Axion cosmology in the presence of non-trivial Nambu-Goldstone modes

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(Dated: May 2, 2022)

Axion cosmology is reexamined taking into account effect of kinetic pseudo Nambu-Goldstone modes, with its importance recently pointed out. When Peccei-Quinn (PQ) symmetry is broken by a chiral U(1) singlet, it is found that the effect of kinetic Nambu-Goldstone mode makes the axion dark matter untenable. When PQ symmetry is extended and is broken by two singlets, we find axion cosmology to work, but there are several differences from the axion cosmology studied in the literature. The differences are (1) ordinary type of dark matter scaling as 1/cosmic scale factor arising from a modulus field and not from the usual angular field, (2) mass of the dark matter quantum in the ultralight range, (10^{-32} \sim 10^{-14}) \text{ eV}, (3) emergence of dark energy with the present density of order (a few meV)^4 consistent with observations, (4) presence of a long range spin-dependent force, (5) slow-roll inflation after PQ symmetry breaking when conformal coupling to gravity is introduced.

I. INTRODUCTION

The invisible axion [1, 2] has been introduced to recover the status of appealing solution to the problem of strong CP violation [3], when the original axion at electroweak scale has been excluded experimentally. Fortunately it was later found that the invisible axion may become one of the leading candidates for dark matter of our universe [4, 5]. The clue for this development is broken chiral Peccei-Quinn (PQ) symmetry which generates pseudo Nambu-Goldstone (NG) field by QCD effect [6]. The symmetry breaking scale $f_a$ may be taken at will, but it is constrained by overclosure of the cosmic energy density and by new cooling mechanism in stellar evolution. If the PQ breaking scale $f_a$ is in the range of order $10^{12}$ GeV, it was argued that the axion may become the major component of dark matter close to the closure density [4].

It was recently pointed out in another context [6] that kinetic Nambu-Goldstone modes, usually not considered in the mean field approach, give rise to a centrifugal repulsion contributing to effective potential, thereby changing the potential minimum to determine the symmetry breaking scale. When this idea is applied to a class of scalar-tensor gravity of cosmology, it is found [7] that the mechanism provides a simultaneous realization of dark energy and dark matter.

It is readily recognized that the same effect of kinetic NG modes changes the standard scenario of axion cosmology. The first part of the present work examines this problem, and indeed we find that the axion cosmology fails in its original form. We then present extension of PQ symmetry breaking scheme by introducing two PQ singlets that enlarge the symmetry of scalar fields. This extension saves the problem of the conventional axion cosmology, but it brings in several new aspects to axion cosmology; most notably, emergent dark energy along with a new form of dark matter, with its quantum mass in the ultralight range less than $O(10^{-14})$ eV. It has been advocated that ultralight dark matter is appealing from a variety of different motivations from ours, [7, 8, 9].

The present paper is organized as follows. In Section II we explain effect of kinetic Nambu-Goldstone modes and show how this effect influences the axion cosmology discussed in the literature. In Section III Peccei-Quinn symmetry is extended for a richer structure of NG modes in order to evade the difficulty of conventional axion cosmology. Section IV is the main part of this work that presents axion cosmology in the extended model, stressing a number of different consequences from the conventional result. In Section V we mention that introduction of conformal coupling to gravity realizes a slow-roll inflation after PQ symmetry breaking. The paper ends with a brief summary.

We use the natural unit of $\hbar = c = 1$ throughout the present work unless otherwise stated.

II. KINETIC NAMBU-GOLDSTONE MODES AND PROBLEM OF AXION COSMOLOGY

A. Nambu-Goldstone modes

Let us ignore QCD effect for simplicity and concentrate on NG modes, in particular their kinetic contributions. PQ chiral symmetry is realized in the simplest invisible axion model by a complex singlet $\phi$ of appropriate U(1) charge, $\phi = \chi e^{i\theta}$ with modulus and angular fields, $\chi, \theta$, both taken real. Potential $V(\phi) = V(\chi)$ is a function of the modulus field $\chi$ and is independent of the angular field $\theta$. Axion field is identified as $\chi \theta$, becoming $f_a \theta$ after spontaneous symmetry breaking where $f_a$ is the field value at PQ symmetry breaking. Kinetic terms, however, include $\theta$ and are written as

$$\frac{(\partial \chi)^2}{2} + \chi^2 (\partial \theta)^2.$$  \hspace{1cm} (1)

Treating two fields, $\chi, \theta$, independently, the field equation for the angular field becomes $\partial(\sqrt{-g} \chi^2 \partial \theta) = 0$, where $g = 1$.
where a positive $-g$ is the metric determinant. This equation is integrated to give $\sqrt{-g}\,\chi^2\partial_\theta = c_\theta$ (integration constant independent of spacetime position). This constant is a quantum number of the angular momentum operator in the abstract two dimensional field space $(\mathcal{R}\phi, 3\phi)$. The generated centrifugal force may be incorporated in an effective potential as $c_\theta^2/(-g\cdot2\chi^2)$ [4]. The effective potential $V_{\text{eff}}$ has the form,

$$V_{\text{eff}}(\chi) = \frac{c_\theta^2}{-g\cdot2\chi^2} + V(\chi). \quad (2)$$

Field location $\chi$ of potential minimum is shifted due to the presence of centrifugal repulsive potential $\propto c_\theta$. In cosmology this shift depends on the cosmic scale factor $R(t)$ since $-g = R^6(t)$ in the Robertson-Walker flat spacetime [10].

### B. Field and Einstein equations

Let us consider axion cosmology in which the potential is given by $V(\chi, \theta) = \lambda\chi^2(\chi^2 - 2f_a^2)/4 - m_a^2\chi^2\cos\theta$. A constant in the potential is fixed by fine-tuning the cosmological constant such that $V(0, \theta) = 0$. We took the QCD potential $\propto \cos\theta$ using the dilute instanton approximation, but the important points are the potential periodicity in $\theta$ variable and the proportionality to a small $m_a^2$, and not detailed form of the QCD potential. This potential has a minimum at $\theta = 2\pi\times$ an integer and $\chi = f_a$ giving the PQ symmetry breaking scale. Field and Einstein equations for spatially homogeneous modes read as

$$\ddot{\chi} + 3\frac{\dot{R}}{R}\chi = \frac{c_\theta^2}{R^6\chi^3} - (\lambda(\chi^2 - f_a^2) - 2m_a^2\cos\theta)\chi, \quad (3)$$

$$\ddot{\theta} + 3\frac{\dot{R}}{R}\dot{\theta} + \frac{1}{\chi}\dot{\chi}\chi = -m_a^2\sin\theta, \quad (4)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{6M_P^2} \left( \frac{\chi^2}{2} + \frac{c_\theta^2}{2R^6\chi^2} + \frac{\lambda}{4}\chi^2(\chi^2 - 2f_a^2) - m_a^2\chi^2\cos\theta + \rho_M \right), \quad (5)$$

where $\rho_M$ is the energy density of nearly massless and massive particles in the standard model. The dot in the present paper means time derivative. These equations are valid when QCD effects are turned on. $M_P = 1/\sqrt{16\pi G_N} \approx 1.72 \times 10^{18}$ GeV is the Planck energy in our convention.

