An analytical derivation of non-local dependence of ionization rate in the glow discharge

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Abstract. Based on the analysis of the kinetic equation for the electron distribution function, variables recorded in full energy-coordinate, showed that the rate of ionization is a function of the mean value of the electric field on the electron mean free path in the case of non-uniform electric field in a glow discharge, that is, there is a non-local of the ionization processes. Results are obtained from the numerical solution of non-local dependence of the ionization rate.

1. Introduction

At present, glow discharges is actively used in practical applications, specifically in treating and modifying the surface of materials, in gas lasers, in plasma chemistry, and other purposes.

In the literature, there are a huge number of publications devoted to the glow discharge, and the modeling of its structure. And in most cases, when modeling the structure of the glow discharge in fluid approximation, using the expression for the ionization rate recorded by the Townsend coefficient [1 and other]:

\[ Q = \alpha(E)n_e \mu_e E, \quad \alpha(E) = Ap \exp\left(-Bp |E|\right), \]

where \( n_e, \mu_e \) – the concentration and mobility of electrons, respectively; \( E \) – the electric field; \( A, B \) – empirical constants depending on the type of gas; \( p \) – gas pressure. In this case ionization rate is a function of the local values of the electric field. Such dependence follows from the solution of the kinetic equation for the distribution function of electrons in uniform electric field [1, 2]. However, in the cathode region, the field is highly non-uniform, and degree of non-uniformity can be such that the field change on the free path of a charged particle is comparable to the value of the field. In this case there is no steady drift motion of charges. Moving in a non-uniform electric field, an electron gains energy, that is different from the energy corresponding to the local value of the electric field \( E(x) \) and should be at least traversed to them of the potential difference, that is, the ionization processes, should be superior to the equilibrium values corresponding to the local field \( E(x) \).

Non-local dependence of ionization processes are often obtained by using different hybrid schemes [2-5]. Non-local model of hollow cathode and glow discharge was obtained in [6].

2. Introduction
To account for the influence of non-uniform electric field on the rate of ionization write a one-dimensional steady-state kinetic equation for the electron distribution function in the two-term and the single-level approximation in variables - full energy and the coordinate [1,7]:

\[
\frac{1}{v} \frac{\partial}{\partial x} \left( \psi D(w) \frac{\partial f_0}{\partial x} \right) + \frac{1}{v} \frac{\partial}{\partial \varepsilon} \left[ \psi \left( D_e(x, w) \frac{\partial f_0}{\partial \varepsilon} + V_e(w) f_0 \right) \right] = -\nu_{in}(w)f_0(x, \varepsilon) + \sqrt{1 + \frac{\Delta w}{w}} \nu_{in}(w + \Delta w)f_0(x, \varepsilon + \Delta w) = 0,
\]

(2)

where \( f_0 \) - the isotropic part of the distribution; \( v(w) \) - velocity of the electron; \( D_e(x, w) \) - the energy diffusion coefficient in the external electric field; \( \nu_e \) - the frequency of elastic collisions; \( \nu_{in} \) - the frequency of inelastic collisions; \( V_e \) - the energy loss in elastic collisions; \( \nu_{in} \) - the frequency of inelastic collisions; \( D(w) = \psi(v(w)\lambda(w))/3 \) - spatial diffusion coefficient for electrons with kinetic energy \( w \); \( \lambda(w) \) - the mean free path of electrons. All other functions of \( w \) are considered dependent on \( \varepsilon, x \) according to \( \varepsilon = w - q_x \varphi(x) \). The first term in (2) describes the spatial diffusion, the second term describes the energy losses associated with elastic collisions, and the third - the energy loss in inelastic collisions, the fourth - the appearance of an electron with kinetic energy due to the loss of their energy in the inelastic process. Analysis of the distribution functions of electrons in inhomogeneous fields done in [8, 9, and others].

