Extension of the Generalized Hydrodynamics to Dimensional Crossover Regime

Frederik Møller\textsuperscript{1}, Chen Li\textsuperscript{1,2}, Igor Mazets\textsuperscript{1,3}, Hans-Peter Stimming\textsuperscript{3,4}, Tianwei Zhou\textsuperscript{2,5}, Zijie Zhu\textsuperscript{6}, Xuzong Chen\textsuperscript{2}, and Jörg Schmiedmayer\textsuperscript{1}

\textsuperscript{1} Vienna Center for Quantum Science and Technology (VCQ), Atominstitut, TU Wien, Vienna, Austria
\textsuperscript{2} School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, China
\textsuperscript{3} Wolfgang Pauli Institut, c/o Fakultät für Mathematik, Universität Wien, 1090 Vienna, Austria
\textsuperscript{4} Fakultät für Mathematik, Universität Wien, 1090 Vienna, Austria
\textsuperscript{5} INO-CNR Istituto Nazionale di Ottica del CNR, Sezione di Sesto Fiorentino, I-50019 Sesto Fiorentino, Italy
\textsuperscript{6} Institute for Quantum Electronics, ETH Zurich, 8093 Zurich, Switzerland

(Dated: June 26, 2020)

In an effort to address integrability-breaking in cold gas experiments, we extend the integrable hydrodynamics of the Lieb-Liniger model with a second component. The additional component represents the population of transversely excited atoms found in quasi-1d condensates. Collisions between different components are accounted for through the inclusion of a Boltzmann-type collision integral in the hydrodynamic equation. We compare predictions of our model to measurements from a quantum Newton’s cradle setup at short to intermediate time scales and observe good agreement.

Over the last decades, the advances in experimentally realizing and manipulating quantum many-body systems in low dimensions have increased the demand for theoretical methods capable of describing their complex dynamics \[1\textsuperscript{-}6\]. Arguably one of the most prominent experimental platforms for studying out-of-equilibrium phenomena is ultracold Bose gases \[7\textsuperscript{-}21\], which upon confinement to one dimension exhibits integrability. Integrable systems abide to an extended set of conservation laws, strongly constraining their dynamics and inhibiting thermalization \[22\textsuperscript{-}24\]. Within the integrable limit, the recent theory of Generalized Hydrodynamics (GHD) has established itself as a powerful and flexible framework by capturing both the transport of all the conserved charges and the Wigner delay time in elastic scattering of particles \[25\] within a single continuity equation \[27\textsuperscript{-}29\]. For earlier applications of the Wigner delay time to the hydrodynamics of one-dimensional, non-degenerate gases see Refs. \[30\textsuperscript{-}31\]. Building upon the framework of GHD, a wide array of extensions have enabled the study of correlations \[32\textsuperscript{-}35\], Drude weights \[36\textsuperscript{-}39\], diffusion constants \[40\textsuperscript{-}43\], and more.

However, real systems realized in even very controlled environments are only approximately integrable. Small experimental imperfections and processes outside the realm of GHD will break the integrability of the system and eventually over time drive it towards thermalization. Recently, the topic of thermalization has attracted a lot of attention \[44\textsuperscript{-}48\], however, a generally applicable theory appears intractable as mechanisms of integrability-breaking often depend on the physical realization of the given system \[49\textsuperscript{-}57\]. Hence, for now, thermalization must be accounted for through considerations of the experimental circumstances \[58\].

In this Letter, we seek to extend the applicability of GHD to the dimensional crossover regime, which is accessed when the collisional energy of atoms exceeds the level spacing of the transverse confinement. Thus, based on heuristic considerations, we introduce a second component to the Lieb-Liniger model, representing atoms in the second transversely excited state. The coupling between components is accounted for by introducing a Boltzmann-type collision integral to the GHD equation. We then compare the predictions of our model to experimental results from a quantum Newton’s cradle-type setup. To demonstrate the effects of the second component, we perform the same calculations using standard GHD and quantitatively compare the two approaches.

The degenerate gas of \(N\) bosonic atoms of mass \(m\) is described by the second-quantized Hamiltonian
\[
\hat{H} = \int dz \left\{ \frac{\hbar^2}{2m} \left( \nabla \hat{\Psi} \right)^2 \right\} + \frac{\hbar^2}{m} \sum_{\mathbf{q}} V_{\perp}(x, \mathbf{q}) \hat{\Psi}^\dagger \hat{\Psi} + \frac{2\pi \hbar a_s}{m} \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi}^\dagger \hat{\Psi}
\]
(1)
where \(\hat{\Psi} = \hat{\Psi}(\mathbf{r})\) is the atom annihilation operator, \(a_s\) is the \(s\)-wave scattering length, \(U(z)\) is the loose trapping potential in the longitudinal direction, \(V_{\perp}(x, \mathbf{q})\) is the tight transverse trapping potential. We assume that \(V_{\perp}(x, \mathbf{q})\) is harmonic and axially symmetric, \(\omega_{\perp}\) being its...
fundamental frequency and \( l_\perp = \sqrt{\hbar/(m\omega_\perp)} \) being the corresponding length scale.

