The origin of the universal optical conductivity form \( \sigma(\omega) \propto (1 - e^{-\omega/T})/\omega \) established in numerical model studies of doped antiferromagnets and consistent with experiments on cuprates near optimal doping is discussed. It is shown that such a behaviour appears quite generally when the single-particle excitations are overdamped and their relaxation follows the marginal Fermi liquid concept. Relation to recent ARPES experiments is also discussed.

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Since the discovery of superconducting cuprates the anomalous linear resistivity law, \( \rho(T) \propto T \), has been the strong indication that cuprates in the normal state are strange metals even near the optimum doping not following the normal Fermi liquid behaviour. Analogous message arises from the analysis of the optical conductivity which does not follow the standard Drude form,

\[
\sigma(\omega) = \frac{i\omega_p^2}{\omega + i/\tau},
\]

with a constant relaxation rate \( 1/\tau \) and plasma frequency \( \omega_p \). Experiments can be however well described within the marginal Fermi liquid (MFL) concept where the generalized frequency-dependent rate is introduced, i.e. \( \tau^{-1}(\omega, T) = \lambda(\omega + \eta T) \). These results suggest that also spectral functions as e.g. measured by the angle-resolved photoemission spectroscopy (ARPES) have to be anomalous, i.e. the quasiparticle (QP) relaxation has to follow the MFL dependence. Only recently the high resolution ARPES experiments on \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8+\delta \) (BSCCO) seem to be in position to confirm beyond this behaviour, obeyed in the optimum-doped materials surprisingly even at \( T \lesssim T_c \) for QP along the nodal direction in the Brillouin zone.

Recently several static and dynamic response functions at finite \( T \) have been studied numerically within the prototype \( t-J \) model, expected to represent well normal-state properties of cuprates. It was established that the MFL concept applies to several dynamic quantities in a broad range of intermediate hole doping 0.1 < \( c_h < 0.3 \), in particular to the dynamical conductivity \( \sigma(\omega) \) and to the QP relaxation rate as obtained from the analysis of spectral functions \( A(\mathbf{k}, \omega) \). Moreover, \( \sigma(\omega) \) has been been found close to a parameter-free form (we use \( k_B = \hbar = 1 \))

\[
\sigma(\omega) = C_0 \frac{1 - e^{-\omega/T}}{\omega},
\]

in a remarkably broad frequency regime \( 0 < \omega < \omega^* \sim 2t \), while \( C_0 \) being essentially \( T \)-independent for \( T < J \). The resulting \( \sigma(\omega < \omega^*) \) is clearly universal, i.e. the conductivity is governed by \( T \) only. Evidently, Eq.\( (2) \) reproduces the linear resistivity law \( \rho = T/C_0 \). The form \( \sigma(\omega) \) is also consistent with the MFL scenario for \( \tau^{-1}(\omega, T) \), however in a very restrictive way since both MFL parameters are essentially fixed. A reasonable overall fit can be thus achieved by \( \lambda \sim 0.6 \) and \( \eta \sim 2.7 \), while for more discussion on MFL parameters emerging from Eq.\( (2) \) we refer to Ref.\( [9] \). When optical experiments on \( \sigma(\omega) \) in cuprates are analysed within the MFL framework quite close values for \( \lambda, \eta \) are in fact reported. Such a behaviour seems not to be restricted to a particular model since it has been found also in the analysis of ladder systems, and the prerequisites for its validity has been partly discussed already in Refs.\( [6][10][12] \).

Our aim is to discuss the relation of \( \sigma(\omega) \) and the associated relaxation rate \( 1/\tau \) with the single-particle damping \( \Gamma \). In the usual case of a weak scattering both are simply related, i.e. \( 1/\tau \sim 2\Gamma \). Recent ARPES measurements in fact are not consistent with a weak scattering but rather with overdamped QP excitations, which needs a reconsideration of usual arguments.

The dynamical conductivity \( \sigma(\omega) \) is within the linear response theory related to the current-current correlation spectral function \( C(\omega) \), which in general replaces the constant \( C_0 \) in Eq.\( (2) \),

\[
C(\omega) = \Re \int_0^\infty dt \, e^{i\omega t} \langle j(t)j^* \rangle,
\]

where \( j \) is the electric current density. In the following we try to establish the conditions for the universal behaviour \( C(\omega) \sim C_0 \). The simplest approach is to decouple \( C(\omega) \) in terms of single-particle spectral functions \( A(\mathbf{k}, \omega) \),

\[
C(\omega) = \frac{2\pi e^2}{N} \sum_\mathbf{k} (\nu^*_k)^2 \int d\omega' \, f(-\omega') f(\omega' - \omega) \, A(\mathbf{k}, \omega') A(\mathbf{k}, \omega' - \omega),
\]

where \( f \) is the Fermi function and \( \nu^*_k \) unrenormalized band velocities. Note that Eq.\( (4) \) should become exact for the local Fermi liquid as realized in the limit of infinite...
dimensions \[15\]. Relevant for cuprates Eq.6 also represents a convolution of the \(A_-\) and \(A_+\) spectral functions so the resulting \(C(\omega)\) should scale with the hole doping \(\epsilon_h\), as expected for doped Mott insulators.

