Connections functorially attached to almost complex product structures

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Abstract

Manifolds endowed with three foliations pairwise transversal are known as 3-webs. Equivalently, they can be algebraically defined as biparacomplex or complex product manifolds, i.e., manifolds endowed with three tensor fields of type (1, 1), $F$, $P$ and $J = F \circ P$, where the two first are product and the third one is complex, and they mutually anti-commute. In this case, it is well known that there exists a unique torsion-free connection parallelizing the structure. In the present paper, we study connections attached to non-integrable almost biparacomplex manifolds.

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1 Introduction

Theory of webs was begun by Blaschke when he introduced a web on a surface as three families of curves pairwise transversal at every point. Similarly, a three-web on a $2n$-dimensional manifold $M$ is given by three $n$-dimensional foliations such that their leaves through any point are pairwise transverse. An $\alpha$-structure on $M$ is a system of three distributions $V_i$ each of dimension $n$ such that $TM = V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3$. Cruceanu obtained an algebraic characterization of these $\alpha$-structures, by means of the almost biparacomplex structures (see also).

An almost biparacomplex structure on a manifold $M$ is given by two tensor fields $F$ and $P$ of type (1, 1) satisfying $F^2 = P^2 = Id$, $F \circ P + P \circ F = 0$. An
almost biparacomplex structure is said to be biparacomplex if the distributions
\( T^\pm_F(M) = \ker(F \mp Id) \) and \( T^\pm_P(M) = \ker(P \mp Id) \) associated to the eigenvalues
\( \pm 1 \) of \( F \) and \( P \) respectively, are involutive. This condition is equivalent to the
vanishing of the Nijenhuis tensors \( N_F \) and \( N_P \).

An almost biparacomplex structure defines an \( \alpha \)-structure on the manifold
by simply setting \( V_1 = T^+_F(M), V_2 = T^-_F(M) \) and \( V_3 = T^+_P(M) \). We call it
the \( \alpha \)-structure associated to the almost biparacomplex structure. In particular, a biparacomplex structure defines a web. Conversely, given an \( \alpha \)-structure
\( (V_1, V_2, V_3) \), there exists a unique almost biparacomplex structure such that
\( T^+_F(M) = V_1, T^-_F(M) = V_2, T^+_P(M) = V_3, T^-_P(M) = F(V_3) \).

The distributions \( T^+_F(M), T^-_F(M) \) are readily seen to be equidimensional
(and then \( M \) is endowed with two almost paracomplex structures, thus being
called an almost biparacomplex manifold). Hence the dimension of \( M \) is even,
and the tensor field \( J = F \circ P \) defines an almost complex structure on \( M \), so
that \( M \) must be orientable.

Then an almost biparacomplex manifold is endowed with three tensor fields of type \((1,1)\), \( F \), \( P \) and \( J = F \circ P \), where the two first are almost product
and the third one is almost complex, and they mutually anti-commute, thus being
also called an almost complex product manifold. Such a structure was studied by Libermann [21] fifty years ago. In this sense, almost biparacomplex
manifolds corresponds to the paraquaternionic numbers, which consist on the
4-dimensional real algebra generated by \( \{1, i, j, k\} \) with the paraquaternionic
relations: \( i^2 = j^2 = -k^2 = 1, ij = -ji = k \).

On the other hand, in recent years 3-webs have been considered in a different
framework. We want to point out the works of Andrada, Barberis, Blažič, Dotti,
Ivanov, Kamada, Ovando, Tsanov, Vukmirović, Zamkovoy [2, 3, 4, 7, 14, 15,
16, 17] and others. Roughly speaking, they call a complex product structure
(also called a para-hypercomplex structure and a hyper-paracomplex structure)
on a manifold a pair of a complex structure \( J \) and a product structure \( P \) with
\( J \circ P = -P \circ J \). Of course, \( F = J \circ P \) is a product structure. They only consider the integrable case, i.e., the case where the Nijenhuis tensors of \( J \) and \( P \) (and
that of \( F \)) vanish. They know that there exists a unique torsion-free connection
parallelizing both structures \( J \) and \( P \). They mainly apply their results to the
study of Lie algebras.

In the present paper we shall study the general case, not only the integrable
one, obtaining significant advances:

- In any case, there exists a canonical connection associated to the almost
biparacomplex structure, defined as the unique connection parallelizing the tensor fields \( F, P, J \) and satisfying \( T(X^+, Y^-) = 0 \), where
\( X^+ \in T^+_F(M), Y^- \in T^-_F(M) \) (Theorem 4), where \( T \) denotes the torsion
tensor field of the connection.

- The canonical connection is torsion-free iff the structure is integrable (Theorem 6).
The $G$-structure defined by the almost biparacomplex structure is integrable iff the canonical connection is locally flat (Theorem 7).

The canonical connection is a functorial connection (Theorem 12).

In any case, there exists another functorial connection, which is called the well-adapted connection (Theorem 13).

The canonical and the well-adapted connection coincide if the structure is integrable (Theorem 18).

As one can see these results enlightens light upon the connection defined in such a manifold and open some questions about the non-integrable case in the aforementioned works.

The organization of the paper is as follows: in section 2 we introduce notations. In section 3 we obtain the main results about the canonical connection of an almost biparacomplex manifold. Also we study functorial connections attached to such a manifold, in particular the so called well-adapted connection. Finally, in the last section we show examples, open problems and the relationship among the quoted recent papers on complex product and para-hypercomplex structures and the present paper.

2 Notations and preliminaries

Manifolds are assumed to be of class $C^\infty$ and satisfying the second axiom of countability. Differentiable maps between manifolds are also assumed to be of class $C^\infty$. We denote by $\otimes^k V, \wedge^k V, S^k V$ the $k$th tensor, exterior and symmetric power of a vector bundle $V$ over a manifold $M$. In particular, we apply this notation to the tangent and cotangent bundles $T(M), T^*(M)$, respectively. We denote by $X(M) = \Gamma(M, T(M))$, $\Omega^k(M) = \Gamma(M, \wedge^k T^*(M))$, $k \in \mathbb{N}$.

If $\varphi: M \to M'$ is a diffeomorphism, for every $X \in X(M)$ we denote by $\varphi \cdot X \in X(M')$ the vector field defined by $(\varphi \cdot X)_{x'} = \varphi_* (X_{\varphi^{-1}(x')})$, $\forall x' \in M'$. More generally, $\varphi$ transforms a tensor field of type $(p,q)$ on $M$ into a tensor field of the same type on $M'$ by imposing

$$\varphi \cdot (\omega_1 \otimes \cdots \otimes \omega_p \otimes X_1 \otimes \cdots \otimes X_q) = \varphi^{-1} \cdot \omega_1 \otimes \cdots \otimes \varphi^{-1} \cdot \omega_p \otimes \varphi \cdot X_1 \otimes \cdots \otimes \varphi \cdot X_q,$$

with $\omega_1, \ldots, \omega_p \in \Omega^1(M)$, $X_1, \ldots, X_q \in X(M)$.

Let $\pi: FM \to M$ be the bundle of linear frames of $M$ and let $G$ be a closed subgroup in $GL(m; \mathbb{R})$, $m = \dim M$. We recall (e.g., see [13], I. pp. 57-58) that the $G$-structures over $M$ are in bijection with the sections of the quotient bundle $\bar{\pi}: F(M)/G \to M$. In fact, the bijection $s \leftrightarrow B_s$ between sections and $G$-structures is given by the formula $B_s = \{ u \in F(M) : u \cdot G = s(\pi(u)) \}$, where $u \cdot G$ denotes the coset of $u \in FM$ in $FM/G$, and the inverse bijection is $s_p: M \to FM/G$, $s_p(x) = u(\mod G)$, where $u$ is any point in the fibre $B_x$.  

