Self-converse mixed graphs are extremely rare

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November 8, 2021

Abstract
A mixed graph is cospectral to its converse, with respect to the usual adjacency matrices. Hence, it is easy to see that a mixed graph whose eigenvalues occur uniquely, up to isomorphism, must be isomorphic to its converse. It is therefore natural to ask whether or not this is a common phenomenon. This note contains the theoretical evidence to confirm that the fraction of self-converse mixed graphs tends to zero.

Keywords: Mixed graph, Digraph, Self-converse

1 Introduction

With the rising interest in spectral characterization of mixed graphs and some of their generalizations came an interesting question, concerning the existence of a fairly obvious pairs of cospectral mixed graphs. At the heart of this issue is the fact that a mixed graph and its converse, obtained from the former by reversing all of the oriented edges, are typically encoded by matrices that are each other’s conjugate transpose. In other words, two mixed graphs that may not be equivalent, are almost trivially cospectral. Thus, in order for a mixed graph to be determined by its spectrum in the traditional way [2], it must be isomorphic to its converse; such mixed graphs are said to be self-converse [1].

This then raises the following question: how rare are self-converse mixed graphs? In [7], numerical evidence (see Table 1, below) suggesting that the fraction of self-converse mixed graphs converges to zero as the number of vertices $n$ goes to infinity was provided, although a formal proof to this claim has not appeared yet. Specifically, while the counting polynomials by [3, 4] are quite easily evaluated, they are relatively unwieldy objects to work with, for arbitrary $n$. In this note, we will present a simple proof, to formally show the desired result.

2 Main result

We recall some terminology. Let $\Gamma = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq \binom{V}{2}$. A mixed graph $X$ is obtained from $\Gamma$ by orienting each edge in $A \subseteq E(\Gamma)$ in some direction; the collection of undirected edges is denoted $E(X)$. $\Gamma$ is said to be the underlying graph of $X$, and the symmetric subgraph $G(X)$ of $X$ is obtained by removing $A(X)$ from $X$.

Two (mixed) graphs $X$ and $Y$ are said to be isomorphic if there exists a bijection $f : V(X) \rightarrow V(Y)$ such that $uv \in A(X)$ if and only if $f(u)f(v) \in A(Y)$, and $\{u, v\} \in E(X)$ if and only if $\{f(u), f(v)\} \in E(Y)$. In case $X$ is mapped onto itself, $f$ is called an automorphism. The converse $X^c$ of $X$ is obtained from $X$ by reversing the direction of every arc in $A(X)$, and $X$ is said to be self-converse if $X^c$ is equal to $X$, up to isomorphism.

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Finally, we recall the Erdős-Rényi random graph \( \Gamma(n,p) \), and its natural mixed analog \( X(n,p) \). \( \Gamma(n,p) \) is the order-\( n \) graph such that every edge occurs with probability \( p \). That is, \( \Pr(\{u,v\} \in E) = p \). Accordingly, \( X(n,p) \) is the order-\( n \) mixed graph whose arcs \( uv \) occur with probability \( p \); if both arcs \( uv \) and \( vu \) occur, we say instead that the edge \( \{u,v\} \) occurs. Lastly, \( X(n,p) \) is said to be asymmetric if it has no non-identity automorphism.

The key argument used in the proof of the main result is the notion that almost all symmetric subgraphs of a random mixed graph \( X(n,1/2) \) have no nontrivial automorphism. For completeness, a proof of this essentially well-known fact for the desired Erdős-Rényi graph \( \Gamma(n,p = 1/4) \) is included, below. For sufficiently large \( n \), the following lemma should be clear.

**Lemma 2.1.** Let \( \Gamma = \Gamma(n,1/4) \) and \( \epsilon > 0 \) be arbitrarily small. For \( n \) sufficiently large, the vertices of \( \Gamma \) have degree at least \( \frac{2}{3}(1-\epsilon) \) and at most \( \frac{2}{3}(1+\epsilon) \) common neighbors, with high probability.

Now, the following is an easy adaptation from [6, Thm. 3.1].

**Theorem 2.2 ([6]).** The probability that \( \Gamma(n,1/4) \) is asymmetric tends to 1 as \( n \to \infty \).

**Proof.** Let \( V = \{1,2,\ldots,n\} \) be the vertex set of \( \Gamma = \Gamma(n,1/4) \) and let \( f : V \to V \) be an automorphism such that \( f(x) = y \) for some vertices \( x \neq y \). Let \( M = \{v \in V : f(v) \neq v\} \) be the set of vertices that are moved by \( f \). Moreover, let \( V' = \binom{V}{2} \), and let \( f' : V' \to V' \) be the permutation defined by \( f'(\{u,v\}) = \{f(u),f(v)\} \).

