Abstract There are various types of global and local spacetime invariant in general relativity. Here I focus on the local invariants obtainable from the curvature tensor and its derivatives. The number of such invariants at each order of differentiation that are algebraically independent will be discussed. There is no universally valid choice of a minimal set. The number in a complete set will also be discussed. The invariants can then be used to characterize solutions of the Einstein equations (locally), to test apparently distinct solutions for equivalence, and to construct solutions. Other applications concern limits of families of spacetimes, and the characterization of horizons and singularities. Further uses are briefly mentioned.

Keywords Invariants · horizons · singularities · exact solutions

PACS 02.40.Ky · 04.20.-q · 04.20.Cv · 04.20.Dw · 04.20.Jb

1 Introduction and motivation

This lecture developed from one given some years ago at the retirement party of my friend, colleague and co-author Dr. Eduard (“Eddie”) Herlt. At the Lahore meeting, time did not permit a complete exposition of all aspects of the subject. This text follows what was said rather than what might have been said in an expanded version. I intend to write up the expanded version as a review article in due course.

I could have been more precise in my title, at the cost of being rather lengthy. There are many occurrences of the word 'invariant' in our field, e.g.: gauge-invariant (in gauge theory) gauge-invariant (in perturbation theory) invariant under a symmetry Lorentz-invariant scale-invariant.

And I could be talking about some global conserved quantity, without a well-defined local density, e.g. Bondi mass.
What I actually meant was local, geometric, invariants of spacetime: essentially curvature invariants.

I started to be seriously interested in this area when we were writing the first edition of the exact solutions book [1] in the late 1970s. One of the big problems we faced was that of identifying the same solution when found with different assumptions or for different reasons, and presented in different coordinate systems. I realised at the time that the work of the Stockholm group on invariant classification, which I will mention later, and which I first heard about in the late 70s, held the key.

This first application, as far as I was concerned, the “equivalence problem”, was what got me started.

2 Mathematics

2.1 Definitions of invariants

Christoffel proved in 1869 that scalars constructed from the metric and its derivatives must be functions of the metric itself and the Riemann tensor and its covariant derivatives.

The first examples to spring to mind are scalar polynomial (s.p.) invariants, such as $R_{ab}R^{ab}$ or $C_{abcd}C^{cdef}C_{ef}^{ab}$. Often when people just say “invariant” they mean “s.p. invariant”.

However, these are not adequate in all circumstances, as one can see by noting that $pp$ waves and flat space both have all scalar polynomial invariants, of all orders, equal to zero [2].

Fortunately they are not the only choice. An important alternative is to use ideas due to Cartan, as follows. As a side benefit, in my view an important one, Cartan invariants require less calculation, in general.

Let $F(\mathcal{M})$ denote a “suitable” frame bundle over a spacetime $\mathcal{M}$ (i.e. take the set of all frames at each point) and $\mathcal{R}^q$ be the set $\{R_{abcd}, R_{abcd;f}, \ldots, R_{abcd;f_1f_2\ldots f_q}\}$ of the components of the Riemann tensor and its derivatives up to the $q$th in a frame.

Choose from $F(\mathcal{M})$ in a canonical and invariant way, e.g. use the principal null directions of the Weyl tensor, when they are distinct, as the basis vectors. The resulting $\{R_{abcd}, R_{abcd;f}, \ldots, R_{abcd;f_1f_2\ldots f_q}\}$ are called the Cartan invariants. They are scalars, because the frames are invariantly defined, e.g. if $a, b, c$ and $d$ are basis vectors (not necessarily all distinct) of the chosen frame, one of the Cartan invariants is $R_{ijkl}a^i b^j c^k d^l$. The idea is like characterizing a symmetric bilinear map (matrix) by its eigenvalues.

2.2 Independence of (s.p.) invariants

In a manifold $\mathcal{M}$ of $n$ dimensions, at most $n$ scalar invariants can be functionally independent, i.e. independent functions on $\mathcal{M}$.

