Phase Space Quantum Mechanics on the Anti-De Sitter Spacetime and its Poincaré Contraction

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ABSTRACT

In this work we propose an alternative description of the quantum mechanics of a massive and spinning free particle in anti-de Sitter spacetime, using a phase space rather than a spacetime representation. The regularizing character of the curvature appears clearly in connection with a notion of localization in phase space which is shown to disappear in the zero curvature limit. We show in particular how the anti-de Sitter optimally localized (coherent) states contract to plane waves as the curvature goes to zero. In the first part we give a detailed description of the classical theory à la Souriau. This serves as a basis for the quantum theory which is constructed in the second part using methods of geometric quantization. The invariant positive Kähler polarization that selects the anti-de Sitter quantum elementary system is shown to have as zero curvature limit the Poincaré polarization which is no longer Kähler. This phenomenon is then related to the disappearance of the notion of localization in the zero curvature limit.

& This work is largely based on the PhD thesis of one of the authors (A.M.E.), presented at Université Paris 7, december 1991.
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1. Introduction

It is a well known fact that the Poincaré group, \( \mathcal{P}_+^{\uparrow}(3,1) \), the kinematical group of Minkowski spacetime, can be obtained by means of a contraction from the anti-de Sitter (AdS) group, \( \text{SO}_0(3,2) \), the kinematical group of anti-de Sitter spacetime [BLL] [LN]. The contraction parameter is the constant positive curvature \( \kappa \) of the anti-de Sitter spacetime. This contraction procedure is thus nothing but a zero curvature limit. Accordingly, one would like to approximate \( \mathcal{P}_+^{\uparrow}(3,1) \)-invariant theories by \( \text{SO}_0(3,2) \)-invariant ones, hoping that such approximations give rise to regularized relativistic theories. Indeed, the nonzero curvature equips the AdS theories with a lengthlike parameter, which is the source of the sought for regularizations.

Up to now, this very stimulating idea has not been fully exploited, though it has received a large amount of attention for its potential implications in the context of quantum field theories [BFFS]. We will concretely implement this idea in connection with the problem of localization. It is well known that no satisfactory notion of space or spacetime localization in Poincaré-invariant quantum mechanics exists [H1] [H2] [NW] [W] (see however [DB1]). In the usual formulation of relativistic quantum field theories, only momentum probability densities are associated to the one-particle states. In quantum theories on anti-de Sitter spacetime, no clear notion of localization has so far been developed be it on spacetime, or in momentum space (for an attempt in this direction see [Fr2]). This makes the interpretation of one-particle states very difficult. We show in this work that the AdS quantum theory of massive particles admits a very natural notion of phase space localization. In addition, we identify certain states of the theory as optimally localized and show that they are -in a sense- the analogs of plane waves on flat spacetime. We show in which sense the appearance of the notion of phase space localization is a manifestation of the regularizing character of the curvature.

More precisely, for a free massive and spinning particle in the 4-dimensional AdS spacetime, the phase space is a Kähler \( \text{SO}_0(3,2) \) homogeneous space, whose (geometric) quantization gives rise to a discrete series representation of \( \text{SO}_0(3,2) \). The latter is known to be a square integrable representation, so the modulus of the wave functions of the quantum states in this realization can be actually interpreted as a probability distribution on phase space. Moreover its Hilbert space contains a particular family of quantum states: the Perelomov [Pe] generalized coherent states. They are the above mentioned optimally localized states in phase space. Here we exhibit the explicit form of these coherent states and we show how their physical interpretation arises. We also stress the disappearance
of this notion of localization in the flat space limit, confirming the effectiveness of the regularizing character of \( \kappa \).

As a byproduct, the methods used here allow one to shed some light on the problem of contracting discrete series representations. The case of the principal series representations was extensively studied using different approaches [MN] [PW]. According to Mackey [Ma] the analogy between these kind of representations, for a given (semi)simple noncompact group \( G \), and those of the semi-direct product group obtained through a Inönü-Wigner contraction [IW] of \( G \), greatly simplifies the contraction. The discrete series representations are far from possessing such an analogy, and so their contraction is more difficult to analyze. The geometric quantization methods we use here give an idea of the kind of difficulties one faces when dealing with the contraction of the discrete series representations. The observations made here and in [DBE] allowed Cishahayo and De Biève [CDB] to treat the \( SU(1,1) \rightarrow P^+_{\uparrow}(1,1) \) case in a rigorous mathematical way.

We shall proceed as follows. In section 2 we introduce the AdS spacetime and we discuss some of its properties. In section 3 we give a careful and pedagogical geometric description (à la Souriau [So]) of the classical theory of a mass \( m \neq 0 \) and spin \( s \) test particle in AdS spacetime. This fixes the physical interpretation of the different quantities that will be used throughout this paper. In the zero curvature limit we recover the original Souriau geometric description of a mass \( m \neq 0 \) and spin \( s \) test particle in Minkowski spacetime [So]. Using methods of geometric quantization [Wo], the quantum theory is explicitly constructed in section 4. Section 5 deals with the notion of localization in phase space. In fact the latter is defined exploiting properties of the generalized coherent states of \( SO_0(3,2) \). The explicit form of those states is derived there. We also show that this notion of optimal localization disappears in the zero curvature limit, relating this fact to the loss of the Kähler character of the polarization in this same limit. In section 6 we explore the behaviour of the discrete series representation, explicitly obtained in section 4, when the curvature tends to zero. This provide us with some information concerning the contraction of this type of representations in the large. Finally section 7 concludes our contribution.

The results presented here are more extensively discussed in an unpublished thesis [E1]. The special case of a massive and spinless free particle constitutes a straightforward generalization of the 1 + 1-dimensional case treated in [DBE]. It will not be considered here.
2. The AdS spacetime

The AdS spacetime [DS] of (constant) curvature $\kappa > 0$ can be viewed as the one sheeted hyperboloid in $(\mathbb{R}^5, \eta)$, $\eta = \text{diag}(-, -, +, +, +)$,

$$y \cdot y \equiv \eta_{\alpha\beta} y^\alpha y^\beta = -(y^5)^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -\kappa^{-2},$$  \hspace{1cm} (2.1)

where $\alpha, \beta \in \{5, 0, 1, 2, 3\}$ and $(y^5, y^0, y^1, y^2, y^3) \in \mathbb{R}^5$. In what follows we shall also denote this spacetime by $M_\kappa$. Figure 2.1 below displays the two dimensional version of $M_\kappa$.

![Figure 2.1](image)

*Anti-de Sitter spacetime*

Alternatively, $M_k$ can be realized through global coordinates $(x^0, \vec{x})$. The latter are related to those in (2.1) by the following relations,

$$y^5 = Y \cos \kappa x^0,$$

$$y^0 = Y \sin \kappa x^0,$$

$$\vec{y} = \vec{x},$$  \hspace{1cm} (2.2a)

where $-\pi \leq \kappa x^0 \leq \pi$, $\vec{x} \in \mathbb{R}^3$ and

$$Y = \sqrt{\kappa^{-2} + (\vec{x})^2}.$$  \hspace{1cm} (2.2b)

In this coordinate system the metric on $M_\kappa$ takes the form,

$$ds_\kappa^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= -(\kappa Y)^2 (dx^0)^2 + (\kappa Y)^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (2.3)
Here \((r, \theta, \phi)\) are the usual spherical coordinates in \(\mathbb{R}^3\). Obviously \(x^0\) is the time coordinate, \(\kappa x^0\) being the rotation angle in the \((y^5, y^0)\) plan of (2.1) and \(\vec{x}\) or \(\vec{y}\) are the usual 3-space coordinates.

Let us mention that the known problems inherent to the intrinsic geometry of \(M_\kappa\), namely the compactness of time and the absence of global hyperboliticity, will not be addressed here. In fact, the first problem can be avoided by considering the AdS spacetime to be the universal covering of \(M_\kappa\) [Fr2] [HE]. A careful study of the second can be found in [AIS], where suggestions for its resolution have also been formulated (see also [Fr2] and [CB]).

The isometry group of the AdS spacetime appears clearly from (2.1) to be the non-compact \(O(3, 2)\) group. The connected component to the identity of the latter, denoted \(SO_0(3, 2)\), is called the AdS group [HE]. Together with the \(SO_0(4, 1)\)-de Sitter spacetime, \(M_\kappa\) is the only non-trivial maximally symmetric solution of Einstein equations (with non zero cosmological constant).

Since in this work we are interested in carrying out a zero curvature limit, it is worth noting that this procedure will actually produce physically relevant quantities provided it is performed using a meaningful parametrization. For example, in the case of \(M_\kappa\), one can see that contrary to the \(y\)-coordinates in (2.1), the \(x\)-coordinates in (2.2) possess a straightforward interpretation for any value of \(\kappa\) and even when \(\kappa \to 0\). In fact, a simple \(\kappa \to 0\) limit in (2.3) shows that \(ds_k^2\) becomes the Minkowski flat metric. Hence one can still interpret \((x^0, \vec{x})\) as the spacetime coordinates. In other words the \(y\) and the \(x\)-coordinates are complementary for the purpose of the present work. The former put in perspective the symmetry of the system, their transformation under the action of \(SO_0(3, 2)\) being obvious. The latter provide the bridge towards a meaningful zero curvature limit, yielding the expected physical quantities.

At the algebraic level, both the Poincaré and AdS Lie algebras have an underlying ten dimensional vector space. The Poincaré Lie algebra can be obtained as the limit of a one parameter-dependent sequence of isomorphic AdS Lie algebras. According to İnönü and Wigner [IW] this singular process is called a contraction. The parameter used in this limiting process can easily be seen to be the curvatutre \(\kappa\) of the AdS spacetime. So this contraction is nothing but the zero curvature limit mentionned above [BLL].

In order to fix the notations, let us explicitly perform the AdS \(\rightarrow\) Poincaré contraction. Let \(V \equiv \mathbb{R}^{10}\) be the vector space underlying the two Lie algebra structures \(p(3, 1)\) and \(so(3, 2)\) \((\dim p(3, 1) = \dim so(3, 2) = 10)\). Let \(\{e_{\alpha\beta}, \alpha, \beta \in \{5, 0, 1, 2, 3\}\}\) be a basis of \(V\)
such that \( so(3, 2) \) is realized in the following way,

\[
[e_{\alpha \beta}, e_{\gamma \rho}] = \eta_{\alpha \gamma} e_{\beta \rho} + \eta_{\beta \rho} e_{\alpha \gamma} - \eta_{\alpha \rho} e_{\beta \gamma} - \eta_{\beta \gamma} e_{\alpha \rho}.
\] (2.4)

Notice that \( e_{\mu \nu}, \mu, \nu \in \{0, 1, 2, 3\} \) realize the Lorentz subalgebra \( so(3, 1) \subset so(3, 2) \). Let \( \phi_\kappa \in GL(V) \) be the contraction map [Do] defined by,

\[
\phi_\kappa : V \ni e_{\alpha \beta} \mapsto \phi_\kappa(e_{\alpha \beta}) \equiv e^\kappa_{\alpha \beta},
\] (2.5a)

where

\[
e^\kappa_{5 \mu} = \kappa e_{5 \mu},
\] (2.5b)

and

\[
e^\kappa_{\mu \nu} = e_{\mu \nu},
\] (2.5c)

for \( \mu, \nu \in \{0, 1, 2, 3\} \). As long as \( \phi_\kappa \) is non-singular, i.e. \( \kappa \neq 0 \), one can define a new Lie algebra structure \([,]_\kappa\) in \( V \), which is isomorphic to the original one. Specifically,

\[
[e_{\alpha \beta}, e_{\gamma \rho}]_\kappa = \phi_\kappa^{-1} [\phi_\kappa(e_{\alpha \beta}), \phi_\kappa(e_{\gamma \rho})].
\] (2.6)

However when \( \kappa \) reaches 0 one obtains a new Lie algebra structure which is no longer isomorphic to \( so(3, 2) \). This is the Poincaré Lie algebra \( p(3, 1) \). Indeed by taking the \( \kappa \to 0 \) limit in (2.6) we get,

\[
[e_{\alpha \beta}, e_{\gamma \rho}]_0 = \lim_{\kappa \to 0} \phi_\kappa^{-1} [\phi_\kappa(e_{\alpha \beta}), \phi_\kappa(e_{\gamma \rho})],
\] (2.7a)

which results in the following explicit commutation relations of \( p(3, 1) \):

\[
[e_{\mu \nu}, e_{\delta \lambda}]_0 = \eta_{\mu \delta} e_{\nu \lambda} + \eta_{\nu \lambda} e_{\mu \delta} - \eta_{\mu \lambda} e_{\nu \delta} - \eta_{\nu \delta} e_{\mu \lambda},
\]

\[
[e_{\mu \nu}, e_{5 \lambda}]_0 = \eta_{\mu \lambda} e_{5 \nu} - \eta_{\nu \lambda} e_{5 \mu},
\] (2.7b)

\[
[e_{5 \mu}, e_{5 \nu}]_0 = 0.
\]

Here \( \mu, \nu, \delta, \lambda \in \{0, 1, 2, 3\} \). The subalgebra \( so(3, 1) \) is clearly preserved in this contraction, and \( SO_0(3, 2) \) is then said to be contracted to \( \mathcal{P}^+_1(3, 1) \) along the Lorentz subgroup \( SO_0(3, 1) \). For more details concerning the procedure of contraction we refer to [IW] [Sa] [Do] [LN] [Gi] (for more recent contributions see [CPSW] and references quoted therein.)
3. The classical theory

As mentioned in the introduction, the theory we are about to develop here describes the free evolution of a mass $m$ and spin $s$ particle on an AdS spacetime of given curvature $\kappa > 0$. Both the classical and the quantum aspects of this theory will be investigated. Here we concentrate on the classical dynamics in order to fix both the notations and the physical interpretation of the quantities that will appear throughout this work. The quantum theory and the associated notion of optimal localization will be discussed in the next sections.

