ON THE POISSON STRUCTURES RELATED TO $\kappa$-POINCARÉ GROUP.

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Abstract. It is shown that the Poisson structure related to $\kappa$-Poincaré group is dual to certain Lie algebroid structure, the related Poisson structure on the (affine) Minkowski space is described in a geometric way.

1. Introduction

It is, more or less, “common knowledge” that the quantum $\kappa$-Poincaré Group [1] exists on a $C^*$-level, it is given by some bicrossproduct construction [2], [3] and it’s a quantization of certain Poisson-Lie structure [4]. Despite these beliefs, no precise and explicit formulae (e.g. for coproduct of generators) are known to the author. This note is a by-product of work on the $C^*$-version of the $\kappa$-Poincaré. It consists of two parts. In the first one, it is shown that really the Poisson structure presented in [4] is dual to certain Lie algebroid structure; this Lie algebroid is the Lie algebroid of a groupoid. The first part describes the Poisson version of the “$\kappa$-Minkowski” space and its relation to the Poisson structure on the Poincaré Group. These are again rather simple observations (essentially this part is almost contained in [7]). In the second part, too, I tried to clarify geometric picture and present results in a coordinate-free form.

Notation for orthogonal Lie algebras. $(V, \eta)$ stands for a real, finite dimensional vector space with a bilinear, symmetric and nondegenerate form $\eta$. An orthonormal basis is a basis $(v_\alpha)$ in $V$ such that $\eta(v_\alpha, v_\beta) = \eta(v_\alpha, v_\alpha)\delta_{\alpha\beta}$, $|\eta(v_\alpha, v_\alpha)| = 1$. For a vector $v$ with $|\eta(v, v)| = 1$ we write $sgn(v)$ for $\eta(v, v)$. By $\eta$ we denote also the isomorphism $V \to V^*$ given by $<\eta(x), y> := \eta(x, y)$. Using this notation, for any orthonormal basis $(v_\alpha)$ and any $x, y \in V$:

$$I = \sum_{\alpha} sgn(v_\alpha)v_\alpha \otimes \eta(v_\alpha), \quad \eta(x, y) = \sum_{\alpha} sgn(v_\alpha)\eta(x, v_\alpha)\eta(v_\alpha, y)$$

A subspace generated by vectors $v_1, \ldots, v_k$ is denoted by $<v_1, \ldots, v_k>$ or $\text{span}\{v_1, \ldots, v_k\}$; for a subset $S \subset V$ the symbol $S^\perp$ denotes the orthogonal complement of $S$, if $S = \{v\}$ we write $v^\perp$ instead of $\{v\}^\perp$; the symbol $S^0 \subset V^*$ stands for the annihilator of $S$.

For vectors $x, y \in V$ let $\Lambda_{xy} := x \otimes \eta(y) - y \otimes \eta(x)$; for a basis $(v_\alpha)$ in $V$ we write $\Lambda_{\alpha\beta}$ instead of $\Lambda_{v_\alpha, v_\beta}$. Operators $\Lambda_{xy}$ satisfy:

$$[\Lambda_{xy}, \Lambda_{zt}] = \eta(x, t)\Lambda_{yz} + \eta(y, z)\Lambda_{xt} - \eta(x, z)\Lambda_{yt} - \eta(y, t)\Lambda_{xz}$$

and $so(\eta) = \text{span}\{\Lambda_{xy} : x, y \in V\}$. If $W \subset V$ is a subspace then $\Lambda_W := \text{span}\{\Lambda_{xy} : x, y \in W\}$ is a Lie subalgebra of $so(\eta)$; for a null vector $f \in V$ and a subspace $W \subset V$ the subspace $\Lambda_{Wf} := \text{span}\{\Lambda_{wf} : w \in W\}$ is also a subalgebra; notice that for $g \in O(\eta)$ we have $\Lambda_{gxyg} = g\Lambda_{xy}g^{-1} =: \text{ad}(g)(\Lambda_{xy})$.

We will use a bilinear, nondegenerate form $k: so(\eta) \times so(\eta) \to \mathbb{R}$ defined by:

$$k(\Lambda_{xy}, \Lambda_{zt}) := \eta(x, t)\eta(y, z) - \eta(x, z)\eta(y, t)$$

It is easy to see that for $g \in O(\eta)$: $\text{ad}(g) \in O(k)$ i.e.

$$k(g\Lambda_{xy}g^{-1}, g\Lambda_{zt}g^{-1}) = k(\Lambda_{xy}, \Lambda_{zt}), \quad g \in O(\eta).$$
(of course $k$ is proportional to the Killing form). By $\text{ad}^\#$ we denote the coadjoint representation of $O(\eta)$ on $so(\eta)^*$: $\text{ad}^\#(g) := \text{ad}(g^{-1})^*$. If $k$ is the isomorphism $so(\eta) \rightarrow so(\eta)^*$ defined by the form $k$ then

$$\text{ad}^\#(g)k(X) = k(\text{ad}(g)X) \ , \ X \in so(\eta)$$

Let us also define a bilinear form $\tilde{k}$ on $so(\eta)^*$ by:

$$k(\varphi, \psi) := k(k^{-1}(\varphi), k^{-1}(\psi)) \ , \ \varphi, \psi \in so(\eta)^*$$

so $\tilde{k}(\varphi, \psi) = <\varphi, k^{-1}(\psi)>$; again it is clear that if $g \in O(\eta)$ then $\text{ad}^\#(g) \in O(\tilde{k})$, and

$$\tilde{k}(\text{ad}^\#(g)k(X), k(Y)) = k(\text{ad}(g)X, Y) \ , \ X, Y \in so(\eta)$$

We will also need one-parameter groups; they are given by formulae:

$$\exp(\Lambda_{\eta f}) = I + \frac{\eta(u, u)}{2}f \otimes \eta(f) \ , \ \eta(u, f) = 0 = \eta(f, f)$$

