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Effective bounds for Faltings’s delta function

JAY JORGENSON\textsuperscript{(1)}, JüRG KRAMER\textsuperscript{(2)}

Dedicated to Christophe Soulé at his sixtieth birthday

\textbf{Abstract. —} In his seminal paper on arithmetic surfaces Faltings introduced a new invariant associated to compact Riemann surfaces $X$, nowadays called Faltings’s delta function and here denoted by $\delta_{\text{Fal}}(X)$. For a given compact Riemann surface $X$ of genus $g_X = g$, the invariant $\delta_{\text{Fal}}(X)$ is roughly given as minus the logarithm of the distance with respect to the Weil-Petersson metric of the point in the moduli space $M_g$ of genus $g$ curves determined by $X$ to its boundary $\partial M_g$. In this paper we begin by revisiting a formula derived in [14], which gives $\delta_{\text{Fal}}(X)$ in purely hyperbolic terms provided that $g > 1$. This formula then enables us to deduce effective bounds for $\delta_{\text{Fal}}(X)$ in terms of the smallest non-zero eigenvalue of the hyperbolic Laplacian acting on smooth functions on $X$ as well as the length of the shortest closed geodesic on $X$. The article ends with a discussion of an application of our results to Parshin’s covering construction.

\textbf{Résumé. —} Dans son article fondateur sur les surfaces arithmétiques Faltings a introduit un nouvel invariant des surfaces de Riemann compactes, que l’on appelle de nos jours l’invariant delta de Faltings et que l’on note $\delta_{\text{Fal}}(\cdot)$. Pour une surface de Riemann compacte $X$ de genre $g_X = g$, l’invariant $\delta_{\text{Fal}}(X)$ est donné à peu de choses près par l’opposé du logarithme de la distance, pour la métrique de Weil-Petersson, du point sur l’espace de modules $M_g$ des courbes de genre $g$ déterminé par $X$ à

\textsuperscript{(1)} Department of Mathematics, City College of New York, Convent Avenue at 138th Street, New York, NY 10031 U.S.A. jjorgenson@mindspring.com

\textsuperscript{(2)} Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany kramer@math.hu-berlin.de

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son bord $\partial M_g$. Dans le présent article nous commençons par un nouvel examen de la formule obtenue dans [14], qui décrit $\delta_{\text{Fal}}(X)$ en termes purement hyperboliques, tout au moins si $g > 1$. Cette formule nous permet ensuite de déduire des bornes effectives pour $\delta_{\text{Fal}}(X)$ en termes de la plus petite valeur propre non-nulle du Laplacien hyperbolique agissant sur les fonctions lisses sur $X$ et du minimum des longueurs des géodésiques fermées sur $X$. L’article se termine par une discussion d’une application de nos résultats à la construction du recouvrement de Parshin.

1. Introduction

In [1] and [2], S. J. Arakelov introduced Green’s functions on compact Riemann surfaces in order to define an intersection theory on arithmetic surfaces, thus initiating a far-reaching mathematical program which bears his name. G. Faltings extended the pioneering work of Arakelov in [8] by defining metrics on determinant line bundles arising from the cohomology of algebraic curves, from which he derived arithmetic versions of the Riemann-Roch theorem, Noether’s formula, and the Hodge index theorem. Although Faltings employed the classical Riemann theta function to define metrics on these determinant line bundles, he does refer to the emerging idea of D. Quillen to use Ray–Singer analytic torsion to define the metrics on these determinant line bundles as being “more direct”. One of the many aspects of the mathematical legacy of Christophe Soulé is the central role he played in developing higher dimensional Arakelov theory where Quillen metrics are fully utilized; see [22] and the references therein.

From Faltings’s theory [8], there appears a naturally defined analytic quantity associated to any compact Riemann surface $X$. The new invariant in [8] became known as Faltings’s delta function, which we denote by $\delta_{\text{Fal}}(X)$. Many of the fundamental arithmetic theorems and formulas in [8], such as those listed above, amount to statements which involve $\delta_{\text{Fal}}(X)$. By comparing the Riemann-Roch theorem from [8] and the arithmetic Riemann-Roch theorem, Soulé expressed in [21] the Faltings’s delta function in terms of the analytic torsion of the trivial line bundle on $X$ when given the Arakelov metric; see equation (3.1) below, as well as [21] and, more recently, [24].

The Polyakov formula allows one to relate values of the analytic torsion for conformally equivalent metrics. As a result, one can use Soulé’s formula for the Faltings’s delta function and obtain an identity which expresses $\delta_{\text{Fal}}(X)$ in terms of the hyperbolic geometry of $X$; see Theorem 3.2 below, which comes from [14]. In [14], we used the relation between the Faltings’s delta function and the hyperbolic geometry in order to study $\delta_{\text{Fal}}(X)$.
Effective bounds for Faltings’s delta function through covers. As an arithmetic application of the analytic bounds obtained for $\delta_{\text{Fal}}(X)$, we derived in [14] an improved estimate for the Faltings height of the Jacobian of the modular curve $X_0(N)$ for square-free $N$ which is not divisible by 6.

After the completion of [14], A. N. Parshin posed the following question to the second named author: Can one derive an effective bound for the Faltings’s delta function $\delta_{\text{Fal}}(X)$ in terms of basic information associated to the hyperbolic geometry of $X$? The purpose of the present article is to provide an affirmative answer to Parshin’s question. More specifically, our main results, given in Theorem 6.1 and Corollaries 6.3, 6.4, explicitly bound $\delta_{\text{Fal}}(X)$ in the case when $X$ is a finite degree covering of a compact Riemann surface $X_0$ of genus bigger than 1, where the bound for $\delta_{\text{Fal}}(X)$ is effectively computable once knowing the genera of $X_0$ and $X$, the smallest non-zero eigenvalues of the hyperbolic Laplacian acting on $X_0$ and $X$, and the length of the shortest closed geodesic on $X_0$ (as well as some ramification data in case the covering is ramified).

An important ingredient in the analysis of the present paper is the algorithm from [9], which provides effective means by which one can bound the Huber constant on $X$, a quantity associated to the error term in the prime geodesic theorem; see [9] and the references therein. As with the main result in [9], it is possible that the effective bound we obtain here may not be optimal, perhaps even far from it. However, the existence of an effective bound for the Faltings’s delta function $\delta_{\text{Fal}}(X)$, albeit a sub-optimal bound, may be a tool by which one can further investigate the application of Arakelov theory to diophantine problems, as originally intended.

The paper is organized as follows: After recalling basic notations in section 2, we express Faltings’s delta function $\delta_{\text{Fal}}(X)$ in hyperbolic terms of $X$ in section 3. Section 4 is devoted to derive effective bounds for the ratio $\mu_{\text{can}}(z)/\mu_{\text{hyp}}(z)$ of the canonical by the hyperbolic metric on $X$ and section 5 gives effective bounds for the Huber constant $C_{\text{Hub},X}$ on $X$. In section 6, we combine the results of the sections 3, 4, 5 to derive effective bounds for $\delta_{\text{Fal}}(X)$. In section 7, we discuss an application of our results to an idea of A. N. Parshin for an attempt giving effective bounds for the height of rational points on smooth projective curves defined over number fields.

Acknowledgements. — We would like to use this opportunity to thank Christophe Soulé for having introduced us into the theory of arithmetic intersections by generously sharing his broad knowledge and deep insights on the subject with us. Furthermore, we would like to thank Alexei Parshin for his interest in our results and for having pointed out to us an application
to his work. Finally, we would like to thank the referee for some of his/her comments.

2. Basic notations

2.1. Hyperbolic and canonical metrics

In this note $X$ will denote a compact Riemann surface of genus $g_X > 1$. By the uniformization theorem, $X$ is isomorphic to the quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma$ is a cocompact and torsionfree Fuchsian subgroup of the first kind of $\text{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} \mid z = x + iy, y > 0 \}$. In the sequel, we will identify $X$ locally with its universal cover $\mathbb{H}$.

We denote by $\mu_{\text{hyp}}$ the $(1, 1)$-form corresponding to the hyperbolic metric on $X$, which is compatible with the complex structure of $X$ and has constant negative curvature equal to $-1$. Locally, we have

$$\mu_{\text{hyp}}(z) = i \frac{dz \wedge d\overline{z}}{\text{Im}(z)^2} = \frac{dx \wedge dy}{y^2}.$$ 

We write $\text{vol}_{\text{hyp}}(X)$ for the hyperbolic volume of $X$; recall that $\text{vol}_{\text{hyp}}(X)$ is given by $4\pi(g_X - 1)$. By $\mu_{\text{shyp}}$, we denote the $(1, 1)$-form corresponding to the rescaled hyperbolic metric, which measures the volume of $X$ to be 1. We write $\text{dist}_{\text{hyp}}(z, w)$ for the hyperbolic distance between two points $z, w \in \mathbb{H}$. We recall the formula

$$\text{dist}_{\text{hyp}}(z, w) = \cosh^{-1} \left( 1 + \frac{|z - w|^2}{2 \text{Im}(z)\text{Im}(w)} \right).$$

We denote the hyperbolic Laplacian on $X$ by $\Delta_{\text{hyp}}$; locally, we have

$$\Delta_{\text{hyp}} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The discrete spectrum of $\Delta_{\text{hyp}}$ is given by the increasing sequence of eigenvalues

$$0 = \lambda_{X,0} < \lambda_{X,1} \leq \lambda_{X,2} \leq \ldots$$

The $(1, 1)$-form $\mu_{\text{can}}$ associated to the canonical metric is defined as follows. Let $\{\omega_1, \ldots, \omega_{g_X}\}$ denote an orthonormal basis of the space $\Gamma(X, \Omega^1_X)$ of holomorphic 1-forms on $X$. Then, $\mu_{\text{can}}$ is locally given by

$$\mu_{\text{can}}(z) = \frac{1}{g_X} \cdot \frac{i}{2} \sum_{j=1}^{g_X} \omega_j(z) \wedge \overline{\omega}_j(z).$$
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We recall that the Arakelov metric on $X$ is induced by means of the residual canonical metric $\| \cdot \|_{\text{Ar}}$ on $\Omega^1_X$, which turns the residue map into an isometry.

