A Direct Estimation of High Dimensional Stationary Vector Autoregressions

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Abstract

The vector autoregressive (VAR) model is a powerful tool in modeling complex time series and has been exploited in many fields. However, fitting high dimensional VAR model poses some unique challenges: On one hand, the dimensionality, caused by modeling a large number of time series and higher order autoregressive processes, is usually much higher than the time series length; On the other hand, the temporal dependence structure in the VAR model gives rise to extra theoretical challenges. In high dimensions, one popular approach is to assume the transition matrix is sparse and fit the VAR model using the “least squares” method with a lasso-type penalty.

In this manuscript, we propose an alternative way in estimating the VAR model. The main idea is, via exploiting the temporal dependence structure, to formulate the estimating problem into a linear program. There is instant advantage for the proposed approach over the lasso-type estimators: The estimation equation can be decomposed into multiple sub-equations and accordingly can be efficiently solved in a parallel fashion. In addition, our method brings new theoretical insights into the VAR model analysis. So far the theoretical results developed in high dimensions (e.g., Song and Bickel (2011) and Kock and Callot (2012)) mainly pose assumptions on the design matrix of the formulated regression problems. Such conditions are indirect about the transition matrices and not transparent. In contrast, our results show that the operator norm of the transition matrices plays an important role in estimation accuracy. We provide explicit rates of convergence for both estimation and prediction. In addition, we provide thorough experiments on both synthetic and real-world equity data to show that there are empirical advantages of our method over the lasso-type estimators in both parameter estimation and forecasting.

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1 Introduction

The vector autoregressive (VAR) model plays a fundamental role in analyzing multivariate time series data and has numerous applications in areas including finance (Tsay, 2005), econometrics (Sims, 1990), and brain imaging data analysis (Valdés-Sosa et al., 2005). For example, in understanding the brain connectivity network, multiple resting-state functional magnetic resonance imaging (rs-fMRI) data are collected by consecutively scanning the same subject at hundreds of time points. This produces a high dimensional dataset with significant amounts of temporary dependency. One common strategy in analyzing such data is via the VAR modeling (see Qiu et al. (2013) and the references therein).

This manuscript considers the problem of estimating the VAR model. Our focus is on the stationary vector autoregression model with the order (or called lag) \( p \) and Gaussian noises. More specifically, let \( x_1, \ldots, x_T \) be observed data points of random vectors \( X_1, \ldots, X_T \) from a stochastic process \( (X_t)_{t=-\infty}^{\infty} \). Each \( X_t \in \mathbb{R}^d \) is a \( d \)-dimensional random vector and satisfies

\[
x_t = \sum_{k=1}^{p} A_k^T X_{t-k} + Z_t, \quad Z_t \sim N_d(0, \Psi),
\]

where \( A_1, \ldots, A_p \) are called the transition matrices and \( (Z_t)_{t=-\infty}^{\infty} \) are independent multivariate Gaussian noises. Under the condition that \( \det(I_d - \sum_{k=1}^{p} A_k^T z^k) \neq 0 \) for all \( z \in \mathbb{C} \) (Here \( \mathbb{C} \) is the set of all complex numbers) with modulus not greater than 1, we then have the process is stationary (check, for example, Section 2.1 in Lütkepohl (2005)) and \( X_t \sim N_d(0, \Sigma) \) for some covariance matrix \( \Sigma \) depending on \( \{A_k\} \) and \( \Psi \).

There are mainly three goals in analyzing the VAR model. One is to estimate the transition matrices \( A_1, \ldots, A_p \). Estimating these transition matrices reveal the temporal dependence in the data sequence and builds a first step in forecasting. Moreover, the zero and nonzero entries in the transition matrices directly incorporate the Granger non-causalities and causalities with regard to the stochastic sequence (see, for example, Corollary 2.2.1 in Lütkepohl (2005)). A second goal is to estimate the error covariance \( \Psi \), which reveals the contemporaneous interactions between the \( d \) time series. Finally, by merely treating the temporal dependence as another measure of the data dependence (in parallel to the mixing conditions (Bradley, 2005)), a third goal is to estimate the covariance matrix \( \Sigma \).

This manuscript focuses on estimating the transition matrices \( A_1, \ldots, A_p \), while noting that the techniques developed here can also be exploited to estimate the covariance matrix.
Σ and the noise covariance Ψ. We first review the transition matrix estimation methods in the literature. Let $A = (A^T_1, \ldots, A^T_p)^T \in \mathbb{R}^{dp \times d}$ be a combination of the transition matrices. Given the data points $x_1, \ldots, x_T$, the perhaps most classic method in estimating $A$ is least squares minimization (Hamilton, 1994):

$$\hat{A}_{\text{LSE}} = \arg\min_{M \in \mathbb{R}^{dp \times d}} \| Y - M^T X \|^2_F,$$

where $\| \cdot \|_F$ is the Frobenius norm, $Y = (x_{p+1}, \ldots, x_T) \in \mathbb{R}^{d \times (T-p)}$, and

$$X = \{(x^T_p, \ldots, x^T_1)^T, \ldots, (x^T_{T-1}, \ldots, x^T_{T-p})^T\} \in \mathbb{R}^{(dp) \times (T-p)}.$$

However, one caveat in (1.2) is that the product of the order of the autoregression $p$ and the number of time series $d$ is in general larger than the time series length $T$. Therefore, the model needed to be carefully regularized to avoid the curse of dimensionality. A common strategy is to add a sparsity-inducing penalty on the transition matrices so that the number of nonzero entries is less than $T$. There has been a large literature studying the incorporation of different penalty terms to (1.2) for regularizing the estimator, including the ridge-penalty, lasso-penalty, and more non-concave penalty terms. For example, Hamilton (1994) discussed the use of the ridge penalty $\| M \|^2_F$ in estimating the transition matrices. Hsu et al. (2008) proposed to add the $L_1$-penalty in estimating the transition matrices, inducing a sparse output. For transition matrix estimation in the VAR mode, Wang et al. (2007) exploited the $L_1$-penalty in simultaneously estimating the regression coefficients and determining the number of lags in a linear regression model with autoregressive errors. In detecting causality, Haufe et al. (2008) transferred the problem to estimating transition matrices in an VAR model and advocated the usage of a group lasso penalty for inducing joint sparsity for blocks of coefficients. In the graphical granger causality study, Shojaie and Michailidis (2010) exploited the VAR model and proposed to estimate the coefficients using a truncated weighted $L_1$-penalty. Song and Bickel (2011) exploited the $L_1$ penalty in a complicated VAR model and aimed to select the variables and lags simultaneously.

The theoretical properties of the $L_1$-regularized estimator has been analyzed in Bento et al. (2010), Nardi and Rinaldo (2011), Song and Bickel (2011), and Kock and Callot (2012) under the assumption that the matrix $A$ is sparse, i.e., the number of nonzero entries in $A$ is much less than the dimension of parameters $pd^2$. Nardi and Rinaldo (2011) provided both support set and parameter estimation consistency under a relatively low dimensional setting with $d = o(n^{1/2})$. Bento et al. (2010) studied the problem of estimating support sets of the transition matrices in the high dimensional settings and proposed an “irrepresentable condition” similar as what has been used in studying linear regression models (Zou, 2006; Zhao and Yu, 2006; Meinshausen and Bühlmann, 2006; Wainwright, 2009) for the $L_1$ regularized estimator to attain the support set selection consistency. In parallel,
Song and Bickel (2011) and Kock and Callot (2012) studied the parameter estimation and support set selection consistency of the $L_1$-regularized estimators in high dimensions.

In this paper, we propose a new approach to estimate the transition matrix $A$. Different from the line of lasso-typo estimators, we exploit the linear programming technique and the proposed method is very fast to solve via parallel computing. Moreover, we do not need the transition matrix $A$ to be exactly sparse and allow it to be only “weakly sparse”. The main idea is to estimate $A$ using the relationship between $A$ and the marginal and lag 1 autocovariance matrices (such a relationship is referred to as the Yule-Walker equation), and thus formulate the estimation procedure to a linear problem. Here we note that the proposed procedure can be considered as a generalization of the Dantzig selector (Candes and Tao, 2007) to the linear regression model with multivariate response. Indeed, our proposed method can also be exploited in conducting multivariate regression (Breiman and Friedman, 1997).

The proposed method enjoys several advantages compared to the existing ones: (i) Computationally, our method can be formulated into $d$ linear programs and can be solved in parallel. Similar ideas have been used in learning high dimensional linear regression (Candes and Tao, 2007; Bickel et al., 2009) and graphical models (Yuan, 2010; Cai et al., 2011). (ii) Methodologically, our method allows $A$ to be only weakly sparse. (iii) Theoretically, most existing analysis of the lasso-type estimators (Song and Bickel, 2011; Kock and Callot, 2012) depends on certain regularity conditions — for examples, the restricted eigenvalue conditions on the design matrix — which are not transparent and do not explicitly reveal the impact of temporal dependence on estimation. In contrast, we provide explicit nonasymptotic analysis, and our analysis shows that the operator norm of the transition matrix $\|A\|_2$ plays an important role. Moreover, for exact sign recovery, our analysis does not need the “irrepresentable condition” which is usually required in the analysis of lasso-type estimators (Bento et al., 2010).

The major theoretical results are briefly stated as follows. We adopt a double asymptotic framework where $d$ is allowed to increase with $T$. We call a matrix $s$-sparse if there are at most $s$ nonzero elements on each of its column. Under mild conditions, we provide the explicit rates of convergence of our estimator $\hat{A}$ under the assumption that $A$ is $s$-sparse (Cai et al., 2011). In particular, for a lag 1 time series, we show that

$$\|\hat{A} - A\|_1 = O_P \left\{ \frac{s\|A\|_1}{1 - \|A\|_2} \left( \frac{\log d}{T} \right)^{1/2} \right\}, \quad \|\hat{A} - A\|_{\max} = O_P \left\{ \frac{\|A\|_1}{1 - \|A\|_2} \left( \frac{\log d}{T} \right)^{1/2} \right\},$$

where $\|\cdot\|_{\max}$ and $\|\cdot\|_q$ represents the matrix elementwise absolute maximum norm ($L_{\max}$ norm) and induced $L_q$ norm (detailed definitions will be provided in §2). Using the $L_{\max}$ norm consistency result, we further provide the sign recovery consistency of the proposed
method. This result is of self interest and sheds light to detecting Granger causality. We also characterize the prediction performance based on the $L_1$ consistency result and show that the element-wise error in prediction can be controlled. Here, for simplicity, we only provide the results when $A$ is exactly sparse and defer the presentation of the results for weakly sparse matrix to Section 4.

