LOCAL LIMIT THEOREMS FOR THE RANDOM CONDUCTANCE MODEL AND APPLICATIONS TO THE GINZBURG-LANDAU $\nabla\varphi$ INTERFACE MODEL

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ABSTRACT. We study a continuous-time random walk on $\mathbb{Z}^d$ in an environment of random conductances taking values in $(0, \infty)$. For a static environment, we extend the quenched local limit theorem to the case of a general speed measure, given suitable ergodicity and moment conditions on the conductances and on the speed measure. Under stronger moment conditions, an annealed local limit theorem is also derived. Furthermore, an annealed local limit theorem is exhibited in the case of time-dependent conductances, under analogous moment and ergodicity assumptions. This dynamic local limit theorem is then applied to prove a scaling limit result for the space-time covariances in the Ginzburg-Landau $\nabla\varphi$ model. This result applies to convex potentials for which the second derivative may be unbounded.

1. INTRODUCTION

1.1. The Model. We consider the graph $G = (\mathbb{Z}^d, E_d)$ of the hypercubic lattice with the set of nearest-neighbour edges $E_d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$ with $d \geq 2$. We place upon $G$ positive weights $\omega = \{\omega(e) \in (0, \infty) : e \in E_d\}$. We will also write $\omega$ for the conductance matrix; $\omega(x, y) = \omega(y, x) = \omega(\{x, y\})$ if the edge $\{x, y\} \in E_d$ and $\omega(x, y) = 0$ otherwise. We define two measures on $\mathbb{Z}^d$,

$$\mu^\omega(x) := \sum_{y \sim x} \omega(x, y), \quad \nu^\omega(x) := \sum_{y \sim x} \frac{1}{\omega(x, y)}.$$ 

Let $(\Omega, \mathcal{F}) := (\mathbb{R}_+^{E_d}, \mathcal{B}(\mathbb{R}_+)^{\otimes E_d})$ be the measurable space of all possible environments. We denote by $\mathbb{P}$ an arbitrary probability measure on $(\Omega, \mathcal{F})$ and $\mathbb{E}$ the respective expectation. The measure space $(\Omega, \mathcal{F})$ is naturally equipped with a group of space shifts $\{\tau_z : z \in \mathbb{Z}^d\}$, which act on $\Omega$ as

$$(\tau_z \omega)(x, y) := \omega(x + z, y + z), \quad \forall \{x, y\} \in E_d.$$ (1.1)

Let $\theta^\omega : \mathbb{Z}^d \to (0, \infty)$ be a positive function which may depend upon the environment $\omega \in \Omega$. The random walk $(X_t)_{t \geq 0}$ defined by the following generator,

$$\mathcal{L}^\omega f(x) := \frac{1}{\theta^\omega(x)} \sum_{y \sim x} \omega(x, y) \left( f(y) - f(x) \right),$$

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acting on bounded functions \( f : \mathbb{Z}^d \to \mathbb{R} \), is reversible with respect to \( \theta^\omega \), and we call this process the random conductance model (RCM) with speed measure \( \theta^\omega \). We denote \( P^\omega_x \) the law of this process started at \( x \in \mathbb{Z}^d \) and \( E^\omega_x \) the corresponding expectation. There are two natural laws on the path space that are considered in the literature - the quenched law \( P^\omega_x(\cdot) \) which concerns \( \mathbb{P} \)-almost sure phenomena, and the annealed law \( \mathbb{E}P^\omega_x(\cdot) \).

The random walk \( X \) chooses its next position with probability \( \omega(x, y)/\mu^\omega(x) \), after waiting an exponential time with mean \( \theta^\omega(x)/\mu^\omega(x) \) at the vertex \( x \). The main results of this paper are statements about the heat kernel of \( X \), which is defined as

\[
p^\omega_x(t, x, y) := \frac{P^\omega_x(X_1 = y)}{\theta^\omega(y)} \quad \text{for } t \geq 0 \text{ and } x, y \in \mathbb{Z}^d.
\]

Perhaps the most natural choice for the speed measure is \( \theta^\omega \equiv \mu^\omega \), for which we obtain the constant speed random walk (CSRW) that spends i.i.d. \( \text{Exp}(1) \)-distributed waiting times at all vertices it visits. Another well-studied process, the variable speed random walk (VSRW) is recovered by setting \( \theta^\omega \equiv 1 \), so called because as opposed to the CSRW, the waiting time at a vertex \( x \) does indeed depend on the location; it is an \( \text{Exp}(\mu^\omega(x)) \)-distributed random variable.

1.2. Main Results on the Static RCM. As our first main results we obtain quenched and annealed local limit theorems for the static random conductance model. A general assumption required in this context is stationarity and ergodicity of the environment. All results on the static RCM are restricted to dimension \( d \geq 2 \).

**Assumption 1.1.**

(i) \( \mathbb{P}[0 < \omega(e) < \infty] = 1 \) and \( \mathbb{E}[\omega(e)] < \infty \) for all \( e \in E_d \).

(ii) \( \mathbb{P} \) is ergodic with respect to spatial translations of \( \mathbb{Z}^d \), i.e. \( \mathbb{P} \circ \tau_x^{-1} = \mathbb{P} \) for all \( x \in \mathbb{Z}^d \) and \( \mathbb{P}(A) \in \{0, 1\} \) for any \( A \in \mathcal{F} \) such that \( \tau_x(A) = A \) for all \( x \in \mathbb{Z}^d \).

(iii) \( \theta \) is stationary, i.e. \( \theta^\omega(x + y) = \theta^\omega(x) \) for all \( x, y \in \mathbb{Z}^d \) and \( \mathbb{P} \)-a.e. \( \omega \in \Omega \). Further, \( \mathbb{E}[\theta^\omega(0)] < \infty \) and \( \mathbb{E}[\theta^\omega(0)/\mu^\omega(0)] \in (0, \infty) \).

In particular, the last condition in Assumption 1.1(iii) ensures that the process \( X \) is non-exlosive. During the last decade, considerable effort has been invested in the derivation of quenched invariance principles or quenched functional central limit theorems (QFCLT), see the surveys [10, 24] and references therein. The following QFCLT for random walks under ergodic conductances is the main result of [4].

**Theorem 1.2** (QFCLT). Suppose Assumption 1.1 holds. Further assume that there exist \( p, q \in (1, \infty] \) satisfying \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d} \) such that

\[
\mathbb{E}[\omega(e)^p] < \infty \quad \text{and} \quad \mathbb{E}[\omega(e)^{-q}] < \infty
\]

for any \( e \in E_d \). For \( n \in \mathbb{N} \), define \( X^{(n)}_t := \frac{1}{n}X_{nt} \), \( t \geq 0 \). Then, for \( \mathbb{P} \)-a.e. \( \omega \), \( X^{(n)}(t) \) converges (under \( P^0_0 \)) in law towards a Brownian motion on \( \mathbb{R}^d \) with a deterministic non-degenerate covariance matrix \( \Sigma^2 \).
Proof. For the VSRW, this is [4, Theorem 1.3]. As noted in [4, Remark 1.5] the QFCLT extends to the random walk with general speed measure $\theta^\omega$ provided $\mathbb{E}[\theta^\omega(0)] \in (0, \infty)$. See [2, Section 6.2] for a proof of this extension in the case of the CSRW.

Recently the moment condition in Theorem 1.2 has been improved in [8].

Remark 1.3. If we let $\bar{\Sigma}^2$ denote the covariance matrix of the above Theorem in the case of the VSRW, the corresponding covariance matrix of the random walk $X$ with speed measure $\theta^\omega$ is given by $\Sigma^2 = \mathbb{E}[\theta^\omega(0)]^{-1} \bar{\Sigma}^2$ – see [4, Remark 1.5].

The local limit theorem roughly describes how the transition probabilities of the random walk $X$ can be rescaled in order to get the Gaussian transition density of the Brownian motion with covariance matrix $\Sigma^2$, which appears as the limit process in the invariance principle in Theorem 1.2. The Gaussian heat kernel associated with that process will be denoted

$$k_t(x) \equiv k_t^2(x) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma^2}} \exp \left( -x \cdot (\Sigma^2)^{-1} x / 2t \right).$$

The quenched local limit theorem will require the following moment condition.

Assumption 1.4. There exist $p, q, r \in (1, \infty]$ satisfying

$$\frac{1}{r} + \frac{1}{p} \frac{r - 1}{r} + \frac{1}{q} \leq \frac{2}{d}$$

such that

$$\mathbb{E} \left[ \left( \frac{\nu^\omega(0)}{\theta^\omega(0)} \right)^p \theta^\omega(0)^q \right] + \mathbb{E} [\nu^\omega(0)^q] + \mathbb{E} [\theta^\omega(0)^{-1}] + \mathbb{E} [\theta^\omega(0)^r] < \infty. \quad (1.4)$$

Note that in the case of the CSRW or VSRW Assumption 1.4 coincides with the moment condition in Theorem 1.2. For $x \in \mathbb{R}^d$ write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor) \in \mathbb{Z}^d$.

Theorem 1.5 (Quenched local limit theorem). Let $T_2 > T_1 > 0$, $K > 0$ and suppose that Assumptions 1.1 and 1.4 hold. Then,

$$\lim_{n \to \infty} \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \left| n^d p^\omega_{\theta^\omega}(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x) \right| = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

with $a := \mathbb{E} [\theta^\omega(0)]^{-1}$.

This result extends the local limit theorem in [5, Theorem 1.11] for the CSRW to the case of a general speed measure. In general, a local limit theorem is a stronger statement than a FCLT. In fact, already in the i.i.d. case, where the QFCLT does hold [2], we see the surprising effect that due to a trapping phenomenon the heat kernel may behave subdiffusively (see [9]), in particular a local limit theorem may fail in general. Nevertheless, it does hold, for instance, in the case of uniformly elliptic conductances, where $\mathbb{P}(c^{-1} \leq \omega(e) \leq c) = 1$ for some $c \geq 1$, or for random walks on supercritical percolation clusters (see [7]). For sharp conditions on the tails of i.i.d. conductances at zero for Harnack inequalities and a local limit theorem to hold we refer to [11]. Hence, it is clear that some moment condition is necessary. In the
case of the CSRW under general ergodic conductances the moment condition in Assumption 1.4 is known to be optimal, see [5, Theorem 5.4]. Local limit theorems have also been obtained in slightly different settings, see [17], where based on the arguments in [7] some general criteria for local limit theorems have been provided.

For the proof of Theorem 1.5 we adapt the techniques employed in [15] to the static, general speed measure case. The idea comes from [28] where it is introduced as a way of deriving Hölder regularity of solutions to parabolic PDEs in continuum. Bounds on the level sets of caloric functions, such as the heat kernel, are derived and used to prove an oscillations inequality, Theorem 2.5, which bounds the oscillations of the function on a time-space cylinder by those on a larger cylinder. By iterating the oscillations bound, a Hölder regularity result is deduced. This, along with the QFCLT, is precisely what is required to prove a local limit theorem. We stress that this approach to show Hölder regularity directly circumvents the need for a parabolic Harnack inequality, in contrast to the proofs in [5, 7], which makes it significantly simpler. However, as a by-product we still obtain the following weak parabolic Harnack inequality. Write $m_\theta = dt \times \theta^\omega$.

**Theorem 1.6** (Weak Harnack inequality). Suppose Assumptions 1.1 and 1.4 hold. For any $x_0 \in \mathbb{Z}^d$, $t_0 \in \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega$, there exists $N_1 = N_1(\omega, x_0)$ such that for all $n \geq N_1$ the following holds. Let $u > 0$ be such that $\Delta u - \mathcal{L}^\theta u = 0$ on $Q(n) := [t_0 - n^2, t_0] \times B(x_0, n)$. Assume there exists $\lambda \in (\frac{1}{2}, 1)$ such that

$$m_\theta \left( (t, x) \in Q(n) : u(t, x) \geq \epsilon \right) \geq \lambda m_\theta \left( Q(n) \right).$$

for some $\epsilon > 0$. Then there exists $\gamma = \gamma(\epsilon, \lambda)$ (also depending on the law of $\omega$ and $\theta^\omega$) such that for any $\sigma' \in [\frac{1}{2}, \lambda)$,

$$u(t, x) \geq \gamma \quad \forall (t, x) \in Q(\sigma'n) := [t_0 - \sigma'n^2, t_0] \times B(x_0, \sigma'n).$$

A natural example for a harmonic function satisfying the condition $u(t, x) \leq 1/\theta^\omega(x)$ in Theorem 1.6 is the heat kernel $p^\omega(t, 0, x)$. Next we provide an annealed local limit theorem under a stronger and non-optimal moment condition.

**Theorem 1.7** (Annealed local limit theorem). Suppose Assumption 1.1 holds. There exist exponents $p, q, r_1, r_2 \in (1, \infty)$ (only depending on $d$) such that if

$$\mathbb{E} \left[ \mu^{\omega}(0)^p \right] + \mathbb{E} \left[ \nu^{\omega}(0)^q \right] + \mathbb{E} \left[ \theta^{\omega}(0)^{-r_1} \right] + \mathbb{E} \left[ \theta^{\omega}(0)^{r_2} \right] < \infty$$

the following holds. For all $K > 0$ and $0 < T_1 \leq T_2$,

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} n^d p^\omega_n (n^2 t, 0, |nx|) - ak_t(x) \right] = 0. \quad (1.5)$$

**Remark 1.8.** In the case of the VSRW, i.e. $\theta^\omega = 1$, the moment condition required in Theorem 1.7 is more explicitly given by

$$\mathbb{E} \left[ \omega(e)^{2(\kappa' \lor p)} \right] < \infty, \quad \mathbb{E} \left[ \omega(e)^{-2(\kappa' \lor q)} \right] < \infty, \quad e \in E_d.$$
for some \( p, q \in (1, \infty) \) such that \( 1/p + 1/q < 2/d \) and \( \kappa' = \kappa'(d, p, q, \infty) \) defined in Proposition 3.1 below. Similarly, in the case of the CSRW, \( \theta^{\omega} = \mu^{\omega} \), the condition reduces to
\[
E[\omega(e)^{4\kappa'\sqrt{2p}}] < \infty, \quad E[\omega(e)^{-4\kappa'\sqrt{2q}}] < \infty, \quad e \in E_d,
\]
again for some \( p, q \in (1, \infty) \) such that \( 1/p + 1/q < 2/d \) and \( \kappa' = \kappa'(d, \infty, q, p) \) again as defined in Proposition 3.1 below.

In general, a QFCLT does imply an annealed FCLT. However, the same does not apply to the local limit theorem. In fact, as mentioned above, the proofs of the quenched local limit theorems in [5] and Theorem 1.5 rely on Hölder regularity estimates on the heat kernel, which involve some random constants depending on the exponential of the conductances. Those constants can be controlled almost surely, but naively taking expectations would require exponential moment conditions stronger than the polynomial moment conditions in Assumption 1.4. To derive the annealed local limit theorem given the corresponding quenched result, one might hope to employ the dominated convergence theorem, which requires that the integrand above can be dominated uniformly in \( n \) by a function of finite expectation. We achieve this using a quenched maximal inequality from [6]. It is precisely the form of the random constants in this inequality that allows us to anneal the result using only polynomial moments, together with a simple probabilistic bound.

1.3. Main Results on the Dynamic RCM. Next we introduce the dynamic random conductance model. As for the static case, we consider \( d \geq 2 \) only. We endow \( \omega = \{\omega_t(e) \in (0, \infty) : e \in E_d, t \in \mathbb{R}\} \) of positive, time-dependent weights. Define, for \( t \in \mathbb{R} \), the measures \( \mu^{\omega}_t, \nu^{\omega}_t \) on \( \mathbb{Z}^d \) by
\[
\mu^{\omega}_t(x) := \sum_{y \sim x} \omega_t(x, y), \quad \nu^{\omega}_t(x) := \sum_{y \sim x} 1/\omega_t(x, y).
\]

We define the dynamic variable speed random walk starting in \( x \in \mathbb{Z}^d \) at \( s \in \mathbb{R} \) to be the continuous-time Markov chain \( (X_t : t \geq s) \) with time-dependent generator
\[
(L^{\omega}_t f)(x) := \sum_{y \sim x} \omega_t(x, y) \left(f(y) - f(x)\right),
\]
acting on bounded functions \( f : \mathbb{Z}^d \to \mathbb{R} \). Note that the counting measure, which is time-independent, is an invariant measure for \( X \). We denote \( P^{\omega}_{s,x} \) the law of this process started at \( x \in \mathbb{Z}^d \) at time \( s \), and \( E^{\omega}_{s,x} \) the corresponding expectation. For \( x, y \in \mathbb{Z}^d \) and \( t \geq s \), we denote \( p^{\omega}(s, t, x, y) \) the heat kernel of \( (X_t)_{t \geq s} \), that is
\[
p^{\omega}(s, t, x, y) := P^{\omega}_{s,x} [X_t = y].
\]
Let now \( \Omega \) be the set of measurable functions from \( \mathbb{R} \) to \((0, \infty)^{E_d} \) equipped with a \( \sigma \)-algebra \( \mathcal{F} \) and let \( \mathbb{P} \) be a probability measure on \((\Omega, \mathcal{F})\). Upon it we consider the \( d + 1 \)-parameter group of translations \((\tau_{t,x})(t,x) \in \mathbb{R} \times \mathbb{Z}^d \) given by
\[
\tau_{t,x} : \Omega \to \Omega, \quad (\omega_s(e))_{s \in \mathbb{R}, e \in E_d} \mapsto (\omega_{t+s}(x + e))_{s \in \mathbb{R}, e \in E_d}.
\]
The required ergodicity and stationarity assumptions on the time-dependent random environment are as follows.

**Assumption 1.9.** (i) \( \mathbb{P} \) is ergodic with respect to time-space translations, i.e. for all \( x \in \mathbb{Z}^d \) and \( t \in \mathbb{R} \), \( \mathbb{P} \circ \tau_{t,x}^{-1} = \mathbb{P} \). Further, \( \mathbb{P}(A) \in \{0,1\} \) for any \( A \in \mathcal{F} \) such that \( \tau_{t,x}(A) = A \) for all \( x \in \mathbb{Z}^d, t \in \mathbb{R} \).

(ii) For every \( A \in \mathcal{F} \), the mapping \( (\omega, t, x) \mapsto \mathbb{1}_{A}(\tau_{t,x}\omega) \) is jointly measurable with respect to the \( \sigma \)-algebra \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{Z}^d) \).

