Emergent Dynamics from Entangled Mixed States

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Entanglement is at the core of quantum physics, playing a central role in quantum phenomena involving composite systems. According to the timeless picture of quantum dynamics, entanglement may also be essential for understanding the very origins of dynamical evolution and the flow of time. Within this point of view, the Universe is regarded as a bipartite entity comprising a clock \( C \) and a system \( R \) (or “rest of the Universe”) jointly described by a global stationary state, and the dynamical evolution of \( R \) is construed as an emergent phenomenon arising from the entanglement between \( C \) and \( R \). In spite of substantial recent efforts, many aspects of this approach remain unexplored, particularly those involving mixed states. In the present contribution we investigate the timeless picture of quantum dynamics for mixed states of the clock-system composite, focusing on quantitative relations linking the clock-system entanglement with the emerging dynamical evolution experienced by the system.

I. INTRODUCTION

One of the goals of Science is to formulate the most economical description possible of natural phenomena. Guided by this desire for conceptual economy, scientists try to develop theories having the least possible number of basic assumptions or primitive elements. In this regard, research into the phenomenon of quantum entanglement has led to remarkable insights. For instance, the study of entanglement clarified the origin of the states describing systems in thermal equilibrium with a heat bath, without the need to invoke the micro-canonical distribution for the system-bath composite \([\text{1, 2}]\). More radically, research work revolving around quantum entanglement also provided a plausible explanation of the origins of dynamical evolution and the flow of time. The concomitant arguments, according to which time and dynamics are emergent phenomena arising from quantum correlations, were first articulated by Page and Wootters (PW) \([\text{3, 4}]\), although related ideas had been previously advanced in the context of the quantum theory of gravity \([\text{5, 6}]\).

Within the PW timeless picture of quantum mechanics \([\text{3, 4}]\), the whole Universe \( U \) is assumed to be in a global stationary state, which is an eigenstate of the total Hamiltonian with zero energy eigenvalue. Dynamical evolution arises from this static state as a result of the quantum entanglement between the degree of freedom of an appropriate subsystem \( C \), called the clock, and the rest of the Universe \( R \). According to this idea, time and dynamics are emergent features of the Universe rooted in the entanglement between two subsystems, \( R \) and \( C \). The Schrödinger time-independent equation describing the global stationary state of the \( R + C \) composite is reminiscent of the celebrated Wheeler-DeWitt equation in quantum cosmology, describing a stationary state with zero eigenvalue for the wave function of the entire (closed) Universe \([\text{5, 6}]\).

The PW timeless approach to quantum mechanics has been elaborated and extended in various directions, from both the theoretical and the experimental points of view \([\text{7–24}]\). Healthy controversy \([\text{9, 11}]\) has invigorated research into the PW proposal, stimulating the exploration of its possibilities. The timeless picture was criticized by Albrecht and Iglesias \([\text{9}]\), who pointed out apparent ambiguities concerning non-equivalent choices for the clock subsystem. Subsequent counter arguments by Marletto and Vedral \([\text{11}]\) showed that these ambiguities do not arise, if one takes carefully into account the properties needed by a subsystem to be acceptable as a clock. Recent work reported in the literature attests to the deep and manifold implications of the timeless picture of quantum mechanics. Research into this subject has led to the re-consideration of well-known foundational issues, such as Pauli’s famous argument for the impossibility of a time observable in quantum mechanics \([\text{13}]\). New facets of time in quantum mechanics have been discovered, such as its basic connection with quantum coherence \([\text{14}]\). Interesting forays into relativistic scenarios have also been made, with the implementation of the PW scheme for Dirac \([\text{15}]\) and scalar \([\text{16}]\) particles. A formalism akin to the one behind the timeless picture has led to the development of new, practical computational techniques.
for problems in quantum dynamics, which are reformulated as ground-state eigenvalue problems [17]. Going beyond theoretical considerations, concrete experiments illustrating the timeless picture have been successfully conducted in recent years [18–20].

As already mentioned, the system-clock entanglement is central to the timeless approach to quantum dynamics. However, the quantitative relation between quantum entanglement and specific, dynamic-related aspects of the evolving system is one that has received relatively little attention by researchers, as most efforts focusing on scenarios where the system-clock composite is in a pure state [21–24]. Our aim in this work is to explore the timeless picture of quantum dynamics for mixed global states of the bipartite system $R + C$. Motivations to study mixed states within the context of the timeless approach are manifold. First, the system $R$ is, in general, itself composite. In realistic scenarios one may have access only to a subsystem $R_a$ of $R$ that, while weakly coupled to other parts of $R$, may nevertheless be entangled to them and, consequently, be in a mixed state [25]. Second, the Universe itself (that is, the whole system $R + C$) may conceivably be in a mixed state [26, 27]. These two motivations are not entirely independent from each other. Last, the analysis of mixed states in connection with the timeless approach to quantum mechanics may shed new light on the problem of the ontological status of mixed states [28]. All these motivations can be encompassed by a single one: it is desirable to formulate the PW picture in a fashion that incorporates the most general description of the dynamics of a closed quantum system, which is the one given by von Neumann’s equation for the evolution of the system-clock composite.

