Interpreting the lattice monopoles
in the continuum terms

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Abstract
We review briefly current interpretation of the lattice monopoles, defined within the
Maximal Abelian Projection, in terms of the continuum theory. We emphasize, in
particular, that the lattice data, at the presently available lattices, indicate that the
monopoles are associated with singular fields. This note is prompted by a recent
analysis [hep-lat/0211005] which is based on an implicit assumption that the fields
are regular.

1 Introduction
Writing these notes was originally motivated by the recent paper “On Nonexistence of
Magnetic Charge in Pure Yang-Mills theories” [1]. The paper appears critical about a
few earlier papers written with our participation and our intention was to appreciate
the critique and summarize our (respectively modified) views. However, a closer reading
of the paper in Ref. [1] reveals that the points which were central for us are not even
mentioned in [1]. In particular, the paper in Ref. [2] reports the following result:

\[ S_{\text{lat mon}} - S_{\text{lat av}} = f(a) \] [lattice units],

(1)

where \( S_{\text{lat av}} \) is the average action per plaquette over the whole lattice, \( S_{\text{lat mon}} \) is the action on
the plaquettes closest to the position of the monopoles (occupying center of a cube) and
\( f(a) \) is a positive function which depends slightly on the lattice spacing \( a \) for the lattices
tested. Note that Eq. (1) is written in lattice units. Moreover the monopoles themselves are defined within the Abelian-projected theory while $S_{av,mon}$ is the full non-Abelian action of the original lattice SU(2) gluodynamics.

The fact that the access of the action associated with the monopoles is finite in the lattice units implies that at the presently available lattices:

$$\left| (G^a_{\mu\nu})_{mon} \right| \sim \frac{1}{a^2},$$

(2)

where $(G^a_{\mu\nu})_{mon}$ is the non-Abelian field-strength tensor associated with the monopole. In other words, in the limit $a \to 0$ the monopole fields appear singular (if we extrapolate the observed behavior to $a \to 0$ limit\(^1\)) and this is the main implication of \(^2\). On the other hand, there is no single line in \(^1\) which mentions singular fields. The critique \(^1\) is written in a bona fide conviction that the authors know the content of the papers criticized without examining them.

Although we cannot compare our views with those of the authors of \(^1\) directly for this reason, we could view the paper \(^1\) in a more general context as a reminder that it is better to have interpretation of the lattice data in terms of the continuum theory. Finding such an interpretation is, indeed, a long overdue and a real challenge, see, e.g., \(^3\). Moreover, saying that the monopole fields are actually singular is no panacea at all and in fact makes the challenge even more acute. Thus, we decided to summarize the current interpretation of the lattice data on the monopoles, as we understand it. There are many open ends yet and there is little doubt that our current understanding misses some important points. However, we find the interplay between the lattice and continuum exciting and productive.

One more remark is in order now. While the authors of \(^1\) discuss (non)existence of the monopoles we prefer to discuss interpretation of the data. It might well be just a matter of language, but the very observation \(^1\) makes the problem of existence of the monopoles redundant. They exist so to say by definition since have a non-vanishing gauge-invariant action\(^1\). However, theoretical interpretation of the results is of course an open issue. In particular, they might not be the “true” monopoles, whatever it means. They cannot be, however, gauge artifacts.

The organization of the paper is as follows. In Sect. 2 we review briefly the ’t Hooft’s definition of the monopoles \(^5\) and its implementation on the lattice. As far as the case of regular fields is concerned, our conclusions are close, albeit not identical, to those of Ref. \(^1\). In Sect. 3 we turn to the realistic case of singular fields and review data on

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\(^1\) Through the whole paper we actually mean only the lattice results valid on the available lattices and consider $1/a$ as the ultraviolet (UV) cut off. We do not speculate here what actually happens or should happen at lattice spacings much smaller than those tested. Thus, if we simply say, for example, that a quantity is UV divergent this actually assumes “if the presently observed dependence on $a$ holds for $a \to 0$. Since we are going to discuss only interpretation of the existing data this reservation is not important for our purposes.

