On the Decomposition of the Laplacian on Metric Graphs

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Abstract

We study the Laplacian on family preserving metric graphs. These are graphs that have a certain symmetry that, as we show, allows for a decomposition into a direct sum of one-dimensional operators whose properties are explicitly related to the structure of the graph. Such decompositions have been extremely useful in the study of Schrödinger operators on metric trees. We show that the tree structure is not essential, and moreover, obtain a direct and simple correspondence between such decompositions in the discrete and continuum case.

1 Introduction

The study of Schrödinger operators on graphs has drawn a considerable amount of attention in the past few decades. So much so, that any attempt at a short comprehensive review is doomed to fail. We can only refer the reader to a few representative surveys and collections [7, 8, 21, 34]. Besides arising naturally in many physical contexts, the setting of graphs offers a wide array of examples where a variety of mathematical phenomena related to the effects of geometry on spectral properties may be studied. Of particular recent interest is the setting of continuum (aka ‘metric’) graphs. In this setting, a graph is seen as a one dimensional simplicial complex, where the line segments have lengths, and are glued to each other at the relevant vertices. Functions are defined on the line segments, and the operator studied is the one dimensional Laplacian on the line segments, with some prescribed boundary conditions at the vertices. This is the model that will be at the focus of our attention here. We shall describe it formally in the next section.

While originating in chemistry and physics in the study of molecules and mesoscopic systems (such as waveguides) (see e.g. [8] and references therein), such continuum models (also known as quantum graphs) have drawn the attention of the spectral theory community and have served as a platform for the study of various topics. These include trace formulas in quantum chaos [22], isospectrality and its association with geometry [24], Anderson localization and extended states [2, 25], Hardy inequalities [19, 35], eigenvalue estimates [20] and others.

A useful method in the context of infinite metric trees (i.e. connected graphs with no cycles), that has been applied in several of the works mentioned above, was introduced by Naimark and Solomyak in [35]. This method requires the tree to be spherically symmetric around a particular vertex (the root) and involves a decomposition of the operator into a direct sum of one-dimensional operators whose properties are explicitly related to the structure of the graph. Such decompositions have been extremely useful in the study of Schrödinger operators on metric trees. We show that the tree structure is not essential, and moreover, obtain a direct and simple correspondence between such decompositions in the discrete and continuum case.

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operators whose structure is directly related to the structure of the tree and to the boundary conditions at the vertices. A similar method exists in the discrete case, where the graph is considered as a combinatorial object and the operator studied is the discrete Laplacian or the adjacency matrix (see, e.g., [3, 10, 11, 13]). While the similarity between the decomposition in the continuous and discrete case is clear and lies in the exploitation of the symmetry properties of the graph, it is important to note there are essential technical differences. Whereas the discrete case involves studying cyclic subspaces generated by specially chosen functions, the continuum case (as presented in [35, 40]) involves defining the relevant invariant subspaces directly and relies heavily on the structure of the tree.

It has recently been realized in [14] that in the discrete case, the tree structure is not essential for this decomposition. It is in fact possible to carry out this procedure for a more general class of graphs that we call ‘family preserving’ and whose definition we give below (see Definition 2.2). The objective of this work is to extend this decomposition for family preserving graphs to the continuum case of metric graphs as well. Since, as remarked above, the standard decomposition technique for metric trees relies heavily on the tree structure, this is not a trivial task. We will therefore approach this task by finding a direct translation of the discrete decomposition to the decomposition in the metric case. Thus as an added bonus, we describe an algorithm for obtaining an invariant space decomposition for metric graphs from the decomposition for the underlying combinatorial structure. As we show, this algorithm allows one to obtain the original Naimark-Solomyak decomposition for metric trees from the decomposition for discrete trees.

The rest of this paper is structured as follows. After presenting some basic definitions, we describe our main result (Theorem 2.12) in the next section. As the proof of Theorem 2.12 relies on a connection between the discrete and continuous case, we devote the first two subsections of Section 3 to some preliminary results on this connection, presenting the proof in Section 3.3. While Theorem 2.12 is the main abstract result, the raison d’être of this paper lies in Section 4, where we describe the structure of the components in the decomposition and their relation to the structure of the graph. Section 5 presents a demonstration that our approach reduces to the Naimark-Solomyak approach when Γ is a tree and an application to the spectral analysis of metric antitrees.

## 2 Definitions and Statement of the Main Result

### 2.1 Some Definitions

We begin with some definitions pertaining to the combinatorial structure of a graph. We shall add the continuous structure later. A rooted graph is a graph $G = (V(G), E(G))$, with a special vertex $o \in V(G)$, which is called the root. We assume that $V(G), E(G)$ (the vertex and edge set, respectively) are infinite, and that for every $v \in V(G)$, $\text{deg}(v) < \infty$, where $\text{deg}(v)$ is the degree of $v$ (=the number of edges incident to $v$). We also assume throughout that our graphs are simple (i.e. there are no multiple edges or loops) and connected.

For every $v \in V(G)$, we denote the set of edges incident to $v$ by $E_v$. A path of length $k$ between two vertices, $v, w \in V(G)$ is a $k$-tuple of vertices $(v = v_1, v_2, v_3 \ldots v_k = w)$ such that for each $j = 1, \ldots, (k-1), (v_j, v_{j+1}) \in E(G)$. The set of all shortest paths between two vertices $u, v \in V(G)$ will be

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1. Actually study a slightly more general class of graphs that they call ‘path commuting’, but as the definition is considerably more involved and less intuitive and since all relevant examples are family preserving we have decided to prefer here simplicity to generality and restrict our attention to family preserving graphs.
denoted as \( \langle u, v \rangle \). For \( u, v, w \in V(G) \), we denote \( u \in \langle v, w \rangle \) if \( u \) lies on some shortest path between \( v \) and \( w \). Given a graph, the relation \( u \leq v \iff u \in \langle o, v \rangle \) induces a partial order on the vertices, with \( o \) being the minimal element. For \( v \in V(G) \), we define \( gen(v) \) to be the length (i.e. the number of steps) of some shortest path between \( v \) and \( o \) and \( G_v \) to be the maximal chain (with respect to \( \leq \)) which contains \( v \). Define \( S_n = \{ v \in V(G) : gen(v) = n \} \) (namely, \( S_n \) is the set of vertices with distance \( n \) from the root). We assume throughout that for any \( n \), there are no edges between vertices in \( S_n \). With this assumption and the partial order structure induced on a graph, an edge \( (u, v) = e \in E(G) \) has an initial vertex, namely the vertex closer to \( o \), which is denoted by \( i(e) \), and a terminal vertex, denoted by \( t(e) \). We also define \( gen(e) = gen(i(e)) \).

The discrete Laplacian on a graph \( G \) is the densely defined operator \( \Delta_d : D(\Delta_d) \to \ell^2(G) \), defined by \( \Delta_d \phi(x) = \sum_{x \sim y} (\phi(x) - \phi(y)) \), where \( x \sim y \) means that \( x \) and \( y \) are neighbors, and \( D(\Delta_d) = \{ \phi \in \ell^2(G) : \Delta \phi \in \ell^2(G) \} \). \( D(\Delta_d) \) is dense because it contains all of the functions with compact (finite) support.

In order to discuss some symmetry properties of a graph, one would want to be able to say when any two vertices of the same sphere (i.e. two vertices in \( S_n \)) “look alike”. The definition of a rooted graph automorphism serves that purpose.

**Definition 2.1.** A rooted graph automorphism is a bijection \( \tau : V(G) \to V(G) \) such that \( \tau(o) = o \), and \( (\tau(v), \tau(u)) \in E \iff (v, u) \in E \).

A rooted graph \( G = \langle V(G), E(G) \rangle \) will be called spherically symmetric if for every \( n \in \mathbb{N} \) and for every \( v, u \in S_n \), there exists a rooted graph automorphism \( \tau \), such that \( \tau(v) = u \). In other words, the group of all rooted graph automorphisms on \( G \) acts transitively on \( S_n \).

Two vertices \( u, v \in S_n \) will be called forward neighbors if there exists some \( w \in S_{n+1} \) such that \( (v, w), (u, w) \in E \). Similarly, \( u, v \in S_n \) will be called backward neighbors if there exists some \( w \in S_{n-1} \) such that \( (v, w), (u, w) \in E \). We can now define the type of symmetry we need.

**Definition 2.2.** A graph \( G \) will be called family preserving if the following conditions hold:

(i) If \( u, v \in S_n \) are backward neighbors, then there exists a rooted graph automorphism \( \tau \), such that \( \tau(u) = v \), and \( \tau|_{S_n - j} = Id \) for all \( n \geq j \geq 1 \).

(ii) if \( u, v \in S_n \) are forward neighbors, then there exists a rooted graph automorphism \( \tau \) such that \( \tau(u) = v \), and \( \tau|_{S_n + j} = Id \) for all \( j \geq 1 \).

**Remark 2.3.** Family preserving graphs were introduced in [14] in the discrete context as graphs on which it is possible to obtain a constructive decomposition of the combinatorial Laplacian into operators over cyclic subspaces. We shall present some examples in Section 5 below, but refer the reader to that paper for a discussion and illustrations. We remark here only that in the case that \( G \) is a tree then Definition 2.2 is equivalent to spherical symmetry, whereas for general graphs, this is a strictly stronger property.

Having presented the basic definitions we need for the underlying combinatorial structure, we can now add a continuous structure on the edges. Consider a rooted graph \( G \) and identify each edge, \( e \in E(G) \), with a non degenerate line segment \( \subseteq \mathbb{R} \). That edge is now a metric space (with the usual metric on \( \mathbb{R} \)), and in particular its length, denoted \( l(e) \), is that of the associated line segment. This metric structure on the edges together with the gluing at the vertices endows the underlying graph with a new metric, denoted by \( d \). To avoid confusion with the underlying combinatorial structure,
we denote this metric space \( \Gamma \) and refer to such a structure as a **metric graph**. For \( x \in \Gamma \) we denote \( |x| = d(x, o) \) (note \( x \) here is not necessarily a vertex of \( G \), the underlying combinatorial graph). In the sequel, when we want to refer to the combinatorial structure of a metric graph, \( \Gamma \), we shall use the notation \( G_{\Gamma} \) (though we may omit the subscript when there is no risk of confusion). For simplicity, we let \( V(\Gamma) = V(G_{\Gamma}) \) and \( E(\Gamma) = E(G_{\Gamma}) \).

**Definition 2.4.** A rooted metric graph \( \Gamma \) will be called spherically homogeneous if for every \( e_1, e_2 \in E(\Gamma) \), \( \text{gen}(e_1) = \text{gen}(e_2) \Rightarrow l(e_1) = l(e_2) \).

In this work, unless stated otherwise, a metric graph \( \Gamma \) will always be spherically homogeneous. For any such graph, the length of an edge and the distance of a vertex from the root depend solely on their generation. Thus, we denote by \( t_k \) the distance of a vertex in \( S_k \) from the root. Also, letting \( l_k = l(e) \) for some \( e \) with \( \text{gen}(e) = k \) and following [35] [40], we may define the height of \( \Gamma \) by

\[
h(\Gamma) = \sum_{k=0}^{\infty} l_k
\]

which may be finite or infinite.

**Definition 2.5.** A rooted metric graph \( \Gamma \) will be called family preserving if it is spherically homogeneous and if \( G_{\Gamma} \) is family preserving.

The underlying metric structure induces a measurable structure on \( \Gamma \). We denote the relevant \( \sigma \)-algebra by \( \mathcal{B} \). The measure of a set \( E \in \mathcal{B} \) is \( \sum_{e \in E(\Gamma)} \lambda(e \cap E) \), where \( \lambda \) is the Lebesgue measure on \( e \).

