Discrete approximations to Dirichlet and Neumann Laplacians on a half-space and norm resolvent convergence

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Abstract
We extend recent results on discrete approximations of the Laplacian in \( \mathbb{R}^d \) with norm resolvent convergence to the corresponding results for Dirichlet and Neumann Laplacians on a half-space. The resolvents of the discrete Dirichlet/Neumann Laplacians are embedded into the continuum using natural discretization and embedding operators. Norm resolvent convergence to their continuous counterparts is proven with a quadratic rate in the mesh size. These results generalize with a limited rate to also include operators with a real, bounded, and Hölder continuous potential, as well as certain functions of the Dirichlet/Neumann Laplacians, including any positive real power.

Keywords: norm resolvent convergence, Dirichlet Laplacian, Neumann Laplacian, lattice.

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1 Introduction
Let \( H^D_0 \) be the Dirichlet Laplacian and let \( H^N_0 \) be the Neumann Laplacian on the half-space \( \mathbb{R}^d_+ \), and let \( \mathcal{H}^+ = L^2(\mathbb{R}^d_+) \). Let \( H^D_0,h \) and \( H^N_0,h \) be the standard finite difference discretizations of \( H^D_0 \) and \( H^N_0 \), defined on \( \mathcal{H}^+_h = \ell^2(h\mathbb{Z}^d_+) \) with a mesh size \( h > 0 \); see section 2.3 for the precise definitions.

Using suitable embedding operators \( J^{ro}_h, J^{re}_h : \mathcal{H}^+_h \to \mathcal{H}^+ \) and discretization operators \( K^{ro}_h, K^{re}_h : \mathcal{H}^+ \to \mathcal{H}^+_h \) (see section 2.2), we prove the following type of norm resolvent convergence with an explicit rate in the mesh size.

Theorem. Let \( K \subset \mathbb{C} \setminus [0, \infty) \) be compact. Then there exists \( C > 0 \) such that
\[
\| J^{ro}_h(H^D_{0,h} - zI^+_h)^{-1}K^{ro}_h - (H^D_{0} - zI^+)^{-1}\|_{\mathcal{B}(\mathcal{H}^+)} \leq Ch^2,
\]
and
\[
\| J^{re}_h(H^N_{0,h} - zI^+_h)^{-1}K^{re}_h - (H^N_{0} - zI^+)^{-1}\|_{\mathcal{B}(\mathcal{H}^+)} \leq Ch^2,
\]
for \( 0 < h \leq 1 \) and \( z \in K \).

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Norm resolvent convergence was first shown for discrete approximations of the Laplacian on \( \mathbb{R}^d \) in \([10]\) and was extended to classes of Fourier multipliers in \([2]\). Recently norm resolvent convergence of discrete approximations to other operators have been considered as well, such as discrete Dirac operators in \([3]\) and quantum graph Hamiltonians in \([4]\).

We prove the above result as Theorem 3.2 in section 3, and also prove several extensions to this result. In section 3.1 we add a real, bounded, and Hölder continuous potential \( V \) to \( H^D_0 \) and \( H^N_0 \), and add a discrete potential \( V_h(n) = V(hn) \) with \( n \in \mathbb{Z}^d \) to \( H^D_{0,h} \) and \( H^N_{0,h} \). The norm resolvent estimates with a potential are given in Theorem 3.6 with a rate that now depends explicitly on the Hölder exponent for \( V \). Such norm resolvent convergence implies much improved spectral results compared to e.g. strong resolvent convergence. This includes convergence of the spectrum in a local Hausdorff distance \([2, \text{Section 5}]\).

Finally in section 3.2 we prove norm resolvent estimates between \( \Psi(H^D_{0,h}) \) and \( \Psi(H^D_0) \), and between \( \Psi(H^N_{0,h}) \) and \( \Psi(H^N_0) \), defined via the functional calculus for certain functions \( \Psi \) that have also been considered in \([2]\) for estimates on the full space \( \mathbb{R}^d \). The results are given in Theorem 3.10. As an example, this includes \( \Psi(\lambda) = \lambda^{s/2} \) for any positive real power \( s \). This example leads to norm resolvent estimates with a rate of \( 1^{\min\{s,2\}} \).

Fractional Laplacians on a half-space (or general domains) with Dirichlet and Neumann boundary conditions have been considered by several authors. See e.g. \([1, 5, 7, 8]\) for some recent results. However, results are scarce for discrete approximations of such operators.

2 Preliminaries

We give the results in dimensions \( d \geq 2 \). The case \( d = 1 \) is obtained by a simple modification of the arguments below.

