).

4. Proofs

Now we are ready to prove our results. The proofs are actually quite simple, but based on the deep results from the theory of complex cobordisms. In particular, the following result from [4] is crucial for us.
Theorem 4.1. (Buchstaber) The coefficient $[\mathcal{B}^n]$ of the Chern-Dold character in complex cobordisms is the cobordism class of a $U$-manifold $M^{2n}$, which is uniquely determined by the following conditions:

$$c^\nu_\omega(M^{2n}) = 0$$

(14)

for any partition $\omega$ of $n$ different from one-part partition $\omega = (n)$, and

$$c^\nu_\omega(M^{2n}) = (n+1)!,$$

(15)

where $c^\nu_\omega(M^{2n})$ are Chern numbers defined by (8).

To prove our Theorem 1.1 we need now only to check that the theta divisor $\Theta^n$ of a general principally polarised abelian variety $A^{n+1}$ satisfies these conditions.

The normal bundle $\nu \Theta^n$ can be identified with $L = i^*(L)$, where $i: \Theta^n \rightarrow A^{n+1}$ is natural embedding and $L$ is the principle polarisation line bundle on $A^{n+1}$. This immediately implies the condition (14).

To prove condition (15) we need only to use the well-known fact that $D^g = g! \in H^2(\mathcal{E}^g, \mathbb{Z}) = \mathbb{Z}$ where $D \in H^2(\mathcal{E}^g, \mathbb{Z})$ is the Poincare dual cohomology class of the theta divisor $\Theta^n$ of any principally polarised abelian variety (see e.g. [3]). Geometrically, this means that the intersection of $g$ shifts of theta divisor $\Theta$ consists of $g!$ points. To see this one can consider the degenerate case when $X^g = \mathcal{E}^g$ is the product of $g$ elliptic curves. In that case the theta divisor is a singular union of $g$ coordinate hypersurfaces, but the calculation of the intersection number is still valid and simple, giving $g!$. This completes the proof of Theorem 1.1.

The formula now follows from comparison of the Chern numbers of both sides.

To prove Theorem 1.2 recall that the cobordism class $\alpha \in U^2(X)$ is called geometric if it belongs to the image of the natural homomorphism $H^2(X, \mathbb{Z}) \rightarrow U^2(X)$ (see [17]).

Novikov proved that for any geometric cobordism class $\alpha$ the Landweber-Novikov operation $S_k(\alpha) = 0$ if $\omega \neq (k)$ for some $k \in \mathbb{N}$, and $S_k(\alpha) = \alpha^{k+1}$ for all $k \in \mathbb{N}$ (see Lemma 5.6 in [17]).

Since $\Theta^n$ realises geometric cobordism class in $U^2(A^{n+1})$ corresponding to the principal polarisation $D = c_1(\mathcal{L}) \in H^2(A^{n+1}, \mathbb{Z})$ we conclude that $S_k[\Theta^n] = 0$ if $\omega \neq (k)$ for some $k \in \mathbb{N}$. When $\omega = (k)$ the cobordism class $S_k[\Theta^n]$ is Poincare dual to $D^{k+1}$, which is represented by the intersection theta divisor $\Theta^{n-k}_k$. This proves Theorem 1.2.

5. Real-analytic elliptic representatives

Consider now the most degenerate case of the abelian variety, when $A^{n+1}$ is the product of $n + 1$ copies of an elliptic curve $\mathcal{E} = \mathbb{C}/\mathcal{L}$, where $\mathcal{L}$ is a lattice with the periods of $2\omega_1, 2\omega_2$.

Let $L$ be the natural line bundle on it with the holomorphic section

$$S_0(u) = \sigma(u_1) \ldots \sigma(u_{n+1}), \quad u = (u_1, \ldots, u_{n+1}) \in \mathcal{E}^{n+1}.$$
where $\sigma$ is the classical Weierstrass elliptic function [22]. The zero set of this section given by the union of the coordinate hypersurfaces $u_i = 0, i = 1, \ldots, n + 1$ is singular. One can actually show that there are no smooth algebraic representatives in the same cohomology class. \footnote{We are grateful to Ivan Cheltsov and Artie Prendergast-Smith for explaining how to do this rigorously.}

We will show now that one can find a smooth real-analytic representative of the same cohomology class, which can be determined in terms of classical elliptic functions.