We will denote that measure by \( \mu \). \( L^2(\Gamma) \) is the space of all measurable functions (with respect to \( \mathcal{B} \)) \( f : \Gamma \to \mathbb{C} \) such that \( \int_{\Gamma} |f|^2 d\mu < \infty \).

Given a metric graph \( \Gamma \), we denote by \( \mathcal{H} \) the space \( L^2(\Gamma) \). Recall that given a domain \( \Omega \subseteq \mathbb{R}^d \) and \( k \in \mathbb{N} \), \( H^k(\Omega) \) is the space of all functions \( f \in L^2(\Omega) \) for which all of the weak derivatives up to order \( k \) are in \( L^2(\Omega) \).

We define a quadratic form \( q : Q \times Q \to \mathbb{C} \) by

\[
Q = \{ f \in L^2(\Gamma) \text{ s.t. } f \text{ is continuous, } f|_e \in H^1(e) \text{ for every } e \in E(\Gamma) \text{ if } f(o) = 0, \text{ and } \int_{\Gamma} |f'(x)|^2 d\mu(x) < \infty \}
\]

and for \( f, g \in Q \), \( q(f, g) = \int_{\Gamma} f \overline{g} d\mu \). The (Dirichlet-Kirchhoff) Laplacian, \( \Delta^\Gamma \), on \( \Gamma \) is the self adjoint operator associated with this quadratic form. The domain of \( \Delta^\Gamma \) is the space of all functions \( f \) such that \( f(o) = 0 \), \( f \) is continuous, \( f|_e \in H^2(e) \) for every \( e \in E(\Gamma) \), \( \int_{\Gamma} |f''(x)|^2 dx < \infty \), and for every \( v \in V(\Gamma) \), \( \sum_{u \sim v} (f|_[u,v])'(v) = 0 \), where \([u,v] \) is the line segment connecting \( u \) and \( v \). These conditions are called the Kirchhoff boundary conditions [35]. When it is clear from the context, we will omit the subscript and superscript and just write \( \Delta \).

### 2.2 The Main Result

Our main result is a decomposition of the Laplacian into invariant subspaces described by the structure of the graph. These spaces are generated by functions arising in the decomposition of the corresponding discrete structure. The discrete decomposition provides the skeleton for the decomposition in the metric case. Thus, before describing our main result, we need to recall the result in the discrete case.
Let $G$ be a family preserving graph. Recall that $G_v$ is the maximal chain containing $v$ with respect to the order relation $\preceq$ introduced in the previous subsection. Given $n,j \in \mathbb{N}$, introduce the following operators:

$$E_n : \ell^2(S_n) \rightarrow \ell^2(S_{n+1}), \ E_n(\phi)(v) = \sum_{x \in S_n \cap G_v} \phi(x) = \sum_{x \in S_n, x \sim v} \phi(x) \quad (2.1)$$

$$\Lambda_{n,j} : \ell^2(S_n) \rightarrow \ell^2(S_n), \ \Lambda_{n,j} = E_n^T \cdots E_{n+j}^T \cdot E_n \cdots E_{n-j} \quad (2.2)$$

and if $j \leq n$, then we may also define

$$\Lambda_{n,j} : \ell^2(S_n) \rightarrow \ell^2(S_n), \ \Lambda_{n,j} = E_{n-1} \cdots E_{n-j} \cdot E_{n-j}^T \cdots E_{n-1}^T \quad (2.3)$$

A decomposition for the Laplacian on family preserving graphs was presented in [14].

**Theorem 2.6.** ([14] Theorem 2.6) Let $G$ be a family preserving graph. Then $\ell^2(G) = \bigoplus_{r=0}^{\infty} H_r$ where $H_r$ is invariant under $\Delta_d$ and such that:

(i) For every $r$ there exists $n(r)$ and a vector $\phi^r_0$ such that $\text{supp}(\phi^r_0) \subseteq S_{n(r)}$, and $H_r = \text{span}\{\phi^r_0, \Delta_d \phi^r_0, \Delta_d^2 \phi^r_0, \ldots\}$.

(ii) The set $\{\phi^r_0, \phi^1_0, \ldots\}$ obtained from $\{\phi^r_0, \Delta_d \phi^r_0, \Delta_d^2 \phi^r_0, \ldots\}$ by applying the Gram-Schmidt process has the property that $\text{supp}(\phi^r_k) \subseteq S_{n(r)+k}$ for every $k \geq 0$.

Furthermore, for every $r,j \in \mathbb{N}$, $\phi^r_0$ is an eigenfunction of $\Lambda_{n(r),\pm j}$.

**Remark 2.7.** Throughout this work, the notation $\phi^r_n$ will refer to the functions presented in Theorem 2.6 (ii).

**Remark 2.8.** The last statement, about $\phi^r_0$ being an eigenfunction of $\Lambda_{n(r),\pm j}$, is not part of the statement of Theorem 2.6 in [14]. However, it is shown in its proof. We include it here in the statement for simplicity of reference.

**Remark 2.9.** It also follows from the proof of [14] Theorem 2.6 that for every $k \in \mathbb{N}$, there exists $c \in \mathbb{C}$ such that $\Delta_d^k(\phi^r_0)|_{S_{n+k}} = c\phi^r_k$.

Now let $G$ be a family preserving metric graph and $G = G_\Gamma$ the associated discrete graph. For $x \in \Gamma$ define $G_x$ to be $G_{i(e)}$ where $x$ lies on the edge $e$ (recall that $i(e)$ is the initial vertex of $e$). Note that for any $v \in V(G)$ and $0 < t < h(\Gamma)$ the set $G_u \cap \{x \in \Gamma : |x| = t\}$ is finite. Moreover, if $|u| = |v|$ then, by symmetry, the size of this set is equal to the size of $G_u \cap \{x \in \Gamma : |x| = t\}$. For $|v| = n$ we denote this number by $g_n(t)$. Finally, we let $g_0(t) = g_0(t) = \# \{x \in \Gamma : |x| = t\}$.

The first step in translating the discrete decomposition into a continuous one is defining a procedure to obtain a function on the metric graph from one defined on the discrete graph. We do this by ‘spreading’ the values of the function over the graph, taking into account the symmetry. To be precise, given $n \in \mathbb{N}$ and $\phi \in \ell^2(S_n)$, let $h^\phi : \Gamma \rightarrow \mathbb{C}$ be defined by

$$h^\phi(x) = \frac{1}{\sqrt{g_n(|x|) \cdot |G_x \cap S_n|}} \sum_{v \in G_x \cap S_n} \phi(v) \quad (2.4)$$

Next we want to use this procedure to define projection operators based on the functions $\phi^r_n$ above. For any $f : \Gamma \rightarrow \mathbb{C}$, let $f_t \in C^0(t)$ be the vector whose entries are the values $f$ takes on points with distance $t$ from the root (with respect to some fixed ordering of these points). For $\phi \in \ell^2(S_n)$, let
$P_{\phi}: L^2(\Gamma) \to L^2(\Gamma)$ be defined by
$$P_{\phi}(f)(x) = \left\langle f_{|x|}, h_{|x|}^{\phi} \right\rangle h^{\phi}(x)$$  \hfill (2.5)
where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^{g_n(|x|)}$. We shall show in the next section that for every $r, n \in \mathbb{N}$, $P_{\phi^n}$ is in fact an orthogonal projection (with image in $L^2(\Gamma)$). We define the space $F_{r,n}$ to be the image of the orthogonal projection $P_{r,n} := P_{\phi^n}$. The decomposition presented in this work will include the image of $P_{r,0}$ for every $r \in \mathbb{N}$. To abbreviate, we shall denote $h^{\phi^n} = h^{r,n}$, $h^{r,0} = h^r$, $P_{r,0} = P_r$, and $F_{r,0} = F_r$.

We want to use the spaces $F_r$ to decompose $\Delta$ on $\Gamma$. There is, however, a ‘local dimension counting’ issue: while functions on $\Gamma$ are determined by the values they take on edges, functions on $G_\Gamma$ are determined by their values on vertices. If $G$ is a tree, this is not an issue since the number of edges of a particular generation is always equal to the number of vertices of the next generation. Generally, however, the local number of vertices and edges does not need to be the same (think of antitrees described in Section 5.2). In order to deal with this problem, we first restrict our attention to graphs that have no edges between vertices of the same generation. In addition, we need the following

**Definition 2.10.** A metric graph $\Gamma$ will be called **locally balanced** if for every $n \in \mathbb{N}$,

$$\#\{e \in E : \text{gen}(e) = n\} = \#\{v \in V : \text{gen}(v) = n\}$$

or

$$\#\{e \in E : \text{gen}(e) = n\} = \#\{v \in V : \text{gen}(v) = n + 1\}$$

It is intuitively clear that any graph can be made into a locally balanced graph by adding vertices in the middle of ‘bad’ edges. Since such vertices come with Kirchhoff boundary conditions, this makes no difference as far as $\Delta$ is concerned. Thus

**Proposition 2.11.** Let $\Gamma$ be a metric family preserving graph, then there exists another family preserving graph $\overline{\Gamma}$ such that the pair $(\Gamma, \Delta_\Gamma)$, $(\overline{\Gamma}, \Delta_{\overline{\Gamma}})$ are unitarily equivalent, and $\overline{\Gamma}$ is locally balanced.

A proof is given in the appendix. Now, let $\Gamma$ be a metric family preserving graph and let $\overline{\Gamma}$ be a corresponding locally balanced family preserving graph. Note that although $\Delta_\Gamma$ is unitarily equivalent to $\Delta_{\overline{\Gamma}}$, the spaces $F_r$ for each one of these graphs may be different.

We are finally ready to state our main result.

**Theorem 2.12.** Let $\Gamma$ be a locally balanced family preserving metric graph. Assume further that there are no edges between vertices of the same generation in $\Gamma$. Then the subspaces $(F_r)_{r \in \mathbb{N}}$ form a decomposition of $L^2(\Gamma)$, which reduces the Laplacian. Meaning:

(i) $L^2(\Gamma) = \bigoplus_r F_r$.

(ii) For every $r$, the projection $P_r$ onto $F_r$ commutes with the Laplacian, i.e. $P_r(D(\Delta)) \subseteq D(\Delta)$, and $P_r \Delta = \Delta P_r$ on $D(\Delta)$.

(iii) $\Delta(F_r \cap D(\Delta)) \subseteq F_r$ for each $r \in \mathbb{N}$.

3 Proof of Theorem 2.12

Before proving Theorem 2.12, we need some results regarding the discrete structure of a family preserving graph, and some properties of the functions presented in Theorem 2.6.
3.1 The Discrete Structure of a Family Preserving Graph

In order to express the type of symmetry family preserving graphs have, we will first need some definitions.

Definition 3.1. Let \( n, k \in \mathbb{N} \) and let \( v, u \in S_n \). A \( k \)-forward path from \( v \) to \( u \) is a path \((x_0, \ldots, x_k, \ldots, x_{2k})\) of length \( 2k \) such that \( x_0 = v, x_{2k} = u \), and \( \text{gen}(x_j) = n + j \) for all \( j \leq k \).

Similarly, a \( k \)-backward path from \( v \) to \( u \) is a path \((x_0, \ldots, x_k, \ldots, x_{2k})\) of length \( 2k \) such that \( x_0 = v, x_{2k} = u \), and \( \text{gen}(x_j) = n - j \) for all \( j \leq k \).

A \( k \)-forward path between \( x, y \in S_n \) is a path that starts at \( x \), takes \( k \) steps forward (i.e. away from the root), and then takes \( k \) steps backwards and reaches \( y \). A \( k \)-backward path starts at \( x \), takes \( k \) steps towards the root and then takes \( k \) steps back to \( y \).