As in [DBE] we shall use here the Souriau scheme [So] (see also [DB2]), which provides the classical dynamics, its phase space and its symmetries in a unified and an efficient way. This scheme can be summarized in the following diagram:

Figure 3.1
Souriau's scheme

The phase space $\Sigma$ of the model is obtained by a symplectic reduction of a presymplectic manifold $E$. Hence, the degenerate closed two-form $\omega_E$ equipping $E$ is the pull-back of the non-degenerate closed two-form $\omega_\Sigma$ equipping $\Sigma$, i.e. $\omega_E = \pi^* \omega_\Sigma$. In other words the projection $\pi$ kills the kernel of $\omega_E$. The presymplectic manifold $E$ is chosen in such a way that the projection $\rho$ on $M$ of the leaves of the foliation generated by the distribution $\ker \omega_E$ gives rise to the dynamics of the theory. More precisely, in the case of a free massive particle on $M$ this should produce the time-like geodesics of $M$. Unfortunately there exists no general theory prescribing the choice of the presymplectic manifold $E$. However the symmetries of the model, if any, provide a precious guide for such a determination. Let us moreover anticipate by indicating that even if the above scheme concerns just the classical theory, it highly simplifies the quantization procedure. This will be shown in section 4.

In what follows we shall sometimes use Souriau's terminology. The presymplectic manifold and the phase space will then be called the evolution space and the space of motions, respectively.
3.1. Souriau’s scheme

For the present case the evolution space, denoted $E_{\kappa}^{m,s}$, will be taken as a principal homogeneous space [LM] of $SO_0(3,2)$, i.e. $E_{\kappa}^{m,s} \cong SO_0(3,2)$. It can be concretely and conveniently realized as a subspace of the cartesian product of five copies of $(\mathbb{R}^5, \eta)$. Actually, let $(y, q, u, v, t)$ be five five-vectors of $(\mathbb{R}^5, \eta)$, then $E_{\kappa}^{m,s}$ is defined as the set of points $w = (y, q, u, v, t) \in \mathbb{R}^{25}$ satisfying the following $SO_0(3,2)$-invariant constraints:

\begin{align}
    y \cdot y &= -\kappa^{-2}, \\
    q \cdot q &= -m^2, \\
    u \cdot u &= 1, \\
    v \cdot v &= 1, \\
    t \cdot t &= m^2 s^2, \\
    y \cdot q &= y \cdot u = y \cdot v = y \cdot t = q \cdot u = q \cdot v = q \cdot t = u \cdot v = u \cdot t = v \cdot t = 0, \\
\end{align}

(3.1a) \quad (3.1b) \quad (3.1c) \quad (3.1d) \quad (3.1e) \quad (3.1f)

\begin{align}
    \epsilon_{\alpha\beta\gamma\rho\sigma} y^\alpha q^\beta u^\gamma v^\rho t^\sigma &= \frac{m^2 s}{\kappa}, \\
\end{align}

(3.1g)

and

\begin{align}
    y^5 q^0 - y^0 q^5 &> 0. \\
\end{align}

(3.1h)

In these equations $m$, $s$ and $\kappa$ are the three original physical ingredients and $\epsilon_{\alpha\beta\gamma\rho\sigma}$ is the completely skew-symmetric tensor associated to $(\mathbb{R}^5, \eta)$, such that $\epsilon_{50123} = \epsilon_{50123} = 1$. The indices of the vectors $(y, q, u, v, t)$ are raised and lowered by $\eta_{\alpha\beta}$ (2.1). Note also that $q$, $u$, $v$ and $t$ will be considered either as points of the tangent or the cotangent space to $M_\kappa (2.1)$.

The physical interpretation of the above constraints is now displayed in some detail:

- (3.1a) defines the AdS spacetime points $y \in M_\kappa (2.1)$;
- (3.1b) from $y \cdot q = 0$ in (3.1f) $q$ appears as the conjugate linear momentum of the position $y$, through (3.1b) it is constrained to the AdS mass shell associated to $m$;
- (3.1e) $t$ is what we call the $AdS$-Pauli-Lubanski five-vector; (3.1e) is the AdS analog of the Pauli-Lubanski constraint appearing in the case of the Poincaré-invariant theory [So], and this is the way the spin enters in our approach; note that $t$ is spacelike, belongs to $T_y M_\kappa$ and is perpendicular to the direction of motion (3.1f);
• (3.1c,d) these two constraints allow a covariant treatment of spin and they will play an important role at the quantum level; we shall see that \( t \) and \( (u,v) \) are equivalent;

• (3.1g,h) these constraints specify a choice of orientation and select one of the four connected components of the manifold defined just by the previous constraints (3.1a-f); in particular (3.1h) selects one of the two mass shells satisfying (3.1b), in the zero curvature limit this corresponds to the positive energy condition.

Since one of our goals is to perform the zero curvature limit, we will introduce, later on in section 3.2, a new coordinate system that will make such a procedure both possible and meaningful. This is related to the remarks we made in section 2 concerning the complementarity of the \( y \) and the \( x \)-coordinates. The preceding constraints will then be rewritten in that new coordinate system and their zero curvature limits will be evaluated, the results we will obtain will confirm the physical interpretation we gave above. At this point it is worth noting that \( E^m_s \) can be viewed as the Lorentz bundle over the AdS spacetime. This makes clear the connection with K¨unzle’s 1972 work [Ku]. There the evolution space for a free massive and spinning particle on a general spacetime is taken as the Lorentz bundle over that spacetime.

Simple arguments of linear algebra allow one to make the identification \( E^m_s \cong SO_0(3,2). \) Concretely, to each \( w \in E^m_s \) we can associate in a 1-1 manner an \( SO_0(3,2) \) element. We must first fix the point \( w \) that is associated to the identity element of \( SO_0(3,2). \) We choose \( w \) as follows:

\[
y^\alpha_{(0)} = \kappa^{-1} \eta^{\alpha_5}, \quad q^\alpha_{(0)} = m \eta^{\alpha_0}, \quad u^\alpha_{(0)} = \eta^{\alpha_1}, \quad v^\alpha_{(0)} = \eta^{\alpha_2}, \quad t^\alpha_{(0)} = ms \eta^{\alpha_3}. \tag{3.2}
\]

From here one can identify the \( SO_0(3,2) \) element \( \Lambda(w) \) associated to a general point \( w \) of \( E^m_s \). In fact \( \Lambda(w) \) is the group element relating \( w \) to \( w \) when \( SO_0(3,2) \) acts on \( E^m_s \) on the left. It is given by,

\[
[\Lambda(w)]^\mu_\nu = -\kappa^2 y^\mu y_{(0)\nu} - \frac{1}{m^2} q^\mu q_{(0)\nu} + u^\mu u_{(0)\nu} + v^\mu v_{(0)\nu} + \frac{1}{m^2 s^2} t^\mu t_{(0)\nu}. \tag{3.3}
\]

The Lie algebra \( so(3,2) \) in (2.4) can now be realized in terms of (left) invariant vector fields on \( (SO_0(3,2) \equiv) E^m_s \). Their expressions are obtained from the formula below,

\[
Y_{\alpha\beta}(w) = \frac{d}{d\tau} \left[ [\Lambda(w)]^\mu_\nu (\exp \tau \epsilon_{\alpha\beta})^\nu^\rho y_{(0)}^\rho \right]_{\tau=0} \frac{\partial}{\partial y^\mu} + \frac{d}{d\tau} \left[ [\Lambda(w)]^\mu_\nu (\exp \tau \epsilon_{\alpha\beta})^\nu^\rho q_{(0)}^\rho \right]_{\tau=0} \frac{\partial}{\partial q^\mu} + \frac{d}{d\tau} \left[ [\Lambda(w)]^\mu_\nu (\exp \tau \epsilon_{\alpha\beta})^\nu^\rho u_{(0)}^\rho \right]_{\tau=0} \frac{\partial}{\partial u^\mu} + \frac{d}{d\tau} \left[ [\Lambda(w)]^\mu_\nu (\exp \tau \epsilon_{\alpha\beta})^\nu^\rho v_{(0)}^\rho \right]_{\tau=0} \frac{\partial}{\partial v^\mu} + \frac{d}{d\tau} \left[ [\Lambda(w)]^\mu_\nu (\exp \tau \epsilon_{\alpha\beta})^\nu^\rho t_{(0)}^\rho \right]_{\tau=0} \frac{\partial}{\partial t^\mu}. \tag{3.4}
\]
Using (3.3), (3.2) and the equation \[ \left[ \frac{d}{d\tau} (\exp \tau e_{\alpha \beta}) \right]_{\tau=0}^\nu = \eta_{\alpha \rho} \eta^\nu_{\beta} - \eta_{\beta \rho} \eta^\nu_{\alpha} \] we get the $Y_{\alpha \beta}$’s explicitly as follows,

\[ Y_{50} = \kappa y \cdot \frac{\partial}{\partial q} - \frac{1}{m \kappa} q \cdot \frac{\partial}{\partial y}, \quad Y_{12} = v \cdot \frac{\partial}{\partial u} - u \cdot \frac{\partial}{\partial v}, \] (3.5a)

\[ Y_{23} = \frac{1}{m s} t \cdot \frac{\partial}{\partial v} - ms v \cdot \frac{\partial}{\partial t}, \quad Y_{31} = ms u \cdot \frac{\partial}{\partial t} - \frac{1}{ms} t \cdot \frac{\partial}{\partial u}, \] (3.5b)

\[ Y_{15} = \frac{1}{\kappa} u \cdot \frac{\partial}{\partial y} + \kappa y \cdot \frac{\partial}{\partial u}, \quad Y_{25} = \frac{1}{\kappa} v \cdot \frac{\partial}{\partial y} + \kappa y \cdot \frac{\partial}{\partial v}, \quad Y_{35} = \frac{1}{ms \kappa} t \cdot \frac{\partial}{\partial y} + ms \kappa y \cdot \frac{\partial}{\partial t}, \] (3.5c)

\[ Y_{01} = -m u \cdot \frac{\partial}{\partial q} - \frac{1}{m} q \cdot \frac{\partial}{\partial u}, \quad Y_{02} = -m v \cdot \frac{\partial}{\partial q} - \frac{1}{m} q \cdot \frac{\partial}{\partial v}, \quad Y_{03} = -\frac{1}{s} t \cdot \frac{\partial}{\partial q} - s q \cdot \frac{\partial}{\partial t}. \] (3.5d)

At each point $w \in E^m_{\kappa}$, these vector fields are linearly independent, they form at this point a basis of $T_w E^m_{\kappa}$.

The identification $E^m_{\kappa} \cong SO_0(3, 2)$ just realized plays a crucial role in the present construction. In particular it allows us to choose an invariant presymplectic form $\omega_E$ in an easy way. In order to show this, let us first introduce the dual basis to \{\(Y_{\alpha \beta}, \alpha, \beta \in \{5, 0, 1, 2, 3\}\)\} denoted \{\(\theta^{\alpha \beta}, \alpha, \beta \in \{5, 0, 1, 2, 3\}\)\}. The $\theta^{\alpha \beta}$’s are (left) invariant one-forms on $E^m_{\kappa}$ and can be viewed as the basis elements of $so^*(3, 2)$. Then $\omega_E$ can be chosen as the exterior derivative of an invariant one-form $\theta_E$, which is some linear combination of the $\theta^{\alpha \beta}$’s with constant coefficients. The two-form $\omega_E \equiv -d\theta_E$ will then be closed and invariant as wanted. But still one needs to make a choice among all possible linear combinations. The only requirement really constraining this choice is that $\ker \omega_E$ should produce the right dynamics on $M_\kappa$. We will choose here $\theta_E$ on the basis of some physical arguments, and we will confirm the validity of this choice subsequently by the evaluation of the dynamics.

We will use here dimensionality and kinematic arguments. First of all, $\theta_E$ must have the dimension of an action. Its expression must contain the three original physical ingredients, namely the mass $m$, the spin $s$ and the curvature $\kappa$. Since the $\theta^{\alpha \beta}$’s are dimensionless, the dimensionality of $\theta_E$ can only arise from $m$, $s$ and $\kappa$. One can easily check that only $m \kappa$ and $s$ have the dimension of an action (in $\hbar = c = 1$ units). On the other hand, it is well known that the mass and the spin are kinematically related to the spacetime translations and space rotations, respectively. Since we are dealing here with the $SO_0(3, 2)$ kinematics, the spacetime translations are pseudo-rotations, with dimensionless parameters. The latter
acquire actual lengthlike dimension when multiplied by $\kappa^{-1}$, the unit of length [BEGG]. All these arguments put together, suggest the following choice for $\theta_E$,

$$\theta_E \equiv \frac{m}{\kappa} \theta^{50} + s \theta^{12}. \quad (3.6)$$

At this point we must mention that even if this choice gives the right dynamics, as we will soon show, it still carries some arbitrariness. The latter is inherent to classical theory and can be compared to the one appearing in the Lagrangian formalism when one tries to fix a Lagrangian. What is important in that formalism is that the equations of motions describing the dynamics of the system under study are obtained correctly from the chosen Lagrangian via the variational principle.

