Pöschl-Teller Hamiltonian: Gazeau-Klauder type coherent states, related statistics and geometry

Mahouton Norbert Hounkonnou,¹,a) Sama Arjika,¹,b) and Ezinvi Baloitcha¹,c)

University of Abomey-Calavi, International Chair in Mathematical Physics and Applications (ICMPA–UNESCO Chair), 072 B.P. 50 Cotonou, Republic of Benin

(Dated: 22 May 2014)

This work mainly addresses a construction of Gazeau-Klauder type coherent states for a Pöschl-Teller model. Relevant characteristics are investigated. Induced geometry and statistics are studied. Then the Berezin - Klauder - Toeplitz quantization of the classical phase space observables is presented.

¹Electronic mail: norbert.hounkonnou@cipma.uac.bj
b)Electronic mail: rjksama2008@gmail.com
c)Electronic mail: ezinvi.baloitcha@cipma.uac.bj
I. INTRODUCTION

The search for exactly solvable models remains in the core of today research interest in quantum mechanics. A reference list of exactly solvable one-dimensional problems (harmonic oscillator, Coulomb, Morse, Pöschl-Teller potentials, etc.) obtained by an algebraic procedure, namely by a differential operator factorization methods, can be found in references therein. This technique, introduced long ago by Schrödinger, was analyzed in depth by Infeld and Hull, who made an exhaustive classification of factorizable potentials. It was reproduced rather recently in supersymmetric quantum mechanics (SUSY QM) approach, initiated by Witten and was immediately applied to the hydrogen potential. This approach gave many new exactly solvable potentials which were obtained as superpartners of known exactly solvable models. Later on, it was noticed by Witten the possibility of arranging the Schrödinger’s Hamiltonians into isospectral pairs called supersymmetric partners. The resulting supersymmetric quantum mechanics revived the study of exactly solvable Hamiltonians.

SUSY QM is also used for the description of hidden symmetries of various atomic and nuclear physical systems. Besides, it provides a theoretical laboratory for the investigation of algebraic and dynamical problems in supersymmetric field theory. The simplified setting of SUSY helps to analyze the problem of dynamical SUSY breaking at full length and to examine the validity of the Witten index criterion. In, it was shown that the reflectionless Pöschl-Teller system possesses a hidden bosonized nonlinear (higher order) supersymmetry. This observation was developed further in, where it was found that due to a hidden nonlinear SUSY, the usually super-extended systems possess a much more reach structure than it is usually thought. In, a nonlinear SUSY of reflectionless PT systems was explained using the ideas of AdS/CFT holography and Aharonov-Bohm effect. Exotic nonlinear (higher order) SUSY in pairs of mutually shifted reflectionless PT systems was studied in. Its relation to kink-antikink crystal appearing in Gross-Neveu model was studied in and. Extension of such class of the systems with higher order supersymmetry for the case of PT-symmetry was investigated recently in and. Besides, in a recent paper, an exotic nonlinear (higher order) supersymmetry was investigated in a much more general class of soliton systems. Recently, Bergeron et al developed the mathematical aspects that have been left apart in (proof of the resolution of unity, detailed calculations of quantized ver-
sion of classical observables and mathematical study of the resulting operators: problems of
domains, self-adjointness or self-adjoint extensions). Some additional questions as asymp-
totic behavior were also studied. Moreover, extensions were discussed to a larger class of
Pöschl-Teller potentials.

This paper is organized as follows. In Section 2, we recall known results and some
intertwining relations. In Section 3, we build the associated Gazeau-Klauder coherent states
(GKCSs). Their main mathematical properties, i.e., the orthogonality, the normalizability,
the continuity in the labels and the resolution of the identity are investigated. Quantum
statistics and geometry of these states are studied in Section 4. Section 5 is devoted to the
Berezin - Klauder - Toeplitz quantization of classical phase space observables. We end with
a conclusion in Section 6.

II. THE PÖSCHL-TELLER HAMILTONIAN AND SUSY-QM
FORMALISM

In this section, for the clarity of our development, we briefly recall the Pöschl-Teller
Hamiltonian model presented inREF2 and summarize main results on the eigenvalue problem
and supersymmetry factorization method of the time independent Schrödinger equation.

A. The model

The physical system is described by the Hamiltonian

\[ H_{\nu,\beta} \phi := \left( -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{\varepsilon_0,\nu,\beta}(x) \right) \phi \quad \text{for} \quad \phi \in D_{H_{\nu,\beta}} \tag{1} \]

in a suitable Hilbert space \( \mathcal{H} = L^2([0, L], dx) \) endowed with the inner product defined by

\[ (u, v) = \int_0^L dx \bar{u}(x)v(x), \quad u, v \in \mathcal{H}, \ [0, L] \subset \mathbb{R} \tag{2} \]

where \( \bar{u} \) denotes the complex conjugate of \( u \). \( M \) is the particle mass and \( D_{H_{\nu,\beta}} \) is the domain
of definition of \( H_{\nu,\beta} \).

\[ V_{\varepsilon_0,\nu,\beta}(x) = \varepsilon_0 \left( \nu^2 + \frac{1}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right) \tag{3} \]

is the Pöschl-Teller potential; \( \varepsilon_0 \) is some energy scale, \( \nu \) and \( \beta \) are dimensionless parameters.
The one-dimensional second-order operator $H_{\nu,\beta}$ has singularities at the end points $x = 0$ and $x = L$ permitting to choose $\varepsilon_0 \geq 0$ and $\nu \geq 0$. Further, since the symmetry $x \to L - x$ corresponds to the parameter change $\beta \to -\beta$, we can choose $\beta \geq 0$. As assumed in \textsuperscript{2}, we consider the energy scale $\varepsilon_0$ as the zero point energy of the energy of the infinite well, i.e. $\varepsilon_0 = \hbar^2 \pi^2 / (2 ML^2)$ so that the unique free parameters of the problem remain $\nu$ and $\beta$ which will be always assumed to be positive. The case $\beta = 0$ corresponds to the symmetric repulsive potentials investigated in \textsuperscript{20}, while the case $\beta \neq 0$ leads to the Coloumb potential in the limit $L \to \infty$.

Let us define the operator $H_{\nu,\beta}$ with the action $-\frac{\hbar^2}{2M} \phi''(x) + \varepsilon_0 \left( \frac{\nu(\nu + 1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right) \phi(x)$ with the domain being the set of smooth functions with a compact support, $C^\infty(0, L)$. The Pöschl-Teller potential is in the limit point case at both ends $x = 0$ and $x = L$, if the parameter $\nu \geq 1/2$, and in the limit circle case at both ends if $0 \leq \nu < 1/2$. Therefore, the operator $H_{\nu,\beta}$ is essentially self-adjoint in the former case. The closure of $H_{\nu,\beta}$ is $H_{\nu,\beta}$, i.e., $D_{H_{\nu,\beta}} = D_{H_{\nu,\beta}}$ and its domain coincides with the maximal one, i.e.,

$$D_{H_{\nu,\beta}} = \left\{ \phi \in ac^2(0, L), \left[ -\frac{\hbar^2}{2M} \phi'' + \varepsilon_0 \left( \frac{\nu(\nu + 1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right) \phi \right] \in H \right\},$$

where $ac^2(0, L)$ denotes the absolutely continuous functions with absolutely continuous derivatives. As mentioned in \textsuperscript{2}, a function of this domain satisfies Dirichlet boundary conditions and in the range of considered $\nu$, the deficiency indices of $H_{\nu,\beta}$ is $(2, 2)$ indicating that this operator is no longer essentially self-adjoint but has a two-parameter family of self-adjoint extensions indeed. As in \textsuperscript{2}, we will restrict only to the extension described by Dirichlet boundary conditions, i.e.,

$$D_{H_{\nu,\beta}} = \left\{ \phi \in ac^2(0, L), \right\}$$

where $D_{H_{\nu,\beta}}$ is dense in $H$ since $H^{2,2}(0, L) \supset D_{H_{\nu,\beta}}$ and $H_{\nu,\beta}$ is self-adjoint, where $H^{m,n}(0, L)$ is the Sobolev space of indice $(m, n)$. Later on, we use the dense domain

$$D_H = \left\{ \phi \in AC^2(0, L), \varepsilon_0 \left( \frac{\nu(\nu + 1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right) \phi \in H \right\},$$

and $AC(0, L)$ is defined as

$$AC(0, L) = \left\{ \phi \in ac(0, L) : \phi' \in H \right\}.$$
B. Eigenvalues and eigenfunctions

