VECTOR FIELDS WITH A GIVEN SET OF SINGULAR POINTS

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Theorems on the existence of vector fields with given sets of Indexes of isolated Singular points are proved for the cases of closed manifolds, pairs of manifolds, manifolds with boundary, and gradient fields. It is proved that, on a two-dimensional manifold, an index of an isolated Singular point of the gradient field is not greater than one.

In the present paper, we consider vector fields on manifolds with isolated singular points. In 1885, Poincare [1] proved that the sum of the indices of the singular points of such a field on a two-dimensional manifold is equal to the Euler characteristic of this manifold. For the n-dimensional case, this fact, called the Poincare-Hopf theorem, was proved by Hopf [2] in 1926 after partial results of Brauer and Hadamard. This theorem holds for a manifold with boundary if the field is directed outside in any point of the boundary. It was established that there exists a vector field without singular points on a manifold with zero Euler characteristic. These facts are proved in [3] and [4].

The aim of the present paper is to prove the existence of a vector field with a given set of indices satisfying the conditions of the Poincare-Hopf theorem. In Sec. 1, we establish the existence of such fields for closed manifolds. There we introduce the following two operations over vector fields: introduction of a pair of singular points and composition of singular points. These operations are also used in the proofs of other theorems. In Sec. 2, we prove that there exists a vector field with two sets of indices on a pair of manifolds. The case where a vector field given on the manifold is tangent to a submanifold is considered in particular. Manifolds with a boundary are investigated in Sec. 3. Sections 4 and 5 deal with gradient fields for functions on manifolds. Theorem 7 is a "well-known fact," the proof of which is not yet published.

All manifolds, functions, and vector fields considered in the present paper are $C^\infty$-differentiable.

1. Singular Points of Differential Equations

Let $M^n$ be a smooth manifold. A vector field $v$ setting the differential equation
\[
\frac{dx}{dt} = v(x)
\]  
(1)

is considered as a cut of a tangent fibre bundle \( TM^n \). In what follows, we assume that the differential equation (1) is given if the vector field \( v \) is given.

**Theorem 1.** Let \( M^n, \ n \leq 2, \) be a smooth connected manifold. Let \( \alpha_1, \ldots, \alpha_k, \ k \leq 1, \) be an integer set

\[
\sum_{i=1}^{k} \alpha_i = \chi(M^n)
\]

where \( \chi(M^n) \) is the Euler characteristic of the manifold \( M^n \). Then there exists a vector field \( v \) on \( M^n \) singular points of which are isolated and have indices \( \alpha_1, \ldots, \alpha_k \).

**Proof.** Let us describe two operations over vector fields that allow one to obtain a vector field with a prescribed set of singular points starting with an arbitrary vector field.

1. Introduction of two singular points with indices +1 and -1. Let \( x_0 \) be a regular point of a vector field \( v_0 \). According to the theorem on rectification of a vector field [5], there exists a map \( U \) at the point \( x_0 \), where the vector field is constant. Thus, there exists a function \( f : U \to \mathbb{R}^1 \) without critical points in a neighborhood \( U \) of the point \( x \) and such that the gradient field of this function in a proper metric coincides with the field \( v_0 \). In a neighborhood \( V \) of the point \( x \) such that \( V \subset U \), we change the function \( f \) by introducing a pair of mutually reducible critical points of adjacent indices. Then by replacing the field \( v_0 \) by the gradient field of the new function in the neighborhood \( U \), we obtain a field that has two extra points of indices +1 and -1 in comparison with the field \( v_0 \).

2. Replacement of two singular points of indices \( \lambda_1 \) and \( \lambda_2 \) by a singular point of index \( \lambda_1 + \lambda_2 \). Let \( x_0 \) and \( x_1 \) be singular points of indices \( \lambda_1 \) and \( \lambda_2 \), respectively, of a vector field \( v_0 \). We select a path

\[
\gamma : [0,1] \to M^n
\]

without self-intersections that joins points \( x_0 \) and \( x_1 \), i.e., a path such that \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). Let \( U \) be a neighborhood of the path \( \gamma([0,1]) \), the closure of which contains no singular points except \( x_0 \) and \( x_1 \). According to the Uryson lemma, there is a smooth function \( f : M^n \to \mathbb{R}^1 \) such that

\[
f(x) = 0 \ for \ x \in \gamma([0,1]),
\]

\[
f(x) = 1 \ for \ x \in M^n \setminus U,
\]
We consider a vector field $v_1 = f v_0$. Let $g : M^n \to M^n/\gamma([0, 1])$ be a mapping that transforms the path $\gamma([0, l])$ into the point $y_0$ and is a one-to-one correspondence on $M^n \setminus \gamma([0, 1])$. Since $v_1 = 0$ for $x \in \gamma([0, 1])$, the field $v_1$ induces the vector field $v_2$ on the manifold $M^n/\gamma([0, 1])$. In this case, the trajectories of the vector field $v_1$ ending in (or starting from) $\gamma([0, 1])$ correspond to trajectories ending in (or starting from) the point $y_0$. Since the manifold $M^n/\gamma([0, l])$ is diffeomorphic to the manifold $M^n$, the field $v_2$ sets a field $v$ on the manifold $M^n$. By virtue of the Poincare-Hopf theorem, the sum of the indices of $v$, just as $v_0$, is the Euler characteristic $\chi(M^n)$. Thus, $y_0$ is a singular point of the index $\lambda_1 + \lambda_2$ of the vector field $v$.