We treat the motion of atoms in the longitudinal direction within the GHD framework, while the transverse motion is accounted for via a collisional integral. The GHD provides a coarse grained theory for the dynamics of systems close to an integrability point [27–28]. Just like the thermodynamics Bethe ansatz, the theory encodes the thermodynamic properties of a local equilibrium macrostate in a distribution of quasiparticles [59, 60]. Each quasiparticle is uniquely labelled by its rapidity, \( \theta \). Following the original paper by Lieb and Liniger [61, 62], we express rapidity in inverse length units. In the thermodynamics limit, the rapidity becomes a continuous variable, with the density of occupied rapidities in the phase \( (z, \theta) \)-space given by the time-dependent quasiparticle density, \( \rho_p(z, \theta, t) \). Similarly, one can introduce a density of holes, \( \rho_h(z, \theta, t) \), describing the density of unoccupied rapidities [63]. Together these two densities describe the density of states and obey the relation \( \rho_p(\theta) + \rho_h(\theta) = (2\pi)^{-1} + \pi^{-1} \int_{-\infty}^{\infty} d\theta' \left( c/\sqrt{c^2 + (\theta' - \theta)^2} \right) \rho_p(\theta') \), where \( c = 2a_s/l_\perp^2 \) is the interaction parameter of the Lieb–Liniger model (we assume \( a_s \ll l_\perp \)). Here we omit the co-ordinate and time arguments when appearing the same in all terms. A quasiparticle with rapidity \( \theta \) propagates at velocity \( v_{\text{eff}} \), which obeys the integral equation
\[
\rho_{\text{eff}}(\theta')(\theta') = \frac{(2\pi)^2\hbar}{m} \int_{\mathcal{R}_\pm} d\theta \left( \theta' - \theta, \rho_{\text{eff}}(\theta') \right) \left( \theta' - \theta, \rho_{\text{eff}}(\theta) \right) \rho_{\text{eff}}(\theta' - \theta, \theta), \quad \theta \neq \theta', \quad \theta' < \theta \quad \text{or} \quad \theta < \theta',
\]
where \( \mathcal{R}_+ = \{ \theta' : \theta' - \theta = \pm \sqrt{2} l_\perp \} \) and \( \mathcal{R}_- = \{ \theta' : \theta - \theta' = \pm \sqrt{2} l_\perp \} \). The integration ranges in Eq. (4) are following: \( \mathcal{R}_+ \) is the whole real axis and \( \mathcal{R}_- \) is comprised of those real values of \( \theta' \), which yield real \( \theta_\pm \) and \( \theta_\pm' \), i.e., \( \mathcal{R}_- = \{ \theta' : \theta < \theta' < \theta + 2\sqrt{2} l_\perp \} \cup \{ \theta' : \theta' > \theta + 2\sqrt{2} l_\perp \} \).

If two atoms with rapidities \( \theta \) and \( \theta' \) collide and the collision energy exceeds \( 2\hbar\omega_\perp \), their transverse states can change, as illustrated in figure 1. Two collision outcomes are equally probable: (i) one atom remains in the transverse ground state and the other one occupies the second excited state; (ii) both of the atoms are transferred to the first excited state. Parity conservation plays an important role here. First of all, transitions of only one atom to the first transversely excited state are forbidden, since this state is odd (has a negative parity), in contrast to the ground and second excited states, which are even. Likewise, the de-excitation of an atom in the transverse second excited state happens due to a collision with an atom in the ground state. This process occurs at a higher rate than the de-excitation of an atom in the first excited state, since a collision with another excited atom is a much more rare event in the regime where most of the atoms are confined to the transverse ground state. In the presence of these processes, our extended model yields
\[
\partial_z \rho_p + \partial_z (v_{\text{eff}} \rho_p) + \hbar^{-1} \partial_\theta (F_{\text{eff}} \rho_p) = \mathcal{I}(\theta).
\]
Eq. (2) differs from the conventional GHD equation by the Boltzmann-type collision integral \( \mathcal{I}(\theta) \) in the right-hand side.

We take into account only collisional transitions between the transverse ground state and the second excited state, since this process leads to fast relaxation of the rapidity distribution. Slow relaxation via double population of the first excited state is neglected in order to make the model computationally fast and efficient. This exclusion of one of the transverse excitation channels is accounted for by reducing the probability of transverse-state changing collisions by a factor \( \zeta \approx 0.5 \). Therefore, to the simplest approximation, the collision integral can be written as
\[
\mathcal{I}(\theta) = -\mathcal{I}_p(\theta) + \mathcal{I}_h(\theta) \nu - \mathcal{I}_p(\theta) \nu + \mathcal{I}_h(\theta), \quad (3)
\]
where \( \nu \) is the probability for an atom to be in the transversely excited state with the excitation energy \( 2\hbar\omega_\perp \). We assume \( \nu \ll 1 \). The terms in Eq. (3) are defined as
\[
\mathcal{I}_p(\theta) = \frac{(2\pi)^2\hbar}{m} \int_{\mathcal{R}_\pm} d\theta \left( \theta - \theta' \right) P_{\pm}(\theta, \theta') \rho_p(\theta') \rho_h(\theta),
\]
where \( \alpha = \hbar = h, \) and \( \nu = \nu(\theta, \theta') = 4\left( 4\pi^2 \theta_1^2 \theta_2^2 / [\theta_1^2 \theta_2^2 + c^2 (\theta_1 + \theta_2)^2] \right) \frac{1}{[\theta - \theta']\sqrt{1 \pm 8\left( |\theta - \theta'|/l_\perp \right)^2}}, \) and \( \nu_\pm = \frac{1}{2}(\theta + \theta') - \frac{1}{2}(\theta - \theta')\sqrt{1 \pm 8\left( |\theta - \theta'|/l_\perp \right)^2} \).

