Generalized Chern-Simons terms in $\mathcal{N} = 1$ supergravity

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Abstract. We discuss general actions for matter-coupled $\mathcal{N} = 1$ supergravity, that include generalized Chern-Simons terms. We clarify the importance of these terms in ensuring gauge and supersymmetry invariance in case axionic shift symmetries are gauged. We also indicate the interplay of Peccei-Quinn terms, generalized Chern-Simons terms and anomalies in the context of $\mathcal{N} = 1$ supergravity.

1. Introduction

In this talk, we report on work concerning the interplay of generalized Chern-Simons terms and anomalies in $\mathcal{N} = 1$ supergravity, that appeared in [1]. 'Generalized Chern-Simons terms' (GCS terms) are a specific kind of terms that often appear in theories of supersymmetry and supergravity. More specifically, they involve vector fields that appear in the theory under consideration, as well as their field strengths. Typically these GCS terms assume the following form:

$$C^{(CS)}_{AB,C} W^C \wedge W^A \wedge F^B ,$$

where we have denoted the different vectors $W^A$ and their field strengths $F^A$ with the index $A = 1, \cdots, n$. Note that these terms depend on a constant tensor $C^{(CS)}_{AB,C}$, whose symmetry structure will be discussed in due course.

The existence and appearance of GCS terms in extended supersymmetry and supergravity is well-known (see for instance [2, 3, 4, 5, 6, 7, 8] and [9] for a more systematic analysis.). As we will review, the GCS terms are associated to certain axionic shift symmetries. The gauging of these symmetries spoils the gauge and supersymmetry invariance of the Lagrangian and the addition of GCS terms may be required to restore these invariances.

Generalized Chern-Simons terms also appear in more phenomenological applications of string theory. They can for instance appear in flux compactifications and generalized Scherk-Schwarz compactifications of higher-dimensional string theories. Indeed, these compactifications are at low energies encoded in the gauging of some of the global symmetries of the four-dimensional Lagrangian. It is thus possible that GCS terms are present in four dimensions, where they again ensure supersymmetry and gauge invariance of the action. Another application was given in [10], where it was emphasized that a combination of GCS terms and the Green-Schwarz mechanism is generically needed to cancel the anomalies in orientifold models with intersecting...
D-branes. It was moreover argued that non-vanishing GCS terms might have observable effects for certain variants of $Z'$ bosons. Finally, GCS terms also play an important role in the manifestly symplectic formulation of gauged supergravity with electric and magnetic potentials and tensor fields that was proposed in [3].

In view of these applications, it is rather surprising that GCS terms have up to now mainly been considered in theories of extended supersymmetry (see however [11] for a discussion of GCS terms in $\mathcal{N} = 1$ rigid supersymmetry). This is even more surprising in view of the fact that there is an important qualitative difference between $\mathcal{N} = 1$ and extended supersymmetry, having to do with the possible presence of quantum anomalies in $\mathcal{N} = 1$ theories. Indeed, the latter are only present in $\mathcal{N} = 1$ theories, since only these exhibit chiral fermions. The presence of these anomalies alters the discussion of how gauge and supersymmetry invariance is restored.

In this talk, we will give a systematic discussion of the structure of general $\mathcal{N} = 1$ supersymmetry with anomaly cancellation and GCS terms.

2. General gaugings in $\mathcal{N} = 1$ supersymmetry
In the following, we will consider $\mathcal{N} = 1$ supersymmetric theories, including vector multiplets and chiral multiplets. We will first focus on rigid supersymmetry and comment on supergravity later.

The kinetic terms for the gauge fields generically assume the following form:

$$L_1 = -\frac{i}{4} \text{Re} f_{AB} F^{A}_{\mu\nu} F^{B}_{\mu\nu} + \frac{i}{4} \text{Im} f_{AB} \tilde{F}^{A}_{\mu\nu} \tilde{F}^{B}_{\mu\nu},$$

(2)

where we have used the non-Abelian field strength $F^{A}_{\mu\nu} = F^{A}_{\mu\nu} + W^{B}_{\mu} W^{C}_{\nu} f^{BCA}$, where $F^{A}_{\mu\nu} = 2 \partial_{[\mu} W^{\nu]}^{A}$ is the Abelian part. We have furthermore denoted the dual of $F^{A}_{\mu\nu}$ by $\tilde{F}^{A}_{\mu\nu}$. In the above equation, the vector fields couple to the complex scalar fields $z^i$ of the chiral multiplets, via the gauge kinetic function $f_{AB}(z)$, which is a holomorphic function of the $z^i$. The second term in (2) is often referred to as the Peccei-Quinn term.

