Effective quantum field theories in general spacetimes

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Abstract

We introduce regular charts as physical reference frames in spacetime, and we show that general spacetimes can always be fully captured by regular charts. Effective quantum field theories (QFTs) can be conveniently defined in regular reference frames, and the definition is independent of specific background metric and independent of specific regular reference frame. As a consequence, coupling to classical gravity is possible in effective QFTs without getting back-reaction effects. Moreover, we present an approach to effective QFTs including quantum gravity.

Keywords: Effective Quantum Field Theory, Curved Spacetime.

1 Introduction

In the past 25 years it has been discussed if quantum field theories (QFTs) really should be seen as fundamental theories or if they rather are effective theories. There exists a list of good reasons that QFTs are not fundamental and that a fundamental theory could look like string theory [1] [2] [3], for example. For this reason, it appears fair to say that to see the experimentally well tested theories like quantum electrodynamics as effective theories is nowadays a valid point of view.

An effective QFT is supposed to be valid on a specific energy scale, and effects from processes at much higher energies are assumed to be suppressed.
For this reason, a cut-off-energy parameter is typically introduced that has a much larger value than typical energies on the energy scale in focus. If one writes down the most general Lagrangian for an effective QFT, which is allowed by the symmetries, then one basically obtains infinitely many interaction terms. Each term is however formally related to a power of the cut-off parameter, and grouping the terms with respect to the order of the cut-off parameter one formally obtains a perturbative expansion. If we restrict the formal expansion to a specific order, \( n \ (n \in \mathbb{N}_0) \), then we obtain an approximate Lagrangian, \( \mathcal{L}_n \), containing finitely many interaction terms. In this way, we obtain a series of approximate Lagrangians, \((\mathcal{L}_n)_{n \in \mathbb{N}_0}\), which basically all describe the same effective QFT.

We have recently discussed effective QFTs in Minkowski spacetime \([4]\) and shown that for each Lagrangian, \( \mathcal{L}_n \), emerging in the perturbative treatment of an effective QFT, a rigorous representation can be derived, and that the representations are unitarily equivalent to each other. We believe that this approach is consistent since the representations all describe the same effective QFT. In particular, each representation is unitarily equivalent to the representation corresponding to the non-interaction Lagrangian, \( \mathcal{L}_0 \), which defines the algebraic properties of the effective QFT.

QFTs are usually formulated in flat Minkowski spacetime, which is based on the assumption that gravitational effects can be neglected. If gravitation should however be considered in an effective QFT then one has to generalize the formulation. From a mathematical point of view, spacetime is a four-dimensional smooth Lorentzian manifold, which is defined through an atlas. The charts belonging to an atlas are usually seen as reference frames of observers in spacetime. Mathematically, a chart, \((U, \phi)\), only has some quite general features, i.e., \( U \) is open and \( \phi \) is a diffeomorphism. We feel however that, from a physical perspective, these features are not sufficient to describe a reference frame of an observer, as open sets can have all kinds of pathological shapes. We define for this reason regular reference frames/charts in Sec. 2 which turn out to be always available in general spacetimes. In fact, a description of smooth Lorentzian manifolds only in terms of regular charts is possible.

We further show in Secs. 3 and 4 that effective QFTs can be defined in a straightforward manner in regular reference frames, and that algebraic structures are actually independent of the background metric and independent of the specific chosen regular reference frame. We therefore can conveniently add gravitation to our effective QFT framework, either classically or quantum mechanically. In Sec. 5 we first discuss a semi-classical treatment of gravitation, and we then present an approach to effective QFTs including quantum gravity. Finally, we summarize and discuss our results in Sec. 6.
2 Classical fields on regular manifolds

We assume in this paper that all manifolds are four-dimensional, smooth, and that they are equipped with the unique torsion-free metric-compatible connection. Moreover, for a globally hyperbolic manifold, \((\mathcal{M}, g)\), we can introduce a "slicing" of \(\mathcal{M}\) by space-like Cauchy surfaces \(\Sigma_t (t \in \mathbb{R})\), where \(t\) denotes the smooth time parameter \([5, 6]\). We will assume throughout this paper that
\[
\mathcal{M} = \{(t, x) : t \in \mathbb{R}, x \in \Sigma_t\}
\]
whenever we refer to a globally hyperbolic manifold, \((\mathcal{M}, g)\).

**Definition 1:** A regular manifold is a globally hyperbolic manifold satisfying \(\Sigma_t = \mathbb{R}^3\) for all \(t \in \mathbb{R}\). An atlas \(\{\Phi_\alpha, U_\alpha\}_{\alpha \in A}\) is regular if each \((\Phi_\alpha(U_\alpha), g_\alpha)\) \((\langle g_\alpha \rangle_p = g_{\Phi_\alpha^{-1}(p)}\) is a regular manifold.