The state of the composite system [the “Universe” $U$] comprising a clock ($C$) and the rest of the Universe ($R$). The Hilbert spaces corresponding to these two subsystems and to the total system are, respectively, $\mathcal{H}_C$, $\mathcal{H}_R$, and $\mathcal{H}_U = \mathcal{H}_R \otimes \mathcal{H}_C$. Global states of $R + C$ are spanned in a product orthonormal basis $\{|x\rangle \otimes |t\rangle = |x\rangle |t\rangle \}$ of $\mathcal{H}_U$, where $\{|t\rangle \}$ and $\{|x\rangle \}$ are orthonormal bases of $\mathcal{H}_C$ and $\mathcal{H}_R$, respectively. The continuous label $t \in \mathbb{R}$ characterizing the basis states of $\mathcal{H}_C$ corresponds to the eigenvalues of an observable $\hat{T}$ associated with the position of the clock’s hands. That is, $\hat{T} |t\rangle = t |t\rangle$. Likewise, the label $x$ characterizing the basis states of $\mathcal{H}_R$ represents the position, or any other degree of freedom, of the particle or particles constituting the system $R$. Throughout the paper we will assume that $x$ is a continuous variable, yet it may also denote a discrete one, provided integrals are properly substituted by discrete sums.

To analyze the behavior of the complete system during a finite time interval $[0, T]$, we assume that $U$ is in the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{T}} \int_{0}^{T} \Psi(x, t) |x\rangle |t\rangle \, dx \, dt,$$

(1)
described by a wave function $\Psi(x, t) = \langle x |t\rangle |\Psi\rangle$ that is spatially normalized, $\int \Psi(x, t)^{2} \, dx = 1$. The state $|\Psi\rangle$ is then properly normalized, both spatially and temporally,

$$\langle \Psi |\Psi\rangle = \frac{1}{T} \int_{0}^{T} \left( \int |\Psi(x, t)|^{2} \, dx \right) \, dt = 1.$$

(2)

The state of $R$ for a given configuration of the clock’s hands (that is, for a particular value of $t$) is described by the Everett relative state [29]

$$|\Phi_{t}\rangle = |t\rangle |\Psi\rangle = \frac{1}{\sqrt{T}} \int \Psi(x, t) |x\rangle \, dx = \frac{1}{\sqrt{T}} |\Phi_{t}\rangle,$$

(3)

obtained by projecting $|\Psi\rangle$ onto $|t\rangle$. In (3), $|\Phi_{t}\rangle$ stands for the normalized relative state, satisfying

$$\langle \Phi_{t} |\Phi_{t}\rangle = T \langle \Phi_{t} |\Phi_{t}\rangle = 1.$$

(4)

Within the timeless formalism it is assumed that

$$\hat{H}_{U} |\Psi\rangle = 0,$$

(5)
where $\tilde{H}_U$ is the total Hamiltonian $\tilde{H}_U = \tilde{H}_R \otimes I_C + I_R \otimes \tilde{H}_C$, with $H_R$ an arbitrary Hamiltonian of $R$, and $H_C$ the Hamiltonian of the clock. Furthermore, it is considered that the clock’s observable $\tilde{T}$ and the Hamiltonian $\tilde{H}_C$ satisfy the commutation relation $[\tilde{T}, \tilde{H}_C] = i\hbar$. Under these conditions, it follows from (5) that the relative state $|\Phi_i\rangle$ (whether normalized or not) obeys the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\Phi_i\rangle = \tilde{H}_R |\Phi_i\rangle.$$  

(6)

We thus see that the usual dynamical scenario—embodied in the (time-dependent) Schrödinger equation—ensues from the static image of the non-evolving state $|\Psi\rangle$. The important point here to be noticed, is that the evolution emerges if and only if $C$ and $R$ are entangled. Otherwise $\Psi(x,t)$ factorizes as $\Psi(x,t) = \Psi_C(t)|\Psi_R(x)\rangle$. $\Psi_R(x)$ is an eigenstate of $\tilde{H}_R$, and therefore $\Psi(x,t)$ is a stationary (non-evolving) state. Such intimate relation between entanglement and time evolution has been explored previously [21–23] in this pure-case scenario. In what follows we will analyze the more general case of mixed states, and show that the relation still holds.

