Rational Elliptic Surfaces and the Trigonometry of Tetrahedra

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To Sonya Pashchevskaya, the bravest person I know

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Abstract

We study the trigonometry of non-Euclidean tetrahedra using tools from algebraic geometry. We establish a bijection between non-Euclidean tetrahedra and certain rational elliptic surfaces. We interpret the edge lengths and the dihedral angles of a tetrahedron as values of period maps for the corresponding surface. As a corollary we show that the cross-ratio of the exponents of the solid angles of a tetrahedron is equal to the cross-ratio of the exponents of the perimeters of its faces. The Regge symmetries of a tetrahedron are related to the action of the Weyl group $W(D_6)$ on the Picard lattice of the corresponding surface.
1 Introduction

1.1 Trigonometry of tetrahedra and rational elliptic surfaces

Trigonometry is a branch of mathematics that studies the relations involving side lengths and angles of a triangle. It seems that these relations are fairly well understood though some questions remain unanswered, see [Kle16, pp. 189-194] and [Tju75, §2]. The situation in higher dimensions is much more complicated.

By a tetrahedron we mean a geodesic tetrahedron in \( S^3 \), \( \mathbb{R}^3 \) or \( \mathbb{H}^3 \). We call a tetrahedron non-Euclidean if it is spherical or hyperbolic. The following problem is a subject of three-dimensional trigonometry:

How can one determine the dihedral angles of a tetrahedron from its edge lengths?

This problem admits a straightforward solution: one can write a complicated explicit formula, presenting the dihedral angles as functions of the lengths of edges. Surprisingly, this is not the end of the story. We start with formulating a theorem in three-dimensional trigonometry, which motivated this work.

Let \( T \) be a tetrahedron with vertices \( A_1, A_2, A_3, A_4 \). Denote by \( l_{ij} \in \mathbb{R} \) for \( 1 \leq i < j \leq 4 \) the length of an edge \( A_i A_j \) and by \( \alpha_{ij} \in \mathbb{R}/2\pi\mathbb{Z} \) the corresponding dihedral angle. Next, consider solid angles \( \Omega_{123}, \Omega_{124}, \Omega_{134}, \Omega_{234} \) and perimeters \( \Pi_{123}, \Pi_{124}, \Pi_{134}, \Pi_{234} \) of its faces. More explicitly,

\[
\begin{align*}
\Pi_{123} &= l_{12} + l_{13} + l_{23}, & \Omega_{123} &= \alpha_{14} + \alpha_{24} + \alpha_{34} - \pi, \\
\Pi_{124} &= l_{12} + l_{14} + l_{24}, & \Omega_{124} &= \alpha_{13} + \alpha_{23} + \alpha_{34} - \pi, \\
\Pi_{134} &= l_{13} + l_{14} + l_{34}, & \Omega_{134} &= \alpha_{12} + \alpha_{23} + \alpha_{24} - \pi, \\
\Pi_{234} &= l_{23} + l_{24} + l_{34}, & \Omega_{234} &= \alpha_{12} + \alpha_{13} + \alpha_{14} - \pi.
\end{align*}
\]

We also consider the following quantities:

\[
\begin{align*}
\Pi_{1234} &= l_{12} + l_{23} + l_{34} + l_{41}, & \Omega_{1234} &= \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41}, \\
\Pi_{1324} &= l_{13} + l_{23} + l_{24} + l_{41}, & \Omega_{1324} &= \alpha_{13} + \alpha_{23} + \alpha_{24} + \alpha_{41}, \\
\Pi_{1243} &= l_{12} + l_{24} + l_{43} + l_{31}, & \Omega_{1243} &= \alpha_{12} + \alpha_{24} + \alpha_{43} + \alpha_{31}.
\end{align*}
\]

We assemble these numbers into a pair of configurations of eight points in \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \). First, we consider a configuration

\[
\Omega(T) = (1, e^{i\Omega_{123}}, e^{i\Omega_{124}}, e^{i\Omega_{134}}, e^{i\Omega_{234}}, e^{i\Omega_{1234}}, e^{i\Omega_{1324}}, e^{i\Omega_{1243}}).
\]

Next, we consider a configuration

\[
\Pi(T) = \begin{cases} 
(1, e^{i\Pi_{123}}, e^{i\Pi_{124}}, e^{i\Pi_{134}}, e^{i\Pi_{234}}, e^{i\Pi_{1234}}, e^{i\Pi_{1324}}, e^{i\Pi_{1243}}) & \text{if } T \text{ is spherical}, \\
(0, \Pi_{123}, \Pi_{124}, \Pi_{134}, \Pi_{234}, \Pi_{1234}, \Pi_{1324}, \Pi_{1243}) & \text{if } T \text{ is Euclidean}, \\
(1, e^{i\Pi_{123}}, e^{i\Pi_{124}}, e^{i\Pi_{134}}, e^{i\Pi_{234}}, e^{i\Pi_{1234}}, e^{i\Pi_{1324}}, e^{i\Pi_{1243}}) & \text{if } T \text{ is hyperbolic}.
\end{cases}
\]

**Theorem 1.1.** For a tetrahedron \( T \) the configurations \( \Omega(T) \) and \( \Pi(T) \) are projectively equivalent.

For distinct \( z_1, z_2, z_3, z_4 \in \mathbb{C} \) consider a cross-ratio \( \left[ z_1, z_2, z_3, z_4 \right] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \in \mathbb{C}^\times \). Projective transformations preserve cross-ratios, so the following corollary holds.
Corollary 1.2. For a tetrahedron $T$ the following equality holds:

$$[e^{I_{123}}, e^{I_{124}}, e^{I_{134}}, e^{I_{234}}] = \begin{cases} [e^{I_{123}}, e^{I_{124}}, e^{I_{134}}, e^{I_{234}}] & \text{if } T \text{ is spherical}, \\ \{I_{123}, I_{124}, I_{134}, I_{234}\} & \text{if } T \text{ is Euclidean}, \\ [e^{I_{123}}, e^{I_{124}}, e^{I_{134}}, e^{I_{234}}] & \text{if } T \text{ is hyperbolic}. \end{cases}$$

We discovered Theorem 1.1 in an attempt to understand a formula of Cho and Kim for volume of a non-Euclidean tetrahedron (see [CK99]) from a motivic perspective. For an elementary proof of Corollary 1.2 see an answer by Petrov in the discussion “A curious relation between angles and lengths of edges of a tetrahedron” on https://mathoverflow.net/q/336464. It is of interest to understand the geometric meaning of the coefficients of the projective transformation sending $\Omega(T)$ to $\Pi(T)$.

The correspondence between the dihedral angles and the lengths of edges of a tetrahedron has a hidden symmetry called Regge symmetry, discovered by Ponzano and Regge in the Euclidean case, see [PR68, Appendix D].

Theorem 1.3. Let $T$ be a tetrahedron. Suppose that there exists a tetrahedron $T'$ in the same space with edge lengths $l'_{ij}$ for $1 \leq i < j \leq 4$ such that

$$l'_{12} = l_{12}, \quad l'_{13} = \frac{l_{14} + l_{23} + l_{24} - l_{13}}{2}, \quad l'_{14} = \frac{l_{13} + l_{23} + l_{24} - l_{14}}{2},$$

$$l'_{34} = l_{34}, \quad l'_{24} = \frac{l_{23} + l_{34} + l_{24} - l_{23}}{2}, \quad l'_{23} = \frac{l_{13} + l_{14} + l_{24} - l_{23}}{2}.$$

Then the corresponding dihedral angles $\alpha'_{ij}$ of $T'$ satisfy

$$\alpha'_{12} = \alpha_{12}, \quad \alpha'_{13} = \frac{\alpha_{14} + \alpha_{23} + \alpha_{24} - \alpha_{13}}{2}, \quad \alpha'_{14} = \frac{\alpha_{13} + \alpha_{23} + \alpha_{24} - \alpha_{14}}{2},$$

$$\alpha'_{34} = \alpha_{34}, \quad \alpha'_{24} = \frac{\alpha_{13} + \alpha_{14} + \alpha_{23} - \alpha_{24}}{2}, \quad \alpha'_{23} = \frac{\alpha_{13} + \alpha_{14} + \alpha_{24} - \alpha_{23}}{2}.$$

Moreover, the volumes of the tetrahedra $T$ and $T'$ coincide.

A geometric proof of Theorem 1.3 in the non-Euclidean case was found by Akopyan and Izmestiev, see [AI19, Theorem 1]. It was noticed in [DL03] that Regge symmetry is a part of a bigger group of order 23040, which is isomorphic to the Weyl group $W(D_6)$.

Our initial goal was to find a conceptual explanation for Theorems 1.1 and 1.3. Our main result is a construction of a correspondence between tetrahedra and certain complex projective surfaces. The cross-ratios from Theorem 1.2 are equal to the classical invariants of the surfaces, called cross-ratios of type $D_4$ in [Nar80, §3]. Regge symmetry is manifested in the Weyl group action on the Picard lattice of the surface.

By a rational elliptic surface $X$ we mean a smooth projective surface over $\mathbb{C}$, which can be obtained as a blow up of $\mathbb{P}^2$ at nine points of intersection of a pair of elliptic curves. The anti-canonical linear system $|-K_X|$ defines a map $X \to \mathbb{P}^1$ with generic fiber of genus 1. A fiber $F$ of the elliptic fibration on $X$ is said to have type $I_2$ if it is isomorphic to a union of two rational curves intersecting transversally at a pair of points. The group $\text{Pic}^0(F)$ can be identified with $\mathbb{C}^\times$. Consider a divisor $D$, which is orthogonal (with respect to the intersection pairing on $\text{Pic}(X)$) to each irreducible component of $F$. We denote by

$$\text{Res}_F(D) \in \text{Pic}^0(F) \cong \mathbb{C}^\times$$

the restriction of $D$ to $F$. The map $\text{Res}_F$ gives a natural way to parametrize rational elliptic surfaces, see [Nar82, Appendix by E. Looijenga], we call it a period map.
Theorem 1.4. For a generic non-Euclidean tetrahedron (see Definition 2.7) there exists a rational elliptic surface $X_T$ with a pair of $I_2$-fibers $F_1$ and $F_2$ and a collection of six classes $e_{ij} \in \text{Pic}(X_T)$ orthogonal to $F_1$ and $F_2$ such that for $1 \leq i < j \leq 4$ we have

\[ \text{Res}_{F_1}(e_{ij}) = \begin{cases} e^{2i \alpha_{ij}} & \text{if } T \text{ is spherical}, \\ e^{2i \alpha_{ij}} & \text{if } T \text{ is hyperbolic}, \end{cases} \]

(1.1)

\[ \text{Res}_{F_2}(e_{ij}) = e^{2i(\pi - \alpha_{ij})}. \]

Remark 1.5. One can show that Theorem 1.4 is true without an extra assumption that $T$ is generic. Moreover, we expect that Theorem 1.4 can be generalized to the case of a Euclidean tetrahedron. The corresponding surface $X_T$ has a fiber $F_1$ of type $I_{11}$ and a fiber $F_2$ of type $I_2$. Then $\text{Pic}^0(F_1) \cong \mathbb{C}$ and we have

\[ \text{Res}_{F_1}(e_{ij}) = l_{ij}, \]

\[ \text{Res}_{F_2}(e_{ij}) = e^{2i(\pi - \alpha_{ij})}. \]

First we explain that Theorem 1.3 follows from Theorem 1.4. The lattice $\text{Pic}(X_T)$ has rank 10; the orthogonal complement to the canonical class $K_X \in \text{Pic}(X_T)$ is an affine root lattice of type $E_8^1$. The orthogonal complement to all components of the fibers $F_1$ and $F_2$ contains the null-vector $K_X$, and its quotient by $K_X$ is a root lattice of type $D_6$, so the Weyl group $W(D_6)$ acts on it. Regge symmetry is an action on this quotient by a particular element of the Weyl group, namely the reflection with respect to the plane perpendicular to the root $\frac{e_{13} + e_{14} + e_{23} + e_{24}}{2}$.

Since the period maps (1.1) in Theorem 1.3 are linear, Regge symmetry transforms lengths of edges and dihedral angles according to the formulas in Theorem 1.3.