Let \( d \geq 2 \) and \( d' = d - 1 \). For \( x \in \mathbb{R}^d \) we write \( x = (x_1, x') \) with \( x_1 \in \mathbb{R} \) and \( x' \in \mathbb{R}^{d'} \). The half-space is denoted by \( \mathbb{R}^d_+ = (0, \infty) \times \mathbb{R}^{d'} \). For \( x = (x_1, x') \in \mathbb{R}^d_+ \) the reflection of \( x \) in the hyperplane \( \{0\} \times \mathbb{R}^{d'} \) is denoted by
\[
\hat{x} = (-x_1, x').
\]

For \( n \in \mathbb{Z}^d \) we write \( n = (n_1, n') \) with \( n_1 \in \mathbb{Z} \) and \( n' \in \mathbb{Z}^{d'} \). We write
\[
\mathbb{Z}^d_+ = \{ n \in \mathbb{Z}^{d'} \mid n_1 \geq 1 \}
\]
for the discrete half-space. We denote the reflection in the discrete hyperplane \( \{0\} \times \mathbb{Z}^{d'} \) by
\[
\hat{n} = (-n_1, n').
\]

2.1 Extension and restriction operators

The continuous Hilbert spaces are denoted by
\[
\mathcal{H} = L^2(\mathbb{R}^d) \quad \text{and} \quad \mathcal{H}_+ = L^2(\mathbb{R}^d_+).
\]
In analogy with the even-odd decomposition of functions in dimension one we introduce the reflection-even and reflection-odd functions in \( \mathcal{H} \) by defining
\[
\mathcal{H}^e = \{ f \in \mathcal{H} \mid f(x) = f(\hat{x}), \ x \in \mathbb{R}^d \}
\]
and
\[
\mathcal{H}^o = \{ f \in \mathcal{H} \mid f(x) = -f(\hat{x}), \ x \in \mathbb{R}^d \},
\]
such that $\mathcal{H} = \mathcal{H}^e \oplus \mathcal{H}^{ro}$ as an orthogonal direct sum.

The discrete Hilbert spaces are given by

$$\mathcal{H}_h = \ell^2(h\mathbb{Z}^d) \quad \text{and} \quad \mathcal{H}_h^+ = \ell^2(h\mathbb{Z}_+^d)$$

with norms

$$\|v_h\|_{\mathcal{H}_h}^2 = h^d \sum_{n \in \mathbb{Z}^d} |v_h(n)|^2 \quad \text{and} \quad \|u_h\|_{\mathcal{H}_h^+}^2 = h^d \sum_{n \in \mathbb{Z}_+^d} |u_h(n)|^2.$$ 

Notice that we use the index $n \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_+^d$ in the notation for $v_h \in \mathcal{H}_h$ and $u_h \in \mathcal{H}_h^+$.

The dependence on the mesh size is given by the subscript $h$.

The reflection-even and reflection-odd sequences are defined by

$$\mathcal{H}_h^e = \{v_h \in \mathcal{H}_h \mid v_h(n) = v_h(\bar{n}), \ n \in \mathbb{Z}^d\}$$

and

$$\mathcal{H}_h^{ro} = \{v_h \in \mathcal{H}_h \mid v_h(n) = -v_h(\bar{n}), \ n \in \mathbb{Z}^d\}.$$ 

We have $\mathcal{H}_h = \mathcal{H}_h^e \oplus \mathcal{H}_h^{ro}$ as an orthogonal direct sum.

The reflection-odd extension operator $O : \mathcal{H}^+ \to \mathcal{H}$ and reflection-even extension operator $E : \mathcal{H}^+ \to \mathcal{H}$ are given by

$$O f(x) = \begin{cases} f(x), & x \in \mathbb{R}^d_+, \\ -f(\bar{x}), & \bar{x} \in \mathbb{R}^d_+, \end{cases} \quad E f(x) = \begin{cases} f(x), & x \in \mathbb{R}^d_+, \\ f(\bar{x}), & \bar{x} \in \mathbb{R}^d_+. \end{cases}$$

In the discrete case the reflection-odd extension operator $O_h : \mathcal{H}_h^+ \to \mathcal{H}_h$ is given by

$$O_h u_h(n) = \begin{cases} u_h(n), & n \in \mathbb{Z}_+^d, \\ 0, & n_1 = 0, n' \in \mathbb{Z}_+^d, \\ -u_h(\bar{n}), & \bar{n} \in \mathbb{Z}_+^d. \end{cases}$$

The discrete reflection-even extension operator $E_h : \mathcal{H}_h^+ \to \mathcal{H}_h$ is defined by

$$E_h u_h(n) = \begin{cases} u_h(n), & n \in \mathbb{Z}_+^d, \\ u_h(1, n'), & n_1 = 0, n' \in \mathbb{Z}_+^d, \\ u_h(\bar{n}), & \bar{n} \in \mathbb{Z}_+^d. \end{cases}$$

The natural restriction operators onto the half-spaces are denoted by

$$\mathcal{R} : \mathcal{H} \to \mathcal{H}^+ \quad \text{and} \quad \mathcal{R}_h : \mathcal{H}_h \to \mathcal{H}_h^+.$$ 

Obviously we have $\mathcal{R} O = \mathcal{R} E = I^+$ and $\mathcal{R}_h O_h = \mathcal{R}_h E_h = I^+_{\mathcal{H}_h}$, where we also introduced the notation for the identity operators on $\mathcal{H}^+$ and $\mathcal{H}_h^+$, respectively.

### 2.2 Embedding and discretization operators

In [2] embedding and discretization operators were defined using a pair of biorthogonal Riesz sequences. Here we consider only the special case of an orthogonal sequence, as in [10], but with the additional assumption that the generating function is reflection-even.
**Assumption 2.1.** Assume \( \varphi_0 \in \mathcal{H}^{re} \) such that \( \{\varphi_0(\cdot - n)\}_{n \in \mathbb{Z}^d} \) is an orthonormal sequence in \( \mathcal{H} \).