Two major differences from axion cosmology in the literature [4] are (1) $c_\theta \neq 0$, (2) $\chi$ allowed to vary away from the constant $f_a$ and presence of $\chi/\chi$ term in eq. [4].

### C. Time evolution of axion and modulus fields

The potential minimum of $V_{\text{eff}}$ adiabatically (namely, slowly varying with time) shifts due to the centrifugal repulsion $c_\theta^2/(2R^6\chi^2)$, with the modulus field $\chi$ following the equation,

$$\chi (\lambda(\chi^2 - f_a^2) - 2m_a^2) = \frac{c_\theta^2}{R^6}\chi^3. \quad (6)$$

We took $\theta = 2\pi\times$ an integer (minimum location of angular variable). Without the centrifugal repulsion ($c_\theta = 0$ case) the minimum is determined by vanishing left hand side. A salient feature of centrifugal repulsion (the case $c_\theta \neq 0$) is an enormous change of the field value at potential minimum, since the scale factor $R$ varies by roughly of order $10^{12}$ at two epochs, QCD and the present.

The most serious problem of conventional axion cosmology concerns the axion angular field equation [4], in particular, the presence of $\chi$ dependent term, $2\dot{\chi}/\chi$. In order to quantify the effect of this term, let us first discuss time evolution of the modulus field $\chi$ governed by [4] neglecting the small term proportional to the axion mass squared, which is justified at epochs prior to QCD. Under the adiabatic approximation, there is a conservation law giving a constant energy $E$:

$$\frac{\chi^2}{2} + \frac{c_\theta^2}{2R^6\chi^2} + \frac{\lambda}{4}\chi^2(\chi^2 - 2f_a^2) = E. \quad (7)$$

The $\chi$ equation is in a closed form for constant $R$, and is analytically integrated, giving a solution,

$$\int_{\chi_0}^{\chi_0(t)} dx \frac{1}{\sqrt{2(E - V_{\text{eff}}(x))}} = t, \quad (8)$$

$$V_{\text{eff}}(\chi_0) = \frac{c_\theta^2}{2R^6\chi_0^2} + \frac{\lambda}{4}\chi_0^2(\chi_0^2 - 2f_a^2). \quad (9)$$

The function $\chi^2(t)$ is given by Jacobi’s elliptic function $\text{sn}(z; k)$ as

$$\chi^2(t) = f_a^2 \left( a - (a - b)\text{sn}^2 \left( \frac{\sqrt{a-c}}{2} f_a t; \sqrt{\frac{a-b}{a-c}} \right) \right), \quad (10)$$

where parameters $a > b > c$ are three real roots of the third order algebraic equation,

$$x^2(x - 2) + d^2 = (x - a)(x - b)(x - c), \quad (11)$$

$$x = \frac{\chi^2}{f_a^2}, \quad d^2 = \frac{1}{2R^6} \frac{c_\theta^2}{f_a^2}. \quad (12)$$

We illustrate three roots $(a, b, c)$ and $k$ parameter of the elliptic function in Fig. [1].

The Jacobi elliptic function $\text{sn}(z; k)$ has double periods in the complex $z$ plane, and the relevant time period is $2/(f_a\sqrt{a - c})$ times

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \quad (13)$$

with slowly increasing $\sqrt{a - c} \to 1.54 d^{1/3}$ as $d \to \infty$, and always larger than 2, as seen in Fig. [1]. Since it is natural to expect $d^2 \ll O(1)$ at the present cosmological
epoch, the time period remains of order $1/f_a$, giving a rapid oscillation of the modulus $\chi$ field.

It would be instructive to show two limiting cases of Jacobi solutions; $d \to 0^+$ and $d \to \infty$ limits. In the $d \to 0^+$ limit, $\sqrt{a - b} \sim \sqrt{2}$ and $k \to 1^-$, which gives

$$\chi^2 \to f_a^2 \left( a - (a - b) \tanh^2 \left( \frac{\sqrt{a - b}}{2} f_a t \right) \right). \quad (14)$$

This is a delta-function with its peak value $a$ over a background $b$. For a finite, but small $d$, the function is a periodic series of delta-functions. On the other hand, in the $d \to \infty$ limit, $\sqrt{a - b} \sim 1.0 \cdot d^{1/6}$ and $k \sim 0.65 \cdot d^{-1/6}$. The Jacobi solution is simplified to a sinusoidal function,

$$\chi^2 \to f_a^2 \left( a - (a - b) \sin^2 \left( \frac{\sqrt{a - b}}{2} f_a t \right) \right). \quad (15)$$

Approaches to these limiting functions are not fast, but these formulas may help to understand the nature of solutions.

With these in mind we analyze the axion angular evolution. In the small axion mass limit it is sufficient to use the approximate equation,

$$\ddot{\theta} + 3 \frac{R}{R_d^3} \dot{\theta} + \frac{2}{\chi^3} c_\theta = -m_a^2 \sin \theta. \quad (16)$$

This is a damped pendulum equation with accelerating force given by the third term $\propto c_\theta$ in the left hand side. Conditions for damped oscillation necessary for dark matter interpretation are $m_a > 3R/R$ (for oscillation start) and $2|c_\theta \chi^3| < m_a^2$ (condition to suppress continued pendulum rotation). The second condition is difficult to satisfy, because $|c_\theta \chi^3| \approx \sqrt{2} d$ at present and as numerically illustrated in Fig(2), $2|c_\theta \chi^3|$ can never be in the appropriate axion squared mass range, namely $< 1 \text{MeV}^2$ for its stability against $e^\pm$ pair annihilation.

The conclusion is robust despite that the adopted adiabatic approximation is not exact, and damped oscillation never occurs when kinetic pseudo Nambu-Goldstone modes are incorporated in the axion cosmology. The result holds even if $\chi$ mode decays.

III. EXTENDED PQ SYMMETRY BREAKING

Since PQ mechanism is attractive as a solution to the strong CP problem, we would like to explore extension of the original model [1]. A possible direction towards extension would be to attribute the potential shift to another field different from the axion. For this purpose we introduce a second spinless field, this time with no PQ-charge.

