THERMODYNAMIC LIMIT IN CHERN-SIMONS SYSTEM OF PARTICLES WITH MB STATISTICS.

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Abstract

The reduced density matrices (RDMs) are calculated in the thermodynamic limit for the Chern-Simons non-relativistic particle system and Maxwell-Boltzmann (MB) statistics. It is established that they are zero outside of a diagonal and well-behaved after a renormalization, depending on an arbitrary real number, if the condition of neutrality holds.

1 Introduction.

The Chern-Simons 2-d quantum system of n nonrelativistic spinless identical particles of unit mass is described by the Hamiltonian $\dot{H}_n$, defined on $C^\infty(\mathbb{R}^{2n}\cup_{j,k}(x_j = x_k))$

$$\dot{H}_n = \frac{1}{2} \sum_{j=1}^{n} \|p_j - a_j(X_n)\|^2, \quad a'_j(X_n) = \epsilon^{\nu\mu}\partial_{\nu,j}U_C(X_n), \quad \nu, \mu = 1, 2,$$

where $\|v\|^2 = (v^1)^2 + (v^2)^2$, $\epsilon$ is the skew symmetric tensor and the repeating index implies a summation, real number $e_j$ (a charge) takes values in a finite set $E_{(c)}$ from $\mathbb{R}$.

Differentiating the equality $f(f^{-1}(x)) = x$ we derive the formula

$$\frac{df^{-1}(x)}{dx} = (\frac{df(y)}{dy})^{-1}, \quad y = f^{-1}(x).$$

From this equality for $f(x) = \tan x$ and the equality $\frac{d}{dx} \tan x = 1 + \tan^2(x)$ the following relation is derived

$$\frac{\partial}{\partial x^\mu} \arctan \frac{x^2}{x^1} = \epsilon^{\nu\mu} x^\nu |x|^{-2} = \epsilon^{\nu\mu} \frac{\partial}{\partial x^\nu} \ln|x|.$$

This means that CS system is almost (quasi-)integrable, that is

$$\dot{H}_n = \epsilon^{i\hat{U}} \hat{H}_n^0 e^{-i\hat{U}},$$

where $-2\hat{H}_n^0$ is the 2n-dimensional Laplacian $-2H_n^0$ restricted to $C^\infty(\mathbb{R}^{2n}\cup_{j,k}(x_j = x_k))$ and $\hat{U}$ is the operator of multiplication by $U(X_n)$,

$$U(X_n) = g \sum_{1 \leq k < j \leq n} e_j e_k \phi(x_j - x_k), \quad \phi(x_j - x_k) = \arctan \frac{x^2_j - x^2_k}{x^1_j - x^1_k}.$$

As a result, there exists the simplest selfadjoint extension $H_n$ of $\dot{H}_n$.
with the domain \( D(H_n) = e^{i \mathbf{U}} D(H_n^0) \). Another selfadjoint extension of \( \tilde{H}_n \) is given by (1.2) in which, instead of \( H_n^0 \), another selfadjoint extension of \( \tilde{H}_n^0 \) is considered.

The CS system of particles with different statistics has been studied by many authors [1-4] since it can be derived (formally) in 3-d topological electrodynamics (its Lagrangian contains CS term) in the limit of the vanishing Maxwell term (the same is true for its relativistic version). There is a hope it can give a new mechanism of superconductivity, superfluidity and \( P,T \) violation (general outlook of modern statistical mechanics allows to expect breakdown only of discrete (hidden) symmetries for quantum systems at nonzero temperature in 3 space-time dimensions).

In this paper the CS system is considered with the Hamiltonian (1.2) and the Maxwell-Boltzmann (MB) statistics. Its thermodynamics coincides with thermodynamics of the free particle system if Dirichlet boundary condition is considered (we do this in this paper). Our calculations show that the Gibbs (grand-canonical) reduced density matrices (RDMs) \( \rho_{m,Y}^B(X_m|Y_m) \) in the sphere \( B_L \) of radius \( L \), centered at the origin, tend to zero if \( x_j \neq y_j \). They are expressed as the product of a function having a non-zero limit and

\[
\exp\left\{ -\frac{N_{r,L}}{4} g^2 \left( \sum e \sum j \right) e^2 \right\} \sum e \sum j \left| x_j - y_j \right|^2, \quad N_{r,L} = \int_{r \leq \left| x \right| \leq L} P_{0(B_L)}(x|x)|x|^{-2}dx.
\]

where \( z_e \) is the activity of the particles with the charge \( e \) and \( N_{r,L} \) diverges as \( 2 \ln L \), \( P_0(B_L)(x|y) \) is the kernel of the semigroup whose infinitesimal generator coincides with the two-dimensional Laplacian with the Dirichlet boundary condition on the boundary of the ball \( B_L \). We propose to renormalize the RDMs by dividing them by this exponent. We show then the important feature of this renormalization: if the condition of the neutrality holds then the resulting RDMs in the thermodynamic limit \( H_{m,Y}^B(X_m|Y_m) \) satisfy the compatibility condition (see the corollary), that is, they define a state of the infinite particle system.