The eigenvalues $E_{n}(\nu, \beta)$ and functions $\phi_{n}(\nu, \beta)$ solving the Sturm-Liouville differential equation (1), i.e., $H_{\nu, \beta} \phi_{n}(\nu, \beta) = E_{n}(\nu, \beta) \phi_{n}(\nu, \beta)$, are given by

$$E_{n}(\nu, \beta) = \varepsilon_{0} \left( (n + \nu + 1)^{2} - \frac{\beta^{2}}{(n + \nu + 1)^{2}} \right)$$

and

$$\phi_{n}(\nu, \beta)(x) = K_{n}^{(\nu, \beta)} \sin^{\nu+n+1} \left( \frac{\pi x}{L} \right) \exp \left( -\frac{\beta \pi x}{L(n + \nu + 1)} \right) P_{n}^{(a_{n}, \bar{a}_{n})} \left( i \cot \frac{\pi x}{L} \right)$$

respectively, where $n \in \mathbb{N}$, $a_{n} = -(n + \nu + 1) + \frac{i\beta}{n+\nu+1}$, $P_{n}^{(a_{n}, \bar{a}_{n})}(z)$ are the Jacobi polynomials and $K_{n}^{(\nu, \beta)}$ is a normalization constant giving by

$$K_{n}^{(\nu, \beta)} = 2^{n+\nu+1} L^{-\frac{1}{2}} T(n; \nu, \beta) O^{-\frac{1}{2}}(n; \nu, \beta) \exp \left( \frac{\beta \pi}{2(n + \nu + 1)} \right)$$

where

$$O(n; \nu, \beta) = \sum_{k=0}^{n} (-n, -2\nu - n - 1)_{k} \frac{(\nu - n - \frac{i\beta}{\nu+n+1})_{k} \Gamma(n + \nu + 2 - k + \frac{i\beta}{\nu+n+1})}{\Gamma(n + \nu + 2 - k + 3)}$$

$$\times \sum_{s=0}^{n} (-n, -2\nu - n - 1)_{s} \frac{(\nu - n + \frac{i\beta}{\nu+n+1})_{s} \Gamma(n + \nu + 2 - s - \frac{i\beta}{\nu+n+1})}{\Gamma(n + \nu + 2 - s - \frac{i\beta}{\nu+n+1})}$$

and

$$T(n; \nu, \beta) = n! \left| -n - \nu + \frac{i\beta}{n+\nu+1} \right|^{-1}.$$

For details on the $K_{n}^{(\nu, \beta)}$, see Appendix A.

For $n = 0$, one can retrieve

$$\phi_{0}^{(\nu, \beta)}(x) = \frac{2^{\nu+1}}{\sqrt{L \times \Gamma(2\nu + 3)}} \sin^{\nu+1} \left( \frac{\pi x}{L} \right) \exp \left\{ \frac{\beta \pi}{\nu + 1} \left( \frac{1}{2} - \frac{x}{L} \right) \right\}.$$

C. Factorization method, shape invariance of the Pöschl-Teller Hamiltonian and intertwining relations

We assume the ground state $\phi_{0}^{(\nu, \beta)}$ and the energy $E_{0}^{(\nu, \beta)}$ are known. By using a Darboux factorization method of the Hamiltonian, one can define the differential operators
\[ A_{\nu, \beta}, \quad A_{\nu, \beta}^\dagger \] factorizing the Pöschl-Teller Hamiltonian \( H_{\nu, \beta} \) and the associated super potential \( W_{\nu, \beta} \) as follows

\[
H_{\nu, \beta} := \frac{1}{2M} A_{\nu, \beta} A_{\nu, \beta}^\dagger + E^{(\nu, \beta)}_0,
\]

where the differential operators \( A_{\nu, \beta} \) and \( A_{\nu, \beta}^\dagger \) are defined as

\[
A_{\nu, \beta} := \hbar \frac{d}{dx} + W_{\nu, \beta}(x), \quad A_{\nu, \beta}^\dagger := -\hbar \frac{d}{dx} + W_{\nu, \beta}(x),
\]

acting in the domains

\[
D_{A_{\nu, \beta}} = \{ \phi \in ac(0, L) | (\hbar \phi' + W_{\nu, \beta} \phi) \in \mathcal{H} \},
\]

and

\[
D_{A_{\nu, \beta}^\dagger} = \{ \phi \in ac(0, L) | \exists \tilde{\phi} \in \mathcal{H} : [\hbar \psi(x) \phi(x)]_0^L = 0, \quad \langle A_{\nu, \beta} \psi, \phi \rangle = \langle \psi, \tilde{\phi} \rangle, \quad \forall \psi \in D_{A_{\nu, \beta}} \},
\]

where \( A_{\nu, \beta}^\dagger \phi = \tilde{\phi} \). The operator \( A_{\nu, \beta}^\dagger \) is the adjoint of \( A_{\nu, \beta} \). Besides, considering their common restriction

\[
D_A = \{ \phi \in AC(0, L) | W_{\nu, \beta} \phi \in \mathcal{H} \},
\]

we have \( A_{\nu, \beta} \uparrow D_A = A_{\nu, \beta} \) and \( A_{\nu, \beta}^\dagger \uparrow D_A = A_{\nu, \beta}^\dagger \) (for more details, see \ref{2}). The super-potential \( W_{\nu, \beta} \) is given by

\[
W_{\nu, \beta}(x) := -\hbar \frac{[\phi_{0}^{(\nu, \beta)}(x)]'}{\phi_{0}^{(\nu, \beta)}(x)} = -\frac{\pi \hbar}{L} \left( (\nu + 1) \cot \frac{\pi x}{L} - \frac{\beta}{\nu + 1} \right).
\]

The superpartner Hamiltonian \( H^{(1)}_{\nu, \beta} \) of \( H_{\nu, \beta} \) is obtained by permuting the operators \( A_{\nu, \beta} \) and \( A_{\nu, \beta}^\dagger \) and we get

\[
H^{(1)}_{\nu, \beta} := \frac{1}{2M} A_{\nu, \beta} A_{\nu, \beta}^\dagger + E^{(\nu, \beta)}_0 = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V^{(1)}_{\nu, \beta}(x),
\]

where the partnerpotential \( V^{(1)}_{\nu, \beta} \) of \( V_{\nu, \beta} \) is defined by the relation

\[
V^{(1)}_{\nu, \beta}(x) := \frac{1}{2M} \left( W^2_{\nu, \beta}(x) + W'_{\nu, \beta}(x) \right) + E^{(\nu, \beta)}_0.
\]

Performing \( \ref{19} \), we arrive at the following relation

\[
V^{(1)}_{\nu, \beta}(x) \equiv V_{\nu+1, \beta}(x).
\]

Therefore, the superpartner Hamiltonian \( \ref{18} \) becomes

\[
H^{(1)}_{\nu, \beta} \equiv H_{\nu+1, \beta}.
\]
This relation specifies that Pöschl-Teller Hamiltonians are shape invariant, i.e.,

\[ A_{\nu,\beta} A_{\nu,\beta}^\dagger = A_{\nu+1,\beta}^\dagger A_{\nu+1,\beta} + 2M(E_0^{(\nu+1,\beta)} - E_0^{(\nu,\beta)}). \]  

(22)

In the equation

\[ H_{\nu,\beta}^{(1)} \phi_n^{(1,\nu,\beta)} = E_n^{(1,\nu,\beta)} \phi_n^{(1,\nu,\beta)}, \]

(23)

the eigenfunction \( \phi_n^{(1,\nu,\beta)} \) and the eigenvalue \( E_n^{(1,\nu,\beta)} \) of \( H_{\nu,\beta}^{(1)} \) are related to those of \( H_{\nu,\beta} \), i.e., \( E_n^{(1,\nu,\beta)} = E_n^{(\nu,\beta)} \), and

\[ |\phi_n^{(\nu+1,\beta)}\rangle = \frac{A_{\nu,\beta} |\phi_n^{(\nu,\beta)}\rangle}{\sqrt{2M(E_{n+1}^{(\nu,\beta)} - E_0^{(\nu,\beta)})}} \quad |\phi_{n+1}^{(\nu,\beta)}\rangle = \frac{A_{\nu,\beta}^\dagger |\phi_n^{(\nu+1,\beta)}\rangle}{\sqrt{2M(E_{n+1}^{(\nu,\beta)} - E_0^{(\nu,\beta)})}} \]

(24)

satisfying

\[ \langle \phi_m^{(\nu,\beta)} | \phi_n^{(\nu,\beta)} \rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{\infty} |\phi_n^{(\nu,\beta)}\rangle \langle \phi_n^{(\nu,\beta)}| = I. \]

(25)

Now let us introduce the positive sequence \( \eta_n^{(\nu,\beta)} = \varepsilon_0^{-1} \left( E_n^{(\nu,\beta)} - E_0^{(\nu,\beta)} \right) \). Then, the representations of the operators \( A_{\nu,\beta} \) and \( A_{\nu,\beta}^\dagger \) are given by

\[ A_{\nu,\beta} = \sqrt{2M\varepsilon_0} \sum_{n=0}^{\infty} \sqrt{\eta_n^{(\nu,\beta)}} |\phi_n^{(\nu+1,\beta)}\rangle \langle \phi_n^{(\nu,\beta)}| \]  

(26)

and

\[ A_{\nu,\beta}^\dagger = \sqrt{2M\varepsilon_0} \sum_{n=0}^{\infty} \sqrt{\eta_n^{(\nu,\beta)}} |\phi_n^{(\nu+1,\beta)}\rangle \langle \phi_n^{(\nu,\beta)}|, \]

(27)

respectively. Any eigenstate \( \phi_n^{(\nu,\beta)}(x) \) \((n = 1, 2, \cdots)\) of \( H_{\nu,\beta} \) may then be constructed from the ground state \( \phi_0^{(\nu+n,\beta)}(x) \) through the repeated application of \( A_{\nu+k,\beta}^\dagger \), \( k = 0, 1, 2, \cdots, n-1 \) operators defined in terms of the superpotential, i.e.,

\[ \phi_n^{(\nu,\beta)}(x) \propto A_{\nu,\beta}^\dagger A_{\nu+1,\beta}^\dagger \cdots A_{\nu+n-1,\beta}^\dagger \phi_0^{(\nu+n,\beta)}(x). \]

(28)