\[ A(\mathbf{k}, \omega) = \frac{1}{\pi} \frac{Z_k \Gamma_k}{(\omega - \epsilon_k)^2 + \Gamma_k^2}, \]  

where QP parameters \(Z_k, \Gamma_k, \epsilon_k\) in general dependent on \(\omega\) and \(T\). In order to reproduce the MFL form of \(\sigma(\omega)\) one has to assume the MFL form for the QP damping, i.e. \(\Gamma = \gamma(|\omega| + \xi T)\), but as well neglect the \(k\) dependence of \(\Gamma\) and \(Z\). We however allow for the asymmetry of damping of electron-like \((\omega > 0)\) and hole-like \((\omega < 0)\) QP excitations as has been clearly established in the numerical analysis \[8\]. We thus consider also cases where \(\gamma_+ \neq \gamma_-\) and \(\xi_+ \neq \xi_-\) where \(\pm\) refer to \(\omega > 0\) and \(\omega < 0\) regimes, respectively. We also neglect the \(Z(\omega)\) dependence within the MFL framework.

Below the cutoff frequency \(\omega < \omega^*\) the behaviour near the Fermi surface should be dominant. Assuming slowly varying density of states \(N(\epsilon)\) one can derive \[2\]

\[ C(\omega) = \bar{C} \int d\omega' f(-\omega') f(\omega' - \omega) \frac{\bar{\Gamma}(\omega, \omega')}{\omega'^2 + \bar{\Gamma}(\omega, \omega')^2}, \]  

where \(\bar{\Gamma}(\omega, \omega') = \Gamma(\omega') + \Gamma(\omega' - \omega)\), depending only on \(\omega/T\) and MFL parameters \(\gamma, \xi\), where \(\xi \sim \pi\) is usually used.

It is evident from Eqs.6 that for \(\gamma \ll 1\) one recovers \(C(\omega)\) strongly peaked at \(\omega = 0\) and consequently \(\sigma(\omega) \propto C(\omega) \propto \Gamma(\omega)/\omega^2 + \Gamma^2\) with \(\bar{\Gamma} = 2\Gamma(\omega/2, T)\). This is just a generalized Drude form with \(1/\tau(\omega) = 2\Gamma(\omega/2)\), invoked in connection with the MFL concept \[2\].

For the regime of overdamped QP with \(\gamma \sim 1\) or more appropriate \(\gamma \xi \sim 1\) we present in Fig. 1 \(C(\omega)\) for several \(\gamma\) fixing \(\xi = \pi\). While for \(\gamma \ll 0.2\) still a pronounced peak shows up at \(\omega \sim 0\), \(C(\omega)\) becomes for larger \(\gamma\) nearly constant or very slowly varying in a wide range of \(\omega/T\). For \(\gamma > 0.3\) one finds \(C(0) < C(\omega \to \infty)\), approaching for \(\gamma \gg 1\) the ratio \(C(0)/C(\infty) = 1/\xi\).

![FIG. 1. Current-current correlation spectra \(C(\omega)\) vs. \(\omega/T\) for various \(\gamma\) at fixed \(\xi = \pi\).](image)

Quite similar is the behaviour in the case of an asymmetric QP damping. In Fig. 2 we present some results of fixed \(\gamma_+ = 0.7\) and \(\xi = \pi\), while varying \(\gamma_+\). From Eq.6 it is evident that essentially \(\gamma_+ + \gamma_-\) matters, so quite constant \(C(\omega)\) appears already for \(\gamma_+ \gtrsim 0.2\).