### III. Evolution and Entanglement for Mixed States of the System-Clock Composite

In order to extend the above ideas beyond scenarios corresponding to pure global states of the $R + C$ system, we shall assume a mixed global state $\rho$ that is stationary under the dynamics determined by the total Hamiltonian $\tilde{H}_U$, and has a definite total energy equal to 0. That is, we shall assume that $\langle \tilde{H}_U \rangle = \text{Tr}(\rho \tilde{H}_U) = 0$ and $\langle \tilde{H}_U^2 \rangle - \langle \tilde{H}_U \rangle^2 = \text{Tr}[(\rho \tilde{H}_U - \langle \tilde{H}_U \rangle)^2] = 0$. The state $\rho$ is then of the form

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|,$$  

(7)

where $p_i \geq 0$ for all $i$, $\sum_i p_i = 1$, and $\{|\Psi_i\rangle\}$ is a set of stationary pure states of $U$ with $\tilde{H}_U|\Psi_i\rangle = 0$ for all $i$. The density matrix (7) describes thus a statistical mixture of the pure states $|\Psi_i\rangle$ with (probability) weights $p_i$.

Using the same notation as in the previous section, we have

$$|\Psi_i\rangle = \frac{1}{\sqrt{T}} \int \Psi_i(x,t)|x\rangle|t\rangle \, dx \, dt,$$  

(8)

and the corresponding relative state

$$|\Phi_{i,t}\rangle = \langle t|\Psi_i\rangle = \frac{1}{\sqrt{T}} \int \Psi_i(x,t)|x\rangle \, dx = \frac{1}{\sqrt{T}}|\tilde{\Phi}_{i,t}\rangle,$$  

(9)

where $|\tilde{\Phi}_{i,t}\rangle = \sqrt{T}|\Phi_{i,t}\rangle$ stands for the normalized relative states ($\langle \tilde{\Phi}_{i,t}|\tilde{\Phi}_{i,t}\rangle = 1$).

Now, in this case the Everett relative state, describing the state of $R$ given that the clock hands’ state is $|t\rangle$, is obtained according to

$$\sigma_{R,t} = \frac{\text{Tr}_C(|t\rangle \langle t| \rho)}{\text{Tr}_C(|t\rangle \langle t|)} = T \sum_i p_i \langle \Phi_{i,t}| \langle \Phi_{i,t}|$$  

(10)

This is a mixture of the states $|\Phi_{i,t}\rangle$, each of which satisfies the Schrödinger equation (6). Therefore, the relative state of $R$ satisfies the von Neumann equation,

$$\frac{d}{dt} \sigma_{R,t} = \frac{1}{i\hbar}[\tilde{H}_R, \sigma_{R,t}].$$  

(11)

We thus verify that the quantum dynamical equations of $R$ are recovered also in the mixed state case.

In order to investigate the relation between the evolution and the entanglement in this more general scenario, we shall use an entanglement criteria based on the reduced, marginal, density matrix $\rho_R$ of the system $R$, obtained by taking the partial trace over $C$ of the global density matrix $\rho$:

$$\rho_R = \text{Tr}_C \rho = \int_0^T \langle t| \rho |t\rangle \, dt = \frac{1}{T} \int_0^T \sigma_{R,t} \, dt = \sigma_{R,t},$$  

(12)

where $\langle \cdot \rangle$ denotes the time average $\langle \cdot \rangle = \frac{1}{T} \int_0^T (\cdot) \, dt$. It is worth to emphasize that the density matrices $\sigma_{R,t}$ and $\rho_R$, though both referring to system $R$, represent different states. The former represents the state of $R$ conditioned to the state $|t\rangle$ of the clock, and is a mixed state that evolves unitarily as a function of the parameter $t$. On the other hand, the (in general) mixed state $\rho_R$ is obtained through taking, on the global state of $R + C$, the partial trace over the degrees of freedom of $C$. It represents a time-averaged state (over the interval $[0,T]$) and does not depend on $t$.

Now, the entropies $S[\rho]$ and $S[\rho_R]$ of the global ($\rho$) and the marginal ($\rho_R$) density matrices, respectively, provide an entanglement criterion for the global state as follows (see [31] [32] and references therein)

$$S[\rho_R] > S[\rho] \Rightarrow \rho \text{ is entangled.}$$  

(13)

That is, if we have less information about the subsystem $R$ than information about the composite system $R + C$, then $R$ and $C$ are entangled. This entropic entanglement criterion can be implemented irrespective of the particular entropic measure used. Possible choices are von Neumann entropy, or the linear entropy defined, for a generic density matrix $\varphi$, as

$$S_L[\varphi] = \frac{1}{2} - \text{Tr} \varphi^2.$$  

(14)

