Parameter optimization of a grounded dynamic vibration absorber with lever and inerter

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Abstract
Mechanical vibration is mostly harmful, and it may not only generate noise but also affect the working life of the equipment. Lever, inerter, and grounded stiffness have good performance in the field of vibration control, but the dynamic vibration absorber simultaneously containing lever, inerter, and grounded stiffness is rarely studied. Based on the grounded damptype dynamic vibration absorber, a dynamic vibration absorber with lever, inerter, and grounded stiffness is presented. And the optimal system parameters are analytically researched in detail. Firstly, the differential equation of motion is established according to Newton’s second law, and the analytical solution of the system is obtained. According to the amplitude-frequency curve of the system, it is obvious that there are two fixed points unrelated to the damping ratio. Meanwhile, the optimal frequency ratio of the dynamic vibration absorber is obtained based on the fixed-point theory. Under the premise of ensuring the system stability, the optimal grounded stiffness ratio is screened out, and the working range of inerter is further calculated. It is found that the inerter ratio has two working ranges when the coupling term values of magnification ratio and mass ratio are different. Furthermore, the approximate optimal damping ratio is derived by minimizing the maximum value of the amplitude-frequency curve. Using MATLAB, the numerical result is analyzed, and the correctness of analytical results is verified. Compared with other dynamic vibration absorbers under harmonic and random excitations, it is known that the model in this paper can evidently reduce the resonance amplitude and broaden the vibration band of the primary system. These results may provide a theoretical basis for the optimal design of similar dynamic vibration absorbers.

Keywords
dynamic vibration absorber, lever, inerter, grounded stiffness, fixed-point theory, parameter optimization

Introduction
The dynamic vibration absorber (DVA) is one of the common devices in vibration control, which relies on attaching a free-to-vibration mass to the primary structure to suppress the system motion. It is widely used in engineering fields such as transportation, civil structures, and industrial machinery. The first DVA without damping was invented by Frahm.¹ However, this DVA only worked in a narrow applicable frequency range. Den Hartog and Ormondroyd² firstly used damping in the designing of a DVA to overcome the above defect. They found this type of DVA can effectively suppress the amplitude of the primary system and broaden the vibration frequency, which was widely known as Voigt type DVA. In addition, they also found that there were two fixed points on the amplitude-frequency response curve of the primary system which were independent of the damping ratio, so the fixed-point theory was proposed. Based on the fixed-point theory, the
optimal frequency ratio and optimal damping ratio of Voigt type DVA were calculated by Hahnkamm and Brock. When damping existed in the primary system, a design method of the DVA was studied and proposed by Nishihara and Asami, and the accurate and optimal solution of the Voigt type DVA was also deduced. The application value of the results in practical engineering was verified by the fixed-point theory. The grounded damping dynamic vibration absorber was studied by Ren and Liu to improve the vibration control effect by optimizing the grounded damping. Viscoelastic materials were widely used in vibration control engineering and had damping properties and stiffness properties concurrently. Therefore, a three-element type DVA with better control performance was presented by Asami, and the designing parameters were optimized. Shen et al. studied the approximate analytical solutions and the parameters optimization of four kinds of semi-active onoff control DVAs. And they proposed a single-degree-of-freedom passive vibration isolation system with the same performance as the active vibration isolation system. A new quasi-zero-stiffness vibration isolation device was proposed by Li et al., and the vibration isolation effect was studied. Yuan et al. studied the influence of different controllers on the performance of a quasi-zero-stiffness hydraulic dynamic vibration absorber. Another quasi-zero-stiffness DVA was proposed by Chang et al., which could effectively suppress the ultra-low frequency vibration of the primary system.

With the increasing requirements for vibration control in the field of high precision, many researchers tried to apply negative stiffness devices to the field of vibration isolation and achieved some remarkable results. Alabuzhev et al. introduced the theory and application of vibration isolation with negative stiffness in their monograph. An undamped DVA with negative stiffness was proposed by Acar et al., which could effectively reduce the amplitude of the primary system. Shen et al. applied negative stiffness elements to a variety of DVAs, and proved that reasonable negative stiffness would improve vibration damping performance. Particularly, it is found in the literature that the DVA with positive grounded stiffness had the best control effect, when the coupling terms of magnification ratio and mass ratio reached a certain value.

The concept of inerter was introduced by Smith, which was a two-terminal mechanical element that the force applied on the terminals was proportional to the relative acceleration between the two terminals. Lazar et al. introduced an inerter into a multi-storey building and found that it could achieve an excellent level of vibration reduction. Chen et al. studied the influence of an inerter on the frequency of a vibration isolation system. Hu et al. and Wang et al. studied DVAs with inerters, which were optimized by using fixed-point theory and numerical algorithm. It was found that installing an inerter could improve the performance of DVA, but the inerter should be placed in a suitable position. Two types of DVA models with inerter and negative stiffness to restrain the transverse vibration of beams were proposed by Chen et al. Marian et al. studied the tuned mass damper-inerter (TMDI) and found that the TMDI is more effective to suppress vibrations close to the natural frequency, while it was more robust to detuning effects. Jang et al. investigated the optimum parameters and performance of the tuned inerter damper (TID) for the base-isolated structure and evaluated the stochastic response under white-noise excitation. Djerouni et al. proposed a double-mass tuned damper-inerter (DTMDI) and optimized the parameters of TMDI and DTMDI using a genetic algorithm. John et al. presented a novel type of frictionless mechanical inerter device which used living-hinges to achieve the motion of the flywheel. Wang et al. presented a semi-active inerter which was used for the vibration isolator. Tang et al. studied the dynamic characteristics of skyhook semi-active inerter-based vibration isolator, which had lower absolute displacement peak and wider vibration isolation frequency band.