2.2. Hyperbolic heat kernel for functions

The hyperbolic heat kernel $K_{\mathbb{H}}(t; z, w)$ ($t \in \mathbb{R}_{>0}; z, w \in \mathbb{H}$) for functions on $\mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t; z, w) := K_{\mathbb{H}}(t; \rho) := \sqrt{2} e^{-t/4} \int_\rho^\infty \frac{r e^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} \, dr,$$

where $\rho = \text{dist}_{\text{hyp}}(z, w)$. The hyperbolic heat kernel $K_X(t; z, w)$ ($t \in \mathbb{R}_{>0}; z, w \in X$) for functions on $X$ is obtained by averaging over the elements of $\Gamma$, namely

$$K_X(t; z, w) := \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; z, \gamma w).$$

The heat kernel $K_X(t; z, w)$ satisfies the equations

$$\left( \frac{\partial}{\partial t} + \Delta_{\text{hyp}, z} \right) K_X(t; z, w) = 0 \quad (z, w \in X),$$

$$\lim_{t \to 0} \int_X K_X(t; z, w) f(w) \mu_{\text{hyp}}(w) = f(z) \quad (z \in X)$$

for all $C^\infty$-functions $f$ on $X$. As a shorthand, we use in the sequel the notation

$$HK_X(t; z) := \sum_{\substack{\gamma \in \Gamma \\gamma \neq \text{id}}} K_{\mathbb{H}}(t; z, \gamma z).$$

2.3. Selberg zeta function

Let $\mathcal{H}(\Gamma)$ denote the set of conjugacy classes of primitive, hyperbolic elements in $\Gamma$. We denote by $\ell_\gamma$ the hyperbolic length of the closed geodesic determined by $\gamma \in \mathcal{H}(\Gamma)$ on $X$; it is well-known that the equality

$$|\text{tr}(\gamma)| = 2 \cosh(\ell_\gamma/2)$$

holds.

For $s \in \mathbb{C}$, $\Re(s) > 1$, the Selberg zeta function $Z_X(s)$ associated to $X$ is defined via the Euler product expansion

$$Z_X(s) := \prod_{\gamma \in \mathcal{H}(\Gamma)} Z_\gamma(s),$$

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where the local factors $Z_\gamma(s)$ are given by

$$Z_\gamma(s) := \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\ell_\gamma}\right).$$

The Selberg zeta function $Z_X(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian; furthermore, $Z_X(s)$ satisfies a functional equation. For our purposes, it suffices to know that the Selberg zeta function $Z_X(s)$ has a simple zero at $s = 1$, so that the quantity

$$\lim_{s \to 1} \left(\frac{Z_X'(s)}{Z_X(s)} - \frac{1}{s - 1}\right)$$

is well-defined.

### 2.4. Prime geodesic theorem

For any small eigenvalue $\lambda_{X,j} \in [0, 1/4)$, we define

$$s_{X,j} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,j}},$$

and note that $1/2 < s_{X,j} \leq 1$. For $u \in \mathbb{R}_{>1}$, we recall the prime geodesic counting function

$$\pi_X(u) := \#\{\gamma \in \mathcal{H}(\Gamma) | e^{\ell_\gamma} < u\}.$$ 

Introducing the logarithmic integral

$$\text{li}(u) := \int_{2}^{u} \frac{d\xi}{\log(\xi)},$$

the prime geodesic theorem states

$$\pi_X(u) = \sum_{0 \leq \lambda_{X,j} < 1/4} \text{li}(u^{s_{X,j}}) + O_X(u^{3/4} \log(u)^{-1/2}) \quad (2.1)$$

for $u > 1$, where the implied constant for all $u > 1$, not just asymptotically, depends solely on $X$. We call the infimum of all possible implied constants the Huber constant and denote it by $C_{\text{Hub},X}$. 

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3. Faltings’s delta function in hyperbolic terms

3.1. Faltings’s delta function

Faltings’s delta function $\delta_{\text{Fal}}(X)$ was introduced in [8], where also some of its basic properties were given. In [10], Faltings’s delta function is expressed in terms of Riemann theta functions, and its asymptotic behavior is investigated; see also [23]. As a by-product of the analytic part of the arithmetic Riemann-Roch theorem for arithmetic surfaces, C. Soulé has shown in [21] that

$$
\delta_{\text{Fal}}(X) = -6D_{\text{Ar}}(X) + a(g_X), \quad (3.1)
$$

where

$$
D_{\text{Ar}}(X) := \log \left( \frac{\det^* (\Delta_{\text{Ar}})}{\text{vol}_{\text{Ar}}(X)} \right)
$$

with $\det^* (\Delta_{\text{Ar}})$ the regularized determinant of the Laplacian, $\text{vol}_{\text{Ar}}(X)$ the volume with respect to the Arakelov metric $\| \cdot \|_{\text{Ar}}$, and

$$
a(g_X) := -2g_X \log(\pi) + 4g_X \log(2) + (g_X - 1)(-24\zeta'_Q(-1) + 1).
$$

It has been shown in [14] how Faltings’s delta function can be expressed solely in hyperbolic terms. Theorem 3.8 therein states:

3.2. Theorem

For $X$ with genus $g_X > 1$, let

$$
F(z) := \int_0^\infty \left( HK_X(t; z) - \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) dt.
$$

Then, we have

$$
\delta_{\text{Fal}}(X) = 2\pi \left( 1 - \frac{1}{g_X} \right) \int_X F(z) \Delta_{\text{hyp}} F(z) \mu_{\text{hyp}}(z) - 6\log \left( Z'_X(1) \right) \quad (3.2)
$$

$$
+ 2 \lim_{s \to 1} \left( \frac{Z'_X(s)}{Z_X} - \frac{1}{s - 1} \right) + c(g_X),
$$

where

$$
c(g_X) := a(g_X) - 6b(g_X) + 2(g_X - 1) \log(4) + 6 \log \left( \text{vol}_{\text{hyp}}(X) \right) - 2
$$

$$
= 2g_X \left( -24\zeta'_Q(-1) - 4 \log(\pi) + \log(2) + 2 \right) + 6 \log \left( \text{vol}_{\text{hyp}}(X) \right)
$$

$$
+ \left( 48\zeta'_Q(-1) + 6 \log(2\pi) - 2 \log(4) - 6 \right)
$$
with \( a(g_X) \) as above and \( b(g_X) \) given by
\[
b(g_X) := (g_X - 1)(4\zeta_Q'(1) - 1/2 + \log(2\pi)).
\]

**Proof.** — The proof is given in [14]. Here we present only a short outline of the proof, which consists of the following three main ingredients:

(i) One starts by using the Polyakov formula to relate the regularized determinants with respect to the Arakelov and the hyperbolic metric, namely
\[
D_{Ar}(X) = D_{hyp}(X) + \frac{g_X - 1}{6} \int_X \phi_{Ar}(z) \left( \mu_{can}(z) + \mu_{hyp}(z) \right),
\]
where \( \phi_{Ar}(z) \) is the conformal factor describing the change from the Arakelov to the hyperbolic metric.

(ii) In a second step, one uses the result [20] by P. Sarnak describing the hyperbolic regularized determinant in terms of the Selberg zeta function, namely
\[
D_{hyp}(X) = \log \left( \frac{Z'_X(1)}{\text{vol}_{hyp}(X)} \right) + b(g_X).
\]

(iii) In order to express the conformal factor \( \phi_{Ar}(z) \) and the canonical metric form \( \mu_{can}(z) \) in hyperbolic terms, we make use of the fundamental relation
\[
\mu_{can}(z) = \mu_{shyp}(z) + \frac{1}{2g_X} \left( \int_0^\infty \Delta_{hyp} K_X(t; z) \, dt \right) \mu_{hyp}(z), \tag{3.3}
\]
which has been proven in Appendix 1 of [14]. \( \square \)

**3.3. Remark**

We note that formula (3.3) has meanwhile been generalized to cofinite Fuchsian subgroups of the first kind of \( \text{PSL}_2(\mathbb{R}) \) without torsion elements in [16], and, as a relation of \((1,1)\)-currents, to cofinite Fuchsian subgroups of the first kind of \( \text{PSL}_2(\mathbb{R}) \) allowing torsion elements in [3].

Based on formula (3.2), the following bound can be derived for \( \delta_{Fal}(X) \) in terms of basic hyperbolic invariants of \( X \). For this we introduce the following notations
\[
\lambda_X := \frac{1}{2} \min \left\{ \lambda_{X,1}, \frac{7}{64} \right\},
\]
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\[ N_{ev,X}^{(0,1/4)} := \#\{\lambda_{X,j} \mid 0 \leq \lambda_{X,j} < 1/4\}, \]
\[ N_{geo,X}^{(0,5)} := \#\{\gamma \in \mathcal{H}(\Gamma) \mid 0 < \ell_{\gamma} < 5\}, \]
\[ S_X := \sup_{z \in X} \left( \frac{\mu_{can}(z)}{\mu_{hyp}(z)} \right), \]

where \( \lambda_{X,1} \) is the smallest non-zero eigenvalue of \( \Delta_{hyp} \), and we recall that \( \ell_X \) denotes the length of the shortest closed geodesic on \( X \) and \( C_{Hub,X} \) is the Huber constant introduced in subsection 2.4.

### 3.4. Corollary

With the above notations, we have the bound

\[ \delta_{Fal}(X) \leq D_1 \left( g_X + \frac{1}{\lambda_X} (g_X(S_X+1)^2 + C_{Hub,X} + N_{ev,X}^{(0,1/4)}) + \left( 1 + \frac{1}{\ell_X} \right) N_{geo,X}^{(0,5)} \right) \]

with an absolute constant \( D_1 > 0 \), which can be taken to be \( 10^3 \).

**Proof.** — The proof is straightforward using Theorem 3.2 in combination with the estimates given in Propositions 4.1, 4.2, 4.3, and Lemma 4.4 in [14]. For the convenience of the reader, we give now a more detailed derivation of the proof.

