On the snappability and singularity-distance of frameworks with bars and triangular plates

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Abstract. In a recent article the author presented a method to measure the snapping capability – shortly called snappability – of bar-joint frameworks based on the total elastic strain energy by computing the deformation of all bars using Hooke’s law and the definition of Cauchy/Engineering strain. Within the paper at hand, we extend this approach to frameworks composed of bars and triangular plates by using the physical concept of Green-Lagrange strain. An intrinsic pseudometric based on the resulting total elastic strain energy density cannot only be used for evaluating the snappability but also for measuring the distance to the closest singular configuration. The presented methods are demonstrated on the basis of the 3-legged planar parallel manipulator.

Keywords: Snapping framework, singularity distance, elastic deformation

1 Introduction

A framework in the Euclidean space $\mathbb{E}^n$ consists of a knot set $\mathcal{K} = \{K_1, K_2, \ldots, K_s\}$ and an abstract graph $G$ on $\mathcal{K}$ fixing the combinatorial structure. We denote the edge connecting $K_i$ to $K_j$ by $e_{ij}$ with $i < j$ and collect all indices of knots edge-connected to $K_i$ in the knot neighborhood $N_i$. Moreover we denote the number of edges in the graph by $b$ and fix the intrinsic metric of the framework by assigning a length $L_{ij} \in \mathbb{R}_{>0}$ to each edge $e_{ij}$. In general this assignment does not determine the shape of the framework uniquely thus a framework has different incongruent realizations. For example, a triangular framework has in general two realizations in $\mathbb{E}^2$, which are not congruent with respect to the group of direct isometries. If we consider the isometry group then this number halves. We denote a framework’s realization by $G(k)$ where the configuration of knots $k := (k_1, \ldots, k_s)$ is composed of the $n$-dimensional coordinate vectors $k_i$ of the knots $K_i (i = 1, \ldots, s)$.

In general we materialize edges $e_{ij}$ by straight bars, but if three edges $e_{ij}, e_{ik}$ and $e_{jk}$ form a triangle structure then the three bars can alternatively be replaced by a triangular plate$^1$. The elements of the framework are linked in the planar case ($n = 2$) by rotational joints and in the spatial case ($n = 3$) either by spherical joints or hinges. We assume that

$^1$A $r$-plate is a compact connected set in $\mathbb{E}^n$ whose affine span is $r$-dimensional according to $\mathbb{S}$. Triangular plates refer to 2-plates of triangular shape.
(I) all bars and triangular plates are uniform made of the same homogeneous isotropic material deforming at constant volume,
(II) all bars have the same cross-sectional area $A$,
(III) all joints are without clearance.

A realization is called a snapping realization if it is close enough to another incongruent realization such that the physical model can snap into this neighboring realization due to non-destructive elastic deformations of material. Shakiness can be seen as the limiting case where two realizations of a framework coincide \cite{12,16}.

In this article we define an intrinsic pseudometric based on the total elastic strain energy density of the framework (Sec. 3) using the physical model of Green-Lagrange (GL) strain (Sec. 2). This metric is then employed to measure (a) the snapping capability (shortly called snappability) of a realization (Sec. 3.1) and (b) the distance to the next shaky (also referred to as singular or infinitesimal flexible) realization (Sec. 3.2).

One can apply the proposed approach to almost all known examples of snapping spatial frameworks, as for example the Siamese dipyramids \cite{4,5}, the four-horn \cite{18} or Wunderlich’s snapping octahedra, icosaedrer and dodekaeder, which are reviewed in \cite{15}. But the presented method is not limited to the listed triangular plate-hinge structures – also known as panel-hinge frameworks\footnote{A body-hinge framework with the property that all hinges of each body are coplanar \cite{6}.} – as it can also handle structures including bars, as for example the 3-legged planar parallel manipulator or an spatial hexapod of octahedral structure, which are both of practical importance. Especially for these mechanical devices also the proposed singularity-distance is of interest (e.g. for path planning), which can be seen as an alternative to the extrinsic metrics presented by the author in \cite{8}. Therefore we demonstrate our methods on the basis of a 3-legged planar parallel manipulator (cf. Ex. 1).