Define
\[
\varphi_{h,n}(x) = \varphi_0((x - hn)/h), \quad h > 0, \ n \in \mathbb{Z}^d, \ x \in \mathbb{R}^d.
\]
Since \( \varphi_0 \) is assumed reflection-even we have the important property
\[
\varphi_{h,n}(\bar{x}) = \varphi_{h,n}(x), \quad h > 0, \ n \in \mathbb{Z}^d, \ x \in \mathbb{R}^d. \tag{2.1}
\]
Define the embedding operators \( J_h : \mathcal{H}_h \to \mathcal{H} \) by
\[
J_h v_h(x) = \sum_{n \in \mathbb{Z}^d} v_h(n) \varphi_{h,n}(x), \quad v_h \in \mathcal{H}_h.
\]
From Assumption 2.1 it follows that \( \{h^{-d/2} \varphi_{h,n}\}_{n \in \mathbb{Z}^d} \) is an orthonormal sequence, hence that \( J_h \) is isometric.

The discretization operators are given by \( K_h = (J_h)^* \). With the convention that inner products are linear in the second entrance, we explicitly have
\[
K_h g(n) = \frac{1}{h^d} \langle \varphi_{h,n}, g \rangle_{\mathcal{H}}, \quad g \in \mathcal{H}.
\]
Let us note that (2.1) implies \( J_h \mathcal{H}_h^{ro} \subseteq \mathcal{H}^{ro} \), \( J_h \mathcal{H}_h^{re} \subseteq \mathcal{H}^{re} \), \( K_h \mathcal{H}_h^{ro} \subseteq \mathcal{H}_h^{ro} \), and \( K_h \mathcal{H}_h^{re} \subseteq \mathcal{H}_h^{re} \).

The half-space embedding operators \( J_h^{ro}, J_h^{re} : \mathcal{H}_h^+ \to \mathcal{H}^+ \) are defined as
\[
J_h^{ro} = \mathcal{R}_h O_h, \quad J_h^{re} = \mathcal{R}_h E_h.
\]
The operators \( J_h^{ro} \) and \( J_h^{re} \) are isometric, as can be seen from the following computation. Let \( u_h \in \mathcal{H}_h^+ \) and use that \( J_h O_h u_h \in \mathcal{H}^{ro} \),
\[
\|J_h^{ro} u_h\|_{\mathcal{H}^+}^2 = \frac{1}{2} \|J_h O_h u_h\|^2_{\mathcal{H}} = \frac{1}{2} \|O_h u_h\|^2_{\mathcal{H}_h} = \|u_h\|^2_{\mathcal{H}_h^+}.
\]
A similar computation holds for \( J_h^{re} \).

The half-space discretization operators \( K_h^{ro}, K_h^{re} : \mathcal{H}^+ \to \mathcal{H}_h^+ \) are defined as
\[
K_h^{ro} = \mathcal{R}_h K_h O, \quad K_h^{re} = \mathcal{R}_h K_h E.
\]
Note that \( K_h^{ro} J_h^{ro} = K_h^{re} J_h^{re} = I_h^+ \). \( J_h^{ro} K_h^{ro} \) is the orthogonal projection onto \( \text{Ran} \ J_h^{ro} \) in \( \mathcal{H}^+ \) and \( J_h^{re} K_h^{re} \) is the orthogonal projection onto \( \text{Ran} \ J_h^{re} \) in \( \mathcal{H}^+ \).

### 2.3 Laplacians

Let \( H_0 = -\Delta \) be the Laplacian in \( \mathcal{H} \) with domain \( \mathcal{D}(H_0) = H^2(\mathbb{R}^d) \).

The Dirichlet Laplacian \( H_0^D \) on \( \mathcal{H}^+ \) is defined as the positive self-adjoint operator given by the Friedrichs extension of \( -\Delta|_{C_0^\infty(\mathbb{R}_+^d)} \). Equivalently, \( H_0^D \) is the variational operator associated with the triple \( (\mathcal{H}^+, H_0^1(\mathbb{R}_+^d), q) \), where the sesquilinear form \( q \) is
\[
q(u, v) = \int_{\mathbb{R}_+^d} \nabla u \cdot \nabla v \ dx.
\]
By [6] Theorem 9.11] the domain of \( H_0^D \) on a half-space simplifies to
\[
\mathcal{D}(H_0^D) = H^2(\mathbb{R}_+^d) \cap H_0^1(\mathbb{R}_+^d) = \{ u \in H^2(\mathbb{R}_+^d) \mid \gamma_0 u = 0 \},
\]
where $\gamma_0$ is the Dirichlet trace operator.

Next we define the Neumann Laplacian $H_0^N$ on $\mathcal{H}^+$ as the positive self-adjoint variational operator associated with the triple $(\mathcal{H}^+, H^1(R^d_+), q)$. On a half-space its domain simplifies via [6, Theorem 9.20] to

$$\mathcal{D}(H_0^N) = \{ u \in H^2(R^d_+ \mid \gamma_1 u = 0 \},$$

where $\gamma_1$ is the Neumann trace operator.

