n-DIMENSIONAL FRACTIONAL BESSEL OPERATORS AND LIOUVILLE THEOREMS

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Abstract. In this paper we extend the results given in [11] to the n-dimensional case the fractional powers of Bessel operators. Moreover, we established a Liouville type theorems for these operators. This extend the result obtained in [6] for Bessel operators.

1. Introduction

Bessel operators appear in the setting of harmonic analysis related with Hankel transformations. In the one dimensional case, Bessel operators appear when we consider the Laplacian operator in polar coordinates. In this work we study the fractional Bessel operator and Liouville theorems of the n-dimensional versions in $\mathbb{R}^n_+ = (0, \infty)^n$ given by

\begin{equation}
S_\mu = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{4\mu_i^2 - 1}{4x_i^2}
\end{equation}

and

\begin{equation}
\Delta_\mu = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + (2\mu_i + 1)(x_i^{-1} \frac{\partial}{\partial x_i})
\end{equation}

where $\mu \in \mathbb{R}^n$, $\mu = (\mu_1, \ldots, \mu_n)$ and $\mu_i > -\frac{1}{2}$.

In [11], were studied the fractional powers of the one dimensional case of (1.1) and (1.2) in the sense of the classical theory of fractional powers developed by Balakrishnan in [1] and using similarity of both operators. Let $X$ and $Y$ Banach spaces. Two linear operators $A$ and $B$, $A : D(A) \subset X \to X$ and $B : D(B) \subset Y \to Y$ are similar if there exists an isomorphism $T : X \to Y$ with inverse $T^{-1} : Y \to X$ such that $D(B) = \{ x \in Y : T^{-1}x \in D(A) \}$ given by

\begin{equation}
B = TAT^{-1}.
\end{equation}

Similar operators have the same spectral properties and also that of being non-negative if one of them has this property. Thus, their powers are similar operators and verifies the same similarity relation, so

\begin{equation}
B^\alpha = T A^\alpha T^{-1}.
\end{equation}

In this work, we generalize the results obtained in [11] to the n-dimensional case obtaining the fractional powers of Bessel operator (1.1) and (1.2) in weighted Lebesgue spaces and in distributional spaces. As in [11] we first study the non-negativity of Bessel operator (1.1) in suitable weighted Lebesgue spaces and by similarity we obtain the non-negativity of (1.2) in the corresponding Lebesgue space. Analogously to the one-dimensional case, we construct a locally convex space $\mathcal{B}$ in which $-S_\mu$ is continuous and non-negative. Next, we can consider the dual space $\mathcal{B}'$ with the strong topology and thus obtaining non-negativity of $-S_\mu$ in this distributional space. $\mathcal{B}'$ is contained in the distributional Zemanian space and contain the weighted Lebesgue spaces in which non-negativity was studied. Consequently, if we denote with $(S_\mu)^{\mathcal{B}}$, the Bessel operator with domain $\mathcal{B}'$, we can consider the powers $(-S_\mu)^{\mathcal{B}}\alpha$ with $\text{Re}(\alpha) > 0$ and it is verified the following relation inherited from the selfadjunction of $S_\mu$

\begin{equation}
((-S_\mu)^\alpha u, \phi) = (u, (-S_\mu)^\alpha \phi),
\end{equation}

for $\phi \in \mathcal{B}$ and $u \in \mathcal{B}'$.

In [6], a Liouville-type theorem was studied for a certain general class of Bessel-type operators. This class of operators contain as a particular case the Bessel operator (1.1), and the Liouville
Theorem 1.1. Let \( u \in B' \) and \( \alpha \in \mathbb{C} \) with Re \( \alpha > 0 \). If \((-S_\mu)^{\alpha} u = 0\) then there exists a polynomial \( p \) such that \( u = x^{\beta + \frac{1}{2}} p(\|x\|^2)\).

For the study of the powers of Bessel operator given by (1.2) we introduce a locally convex space \( \mathcal{F} \). This space verifies that its dual space \( \mathcal{F}' \) with the strong topology is a suitable distributional space for the study of fractional powers \((-\Delta_\mu)^{\alpha} \) and from similarity we conclude the following result

Theorem 1.2. Let \( u \in \mathcal{F}' \) and \( \alpha \in \mathbb{C} \) with Re \( \alpha > 0 \). If \((-\Delta_\mu)^{\alpha} u = 0\) then there exists a polynomial \( p \) such that \( u = x^{2\beta + 1} p(\|x\|^2)\).

This paper is organized as follow. In section 2 we summarize basic results related with harmonic analysis in the Hankel setting. Section 3 contain a brief review of non-negative operators in Banach and locally convex spaces and properties of fractional powers of similar operators. In sections 4, 5 and 6 we study the non-negativity of Bessel operator (1.1) and (1.2). Finally, sections 6 and 7 contains Liouville’s theorems for the two fractional Bessel operators.

2. Preliminaries

In this section we introduce the Lebesgue and distributional spaces necessary for our purposes.

We now present some notational conventions that will allow us to simplify the presentation of our results. Let \( \mathbb{R}^n \) be the \( n \)-dimensional euclidean space, \( \mathbb{R}^+ = (0, \infty)^n \) the \( n \)-tuples of real positive numbers. We denote by \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) to the elements of \((0, \infty)^n\) or \( \mathbb{R}^n \) and let \( \mathbb{N} \) be the set \( \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( x \in \mathbb{R}^n \), the norm is given by \( \|x\| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \). The notations \( x < y \) and \( x \leq y \) mean, respectively \( x_i < y_i \) and \( x_i \leq y_i \), for \( i = 1, \ldots, n \). Moreover if \( x \in \mathbb{R}^n \) and \( a \in \mathbb{R} \), \( x = a \) means \( x_1 = x_2 = \ldots = x_n = a \). We denote \( e_j \), for \( j = 1, \ldots, n \), the elements of the canonical basis \( \mathbb{R}^n \). An element \( k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n = \mathbb{N}_0 \times \mathbb{N}_0 \times \ldots \times \mathbb{N}_0 \) is called multi-index. For \( k, m \) multi-index we set \( |k| = k_1 + \ldots + k_n \).

Also we will note

\[
\begin{align*}
  k! &= k_1! \ldots k_n! \\
  \binom{k}{m} &= \binom{k_1}{m_1} \ldots \binom{k_n}{m_n} \\
\end{align*}
\]

for \( k, m \in \mathbb{N}_0^n \).

If \( x \in \mathbb{R}^n \) and \( \beta \in \mathbb{R}^n \), we define

\[
x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}.
\]

In particular if \( a \in \mathbb{R} \), \( a^\beta \) means

\[
a^\beta = a^{\beta_1} \ldots a^{\beta_n},
\]

and if \( \beta \) is a multi-index, \( \beta \in \mathbb{N}_0^n \),

\[
a^\beta = a^{|eta|}.
\]

For \( \alpha \in \mathbb{R} \) let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), then for \( a \in \mathbb{R} \) and \( x \in \mathbb{R}^n \)

\[
a^\alpha = (a_1)^{\alpha_1} \ldots (a_n)^{\alpha_n} \quad x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x^\alpha.
\]

If \( \beta \) is a multi-index, \( \beta = (\beta_1, \ldots, \beta_n) \) \( \in \mathbb{R} \) let

\[
\beta - \alpha = (\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n).
\]

If \( D_j = \frac{\partial}{\partial x_j}, \ j = 1, \ldots, n \) then the partial derivatives respect to \( x \) is denoted by

\[
D^k = D_1^{k_1} \ldots D_n^{k_n},
\]

where \( k \) is a multi-index. We define the operators

\[
T_j = x_j^{-1} D_j
\]

for \( j = 1, \ldots, n \). For a multi-index \( k \) we shall write

\[
T^k = T_n^{k_n} \circ T_{n-1}^{k_{n-1}} \circ \ldots \circ T_1^{k_1}.
\]
Remark 2.1. Let $k$ be a multi-index and $\theta, \varphi$ differentiable functions up to order $|k|$, the following equality is valid
\begin{equation}
T^k \{ \theta \cdot \varphi \} = \sum_{j=0}^{k} \binom{k}{j} T^{k-j} \theta T^j \varphi,
\end{equation}
where " $\cdot$ " denote the usual product of functions.

Hankel transformation appears in mathematical literature in various forms, two classical versions correspond to the versions studied by A. H. Zemanian [17, 18] and I. I. Hirschman [7]
\begin{equation}
(h_{\alpha}f)(t) = \int_0^{\infty} f(x)\sqrt{xt}J_{\alpha}(xt) \, dx, \quad t \in (0, \infty)
\end{equation}
and
\begin{equation}
(H_{\alpha}f)(t) = \int_0^{\infty} f(x)(xt)^{-\alpha}J_{\alpha}(xt) x^{2n+1} \, dx, \quad t \in (0, \infty)
\end{equation}
where $\alpha > -\frac{1}{2}$ and $J_{\alpha}$ is the well-known Bessel function of first kind and order $\alpha$.

S. Molina y S. Trione studied in [12, 13] a $n$-dimensional generalization of (2.6), given by $h_{\mu}$ and defined by
\begin{equation}
(h_{\mu} \phi)(y) = \int_{\mathbb{R}^n_+} \phi(x_1, \ldots, x_n) \prod_{i=1}^{n} (\sqrt{x_i y_i} J_{\mu_i}(x_i y_i)) \, dx_1 \ldots dx_n.
\end{equation}
Analogously it is possible to define a $n$-dimensional generalization for (2.7), given by $H_{\mu}$ and defined by
\begin{equation}
(H_{\mu} \phi)(y) = \int_{\mathbb{R}^n_+} \phi(x_1, \ldots, x_n) \left\{ \prod_{i=1}^{n} (x_i y_i)^{-\mu_i} J_{\mu_i}(x_i y_i) x_i^{2\mu_i+1} \right\} \, dx_1 \ldots dx_n.
\end{equation}
In both, (2.8) and (2.9), $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i > -\frac{1}{2}$ and $J_{\mu_i}$ represents the Bessel function of first kind and order $\mu_i$ for $i = 1, \ldots, n$.

Next we define certain weighted $L^p$-spaces for $1 \leq p \leq \infty$. Let
\begin{equation}
s(x) = \frac{x^{2\mu+1}}{C_{\mu}},
\end{equation}
\begin{equation}
r(x) = x^{-\mu - \frac{1}{2}}
\end{equation}
where $\mu = (\mu_1, \ldots, \mu_n)$, $x \in \mathbb{R}^n_+$, $C_{\mu} = 2^{\mu} \Gamma(\mu_1 + 1) \ldots \Gamma(\mu_n + 1)$ and $dx$ is the usual $n$-dimensional Lebesgue and the powers $x^{2\mu+1}$ and $x^{-\mu - \frac{1}{2}}$ are given by (2.1) and $2^{\mu}$ is given by (2.2). Let $L^p(\mathbb{R}^n_+, sr^p)$, $1 \leq p < \infty$ the space of measurable functions $f$ defined over $\mathbb{R}^n_+$ with norm
\begin{equation}
\|f\|_{L^p(\mathbb{R}^n_+, sr^p)} = \left( \int_{\mathbb{R}^n_+} |f(x)|^p s(x)^r(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty.
\end{equation}
Moreover, $L^\infty(\mathbb{R}^n_+, s)$, is the space of measurable functions over $\mathbb{R}^n_+$ such that
\begin{equation}
\|f\|_{L^\infty(\mathbb{R}^n_+, s)} = \text{ess sup}_{x \in \mathbb{R}^n_+} |r(x)f(x)| < \infty.
\end{equation}
In particular, if $p = 2$, $L^2(\mathbb{R}^n_+, sr^2) = L^2(\mathbb{R}^n_+)$.

For simplicity sometimes we write $L^p(sr^p)$ and $L^\infty(s)$ instead of $L^p(\mathbb{R}^n_+, sr^p)$ and $L^\infty(\mathbb{R}^n_+, s)$.

By $\mathcal{D}(\mathbb{R}^n_+)$ we denote the space of functions in $C^\infty(\mathbb{R}^n_+)$ with compact support in $\mathbb{R}^n_+$ with the usual topology, and by $\mathcal{D}'(\mathbb{R}^n_+)$ the space of classical distributions in $\mathbb{R}^n_+$.

We consider the Zemanian space $\mathcal{H}_\mu$ of the functions $\phi \in C^\infty(\mathbb{R}^n_+)$ such that
\begin{equation}
\sup_{x \in \mathbb{R}^n_+} |(1 + \|x\|^2)^{m} T^k \{ x^{-\mu - 1/2} \phi(x) \}| < \infty, \quad m \in \mathbb{N}_0, \quad k \in \mathbb{N}_0
\end{equation}
endowed with the topology generated by the family of seminorms \( \{ \nu^\mu_{m,k} \} \), given by
\[
\nu^\mu_{m,k}(\phi) = \sup_{x \in \mathbb{R}^n_+} [(1 + \|x\|^2)^m T^k \{x^{-\mu-1/2} \phi(x)\}]
\]
where \(-\mu - 1/2 = (-\mu_1 - 1/2, \ldots, -\mu_n - 1/2)\) and the operators \( T^k \) are given by

\[
T^k = T^{k_n} \circ T^{k_{n-1}} \circ \ldots \circ T^{k_1},
\]
where \( T_i = x_i^{\frac{1}{\mu_i}} \) and \( k = (k_1, \ldots, k_n) \). \( \mathcal{H}_\mu \) is a Fréchet space (see [12]). The dual space of \( \mathcal{H}_\mu \) is denoted by \( \mathcal{H}'_\mu \).

**Remark 2.2.** Sometimes we will consider the family of seminorms
\[
\gamma^\mu_{m,k}(\phi) = \sup_{x \in \mathbb{R}^n_+} |x^m T^k \{x^{-\mu-1/2} \phi(x)\}|
\]
with \( m, k \in \mathbb{N}_0^n \), which are equivalent to \( \nu^\mu_{m,k} \).

**Lemma 2.3.** The following inclusions hold
\[
\mathcal{H}_\mu \subset L^1(\mathbb{R}^n_+, sr) \cap L^\infty(\mathbb{R}^n_+, r) \subset L^p(\mathbb{R}^n_+, sr^p), \quad 1 \leq p < \infty
\]
where \( s \) and \( r \) are given by (2.10) and (2.11) respectively.

**Proof.** Let \( \phi \in \mathcal{H}_\mu \),
\[
\| \phi \|_{L^\infty(\mathbb{R}^n_+, r)} = \sup_{x \in \mathbb{R}^n_+} |x^{-\mu-1/2} \phi(x)| = \gamma^{\mu}_{0,0}(\phi),
\]
then \( \phi \in L^\infty(\mathbb{R}^n_+, r) \).