2 Elastic GL strain energy of bars and triangular plates

In \cite{9} the elastic strain energy stored in a deformed bar $e_{ij}$ was computed by

$$U_{ij} = \frac{EA}{2L_{ij}} (L'_{ij} - L_{ij})^2 \quad (1)$$

where $E$ denotes the modulus of elasticity\footnote{In this paper we assume $E > 0$ as for conventional structural material $E$ is positive.}, $L'_{ij}$ is the deformed length of the bar and $L_{ij}$ its original one. The formula (1) is based on the so-called Cauchy/Engineering (CE) strain, which can also be extended to triangular elements playing a central role in the plane stress analysis within the finite element method (e.g. see \cite{7} Chapter 6). But the resulting elastic strain energy of a triangular plate is not invariant under rotations; i.e. a pure rotation already implies a deformation energy. Therefore this formulation is not suited for kinematic considerations. As a consequence we follow a more sophisticated approach; namely the GL strain (e.g. see \cite{10} Sec. 2.4.2)), which is summarized in the remainder of this section.

\footnote{For a detailed review please see \cite{9} and the references therein.}

\footnote{The shear stress and normal stress perpendicular to the plane of the triangle is zero.}
Let \( K_i, K_j, K_k \) denote the vertices of the triangular plate in the given undeformed configuration and \( K'_i, K'_j, K'_k \) in the deformed one. Then there exists a uniquely defined \( 2 \times 2 \) matrix \( A \) which has the property

\[
A(\hat{k}_j - \hat{k}_i) = \hat{k}'_j - \hat{k}'_i, \quad A(\hat{k}_k - \hat{k}_i) = \hat{k}'_k - \hat{k}'_i.
\]

where \( \hat{k}_z \) (resp. \( \hat{k}'_z \)) is a 2-dimensional vector of \( K_z \) (resp. \( K'_z \)) for \( z \in \{i, j, k\} \) with respect to a planar Cartesian frame \( F \) (resp. \( F' \)) attached to the carrier plane of the triangle \( K_i, K_j, K_k \) (resp. \( K'_i, K'_j, K'_k \)). Then the GL normal strains \( \varepsilon_x \) and \( \varepsilon_y \), respectively, and the GL shear strain \( \gamma_{xy} \) can be computed as

\[
\begin{pmatrix}
\varepsilon_x & \gamma_{xy} \\
\gamma_{xy} & \varepsilon_y
\end{pmatrix} = \frac{1}{2} (A^T A - I).
\]

We reassemble these quantities in the vector \( \mathbf{e} = (\varepsilon_x, \varepsilon_y, \gamma_{xy})^T \). Using this notation the elastic GL strain energy of the deformation can be calculated as

\[
U_{ijk} = V_{ijk} \frac{1}{2} \mathbf{e}^T \mathbf{D} \mathbf{e}
\]

where \( V_{ijk} \) denotes the volume of the triangular plate and \( \mathbf{D} \) the planar stress/strain matrix (constitutive matrix), which reads as:

\[
\mathbf{D} = \frac{E}{1 - \nu^2} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{pmatrix}.
\]

We can set the Poisson’s ratio \( \nu \) to one-half due to the assumed invariance of the volume \( V_{ijk} \) under deformation (cf. assumption I), which is used later on for the computation of the total elastic GL strain energy density. Moreover we set \( E = 1 \) as done in [9] in order to reduce the physical formulation to its geometric core.

Following the same approach the elastic GL strain energy of a deformed bar can be computed as

\[
U_{ij} = \frac{A}{6E_{ij}} (L'_{ij} - L_{ij})^2 (L'_{ij} + L_{ij})^2.
\]

### 3 The framework’s total elastic GL strain energy and its density

The total elastic GL strain energy \( U \) of a framework composed of bars and triangular plates results from the summation of the plate energies (4) and the bar energies (6).