Consider the classical Weierstrass elliptic function $\eta(z)$ with the simple pole at the lattice points and the transformation properties

$$
\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2,
$$

where $\eta_i = \zeta(\omega_i), i = 1, 2$. Let us introduce the following non-holomorphic function (cf. [12, 21])

$$
\xi(z) = \zeta(z) + az + b\bar{z}, \quad (16)
$$

where $a, b$ is the (unique) solution of the following linear system

$$
a\omega_1 + b\bar{\omega}_1 + \eta_1 = 0, \quad a\omega_2 + b\bar{\omega}_2 + \eta_2 = 0, \quad (17)
$$

or, explicitly

$$
a = -\frac{\eta_1\omega_2 - \eta_2\omega_1}{\omega_1\omega_2 - \omega_2\omega_1} = -\frac{\eta_1\omega_2 - \eta_2\omega_1}{23 (\omega_1\omega_2)}, \quad b = \frac{\eta_1\omega_2 - \eta_2\omega_1}{\omega_1\omega_2 - \omega_2\omega_1} = \frac{\pi}{43 (\omega_1\omega_2)}, \quad (18)
$$

where we have used the Legendre identity [22]

$$
\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}.
$$

Lemma 5.1. The function $\xi$ is an odd doubly-periodic real-analytic function and with the asymptotic behaviour $\xi \sim 1/z$ at zero and with zeroes at all 3 half periods $\omega_1, \omega_2, \omega_3 = \omega_1 + \omega_3$.

In the lemniscatic case with $\omega_1 = \omega \in \mathbb{R}, \omega_2 = i\omega$ we have

$$
\xi(z) = \zeta(z) - \frac{\pi i}{4\omega^2} \bar{z}, \quad (19)
$$

which has zeros precisely at 3 half-periods.

Proof. The property $\xi(-z) = -\xi(z)$ follows from the same property of $\zeta(z)$.

The double-periodicity follows from the transformation properties of function $\zeta$:

$$
\xi(z + 2\omega_1) = \zeta(z) + 2\eta_1 + 2a\omega_1 + 2b\bar{\omega}_1 = \xi(z),
$$

$$
\xi(z + 2\omega_2) = \zeta(z) + 2\eta_2 + 2a\omega_2 + 2b\bar{\omega}_2 = \xi(z)
$$

due to (17). Combining these two properties we have $\xi(\omega_1) = -\xi(-\omega_1) = -\xi(\omega_1)$, so $\xi(\omega_1) = 0$ for all half-periods.

In the lemniscatic case with $\omega_1 = \omega \in \mathbb{R}, \omega_2 = i\omega$ we have $\eta(iz) = -i\eta(z)$ and thus $\eta_2 = -i\eta_1, \eta_1 \in \mathbb{R}$. Corresponding $\wp$-function satisfies equation

$$
(\wp')^2 = 4\wp(\wp^2 - e^2), \quad e = \wp(\omega) = \frac{\Gamma^4(1/4)}{32\pi \omega^2}
$$

where $\Gamma$ is the classical Euler’s $\Gamma$-function.
From Legendre identity we have $i\omega \eta_1 - \omega \eta_2 = 2i\omega \eta_1 = \frac{\pi i}{2}$, so in this case $\eta_1 = \frac{\pi}{4\omega}$, $\eta_2 = -i\frac{\pi}{4\omega}$, which gives $a = 0$, $b = -\frac{\pi}{4\omega}$ and the relation (19).

Note that the function $\xi(z)$ satisfies the equation $\partial \bar{\partial} \xi(z, \bar{z}) = 0$ and thus is a complex-valued harmonic function. The zeros of such functions were extensively studied, see [19] and references therein.