To show that \( \mathcal{H}_\mu \subset L^1(\mathbb{R}^n_+, sr) \), let \( \phi \in \mathcal{H}_\mu \), \( m, n \in \mathbb{N} \) such that \( m > 2\mu_i + 2 \), for \( i = 1, \ldots, n \), then
\[
\int_{\mathbb{R}^n_+} |\phi(x)| s(x) r(x) \, dx = \int_{[0,1]^n} |x^{-\mu-1/2} \phi(x)| \frac{2^{\mu+1}}{C\mu} \, dx + \int_{\mathbb{R}^n_+ \setminus [0,1]^n} x^m |x^{-\mu-1/2} \phi(x)| \frac{2^{\mu+1-m}}{C\mu} \, dx
\]
\[
\leq \gamma^{\mu}_{0,0}(\phi) C^{\mu-1}_\mu \int_{[0,1]^n} x^{2^{\mu+1}} \, dx + \gamma^\mu_{m,0}(\phi) C^{\mu-1}_\mu \int_{\mathbb{R}^n_+ \setminus [0,1]^n} x^{2^{\mu+1}-m} \, dx < \infty.
\]
Thus
\[
\| \phi \|_{L^1(\mathbb{R}^n_+, sr)} \leq C\{ \gamma^{\mu}_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi) \}, \quad \phi \in \mathcal{H}_\mu.
\]

Now let us see that \( L^1(\mathbb{R}^n_+, sr) \cap L^\infty(\mathbb{R}^n_+, r) \subset L^p(\mathbb{R}^n_+, sr^p) \). Let \( \phi \in L^1(\mathbb{R}^n_+, sr) \cap L^\infty(\mathbb{R}^n_+, r) \)
\[
\int_{\mathbb{R}^n_+} |\phi(x)|^p s(x) r^{p}(x) \, dx = \int_{\mathbb{R}^n_+} |\phi(x)|^{p-1} r(x) |\phi(x)| s(x) r(x) \, dx
\]
\[
= \int_{\mathbb{R}^n_+} |r(x)| \phi(x) |\phi(x)|^{p-1} |\phi(x)| s(x) r(x) \, dx
\]
\[
\leq \| \phi \|_{L^\infty(\mathbb{R}^n_+, r)} \| \phi \|_{L^1(\mathbb{R}^n_+, sr)}^p \]
from where
\[
\| \phi \|_{L^p(\mathbb{R}^n_+, sr^p)} \leq \| \phi \|_{L^\infty(\mathbb{R}^n_+, r)} \| \phi \|_{L^1(\mathbb{R}^n_+, sr)}^p.
\]

From (2.15) and (2.16) we can consider that there exist constants \( C_1 \) y \( C_2 \) such that
\[
\| \phi \|_{L^\infty(\mathbb{R}^n_+, r)} \leq C_1 \{ \gamma^{\mu}_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi) \}, \quad \phi \in \mathcal{H}_\mu.
\]
\[
\| \phi \|_{L^1(\mathbb{R}^n_+, sr)} \leq C_2 \{ \gamma^{\mu}_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi) \}, \quad \phi \in \mathcal{H}_\mu.
\]

Then from (2.17), (2.18) y (2.19) we can consider a constant \( C_3 \) such that
(2.20) \[ \|\phi\|_{L^p(\mathbb{R}_+^n, sr^p)} \leq C_3 \{ \gamma_{m,0}^\mu(\phi) + \gamma_{m,0}^\nu(\phi) \}, \quad \phi \in H_\mu. \]

**Remark 2.4.** If \( \phi \in L^1(\mathbb{R}_+^n, sr) \), then Hankel transform \( h_\mu \phi \) is well defined because the kernel \( (x,y)^{-\mu}J_\mu(x,y) \) is bounded for \( \mu > -\frac{1}{2}, \ i = 1, \ldots, n \) (see [16, (1), pp.49]),

\[
\int_{\mathbb{R}_+^n} |\phi(x)| \prod_{i=1}^n \{(x_iy_i)^{\mu+1/2}(x_iy_i)^{-\mu}J_{\mu}(x_iy_i)\} \, dx \\
\leq y^{\mu+1/2}M^n \int_{\mathbb{R}_+^n} |\phi(x)| x^{\mu+1/2} \, dx = Cy^{\mu+1/2} \|\phi\|_{L^1(\mathbb{R}_+^n, sr)} < \infty
\]

By Lemma 2.3, \( h_\mu \phi \) is well defined for all \( \phi \in H_\mu \) and is an automorphism of \( H_\mu \) (see [18] for the 1-dimensional case and [12] for the n-dimensional case.)

The space of continuous linear functions \( T : H_\mu \to \mathbb{C} \) is denoted by \( H'_\mu \). We call a function \( f \in L^1_{\text{loc}}(\mathbb{R}_+^n) \) a regular element of \( H'_\mu \) if the application \( T_f \in H'_\mu \) where \( T_f(\phi) = \int_{\mathbb{R}_+^n} f\phi \), with \( \phi \in H_\mu \).

**Lemma 2.5.** Let \( 1 \leq p < \infty \). A function in \( L^p(\mathbb{R}_+^n, sr^p) \) or in \( L^\infty(\mathbb{R}_+^n, r) \) is a regular element of \( H'_\mu \). In particular, the functions in \( H_\mu \) can be considered as regular elements of \( H'_\mu \).

**Proof.** Let \( f \in L^\infty(\mathbb{R}_+^n, r) \) and \( \phi \in H_\mu \). Since \( H_\mu \subset L^1(\mathbb{R}_+^n, sr) \), then \( \phi \in L^1(\mathbb{R}_+^n, r^{-1}) \) and \( (T_f, \phi) = \int_{\mathbb{R}_+^n} f\phi \) is well defined. So, by (2.16)

\[
|(T_f, \phi)| \leq \|f\|_{L^\infty(\mathbb{R}_+^n, r)} \|\phi\|_{L^1(\mathbb{R}_+^n, r^{-1})} = C_\mu \|f\|_{L^\infty(\mathbb{R}_+^n, r)} \|\phi\|_{L^1(\mathbb{R}_+^n, sr)} \leq C_\mu \|f\|_{L^\infty(\mathbb{R}_+^n, r)} \{\gamma_{m,0}^\mu(\phi) + \gamma_{m,0}^\nu(\phi)\},
\]

consequently, \( f \) is a regular element of \( H'_\mu \).

Now, let \( f \in L^p(\mathbb{R}_+^n, sr^p) \) with \( 1 \leq p < \infty \) and \( \phi \in H_\mu \), then

\[
|(T_f, \phi)| \leq \int_{\mathbb{R}_+^n} |f(x)\phi(x)| \, dx = \int_{\mathbb{R}_+^n} |r(x)f(x)| |s^{-1}(x)r^{-1}(x)\phi(x)| s(x) \, dx
\]

(2.21)

\[
= \int_{\mathbb{R}_+^n} |r(x)f(x)| C_\mu |r(x)\phi(x)| s(x) \, dx.
\]

Since \( r|f| \in L^p(\mathbb{R}_+^n, s) \) and \( r|\phi| \in L^q(\mathbb{R}_+^n, s) \), being \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then due to Hölder’s inequality and (2.20) we obtain that

\[
|(T_f, \phi)| \leq C_\mu \|f\|_{L^p(\mathbb{R}_+^n, sr^p)} \|\phi\|_{L^q(\mathbb{R}_+^n, sr^q)} \leq C_\mu \|f\|_{L^p(\mathbb{R}_+^n, sr^p)} \{\gamma_{m,0}^\mu(\phi) + \gamma_{m,0}^\nu(\phi)\}
\]

with \( m > 2\mu_i + 2, \ i = 1, \ldots, n \). Therefore \( f \) is a regular element of \( H'_\mu \). \( \blacksquare \)

**Remark 2.6.** In particular if \( p = 2 \), \( L^p(\mathbb{R}_+^n, sr^p) = L^2(\mathbb{R}_+^n) \) and from the previous Lemma we have that the functions in \( L^2(\mathbb{R}_+^n) \) can be considered as regular elements of \( H'_\mu \).

Given \( f, g \) defined on \( \mathbb{R}_+^n \), the Hankel convolution associated to the transformation \( h_\mu \) is defined formally by

\[
(f* g)(x) = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} D_\mu(x, y, z) f(y)g(z) \, dy \, dz
\]

where for every \( x, y, z \in \mathbb{R}_+^n \),

\[
D_\mu(x, y, z) = \prod_{i=1}^n D_{\mu_i}(x_i, y_i, z_i)
\]

where \( D_{\mu_i} \) is the Delsarte kernel defined in [4], given by
Let $L$ be the area of the triangle with sides $u, v, w \in \mathbb{R}$ and $\alpha > -\frac{1}{2}$.

Note that $|u - v| < w < u + v$ is the condition for such triangle to exist, and in this case

$$A(u, v, w) = \begin{cases} \frac{1}{4} \sqrt{[(u + v)^2 - w^2][w^2 - (u - v)^2]} & |u - v| < w < u + v \\ 0 & 0 < w < |u - v| \text{ or } w > u + v, \end{cases}$$

Remark 2.7. If $u, v$ and $w$ are the sides of a triangle and $\theta$ is the angle opposite the side $w$, then

$$A(u, v, w) = \frac{w \sin \theta}{2}$$

Proposition 2.8.

(i) $D_\mu(x, y, z) \geq 0, \quad x, y, z \in \mathbb{R}_+^n$.

(ii) $\int_{\mathbb{R}_+^n} D_\mu(x, y, z) \prod_{i=1}^{n} \{\sqrt{x_i} J_{\mu_i}(z_i t_i)\} \, dz = t^{-\mu - 1/2} \prod_{i=1}^{n} \{\sqrt{x_i} J_{\mu_i}(x_i t_i)\}$

(iii) $\int_{\mathbb{R}_+^n} x^{\mu+1/2} D_\mu(x, y, z) \, dz = C^{-1}_\mu x^{\mu+1/2} y^{\mu+1/2}$

Proof. For the proof of this result, we refer the reader to the Appendix, page 23.

Lemma 2.9. Let $f \in L^1(\mathbb{R}_+^n, sr)$.

(i) If $g \in L^\infty(\mathbb{R}_+^n, r)$, then the convolution $f * g(x)$ exists for every $x \in \mathbb{R}_+^n$, $f * g(x) \in L^\infty(\mathbb{R}_+^n, r)$ and

$$\|f * g\|_{L^\infty(\mathbb{R}_+^n, r)} \leq \|f\|_{L^1(\mathbb{R}_+^n, sr)} \|g\|_{L^\infty(\mathbb{R}_+^n, r)}.$$

(ii) If $g \in L^p(\mathbb{R}_+^n, sr^p)$, $1 \leq p < \infty$, then the convolution $f * g(x)$ exists for almost every $x \in \mathbb{R}_+^n$, $f * g(x) \in L^p(\mathbb{R}_+^n, sr^p)$ and

$$\|f * g\|_{L^p(\mathbb{R}_+^n, sr^p)} \leq \|f\|_{L^1(\mathbb{R}_+^n, sr^p)} \|g\|_{L^p(\mathbb{R}_+^n, sr^p)}.$$

Proof. For the proof of Lemma 2.9, refer to the Appendix, page 25.

The proof of the following results uses standard arguments and it will be omitted.

Lemma 2.10. Let $f, g \in L^1(\mathbb{R}_+^n, sr)$, then

$$h_\mu(f * g) = r h_\mu(f) h_\mu(g).$$

Lemma 2.11. Let $f \in L^1(sr)$, then the Hankel transform $h_\mu f \in L^\infty(r)$ and

$$\|h_\mu f\|_{L^\infty(r)} \leq \|f\|_{L^1(sr)}.$$

Remark 2.12. Given $f \in L^1(\mathbb{R}_+^n)$ we have that $h_\mu f$ is continuous, is in $L^\infty(\mathbb{R}_+^n)$ and

$$\|h_\mu f\|_{L^\infty(\mathbb{R}_+^n)} \leq C\|f\|_1.$$

Proposition 2.13. $h_\mu(L^1(\mathbb{R}_+^n)) \subset C_0(\mathbb{R}_+^n)$.

Proof. First, we observe that

$$L^1(\mathbb{R}_+^n, sr) \cap L^\infty(\mathbb{R}_+^n, r) \subset L^1(\mathbb{R}_+^n).$$

Let $Q$ be the cube $Q = [0, 1]^n$, then

$$\int_{\mathbb{R}_+^n} |f(x)| \, dx = \int_{\mathbb{R}_+^n} |f(x)| r(x) r^{-1}(x) \, dx$$

$$= \int_{Q \cap \mathbb{R}_+^n} |f(x)| r(x) r^{-1}(x) \, dx + \int_{Q^c \cap \mathbb{R}_+^n} |f(x)| r(x) r^{-1}(x) \, dx$$

$$\leq \|f\|_{L^\infty(\mathbb{R}_+^n, r)} \int_{Q \cap \mathbb{R}_+^n} r^{-1}(x) \, dx + \int_{Q^c \cap \mathbb{R}_+^n} |f(x)| r^{-1}(x) \, dx$$

$$\leq C \|f\|_{L^\infty(\mathbb{R}_+^n, r)} + C_\mu \|f\|_{L^1(\mathbb{R}_+^n, sr)},$$

because $r(x) < 1$ for $||x|| > 1$, $\mu_i > -\frac{1}{2}$, $i = 1, \ldots, n$ and $r(x)s(x) = C^{-1}_\mu r^{-1}(x)$.
By (2.14) and (2.29) we deduce that \( \mathcal{H}_\mu \subset L^1(\mathbb{R}^n_+) \). Since \( \mathcal{D}(\mathbb{R}^n_+) \subset \mathcal{H}_\mu \) then \( \mathcal{H}_\mu \) is dense in \( L^1(\mathbb{R}^n_+) \). Given \( f \in L^1(\mathbb{R}^n_+) \) and \( \{ \phi_m \} \in \mathcal{H}_\mu \) such that \( \phi_m \to f \) in \( L^1(\mathbb{R}^n_+) \), then by Remark 2.12 \( h_\mu(\phi_m) \to h_\mu(f) \) uniformly. Since \( h_\mu(\phi_m) \in C_0(\mathbb{R}^n_+) \) then \( h_\mu(f) \in C_0(\mathbb{R}^n_+) \).

\[ \blacksquare \]
Lemma 2.14. Let \( \{ \phi_m \} \subset L^1(\mathbb{R}^n_+, sr) \) such that

1. \( \phi_m \geq 0 \) in \( \mathbb{R}^n_+ \),
2. \( \int_{\mathbb{R}^n_+} \phi_m(x) s(x) r(x) \, dx = 1 \) for all \( m \in \mathbb{N} \),
3. For all \( \eta > 0 \), \( \lim_{m \to \infty} \int_{\|z\| > \eta} \phi_m(x) \, r(x) s(x) \, dx = 0 \).