The rest of the paper is organized as follows. In §2, we briefly review the vector autoregressive model. In §3, we introduce the proposed method for estimating the transition matrices of the vector autoregressive model. In §4, we provide the main theoretical results. In §5, we apply the new method to both synthetic and real equity data to illustrate its effectiveness. More discussions are provided in the last section. Detailed technical proofs are provided in the appendix\(^1\).

### 2 Background

In this section, we briefly overview the vector autoregressive model. Let $M = (M_{jk}) \in \mathbb{R}^{d \times d}$ and $v = (v_1, ..., v_d)^T \in \mathbb{R}^d$. We denote $v_I$ to be the subvector of $v$ whose entries are indexed by a set $I \subset \{1, \ldots, d\}$. We also denote $M_{I,J}$ to be the submatrix of $M$ whose rows are indexed by $I$ and columns are indexed by $J$. We denote $M_{I,*}$ to be the submatrix of $M$ whose rows are indexed by $I$, $M_{*,J}$ to be the submatrix of $M$ whose columns are indexed by $J$. For $0 < q < \infty$, we define the $L_0$, $L_q$, and $L_\infty$ vector (pseudo-)norms to be

$$
\|v\|_0 = \sum_{j=1}^{d} I(v_j \neq 0), \quad \|v\|_q = \left( \sum_{j=1}^{d} |v_j|^q \right)^{1/q}, \quad \text{and} \quad \|v\|_\infty = \max_{1 \leq j \leq d} |v_j|,
$$

where $I(\cdot)$ is the indicator function. Letting $M$ be a matrix, we denote the matrix $L_q$, $L_{\max}$, and Frobenius norms to be

$$
\|M\|_q = \max_{\|v\|_q = 1} \|Mv\|_q, \quad \|M\|_{\max} = \max_{jk} |M_{jk}|, \quad \text{and} \quad \|M\|_F = \left( \sum_{jk} |M_{jk}|^2 \right)^{1/2}.
$$

Let $1_d = (1, \ldots, 1)^T \in \mathbb{R}^d$. Let $\sigma_1(M) \geq \cdots \geq \sigma_d(M)$ be the singular values of $M$.

Let $p \geq 1$ be an integer. A lag $p$ vector autoregressive process can be elaborated as follows: Let $(X_t)_{t=-\infty}^{\infty}$ be a stationary sequence of random vectors in $\mathbb{R}^d$ with mean 0 and covariance matrix $\Sigma$. We say that the stochastic process $(X_t)_{t=-\infty}^{\infty}$ follows a lag $p$ vector autoregressive model if and only if it satisfies

$$
X_t = \sum_{k=1}^{p} A_k^T X_{t-k} + Z_t \quad (t \in \mathbb{Z}). \tag{2.1}
$$

\(^1\)Some of the results in this paper were first stated without proof in a conference version (Han and Liu, 2013)
Here $A_1, \ldots, A_p$ are called transition matrices. We denote $A = (A_1^T, \ldots, A_p^T)^T$ to be the combination of the transition matrices. We assume that $Z_t$ are independently and identically generated from a Gaussian distribution $N_d(0, \Psi)$. Moreover, $Z_t$ and $\{X_s\}_{s < t}$ are independent for any $t \in \mathbb{Z}$. We pose an additional assumption that $\det(I_d - \sum_{k=1}^p A_k^T z_k) \neq 0$ for all $z \in \mathbb{C}$ with modulus no larger than one. This guarantees that the sequence is stationary and we have, for any $t \in \mathbb{Z}$, $X_t$ follows a Gaussian distribution $N_d(0, \Sigma)$.

We denote $\Sigma_i(\cdot)$ to be an operator on the process $(X_t)_{-\infty}^{\infty}$. In particular, we define $\Sigma_i\{(X_t)\} = \text{Cov}(X_0, X_t)$. It is easy to see that $\Sigma_0\{(X_t)\} = \Sigma$. If the lag of the vector autoregressive model is 1 (i.e., $X_t = A_1^T X_{t-1} + Z_t$, for any $t \in \mathbb{Z}$), by simple calculation we get the “Yule-Walker Equation”:

$$\Sigma_i\{(X_t)\} = (A_1^T)^i \Sigma_0\{(X_t)\}, \quad (2.2)$$

which further implies that

$$A_1 = [\Sigma_0\{(X_t)\}]^{-1} [\Sigma_1\{(X_t)\}]^T. \quad (2.5)$$

The results for lag 1 vector autoregressive model can be extended to the lag $p$ vector autoregressive model by appropriately redefining the random vectors. In detail, the autoregressive model with lag $p$ shown in (2.1) can be reformulated as an autoregressive model with lag 1:

$$\tilde{X}_t = \tilde{A}^T \tilde{X}_{t-1} + \tilde{Z}_t, \quad (2.3)$$

where

$$\tilde{X}_t = \begin{pmatrix} X_{t+p-1} \\ X_{t+p-2} \\ \vdots \\ X_t \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A_1 & I_d & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ A_{p-1} & 0 & 0 & \ldots & I_d \\ A_p & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \tilde{Z}_t = \begin{pmatrix} Z_{t+p-1} \\ \vdots \\ 0 \end{pmatrix}. \quad (2.4)$$

Here $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix, $\tilde{X}_t \sim N_{dp}(0, \tilde{\Sigma})$ for $t = 1, \ldots, T$ and $\tilde{Z}_t \sim N_{dp}(0, \tilde{\Psi})$ with $\tilde{\Sigma} = \text{Cov}(\tilde{X}_t)$ and $\tilde{\Psi} = \text{Cov}(\tilde{Z}_t)$. Therefore, we also have

$$\tilde{A} = [\Sigma_0\{((\tilde{X}_t))\}]^{-1} [\Sigma_1\{((\tilde{X}_t))\}]^T. \quad (2.5)$$

This is similar to the relationship for the lag 1 vector autoregressive model.

## 3 Methods and Algorithms

We provide a new formulation to estimate $A_1, \ldots, A_p$ for the vector autoregressive model. Let $x_1, x_2, \ldots, x_T$ be observed data points of $X_1, \ldots, X_T$ from a lag $p$ vector autoregressive
process \( (X_t)_{t=-\infty}^{\infty} \). Let \( \tilde{x}_t = (x_{t+p-1}^T, \ldots, x_t^T)^T \) for \( t = 1, \ldots, T - p + 1 \). We denote \( S \) and \( S_1 \) to be the marginal and lag 1 sample covariance matrices of \( (\tilde{x}_t)_{t=1}^{T-p+1} \):

\[
S = \frac{1}{T - p + 1} \sum_{t=1}^{T-p+1} \tilde{x}_t \tilde{x}_t^T, \quad S_1 = \frac{1}{T - p} \sum_{t=1}^{T-p} \tilde{x}_t \tilde{x}_{t+1}^T.
\]

(3.1)

Using the connection between \( \Sigma_0 \{ (\tilde{X}_t) \} \), \( \Sigma_1 \{ (\tilde{X}_t) \} \) and \( \tilde{A} \) in (2.5), we know that a good estimator \( \hat{\Omega} \) of \( \tilde{A} \) shall satisfy that

\[
\| \Sigma_0 \{ (\tilde{X}_t) \} \hat{\Omega} - [\Sigma_1 \{ (\tilde{X}_t) \}]^T \|\]

is small enough under certain matrix norm \( \| \cdot \| \). Moreover, using the fact that \( A = (A_1^T, \ldots, A_p^T)^T = \tilde{A}_{*,J} \), where \( J = \{1, \ldots, d\} \), by (3.2) we have that a good estimate \( \hat{A} \) of \( A \) shall satisfy

\[
\| \Sigma_0 \{ (\tilde{X}_t) \} \hat{A} - (\Sigma_1 \{ (\tilde{X}_t) \})^T_{*,J} \|
\]

is small enough.

Motivated by (3.3), we estimate \( A_1, \ldots, A_p \) via replacing \( \Sigma_0 \{ (\tilde{X}_t) \} \) and \( (\Sigma_1 \{ (\tilde{X}_t) \})^T_{*,J} \) with their empirical versions. To formulate the estimation equation into a linear program, we use the \( L_{\max} \) norm. Accordingly, we get the following convex optimization program:

\[
\hat{\Omega} = \arg\min_{M \in \mathbb{R}^{dp \times p}} \sum_{jk} |M_{jk}|, \quad \text{subject to} \quad \|SM - (S_1^T)_{*,J}\|_{\max} \leq \lambda_0,
\]

(3.4)

where \( \lambda_0 > 0 \) is a tuning parameter. In (3.4), the constraint part aims to find an estimate that approximates the true parameter well, and combined with the minimization part aims to induce certain sparsity. Let \( \hat{\Omega}_{*,j} = \hat{\beta}_j \), it is easy to see that (3.4) decomposes into many subproblems and each \( \hat{\beta}_j \) can be solved by

\[
\hat{\beta}_j = \arg\min_{v \in \mathbb{R}^d} \|v\|_1, \quad \text{subject to} \quad \|Sv - (S_1^T)_{*,j}\|_{\infty} \leq \lambda_0.
\]

(3.5)

Accordingly, compared with the lasso-type procedures, the proposed method can be solved efficiently in parallel and therefore is computationally more efficient.

Once \( \hat{\Omega} \) is obtained, the estimator of the transition matrix \( A_k \) can then be written as

\[
\hat{A}_k = \hat{\Omega}_{J_k,*},
\]

(3.6)

where \( J_k = \{j : d(k-1) + 1 \leq j \leq dk\} \).