Let $f : M^n \to R^1$ be a Morse function on the manifold $M^n$. We transform this function by introducing pairs of mutually reducible critical points in such a way that the number of critical points is no less than $\lambda$. Then the gradient field of the function $f$ consists of singular points of indices $1$ and $-1$ corresponding to critical points of the function $f$. We decompose the set of these critical points into groups such that the sum of the indices of the $i$-th group is $\alpha_i$ and then apply operation 2 to the points of every group. As a result, we get the desired field $v$.

2. Differential Equations on Pairs of Manifolds

Definition 1. Assume that $M^n$ is a smooth manifold, $N^k$ is its submanifold, $\rho$ is a Riemann metric, and $v$ is a vector field on the manifold $M^n$. We say that a vector field $u$ on the submanifold $N^k$ is induced by the vector field $v$ in the Riemann metric $\rho$ if, for any point $x \in N^k$, the vector $u(x) \in T_x N^k$ is the projection orthogonal in the metric $\rho$ of the vector $v(x) \in T_x M^n$ to the subspace $T_x N^k$.

If $v$ is the gradient field of the function $f$ on the manifold $M^n$ then the induced vector field $u$ on the submanifold $N^k$ coincides with the gradient $g$, where $g$ is the restriction of the function $f$ to the submanifold $N^k$.

Theorem 2. Assume that $N^k$ is a submanifold of a smooth manifold $M^n$, $\rho$ is a metric on the manifold $M^n$ and $\alpha_1, ..., \alpha_p (p \leq 1)$, $\beta_1, ..., \beta_s (s \leq 1)$ are integer sets such that

$$
\sum_{i=1}^{p} \alpha_i = \chi(M^n), \quad \sum_{i=1}^{s} \beta_i = \chi(N^k),
$$

where $\chi(M^n)$ and $\chi(N^k)$ are the Euler characteristics of the manifolds $M^n$ and $N^k$ respectively. If $n - k \leq 2$, then on the manifold $M^n$ there exists a vector field $v$ with singular points of indices $\alpha_1, ..., \alpha_p$ such that the vector field $u$ on
the submanifold \( N \) induced by the vector field \( v \) in the metric \( \rho \) has singular points of indices \( \beta_1, \ldots, \beta_s \).

**Proof.** By using Theorem 1, we construct a vector field \( v \) on \( M^n \) with singular points of indices \( \alpha_1, \ldots, \alpha_p \). Then we consider this vector field as a section of the tangent fiber bundle and present it in a general position with respect to the submanifold \( N^k \) located in the zero section of the tangent fiber bundle \( TM^n \). The obtained vector field \( v_1 \) has singular points with the same indices \( \alpha_1, \ldots, \alpha_p \); these points are not located on the submanifold \( N^k \). Let \( A \) be a tubular neighborhood of \( N^k \) that contains no singular points of the vector field \( v_1 \). We consider the restriction of the tangent fiber bundle \( TM^n \) to the manifold \( N^k \). The obtained fiber bundle \( \xi \) has dimension \( n+k \) as a manifold. The tangent fiber bundle \( TN^k \) is a subbundle of \( \xi \) and has dimension \( 2k \) as a manifold. The restriction of the vector field \( v \) to the submanifold \( N^k \) considered as a section of the fiber bundle \( \xi \) is of dimension \( k \). In general position, its intersection with the tangent fiber bundle \( TN^k \) is a submanifold \( L \) of dimension \( 2k-n \).

Let \( u_1 \) be a vector field on the manifold \( N^k \) induced by the vector field \( v_1 \). Evidently \( u_1(x) = v_1(x) \) if and only if the point \( x \) belongs to the submanifold \( L \), and \( u_1(x) = 0 \) if and only if the vector \( v_1(x) \) is perpendicular to the submanifold \( N^k \) with respect to the metric \( \rho \). Thus, singular points of the vector field \( u_1 \) are not on the submanifold \( L \). By using the vector field \( u_1 \), a vector field \( u \) on the manifold \( N^k \) can be constructed such that \( u(x) = u_1(x) \) for all points \( x \) of some neighborhood of the submanifold \( L \). In fact, two singular points of indices \( +1 \) and \( -1 \) can be introduced in a neighborhood \( V \) of any nonsingular point \( x_0 \) if \( V \) contains no singular points and the closure of \( V \) does not intersect the submanifold \( L \). Since the codimension of the submanifold \( L \) in the manifold \( N \) is at least two, we can join any two singular points in a way that does not intersect \( L \) and choose a neighborhood \( W \) of this path such that the closure of the neighborhood does not intersect the submanifold \( L \) and contains no other singular points. Let us replace, as was done in Theorem 1, these two singular points with indices \( \lambda_1 \) and \( \lambda_2 \) by a singular point of index \( \lambda_1 + \lambda_2 \). In this case, the vector field is changed only on the set \( W \).