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For every diffeomorphism \( \varphi: M \to M' \), let \( \bar{\varphi}: FM \to FM' \) be the \( GL(m; \mathbb{R}) \)-principal bundle isomorphism given by (cf. [19] Chapter VI, p. 226):

\[
\bar{\varphi}(X_1, \ldots, X_m) = (\varphi_* X_1, \ldots, \varphi_* X_m), \quad (X_1, \ldots, X_n) \in FM.
\]

Two \( G \)-structures \( \pi: B \to M, \pi: B' \to M' \) are said to be equivalent if there exists a diffeomorphism \( \varphi: M \to M' \) such that \( \bar{\varphi}(P) = P' \).

A diffeomorphism \( \varphi: M \to M' \) transforms each section \( s: M \to F(M)/G \) of \( \bar{\pi}: F(M)/G \to M \) into a section of \( \bar{\pi}' : F(M)/G \to M \) by setting

\[
\varphi \cdot s = \bar{\varphi} \circ s \circ \varphi^{-1}.
\]

The \( G \)-structure corresponding to \( \varphi \cdot s \) is said to be obtained by transporting \( B \) to \( M' \) via \( \varphi \).

Two \( G \)-structures \( B, B' \) are equivalent if and only if their corresponding sections are related by a diffeomorphism; more precisely, \( \bar{\varphi}(B) = B' \) if and only if \( \varphi \cdot s_P = s_{P'} \).

By passing \( \bar{\varphi}: FM \to FM' \) to the quotient modulo \( G \), \( \bar{\varphi} \) induces a diffeomorphism \( \bar{\varphi}: F/M/G \to F'/M'/G \), such that \( \bar{\pi}' \circ \bar{\varphi} = \varphi \circ \bar{\pi} \).

We denote by \( C(M) \to M \) the bundle of linear connections on \( M \). This is an affine bundle modelled over the vector bundle \( T^*(M) \otimes T^*(M) \otimes T(M) \) whose global sections are identified to the linear connections on \( M \) (cf. [20]).

Let \( G \subseteq GL(m; \mathbb{R}) \) be a Lie subgroup and let \( \mathfrak{g} \) be its Lie algebra. We denote by \( \mathfrak{g}^{(1)} \) its first prolongation; that is,

\[
\mathfrak{g}^{(1)} = \{ T \in \text{Hom}(\mathbb{R}^m, \mathfrak{g}) : T(u)v - T(v)u = 0, \forall u, v \in \mathbb{R}^m \},
\]

where we identify \( \mathfrak{g} \) to its natural image in \( \text{gl}(m; \mathbb{R}) = \text{Hom}(\mathbb{R}^m, \mathbb{R}^m) \).

### 3 The canonical connection

Let \( (M, F, P) \) be an almost biparacomplex manifold. Then the following can be proved:

**Proposition 1** If \( \{X_1, \ldots, X_n\} \) is a basis of the distribution \( T^+_x(M) \) at a point \( x \in M \), then \( \{PX_1, \ldots, PX_n\}, \{X_1 + PX_1, \ldots, X_n + PX_n\} \) and \( \{X_1 - PX_1, \ldots, X_n - PX_n\} \) are basis of the distributions \( T^-_x(M), T^+_p(M), T^-_p(M) \) at \( x \in M \), respectively.

Now, taking Proposition 1 into account one can prove that the \( G \)-structure determined by an almost biparacomplex structure is given by \( G = \Delta GL(n; \mathbb{R}) \), where

\[
\Delta GL(n; \mathbb{R}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in GL(n; \mathbb{R}) \right\},
\]

is a closed subgroup of the Lie group \( GL(n; \mathbb{R}) \).

Then, we can state the following:
Proposition 2 The set of all almost biparacomplex structures on a manifold $M$ can be identified with the sections of the quotient bundle $FM/\Delta GL(n; \mathbb{R}) \to M$.

Now we shall define a canonical connection on an almost biparacomplex manifold $(M, F, P)$. This connection $\nabla$ has the following properties: (1) $\nabla$ is torsion-free iff the distributions associated to the almost biparacomplex structure are involutive; (2) $\nabla$ is locally flat (i.e., its torsion and curvature tensor fields vanish) iff the $\Delta GL(n; \mathbb{R})$-structure defined by $(F, P)$ is integrable.

Lemma 3 Let $(F, P)$ be an almost biparacomplex structure on a manifold $M$, let $(V_1, V_2, V_3)$ be its associated $\alpha$-structure, and let $\nabla$ be a linear connection on $M$. The following conditions are equivalent:

1. $\nabla X_i Y_i \in V_i, \forall X \in \mathfrak{X}(M), \forall Y_i \in V_i, i = 1, 2, 3.$
2. $\nabla F = 0 = \nabla P.$

One can prove the previous lemma by direct calculations taking into account that $F$ is an isomorphism between the distributions $V_3$ and $V_4$ and the definition of the tensor fields $\nabla P$ and $\nabla F$ of type $(1, 1)$.

Theorem 4 Let $(F, P)$ be an almost biparacomplex structure on $M$ and let $(V_1, V_2, V_3)$ be its associated $\alpha$-structure. Then there exists a unique linear connection $\nabla$ on $M$ verifying the following conditions:

1. $\nabla F = 0 = \nabla P.$
2. $T(X, Y) = 0$, for all $X \in V_1, Y \in V_2$, where $T$ denotes the torsion tensor field of $\nabla$.

Then $\nabla$ is called the canonical connection of $(F, P)$.

Proof. Let us assume that $\nabla$ is a linear connection satisfying the above conditions. If $X \in V_1, Y \in V_2$ one has

$$0 = T(X, Y) = \nabla_X Y - \nabla Y X - [X, Y] = \nabla_X Y - \nabla_Y X - F^+[X, Y] - F^-[X, Y],$$

where $F^+$ (resp. $F^-$) denotes the projection over $T^+_F(M)$ (resp. $T^-_F(M)$). On the other hand, as we assume $\nabla$ parallelize both $F$ and $P$, then by Lemma 3 we have $\nabla_X Y \in V_2, \nabla_Y X \in V_1$, $X$ belonging to $V_1$ and $Y$ to $V_2$, and then one has $\nabla_X Y - F^- [X, Y] = 0, \ -\nabla_Y X - F^+ [X, Y] = 0$, i.e.,

$$\nabla_X Y = F^- [X, Y], \ \nabla_Y X = F^+ [Y, X].$$

Now we prove that this equation determines completely $\nabla$. Let $X, Y \in \mathfrak{X}(M)$. Then one can decompose $X = F^+ X + F^- X, \ Y = F^+ Y + F^- Y$, and then

$$\nabla_X Y = \nabla_{F^+ X} F^+ Y + \nabla_{F^+ X} F^- Y + \nabla_{F^- X} F^+ Y + \nabla_{F^- X} F^- Y.$$
Taking into account the above equation (1) we obtain

(3) \[ \nabla F + X F - Y = F - [F + X, F - Y], \quad \nabla F - X F + Y = F + [F - X, F + Y], \]

which allows us to deduce the following equations, taking into account that \( \nabla P = 0 \):

\[ \nabla F + X F + Y = \nabla F + X P F + Y, \]
\[ \nabla F - X F - Y = \nabla F - X P F - Y, \]

Then, by using the relations \( P \circ F - = F - \circ P, P \circ F + = F + \circ P \), we can conclude:

(4) \[ \nabla F + X F + Y = F + P[F + X, P F + Y], \quad \nabla F - X F - Y = F - P[F - X, P F - Y], \]

for all \( X, Y \in \mathcal{X}(M) \). Finally, joining the above equations (2), (3) and (4) we obtain the general expression of the derivation law of \( \nabla \):

(5) \[ \nabla X Y = F^+ \left( [F - X, F + Y] + P[F + X, P F + Y] \right) \]
\[ + F^- \left( [F + X, F - Y] + P[F - X, P F - Y] \right), \]

for all vector fields \( X, Y \) on \( M \).