The experiment (for a detailed description see [65]) studies the dynamics of \(^{85}\text{Rb} \) Bose–Einstein condensates in a 2d lattice of independent 1d traps with a tight transverse confinement of \( \omega_\perp /2\pi = 31 \text{kHz} \) and weak longitudinal confinement of \( \omega_\parallel /2\pi = 83.3 \text{Hz} \) (oscillation period
\[ \tau = 12 \text{ ms}. \] Owing to the Gaussian profile of the trapping beams (beam-waist of \( \sigma = 145 \, \mu \text{m} \)), the longitudinal potential is slightly anharmonic. The weighted-average atom number per tube is between 80 and 120, providing a coupling strength in the intermediate regime.

The dynamics in the longitudinal direction are initiated by two Bragg pulses, imparting opposite momenta of \( \pm 2 \hbar k_{\text{Bragg}} \) to each half of the atomic cloud, with \( k_{\text{Bragg}} = 2\pi / 852 \, \text{nm} \). The momentum kick corresponds to \( \sim 40\% \) of the excitation energy \( (2\hbar \omega_{\perp}) \).

After the Bragg pulses, the atoms oscillate and collide with each other under the 1d confinements for some duration. At the end of the evolution, the atoms are released from the optical lattice and detected after time-of-flight (TOF) providing us with the momentum distribution function in the longitudinal direction. Very low heating rate, about 40 nK/s, and negligible atom loss, smaller than 5% within the concerned time scale in this Letter, are observed in experiments.

The initial state of the simulation is obtained in a manner similar to Ref. \[57, 66\]: we assume the pre-pulse quasiparticle density to be a thermal state and model the Bragg pulse sequence as shifts of the distribution along the rapidity axis. In the experiment we observe a small number of atoms leftover at low momenta after the pulse sequence. We account for those by leaving a fraction \( \eta = 0.07 \) of the quasiparticle density un-shifted. Thus, our initial state reads

\[
\rho_p^{\text{init}}(\theta) = \frac{1}{2}(1-\eta)\rho_p^{\text{th}}(\theta + 2k_{\text{Bragg}}) + \frac{1}{2}(1-\eta)\rho_p^{\text{th}}(\theta - 2k_{\text{Bragg}}) + \eta \rho_p^{\text{th}}(\theta).
\]

In the following, we study the case of \( N = 80 \) atoms per tube at a temperature of 80nK, leaving around 1.5\% of the atoms above the excitation threshold. The same analysis was also performed for a data set with \( N = 120 \) yielding similar results. We focus on the first 60 oscillation periods of the cradle (720ms), where interactions of the atoms play a significant role and previous studies have shown dephasing processes to dominate the dynamics \[55, 65\]. Thus, methods like GHD are required to properly describe this regime. At later times the gas is sufficiently deep in the non-degenerate regime, enabling the use of molecular dynamics simulations. For discussions of the numerical methods used for the simulation, see Supplemental Material and Ref. \[67\].

Figure 2 shows a side-by-side comparison of the quasiparticle density of the standard and two-component Lieb-Liniger model. While the Bragg peaks of the initial state persist throughout the evolution when propagated using the standard GHD equation, the inclusion of the collision integral enables quasiparticles to distribute across the phase space. Hence, the additional component appears to accelerate (or in this case enable) the dephasing. This is contrary to Ref. \[57\], where the anharmonic trapping potential was sufficient to induce dephasing. However, the dephased state there was distinctly different from thermal. Further, the initial population of atoms at low rapidities is depleted in the standard GHD, giving rise to ‘self-stabilization’ of the Bragg peaks. Reducing the interaction between atoms or increasing the degree of anharmonicity in our simulation yields results similar to Ref. \[57\]. Thus, we attribute our observations to a certain combination of parameters, further exemplifying the complex dynamics possible in the quantum Newton’s cradle. Nevertheless, tuning these parameters has only little influence on the final state obtained via the
The quantity \( \text{variance of the RDF (and MDF) profiles over one period.} \)

Simulations and the experiment, we plot in figure 3 the prominent high-rapidity tails.

Two-component model, although our model features more good agreement between the MDF and RDF of the not completely non-degenerate. Nevertheless, we observe this in figure 2, where the measured density of atoms, thus bringing the gas towards the non-degenerate regime. After 60 oscillation periods (720ms), the MDF approaches the RDF over time. The main driver of this transition is the dephasing, which lowers the density probability in the extended model. The shaded area indicates the noise floor, where the variance is dominated by noise in the experiment.

Two-component model, which apart from the now barely visible Bragg peaks strongly resembles a thermal state.

