Consistency Conditions for AdS/CFT Embeddings

Paul H. Frampton\(^{(a)}\) and Thomas W. Kephart\(^{(b)}\)

\(^{(a)}\)Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599.

\(^{(b)}\)Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37325.

Abstract

The group embeddings used in orbifolding the AdS/CFT correspondence to arrive at quiver gauge field theories are studied for both supersymmetric and non-supersymmetric cases. For an orbifold $AdS_5 \times S^5/\Gamma$ the conditions for embeddings of the finite group $\Gamma$ in the $SU(4) \sim O(6)$ isotropy of $S^5$ are stated in the form of consistency rules, both for Abelian and Non-Abelian $\Gamma$. 
Introduction

Independently of the important question of whether or not superstring theory, or M theory, is the correct theory of quantum gravity, superstring theory can be used through the AdS/CFT correspondence [1] to arrive at interesting gauge field theories in four spacetime dimensions. According to the AdS/CFT correspondence, a Type IIB superstring compactified on an $AdS_5 \times S^5$ background is dual to an $\mathcal{N} = 4$ supersymmetric gauge field theory in flat 4-dimensional spacetime. The $\mathcal{N} = 4$ theory is much too symmetric to be of phenomenological interest. What makes the AdS/CFT correspondence far more interesting is the possibility of orbifolding $S^5 \rightarrow S^5/\Gamma$ where $\Gamma$ is a finite group embedded in the isometry $SU(4) \sim O(6)$ of $S^5$ in the 3-dimensional complex space $\mathbb{C}^3 \sim \mathbb{R}^6$. Although $O(6)$ and $SU(4)$ have the same local structure, i.e. Lie algebra, globally $O(6)$ is doubly-covered by $SU(4)$ and this distinction will be important to us here.

Another consideration, already discussed fully in [2], is the survival of chiral fermions. The correct statement of the necessary and sufficient condition is that the $4$ of $SU(4)$ must be neither real nor pseudoreal for chiral fermions to be present in the quiver gauge field theory.

It is often stated that the number of surviving supersymmetries is $\mathcal{N} = 2, 1, 0$ according to whether $\Gamma \subset SU(2), SU(3), SU(4)$. While this statement essentially survives our analysis, the statement itself needs sharpening, in particular by a more careful definition of the embedding. The present article is devoted to addressing that issue.

$\mathcal{N} = 1$ Embeddings of Abelian $\Gamma$ in $SU(4)$ and $O(6)$.

To preserve $\mathcal{N} = 1$ supersymmetry one must keep exactly one invariant spinor under the joint action of the finite symmetry $\Gamma$ and the quiver gauge group, which for Abelian $\Gamma$ is $SU(N)^p$ where $p$ is the order of $\Gamma$. This implies that one component of the $4$ of $SU(4)$ is the trivial singlet representation of $\Gamma$. 
The most general Abelian $\Gamma$ of order $g = \sigma(\Gamma)$ is $[3,4]$ made up of the basic units $Z_p$, the order $p$ cyclic groups formed from the $p^{th}$ roots of unity. It is important to note that the product $Z_pZ_q$ is identical to $Z_{pq}$ if and only if $p$ and $q$ have no common prime factor.

If we write the prime factorization of $g$ as:

$$ g = \prod_i p_i^{k_i} $$

where the product is over primes, it follows that the number $N_a(g)$ of inequivalent abelian groups of order $g$ is given by:

$$ N_a(g) = \prod_i P(k_i) $$

where $P(x)$ is the number of unordered partitions of $x$. For example, for order $g = 144 = 2^43^2$ the value would be $N_a(144) = P(4)P(2) = 5 \times 2 = 10$. For $g \leq 31$ it is simple to evaluate $N_a(g)$ by inspection. $N_a(g) = 1$ unless $g$ contains a nontrivial power ($k_i \geq 2$) of a prime. These exceptions are: $N_a(g = 4, 9, 12, 18, 20, 25, 28) = 2; N_a(8, 24, 27) = 3; \text{and } N_a(16) = 5$. This confirms that:

$$ \sum_{g=1}^{31} N_a(g) = 48 $$

Let us define $\alpha = \exp(2\pi i/p)$ and write the embedding as $4 = (\alpha^{A_1}, \alpha^{A_2}, \alpha^{A_3}, \alpha^{A_4})$. For $N = 1$ we have agreed that one $A_\mu$ must vanish so let us put $A_4 = 0$ and denote the embedding by the shorthand $\mathbf{A} \equiv (A_1, A_2, A_3)$.