The simplest scheme toward improvements is to introduce two complex spinless fields, $\phi_a = \phi_1 + i\phi_2$ and $\phi_d = \phi_3 + i\phi_4$ with $\phi_i, i = 1, \cdots 4$ real fields, and identify PQ U(1) symmetry transformation as a subgroup of O(4) among four $\phi_i$. $\phi_a \to e^{i\alpha X} \phi_a, \phi_d \to \phi_d$. PQ charge $X$ is defined as in [1]. $\phi_d$ does not transform under PQ symmetry transformation, hence no direct coupling to two Higgs doublets, $H_u, H_d$. But $\phi_d$ field has a non-trivial coupling to $\phi_a$ field in the interaction lagrangian, $|\phi_a|^2 |\phi_d|^2$ arising O(4)-symmetric $(|\phi_a|^2 + |\phi_d|^2)^2$. We impose O(4) symmetry for the potential $V(\phi_i)$. To quartic orders, we choose

$$V(\chi) = \frac{\lambda_\phi}{4} \chi^2 (\chi^2 - 2f_a^2), \quad (17)$$

$$\phi_a = \chi_a e^{i\theta_a}, \quad \phi_d = \chi_d e^{i\theta_d}, \quad \chi^2 = \chi_a^2 + \chi_d^2. \quad (18)$$

The coupling potential to the Higgs sector $V_{\phi H} = V_{\phi H}(H_u, H_d, \phi_a)$ is assumed to have a partial O(4) symmetry transforming under U(1) PQ transformation,
Pseudo NG kinetic terms may be incorporated in the an effective potential using a few quadratic terms of six angular momentum components, $L_{ij}$, $i \neq j$, $i, j = 1 \sim 4$:

\[
V_{\text{NG}} = \frac{\sum_{(ij)} L_{ij}^2}{2 R^6 \chi^2} + \frac{L_{12}^2}{2 R^6 \chi_a^2} + \frac{L_{34}^2}{2 R^6 \chi_d^2}.
\]

This particular pattern of symmetry breaking was chosen for our purpose. The remaining symmetry is rotation among $(3,4)$ and $(1,2)$ components, with the latter being broken by QCD effect. The rest of kinetic terms is $\dot{\chi}_a^2/2 + \dot{\chi}_d^2/2$. We denote quantum numbers of angular momentum components by $c_a$, $c_d$, $c_\psi$ for three of $L_{12}^2$, $L_{34}^2$, $\sum_{(ij)} L_{ij}^2$.

\[
V_{\text{eff}}(\chi_a, \chi_d) = V_{\text{NG}} + V(\chi),
\]

\[
IV. \text{ COSMOLOGY OF EXTENDED AXION MODEL}
\]

A. Preliminary

Field equations for $\chi_i$, $i = a, d$ and the Einstein equation are

\[
\ddot{\chi}_a + \frac{3R}{R} \dot{\chi}_a = -\partial_{\chi_a} V_{\text{eff}}(\chi_a, \chi_d, \theta_a),
\]

\[
\ddot{\chi}_d + \frac{3R}{R} \dot{\chi}_d = -\partial_{\chi_d} V_{\text{eff}}(\chi_a, \chi_d, \theta_a),
\]

\[
\left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{6 M_p^2} \left(\frac{\dot{\chi}_a^2}{2} + \frac{\dot{\chi}_d^2}{2} + V_{\text{eff}}(\chi_a, \chi_d, \theta_a)\right),
\]

\[
V_{\text{eff}}(\chi_a, \chi_d, \theta_a) = \frac{c_a^2}{2 R^6 \chi_a^2} + \frac{c_d^2}{2 R^6 \chi_d^2} + \frac{c_\psi^2}{2 R^6 \chi_d^2} + \frac{\lambda_a}{4} \chi_a^2 \left(\chi_a^2 - 2 f_\chi^2\right) - m_a^2 \chi_a^2 \cos \theta_a.
\]

These field equations are for spatially homogeneous modes. For spatially inhomogeneous modes $\chi_i(\vec{x}, t)$, $i = a,d$ there are additional derivative terms $-\nabla^2 \chi_i/R^3$, which are separately discussed later. QCD term, $-m_a^2 \chi_a^2 \cos \theta_a$, is proportional to a small squared mass $m_a^2$, and approximate this vanishing for some parts of subsequent discussion. Angular fields obey $\dot{\theta}_d = c_d/(R^3 \chi_d^2), \psi = c_\psi/(R^3 \chi_d^2)$, while the equation for the axion angular field $\theta_a$ is more complicated and shall be discussed later.

We focus on cosmic time evolution at epochs when ordinary radiation and matter particle contributions $\rho_m$ are negligible compared to dark fields. The method we resort in these regions is the adiabatic approximation, namely we assume that the scale factor $R$ changes slowly, and approximate it as a constant for some finite time.

![Figure 2](image-url)
region. Potential minimum in this extended model is then given by two equations, $\partial \chi_a V_{\text{eff}} = \partial \chi_a V_{\text{eff}} = 0$, and reads as

$$\chi_a \left( \lambda_\phi (\chi^2 - f_a^2) - 2m_a^2 \right) = \frac{1}{R^6} \left( \frac{c_i^2}{\chi_a} + \frac{c_d^2}{\chi_d} \right), \quad (25)$$

$$\chi_d \lambda_\phi (\chi^2 - f_d^2) = \frac{1}{R^6} \left( \frac{c_i^2}{\chi_a} + \frac{c_d^2}{\chi_d} \right). \quad (26)$$

We included the major QCD term, $-2m_a^2$. Dividing each of these by $\chi_i$, $i = a, d$ and subtracting the second from the first or adding both, one derives

$$-2m_a^2 = \frac{1}{R^6} \left( \frac{c_i^2}{\chi_a} - \frac{c_d^2}{\chi_d} \right), \quad (27)$$

$$2 \left( \lambda_\phi (\chi^2 - f_d^2) - m_a^2 \right) = \frac{1}{R^6} \left( \frac{c_i^2}{\chi_a} + \frac{c_d^2}{\chi_d} + 2\frac{c_i^2}{\chi_a} \right). \quad (28)$$

Thus, the quantity to be fine-tuned to the small axion mass is the difference $c_i^2/\chi_a - c_d^2/\chi_d$, and the total sum, the right hand side of eq. (28), should be large of order, $\lambda_\phi (\chi^2 - f_d^2)$. In $m_a \to 0$ limit the global bifurcation into two modes is given by

$$\chi_1^2 = \lambda_\phi^4 (\chi^2 - f_d^2) = \frac{c_i^2}{c_a + c_d} (c_i + c_d)^2 \frac{c_i^2}{R^6}, \quad (29)$$

$$\lambda_\phi \chi_4^2 (\chi^2 - f_d^2) = \lambda_\phi \chi_4^2 (\chi^2 - f_d^2) = \frac{2 \sqrt{t_i}}{c_d}, \quad (30)$$

For a finite $m_a$ there are additional terms, $\delta \chi_i^2 \propto m_a^2$. One needs $c_a > c_i$ for $\chi_2^2 \propto \chi_4^2$. The limit $c_a \to 0$ seems to give back the original axion cosmology. But there are a number of differences, for instance $\lambda_\phi \propto \chi_2^2$. We shall explain these as we proceed. We denote extrema thus found by $(\chi_1^{(0)}(R), \chi_4^{(0)}(R))$.

Away from extrema, two field equations for $\chi_a$ and $\chi_d$ are separately integrated when $m_a = 0$, to give two integration constants, $E_i, i = a, d$.

$$\chi_a^2 - \frac{c_a}{2R^6 \chi_a^2} + \frac{c_i^2}{2R^6 \chi_a^2} + \frac{\lambda_\phi c_i^2}{4 \chi_a^2} (\chi^2 - 2f_d^2) = E_a, \quad (31)$$

$$\chi_d^2 - \frac{c_d}{2R^6 \chi_d^2} + \frac{c_i^2}{2R^6 \chi_d^2} + \frac{\lambda_\phi c_i^2}{4 \chi_d^2} (\chi^2 - 2f_d^2) = E_d. \quad (32)$$

There are two ways to analyze this $(\chi_a, \chi_d)$ system: (1) expansion of the effective potential $V_{\text{eff}}$ around extrema, (2) potential dominance approximation. The second method is most useful at latest epochs of evolution.