Let \( f \in L^\infty(\mathbb{R}^n_+, r) \) and continuous in \( x_0 \in \mathbb{R}^n_+ \), then \( \lim_{m \to \infty} \int f \phi_m(x) = f(x_0) \). Moreover, if \( rf \) is uniformly continuous in \( \mathbb{R}^n_+ \) then

\[
\lim_{m \to \infty} \|f \phi_m(x) - f(x)\|_{L^\infty(\mathbb{R}^n_+, r)} = 0.
\]

Proof. First let us observe that

\[
\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} x_0^{-\mu-1/2} y^{\mu+1/2} D_\mu(x_0, y, z) \, \phi_m(z) \, dydz
\]

\[
= \int_{\mathbb{R}^n_+} x_0^{-\mu-1/2} \phi_m(z) \left( \int_{\mathbb{R}^n_+} y^{\mu+1/2} D_\mu(x_0, y, z) \, dy \right) dz
\]

\[
= \int_{\mathbb{R}^n_+} x_0^{-\mu-1/2} \phi_m(z) C_\mu^{-1} x_0^{\mu+1/2} y^{\mu+1/2} \, dz
\]

\[
= \int_{\mathbb{R}^n_+} \phi_m(z) s(z) r(z) \, dz = 1.
\]

Let \( \varepsilon > 0 \), then since \( f \) is a continuous function in \( x_0 \), there exists \( \delta > 0 \) such that if \( \|y - x_0\| < \delta \), then \( |y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)| < \frac{\varepsilon}{2x_0^{\mu+1/2}} \).

\[
f \phi_m(x_0) - f(x_0) = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz
\]

\[
= \int_{\|z\| > \delta} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz
\]

\[
+ \int_{\|z\| < \delta} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz.
\]

Calling

\[
I_1 = \int_{\|z\| > \delta} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz
\]

\[
I_2 = \int_{\|z\| < \delta} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz,
\]

from where

\[
|f \phi_m(x_0) - f(x_0)| \leq |I_1| + |I_2|.
\]

Since \( f \in L^\infty(r) \),

\[
|y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)| \leq 2\|f\|_{L^\infty(r)}.
\]

Moreover, since \( \lim_{m \to \infty} \int_{\|z\| > \delta} \phi_m(z) s(z) r(z) \, dz = 0 \), there exists \( N_0 \in \mathbb{N} \) such that

\[
\int_{\|z\| > \delta} \phi_m(z) s(z) r(z) \, dz < \frac{\varepsilon}{4\|f\|_{L^\infty(r)} x_0^{\mu+1/2}}, \quad \forall m > N_0.
\]

\[
|I_1| \leq \int_{\|z\| > \delta} \int_{\mathbb{R}^n_+} y^{\mu+1/2} \phi_m(z) D_\mu(x_0, y, z) [y^{-\mu-1/2} f(y) - x_0^{-\mu-1/2} f(x_0)] \, dy \, dz
\]

\[
= 2\|f\|_{L^\infty(r)} \int_{\|z\| > \delta} \phi_m(z) \left( \int_{\mathbb{R}^n_+} y^{\mu+1/2} D_\mu(x_0, y, z) \, dy \right) dz
\]
\[ = 2 \| f \|_{L^\infty(z)} \int_{\| z \| > \frac{\varepsilon}{2}} \phi_m(z) C_{\mu}^{-1} x_0^{\mu - 1/2} z^{\mu + 1/2} dz \]

\[ = 2 \| f \|_{L^\infty(z)} \int_{\| z \| > \frac{\varepsilon}{2}} \phi_m(z) s(z) r(z) dz \]

\[ < 2 \| f \|_{L^\infty(z)} x_0^{\mu - 1/2} \frac{\varepsilon}{4} \| f \|_{L^\infty(z)} x_0^{\mu + 1/2} = \frac{\varepsilon}{2} \]

On the other hand, if \( \| z \| < \frac{\varepsilon}{\sqrt{n}} \), then \( z_i \in (0, \delta/\sqrt{n}) \) for all \( i = 1, \ldots, n \). Moreover, if we consider \( D_\mu(x_0, y, z) \) as a function depending on \( y \) then

\[ \text{supp } D_\mu(x_0, y, z) \subset (|x_0^0 - z_1|, x_0^0 + z_1) \times \cdots \times (|x_n^0 - z_n|, x_n^0 + z_n) \]

\[ \subset (x_0^0 - \delta/\sqrt{n}, x_0^0 + \delta/\sqrt{n}) \times \cdots \times (x_n^0 - \delta/\sqrt{n}, x_n^0 + \delta/\sqrt{n}). \]

So, if \( y \in \text{supp } D_\mu(x_0, y, z) \) then \( |y_i - x_i^0| < \delta/\sqrt{n} \), for all \( i = 1, \ldots, n \). Thus

\[ \| y - x_0 \|_{\infty} \leq \max_{1 \leq i \leq n} |y_i - x_i^0| < \delta/\sqrt{n}. \]

Since the euclidean norm is equivalent to the uniform norm and \( \| \cdot \|_{\infty} \leq \| \cdot \| \leq \sqrt{n} \| \cdot \|_{\infty} \), then we obtain that \( \| z \| < \frac{\varepsilon}{\sqrt{n}} \) implies \( \| y - x_0 \| < \delta \), then

\[ |I_2| \leq \int_{\| z \| < \frac{\varepsilon}{\sqrt{n}}} \int_{\mathbb{R}_+^n} y^{\mu + 1/2} \phi_m(z) D_\mu(x_0, y, z) |y^{\mu - 1/2} f(y) - x_0^{\mu - 1/2} f(x_0)| \, dy \, dz \]

\[ \leq \int_{\| z \| < \frac{\varepsilon}{\sqrt{n}}} \int_{\mathbb{R}_+^n} y^{\mu + 1/2} \phi_m(z) D_\mu(x_0, y, z) \frac{\varepsilon}{2x_0^{\mu + 1/2}} \, dy \, dz \]

\[ = \frac{\varepsilon}{2x_0^{\mu + 1/2}} \int_{\| z \| < \frac{\varepsilon}{\sqrt{n}}} \phi_m(z) \left( \int_{\mathbb{R}_+^n} y^{\mu + 1/2} D_\mu(x_0, y, z) \, dy \right) \, dz \]

\[ = \frac{\varepsilon}{2x_0^{\mu + 1/2}} \int_{\| z \| < \frac{\varepsilon}{\sqrt{n}}} \phi_m(z) C_{\mu}^{-1} x_0^{\mu + 1/2} z^{\mu + 1/2} \, dz \]

\[ = \frac{\varepsilon}{2} \int_{\| z \| < \frac{\varepsilon}{\sqrt{n}}} \phi_m(z) s(z) r(z) \, dz \leq \frac{\varepsilon}{2} \int_{\mathbb{R}_+^n} \phi_m(z) s(z) r(z) \, dz = \frac{\varepsilon}{2}. \]

From where we have proved that given \( \varepsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that \( |f_t^\# \phi_m(x_0) - f(x_0)| < \varepsilon \), for all \( n > N_0 \). The uniform convergence is obtained analogously to the uniform continuity of \( rf \).

We are going to consider Bessel operators in \( \mathbb{R}_+^n \) given by (1.1) and (1.2) which are related through

(2.31) \[ S_\mu = x^{\mu + 1/2} \Delta_\mu x^{-\mu - 1/2}, \]

see remark C.1 for a proof.

Bessel operator (1.1) and Hankel transform (2.8) were studied in the distributional setting over the Zemanian spaces \( \mathcal{H}_\mu \) and \( \mathcal{H}'_\mu \) in [17] (1-dimensional case), [10] and [12] (n-dimensional case).

\( S_\mu \) is a continuous operator in \( \mathcal{H}_\mu \) and selfadjoint, so the generalized Bessel operator \( S_\mu \) can be extended to \( \mathcal{H}'_\mu \) by transposition

\( (S_\mu f, \phi) = (f, S_\mu \phi), \quad f \in \mathcal{H}'_\mu, \quad \phi \in \mathcal{H}_\mu. \)

Analogously, generalized Hankel transform \( h_\mu f \) can be extended to \( \mathcal{H}'_\mu \) by

\( (h_\mu f, \phi) = (f, h_\mu \phi), \quad f \in \mathcal{H}'_\mu, \quad \phi \in \mathcal{H}_\mu \)

for \( \mu = (\mu_1, \ldots, \mu_n), \mu_i > -\frac{1}{2}, \ i = 1, \ldots, n. \) Then \( h_\mu \) is an automorphism over \( \mathcal{H}_\mu \) and \( \mathcal{H}'_\mu \).
There exist different proofs for the inversion theorem of the Hankel transform for the 1-dimensional case. In this work we present a proof for the inversion theorem for the $n$-dimensional case, in the same way of the classic versions of the results known for the inversion of the Fourier transform in Lebesgue spaces.

**Theorem 2.15.** Let $f \in L^1(\mathbb{R}^n_+, x^{\mu+1/2})$ and $h_\mu f \in L^1(\mathbb{R}^n_+, x^{\mu+1/2})$ where $x^{\mu+1/2}$ is given by (2.1). Then $f(x)$ may be redefined on a set of measure zero so that it is continuous on $\mathbb{R}^n_+$ and

\[(2.32) \quad f(x) = h_\mu(h_\mu f)(x),\]

for almost every $x \in \mathbb{R}^n_+$.

**Proof.** For the proof of this result we refer the reader to the Appendix. Details can be found in page 29. \hfill $\blacksquare$

**Remark 2.16.** From Theorem 2.15 we deduce immediately the validity of equality (2.32) in $\mathcal{H}_\mu$ and $\mathcal{H}_\mu'$.

For the proof of the next results we refer the reader to [12].

**Lemma 2.17.** Let $\phi \in \mathcal{H}_\mu$, then

1. $h_\mu S_\mu \phi = -\|y\|^2 h_\mu \phi$.
2. $S_\mu h_\mu \phi = h_\mu (-\|x\|^2 \phi)$.

**Lemma 2.18.** If $u \in \mathcal{H}_\mu'$, then

1. $h_\mu S_\mu u = -\|x\|^2 h_\mu u$.
2. $S_\mu h_\mu u = h_\mu (-\|y\|^2 u)$.

**Remark 2.19.** According to Lemma 3.2 in [10] the functions $(\lambda + \|x\|^2)$ for $\lambda \geq 0$ and $(\lambda + \|x\|^2)^{-1}$ for $\lambda > 0$ belong to the space of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}_\mu'$.

So, the next result holds.

**Lemma 2.20.** The following equalities are valid in $\mathcal{H}_\mu$ and in $\mathcal{H}_\mu'$ for $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

1. $(-S_\mu + \lambda)^m h_\mu = h_\mu(\lambda + \|y\|^2)^m$.
2. $h_\mu(-S_\mu + \lambda)^{-m} = (\lambda + \|y\|^2)^{-m} h_\mu$.
3. $h_\mu(-S_\mu - S_\mu + \lambda)^{-m} = \|y\|^{2m}(\lambda + \|y\|^2)^{-m} h_\mu$.

**Proof.** We refer the reader to page 30 in the Appendix for details. \hfill $\blacksquare$

3. **Non-negativity and fractional powers of similar operators**

In this section we include a brief review of non-negative operators in Banach spaces and locally convex spaces. Let $X$ be a Banach space (real or complex). Let $A$ be a closed linear operator $A : D(A) \subset X \to X$ and $\rho(A)$ the resolvent set of $A$. We say that $A$ is non-negative if $(-\infty, 0) \subset \rho(A)$ and

$$\sup_{\lambda > 0} \{\|\lambda(\lambda + A)^{-1}\|\} < \infty.$$

Now, let $X$ is a locally convex space with a Hausdorff topology generated by a directed family of seminorms $\{\|\|_\alpha\}_{\alpha \in A}$. A family of linear operators $\{A_\lambda\}_{\lambda \in \Gamma}$, $A_\lambda : D(A_\lambda) \subset X \to X$, is equicontinuous if for each $\alpha \in \Lambda$ there are $\beta = \beta(\alpha) \in \Lambda$ and a constant $C = C_\alpha \geq 0$ such that for all $\lambda \in \Gamma$

$$\|A_\lambda \phi\|_\alpha \leq C \|\phi\|_\beta, \quad \phi \in X.$$

Under the above conditions, we say that a closed linear operator $A : D(A) \subset X \to X$ is non-negative if $(-\infty, 0) \subset \rho(A)$ and the family of operators

$$\{\lambda(\lambda + A)^{-1}\}_{\lambda > 0}$$

is equicontinuous.
Now, we will briefly describe the theory of fractional powers of operators. According to [9, Proposition 3.1.3], we can define the Balakrishnan operator $J^\alpha$ in the following way.

Let $A$ be a non-negative operator in a Banach space or in a locally convex and sequentially complete space. Let $\alpha \in \mathbb{C}_+$ and $n > \Re \alpha$, $n \in \mathbb{N}$. If $\phi \in D(A^n)$ and $m \geq n$ is a positive integer, then

$$J^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m - \alpha)} \int_0^\infty \lambda^{n-1} [A(\lambda + A)^{-1}]^m \phi \, d\lambda.$$  

(3.1)

If $A$ is bounded, $J^\alpha A$ can be considered as the fractional power of $A$. In other cases we can consider the following representation for the fractional power stated in [9, Theorem 5.2.1]

**Theorem 3.1.** Let $A$ be a non-negative operator, $\alpha \in \mathbb{C}_+$, $\lambda \in \rho(-A)$ and $n \in \mathbb{N}$. Then

$$A^\alpha = (A + \lambda)^n J^\alpha \lambda^n (A + \lambda)^{-n}.$$  

(3.2)

(If $n > \Re \alpha$, the operator $J^\alpha A$ can be replaced by $J^\alpha A$ in the preceding formula.)

Similar operators have been described in the introduction. Let $A$ and $B$ similar operators and $T$ the isomorphism that verifies (1.3) then

$$(z \text{Id} - B)^{-1} = T(z \text{Id} - A)^{-1}T^{-1},$$

for $z$ a complex number, from where we deduce immediately that $A$ is non-negative operator if and only if so is $B$.

When two operators are similar, the fractional powers also meet this property. Thus we have the following result which holds in Banach spaces and in locally convex and sequentially complete spaces.

**Proposition 3.2.** Let $A$ and $B$ be similar non-negative operators. If $\alpha \in \mathbb{C}_+$ then

$$J^\alpha_B = TJ^\alpha_AT^{-1},$$

(3.3)

and

$$B^\alpha = T A^\alpha T^{-1},$$

(3.4)

where $T$ is the isometric isomorphism that verifies $B = TAT^{-1}$.

4. **Fractional powers of $S_\mu$ in Lebesgue spaces**

Let $s$ and $r$ as in Section 2 and let $1 \leq p < \infty$. We will denote by $S_{\mu,p}$ the part of $S_\mu$ in $L^p(\mathbb{R}_+, sr^p)$, that is to say, the operator $S_\mu$ with domain

$$D(S_{\mu,p}) = \{ f \in L^p(\mathbb{R}^n_+, sr^p) : S_{\mu}f \in L^p(\mathbb{R}^n_+, sr^p) \}$$

and given by $S_{\mu,p}f = S_{\mu}f$.

Analogously, with $S_{\mu,\infty}$ we will denote the part of $S_\mu$ in $L^\infty(\mathbb{R}^n_+, r)$, $\Delta_{\mu,p}$ and $\Delta_{\mu,\infty}$ the part of $\Delta_\mu$ in $L^p(\mathbb{R}^n_+, s)$ and $L^\infty(\mathbb{R}^n_+, s)$ respectively.

Let $L_\nu$ the isometric isomorphism

$$L_\nu : L^p(\mathbb{R}_+, sr^p) \to L^p(\mathbb{R}_+, s), \quad \text{with} \quad 1 \leq p < \infty$$

(or $L_\nu : L^\infty(\mathbb{R}_+, r) \to L^\infty(\mathbb{R}_+, s)$) given by

$$L_\nu(f) = r f.$$  

(4.1)

Let then

$$S_{\mu,p} = L^{-1}_\nu \circ \Delta_{\mu,p} \circ L_\nu.$$  

Consequently it is enough to study the operator $S_\mu$ in the spaces $L^p(\mathbb{R}_+, sr^p)$ (or $L^\infty(\mathbb{R}_+, r)$). In order to study the non-negativity of operators $-S_{\mu,p}$ and $-S_{\mu,\infty}$ we consider the following function given by

$$N_\nu(w) = \int_0^\infty e^{-t - \frac{w^2}{4t}} \frac{dt}{t^{\nu+1}}.$$
which is defined for all \( \nu \in \mathbb{R} \) and \( w \in \mathbb{R}_+ \).