Since this latter has some computational advantages, we will choose it for our calculations, and thus compare $S_L[\rho]$.
with $S_L[\rho_R]$. Our entanglement indicator is thus
\begin{equation}
\Delta S \equiv S_L[\rho_R] - S_L[\rho],
\end{equation}
in terms of which the entanglement criterion reads
\begin{equation}
\Delta S > 0 \Rightarrow \rho \text{ is entangled.}
\end{equation}

The linear entropy of the global state $\rho$ is given by
\begin{align}
S_L[\rho] &= 1 - \text{Tr} \rho^2 = 1 - \sum_{ij} p_i p_j |\langle \Psi_i | \Psi_j \rangle|^2 \\
&= 1 - \sum_{ij} p_i p_j \frac{1}{T} \int_0^T \langle \tilde{\Phi}_{i,t} | \tilde{\Phi}_{j,t} \rangle dt^2.
\end{align}

Since the inner product $\langle \tilde{\Phi}_{i,t} | \tilde{\Phi}_{j,t} \rangle$ is invariant under the unitary evolution determined by the Schrödinger equation, we can substitute $\langle \tilde{\Phi}_{i,t} | \tilde{\Phi}_{j,t} \rangle = \langle \tilde{\Phi}_{i,0} | \tilde{\Phi}_{j,0} \rangle$ in the above equation and get
\begin{align}
S_L[\rho] &= 1 - \sum_{ij} p_i p_j |\langle \tilde{\Phi}_{i,0} | \tilde{\Phi}_{j,0} \rangle|^2 \\
&= 1 - \sum_{ij} p_i p_j |\langle \tilde{\Phi}_{i,t} | \tilde{\Phi}_{j,t} \rangle|^2 \\
&= 1 - \text{Tr} \sigma_{R,t}^2 = S_L[\sigma_{R,t}] = S_L[\sigma_{R,t}],
\end{align}
where the last equality is due to the fact that (as follows from the first two lines) $S_L[\sigma_{R,t}]$ is a time-independent quantity.

As for the linear entropy of the marginal state $\rho_R$, Eq. (12) gives
\begin{equation}
S_L[\rho_R] = S_L[\sigma_{R,t}].
\end{equation}

It follows from the above expressions that comparing the entropies of $\rho$ and $\rho_R$ amounts to compare the (time) average entropy of $\sigma_{R,t}$ with the entropy of the (time) average of $\sigma_{R,t}$.

Now, given a time-dependent density matrix $\rho_t$ and a concave function $f(x)$, the following inequality holds:
\begin{equation}
\text{Tr} [f(\rho_t)] \geq \text{Tr} [f(\rho)],
\end{equation}
with the equality satisfied only if $\rho_t$ is constant in time [30]. In particular, for $f(x) = x - x^2$, we get $S_L(\rho_t) = \text{Tr} f(\rho_t)$, the inequality (20) leads to
\begin{equation}
S_L[\rho_t] \geq S_L[\rho],
\end{equation}
and therefore, putting $\rho_t = \sigma_{R,t}$ it follows from (18)-(19) that
\begin{equation}
S_L[\rho_R] \geq S_L[\rho].
\end{equation}

Since Eq. (22) holds for all states belonging to the subspace spanned by the eigenstates of zero energy of the total Hamiltonian $H_T$, it follows from the criterion [16] that all these states are entangled, provided $\sigma_{R,t}$ evolves in time. In other words, if the (relative) state of $R$ changes with the ticking of the clock’s hands, then $R$ is necessarily entangled with $C$. Put another way, in the absence of entanglement between the clock and the system $R$, the state of $R$ remains independent of $t$, and no evolution occurs. This means that quantum correlations other than entanglement, that may be present in mixed states, such as quantum discord, are not enough for dynamics and the flow of time to arise. Therefore, the study of mixed states within the timeless approach to quantum dynamics provides further evidence for the intimate link existing between entanglement and evolution.

\section{IV. Upper Bound and Asymptotic Limit of the Entanglement Indicator}

Now we shall determine an upper bound for the indicator $\Delta S$ of entanglement between the system and the clock, and also its asymptotic limit for large lengths of the interval $[0,T]$ within which the joint state of the system-clock composite is defined. We consider a $d$-level system with a Hamiltonian $H_R$ having eigenstates $\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$ with corresponding eigenvalues $\{E_1, E_2, \ldots, E_d\}$. The relative state $\sigma_{R,t}$ evolves according to Eq. (11), and its matrix elements in the basis $\{|n\rangle\}$ (with $n = 1, \ldots, d$) can thus be written as
\begin{equation}
\sigma_{nm}(t) \equiv \langle n|\sigma_{R,t}|m \rangle = e^{-i(E_n-E_m)t/\hbar} \sigma_{nm}(0).
\end{equation}