From [6, Theorem 9.2] the trace maps satisfy $\gamma_j \in \mathcal{B}(H^m(R^d_+), H^{m-j-\frac{1}{2}}(\partial R^d_+))$ for $m \in \mathbb{N}$ and $j \leq m - 1$.

We need the following lemma. The result is a consequence of e.g. [9, Proposition 2.2]. We give a shorter proof for the sake of completeness.

**Lemma 2.2.**

(i) Let $f \in \mathcal{D}(H_0^D)$. Then $\mathcal{O} f \in \mathcal{D}(H_0)$ and $\mathcal{O} H_0^D f = H_0 \mathcal{O} f$. Furthermore, for $z \in C \setminus [0, \infty)$ and all $g \in \mathcal{H}^+$ we have

$$\mathcal{O}(H_0^D - z I^+)^{-1} g = (H_0 - z I)^{-1} \mathcal{O} g. \quad (2.2)$$

(ii) Let $f \in \mathcal{D}(H_0^N)$. Then $\mathcal{E} f \in \mathcal{D}(H_0)$ and $\mathcal{E} H_0^N f = H_0 \mathcal{E} f$. Furthermore, for $z \in C \setminus [0, \infty)$ and all $g \in \mathcal{H}^+$ we have

$$\mathcal{E}(H_0^N - z I^+)^{-1} g = (H_0 - z I)^{-1} \mathcal{E} g. \quad (2.3)$$

**Proof.** (i): Let $f \in \mathcal{D}(H_0^D)$. Since $C_0^\infty(R^d_+)$ is a core for $H_0^D$, we can find a sequence $\psi_n \in C_0^\infty(R^d_+)$ such that $\psi_n \to f$ and $H_0^D \psi_n \to H_0^D f$ in $\mathcal{H}^+$, as $n \to \infty$. We have $\mathcal{O} \psi_n \in C_0^\infty(R^d)$, such that $\mathcal{O} \psi_n \to \mathcal{O} f$ and $\mathcal{O} H_0^D \psi_n \to \mathcal{O} H_0^D f$ in $\mathcal{H}$, as $n \to \infty$. Note that

$$\mathcal{O} H_0^D \psi_n = \mathcal{O}(-\Delta \psi_n) = -\Delta \mathcal{O} \psi_n,$$

since $-\Delta$ commutes with orthogonal coordinate transformations and since $\psi_n$ is supported in $R^d_+$, i.e. away from the hyperplane $\{0\} \times R^d$. Thus

$$\mathcal{O} \psi_n \to \mathcal{O} f \quad \text{and} \quad -\Delta \mathcal{O} \psi_n \to \mathcal{O} H_0^D f$$

in $\mathcal{H}$. Since $H_0$ is a closed operator we conclude that $\mathcal{O} f \in \mathcal{D}(H_0)$ and $H_0 \mathcal{O} f = \mathcal{O} H_0^D f$. The second part of the statement follows by using $\mathcal{D}(H_0^D) = \text{Ran}((H_0^D - z I^+)^{-1})$ for $z \in C \setminus [0, \infty)$.

(ii): Let $f \in \mathcal{D}(H_0^N)$. Restrictions of $\mathcal{E} f$ to either side of $\{0\} \times R^d$ has coinciding Dirichlet and Neumann traces, so at least $\mathcal{E} f \in H^1(R^d)$. We can approximate $f$ in $H^1(R^d_+)$ by a sequence $\psi_n \in C^\infty(R^d_+)$ with $\gamma_1 \psi_n = 0$. Now $\mathcal{E} \psi_n \in C^1(R^d)$ implies the identity

$$\partial_1 \mathcal{E} f \to \mathcal{O} \partial_1 f.$$

However we have that $\partial_1 f \in H^1_0(R^d_+)$, which has a zero-extension $E_0(\partial_1 f) \in H^1(R^d)$. Since

$$\partial_1 \mathcal{E} f(x) = \mathcal{O} \partial_1 f(x) = E_0(\partial_1 f)(x) - E_0(\partial_1 f)(\bar{x}),$$

then $\partial_1 \mathcal{E} f \in H^1(R^d)$ and as a consequence $\mathcal{E} f \in H^2(R^d) = \mathcal{D}(H_0)$, since there was no contention regarding the square integrability of the other partial derivatives. The rest of the proof follows by using that $-\Delta$ on $H^2(R^d)$ commutes with orthogonal coordinate transformations, and that $\mathcal{D}(H_0^N) = \text{Ran}((H_0^N - z I^+)^{-1})$ for $z \in C \setminus [0, \infty)$. \hfill \Box
The discrete Laplacian $H_{0,h}$ on $\mathcal{H}_h$ is given by

$$H_{0,h}v_h(n) = \frac{1}{h^2} \sum_{j=1}^d \left( 2v_h(n) - v_h(n + e_j) - v_h(n - e_j) \right), \quad v_h \in \mathcal{H}_h, \ n \in \mathbb{Z}^d.$$ 