We now show that the optimization in (3.5) can be formulated into a linear program. Recall that any real number \( a \) takes the decomposition \( a = a^+ - a^- \), where \( a^+ = a \cdot I(a \geq 0) \).
and $a^- = -a \cdot I(a < 0)$. For any vector $v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d$, let $v^+ = (v_1^+, \ldots, v_d^+)^T$ and $v^- = (v_1^-, \ldots, v_d^-)^T$. We denote $v \geq 0$ if $v_1, \ldots, v_d \geq 0$ and $v < 0$ if $v_1, \ldots, v_d < 0$, similarly, $v_1 \geq v_2$ if $v_1 - v_2 \geq 0$, and $v_1 < v_2$ if $v_1 - v_2 < 0$. Letting $v = (v_1, \ldots, v_d)^T$, the problem in (3.5) can be further relaxed to the following problem:

$$
\hat{\beta}_j = \arg\min_{v^+, v^-} 1_d^T (v^+ + v^-),
$$
subject to $\|Sv^+ - Sv^- - (S_1^T)_{*,j}\|_\infty \leq \lambda_0$, $v^+ \geq 0, v^- \geq 0$. \hspace{1cm} (3.7)

To minimize $1_d^T (v^+ + v^-)$, $v^+$ or $v^-$ can not be both nonzero. Therefore, the solution to (3.7) is exactly the solution to (3.5). The optimization problem in (3.7) can be written as

$$
\hat{\beta}_j = \arg\min_{v^+, v^-} 1_d^T (v^+ + v^-),
$$
subject to $Sv^+ - Sv^- - (S_1^T)_{*,j} \leq \lambda_0 1_d$,

$$
-Sv^+ + Sv^- + (S_1^T)_{*,j} \leq \lambda_0 1_d,
$$
$v^+ \geq 0, v^- \geq 0$.

This is equivalent to

$$
\hat{\beta}_j = \arg\min_\omega 1_{2d}^T \omega, \text{ subject to } \theta + W \omega \geq 0, \omega \geq 0, \hspace{1cm} (3.8)
$$

where

$$
\omega = \begin{pmatrix} v^+ \\ v^- \end{pmatrix}, \quad \theta = \begin{pmatrix} (S_1^T)_{*,j} + \lambda_0 1_d \\ -(S_1^T)_{*,j} + \lambda_0 1_d \end{pmatrix}, \quad W = \begin{pmatrix} -S & S \\ S & -S \end{pmatrix}.
$$

The optimization (3.8) is a linear program. We can solve it using the simplex algorithm (Murty, 1983).

### 4 Theoretical Properties

In this section, under the double asymptotic framework, we provide the nonasymptotic rates of convergence in parameter estimation under the matrix $L_1$ and $L_{\max}$ norms.

We first present the rates of convergence of the estimator $\hat{\Omega}$ in (3.4) under the vector autoregressive model with lag 1. This result allows us to sharply characterize the impact of the temporal dependence of the time series on the obtained rate of convergence. In particular, we show that the rates of convergence are closely related to the $L_1$ and $L_2$ norms of the transition matrix $A_1$, where $\|A_1\|_2$ is the key part in characterizing the impact of temporal dependence on estimation accuracy. Secondly, we present the sign recovery
consistency result of our estimator. Compared to the lasso-type estimators, our result does not require the irrepresentable condition. These results are combined together to show that we have the prediction consistency, i.e., the term \( \| A_1 x_T - \hat{A}_1 x_T \| \) goes to zero with regard to certain norms \( \| \cdot \| \). In the end, we extend these results from the vector autoregressive model with lag 1 to lag \( p \) with \( p > 1 \).

We start with some additional notation. Let \( M_d \in \mathbb{R} \) be a quantity which may scale with the time series length and dimension \((n, d)\). We define the set of square matrices in \( \mathbb{R}^{d \times d} \), denoted by \( \mathcal{M}(q, s, M_d) \), as

\[
\mathcal{M}(q, s, M_d) = \left\{ M \in \mathbb{R}^{d \times d} : \max_{1 \leq j \leq d} \frac{d}{\min_j(\Sigma_{jj})} \sum_{i=1}^d |M_{ij}|^q \leq s, \| M \|_1 \leq M_d \right\}.
\]

For \( q = 0 \), the class \( \mathcal{M}(0, s, M_d) \) contains all the \( s \)-sparse matrices with bounded \( L_1 \) norms.

There are two general remarks about the model \( \mathcal{M}(q, s, M_d) \): (i) \( \mathcal{M}(q, s, M_d) \) can be considered as the matrix version of the vector “weakly sparse set” explored in Raskutti et al. (2011) and Vu and Lei (2012). Such a way to define the weakly sparse set of matrices has also been investigated in Cai et al. (2011). (ii) For \( \mathcal{M}(0, s, M_d) \), the sparsity level \( s \) represents the largest number of nonzero entries in each column of the matrix. In contrast, the sparsity level \( s' \) exploited in Kock and Callot (2012) is the total number of nonzero entries in the matrix. We must have \( s' \geq s \) and in general \( s' \gg s \) (i.e., \( s/s' \to 0 \)).

The next theorem presents the \( L_1 \) and \( L_{\text{max}} \) rates of convergence of our estimator under the vector autoregressive model with lag 1.

**Theorem 4.1.** Suppose that \((x_t^T)_{t=1}^T\) are realizations of \((X_t^T)_{t=1}^\infty\) from a lag 1 vector autoregressive process \((X_t)_{t=-\infty}^\infty\) as described in (2.1). We assume the transition matrix \( A_1 \in \mathcal{M}(q, s, M_d) \) for some \( 0 \leq q < 1 \). Let \( \hat{A}_1 \) be the optimum to (3.4) with a tuning parameter

\[
\lambda_0 = \frac{32\|\Sigma\|_2 \max_j(\Sigma_{jj})}{\min_j(\Sigma_{jj})(1 - \|A\|_2)} (2M_d + 3) \left( \frac{\log d}{T} \right)^{1/2}.
\]

For \( T \geq 6 \log d + 1 \) and \( d \geq 8 \), we have, with probability no smaller than \( 1 - 14d^{-1} \),

\[
\| \hat{A}_1 - A_1 \|_1 \leq 4s \left\{ \frac{32\|\Sigma^{-1}\|_1 \max_j(\Sigma_{jj})\|\Sigma\|_2}{\min_j(\Sigma_{jj})(1 - \|A\|_2)} (2M_d + 3) \left( \frac{\log d}{T} \right)^{1/2} \right\}^{1-q}.
\]

Moreover, with probability no smaller than \( 1 - 14d^{-1} \),

\[
\| \hat{A}_1 - A_1 \|_{\text{max}} \leq 64\|\Sigma^{-1}\|_1 \max_j(\Sigma_{jj})\|\Sigma\|_2 (2M_d + 3) \left( \frac{\log d}{T} \right)^{1/2}.
\]

In the above results, \( \Sigma \) is the marginal covariance matrix of \( X_t \).
Similar as the lasso and Dantzig selector (Candes and Tao, 2007; Bickel et al., 2009), the tuning parameter \( \lambda_0 \) here depends on \( \Sigma \). In practice, the same as most preceded developments (see, for example, Song and Bickel (2011)), we can use a data-driven way to select the tuning parameter. In the numerical study section, we use the cross validation to choose \( \lambda_0 \). In Section 5 we will show that the procedure of selecting the tuning parameter via cross validation gives reasonable results.

Here \( A_1 \) is assumed to be at least weakly sparse and belong to the set \( \mathcal{M}(q,s,M_d) \). This makes the model identifiable. Otherwise, we will have multiple global optimum in the optimization problem.

The obtained rates of convergence in Theorem 4.1 depend on both \( \Sigma \) and \( A_1 \) with \( \| A_1 \|_2 \) characterizing the temporal dependence. In the following, we list two examples to provide more insights about the results in Theorem 4.1.

**Example 4.2.** We consider the case when \( \Sigma \) is a strictly diagonal dominant (SDD) matrix (Horn and Johnson, 1990) with the property

\[
\delta_i := |\Sigma_{ii}| - \sum_{j \neq i} |\Sigma_{ij}| \geq 0, \quad (i = 1, \ldots, d).
\]

This corresponds to the cases when the \( d \) entries in any \( X_t \) with \( t \in \{1, \ldots, T\} \) are weakly dependent. In this setting, Ahlberg and Nilson (1963) showed that

\[
\| \Sigma^{-1} \|_1 = \| \Sigma^{-1} \|_{\infty} \leq \left\{ \min_i \left( |\Sigma_{ii}| - \sum_{j \neq i} |\Sigma_{ij}| \right) \right\}^{-1} = \max_i (\delta_i^{-1}). \tag{4.3}
\]

Moreover, by algebra,

\[
\| \Sigma \|_2 \leq \| \Sigma \|_1 = \max_i \left( |\Sigma_{ii}| + \sum_{j \neq i} |\Sigma_{ij}| \right) \leq 2 \max_i (|\Sigma_{ii}|). \tag{4.4}
\]

Equations (4.3) and (4.4) suggest that, when \( \max_i (\Sigma_{ii}) \) is upper bounded, and both \( \min_i (\Sigma_{ii}) \) and \( \delta_i \) are lower bounded by a fixed constant, we have both \( \| \Sigma^{-1} \|_1 \) and \( \| \Sigma \|_2 \) do not scale with \( (T,d,s) \), and the obtained rates of convergence in (4.1) and (4.2) can be simplified as:

\[
\| \hat{A}_1 - A_1 \|_1 = O_P \left[ s \left\{ \frac{M_d}{1 - \| A_1 \|_2} \left( \frac{\log d}{T} \right)^{1/2} \right\}^{1-q} \right],
\]

\[
\| \hat{A}_1 - A_1 \|_{\text{max}} = O_P \left\{ \frac{M_d}{1 - \| A_1 \|_2} \left( \frac{\log d}{T} \right)^{1/2} \right\}.
\]
Example 4.3. We can generalize the “entry-wise weakly dependent” structure in Example 4.2 to a “block-wise weakly dependent” structure. More specifically, we consider the case when $\Sigma = (\Sigma^{b}_{jk})$ with blocks $\Sigma^{b}_{jk} \in \mathbb{R}^{d_j \times d_k}$ ($1 \leq j, k \leq K$) is a strictly block diagonal dominant (SBDD) matrix with the property

$$
\delta^{b}_{i} = \| (\Sigma^{b}_{ii})^{-1} \|_{\infty}^{-1} - \sum_{j \neq i} \| \Sigma^{b}_{ij} \|_{\infty} > 0 \quad (i = 1, \ldots, K).
$$