Consider now the $6$ of $SU(4)$ which is the antisymmetric part in $(4 \times 4)$. The $6$ is necessarily real, and in the double covering of $O(6)$ is identified with the defining "vector" representation of $O(6)$. In the shorthand notation already adopted for the $4$, the $6$ is $6 = (A_1, A_2, A_3, A_1 + A_2, A_2 + A_3, A_3 + A_1)$.

For general $\mathbf{A}$, the $6$ is not real but we must now impose the condition that the diagonal $3 \times 3$ matrix with diagonal elements $\alpha A_\mu$ be a transformation belonging to $SU(3)$, i.e. a matrix which is unitary and of determinant one. This requires that $\sum_i A_i = 0 \mod p$. But with that condition we can rewrite (up to mod $p$) $6 \equiv (A_1, A_2, A_3, -A_3, -A_2, -A_1)$ which is manifestly real and hence we have a consistent embedding.
Thus, in this the simplest case, the embedding is uniquely defined for the 4 of $SU(4)$ by the three components (recall $A_4 \equiv 0$) of $\mathbf{A} \equiv (A_1, A_2, A_3)$ which for consistency must satisfy $A_1 + A_2 + A_3 = 0 \pmod{p}$. This will always lead to a consistent $\mathcal{N} = 1$ chiral quiver gauge theory.

$\mathcal{N} = 0$ Embeddings of Abelian $\Gamma$ in $SU(4)$ and $O(6)$.

For $\mathcal{N} = 0$ the discussion is similar to the above except that $A_4$ does not vanish. Therefore the considerations involve not $\mathbf{A}$ with three components but $A_\mu = (A_1, A_2, A_3, A_4)$ with four components.

Consider now the 6 of $SU(4)$ which is the antisymmetric part in $(4 \times 4)$. The 6 can be written $6 = (A_1 + A_4, A_2 + A_4, A_3 + A_4, A_1 + A_2, A_2 + A_3, A_3 + A_1)$.

But here for consistency the $4 \times 4$ diagonal matrix with diagonal elements $\alpha^{A_\mu}$ must be an $SU(4)$ transformation which requires $\sum_\mu A_\mu = 0 \pmod{p}$ in which case the 6 is again manifestly real.

For a consistent $\mathcal{N} = 0$ quiver gauge field theory, therefore, the 4 of $SU(4)$ is defined by four non-vanishing integers $A_\mu$ satisfying $A_1 + A_2 + A_3 + A_4 = 0 \pmod{p}$. This is the necessary and sufficient condition for consistent Abelian orbifolding to $\mathcal{N} = 0$. Similar results hold for products of abelian groups, e.g., for $\Gamma = Z_p Z_q$ we can write $4 = (e^{A_1} e^{B_1}, e^{A_2} e^{B_2}, e^{A_3} e^{B_3}, e^{A_4} e^{B_4})$ where we require $\sum_\mu A_\mu = 0 \pmod{p}$ and $\sum_\mu B_\mu = 0 \pmod{q}$.

$\mathcal{N} = 1$ Embeddings of Non-Abelian $\Gamma$ in $SU(4)$ and $O(6)$.

As usual, going from Abelian $\Gamma$ to Non-Abelian $\Gamma$ makes everything more complicated. This is why paper [2] is necessarily much longer than e.g. paper [5]!

The complication arises because for Non-Abelian $\Gamma$, the irreducible representations $\rho_i$ can have dimensions $d_i$ greater than one. If the order of $\Gamma$ is $g$ the dimensions must satisfy $\sum_i d_i^2 = g$. The resultant gauge group for the quiver theory is $\otimes_i SU(nd_i)$. Because of this
more complex quiver theories emerge from Non-Abelian $\Gamma$ than are possible for Abelian $\Gamma$. On the other hand, as shown in [2], the constraints imposed phenomenologically are also more complex to such an extent that for all $g \leq 31$ only one acceptable theory occurs amongst many hundreds of candidates.