**B. Expansion around extrema**

We shall first discuss intermediate epoch of scalar dominance using the method (1). Deviation from extrema $(\chi_a^{(0)}(R), \chi_d^{(0)}(R))$ are denoted by $(\delta \chi_a, \delta \chi_d)$ in terms of which the conservation equation is approximately

$$\frac{(\delta \chi_a)^2}{2} + \delta \chi_a \left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a^2} \right)_0 \delta \chi_a + \delta \chi_a \left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \chi_d} \right)_0 \delta \chi_d = E_a, \quad (33)$$

$$\frac{(\delta \chi_d)^2}{2} + \delta \chi_d \left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d^2} \right)_0 \delta \chi_d + \delta \chi_d \left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d \partial \chi_d} \right)_0 \delta \chi_d = E_d, \quad (34)$$

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a^2} \right)_0 = \frac{1}{R^6} \left( 3 \frac{c_i^2}{\chi_a} + \frac{(c_a + c_d)^2}{4} \right) + 4(\lambda_\phi + \frac{c_i^2}{\chi_a}), \quad (35)$$

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d^2} \right)_0 = \frac{1}{R^6} \left( 3 \frac{c_i^2}{\chi_d} + \frac{(c_a + c_d)^2}{4} \right) + 4(\lambda_\phi + \frac{c_i^2}{\chi_d}), \quad (36)$$

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \chi_d} \right)_0 = 4 \chi_a \chi_d (\lambda_\phi + \frac{c_i^2}{\chi_a}). \quad (37)$$

All quantities in $(0)$ are taken at $\chi^{(0)}(R)$.

Simplification of these formulas is made under the hierarchical parameter relation, $c_\psi \gg c_d \gg c_a$:

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d^2} \right)_0 \approx 4 \lambda_\phi f_a^2 + 8 \frac{c_i^2}{c_d R^6}, \quad (38)$$

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \chi_d} \right)_0 \approx 4 \frac{f_a}{c_d} (\lambda_\phi + \frac{c_i^2}{\chi_a}) \chi_d, \quad (39)$$

$$\left( \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a^2} \right)_0 \approx 4 \frac{c_i}{c_d} \lambda_\phi \chi_a^2, \quad (40)$$

by keeping at least one term for each of elements and using the simplified form of eq. (30):

$$\lambda_\phi \chi_d^2 (\chi_d^2 - f_d^2) \approx \frac{c_i^2}{R^6}. \quad (41)$$

This corresponds to eq. (6) in the usual one-singlet model.

The conservation law for $\delta \chi_d$ is readily integrated, to give oscillating solution,

$$\sqrt{\frac{E_d}{4(\lambda_\phi f_a^2 + 2c_i^2)}} \sin \left( \frac{t}{\sqrt{\lambda_\phi f_a^2 + 2c_i^2}} \right). \quad (42)$$

Related $\delta \chi_a$ is solved directly from the equation involving second derivatives,

$$\delta \chi_a = \frac{c_i}{c_d} \left( \chi_a + \frac{c_i^2}{c_d} \right) \delta \chi_d, \quad (43)$$

giving oscillatory behavior. A possible linear growth term $\propto t$ is forbidden by the conservation equation, eq. (33), and $\delta \chi_a$ oscillates with the same frequency as $\delta \chi_d$. These indicate stability of adiabatic solutions $(\chi_a^{(0)}, \chi_d^{(0)})$ with given constant scale factor $R$. The governing equation at the present epoch of $R(t_f) = 1$ may be written in terms of dimensionless quantities:

$$y^4(y^2 - 1) = e^2, \quad y = \frac{\chi_a}{f_a}, \quad e = \frac{c_i}{\sqrt{\lambda_\phi f_a}}. \quad (44)$$
Numerical solutions of this algebraic equation are illustrated in Fig. [3]. For $e \leq 0.1$ the solution gives $\chi_d/f_a \sim 1$. If $e$ is close to, or smaller than, unity, the adiabatic solution indicates that $\chi_d$ approaches to $f_a$ at the present epoch.

C. Masses of heavy and light quanta

We shall discuss this problem in general terms, going back to $(\delta \chi_d, \delta \chi_a)$ part of sub-matrix:

$$\mathcal{W} = -\begin{pmatrix} \partial^2 \chi_d V_{\text{eff}} & \partial \chi_d \partial \chi_a V_{\text{eff}} \\ \partial \chi_a \partial \chi_d V_{\text{eff}} & \partial^2 \chi_a V_{\text{eff}} \end{pmatrix}^{\text{background}}.$$  \hspace{1cm} (45)

This separation of two modes from angular modes is allowed in $m_a \to 0$ limit. The effective potential is a sum of three functions, $V_a(\chi_d^2) + V_q(\chi_a^2) + V_0(\chi_a^2 + \chi_d^2)$, hence

$$\partial^2 \chi_d V_{\text{eff}} = 4(V_q')\chi_d^2 + 2V_q' + 4(V_0')\chi_d^2 + 2V_0',$$  \hspace{1cm} (46)

$$\partial \chi_a \partial \chi_d V_{\text{eff}} = 4(V_q')\chi_a \chi_d.$$  \hspace{1cm} (47)

First derivative terms vanish at extrema, hence $2\chi_d(V_q' + V_0') = 0$ gives $V_q' + V_0' = 0$ unless $\chi_d = 0$. The secular equation of $\det(z + \mathcal{W}) = 0$ results in

$$z^2 - 4((V_q')\chi_d^2 + (V_0')\chi_d^2)z + 16\lambda_2 \lambda_1 (V_q')^2 (V_0')^2 (V_q''(V_0')' + (V_q')'V_0' + (V_q')V_0'') = 0.$$  \hspace{1cm} (48)

To go further, we must remember the hierarchical parameter relation $c_\phi \gg c_d \gg c_a$, which states that $z^2$ order terms are sub-dominant. It reduces the secular equation to an approximate one;

$$z^2 - 4((V_q')\chi_d^2 + (V_0')\chi_d^2)z = 0,$$  \hspace{1cm} (49)

to leading and the next leading orders. This result demonstrates that the first eigen-mode gives a heavy particle a mass $2\sqrt{(V_q')} = \sqrt{2}\lambda_2 f_a$, while the second has zero mass, meaning actually a small mass much less than $f_a$. More precisely, the light particle has a squared mass,

$$\chi_d \sim 4\lambda_2 \lambda_1 (V_q')^2 (V_0')^2 (V_q''(V_0')' + (V_q')'V_0' + (V_q')V_0'') \sim 4c_a c_d f_a.$$  \hspace{1cm} (50)

Near the present epoch this value is $\approx 4c_a c_d f_a$. We derive later an upper bound on this mass in eq. (72).

As we shall show in the next subsection, the heavy particle and the corresponding field mode quickly decay. This has an important consequence of evading the over-closure of cosmic density caused by heavy dark matter particles.