As a simple mechanical element with the function of amplifying force, a lever could be used to suppress vibration in systems. The lever was introduced into the vibration isolation system by Flannelly. And a dynamic anti-resonance vibration isolator was designed, which amplified the inertial force generated by the mass to offset the spring force to achieve anti-resonance. It was found that when the same vibration isolation effect was obtained, the mass with an amplifying mechanism would be smallest. Xing et al. applied the lever to the DVA with negative stiffness, and studied the optimal parameters. Sui et al. introduced the amplifying mechanism, inerter, and grounded stiffness into Voigt type DVA, and it was found that the system would be unstable if the inerter was inappropriate. Yang et al. optimized the design of lever-type tuned mass dampers, which were used to control the wake-induced vibrations of coupled twin-cable hangers.

According to the above-mentioned explanation, it is obvious that the lever, inerter, and grounded stiffness all have wide applications in vibration control engineering. This paper presents a DVA model simultaneously containing the above three components. The proposed model parameters are optimized by using the fixed-point theory. The numerical result shows that the calculation process of analytical solution is accurate. The obtained results are compared with other DVAs, and it is proved that the presented model in this paper shows better control performance, which provides a choice for the design of a new DVA.

The rest of this paper is organized by the following sections. The Model of DVA and Parameters Optimization section shows the optimal parameters of the primary system based on the fixed-point theory, and discusses the working range of inerter. In Section 3, the correctness of the analytical solution is verified by numerical simulation, and the presented model is compared with other DVAs under harmonic and random excitations. Finally, the conclusion is presented in Section 4.
The model of DVA and parameters optimization

The model presented in this paper is shown in Figure 1. It is a grounded dynamic vibration absorber containing an amplifying mechanism (a lever is taken as an example) and inerter. The primary system and subsystem are connected by a lever to form a complete system. \(m_1\) and \(m_2\) present the masses of the primary system and subsystem, respectively. \(k_1\) and \(k_2\) denote the stiffnesses of the primary system and subsystem, respectively. The grounded spring stiffness is \(k_3\), the subsystem damping coefficient is \(c\), and the inerter coefficient is \(b\). \(r_1\) and \(r_2\) denote the distances from the lever support point \(O\) to the hinge points \(M\) and \(N\) of the two sliding blocks, respectively. The amplitude and frequency of the exciting force are \(F_0\) and \(\omega\). \(x_1\) and \(x_2\) are regarded as the displacements of primary system and subsystem. Here, all the devices are rigid enough, and the hinged parts associated with the sliders have good lubrication. According to the principle of similitude and the lever principle, \(L = r_2/r_1\) is considered as the magnification ratio. From the perspective of actual engineering, the grounded stiffness, lever, and inerter are general mechanical devices and can be easily realized. Therefore, the configuration of the presented model can be used in practical engineering fields.

By ignoring the mass of the amplifying mechanism and the friction in the motion procedure, the dynamic equation of the system can be established by Newton’s second law as

\[
\begin{align*}
\begin{cases}
 m_1 \ddot{x}_1 + k_2 L (x_1 - x_2) + b L (\dot{x}_1 - \dot{x}_2) + k_1 x_1 &= F_0 \cos(\omega t) \\
 m_2 \ddot{x}_2 + k_2 (x_2 - L x_1) + b (\dot{x}_2 - L \dot{x}_1) + c \dot{x}_2 + k_3 x_2 &= 0
\end{cases}
\end{align*}
\]

(1)

Using the following parametric transformation

\[
\begin{align*}
 \mu &= \frac{m_2}{m_1}, \quad \omega_1 = \sqrt{\frac{k_1}{m_1}}, \quad \omega_2 = \sqrt{\frac{k_2}{m_2}}, \quad \xi = \frac{c}{2 m_2 \omega_2}, \quad f = \frac{F_0}{m_1}, \quad \alpha = \frac{k_3}{k_2}, \quad \delta = \frac{b}{m_2}
\end{align*}
\]

Equation (1) can be simplified to

\[
\begin{align*}
\begin{cases}
 \ddot{x}_1 + L \mu \omega_2^2 (x_1 - x_2) + L \mu \dot{\omega}_2 (\dot{x}_1 - \dot{x}_2) + \omega_1^2 x_1 &= f \cos(\omega t) \\
 \ddot{x}_2 + \omega_2^2 (x_2 - L x_1) + \delta (\dot{x}_2 - L \dot{x}_1) + 2 \xi \omega_2 \dot{x}_2 + a \omega_2^2 x_2 &= 0
\end{cases}
\end{align*}
\]

(2)

The analytical solution

The steady-state solution of equation (2) has the following form

\[x_1 = X_1 e^{j \omega t}, \quad x_2 = X_2 e^{j \omega t}\]