Let us consider the vector field \( u - u_1 \) on the manifold \( N \). We extend this vector field to a vector field \( w \) in a tubular neighborhood \( A \) of the submanifold \( N^k \) as follows: the coordinates of the vector \( w(x) \) are equal to the coordinates of the vector \( u(x_0) - u_1(x_0) \) in some map if the point \( x \) is located in a fiber bundle of the tubular neighborhood over the point \( x_0 \). Let us take the tubular neighborhood \( A \) small enough so that vectors \( w(x) \) will not be parallel to the vectors \( v(x) \) if \( w(x) \neq 0 \).

By the Uryson lemma, there is a smooth function \( f : M^n \rightarrow \mathbb{R}^1 \) such that

\[
f(x) = 1 \quad \text{for} \quad x \in N^k
\]
\[
f(x) = 0 \quad \text{for} \quad x \in M^n \setminus A,
\]
We consider a vector field \( v = v_1 + f w \). According to the construction, it has the same set of singular points on \( M^n \) as the field \( v_1 \) does. The field induced by \( v_1 \) on the manifold \( N^k \) coincides with the field \( u = u_1 + 1(u - u_1) = v_1 + f w \). Theorem 2 is proved.

Now let us consider the case of vector fields on a pair of manifolds \((M^n, N^k)\) such that the manifold \( N^k \) is imbedded into the manifold \( M^n \), where the restriction of a field \( v \), given on the manifold \( M^n \), to the manifold \( N^k \) is a vector field tangent to \( N \), i.e., the vectors induced by the field \( v \) coincide with the corresponding vectors of the field \( v \). It is evident that the induced vector field in this case does not depend on the metric \( \rho \) on the manifold \( M^n \) and every singular point of the vector field \( v \), located on the submanifold \( N^k \), is a singular point of the vector field \( v \) set on the manifold \( M^n \).

**Theorem 3.** Let \( N^k \) be a submanifold of a smooth manifold \( M^n \), \( n-k \leq 1 \), and let \( \alpha_1, ..., \alpha_p \) and \( \beta_1, ..., \beta_s \) be integer sets such that

\[
\sum_{i=1}^{p} \alpha_i = \chi(M^n), \quad \sum_{i=1}^{s} \beta_i = \chi(N^k), \quad 1 \leq s \leq p.
\]

Then, there exists a vector field \( v \) on the manifold \( M^n \) with singular points of indices \( \alpha_1, ..., \alpha_p \) tangent to the submanifold \( N^k \), and such that the vector field \( v \) on the submanifold \( N^k \) has singular points of indices \( \beta_1, ..., \beta_s \).

**Proof.** Let us construct, by analogy with the proof of Theorem 1, a vector field \( u_1 \) on the submanifold \( N^k \) with singular points of indices \( \beta_1, ..., \beta_s \). We select on the manifold \( M^n \) a vector field \( u_2 \) which is normal to the submanifold \( N^k \) and nonzero in the singular points of the vector field \( u_1 \). Let \( U \) be a tubular neighborhood of the submanifold \( N \). According to the Uryson lemma, there exists a smooth function \( f \) on the manifold \( M^n \) such that

\[
f(x) = 0 \quad \text{for} \quad x \in N^k,
\]

\[
f(x) = 1 \quad \text{for} \quad x \in M^n \setminus U,
\]

\[
0 < f(x) < 1 \quad \text{for} \quad x \in U \setminus N^k.
\]

Let \( u = (l-f)u_1 + f u_2 \) be a vector field on \( U \), where \( u_1 \) and \( u_2 \) are vector fields on \( U \), the vectors of which in every fiber bundle on a tubular neighborhood have the same coordinates in some map as the corresponding vectors on \( N^k \) do. Let us arbitrarily extend the field \( u \) onto the manifold \( M^n \). When considering \( u \) as
a section of the tangent fiber bundle, we lead it to the general position with the zero section and preserve it without changes on the set \( U \). We introduce pairs of singular points with indices +1 and -1 and add singular points along paths the inner parts of which do not intersect the submanifold \( N^k \). The addition of two points located on the submanifold is not assumed. The vector field thus constructed is a tangent field for the submanifold \( N^k \) and satisfies all the conditions of the theorem.

**Remark 1.** For every singular point on the submanifold \( N^k \), any two integer numbers can be the indices of the vector field \( v \) and the indices of the vector field \( u \).

