LARGE DEVIATIONS OF THE GREEDY INDEPENDENT SET ALGORITHM ON SPARSE RANDOM GRAPHS

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ABSTRACT. We study the greedy independent set algorithm on sparse Erdős–Rényi random graphs $G(n, c/n)$. This range of $p$ is of interest due to the threshold at $c = e$, beyond which it appears that greedy algorithms are affected by a sudden change in the independent set landscape. A large deviation principle was recently established by Bermolen et al., however, the proof and rate function are somewhat involved. Using discrete calculus, we identify the optimal trajectory realizing a given large deviation and obtain the rate function in a simple closed form. The proof is brief and elementary. We think the methods presented here will be useful in analyzing other random growth and exploration processes.

1. INTRODUCTION

We investigate the size of the independent set found by the greedy algorithm on the sparse Erdős–Rényi [6] graph $G(n, c/n)$. Each pair of vertices in $G$ is joined by an edge independently with probability $p = c/n$, where $c \in (0, \infty)$ is a constant, fixed throughout this work. Recall that a set $I \subset [n]$ is independent in $G$ if no two vertices in $I$ are neighbors.

The greedy algorithm is a local exploration process. Initially, all vertices are available, $A_0 = [n]$. Beginning with $I_0 = \emptyset$, in each step $k \geq 1$, an independent set $I_k$ is formed by adding to $I_{k-1}$ a random vertex $v_k$ in the set $A_{k-1}$ of currently available vertices with no neighbors in $I_{k-1}$. Edges from $v_k$ are then revealed, and neighbors of $v_k$ in $A_{k-1}$ are removed to obtain $A_k$. The size of the independent set eventually obtained is the first step $S_g = \min\{k \geq 0 : A_k = \emptyset\}$ in which $I_k$ is maximal.

Typically, a set of size approximately $s_c n$ is found, where $s_c = (1/c) \log(1+c)$. See Section 2.1 below. In this work, we focus on the atypical behavior. For $s \neq s_c$, we let $P_s$ denote $P(S_g \leq s n)$ if $s < s_c$ and $P(S_g \geq s n)$ if $s > s_c$.

Theorem 1. Fix $s \neq s_c$. Suppose that $s_n \to s$ as $n \to \infty$. Then

$$\lim_{n \to \infty} \frac{1}{n} \log P_{s_n} = \log a_s + \frac{1}{c} \int_{a_s}^{b_s} \frac{\log u}{1-u} du$$

where $b_s = a_s e^{((1-1/a_s))}$ and $a_s > 0$ uniquely satisfies

$$s = \frac{1}{c} \log \frac{b_s - 1}{a_s - 1}.$$  \hspace{1cm} (1)
The right hand side of (1) converges to \( s_c \) as \( a \to 1 \). Otherwise, \( a \in (0, 1) \) if \( s < s_c \) and \( a \in (1, \infty) \) if \( s > s_c \).

This result is obtained in part by identifying the least-cost trajectory realizing a given deviation from \( s_c \).

**Theorem 2.** Fix \( s \neq s_c \). Asymptotically,

\[
\hat{y}_s(x) = \frac{1 + (a_s - 1)e^{cx}}{e^{cx}} \left[ a_s - \frac{1}{c(a_s - 1) \log 1 + (a_s - 1)e^{cx}} \right]
\]

is the optimal trajectory for \( |A_{Sn}|/n \) amongst those decreasing from 1 to 0 over \([0, s]\). More specifically, if \( s_n \to s \), then

\[
\lim_{n \to \infty} \frac{1}{n} \log P_{s_n} = \int_0^s \ell_s(x) \, dx
\]

where

\[
\ell_s(x) = -(1 + \hat{y}_s'(x))\left[ 1 + c\hat{y}_s(x) \right] + \log \left( 1 + \frac{c\hat{y}_s(x)}{1 + \hat{y}_s'(x)} \right)
\]

\[
= \left[ \log \frac{1 + (a_s - 1)e^{cx}}{a_s} - \frac{c(a_s - 1)}{a_s} \right] \left[ 1 - \log (1 + (a_s - 1)e^{cx}) \right]
\]

is the cost function associated with \( \hat{y}_s \).

Taking \( a \to 1 \), we recover the typical trajectory

\[
\bar{y}_c(x) = (1 + 1/c)e^{-cx} - 1/c
\]

which decreases from 1 to 0 over \([0, s_c]\). See also Section 2.1 below.