If, under a gauge transformation with gauge parameter $\Lambda^A(x)$, some of the $z^i$ transform nontrivially, this may induce a corresponding gauge transformation of $f_{AB}(z)$. In case this transformation is of the form of a symmetric product of two adjoint representations of the gauge group,

$$\delta(\Lambda) f_{AB} = \Lambda^C \delta_C f_{AB}, \quad \delta_C f_{AB} = f_{CA}^D f_{BD} + f_{CB}^D f_{AD},$$

(3)

with $f_{CA}^B$ the structure constants of the gauge group, the kinetic term (2) is obviously gauge invariant. Based on arguments relying on the embedding of the gauge transformations in the symplectic group of electric-magnetic duality transformations (see [12, 13, 14, 15]), one can allow a more general transformation rule for the gauge kinetic function under gauge transformations. Apart from the transformation (3), the imaginary part of the gauge kinetic function then transforms with a constant shift:

$$\delta_C f_{AB} = i C_{AB,C} + f_{CA}^D f_{BD} + f_{CB}^D f_{AD}.$$  

(4)

The $C_{AB,C}$ are real constants, that are symmetric in the first two indices. For non-zero $C_{AB,C}$ and rigid parameters $\Lambda^A$ the above transformations represent a global symmetry, as they leave the Lagrangian invariant up to a total derivative:

$$\delta(\Lambda) L_1 = \frac{i}{4} C_{AB,C} \Lambda^C F^{A}_{\mu\nu} \tilde{F}^{B}_{\mu\nu}.$$  

(5)

For local gauge parameters $\Lambda^A(x)$, the above term no longer represents a total derivative and gauge invariance is broken.
In a full supersymmetric context, the gauge non-invariance stems from other terms as well. Consider the full kinetic terms of the vector multiplets in $\mathcal{N} = 1$ supersymmetry, including the contributions of the gauginos $\lambda^A$:

\[
S_{f,\text{kin}} = \int d^4x \left[ -\frac{1}{4} \text{Re} \ f_{AB} \mathcal{F}^A_{\mu\nu} \mathcal{F}^{\mu\nu B} - \frac{1}{2} \text{Re} \ f_{AB} \bar{\lambda}^A \partial^{\mu} \lambda^B \\
+ \frac{1}{4} \text{Im} \ f_{AB} \mathcal{F}^A_{\mu\nu} \bar{\mathcal{F}}^{\mu\nu B} + \frac{1}{4} i (D_\mu \text{Im} f_{AB}) \bar{\lambda}^A \gamma^5 \gamma^\mu \lambda^B \right],
\]

where

\[
D_\mu f_{AB} = \partial_\mu f_{AB} - 2W^C_\mu f_{C(AB)D} f_{D}\ .
\]

The gauge non-invariance of the action now stems from the last two terms. Note however that by adding an extra term to the action, one can obtain an action $\hat{S}_f$, whose gauge non-invariance again purely stems from the Peccei-Quinn term:

\[
\hat{S}_f = S_f + S_{\text{extra}}, \quad S_{\text{extra}} = \int d^4x \left( -\frac{1}{4} i W^C_\mu C_{AB,C} \bar{\lambda}^A \gamma^5 \gamma^\mu \lambda^B \right).
\]

Indeed, the gauge non-invariance of the action $\hat{S}_f$ is again given by (5).

As a consequence of the supersymmetry algebra, this gauge non-invariance also implies a non-invariance under supersymmetry. Indeed, the anticommutation relation satisfied by the supercharges:

\[
\left\{ Q_\alpha, Q^\dagger_{\dot{\alpha}} \right\} = \sigma^\mu_{\alpha\dot{\alpha}} D_\mu = \sigma^\mu_{\alpha\dot{\alpha}} (\partial_\mu - W^A_\mu \delta_A) \ ,
\]

implies that if an action is invariant under supersymmetry, it should be gauge invariant as well. An explicit calculation shows that the non-invariance under supersymmetry is given by

\[
\delta(\epsilon) \hat{S}_f = \int d^4x \text{Re} \left( \frac{1}{2} C_{AB,C} \epsilon^\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma \left( W^C_\mu \mathcal{F}^A_{\nu\rho} \bar{\mathcal{F}}^{\nu\rho B} + \frac{1}{4} f_{DE} W^D_\mu W^E_\nu W^C_\rho W^B_\sigma \right) \right).
\]

Note that this expression involves only fields of the vector multiplets and none of the chiral multiplets. In the following, we will try to restore gauge and supersymmetry invariance by considering anomalies and GCS terms. Let us first comment on these two ingredients. We will first consider the GCS terms in more detail and then focus on the structure of the anomalies.