The number of algebraically independent scalar polynomial invariants, i.e. s.p. invariants not satisfying any polynomial relation (called a syzygy), is rather larger, as we shall see next. (A set of algebraically independent Cartan invariants in a general spacetime, written using the Newman-Penrose spinor formalism, is given in [3]: it takes fully into account the Ricci and Bianchi identities.)
Larger still is the size of a complete set of s.p. invariants (a finite Hilbert basis): such a set \( \{ I_1, I_2, \ldots, I_k \} \) is complete if any other such s.p. invariant can be written as a polynomial in the \( I_j \) but no invariant in the set can be so expressed in terms of the others.

One way to find the number of algebraically independent scalar invariants is to consider Taylor expansions of the metric and of the possible coordinate transformations \([4]\); another way follows Hilbert \([5]\).

The result \([6]\) is that in a general \( V_n \) the number of algebraically independent scalars constructible from the metric and its derivatives up to the \( p \)th order (the Riemann tensor and its derivatives up to the \((p - 2)\)th) is 0 for \( p = 0 \) or \( p = 1 \) and

\[
N(n, p) = \frac{n(n + 1)![(n + p)!]}{2n^2p!} - \frac{(n + p + 1)!}{(n - 1)!(p + 1)!} + n,
\]

for \( p \geq 2 \), except for \( N(2, 2) = 1 \). Thus in a general space-time the Riemann tensor has \( N(4, 2) = 14 \) algebraically independent scalar invariants. In particular cases the number is reduced.

The origin of many syzygies can be understood in terms of the vanishing of any object skewed over \((n + 1)\) indices in \( n \) dimensions \([7, 8, 9]\). In a series of papers \([10, 11, 12]\) Carminati and Lim have given a detailed discussion of the syzygies for scalar polynomial (s.p.) invariants of the Riemann tensor, using graph-theoretic techniques.

A given invariant may be written in more than one way, due to symmetries, and other relations between components, of the Riemann tensor and its derivatives. This is essentially a problem in representations of the permutation group. The issue is to select a canonical representative of each equivalence class in the orbit under permutations.

One wants to select a canonical set of invariants and then express any other invariant in terms of canonized members of the algebraically independent set.

Several methods have been used to do this for s.p. invariants, e.g. by Hörnfeldt, by Fulling et al., by Ilyin and Kryukov, and by Dresse. The method most readily available is due to Portugal \([13]\). It has been implemented in Maple\textsuperscript{TM} and Mathematica\textsuperscript{®} by Martin-Garcia et al, and is distributed in xAct, a package for use with Mathematica\textsuperscript{®}. It has (e.g.) been applied to all objects with up to 12 derivatives of the metric \([14]\).

Any complete set of s.p. invariants of the Riemann tensor, and any set which always contains a maximal set of independent scalars, contains redundant elements. Hence all the old papers giving specific sets of 14, in 4 dimensions, are inadequate.

The smallest known set of s.p. invariants for a general Riemann tensor that always contains a maximal set of algebraically independent scalars consists of 17 polynomials \([15]\), though 16 suffice for perfect fluids and Einstein-Maxwell fields. There is a special subclass of spacetimes \([16]\) which require 18.

The smallest set of s.p. invariants known to be complete contains 38 scalars \([17]\), and Lim and Carminati \([12]\) proved that it is minimal.

2.3 Computation of invariants

The s.p. invariants are expressions whose contractions hide very large numbers of individual terms, and are therefore hard to calculate. It is thus very useful to adopt a method that reduces the number of terms. One can use a bivector formalism (see e.g. \([15]\)). Another way, in 4 dimensions, is to use the Newman-Penrose complex spinor (or null tetrad) formalism, the GHP formalism, or related ideas: this has analogues in other dimensions (e.g. in three dimensions one can use real two-component spinors: see \([18]\)).
For higher dimensions it may be efficient to use parallel computing, i.e. to split the calculation into subproblems e.g. by fixing some of the dummy indices, and then sending the subproblems to separate processors [19].

A simple practical way in many cases is to use Cartan invariants instead. They are quite cheap to calculate provided the frame choice is easy to calculate with, in particular because one never has to multiply curvature components, or their derivatives, together. In a general case, either s.p. or Cartan invariants would characterize the spacetime, as we shall see below, so the choice is really a question of convenience or purpose. There is an open research problem about which is actually more computationally efficient (or for which cases).