Definition 3.2. Let \( n, k \in \mathbb{N} \) and let \( u \in S_n, v \in S_{n+k} \). A descending path between \( u \) and \( v \) is a path \((x_0 = u, \ldots, x_k = v)\), where \( \text{gen}(x_i) = \text{gen}(x_{i-1}) + 1 \) for every \( 1 \leq i \leq k \). An ascending path is a path \((y_0 = v, \ldots, y_k = u)\) where \( \text{gen}(y_i) = \text{gen}(y_{i-1}) - 1 \) for every \( 1 \leq i \leq k \).

We will use the following lemma, proved in [14].

Lemma 3.3. ([14] Lemma 3.4) Let \( G \) be a family preserving graph. Let \( x, y \in S_n \), and assume there is a \( k \)-forward path between \( x \) and \( y \). Then there is a rooted graph automorphism \( \tau \), such that \( \tau(x) = y \), and \( \tau|_{S_{n+k}} = \text{Id} \).

Similarly, assume there is a \( k \)-backward path between \( x \) and \( y \). Then there is a rooted graph automorphism \( \tau \), such that \( \tau(x) = y \), and \( \tau|_{S_{n-k}} = \text{Id} \).

Corollary 3.4. Let \( G \) be a family preserving graph. Let \( x, y \in S_n \), and assume there exists a \( k \)-forward path between \( x \) and \( y \). Assume also that \( x_k = v \in S_{n+k} \). Then the number of descending paths between \( x \) and \( v \) is equal to the number of descending paths between \( y \) and \( v \).

Similarly, assume there exists a \( k \)-backward path between \( x \) and \( y \) and assume that \( x_k = v \in S_{n-k} \). Then the number of ascending paths between \( x \) and \( v \) is equal to the number of ascending paths between \( y \) and \( v \).

Proof. We will prove the first part. The second one is analogous. Let \( \tau \) be a rooted graph automorphism such that \( \tau(x) = y \), and \( \tau|_{S_{n+k}} = \text{Id} \). Then \( \tau \) takes every descending path between \( x \) and \( v \) to a descending path between \( y \) and \( v \). The fact that \( \tau \) is bijective implies that this correspondence is also bijective.

Lemma 3.5. Let \( \Gamma \) be a family preserving graph, and let \( n \in \mathbb{N} \). For every \( k \in \{-n, -n+1, \ldots, 0, 1, 2, \ldots\} \) and \( v, u \in S_{n+k} \), either

\[(i) \ (G_v \cap S_n) \bigcap (G_u \cap S_n) = \emptyset\]

or

\[(ii) \ G_v \cap S_n = G_u \cap S_n.\]

Proof. Let \( v, u \in S_{n+k} \). Suppose \((G_v \cap S_n) \bigcap (G_u \cap S_n) \neq \emptyset \), and let \( w \in (G_v \cap S_n) \bigcap (G_u \cap S_n) \). Let \( x \in G_v \cap S_n \). If \( k \geq 0 \) then, by the assumption, there is a \( k \)-forward path between \( x \) and \( w \). Thus by Lemma 3.3 there is a rooted graph automorphism \( \tau \) such that \( \tau(w) = x \), and \( \tau|_{S_{n+k}} = \text{Id} \).
There is a descending path \((w = x₀, x₁, \ldots, xₖ = u)\), which translates under \(τ\) to a descending path \((x = τ(x₀), τ(x₁), \ldots, τ(xₖ) = u)\), which implies that \(x \in G_u \cap S_u\). Since \(x\) was chosen arbitrarily from \(G_v \cap S_n\), we have that \(G_v \cap S_n \subseteq G_u \cap S_n\). It can be similarly proved that \(G_u \cap S_n \subseteq G_v \cap S_n\), and we have \(G_v \cap S_n = G_u \cap S_n\), as required.

If \(k \leq 0\) the proof is similar. \(\Box\)

Lemma 3.6 is the first step in obtaining the orthogonality of the spaces \((F_r)_{r \in \mathbb{N}}\) as a consequence of the orthogonality of the functions in the set \(\{φₙ^r : r, n \in \mathbb{N}\}\).

**Lemma 3.6.** Let \(G\) be a family preserving graph, \(j, n \in \mathbb{N}\), and \(φ \in \ell²(S_n)\). Then \(∀u \in S_{n+j+1}\),

\[
(E_{n+j} \cdots E_n)(φ)(u) = \sum_{x \in S_n \cap G_u} k_u(x)φ(x)
\]

where \(k_u(x)\) is the number of different descending paths from \(x\) to \(u\).

**Proof.** By induction on \(j\). For \(j = 1\), the claim is trivial. Assume the claim is true for \(j = k\). Then for \(j = k + 1\),

\[
(E_{n+k+1} \cdots E_n)(φ)(u) = E_{n+k+1}(E_{n+k} \cdots E_n)(φ)(u) = \sum_{y \in S_{n+k} \cap G_u} \sum_{x \in S_n \cap G_y} k_y(x)φ(x) = \sum_{x \in S_n \cap G_u} k_u(x)φ(x).
\]

**Lemma 3.7.** Let \(G\) be a family preserving graph, \(j, n \in \mathbb{N}\), and \(φ \in \ell²(S_{n+j})\). Then \(∀v \in S_n\),

\[
(E_n^T \cdots E_{n+j}^T)(φ)(v) = \sum_{x \in S_{n+j} \cap G_v} k^v(x)φ(x)
\]

where \(k^v(x)\) is the number of different ascending paths from \(x\) to \(v\).

The proof is symmetric to the proof of Lemma 3.6.

**Remark 3.8.** It is easy to see that by the definition of \(Δ_d\), given \(φ \in \ell²(S_n), k \in \mathbb{N}\), for \(v \in S_{n+k}\) we have \(Δ_k^d(φ)(v) = (E_{n+k-1} \cdots E_n)(φ)(v)\).

The fact that for every \(r, k \in \mathbb{N}\), \(φ^r_k\) is an eigenvector of \(Λ_{n(r),+k}\) means that \(F_r = F_{r,n}\) for every \(n \in \mathbb{N}\). This is proved in the following lemmas, and is necessary in order to prove that for every family preserving metric graph \(Γ\), the spaces \((F_r)_{r \in \mathbb{N}}\) span \(L²(Γ)\). The main idea is that on family preserving graphs, for every \(φ \in \ell²(S_n)\), \(Λ_{n,±k}(φ)\) takes the same values on vertices that share a \(k\)-forward/backward path. This is shown in the following lemma.

**Lemma 3.9.** Let \(G\) be a family preserving graph and let \(v, u \in S_n\) such that there exists a \(k\)-forward path between \(v\) and \(u\). Then for every \(φ \in \ell²(S_n)\), \(Λ_{n,+k}(φ)(v) = Λ_{n,+k}(φ)(u)\). Furthermore, denote by \(K\) the set of vertices in \(S_n\) that share a \(k\)-forward path with \(v\) (or \(u\)). Then \(Λ_{n,+k}(φ)|_K ≡ c \sum_{x \in K} φ(x)\), where \(c\) is some natural number greater than 1, which depends only on \(n, k\). The analogous statement for \(Λ_{n,-k}\) and for \(v, u \in S_n\) that have a \(k\)-backward path between them is also true.
Proof. We prove the lemma for \( v, u \in S_n \) that share a \( k \)-forward path. The proof for the case that they share a \( k \)-backward path is symmetric.

By Lemma 3.5 we have
\[
G_v \cap S_{n+k} = G_u \cap S_{n+k} := A_{v,u}.
\]
In addition, by the same lemma, for every \( w_1, w_2 \in A_{v,u} \),
\[
G_{w_1} \cap S_n = G_{w_2} \cap S_n := B_{v,u}.
\]
Denote \( \tilde{\phi} = E_{n+k} \cdots E_n(\phi) \in \ell^2(S_{n+k}) \). By Lemma 3.6 for every \( w \in A_{v,u} \),
\[
\tilde{\phi}(w) = \sum_{x \in B_{v,u}} k_w(x)\phi(x).
\]
Now, by Corollary 3.4 the factor \( k_w(x) \) in the sum is constant, so we may write
\[
\tilde{\phi}(w) = c_1 \sum_{x \in B_{v,u}} \phi(x)
\]
where \( c_1 \) is \( k_w(x) \) for some \( x \in B_{v,u} \). Now, again by Corollary 3.4 the constant \( c_1 \) is the same for every \( w_1, w_2 \in A_{v,u} \), and that means that \( \tilde{\phi} \) is constant on \( A_{v,u} \). By definition, we have
\[
\Lambda_{n,k}(\phi)(v) = E_n^T \cdots E_{n+k}^T (\tilde{\phi})(v) = \sum_{x \in A_{v,u}} k^v(x)\tilde{\phi}(x).
\]
We apply Corollary 3.4 twice again, and get that \( k^v(x) \) is constant on \( A_{v,u} \), and that constant is the same for every \( w \in B_{v,u} \), so we have
\[
\sum_{x \in A_{v,u}} k^v(x)\tilde{\phi}(x) = \sum_{x \in A_{v,u}} c_2\tilde{\phi}(x) = \sum_{x \in A_{v,u}} k^u(x)\tilde{\phi}(x) = E_n^T \cdots E_{n+k}^T (\tilde{\phi})(u) = \Lambda_{n,k}(\phi)(u)
\]
which proves the first part of the lemma. The second part follows by observing that \( c_1, c_2 \) are natural constants greater than 0, and the same procedure can be applied to every vertex that shares a \( k \)-forward path with \( v \).

\[\square\]

Lemma 3.10. Let \( G \) be a family preserving graph, and let \( r \in \mathbb{N} \). Then for every \( k \in \{-n(r),-n(r)+1,\ldots,0,1,\ldots\} \) and for every \( v \in S_{n(r)+k} \), either \( \sum_{x \in G_v \cap S_{n(r)}} \phi_0(x) = 0 \) or \( \phi_0|_{G_v \cap S_{n(r)}} \equiv c \) for some constant \( c \in \mathbb{C} \).

Proof. We prove the lemma for \( k \geq 0 \), the proof for \( k < 0 \) is symmetric. By Theorem 2.6 \( \phi_0 \) is an eigenvector of the operator \( \Lambda_{n,k} \). Assume \( \Lambda_{n,k}(\phi_0) = \lambda_{n,k}\phi_0 \) for some \( \lambda_{n,k} \in \mathbb{C} \). Let \( x, y \in G_v \cap S_{n(r)} \). The set \( G_v \cap S_{n(r)} \) is exactly the set of vertices in \( S_{n(r)} \) which share a \( k \)-forward path with \( x \) (or with \( y \)), which means (by Lemma 3.9) that \( \Lambda_{n,k}(\phi_0) \) is constant on that set. Combining that with the fact that \( \phi_0 \) is an eigenvector of \( \Lambda_{n,k} \), we get that for every \( x \in G_v \cap S_{n(r)} \),
\[
\lambda_{n,k}\phi_0(x) = \Lambda_{n,k}(\phi_0)(x) = c \sum_{y \in G_v \cap S_{n(r)}} \phi_0(y)
\]
(3.2)
Where \( c \) is a natural number greater than 0. If \( \lambda_{n+k} = 0 \), we get \( \sum_{y \in G_v \cap S_{n(r)}} \phi_0^y(y) = 0 \). Otherwise, dividing (3.4) by \( \lambda_{n+k} \) proves the result.

**Lemma 3.11.** Let \( G \) be a family preserving graph, and let \( r, k \in \mathbb{N} \). Assume that \( \phi_k^r \neq 0 \). Then for every \( v \in S_{n(r)+k} \), \( \phi_0^r \) is constant on \( G \cap S_{n(r)} \).