**Proposition 1:** Let \((\mathcal{M}, g)\) be a paracompact Lorentzian manifold, then there exists a countable regular atlas.

**Proof:** Let \(x \in \mathcal{M}\). We can choose Riemann normal coordinates, which are defined for a convex open neighborhood \(U_x\) of \(x\) by a diffeomorphism \(\exp_x : N_x \rightarrow U_x\), where \(N_x\) is a convex open neighborhood of the origin of the tangent space at \(x\) \([5, 7]\). We label coordinates in \(N_x\) so that \(y_0\) denotes the time coordinate and that \((y_1, y_2, y_3)\) denote space coordinates. The corresponding basis vectors are further denoted by \(E_{\mu}(0 \leq \mu \leq 3)\). For \(y \in N_x\) let \(n_y^\mu = g_y^{\mu\nu}E_{\nu}(0)\), where \(g_y\) denotes the metric at \(\exp_x(y)\), \(g_{\mu\nu,y} = g_y(E_\mu, E_\nu)\), and \(g_y^{\mu
u}\) denotes the inverse of the matrix \((g_{\mu\nu,y})\). Note that \(g_{\mu\nu,y}\) is Minkowskian. Due to smoothness there exists an open cube, \(C_x = \{y \in N_x : \max |y_\mu| < r_x\} \subset N_x\), so that for all \(y \in C_x\) the set \(\{n_y, E_1, E_2, E_3\}\) is linear independent and \(g_y(n_y, n_y) < 0\). For \(y \in C_x\), the set \(\{n_y, E_1, E_2, E_3\}\) is therefore a basis of the tangent space at \(\exp_x(y)\). With respect to this basis the metric has the components
\[
(g_{\mu\nu,y}) = \begin{pmatrix} g_y(n_y, n_y) & 0 \\ 0 & G_y \end{pmatrix} \quad (G_{\mu\nu,y} = g_y(E_\mu, E_\nu), 1 \leq \mu, \nu \leq 3),
\]
where the matrix \(G_y\) is positive definite since \((g_{\mu\nu,y})\) has one negative eigenvalue only. Note that \(n_y\) is orthogonal to the vectors \(E_1, E_2, E_3\) and that the manifolds
\[
\Sigma_x = \{\exp_x(y) : y \in C_x, y_0 = \tau\} \quad (-r_x < \tau < r_x)
\]
are space-like hypersurfaces. Let \(V_x = \exp_x(C_x)\), let \(\phi_x : C_x \rightarrow \mathbb{R}^4\) be the diffeomorphism defined by
\[
\phi_x(y)_\mu = \tan \left( \frac{\pi y_\mu}{2r_x} \right) \quad (y \in C_x)
\]
and let \( \psi_x = (\phi_x \circ \exp^{-1}_x)|_{V_x} \). The constructed manifold \((\psi_x(V_x), g_x) \) is a regular manifold with time parameter \( t = \phi_x(y)_0 \) \((y \in C_x)\). Moreover, let us assume for each \( x \in \mathcal{M} \) this construction, then \((V_x)_{x \in \mathcal{M}} \) is an open cover of \( \mathcal{M} \), and since \((\mathcal{M}, g)\) is paracompact it is also a Lindelöf space, i.e., there exists a countable subcover \((V_{\alpha_n})_{n \in \mathbb{N}} \). Thus, \((\psi_{\alpha_n}, V_{\alpha_n})_{n \in \mathbb{N}} \) is a regular atlas of \((\mathcal{M}, g)\). ■

Proposition 1 guarantees that for a general manifold a regular atlas always exists. If we apply this result to spacetimes then we can basically restrict the reference frames of local observers to be regular manifolds. We therefore call such reference frames regular, and we call an observer in spacetime a regular observer, if his reference frame is regular.

**Lemma 1:** Let \((\mathcal{M}, g)\) be a regular Lorentzian manifold and let \( t_1 < t_2 \). There exists a smooth metric \( g' \) so that \( g' \) is Minkowskian for \( t < t_1 \) and \( g' = g \) for \( t > t_2 \).

**Proof:** Let us choose coordinates in \( \mathcal{M} \) so that \( y_0 \) denotes the time coordinate and that \((y_1, y_2, y_3)\) denote space coordinates. With respect to these coordinates, we denote the canonical basis of the tangent space at \( p \in \mathcal{M} \) by \((E_\mu)_{0 \leq \mu \leq 3}\). For \( p \in \mathcal{M} \) let \( n_p = g_p^{\mu} E_\mu \), where \( g_p \) denotes the metric at \( p \), \( g_{\mu \nu, p} = g_p(E_\mu, E_\nu) \), and \((g_p^{\mu \nu})\) denotes the inverse of the matrix \((g_{\mu \nu, p})\). Since \((\mathcal{M}, g)\) is globally hyperbolic, \( g_p(n_p, n_p) < 0 \), \( \{n_p, E_1, E_2, E_3\} \) is a basis of the tangent space at \( p \), and with respect to this basis the metric \( g \) has the form