In four dimensions we can always choose a frame, for example by the principal null directions of the Weyl tensor, and thus compute Cartan invariants, but in general this may entail using the unpleasant algebraic expressions which arise from the general formula for solutions of quartics. However, in practice many solutions give way easily. Note that in 5 dimensions or more there is no guarantee we can calculate frames of a specified type even if we can show they must exist, as there is no general algebraic formula for solutions of quintics.

2.4 Do scalar polynomial invariants suffice?

Until 2009, I would have said definitely not. I mentioned earlier that pp waves and flat space both have all scalar polynomial invariants, of all orders, equal to zero. In fact all vacuum type N and III metrics with $\rho = 0$ have this property [20]. There are also metrics which have equal non-zero s.p. invariants e.g. [21, 22, 23].

These ambiguities are associated with the indefiniteness of the metric and the non-compactness of the Lorentz group [24].

Coley et al. [25] gave an argument that spacetimes are completely characterized by their s.p. invariants, except for spacetimes in the Kundt class. Although their proof seems to me to have a gap, and I have not yet been able to see if it has been filled, the result seems correct.

In particular Coley and collaborators have followed up by a substantial number of papers, which I did not have time to review comprehensively in my Lahore talk. These consider not only 4D spacetimes, but higher dimensional spaces and various signatures. I just mentioned a few to give the flavour. In the initial paper [25], they also discuss what properties a given set of invariants characterize.

- Hervik and Coley [26] show that any metric with an analytic continuation to the pure Riemannian case is completely characterized by its s.p. invariants
- Coley et al. [27] give a detailed analysis of the generic 5D spacetimes.
- Hervik [28] gave arguments that spacetimes not characterized by s.p. invariants are limits of families that are so characterized.

It may be worth noting that since it is obvious that the algebraically independent set of Cartan invariants given in [3] completely and uniquely specify the s.p. invariants, the result of [25] amounts to saying that in general the converse is true, i.e. that the s.p. invariants uniquely determine the Cartan invariants (given the procedure for the choice of frame).

The work just described picked out the Kundt class as the exceptions. These are spacetimes which have a null vector field which is geodesic, and expansion-, shear- and twist-free. In a paper [29] which appeared just before [25] this class was discussed in detail. For it, one has to use the Cartan approach or some equivalent.
In [29] Coley et al. discussed the Petrov D examples in the Kundt class. Podolsky and Svarc [30] gave a classification of Kundt metrics in arbitrary dimension, and later discussed their physical interpretation. A subclass of “universal spacetimes” was discussed by Hervik et al. [31]: universal in that they solve vacuum equations of all gravitational theories with Lagrangian constructed from the metric, the Riemann tensor and its derivatives of arbitrary order. Hervik et al. [32] found all cases where there exists a continuous family of metrics having identical [scalar] polynomial invariants.

3 Uses of spacetime invariants

3.1 The Equivalence Problem

Now let me return to my original motivation, i.e. identifying the same solution when found with different motivations and assumptions and presented in different coordinate systems. It could be considered either a mathematical or a physical problem. It is essentially resolved by the following theorem (in terms of Cartan invariants).

**Theorem 1** (Sternberg [33], Ehlers [34]) Given two spacetimes, $\mathcal{M}$ and $\overline{\mathcal{M}}$, each expressed using some frames $E$ and $\overline{E}$, then there is an isometry which maps $(x, E)$ to $(\overline{x}, \overline{E})$ if and only if $\mathcal{R}^{p+1}$ for $\mathcal{M}$ and $\overline{\mathcal{R}}^{p+1}$, where $p$ is the last derivative at which a new functionally independent entry in $\mathcal{R}^{p+1}$ arises.

See [35] for a more careful statement.

One could therefore calculate everything on a suitable frame bundle.