The operators \( A_{\nu,\beta} \) and \( A_{\nu,\beta}^\dagger \) do not commute with the Pöschl-Teller Hamiltonian \( H_{\nu,\beta} \), but satisfy the intertwining relations

\[ H_{\nu,\beta} A_{\nu,\beta}^\dagger = A_{\nu,\beta}^\dagger H_{\nu,\beta+1,\beta}, \quad A_{\nu,\beta} H_{\nu,\beta} = H_{\nu+1,\beta} A_{\nu,\beta}. \]

(29)

More generally,

\[ H_{\nu,\beta} B_n = B_n H_{\nu,\beta+n,\beta}, \quad B_n^\dagger H_{\nu,\beta} = H_{\nu,\beta+n,\beta} B_n^\dagger, \]

(30)
where $B_n$ and $B_n^\dagger$ are operators of degree $n$, i.e.,

$$B_n := A_{\nu,\beta} A_{\nu+1,\beta} \cdots A_{\nu+n-1,\beta}, \quad B_n^\dagger := A_{\nu+n-1,\beta} \cdots A_{\nu+1,\beta} A_{\nu,\beta}. \quad (31)$$

Therefore, for any positive integers $n, m$, the following result holds:

$$B_m^\dagger B_n = (2M)^n m \Lambda_{n,\nu,\beta} \prod_{k=0}^{n-1} \left( H_{\nu+n,\beta} - E_{0}^{(\nu+k,\beta)} \right) \quad (32)$$

if $n < m$,

$$B_m^\dagger B_n = (2M)^m \prod_{k=0}^{m-1} \left( H_{\nu+m,\beta} - E_{0}^{(\nu+k,\beta)} \right) \Theta_{n,\nu,\beta}^m \quad (33)$$

if $n > m$, where the operators $m \Lambda_{n,\nu,\beta}$ and $\Theta_{n,\nu,\beta}^m$ are given by

$$m \Lambda_{n,\nu,\beta} := A_{\nu+m-1,\beta} A_{\nu+m-2,\beta} \cdots A_{\nu+n,\beta}, \quad \Theta_{n,\nu,\beta}^m := A_{\nu+n,\beta}^\dagger A_{\nu+m+1,\beta} \cdots A_{\nu+n-1,\beta}. \quad (34)$$

In particular, for $n = m$, we have

$$B_n B_n^\dagger = (2M)^n \prod_{k=0}^{n-1} \left( H_{\nu+k,\beta} - E_{0}^{(\nu+k,\beta)} \right), \quad B_n^\dagger B_n = (2M)^n \prod_{k=0}^{n-1} \left( H_{\nu+n,\beta} - E_{0}^{(\nu+k,\beta)} \right). \quad (35)$$

The operators $m \Lambda_{n,\nu,\beta}$ and $\Theta_{n,\nu,\beta}^m$ satisfy the following identities

$$m \Lambda_{n,\nu,\beta} m \Lambda_{n,\nu,\beta} = (2M)^{m-n} \prod_{k=n}^{m-1} \left( H_{\nu+n,\beta} - E_{0}^{(\nu+k,\beta)} \right), \quad (36)$$

$$\Theta_{n,\nu,\beta}^m \Theta_{n,\nu,\beta}^m = (2M)^{n-m} \prod_{k=m}^{n-1} \left( H_{\nu+m,\beta} - E_{0}^{(\nu+k,\beta)} \right). \quad (37)$$

Indeed, from (28) and (31), one can see that the actions of the operators $B_n^\dagger$ and $B_n$ on the normalized eigenfunctions $\phi_{\nu,\beta}^{(v,\beta)}$ and $\phi_{0}^{(\nu+n,\beta)}$ of $H_{\nu,\beta}$ are given by

$$B_n^\dagger \phi_{\nu,\beta}^{(v,\beta)}(x) = (\pi \hbar L)^n M^{1/2}(n; \nu, \beta) \phi_{0}^{(\nu+n,\beta)}(x) \quad (38)$$

and

$$B_n \phi_{0}^{(\nu+n,\beta)}(x) = (\pi \hbar L)^n M^{1/2}(n; \nu, \beta) \phi_{n}^{(\nu,\beta)}(x), \quad (39)$$

respectively, where

$$M(n; \nu, \beta) = \prod_{k=0}^{n-1} \left( E_n^{(\nu,\beta)} - E_k^{(\nu,\beta)} \right). \quad (40)$$
The mean values of the operators $B_n B^\dagger_n$, $B^\dagger_n B_n$, $m\Lambda^\dagger_{n,\nu,\beta} n\Lambda_{n,\nu,\beta}$ and $\Theta^m_{n,\nu,\beta} \Theta^{m\dagger}_{n,\nu,\beta}$ in the states $|\phi^{(\nu,\beta)}_n\rangle$ are derived by using (35). We obtain

$$\langle B_n B^\dagger_n \rangle_{\phi^{(\nu,\beta)}_n} = (\pi \hbar L^{-1})^{2n} M(n; \nu, \beta), \quad \langle B^\dagger_n B_n \rangle_{\phi^{(\nu,\beta)}_n} = (\pi \hbar L^{-1})^{2n} T(n; \nu, \beta),$$

(41)

$$\langle m\Lambda^\dagger_{n,\nu,\beta} n\Lambda_{n,\nu,\beta} \rangle_{\phi^{(\nu,\beta)}_n} = (\hbar \pi L^{-1})^{2m-2n} \prod_{k=n}^{m-1} \left( E^{(\nu,\beta)}_{2n} - E^{(\nu,\beta)}_{k} \right) \quad n < m,$$

(42)

and

$$\langle \Theta^m_{n,\nu,\beta} \Theta^{m\dagger}_{n,\nu,\beta} \rangle_{\phi^{(\nu,\beta)}_n} = (\hbar \pi L^{-1})^{2n-2m} \prod_{k=m}^{n-1} \left( E^{(\nu,\beta)}_{n+m} - E^{(\nu,\beta)}_{k} \right) \quad n > m,$$

(43)

where

$$T(n; \nu, \beta) = \prod_{k=0}^{n-1} \left( E^{(\nu,\beta)}_{2n} - E^{(\nu,\beta)}_{k} \right)$$

(44)

and

$$\langle A^{(\nu,\beta)}_\nu \rangle_{\phi^{(\nu,\beta)}_n} := \int_0^L dx \, \phi^{(\nu,\beta)}_n(x) A^{(\nu,\beta)}_\nu \phi^{(\nu,\beta)}_n(x).$$

(45)

In the sequel, the parameter $\beta = 0$.

### III. Gazeau-Klauder Coherent States (GKCSS)

Without loss of generality, let us consider the Pöschl-Teller Hamiltonian (1) with the parameter $\beta = 0$. The resulting one-dimensional quantum mechanical system has a finite or infinite number of discrete energy levels $\mathcal{E}^{(\nu)}_n$, $n \in \mathbb{N}$,

$$\mathcal{E}^{(\nu)}_n := 2M(E^{(\nu)}_n - E^{(\nu)}_0)$$

(46)

chosen by adjusting the constant part of the Pöschl-Teller Hamiltonian $H_{\nu,0}$, and satisfying $\mathcal{E}^{(\nu)}_n < \mathcal{E}^{(\nu)}_{n+1}$. Then the resulting positive definite Hamiltonian $H_\nu$ is expressed in a factorized form as follows

$$H_\nu := A^\dagger_\nu A_\nu,$$

(47)

where the differential operators $A_\nu$ and $A^\dagger_\nu$ are defined as

$$A_\nu := A_{\nu,0} \quad \text{and} \quad A^\dagger_\nu := A^\dagger_{\nu,0}$$

(48)
and their representations are given, respectively, by

\[ A_\nu = \sum_{n=0}^{\infty} \sqrt{\mathcal{E}_n^{(\nu)}} |\phi_n^{(\nu+1,0)}\rangle \langle \phi_n^{(\nu,0)}|, \]

(49)

\[ A_\nu^\dagger = \sum_{n=0}^{\infty} \sqrt{\mathcal{E}_n^{(\nu)}} |\phi_n^{(\nu,0)}\rangle \langle \phi_n^{(\nu+1,0)}|, \]

(50)

and

\[ \mathcal{H}_\nu = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\nu)} |\phi_n^{(\nu,0)}\rangle \langle \phi_n^{(\nu,0)}|. \]

(51)

The eigenvalues \( \mathcal{E}_n^{(\nu)} := \mathcal{E}^{(\nu)}(n) \) that solve the eigenvalue problem related to (47), i.e., \( \mathcal{H}_\nu |\nu\rangle = \mathcal{E}^{(\nu)}(n) |\nu\rangle \), are given by

\[ \mathcal{E}^{(\nu)}(n) = 2M \varepsilon_0 n(n + 2\nu + 2). \]

(52)

Let \( \mathcal{F} \) be the Fock space spanned by \( \{|n\rangle_\nu, n = 0, 1, \ldots, \} \) satisfying the useful conditions

\[ \nu \langle m|n\rangle_\nu = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle_\nu \langle n| = 1 \]

(53)

such that

\[ |n\rangle_\nu := Z_{n,\nu} \sin^{\nu+1} \left( \frac{\pi x}{L} \right) C_n^{\nu+1} \left( \cos \frac{\pi x}{L} \right), \quad n \in \mathbb{N}, \]

(54)

where \( C_n^{(\theta)}(z) \) is a Gegenbauer polynomial and \( Z_{n,\nu} \) is a normalization constant. We assume that there exists a real number \( \gamma \) such that the actions of the \( \gamma \)-depending annihilation and creation-like operators \( \gamma A_\nu \) and \( \gamma A_\nu^\dagger \) on (51) are given by

\[ \gamma A_\nu |n\rangle_\nu := \sqrt{\mathcal{E}^{(\nu)}(n)} e^{i\gamma(\mathcal{E}^{(\nu)}(n)-\mathcal{E}^{(\nu)}(n-1))} |n - 1\rangle_\nu \]