**Theorem 1.10** (Quenched FCLT and local limit theorem). Suppose Assumption 1.9 holds and that there exist \( p, q \in (1, \infty) \) satisfying

\[
\frac{1}{p-1} + \frac{1}{(p-1)q} + \frac{1}{q} < \frac{2}{d}
\]

such that

\[
\mathbb{E}[\omega_0(e)^p] < \infty \quad \text{and} \quad \mathbb{E}[\omega_0(e)^{-q}] < \infty
\]

for any \( e \in E_d \) and \( t \in \mathbb{R} \).

(i) For \( \mathbb{P}\)-a.e. \( \omega \), \( X^{(n)} \), defined as \( X^{(n)}_t := \frac{1}{n} X_{n^2 t, 0}, t \geq 0 \), converges (under \( \mathbb{P}^\omega_0 \)) in law towards a Brownian motion on \( \mathbb{R}^d \) with a deterministic non-degenerate covariance matrix \( \Sigma^2 \).

(ii) For any \( T_2 > T_1 > 0 \) and \( K > 0 \),

\[
\lim_{n \to \infty} \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \left| n^d p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) - k_t(x) \right| = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega,
\]

where \( k_t \) still denotes the heat kernel of a Brownian motion on \( \mathbb{R}^d \) with covariance \( \Sigma^2 \).

**Proof.** The QFLCT in (i) has been proven in [3], for the quenched local limit theorem in (ii) we refer to [15]. \( \square \)

Similarly as in the static case we establish an annealed local limit theorem for the dynamic RCM under a stronger, non-optimal but polynomial moment condition.

**Theorem 1.11** (Annealed local limit theorem). Suppose Assumption 1.9 holds. There exist exponents \( p, q \in (1, \infty) \) (specified more explicitly in Assumption 4.2 below) such that if

\[
\mathbb{E}[\omega_0(e)^p] < \infty \quad \text{and} \quad \mathbb{E}[\omega_0(e)^{-q}] < \infty
\]

for any \( e \in E_d \), the following holds. For all \( K > 0 \) and \( 0 < T_1 \leq T_2 \),

\[
\lim_{n \to \infty} \mathbb{E}\left[ \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \left| n^d p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) - k_t(x) \right| \right] = 0. \quad (1.6)
\]
1.4. Application to the Ginzburg-Landau $\nabla \varphi$ Model. A somewhat unexpected context in which one encounters (dynamic) RCMs is that of gradient Gibbs measures describing stochastic interfaces in statistical mechanical systems. One well-established model is the Ginzburg-Landau model, where an interface is described by a field of height variables $\{\phi_t(x), x \in \mathbb{Z}^d, t \geq 0\}$, whose stochastic dynamics are given by the following infinite system of stochastic differential equations involving nearest neighbour interaction:

$$\phi_t(x) = \phi_0(x) - \int_0^t \sum_{y : |x-y|=1} V'(\phi_t(y) - \phi_t(x)) \, dt + \sqrt{2} w_t(x), \quad x \in \mathbb{Z}^d.$$ 

Here $\{w(x), x \in \mathbb{Z}^d\}$ is a collection of independent Brownian motions and the potential $V \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^+)$ is even and convex. The formal equilibrium measure for the dynamic is given by the Gibbs measure $Z^{-1} \exp(-H(\phi)) \prod_x d\phi(x)$ on $\mathbb{R}^{\mathbb{Z}^d}$ with formal Hamiltonian $H(\phi) = \frac{1}{4} \sum_{x-y} V(\phi(x) - \phi(y))$. Investigating the fluctuations of the macroscopic interface has been quite an active field of research, see [19] for a survey.

We are interested in the decay of the space-time covariances of height variables under an equilibrium Gibbs measure. By the Helffer-Sjöstrand representation [22] (cf. also [18, 21]) such covariances can be written in terms of the annealed heat kernel of a random walk among dynamic random conductances. More precisely,

$$\text{Cov}_\mu(\phi_0(0), \phi_t(y)) = \int_0^\infty \mathbb{E}_\mu \left[ p_{\omega(t)}(0, y) \right] ds,$$

where the covariance and expectation are taken with respect to an ergodic Gibbs measure $\mu$ and $p_{\omega}$ denotes the heat kernel of the dynamic RCM with time-dependent conductances given by

$$\omega_t(x, y) := V''(\phi_t(y) - \phi_t(x)), \quad \{x, y\} \in E_d, t \geq 0.$$ 

Thus far, all applications of the aforementioned Helffer-Sjöstrand relation have been restricted to gradient models with strictly convex potential function, which corresponds to uniformly elliptic conductances in the random walk picture. However, recent developments in the degenerate setting will also allow some potentials that are convex but not strictly convex. As an example in this direction, we use the annealed local limit theorem in Theorem 1.11 to derive a scaling limit for the space-time covariances of the $\varphi$-field for a wider class of potentials.

**Theorem 1.12.** Suppose $d \geq 3$ and let $V \in \mathcal{C}^2(\mathbb{R})$ be even with $V'' \geq c_- > 0$. Then for all $h \in \mathbb{R}$ there exists a stationary, shift-invariant, ergodic $\varphi$-Gibbs measure $\mu \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$ of mean $h$, i.e. $\mathbb{E}_\mu[\phi(x)] = h$ for all $x \in \mathbb{Z}^d$. Further, assume that

$$\mathbb{E}_\mu \left[ V''(\phi_t(y) - \phi_t(x)) \right] < \infty, \quad \text{for any } \{x, y\} \in E_d,$$

(1.7)
with \( p(d) := (2 + d)(1 + 2/d + \sqrt{1 + 1/d^2}) \). Then for all \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} n^{d-2} \text{Cov}_\mu \left( \phi_0(0), \phi_{n^2}([nx]) \right) = \int_0^\infty k_{t+s}(x) \, ds,
\]

where \( k_t \) is the heat kernel of a Brownian motion on \( \mathbb{R}^d \) with a deterministic non-degenerate covariance matrix.

Theorem 1.12 extends the scaling limit result of [1, Theorem 5.2] to hold for potentials \( V \) for which \( V'' \) may be unbounded above.

**Example 1.13.** The moment condition in (1.7) on the potential \( V \) is satisfied for any \( V \) with \( V'' \) having polynomial growth (see Proposition 5.10 below). Hence, Theorem 1.12 applies, for instance, to the anharmonic crystal potential \( V(x) = x^2 + \lambda x^4 \) (\( \lambda > 0 \)), for which the decay of spatial correlations is discussed in [12].

### 1.5. Notation

We finally introduce some further notation used in the paper. We write \( c \) to denote a positive, finite constant which may change on each appearance. Constants denoted by \( c_i \) will remain the same. For a number \( p \in [1, \infty] \) we write \( p_* := p/(p-1) \in [1, \infty] \) for its Hölder conjugate. We endow the graph \( G = (\mathbb{Z}^d, E_d) \) with the natural graph distance \( d \), i.e. \( d(x, y) \) is the minimal length of a path between \( x \) and \( y \). Denote \( B(x, r) := \{ y \in \mathbb{Z}^d : d(x, y) \leq r \} \) the closed ball with centre \( x \) and radius \( r \). For a non-empty, finite, connected set \( A \subseteq \mathbb{Z}^d \), we denote by \( \partial A := \{ x \in A : d(x, y) = 1 \text{ for some } y \in A^c \} \) the inner boundary and by \( \partial^+ A := \{ x \in A^c : d(x, y) = 1 \text{ for some } y \in A \} \) the outer boundary of \( A \). We write \( \mathcal{A} = A \cup \partial^+ A \) for the closure of \( A \). The graph is given the counting measure, i.e. the measure of \( A \subseteq \mathbb{Z}^d \) is the number \( |A| \) of elements in \( A \). For \( f : \mathbb{Z}^d \to \mathbb{R} \) we define the operator \( \nabla \) by

\[
\nabla f : E \to \mathbb{R}, \quad E \ni e \longmapsto \nabla f(e) := f(e^+) - f(e^-),
\]

where for each non-oriented edge \( e \in E_d \) we specify one of its two endpoints as its initial vertex \( e^+ \) and the other one as its terminal vertex \( e^- \). Further, the corresponding adjoint operator \( \nabla^* F : \mathbb{Z}^d \to \mathbb{R} \) acting on functions \( F : E_d \to \mathbb{R} \) is defined in such a way that \( \langle \nabla f, F \rangle_{\ell^2(E_d)} = \langle f, \nabla^* F \rangle_{\ell^2(\mathbb{Z}^d)} \) for all \( f \in \ell^2(\mathbb{Z}^d) \) and \( F \in \ell^2(E_d) \). Notice that in the discrete setting the product rule reads

\[
\nabla (fg) = \text{av}(f) \nabla g + \text{av}(g) \nabla f, \quad (1.8)
\]

where \( \text{av}(f)(e) := \frac{1}{2} (f(e^+) + f(e^-)) \). We denote inner products as follows; for \( f, g : \mathbb{Z}^d \to \mathbb{R} \) and a weighting function \( \phi : \mathbb{Z}^d \to \mathbb{R} \), \( \langle f, g \rangle_{\ell^2(\mathbb{Z}^d, \phi)} := \sum_{x \in \mathbb{Z}^d} f(x)g(x)\phi(x) \) and if \( f, g : E_d \to \mathbb{R} \), \( \langle f, g \rangle_{\ell^2(E)} := \sum_{e \in E_d} f(e)g(e) \). The corresponding weighted norm is denoted \( \| f \|_{\ell^2(\mathbb{Z}^d, \phi)} \). Given a bounded Lipschitz function \( F \) on \( \ell^2(\mathbb{Z}^d, \phi) \), its Lipschitz semi-norm will be written

\[
\| F \|_{\text{lip}, \phi} := \sup_{f \neq g} \frac{|F(f) - F(g)|}{\| f-g \|_{\ell^2(\mathbb{Z}^d, \phi)}^{-1}}.
\]
The Dirichlet form associated with the operator $\mathcal{L}_0^\omega$ is

$$\mathcal{E}_\omega(f, g) := \langle f, -\mathcal{L}_0^\omega g \rangle_{\mathcal{E}(\mathbb{Z}^d, \theta)} = \langle \nabla f, \omega \nabla g \rangle_{\mathcal{E}(E)} ,$$

acting on bounded $f, g : \mathbb{Z}^d \to \mathbb{R}$. For non-empty, finite $B \subseteq \mathbb{Z}^d$ and $p \in (0, \infty)$, space-averaged $\ell^p$-norms on functions $f : B \to \mathbb{R}$ will be used,

$$\|f\|_{p,B} := \left( \frac{1}{|B|} \sum_{x \in B} |f(x)|^p \right)^{1/p} \quad \text{and} \quad \|f\|_{\infty,B} := \max_{x \in B} |f(x)| .$$

Now let $Q = I \times B$ where $I \subseteq \mathbb{R}$ is compact. Let $u : Q \to \mathbb{R}$ and denote $u_t : B \to \mathbb{R}$, $u_t(\cdot) := u(t, \cdot)$ for $t \in I$. For $p' \in (0, \infty)$, we define norms averaged over space and time:

$$\|u\|_{p,p',Q} := \left( \frac{1}{|I|} \int_I \|u_t\|_{p,B}^{p'} \, dt \right)^{1/p'} \quad \text{and} \quad \|u\|_{p,\infty,Q} := \max_{t \in I} \|u_t\|_{p,B} .$$

Furthermore, we will work with two varieties of weighted norms

$$\|f\|_{p,B,\phi} := \left( \frac{1}{\phi(B)} \sum_{x \in B} f(x)^p \phi(x) \right)^{1/p} , \quad |f|_{p,B,\phi} := \left( \frac{1}{|B|} \sum_{x \in B} f(x)^p \phi(x) \right)^{1/p} ,$$

$$\|u\|_{p,p',Q,\phi} := \left( \frac{1}{|I|} \int_I \|u_t\|_{p,B,\phi}^{p'} \, dt \right)^{1/p'} , \quad \|u\|_{p,\infty,Q,\phi} := \max_{t \in I} \|u_t\|_{p,B,\phi} ,$$

for a weighting function $\phi : B \to (0, \infty)$, where $\phi(B) := \sum_{x \in B} \phi(x)$.

In Section 5, we employ also the following notation. We write $\Lambda \subseteq \mathbb{Z}^d$ for $\Lambda$ a finite subset of $\mathbb{Z}^d$. $\Lambda^*$ denotes the set of all directed edges in $\Lambda$, i.e. $\Lambda^* = \{(x, y) \in E_d : x, y \in \Lambda\}$. Write $\mathcal{P}(S)$ for the family of Borel probability measures on a topological space $S$. Given a measure $\mu \in \mathcal{P}(S)$, $\mathbb{E}_{\mu}[X]$ denotes the expectation of a random variable $X$ under $\mu$ and $\text{var}_{\mu}(X)$ its variance. We denote the covariance of two random variables $X, Y$ under $\mu$, $\text{Cov}_{\mu}(X, Y)$.

1.6. Structure of the Paper. Section 2 is devoted to the proof of the quenched local limit theorem for general speed measures - Theorem 1.5. The annealed local limit theorems for the static and dynamic RCM in Theorem 1.7 and Theorem 1.11 are shown in Section 3 and Section 4, respectively. Finally, the application to the Ginzburg-Landau interface model is discussed in Section 5.

2. Local Limit Theorem for the Static RCM under General Speed Measure

For the purposes of this section, we work with space-time cylinders defined as follows. For any $x \in \mathbb{Z}^d$ and $t_0 \in \mathbb{R}$ let $I_x := [t_0 - \tau n^2, t_0]$ and $B_\sigma := B(x, \sigma n)$ for $\sigma \in (0, 1], \tau \in (0, 1]$. We write

$$Q_{\tau,\sigma}(n) := I_{\tau} \times B_\sigma \quad \text{and} \quad Q_\sigma := Q_\sigma(n) := Q_{\sigma,\sigma}(n).$$

Note that throughout this section we assume the dimension $d \geq 2$. 
2.1. Maximal Inequality. We first derive a maximal inequality for caloric functions under a general speed measure using a De Giorgi iteration scheme. We require the following energy estimate, cf. [6, Lemma 3.7].

**Lemma 2.1.** Suppose \( Q = I \times B \) where \( I = [s_1, s_2] \subseteq \mathbb{R} \) is an interval and \( B \subset \mathbb{Z}^d \) is finite and connected. Let \( u \) be a non-negative solution of \( \partial_t u - \Delta^\theta u \leq 0 \) on \( Q \). Let \( \eta: \mathbb{Z}^d \rightarrow [0, 1] \) and \( \xi: \mathbb{R} \rightarrow [0, 1] \) be cutoff functions such that \( \text{supp} \, \eta \subseteq B \), \( \text{supp} \, \xi \subseteq I \) and \( \eta \equiv 0 \) on \( \partial B \), \( \xi(s_1) = 0 \). Then there exists \( c_1 \) such that for any \( k \geq 0 \) and \( p, p_* \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{p_*} = 1 \),

\[
\frac{1}{|I|} \left\| \xi \eta^2 (u - k)^2 \right\|_{1, \infty, Q, \theta} + \frac{1}{|I|} \int_I \xi(t) \frac{E^\omega(\eta v)}{\theta^\omega(B)} \, dt \\
\leq c_1 \left( \left\| \mu^\omega/\theta^\omega \right\|_{p, B, \theta} \left\| \nabla \eta \right\|^2_{L^\infty(I)} + \left\| \xi \right\|^2_{L^\infty(I)} \right) \left\| (u - k)^2 \right\|_{p, 1, Q, \theta}. \tag{2.1}
\]

**Proof.** Set \( v = (u - k)_+ \) for abbreviation. Then by the chain rule \( \partial_t v^2 = 2v \partial_t u \leq 2v \Delta^\theta u \). Further, note that \( \nabla ((\eta^2 v_1) \nabla v_t) \leq \nabla ((\eta^2 v_1) \nabla u_t) \), which can be verified by distinguishing several cases. Thus, a summation by parts gives for any \( t \in [s_1, s_2] \),

\[
\frac{1}{2} \partial_t \left\| \eta v_t \right\|_{t^2(\mathbb{Z}^d, \theta)}^2 \leq - \left\langle \nabla (\eta^2 v_1), \omega \nabla v_t \right\rangle_{\ell^2(A_d)}.
\]

Further, by the product rule (1.8)

\[
\left\langle \nabla (\eta v_t), \omega (\eta v_t) \right\rangle_{\ell^2(A_d)} \leq \left\langle \nabla (\eta^2 v_1), \omega \nabla v_t \right\rangle_{\ell^2(A_d)} + \left\langle \text{av}(v t)^2, \omega (\nabla \eta)^2 \right\rangle_{\ell^2(A_d)},
\]

where we used that \( \text{av}(\eta)^2 \leq \text{av}(\eta^2) \) by Jensen's inequality. By combining the last two inequalities we get

\[
\frac{1}{2} \partial_t \left\| \eta v_t \right\|_{t^2(\mathbb{Z}^d, \theta)} + E^\omega(\eta v_t) \leq \left\langle \text{av}(v t)^2, \omega (\nabla \eta)^2 \right\rangle_{\ell^2(A_d)},
\]

therefore by Hölder's inequality

\[
\frac{1}{2} \partial_t \left\| \eta v_t^2 \right\|_{1, B, \theta} + E^\omega(\eta v_t) \leq \left\| \mu^\omega/\theta^\omega \right\|_{p, B, \theta} \left\| \nabla \eta \right\|^2_{L^\infty(I)} \left\| v_t^2 \right\|_{p, B, \theta}. \tag{2.2}
\]

Finally, since \( \xi(s_1) = 0 \), applying integration by parts and Jensen's inequality

\[
\begin{align*}
\int_{s_1}^s \xi(t) \partial_t \left\| (\eta v_t)^2 \right\|_{1, B, \theta} \, dt &= \int_{s_1}^s \left( \partial_t (\xi(t)) \left\| (\eta v_t)^2 \right\|_{1, B, \theta} - \xi \left( (\eta v_t)^2 \right\|_{1, B, \theta} \right) \, dt \\
&\geq \xi(s) \left\| (\eta v_s)^2 \right\|_{1, B, \theta} - \left\| \xi \right\|_{L^\infty(I)} |I| \left\| v_t^2 \right\|_{p, 1, Q, \theta}
\end{align*}
\]

for any \( s \in (s_1, s_2] \). Thus, by multiplying both sides of (2.2) with \( \xi(t) \) and integrating the resulting inequality over \( [s_1, s] \) for any \( s \in I \), the assertion (2.1) follows. \( \square \)

We will also need a modification of the Sobolev inequality derived in [4].