Finally, we prove that the connection \( \nabla \) defined by the above equation (5) verifies both conditions of the present theorem. It is an easy exercise to check that \( \nabla \) is a linear connection and that \( \nabla \) verifies condition (ii), because this condition was the starting point of our construction.

In order to prove condition i), one can prove, by Lemma 8 that \( \nabla \) preserves the distributions \( V_i, i = 1, 2, 3 \). Let \( X \) a vector field on \( M \), and let \( Z_i \in V_i, i = 1, 2, 3 \). Then we have

\[ \nabla X Z_1 = F^+ \left( [F - X, Z_1] + P[F + X, P Z_1] \right) \Rightarrow \nabla X Z_1 \in V_1; \]

\[ \nabla X Z_2 = F^- \left( [F + X, Z_2] + P[F - X, P Z_2] \right) \Rightarrow \nabla X Z_2 \in V_2; \]

and, finally,

\[ \nabla X Z_3 = F^+ \left( [F - X, F + Z_3] + P[F + X, P F + Z_3] \right) \]
\[ + F^- \left( [F + X, F - Z_3] + P[F - X, P F - Z_3] \right). \]

Taking into account the equations \( P \circ F^+ = F^- \circ P \) and \( P \circ F^- = F^+ \circ P \), we obtain

\[ P(\nabla X Z_3) = F^- \left( P[F - X, F + Z_3] + [F + X, F - Z_3] \right) \]
\[ + F^+ \left( P[F + X, F - Z_3] + [F - X, F + Z_3] \right). \]
Then, \( P(\nabla_X Z_3) = \nabla_X Z_3 \Rightarrow \nabla_X Z_3 \in V_3 \), as wanted. ■

Now we shall show that \( \nabla \) measures the involutiveness of the distributions associated to an almost biparacomplex structure and the integrability of the \( \Delta GL(n; \mathbb{R}) \)-structure. In the first case, the Frölicher-Nijenhuis tensor field of the pair \((F, P)\) is also useful. Taking the equation \( F \circ P + P \circ F = 0 \) into account we have that the Nijenhuis bracket of \( F \) and \( P \) is

\[
[F, P](X, Y) = [FX, PY] + [PX, FY] + PF[X, Y] + FP[X, Y]
- F[PX, Y] - F[X, PY] - P[FX, Y] - P[X, FY]
= [FX, PY] + [PX, FY] - F[PX, Y]
- F[X, PY] - P[FX, Y] - P[X, FY].
\]

**Lemma 5** Let \( M \) be a manifold endowed with an almost biparacomplex structure \((F, P)\) and let \( T \) be the torsion tensor field of the canonical connection. Then the following relations hold

\[
[F, P](X, Y) = 2PT(X, Y), \forall X, Y \in T^+_F(M),
\]

\[
[F, P](X, Y) = -2PT(X, Y), \forall X, Y \in T^-_F(M),
\]

\[
[F, P](X, Y) = 2F^-[X, PY] - 2F^+[PX, Y], \forall X \in T^+_F(M), Y \in T^-_F(M).
\]

This technical result follows from an easy—but rather long—calculation. Then, we can state

**Theorem 6** Let \((F, P)\) be an almost biparacomplex structure on \( M \) and let \( \nabla \) be its canonical connection. Then the following three conditions are equivalent:

(i) \((F, P)\) is a biparacomplex structure.

(ii) \([F, P] = 0\).

(iii) \( T = 0 \), i.e., \( \nabla \) is torsion-free.

The proof of this result follows directly from the properties of the canonical connection of \((F, P)\) and the equations of above lemma.

**Theorem 7** Let \( M \) be a 2n-dimensional manifold endowed with an almost biparacomplex structure \((F, P)\) and let \( \nabla \) be its canonical connection. Then, the following three conditions are equivalent:

(i) \( T = 0 \) and \( R = 0 \), \( T \) and \( R \) being the torsion and curvature tensor fields of \( \nabla \); i.e., the canonical connection is locally flat.
(ii) For every point \( x \in M \) there exists an open neighbourhood \( U \) of \( x \) and local coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) on \( U \) such that
\[
F (\partial / \partial x_i) = \partial / \partial y_i, \quad F (\partial / \partial y_i) = \partial / \partial x_i,
\]
\[
P (\partial / \partial x_i) = \partial / \partial x_i, \quad P (\partial / \partial y_i) = -\partial / \partial y_i,
\]
for all \( i = 1, \ldots, n \).

(iii) The \( \Delta GL(n; \mathbb{R}) \)-structure on \( M \) defined by \((F, P)\) is integrable.

Proof. The equivalence (ii) \( \Leftrightarrow \) (iii) consists on the adaptation of the general result about the integrability of a \( G \)-structure to our case of \( G = \Delta GL(n; \mathbb{R}) \). Thus, we only prove (i) \( \Rightarrow \) (ii).

(i) \( \Rightarrow \) (ii). As \( \nabla \) is locally flat, for every \( x \in M \) there exists a coordinate neighbourhood \((U; x_1, \ldots, x_{2n})\) such that \( \nabla_{\partial / \partial x_i}(\partial / \partial x_j) = 0 \), \( 1 \leq i, j \leq 2n \).

Let
\[
F = \sum_{i,j=1}^{2n} f_{ij} \frac{\partial}{\partial x_i} \otimes dx_j, \quad P = \sum_{i,j=1}^{2n} g_{ij} \frac{\partial}{\partial x_i} \otimes dx_j, \quad f_{ij}, g_{ij} \in C^\infty(U)
\]
be the local expressions of \( F \) and \( P \) on \( U \). First, we shall show that the functions \( f_{ij}, g_{ij} \) are constant functions on \( U \). Let \( X = \sum_{k=1}^{2n} X^k(\partial / \partial x_k), X^k \in C^\infty(U) \), be any vector field on \( U \). We have
\[
\nabla_X F = \sum_{i,j=1}^{2n} X(f_{ij}) \frac{\partial}{\partial x_i} \otimes dx_j + \sum_{i,j=1}^{2n} f_{ij} \nabla_X \frac{\partial}{\partial x_i} \otimes dx_j + \sum_{i,j=1}^{2n} f_{ij} \frac{\partial}{\partial x_i} \otimes \nabla_X dx_j.
\]

As the Christoffel symbols of \( \nabla \) vanish on \( U \) one has \( \nabla_X(\partial / \partial x_i) = 0 \) and \( \nabla_X dx_i = 0 \), for all \( i \in \{1, \ldots, 2n\} \), and then the above equation reduces to
\[
\nabla_X F = \sum_{i,j=1}^{2n} X(f_{ij}) (\partial / \partial x_i) \otimes dx_j.
\]

Now, taking into account that \( \nabla \) parallelizes \( F \), we have \( \nabla_X F = 0 \), thus proving \( X(f_{ij}) = 0 \), and then \( f_{ij} \) are constant functions. A similar proof runs for \( g_{ij} \).

Now we shall define a new local chart verifying condition (ii) of the theorem.