Using Proposition 4.1 of [14] in combination with the inequalities \( \lambda_{X,1} \geq \lambda_X \) and \( \text{vol}_{hyp}(X) \leq 4\pi g_X \), the integral in (3.2) can be bounded as

\[ \left| \int_X F(z) \Delta_{hyp} F(z) \mu_{hyp}(z) \right| \leq \frac{(S_X + 1)^2 \text{vol}_{hyp}(X)}{\lambda_{X,1}} \leq \frac{4\pi g_X}{\lambda_X} (S_X + 1)^2. \quad (3.4) \]

In order to bound the absolute value of the second summand in (3.2), we first observe that we have to take the second bound in Proposition 4.3 of [14], since the first one being logarithmic in \( g_X \) is too small; choosing \( \varepsilon = \lambda_X \), we obtain

\[ |\log (Z_X'(1))| \leq - \sum_{\gamma \in \mathcal{H}(\Gamma) \atop \ell_{\gamma} < 5} \log (Z_{\gamma}(1)) + 12 \left( 5 + \frac{1}{\lambda_X} \right) (C_{Hub,X} + N_{ev,X}^{(0,1/4)} + 1). \]

Using Lemma 4.4 (i) of [14], we derive from this the bound

\[ |\log (Z_X'(1))| \leq \sum_{\gamma \in \mathcal{H}(\Gamma) \atop \ell_{\gamma} < 5} \frac{\pi^2}{6\ell_{\gamma}} + 72 \left( C_{Hub,X} + N_{ev,X}^{(0,1/4)} + 1 \right) \]

\[ \leq \frac{\pi^2}{6\ell_X} N_{geo,X}^{(0,5)} + \frac{144}{\lambda_X} (C_{Hub,X} + N_{ev,X}^{(0,1/4)}). \quad (3.5) \]
Finally, in order to bound the absolute value of the third summand in (3.2), we again observe that we have to take the second bound in Proposition 4.2 of [14], since the first one being logarithmic in $g_X$ is too small; choosing again $\varepsilon = \lambda_X$, we obtain

$$
\left| \lim_{s \to 1} \left( \frac{Z'_X}{Z_X}(s) - \frac{1}{s - 1} \right) \right| \leq \sum_{\gamma \in \mathcal{H}(\Gamma)} \frac{Z'_X}{Z_X}(1) + \frac{6}{\lambda_X} \left( C_{\text{Hub},X} + N_{\text{ev},X}^{[0,1/4]} \right) + 2.
$$

Using Lemma 4.4 (ii) of [14], we derive from this the bound

$$
\left| \lim_{s \to 1} \left( \frac{Z'_X}{Z_X}(s) - \frac{1}{s - 1} \right) \right| \leq \sum_{\gamma \in \mathcal{H}(\Gamma)} \left( 3 + \log \left( \frac{1}{\ell_\gamma} \right) \right)
+ \frac{6}{\lambda_X} \left( C_{\text{Hub},X} + N_{\text{ev},X}^{[0,1/4]} \right) + 2
\leq \sum_{\gamma \in \mathcal{H}(\Gamma)} \left( 3 + \frac{1}{\ell_\gamma} \right) + \frac{6}{\lambda_X} \left( C_{\text{Hub},X} + N_{\text{ev},X}^{[0,1/4]} \right) + 2
\leq \left( 3 + \frac{1}{\ell_X} \right) N_{\text{geo},X}^{(0,5)}
+ \frac{6}{\lambda_X} \left( C_{\text{Hub},X} + N_{\text{ev},X}^{[0,1/4]} \right) + 2.
\quad (3.6)
$$

The quantity $c(g_X)$ in (3.2) is easily bounded as

$$
c(g_X) \leq 11g_X + 10.
\quad (3.7)
$$

Adding up the bounds (3.4)–(3.7), using that $g_X > 1$, and by crudely estimating the arising integral constants by $D_1 = 10^3$, yields the claimed bound. Note that, estimating more rigorously, $D_1$ can in fact be taken to be 876.

\[\square\]

### 4. Effective bounds for the sup-norm

#### 4.1. Hyperbolic heat kernel for forms

In addition to the hyperbolic heat kernel on $\mathbb{H}$, resp. $X$, introduced in subsection 2.2, we also need the hyperbolic heat kernel for forms of weight 1 on $\mathbb{H}$, resp. $X$. The hyperbolic heat kernel for forms of weight 1 on $\mathbb{H}$ is defined as in [13], namely we have

$$
K_{\mathbb{H}}^{(1)}(t; z, w) := K_{\mathbb{H}}^{(1)}(t; \rho) := \sqrt{2} e^{-t/4} \int_0^\infty \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr,
$$

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where $T_2$ is the Chebyshev polynomial given by $T_2(r) := 2r^2 - 1$. The hyperbolic heat kernel for forms of weight 1 on $X$ on the diagonal is then given as

$$K^{(1)}_X(t; z) := \sum_{\gamma \in \Gamma} c(\gamma; z) K^{(1)}_{\mathbb{H}}(t; z, \gamma z),$$

where $c(\gamma, z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as

$$c(\gamma, z) := \frac{cz + d}{cz + \gamma z} \cdot \frac{z - \gamma z}{\gamma z - \bar{z}}.$$

We note that $|c(\gamma, z)| = 1$. From [13], we recall the crucial relation

$$\lim_{t \to \infty} K^{(1)}_X(t; z) = g_X \mu_{\text{can}}(z) \mu_{\text{hyp}}(z).$$

(4.1)

4.2. Lemma

With the above notations, we have the bound

$$K^{(1)}_{\mathbb{H}}(t; \rho) \leq \frac{17\sqrt{2} e^{-t/4} (\rho + \log(4))e^{-\rho^2/(4t)}}{(4\pi t)^{3/2}} \sinh^{1/2}(\rho) + \frac{4 \sqrt{2} e^{-(\rho/(2\sqrt{t}) + \sqrt{t}/2)^2}}{\pi^{3/2} \sqrt{t} + 4 \sqrt{2} e^{-(\rho/(2\sqrt{t}) + \sqrt{t}/2)^2} \int_{\rho/(2\sqrt{t}) - \sqrt{t}/2}^{\rho + \log(4)} e^{-r^2} dr'}$$

for any $t > 0$ and $\rho > 0$.

Proof. — Starting with the defining formula

$$K^{(1)}_{\mathbb{H}}(t; \rho) := \sqrt{2} e^{-t/4} \int_{\rho}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr,$$

we decompose the integral under consideration as

$$\int_{\rho}^{\rho + \log(4)} \ldots = \int_{\rho}^{\rho + \log(4)} \ldots + \int_{\rho + \log(4)} \ldots$$

(4.3)

We start by estimating the first integral on the right-hand side of (4.3). Using the mean value theorem for the function $\cosh(r)$ with $r \in [\rho, \rho + \log(4)]$, we obtain the bound

$$\cosh(r) - \cosh(\rho) = (r - \rho) \sinh(r_*) \geq (r - \rho) \sinh(\rho),$$

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where $r_* \in [\rho, \rho + \log(4)]$. With this in mind, we have the estimate
\[
\int_{\rho}^{\rho + \log(4)} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr
\leq \frac{(\rho + \log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)} T_2 \left( \frac{\cosh((\rho + \log(4))/2)}{\cosh(\rho/2)} \right) \int_{\rho}^{\rho + \log(4)} (r - \rho)^{-1/2} dr
\leq \frac{2 \log(4)^{1/2}(\rho + \log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)} T_2 \left( \frac{\cosh((\rho + \log(4))/2)}{\cosh(\rho/2)} \right).
\]

Since, for any $r_1, r_2 \in \mathbb{R}_{>0}$, we have
\[
\frac{\cosh(r_1 + r_2)}{\cosh(r_1)} = \frac{\cosh(r_1) \cosh(r_2)}{\cosh(r_1)} + \frac{\sinh(r_1) \sinh(r_2)}{\cosh(r_1)} \leq \cosh(r_2) + \sinh(r_2) = e^{r_2},
\]
we can estimate the Tshebyshev polynomial contribution as
\[
T_2 \left( \frac{\cosh((\rho + \log(4))/2)}{\cosh(\rho/2)} \right) \leq T_2(e^{\log(4)/2}) = 7.
\]

In summary, we find the following bound for the integral in question
\[
\int_{\rho}^{\rho + \log(4)} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr \leq \frac{17(\rho + \log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)}. \tag{4.4}
\]

We now estimate the second integral on the right-hand side of (4.3). Since $r \geq \rho + \log(4)$, we have
\[
\frac{\cosh(r)}{2} \geq \frac{\cosh(\rho + \log(4))}{2} \geq \frac{\cosh(\rho) \cosh(\log(4))}{2} \geq \cosh(\rho),
\]
whence
\[
\cosh(r) - \cosh(\rho) \geq \frac{\cosh(r)}{2} \geq \frac{e^r}{4}.
\]
Therefore, using the estimate $T_2(r) \leq 2r^2$ in combination with
\[
\cosh(r/2) \leq e^{r/2} \quad \text{and} \quad \cosh(\rho/2) \geq \frac{e^{\rho/2}}{2},
\]
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we derive the bound

\[
\int_{\rho+\log(4)}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr \leq \\
\int_{\rho+\log(4)}^{\infty} \frac{2re^{-r^2/(4t)}}{e^{\rho/2}} e^{\rho} dr = 16 e^{-\rho} \int_{\rho+\log(4)}^{\infty} r e^{r/2} e^{-r^2/(4t)} dr . (4.5)
\]

In order to complete the proof, we will further estimate the integral in (4.5). Keeping in mind that we finally have to multiply (4.5) by the factor \(e^{-t/4}\), we estimate the quantity

\[
e^{-t/4} \int_{\rho+\log(4)}^{\infty} r e^{r/2} e^{-r^2/(4t)} dr \leq \int_{\rho}^{\infty} r e^{-(r/(2\sqrt{t})-\sqrt{t}/2)^2} dr \\
= 2 \sqrt{t} \int_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} (2\sqrt{t}r' + t) e^{-r^2} dr' \\
= 2 t e^{-(\rho/(2\sqrt{t})-\sqrt{t}/2)^2} + 2 t^{3/2} \int_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} e^{-r^2} dr'.
\]

Multiplying by the remaining factor

\[
\frac{16 \sqrt{2} e^{-\rho}}{(4\pi t)^{3/2}} = \frac{2 \sqrt{2} e^{-\rho}}{(\pi t)^{3/2}},
\]

yields the following bound involving the second integral

\[
\frac{\sqrt{2} e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho+\log(4)}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr \leq \\
\frac{4 \sqrt{2} e^{-(\rho/(2\sqrt{t})+\sqrt{t}/2)^2}}{\pi^{3/2} \sqrt{t}} + \frac{4 \sqrt{2} e^{-\rho}}{\pi^{3/2}} \int_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} e^{-r^2} dr'. (4.6)
\]

Adding up the bounds (4.4) and (4.6) yields the claimed upper bound for \(K_{H}^{(1)}(t; \rho)\). \(\square\)
4.3. Lemma

Let $X \rightarrow X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0 \backslash \mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$, and let $\ell_{X_0}$ denote the length of the shortest closed geodesic on $X_0$. Then, the quantity $S_X$ can be bounded as

$$S_X \leq 4\pi \int_{\ell_{X_0}/4}^{\infty} K^{(1)}_{\mathbb{H}}(t_0; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho$$

$$+ 4\pi K^{(1)}_{\mathbb{H}}(t_0; \ell_{X_0}/4) \left( \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1 \right)$$

(4.7)

for any $t_0 > 0$.