Note that \( \ell_s(x) = -\Gamma^*_\lambda(\xi) \), where \( \Gamma^*_\lambda(\xi) = \sup_\vartheta \left[ \vartheta \xi - \Gamma_\lambda(\vartheta) \right] \) is the Legendre–Fenchel transformation of the cumulant-generating function \( \Gamma_\lambda(\vartheta) = \lambda(e^{\vartheta} - 1) \) of a rate \( \lambda \) Poisson random variable, evaluated at \( \lambda_x = c\hat{y}_s(x) \) and \( \xi_x = -(1 + \hat{y}_s'(x)) \). See Section 2.2 below for a detailed heuristic.

**Figure 1.** In both figures, \( c = e \). The rate function is at left, intersecting the \( s \)-axis at \( s_c \approx 0.483 \). At right, the expected trajectory \( \bar{y}_c \) is dotted, between two least-cost deviating trajectories \( \hat{y}_s \), associated with the values \( a = 1/2 \) (\( s \approx 0.243 \)) and \( a = 2 \) (\( s \approx 0.704 \)).
1.1. **Discussion.** Recent work by Bermolen et al. [4] proves a large deviation principle for the trajectory of the greedy algorithm and the size $S_g$ of the independent set obtained. This is based on the approach of Feng and Kurtz [7] to large deviations by the theory of viscosity solutions and convergence of nonlinear semigroups. The arguments in [4] are somewhat lengthy and complicated and the rate function given there, although explicit, is not as convenient as possible.

In this work, we employ Guseinov’s [9] discrete analogue of the Euler–Lagrange equation. The proof is simple and natural. The heuristic in Section 2.2 correctly identifies the optimal trajectory $\hat{y}_s$ using the continuous Euler–Lagrange equation, and [9] provides the means to make this rigorous (Section 3).

We used [9] recently in [1] to derive sharp tail estimates for a certain percolation model on $G_{n,p}$, which are key to locating the precise leading order asymptotics of the critical $p_c$ involved. We suspect that the utility of Guseinov’s equation is wide-ranging, although perhaps not well-known in the context of discrete probability. In particular, we think the methods of the current article will be of use in studying the large deviations of a variety of random growth and exploration processes.

1.2. **Motivation.** The range $p = \Theta(1/n)$ is of interest in relation to greedy algorithms due to the so-called $e$-cutoff phenomenon. As discussed in [4], one might hope [15] that further analysis of the large deviations of such algorithms could bring this threshold into clearer light. See [4] for some loose observations in this direction. It is known [12, 2] that a slight modification of the greedy algorithm, called the degree-greedy algorithm, finds an optimal independent set for $c \leq e$. However, for larger values for $c$, the situation is more complicated. For instance, even showing that the scaled expected independence number $E(I_{c,n}/n)$ of $G(n,c/n)$ has a limiting value remained open for some time, until the breakthrough [3]. See also [8], where a more amenable weighted version is analyzed.

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2. **Heuristics**

2.1. **Typical behavior.** A simple heuristic for $s_c$ and $\bar{y}_c$ can be seen by a graphical description of the greedy algorithm. Although the proof in Section 3 does not rely on this description, it lends some useful intuition. Similar arguments are used by Nachmias and Peres [14], for instance, in analyzing the size of the largest component in the critical graph $G(n,1/n)$. This analysis is based on the exploration processes of Karp [11] and Martin-Löf [13].

Start with a row of $n$ particles, one for each vertex in $G_{n,p}$. In step $k$, and unmarked particle is marked, and then all particles move up to the next row independently with probability $1 - p$. The independent set obtained by the greedy algorithm consists of all marked particles, its size $S_g$ being the index of the first row without any unmarked particles, and the available vertices $A_k$ those in the $k$th row that are unmarked.
In the $k$th row, we expect $n(1-p)^k$ particles, and $\sum_{k=1}^{k} (1-p)^k$ of these to be marked. Therefore, we expect

$$n(1-p)^k - (1/p - 1)[1 - (1-p)^k] \sim n\bar{y}_x(k/n)$$

unmarked particles in the $k$th row. This diverges to either $+\infty$ or $-\infty$, depending on whether $k/n < s_c$ or $k/n > s_c$. Hence it can be shown that $E(S_k/n) \sim s_c$. Indeed, a central limit for $S_k$ is proved in [5].