In order to write (3.6) in a concrete form, let us first find the $\theta^{\alpha\beta}$'s in terms of the coordinates on $E^m_{\kappa,s}$. This is done using (3.5) and the duality relation,

$$\gamma_{\alpha\beta} \theta^{\alpha\beta} = \theta^{\alpha\beta}(\gamma_{\alpha\beta}) = \eta^{\alpha}_{\gamma} \eta^{\beta}_{\rho} - \eta^{\alpha}_{\rho} \eta^{\beta}_{\gamma}, \quad \forall \alpha, \beta, \gamma, \rho \in \{5, 0, 1, 2, 3\}. \quad (3.7)$$

We display here only the results that will be needed subsequently,

$$\theta^{50} = \frac{\kappa}{m} q \cdot dy = -\frac{\kappa}{m} y \cdot dq \quad \text{and} \quad \theta^{12} = v \cdot du = -u \cdot dv; \quad (3.8)$$

note that (3.1f) is at the origin of the second equalities in (3.8). Finally, we obtain for (3.6)

$$\theta_E = q \cdot dy + s v \cdot du. \quad (3.9)$$

The expression of $\theta_E$ in (3.6) has a group theoretic character. Its translation in geometric terms given in (3.9) is based, once again, on the identification $E^m_{\kappa,s} \cong SO_0(3, 2)$, and it allows us to view it as the restriction to the case of a constant curvature manifold of the one-form used in [Ku].

Let us now give the expression of the invariant presymplectic form $\omega_E$,

$$\omega_E \equiv -d\theta_E = dy \wedge dq + s du \wedge dv; \quad (3.10)$$

the convention used here is $dy\wedge dq \equiv dy^\alpha \wedge dq_\alpha$, where Einstein’s summation rule is assumed. The canonical character of (3.10) confirms the physical interpretation we assigned to the coordinates $(y, q, u, v, t) \in E^m_{\kappa,s}$. Another useful formula for $\omega_E$ can be obtained starting with (3.6) and using the Maurer-Cartan equations of $so^*(3, 2)$. Hence,

$$\omega_E = \frac{m}{\kappa} (\theta^{01} \wedge \theta^{15} + \theta^{02} \wedge \theta^{25} + \theta^{03} \wedge \theta^{35}) - s (\theta^{15} \wedge \theta^{25} + \theta^{01} \wedge \theta^{02} + \theta^{31} \wedge \theta^{23}). \quad (3.11)$$
In order to complete the description of Souriau’s scheme, we come now to the evaluation of the kernel of $\omega_E$. The latter is defined as follows,

$$\ker \omega_E = \bigcup_{w \in E^{m,s}_\kappa} \ker_w \omega_E$$

(3.12a)

where

$$\ker_w \omega_E = \{ Y \in T_w E^{m,s}_\kappa \mid \omega_E(w)(Y,Y') = 0 \quad \forall \ Y' \in T_w E^{m,s}_\kappa \}.$$  

(3.12b)

Evaluating $\ker \omega_E$ is equivalent to solve the equation,

$$K_E \omega_E = 0,$$

(3.13)

for $K_E$ a $\ker \omega_E$-valued vector field on $E^{m,s}_\kappa$, i.e. $K_E(w) \in \ker_w \forall w \in E^{m,s}_\kappa$. The symbol “$\|$” denotes the interior product. Since the $Y_{\alpha\beta}$’s form a basis of $T_w E^{m,s}_\kappa$ at each point $w \in E^{m,s}_\kappa$, $K_E$ can be expanded in the following way,

$$K_E = \frac{1}{2} f^{\alpha\beta} Y_{\alpha\beta},$$

(3.14)

with $f^{\alpha\beta} \in C^\infty(E^{m,s}_\kappa)$. Using (3.14) and (3.11) it is easy to solve (3.13). A very interesting situation appears. In fact two cases must be considered. They are summarized as follows:

(i) $m_\kappa = s \implies \dim \ker \omega_E = 4$, $\ker \omega_E$ is spanned by four linearly independent vector fields on $E^{m,s}_\kappa$,

(ii) $m_\kappa \neq s \implies \dim \ker \omega_E = 2$, $\ker \omega_E$ is spanned by two linearly independent vector fields on $E^{m,s}_\kappa$.

We consider in this work only the second case, the first one deserves a separate treatment and it will be discussed elsewhere [E2]. However let us say a few words about this phenomenon. For the case (i) the symplectic reduction $\pi : E^{m,s}_\kappa \longrightarrow E^{m,s}_\kappa / \ker \omega_E$ will yield a six-dimensional phase space. In the case of a $P^+_1(3,1)$-invariant free theory, six-dimensional phase spaces describe either massive spinless free particles or massless spinning ones [Ar] [So]. Since the Lorentz subgroup of $SO_0(3,2)$ is preserved when contracting, the spin part of any physical $SO_0(3,2)$-invariant theory will also be preserved in the zero curvature limit. So the second possibility above, namely mass = 0 and spin $\neq 0$, is the only one that could arise from a zero curvature limit starting from the six-dimensional $SO_0(3,2)$-phase space corresponding to the case $m_\kappa = s$. Thus, in a sense to be defined, the case (i)
corresponds to an $SO_0(3,2)$-invariant free massless elementary system. For more details we refer to [E2].

The symplectic reduction in case (ii), which is our main concern here, gives rise to an eight-dimensional phase space. For a $\mathcal{P}_\pm^\uparrow(3,1)$-invariant free theory, only a massive spinning free particle on Minkowski spacetime is described by an eight-dimensional phase space. So, as we will show subsequently, the latter is the zero curvature limit of the former. Or the other way around, the former is the AdS-deformation of the latter. This suggests calling the $m_\kappa \neq s$ case, the massive case.

When $m_\kappa \neq s$, one can show that $\ker \omega_E$ is spanned by the two vector fields $Y_{50}$ and $Y_{12}$ given in (3.5a). Hence, $K_E$ is of the form,

$$K_E = f^{50}Y_{50} + f^{12}Y_{12}, \quad (3.15)$$

for $f^{50}$ and $f^{12}$ arbitrary elements of $C^\infty(E^{m,s}_E)$. Recall now that $Y_{50}$ and $Y_{12}$ together, as $so(3,2)$ basis elements, generate the subgroup $SO(2) \times SO(2)$. As vector fields on $E^{m,s}_E$ they have closed integral curves. The integral manifold or the leaf of $\ker \omega_E$ through each point $w \in E^{m,s}_E$ is then a torus $T = S^1 \times S^1$. The symplectic reduction allows then to identify the phase space of our physical system as $\Sigma^{m,s}_E \equiv E^{m,s}_E/S^1 \times S^1 \cong SO_0(3,2)/SO(2) \times SO(2)$. Hence, $\Sigma^{m,s}_E$ is a homogeneous space of $SO_0(3,2)$. Moreover, as we will show in section 4, $\Sigma^{m,s}_E$ is a Kähler homogeneous space of $SO_0(3,2)$. The symplectic reduction stressed here can be explicitly carried out, namely $\Sigma^{m,s}_E$ can be given coordinates with clear physical interpretation. Formulas with this respect will be displayed later on.

In order to evaluate the dynamics of our system let us now make a short detour and complete Souriau’s scheme. To this end we must first integrate $\ker \omega_E$ and then project its leaves on $M_\kappa$. Just by looking at the form of $Y_{50}$ and $Y_{12}$ in (3.5a) one can easily see that only the integral curves of $Y_{50}$ project non-trivially on $M_\kappa$, giving birth to worldlines. The equations of the latter result from the integration of the flows of $Y_{50}$. Concretely, this gives the following solution,

$$\begin{align*}
y^\alpha(\tau) &= y^\alpha(0) \cos \tau - i \frac{1}{m_\kappa} q^\alpha(0) \sin \tau, \\
q^\alpha(\tau) &= m_\kappa y^\alpha(0) \sin \tau + q^\alpha(0) \cos \tau
\end{align*} \quad \alpha \in \{5, 0, 1, 2, 3\}, \quad (3.16)$$

$y^\alpha(0)$ and $q^\alpha(0)$ are the initial conditions and $\tau \in \mathbb{R}$. It is easy to see that for each such initial condition, $y^\alpha(\tau)$ traces out a timelike geodesic on $M_\kappa$. Hence, our choice for $\theta_E$ in (3.6) generates the expected dynamics of our free AdS system. Finally, note that
the position and the spin degrees of freedom behave in an independent way. This is a consequence of the fact that $Y_{0}\circ Y_{12}$ commute.

Let us now come back to the symplectic reduction sketched above. Concretely, we define the phase space $\Sigma_{m,s}^{k,s}$ in the following way,

$$\Sigma_{m,s}^{k,s} = \left\{ (y, q, t) \in F_{m,s}^{k,s} | y^0 = 0 \text{ and } y^5 > 0 \right\}.$$  \hfill (3.17)

Here $F_{m,s}^{k,s}$ is a 9-dimensional manifold obtained through the partial reduction of $E_{m,s}^{k,s}$:

$$\pi_1 : E_{m,s}^{k,s} \ni (y, q, u, v, t) \longmapsto (y, q, t) \in F_{m,s}^{k,s}.$$  \hfill (3.18)

Clearly $F_{m,s}^{k,s}$ is a submanifold of the cartesian product of three copies of $(\mathbb{R}^5, \eta)$, $F_{m,s}^{k,s} \subset \mathbb{R}^{15}$. This manifold is presymplectic, its presymplectic form $\omega_F$ is obtained from the pull-back:

$$\omega_F = \pi_1^* \omega_\Sigma.$$  \hfill (3.19)

More explicitly,

$$\omega_F = dy \wedge dq - \frac{\kappa}{2m^2} \epsilon_{\alpha \beta \gamma \rho \sigma} y^\alpha q^\beta t^\gamma \left[ \kappa^2 dy^\rho \wedge dq^\sigma + \frac{1}{m^2} dq^\rho \wedge dq^\sigma - \frac{1}{m^2 m} dt^\rho \wedge dt^\sigma \right].$$  \hfill (3.20)

Moreover $F_{m,s}^{k,s}$ is a homogeneous space of $SO_0(3,2)$, namely

$$F_{m,s}^{k,s} \cong SO_0(3,2)/SO(2).$$  \hfill (3.21)

Hence, the reduction $\pi_1 : E_{m,s}^{k,s} \rightarrow F_{m,s}^{k,s}$ partially kills $\ker \omega_F$. It kills the spin part represented by the flows of the vector field $Y_{12}$. The symplectic reduction process is completed by killing $\ker \omega_F$. This is actually done through choosing a section of the bundle $\pi_2 : F_{m,s}^{k,s} \rightarrow \Sigma_{m,s}^{k,s} \equiv F_{m,s}^{k,s} / \ker \omega_F$. The same choice as in [DBE] is made here (see (3.17)). It’s a simple one from the computational point of view. Each leaf of $\ker \omega_F$, when projected on $M_\kappa$, gives rise to a timelike geodesic of $M_\kappa$ (as previously described for $\ker \omega_\Sigma$). This leaf can be uniquely represented by the inverse image of the point of the geodesic that intersects the half plane $y^0 = 0, y^5 > 0$. Note that the timelike geodesics of $M_\kappa$ are all closed curves [Wi2]. Physically this choice of section corresponds to choosing initial conditions at time zero ($y^0 = 0 \Leftrightarrow x^0 = 0$ see (2.2a)). The symplectic form $\omega_\Sigma$ is such that $\omega_F = \pi_2^* \omega_\Sigma$ or equivalently $\omega_F = \pi^* \omega_\Sigma$, where $\pi = \pi_2 \circ \pi_1$. More precisely,

$$\omega_\Sigma = \omega_F |_{y^0=0}.$$  \hfill (3.22)
At this point a precision is in order. One could argue that we should have started the construction by considering $F_{\kappa}^{m,s}$, instead of $E_{\kappa}^{m,s}$, as the evolution space. One can easily see from (3.20) that this is not an easy task. In fact $\omega_{\kappa}$ can not be easily guessed, whereas constructing $\omega_{\kappa}$ was relatively straightforward.

We end this subsection by summarizing Souriau’s scheme in a diagram.

**Figure 3.2**

*Souriau’s scheme for the AdS massive spinning free particle*

### 3.2. Contraction adapted coordinates

In this subsection we introduce new coordinates on $E_{\kappa}^{m,s}$. They are needed in order to concretely carry out the zero curvature limit. Up to now we used coordinates based on the $y$-coordinatization of $M_\kappa$, the ones we introduce here are based on the $x$-coordinatization of $M_\kappa$. Thus, in the same way as the position $x$ replaces the position $y$ (see (2.2)), the linear momentum $p$, the spin $s$ and the $a$ and $b$ four-vectors will replace $q$, $t$, $u$ and $v$, respectively. Subsequently we shall call the old and the new coordinates on $E_{\kappa}^{m,s}$, the $y$ and the $x$-coordinates, respectively. One obtains relations linking old and new coordinates, as in (2.2), through equations of the type:

$$q \cdot dy \equiv p_\mu \cdot dx^\mu = g_{\mu\nu}(x)p^\mu dx^\nu,$$

and their analogs for the other pairs of coordinates. In (3.23) $g$ is the metric on $M_\kappa$ given in (2.3). We will here only display the solutions of (3.23) for the $(q,p)$ pair, the others arise through obvious modifications. We obtain,

$$p_0 = \kappa \left(y^5q_0 - y^0q_5\right),$$

$$p_i = q_i + \left(\vec{y} \cdot \vec{q}\right) \frac{y^i}{Y^2}, \quad i \in \{1, 2, 3\};$$
here $\vec{y} \cdot \vec{q} = y^i q_i = \Sigma_{i=1}^3 y^i q_i$. When inverting (3.24a-b) one gets,

$$q_5 = -\kappa (\vec{x} \cdot \vec{p}) (\kappa y^5) - \frac{p_0}{(\kappa Y)^2} (\kappa y^0), \quad (3.25a)$$

$$q_0 = -\kappa (\vec{x} \cdot \vec{p}) (\kappa y^0) + \frac{p_0}{(\kappa Y)^2} (\kappa y^5), \quad (3.25b)$$

$$q_i = p_i + \kappa^2 (\vec{x} \cdot \vec{p}) x^i, \quad i \in \{1, 2, 3\}; \quad (3.25c)$$

here $\vec{x} \cdot \vec{p} = x^i p_i$. Note also the interesting relation $\vec{y} \cdot \vec{q} = (\kappa Y)^2 \vec{x} \cdot \vec{p}$.