\[
(g_{\mu \nu, p}) = \begin{pmatrix} g_p(n_p, n_p) & 0 \\ 0 & G_p \end{pmatrix} \quad (G_{\mu \nu, p} = g_p(E_\mu, E_\nu), 1 \leq \mu, \nu \leq 3),
\]

where the matrix \( G_p \) is positive definite since \((g_{\mu \nu, p})\) has one negative eigenvalue only \( \mathbb{E} \). We can further write \( n_p = \sum_\mu a_{\mu, p} E_\mu \), where \( a_{\mu, p} \) are the coefficients of \( n_p \) with respect to the basis \((E_\mu)_{0 \leq \mu \leq 3}\). Let \( f(t) \) be a smooth function with \( f(t) = 0 \) for \( t < t_1 \), \( 0 \leq f(t) \leq 1 \) for \( t_1 \leq t \leq t_2 \), and \( f(t) = 1 \) for \( t > t_2 \). We define

\[
m_p = \text{sgn}(a_{0, p}) |a_{0, p}| f(t) E_0 + \sum_{\mu=1}^3 a_{\mu, p} f(t) E_\mu \quad (p \in \Sigma_t),
\]

where \( \text{sgn}(a_{0, p}) \) is the sign of \( a_{0, p} \). Note that \( a_{0, p} \neq 0 \), \( m_p = n_p \) for \( t < t_2 \), and \( m_p = \pm E_0 \) for \( t \leq t_1 \). Moreover, the set \( \{m_p, E_1, E_2, E_3\} \) is a basis of the tangent space at \( p \), and with respect to this basis we define

\[
(g'_{\mu \nu, p}) = \begin{pmatrix} -(-g_p(n_p, n_p)) f(t) & 0 \\ 0 & G_p^{f(t)} \end{pmatrix} \quad (p \in \Sigma_t).
\]
Note that \((g'_{\mu\nu,p}) = (g_{\mu\nu,p})\) if \(t \geq t_2\) and that \((g'_{\mu\nu,p})\) defines a Minkowski metric for \(t \leq t_1\). The corresponding metric \(g'\) on \(\mathcal{M}\) has therefore the required properties.

**Proposition 2:** Let \(\{(\mathcal{M},g)\}\) be the set of regular Lorentzian manifolds including Minkowski spacetime. Assume a covariant partial differential equation, which is defined on each \((\mathcal{M},g)\), and assume that it has a well-defined initial-value formulation with respect to the time coordinate, \(t\) (c.f. Eq. (1)). Assume further that for each \((\mathcal{M},g)\) there exists a Hilbert space of solutions, \(\mathcal{H}_g\), for which the scalar product is defined on the spacelike hypersurfaces and for which the scalar product is independent of \(t\). Then \(\mathcal{H}_g\) is unitarily equivalent to the Hilbert space of solutions in Minkowski spacetime, \(\mathcal{H}_0\).

**Proof:** Let \((\mathcal{M},g)\) be a regular Lorentzian manifold, let \(t_1 < t_2\), and let \(g'\) be a metric according to lemma 1. The manifold \((\mathcal{M},g')\) is a regular manifold, which equals \((\mathcal{M},g)\) for times \(t > t_2\) and which equals Minkowski space for times \(t < t_1\). Since the covariant partial differential equation has a well-posed initial-value formulation on regular Lorentzian manifolds we can construct the following maps: Let \(f \in \mathcal{H}_0\), then there exists a unique \(f' \in \mathcal{H}_{g'}\) which coincides with \(f\) for \(t < t_1\), and there exists a unique \(F \in \mathcal{H}_g\) which coincides with \(f'\) for \(t > t_2\). We can thus define maps \(U_{g'} : \mathcal{H}_0 \rightarrow \mathcal{H}_{g'}\) and \(U_g : \mathcal{H}_0 \rightarrow \mathcal{H}_g\). Since the scalar products in \(\mathcal{H}_0\), \(\mathcal{H}_{g'}\), and \(\mathcal{H}_g\) are defined on the spacelike hypersurfaces and since they are independent of \(t\), we obtain

\[
\langle f, h \rangle_0 = \langle U_{g'}(f), U_{g'}(h) \rangle_{g'} = \langle U_g(f), U_g(h) \rangle_g \quad (f, h \in \mathcal{H}_0).
\]

\(U_g\) is therefore a unitary operator from \(\mathcal{H}_0\) into a subspace of \(\mathcal{H}_g\). However, by inverting the initial-value argumentation, we see that for each \(F \in \mathcal{H}_g\) there exists a unique \(f \in \mathcal{H}_0\) so that \(F = U_g(f)\), i.e., \(\mathcal{H}_0\) and \(\mathcal{H}_g\) are unitarily equivalent.