In the final row of figure 2, we compare the rapidity distribution (RDF) of the quasiparticles \( f(\theta) = \int d\rho_{\text{z}}(\theta, z) \) with the measured momentum distribution (MDF) of the Bose gas. It is important to note that the two quantities are inherently different; while evolving in the 1d cradle the atoms follow the rapidity distribution, however, upon measurement the entire lattice is ramped down causing the atoms to expand in three dimensions. After time-of-flight expansion in 3d, the recorded density profile of the atoms corresponds to the MDF [68]. The relation between the different distributions is not easily obtained. For short times we can estimate the MDF from the rapidity distribution. Newton’s cradle setup. Comparing the dynamics of the extended model with experimental data from a quantum harmonic trap can have up to 30% of its total energy stored in the transverse degrees of freedom. Hence, our extended model provides a useful tool for further studies of the crossover regime.

In conclusion, we have extended the theory of Generalized Hydrodynamics with a two-component Lieb-Liniger model in order to address the question of thermalization in the dimensional crossover. Our model takes into account collisions with transversely excited atoms through a Boltzmann-type collision integral. We have compared the predictions of both the standard GHD and our extended model with experimental data from a quantum Newton’s cradle setup. Comparing the dynamics of the two models reveals significant differences over time, with the standard GHD clearly deviating from experimental observations. Despite the simplicity of our model, the observed qualitative agreement with the experimental results is surprisingly good. We believe even more accurate predictions can be achieved through the inclusion of additional sources of integrability-breaking along with better knowledge of the initial state.

This work was supported by the Austrian Science Fund (FWF) via the SFB 1225 ISOQUANT (I 3010-N27). I.M. acknowledges the support by the Wiener Wissenschafts- und Technologiefonds (WWTF) via Grant No. MA16-066 (SEQUEX). X.C. acknowledges the support from the National Natural Science Foundation of China (Grant No. 11920101004, 91736208). H.P.S. acknowledges the support from the FWF via Grant SFB F65 (Complex-...
Here, $\Phi(\theta)$ describes changes in the rapidity distribution in the presence of inhomogeneous interactions, while the effective velocity is given by

$$v^\text{eff}(\theta) = \frac{\hbar \theta}{m} + \int_{-\infty}^{\infty} d\theta' \Phi(\theta, \theta') \rho_p(\theta') \left( v^\text{eff}(\theta') - v^\text{eff}(\theta) \right).$$

(7)

| \hline
| where $\Phi(\theta, \theta') = \frac{2\epsilon}{c^2 + (\theta - \theta')^2}$ is the Lieb-Liniger two-body scattering kernel. From the quasiparticle density, one can extract the expectation values of the conserved charges and their associated current, respectively, via

$$q_i = \int d\theta \ h_i(\theta) \rho(\lambda)$$

and

$$j_i = \int d\theta \ h_i(\theta) v^\text{eff}(\lambda) \rho(\lambda),$$

where $h_i(\lambda)$ being the one-particle eigenvalue of the $i$'th conserved charge.

As an alternative to the quasiparticle density, one can encode the thermodynamic properties of the system in the filling function

$$\vartheta(\theta) = \frac{\rho_p(\theta)}{\rho_p(\theta) + \rho_h(\theta)},$$

where the density of states is given by

$$\rho_p(\theta) + \rho_h(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-\infty}^{\infty} d\theta' \frac{c}{c^2 + (\theta - \theta')^2} \rho_p(\theta').$$

(11)

The quasiparticles of the Lieb-Liniger model follow Fermionic statistics. Thus, a thermal state can be calculated from

$$\vartheta(\theta) = \frac{1}{1 + e^{(\theta)/\beta}},$$

where $\beta$ is the inverse temperature and the pseudoenergy $\epsilon(\theta)$ is acquired from solving the equation

$$\epsilon(\theta) = \frac{\hbar^2 \theta^2}{2m} - \mu - \frac{1}{\pi \beta} \int_{-\infty}^{\infty} d\theta' \frac{c}{c^2 + (\theta - \theta')^2} \ln \left( 1 + e^{\epsilon(\theta')/\beta} \right).$$

(13)

The chemical potential $\mu(z) = \mu_0 - U(z)$ accounts for the external potential.

### Numerically solving the propagation equation

For the numerical GHD computations we employ the iFluid library [67]. In order to solve Eq. (2) we employ a first order split step propagation scheme.

First, we evaluate the collision integral, which requires quantities readily available from GHD. Throughout the entire calculation we maintain the same rapidity and collision grids. Consider the rapidity discretized on a grid $\theta_i$ with $i = 1, \ldots, i_{\text{max}}$. The collision grids then read

$$\theta_{\pm}[i;j] = \frac{1}{2} (\theta_i + \theta_j) + \text{sgn} (\theta_i - \theta_j) \sqrt{1 + 8/[(\theta_i - \theta_j) l_{\pm}]^2}$$

(14)

where $\theta_{\pm}'[i;j] = \theta_{\pm}[j;i]$. To obtain the particle and hole densities on the collision grids we use interpolation, which can be expressed in matrix form as

$$\rho_{p,h}(\theta_{\pm}[i;j]) = \sum_k \Xi_{\pm}(i;j,k) \rho_{p,h}(\theta_k).$$

(15)

Throughout the simulation we maintain constant rapidity and collision grids. Thus, the interpolation matrix $\Xi_{\pm}$ can be calculated beforehand, greatly reducing the computational time needed. For linear interpolation, the interpolation matrix can be constructed as follows

$$\Xi_{\pm}(i;j,k) = \frac{\theta_k - \theta_{\pm}[i;j]}{\theta_k - \theta_{\pm}[i-1;j]},$$

(16)

where $k$ is the index minimizing $\min_k |\theta_k - \theta_{\pm}[i;j]|$. This matrix structure is sparse, allowing for very fast interpolation.