(55)

and

\[ \gamma A_\nu^\dagger |n\rangle_\nu := \sqrt{\mathcal{E}^{(\nu)}(n + 1)} e^{-i\gamma(\mathcal{E}^{(\nu)}(n+1)-\mathcal{E}^{(\nu)}(n))} |n + 1\rangle_\nu, \]

(56)

respectively. Indeed,

\[ \gamma A_\nu \gamma A_\nu^\dagger |n\rangle_\nu = \mathcal{E}^{(\nu)}(n + 1) |n\rangle_\nu, \quad \gamma A_\nu^\dagger \gamma A_\nu |n\rangle_\nu = \mathcal{E}^{(\nu)}(n) |n\rangle_\nu. \]

(57)

Formally,

\[ N|n\rangle_\nu = n|n\rangle_\nu. \]

(58)
The relations in (57) lead to
\[ \gamma A_\nu \gamma A_\nu^\dagger = \mathcal{E}^{(\nu)}(N + 1), \quad \gamma A_\nu^\dagger \gamma A_\nu = \mathcal{E}^{(\nu)}(N) \] (59)
and we arrive at the following set of non-null commutation relation of the algebra of \( \mathcal{H}_\nu \)
\[ \gamma A_\nu \gamma A_\nu^\dagger - \gamma A_\nu^\dagger \gamma A_\nu = f_\nu(N), \quad [N, \gamma A_\nu] = - \gamma A_\nu, \quad [N, \gamma A_\nu^\dagger] = \gamma A_\nu^\dagger, \] (60)
\[ [\mathcal{H}_\nu, A_\nu] = - \gamma A_\nu f_\nu(N - 1), \quad [\mathcal{H}_\nu, A_\nu^\dagger] = \gamma A_\nu^\dagger f_\nu(N) \] (61)
where the function \( f_\nu(x) = 2M_0(2x + 2\nu + 3) \). By setting
\[ \gamma a_\nu := \gamma A_\nu \sqrt{\frac{N}{\mathcal{E}^{(\nu)}(N)}} e^{-i\gamma(\mathcal{E}^{(\nu)}(N) - \mathcal{E}^{(\nu)}(N - 1))}, \quad \gamma a_\nu^\dagger := e^{i\gamma(\mathcal{E}^{(\nu)}(N) - \mathcal{E}^{(\nu)}(N - 1))} \sqrt{\frac{N}{\mathcal{E}^{(\nu)}(N)}} \gamma A_\nu^\dagger, \] (62)
the algebra \( (60) \) becomes the Weyl-Heisenberg algebra, i.e.,
\[ \gamma a_\nu \gamma a_\nu^\dagger - \gamma a_\nu^\dagger \gamma a_\nu = 1, \quad [N, \gamma a_\nu] = - \gamma a_\nu, \quad [N, \gamma a_\nu^\dagger] = \gamma a_\nu^\dagger, \quad \gamma a_\nu^\dagger \gamma a_\nu := N. \] (63)

**Definition III.1** The GKCSs associated with the annihilation operator \( (48) \) are defined as follows\(^{25,27,33,34} \)
\[ |z, \gamma\rangle_\nu := N_\nu^{-1/2} (|z|^2)^{\nu} \sum_{n=0}^{\infty} \frac{z^n e^{-i\gamma \mathcal{E}^{(\nu)}(n)}}{\sqrt{\rho_n}} |n\rangle_\nu, \quad z \in D_R, \] (64)
where the normalization factor \( N_\nu(x) \) is given by
\[ N_\nu(x) := \sum_{n=0}^{\infty} \frac{1}{(2\nu + 3)_n} \left( \frac{x(2M_0)^{-1}}{n!} \right)^n = \frac{\Gamma(2\nu + 3)}{(x/2M_0)^{2\nu+2}} I_{2\nu+2} \left( \frac{x}{M_0} \right), \quad x = |z|^2 \] (65)
with
\[ \rho_n := \mathcal{E}^{(\nu)}(n)!, \quad \mathcal{E}^{(\nu)}(0)! = 1, \quad D_R = \{ z \in \mathbb{C} : |z| < R \}. \] (66)
\( R = \limsup_{n \to \infty} \sqrt[2n]{\rho_n} \) is the radius of convergence of the series \( (65) \) and \( I_\nu(x) \) is the modified Bessel function of order \( \nu \) (for more details, see\(^{28} \)). The GKCSs \( (64) \) can be re-expressed in the following form
\[ |z, \gamma\rangle_\nu = \frac{|z|^{2\nu+2}}{(2M_0)^{\nu+1} \sqrt{I_{2\nu+2}(|z|^2/M_0)}} \sum_{n=0}^{\infty} \frac{e^{-i\gamma \mathcal{E}^{(\nu)}(n)}}{\sqrt{\Gamma(n + 1) \Gamma(2\nu + 3 + n)}} \left( \frac{z}{2M_0} \right)^n |n\rangle_\nu. \] (67)
We now aim at showing that the coherent states \( (67) \) satisfy the Klauder’s criteria\(^{29,30} \). To this end let us first prove the following lemma.
Lemma III.1

\[
\int_{c-\beta-i\infty}^{c-\beta+i\infty} \Gamma(\beta + x)\Gamma(x)a^{-x}dx = 4i\pi a^{\beta/2}K_\beta(2a^{1/2}), \quad \text{Re}(\beta) < c, \ a \in \mathbb{R}
\]  

(68)

where \( K_m(x) \) is the modified Bessel function of the second kind\(^{28}\)

\[
K_m(x) = \frac{\pi}{2} \frac{I_{-m}(x) - I_m(x)}{\sin(m\pi)}
\]  

(69)

and \( I_m(|z|) \) is the modified Bessel function of the first kind\(^{28}\)

\[
I_m(2|z|) = \sum_{n=0}^{+\infty} \frac{|z|^{n+m}}{\Gamma(n+1)\Gamma(n+m+1)}.
\]  

(70)

**Proof.** From the formula\(^{31}\) (see eq. EH II 83(34) on page 685) for \( z = i\sqrt{t} \), we have

\[
\int_{c-\beta-i\infty}^{c-\beta+i\infty} \Gamma(-\beta - s)\Gamma(-s) \left( \frac{t}{4} \right)^{\beta/2+s} ds = -2\pi^2 e^{i\pi\beta/2} H_\beta^{(1)}(it^{1/2}).
\]  

(71)

By setting \(-s = x + \beta \) and \( a = t/4 \), the latter formula becomes

\[
\int_{c-\beta-i\infty}^{c-\beta+i\infty} \Gamma(\beta + x)\Gamma(x)a^{-x}dx = -2\pi^2 e^{i\pi\beta/2}a^{\beta/2}H_\beta^{(1)}(2ia^{1/2}).
\]  

(72)

The proof is achieved by replacing \( H_\beta^{(1)}(it) = 2\pi i^{-\beta-1}K_\beta(t) \). \( \square \)

**Proposition III.1** The GKCSs defined in \([67]\)

1. are not orthogonal to each other, i.e.,

\[
\nu \langle z', \gamma | z, \gamma \rangle_\nu \neq \delta(z - z'),
\]  

(73)

2. are normalized,

\[
\nu \langle z, \gamma | z, \gamma \rangle_\nu = 1,
\]  

(74)

3. are continuous in their labels, i.e.,

\[
\forall z, z' \in \mathbb{C}, \quad \| |z, \gamma\rangle_\nu - |z', \gamma\rangle_\nu \|^2 \to 0 \text{ as } |z - z'| \to 0,
\]  

(75)
4. solve the unity, i.e.,
\[ \int_{\mathbb{C}} d\mu_{\nu}(\lvert z \rvert^2) \langle z, \gamma \rangle_{\nu\nu} \langle z, \gamma \rangle = 1, \] (76)
where the measure \( d\mu_{\nu}(\lvert z \rvert^2) \) is given by
\[ d\mu_{\nu}(\lvert z \rvert^2) = \frac{(2M\varepsilon_0)^\nu}{\lvert z \rvert^{2\nu+2}} I_{2\nu+2} \left( \frac{\lvert z \rvert^2}{M\varepsilon_0} \right) K_{2\nu+2} \left( \lvert z \rvert \sqrt{\frac{2}{M\varepsilon_0}} \right) \frac{d^2z}{\pi}. \] (77)

5. are temporarily stable, i.e.,
\[ e^{-itH_\nu} \langle z, \gamma \rangle_\nu = \left| z, \gamma + t \right>_\nu \quad t \in \mathbb{R}, \] (78)

6. satisfy the action identity, i.e.,
\[ \nu \langle \gamma, z | H_\nu | z, \gamma \rangle_\nu = \lvert z \rvert^2. \] (79)

Proof.
• The product of two GKCSs \( \langle z, \gamma \rangle_\nu \) and \( \langle z', \gamma \rangle_\nu \) is given by
\[ \nu \langle z', \gamma | z, \gamma \rangle_\nu = \left[ N_\nu(\lvert z \rvert^2) N_\nu(\lvert z' \rvert^2) \right]^{-1/2} \sum_{n=0}^{\infty} \frac{((2M\varepsilon_0)^{-1} z \bar{z}')^n}{(2\nu + 3)_n n!} I_{2\nu+2} \left( \frac{z \bar{z}}{M\varepsilon_0} \right) \frac{I_{2\nu+2} \left( \frac{z' \bar{z}}{M\varepsilon_0} \right)}{\sqrt{I_{2\nu+2} \left( \frac{\lvert z \rvert^2}{M\varepsilon_0} \right) I_{2\nu+2} \left( \frac{\lvert z' \rvert^2}{M\varepsilon_0} \right)}}. \] (80)
Therefore, the GKCSs (64) are not orthogonal but for \( z' = z \), the product (80) is equal to 1.

