Data-driven integration of regularized mean-variance portfolios

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Abstract

Mean-variance optimization (MVO) is known to be highly sensitive to estimation error in its inputs. Recently, norm penalization of MVO programs has proven to be an effective regularization technique that can help mitigate the adverse effects of estimation error. In this paper, we augment the standard MVO program with a convex combination of parameterized $L_1$ and $L_2$ norm penalty functions. The resulting program is a parameterized penalized quadratic program (PPQP) whose primal and dual form are shown to be constrained quadratic programs (QPs). We make use of recent advances in neural-network architecture for differentiable QPs and present a novel, data-driven stochastic optimization framework for optimizing parameterized regularization structures in the context of the final decision-based MVO problem. The framework is highly flexible and capable of jointly optimizing both prediction and regularization model parameters in a fully integrated manner. We provide several historical simulations using global futures data and highlight the benefits and flexibility of the stochastic optimization approach.

Keywords: Data driven stochastic-programming, differentiable neural networks, mean-variance optimization, robust optimization, regularization

1 Introduction

The fundamental objective of portfolio optimization is the determination of next period’s optimal asset allocation under conditions of uncertainty. Markowitz [52], a pioneer of Modern Portfolio Theory, proposed that investors’ preferences for return and risk are characterized by quadratic utility: mean-variance optimization (MVO). The economic foundation for expected mean-variance utility as a rational choice theory are strong. In practice, however, the out-of-sample performance of MVO is underwhelming and can lead to portfolios that often underperform simple linear models [22, 56].

Computing optimal mean-variance portfolios requires, at a minimum, an estimate of expected (mean) returns and covariances. Generating a reliable estimator for asset mean returns is particularly difficult as asset returns are characterized as both nonstationary and heterogeneous [29, 37, 58, 64]. Moreover, unlike asset prices, asset returns themselves typically display no significant auto-correlation over any lag [26] and are generally estimated with a high degree of uncertainty [51]. While there exists several sophisticated and highly effective models for accurately estimating the second moment of asset returns, (see for example [13, 14]), providing an accurate estimate of the asset covariance
matrix, however, can also be challenging. Indeed, a covariance matrix of $N$ assets requires estimation of $N(N + 1)/2$ cross-covariance terms, and when the number of assets is large relative to the relevant number of historical observations available for estimation, then the resulting sample covariance matrix can be ill-conditioned, numerically unstable and a poor estimate of future covariance. As such, the sample covariance matrix and the action of its inverse is also often estimated with a high degree of uncertainty [10][48]. In practice we find that forecasting both the relative magnitude and direction of returns is difficult and leaves MVO portfolios highly susceptible to estimation error.

In this paper, we address the adverse effects of estimation error by augmenting the mean-variance objective with a norm penalty on portfolio weights. Specifically, we follow the work of Ho et al. [36] and consider a convex combinations of general $L_1$ and $L_2$ norm penalty functions; equivalent to elastic net regularization from regression [74], presented below.

$$P(z) = \alpha \gamma_1 \| E(\theta) z \|_1 + (1 - \alpha) \frac{\gamma_2}{2} \| D(\theta) z \|_2^2. \quad (1)$$

Here, $E(\theta)$ and $D(\theta)$ are parameterized sparsifying and regularization transforms and therefore Equation (1) generalizes the standard elastic net formulation. As outlined in Section 2, our choice of the generalized elastic net penalty function is deliberate and motivated by prior work that establishes the connection between norm penalized portfolio weights with covariance shrinkage and robust optimization [23, 36].

Mitigating the sensitivity of MVO portfolios to estimation error typically requires a user-defined regularization structure. The ‘optimal’ amount of regularization required is then estimated from the historical data in a variety of ways. For example, Ledoit and Wolf [49] proposed a linear shrinkage covariance estimate that is a convex weighting between the sample covariance estimate and a structured covariance matrix. The ‘optimal’ amount of shrinkage is then determined by minimizing the mean-square error of the estimator to the realized covariance. DeMiguel et al. [23], on the other hand, consider minimum-variance portfolios augmented by standard $L_1$ and $L_2$ norm penalty functions and propose optimizing the amount of norm-penalization by cross-validation. Alternatively Tutuncu and Koenig [69] consider robust MVO with a box uncertainty set structure and proposes calibrating the uncertainty bounds by bootstrapping the historical return distribution. In most cases, however, we find that the regularization structure is determined in advance by the user and the desired amount of regularization is often estimated independently from the original decision-based optimization problem, its inputs and constraints.

Recently, Butler and Kwon [18, 19] provide compelling evidence and advocate for a parameter estimation process that is made aware of it’s final decision-based optimization problem. Specifically, they present a stochastic optimization framework for integrating expected return and covariance predictions in a portfolio optimization setting. This is in contrast to a more traditional ‘predict, then optimize’ approach, in which prediction model parameters are optimized independently from the downstream decision-based optimization problem. They demonstrate that their integrated prediction and optimization (IPO) framework can result in lower out-of-sample costs and higher absolute and risk-adjusted performance in comparison to the decoupled alternative.

Motivated by the above mentioned work, in this paper we provide a fully-integrated, stochastic optimization framework for optimizing parameterized penalized quadratic programs (PPQPs), with direct applications to the regularized norm penalized MVO problem. The framework is flexible, and the choice of regularization structure (parameterized or not) is ultimately left to the user. The primary
advantage of the approach is that both the regularization structure and amount of regularization required are optimized in the context of the penalized MVO program such that the resulting portfolio decisions induce an optimal nominal mean-variance policy. As such, the regularization structures need not be fully-specified in advance and instead are optimized in a systematic, data-driven manner. To our knowledge, our stochastic optimization framework is the first to directly optimize parameterized regularization structures in a portfolio optimization setting. Our primary contributions and remaining of the paper is outlined as follows.

- In the following subsection we provide a brief review of the relevant literature in the field of robust and norm-constrained portfolio optimization and draw connections to related work from the fields of statistics and machine learning.

- In Section 2 we consider the norm-penalized MVO and re-cast the program as a parameterized penalized quadratic program (PPQP). We establish connections between our norm-penalized MVO problem and alternative forms of portfolio regularization such as covariance shrinkage, robust optimization under box and ellipsoidal uncertainty and regularized least-squares. We then present the stochastic optimization framework for optimization of PPQPs and structure the resulting problem as a neural-network whereby problem parameters, \((\theta, \lambda)\), are updated efficiently by backpropogating through differentiable QP layer(s).

- In Section 2.4 we demonstrate that the primal and dual form of our norm-constrained quadratic program can be re-cast as a constrained quadratic program (QP). We show that the QP solution is differentiable with respect to all relevant program variables and thus can be readily integrated as specialized layers in a larger neural-network structure. For the primal form, we consider a second-order approximation of the \(L_1\) norm and provide an efficient and numerically stable modified backward pass routine.

- In Section 3 we present a simulation study using global futures data, considering both unconstrained and constrained nominal MVO programs, and demonstrate the flexibility and effectiveness of the approach. We also consider a fully integrated setting whereby both prediction model and regularization parameters are jointly optimized with the objective of inducing optimal mean-variance decision-making. To our knowledge this is the first empirical study to consider the joint optimization of prediction and regularization variables in a fully integrated portfolio setting. In Section 4 we provide concluding statements and briefly discuss future areas of research.

- In summary, we find that in almost every case, the norm-penalized MVO portfolios provide improved out-of-sample MVO costs and higher risk-adjusted returns. In general, the parameterized norm models, in which the exact regularization structure is learned from the data, provide superior MVO costs and risk-adjusted performance in comparison to standard user-defined regularization methods. Lastly, we find that the joint integrated optimization of prediction and regularization result in improved economic outcomes over the standard ‘predict, then optimize’ approach.
1.1 Relevant literature:

It is well documented that MVO portfolios are highly sensitive to estimation error in its inputs (see for example [21, 41, 54]). Several attempts to mitigate the adverse effects of estimation error have been explored in the portfolio optimization literature. For example, Black and Litterman [12] developed a Bayesian framework for altering MVO inputs based on prior estimates of means and covariances. However, the extent in which to favour the prior estimate is generally unclear. As a result, portfolios tend to systematically over (or under) weight the in-sample estimates, irrespective of the magnitude of sampling error [55]. Michaud and Michaud [53] proposed the concept of resampled mean-variance optimization; a heuristic bootstrap technique that averages the optimal weights derived from multiple resampled realizations of asset returns. The estimates, however, are assumed to follow a multivariate normal distribution, with population moments equal to the in-sample estimates. Moreover, the impact of sampling error is assumed to always be in proportion to the variance of the sample mean. Such assumptions might be unrealistic, and in practice, offers little improvement over traditional MVO [61].

Ledoit and Wolf [49, 50], and Benichou et al. [8], demonstrate that when the covariance matrix is poorly conditioned, the portfolio construction process can become highly unstable, resulting in concentrated ‘error-maximizing’ portfolio weights. Their methods focus on providing improved covariance estimates based on James-Stein shrinkage [65], $L_2$-regularization [68] and rotationally invariant estimators [17]. In most cases, however, the covariance estimator is determined by minimizing an independent prediction-based loss function, irrespective of its final decision-based optimization problem. This is in contrast to an integrated ‘predict then optimize’ approach, whereby prediction model parameters are optimized based on the decision errors induced by the estimate (see for example Butler and Kwon [18] for more details).

Alternatively, robust portfolio optimization attempts to insulate the portfolio construction process from estimation error by minimizing the objective under worst-case realizations of the estimates [34]. The effectiveness of robust optimization techniques relies heavily on appropriately defining and calibrating the uncertainty structure. Indeed, poorly calibrated uncertainty structures will often result in overly conservative or computationally intractable portfolio solutions [11]. While the seminal work in the field of robust optimization provide several theoretical methods for constructing good uncertainty sets (e.g. [5, 6, 10, 34]), in quantitative finance, defining appropriate uncertainty structures remains a relatively ad-hoc and problem specific process.