Proof. — From the spectral expansion, one immediately sees that the function $K^{(1)}_{\mathbb{H}}(t; z)$ is monotone decreasing in $t$. Using relation (4.1) together with the triangle inequality, we then obtain for any $t_0 > 0$, the bound

$$g_{X_0} \frac{\mu_{\text{can}}(z)}{\mu_{\text{hyp}}(z)} \leq \sum_{\gamma \in \Gamma} K^{(1)}_{\mathbb{H}}(t_0; z, \gamma z) \leq \sum_{\gamma \in \Gamma_0} K^{(1)}_{\mathbb{H}}(t_0; z, \gamma z).$$

Using the counting function

$$N_{X_0}(\rho; z) := \# \{ \gamma \in \Gamma_0 \mid \text{dist}_{\text{hyp}}(z, \gamma z) < \rho \},$$

we can express the latter bound in terms of the Stieltjes integral

$$g_{X_0} \frac{\mu_{\text{can}}(z)}{\mu_{\text{hyp}}(z)} \leq \int_{\ell_{X_0}/4}^{\infty} K^{(1)}_{\mathbb{H}}(t_0; \rho) \, dN_{X_0}(\rho; z).$$

With the notation of Lemma 4.6 of [11], we put $u := \rho$, $a := \ell_{X_0}/4$, and further

$$F(u) := K^{(1)}_{\mathbb{H}}(t_0; \rho),$$

$$g_1(u) := N_{X_0}(\rho; z),$$

$$g_2(u) := \frac{\sinh^2((\rho + 2r)/2) - \sinh^2((\rho_0 - 2r)/2)}{\sinh^2(r/2)} + N_{X_0}(\rho_0; z),$$

where $r := \ell_{X_0}/4$ and $\rho_0 := 3\ell_{X_0}/4$. By the latter choices for $r$ and $\rho_0$, the inequalities

$$2r < \ell_{X_0}, \quad 2r < \rho_0 < \ell_{X_0},$$

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Effective bounds for Faltings’s delta function hold, which enables us to apply Lemma 2.3 (a) of [17] to derive the inequality

$$g_1(u) \leq g_2(u).$$

This in turn allows us to apply Lemma 4.6 of [11], in particular taking into account that $K_{\mathbb{H}}^{(1)}(t_0; \rho)$ is strictly monotone decreasing in $\rho$ by Proposition A.2, namely the inequality of Stieltjes integrals

$$\int_a^\infty F(u) \, d g_1(u) + F(a) \, g_1(a) \leq \int_a^\infty F(u) \, d g_2(u) + F(a) \, g_2(a). \tag{4.8}$$

Using the above notation, we get

$$F(a) \, g_1(a) = K_{\mathbb{H}}^{(1)}(t_0; \ell_{X_0}/4) \, N_{X_0}(\ell_{X_0}/4; z) = K_{\mathbb{H}}^{(1)}(t_0; \ell_{X_0}/4),$$

$$F(a) \, g_2(a) = K_{\mathbb{H}}^{(1)}(t_0; \ell_{X_0}/4) \frac{\sinh^2(3\ell_{X_0}/8) - \sinh^2(\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} + K_{\mathbb{H}}^{(1)}(t_0; \ell_{X_0}/4).$$

Furthermore, we compute

$$g_2(u) = \frac{1}{\frac{1}{2} \left( e^{\rho+\ell_{X_0}/2} - e^{-\rho-\ell_{X_0}/2} - \sinh^2(\ell_{X_0}/8) \right) + 1},$$

hence

$$\frac{dg_2(u)}{du} = \frac{d}{d\rho} \frac{1}{\frac{1}{2} \left( e^{\rho+\ell_{X_0}/2} - e^{-\rho-\ell_{X_0}/2} - \sinh^2(\ell_{X_0}/8) \right)} = \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)}.$$

Inserting all of the above into (4.8), we arrive at the bound

$$\int_{\ell_{X_0}/4}^{\infty} K_{\mathbb{H}}^{(1)}(t_0; \rho) \, d N_{X_0}(\rho; z) \leq \int_{\ell_{X_0}/4}^{\infty} K_{\mathbb{H}}^{(1)}(t_0; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho + K_{\mathbb{H}}^{(1)}(t_0; \ell_{X_0}/4) \left( \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1 \right).$$

Observing the inequality

$$\frac{\mu_{\text{can}}(z)}{\mu_{\text{shyp}}(z)} \leq 4\pi \frac{g_{X\mu_{\text{can}}}(z)}{\mu_{\text{hyp}}(z)},$$

proves the claimed bound. □
4.4. Proposition

Let $X \to X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0 \backslash \mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$, and let $\ell_{X_0}$ denote the length of the shortest closed geodesic on $X_0$. Then, the quantity $S_X$ can be bounded as

$$S_X \leq \frac{D_2 e^{\ell_{X_0}/2}}{(1 - e^{-\ell_{X_0}/4})^{5/2}}$$

with an absolute constant $D_2 > 0$, which can be taken to be $1.2 \cdot 10^3$.

Proof. — We work from the estimate (4.7) for $S_X$ given in Lemma 4.3 and insert therein the bound (4.2) for $K_{\mathbb{H}}^{(1)}(t_0; \rho)$ obtained in Lemma 4.2, which we rewrite as

$$K_{\mathbb{H}}^{(1)}(t_0; \rho) \leq A_1(t_0; \rho) + A_2(t_0; \rho) + A_3(t_0; \rho),$$

where

$$A_1(t_0; \rho) := \frac{17\sqrt{2} e^{-t_0/4} (\rho + \log(4)) e^{-\rho^2/(4t_0)}}{(4\pi t_0)^{3/2} \sinh^{1/2}(\rho)},$$
$$A_2(t_0; \rho) := \frac{4\sqrt{2} e^{-\rho/(2\sqrt{t_0})+\sqrt{t_0}/2}}{\pi^{3/2} \sqrt{t_0}},$$
$$A_3(t_0; \rho) := \frac{4\sqrt{2} e^{-\rho}}{\pi^{3/2}} \int_{\rho/(2\sqrt{t_0})-\sqrt{t_0}/2}^{\infty} e^{-r'^2} \, dr'.$$

With this notation and keeping in mind that our bounds are valid for all $t_0 > 0$, we can rewrite (4.7) in the form

$$S_X \leq B_1(t_0; \ell_{X_0}) + B_2(t_0; \ell_{X_0}) + B_3(t_0; \ell_{X_0}),$$

where

$$B_j(t_0; \ell_{X_0}) := 4\pi \int_{\ell_{X_0}/4}^{\infty} A_j(t_0; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho$$

$$+ 4\pi A_j(t_0; \ell_{X_0}/4) \left( \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1 \right)$$

for $j = 1, 2, 3$. In order to obtain a precise, effective upper bound for $S_X$, we will evaluate the expression under consideration at $t_0 = 10$; there is no particular reason for this choice of $t_0$ except to derive an explicit bound for $S_X$. 

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For the first summand of \( B_1(t_0; \ell X_0) \) involving the integral, since \( \sinh(\rho + \ell X_0/2) \leq e^{\rho + \ell X_0/2} \) and

\[
\frac{1}{\sinh^{1/2}(\rho)} \leq \frac{\sqrt{2} e^{-\rho/2}}{(1 - e^{-2\rho})^{1/2}} \leq \frac{\sqrt{2} e^{-\rho/2}}{(1 - e^{-\ell X_0/2})^{1/2}}
\]

for \( \rho \geq \ell X_0/4 \), we have the bound

\[
\int_{\ell X_0/4}^{\ell X_0/2} A_1(t_0; \rho) \frac{\sinh(\rho + \ell X_0/2)}{2 \sinh^2(\ell X_0/8)} d\rho 
\leq \frac{17 e^{\ell X_0/2}}{(4\pi t_0)^{3/2} \sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} \int_{\ell X_0/4}^{\ell X_0/2} (\rho + \log(4)) e^{-(\rho/\sqrt{t_0}) - \sqrt{t_0}/2}^2 d\rho 
\leq \frac{34 e^{\ell X_0/2}}{(4\pi t_0)^{3/2} \sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} \int_{-\infty}^{\ell X_0/2} (t_0^{3/2} + 2t_0|\rho'| + \log(4)t_0^{1/2}) e^{-\rho'^2} d\rho'
\]

\[
= \frac{34 e^{\ell X_0/2}}{(4\pi)^{3/2} \sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} \left( \left( 1 + \frac{\log(4)}{t_0} \right) \sqrt{\pi} + \frac{2}{\sqrt{t_0}} \right)
\]

hence we obtain for \( t_0 = 10 \)

\[
\int_{\ell X_0/4}^{\ell X_0/2} A_1(10; \rho) \frac{\sinh(\rho + \ell X_0/2)}{2 \sinh^2(\ell X_0/8)} d\rho \leq \frac{3 e^{\ell X_0/2}}{\sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}}.
\]