The unusual property of the RDMs to be zero almost everywhere was found by us in 1-d (integrable) systems of particles with the Hamiltonian (1.1) when \( a_j \) is expressed through a pair (scalar magnetic) long-range potential [5-6]. We expect that the similar property is true for bosonic and fermionic CS systems for small values of activities of particles (see also [7]).

The results of our calculations are formulated in the theorem and corollary in the next paragraph.

2 Main result.

Let \( \Lambda \in \mathbb{R}^2 \) be a compact set and assume the Dirichlet boundary conditions on the boundary \( \partial \Lambda \). For the inverse temperature \( \beta \), and the activity \( z_{es} \) of the particles with the charge \( e \), the Gibbs (grand-canonical, equilibrium) RDMs are given by

\[
\rho^\Lambda(X_m|Y_m) = \frac{Z_{(e)m}}{\Xi_{\Lambda}^{-1}} \prod_{n \geq 0} \sum_{s=1}^n \Xi_{\Lambda}^{-1} \int dX_n \exp\{i[U(X_m,X_n') - U(Y_m,X_n')]\} \rho_0^\Lambda(X_m,X_n'|Y_m,X_n'),
\]

where \( \Xi_{\Lambda} \) is the grand partition function (it coincides with the numerator in the r.h.s. of this equality for the case "\( m=0 \)", i.e. when there are no \( X_m \) and \( Y_m \)), \( P_{0(\Lambda)}^\Lambda(X_m|Y_m) \) is the kernel of the semigroup, whose infinitesimal generator coincides with \( H_{n,\Lambda}^0 \) (\( -2H_{n,\Lambda}^0 \) is the Laplacian in \( \Lambda \) with the Dirichlet boundary condition on the boundary \( \partial \Lambda \)), \( Z_{(e)m} = \Xi_{\Lambda}^{-1} \), and the
summation in $e_s$ is performed over $E_{(c)}$.

$$P^\beta_{0(\Lambda)}(X_n|Y_n) = \prod_{j=1}^n P^\beta_{0(\Lambda)}(x_j|y_j), \quad X_m = (X^1_m, X^2_m), Y_m = (Y^1_m, Y^2_m) \in \mathbb{R}^{2m}, \quad (2.1)$$

$P^\beta_{0(\Lambda)}(x|y)$ is the transition probability of the 2-dimensional free diffusion process with the Dirichlet boundary condition on $\partial \Lambda$.

$$P^\beta_{0(\Lambda)}(x|y) = \int P^\beta_{x,y}(d\omega)\chi_{\Lambda}(\omega),$$

$P_{x,y}(d\omega)$ is the conditional Wiener measure and $\chi_{\Lambda}(\omega)$ is the characteristic function of the paths that are inside $\Lambda$.

From the equality

$$U(X_m, X_n') = U(X_m) + \sum_{j=1}^m \sum_{k=1}^n \phi(x_j - x'_k) e_j e'_k + U(X'_n)$$

we obtain

$$\rho^A(X_m|Y_m) = Z_{(e)} \Xi_{\Lambda}^{-1} \exp\{i[U(X_m) - U(Y_m)]\} P^\beta_{0(\Lambda)}(X_m|Y_m) \sum_{n \geq 0} \sum e_n \frac{e_{n+1}}{n+1} \times$$

$$\times \int_{\Lambda^0} dx' \prod_{k=1}^n \exp\{i \sum_{j=1}^m e_j e_k (\phi(x_j - x'_k) - \phi(y_j - x'_k))\} P^\beta_{0(\Lambda)}(x'_k|x_k),$$

As a result

$$\rho^A(X_m|Y_m) = Z_{(e)} \exp\{i[U(X_m) - U(Y_m)] + G_{\Lambda}(X_m|Y_m)\} P^\beta_{0(\Lambda)}(X_m|Y_m), \quad (2.2)$$

$$G_{\Lambda}(X_m|Y_m) = \sum e \int_{\Lambda} P^\beta_{0(\Lambda)}(x|x) [\exp\{i \sum_{j=1}^m e e_j (\phi(x_j - x) - \phi(y_j - x))\} - 1] dx. \quad (2.3)$$

Here we used the equality $\Xi_{\Lambda} = \exp\{\sum e \int_{\Lambda} P^\beta_{0(\Lambda)}(x|x) dx\}$.