For example, Goldfarb and Iyengar [34] propose calibrating the uncertainty structure for both the mean and covariance by considering the distribution of residuals from a linear regression model. They demonstrate that under these conditions the nominal MVO program can be re-cast as a convex second-order cone program and solved efficiently by interior-point methods. Tutuncu and Koenig [69] considers a robust MVO counterpart with box uncertainty sets and proposes calibrating the uncertainty bounds by bootstrapping the historical return distribution. Alternatively, Zhu et al. [72] investigates robust MVO methods under ellipsoidal uncertainty in estimates of the mean. They show that when the uncertainty matrix is proportional to the asset covariance matrix then the robust MVO counterpart is equivalent to a standard MVO model based on the nominal mean estimates but with a larger risk aversion parameter. More recently, Yin et al. [71] make the case for the preference of quadratic uncertainty sets over the more restrictive box uncertainty. Furthermore, they provide evidence for a diagonal uncertainty structure based on asset variances and propose calibrating the level of uncertainty as a function of the underlying asset Sharpe ratios.
As mentioned previously, augmenting the nominal MVO optimization with a norm penalty on portfolio weights has been shown to be directly related to covariance shrinkage and robust optimization methods \[23, 36\]. This is discussed in more detail in Section 2. For example, DeMiguel et al. \[23\] considers the minimum-variance portfolio optimization augmented with $L_1$ and $L_2$ norm penalty functions. They demonstrate that the norm penalty is equivalent to a shrinkage estimator on the sample covariance and propose optimizing the regularization amount by heuristic cross-validation. More recently, Ho et al. \[36\] proposed regularizing MVO portfolio weights by augmenting the program with an elastic net penalty function. They demonstrate that the elastic net formulation is equivalent to a robust MVO counterpart with a particular uncertainty structure and propose tuning the amount of regularization by bootstrapping the historical distribution. Similarly, Kremer et al. \[46\] augment the nominal MVO program with a sorted $L_1$ norm penalty. They show that by varying the intensity of the penalty their approach can span the entire efficient frontier. In practice, however, they propose tuning the penalty parameter in order to achieve a desired target percentage of active positions.

In contrast, the stochastic optimization framework, presented in Section 2, can be viewed as both an extension and generalization of the portfolio regularization methods discussed above. We show that our choice of norm penalty can be interpreted as both a covariance shrinkage method and a robust MVO counterpart with arbitrary shrinkage target and uncertainty structure, respectively. Moreover, the general $L_1$ penalty term can be defined in order to generate sparse portfolio solutions with asset specific levels of regularization intensity. More generally, the parameterized form of the elastic net penalty, provided in Equation (1), obviates the need to pre-define a particular regularization structures. Instead, we work with a parameterized regularization model whose parameters, $\theta$, define the regularization structure and are optimized directly in the context of the decision-based optimization problem. The primary advantage of the integrated approach is that it takes into consideration all other problem dimensions such as portfolio objectives, constraints, as well as the efficacy and implied uncertainty in the estimates of mean and covariance.

Integrated predict, then optimize (IPO) methods dates back to the work of Vapnik \[70\] who proposed training prediction models based on empirical risk minimization objectives rather than more traditional prediction-based objectives such as least-squares or maximum-likelihood. In quantitative finance, IPO methods were first presented by Bengio \[7\] who proposed training prediction models based on financial risk and return objectives. More recently, with the rise of big-data and high performance computing, many novel and highly effective IPO frameworks have been proposed. For example, Donti et al. \[25\] present end-to-end stochastic programming approach for estimating the parameters of probability density functions in the context of their final task-based loss function. They consider applications from power scheduling and battery storage and focus specifically on parametric models in a stochastic programming setting. They demonstrate that their task-based end-to-end approach can result in lower realized costs in comparison to traditional maximum likelihood estimation and a black-box neural network. Similarly, Butler and Kwon \[18, 19\] present a stochastic optimization framework for integrating expected return and covariance predictions in a portfolio optimization setting. They demonstrate that their IPO framework can result in lower out-of-sample costs and higher absolute and risk-adjusted performance in comparison to the standard ‘predict, then optimize’ approach.

Recently, Bertsimas and Kallus \[9\] present a general conditional stochastic approximation framework, capable of supporting a variety of parametric and nonparametric machine learning methods. Their approach generates locally optimal decision policies within the context of a nominal optimization
problem and considers the setting where the decision policy affects subsequent realizations of the uncertainty variable. Alternatively, Elmachtoub and Grigas [27] proposed replacing the prediction-based loss function with a convex surrogate loss function that optimizes prediction variables based on the decision error induced by the prediction model. They demonstrate that optimizing prediction models based on their ‘smart predict, then optimize’ (SPO) loss function can lead to improved overall decision error. Subsequently, Elmachtoub et al. [28], propose using the SPO loss for optimizing decision-tree based prediction models and demonstrate a reduction in decision-making errors in comparison to the standard regression trees and random forests based approaches. Similarly, Kallus and Mao [42] explore contextual trees, a data-driven approach for training forest decision policies in the context of the nominal optimization problem. Their framework is efficient in that they approximate the greedy determination of optimal tree split points by perturbation analysis at the optimality conditions of candidate splits.

The IPO stochastic optimization framework, presented in Section 2, is most similar to, and in part inspired by, the joint estimation and robustness optimization (JERO) framework, provided by Zhu et al. [73]. Specifically, the JERO framework mitigates estimation error in downstream optimization problems by constructing a robustness uncertainty set whereby the size of the uncertainty set is based on the efficacy of a particular parameter estimation process. They consider several estimation procedures, namely: least-squares regression, least absolute shrinkage and selection operator (LASSO), and maximum likelihood estimation. The JERO framework, however, differs from the stochastic optimization framework in several ways. First, the primary objective of the JERO framework is to maximize the size of the uncertainty set subject to feasibility conditions on the decision-based variables. Furthermore, optimization is performed by a binary-search algorithm. This is in contrast to the stochastic optimization framework, which optimizes both the size and shape of a parameterized regularization structure. Regularization model parameters are optimized directly in the context of the final decision problem, with the objective of minimizing the expected downstream decision cost value. Lastly, the approach is general, and capable of optimizing regularization and all other model parameters jointly by first-order gradient descent.

Another closely related area of work is the optimization of regularization (hyper)parameters in general machine learning prediction problems. Traditionally, model hyperparameters are tuned by computationally expensive cross-validation procedures [35]. More recently, gradient descent based methods have been proposed for joint optimization of prediction model parameters and regularization model hyperparameters. For example, Pedregosa [60] propose an algorithm for the optimization of continuous hyperparameters by implicit differentiation and provide sufficient conditions for global convergence. Feng and Simon [32] consider norm penalized regression problems whereby penalty parameters are optimized in order to minimize cross-validation errors. They show that for many penalized regression problems, the validation loss is smooth almost-everywhere with respect to the penalty parameters and propose a gradient descent algorithm for parameter optimization. Lastly, Lorraine and Duvenaud [51] provides a general stochastic optimization framework for joint optimization of prediction model parameters and hyperparameters. Their neural network based approach is flexible and capable of supporting large scale problems with thousands of hyperparameters. In contrast to the methods mentioned above, our stochastic optimization framework considers the more general parameterized norm penalty functions and considers a fully integrated setting whereby the prediction and regularization models are optimized in the context of a final downstream decision-based optimization.
2 Methodology

2.1 Mean-Variance Optimization

We consider a universe of $d_z$ assets and denote the matrix of (excess) return observations as $Y = [y^{(1)}, y^{(2)}, ..., y^{(m)}] \in \mathbb{R}^{m \times d_z}$ with $m > d_z$. We define the portfolio $z \in \mathbb{R}^{d_z}$, where the element, $z_j$, denotes the proportion of total capital invested in the $j$th asset. Therefore, for a particular return observation $y^{(i)}$, the optimal portfolio weights, $z^*(y^{(i)})$, is given by the solution to the following convex quadratic program:

$$
\begin{align*}
\text{minimize} \quad & c(z, y^{(i)}) = -z^T y^{(i)} + \delta z^T V^{(i)} z \\
\text{subject to} \quad & A z = b, \quad G z \leq h
\end{align*}
$$

where $V^{(i)}$ is a positive definite covariance matrix and $\delta \in \mathbb{R}_+$ is a risk-aversion parameter that controls the trade-off between minimizing variance and maximizing return. In reality, we do not know the true value of $y^{(i)}$ at decision time. We therefore estimate the values $y^{(i)}$ through associated auxiliary variables $x^{(i)} \in \mathbb{R}^{d_x}$. In this paper we consider regression based predictions models of the form:

$$
\hat{y}^{(i)} = \beta^T x^{(i)}
$$

where $\beta \in \mathbb{R}^{d_x \times d_y}$ is a matrix of regression coefficients. For simplicity, we consider a traditional parameter estimation process, and choose $\beta$ that minimizes the least-squares loss function $\ell(y, \hat{y}) = \|y - \hat{y}\|^2$. Therefore, given training data set $D = \{(x^{(i)}, y^{(i)})\}_{i=1}^m$, we choose $\hat{\theta}$ such that:

$$
\hat{\beta} = \arg\min_\beta E_D[\ell(\beta^T x^{(i)}, y^{(i)})], 
$$

where $E_D$ denotes the expectation operator with respect to the training distribution $D$. In practice, however, we note that the prediction model parameters, $\beta$, can easily be integrated and optimized by the stochastic optimization framework presented in Section 2.3. The resulting least-squares estimators are therefore given by:

$$
\begin{align*}
\hat{y}^{(i)} &= \hat{\beta}^T x^{(i)} \\
\hat{V}^{(i)} &= \hat{\beta}^T \hat{W} \hat{\beta} + \hat{F}
\end{align*}
$$

where $\hat{W}$ and $\hat{F}$ denote the estimated covariance matrix of auxiliary variables and the diagonal matrix of residual variances, respectively. Note that in many applications, the eigenvalues of $\hat{F}$ are typically much smaller that that of $\hat{\beta}^T \hat{W} \hat{\beta}$, and therefore the latter is a good low-rank approximation to the asset covariance matrix $\hat{\Sigma}$.