(5) \[ \exp(\nu \Lambda_{st}) = I - P_{st} + \cosh(\nu)P_{st} + \sinh(\nu)\Lambda_{st} \ , \ \eta(s, s) = -1 = \eta(t, t) , \ \eta(s, t) = 0 \ , \ \nu \in \mathbb{R} \]

$$\exp(\nu \Lambda_{xy}) = I - P_{xy} + \cos(\nu)P_{xy} + \sin(\nu)\Lambda_{xy} \ , \ \eta(x, x) = \eta(y, y) = \pm 1 \ , \ \eta(x, y) = 0 \ , \ \nu \in \mathbb{R}$$

where $P_{vw}$ denotes the orthogonal projection onto $<v, w>$. If vectors $v, w$ are orthogonal and $|\eta(v, v)| = |\eta(u, w)| = 1$, then:

$$P_{vw} = \text{sgn}(v)v \otimes \eta(v) + \text{sgn}(w)w \otimes \eta(w) = -\text{sgn}(v)\text{sgn}(w)\Lambda_{vw}^2$$

**Poincaré Group.** Let $(V, \eta)$ be a vector Minkowski space (signature of $\eta$ is $(+, -, -, -)$). For a vector $v \in V$ with $\eta(v, v) \neq 0$ let $R_v$ denote the reflection across the hyperplane $v^\perp$, i.e. $R_v = I - \frac{2}{\eta(v, v)}v \otimes \eta(v)$. The full orthogonal group $O(\eta)$ has four connected components: $SO_0(\eta)$ – the connected component of identity; $R_vSO_0(\eta)$, $\eta(t, t) > 0$ – the component containing time reflection; $R_vSO_0(\eta)$, $\eta(s, s) < 0$ – the component containing space reflection and $SO_1(\eta) := R_tR_vSO_0(\eta)$, $\eta(t, t) > 0, \eta(s, s) < 0$ – the component reversing time and space orientation (but keeping the space-time orientation intact). In this paper the Poincaré Group $P(\eta)$ will mean the semidirect product $V \rtimes O(\eta)$ and the restricted Poincaré Group $P_0(\eta)$ is $V \rtimes SO_0(\eta)$. Elements $(w, g)$ of $P(\eta)$ act on $V$ by affine mappings: $(w, g)(v) := w + gv$ and the group law is just the composition of these mappings: $(w, g)(u, h) := (w + gu, gh)$. Since $P(\eta)$ depends only on dimension $n$ of $V$ it will be also denoted by $P(n)$; also $O(\eta)$ will be denoted by $O(1, n - 1)$.

2. **Poisson-Poincaré Group.**

The particular Lie-Poisson structure on Poincaré Group we are interested in was defined in [4]: it is dual to a certain Lie algebroid structure. The construction is as follows.

Let $(V, \eta)$ be a vector Minkowski space of dimension $n + 2$, $n > 1$ and $G := SO_0(\eta)$. Our Poisson-Poincaré Group will be realized as a subgroup of the semidirect product $g^* \rtimes G$:

$$\langle \varphi, g \rangle(\psi, h) := (\varphi + \text{ad}^\#(g)\psi, gh)$$

(7) \[ \langle \varphi, g \rangle(\psi, h) := (\varphi + \text{ad}^\#(g)\psi, gh) \]

Notice that if $H \subset G$ is a subgroup with a Lie algebra $\mathfrak{h} \subset g$ then $\mathfrak{h}^0 \times H$ is a subgroup of $g^* \rtimes G$.

Let us choose a (spacelike) vector $s \in V$ with $\eta(s, s) = -1$ and define a subalgebra:

$$a := \text{span}\{\Lambda_{xy}, x, y \in s^\perp\} = \{Y \in so(\eta) : Ys = 0\}$$

(8) \[ a := \text{span}\{\Lambda_{xy}, x, y \in s^\perp\} \]

It is straightforward to see that: $a^\perp = \text{span}\{\Lambda_{xs}, x \in s^\perp\}$ and $k(\Lambda_{xs}, \Lambda_{sy}) = \eta(x, y), x, y \in s^\perp$ i.e. $(a^\perp, k)$ is an $n + 1$ dimensional vector Minkowski space and the same is true for $(a^0, k)$.

Let $\tilde{A}$ be the connected subgroup of $G$ with Lie algebra $a$: $\tilde{A} = \{g \in G : gs = s\} \simeq SO_0(1, n)$ (i.e $\tilde{A}$ is the proper, orthochronous Lorentz group). Therefore the subgroup $a^0 \times \tilde{A}$ is $P_0(n + 1)$; this way we have identified $P_0(n + 1)$ as a subgroup of the semidirect product $g^* \rtimes G$. For reasons which are related to the “quantum version” of our Poisson-Poincaré group, we will also consider the normalizer of $\tilde{A}$ in $G$ which will be denoted by $A$. It is easy to see that

$$A := \{g \in G : gs = d(g)s, d(g) = \pm 1\} = \tilde{A} \cup \exp(\pi\Lambda_{us})\tilde{A} = \tilde{A} \cup \tilde{A}\exp(\pi\Lambda_{us})$$

(9) \[ A := \{g \in G : gs = d(g)s, d(g) = \pm 1\} = \tilde{A} \cup \exp(\pi\Lambda_{us})\tilde{A} = \tilde{A} \cup \tilde{A}\exp(\pi\Lambda_{us}) \]
for any spacelike, normalized vector $u \in s^\perp$.