From now on we will reexpress the most important equations derived in the previous subsection in terms of the $x$-coordinates and then we will investigate their zero curvature limit. We start with the constraints defining $E_{\kappa}^{m,s}$ in (3.1a-h). We present the results in the following form,

$$y \cdot y = -\kappa^{-2} \quad \longrightarrow \quad x \in M_{\kappa} \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad x \in M_0 \quad \text{Minkowski spacetime}$$

$$q \cdot q = -m^2 \quad \longrightarrow \quad g_{\mu\nu}(x) p^\mu p^\nu = -m^2 \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad p_\mu p^\mu = -m^2 \quad \text{Poincaré-mass shell}$$

$$u \cdot u = 1 \quad \longrightarrow \quad g_{\mu\nu}(x) a^\mu a^\nu = 1 \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad a_\mu a^\mu = 1$$

$$v \cdot v = 1 \quad \longrightarrow \quad g_{\mu\nu}(x) b^\mu b^\nu = 1 \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad b_\mu b^\mu = 1$$

$$t \cdot t = m^2 s^2 \quad \longrightarrow \quad g_{\mu\nu}(x) s^\mu s^\nu = m^2 s^2 \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad s_\mu s^\mu = m^2 s^2 \quad \text{Pauli-Lubanski cdt.}$$

$$q \cdot t = 0 \quad \longrightarrow \quad g_{\mu\nu}(x) p^\mu s^\nu = 0 \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad p_\mu s^\mu = 0 \quad \text{orthogonality cdts.}$$

$$y^5 q^0 - y^0 q^5 > 0 \quad \longrightarrow \quad p^0 > 0 \quad (3.24a) \quad \overset{\kappa \rightarrow 0}{\longrightarrow} \quad p^0 > 0 \quad \text{positive energy.}$$

Here the third column concerns the Poincaré-invariant theory. The scalar product appearing there is the one associated to the Minkowski flat metric, which is the zero curvature of $g$ in (2.3). In the next to last row we displayed only one example of the pseudo-orthogonality relations. The constraint (3.1g) becomes, when $\kappa \rightarrow 0$ , $\epsilon_{\mu\nu\lambda\delta} p^\mu a^\nu b^\lambda s^\delta = m^2 s$. Here $\epsilon_{\mu\nu\lambda\delta}$ is the completely skew-symmetric on the Minkowski spacetime. This equation is also valid on $E_{\kappa}^{m,s}$ with its new coordinates, $\epsilon_{\mu\nu\lambda\delta}$ will be then the completely skew-symmetric tensor on $M_\kappa$ (det $g = -1$).

We show now how we can recover in the zero curvature limit the evolution space used by Souriau [So] for the case of a mass $m$ and spin $s$ free particle on $M_\kappa$. Actually, the latter appears as the $\kappa \rightarrow 0$ limit of $F_{\kappa}^{m,s}$ (3.18). In fact, translating the constraints defining $F_{\kappa}^{m,s}$, in the $x$-coordinates and using the limits displayed above one can easily see that
they become in the $\kappa \to 0$ limit those used by Souriau [So]. In order to apply the same procedure to $\omega_{\nu}$, we first rewrite it in the $x$-coordinates,

$$\omega_{\nu} = dx \wedge dp - \frac{1}{2m^2} \epsilon_{\mu \nu \rho \sigma} p^\mu s^\nu \left[ \kappa^2 dx^\rho \wedge dx^\sigma + \frac{1}{m^2} Dp^\rho \wedge Dp^\sigma - \frac{1}{m^2 s^2} Ds^\rho \wedge Ds^\sigma \right]. \quad (3.26)$$

Here $\epsilon_{\mu \nu \rho \sigma}$ is the completely skew-symmetric tensor on $M_\kappa$, $dx \wedge dp = dx^\mu \wedge dp^\mu$ with $p^\mu = g^{\mu \nu} p^\nu$ and $Dp$ is the covariant differential of $p$, i.e.

$$Dp_\mu = dp_\mu - \Gamma^\rho_{\mu \sigma} p_\rho dx^\sigma, \quad \mu, \nu, \rho \text{ et } \sigma \in \{0, 1, 2, 3\}; \quad (3.27)$$

$\Gamma^\mu_{\rho \sigma}$ is the affine connection corresponding to the AdS metric (2.3). Then we take the $\kappa \to 0$ limit. We obtain,

$$\omega^0_{\nu} = \lim_{\kappa \to 0} \omega_{\nu} = dx \wedge dp - \frac{1}{2m^2} \epsilon_{\mu \nu \rho \sigma} p^\mu s^\nu \left[ \frac{1}{m^2} dp^\rho \wedge dp^\sigma - \frac{1}{m^2 s^2} ds^\rho \wedge ds^\sigma \right]. \quad (3.28)$$

This is exactly the presymplectic form Souriau used in his work [So]. Thus, $F^{m,s}_\kappa$ can be considered as the AdS deformation of the evolution space describing the theory of a free massive and spinning particle on the Minkowski spacetime.

Finally, note that all the limits we evaluated in this subsection confirm the physical interpretation we gave to the AdS quantities previously introduced.

### 3.3. The coadjoint orbit contraction

In the previous subsection we showed that the presymplectic manifold $F^{m,s}_\kappa$ is the AdS deformation of the evolution space used by Souriau in its description of a massive and spinning free particle on Minkowski spacetime. Here we investigate the zero curvature behaviour of the phase space $\Sigma^{m,s}_\kappa$. In order to stay close to the group theoretical meaning of contraction, this point is discussed in the language of coadjoint orbits. In fact $\Sigma^{m,s}_\kappa$ is diffeomorphic to a coadjoint orbit $O^{m,s}_\kappa$ of $SO_0(3,2)$ [Ko] [SW].

We first start by identifying $O^{m,s}_\kappa$. This is achieved through the moment map in the following way. Usually the momentum map is defined as a map from the phase space into the dual of the Lie algebra. However in the present case it can be defined as a map from the evolution space into $so^*(3,2)$. The two constructions are equivalent as shown below. In fact, let the map $L$ be defined as follows:

$$L : E^{m,s}_\kappa \ni w \mapsto L(w) \in so^*(3,2) \quad (3.29a)$$
such that,
\[ \langle L(w), e_{\alpha\beta} \rangle = X_{\alpha\beta} |_{\theta_E} \equiv L_{\alpha\beta}(w), \quad \alpha, \beta \in \{5, 0, 1, 2, 3\}. \] (3.29b)

The symbol \( \langle , \rangle \) denotes the duality \( so(3, 2) \)-\( so^*(3, 2) \) and the \( e_{\alpha\beta} \)'s are the basis elements of the abstract \( so(3, 2) \) algebra (see (2.4)). The \( X_{\alpha\beta} \)'s are the fundamental vector fields associated to the (left) action of \( SO_0(3, 2) \) on \( E^m_s \). The \( L_{\alpha\beta} \)'s introduced in (3.29b) are the classical observables associated to the action of \( SO_0(3, 2) \) on \( E^m_s \). Their explicit form is easily obtained:
\[ L_{\alpha\beta} = y_{\alpha} q_{\beta} - y_{\beta} q_{\alpha} + s (u_{\alpha} v_{\beta} - u_{\beta} v_{\alpha}) \]
\[ = y_{\alpha} q_{\beta} - y_{\beta} q_{\alpha} + \frac{k}{m^2} \epsilon_{\alpha\beta\gamma\rho\sigma} y^\gamma q^\rho t^\sigma, \quad \forall \alpha, \beta \in \{5, 0, 1, 2, 3\}. \] (3.30a)

The second expression is a consequence of the orientation condition in (3.1g). These observables are constants of the motion, i.e. \( dL_{\alpha\beta} |_{K_S} = 0 \), where we recall that \( K_S(w) \in \ker \omega_E \), \( \forall w \in E^m_s \) is given in (3.15). As a result, the usual moment map \( \tilde{L} : \Sigma^m_s \rightarrow so^*(3, 2) \) is obtained through the symplectic reduction \( \pi : E^m_s \rightarrow \Sigma^m_s \), i.e. \( L = \tilde{L} \circ \pi \).

The associated classical observables \( \tilde{L}_{\alpha\beta} \in C^\infty (\Sigma^m_s) \) are then given by \( \tilde{L}_{\alpha\beta}(\tilde{w}) = L_{\alpha\beta}(w) \), where \( \tilde{w} = \pi(w) \). The \( \tilde{L}_{\alpha\beta} \)'s realize \( so(3, 2) \) through the Poisson bracket defined by the symplectic form \( \omega_\Sigma \). Since \( SO_0(3, 2) \) is a simple Lie group, its action on \( \Sigma^m_s \) is strongly Hamiltonian, the momentum map is uniquely defined and also equivariant [LM]. The image of \( \Sigma^m_s \) under \( \tilde{L} \) is then an orbit in \( so^*(3, 2) \). This is the coadjoint orbit we shall denote \( O^m_s \). Note that for \( w_{(0)} \in E^m_s \) given in (3.2)
\[ L(w_{(0)}) = \frac{m}{\kappa} \theta^{50} + s \theta^{12}; \] (3.31)

hence \( O^m_s \) passes through \( \theta_E \). This is not a coincidence, it is just a consequence of the general theory [SW]. The orbit \( O^m_s \), can be realized through constraint equations in \( V \equiv \mathbb{R}^{10} \) (the vector space underlying \( so(3, 2) \) or \( so^*(3, 2) \)). These equations are provided by the Casimir invariants of \( so(3, 2) \). Actually, a straightforward computation gives the two identities,
\[ \frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} = \frac{m^2}{\kappa^2} + s^2, \] (3.32a)
\[ \Pi_{\alpha} \Pi^\alpha = \frac{m^2 s^2}{\kappa^2}, \] (3.32b)
where \( \Pi_{\alpha} = \frac{1}{8} \epsilon_{\alpha\mu\nu\rho\sigma} L_{\mu\nu} L_{\rho\sigma} \). Notice that these equations do not define a connected submanifold of \( V \). However, the connected component \( O^m_s \) is uniquely specified by imposing that it passes through \( \theta_E \in V \). Two remarks are now in order:
1. The invariants above have the same value for two distinct physical systems \((m, s, \kappa)\) and \((m', s', \kappa')\), such that \(\frac{m}{\kappa} = s'\) and \(\frac{m'}{\kappa'} = s\), however the corresponding \(\theta_E\) belong then to two distinct orbits. This phenomenon can be viewed as a consequence of a classical counterpart of the Weyl symmetry [GH]. In order to restrict ourselves to only one of the two types of orbits, we will from now on consider as physical only triplets \((m, s, \kappa)\) such that \(\frac{m}{\kappa} > s > 0\). At first sight this condition seems to be weakly justified. Its physical origin will be discussed in connection with previous works in the next subsection.

2. The same orbit, \(O_{m,s}^\kappa\), is associated to two distinct physical systems: \((m, s, \kappa)\) and \((m', s', \kappa')\), such that \(\frac{m}{\kappa} = \frac{m'}{\kappa'}\) and \(s = s'\). Hence, to the contrary of the Poincaré group, the correspondence between physical systems and elementary systems (i.e. coadjoint orbits) is not one-to-one. Once again, this point will be discussed in the next subsection. In fact we will show that 1 and 2 are related.

We come now to the contraction of the orbit \(O_{m,s}^\kappa\). To this end we will use a sequence of \(\kappa\)-dependent transformations, such that when \(\kappa\) tends to zero the transformed orbit (which is no longer an orbit) tends to a Poincaré-coadjoint orbit (see [Do]). Recall that the contraction map \(\phi_{\kappa}\) introduced in (2.5) allowed us to reach \(p(3,1)\) starting from \(so(3,2)\). Using \(\phi_{\kappa}\), we define in a natural way the following family of maps:

\[
\tilde{L}^\kappa = \phi_{\kappa}^* \circ \tilde{L} : \Sigma_{m,s}^{\kappa} \rightarrow V^*. \tag{3.33}
\]

In analogy with (3.29), we shall write,

\[
\tilde{L}^\kappa(\tilde{w}) = \frac{1}{2} \tilde{L}_{\alpha\beta}(\tilde{w}) \vartheta_{\alpha\beta}. \tag{3.34}
\]

where \(\{\vartheta_{\alpha\beta}\}\) is the basis of \(so^*(3,2)\) dual to \(\{e_{\alpha\beta}\}\). More precisely, the \(\tilde{L}_{\alpha\beta}^\kappa\)'s are related to the \(\tilde{L}_{\alpha\beta}\)'s in the same way the \(e_{\alpha\beta}\)'s are related to \(e_{\alpha\beta}\)'s in (2.5). In order to evaluate the \(\kappa \rightarrow 0\) limit, we first express the \(\tilde{L}_{\alpha\beta}^\kappa\)'s in the contraction adapted coordinates. Concretely, using (3.30), the results of subsection 3.2 and the explicit symplectic reduction of subsection 3.1, one finds

\[
\tilde{L}^{50}_5 = -p_0 + \frac{\kappa^2}{m^2} \sum_{i,j,k=1,2,3} \epsilon_{ijk} (x^j p_j s_k) \tag{3.35a}
\]

\[
\tilde{L}^{i5}_5 = \kappa Y p_i + \frac{\kappa^2}{m^2 (\kappa Y)} \sum_{j,k=1,2,3} \epsilon_{ijk} \left[ x^j (p_0 s_k - s_0 p_k) \right], \tag{3.35b}
\]
\[
\tilde{L}^\kappa_{0i} = -\frac{p_0}{(\kappa Y)} x^i + \frac{\kappa Y}{m^2} \sum_{j,k=1,2,3} \epsilon_{ijk} p_j s_k,
\]

(3.35c)

\[
\tilde{L}^\kappa_{ij} = (x^i p_j - x^j p_i) - \frac{1}{m^2} \sum_{k=1,2,3} \epsilon_{ijk} (p_0 s_k - s_0 p_k).
\]