\(^2\) The authors of \(^1\) might have argued that their non-existence theorem implies the data \(^1\) to be wrong. It is worth mentioning therefore that non-vanishing of the r.h.s. of \(^1\) in the lattice units can be checked against quite a few publications, see, in particular, \(^3\). Ref. \(^2\) addresses mainly more subtle points to be mentioned below.
fine tuning of the monopoles. In Sect. 4 we confront some further theoretical problems brought in by the fine tuning.

2 Defining monopoles in pure gauge theories

2.1 ’t Hooft’s procedure

Generically, all the definitions of the monopoles on the lattice can be traced back to the seminal paper by ’t Hooft [3]. For simplicity we will consider only the $SU(2)$ case. Then one introduces a color vector $\chi^a$ ($a = 1, 2, 3$) and partially fixes the gauge by rotating the vector to the third direction in the color space. This leaves $U(1)$ gauge freedom untouched since the gauge rotations around the third axis are still allowed. Then one can demonstrate that the world-lines where all the three components of $\chi^a$ vanish,

$$\chi^a = 0, \quad a = 1, 2, 3,$$

are in fact monopole trajectories where the monopole charge is defined with respect to the $U(1)$ group which is the remaining gauge freedom, see above. Moreover, the magnetic flux can be evaluated in terms of the ’t Hooft tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = G_{\mu\nu}^a n^a + \varepsilon^{abc} n^a D_\mu n^b D_\nu n^c,$$

where $n^a \equiv \chi^a/|\chi^a|$. Note that the field $\chi^a$ in many respect plays the role of the standard Higgs field of the Georgi-Glashow model.

This procedure fixes uniquely a definition of a monopole and its position is gauge invariant since vanishing of a color vector is a gauge invariant condition. However, the interaction of such monopoles between themselves and with, say, Wilson loop is an open question since $F_{\mu\nu}$ does not enter directly the action. Moreover, unlike the Georgi-Glashow model, the vacuum expectation of $\chi^a$ is to vanish,

$$\langle \chi^a \rangle = 0,$$

because of the color conservation. The best that one can do to link the dynamics of the monopoles defined in terms of (3) to the standard dynamics of the Georgi-Glashow model is to assume a domain structure such that $|\chi^a| \approx \text{const}$ within a domain. As far as we know, neither the domain picture nor the mechanism of the Abelian dominance in terms of the monopoles defined this way has ever been analyzed in any detail.

2.2 C-parity

In case of $SU(2)$ gluodynamics the simplest composite (pseudoscalar) field $\chi^a$ is of the form:

$$\chi^a = \varepsilon^{ade} \varepsilon^{bcd} \varepsilon_{\mu\nu\lambda\rho} G^b_{\mu\alpha} G^c_{\nu\beta} G^e_{\rho\lambda},$$

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while the simplest scalar field can be constructed only on four gluonic field-strength tensors.

Moreover, the field (3) has a negative C-parity,

\[ C(\chi^3) = -1 , \]

while the C-parity of the standard Higgs field of the Georgi-Glashow model is positive. This observation about the C-parity can be generalized on any composite field \( \chi^a \) in the \( SU(2) \) case. Indeed, G-parity of the triplet of the gluon fields is positive while the G-parity of the standard Higgs is negative. This is the central point of the paper [1] which concludes that in the \( SU(2) \) case monopoles cannot be defined.

While the theorem that all the composite \( \chi^a \) fields have positive G-parity is certainly true, we are less sure that it precludes us from using the composite fields for the monopole definition as a matter of principle. Indeed, usually, the G-parity of the Higgs field is fixed by the condition that the final theory should conserve C-parity and by the observation that the v.e.v. \( <\chi^3> \neq 0 \). Since in our case the v.e.v. vanishes anyhow (see (3)) the G-parity of the effective Higgs field might be not fixed either. There is no discussion of this point in [1]. From the overview above it does not follow, we feel, that the field \( \chi^a \) should have negative G-parity. However, we would not discuss the issue in detail either since the very possibility that a composite Higgs plays a significant dynamical role seems to us to be too remote from the physics of the lattice monopoles.

### 2.3 Maximal Abelian Projection

By “monopoles”, without further specifications, we mean in this note the monopoles of the Maximal Abelian Projection (MAP), for review see, e.g., [6]. To define the monopoles, one fixes first the maximal Abelian gauge by minimizing

\[ R = \sum_l \left[ (A^1_l)^2 + (A^2_l)^2 \right] , \]

where the sum is taken over all links on the lattice and the upper index of the gauge potential indicates the color. As usual for the procedures following the pattern proposed in [5], the gauge is fixed up to U(1) rotations around the third axis.