We assume that:
1) the cross section of the elementary processes are independent on the energy \( w \), that is, the frequency of elastic and inelastic collisions are defined, respectively \( \nu_{el}(w) = n_0 \nu \sigma_{el} = 2w/m \sigma_{el} \), \( \nu_{in}(w) = n_0 \nu \sigma_{in} = 2w/m \sigma_{in} \), where \( n_0 \) - the concentration of neutral particles, \( \sigma_{el} \) - cross section of elastic collisions, \( \sigma_{in} \) - the cross section of inelastic collisions;
2) the electric field is a linear function in the cathode region of the glow discharge;
3) scale spatial heterogeneity is more than the distance at which the electron acquires the mean energy;
4) the loss of energy by electrons in elastic collisions with neutral gas particles are small compared to the energy losses in inelastic collisions. In this case, we assume that the value of the transmitted energy \( \Delta w \) is comparable to the finite energy of the electron \( w \), i.e., \( \Delta w \approx w \).

Based on the above equation (2) can be rewritten as:

\[
-\frac{d}{dx} \left( \frac{2w^{3/2}}{3m\nu_{el}} \frac{df_0(x, \varepsilon)}{dx} \right) + \nu_{in}(w) \sqrt{2w} f_0(x, \varepsilon) = \sqrt{2w} \nu_{in}(w + \Delta w)f_0(x, \varepsilon + \Delta w).
\]

(3)

The boundary conditions for equation (3) are in the requirement for the limited solutions at \( x \to \infty \) and \( x \to x_e \) where \( x_e \) is determined from the condition \( w = \varepsilon + q_x \varphi(x) = 0 \). Using the Green function \( G(x, x') \) of the differential operator of the left side of equation (2) represent the solution in the form of:

\[
f_0(x) = \int_{x_e}^{\infty} G(x, x') \sqrt{2w} \nu_{in}(w + \Delta w)f_0(x, \varepsilon + \Delta w)dx'.
\]

(4)

The Green's function is given by \( G(x, x') = f_1(x')f_2(x) \), where \( f_1 \) and \( f_2 \) - solutions of the homogeneous equation

\[
-\frac{d}{dx} \left( a(w) \frac{df_0}{dx} \right) + b(w) f_0(x, \varepsilon) = 0.
\]

2
with the boundary conditions \( x \rightarrow x_e \) and \( x \rightarrow \infty \), respectively. Here \( a(w) = 2w^{3/2} / 3m v_{el} \), \( b(w) = v_{el} (w) \sqrt{w} \). By demanding the continuity solutions to the variable at the point \( x = s \), and the conditions on leap of the derivative, we obtain the following conditions on \( f_1 \) and \( f_2 \):

\[
c_1 f_1(s) = f_2(s) \quad \text{and} \quad \frac{df_1}{dx} \bigg|_{x=s} - \frac{df_2}{dx} \bigg|_{x=s} = \frac{1}{a(w)}. \tag{5}
\]

Determine the form of the function. Suppose \( \langle w \rangle \) – mean energy of the electron on the mean free path \( \lambda_{free} \), \( E^*_{free} \) – the mean value of the electric field on \( \lambda_{free} \). Considering that \( dw = -q_e E_{free} dx \), the expression for the mean energy of an electron can be written as follows \( w = \langle w \rangle = q_e E^*_{free} (x - x_e) \).

Then

\[
a(w) = \frac{2w \sqrt{w}}{3m v_{el}} = \frac{2}{3} \sqrt{ \frac{m}{2} } - \frac{2q_e E^*_{free} (x - x_e)}{3m n_0 \sigma_{el}} \sqrt{ \frac{m}{2} },
\]

\[
b(w) = v_{el} \sqrt{w} = \sqrt{ \frac{2}{m} n_0 \sigma_{el} w } = \sqrt{ \frac{2}{m} n_0 \sigma_{el} q_e E^*_{free} (x - x_e) }.
\]

and equation (4) becomes:

\[
- \frac{d}{dx} \left( (x - x_e) \frac{df_0}{dx} \right) + 3n_0^2 \sigma_{el} (x - x_e) f_0 = 0. \tag{6}
\]

It’s solution under the condition \( x \rightarrow x_e \) and satisfies the condition (5) is

\[
f_1(x) = \frac{\sqrt{m}}{2} \frac{3n_0 \sigma_{el}}{q_e E^*_{free}} I(\kappa(x)), \tag{7}
\]

where \( I(\kappa(x)) \) – the Bessel function of imaginary argument, \( \kappa(x) = n_0 \sqrt{3\sigma_{el} E^*_{free} (x - x_e)} \).