For the Non-Abelian embeddings we must again study whether the $6 \equiv (4 \times 4)_a$ is consistently real. Let us, for the $\mathcal{N} = 1$ case write $4 = (\sum_i' \rho_i', 1)$ where $\sum d_i' = 3$ and where none of the $\rho_i'$ is the identity representation. (Here the set $\{i\}'$ is a subset of the $\{i\}$). This will generically lead to an $\mathcal{N} = 1$ supersymmetric quiver gauge theory.

The consistency requirements for this Non-Abelian embedding are very demanding and may be stated as the three rules for $\mathcal{N} = 1$:

1. Exactly one component of the $4$ of $SU(4)$ must be the identity of $\Gamma$. That is, $4 = (\sum_i' \rho_i', 1)$ where the $\rho_i'$ are irreducible representations (not the identity) of $\Gamma$ satisfying $\sum d_i' = 3$.

2. Written as a block-diagonal matrix the determinant of the $3 \times 3$ matrix $\otimes_i' \rho_i'$ must be real and equal to one. Note that this reduces to the vanishing trace condition if we were to break $\Gamma$ to an abelian subgroup.

To see just how demanding these three rules are consider even the simplest Non-Abelian finite group $\Gamma = S_3 \equiv D_3$. The order is $g = 6$ and the available irreducible representations are $\rho = 1, 1', 2$. The only choice satisfying Rules 1 and 2 is $4 = (1', 2, 1)$ and this satisfies Rule 2 because 2 is an $SU(2)$ matrix which necessarily has determinant equals one.
$\mathcal{N} = 0$ Embeddings of Non-Abelian $\Gamma$ in $SU(4)$ and $O(6)$.

The consistency requirements for Non-Abelian embedding to give $\mathcal{N} = 0$ are similar to but different from those given for $\mathcal{N} = 0$ and may be stated as the two new rules:

1. No component of the 4 of $SU(4)$ may be the identity of $\Gamma$. That is, $4 = (\sum \rho_i')$ where the $\rho_i'$ are irreducible representations (not the identity) of $\Gamma$ satisfying $\sum d_i' = 4$.

2. Written as a block-diagonal matrix the determinant of the $4 \times 4$ matrix $\otimes_i \rho_i'$ must be real and equal to one.

To check how these rules work, consider again the simplest Non-Abelian finite group $\Gamma = S_3 \equiv D_3$. The only choice satisfying Rules 1 and 2 is $4 = (2, 2)$. Therefore $\Gamma = S_3$ can be used to obtain either an $\mathcal{N} = 1$ or an $\mathcal{N} = 0$ quiver gauge field theory. [The only other consistent embeddings of $S_3$ in $SU(4)$ are $4 = (1, 1', 1')$ which leads to $\mathcal{N} = 2$ and the trivial embedding which leaves $\mathcal{N} = 4$.]
Summary and Discussion.

Let us summarize the consistency conditions for the four cases considered:

**Abelian** \( \Gamma; \mathcal{N} = 1 \)

\[ A_4 = 0 \pmod{p} \text{ and } A_1 + A_2 + A_3 = 0 \pmod{p}. \]

**Abelian** \( \Gamma; \mathcal{N} = 0 \)

\[ A_1 + A_2 + A_3 + A_4 = 0 \pmod{p}. \]

**Non-Abelian** \( \Gamma; \mathcal{N} = 1 \)

1. Exactly one component of the 4 of \( SU(4) \) must be the identity of \( \Gamma \).
2. The determinant of the 3 \( \times \) 3 matrix \( \otimes_{\iota'} \rho_{\iota'} \) must be real and equal to one.

**Non-Abelian** \( \Gamma; \mathcal{N} = 0 \)

1. No component of the 4 of \( SU(4) \) may be the identity of \( \Gamma \).
2. The determinant of the 4 \( \times \) 4 matrix \( \otimes_{\iota'} \rho_{\iota'} \) must be real and equal to one.