Recall that we can model a triangular structure either as bar-joint framework or as triangular plate. In order to ensure a fair comparability of both approaches, the used amount of material has to be the same; i.e. \( V_{ijk} = A(L_{ij} + L_{ik} + L_{jk}) \). Taking this relation into account the following lemma holds:

**Lemma 1.** The total elastic GL strain energy \( U \) of a framework composed of bars and triangular plates is a rational polynomial function with respect to the intrinsic metric of the framework. The polynomial in the denominator only depends on the undeformed edge lengths \( L_{ij} \). The polynomial in the numerator also includes the deformed edge lengths \( L'_{ij} \) and it is of degree 4 with respect to these variables \( L'_{ij} \) which only appear with even powers.
Proof. We choose the planar Cartesian frame $\mathcal{F}$ in a way that its origin equals $K_i$ and that $K_j$ is located on its positive x-axis; i.e. $\mathbf{k}_i = (0, 0)^T$, $\mathbf{k}_j = (L_{ij}, 0)^T$ and

$$\mathbf{\tilde{k}}_k = \left(\frac{L_{ik}^2 + L_{jk}^2 - L_{ijk}^2}{2L_{ij}}, \pm \sqrt{\frac{(L_{ij} + L_{ik} + L_{jk})(L_{ij} - L_{ik} - L_{jk})(L_{ij} + L_{ik} - L_{jk})(L_{ik} + L_{jk} - L_{ij})}{2L_{ij}}}\right)^T$$

(7)

where the y-coordinate can have positive or negative sign for the dimension $n = 2$ depending on the orientation of the triangle $K_i, K_j, K_k$. For the dimension $n > 2$ one can always assume a positive sign. Similar considerations can be done for the planar Cartesian frame $\mathcal{F}'$ with respect to the triangle $K'_i, K'_j, K'_k$ where one ends up with exactly the same coordinatisation as above but only primed. Inserting these coordinates of the six vectors $\mathbf{\tilde{k}}_i, \mathbf{\tilde{k}}_j, \mathbf{\tilde{k}}_k, \mathbf{\tilde{k}}'_i, \mathbf{\tilde{k}}'_j, \mathbf{\tilde{k}}'_k$ into Eq. (4) shows the result for plates. For bars this result is directly visible from Eq. (6), which concludes the proof. $\square$

As in our case the undeformed lengths $L_{ij}$ are given we can interpret $U$ as a function of the bar lengths $L' = (\ldots, L'_{ij}, \ldots)^T \in \mathbb{R}^b$ of a realization $G(k')$; i.e. $U(L')$. Note that due to Lemma 1 the formula for $U(L')$ can be written in matrix formulation as $U(L') = L'^T ML'$ where $M$ is a symmetric $(b+1)$-matrix and $L' := (1, \ldots, L'_{ij}, \ldots)^T$ is composed of the $b$ squared edge lengths and the number 1.

From the underlying physical interpretation it seems to be clear that the elastic strain energy $U_{ij}$ given in Eq. (4) is positive semi-definite; but one can also prove this mathematically by decomposing it into a sum of squares (see e.g. [11]). For $U_{ij}$ given in Eq. (6) it can immediately be seen that it is positive semi-definite. Due to the resulting positive semi-definiteness of $U(L')$ its density $D(L') := U(L')/(AL)$, where $L$ is the total length $L = \sum_{i<j} L_{ij}$ of the framework, can be used for building up the following intrinsic pseudometric $d_p$ for framework realizations:

$$d_p : \mathbb{R}^b \times \mathbb{R}^b \rightarrow \mathbb{R}_{\geq 0} \quad \text{with} \quad (L', L'') \rightarrow |D(L') - D(L'')|$$

(8)

where $L'' = (\ldots, L''_{ij}, \ldots)^T \in \mathbb{R}^b$ collects the bar lengths of another realization $G(k'')$.

3.1 Snappability

Theorem 1. The critical points of the total elastic GL strain energy $U(k')$ correspond to realizations $G(k')$ that are either undeformed or deformed and shaky.

Proof. The proof is based on the following characterization of shakiness in terms of self-stress (e.g. [2]): If one can assign to each edge $e_{ij}$ of $G(k')$ a stress $\omega_{ij} \in \mathbb{R}$ in a way that for each knot the so-called equilibrium condition

$$\sum_{i<j \in N_i} \omega_{ij}(k'_i - k'_j) + \sum_{i>j \in N_i} \omega_{ji}(k'_i - k'_j) = 0$$

(9)

is fulfilled, where $\omega$ denotes the $n$-dimensional zero vector, then the $b$-dimensional vector $\omega = (\ldots, \omega_{ij}, \ldots)^T$ is referred as self-stress. If $\omega$ differs from the zero vector, then the framework realization $G(k')$ is shaky (e.g. [3][12]).