We thus get the bound

\[
\frac{B_1(10, \ell X_0)}{4\pi} \leq \frac{3 e^{\ell X_0/2}}{\sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} + A_1(10; \ell X_0/4) \left( \frac{\sinh^2(3\ell X_0/8)}{\sinh^2(\ell X_0/8)} - 1 \right)
\leq \frac{3 e^{\ell X_0/2}}{\sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} + \frac{17 \sqrt{2} (1 + \log(4))}{(40\pi)^{3/2} \sinh^{1/2}(\ell X_0/4)} \frac{\sinh^2(3\ell X_0/8)}{\sinh^2(\ell X_0/8)}
\leq \frac{3 e^{\ell X_0/2}}{\sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}} + \frac{e^{5\ell X_0/8}}{\sinh^2(\ell X_0/8)(1 - e^{-\ell X_0/2})^{1/2}}.
\]
For the first summand of $B_2(t_0; \ell_{X_0})$ involving the integral, we have the bound
\[
\int_{\ell_{X_0}/4}^{\infty} A_2(t_0; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho \\
\leq \frac{2 \sqrt{2} e^{\ell_{X_0}/2}}{\pi^{3/2} \sqrt{t_0} \sinh^2(\ell_{X_0}/8)} \int_{\ell_{X_0}/4}^{\infty} e^{-(\rho/(2\sqrt{t_0}) - \sqrt{t_0}/2)^2} \, d\rho,
\]
hence we obtain for $t_0 = 10$
\[
\int_{\ell_{X_0}/4}^{\infty} A_2(10; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho \leq \frac{4 \sqrt{2} e^{\ell_{X_0}/2}}{\pi \sinh^2(\ell_{X_0}/8)} \leq \frac{2 e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)}.
\]
We thus get the bound
\[
\frac{B_2(10, \ell_{X_0})}{4\pi} \leq \frac{2 e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + A_2(10; \ell_{X_0}/4) \left( \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1 \right) \\
\leq \frac{2 e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} \leq \frac{3 e^{3\ell_{X_0}/4}}{\sinh^2(\ell_{X_0}/8)}. \tag{4.10}
\]
For the first summand of $B_3(t_0; \ell_{X_0})$ involving the integral, we have the bound
\[
\int_{\ell_{X_0}/4}^{\infty} A_3(t_0; \rho) \frac{\sinh(\rho + \ell_{X_0}/2)}{2 \sinh^2(\ell_{X_0}/8)} \, d\rho \\
\leq \frac{2 \sqrt{2} e^{\ell_{X_0}/2}}{\pi^{3/2} \sinh^2(\ell_{X_0}/8)} \int_{\ell_{X_0}/4}^{\infty} \int_{\ell_{X_0}/4}^{\infty} \int_{\ell_{X_0}/4}^{\infty} e^{-r^2} \, dr' \, d\rho \\
= \frac{2 \sqrt{2} e^{\ell_{X_0}/2}}{\pi^{3/2} \sinh^2(\ell_{X_0}/8)} \int_{\ell_{X_0}/4}^{\infty} \int_{\ell_{X_0}/4}^{\infty} \int_{\ell_{X_0}/4}^{\infty} e^{-r^2} \, d\rho \, dr' \\
= \frac{2 \sqrt{2} e^{\ell_{X_0}/2}}{\pi^{3/2} \sinh^2(\ell_{X_0}/8)} \int_{\ell_{X_0}/4}^{\infty} \int_{\ell_{X_0}/4}^{\infty} \left( 2 \sqrt{t_0} r' + t_0 - \frac{\ell_{X_0}}{4} \right) e^{-r^2} \, dr' \\
\leq \frac{2 \sqrt{2} e^{\ell_{X_0}/2}}{\pi^{3/2} \sinh^2(\ell_{X_0}/8)} \left( 2 \sqrt{t_0} + \sqrt{\pi} t_0 \right),
\]
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hence we obtain for $t_0 = 10$

$$\int_{\ell X_0 / 4}^{\infty} A_3(10; \rho) \frac{\sinh(\rho + \ell X_0 / 2)}{2 \sinh^2(\ell X_0 / 8)} \, d\rho \leq \frac{13 e^{\ell X_0 / 2}}{\sinh^2(\ell X_0 / 8)}.$$

We thus get the bound

$$\frac{B_3(10, \ell X_0)}{4\pi} \leq \frac{13 e^{\ell X_0 / 2}}{\sinh^2(\ell X_0 / 8)} + A_3(10; \ell X_0 / 4) \left( \frac{\sinh^2(3\ell X_0 / 8)}{\sinh^2(\ell X_0 / 8)} - 1 \right)$$

$$\leq \frac{13 e^{\ell X_0 / 2}}{\sinh^2(\ell X_0 / 8)} + \frac{2 \sinh^2(3\ell X_0 / 8)}{\sinh^2(\ell X_0 / 8)} \leq \frac{15 e^{3\ell X_0 / 4}}{\sinh^2(\ell X_0 / 8)}. \quad (4.11)$$

Adding up the bounds (4.9) – (4.11), we obtain

$$S_X \leq \frac{88 \pi}{e^{3\ell X_0 / 4}} \frac{\sinh^2(\ell X_0 / 8)(1 - e^{-\ell X_0 / 2})^{1/2}}{1 - e^{-\ell X_0 / 4})^{5/2}},$$

which proves the claim. $\square$

4.5. Remark

In addition to the cartesian coordinates $x, y$, we introduce the euclidean polar coordinates $\rho = \rho(z), \theta = \theta(z)$ of the point $z$ centered at the origin. These are related to $x, y$ by the formulae

$$x := e^\rho \cos(\theta), \quad y := e^\rho \sin(\theta). \quad (4.12)$$

Given $\gamma \in H(\Gamma)$, then there exists $\sigma_\gamma \in PSL_2(\mathbb{R})$ such that

$$\sigma^{-1}_\gamma \gamma \sigma_\gamma = \begin{pmatrix} e^{\ell_{\gamma} / 2} & 0 \\ 0 & e^{-\ell_{\gamma} / 2} \end{pmatrix} \iff \gamma = \sigma_\gamma \begin{pmatrix} e^{\ell_{\gamma} / 2} & 0 \\ 0 & e^{-\ell_{\gamma} / 2} \end{pmatrix} \sigma^{-1}_\gamma.$$

For $s \in \mathbb{C}, \text{Re}(s) > 1$, the hyperbolic Eisenstein series $E_{hyp, \gamma}(z, s)$ associated to $\gamma$ is defined by the series

$$E_{hyp, \gamma}(z, s) := \sum_{\eta \in \langle \gamma \rangle \setminus \Gamma} \sin \left( \theta(\sigma^{-1}_\gamma \eta z) \right)^s \quad (4.13)$$

using the polar coordinates (4.12). The hyperbolic Eisenstein series (4.13) is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with
Re(s) > 1; it is invariant under the action of $\Gamma$ and satisfies the differential equation

$$\left(\Delta_{\text{hyp}} - s(1-s)\right)e_{\text{hyp},\gamma}(z, s) = s^2 e_{\text{hyp},\gamma}(z, s + 2).$$

For proofs of these facts and further details, we refer to [18].

By means of the hyperbolic Eisenstein series the following alternative bound for the quantity $S_X$, namely

$$S_X \leq 8 \left[ \sum_{\gamma \in \mathcal{H} \backslash \Gamma_0} \left( \sup_{z \in X_0} \left| \Delta_{\text{hyp}} e_{\text{hyp},\gamma}(z, 2) \right| e^{-\ell_X} + \frac{24}{(1-e^{-\ell_X})^3} \sup_{z \in X_0} \left| e_{\text{hyp},\gamma}(z, 2) \right| e^{-2\ell_X} \right) + 170 \right],$$

has been obtained in [15]. This upper bound for $S_X$ involves special values of hyperbolic Eisenstein series in the half-plane of convergence of the series. As such, it is possible to use various counting function arguments, as above, to complete this approach to obtaining an upper bound for the quantity $S_X$ analogous to the one given in Proposition 4.4.

5. Effective bounds for the Huber constant

5.1. Remark

In Table 2 of the recent joint work [9] with J. S. Friedman, an algorithm was given to bound the Huber constant $C_{\text{Hub},X}$ for $X$ effectively in terms of our basic quantities $g_X$, $d_X$, $\ell_X$, $\lambda_{X,1}$, and $N_{ev,X}^{[0,1/4]}$; here the newly introduced quantity $d_X$ denotes the diameter of $X$. In the subsequent proposition, we will summarize the result of this algorithm by utilizing convenient yet possibly crude estimates.

5.2. Proposition

The Huber constant $C_{\text{Hub},X}$ for $X$ can be bounded as

$$C_{\text{Hub},X} \leq D_3 g_X e^{8\pi g_X/\ell_X + \ell_X/2} \frac{e^{\ell_X/2}}{(1 - s_{X,1})(1 - e^{-\ell_X/2})^2};$$

here $\ell_X$ denotes the length of the shortest closed geodesic on $X$,

$$s_{X,1} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,1}}$$

with $\lambda_{X,1}$ denoting the smallest non-zero eigenvalue of $\Delta_{\text{hyp}}$, and $D_3 > 0$ is an absolute constant, which can be taken to be $10^{11}$. 
Proof. — As mentioned in 5.1, we follow the algorithm given in Table 2 of [9]. In the sequel we also use the definitions of the quantities $A$, $B$, $C$, $C_j$ ($j = 1, 2, 3, \ldots$) therein.

Recalling from [6] the bound for $N_{ev,X}^{[0,1/4]}$, we obtain for the quantity $A$ the estimate

$$A := N_{ev,X}^{[0,1/4]} \leq 4g_X - 2 \leq 4g_X.$$

Using the inequality (2) from the main theorem of [7], namely

$$2 \frac{\ell_X}{4} d_X \leq 2 \sinh \left( \frac{\ell_X}{4} \right) d_X \leq 4\pi (g_X - 1),$$

we obtain the following bound for the diameter $d_X$ of $X$

$$d_X \leq \frac{8\pi g_X}{\ell_X}.$$

Hence, the quantity $B$ can be estimated by

$$B := \frac{2\pi e^{d_X}}{4\pi(g_X - 1)} \leq \frac{e^{8\pi g_X/\ell_X}}{2}.$$

For the quantity $C$, we have

$$C := 3 \left( \frac{4\pi(g_X - 1)}{4\pi} + 745B \right) \leq 3g_X + 1118e^{8\pi g_X/\ell_X}.$$

Next, we have

$$C_1 := 2e - 2 \leq 4,$$

and

$$C_{10} := 8480\sqrt{\frac{e}{2\pi}} \leq 5578.$$

From this we derive

$$C_{12} := (A - 1) \left( 1 + 3C_1 + \frac{2}{1 - s_{X,1}}(1 + C_1) \right) + 2C_1 + 2$$

$$\leq 4g_X \left( 13 + \frac{10}{1 - s_{X,1}} \right) + 10$$

$$\leq \frac{92g_X}{1 - s_{X,1}} + 10 \leq \frac{102g_X}{1 - s_{X,1}},$$

and

$$C_{13} := \frac{41}{6} \cdot C \cdot C_{10} \leq \frac{41 \cdot 5578}{6} \left( 3g_X + 1118e^{8\pi g_X/\ell_X} \right)$$

$$\leq 114349g_X + 42614061e^{8\pi g_X/\ell_X}.$$
From this we obtain
\[ C_{16} := C_{12} + C_{13} + \frac{3}{2\pi} 4\pi (g_X - 1) C_{10} \]
\[ \leq \frac{102 g_X}{1 - s_{X,1}} + 114 349 g_X + 42 614 061 e^{8\pi g_X/\ell_X} + 6 \cdot 5578 g_X \]
\[ \leq \frac{147 919 g_X}{1 - s_{X,1}} + 42 614 061 e^{8\pi g_X/\ell_X} \leq \frac{42 761 980 g_X e^{8\pi g_X/\ell_X}}{1 - s_{X,1}}. \]

For notational convenience, let us keep the constant \( C_{16} \) without replacing it with the above bound for the next few computations. We further have

\[ C_{17} := 4 A + 4 C_{16} \leq 16 g_X + 4 C_{16}, \]
\[ C_{18} := 4 A + 5 C_{16} \leq 16 g_X + 5 C_{16}. \]