In its turn, the matrix elements of the reduced state $\rho_R$ are given, according to Eqs. (12) and (23), by
\begin{equation}
\langle n|\rho_R|m \rangle = \langle n|\sigma_{R,t}|m \rangle = \sigma_{nm}(0) e^{i(E_n-E_m)t/\hbar} \sin((E_n-E_m)t/2\hbar),
\end{equation}
with $\sin x = x - x^3/6 + \cdots$. From these expressions the linear entropies $S_L[\rho]$ and $S_L[\rho_R]$ can be computed directly as follows
\begin{align}
S_L[\rho] &= S_L[\sigma_{R,t}] = 1 - \text{Tr} \sigma_{R,t}^2 = 1 - \sum_{nm} |\sigma_{nm}(0)|^2, \\
S_L[\rho_R] &= 1 - \text{Tr} \rho_R^2 = 1 - \sum_{nm} \langle n|\rho_R|m \rangle \langle m|\rho_R|n \rangle = 1 - \sum_{nm} |\sigma_{nm}(0)|^2 \sin^2(\omega_{nm}T/2),
\end{align}
where $\omega_{nm} = |E_n-E_m|/\hbar$. Decomposing the sum in (26) into those terms for which $\omega_{nm} = 0$ and those for which...
\( \omega_{nm} \neq 0 \) we get
\[
S_L[\rho_R] = \left( 1 - \sum_{(\omega_{nm}=0)} |\sigma_{nm}(0)|^2 \right) - \sum_{(\omega_{nm} \neq 0)} |\sigma_{nm}(0)|^2 \text{sinc}^2 (\omega_{nm} T/2). \tag{27}
\]
The entanglement indicator is thus
\[
\Delta S = S_L[\rho_R] - S_L[\rho] = \sum_{nm} |\sigma_{nm}(0)|^2 [1 - \text{sinc}^2 (\omega_{nm} T/2)]
= \sum_{nm} |\sigma_{nm}(0)|^2 [1 - \text{sinc}^2 (\omega_{nm} T/2)] \tag{28}
\leq \sum_{nm} |\sigma_{nm}(0)|^2, \tag{29}
\]
and its maximum value—which coincides with its asymptotic value when \( T \to \infty \)—is
\[
\Delta S_{\text{max}} = \sum_{nm} |\sigma_{nm}(0)|^2. \tag{30}
\]

Now, let us denote with \( \sigma_{R|M} \) the state of \( R \) obtained when a non-selective energy measurement is performed on \( R \), that is
\[
\sigma_{R|M} = \sum_E p_E \sigma_{R|E} = \sum_E \Pi_E \sigma_{R,t} \Pi_E, \tag{31}
\]
where \( p_E = \text{Tr}(\Pi_E \sigma_{R,t}) \) is the probability of obtaining the result \( E \) when measuring the energy of \( R \) when it is in the state \( \sigma_{R,t} \), \( \sigma_{R|E} = \Pi_E \sigma_{R,t} \Pi_E / p_E \) is the (collapsed) state of \( R \) obtained when the energy measurement yields the result \( E \), and \( \Pi_E = \sum_{(E_n=E)} |n\rangle \langle n| \) is the projector onto the subspace spanned by the degenerated eigenstates \( |n\rangle \) that correspond to the same energy eigenvalue \( E \). The projector satisfies
\[
\Pi_E \Pi'_E = \sum_{nm} |n\rangle \langle n| \langle m| = \delta_{EE'} \sum_{n} |n\rangle \langle n| = \delta_{EE'} \Pi_E, \tag{32}
\]
so that
\[
\sigma_{R|E} \sigma_{R'|E'} = \sigma_{R|E}^2 \delta_{EE'}. \tag{33}
\]
Taking into account the second equality in \( \text{(31)} \) and \( \text{(32)} \), we get for the linear entropy of the state \( \sigma_{R|M} \),
\[
S_L[\sigma_{R|M}] = 1 - \text{Tr} \sigma_{R|M}^2 = 1 - \sum_E \text{Tr}(\sigma_{R,t} \Pi_E \sigma_{R,t} \Pi_E)
= 1 - \sum_{nm} |\sigma_{nm}(0)|^2, \tag{34}
\]
which combined with Eqs. \( \text{(25)} \) and \( \text{(30)} \) leads to
\[
\Delta S_{\text{max}} = S_L[\sigma_{R|M}] - S_L[\sigma_{R,t}]. \tag{35}
\]
This relation shows that the asymptotic value of the entanglement indicator is given by the difference between the entropy of the state of \( R \) after and before a non-selective energy measurement is performed.