There exists a variety of classical-field equations, which have a well-defined initial-value formulation on globally hyperbolic spacetimes. Scalar products can be defined for solutions to these equations on the basis of symplectic forms, which are conserved with respect to the time variable of the respective spacetime [8]. We will discuss in the following two basic examples.

For the covariant Klein-Gordon equation for particles of mass \(m\) on a globally hyperbolic spacetime there exists a Hilbert space of positive-frequency solutions. The corresponding Klein-Gordon scalar product is defined on each Cauchy surface and it is conserved with respect to the time coordinate [7,9].

**Corollary 1:** Let \(m \geq 0\) and let \(\mathcal{H}_{m,g}\) denote the Hilbert space of positive-frequency solutions of the Klein-Gordon equation for particles of mass \(m\) on a regular Lorentzian manifold \((\mathcal{M},g)\). \(\mathcal{H}_{m,g}\) is unitarily equivalent to the
corresponding Hilbert space of solutions in Minkowski spacetime, $\mathcal{H}_{m,0}$.

**Proof.** As outlined in Ref. [7], the covariant Klein-Gordon equation has a well-posed initial-value formulation on globally hyperbolic spacetimes, so the assertion follows from proposition 2.

Let $H_m = \{ (p_0, p) \in \mathbb{R}^4 : p_0^2 - p^2 = m^2, p_0 > 0 \}$ and let $j_m : \mathbb{R}^3 \to H_m$ be defined as $j_m(p) = (\sqrt{m^2 + p^2}, p) \; (p \in \mathbb{R}^3)$, then

$$\mu_m(\Omega) = \int_{j_m^{-1}(\Omega)} \frac{d^3p}{\sqrt{m^2 + p^2}}$$

defines a measure on the Borel sets on $H_m$, and there exits a unitary transform $J_m : L^2(\mathbb{R}^3) \to L^2(H_m, \mu_m)$. As outlined in Ref. [9], every element in $L^2(H_m, \mu_m)$ can be associated in a one-to-one linear manner with a solution of the Klein-Gordon equation for particles with mass $m$ in flat spacetime by

$$\hat{K}_m(f)(p_0, p) = f(p_0, p)\delta(p_0^2 - p^2 - m^2) \; (f \in L^2(H_m, \mu_m))$$

where $\hat{K}_m(f)$ denotes the Fourier transform of $K_m(f)$. For functions $K_m(f)$ and $K_m(g)$ the Klein-Gordon scalar product is defined, which equals the scalar product of $f$ and $g$ in $L^2(H_m, \mu_m)$, and $K_m$ thus defines a unitary transform of $L^2(H_m, \mu_m)$ to the space of positive-frequency solutions of the Klein-Gordon equation in flat spacetime, $\mathcal{H}_{m,0}$. In particular, the Klein-Gordon scalar product is defined with respect to any spacelike hypersurface of Minkowski space and it is constant in time.

**Corollary 2:** Let $m \geq 0$ and let $H_{m,g}$ denote the Hilbert space of positive-frequency solutions of the Klein-Gordon equation for particles of mass $m$ on a regular Lorentzian manifold $(M, g)$. $H_{m,g}$ is unitarily equivalent to $L^2(\mathbb{R}^3)$ and to $L^2(H_m, \mu_m)$.

The Dirac equation for particles of mass $m$ can be formulated in a covariant way, and for spinor solutions on a globally hyperbolic spacetime a scalar product can be introduced, which is defined on each Cauchy surface and which is conserved with respect to the time coordinate. We note that these statements are generally derived in Ref. [7] for classical spinor fields having spin $s = 0, \pm 1/2, \pm 1$.

**Corollary 3:** Let $m \geq 0$ and let $W_{m,g}$ denote the Hilbert space of spinor solutions of the Dirac equation for particles of mass $m$ on a regular Lorentzian manifold $(M, g)$. $W_{m,g}$ is unitarily equivalent to the Hilbert space of spinor solutions in Minkowski spacetime, $W_{m,0}$.

**Proof.** As outlined in Ref. [7], the covariant Dirac equation has a well-posed initial-value formulation on globally hyperbolic spacetimes, so the assertion
follows from proposition 2. ■

In Minkowski spacetime, solutions to the Dirac equation are distinguished by positive-energy solutions $\psi^+_m(x) = u_{s,m}(p)e^{-ixp}$ and negative-energy solutions $\psi^-_m(x) = v_{s,m}(p)e^{ixp}$ ($s = \pm 1/2$) [10]. $u_{s,m}(p)$ and $v_{s,m}(p)$ are spinors, and in order to make $\psi^\pm_m L^2$-integrable over spacelike hypersurfaces one needs to multiply $u_{s,m}(p)$ and $v_{s,m}(p)$ by functions in $L^2(H_m, \mu_m)$, respectively.