Let us now compute the action of $\exp(\pi \Lambda_{us})$ on $a^0$:
\[ \text{ad}^g(\exp(\pi \Lambda_{us}))k(\Lambda_{xs}) = k(\exp(\pi \Lambda_{us})\Lambda_{xs} \exp(-\pi \Lambda_{us})), \] and
\[ \exp(\pi \Lambda_{us})\Lambda_{xs} \exp(-\pi \Lambda_{us}) = -\Lambda_{xs} - 2\eta(x, u)\Lambda_{us}, \eta(u, u) = -1, x, u \in s^\perp. \]

**In this way what exactly is $a^0 \times A$ depends on the dimension of $V$: for $n + 1$ – even this is $P_0(n + 1)$ extended by time reflection; for $n + 1$ – odd this is $P_0(n + 1)$ extended by space and time reflection.**

The Lie-Poisson structure on $a^0 \times A$ depends on a choice of a timelike vector $t \in V$ or, equivalently, on a splitting of $a^0$ into space: $\text{span}\{k\Lambda_{us}, u \in s^\perp, t >^\perp\}$ and time: $< k\Lambda_{ts} >$. So let us choose a (timelike) vector $t \in s^\perp, \eta(t, t) = 1$; denote $f := t - s$ and let us define subalgebras:
\[
\begin{align*}
\mathfrak{c} & := \text{span}\{\Lambda_{xt} : x \in s^\perp\} = \text{span}\{\Lambda_{yt} : y \in t^\perp\} \\
\mathfrak{b} & := \text{span}\{\Lambda_{xy}, x, y \in t^\perp\} = \{Y \in so(\eta) : Yt = 0\}
\end{align*}
\]

The Lie algebra $so(\eta)$ can be decomposed as (direct sums of vector spaces):
\[
so(\eta) = \mathfrak{c} \oplus \mathfrak{b} = \mathfrak{c} \oplus a
\]

Let $B, C$ be connected subgroups of $G$ with Lie algebras $\mathfrak{b}, \mathfrak{c}$ respectively; then $B = \{g \in G : gt = t\} \simeq SO(n + 1)$. Denote $U := \langle s, t >^\perp \subset V$; then $(U, -\eta)$ is an $n$ dimensional (vector) Euclidean space. The subalgebra $\mathfrak{c}$ can be decomposed further as
\[ \mathfrak{c} = \Lambda_U \oplus \Lambda_{ts}, \]
where by (2) the first summand is an abelian ideal (in $\mathfrak{c}$).
Using (5) we obtain:
\[ \exp(\nu \Lambda_{ts}) \exp(\Lambda_{uf}) \exp(-\nu \Lambda_{ts}) = \exp(\Lambda_{(x^\mu, \nu)f}), u \in U, \nu \in \mathbb{R} \]
Therefore $C = \{\exp(\Lambda_{uf}) \exp(\nu \Lambda_{ts}) : u \in U, \nu \in \mathbb{R}\}$ and is isomorphic to the semidirect product $U \rtimes \mathbb{R}$ with group operation:
\[ (u, \mu)(v, \nu) := (u + e^\mu v, \mu + \nu), u, v \in U, \mu, \nu \in \mathbb{R} \]

The group $C$ is the AN group in the Iwasawa decomposition $SO_0(1, n + 1) = SO(n + 1)(AN)$ i.e there is the equality $G = BC$.

The open set $\Gamma := AC \cap CA$ carries two differential groupoid structures over $A$ and $C$ [8]. The groupoid “responsible” for our Lie-Poisson structure is the groupoid $\Gamma_A : \Gamma \rightarrow A$. Namely, the bundle $(\mathcal{T}A)^0 \subset T^*G$ is dual to the Lie algebroid $\mathcal{L}(\Gamma_A)$, which we realize as a vectors tangent in points of $A$ to left fibers with bracket coming from left invariant vector fields. In this way $(\mathcal{T}A)^0$ carries the canonical Poisson structure; on the other hand via right translation we can identify $(\mathcal{T}A)^0$ with $d^0 \times A$ i.e. with the Poincaré group; it turns out this is a Poisson structure described in [4]. Let us compute Poisson brackets explicitly.

**Algebroid structure.** The map $A \times \mathfrak{c} \ni (a, p) \mapsto ap \in T_a \Gamma_A$ is a global trivialization of $\mathcal{L}(\Gamma_A)$. For $p \in \mathfrak{c}$ let $X^L_p$ be the left invariant vector field on $\Gamma_A$ defined by:
\[ X^L_p(a) := ap \]
These vector fields satisfy:
\[ [X^L_p, X^L_q] = X^L_{[p, q]}, p, q \in \mathfrak{c} \]
and the anchor map $\Pi^L : \mathcal{L}(\Gamma_A) \rightarrow TA$ is given by
\[ \Pi^L(X^L_p)(a) = P_a(\text{ad}(a)p)a, \]
where $P_a$ is the projection onto $a$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{c} \oplus a$. Short computations give:
\[ P_a\text{ad}(a)\Lambda_{xt} = \text{ad}(a)\Lambda_{xt} - d(a)\Lambda_{(ax)t}, x \in s^\perp, a \in A, as =: d(a)s \]
The Poisson structure. Sections of $\mathcal{L}(\Gamma A)$ define linear functions on $(TA)^0$, if $X$ is a section of $\mathcal{L}(\Gamma A)$, the corresponding function will be denoted by $\tilde{X}$. Explicit form of this function for $X^L_p$ is: 

$$\tilde{X}^L_p(\varphi, a) = \langle \varphi a, X^L_p(a) \rangle = \langle \varphi, \text{ad}(a)p \rangle = \langle \varphi, \text{ad}^\#(a)k(p) \rangle, \ \varphi \in \mathfrak{a}^0, \ p \in \mathfrak{e}$$

(in this formula $(TA)^0 \simeq \mathfrak{a}^0 \times A$ via right translations). The Poisson structure on $(TA)^0$ is defined by the brackets:

$$(17) \quad \{X_1, X_2\} = [X_1, X_2], \ \{X, \pi^*(f_1)\} = \pi^*(\Pi L^*(X)f_1), \ \{\pi^*(f_1), \pi^*(f_2)\} = 0,$$

where $f_1, f_2$ are smooth functions on $A$, $\pi : T^*G \to G$ is the cotangent bundle projection and $\pi^*$ denotes the pullback of functions.