2.2 Motivation for Norm Penalty

Augmenting the objective of Program Equation (2) with the norm penalty (1) results in a parameterized penalized quadratic programs (PPQP), presented below:
\[
\min_{z} \quad c_P(z, \hat{y}^{(i)}, \theta, \gamma) = -z^T \hat{y}^{(i)} + \frac{\delta}{2} z^T V^{(i)} z + \alpha \gamma_1 \|E(\theta) z\|_1 + (1 - \alpha) \frac{\gamma_2}{2} \|D(\theta) z\|_2^2
\]
subject to \quad \mathbf{A} z = \mathbf{b}, \quad \mathbf{G} z \leq \mathbf{h},
\]

where \(\|\cdot\|_p\) denotes the standard \(L_p\) norm and \(0 \leq \alpha \leq 1\). The matrices, \(E(\theta)\) and \(D(\theta)\) are parameterized sparsifying and regularization transforms, respectively. Here we require that \(E(\theta)\) preserves the sign of \(z\) under the linear transform \(E(\theta) z\), and in general that \(E(\theta)^T E(\theta) \succeq 0\) and \(D(\theta)^T D(\theta) \succeq 0\) in order to ensure convexity of Program (5). Observe, that when \(\alpha = 1\) and \(E(\theta) = \mathbf{I}\) then \(P(z, \theta, \gamma)\) is the standard \(L_1\) penalty common to many applications in statistics and engineering [66, 43]. Conversely, when \(\alpha = 0\) and \(D(\theta) = \mathbf{I}\) then \(P(z, \theta, \gamma)\) is the standard \(L_2\) penalty or Tikhonov regularization for approximations to ill-posed problems [68]. In its general form, \(P(z, \theta, \gamma)\) resembles a generalization of the elastic net penalty for joint regularization and feature selection in least-squares regression [74]. Indeed, the choice of this specific norm penalty is deliberate and motivated by prior work that establishes the connection between norm penalized portfolio weights with covariance shrinkage and robust optimization.

### 2.2.1 Covariance Shrinkage Estimator

The linear covariance shrinkage estimator presented by Ledoit and Wolf [48] is a convex combination of the sample covariance matrix, \(V\) and a structured estimator \(S\):

\[
\hat{V} = (1 - \epsilon) V + \epsilon S = V + \epsilon (S - V),
\]

where \(0 \leq \epsilon \leq 1\). As we will demonstrate shortly, it is easy to see that our choice of norm penalty conforms to a linear covariance shrinkage estimator (see Equations (17) and (25)). Specifically, for a particular choice of \(\alpha, \gamma_1\) and \(\gamma_2\), we have the following covariance shrinkage estimator:

\[
\hat{V} = V + \alpha \gamma_1 \text{diag}(|E(\theta) z^*| + \tau)^{-1} E(\theta)^T E(\theta) + (1 - \alpha) \gamma_2 D(\theta)^T D(\theta),
\]

where \(E(\theta)^T E(\theta) \succeq 0\), \(D(\theta)^T D(\theta) \succeq 0\) and \(\tau > 0\) to ensure positive definiteness of \(\hat{V}\).

### 2.2.2 Robust Optimization

Following Fabozzi et al. [30], we assume that the error in our estimate of expected returns is no larger than \(\gamma \geq 0\) and consider the following box uncertainty set:

\[
\mathbf{U}_\gamma = \{\mathbf{y} \mid \|\mathbf{y}_i - \bar{\mathbf{y}}_i\|_1 \leq \gamma_i, i = 1, ..., d_y\}.
\]

The robust MVO counterpart is given as:

\[
\min_{z} \quad \max_{\hat{y}^{(i)} \in \mathbf{U}_\gamma} (-z^T \hat{y}^{(i)}) + \frac{\delta}{2} z^T V^{(i)} z
\]
subject to \quad \mathbf{A} z = \mathbf{b}, \quad \mathbf{G} z \leq \mathbf{h}
\]

8
Under box uncertainty, Program (9) simplifies to the following $L_1$-norm penalized quadratic program:

$$\text{minimize } -z^T \hat{y}(i) + \gamma^T |z| + \frac{\delta}{2} z^T V(i) z$$

subject to $A z = b, \quad G z \leq h$ \hspace{2cm} (10)

It is then easy to verify that Program (10) is an $L_1$-norm PPQP with sparsifying matrix $E(\theta) = \text{diag}(\gamma)$.

Alternatively, following Ceria and Stubbs [20] we now consider the ellipsoidal uncertainty set on the vector of expected returns:

$$U_y = \{y | (y - \bar{y})^T \Omega^{-1} (y - \bar{y}) \leq \kappa^2\},$$

where $\Omega$ is a symmetric positive-definite uncertainty matrix and $\kappa^2$ defines the level of uncertainty.

Under ellipsoidal uncertainty, Program (9) simplifies to the following constrained second-order cone program:

$$\text{minimize } -z^T \hat{y}(i) + \frac{\delta}{2} z^T V(i) z$$

subject to $A z = b, \quad G z \leq h, \quad \|\Omega^{1/2} z\|_2 \leq \kappa$ \hspace{2cm} (12)

Relaxing the second-order cone constraint and following the results presented in Zhu et al. [72], it follows that there exists a $\gamma_2 \geq 0$ such that the following Program is equivalent to Program (12):

$$\text{minimize } -z^T \hat{y}(i) + \frac{\delta}{2} z^T V(i) z + \gamma_2 \|\Omega^{1/2} z\|_2^2$$

subject to $A z = b, \quad G z \leq h$. \hspace{2cm} (13)

It is easy to see that Program (13) is an $L_2$-norm PPQP with uncertainty structure $D(\theta) = \Omega^{1/2}$.

2.3 Stochastic Optimization Framework

As stated earlier, the prediction model parameters, $\beta$, are typically estimated by minimizing a prediction-based loss function. Furthermore, the norm penalty parameters: $\theta$ and $\gamma$, are typically selected by heuristic cross-validation or resampling methods. Here, we follow the work of Butler and Kwon [18, 19] and present a stochastic optimization framework for optimizing the prediction model and norm penalty parameters in the context of the final decision-based optimization, its inputs and constraints.

Specifically, we are interested in optimizing $\beta, \theta$ and $\gamma$ in the context of the norm penalized mean-variance cost, $c_P$, and its constraints, $Z = \{z \in \mathbb{R}^d | A z = b, G z \leq h\}$, in order to minimize the average realized nominal cost of the policy $z^*(\hat{y}(i), \theta, \gamma)$ induced by this parameterization. For a fixed instantiation $(\hat{y}(i), \theta, \gamma)$, we solve Program (5) in order to determine the optimal portfolio weights: $z^*(\hat{y}(i), \theta, \gamma)$.
Our objective is to therefore choose $\beta, \theta$ and $\gamma$ in order to minimize the average realized nominal cost, $c$, induced by the decision policy $z^*(\hat{y}(i), \theta, \gamma)$. The resulting problem can be posed as a stochastic optimization problem, presented in program (15).

$$\min_{\theta, \gamma} E_D[c(z^*(\hat{y}, \theta, \gamma), y)]$$

subject to $z^*(\hat{y}, \theta, \gamma) = \text{argmin}_{z \in \mathbb{Z}} c_P(z, \hat{y}(i), \theta, \gamma)$ \quad \gamma \geq 0. \quad (15)$

In practice, we are typically presented with discrete observations $D = \{(x^{(i)}, y^{(i)})\}_{i=1}^{m}$ and therefore we can approximate the expectation by its sample average approximation [63]. The full discrete stochastic optimization program is presented in program (16):

$$\min_{\theta, \gamma} \frac{1}{m} \sum_{i=1}^{m} c(z^*(\hat{y}(i), \theta, \gamma), y^{(i)})$$

subject to $z^*(\hat{y}(i), \theta, \gamma) = \text{argmin}_{z \in \mathbb{Z}} c_P(z, \hat{y}(i), \theta, \gamma)$ \quad \forall i \in 1, ..., m \quad \gamma \geq 0. \quad (16)$

Observe that the stochastic optimization framework results in a complicated dependency of the parameters, $\beta, \theta$ and $\gamma$, on the optimized values, $z^*(\hat{y}, \theta, \gamma)$, connected through the argmin function. Fortunately, the implicit differentiation techniques, described in more detail below, provide an efficient framework for computing the gradient of the nominal cost with respect to all relevant problem variables. We propose optimizing the penalty parameters by first-order gradient descent in an end-to-end differentiable neural network, depicted in Figure 1. For more detail we refer the reader to Butler and Kwon [18, 19].

![Figure 1: Stochastic optimization framework represented as an end-to-end neural-network with predictive model layer, differentiable quadratic programming layer and realized nominal cost loss function.](image)

Lastly, we choose to search for locally optimal solutions by stochastic gradient descent (SGD). The descent direction, $g_\theta$, at each iteration approximates the gradient of the nominal cost with respect to $\theta$ and is given by:

$$g_\theta = \sum_{i \in B} \left( \frac{\partial c}{\partial \theta} \right)_{(z^*(\hat{y}(i), y^{(i)}))}$$
where in standard stochastic gradient descent, \( B \) represents a randomly drawn sample batch.

### 2.4 Differentiating Penalized Quadratic Programs

The stochastic optimization framework outlined in Section 2.3 requires differentiating the solutions, \( z^*(\hat{y}, \theta, \gamma) \), to the PPQP (5) with respect to the relevant problem variables. Note that norm-constrained QPs can be cast as standard second-order cone programs and subsequently integrated in an end-to-end system by implicit differentiation of the corresponding solution map provided by the homogeneous self-dual embedding \[1, 2\]. In this paper, however, we choose to re-cast Program (5) as a standard quadratic program and implicitly differentiate through the system of equations provided by the Karush–Kuhn–Tucker (KKT) conditions at optimality. This is similar to the approach taken in Butler and Kwon \[18, 19\] and we refer to Amos and Kolter \[3\] for full implementation details. Once the PPQP problem has been cast as a standard QP, the solution and surrounding problem variables can be embedded as a specialized differentiable quadratic programming layer in a fully end-to-end trainable neural network. We consider two formulations: the first based on a primal reformulation with a modified backward pass, and the other based on a standard dual reformulation. As we will show in Section 3, both formulations have their advantages under specific circumstances.