Once the collision integral has been obtained, we solve the equations

$$\frac{d}{dt} \rho_p(\theta, z, t) = \mathcal{I}(\theta, z, t)$$

(18)

and

$$\frac{d}{dt} \nu(t) = \Gamma^p(t) - \Gamma^h(t) \nu(t) + \gamma$$

(19)

using the two-step AdamsBashforth method. Next, we solve the standard GHD equation [69] without collision integral using the method of characteristics. Here we employ the second order scheme detailed in Ref. [29].
Details of the simulation

The quantum Newtons cradle is realized experimentally in a red detuned optical lattice consisting of many 1D tubes. By preparing a specific number of atoms before the lattice is ramped up we can adjust the number of atoms per tube. Nevertheless, the atom number will not be the same for all tubes. Particularly between inner and outer tubes will differences occur. Upon measurement, the contribution from all tubes are averaged.

In the GHD simulation we treat only a single tube, using parameters obtained from the weighted average over all the tubes. Thus, we study the case of a tube with $N = 80$ atoms at an initial temperature $80 \text{nK}$. The longitudinal potential emerges naturally from the Gaussian intensity profile of the lattice beams

$$U(z) = \frac{m \omega_{\parallel}^2 \sigma^2}{2} \left(1 - e^{-2z^2/\sigma^2}\right),$$

where $\omega_{\parallel}/2 \pi = 83.3 \text{ Hz}$ is the longitudinal trapping frequency, and $\sigma = 145 \mu \text{m}$ is the beam-waist of the lattice beams. We note that the $\sigma$ of a given tube will be slightly larger, if the optical lattice is not perfectly aligned.

The heating process is studied by observing the evolution of a cloud held in the optical lattice without the Bragg-pulse excitation. Over time, heating effects from the trapping laser will cause the momentum peak of the cloud to expand over time. From the MDF we can compute the kinetic energy of the gas, and any increase in kinetic energy is attributed heating effects. For $N = 80$ atoms, we estimate a heating rate of $40 \text{nK/s}$. We do not take into account atom losses in the simulation.

We performed several simulations with slightly perturbed parameters and observed no significant change in the outcome.

Setting up the two-component model

To construct a numerically tractable extension of the GHD, we need to assume several simplifications and approximations. One possible path towards thermalization is through collisions with transversely excited atoms. Parity conserving collisions of atoms in the transverse ground state with sufficiently high collision energy can lead to excitation of either one atom into the second transversely excited state or two atoms into the first excited state. We will neglect this distinction and assume that the system contains only two components: atoms in the transverse ground state (denoted as the pseudospin state) and atoms in the axially symmetric transverse state with the excitation energy $2\hbar \omega_{\perp}$ (denoted by $| \uparrow \rangle$). The Bethe-ansatz solution for an integrable two-component 1D Bose gas was first proposed by Yang [83]. Eigenstates of the two-component 1D Bose gas are characterized not only by the quasiparticle rapidities, but also by rapidities $\lambda$ of pseudospin waves.

The bosonic wave function of $N$ bosons is symmetric with respect to permutations of atoms. For an eigenstate, it can be written as an irreducible tensor product of the pseudospin and coordinate parts, each of them belonging to the same irreducible representation of the symmetric group $S_N$. An irreducible representation of $S_N$ is denoted by the corresponding Young diagram. Since only two pseudospin states are present, the Young diagram can contain maximally 2 rows, i.e., has the form $\{N - M, M\}$ where $M$ is an integer from 0 to the integer part of $N/2$; Note that $M$ is not the number of atoms in the state $| \uparrow \rangle$; the latter number is larger than or equal to $M$.

In the general case, pseudospin rapidities can be complex, forming so-called Bethe strings. However, since the fraction of atoms in the $| \uparrow \rangle$ state is small, we can assume $1 \lambda = 0$. Thus, we can introduce quasiparticle (p) and hole (h) distributions $\sigma_{p,h}(\lambda)$ for the pseudospin rapidities as well. Because $M \ll N$, the contribution of the pseudospin component to the quasimomenta density of states is negligible. Therefore, we can roughly estimate

$$\sigma_p(\lambda) + \sigma_h(\lambda) \approx \rho_p|_{\theta=\lambda}.$$

Eq. 21 has a clear physical meaning: each atom can bear, additionally to its quasimomentum, a pseudospin component to the quasimomenta density of states is negligible. Therefore, we can roughly estimate

$$\nu(\theta) = \frac{\sigma_p}{\sigma_p + \sigma_h}_{\lambda=0} \approx \frac{\sigma_p}{\rho_p}_{\lambda=0}.$$

and assume in the following that $\nu \ll 1$.

Finally, we can formally account for the excitation component within the framework of GHD by introducing a Boltzmann-type collision integral to the hydrodynamic equation

$$\partial_t \rho_p + \partial_x (v_{\text{eff}} \rho_p) + \hbar^{-1} \partial_p (F_{\text{eff}} \rho_p) = \mathcal{I}(\theta).$$

In the following, we will derive an expression for the collision integral.