– Christoffel used coordinate frames. This works but the dimension of $F(\mathcal{M})$ is large.
– Cartan proposed using frames with constant scalar products, e.g. null frames or orthonormal frames, which is better (hence the credit for Cartan invariants).
– In 1965 Brans [36] proposed practical use of this method via lexicographic indexing of components.
– Later Brans and, more fully, Karlhede [37, 38] realised that a more efficient way to implement the idea was to restrict the frames wherever possible by only allowing canonical forms of the curvature and its derivatives (i.e. go to Cartan invariants).
– This makes the whole thing manageable (with a computer) and it was first implemented by Åman in 1979 [39].

Given the Coley et al. results, we now know we could in general work only with s.p. invariants, but the present software [40, 41] uses the Cartan method. In an algebraically general spacetime, one can choose the principal null directions of the Weyl tensor to fix the frame, or in a conformally flat spacetime one can begin with eigenvectors of the Ricci tensor. In the actual implementation we use the Weyl PNDs, where they exist, as the first choice.

Many details have been considered (by Åman, myself, Skea, d’Inverno, McLenaghan, Pollney and others) in specifying frames and in making the computations more efficient. The main aspects are the enumeration of canonical forms, the tests of which canonical form applies, and the transformation of the non-canonical to the canonical. The literature cited in the exact solutions book gives more information.

A first extension is to maps more general than isometries, e.g. homotheties [42] or conformal equivalence [43]. One can of course tackle Euclidean metrics and metrics in any dimension, though the details of making the method work efficiently are different (e.g. [44]).
The above ideas are in fact a special case of Cartan’s general procedures (see e.g. [45]), which apply to other situations which can be expressed in differential forms and a connection or similar structures. In particular it applies to the equivalence of (systems of) differential equations, under coordinate transformations (see the books of Gardner [46] and Olver [47]).

Another context is that of gauge theories of physics, the gauge potential being just a connection [48].

One interesting issue for spacetimes is how large \( p \) in the theorem has to be. Karlhede [38] showed \( p \leq 7 \). For a long time examples suggested \( p \leq 3 \). At the time of the second edition of the exact solutions book [35] an example with \( p = 5 \) was known, as were a number of bounds for subclasses. We quoted results suggesting that \( p \leq 5 \) was generally true.

In 2009 (the result was actually first announced in 2007), Milsom and Pelavas [49] showed that the original bound of 7 is sharp, by giving a (unique) example. This is still small enough for calculation to be practical. The example is of course in Kundt’s class, as are those further examples they found where \( p = 6 \).

As well as giving a way to compare two spacetimes, the theorem above also implies that the Cartan invariants uniquely classify and determine the local geometry. This implies that all local properties are encoded in this information. I will rather briefly discuss some of the possible resulting applications. Aspects I was not able to cover in Lahore included: applications to three-dimensional spaces, incuding the use in checking junction conditions; applications in numerical relativity; applications to cosmology and to perturbations; use in constructing alternative theories of gravity; and applications to internal state spaces of thermodynamics. I think there must be more we have not yet explored.

Note that global topology is of course not determined by the invariants we have been considering (local invariants cannot distinguish a plane and a flat torus) and nor are continuations which do not respect the required differentiability for the theorem above.

My original motivation can now be carried out for many cases. I used these methods, for example, to disentangle the known exact solutions for inhomogeneous cosmologies with a symmetry group \( G_2 \). It would be good to have a complete online catalogue of (at least) the solutions in [35]. Skea and Lake have independently implemented systems to facilitate this. Properties such as the isometry group can also be found [50, 51, 52]: for examples see e.g. [53, 54]. Similar results apply for homotheties [42].

### 3.2 Finding solutions

The same ideas can be used to find solutions, by working out consistent sets of Cartan invariants and then integrating to get a metric. Examples are mentioned in the exact solutions book. Machado Ramos and Edgar [55, 56] used these ideas, implemented in their invariant operator formalism, to find pure radiation solutions of types N and O with a cosmological constant.

Coley and collaborators looked for all spacetimes in which all s.p. invariants vanish (which they called VSI spacetimes although not all Cartan scalar invariants vanish). In the 4-d Lorentzian case these give all spacetimes indistinguishable from flat space by s.p. invariants [57]. The higher dimensional cases give examples in supergravity and string theory. A closely related series of papers has studied the spacetimes with constant s.p. invariants (CSI spacetimes). For a review of this work see [58].
3.3 Limits of families of spacetimes

Another aspect, described first by Paiva et al. [59], is the use of invariants to work out the limits of families described by parameters. This enables one to find all possible limits in a coordinate-free way, whereas previous treatments were in effect trial and error. This is another area where further work would be useful.