With the help of the equality

$$\exp\{i \arctan \frac{x^2}{x^1}\} = \frac{x}{|x|} = (\frac{x}{x^1})^{\frac{1}{2}}, \quad x = x^1 + ix^2,$$

we derive

$$\exp\{i \sum_{j=1}^m e e_j (\phi(x_j - x) - \phi(y_j - x))\} = \prod_{j=1}^m \left(\frac{(x - x_j)(x^*_j - y^*_j)}{(x^*_j - x_j)(x - y_j)}\right)^{\frac{1}{2} g e e_j} = G_x(X_m|Y_m).$$

We have to use the Taylor expansions for $|x| < 1$

$$(1 - x)^g = 1 - gx + \frac{g(g-1)}{2} x^2 + \sum_{n \geq 3} C^g_n x^n = \sum_{n \geq 0} C^g_n x^n,$$
\[
\begin{align*}
\left( \frac{x - x_j}{x^*-x_j} \right)^{g'} & = \frac{x}{x^*} \left( \frac{1 - \frac{x_j}{x}}{1 - \frac{x_j}{x^*}} \right)^{g'} = \frac{x}{x^*} \left\{ 1 + g' \left( -\frac{x_j}{x} + \frac{x^*_j}{x^*} \right) + \\
& + \frac{g'(g' - 1)}{2} \left( \frac{x_j^2}{x^2} + \frac{x_j^2}{x^*} \right) - g'^2 \left| x_j \right|^2 + \sum_{n_1, n_1 \geq 3} C_{n_1}^{-g'} C_{n_1}^{-g'} C_{n_2}^{-g'} C_{n_2}^{-g'} \left( \frac{x_j}{x} \right)^{n_1} \left( \frac{x_j}{x^*} \right)^{n_2} \left( \frac{y_j}{x} \right)^{n_2} \left( \frac{y_j}{x^*} \right)^{n_1},
\end{align*}
\]

Applying this formula we deduce
\[
\begin{align*}
\left( \frac{x - x_j}{x^*-x_j} (x^* - y_j^*) \right)^{g'} & = \\
& = 1 + G_x(x_j | y_j) + \sum_{n_1, n_1 \geq 3} C_{n_1}^{-g'} C_{n_1}^{-g'} C_{n_2}^{-g'} C_{n_2}^{-g'} \left( \frac{x_j}{x} \right)^{n_1} \left( \frac{x_j}{x^*} \right)^{n_2} \left( \frac{y_j}{x} \right)^{n_2} \left( \frac{y_j}{x^*} \right)^{n_1}.
\end{align*}
\]

where
\[
G_x'(x_j | y_j) = g' \left( -\frac{x_j - y_j}{x} + \frac{x_j^* - y_j^*}{x^*} \right) - g'^2 \left( \frac{x_j}{x} - \frac{x_j^*}{x^*} \right) \left( \frac{y_j}{x} - \frac{y_j^*}{x^*} \right) + \\
+ \frac{g'(g' - 1)}{2} \left( \frac{x_j^2 + y_j^2}{x^2} + \frac{x_j^2 + y_j^2}{x^*} \right) - g'^2 \left| x_j \right|^2 + \left| y_j \right|^2.
\]

As a result
\[
G_x(X_m | Y_m) = 1 + G_{x,e}^0(X_m | Y_m) + \sum_{j=1}^m G_x'(x_j | y_j), \quad (2.4)
\]

where
\[
G_{x,e}^0(X_m | Y_m) = \sum_{n_1, n_1 \geq 3} \prod_{j=1}^m C_{n_1}^{-g'} C_{n_1}^{-g'} C_{n_2}^{-g'} C_{n_2}^{-g'} \left( \frac{x_j}{x} \right)^{n_1} \left( \frac{x_j}{x^*} \right)^{n_2} \left( \frac{y_j}{x} \right)^{n_2} \left( \frac{y_j}{x^*} \right)^{n_1}.
\]

It can be checked that
\[
\begin{align*}
-g'^2 \left( \frac{x_j}{x} - \frac{x_j^*}{x^*} \right) (y_j - y_j^*) + \frac{g'^2}{2} \left( \frac{x_j^2 + y_j^2}{x^2} + \frac{x_j^2 + y_j^2}{x^*} \right) - g'^2 \left| \frac{x_j}{x} \right|^2 + \left| \frac{y_j}{x} \right|^2 = \\
= -g'^2 \left| \frac{x_j - y_j}{x} \right|^2 + \frac{g'^2}{2} \left( \frac{(x_j - y_j)^2}{x^2} + \frac{(x_j^* - y_j^*)^2}{x^*} \right), \quad (2.5)
\end{align*}
\]