- \[ ||\langle z, \gamma \rangle_\nu - \langle z', \gamma \rangle_\nu||^2 = 2 \left( 1 - Re(\nu \langle z', \gamma | z, \gamma \rangle_\nu) \right). \] (81)
So, \( ||\langle z, \gamma \rangle_\nu - \langle z', \gamma \rangle_\nu||^2 \to 0 \) as \( \lvert z - z' \rvert \to 0 \), since \( \nu \langle z', \gamma | z, \gamma \rangle_\nu \to 1 \) as \( \lvert z - z' \rvert \to 0 \). The GKCSs (64) are continuous in their labels.

- To investigate the resolution of unity (76), we assume the existence of a positive weight \( \mu_{\nu}(\lvert z \rvert^2) \) such that the resolution of the identity reads
\[ \int_{\mathbb{C}} d\mu_{\nu}(\lvert z \rvert^2) \langle z, \gamma \rangle_{\nu\nu} \langle z, \gamma \rangle = \sum_{n=0}^{\infty} |n\rangle_{\nu\nu} \langle n| = 1. \] (82)
Upon passing to polar coordinates, \( z = re^{i\theta} \), \( d\mu(\nu) = \omega_{\nu}(|z|^2)d^2z \), \( d^2z = d(Rez)d(Imz) \) and integrating with respect to \( \theta \), the function \( \omega_{\nu}(x) \) is required to be in the form

\[
\omega_{\nu}(x) = \frac{1}{\pi} \tilde{\omega}_{\nu}(x) N_{\nu}(x), \quad x = r^2,
\]

where the function \( \tilde{\omega}_{\nu}(x) \) is to be determined from the equation

\[
\int_0^{\infty} x^n \tilde{\omega}_{\nu}(x) dx = \rho_n, \quad n = 0, 1, 2, \ldots.
\]

If \( n \) in (84) is extended to \( s - 1 \), where \( s \in \mathbb{C} \), then the problem can be formulated in terms of the Mellin and inverse Mellin transforms that have been extensively used in the context of various kinds of generalized CSs. By setting \( \rho_{\nu}(n) = \rho_n \), here \( \rho_{\nu}(s - 1) \) is the Mellin transform \( \mathcal{M}[\tilde{\omega}_{\nu}(x); s] \) of \( \tilde{\omega}_{\nu}(x) \), i.e.,

\[
\rho_{\nu}(s - 1) = \mathcal{M}[\tilde{\omega}_{\nu}(x); s] \equiv \int_0^{\infty} x^{s-1} \tilde{\omega}_{\nu}(x) dx,
\]

and \( \tilde{\omega}_{\nu}(x) \) is the inverse Mellin transform \( \mathcal{M}^{-1}[\rho_{\nu}(s - 1); x] \) of \( \rho_{\nu}(s - 1) \), i.e.,

\[
\tilde{\omega}_{\nu}(x) = \mathcal{M}^{-1}[\rho_{\nu}(s - 1); x] \equiv \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \rho_{\nu}(s - 1) ds = \frac{1}{4i\pi M\varepsilon_0 (2\nu + 3)} \int_{c-i\infty}^{c+i\infty} a^{-s} \Gamma(s) \Gamma(2\nu + 2 + s) ds,
\]

where \( a = x/2M\varepsilon_0 \). By using the Lemma III.1 for \( \beta = 2\nu + 2 \), we deduce that

\[
\tilde{\omega}_{\nu}(x) = \frac{2}{(2M\varepsilon_0)^{\nu/2} \Gamma(2\nu + 3)} K_{2\nu+2} ((2x/M\varepsilon_0)^{1/2})
\]

and

\[
d\omega_{\nu}(x) = \frac{2(2M\varepsilon_0)^{\nu}}{\pi x^{\nu+1}} I_{2\nu+2}(x/M\varepsilon_0) K_{2\nu+2}(\sqrt{2x/M\varepsilon_0}) dx.
\]

Since \( \mathcal{H}_\nu|n\rangle_\nu = \mathcal{E}(\nu)(n)|n\rangle_\nu \), the stability and action identity are obviously true.  \( \square \)

### IV. STATISTICS AND GEOMETRY OF THE GKCSS \( |z, \gamma\rangle_\nu \)

The conventional boson operators \( \gamma a_{\nu} \) and \( \gamma a_{\nu}^\dagger \) have their actions on the states \( |n\rangle_\nu \) given by

\[
\gamma a_{\nu} |n\rangle_\nu = \sqrt{n} |n - 1\rangle_\nu \quad \text{and} \quad \gamma a_{\nu}^\dagger |n\rangle_\nu = \sqrt{n + 1} |n + 1\rangle_\nu.
\]

Besides,

\[
(\gamma a_{\nu})^r |n\rangle_\nu = \sqrt{n! \over (n-r)!} |n - r\rangle_\nu, \quad 0 \leq r \leq n
\]

(89)
and
\[ (\gamma a_\nu^\dagger)^s |n\rangle_\nu = \sqrt{\frac{(n+s)!}{n!}} |n+s\rangle_\nu. \] (91)

### A. Statistics of the GKCSs $|z,\gamma\rangle_\nu$

**Proposition IV.1** The expectation value of $(\gamma a_\nu^\dagger)^s \gamma a_\nu^r$ in the coherent states $|z,\gamma\rangle_\nu$ is given by

\[ \langle (\gamma a_\nu^\dagger)^s \gamma a_\nu^r \rangle = z^s z^r S_{\nu,\gamma}^{(s,r)}(|z|^2), \quad s, r = 0, 1, 2, \ldots \] (92)

where
\[ S_{\nu,\gamma}^{(s,r)}(x) = \frac{1}{N_\nu(x)} \sum_{m=0}^{\infty} e^{i\gamma (E^{(\nu)}(m+s) - E^{(\nu)}(m+r))} \sqrt{\frac{(m+r)!(m+s)!}{\rho_{m+s} \rho_{m+r}}} \frac{x^m}{m!}. \] (93)

In particular,
\[ \langle (\gamma a_\nu^\dagger)^r \gamma a_\nu^r \rangle = \frac{x^r}{N_\nu(x)} \left( \frac{d}{dx} \right)^r N_\nu(x), \quad x = |z|^2, \quad r = 0, 1, 2, \ldots, \] (94)

and
\[ \langle N \rangle = \frac{x N_\nu'(x)}{N_\nu(x)}, \] (95)

where $()'$ denotes the derivative with respect to $x$.

**Proof.** Indeed, for $s = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$, we have

\[ \langle (\gamma a_\nu^\dagger)^s \gamma a_\nu^r \rangle = \nu \langle z,\gamma | (\gamma a_\nu^\dagger)^s \gamma a_\nu^r | z,\gamma \rangle_\nu \]
\[ = \frac{1}{N_\nu(|z|^2)} \sum_{m=0}^{\infty} \sum_{n=r}^{\infty} e^{i\gamma (E^{(\nu)}(m+s) - E^{(\nu)}(n))} \sqrt{\frac{n!(n-r+s)!}{\rho_{m+s} \rho_{n+r}}} \frac{z^m}{m!} \frac{z^n}{n!} \langle m | n+s-r \rangle_\nu \]
\[ = \frac{1}{N_\nu(|z|^2)} \sum_{n=r}^{\infty} e^{i\gamma (E^{(\nu)}(n+s) - E^{(\nu)}(n+r))} \sqrt{\frac{n!(n-r+s)!}{\rho_{n+s+r} \rho_{n+r}}} \frac{z^{n+s-r}}{(n+r)!} \frac{z^n}{n!}, \] (96)

In the special case $s = r$, we have
\[ \langle (\gamma a_\nu^\dagger)^r \gamma a_\nu^r \rangle = \frac{x^r}{N_\nu(x)} \sum_{m=0}^{\infty} \frac{(m+r)!}{\rho_{m+r} m!} \frac{x^m}{m!} \]
\[ = \frac{x^r}{N_\nu(x)} \sum_{m=r}^{\infty} \frac{m!}{\rho_{m} (m-r)!} \frac{x^{m-r}}{(m-r)!} \]
\[ = \frac{x^r}{N_\nu(x)} \left( \frac{d}{dx} \right)^r N_\nu(x), \quad x = |z|^2. \]

In particular, for \( r = 1 \) the latter expression takes the form

\[ \langle N \rangle \equiv \langle \gamma a_\nu^\dagger \gamma a_\nu \rangle = x \frac{N'_\nu(x)}{N_\nu(x)}. \tag{97} \]

The probability of finding \( n \) quanta in the deformed state \( |z, \gamma \rangle_\nu \) is given by

\[ P_\nu(x, n) := |\langle n |z, \gamma \rangle_\nu|^2 = \frac{x^n}{E^{(\nu)}(n)! N_\nu(x)}, \quad x = |z|^2. \tag{98} \]

Since for the non-deformed CS the variance of the number operator \( N \) is equal to its average, deviations from Poisson distribution can be measured with the Mandel parameter defined by the quantity \(^3^3\)

\[ Q_\nu := \frac{(\Delta N)^2 - \langle N \rangle}{\langle N \rangle} \tag{99} \]

where \( \langle N \rangle \) is the average counting number, \((\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2\) is the corresponding square variance. Moreover, the Mandel parameter \( Q_\nu \)

\[ Q_\nu \equiv F - 1 \tag{100} \]

is closely related to the normalized variance, also called the quantum Fano factor \( F \), \(^3^5,^3^6\), given by \( F = (\Delta N)^2/\langle N \rangle \), of the photon distribution. For \( F < 1 (Q_\nu < 0) \), the emitted light is referred to as sub-Poissonian, \( F = 1, Q_\nu = 0 \) corresponds to the Poisson distribution while for \( F > 1, (Q_\nu > 0) \) it corresponds to super-Poissonian \(^3^7,^3^8\).