From the condition that the amount of transmitted energy is comparable in magnitude to the final electron energy at which there is ionization of neutral particles, it follows that the right-hand side of (3) is a function that is localized near the point \( x_e \). Outside this interval is negligibly small compared to the other terms in (3), i.e.

\[
\sqrt{2w v_{el}} (w + \Delta w) f_0(x, \varepsilon + \Delta w) / \sqrt{w v_{el}} (w) f_0(x, \varepsilon) << 1. \tag{8}
\]

From the last relation follows that the integral in (4) is dialed from the neighborhood, then the asymptotic behavior of the function \( f_0(x) \) under the condition \( x \rightarrow \infty \) can be defined by the expression

\[
f_0(x) = f_2(x) \int_{x_e}^{\infty} f_1(x') \sqrt{2w v_{el}} (w + \Delta w) f_0(x', \varepsilon + \Delta w) dx'. \tag{9}
\]

In the region where the valid such representation \( f_2(x) \) has of a quasi-classical form (Wentzel-Kramers-Brillouin approximation) [7]:

\[
f_2(x) \sim \frac{1}{\sqrt{w}} \exp \left( - \int_{x_e}^{x} \chi(w) dx' \right), \tag{10}
\]

where \( \chi(w) = 3m v_{el} v_{in} / 2w \). Thus, in account of (6) and (9) and introducing the notation
\[
C(E_w^*) = \int_{x_e}^{\infty} \frac{m n_e \sigma_{el}}{2 q_e \epsilon_{\text{free}}} \cdot I(\kappa(x)) \sqrt{2w} \cdot \nu_{in}(w + \Delta w) f_0(x, \varepsilon + \Delta w) dx',
\]

expression (9) takes the form
\[
f_0(x) = C(E_{\text{free}}^*) f_2(x).
\]

Taking into account condition 2 the mean value of electric field \( E_i^*(x, w) \) at the interval \((x_e, x) = (x - \Delta x, x)\) corresponding to the length of the path of an electron \( \lambda_i \) (index \( i = \text{ion} - \)ionization mileage \( i = \text{free} - \)free path) as follows [8, 11]:
\[
E_i^*(x, w) = \left( E - \frac{dE \lambda_i}{dx} \right).
\]

Taking into account \( dw = -q_e Edx \) an expression for the distribution function can be rewritten as:
\[
f_0(x) = \frac{C(E_{\text{free}}^*)}{\sqrt{w}} \exp\left( - \frac{1}{-q_e E_{\text{ion}}^*} \int_0^w \chi(w^*)dw^* \right).
\]

The ionization rate is known to be defined by the equation
\[
Q(x) = \int_{x_e}^{\infty} v(w) f_0(w, x) dw \quad \text{or with (13)}.
\]

Denoting, \( B = \int_0^l (\chi(e)/p) d\varepsilon / q_e \) we obtain
\[
Q(x) = C(E_{\text{free}}^*) \exp\left( - \frac{Bp}{E_{\text{ion}}^*} \int_0^w \nu_{ion}(w) \frac{f_0(w, x)}{f_{ion}(w_{ion}, x)} dw \right).
\]

From (14) it is clear that the structure of the expression for the ionization rate in the case of non-uniform electric field is similar to (1) with the difference that in the exponent, the value of the local field is replaced by its mean value over the length of the ionization path of an electron, and in the pre-exponential factor - mean the value of the mean free path of electrons.

3. Appendices
To use the latter expression for practical calculations in the system of equations describing the structure of glow discharge in the fluid approximation it can be rewritten as follows:
\[
Q = A p n, \mu, \frac{E - \frac{dE \lambda_{\text{free}}}{dx}}{2} \exp\left( - \frac{Bp}{E - \frac{dE \lambda_{\text{ion}}}{dx}} \right).
\]

Expression (15) allows us to formulate the problem as a system of differential equations, in contrast to the system of integral-differential allows you to simplify the numerical algorithm for solving equations, taking into account the most important phenomena (non-locality of the ionization processes) occurring in the glow discharge, and using the known experimental data \( A \) and \( B \) characterizing the type of gas.
The resulted analysis confirms the validity of the applied in [7,10] the expression for the ionization rate as a function of the mean value of the electric field on the electron path length to account for non-locality of the ionization processes in the glow discharge.

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