**Lemma 2.2.** Let \( B \subset \mathbb{Z}^d \) be finite and connected. For any \( q \in [1, \infty] \) there exists \( c_2 = c_2(d, q) \) such that for any \( v: \mathbb{Z}^d \rightarrow \mathbb{R} \),

\[
\left\| v^2 \right\|_{\rho, B} \leq c_2 n^2 \left\| v^2 \right\|_{q, B} \left\| \theta^\omega \right\|_{1, B} \frac{E^\omega(v)}{\theta^\omega(B)}.
\]

where
\[
\rho := \frac{qd}{q(d - 2) + d}.
\] (2.3)

Proof.  By [4, equation (28)],
\[
\|v^2\|_{p,B} \leq c_2 n^2 \|v^2\|_{q,B} \frac{E^v(v)}{|B|},
\]

and since \(\|\theta^c\|_{1,B} = \theta^c(B)/|B|\) this gives the claim. \(\Box\)

For the rest of this section we fix \(p, q, r \in (1, \infty)\) such that
\[
\frac{1}{r} + \frac{1}{p} - \frac{1}{q} < \frac{2}{d}.
\]

**Theorem 2.3.** Let \(t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d\) and \(u > 0\) be such that \(\partial_t u - \mathcal{L}^u \leq 0\) on \(Q(n)\) for any \(n \geq 1\). Then, for any \(0 \leq \Delta < 2/(d + 2)\) there exists \(N_2 = N_2(\Delta) \in \mathbb{N}\) and \(c_3 = c_3(d, p, q, r)\) such that for all \(n \geq N_2, h \geq 0\) and \(1/2 < \sigma' < \sigma \leq 1\) with \(\sigma - \sigma' > n^{-\Delta},\)
\[
\max_{(t,x) \in \Omega_0(n)} u(t,x) \leq h + c_3 \left( \frac{\mathcal{A}^c(n)}{(\sigma - \sigma')^2} \right) \|u - h\|_{2p,2Q_0(n),\theta}
\]

where \(\kappa := 1 + p_+\rho/2(p_+ - r_\ast\rho)\) with \(\rho\) as in (2.3) and
\[
\mathcal{A}^c(n) := \| \max\{1 \wedge (\mu^c/\theta^c)\} \|_{p,B_0(n),\theta} \| \max\{1 \wedge \nu^c\} \|_{q,B_0(n)} \| \max\{1 \wedge \theta^c\} \|_{r,B_0(n)} \| \max\{1 \wedge (1/\theta^c)\} \|_{1,B_0(n)}.
\] (2.4)

Proof. The proof is based on an iteration argument and will be divided into two steps. First we will derive the estimate needed in a single iteration step, while the actual iteration is carried out in a second step.

**Step 1:** Let \(1/2 < \sigma' < \sigma \leq 1\) and \(0 \leq k < l\) be fixed and set \(\alpha := 1 + \frac{1}{p_\ast} - \frac{r_\ast}{p}\). Note that, due to the discrete structure of the underlying space \(\mathbb{Z}^d\), the balls \(B_\sigma\) and \(B_{\sigma'}\) may coincide. To ensure that \(B_{\sigma'} \subseteq B_{\sigma}\) we assume in this step that \((\sigma - \sigma')n \geq 1\). Then, it is possible to define a spatial cut-off function \(\eta : \mathbb{Z}^d \to [0, 1]\) such that \(\text{supp}\ \eta \subseteq B_{\sigma'}, \eta \equiv 1\) on \(B_{\sigma'}, \eta \equiv 0\) on \(\partial B_{\sigma}\) and \(\|\nabla \eta\|_{L^\infty(E)} \leq 1/(\sigma - \sigma')n\). Further, let \(\xi \in C^\infty(\mathbb{R})\) be a cut-off in time satisfying \(\text{supp}\ \xi \subseteq I_{\sigma'}, \xi \equiv 1\) on \(I_{\sigma'}, \xi(t_0 - \sigma n^2) = 0\) and \(\xi' \|_{L^\infty([0,\infty))} \leq 1/(\sigma - \sigma')n^2\). By Hölder’s inequality, followed by applications of Hölder’s and Young’s inequalities,
\[
\|u - l\|_{p_\ast,1,Q_{\sigma',\theta}} \leq \|u - k\|_{p_\ast,\alpha,Q_{\sigma',\theta}} \|1_{\{u \geq l\}}\|_{p_\ast,\alpha,Q_{\sigma',\theta}} \|1_{\{u \geq l\}}\|_{p_\ast,1,Q_{\sigma',\theta}}.
\] (2.5)

Note that by Jensen’s inequality
\[
\frac{\theta^c(B_{\sigma'})}{\theta^c(B_{\sigma'})} \leq c \frac{\theta^c(B_{\sigma})}{\theta^c(B_{\sigma'})} \frac{1}{\theta^c(B_{\sigma'})}.
\] (2.6)
We use Hölder's inequality, the Sobolev inequality in Lemma 2.2, the fact that \(r_*/\rho < 1\) and Lemma 2.1 to obtain

\[
\| (u - k)^2 \|_{r_*/1, Q_{\sigma, \theta}} \leq c \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right)^{r_*/2} \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right) \xi \int_{I_\sigma} \frac{1}{|I_\sigma|} \frac{\xi(t) \varphi(\eta (u - k) + \rho)}{\theta^\omega(B_{\sigma})} dt
\]

with \( \varphi(x) := A_1^\omega(n) / (x + 1) \). Further, again by (2.6) and Lemma 2.1,

\[
\| (u - k)^2 \|_{1, Q_{\sigma, \theta}} \leq c \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right)^{r_*/2} \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right) \xi \int_{I_\sigma} \frac{1}{|I_\sigma|} \frac{\xi(t) \varphi(\eta (u - k) + \rho)}{\theta^\omega(B_{\sigma})} dt
\]

Moreover, note that

\[
\| \mathbb{I}_{\{u \geq k\}} \|_{p_*, Q_{\sigma, \theta}} \leq c \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right)^{r_*/2} \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right) \xi \int_{I_\sigma} \frac{1}{|I_\sigma|} \frac{\xi(t) \varphi(\eta (u - k) + \rho)}{\theta^\omega(B_{\sigma})} dt
\]

Therefore, combining (2.5) with (2.7), (2.8) and (2.9) yields

\[
\| (u - l)^2 \|_{p_*, Q_{\sigma, \theta}} \leq c \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right)^{r_*/2} \left( \| \theta^{\omega} \|_{1, B_{\sigma}} \right) \xi \int_{I_\sigma} \frac{1}{|I_\sigma|} \frac{\xi(t) \varphi(\eta (u - k) + \rho)}{\theta^\omega(B_{\sigma})} dt
\]

Introducing \( \varphi(l, \sigma') := \| (u - l)^2 \|_{p_*, Q_{\sigma, \theta}} \) and setting \( M := c \varphi(\sigma) \) the above inequality reads

\[
\varphi(l, \sigma') \leq \frac{M}{(l - k)^{2/\alpha + \frac{1}{\alpha_e}} (\sigma - \sigma')^2} \varphi(k, \sigma')^{1 + \frac{1}{\alpha_e}}
\]

and holds for any \( 0 \leq k < l \) and \( 1/2 \leq \sigma' < \sigma \leq 1 \).

**Step 2:** For any \( \Delta \in [0, 2/(d + 2)] \) let \( n \geq N_2(\Delta) \) where \( N_2(\Delta) < \infty \) is such that \( n^{2/(d + 2) - \Delta} \geq 2 \) for all \( n \geq N_2 \). Let \( h \geq 0 \) be arbitrary and \( 1/2 \leq \sigma' < \sigma \leq 1 \) be chosen in such a way that \( \sigma - \sigma' > n^{-\Delta} \). Further, for \( j \in \mathbb{N} \) we set

\[
\sigma_j := 2^{-j} (\sigma - \sigma'), \quad k_j := h + K \left( 1 - 2^{-j} \right),
\]

where \( K := 2^{(1+\alpha_e)^2} \left( \frac{M}{(\sigma - \sigma')^{\alpha_e/2}} \right) \phi(h, \sigma)^{1/2}, \) and \( J := \lfloor d \ln n / 2 \alpha_e \ln 2 \rfloor \). Since \( \alpha_e \geq (d + 2)/2 \), we have

\[
(\sigma_{j-1} - \sigma_j)n = 2^{-j} (\sigma - \sigma')n > 1, \quad \forall j = 1, \ldots, J.
\]
Next we claim that, by induction,
\[ \varphi(k_j, \sigma_j) \leq \frac{\varphi(h, \sigma)}{r^j}, \quad \forall j = 1, \ldots, J, \quad (2.11) \]
where \( r = 2^{4(1+\alpha_s)}. \) Indeed for \( j = 0 \) the bound \( (2.11) \) is trivial. Now assuming that \( (2.11) \) holds for any \( j - 1 \in \{0, \ldots, J\} \), we obtain by \( (2.10) \) that
\[ \varphi(k_j, \sigma_j) \leq M \left( \frac{2^j}{K} \right)^{2/\alpha_s} \left( \frac{2^j}{(\sigma - \sigma')} \right)^2 \varphi(k_{j-1}, \sigma_{j-1})^{1+\frac{1}{\alpha_s}} \]
\[ \leq M \left( \frac{2^j}{K} \right)^{2/\alpha_s} \left( \frac{2^j}{(\sigma - \sigma')} \right)^2 \left( \frac{\varphi(h, \sigma)}{r^{j-1}} \right)^{1+\frac{1}{\alpha_s}} \leq \frac{\varphi(h, \sigma)}{r^j}, \]
which completes the proof of \( (2.11) \). Note that by the choice of \( J \), \( (n^{2d} 2^{2J})/r^J \leq 1 \) and \( (\sigma_j - \sigma_{j+1})/n \geq 1 \).

By using the Cauchy-Schwarz inequality, \( (2.8) \) and \( (2.11) \), we have that
\[ \max_{(t,x) \in Q_{J+1}} (u(t, x) - k_{J+1}^\ast) \leq c n^{-d} \|1/\theta^\ast\|^{1/2}_{1,B_\theta} \|u - k_{J+1}^\ast\|^{1/2}_{1,\infty,Q_{J+1},\theta} \]
\[ \leq c \|1/\theta^\ast\|^{1/2}_{1,B_\theta} \left( \frac{\tilde{A}_1^\ast(n)}{(\sigma - \sigma')^2} \varphi(k_J, \sigma_J) \right)^{1/2} \leq c \left( \frac{\tilde{A}_1^\ast(n)}{(\sigma - \sigma')^2} \varphi(h, \sigma) \right)^{1/2} \]
\[ = c \left( \frac{\tilde{A}_1^\ast(n)}{(\sigma - \sigma')^2} \right)^{1/2} \|(u - h)\|_{2p,2,Q_{\sigma}(n),\theta}. \]
Hence,
\[ \max_{(t,x) \in Q_{\sigma}(n)} u(t, x) \leq h + K + c \left( \frac{\tilde{A}_1^\ast(n)}{(\sigma - \sigma')^2} \right)^{1/2} \|(u - h)\|_{2p,2,Q_{\sigma}(n),\theta}, \]
and the claim follows with \( \kappa = (1 + \alpha_s)/2 \) as in the statement. \( \square \)

The proof of the weak Harnack inequality in Section 2.4 below will require a stronger version of the maximal inequality, as follows.

**Corollary 2.4.** Let \( t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d \) and \( 0 \leq \Delta < 2/(d + 2) \) be fixed. Let \( u > 0 \) be such that \( \partial_t u - \mathcal{L}_\sigma^\alpha u \leq 0 \) on \( Q(n) \). Then there exists \( c_4 = c_4(d, p, q, r) \) such that for all \( n \geq N_2(\Delta) \) and \( 1/2 \leq \sigma' < \sigma \leq 1 \) with \( \sigma - \sigma' > n^{-\Delta} \),
\[ \max_{(t,x) \in Q_{\sigma'}(n)} u(t, x) \leq c_4 \left( \frac{\tilde{A}_1^\ast(n)}{(\sigma - \sigma')^2} \right)^{1/2} \|(u - h)\|_{2p,2,Q_{\sigma}(n),\theta}. \]
where \( \beta_n := \gamma \sum_{k=0}^{K_n-1} (1 - e^k) \) with \( \gamma := 1/2p_\ast \in (0, 1) \) and \( K_n := \left[ \ln 2^{-n + \ln(\sigma - \sigma')}/\ln 2 \right]. \)

**Proof.** For abbreviation we set \( \sigma_k := \sigma - (\sigma - \sigma')2^{-k} \) for \( k \in \mathbb{N} \). By Hölder’s inequality
\[ \|u\|_{2p,2,Q_{\sigma_k},\theta} \leq \|u\|_{1,2\gamma,Q_{\sigma_k},\theta} \|u\|^{1-\gamma}_{\infty,\infty,Q_{\sigma_k}} \]
\[ \leq c \|\theta^\ast\|_{1,B_\theta}^{1/\gamma} \|u\|_{1,2\gamma,Q_{\sigma_k},\theta} \|u\|^{1-\gamma}_{\infty,\infty,Q_{\sigma_k}}, \]
where the second step follows by Jensen’s inequality as in (2.6). Note that by the definition of \( K_n \) we have \( \sigma_k - \sigma_{k-1} > n^{-\Delta} \) for all \( k \in \{1, \ldots, K_n\} \). By choosing \( h = 0 \) in Theorem 2.3, we have for \( n \geq N_2 \) and \( k \in \{1, \ldots, K_n\} \),

\[
\|u\|_{\infty, \sigma_k} \leq c \left( \frac{A_\gamma^k(n)}{(\sigma_k - \sigma_{k-1})^2} \right) \|u\|_{2^\mu, 2Q_{\sigma_k}, \theta}.
\]

Combining the above equations yields

\[
\|u\|_{\infty, \sigma_k} \leq 2^{2nk} J \|u\|_{1, 2\gamma, \sigma_k, \theta} \|u\|_{\infty, \sigma_k}^{1-\gamma},
\]

where we set \( J := c \|1 \vee \theta^\omega\|_{1, B_\sigma} \|1 \vee (1/\theta^\omega)\|_{1, B_\sigma} (A_\gamma^k(n)/(-\sigma)^2)^{\kappa} \geq 1 \) for brevity. By iteration

\[
\|u\|_{\infty, \sigma_k} \leq 2^{2nK_n^{-1}(k+1)(1-\gamma)^k} \left( J \|u\|_{1, 2\gamma, \sigma_k, \theta} \right)^{\sum_{k=0}^{K_n-1}(1-\gamma)^k} \|u\|_{\infty, \sigma_k}. \]

Therefore,

\[
\|u\|_{\infty, \sigma_k} \leq 2^{2nK_n^{-1}(1-\gamma)} \|1 \vee (1/\theta^\omega)\|_{1, B(\gamma)} \|u\|_{1, 2\gamma, \sigma_k, \theta} \|u\|_{\infty, \sigma_k},
\]

which completes the proof.

\[\square\]

2.2. Hölder Regularity. The next significant result allows us to control the oscillations of a caloric function on a time-space cylinder, by those on a larger cylinder. We denote the oscillation of a function \( u \) on a cylinder \( Q \subseteq \mathbb{R} \times \mathbb{Z}^d \) by

\[
\text{osc}_Q u := \max_{(t, x) \in Q} u(t, x) - \min_{(t, x) \in Q} u(t, x).
\]

Recall the definition of \( A_\gamma^k(n) \) in (2.4). For \( n \geq 4 \) we also set for abbreviation

\[
A_\gamma^k(n) := A_\gamma^k(x_0, n) := \left\|1/\theta^\omega\right\|_{1, B(\gamma)} \left\|\theta^\omega\right\|_{1, B(\gamma)},
A_\gamma^k(n) := A_\gamma^k(x_0, n) := \left\|1/\theta^\omega\right\|_{1, B(n)} \left\|\theta^\omega\right\|_{1, B(n)} \left\|\mu^\omega\right\|_{1, B(n)}.
\]

**Theorem 2.5 (Oscillations Bound).** Fix \( t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d \). Let \( u > 0 \) be such that \( \partial_t u - \mathcal{L}_0^\gamma u = 0 \) on \( Q(n) \) for \( n \geq 1 \). For any \( \Delta \in [0, 2/(d+2)] \) there exists \( N_3 = N_3(\Delta) \) (independent of \( x_0 \)) such that for all \( n \geq N_3 \) the following holds. There exists

\[
\gamma^\omega(x_0, n) = \gamma(A_\gamma^k(n), A_\gamma^k(n), A_\gamma^k(n), \|\mu^\omega\|_{1, B(n)}, \|\theta^\omega\|_{1, B(n)}, \|1/\theta^\omega\|_{1, B(n)}),
\]

which is continuous and increasing in all components, such that

\[
\text{osc}_{Q(n/4)} u \leq \gamma^\omega(x_0, n) \text{osc}_{Q(n)} u.
\]

In the remainder of this subsection we will prove Theorem 2.5 by following the method in [15], originally used in [28] to prove Hölder regularity for parabolic equations in continuous spaces. Consider the function \( g : (0, \infty) \to [0, \infty) \), which may be regarded as a continuously differentiable version of the function \( x \mapsto (-\ln x)_+ \), defined by

\[
g(\varepsilon) := \begin{cases} 
- \ln z & \text{if } z \in (0, \varepsilon], \\
\frac{(z-1)^2}{2e(1-e)} & \text{if } z \in (\varepsilon, 1], \\
0 & \text{if } z \in (1, \infty),
\end{cases}
\]
where \( \bar{c} \in \left[ \frac{1}{3}, \frac{1}{2} \right] \) is the smallest solution of the equation \( 2c \ln(1/c) = 1 - c \). Note that \( g \in C^1(0, \infty) \) is convex and non-increasing. Although \( g \) is not caloric, we can still bound its Dirichlet energy as follows.