(3.35d)

Here \(i, j \in \{1, 2, 3\}\) and \(\epsilon_{ijk}\) is the completely skew-symmetric tensor of \(\mathbb{R}^3\). When \(\kappa \to 0\) they become respectively, \(H, P_i, K_i\) and \(J_i\) given by,

\[
H = p^0 = \sqrt{(\vec{p})^2 + m^2},
\]

(3.36a)

\[
\vec{P} = \vec{p},
\]

(3.36b)

\[
\vec{K} = H \vec{x} + \frac{\vec{p} \times \vec{s}}{m^2},
\]

(3.36c)

\[
\vec{J} = \vec{x} \times \vec{p} + \frac{p^0 \vec{s} - s^0 \vec{p}}{m^2}.
\]

(3.36d)

One can easily recognize these observables as those associated to the Poincaré group, even if they are written in an unusual representation. In fact the position \(\vec{x}\) arising here, is naturally the world line position (\(x^\mu\) is a four vector) but not the canonical position usually appearing in the literature [SM] [He] [M]. The usual realization is recovered through the use of the following transformation,

\[
\vec{X} = \vec{x} + \frac{\vec{p} \times \vec{s}}{m^2(p^0 + m)},
\]

(3.37a)

\[
\vec{S} = \frac{\vec{s}}{m} - \frac{s^0 \vec{p}}{m(p^0 + m)};
\]

(3.37b)

\(\vec{X}\) is the canonical position, \(\vec{S}\) lies on the sphere of radius \(s\), \(\vec{S} \cdot \vec{S} = s^2\) and \(s^\mu\) is, according to subsection 3.2, the Pauli-Lubanski four-vector. In terms of \(\vec{X}\) and \(\vec{S}\), \(\vec{K}\) and \(\vec{J}\) are then

\[
\vec{K} = p^0 \vec{X} + \frac{\vec{p} \times \vec{S}}{p^0 + m};
\]

(3.38a)

\[
\vec{J} = \vec{X} \times \vec{p} + \vec{S}.
\]

(3.38b)

Equations (3.36a-b) and (3.38a-b) are the usual Poincaré observables associated to a mass \(m\) and spin \(s\) free particle on Minkowski spacetime. The casimir invariants that allow the identification of the \(P^\dagger_+ (3,1)\)-coadjoint orbit obtained here arise also as a \(\kappa \to 0\) limit of (3.32a-b). Actually, using (3.36a-d) we show that,

\[
\frac{\kappa^2}{2} \tilde{L}_{\alpha\beta} \tilde{L}^{\alpha\beta} = m^2 + \kappa^2 s^2 \quad \xrightarrow{\kappa \to 0} \quad H^2 - \vec{P} \cdot \vec{P} = m^2,
\]

(3.39)
\begin{align}
\lim_{\kappa \to 0} \tilde{\Pi}^\kappa_5 &= 0, & \lim_{\kappa \to 0} \tilde{\Pi}^\kappa_0 &= \vec{P} \cdot \vec{J} = s^0, & \lim_{\kappa \to 0} \tilde{\Pi}^\kappa = H \vec{J} - \vec{K} \times \vec{P} = \vec{s}. \tag{3.40}
\end{align}

The tilde in (3.40) means that we consider (3.32b) with the $\tilde{L}_{\alpha\beta}$'s instead of the $L_{\alpha\beta}$'s. Equations in (3.40) are the well known expressions defining the Pauli-Lubanski four-vector $s^\mu$ in terms of the Poincaré generators [SM], in fact

$$-(s^0)^2 + (\vec{s})^2 = m^2 s^2. \tag{3.41}$$

We conclude that in the zero curvature limit, the surfaces $\tilde{L}^\kappa(\Sigma_{m,s}^\kappa) \subset V^*$ tend to the $\mathcal{P}^\dagger_+(3,1)$-orbit $\mathcal{O}_P(m,s)$, corresponding to a mass $m$ and spin $s$ free particle on Minkowski spacetime, which passes through $m \vartheta^{50} + s \vartheta^{12} \in V^*$. In other words, $\tilde{L}^0$ is a $\mathcal{P}^\dagger_+(3,1)$-equivariant moment map.

### 3.4. Discussion

In subsection 3.3 we restricted the notion of a physical system to those triplets $(m, s, \kappa)$ satisfying $\frac{m}{\kappa} > s > 0$. We imposed this condition in order to associate to physical systems a unique type of coadjoint orbits, namely those passing through $a \vartheta^{50} + b \vartheta^{12}$ such that $a > b$. One can end up with this condition from other considerations. In fact, as we noticed in remark 2 of subsection 3.3, the same coadjoint orbit can be associated to different physical systems. This is clearly due to the fact that a physical system is specified by three parameters while the corresponding coadjoint orbit is specified by two parameters. In order to compare this situation with the one appearing in the flat case one must first fix the AdS spacetime, i.e. fix the curvature $\kappa$, and then investigate the one-to-one character of the correspondence. Doing so, one has to solve the system of equations arising from (3.32a-b), where $m$ and $s$ are the unknowns for given values of $\frac{m^2}{\kappa^2} + s^2$ and $\frac{m^2 s^2}{\kappa^2}$. This problem has two possible solutions satisfying either $\frac{m}{\kappa} > s$ or $\frac{m}{\kappa} < s$. By considering only one of the latter as physically realizable, we obtain an AdS analog of the one-to-one correspondence (physical system $\leftrightarrow$ coadjoint orbit) occurring in the case of the Poincaré group. Here we choose $\frac{m}{\kappa} > s$ as the physical condition. This choice is encouraged by concording arguments used by different authors, see for instance [Di] [Ku] [Wo].

A purely classical argument can be found in [Di]. In fact, there the author considers a free extended object on $M_\kappa$. The condition $\frac{m}{\kappa} > s$ appears as a reasonable physical one, since $\frac{m}{\kappa} \leq s$ implies that the extended object spin with a speed larger than that of light and has dimensions greater than the radius $\kappa^{-1}$ of the universe!

The arguments found in [Ku] and [Wo] can be applied to our present work since they arise from the study of the classical dynamics of an elementary particle in general relativity.
The first author imposes the general condition \((\text{spin}) \times (\text{curvature}) < (\text{mass})\) in order to avoid the region where the equality holds, since then the dimension of the kernel of the presymplectic form becomes larger (as it happened here, see (ii) in subsection 3.1). He shows also that this condition is valid even in the extreme situation of an electron near the horizon of a Schwarzschild black hole. The second author used the following argument: for spin values of the order of \(\hbar\) the condition \(\frac{m}{\kappa} > s\) (with \(\hbar\) and \(c\) no longer equal to 1) is violated by particles having a Compton wavelength \(\frac{\hbar}{mc}\) of the same order or larger than the radius of the universe. The one particle theory fails then, since in that case the gravitational force is strong enough to induce the creation of pairs of particles.

Finally, let us mention that this constraint fits very well with the fact noticed in (ii) (subsection 3.1). In fact, as for the Poincaré case where \(m > 0\) specifies a massive elementary system and the lower limit on \(m\) a massless one, the lower limit \(\frac{m}{\kappa} = s\) here corresponds to a massless AdS elementary system [E2]. Observe also that when \(\kappa \to 0\) the condition \(m > \kappa s\) becomes simply \(m > 0\). Moreover, it will appear in section 4 that \(\frac{m}{\kappa} > s\) is a necessary condition for the unitarity of the representation obtained.

4. The quantum theory

The aim of this section is to construct the quantum theory of a mass \(m\) and spin \(s\) free particle on AdS spacetime of curvature \(\kappa > 0\). To this end we shall quantize the classical theory described in the previous section. More precisely, we will use geometric quantization techniques, which exploit in an efficient way the geometric constructions of section 3. It is well known the quantum theory we are looking for is described by a unitary irreducible representation (UIR) of \(SO_{0}(3, 2)\) [Fr2] [Wi1]. This representation can be obtained by geometric quantization, known also as the orbit method of Kirillov [Ki1] [Ki2]. This particular method has the advantage of allowing us to identify the physical interpretation of the quantities appearing in the quantum theory, such as the quantum states and the observables, since it is based on the classical theory, where the physical interpretations are already established. Note that the spacetime realization of the representation do not have this feature [Fr2].

The geometric quantization proceeds in two steps. First, one follows a prequantization procedure that identifies a unitary but reducible representation of \(SO_{0}(3, 2)\) (section 4.1). Then one uses polarization conditions that select an irreducible subrepresentation (section 4.2). Since this programme has already been carried out in [DBE] for the 1+1 dimensional case, we will omit here unnecessary details. For the general theory of geometric quantization
we refer to [Ko] [SW] and [Wo].

Finally let us mention that the results of section 3 greatly simplify the computations. In fact, we shall base our construction on \( E^m_s \kappa \), i.e. all relevant quantities will be defined on \( E^m_s \kappa \), keeping in mind that the quantities on the phase space \( \Sigma^m_s \kappa \) can be derived making use of the symplectic reduction of subsection 3.1.

4.1. Prequantization

The prequantum Hilbert space \( \mathcal{H} \), when it exists, is generally defined as the space of square integrable sections of a Hermitian line bundle-with-connection over the phase space, such that the symplectic form of the latter is the curvature of that connection. The existence of \( \mathcal{H} \) requires that the symplectic form satisfies an integrability condition [Wo].

The identification of \( \mathcal{H} \) in the present case is greatly simplified because of the principal bundle structure \( \pi : E^m_s \kappa \rightarrow SO_0(3, 2) \rightarrow SO_0(3, 2)/SO(2) \approx \Sigma^m_s \kappa \), explicitly realized in section 3. In fact, \( \mathcal{H} \) consists then of functions \( \psi \in L^2(E^m_s \kappa, d\mu^m_s \kappa) \), satisfying the condition:

\[
(K\psi)(w) = i(K \theta_E \psi)(w), \tag{4.1}
\]

for all vector field \( K \) on \( E^m_s \kappa \), such that \( K(w) \in \ker w \omega_E \forall w \in E^m_s \kappa \) [Wo]. Note that \( d\mu^m_s \kappa \) is the invariant measure on \( E^m_s \kappa \), obtained from the left Haar measure on \( SO_0(3, 2) \approx E^m_s \kappa \). Recalling from (3.15) that \( \ker \omega_E \) is generated by \( Y_{50} \) and \( Y_{12} \) given in (3.5a) we can explicitly write,

\[
\mathcal{H} = \left\{ \psi : E^m_s \kappa \rightarrow \mathbb{C} \mid \int_{E^m_s \kappa} |\psi|^2 d\mu^m_s \kappa < \infty, \quad Y_{50} \psi = i \frac{m}{\kappa} \psi \text{ and } Y_{12} \psi = is \psi \right\}. \tag{4.2}
\]

The integrability condition mentioned in the first paragraph above appears here as a condition for the integrability of the equations in (4.2) to the global group action. Hence, since \( Y_{50} \) and \( Y_{12} \) generate a compact subgroup of \( SO_0(3, 2) \), this implies that \( \frac{m}{\kappa} \) and \( s \) must be integers. One can also view this condition in terms of the integrability of \( \theta_E \) given in (3.6).

Clearly the quantum theory that will arise from the present quantization will only describe integer spin elementary systems. In order to take also into account the half integer spin particles one should, from the beginning, consider as the symmetry group of the theory \( Sp(4, \mathbb{R}) \) instead of \( SO_0(3, 2) \). The former is the double covering of the latter (see for instance [BEGG]). This can also be achieved by considering the universal covering group of \( SO_0(3, 2) \) as the symmetry group, in this case \( \frac{m}{\kappa} \) will no longer be an integer, it will then take its values in \( \mathbb{R}_+^* \).
The Hilbert space $\mathcal{H}$ carries a unitary representation of $SO_0(3,2)$. In fact, since $E^m_s \cong SO_0(3,2)$, there exists a natural action of $SO_0(3,2)$ in $L^2(E^m_s, d\mu^m_s)$. This yields the left regular representation of $SO_0(3,2)$ denoted $U(\Lambda)$ and given by,
\[
(U(\Lambda)\psi)(w) = \psi(\Lambda^{-1} \cdot w), \quad \text{where} \quad (\Lambda^{-1} \cdot w)^\mu = (\Lambda^{-1})^\mu_{\nu} w^\nu.
\] (4.3)

When restricted to $\mathcal{H}$ it provides us with a unitary representation of $SO_0(3,2)$ denoted by $(\mathcal{H}, U)$. In the language of induced representations $(\mathcal{H}, U)$ is a representation of $SO_0(3,2)$ induced from the unitary character $\exp i(m_\kappa \tau + s_\kappa')$ of the subgroup $SO(2) \times SO(2) \subset SO_0(3,2)$.

From (4.3) we can obtain the expression of the (pre)quantum operators $\hat{L}_{\alpha\beta}$, i.e. the quantum analogs of the $L_{\alpha\beta}$’s given in (3.30). Actually,
\[
\hat{L}_{\alpha\beta} \equiv i \frac{d}{d\tau} \left. U(\exp \tau e_{\alpha\beta})\psi\right|_{\tau=0} = -i (X_{\alpha\beta}\psi)(w).
\] (4.4)

The $\hat{L}_{\alpha\beta}$’s are then nothing but $(-i)$ times the fundamental vector fields $X_{\alpha\beta}$ associated to the action of $SO_0(3,2)$ on $E^m_s$.

4.2. Polarization

Exploiting once again the principal bundle structure $\pi : E^m_s \longrightarrow \Sigma^m_s$, we can use an algebraic characterization of the polarization on $\Sigma^m_s$ in order to concretely evaluate it [Ra1] [Re] [Wo]. Actually this characterization determines a prepolarization on $E^m_s$ the projection of which on $\Sigma^m_s$ produces an invariant polarization. Since we are interested in evaluating the quantum theory at the level of $E^m_s$, we will only need the prepolarization [Wo].