**Remark 2.** If \( n - k > 2 \) or \( n - k = 1 \) and the submanifold \( N^k \) does not divide the manifold \( M^n \), then the theorem also holds for \( s = 0 \). That is, if \( \chi(N^k) = 0 \), then there exists a vector field \( v \) on the manifold \( M^n \) that is tangent to the submanifold \( N^k \) has no singular points on \( N^k \), and possesses the given set of singular points \( \alpha_1, ..., \alpha_p \) on the submanifold \( M^n (\sum_{i=1}^{p} \alpha_i = \chi(M^n)) \).

If the submanifold \( N^k \) decomposes the manifold \( M^n \) into two submanifolds \( M_1 \) and \( M_2 \) with the boundary \( \partial M_1 = \partial M_2 = N^k \), then the problem of existence of a vector field \( v \) on the manifold \( M^n \) with a given set of indices and tangent to the submanifold \( N^k \) and without singular points on \( N \) is equivalent to the problem of existence on a manifold with a boundary of a vector field tangent to the boundary and the singular points of which are not located on the boundary and have the preassigned set of indices.

### 3. Differential Equations on Manifolds with Boundaries

**Proposition 1.** Let \( M^n \) be a smooth manifold with the boundary \( N \), \( \chi(N) = 0 \). On the manifold \( M \), there exists a vector field tangent to the manifold \( N \), and all its singular points are inner and have indices \( \alpha_1, ..., \alpha_p \) if and only if

\[
\sum_{i=1}^{k} \alpha_i = \chi(M^n),
\]

where \( \chi(M^n) \) is the Euler characteristic of the manifold \( M \) with the boundary \( N \).

**Proof.** Since the Euler characteristic of the manifold \( N \) is 0, there exists a vector field \( u \) on \( N \) without singular points. Let \( w \) be a vector field on \( N \), normal to \( N \), without singular points, and directed outside of the manifold \( M \). We extend this field onto the collar \( N \times [0, 1] \) according to the formula

\[
v(x, t) = tu(x) + (1 - t)w(x),
\]

where the point \( x \in N \), \( t \in [0, 1] \). We extend the field \( v \) onto the manifold \( M^n \) in such a manner that all singular points are isolated. Then the sum of the
indices of these singular points is equal to the Euler characteristic $\chi(M^n)$. By using the processes of the introduction of new pairs of singular points of indices +1 and -1 and the composition of singular points, we obtain the desired vector field.

Let us consider an arbitrary vector field $v$ on the manifold $M^n$ with the boundary $\partial M^n = N$. We assume that the vector field $v$ has no singular point on the boundary $N$. Let $u$ be a vector field induced by the vector field $v$ on the boundary $N$. We consider a manifold $M'$ obtained from the manifold $M^n$ by gluing the collar $N \times [0, 1]$; to do this, we identify points $x \in N = \partial M^n$ with points $(x, 0) \in N \times [0, 1]$. We construct a vector field $w$ on $M'$ which is an extension of the field $v$. Let $a$ be a vector field given on $N \times \{1\}$, orthogonal to $M'$ directed outside of the manifold $M'$. For a point $x \in N$, there exists $t \in (0, 1)$ such that $w(x, t) = 0$ if and only if $u(x) = 0$ and the vector $v(x, 0)$ is directed inside of the manifold $M^n$. The indices of the corresponding singular points have the same absolute value and different signs. We denote by $\delta_+(v)$ and $\delta_-(v)$, respectively, the sum of the indices of singular points of the vector field $u$ where the vector field $v$ is directed inside of and out of the manifold $M^n$. Then the following equalities hold:

$$\chi(M^n) = \sum \text{ind } v - \delta_+(v),$$

$$\chi(M^n) = \sum \text{ind } v + \delta_-(v) \text{ for even } n,$$

$$\chi(M^n) = -\sum \text{ind } v - \delta_-(v) \text{ for odd } n.$$

Here $\chi(M^n) = \chi(M') = \sum \text{ind } w$ is the Euler characteristic of the manifold $M^n$ with the boundary $N$.

**Theorem 4.** Let $M^n$ be a smooth manifold with the boundary $N$, let $(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_{p+1})$ be integer sets. There exists a vector field $v$ with inner singular points of indices $\alpha_1, \ldots, \alpha_s$ and with the induced vector field $u$ on the manifold $N$ such that the vector field $v$ is directed inside of the manifold $M^n$ at the singular points of the field $u$ with indices $\beta_1, \ldots, \beta_k$ and
outside of this manifold at the singular points of the field $u$ with indices $\beta_{k+1}, \ldots, \beta_p$, if and only if

$$\sum_{i=1}^{p} \beta_i = \chi(N)$$

$$\chi(M^n) = \sum_{i=1}^{s} \alpha_i - \sum_{i=1}^{k} \beta_i \text{ for even } n,$$

$$\chi(M^n) = \sum_{i=1}^{s} \alpha_i - \sum_{i=k+1}^{p} \beta_i \text{ for odd } n.$$ 