**Corollary 4:** Let $m \geq 0$ and let $\mathcal{W}_{m,g}$ denote the Hilbert space of spinor solutions of the Dirac equation for particles of mass $m$ on a regular Lorentzian manifold $(\mathcal{M}, g)$. $\mathcal{W}_{m,g}$ is unitarily equivalent to $\bigoplus^4 L^2(\mathbb{R}^3)$ and to $\bigoplus^4 L^2(H_m, \mu_m)$.

Proposition 2 leads to a remarkable conclusion: If we assume a covariant partial differential equation on a regular Lorentzian manifold and a Hilbert space, $\mathcal{H}$, of corresponding classical field solutions, then, up to unitary equivalence, the Hilbert-space structure is actually independent of the background metric. So if we assume such an equation on a general spacetime manifold, then regular observers, i.e., observers whose reference frames are regular Lorentzian manifolds, all detect the same Hilbert-space structure of classical field solutions. In this sense, the Hilbert-space structure of the solution space is:

1. Independent of background metric of spacetime
2. Independent of chosen regular reference frame

In particular, the second statement can also be interpreted as general covariance. Moreover, if we use $\mathcal{H}$ as single-particle space in a Fock-space construction, then the structure of the resulting Fock space, $\mathcal{F}(\mathcal{H})$, has the same properties.

### 3 Representation of conventional free quantum field theories

We will use the results of Sec. 2 in the sequel to define conventional free QFTs in regular reference frames. To this end, we use the general model of Ref. [4].

**Definition 2:** A particle system is a finite set, $S$, on which a conjugation is defined,

$$p \to \bar{p}, \quad (p, \bar{p} \in S),$$

$$\bar{\bar{p}} = p.$$
The elements of $S$ are called particles and $\bar{p}$ is called the anti-particle of a particle, $p \in S$.

We associate a conventional free QFT with the particle system $S$ as follows. Let us assume for each particle, $p \in S$, a mass, $m_p \in \mathbb{R}$, and a Hilbert space, $\mathcal{H}_p$, which is unitarily equivalent to $L^2(H_{m_p}, \mu_{m_p})$. We further assume that each particle, $p \in S$, can either be classified as Boson or as Fermion. If $p$ is a Fermion then we assume that the elements in $\mathcal{H}_p$ are classical solutions to the Dirac equation for particles of mass $m_p$. In particular, elements in $\mathcal{H}_p$ are spinors having spin $s = \pm 1/2$. If $p$ is a massive Boson, $m_p \neq 0$, then we assume spin $s = 0$ or $s = \pm 1$, and unitary equivalence of $\mathcal{H}_p$ to $L^2(H_{m_p}, \mu_{m_p})$ basically means that the spinors also satisfy the Klein-Gordon equation in Minkowski spacetime. If $p$ is a massless Boson, $m_p = 0$, then we do not associate spin with elements in $\mathcal{H}_p$. They rather are polarization vectors satisfying the wave equation $\Box v = 0$ in Minkowski spacetime.

However, let $\mathcal{F}_p$ denote either the fermionic Fock space or the Bosonic Fock space built upon $\mathcal{H}_p$. The total Fock space is defined by

$$\mathcal{F}_0 = \bigotimes_{p \in S} \mathcal{F}_p.$$ 

For a unitary operator, $U_p$, acting on $\mathcal{H}_p$ the unitary operator, $\Gamma_p(U_p)$, on $\mathcal{F}_p$ is defined by

$$(\Gamma_p(U_p)u_p)^{(n)} = \left( \bigotimes_{k=1}^{n} U_p \right) u_p^{(n)}, \quad (\Gamma_p(U_p)u_p)^{(0)} = u_p^{(0)},$$

for $u_p \in \mathcal{F}_p$.

Local raising and lowering ’operators’ are defined as quadratic forms on the bosonic and fermionic Fock space over $L^2(\mathbb{R}^3)$, respectively. These quadratic forms can be promoted to operators by smearing with smooth functions, and for any unitary transform of $L^2(\mathbb{R}^3)$ we thus can define an associated transform of the local raising and lowering ’operators’. In particular, $L^2(\mathbb{R}^3)$ is unitarily equivalent to each $L^2(H_{m_p}, \mu_{m_p}) (m_p \geq 0)$, and the local raising and lowering ’operators’ can further be transformed to quadratic forms on each $\mathcal{F}_p$ ($s \in I$) using the transformations $J_{m_p} : L^2(H_{m_p}, \mu_{m_p}) \rightarrow L^2(\mathbb{R}^3)$ given in Ref. [11]. We denote these quadratic forms in the following by $a^\dagger_p(p)$ and $a_p(p)$ ($p \in \mathbb{R}^3$), respectively. Free local fields are further defined for $p = \bar{p}$ by

$$\Phi_{0,p}(x,t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{\mu_p(p)}} e^{i(\mu_p(p)t - xp)} a^\dagger_p(p) + e^{-i(\mu_p(p)t - xp)} a_p(p),$$
and for $p \neq \bar{p}$ by

$$
\Phi_{0,p}(x, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{\mu_p(p)}} e^{i\mu_p(p) t - x p} a^\dagger_p(p) + e^{-i\mu_p(p) t + x p} a_p(p).
$$