Our Poincaré Group was identified with $\mathfrak{a}^0 \times A \simeq (TA)^0$ (via right translations). For $\varphi, \psi \in \mathfrak{a}^0$ let us define smooth functions $k_\varphi, k_\psi$ on $\mathfrak{a}^0 \times A$:

$$(18) \quad k_\varphi(\rho, a) := k(\varphi, \rho)$$

$$k_\psi(\rho, a) := k(\psi, \text{ad}^\#(a)\rho)$$

Any Poisson structure on $\mathfrak{a}^0 \times A$ is determined by brackets:

$$\{k_\varphi, k_\psi\}, \ \{k_{\varphi, \rho}, k_{\psi, \rho}\}, \ \{k_{\varphi, \lambda}, k_{\psi, \rho}\}, \ \varphi, \psi, \rho, \lambda \in \mathfrak{a}^0$$

For the Poisson structure given by (17) we immediately get:

$$(19) \quad \{k_\varphi, k_\psi\} = 0.$$

Let us now compute remaining brackets and compare them with the ones presented in [4]. To this end we will relate functions $\tilde{k}_\psi$ and $\tilde{X}^L_p$.

**Lemma 2.1.** Let $(\rho_\alpha)$ be an orthonormal basis in $\mathfrak{a}^0$ and assume that elements $c_\alpha \in \mathfrak{c}$ satisfy $k(\psi, \text{ad}^\#(a)\rho_\alpha) = \langle \psi, \text{ad}(a)c_\alpha \rangle > 0$ for any $\psi \in \mathfrak{a}^0$ and any $a \in A$. Then:

$$\tilde{k}_\varphi = \sum_\alpha \text{sgn}(\rho_\alpha)\tilde{k}_{\varphi, \rho_\alpha} \tilde{X}_c^L$$

**Proof:** Indeed, using (1) let us compute:

$$\tilde{k}_\varphi(\psi, a) = \tilde{k}(\varphi, \psi) = k(\text{ad}^\#(a^{-1})\varphi, \text{ad}^\#(a^{-1})\psi) = \sum_\alpha \text{sgn}(\rho_\alpha)k(\text{ad}^\#(a^{-1})\varphi, \rho_\alpha)k(\rho_\alpha, \text{ad}^\#(a^{-1})\psi) =$$

$$\sum_\alpha \text{sgn}(\rho_\alpha)k(\varphi, \text{ad}^\#(a)\rho_\alpha)k(\psi, \text{ad}^\#(a)\rho_\alpha)$$

$$= \sum_\alpha \text{sgn}(\rho_\alpha)\tilde{k}_{\varphi, \rho_\alpha}(\psi, a) < \psi, \text{ad}(a)c_\alpha \rangle > \sum_\alpha \text{sgn}(\rho_\alpha)\tilde{k}_{\varphi, \rho_\alpha}(\psi, a)X_{c_\alpha}^L =$$

$$= \left(\sum_\alpha \text{sgn}(\rho_\alpha)\tilde{k}_{\varphi, \rho_\alpha} X_{c_\alpha}^L\right)(\psi, a)$$

Let $(v_\alpha)$ be an orthonormal basis in $\mathfrak{s}^\perp$, then $(\rho_\alpha) := (k(L_{v_\alpha}))$ is an orthonormal basis in $\mathfrak{a}^0$. Straightforward computations prove that elements $c_\alpha := -L_{v_\alpha\varphi}$ satisfy condition stated in the lemma above. Now, using (17) and the decomposition above, we have:

$$(20) \quad \{\tilde{k}_\varphi, \tilde{k}_\psi\} = \sum_{\alpha, \beta} \text{sgn}(\rho_\alpha)\text{sgn}(\rho_\beta)\{\tilde{k}_{\varphi, \rho_\alpha} \tilde{X}_{c_\alpha}^L, \ \tilde{k}_{\psi, \rho_\beta} \tilde{X}_{c_\beta}^L\} =$$

$$= \left(\sum_{\alpha, \beta} \text{sgn}(\rho_\alpha)\text{sgn}(\rho_\beta)\left[\tilde{X}_{c_\alpha}, \tilde{k}_{\psi, \rho_\beta}\right] \tilde{k}_{\varphi, \rho_\alpha} \tilde{X}_{c_\beta}^L - \left[\tilde{X}_{c_\beta}, \tilde{k}_{\varphi, \rho_\alpha}\right] \tilde{k}_{\psi, \rho_\beta} \tilde{X}_{c_\alpha}^L + \tilde{X}_{c_\alpha}^L \tilde{X}_{c_\beta}^L \tilde{k}_{\varphi, \rho_\alpha} \tilde{k}_{\psi, \rho_\beta}\right) =$$

$$= \mathbf{I} + \mathbf{II} + \mathbf{III}$$
To end our computations we need formula for $\Pi^L(X^L_p)(\tilde{k}_{\phi \psi})$ for $p := \Lambda_{xt} \in e_x, x \in s^\perp$ (note that the same symbol $\tilde{k}_{\phi \psi}$ is used for function on $a^0 \times A$ and on $A$). By (13) and (19):

$$\Pi^L(X^L_p)(a) = Za , \text{ where } Z := \text{ad}(a)\Lambda_{xt} - d(a)\Lambda_{(ax)t} , a \in A, \text{ as } =: d(a)s$$
and

$$\Pi^L(X^L_p)(\tilde{k}_{\phi \psi})(a) = \frac{d}{dt} \bigg|_{t=0} \tilde{k}_{\phi \psi}(\exp(Zt)a) = \frac{d}{dt} \bigg|_{t=0} \tilde{k}(\phi, \text{ad}^\#(\exp(Zt)a)\psi) =$$

$$= \frac{d}{dt} \bigg|_{t=0} k(\Lambda_{vs}, \text{ad}(\exp(Zt))\text{ad}(a)\Lambda_{us}) = \frac{d}{dt} \bigg|_{t=0} k(\text{ad}(\exp(-Zt))\Lambda_{vs}, \text{ad}(a)\Lambda_{us}),$$

where we put $\phi = k(\Lambda_{vs}), \psi = k(\Lambda_{us})$ for $v, w \in s^\perp$. We have the equality $\text{ad}(\exp(-Zt))\Lambda_{vs} = \Lambda(\exp(-Zt))v, \exp(-Zt)s)$, where for a while we use $\Lambda(x, y)$ for $\Lambda_{xy}$. Since we are interested only in derivative in $t = 0$ we can replace $\exp(-Zt)$ by $I - Zt$ and get:

$$\frac{d}{dt} \bigg|_{t=0} k(\text{ad}(\exp(-Zt))\Lambda_{vs}, \text{ad}(a)\Lambda_{us}) = k(-\Lambda_{(Zv)s} - \Lambda_{v(Zs)}, \text{ad}(a)\Lambda_{us})$$