Collision integral

We consider atoms in a waveguide under assumption that the collision energy may exceed $2\hbar \omega_{\perp}$, but is certainly below $4\hbar \omega_{\perp}$. We extend Olshanii’s treatment [74] to collision energies high enough to excite the transverse degrees of freedom. The renormalized coupling strength is then

$$\tilde{c} = \frac{c}{1 - \frac{1}{2} c \mathcal{C}(\epsilon)},$$

where

$$\mathcal{C}(\epsilon) \approx 2 \sqrt{1 - \frac{1}{2} \epsilon} - \frac{1}{2 \sqrt{2 \sqrt{1 - \epsilon}}} - \frac{1}{2 \sqrt{2 \sqrt{1 - \epsilon}}},$$

and

$$\mathcal{C}(\epsilon) \approx 2 \sqrt{1 - \frac{1}{2} \epsilon} - \frac{1}{2 \sqrt{2 \sqrt{1 - \epsilon}}} - \frac{1}{2 \sqrt{2 \sqrt{1 - \epsilon}}}.$$
\[ \epsilon = \frac{1}{2}(k_1 - k_2)^2 l_\perp^2, \quad \text{and} \quad \hbar k_1, \hbar k_2 \text{ are the momenta of colliding bosonic atoms. Eq. (25) is derived using the simplest approximation to the sum} \]
\[ \sum_{n=2}^\infty \frac{\exp(-\sqrt{2\sqrt{n-\epsilon}}|z|/l_\perp)}{\sqrt{2\sqrt{n-\epsilon}}} \approx \int_2^\infty \frac{\exp(-\sqrt{2\sqrt{n-\epsilon}}|z|/l_\perp)}{\sqrt{2\sqrt{n-\epsilon}}} + \frac{1}{2} \exp(-\sqrt{2\sqrt{2-n|z|/l_\perp}}) \text{ according to the Euler-Maclaurin formula. This approximation is quite good, since Eq. (25) yields \( C(0) \approx 1.04 \), while the exact result is \( C(0) = 1.06 \ldots \).} \]

For \( c_\perp \ll 1 \) the real part of \( c \) is close to \( c \) for almost all collision energies, except of a narrow interval near the excitation threshold \( \epsilon = 1 \). If \( \epsilon > 1 \), the imaginary part of \( c \) is non-zero, which corresponds to the probability of excitation of the transverse degrees of freedom

\[ \mathcal{P}(k, q) = \frac{4\epsilon^2 k q}{k^2 q^2 + c^2(k + q)^2}, \quad (26) \]

We will use the dimensional coupling constant \( c \) and the excitation probability \( \mathcal{P} \) as basic building blocks for our extended GHD. The effective three-body elastic scattering of atoms via virtual excitation of the transverse degrees of freedom will be fully neglected in our theory.

A collision that leads to excitation of transverse degrees of freedom with the energy transfer \( 2\hbar \omega_\perp \) has two equally probable outcomes: (i) one atom remains in the transverse ground state and the other atom occupies the second excited state, (ii) both the atoms occupy the first excited state. In the former case, the excited atom can be de-excited by a collision with an atom in the ground state. In the latter case, de-excitation requires a collision of two transversely excited atoms, which is much less probable, since, by assumption, the population of transversely excited states is small. Therefore, the channel (ii) leads to much slower relaxation (change of the rapidity distribution) than the channel (i). To simplify our equations and to make their numerical implementation more efficient, we neglect the channel (ii) and, respectively, reduce the probability of the transverse-state changing collision by a factor \( \zeta \approx 0.5 \).

\[ P_\perp(k, q) = \zeta \mathcal{P}(k, q). \quad (28) \]

We also introduce the quasimomenta of the atoms after a collisional excitation or de-excitation of the transverse state as \( \theta_\perp = \frac{1}{2}(\theta + \theta') + \frac{1}{2}(\theta - \theta') \sqrt{1 + 8/[(\theta - \theta') l_\perp]^2} \), and \( \theta'_\perp = \frac{1}{2}(\theta + \theta') - \frac{1}{2}(\theta - \theta') \sqrt{1 + 8/[(\theta - \theta') l_\perp]^2} \). The microscopic collision velocity is \( \hbar \theta - \theta'|/m \). Knowing the scattering probability \( P_\perp \), we can write the Boltzmann-type collision integral as

\[ I(\theta) = (2\pi)^2 \frac{\hbar}{m} \int_{-\infty}^{\infty} d\theta' P_\perp(|\theta - \theta'|, |\theta_\perp - \theta'_\perp|) |\theta - \theta'| \Theta(|\theta - \theta'| l_\perp - 2\sqrt{2}) \left\{ -\rho_p(\theta') \rho_p(\theta') \rho_h(\theta_\perp) \rho_h(\theta'_\perp) + \frac{1}{2} \rho_h(\theta') \rho_p(\theta') \rho_p(\theta_\perp) \rho_h(\theta'_\perp) [\nu(\theta_\perp) + \nu(\theta'_\perp)] \right\} + \]

\[ (2\pi)^2 \frac{\hbar}{m} \int_{-\infty}^{\infty} d\theta' P_\perp(|\theta - \theta'|, |\theta_\perp - \theta'_\perp|) |\theta - \theta'| \left\{ -\frac{1}{2} \rho_p(\theta') \rho_p(\theta') \rho_h(\theta_\perp) \rho_h(\theta'_\perp) [\nu(\theta) + \nu(\theta')] + \rho_h(\theta') \rho_p(\theta_\perp) \rho_p(\theta'_\perp) \right\}, \]