This yields
\[
G_x'(x_j | y_j) = -g'^2 \left| \frac{x_j - y_j}{x} \right|^2 + G_x^{-}(x_j | y_j), \quad G_x^{-}(x_j | y_j) = \\
= g' \left( -\frac{x_j - y_j}{x} + \frac{x_j^* - y_j^*}{x^*} \right) + \frac{g'^2}{2} \left( \frac{(x_j - y_j)^2}{x^2} + \frac{(x_j^* - y_j^*)^2}{x^*} \right) - g'^2 \left( \frac{x_j^2 + y_j^2}{x^2} + \frac{x_j^2 + y_j^2}{x^*} \right). \quad (2.6)
\]

Let \( l_m^+ = \text{max}(|x_j|, |y_j|, j = 1, \ldots, m) \). Let \( \Lambda = B_L \) then
\[
G_x(X_m | Y_m) = \sum_{x} \mathcal{P}_{\Lambda}(x) \left| \prod_{j=1}^m \left( \frac{x - x_j - x^* - y_j}{x - y_j} \right)^{\frac{1}{2}} \right|, \quad 1 | dx + \]
\[+ \int_{2t_m^+ \leq |x| \leq L} P_{0(\Lambda)}^\beta(x|x)G_x^0(X_m|Y_m)dx + \int_{2t_m^+ \leq |x| \leq L} P_{0(\Lambda)}^\beta(x|x)G_x'(X_m|Y_m)dx \{2.7\}

For \(|x| \geq 2t_m^+\) we have the bound

\[G_{x,e}^0(X_m|Y_m) \leq \frac{(2t_m^+)^3}{|x|^3} 2^{4|g|}\m.

Here we used the inequalities

\[\left| \left(\frac{x^*}{x} \right)^{n_{+},j} \left(\frac{x}{x^*} \right)^{n_{-},j} \left(\frac{y^*}{x} \right)^{n_3,j} \left(\frac{y}{x^*} \right)^{n_2,j} \right| \leq \frac{(2t_m^+)^3}{|x|^3} 2^{-(n_{+},j+n_{-},j+n_3,j+n_2,j)}, \quad |C_n^*(-g)| \leq C_n^{-|g|} > 0.

After applying them we enlarge the sum in (2.5) to the sum over \((Z^+)^{4m}\).

Since \(P_\Lambda(x|x)\) tends to \((2\pi\beta)^{-1}\) the first and the second terms in (2.7) have limits when \(L\) tends to infinity. We have only to calculate the third term. Let’s show that

\[\int_{r \leq |x| \leq L} G_x^-(x'|y')dx = 0. \quad \{2.8\}

For arbitrary \(r, L, v = v^1 + iv^2, B = B_L \setminus B_r\) we have

\[\int_{B} (\frac{v}{x} - \frac{v^*}{x^*})dx = -2i[v^1 \int_{B} \frac{x^2}{|x|^2}dx - v^2 \int_{B} \frac{x}{|x|^2}dx] = 0,

\[\int_{B} (\frac{v}{x^2} + \frac{v^*}{x^2})dx = 2(v^1 \int_{B} \frac{(x^1)^2}{|x|^4}dx + 2v^2 \int_{B} \frac{x^1 x^2}{|x|^4}dx) = 0.

All the above integrals are zero since the all the functions change signs when either a sign of one of the variables is changed , or a permutation is done.

As a result

\[\int_{r \leq |x| \leq L} G_x'(X_m|Y_m)dx = -\frac{1}{4}\left(\sum_{e} z_e e^2\right) N_{r,L} \sum_{j=1}^{m} e_j^2 |x_j - y_j|^2. \quad \{2.9\}

The integral in the right-hand-side of this equality diverges as \(2lnL\). For \(g' = k, k \in \mathbb{Z}\) we obtain (2.4) in which we have to put \(C_n^k = 0\) for \(n > k, C_n^k = \frac{k!}{(n-k)!n!}\).