Let \( t \in \mathbb{R}_+ \), if \( \mu = (\mu_1, \ldots, \mu_n) \), then \( t^{\mu+1} \) means
\[
\mu^{n+1} = t^{\mu_1+1} \cdots t^{\mu_n+1} = t^{\mu_1 + \cdots + \mu_n + n},
\]
from where
\[
(4.2) \quad \mathcal{N}_{\mu_1 + \cdots + \mu_n + n-1}(\|x\|) = \int_0^\infty e^{-\frac{\|x\|^2}{t}} \frac{dt}{t^{\mu_1 + \cdots + \mu_n + n+1}} = \int_0^\infty e^{-\frac{\|x\|^2}{t}} \frac{dt}{t^{\mu+1}}.
\]

Given \( \lambda > 0 \), let us consider the function
\[
(4.3) \quad N_\lambda(x) = 2^{-\mu - 1} x^{\mu + 1/2} \lambda^{\mu - 1} \mathcal{N}_{\mu_1 + \cdots + \mu_n + n-1}(\|\sqrt{\lambda}x\|), \quad x \in \mathbb{R}_+^n.
\]

**Lemma 4.1.** Given \( \mu = (\mu_1, \ldots, \mu_n) \), \( \mu_i > -\frac{1}{2} \) and \( \lambda > 0 \) then

(a) \( N_\lambda \in L^1(\mathbb{R}_+^n, s\nu) \) and
\[
\|N_\lambda\|_{L^1(\mathbb{R}_+^n, s\nu)} = \frac{1}{\lambda}
\]

(b) \[
h_\mu N_\lambda(y) = \frac{y^{\mu+1/2}}{\lambda + \|y\|^2}
\]

**Proof (a).**
\[
\|N_\lambda\|_{L^1(\mathbb{R}_+^n, s\nu)} = \int_{\mathbb{R}_+^n} |N_\lambda(x)| \frac{x^{\mu+1/2}}{C_\mu} dx
\]
\[
= \int_{\mathbb{R}_+^n} 2^{-\mu - 1} x^{\mu + 1/2} \lambda^{\mu - 1} \mathcal{N}_{\mu_1 + \cdots + \mu_n + n-1}(\|\sqrt{\lambda}x\|) \frac{x^{\mu+1/2}}{C_\mu} dx
\]
\[
= 2^{-\mu - 1} \lambda^{\mu - 1} \int_{\mathbb{R}_+^n} \left\{ \int_0^\infty e^{-\frac{\|x\|^2}{t}} dt \right\} x^{2\mu + 1} dx
\]
\[
= 2^{-\mu - 1} \lambda^{\mu - 1} \int_{\mathbb{R}_+^n} \left\{ \int_0^\infty e^{-\frac{\|x\|^2}{2t}} x^{2\mu + 1} dx \right\} e^{-t} dt
\]
\[
= 2^{-\mu - 1} \lambda^{\mu - 1} \int_{\mathbb{R}_+^n} \left\{ \prod_{i=1}^n \left\{ \int_0^\infty e^{-\frac{\|x_i\|^2}{4t}} x_i^{2\mu_i + 1} dx_i \right\} e^{-t} dt \right\}
\]
\[
= 2^{-\mu - 1} \lambda^{\mu - 1} \int_{\mathbb{R}_+^n} \left\{ \prod_{i=1}^n \left\{ 2^{\mu_i} \Gamma(\mu_i + 1) \left( \frac{2}{\lambda} \right)^{\mu_i + 1} \right\} e^{-t} \right\} dt
\]
\[
= 2^{-\mu - 1} \lambda^{\mu - 1} \int_{\mathbb{R}_+^n} \left\{ \prod_{i=1}^n \left\{ 2^{\mu_i} \Gamma(\mu_i + 1) \right\} e^{-t} \right\} dt
\]
where we have used the formula (A.6).

**Proof (b).**
\[
h_\mu N_\lambda(y) = \int_{\mathbb{R}_+^n} N_\lambda(x) \prod_{i=1}^n \{ \sqrt{x_i y_i} \mu(x_i y_i) \} dx
\]
\[
= \int_{\mathbb{R}_+^n} 2^{-\mu - 1} x^{\mu + 1/2} \lambda^{\mu - 1} \left\{ \int_0^\infty e^{-\frac{\|x\|^2}{2t}} dt \right\} \prod_{i=1}^n \{ \sqrt{x_i y_i} \mu(x_i y_i) \} dx
\]
\[
= 2^{-\mu - 1} y^{\mu + 1/2} \lambda^{\mu - 1} \int_0^\infty \left\{ \int_{\mathbb{R}_+^n} x^{\mu + 1} e^{-\frac{\|x\|^2}{2t}} \prod_{i=1}^n \{ J_{\mu_i}(x_i y_i) \} dx \right\} e^{-t} dt
\]
\[
= 2^{-\mu - 1} y^{\mu + 1/2} \lambda^{\mu - 1} \int_0^\infty \prod_{i=1}^n \left\{ \int_0^\infty x_i^{\mu_i + 1} e^{-\frac{\|x_i\|^2}{2t}} J_{\mu_i}(x_i y_i) dx_i \right\} e^{-t} dt
\]
\[
\begin{align*}
&= 2^{-\mu - 1} y^{1/2} \lambda^\mu \lambda^{n-1} \int_0^\infty \prod_{i=1}^n \left\{ \left( \frac{\lambda}{2t} \right)^{-\mu - 1} y_t^\mu e^{-\frac{\lambda t^2}{2}} \right\} e^{-t} \frac{dt}{\mu + 1} \\
&= 2^{-\mu - 1} y^{1/2} \lambda^\mu \lambda^{n-1} 2^{\mu + 1} \lambda^{-\mu - 1} y^{\mu} \int_0^\infty \mu^{\mu + 1} e^{-\frac{\mu t^2}{2}} e^{-t} \frac{dt}{\mu + 1} \\
&= y^{\mu + 1/2} \lambda^\mu \lambda^{-1} \int_0^\infty e^{-t(1 + \frac{\mu}{\lambda})} dt \\
&= y^{\mu + 1/2} \lambda^\mu \lambda^{-1} \int_0^\infty e^{-s} ds \\
&= \frac{y^{\mu + 1/2}}{\lambda + \|y\|^2 }
\end{align*}
\]
where we have used (A.4).

\textbf{Lemma 4.2.} Let \(1 \leq p < \infty\). If \(f \in L^p(\mathbb{R}_+, sr^\mu)\) or \(f \in L^\infty(\mathbb{R}_+, r)\) then the following equality holds on \(\mathcal{H}_\mu^r\)

\[
h_\mu(N_\lambda f) = \frac{1}{\lambda + \|y\|^2} h_\mu f
\]

\textbf{Proof.} Suppose that \(f \in L^p(\mathbb{R}_+, sr^\mu)\) and \(\psi \in \mathcal{H}_\mu\), we claim that

\[
\int_{\mathbb{R}_+^n} (N_\lambda \psi)(x) dx = \int_{\mathbb{R}_+^n} f(z)(N_\lambda \psi)(z) dz
\]

(4.5)

\[
\int_{\mathbb{R}_+^n} f(z)(N_\lambda \psi)(z) dz = \int_{\mathbb{R}_+^n} f(z) \left\{ \int_{\mathbb{R}_+^n} N_\lambda(y) \psi(y) D_\mu(x, y, z) dy \right\} dz
\]

(4.6)

Let us see that \(\int_{\mathbb{R}_+^n} |f(z)| \left\{ \int_{\mathbb{R}_+^n} |N_\lambda(y)| |\psi(y)| D_\mu(x, y, z) dy \right\} dz\) is finite. Let

\[
G(z) = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |N_\lambda(y)| |\psi(y)| D_\mu(x, y, z) dy dx
\]

and let \(q\) such that \(\frac{1}{p} + \frac{1}{q} = 1\). The function \(G\) is the convolution of \(|N_\lambda|\) and \(|\psi|\). From Lemma 2.9, since \(|N_\lambda| \in L^1(\mathbb{R}_+, sr)\) and \(|\psi| \in L^q(\mathbb{R}_+, sr^\mu)\) we have that \(G \in L^q(\mathbb{R}_+, sr^\mu)\), then

\[
\int_{\mathbb{R}_+^n} |f(z)| |G(z)| dz = \int_{\mathbb{R}_+^n} (r|f(z)|) (r^{-1}s^{-1}|G(z)|) s dz
\]

\[
= \int_{\mathbb{R}_+^n} (r|f(z)|) (C_\mu r|G(z)|) s dz
\]

\[
= C_\mu \int_{\mathbb{R}_+^n} |rf(z)| |rG(z)| s dz
\]

\[
\leq C_\mu \|rf\|_{L^p(\mathbb{R}_+, sr)} \|rG\|_{L^q(\mathbb{R}_+, sr^\mu)}
\]

then it is possible to change the order of integration in (4.6).

\[
\int_{\mathbb{R}_+^n} f(z) (N_\lambda \psi)(z) dz = \int_{\mathbb{R}_+^n} \left\{ \int_{\mathbb{R}_+^n} N_\lambda(y) f(z) D_\mu(x, y, z) dy \right\} \psi(x) dx
\]

(4.6)

So, we have proved (4.6).
Now let \( f \in L^\infty(\mathbb{R}^n_+, r) \). To see that (4.5) holds, it will be enough to see that

\[
\int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} |f(z)| \, |N_\lambda(y)| \, |\psi(x)| \, D_\mu(x, y, z) \, dz \right\} \, dy \right\} \, dx
\]

\[
\leq \|rf\|_{L^\infty(\mathbb{R}^n_+, r)} \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} |N_\lambda(y)| \, |\psi(x)| \right\} \left\{ \int_{\mathbb{R}^n_+} z^{\mu+1/2} D_\mu(x, y, z) \, dz \right\} \, dy \, dx
\]

\[
= C_\mu \|f\|_{L^\infty(\mathbb{R}^n_+, r)} \|N_\lambda\|_{L^1(\mathbb{R}^n_+, sr)} \|\psi\|_{L^1(\mathbb{R}^n_+, sr)} < \infty.
\]

Let \( \phi \in \mathcal{H}_\mu \) and \( f \in L^p(\mathbb{R}^n_+, sr) \) or \( f \in L^\infty(\mathbb{R}^n_+, r) \), from (4.5) we have that

\[
(4.7) \quad (h_\mu(N_\lambda \phi), \phi) = ((N_\lambda \phi), h_\mu(\phi)) = \int_{\mathbb{R}^n_+} (N_\lambda \phi)(x) \, (h_\mu(\phi))(x) \, dx = \int_{\mathbb{R}^n_+} f(z) \, (N_\lambda \phi)(z) \, dz.
\]

From Lemma 2.10, Theorem 2.15 and item (b) of Lemma 4.1 we obtain that

\[
(h_\mu(N_\lambda \phi), \phi) = r(h_\mu(N_\lambda h_\mu(\phi)))(y) = y^{-\mu-1/2} \frac{\lambda^{\mu+1/2}}{\lambda + \|y\|^2} \frac{\phi(y)}{\lambda + \|y\|^2}.
\]

Then

\[
(4.8) \quad N_\lambda \phi = \frac{(h_\mu(N_\lambda \phi), \phi)}{(h_\mu(h_\mu(\phi), \phi)).}
\]

Finally, from (4.7) and (4.8) we obtain that for \( \phi \in \mathcal{H}_\mu \) that

\[
(h_\mu(N_\lambda \phi), \phi) = \int_{\mathbb{R}^n_+} f(x)(N_\lambda \phi)(x) \, dx
\]

\[
= \int_{\mathbb{R}^n_+} f(x) \, h_\mu \left( \frac{\phi}{\lambda + \|y\|^2} \right) \, (x) \, dx
\]

\[
= \int_{\mathbb{R}^n_+} \frac{1}{\lambda + \|x\|^2} h_\mu f(x) \, \phi(x) \, dx
\]

\[
= \left( \frac{h_\mu f}{\lambda + \|x\|^2} \right) \phi
\]

**Theorem 4.3.** Given \( \mu = (\mu_1, \ldots, \mu_n) \), \( \mu_i > -\frac{1}{2} \), then \( S_{\mu,p} \) and \( S_{\mu,\infty} \) are closed and non-negative operators.

**Proof.** Since convergence in \( L^\infty(\mathbb{R}^n_+, r) \) and \( L^p(\mathbb{R}^n_+, sr^p) \) implies convergence in \( D'(\mathbb{R}^n_+) \), then \( S_{\mu,\infty} \) and \( S_{\mu,p} \) are closed.

Now let \( \lambda > 0 \) and \( f \in D(S_{\mu,\infty}) \) such that \( (\lambda - S_{\mu,\infty})f = 0 \). So,

\[
h_\mu(\lambda - S_{\mu,\infty})f = 0
\]

in \( \mathcal{H}_\mu \). By Lemma 2.18 we obtain that

\[
(\lambda + \|y\|^2)h_\mu f = 0
\]

in \( \mathcal{H}_\mu \) and hence by Lemma 2.19

\[
h_\mu f = (\lambda + \|y\|^2)^{-1}(\lambda + \|y\|^2)h_\mu f = 0.
\]

Then, \( f = 0 \) as element of \( \mathcal{H}_\mu \) and we conclude that \( f = 0 \) a.e. in \( \mathbb{R}^n_+ \) and \( \lambda - S_{\mu,\infty} \) is injective.

Let \( f \in L^\infty(\mathbb{R}^n_+, r) \) and \( g = N_\lambda f \). Then, by Lemma 2.9 \( g \in L^\infty(\mathbb{R}^n_+, r) \) and

\[
h((\lambda - S_{\mu,\infty})g) = (\lambda + \|y\|^2)h_\mu g = (\lambda + \|y\|^2)h_\mu (N_\lambda f) = h_\mu f.
\]

By injectivity of Hankel transform in \( \mathcal{H}_\mu \) we obtain that

\[
(\lambda - S_{\mu,\infty})g = f,
\]

so, \( \lambda - S_{\mu,\infty} \) is onto. Also

\[
\| (\lambda - S_{\mu,\infty})^{-1} f \|_{L^\infty(\mathbb{R}^n_+, r)} = \| g \|_{L^\infty(\mathbb{R}^n_+, r)} = \| N_\lambda f \|_{L^\infty(\mathbb{R}^n_+, r)} \leq \| N_\lambda \|_{L^1(\mathbb{R}^n_+, sr)} \| f \|_{L^\infty(\mathbb{R}^n_+, r)}
\]
Thus we have that
\[
\frac{1}{\lambda} \| f \|_{L^\infty(Y,r)}.
\]
hence
\[
\| \lambda(1 - S_{\mu,\infty})^{-1}f \|_{L^\infty(Y,r)} \leq \| f \|_{L^\infty(Y,r)}
\]
and \(-S_{\mu,\infty}\) is non-negative.

The proof of the non-negativity of \(S_{\mu,p}\) is similar.

Since we have proved that both \(-S_{\mu,p}\) and \(-S_{\mu,\infty}\) are non-negative we can consider the fractional powers of them. If \(\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0\) and \(n > \text{Re}(\alpha)\) then the fractional power of \(-S_{\mu,\infty}\) can be represented from (3.2) by:
\[
(-S_{\mu,\infty})^\alpha = (-S_{\mu,\infty} + 1)^\alpha J_\infty^\alpha (-S_{\mu,\infty} + 1)^{-n},
\]
where with \(J_\infty^\alpha\) we denote the Balakrishnan operator associated to \(-S_{\mu,\infty}\) given by:
\[
J_\infty^\alpha \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n - \alpha)} \int_0^\infty \lambda^{\alpha - 1}(-S_{\mu,\infty}(\lambda - S_{\mu,\infty})^{-1})^n \phi \, d\lambda,
\]
for \(\alpha \in \mathbb{C}, 0 < \text{Re}(\alpha) < n\) and \(\phi \in D([-S_{\mu,\infty}]^n)\).

Analogously for the representation of fractional powers of \(-S_{\mu,p}\).

5. Non-negativity of Bessel operator \(S_{\mu}\) in the space \(\mathcal{B}\)

Remark 5.1. The operator \(-S_{\mu}\) is not non-negative in \(\mathcal{H}_{\mu}\).