The number of the zeros of such functions depends on the position of zero in relation with the caustic defined as the image $\xi(\Sigma) \subseteq \mathbb{C}$ of the critical set $\Sigma := \{ z \in \mathbb{C} : J_{\xi}(z, \bar{z}) = 0 \}$, where $J_{\xi}$ is the Jacobian of the map $\xi : \mathbb{R}^2 \to \mathbb{R}^2$. The real Jacobian of the harmonic function $f(z, \bar{z}) = g(z) + h(\bar{z})$ with holomorphic $g, h$ is $J_f(z, \bar{z}) = |g'(z)|^2 - |h'(z)|^2$.

Thus in our case the critical set is $\Sigma := \{ z \in \mathbb{C} : |\varphi(z)| = \frac{\pi}{4\omega^2} \}$.

Note that this level of the real function $F(z, \bar{z}) = |\varphi(z)|$ is non-singular. Indeed, if $\varphi'(z) = 0$ then $z$ is a half-period and thus $\varphi(z) = 0$, or $\varphi(z) = \pm e$. Since $3.6 < \Gamma(1/4) < 3.7$ we have

$e = |\varphi(z)| = \frac{\Gamma^4(1/4)}{32\pi\omega^2} > \frac{\pi}{4\omega^2}$.

Thus by general theory [19] the equation $\xi(z) = c$ has one solution if $c$ lies outside the caustic and 3 solutions if $c$ is inside the caustic. Since $c = 0$ is inside the equation $\xi(z) = 0$ has 3 solutions, which are precisely the half-periods.

Note that the corresponding Jacobians at the half-periods $\omega_1 = \omega$, $\omega_2 = i\omega$ are

$J(z, \bar{z}) = |\varphi(\omega)|^2 - \left(\frac{\pi}{4\omega^2}\right)^2 = \left(\frac{\Gamma^4(1/4)}{32\pi\omega^2}\right) - \left(\frac{\pi}{4\omega^2}\right)^2 > 0$,

while at $z = \omega + i\omega$ we have

$J(z, \bar{z}) = |\varphi(z)|^2 - \left(\frac{\pi}{4\omega^2}\right)^2 = -\left(\frac{\pi}{4\omega^2}\right)^2 < 0$,

so the sum of the indices is 1, as it is expected since the degree of the map $\xi : \mathcal{E} \to \mathbb{C}P^1$ is 1.

Let $I \subseteq [n + 1] := \{1, 2, \ldots, n+1\}$ be a finite subset and denote $\xi_I(u) := \prod_{i \in I} \xi(u_i)$, $u \in \mathcal{E}^{n+1}$.

Consider now the following family of real-analytic (but non-holomorphic) sections of the line bundle $L$ given by

$S(u, a) = S_0 + S_0 \sum\limits_{I, J \subseteq [n+1], I \cap J = \emptyset} a_{IJ} (\xi_I(u) + \xi_J(u))$ (20)

with arbitrary coefficients $a_{IJ} = a_{JI} \in \mathbb{C}$.
**Theorem 5.2.** For generic coefficients $a_{IJ}$ the zero set of this section

$$V^n_a = \{ u \in E^{n+1} : S(u, a) = 0 \} \subset E^{n+1}$$

is a smooth connected real-analytic $U$-manifold, which can be used as a representative of the cobordism class $[B^n]$ from the Chern-Dold character.