Let \( \{X_1(x), \ldots, X_n(x)\} \) be a basis of \( T^\perp_E(M) \) with \( X_i(x) = \sum_{j=1}^{2n} \alpha^i_j (\partial / \partial x_j) \), \( 1 \leq i \leq n \). As the local coefficients of \( F \) on the above chart \((U; x_1, \ldots, x_{2n})\) are constant, then the following \( n \) vector fields define a local basis of \( T^\perp_E(M) \) on \( U \):
\[
X_i(y) = \sum_{j=1}^{2n} \alpha^i_j (\partial / \partial x_j)y, \quad \forall y \in U, \quad 1 \leq i \leq n.
\]

Let us consider the vector fields on \( U \) defined by \( U_i = X_i + PX_i, \quad V_i = X_i - PX_i, \quad 1 \leq i \leq n \). By Proposition \( \{U_1, \ldots, U_n, V_1, \ldots, V_n\} \) is a local basis of \( TM \) at \( U \). One easily check that \([U_i, U_j] = 0, [U_i, V_j] = 0, [V_i, V_j] = (...)\)
0, 1 ≤ i, j ≤ n, taking into account that F and P have constant coefficients on 
(U; x_1, . . . , x_{2n}). Then there exist coordinates (u_1, . . . , u_n, v_1, . . . , v_n) on U 
such that U_i = ∂/∂u_i, V_i = ∂/∂v_i, 1 ≤ i ≤ n. Then we have
\[
F (∂/∂u_i) = F (X_i + PX_i) = X_i + FPX_i = X_i - PX_i = ∂/∂v_i,
\]
\[
F (∂/∂v_i) = F (X_i - PX_i) = X_i - FPX_i = X_i + PX_i = ∂/∂u_i,
\]
\[
P (∂/∂u_i) = P (X_i + PX_i) = PX_i + X_i = ∂/∂u_i,
\]
\[
P (∂/∂v_i) = P (X_i - PX_i) = PX_i - X_i = -∂/∂v_i,
\]
for all 1 ≤ i ≤ n, as wanted.

(ii)⇒(i). The existence of such a local chart implies that the almost para-
complex structures defined by F and P are in fact paracomplex, and then the
associated distributions are involutive (see [18, Prop. 1.2]). Then the struc-
ture is biparacomplex and, by the above Theorem 6, the canonical con-
nection is torsion-free, thus proving
\[
T (∂/∂x_i, ∂/∂y_j) = \nabla_{∂/∂x_i}(∂/∂y_j) - \nabla_{∂/∂y_j}(∂/∂x_i) = 0.
\]
We only need to prove that R also vanishes. As the structure is biparacomplex,
we know that ∇F = 0 = ∇P. By Lemma 4 we can deduce that ∇_{∂/∂x_i}(∂/∂y_j) ∈
T_P(M) and that ∇_{∂/∂y_j}(∂/∂x_i) ∈ T_P(M). Taking into account that TM =
T_P(M) ⊕ T_P(M), from the above equation (7) we obtain
\[
\nabla_{∂/∂x_i}(∂/∂y_j) = 0, \nabla_{∂/∂y_j}(∂/∂x_i) = 0, \quad 1 ≤ i, j ≤ n.
\]
Moreover, as F^2 = Id and ∇F = 0, for all i, j ∈ {1, . . . , n} we can deduce
\[
\nabla_{∂/∂y_j}(∂/∂x_i) = F (∇_{∂/∂y_j}(∂/∂x_i)) = F (∇_{∂/∂x_i}(∂/∂y_j)) = 0,
\]
\[
\nabla_{∂/∂y_j}(∂/∂x_i) = F (∇_{∂/∂y_j}(∂/∂x_i)) = F (∇_{∂/∂x_i}(∂/∂y_j)) = 0,
\]
for all i, j = 1, . . . , n. So we have proved that all the Christoffel symbols vanish,
and hence ∇ is locally flat.

We end this study obtaining the expression of the canonical connection on an
adapted local frame. This result will be useful in order to compare the canonical
connection with other connections. Let M be a manifold endowed with an
almost biparacomplex structure (F, P). Let ∇ be its canonical connection. Let
U be an open subset of M and let \{X_1, . . . , X_n, Y_1, . . . , Y_n\} be a local frame
on U adapted to (F, P). Let us denote by \{ω_1, . . . , ω_n, η_1, . . . , η_n\} its dual
coframe. Then, ∇ is determined on U by \nabla_{X_a}X_a, \nabla_{Y_a}X_a, a, h = 1, . . . , n,
because \nabla_{X_a}Y_a = P \nabla_{X_a}X_a, \nabla_{Y_a}X_a = P \nabla_{Y_a}X_a, a, h = 1, . . . , n.

As ∇_{X_a}X_a, ∇_{Y_a}X_a ∈ T_P(M) for all h, a = 1, . . . , n, we can write
\[
\nabla_{X_a}X_a = \sum_{i=1}^{n} \Gamma_{ha}^{i} X_i, \quad \nabla_{Y_a}X_a = \sum_{i=1}^{n} \Gamma_{ha}^{i} X_i, \quad a, h = 1, . . . , n,
\]
and from $T(X_h, Y_a) = 0$ we obtain $\sum_{i=1}^{n}(\Gamma_{ha}^i Y_i - \bar{\Gamma}_{ah}^i X_i) = [X_h, Y_a]$. Hence

$$\sum_{i=1}^{n}\Gamma_{ha}^i Y_i = \sum_{i=1}^{n}\eta_i([X_h, Y_a])Y_i, \quad -\sum_{i=1}^{n}\bar{\Gamma}_{ah}^i X_i = \sum_{i=1}^{n}\omega_i([X_h, Y_a])X_i,$$

and finally,

$$(8) \quad \Gamma_{ha}^i = \eta_i([X_h, Y_a]), \quad \bar{\Gamma}_{ah}^i = \omega_i([Y_a, X_h]), \quad a, h, i = 1, \ldots, n.$$

### 3.1 Functorial connections

Now we shall prove that the canonical connection of an almost biparacomplex structure is a functorial connection. Roughly speaking, a functorial connection associated to a $G$-structure is a family of reducible connections, one for each concrete $G$-structure, which is natural with respect to the isomorphisms of the $G$-structure. We shall also show that in our case $G = \Delta GL(n; \mathbb{R})$ there exists, at least, another functorial connection, which will be called the well-adapted connection. Both connections, the canonical and the well-adapted, coincide iff the manifold is a biparacomplex manifold.

Now, we shall present general results concerning functorial connections.

**Definition 8** Let $G$ be a Lie subgroup of $GL(m; \mathbb{R})$, with $m = \dim M$. A functorial connection attached to the $G$-structures over $M$ is a presheaf morphism $s \mapsto \nabla(s)$ from the sheaf of sections of $FM/G \to M$ into the presheaf of sections of $C(M) \to M$, such that:

1. If $s: U \to FM$ is a section on an open subset $U \subseteq M$, then $\nabla(s)$ is a linear connection on $U$ adapted to the $G$-structure defined by $s$; i.e., $\nabla(s)$ is reducible to the subbundle $B_s$ (see [3]).

2. If $\varphi: U \to U'$ is a diffeomorphism, then $\nabla(\varphi \cdot s) = \varphi \cdot \nabla(s)$, where $\varphi \cdot \nabla(s)$ denotes the direct image of $\nabla(s)$ by $\varphi$ (cf. [14, II. Prop. 6.1]).

3. There exists a non-negative integer $r$ such that $\nabla$ factors smoothly through $j^r(FM/G)$.

The third item above means that the value of the section $\nabla(s)$ of $C(U) \to U$ at a point $x \in U$ depends only on $j^r_x s$, and that the induced map

$$\nabla^r: j^r(FM/G) \to C(M)$$

$$\nabla^r(j^r_x s) = \nabla(s)(x)$$

is differentiable. Also note that $\nabla = \nabla^r \circ j^r$. This item can be substituted by apparently less restrictive conditions; for example, by only imposing that $\nabla(s)(x)$ depends on the germ of $s$ at $x$, not on the $r$-jet of the section. Nevertheless, standard techniques working in a very general setting (see [20, Chapter V]) readily shows the equivalence of both conditions.
Theorem 9 [26, Th. 1.1] The following conditions are equivalent:

(i) For every $G$-structure $B \to M$ there exists a unique connection $\nabla'$ adapted to the $G$-structure such that, for every endomorphism $S$ given by a section of the adjoint bundle $\text{ad}B$ and every vector field $X \in \mathfrak{X}(M)$, one has $\text{trace}(S \circ i_X \circ T') = 0$, where $T'$ denotes the torsion of $\nabla'$. Moreover, this connection only depends on the first contact of the $G$-structure.