#### 2.4.1 Primal Formulation

We begin by first simplifying with respect to the \( L_2 \) norm penalty. Specifically we let:

\[
V_{\gamma_2} = \delta V + (1 - \alpha) \gamma_2 D(\theta)^T D(\theta)
\]  \hspace{1cm} (17)

and therefore the norm-penalized objective of Program (5) can be simplified as follows:

\[
c_P(z, \hat{y}, \theta, \gamma) = -z^T \hat{y} + \frac{1}{2} z^T V_{\gamma_2} z + \alpha \gamma_1 \| E(\theta) z \|_1.
\]  \hspace{1cm} (18)

Observe that if \( \alpha = 0 \) then Program (5) is a standard quadratic program and the relevant gradients with respect to the \( L_2 \) norm penalty are given as:

\[
\frac{\partial c_P}{\partial \gamma_2} = (1 - \alpha) \text{Tr} \left( \frac{\partial c_P}{\partial V_{\gamma_2}} D(\theta)^T D(\theta) \right) \quad \quad \frac{\partial c_P}{\partial D} = 2 \gamma_2 (1 - \alpha) D(\theta) \frac{\partial c_P}{\partial V_{\gamma_2}}
\]  \hspace{1cm} (19)

In the general case, Program (5) is a \( L_1 \) norm penalized quadratic program. Observe that when \( V_{\gamma_2}(\theta, \gamma_2) > 0 \) then Program (5) is strictly convex and therefore has a unique minimizer \[67\]. There are several known methods for optimizing \( L_1 \) norm penalized quadratic programs (see Schmidt \[62\] for a comprehensive overview). For problems of moderate size (\( d_z < 1000 \)), it is often convenient to re-cast Program (5) by standard separation of variables and the introduction of \( 2d_z \) inequality constraints (see for example Tibshirani \[66\], Gaines and Zhou \[33\]). Specifically, we let \( z = z^+ - z^- \) where \( z^+ \in \mathbb{R}_{++}^{d_z} \) and \( z^- \in \mathbb{R}_{++}^{d_z} \) denote the positive and negative components of \( z \), respectively. Then Program (5) can be re-cast as a convex quadratic program, presented in Program (20).
\[
\begin{align*}
\text{minimize} \quad & \frac{1}{2} \begin{bmatrix} z^+ & z^- \end{bmatrix}^T \begin{bmatrix} \gamma_2 & -\gamma_2 \\ -\gamma_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} z^+ \\ z^- \end{bmatrix} - \begin{bmatrix} \hat{y} + \gamma_1 1^T E(\theta) \gamma_2^T \\ -\hat{y} + \gamma_1 1^T E(\theta) \gamma_2^T \end{bmatrix}^T \begin{bmatrix} z^+ \\ z^- \end{bmatrix} \\
\text{subject to} \quad & \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} z^+ \\ z^- \end{bmatrix} = b \\
& \begin{bmatrix} G & -G \end{bmatrix} \begin{bmatrix} z^+ \\ z^- \end{bmatrix} \leq h \\
& z^+ \geq 0, \quad z^- \geq 0
\end{align*}
\] (20)

The relevant gradients with respect to the augmented problem variables found in Program (20) are determined by implicit differentiation of the KKT system of equations at optimality. Firstly, we note that the $L_1$ norm in Program (5) is not differentiable everywhere and certainly not twice differentiable. Moreover, while Program (20) is convex whenever $V_{\gamma_2} \succ 0$, the augmented covariance matrix invariably contains $d_z$ eigenvalues that are exactly zero. As a result, the solution mapping provided by the optimality conditions of Program (20) do not satisfy the necessary nonsingularity condition for implicit differentiation [24].

To overcome these challenges we present a modified backward pass gradient based on a second-order approximation to the $L_1$ norm at optimality. First, let $z^* = z^{++} - z^{--}$ be the optimal solution to Program (20). We follow [31, 39] and consider the following quadratic approximation to the $L_1$ penalty at a point $z$ in the neighbourhood of $z^*$:

\[
\|E(\theta) z\|_1 \approx \|E(\theta) z^*\|_1 + \frac{1}{2} \frac{\|E(\theta) z\|_2^2 - \|E(\theta) z^*\|_2^2}{\|E(\theta) z^*\|_1 + \tau}
\] (21)

for some small scalar $\tau \geq 0$. The second order approximation to the objective function of Program (5) is given below:

\[
c_\tau(z, \hat{y}, \theta, \gamma) = -z^T \hat{y} + \frac{1}{2} z^T V_{\gamma_2} z + \alpha \gamma_1 \left( \|E(\theta) z^*\|_1 + \frac{1}{2} \frac{\|E(\theta) z\|_2^2 - \|E(\theta) z^*\|_2^2}{\|E(\theta) z^*\|_1 + \tau} \right).
\] (22)

Observe that $c_\tau(z, \hat{y}, \theta, \gamma)$ is quadratic in decision variable $z$. Ignoring constant terms, then Program (23) is an Iteratively Reweighted Least-Squares (IRLS) [47, 59] approximation to Program (5):

\[
\begin{align*}
\text{minimize} \quad & -z^T \hat{y} + \frac{1}{2} z^T V_{\gamma_2} z + \alpha \gamma_1 \frac{1}{2} \frac{\|E(\theta) z\|_2^2 - \|E(\theta) z^*\|_2^2}{\|E(\theta) z^*\|_1 + \tau} \\
\text{subject to} \quad & A z = b, \quad G z \leq h
\end{align*}
\] (23)

with weight matrix:

\[
W(\theta) = \text{diag}(\|E(\theta) z^*\|_1 + \tau)^{-1} E(\theta)^T E(\theta).
\] (24)

**Proposition 1.** Let $z^*_\tau$ denote the solution to Program (23). Let $V_{\gamma_2} \succ 0$ and $E(\theta)^T E(\theta) \succ 0$, then Program (23) is strictly convex and

\[
\lim_{\tau \to 0} z^*_\tau = z^*.
\]
Corollary 1. Let $z^*$ and $z^*_\tau$ denote the solutions to Program (5) and Program (23), respectively. It follows then that:

$$\lim_{\tau \to 0} c_\tau (z^*_\tau, \theta) = c(z^*, \theta).$$

By Proposition 1, for a sufficiently small $\tau$, the unique solution to Program (23), $z^*_\tau$, converges to the nominal solution $z^*$. Furthermore, Program (23) is a standard convex constrained QP, whose solution mapping at optimality satisfies the conditions for implicit differentiation. We therefore propose computing approximate gradients by applying the second order approximation:

$$V_{\gamma_2} z = V_{\gamma_2} + \alpha \gamma_1 \text{diag}(E(\theta) z^* \mid + \tau)^{-1} E(\theta)^T E(\theta),$$

and implicitly differentiating the resulting KKT system of equations. Appendix A provides a proof of Proposition 1 and numerical validation of the approximate solution $z^*_\tau$ and its gradient with respect to $\theta$. In all experiments we find that a value of $\tau = 10^{-4}$ provides consistent and numerical stable results.

2.4.2 Dual Formulation

We now consider the dual of Program (5). Let $w = E(\theta) z$ then we have the following equivalent program:

$$\begin{align*}
\text{minimize} \quad & -z^T \hat{y} + \frac{1}{2} z^T V_{\gamma_2} z + \alpha \gamma_1 \|w\|_1 \\
\text{subject to} \quad & A z = b, \quad G z \leq h \\
& w = E(\theta) z.
\end{align*}$$

(26)

The resulting Lagrange is presented below:

$$L(z, w, v, \eta, \mu) = -z^T \hat{y} + \frac{1}{2} z^T V_{\gamma_2} z + \alpha \gamma_1 \|w\|_1 + v^T (E(\theta) z - w)$$

$$+ \eta^T (A z - b) + \mu^T (G z - h)$$

(27)

with dual variables $v \in \mathbb{R}^d_z$, $\eta \in \mathbb{R}^{d_{eq}}$ and $\mu \in \mathbb{R}^{d_{iq}}$. Following Kim et al. [43], the Lagrange dual problem associated with Program (26) is given by:

$$\begin{align*}
\text{maximize} \quad & \inf_{z, w} L(z, w, v, \eta, \mu) \\
\text{subject to} \quad & \mu \geq 0.
\end{align*}$$

(28)

Observe that Program (26) is convex and therefore strong duality holds [15]. The first-order optimality conditions with respect to primal variable $z$ results in the following linear system of equations:

$$V_{\gamma_2} z = \hat{y} - E(\theta)^T v - A^T \eta - G^T \mu.$$  

(29)
Furthermore, dual feasibility is given by boundedness with respect to the dual variable \( v \), specifically:
\[
\inf_w \alpha \gamma_1 \|w\|_1 - v^T w = -\sup_w v^T w - \alpha \gamma_1 \|w\|_1
\]
\[
= \begin{cases} 
0 & \text{if } \|v\|_\infty \leq \alpha \gamma_1 \\
-\infty & \text{otherwise} 
\end{cases}
\]
(30)

Combining the results from Equations (29) and (30) results in the following constrained quadratic program:
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} r^T V^{-1} r + \eta^T b + \mu^T h \\
\text{subject to} & \quad -\alpha \gamma_1 \leq v \leq \alpha \gamma_1 \\
& \quad \mu \geq 0,
\end{align*}
\]
(31)

where \( r = \hat{y} - E(\theta)^T v - A^T \eta - G^T \mu \). Program (31) can be solved by standard low-rank interior point quadratic programming methods or more recently by efficient cyclical coordinate descent algorithm [38]. The primal solution, \( z^* \), is then determined by solving Equation (29) with respect to the optimal dual variables.

Finally, note that in the absence of constraints, Program (31) reduces to a convex constrained quadratic program in the dual variable \( v \):
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} v^T E(\theta) V^{-1} \gamma_2 E(\theta)^T v - v^T E(\theta) Q^{-1} \gamma_2 \hat{y} \\
\text{subject to} & \quad -\alpha \gamma_1 \leq v \leq \alpha \gamma_1,
\end{align*}
\]
(32)

When \( E(\theta) \) has full column rank then Program (32) is strictly convex and can be solved efficiently by standard constrained quadratic programming methods (see for example [43]). Furthermore, the KKT conditions at optimality satisfy the necessary conditions for implicit differentiation. Lastly, we note that once the optimal dual variable has been determined, it is trivial to recover the optimal primal variable by solving Equation (29).

3 Computational experiments

Experiment Setup:
We consider an asset universe of 24 U.S. commodity futures markets, described in Table 1. The daily price data is given from March 1986 through December 2020, and is provided by Commodity Systems Inc. Futures contracts are rolled on, at most, a monthly basis in order to remain invested in the most liquid contract, as measured by open-interest and volume. Arithmetic returns are computed directly from the price data and are in excess of the risk-free rate.