By using the expectation value of the operator \( N^2 = (\gamma a_\nu^\dagger \gamma a_\nu^2 + N \) provided by

\[ \langle N^2 \rangle = x^2 S^{(2,2)}_{\nu,\gamma}(x) + x S^{(1,1)}_{\nu,\gamma}(x), \quad x = |z|^2 \tag{101} \]

one readily finds

\[ Q_\nu(x) = x \left( \frac{N''_\nu(x)}{N'_\nu(x)} - \frac{N'_\nu(x)}{N_\nu(x)} \right). \tag{102} \]

For \( x << 1 \), the mandel parameter \((99)\) is reduced to

\[ Q_\nu(x) = -\frac{x}{2M\varepsilon_0(2\nu + 3)(2\nu + 4)} + o(x^2) < 0 \tag{103} \]

which yields the sub-Poissonian distribution.
The second order correlation function defined as

\[ g^{(2)}(\nu)(x) := \frac{\langle N^2 \rangle - \langle N \rangle}{\langle N \rangle^2}, \quad x = |z|^2 \]  

(104)
is explicitly given by

\[ g^{(2)}(\nu)(x) = \frac{N''(x)\mathcal{N}_\nu(x)}{\mathcal{N}_\nu'(x)^2}. \]  

(105)

For \( x \ll 1 \), the second order correlation function (105) is reduced to

\[ g^{(2)}(\nu)(x) = \frac{2\nu + 3}{2\nu + 4} \left( 1 + \frac{x}{M\varepsilon_0(2\nu + 3)(2\nu + 4)(2\nu + 5)} \right) + o(x^2). \]  

(106)
The Hermitian operators \( X_\nu \) and \( P_\nu \) defined as follows

\[ X_\nu := \gamma a_\nu^\dagger + \gamma a_\nu \sqrt{2}, \quad P_\nu := i\gamma a_\nu^\dagger - \gamma a_\nu \sqrt{2} \]  

(107)
lead to the following uncertainty relation

\[ (\Delta X_\nu)^2 (\Delta P_\nu)^2 \geq \frac{1}{4} |\langle [X_\nu, P_\nu] \rangle|^2. \]  

(108)
When the expectation values are evaluated in the CSs \( |z, \gamma\rangle_\nu \) and we find that \( \sigma_{X_\nu} < \Delta_{H_\nu} < \sigma_{P_\nu} \) (\( \sigma_{P_\nu} < \Delta_{H_\nu} < \sigma_{X_\nu} \)), we say that \( |z, \gamma\rangle_\nu \) is an \( X_\nu \)-squeezed state (\( P_\nu \)-squeezed state)\(^{40}\).

Using the relation (92), the variances of the operators \( X_\nu \) and \( P_\nu \) are evaluated in the state \( |z, \gamma\rangle_\nu \) as

\[ \sigma_{X_\nu}(z) = \text{Re} \left[ \bar{z}^2 \left( S^{(2,0)}_{\nu,\gamma}(|z|^2) - (S^{(1,0)}_{\nu,\gamma}(|z|^2))^2 \right) \right] 
\]  

\[ + |z|^2 \left( S^{(1,1)}_{\nu,\gamma}(|z|^2) - (S^{(1,0)}_{\nu,\gamma}(|z|^2))^2 \right) + \frac{1}{2} \]  

(109)
and

\[ \sigma_{P_\nu}(z) = -\text{Re} \left[ \bar{z}^2 \left( S^{(2,0)}_{\nu,\gamma}(|z|^2) - (S^{(1,0)}_{\nu,\gamma}(|z|^2))^2 \right) \right] 
\]  

\[ + |z|^2 \left( S^{(1,1)}_{\nu,\gamma}(|z|^2) - (S^{(1,0)}_{\nu,\gamma}(|z|^2))^2 \right) + \frac{1}{2}, \]  

(110)
where \( S^{(s,r)}_{\nu,\gamma}(x) \) is defined in (93). From the relations (109) and (110), one can show that

\[ \sigma_{P_\nu}(z) = \sigma_{X_\nu}(e^{i\pi/2} z). \]  

(111)
This relation means that to obtain the representation of \( \sigma_{P_\nu}(z) \) in the same plane we only need to apply a positive rotation of \( \pi/2 \) to the \( [\text{Re}(z), \text{Im}(z)] \)-plane representation of \( \sigma_{X_\nu}(z) \).
B. Geometry of the states $|z, \gamma\rangle_{\nu}$

The geometry of a quantum state space can be described by the corresponding metric tensor. This real and positive definite metric is defined on the underlying manifold that the quantum states form, or belong to, by calculating the distance function (line element) between two quantum states. It is also known as a Fubini-Study metric of the ray space. The knowledge of the quantum metric enables to calculate quantum mechanical transition probability and uncertainties.\(^4\)

The map $z \mapsto |z, \gamma\rangle_{\nu}$ defines a map from the space $\mathbb{C}$ of complex numbers onto a continuous subset of unit vectors in Hilbert space and generates in the latter a two-dimensional surface with the following Fubini-Study metric:

$$d\sigma^2 := ||d|z, \gamma\rangle_{\nu}|^2 - |\nu\langle z, \gamma|d|z, \gamma\rangle_{\nu}|^2. \quad (112)$$

**Proposition IV.2** The Fubini-Study metric (112) is reduced to

$$d\sigma^2 = W_{\nu}(x)d\bar{z}dz, \quad (113)$$

where $x = |z|^2$ and

$$W_{\nu}(x) = \left( x \frac{N'_{\nu}(x)}{N_{\nu}(x)} \right)' = \frac{d}{dx} \langle N \rangle. \quad (114)$$

**Proof.** Computing $d|z, \gamma\rangle_{\nu}$ by taking into account the fact that any change of the form $d|z, \gamma\rangle_{\nu} = \alpha |z, \gamma\rangle_{\nu}, \alpha \in \mathbb{C}$, has zero distance, we get

$$d|z, \gamma\rangle_{\nu} = N^{-1/2}_{\nu}(|z|^2) \sum_{n=0}^{\infty} \frac{n z^{n-1} e^{-i \gamma \xi^{(\nu)}(n)}}{\sqrt{\rho_n}} |n\rangle_{\nu} d\bar{z}dz. \quad (115)$$

Then,

$$||d|z, \gamma\rangle_{\nu}|^2 = N^{-1}_{\nu}(|z|^2) \sum_{n=0}^{\infty} \frac{n^2 |z|^{2(n-1)}}{\rho_n} d\bar{z}dz$$

$$= N^{-1}_{\nu}(|z|^2) \left( \sum_{n=0}^{\infty} \frac{n |z|^{2(n-1)}}{\rho_n} + |z|^2 \sum_{n=0}^{\infty} \frac{n(n-1) |z|^{2(n-2)}}{\rho_n} \right) d\bar{z}dz$$

$$= N^{-1}_{\nu}(x) (N'_{\nu}(x) + xN''_{\nu}(x)) d\bar{z}dz$$

$$= N^{-1}_{\nu}(x) (xN'_{\nu}(x))' d\bar{z}dz \quad (116)$$
\[ |\nu(z, \gamma|d|z, \gamma)\nu|^2 = \left| \mathcal{N}_\nu^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{n|z|^{2(n-1)}}{\rho_n} \bar{z}dz \right|^2 = x\mathcal{N}_\nu^{-2}(x) (\mathcal{N}_\nu'(x))^2 d\bar{z}dz. \] (117)

Therefore,
\[
d\sigma^2 = \left( \mathcal{N}_\nu'(x) + x\mathcal{N}_\nu''(x) \right) \mathcal{N}_\nu'(x) d\bar{z}dz \\
= \left( \frac{x\mathcal{N}_\nu'(x)}{\mathcal{N}_\nu(x)} \right)' d\bar{z}dz = \left( \frac{d}{dx} \langle N \rangle \right) d\bar{z}dz. \] (118)

For \( x \ll 1 \), we have
\[
W_\nu(x) = \frac{1}{2M\varepsilon_0(2\nu + 3)} \left( 1 - \frac{1}{M\varepsilon_0 (2\nu + 3)(2\nu + 4)} x \right) + o(x^2). \] (119)

V. QUANTIZATION WITH THE GKCSS

The Berezin-Klauder-Toeplitz quantization, (also called "anti-Wick" or coherent states quantization), of phase space observables of the complex plane, \( D_R \), uses the resolution of the identity (76) and is performed by mapping a function \( f \) that satisfies appropriate conditions, to the following operator in the Hilbert space (see[44,45] and references therein for more details):
\[
f \mapsto A_f = \int_{D_R} f(z, \bar{z})|z, \gamma\rangle_\nu \langle z, \gamma|d\mu_\nu(|z|^2) = \sum_{n,n'=0}^{\infty} (A_f)_{nn'} |n\rangle_\nu \langle n'|, \] (120)

where this integral is understood in the weak sense, i.e., it defines in fact a sesquilinear form (eventually only densely defined)
\[
B_f(\psi_1, \psi_2) = \int_{D_R} f(z, \bar{z}) \langle \psi_1|z, \gamma\rangle_\nu \langle z, \gamma|\psi_2 \rangle d\mu_\nu(|z|^2), \] (121)

with the matrix elements
\[
(A_f)_{nn'} = \frac{e^{i\gamma(\mathcal{E}(\nu)(n) - \mathcal{E}(\nu')(n'))}}{\sqrt{\rho_n \rho_{n'}}} \int_{D_R} f(z, \bar{z}) z^n \bar{z}^{n'} d\mu_\nu(|z|^2). \] (122)