(3)

where \(j\) is the imaginary unit. Substituting equation (3) into (2), one can get
\[
X_1 = \frac{f(jA_1 + B_1)}{fC_1 + D_1}, \quad X_2 = \frac{f(jA_2 + B_2)}{fC_1 + D_1}
\]

(4)

where

\[
\begin{align*}
A_1 &= 2\xi_1 \omega_0 \\
B_1 &= -(1 + \delta)\omega^2 + (1 + \alpha)\omega_1^2 \\
C_1 &= 2\xi_1 \omega_0 \left[ -\omega^2(1 + L^2\delta \mu) + L^2\mu \omega_1^2 + \omega^2 \right] \\
D_1 &= \omega^4(1 + \delta + L^2\delta \mu) - \omega^2 \omega_1^2 \left[ 1 + \alpha + L^2 \mu(1 + a\delta) \right] \\
&\quad + L^2 \mu \omega^2 + \omega_1^2 \left[ -\omega^2(1 + \delta) + \omega_1^2(1 + \alpha) \right] \\
A_2 &= 0 \\
B_2 &= L(\omega_2^2 - \delta \omega^2)
\end{align*}
\]

Introducing the parameters

\[
\lambda = \frac{\omega}{\omega_1}, \quad \nu = \frac{\omega_2}{\omega_1}, \quad X_{st} = \frac{F_0}{k_1}
\]

and defining \( A \) as the normalized amplitude amplification factor of the primary system, one could obtain

\[
A^2 = \frac{X_{st}^2}{X_{st}} = \frac{\xi^2 A_1^2 + B_1^2}{\xi^2 C_2^2 + D_2^2}
\]

(6)

where

\[
\begin{align*}
A_1 &= 2\lambda \nu \\
B_3 &= -(1 + \delta)\lambda^2 + (1 + \alpha)\nu^2 \\
C_2 &= 2\lambda \nu \left[ 1 - \lambda^2(1 + L^2\delta \mu) + L^2 \mu \nu^2 \right] \\
D_2 &= \lambda^2 \left[ 1 + \delta - \lambda^2(1 + \delta + L^2\delta \mu) \right] \\
&\quad + \nu^2 \left[ -1 - \alpha + \lambda^2 + \alpha \lambda^2 + L^2 \lambda^2 \mu(1 + a\delta) \right] - L^2 \mu \nu \lambda^2
\end{align*}
\]

(7)

The natural frequency of the system can be obtained by setting the denominator of equation (6) equal to zero, so that one can get

\[
\omega_{i2}^2 = \frac{(1 + \delta)\omega_1^2 + (1 + \alpha + L^2 \mu + L^2 \delta \mu)\omega_1^2 \pm \sqrt{\gamma}}{2(1 + \delta + L^2 \delta \mu)}
\]

(8)

where

\[
\gamma = (1 + \delta)^2 \omega_1^4 + 2\omega_1^2 \omega_2^2 \left[ -(1 + \alpha)(1 + \delta + L^2 \delta \mu) + L^2 \mu(1 + a\delta^2) \right] \\
+ \omega_2^2 \left[ (1 + L^2 \mu)^2 + (\alpha + L^2 \delta \mu)^2 - 2\alpha(-1 + L^2 \mu + L^2 \delta \mu + L^4 \delta^2 \mu^2) \right]
\]

**Parameters’ optimization**

According to the fixed-point theory, it is easily found that there are two fixed points on the normalized amplitude-frequency curve that have nothing to do with the damping ratio when considering equation (6). Some typical normalized amplitude-frequency curves for different damping ratios such as 0.2, 0.3, and 0.6 are given in Figure 2. It can be clearly seen that all the curves pass through two fixed points \( P \) and \( Q \) for different damping ratios. And the point \( Z \) is a stationary point that corresponds to zero frequency and is also independent of damping ratios.
Since the two fixed points \( P \) and \( Q \) are unrelated to the damping ratio, the response values of the primary system are equal when the damping ratio approaches zero and infinity in equation (6). That is

\[
\frac{A_3}{C_2} = \frac{B_3}{D_2}
\]  

(9)

Substituting equation (7) into equation (9), one can obtain

\[
\frac{-\lambda^2(1 + \delta) + \nu^2(1 + \alpha)}{\lambda^2[1 + \delta - \lambda^2(1 + \delta + L^2\delta\mu)] + \nu^2[(1 + \alpha)(-1 + \lambda^2l^2\mu(1 + a\delta)] - L^2 a\mu^d}
\]

\[
= \frac{1}{1 - \lambda^2(1 + L^2\delta\mu) + L^2\mu^2}
\]

(10)

Considering meaningful situations and simplifying equation (9), one can get

\[
\lambda^4\left[(L^2\delta\mu + 1)(2 + \delta) + \delta\right] + \nu^2\left[1 + (1 + L^2\mu^2)(1 + 2\alpha)\right]
- 2\lambda^2\left[1 + \delta + L^2\delta\mu^2(1 + \alpha) + \nu^2(1 + \alpha + L^2\mu)\right] = 0
\]  

(11)

Assuming that equation (11) has two real roots, that is, \( \lambda_P \) and \( \lambda_Q \), we can get the following formula according to Veda theorem

\[
\lambda_P^2 + \lambda_Q^2 = \frac{2[1 + \delta + L^2\delta\mu + \nu^2 + (1 + \alpha + L^2\mu)]}{(L^2\delta\mu + 1)(2 + \delta) + \delta}
\]

(12)

The purpose of optimization is to minimize the maximum value of the amplitude-frequency curve, that is, the two fixed points should be adjusted to the same height, and the highest point of the amplitude-frequency curve exactly fall on the two fixed points.