**Proof.** The previous reasoning demonstrates that the conditions of the theorem are satisfied for any vector field $v$. Let us prove the inverse statement i.e., if the sets of indices satisfy the conditions of the theorem, then there exists a vector field with given sets of indices. In fact, let $u_0$ be an arbitrary vector field tangent to the manifold $N = \partial M^n$, singular points of which have indices $\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_p$. Such a vector field exists by virtue of Theorem 1. Let $u_1$ be a vector field given on the boundary $N$, orthogonal to $N$, directed inside of the manifold $M^n$ in singular points with indices $\beta_1, \ldots, \beta_k$ and outside of $M^n$ in singular points with indices $\beta_{k+1}, \ldots, \beta_p$. We consider a field

$$u_2 = u_0 + u_1.$$ 

Let us arbitrarily extend this field up to the field $v_0$ on the whole of the manifold $M^n$. Then the field $v_0$ induces a vector field $u_2$ on the manifold $N$ with the desired set of indices. The conditions of the theorem are satisfied for this field. By applying the processes of introduction of a pair of singular points of indices +1 and -1 and composition of singular points described in Theorem 1 to the field $v_0$, we get the desired vector field $v$.

4. Gradient Field of a Smooth Function

**Definition 2.** Let $v$ be a vector field on a manifold $M^n$ that has only isolated singular points. Let us consider an oriented graph, the vertices $\alpha_i$ of which correspond one to one with singular points $x_i$ of the vector field $v$, and two vertices are joined by an arc if there exists an integral trajectory that starts and ends at the corresponding points of the vector field. We call such a graph the graph $G(v)$ of the vector field $v$.

**Definition 3.** The contour of a graph $G$ is a sequence $S = (\alpha_0, \gamma_1, \alpha_1, \gamma_2, \ldots, \alpha_{n-1}, \gamma_n, \alpha_n)$ of its vertices $\alpha_i$ and arcs $\gamma_i$ alternating in such a manner that $\alpha_i - 1$ is the beginning and $\alpha_i$ is the end of the arc $\gamma_i$, $i = 1, n$, and $\alpha_0 = \alpha_n$. 

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Theorem 5. Let \( v \) be a smooth vector field on a smooth manifold \( M^n \) that satisfies the following conditions:

(i) all singular points are isolated; limit set of each trajectories consist of singular points; for every singular point \( x_i \) there is a neighborhood \( U_i \) and a smooth function \( f_i \) determined on \( U_i \) and such that \( v \) is a gradient field of the function \( f_i \) in some metric \( \rho_i \) on \( U_i \);

(ii) the graph \( g(v) \) of the vector field \( v \) has no contours.

Then there is a Riemann metric \( \rho \) on the manifold \( M^n \) and a function \( f : M^n \rightarrow \mathbb{R}^1 \) such that \( \text{grad} \ (f) = v \).

Proof. Let \( \rho_0 \) be a fixed metric on the manifold \( M^n \). Since the graph of the vector field has no contours, the singular points \( x_i \) can be ordered in such a manner that if \( i < j \), then the graph \( G(v) \) does not have a path originating at the vertex \( \alpha_i \) and ending at the vertex \( \alpha_j \). Without any loss of generality, we assume that any neighborhood \( U_i \) is homeomorphic to an open disk and \( U_i \cap U_j = \emptyset \), \( i \neq j \). We define the function \( f \) on \( U_i \) in such a way that \( f(x_i) = i \), \( f(y) = f_i(y) + i - f_i(x_i) \), \( y \in U_i \). We here take \( U_i \) sufficiently small such that

\[ |f(y) - f(x_i)| < \frac{1}{3}, \quad y \in U_i. \]

We define \( f \) on \( M^n \setminus \bigcup_i U_i \). Let \( x \in M^n \setminus \bigcup_i U_i \), \( \gamma_x \) be a trajectory of the vector field \( v \) passing through the point \( x \). Let \( x_i \) and \( x_j \) be points where the trajectory \( \gamma_x \) starts and ends, \( y_i = \gamma_x \partial U_i \), \( y_j = \gamma_x \partial U_j \) and \( y_i = \gamma_x(t_i), y_j = \gamma_x(t_j) \). We set

\[ f(x) = f(y_i) + \frac{S_\gamma(t_i, 0)}{S_\gamma(t_i, t_j)} (f(y_i) - f(y_j)), \]

where \( S_\gamma(t_i, t) \) is the length of the trajectory \( \gamma_x \) in the metric \( \rho_0 \) between \( t_i \) and \( t \). Thus, the function \( f \) in-creases from \( f(y_i) \) to \( f(y_j) \) along \( \gamma_x \) in proportion to the length of the arc \( \gamma_x \). Let us smooth the function \( f \) on the boundary \( \partial U_i \) as was done in [4]. The metric \( \rho \) can be taken as follows: for any point \( x \), \( x^1, x_i \), we choose a coordinate system \( x^1, x^2, ..., x^n \), we directs \( x^1 \) along the integral trajectory passing through the point \( x \), and we take \( x^2, ..., x^n \) on the level surface of the function \( f \) that passes through the point \( x \). The scalar product at a point \( x \) is taken to be in proportion to the standard one, namely,

\[ \rho(x, y) = \frac{|v(x)|}{|\partial f(x)/\partial x|} \sum_{i=1}^n x^i y^i. \]

By using a partition of the unit, we glue this metric with the metrics \( \rho_i \) given on \( U_i \). The obtained metric is the desired one.
Definition 4. A smooth function $f : M^n \to R^1$ is called minimal if any other smooth function $g : M^n \to R^1$ has no fewer critical points than $f$ does. We denote it by $q(M^n)$.