If \(-S_{\mu}\) were non-negative in \(\mathcal{H}_{\mu}\), since \(-S_{\mu}\) is continuous in \(\mathcal{H}_{\mu}\), given \(\alpha \in \mathbb{C}, 0 < \alpha < 1\) and according to (3.1) and (A.8), we have that fractional power \((-S_{\mu})^\alpha\) would be given by
\[
(-S_{\mu})^\alpha = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha - 1}(-S_{\mu})(\lambda - S_{\mu})^{-1} \phi \, d\lambda,
\]
and \(D([-S_{\mu}]^\alpha) = D(-S_{\mu}) = \mathcal{H}_{\mu}\). Applying the Hankel transform in (5.1) we obtain
\[
h_{\mu}(-S_{\mu})^\alpha \phi = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha - 1}h_{\mu}([-S_{\mu})(\lambda - S_{\mu})^{-1} \phi \, d\lambda,
\]
where we have interchanged the Bochner integral with the Hankel transform, and the we have applied item (iii) of Lemma 2.2.20 and [9, Remark 3.1.1]. This would imply that \((\|y\|^2)^{\alpha}h_{\mu} \phi(y) \in \mathcal{H}_{\mu}\) which is not true in general.

Now we consider the Banach space \(Y = L^1(\mathbb{R}^n_+,sr) \cap L^\infty(\mathbb{R}^n_+,r)\), with norm
\[
\| f \|_Y = \max \{ \| f \|_{L^1(\mathbb{R}^n_+,sr)}, \| f \|_{L^\infty(\mathbb{R}^n_+,r)} \},
\]
and the part of the Bessel operator in \(Y\), \((S_{\mu})_Y\), with domain given by \(D[(S_{\mu})_Y] = \{ f \in Y : S_{\mu}f \in Y \}\).

From Theorem 4.3 we have that \(-(S_{\mu})_Y\) is closed and non-negative.
Proposition 5.2. If \( k > \frac{n}{2} \) then \( D[((S_\mu)Y)^{k+1}] \subset C_0(\mathbb{R}_+^n) \).

Proof. Let \( f \in D[((S_\mu)Y)^{k+1}] \), then \( f \) and \( (S_\mu)Y^k f \) are in \( D[(S_\mu)Y] \).

From Lemma 2.3 and (2.29) we have that
\[
L^1(\mathbb{R}_+^n, sr) \cap L^\infty(\mathbb{R}_+^n, r) \subset L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n).
\]
Then \( f \) and \( (S_\mu)Y^k f \) are in \( L^1(\mathbb{R}_+^n) \). From Remark 2.12 we obtain that \( h_\mu f \) and \( h_\mu ((S_\mu)Y^k f) \) are in \( L^\infty(\mathbb{R}_+^n) \), that is to say that there exist \( M > 0 \) such that
\[
|1 + \|y\|^{2k}| h_\mu f| \leq M.
\]
Since for \( k > \frac{n}{2} \), \( (1 + \|y\|^{2k})^{-1} \) is integrable in \( \mathbb{R}_+^n \), then \( h_\mu f \in L^1(\mathbb{R}_+^n) \). Then, we have proved that \( f \in D[((S_\mu)Y)^{k+1}] \), \( f \) and \( h_\mu f \in L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n) \).

From Remark 2.6 we have that \( L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n) \subset \mathcal{H}_\mu \) and from Remark 2.16 we have
\[
h_\mu(h_\mu f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}_+^n,
\]
considering \( f \) as a regular distribution in \( \mathcal{H}_\mu \). Since \( h_\mu f \in L^1(\mathbb{R}_+^n) \), then by Proposition 2.13 we have that \( f = g \) a.e. in \( \mathbb{R}_+^n \) with \( g \in C_0(\mathbb{R}_+^n) \).

We now consider the following space:

\[
B = \{ f \in Y : (S_\mu)^k f \in Y \quad \text{for} \quad k = 0, 1, 2, \ldots \} = \bigcap_{k=0}^{\infty} D[((S_\mu)Y)^k],
\]
with seminorms
\[
\rho_m(f) = \max_{0 \leq k \leq m} \{ \| (S_\mu)^k f \|_Y, \quad m = 0, 1, 2, \ldots \}.
\]

Remark 3.3. From proposition 5.2 is evident that \( B \subset C_0(\mathbb{R}_+^n) \). Moreover, from Lemma 2.3 we obtain that \( B \subset L^p(\mathbb{R}_+^n, sr^p) \) for all \( 1 \leq p < \infty \), and considering that \( S_\mu \) is a continuous operator from \( \mathcal{H}_\mu \) in itself then \( \mathcal{H}_\mu \subset B \) and the topology of \( \mathcal{H}_\mu \) induced by \( B \) is weaker than the usual topology generated by the seminorms given by (2.12). In fact, from (2.18) and (2.19) we have that

\[
\| \phi \|_Y \leq C\{ \gamma_{i,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi) \}, \quad \phi \in \mathcal{H}_\mu.
\]

for \( m > 2\mu_i + 2, i = 1, \ldots, n \) and by the continuity of \( S_\mu \) in \( \mathcal{H}_\mu \) we deduce that given a seminorm \( \rho_m \), there exists a finite set of seminorms \( \{ \gamma_{m,k_i}^\mu \}_{i=1}^{r} \) and constants \( c_1, \ldots, c_r \) such that

\[
\rho_m(\phi) \leq \sum_{i=1}^{r} c_i \gamma_{m,k_i}^\mu(\phi), \quad \phi \in \mathcal{H}_\mu.
\]

From the density of \( D(\mathbb{R}_+^n) \) in \( B \) we deduce the density of \( \mathcal{H}_\mu \) in \( B \).

We denote with \( (S_\mu)_B \) the part of Bessel operator \( S_\mu \) in \( B \), so the domain of the operator \( (S_\mu)_B \) is \( B \) and the following result holds.

Theorem 5.4. \( B \) is a Fréchet space and \( -(S_\mu)_B \) is a continuous and non-negative operator on \( B \).

Proof. Let \( \{ \phi_k \} \) a Cauchy sequence in \( B \), then the convergence of \( \{ \phi_k \} \) follows considering the seminorm \( \rho_0 \) and the completeness of \( L^1(\mathbb{R}_+^n, sr) \) and \( L^\infty(\mathbb{R}_+^n, r) \).

Since \( \rho_m(S_\mu \phi) = \rho_{m+1}(\phi) \) then \( (S_\mu)_B \) is continuous. The non-negativity follows from Proposition 1.4.2 in [9].
6. NON-NEGATIVITY OF BESSEL OPERATOR $S_\mu$ IN THE DISTRIBUTIONAL SPACE $\mathcal{B}'$

We will study the non-negativity of Bessel operator in the topological dual space of $\mathcal{B}$ with the strong topology, that is to say, the space $\mathcal{B}'$ endowed with the topology generated by the family of seminorms $\{ |·|_B \}$, where the sets $B$ are bounded sets in $\mathcal{B}$, and the seminorms are given by

$$|T|_B = \sup_{\phi \in B} |(T, \phi)|, \quad T \in \mathcal{B}'.$$

**Proposition 6.1.** $\mathcal{B}'$ is sequentially complete.

**Proof.** Let $\{T_m\} \subset \mathcal{B}'$ a Cauchy sequence, then for all bounded set $B \subset \mathcal{B}$,

$$|T_k - T_m|_B \to 0, \quad \text{for } k, m \to \infty,$$

i.e., for all $\varepsilon > 0$ there exists $N$ such that for all $k, m \geq N$ then

$$|T_k - T_m|_B < \varepsilon.$$

So, in particular, since the unit sets $\{\phi\} \subset \mathcal{B}$ are bounded,

$$|T_k - T_m|_{\{\phi\}} = |(T_k - T_m, \phi)| < \varepsilon, \quad \forall k, m \geq N,$$

then $\{(T_m, \phi)\}$ is a Cauchy sequence in $\mathbb{C}$, with which is convergent and there exists $T : \mathcal{B} \to \mathbb{C}$ such that

$$(T, \phi) = \lim_{m \to \infty} (T_m, \phi).$$

Since $\mathcal{B}$ is barrelled, for being a Fréchet space, and from a generalization of the BanachâŚSteinhaus theorem (see [15, Theorem 4.7, pp.86]), it has to $T \in \mathcal{B}'$. \hfill \blacksquare

**Remark 6.2.** $L^p(\mathbb{R}^n_+, sr^p)$ and $L^\infty(\mathbb{R}^n_+, r)$ are included in $\mathcal{B}'$, \quad (1 \leq p < \infty).

Let $f \in L^p(\mathbb{R}^n_+, sr^p)$, $\phi \in \mathcal{B}$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \int_{\mathbb{R}^n_+} f(x)\phi(x) \, dx \right| \leq \|f\|_{L^p(\mathbb{R}^n_+, sr^p)} \|\phi\| \| s^{-1}r^{-p}\|_{L^q(\mathbb{R}^n_+, sr^p)}$$

(6.1)

and

$$\|\phi\|_{L^q(\mathbb{R}^n_+, sr^p)} = \left\{ \int_{\mathbb{R}^n_+} |\phi|^{q} \, dx \right\}^{\frac{1}{q}} = \left\{ \int_{\mathbb{R}^n_+} |\phi|^{q} (C\mu r^2 r^{-p})^q \, dx \right\}^{\frac{1}{q}}$$

(6.2)

Furthermore, from (2.17)

$$\|\phi\|_{L^q(\mathbb{R}^n_+, sr^p)} \leq \left\{ \|\phi\|_{L^\infty(\mathbb{R}^n_+, r)} \right\}^{\frac{1}{q}} \left\{ \|\phi\|_{L^1(\mathbb{R}^n_+, r)} \right\}^{\frac{1}{q}},$$

from where

(6.3)

$$\|\phi\|_{L^q(\mathbb{R}^n_+, sr^p)} \leq \rho_0(\phi).$$

Then, from (6.1), (6.2) and (6.3) we obtain that $f \in \mathcal{B}'$.

Now, let $B$ a bounded set in $\mathcal{B}$, then

$$|f|_B = \sup_{\phi \in B} \left| \int_{\mathbb{R}^n_+} f \phi \right| \leq C_\mu \|f\|_{L^p(\mathbb{R}^n_+, sr^p)} \sup_{\phi \in B} \|\phi\|_{L^1(\mathbb{R}^n_+, sr^p)} \leq C_\mu \|f\|_{L^p(\mathbb{R}^n_+, sr^p)} \sup_{\phi \in B} \rho_0(\phi).$$

Thus, the topology in $L^p(\mathbb{R}^n_+, sr^p)$ induced by $\mathcal{B}'$ with the strong topology is weaker than the usual topology.
To prove the injectivity, let
\[ (S_\mu T, \phi) = (T, S_\mu \phi), \quad T \in B', \phi \in B, \]
and we denote with \((S_\mu)_{B'}\) the part of Bessel operator in \(B'\).

**Theorem 6.4.** The operator \((S_\mu)_{B'}\) is continuous and non-negative considering the strong topology in \(B'\).

**Proof.** Given a bounded set \(B \subset B\) and \(T \in B'\), then
\[
|(S_\mu)_{B'}T|_B = \sup_{\phi \in B} |((S_\mu)_{B'}T, \phi)| = \sup_{\phi \in B} |(T, (S_\mu)_{B'}\phi)| = |T|_B
\]
where the set \(E = \{(S_\mu)_{B'}\phi : \phi \in B\}\) is also bounded. Then it follows that \((S_\mu)_{B'}\) is continuous.

Let now \(\lambda > 0\) and \(T \in B'\). It is not difficult to see that the linear map \(G : \psi \to (T, (\lambda - (S_\mu)_{B'})^{-1}\psi)\) is continuous and \((\lambda - (S_\mu)_{B'})G = T\). Therefore \((\lambda - (S_\mu)_{B'})\) is surjective.

To prove the injectivity, let \(T \in B'\) be such that \((\lambda - (S_\mu)_{B'})T = 0\). Then, for all \(\phi \in B\),
\[
((\lambda - (S_\mu)_{B'})T, \phi) = (T, (\lambda - (S_\mu)_{B'})\phi) = 0,
\]
and thus \(T = 0\) as \(R(\lambda - (S_\mu)_{B'}) = B\), due to the fact that \(-(S_\mu)_{B'}\) is a non-negative operator.

To see that \((\lambda - (S_\mu)_{B'})^{-1}\) is continuous let \(T \in B'\), \(B \subset B\) a bounded set and let us consider the set \(F = \{(\lambda - (S_\mu)_{B'})^{-1}\phi : \phi \in B\}\), then
\[
|\lambda (\lambda - (S_\mu)_{B'})^{-1}T|_B = |G|_B = \sup_{\psi \in B} |(G, \psi)| = \sup_{\phi \in B} |(T, (\lambda - (S_\mu)_{B'})^{-1}\psi)|_B = |T|_F.
\]
For every bounded set \(B \subset B\) and \(T \in B'\), since \(-(S_\mu)_{B'}\) is non-negative, the set \(D = \{\eta(\eta - (S_\mu)_{B'})^{-1}\phi : \phi \in B, \eta > 0\}\) is also bounded and thus, for \(\lambda > 0\),
\[
|\lambda (\lambda - (S_\mu)_{B'})^{-1}T|_B = \sup_{\phi \in B} |(\lambda (\lambda - (S_\mu)_{B'})^{-1}\phi)|
= \sup_{\phi \in B} |(T, \lambda (\lambda - (S_\mu)_{B'})^{-1}\phi)|
\leq |T|_D.
\]
We now conclude that the operator \(-(S_\mu)_{B'}\) is non-negative. \(\square\)

**Remark 6.5.** The operator \((S_\mu)_{B'}\) is not injective because the function \(x^{\alpha + \frac{1}{2}}\) is solution of \(S_\mu u = 0\) and belongs to \(B'\), in fact
\[
\|(x^{\alpha + \frac{1}{2}}, \phi)| \leq C_\mu \|\phi\|_{L^1([\gamma_2, \gamma_3])} \leq C_\mu \rho_0(\phi), \quad \phi \in B.
\]

According to the representation of fractional powers of operators in locally convex spaces given in [9], it has to \(-(S_\mu)_{B'}\)^\alpha is given by
\[
(-(S_\mu)_{B'})^\alpha T = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n - \alpha)} \int_0^\infty \lambda^{\alpha - 1}[-(S_\mu)_{B'}(\lambda - (S_\mu)_{B'})^{-1}]^n T d\lambda.
\]
for Re\(\alpha > 0\), \(n > \text{Re}\alpha\), \(T \in B'\).

From the general theory of fractional powers in sequentially complete locally convex spaces (see [9], pp.134), we deduce some properties of powers such as multiplicativity and

(1) If Re\(\alpha > 0\) then
\[
((-(S_\mu)_{B'})^\alpha)^* = ((-(S_\mu)_{B'})^\alpha)^\alpha
\]
Since \(-(S_\mu)_{B'}^* = -(S_\mu)_{B'}\) then from (6.4) we obtain the following duality formula
\[
((-(S_\mu)_{B'})^\alpha T, \phi) = (T, (-(S_\mu)_{B'})^\alpha \phi), \quad \phi \in B, T \in B'.
\]
(2) Since the usual topology in $L^p(\mathbb{R}_+^n, sr^p)$ is stronger than the topology induced by $\mathcal{B}'$ we can deduce that

$$((-(S_\mu)\mathcal{B})^\alpha)_{L^p(\mathbb{R}_+^n, sr^p)} = (-(S_{\mu,p}))^\alpha,$$

for $\text{Re}\, \alpha > 0$, (see [9, Theorem 12.1.6, pp.284]).

This last property expresses a very desirable property in the theory of powers since it tells us that the restriction of the distributional power of $-S_\mu$ to $L^p(\mathbb{R}_+^n, sr^p)$ coincides with the power of $-S_\mu$ in $L^p(\mathbb{R}_+^n, sr^p)$.

7. DISTRIBUTIONAL LIOUVILLE THEOREM FOR $(-(S_\mu))^\alpha$.

In this section we include the proof of Theorem 1.1. Before that, we will show the following Lemma.