The constant \( c \) must satisfy \( 1 < c < e^{\ell_X} \), so we may take \( c := e^{\ell_X/2} \), and hence \( \mu := \ell_X/2 \). With this choice, we find

\[ C_{19} := C_{18} + \frac{8 A + 4 C_{18}}{1 - 1/c} \leq 16 g_X + 5 C_{16} + \frac{96 g_X + 20 C_{16}}{1 - e^{-\ell_X/2}} \]
\[ \leq \frac{112 g_X + 25 C_{16}}{1 - e^{-\ell_X/2}}. \]

Observing that
\[ f(r) := \frac{r}{1 - e^{-r}} \geq 1 \]
for \( r \in \mathbb{R}_{\geq 0} \), we find
\[ \frac{1}{\mu} = \frac{2}{\ell_X} \leq \frac{1}{1 - e^{-\ell_X/2}}. \]

Thus, we obtain

\[ C_{20} := C_{19} + \frac{8 A + 4 C_{18}}{\mu} \leq \frac{112 g_X + 25 C_{16}}{1 - e^{-\ell_X/2}} + \frac{8 A + 4 C_{18}}{1 - e^{-\ell_X/2}} \]
\[ \leq \frac{112 g_X + 25 C_{16}}{1 - e^{-\ell_X/2}} + \frac{96 g_X + 20 C_{16}}{1 - e^{-\ell_X/2}} \]
\[ = \frac{208 g_X + 45 C_{16}}{1 - e^{-\ell_X/2}}. \]

For the quantity \( C_{21} \), we find the estimate
\[ C_{21} := \frac{|c - 2|}{\log(2)} + \frac{2|2 - \sqrt{c}|}{\log(c)} \leq \frac{c + 2}{\log(2)} + \frac{2(\sqrt{c} + 2)}{\log(c)} \]
\[ \leq \frac{e^{\ell_X/2} + 2}{\log(2)} + \frac{4(e^{\ell_X/4} + 2)}{\ell_X} \leq \frac{3 e^{\ell_X/2}}{1/2} + \frac{4 \cdot 3 e^{\ell_X/2}}{\ell_X} \]
\[ \leq 12 e^{\ell_X/2} \left( 1 + \frac{1}{\ell_X} \right) \leq \frac{18 e^{\ell_X/2}}{1 - e^{-\ell_X/2}}. \]
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At this point, we have to correct the statement about the constant $C_{22}$, which comes from Lemma 4.14 in [9]. The correct assertion is that

$$C_{22} := \frac{1}{1 + 1/\log(2)}.$$  

In fact, $C_{22}$ has to be such that for any $r \geq 2$, we have the inequality

$$\text{li}(r) \leq C_{22} \frac{r}{\log(r)}.$$  

For a proof we consider the function

$$f(r) := \text{li}(r) - d \frac{r}{\log(r)}$$

for some positive constant $d$, which we determine such that $f(r)$ is negative for $r \geq 2$. Obviously, $f(2) < 0$, so we have to determine $d$ such that $f(r)$ becomes a decreasing function. We have

$$f'(r) = \frac{1}{\log(r)} \left( 1 - d + \frac{d}{\log(r)} \right),$$

hence, we need to have

$$1 - d + \frac{d}{\log(r)} \leq 0 \iff 1 - \frac{1}{d} \geq \frac{1}{\log(r)}$$

for $r \geq 2$, which holds for

$$1 - \frac{1}{d} \geq \frac{1}{\log(2)} \iff d \geq \frac{1}{1 + 1/\log(2)}$$

giving the claimed value of $C_{22}$. (Note that the error in the proof of Lemma 4.14 of [9] arose by dividing by a constant which is negative, so then the inequality has to change directions.) Continuing with this value of $C_{22}$, we have

$$C_{22} = \frac{1}{1 + 1/\log(2)} \leq \frac{1}{2}.$$  

Finally, we are in a position to compute $C_u$; we have

$$C_u := C_{21} A + C_{20} \frac{e^{3/4}}{\log(c)} + C_{20}(1 + C_{22}) + \frac{3}{4} C_{20} C_{21}$$

$$\leq \frac{72 g_X e^{\ell_X/2}}{1 - e^{-\ell_X/2}} + \frac{(208 g_X + 45 C_{16}) e^{\ell_X/2}}{(1 - e^{-\ell_X/2})^2}$$

$$+ \frac{312 g_X + 69 C_{16}}{1 - e^{-\ell_X/2}} + \frac{(3744 g_X + 810 C_{16}) e^{\ell_X/2}}{(1 - e^{-\ell_X/2})^2}.$$
Employing finally the bound for $C_{16}$ yields the estimate

$$C_u \leq \frac{384 g_X e^{\ell_X/2}}{1 - e^{-\ell_X/2}} + \frac{69 C_{16}}{1 - e^{-\ell_X/2}} + \frac{(3952 g_X + 855 C_{16}) e^{\ell_X/2}}{(1 - e^{-\ell_X/2})^2}$$

$$\leq \frac{39512073856 g_X e^{8g_X/\ell_X + \ell_X/2}}{(1 - s_{X,1})(1 - e^{-\ell_X/2})^2}.$$ 

This completes the proof of the proposition. □

6. Effective bounds for Faltings’s delta function

The main result proven in this paper consists in simplifying the bound obtained in Corollary 3.4 and making it effective.

6.1. Theorem

Let $X \rightarrow X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0 \backslash \mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$. Let $\ell_{X_0}$ denote the length of the shortest closed geodesic on $X_0$ and $\lambda_{X,1}$, $\lambda_{X_0,1}$ the smallest non-zero eigenvalues of $\Delta_{\text{hyp}}$ on $X$, $X_0$, respectively, and

$$\lambda_X = \frac{1}{2} \min \left\{ \lambda_{X,1}, \frac{7}{64} \right\}, \quad s_{X_0,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X_0,1}}.$$ 

Then, we have the effective bound

$$\delta_{\text{Fal}}(X) \leq \frac{D_4 g_X e^{8g_X/\ell_{X_0} + \ell_{X_0}}}{(1 - e^{-\ell_{X_0}/4})^5 (1 - s_{X_0,1})} \frac{g_X}{\lambda_X}$$

(6.1)

with an absolute constant $D_4 > 0$, which can be taken to be $10^{15}$.

Proof. — We work from the bound

$$\delta_{\text{Fal}}(X) \leq 876 \left( g_x + \frac{1}{\lambda_X} \left( g_x (S_X + 1)^2 + C_{\text{Hub},X} + N_{\text{ev},X}^{[0,1/4]} \right) + \left( 1 + \frac{1}{\ell_X} \right) N_{\text{geo},X}^{(0,5)} \right).$$

(6.2)

obtained in the proof of Corollary 3.4 (using the notation therein). We will next bound the quantities

$$\ell_X, \quad N_{\text{ev},X}^{[0,1/4]}, \quad S_X, \quad C_{\text{Hub},X}, \quad N_{\text{geo},X}^{(0,5)}$$

in terms of the underlying compact Riemann surface $X_0$.

(i) We start by observing that the trivial inequality

$$\ell_X \geq \ell_{X_0}$$

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holds true for the lengths of the shortest closed geodesics on \(X, X_0\), respectively.

(ii) In order to estimate \(N_{\text{ev,}X}^{[0,1/4]}\), we recall as in the proof of Proposition 5.2 from [6] the bound

\[
1 \leq N_{\text{ev,}X}^{[0,1/4]} \leq 4g - 2 \leq 4g_X. \quad (6.4)
\]

(iii) From Proposition 4.4, we recall the bound

\[
S_X \leq \frac{1200 e^{\ell_X/2}}{(1 - e^{-\ell_X/4})^{5/2}} \quad (6.5)
\]

with \(\ell_X\) as in the statement of the theorem.

(iv) Next, we have to estimate \(C_{\text{Hub,}X}\). We start by citing Theorem 3.4 of [12] and use the Artin formalism for the covering \(X \rightarrow X_0\), to derive the bound

\[
C_{\text{Hub,}X} \leq [\Gamma_0 : \Gamma] C_{\text{Hub,}X_0},
\]

From the Riemann–Hurwitz formula we now easily derive the bound

\[
[\Gamma_0 : \Gamma] \leq \frac{g_X - 1}{g_X - 1} \leq g_X,
\]

from which we get

\[
C_{\text{Hub,}X} \leq g_X C_{\text{Hub,}X_0}, \quad (6.6)
\]

where the proof of Proposition 5.2 shows

\[
C_{\text{Hub,}X_0} \leq \frac{39512073856 g_X e^{8\pi g_X/\ell_X + \ell_X/2}}{(1 - s_{X_0,1})(1 - e^{-\ell_X/2})^2} \quad (6.7)
\]

with \(\ell_X\) and \(s_{X_0,1}\) as in the statement of the theorem.

(v) Finally, we need to bound \(N_{\text{geo}}^{(0,5)}\). With the above notation, using arguments from the proof of Theorem 4.11 in [11] (as well as the notation \(r_{\Gamma_0,\Gamma}\) therein), we find (as above)

\[
N_{\text{geo,}X}^{(0,5)} \leq \frac{5 r_{\Gamma_0,\Gamma}}{\ell_X} N_{\text{geo,}X_0}^{(0,5)} \leq \frac{5 [\Gamma_0 : \Gamma]}{\ell_X} N_{\text{geo,}X_0}^{(0,5)} \leq \frac{5 g_X}{\ell_X} N_{\text{geo,}X_0}^{(0,5)}.
\]

Applying the prime geodesic theorem (2.1) to \(X_0\) and recalling the monotonicity of the logarithmic integral for \(u > 0\), we find

\[
N_{\text{geo,}X_0}^{(0,5)} = \pi_X \left( \log(5) \right) \leq N_{\text{ev,}X_0}^{[0,1/4]} \text{li} \left( \log(5) \right) + C_{\text{Hub,}X_0} \frac{\log(\log(5))^{3/4}}{\log \left( \log(5) \right)^{1/2}}
\]

\[
\leq N_{\text{ev,}X_0}^{[0,1/4]} + 3C_{\text{Hub,}X_0} \leq 4g_X + 3C_{\text{Hub,}X_0}, \quad (6.8)
\]
where \( C_{\text{Hub}, X_0} \) can be effectively bounded using Proposition 5.2.