D. Induced $\chi_d$ coupling to standard model particles and its consequences

$\chi_d - \chi_a$ mixing via $\chi_a$ coupling to Higgs doublets $H_u, H_d$ is expected to induce an effective coupling of $\chi_d$ to standard model particles. These are important because inhomogeneous mode of $\chi_d$ as discussed in the following subsection has a mass very large and it may have a serious difficulty of over-closing the universe. In another word, $\chi_d$ matter, both homogeneous and inhomogeneous, must decay sufficiently fast.

First, note that our extended model has coupling term $\lambda_\phi|\phi_d|^2|\phi_a|^2/2$, which gives a three-point vertex $\lambda_\phi \chi_d \phi \chi_d \phi$, and particle mixing $4\lambda_\phi \chi_d \phi \chi_d \phi \chi_d$. U(1) PQ non-singlet $\phi_a$ has coupling to Higgs doublet, in the form $\chi_a \phi \chi_d |H_1|^2$, $i = u, d$, for simplicity. In these formulas $\chi_i$ are background fields and $\delta \chi_i$ are their quantum fields. Thus, there is an induced $\delta \chi_d$ coupling to Higgs doublets of the form,

$$g_a \delta \chi_d H_u^i c_d H_d^j, \quad g_a = 4\lambda_\phi \sqrt{c_a/c_d} f_a.$$  \hspace{1cm} (51)

This gives rise to $\delta \chi_d$ (of mass $M \sim 2\sqrt{\lambda_\phi} f_a$) decay into a Higgs pair, $H_u H_d$. Its decay rate is given by the formula,

$$\gamma_d = \frac{\lambda_\phi c_a}{16\pi c_d} M,$$  \hspace{1cm} (52)

ignoring small Higgs boson masses $\ll M$. Placing constraint that decay is completed prior to QCD epoch, $\gamma_d \geq H(\Lambda_{\text{QCD}}) \approx O(4)\Lambda_{\text{QCD}}^2/M_T$, gives

$$\frac{c_a}{c_d} \geq \frac{8\pi}{\lambda_\phi^2} O(4)\Lambda_{\text{QCD}}^2/M_T f_a.$$  \hspace{1cm} (53)

This is a condition that can be satisfied readily if $f_a$ is not too small, for instance, $f_a > 10^9$ GeV.

Finally, we consider induced coupling of the massless mode $\theta_d$ previously judged as decoupled from the rest. The question is that this mode couples to standard particles via the modulus $\chi_d$, inducing a pseudo-scalar long range force. The following field equation for this angular mode

$$\theta_d + 3\frac{R}{R} \dot{\theta}_d + 2\chi_d \dot{\theta}_d = 0,$$  \hspace{1cm} (54)

indicates its coupling to $\chi_d$ field, which in turn gives an induced coupling to Higgs doublets via $\chi_d - \chi_a$ mixing. Detailed form of this induced coupling and its magnitude is left to future work.

E. Potential dominated epoch and dark energy

So far we relied on the adiabatic approximation. We now turn to another approximation which focuses on time evolution at latest times near the present. When kinetic energy $\chi_a^2/2 + \chi_d^2/2$ is much less than the potential energy $V_{\text{eff}}$, one can approximate the exact broken conservation equation,

$$\frac{d}{dt} \left( \frac{1}{2} \sum_i \chi_i^2/2 + V_{\text{eff}} \right) = -3 \frac{R}{R} \sum_i \frac{\chi_i^2}{2},$$  \hspace{1cm} (55)
by replacing \( \sum \chi_i^2/2 \) in the right hand side by \( E - V_{\text{eff}}, E = E_a + E_d \) and dropping time variation of kinetic terms in the left hand side. This leads to an approximate differential equation for the effective potential, when it is supplemented by the Einstein equation,

\[
\frac{d}{dt} V_{\text{eff}} = -3 \frac{1}{\sqrt{6} M_P} \sqrt{V_{\text{eff}}^2 (E - V_{\text{eff}})},
\]

This equation is readily integrated, to give for \( t \geq t_i \)

\[
\sqrt{\frac{V_{\text{eff}}(t)}{E}} = \tanh \left( A_i - \sqrt{\frac{3}{2} M_P} (t - t_i) \right),
\]

\[
R(t) = R(i) \cosh^{-1/3} \left( A_i - \sqrt{\frac{3}{2} M_P} (t - t_i) \right),
\]

\[
A_i = \arctanh \left( \frac{V_{\text{eff}}(t_i)}{E} \right).
\]

R(\( t_i \)) is determined by the present value of scale factor \( R(t_o) = 1 \).

This is the solution relevant to the accelerating phase of late universe. The Hubble constant at present \( R/R = H_0 \) may be used to relate the dark energy density \( E \) to other parameters. By calculating

\[
H_0 = \left( \frac{d}{dt} \ln R \right)_{t=t_0} = \sqrt{\frac{E}{6 M_P}} \sqrt{\frac{V_{\text{eff}}(t_0)}{E}},
\]

one derives

\[
E^{1/4} \sim 6^{1/4} (1 - R^{-6}(i))^{-1/4} \sqrt{H_0 M_P}
\sim 6^{1/4} \sqrt{H_0 M_P}, \quad \sqrt{H_0 M_P} \approx 2 \text{ meV}.
\]

Thus, independent of detailed evolution of scale factor and field, the present dark energy density is found of order (a few meV)\(^4\) consistent with observations. Time scale of variation is given by

\[
\sqrt{\frac{2}{3}} \frac{M_P}{\sqrt{E}} (1 - R^{-6}(i))^{1/2} \frac{1}{3 H_0} \approx \frac{1}{3 H_0},
\]

roughly of order the cosmic age. This should be compared to the oscillation period \( T_o \) discussed in Subsection B and C, which gave

\[
T_o = \frac{\pi f_a^2 R^3}{\sqrt{c_a c_d}}.
\]

Comparison of two formulas places a constraint,

\[
\frac{\sqrt{c_a c_d}}{f_a^2} > 3 \pi H_0, \quad H_0 \approx 10^{-33} \text{eV}.
\]

The dark energy density of the same order has also been derived by using a related, but somewhat different approach in the scalar-tensor gravity \[6\]. Results of two approaches are similar.