The MA projection is defined then by the putting \( A^{1,2}_l \equiv 0 \). Finally, the MAP monopoles are the monopoles associated with \( A^3_l \).

Locally and in the continuum limit the procedure described corresponds to choosing the gauge

\[ (\partial_\mu \mp igA^3_\mu)A^\pm_\mu = 0 . \]

Note that we do not introduce any composite Higgs field now. The fixation of the gauge can fail only because of the singularities in \( A^3_\mu \). As usual, the lines where the fixation of the gauge fails are associated with the monopoles which are the topological defects in MA projected theory.
The magnetic current $j_\nu$ is defined in terms of violations of the Bianchi identity:

$$\partial_\mu \tilde{F}_{\mu\nu} \equiv j_\nu,$$

(10)

where $\tilde{F}_{\mu\nu} \equiv 1/2 \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$. By the field strength one understands here the $F_{\mu\nu}$ of the projected theory. Moreover, since violations of the Bianchi identity assume singular field, Eq. (10) is to be understood as regularized on the lattice [7].

There are no problems with the C-parity of the current defined through (10) – the issue of concern for the authors of [1].

Thus, it is easy to argue that the MAP monopoles are associated with singular gauge potentials. This is an alternative to introducing composite Higgs fields. However, one might have thought that this singularity in the (projected) potential is not related to any original, non-Abelian action. The data (1) show that in fact there is a singularity in the non-Abelian action as well. To accommodate for such a possibility we should readjust considerably our way of thinking.

3 Fine tuning

3.1 Tuning the action and entropy factors

There are quite a few reasons why one would be inclined to disregard singular fields. First of all, apparently, such fields are associated with an infinite action. Indeed, if the action density associated with the monopoles, see (1), is in the lattice units the field-theoretical mass of the monopole is also in the lattice units, that is:

$$M_{mon}(a) \sim \int d^3 r (B^a)^2 \sim \frac{1}{a},$$

(11)

where the lattice spacing plays the role of the UV cut off in the continuum limit, $B^a$ is the color magnetic field. Thus, the action suppression is infinitely strong in the limit $a \to 0$:

$$e^{-S} \sim e^{-\text{const} \cdot L/a},$$

(12)

where $L$ is the length of the monopole trajectory. At presently available lattices one gets an estimate for the monopole mass:

$$M_{mon} \gtrsim 5 \text{ GeV},$$

(13)

which corresponds to the smallest available $a \approx 0.06$ fm.

Naively, excitations with such mass would be mere lattice artifacts. However, the monopole trajectories have scaling properties. Without reviewing all the data, let us mention that in the confining phase there is a single percolating cluster which fills in the whole of the lattice. Moreover, there are self-intersections in the percolating cluster. Associating intersections with interactions, one can introduce a “free-path length” of the
monopole, $L_{\text{free}}$ as the length of the segments of the trajectories between the intersections. According to the measurements $[8]$ $L_{\text{free}}$ scales and does not depend on $a$:

$$L_{\text{free}} \approx 1.6 \text{ fm}.$$ \hspace{1cm} (14)

In other words:

$$L_{\text{free}} \cdot M_{\text{mon}} \gtrsim 40,$$ \hspace{1cm} (15)

where the numerical factor on the r.h.s. is related to the smallest $a$ available.

We are in haste to add that Eq. (15) is in no contradiction with Quantum Mechanics. The point is that $M_{\text{mon}}(a)$ entering the action, $S_{\text{mon}} = M_{\text{mon}} \cdot L$, is not the propagating mass in the lattice formulation of the field theory. For a free particle the propagating mass $m_{\text{prop}}$ is related to it as:

$$m_{\text{prop}}^2 \cdot a \approx 8\left[ M_{\text{mon}}(a) - \frac{\ln 7}{a} \right].$$ \hspace{1cm} (16)

The factor proportional to the ln 7 is due to the entropy, i.e. to the number of trajectories of the same length $L$. Derivations of (16) can be found in textbooks, see, e.g. [4]. (It might worth noting that ln 7 is specific for the monopoles on the cubic lattice).