2.2. Large deviations. The optimal deviating trajectory $\hat{y}_x(x)$ given in Theorem 2 can be guessed by Poisson approximation and the Euler–Lagrange equation.

Consider some trajectory $y_s(x)$ for $|A_{in}|/n$, decreasing from 1 to 0 over $[0,s]$, leading to an independent set of size $sn$. Suppose that $|A_{in}|$ has followed this trajectory up until $x = k/n$. Before proceeding to the next step, one of the $ny_s(x)$ available particles $v_k$ is marked (see Section 2.1 above). Approximately a Poisson with rate $\hat{\lambda}_x = cy_s(x)$ of the rest are neighbors with $v_k$. Hence, to continue along this trajectory until $x' = x + 1/n$, we require this random variable to take the value $ny_s(x) - 1 - ny_s(x') \approx -(1 + y_s'(x)) = \xi_x$. The log probability of this event is approximately $-\Gamma^*_{\hat{\lambda}}(\xi_x)$. Hence the log probability that $|A_{in}|/n$ follows $y_s(x)$ over $[0,s]$ is approximately $-n \int_0^s \Gamma^*_{\hat{\lambda}}(\xi_x)dx$. Therefore, by the Euler–Lagrange equation, we expect the optimal trajectory $\hat{y}_s(x)$ to satisfy

$$\frac{1 + y_s'(x)}{\hat{y}_s(x)} + c = \frac{d}{dx} \log^\star \left( \frac{c\hat{y}_s(x)}{1 + \hat{y}_s'(x)} \right).$$

It can be seen that $\hat{y}_s(x)$ solves this equation subject to $y_s(0) = 1$ and $y_s(s) = 0$.

3. The proof

First we show that, in the limit, the tail probability $P_x$ is dominated by a single optimal trajectory $\hat{y}_s$, which we identify by discrete calculus of variations. This establishes the upper bound. The matching lower bound follows by considering any given trajectory that is sufficiently close to $\hat{y}_s$.

3.1. Upper bound. Let $s_n \to s \neq s_c$ be given. Let us assume that $s > s_c$ and so $a_x > 1$ in [1]. The same argument works for $s < s_c$, with only minor changes.

For any $t \geq s_n$, let $0 = x_0 < x_1 < \cdots < x_m = t$ be evenly spaced points (to the extent possible subject to all $x,n \in \mathbb{Z}$) where $m$ is chosen so that all $\Delta x_i = x_{i+1} - x_i = \Theta((\log n)^2/n)$. The choice of $(\log n)^2$ is not important; only that $m \log n \ll n$.

Let $\mathcal{Y} \subset \mathbb{Z}^{m+1}$ denote the set of possible values $n = Y_0 > Y_1 > \cdots > Y_m = 0$ taken by the sequence of $|A_{in}|$. All relevant trajectories are strictly decreasing, since at least one available vertex is removed in each step. Put $\mathcal{Y}_s^+ = \bigcup_{s \geq s_n} \mathcal{Y}$. Then, taking a union bound,

$$P_{s_n} \leq \sum_{Y \in \mathcal{Y}_s^+} \prod_{i=0}^{m-1} P(Y)$$

where

$$P(Y) = P(|A_{x_i+1}| = Y_{i+1} \mid |A_{x_i}| = Y_i).$$
Moreover, since $|\mathcal{Y}_n^+| \leq O(n^m)$, it follows, by the choice of $m$, that

$$\frac{1}{n} \log P_{s_n} \leq o(1) + \frac{1}{n} \sum_{i=0}^{m-1} \log P_i(\hat{Y})$$

(2)

where $\hat{Y}$ maximizes $\sum_i \log P_i(Y)$ over $Y \in \mathcal{Y}_n^+$.