By (21) $Zs = 0$ and :

$$Zv = [\Lambda_{(ax)(at)} - d(a)\Lambda_{ax}] v = [\eta(\phi, v) - d(\eta(\phi, v)](ax) - \eta(\phi, v)(at) + d(a)\eta(\phi, v)t.$$ 
Therefore

$$\Lambda_{(Zv)s} + \Lambda_{v(Zs)} = \Lambda_{(Zv)s} = [\eta(\phi, v) - d(\eta(\phi, v)](ax) - \eta(\phi, v)(at) + d(a)\eta(\phi, v)t.$$

So (22) is equal to

$$\frac{d}{dt} \bigg|_{t=0} k(\text{ad}(\exp(-Zt))\Lambda_{vs}, \text{ad}(a)\Lambda_{us}) = k(-\Lambda_{(Zv)s} - \Lambda_{v(Zs)}, \text{ad}(a)\Lambda_{us})$$

Let us define $\rho := k(\Lambda_{v})$ and $\rho := k(\Lambda_{us})$, then we have (recall that $\phi = k(\Lambda_{us}), \psi = k(\Lambda_{us})$):

$$k(\Lambda_{xs}, \Lambda_{us}) = \tilde{k}(\rho, \psi), k(\Lambda_{ts}, \Lambda_{us}) = \tilde{k}(\rho, \phi), k(\Lambda_{ts}, \text{ad}(a)\Lambda_{us}) = \tilde{k}(\rho, \text{ad}^\#(\alpha)\psi) = \tilde{k}_{\rho(\psi)}(\alpha),$$

$$d(a)\eta(\phi, v) = d(a)k(\Lambda_{(ax)(at)}\Lambda_{(as)}, \Lambda_{(as)}) = k(\text{ad}(a)\Lambda_{ax}) = \tilde{k}(\text{ad}^\#(\alpha)\phi, \psi) = \tilde{k}_{\rho(\psi)}(\alpha)$$
and (23) is equal to:

$$[\tilde{k}(\rho, \phi) - \tilde{k}_{\rho(\psi)}(\alpha)] \tilde{k}(\rho, \psi) + \tilde{k}_{\rho(\psi)}(\alpha) \tilde{k}(\rho, \phi) - \tilde{k}_{\rho(\psi)}(\alpha)] =$$

$$\left\{ \tilde{k}(\rho, \psi) \left[ \tilde{k}(\rho, \phi)I - \tilde{k}_{\rho(\psi)} \right] + \tilde{k}_{\rho(\psi)} \left[ \tilde{k}(\rho, \phi)I - \tilde{k}_{\rho(\psi)} \right] \right\}(a)$$

In this way we finally get:

$$\Pi^L(X^L_p)(\tilde{k}_{\phi \psi}) = \tilde{k}(\rho, \phi)I - \tilde{k}_{\rho(\psi)} + \tilde{k}_{\rho(\psi)} I - \tilde{k}_{\rho(\psi)}$$

where $p := \Lambda_{xt}, x \in s^\perp, \rho := k(\Lambda_{xs})$ and $\rho := k(\Lambda_{ts})$.

Now we return to computations of (20). Choose an orthonormal basis $(e_a)$ in $s^\perp$ with $e_0 := t$. Then we have orthonormal basis $\rho_\alpha := k(\Lambda_{e_\alpha}) = k(\Lambda_{os})$ in $a^0$ with $\rho_0 = \rho$ and corresponding elements $e_\alpha := -\Lambda_{e_\alpha} := -\Lambda_{at}$. Using (21) we obtain:

$$\left\{ X^L_{e_\alpha}(\tilde{k}_{\phi \psi}) \right\} = \Pi^L(X^L_{e_\alpha})(\tilde{k}_{\phi \psi}) = \tilde{k}(\Lambda_{(e_\alpha)s}) = \tilde{k}(\Lambda_{(e_\alpha)s}) \left[ \tilde{k}(\rho, \psi)I - \tilde{k}_{\rho(\psi)} \right] =$$

$$= -\tilde{k}(\rho_\alpha, \rho_\beta) \left[ \tilde{k}(\rho, \psi)I - \tilde{k}_{\rho(\psi)} \right] - \tilde{k}_{\rho_\alpha}(\rho, \psi)I - \tilde{k}_{\rho_\beta}(\rho, \psi)I$$

$$= -\delta(\alpha, \beta) \tilde{k}(\rho, \psi)I - \tilde{k}_{\rho(\psi)}$$
In this way the first term in the sum (20) is equal to:

\[ \mathbf{I} = \sum_{\alpha \beta} sgn(\rho_\alpha)sgn(\rho_\beta)\{\Xi^{L}_{\alpha \beta}, \tilde{k}_{\psi p} \} \tilde{k}_{\varphi p a} \Xi^{L}_{\epsilon a} = -\tilde{k}_\varphi \left( \tilde{k}(\rho, \psi)I - \tilde{k}_\psi \rho \right) - \sum_{\alpha} sgn(\rho_\alpha) \tilde{k}_{\psi p a} \tilde{k}_{\varphi p a} \Xi^{L}_{\epsilon a} + \tilde{k}_\rho \sum_{\alpha} sgn(\rho_\alpha) \tilde{k}_{\psi p a} \tilde{k}_{\varphi p a} \]