In each experiment, described below, we consider two methods for estimating asset returns:

1. **OLS**: ordinary-least squares method, with prediction coefficients, \( \beta \), provided by Equation (4).
2. **IPO**: integrated prediction and optimization method, where \( \beta \) is optimized (along with regularization parameters) via the stochastic optimization Program (16). See Butler and Kwon [19] for more detail.
The asset covariance matrix is measured using an exponential moving average with a decay rate of 0.98. We consider both univariate and multivariate prediction models. The auxiliary feature, $x$, for univariate models is the 252-day average return, or trend, for each market. The feature therefore represents a measure of the well-documented ‘trend’ factor, popular to many Commodity Trading Advisors (CTAs) and Hedge Funds (see for example [4], [16], [57]). The auxiliary feature for multivariate models is the 252-day trend and the ‘carry’ for each market. We follow Koijen et al. [45] and define the ‘carry’ as the expected convenience yield, or cost, for holding that commodity, and is estimated by the percent difference in price between the two futures contracts closest to expiry.

We consider 4 experiments:

1. Unconstrained MVO program with univariate regression.
2. Unconstrained MVO program with multivariate regression.
3. Constrained MVO program with univariate regression.
4. Constrained MVO program with multivariate regression.

In all experiments we fix the risk-aversion parameter to $\delta = 10$. The constrained MVO programs are market-neutral, with upper bound and lower bound market exposure constraints of $\pm 25\%$. Formally, the feasible region is given by:

$$Z = \{ z^T 1 = 0, -0.25 \leq z \leq 0.25 \}.$$

In the absence of portfolio constraints, we choose to solve and differentiate through the dual PPQP, as outlined in Section 2.4.2. Conversely, in the presence of portfolio constraints, the forward pass is computed by solving the augmented PPQP (Program (20)) and applying the modified quadratic approximate gradient during the backward pass, as outlined in Section 2.4.1.

In an effort to mitigate turnover and provide more realistic results, the dependent variable, $y^{(i)}$, is the average realized 5-day forward return. Therefore the portfolio weights, $z^*(\hat{y}, \theta, \gamma)$, represents the optimal holding for the next 5 days. In practice we reform the portfolio at the close of each day, and rebalance $1/5^{th}$ of the exposure on a daily basis. All experiments start in January 2000 and end in December 2020. For each experiment, the first 14 years (March 1986 through December 1999) is used to perform the initial parameter estimation. Thereafter, we apply a walk-forward training and testing methodology. The optimal regularization and prediction model coefficients are updated every 2 years using all available data for parameter estimation and the optimal policy, $z^*(\hat{y}, \theta, \gamma)$, is then applied to the next out-of-sample 2 year segment. All performance is gross of trading costs and in excess of the risk-free rate.

For each prediction model (OLS and IPO) we consider several norm penalty regularization models, summarized below.

- **Nominal**: the nominal MVO program, without regularization:
  $$P(z) = 0.$$

- **L2**: standard $L_2$-norm regularization:
  $$P(z) = \frac{\gamma^2}{2} \|z\|^2_2.$$
• **L2-COV**: follow Goldfarb and Iyengar [34] and model the uncertainty structure according to the low-rank asset covariance approximation provided by Equation (4), namely $D = \hat{\beta}^T \hat{W} \hat{\beta}$ and:

$$P(z) = \frac{\gamma_2}{2} \|Dz\|_2^2.$$  

• **L1**: standard $L_1$-norm regularization:

$$P(z) = \gamma_1 \|z\|_1.$$  

• **EN**: standard elastic-net regularization with mixing coefficient $\alpha = 0.5$:

$$P(z) = \alpha \gamma_1 \|z\|_1 + (1 - \alpha) \frac{\gamma_2}{2} \|z\|_2^2.$$  

• **L2-P**: parameterized $L_2$-norm regularization, where $D(\theta) = \text{Diag}(\theta)$ for $\theta \in \mathbb{R}^d_+$:

$$P(z) = \frac{\gamma_2}{2} \|D(\theta)z\|_2^2.$$  

• **L1-P**: parameterized $L_1$-norm regularization, where $E(\theta) = \text{Diag}(\theta)$ for $\theta \in \mathbb{R}^d_+$:

$$P(z) = \gamma_1 \|E(\theta)z\|_1.$$  

• **EN-P**: parameterized elastic-net regularization, with mixing coefficient $\alpha = 0.5$ and $D(\theta)$ and $E(\theta)$ as described previously:

$$P(z) = \alpha \gamma_1 \|E(\theta)z\|_1 + (1 - \alpha) \frac{\gamma_2}{2} \|D(\theta)z\|_2^2.$$  

We refer the reader to Appendix [C] for full implementation details.

Each model is evaluated on absolute and relative terms, with a focus on out-of-sample MVO cost and out-of-sample Sharpe ratios, provided by Equation (33).

$$c_{\text{MVO}}(z, y) = \mu(z, y) + \frac{\delta}{2} \sigma^2(z, y), \quad \text{and} \quad c_{\text{SR}}(z, y) = \frac{\mu(z, y)}{\sigma(z, y)}$$  

where

$$\mu(z, y) = \frac{1}{m} \sum_{i=1}^{m} z^{(i)} y^{(i)} \quad \text{and} \quad \sigma^2(z, y) = \frac{1}{m} \sum_{i=1}^{m} (z^{T(i)} y^{(i)} - \mu(z, y))^2,$$

denote the mean and variance of realized daily returns. To quantify the magnitude and consistency of observed performance metrics, and to ensure our results are robust to potential outliers in the out-of-sample periods, boxplots are created by bootstrapping the out-of-sample distribution using 1000 samples as follows:

1. For each $k \in \{1, 2, ..., 1000\}$, sample, without replacement, a batch, $B_k$, with $|B_k| = 252$ observations (1 year) from the out-of-sample period.
2. For each model, compute the average realized costs and Sharpe over the sample, using Equation (33).

Note, in each sample draw we use the same observation dates across all models in order to fairly compare the realized nominal costs over the resulting sample.

Our primary objective is to analyze, for each prediction model (OLS and IPO), the effectiveness of the stochastic optimization framework in optimizing the regularization parameters, \((\gamma, \theta)\). We demonstrate, across 4 different experiments, that in almost every case the regularized MVO portfolios produce a mean-variance policy, \(z^*(\hat{y}, \theta, \gamma)\), with lower out-of-sample MVO costs and improved economic performance in comparison to the nominal MVO counterpart. In general, portfolios with OLS prediction models exhibit the largest relative improvement in out-of-sample MVO cost and realized Sharpe ratios. This is in contrast to portfolios with IPO prediction models, which display a more modest and less consistent improvement in MVO cost and realized economic performance.

Of secondary importance, we compare the relative performance across the various different regularization models. The regularization models, described above, can be divided into two groups:

1. **User-defined:** the regularization structure is pre-defined by the user and the amount of regularization, \(\gamma\), is optimized by the stochastic optimization framework. Models: L2, L2-COV, L1, EN.

2. **Parameterized:** the regularization structure is parameterized and therefore both the structure parameters, \(\theta\), and amount of regularization \(\gamma\), are optimized by the stochastic optimization framework. Models: L2-P, L1-P, EN-P.

We evaluate the effectiveness of traditional user-defined regularization models in comparison to the parameterized structures, in which the exact regularization model is learned from the data. In general, we observe that parameterized regularization models exhibit lower MVO costs and higher Sharpe ratios. This is particularly true when the prediction model is OLS, in which we observe, in almost every case, lower MVO costs and improved economic outcomes when comparing the parameterized model to its corresponding user-defined counterpart.

Thirdly, we compare across the predictive model methods: OLS and IPO. In the absence of portfolio constraints, the nominal IPO models exhibit significantly lower MVO costs and higher Sharpe ratios in comparison to the nominal OLS models, suggesting that the IPO prediction framework is already fairly robust to prediction uncertainty. However, the regularized IPO models, in general, exhibit marginally higher out-of-sample MVO costs in comparison to the corresponding regularized OLS models. In general, however, the difference in realized MVO costs is small, and does not have a material negative impact on economic outcomes. In fact, we observe that the regularized IPO models produce MVO policies with higher mean returns and Sharpe ratios. The regularized IPO models also exhibit higher volatility, suggesting that the amount of regularization is less that that of the corresponding OLS models. In the presence of constraints, the IPO models exhibit, both lower average MVO costs and higher average out-of-sample Sharpe ratios.

Our experiments should be interpreted as a proof-of-concept, rather than a fully comprehensive financial study. That said, we believe that the results presented below provide evidence that regularization models can be trained effectively by a stochastic optimization framework that seeks to minimize the final downstream decision-based optimization problem. In general, the regularized MVO portfolios
exhibit lower out-of-sample MVO costs and improved economic outcomes. Moreover, we demonstrate that the exact regularization structure does not need to be fully specified in advance and can instead be parameterized and learned from the data. Lastly, the stochastic optimization framework is flexible, and allows for joint optimization of prediction and regularization parameters, as demonstrated by the regularized IPO models. In general we observe that the joint prediction and regularization models lead to competitive out-of-sample MVO costs and improved out-of-sample economic outcomes.

| Asset Class | Market (Symbol) |
|-------------|-----------------|
| Energy      | WTI crude (CL)  |
|             | RBOB gasoline (XB) |
|             | Heating oil (HO) |
|             | Gasoil (QS)     |
| Grain       | Bean oil (BO)   |
|             | Corn (C)        |
|             | Soybean (S)     |
|             | Soy meal (SM)   |
|             | KC Wheat (KW)   |
|             | Wheat (W)       |
| Livestock   | Feeder cattle (FC) |
|             | Live cattle (LC) |
|             | Lean hogs (LH)  |
| Metal       | Gold (GC)       |
|             | Copper (HG)     |
|             | Platinum (PL)   |
|             | Silver (SI)     |
| Soft        | Cocoa (CC)      |
|             | Cotton (CT)     |
|             | Coffee (KC)     |
|             | Canola (RS)     |
|             | Robusta Coffee (DF) |
|             | Sugar (SB)      |

Table 1: Futures market universe. Symbols follow Bloomberg market symbology. Data is provided by Commodity Systems Inc (CSI).