Inserting the bounds (6.3) – (6.8) into the estimate (6.2) yields the following bound for \( \delta_{\text{Pal}}(X) \):

\[
876 \left( g_X + \frac{1}{\lambda_X} \right) \left( g_X \left( \frac{1200 e^{\ell_X_0/2}}{(1 - e^{-\ell_X_0/4})^{1/2}} + 1 \right)^2 + g_X C_{\text{Hub}, X_0} + 4 g_X \right) + \frac{5 g_X}{\ell_X_0} \left( 1 + \frac{1}{\ell_X_0} \right) N_{\text{geo}, X_0}^{(0, 5)} \leq 876 g_X \left( 1 + \frac{1}{\lambda_X} \left( \frac{1201^2 e^{\ell_X_0}}{(1 - e^{-\ell_X_0/4})^{2}} + C_{\text{Hub}, X_0} + 4 \right) \right) + \frac{10 N_{\text{geo}, X_0}^{(0, 5)}}{\ell_X_0 (1 - e^{-\ell_X_0/2})} \leq 876 g_X \left( \frac{1}{\lambda_X} \left( \frac{1442401 e^{\ell_X_0}}{(1 - e^{-\ell_X_0/4})^{5}} + C_{\text{Hub}, X_0} + 5 \right) \right) + \frac{20 g_X + 15 C_{\text{Hub}, X_0}}{(1 - e^{-\ell_X_0/2})^2} \leq \frac{876}{(1 - e^{-\ell_X_0/4})^5} \frac{g_X}{\lambda_X} \left( 1442426 g_X e^{\ell_X_0} + \frac{632193181696 g_X e^{8\pi g_X/\ell_X_0 + \ell_X_0/2}}{1 - s_{X_0, 1}} \right) \leq 553802490730872 \frac{g_X e^{8\pi g_X/\ell_X_0 + \ell_X_0}}{(1 - e^{-\ell_X_0/4})^5 (1 - s_{X_0, 1})} \frac{g_X}{\lambda_X}.
\]

This completes the proof of the theorem. \( \square \)

### 6.2. Remarks

(i) We can further refine the lower bound for \( \ell_X_0 \) provided that \( X_0 \) has a model defined over some number field. In fact, by Bézout’s theorem, we then have \( X_0 \cong \Delta_0 \backslash \mathbb{H} \), where \( \Delta_0 \) is a subgroup of finite index in \( \Gamma(2) \). Therefore, we have the estimate

\[
2 \cosh(\ell_X_0/2) = |\text{tr}(\delta_0)| \geq 4,
\]

where \( \delta_0 \in \Delta_0 \) is such that \( \ell_\delta = \ell_X_0 \); this gives \( \ell_X_0 \geq 2 \arccosh(2) \). The factor depending on \( X_0 \) in (6.1) can thus be bounded as

\[
\frac{D_4 g_X e^{8\pi g_X/\ell_X_0 + \ell_X_0}}{(1 - e^{-\ell_X_0/4})^5 (1 - s_{X_0, 1})} \leq \frac{40 D_4 g_X e^{10 g_X + \ell_X_0}}{1 - s_{X_0, 1}} \leq \frac{10^{17} g_X e^{10 g_X + \ell_X_0}}{1 - s_{X_0, 1}}. \tag{6.9}
\]

(ii) On the other hand, if \( X_0 \) can be covered by a modular curve \( \overline{\Gamma(N)} \backslash \mathbb{H} \) for the full congruence subgroup \( \Gamma(N) \) for some \( N \in \mathbb{N} \), a result of R. Brooks in [5] shows that \( \lambda_{X_0, 1} \geq 5/36 \), which gives the estimate

\[
\frac{1}{1 - s_{X_0, 1}} \leq 6.
\]
In addition, assuming that $X_0$ has a model defined over some number field, case (i) above also applies and the bound (6.9) simplifies to
\[
D_4 g_{X_0} e^{8\pi g_{X_0}/\ell_{X_0} + \ell_{X_0}} \frac{1}{(1 - e^{-\ell_{X_0}/4})^5 (1 - s_{X_0,1})} \leq 10^{18} g_{X_0} e^{10 g_{X_0} + \ell_{X_0}}.
\]

6.3. Corollary

Let $X$ be a compact Riemann surface of genus $g_X > 1$. Let $\ell_X$ denote the length of the shortest closed geodesic on $X$, $\lambda_{X,1}$ the smallest non-zero eigenvalue of $\Delta_{\text{hyp}}$ on $X$, and
\[
\lambda_X = \frac{1}{2} \min \left\{ \lambda_{X,1}, \frac{7}{64} \right\}, \quad s_{X,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,1}}.
\]
Then, we have the effective bound
\[
\delta_{\text{Fal}}(X) \leq \frac{D_4 g_X e^{8\pi g_X/\ell_X + \ell_X}}{\lambda_X (1 - s_{X,1})} \leq \frac{10^{18} g_{X_0} e^{10 g_{X_0} + \ell_{X_0}}}{\lambda_X (1 - s_{X,1})}
\]
with an absolute constant $D_4 > 0$, which can be taken to be $10^{15}$.

Proof. — The proof follows immediately from an analysis of the proof of Theorem 6.1 for the trivial covering $X_0 = X$. \qed

Using Corollary 6.3, we can now also give a variant of the bound (6.1) in the case that $X$ is a ramified covering of finite degree of a compact Riemann surface $X_0$ of genus $g_{X_0} > 1$. For this, let $\text{Ram}(X/X_0) \subset X_0$ denote the ramification locus of the given covering.

6.4. Corollary

Let $X \longrightarrow X_0$ be a ramified covering of finite degree of compact Riemann surfaces of genera $g_X, g_{X_0} > 1$, respectively. With $\ell_{X_0}$ denoting the length of the shortest closed geodesic on $X_0$, put
\[
r_{X_0} := \min \left\{ \ell_{X_0}, \min_{z, w \in \text{Ram}(X/X_0), z \neq w} \text{dist}_{\text{hyp}}(z, w) \right\},
\]
\[
R_{X_0} := \max \left\{ \ell_{X_0}, \max_{z, w \in \text{Ram}(X/X_0), z \neq w} \text{dist}_{\text{hyp}}(z, w) \right\}.
\]
Furthermore, with $\lambda_{X,1}$ denoting the smallest non-zero eigenvalue of $\Delta_{\text{hyp}}$ on $X$, put
\[
\lambda_X = \frac{1}{2} \min \left\{ \lambda_{X,1}, \frac{7}{64} \right\}, \quad s_{X,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,1}}.
\]
Then, we have the effective bound
\[
\delta_{\text{Fal}}(X) \leq \frac{D_4 g_X e^{8\pi g_X/r_{X_0}} + g_X R_{X_0}}{(1 - e^{-r_{X_0}/4})^5} \frac{1}{\lambda_X(1 - s_{X, 1})}
\]
with an absolute constant \( D_4 > 0 \), which can be taken to be \( 10^{15} \).

Proof. — We work from the effective bound obtained in Corollary 6.3 and estimate the length of the shortest closed geodesic \( \ell_X \) from below and above by quantities depending on the base \( X_0 \).

In order to estimate \( \ell_X \) from below, we observe that the length of closed geodesics on \( X \), which do not pass through ramification points, can be bounded from below by \( \ell_{X_0} \); the same estimate holds true, if the closed geodesic passes through a single ramification point. However, if the closed geodesic happens to pass through at least two ramification points lying above two distinct points of \( \text{Ram}(X/X_0) \), we additionally have to take into account the distances between mutually distinct points of \( \text{Ram}(X/X_0) \) in our estimate. All in all this leads to the lower bound
\[
\ell_X \geq r_{X_0}. \tag{6.11}
\]
Similarly, we find that the length of closed geodesics on \( X \), which do not pass through ramification points, can be bounded from above by \( \deg(X/X_0) \ell_{X_0} \), and the same estimate holds true, if the closed geodesic passes through a single ramification point. Again, if the closed geodesic happens to pass through at least two ramification points lying above two distinct points of \( \text{Ram}(X/X_0) \), we additionally have to take into account the distances between mutually distinct points of \( \text{Ram}(X/X_0) \) in our estimate. This leads to the upper bound
\[
\ell_X \leq \deg(X/X_0) R_{X_0} \leq g_X R_{X_0}. \tag{6.12}
\]
Inserting the bounds (6.11) and (6.12) into the estimate (6.10) completes the proof of the corollary.

7. Application to Parshin’s covering construction

7.1. The set-up

Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \) and \( S := \text{Spec} \left( \mathcal{O}_K \right) \). In contrast to the previous sections, let \( X \) denote a smooth projective curve defined over \( K \) of genus \( g_X > 1 \), and let \( \mathcal{X}/S \) be a minimal regular model of \( X/K \), which is semistable. Denote by \( \mathcal{O}_{\mathcal{X}/S} \) the relative dualizing sheaf of
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\( \mathcal{X}/S \) equipped with the Arakelov metric. For \( p \in S \), we let \( \delta_p \) be the number of singular points in the fiber above \( p \). For an archimedean place \( v \), we put

\[
X_v := X \times_v \mathbb{C},
\]

whose complex points \( X_v(\mathbb{C}) \) constitute a compact Riemann surface of genus equal to \( g_X \). In order to simplify our notation, we allow ourselves subsequently to write \( X_v \) instead of \( X_v(\mathbb{C}) \).

In his quest for an arithmetic version of the van de Ven–Bogomolov–Miyaoka–Yau inequality, A. N. Parshin proposed the following inequality (see [19])

\[
\mathcal{O}^2_{\mathcal{X}/S} \leq c_1 \left( \sum_p \delta_p \log (N_{K/\mathbb{Q}}(p)) + \sum_v \varepsilon_v \delta_{\text{Fal}}(X_v) \right) + c_2 (2g_X - 2) \log |\text{disc}(K/\mathbb{Q})| + c_3 [K: \mathbb{Q}]; \tag{7.1}
\]

here \( c_j \) are positive constants depending solely on \( K \) \((j = 1, 2, 3)\), \( N_{K/\mathbb{Q}}(p) \) denotes the absolute norm of \( p \), and \( \text{disc}(K/\mathbb{Q}) \) is the discriminant of the field extension \( K/\mathbb{Q} \). As is well known by subsequent work of J.-B. Bost, J.-F. Mestre, and L. Moret-Bailly (see [4]), the inequality (7.1) does not hold true in general.

7.2. The covering construction

Assuming the validity of the inequality (7.1), A. N. Parshin proposed in [19], how to bound the height of \( K \)-rational points \( P \in X(K) \) as effective as possible using the following ramified covering construction.

Given the smooth projective curve \( X/K \) of genus \( g_X > 1 \), and \( P \in X(K) \) a \( K \)-rational point, there exists a finite covering \( X_P/K_P \) over \( X \) with the following properties:

(i) The field extension \( K_P/K \) is a finite extension of degree effectively bounded as \( O(g_X) \) with prescribed ramification.