Definition 5. An integer set $\alpha_1, ..., \alpha_k$ is called admissible for a manifold $M^n$ if $\alpha_1 = 1$, $\alpha_k = (-1)^n$, and $\sum_{i=1}^n \alpha_i = \chi(M^n)$.

Definition 6. An integer set $\alpha_1, ..., \alpha_k$ is called realizable by a smooth function $f$ if the gradient field of the function $f$ in some metric $\rho$ has $k$ isolated singular points and their indices are $\alpha_1, ..., \alpha_k$.

Evidently, if $\alpha_1, ..., \alpha_k$ are indices of singular points $x_1, ..., x_k$ of a gradient vector field of a function $f$ and $f(x_i) \leq f(x_j)$ if $i \leq j$, then the set $\alpha_1, ..., \alpha_k$ is admissible.

Theorem 6. Let $M^n$ be a smooth manifold, $n \leq 4$. Let $\alpha_1, ..., \alpha_k$ be an admissible set. Then there exists a smooth function $f$ realizing the set if and only if $k > q(M^n)$.

Proof. Let $f$ be a minimal function. We construct a function $f_1$ having the same number of critical points as $f$ and such that its gradient field has an integral trajectory starting at the point of minimum and ending at the point of maximum of $f_1$. Let $y_1, y_2, ..., y_n$ be critical values of the function $f$ such that $y_i < y_j$ if $i < j$, and $x_0$ and $x_s$ are points of minimum and maximum of the function $f$. We set

$$y_{i+1/2} = \frac{y_i + y_{i+1}}{2}.$$

Let $z_1 \in f^{-1}(y_1)$ be a regular point of the mapping $f$, and let $\gamma_1$ be the trajectory of the gradient field that passes through the point $z_1$. It is evident that $\gamma_1$ starts at the point of the minimum of the function $f$. If $\gamma_1$ ends at the point of the maximum, then $f_1 = f$. Assume that the trajectory $\gamma_1$ is ending at a critical point $x_i$, $f(x_i) = y_i$, $i < s$. Let $z_i \in f^{-1}(y_i)$,

$$p_i = \gamma_1 \cap f^{-1}(y_{i-1/2}),
q_i = \gamma_1 \cap f^{-1}(y_{i-1/2}).$$

Here, $\gamma_i$ is the integral trajectory passing through the point $z_i$. The point $z_i$ is chosen in such a way that the points $p_i$ and $q_i$ are in the same connected component of the submanifold $f^{-1}(y_{i-1/2})$. Then there exists an ambient isotopy of the manifold $f^{-1}(y_{i-1/2})$ that transfers the point $q_i$ into the point $p_i$. This isotopy sets a vector field (that is a set of integral trajectories) on the submanifold $f^{-1}([y_{i-1/2} - \varepsilon, y_{i-1/2} + \varepsilon])$ for $\varepsilon > 0$ sufficiently small. By smoothing the vector field on the boundary $f^{-1}(y_{i-1/2} - \varepsilon)$ and $f^{-1}(y_{i-1/2} + \varepsilon)$ we obtain that the integral trajectory $\gamma_1$ passes through the point $z_i$. Applying the same reasoning
to all points \( z_j, f(z_j) < y_s \), we construct a vector field, the trajectory \( \gamma_1 \) of which starts at the point \( z_0 \) and ends at the point \( z_s \). By applying Theorem 5 to this vector field, we construct a function \( f \), the gradient field of which has trajectories starting and ending at \( z_0 \) and \( z_s \) respectively, and its set of critical points coincides with that of the function \( f \).

Evidently, the integral trajectories sufficiently close to the trajectory \( \gamma_1 \) also start and end at the points \( z_0 \) and \( z_s \) respectively. By introducing pairs of singular points of indices +1 and -1 along these trajectories and composing singular points on the corresponding critical levels as was done in [6], we get the desired function \( f \).

5. Gradient Fields on Two-Dimensional Manifolds

**Theorem 7.** Let \( y_0 \) be a singular point of a gradient vector field \( v \) of the function \( f \) on the manifold \( M^2 \). Then its index is less than two.