The dark energy density \( E \) given here may in principle include both of spatially homogeneous \( \chi_a \) and \( \chi_a \) modes, but as discussed above, the heavy \( \chi_d \) mode quickly decays if the inequality \( \| \chi_a \| \) is obeyed. Thus, the dark energy consists of stable light \( \chi_a \) mode. Its field value \( \chi_a \) is calculated by using the relation \( E = V_{\text{eff}}(\chi_a) \) (effective potential containing \( \chi_a \) mode alone). Under the inequality constraint below, the present value of \( \chi_a \equiv \chi_0 \) satisfies the equation,

\[
\left( \frac{c_a}{\chi_0} \right)^2 = \left( \frac{m_a}{\chi_0} \right)^2 + \lambda_\phi,
\]

\[
\left( \frac{m_a}{\chi_0} \right)^2 + \frac{3}{4} \lambda_\phi < \frac{6 (H_0 M_P)^2}{\chi_0^4}.
\]

Since \( \sqrt{H_0 M_P} \) is of order a few meV, solutions in the same energy range is most natural, giving both \( \chi_0^2, (c_a)^{1/3} \text{ of order meV, and } m_a \leq \text{ a few meV under the assumption, } \chi_\phi \text{ of order unity. The interesting possibility of } m_a \text{ much smaller than } O(\mu \text{eV) is not excluded.}

We now discuss field variation \( \chi_a \) which becomes important to discussion of axion dark matter in the next subsection. The best way to derive a relation between \( \chi_a/\chi_a \) and the Hubble rate \( R/R \) is to use the adiabatic
relation at late times, \( V_{\text{eff}}(\chi_a; R) = E \) (a constant), to obtain

\[
\frac{\dot{\chi}_a}{\chi_a} = \frac{6c_a^2}{R^6 \lambda_\phi \chi_a^6 R^3} \frac{\dot{R}}{R} + \frac{6\dot{\chi}_a^2}{\lambda_\phi \chi_a^6} \sim 6(1 + \frac{m_a^2}{\lambda_\phi \chi_a^2}). \tag{67}
\]

This implies that rate of \( \dot{\chi}_a \) variation is faster than the Hubble rate. This peculiar situation of seemingly kinetic dominance does not preclude an approximate constancy of dark energy density, which shall be discussed after we introduce our final model of conformal gravity in Section V.

**F. Spatially inhomogeneous modes as dark matter**

We next investigate time evolution of spatially inhomogeneous modes. By taking homogeneous modes \( \chi_{i,0}, \theta_{i,0}, i = a, d \) as a background, one analyzes inhomogeneous modes by treating them in perturbative expansion or performing the linearized approximation. Linearized modes are decomposed into modes of definite wave vectors: \( \delta \chi_{i,k}, \delta \theta_{i,k} \propto e^{i k \cdot \vec{r}} \). This is a valid procedure due to translational invariance of Robertson-Walker metric and homogeneous solutions. The field equations for amplitudes, \( \delta \chi_{i,k}, \delta \theta_{i,k}, i = a, d, \) are

\[
\left( \frac{d^2}{dt^2} + \frac{3}{R} \frac{dR}{dt} \frac{d}{dt} + \frac{k^2}{R^2} \right) \begin{pmatrix} \delta \chi_{a,k} \\ \delta \theta_{a,k} \\ \delta \chi_{d,k} \\ \delta \theta_{d,k} \end{pmatrix} = W \begin{pmatrix} \delta \chi_{a,k} \\ \delta \theta_{a,k} \\ \delta \chi_{d,k} \\ \delta \theta_{d,k} \end{pmatrix}, \tag{68}
\]

where

\[
\begin{pmatrix} V_{\chi a} \\ V_{\theta a} \\ V_{\chi d} \\ V_{\theta d} \end{pmatrix} = \begin{pmatrix} \frac{\chi_{a,k}}{\chi^2_a} \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a^2} & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \chi_d} & 0 & 0 \\ \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \theta_a} & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a^2} & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_a \partial \chi_d} & 0 \\ 0 & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d \partial \chi_a} & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d^2} & \frac{\partial^2 V_{\text{eff}}}{\partial \chi_d \partial \chi_d} \\ 0 & 0 & \frac{\partial^2 V_{\text{eff}}}{\partial \theta_a^2} & \frac{\partial^2 V_{\text{eff}}}{\partial \theta_a \partial \theta_d} \end{pmatrix} \begin{pmatrix} \delta \chi_{i,k} \\ \delta \theta_{i,k} \\ \delta \chi_{i,k} \\ \delta \theta_{i,k} \end{pmatrix}. \tag{69}
\]

The \( \delta \theta_d \) mode appears to decouple from the rest, giving a long range massless field. As discussed in Subsection D, there exists, however, an induced coupling of \( \delta \theta_d \) to standard model particles via Higgs doublets, leading to a long range spin-dependent force.

The rest of three components (\( \delta \chi_{a,k}, \delta \theta_{a,k}, \delta \chi_{d,k} \)) may or may not be coupled, and this system is analyzed using \( 3 \times 3 \) sub-matrix \(-W\) in the form,

\[
\begin{pmatrix} 4\sqrt{c_d/c_a} \chi_a & \frac{c_d}{c_a} f_a & 0 \\ \frac{c_d}{c_a} f_a & -m_a^2 & 0 \\ 0 & 4\chi_a & 4\chi_a + 8\sqrt{c_d/c_a} f_a \end{pmatrix} \begin{pmatrix} 4\sqrt{c_d/c_a} \chi_a & \frac{c_d}{c_a} f_a & 0 \\ \frac{c_d}{c_a} f_a & -m_a^2 & 0 \\ 0 & 4\chi_a & 4\chi_a + 8\sqrt{c_d/c_a} f_a \end{pmatrix}. \tag{70}
\]

The three component system is thus further decomposed into a single angular mode \( \delta \theta_{a,k} \) and two-component coupled system (\( \delta \chi_{a,k}, \delta \chi_{d,k} \)).

The inhomogeneous angular field \( \delta \theta_{a,k} \) satisfies the differential equation,

\[
\delta \theta_{a,k} + (3 \frac{\dot{R}}{R} + 2 \frac{\dot{\chi}_a}{\chi_a}) \delta \theta_{a,k} = -m_a^2 \frac{\chi_a}{\chi^2_a} \delta \theta_{a,k}. \tag{71}
\]

The presence of the third term \( \propto \dot{\chi}_a/\chi_a \) is characteristic of pseudo Nambu-Goldstone mode. As discussed in the preceding subsection, this term, with eq. (67), damps the oscillation faster than the Hubble expansion rate \( 3R/R \). The modified picture of dark axion field at latest times is as follows. When \( 2\chi_a/\chi_a \) becomes smaller than \( m_a \), the field starts to oscillate with frequency \( \sqrt{m_a^2 + \chi^2}/R^2 \) and amplitude \( \propto 1/R^6 \). This inverse power \(-6\) is subject to the precise \( \chi_a \) variation rate at late times, but it appears that the decrease rate is faster than that of radiation. Note, however, that there exists another candidate of cold dark matter behaving like an ordinary one with its number decrease rate \( \propto 1/R^3 \), as we shall see shortly.

The symmetry breaking bound \( f_a < O(10^{13}) \text{GeV} \) derived in the literature \([4]\) is based on a few assumptions: the epoch of axion damped oscillation starting at \( m_a = 3R/R \) with the number density decrease \( \propto 1/R^3 \), and the current algebra estimate relating axion parameters to pion parameters, \( f_a m_a \approx f_a m_{\pi} \) \([1\text{,} 12]\). Since the initial time of damped oscillation is changed at \( m_a \sim 6R/R \) and the damping rate \( \propto 1/R^6 \) is faster in our extended model, this quoted bound on \( f_a \) is no longer applicable.