I did not give more detail in my Lahore lecture, but passed quickly on to limiting points within spacetimes.

3.4 Black hole and other horizons

Karlhede et al. [60] first noted that $R_{abcd,e}R^{abcd,e} = 0$ at the Schwarzschild horizon (so a prudent space traveller might monitor that). Skea in his thesis [61] noted that this is not true for other horizons (a point rediscussed by Saa [62] for higher-dimensional static cases; Saa also found points where $R_{abcd,e}R^{abcd,e} = 0$ but which are not horizons). Lake [63] continued the work on Kerr by considering first derivative invariants, and found their vanishing characterized the horizons.

Related to my arguments against the claims of Antoci and others that the Schwarzschild horizon was singular, I proposed in 2006 [64] a new test for occurrence of a horizon using ratios of Cartan invariants which works in all cases of Petrov type D. I believe it works more generally, but this needs further study. Maybe it could also be useful to numerical relativists (and space travellers).

More recently Abdelqader and Lake [65, 66] have given an invariant characterization of the Kerr horizon, and, prompted by that, Page and Shoom [67] have given another for stationary black holes. I have not yet had the opportunity to check how these relate to earlier characterizations.

Moffat and Toth [68] considered the relation of the “Karlhede invariant” (i.e. $R_{abcd,e}R^{abcd,e}$) to discussions of a “firewall” at the horizon.

3.5 Singularities

Singularities in general relativity are defined to occur when a causal geodesic cannot be continued to infinite affine parameter values even when the spacetime is maximally extended – “geodesic incompleteness” (for the reasons for this definition see [69]).

It is well-known that this happens if an s.p. invariant of the Riemann tensor itself (not its derivatives) blows up along the geodesic. This does not however mean that:
(i) an “infinite” Riemann tensor implies a singularity, or
(ii) if an invariant blows up, there is a singularity, or
(iii) at all singularities, an s.p. invariant of the Riemann tensor blows up.

1. An “infinite” Riemann tensor does not imply a singularity. Geodesics can be continued across a delta function curvature modelling a thin shell or an impulsive gravitational wave.

2. The blowing up of an invariant does not imply a singularity. For example [64] the invariant $1/R_{abcd,e}R^{abcd,e}$ blows up at the Schwarzschild horizon, but the horizon is not singular.
3. There are singularities at which no s.p. invariant of the Riemann tensor blows up. At “whimper” singularities an invariant involving first derivatives does, though [70]. An example was studied by Podolsky and Belan [71].

Question: when does blow up of higher derivative invariants imply a singularity?

Another application is to “directional” singularities, where a singular point apparently has directionally dependent limits. Scott and Szekeres [72, 73] showed that the directional singularity of the Curzon metric hid more extended regions at whose boundary the original coordinates broke down. My student Taylor [74] showed that such cases could be appropriately “unravelled” by using level surfaces of Cartan invariants to define new coordinates. Lake [63] used the Weyl tensor s.p. invariants to show that the Kerr singularity was not directional.

With Coley and others [75], I considered “kinematic singularities” (where fluid expansion blows up), giving examples in which, given an integer $p$, the Cartan (and hence s.p.) scalars can be finite up to the $p$-th derivative, but not the $(p + 1)$-th.

Geodesic continuation needs a $C^2$ metric. In invariantly-defined frames the connection coefficients (which would be $C^1$) are typically expressible as ratios of first derivative Cartan invariants to zeroth derivative ones. We know that there are “intermediate” or “whimper” singularities where s.p. invariants of the Riemann tensor do not blow up, while s.p. invariants of the first derivatives of the Riemann tensor do. Hence I conjecture that under some suitable differentiability conditions:

**Conjecture:** Spacetime singularities are either locally extensible or at least one Cartan invariant in $\mathcal{R}^1$ has an infinite limit along any curve approaching the singularity.

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