Theorem 1.1 also follows from Theorem 1.4. A rational elliptic surface $X_T$ carries an admissible conic bundle: a map $b: X_T \rightarrow \mathbb{P}^1$ with generic fiber of genus 0 such that $b$ sends each irreducible component of $F_1$ and $F_2$ isomorphically to $\text{Im}(b) \cong \mathbb{P}^1$. Map $b$ has eight critical values $p_1, \ldots, p_8 \in \text{Im}(b)$. We choose $b$ such that $b^{-1}(p_i)$ intersects each component of $F_1$ and $F_2$ in exactly one point. For a certain choice of a conic bundle the eight points in which $b^{-1}(p_i)$ intersect a component of $F_1$ form a configuration $\Pi(T)$ and the eight points in which $b^{-1}(p_i)$ intersect a component of $F_2$ form a configuration $\Omega(T)$. The map $b$ defines a projective transformation sending $\Pi(T)$ to $\Omega(T)$. This proves Theorem 1.1.

1.2 Projective tetrahedra and the $E_8$ lattice

We start with introducing an algebro-geometric avatar of a non-Euclidean tetrahedron. A projective tetrahedron $T = (Q, H)$ is a configuration of an irreducible quadric $Q$ and an ordered set of four planes $H = \{H_1, H_2, H_3, H_4\}$ in $\mathbb{P}^3$ satisfying a certain non-degeneracy condition, see Definition 2.1.

Every non-Euclidean tetrahedron defines a projective tetrahedron, see [Gon99, §1.5]. We describe here the hyperbolic case. In Klein’s model the hyperbolic space $\mathbb{H}^3$ is identified with the interior of the unit ball in $\mathbb{R}^3$ and geodesic subspaces of $\mathbb{H}^3$ are intersections of lines and planes in $\mathbb{R}^3$ with $\mathbb{H}^3$. We view $\mathbb{R}^3$ as the set of real points of an affine space inside the complex projective space $\mathbb{P}^3$. Let $Q$ be a quadric, obtained as a projectivization of the complexification of the ideal boundary $\partial \mathbb{H}^3$. Let $H_1, H_2, H_3, H_4$ be the projectivizations of the complexifications of the faces of the tetrahedron. In this way a geodesic tetrahedron in $\mathbb{H}^3$ determines the projective tetrahedron $(Q, \{H_1, H_2, H_3, H_4\})$. A marking of a projective tetrahedron is a choice of a family of lines on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and ordering of points $\{x, y\} = Q \cap (A_iA_j)$. We denote the first point in $Q \cap (A_iA_j)$ by $E_{ij}$ and the second by $E_{ji}$.
A projective tetrahedron obtained from a non-Euclidean tetrahedron admits a canonical marking. In the hyperbolic case we have

\[ \log ([A_i, x, A_j, y]) = \pm 2l_{ij} \]

and we choose the ordering of \( \{x, y\} \) such that \( \log ([x, A_i, y, A_j]) > 0 \). In the spherical case

\[ \log ([A_i, x, A_j, y]) = \pm 2\alpha_{ij} \]

and we choose the ordering of \( \{x, y\} \) such that \( \frac{1}{4} \log ([x, A_i, y, A_j]) \in (0, \pi) \).

A non-Euclidean tetrahedron is uniquely determined by its edge lengths. One might expect that a marked projective tetrahedron is uniquely determined by the quantities

\[ [A_i, E_{ij}, A_j, E_{ji}] \in \mathbb{C}^\times \]  \hspace{1cm} (1.2)

for \( 1 \leq i < j \leq 4 \), but this is not the case. The reason is that quantities like \( e^{\Pi_{123}} \) appearing in \( \Pi(T) \) are well defined, but are not rational functions of expressions like (1.2).

Here the lattice \( Q(E_q) \) of a root system of type \( E_8 \) comes into play. It is known that it contains a set of eight pairwise orthogonal roots, which we label by subsets of a set \( I = \{1, 2, 3, 4\} \) of even cardinality. The stabilizer of a set of eight orthogonal roots acts transitively on them. After fixing roots \( e_\varnothing \) and \( e_I \) the remaining six could be labelled by 2-subsets of the set \( \{1, 2, 3, 4\} \). The stabilizer of the set of eight roots, \( e_\varnothing \) and \( e_I \), acts as \( S_4 \) on this set. We call the remaining roots \( e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34} \).

The collection of roots orthogonal to \( e_I \) spans a lattice of type \( E_7 \), which we denote by \( Q(E_7^1) \). In §2.3 we construct a homomorphism

\[ L_T: Q(E_q^1) \longrightarrow \mathbb{C}^\times, \]

such that \( L_T(e_\varnothing) = 1 \) and

\[ L_T(e_{ij}) = [A_i, E_{ij}, A_j, E_{ji}] \]

for \( 1 \leq i < j \leq 4 \). If \( T \) is obtained from a non-Euclidean tetrahedron, the coordinates of the vector \( \Pi(T) \) are values of the map \( L_T \) on certain roots in \( E_q^1 \). We call \( T \) generic if the only roots \( r \in R(E_q^1) \) for which \( L_T(r) = 1 \) equal to \( \pm e_\varnothing \).

Similarly, denote by \( Q(E_7^1) \) an orthogonal complement to the root \( e_I \) in \( Q(E_8) \). In §2.3 we define an angle function

\[ A_T: Q(E_7^1) \longrightarrow \mathbb{C}^\times. \]

This function is closely related to the length function of the dual tetrahedron \( T^\dual \). If \( T \) is obtained from a non-Euclidean tetrahedron we have \( A_T(e_\varnothing) = 1 \) and

\[ A_T(e_{ij}) = e^{2i(\pi - \alpha_{ij})} \text{ for } 1 \leq i < j \leq 4. \]

Similarly to \( \Omega(T) \) and \( \Pi(T) \) we define configurations

\[ A(T) = (1, A_{123}, A_{124}, A_{134}, A_{234}, A_{1234}, A_{1324}, A_{1423}), \]

\[ L(T) = (1, L_{123}, L_{124}, L_{134}, L_{234}, L_{1234}, L_{1324}, L_{1423}). \]

of eight points in \( \mathbb{R}^4 = \mathbb{C} \cup \{\infty\} \), see §2.3.
Definition 1.6. Let \( T \) be a marked projective tetrahedron. The following two rational functions

\[
\begin{align*}
\CK^L_T(t) &= \frac{(t - L_{123})(t - L_{124})(t - L_{134})(t - L_{234})}{(t - 1)(t - L_{123}4)(t - L_{1324})(t - L_{1234})}, \\
\CK^A_T(t) &= \frac{(t - A_{123})(t - A_{124})(t - A_{134})(t - A_{234})}{(t - 1)(t - A_{1234})(t - A_{1324})(t - A_{1243})}
\end{align*}
\]

are called the Cho-Kim function of \( T \) and the dual Cho-Kim function of \( T \) respectively.

It is easy to see that \( \CK^L_T(0) = \CK^L_T(\infty) = 1 \). There exist two more points \( p_1, p_2 \in \mathbb{P}^1 \) such that \( \CK^L_T(p_1) = \CK^L_T(p_2) = 1 \). These numbers are called the principal parameters of \( T \). The Cho-Kim function first appeared in a surprising formula for the volume of a hyperbolic tetrahedron in [MY05].

Our main result about trigonometry of a projective tetrahedron is the following theorem, which immediately implies Theorem 1.1.

Theorem 1.7. There exists a unique fractional linear transformation \( \psi \in \text{PSL}_2(\mathbb{C}) \) such that

\[
\CK^L_T(t) = \CK^A_T(\psi(t)).
\]

For a certain order of the principal parameters \( p_1 \) and \( p_2 \) we have

\[
\psi(t) = \frac{(t - p_1)(1 - p_2)}{(t - p_2)(1 - p_1)}.
\]

In particular, the configurations \( L(T) \) and \( A(T) \) are projectively equivalent.

Theorem 1.7 allows one to solve a tetrahedron: compute its dihedral angles in terms of edge lengths.

Example 1.8. Consider a spherical tetrahedron with all dihedral angles equal to \( \frac{\pi}{2} \). Let \( T \) be the corresponding projective tetrahedron. The dual Cho-Kim function of \( T \) is equal to

\[
\CK^A_T(t) = \frac{(t - i)^4}{(t - 1)^4}
\]

The principal parameters of \( T \) are \( p_1 = 1 + i \) and \( p_2 = \frac{1 + i}{2} \), so \( \psi(t) = \frac{t(1-i)-1}{t-(1+i)} \). By Theorem 1.7 we have

\[
\CK^L_T(t) = \frac{(t + i)^4}{(t - 1)^4}
\]

and the edge lengths of \( T \) equal to \( \frac{3\pi}{2} \).

1.3 Projective tetrahedra and \( D_6 \)-surfaces

We call a rational elliptic surface \( X \) with a pair of \( I_2 \)-fibers \( F_1 \) and \( F_2 \) a \( D_6 \)-surface \( (X, F_1, F_2) \). A \( D_6 \)-surface is called generic if all other fibers are irreducible. Fix an ordering of the components and of the singular points of each fiber \( F_1 \) and \( F_2 \). After that we can identify groups \( \text{Pic}^0(F_1) \) and \( \text{Pic}^0(F_2) \) with \( \mathbb{C}^* \). Lattice \( \text{Pic}(X) \) is isomorphic to \( \mathbb{Z}^{10} \), moreover \( K_X^2 = 0 \). It is known that the lattice \( K_X^2/K_X \) is a root lattice of type \( E_8 \). Fix an isomorphism \( K_X^2/K_X \cong \mathbb{Q}(E_8) \) sending the classes of the chosen components of the fibers \( F_1 \) and \( F_2 \) to the
roots $e_I$ and $e_∅$ respectively. We call a choice of this isomorphism together with an ordering of the components and of the singular points of $F_1$ and $F_2$ a marking of $X$. The Weyl group $W(D_6)$ of order 23040 is a stabilizer of roots $e_I$ and $e_∅$ in $W(E_8)$ and thus acts on the set of markings of $X$ by changing the isomorphism $K_X^7/K_X \cong Q(E_8)$. For a marked $D_6$-surface the following period maps are defined:

$$\text{Res}_{F_1} : Q(E_7^I) \longrightarrow \text{Pic}^0(F_1),$$

$$\text{Res}_{F_2} : Q(E_7^∅) \longrightarrow \text{Pic}^0(F_2).$$

The main result of our paper is Theorem 1.9, the generalization of Theorem 1.4 below.

**Theorem 1.9.** There exists a $W(D_6)$-equivariant one-to-one correspondence $T \longleftrightarrow (X_T, F_1, F_2)$ between generic marked projective tetrahedra and generic marked $D_6$-surfaces such that $L_T = \text{Res}_{F_1}$ and $A_T = \text{Res}_{F_2}$. The construction of the surface $X_T$ from the tetrahedron $T$ consists of two steps. First we blow up the quadric $Q$ at the twelve points $\{H_i \cap H_j \cap Q\}$ and obtain a rational surface $R_T$. Next, we blow down a certain set of four non-intersecting $(-1)$-curves and obtain a rational elliptic surface $X_T$. The last step is not canonical. Surprisingly, different choices of four $(-1)$-curves result in isomorphic rational elliptic surfaces. Unfortunately, we do not have a clear explanation of this fact yet; its proof is based on the Torelli theorem for anti-canonical pairs. The details of the construction are presented in §4.2. In the sequel to this paper we will give a “motivic” proof of Theorem 1.9 based on an isomorphism of mixed Hodge structures

$$H^3(\mathbb{P}^3 \setminus Q, \bigcup_{i=1}^{4} H_i) \cong H^2(X_T \setminus F_1, F_2).$$

Theorem 1.7 is a corollary of Theorem 1.9. For the generic $D_6$-surface $X_T$ there exists a map $b : X_T \longrightarrow \mathbb{P}^1$ such that the restriction of $b$ to each of the four irreducible components of the fibers $F_1$ and $F_2$ is an isomorphism. For almost every point $p \in \mathbb{P}^1$ the fiber $b^{-1}(p)$ is a rational curve with four marked points of intersection with the components of the fibers $F_1$ and $F_2$. The cross-ratio of the points is a rational function on the target space $\mathbb{P}^1$ of $b$. To write it down explicitly we need to fix three points $0, 1$ and $∞$ on the target. We will see that for one such choice this rational function is equal to the Cho-Kim function $\text{CK}_{L_T}^I$, and for another, the dual Cho-Kim function $\text{CK}_{A_T}^I$. This immediately implies Theorem 1.7.