In this case, Varah (1975) showed that

$$
\| \Sigma^{-1} \|_1 = \| \Sigma^{-1} \|_{\infty} \leq \left\{ \min_{i} \left( \| (\Sigma^{b}_{ii})^{-1} \|_{\infty}^{-1} - \sum_{j \neq i} \| \Sigma^{b}_{ij} \|_{\infty} \right) \right\}^{-1} = \max \{ (\delta^{b}_{i})^{-1} \}.
$$

Moreover, we have

$$
\| \Sigma \|_2 \leq \| \Sigma \|_1 \leq \max_{i} (\| (\Sigma^{b}_{ii})^{-1} \|_{\infty} + \| \Sigma^{b}_{ii} \|_{\infty}).
$$

In general, $(\| (\Sigma^{b}_{ii})^{-1} \|_{\infty} - \| \Sigma^{b}_{ii} \|_{\infty})$ is in the scale of $\max_{i} (d_i \ll d)$, and when $\delta^{b}_{i}$ is lower bounded for any $i \in \{1, \ldots, K\}$ and the condition number of $\Sigma$ is upper bounded, we simplify the obtained rates of convergence as:

$$
\| \hat{A}_1 - A_1 \|_1 = O_P \left[ s \left\{ \frac{M_{d} \cdot \max_{i} (d_i) \left( \log \frac{d}{T} \right)^{1/2}}{1 - \| A_1 \|_2} \right\}^{1-q} \right],
$$

$$
\| \hat{A}_1 - A_1 \|_{\max} = O_P \left\{ \frac{M_{d} \cdot \max_{i} (d_i) \left( \log \frac{d}{T} \right)^{1/2}}{1 - \| A_1 \|_2} \right\}.
$$

We then present the results on feature selection. If $A_1 \in \mathcal{M}(0, s, M_{d})$, from the element-wise $L_{\infty}$ norm convergence, a sign recovery result can be obtained. In detail, let $\hat{A}_1$ be a truncated version of $A_1$ with level $\gamma$:

$$
(\hat{A}_1)_{ij} = (\hat{A}_1)_{ij} I \{ |(\hat{A}_1)_{ij}| \geq \gamma \}. \quad (4.5)
$$

The following corollary shows that $\hat{A}_1$ recovers the sign of $A_1$ with overwhelming probability.

Corollary 4.4. Suppose that the conditions in Theorem 4.1 hold and $A_1 \in \mathcal{M}(0, s, M_{d})$. If we choose the truncation level

$$
\gamma = \frac{64 \| \Sigma^{-1} \|_1 \max_{j} (\Sigma_{jj}) \| \Sigma \|_2 \left( 2M_{d} + 3 \right) \left( \frac{\log d}{T} \right)^{1/2}}{\min_{j} (\Sigma_{jj})(1 - \| A_1 \|_2)}
$$

in (4.5) and with the assumption that

$$
\min_{\{ (j,k): (A_1)_{jk} \neq 0 \}} |(A_1)_{jk}| \geq 2\gamma,
$$

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we have, with probability no smaller than \(1 - 14d^{-1}\), \(\text{sign}(A_1) = \text{sign}(\hat{A}_1)\). Here for any matrix \(M\), \(\text{sign}(M)\) is a matrix with each element representing the sign of the corresponding entry in \(M\).

Here we note that Corollary 4.4 sheds lights on detecting Granger causality. For any two processed \(\{y_t\}\) and \(\{z_t\}\), Granger (1969) defined the causal relationship in principle as follows: Provided that we know everything in the universe, \(\{y_t\}\) is said to cause \(\{z_t\}\) in Granger’s sense if removing the information about \(\{y_s\}_{s \leq t}\) from the whole knowledge base built by time \(t\) will increase the prediction error about \(z_t\). It is known that the noncausalities are determined by the transition matrices in the stable VAR process (Lütkepohl, 2005). Therefore, detecting the nonzero entries of \(A_1\) consistently means that we can estimate the Granger-causality network consistently.

We then turn to evaluate the prediction performance of the proposed method. Given a new data point \(X_{T+1}\) in time point \(T + 1\), based on the observations \(\{X_t\}_{t=1}^T\), the next corollary quantifies the distance between \(X_{T+1}\) and \(\hat{A}_1X_T\) in terms of \(L_\infty\) norm.

**Corollary 4.5.** Suppose that the conditions in Theorem 4.1 hold and let

\[
\Psi_{\max} := \max_i (\Psi_{ii}) \quad \text{and} \quad \Sigma_{\max} := \max_i (\Sigma_{ii}).
\]

Then for a data point \(x_{T+1}\) at time point \(T + 1\) and any constant \(\alpha > 0\), with probability greater than

\[
1 - 2\left(d^{\alpha/2-1} \sqrt{\pi/2 \cdot \alpha \log d}\right)^{-1} - 14d^{-1},
\]

we have

\[
\|x_{T+1} - \hat{A}_1^+ x_T\|_\infty \leq (\Psi_{\max} \cdot \alpha \log d)^{1/2} + 4s \left\{ \frac{32\|\Sigma\|_1}{\min_j (\Sigma_{jj})} \frac{\|\Sigma\|_2}{(1 - \|A_1\|_2)} (2M_d + 3) \left( \frac{\log d}{T} \right)^{1/2} \right\}^{1-q} \cdot (\Sigma_{\max} \cdot \alpha \log d)^{1/2}, \quad (4.6)
\]

where \(\hat{A}_1\) is calculated based on \(\{x_t\}_{t=1}^T\).

Here we note that the first term in the right-hand side of Equation (4.6), \((\Psi_{\max} \cdot \alpha \log d)^{1/2}\), is present due to the diverges of the new data point from its mean caused by an unpredictable noise perturbation term \(Z_{T+1} \sim N_d(0, \Psi)\). This term is unable to be canceled out even if we have almost infinite data points. The second term in the right-hand side of Equation (4.6) depends on the estimation accuracy of \(\hat{A}_1\) to \(A_1\) and will converge to zero under certain conditions. In other words, the term

\[
\|A_1^+ x_T - \hat{A}_1^+ x_T\|_\infty \to 0, \quad (4.7)
\]
in probability.

Although $A_1$ is in general asymmetric, there exist cases in which a symmetric transition matrix is of more interest. It is known that the off-diagonal entries in the transition matrix represent the influence of one state on the others and such influence might be symmetric or not. Weiner et al. (2012) provided several examples where a symmetric transition matrix is more appropriate for modeling the data.

If we further suppose that the transition matrix $A_1$ is symmetric, we can exploit this information and obtain a new estimator $\bar{A}_1$ as

$$(\bar{A}_1)_{jk} = (\hat{A}_1)_{jk}I(|(\hat{A}_1)_{jk}| \leq (\hat{A}_1)_{kj}) + (\hat{A}_1)_{kj}I(|(\hat{A}_1)_{kj}| \leq (\hat{A}_1)_{jk}).$$

In other word, we always pick the entry with smaller magnitudes. Then using Theorem 4.1, we have

$$\|\bar{A}_1 - A_1\|_1$$

and

$$\|\bar{A}_1 - A_1\|_{\infty}$$

can be upper bounded by the same number presented in the right-hand side of (4.1). In this case, because both $A_1$ and $\bar{A}_1$ are symmetric, we have

$$\|\bar{A}_1 - A_1\|_2 \leq \|\bar{A}_1 - A_1\|_1 = \|\bar{A}_1 - A_1\|_{\infty}.$$ We then proceed to quantify the prediction accuracy under $L_2$ norm in the next corollary.

**Corollary 4.6.** Suppose that the conditions in Theorem 4.1 hold and $A_1$ is a symmetric matrix. Then for a new data point $x_{T+1}^{\text{T}}$ at time point $T+1$, with probability greater than $1 - \frac{1}{18}d$, we have

$$\|x_{T+1} - \bar{A}_1^{\text{T}}x_{T}\|_2 \leq \sqrt{2\|\Psi\|_2 \log d} + \sqrt{\text{tr}(\Psi)} +$$

$$4s \left\{ \frac{32\|\Sigma^{-1}\|_1 \max_j (\Sigma_{jj}) \|\Sigma\|_2}{\min_j (\Sigma_{jj})(1 - \|A_1\|_2)} (2Md + 3) \left( \frac{\log d}{T} \right)^{1/2} \right\}^{1-q} \cdot \left\{ \sqrt{2\|\Sigma\|_2 \log d} + \sqrt{\text{tr}(\Sigma)} \right\}. \quad (4.8)$$

Based on Corollary 4.6, we have, similar to what has been discussed in Corollary 4.5, the term $\|A_1^{\text{T}}x_{T} - \bar{A}_1^{\text{T}}x_{T}\|_2$ vanishes to zero when the second term in the left hand side of (4.8) converges to zero. This is true when, for example, $\Sigma$ is (near-)low rank with many singular values close to zero.

Using the augmented formulation of the lag $p$ vector autoregressive model in (2.3), we can extend the results of Theorem 4.1 from lag 1 to the more general lag $p$ model with $p > 1$.