### 3. The structure of generalized Chern-Simons terms and anomalies

The structure of the GCS terms was already considered in [2, 9, 11] in the context of extended supergravity and rigid $\mathcal{N} = 1$ supersymmetry. Explicitly, the GCS terms are of the form

\[
S_{\text{CS}} = \int d^4x \frac{1}{2} C_{AB,C}^{(\text{CS})} \epsilon^\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma \left( \frac{1}{3} W^C_\mu W^A_\nu \mathcal{F}^{\mu B} + \frac{1}{2} f_{DE} W^D_\mu W^E_\nu W^C_\rho W^B_\sigma \right).
\]

These terms are proportional to a tensor $C_{AB,C}^{(\text{CS})}$ that is symmetric in $(A, B)$. Since a completely symmetric part in $C_{AB,C}^{(\text{CS})}$ would drop out of $S_{\text{CS}}$, we can restrict $C_{AB,C}^{(\text{CS})}$ to be a tensor of mixed symmetry structure, i.e. with

\[
C_{(AB,C)}^{(\text{CS})} = 0.
\]

Another point that we should mention is that for semi-simple gauge algebras, the GCS terms do not bring anything new, at least in the classical theory. Indeed, it was shown in [9] that for semi-simple gauge algebras the GCS terms can be replaced by an appropriate redefinition of the gauge kinetic function $f_{AB}$.
We also stress that at this point, the constants $C^{(CS)}_{AB,C}$ are not yet related to the constants $C_{AB,C}$ introduced in the previous section. Later on, we will choose them appropriately, in order to get gauge invariant actions.

Let us now discuss the structure of anomalies in $N=1$ supergravity, that can arise since $N=1$ theories contain chiral fermions. Anomalies indicate a non-invariance of the quantum effective action $\Gamma[W_\mu]$:

$$e^{-\Gamma[W_\mu]} = \int D\tilde{\phi} D\phi e^{-S(W_\mu, \tilde{\phi}, \phi)}.$$  

(13)

Even if the classical action is gauge invariant, a non-invariance of the path integral measure may occur, leading to a quantum anomaly. The non-invariance of the quantum effective action is then encoded in an anomaly $A_{A}$, that is a local polynomial functional of the vector fields $W_\mu$:

$$\delta(\Lambda)\Gamma[W] = -\int d^4x \Lambda^A \left( D_\mu \frac{\delta \Gamma[W]}{\delta W_\mu} \right)_A \equiv \int d^4x \Lambda^A A_A,$$  

(14)

As discussed above, the appearance of gauge anomalies also indicates that a supersymmetry anomaly will be present. The final non-invariance of the quantum effective action will thus involve both a gauge anomaly $A_A$ and a supersymmetry anomaly $\bar{\epsilon} A_e$:

$$A = \delta \Gamma(W) = \delta(\Lambda)\Gamma[W] + \delta(\epsilon)\Gamma[W] = \int d^4x \left( \Lambda^A A_A + \bar{\epsilon} A_e \right).$$  

(15)

The anomaly should satisfy the Wess-Zumino consistency conditions, which is just the statement that the anomalies satisfy the gauge algebra. For the gauge anomalies, these consistency conditions are for instance given by:

$$\delta(\Lambda_1) \left( \Lambda_2^A A_A \right) - \delta(\Lambda_2) \left( \Lambda_1^A A_A \right) = \Lambda_1^B \Lambda_2^C f_{BC}^A A_A.$$  

(16)

A full cohomological analysis of anomalies in supersymmetry and supergravity was made by Brandt in [16, 17]. His result is that the total anomaly should be of the form (15) with

$$A_C = -\frac{1}{4} \left[ d_{ABC} F_\mu^B + \left( d_{ABD} f_{CE}^B + \frac{3}{2} d_{ABC} f_{DE}^B \right) W_\mu^D W_\nu^E \right] \tilde{F}^{\mu\nu A},$$  

(17)

$$\bar{\epsilon} A_e = \text{Re} \left[ \frac{3}{2} i d_{ABC} \epsilon_{R} A_R^C A_{L}^{\mu} + i d_{ABC} W_\nu^C \tilde{F}^{\mu\nu A} \epsilon_{L} \gamma_\mu A_R^B \right. $$

$$\left. + \frac{3}{8} d_{ABC} f_{DE}^A \epsilon^{\mu\nu\rho\sigma} W_\mu^D W_\nu^E W_\rho^C \epsilon_{L} \gamma_\mu A_R^B \right].$$  