The structure of possible configurations of degenerate fibers of a rational elliptic surface is well understood, see [OS91], [Per90]. It is interesting to extend Theorem 1.9 to non-generic $D_6$-surfaces and non-generic non-Euclidean tetrahedra: tetrahedra with ideal vertices, Euclidean tetrahedra, disphenoids etc.

**Example 1.10.** Consider the spherical tetrahedron from Example 1.8. Both $L_T$ and $A_T$ take values $±1$ on the roots. The corresponding rational elliptic surface has four singular fibers: two of type $I_2$ and two of type $I_4$. There is a unique surface $X_{4422}$ with these types of fibers and it has eight $(-1)$-curves, see [MP86, §5].
1.4 Notation and conventions

Throughout this paper we work over \( \mathbb{C} \). For distinct points \( P_1, P_2 \in \mathbb{P}^3 \) we denote by \((P_1P_2)\) the line containing them. Similarly, for three points \( P_1, P_2, P_3 \) in general position we denote by \((P_1P_2P_3)\) the plane containing them. For a point \( P \) and a line \( l \) not containing \( P \) we denote by \((P,l)\) the plane spanned by \( P \) and \( l \). Finally, for a pair of intersecting distinct lines \( l_1, l_2 \) we denote by \((l_1,l_2)\) the plane that contains them.

The cross-ratio of four points \( P_1, P_2, P_3, P_4 \) on a rational curve \( C \) is denoted \([P_1, P_2, P_3, P_4]_C\). Consider a birational isomorphism \( \varphi: X \to Y \) of smooth projective surfaces \( X \) and \( Y \). For a smooth projective curve \( C \) in \( X \) we define

\[
\varphi(C) = (C \setminus \text{Ind}(\varphi)),
\]

where \( \text{Ind}(\varphi) \) is the locus of indeterminacy of \( \varphi \). Then \( \varphi(C) \) is a point or the map \( \varphi \) defines an isomorphism \( C \to \varphi(C) \), which we denote by the same letter. If \( C \) is rational and \( \varphi(C) \) is not a point, then for every four points \( P_1, P_2, P_3, P_4 \in C \) we have

\[
[P_1, P_2, P_3, P_4]_C = [\varphi(P_1), \varphi(P_2), \varphi(P_3), \varphi(P_4)]_{\varphi(C)}.
\]

If for \( E \in \text{Pic}(X) \) there exists a unique curve \( C \) on \( X \) such that \([C] = E\), then we will denote the curve by the same letter \( E \).

Finally, for a root system \( \mathcal{R} \) we denote the corresponding lattice by \( \mathbb{Q}(\mathcal{R}) \) and the set of roots by \( \mathbb{R}(\mathcal{R}) \). We adopt a convention, according to which we have \( r^2 = -2 \) for \( r \in \mathbb{R}(\mathcal{R}) \). We mainly work with a concrete root system of type \( E_8 \) and its subsystems (see §2.2), which we denote by roman \( E_8 \).

Acknowledgments I would like to thank F. Brown, I. Dolgachev, O. Martin, E. Looijenga, and A. Goncharov for motivating discussions and invaluable help with preparing this manuscript. I am very grateful to P. Deligne who read a preliminary version of this paper and made a lot of useful comments and suggestions. I also thank Gerhard Paseman for checking some of the details in an earlier draft.

2 Trigonometry of projective tetrahedra

2.1 Projective tetrahedra

Definition 2.1. A projective tetrahedron \( T = (Q, \mathcal{H}) \) is a configuration, consisting of a smooth quadric \( Q \) in \( \mathbb{P}^3 \) and an ordered set of four planes \( \mathcal{H} = \{H_1, H_2, H_3, H_4\} \) in general position such that conics \( Q \cap H_i \) are smooth and points \( A_i = \bigcap_{j \neq i} H_j \) do not lie on \( Q \). Planes \( H_i \) are called faces of \( T \), lines \( H_i \cap H_j \) are called edges of \( T \), and points \( A_i \) are called vertices of \( T \).

A smooth quadric in \( \mathbb{P}^3 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) and defines a duality between subspaces of \( \mathbb{P}^3 \) known as the polar duality. Points, lines, and planes are dual to planes, lines, and points respectively. A general line \( l \) intersects \( Q \) in two points; ordering of these points is called an orientation of \( l \). Lines contained in \( Q \) are called generators of the quadric; there are two families of generators. An ordering of these families is called an orientation of \( Q \).

Every point \( p \in Q \) is contained in exactly two generators, which belong to the different families. After an orientation of \( Q \) is fixed, we denote them \( L_p \) and \( R_p \) and call the left generator and the right generator respectively. Generators of the quadric are self-dual lines. Let \( l \) be a line in \( \mathbb{P}^3 \) which intersects \( Q \) transversally at points \( x \) and \( y \). Then \( Q \cap l = \{x, y\} \) and the dual line \( l^\perp \) is the unique line passing through the points \( L_x \cap R_y \) and \( R_x \cap L_y \).

Let \( T = (Q, \mathcal{H}) \) be a projective tetrahedron in \( \mathbb{P}^3 \). The dual tetrahedron \( T^\perp = (Q, \mathcal{A}) \) is given by the configuration consisting of the same quadric \( Q \) and planes \( \mathcal{A} = \{A_1^\perp, A_2^\perp, A_3^\perp, A_4^\perp\} \) in \( \mathbb{P}^3 \) dual to the vertices of \( T \). Notice that the edges of \( T \) are dual to the edges of \( T^\perp \).
Definition 2.2. A marking of a projective tetrahedron \( T = (Q, \mathcal{H}) \) is combinatorial data consisting of an orientation of the quadric \( Q \) and orientations of the edges of \( T \).

Denote by \( E_{ij} \) the first point of \( (A_i, A_j) \cap Q \) and by \( E_{ji} \) the second point. Every marking of a tetrahedron \( T \) determines a marking of the dual tetrahedron \( T^\vee \) in the following way. Points \( (A_i, A_j) \cap Q \) are ordered so that \( E_{ij} = L_{E_{ij}} \cap R_{E_{ij}} \) is the first and \( E_{ji} = R_{E_{ji}} \cap L_{E_{ji}} \) is the second. It is easy to see that projective duality is an involution on marked projective tetrahedra.

2.2 The root system \( E_8 \)

A root system of type \( E_8 \) consists of 240 vectors in \( \mathbb{R}^8 \), see [Bou68, §6.4.10]. It contains a set of 8 orthogonal roots and the Weyl group \( W(E_8) \) acts transitively on such sets, see [DM10, Proposition 2.1].

Proposition 2.3. In a root system of type \( E_8 \), let \( S \) be a set of 8 orthogonal roots. Then \( S \) has a natural structure of an affine space of dimension 3 over \( \mathbb{F}_2 \). The planes for this structure are sets \( P \) of 4 elements of \( S \) such that \( \frac{1}{2} \sum_{\alpha \in P} \alpha \) is a root of \( E_8 \) and the stabilizer of \( S \) in \( W(E_8) \) is the group of affine transformations.

Proof. See [DM10, Theorem 2.5].

There is a way to construct an \( E_8 \) root system out of an affine space \( S \) over \( \mathbb{F}_2 \) of dimension 3. Consider the following subset \( C \subset (\mathbb{Z}/2\mathbb{Z})^S \):

\[
C = \left\{ 0, \sum_{s \in S} e_s, \sum_{s \in P} e_s \text{ for } P \text{ a plane in } S \right\}.
\]

Then \( C \) is a subgroup: if \( P, P' \) are planes in \( S \), their symmetric difference is the empty set, \( S \), or a plane in \( S \). In a lattice \((\frac{1}{2}\mathbb{Z})^S\) with quadratic form \(-2(\sum x_i^2)\) consider the subspace \( Z^S \subset E \subset (\frac{1}{2}\mathbb{Z})^S \) given by

\[
E = \{ x \mid 2x \text{ has an image in } (\mathbb{Z}/2\mathbb{Z})^S \text{ which is in } C \}.
\]

Then \( E \) is a lattice of type \( E_8 \). The roots are vectors \( \pm e_s \) for \( s \in S \) and \( e_a \pm e_b \pm e_c \pm e_d \) for \( \{a, b, c, d\} \) – an affine plane.

For a set \( I = \{1, 2, 3, 4\} \) consider an affine space \( S \) of even subsets of \( I \). Explicitly,

\[
S_I = \left\{ \emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, I \right\}.
\]

From now on we denote by (roman) \( E_8 \) the root system obtained from the affine space \( S_I \) by the construction above. We put \( e_{ij} := e_{(i,j)} \) for \( i < j \). Next, we denote the orthogonal complement to the root \( e_I \) in \( E_8 \) by \( E^I_8 \) and the orthogonal complement to \( e_s \) by \( E^s_8 \). These are root systems of type \( E_7 \) and their intersection \( D_6 = E^I_8 \cap E^s_8 \) is a root system of type \( D_6 \). A map \( D: S_I \longrightarrow S_I \) sending a subset of \( I \) to its complement is affine and so lies in the Weyl group \( W(E_8) \). It defines an isometry \( D: Q(E^I_7) \longrightarrow Q(E^s_8) \) which leaves the lattice \( Q(D_6) \) invariant.

2.3 Edge function and angle function

We start with discussing an algebro-geometric counterpart of the Poincare model of hyperbolic geometry. Consider a double cover of the projective space \( \mathbb{P}^3 \) ramified at \( Q \). This is a 3–dimensional smooth quadric \( \tilde{Q} \); \( \mathbb{P}^3 \) is the quotient of \( \tilde{Q} \) by an involution fixing \( Q \). For any line \( l \) in \( \mathbb{P}^3 \) the inverse image \( \tilde{l} \) is a “straight line” of \( \tilde{Q} \), that is a linear section of \( \tilde{Q} \) stable under
the involution. We have a double cover \( \tilde{\ell} \rightarrow \ell \) ramified at \( \ell \cap Q \). Suppose that \( \ell \) is transversal to \( Q \) and choose an orientation of \( \ell \) and assume that \( \ell \cap Q = \{ P_{12}, P_{21} \} \). Then
\[
\ell \setminus (\ell \cap Q) \cong \mathbb{P}^1 \setminus \{ 0, \infty \}
\]
becomes a \( \mathbb{C}^\times \)-principal homogeneous space. Consider a pair of points \( p_1, p_2 \in \ell \setminus \{ \ell \cap Q \} \). As \( \tilde{\ell} \rightarrow \ell \) is isomorphic to \( z \mapsto z^2 : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), for \( \tilde{p}_1 \) above \( p_1 \) and \( \tilde{p}_2 \) above \( p_2 \) we have
\[
[\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}]^2 = [p_1, P_{12}, p_2, P_{21}]
\]
and replacing \( \tilde{p} \) with the other preimage of \( p \) changes \( [\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}] \) to \(-[\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}]\). Same for \( p_2 \).

**Lemma 2.4.** Consider a triple of distinct points \( p_1, p_2, p_3 \in \mathbb{P}^3 \setminus Q \). Suppose that lines \( (p_ip_j) \cap Q = \{ P_{ij}, P_{ji} \} \) are oriented. Then for \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \) lying over \( p_1, p_2, p_3 \) we have
\[
[\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}][\tilde{p}_2, P_{23}, \tilde{p}_3, P_{32}][\tilde{p}_1, P_{13}, \tilde{p}_3, P_{31}] = [p_3, P_{31}, p_1, (p_1p_3) \cap (P_{12} P_{23})]_{(p_1p_3)}.
\]

**Proof.** Equality
\[
([\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}][\tilde{p}_2, P_{23}, \tilde{p}_3, P_{32}][\tilde{p}_1, P_{13}, \tilde{p}_3, P_{31}])^2
= [p_1, P_{12}, p_2, P_{21}][p_2, P_{23}, p_3, P_{32}][p_1, P_{13}, p_3, P_{31}]
= [p_3, P_{31}, p_1, (p_2p_3) \cap (P_{12} P_{23})]^2_{(p_2p_3)}
\]
is a version of the Menelaus’s theorem and can be easily checked directly, so we have
\[
[\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}][\tilde{p}_2, P_{23}, \tilde{p}_3, P_{32}][\tilde{p}_1, P_{13}, \tilde{p}_3, P_{31}] = \pm [p_3, P_{31}, p_1, (p_2p_3) \cap (P_{12} P_{23})]_{(p_2p_3)}.
\]
To fix the sign, consider a case, when \( p_1, p_2, p_3 \) lie in \( \mathbb{H}^3 \), see §1.2. In this case we easily see that numbers
\[
[\tilde{p}_1, P_{12}, \tilde{p}_2, P_{21}], [\tilde{p}_2, P_{23}, \tilde{p}_3, P_{32}], [\tilde{p}_1, P_{13}, \tilde{p}_3, P_{31}], [p_3, P_{31}, p_1, (p_1p_3) \cap (P_{12} P_{23})]_{(p_1p_3)}
\]
are positive. From here the statement follows. \( \square \)

Consider a marked projective tetrahedron \( T = (Q, \mathcal{H}) \). Denote by \( \tilde{A}_i \in \tilde{Q} \) lifts of its vertices. Next consider a map \( \tilde{L}_T : \left( \frac{1}{2} \mathbb{Z} \right)^S \rightarrow \mathbb{C}^\times \) defined by the rule
\[
\tilde{L}_T \left( \frac{1}{2} e_{ij} \right) = [\tilde{A}_i, E_{ij}, \tilde{A}_j, E_{ji}] \text{ for } 1 \leq i < j \leq 4,
\]
\[
\tilde{L}_T \left( \frac{1}{2} e_\emptyset \right) = \tilde{L}_T \left( \frac{1}{2} e_t \right) = 1.
\]
Denote \( L_T \) the restriction of \( \tilde{L}_T \) to \( \text{Q}(E^1_T) \subset \text{Q}(E_8) \subset \left( \frac{1}{2} \mathbb{Z} \right)^S \).