For all relevant $Y$, we have that

$$P_i(Y) \leq \left( \frac{Y_i - n\Delta x_i}{Y_{i+1}} \right) [(1-p)^{n\Delta x_i} Y_{i+1} [1 - (1-p)^{n\Delta x_i}]^{-(n\Delta x_i + \Delta Y_i)}].$$

Therefore, since $1 - (1-p)^{n\delta} = c\delta(1 + O(\delta))$, and using the standard bounds $(\frac{k}{n}) \leq (ek/(k-\ell))^k/\ell$ and $(1-x)^n \leq e^{-ny}$, we find that

$$P_i(Y) \leq \left[ -e^{cY_i/\Delta x_i} (1 + O(\Delta x_i)) \right]^{-(n\Delta x_i + \Delta Y_i)} e^{-cY_i/\Delta x_i}.$$ (3)

Observe that

$$\frac{1}{n} \sum_{i=1}^{m} (n\Delta x_i + \Delta Y_i) \log(1 + O(\Delta x_i)) \leq O[\log(1 + O(1/m))] \ll 1.$$

Therefore

$$\frac{1}{n} \sum_{i=0}^{m-1} \log P_i(Y) \leq o(1) + \sum_{i=0}^{m-1} f(y_i, \Delta y_i/\Delta x_i) \Delta x_i$$

(4)

where $y = Y/n$ and

$$f(u, w) = -(1 + w)\left[ 1 + \frac{cu}{1+w} + \log\left( \frac{cu}{1+w} \right) \right].$$

In upper bounding the righthand side of (4), we may relax the restriction that all $y/n \in \mathbb{Z}$, and instead optimize over $y \in \mathbb{R}$. Then, applying Theorem 5 in [9], we find that the maximizer $\hat{y}$ satisfies

$$f_u(\hat{y}_{i+1}, \Delta \hat{y}_{i+1}/\Delta x_{i+1}) = \Delta f_w(\hat{y}_i, \Delta \hat{y}_i/\Delta x_i)/\Delta x_i.$$ (5)

Note that

$$f_u = -\frac{1+w}{u} - c, \quad f_w = -\log\left( \frac{cu}{1+w} \right).$$

Standard results on Euler's method (see e.g. Section I.7 in [10], Theorems 7.3 and 7.5 in particular), imply that $\hat{y}_i$ and $\Delta \hat{y}_i/\Delta x_i - 1$ are within $O(1/m)$ of $\hat{y}_i$ and $\hat{y}_i'$, where $\hat{y}_i$ is the limiting trajectory satisfying

$$\frac{1 + \hat{y}_i'(x)}{\hat{y}_i(x)} + c = \frac{d}{dx} \log\left( \frac{-c\hat{y}_i(x)}{1 + \hat{y}_i'(x)} \right)$$

subject to $\hat{y}_i(0) = 1$ and $\hat{y}_i(1) = 0$. Therefore, by (2), (4) and (5),

$$\lim_{n \to \infty} \frac{1}{n} \log P_{s_n} \leq \max_{t \in [0,1]} \int_0^t f(\hat{y}_i(x), \hat{y}_i'(x)) dx.$$ (7)

To solve (6), we first observe that

$$c(z(x) + 1) = \frac{d}{dx} \log(-1/z(x)) \iff z(x) = -\frac{1}{1+(a-1)xe^x}.$$
and then that
\[
1 + \hat{y}'(x) = -\frac{c\hat{y}(x)}{1 + (a - 1)e^{cx}}
\]
\[
\implies \hat{y}(x) = \frac{1 + (a - 1)e^{cx}}{e^{cx}} \left[ b - \frac{1}{c(a - 1)} \log(1 + (a - 1)e^{cx}) \right].
\]
The boundary conditions imply that
\[
a = a_{t}, \quad b = \frac{\log a_{t}}{c(a - 1)} + \frac{1}{a_{t}}.
\]
Hence
\[
\hat{y}_t(x) = \frac{1 + (a_{t} - 1)e^{cx}}{e^{cx}} \left[ 1 - \frac{1}{a_{t}} - \frac{1}{c(a_{t} - 1)} \log \frac{1 + (a_{t} - 1)e^{cx}}{a_{t}} \right]
\]
as appears in [Theorem 2].