Operator \( A_f \) is symmetric if \( f(z, \bar{z}) \) is real-valued, and is bounded (resp. semi-bounded) if \( f(z, \bar{z}) \) is bounded (resp. semi-bounded). In particular, the Friedrich extension allows to define \( A_f \) as a self-adjoint operator if \( f(z, \bar{z}) \) is a semi-bounded real-valued function. Note
that the original \( f(z, \bar{z}) \) is a “upper or contravariant symbol”, usually non-unique, for the operator \( A_f \). This problem involving the property of the function \( f \) and the self-adjointness criteria of operators is thoroughly discussed in a recent work by Bergeron et al and does not deserve further development here. So, without loss of generality, let us immediately examine different concrete expressions for the function \( f \) in the line of \(^{45}\) as matter of result comparison:

1. The function \( f \) only depends on \(|z|^2 = x\) : the matrix elements \(^{122}\) take the form

\[
(A_f)_{nn'} = \frac{2 \delta_{n,n'}}{(2M\varepsilon_0)^{\nu+2}} \Gamma(2\nu + 3) \int_0^\infty x^{n+1+\nu} f(x) K_{2\nu+2}((2x/M\varepsilon_0)^{1/2}) dx
\]

(123)

2. The function \( f \) only depends on the angle \( \theta = \arg z \), i.e., \( f(z, \bar{z}) = F(\theta) \) : the matrix elements \(^{122}\) are given by

\[
(A_f)_{nn'} = c_{n'-n}(F) \frac{e^{i\gamma (\varepsilon^{(\nu)}(n) - \varepsilon^{(\nu)}(n'))} (2M\varepsilon_0)^{\frac{n+n'}{2}} \Gamma(2\nu + 3 + \frac{n+n'}{2}) \Gamma(1 + \frac{n+n'}{2})}{\Gamma(2\nu + 3) \sqrt{\rho_n \rho_{n'}}}
\]

(124)

where \( c_n(F) \) are the Fourier coefficients of the function \( F \)

\[
c_n(F) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} F(\theta) d\theta
\]

(125)

and the integral

\[
\int_0^\infty dx x^\mu K_\nu(ax) = 2^{\mu-1}a^{-\mu-1} \Gamma \left( \frac{1 + \mu + \nu}{2} \right) \Gamma \left( \frac{1 + \mu - \nu}{2} \right),
\]

\([Re(\mu + 1 + \nu) > 0, Re(a) > 0]\).

(126)

is used\(^{31}\).

3. The function \( f(z, \bar{z}) = z \) and \( f(z, \bar{z}) = \bar{z} \) : the operator \(^{120}\) gives

\[
A_z = \gamma A_\nu, \quad \bar{A}_z = \gamma A^\dagger_\nu
\]

(127)

which act on the states as \(|n\rangle_\nu\)

\[
\gamma A_\nu |n\rangle_\nu = \sqrt{\varepsilon^{(\nu)}_n} e^{i\gamma (\varepsilon^{(\nu)}(n) - \varepsilon^{(\nu)}(n-1))} |n - 1\rangle_\nu, \quad \gamma A_\nu |0\rangle_\nu = 0,
\]

\[
\gamma A^\dagger_\nu |n\rangle_\nu = \sqrt{\varepsilon^{(\nu)}_{n+1}} e^{-i\gamma (\varepsilon^{(\nu)}(n+1) - \varepsilon^{(\nu)}(n))} |n + 1\rangle_\nu.
\]

(128)

(129)
The state $|z,\gamma\rangle_\nu$ is eigenvector of $A_z = \gamma A_\nu$ with eigenvalue $z$ like for standard CSs and the operators $A_z$ and $A_\bar{z}$ satisfy the algebra (60), i.e

$$[A_z, A_\bar{z}] = f_\nu(N),$$

as required.

4. Let $f$ be the function defined as $f(z, \bar{z}) = z^\alpha \bar{z}^\sigma$, $\alpha, \sigma \in \mathbb{N} \cup \{0\}$: the matrix elements (122) of $A_f$ are given by

$$\langle A_f \rangle_{nn'} = \frac{\exp[i(\epsilon^{(\nu)}(n) - \epsilon^{(\nu)}(n'))](2M\varepsilon_0)^{\frac{n+n'+\alpha+\sigma}{2}}}{\Gamma(2\nu + 3) \sqrt{\rho_n \rho_{n'}} \Gamma\left(2\nu + 3 + \frac{n+n'+\sigma+\alpha}{2}\right) \Gamma\left(1 + \frac{n+n'+\sigma+\alpha}{2}\right) \delta_{n-n'-\alpha}}\delta_{n',n},$$

where (126) is used.

To end this section, let us turn back for the cases where $f(z, \bar{z}) = z$ and $f(z, \bar{z}) = \bar{z}$. Interesting results emerging from this context are given by:

$$\nu\langle \gamma, z | A_z | z, \gamma \rangle_\nu = z,$$  

$$\nu\langle \gamma, z | A_\bar{z} | z, \gamma \rangle_\nu = \bar{z},$$  

$$\nu\langle \gamma, z | A_z^2 | z, \gamma \rangle_\nu = z^2,$$  

$$\nu\langle \gamma, z | A_\bar{z}^2 | z, \gamma \rangle_\nu = \bar{z}^2,$$  

and

$$\nu\langle \gamma, z | A_z A_\bar{z} | z, \gamma \rangle_\nu = |z|^2 \left(1 + 4M\varepsilon_0 \frac{\mathcal{N}_0(|z|^2)}{\mathcal{N}_\nu(|z|^2)} + 2M\varepsilon_0(2\nu + 3)\right).$$

VI. CONCLUSION

In this work, we have constructed Gazeau-Klauder type coherent states for a Pöschl-Teller model. Relevant characteristics have been investigated. Induced geometry and statistics have been studied. Finally the coherent states quantization of the classical phase space observables has been presented and discussed.

ACKNOWLEDGEMENTS

This work is partially supported by the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the Office of External Activities (OEA) - Prj-15. The ICMPA is also in partnership with the Daniel Iagolnitzer Foundation (DIF), France.
APPENDIX A. THE NORMALIZATION CONSTANT OF THE EIGENVECTOR $|\phi_{n}^{(\nu,\beta)}\rangle$

By using the property of the eigenstates, we have

$$\delta_{n,m} = \langle \phi_{n}^{(\nu,\beta)} | \phi_{m}^{(\nu,\beta)} \rangle$$

$$= K_{n}^{(\nu,\beta)} K_{m}^{(\nu,\beta)} \int_{0}^{L} dx \sin^{2\nu+m+2} \left( \frac{\pi x}{L} \right) \exp \left\{ - \frac{\beta \pi x}{L} \left( \frac{1}{\nu + n + 1} + \frac{1}{\nu + m + 1} \right) \right\}$$

$$\times P_{n}(a_{m}, \bar{a}_{m}) \left( i \cot \frac{\pi x}{L} \right) P_{m}(a_{n}, \bar{a}_{n}) \left( i \cot \frac{\pi x}{L} \right)$$

$$= K_{n}^{(\nu,\beta)} K_{m}^{(\nu,\beta)} \frac{(\bar{a}_{n} + 1)_{n} (a_{m} + 1)_{m}}{n! m!} \sum_{k=0}^{m} \sum_{s=0}^{n} (-m, a_{m} + \bar{a}_{m} + m + 1)_{k} (-n, a_{n} + \bar{a}_{n} + n + 1)_{s} \mathcal{J}$$

where

$$\mathcal{J} = 2^{-k-s} \int_{0}^{L} dx \left[ \sin^{2\nu+m+n+2} \left( \frac{\pi x}{L} \right) \exp \left\{ - \frac{\beta \pi x}{L} \left( \frac{1}{\nu + n + 1} + \frac{1}{\nu + m + 1} \right) \right\} \right.$$\n
$$\times \left( 1 + i \cot \frac{\pi x}{L} \right)^{k} \left( 1 - i \cot \frac{\pi x}{L} \right)^{s} \right]$$

In [2] it is shown that

$$\int_{0}^{1} dx \sin^{2\xi} (\pi x) e^{\xi x} = \frac{\Gamma(2\delta + 3) e^{\xi/2}}{4^{\delta+1} \Gamma(\delta + 2 + i \frac{\xi}{2\pi}) \Gamma(\delta + 2 - i \frac{\xi}{2\pi})}, \quad n + \nu - \frac{k}{2} - \frac{s}{2} = \delta > -\frac{3}{2}.$$\

Therefore,

$$\delta_{n,m} = \frac{L}{K_{n}^{(\nu,\beta)} K_{m}^{(\nu,\beta)}} \frac{(-\nu - m - \frac{i\beta}{\nu + m + 1})_{n} (-\nu - n + \frac{i\beta}{\nu + n + 1})_{m}}{n! m!} \exp \left\{ - \frac{\beta \pi x}{2L} \left( \frac{1}{\nu + n + 1} + \frac{1}{\nu + m + 1} \right) \right\} 2^{2\nu+m+n+2}$$

$$\times \sum_{k=0}^{m} (-\nu - m - \frac{i\beta}{\nu + m + 1})_{k} \Gamma \left( \frac{n + m}{2} + \nu + 2 - k + \frac{i\beta}{\nu + n + 1} \right)$$

$$\times \sum_{s=0}^{n} \left\{ (-n, -2\nu - n - 1)_{s} \Gamma(n + m + 2\nu - s - k + 3) \right.$$\n
$$\times \left( -\nu - n + \frac{i\beta}{2} \left( \frac{1}{\nu + n + 1} + \frac{1}{\nu + m + 1} \right) \right)_{s} \mathcal{J}$$

$$\times \frac{1}{\Gamma \left( \frac{n + m}{2} + \nu + 2 - s - \frac{i\beta}{2} \left( \frac{1}{\nu + n + 1} + \frac{1}{\nu + m + 1} \right) \right)}.$$

The proof is achieved by taking $n = m$.