**Lemma 2.5.** The function \( L_T : \text{Q}(E^1_T) \rightarrow \mathbb{C}^\times \) does not depend on the choice of the lifts \( \tilde{A}_i \in \tilde{Q} \).

**Proof.** Consider a linear map \( p : (\mathbb{Z}/2\mathbb{Z})^S \rightarrow S \) sending \( e_\emptyset \) to \( s \) (we view \( S \) as a vector space with \( e_\emptyset \) as an origin). Then \( Q(E^1_T) \) is contained in \( \text{Ker}(p) \), as one can easily check. For \( r \in \text{Ker}(p) \) the value \( \tilde{L}_T(r) \) is a product of expressions \([\tilde{A}_i, E_{ij}, \tilde{A}_j, E_{ji}]^{\pm 1} \) such that any lift \( \tilde{A}_i \) occurs an even number of times and thus \( \tilde{L}_T(r) \) does not depend on the choice of lifts \( \tilde{A}_i \in \tilde{Q} \). \( \square \)

**Definition 2.6.** The function \( L_T : \text{Q}(E^1_T) \rightarrow \mathbb{C}^\times \) is called the length function of the marked projective tetrahedron \( T \). The function \( A_T : \text{Q}(E^8_T) \rightarrow \mathbb{C}^\times \) defined by \( A_T = L_T \circ D \) is called the angle function of \( T \).
It is easy to see that
\[ A_T(e_{ij}) = [H^\vee_k, \tilde{E}_{ij}, H^\vee_i, \tilde{E}_{jl}] \]
for \( k, l \in I \) such that permutation \( \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix} \) is even. We denote
\[
\begin{align*}
L_{123} & = L_T \left( \frac{e_{12} + e_{13} + e_{23} + e_{34}}{2} \right), & A_{123} & = A_T \left( \frac{e_{12} + e_{13} + e_{23} + e_{4} }{2} \right), \\
L_{124} & = L_T \left( \frac{e_{12} + e_{14} + e_{24} + e_{34}}{2} \right), & A_{124} & = A_T \left( \frac{e_{12} + e_{14} + e_{24} + e_{4} }{2} \right), \\
L_{134} & = L_T \left( \frac{e_{13} + e_{14} + e_{34} + e_{23}}{2} \right), & A_{134} & = A_T \left( \frac{e_{13} + e_{14} + e_{34} + e_{23}}{2} \right), \\
L_{234} & = L_T \left( \frac{e_{23} + e_{24} + e_{34} + e_{12}}{2} \right), & A_{234} & = A_T \left( \frac{e_{23} + e_{24} + e_{34} + e_{12}}{2} \right), \\
L_{1234} & = L_T \left( \frac{e_{12} + e_{14} + e_{23} + e_{34}}{2} \right), & A_{1234} & = A_T \left( \frac{e_{12} + e_{14} + e_{23} + e_{34}}{2} \right), \\
L_{1324} & = L_T \left( \frac{e_{13} + e_{14} + e_{23} + e_{24}}{2} \right), & A_{1324} & = A_T \left( \frac{e_{13} + e_{14} + e_{23} + e_{24}}{2} \right), \\
L_{1243} & = L_T \left( \frac{e_{12} + e_{13} + e_{24} + e_{34}}{2} \right), & A_{1243} & = A_T \left( \frac{e_{12} + e_{13} + e_{24} + e_{34}}{2} \right).
\end{align*}
\]

Now we have defined all notions involved in the formulation of Theorem 1.7.

**Definition 2.7.** A projective tetrahedron \( T \) is called generic if the only roots \( r \in R(E^L_7) \) such that \( L_T(r) = 1 \) are \( \pm e_{ij} \).

### 2.4 Moduli space of generic marked projective tetrahedra

For a function \( \tilde{L} : \left( \frac{1}{2}\mathbb{Z} \right)^{S_T} \to \mathbb{C}^\times \) such that \( \tilde{L}(e_{ij}) = \tilde{L}(e_{il}) = 1 \) consider the determinant
\[
\det(\tilde{L}) = \left| \begin{array}{cccc}
1 & \tilde{L}(\frac{e_{12}}{2}) + \tilde{L}(\frac{-e_{12}}{2}) & \tilde{L}(\frac{e_{13}}{2}) + \tilde{L}(\frac{-e_{13}}{2}) & \tilde{L}(\frac{e_{14}}{2}) + \tilde{L}(\frac{-e_{14}}{2}) \\
\tilde{L}(\frac{e_{12}}{2}) + \tilde{L}(\frac{-e_{12}}{2}) & 2 & \tilde{L}(\frac{e_{13}}{2}) + \tilde{L}(\frac{-e_{13}}{2}) & \tilde{L}(\frac{e_{14}}{2}) + \tilde{L}(\frac{-e_{14}}{2}) \\
\tilde{L}(\frac{e_{13}}{2}) + \tilde{L}(\frac{-e_{13}}{2}) & \tilde{L}(\frac{e_{14}}{2}) + \tilde{L}(\frac{-e_{14}}{2}) & 2 \\
\tilde{L}(\frac{e_{14}}{2}) + \tilde{L}(\frac{-e_{14}}{2}) & 2 & 1
\end{array} \right|.
\]

**Lemma 2.8.** Determinant \( \det(\tilde{L}) \) depends only on the restriction \( L \) of \( \tilde{L} \) to \( Q(E^L_7) \subset \left( \frac{1}{2}\mathbb{Z} \right)^{S_T} \); we denote it by \( \det(L) \). We have
\[
\det(L) = \det(L \circ w)
\]
for any \( w \in W(D_6) \).

**Proof.** Expanding the determinant (2.1), we obtain an explicit formula
\[
\det(\tilde{L}) = -\frac{3}{2} + \frac{1}{24} \sum_{r \in R(E^L_7) \setminus R(D_6)} \tilde{L}(r) - \frac{1}{24} \sum_{r \in R(D_6)} \tilde{L}(r) + \frac{1}{24} \sum_{r \in R(E^L_7) \setminus (R(E^L_7) \setminus R(E^A_7))} \tilde{L}(2r).
\]

A root \( r \in R(E^L_8) \setminus (R(E^L_7) \cup R(E^A_7)) \) is equal to \( \frac{\pm e_{ij} \pm e_{kl} \pm e_{ij} \pm e_{kl}}{2} \) for \( \{i, j\} \cup \{k, l\} = I \), so
\[
\tilde{L}(2r) = L(\pm e_{ij})L(\pm e_{kl}).
\]
From here the first statement of the lemma follows. The second statement follows from the fact that \( w \in W(D_6) \subset W(E_8) \) leaves both sets \( R(E_7^1) \) and \( R(E_7^4) \) invariant.

Consider a quasi-affine subset \( \mathbb{T} \) of Hom \((Q(E_7^1), \mathbb{C}^\times) \) consisting of maps \( L \) such that \( \det(L) \neq 0 \) and \( L(r) = 1 \) for a root \( r \in R(E_7^1) \) if and only if \( r = \pm e_6 \).

**Proposition 2.9.** The set of isometry classes of generic marked projective tetrahedra \( \text{M}_{tet} \) has a structure of an algebraic variety, which is an unramified double cover of \( \mathbb{T} \).

**Proof.** Consider a map \( \text{M}_{tet} \rightarrow \text{Hom}(Q(E_7^1), \mathbb{C}^\times) \) sending a tetrahedron \( T \) to its length function \( L_T \). Fix homogeneous coordinates in \( \mathbb{P}^3 \) so that

\[
A_1 = [1, 0, 0, 0], \quad A_2 = [0, 1, 0, 0], \quad A_3 = [0, 0, 1, 0], \quad A_4 = [0, 0, 0, 1].
\]

Let \( q = \sum a_{ij}x_ix_j \) be an equation of \( Q \). Since \( A_i \neq Q \) we have \( a_{ii} \neq 0 \). After a change of coordinates, we can assume that \( a_{ii} = 1 \) for \( i \in \{1, 2, 3, 4\} \). Then \( \det(L_T) \) is the determinant of the matrix of \( Q \) and so \( \det(L_T) \neq 0 \), because \( Q \) is smooth. It follows that \( L_T \in \mathbb{T} \).

Given a point \( L \in \mathbb{T} \) consider a projective tetrahedron \( T \) with vertices

\[
A_1 = [1, 0, 0, 0], \quad A_2 = [0, 1, 0, 0], \quad A_3 = [0, 0, 1, 0], \quad A_4 = [0, 0, 0, 1].
\]

and a quadric

\[
x_1^2 + L(e_{12})x_2^2 + L(e_{13})x_3^2 + L(e_{14})x_4^2
- (L(e_{12}) + 1) x_1x_2 + (L(e_{13}) + 1) x_1x_3 + (L(e_{14}) + 1) x_1x_4
+ \left( L\left(\frac{e_{12} + e_{13} + e_{23} + e_{24}}{2}\right) + L\left(\frac{e_{12} + e_{13} - e_{23} + e_{24}}{2}\right)\right) x_2x_3
+ \left( L\left(\frac{e_{12} + e_{14} + e_{24} + e_{34}}{2}\right) + L\left(\frac{e_{12} + e_{14} - e_{24} + e_{34}}{2}\right)\right) x_2x_4
+ \left( L\left(\frac{e_{13} + e_{14} + e_{34} + e_{24}}{2}\right) + L\left(\frac{e_{13} + e_{14} - e_{34} + e_{24}}{2}\right)\right) x_3x_4 = 0.
\]

It is easy to see that coordinates of points \( E_{ij} \) could be computed explicitly from the values of \( L \) on the roots and there exists a canonical labelling of these points so that \( L_T = L \). Thus for every point \( L \in \mathbb{T} \) there exist exactly two marked projective tetrahedra with \( L_T = L \) with different orientations of \( Q \). From here the statement follows.

The group \( W(D_6) \) acts on \( \mathbb{T} \) and this action extends to the action on \( \text{M}_{tet} \). The stabilizer of the set of twelve roots \( \pm e_{ij} \) is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^6 \times \mathbb{S}_4 \) and acts by changing the marking of a generic projective tetrahedron. The group \( W(D_6) \) is generated by this subgroup and a Regge symmetry.

3 Rational elliptic surfaces and period maps

3.1 Rational elliptic surfaces

A rational elliptic surface is a smooth complete rational surface \( X \), which admits an elliptic fibration, see [HL02], [Man86], [SS10]. The elliptic fibration on \( X \) is unique and is given by the anti-canonical linear system \( | - K_X | \). We assume that the elliptic fibration is relatively minimal and has a section. It is known that every rational elliptic surface can be obtained by blowing up the nine base points of a pencil of plane cubic curves having at least one smooth member.