By Lemma 2.1, for sufficiently large \( n \), there exist at least \( \lceil \frac{n}{8}(1-\epsilon) - \frac{n}{8}(1+\epsilon) \rceil = \lceil \frac{n}{8}(1-3\epsilon) \rceil \) vertices that are connected by an edge to \( x \) but not to \( y \). All of these vertices are moved by the automorphism \( f \). Therefore, \( |M| \geq cn \) for \( c = (1-3\epsilon)/8 \) with \( \epsilon \) small. Thus the number of pairs of vertices that are moved by this automorphism is at least \( \binom{cn}{2} - n \geq c'n^2/2 \).

Combining the above, it follows that the probability that \( \Gamma(n,1/4) \) has a non-identity automorphism is at most

\[
\frac{n!2(\epsilon^2/2)\cdot c'n^2/2}{2(\epsilon^2/2)} \leq \frac{n^n}{2^{c'n^2/2}},
\]

which tends to 0 as \( n \to \infty \). Indeed, note that

\[
\log \left( \frac{n^n}{2^{c'n^2/2}} \right) = n \log n - \frac{1}{2}c'n^2 \log 2 \to -\infty, \quad n \to \infty
\]

for all \( c' > 0 \).

The next result now follows naturally, by observing that any relabeling of the vertices that maps a mixed graph \( X \to X^c \) simultaneously maps its symmetric subgraph onto itself. Indeed, since the latter implies with high probability that said mapping is, in fact, the identity mapping, a contradiction follows.

**Proposition 2.3.** The probability that \( X(n,1/2) \) is is self-converse tends to zero as \( n \to \infty \).

**Proof.** Let \( n \to \infty \), and let \( X \) be an order-\( n \) mixed graph whose symmetric subgraph is \( G = G(X) \). If \( X = X(n,1/2) \), then \( G \) is the Erdős-Rényi graph with edge probability \( 1/4 \). By Theorem 2.2 \( G \) has no nontrivial automorphism with probability tending to 1. Now, since any isomorphism from \( X \) to \( X^c \) is an automorphism of \( G \), said isomorphism must be the identity map. However, with a probability tending to 1, there is a pair \( (x,y) \in V \times V \) such that \( X \) contains the arc \( xy \) but not its converse arc \( yx \). Therefore, the identity map is no isomorphism from \( X \) to \( X^c \) (with high probability), thus yielding a contradiction.
One should be somewhat mindful of what is being counted. Proposition 2.3 implies that the fraction of self-converse labeled mixed graphs tends to zero, whereas we are interested in its unlabeled counterpart, i.e., the fraction of all non-isomorphic mixed graphs. Note the significant distinction: any mixed graph with only the identity automorphism has \( n! \) labeled versions, whereas (e.g.) the complete graph only has one. In other words, the former is weighted much more heavily than the latter, by a probabilistic argument. Fortunately, this does not invalidate the approach. In their extensive book, Harary and Palmer [5] prove that almost all graphs of order \( n \) can be labeled in \( n! \) ways, and observe:

**Theorem 2.4** ([5]). Most labeled graphs have property “\( P \)” if and only most unlabeled graphs have property “\( P \)”.

It should be clear that the argumentation would directly carry over to mixed graphs. Hence, the desired result follows from Proposition 2.3.

**Proposition 2.5.** The fraction of order-\( n \) self-converse mixed graphs tends to zero as \( n \to \infty \).

### 3 Convergence rate

To give some idea as to the rate at which the fraction of self-converse mixed graphs tends to zero, we include Table 1 from [7], below. Here, \( f(n) \) denotes said fraction of the non-isomorphic mixed graph of order \( n \), obtained by evaluation of counting polynomials from [3, 4].

| \( n \) | \( f(n) \) |
|-------|----------|
| 3     | 6.25\cdot 10^{-1} |
| 4     | 3.21\cdot 10^{-1} |
| 5     | 7.36\cdot 10^{-2} |
| 6     | 9.87\cdot 10^{-3} |
| 7     | 6.16\cdot 10^{-4} |
| 8     | 2.20\cdot 10^{-5} |

| \( n \) | \( f(n) \) |
|-------|----------|
| 9     | 3.89\cdot 10^{-7} |
| 10    | 3.79\cdot 10^{-9} |
| 11    | 1.85\cdot 10^{-11} |
| 12    | 4.89\cdot 10^{-14} |
| 13    | 6.50\cdot 10^{-17} |
| 14    | 4.58\cdot 10^{-20} |

| \( n \) | \( f(n) \) |
|-------|----------|
| 15    | 1.63\cdot 10^{-23} |
| 16    | 3.06\cdot 10^{-27} |
| 17    | 2.90\cdot 10^{-31} |
| 18    | 1.43\cdot 10^{-35} |
| 19    | 3.59\cdot 10^{-40} |
| 20    | 4.64\cdot 10^{-45} |

Table 1: The fraction \( f(n) \) of mixed graphs of order \( n \) that is self-converse.

### Acknowledgements

The author would like to express his thanks to Oleg Verbitsky, for the useful observation that formed the core idea of this note.

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