We note that $\mu_p(p) = \sqrt{m_p^2 + p^2}$ and that

$$
\Phi_{0,\bar{p}}(x, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{\mu_p(p)}} e^{i\mu_p(p) t - x p} a^\dagger_{\bar{p}}(p) + e^{-i\mu_p(p) t - x p} a_{\bar{p}}(p)
$$

for $p \neq \bar{p}$. Further, the total free Hamiltonian is given by

$$
A_0 = \sum_{p \in S} A_{0,p} = \sum_{p \in S} \int dp \mu_p(p) a_p(p) a_p(p). \tag{2}
$$

Let us denote the free QFT introduced above for the particle system $S$ by $\Phi(S)$. The construction given above basically comprises the Fock space and the local fields with their transformation laws under the Poincaré group, i.e., the construction yields a representation of the conventional free QFT, $\Phi(S)$, satisfying the Wightman axioms [12].

The representation of $\Phi(S)$ given above is defined on Minkowski spacetime. From a more general perspective, we can consider regular observers on a general spacetime, and the Fock space $\mathcal{F}_0$ then unitarily transforms when switching from a flat metric to a curved metric. Therefore, each background metric in a regular reference frame gives rise to a representation of $\Phi(S)$, which is unitarily equivalent to the one in Minkowski spacetime. With respect to regular observers, $\Phi(S)$ is therefore:

1. Independent of background metric of spacetime
2. Independent of chosen (regular) reference frame

In particular, the second statement can also be interpreted as general covariance.

4 Effective quantum field theories

4.1 Representation of effective quantum field theories

As outlined in Sec. 1, effective QFTs are described by the most general Lagrangian, $\mathcal{L}$, which is consistent with the symmetries of the theory [1] [2], and consequently $\mathcal{L}$ contains infinitely many interaction terms. An effective
QFT is however assumed to be valid only on a specific energy scale, and one typically introduces a cut-off-energy parameter into the theory, which has a much higher value as compared to energies on the energy scale in focus. The interaction terms in $\mathcal{L}$ can formally be grouped with respect to the order of the cut-off parameter, and one formally obtains a perturbative expansion of $\mathcal{L}$. Depending on up to which order, $n (n \in \mathbb{N}_0)$, one wants to consider in the expansion of $\mathcal{L}$ one obtains in this way approximate Lagrangians, $\mathcal{L}_n$. In particular, the non-interaction approximation Lagrangian, $\mathcal{L}_0$, describes a conventional free QFT. However, as the approximate Lagrangians, $\mathcal{L}_n$, all relate to the same effective QFT, we believe that it is consistent if we define for each $\mathcal{L}_n$ a representation $R_n$ so that the representations are unitarily equivalent to each other, $R_n \sim R_m (n, m \in \mathbb{N}_0)$.

Let us assume a particle system $S$ and a Lagrangian $L_a(S) = L_0(S) + L_i(S)$, which is the sum of the Lagrangian $L_0(S)$ of a conventional free-field theory and the term $L_i(S)$, which represents higher-order interaction terms, so that $L_a(S)$ could be one of the approximate Lagrangians, $\mathcal{L}_n$, occurring in the formal perturbative expansion of the Lagrangian of an effective QFT. Corresponding expressions for Hamiltonians are typically obtained by means of canonical quantization. In particular, the Hamiltonian, $A_0$, corresponding to $L_0(S)$ is given by Eq. (2).

Let $R_0$ denote the representation of the conventional free field theory corresponding to $L_0(S)$ as described in Sec. 3. We can use conventional scattering theory to derive a representation, $R_a$, related to $L_a(S)$ as follows. We assume that the quantum system is complete and that the Möller wave operators, $W_{\pm}$, exist. The operators can be extended to partial isometries by defining that they leave the vacuum state, $w_0$, of the free field theory invariant, $W_{\pm}w_0 = w_0$. We can define the representation $R_a$ by applying one of the Möller wave operators, for example $W_+$, to the representation $R_0$. The Hamiltonian in $R_a$ is then given by $A = W_+A_0W_+^\dagger$.

This approach is more or less well-known from physics text books, but criticized from the viewpoint of axiomatic quantum field theory [12]. Nevertheless, the approach is mathematically rigorous, and either one approximately treats the interacting part of the Hamiltonian, or one switches to a nonstandard framework where an exact formulation is possible [4]. In any case, one obtains two unitarily equivalent representations of the same QFT: The trivial representation, $R_0$, of the effective QFT, which is related to the free part of the Lagrangian, $L_0(S)$, and the representation, $R_a$, which is related to the full Lagrangian, $L_a(S)$. Both representations refer to the same Fock space, but they implement different particle pictures: Single-particle states in $R_a$ typically are multi-particle states in $R_0$, for example. Let us illustrate this idea for the examples of quantum electrodynamics (QED) and
quantum chromodynamics (QCD).