Because the response values at \( \lambda_P \) and \( \lambda_Q \) are equal, one can get

\[
\frac{1}{1 - \lambda_P^2(1 + L^2\delta\mu) + L^2\mu^2} = \frac{1}{1 - \lambda_Q^2(1 + L^2\delta\mu) + L^2\mu^2}
\]  

(13)

Then, equation (13) can be reorganized as

\[
\lambda_P^2 + \lambda_Q^2 = \frac{2 + 2L^2\mu^2}{1 + L^2\delta\mu}
\]

(14)
Combining equations (12) with (14), one can find

\[ [1 + \alpha + L^2\mu(-1 + 2a\delta) + L^4\delta^2\mu(-1 + a\delta)]v^2 = 1 + \delta + L^2\delta\mu \]  \hspace{1cm} (15)

Solving equation (15), the optimal natural frequency ratio can be obtained as

\[ v_{\text{opt}} = \sqrt{\frac{1 + \delta + L^2\delta\mu}{1 + (1 + L^2\delta\mu)[\alpha + L^2\mu(\alpha\delta - 1)]}} \]  \hspace{1cm} (16)

The abscissas at the two fixed points can be obtained by substituting equation (16) into (10)

\[ \lambda_p^2 = \frac{(1 + \alpha + L^2\alpha\delta\mu)\sqrt{\Delta} - L(1 - \alpha\delta)(1 + L^2\delta\mu)\sqrt{\bar{\mu}}}{1 - L^2\mu(1 + L^2\delta\mu) + \alpha(1 + L^2\delta\mu)^2\sqrt{\bar{\mu}}} \]  \hspace{1cm} (17a)

\[ \lambda_Q^2 = \frac{(1 + \alpha + L^2\alpha\delta\mu)\sqrt{\Delta} + L(1 - \alpha\delta)(1 + L^2\delta\mu)\sqrt{\bar{\mu}}}{1 - L^2(1 + L^2\delta\mu) + \alpha(1 + L^2\delta\mu)^2\sqrt{\bar{\mu}}} \]  \hspace{1cm} (17b)

where \( \Delta = (2 + \delta)(1 + L^2\alpha\delta) + \delta \).

When the frequency ratio achieves the optimal value, the response at the two fixed points \( P \) and \( Q \) can be found

\[ A|_{(\nu,\lambda)} = \frac{\left[1 - L^2\mu - L^4\delta^2\mu + \alpha(1 + L^2\delta\mu)^2\right]\sqrt{\Delta}}{L(1 - \alpha\delta)(1 + L^2\delta\mu)^2\sqrt{\bar{\mu}}} \]  \hspace{1cm} (18)

It can be known from equation (18) that the control performance of the system is only determined by the grounded stiffness ratio, when mass ratio \( \mu \), magnification ratio \( L \), and inerter ratio \( \delta \) are constant. In order to minimize the peak value of the amplitude-frequency curve, the response values at the two fixed points \( P, Q \), and point \( Z \) should be equal, that is,

\[ A^2|_{\lambda=0} = A^2|_{(\nu,\lambda)} \]  \hspace{1cm} (19)

that is

\[ \frac{\left[-1 + L^2\mu(1 + L^2\delta\mu) - \alpha(1 + L^2\delta\mu)^2\right]^2\Delta}{L^2(-1 + \alpha\delta)^2\mu(1 + L^2\delta\mu)^4} = \frac{(1 + \alpha)^2\left[1 - L^2\mu(1 + L^2\delta\mu) + \alpha(1 + L^2\delta\mu)^2\right]^2}{\left[(1 + \alpha)^2 + L^2\mu(-1 + 3a\delta + 2a^2\delta) + L^4\delta^2\mu(-1 + a\delta + a^2\delta)\right]^2} \]  \hspace{1cm} (20)

Solving equation (20), one can get all possible optimal grounded stiffness ratios

\[ \alpha_{1,2} = \frac{-1 + L^2\mu + L^4\delta^2\mu}{(1 + L^2\delta\mu)^2} \]  \hspace{1cm} (21a)

\[ \alpha_{3,4} = \frac{-1 \pm L\sqrt{\mu\Delta}}{1 + L^2\delta\mu} \]  \hspace{1cm} (21b)

\[ \alpha_{5,6} = \frac{1}{2} \frac{1 \pm L\sqrt{\mu\Delta}}{2(1 + L^2\delta\mu)} \]  \hspace{1cm} (21c)