**Theorem 4.7.** Suppose that $(x_t)_{t=1}^{T}$ are realizations of $(X_t)_{t=1}^{T}$ from a lag $p$ vector autoregressive process $(X_t)_{t=-\infty}^{\infty}$ as described in (2.1). Let $\bar{A}$ and $\tilde{\Sigma}$ be defined as in §2. We assume that $\bar{A} \in \mathcal{M}(q,s,M_{dp})$ for some $0 \leq q < 1$. Let $\hat{\Omega}$ be the optimum to (3.4) with the tuning parameter

$$\lambda_0 = \frac{C\|\tilde{\Sigma}\|_2 \max_j (\tilde{\Sigma}_{jj}) \max(M_{dp}, 1)}{\min_j (\tilde{\Sigma}_{jj})(1 - \|\bar{A}\|_2)} \left( \frac{\log d + \log p}{T - p} \right)^{1/2},$$

with probability greater than $1 - \frac{1}{2d}$.
where $C$ is a generic constant. Then we have,

$$\sum_{k=1}^{p} \| \hat{A}_k - A_k \|_1 = \| \hat{\Omega} - A \|_1 = O_P \left[ s \left\{ \frac{\| \hat{\Sigma} \|_2 \max_j (\hat{\Sigma}_{jj}) \max(M_{dp}, 1)}{\min_j (\hat{\Sigma}_{jj}) (1 - \| \hat{A} \|_2)} \left( \frac{\log d + \log p}{T - p} \right)^{1/2} \right\}^{1-q} \right].$$

$$\max_k \| \hat{A}_k - A_k \|_{\max} = \| \hat{\Omega} - A \|_{\max} = O_P \left\{ \frac{\| \hat{\Sigma} \|_2 \max_j (\hat{\Sigma}_{jj}) \max(M_{dp}, 1)}{\min_j (\hat{\Sigma}_{jj}) (1 - \| \hat{A} \|_2)} \left( \frac{\log d + \log p}{T - p} \right)^{1/2} \right\}. \]$$

Here we remind that $A = (A_1^T, \ldots, A_p^T)^T$ with $\hat{A}_k$ is defined in (3.6) for $k = 1, \ldots, p$.

Using the augmented formulation, we can show that similar arguments as in Corollaries 4.4, 4.5, and 4.6 also hold.

## 5 Experiments

We conduct numerical experiments on both synthetic and real data to illustrate the effectiveness of our proposed method compared to the competing ones. In detail, we consider three methods:

- (i) The least square estimation using a ridge penalty (The method in Hamilton (1994) by adding a ridge penalty $\| M \|_F^2$ to the least squares loss function in (1.2)).
- (ii) The least square estimation using a $L_1$ penalty (The method in Hsu et al. (2008) by adding a $L_1$ penalty $\sum_{ij} |M_{ij}|$ to (1.2)).
- (iii) Our method (The estimator described in (3.4)).

Here we consider including the procedure discussed in Hamilton (1994) because it is a commonly explored baseline, which can show how bad the classic procedure can be when the dimension is very high. We only consider the competing procedure proposed in Hsu et al. (2008) because this is the only method that is specifically designed for the same simple VAR as what we studied. We do not consider other aforementioned procedures (e.g., Haufe et al. (2008), Shojaie and Michailidis (2010)) because they are designed for more specific models with more assumptions.

### 5.1 Synthetic Data

We compare our method with the other two methods on several synthetic data. We consider the settings where the sample size $T$ varies from 50 to 100 and the dimension $d$ varies from 50 to 200. For simplicity, we focus on vector autoregressive model described in (2.1) with lag one.
We create the transition matrix $A_1$ according to five different patterns: band, cluster, hub, random, and scale-free. Typical realizations of these patterns are illustrated in Figure 1 and are generated using the “flare” package in R. In those plots, the gray points represent the zero entries and the black points represent the nonzero entries. We then rescale $A_1$ such that $\|A_1\|_2 = 0.5$. Once $A_1$ is obtained, $\Sigma$ is then generated as $\Sigma = 2\|A_1\|_2 \Sigma_0$, where $\Sigma_0$ can be a strict diagonal dominant (SDD) or a strictly block diagonal dominant (SBDD) matrix as discussed in Examples 4.2 and 4.3. Here for simplicity we only present the settings with $\Sigma$ to be diagonal: $\Sigma = 2\|A_1\|_2 I_d$. We found that similar observations hold for other complicated settings. We then calculate the covariance matrix of the Gaussian noise vector $Z_t$ as $\Psi = \Sigma - A_1^T \Sigma A_1$. With $A_1, \Sigma$, and $\Psi$, we simulate a time series $(x_1, \ldots, x_T)^T \in \mathbb{R}^{T \times d}$ according to the model described in (2.1).

We construct 1,000 replicates and compare the three methods described above. The averaged estimation errors under different matrix norms are illustrated in Tables 1 to 5. The standard deviations of the estimation errors are provided in the parentheses. The tuning parameters for the three methods are selected using cross validation by minimizing the averaged prediction errors. In detail, each time we randomly select a length $T/2$ chunk...
Table 1: Comparison of estimation performance of three methods over 1,000 replications. The standard deviations are present in the parentheses. Here $L_F$, $L_2$ and $L_1$ represent the Frobenius, $L_2$ and $L_1$ matrix norms respectively. The pattern of the transition matrix is “band”

| $d$ | $T$ | ridge method | | lasso method | | our method |
|---|---|---|---|---|---|---|
| | | $L_F$ | $L_2$ | $L_1$ | $L_F$ | $L_2$ | $L_1$ | $L_F$ | $L_2$ | $L_1$ |
| 50 | 100 | 6.95 | 0.91 | 7.64 | 2.34 | 0.50 | 1.48 | 2.09 | 0.49 | 0.57 |
| | | (0.15) | (0.04) | (0.40) | (0.06) | (0.02) | (0.13) | (0.03) | (0.02) | (0.03) |
| 100 | 50 | 9.83 | 1.24 | 10.53 | 5.52 | 0.75 | 3.06 | 3.27 | 0.53 | 1.06 |
| | | (0.17) | (0.02) | (0.85) | (0.07) | (0.03) | (0.19) | (0.04) | (0.02) | (0.27) |
| 200 | 100 | 14.30 | 1.25 | 14.16 | 6.38 | 0.64 | 2.74 | 4.26 | 0.49 | 0.69 |
| | | (0.11) | (0.01) | (0.39) | (0.05) | (0.02) | (0.18) | (0.03) | (0.03) | (0.05) |

Table 2: Comparison of estimation performance of three methods over 1,000 replications. The standard deviations are present in the parentheses. Here $L_F$, $L_2$ and $L_1$ represent the Frobenius, $L_2$ and $L_1$ matrix norms respectively. The pattern of the transition matrix is “cluster”

| $d$ | $T$ | ridge method | | lasso method | | our method |
|---|---|---|---|---|---|---|
| | | $L_F$ | $L_2$ | $L_1$ | $L_F$ | $L_2$ | $L_1$ | $L_F$ | $L_2$ | $L_1$ |
| 50 | 100 | 7.00 | 0.82 | 7.66 | 2.13 | 0.45 | 1.50 | 1.48 | 0.49 | 0.68 |
| | | (0.19) | (0.04) | (0.52) | (0.06) | (0.04) | (0.17) | (0.02) | (0.03) | (0.02) |
| 100 | 50 | 9.61 | 1.13 | 10.36 | 5.22 | 0.65 | 3.12 | 2.29 | 0.48 | 1.00 |
| | | (0.14) | (0.02) | (0.92) | (0.07) | (0.03) | (0.20) | (0.12) | (0.02) | (0.15) |
| 200 | 100 | 13.98 | 1.15 | 14.52 | 5.78 | 0.56 | 2.82 | 2.81 | 0.49 | 0.75 |
| | | (0.13) | (0.01) | (0.58) | (0.05) | (0.02) | (0.17) | (0.01) | (0.04) | (0.05) |
Table 3: Comparison of estimation performance of three methods over 1,000 replications. The standard deviations are present in the parentheses. Here $L_F$, $L_2$ and $L_1$ represent the Frobenius, $L_2$ and $L_1$ matrix norms respectively. The pattern of the transition matrix is “hub”

| $d$ | $T$ | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  |
|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 50  | 100 | 6.93   | 0.78   | 7.51   | 1.92   | 0.39   | 1.46   | 1.13   | 0.38   | 1.04   |
|     |     | (0.15) | (0.04) | (0.42) | (0.07) | (0.05) | (0.13) | (0.15) | (0.06) | (0.07) |
| 100 | 50  | 9.59   | 1.10   | 10.47  | 5.06   | 0.64   | 2.96   | 1.90   | 0.50   | 1.40   |
|     |     | (0.16) | (0.01) | (0.52) | (0.09) | (0.03) | (0.15) | (0.05) | (0.01) | (0.12) |
| 200 | 100 | 13.89  | 1.11   | 14.21  | 5.49   | 0.51   | 2.68   | 2.09   | 0.49   | 1.24   |
|     |     | (0.10) | (0.01) | (0.48) | (0.05) | (0.02) | (0.18) | (0.04) | (0.06) | (0.03) |

of the time series $(x_t)_{t=1}^T$ as training set and use the immediate next time point as testing data. The cross-validation score is obtained by averaging the testing errors out of many trials.

We see that our method nearly uniformly outperforms the methods in Wang et al. (2007) and Hamilton (1994) significantly under different norms (Frobenius, $L_2$, and $L_1$ norms). In particular, the improvement over the method in Wang et al. (2007) tends to be more significant when the dimension $d$ is large. Our method also has slightly less standard deviations compared to the method in Wang et al. (2007), but overall the difference is not significant. The method in Hamilton (1994) has worse performance than the other two methods. Therefore, it is not appropriate to handle very high dimensional data.

### 5.2 Real Data

We further compare the three methods on the equity data collected from Yahoo! Finance. The task is to predict the stock prices. We collect the daily closing prices for 452 stocks that are consistently in the S&P 500 index between January 1, 2003 through January 1, 2008. This gives us altogether 1,258 data points, each of which corresponds to the vector of closing prices on a trading day.