**Lemma 2.6.** Suppose \( u > 0 \) satisfies \( \partial_t u - \mathcal{L}^\omega_{\theta} u = 0 \) on \( Q = I \times B \) with \( I \) and \( B \) as in Lemma 2.1. Let \( \eta : \mathbb{Z}^d \to [0, 1] \) be a cut-off function with \( \text{supp}\ \eta \subseteq B \) and \( \eta \equiv 0 \) on \( \partial B \). Then

\[
\partial_t \| \eta^2 g(u_t) \|_{1,B,\theta} + \frac{\mathcal{E}^\omega,\eta^2(g(u_t))}{6 \theta^\omega(B)} \leq 6 \left\| 1 \vee \mu^\omega \right\|_{1,B} \text{osr}(\eta)^2 \| \nabla \eta \|^2_{L^\infty(E_d)}, \tag{2.13}
\]

where \( \text{osr}(\eta) := \max \{ \eta(y)/\eta(x) \vee 1 \ | (x, y) \in E_d \} \) and

\[
\mathcal{E}^\omega,\eta^2(f) := \sum_{e \in E_d} (\eta^2(e^+) \wedge \eta^2(e^-)) \omega(e) (\nabla f)^2(e).
\]

**Proof.** Since \( \partial_t u - \mathcal{L}^\omega_{\theta} u = 0 \) on \( Q = I \times B \),

\[
\partial_t \langle \eta^2, g(u_t) \rangle_{L^2(\mathbb{Z}^d, \theta)} = \langle \eta^2 g'(u_t), \partial_t u_t \rangle_{L^2(\mathbb{Z}^d, \theta)}
\]

\[
= \langle \eta^2 g'(u_t), \mathcal{L}^\omega_{\theta} u_t \rangle_{L^2(\mathbb{Z}^d, \theta)} = -\langle \nabla (\eta^2 g'(u_t)), \omega \nabla u_t \rangle_{L^2(E_d)}.
\]

Now, \( g' \) is piecewise differentiable and \( 1/3g''(z)^2 \leq g''(z) \) for a.e. \( z \in (0, \infty) \). In particular, \( -zg''(z) \leq 4/3 \) for any \( z \in (0, \infty) \). So by [15, Lemma A.1],

\[
-\langle \nabla (\eta^2 g'(u_t)), \omega \nabla u_t \rangle_{L^2(E_d)} \leq -\frac{1}{6} \mathcal{E}^\omega,\eta^2(g(u_t)) + 6 \text{osr}(\eta)^2 \langle \nabla \eta, \omega \nabla \eta \rangle_{L^2(E_d)}.
\]

The result follows by combining the above two estimates. \( \square \)

Now, define

\[
M_n := \sup_{(t,x) \in Q(n)} u(t,x) \quad \text{and} \quad m_n := \inf_{(t,x) \in Q(n)} u(t,x). \tag{2.14}
\]

For the purposes of the next lemma, given \( k_0 \in \mathbb{R} \), we denote

\[
k_j := M_n - 2^{-j}(M_n - k_0), \quad j \in \mathbb{N}. \tag{2.15}
\]

**Lemma 2.7.** Let \( t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d \), and \( u > 0 \) be such that \( \partial_t u - \mathcal{L}^\omega_{\theta} u = 0 \) on \( Q(n) \) for \( n \geq 4 \). Let \( \eta : \mathbb{Z}^d \to [0, 1] \) be the spatial cut-off function \( \eta(x) := [1 - 2d(x_0, x)/n]^+ \). Suppose, for some \( k_0 \in \mathbb{R} \),

\[
\frac{1}{n^2} \int_{t_0-n^2}^{t_0} \| \mathbb{1}_{\{u_t \leq k_0\}} \|_{1,B(n),\eta^2 \theta} dt \geq \frac{1}{2}. \tag{2.16}
\]

Then there exist \( c_5, c_6 > 0 \) such that for any \( \delta \in (0, 1/4c_6A_2^\omega(n)) \) and any

\[
j \geq 1 + \frac{c_5 \| 1 \vee \mu^\omega \|_{1,B(n)} \| 1 \vee (1/\theta^\omega) \|_{1,B(n)}}{4 - c_6 \delta A_2^\omega(n)}
\]

we have that

\[
\| \mathbb{1}_{\{u_t \leq k_j\}} \|_{1,B(n/2),\theta} \geq \delta, \quad \forall t \in \left[ t_0 - \frac{1}{4}n^2, t_0 \right].
\]
Proof. Set
\[ v_t(x) := \frac{M_n - u_t(x)}{M_n - k_0}, \quad h_j = \epsilon_j := 2^{-j}, \quad j \in \mathbb{N}. \]
Then \( \partial_t (v + \epsilon_j) - \mathcal{L}_n^\omega (v + \epsilon_j) = 0 \) on \( Q(n) \) for all \( j \in \mathbb{N} \) and, for any \( x \in \mathbb{Z}^d, u_t(x) > k_j \) if and only if \( v_t(x) < h_j \). By (2.16) there exists \( s_0 \in [t_0 - n^2, t_0 - \frac{4}{3} n^2] \) such that
\[ \| \mathbb{I}_{\{v_{s_0} < 1\}} \|_{1, B(n), \eta^2 \theta} \leq \frac{3}{4}. \] (2.17)
To see this, assume the contrary is true, that is \( \| \mathbb{I}_{\{v_s < 1\}} \|_{1, B(n), \eta^2 \theta} > \frac{3}{4} \) for all \( s \in [t_0 - n^2, t_0 - \frac{1}{3} n^2] \). Then
\[
\frac{1}{2} \geq \frac{1}{n^2} \int_{t_0-n^2}^{t_0} \| \mathbb{I}_{\{u_t > k_0\}} \|_{1, B(n), \eta^2 \theta} dt = \frac{1}{n^2} \int_{t_0-n^2}^{t_0} \| \mathbb{I}_{\{v_t < 1\}} \|_{1, B(n), \eta^2 \theta} dt
\]
\[ > \frac{1}{n^2} \int_{t_0-n^2}^{t_0-\frac{1}{3} n^2} \frac{3}{4} dt = \frac{1}{2}, \]
which is a contradiction. Let \( t \in [t_0 - \frac{1}{3} n^2, t_0] \). By integrating the estimate (2.13) over the interval \([s_0, t] \), noting that \( \| \nabla \eta \|_{\ell^\infty(E)} \leq 2/n, \operatorname{osr}(\eta) \leq 2 \) and \( t - s_0 \leq n^2 \),
\[ \| g(v_t + \epsilon_j) \|_{1, B(n), \eta^2 \theta} \leq \| g(v_{s_0} + \epsilon_j) \|_{1, B(n), \eta^2 \theta} + c \| 1 \vee \mu^\omega \|_{1, B(n)} \| \theta^\omega \|_{1, B(n)}^{-1}. \]
Since \( g \) is non-increasing and identically zero on \([1, \infty) \), using (2.17) we have
\[ \| g(v_{s_0} + \epsilon_j) \|_{1, B(n), \eta^2 \theta} \leq g(\epsilon_j) \| \mathbb{I}_{\{v_{s_0} < 1\}} \|_{1, B(n), \eta^2 \theta} \leq \frac{3}{4} g(\epsilon_j), \]
and
\[ \| g(v_t + \epsilon_j) \|_{1, B(n), \eta^2 \theta} \geq g(h_j + \epsilon_j) \| \mathbb{I}_{\{v_t < h_j\}} \|_{1, B(n), \eta^2 \theta}. \]
So for \( j \geq 2, \)
\[ \| \mathbb{I}_{\{v_t < h_j\}} \|_{1, B(n), \eta^2 \theta} \leq \frac{3}{4} \frac{g(\epsilon_j)}{g(h_j + \epsilon_j)} + \frac{c}{g(h_j + \epsilon_j)} \| 1 \vee \mu^\omega \|_{1, B(n)} \| \theta^\omega \|_{1, B(n)}^{-1} \]
\[ \leq \frac{3}{4} \left( 1 + \frac{1}{j-1} \right) + \frac{c}{j-1} \| 1 \vee \mu^\omega \|_{1, B(n)} \| 1 \vee (1/\theta^\omega) \|_{1, B(n)}. \]
Then, since \( \eta \equiv 0 \) on \( B(n/2)^c \),
\[ \| \mathbb{I}_{\{u_t \leq k_j\}} \|_{1, B(n/2), \theta} = \frac{\langle \eta^2 \theta^\omega, 1 \rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))} \left( 1 - \| \mathbb{I}_{\{v_t < h_j\}} \|_{1, B(n), \eta^2 \theta} \right) \]
\[ \geq \frac{\langle \eta^2 \theta^\omega, 1 \rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))} \left( 1 - \frac{c}{j-1} \| 1 \vee \mu^\omega \|_{1, B(n)} \| 1 \vee (1/\theta^\omega) \|_{1, B(n)} \right). \] (2.18)
Note that \( \langle \eta^2 \theta^\omega, 1 \rangle_{\ell^2(\mathbb{Z}^d)}/\theta^\omega(B(n/2)) \in (0, 1) \) and since \( \eta \geq 1/2 \) on \( B(n/4), \)
\[ \frac{\langle \eta^2 \theta^\omega, 1 \rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))} \geq c \| \theta^\omega \|_{1, B(n/4)} \| \theta^\omega \|_{1, B(n/2)}^{-1}. \]
By combining this inequality above with (2.18) and using that
\[
j - 1 \geq \frac{c_5 \|1 \vee \mu^\omega\|_{1,B(n)} \|1 \vee (1/\theta^\omega)\|_{1,B(n)}}{\frac{1}{4} - c_6 \|\theta^\omega\|_{1,B(n/4)}^{1/2} \|\theta^\omega\|_{1,B(n/2)}^{1/2}}
\]
by Jensen’s inequality, we get the claim. \(\square\)

For any finite \(B \subset \mathbb{Z}^d\) and any \(u : \mathbb{Z}^d \to \mathbb{R}\), let
\[
(u)_{B,\theta} := \frac{1}{\theta^\omega(B)} \sum_{x \in B} u(x) \theta^\omega(x)
\]
denote the weighted average of \(u\) over the subset \(B\), and write \((u)_B := (u)_{B,1}\).

**Proposition 2.8 (Weighted Local Poincaré Inequality).** There exists \(c_7 = c_7(d) < \infty\) such that for any ball \(B(n) := B(x_0, n)\) with \(x_0 \in \mathbb{Z}^d\) and \(n \geq 1\), any non-empty \(\mathcal{N} \subseteq B\) and \(u : \mathbb{Z}^d \to \mathbb{R}\),
\[
\|u - (u)_{B(n),\theta}\|_{2,B(n),\theta}^2 \leq c_7 A_3^\omega(n) \frac{n^2}{|B(n)|} \sum_{x,y \in B(n), x \sim y} \omega(x,y) \left( u(x) - u(y) \right)^2,
\]
and
\[
\|u - (u)_{\mathcal{N},\theta}\|_{1,B(n),\theta}^2 \leq c_7 A_3^\omega(n) \left( 1 + \frac{\theta^\omega(B(n))}{\theta^\omega(\mathcal{N})} \right) \frac{n^2}{|B(n)|} \sum_{x,y \in B(n), x \sim y} \omega(x,y) \left( u(x) - u(y) \right)^2,
\]
with \(A_3^\omega(n) = \|1/\theta^\omega\|_{1,B(n)}^2 \|\theta^\omega\|_{r,B(n)}^2 \|\nu^\omega\|_{q,B(n)}\) as defined in (2.12).

**Proof:** By a discrete version of the co-area formula the classical local \(\ell^1\)-Poincaré inequality on \(\mathbb{Z}^d\) can be easily established, see e.g. [27, Lemma 3.3], which also implies an \(\ell^\alpha\)-Poincaré inequality for any \(\alpha \in [1, d)\). Note that, by [16, Théorème 4.1], the volume regularity of balls and the local \(\ell^\alpha\)-Poincaré inequality on \(\mathbb{Z}^d\) implies that for \(d \geq 2\) and any \(u : \mathbb{Z}^d \to \mathbb{R}\),
\[
\inf_{a \in \mathbb{R}} \|u - a\|_{\ell^\alpha,B(n)} \leq c n \left( \frac{1}{|B(n)|} \sum_{x,y \in B(n), x \sim y} |u(x) - u(y)|^\alpha \right)^{1/\alpha}.
\]
Further, for any \(\alpha \in [1, 2)\), Hölder’s inequality yields
\[
\left( \frac{1}{|B(n)|} \sum_{x,y \in B(n), x \sim y} |u(x) - u(y)|^\alpha \right)^{1/\alpha} \leq \|\nu^\omega\|_{2,\alpha}^{1/2} \left( \frac{1}{|B(n)|} \sum_{x,y \in B(n), x \sim y} \omega(x,y) \left( u(x) - u(y) \right)^2 \right)^{1/2}.
\]
Now we prove (2.19) where we distinguish two cases. In the case \( r \geq 2 \) we have by Cauchy-Schwarz,
\[
\|u - a\|_{1,B(n),\theta} \leq \|\theta^{\omega}\|^{-1}_{1,B(n)} \|\theta^{\omega}\|_{2,B(n)} \|u - a\|_{2,B(n)}.
\]
Hence, since \( \inf_{a \in \mathbb{R}} \|u - a\|_{1,B(n),\theta} = \|u - (u)_{B(n),\theta}\|_{1,B(n),\theta} \), we obtain the assertion (2.19) by using (2.21) and (2.22) with the choice \( \alpha = 2d/(d + 2) \) and Jensen’s inequality. Note that \( \alpha/(2 - \alpha) = d/2 < q \).

Similarly, in the case \( r \in [1,2) \) we have by Hölder’s inequality
\[
\|u - a\|_{1,B(n),\theta} \leq \|\theta^{\omega}\|^{-1}_{1,B(n)} \|\theta^{\omega}\|_{r,B(n)} \|u - a\|_{r,B(n)},
\]
and we may use (2.21) and (2.22) with the choice \( \alpha = dr_s/(d + r_s) \). Notice that \( d\alpha/(d - \alpha) = r_s, \alpha/(2 - \alpha) \leq q \) and \( \alpha \in [1,2) \) since \( r \in [1,d] \) and satisfies (1.4).

This finishes the proof of (2.19).

To see (2.20), note that by the triangle inequality
\[
\|u - (u)_{\mathcal{N},\theta}\|_{1,B(n),\theta} \leq \|u - (u)_{B(n),\theta}\|_{1,B(n),\theta} + \|(u)_{\mathcal{N},\theta} - (u)_{B(n),\theta}\|_{1,B(n),\theta} \leq \|u - (u)_{B(n),\theta}\|_{1,B(n),\theta} + \frac{1}{\theta^{\omega}(\mathcal{N})} \sum_{y \in \mathcal{N}} |u(y) - (u)_{B(n),\theta}| \theta^{\omega}(y)
\]
so that (2.20) follows from (2.19).

\[\Box\]

**Lemma 2.9.** Set \( \tau := 1/4 \) and \( \sigma := 1/2 \). Let \( t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d, n \geq 4, \) and suppose \( u > 0 \) satisfies \( \partial_t u - \mathcal{L}_u u = 0 \) on \( Q(n) \). Assume there exist \( \delta > 0 \) and \( i_0 \in \mathbb{N} \) such that
\[
\|\mathbb{I}_{\{u \leq k_{i_0}\}}\|_{1,B(x_0,\sigma n),\theta} \geq \delta, \quad \forall t \in I_{\tau} = [t_0 - \frac{1}{4}n^2, t_0]. \tag{2.23}
\]
Let \( \epsilon \in (0,1) \) be arbitrary. Then there exists
\[
j_0 = j_0(\epsilon, \delta, i_0, A^\omega(n), \|\mu^{\omega}\|_{1,B(n)}, \|\theta^{\omega}\|_{1,B(n)}) \in \mathbb{N} \quad \text{with} \quad j_0 \geq i_0,
\]
which is continuous and decreasing in the first two components and continuous and increasing in the other components, such that
\[
\|\mathbb{I}_{\{u > k_{j}\}}\|_{1,1,Q_{\tau,\sigma}(n),\theta} \leq \epsilon, \quad \forall j \geq j_0.
\]

**Proof.** Let \( \eta : \mathbb{Z}^d \rightarrow [0,1] \) be a cut-off function such that \( \text{supp} \eta \subseteq B(n), \text{ supp } \eta \equiv 1 \) on \( B_\sigma \) and \( \eta \equiv 0 \) on \( \partial B(n) \) with linear decay on \( B(n) \setminus B_\sigma \). So \( \|\nabla \eta\|_{L^\infty(E_d)} \leq 2/n \) and \( \text{osr}(\eta) \leq 2 \). Now, let
\[
w_t(x) := \frac{M_n - u_t(x)}{M_n - k_{i_0}} \quad \text{and} \quad h_j = \epsilon_j := 2^{-j}.
\]
Then \( w \geq 0 \) and \( \partial_t (w + \epsilon_j) - \mathcal{L}_u (w + \epsilon_j) = 0 \) on \( Q(n) \) for \( j \in \mathbb{N} \). For any \( t \in I_{\tau}, \) let \( \mathcal{N}_t := \{x \in B_\sigma : g(w_t(x) + \epsilon_j) = 0\} \). Since by its definition \( g \equiv 0 \) on \((1,\infty)\), we have

\[\Box\]
by (2.23),
\[
\frac{\theta^\omega(N_t)}{\theta^\omega(B_\sigma)} = \left\| \mathbb{I}_{\{g(w_t+\epsilon) < 0\}} \right\|_{1, B_\sigma, \theta} \geq \left\| \mathbb{I}_{\{w_t \leq k_0\}} \right\|_{1, B_\sigma, \theta} = \left\| \mathbb{I}_{\{w_t > k_0\}} \right\|_{1, B_\sigma, \theta} \geq \delta.
\]
By Proposition 2.8 we have
\[
\left\| g(w_t + \epsilon_j) \right\|^2_{1, B_\sigma, \theta} \leq c_T n^2 A_3^\omega(\sigma n) \left( 1 + \frac{\theta^\omega(B_\sigma)}{\theta^\omega(N_t)} \right)^2 \frac{\mathcal{E}^\omega, \eta^2(g(w_t + \epsilon_j))}{|B_\sigma|},
\]
so that by Jensen’s inequality and by integrating (2.13) over \( I_\tau \),
\[
\left\| g(w + \epsilon_j) \right\|^2_{1, Q_\tau, \theta} \leq \frac{1}{\tau n^2} \int_{I_\tau} \left\| g(w_t + \epsilon_j) \right\|^2_{1, B_\sigma, \theta} dt \\
\leq \frac{c}{\delta^2} A_3^\omega(\sigma n) \left( \left\| \theta^\omega \right\|_{1, B(n)} \int_{I_\tau} \frac{\mathcal{E}^\omega, \eta^2(g(w_t + \epsilon_j))}{\theta^\omega(B(n))} dt \right) \\
\leq \frac{c}{\delta^2} A_3^\omega(\sigma n) \left( \left\| \theta^\omega \right\|_{1, B(n)} \left( \left\| \eta^2(g(w_{t_0} - r n^2 + \epsilon_j) \right\|_{1, B(n), \theta} + \left\| 1 \vee \mu^\omega \right\|_{1, B(n)} \right) \right).
\]
Since \( g \) is non-increasing and \( w_t > 0 \) for all \( t \in I_\tau \),
\[
\left\| \mathbb{I}_{w < h_j} \right\|^2_{1, Q_\tau, \theta} \leq \frac{\left\| g(w + \epsilon_j) \right\|^2_{1, Q_\tau, \theta}}{g(h_j + \epsilon_j)^2} \\
\leq \frac{c}{\delta^2} A_3^\omega(\sigma n) \left( \left\| \theta^\omega \right\|_{1, B(n)} \frac{g(\epsilon_j)}{g(h_j + \epsilon_j)^2} + \left\| 1 \vee \mu^\omega \right\|_{1, B(n)} \frac{1}{g(h_j + \epsilon_j)^2} \right) \\
\leq \frac{c}{\delta^2} A_3^\omega(\sigma n) \left( 1 \vee \theta^\omega \right)_{1, B(n)} \left( 1 \vee \mu^\omega \right)_{1, B(n)} \left( \frac{j}{(j - 1)^2} + \frac{1}{(j - 1)^2} \right).
\]
Thus, for any \( \epsilon > 0 \), there exists some \( j_0 \geq i_0 \) as in the statement such that
\[
\left\| \mathbb{I}_{\{u > k_j\}} \right\|_{1, Q_\tau, \theta} = \left\| \mathbb{I}_{\{w < h_j - i_0\}} \right\|_{1, Q_\tau, \theta} \leq \epsilon \text{ for all } j \geq j_0.
\]
\[\square\]