Here $\{e_j\}_{j=1}^d$ denotes the canonical basis for $\mathbb{R}^d$. The discrete Dirichlet Laplacian on $\mathcal{H}_h^+$ is given by

$$H_{0,h}^{D}v_h(n) = \left\{ \begin{array}{ll} \frac{1}{h^2} \sum_{j=1}^d \left( 2v_h(n) - v_h(n + e_j) - v_h(n - e_j) \right) & \text{if } n_1 = 1, \\ + \frac{1}{h^2} \left( 2v_h(n) - v_h(n + e_1) \right) & \text{if } n_1 \geq 2. \end{array} \right.$$ 

Let $u_h \in \mathcal{H}_h^+$. Then using the definitions one can verify that $\mathcal{O}_h H_{0,h}^{D}u_h = H_{0,h} \mathcal{O}_h u_h$ and then

$$\mathcal{O}_h(H_{0,h}^{D} - zI_h^+)^{-1}u_h = (H_{0,h} - zI_h)^{-1}\mathcal{O}_h u_h. \quad (2.4)$$

The discrete Neumann Laplacian on $\mathcal{H}_h^+$ is given by

$$H_{0,h}^{N}v_h(n) = \left\{ \begin{array}{ll} \frac{1}{h^2} \sum_{j=1}^d \left( 2v_h(n) - v_h(n + e_j) - v_h(n - e_j) \right) & \text{if } n_1 = 1, \\ + \frac{1}{h^2} \left( v_h(n) - v_h(n + e_1) \right) & \text{if } n_1 \geq 2. \end{array} \right.$$ 

Let $u_h \in \mathcal{H}_h^+$. Similar to the above, $\mathcal{E}_h H_{0,h}^{N}u_h = H_{0,h} \mathcal{E}_h u_h$ and then

$$\mathcal{E}_h(H_{0,h}^{N} - zI_h^+)^{-1}u_h = (H_{0,h} - zI_h)^{-1}\mathcal{E}_h u_h. \quad (2.5)$$

**Remark 2.3.** Since we use homogeneous Dirichlet and Neumann conditions, the discrete Laplacians have a very similar finite difference structure. The discrete Neumann Laplacian only differs from the discrete Dirichlet Laplacian at the indices where $n_1 = 1$. Here the contributions from the boundary conditions either mean that $v_h(n - e_1) = 0$ (Dirichlet case) or that $v_h(n - e_1) = v_h(n)$ (Neumann case). This subtle difference also implies the connections to odd and even reflections in (2.4) and (2.5).

### 3 Results

Additional assumptions on the function $\varphi_0$ are needed to obtain our results, cf. [2] Assumption 2.8] or [10] Assumption B]. Let $\hat{\varphi}_0$ denote the Fourier transform of $\varphi_0$, defined as

$$\hat{\varphi}_0(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi_0(x) \, dx.$$
Assumption 3.1. Let $\phi_0$ satisfy Assumption 2.2 and assume that $\hat{\phi}_0$ is essentially bounded. Assume there exists $c_0 > 0$ such that

$$\text{supp}(\hat{\phi}_0) \subseteq [-3\pi/2, 3\pi/2]^d,$$

and $|\hat{\phi}_0(\xi)| \geq c_0, \quad \xi \in [-\pi/2, \pi/2]^d$.  

Theorem 3.2. Let $J^\text{ro}_h$, $J^\text{re}_h$, $K^\text{ro}_h$, and $K^\text{re}_h$ be as above, with $\phi_0$ satisfying Assumption 3.1. Let $K \subset C \setminus [0, \infty)$ be compact. Then there exists $C > 0$ such that

$$\|J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h - (H^D_0 - zI^+)^{-1}\|_{\mathcal{B}(\mathcal{H}^+)} \leq Ch^2, \quad (3.1)$$

and

$$\|J^\text{re}_h(H^N_0 - zI^+_h)^{-1}K^\text{re}_h - (H^N_0 - zI^+)^{-1}\|_{\mathcal{B}(\mathcal{H}^+)} \leq Ch^2, \quad (3.2)$$

for $0 < h \leq 1$ and $z \in K$.

Proof. Let $f \in \mathcal{H}^+$. Then

$$\|J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h f - (H^D_0 - zI^+)^{-1} f\|_{\mathcal{H}^+}^2 = \frac{1}{2}\|\mathcal{O}J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h f - \mathcal{O}(H^D_0 - zI^+)^{-1} f\|_{\mathcal{H}}^2.$$

We have $\mathcal{O}J^\text{ro}_h = \mathcal{O}R_{\mathcal{H}} \mathcal{O}_h = J_h \mathcal{O}_h$, since $J_h \mathcal{O}_h u_h$ is a reflection-odd function for all $u_h \in \mathcal{H}_+^h$. Thus using (2.4) we get

$$\mathcal{O}J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h f = J_h \mathcal{O}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h f = J_h(H_0 - zI)^{-1} \mathcal{O}_h K^\text{ro}_h f.$$

Now $\mathcal{O}_h K^\text{ro}_h f = \mathcal{O}_h R_{\mathcal{H}} K^\text{ro}_h f = K^\text{ro}_h \mathcal{O} f$, since $K^\text{ro}_h \mathcal{O} f$ is a reflection-odd sequence. Thus we have shown

$$\mathcal{O}J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h f = J_h(H_0 - zI)^{-1} K^\text{ro}_h \mathcal{O} f.$$

Using this result together with (2.2) we have shown that

$$\|(J^\text{ro}_h(H^D_0 - zI^+_h)^{-1}K^\text{ro}_h - (H^D_0 - zI^+)^{-1}) f\|_{\mathcal{H}^+}^2 = \frac{1}{2}\|(J_h(H_0 - zI)^{-1} - (H_0 - zI)^{-1}) \mathcal{O} f\|_{\mathcal{H}}^2.$$

Thus we can use the results in [2] or [10] to obtain (3.1).