In QED, the free particles at low energies approximately are the electron and the photon, but at higher energies these particles can never be observed alone, since other electrons or photons are always in their neighborhood. We therefore cannot consider electrons and photons as free particles at higher energies, since we rather observe composite states of these particles. So if we derive representations $R_0$ and $R_a$ as described above for QED at higher energies, then representation $R_a$ yields physical single-particle states that are interpreted as multi-particle states with respect to the trivial representation, $R_0$. In particular, $R_0$ relates to the particle picture at low energies, which basically serves as a benchmark particle picture.

Another example is QCD, which exhibits the feature of asymptotic freedom. Quarks can (approximately) be seen as free particles in QCD at very large energies. For lower energies, however, their coupling becomes stronger and we observe only composite quark states. Accordingly, if we derive representations $R_0$ and $R_a$ for QCD at moderate energies, then $R_0$ yields the benchmark particle picture, which relates to QCD at very large energies. States in $R_a$ are interpreted with respect to the benchmark particle picture, i.e., single-particle states in $R_a$ are seen as multi-particle states in $R_0$.

As mentioned above, the trivial representation, $R_0$, of an effective QFT is mathematically well-defined, and it serves to determine the algebraic properties of the theory. It is however important to note that $R_0$ does not necessarily implement a physical particle picture, which is strictly valid at some energy. For example, in QED the interaction between electrons and photons never vanishes as far as we know.

4.2 Categories of effective quantum field theories

As outlined in Sec. 4.1, effective QFTs can be defined through a formal perturbative expansion of the Lagrangian with respect to a cut-off-energy parameter. The cut-off energy is much higher than energies on the energy scale for which the effective QFT is supposed to be valid. For the approximate Lagrangians, $\mathcal{L}_n \ (n \in \mathbb{N}_0)$, emerging in the expansion representations, $R_n$, of the effective QFT can be defined, which are unitarily equivalent to each other. In particular, each $R_n$ is unitarily equivalent to the trivial representation, $R_0$. The trivial representation therefore defines the algebraic properties of an effective QFT.

As the parameters of an effective QFT, like for example physical masses, are energy dependent, we add the energy index $E$ from now on to our notation. Let $S$ be a particle system and let $\Phi_E(S) \ (E \geq 0)$ be an effective QFT defined through its trivial representation. The quantum field theories, $\Phi_E(S)$,
can be further subsumed into the category $\mathcal{K}(S) = \{ \Phi_E(S) : E \geq 0 \}$ with morphisms $m(E_1, E_2) = \Phi_{E_1} \circ \Phi_{E_2}^{-1}$, i.e. $\Phi_{E_2}(S) = m(E_1, E_2)(\Phi_{E_1}(S))$. $\mathcal{K}(S)$ comprises the quantum field theories at all energy scales with Lagrangians $L_E(S)$ ($E \geq 0$). $\mathcal{K}(S)$ therefore represents the energy-independent structure of the effective QFT.

The QFTs $\Phi_E(S)$ have been defined in Minkowski spacetime, but, as argued at the end of Sec. 3, they are invariant when considered by regular observers in general spacetimes. In particular, their representations unitarily transform when changing the background metric in regular reference frames. For this reason, we can state that, for regular observers, $\mathcal{K}(S)$ is:

1. Independent of background metric of spacetime
2. Independent of chosen regular reference frame

Again, the second statement can also be interpreted as general covariance.

5 Gravitation coupling

5.1 Semi-classical gravitation coupling

We discuss in the following how effective QFTs can be used in a semi-classical description of gravitation. In the case of presence of matter fields the stress-energy tensor is given by

$$T_{ab} = -\frac{1}{\sqrt{-g}} \frac{\delta (L_M \sqrt{-g})}{\delta g^{ab}} = -\frac{\delta L_M}{\delta g^{ab}} + \frac{1}{2} g_{ab} L_M.$$  

In a semi-classical approach one formally replaces classical fields in $T_{ab}$ by operators and considers the expectation value $\langle T_{ab} \rangle$ in the Einstein field equations. The stress-energy tensor can formally be written as $T = \sum_i F_i(g) O_i$, where the $F_i(g)$ denote functions of the metric $g$ and the $O_i$ are operators, and where we have suppressed indices for the sake of clearness. Using this notation, the expectation value is formally given as $\langle T \rangle = \sum_i F_i(g) \langle O_i \rangle$. If we use an effective QFT as defined in Sec. 4 to define the expectation values $\langle O_i \rangle$, then we immediately notice that they are independent of the metric $g$. Therefore, we can solve the Einstein field equations using $\langle T \rangle$ without getting back-reaction effects [8], i.e., the metric $g$ can be determined without affecting the the expectation values $\langle O_i \rangle$. In this way, classical spacetime curvature emerges from effective QFT. We note that for a rigorous treatment regularization of singular expressions in $\langle T \rangle$ is necessary [8], which does however not affect the validity of our argumentation.
5.2 An approach to quantum gravity

The basic idea of our approach of defining effective QFTs in general spacetimes is to restrict the definition to regular reference frames. The formulation of effective QFTs is then independent of the background metric and independent of a specific regular reference frame. In this way, we decouple the definition of effective QFTs from the treatment of gravity. As pointed out in Sec. 5.1, a semi-classical formulation of Einstein field equations is possible without getting back-reaction effects.