**Proof.** Let \( v \in S_{n(r)+k} \) be such that \( \phi_k^r(v) \neq 0 \). By Remark 2.9 this means that \( \Delta_d^k(\phi_0^r)(v) \neq 0 \), and by Remark 3.8 and (3.1), this means that \( \sum_{x \in G_v \cap S_{n(r)}} \phi_0^x \neq 0 \). By the previous lemma, we get that \( \phi_0^r \) is constant on \( G_v \cap S_{n(r)} \). By Theorem 2.6 and the second part of Lemma 3.9, \( \phi_0^r \) is an eigenvector of \( \Lambda_{n+r} \) with an eigenvalue \( \lambda_{n+k} \neq 0 \). Let \( w \in S_{n(r)+k} \) be such that \( G_w \cap S_{n(r)} \neq G_v \cap S_{n(r)} \). Assume that \( \phi_0^r \) is not constant on \( G_w \cap S_{n(r)} \), and let \( x \in G_w \cap S_{n(r)} \) be such that \( \phi_0^r(x) \neq 0 \). By Lemma 3.10 \( \sum_{y \in G_w \cap S_{n(r)}} \phi_0^y(y) = 0 \). By the second part of Lemma 3.9 \( \Lambda_{n+k}(\phi_0^r)(x) = c \sum_{y \in G_w \cap S_{n(r)}} \phi_0^y(y) = 0 \).

On the other hand, \( \Lambda_{n+k}(\phi_0^r)(x) = \lambda_{n+k,0}^r \phi_0^r(x) \neq 0 \), which is a contradiction. Thus we have that \( \phi_0^r \) is constant on \( G_w \cap S_{n(r)} \), as required.

**Proposition 3.12.** Let \( G \) be a family preserving graph and \( r, k \in \mathbb{N} \) such that \( \phi_k^r \neq 0 \). Then \( F_r = F_r,k \).

**Proof.** It is sufficient to show that there exists \( c \in \mathbb{C} \) such that for every \( v \in S_{n(r)+k} \), \( \phi_0^r|G_v \cap S_{n(r)}(v) = c \phi_k^r(v) \), and by Remark 2.9 we may examine \( \Delta_d(\phi_0^r) \). By Lemma 3.11 for every \( v \in S_{n(r)+k} \), \( \phi_0^r \) is constant on \( G_v \cap S_{n(r)} \). Furthermore, by (3.2) and remark 3.8 \( \Delta_d(\phi_0^r)(v) = c \sum_{x \in G_v \cap S_{n(r)}} \phi_0^x(v) \), where \( c \) is the number of descending paths from some \( x \in G_v \cap S_{n(r)} \) to \( v \). By spherical symmetry of \( G \), for every \( w \in S_{n(r)+k} \), the number of descending paths from some \( x \in G_w \cap S_{n(r)} \) is exactly \( c \), and we get the result.
where $v_x$ is some representative of $G_x \cap S_n$. Now, note that

$$\langle \phi_0^r, \phi_0^l \rangle_{\ell^2} = \sum_{v \in S_n} \phi_0^r(v)\overline{\phi_0^l(v)}$$

and if we divide $S_n$ to the classes of $G_x \cap S_n$ for $x$ with $|x| = t$, say $S_n = \bigsqcup_{i=1}^k G_{x_i} \cap S_n$, then

$$\sum_{v \in S_n} \phi_0^r(v)\overline{\phi_0^l(v)} = \sum_{i=1}^k (G_{x_i} \cap S_n)\phi_0^r(v_{x_i})\overline{\phi_0^l(v_{x_i})}$$

(3.4)

Now, each summand in the right hand side of (3.4) appears $g_n(t)$ times in (3.3), and we have

$$\langle h_t^r, h_t^l \rangle_{\ell^2} = \langle \phi_0^r, \phi_0^l \rangle_{\ell^2} = 1$$

as required. □

**Lemma 3.14.** Let $\phi \in \ell^2(S_n)$ such that $||\phi||_{\ell^2} = 1$. Then the image of the transformation defined by (2.3) is contained in $L^2(\Gamma)$. Furthermore, $P_\phi$ is an orthogonal projection.

**Proof.** The first part of the lemma follows from the fact that for every $0 \leq t < h(\Gamma)$ we have $||P_\phi(f)_t||_{\ell^2(\Gamma)} \leq ||f_t||_{\ell^2(\Gamma)}$. Additionally, for every $x \in \Gamma$ we have

$$P_\phi(P_\phi(f)(x)) = \left\langle \left\langle f_{[x]}, h_t^\phi \right\rangle_{\ell^2} h_t^\phi, h_t^\phi \right\rangle_{\ell^2} h_t^\phi(x) = \left\langle f_{[x]}, h_t^\phi \right\rangle_{\ell^2} h_t^\phi(x)$$

thus $P_\phi$ is indeed a projection. Finally, we need to show that $P_\phi$ is self adjoint. Let $f \in L^2(\Gamma)$. We need to show that $\langle P_\phi(f), f - P_\phi(f) \rangle_{L^2(\Gamma)} = 0$. Indeed

$$\langle P_\phi(f), f - P_\phi(f) \rangle = \int_{\Gamma} P_\phi(f)(x)(f - P_\phi(f)(x))d\mu(x)$$

$$= \int_0^{h(\Gamma)} \sum_{|x|=t} P_\phi(f)(x)(f - P_\phi(f)(x))dt$$

$$= \int_0^{h(\Gamma)} \left\langle f_{[x]}, h_t^\phi \right\rangle_{\ell^2} h_t^\phi(x)\overline{f(x) - \left\langle f_{[x]}, h_t^\phi \right\rangle_{\ell^2} h_t^\phi(x)}dt$$

$$= \int_0^{h(\Gamma)} \langle f_{[x]}, h_t^\phi h_t^\phi, f_t - \langle f_{[x]}, h_t^\phi \rangle h_t^\phi \rangle dt$$

$$= \int_0^{h(\Gamma)} 0 dt = 0$$

where the next to last equality follows since $||h_t^\phi||_{\ell^2(\Gamma)} = 1$. □

### 3.3 Proof of Theorem 2.12

**Proof.** Let $\Gamma$ be a locally balanced family preserving metric graph.

(i) We first prove orthogonality. Note that for every $t \geq 0$, $(P_t(f))_t$ is the orthogonal projection of $f_t$ to the span of $h_t^r$. In addition, for every $r_1 \neq r_2$, the orthogonality of $\phi_0^r_1$ and $\phi_0^r_2$ implies that $h_t^{r_1}$


and $h^r_\ell$ are also orthogonal in $\mathbb{C}^{n(\ell)}$. So given $f \in F_{r_1}$, $f = P_{r_1}(f)$, $g \in F_{r_2}$, $g = P_{r_2}(g)$, we have that

$$\sum_{|x|=t} P_{r_1}(f)(x)\overline{P_{r_2}(g)(x)} = 0$$

which implies that $\langle f, g \rangle_{L^2} = 0$, as required.

Now, let $f \in L^2(\Gamma)$ such that $\text{supp}(f) \subseteq e = (u, v)$, $\text{gen}(e) = n$. For every $e' \in E(\Gamma)$ such that $\text{gen}(e') = \text{gen}(e)$, define $f_{e'}$ on $e'$ to be the translation of $f$ to the edge $e'$. That is, for $x$ on $e'$, $f_{e'}(x) = f(y)$ where $y \in e$ such that $|y| = |x|$.

Define $H_f$ to be the subspace of $L^2(\Gamma)$ spanned by the set $\{f_{e'} : \text{gen}(e') = \text{gen}(e)\}$. Since $\Gamma$ is locally balanced, either

$$\# \{e \in E : \text{gen}(e) = n\} = \# \{v \in V : \text{gen}(v) = n\}$$

or

$$\# \{e \in E : \text{gen}(e) = n\} = \# \{v \in V : \text{gen}(v) = n + 1\}.$$

Assume first that $\# \{e \in E : \text{gen}(e) = n\} = \# \{v \in V : \text{gen}(v) = n\}$ and let $\{\phi_1, \ldots, \phi_{|S_n|}\} \subseteq \{\phi_i : j, r \in \mathbb{N}\}$ be a basis of $\ell^2(S_n)$. For every $1 \leq i \leq |S_n|$, define $g^i \in H_f$ in the following way:

$$g^i(x) = f_{e_x}(x)\phi_i(x)$$

where $e_x$ is the edge containing $x$, and $v_x = i(e_x)$ is the initial vertex of $e_x$. In other words,

$$g^i = \sum_{\text{gen}(e') = n} \phi_i(i(e))f_{e'},$$

so clearly, $g^i \in H_f$. Now, we have (note that if $\text{gen}(e') = n$ then $|G_x \cap S_n| = 1$)

$$P_{\phi_i}(g^i)(x) = \langle g_{[x]}^i, h^\phi_{\phi_i} \rangle h^\phi_i(x) =$$

$$= \frac{1}{g_{[x]}(x)} \left( \sum_{|y|=|x|} f_{e_y}(y)\overline{\phi_i(v_y)}\overline{\phi_i(v_y)} \right) \phi_i(v_y) =$$

$$= f_{e_x}(x) \left( \sum_{v \in S_n} \phi_i(v)\overline{\phi_i(v)} \right) \phi_i(v_x) =$$

$$= f_{e_x}(x)\phi^i(v_x) = g^i(x)$$

where the third equality follows from the fact that $|x| = |y| \Rightarrow f_{e_x}(x) = f_{e_y}(y)$, and the fourth from the fact that $||\phi_i||_{\ell^2} = 1$. This implies that $g^i \in \text{Im}(P_{\phi_i})$ and so that $g^i \in \oplus F_r$. Moreover, by the independence of the $\phi_i$, it is clear that the functions $\{g^1, \ldots, g^{|S_n|}\}$ are linearly independent. Thus, by dimension considerations it follows that $H_f$ is spanned by $\{g^1, \ldots, g^{|S_n|}\}$. This implies that $H_f \subseteq \oplus F_r$, and in particular $f \in \oplus F_r$. Since linear combinations of such $f$’s are dense in $L^2(\Gamma)$ we obtain $L^2(\Gamma) = \oplus_{r \in \mathbb{N}} F_r$, as required.

The proof in the case that $\# \{e \in E : \text{gen}(e) = n\} = \# \{v \in V : \text{gen}(v) = n + 1\}$ is similar.

(ii) First, since differentiation is a local action and $h^r$ is constant on edges, it is easy to see that $\Delta P_r = P_r\Delta$ on $D(\Delta)$. Moreover, for the same reason differentiability properties of $\varphi$ on the edges are unchanged by $P_r$. Therefore to show that $P_r(D(\Delta)) \subseteq D(\Delta)$ we only need to check the gluing conditions at the vertices. Thus let $\varphi \in D(\Delta)$, $f = P_r(\varphi)$, and let $v \in S_{n(r)+k}$, where $k \in \{-n(r)+1, -n(r), \ldots, 0, 1, \ldots\}$. We divide into cases.
Assume first that $\sum_{u \in G_v \cap S_n(r)} \phi_0^r(u) = 0$ for every $w \in S_n(r)+k$. In that case $k \neq 0$. We treat the case that $k \geq 1$. The proof for $k \leq -1$ is symmetric. Note that for $x \in G_v$ such that $|x| > |v|$, $G_v \cap S_n(r) \subseteq G_x \cap S_n(r)$. Thus, by Lemma 3.10 and the definition of $P_r$, $f$ vanishes on the set \(\{x \in \Gamma \mid |x| \geq |v|\}\).