**Proof of Theorem 2.5.** Set \( \tau = 1/4, \sigma = 1/2 \) as before in Lemma 2.9. Define \( k_0 := (M_n + m_n)/2 \) with \( M_n \) and \( m_n \) as in (2.14) and let \( k_j \) be defined by (2.15). Further, let \( \eta : \mathbb{Z}^d \rightarrow \mathbb{R} \) be the cut-off function \( \eta(x) := [1 - d(x_0, x)/\sigma n]_+ \). We may assume
\[
\frac{1}{n^2} \int_{I_1} \left\| \mathbb{I}_{\{u \leq k_0\}} \right\|_{1, B(n), \eta^2 \theta} dt \geq \frac{1}{2}
\]
Otherwise, consider \( M_n + m_n - u \) in place of \( u \). Set \( \epsilon := (2c_3 (4A_2^\omega(\sigma n))^2)^{-2p_\ast} \) with \( A_2^\omega(\sigma n) \) as in Theorem 2.3. Fix \( \Delta \in (0, \frac{1}{2}) \) and \( N_3 \geq 2N_2(\Delta) \) such that \( \frac{1}{2} > (\sigma N_3)^{-\Delta} \). Now for all \( n \geq N_3 \), applying consecutively Lemma 2.7 and Lemma 2.9, there exists
\[
l = l^\omega(x_0, n) = l(A_1^\omega(n), A_2^\omega(n), A_3^\omega(n), \left\| \mu^\omega \right\|_{1, B(n)}, \left\| \theta^\omega \right\|_{1, B(n)}, 1/\theta^\omega_{1, B(n)}),
\]
which is continuous and increasing in all components, such that
\[
\left\| \mathbb{I}_{\{u > k_j\}} \right\|_{1, Q_\tau, \theta} \leq \epsilon, \quad \forall j \geq l.
\]
By an application of Jensen’s inequality,
\[
\left\| (u - k_l)^+ \right\|_{2p^*, 2Q_1(\sigma_n), \theta} \leq (M_n - k_l) \left\| \mathbb{I}_{\{u > k_l\}} \right\|_{2p^*, 2Q(\sigma_n), \theta} \\
\leq (M_n - k_l) \left\| \mathbb{I}_{\{u > k_l\}} \right\|_{1, 1, Q_1(\sigma_n), \theta}^{1/2p^*} \leq (M_n - k_l) \epsilon^{1/2p^*}.
\]

Now, let \( \vartheta = \frac{\epsilon}{2} = \frac{1}{4} \). Then Theorem 2.3 implies that
\[
M_{\vartheta n} \leq \max_{Q_{1/2}(\sigma_n)} u(t, x) \leq k_l + c_3 \left( 4A^\vartheta(\sigma_n) \right)^{\kappa} \left\| (u - k_l)^+ \right\|_{2p^*, 2Q_1(\sigma_n), \theta} \leq k_l + \frac{1}{2} (M_n - k_l) = M_n - 2^-(l+2) (M_n - m_n).
\]
Hence
\[
M_{\vartheta n} - m_{\vartheta n} \leq M_n - 2^-(l+2) (M_n - m_n) - m_{\vartheta n} \leq (1 - 2^-(l+2)) (M_n - m_n),
\]
and the theorem is proven.

2.3. **Quenched Local Limit Theorem.** We are following a method first developed in [17] and [7], which has the QFCLT and Hölder regularity of the heat kernel as its main ingredients. First, we will derive the latter from the oscillation inequality in Theorem 2.5. This requires some preparation.

**Lemma 2.10.** Let \( f : \mathbb{R}^d \to [0, \infty) \) and let \( \epsilon > 0 \) and \( r_0 > 0 \) be arbitrary. Assume
\[
\sup_{x \in \mathbb{R}^d} \limsup_{n \to \infty} \left\| f \right\|_{1, B([nx], r)} =: K < \infty.
\]
Then there exists \( N_4 = N_4(x, r_0, \epsilon) > 0 \) such that for all \( x \in \mathbb{R}^d \) and all \( n \geq N_4 \),
\[
\sup_{r \geq r_0} \left\| f \right\|_{1, B([nx], r)} < K (1 + \epsilon).
\]
**Proof.** See [14, Lemma 6.1.7] for a proof for functions on \( \mathbb{R}^d \). The arguments directly transfer into the discrete setting.

**Lemma 2.11.** Suppose Assumptions 1.1 and 1.4 hold. Let \( \gamma^\omega \) be as in Theorem 2.5. For any \( x \in \mathbb{R}^d \) and \( \delta > 0 \) we have for \( \mathbb{P} \)-a.e. \( \omega \),
(i) \( \lim_{n \to \infty} \left\| \theta^\omega \right\|_{1, B([nx], \delta n)} = \mathbb{E} \left[ \theta^\omega(0) \right] \),
(ii) there exist \( N_5 = N_5(\omega, x, \delta) \) and \( \hat{\gamma} \in (0, 1) \) independent of \( \omega, x \) and \( \delta \), such that \( \gamma^\omega([nx], \delta n) \leq \hat{\gamma} \) for all \( r \geq \delta \) and \( n \geq N_5 \).

**Proof.** (i) Since \( \theta^\omega \in L^1(\mathbb{P}) \) and stationary by Assumption 1.1, this is a direct consequence of the spatial ergodic theorem in [23, Theorem 2.8 in Chapter 6]. All that remains is to check that \( \left( B([nx], \delta n) \right)_{n \in \mathbb{N}} \) is a regular sequence. For this purpose, consider the increasing sequence \( \left( B(0, [nx, \delta n]) \right)_{n \in \mathbb{N}} \) with the property that \( B([nx], \delta n) \subset B(0, [nx, \delta n]) \) for all \( n \in \mathbb{N} \). Moreover, there exists \( K = K(x, \delta) < \infty \) such that
\[
|B(0, [nx, \delta n])| \leq K \left| B([nx], \delta n) \right|, \quad \forall n \in \mathbb{N}.
\]
Thus, the sequence is regular.
(ii) Recall that $\gamma^\omega$ is continuous and increasing in all components. Arguing as in (i) we get by the moment condition in Assumption 1.4 and again the spatial ergodic theorem in [23, Theorem 2.8 in Chapter 6] that for $\mathbb{P}$-a.e. $\omega$,
\[
\lim_{n\to\infty} A_i^\omega([nx], n\delta) =: \bar{A}_i < \infty, \quad \forall i \in \{1, 2, 3\},
\]
with $\bar{A}_i$ constant independent of $x$ and $\delta$. Furthermore,
\[
\lim_{n\to\infty} \|\mu^\omega\|_{1,B([nx], \delta n)} = \mathbb{E}[\mu^\omega(0)] =: \bar{\mu}, \quad \lim_{n\to\infty} \|\theta^\omega\|_{1,B([nx], \delta n)} = \mathbb{E}[\theta^\omega(0)] =: \bar{\theta}_1,
\]
and
\[
\lim_{n\to\infty} \|1/\theta^\omega\|_{1,B([nx], \delta n)} = \mathbb{E}[1/\theta^\omega(0)] =: \bar{\theta}_2.
\]
Now we apply Lemma 2.10, which yields that for $\bar{A}_i' > \bar{A}_i$, $\bar{\mu}' > \bar{\mu}$ and $\bar{\theta}_i' > \bar{\theta}_i$ sufficiently small there exists $N_5 = N_5(\omega, \bar{x}, \delta, \bar{A}_i', \bar{\mu}', \bar{\theta}_i')$ such that
\[
\gamma^\omega([nx], r) \leq \gamma(\bar{A}_i', \bar{A}_i', \bar{\mu}', \bar{\theta}_1', \bar{\theta}_2') =: \bar{\gamma} \in (0, 1),
\]
for all $r \geq \delta$ and $n \geq N_5$. \hfill \Box

**Lemma 2.12.** Suppose Assumptions 1.1 and 1.4 hold. For $\mathbb{P}$-a.e. $\omega$, any $\lambda > 0$ and $x \in \mathbb{Z}^d$ there exist $c_8 = c_8(d, p, q, r, \lambda)$ and $N_6 = N_6(x, \omega)$ such that for any $t$ with $\sqrt{t} \geq N_6$ and all $y \in B(x, \lambda \sqrt{t})$,
\[
p_\theta^\omega(t, x, y) \leq c_8 t^{-d/2}.
\]

**Proof.** This can be directly read off [6, Theorem 3.2]. \hfill \Box

**Proposition 2.13.** Let $\delta > 0$, $\sqrt{t}/2 \geq \delta$ and $x \in \mathbb{R}^d$ be fixed. Then, there exists $c_9 > 0$ such that for $\mathbb{P}$-a.e. $\omega$,
\[
\limsup_{n\to\infty} \sup_{y_1, y_2 \in B([nx], \delta n)} n^d \left| p_\theta^\omega(n^2s_1, 0, y_1) - p_\theta^\omega(n^2s_2, 0, y_2) \right| \leq c_9 \left( \frac{\delta}{\sqrt{t}} \right)^{\varphi} t^{-d/2},
\]
where $\varphi = \ln(\bar{\gamma})/\ln(1/4)$.

**Proof.** Set $\delta_k := 4^{-k}\sqrt{t}/2$ and with a slight abuse of notation let
\[
Q_k := n^2[t - \delta_k^2, t] \times B([nx], \delta_k n), \quad k \geq 0.
\]
Choose $k_0 \in \mathbb{N}$ such that $\delta_{k_0} \geq \delta \geq \delta_{k_0+1}$. In particular, for every $k \leq k_0$ we have $\delta_k \in [\delta, \sqrt{t}]$. Now we apply Theorem 2.5 and Lemma 2.11-(ii), which gives that there exists $N_5 = N_5(\omega, x, \delta) \geq \delta_{k_0}$, where $Q_0 \supset Q_1 \supset \cdots \supset Q_{k_0}$ to obtain
\[
\text{osc}_{Q_k} p_\theta(y, 0, \cdot) \leq \bar{\gamma} \text{osc}_{Q_{k_0}} p_\theta(y, 0, \cdot), \quad \forall k = 1, \ldots, k_0.
\]
We iterate the above inequality on the chain $Q_0 \supset Q_1 \supset \cdots \supset Q_{k_0}$ to obtain
\[
\text{osc}_{Q_{k_0}} p_\theta(y, 0, \cdot) \leq \bar{\gamma}^{k_0} \max_{Q_0} \text{osc}_{Q_0} p_\theta(y, 0, \cdot)
\]
Note that
\[
Q_{k_0} = n^2[t - \delta_{k_0}^2, t] \times B([nx], \delta_{k_0} n) \supset n^2[t - \delta^2, t] \times B([nx], \delta n).
\]
Hence, since $\eta_{\theta}\leq c(\delta/\sqrt{t})^{\theta}$, the claim follows from (2.25) and Lemma 2.12. □

**Proposition 2.14.** Suppose Assumptions 1.1 and 1.4 hold. For any $x \in \mathbb{Z}^d$ and $t > 0$,

$$\lim_{n \to \infty} \left| n^d p_{\theta}^x (n^2 t, 0, |nx|) - a k_t(x) \right| = 0, \quad \mathbb{P}\text{-a.s.}$$

with $k_t$ as defined in (1.2) and $a := \mathbb{E} [\theta^\omega(0)]^{-1}$.

**Proof.** For any $x \in \mathbb{R}^d$ and $\delta > 0$ let $C(x, \delta) := x + [-\delta, \delta]^d$ and $C^n(x, \delta) := n C(x, \delta) \cap \mathbb{Z}^d$, i.e. $C(x, \delta)$ is a ball in $\mathbb{R}^d$ with respect to the supremum norm. Note that the balls $C^n(x, \delta)$ are comparable with $B(|nx|, \delta n)$ and we may apply Lemma 2.11 and Proposition 2.13 with $B(|nx|, \delta n)$ replaced by $C^n(x, \delta)$ as they also form a regular sequence. Let

$$J := \left( p_{\theta}^x (n^2 t, 0, |nx|) - n^{-d} a k_t(x) \right) \theta^\omega \left[ C^n(x, \delta) \right].$$

We can rewrite this, for any $\delta > 0$, as $J = \sum_{i=1}^4 J_i$ where

$$J_1 := \sum_{z \in C^n(x, \delta)} \left( p_{\theta}^x (n^2 t, 0, |nx|) - p_{\theta}^y (n^2 t, 0, z) \right) \theta^\omega (z),$$

$$J_2 := P_0^x \left[ X^{(n)}_t \in C(x, \delta) \right] - \int_{C(x, \delta)} k_t(y) \, dy,$$

$$J_3 := k_t(x) \left( (2\delta)^d - \theta^\omega \left[ C^n(x, \delta) \right] n^{-d} a \right),$$

$$J_4 := \int_{C(x, \delta)} (k_t(y) - k_t(x)) \, dy,$$

with $X^{(n)}_t := \frac{1}{n} X_{nt}$, $t \geq 0$, being the rescaled random walk. It suffices to prove that, for each $i = 1, \ldots, 4$, as $n \to \infty$, $|J_i|/n^{-d}\theta^\omega[C^n(x, \delta)]$ converges $\mathbb{P}$-a.s. to a limit which is small with respect to $\delta$.

First note that $J_2 \to 0$ by Theorem 1.2 and $n^{-d}\theta^\omega[C^n(x, \delta)] \to (2\delta)^d/a$ by Lemma 2.11(i). Thus, $\lim_{n \to \infty} |J_i|/n^{-d}\theta^\omega[C^n(x, \delta)] = 0$ for $i = 2, 3$. Further, by the Lipschitz continuity of the heat kernel $k_t$ in its space variable it follows that $\lim_{n \to \infty} |J_4|/n^{-d}\theta^\omega[C^n(x, \delta)] = O(\delta^d)$. To deal with the remaining term, we apply Proposition 2.13, which yields

$$\limsup_{n \to \infty} \max_{z \in C^n(x, \delta)} n^d \left| p_{\theta}^x (n^2 t, 0, z) - p_{\theta}^y (n^2 t, 0, |nx|) \right| \leq c \delta^d t^{-\frac{d}{2}-\frac{\theta}{2}}.$$ 

Hence, $\limsup_{n \to \infty} |J_i|/n^{-d}\theta^\omega[C^n(x, \delta)] = O(\delta^d)$, $\mathbb{P}$-a.s. Finally, the claim follows by letting $\delta \to 0$. □

**Proof of Theorem 1.5.** Having proven the pointwise result Proposition 2.14, the full local limit theorem follows by extending over compact sets in $x$ and $t$. This is done using a covering argument, exactly as in the proof of [15, Proposition 3.1], which is a slight modification of the proofs [17] and [7]. □
2.4. Weak Parabolic Harnack Inequality. In this subsection, we adapt the techniques of [28] to prove Theorem 1.6. The proof requires the $L^1$-version of the maximal inequality in Corollary 2.4 and the following auxiliary estimate on the level sets of a caloric function under the measure $\theta^\omega$. Recall that $m_\theta = dt \times \theta^\omega$.