The Picard lattice \( \text{Pic}(X) \) has rank 10 and signature \((1, 9)\). Denote by \( f \) the class of a fiber, which is equal to \(-K_X\). The orthogonal complement \( f^\perp \subset \text{Pic}(X) \) contains an isotropic vector
Figure 1: The elliptic fibration on a $D_6$-surface with fibers $F_1$ and $F_2$ of type $I_2$. $f$ and is an affine root lattice of type $E_8^1$. The quotient $f^\perp/f$ is a lattice of type $E_8$; we have a projection

$$\pi: f^\perp \rightarrow f^\perp/f.$$ 

An element $r \in \text{Pic}(X)$ is called a root if $r \in f^\perp$ and $r^2 = -2$. If $s_0$ is a section of $X$ then the lattice $\langle s_0, f \rangle$ is unimodular so

$$\text{Pic}(X) = \langle s_0, f \rangle \oplus \langle s_0, f \rangle^\perp.$$ 

The map $\langle s_0, f \rangle^\perp \rightarrow f^\perp/f$ is an isometry.

From the adjunction formula it follows that the self-intersection number of a smooth rational curve $C$ on a rational elliptic surface is greater or equal than $-2$. Smooth rational curves with self-intersection number $-1$ are sections of the elliptic fibration. Smooth rational curves with self-intersection number $-2$ are irreducible components of reducible fibers of the elliptic fibration. The classification of singular fibers of an elliptic fibration goes back to Kodaira. The Euler characteristic of a rational elliptic surface is equal to 12, so it can have at most twelve singular fibers. The simplest type of a singular fiber is called $I_n$; such fiber is a “wheel” made up of $n$ smooth rational curves intersecting transversally.

**Definition 3.1.** Let $X$ be a rational elliptic surface with a pair $F_1, F_2$ of singular fibers of type $I_2$. We call a triple $(X, F_1, F_2)$ a $D_6$-surface. A $D_6$-surface is called generic if $F_1$ and $F_2$ are the only reducible fibers of the elliptic fibration.

The term “$D_6$-surface” is justified by the fact that the orthogonal complement in $f^\perp/f$ to the classes of the components of the fibers $F_1$ and $F_2$ is a root lattice of type $D_6$. Recall that in §2.2 we defined a root system $E_8$.

**Definition 3.2.** A marking of a $D_6$-surface $X$ consists of the following data.

1. A choice of a section $s_0$ of the elliptic fibration (zero section).

2. An isometry $m: \text{Q}(E_8) \rightarrow f^\perp/f$ sending $e_1$ and $e_\infty$ to the classes of components of fibers $F_1$ and $F_2$, which intersect $s_0$.

3. A choice of a nodal point in $F_1$ and in $F_2$.

The Weyl group $W(D_6)$ acts on the set of markings by changing the isomorphism $m$. 

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We will usually omit \( m \) from the notation and simply put \( Q(E_b) = f^\perp/f \). We denote the components of \( F_1 \) and \( F_2 \) intersecting \( s_0 \) by \( F_{11} \) and \( F_{21} \). The other components are denoted \( F_{12} \) and \( F_{22} \), see Figure 1. Thus

\[
\begin{align*}
\pi ([F_{11}]) &= e_1, \\
\pi ([F_{12}]) &= -e_1, \\
\pi ([F_{21}]) &= e_3, \\
\pi ([F_{22}]) &= -e_3.
\end{align*}
\]

The first nodal point of \( F_1 \) is denoted by \( F_1^x \), the second is denoted by \( F_1^y \), and similarly for \( F_2 \).

### 3.2 Period maps

In this section we adapt the ideas of [Loo81, §5] to the case of a marked \( D_6 \)-surface \( X \). A choice of a nodal point in \( F_1 \) and \( F_2 \) fixes isomorphisms

\[
\text{Pic}^0(F_1) \cong \mathbb{C}^\times, \quad \text{Pic}^0(F_2) \cong \mathbb{C}^\times.
\]

Let \( d \in \text{Pic}(X) \) be a line bundle, which restricts trivially to both components of \( F_1 \). In this case \( d \in f^\perp \) and \( \pi(d) \in Q(E_1^F) \). The restriction of \( d \) to \( F_1 \) determines an element \( \text{Res}_{F_1}(d) \in \text{Pic}^0(F_1) \). Since the restriction of \( f \in f^\perp \) is trivial, the image of \( d \) in \( \text{Pic}^0(F_1) \) depends only on \( \pi(d) \in Q(E_1^F) \).

Thus we have constructed a period map

\[
\text{Res}_{F_1} : Q(E_1^f) \longrightarrow \mathbb{C}^\times.
\]

Similarly, by restricting to \( F_2 \) we define a second period map

\[
\text{Res}_{F_2} : Q(E_1^f) \longrightarrow \mathbb{C}^\times.
\]

Here is a more explicit description of the map \( \text{Res}_{F_1} \) (similar statements hold for \( \text{Res}_{F_2} \)). Consider a root \( r \in \text{Pic}(X) \), which can be represented as a difference of classes of sections \( s_1 \) and \( s_2 \) intersecting the fiber \( F_1 \) in the same component. If this component is \( F_{11} \), then

\[
\text{Res}_{F_1}(r) = \text{Res}_{F_1}([s_1] - [s_2]) = [F_1^x, s_1 \cap F_{11}, F_1^y, s_2 \cap F_{11}]_{F_{11}}.
\]

If this component is \( F_{12} \), then

\[
\text{Res}_{F_1}(r) = \text{Res}_{F_1}([s_1] - [s_2]) = [F_1^y, s_1 \cap F_{12}, F_1^x, s_2 \cap F_{12}]_{F_{12}}.
\]

**Lemma 3.3.** Let \( X \) be a \( D_6 \)-surface. The following conditions are equivalent.

1. \( X \) is generic.
2. We have \( \text{Res}_{F_1}(r) = 1 \) for a root \( r \in R(E_1^f) \) if and only if \( r = \pm e_3 \).
3. We have \( \text{Res}_{F_2}(r) = 1 \) for a root \( r \in R(E_1^f) \) if and only if \( r = \pm e_1 \).

**Proof.** Let \( D \) be an irreducible component of a reducible fiber, distinct from \( F_1 \) and \( F_2 \). Then \( \pi([D]) \in R(D_6) \) and we have

\[
\text{Res}_{F_1}([D]) = \text{Res}_{F_2}([D]) = 1.
\]

This proves that \( 2 \) implies \( 1 \) and that \( 3 \) implies \( 1 \). To show that \( 1 \) implies \( 2 \) assume that there exists a root \( r \in R(E_1^f) \) such that \( \text{Res}_{F_1}(r) = 1 \) and \( r \neq \pm e_3 \). By [Loo81, Proposition 5.2] there exists a component \( D \) of a reducible fiber \( F \neq F_1, F_2 \) of the elliptic fibration such that \( r = \pi([D]) \). Thus \( X \) is not generic. Similarly, \( 1 \) implies \( 3 \).
3.3 Admissible conic bundles

Definition 3.4. A conic bundle on a $D_5$-surface $X$ is a surjective morphism $b : X \to \mathbb{P}^1$ such that the generic fiber is a smooth rational curve. The conic bundle $b$ is called admissible if the irreducible components of fibers $F_1$ and $F_2$ are sections of $b$. The fiber $b^{-1}(p)$ will be denoted $b_p$ and its class $[b_p] \in \text{Pic}(X)$ will be denoted by $B$.

Lemma 3.5. Every singular fiber of an admissible conic bundle on a $D_5$-surface $X$ is a chain of smooth rational curves $Y_1, \ldots, Y_k$ for $k \geq 2$, where $Y_1, Y_k$ are $(-1)$-curves and $Y_2, \ldots, Y_{k-1}$ are $(-2)$-curves.

Proof. This result follows immediately from the classification of singular fibers of conic bundles on rational elliptic surfaces, see [GS19, Proposition 5.1].

Assume that $b : X \to \mathbb{P}^1$ is an admissible conic bundle on $X$. Fix a zero section $s_0$ on $X$ and label the components of fibers $F_1$ and $F_2$ as in §3.1. A computation of Euler characteristic shows that a conic bundle $b : X \to \mathbb{P}^1$ has at most eight singular fibers. Since curves $F_1, F_2, F_{21}, F_{22}$ are sections of the conic bundle, for every singular fiber $b_p = Y_1 \cup \ldots \cup Y_k$ the points $Y_1 \cap F_1$ lie in different components of $F_1$ and the points $Y_1 \cap F_2$ lie in different components of $F_2$; similarly for $Y_2$. We denote the component of $b_p$ intersecting $F_{11}$ by $b_{p1}^1$ and the component intersecting $F_{21}$ by $b_{p1}^2$. Notice that $b_{p1}^1$ and $b_{p2}^2$ are not necessarily distinct.

Definition 3.6. The conic bundle function $C_{X,b} \in \mathbb{C}(\mathbb{P}^1)$ is a rational function on the target space of $b$ which associates to a point $p \in \mathbb{P}^1$ the cross-ratio of the four points in which the fiber $b_p$ intersects $F_1$ and $F_2$, namely

$$C_{X,b}(p) = [b_p \cap F_{11}, b_p \cap F_{21}, b_p \cap F_{12}, b_p \cap F_{22}]_{b_p}.$$  \hfill (3.3)

Lemma 3.7. 1. $C_{X,b} \in \mathbb{C}(\mathbb{P}^1)$ is a rational function of degree 4.

2. Assume that fiber $b_p$ has $k$ irreducible components. Then

$$\text{ord}_p(C_{X,b}) = \begin{cases} 
0 & \text{if } k = 1, \\
k - 1 & \text{if } k > 1 \text{ and } b_{p1}^1 = b_{p2}^2, \\
-(k - 1) & \text{if } k > 1 \text{ and } b_{p1}^1 \neq b_{p2}^2.
\end{cases}$$

3. $C_{X,b}$ takes the value $1$ at four points

$$b(F_1^x), b(F_1^y), b(F_2^x), b(F_2^y) \in \mathbb{P}^1.$$

Proof. This statement follows directly from (3.3) and Lemma 3.5.

Assume that

$$\text{div} \ C_{X,b} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8$$  \hfill (3.4)

for points $p_1, \ldots, p_8 \in \mathbb{P}^1$, which are not necessarily distinct. Then $b_{p1}^i = b_{p2}^i$ for $i \in \{1, 2, 3, 4\}$ and $b_{p1}^i \neq b_{p2}^i$ for $i \in \{5, 6, 7, 8\}$.

Our next goal is to write down the conic bundle function explicitly. For that we choose a coordinate on $\mathbb{P}^1$. There are two natural ways to do it. Consider coordinates $t_1, t_2$ on $\mathbb{P}^1$ such that

$$t_1(b(F_1^x)) = t_2(b(F_1^y)) = 0,$$

$$t_1(b(F_2^x)) = t_2(b(F_2^y)) = \infty,$$

$$t_1(p_8) = t_2(p_8) = 1.$$

Denote by $\psi_{X,b} \in \text{PGL}_2(\mathbb{C})$ the Möbius transformation such that $t_2 = \psi_{X,b}(t_1)$.
Lemma 3.8. For $1 \leq i \leq 8$ we have $t_1(p_i) = \text{Res}_{F_1}([b^1_{p_i} - b^1_{p_s}])$ and $t_2(p_i) = \text{Res}_{F_2}([b^2_{p_i} - b^2_{p_s}])$.

Proof. The morphism $b: X \to \mathbb{P}^1$ induces an isomorphism between $F_{11}$ and $\mathbb{P}^1$, so the conic bundle function can be viewed as a rational function on $F_{11}$. The first statement follows from Lemma 3.7 and (3.1). The proof of the second statement is similar. □

Corollary 3.9. The function $C_{X,b} \in \mathbb{C}(\mathbb{P}^1)$ can be written as a rational function of $t_1$ or as a rational function of $t_2$ :

$$C_{X,b} = C^L_{X,b}(t_1) = C^A_{X,b}(t_2).$$

We have

$$C^L_{X,b}(t) = \left(\frac{t - \text{Res}_{F_1}([b^1_{p_1} - b^1_{p_s}])}{t - \text{Res}_{F_1}([b^1_{p_2} - b^1_{p_s}])} \frac{t - \text{Res}_{F_1}([b^1_{p_3} - b^1_{p_s}])}{t - \text{Res}_{F_1}([b^1_{p_4} - b^1_{p_s}])}\right)$$

We have $C^L_{X,b}(t) = C^A_{X,b}(\psi_{X,b}(t))$.

3.4 Conic bundles on a generic $D_6$-surface

Proposition 3.10. If $X$ is a generic $D_6$-surface then there exists an admissible conic bundle $b: X \to \mathbb{P}^1$.