**Lemma 7.1.** Let $\psi \in \mathcal{H}_\mu$ such that $\text{supp}\, \psi \subset \mathbb{R}_+^n \cap \{x : \|x\| \geq a\}$ with $a > 0$ and $\alpha \in \mathbb{C}$ with $\text{Re}\, \alpha > 0$. Then $\|x\|^{-2\alpha}\psi(x) \in C^\infty(\mathbb{R}_+^n)$. We are going to see that

$$\sup_{x \in \mathbb{R}_+^n} \left| x^m T^k \{x^{-\mu - \frac{n}{2}}\|x\|^{-2\alpha}\psi(x)\} \right| < \infty,$$

with $k,m \in \mathbb{N}_0^n$. Since $\psi \in \mathbb{R}_+^n \cap \{x : \|x\| \geq a\}$ with $a > 0$, then

$$\sup_{x \in \mathbb{R}_+^n \|x\| \geq a} \left| x^m T^k \{x^{-\mu - \frac{n}{2}}\|x\|^{-2\alpha}\psi(x)\} \right| \leq \sup_{a \leq \|x\| \leq 1} \left| x^m T^k \{x^{-\mu - \frac{n}{2}}\|x\|^{-2\alpha}\psi(x)\} \right| + \sup_{\|x\| \geq 1} \left| x^m T^k \{x^{-\mu - \frac{n}{2}}\|x\|^{-2\alpha}\psi(x)\} \right|$$

The first term in the last inequality is bounded because it is a continuous function over a compact set. On the other hand, since equality (2.5) holds then,

$$\sup_{\|x\| \geq 1} \left| x^m T^k \{x^{-\mu - \frac{n}{2}}\|x\|^{-2\alpha}\psi(x)\} \right| \leq \sup_{\|x\| \geq 1} \left| \sum_{j=0}^{k} \binom{k}{j} T^{k-j} \{x^{-\mu - \frac{n}{2}}\psi(x)\} \cdot T^j \|x\|^{-2\alpha} \right| \leq \sum_{j=0}^{k} \binom{k}{j} C(j, \alpha) \gamma_{m,k-j}^\mu(\psi),$$

where $C(j, \alpha)$ are constants depending on $\alpha$ and $j$ such that $\sup_{\|x\| \geq 1} |T^j \|x\|^{-2\alpha}| \leq C(j, \alpha)$.

**Proof of Theorem 1.1.** Let $u \in \mathcal{B}'$ such that $(-(S_\mu)\mathcal{B})^\alpha u = 0$. Then for all $\phi \in \mathcal{B}$

$$((-(S_\mu)\mathcal{B})^\alpha u, \phi) = (u, -(S_\mu)\mathcal{B})^\alpha \phi = 0.$$  \hspace{1cm} (7.1)

Since $S_\mu$ is a continuous operator in $\mathcal{B}$ (see Theorem 5.4), then $(-(S_\mu)\mathcal{B})^\alpha \phi$ is given by the Balakrishnan operator as:

$$(-(S_\mu)\mathcal{B})^\alpha \phi = \frac{\Gamma(\alpha)\Gamma(m - \alpha)}{\Gamma(m)} \int_0^\infty \lambda^{\alpha-1}[-(S_\mu)\mathcal{B}(\lambda - (S_\mu)\mathcal{B})^{-1}]^m \phi \, d\lambda.$$  \hspace{1cm} (7.2)

By definition of $\mathcal{B}$ and the fact that $L^1(\mathbb{R}_+^n, sr) \cap L^\infty(\mathbb{R}_+^n, r) \subset L^p(\mathbb{R}_+^n, sr^p)$ for all $1 \leq p \leq \infty$ then $\mathcal{B} \subset D(S_{\mu,p})$ for all $1 \leq p \leq \infty$, in particular, $\mathcal{B} \subset D(S_{\mu,2})$. Then from Propositions 8.3 and 8.4 in [12] we obtain that:

$$(-(S_{\mu,2})^\alpha \phi = \frac{\Gamma(\alpha)\Gamma(m - \alpha)}{\Gamma(m)} \int_0^\infty \lambda^{\alpha-1}[-S_{\mu,2}(\lambda - (S_{\mu,2})^{-1}]^m \phi \, d\lambda.$$ \hspace{1cm} (7.3)

Since for $\phi \in \mathcal{B}$, the integrating into the expressions are equal and the fact that the convergence in $\mathcal{B}$ implies the convergence in $L^2(\mathbb{R}_+^n)$ (see Lemma 2.1 and Remark 5.3 in [11]) we obtain the equality of (7.2) and (7.3) as functions.
We conclude that
\[ (-S_\mu)_B^\alpha \phi = h_\mu \|y\|^{2\alpha} h_\mu \phi, \quad \phi \in B, \]
(see [11, Proposition 8.4]). From the last equality and (7.1), we have that
\[
(7.4) \quad ((-S_\mu)_{B'}^\alpha u, \phi) = (u, h_\mu \|y\|^{2\alpha} h_\mu \phi) = 0,
\]
for all \( \phi \in B'. \)

Since \( B' \subset H_\mu' \) (see [11, Remark 6.2]), we can consider the Hankel transform in \( B' \). We are going to see that the following affirmation holds:

"If \( u \in B' \) is such that (7.4) is verified, then \( (h_\mu u, \psi) = 0 \) for all \( \psi \in H_\mu \) such that \( \text{supp} \, \psi \subset \mathbb{R}^n_+ \cap \{x: \|x\| \geq a\} \) with \( a > 0 \)."

Let \( u \in B' \) such that (7.4) is valid and \( \psi \in H_\mu \) such that \( \text{supp} \, \psi \subset \mathbb{R}^n_+ \cap \{x: \|x\| \geq a\} \) with \( a > 0 \). Then, by Lemma 7.1, \( \|x\|^{-2\alpha} \psi(x) \in H_\mu \) and since the Hankel transform is an isomorphism in \( H_\mu \), there exists \( \phi \in H_\mu \) such that \( h_\mu \phi = \|x\|^{-2\alpha} \psi(x) \). So,
\[
(h_\mu u, \psi) = (h_\mu u, \|x\|^{2\alpha} \psi) = (h_\mu u, \|x\|^{2\alpha} h_\mu \phi) = (u, h_\mu \|x\|^{2\alpha} h_\mu \phi).
\]

Consequently, from (7.4) we conclude that \( (h_\mu u, \psi) = 0 \), then the assertion is valid. Thus by [6, Theorem 4.1], there exist \( N \in \mathbb{N}_0 \) and scalars \( c_k \) with \( |k| < N \) such that \( h_\mu u = \sum_{|k| < N} c_k S_\mu^k \delta_\mu \)
where \( \delta_\mu \) is given by [6, equation (2.3)] for \( k = 0 \). Then,
\[
u = x^{\mu+\frac{1}{2}} \sum_{|k| \leq N} c_k (-1)^{|k|} \|x\|^{2k}
\]

\[ \Box \]

**Remark 7.2** (Regular distributions in \( B' \)).

If \( f \in L^1_{loc}(\mathbb{R}^n_+) \) and \( f = O(x^{\mu+\frac{1}{2}}) \) then \( f \) is a regular distribution in \( B' \) given by
\[
(f, \phi) = \int_{\mathbb{R}^n_+} f(x) \phi(x) \, dx, \quad \phi \in B,
\]
and
\[
|(f, \phi)| \leq \left| \int_{\|x\| \leq 2} f(x) \phi(x) \, dx \right| + \left| \int_{\|x\| \geq 2} f(x) \phi(x) \, dx \right|
\leq \int_{\|x\| \leq 2} |x^{-1}(x)f(x)| \, dx \|\phi\|_{L^\infty(r)} + \int_{\|x\| \geq 2} \|x^{\mu+\frac{1}{2}}\| \phi(x) \, dx
\leq C\|\phi\|_{L^\infty(r)} + c C_\mu \|\phi\|_{L^1(r)} \leq C' \rho_0(\phi).
\]

**Corollary 7.3.** If \( f \in L^1_{loc}(\mathbb{R}^n_+) \), \( f = O(x^{\mu+\frac{1}{2}}) \) and \( (-S_\mu)_{B'}^\alpha f = 0 \) then \( f = C x^{\mu+\frac{1}{2}} \).

8. **DISTRIBUTIONAL LIOUVILLE THEOREM FOR \( (-\Delta_\mu)^\alpha \).**

From theory of similar operators given in [11], by the similarity of \( S_\mu \) and \( \Delta_\mu \), and by the non-negativity of the part of \(-S_\mu\) in \( L^1(\mathbb{R}^n_+, sr) \) and \( L^\infty(\mathbb{R}^n_+, r) \) we deduce the non-negativity of the part of \(-\Delta_\mu\) in \( L^1(\mathbb{R}^n_+, s) \) and \( L^\infty(\mathbb{R}^n_+) \). Consequently, we infer the non-negativity of the part of \(-\Delta_\mu\) in the Banach space \( Z = L^1(\mathbb{R}^n_+, s) \cap L^\infty(\mathbb{R}^n_+) \) with norm
\[
\|f\|_Z = \max \left\{ \|f\|_{L^1(\mathbb{R}^n_+, s)}, \|f\|_{L^\infty(\mathbb{R}^n_+)} \right\}.
\]
Thus, if we consider \( Y \) as in Section 5 and \( L_r : Y \to Z \) given by \( L_rf = rf \) then
\[
\|rf\|_Z = \max \left\{ \|rf\|_{L^1(\mathbb{R}^n_+, s)}, \|rf\|_{L^\infty(\mathbb{R}^n_+)} \right\}
= \max \left\{ \|f\|_{L^1(\mathbb{R}^n_+, rs)}, \|f\|_{L^\infty(\mathbb{R}^n_+, r)} \right\}
= \|f\|_Y,
\]
so, $L_r$ is an isometric isomorphism.

Moreover, we can consider the locally convex space $F$ given by:

$$F = \{ f \in Z : (\Delta_\mu)^k f \in Z \quad \text{for} \quad k = 0, 1, 2, \cdots \} = \bigcap_{k=0}^{\infty} D[(\Delta_\mu)^k],$$

where with $(\Delta_\mu)^k$ we denote the part of $\Delta_\mu$ in $Z$. The space $F$ is endowed with the topology generated by the family of seminorms given by

$$\gamma_m(f) = \max_{0 \leq k \leq m} \{ ||(\Delta_\mu)^k f||_Z \}, \quad m = 0, 1, 2, \cdots$$

Thus, the space $F$ verifies that is a Fréchet space and from Remarks 5.3 and 6.2 we deduce that $F \subset C_0(\mathbb{R}_+^n)$, $F \subset L^p(\mathbb{R}_+^n, s)$ for all $1 \leq p < \infty$, $F \subset L^\infty(\mathbb{R}_+^n)$, $\mathcal{H}_\mu \subset F$ and the topology of $\mathcal{H}_\mu$ induced by $F$ is weaker than the usual topology in $\mathcal{H}_\mu$. Moreover, the operator $\Delta_\mu$ verifies that

$$\gamma_m(\Delta_\mu f) = \max_{0 \leq k \leq m} \{ ||(\Delta_\mu)^k \Delta_\mu f||_Z \} = \gamma_{m+1}(f),$$

for all $f \in F$. Then $(\Delta_\mu)_F$ the part of $\Delta_\mu$ in $F$, is a continuous

$$(\Delta_\mu)_F : F \rightarrow F.$$

If $f \in B$, (see (5.2)), then $r \in F$ and

$$\gamma_m(r f) = \max_{0 \leq k \leq m} \{ ||(\Delta_\mu)^k r f||_Z \} = \max_{0 \leq k \leq m} \{ ||r(\Delta_\mu)^k r^{-1} f||_Z \}$$

$$= \max_{0 \leq k \leq m} \{ ||r(\Delta_\mu)^k||_Z \} = \gamma_m(f),$$

where we have consider (2.31). So, the application $L_r : B \rightarrow F$, given by $L_r f = r f$ is an isomorphism of locally convex spaces with inverse given by $L_{r^{-1}} : F \rightarrow B$.

**Remark 8.1.** Since $B$ and $F$ are isomorphic then we can deduce that $F'$ is sequentially complete as $B'$ is also sequentially complete (see Proposition 6.1).

So, if we consider the continuous operator $(S_\mu)_B : B \rightarrow B$, then by (2.31) we obtain the similarity relation,

$$(8.1) \quad (\Delta_\mu)_F = L_r (S_\mu)_B L_{r^{-1}}.$$

We deduce by (8.1) the non-negativity of $-(\Delta_\mu)_F$ and by [11, Proposition 1.1], for $\alpha \in \mathbb{C}$, $\Re \alpha > 0$, we have that

$$(8.2) \quad (-(\Delta_\mu)_F)^{\alpha} = L_r (-(S_\mu)_B)^{\alpha} L_{r^{-1}}.$$

Consequently,

$$(-(\Delta_\mu)_F)^{\alpha} = ((-(\Delta_\mu)_F)^{\ast})^{\alpha} = ((-(\Delta_\mu)_F)^{\ast})^{\ast},$$

where we have considered in the second equality that $F$ is a Fréchet space, (see [9, pp.134]). Thus,

$$(8.3) \quad (-(\Delta_\mu)_F)^{\alpha} = (L_{r^{-1}})^{\ast} ((-(S_\mu)_B)^{\alpha} (L_r)^{\ast} = (L_{r^{-1}})^{\ast} (-(S_\mu)_B^{\ast})^{\alpha} (L_r)^{\ast},$$

and for $T \in F'$, $\phi \in F$

$$((-(\Delta_\mu)_F)^{\alpha} T, \phi) = ((L_{r^{-1}})^{\ast} ((-(S_\mu)_B)^{\alpha} (L_r)^{\ast} T, \phi)$$

$$= (T, L_r ((-(S_\mu)_B)^{\alpha} L_{r^{-1}} \phi)).$$

Thus,

$$(8.4) \quad (-(\Delta_\mu)_F)^{\alpha} = x^{-\mu-\frac{1}{2}} ((S_\mu)_B x^{\alpha + \frac{1}{2}},$$

$$(-(\Delta_\mu)_F)^{\alpha} = x^{-\mu-\frac{1}{2}} ((S_\mu)_B^{\ast} x^{\alpha + \frac{1}{2}}$$
and
\[ (-\Delta_{\mu})^{\alpha} = x^{\mu+\frac{\alpha}{2}} (-\frac{\partial}{\partial x} + B) x^{-\mu-\frac{\alpha}{2}} \]
to refer to (8.1), (8.2) and (8.3). In the last equation, the operators \( x^{\mu+\frac{\alpha}{2}} \) and \( x^{-\mu-\frac{\alpha}{2}} \) represent \( (L_{\alpha-1})^\ast \) and \( (L_\alpha)^\ast \), so,
\[
\begin{align*}
x^{\mu+\frac{\alpha}{2}}: B' &\to F', \\
x^{-\mu-\frac{\alpha}{2}}: F' &\to B',
\end{align*}
\]
are given by
\[
\begin{align*}
(x^{\mu+\frac{\alpha}{2}}T_1, \phi) &= (T_1, x^{\mu+\frac{\alpha}{2}}\phi), \quad (T_1 \in B'), (\phi \in F) \\
(x^{-\mu-\frac{\alpha}{2}}T_2, \psi) &= (T_2, x^{-\mu-\frac{\alpha}{2}}\psi), \quad (T_2 \in F'), (\psi \in B)
\end{align*}
\]

Now we are able to establish the following theorem:

**Theorem 8.2.** Let \( u \in F' \) and \( \alpha \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \). If \(-\Delta_{\mu})^{\alpha} u = 0\) then there exists a polynomial \( p \) such that \( u = x^{2\mu+1} p(\|x\|^2) \).