**Lemma 2.15.** Suppose Assumptions 1.1 and 1.4 hold. For any $x_0 \in \mathbb{Z}^d$, $t_0 \in \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega$, there exists $N^\ast = N^\ast(\omega, x_0) \in \mathbb{N}$ such that for all $n \geq N^\ast$ the following holds. Assume there exists $\lambda \in (0, 1)$ such that

$$m_\theta(t, x) \in Q(n) : u(t, x) \geq 1 \geq \lambda m_\theta(Q(n)).$$

Then, for any $\sigma_1 \in (0, \lambda)$ and $\sigma_2 \in (\lambda, 1)$, there exists $h = h(d, \lambda) \in (0, 1)$ (also depending on the law of $\omega$ and $\theta^\omega$) such that

$$\theta^\omega(\{x \in B_{\sigma_2} : u(t, x) \geq h\}) \geq \frac{1}{\lambda} \theta^\omega(B_{\sigma_2}) \quad \forall t \in [t_0 - \sigma_1 n^2, t_0].$$

**Proof.** We denote $N^\ast_\lambda := \{x \in B_{\sigma_2} : u(t, x) \geq h\}$ with $h > 0$ to be chosen later. Let $w := g(u + h)$ and $\eta : \mathbb{R} \rightarrow [0, 1]$ be a cut-off function such that $\text{supp } \eta \subseteq B_1$, $\eta \equiv 1$ on $B_{\sigma_2}$, $\eta \equiv 0$ on $B_1$ and $\|\nabla \eta\|_{L^\infty(E_d)} \leq c n^{-1}$. For any $0 < t_1 \leq t_2$ the same arguments as in Lemma 2.6 give

$$\|\eta^2 w_{t_2}\|_{1, B_1, \theta} \leq \|\eta^2 w_{t_1}\|_{1, B_1, \theta} + \int_{t_1}^{t_2} \frac{\mathcal{E} \omega \eta^2(w_{t_1})}{6 \theta^\omega(B_{\sigma_1})} dt \leq c \frac{\|1 \vee \mu^\omega\|_{1, B_1}}{\|\theta^\omega\|_{1, B_1}} \|\nabla \eta\|^2_{L^\infty(E_d)}(t_2 - t_1).$$

Set $\Lambda^\lambda := \{x \in B_1 : u(t, x) \geq 1\}$. Then, the assumption in (2.26) can be rewritten as $\int_{t_0 - n^2}^{t_0} \|\Lambda^\lambda\|_{1, B_1, \theta} dt \geq \lambda n^2$. Further, trivially $\int_{t_0 - \sigma_1 n^2}^{t_0} \|\Lambda^\lambda\|_{1, B_1, \theta} dt \leq \sigma_1 n^2$, so

$$\int_{t_0 - n^2}^{t_0 - \sigma_1 n^2} \|\Lambda^\lambda\|_{1, B_1, \theta} dt \geq (\lambda - \sigma_1) n^2.$$

By the mean value theorem there exists $\tau \in [t_0 - n^2, t_0 - \sigma_1 n^2]$ such that

$$\|\Lambda^\lambda\|_{1, B_1, \theta} \geq \frac{\lambda - \sigma_1}{1 - \sigma_1},$$

that is

$$\frac{\theta^\omega(\Lambda^\lambda)}{\theta^\omega(B_1)} \geq \frac{\lambda - \sigma_1}{1 - \sigma_1}.$$  

(2.29)

Take $t_1 = \tau$ and $t_2 \in [t_0 - \sigma_1 n^2, t_0]$ arbitrary. Then, by (2.28) we have

$$\|\eta^2 w_{t_2}\|_{1, B_1, \theta} \leq \|\eta^2 w_{\tau}\|_{1, B_1, \theta} + c \frac{\|1 \vee \mu^\omega\|_{1, B_1}}{\|\theta^\omega\|_{1, B_1}}$$

(3.20)

since $\|\nabla \eta\|^2_{L^\infty(E_d)}(t_2 - \tau) \leq c$. Further, note that

$$\|\eta^2 w_{t_2}\|_{1, B_1, \theta} \geq \frac{1}{\theta^\omega(B_1)} \sum_{x \in B_{\sigma_2} \setminus N^\ast_2} w(t_2, x) \theta^\omega(x) \geq g(2h) \frac{\theta^\omega(B_{\sigma_2} \setminus N^\ast_2)}{\theta^\omega(B_1)}.$$
and, using that \( g(s) = 0 \) for all \( s \geq 1 \), we get by the definition of \( \Lambda_r \) and (2.29),

\[
\| \eta^2 w_r \|_{1, B_1, \theta} \leq \| w_r \|_{1, B_1, \theta} = \frac{1}{\theta^\omega(B_1)} \sum_{x \in B_1 : u(x) < 1} w(x) \theta^\omega(x) \\
\leq g(h) \frac{\theta^\omega(B_1 \setminus \Lambda_r)}{\theta^\omega(B_1)} \leq g(h) \left( 1 - \frac{\lambda - \sigma_1}{1 - \sigma_1} \right).
\]

Substituting the above into (2.30) yields

\[
g(2h) \frac{\theta^\omega(B_{\sigma_2} \setminus N_{t_2})}{\theta^\omega(B_1)} \leq g(h) \left( 1 - \frac{\lambda - \sigma_1}{1 - \sigma_1} \right) + c \frac{\| 1 \lor \mu^\omega \|_{1, B_1}}{\| \theta \|_{1, B_1}}.
\]

Hence,

\[
\theta^\omega(B_{\sigma_2} \setminus N_{t_2}) \leq c \left( g(h) + \frac{1}{g(2h)} \frac{\| 1 \lor \mu^\omega \|_{1, B_1}}{\| \theta \|_{1, B_1}} \right) \theta^\omega(B_{\sigma_2}),
\]

Since Assumptions 1.1 and 1.4 hold, we can choose \( N_7(\omega, x_0) \) such that for all \( n \geq N_7(\omega, x_0) \),

\[
\theta^\omega(B_{\sigma_2} \setminus N_{t_2}) \leq c \left( g(h) + \frac{1}{g(2h)} \right) \theta^\omega(B_{\sigma_2}) \leq \frac{7}{4} \theta^\omega(B_{\sigma_2}),
\]

provided \( h \in (0, 1) \) is small enough. Thus, \( \theta^\omega(N_{t_2}) \geq \frac{4}{7} \theta^\omega(B_{\sigma_2}). \)

\[\square\]

**Proof of Theorem 1.6.** Without loss of generality, assume \( \epsilon = 1 \) (otherwise, replace \( u \) by \( u/\epsilon \)). Let \( G(s) := g(\frac{s+\lambda}{\lambda}) \) for \( s \in \mathbb{R} \), with \( 0 < k < h \) to be specified later. We write \( W := G(u) \). Then \( \partial_t W = G'(u) \partial_t u \) and by Taylor expansion

\[
\mathcal{L}_\theta^w W(t, x) = \frac{1}{\theta^\omega(x)} \sum_{y \sim x} \omega(x, y) \left( G(u(t, y) - G(u(t, x)) \right) \\
= \frac{1}{\theta^\omega(x)} \sum_{y \sim x} G'(u(t, x)) \omega(x, y) \left( u(t, y) - u(t, x) \right) \\
+ \frac{1}{2 \theta^\omega(x)} \sum_{y \sim x} G''(\tilde{u}_{x, y}) \omega(x, y) \left( u(t, y) - u(t, x) \right)^2
\]

for some \( \tilde{u}_{x, y} \in \mathbb{R} \). Since the latter term in the above is non-negative, this implies \( \partial_t W - \mathcal{L}_\theta^w W \leq G'(u) (\partial_t u - \mathcal{L}_\theta^w u) = 0 \). So we can apply Corollary 2.4, which gives that for \( \frac{1}{2} \leq \sigma' < \sigma_1 < \sigma_2 \leq 1 \) with \( \sigma_1 < \lambda < \sigma_2 \) and large enough \( n \),

\[
\max_{(t, x) \in Q^w_{\sigma_1}(n)} W(t, x)^2 \leq c A_1^\omega(n)^{4p_1} \| W \|_{1, \frac{1}{p_1}, Q_{\sigma_1}(n)}^2 \| W \|_{\infty, \infty, Q_{\sigma_1}(n)}^{2(1-\gamma)k_{\sigma_1}}. \tag{2.31}
\]

Let \( \eta \) be a linear cut-off function between \( B_{\sigma_1} \) and \( B_1 \) such that \( \| \nabla \eta \|_{L^\infty(E_d)} \leq 1/n \). Then, by the same arguments as in Lemma 2.6,

\[
\int_{t_0}^{t_0 - \sigma_1 n^2} \partial_t \| \eta^2 W_t \|_{1, B_1, \theta} dt + \int_{t_0 - \sigma_1 n^2}^{t_0} \mathcal{L}^\omega \eta^2(W_t) dt \leq c \frac{\| 1 \lor \mu^\omega \|_{1, B_1}}{\| \theta \|_{1, B_1}}. \tag{2.32}
\]
This implies
\[
\int_{t_0-T_n^2}^{t_0} \mathcal{E}_{\omega, n^2}^2(W_t) \, dt \leq c \theta^\omega(B_1) \left( \|1 \vee \mu^\omega\|_{1, B_1} + \|\gamma^2 W_{t_0-T_n^2}\|_{1, B_1, \theta} \right) \\
\leq c \theta^\omega(B_1) \left( \|1 \vee \mu^\omega\|_{1, B_1} + \|\gamma^2 W_{t_0-T_n^2}\|_{1, B_1, \theta} \right),
\]
where we have used that \( \|\gamma^2 W_{t_0-T_n^2}\|_{1, B_1, \theta} \leq \|W_{t_0-T_n^2}\|_{1, B_1, \theta} \leq g(\frac{k}{n}) \) since \( u > 0 \) and \( g \) is decreasing.

Now, let \( N_t := \{ x \in B_{\sigma_2} : W(t, x) = 0 \} \). Then, by Lemma 2.15 we have \( \theta^\omega(N_t) \geq \frac{1}{4} \theta^\omega(B_{\sigma_2}) \), so that by the Poincaré inequality in Proposition 2.8,
\[
\|W_t\|_{1, B_{\sigma_2}, \theta}^2 \leq c A_3^\omega(n) \frac{n^2}{B_{\sigma_2}} \int_{x,y \in B_{\sigma_2}} \omega(x, y) \left(W_t(x) - W_t(y)\right)^2.
\]

Hence, noting that \( \theta^\omega(B_1)/\|B_{\sigma_2}\| \leq c \|\theta^\omega\|_{1, B_1} \), we get
\[
\frac{1}{n^2} \int_{t_0-T_n^2}^{t_0} \|W_t\|_{1, B_{\sigma_2}, \theta}^2 dt \leq c A_3^\omega(n) \left(1 \vee \mu^\omega\|_{1, B_1} + \|\theta^\omega\|_{1, B_1} g(\frac{k}{n})\right).
\]

Returning to (2.31), we use Jensen’s inequality, the fact that \( W \leq g\left(\frac{k}{n}\right) \) and \( \beta_n \leq 1 \) to obtain for \( n \geq N_1(\omega, x_0) \),
\[
\max_{(t,x) \in Q_{\sigma_0}(n)} W(t, x)^2 \leq \frac{c A_1^\omega(n)^{4p}}{n^2} \left(1 \vee \mu^\omega\|_{1, B_1} + \|\theta^\omega\|_{1, B_1} g\left(\frac{k}{n}\right)\right)^{2p \beta_n} W_{2(1-\gamma)K_n}^2 \|
\]
\[
\leq \frac{c A_1^\omega(n)^{4p}}{n^2} \left(1 \vee \mu^\omega\|_{1, B_1} + \|\theta^\omega\|_{1, B_1} g\left(\frac{k}{n}\right)\right)^{2(1-\gamma)K_n} \|
\]
\[
\leq c \left(1 + g\left(\frac{k}{n}\right)\right)^{\beta_n} g\left(\frac{k}{n}\right)^{\frac{1}{4}} \leq c \left(1 + g\left(\frac{k}{n}\right)\right)^{\frac{3}{2}},
\]
where we have chosen \( N_1(\omega, x_0) \) large enough to control the random constants and such that \( (1-\gamma)K_n < \frac{1}{4} \). Now, choose \( \gamma > 0 \) small enough such that \( 2\gamma < h \) and
\[
g\left(\frac{2\gamma}{h}\right)^2 > c \left(1 + g\left(\frac{\gamma}{h}\right)\right)^{\frac{3}{2}}
\]
with \( c \) as above. Take \( k = \gamma \). The weak Harnack inequality now follows by contradiction. Suppose there exists \( (\tilde{t}, \tilde{x}) \in Q_{\sigma_0}(n) \) such that \( u(\tilde{t}, \tilde{x}) < \gamma \). Then
\[
g\left(\frac{2\gamma}{h}\right)^2 \leq g\left(u(\tilde{t}, \tilde{x}) + k\right)^2 = W(\tilde{t}, \tilde{x})^2 \leq c \left(1 + g\left(\frac{\gamma}{h}\right)\right)^{\frac{3}{2}},
\]
which is a contradiction to (2.33). Therefore \( u \geq \gamma \) on \( Q_{\sigma_0}(n) \). \(\square\)
3. Annealed Local Limit Theorem Under General Speed Measure

3.1. Maximal Inequality for the Heat Kernel. The first step to show the annealed local limit theorem in Theorem 1.7 is to establish an $L^1$ form of the maximal inequality in [6], which involves space-time cylinders of a form more convenient for our purposes. So for $\epsilon \in (0, 1/4)$, $x_0 \in \mathbb{Z}^d$, we redefine

$$Q_\sigma(n):= \left[(1 - \sigma)\epsilon n^2, (1 - \sigma)(1 - \epsilon)n^2 + \sigma n^2\right] \times B(x_0, \sigma n)$$

where $n \in \mathbb{N}$ and $\sigma \in \left[\frac{1}{2}, 1\right]$.

**Proposition 3.1.** Fix $\epsilon \in (0, 1/4)$, $x_0 \in \mathbb{Z}^d$ and let $p, q, r \in (1, \infty]$ be such that (1.4) holds. There exists $C_{10} = C_{10}(d, p, q, r)$ such that for all $n \geq 1$ and $1/2 \leq \sigma' < \sigma \leq 1$,

$$\max_{(t,x) \in Q_\sigma(n)} \rho_0^\omega(t, 0, x) \leq C_{10} \|1 \vee (1/\theta^\omega)\|_{1,B(n)} \left(\frac{A^\omega_4(n)}{\epsilon(\sigma - \sigma')^2}\right) \|\rho_0^\omega(\cdot, 0, \cdot)\|_{1,1/p_\omega, Q_\sigma(n), \theta},$$

where $\kappa' = \kappa'(d, p, q, r) := p_\ast + p_\ast^2 p_\ast / (\rho - r_\ast p_\ast)$ with $\rho$ as in (2.3) and

$$A^\omega_4(n) := \left[1 \vee (1/\theta^\omega)\right]_{p,B(n), \theta} \|1 \vee \theta^\omega\|_{q,B(n)} \|1 \vee (1/\theta^\omega)\|_{q,B(n)}.$$  

(3.1)

**Proof.** For abbreviation we set $u = \rho_0^\omega(\cdot, 0, \cdot)$ and $\sigma_k := \sigma - (\sigma - \sigma')2^{-k}$. Further, write $B_k := B(x_0, \sigma_k n)$ and $Q_k := Q_{\sigma_k}(n)$. Note that $|B_k|/|B_{k+1}| \leq c_{2^d}$. Let $\gamma = 1/2p_\ast$. Then by Hölder’s inequality

$$\|u\|_{2p_\ast, 2Q_k, \theta} \leq \|u\|_{1, 2\gamma, Q_k, \theta} \|u\|_{1, \infty, Q_k}^{1 - \gamma},$$

and by the proof of [6, Proposition 3.8] (cf. last line on page 14), setting $\phi = 1$ and $\delta = 1$ there, we have

$$\|u\|_{\infty, \infty, Q_{k-1}} \leq c \left(\frac{A^\omega_4(n)}{\epsilon(\sigma_k - \sigma_{k-1})^2}\right)^{\kappa/p_\ast} \|u\|_{2p_\ast, 2Q_k, \theta}$$

with $\kappa = \kappa(d, p, q, r)$ as throughout [6]. Combining the above equations yields

$$\|u\|_{\infty, \infty, Q_{k-1}} \leq 2^{2\kappa/2p_\ast} J \|u\|_{1, 2\gamma, Q_{\sigma_k}, \theta} \|u\|_{1, \infty, Q_k}^{1 - \gamma},$$

where we have introduced $J := c(A^\omega_4(n)/\epsilon(\sigma - \sigma')^2)^{\kappa/p_\ast} \geq 1$ for brevity. By iteration

$$\|u\|_{\infty, \infty, Q_{\sigma_k}} \leq 2^{2\kappa/p_\ast} \sum_{k=0}^{K-1} (1 - \gamma)^k \left(\|u\|_{1, 2\gamma, Q_{\sigma_k}, \theta}\right) \sum_{k=0}^{K-1} (1 - \gamma)^k \|u\|_{1, \infty, Q_k}^{(1 - \gamma)^K}.$$  

(3.2)

Recall that $p_\ast^\omega(t, 0, x) \leq 1$ for all $t > 0$ and $x \in \mathbb{Z}^d$. Therefore,

$$\|u\|_{\infty, \infty, Q_K} \leq \max_{x \in B_K} \theta^\omega(x)^{-1} \max_{(t,x) \in Q(n)} u(t, x) \theta^\omega(x) \leq C |B_K| \|1/\theta^\omega\|_{1, B_K}.$$  

Since $|B_K|^{-1} \|1/\theta^\omega\|_{1, B_K} \leq c \|1 \vee (1/\theta^\omega)\|_{1, B(n)}$, we obtain by letting $K \to \infty$ in (3.2),

$$\|u\|_{\infty, \infty, Q_{\sigma_k}} \leq 2^{2\kappa/2p_\ast} \|1 \vee (1/\theta^\omega)\|_{1, B(n)} J^{1/\gamma} \|u\|_{1, 2\gamma, Q_{\sigma_k}, \theta},$$

which completes the proof, with $\kappa' := 2\kappa$. \qed
3.2. Proof of Theorem 1.7. Here we anneal the results of Section 2 to derive the annealed local limit theorem for the static RCM under a general speed measure stated in Theorem 1.7. This will require a stronger moment condition. For any \( p, q, r_1, r_2 \in [1, \infty] \) set
\[
M(p, q, r_1, r_2) := \mathbb{E}\left[\mu^\omega(0)^p\right] + \mathbb{E}\left[\theta^\omega(0)^{-r_1}\right] + \mathbb{E}\left[\theta^\omega(0)^{r_2}\right] \in (0, \infty].
\]

**Proposition 3.2.** Suppose Assumption 1.1 holds. Then there exist \( p, q, r_1, r_2 \in (0, \infty) \) (only depending on \( d \)) such that, under the moment condition \( M(p, q, r_1, r_2) < \infty \), for all \( K > 0 \) and \( 0 < T_1 \leq T_2 \),
\[
\mathbb{E}\left[\sup_{n \geq 1, |x| \leq K, t \in [T_1, T_2]} n^d p^\omega_0(n^2 t, 0, [nx])\right] < \infty.
\]

Before we prove Proposition 3.2 we remark that it immediately implies the annealed local limit theorem.

**Proof of Theorem 1.7.** The statement follows from the corresponding quenched result in Theorem 1.5 and Proposition 3.2 by an application of the dominated convergence theorem. \( \square \)

As a by-product we obtain an annealed on-diagonal estimate on the heat kernel.

**Corollary 3.3.** Under the assumptions of Proposition 3.2, for any \( \lambda > 0 \) there exists \( c_{11} = c_{11}(d, p, q, r) \) such that for all \( t \geq 1 \),
\[
\mathbb{E}\left[\sup_{x \in B(0, \lambda \sqrt{t})} p^\omega_0(t, 0, x)\right] \leq c_{11} t^{-d/2}.
\]  
(3.3)

**Proof.** Choosing \( K = 2\lambda, n = [\sqrt{t}], T_2 = 1 \) and any \( T_1 \in (0, 1) \), this follows directly from Proposition 3.2. \( \square \)

The rest of this section is devoted to the proof of Proposition 3.2. We start with a consequence of the maximal inequality in Proposition 3.1.

**Lemma 3.4.** Let \( p, q, r \in (1, \infty) \) be such that (1.4) holds. For all \( K > 0 \), \( 0 < T_1 \leq T_2 \), there exist \( c_{12} = c_{12}(d, p, q, r, K, T_1, T_2) \) and \( c_{13} = c_{13}(K, T_2) \) such that
\[
\sup_{|x| \leq K, t \in [T_1, T_2]} n^d p^\omega_0(n^2 t, 0, [nx]) \leq c_{12} \|1 \vee (1/\theta^\omega)\|_{1, B(n)} A_{n}^\omega(c_{13}n)^{n'}, \quad \forall n \geq 1,
\]
with \( A_{n}^\omega \) as in (3.1).