According to the direct measurements [2] the monopole action is indeed close to $S = \ln 7 \cdot L/a$. However, for a number of reasons it is difficult to measure directly how fine the tuning is. Indirect evidences, like (15), are more powerful.

Thus, the lattice data strongly indicate [10] that the monopoles are fine-tuned objects with a huge cancellation between the suppressing factor due to the action and enhancing factor due to the entropy. Without taking this fine tuning into consideration, any discussion of the physics of the lattice monopoles is abstract.

### 3.2 Gauge-invariant “definition” of the monopole

Since the fine tuning is formulated in terms of the non-Abelian action, one could use it to define the monopoles. Namely, the monopole currents could be defined as closed trajectories along which the average action is fine tuned to the entropy factor. Note that the entropy factor would change with a change of the lattice.

Of course such a definition can be discussed only as a matter of principle. For an individual plaquette associated with a monopole the action can be far from its average value. Indeed, we deal with a quantum process of measurement.

### 4 Implications of the fine tuning

Realization that the lattice monopoles are fine tuned brings actually further difficult questions. First of all, the fine tuning above was discussed in terms of light particles, but this cannot be true – everybody feels this way – that we have new light particles. Since

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3 A similar remark was made independently by T. Suzuki.
the monopole fine tuning is quite a recent notion, the answers to this and other questions are very provisional at best. Nevertheless we would discuss, in an exploratory way, a few further points.

4.1 Two facets of the fine tuning

Now, we turn to the question whether the fine tuning implies introduction of new light particles (and is unacceptable for this reason). A simple argument shows that it is not necessarily so [10].

Indeed, imagine that (see Eq. (16))

\[ M_{\text{mon}}(a) - \ln \frac{7}{a} \sim \Lambda_{QCD} . \]  \hspace{1cm} (17)

This kind of fine tuning is suggested by the scaling of \( L_{\text{free}} \), see Eqs. (14,15). On the other hand, the corresponding “mass” of the monopole is then of order

\[ m^2_{\text{prop}} \sim \frac{\Lambda_{QCD}}{a} , \]  \hspace{1cm} (18)

and the monopoles are removed from the spectrum in the limit \( a \to 0 \).

There is another relation which indicates that interpretation of the monopoles in terms of a magnetically charged field reveals a non-standard picture. The point is that the density of the monopoles in the percolating cluster scales:

\[ \rho_{\text{perc}} \sim \Lambda_{QCD}^{-3} , \]  \hspace{1cm} (19)

see in particular [11]. Here \( \rho_{\text{perc}} \) is defined in terms of the length of the percolating cluster:

\[ L = \rho_{\text{perc}} V_4 , \]  \hspace{1cm} (20)

where \( V_4 \) is the 4-volume of the lattice. If one considers the percolating cluster of the monopoles as representing the ground state of the condensed monopoles, then

\[ \langle |\phi|^2 \rangle \sim \rho_{\text{perc}} \cdot a , \]  \hspace{1cm} (21)

where \( \phi \) is the magnetically charged field.

According to (21) the condensate \( \langle |\phi|^2 \rangle \) vanishes in the limit \( a \to 0 \). However, the vacuum expectation value of the product \( m^2|\phi|^2 \) scales, compare (18). Generalizing this observation one can speculate that the vacuum expectation value of the whole potential energy scales similarly:

\[ \langle V(|\phi|^2) \rangle \sim \Lambda_{QCD}^4 . \]  \hspace{1cm} (22)

Eq. (22), if confirmed by further analysis, would make manifest the relevance of the field \( \phi \) to the non-perturbative QCD.

We pause here to emphasize a two-facet nature of the monopole fine tuning. We do introduce singular fields [14] but this is necessary to cancel the singular entropy factor.
The singularity is specific for the Euclidean (and lattice) formulation. If, for example, the geometry of the lattice is changed, the singular fields are predicted to change as well, in such a way as to maintain the cancellation with the entropy factor. Eq. 22 reveals this facet of the fine tuning even more convincingly. Namely, one could postulate from the very beginning that the vacuum expectation value of the magnetically charged field cannot not diverge in the ultraviolet. This is now a manifestation, in the field theoretical language, of the fact that φ is an effective field describing non-perturbative effects (although in the language of fine tuning it is associated with singular original fields).