Fortunately, these consistency conditions are respected by most of the earlier work on Abelian \( \Gamma \). For example, the TeV unification model in [6] which provides an alternative to SUSYGUTs [7] is consistent. Most of the Abelian models in [8] are consistent but 7 of the 60 which are called non-partition are not. Even though the embeddings of these seven models are inconsistent, the models are themselves consistent gauge theories and of interest in their own right. Despite the fact that they are not derivable from orbifolding \( AdS \times S^5 \), they have many of the features of orbifolded \( AdS \times S^5 \) models, e.g., vanishing first-order (and perhaps higher order) \( \beta \) functions, similar boson-fermion counting providing \( \mathcal{N} = 1 \) SUSY, etc. It can be shown that these are the only models besides the partition models with a real 6, and that they fall into three classes. Recall first that the \( \mathcal{N} = 1 \) partition models for \( \Gamma = Z_p \) can be written as \( M_{p_1,p_2,p_3}^p \) where \( p_1 + p_2 + p_3 = 0 \pmod{p} \). In this notation the nonpartition models are \( M_{p,p,2p}^{3p}, M_{p,2p,4p}^{6p}, \) and \( M_{p,4p,7p}^{9p} \). Along with their own phenomenological interest, these models can probably be used in comparison with properly embedded partition orbifolded models to investigate violation of conformal invariance, etc. In the \( \mathcal{N} = 0 \) case, again one finds partition models \( M_{p_1,p_2,p_3,p_4}^p \) with \( p_1 + p_2 + p_3 + p_4 = 0 \pmod{p} \) are properly embedded. But
again there exist improperly-embedded, but phenomenologically-interesting, non-partition models with complex 4 and real 6 [9].

For Non-Abelian $\Gamma$ the most extensive work is in our earlier paper [2]. The only phenomenologically-acceptable model out of the many considered in [4] was for the choice $\Gamma \equiv 24/7$, in the notation of Thomas and Wood [3] for $D_4 \times Z_3$. The embedding used is $4 = (1\alpha, 1\alpha', 2\alpha)$ and this satisfies Rules 1 and 2 for Non-Abelian $\Gamma$ and $N = 0$. It is therefore a quiver gauge theory which arises from a consistent embedding of $\Gamma$ in $AdS_5 \times S^5/\Gamma$. 
Acknowledgements

TWK thanks the Department of Physics and Astronomy at UNC Chapel Hill and PHF thanks the Department of Physics and Astronomy at Vanderbilt University for hospitality while this work was in progress. This work was supported in part by the US Department of Energy under Grants No. DE-FG02-97ER-41036 and No. DE-FG05-85ER40226.
REFERENCES

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998). hep-th/9711200

S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. B428, 105 (1998).
hep-th9802109. E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998). hep-th/9802150.

[2] P.H. Frampton and T.W. Kephart, Phys. Rev. D64, 086007 (2001). hep-th/0011186.

[3] Useful sources of information on the finite groups include:

D. E. Littlewood, The Theory of Group Characters and Matrix Repesentations of Groups, Oxford (1940).

M. Hamermesh, Group Theory and Its Applications to Physical Problems, Addison-Wesley (1962).

J. S. Lomont, Applications of Finite Groups, Academic Press (1959), reprinted by Dover (1993).

A. D Thomas and G. V. Wood, Group Tables, Shiva Publishing (1980).

[4] P.H. Frampton and T.W. Kephart, Int. J. Mod. Phys. A10, 4689 (1995).

[5] P.H. Frampton, Phys. Rev. D60, 121901 (1999). hep-th/9907051.

[6] P.H. Frampton, Mod. Phys. Lett. A (in press). hep-ph/0208044.

P.H. Frampton and T.W. Kephart. hep-ph/0306053.

[7] U. Amaldi, et al. Phys. Lett. B281, 374 (1992).

[8] T.W. Kephart and H. Päs, Phys. Lett. B522, 315 (2001). hep-ph/0109111.

[9] T. W. Kephart and H. Päs, to appear.