In order to check the gluing conditions, we first check continuity at $v$. We need to verify that $\lim_{x \to v} f(x) = 0$. For convenience, we denote $g_n(r)(|x|)$ by $g_n(r)$, and $|G_x \cap S_n(r)|$ by $w_n(r)$ since they are constant along segments between vertices.

\[
\lim_{x \to v} f(x) = \lim_{x \to v} \left( \sum_{|y|=|x|} \varphi(y) \cdot \frac{1}{\sqrt{g_n(r) \cdot w_n(r)}} \sum_{w \in G_v \cap S_n(r)} \phi_0^r(w) \right) \cdot \frac{1}{\sqrt{g_n(r) \cdot w_n(r)}} \sum_{u \in G_v \cap S_n(r)} \phi_0^r(u) = \lim_{x \to v} \frac{1}{g_n(r) \cdot w_n(r)} \left( \sum_{|y|=|x|} \varphi(y) \sum_{w \in G_v \cap S_n(r)} \phi_0^r(w) \right) \sum_{u \in G_v \cap S_n(r)} \phi_0^r(u) = \lim_{x \to v} \frac{1}{g_n(r) \cdot w_n(r)} \left( \sum_{s \in S_n(r)+k} \varphi(s) \sum_{y \in G_x} \sum_{w \in G_y \cap S_n(r)} \phi_0^r(w) \right) \sum_{u \in G_v \cap S_n(r)} \phi_0^r(u) = \lim_{x \to v} \frac{1}{g_n(r) \cdot w_n(r)} \left( \sum_{s \in S_n(r)+k} \varphi(s) \sum_{y \in G_x} \sum_{w \in G_y \cap S_n(r)} \phi_0^r(w) \right) \sum_{u \in G_v \cap S_n(r)} \phi_0^r(u) = 0
\]

where the next to last equality follows from continuity of $\varphi$, and the rest follow from changing the order of summation. As for the derivative matching condition, let $e \in E(\Gamma)$ such that $t(e) = v$. Note that $h^r$ is constant on $e$, so for $x(e) \in e$ we denote $h^r(x(e)) = c_e$. Now, for every $x \in e$ $(f|_x)(x) = c_e \left( \varphi_{|x|}, h^r_{|x|} \right)$. Thus, we may write

\[
\sum_{e : t(e) = v} (f|_e)'(v) = \left( \left( \varphi_{|v|}, h^r_{|v|} \right) \right)' \cdot \sum_{e : t(e) = v} c_e
\]

Now note that

\[
\sum_{e : t(e) = v} c_e = 0
\]

as it is some multiple of $\sum_{u \in G_v \cap S_n(r)} \phi_0^r(u)$. This is a consequence of the fact that $G_v \cap S_n(r) = \bigcup_{e : t(e) = v} G_x(e) \cap S_n(r)$ and for every $e_1, e_2 \in E(\Gamma)$ such that $t(e_1) = t(e_2) = v$,

\[
|\{e : G_x(e) \cap S_n(r) = G_x(e_1) \cap S_n(r), t(e) = v\}| = |\{e : G_x(e) \cap S_n(r) = G_x(e_2) \cap S_n(r), t(e) = v\}|\]

which in turn follows from Corollary 3.4. Thus, by our assumption, the above sum is 0 as required.

We now treat the case in which there exists some $w \in S_n(r)+k$ for which $\sum_{u \in G_w \cap S_n(r)} \phi_0^r(u) \neq 0$. This is impossible for $k < 0$ since $H_r$ is orthogonal to $\ell^2(S_n)$ for every $n < n(r)$, and the above sum is a
multiple of $\langle \Delta^{|k|}_q \phi_0^r, \delta_w \rangle$. Thus $k \geq 0$. The fact that the above sum is not 0 implies that $\phi_k^r \neq 0$. Thus, by Lemma 3.12, $F_r = F_{r,k}$. By the uniqueness of the orthogonal projection and by Lemma 3.14 we may write $f = P_{r,k}(\varphi)$. For sufficiently small $\epsilon > 0$, for every $x \in \Gamma$ with $|x| = |v| \pm \epsilon$ it holds that $|G_x \cap S_n(r) + k| = 1$, thus

$$
f^{\phi_k^r}(x) = \frac{1}{\sqrt{g_n(r) + k(|x|)}} \phi_k^r(v_x)
$$

where $v_x \in S_n(r) + k$ and $x$ lies on the edge $e$ for which either $i(e) = v_x$ or $t(e) = v_x$. Thus, for $v \in S_n(r) + k$ and $x \in \Gamma$ which lies on an edge $(u, v) \in E(\Gamma)$, we have $v_x = v$ and so

$$
f(x) = P_{r,k}(\varphi)(x) = \left( \sum_{|y| = |x|} \varphi(y) \frac{1}{\sqrt{g_n(r) + k(|y|)}} \phi_k^r(v_y) \right) \frac{1}{\sqrt{g_n(r) + k(|x|)}} \phi_k^r(v) = 
$$

$$
= \left( \sum_{w \in S_n(r) + k} \phi_k^r(w) \sum_{|y| = |x|, y \in g_n(r) + k(|y|)} \varphi(y) \phi_k^r(v) \right) \phi_k^r(v)
$$

Now, when $x$ tends to $v$, $\varphi(y)$ tends to $\varphi(w)$ and so

$$
\sum_{|y| = |x|, y \in g_n(r) + k(|y|)} \varphi(y) \phi_k^r(v) \rightarrow \varphi(w).
$$

It follows that

$$
P_{r,k}(\varphi)(x) \xrightarrow{x \rightarrow v} \phi_k^r(v) \sum_{w \in S_n(r) + k} \phi_k^r(w) \varphi(w)
$$

which means that $f$ is continuous.

We now prove that the derivative matching condition holds. In order to do so, we divide the edges of which $v$ is a part into two sets. The set of edges which terminate in $v$ will be denoted as $E_t$, and the set of edges emanating from $v$ will be denoted as $E_s$. We also denote $g_n(r) + k(|x|) \equiv g_n(r) + k$ for $|x| < |v|$, and $g_n(r) + k(|x|) \equiv g_1^n(r) + k$ for $|x| > |v|$. Finally, for every edge $e \in E(\Gamma)$ denote by $x(e)$ the selection of some point $x \in e$. Now, we calculate the sum $\sum_{u \sim v} (P_{r,k}(\varphi)(u,v))(x(u,v))$:

$$
\sum_{u \sim v} (P_{r,k}(\varphi)(u,v))(x(u,v)) = \sum_{e \in E_t} (P_{r,k}(\varphi)(e))(x(e)) + \sum_{e \in E_s} (P_{r,k}(\varphi)(e))(x(e)) = 
$$

$$
= \sum_{e \in E_t} \left( \sum_{w \in S_n(r) + k} \sum_{y \in g_n (|y|) \in g_n(r) + k} \frac{1}{\sqrt{g_n(r) + k}} \phi_k^r(w) \varphi(y) \right) \frac{1}{\sqrt{g_n(r) + k}} \phi_k^r(v) + 
$$

$$
+ \sum_{e \in E_s} \left( \sum_{w \in S_n(r) + k} \sum_{y \in g_n (|y|) \in g_n(r) + k} \frac{1}{\sqrt{g_n(r) + k}} \phi_k^r(w) \varphi(y) \right) \frac{1}{\sqrt{g_n(r) + k}} \phi_k^r(v)
$$

Note that if we pick $|x(e_1)| = |x(e_2)| = t_1$ for every $e_1, e_2 \in E_t$ and $|x(e_3)| = |x(e_4)| = t_2$ for every
$e_3, e_4 \in E_x$, then we may write the above sum as

$$
g^0_n(r+k) \left( \sum_{w \in S_{n(r)+k}} \sum_{y \in G_w} \frac{1}{\sqrt{g^0_n(r+k)}} \phi_k^r(w) \varphi(y) \right) \frac{1}{\sqrt{g^0_n(r+k)}} \phi_k^r(v)$$

$$
+ g^1_n(r+k) \left( \sum_{w \in S_{n(r)+k}} \sum_{y \in G_w} \frac{1}{\sqrt{g^1_n(r+k)}} \phi_k^r(w) \varphi(y) \right) \frac{1}{\sqrt{g^1_n(r+k)}} \phi_k^r(v)$$

$$= \left( \sum_{w \in S_{n(r)+k}} \phi_k^r(w) \sum_{y \in G_w} \varphi(y) \right) \phi_k^r(v) +$$

$$+ \left( \sum_{w \in S_{n(r)+k}} \phi_k^r(w) \sum_{y \in G_w} \varphi(y) \right) \phi_k^r(v)$$

$$= \phi_k^r(v) \sum_{w \in S_{n(r)+k}} \phi_k^r(w) \left( \sum_{y \in G_w} \varphi(y) + \sum_{y \in G_w} \varphi(y) \right)$$

Since $\varphi \in D(\Delta)$, by taking the limit $|t_1|$ and $|t_2|$ to $|v|$ and differentiating, we see that the derivative matching condition holds.

(iii) Finally, for every $f \in F_r \cap D(\Delta)$, $f = P_r(f)$, and we have

$$\Delta(P_r(f)) = P_r(\Delta(f))$$

(3.5)

and $P_r(\Delta(f)) \in F_r$ since the image of $\Delta$ is contained in $L^2(\Gamma)$. Thus, we have $\Delta(F_r \cap D(\Delta)) \subseteq F_r$ as required.

4 The Operators $(\Delta |_{F_r})_{r \in \mathbb{N}}$

Let $r \in \mathbb{N}$ and let

$$a_r = \inf \left\{ |x| \mid h^{\delta_0}(x) \neq 0 \right\}$$

and

$$b_r = \sup \left\{ |x| \mid h^{\delta_0}(x) \neq 0 \right\} .$$

Now let $f \in F_r$ (so that $f = P_r(f)$), and let $\varphi_f(x) = (f|_{[x]}, h^r_{[x]}|_{\mathcal{W}(r)})$. Then for every $f \in F_r$, $\text{supp}(\varphi_f) \subseteq \{ x \in \Gamma : |x| \in [a_r, b_r] \}$. Clearly $|x| = |y| \Rightarrow \varphi_f(x) = \varphi_f(y)$. Let $U_r : F_r \rightarrow L^2(a_r, b_r)$ be defined by

$$U_r(f)(t) = \varphi_f(x)$$

(4.1)
for some \( x \in \Gamma \) with \(|x| = t\). By a direct computation, for \( \psi, \eta \in F_r \), we have

\[
\langle \psi, \eta \rangle_{L^2(\mu)} = \int_{r} (\psi|_{x}, h_{r,x}^* \bigotimes_{W} h_{r,x}^*(x)) d\mu(x) = \int_{0}^{r} \varphi_{\psi}(t) \varphi_{\eta}(t) \sum_{|x|=t} h_{r,x}(x) h_{r,x}(x) dt = \int_{a_r}^{b_r} \varphi_{\psi}(t) \varphi_{\eta}(t) dt = \langle U_r(\psi), U_r(\eta) \rangle_{L^2(\mathbb{R})}.
\]

In addition, by the same reasoning as in the proof of Theorem 4.12 (ii), we have \( U_r \Delta = \Delta_r U_r \), where \( \Delta_r \) is the unbounded operator whose domain is \( U_r(F_r \cap D(\Delta)) \), and on that domain, \( \Delta_r(f) = -f'' \).

We conclude that \( \Delta |_{F_r} \) is unitarily equivalent to \( \Delta_r \), which means that \( \Delta \sim \oplus_{r \in \mathbb{N}} \Delta |_{F_r} \sim \oplus_{r \in \mathbb{N}} \Delta_r \). So in order to analyze the spectral properties of \( \Delta \), it is sufficient to examine the operators \( \Delta_r \).