It has been pointed out that the above six alternative optimal solutions will make the system unstable when the inerter ratio is inappropriate. Therefore, when selecting the appropriate stiffness ratio, it should be carried out under the premise of ensuring system stability. The above grounded stiffness ratios are substituted into equation (16), respectively. Then, setting it to be greater than zero to solve the inerter ratio \( \delta \), the inerter working ranges to every optimal grounded stiffness...
can be obtained. Furthermore, the above optimal grounded stiffness ratios are substituted into equation (8) to calculate the systems’ natural frequencies in their respective inerter working ranges. It is found that only $\alpha_3$ can guarantee the natural frequencies of the system are positive, and the abscissas of the two fixed points $P$ and $Q$ are greater than zero, so the optimal grounded stiffness ratio in this paper is selected as $\alpha_3$

$$\alpha_{\text{opt}} = \alpha_3 = -1 + \frac{L\sqrt{\mu \Delta}}{1 + L^2 \delta \mu}$$

(22)

So far, the optimal frequency ratio and optimal grounded stiffness ratio are obtained. Simultaneously, the two fixed points $P$ and $Q$ have been adjusted to the same height. As the coordinates of the two fixed points $P$ and $Q$ are independent of the damping ratio, the two fixed points should become the highest point of the amplitude-frequency curve so as to achieve the optimal control performance. According to the extreme value condition, it could be achieved if the derivatives of the amplitude-frequency curve at the two fixed points are zero, that is

$$\frac{\partial A^2}{\partial \lambda_P} = 0, \quad \frac{\partial A^2}{\partial \lambda_Q} = 0$$

(23)

By solving equation (23), one can get the damping ratio when the two fixed points $P$ and $Q$ become the highest points of the amplitude-frequency curve. Then, the optimal damping ratio $\zeta_{\text{opt}} = (\xi_1 + \xi_2)/2$ can be found, where $\xi_1$ is for $\lambda_P$ and $\xi_2$ is for $\lambda_Q$. But this method is difficult to get the analytical results when the formula is very complicated. Liu$^9$ used another method to get an approximation of the damping ratio directly without using differentiation.

In order to make the amplitude-frequency curve pass through point $P$ levelly, we can assume that it passes through the adjacent point $P'$ whose coordinate is $(\lambda_P', A_P')$

$$\lambda_P' = \lambda_P + \varepsilon$$

$$A_P' = A = \left[1 - L^2 \mu - L^4 \delta \mu^2 + \alpha(1 + L^2 \delta \mu)^2 \frac{\sqrt{\Delta}}{1 - a \delta} \frac{1}{(1 + L^2 \delta \mu)^2} \frac{1}{\sqrt{\mu}} \right]$$

(24)

where $\varepsilon \rightarrow 0$

By substituting equations (24) into (6), one can get

$$\xi_1^2 = -\frac{(ad + e)(af + e)}{g}$$

(25)

where

$$\begin{align*}
a &= v^2(1 + \alpha) - \lambda_P^2(1 + \delta) \\
d &= -1 - A_P'( -1 + \lambda_P') \\
e &= L^2 \mu A_P' (v^2 - \lambda_P^2 \delta)(\alpha v^2 - \lambda_P^2) \\
f &= 1 - A_P'( -1 + \lambda_P') \\
g &= 4 \lambda_P^2 v^2 \left\{ -1 + A_P' \left[ 1 + L^2 v^2 \mu - \lambda_P^2 (1 + L^2 \delta \mu) \right] \right\}^{-1}
\end{align*}$$

(26)

After expanding equation (25), it can be written as

$$\xi_1^2 = \frac{a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3 + \ldots}{d_0 + d_1 \varepsilon + d_2 \varepsilon^2 + d_3 \varepsilon^3 + \ldots}$$

(27)

Equation (25) will assume the indeterminate with the form $0/0$ if $\varepsilon = 0$. And according to equation (27), $a_0 = d_0 = 0$ is given. Since $\varepsilon$ is a very small quantity whose higher-order terms can be ignored in equation (27), the damping ratio at this time is

$$\zeta_{\text{opt}} = \left(\frac{a_1 + a_2 \varepsilon + a_3 \varepsilon^2 + \ldots}{d_1 + d_2 \varepsilon + d_3 \varepsilon^2 + \ldots}\right)$$

(28)
\[ \dot{\varphi}_i^2 = \frac{a_i}{d_i} \] (28)

Therefore, we only need to find all the coefficients of the first-order term in the numerator and denominator of equation (27)

\[
a_i = -4\left(1 + \delta + L^2 \dot{\delta} \mu\right)^2 A_p^2 \alpha_0^6 + 2 \delta^2 \mu(1 + \delta)^2 - 2 \nu^2 (1 + a)(1 + \delta) + 6 A_p^2 \mu \nu^2 (1 + \delta + L^2 \dot{\delta} \mu) \left[1 + \delta + \nu^2 (1 + a + L^2 \mu + L^2 \mu \alpha \delta)\right]
\]

\[+ 2 \nu^2 A_p^2 (1 + a + L^2 \alpha \nu^2) \left[1 + \delta + \nu^2 (1 + a + L^2 \mu + L^2 \alpha \delta)\right]\]

\[= 2 A_p^2 \mu \nu^2 \left\{1 + 2 \delta + \delta^2 + 4(1 + a) \nu^2 + 2 \nu^2 (2 \delta + 2 a \alpha) + 4 L^2 \nu^2 (1 + a \alpha \nu^2)\right\}\]

\[d_i = -4 \nu^2 + 4 A_p^2 \left[1 + L^2 \mu \nu^2\right]^2 + 3 A_p^2 \left[1 + L^2 \mu \nu^2\right]^2\]

\[= -4 \nu^2 + 4 A_p^2 \left[1 + L^2 \mu \nu^2\right]\]