### 3.1 Experiment 1: unconstrained and univariate

Economic performance metrics and average out-of-sample MVO costs are provided in Table 2 for the time period of 2000-01-01 to 2020-12-31 for the unconstrained MVO portfolios with univariate prediction models. Equity growth charts for the same time period are provided in Figure 3. We first observe that, for the OLS prediction models, all regularized MVO portfolios result in lower MVO costs and, with the exception of the OLS-L2-COV model, produce higher Sharpe ratios than the nominal OLS model. Furthermore, the regularized OLS models provide more conservative risk metrics, as measured by portfolio volatility, value-at-risk (VaR), and average drawdown (Avg DD). The user-defined regularization models exhibit smaller excess mean returns than that of the nominal OLS model. However, the reduction in mean return is more than offset by the reduction in portfolio volatility, thus resulting in Sharpe ratios that are, on average, 40% larger than the nominal OLS Sharpe ratio. This is in contrast to the parameterized regularization models, which exhibit mean excess returns that are equivalent or greater than that of the nominal model OLS; resulting in Sharpe ratios that are over 100% larger than the nominal OLS Sharpe ratio. This result is highly encouraging for the parameterized regularization approach and demonstrates the benefit of learning the regularization structure from the data.

For the IPO prediction models, the improvement in out-of-sample MVO costs and Sharpe ratios over the nominal model is more modest and less consistent than the improvement exhibited by the
OLS prediction models. Indeed, the nominal IPO model exhibits a 135% increase in average out-of-sample Sharpe ratio and a meaningful reduction in realized MVO costs in comparison to the nominal OLS model. This is consistent with the view that, even in the absence of regularization, the nominal IPO models tend to be more resilient to prediction uncertainty. Furthermore, the equity growth charts in Figure 3 demonstrate that the character of the IPO models can, at times, be very different than the OLS models. This observation is consistent with the findings in Butler and Kwon [19], which demonstrate that the IPO prediction model parameters, $\beta$, can be very different than the corresponding OLS parameters.

With the exception of OLS-L1-P model, all other regularized IPO models exhibit lower average realized MVO costs in comparison to the nominal IPO model. Moreover, the regularized IPO models also have marginally lower portfolio risk metrics, again demonstrating the benefit of portfolio regularization from a risk-management perspective. Furthermore, we observe that the parameterized IPO models exhibit marginally lower MVO costs and approximately 10% higher average Sharpe ratios in comparison to the nominal and user-defined models. We observe that the IPO-L2-P and IPO-EN-P models have the highest average annualized Sharpe ratio, of 0.9489 and 0.9137, respectively, and also exhibit the lowest MVO costs across all other IPO models.

Figure 2 compares the distribution of out-of-sample realized MVO costs and Sharpe ratios for all models, using the bootstrap procedure as defined in Section 3. Again, we observe that the regularized models exhibit materially lower realized MVO costs in comparison to the nominal OLS model. As stated earlier, the regularized OLS models exhibit, on average, lower MVO costs than the corresponding IPO models. The difference, however, is marginal, with considerable overlap in their sampling distributions. Furthermore, the dispersion of MVO costs is generally higher for IPO models than that of the OLS models. This is consistent with the fact that the volatility of the IPO models is larger, again suggesting that the OLS models have a greater amount of portfolio regularization. Indeed, it is possible that the regularized IPO models are overfitting the training data and that further portfolio regularization would result in lower out-of-sample MVO costs and portfolio volatility. We conjecture that a training framework that applies the stochastic optimization framework on validation data, as proposed by Feng and Simon [32] may help to attenuate the potential overfitting. Nonetheless, as shown in Figure 2(b), the regularized IPO models still exhibit consistently higher Sharpe ratios, in comparison to the corresponding OLS models; a result that we view as encouraging for the joint estimation and regularization approach.
| Model         | Mean  | Volatility | Sharpe | VaR    | Avg DD | MVO Cost |
|--------------|-------|------------|--------|--------|--------|----------|
| Nominal OLS  | 0.1160| 0.3410     | 0.3401 | -0.0298| -0.0849| 0.4654   |
| Nominal IPO  | 0.1829| 0.2258     | 0.8098 | -0.0179| -0.0460| 0.0720   |
| OLS-L2       | 0.0732| 0.1321     | 0.5539 | -0.0130| -0.0358| 0.0141   |
| IPO-L2       | 0.1558| 0.2041     | 0.7631 | -0.0183| -0.0400| 0.0525   |
| OLS-L2-COV   | 0.0458| 0.1588     | 0.2883 | -0.0153| -0.0513| 0.0803   |
| IPO-L2-COV   | 0.1422| 0.1921     | 0.7400 | -0.0177| -0.0379| 0.0424   |
| OLS-L1       | 0.0695| 0.1519     | 0.4577 | -0.0134| -0.0371| 0.0458   |
| IPO-L1       | 0.1878| 0.2183     | 0.8605 | -0.0186| -0.0454| 0.0504   |
| OLS-EN       | 0.0730| 0.1316     | 0.5545 | -0.0130| -0.0342| 0.0136   |
| IPO-EN       | 0.1594| 0.2072     | 0.7692 | -0.0185| -0.0396| 0.0552   |
| OLS-L2-P     | 0.1279| 0.1566     | 0.8170 | -0.0141| -0.0359| -0.0054  |
| IPO-L2-P     | 0.2065| 0.2176     | 0.9489 | -0.0192| -0.0400| 0.0303   |
| OLS-L1-P     | 0.1207| 0.1705     | 0.7079 | -0.0151| -0.0416| 0.0246   |
| IPO-L1-P     | 0.1958| 0.2371     | 0.8260 | -0.0194| -0.0437| 0.0853   |
| OLS-EN-P     | 0.1159| 0.1596     | 0.7265 | -0.0142| -0.0389| 0.0114   |
| IPO-EN-P     | 0.1810| 0.1981     | 0.9137 | -0.0179| -0.0362| 0.0152   |

Table 2: Out-of-sample MVO costs and economic performance metrics for unconstrained mean-variance portfolios with univariate OLS and IPO prediction models.

Figure 2: Bootstrapped out-of-sample MVO costs and Sharpe ratios for unconstrained mean-variance portfolios with univariate OLS and IPO prediction models.
Figure 3: Out-of-sample equity growth for unconstrained mean-variance portfolios with univariate OLS and IPO prediction models.

3.2 Experiment 2: unconstrained and multivariate

Economic performance metrics are provided in Table 2 for the time period of 2000-01-01 to 2020-12-31 for the unconstrained MVO portfolios with multivariate prediction models. As before, we observe
that, for the OLS prediction models, all regularized MVO portfolios result in lower MVO costs and higher Sharpe ratios than the nominal OLS model; a result that we find encouraging for the stochastic optimization framework. Moreover, the regularized OLS models have more conservative risk metrics, with an over 50% reduction in realized volatility in comparison to the nominal model. All regularized models also exhibit smaller excess mean returns than that of the nominal OLS model. However, the reduction in mean return is more than offset by the reduction in portfolio volatility, thus resulting in Sharpe ratios that range from 10% to 70% larger than that of the nominal OLS model. Once again, we observe the benefit of the parameterized regularization models over the traditional user-defined models. Specifically, the parameterized regularization models exhibit mean returns and Sharpe ratios that are 15% – 20% larger than their user-defined counterpart.

As in the univariate prediction case, the regularized IPO prediction models in general exhibit only a marginal reduction in out-of-sample MVO costs in comparison to the nominal IPO model. Again, this suggests that the nominal IPO model is more resilient to estimation uncertainty than the corresponding nominal OLS model. Indeed, the nominal IPO model exhibits a substantially lower realized MVO costs and 70% greater out-of-sample Sharpe ratio than the nominal OLS model. In this particular experiment, we find that, when evaluating by MVO costs and Sharpe ratio, the user-defined IPO models are competitive with the corresponding parameterized IPO models. When comparing across all models (IPO and OLS), however, we find that the L2-P and EN-P regularized portfolios provide the most consistent performance, with average Sharpe ratios of 0.9412 and 0.8824, respectively.

Figure 4 compares the distribution of out-of-sample realized MVO costs and Sharpe ratios for the unconstrained MVO portfolios with multivariate prediction models. Observe that portfolio regularization models optimized by the stochastic optimization framework result in consistently lower MVO costs and improved Sharpe ratios, irrespective of the choice of regularization model. When comparing across prediction model methods, we find that the IPO model exhibit marginally higher MVO costs, again suggesting the potential for model overfit. We note, however, the difference, is relatively small, with considerable overlap in their distributions. Furthermore, the out-of-sample Sharpe ratios, displayed in Figure 4(b), are generally higher for the models with IPO predictions.
| Model         | Mean  | Volatility | Sharpe | VaR   | Avg DD | MVO Cost |
|--------------|-------|------------|--------|-------|--------|----------|
| Nominal OLS  | 0.2627| 0.4680     | 0.5613 | -0.0395| -0.0963| 0.8321   |
| Nominal IPO  | 0.2889| 0.3084     | 0.9368 | -0.0241| -0.0507| 0.1866   |
| OLS-L2       | 0.1298| 0.1642     | 0.7902 | -0.0159| -0.0343| 0.0051   |
| IPO-L2       | 0.2687| 0.2884     | 0.9318 | -0.0240| -0.0521| 0.1471   |
| OLS-L2-COV   | 0.1091| 0.1842     | 0.5923 | -0.0177| -0.0506| 0.0605   |
| IPO-L2-COV   | 0.2697| 0.2956     | 0.9124 | -0.0244| -0.0512| 0.1671   |
| OLS-L1       | 0.1217| 0.1958     | 0.6214 | -0.0173| -0.0492| 0.0700   |
| IPO-L1       | 0.2157| 0.2517     | 0.8569 | -0.0220| -0.0499| 0.1011   |
| OLS-EN       | 0.1244| 0.1676     | 0.7421 | -0.0160| -0.0359| 0.0161   |
| IPO-EN       | 0.2112| 0.2297     | 0.9198 | -0.0215| -0.0481| 0.0525   |
| OLS-L2-P     | 0.1805| 0.1917     | 0.9418 | -0.0181| -0.0342| 0.0032   |
| IPO-L2-P     | 0.2671| 0.2840     | 0.9406 | -0.0254| -0.0517| 0.1361   |
| OLS-L1-P     | 0.1559| 0.2100     | 0.7423 | -0.0196| -0.0446| 0.0646   |
| IPO-L1-P     | 0.2585| 0.3244     | 0.7967 | -0.0274| -0.0642| 0.2677   |
| OLS-EN-P     | 0.1659| 0.1850     | 0.8969 | -0.0178| -0.0348| 0.0052   |
| IPO-EN-P     | 0.2369| 0.2729     | 0.8679 | -0.0243| -0.0555| 0.1356   |

Table 3: Out-of-sample MVO costs and economic performance metrics for unconstrained mean-variance portfolios with multivariate OLS and IPO prediction models.