To prove (3.2) note that $\mathcal{E}J^\text{re}_h = J_h \mathcal{E}_h$ and $\mathcal{E}_h K^\text{re}_h = K^\text{ro}_h \mathcal{E}_h$, and then use (2.3) instead of (2.2) and (2.5) instead of (2.4). This leads to:

$$\|(J^\text{re}_h(H^N_0 - zI^+_h)^{-1}K^\text{re}_h - (H^N_0 - zI^+)^{-1}) f\|_{\mathcal{H}^+}^2 = \frac{1}{2}\|(J_h(H_0 - zI)^{-1} - (H_0 - zI)^{-1}) \mathcal{E} f\|_{\mathcal{H}}^2,$$

which together with the results in [2] or [10] completes the proof of (3.2). □

3.1 Adding a potential

Next we add a potential. To obtain the results we introduce two assumptions.

Assumption 3.3. Let $\tau > d$. Assume that there exists $C > 0$ such that

$$|\phi_0(x)| \leq C(1 + |x|)^{-\tau}, \quad x \in \mathbb{R}^d.$$
**Assumption 3.4.** Let $V: \mathbb{R}^d_+ \to \mathbb{R}$ be a bounded function which is uniformly Hölder continuous of order $\theta \in (0, 1]$.

Note that $\mathbb{R}^d_+$ denotes the closed half-space, so the conditions hold up to the boundary.

**Lemma 3.5.** Let $V$ satisfy Assumption 3.4. Then $\mathcal{E}V$ is bounded and uniformly Hölder continuous of order $\theta$ on $\mathbb{R}^d$.

**Proof.** Boundedness is clear, and for the Hölder continuity we only need to consider points $x, y \in \mathbb{R}^d$ such that $x, \tilde{y} \in \mathbb{R}^d_+$. Now Assumption 3.4 and $|x - \tilde{y}| \leq |x - y|$ imply

$$|\mathcal{E}V(x) - \mathcal{E}V(y)| = |V(x) - V(\tilde{y})| \leq C|x - \tilde{y}|^\theta \leq C|x - y|^{\theta}.$$ 

We define the discretized potential as $V_h(n) = V(hn), h > 0, n \in Z^d$. Then we define $H^D = H^D_0 + V$ and $H^N = H^N_0 + V$ on $\mathcal{H}^+$, and $H^D_h = H^D_0 + V_h$ and $H^N_h = H^N_0 + V_h$ on $\mathcal{H}^+_h$.

**Theorem 3.6.** Let $J^{ro}_h, J^{re}_h, K^r_h$, and $K^e_h$ be as above, with $\varphi_0$ satisfying Assumptions 3.1 and 3.3. Let $V$ satisfy Assumption 3.4. Define

$$\frac{1}{\theta'} = \frac{1}{\theta} + \frac{1}{\tau - d}.$$ 

Let $K \subset C \setminus \mathbb{R}$ be compact. Then there exists $C > 0$ such that

$$\|J^{ro}_h(H^D_h - z I^+_h)^{-1}K^{ro}_h - (H^D - z I^+)^{-1}\|_{B(\mathcal{H}^+)} \leq Ch^{\theta'},$$

and

$$\|J^{re}_h(H^N_h - z I^+_h)^{-1}K^{re}_h - (H^N - z I^+)^{-1}\|_{B(\mathcal{H}^+)} \leq Ch^{\theta'},$$

for $0 < h \leq 1$ and $z \in K$.

**Proof.** Let $f \in \mathcal{H}^+$ and $x \in \mathbb{R}^d_+$. Then

$$\mathcal{O}(Vf)(x) = V(x)f(x) = (\mathcal{E}V)(x)(\mathcal{O}f)(x),$$

and

$$\mathcal{O}(Vf)(\tilde{x}) = -(Vf)(x) = V(x)(-f(x)) = (\mathcal{E}V)(\tilde{x})(\mathcal{O}f)(\tilde{x}).$$

Thus we have $\mathcal{O}V = (\mathcal{E}V)\mathcal{O}$ as operators from $\mathcal{H}^+$ to $\mathcal{H}$, where $\mathcal{E}V$ denotes the operator of multiplication in $\mathcal{H}$ by $(\mathcal{E}V)(x), x \in \mathbb{R}^d$. Let $H = H_0 + \mathcal{E}V$ on $\mathcal{H}$. Then combining the above result with the arguments leading to (2.2) we get for $f \in \mathcal{H}^+$

$$\mathcal{O}(H^D - z I^+)^{-1}f = (H - z I)^{-1}\mathcal{O}f.$$ 

We can repeat these arguments in the discrete case, leading to

$$\mathcal{O}_h(H^D_h - z I^+_h)^{-1}u_h = (H_h - z I_h)^{-1}\mathcal{O}_h u_h$$

for $u_h \in \mathcal{H}^+_h$. Here we have defined $H_h = H_0 + \mathcal{E}_h V_h$ on $\mathcal{H}_h$. Note that $\mathcal{E}_h V_h$ and $(\mathcal{E}V)_h$ may differ only at $n_1 = 0$. Thus replacing $\mathcal{E}_h V_h$ by $(\mathcal{E}V)_h$ introduces an error of order $h^{\theta}$, due to Assumption 3.4, and this error can be absorbed in the final estimate below.