The entropy \( S_L[\sigma_{R|M}] \) bears information regarding the possible states \( \sigma_{R|E} \) that can be obtained after an energy measurement, and also regarding the energy probability distribution \( \{ p_E \} \). Such information can be extracted by recourse to the first equality in Eq. \( \text{(31)} \) and to \( \text{(33)} \), obtaining
\[
S_L[\sigma_{R|M}] = 1 - \text{Tr} \sigma_{R|M}^2 = 1 - \text{Tr} \sum_{EE'} p_E p'_{E'} \sigma_{R|E} \sigma_{R|E'}
= 1 - \sum_E p_E^2 \text{Tr} \sigma_{R|E}^2
= S_L[\{ p_E \}] + \sum_E p_E^2 S_L[\sigma_{R|E}], \tag{36}
\]
where \( S_L[\sigma_{R|E}] \) stands for the linear entropy associated to the state \( \sigma_{R|E} \), and \( S_L[\{ p_E \}] = 1 - \sum_E p_E^2 \) is the linear entropy corresponding to the energy probability distribution \( \{ p_E \} \).

It is instructive to consider particular cases of the bound \( \text{(35)} \). When the spectrum of \( H_R \) has no degeneracy, one has \( S_L[\sigma_{R|E}] = 0 \), hence \( S_L[\sigma_{R|M}] \) becomes \( S_L[\{ p_E \}] \), and the upper bound reduces to
\[
(\Delta S_{\text{max}})_{\text{non-degenerate}} = S_L[\{ p_E \}] - S_L[\sigma_{R,t}], \tag{37}
\]
It is also particularly interesting to see what happens if the global state \( \rho \) is pure, so that \( S_L[\rho] = S_L[\sigma_{R,t}] = 0 \). In this case also \( \sigma_{R|E} \) is a pure state, whence \( S_L[\sigma_{R|E}] = 0 \), and again \( S_L[\sigma_{R|M}] = S_L[\{ p_E \}] \). Consequently, for pure states one recovers the expression \( \text{(23)} \)
\[
(\Delta S_{\text{max}})_{\text{pure}} = S_L[\{ p_E \}], \tag{38}
\]
meaning that the upper bound of the \( S_L \)-based indicator of entanglement is given by the spread of the energy probability distribution \( p_E \), as measured by its linear entropy. This is no longer the case for mixed states. In an extreme case, for example, in which \( \rho \) is diagonal in the energy eigenbasis one has \( \sigma_{nm} \sim \delta_{nm} \), and Eq. \( \text{(30)} \) leads straightforward to
\[
(\Delta S_{\text{max}})_{\text{diagonal}} = 0. \tag{39}
\]
Now, when \( \rho \) is diagonal in an energy eigenbasis, all the spread in the energy probability distribution is purely
classical, whereas for pure states (that have no energy eigenstates) all the spread in the energy probability distribution is of a quantum nature. These observations, together with Eqs. 38 and 39, indicate that only the quantum component of the spread in the energy probability distribution contributes to the upper bound of the system-clock entanglement.

A. An example. The qubit case

As an illustration of our previous results we consider now a qubit (two-level) system with a Hamiltonian \( H_R \) having eigenvalues \( |E_1 \rangle \) and \( |E_2 \rangle \), with corresponding eigenvalues \( E_1 \) and \( E_2 \). Following Eq. (23), the relative state \( \sigma_{R,t} \) in the basis \( \{ |1 \rangle, |2 \rangle \} \) reads

\[
\sigma_{R,t} = \left( \frac{\sigma_{11}(0)}{e^{i\epsilon t/\hbar} \sigma_{12}(0)} \begin{pmatrix} e^{i\epsilon t/\hbar} \sigma_{12}(0) \\ 1 - \sigma_{11}(0) \end{pmatrix} \right),
\]

where we wrote \( \epsilon = E_2 - E_1 \). The reduced density matrix \( \rho_R \) is given, according to Eq. (24), by

\[
\rho_R = \sigma_{R,t}^R = \left( \begin{pmatrix} \sigma_{11}(0) \\ \sigma_{12}(0)e^{i\epsilon x \sin x} \end{pmatrix} \begin{pmatrix} \sigma_{12}(0)e^{-i\epsilon x \sin x} \\ 1 - \sigma_{11}(0) \end{pmatrix} \right),
\]

with \( x = \epsilon T/2\hbar \). The corresponding linear entropies are (see Eqs. 25 and 26)

\[
S_L[\rho_R] = 2\left\{ \sigma_{11}(0)[1 - \sigma_{11}(0)] - |\sigma_{12}(0)|^2 \sin^2 x \right\},
\]

and

\[
S_L[\rho] = S_L[\sigma_{R,t}]
\]

\[
= 2\left\{ \sigma_{11}(0)[1 - \sigma_{11}(0)] - |\sigma_{12}(0)|^2 \right\}.
\]

The entanglement indicator is thus

\[
\Delta S = 2|\sigma_{12}(0)|^2(1 - \sin^2 x),
\]

which is greater than zero for \( x > 0 \), provided \( \sigma_{12}(0) \neq 0 \). That is, for any nonzero \( T \), the evolution of \( R \) reflects its entanglement with the clock.