The second term in (20) we get by interchanging in \( \mathbf{I} \) \( \alpha \) with \( \beta, \varphi \) with \( \psi \) and changing the sign:

\[ \mathbf{II} = \tilde{k}_\psi \left( \tilde{k}(\rho, \varphi)I - \tilde{k}_\varphi \rho \right) + \sum_{\beta} sgn(\rho_\beta) \tilde{k}_{\psi p b} \tilde{k}_{\varphi p b} \Xi^{L}_{\epsilon a} - \tilde{k}_\rho \sum_{\beta} sgn(\rho_\beta) \tilde{k}_{\psi p b} \tilde{k}_{\varphi p b} \]

and their sum is

\[ \mathbf{I} + \mathbf{II} = \tilde{k}_\psi \left( \tilde{k}(\rho, \varphi)I - \tilde{k}_\varphi \rho \right) - \tilde{k}_\varphi \left( \tilde{k}(\rho, \psi)I - \tilde{k}_\psi \rho \right) \]

It remains to compute \( \mathbf{III} \):

\[ [c_\alpha, c_\beta] = [-\Lambda_\alpha t, -\Lambda_\beta t] = \eta(f, v_\beta)\Lambda_\alpha t - \eta(f, v_\alpha)\Lambda_\beta t = \delta_{0 \alpha} c_\beta - \delta_{0 \beta} c_\alpha \]

therefore

\[ \Xi^{L}_{[c_\alpha, c_\beta]} = \delta_{0 \alpha} \Xi^{L}_{\epsilon \beta} - \delta_{0 \beta} \Xi^{L}_{\epsilon \alpha} \]

and

\[ \mathbf{III} = \sum_{\alpha \beta} sgn(\rho_\alpha)sgn(\rho_\beta)\tilde{k}_{\varphi p a} \tilde{k}_{\psi p b} \left[ \delta_{0 \alpha} \Xi^{L}_{\epsilon \beta} - \delta_{0 \beta} \Xi^{L}_{\epsilon \alpha} \right] = \tilde{k}_\psi k_{\varphi p} - \tilde{k}_\varphi k_{\psi p} \]

Finally:

\[ \{ \tilde{k}_\varphi, \tilde{k}_\psi \} = \mathbf{I} + \mathbf{II} + \mathbf{III} = \tilde{k}(\rho, \varphi)\tilde{k}_\psi - \tilde{k}(\rho, \psi)\tilde{k}_\varphi \]

In the similar way, using lemma 2.1 and formulae (19) and (24) we obtain:

\[ \{ \tilde{k}_\lambda, \tilde{k}_\varphi \} = \tilde{k}_\lambda \varphi (\tilde{k}_{\psi p} - \tilde{k}(\rho, \varphi)I) + \tilde{k}(\lambda, \varphi)(\tilde{k}_{\psi p} - \tilde{k}(\rho, \psi)I) \]

Now we have all the brackets:

\[ \{ \tilde{k}_\varphi, \tilde{k}_\psi \} = \tilde{k}(\rho, \varphi)\tilde{k}_\psi - \tilde{k}(\rho, \psi)\tilde{k}_\varphi, \]

\[ \{ \tilde{k}_\lambda, \tilde{k}_\varphi \} = \tilde{k}_\lambda \varphi (\tilde{k}_{\psi p} - \tilde{k}(\rho, \varphi)I) + \tilde{k}(\lambda, \varphi)(\tilde{k}_{\psi p} - \tilde{k}(\rho, \psi)I), \]

\[ \{ k_{\varphi \lambda}, k_{\psi \varphi} \} = 0 \quad \text{for} \quad \varphi, \lambda, \psi, \rho \in a^0 \quad \text{and} \quad \rho := k(\Lambda_{ts}). \]

The Poincaré group in [4] was identified with matrices \( g = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix} \), where \( \Lambda \) is a Lorentz matrix of dimension \( n + 1 \) and \( v \in \mathbb{R}^{n+1} \). Poisson brackets for matrix elements of \( g \) are given by:

\[ \{ \Lambda_{\mu \nu}, v_\beta \} = h \left[ (\Lambda_{\mu \delta} - \delta_{\mu \delta})\Lambda_{\beta \nu} + \eta_{\mu \beta}(\Lambda_{0 \nu} - \delta_{0 \nu}) \right], \]

\[ \{ v_\alpha, v_\beta \} = h(v_\alpha \delta_{\beta 0} - v_\beta \delta_{\alpha 0}), \]

\[ \{ \Lambda_{\mu \nu}, \Lambda_{\alpha \beta} \} = 0, \]

where \( \eta_{\alpha \beta} := \text{diag}(1, -1, \ldots, -1) \) and \( h \) is a real parameter (Note: here \( \Lambda_{\alpha \beta} \) are matrix elements not operators). To compare the brackets, let us choose an orthonormal basis \( (\rho_\alpha) \in a^0 \) with \( \rho_0 = \rho \). We have

\[ \tilde{k}(\rho_\alpha, \rho_\beta) = \text{diag}(1, -1, \ldots, -1) = \eta_{\alpha \beta}, \quad v_\alpha = sgn(\rho_\alpha)\tilde{k}_{\rho_\alpha}, \quad \text{and} \quad \Lambda_{\alpha \beta} = sgn(\rho_\alpha)\tilde{k}_{\rho_\alpha \rho_\beta}. \]

Short computations show that brackets (27) coincide with (28) for \( h = -1 \).
3. Poisson Minkowski space

Let \((V, \eta)\) be a real, \(n\)-dimensional \((n > 2)\) vector space with a symmetric, bilinear, nondegenerate form \(\eta\). For a basis \(\{e_\alpha\}\) of \(V\) let \(\eta_{\alpha\beta} := \eta(e_\alpha, e_\beta)\) be the corresponding matrix of \(\eta\) and \(\eta^{\alpha\beta}\) stands for the inverse matrix. Note that despite the title of the section, \((V, \eta)\) need not to be a (vector) Minkowski space. In this section \(G\) denotes any subgroup of \(O(\eta)\) containing \(SO_0(\eta)\) and \(IG := V \rtimes G\) is the semi-direct product. The Lie algebra of \(IG\) is \(iso(\eta) := V \times so(\eta)\) and the bracket is:

\[
[(v, A), (w, B)] = (Aw - Bv, [A, B])
\]