Let $E = (E_{t,j}) \in \mathbb{R}^{1258 \times 452}$ with $E_{t,j}$ denoting the closing price of the stock $j$ on day $t$. We screen out all the stocks with low marginal standard deviations and only keep 50 stocks which vary the most. We center the data so that the marginal mean of each time series is zero. The resulting data matrix is denoted by $\bar{E}$. We apply the three methods
Table 4: Comparison of estimation performance of three methods over 1,000 replications. The standard deviations are present in the parentheses. Here $L_F$, $L_2$ and $L_1$ represent the Frobenius, $L_2$ and $L_1$ matrix norms respectively. The pattern of the transition matrix is “random”

| $d$ | $T$ | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  |
|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 50  | 100 | 6.95   | 0.86   | 7.31   | 2.18   | 0.44   | 1.46   | 1.63   | 0.48   | 0.67   |
|     |     | (0.15) | (0.05) | (0.51) | (0.06) | (0.03) | (0.13) | (0.04) | (0.04) | (0.05) |
| 100 | 50  | 9.73   | 1.17   | 10.90  | 5.18   | 0.67   | 3.05   | 2.43   | 0.48   | 0.96   |
|     |     | (0.16) | (0.01) | (0.50) | (0.08) | (0.03) | (0.19) | (0.06) | (0.02) | (0.11) |
| 200 | 100 | 13.98  | 1.16   | 14.30  | 5.74   | 0.53   | 2.69   | 2.78   | 0.48   | 0.89   |
|     |     | (0.11) | (0.01) | (0.40) | (0.05) | (0.01) | (0.16) | (0.04) | (0.06) | (0.08) |

Table 5: Comparison of estimation performance of three methods over 1,000 replications. The standard deviations are present in the parentheses. Here $L_F$, $L_2$ and $L_1$ represent the Frobenius, $L_2$ and $L_1$ matrix norms respectively. The pattern of the transition matrix is “scale-free”

| $d$ | $T$ | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  | $L_F$  | $L_2$  | $L_1$  |
|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 50  | 100 | 6.90   | 0.80   | 7.44   | 1.99   | 0.40   | 1.43   | 1.16   | 0.36   | 0.89   |
|     |     | (0.13) | (0.03) | (0.42) | (0.07) | (0.05) | (0.14) | (0.08) | (0.09) | (0.16) |
| 100 | 50  | 9.59   | 1.10   | 10.64  | 5.05   | 0.61   | 2.97   | 1.78   | 0.39   | 1.15   |
|     |     | (0.16) | (0.01) | (0.70) | (0.08) | (0.03) | (0.16) | (0.16) | (0.08) | (0.23) |
| 200 | 100 | 13.89  | 1.10   | 14.18  | 5.41   | 0.47   | 2.64   | 1.71   | 0.39   | 1.18   |
|     |     | (0.11) | (0.01) | (0.42) | (0.05) | (0.02) | (0.18) | (0.15) | (0.07) | (0.11) |
Table 6: The averaged prediction errors for the three methods on the equity data. The standard deviations are present in the parentheses.

| lag  | ridge method       | lasso method      | our method        |
|------|--------------------|-------------------|-------------------|
| p=1  | 14.22 (9.43)       | 13.72 (9.67)      | 12.34 (9.80)      |
| p=3  | 14.79 (14.11)      | 13.45 (8.24)      | 13.20 (8.95)      |
| p=5  | 14.82 (14.13)      | 12.79 (8.52)      | 12.69 (7.43)      |
| p=7  | 14.74 (14.09)      | 12.24 (7.94)      | 11.73 (6.57)      |
| p=9  | 14.57 (14.03)      | 11.71 (7.35)      | 10.66 (5.82)      |

on $\bar{E}$ with different lags $p$ changing from 1 to 9. To evaluate the performance of the three methods, for $t = 1, \ldots, 258$, we select the dataset $\bar{E}_{t,*}$, where $J_t = \{j : t \leq j \leq t + 999\}$, as the training set for calculating the transition matrices estimates $\hat{A}_1^{(t)}, \ldots, \hat{A}_p^{(t)}$. The tuning parameters are selected by cross validation as described in §5.1. We then use the obtained estimates to predict the stock price in day $t + 1000$. The averaged prediction errors are calculated as

$$\frac{1}{258} \sum_{t=1}^{258} \| \bar{E}_{t+1000,*} - \sum_{k=1}^{p} \hat{A}_k^{(t)} \hat{E}_{t+1000-k,*} \|_2.$$ 

In Table 6, we present the averaged prediction errors for the three methods with different lags. The standard deviations of the prediction errors are present in the parentheses. Our method uniformly outperforms the other two methods in terms of prediction accuracy.

6 Discussion

Estimating vector autoregressive models is a fundamental problem in multivariate time series analysis. However, high dimensionality brings new challenges and viewpoints to this classic problem. In this paper, we propose a new linear programming based formulation to estimate VAR models. As has been shown in this draft, this new formulation brings new advantages over the least squares method. Moreover, our theoretical analysis brings new insights into the problem of transition matrix estimation and shows that $\| A \|_2$ plays a pivotal role in characterizing the estimation performance of the obtained estimator.

In this paper we did not discuss estimating the covariance matrix $\Sigma$ and $\Psi$. Lemma A.1 builds the $L_{\text{max}}$-rate of convergence for estimating $\Sigma$. Under the additional assumption that $\Sigma$ is sparse in some sense, we can leverage the existing methods from the literature (including “banding” (Bickel and Levina, 2008b), “tapering” (Cai et al., 2010), and “thresh-
olding" (Bickel and Levina, 2008a)) to estimate the covariance matrix $\Sigma$ and establish the consistency results under matrix $L_1$ and $L_2$ norms. Under an VAR model with lag 1, with both $\Sigma$ and $A_1$ estimated by some consistent estimators $\hat{\Sigma}$ and $\hat{A}_1$, an estimator $\hat{\Psi}$ of $\Psi$ can be obtained as:

$$\hat{\Psi} = \hat{\Sigma} - \hat{A}_1^T \hat{\Sigma} \hat{A}_1,$$

and a similar estimator can be obtained for lag $p$ VAR model using the augmented formulation shown in Equation (2.3).

In this manuscript we focus on the stationary vector autoregressive model and our method is designed for such stationary process. The stationary requirement is a common assumption in analysis and is adopted by most recent works, for example, Kock and Callot (2012) and Song and Bickel (2011). We noticed that there are works in handling unstable VAR models, checking for example Song et al. (2013) and Kock (2012). This could be a topic left for future investigation. Another unexplored topic is on determining the order of the vector autoregression. There have been results in this area (e.g., Song and Bickel (2011)). We plan to develop more data-dependent model selection methods in the future.

A Proofs of Main Results

In this section we provide the proofs of the main results in the manuscript.

A.1 Proof of Theorem 4.1

Before proving the main result in Theorem 4.1, we first establish several lemmas. In the sequel, because we only focus on the lag 1 autoregressive model, for notation simplicity, in $\Sigma_i(\{(X_t)\})$ we remove $\{(X_t)\}$ and simply denote the lag $i$ covariance matrix to be $\Sigma_i$.

The following lemma describes the $L_{\max}$ rate of convergence $S$ to $\Sigma$. This result generalizes the upper bound derived when data are independently generated (see, for example, Bickel and Levina (2008a)).

Lemma A.1. Letting $S$ be the marginal sample covariance matrix defined in (3.1), when $T \geq \max(6 \log d, 1)$, we have, with probability no smaller than $1 - 6d^{-1}$,

$$\|S - \Sigma\|_{\max} \leq \frac{16\|\Sigma\|_{2} \max_j(\Sigma_{jj})}{\min_j(\Sigma_{jj})(1 - \|A_1\|_2)} \left\{ \left( \frac{6 \log d}{T} \right)^{1/2} + 2 \left( \frac{1}{T} \right)^{1/2} \right\}.$$

Proof. For any $j, k \in \{1, 2, \ldots, d\}$, we have

$$\mathbb{P}(|S_{jk} - \Sigma_{jk}| > \eta) = \mathbb{P} \left( \left| \frac{1}{T} \sum_{t=1}^{T} x_{tj} x_{tk} - \Sigma_{jk} \right| > \eta \right).$$
Letting $y_t = \{x_{t1}(\Sigma_{11})^{-1/2}, \ldots, x_{td}(\Sigma_{dd})^{-1/2}\}^T$ for $t = 1, \ldots, T$ and $\rho_{jk} = \Sigma_{jk}(\Sigma_{jj}\Sigma_{kk})^{-1/2}$, we have

$$P(|S_{jk} - \Sigma_{jk}| > \eta) = \mathbb{P}\left\{\frac{1}{T} \sum_{t=1}^{T} y_{tj} y_{tk} - \rho_{jk} > \eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}$$

$$= \mathbb{P}\left\{\frac{1}{T} \sum_{t=1}^{T} (y_{tj} + y_{tk})^2 - \frac{1}{4T} \left(\sum_{t=1}^{T} (y_{tj} - y_{tk})^2\right) - \rho_{jk} > \eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}$$

$$\leq \mathbb{P}\left\{\frac{1}{T} \sum_{t=1}^{T} (y_{tj} + y_{tk})^2 - 2(1 + \rho_{jk}) > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}$$

$$+ \mathbb{P}\left\{\frac{1}{T} \sum_{t=1}^{T} (y_{tj} - y_{tk})^2 - 2(1 - \rho_{jk}) > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}. \quad (A.1)$$

Using the property of Gaussian distribution, we have $(y_{1j} + y_{1k}, \ldots, y_{Tj} + y_{Tk})^T \sim N_T(0, Q)$ for some positive definite matrix $Q$. In particular, we have

$$|Q_{il}| = |\text{Cov}(y_{ij} + y_{ik}, y_{lj} + y_{lk})| = |\text{Cov}(y_{ij}, y_{lj}) + \text{Cov}(y_{ij}, y_{lk}) + \text{Cov}(y_{ik}, y_{lj}) + \text{Cov}(y_{ik}, y_{lk})|$$

$$\leq \frac{1}{\min_j(\Sigma_{jj})} |\text{Cov}(x_{ij}, x_{ij}) + \text{Cov}(x_{ij}, x_{lk}) + \text{Cov}(x_{ik}, x_{lk}) + \text{Cov}(x_{ik}, x_{lj})|$$

$$\leq \frac{4}{\min_j(\Sigma_{jj})} \|\Sigma_{l-i}\|_{\text{max}} \leq \frac{8\|\Sigma\|_2 \|A_1\|_2^{l-i}}{\min_j(\Sigma_{jj})},$$

where the last inequality follows from (2.2).