![FIG. 2. \(C(\omega)\) vs. \(\omega/T\) for various \(\gamma_+\) at fixed \(\gamma_- = 0.7\).](image)

The main message of such a simple analysis is that for systems with overdamped QP excitation of the MFL form Eq.6 describes quite well \(\sigma(\omega)\) for a wide range of parameters. Nearly constant \(C(\omega < \omega^*)\) also means that the current relaxation rate \(1/\tau^*\) is very fast, \(1/\tau^* \sim \omega^* \gg 1/\tau\), i.e. much faster than the conductivity relaxation scale apparent from Eqs.6, where \(1/\tau \propto T\) is determined solely by thermodynamics. It should be mentioned that one can fit results for large \(\gamma\) also with Eq.6 with large \(\lambda\), but in this regime the form of \(\sigma(\omega)\) is quite insensitive to \(\lambda\).
It is evident that the preceding treatment of $\sigma(\omega)$ is oversimplified. Vertex corrections seem to be rather unavoidable in the treatment of strongly correlated electrons in lower dimensions. Delicate appears the assumption that QP parameters $\Gamma_{\mathbf{k}}, Z_{\mathbf{k}}$ are independent of $\mathbf{k}$ near the Fermi surface. However, for the validity of Eq.(3) it is essentially enough to assume that $\Gamma_{\mathbf{k}}(\omega)$ is independent of deviations $\Delta \mathbf{k}_\perp$ perpendicular to the Fermi surface. In fact this is just what is observed in recent ARPES studies of BSCCO [3]. A smooth dependence on $\Delta \mathbf{k}_\parallel$ which on the other hand appears to be quite pronounced [3] could be presumably accommodated not spoiling general conclusions. One should however realize the importance of the upper cutoff scale $\epsilon^*$ for the validity of the MFL QP behaviour in Eqs.(4,5). In the regime of large $\gamma \xi \gg 1$, which appears in the case in cuprates, even at moderately low $T$ $C(\omega)$ could be influenced by cutoff $\epsilon^*$.

It seems that more complete calculations of $C(\omega)$, as e.g. performed numerically [3] lead to even flatter $C(\omega < \omega^*)$, than found in Figs. 1.2 e.g. for $\gamma \gg 1$. In fact the condition for $C(\omega < \omega^*) \sim C_0$ is that the decay of $C(t)$ is fast and monotonous. Within the planar $t$-$J$ model at the intermediate doping it was found that $\omega^* \sim 2t$, allowing for an effective mean free path $l^*$ of only few cells. Such a short $l^*$ can be plausibly explained by assuming that charge carriers - holes entirely loose the phase coherence in collisions with other due to randomizing effect of an incoherent spin background. Note again that short correlation length (even at $T \lesssim T_\varepsilon$) appears also from the analysis of ARPES spectral functions $A(k, \omega)$ varying $\Delta k_{||}$ along the Fermi surface [3].

Let us finally discuss in more detail the relevance of the above analysis to the situation in cuprates, as well as the relation to numerical model calculations performed for the $t$-$J$ model at $T > 0$. Recent analysis of ARPES results in BSCCO gives for hole-like excitations in the particular (nodal) direction ($0, 0$)$\rightarrow(\pi, \pi)$ the MFL form with $\Gamma \sim 0.75\omega$ for $\omega > T$ and $\Gamma \sim 2.5T$ for $\omega < T$ [3]. Similar is the analysis of spectra in Refs. [4]. This means definitely an overdamped character of hole excitations, since the full width at half maximum (FWHM) $\Delta \sim 2\Gamma(\epsilon) > \epsilon$ is always larger than the QP (binding) energy $\epsilon$. Also one should note that the QP damping appears even larger along other parts of the FS. Quite similar QP damping was established within the $t$-$J$ model [3] where at intermediate doping the hole-part self energy was found to be of the MFL form, i.e. $\text{Im}\Sigma \sim -\gamma(\omega + \xi T)$ with $\gamma \sim 1.4$ and $\xi \sim 3.5$. In making the comparison one should take into account that $\Gamma = Z\text{Im}\Sigma$. Since at the peak position we find $Z \sim 0.5$ experimental and model values appear reasonably close. If one takes any of these data they do not satisfy the weak-scattering relation $\lambda \sim 2t$, since optical experiments give $\lambda \sim 0.5 - 0.7 < 2t$ [3,4].

In conclusion, we have investigated the origin of the universal conductivity Eq.(2) which represents a novel diffusion law, being a counterpart of the usual Drude form for weak scattering, but here determined by $T$ only. From the analysis it follows that such a behaviour generally appears when the QP are overdamped and their relaxation also follows the MFL behaviour. The calculated spectra $C(\omega)$ represent a correction to the universal form but show rather insignificant variation under the latter conditions. Moreover one can expect its validity even in a more elaborate treatment. While the case of cuprates at optimum doping is striking due to a large range of validity (both in frequency and temperature) of the novel dynamics the situation is valid in more restricted range also outside the optimum doping regime (where additional low energy scales appear) and possibly also in other strongly correlated systems.

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