Proof. By [Fus06, Theorem 3.5] a generic $D_6$-surface $X$ is obtained as a blow up of $\mathbb{P}^2$ at the nine base points of a pencil of plane cubes generated by curves $C_1 \cup l_1$ and $C_2 \cup l_2$ for conics $C_1, C_2$ and lines $l_1, l_2$ such that $C_1 \cap C_2 \cap l_1 = \emptyset$ and $C_1 \cap C_2 \cap l_2 = \emptyset$. Pick a point $p \in C_1 \cap C_2$. The pencil of lines passing through $p$ defines an admissible conic bundle on $X$. 

Lemma 3.5 implies that $b$ has exactly eight singular fibers, because otherwise a component $Y_2$ of a singular fiber would be a component of a reducible fiber $F$ of the elliptic fibration, such that $F$ is distinct from $F_1$ and $F_2$. It follows that points $b_1, \ldots, b_8$ are distinct. Let us fix their order and assume that (3.4) holds.

We are going to define a marking of the surface $X$. Since we have already chosen a zero section and ordered singular points of the fibers, we just need to construct an isometry between $\mathbb{F}_f$ and $\mathbb{Q}(E_8)$. For this we choose a base of roots in $\mathbb{F}_f$ and in $E_8$ as in Figure 2; there exists an isometry identifying the corresponding simple roots. To check that this defines a marking in the sense of Definition 3.2 we need to prove that $e_{\mathbb{F}} = \pi([F_{21}])$; Lemma 3.11 does this.

Lemma 3.11. The following equalities hold in $\text{Pic}(X)$ :

$$2B = \sum_{i=1}^{4} [b^1_{p_i}] + [F_{11}] - [F_{22}].$$

Proof. Consider eight $(-1)$-curves $b^1_{p_i}, 1 \leq i \leq 8$. These curves are pairwise disjoint, so could be blown down simultaneously by a morphism $r: X \to Y$. Rational surface $Y$ has Picard number 2 and contains a $(-2)$-curve $r(F_{12})$, hence $Y$ is isomorphic to the Hirzenbruch surface $\Sigma_2$. Let $s_{Y} = [r(F_{12})] \in \text{Pic}(Y)$ be the class of the zero section of $Y$ and $f_Y \in \text{Pic}(Y)$ be the classes of a fiber. It is known that $\text{Pic}(Y) = \langle s_{Y}, f_{Y} \rangle$ and $s_{Y}^2 = -2, s_{Y}f_{Y} = 1, f_{Y}^2 = 0$. Computing the intersection indices one can see that $r_{*}([F_{11}]) = s_{Y} + 4f_{Y}$ and $r_{*}([F_{22}]) = s_{Y} + 2f_{Y}$. It follows
Figure 2: Matching root bases of lattices $f^+/f$ and $Q(E_8)$. We use the same notation for a class $r \in f^+$ and its projection $\pi(r) \in f^+/f$.

that

$$r^*r_*([F_{11}]) = [F_{11}] + \sum_{i=1}^{8} [b^1_{p_i}],$$

$$r^*r_*([F_{22}]) = [F_{22}] + \sum_{i=5}^{8} [b^1_{p_i}],$$

so

$$\sum_{i=1}^{4} [b^1_{p_i}] + [F_{11}] - [F_{22}] = 2r^*\tau(f_Y) = 2B.$$

From here one can deduce that the following equalities hold:

$$\frac{e_{23} + e_{24} + e_{34} - e_5}{2} = \pi((b^1_{p_1} - b^1_{p_5})),$$

$$\frac{e_{12} + e_{13} + e_{14} + e_I}{2} = -\pi((b^2_{p_1} - b^2_{p_5})),$$

$$\frac{e_{13} + e_{14} + e_{34} - e_5}{2} = \pi((b^1_{p_2} - b^1_{p_8})),$$

$$\frac{e_{12} + e_{23} + e_{24} + e_I}{2} = -\pi((b^2_{p_2} - b^2_{p_8})),$$

$$\frac{e_{12} + e_{14} + e_{23} - e_5}{2} = \pi((b^1_{p_3} - b^1_{p_9})),$$

$$\frac{e_{13} + e_{23} + e_{34} - e_5}{2} = -\pi((b^2_{p_3} - b^2_{p_9})),$$

$$\frac{e_{12} + e_{13} + e_{24} - e_5}{2} = \pi((b^1_{p_4} - b^1_{p_{10}})),$$

$$\frac{e_{12} + e_{14} + e_{23} + e_{34}}{2} = -\pi((b^2_{p_4} - b^2_{p_{10}})),$$

$$\frac{e_{13} + e_{24} - e_{14}}{2} = \pi((b^1_{p_5} - b^1_{p_{11}})),$$

$$\frac{e_{12} + e_{13} + e_{24} + e_{34}}{2} = -\pi((b^2_{p_5} - b^2_{p_{11}})).$$

3.5 The moduli space of generic marked $D_6$-surfaces

Let $X$ be a marked $D_6$-surface. The period map is a point $\text{Res}_{F_1} \in \text{Hom}(Q(E_7^1), \mathbb{C}^\times)$. In §2.4 we defined an open subset $\mathcal{T} \subseteq \text{Hom}(Q(E_7^1), \mathbb{C}^\times)$. 

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Proposition 3.12. For a generic marked $D_6$-surface $(X, F_1, F_2)$ we have $\text{Res}_{F_1} \in T$.

Proof. From Lemma 3.3 we see that $\text{Res}_{F_1}(r) = 1$ for a root $r \in \mathbb{R}(E_f^1)$ if and only if $r = \pm e_f$. It remains to prove that $\det(\text{Res}_{F_1}) \neq 0$. By Lemma 2.8 this statement does not depend on the choice of the marking of $X$. It is convenient to choose the marking introduced in §3.4. The points $t_1(b(F_2^0))$ and $t_1(b(F_3^0))$ are distinct, so the discriminant of the quadratic polynomial

$$
\begin{align*}
\frac{1}{t} & \left( (t - \text{Res}_{F_1}(b_{p_1}^1 - b_{p_1}^1)) (t - \text{Res}_{F_1}(b_{p_2}^1 - b_{p_2}^1)) (t - \text{Res}_{F_1}(b_{p_4}^1 - b_{p_4}^1)) \\
& - (t - \text{Res}_{F_1}(b_{p_5}^1 - b_{p_5}^1)) (t - \text{Res}_{F_1}(b_{p_6}^1 - b_{p_6}^1)) (t - \text{Res}_{F_1}(b_{p_7}^1 - b_{p_7}^1)) (t - \text{Res}_{F_1}(b_{p_8}^1 - b_{p_8}^1)) \right)
\end{align*}
$$

is not equal to zero. One can check by a direct but tedious computation that the discriminant of a quadratic polynomial

$$
\begin{align*}
\frac{1}{t} & \left( (t - a_{12}a_{23}a_{13})(t - a_{12}a_{24}a_{14})(t - a_{13}a_{34}a_{14})(t - a_{24}a_{34}a_{23}) \\
& - (t - a_{12}a_{23}a_{34}a_{14})(t - a_{13}a_{24}a_{14})(t - a_{12}a_{24}a_{34}a_{13})(t - 1) \right)
\end{align*}
$$

is equal to

$$
16 \left( \prod_{i < j} a_{ij} \right)^2 \left| \begin{array}{cccc}
1 & a_{12} & a_{13} & a_{14} \\
a_{12} & a_{12}^2 & a_{13} & a_{14} \\
a_{13} & a_{13} & a_{13}^2 & a_{14} \\
a_{14} & a_{14} & a_{14} & a_{14}^2 \\
\end{array} \right|,
$$

which implies that $\det(\text{Res}_{F_1}) \neq 0$.

Proposition 3.13. The set $M_{\text{surf}}$ of isomorphism classes of generic marked $D_6$-surfaces has a structure of an algebraic variety, which is an unramified double cover of $T$.

Proof. Consider the map that associates to a marked $D_6$-surface $X$ the period map $\text{Res}_{F_1} \in T$. From [Loo81, Proposition 5.5] it follows that this map is surjective.

Next, consider two marked $D_6$-surfaces $X$ and $X'$ with $\text{Res}_{F_1} = \text{Res}_{F_1'}$. Recall the orthogonal decompositions

$$
\text{Pic}(X_1) = \langle s_0, f \rangle \oplus \langle f^\perp, f \rangle, \quad \text{Pic}(X_2) = \langle s_0', f' \rangle \oplus \langle f'^\perp, f' \rangle.
$$

Define an isometry $\psi: \text{Pic}(X_1) \rightarrow \text{Pic}(X_2)$ by a rule $\psi(s_0) = s_0', \quad \psi(f) = f'$ and $\psi = m_2 \circ m_1^{-1}$ on $f^\perp/f$. Now we are in a position to apply the Torelli theorem, see [Loo81, Theorem 5.3]. Indeed, $(X, F')$ and $(X, F')$ are smooth rational surfaces endowed with negative oriented anticanonical cycles $F_{11} + F_{12}$ and $F_{11}' + F_{12}'$. The isometry $\psi$ sends roots to roots, the positive cone to the positive cone, and $\psi$ commutes with period maps. Since $X$ is a generic $D_6$-surface, it has only four $-2$-curves, so the lattice $\langle F_{11}, F_{12} \rangle^\perp$ has two nodal roots, namely $(f, m_1(e_f))$ and $(0, -m_1(e_f))$; the same for $X'$. Since $\psi(f, m_1(e_f)) = (f', m_2(e_f))$ and $\psi(0, -m_1(e_f)) = (0, -m_2(e_f))$, isometry $\psi$ sends nodal roots to nodal roots. By [Loo81, Theorem 5.3] $\psi$ must be induced by an isomorphism $\Psi: X_1 \rightarrow X_2$ such that $\Psi(F_{11}') = (F_{11}')'$ and $\Psi(F_{12}') = (F_{12}')'$. The only ambiguity is in the action of $\Psi$ on the marking of nodal points of the second special fiber. From here the statement follows.
4 Correspondence between tetrahedra and $D_6$-surfaces

4.1 Proofs of Theorem 1.7 and Theorem 1.9

In §2.4 we defined the moduli space of generic marked projective tetrahedra $M_{tet}$ and in §3.5 we defined the moduli space of generic marked $D_6$-surfaces $M_{surf}$. Propositions 2.9 and 3.13 imply that both $M_{tet}$ and $M_{surf}$ are unramified double covers of the same variety $T$. In §4.2 we construct a rational map

$$\text{Cor}: M_{tet} \to M_{surf}$$

sending $T$ to $(X_T, F_1, F_2)$. In §4.3 we show that this map commutes with projections to $T$. It is easy to see that a rational map $U \to V$, which commutes with étale maps $U \to X$ and $V \to X$ can be extended to a morphism, so the rational map $\text{Cor}$ can be extended to a morphism

$$\text{Cor}: M_{tet} \to M_{surf}.$$ 

From the construction it follows immediately that Cor is equivariant with respect to the deck transformations of the coverings, so Cor is an isomorphism. In §4.4 we prove the equality $\text{Res}_{F_2} = A_T$, which finishes the proof of Theorem 1.9.

Finally, we show that Theorem 1.9 implies Theorem 1.7. Indeed, Cho-Kim functions of a tetrahedron (see Definition 1.6) coincide with conic bundle functions of the corresponding $D_6$-surface, so Theorem 1.7 follows from Corollary 3.9.

4.2 The construction of the correspondence

Our goal in this section is to construct a rational map

$$\text{Cor}: M_{tet} \to M_{surf}.$$ 

Since we are constructing only a rational map, we assume that vertices of $T$ are in general position in the sense specified in the course of the proof.

Consider a marked projective tetrahedron $T = (Q, \{H_1, \ldots, H_4\})$. We construct the rational elliptic surface $(X_T, F_1, F_2)$ in two steps. First, we blow up the quadric $Q$ at the twelve points $E_{ij}$ and obtain a rational surface $R_T$. We denote the class of the preimage of the exceptional divisor associated to the blow up at $E_{ij}$ by $[E_{ij}]$ and the strict transforms of the generators of the quadric by $[L]$ and $[R]$. The surface $X_T$ is obtained from $R_T$ by blowing down the following four $(-1)$-curves:

$$[E_{21}],$$
$$[L] - [E_{12}],$$
$$[R] - [E_{12}],$$
$$[L] + [R] - [E_{34}] - [E_{43}] - [E_{12}].$$

We may assume that these lines are pairwise disjoint. Denote by $g: Q \to X_T$ the resulting rational map.