The two-component system (\( \delta \chi_{a,k}, \delta \chi_{d,k} \)) consists of two particles moving with the momentum \( k/R \); a decaying heavy particle and a light stable particle forming a candidate of dark matter. The mass of light particle predominantly made of \( \chi_a \) mode is given by a time dependent function, eq. (50), and its value \( \sim 2\sqrt{c_a/c_d}/f_a^2 \) at the present epoch. In Subsection E it was suggested that \( c_a^{1/3} < f_a \) is of order, a few meV, which implies that the mass \( m_D \) of light particle is constrained by

\[
m_D = \sqrt{c_a/c_d} f_a \lesssim O(10^{-14}\text{eV})(\frac{10^{9}\text{GeV}}{f_a})^{1/2}, \tag{72}
\]

with \( c_d \ll c_a \approx f_a^2 \). As a reference \( f_a \) value we took astrophysics bound by stellar cooling derived in the standard axion cosmology \([13]\). This should be re-examined by using the coupling scheme of new dark matter different from the original axion model. The lower bound, eq. (64), gives the inequality, \( m_D > 3\pi H_0 \approx 10^{-32}\text{eV} \).

According eq. (64), the light dark matter particle has its mass inversely proportional to the scale factor to cube, hence at nucleo-synthesis it is likely that the mass is of order 100 GeV or slightly less, taking the Peccei-Quinn symmetry breaking scale at \( 10^{12}\text{GeV} \). There is no abundant particle close to thermal values at nucleo-synthesis from which this massive particle can be produced. The problem persists even at higher temperatures, because the heaviest particle in standard particle physics is Higgs boson. A likely production process is then radiative emission from excited ion states in cosmic plasma after nucleo-synthesis:

\[
\text{He}^+ + \gamma \rightarrow \text{He}^{+} + \chi, \tag{73}
\]

when the mass is below the ionization energy \( \sim 20\text{eV} \), and at redshifts below \( \sim 10^5 \). Produced energy spectrum
is non-thermal, and there is no significant interaction of produced dark matter particle with the rest of cosmic constituents. It is thus expected that a significant part of dark matter particles is in the non-relativistic region. Even if there exists a significant relativistic component, this part loses its role in the energy budget compared to the cold component due to its slower decrease rate.

Precise estimate of produced cold dark matter amount requires detailed calculation of radiative emission rate, which is beyond the scope of the present work.

An interesting possibility of detecting ultralight dark matter clouds surrounding black holes by using gravitational wave emission has been pointed out [5]. The method appears to work due to a recent announcement 

 aunual wave emission has been pointed out [5]. The

moral recombination epoch.

In conformal gravity the role of $w$ factor is less obvious due to a modified form of the energy-momentum conservation. In the Einstein frame of conformal gravity they take different forms:

$$T^{(E)}_{\mu\nu}(\chi) = \frac{5}{F} \left( \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \chi \partial^\alpha \chi \right) + g_{\mu\nu} \frac{V_\chi(\chi^2)}{F^2},$$

$$\rho_\chi^{(E)} = -(3 \frac{\dot{R}}{R} + \frac{\partial_\chi F}{F} \chi) \rho_\chi^{(E)}(1 + w_\chi^{(E)}) + \frac{g_{\mu\nu} V_\chi(\chi^2)}{F^2},$$

$$w_\chi^{(E)} \rho_\chi^{(E)} = \rho_\chi^{(E)} = \frac{5}{2F} \left( \chi^2 - \frac{(\nabla \chi)^2}{3R^2} \right) - \frac{V_\chi(\chi^2)}{F^2}. $$

Derivation of the energy-momentum tensor in the Einstein frame, eq. (74), is given for instance in [19]. These relation holds separately for heavy and light inflaton, $\chi_d, \chi_a$.

With the presence of a large suppression due to the conformal factor $1/F(\chi)$ in the right-hand side of eq. (77), the energy density $\rho_\chi^{(E)}$ can decrease towards a constant, irrespective of $w_\chi^{(E)}$ values of order unity (but not necessarily equal to $-1$). Indeed, one can derive, by neglecting complicated contribution $3\dot{R}/(RF)$ in the right-hand side (which actually gives a faster approach towards the constant behavior), $\rho_\chi = \text{const.} \times \exp[5(1 + w_\chi^{(E)})/F]$ for a constant $w_\chi^{(E)}$ in the range $-1 \leq w_\chi^{(E)} \leq 1$. This explains why the energy density stays nearly constant, although there may be a non-trivial contribution from kinetic energy $\chi^2/2$ in $\rho_\chi^{(E)}$. Indeed, our model does not always give a stable value of $w_\chi^{(E)} = -1$ at some times of cosmic evolution.

Nevertheless, CMB anisotropy is sensitive to the pressure contribution $\rho_\chi^{(E)}$ around the recombination epoch. We shall therefore directly calculate the pressure and $w_\chi^{(E)}$ for the dark energy and CDM proposed in preceding subsections. Around the recombination epoch the dark energy is made of a light inflaton $\chi_a$, its homogeneous component, while CDM is made of inhomogeneous modes around this component $\delta \chi_a$. The dark energy field $\chi_a$ is given by eq. (29) and (41):

$$\chi_a^2 \sim \frac{c_a^c}{c_d^d} \frac{\chi^2}{R^6 \lambda_\phi} \frac{1}{\lambda_\phi^{1/3} c_d R^2}. \quad (78)$$

As argued around (65) and (66), this quantity $\chi_a$ is of order meV. The potential term contributes to the dark energy density, which is calculated by balancing the Nambu-Goldstone kinetic repulsion. The result is $3\lambda_\phi \chi_a^4/4$. The kinetic contribution from (67) is much smaller than this value. Therefore,

$$\rho_{DE} \sim \lambda_\phi \rho(\text{meV})^4, \quad (79)$$

and the equation-of-state factor $w_{DE} = -1$.

On the other hand, CDM inflaton $\delta \chi_a$ has a mass $m_D$ given by (72), and its Fourier-mode equation is given by

$$\delta \chi_a + 3 \frac{\dot{R}}{R} \delta \chi_a + \frac{q^2}{R^2} \delta \chi_a = -m_D^2 \delta \chi_a. \quad (80)$$

CDM energy density $\rho_{CDM}$ varies with time according to

$$\frac{d}{dt}(R^3 \rho_{CDM}) = \frac{1}{2} (\delta \chi_a)^2 \frac{d}{dt} \left( R^3 (m_D^2 + \frac{q^2}{R^2}) \right). \quad (81)$$

The right-hand side decreases fast enough to prove the CDM energy density decrease $\propto 1/R^3$ and effectively $w_{CDM} = 0$. Thus, CMB anisotropy spectrum is well described by $\Lambda$ CDM model in the present work as well.

V. OTHER IMPORTANT ISSUE OF COSMOLOGY

As discussed in [6], the scalar-tensor gravity including conformal coupling realizes both inflation and accelerating universe at late times. This brings in a new aspect to axion cosmology when applied to our extended O(4) model: inflation after PQ-symmetry breaking. We shall discuss this issue.