Therefore, using the matrix norm inequality,

$$\|Q\|_2 \leq \max_{1 \leq i \leq T} \sum_{t=1}^{T} |Q_{il}| \leq \frac{8\|\Sigma\|_2}{\min_j(\Sigma_{jj})(1 - \|A_1\|_2)}.$$

Then applying Lemma B.1 to (A.1), we have

$$P\left\{\frac{1}{T} \sum_{t=1}^{T} (y_{tj} + y_{tk})^2 - 2(1 + \rho_{jk}) > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}$$

$$\leq 2 \exp\left[\frac{-T}{2} \left\{\frac{\eta \min_j(\Sigma_{jj})(1 - \|A_1\|_2)}{16\|\Sigma\|_2(\Sigma_{jj}\Sigma_{kk})^{1/2}} - 2T^{-1/2}\right\}^2\right] + 2 \exp\left(-\frac{T}{2}\right). \quad (A.2)$$

Using a similar argument, we have

$$P\left\{\frac{1}{T} \sum_{t=1}^{T} (y_{tj} - y_{tk})^2 - 2(1 - \rho_{jk}) > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\}$$

$$\leq 2 \exp\left[\frac{-T}{2} \left\{\frac{\eta \min_j(\Sigma_{jj})(1 - \|A_1\|_2)}{16\|\Sigma\|_2(\Sigma_{jj}\Sigma_{kk})^{1/2}} - 2T^{-1/2}\right\}^2\right] + 2 \exp\left(-\frac{T}{2}\right). \quad (A.3)$$
Combining (A.2) and (A.3), then applying the union bound, we have
\[ P(\|S - \Sigma\|_{\text{max}} > \eta) \leq 3d^2 \exp\left(\frac{-T}{2}\right) + \frac{3d^2}{16\|A_1\|_2} \eta \left(\frac{1}{T}\right)^{-1/2} \].

The proof thus completes by choosing \( \eta \) as the described form.

In the next lemma we try to quantify the difference between \( S_1 \) and \( \Sigma_1 \) with respect to the matrix \( L_{\text{max}} \) norm. Remind that \( \Sigma_1 \{(X_t)\} \) is simplified to be \( \Sigma_1 \).

**Lemma A.2.** Letting \( S_1 \) be the lag 1 sample covariance matrix, when \( T \geq \max(6\log d + 1, 2) \), we have, with probability no smaller than \( 1 - 8d^{-1} \),
\[ \|S_1 - \Sigma_1\|_{\text{max}} \leq \frac{32\|\Sigma\|_2 \max_j(\Sigma_{jj})}{\min_j(\Sigma_{jj})(1 - \|A_1\|_2)} \left(\frac{3\log d}{T}\right)^{1/2} + \left(\frac{2}{T}\right)^{1/2} \].

**Proof.** We have, for any \( j, k \in \{1, 2, \ldots, d\} \),
\[ P((S_1)_{jk} - (\Sigma_1)_{jk} > \eta) = P\left(\left|\frac{1}{T-1} \sum_{t=1}^{T-1} x_{tj} x_{(t+1)k} - (\Sigma_1)_{jk}\right| > \eta \right) \].

Letting \( y_t = \{x_{t1}(\Sigma_{11})^{-1/2}, \ldots, x_{td}(\Sigma_{dd})^{-1/2}\}^T \) and \( \rho_{jk} = (\Sigma_1)_{jk}(\Sigma_{jj}\Sigma_{kk})^{-1/2} \), we have
\[ P((S_1)_{jk} - (\Sigma_1)_{jk} > \eta) = P\left\{\left|\frac{1}{T-1} \sum_{t=1}^{T-1} y_{tj} y_{(t+1)k} - \rho_{jk}\right| > \eta(\Sigma_{jj}\Sigma_{kk})^{-1/2}\right\} \]
\[ = P\left[ \frac{\sum_{t=1}^{T-1}(y_{tj} + y_{(t+1)k})^2 - \sum_{t=1}^{T-1}(y_{tj} - y_{(t+1)k})^2}{4(T - 1)} - \rho_{jk} \right| > \eta(\Sigma_{jj}\Sigma_{kk})^{-1/2} \]
\[ \leq P\left[ \frac{\sum_{t=1}^{T-1}(y_{tj} + y_{(t+1)k})^2}{T - 1} - 2(1 + \rho_{jk}) \right| > 2(\Sigma_{jj}\Sigma_{kk})^{-1/2} \]
\[ + P\left[ \frac{\sum_{t=1}^{T-1}(y_{tj} - y_{(t+1)k})^2}{T - 1} - 2(1 - \rho_{jk}) \right| > 2(\Sigma_{jj}\Sigma_{kk})^{-1/2} \]. \]

Using the property of Gaussian distribution, we have \( \{y_{1j} + y_{2k}, \ldots, y_{(T-1)j} + y_{Tk}\}^T \sim \).
$N_{T-1}(0, Q)$, for some positive definite matrix $Q$. In particular, we have

$$|Q_{il}| = |\text{Cov}(y_{ij} + y_{(i+1)k}, y_{lj} + y_{(l+1)k})|$$

$$= |\text{Cov}(y_{ij}, y_{lj}) + \text{Cov}(y_{ij}, y_{(l+1)k}) + \text{Cov}(y_{(i+1)k}, y_{lj}) + \text{Cov}(y_{(i+1)k}, y_{(l+1)k})|$$

$$\leq \frac{1}{\min_j(\Sigma_{jj})} |\text{Cov}(x_{ij}, x_{lj}) + \text{Cov}(x_{ij}, x_{(l+1)k}) + \text{Cov}(x_{(i+1)k}, x_{lj}) + \text{Cov}(x_{(i+1)k}, x_{(l+1)j})|$$

$$\leq 2\|\Sigma_{t-1}\|_{\max} + \|\Sigma_{t+1-i}\|_{\max} + \|\Sigma_{t-1-i}\|_{\max}$$

$$\min_j(\Sigma_{jj})$$

$$\leq \frac{\|\Sigma\|_2(2\|A_1\|_2^{|l-i|} + \|A_1\|_2^{2+|l-i|} + \|A_1\|_2^{2-|l-i|})}{\min_j(\Sigma_{jj})}.$$ 

Therefore, using the matrix norm inequality,

$$\|Q\|_2 \leq \max_{1 \leq i \leq (T-1)} \sum_{t=1}^{T-1} |Q_{il}| \leq \frac{8\|\Sigma\|_2}{\min_j(\Sigma_{jj})(1 - \|A_1\|_2)}.$$ 

Then applying Lemma B.1 to (A.4), we have

$$\mathbb{P}\left[ \left| \frac{1}{T-1} \sum_{t=1}^{T-1} \{y_{ij} + y_{(t+1)k}\}^2 - 2(1 + \rho_{jk}) \right| > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2} \right] \leq$$

$$2 \exp \left[ -\frac{(T - 1)}{2} \left\{ \frac{\eta \min_j(\Sigma_{jj})(1 - \|A_1\|_2)}{16\|\Sigma\|_2(\Sigma_{jj}\Sigma_{kk})^{1/2}} \right\}^2 - 2(T - 1)^{-1/2} \right] + 2 \exp \left( -\frac{T - 1}{2} \right). \quad \text{(A.5)}$$

Using a similar technique, we have

$$\mathbb{P}\left[ \left| \frac{1}{T-1} \sum_{t=1}^{T-1} \{y_{ij} - y_{(t+1)k}\}^2 - 2(1 - \rho_{jk}) \right| > 2\eta(\Sigma_{jj}\Sigma_{kk})^{-1/2} \right] \leq$$

$$2 \exp \left[ -\frac{(T - 1)}{2} \left\{ \frac{\eta \min_j(\Sigma_{jj})(1 - \|A_1\|_2)}{16\|\Sigma\|_2(\Sigma_{jj}\Sigma_{kk})^{1/2}} \right\}^2 - 2(T - 1)^{-1/2} \right] + 2 \exp \left( -\frac{T - 1}{2} \right). \quad \text{(A.6)}$$

Combining (A.5) and (A.6), and applying the union bound across all pairs $(j, k)$, we have

$$\mathbb{P}(|S_1 - \Sigma_j|_{\max} > \eta) \leq$$

$$4d^2 \exp \left[ -\frac{(T - 1)}{2} \left\{ \frac{\eta \min_j(\Sigma_{jj})(1 - \|A_1\|_2)}{16\|\Sigma\|_2 \max_j(\Sigma_{jj})} \right\}^2 - 2(T - 1)^{-1/2} \right] + 4d^2 \exp \left( -\frac{T - 1}{2} \right). \quad \text{(A.7)}$$

Finally noting that when $T \geq 3$, we have $1/(T - 1) < 2/T$. The proof thus completes by choosing $\eta$ as stated.

Using the above two technical lemmas, we can then proceed to the proof of the main results in Theorem 4.1.
Proof of Theorem 4.1. With Lemmas A.1 and A.2, we proceed to prove Theorem 4.1. We first denote
\[ \zeta_1 = 16\|\Sigma\|_2 \max_j (\Sigma_{jj}) \left\{ \left( \frac{6 \log d}{T} \right)^{1/2} + 2 \left( \frac{1}{T} \right)^{1/2} \right\} , \]
\[ \zeta_2 = 32\|\Sigma\|_2 \max_j (\Sigma_{jj}) \left\{ \left( \frac{3 \log d}{T} \right)^{1/2} + \left( \frac{2}{T} \right)^{1/2} \right\} . \]
Using Lemmas A.1 and A.2, we have, with probability no smaller than 1, \( S - \Sigma \) is feasible in the optimization equation, by checking the equivalence between (3.4) and (3.5), we have \( \|S - \Sigma\|_{\max} \leq \zeta_1, \|S_1 - \Sigma_1\|_{\max} \leq \zeta_2. \)

We firstly prove that population quantity \( A_1 \) is a feasible solution to the optimization problem in (3.4) with probability no smaller than 1:
\[ \|SA_1 - S_1^T\|_{\max} = \|S\Sigma^{-1}\Sigma_1^T - S_1^T\|_{\max} \]
\[ = \|S\Sigma^{-1}\Sigma_1^T - \Sigma_1^T + \Sigma_1^T - S_1^T\|_{\max} \]
\[ \leq \|(S\Sigma^{-1} - I_d)\Sigma_1^T\|_{\max} + \|\Sigma_1^T - S_1^T\|_{\max} \]
\[ \leq \|(S - \Sigma)\Sigma^{-1}\Sigma_1^T\|_{\max} + \zeta_2 \]
\[ \leq \zeta_1\|A_1\|_1 + \zeta_2 \]
\[ \leq \lambda_0. \]

The last inequality holds by using the condition that \( d \geq 8 \) implies that \( 1/T \leq \log d/(2T) \).