(ii) If $L \in \text{Hom}(\mathbb{R}^n, g)$ verifies $i_v \circ \text{Alt}(L) \in g^\perp$ for every $v \in \mathbb{R}^n$ then $L = 0$, where $g$ is the Lie algebra of the group $G$, $g^\perp$ is the orthogonal subspace of $g$ in $\mathfrak{gl}(n; \mathbb{R})$ respect to the Killing-Cartan metric, and $\text{Alt}(L)(u, v) = L(v)u - L(u)v$, $\forall u, v \in \mathbb{R}^n$.

If there exists, we shall call this connection the well-adapted connection to the $G$-structure $\pi : B \to M$.

The well-adapted connection is a functorial connection and measures the integrability of the $G$-structure in the sense that the $G$-structure is integrable iff the well-adapted connection is locally flat (cf. [26, Th. 2.3]). The above Theorem 9 gives us an explicit way of obtaining a functorial connection. Moreover, one has

Theorem 10 [26, Th. 2.1] Let $G$ be a Lie group with Lie algebra $g$. If $g^{(1)} = 0$ and $g$ is invariant under transposition, then condition ii) of Theorem 9 holds.

Then, we have finished the study of the general framework. Now, we specialize to the case $G = \Delta \text{GL}(n; \mathbb{R})$, showing that (1) the canonical connection is a functorial connection; (2) there exists the well-adapted connection; and (3) the canonical and the well-adapted connections coincide iff the manifold is biparacomplex.

The following result will be useful to prove that the canonical connection is functorial. In fact, this is a property stronger than that of a functorial connection.

Theorem 11 Let $M, M'$ be two manifolds endowed with almost biparacomplex structures $(F, P), (F', P')$, respectively and let $\varphi : M \to M'$ be a diffeomorphism between both structures, i.e.,

$$\varphi_* \circ F = F' \circ \varphi_*, \quad \varphi_* \circ P = P' \circ \varphi_*.$$  

(9)

Let $\nabla, \nabla'$ be the canonical connections of $(F, P), (F', P')$, respectively. Then $\nabla'$ is the direct image of $\nabla$ via $\varphi$.

Proof. It is a direct consequence of equation (9), which states $\nabla$ in terms of $F$ and $P$. $\blacksquare$

Theorem 12 The assignment of the canonical connection of $(F_\sigma, P_\sigma)$ to each section $\sigma$ of the fiber bundle $F(M)/\Delta \text{GL}(n; \mathbb{R}) \to M$ is a functorial connection, $(F_\sigma, P_\sigma)$ being the almost biparacomplex structure associated to the section $\sigma$.  

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Proof. Obviously $\nabla$ is an adapted connection and then it satisfies the first condition in Definition 8. The above Theorem 11 shows that the second condition is also satisfied. Finally, from equation (8) we obtain that $\nabla(x)$ depends only on $f_{ij}(x)$, $g_{ij}(x)$, $\partial f_{ij}/\partial x_k(x)$, $\partial g_{ij}/\partial x_k(x)$, $i, k = 1, \ldots, 2n$, $1 \leq j \leq n$, and then, taking $r = 1$, the third condition is satisfied.

The following result proves the existence of the well-adapted connection.

**Proposition 13** The first prolongation of the Lie algebra $\Delta_*\mathfrak{gl}(n; \mathbb{R})$ of the Lie group $\Delta GL(n; \mathbb{R})$ vanishes and $\Delta_*\mathfrak{gl}(n; \mathbb{R})$ is invariant under transposition. Then, there exists the well-adapted connection.

The proof of this result is an exercise of Linear Algebra and therefore is omitted.

Now we shall obtain the local expression of the well-adapted connection $\nabla'$ of an almost biparacomplex structure $(F, P)$. We are looking for comparing with the expression of the canonical connection obtained in equation (8). The first step consists on determining the well-adapted connection by means of Theorem 9, so we must obtain information about the adjoint bundle.

Let $B \to M$ be the $\Delta GL(n; \mathbb{R})$-structure defined by $(F, P)$. We denote by $\text{ad}B$ the associated fiber bundle to $B$ via the adjoint representation of $\Delta GL(n; \mathbb{R})$ on $\Delta_* GL(n; \mathbb{R})$, i.e., $\text{ad}B = (B \times \Delta_*\mathfrak{gl}(n; \mathbb{R}))/\Delta GL(n; \mathbb{R})$, where the action of $\Delta GL(n; \mathbb{R})$ on $B \times \Delta_*\mathfrak{gl}(n; \mathbb{R})$ is given by

$$\left( u, \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right) \cdot \left( B \begin{pmatrix} 0 \\ 0 \end{pmatrix} B \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left( u \cdot \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \begin{pmatrix} 0 & B^{-1}AB \\ B^{-1}AB & 0 \end{pmatrix} \right),$$

where $u \in B$, $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \in \Delta GL(n; \mathbb{R})$, $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in \Delta_*\mathfrak{gl}(n; \mathbb{R})$.

One observes that the adjoint bundle $\text{ad}B$ is a subbundle of the bundle of endomorphisms of the tangent bundle, $\text{ad}B \subset \text{End}(TM) \cong T^1_1(M)$, because $\Delta_*\mathfrak{gl}(n; \mathbb{R}) \subset \mathfrak{gl}(2n; \mathbb{R}) \equiv \text{End}(\mathbb{R}^{2n})$. The following result characterizes the endomorphisms of the tangent bundle which belongs to the adjoint bundle. Let us introduce some notations: let $x \in M$ and let $U$ an open neighbourhood of $x$. Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ be a local frame adapted to $(F, P)$ on $U$. Then, an element of $(\text{ad}B)_x$ is, by definition, an endomorphism $S: T_x M \to T_x M$

$$(10) \quad S(X_j) = \sum_{i=1}^n a_{ij} X_i, \quad S(Y_j) = \sum_{i=1}^n a_{ij} Y_i, \quad j = 1, \ldots, n,$$

where $A = (a_{ij})$ is any matrix of $\mathfrak{gl}(n; \mathbb{R})$.

**Lemma 14** An endomorphism $S$ of the tangent bundle can be written as in (10), for every point $x \in M$, iff $S$ commutes with $F$ and $P$. 

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Proof. Let 
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]
be the matrix of \(S\) respect to the basis \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\). Respect to such basis, the expressions of \(F\) and \(P\) are:
\[
\begin{pmatrix}
I_n & 0 \\
0 & -I_n
\end{pmatrix},
\begin{pmatrix}
0 & I_n \\
I_n & 0
\end{pmatrix}.
\]
Then \(F \circ S = S \circ F\) and \(P \circ S = S \circ P\) iff \(B = C = 0\) and \(D = A\). i.e., \(S \in \Delta, gl(n; \mathbb{R})\).

Then we can deduce:

**Proposition 15** The sections of the adjoint bundle of a \(\Delta GL(n; \mathbb{R})\)-structure over \(M\) defined by an almost biparacomplex structure \((F, P)\) on \(M\) are the endomorphisms of \(T M\) which commute with \(F\) and \(P\).