Basic features of the connection between the evolution of the qubit and its entanglement with the clock can be appreciated in Figure 1. The dependence of the entropies \( S_L[\rho_R] \) and \( S_L[\rho] \) on the parameter \( \sigma_{12}(0) \) is depicted in the left panel of the figure. The entanglement indicator \( \Delta S \) as a function of \( x = \epsilon T/2\hbar \) for two values of the parameter \( \sigma_{12}(0) \), is shown on the right panel.

V. RELATION BETWEEN THE ENTANGLEMENT INDICATOR AND ENERGY DISPERSION

As we have seen in the previous sections, the entanglement between the system \( R \) and the clock \( C \) is linked to the time evolution of \( R \). On the other hand, the evolution of a quantum system is closely related to the system’s energy uncertainty. Consequently, there has to be a connection between the energy uncertainty of \( R \), and the entanglement between \( R \) and \( C \). In this section we shall investigate such connection for mixed joint states of \( R + C \).

By recourse to Eq. 28 and to the Taylor series of the \( \text{sinc} \) function

\[
\text{sinc} z = \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l+1)!},
\]

it can be verified that to lowest order in \( T \) the entanglement indicator \( \Delta S \) is

\[
\Delta S = \frac{T^2}{12\hbar^2} \sum_{nm} |\sigma_{nm}(0)|^2 (E_n - E_m)^2
\]

\[
= -\frac{T^2}{12\hbar^2} \text{Tr} \left( \left[ \hat{H}_R, \sigma_{R,t} \right]^2 \right).
\]

Here we face a situation similar to the one analyzed in the previous Section, but now referred to the energy dispersion

\[
\sigma_E^2 \equiv \langle \hat{H}_R^2 \rangle - \langle \hat{H}_R \rangle^2
\]

\[
= \text{Tr} \left( \hat{H}_R^2 \sigma_{R,t} \right) - \text{Tr}^2 \left( \hat{H}_R \sigma_{R,t} \right)
\]

instead of the spread in the energy probability distribution, as measured by \( S_L[|\rho_E \rangle] \). This can be seen as follows. For pure states \( \sigma_{R,t} = |\Phi_\ell \rangle \langle \Phi_\ell | \) we have

\[
\text{Tr} \left( \left[ \hat{H}_R, \sigma_{R,t} \right]^2 \right) = \text{Tr} \left( \left[ \hat{H}_R, |\Phi_\ell \rangle \langle \Phi_\ell | \right]^2 \right)
\]

\[
= 2(|\langle \Phi_\ell | \hat{H}_R | \Phi_\ell \rangle |^2 - 2|\langle \Phi_\ell | \hat{H}_R^2 | \Phi_\ell \rangle |^2 - 2|\langle \Phi_\ell | \hat{H}_R | \Phi_\ell \rangle |^2)
\]

\[
= -2\sigma_E^2,
\]

and we obtain the expression

\[
(\Delta S)_{\text{pure}} = \frac{T^2}{6\hbar^2} \sigma_E^2
\]
relating, for pure states of \( R + C \), the lowest-order expansion of the \( S_L \)-based entanglement indicator (describing its behavior for short-time intervals), with the energy dispersion. In the other extreme situation, for mixed states that are diagonal in the basis of eigenvectors of \( \hat{H}_R \), one has \( [\hat{H}_R, \sigma_{R,t}] = 0 \), and Eq. \((46)\) gives
\[
(\Delta S)_{\text{diagonal}} = 0.
\]
Equations \((49)\) and \((50)\) are analogous to Eqs. \((38)\) and \((39)\). As happens with the spread of the energy probability distribution, the energy dispersion has both classical and quantum components. For pure states, all the energy dispersion is of quantum nature, whereas for mixed states that are diagonal in an energy eigenbasis, it is purely classical. Thus, the quantity
\[
\mathcal{D} = -\text{Tr} \left( |\hat{H}_R, \sigma_{R,t} \rangle \langle \hat{H}_R, \sigma_{R,t} | \right)
\]
can be interpreted as a measure of the quantum contribution to the energy dispersion of the state \( \sigma_{R,t} \).