where \( \Theta(x) \) is the Heaviside step function. For \( \nu \) that does not depend on \( \theta \) we obtain

\[ I(\theta) = (2\pi)^2 \frac{\hbar}{m} \int_{-\infty}^{\infty} d\theta' P_\perp(|\theta - \theta'|, |\theta_\perp - \theta'_\perp|) |\theta - \theta'| \Theta(|\theta - \theta'| l_\perp - 2\sqrt{2}) \left\{ -\rho_p(\theta') \rho_p(\theta') \rho_h(\theta_\perp) \rho_h(\theta'_\perp) + \rho_h(\theta') \rho_p(\theta') \rho_p(\theta_\perp) \rho_h(\theta'_\perp) \right\} + \]

\[ (2\pi)^2 \frac{\hbar}{m} \int_{-\infty}^{\infty} d\theta' P_\perp(|\theta - \theta'|, |\theta_\perp - \theta'_\perp|) |\theta - \theta'| \left\{ -\rho_p(\theta') \rho_p(\theta') \rho_h(\theta_\perp) \rho_h(\theta'_\perp) \nu + \rho_h(\theta') \rho_p(\theta'_\perp) \rho_p(\theta_\perp) \right\}. \]

The key idea behind the expression for the collision integral is that in the quantum degenerate regime the scattering is affected by the Pauli blocking: scattered atoms can acquire only those values of quasimomentum, which
were not occupied before the collision. Therefore, the collision integrals must contain not only particle distribution functions, but also hole distribution functions. Here the fermionic nature of particles and holes in the Lieb-Liniger model is manifested. Note, that $\rho_h(\theta_{ \pm})$ is the Pauli blocking factor (1 minus the population) times the density of states for the scattering products. Factor $(2\pi)^2$ arises from the normalization. One factor $2\pi$ arises from $\int dt \exp[-i(E_i-E_f)t/\hbar] = 2\pi\delta(E_i-E_f)$, where $E_i$, $E_f$ are the energies of the initial and final states, respectively. Another factor $2\pi$ appears when we switch from summation over discrete rapidities defined by the periodic boundary conditions over the length $L$ to the integration over continuous $\theta_{ \pm}$: the Kronecker delta-symbol for discretized total momentum, $\delta_p, p_j = \sin((\theta_{ \pm} + \theta_{ \pm} - \theta_j)L/2)$, where $sinc x = \sin x/x$, transforms to a $2\pi\delta(\theta_{ \pm} + \theta_{ \pm} - \theta_j)/L$, when we replace the discrete sum by $\int d\theta \ldots$, recall the normalization $L \int d\rho_p(\theta) = N$.

The kinetics of pseudospin waves is more complicated. However, we can expect that the pseudospin transfer occurs on time scales much shorter than excitation of transverse modes. Therefore, we can assume that $\nu$ does not depend on $\theta$. As a further simplification, we assume that $\nu$ is also spatially uniform and depends on $t$ only. In this approximation,

$$
\frac{d \nu(t)}{dt} = \frac{(2\pi)^2}{2mN} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\theta' P_{\pm}(|\theta - \theta'|, |\theta_+ - \theta'_+|)\theta - \theta' \rho_p(\theta_+)\rho_p(\theta) - \frac{(2\pi)^2}{2mN} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\theta' P_{\pm}(|\theta - \theta'|, |\theta_+ - \theta'_+|)\theta - \theta' \rho_h(\theta) \rho_p(\theta'\nu(t)).
$$

The collision integral $I(\theta)$ is identically zero when rapidities obey the Fermi–Dirac distribution and the classical (Boltzmann) statistics holds for transverse excitations (recall that $\nu \ll 1$ by assumption). The temperature-dependent collective correction to the quasiparticle energy that appears in the thermodynamic Bethe ansatz is dependent collective correction to the quasiparticle energy distribution functions, but also hole distribution functions. The collision integrals take the limit (Boltzmann) limit

$$
I_{cl}(\theta) = \frac{\hbar m}{\pi} \int_{-\infty}^{\infty} d\theta' P_{\pm}(|\theta - \theta'|, |\theta_+ - \theta'_+|)\theta - \theta' \times 
\Theta(\theta - \theta')|\theta_+ - \theta'_+| \times 
\left[ - \rho_p(\theta)\rho_p(\theta') + \rho_p(\theta_-)\rho_p(\theta'_+)\right] + 
\frac{\hbar m}{\pi} \int_{-\infty}^{\infty} d\theta' P_{\pm}(|\theta - \theta'|, |\theta_+ - \theta'_+|)\theta - \theta' \times 
\left[ - \rho_p(\theta)\rho_p(\theta')\nu + \rho_p(\theta_-)\rho_p(\theta'_+)\right].
$$

Estimation of the bosonic MDF

The calculation of the bosonic MDF in Lieb-Liniger model is not accessible with any general analytic method, and has only been done in numerical works [57, 68, 75–80]. Within the scope of this Letter, a rough estimation of MDF is sufficient to help us on demonstrating the applicability of our model and comparing the results with experimental measurements.