Figure 4: Bootstrapped out-of-sample MVO costs and Sharpe ratios for unconstrained mean-variance portfolios with multivariate OLS and IPO prediction models.
Figure 5: Out-of-sample equity growth for unconstrained mean-variance portfolios with multivariate OLS and IPO prediction models.

3.3 Experiment 3: constrained and univariate

Economic performance metrics and average out-of-sample MVO costs are provided in Table 4 for the time period of 2000-01-01 to 2020-12-31 for the constrained MVO portfolios with univariate prediction.
models. First, observe that the MVO costs for the nominal models in the presence of portfolio constraints are substantially smaller than in the previous unconstrained experiments. This is consistent with the fact that portfolio constraints are themselves a form of model regularization. This is demonstrated analytically in Section 2.4, where we observe that the dual of the $L_1$-norm is the $L_\infty$-norm; and therefore box constrained MVO portfolios can be interpreted as a $L_\infty$ norm-penalized quadratic program.

Nonetheless, we observe that for OLS prediction models, all regularized MVO portfolios result in lower or equivalent MVO costs and produce on average higher Sharpe ratios than the nominal OLS counterpart. The increase in out-of-sample Sharpe ratios over the nominal OLS model ranges from 0% - 70%, and as before, the regularized OLS models provide more conservative risk metrics. Once again we observe that the parameterized regularization models produce higher Sharpe ratios and lower MVO costs in comparison to their corresponding user-defined regularization models. Moreover, we note that the OLS models with L2-P and EN-P regularization produce meaningfully higher Sharpe ratios, with values of 0.5837 and 0.5329, respectively. This is in contrast to the nominal program, with a out-of-sample a Sharpe ratio of 0.3416. Again, we view these results as being highly encouraging for the parameterized regularization approach, in which the exact regularization structure is ultimately learned from the training data.

For the IPO prediction models, we observe that regularization results in lower out-of-sample MVO costs and higher Sharpe ratios over the nominal IPO. The improvement in Sharpe ratio ranges from 7% - 20%, with both the IPO-L2 and IPO-L2-P providing the lowest MVO costs and highest Sharpe ratios. We also observe that the regularized models have substantially lower portfolio volatility, value-at-risk and average drawdown. This is consistent with the objective of the norm-penalty, which if properly tuned will provide protection and robustness against estimation uncertainty. We observe this visually in Figure 7, whereby the regularized models exhibit more stable equity growth in comparison to their nominal counterpart.

Figure 6 compares the distribution of out-of-sample realized MVO costs and Sharpe ratios. In the presence of constraints, we observe a marginal reduction in MVO costs, with considerable overlap across all distributions, irrespective of the model. In general, however, we observe that the IPO models have lower out-of-sample MVO costs and higher Sharpe ratios in comparison to their corresponding OLS models. Recall, the OLS prediction model parameters are optimized for prediction accuracy only, and are therefore unaware of the final downstream decision-based optimization and constraints. In contrast, the IPO prediction model parameters are optimized in the context of the decision-based optimization in order to minimize the average realized MVO costs. We observe here that having both the prediction model and regularization parameters aware of the final decision based optimization problem can result in consistently improved economic outcomes. This again is evidence in support of the joint prediction estimation and regularization optimization framework.
| Model        | Mean  | Volatility | Sharpe | VaR   | Avg DD | MVO Cost |
|-------------|-------|------------|--------|-------|--------|----------|
| Nominal OLS | 0.0438| 0.1283     | 0.3416 | -0.0127 | -0.0399 | 0.0385   |
| Nominal IPO | 0.0991| 0.1832     | 0.5411 | -0.0182 | -0.0473 | 0.0687   |
| OLS-L2      | 0.0451| 0.1097     | 0.4115 | -0.0109 | -0.0336 | 0.0151   |
| IPO-L2      | 0.0774| 0.1175     | 0.6589 | -0.0115 | -0.0267 | -0.0084  |
| OLS-L2-COV  | 0.0438| 0.1283     | 0.3415 | -0.0127 | -0.0399 | 0.0385   |
| IPO-L2-COV  | 0.0713| 0.1188     | 0.6000 | -0.0115 | -0.0277 | -0.0007  |
| OLS-L1      | 0.0449| 0.1045     | 0.4298 | -0.0105 | -0.0318 | 0.0097   |
| IPO-L1      | 0.0682| 0.1124     | 0.6064 | -0.0110 | -0.0275 | -0.0050  |
| OLS-EN      | 0.0454| 0.1095     | 0.4145 | -0.0110 | -0.0327 | 0.0145   |
| IPO-EN      | 0.0773| 0.1330     | 0.5809 | -0.0128 | -0.0357 | 0.0112   |
| OLS-L2-P    | 0.0611| 0.1047     | 0.5837 | -0.0104 | -0.0294 | -0.0063  |
| IPO-L2-P    | 0.0718| 0.1108     | 0.6490 | -0.0107 | -0.0261 | -0.0105  |
| OLS-L1-P    | 0.0470| 0.1044     | 0.4500 | -0.0105 | -0.0305 | 0.0075   |
| IPO-L1-P    | 0.0720| 0.1167     | 0.6172 | -0.0115 | -0.0294 | -0.0039  |
| OLS-EN-P    | 0.0554| 0.1040     | 0.5329 | -0.0103 | -0.0273 | -0.0014  |
| IPO-EN-P    | 0.0700| 0.1171     | 0.5981 | -0.0114 | -0.0284 | -0.0015  |

Table 4: Out-of-sample MVO costs and economic performance metrics for constrained mean-variance portfolios with univariate OLS and IPO prediction models.

![Bootstrap Out-of-sample MVO Costs and Sharpe Ratios](image_url)

(a) MVO cost  
(b) Sharpe ratio cost

Figure 6: Bootstrapped out-of-sample MVO costs and Sharpe ratios for constrained mean-variance portfolios with univariate OLS and IPO prediction models.
3.4 Experiment 4: constrained and multivariate

Economic performance metrics and average out-of-sample MVO costs are provided in Table 5 for the time period of 2000-01-01 to 2020-12-31 for the constrained MVO portfolios with multivariate prediction models. As we saw in Experiment 3, the portfolio constraints act as a form of portfolio regularization and results in lower MVO costs and higher Sharpe ratios in comparison to the corresponding unconstrained nominal models. We observe that for the OLS prediction models, all user-defined regularized MVO portfolios provide very similar MVO costs and economic performance.
metrics to that of the nominal OLS counterpart. This is observed in Figure 9 in which the nominal and user-defined models all produce effectively the same equity growth. The parameterized models, on the other hand, exhibit lower out-of-sample MVO costs and, with the exception of the L1-P model, exhibit marginally higher Sharpe ratios. Unlike in experiment 3, the regularized OLS models do not provide materially different risk metrics, suggesting perhaps, that the multivariate prediction model is better specified and therefore is less susceptible to estimation uncertainty. This is further justified by the fact that the nominal OLS models realize a Sharpe ratio of 0.6746, which is greater than the out-of-sample Sharpe ratio of all constrained univariate models. Once again we observe that the L2-P and EN-P parameterized regularization models produce the highest Sharpe ratios and lowest MVO costs.

For the IPO prediction models, we observe that all regularized models exhibit a lower out-of-sample MVO costs and higher or equivalent Sharpe ratios than the nominal IPO counterpart. The improvement in Sharpe ratio ranges from 0% – 38%. Once again, we observe that the parameterized regularization models exhibit lower MVO costs and higher Sharpe ratios in comparison to their corresponding user-defined models; further evidence in support of the parameterized approach.

Finally Figure 8 compares the distribution of out-of-sample realized MVO costs and Sharpe ratios for the constrained MVO portfolios with multivariate prediction models. As we observed in Experiment 3, in the presence of constraints, the regularized models exhibit a marginal improvement in MVO costs and Sharpe ratios in comparison to the nominal counterparts. With the exception of the nominal models, we observe that all other IPO models realize a small, but consistently lower out-of-sample MVO costs and, in general, higher Sharpe ratios in comparison to their corresponding OLS models. Again these results suggest that it is possible to produce an optimal MVO policy that results in improved economic outcomes by making all model parameters aware of the final decision-based objectives and constraints.
## Table 5: Out-of-sample MVO costs and economic performance metrics for constrained mean-variance portfolios with multivariate OLS and IPO prediction models.

| Model         | Mean | Volatility | Sharpe | VaR | Avg DD | MVO Cost |
|---------------|------|------------|--------|-----|--------|----------|
| Nominal OLS   | 0.1195 | 0.1772 | 0.6746  | -0.0178 | -0.0384 | 0.0375   |
| Nominal IPO   | 0.0978 | 0.1687 | 0.5799  | -0.0167 | -0.0470 | 0.0444   |
| OLS-L2        | 0.1201 | 0.1760 | 0.6824  | -0.0176 | -0.0389 | 0.0347   |
| OLS-L2-COV    | 0.1195 | 0.1772 | 0.6746  | -0.0178 | -0.0384 | 0.0375   |
| OLS-L1        | 0.1196 | 0.1768 | 0.6765  | -0.0177 | -0.0384 | 0.0366   |
| OLS-EN        | 0.1199 | 0.1761 | 0.6810  | -0.0176 | -0.0382 | 0.0351   |
| OLS-L2-P      | 0.1250 | 0.1587 | 0.7876  | -0.0158 | -0.0326 | 0.0010   |
| OLS-EN-P      | 0.1125 | 0.1619 | 0.6948  | -0.0158 | -0.0360 | 0.0186   |
| OLS-L1-P      | 0.0970 | 0.1589 | 0.6103  | -0.0158 | -0.0337 | 0.0293   |
| OLS-L1-COV    | 0.1170 | 0.1446 | 0.8091  | -0.0137 | -0.0318 | -0.0124  |
| OLS-L2-P      | 0.1120 | 0.1446 | 0.7747  | -0.0142 | -0.0311 | -0.0075  |

Figure 8: Bootstrapped out-of-sample MVO costs and Sharpe ratios for constrained mean-variance portfolios with multivariate OLS and IPO prediction models.
Figure 9: Out-of-sample equity growth for constrained mean-variance portfolios with multivariate OLS and IPO prediction models.