Lemma 4.1. The triple $\left( X_T, g(H_3 \cap H_4 \cap Q), g(H_1 \cap H_2 \cap Q) \right)$ is a $D_6$-surface.

Proof. The eight $(-1)$-curves

$$[E_{13}], [E_{31}], [E_{14}], [E_{41}], [E_{23}], [E_{32}], [E_{24}], [E_{42}],$$

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Figure 3: A pencil of \((2,2)\) curves on \(\mathbb{P}^1 \times \mathbb{P}^1\). Surface \(X_T\) is obtained by blowing up points \(u_{ij}\).

are mutually disjoint on \(X_T\), so they can be blown down simultaneously. By doing so, we obtain a del Pezzo surface with Picard number 1. The image of the strict transform of \(H_1 \cap Q\) has self-intersection number equal to 2, so the surface is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\). The images of the curves \([E_{ij}]\) lie on a pair of reducible \((2,2)\)-curves, which are the strict transforms of \((H_1 \cup H_2) \cap Q\) and \((H_3 \cup H_4) \cap Q\), so \(X_T\) is a rational elliptic surface with a pair of \(I_2\)-fibers.

We introduce the following notation for the components of the fibers:

\[
\begin{align*}
F_{11} & := g(H_3 \cap Q), \\
F_{12} & := g(H_4 \cap Q), \\
F_{21} & := g(H_1 \cap Q), \\
F_{22} & := g(H_2 \cap Q).
\end{align*}
\]

and singular points

\[
\begin{align*}
F_{11}^x & := g([E_{21}]), \\
F_{12}^y & := g([L] + [R] - [E_{12}] - [E_{34}] - [E_{43}]), \\
F_{21}^x & := g([L] - [E_{12}]), \\
F_{22}^y & := g([R] - [E_{12}]).
\end{align*}
\]

**Lemma 4.2.** The Picard lattice of \(X_T\) is generated by the classes

\[
\begin{align*}
l & := g([L] + [R] - [E_{12}] - [E_{34}]), \\
r & := g([L] + [R] - [E_{12}] - [E_{43}]), \\
u_{13} & := g([E_{13}]), \quad u_{31} := g([E_{31}]), \quad u_{14} := g([E_{14}]), \quad u_{41} := g([E_{41}]), \\
u_{23} & := g([E_{23}]), \quad u_{42} := g([E_{42}]), \quad u_{24} := g([E_{24}]), \quad u_{43} := g([E_{43}]).
\end{align*}
\]

The pairing is given on the classes by

\[
l^2 = r^2 = 0, \quad l.r = 1, \quad l.u_{ij} = r.u_{ij} = 0,
\]

\[
u_{ij}.u_{kl} = \begin{cases} 
-1 & \text{if } u_{ij} = u_{kl} \\
-1 & \text{if } u_{ij} \neq u_{kl}.
\end{cases}
\]

The canonical class is equal to

\[
K_X = -2l - 2r + u_{13} + u_{34} + u_{14} + u_{41} + u_{23} + u_{32} + u_{24} + u_{42}.
\]
allow us to reduce that even further and check the equality for only two roots.

\[\begin{align*}
-l - r + u_{24} + u_{42} + u_{14} + u_{41} &\quad -l - u_{24} - u_{13} \\
l - r - u_{23} - u_{31} &\quad l + r - u_{31} - u_{32} - u_{41} - u_{42} \\
l + r - u_{31} - u_{32} - u_{41} - u_{42} &\quad l + r - u_{23} - u_{31} - u_{41} - u_{24} \\
l - u_{23} - u_{14} &\quad l - u_{32} - u_{14}
\end{align*}\]

Proof. It is easy to see that the lattice \(Q(T)\) comes from the presentation of \(X_T\) as a blow up of \(\mathbb{P}^1 \times \mathbb{P}^1\) in eight points described in the proof of Lemma 4.1, see Figure 3.

To define the marking of \(X_T\) it remains to construct an isometry between \(Q(E_8)\) and \(f/f\). We do that by identifying root bases, as in Figure 4. This finishes the construction of a rational map \(\text{Cor.}\)

4.3 The correspondence commutes with length function

Proposition 4.3. The map \(M_{\text{tet}} \rightarrow M_{\text{surf}}\) commutes with the projections to \(T\). In other words, for a (general) marked projective tetrahedron \(T\) and the corresponding surface \((X_T, F_1, F_2)\) maps \(\text{LT}\) and \(\text{Res}_{F_1}\) coincide.

Proof. It is easy to see that the lattice \(Q(E_8)\) is generated by the roots

\[\begin{align*}
\pm e_{23} \pm e_{24} \pm e_{34} \pm e_{32},
\pm e_{13} \pm e_{14} \pm e_{34} \pm e_{32},
\pm e_{12} \pm e_{13} \pm e_{23} \pm e_{24},
\pm e_{12} \pm e_{14} \pm e_{24} \pm e_{32},
\end{align*}\]

so it is enough to check equality of \(\text{LT}\) and \(\text{Res}_{F_1}\) on them. The symmetries of the construction in §4.2 allow us to reduce that even further and check the equality for only two roots \(\frac{e_{23} + e_{24} + e_{34} + e_{32}}{2}\) and \(\frac{e_{12} + e_{13} + e_{23} + e_{24}}{2}\).

Lemma 4.4. We have \(\text{LT}\left(\frac{e_{23} + e_{24} + e_{34} + e_{32}}{2}\right) = \text{Res}_{F_1}\left(\frac{e_{23} + e_{24} + e_{34} + e_{32}}{2}\right)\).

Proof. First, we have

\[\frac{e_{23} + e_{24} + e_{34} + e_{32}}{2} = \pi(l - u_{23} - u_{42}).\]

Consider the sections \([l - u_{23}]\) and \([u_{42}]\) of the elliptic fibration on \(X_T\). They both intersect the component \(F_{11}\) of the fiber \(F_1\). By (3.1) we have

\[\text{Res}_{F_1}\left(\frac{e_{23} + e_{34} + e_{24} + e_{32}}{2}\right) = \text{Res}_{F_1}([l - u_{23}] - [u_{42}]) = \left[F^r_{11}, F_{11} \cap [l - u_{23}], F^y_{11}, F_{11} \cap [u_{42}]\right].\]
Our goal is to compute the images of the points $F_1^x, F_{11} \cap [l - u_{23}], F_1^y$, and $F_{11} \cap [u_{42}]$ under the birational isomorphism $g^{-1}$ restricted to $F_{11}$. Denote by $U$ the intersection point of the plane $(A_3A_4E_{12})$ with $H_3 \cap Q$ which is different from $E_{12}$. Similarly, denote by $V$ the intersection point of the plane $(E_{12}E_{34}E_{23})$ with $H_3 \cap Q$ which is different from $E_{12}$. Under the birational isomorphism $g^{-1}$ the curve $F_{11}$ is mapped to the conic $H_3 \cap Q$. Moreover, the restriction of $g^{-1}$ to $F_{11}$ sends $F_1^x$ to $E_{21}$, $F_1^y$ to $U$, $F_{11} \cap [l - u_{23}]$ to $V$, and $F_{11} \cap [u_{42}]$ to $E_{42}$. Since $g^{-1}$ is a birational isomorphism,

$$[F_1^x, F_{11} \cap [l - u_{23}], F_1^y, F_{11} \cap [u_{42}]] = [E_{21}, V, U, E_{42}]_{H_3 \cap Q}.$$

Next, consider the projection from the point $E_{12}$ of the conic $H_3 \cap Q$ to the line $(A_2A_4)$. We get that

$$\text{Res}_{F_1} \left(\frac{e_{23} + e_{24} + e_{34} + e_\emptyset}{2}\right) = [E_{21}, V, U, E_{42}]_{H_3 \cap Q}$$

(4.1)

On the other hand, Lemma 2.4 implies that

$$L_T \left(\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2}\right) = [A_2, (A_2A_4) \cap (E_{23}E_{34}), A_4, E_{42}]_{(A_2A_4)}.$$

(4.2)

The statement of the lemma follows from (4.1) and (4.2). \hfill \Box

**Lemma 4.5.** We have $L_T \left(\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2}\right) = \text{Res}_{F_1} \left(\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2}\right)$.

**Proof.** The proof is similar to the proof of the previous lemma. First, observe that

$$\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2} = \pi(u_{31} - u_{23}).$$

Next, consider the sections $u_{31}$ and $u_{23}$, which both intersect $F_{12}$. By (3.2) we have

$$\text{Res}_{F_1} \left(\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2}\right) = \text{Res}_{F_1}([u_{31}] - [u_{23}])$$

$$= [F_1^y, F_{12} \cap [u_{31}], F_1^x, F_{12} \cap [u_{23}]].$$

First, we compute the images of the points $F_1^y, F_{12} \cap [u_{31}], F_1^x$, and $F_{12} \cap [u_{23}]$ under the birational isomorphism $g^{-1}$ restricted to $F_{12}$. Denote by $W$ the intersection point of the plane $(A_3A_4E_{12})$ with the conic $H_4 \cap Q$ which is different from $E_{12}$. Under the birational isomorphism $g^{-1}$ the curve $F_{12}$ is mapped to the conic $H_4 \cap Q$, $F_1^y$ to $W$, $F_{12} \cap [u_{31}]$ to $E_{31}$, $F_1^x$ to $E_{21}$, and $F_{12} \cap [u_{23}]$ to $E_{23}$. Since $g^{-1}$ is a birational isomorphism, we have

$$[F_1^y, F_{12} \cap [u_{31}], F_1^x, F_{12} \cap [u_{23}]] = [W, E_{31}, E_{21}, E_{23}]_{H_4 \cap Q}.$$

Next, consider the projection from the point $E_{12}$ of the conic $H_4 \cap Q$ to the line $(A_2A_3)$. We get that

$$\text{Res}_{F_1} \left(\frac{e_{12} + e_{13} + e_{23} + e_\emptyset}{2}\right) = [W, E_{31}, E_{21}, E_{23}]_{H_4 \cap Q}$$

(4.3)

$$= [A_3, (E_{12}E_{31}) \cap (A_2A_3), A_2, E_{23}]_{(A_2A_3)}.$$
Lemma 2.4 implies that
\[
L_T \left( \frac{e_{12} + e_{13} + e_{23} + e_0}{2} \right) \\
= \left[ \tilde{A}_1, E_{12}, \tilde{A}_2, E_{21} \right] \left[ \tilde{A}_1, E_{13}, \tilde{A}_3, E_{31} \right] \left[ \tilde{A}_2, E_{23}, \tilde{A}_3, E_{32} \right]
= \left[ A_2, E_{23}, A_3, (E_{12}E_{31}) \cap (A_2A_3) \right]_{(A_2A_3)}
= \left[ A_3, (E_{12}E_{31}) \cap (A_2A_3), A_2, E_{23} \right]_{(A_2A_3)}.
\]

From (4.3) and (4.4) we get the result. □

Proposition 4.3 follows from Lemmas 4.4 and 4.5.

4.4 The correspondence commutes with angle function

Proposition 4.6. For a (general) marked projective tetrahedron \( T \) and the corresponding surface \((X_T, F_1, F_2)\) maps \( A_T \) and \( \text{Res}_{F_2} \) coincide.

Proof. Similarly to Proposition 4.3 in §4.3 it is sufficient to check the equality \( A_T = \text{Res}_{F_2} \) on the two roots
\[
\frac{e_{12} + e_{13} + e_{14} + e_I}{2}, \quad \frac{e_{14} + e_{24} + e_{34} + e_I}{2}.
\]
We do that in Lemmas 4.7 and 4.8.