(ii) The covering \( X_P/X \) is finite of degree effectively bounded as \( O(g_X) \) and ramified only at \( P \) of ramification index effectively bounded as \( O(g_X) \); by the Riemann–Hurwitz formula, the genus \( g_{X_P} \) of \( X_P \) is then also effectively bounded as \( O(g_X) \).

(iii) For each archimedean place \( v \) of \( K \) and each archimedean place \( v' \) of \( K_P \) lying above \( v \), there exists a smooth projective complex surface \( Y_v \) together with a smooth morphism \( \varphi_v: Y_v \to X_v \) such that

\[
\varphi^{-1}_v(P) \cong X_{P,v'} := X_P \times_{v'} \mathbb{C}.
\]
Denoting by $\mathcal{O}_{K_P}$ the ring of integers of $K_P$, setting $S_P := \text{Spec} (\mathcal{O}_{K_P})$, letting $\mathcal{X}_P/S_P$ be a minimal regular model of $X_P/K_P$, which is semistable, and denoting by $\mathcal{X}_{\mathcal{X}_P/S_P}$ the relative dualizing sheaf of $\mathcal{X}_P/S_P$ equipped with the Arakelov metric, the height $h(P)$ of $P$ can be bounded by the arithmetic self-intersection number

$$h(P) \ll \mathcal{O}_{\mathcal{X}_P/S_P}, \quad (7.2)$$

which, in turn, can then be bounded using (7.1), after replacing $\mathcal{O}_{\mathcal{X}/S}$ by $\mathcal{O}_{\mathcal{X}_P/S_P}$. In [19], the quantities $\delta P (\mathcal{P} \in S_P)$, $\text{disc}(K_P/\mathbb{Q})$, and $[K_P: \mathbb{Q}]$ are then effectively bounded in terms of the genus $g_X$ of $X$. The contribution from Faltings’s delta function $\delta_{\text{Fal}} (X_{P,v'}) (v'|v)$ is bounded in terms of $X$ by arguing that, as $P$ is moving through the set of $K$-rational points $X(K)$, the function $\delta_{\text{Fal}} (X_{P,v'})$ can be viewed as the restriction of a real-analytic function on $X_v$, which takes its maximum on the compact Riemann surface $X_v$.

### 7.3. Parshin’s question

After having presented our estimate (6.2) for Faltings’s delta function obtained in Corollary 3.4, Parshin proposed to apply our bound to $\delta_{\text{Fal}} (X_{P,v'})$ in order to obtain a more explicit bound than his.

Indeed, applying the bound obtained in Corollary 6.4 to the ramified covering $X_{P,v'} \to X_v$ of finite degree, observing that the ramification locus $\text{Ram}(X_{P,v'}/X_v)$ consists of only one point, we are led to the bound

$$\delta_{\text{Fal}} (X_{P,v'}) \leq \frac{D_4 g_{X_P} e^{8\pi g_{X_P}/\ell_{X_v} + g_{X_P} \ell_{X_v}}}{(1 - e^{-\ell_{X_v}/4})^5} \frac{1}{\lambda_{X_{P,v'}} (1 - s_{X_{P,v'},1})}, \quad (7.3)$$

where

$$\lambda_{X_{P,v'}} = \frac{1}{2} \min \left\{ \lambda_{X_{P,v'},1}, \frac{7}{64} \right\}, \quad s_{X_{P,v'},1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X_{P,v'},1}}$$

with $\ell_{X_v}$ denoting the length of the shortest closed geodesic on $X_v$ and $\lambda_{X_{P,v'},1}$ denoting the smallest non-zero eigenvalue of $\Delta_{\text{hyp}}$ on $X_{P,v'}$.

As $P$ is moving through the set of $K$-rational points $X(K)$ or, more generally, through the compact Riemann surface $X_v$, the Riemann surfaces $X_{P,v'}$ (or, rather their isomorphism classes) cover a compact region $\mathcal{D}$ in the moduli space $\mathcal{M}_{g_{X_P}}$ of curves of genus $g_{X_P}$. While $P$ is ranging over $X_v$, the function

$$\lambda_{X_{P,v'}} (1 - s_{X_{P,v'},1})$$

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takes its minimum on $\mathcal{D}$, which we denote by $\lambda_{v,\text{min}}$. Keeping in mind that $X_v$ is defined over a number field, Remark 6.2 (i) allows us to simplify the bound (7.3) to

$$\delta_{\text{Fal}}(X_P, v') \leq \frac{10^{17} g_X e^{10g_X} + g_X \ell_X v}{\lambda_{v,\text{min}}};$$

here we recall that the genus $g_X$ can be effectively bounded in terms of the genus $g_X$.

We conclude by emphasizing that our results do not lead to an effective bound for the height $h(P)$ of $K$-rational points $P \in X(K)$, since the bound (7.2) as well as the determination of the minimum $\lambda_{v,\text{min}}$ are not effective.

A. Appendix

In order to apply the inequality of Stieltjes integrals (4.8), we need that the function $K_{\mathbb{H}}^{(1)}(t; \rho)$ is monotone decreasing in $\rho$. The purpose of this appendix is to provide a proof of this claim.

A.1. Lemma

For $t > 0$, $\rho > 0$, and $r \geq \rho$, let

$$F(t; \rho, r) := \frac{r e^{-r^2/(4t)}}{\sinh(r)} T_2 \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right).$$

Then, for all values of $t$, $\rho$, $r$ in the given range, we have

$$\sinh(r) \frac{\partial}{\partial \rho} F(t; \rho, r) + \sinh(\rho) \frac{\partial}{\partial r} F(t; \rho, r) < 0.$$

Proof. — We set

$$X := \frac{\cosh(r/2)}{\cosh(\rho/2)},$$

and compute

$$\frac{\partial}{\partial \rho} F(t; \rho, r) = -\frac{r e^{-r^2/(4t)}}{\sinh(r)} \frac{2 \cosh^2(r/2) \sinh(\rho/2)}{\cosh^3(\rho/2)} =$$

$$- \frac{r e^{-r^2/(4t)}}{\sinh(r)} 2X^2 \tanh(\rho/2) = -F(t; \rho, r) \frac{2X^2}{2X^2 - 1} \tanh(\rho/2),$$

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and
\[ \frac{\partial}{\partial r} F(t; \rho, r) = F(t; \rho, r) \left( \frac{1}{r^2} - \frac{r}{2t} + \frac{2X^2}{2X^2 - 1} \frac{\tanh(r/2) - \cosh(r)}{\sinh(r)} \right). \]

From this we deduce
\[ \sinh(r) \frac{\partial}{\partial \rho} F(t; \rho, r) + \sinh(\rho) \frac{\partial}{\partial r} F(t; \rho, r) = \]
\[ -F(t; \rho, r) \sinh(\rho) \left( \frac{r}{2t} + \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r} \right) - F(t; \rho, r) \frac{2X^2}{2X^2 - 1} h_\rho(r), \quad (7.4) \]
where
\[ h_\rho(r) := \frac{\tanh(\rho/2) \sinh(r) - \sinh(\rho) \tanh(r/2)}{\sinh(r)}. \]

For \( r > 0 \), we now have the estimate
\[ \frac{r}{2t} + \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r} > \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r} = \frac{r \cosh(r) - \sinh(r)}{r \sinh(r)} > 0, \]
using the power series expansions for \( \cosh(r) \) and \( \sinh(r) \). Next, we compute and estimate for \( r \geq \rho > 0 \)
\[ h'_\rho(r) = \tanh(\rho/2) \cosh(r) - \frac{\sinh(\rho)}{2 \cosh^2(r/2)} \]
\[ = \tanh(\rho/2) (2 \cosh^2(r/2) - 1) - \frac{\sinh(\rho/2) \cosh(\rho/2)}{\cosh^2(r/2)} \]
\[ = \frac{\tanh(\rho/2)}{\cosh^2(r/2)} (2 \cosh^4(r/2) - \cosh^2(r/2) - \cosh^2(\rho/2)) \]
\[ \geq \frac{2 \tanh(\rho/2)}{\cosh^2(r/2)} (\cosh^4(r/2) - \cosh^2(r/2)) \]
\[ = 2 \tanh(\rho/2) (\cosh^2(r/2) - 1) > 0. \]

Since \( h_\rho(\rho) = 0 \), this shows that \( h_\rho(r) \geq 0 \) for \( r \geq \rho > 0 \). Recalling (7.4) the claim of the lemma follows from the above estimates. \( \Box \)

**A.2. Proposition**

*For any \( t > 0 \), the heat kernel \( K_{\mathbb{H}}^{(1)}(t; \rho) \) for forms is strictly monotone decreasing for \( \rho > 0 \).*

**Proof.** — We will prove that \( \partial/\partial \rho K_{\mathbb{H}}^{(1)}(t; \rho) < 0 \) for \( \rho > 0 \). To simplify notations, we put
\[ c(t) := \sqrt{2e^{-t/4}} \]
\[ = \frac{\sqrt{2}}{(4\pi t)^{3/2}}. \]
Effective bounds for Faltings’s delta function

In the notation of Lemma A.1, we then have, using integration by parts,

\[
K^{(1)}_{\mathbb{H}}(t; \rho) = c(t) \int_\rho^\infty F(t; \rho, r) \frac{\sinh(r)}{\sqrt{\cosh(r) - \cosh(\rho)}} \, dr
\]

\[
= -2c(t) \int_\rho^\infty \frac{\partial}{\partial r} F(t; \rho, r) \sqrt{\cosh(r) - \cosh(\rho)} \, dr.
\]

We now apply the Leibniz rule of differentiation to write

\[
\frac{\partial}{\partial \rho} K^{(1)}_{\mathbb{H}}(t; \rho) = -2c(t) \int_\rho^\infty \frac{\partial^2}{\partial r \partial \rho} F(t; \rho, r) \sqrt{\cosh(r) - \cosh(\rho)} \, dr
\]

\[
+ c(t) \int_\rho^\infty \frac{\partial}{\partial r} F(t; \rho, r) \frac{\sinh(\rho)}{\sqrt{\cosh(r) - \cosh(\rho)}} \, dr.
\]

Using integration by parts on the first term once again, yields the identity

\[
\frac{\partial}{\partial \rho} K^{(1)}_{\mathbb{H}}(t; \rho) = c(t) \int_\rho^\infty \left( \sinh(r) \frac{\partial}{\partial \rho} F(t; \rho, r) + \sinh(\rho) \frac{\partial}{\partial r} F(t; \rho, r) \right) \frac{dr}{\sqrt{\cosh(r) - \cosh(\rho)}}.
\]

From Lemma A.1, we conclude that \(\partial/\partial \rho K^{(1)}_{\mathbb{H}}(t; \rho) < 0\) for \(\rho > 0\), which proves the claim. \(\square\)

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