However, it is tempting now to go one step further and to also include graviton fields in an effective theory. In fact, general relativity can already be viewed as an effective theory of gravitation at low energies [13], and the semi-classical gravitation coupling discussed in Sec. 5.1 fits very well into the framework of effective field theory. Let us first assume a flat background metric on which graviton fields are defined in linear approximation to general relativity [14], then the correction terms to the free Hamiltonian are obtained by expanding the Einstein-action Lagrangian [13]. In fact, we obtain a formal perturbative expansion, which is typical for an effective QFT. As described in Sec. 4.2, we can now define a category, $\mathcal{K}(S)$, of quantum field theories, $\Phi_E(S)$, where $S$ denotes a particle system including gravitons. $\mathcal{K}(S)$ and each $\Phi_E(S)$ are independent of the background metric and independent of a specific regular reference frame, and we basically obtain a background-independent quantum field theory including quantum gravity.

6 Discussion

We discuss in this paper effective QFTs in general spacetimes. An effective QFT is described by the most general Lagrangian, which complies with the symmetries of the theory [1, 2]. Since an effective QFT is valid only on a specific energy scale, it typically contains a cut-off-energy parameter. The Lagrangian of an effective QFT can formally be expanded with respect to the cut-off parameter, and one obtains in this way a series of approximate Lagrangians. We have recently established rigorous representations with respect to the approximate Lagrangians of an effective QFT [4], which are unitarily equivalent to each other. We believe that this is consistent as the Lagrangians all refer to the same effective QFT. In particular, the Lagrangian, $\mathcal{L}_0$, in the non-interaction approximation yields a conventional free field theory, which is the trivial representation of the effective QFT. In fact, the trivial representation defines the algebraic properties of an effective QFT.

However, QFTs are usually defined in Minkowski spacetime. From a
mathematical point of view, spacetime is a four-dimensional smooth Lorentzian manifold, \((\mathcal{M}, g)\), which is defined through an atlas, and a chart, \((U, \phi)\), in an atlas consists of an open set \(U\) and a diffeomorphism \(\phi\). From a physical perspective, charts are seen as reference frames of observers. As an open set can have all kinds of pathological shapes, we believe that reference frames should be more specifically defined. For this reason, we introduce regular reference frames/charts in Sec. 2. We show that regular charts of spacetimes always exist, and that spacetimes can actually be described only by regular charts.

Regular manifolds have the convenient property that classical wave equations have a well-defined initial-value formulation on them. For this reason, conventional free QFTs can be defined in a straightforward manner in regular reference frames. However, as shown in Sec. 2 and 3, algebraic properties of conventional free QFTs in regular reference frames are actually independent of background metric and independent of the specific chosen reference frame. This is consequently also valid for effective QFTs, as an effective QFT is algebraically defined through its trivial representation, which is a conventional free QFT.

We define effective QFTs in this paper with respect to a given particle system, \(S\), which represents a finite set of fundamental particles. The definition however depends on the energy, for which the effective QFT is applicable. In Sec. 4.2 we further show that the energy-dependent effective QFTs, which are based on the same particle system, can be subsumed into a category. The category represents the energy-independent structure of the effective QFT.

The basic advantage of our approach to defining effective QFTs in general spacetimes is to restrict the definition to regular reference frames. In this way, we decouple the definition of effective QFTs from the treatment of gravity. We can therefore conveniently include gravity in our effective QFT framework, both semi-classically and quantum mechanically. In Sec. 5 we first demonstrate that a semi-classical treatment is possible without getting back-reaction effects. We then outline a purely quantum-mechanical approach, which leads to an effective QFT including quantum gravity.

Let us finally take a look at the validity range of effective QFTs, noting that experimentally well tested theories like quantum electrodynamics are nowadays considered as effective theories. These theories are supposed to break down at high energies near the Planck scale, where a fundamental theory including quantum gravity is required. In particular, fundamental approaches to quantum gravity propose relativity violations at high energies, which are associated with the breaking of Lorentz symmetry. Today’s experimental situation however puts rather tight bounds on the occurrence of such effects \([15, 16]\), and we can therefore expect that effective QFTs have a
rather large validity range.

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