Substituting equations (16), (17a), (18), and (22) into equation (28) yields

\[
\dot{\varphi}_i^2 = \frac{L^2 \mu (-1 + a a \delta)^2 (3 \Delta + 4 L \delta \sqrt{\Delta \mu} + L^2 \delta^2 \mu)}{8 L \sqrt{\Delta \mu} (-1 + a a \delta) (1 + L^2 \delta \mu) + 8 \Delta (1 + a + L^2 a \delta) \mu)}
\] (30)

Similarly, the damping ratio when the highest point of the amplitude-frequency curve is located at the point \(Q\) can also be obtained as

\[
\dot{\varphi}_2^2 = \frac{L^2 \mu (-1 + a a \delta)^2 (3 \Delta - 4 L \delta \sqrt{\Delta \mu} + L^2 \delta^2 \mu)}{-8 L \sqrt{\Delta \mu} (-1 + a a \delta) (1 + L^2 \delta \mu) + 8 \Delta (1 + a + L^2 a \delta) \mu)}
\] (31)

According to the above calculation, the approximate optimal damping ratio of the model is finally obtained as

\[
\xi_{opt} \approx \frac{\dot{\varphi}_1^2 + \dot{\varphi}_2^2}{2} = \frac{L^2 \mu (-1 + a a \delta)^2 \left[1 + \left(1 + L^2 \delta \mu + 1\right) (2 + 3 a)\right]}{8 (1 + \mu)^2 + 4 L \mu (-1 + \Delta + 6 a a \delta + 4 a a \delta) + 4 L^2 a \delta^2 (-1 + 2 a a + 2 a^2 \delta)}
\] (32)

So far, all the parameters of this model have been optimized.

The working range of inerter

Observing the optimization parameters and calculation process, the working range of the inerter should satisfy the following conditions at the same time: the natural frequencies of the system are positive, the lower part of the radical is positive, the denominator of each expression is not equal to zero, and the optimal frequency ratio \(\nu_{opt}\) and optimal damping ratio \(\xi_{opt}\) are greater than zero. When selecting the optimal grounded stiffness ratio, it has been satisfied that the natural frequencies of the system are greater than zero, so the inerter just needs to follow the last three conditions. According to the physical characteristics of the system components, the magnification ratio and mass ratio are both positive. Substituting the optimal grounded stiffness ratio \(\alpha_{opt}\) into the optimal frequency ratio \(\nu_{opt}\) to calculate the working range of inerter, that is

\[
\nu_{opt}\bigg|_{\alpha_{opt}} = \sqrt{\frac{1 + \Delta + L^2 \delta \mu}{L (1 + L^2 \delta \mu) \sqrt{\mu \Delta} - L^2 \mu \left[1 + L^2 \delta \mu (1 + \delta) + \delta\right]} \}
\] (33)

Therefore, inerter ratio \(\delta\) should satisfy

...
Solving equation (34), one can get the working ranges of the inerter as
\[
\delta > 0
\]
for \( 0 < \mu L^2 \leq 1 \) and
\[
0 < \delta < \frac{2 - \mu L^2}{\mu L^2 (\mu L^2 - 1)}
\]
for \( 1 < \mu L^2 < 2 \).

The results show that the working range of the inerter is different when the coupling term of the magnification ratio and mass ratio are different values.

**Analysis of results**

**Numerical simulation**

To validate the correctness of the above calculation results, it will be verified for the two working ranges of inerter ratios, respectively. The parameters selected are shown in Table 1, where the two cases correspond to the two ranges.

The numerical solution is presented by the fourth-order Runge-Kutta method. The analytical and numerical solution curves of the primary system for two cases are shown in Figure 3, where (a) denotes the analytical solution and (b) represents the numerical solution. The numerical and analytical solutions for two cases are completely consistent, which verifies the correctness and higher precision of the above solution process.

**Analysis of the influence of system parameters on response characteristics**

The influence of system parameters on optimal parameters. Taking the coupling term of the mass ratio and magnification ratio as the abscissa and the inerter ratio as the ordinate, the relationship between optimal parameters and system parameters \( \mu, L, \) and \( \delta \) are shown in Figure 4. Based on the figure, the system remains stable within the selected parameter range.

The influence of magnification ratio on the response of the primary system. The mass ratio \( \mu = 0.1 \) is chosen. In the first working range, the magnification ratios are selected as 1, 2, and 3. In the second working range, the magnification ratios are selected as 3.5, 3.8, and 4. By calculating equation (35b), we can get the working ranges of inerter ratio as \((0,2.8),(0,0.8),(0,0.4)\), respectively. Then, \( \delta = 0.3 \) is taken, and the normalized displacement amplitude-frequency response curves of the primary system are plotted in Figure 5. It can be seen from Figure 5 that the larger the magnification ratio, the better the vibration reduction effect when the other parameters are the same.