**Proof.** We take a neighborhood \( U \) of the point \( y_0 \) and coordinates \((u, v)\) in it in such a manner that \( y_0 \) is the origin and there are no other critical points in \( B = \{(u, v) : u^2 + v^2 \leq 1\} \) except the point \( y_0 \). On the boundary of the circle \( B^2 \), we introduce a parametrization \( \partial B^2 = S^1 = \{(u, v) : u^2 + v^2 = 1\} = \{(u, v) : u = \cos t, v = \sin t, 0 \leq t \leq 2\pi\} \). We denote by \( x_t \) and \( \tau(t) \), respectively, a point on the circumference \( S^1 \) with coordinates \((\cos t, \sin t)\) and the vector tangent to \( S^1 \) at the point \( x_t \)

\[
\tau(t) = \{-\sin t, \cos t\}.
\]

We also denote by \( \alpha(t) \) a continuous function of an angle put counterclockwise between the vector \( \tau(t) \) and the vector of the field \( v \) at a point \( x_t \). By virtue of the index of a singular point on a two-dimensional manifold, \( \alpha(2\pi) = \alpha(0) + 2(k - 1)\pi \), where \( k \) is the index of the singular point \( y_0 \).

**Definition 7.** We say that a function \( \alpha(t) \) at a point \( t_0 \), increasing (or decreasing), passes a level \( \alpha_0 \) if \( \alpha(t_0) = \alpha_0 \) and there is a neighborhood \( V = (t_0 - \varepsilon_1, t_0 + \varepsilon_2) \) of the point \( t_0 \) (here, \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \)) such that the function \( \alpha \) is monotonically nondecreasing (or nonincreasing) in the interval \( V \) and

\[
\alpha(t_0 - \varepsilon_1) < \alpha_0 < \alpha(t_0 + \varepsilon_2) \quad \text{(or } \alpha(t_0 - \varepsilon_1) > \alpha_0 > \alpha(t_0 + \varepsilon_2)) \).
\]

**Lemma 1.** Assume that a trajectory \( \gamma(s) \) of a vector field \( v \) passes through a point \( x_{t_0} \) of a circumference \( S^1 \) such that at the point \( t_0 \) the function \( \alpha(t) \), increasing, passes the level \( \pi n(n \in \mathbb{Z}) \). Then this trajectory is situated in the circle \( B^2 \) locally in a neighborhood of the point \( x_{t_0} \).

**Proof.** Assume the contrary. Let the trajectory \( \gamma(s) \) pass through the point \( x_{t_0} \) (i.e., \( \gamma(s_0) = x_{t_0} \)) and leave the circle \( B^2_{t_0 + \varepsilon} = \{(u, v) : u^2 + v^2 \leq t_0 + \varepsilon\} \) for
some $\varepsilon > 0$. For definiteness, we assume that $n = 0$ and $\gamma(s)$ leaves the circle as the parameter $s$ increases. We take $\varepsilon$ sufficiently small (this can be done because of the continuity of the vector field $v$) for the inequality

$$\alpha_\varepsilon(t_0 - \varepsilon_1) < 0 < \alpha_\varepsilon(t_0 + \varepsilon_2)$$

to be satisfied, where $\alpha_\varepsilon$ is the function of the angle between the tangent vector to the boundary $\partial B_1^{1+\varepsilon} = S_1^{1+\varepsilon}$ and the corresponding vector of the field $v$, the circle $B_1^{1+\varepsilon}$ contains no other singular points except $y_0$, and the trajectory passing through a point $x_{t_0 + \varepsilon_2}$ intersects $S_1^{1+\varepsilon}$ at the point $y$ with the parameter $t < t_0 + \varepsilon_2$ (for this purpose, $\varepsilon_2$ should satisfy the inequality $\alpha(t_0 + \varepsilon_2) < \pi/2$).

Then there exists $t_1 \in (t_0 - \varepsilon_1, t_0 + \varepsilon_2)$ such that $\alpha_\varepsilon(t_1) = 0$. We denote this point on the circumference $S_1^{1+\varepsilon}$ by $y_1$. Similarly, for any $\varepsilon_1, 0 < \varepsilon_1 < \varepsilon$, there is $t_i$ such that $\alpha_\varepsilon(t_i) = 0$ and $t_i$ continuously depends on $\varepsilon_i$. We consider a path $\beta$ that consists of the following three parts:

(i) the point $x_i$, on $S_1^{1}(\alpha_{\varepsilon_i}(t_i) = 0)$;

(ii) the arc of the circumference $S^1_2$ from the point $y_1$ to the point $y$; 

(iii) the arc of the trajectory of the vector field $v$ from the point $y$ to the point $x_{t_0 + \varepsilon_2}$.

Then the trajectory $y$ leaves the circle $B_1^{1+\varepsilon}$, and must pass through a point of the path $\beta$. This is impossible since all vectors of the field $v$ are directed into $B_2^{1+\varepsilon}$ at points of the path $\beta$. The contradiction obtained proves Lemma 1.

One can similarly prove the following fact. If a function $\alpha(t)$, decreasing, passes through a level $\pi n, n \in \mathbb{Z}$, at a point $t_0$, then, locally in a neighborhood of the point $x_{t_0}$, the integral trajectory passing through this point does not intersect the interior of the circle $B^2$.