Repeating the computations in the proof of Theorem 3.2 we get for $f \in \mathcal{H}^+$
\[
\left\| \left( J_h^o (H_h^D - z I_h^+)^{-1} K_h^o - (H^D - z I^+)^{-1} \right) f \right\|^2_{H^+} = \frac{1}{2} \left\| \left( J_h (H_h - z I_h) - (H - z I)^{-1} \right) \mathcal{O} f \right\|^2_{H}\]

We can then use [2, Theorem 4.4] to complete the proof.

The proof for the Neumann Laplacian is analogous, using instead that \( \mathcal{E}(V f) = (\mathcal{E} V)(\mathcal{E} f) \), which for \( f \in H^+ \) and \( u_h \in H^+_h \) gives

\[
\mathcal{E}(H^N - z I^+)^{-1} f = (H - z I)^{-1} \mathcal{E} f,
\]

\[
\mathcal{E}_h (H_h^N - z I_h^+)^{-1} u_h = (H_h - z I_h)^{-1} \mathcal{E}_h u_h.
\]

This leads to

\[
\left\| \left( J_h^o (H_h^N - z I_h^+)^{-1} K_h^o - (H^N - z I^+)^{-1} \right) f \right\|^2_{H^+} = \frac{1}{2} \left\| \left( J_h (H_h - z I_h) - (H - z I)^{-1} \right) \mathcal{E} f \right\|^2_{H},
\]

which can also be estimated by [2, Theorem 4.4].

3.2 Functions of Dirichlet and Neumann Laplacians

Now we extend the approximation results given in Theorem 3.2 to functions of the Dirichlet and Neumann Laplacians on the half-space. Let \( \Psi: [0, \infty) \to \mathbb{R} \) be a Borel function. Using the functional calculus we can define the operators \( \Psi(H_0^D), \Psi(H_0^D), \Psi(H_0^N), \) and \( \Psi(H_0^N_h) \).

We need the following lemma, which is an immediate consequence of [11, Proposition 5.15]; see also [9]. For operators \( S \) and \( T \) the notation \( S \subset T \) means that \( T \) is an extension of \( S \).

Lemma 3.7. For \( j = 1, 2 \) assume that \( A_j \) is a self-adjoint operator on a Hilbert space \( \mathcal{H}_j \). Assume that \( B: \mathcal{H}_1 \to \mathcal{H}_2 \) is a bounded operator such that

\[
BA_1 \subset A_2 B.
\]

Let \( \Psi \) be a Borel function on \( \mathbb{R} \). Then we have

\[
B \Psi(A_1) \subset \Psi(A_2) B \quad (3.5)
\]

If \( \Psi \) is a bounded function then equality holds in (3.5).

In the following assumption the parameters are chosen to be compatible with the ones in [2, Assumption 3.1].

Assumption 3.8. Assume

\[
\alpha > \frac{1}{2}, \quad \beta > -\frac{1}{2}, \quad \text{and} \quad \alpha \leq 1 + \beta < 2\alpha \leq 3 + \beta.
\]

Let \( \Psi: [0, \infty) \to \mathbb{R} \) be a continuous function which is continuously differentiable on \( (0, \infty) \) and satisfies the following conditions:

1. \( \Psi(0) = 0 \),
2. there exist \( c_0 > 0 \) and \( c_1 > 0 \) such that \( \Psi(\lambda) \geq c_0 \lambda^{\alpha/2} \) for \( \lambda \geq c_1 \).
(3) there exists \( c > 0 \) such that \( |\Psi'(\lambda)| \leq c \lambda^{(\beta-1)/2} \) for \( \lambda > 0 \).

We omit the straightforward proof of the following lemma.

**Lemma 3.9.** Let \( \Psi \) satisfy Assumption 3.8 with parameters \( \alpha \) and \( \beta \). Define \( G_0(\xi) = \Psi(|\xi|^2) \), \( \xi \in \mathbb{R}^d \). Then \( G_0 \) satisfies Assumption 3.1 in [2] with the same parameters \( \alpha \) and \( \beta \).

Next we define

\[
G_{0,h}(\xi) = G_0\left(\frac{2}{h} \sin\left(\frac{h}{2} \xi_1\right), \frac{2}{h} \sin\left(\frac{h}{2} \xi_2\right), \ldots, \frac{2}{h} \sin\left(\frac{h}{2} \xi_d\right)\right), \quad h > 0, \quad \xi \in \mathbb{R}^d.
\]

Using these definitions it follows that \( \Psi(H_0) \) is the Fourier multiplier with symbol \( G_0 \) on \( \mathcal{H} \), and \( \Psi(H_{0,h}) \) is the Fourier multiplier with symbol \( G_{0,h} \) on \( \mathcal{H}_h \). The operators \( \Psi(H_0^D) \) and \( \Psi(H_0^N) \) on \( \mathcal{H}^+ \), and the operators \( \Psi(H_{0,h}^D) \) and \( \Psi(H_{0,h}^N) \) on \( \mathcal{H}_h^+ \), are defined using the functional calculus.