Lemma 4.7. We have
\[
A_T \left( \frac{e_{12} + e_{13} + e_{14} + e_I}{2} \right) = \text{Res}_{F_2} \left( \frac{e_{12} + e_{13} + e_{14} + e_I}{2} \right).
\]

Proof. Observe that
\[
\frac{e_{12} + e_{13} + e_{14} + e_I}{2} = \pi(u_{31} - u_{41}).
\]
Sections \( u_{31} \) and \( u_{41} \) intersect the component \( F_{22} \) of the fiber \( F_2 \). We have
\[
\text{Res}_{F_2} \left( \frac{e_{12} + e_{13} + e_{14} + e_I}{2} \right) = \text{Res}_{F_2}([u_{31}] - [u_{41}])
= \left[ F_2^y, F_{22} \cap [u_{31}], F_2^x, F_{22} \cap [u_{41}] \right]_{F_{22}}.
\]

Our goal is to compute the images of the points \( F_2^y, F_{22} \cap [u_{31}], F_2^x \) and \( F_{22} \cap [u_{41}] \) under the birational isomorphism \( g^{-1} \) restricted to \( F_{22} \). The curve \( F_{22} \) is mapped to the conic \( H_2 \cap Q, F_2^y \) to \( H_2 \cap R_{E_{12}}, F_{22} \cap [u_{31}] \) to \( E_{31}, F_2^x \) to \( L_{E_{12}} \cap H_2 \), and \( F_{22} \cap [u_{41}] \) to \( E_{41} \). Since \( g^{-1} \) is a birational isomorphism, we have
\[
\left[ F_2^y, F_{22} \cap [u_{31}], F_2^x, F_{22} \cap [u_{41}] \right]_{F_{22}} = \left[ R_{E_{12}} \cap H_2, E_{31}, L_{E_{12}} \cap H_2, E_{41} \right]_{H_2 \cap Q}.
\]

Consider the map from \( H_2 \cap Q \) to \( R_{E_{12}} \) sending the point \( X \in H_2 \cap Q \) to \( L_X \cap R_{E_{12}} \). This map is an isomorphism, so
\[
\left[ R_{E_{12}} \cap H_2, E_{31}, L_{E_{12}} \cap H_2, E_{41} \right]_{H_2 \cap Q} = \left[ H_2 \cap R_{E_{12}}, L_{E_{31}} \cap R_{E_{12}}, E_{12}, L_{E_{41}} \cap R_{E_{12}} \right]_{R_{E_{12}}},
\]

Next we apply the duality with respect to \( Q \). Since points of \( Q \) are dual to the tangent planes at these points, we obtain the following equality of cross-ratios:
\[
\left[ H_2 \cap R_{E_{12}}, L_{E_{31}} \cap R_{E_{12}}, E_{12}, L_{E_{41}} \cap R_{E_{12}} \right]_{R_{E_{12}}}
= \left[ \left( H_2^y, R_{E_{12}} \right), \left( L_{E_{31}}, R_{E_{12}} \right), \left( L_{E_{12}}, R_{E_{12}} \right), \left( L_{E_{41}}, R_{E_{12}} \right) \right]_{R_{E_{12}}}.
\]

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Recall that for all $i,j$ we have $L_{E_{ij}} = L_{\tilde{E}_{ij}}$ and $R_{E_{ij}} = R_{\tilde{E}_{ij}}$, so

$$[\langle H^\gamma_2, R_{E_{12}} \rangle, \langle L_{E_{31}}, R_{E_{12}} \rangle, \langle L_{E_{12}}, R_{E_{12}} \rangle, \langle L_{E_{41}}, R_{E_{12}} \rangle]_{R_{E_{12}}} = [\langle H^\gamma_2, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{31}}, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{12}}, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{41}}, R_{\tilde{E}_{21}} \rangle]_{R_{\tilde{E}_{21}}}. \tag{4.9}$$

The last cross-ratio can be computed by intersecting the four planes with the line $(H^\gamma_2 H^\gamma_4)$:

$$[\langle H^\gamma_2, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{31}}, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{12}}, R_{\tilde{E}_{21}} \rangle, \langle L_{\tilde{E}_{41}}, R_{\tilde{E}_{21}} \rangle]_{R_{\tilde{E}_{21}}} = [H^\gamma_2, \tilde{E}_{31}, H^\gamma_4, (\tilde{E}_{41} \tilde{E}_{21}) \cap (H^\gamma_2 H^\gamma_4)]_{(H^\gamma_2 H^\gamma_4)}. \tag{4.10}$$

Combining (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) we get

$$\text{Res}_{F_2} \left( \frac{e_{12} + e_{13} + e_{14} + e_I}{2} \right) = [H^\gamma_2, \tilde{E}_{31}, H^\gamma_4, (\tilde{E}_{41} \tilde{E}_{21}) \cap (H^\gamma_2 H^\gamma_4)]_{(H^\gamma_2 H^\gamma_4)}. \tag{4.11}$$

From Lemma 2.4 we conclude that

$$A_T \left( \frac{e_{12} + e_{13} + e_{14} + e_I}{2} \right) = [H^\gamma_2, \tilde{E}_{31}, H^\gamma_4, (\tilde{E}_{41} \tilde{E}_{21}) \cap (H^\gamma_2 H^\gamma_4)]_{(H^\gamma_2 H^\gamma_4)}. \tag{4.12}$$

The lemma follows from (4.11) and (4.12).

**Lemma 4.8.** We have $A_T \left( \frac{e_{14} + e_{24} + e_{34} + e_I}{2} \right) = \text{Res}_{F_2} \left( \frac{e_{14} + e_{24} + e_{34} + e_I}{2} \right)$.

**Proof.** It is easy to see that

$$\frac{e_{14} + e_{24} + e_{34} + e_I}{2} = \pi (l - u_{42} - u_{41}).$$

Consider the sections $[l - u_{41}]$ and $[u_{42}]$ intersecting $F_{21}$. We have

$$\text{Res}_{F_2} \left( \frac{e_{14} + e_{24} + e_{34} + e_I}{2} \right) = \text{Res}_{F_2}([l - u_{41}] - [u_{42}]) = [F^x_{21}, F_{21} \cap [l - u_{41}], F^y_{21}, F_{21} \cap [u_{42}]]_{F_{21}}. \tag{4.13}$$

Our goal is to compute the images of the points $F^x_{21}, F_{21} \cap [l - u_{41}], F^y_{21}$ and $F_{21} \cap [u_{42}]$ under the birational isomorphism $g^{-1}$ restricted to $F_{21}$. Denote by $U$ the point of intersection of the plane $(E_{12} E_{34} E_{41})$ with $H_1 \cap Q$, which is different from $E_{34}$. Under the birational isomorphism $g^{-1}$ the curve $F_{21}$ is mapped to the conic $H_1 \cap Q$, $F^x_{21}$ to $L_{E_{12}} \cap H_1$, $F^y_{21}$ to $l - u_{41}$ to $U$, $F^y_{21}$ to $H_1 \cap R_{E_{12}}$, and $F_{21} \cap [u_{42}]$ to $E_{24}$. Since $g^{-1}$ is a birational isomorphism we have

$$[F^x_{21}, F_{21} \cap [l - u_{41}], F^y_{21}, F_{21} \cap [u_{42}]]_{F_{21}} = [L_{E_{12}} \cap H_1, U, H_1 \cap R_{E_{12}}, E_{42}]_{H_1 \cap Q}. \tag{4.14}$$

Next, consider the map from $H_1 \cap Q$ to $R_{E_{12}}$ sending $X \in H_1 \cap Q$ to $L_X \cap R_{E_{12}}$. It is an isomorphism, so

$$[L_{E_{12}} \cap H_1, U, H_1 \cap R_{E_{12}}, E_{42}]_{H_1 \cap Q} = [E_{12}, L_U \cap R_{E_{12}}, H_1 \cap R_{E_{12}}, L_{E_{42}} \cap R_{E_{12}}]_{R_{E_{12}}}. \tag{4.15}$$
We apply the duality with respect to $Q$:
\[
[E_{12}, L_U \cap R_{E_{12}}, H_1 \cap R_{E_{12}}, L_{E_{42}} \cap R_{E_{12}}]_{E_{12}} = [(L_{E_{12}}, R_{E_{12}}), \langle L_U, R_{E_{12}} \rangle, \langle H_{1}^{\gamma}, R_{E_{12}} \rangle, \langle L_{E_{42}}, R_{E_{12}} \rangle]_{E_{12}}.
\]
(4.16)
We have $L_{E_{ij}} = L_{E_{ij}}$ and $R_{E_{ij}} = R_{E_{ij}}$, so
\[
[(L_{E_{12}}, R_{E_{12}}), \langle L_U, R_{E_{12}} \rangle, \langle H_{1}^{\gamma}, R_{E_{12}} \rangle, \langle L_{E_{42}}, R_{E_{12}} \rangle]_{E_{12}} = \langle \langle L_{E_{12}}, R_{E_{12}} \rangle, \langle L_U, R_{E_{12}} \rangle, \langle H_{1}^{\gamma}, R_{E_{12}} \rangle, \langle L_{E_{42}}, R_{E_{12}} \rangle \rangle_{E_{12}}.
\]
(4.17)
The last cross-ratio can be computed by intersecting the four planes with the line $(H_{1}^{\gamma} H_{3}^{\gamma})$:
\[
[(L_{E_{12}}, R_{E_{21}}), \langle L_U, R_{E_{21}} \rangle, \langle H_{1}^{\gamma}, R_{E_{21}} \rangle, \langle L_{E_{42}}, R_{E_{21}} \rangle]_{E_{21}} = [H_{3}^{\gamma}, \langle U, R_{E_{12}} \rangle \cap (H_{1}^{\gamma} H_{3}^{\gamma}), H_{1}^{\gamma}, \tilde{E}_{42}](H_{1}^{\gamma} H_{3}^{\gamma}).
\]
(4.18)
Combining (4.13), (4.14), (4.15), (4.16), (4.17), and (4.18) we get
\[
\text{Res}_{E_{2}} \left( \frac{e_{14} + e_{24} + e_{34} + e_{1}}{2} \right) = [H_{3}^{\gamma}, \langle U, R_{E_{12}} \rangle \cap (H_{1}^{\gamma} H_{3}^{\gamma}), H_{1}^{\gamma}, \tilde{E}_{42}](H_{1}^{\gamma} H_{3}^{\gamma}).
\]
(4.19)
On the other hand, we have
\[
A_T \left( \frac{e_{14} + e_{24} + e_{34} + e_{1}}{2} \right) = [H_{3}^{\gamma}, \tilde{E}_{41}, H_{2}^{\gamma}, \tilde{E}_{14}][H_{2}^{\gamma}, \tilde{E}_{43}, H_{1}^{\gamma}, \tilde{E}_{34}][H_{1}^{\gamma}, \tilde{E}_{24}, H_{1}^{\gamma}, \tilde{E}_{42}]
\]
\[
= [H_{3}^{\gamma}, (\tilde{E}_{41} \tilde{E}_{43}) \cap (H_{1}^{\gamma} H_{3}^{\gamma}), H_{1}^{\gamma}, \tilde{E}_{42}](A_{2} A_{3}).
\]
To prove the lemma we need to show that the lines $(\tilde{E}_{41} \tilde{E}_{43})$, $(H_{1}^{\gamma} H_{3}^{\gamma})$ and the plane $\langle U, R_{E_{12}} \rangle$ intersect in one point. Consider point $V = (E_{12} E_{41}) \cap (U E_{34})$. The plane $V^\gamma$ passes through the line $(H_{1}^{\gamma} H_{3}^{\gamma})$. Since $V$ is contained in the plane $\langle R_{E_{12}}, L_{E_{41}} \rangle$, the plane $V^\gamma$ contains the point $W_1 = R_{E_{21}} \cap L_{E_{41}}$. Similarly, $V$ is contained in the plane $\langle L_U, R_{E_{34}} \rangle$, so $V^\gamma$ contains $W_2 = L_U \cap R_{E_{43}}$.

Consider the point $W = (W_1 W_2) \cap (H_{1}^{\gamma} H_{3}^{\gamma})$. We claim that
\[
W = (\tilde{E}_{41} \tilde{E}_{43}) \cap (H_{1}^{\gamma} H_{3}^{\gamma}) \cap \langle U, R_{E_{12}} \rangle.
\]
(4.20)
Indeed, clearly $W \in (H_{1}^{\gamma} H_{3}^{\gamma})$. Since
\[
\langle W_1 W_2 \rangle = \langle L_U, R_{E_{21}} \rangle \cap \langle R_{E_{43}}, L_{E_{41}} \rangle,
\]
the point $W$ is contained in the plane $\langle L_U, R_{E_{21}} \rangle = \langle U, R_{E_{12}} \rangle$. By the same reason, $W$ is contained in the plane $\langle R_{E_{43}}, L_{E_{41}} \rangle$. Since $W \in (H_{1}^{\gamma} H_{2}^{\gamma} H_{3}^{\gamma})$, we have
\[
W \in \langle R_{E_{43}}, L_{E_{41}} \rangle \cap (H_{1}^{\gamma} H_{2}^{\gamma} H_{3}^{\gamma}) = (\tilde{E}_{41} \tilde{E}_{43}),
\]
which implies (4.20). From here the lemma follows.

Proposition 4.6 follows from Lemmas 4.7 and 4.8.
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