The first step is to examine the domain of the operators \( \Delta_r \). Fix \( r \in \mathbb{N} \). Note first that for fixed \( n \) and every \( x \in \Gamma \), \( |G_x \cap S_n| \) depends only on \(|x|\). For \(|x| = t \), let \( w_n(t) = |G_x \cap S_n| \). Now, by (4.1), and using (2.4), we may write explicitly

\[
U_r(f)(t) = \sum_{|x|=t} f(x) \frac{1}{\sqrt{g_{n(r)}(t) w_{n(r)}(t)}} \sum_{u \in G_x \cap S_{n(r)}} \phi_{\theta}(u).
\] (4.2)

Recall that \( t_k \) is the distance in \( \Gamma \) of a vertex \( S_k \) from the root \( o \). Let \( k \in \mathbb{N} \) be such that \( a_r = t_k \). If \( b_r = \infty \), let \( l = \infty \), and otherwise let \( l \) be the maximal \( j \in \mathbb{N} \) such that \( t_j \leq b_r \). Denote \( A_r = \{ k, k + 1, \ldots, l - 1 \} \) and divide the segment \( (a_r, b_r) \) into the disjoint collection of segments

\[
\{(t_j, t_{j+1})\}_{j \in A_r} = \{I_j\}_{j \in A_r}.
\]

By the fact that \( \text{supp}(\phi_{\theta}) \subseteq S_{n(r)+k} \), one can conclude that for every \( n < n(r) \), and every \( v \in S_n \), \( \sum_{u \in G_v \cap S_{n(r)}} \phi_{\theta}(u) = 0 \). Taking this and the definition of \( P_r \) into consideration, we conclude that \( A_r = \{ n(r)-1, n(r), \ldots, l-1 \} \).

Claim 4.1. For \( n(r) \leq k \in A_r \), and for \( v \in S_k \), \( \phi_{\theta} \) is constant on \( G_v \cap S_{n(r)} \). For \( v \in S_{n(r)-1} \), it holds that

\[
\sum_{u \in G_v \cap S_{n(r)}} \phi_{\theta}(u) = 0.
\]

Proof. The second part of the claim follows from the fact that \( H_r \) is orthogonal to \( \ell^2(S_{n(r)-1}) \). As for the first part, it is enough to show that there exists \( v \in S_k \) such that \( \sum_{u \in G_v \cap S_{n(r)}} \phi_{\theta}(u) \neq 0 \). Indeed, if this is true, then \( \phi_{\theta} \) is an eigenvector of \( \Lambda_{n(r),+(n(r)-k)} \) with an eigenvalue \( \lambda \neq 0 \). Now, if there exists some \( w \) such that \( \phi_{\theta} \) is not constant on \( G_w \cap S_{n(r)} \), then by Lemma 3.10 \( \sum_{u \in G_w \cap S_{n(r)}} \phi_{\theta}(u) = 0 \). This means that \( \phi_{\theta} \) is an eigenvector of \( \Lambda_{n(r),+(n(r)-k)} \) with 0 as an eigenvalue, which is a contradiction. Now assume that for every \( v \in S_k \), \( \sum_{u \in G_v \cap S_{n(r)}} \phi_{\theta}(u) = 0 \). Combining Lemma 3.10 and the fact that for \( x, y \in \Gamma \) such that \(|x| > |y| > t_{n(r)} \) it holds that \( G_y \cap S_{n(r)} \subseteq G_x \cap S_{n(r)} \), we conclude that for every \( x \in \Gamma \) with \(|x| > t_k \), \( U_r(f)(x) = 0 \). This contradicts the fact that \( k \in A_r \).
Note that for every $j \in A_r$, the functions $g_{n(r)}$ and $w_{n(r)}$ are constant on $I_j$ as they change their values only at the $t_j$'s. Thus, $U_r(f)|_{I_j} = \varphi f|_{I_j}$ is a linear combination of functions in $H^2(I_j)$, so we have $U_r(f)|_{I_j} \in H^2(I_j)$. In addition, we have

$$\int_{a_r}^{b_r} |(U_r(f))''|^2 \, dx = \int_I |\Delta f|^2 \, d\mu < \infty.$$  

Functions in $D(\Delta)$ satisfy certain matching conditions on vertices, which transform under $U_r$ into matching conditions on the connection points of the segments $(I_j)_{j \in A_r}$. In order to express these matching conditions, for every $j \in A_r$ we compute

$$\varphi_f(t_j + \epsilon) = \lim_{\epsilon \to 0} \varphi_f(t_j + \epsilon), \quad \varphi_f(t_j - \epsilon) = \lim_{\epsilon \to 0} \varphi_f(t_j - \epsilon)$$

for $f \in F_r$. Let $\epsilon > 0$.

$$\varphi_f(t_j + \epsilon) = (f_{t_j+\epsilon}, h_{t_j+\epsilon}) = \sum_{x = t_j + \epsilon} f(x) \frac{1}{\sqrt{g_n(t_j + \epsilon)w_n(t_j + \epsilon)}} \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u) = \sum_{u \in S_j} \sum_{x : i(x) = u} f(x) \frac{1}{\sqrt{g_n(t_j + \epsilon)w_n(t_j + \epsilon)}} \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u)$$

The functions $g_n, w_n$ are constant on $(t_j, t_{j+1})$, so we may write $g_n(t_j + \epsilon) = g_n^j$ and $w_n(t_j + \epsilon) = w_n^j$ for sufficiently small $\epsilon$. Now, taking $\epsilon \to 0$, we have

$$\varphi_f(t_j + \epsilon) = \frac{1}{\sqrt{g_n^j w_n^j}} \sum_{v \in S_j : i(v) = u} \lim_{x \to v} (f|_{x \to v})(x) \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u) = \frac{1}{\sqrt{g_n^j w_n^j}} \sum_{v \in S_j} \sum_{c : i(c) = v} f(v) \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u)$$

Now, $\star$ is constant on every edge, so for every edge $e$ we pick $x(e) \in e$ and we have

$$\varphi_f(t_j + \epsilon) = \frac{1}{\sqrt{g_n^j w_n^j}} \sum_{v \in S_j} \sum_{c : i(c) = v} f(v) \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u)$$

Now note that for every $v \in S_j$ we have that $\sum_{c : i(c) = v} \sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u)$ is some multiple of $\sum_{u \in G_x \cap S_{n(r)}} \phi^*_0(u)$. Thus, for $j = n(r) - 1$, by the second part of Claim 4.1 we have that $\varphi_f(t_j + \epsilon) = 0$.

For $j \geq n(r)$, again by Claim 4.1 we have that $\phi^*_0$ is constant on $G_v \cap S_{n(r)}$, and for every $e$ such that $i(e) = v$ and $x \in e$, $G_x \cap S_{n(r)} = G_v \cap S_{n(r)}$. Thus, $\phi^*_0|_{G_v \cap S_{n(r)}} = \phi^*_0(u_v)$ for some $u_v \in G_v \cap S_{n(r)}$.

Now, if we denote by $b^out_j$ the number of edges emanating from vertices in $S_j$ (which is the same for
every vertex due to the symmetry of the graph), we have

\[ \varphi_f(t_j+) = \frac{1}{\sqrt{g_{n(r)}w_{n(r)}^j}} \sum_{v \in S_j} f(v) \cdot b_{j}^{\text{out}} \cdot w_{n(r)}^j(u_v) = \]

\[ = \frac{w_{n(r)}^j b_{j}^{\text{out}}}{\sqrt{g_{n(r)}w_{n(r)}^j}} \sum_{v \in S_j} f(v)\phi_0^r(u_v) = \]

\[ = \frac{\sqrt{w_{n(r)}^j b_{j}^{\text{out}}}}{\sqrt{g_{n(r)}w_{n(r)}^j}} \sum_{v \in S_j} f(v)\phi_0^r(u_v) \]

By a similar computation, it can be shown that \( \varphi_f(b_r) = 0 \) (in the case that \( b_r < \infty \)), and (with the proper notations) that for \( j \geq n(r) \)

\[ \varphi_f(t_j-) = \frac{\sqrt{w_{n(r)}^{j-1} b_{j}^{\text{in}}}}{\sqrt{g_{n(r)}^{j-1}}} \sum_{v \in S_j} f(v)\phi_0^r(u_v) \]

Thus, for \( j \geq n(r) \),

\[ U_r(f)(t_j+) = d_j^r U_r(f)(t_j-) \]

where

\[ d_j^r = \frac{\sqrt{w_{n(r)}^{j-1} g_{n(r)}^{j-1} b_{j}^{\text{out}}}}{\sqrt{w_{n(r)}^{j-1} g_{n(r)}^{j-1} b_{j}^{\text{in}}}}. \]

By a similar computation, using the matching condition of the derivative, we also have

\[ U_r(f)'(t_j+) = c_j^r U_r(f)'(t_j-) \]

where

\[ c_j^r = \frac{\sqrt{w_{n(r)}^{j-1} g_{n(r)}^{j-1}}}{\sqrt{w_{n(r)}^{j-1} g_{n(r)}^{j-1}}}. \]

**Remark 4.2.** In the case that \( r = 0, n(r) = 0 \) as the function \( \phi_0^r \) is chosen to be \( \delta_0 \). Thus \( w_{n(r)}^j = 1 \) for every \( j \in \mathbb{N} \). In addition, it can be seen that \( b_{j}^{\text{in}} = \frac{g_{n(r)}^{j-1}}{|S_j|} \) and \( b_{j}^{\text{out}} = \frac{g_{n(r)}^j}{|S_j|} \). From here, it can be seen that

\[ d_j^0 = \frac{\sqrt{g_0}}{g_0^{j-1}} \]

and that \( c_j = \frac{1}{b_j} \).

To conclude, \( D(\Delta_r) \) consists of all of the functions in \( L^2(a_r, h(\Gamma)) \) which satisfy the following conditions:
1. \( f(a_r) = 0 \) if \( b_r < \infty \), then also \( f(b_r) = 0 \).

2. \( \forall j \in A_r \ f|_{I_j} \in H^2(I_j) \).

3. \( \sum_{j \in A_r} \int_{I_j} |f''(t)| dt < \infty \).

4. \( f(t_j-) = d_j f(t_j+) \) for every \( 2 \leq j \in A_r \).

5. \( f'(t_j-) = c_j f'(t_j+) \) for every \( j \in A_r \).

In the case that \( A_r \) is finite, functions in \( D(\Delta_r) \) also satisfy the condition \( f(b_r) = 0 \).

To conclude, the operator \( \Delta_r \) is a Sturm-Liouville operator on a weighted \( L^2 \) space where the weight is singular and localized at points determined by the structure of the graph \( \Gamma \). Since locally this operator acts as a second derivative, we henceforth refer to such operators as ‘weighted Laplacians’.

5 Examples

In this section we first describe how the results of [35, 40] are a special case of our analysis here. Then, in order to demonstrate the utility of our method, we apply it to the study of metric antitrees.

5.1 Radial Trees

A metric tree \( \Gamma \) is called radial (or regular) if for every \( x, y \in S_n \), \( \deg(x) = \deg(y) \), and for any two edges of the same generation \( e_1, e_2 \), \( l(e_1) = l(e_2) \).

Claim 5.1. Every regular tree is family preserving.

Proof. Let \( \Gamma \) be a regular tree, and let \( v, u \in S_n \) be backward neighbors. \( \Gamma \) is a tree, so the sets \( A_v = \{ w \in V(\Gamma) : v \geq w \} \), \( A_u = \{ w \in V(\Gamma) : u \geq w \} \) are disjoint. Furthermore, regularity of \( \Gamma \) implies that the graphs induced by those sets are isomorphic as rooted graphs, with \( u, v \) as roots. Let \( \sigma : A_v \rightarrow A_u \) be an isomorphism between the induced graphs. Define

\[
\tau(w) = \begin{cases} 
\sigma(w) & w \in A_v \cup A_u \\
w & \text{else}
\end{cases}
\]

It is easy to verify that \( \tau \) is a rooted graph automorphism which satisfies \( \tau|_{S_{n-j}} = Id \) for every \( j \geq 1 \). \( \Gamma \) is a tree, so there are no forward neighbors, and that finishes the proof.

A decomposition for the Laplacian on regular trees was presented in [35, 40]. We will present the decomposition here, and show that it is a special case of the decomposition presented in the proof of Theorem 2.12.