$^6$The obtained expression is independent of the sign of the y-coordinate of $\mathbf{\tilde{k}}_k$ and $\mathbf{\tilde{k}}'_k$. 


Now we consider $U$ in dependence of the configuration of knots $k'$, i.e. $U(k')$ and compute the system of equations characterizing its critical points as

$$
\nabla_i U(k') = 0 \quad \text{with} \quad \nabla_i U(k') = \left( \frac{\partial U}{\partial k'_{i_1}}, \ldots, \frac{\partial U}{\partial k'_{i_n}} \right) \quad \text{and} \quad i = 1, \ldots, s \quad (10)
$$

where $(k'_{i_1}, \ldots, k'_{i_n})$ is the coordinate vector of $k'$. Due to the sum rule for derivatives we only have to investigate $\nabla_i$ of $U_{ijk}(k')$ and $U_{ij}(k')$ given in Eqs. (4) and (6).

1. Due to $\nabla_i U_{ij}(k') = \frac{2(U_{ij} - L_{ij})}{3L_{ij}} (k'_i - k'_j)$ Theorem 1 is valid for frameworks, which only consist of bars, as $\nabla_i U(k')$ equals Eq. (9) with $\omega_j = 2(U_{ij} - L_{ij})/(3L_{ij})$.

2. If triangular plates are involved we consider $\nabla_i U_{ijk}(k')$, $\nabla_j U_{ijk}(k')$ and $\nabla_k U_{ijk}(k')$.

Straightforward symbolic computations (e.g. using Maple) show that the following overdetermined system of equations

$$
\begin{align*}
\omega_{ij}(k'_i - k'_j) + \omega_{ik}(k'_i - k'_k) - \nabla_i U_{ijk}(k') &= 0 \\
\omega_{ij}(k'_i - k'_j) + \omega_{jk}(k'_j - k'_k) - \nabla_j U_{ijk}(k') &= 0 \\
\omega_{ik}(k'_i - k'_k) + \omega_{jk}(k'_k - k'_j) - \nabla_k U_{ijk}(k') &= 0
\end{align*}
\quad (11)
$$

has a unique solution for $\omega_{ij}$, $\omega_{ik}$ and $\omega_{jk}$ if $K_i', K_j', K_k'$ generate a triangle. If these points are collinear we get a positive dimensional solution set. Hence, one can replace $\nabla_i U_{ijk}(k')$ by a linear combination $\omega_{ij}(k'_i - k'_j) + \omega_{ik}(k'_i - k'_k)$ where the coefficients $\omega_{ij}$ and $\omega_{ik}$ are compatible with the other equations of (11). As a consequence $\nabla_i U(k')$ can again be written in the form of Eq. (9).

This result implies that the Theorems 1 and 2 of [9] also hold true for frameworks with bars and triangular plates. They can be summed up as follows:

**Theorem 2.** If a framework snaps out of a stable realization $G(k)$ by applying the minimum GL strain energy needed to it, then the corresponding deformation of the realization has to pass a shaky realization $G(k')$ at the maximum state of deformation. Such a snap of a framework ends up in a realization $G(k'')$ which is either undeformed or deformed and shaky.

Therefore the snappability $s(k)$ of an undeformed realization $G(k)$ can be measured by $d_p(L', L) = D(L')$ of Eq. (8). In the following we present the procedure for determining $G(k')$, which is similar to the one given in [9]. As preparatory work for this algorithm we define the quotient set $\mathcal{R} := \mathcal{S}/SE(n)$ where $SE(n)$ denotes the group of direct isometries of $\mathbb{R}^n$ and $\mathcal{S}$ the set of real saddle points of $U(k')$, which can be selected from the critical points via the second derivative test.

Let us assume that $G(k') \in \mathcal{R}$ yields the minimal value for $d_p(L', L) = D(L')$ where $G(k)$ is the given undeformed realization. The following equation

$$
\bar{L} := \bar{L} + t(\bar{L}' - \bar{L}) \quad \text{with} \quad t \in [0, 1]
\quad (12)
$$

A realization $G(k)$ is called stable if it corresponds to the local minimum of $U(k)$.