The latter is a subalgebra $h$ of $so^C(3,2)$, the complexified $so(3,2)$, which satisfies the following conditions:

(i) $Y_{50}$ and $Y_{12} \in h$.

(ii) $\dim_{\mathbb{C}} h = \frac{1}{2} (\dim so(3,2) + \dim \ker \omega_E) \implies \dim_{\mathbb{C}} h = 6$ since $\dim \ker \omega_E = 2$.

(iii) $\theta_E ([Y, Y']) = 0, \quad \forall Y, Y' \in h$.

(iv) $h + \tilde{h}$ is a subalgebra of $so^C(3,2)$.

The prepolarization projects on $\Sigma^m_s$ to a Kähler (resp. positive) polarization if $h \cap \tilde{h} = \{Y_{50}, Y_{12}\}$ (resp. $i\theta_E ([Z, Z]) \geq 0, \quad \forall Z \in h$).

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It is easy to check that the following subalgebra $h$ of $so^C(3,2)$,

$$h = \text{span}\{Y_{50}, Y_{12}, Z_1, Z_2, Z_3, \Xi\}, \quad (4.5a)$$

where

$$Z_i = Y_{0i} + iy_{i5}, \quad i \in \{1, 2, 3\} \quad \text{and} \quad \Xi = Y_{23} + iy_{31}, \quad (4.5b)$$

is a prepolarization on $E^m_s$. Moreover, its projection on $\Sigma^m_s$ is a Kähler, positive and invariant polarization. Notice that $\Xi$ in (4.5b) is, at least algebraically, the usual Kähler polarization of a sphere. The latter is in our case the homogeneous space for the subgroup $SO(3) \subset SO_0(3,2)$ generated by the $Y_{ij}, i, j \in \{1, 2, 3\}$. Hence, $\Xi$ characterizes the spin contribution in $E^m_s$.

We are now able to select in $(H, U)$ the unitary irreducible subrepresentation $U^m_s$ we are looking for [Wo]. This is just the restriction of $U$ to $H^m_s$, where

$$H^m_s = \{\psi \in H \mid Z_i \psi = 0, \quad i \in \{1, 2, 3\} \quad \text{and} \quad \Xi \psi = 0\}. \quad (4.6)$$

The way we obtained the UIR $(H^m_s, U^m_s)$ is called in mathematics literature a holomorphic induction [Hu] [Sc]. It produces discrete series representations of noncompact semi-simple Lie groups. The one we obtained here is the quantization of the orbit $O^m_s$ of section 3. It is the discrete series representation of $SO_0(3,2)$ characterized by the highest weight $(m, s)$ associated to the Cartan subalgebra generated by $e_{50}$ and $e_{12}$. Moreover, it is also known that the unitarity of $U^m_s$ requires the necessary condition $m > s$ [Fr1] [Ev]. Notice that the physical constraint $m > s$ we imposed in the classical theory (see section 3.4) finds in the quantum paradigm an interpretation in terms of unitarity. A more restrictive necessary and sufficient unitarity condition can be found in [Fr2] [FH]. It can be recovered in the present construction using the same method we used in [DBE].

The quantum states are represented by well defined wave functions belonging to $H^m_s$. The physical interpretation of their modulus as probability densities on $\Sigma^m_s$ is also well defined. Notice that this very important quantum property is inherent to the phase space representation, it lacks in the other known representations (spacetime or momentum space representations), and it clearly arises from the square integrability of the representation $(H^m_s, U^m_s)$. This property constitutes the basic ingredient necessary for the definition of the notion of optimal localization that will be given in the next section.

Finally, let us write the complex coordinates induced by the Kähler polarization on $E^m_s$. These are $z$ and $\xi \in (\mathbb{C}^5, \eta)$ given by,

$$z = \kappa y - im^{-1} q \quad \text{and} \quad \xi = u - iv. \quad (4.7)$$
Here $y$, $q$, $u$ and $v$ are the vectors introduced in section 3.1. The presymplectic form $\omega_E$ in (3.10) becomes in terms of these new coordinates,

$$\omega_E = i\frac{m}{\kappa} dz \wedge d\bar{z} + is \, d\xi \wedge d\bar{\xi}. \quad (4.8)$$

All the quantities introduced up to now can be reexpressed in terms of $z$ and $\xi$. Moreover, the phase space $\Sigma_{\kappa}^{m,s}$ can be viewed as a symplectic reduction of $(T\mathbb{C}^5, \eta)$ equipped with the canonical symplectic form $\omega_E$ given above.

5. Optimal localization and its zero curvature limit

In section 3 we constructed the classical theory and we investigated its zero curvature limit. More precisely, we evaluated the $\kappa \to 0$ limit of the orbit $O_{\kappa}^{m,s}$ and we found that it gives rise to a $\mathcal{P}^1_{\kappa}(3,1)$-coadjoint orbit $O_P(m, s)$. In section 4 we quantized $O_{\kappa}^{m,s}$. In order to complete the picture we want now to investigate the zero curvature limit of this quantum theory. We start in this section by studying the $\kappa \to 0$ behaviour of a particular family of quantum states $\varphi_{\tilde{w}} \in \mathcal{H}_{\kappa}^{m,s}$ indexed by the points $\tilde{w} \in \Sigma_{\kappa}^{m,s}$. We shall identify the $\varphi_{\tilde{w}}$ through the notion of optimal localization on $\Sigma_{\kappa}^{m,s}$ that we now introduce. First, recall that since $\Sigma_{\kappa}^{m,s}$ is an 8-dimensional $SO_0(3,2)$-homogeneous space, a point $\tilde{w} \in \Sigma_{\kappa}^{m,s}$ is completely specified by giving the values of $L_{31}, L_{23}, L_{0i}$ and $L_{i5}$ $i \in \{1, 2, 3\}$ at $\tilde{w}$. This suggests the following definition. We shall say a state $\varphi \in \mathcal{H}_{\kappa}^{m,s}$ is localized at the point $\tilde{w}$ in phase space, if the quantum expectation values of the eight observables $\hat{L}_{\alpha\beta}$’s with $(\alpha, \beta) = (0, i), (i, 5), (3, 1)$ or $(2, 3)$ equal the corresponding classical values, i.e.

$$\langle \varphi \mid \hat{L}_{\alpha\beta} \mid \varphi \rangle = L_{\alpha\beta}(w), \quad (5.1)$$

when $\tilde{w} = \pi(w)$. It is not hard to see that these eight conditions do not specify $\varphi$ uniquely. If on the other hand we require (5.1) to hold for all ten group generators, then $\varphi$ is uniquely determined (up to a phase) and we write $\varphi_{\tilde{w}}$ for the solution. For reasons explained shortly, we shall say $\varphi_{\tilde{w}}$ is the state optimally localized at $\tilde{w}$. We now compute the $\varphi_{\tilde{w}}$ explicitly. First, consider the state $\varphi_0 \equiv \varphi_{w^{(0)}}$ with $w^{(0)} \in E_{\kappa}^{m,s}$ defined in (3.2). Equation (5.1) yields,

$$\langle \varphi_0 \mid \hat{L}_{50} \mid \varphi_0 \rangle = \frac{m}{\kappa}, \quad \langle \varphi_0 \mid \hat{L}_{12} \mid \varphi_0 \rangle = s \quad \text{and} \quad \langle \varphi_0 \mid \hat{L}_{\alpha\beta} \mid \varphi_0 \rangle = 0 \text{ otherwise.} \quad (5.2)$$

Hence $\varphi_0$ is the highest weight vector in $\mathcal{H}_{\kappa}^{m,s}$. It is then immediately clear that,

$$\varphi_{\tilde{w}} = U_{\kappa}^{m,s}(\Lambda(w)) \varphi_0, \quad (5.3)$$

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where \( \Lambda(w) \) is the \( SO_0(3,2) \) element given in (3.3). We conclude that the states \( \{ \varphi_w \mid \tilde{w} \in \Sigma_{m,s}^{m,s} \} \) belong to the orbit \( \mathcal{O}_{\varphi_0} \) of the action of \( U_{m,s}^m \) on the highest weight vector \( \varphi_0 \).

The optimal character of the localization of the states \( \varphi_w \in \mathcal{O}_{\varphi_0} \) arises from a known property of these states. In fact, they minimize the dispersion relations associated to the Casimir invariants of \( (H_{m,s}, U_{m,s}^m) \) [De] [DF] [Pe]. The explicit derivation of these results will not be given here since it is a straightforward generalization of those obtained in [DBE]. According to Perelomov [Pe] the states in \( \mathcal{O}_{\varphi_0} \) are called generalized coherent states. Note that in the subsequent we will denote \( \varphi_w \) equivalently by \( \varphi_w \).

Let us now evaluate \( \varphi_w \in \mathcal{O}_{\varphi_0}. \) To this end we first need to determine \( \varphi_0. \) This is actually realized through solving the following system of equations, which arise from (4.2), (4.6) and (5.2),

\[
Y_{50} \varphi_0 = i \frac{m}{k} \varphi_0 \implies (\bar{z} \cdot \frac{\partial}{\partial \bar{z}} - z \cdot \frac{\partial}{\partial z}) \varphi_0 = \frac{m}{k} \varphi_0, \tag{5.4a}
\]

\[
Y_{12} \varphi_0 = is \varphi_0 \implies (\bar{\xi} \cdot \frac{\partial}{\partial \bar{\xi}} - \xi \cdot \frac{\partial}{\partial \xi}) \varphi_0 = -s \varphi_0, \tag{5.4b}
\]

\[
\mathcal{Z}_+ \varphi_0 = 0 \implies (\bar{\xi} \cdot \frac{\partial}{\partial \bar{z}} + z \cdot \frac{\partial}{\partial \xi}) \varphi_0 = 0, \tag{5.4c}
\]

\[
\mathcal{Z}_- \varphi_0 = 0 \implies (\xi \cdot \frac{\partial}{\partial \bar{z}} + \bar{z} \cdot \frac{\partial}{\partial \xi}) \varphi_0 = 0, \tag{5.4d}
\]

\[
\mathcal{Z}_3 \varphi_0 = 0 \implies t \cdot \frac{\partial}{\partial \bar{z}} \varphi_0 = 0, \tag{5.4e}
\]

\[
\Xi \varphi_0 = 0 \implies t \cdot \frac{\partial}{\partial \xi} \varphi_0 = 0, \tag{5.4f}
\]

\[
\hat{L}_{50} \varphi_0 = \frac{m}{k} \varphi_0 \implies [(z^5 \frac{\partial}{\partial z^5} - z_0 \frac{\partial}{\partial z^0}) + (\bar{z}^5 \frac{\partial}{\partial \bar{z}^5} - \bar{z}_0 \frac{\partial}{\partial \bar{z}^0}) + (z \rightarrow \xi)] \varphi_0 = \frac{i}{k} \varphi_0, \tag{5.4g}
\]

\[
\hat{L}_{12} \varphi_0 = s \varphi_0 \implies [(z_1 \frac{\partial}{\partial z^2} - z_2 \frac{\partial}{\partial z^1}) + (\bar{z}_1 \frac{\partial}{\partial \bar{z}^2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}^1}) + (z \rightarrow \xi)] \varphi_0 = is \varphi_0. \tag{5.4h}
\]

Some notational precisions are in order. Actually, \( \mathcal{Z}_+ = \frac{i}{2}(\mathcal{Z}_1 + i\mathcal{Z}_2) \) and \( \mathcal{Z}_- = \frac{i}{2}(\mathcal{Z}_1 - i\mathcal{Z}_2) \), for \( \mathcal{Z}_1 \text{ and } \mathcal{Z}_2 \) given in (4.5b). The vector fields in (5.4) do not contain any derivatives with respect to \( t^\alpha. \) This is due to a transformation we made in order to express the equations above only in terms of the complex coordinates \( z \) and \( \xi. \) Thus, \( \varphi_0(z, \bar{z}, \xi, \bar{\xi}) \equiv \psi_0(z, \bar{z}, \xi, \bar{\xi}, t(z, \bar{z}, \xi, \bar{\xi})) \), where \( \psi_0 \) is the vector that originally appears in (4.6) and \( t^\alpha = -\frac{ms}{4} \epsilon^\alpha_{\beta\gamma\rho} z^\beta \bar{z}^\gamma \xi^\rho \xi^\sigma \) (see (3.1g)).
It is easy to check that the solution of the above system of equations is given by,

$$\varphi_0(z, \xi) = N \left( \bar{z}_{(0)} \cdot z \right)^{-\frac{m}{\kappa} - s} \left[ (\bar{z}_{(0)} \cdot z)(\bar{\xi}_{(0)} \cdot \xi) - (\bar{z}_{(0)} \cdot \xi)(\bar{\xi}_{(0)} \cdot z) \right]^s; \quad (5.5)$$