Let \( v \in \Gamma \). Denote by \( b(v) \) the number of edges emanating from \( v \) (i.e., the edges going ‘away’ from the root), and order the edges \( (e^v_1, \ldots, e^v_{b(v)}) \). Let \( \omega \) be the root of unity of order \( k \) (i.e. \( \omega = e^{2\pi i / k} \)), and let \( s \in \mathbb{N}, 1 \leq s \leq b(v) - 1 \). The space \( M_i^{(s)}(\Gamma) \) is the space of all functions \( f \in L^2(\Gamma) \) such that there exists a measurable function \( g : (t_{\text{gen}(v)}, \infty) \rightarrow \mathbb{C} \) for which \( f(x) = g(|x|)\omega^{js} \iff x \in G_e^v \). Also, denote by \( M_\Gamma \) the space of all functions in \( L^2(\Gamma) \) that are symmetric.
Theorem 5.2. Let $\Gamma$ be a radial metric tree such that there is only 1 edge emanating from its root. Then the spaces $\{M_v\} \cup \bigcup_{w \in V(\Gamma) \setminus \{\alpha\}} \bigcup_{1 \leq b(v) - 1} \{M_v^{s(b)}\}$ form a decomposition of the Laplacian on $\Gamma$.

A generalized version of Theorem 5.4 was proved in [35].

Lemma 5.3. For every $v \in V(\Gamma)$, and every $s = 1, \ldots, b(v) - 1$, the space $M_v^{s}$ is equal to the spaces $F_{\phi_v^s}$, where $\phi_v^s$ is defined in the following way:

$$\phi_v^s(u) = \begin{cases} \omega^j s & e^j = (v, u) \\ 0 & \text{else} \end{cases}$$

where $\omega = e^{\frac{2\pi}{b(v)}}$.

Proof. Let $f \in M_v^{s}$, and let $g : (t_{gen(v)}, \infty)$ be such that $f(x) = g(|x|)\omega^j s \iff x \in G_{e^j}$. In particular, $g$ is a symmetric function on $\Gamma$, and we have $T_{\phi_v^s}(g) = f$ directly from the definition of $T_{\phi_v^s}$. Thus, we have $M_v^{s} \subseteq F_{\phi_v^s}$. The inclusion in the opposite direction is symmetric.

We want to show that the collection $\{\delta_v\} \cup \{\phi_v^s : v \in V(\Gamma), 1 \leq s \leq b(v) - 1\}$ can play the role of $\{\phi_v^s : r \in \mathbb{N}\}$ in Theorem 2.0. The following propositions will demonstrate this.

Proposition 5.4. Let $v \in V(\Gamma)$, $1 \leq s \leq b(v) - 1$. Then the set $\{\phi_v^{0, s}, \phi_v^{v, s}, \ldots\}$ obtained by applying the Gram-Schmidt process on $\{\phi_v^0, \Delta_d(\phi_v^s), \ldots\}$ satisfies the following:

1. $\text{supp}(\phi_v^{0, s}) \subseteq G_v \cap S_{gen(v)+k+1}$.

2. There exists $c_k \in \mathbb{R}$ such that for every $u \in G_v \cap S_{gen(v)+k+1}$, $\phi_v^{0, s}(u) = c_k \phi_v^{0, s}(v_u)$, where $v_u$ is the only neighbor of $u$ in $S_{gen(v)+1}$.

Proof. We show this by induction on $k$. For $k = 0$ the claim follows from the definition of $\phi_v^s$.

Now, we apply the Gram-Schmidt process on the set $\{\phi_v^{0, s}, \phi_v^{v, s}, \ldots\}$. Note that by the induction hypothesis, the first $k$ elements in this set are already orthonormal. In addition, by the definition of $\Delta_d$ and the induction hypothesis, $\text{supp}(\Delta_d(\phi_v^{0, s})) \subseteq (G_v \cap S_{gen(v)+k}) \cup (G_v \cap S_{gen(v)+k+1}) \cup (G_v \cap S_{gen(v)+k+2})$. Note that for $u \in G_v \cap S_{n(r)+k+1}$ and $w \in G_v \cap S_{n(r)+k}$, $v_u = v_w$. Thus,

$$\langle \Delta_d(\phi_v^{0, s})(w) \rangle = b(w)\phi_v^{0, s}(u) = b(w)c_k \phi_v^{0, s}(v_u) = b(w)c_k \phi_v^{0, s}(v_w)$$

Now, due to the symmetry of the graph, for every $n \in \mathbb{N}$ and every $w_1, w_2 \in S_n$ we have that $b(w_1) = b(w_2)$. Thus we get that $\langle \Delta_d(\phi_v^{0, s}) \rangle |_{S_{gen(v)+k}}$ is some multiple of $\phi_v^{0, s}$. It is also easy to see that $\langle \Delta_d(\phi_v^{0, s}) \rangle |_{S_{gen(v)+k+1}}$ is also some multiple of $\phi_v^{0, s}$. Thus, the orthogonalization process leads to the fact that $\text{supp}(\phi_v^{k+1}) \subseteq S_{gen(v)+k+2}$. As for the second part of the claim, let $w \in G_v \cap S_{gen(v)+k+2}$. Suppose that $u$ is the neighbor of $w$ in $S_{gen(v)+k+1}$.

$$\langle \Delta_d(\phi_v^{0, s})(w) \rangle = -\phi_v^{0, s}(u) = -c_k \phi_v^{0, s}(v_u) = -c_k \phi_v^{0, s}(v_w).$$

Thus, the constant $c_{k+1}$ is obtained by normalizing $\langle \Delta_d(\phi_v^{0, s}) \rangle$. □
From now on, we denote $H^s_v = \{ \text{span} \phi_k^v | k \in \mathbb{N} \}$.

**Proposition 5.5.** Let $v_1, v_2 \in V(\Gamma)$ and let $1 \leq s_1 \leq b(v_1) - 1$, $1 \leq s_2 \leq b(v_2) - 1$. Assume that either $v_1 \neq v_2$ or $s_1 \neq s_2$. Then $H^s_{v_1} \perp H^s_{v_2}$.

**Proof.** Let $\phi \in H^s_{v_1}$, $\psi \in H^s_{v_2}$. $\phi$ and $\psi$ are supported on a single sphere. Thus, the only case that we need to check is when they are supported on the same sphere. This is possible if $v_1 = v_2$ and $s_1 \neq s_2$, or $v_2 < v_1$. In the first case, orthogonality follows from the orthogonality of $\phi^v_{s_1}$ and $\phi^v_{s_2}$ and part 2 of the previous proposition. In the second case, $\phi$ is symmetric on the spheres on which $\psi$ is supported. Now, the claim follows from the fact that $\psi$ is orthogonal to the symmetric functions. \qed

**Proposition 5.6.** For every $v \in V(\Gamma)$, $j \in \mathbb{N}$ and $1 \leq s \leq b(v) - 1$, $\phi^v_s$ is an eigenfunction of $\Lambda_{n,\pm j}$ (for $\Lambda_{n, \pm j}$ we require $j \leq \text{gen}(v) + 1$).

**Proof.** In every spherically symmetric tree, for every $n \in \mathbb{N}$ and every $\phi \in \ell^2(S_n)$, $\phi$ is an eigenfunction of $\Lambda_{n, \pm j}$. As for $\Lambda_{n, -j}$, note that $E^T_{\text{gen}(v)}(\phi^v_s)(v) = \sum_{u \in G, v \in S_{\text{gen}(v) + 1}} \phi^v_s(u) = 0$. Thus, $\phi^v_s$ is an eigenvector of $\Lambda_{n, -j}$ with eigenvalue 0. \qed

**Proposition 5.7.** $\ell^2(V(\Gamma)) = \left( \bigoplus_{v \in V(\Gamma) \setminus \{o\}} \bigoplus_{1 \leq s \leq b(v) - 1} H^s_v \right) \oplus H_0 := H$, where $H_0$ is the space spanned by $\delta_o$ and $\Delta d$.

**Proof.** We prove that for every $n \in \mathbb{N}$, $\ell^2(S_n) \subseteq H$. This is sufficient due to the fact that the compactly supported functions are dense in $\ell^2(V(\Gamma))$. Let $n \in \mathbb{N}$. Note that for every $j < n$, $v \in S_j$ and $1 \leq s \leq b(v) - 1$, $\dim(H^s_v \cap \ell^2(S_n)) = 1$ (this follows from part 1 in Proposition 5.4). In addition, $\dim(H_0 \cap \ell^2(S_n)) = 1$. In addition, $|S_j| = b_1 \cdots b_{j-1}$ where $b_i$ is the number of edges emanating from a vertex in $S_i$. Thus, by a direct computation (in analogy to the computation done in the proof of [35 Theorem 2.2]), we have that

$$\dim(\ell^2(S_n) \cap H) = 1 + (b_1 - 1) + \ldots + b_1 \cdots b_{n-1}(b_n - 1) = b_1 \cdots b_n = \dim(\ell^2(S_n))$$

as required. \qed

### 5.2 Antitrees

An antitree is a rooted graph, $G$, with all possible edges between any two neighboring spheres (and no other edges). Formally, a rooted graph is an antitree if $\forall n \in \mathbb{N}$, $\forall u \in S_n$, $\forall v \in S_{n+1}$, $(u, v) \in E$. In a sense, an antitree is a bipartite antithesis to a tree, as it has all available cycles between vertices of a different generation. Works exploiting this structure in the context of spectral properties of the Laplace operator include [23, 26, 42]. The Anderson model on antitrees has been studied in [38, 39].

It is shown in [14] that antitrees are family preserving. A metric graph, $\Gamma$, is called an antitree if $G_\Gamma$ is an antitree. Let $\Gamma$ be a spherically homogeneous metric antitree.

Antitrees are (generally) not locally balanced, as for every $n \in \mathbb{N}$ we have $|S_n| \cdot |S_{n+1}|$ edges of generation $n$ (which, if $|S_n|, |S_{n+1}| \geq 2$, is greater than both $|S_n|$ and $|S_{n+1}|$). Thus, given an antitree $\Gamma$, we consider its unitarily equivalent graph $\hat{\Gamma}$ which is obtained by adding vertices where necessary. For simplicity of notation, we assume here that for every $n \geq 1$, $|S_n| \geq 2$ (note that $|S_0| = 1$ since $S_0 = \{\delta_o\}$—the root). The analysis of the general case is not fundamentally different, but more cumbersome to describe.
Write \( V(\bar{\Gamma}) = V(\Gamma) \cup \bar{V} \), and denote by \( A := (a_n)_{n \in \mathbb{N}} \) the set of indices for which \( S_{a_n} \subseteq V(\Gamma) \) (note that \( a_1 = 0, a_2 = 1, a_3 = 3 \) etc.). Also, denote by \( B = (b_n)_{n \in \mathbb{N}} \) the set of indices for which \( S_{b_n} \subseteq \bar{V} \) (here, \( b_1 = 2, b_2 = 4 \) etc.). Finally, let \( \tilde{G} \) be the discrete structure of \( \bar{\Gamma} \) and let \( (H_r)_{r = 0}^{\infty} \) be the decomposition of \( \ell^2(\tilde{G}) \) described in Theorem 2.6 such that \( H_0 = \text{span} \{ \delta_0, \Delta_d(\delta_0), \Delta^{2}_d(\delta_0), \ldots \} \) and for every \( r \in \mathbb{N} \), \( H_r = \text{span} \{ \phi_r^0, \Delta_d(\phi_r^0), \Delta^{2}_d(\phi_r^0), \ldots \} \) for \( \phi_r^0 \in \ell^2(S_{n(r)}) \) (i.e. \( n(r) \) denotes the sphere on which \( \phi_r^0 \) is supported). Recall that for every \( r \in \mathbb{N} \), \( \phi_k^{(r)} \) is the set obtained by applying the Gram-Schmidt process on \( (\Delta^k(\phi_r^0))_{k = 0}^{\infty} \). We want to describe the decomposition of \( \Delta \) on \( \bar{\Gamma} \). In order to do this, we first need to focus on the corresponding discrete decomposition.