But improved due to the final remark (a) given later on.
implies a path \( L_t \) in \( \mathbb{R}^b \) between \( L \) and \( L' \). Along this path the deformation energy \( U_{ijk} \) of each triangular plate and the deformation energy \( U_{ij} \) of each bar is monotonic increasing with respect to the curve parameter \( t \), which ensures that the minimum mechanical work needed is applied on the framework to reach \( G(k') \). This results from Lemma \[1\] as \( U_{ijk}(L_t) \) as well as \( U_{ij}(L_t) \) are quadratic functions in \( t \), which are at their minima for \( t = 0 \). The path \( L_t \) corresponds to different 1-parametric deformations of realizations in \( \mathbb{R}^n \). If among these a deformation \( G(k_t) \) with the property

\[
G(k)|_{t=0} = G(k), \quad G(k)|_{t=1} = G(k')
\]

exists, then the given realization \( G(k) \) is deformed into \( G(k') \) under \( L_t \). Computationally the property \[13\] can be checked by a user defined homotopy approach relying on the software Bertini \[1, \text{Sec. 2.3}\]. If such a deformation does not exist then we redefine \( \mathcal{R} \) as \( \mathcal{R} \setminus \{ G(k') \} \) and run again the procedure explained in this paragraph until we get the sought-after realization \( G(k') \) implying \( s(k) \) by \( U(k') \). If we end up with \( \mathcal{R} = \emptyset \) then we set \( s(k) = \infty \).

### 3.2 Singularity-distance

One can rewrite the \( s \) equations given in \[9\] in matrix form as \( R_{G(k')} \mathbf{0} = \mathbf{0} \), where the \((sn \times b)\)-matrix \( R_{G(k')} \) is the so-called rigidity matrix. It is well known (e.g. \[13\]) that a realization \( G(k') \) is shaky if and only if \( rk(R_{G(k')}) < r \) with \( r := sn - (n^2 + n)/2 \). Clearly, based on this rank condition one can characterize all shaky realizations \( G(k') \) algebraically by the variety \( V(J) \) where \( J \) denotes the ideal generated by all minors of \( R_{G(k')} \) of order \( r \times r \). Let us assume that the polynomials \( f_1, \ldots, f_s \) form the Gröbner basis of the ideal \( J \).

In the following we want to determine the real point \( k' \) of this shakiness variety \( V(J) \subset \mathbb{C}^n \) which minimizes the value \( d_p(L', L) = D(L') \), where \( G(k) \) denotes the given undeformed realization. Moreover there should again exist a 1-parametric deformation \( L_t \) implied by Eq. \[12\] such that the properties of Eq. \[13\] hold. If this is the case then we call \( d_p(L', L) = D(L') \) the singularity-distance \( \sigma(k) \).

**Theorem 3.** For an undeformed realization \( G(k) \), which is not shaky, the singularity-distance \( \sigma(k) \) equals the snappability \( s(k) \).

**Proof.** \( \sigma(k) \leq s(k) \) has to hold, as the realization \( G(k') \) of Theorem \[2\] which implies the snappability \( s(k) \), is shaky. We show the equality indirectly by assuming \( \sigma(k) < s(k) \). We denote the shaky realization implying \( \sigma(k) \) by \( G(k'') \) which corresponds to \( L'' \in \mathbb{R}^b \). In analogy to Eq. \[12\] we consider the relation

\[
\tilde{L}_t := \tilde{L} + t(L'' - \tilde{L}) \quad \text{with} \quad t \in [0, 1]
\]

defining a path \( L_t \) in \( \mathbb{R}^b \) between \( L \) and \( L'' \), which corresponds to a set of 1-parametric deformations \( \{ G(k'), G(k_2'), \ldots \} \). A subset \( \mathcal{D} \) of this set has the property \( G(k')|_{t=1} = G(k'') \) where \( \#\mathcal{D} > 1 \) holds as \( G(k'') \) is shaky \[17,15\]. Therefore the framework can snap out of \( G(k) \) over \( G(k'') \) which contradicts \( \sigma(k) < s(k) \) \( \iff \sigma(k) = s(k) \). \( \square \)