From (2.7), (2.9) we derive

\[G_\Lambda(X_m|Y_m) = \sum_{e} z_e \left\{ \int_{|x| \leq 2t_m^+} P_{0(\Lambda)}^\beta(x|x) \prod_{j=1}^{m} \left(\frac{x^* - x_j^*}{x - x_j} \frac{x - y_j}{x^* - y_j^*}\right)^{\frac{1}{2}g e e_j} - 1\right\} dx + \int_{2t_m^+ \leq |x| \leq L} P_{0(\Lambda)}^\beta(x|x)G_x^0(X_m|Y_m)dx - \frac{1}{4}\left(\sum_{e} z_e e^2\right) N_{2t_m^+,L} \sum_{j=1}^{m} e_j^2 |x_j - y_j|^2. \quad \{2.10\}

This equality together with (2.8-9), in which \(2t_m^+\) is substituted instead of \(L\), yields for arbitrary \(0 < r < \infty\)
Moreover, the following equality is valid

\[
G_A(X_m|Y_m) = \int \sum_e z^e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1) P_{0(B_L)}(x|x) dx + 
\]

\[
+ \int \sum_{|x| < r} z_e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1 - \sum_{j=1}^m G'_x(x_j|y_j)] P_{0(B_L)}(x|x) dx - 
\]

\[
- \frac{1}{4} g^2 (\sum_e z^e e^2) N_{r,L} \sum_{j=1}^m e^2_j |x_j - y_j|^2. \tag{2.11}
\]

Using the equality

\[
\lim_{L \to \infty} P_{0(B_L)}(x|x) = (2\pi\beta)^{-1}
\]

we deduce that the function \( G \) is well defined

\[
G(X_m|Y_m) = \lim_{L \to \infty} [G_A(X_m|Y_m) + \frac{1}{4} g^2 (\sum_e z^e e^2) N_{r,L} \sum_{j=1}^m e^2_j |x_j - y_j|^2] = 
\]

\[
= (2\pi\beta)^{-1} \int \sum_e z^e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1) dx + 
\]

\[
+ (2\pi\beta)^{-1} \int \sum_{|x| > r} z_e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1 - \sum_{j=1}^m G'_x(x_j|y_j)] dx. \tag{2.12}
\]

**THEOREM**

Let \( \Lambda \) be the ball \( B_L \), centered at the origin, of radius \( L \). Then for all the values of the activities \( z^e \) and charges \( e_j \) the thermodynamic limit for the RDMs exists and is zero if \( x_j \neq y_j \). Moreover, the following equality is valid

\[
\rho(X_m|Y_m) = \lim_{L \to \infty} \exp \left\{ g^2 N_{r,L} \sum_{j=1}^m e^2_j |x_j - y_j|^2 \right\} \rho^{B_L}(X_m|Y_m) = 
\]

\[
= Z_{e_m} \exp \left\{ i[U(X_m) - U(Y_m)] \right\} P_{0}^\beta(X_m|Y_m) e^{G(X_m|Y_m)},
\]

where \( G \) is defined by (2.12) and

\[
P_{0}^\beta(X_m|Y_m) = (2\pi\beta)^{-m} \exp \left\{ - \frac{|x_j - y_j|^2}{2\beta} \right\}.
\]

Now, let the condition of neutrality holds \( \sum_e z^e e = 0 \), then from (2.12) it follows that

\[
G(X_m|Y_m) = (2\pi\beta)^{-1} \int \sum_e z^e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1) + 
\]

\[
+ (2\pi\beta)^{-1} \int \sum_{|x| > r} z_e \prod_{j=1}^m \left( \frac{x^e - x_j^e}{x^e - y_j} \right) \frac{1}{|x^e|^2} - 1) + 
\]
+ \frac{y^2}{\epsilon} \left( \sum_{e} \epsilon_e^2 \sum_{j=1}^{m} \epsilon_j^2 |x_j - y_j|^2 - \frac{1}{2} \left( \frac{(x_j - y_j)^2}{x^2} + \frac{(x_j^* - y_j^*)^2}{x^*2} \right) \right) dx.

(2.13)

From (2.13) it follows that \( G(x, X_m|x, Y_m) = G(X_m|Y_m) \). Repeating the above arguments the equality

\[
\lim_{L \to \infty} \frac{1}{\pi L^2} \int_{B_L} \exp \left\{ \sum_{j=1}^{m} \epsilon_j^2 \left( \phi(x - x_j) - \phi(x - y_j) \right) \right\} dx = 1
\]

is proved. So, we see that the following corollary is true.

COROLLARY. If the condition of neutrality holds then the compatibility condition for the renormalized RDMs in the thermodynamic limit holds

\[
\lim_{L \to \infty} \frac{2\beta}{\pi L^2} \int_{B_L} \rho(x, X_m|x, Y_m) dx = \rho(X_m|Y_m).
\]

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