**Claim 5.8.** For every \( n \geq 3 \) and for every \( r \in \mathbb{N}, a_n \neq n(r) \).

**Proof.** By Theorem 2.6 for every \( \phi \in H_r \) we have that \( \text{supp}(\phi) \cap S_{n(r)−1} = \emptyset \). Assume that there exists \( n \geq 3 \) and \( r \in \mathbb{N} \) such that \( n(r) = a_n \). Note that, as the sphere \( S_{a_n−1} \) consists of added vertices, every \( v \in S_{a_n−1} \) is connected to exactly one \( w \in S_{a_n} \). Thus, for \( w \in S_{a_n} \) such that \( \phi_r^0(w) \neq 0 \) and a neighbor of \( w, v \in S_{a_n−1}, \Delta_d(\phi_r^0)(v) = −\phi_r^0(w) \) which is a contradiction.

**Claim 5.9.** For every \( 3 \leq n \in \mathbb{N} \) and for every \( r \in \mathbb{N} \) such that \( 0 \leq n(r) \leq a_n - 2, H_r \cap \ell^2(S_{a_n}) = \emptyset \).

**Proof.** For every \( 0 \leq k \leq a_n - 2 \) and \( v \in S_{a_n}, G_v \cap S_k = S_k \). In addition, \( \phi_r^0 \) is orthogonal to the symmetric functions. Combining this with the fact that \( \Delta^{n−n(r)}_d(\phi_r^0)(v) = c \sum_{w \in G_v \cap S_{n(r)}} \phi_r^0(w) \) for some \( c \in \mathbb{N} \), we get the result.

**Corollary 5.10.** For every \( 3 \leq n \in \mathbb{N} \), \( \ell^2(S_{a_n}) = \text{span} \{ \phi^0_{a_n}, \phi_r^1, \phi_r^2, \ldots, \phi_r^{[S_{a_n}]−1} \} \) for \( r, r \mid S_{a_n}−1 \mid \) such that \( n(r_i) = a_n − 1 \) for every \( 1 \leq i \leq |S_{a_n} − 1| − 1 \). In addition, for every such \( i, \Delta_d(\phi_r^i)|_{S_{a_n+1}} \neq 0 \).

**Proof.** The first part follows from the previous claims and from the fact that \( \ell^2(S_{a_n}) \subseteq \bigoplus_{r \in \mathbb{N}} H_r \). The second part follows from the fact that for \( 0 \neq \phi \in \ell^2(S_{a_n}), \Delta_d(\phi)|_{S_{a_n+1}} \neq 0 \), as again, every vertex in \( S_{a_n+1} \) is connected to exactly one vertex in \( S_{a_n} \).

Corollary 5.3 implies that for every \( n \geq 3 \), a subspace of \( \ell^2(S_{a_n+1}) \), of dimension \( |S_{a_n}| − 1 \), is spanned by \( \{ \phi_1^1, \ldots, \phi_{[S_{a_n}]−1}^1 \} \). For every \( r \in \mathbb{N} \) such that \( n(r) < a_n − 1, H_r \cap \ell^2(S_{a_n+1}) = \emptyset \).

This follows, as before, from the fact that for every \( v \in S_{a_n+1}, G_v \cap S_{n(r)} = S_{n(r)} \). Another one-dimensional space is spanned by \( \phi_{a_n+1}^0 \). Recalling that \( |S_{a_n+1}| = |S_{a_n}| \cdot |S_{a_n+1}| \), this means that \( |\{ r \in \mathbb{N} | n(r) = a_n + 1 \} | = |S_{a_n+1}| \cdot |S_{a_n}| - |S_{a_n}| = |S_{a_n}| \cdot (|S_{a_n+1}| - 1) \) (as these span the rest of \( \ell^2(S_{a_n+1}) \)). Noting that for every \( n \geq 2, a_n + 1 = b_n - 1 \), we conclude the following.

**Proposition 5.11.** Denote \( k_n = |S_{a_n}| \cdot (|S_{a_n+1}−1|) \), and for \( n \geq 2, B_n = \{ r \in \mathbb{N} | n(r) = b_n−1 \} \). Then for every \( n \geq 2, |B_n| = k_n \). Furthermore, define \( K_n := \{ r \in B_n | \text{dim}(H_r) = 3 \} \). Then \( |K_n| = |S_{a_n+1}−1| − 1 \), and for every \( r \in B_n \setminus K_n, \text{dim}(H_r) = 1 \).

Denote by \( Q \) the set \( \{ r \in \mathbb{N} | \exists n \in \mathbb{N} \text{ s.t. } n(r) = b_n \} \cup \{ 0 \} \), and define \( W = \bigoplus_{r \in Q} H_r \). From all of the above, we conclude the following:

- For \( n \geq 3, \ell^2(S_n) \subseteq W \).
- For \( n = 2, \text{dim} (\ell^2(S_n) \cap W) = |S_1| \cdot |S_3| - |S_1| + 1 \).
• For $n = 0, 1$, the intersection of $W$ with $\ell^2(S_n)$ consists of the symmetric functions on $S_n$ (for $S_0$ this is actually the whole space, as $|S_0| = 1$).

• Finally, for $r \notin Q H_r$ is orthogonal to $W$. Thus, the intersection of $H_r$ with $\ell^2(S_n)$ is trivial for every $n \geq 3$ and for $n = 0$. This implies that for every $r \notin Q$, $n(r) = 1$ and there will be exactly $|S_1| - 1$ such $r$’s.

By considering the spaces $(F_r)_{r=0}^\infty$ given by Theorem 1.3 we can now state the analog of [14, Theorem 1.3]

**Theorem 5.12.** Let $\Gamma$ be an infinite spherically homogeneous metric antitree. Then the spectrum of $\Delta$ on $\Gamma$ is given by the spectrum of a weighted Laplacian on the whole line and a sequence of compactly supported eigenfunctions.

**Proof.** Recall that for $n \in \mathbb{N}$, $t_n$ denotes the metric distance of $v$ from $o$ for some $v \in S_n$. In addition, recall that $a_r$ and $b_r$ denote the part of the real line on which functions in $F_r$ are supported.

Now, note that for every $x \in \Gamma$, $G_x \cap S_0 = \{o\}$. This means that $h^{b_n}(x) \neq 0$. Thus, $b_0 = \infty$ and $\Delta$ restricted to $F_0$ is unitarily equivalent to a weighted Laplacian on the positive real line, with matching conditions as described in the previous section on the points $(t_n)_{n \in \mathbb{N}}$.

For every $r_0 \in \{r \in \mathbb{N} | n(r) = 1\}$, $F_{r_0}$ is orthogonal to the symmetric functions. Thus, as every $v \in S_3$ is connected to every vertex in $S_1$ (through a vertex in $S_2$), $\Delta$ restricted to $F_{r_0}$ is unitarily equivalent to a weighted Laplacian on the compact segment $[0,t_3]$.

Finally, let $n \in \mathbb{N}$ and let $r_0 \in \{r \in \mathbb{N} | n(r) = b_n\}$ and consider two cases.

- **dim($H_{r_0}$) = 1:** In this case, $b_r = t_{b_n+1}$, and we get a copy of a weighted Laplacian on the compact segment $[t_{b_n-1}, t_{b_n+1}]$.

- **dim($H_{r_0}$) = 3:** In this case, $b_r = t_{b_n+3}$, and we get a copy of a weighted Laplacian on the compact segment $[t_{b_n-1}, t_{b_n+3}]$.

\[\square\]

**Remark 5.13.** In an antitree, there is an explicit formula for $g_0'$ which is given by $g_0' = |S_j| \cdot |S_{j+1}|$. Thus, with Remark 1.2 in mind, $d_0' = \frac{|S_{j-1}|}{|S_{j+1}|}$ and $c_j = \frac{1}{d_j}$. As a demonstration of the usefulness of an explicit decomposition to one dimensional operators, we mention Remling’s Theorem [37], whose main message is the fact that absolutely continuous spectrum for one dimensional operators is extremely restrictive. In the context of antitrees, following the discussion above and applying [13, Theorem 6] we immediately obtain that

**Theorem 5.14.** Let $\Gamma$ be a spherically homogeneous metric antitree and assume that

$$\inf_n (t_{n+1} - t_n) > 0$$

and

$$\liminf_{n \to \infty} \frac{|S_{n-1}|}{|S_{n+1}|} > 1$$

Then if $\lim_{n \to \infty} (t_{n+1} - t_n) = \infty$ then the absolutely continuous spectrum of $\Delta$ on $\Gamma$ is empty.
6 Appendix - Proof of Proposition 2.11

Proof of Proposition 2.11. We say that \( n \in \mathbb{N} \) is “bad” if the number of edges of generation \( n \) is greater than the number of vertices of generation \( n \) and of the number of vertices of generation \( n + 1 \). For such \( n \), and \( e = (i(e), t(e)) \) such that \( \text{gen}(e) = n \), we add a vertex \( w \), the edges \( e_1 = (i(e), w) \) and \( e_2 = (w, t(e)) \) such that \( l(e_i) = \frac{1}{2} l(e) \), and remove \( e \) from \( E(\Gamma) \). In simple words, we divide the edge into two edges with equal length. Note that in the resulting graph, the numbers \( n, n + 1 \) are not bad, because for every new edge \( e \), there is a unique vertex \( w \) which was added with \( e \). In addition, none of the other generations became bad, because we did not remove any vertices from \( V(\Gamma) \). Thus, if we follow this procedure for every bad \( n \), the resulting graph will be locally balanced. Denote that graph by \( \Gamma \), and let \( \Delta = \Delta_{\Gamma} \). Note that as measure spaces, there is no difference between \( \Gamma \) and \( \Delta \).

The Kirchhoff boundary conditions assure us that the transformation \( T : D(\Delta) \to D(\Delta) \) defined by \( T(\varphi) = \varphi \) is unitary, and maps \( D(\Delta) \) bijectively onto \( D(\Delta) \).

It is left to prove that \( \Gamma \) is family preserving. Denote by \( S_n \) the n-th sphere in \( \Gamma \). Let \( v, u \in S_n \) such that \( v \) and \( u \) are forward neighbors. Consider the following cases:

(i) \( S_n \) is not a part of \( \Gamma \), meaning we added that sphere during the above process. In this case, \( v \) and \( u \) were added in the middle of the edges \((w_1, z), (w_2, z)\) which means that \( w_1 \) and \( w_2 \) are forward neighbors in \( \Gamma \). That implies that there exists a rooted graph automorphism \( \tau : V(\Gamma) \to V(\Gamma) \) such that \( \tau(w_1) = \tau(w_2) \), and \( \forall k \geq 0 \tau|_{S_{n+k}} = \text{Id} \). We define \( \tau : V(\Gamma) \to V(\Gamma) \) by \( \tau|_{V(\Gamma)} = \tau \), and if \( x \) was added in the middle of the edge \((v_1, v_2)\), \( \tau(x) \) will be the vertex that was added in the middle of \((\tau(v_1), \tau(v_2))\) (such vertex exists because of spherical symmetry). Now, we have that \( \tau(v) = u \), and \( \forall k \geq 0 \tau|_{S_{n+k}} = \text{Id} \), as required.

(ii) \( S_n \) is a part of \( \Gamma \). In that case, \( S_{n+1} \) is also a part of \( \Gamma \), because an added vertex cannot connect vertices from the preceding sphere, so we can define \( \tau \) exactly as in case (i), and get the desired automorphism.

The proof for backward neighbors is exactly the same. ∎

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