4 Conclusion and future work

In this paper, we augmented the standard MVO program with a convex combination of parameterized $L_1$ and $L_2$ norm penalty functions. Our choice of the generalized elastic net penalty function was deliberate and motivated by prior work that established the connection between norm penalized portfolio
weights with covariance shrinkage and robust optimization. The resulting program is a parameterized penalized quadratic program (PPQP) whose primal and dual form are shown to be constrained quadratic programs (QPs). The norm-penalty acts to regularize the nominal MVO program, with the objective of immunizing the portfolio construction process to estimation uncertainty. Traditionally, regularization structures are pre-defined by the user and the amount of regularization is determined by heuristic cross-validation procedures. Instead, we proposed a data-driven stochastic optimization framework for optimizing parameterized regularization structures in the context of the final decision-based MVO problem. The objective of the stochastic optimization framework is to produce an optimal mean-variance policy that minimizes the realized mean-variance cost. The framework is highly flexible and capable of jointly optimizing both the prediction and regularization model parameters.

We structure the problem as an end-to-end trainable neural-network whereby the PPQP is embedded as differentiable quadratic programming layer. It is straightforward to recast the $L_2$ norm penalty QP as a standard QP. The $L_1$ norm penalized QP, on the other hand, is not everywhere differentiable, and therefore computing gradients of the $L_1$ penalized QP is difficult. In this paper, we provide two methods for overcoming this challenge: the first is based on a quadratic approximation to the $L_1$ penalized QP at optimality, whereas the second approach solves the dual $L_1$ penalized QP, which is shown to be a standard box-constrained QP.

Numerical results, using real futures price data, demonstrates the efficacy of the stochastic optimization framework in optimizing both the parameterized structure and regularization amount. Regularization parameters are optimized locally using the first-order method proposed by Butler and Kwon [19]. We consider linear prediction models, optimized in two ways: standard ordinary-least-squares (OLS) and a fully integrated approach (IPO), which optimizes the prediction and regularization model parameters jointly. The models are evaluated based on 20 years of out-of-sample performance, with a focus on realized MVO cost and out-of-sample Sharpe ratios. We demonstrate, across 4 different experiments, that in almost every case the norm-penalized MVO portfolios produce a mean-variance policy that exhibit lower out-of-sample MVO costs and higher Sharpe ratios, in comparison to the corresponding nominal MVO counterpart. We observe the largest relative improvement in regularized OLS prediction models. This result is encouraging as it suggests that the decision-errors induced by the OLS predictions can largely be corrected by appropriate portfolio regularization. IPO prediction models, on the other hand, are shown to already be relatively robust to estimation uncertainty. Nonetheless, the combination of IPO predictions and norm-penalized portfolio regularization can result in marginally lower MVO costs and improved economic outcomes.

We also evaluate the effectiveness of the parameterized regularization models, in which the exact regularization structure is learned from the data, and compare to several user-defined regularization models. In general, we observe that the parameterized regularization models, in particular the parameterized $L_2$ (L2-P) and parameterized elastic net (EN-P) models exhibit the lowest MVO costs and highest Sharpe ratios, on average. This result is highly encouraging as it demonstrates that the user does not necessarily need to specify the exact regularization structure in advance and instead, parameterized regularization structures can be learned directly from the data.

Our experiments should be interpreted as a proof-of-concept, rather than a fully comprehensive financial study and we acknowledge that further testing with alternative data sets, and under varying prediction model and portfolio constraint assumptions is required in order to better determine the efficacy of the stochastic optimization framework. In particular, there is a risk of model overfit
when both prediction and regularization model parameters are optimized in a joint manner. We are interested in determining, under what conditions are we most likely to observe negative economic outcomes as a result of an MVO policy that overfits the training data. Furthermore, we believe that a stochastic optimization framework whereby model parameters are optimized in order to minimize the realized cross-validation cost is interesting and may attenuate the propensity for model overfit. Alternatively, methods for regularizing the prediction model parameters in an fully integrated setting, as well as methods for choosing the ‘best’ feature subsets for large multivariate regression problems are an interesting area of future research.

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A Proof of Proposition 1

Without loss of generality we consider the following convex program:

\[
\begin{align*}
\text{minimize} \quad & c(z) = \frac{1}{2} z^T Q z + z^T p + \gamma_1 \|Ez\|_1 \\
\text{subject to} \quad & Az = b, \quad Gz \leq h.
\end{align*}
\] (34)
Let \( Q \succ 0 \) and \( E \) be such that \( Ez \) preserves the sign of \( z \) and \( E^T E \succeq 0 \). It follows then that Program (34) is strictly convex with a unique minimize \( z^* \).

The sub-differential of \( c(z) \) is given as:

\[
\partial c = Qz + p + ET \phi(Ez),
\]

(35)

where

\[
\phi(z) = \begin{cases} 
\text{sign}(z), & \text{if } z \neq 0 \\
[-1, 1], & z = 0
\end{cases}.
\]

Let \((z^*, \lambda^*, \nu^*)\) denote the optimal primal-dual pair. The KKT optimality conditions of Program (34) therefore must satisfy:

\[
p + Qz^* + G^T \lambda^* + A^T \nu^* + \gamma_1 E^T \phi(Ez^*) = 0
\]

\[
(Gz^* - h) \leq 0
\]

\[
\lambda^* \geq 0
\]

\[
\lambda^* \odot (Gz^* - h) = 0
\]

\[
A z^* = b
\]

(37)

The IRLS approximation to Program (34) is a strictly convex quadratic program given as:

\[
\text{minimize } c_\tau(z) = \frac{1}{2} z^T Qz + z^T p + \frac{1}{2} \|Ez\|_1 + \frac{1}{2} \|Ez\|_2^2 - \tau \|Ez\|_2^2
\]

subject to \( A z = b, \quad G z \leq h \)

(38)

Let \( z^*_\tau \) denote the optimal solution to Program (38). Clearly, the primal-dual solution, \((z^*, \lambda^*, \nu^*)\), satisfies primal-dual feasibility and complimentary slackness conditions for Program (34). Evaluating primal stationarity gives:

\[
p + Qz^* + G^T \lambda^* + A^T \nu^* + \gamma_1 (\|Ez^*\|_1 + \tau)^{-1} E^T Ez
\]

\[
= \gamma_1 E^T \left( \text{diag}(|Ez^*| + \tau)^{-1} Ez^* - \phi(Ez^*) \right)
\]

(39)

Evaluating at the limit as \( \tau \to 0 \):

\[
\lim_{\tau \to 0} \left( \text{diag}(|Ez^*| + \tau)^{-1} Ez^* - \phi(Ez^*) \right) = 0.
\]

(40)

Therefore as \( \tau \to 0 \), then \((z^*, \lambda^*, \nu^*)\) satisfies the necessary and sufficient conditions for optimality of Program (34) and by strict convexity it follows that \( z^*_\tau = z^* \). Direct substitution of the primal solutions gives: \( c_\tau(z^*_\tau) = c(z^*) \).
B Numerical Validation of Proposition 1

Figure 10 plots the absolute error between $z^*$ and the approximation $z^*_\tau$ as a function of $\tau$. The boxplots provide the distribution of errors over the full sample data (8983 observations) for the standard $L_1$ norm and parameterized $L_1$ norm implementations. Observe that the absolute error becomes vanishingly small as $\tau$ goes to zero. In most cases, the choice of $\tau$ will be problem specific and in our case, we find that a value of $\tau = 10^{-4}$ provides an absolute error less than $10^{-5}$; which is more than sufficient level of accuracy.

Figure 10: Absolute error, $\|z^*_\tau - z^*\|_1$, of optimal $L_1$ norm penalized portfolio solutions and its second-order approximation as a function of $\tau$.

The convergence plot presented in Figure 11 demonstrate that the second-order approximation to the $L_1$ norm is sufficiently accurate and smooth and thus amenable to optimization by gradient descent as per the methods outlined in Section 2.

Figure 11: Training error convergence plot of the second-order approximation to the $L_1$ norm and parameterized $L_1$ norm implementations.
C Implementation details

For IPO models, the linear prediction model is implemented using a single-layer linear neural network with the appropriate input and output layer dimensions. The norm-penalty coefficient, $\gamma$, is constrained to the positive orthant and is satisfied by applying following exponential transformation:

$$\gamma \leftarrow 10^\gamma.$$

The diagonal parameterized regularization matrices are uniquely defined as: $D(\theta_D) = \text{diag}(\theta_D)$ and $E(\theta_E) = \text{diag}(\theta_E)$. In order to ensure convexity of the norm-constrained MVO program, the regularization parameters, $\theta_D$ and $\theta_E$ and constrained to be positive, which we implement by applying a standard sigmoid transformation:

$$\theta_D \leftarrow \frac{1}{1 + e^{-\theta_D}} \quad \text{and} \quad \theta_E \leftarrow \frac{1}{1 + e^{-\theta_E}}$$

All model parameters are initialized randomly from the following uniform distributions:

$$\beta \in [-3, 3]^{d_z \times d_x}, \quad \gamma \in [-3, 3], \quad \theta_D \in [-3, 3]^{d_z}, \quad \theta_E \in [-3, 3]^{d_z}$$

All parameters are optimized by employing the ADAM stochastic gradient descent routine \[44\]. We find that using 100% of the training observations with a learning rate of 0.10 results in numerically stable and consistent parameter estimation.

All experiments were conducted on an Apple Mac Pro computer (2.7 GHz 12-Core Intel Xeon E5, 128 GB 1066 MHz DDR3 RAM) running macOS ‘Catalina’. The software was written using the R programming language (version 4.0.0) and torch (version 0.2.0).