Let us prove the following statement: if $y_1$ is a singular point of index two and there are only two points $x_1$ and $x_2$ on the circumference $S^1$ where the function $\alpha$, increasing, passes through the level $\pi n, n \in \mathbb{Z}$, then trajectories of the vector field $v$ passing through the points $x_1$ and $x_2$ both start and end at the singular point $y_0$. In fact, let $\gamma_1(s)$ and $\gamma_2(s)$ be respectively trajectories passing through $x_1$ and $x_2$. Let $\beta_1$ be an arc of the circumference $S^1$ between the points $x_1$ and $x_2$, in the points of which the vectors of the field $v$ are directed inside $B^2$ or are tangent to the circumference $S^1$, and let $\beta_2$ be an arc of the circumference $S^1$ between the points $x_2$ and $x_1$, in the points of which the vectors of the field $v$ are directed outside the circle $B^2$ or are tangent to the circumference $S^1$.

If the trajectory $\gamma_1$ leaves the circle $B^2$ as the parameter $s$ increases (or, similarly, the parameter $s$ decreases), then it intersects the arc $\beta_2$ in a point $x_3$.

Thus, this trajectory decomposes the circle $B^2$ into two parts $A_1$ and $A_2$, $\gamma_1$ the arc between points $x_1$ and $x_3$.

Assume that the points $y_0$ and $x_2$ are located in the same part $A_1$. Since the trajectory $\gamma_1$ cannot enter the interior of the domain $A_2$ through an arc of the
function $\alpha$ passes through the levels $\pi n$ at the point $y_0$. We similarly consider a trajectory $\gamma_2$. One can prove that it starts or ends at the singular point $y_0$. However, if the points $y_0$ and $x_2$ are located in distinct domains $A_1$ and $A_2$, then the trajectories $\gamma_1$ and $\gamma_2$ have to intersect one another, which is impossible.

Thus, both trajectories $\gamma_1$ and $\gamma_2$ are completely located in the circle $B^2$ and start and end at the singular point $y_0$. However, if $\gamma_1$ and $\gamma_2$ are trajectories of the gradient field of the function $f$, then this function strictly increases along these trajectories, which is impossible because

$$f(y_0) = f(\gamma_1(-\infty)) = f(\gamma_1(+\infty)) = f(\gamma_2(-\infty)) = f(\gamma_2(+\infty))$$

Now let us consider the case where there are more than two points where the function $\alpha(t)$ passes through levels $\pi n$, $n \in Z$. Note that the difference between the number of points where the function $\alpha(t)$ increases and the number of points where it decreases when passing the levels $\pi n$, $n \in Z$, is $2(k-l)$, where $k$ is the index of the singular point $y_0$.

Let $x_i$ be a point on the circumference $S^1$, where the function $\alpha(t)$, increasing, passes a level $\pi n$, $n \in U$, and let $\gamma_i(s)$ be the integral trajectory passing through the point $x_i$. An integral trajectory of a gradient field cannot start and end at the same singular point $y_0$. Hence, there is a point $y_i$ where the trajectory $\gamma_i(s)$ leaves the circle $B^2$. Then the arc of the trajectory $\gamma_i(s)$ between the points $x_i$ and $y_i$ decomposes the circle $B^2$ into two parts. We denote the part containing the point $y_0$ by $A$. The boundary of the domain $A$ at the point $x_i$ must be smooth. If this is not the case, we take another direction along the trajectory $\gamma_i(s)$ from the point $x_i$. Let us smooth out the boundary $\partial A$ at the point $y_i$ and denote the obtained curve by $S^1_i$, and the corresponding domain bounded by this curve by $B^2_i$.

The pair $(B^2_i, S^1_i)$ is evidently diffeomorphic to the pair $(B^2, S^1)$. It is possible to introduce coordinates $(u^2, v^1)$ on it such that $S^1_i$ is the unit circle relative to these coordinates. When considering a parametrization of this circle and introducing the function of the angle $\alpha^1$ between a vector tangent to $S^1_i$ and the vector field $v$, one can see that this function has at least two points less where it passes through the levels $\pi n$, in comparison with the function $\alpha$ (at the point $x_i$ the function $\alpha^1$ does not pass the levels $\pi n$; there are no points of passage through the levels $\pi n$ on the arc of the circle $S^1_i$ between the points $x_i$ and $y_i$).

In turn, the pair $(B^2_i, S^1_i)$ is diffeomorphic to the pair $(B^2_j, S^1_j)$, for which the function $\alpha^2$ has at least two points less where it passes through the levels $\pi n$, in comparison with the function $\alpha^1$. We repeat this process until there
are two points left where a function $\alpha^i$ passes through the levels $\pi n$. Since the trajectories passing through these points start and end at the singular point $y_0$, we have proved the following fact: There is no smooth function $f$ on a two-dimensional closed manifold $M^n$ the gradient field of which has a singular point of index $k = 2$.

For the case $k > 2$, one can demonstrate that there exists a trajectory of the vector field originating and ending at the singular point $y_0$, which is impossible for the gradient field of a function $f$. Theorem 7 is completely proved.

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