**Proof.** Let \( u \in F' \) such that \(-\Delta_{\mu})^{\alpha} u = 0\). Then
\[
((-\Delta_{\mu})^{\alpha} u, \phi) = (x^{\mu+\frac{\alpha}{2}} (-\frac{\partial}{\partial x} + B) x^{-\mu+\frac{\alpha}{2}} u, \phi) = 0
\]
for all \( \phi \in F \). Since \(-\frac{\partial}{\partial x} + B \) is a bounded operator on \( B' \), then given \( \psi \in B \) and considering (8.5), we obtain that
\[
((-\frac{\partial}{\partial x} + B) \alpha x^{\mu+\frac{\alpha}{2}} u, \psi) = ((-\frac{\partial}{\partial x} + B) \alpha x^{-\mu+\frac{\alpha}{2}} u, x^{\mu+\frac{\alpha}{2}} x^{-\mu-\frac{\alpha}{2}} \psi)
\]
\[
= (x^{\mu+\frac{\alpha}{2}} (-\frac{\partial}{\partial x} + B) \alpha x^{-\mu+\frac{\alpha}{2}} u, x^{-\mu+\frac{\alpha}{2}} \psi) = 0.
\]
By Theorem 1.1 we deduce that there exists a polynomial \( p \) such that \( x^{-\mu+\frac{\alpha}{2}} u = x^{\pm\frac{\mu}{2}} p(\|x\|^2) \) and consequently \( u = x^{2\mu+1} p(\|x\|^2) \).

**Corollary 8.3.** If \( f \in L^1_{\text{loc}}(\mathbb{R}^n_+) \), \( f = O(x^{2\mu+1}) \) and \(-\frac{\partial}{\partial x} + B\alpha^\alpha f = 0\) then \( f = C x^{2\mu+1} \).

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Appendix A. Special functions

In this appendix we summarize properties of the Bessel function of the first species and order \( \alpha \) given by:

\[
J_\alpha(z) = \left( \frac{z}{2} \right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n}}{n! \Gamma(\alpha + n + 1)}.
\]

According to [7, p.310], for \( \alpha \in \mathbb{R}, \alpha > -\frac{1}{2} \), the Bessel function verifies that

\[
|C_\alpha z^{-\alpha} J_\alpha(z)| \leq 1
\]

where \( C_\alpha = 2^\alpha \Gamma(\alpha + 1) \).

The following equalities are also verified:

\[
\int_0^{\pi} J_\alpha \left( \sqrt{y^2 + z^2 - 2yz \cos \phi} \right) \sin^{2\alpha} \phi \, d\phi = 2^\alpha \Gamma(\alpha + 1/2) \Gamma(1/2) J_\alpha(y) \frac{J_\alpha(z)}{y^\alpha},
\]

for \( \alpha > -\frac{1}{2} \), see [16, pp.367] and

\[
\int_0^\infty e^{-\frac{y^2}{2}} J_\alpha(y) y^{\alpha+1} \, dy = r^\alpha a^{-\alpha-1} e^{-\frac{r^2}{2a}},
\]

for \( \alpha > -1 \) and \( a > 0 \), see [14, pp.46].

The following equalities are valid for integrals that involve the Gamma function: Let \( a > 0 \) and \( \mu > -1 \), then the following equalities are valid

\[
\int_0^\infty e^{-\frac{x^2}{2}} x^{2\mu+1} \, dx = 2\mu \Gamma(\mu + 1).
\]

\[
\int_0^\infty e^{-\frac{x^2}{2}} x^{2\mu+1} \, dx = 2^\mu \Gamma(\mu + 1) a^{\alpha+1}.
\]

\[
\int_0^{\pi/2} \sin^{2\mu} \theta \, d\theta = \frac{\Gamma(1/2) \Gamma(r+1/2)}{2\Gamma(r+1)} = \frac{\sqrt{\pi} \Gamma(r+1/2)}{2\Gamma(r+1)}.
\]

Another important equation is the Euler Complements Formula

\[
\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi \alpha} \quad (0 < \text{Re} \alpha < 1).
\]

Appendix B. Some results on Hankel transforms, convolution and the Inversion theorem

Proof of Proposition 2.8. Assertion (i) follows immediately.

To prove (ii) let us see the integrability of \( D_\mu(x,y,z) \prod_{i=1}^n \{ \sqrt{z_i t_i} J_\mu(z_i t_i) \} \). First, we observe that

\[
\int_{\mathbb{R}_+^n} D_\mu(x,y,z) \prod_{i=1}^n \{ \sqrt{z_i t_i} J_\mu(z_i t_i) \} \, dz = \int_{\mathbb{R}_+^n} \prod_{i=1}^n \{ D_\mu(x_i,y_i,z) \sqrt{z_i t_i} J_\mu(z_i t_i) \} \, dz
\]

To proof this, it will be enough to see that

\[
\int_0^\infty D_\alpha(u,v,w) \sqrt{ut} J_\alpha(wt) \, dw = t^{-\alpha-1/2} \sqrt{ut} J_\alpha(u t) \sqrt{ut} J_\alpha(v t),
\]

where \( u, v, w \) and \( t \in (0, \infty) \) and \( \alpha > -\frac{1}{2} \). Since
Here we have used that the function $w^{-\alpha}J_\alpha(w)$ is bounded for $\alpha \in \mathbb{R}$, greater than $-\frac{1}{2}$. In fact, $|w^{-\alpha}J_\alpha(w)| \leq \frac{\sqrt{\pi}}{\Gamma(\alpha+1/2)}$. Moreover supp $A(u, v, w) \subset [|u-v|, u+v]$, so

$$
\int_0^\infty |D_\alpha(u, v, w)\sqrt{w}J_\alpha(w)| \leq C \int_0^\infty w[(u+v)^2 - w^2]^{\alpha} J_\alpha(w) \, dw,
$$

rewriting the last integral

$$
\int_{\alpha}^{u+v} \frac{w}{(|u-v|)^{\alpha}} \int_{\alpha}^{u+v} \frac{w}{(|u-v|)^{\alpha}} \, dw = (B.1)
$$

To analyze integrability, we separate the region of integration $[|u-v|, u+v]$ considering $c$, arbitrary and fix such that $|u-v| \leq c \leq u+v$. Let us note that it is possible to write (B.1) as

$$
\int_{|u-v|}^{u+v} \frac{w}{(|u-v|)^{\alpha}} \int_{|u-v|}^{u+v} \frac{w}{(|u-v|)^{\alpha}} \, dw = (B.2)
$$

where

$$
f_1(w) = \frac{f_1(w)}{\Gamma(u-v)^{\alpha}} \text{ and } f_2(w) = \frac{f_2(w)}{\Gamma(u-v)^{\alpha}}.
$$

Since $f_1(w)$ is a continuous function in $[|u-v|, c]$, it results bounded. Then there is a constant $C_1 > 0$ such that $|f_1(w)| \leq C_1$ for all $w \in [|u-v|, c]$. The same goes for $f_2$ in $|c, u+v|$. The problem is then reduced to studying the integrability of $\frac{1}{u-v} \int_{|u-v|}^{u+v} \frac{w}{(|u-v|)^{\alpha}} J_\alpha(w) \, dw$ in $[|u-v|, c]$ and $\frac{1}{(u-v)^{\alpha}} \int_{|u-v|}^{u+v} \frac{w}{(|u-v|)^{\alpha}} J_\alpha(w) \, dw$ in $[c, u+v]$.

Thus, using the change of variables

$$
T : (0, \pi) \to (0, \infty) \quad \text{con} \quad T(\theta) = \sqrt{u^2 + v^2 - 2uv \cos \theta},
$$

where $\frac{d}{d\theta}T(\theta) > 0$, $T(0) = |u-v|$, $T(\pi) = u+v$ and

$$
\frac{d}{d\theta}T(\theta) = \frac{uv \sin \theta}{\sqrt{u^2 + v^2 - 2xy \cos \theta}}.
$$

So,

$$
\int_0^\infty D_\alpha(u, v, w)\sqrt{w}J_\alpha(w) \, dw = \frac{2^{(u-v)^{\alpha}}}{\Gamma(u-v)^{\alpha}} \int_{\alpha}^{u+v} \frac{w}{(|u-v|)^{\alpha}} A(u, v, w)2^{(u-v)^{\alpha}+1/2}\sqrt{u^2 + v^2 - 2uv \cos \theta} \, dw.
$$

$\square$
Let us analyze the existence of the next iterated integral

$$\int_{\mathbb{R}^n_+} \zeta^{\mu+1/2} \mathbf{D}_\mu(x, y, z) \, dz = \int_{\mathbb{R}^n_+} \prod_{i=1}^n \{z_i^{\mu+1/2} \mathbf{D}_\mu(x_i, y_i, z_i)\} \, dz_1 \ldots dz_n.$$ 

Suffice it to see then that

$$\int_0^\infty u^{\alpha+1/2} D_\alpha(u, v, w) \, dw = C_\alpha^{-1} u^{\alpha+1/2} v^{\alpha+1/2},$$

where \(u, v, w \in (0, \infty), \alpha > -\frac{1}{2}\). Then

$$\int_0^\infty u^{\alpha+1/2} D_\alpha(u, v, w) \, dw = \frac{2^{\frac{\alpha-1}{2}}}{\Gamma(\alpha + 1/2) \sqrt{\pi}} \int_0^\infty u^{1/2} \sqrt{u^2 + v^2 - 2uv \cos \theta} \left(\frac{uv \sin \theta}{2}\right)^{\alpha-1} \frac{uv \sin \theta}{\sqrt{u^2 + v^2 - 2uv \cos \theta}} \, d\theta$$

$$= \frac{2^{\alpha-1} (uv)^{\alpha+1/2}}{\Gamma(\alpha + 1/2) \sqrt{\pi}} \int_0^\pi \sin^{2\alpha} \theta \, d\theta = \frac{2^{-\alpha} (uv)^{\alpha+1/2}}{\Gamma(\alpha + 1)} = c_\alpha^{-1} u^{\alpha+1/2} v^{\alpha+1/2},$$

where we have used \((A.7)\) from Appendix. \(\blacksquare\)

**Proof of Lemma 2.9 (i).** Let \(f \in L^1(sr)\) and \(g \in L^\infty(r)\), let us see that \(f \sharp g \in L^\infty(r)\). Since

$$f \sharp g(x) = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} f(y) g(z) \mathbf{D}_\mu(x, y, z) \, dy \, dz,$$

then

$$|f \sharp g(x)| \leq \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} |f(y)| |g(z)| \mathbf{D}_\mu(x, y, z) s(y)s(z) \, dy \, dz.$$ 

Let us analyze the existence of the next iterated integral

$$\int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} |f(y)| |g(z)| \mathbf{D}_\mu(x, y, z) \, dz \right\} \, dy$$

$$= \int_{\mathbb{R}^n_+} |f(y)| \left\{ \int_{\mathbb{R}^n_+} |g(z)| \mathbf{D}_\mu(x, y, z) \, dz \right\} \, dy$$

$$= \int_{\mathbb{R}^n_+} |f(y)| \left\{ \int_{\mathbb{R}^n_+} |r(z)g(z)| r^{-1}(z) \mathbf{D}_\mu(x, y, z) \, dz \right\} \, dy$$

$$\leq \|g\|_{L^\infty(r)} \int_{\mathbb{R}^n_+} |f(y)| \left\{ \int_{\mathbb{R}^n_+} z^{\mu+1/2} \mathbf{D}_\mu(x, y, z) \, dz \right\} \, dy$$

$$= \|g\|_{L^\infty(r)} \int_{\mathbb{R}^n_+} |f(y)| C^{-1}_\mu x^{\mu+1/2} y^{\alpha+1/2} \, dy$$

$$\leq \|g\|_{L^\infty(r)} \int_{\mathbb{R}^n_+} |f(y)| s(y)s(y) \, dy = x^{\mu+1/2} \|g\|_{L^\infty(r)} \|f\|_{L^1(sr)}$$

(B.7)

Since the integral \(\int_{\mathbb{R}^n_+} |f(y)| \left\{ \int_{\mathbb{R}^n_+} |g(z)| \mathbf{D}_\mu(x, y, z) \, dz \right\} \, dy < \infty\) for \(f \in L^1(sr)\) and \(g \in L^\infty(r)\), then Tonelli’s Theorem allows us to affirm that the integral in \((B.5)\) exists for all \(x \in \mathbb{R}^n_+\), and from \((B.6)\) and \((B.7)\) the desired result is obtained.
(ii). Let

\[ K(x, z) = C_{\mu} x^{-\mu - 1/2} z^{-\mu - 1/2} \int_{\mathbb{R}^n_+} f(y) D_\mu(x, y, z) \, dy \]

(B.8)

\[
\int_{\mathbb{R}^n_+} |K(x, z)| \, s(x) \, dx = \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} f(y) D_\mu(x, y, z) \, dy \right| \frac{x^{2\mu + 1}}{C_{\mu}} \, dx
\]

\[
\leq \int_{\mathbb{R}^n_+} z^{-\mu - 1/2} \left\{ \int_{\mathbb{R}^n_+} |f(y)| D_\mu(x, y, z) \, dy \right\} x^{\mu + 1/2} \, dx
\]

\[
= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} z^{-\mu - 1/2} |f(y)| x^{\mu + 1/2} D_\mu(x, y, z) \, dy \right\} \, dx
\]

\[
= \int_{\mathbb{R}^n_+} z^{-\mu - 1/2} |f(y)| \left\{ \int_{\mathbb{R}^n_+} x^{\mu + 1/2} D_\mu(x, y, z) \, dy \right\} \, dx
\]

\[
= \int_{\mathbb{R}^n_+} z^{-\mu - 1/2} |f(y)| C_{\mu}^{-1} y^{\mu + 1/2} z^{\mu + 1/2} \, dy = \|f\|_{L^1(s\nu)} < \infty
\]

Thus

(B.9)

\[
\int_{\mathbb{R}^n_+} |K(x, z)| \, s(x) \, dx \leq \|f\|_{L^1(s\nu)} = \|rf\|_{L^1(s)}
\]

and similarly

(B.10)

\[
\int_{\mathbb{R}^n_+} |K(x, z)| \, s(z) \, dx \leq \|f\|_{L^1(s\nu)} = \|rf\|_{L^1(s)}.
\]

So, from [?, Theorem 6.18 - pp.193] if \( h \in L^p(s) \) then

\[
Th(x) = \int_{\mathbb{R}^n_+} h(z) K(x, z) \, s(z) \, dz
\]

exists for almost every \( x \in \mathbb{R}^n_+ \) and

\[
\|Th\|_{L^p(s)} \leq \|rf\|_{L^1(s)} \|h\|_{L^p(s)}
\]

In particular, for \( h = rg \), since \( g \in L^p(s\nu^p) \)

\[
\|g\|_{L^p(s\nu^p)} = \int_{\mathbb{R}^n_+} |g(z)|^p s(z) \, r^p(z) \, dz = \int_{\mathbb{R}^n_+} |r(z)g(z)|^p s(z) \, dz = \|rg\|_{L^p(s)}.
\]

then \( h \in L^p(s) \).

\[
T(h)(x) = T(rg)(x) = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} r(z)g(z) c_{\mu} x^{-\mu - 1/2} z^{-\mu - 1/2} f(y) D_\mu(x, y, z) \, dy \, dz
\]

\[
= x^{-\mu - 1/2} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} f(y) g(z) D_\mu(x, y, z) \, dy \, dz
\]

\[
= x^{-\mu - 1/2} f^g(x)
\]

\[
\|f^g\|_{L^p(s\nu^p)} = \|r(f^g)\|_{L^p(s)} = \|T(rg)\|_{L^p(s)} \leq \|rf\|_{L^1(s)} \|rg\|_{L^p(s)} = \|f\|_{L^1(s)} \|g\|_{L^p(s\nu^p)}
\]

\[ \blacksquare \]
Hirschman defined in [7] for the 1-dimensional case a kernel $\mathcal{D}_\alpha$ which is defined for $u, v, w \in (0, \infty)$, $\alpha > -\frac{1}{2}$, by

$$\mathcal{D}_\alpha(u, v, w) = \frac{2^{4\alpha-1}\Gamma^2(\alpha+1)}{\Gamma(\alpha+1/2)}\sqrt{\pi} (uvw)^{-2\alpha} A(u, v, w)^{2\alpha-1}$$  \hspace{1cm} (B.11)$$
where $A(u, v, w)$ is the area of a triangle of sides $u, v, w \in \mathbb{R}_+$ defined by (2.25).