**Remarks.** Our theoretical considerations end with the following four final remarks:

1. **Remarks:**
(a) Due to Theorem 3, one can narrow the set \( \mathcal{R} \) down to the set \( \mathcal{D} \), which improves the algorithm for the computation of \( s(\mathbf{k}) \) considerably (cf. Ex. 1). This can be done as follows: As \( \sigma(\mathbf{k}) = s(\mathbf{k}) \) has to hold, the sought-after realization \( G(\mathbf{k}') \in \mathcal{R} \) must also be a local minum of the Lagrangian

\[
F(\mathbf{k}', \lambda) = U(\mathbf{k}') - \lambda_1 f_1 - \ldots - \lambda_g f_g \quad \text{with} \quad \lambda := (\lambda_1, \ldots, \lambda_g).
\]  
(15)

Thus the set \( \mathcal{R} \) can be filtered out from \( \mathcal{R} \) by using a second-derivative test based on the bordered Hessian \([14]\) of \( F(\mathbf{k}', \lambda) \) under consideration of \( \lambda_1 = \ldots = \lambda_g = 0 \).

(b) The deformation \( G(\mathbf{k}) \) implying \( \sigma(\mathbf{k}) = s(\mathbf{k}) \) has to be real for \( t \in [0, 1] \), as a real solution can only change over into a complex one through a double root, which corresponds to a shaky realization (\( \Rightarrow \) contradiction to definition of \( \sigma(\mathbf{k}) \)).

(c) The results of Sec. 3.2 also hold if one uses the CE strain approach (cf. [9]). Therefore item (b) gives an answer to the open problem stated in [9, Remark 3], but our concept still ignores the collision of bars and/or plates during the deformation.

(d) The results of this paper also hold for pinned frameworks (cf. [9, Sec. 3.3]).

**Example 1.** We consider a 3-legged planar parallel manipulator (cf. Fig. 1) with a pinned base given by \( \mathbf{k}_1 = (0, 0)^T \), \( \mathbf{k}_2 = (9, 0)^T \), \( \mathbf{k}_3 = (7, 4)^T \) which is equipped with the intrinsic metric \((L_{14}, L_{25}, L_{36}, L_{45}, L_{46}, L_{56}) = (11, 10, 5, 8, 5, 5)\). The two undeformed realizations \( G(\mathbf{k}) \) and \( G(\mathbf{k}) \) snap into each other over \( G(\mathbf{k}') \) which was computed based on a framework consisting of (a) six bars using GL/CE strain (cyan/magenta dotted), (b) three bars and one triangular plate using GL strain (blue dashed). The computation of the critical points of \( U(\mathbf{k}') \) was performed with the software Bertini [1]. The number of tracked paths for each approach is given in the table displayed in Fig. 1 right as well as the number of paths ending up in finite solutions (over \( \mathbb{C} \)) under the homotopy continuation. Moreover, the cardinal numbers of \( \mathcal{R} \) and \( \mathcal{D} \) are given as well as the value for \( \sigma(\mathbf{k}) = s(\mathbf{k}) \). The coordinates of the corresponding configurations \( G(\mathbf{k}') \) are printed above this table. These three configurations are very close together, especially the cyan and magenta one (they are identical up to the first three digits after the comma). The latter difference is even not visible in the blow-up provided in Fig. 1 left.

![Fig. 1. 3-legged planar parallel manipulator of Ex. 1 Illustration (left) and data (right).](image)
4 Conclusion

We presented the computation of the snappability and the singularity-distance of frameworks composed of bars and triangular plates based on the total elastic strain energy density using the physical concept of GL strain. This measure enables the fair comparison of frameworks, which differ in the number of knots, the combinatorial structure, the intrinsic metric and the realization of triangular structures (triangular plates vs. joint-bar triangles). Our methods are demonstrated on the basis of the example of a 3-legged planar parallel manipulator, which also points out the computational efficiency of the proposed approach compared to the one using the definition of CE strain [9].

The extension/generalization of this approach to frameworks involving polygonal plates and/or polyhedrons is dedicated to future research.

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