The influence of inerter ratio on the response of the primary system. The mass ratio \( \mu = 0.1 \) and magnification ratio \( L = 3 \) are chosen. Furthermore, \( \delta = 0.3, 0.6, 0.9, 1.2, \) and \( 1.5 \) are selected, and the corresponding amplitude-frequency curves are compared.

| Mass ratio, \( \mu \) | Magnification ratio, \( L \) | \( \mu L^2 \) | Inerter ratio, \( \delta \) | Frequency ratio, \( \nu \) | Damping ratio, \( \zeta \) | Grounded stiffness ratio, \( \alpha \) |
|----------------------|--------------------------|-------------|-----------------|-----------------|-----------------|-----------------|
| Case 1 0.1            | 3                        | 0.9         | 0.8             | 6.9902          | 0.1676          | 0.3071          |
| Case 2 0.1            | 3.5                      | 1.225       | 0.1             | 3.9128          | 0.3395          | 0.5456          |

Table 1. The parameters of the system for two cases.
plotted in Figure 6. It can be found that the response amplitude of the primary system is decreased with the increase of the inerter ratio.

Comparison with other DVAs

In order to verify the control performance of the presented DVA for two cases in this paper (called as LB type DVA), this model is compared with the typical Voigt type DVA, the grounded damping type DVA, the grounded inerter type DVA, and the grounded damping type DVA with lever and negative stiffness element in the literature (called as LN type DVA). Four comparative vibration absorber models are shown in Figure 7. The optimal parameters formula for each model is given in Table 2.
The response of the primary system to harmonic excitation. The mass ratio is selected as $\mu = 0.1$ for each model. The parameters in Table 1 are selected for the presented model. The magnification ratio of the LN type DVA is $L = 3$. The inerter ratio of the grounded inerter type DVA is $\beta = 0.1$. Other corresponding parameters of each model are calculated according to Table 2. The normalized displacement amplitude-frequency curves under the optimal parameters are shown in Figure 8. There is a conclusion from Figure 8 that the DVA for two cases in this paper and the LN type DVA can greatly reduce the resonance amplitude and broaden the vibration band of the primary system compared with the classic DVAs, that is, the Voigt type DVA, grounded damping type DVA, and grounded inerter type DVA. Compared with the LN type DVA, the resonance amplitude of the model in this paper is much lower after adding the inerter when the same mass ratio and amplification ratio are selected.

The response of the primary system under random excitation. In actual engineering, the source of external excitation is mostly random or with strong randomness. Therefore, it is important to study the response of the primary system under random excitation. Assuming the primary system is subjected to random excitation with zero mean and the power spectral as $S(\omega) = S_0$, the power spectral density functions of the absolute displacement responses of the primary system in classical Voigt type DVA, grounded damping type DVA, grounded inerter type DVA, LN type DVA, and the presented model in this paper are, respectively, for
Table 2. Optimal parameter formulas for DVAs.

| Model of DVA          | $\nu_{opt}$ | $\xi_{opt}$ | $n_{opt}$ |
|-----------------------|-------------|-------------|-----------|
| Voigt DVA             | $1/1 + \mu$ | $\sqrt{3\mu/8(1 + \mu)}$ | -         |
| Grounded damping DVA  | $\sqrt{1/1 - \mu}$ | $\sqrt{3\mu/8(1 - 0.5\mu)}$ | -         |
| Grounded inerter DVA  | $1/1 + \beta + \mu$ | $\sqrt{3(\beta + \mu)/8(1 + \beta + \mu)}$ | -         |
| LN type DVA           | $\sqrt{1/1 + n - \mu L^2}$ | $3\mu L^2 (1 + n)/8(1 + n)^2 - 4\mu L^2$ | $-1 + L\sqrt{2\mu}$ |

Notes: The definition of grounded stiffness ratio of LN type DVA is $n = k_3/k_2$, and the $\beta$ in grounded inerter type DVA is $\beta = b/m_1$
where the subscripts V, R, GI, LN, and LB represent the Voigt type DVA, grounded damping type DVA, grounded inerter type DVA, LN type DVA, and the DVA presented, respectively. The mean square displacements for the primary system of the five DVAs are

\[
\begin{align*}
\sigma_V^2 &= \int_{-\infty}^{\infty} S_V(\omega) \, d\omega = S_0 \int_{-\infty}^{\infty} |X_V|^2 \, d\omega = \frac{\pi S_0 Y_1}{2 \mu \xi_{VOV}} \\
\sigma_R^2 &= \int_{-\infty}^{\infty} S_R(\omega) \, d\omega = S_0 \int_{-\infty}^{\infty} |X_R|^2 \, d\omega = \frac{\pi S_0 Y_2}{2 \mu \xi_v \omega_1} \\
\sigma_{GI}^2 &= \int_{-\infty}^{\infty} S_{GI}(\omega) \, d\omega = S_0 \int_{-\infty}^{\infty} |X_{GI}|^2 \, d\omega = \frac{\pi S_0 Y_3}{2 (\mu + \beta) \xi_{VOV}} \\
\sigma_{LN}^2 &= \int_{-\infty}^{\infty} S_{LN}(\omega) \, d\omega = S_0 \int_{-\infty}^{\infty} |X_{LN}|^2 \, d\omega = \frac{\pi S_0 Y_4}{2 \mu \xi_L^2 \nu^2 \omega_1^4 (1 + \alpha + \alpha \nu^2 L^2)} \\
\sigma_{LB}^2 &= \int_{-\infty}^{\infty} S_{LB}(\omega) \, d\omega = S_0 \int_{-\infty}^{\infty} |X_{LB}|^2 \, d\omega = \frac{\pi S_0 Y_5}{2 \mu \xi_L^2 \nu^2 \omega_1^4 (v^2 - \delta)^2 (1 + \alpha + \alpha \nu^2 L^2)}
\end{align*}
\]