For the $n$-dimensional case, let $x, y, z \in \mathbb{R}^n_+$ and $\mu = (\mu_1, \ldots, \mu_n)$ such that $\mu_i > -\frac{1}{2}$ for all $i = 1, \ldots, n$. We define

$$\mathcal{D}_\mu(x, y, z) = \prod_{i=1}^n \mathcal{D}_{\mu_i}(x_i, y_i, z_i)$$  \hspace{1cm} (B.12)$$
where $\mathcal{D}_{\mu_i}$ is given by (B.11).

A convolution operation associated to the $n$-dimensional Hankel transform $H_\mu$ can be defined. Given $f, g$ defined on $\mathbb{R}^n_+$, the Hankel convolution associated to the transformation $H_\mu$ is defined formally by

$$f \ast g(x) = \int_{\mathbb{R}^n_+} f(y) g(z) \mathcal{D}_\mu(x, y, z) s(y) s(z) dy dz$$  \hspace{1cm} (B.13)$$
where $x, y, z \in \mathbb{R}^n_+$.

**Remark B.1** (Relation between $\mathcal{D}_\mu$ and $\mathcal{D}_{\mu_i}$).

$$\mathcal{D}_\mu(x, y, z) = C_{\mu}^{-2}(xyz)^{1/2-\mu} \mathcal{D}_{\mu}(x, y, z)$$  \hspace{1cm} (B.14)$$
where $\mathcal{D}_{\mu}(x, y, z)$ is given by (B.12) and $\mathcal{D}_{\mu_i}(x_i, y_i, z_i)$ is given by (2.23).

**Proposition B.2.** In this proposition we summarize some properties for the kernel $\mathcal{D}_\mu(x, y, z)$ given by (B.12).

(i) $\mathcal{D}_\mu(x, y, z) > 0$,

(ii) $\int_{\mathbb{R}^n_+} \mathcal{D}_\mu(x, y, z) \prod_{i=1}^n \{(z_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(z_i t_i)\} s(z) dz = C_{\mu} \prod_{i=1}^n \{(x_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(x_i t_i)\} \prod_{i=1}^n \{(y_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(y_i t_i)\}$

(iii) $\int_{\mathbb{R}^n_+} \mathcal{D}_\mu(x, y, z) s(z) dz = 1$

where $x, y, z, t \in \mathbb{R}^n_+$ and $\mathcal{J}_{\mu_i}$ denotes the well known Bessel function of first kind and order $\mu_i$ given by (A.1) for all $i = 1, \ldots, n$.

**Proof.** The proof of (ii) It is analogous to that of the Proposition 2.8, and it will be enough to observe that

$$\mathcal{D}_\mu(x, y, z) \prod_{i=1}^n \{(z_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(z_i t_i)\} z^{2\mu+1} = \prod_{i=1}^n \{(z_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(z_i t_i) \mathcal{D}_{\mu_i}(x_i, y_i, z_i) z_i^{2\mu_i+1}\},$$

is a product of functions in $z_i$ which are integrables in $\mathbb{R}_+$.

To see (iii), let us note that since $\mathcal{D}_{\mu_i}(x_i, y_i, z_i) \in L^1_{\infty}((0, \infty), s(z_i))$ for all $i = 1, \ldots, n$ then

$$\mathcal{D}_\mu(x, y, z) \in L^1_{\infty}((0, \infty), s(z)),$$

and

$$\left| \mathcal{D}_\mu(x, y, z) \prod_{i=1}^n \{(z_i t_i)^{-1-\mu_i} \mathcal{J}_{\mu_i}(z_i t_i)\} \right| \leq \frac{1}{C_{\mu}} \mathcal{D}_\mu(x, y, z).$$

The result follows from the Dominated Convergence Theorem.

**Theorem B.3.** Let $\{\phi_m\} \subset L^1_{\infty}(\mathbb{R}_+, s)$ a sequence of functions such that:

1. $\phi_m(x) \geq 0$ in $\mathbb{R}_+$.
2. $\int_{\mathbb{R}^n_+} \phi_m(x) s(x) dx = 1$ for all $m \in \mathbb{N}$,
3. For all $\eta > 0$, $\lim_{m\to\infty} \int_{\|x\| > \eta} \phi_m(x) s(x) dx = 0$.

If $f \in L^1_1(\mathbb{R}_+, s)$ then $\lim_{n\to\infty} \|f \# \phi_m - f\|_{L^1_1(\mathbb{R}_+, s)} = 0$. 
We consider the sequence \( \{\phi_m\}_{m \in \mathbb{N}} \) defined by
\[
\phi_m(x) = m^{n+1} e^{-\frac{m^2}{2n}}.
\]
This sequence verifies conditions (1), (2) and (3) of Theorem B.3, then if \( f \in L^1(\mathbb{R}_+^n, s) \),
\[
\lim_{n \to \infty} \| f \# \phi_m - f \|_{L^1(\mathbb{R}_+^n, s)} = 0.
\]
Let us show that:
\[
\phi_m \# f(x) = \int_{\mathbb{R}_+^n} H_\mu(f)(z) e^{-\frac{z^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{-\mu_i} J_{\mu_i}(x_i z_i) \right\} z^{2\mu+1} \, dz.
\]
To see this we define
\[
G_x(z) = e^{-\frac{z^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{-\mu_i} J_{\mu_i}(x_i z_i) \right\}.
\]
Clearly, \( G_x(z) \in L^1(\mathbb{R}_+^n, s) \) and from Lemma B.4 we have
\[
\int_{\mathbb{R}_+^n} H_\mu(f)(z) e^{-\frac{z^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{-\mu_i} J_{\mu_i}(x_i z_i) \right\} z^{2\mu+1} \, dz
\]
\[
= \int_{\mathbb{R}_+^n} H_\mu(f)(z) G_x(z) z^{2\mu+1} \, dz
\]
\[
= \int_{\mathbb{R}_+^n} f(t) H_\mu(G_x(z))(t) t^{2\mu+1} \, dt
\]
Moreover,
\[
H_\mu(G_x(z))(t) = \int_{\mathbb{R}_+^n} G_x(z) \left\{ \prod_{i=1}^n (z_i t_i)^{-\mu_i} J_{\mu_i}(z_i t_i) \right\} z^{2\mu+1} \, dz
\]
\[
= \int_{\mathbb{R}_+^n} e^{-\frac{z^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{-\mu_i} J_{\mu_i}(x_i z_i) \right\} \left\{ \prod_{i=1}^n (z_i t_i)^{-\mu_i} J_{\mu_i}(z_i t_i) \right\} z^{2\mu+1} \, dz
\]
\[
= \int_{\mathbb{R}_+^n} e^{-\frac{z^2}{2m}} \left\{ \int_{\mathbb{R}_+^n} D_\mu(x, t, \xi) \left\{ \prod_{i=1}^n (\xi_i z_i)^{-\mu_i} J_{\mu_i}(\xi_i z_i) \right\} s(\xi) \, d\xi \right\} s(z) \, dz.
\]
Since
\[
\int_{\mathbb{R}_+^n} e^{-\frac{z^2}{2m}} \left\{ \int_{\mathbb{R}_+^n} D_\mu(x, t, \xi) \left\{ \prod_{i=1}^n (\xi_i z_i)^{-\mu_i} J_{\mu_i}(\xi_i z_i) \right\} s(\xi) \, d\xi \right\} s(z) \, dz
\]
\[
\leq C_{\mu}^{-1} \int_{\mathbb{R}_+^n} e^{-\frac{z^2}{2m}} \left\{ \int_{\mathbb{R}_+^n} D_\mu(x, t, \xi) s(\xi) \, d\xi \right\} s(z) \, dz < \infty
\]
it is possible change the order of integration in (B.20), then
On the other hand, by (B.23)

\[
\lim_{\xi \to \infty} \phi_m \# f(x)
\]

and for hypothesis

\[
h_{\mu} f \in L^1(\mathbb{R}_+^n, x^\mu + 1/2)
\]

then

\[
H_{\mu}(x^{-\mu - 1/2} f) = L^1(\mathbb{R}_+^n, x^{\mu + 1/2})
\]

Then the result continues to apply the Theorem B.5 to $x^{-\mu - 1/2} f$ and obtain (2.32).
Let $f, g$ be functions in $L^1(\mathbb{R}_+^n, sr)$, then
\[
\int_{\mathbb{R}_+^n} h_\mu f(t) g(t) \, dt = \int_{\mathbb{R}_+^n} f(t) h_\mu g(t) \, dt.
\]

**Proof of Lemma 2.20.** Let $\phi \in \mathcal{H}_\mu$.

(i). It follows easily from induction on $m$.

(ii). The proof of this result follows for induction on $m$. First let us observe that
\[
(h_\mu(||y||^2 + \lambda)^{-1}h_\mu)(-S_\mu + \lambda)\phi = h_\mu(||y||^2 + \lambda)^{-1}[-h_\mu S_\mu \phi + \lambda h_\mu \phi]
\]
\[
= h_\mu(||y||^2 + \lambda)^{-1}[-(-\cdot)(||y||^2 + \lambda)\phi] = h_\mu(||y||^2 + \lambda)^{-1}(||y||^2 + \lambda)h_\mu \phi = h_\mu(h_\mu \phi) = \phi
\]
So,
\[
(h_\mu(||y||^2 + \lambda)^{-1}h_\mu)(-S_\mu + \lambda) = Id.
\]
On the other hand
\[
(-S_\mu + \lambda)(h_\mu(||y||^2 + \lambda)^{-1}h_\mu)\phi = -S_\mu h_\mu(||y||^2 + \lambda)^{-1}h_\mu \phi + \lambda h_\mu(||y||^2 + \lambda)^{-1}h_\mu \phi
\]
\[
= -h_\mu[-||y||^2(||y||^2 + \lambda)^{-1}h_\mu \phi] + h_\mu[\lambda(||y||^2 + \lambda)^{-1}h_\mu \phi] = h_\mu(||y||^2 + \lambda)(||y||^2 + \lambda)^{-1}h_\mu \phi = h_\mu(h_\mu \phi) = \phi
\]
So,
\[
h_\mu(||y||^2 + \lambda)^{-1}h_\mu = (-S_\mu + \lambda)^{-1},
\]
then
\[
h_\mu(-S_\mu + \lambda)^{-1} = (||y||^2 + \lambda)^{-1}h_\mu.
\]
Now, suppose that $h_\mu(-S_\mu + \lambda)^{-m} = (\lambda + ||y||^2)^{-m}h_\mu$,
\[
h_\mu(-S_\mu + \lambda)^{-(m+1)}\phi = h_\mu(-S_\mu + \lambda)^{-m}(-S_\mu + \lambda)^{-1}\phi = (||y||^2 + \lambda)^{-m}h_\mu(-S_\mu + \lambda)^{-1}\phi = (||y||^2 + \lambda)^{-m}h_\mu \phi = (||y||^2 + \lambda)^{-(m+1)}h_\mu \phi
\]
(iii). By induction on $m$,
\[
h_\mu[-S_\mu(-S_\mu + \lambda)^{-1}] \phi = -h_\mu S_\mu[(-S_\mu + \lambda)^{-1}] \phi = (-1)[-||y||^2h_\mu(-S_\mu + \lambda)^{-1}] \phi = ||y||^2h_\mu(-S_\mu + \lambda)^{-1}\phi = ||y||^2(||y||^2 + \lambda)^{-1}h_\mu \phi
\]
Suppose that $h_\mu(-S_\mu(-S_\mu + \lambda)^{-1})^m = ||y||^{2m}(\lambda + ||y||^2)^{-m}h_\mu$.
\[
h_\mu[-S_\mu(-S_\mu + \lambda)^{-1}]^{m+1} \phi = h_\mu[-S_\mu(-S_\mu + \lambda)^{-1}] [-S_\mu(-S_\mu + \lambda)^{-1}]^m \phi = ||y||^2(||y||^2 + \lambda)^{-1}h_\mu||y||^{2m}(||y||^2 + \lambda)^{-m}h_\mu \phi = ||y||^{2(m+1)}(||y||^2 + \lambda)^{-(m+1)}h_\mu \phi
\]
The equalities for the case $u \in \mathcal{H}_\mu'$ are followed by transposition.
APPENDIX C. SIMILARITY OF BESSEL OPERATORS

Remark C.1. The operators $\Delta_\mu$ and $S_\mu$ given by (1.2) and (1.1) respectively are related through (C.1)

$$S_\mu = x^{\mu+1/2} \Delta_\mu x^{-\mu-1/2}.$$

Let $\phi \in C^2(\mathbb{R}_+^n)$, then

$$x^{-\mu-1/2} S_\mu x^{\mu+1/2} \phi(x) = x^{-\mu-1/2} \left( \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \frac{4\mu_i^2 - 1}{4x_i^2} \right) x^{\mu+1/2} \phi(x)$$

and also

$$x^{-\mu-1/2} \frac{\partial^2}{\partial x_i^2} \{ x^{\mu+1/2} \phi(x) \} = x^{-\mu-1/2} x^{\mu+1/2} x_i^{-\mu_i+1/2} \frac{\partial^2}{\partial x_i^2} \{ x_i^{\mu_i+1/2} \phi(x) \} = x_i^{-\mu_i-1/2} \frac{\partial}{\partial x_i} \left( (\mu_i + 1/2) x_i^{\mu_i+1/2} \phi(x) + x_i^{\mu_i+1/2} \frac{\partial}{\partial x_i} \phi(x) \right)$$

$$= x_i^{-\mu_i-1/2} \left( (\mu_i + 1/2)(\mu_i - 1/2) x_i^{\mu_i-3/2} \phi(x) + (\mu_i + 1/2) x_i^{\mu_i-1/2} \frac{\partial}{\partial x_i} \phi(x) + \frac{\mu_i^2 - 1}{4x_i^2} \phi(x) \right)$$

$$= (\mu_i^2 - 1/4) x_i^{-2} \phi(x) + 2(\mu_i + 1/2) x_i^{-1} \frac{\partial}{\partial x_i} \phi(x) + \frac{\mu_i^2 - 1}{4x_i^2} \phi(x),$$

from where

$$x^{-\mu-1/2} S_\mu x^{\mu+1/2} \phi(x) = \sum_{i=1}^n x^{-\mu-1/2} \frac{\partial^2}{\partial x_i^2} x^{\mu+1/2} \phi(x) - \sum_{i=1}^n \frac{4\mu_i^2 - 1}{4x_i^2} \phi(x)$$

$$= \sum_{i=1}^n (\mu_i^2 - 1/4) x_i^{-2} \phi(x) + 2(\mu_i + 1/2) x_i^{-1} \frac{\partial}{\partial x_i} \phi(x) + \frac{\partial^2}{\partial x_i^2} \phi(x) - \sum_{i=1}^n \frac{4\mu_i^2 - 1}{4x_i^2} \phi(x)$$

$$= \sum_{i=1}^n 2(\mu_i + 1/2) x_i^{-1} \frac{\partial}{\partial x_i} \phi(x) + \frac{\partial^2}{\partial x_i^2} \phi(x) = \Delta_\mu \phi(x)$$

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