The Poisson bracket for \(\kappa\)-Poincaré in [4] is an example of a more general situation [6]. For a vector \(v \in V\) let us define

\[
b_v := \sum \eta^{ik} e_j \wedge \Lambda_{v,e_k} \in iso(\eta) \wedge iso(\eta),
\]

where \(\{e_k\}\) is any basis in \(V\). Direct computation proves that, for \(u, v \in V\), elements \(b_u, b_v\) satisfy:

\[
[b_v, b_u] = -\eta(v, u)\Omega,
\]

where \(\Omega := \sum \eta^{ik} \eta^{mn} e_j \wedge e_m \wedge \Lambda_{e_k, e_n}\) is the canonical invariant element in \(iso(\eta) \wedge iso(\eta) \wedge iso(\eta)\), and

\[
[a \wedge b, c \wedge d] := a \wedge [b, c] \wedge d - a \wedge [b, d] \wedge c - b \wedge [a, c] \wedge d + b \wedge [a, d] \wedge c
\]

is the (algebraic) Schouten bracket. Therefore \(b_v\) defines a Poisson-Lie structure \(\hat{\Pi}_v\) on \(IG\) by:

\[
\hat{\Pi}_v(g) = b_v g - gb_v
\]

The structure in [4] is of this type for \(v\) being a timelike vector. Moreover, it is easy to see that

\[
[b_v, x \wedge u] = 2u \wedge x \wedge v, \quad \text{for any } x, u \in V
\]

so we can replace \(b_v\) in [33] by \(b_v + x \wedge v\) and we obtain another Poisson-Lie structure on \(IG\) which will be denoted by \(\hat{\Pi}_{v,x}\). The adjoint representation of \(IG\) on \(iso(\eta)\) is given by:

\[
\text{ad}_{(w, A)}(v, X) = (w + Av - AXA^{-1}w, AXA^{-1}), \quad v, w \in V, \ A \in O(\eta), \ X \in so(\eta);
\]

by the same symbol we will denote this representation canonically extended to \(iso(\eta) \wedge iso(\eta)\).

Straightforward computations give:

\[
\text{ad}_{(w, A)}(b_v) = w \wedge Av + b_{Av}, \quad \text{ad}_{(w, A)}(x \wedge v) = Ax \wedge Av
\]

Let \((M, V, \eta)\) be an affine space modeled on \((V, \eta)\). Let \(Aff(G)\) be the group of those affine isometries of \(M\) that have \(G\) as their linear part. Any point \(m \in M\) defines the isomorphism \(\phi_m : IG \to Aff(G)\) given by:

\[
\phi_m(w, A)(m + v) := m + w + Av, \quad v \in V
\]

For two points \(m, n \in M\) we have: \(\phi_m^{-1} \phi_n = Ad_{n-m} : IG \ni g \mapsto (n-m)g(n-m)^{-1} \in IG\) — the inner automorphism given by \(n - m \in V\). In this way for a point \(m \in M\) and a vector \(v \in V\) we have the Poisson-Lie structure on \(Aff(G)\) defined by:

\[
\Pi_{m,v} := \phi_m(\hat{\Pi}_v)
\]

Proposition 3.1. Let \(\Pi_{m,v}\) be the Poisson structure defined in [30]. Then:

- \(\Pi_{m,0} = \lambda \Pi_{m,v}\), \(\Pi_{m+\lambda v} = \Pi_{m,v}\) i.e. the bivector \(\Pi_{m,v}\) depends only on a parametrized line \(l := \{m + tv\}, t \in \mathbb{R}\); we will write \(\Pi_l\) for this Poisson structure.
- Let \(l, k\) be two parametrized lines then \(\Pi_l = \Pi_k\) iff \(l = k\).
- If \(\dim(V) > 3\) then \(\Pi_l\) and \(\Pi_k\) are compatible iff \(l\) and \(k\) intersect or are parallel; for \(\dim(V) = 3\): if \(G \subset SO(\eta)\) then \(\Pi_l\) and \(\Pi_k\) are compatible; otherwise the statement is as for \(\dim(V) > 3\).
**Proof:** The equality $\Pi_{m,\lambda w} = \lambda \Pi_{m,v}$ is obvious. Let $m, n \in M$, $v, u \in V$ and $x := n - m \in V$. We can transfer $\Pi_{n,u}$ to $IG$ by $\phi_{m}^{-1}$ and get $\phi_{m}^{-1}\phi_{n}(b_{u}) = \text{ad}_{x}(b_{u}) = x \wedge u + b_{u}$ by (35). Taking $n := m + \lambda v$ we get the second equality.

Let lines $l, k$ be given by $(m, v)$ and $(n, u)$ respectively. Then $l \neq k$ means that $v \neq u$ or if $v = u$ then $x := n - m \neq 0$ and $x, v$ are linearly independent. Using the definition (30) and the formula above it is easy to prove the second statement.

Poisson structures $\Pi_{l}$ and $\Pi_{k}$ are compatible iff $\hat{\Pi}_{v} + \hat{\Pi}_{u,x}$ ($x := n - m$) is a Poisson structure on $IG$, i.e. the Schouten bracket $[\hat{\Pi}_{v} + \hat{\Pi}_{u,x}, \hat{\Pi}_{v} + \hat{\Pi}_{u,x}] = 0$ and (since $\hat{\Pi}_{v}$ and $\hat{\Pi}_{u,x}$ are Poisson) this is equivalent to $[\hat{\Pi}_{v}, \hat{\Pi}_{u,x}] = 0$. By (33) this, in turn, is equivalent to $[b_{v}, x \wedge u + b_{u}]$ being $IG$ invariant (with respect to adjoint action). Using (31) and (34) we get that $x \wedge u \wedge v$ must be $G$ invariant. Clearly this element is 0 for intersecting or parallel lines $l$ and $k$. For $\text{dim}(V) > 3$, invariance of $x \wedge u \wedge v$ forces it to be 0, i.e. lines $l$ and $k$ intersect or are parallel; for $\text{dim}(V) = 3$ the element $x \wedge u \wedge v$ is invariant if $G$ preserves orientation.