**Proof.** First note that by definition of the heat kernel \( p^\omega_0 \),
\[
\left|p^\omega_0(\cdot, 0, \cdot)\right|_{1,1/p^*, Q_1(n), \theta} = \left(\frac{1}{|I_1|} \int_{I_1} \left(\frac{1}{B(n)} \sum_{y \in B(n)} p^\omega_0(t, 0, y) \theta^\omega(y)\right)^{1/p^*} dt\right)^{p^*} = c n^{-d} \left(\frac{1}{|I_1|} \int_{I_1} P^\omega_0 \left[X_t \in B(n)\right] dt\right)^{p^*} \leq c n^{-d},
\]  
(3.4)
for all $n \in \mathbb{N}$. Choose $x_0 = 0$ and set $N = c_{13}n$ for any $c_{13} > 2[K \vee \sqrt{T_2}]$. Then we can find $\epsilon \in (0, 1/4)$ such that
\[
\{(n^{2}t, \lfloor nx \rfloor) : t \in [T_1, T_2], |x| \leq K\} \subseteq Q_{1/2}(N) = \left[\frac{1}{2}N^2, (1 - \frac{1}{2})N^2\right] \times B(0, N/2).
\]
The claim follows now from Proposition 3.1 with the choice $\sigma = 1$ and $\sigma' = 1/2$ together with (3.4).

Another ingredient in the proof of Proposition 3.2 will be the following version of the maximal ergodic theorem, which we recall for the reader's convenience.

**Lemma 3.5.** Suppose Assumption 1.1 holds. Then for all $p \in (1, \infty)$ and $f \in L^p(\Omega)$,
\[
\mathbb{E} \left[ \sup_{n \geq 1} \left( \frac{1}{|B(n)|} \sum_{x \in B(x_0, n)} f \circ \tau_x \right)^p \right] \leq \left( \frac{p}{p-1} \right)^p \mathbb{E} [f^p]. \tag{3.5}
\]

**Proof.** See [23, Chapter 1 Theorem 6.3, p.52]. □

**Proof of Proposition 3.2.** By Lemma 3.4 it suffices to show that, under a suitable moment condition, $\mathbb{E} \left[ \sup_{n \geq 1} \left\| 1 \vee (1/\theta^\omega) \right\|_{L^1(\Omega)} A^\omega_{4}(c_{13}n^{4\kappa'}) < \infty$. Recall that
\[
A^\omega_{4}(n) = \left\| 1 \vee (\mu^\omega/\theta^\omega) \right\|_{L^p(\Omega)} \left\| 1 \vee \nu^\omega \right\|_{L^q(\Omega)} \left\| 1 \vee \theta^\omega \right\|_{L^r(\Omega)} 1 \vee (1/\theta^\omega) \right\|_{L^q(\Omega)},
\]
for any $p, q, r \in (1, \infty]$ satisfying (1.3). After an application of Hölder's inequality it suffices to show
\[
\mathbb{E} \left[ \sup_{n \geq 1} \left\| \nu^\omega \right\|_{L^q(\Omega)}^{4\kappa'} \right] < \infty
\]
and similar moment bounds on the other terms. Now suppose that $\mathbb{E} \left[ \nu^\omega(0)^{4\kappa'/q'} \right] < \infty$ for any $q' > q$. Then, if $4\kappa' > q$, given Assumption 1.1, we can apply Lemma 3.5 to deduce
\[
\mathbb{E} \left[ \sup_{n \geq 1} \left\| \nu^\omega \right\|_{L^q(\Omega)}^{4\kappa'} \right] \leq c \mathbb{E} \left[ \nu^\omega(0)^{4\kappa'} \right] < \infty.
\]
In the case $4\kappa' \leq q < q'$, we have by Jensen’s inequality followed by Lemma 3.5,
\[
\mathbb{E} \left[ \sup_{n \geq 1} \left\| \nu^\omega \right\|_{L^q(\Omega)}^{4\kappa'} \right] \leq \mathbb{E} \left[ \sup_{n \geq 1} \left( \frac{1}{|B(n)|} \sum_{x \in B(0, n)} \nu^\omega(x)^q \right)^{\frac{4\kappa'}{q'}} \right]^{\frac{q}{q'}} \leq c \mathbb{E} \left[ \nu^\omega(0)^{q'\frac{4\kappa'}{q'}} \right]^{\frac{q}{q'}} < \infty.
\]
The other terms involving $\left\| \theta^\omega \right\|_{L^r(\Omega)}$ etc. can be treated similarly. □
4. annealed local limit theorem for the dynamic RCM

Similarly as in the static case our starting point is establishing an $L^1$ maximal inequality for space-time harmonic functions. Once again, we redefine our space-time cylinders. For $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$, $n \in \mathbb{N}$, and $\sigma \in (0, 1]$, let

$$Q_\sigma(n) := [t_0, t_0 + \sigma n^2] \times B(x_0, \sigma n).$$

Throughout this section we fix $p, q \in (1, \infty]$ satisfying

$$\frac{1}{p} - \frac{q + 1}{q} + \frac{1}{q} < \frac{2}{d}. \quad (4.1)$$

**Proposition 4.1.** Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$ and $\Delta \in (0, 1)$. There exist $N_8 = N_8(\Delta) \in \mathbb{N}$ and $c_{14} = c_{14}(d, p, q)$ such that for all $n \geq N_8$ and $1/2 \leq \sigma' < \sigma \leq 1$ with $\sigma - \sigma' > n^{-\Delta}$,

$$\max_{(t, x) \in Q_\sigma(n)} p^{\nu}(t, 0, x) \leq c_{14} \left( \frac{A_{\sigma}^w(n)}{(\sigma - \sigma')^2} \right)^{\kappa'} \|p^{\nu}(0, \cdot, 0, \cdot)\|_{1,1,Q_\sigma(n)},$$

where $\kappa' := \alpha^2 p_*/(\alpha - 1)$ with $\alpha := \frac{1}{p_*} + \frac{1}{q} (1 - \frac{1}{p_*}) \frac{q}{q+1}$, $\sigma$ as in (2.3), and

$$A_{\sigma}^w(n) := \|1 \vee \mu\|_{p,p,Q(n)} \|1 \vee \nu\|_{q,q,Q(n)}, \quad \beta_n := \vartheta \sum_{k=0}^{K_n-1} (1 - \vartheta)^k,$$

with $\vartheta := 1/2p_*$ in $(0, 1)$ and $K_n := \lfloor \frac{\Delta \ln n - \ln (\sigma - \sigma')}{\ln 2} \rfloor$.

**Proof.** Write $u(\cdot, \cdot) = p^{\nu}(0, \cdot, 0, \cdot)$ and $\sigma_k := \sigma - (\sigma - \sigma')2^{-k}$ for $k \in \mathbb{N}$. Then, by Hölder’s inequality

$$\|u\|_{2(\sigma_k, 2(\sigma_k - 1))} \leq \|u\|_{1,1,Q_{\sigma_k}} \|1 \vee \mu\|_{\infty,\infty,Q_{\sigma_k}}.$$

Note that by the definition of $K_n$ we have $\sigma_k - \sigma_{k-1} > n^{-\Delta}$ for all $k \in \{1, \ldots, K_n\}$. By [3, Theorem 5.5] (notice that if $f = 0$ in the present setting which leads to $\gamma = 1$ therein), there exist $c(d) \in (1, \infty)$, $N_8(\Delta) \in \mathbb{N}$ such that for $n \geq N_8(\Delta)$ and $k \in \{1, ..., K_n\}$,

$$\|u\|_{\infty,\infty,Q_{\sigma_{k-1}}} \leq c \left( \frac{A_{\sigma}^w(n)}{(\sigma_k - \sigma_{k-1})^2} \right)^{\kappa} \|u\|_{2(\sigma_0, 2(\sigma_0 - 1))} \leq 2^{2\kappa K_n} \|u\|_{1,1,Q_{\sigma_k}} \|u\|_{\infty,\infty,Q_{\sigma_k}},$$

with $\kappa := \frac{\alpha}{2(\sigma - \sigma')}$. Then by iteration,

$$\|u\|_{\infty,\infty,Q_{\sigma_{k}}} \leq 2^{2\kappa \sum_{k=0}^{K_n-1} (k+1)(1-\vartheta)^k} \left( J \|u\|_{1,1,Q_{\sigma_k}} \right)^{\sum_{k=0}^{K_n-1} (1-\vartheta)^k} \|u\|_{1,1,Q_{\sigma_k}} \|u\|_{\infty,\infty,Q_{\sigma_k}},$$

where we used that $u \leq 1$.

**Assumption 4.2.** Suppose that

$$E[\omega_0(e)^{2(\kappa' + \vartheta)}] < \infty, \text{ and } E[\omega_0(e)^{-2(\kappa' + \vartheta)}] < \infty$$

for any $e \in E_d$ with $p, q \in (1, \infty]$ satisfying (4.1) and $\kappa'$ as in Proposition 4.1.
Proposition 4.3. Suppose Assumption 1.9 and Assumption 4.2 hold. Then for all \( K > 0 \) and \( 0 < T_1 \leq T_2 \), there exists \( c_{15} = c_{15}(d, p, q, K, T_1, T_2) \) such that
\[
E \left[ \sup_{n \in \mathbb{N}, |x| \leq K, t \in [T_1, T_2]} n^d p^\omega(x, n^2 t, 0, \lfloor nx \rfloor) \right] \leq c_{15}.
\]

The proof is postponed to the end of this section.

Proof of Theorem 1.11. The statement follows from the corresponding quenched result, see Theorem 1.10-(ii) above, together with Proposition 4.3 by an application of the dominated convergence theorem. Note that the moment condition in Assumption 4.2 is stronger than the one required in Theorem 1.10. \( \square \)

As in the static case Proposition 4.3 also directly implies an annealed on-diagonal heat kernel estimate (cf. Corollary 3.3 above).

Corollary 4.4. Suppose Assumptions 1.9 and 4.2 hold. Then for any \( \lambda > 0 \) there exists \( c_{16} = c_{16}(d, p, q) \) such that for all \( t \geq 1 \),
\[
E \left[ \sup_{x \in B(0, \lambda \sqrt{T})} p^\omega(0, t, 0, x) \right] \leq c_{16} t^{-d/2}.
\]

The proof of Proposition 4.3 begins with a consequence of the above maximal inequality in Proposition 4.1.

Lemma 4.5. For all \( K > 0, 0 < T_1 \leq T_2 \), there exist \( N_9 = N_9(T_2, K) \in \mathbb{N} \) and constants \( c_{17} = c_{17}(d, p, q, K, T_1, T_2) \) and \( c_{18} = c_{18}(K, T_2) \) such that for all \( n \geq N_9 \),
\[
\sup_{|x| \leq K, t \in [T_1, T_2]} n^d p^\omega(x, n^2 t, 0, \lfloor nx \rfloor) \leq c_{17} A_0^\omega(c_{18} n)^{\kappa'}
\]
with \( A_0^\omega \) and \( \kappa' \) as in Proposition 4.1.

Proof. First note that by definition of the heat kernel \( p^\omega \),
\[
\|p^\omega(0, \cdot, \cdot, 0)\|_{1,1,Q(n)} = \frac{1}{|I_1|} \int_{I_1} \frac{1}{B(n)} \sum_{y \in B(n)} p^\omega(0, t, 0, y) \, dt = c_n n^{-d} \int_{I_1} \frac{1}{|I_1|} \sum_{y \in B(n)} P_{0,0}^\omega(X_t \in B(n)) \, dt \leq c_n n^{-d},
\]
for all \( n \in \mathbb{N} \). Set \( x_0 = 0, t_0 = T_1 \) and let \( N = c_{18} n \) with \( c_{18} \) chosen such that
\[
\{ (n^2 t, \lfloor nx \rfloor) : t \in [T_1, T_2], |x| \leq K \} \subseteq Q_{1/2}(N) = [t_0, t_0 + \frac{1}{2} N^2] \times B(0, N/2).
\]
Then by applying Proposition 4.1 with the choice \( \Delta = 1/2, \sigma = 1 \) and \( \sigma' = 1/2 \) we get that for all \( n \geq \lceil \frac{N}{c_{18}} \rceil \vee 4 \),
\[
\sup_{|x| \leq K, t \in [T_1, T_2]} n^d p^\omega(x, n^2 t, 0, \lfloor nx \rfloor) \leq c A_0^\omega(c_{18} n)^{\kappa'} n^{d(1-\beta_n)}.
\]
Since \( n^{d(1-\beta_n)} \to 1 \) as \( n \to \infty \) the claim follows. \( \square \)

For the proof of Proposition 4.3 we also require a maximal ergodic theorem for space-time ergodic environments.
Lemma 4.6. Suppose Assumption 1.9 holds. For all $x_0 \in \mathbb{Z}^d$, $t_0 \geq 0$, $p \geq 1$ and $f \in L^p(\Omega)$,
\[
\lim_{n \to \infty} \frac{1}{n^2} \int_{t_0}^{t_0+n^2} \frac{1}{|B(n)|} \sum_{x \in B(x_0,n)} f \circ \tau_{t,x} \, dt = \mathbb{E}[f] \quad \mathbb{P}\text{-a.s. and in } L^p(\Omega),
\]
and there exists $c_{19} = c_{19}(p) > 0$ such that
\[
\mathbb{E} \left[ \left( \sup_{n \geq 1} \frac{1}{n^2} \int_{t_0}^{t_0+n^2} \frac{1}{|B(n)|} \sum_{x \in B(x_0,n)} f \circ \tau_{t,x} \, dt \right)^p \right] \leq c_{19} \mathbb{E} [f^p]. \tag{4.3}
\]

Proof. For the proofs see [23, Chapter 6, Theorem 4.4, p.224] and the discussion following that theorem.

Proof of Proposition 4.3. Notice $\sup_{|x| \leq K, t \in [t_1,t_2]} n^d \rho^*(0,n^2 t,0,|n x|) \leq N_0^d$ for all $n \leq N_0$. So by Lemma 4.5, it suffices to bound $\mathbb{E} \left[ \sup_{n \geq N_0} A^\psi_n(c_{18} n)^{\omega'} \right]$ under the moment condition of Assumption 4.2. The claim now follows by utilising the maximal ergodic theorem in (4.3), similarly as in the proof of Proposition 3.2.

5. Applications to the Ginzburg-Landau $\nabla \varphi$ Model

In this section we connect the annealed local limit theorem for the dynamic RCM with a stochastic interface model, the Ginzburg-Landau $\nabla \varphi$ model. The survey [19] provides a nice review of this class of models. The Ginzburg-Landau $\nabla \varphi$ model describes a hypersurface (interface) embedded in $d+1$-dimensional space, $\mathbb{R}^{d+1}$, which separates two pure thermodynamical phases. The interface is represented by a field of height variables $\varphi = \{ \varphi(x) : x \in \Gamma \}$, which measure the vertical distances between the interface and $\Gamma \subseteq \mathbb{Z}^d$, a fixed $d$-dimensional reference hyperplane. The Hamiltonian $H$ represents the energy associated with the field of height variables $\varphi$. In general, for $\Gamma = \mathbb{Z}^d$ or $\Gamma \subseteq \mathbb{Z}^d$,
\[
H(\varphi) \equiv H_\Gamma^\psi(\varphi) = \frac{1}{2} \sum_{\{x,y\} \in \Gamma^*} V(\varphi(x) - \varphi(y)). \tag{5.1}
\]
Note that boundary conditions $\psi = \{ \psi(x) : x \in \partial^+ \Gamma \}$ are required to define the sum in the case $\Gamma \subseteq \mathbb{Z}^d$, i.e. we set $\varphi(x) = \psi(x)$ for $x \in \partial^+ \Gamma$. The sum in (5.1) is merely formal when $\Gamma = \mathbb{Z}^d$. The dynamics of the $\nabla \varphi$ model are governed by the following infinite system of SDEs for $\phi_t = \{ \phi_t(x) : x \in \Gamma \} \in \mathbb{R}^\Gamma$,
\[
d\phi_t = -\frac{\partial H}{\partial \phi_t(x)}(\phi_t) \, dt + \sqrt{2} \, dw_t(x), \quad x \in \Gamma, \quad t > 0,
\]
where $w_t = \{ w_t(x) : x \in \mathbb{Z}^d \}$ is a collection of independent one-dimensional standard Brownian motions. Due to the form of the Hamiltonian, only nearest neighbour interactions are involved. Equivalent to the above in the case $\Gamma = \mathbb{Z}^d$ is
\[
\phi_t(x) = \phi_0(x) - \int_0^t \sum_{y : |x-y|=1} V'(\phi_t(x) - \phi_t(y)) \, dt + \sqrt{2} \, w_t(x), \quad x \in \mathbb{Z}^d. \tag{5.2}
\]
Similarly, if $\Gamma \Subset \mathbb{Z}^d$, we define the finite volume process by
\[
\phi_t^{\Gamma,\psi}(x) = \phi_0^{\Gamma,\psi}(x) - \int_0^t \sum_{y \in \Gamma: |x-y|=1} V'(\phi_t^{\Gamma,\psi}(x) - \phi_t^{\Gamma,\psi}(y)) \, dt + \sqrt{2} \, \omega_t(x), \quad x \in \Gamma,
\]
subject to the boundary conditions $\phi_t^{\Gamma,\psi}(y) = \psi(y), \ y \in \partial^+ \Gamma$. The evolution of $\phi_t$ is designed such that it is stationary and reversible under the equilibrium $\psi$-Gibbs measure $\mu_1^\psi$ or $\mu$ (see (5.4) below). We denote by $\mathbb{P}_\mu$ the law of the process $\phi_t$ started under the distribution $\mu$ (and by $\mathbb{E}_\mu$ the corresponding expectation).

Most of the mathematical literature on the $\nabla \varphi$ model treats the case of a suitably smooth, even and strictly convex interaction potential $V$ such that $V''$ is bounded above. However, we will relax these conditions; throughout the rest of this section we work with $V$ as in the following assumption.

**Assumption 5.1.** The potential $V \in C^2(\mathbb{R})$ is even and there exists $c_- > 0$ such that
\[
c_- \leq V''(x), \quad \text{for all } x \in \mathbb{R}. \tag{5.3}
\]
Note that under Assumption 5.1, the coefficients of the SDE (5.2) are not necessarily globally Lipschitz continuous. However, it is still possible to construct an almost surely continuous solution $\phi_t$, see Proposition 5.3. The assumption that the potential has second derivative bounded away from zero is required for the existence of an equilibrium $\psi$-Gibbs measure. For $\Gamma \Subset \mathbb{Z}^d$, the finite volume $\psi$-Gibbs measure for the field of heights $\phi \in \mathbb{R}^d$ is defined as
\[
\mu(d\phi) \equiv \mu_1^\psi(d\phi) = \frac{1}{Z_\Gamma^\psi} \exp \left(-H_\Gamma^\psi(\phi)\right) \, d\phi_\Gamma, \tag{5.4}
\]
with boundary condition $\psi \in \mathbb{R}^{\partial^+ \Gamma}$, where $d\phi_\Gamma$ is the Lebesgue measure on $\mathbb{R}^\Gamma$ and $Z_\Gamma^\psi$ is a normalisation constant. Note that the condition (5.3) implies $Z_\Gamma^\psi < \infty$ for every $\Gamma \Subset \mathbb{Z}^d$ and hence $\mu_1^\psi \in \mathcal{P}(\mathbb{R}^\Gamma)$ is a probability measure. In the infinite volume case $\Gamma = \mathbb{Z}^d$, (5.4) has no rigorous meaning but one can still define Gibbs measures as follows.