Therefore, \( A_1 \) is feasible in the optimization equation, by checking the equivalence between (3.4) and (3.5), we have \( \|\hat{\Omega}\|_1 \leq \|A_1\|_1 \) with probability no smaller than 1. We then have
\[ \|\hat{\Omega} - A_1\|_{\max} = \|\Sigma^{-1}\Sigma_1^T\|_{\max} \]
\[ = \|\Sigma^{-1}(\hat{\Sigma} - \Sigma_1^T)\|_{\max} \]
\[ = \|\Sigma^{-1}(\hat{\Sigma} - S_1^T + S_1^T - \Sigma_1^T)\|_{\max} \]
\[ = \|\Sigma^{-1}(\hat{\Sigma} - S_1^T + S_1^T - \Sigma_1^T)\|_{\max} \]
\[ \leq \|(I_d - S^{-1}S)\hat{\Omega}\|_{\max} + \|\Sigma^{-1}(S_1^T - S_1^T)\|_{\max} \]
\[ \leq \|\Sigma^{-1}\|_1\|\Sigma - S\|\hat{\Omega}\|_{\max} + \|\Sigma^{-1}\|_1\|S_1^T - S_1^T\|_{\max} \]
\[ \leq \|\Sigma^{-1}\|_1(\|A_1\|_1\zeta_1 + \lambda_0 + \zeta_2) \]
\[ = 2\lambda_0\|\Sigma^{-1}\|_1. \]

Let \( \lambda_1 \) be a threshold level and we define
\[ s_1 = \max_{1 \leq j \leq d} \sum_{i=1}^d \min \{(A_{ij})/|A_{ij}|, 1\}, \quad T_j = \{ i : |(A_{ij})| \geq \lambda_1 \}. \]
We have, with probability no smaller than $1 - 14d^{-1}$, for all $j \in \{1, \ldots, d\}$,

$$\|\hat{\Omega}_{*,j} - (A_1)_{*,j}\|_1 \leq \|\hat{\Omega}^{T_{c,j}\,j}\|_1 + \|(A_1)^{T_{c,j}\,j}\|_1 + \|\hat{\Omega}^{T_{c,j},j} - (A_1)_{T_{c,j}\,j}\|_1$$

$$= \|\hat{\Omega}_{*,j}\|_1 - \|\hat{\Omega}^{T_{c,j}\,j}\|_1 + \|(A_1)^{T_{c,j}\,j}\|_1 + \|\hat{\Omega}^{T_{c,j},j} - (A_1)_{T_{c,j}\,j}\|_1$$

$$\leq 2\|(A_1)^{T_{c,j}\,j}\|_1 + 2\|\hat{\Omega}^{T_{c,j}\,j} - (A_1)_{T_{c,j}\,j}\|_1$$

$$\leq 2\|(A_1)^{T_{c,j}\,j}\|_1 + 4\lambda_0\|\Sigma^{-1}\|_1 |T_j|$$

$$\leq (2\lambda_1 + 4\lambda_0\|\Sigma^{-1}\|_1)s_1.$$

Suppose $\max_j \sum_{i=1}^d |(A_1)_{ij}|^q \leq s$ and setting $\lambda_1 = 2\lambda_0\|\Sigma^{-1}\|_1$, we have

$$\lambda_1 s_1 = \max_{1\leq j\leq d} \sum_{i=1}^d \min\{(A_1)_{ij}, 1\} \leq \lambda_1 \max_{1\leq j\leq d} \sum_{i=1}^d \min\{(A_1)_{ij}^q/\lambda_1^q, 1\} \leq \lambda_1^{-q} s.$$

Therefore, we have

$$\|\hat{\Omega}_{*,j} - (A_1)_{*,j}\|_1 \leq 4\lambda_1 s_1 \leq 4\lambda_1^{1-q} s = 4s(2\lambda_0\|\Sigma^{-1}\|_1)^{1-q}.$$

Noting that when the lag of the time series $p = 1$, by definition in (3.6), we have $\hat{\Omega} = \hat{A}_1$. This completes the proof. \qed

### A.2 Proof of the Rest Results

**Proof of Corollary 4.4.** Corollary 4.4 directly follows from Theorem 4.1, so its proofs is omitted. \qed

**Proof of Corollary 4.5.** Using the generating model described in Equation (2.1), we have

$$\|x_{T+1} - \tilde{A}_r^ix_T\|_\infty = \|(A_1^r - \tilde{A}_1^r)x_T + Z_{T+1}\|_\infty$$

$$\leq \|A_1^r - \tilde{A}_1^r\|_\infty \|x_T\|_\infty + \|Z_{T+1}\|_\infty$$

$$= \|A_1 - \tilde{A}_1\|_1 \|x_T\|_\infty + \|Z_{T+1}\|_\infty$$

Using Lemma B.2 in Appendix B, we have

$$\mathbb{P}(\|x_T\|_\infty \leq (\Sigma_{\max} \cdot \alpha \log d)^{1/2}, \|Z_{T+1}\|_\infty \leq (\Psi_{\max} \cdot \alpha \log d)^{1/2}) \geq 1 - 2(d^{n/2-1} \sqrt{\pi/2 \cdot \alpha \log d})^{-1}.$$ 

This, combined with Theorem 4.1, gives Equation (4.6). \qed
Proof of Corollary 4.6. Similar as the proof in Corollary 4.5, we have
\[ \|x_{T+1} - \bar{A}_1^T x_T\|_2 = \|(A_1^T - \bar{A}_1^T) x_T + Z_{T+1}\|_2 \leq \|A_1 - \bar{A}_1\|_2 \|x_T\|_2 + \|Z_{T+1}\|_2. \]

For any Gaussian random vector \( Y \sim N_d(0, Q) \), we have \( Y = \sqrt{Q} Y_0 \) where \( Y_0 \sim N_d(0, I_d) \).
Using the concentration inequality for Lipchitz functions of standard Gaussian random vector (see, for example, Theorem 3.4 in Massart (2007)), we have
\[ P(\|\sqrt{Q} Y_0\|_2 - E\|\sqrt{Q} Y_0\|_2 \geq t) \leq 2 \exp\left( -\frac{t^2}{2\|Q\|_2} \right). \tag{A.8} \]

Here the inequality exploits the fact that for any vectors \( x, y \in \mathbb{R}^d \),
\[ \|\sqrt{Q} x\|_2 - \|\sqrt{Q} y\|_2 \leq \|\sqrt{Q}(x - y)\|_2 \leq \|\sqrt{Q}\|_2 \|x - y\|_2, \]
and accordingly the function \( x \rightarrow \|\sqrt{Q} x\|_2 \) has the Lipschitz norm no greater than \( \sqrt{\|Q\|_2} \).
Using Equation (A.8), we then have
\[ P(\|x_T\|_2 \leq \sqrt{2\|\Sigma\|_2 \log d} + E\|x_T\|_2, \|Z_{T+1}\|_2 \leq \sqrt{2\|\Psi\|_2 \log d} + E\|Z_{T+1}\|_2) \geq 1 - 4d^{-1}. \]

Finally, we have
\[ (E\|Y\|_2)^2 \leq E\|Y\|_2^2 = \text{tr}(Q). \]
Combined with Theorem 4.1 and the fact that \( \|A_1 - \bar{A}_1\|_2 \leq \|A_1 - \bar{A}_1\|_1 \), we have the desired result. \( \square \)

Proof of Theorem 4.7. Theorem 4.7 follows from the connection between autoregressive model with lag 1 and lag \( p \) shown in (2.3). The proof technique is similar to that of Theorem 4.1, thus is omitted. \( \square \)

B Supporting Lemmas

Lemma B.1 (Negahban and Wainwright (2011)). Suppose that \( Y \sim N_T(0, Q) \) is a Gaussian random vector. We have, for \( \eta > 2T^{-1/2} \),
\[ P\left( \|Y\|_2^2 - E(\|Y\|_2^2) > 4T\eta \|Q\|_2 \right) \leq 2 \exp \left\{ -T(\eta - 2T^{-1/2})^2/2 \right\} + 2 \exp(-T/2). \]
Proof. This can be proved by first using the concentration inequality for the Lipchitz functions \( \|Y\|_2 \) of Gaussian random variables \( Y \). Then combining with the result
\[ \|Y\|_2^2 - E(\|Y\|_2^2) = (\|Y\|_2 - E\|Y\|_2) \cdot (\|Y\|_2 + E\|Y\|_2), \]
we have the desired concentration inequality. \( \square \)
Lemma B.2. Suppose that $Z = (Z_1, \ldots, Z_d)^T \in N_d(0, Q)$ is a Gaussian random vector. Letting $Q_{\max} := \max_i(Q_{ii})$, we have

$$\mathbb{P}\{\|Z\|_\infty > (Q_{\max} \cdot \alpha \log d)^{1/2}\} \leq \left(d^{n/2-1} \sqrt{\pi/2} \cdot \alpha \log d\right)^{-1}.$$

Proof. Simply using the Gaussian tail probability, we have

$$\mathbb{P}(\|Z\|_\infty > t) \leq \sum_{i=1}^d \mathbb{P}(\|Z_i\|_\infty^{-1/2} > t \cdot Q_{ii}^{-1/2}) \leq \sum_{i=1}^d \frac{2 \exp(-t^2/2Q_{ii})}{t \cdot Q_{ii}^{-1/2} \cdot \sqrt{2\pi}} \leq \frac{2d \exp(-t^2/2Q_{\max})}{t \cdot Q_{\max}^{-1/2} \cdot \sqrt{2\pi}}.$$

Taking $t = (Q_{\max} \cdot \alpha \log d)^{1/2}$ into the upper equation, we have the desired result. \qed

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