A parametrized line $l := \{m + tw, t \in \mathbb{R}\}$ defines also a bivector $\pi_{l}$ on $M$:

$$\pi_{l}(m + v) := v \wedge w,$$

in the formula above we identify $TM$ with $M \times V$; it is easy to see that really $\pi_{l}$ depends only on $l$ and not on the chosen point $m \in l$.

**Proposition 3.2.**

- $\pi_{l}$ is a Poisson bivector on $M$.
- $\pi_{l}$ and $\pi_{k}$ are compatible iff lines $l, k$ intersect or are parallel.
- The canonical action of $(\text{Aff}(G), \Pi_{k})$ on $(M, \pi_{l})$ is Poisson iff $l = k$.

**Proof:** Let $l := \{m + tw, t \in \mathbb{R}\}$ and define the vector field $\hat{V}^{m}$ by $\hat{V}^{m}(m + v) := v$; let $\hat{w}$ be the constant vector field: $\hat{w}(m + v) := w$; with this notation we have: $\pi_{l} = \hat{V}^{m} \wedge \hat{w}$. If $\pi_{k}$ is defined by the line $k := \{n + tu, t \in \mathbb{R}\}$ then

$$\pi_{k}(m + v) = \pi_{k}(n + (m - n) + v) = (m - n) \wedge u + v \wedge u = (\hat{x} \wedge \hat{u} + \hat{V}^{m} \wedge \hat{u})(m + v),$$

where $x := m - n$, i.e. $\pi_{k} = \hat{x} \wedge \hat{u} + \hat{V}^{m} \wedge \hat{u}$. Let us compute:

$$[\pi_{l}, \pi_{k}] = [\hat{V}^{m} \wedge \hat{w}, \hat{x} \wedge \hat{u} + \hat{V}^{m} \wedge \hat{u}] = [\hat{V}^{m} \wedge \hat{w}, \hat{x} \wedge \hat{u} + \hat{V}^{m} \wedge \hat{u}] =$$

$$= -\hat{w} \wedge [\hat{V}^{m}, \hat{x}] \wedge \hat{u} + \hat{w} \wedge [\hat{V}^{m}, \hat{u}] \wedge \hat{x} + \hat{V}^{m} \wedge \hat{u} \wedge \hat{x} + \hat{V}^{m} \wedge \hat{u}$$

But for any constant vector field $\hat{y}$ we have: $[\hat{V}^{m}, \hat{y}] = -\hat{y}$, therefore:

$$[\pi_{l}, \pi_{k}] = 2\hat{w} \wedge \hat{x} \wedge \hat{u}.$$

In this way $[\pi_{l}, \pi_{l}] = 0$ and $\pi_{l} + \pi_{k}$ is Poisson iff $w \wedge x \wedge u = 0$. Now first and the second statement are clear.

Let the lines $l, k$ be defined by $(m, w)$ and $(n, u)$, respectively; let $\psi_{n} : V \ni v \mapsto n + v \in M$. Using $\phi_{n}$ and $\psi_{n}$ we can transfer problem to the action on $(IG, \hat{\Pi}_{u})$ on $(V, \hat{\pi}_{l})$, where $\hat{\Pi}_{u}$ is defined by (33) and $\pi_{\psi_{n}}(\hat{\pi}_{l}) = \hat{\pi}_{l}$ i.e. $\hat{\pi}_{l}(v) = (x + v) \wedge w$, $x := n - m$. The action is $IG \times V \ni (y, A; v) \mapsto y + Av \in V$

This action is Poisson iff

$$\hat{\pi}_{l}(gv) = \hat{g}\hat{\pi}_{l}(v) + \hat{\Pi}_{u}(g)\hat{v},\ g := (y, A) \in IG,$$

where $\hat{g}$ is (the extension of) the mapping $V \ni v \mapsto gv \in V$ and $\hat{v}$ (the extension of) $IG \ni g \mapsto gv \in V$.

We have:

$$\hat{\pi}_{l}(gv) = \hat{\pi}_{l}(y + Av) = (x + y + Av) \wedge w$$

$$\hat{g}(\hat{\pi}_{l}(v)) = (Ax + Av) \wedge Aw$$

$$\hat{\Pi}_{u}(g)\hat{v} = (b_{u}g - gb_{u})\hat{v} = (b_{u})\hat{g}\hat{v} - \hat{g}(b_{u}\hat{v})$$
It is straightforward, that for \((\dot{x}, \dot{A}) \in T_e IG:\ (\dot{x}, \dot{A}) \hat{z} = \dot{x} + \dot{A}z; \) so
\[
(b_u)\hat{z} = \sum \eta^{jk} e_j \wedge (\Lambda u e_k z) = \sum \eta^{jk} e_j \wedge (\eta(e_k, z) u - \eta(u, z) e_k) = z \wedge u
\]
therefore
\[
(b_u)\hat{g}v = (y + Av) \wedge u , \quad (b_u)\hat{v} = v \wedge u
\]
and
\[
\dot{g}(b_u \hat{v}) = \dot{g}(v \wedge u) = Av \wedge Au
\]
In this way equality (38) reads:
\[
(x + y + Av) \wedge w = (Ax + Av) \wedge Aw + (y + Av) \wedge u - Av \wedge Au , \text{ for any } y, v \in V , \ A \in G
\]
If \(l = k \) i.e. \(x = 0, w = u\) this condition is fulfilled. On the other hand, setting \(v = 0, A = I\) we get (for any \(y\)) \(y \wedge w = y \wedge u\), so \(w = u\) and the equality reduces to
\[
x \wedge w = Ax \wedge Aw \text{ for any } A \in G
\]
Therefore \(x \wedge w = 0\) and \(l = k\).

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