where

\[
\begin{align*}
Y_1 &= 1 + v^2 (1 + \mu)^2 + v^2 (4 \mu \xi^2 + 4 \xi^2 - \mu - 2) \\
Y_2 &= 1 + v^2 + v^2 (\mu + 4 \xi^2 - 2) \\
Y_3 &= 1 + v^2 (1 + \beta + \mu)^2 + v^2 [4 \xi^2 (1 + \beta + \mu) - \beta - \mu - 2] \\
Y_4 &= 1 + \alpha + v^2 \left[ (1 + \alpha)^2 - 2 \alpha \mu L^2 (1 + \alpha - 2 \xi^2) + \alpha \mu L^2 \right] + v^2 \left[ -2 + 4 \xi^2 + \mu L^2 + 2 \alpha (2 \xi^2 + \mu L^2) \right] \\
Y_5 &= (1 + \alpha) \left[ (1 - v^2 - \alpha v^2 + \delta)^2 + 4 \nu^2 \xi^2 \right] + L^2 \nu^2 \mu \left[ -1 + \alpha (1 + \alpha) (2 + 2 \nu^2 + \nu^2 \alpha - \delta) \right] (-1 + a \delta) + 4 \xi^2 \left[ \delta + \nu^2 \alpha + a \delta \right] \\
&+ L^4 \nu^4 \alpha \mu^2 \left[ (-1 + a \delta)^2 + 4 \delta \xi^2 \right]
\end{align*}
\]

The parameters of the primary system are the same as in Section 3.3.1, and the mean squares of the primary systems of all the DVAs could be calculated as

\[
\begin{align*}
\sigma_V^2 &= \frac{6.401 \pi S_0}{\omega_1^5}, & \sigma_R^2 &= \frac{5.780 \pi S_0}{\omega_1^5}, & \sigma_{GI}^2 &= \frac{4.5 \pi S_0}{\omega_1^5} \\
\sigma_{LN}^2 &= \frac{0.755 \pi S_0}{\omega_1^5}, & \sigma_{LB}^2 &= \frac{0.645 \pi S_0}{\omega_1^5}, & \sigma_{LB}^2 &= \frac{0.375 \pi S_0}{\omega_1^5}
\end{align*}
\]

Table 3. The variances and reduction ratios of the primary system.

| Model of DVA                      | Displacement variances/m² | Reduction ratio/% |
|----------------------------------|---------------------------|-------------------|
| Without DVA                     | 2.0083 \times 10^{-4}     | -                 |
| Voigt DVA                        | 3.1235 \times 10^{-5}     | 84.47             |
| DVA of grounded damping          | 2.8560 \times 10^{-5}     | 85.77             |
| DVA of grounded inerter          | 2.0931 \times 10^{-5}     | 89.58             |
| DNA of LN type                   | 3.5818 \times 10^{-6}     | 97.78             |
| DVA in this paper (case 1)       | 2.1186 \times 10^{-6}     | 98.95             |
| DVA in this paper (case 2)       | 1.8141 \times 10^{-6}     | 99.10             |
where $\sigma_{LB1}^2$ and $\sigma_{LB2}^2$ represented the mean squares of the presented DVA for case 1 and case 2, respectively. The results show that the model presented has a good vibration reduction performance even under random excitation when the system has the same original parameters.

In order to simulate the actual project more realistically, 50s random excitation with zero mean value and unit variance is constructed. Here, the primary system mass $m_1 = 1$ kg and the stiffness of the primary system $k_1 = 100$ N/m are chosen. The displacement variances and their reduction ratios of the primary system for different models are summarized in Table 3.

It can be seen from Table 3 that although the presented model was designed according to the $H_\infty$ optimization criterion, it still can greatly reduce the vibration energy of the primary system in the entire frequency range under random excitation. That means the presented model has better vibration damping performance than others. The primary system will have better vibration damping performance when the coupling terms $\mu L^2$ is in the range of 1 to 2.

Conclusions

A novel type of DVA with lever, inerter, and grounded stiffness is investigated. The parameters of this model are optimized based on fixed-point theory. And the optimal frequency ratio, optimal grounded stiffness ratio, approximate optimal damping ratio, and working range of inerter are obtained. The results show that the working range of the inerter is determined by the coupling term of magnification ratio and mass ratio. The vibration reduction effect is better when the coupling term is in the second range. Compared with other DVAs under harmonic and random excitations, the introduction of a reasonable grounded stiffness, inerter, and lever can greatly reduce the response amplitude and effectively broaden the vibration damping frequency of the primary system.

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