Next, in order to evaluate the integral in (7), note that
\[
-\frac{c\hat{y}(x)}{1 + \hat{y}'(x)} = 1 + (a_{t} - 1)e^{cx}
\]
and
\[
1 + \hat{y}'(x) = \frac{1}{(a_{t} - 1)e^{cx}} \left[ \log \frac{1 + (a_{t} - 1)e^{cx}}{a_{t}} - \frac{c(a_{t} - 1)}{a_{t}} \right].
\]
Hence
\[
f(\hat{y}(x), \hat{y}'(x)) = \left[ \log \frac{1 + (a_{t} - 1)e^{cx}}{a_{t}} - \frac{c(a_{t} - 1)}{a_{t}} \right] \left[ 1 - \frac{\log(1 + (a_{t} - 1)e^{cx})}{(a_{t} - 1)e^{cx}} \right].
\]
Note that this is \(\ell_{t}(x)\) in [Theorem 2]. Then, by basic calculus,
\[
\int f(\hat{y}(x), \hat{y}'(x))dx = -\hat{y}(x) \log(1 + (a_{t} - 1)e^{cx}) + \frac{1}{c} \int_{1}^{1 + (a_{t} - 1)e^{ct}} \frac{\log u}{1 - u} du.
\]
Therefore, since \(1 + (a_{t} - 1)e^{ct} = b_{t}\) by (1), it follows by (7) and (8) that
\[
\lim_{n \to \infty} -\frac{1}{n} \log P_{s_n} \leq \max_{t \in [s, 1]} \left[ \log a_{t} + \frac{1}{c} \int_{a_{t}}^{b_{t}} \frac{\log u}{1 - u} du \right].
\]
Finally, to complete the proof we show that \(t = s\) is the maximizing case, as is expected, since \(s\) is the least extreme deviation from \(s_{c}\) in \([s, 1]\). Note that \(t > s_{c}\) correspond with \(a_{t} \in (1, \infty)\) in (1), which are increasing in \(t\). Hence we show that the right hand side in (9) is decreasing in \(a > 1\). Observe that, with \(b = ace^{(1 - 1/a)}\),
\[
\frac{d}{da} \left[ \log a + \frac{1}{c} \int_{a}^{b} \frac{\log u}{1 - u} du \right] = \frac{(1 - c(1 - 1/a))(b/a) - 1}{b - 1} \left( \frac{\log a}{c(a - 1)} + \frac{1}{a} \right).
\]
Since
\[
\frac{\log a}{c(a - 1)} + \frac{1}{a} > 0
\]
for all \(a > 0\), and \(b > a > 1\) for \(a > 1\), we need only check that
\[
(1 - \frac{c(a - 1)}{a}) \frac{b}{a} < 1.
\]
This follows noting that the left hand side is equal to 1 when \( a = 1 \), since then also \( b = 1 \), and that
\[
\frac{d}{da} \left( \left( 1 - \frac{c(a-1)}{a} \right) x \right) = \frac{(1-a)bc^2}{a^3} < 0.
\]
This proves the claim.

Altogether, by \((9)\), we find that
\[
\lim_{n \to \infty} \frac{1}{n} \log P_{s_n} \leq \log a_s + \frac{1}{c} \int_{a_s}^{b_s} \log u \frac{du}{1-u}
\]
as required.

### 3.2. Lower bound

Having identified \( \hat{y}_s \), the lower bound follows easily by considering any trajectory of \(|A_{2n}|/n\) sufficiently close to \( \hat{y}_s \). We sketch the argument (using the notation from [Section 3.1]).

The first step is to find a lower bound for \( P(Y) \) which agrees with the upper bound \((3)\) up to small error. For all relevant \( Y \),
\[
P(Y) \geq \left( \frac{Y_i - n\Delta_i}{Y_i+1} \right) (1 - p)^{n\Delta_i + 1} \left[ (1 - p)^{n\Delta_i} Y_i^{n\Delta_i} (1 - p)^{n\Delta_i - 1} \right]^{-1}.
\]
Using the bound \((6)\) \( \geq (e(k/\ell - 1))^1/e\ell \), it follows that
\[
P(Y) \geq \left( -e \frac{ey_i+1}{n + \Delta_i} \right)^{-1} \left( 1 - p \right)^{y_i+1} \left( 1 - p \right)^{-y_i-1} = e(n\Delta_i + \Delta_i).
\]
Therefore, by the choice of \( m \), this implies that
\[
\frac{1}{n} \sum_{i=0}^{m-1} \log P(Y) \geq o(1) + \sum_{i=0}^{m-1} f(y_i + 1, \Delta_y / \Delta_i) \Delta_i.
\]
Hence, comparing this with \((4)\), it is not hard to see, by considering any possible trajectory \( y \) sufficiently close to \( \hat{y}_s \), that
\[
\lim_{n \to \infty} \frac{1}{n} \log P_{s_n} \geq \log a_s + \frac{1}{c} \int_{a_s}^{b_s} \log u \frac{du}{1-u}.
\]

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