**Proof.** Consider the set $M \subset E^n \times \mathbb{C}^N(a)$ defined by $S(u, a) = 0$. We claim that this is a smooth submanifold of this product. Indeed, assume that

$$\frac{\partial S}{\partial a_{IJ}} = S_0(\xi_I(u) + \xi_J(u)) = 0$$

for all pairs of non-intersecting subsets $I, J \subset [n]$. Then, in particular, we have

$$\prod_{i=1}^n \sigma(u_i) = 0, \prod_{i=1}^n \sigma(u_i)\xi(u_i) = 0,$$

so some of the coordinates of the potential singularities equal to 0 and some to a half-period. Let

$$I = \{ i \in [n] : \sigma(u_i) = 0 \}, J = \{ j \in [n] : \xi(u_i) = 0 \}, I \cap J = \emptyset,$$

then the corresponding $S_0(\xi_I(u) + \xi_J(u)) \neq 0$, which means that the subset $S(u, a) = 0$ is non-singular. Now the claim follows from Sard’s Lemma, saying that the set of critical values of the natural projection $\pi : M \to \mathbb{C}^N(a)$ has measure zero. □

### 6. Discussion: Milnor-Hirzebruch Problem

The Milnor-Hirzebruch problem was first posed by Hirzebruch in his ICM-1958 talk [11]. Its algebraic version can be formulated in our notations as follows:

Which sets of $\pi(n)$ integers $c_{\omega}$ can be realised as the Chern numbers $c_{\omega}(M^n)$ of some smooth irreducible complex algebraic variety?

In this version it still remains largely open, although some restrictions are known since the work of Milnor and Hirzebruch. In particular, in (complex) dimension $n = 1, 2, 3$ we have the following congruences for the usual Chern numbers of any almost complex manifold (see [11], section 7):

- $n = 1$ : $c_1 \equiv 0 \mod 2$,
- $n = 2$ : $c_2 + c_1^2 \equiv 0 \mod 12$,
- $n = 3$ : $c_1c_2 \equiv 0 \mod 24, c_3 \equiv c_1^3 \equiv 0 \mod 2$.
- $n = 4$ : $-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 \equiv 0 \mod 720, c_1^2c_2 + 2c_1^4 \equiv 0 \mod 12$,
  $$-2c_4 + c_1c_3 \equiv 0 \mod 4.$$

Since the total Chern class of $\Theta^n$ satisfies the relation (10) with $D^n = (n+1)!$ all the characteristic numbers of $\Theta^n$ equal $\pm(n+1)!$.

**Remark 6.1.** It is interesting to note that Chern numbers $c_{\omega}$ of the normal bundle $\nu\Theta^n$ is delta-like function of $\omega$ (see (14), (15)) while the Chern numbers of the tangent bundle $T\Theta^n$ are equidistributed. The analogy with the Fourier transform might be worthy to explore.
In particular, for $n = 1$ we have $c_1 = -2$,

\[ n = 2 : \quad c_1^2 = c_2 = 6, \]

\[ n = 3 : \quad c_1^3 = -c_1c_2 = c_3 = 24, \]

\[ n = 4 : \quad c_4 = c_1c_2 = c_1c_3 = c_2^2 = c_4 = 120. \]

We see that the first Hirzebruch congruence is sharp for the theta divisors, which means that it cannot be improved in the algebraic setting. This is related to the fact that the Todd genus of the theta divisor

\[ Td(\Theta^n) = (-1)^n, \]

which follows from formula [5] (see also [4]).

Recall that the divisibility conditions in terms of characteristic classes in $K$-theory were described by Hattori [10] and Stong [20]. Applying to formula [4] the Landweber-Novikov operations $S_\omega$ and taking the Todd genus we get all divisibility conditions on the Chern numbers $c_\nu(M^{2n})$ of $U$-manifolds in a different, more effective way. It is natural to ask if they can be improved for irreducible algebraic varieties.

We are planning to discuss this problem in a separate publication.

7. Acknowledgements

We are very grateful to S.P. Novikov for his interest and encouragement, and to I. Cheltsov, S. Grushevsky, A. Prendergast-Smith and Yu. Prokhorov for useful discussion of algebro-geometric aspects of this work.

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Steklov Mathematical Institute and Moscow State University, Russia
E-mail address: buchstab@mi-ras.ru

Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK, Moscow State University and Steklov Mathematical Institute, Russia
E-mail address: A.P.Veselov@lboro.ac.uk