We have the following extension of Theorem 3.2.

**Theorem 3.10.** Let \( \Psi \) satisfy Assumption 3.8 with parameters \( \alpha \) and \( \beta \). Let

\[
\gamma = \min\{2\alpha - 1, 2\alpha - \beta - 1\}.
\]

Let \( J_h^\alpha \), \( J_h^\tau \), \( K_h^\alpha \), and \( K_h^\tau \) be as above, with \( \varphi_0 \) satisfying Assumption 3.1. Let \( K \subset \mathbb{C} \setminus [0, \infty) \) be compact. Then there exists \( C > 0 \) such that

\[
\|J_h^\alpha(\Psi(H_{0,h}^D) - zI_h)^{-1} - (\Psi(H_0^D) - zI)^{-1}\|_{B(\mathcal{H}^+)} \leq C h^\gamma,
\]

and

\[
\|J_h^\tau(\Psi(H_{0,h}^N) - zI_h^+)^{-1} - (\Psi(H_0^N) - zI^+)^{-1}\|_{B(\mathcal{H}^+)} \leq C h^\gamma,
\]

for \( 0 < h \leq 1 \) and \( z \in K \).

**Proof.** We prove the result for the Dirichlet Laplacians. Assumption 3.8 and Lemma 3.9 together with [2, Proposition 3.5] imply that we have the estimate

\[
\|J_h(\Psi(H_{0,h}) - zI_h)^{-1}K_h - (\Psi(H_0) - zI)^{-1}\|_{B(\mathcal{H})} \leq C h^\gamma
\]

for \( 0 < h \leq 1 \) and \( z \in K \), with \( K \) satisfying the assumption in the theorem.

Combine Lemma 2.2 with Lemma 3.7 to get the result

\[
\mathcal{O}(\Psi(H_0^D) - zI^+)^{-1} = (\Psi(H_0) - zI)^{-1}\mathcal{O}, \quad z \in K.
\]

Analogously, using [2.4] and Lemma 3.7 we get

\[
\mathcal{O}_h(\Psi(H_{0,h}^D) - zI_h^+)^{-1} = (\Psi(H_{0,h}) - zI_h)^{-1}\mathcal{O}_h, \quad z \in K.
\]

Using the results (3.8)–(3.10) we can repeat the arguments in the proof of Theorem 3.2 to get the result in the Dirichlet case. The proof in the Neumann case is almost the same, so we omit it.

**Remark 3.11.** By repeating the proof of Theorem 3.6 we may also add a potential \( V \) to the operators \( \Psi(H_0^D) \) and \( \Psi(H_0^N) \) and add a discrete potential \( V_h \) to the operators \( \Psi(H_{0,h}^D) \) and \( \Psi(H_{0,h}^N) \). The resulting estimates, replacing those in (3.6) and (3.7), will have the rate \( h^{\min\{\gamma, \theta^\prime\}} \).
Of particular interest are the functions $\Psi$ that give the powers of the Laplacian $H_0$. Let $s > 0$ and define $\Psi_s(\lambda) = \lambda^{s/2}$, $\lambda \geq 0$. Then $G_0(\xi) = |\xi|^s$ and $\Psi_s(H_0) = (-\Delta)^{s/2}$. For $s \geq 2$ we can take $\alpha = (s+2)/2$ and $\beta = s-1$ to satisfy the conditions in Assumption 3.8. Then the estimate (3.5) holds with $\gamma = 2$.

For $\frac{1}{2} < s < 2$ the conditions in Assumption 3.8 are satisfied with $\alpha = s$ and $\beta = s-1$. We get $\gamma = s$ for $1 \leq s < 2$. For $0 < s < 1$ we can use the result in [2, Proposition 3.11] which yields the estimate (3.5) for $\Psi_s(H_0)$ and $\Psi_s(H_0, h)$ with $\gamma = s$.

We summarize the results above as a Corollary to both Theorem 3.10 and the results in [2].

**Corollary 3.12.** Let $\Psi_s(\lambda) = \lambda^{s/2}$, $\lambda \geq 0$, $s > 0$. Then the estimates (3.6) and (3.7) hold for $\gamma = \min\{s, 2\}$.

**Remark 3.13.** The operators $\Psi_s(H_0^D)$ defined here do not agree with the fractional Dirichlet Laplacians on a half-space defined in [7, 8]. Let $u \in D(\Psi_s(H_0^D))$, then Lemmas 2.2 and 3.7 imply $O\Psi_s(H_0^D)u = \Psi_s(H_0)Ou$, such that $\Psi_s(H_0^D)u = R\Psi_s(H_0)E_0u$. Whereas in [7, 8] the definition is based on the operator $R\Psi_s(H_0)E_0$ applied to suitable functions in $H^+$, where $E_0$ is the operator for extension by zero. Hence the two approaches differ by the type of extension operator that is used.

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