Condition (i) of Theorem 9 is given by \(\text{trace}(S \circ i_X \circ T') = 0, \forall S \in \Gamma(\text{ad}\mathcal{B}), \forall X \in \mathfrak{X}(M),\) \(T'\) being the torsion tensor of the well-adapted connection \(\nabla'\). Taking into account the definition of the trace, one has:
\[
0 = \sum_{i=1}^{n} \omega_i((S \circ i_X \circ T')(X_i)) + \sum_{i=1}^{n} \eta_i((S \circ i_X \circ T')(Y_i))
\]
\[
= \sum_{i=1}^{n} \omega_i(S(T')(X, X_i)) + \sum_{i=1}^{n} \eta_i(S(T')(X, Y_i)),
\]
where \(T'\) denotes the torsion tensor of \(\nabla'\) and \(\{\omega_1, \ldots, \omega_n, \eta_1, \ldots, \eta_n\}\) denotes the dual coframe of \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\) on \(U\). We can locally determine \(\nabla'\) choosing a local basis of sections of the adjoint bundle \(\text{ad}\mathcal{B}\) and a local basis of \(\mathfrak{X}(M)\). Let us take the family \(\{\omega_h \otimes X_a + \eta_h \otimes Y_a : a, b = 1, \ldots, n\}\) as local basis of \(\Gamma(\text{ad}\mathcal{B})\) and let us take the local adapted frame as a local basis of \(\mathfrak{X}(U)\). Then, for \(S = \omega_h \otimes X_a + \eta_h \otimes Y_a\) and \(X = X_h\), we have
\[
\sum_{i=1}^{n} \omega_i((\omega_h \otimes X_a + \eta_h \otimes Y_a)(T'(X_h, X_i)) + \sum_{i=1}^{n} \eta_i((\omega_h \otimes X_a + \eta_h \otimes Y_a)(T'(X_h, Y_i)) = 0,
\]
and, consequently,
\[
(11)\quad \omega_h(T'(X_h, X_a)) + \eta_h(T'(X_h, Y_a)) = 0.
\]
On the other hand, for \(X = Y_h\) and \(S = \omega_h \otimes X_a + \eta_h \otimes Y_a\), we have
\[
\sum_{i=1}^{n} \omega_i((\omega_h \otimes X_a + \eta_h \otimes Y_a)(T'(Y_h, X_i)) + \sum_{i=1}^{n} \eta_i((\omega_h \otimes X_a + \eta_h \otimes Y_a)(T'(Y_h, Y_i)) = 0,
\]

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and then,

\[ \omega_b(T'(Y_h, X_a)) + \eta_b(T'(Y_h, Y_a)) = 0. \]

As \( \nabla' \) is a functorial connection, \( \nabla' \) parallelizes \( F \) and \( P \) and preserves the distributions associated to the eigenvalues \( \pm 1 \) of \( F \) and \( P \) (see Lemma 3). Then, \( \nabla'_X h X_a, \nabla'_Y h X_a \in T^+_F (M) \), \( \nabla'_X h Y_a, \nabla'_Y h Y_a \in T^-_F (M) \). Moreover, one observes that \( \nabla' \) is completely determined by the values \( \nabla'_X h X_a, \nabla'_Y h X_a, h, a = 1, \ldots, n \).

Let us denote

\[ \nabla' X h X_a = \sum_{i=1}^n \Gamma'^i_{ha} X_i, \quad \nabla' Y h X_a = \sum_{i=1}^n \bar{\Gamma}'^i_{ha} X_i. \]

Next, we show that the equations (11) and (12) allow us to obtain the Christoffel symbols \( \Gamma'^i_{ha}, \bar{\Gamma}'^i_{ha}, h, a, i = 1, \ldots, n \) on \( U \).

From equation (11), we have

\[ 0 = 2\Gamma'^i_{ha} - \Gamma'^i_{ah} = \omega_b([X_h, X_a]) + \eta_b([X_h, Y_a]). \]

and permuting the indices \( a \) and \( h \), we obtain

\[ 0 = 2\Gamma'^i_{ah} - \Gamma'^i_{ha} = \omega_b([X_a, X_h]) + \eta_b([X_a, Y_h]). \]

From equations (13) and (14) above we conclude

\[ \Gamma'^i_{ab} = \frac{1}{3} \left( \omega_b([X_a, X_h]) + 2\omega_b([X_a, Y_h]) + \eta_b([X_a, Y_h]) \right). \]

Moreover, by using equation (12), and by a similar argument to the one above, we can obtain the remaining Christoffel symbols:

\[ \bar{\Gamma}'^i_{ah} = \frac{1}{3} \left( \omega_b([Y_h, X_a]) + 2\omega_b([Y_a, X_h]) + \eta_b([Y_a, X_h]) \right). \]

**Remark 16** Let \((F, P)\) be an almost biparacomplex structure on a manifold \( M \) and let \( \nabla \) (resp. \( \nabla' \)) be its canonical (resp. well-adapted) connection. In general, these connections do not coincide. The proof is very easy: one only must compare equations (8) of \( \nabla \) with equations (15) and (16) of \( \nabla' \).

Moreover, we can obtain the expression of the tensor field of type \((1, 2)\) given by the difference of both connections: \( A = \nabla - \nabla' \).

**Theorem 17** Let \((F, P)\) be an almost biparacomplex structure on a manifold \( M \) and let \( \nabla \) (resp. \( \nabla' \)) be its canonical (resp. well-adapted) connection. Let \( A \) be the tensor defined by

\[ A(X, Y) = \nabla_X Y - \nabla'_X Y, \quad \forall X, Y \in X(M). \]

Then,

\[ A(X, Y) = \frac{1}{3} \left( F^+T(F^+X, F^+Y) + PF^+T(F^+X, PF^-Y) + PF^-T(F^-X, PF^+Y) + F^-T(F^-X, F^-Y) \right). \]

for all vector fields \( X, Y \) on \( M \), where \( T \) denotes the torsion tensor of the canonical connection.
Proof. From equations (8) and (15) we deduce
\[ A(X_h, X_a) = \nabla X_h X_a - \nabla' X_h X_a \]
\[ = \frac{1}{3} \left( PF^- [X_h, PX_a] - F^+ [X_h, X_a] + PF^- [PX_h, X_a] \right) \]
\[ = \frac{1}{3} F^+ T(X_h, X_a), \quad \forall h, a = 1, \ldots, n. \]

Then
\[ A(X', X'') = \frac{1}{3} \left( PF^- [X', PX''] - F^+ [X', X''] + PF^- [PX', X''] \right) \]
\[ = \frac{1}{3} F^+ T(X', X''), \quad \forall X', X'' \in T^+_F(M). \]

Moreover, we have
\[ A(X_h, Y_a) = \nabla X_h Y_a - \nabla' X_h Y_a = P(\nabla X_h X_a - \nabla' X_h X_a) = PA(X_h, PY_a), \]
for all \( h, a = 1, \ldots, n \). Then
\[ A(X', Y') = \frac{1}{3} \left( F^- [X', Y'] - PF^+ [X', PY'] + F^- [PX', PY'] \right) \]
\[ = \frac{1}{3} PF^+ T(X', PY'), \quad \forall X' \in T^+_F(M), Y' \in T^-_F(M). \]

Similar arguments allow us to obtain the following equalities:
\[ A(Y', X') = \frac{1}{3} \left( F^+ [Y', X'] + F^+ [PY', PX'] - PF^- [Y', PX'] \right) \]
\[ = \frac{1}{3} PF^- T(Y', PX'), \quad \forall X' \in T^+_F(M), Y' \in T^-_F(M); \]
\[ A(Y', Y'') = \frac{1}{3} \left( PF^+ [Y', PY''] + PF^+ [PY', Y''] - F^- [Y', Y''] \right) \]
\[ = \frac{1}{3} F^- T(Y', Y''), \quad \forall Y', Y'' \in T^-_F(M). \]

From the above equations (17), (18), (19) and (20) we obtain two expressions for the tensor \( A \); in the first one \( A \) is refereed to the tensors \( F, P, F^+ \) and \( F^- \):
\[ A(X, Y) = \frac{1}{3} \left( PF^- [FX, PF^+ Y] + PF^+ [PF^+ X, F^+ Y] - F^+ [FX, F^+ Y] \right. \]
\[ + F^+ [PF^- X, PF^+ Y] + F^- [FX, F^- Y] + F^- [PF^+ X, PF^- Y] \]
\[ - PF^+ [FX, PF^- Y] + PF^+ [PF^- X, F^- Y] \right). \]
and in the second one, $A$ is refereed to the torsion $T$ of the canonical connection:

$$
A(X, Y) = \frac{1}{3} \left( F^+ T(F^+ X, F^+ Y) + PF^+ T(F^+ X, PF^- Y) \\
+ PF^- T(F^- X, PF^+ Y) + F^- T(F^- X, F^- Y) \right).
$$

As a direct consequence of this result we obtain

**Theorem 18** Let $(F, P)$ be an almost biparacomplex structure on a manifold $M$ and let $\nabla$ (resp. $\nabla'$) be its canonical (resp. well-adapted) connection. If $(F, P)$ is a biparacomplex structure then $\nabla = \nabla'$.