The conformal coupling of $\phi_d$ field to gravity is introduced in the Jordan-Brans-Dicke metric frame as [13]
\[ \mathcal{L} = \sqrt{-g} (\mathcal{L}_{\phi} + \mathcal{L}_{\text{EW}}(H_u, H_d, \phi, \psi, g_{\mu\nu}) + \mathcal{L}_{\text{QCD}}) \]  
\[ \mathcal{L}_{\phi} = -M_{\phi}^2 F(\chi_d) R + \frac{1}{2} \sum_{i=a,d} \partial \phi_i \partial \phi_i - V_\phi(\chi_d^2 + \chi_d^2) \] 
\[ F(\chi_d) = 1 + \xi \frac{\chi_d^2}{f_a^2} , \quad \xi > 0 , \] 
where \( R(g_{\mu\nu}) \) is the Ricci scalar curvature. If \( \xi = 0 \), this model coincides with the extended model in preceding sections. In a sense this new model is a merger of O(2) scalar-tensor gravity [6] and O(2) axion model into an enlarged O(4) symmetric scheme. It is often convenient to Weyl rescale into the Einstein metric frame by using a new metric \( g_{\mu\nu} = F(\chi_d) g_{\mu\nu} \):

\[ \sqrt{-g} \mathcal{L}_{\phi} = \sqrt{-g} \left( -M_{\phi}^2 R(g_{\mu\nu}) + \frac{5}{2F^2} \sum_{i=a,d} \partial \phi_i \partial \phi_i - \frac{1}{F^2} V_\phi \right) \]  

At inflationary epochs the \( \phi_d \) field is dominant over \( \phi_a \) field, hence we shall drop contributions from the \( \phi_a \) field. Scalar kinetic term \( \propto \partial \phi_i \partial \phi_i \) is transformed to the standard form by a field re-definition that introduces \( \chi_d \),

\[ \chi_d = \sqrt{\frac{5}{\xi}} \int_0^{\chi_d} \frac{du}{\sqrt{F(u)}} , \quad \chi_d = \frac{f_a}{\sqrt{\xi}} \sinh \left( \frac{\sqrt{\xi} \chi_d}{5 f_a} \right) . \]  

For small \( \chi_d, \chi_d \approx \sqrt{5} \chi_d \). Using \( V_\phi = \lambda_\phi \chi_d^2 (\chi_d^2 - 2f_a^2)/4 \), one has

\[ \frac{1}{F^2} V_\phi = \frac{\lambda_\phi f_a^4}{4\xi^2} (1 - \frac{1}{F}) (A - \frac{B}{F}) , \quad A = 1 , \quad B = 1 + 2\xi . \]  

The positivity of potential \( V(\chi_d) \) requires \( A - \frac{B}{F} > 0 \) or

\[ 1 + \xi \frac{\chi_d^2}{f_a^2} > \frac{B}{A} (= 1 + 2\xi) , \]  

which is readily satisfied if \( \chi_d > \sqrt{2} f_a \).

At earliest epochs of \( \chi_d > O(f_a/\sqrt{\xi}) \) the slow-roll inflation [17] may be realized as discussed in [6]. The slow-roll conditions are imposed on the potential derivatives [10]

\[ V_{\text{eff}}' \ll \frac{1}{\sqrt{\xi}} \frac{f_a}{M_P} , \quad \frac{V_{\text{eff}}''}{V_{\text{eff}}} \ll \frac{3}{2\xi} \left( \frac{f_a}{M_P} \right)^2 , \]  

\[ V_{\text{eff}}(\eta) = \frac{\lambda_\phi f_a^4}{4\xi^2} \frac{1}{(1 + \eta^2)^2} \times \left( A - \frac{B}{F} + \frac{B}{F^2} + \frac{2\xi^3 (c_0^2 + c_2^2)}{\lambda_\phi R^6 f_a^2} \frac{1}{\eta^2} \right) , \]  

where the prime \( ' \) indicates field derivative with respect to the dimensionless variable \( \eta = \sqrt{\xi} \chi_d/f_a (> 0) \). The two derivative conditions reduce to an identical inequality,

\[ \chi_d \gg 2M_P . \]  

If \( \chi_d \) at inflation is of order \( f_a \), we derive \( f_a \gg 2M_P \). The slow-roll inflation ends around local minimum producing standard model particles in thermal equilibrium.

Thus, inflation, dark energy, and dark matter can be a manifested pattern of bifurcated symmetry breaking with cosmological evolution, as pointed out in [6], but this time solving the strong CP problem at the same time. Nevertheless, the fine-tuning of cosmological constant must be artificially introduced as usual.

Skeptics might question why one has to consider a higher energy state of non-vanishing centrifugal repulsion. Inflation after PQ symmetry breaking gives an answer, because our universe after inflation happens to be a fluctuation from the lowest energy state, as in the chaotic inflationary scenario. This view has a sort of anthropic flavor.

**VI. SUMMARY**

Kinetic Nambu-Goldstone modes have drastic effect to axion cosmology: the potential minimum that determines the field value of Peccci-Quinn symmetry breaking is shifted due to time dependent centrifugal repulsion caused by NG modes. This change makes the cosmology of original one-singlet model untenable, but its extension to two singlet models saves axion cosmology without losing the attractive feature as a solution to the strong CP problem. Nevertheless, axion cosmology based on the extended model is very different so that the major detection strategy of dark matter focused on \( \mu \)eV range [18] should be reconsidered, shedding more light on the ultralight mass range, \( 10^{-32} \sim 10^{-14} \) eV.

**Note added in proof**

After submitting this work for publication, we calculated the spectral index \( n_s \) and the tensor-to-scalar ratio \( r \) of curvature fluctuation in an extended model that incorporates a bare cosmological constant \( \Lambda \) by changing \( M_{\phi}^2 R \) in eq.(83) to \( M_{\phi}^2 (R + 2\Lambda) \). The fine-tuning condition of cosmological constant in the Einstein frame is given by

\[ 2\Lambda M_P^2 + \lambda_\phi f_a^4 \left( \frac{f_a}{M_P} \right)^2 \left( \frac{f_a}{M_P} \right)^2 - 1 = 0 , \]  

without any problem to impose.

The spectral index \( n_s \) and the tensor-to-scalar ratio \( r \) are given in terms of inflaton potential derivatives, and we calculated these, using formulas given in [20]. Results of the above model are as follows:

\[ r = \left( 2\lambda_\phi \xi \right)^2 \frac{f_a^4 \lambda_\phi^2}{\Lambda^2 M_P^2} ; \]  

\[ n_s - 1 + \frac{3}{8} r = 2\lambda_\phi \xi \frac{f_a^4 \lambda_\phi^2}{\Lambda^2 M_P^2} . \]  

Formulas are valid approximately at large values of field \( \chi_d \) during the slow-roll inflation. BICEP/Keck [21] and
Planck [22] observations indicate that \(n_s - 1 \sim 0.035, r \leq 0.036\). Our result is made consistent with these data by properly arranging field magnitude \(\chi_d\) during inflation for given parameters \(\Lambda, f_a\) and couplings \(\lambda, \phi, \xi\). The result is however model dependent, and if one takes a quartic form for conformal function \(F(\chi)\), results change, leaving more freedom. Detailed analysis shall be published elsewhere.

This result on \(n_s\) and \(r\) differs from interesting and related conformal \(\alpha\) attractor model of \[23].

ACKNOWLEDGMENTS

This research was partially supported by Grant-in-Aid 21K03575 from the Japanese Ministry of Education, Culture, Sports, Science, and Technology.

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