REFERENCES

1. E. Schrödinger, Supersymmetrical separation of variables in two-dimensional quantum mechanics, Proc. Roy. Irish Acad. A 46 183 (1940).
E. Schrödinger, A method of determining quantum-mechanical eigenvalues and eigenfunctions, *Proc. Roy. Irish Acad. Sect. A* 46, 9 - 16 (1940).

E. Schrödinger, Further studies on solving eigenvalue problems by factorization, *Proc. Roy. Irish Acad. Sect. A* 46, 183-206 (1940).

E. Schrödinger, The factorization of the hypergeometric equation, *Proc. Roy. Irish Acad. Sect. A* 47, 53-54 (1941).

2H. Bergeron, P. Siegl and A. Youssouf, New SUSYQM coherent states for Pöschl-Teller potentials: a detailed mathematical analysis, *J. Phys. A: Math. Theor.* 45, 244028 (2012).

3L. Infeld and T. E. Hull, The Factorization method, *Rev. Mod. Phys.* 23, 21 (1951).

4F. Cooper, A. Khare and U. P. Sukhatme, *Supersymmetry in quantum mechanics*, (World Scientific Publishing Company, Singapore, 2002).

5E. Witten, Dynamical breaking of supersymmetry, *Nuclear Phys. B* 185, 513 - 554 (1981).

6D. J. Fernández, New hydrogen-like potentials, *Lett. Math. Phys.* 8, 337-343 (1984).

7L. F. Urrutia, E. Hernández, Long-Range Behavior of Nuclear Forces as a Manifestation of Supersymmetry in Nature, *Phys. Rev. Lett.* 51, 755 (1983).

8L. E. Gendenstein and I. V. Krive, *Dynamical groups and spectrum generating algebras* Eds. A. Bohm, A. O. Barut and Y. Ne’eman World Scientific, Singapore, 1988. Sov. J. Usp. Phys. 28, 645 (1985).

9F. Correa and M. S. Plyushchay, Hidden supersymmetry in quantum bosonic systems, *Annals Phys.* 322, 2493-2500 (2007).

10F. Correa, V. Jakubsky, L.-M. Nieto and M. S. Plyushchay, Self-isospectrality, special supersymmetry and their effect on the band structure, *Phys. Rev. Lett.* 101, 030403 (2008).

11F. Correa, V. Jakubsky and M. S. Plyushchay, Finite-gap systems, tri-supersymmetry and self-isospectrality, *J. Phys. A* 41, 485303 (2008).

12F. Correa, V. Jakubsky and M. S. Plyushchay, Aharonov-Bohm effect on AdS(2) and nonlinear supersymmetry of reflectionless Pöschl-Teller system, *Annals Phys.* 324, 1078-1094 (2009).

13M. S. Plyushchay and L.-M. Nieto, Self-isospectrality, mirror symmetry and exotic nonlinear supersymmetry, *Phys. Rev. D* 82, 065022 (2010).

14M. S. Plyushchay, A. Arancibia A and L.-M. Nieto, Exotic supersymmetry of the kink-antikink crystal and the infinite period limit, *Phys. Rev D* 83, 065025 (2011).
15. A. Arancibia A and M. S. Plyushchay, Extended supersymmetry of the self-isospectral crystalline and soliton chains, *Phys. Rev. D* **85**, 045018 (2012).

16. F. Correa F and M. S. Plyushchay, Self-isospectral tric-supersymmetry in PT-symmetric quantum systems with pure imaginary periodicity, *Annals Phys.* **327**, 1761-1783 (2012).

17. F. Correa F and M. S. Plyushchay, Spectral singularities in PT-symmetric periodic finite-gap systems, *Phys. Rev. D* **86**, 085028 (2012).

18. A. Arancibia A, J. M. Guilarte and M. S. Plyushchay, Effect of scalings and translations on the supersymmetric quantum mechanical structure of soliton system *Phys. Rev.* **87**, 045009 (2013).

19. A. Arancibia, J. M. Guilarte and M. S. Plyushchay, arXiv: 1309.1816.

20. H. Bergeron, J.-P. Gazeau and A. Youssouf, Semi-classical behavior of Pöschl-Teller coherent states, *Europhys. Lett* **92**, 60003 (2010).

21. R. Dautry and J.-L. Lions, *Mathematical Analysis and Numerical methods for Science and Technology*, (Springer-Verlag Berlin Heidelberg vol 2, 1988).

22. R. Koekoek and R. F. Swarttouw, *The Askey-scheme of orthogonal polynomials and its q—analogue*, (Report 98-17, TU Delft, 1998).

23. D. J. Fernández and N. Fernández-Garcia, Higher-order supersymmetric quantum mechanics, *AIP. Conf. Proc.* **744**, 236 (2005).

24. D. J. Fernández, Véronique Hussin and Oscar Rosas-Ortiz, Coherent states for Hamiltonians generated by supersymmetry, arXiv: 0705.0316.

25. A. H. El Kinani and M. Daoud, *Phys. Lett. A*, Vol. 283, no. 5, pp. 291-299 (2001).

A. H. El Kinani and M. Daoud, Generalized intelligent states for an arbitrary quantum system, *J. Phys. A : Math. Gen* **43**, 5373-5387 (2001).

A. H. El Kinani and M. Daoud, Generalized coherent and intelligent states for exact solvable quantum systems, *J. Math. Phys.* Vol 43, 714-733 (2002).

26. J.- P. Gazeau and J. R. Klauder, *J. Phys. A* **32**, 123-132 (1999).

27. F. Bagarello, Extended SUSY quantum mechanics, intertwining operators and coherent states, arXiv: 0904.0199.

28. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, (Springer-Verlag, New York, 1966).

29. J. R. Klauder and B. S. Skagerstam, *Coherent states: Applications in Physics and Mathematical Physics*, (World Scientific, Singapore, 1985).
J. R. Klauder, K. A. Penson and J.-M. Sixderniers, Constructing coherent states through solutions of Stieljes and Hausdorff moment problems, Phys. Rev. A 64 (2001).

I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrales, Series and Products, (Academic, New York, 1980).

J. Bertrand, P. Bertrand and J.-P. Ovarlez, The Mellin transform, ”The Transform and Applications Handbook”, Ed. A.D.Poularikas, Volume of ”The Electrical Engineering Handbook” series, CRC Press inc, 1995).

S. Dey and A. Fring, Bohmian quantum trajectories from coherent states, Phys. Rev. A 88, 022116 (2013).

S. T. Ali, J.-P. Antoine and J.-P. Gazeau, Coherent states, wavelets and their generalizations, (Springer, NY, 2000, 2nd ed. 2013).

I. Aremua, M. N. Hounkonnou and E. Balolitcha, Coherent states for Landau levels: algebraic and thermodynamical properties, arXiv: 1301.6280.

J. Bajer and A. Miranowicz, J. Opt. B 2, L10 (2000).

L. Mandel, Opt. Lett. 4, 205 (1979).

X.-Z. Zhang, Z.-H. Wang, H. Li, Q. Wu, B.-Q. Tang, F. Gao and J.-J. Xu, Chin. Phys. Lett. 25, 3976 (2008).

J.-P. Antoine, J.-P. Gazeau, P. Monceau, J. R. Klauder and K. A. Penson, Temporally stable coherent states for infinite well and Pöschl-Teller potentials, arXiv: 0012044 [math-ph].

A. N. F. Aleixo and A. B. Balantekin, Generalized coherent, squeezed and intelligent states for exactly solvable quantum systems and the analogue of the displacement and squeezing operators, J. Phys. A: Math. Theor. 46, 315303 (2013).

M. N. Hounkonnou and J. D. Bukweli Kyemba, Generalized \((\mathcal{R}, p, q)\)-deformed Heisenberg algebras: coherent states and special functions, J. Math. Phys. 51, 063518 (2010).

M. N. Hounkonnou and E. B. Ngompe Nkouankam, New \((p, q, \mu, \nu, f)\)-deformed states, J. Phys. A: Math. Theor. 40, 12113-12130 (2007).

M. N. Hounkonnou and K. Sodoga, Generalized coherent states for associated hypergeometric-type functions, J. Phys. A: Math. Gen. 38, 7851-7857 (2005).

J. R. Klauder, J. Math. Phys. 4, 1055–1058 and 1058–1073 (1963).

J.-P. Gazeau and M. A. del Olmo, Pisot \(q\)-Coherent states quantization of the harmonic oscillator, arXiv: 1207.1200.