**Proof.** By Theorem 6, if $(F, P)$ is biparacomplex, then $\nabla$ is torsion-free and hence, from equation (21), we obtain $A = 0$, thus proving $\nabla = \nabla'$.

4 Examples and Final Remarks

We finish the paper showing some examples and point out some aspects of the theory.

4.1 Equivalence of $\Delta GL(n; \mathbb{R})$-structures

The equation (9) of Theorem 11 is the definition of the equivalence of the almost biparacomplex structures $(F, P)$ and $(F', P')$ over the manifolds $M$ and $M'$: we say that $(F, P)$ and $(F', P')$ are equivalent if there exists a diffeomorphism $\varphi: M \to M'$ such that $\varphi_* \circ F = F' \circ \varphi_*$, $\varphi_* \circ P = P' \circ \varphi_*$.

This condition is equivalent to the classical definition of $G$-structures in the case of the Lie group $\Delta GL(n; \mathbb{R})$.

**Proposition 19** [25] Let $\pi: \mathcal{B} \to M$ y $\pi': \mathcal{B}' \to M'$ be the $\Delta GL(n; \mathbb{R})$-structures over the $n$-dimensional manifolds $M$ and $M'$ defined by the almost biparacomplex structures $(F, P)$ and $(F', P')$ respectively. Then we have that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent iff the structures $(F, P)$ and $(F', P')$ are equivalent.

Theorem 11 establishes that if the almost biaparacomplex structures $(F, P)$ and $(F', P')$ are equivalent then the canonical connections $\nabla$ and $\nabla'$ are equivalent; i.e., $\nabla'$ is the direct image of $\nabla$ via $\varphi$. This result is the key that allow us to characterize the equivalence problem of $\Delta GL(n; \mathbb{R})$-structures in terms of the linear connections in the analytic case. We have that

**Theorem 20** [25] Let $(F, P), (F', P')$ be two analytic almost biparacomplex structures over the analytic manifolds $M, M'$ respectively, and let $\varphi: M \to M'$ be an analytic diffeomorphism verifying that:
i) there exists a point \( x_0 \in M \) such that \( (F, P) \) and \( (F', P') \) are equivalent by \( \varphi \):

\[
F'_{\varphi(x_0)} \circ (\varphi_*)_{x_0} = (\varphi_*)_{x_0} \circ F_{x_0}, \quad P'_{\varphi(x_0)} \circ (\varphi_*)_{x_0} = (\varphi_*)_{x_0} \circ P_{x_0},
\]

ii) the canonical connections \( \nabla \) and \( \nabla' \) are equivalent by \( \varphi \).

In these conditions, there exists an open neighbourhood \( U \) of \( x_0 \) such that \( (F, P) \) and \( (F', P') \) are equivalent \( \varphi \) on \( U \); i. e.,

\[
F'_{\varphi(x)} \circ (\varphi_*)_{x} = (\varphi_*)_{x} \circ F_{x}, \quad P'_{\varphi(x)} \circ (\varphi_*)_{x} = (\varphi_*)_{x} \circ P_{x}, \quad \forall x \in U.
\]

Moreover, if \( M \) is a connected manifold then the open \( U \) coincides with \( M \).

We leave out the proofs of Proposition 19 and Theorem 20 of this work, which can be found in [25].

The main objective of [25] was to solve the equivalence problem of \( \Delta GL(n; \mathbb{R}) \)-structures, finding the generators of rings of differential invariants of the structure (this is a general technique for \( G \)-structures; see details in [22] y [26]). The construction of these differential invariants leads to functorial connections attached to the \( G \)-structure which allows to construct differential invariants in a natural way. About the rings of differential invariants of \( \Delta GL(n; \mathbb{R}) \)-structures we can establish the following result:

**Theorem 21** [25] Let \( r \in \mathbb{N}, \ r \geq 1 \). The rings of differential invariants of \( r \)-order the \( \Delta GL(n; \mathbb{R}) \)-structures of a manifold \( M \) are locally differentially generated over an open dense subset of \( J^r(FM/\Delta GL(n; \mathbb{R})) \) by exactly \( N_{2n,r} \) functions, where

\[
N_{2n,r} = 2n + \binom{2n+r}{r} \frac{(3r-1)n^2 - 2(r+1)n}{r+1},
\]

If \( r = 0 \) then \( N_{2n,0} = 0 \); i. e., the constant functions are the unique differential invariants of \( 0 \)-order.

In the case of \( n = 1 \), the concept of the \( \Delta GL(2; \mathbb{R}) \)-structure over \( M \) coincides with that of \( \mathbb{R}^* \)-structure (see [24]). Both types of \( G \)-structures define a 3-web over a surface. By the formula (22), one has:

\[
N_{2, r} = \frac{(r+1)(r-2)}{3},
\]

thus re-obtaining the result obtained by Valdés in [24].

The explicit expression of a local basis of functions of the rings of differential invariants of the \( \Delta GL(n; \mathbb{R}) \)-structures is still an open problem. We hope to obtain such invariants from the canonical connection of an almost biparacomplex structure in a future work.
4.2 Uniqueness of the functorial connection in the integrable case

If \( \dim M = 2 \), an \( \alpha \)-structure is always integrable thus defining a web. The canonical connection is the Blaschke’s connection. In the paper [23] the authors have proved that Blaschke’s connection is the only functorial connection which can be attached to two-dimensional three-webs. An open problem consists on proving the following

**Conjecture** In the biparacomplex case there exists only one functorial connection.

4.3 Biparacomplex structures on Lie algebras and groups

Let \( g \) be a Lie algebra. An *almost complex structure* \( J \) on \( g \) is a linear endomorphism such that \( J \circ J = -I \), where \( I \) stands for the identity map. The structure is said *integrable* or *complex structure* if the corresponding Nijenhuis-type operator vanishes:

\[
J[X, Y] = [JX, Y] + [X, JY] + J[JX, JY]
\]

for all \( X, Y \in g \), where \([, ,]\) denotes the Lie bracket of the Lie algebra.

An *almost product structure* \( E \) on \( g \) is a linear endomorphism such that \( E \circ E = I \). The structure is said *integrable* or *product structure* if the corresponding Nijenhuis-type operator vanishes:

\[
E[X, Y] = [EX, Y] + [X, EY] - E[EX, EY]
\]

This condition is equivalent to \( g_+ \) and \( g_- \) being subalgebras, \( g_\pm \) being the eigenspace corresponding to the eigenvalue \( \pm 1 \) of the product structure \( E \). Then \((g, g_+, g_-)\) is a double Lie algebra.