4.2 Monopole current

If we introduce singular fields in the continuum limit, the monopole current $j^a_\mu$ is naturally defined in terms of the fields themselves:

$$ (D_\mu \tilde{G}_{\mu\nu})^a = j^a_\nu. \quad (23) $$

In other words the monopole current is associated with the violation of the Bianchi identity which may arise because of the singularities in the fields. It goes without saying that (23) is rather postulated than derived. Unfortunately, there exists no helpful lattice implementation of Eq. (23). Therefore, one cannot actually check that the definition (23) corresponds to the monopoles observed on the lattice, although there is a little doubt that Eqs. (10, 23) are indeed related to each other.

Moreover, the standard properties of the monopole current, as are observed on the lattice, cannot be derived from (23) alone but imply also some non-local constraints on the Yang-Mills fields [12, 13]. Namely treating the continuum theory as a limiting case of the lattice gauge models one can argue [13] that:

$$ j^a_\mu(x) = \int d\tau \dot{x}_\mu n^a(\tau) \delta^4(x - \tilde{x}(\tau)) = n^a(x) j_\mu(x), \quad (24) $$

where the colorless current $j_\mu$ is conserved, $\partial_\mu j_\mu = 0$, and $n^a(x)$ is determined in terms of the non-Abelian field strength tensor $G^a_{\mu\nu}$.

Moreover, it follows from Eq. (23) that the current $j^a_\mu$ satisfies the condition

$$ D_\mu j^a_\mu = 0, \quad (25) $$

which means in turn that along the current trajectory the current can be rotated to a certain direction in the color space:

$$ j^a_\mu \rightarrow \delta^{a3} j_\mu. \quad (26) $$

The whole construction was realized explicitly in case of a single external magnetic current introduced via the the ’t Hooft loop and allowed for calculation of the heavy monopole potential, with the gluon loop corrections included.

Also, Eq. (26) is the closest point to which theory can be brought in attempt to accommodate the monopole dominance observed in the lattice simulations [14]. Indeed, all
the magnetic currents can be aligned with the third direction in the color space. However, the interaction of the currents is still sensitive, generally speaking, to the “charged” gluons and the Abelian dominance remains empirical in nature.

To summarize, the attempt to define magnetic currents in the continuum involves both singular fields, which are responsible for the violations of the Bianchi identities, and non-local constraints which should reflect the non-perturbative dynamics of the Yang-Mills theory but at present cannot be checked independently. Field theory is only applicable in cases when only one kind of charges (magnetic or color) are dynamical while the dual charges are external (infinitely heavy). Indeed, only in this case the non-localities might be avoided. On the other hand, it is clear that the consideration of full charge-monopole dynamics goes beyond the local field theory context.

5 Conclusions

To summarize, we did not have much theoretical difficulties with the definition of monopoles, or with the C-parity of the corresponding current. Moreover the fine tuning observed at the presently available lattices demonstrates once again the physical nature of the monopoles. However, the description of the magnetic monopole dynamics in the continuum terms remains mostly unsolved problem. The fine tuning of the monopoles exhibits their double-face nature. Indeed, the monopoles are associated with singular fields and their formal definitions Eqs. (10,23) look completely local. However, if this would be the case, the infrared QCD scale, Eq. (17), could not appear. And, indeed, in both cases the definitions involve non-localities as well since require either the gauge fixing over the whole lattice volume or introduction of non-local constraints. The intrinsic non-locality of the monopoles made also manifest, for example, by the fact that their contribution to the vacuum energy appears of order $\Lambda_{QCD}^4$, with no ultraviolet divergences involved.

The very fact that the dynamics of the monopoles is “complicated” is of no surprise of course. Indeed, one searches for the monopoles in an attempt to identify degrees of freedom of the theory dual to the Yang-Mills theories. Theory dual to the U(1) gauge theory is a field theory again. But this case covers only the Abelian dominance (which cannot be exact [15]). As for the SU(2) case there is no much hope that the dual theory is a field theory again [16]. On the positive side, one might say that the lattice monopoles have already exhibited many unexpected and remarkable features which prove their relevance to the confinement mechanism. Therefore, one could hope that further phenomenological studies of the lattice monopoles could provide us with a key to understand the structure of the theory dual to the SU(2) gluodynamics.

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