(18)

The coefficients $d_{ABC}$ form a totally symmetric tensor. In terms of the generators $T_A$ of the gauge group, they are explicitly given by:

$$d_{ABC} \sim \text{Tr} \left( \{T_A, T_B\} T_C \right).$$  

(19)

4. The cancellation

Let us now discuss how, using the ingredients described in the previous section, we can cure the gauge and supersymmetry non-invariances (5) and (10). In order to achieve this goal, we identify the $C^{(CS)}_{AB,C}$ with the part of mixed symmetry of the $C_{AB,C}$ coefficients that appeared in the gauge transformation law of the gauge kinetic function:

$$C^{(CS)}_{AB,C} = C^{(m)}_{AB,C} = C_{AB,C} - C_{(AB,C)}.$$  

(20)
Using this identification, the sum of $\hat{S}_f + S_{CS}$ is still not gauge- and supersymmetry-invariant. However:

$$\delta (A) (\hat{S}_f + S_{CS}) = - \int d^4 x \, A^A A_A^{(s)},$$
$$\delta (\epsilon) (\hat{S}_f + S_{CS}) = - \int d^4 x \, \bar{\epsilon} A^{(s)} ,$$

where $A^{(s)}$ represents the expression for the anomaly with $d_{ABC}$ replaced by $C_{(AB,C)}$. One thus concludes that, when the transformation law of the gauge kinetic function is such that the symmetric part of $C_{AB,C}$ cancels the anomaly, gauge and supersymmetry invariance can be restored by adding appropriate generalized Chern-Simons terms. Note however, that a gauge kinetic function with such a transformation law may simply not exist.

5. Discussion
In this talk, we have discussed the consistency conditions that ensure the gauge and supersymmetry invariance of matter coupled $\mathcal{N} = 1$ supergravity theories with Peccei-Quinn terms, generalized Chern-Simons terms and quantum anomalies. Each of these three ingredients defines a constant three index tensor:

(i) The gauge non-invariance of the Peccei-Quinn terms is proportional to a constant imaginary shift of the gauge kinetic function parameterized by a tensor $C_{AB,C}$. This tensor in general splits into a completely symmetric part and a part of mixed symmetry, $C_{(AB,C)} + C_{(m)}^{(AB,C)}$.

(ii) Generalized Chern-Simons terms are defined by a tensor, $C_{(CS)}^{AB,C}$, of mixed symmetry.

(iii) Quantum gauge anomalies of chiral fermions are proportional to a tensor $d_{ABC}$, which, in the appropriate regularization scheme, can be chosen to be completely symmetric, $d_{ABC} \propto \text{Tr} (\{T_A, T_B\} T_C).$

We find the full quantum effective action to be gauge invariant and supersymmetric if

$$C_{AB,C} = C_{(CS)}^{(AB,C)} + d_{ABC}. \quad (22)$$

Let us note that the inclusion of the quantum anomalies encoded in a non-trivial tensor $d_{ABC}$ is the key feature that distinguishes $\mathcal{N} = 1$ theories from theories with extended supersymmetry. Because of their possible presence, the Peccei-Quinn shift tensor $C_{AB,C}$ can now have a nontrivial symmetric part, $C_{(AB,C)}$. In the context of $\mathcal{N} = 2$ supergravity, the absence of such a completely symmetric part can be directly proven for theories for which there exists a prepotential [2].

The analysis discussed in this talk was performed in rigid supersymmetry. Using techniques of superconformal tensor calculus, one can however show that a similar result holds in supergravity. It turns out that the Chern-Simons term does not need any gravitino corrections and can thus be added as such to the matter-coupled supergravity actions. Our research thus also provides an extension to the general framework of coupled chiral and vector multiplets in $\mathcal{N} = 1$ supergravity [18, 19].

6. Acknowledgments
We are grateful to J. De Rydt, T. Schmidt, A. Van Proeyen and M. Zagermann for collaboration on the original paper that lead to this talk. Furthermore, we would like to thank the organizers of the HEP2007 conference for the opportunity to present this work. This work is supported in part by the European Community’s Human Potential Programme under contract MRTN-CT-2004-005104 ‘Constituents, fundamental forces and symmetries of the universe’ and by the FWO - Vlaanderen, project G.0235.05 and by the Federal Office for Scientific, Technical and Cultural Affairs through the ‘Interuniversity Attraction Poles Programme – Belgian Science Policy’ P6/11-P.
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