Observe that a Lie algebra \( g \) is a real vector space \( \mathbb{R}^n \) endowed with a Lie bracket \([, ,]\). As a real manifold, the Lie derivative of vector fields vanish, i.e., the Lie bracket of vector fields vanish. Then, an almost complex (resp. product) structure is always integrable. But this is not the situation for the Lie bracket of the Lie algebra.

One can recover some results of Andrada and Salamon [4]. They consider a Lie algebra \( g \) endowed with a pair \( \{J, E\} \) where \( J \) is a complex structure on \( g \), \( E \) a product structure on \( g \) and \( J \circ E = -E \circ J \). Of course, \( P = J \circ E \) is also a product structure on \( g \), and \( (g, J, E, P) \) is a biparacomplex manifold. They characterize double Lie algebras \((g, g_+, g_-)\) which are associated to a complex product structure [4 Prop. 2.5] and prove that the complexification of a Lie algebra endowed with a complex product structure has a hypercomplex structure [4 Th. 3.3]. Explicit examples of complex product structures on 4-dimensional Lie algebras are given. All the results through the paper of Andrada and Salamon are given only in the integrable case. They prove [4 Prop. 5.1] that a Lie algebra carrying a complex product structure admits a unique torsion-free
connection parallelizing $J$ and $E$. This can be obtained as a consequence of Theorems 4 and 6 of our paper.

Four-dimensional Lie algebras admitting a biparacomplex structure has been recently classified by Blažič and Vukmirović [7] and by Andrada, Barberis, Dotti and Ovando [2]. The authors of the first paper use the name para-hypercomplex for such a structure.

Finally, we point out the work [14], where the authors consider a Lie group endowed with a biparacomplex structure invariant respect to left translations. Then the Lie group is said to admit a homogeneous complex product structure. They prove that the Lie groups $SL(2m - 1, \mathbb{R})$ and $SU(m, m - 1)$ admit homogeneous product structures.

The authors of the present paper think that the results obtained through the paper about almost complex product structures on manifolds can be translated to the study of non-integrable complex product structures on Lie algebras.

### 4.4 Triple structures

Almost biparacomplex manifolds are example of triple structures, i.e., of manifolds endowed with three $(1, 1)$-tensor fields $F$, $P$ and $J$ satisfying

$$F^2 = \pm Id, \quad P^2 = \pm Id, \quad J = P \circ F, \quad P \circ F \pm F \circ P = 0.$$  

In fact, one can define four different triple structures, namely

- **Almost biparacomplex structure**: $F^2 = Id, P^2 = Id, P \circ F \mp F \circ P = 0$.
- **Almost hyperproduct structure**: $F^2 = Id, P^2 = Id, P \circ F - F \circ P = 0$.
- **Almost bicomplex structure**: $F^2 = -Id, P^2 = -Id, P \circ F - F \circ P = 0$.
- **Almost hypercomplex structure**: $F^2 = -Id, P^2 = -Id, P \circ F + F \circ P = 0$.

Almost hyperproduct and almost bicomplex structures do not admit functorial connections (see [11] for a proof), whereas almost biparacomplex and almost hypercomplex ones do admit: those studied in this paper for almost biparacomplex structures; Obata connection in the hypercomplex case, being the unique torsion-free connection parallelizing the structure. As we have said in the above subsection, a biparacomplex structure on a Lie algebra $\mathfrak{g}$ defines a hypercomplex structure on its complexification $\mathfrak{g}^\mathbb{C}$. Moreover, the Obata connection on $\mathfrak{g}^\mathbb{C}$ is flat iff the canonical connection on $\mathfrak{g}$ is flat (see [11] Cor. 5.3).

### 4.5 Biparacomplex metric structures

Let $(M, F, P)$ be an almost biparacomplex manifold. One of us has defined four different kinds of metrics adapted to the biparacomplex structure

**Definition 22** (see [24]). Let $(M, F, P)$ be a biparacomplex manifold, and let $g$ be a pseudo-Riemannian metric on $M$. Then, $(M, F, P, g)$ is said to be an $(\varepsilon_1, \varepsilon_2)$
pseudo-Riemannian almost biparacomplex manifold, where \( \varepsilon_1, \varepsilon_2 \in \{+,-\} \) according to the following relations:

\[
g(FX, FY) = \varepsilon_1 g(X, Y); \quad g(PX, PY) = \varepsilon_2 g(X, Y).
\]

Each one of the four possibilities of the signs determines the sign of \( g(JX, JY) \).

In the cases \((+, -)\) and \((-+, +)\) the metric \( g \) is neutral of signature \((n, n)\) and in the case \((--, -)\) is neutral of signature \((2n, 2n)\). In \([6]\), \((--, -)\) pseudo-Riemannian almost biparacomplex manifold are called paraquaternionic Hermitian. In that paper the Blažič studies the paraquaternionic projective space, which is an example of this structure.

A hypersymplectic \([13]\), hyper-Hermitian \([15]\) or neutral hyperkähler \([17]\) manifold is a \(4n\)-dimensional biparacomplex manifold endowed with a neutral metric of signature \((2n, 2n)\) such that \(g(JX, JY) = g(X, Y), \; g(FX, FY) = -g(X, Y)\). With the above notation, it is a \((-,-)\)-metric biparacomplex manifold. Then, it is Kähler and Ricci flat. Moreover, hypersymplectic structures are used in string theories.

Recently, Andrada \([1]\) has classified hypersymplectic structures on four-dimensional Lie algebras and Andrada and Dotti \([3]\) have studied hypersymplectic structures on \(\mathbb{R}^{4n}\), showing significant examples.

On the other hand, a connection with torsion attached to a \((-,-)\)-metric biparacomplex manifold have been considered in \([15]\), where the authors define a hyperparaKähler with torsion as a \((-,-)\)-metric \(g\) such that there exists a linear connection \(\nabla\) satisfying the following relations:

\[
\nabla g = \nabla F = \nabla P = \nabla J = 0; \quad T(X, Y, Z) \equiv g(T(X, Y), Z) = -T(X, Y, Z)
\]

where \(T\) denotes the torsion tensor of \(\nabla\). The last relation can be read saying that the torsion tensor of type \((0,3)\) is totally skew-symmetric. Moreover, they obtain a lot of examples.

Finally, relationships between almost biparacomplex structures and almost bi-Lagrangian ones have been found by the authors. Let us remember that an almost bi-Lagrangian structure on a symplectic manifold \((M, \omega)\) is given by two transversal Lagrangian foliations \(D_1, D_2\). Equivalently, it is an almost para-Kähler structure on \(M\) (see \([10]\) or \([12]\) for the details). Then one can prove (see \([9], \[12]\))

Proposition 23 Let \((M, \omega, D_1, D_2)\) be an almost bi-Lagrangian manifold and let \((M, F, g)\) be its associated almost para-Kähler structure. For each Riemannian metric \(G\) such that \(D_1, D_2\) are \(G\)-orthogonal, we define the almost complex structure \(J\) associated with \(G\) and \(\omega\) (i.e., \(\omega(X, Y) = G(JX, Y)\)). Then:

1. \((M, F, P = J \circ F)\) is an almost biparacomplex manifold;
2. \((M, J, g)\) is a Norden manifold;
3. \((M, F, G)\) is a Riemannian almost product manifold;
4. \((M, F, P, g)\) is a \((-,-)\) pseudo-Riemannian almost biparacomplex manifold;
5. \((M, F, P, G)\) is a \((+,-)\) Riemannian almost biparacomplex manifold.
Such a metric always exists: if $H$ is any Riemannian metric on $M$, then one can define a new Riemannian metric $G$ by $G(X,Y) = H(X,Y) + H(FX,FY)$ obtaining that $(M,F,G)$ is a Riemannian almost product manifold, i.e., the two distributions $D_1$ and $D_2$ are $G$-orthogonal.

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