$N$ is a normalization constant and $z_{(0)} = \kappa y_{(0)} - im^{-1}q_{(0)}$ and $\xi_{(0)} = u_{(0)} - iv_{(0)}$. This solution is well defined on $E^m_{\kappa,s}$. In fact, one easily verifies that $\bar{z}_{(0)} \cdot z = z_5 + i z_0$ vanishes only when $y^5 q^0 - y^0 q^5 < 0$. Because of equation (3.1h) this can never happen for points of $E^m_{\kappa,s}$. Using (5.3), $\varphi_{w'} \in \mathcal{O}_{\varphi_0}$ is obtained as follows,

$$\varphi_{w'}(z, \xi) = U^m_{\kappa,s}(\Lambda(w')) \varphi_0(z, \xi)$$

$$= \varphi_0 (\Lambda^{-1}(w') z, \Lambda^{-1}(w') \xi)$$

$$= N (\bar{z}_{(0)} \cdot \Lambda^{-1} z)^{-\frac{m}{\kappa} - s} \left[ (\bar{z}_{(0)} \cdot \Lambda^{-1} z)(\bar{\xi}_{(0)} \cdot \Lambda^{-1} \xi) - (\bar{z}_{(0)} \cdot \Lambda^{-1} \xi)(\bar{\xi}_{(0)} \cdot \Lambda^{-1} z) \right]^s$$

$$= N (\Lambda \bar{z}_{(0)} \cdot z)^{-\frac{m}{\kappa} - s} \left[ (\Lambda \bar{z}_{(0)} \cdot z)(\Lambda \bar{\xi}_{(0)} \cdot \xi) - (\Lambda \bar{z}_{(0)} \cdot \xi)(\Lambda \bar{\xi}_{(0)} \cdot z) \right]^s$$

$$= N (\bar{z}' \cdot z)^{-\frac{m}{\kappa} - s} \left[ (\bar{z}' \cdot z)(\bar{\xi}' \cdot \xi) - (\bar{z}' \cdot \xi)(\bar{\xi}' \cdot z) \right]^s. \quad (5.6)$$

Here we used (4.3) and also the fact that $\Lambda(w')w_{(0)} = w'$ (see (3.3)). Notice that we can equally well write $\varphi_{w'}(z, \xi)$ as $\varphi_{(z', \xi')}(z, \xi)$. The normalization is fixed by imposing that $\varphi_{(z', \xi')}(z, \xi) = 1$. This gives $N = (-2)\frac{m}{\kappa}(2)^{-s}$. Finally the optimally localized state at $(z', \xi') \in E^m_{\kappa,s}$ is given by,

$$\varphi_{z', \xi'}(z, \xi) = (-2)\frac{m}{\kappa}(2)^{-s} (\bar{z}' \cdot z)^{-\frac{m}{\kappa} - s} \left[ (\bar{z}' \cdot z)(\bar{\xi}' \cdot \xi) - (\bar{z}' \cdot \xi)(\bar{\xi}' \cdot z) \right]^s. \quad (5.7)$$

The optimal localization property can be read from (5.7). In fact, the modulus of $\varphi_{z', \xi'}(z, \xi)$ reaches its maximal value only when $z = z'$ and $\xi = \xi'$. Combining this with the physical interpretation of the modulus of the states of $\mathcal{H}^m_{\kappa,s}$ as probability densities on $\Sigma^m_{\kappa,s}$ one sees that $\varphi_{z', \xi'}(z, \xi)$ is actually optimally localized at $\bar{w}' \in \Sigma^m_{\kappa,s}$. If we consider, instead of the phase space realization, the spacetime one we find that the state corresponding to $\varphi_{z', \xi'}(z, \xi)$ is localized along the timelike geodesic that arises from the projection on $M_{\kappa}$ of the leave of ker $\omega_{\kappa}$ passing through $w' \in E^m_{\kappa,s}$. This has been shown for the 1 + 1 dimensional case in [DBEG].

Rewriting (5.7) in terms of the contraction adapted coordinates of section 3.2, we are able to evaluate its zero curvature limit. Using the same techniques as in [DBE], we obtain

$$\lim_{\kappa \to 0} \left( \frac{m}{4\pi\kappa} \right)^{\frac{s}{2}} \varphi_{z', \xi'}(z, \xi) = m^2 p^0 \delta(\bar{p} - \bar{p}') e^{-ip_\mu(x'\nu - x^\nu)} \left( \frac{\bar{\xi}' \cdot \xi}{2} \right)^s. \quad (5.8)$$
Here, $\zeta_{\mu} = a_{\mu} - ib_{\mu}$, $\mu \in \{0, 1, 2, 3\}$ and $(x, p, a, b, s) \in E^{m,s}_0$, $E^{m,s}_0$ being the Lorentz bundle over Minkowski spacetime which is the $\kappa \to 0$ limit of $E^{m,s}_\kappa$, see section 3.2.

The limiting state is clearly a distribution in the space of $\mathbb{C}$-valued functions on $E^{m,s}_0$. Moreover, notice that the spin part $(\frac{\zeta_{\mu} \bar{\zeta}_{\mu}}{2})^s$ and the orbital part factorize separately. More precisely, this state is perfectly localized in momentum space, completely delocalized in spacetime though still optimally localized in spin coordinates. A further analysis of this result will be given in the next section where we will show how the notion of optimal localization is intimately related to the Kähler character of the $SO_0(3, 2)$-invariant polarization. We shall see that the zero curvature limit of the latter produces a $\mathcal{P}^\dagger_+(3, 1)$-invariant polarization, which is no longer Kähler, and relate this to the disappearance of the notion of phase space localization.

6. About the contraction of the discrete series

Exploiting the fact that the construction of $(\mathcal{H}^{m,s}_\kappa, U^{m,s}_\kappa)$ in section 4 is $\kappa$-dependent, we will investigate here the $\kappa \to 0$ limit of that construction. Knowing that the $SO_0(3, 2)$-coadjoint orbit $\mathcal{O}^{m,s}_\kappa$ becomes in the zero curvature limit the $\mathcal{P}^\dagger_+(3, 1)$-coadjoint orbit $\mathcal{O}_P(m, s)$ (see section 3.3), we expect that the $\kappa \to 0$ limit of the construction mentioned above will produce the irreducible unitary representations of $\mathcal{P}^\dagger_+(3, 1)$, obtained by quantization of $\mathcal{O}_P(m, s)$. Notice that the use of ideas from geometric quantization to study the contraction of Lie groups representations was first proposed by Dooley [Do], and used explicitly in [DBE] and [CDB].

Let us first start by fixing some notations. The contraction adapted coordinates $(x, p, a, b, s)$ will describe $E^{m,s}_\kappa$ when $x \in M_\kappa$ and $E^{m,s}_0$ when $x$ belongs to Minkowski spacetime. The presymplectic form on $E^{m,s}_0$ is given by $\omega^0_E = -d\theta^0_E$, where $\theta^0_E$ is obtained from $\theta_E$ in (3.9) by contraction. Actually, $\theta^0_E = p \cdot dx + s b \cdot da$. The dot denotes here and throughout this section the flat metric scalar product.

The kernel of $\omega^0_E$ is spanned by the vector fields $Y_H$ and $Y^0_{12}$ obtained by contraction of $Y_{50}$ and $Y_{12}$ respectively. More precisely,

$$Y_H \equiv \lim_{\kappa \to 0} (\kappa Y_{50}) = -\frac{p}{m} \frac{\partial}{\partial x} \quad \text{and} \quad Y^0_{12} \equiv \lim_{\kappa \to 0} Y_{12} = b \frac{\partial}{\partial a} - a \frac{\partial}{\partial b}. \quad (6.1)$$

The vector fields $Y_H$ and $Y^0_{12}$ generate the right action on $E^{m,s}_0$ of the subgroup $T \times SO(2) \subset \mathcal{P}^\dagger_+(3, 1)$, where $T$ stands for the time translations subgroup. The phase space is then $\Sigma^{m,s}_0 \equiv \mathcal{P}^\dagger_+(3, 1)/T \times SO(2) \cong \mathbb{R}^6 \times S^2$. 

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When trying to prequantize as in 4.2, using \( \ker \omega^0 \), one is faced with the following problem. The space of \( L^2 \) functions on \( E_{m,s}^0 \), such that,

\[
(Y_H \psi)(x, p, a, b, s) = im \psi(x, p, a, b, s) \quad \text{and} \quad (Y_{12}^0 \psi)(x, p, a, b, s) = is \psi(x, p, a, b, s),
\]

contains only the zero function. This is a consequence of the non-compact character of the subgroup generated by \( Y_H \). So we will here proceed without requiring that (6.2) holds for \( L^2 \) functions on \( E_{m,s}^0 \). A solution avoiding this problem will arise subsequently. As in section 4.1, the integrability condition restricts \( s \) to integer values, however \( m \in \mathbb{R}_+ \).

Let us now evaluate the \( \kappa \to 0 \) limit of the polarization vector fields given in (4.5b),

\[
\lim_{\kappa \to 0} (\kappa Z_i) = i Y_{P_i} \quad \text{where} \quad Y_{P_1} = a \cdot \frac{\partial}{\partial x}, \quad Y_{P_2} = b \cdot \frac{\partial}{\partial x} \quad \text{and} \quad Y_{P_3} = \frac{1}{ms} s \cdot \frac{\partial}{\partial x}. \quad (6.3)
\]

The notation \( Y_{P_i} \) originates from the fact that these vector fields generate the right action of the space translations subgroup of \( P_\uparrow(3,1) \) on \( E_{m,s}^0 \). Note here that the complex character of the vector fields \( Z_i \) disappears when \( \kappa \to 0 \). For the spin part \( \Xi \), the contraction gives,

\[
\Xi^0 \equiv \lim_{\kappa \to 0} \Xi = \frac{2i}{ms} s \cdot \frac{\partial}{\partial \zeta} + ims \bar{\zeta} \cdot \frac{\partial}{\partial s}. \quad (6.4)
\]

The set of contracted vector fields \( \{ Y_H, Y_{12}^0, Y_{P_1}, Y_{P_2}, Y_{P_3}, \Xi^0 \} \) spans a \( P_\uparrow(3,1) \)-invariant prepolarization on \( E_{m,s}^0 \). In fact, algebraically, this is the prepolarization obtained by Renuoard \( \text{[Re]} \) in his quantization of \( \mathcal{O}_P(m, s) \). When projected on \( \Sigma_0^{m,s} \), the latter produces a \( P_\uparrow(3,1) \)-invariant polarization which is neither Kähler nor real. Its complex part \( \Xi^0 \) corresponds, as in the \( \kappa \neq 0 \) case, to the Kähler structure on the sphere \( S^2 \) in \( \Sigma_0^{m,s} \cong \mathbb{R}^6 \times S^2 \).

It is then natural to consider the UIR of \( P_\uparrow(3,1) \) that arises when using the previous prepolarization as the \( \kappa \to 0 \) limit of \( (H_{m,s}^\kappa, U_{m,s}^\kappa) \). In order to concretely identify that representation, let us write down all the constraint equations that the quantum states \( \psi \in C^\infty(E_{m,s}^0) \) must satisfy. The appropriate Hilbert space structure on these states will be considered later on. Taking into account the fact that \( s^\mu \) can be expressed in terms of \( p, a, \) and \( b \) using \( \epsilon_{\mu \nu \rho \sigma} p^\mu a^\nu b^\rho s^\sigma = m^2 s \), we first obtain for (6.2),

\[
(Y_H \varphi)(x, p, \zeta, \bar{\zeta}) = im \varphi(x, p, \zeta, \bar{\zeta}) \Rightarrow -\frac{p}{m} \frac{\partial}{\partial x} \varphi(x, p, \zeta, \bar{\zeta}) = im \varphi(x, p, \zeta, \bar{\zeta}) \quad (6.5a)
\]

\[
(Y_{12}^0 \varphi)(x, p, \zeta, \bar{\zeta}) = is \varphi(x, p, \zeta, \bar{\zeta}) \Rightarrow i(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}) \varphi(x, p, \zeta, \bar{\zeta}) = is \varphi(x, p, \zeta, \bar{\zeta}) \quad (6.5b)
\]

and then, for the polarization conditions (as in 4.6), we obtain,

\[
(Y_{P_1} \varphi)(x, p, \zeta, \bar{\zeta}) = 0 \Rightarrow a \cdot \frac{\partial}{\partial x} \varphi(x, p, \zeta, \bar{\zeta}) = 0, \quad (6.6a)
\]
(Y_{P_2}\varphi)(x, p, \zeta, \bar{\zeta}) = 0 \Rightarrow b \frac{\partial}{\partial x} \varphi(x, p, \zeta, \bar{\zeta}) = 0, \quad (6.6b)

(Y_{P_3}\varphi)(x, p, \zeta, \bar{\zeta}) = 0 \Rightarrow s \frac{\partial}{\partial x} \varphi(x, p, \zeta, \bar{\zeta}) = 0, \quad (6.6c)

(\Xi^0\varphi)(x, p, \zeta, \bar{\zeta}) = 0 \Rightarrow s \frac{\partial}{\partial \zeta} \varphi(x, p, \zeta, \bar{\zeta}) = 0. \quad (6.6d)

Clearly conditions (6.5a) and (6.6a-c) fix the $x$-dependence of $\varphi$ to be the phase factor $e^{ip\cdot x}$. Hence the general solution of the previous system is of the following form,

$$\Phi(x, p, \zeta) = e^{ip\cdot x} \phi(p, \zeta) \quad \text{such that} \quad \zeta \frac{\partial}{\partial \zeta} \phi(p, \zeta) = s\phi(p, \zeta). \quad (6.7)$$

The identification of the limiting $\mathcal{P}_+^1(3, 1)$ representation is simplified by the following observation. The conditions (6.6a-c) give rise, as conditions (6.5a-b), to infinitesimal unitary characters of the one dimensional subgroups of space translations. They are clearly trivial. Hence, the limiting representation is nothing but the UIR of $\mathcal{P}_+^1(3, 1)$ induced from the unitary character $e^{i(m\tau+ns\tau')}$ of the subgroup $SO(2) \otimes_s T_{3,1}$. Here $T_{3,1}$ stands for the subgroup of spacetime translations and $\otimes_s$ denotes the semi-direct product. The representation space is then the space of square integrable functions on $SO_0(3,1)/SO(2)$.