We shall now illustrate the above results considering states of the form
\[
\sigma_{R,t} = \alpha |\psi(t)\rangle \langle \psi(t)| + \frac{(1-\alpha)}{d} \mathbb{I}_d,
\]
where \( 0 \leq \alpha \leq 1 \), and \( \mathbb{I}_d \) is the \( d \times d \) identity matrix (recall that \( d \) is the dimension of \( \mathcal{H}_R \)). These states can be regarded as pure states perturbed by white noise. We decompose \( |\psi(t)\rangle \) as
\[
|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\phi_n\rangle,
\]
where \( \{ |\phi_n\rangle \} \) is the set of (orthonormal) eigenstates of \( \mathcal{H}_R \) with corresponding eigenvalues \( E_n \), and the normalization condition \( \sum_n |c_n|^2 = 1 \) is satisfied.

Direct calculation gives
\[
\text{Tr} \sigma_{R,t}^2 = \alpha^2 + \frac{1}{d} - \frac{1}{\alpha^2}
\]
and
\[
\text{Tr} \sigma_{R,M}^2 = \sum_E \left[ \alpha^2 \sum_{(E_n = E)} |c_n|^2 \sum_{(E_m = E)} |c_m|^2 \right] + 2 \alpha(1-\alpha) \sum_{E, \phi_E = 1} p_E \frac{(1 - \alpha^2)}{d}
\]
\[
= \sum_E \left[ \alpha^2 p_E^2 \right] + \frac{1}{d} - \frac{1}{\alpha^2},
\]
where \( p_E = \sum_{(E_n = E)} |c_n|^2 \).

Using Eq. \((34)\) we thus get
\[
\Delta S_{\text{max}} = \text{Tr} \sigma_{R,M}^2 - \text{Tr} \sigma_{R,M}^2 = S_L[\alpha p_E] + (\alpha^2 - 1) = \alpha^2 S_L[\langle p_E \rangle],
\]
and therefore recover the result \((38)\) for \( \alpha = 1 \).

On the other hand, one also has
\[
\text{Tr} \left( |\hat{H}_R, \sigma_{R,t} \rangle \langle \hat{H}_R, \sigma_{R,t} | \right) = \alpha^2 \text{Tr} \left( |\hat{H}_R, |\psi\rangle \langle \psi| \right)
\]
\[
= -2\alpha^2 \sigma_E^2,
\]
which reduces for \( \alpha = 1 \) to the expression \((48)\) corresponding to pure states.

In summary, the entropic indicator \( \Delta S \) that detects entanglement between the system \( R \) and the clock \( C \) is given, to lowest order in the length \( T \) of the interval within which the state of \( R + C \) is defined, by a quantity representing the quantum contribution to the energy uncertainty of \( R \).

VI. CONCLUDING REMARKS

In quantum mechanics, as in life, it takes two to tango. Time evolution requires a composite consisting of at least two parts: a system \( R \) that evolves, and a system \( C \), the clock, that keeps track of time. All the properties of the dynamical evolution of \( R \) can be encoded in the correlations (entanglement) exhibited by a stationary quantum state jointly describing the complete system \( R + C \). In this sense, the origins of dynamics and of the flow of time are, perhaps, the most radical instances of the central role played by entanglement in the physics of composite quantum systems. These considerations constitute the gist of the timeless picture of quantum dynamics. According to this viewpoint, there have to be quantitative relations connecting the amount of entanglement between the clock and the evolving system, on the one hand, with specific features of the dynamical evolution of the system, on the other one. In the present contribution we explore these relations for mixed states of the \( R + C \) composite. By recourse to an entanglement indicator for the global state of \( R + C \), it is possible to elucidate how entanglement relates to the time evolution of the system \( R \). It turns out that entanglement is indeed necessary for \( R \) to exhibit evolution. That is, mild forms of quantum correlations, such as quantum discord without entanglement, are not enough to give rise to time and dynamics. This conclusion follows from an entropic sufficient criterion for entanglement satisfied by the state (pure or mixed) of \( R + C \) whenever the system \( R \) exhibits dynamical evolution.

It is a fact of the quantum world that dynamical evolution is always accompanied by energy uncertainty. Consistently, the system-clock entanglement is related to energy uncertainty as well. Indeed, the aforementioned entanglement indicator for global states of \( R + C \) admits an
upper bound and an asymptotic limit, both expressible in terms of the spread of the energy probability distribution associated with the system $R$, as measured by an entropic measure evaluated on that distribution. The entanglement indicator is also related to the energy dispersion of system $R$, in a way reminiscent of a time-energy uncertainty relation.

The results reported in the present work, concerning mixed states within the PW timeless picture, may contribute to elucidate the way in which the PW approach is related to quantum thermodynamics, and to quantum coherence [14]. These lines of enquiry, in turn, may be enriched by including relativistic effects, along the lines pioneered in [15] [16].

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