Supposing that we have a density distribution of quasiparticles $\rho_p(z, \theta)$ obtained from GHD. We regard it as a target distribution and fit it with a sum of three thermal distributions $\rho(z, \theta) = \sum \rho_j(z, \theta)$, which are calculated by solving the Bethe-ansatz equations in the identical confinement. One of the three distributions ($\rho_1$) is centered at the origin of phase space, and the other two ($\rho_2$ and $\rho_3$) are shifted by the mean rapidity boosts $\langle \theta_j \rangle$. For each of these distributions which occur to be quite close to boosted thermal ones, we find chemical potentials $\mu_j$ and temperatures $T_j$, so that we can estimate the MDFs $w_j(k)$ with the following equations.

In the degenerate limit, the MDF for Luttinger liquid, which is the Fourier transform of the correlation function, is expressed via Euler beta-function [81]

$$
w(k) = \frac{C_0 2\pi \sqrt{\pi}}{2\pi k_T} \text{Re}[B\left(\frac{ik}{2\pi k_T} + 1 - \frac{1}{2K}\right)],
$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, $C_0 \sim 1$, $c_s$ is the speed of sound, $k_T = k_B T/(\hbar c_s)$ and $K = \pi nUnd/(mc_s)$ is the Luttinger liquid parameter. This profile includes a Lorentzian-shape peak and a pedestal decrease $\sim k^{-1}$ for $k \gg k_T$. For much larger momenta, the MDF is determined by Tan’s contact. There are known approaches to a precise calculation of the value of Tan’s contact using, see, e.g., Ref. [82]. However, due to experimental uncertainties, we use as a reasonable approximation the following modification of the MDF:

$$
\tilde{w}(k) = \frac{\sqrt{1 + \frac{1}{4}(k\xi_h)^2[1 + \frac{1}{2}(k\xi_h)^2 + k\xi_h\sqrt{1 + \frac{1}{4}(k\xi_h)^2}].}}{
\sqrt{1 + \frac{1}{4}(k\xi_h)^2[1 + \frac{1}{2}(k\xi_h)^2 + k\xi_h\sqrt{1 + \frac{1}{4}(k\xi_h)^2}].}}
$$

(30)
FIG. 4: Estimation of the mean momentum distribution function (MDF) over the first oscillation period of the cradle with \( N = 120 \) atoms. The estimation is obtained from the rapidity distribution (RDF) and is compared with the experimentally measured profile. On the x-axis \( \theta \) is the rapidity for the RDF, while we let \( \theta = k \) for the MDF in order to compare the two.

When \( k \) strongly exceeds the inverse healing length \( \xi^{-1}_h \), \( \tilde{w}(k) \propto k^{-4} \).

In the non-degenerate limit, the MDF coincides with the quasimomentum distribution and approaches the Maxwell-Boltzman distribution at temperature \( T \).

On the basis of our experimental condition, the best-fit thermal distribution \( \rho_1 \) is close to the non-degenerate limit, so we take \( w_1(k) = \int dz \rho_1(z, k) \). While in the early stage of evolution, \( \rho_2 \) and \( \rho_3 \) is deep in the degenerate regime, and the MDFs \( w_2(k) \) and \( w_3(k) \) are written following Eq. (29), (30). Since the mean quasimomentum equals to the mean momentum, these peaks are shifted to be centered at \( k_j = \langle \theta_j \rangle \). The three MDFs are normalized to the respective particle numbers \( N_j \), subject to the restriction \( \sum N_j = N \). And the MDF \( w(k) = \sum w_j(k) \).

In practice, discerning which atoms correspond to which peak can be difficult, especially at the onset of dephasing. Thus, we can only use the MDF estimation to check whether our initial state matches experimental observations. Additionally, we can estimate the period-mean profile by fitting the radial rapidity distribution in the \((x, \theta)\) phase-space. In theory this enables direct comparison between GHD and experiment, however, measurements were performed at time intervals of 1ms, yielding 12 different profiles per period. This results in a high degree of symmetry, effectively only probing 4 different positions of the peaks (before dephasing). Thus, at short time-scales where the MDF estimation is valid, the measured mean MDF does not reflect the true mean. However, for \( N = 120 \) atoms we do have measurements of the very first period taken with a much finer time resolu-
jan, Physical Review A 78, 053602 (2008), ISSN 1050-2947, URL https://link.aps.org/doi/10.1103/PhysRevA.78.053602

[79] W. Xu and M. Rigol, Phys. Rev. A 92, 063623 (2015), URL https://link.aps.org/doi/10.1103/PhysRevA.92.063623

[80] A. S. Campbell, D. M. Gangardt, and K. V. Kheruntsyan, Physical Review Letters 114, 125302 (2015), ISSN 0031-9007, URL https://link.aps.org/doi/10.1103/PhysRevLett.114.125302

[81] M. A. Cazalilla, Journal of Physics B: Atomic, Molecular and Optical Physics 37, S1 (2004), ISSN 0953-4075, 0307033, URL http://stacks.iop.org/0953-4075/37/i=7/a=051?key=crossref.a9061e0946864bb3c5d42734e161f51fhttps://iopscience.iop.org/article/10.1088/0953-4075/37/7/051

[82] H. Yao, D. Clément, A. Minguzzi, P. Vignolo, and L. Sanchez-Palencia, Phys. Rev. Lett. 121, 220402 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.220402

[83] Note, that the cited paper deals mainly with 1D Fermi systems; the result for bosons is only briefly presented.