We actually have a realization of this representation. In fact, the Hilbert space $\mathcal{H}_0^{m,s}$ is the space of $L^2$ functions on $SO_0(3,1)$ satisfying (6.7). The measure is the left Haar measure on $SO_0(3,1)$. Clearly, the phase factor $e^{ip\cdot x}$ in (6.7) does not influence the square integrability, however it is a crucial ingredient for the realization of the unitary action of $\mathcal{P}_+^1(3,1)$ in $\mathcal{H}_0^{m,s}$. The generators of this action are explicitly obtained through the contraction of the quantum observables $\hat{L}_{\alpha\beta}$ given in (4.4). Actually,

$$\hat{H} \equiv \lim_{\kappa \to 0} (\kappa \hat{L}_{50}) = i \frac{\partial}{\partial x^0}, \quad \hat{P}_i \equiv \lim_{\kappa \to 0} (\kappa \hat{L}_{i5}) = -i \frac{\partial}{\partial x^i}, \quad (6.8)$$

and

$$\hat{L}_{\mu\nu}^0 \equiv \lim_{\kappa \to 0} \hat{L}_{\mu\nu} = -i \left[(x_{\mu} \frac{\partial}{\partial x^\nu} - x_{\nu} \frac{\partial}{\partial x^\mu}) + (x \to p) + (x \to s) + (x \to a) + (x \to b)\right], \quad (6.9)$$

$i \in \{1, 2, 3\}$ and $\mu, \nu \in \{0, 1, 2, 3\}$. One then easily verifies that $\hat{H}$, the $\hat{P}_i$'s and the $\hat{L}_{\mu\nu}$'s realize the Poincaré Lie algebra. The action of these operators on the functions $\Phi$ in (6.7) is as follows,

$$\hat{H}\Phi(x, p, \zeta) = e^{ip\cdot x} (\rho^0 \phi)(p, \zeta), \quad (6.10a)$$

$$\hat{P}_i \Phi(x, p, \zeta) = e^{ip\cdot x} (\rho^i \phi)(p, \zeta), \quad (6.10b)$$

$$\hat{L}_{\mu\nu}^0 \Phi(x, p, \zeta) = -ie^{ip\cdot x} \left[(p_{\mu} \frac{\partial}{\partial p^\nu} - p_{\nu} \frac{\partial}{\partial p^\mu}) + (\zeta_{\mu} \frac{\partial}{\partial \zeta^\nu} - \zeta_{\nu} \frac{\partial}{\partial \zeta^\mu})\right] \phi(p, \zeta); \quad (6.10c)$$

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\(i \in \{1, 2, 3\}\) et \(\mu, \nu \in \{0, 1, 2, 3\}\).

Let us summarize. By contracting the UIR \((\mathcal{H}^{m,s}_\kappa, \mathcal{U}^{m,s}_\kappa)\) of \(SO_0(3, 2)\) we obtain a UIR representation of \(\mathcal{P}_+^\dagger(3, 1)\) by means of a reinterpretation of the Kähler polarization conditions that become real in the \(\kappa \to 0\) limit. In other words the holomorphic induction becomes the usual induction. In fact, the limiting representation is induced from the unitary character \(e^{i(m\tau + s\tau')}\) of the subgroup \(SO(2) \otimes_s T_{3,1}\), which is trivial for the subgroup of space translations. This result is in perfect agreement with the one obtained by direct geometric quantization of \(\mathcal{O}_P(m, s) [\text{Re}]\) (see also [Ra2]).

7. Conclusions

In this concluding section let us first make some comments about the limiting states obtained in (5.8) using the results of section 6. Clearly, they are of the form (6.7). In fact,

\[
\Phi_{x', p', \zeta'}(x, p, \zeta) = e^{ip\cdot x} \left[ m^2 p^0 \delta(p - p') e^{-ip'\cdot x'} \left( \frac{\bar{\zeta}' \cdot \zeta}{2} \right)^s \right].
\]

One easily verifies that they are generalized eigenstates of \(\hat{H}\) and \(\hat{P}_i\) and that they satisfy (6.10c). They are generalized states defined on \(\mathcal{H}^{m,s}_0\). In particular, the state \(\Phi_0 \equiv \Phi_{x', p(0), \zeta(0)}\), for \(p(0) = (m, 0, 0, 0)\) and \(\zeta(0) = (0, 1, -i, 0)\), is a generalized eigenstate of \(\hat{H}\) and \(\hat{P}_{12}\), with eigenvalue \(m\) and \(s\), respectively. All states of the form given above belong to an orbit (of generalized states on \(\mathcal{H}^{m,s}_0\)) of the UIR obtained by contraction. This orbit is the zero curvature limit of \(\mathcal{O}_{\varphi_0}\).

The loss of the notion of optimal localization in the zero curvature limit, reflects the non-existence of such a notion for a \(\mathcal{P}_+^\dagger(3, 1)\)-invariant theory. Moreover, it clearly arises as a consequence of the break down of the Kähler charater of the \(SO_0(3, 2)\)-invariant polarization when \(\kappa \to 0\). In order to recover this notion, one needs to introduce a fundamental length, a positive constant curvature in our case. This observation confirms the regularizing role of the \(SO_0(3, 2)\)-invariant theories as alternatives to the \(\mathcal{P}_+^\dagger(3, 1)\)-invariant ones.

Acknowledgements

The authors thank J.-P. Gazeau for countless stimulating discussions. A. M. E. thanks S.T. Ali and V. Hussin for their hospitality at Concordia University and Université de Montréal where this paper has been completed.

References
[Ar] R. Arens, *Classical Lorentz invariant particles*, J. Math. Phys. **12**, 2415-2422, 1971.

[AIS] S. J. Avis, C. J. Isham and D. Storey, *Quantum field theory in anti-de Sitter space-time*, Phys. Rev. **D18**, 3565-3576, 1978.

[BEGG] R. Balbinot, M.A. El Gradechi, J.-P. Gazeau and B. Giorgini, *Phase spaces for quantum elementary systems in de Sitter and Minkowski spacetimes*, J. Phys. A: Math. Gen. **25**, 1185-1210, 1992.

[BFFS] B. Binegar, M. Flato, C. Fronsdal and S. Salamó, *De Sitter and conformal field theories*, Czech. J. Phys. **B 32**, 439-471, 1982.

[BLL] H. Bacry and J.M. Lévy-Leblond, *Possible kinematics*, J. Math. Phys. **9**, 1605-1614, 1968.

[CB] Y. Choquet-Bruhat, *Global solutions of Yang-Mills equations on anti-de Sitter spacetime*, Class. Quantum Grav. **6**, 1781-1789, 1989.

[CDB] C. Cishahayo and S. De Bièvre, *On the contraction of the discrete series of $SU(1,1)$*, to appear in Ann. Inst. Fourier.

[CPSW] M. Couture, J. Patera, R. T. Sharp and P. Winternitz, *Graded contractions of $sl(3,\mathbb{C})$*, J. Math. Phys. **32**, 2310-2318, 1991.

[De] R. Delbourgo, *Minimal uncertainty states for the rotation and allied groups*, J. Phys. A: Math. Gen. **10**, 1837-1846, 1977.

[Di] W. G. Dixon, *Dynamics of extended bodies in general relativity : I. Momentum and angular momentum*, Proc. Roy. Soc. London **A314**, 499-527, 1970.

[Do] A. H. Dooley, *Contractions of Lie groups and applications to analysis*, in: Topics in modern harmonic analysis, Vol. I, 483-515, (Instituto Nazionale di Alta Matematica Francesco Severi, Roma 1983).

[DB1] S. De Bièvre, *Causality and localization in relativistic quantum mechanics*, in Proceedings of the conference Trobades Scientifiques de la Mediterrania, 234-241, 1985.

[DB2] S. De Bièvre, *Scattering in Relativistic Particle Mechanics*, PhD thesis, University of Rochester, 1986.

[DBE] S. De Bièvre and M. A. El Gradechi, *Quantum mechanics and coherent states on
the anti-de Sitter spacetime and their Poincaré contraction, to appear in Ann. Inst. H. Poincaré: Phys. Théo., 1992.

[DBEG] S. De Bièvre, M. A. El Gradechi and J.-P. Gazeau, *Phase space description of a quantum elementary system on the anti-de Sitter spacetime and its contraction*; to appear in Proceedings of the 18th ICGTMP, Moscow 1990.

[DF] R. Delbourgo and J. R. Fox, *Maximum weight vectors possess minimal uncertainty*, J. Phys. A: Math. Gen. 10, L223-L235, 1977.

[DS] W. De Sitter, *On the relativity of inertia. Remarks concerning Einstein’s latest hypothesis*, Proc. K. Akad. Wet. Amsterdam 19, 1217-1225, 1917 and *On the curvature of space*, Proc. K. Akad. Wet. Amsterdam 20, 229-243, 1918.

[E1] M. A. El Gradechi, *Théories classique et quantique sur l’espace-temps anti-de Sitter et leurs limites à courbure nulle*, Thèse de Doctorat de l’Université Paris 7, Paris 1991 (unpublished).

[E2] M. A. El Gradechi, *A geometric characterization of masslessness for the anti-de Sitter kinematics*, in preparation.

[Ev] N. T. Evans, *Discrete series for the universal covering group of the 3+2 de Sitter group*, J. Math. Phys. 8, 170-184, 1967.

[Fr1] C. Fronsdal, *Elementary particles in a curved space*, Rev. Mod. Phys. 37, 221-224, 1965.

[Fr2] C. Fronsdal, *Elementary particles in a curved space. II*, Phys. Rev. D10, 589-598, 1974.

[FH] C. Fronsdal and B. Haugen, *Elementary particles in a curved space. III*, Phys. Rev. D12, 3810-3818, 1975.

[Gi] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, (Wiley, New York, 1974).

[GH] J.-P. Gazeau and M. Hans, *Integral-spin fields on (3+2)-de Sitter space*, J. Math. Phys. 29, 2533-2552, 1988.

[H1] G. C. Hegerfeldt, *Remark on causality and particle localization*, Phys. Rev. D10, 3320-3321, 1975.

[H2] G. C. Hegerfeldt, *Violation of causality in relativistic quantum theory?*, Phys. Rev. Lett. 54, 2395-2398, 1985.
| Reference | Author(s) | Title and Source |
|-----------|-----------|------------------|
| He | A. Heslot | *Observables et Structure Sympléctique en Mécanique Classique et en Mécanique Quantique*, Thèse de Doctorat d’Etat, Université Paris VI, 1988. |
| Hu | N. E. Hurt | *Geometric Quantization in Action*, (D. Reidel Publishing Company, 1983). |
| HE | S. W. Hawking and G. F. R. Ellis | *The Large Scale Structure of Space-Time*, (Cambridge University Press, Cambridge 1973). |
| IW | E. Inönü and E. P. Wigner | *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U. S. 39, 510-524, 1953. |
| Ki1 | A. Kirillov | *Eléments de la Théorie des Représentations*, (Editions Mir, Moscou 1974). |
| Ki2 | A. Kirillov | *The method of orbits in representation theory*, in: Lie Groups and Their Representations, I. M. Gelfand (Ed.), 219-230, (Akadémiai Kiadó, Budapest 1975). |
| Ko | B. Kostant | *Quantization and unitary representations*, Lect. Not. Math. 170, 87-207, (Springer-Verlag, New York 1970). |
| Ku | H. P. Künzle | *Canonical dynamics of spinning particles in gravitational and electromagnetic fields*, J. Math. Phys. 13, 739-744, 1972. |
| LM | P. Libermann and C.M. Marle | *Symplectic Geometry and Analytical Mechanics*, (D. Reidel Publishing Company, 1987). |
| LN | M. Lévy-Nahas | *Deformation and contraction of Lie algebras*, J. Math. Phys. 8, 1211-1222, 1967. |
| M | L. Martinez-Alonso | *Group-theoretical foundations of classical and quantum mechanics. II. Elementary systems*, J. Math. Phys. 20, 219-230, 1979. |
| Ma | G. W. Mackey | *On the analogy between semisimple Lie groups and certain related semi-direct product groups*, in: Lie Groups and Their Representations, I. M. Gelfand (Ed.), 339-364, (Akadémiai Kiadó, Budapest 1975). |
| MN | J. Mickelsson and J. Niederle | *Contractions of representations of de Sitter groups*, Comm. Math. Phys. 27, 167-180, 1972. |
| NW | T. D. Newton and E. P. Wigner | *Localized states for elementary systems*, Rev. Mod. Phys. 21, 400-406, 1949. |
[Pe] A. Perelomov, *Generalized Coherent States and their Applications* (Springer Verlag, Berlin 1986).

[PW] T. O. Philips and E. P. Wigner, *De Sitter space and positive energy*, in: Group Theory and Its Applications, E. M. Loebel (Ed.), 631-676, (Academic Press, New York 1968).

[Ra1] J. H. Rawnsley, *De Sitter symplectic spaces and their quantizations*, Proc.Camb. Phil. Soc. **76**, 473-480, 1974.

[Ra2] J. H. Rawnsley, *Representations of a semi-direct product by quantization*, Math. Proc. Camb. Phil. Soc. **78**, 345-350, 1975.

[Re] P. Renouard, *Variétés Symplectiques et Quantification*, Thèse Orsay, 1969.

[Sa] J. Saletan, *Contraction of Lie groups*, J. Math. Phys. **2**, 1-21, 1961.

[Sc] W. Schmid, *$L^2$-cohomology and the discrete series*, Ann. Math. **103**, 375-394 (1976).

[So] J. M. Souriau, *Structure des Systèmes Dynamiques*, (Dunod 1970).

[SM] E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective*, (J. Wiley & Sons, 1974).

[SW] D. J. Simms and N. M. J. Woodhouse, *Lectures on Geometric Quantization*, Lecture Notes in Physics **53**, (Springer-Verlag 1976).

[W] A. S. Wightman, *On the localizability of quantum mechanical systems*, Rev. Mod. Phys. **34**, 845-872, 1962.

[Wi1] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40**, 149-204, 1939.

[Wi2] E. P. Wigner, *Some remarks on the infinite de Sitter space*, Proc. Nat. Acad. Sci. U. S. **36**, 184-188, 1950.

[Wo] N. M. J. Woodhouse, *Geometric Quantization*, (Clarendon Press, Oxford 1980).