**Definition 5.2.** A probability measure $\mu \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$ is a $\psi$-Gibbs measure if its conditional probability on $\mathcal{F}_{\Gamma^c} = \sigma\{\phi(x) : x \notin \Gamma\}$ satisfies the DLR (Dobrushin-Lanford-Ruelle) equation
\[
\mu(\cdot | \mathcal{F}_{\Gamma^c})(\psi) = \mu_\Gamma^\psi(\cdot), \quad \text{for } \mu\text{-a.e. } \psi, \tag{5.5}
\]
for all $\Gamma \Subset \mathbb{Z}^d$.

In order to study the properties of solutions to the system of SDES (5.2), it is necessary to restrict to a suitable class of initial configurations. Let $S := \{ (\phi(x))_{x \in \mathbb{Z}^d} : |\phi(x)| \leq a + |x|^n, \text{ for some } a \in \mathbb{R}, n \in \mathbb{N} \}$ denote the configurations of heights with at most polynomial growth.

**Proposition 5.3.** Given any initial configuration $\phi_0 \in S$, there exists a unique solution to the system of SDES (5.2) such that for any $x \in \mathbb{Z}^d$ the process $\phi_t(x)$ is almost surely
continuous and for all \( t > 0 \) the configuration \( \phi_t \in \mathcal{S} \) almost surely. Any Gibbs measure on \( \mathcal{S} \) is stationary and reversible with respect to the process \( \phi_t \).

**Proof.** The proof follows by similar arguments as for the Ising model case of [26, Theorem 4.2.13]. The key observations are that equation (4.2.5) there holds for our Hamiltonian and the relation (4.2.12b) holds for our interaction potential \( V \), where \( p \) is defined as \( p(x) = c_- \mathbb{1}_{|x|=1} \) for \( x \in \mathbb{Z}^d \).

Brascamp-Lieb inequalities state that for every \( \Gamma \subset \mathbb{Z}^d \), covariances under the aforementioned \( \varphi \)-Gibbs measure \( \mu^\varphi_\Gamma \) are bounded by those under \( \mu^\varphi_{\Gamma, m} \), the Gaussian finite volume \( \varphi \)-Gibbs measure determined by the quadratic potential \( V \). Brascamp-Lieb inequalities are pivotal in proving the following existence result. We shall also employ the massive Hamiltonian, for \( m > 0 \),

\[
H_m(\phi) \equiv H^\varphi_{\Gamma, m}(\phi) := H^\varphi_{\Gamma} + \frac{m^2}{2} \sum_{x \in \Gamma} \phi(x)^2.
\] 

(5.8)

**Remark 5.5.** Note that Proposition 5.4 also holds for the massive Hamiltonian \( H^\varphi_{\Gamma, m} \) and in that case the Gaussian potential is given by \( V^*(x) = \frac{c_- + m^2}{2} x^2 \).

**Theorem 5.6 (Existence of \( \varphi \)-Gibbs measures).** If \( d \geq 3 \) then for all \( h \in \mathbb{R} \) there exists a stationary, shift-invariant, ergodic \( \varphi \)-Gibbs measure \( \mu \in \mathcal{P}(\mathbb{R}^\mathbb{Z}^d) \) of mean \( h \), i.e. \( \mathbb{E}_\mu[\phi(x)] = h \) for all \( x \in \mathbb{Z}^d \).

**Proof.** Let \( m > 0 \) and first take a sequential limit as \( n \to \infty \) of finite volume \( \varphi \)-Gibbs measures \( \mu^0_{m, \Gamma_n} \) with periodic boundary conditions, corresponding to the massive Hamiltonian \( H_{\Gamma_n, m}^0 \) on the torus \( \Gamma_n := (\mathbb{Z}/n\mathbb{Z})^d \). Since \( V \) is even, \( \mathbb{E}_{\mu^0_{m, \Gamma_n}}[\phi(x)] = 0 \). When \( d \geq 3 \), the variance of the Gaussian system corresponding to the potential \( V^*(x) = \frac{c_- + m^2}{2} x^2 \) is uniformly bounded in \( n \), so by the Brascamp-Lieb inequality,

\[
\sup_{x \in \mathbb{Z}^d, n \in \mathbb{N}} \mathbb{E}_{\mu^0_{m, \Gamma_n}}[\exp(\lambda \phi(x))] < \infty, \quad \forall \lambda > 0.
\]

Therefore \( \mu^0_{m, \Gamma_n} \) is tight and along some proper subsequence there exists a limit \( \mu^0_{\infty} := \lim_{n \to \infty} \mu^0_{m, \Gamma_n} \), a shift-invariant Gibbs measure on \( \mathbb{Z}^d \) of mean 0. Now for all \( m > 0 \) and \( x \in \mathbb{Z}^d \),

\[
\frac{\partial^2 H_m(\phi)}{\partial \phi(x)^2} \geq c_-
\]

so by Brascamp-Lieb again the limit
\( \mu^0 = \lim_{n \to 0} \mu^0_m \) exists. The distribution of \( \phi + h \) where \( \phi \) is \( \mu^0 \) distributed is a shift-invariant \( \varphi \)-Gibbs measure on \( \mathbb{Z}^d \) under which \( \phi(x) \) has mean \( h \) for all \( x \in \mathbb{Z}^d \). Having shown that the convex set of shift-invariant \( \varphi \)-Gibbs measures of mean \( h \) is non-empty, there exists an extremal element of this set which is ergodic, see [20, Theorem 14.15]. Finally, by Proposition 5.3 this Gibbs measure is reversible and hence stationary for the process \( \phi_t \). 

**Remark 5.7.** The \( \varphi \)-Gibbs measures exist when \( d \geq 3 \) but not for \( d = 1, 2 \). An infinite volume (thermodynamic) limit for \( \mu^0_1 \) as \( \Gamma \uparrow \mathbb{Z}^d \) exists only when \( d \geq 3 \).

Our aim is to investigate the decay of the space-time correlation functions under the equilibrium Gibbs measures. The idea, originally from Helffer and Sjöstrand [22], is to describe the correlation functions in terms of a certain random walk in a dynamic random environment (cf. also [18, 21]). Let \( (X_t)_{t \geq 0} \) be the random walk on \( \mathbb{Z}^d \) with jump rates given by the random dynamic conductances

\[
\omega_t(e) := V''(\nabla \phi_t) = V''(\phi_t(y) - \phi_t(x)), \quad e = \{x, y\} \in E_d. \tag{5.9}
\]

Note that the conductances are positive by Assumption 5.1 and, since \( V \) is even, the jump rates are symmetric, i.e. \( \omega_t(\{x, y\}) = \omega_t(\{y, x\}) \). Further, let \( p^\omega(s, t, x, y) \), \( x, y \in \mathbb{Z}^d \), \( s \leq t \), denote the transition densities of the random walk \( X \). Then the Helffer-Sjöstrand representation (see [19, Theorem 4.2] or [18, Equation (6.10)]) states that if \( F, G \in C^1_b(S) \) are differentiable functions with bounded derivatives depending only on finitely many coordinates then for all \( t > 0 \),

\[
\text{Cov}_\mu(F(\phi_0), G(\phi_t)) = \int_0^\infty \sum_{x, y \in \mathbb{Z}^d} \mathbb{E}_\mu \left[ \frac{\partial F}{\partial \phi(x)}(\phi_0) \frac{\partial G}{\partial \phi(y)}(\phi_t) p^\omega(0, t + s, x, y) \right] ds, \tag{5.10}
\]

where \( \mu \) is a stationary, ergodic, shift-invariant \( \varphi \)-Gibbs measure. Note that in \( d \geq 3 \) the integral in (5.10) is finite due to an on-diagonal heat kernel estimate.

**Lemma 5.8.** There exists deterministic \( c_{20} = c_{20}(d, c_-) < \infty \) such that

\[
p^\omega(0, t, x, y) \leq c t^{-d/2}, \quad \forall t \geq 1, \forall x, y \in \mathbb{Z}^d. \tag{5.11}
\]

**Proof.** Note that by Assumption 5.1, \( \omega_t(e) \geq c_- \) for all \( t \geq 0 \) and \( e \in E_d \), which implies the Nash inequality, i.e. for any \( f : \mathbb{Z}^d \to \mathbb{R} \),

\[
\|f\|_{\mathcal{B}(\mathbb{Z}^d)} \leq c \|\omega_t^{1/2} \nabla f\|_{\mathcal{B}(E_d)} \|f\|_{H^1(\mathbb{Z}^d)},
\]

from which the statement follows by standard arguments, see [13] and [25]. \( \square \)

A consequence of the above is the following variance estimate, an example of algebraic decay to equilibrium, in contrast to the exponential decay to equilibrium which would follow from a spectral gap estimate or Poincaré inequality. For this model, these inequalities hold on finite boxes but fail on the whole lattice.

**Corollary 5.9.** Suppose \( d \geq 3 \) and let \( \mu \in \mathcal{P}(\mathbb{R}^\mathbb{Z}^d) \) be any ergodic, shift-invariant, stationary \( \varphi \)-Gibbs measure.
(i) There exists $c_{21} = c_{21}(d, c_\omega) > 0$ such that for all $F, G \in C^1_b(S)$ and $t > 0$,
\[ \left| \text{Cov}_\mu(F(\phi_0), G(\phi_t)) \right| \leq \frac{c_{21}}{(t + 1)^{\frac{d}{2} - 1}} \sum_{x, y \in \mathbb{Z}^d} \mathbb{E}_\mu \left[ \left( \frac{\partial F}{\partial \phi(x)}(\phi_0) \right)^2 \right]^{\frac{1}{2}} \mathbb{E}_\mu \left[ \left( \frac{\partial G}{\partial \phi(y)}(\phi_0) \right)^2 \right]^{\frac{1}{2}}. \]

(ii) $\mathbb{E}_\mu[\phi_0(0)^2] < \infty$.

Proof: (i) follows by the Cauchy-Schwarz inequality applied to (5.10) together with Lemma 5.8 and stationarity of $\mu$. Further, by taking $t \downarrow 0$ we deduce from (i) and dominated convergence that $\sup_M \mathbb{E}_\mu[(\phi_0(0) \wedge M)^2] < \infty$, which gives (ii).

We are now in a position to prove our main result on the scaling limit of covariances of the random heights in the $\nabla \phi$-model.

Proof of Theorem 1.12. Recall that the existence of a stationary, shift-invariant, ergodic $\phi$-Gibbs measure $\mu$ has been shown in Theorem 5.6 above. Further, the environment $\omega$ defined in (5.9) satisfies Assumption 1.9 by the ergodicity of $\mu$. Note also that $\omega_\ell(e) \geq c_\omega > 0$ for any $e \in E_d$ and $t > 0$ by Assumption 5.1. Thus we may set $q = \infty$ in Assumption 4.2, which then reduces to (1.7). The Helffer-Sjöstrand relation (5.10) gives
\[ \text{Cov}_\mu(\phi_0(0), \phi_t(x)) = \int_0^\infty \mathbb{E}_\mu \left[ \rho^\mu(0, t + s, 0, x) \right] ds. \]

Now, applying Theorem 1.11,
\[ n^{-d-2} \text{Cov}_\mu(\phi_0(0), \phi_{n^2t}([nx])) = \int_0^\infty \mathbb{E}_\mu \left[ n^{-d} \rho^\mu(0, n^2(t + s), 0, [nx]) \right] ds \]
\[ \xrightarrow{n \to \infty} \int_0^\infty k_{t+s}(x) ds, \]
which is the claim. Note that Theorem 1.11 gives uniform convergence of the integrand on any compact interval $[0, T]$ and Corollary 4.4 gives that $g(s) = cs^{-\frac{d}{2}}$ is a dominating function, integrable on $[T, \infty)$ since $d \geq 3$. Therefore, by dominated convergence we are justified in interchanging the limit and the integral.

Having proven Theorem 1.12 we finally provide polynomial moment bounds on the heights $\phi$ under any ergodic, shift-invariant, stationary $\phi$-Gibbs measure. This may be useful in verifying the moment condition of Theorem 1.12.

Proposition 5.10. Suppose $d \geq 3$. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be any ergodic, shift-invariant, stationary $\phi$-Gibbs measure. Then for all $p > 0$, $\mathbb{E}_\mu[|\phi_t(x)|^p] < \infty$ for any $x \in \mathbb{Z}^d$ and $t \geq 0$.

The proof will require the following comparison estimate for $\phi_t$ and $\phi_{t+n}^L$ where $L_n := [-n, n]^d \cap \mathbb{Z}^d$ for $n \in \mathbb{N}$.

Lemma 5.11. Let $\mu$ be a shift-invariant Gibbs measure. There exists a positive, symmetric, summable sequence $\alpha = (\alpha(x))_{x \in \mathbb{Z}^d}$ such that the following holds. There exist
constants $c_{22}, c_{23}, c_{24} > 0$ such that for any $n \in \mathbb{N}$, $t > 0$ and any bounded Lipschitz function $f$ on $l^2(\mathbb{Z}^d, \alpha)$,
\[
\mathbb{E}_\mu \left[ (f(\phi_t) - f(\phi_{t, L_n}^{L_n}))^2 \right] \leq c_{22} \|f\|_{\Lip, \alpha}^2 e^{c_{22}t - c_{24}n} \left( 1 + \mathbb{E}_\mu [\phi_0(0)^2] \right).
\]  
(5.13)

**Proof.** By the same arguments as the Ising model case in [26], see (4.2.14), (5.1.5) and the proof of Corollary 5.1.4 there, we have for any $c_{23} > 1 + 8dc_-$,
\[
\alpha(0) \mathbb{E}_\mu \left[ (f(\phi_t) - f(\phi_{t, L_n}^{L_n}))^2 \right] \leq e^{c_{23}t} \|f\|_{\Lip, \alpha}^2 \mathbb{E}_\mu \left[ \sum_{x \not\in L_n} \alpha \alpha(x) \phi_0(x)^2 + c|\alpha| \alpha(x) \right],
\]
where in our setting $\alpha := \sum_{k=0}^\infty (\sigma')^{-k} (\sigma')^k$ is constructed from the sequence $p(x) = c_- \mathbb{1}_{|x|=1}$ for $\sigma' > 2dc_-$. Then by the shift-invariance of $\mu$ and exponential decay of $\alpha$ and $\alpha \ast \alpha$ on $\mathbb{Z}^d$, we get (5.13) (see also [29, Section 1.1]). □

**Proof of Proposition 5.10.** Since $\mu$ is stationary and shift-invariant it suffices to show that $\mathbb{E}_\mu \left[ |\phi_0(0)|^p \right] < \infty$ for all $p > 0$. By Jensen’s inequality it is enough to consider $p > 2$. For any $M > 1$ let $f_M(\phi) := (|\phi(0)| \wedge M)^{p/2}$ which is Lipschitz continuous on $l^2(\mathbb{Z}^d, \alpha)$ with $\|f_M\|_{\Lip, \alpha}^2 \leq c M^{p-2}$. For arbitrary $t > 0$ and $n \in \mathbb{N}$,
\[
\mathbb{E}_\mu \left[ f_M(\phi_t)^2 \right] \leq \mathbb{E}_\mu \left[ (f_M(\phi_t) - f_M(\phi_{t, L_n}^{L_n}))^2 \right] + \mathbb{E}_\mu \left[ f_M(\phi_{t, L_n}^{L_n})^2 \right].
\]  
(5.14)

To control the first term on the right hand side of (5.14), we fix $\epsilon > 0$. As argued in [26, Theorem 5.1.3], for arbitrary $\lambda > 0$ we introduce an increasing sequence of boxes $L_n(t)$ such that $c_{25} t \leq n(t) \leq c_{25} (t + 1)$ where $c_{25} > 0$ is chosen such that $c_{23} t - c_{24} n(t) < -\lambda t$, with $c_{23}, c_{24}$ as in Lemma 5.11. Therefore, by (5.13) and Corollary 5.9-(ii), there exists $T_{\epsilon, M} > 0$ such that
\[
\mathbb{E}_\mu \left[ (f_M(\phi_t) - f_M(\phi_{t, L_n}^{L_n}))^2 \right] \leq c M^{p-2} \left( 1 + \mathbb{E}_\mu [\phi_0(0)^2] \right) e^{-\lambda t} \leq \epsilon
\]  
(5.15)
for all $t > T_{\epsilon, M}$. For the latter term in (5.14), the constant zero boundary condition allows us, via the DLR equation (5.5), to reduce the expectation to that over a finite Gibbs measure as defined in (5.4),
\[
\mathbb{E}_\mu \left[ f_M(\phi_{t, L_n}^{L_n})^2 \right] = \mathbb{E}_{\mu_{L_n}}^0 \left[ f_M(\phi_{t, L_n}^{L_n})^2 \right].
\]  
(5.16)

Now, the finite volume process $\phi_{L_n, 0}$ is stationary with respect to $\mu_{L_n}^0$ so by the Brascamp-Lieb inequality, as argued in Theorem 5.6,
\[
\sup_{n \in \mathbb{N}, M > 0, t > 0} \mathbb{E}_{\mu_{L_n}}^0 \left[ f_M(\phi_{t, L_n}^{L_n})^2 \right] = \sup_{n \in \mathbb{N}, M > 0} \mathbb{E}_{\mu_{L_n}}^0 \left[ f_M(\phi_{0, L_n}^{L_n})^2 \right] < \infty.
\]  
(5.17)

Substituting (5.17) and (5.15) into (5.14) gives
\[
\mathbb{E}_\mu \left[ (|\phi_t(0)| \wedge M)^p \right] < \epsilon + c,
\]  
(5.18)
for all $t > T_{\epsilon, M}$, with constant $c$ independent of $M$. However, $\phi_t$ is stationary with respect to $\mu$ so (5.18) in fact holds for all $t \geq 0$. We conclude by the monotone convergence theorem, letting $M \uparrow \infty$, that $\mathbb{E}_\mu \left[ |\phi_t(0)|^p \right] < \infty.$ □
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