Frequency Regulation with Thermostatically Controlled Loads:
Aggregation of Dynamics and Synchronization

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Abstract—Thermostatically controlled loads (TCLs) can provide ancillary services to the power network by aiding existing frequency control mechanisms. TCLs are, however, characterized by an intrinsic limit cycle behavior which raises the risk that these could synchronize when coupled with the frequency dynamics of the power grid, i.e. simultaneously switch, inducing persistent and possibly catastrophic power oscillations. To address this problem, schemes with a randomized response time in their control policy have been proposed in the literature. However, such schemes introduce delays in the response of TCLs to frequency feedback that may limit their ability to provide fast support at urgencies. In this paper, we present a deterministic control mechanism for TCLs such that those switch when prescribed frequency thresholds are exceeded in order to provide ancillary services to the power network. For the considered scheme, we provide analytic conditions which ensure that synchronization is avoided. In particular, we show that as the number of loads tends to infinity, there exist arbitrarily long time intervals where the frequency deviations are arbitrarily small. Our analytical results are verified with simulations on the Northeast Power Coordinating Council (NPCC) 140-bus system, which demonstrate that the proposed scheme offers improved frequency response compared with existing implementations.

I. INTRODUCTION

Motivation and literature review: A significant growth in the penetration of renewable sources of generation in power networks is expected over the following years [2], [3], driven by environmental concerns. This will result in increasingly intermittent generation, endangering power quality and potentially the stability of the power network. Controllable loads are considered to be a way to counterbalance intermittent generation, due to their ability to provide a fast response at urgencies by accordingly adapting their demand. The use of loads as ancillary services, in conjunction with a large penetration of renewable sources of generation will significantly increase the number of active devices in the network making its electromechanical response difficult to predict and encouraging the analytical study of its behavior. Along these lines, various research studies in recent years have considered controllable demand as a means of providing support to primary [4]–[8], and secondary [9], [10], [11], frequency control mechanisms, where the objective is to ensure that generation and demand are balanced and that the frequency converges to its nominal value (50Hz or 60Hz) respectively.

Thermostatically controlled loads (TCLs) comprise a significant portion of the total demand. A recent survey in the EU [12] showed that TCLs exceeded 80% and 40% of the total consumption in households with and without electric heating respectively. TCLs have an intrinsic limit cycle behavior whereby they need to periodically turn on and off in order to maintain the temperature within a prescribed range. This significantly complicates their use for frequency control, in comparison with loads that are not thermostatically controlled [11], [13], [14]. In particular, the coupling of the individual limit cycles in TCLs with the grid frequency, could lead to a synchronization of these limit cycles thus resulting to highly undesirable oscillations in the aggregate load profile. Therefore dedicated analysis tools and studies are needed for the efficient integration of TCLs to the grid such that they provide support to frequency regulation.

The use of TCLs for frequency control has been considered in [15], where the authors suggested temperature thresholds in TCLs to be linearly dependent on frequency and demonstrated with simulations that this resulted in improved performance. However, it was demonstrated in [16] that such control schemes could potentially result to load synchronization. As a remedy to this problem, the authors proposed a randomized control scheme which ensured that TCLs would not synchronize. Various other studies considered similar problems by proposing schemes with randomization in the control policy. In [17], safety constraints in the operation of TCLs are additionally included, and [18], [19] incorporate stochastic switching in the TCL operation so as to achieve a prescribed power profile. However, schemes with a randomized delay in their control policies may limit the ability of TCLs to respond to unforeseen frequency fluctuations and provide ancillary support at fast timescales. The latter, motivates the study of alternative schemes for the control of thermostatic loads, such that a faster response can be achieved at urgencies, while at the same time avoiding load synchronization.

Contribution: This study considers a deterministic approach for the control of thermostatic loads, such that ancillary services with a fast response are provided at urgencies. Our main analytic results concern the case where the number of

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A preliminary version of this work has appeared in [1]. This manuscript extends the analysis to a broad class of linear generation dynamics and includes the analytic proofs of the main results, additional discussion and simulations that demonstrate the impact of the proposed analysis.
loads tends to infinity, a condition justified by the large number of thermostatic appliances in power networks.

More precisely, we propose a control scheme for TCLs, such that loads switch when certain frequency thresholds are exceeded in order to support existing secondary frequency control schemes. For the considered scheme, we provide design conditions for the frequency thresholds that bound the coupling between the frequency and the load dynamics so as to avoid load synchronization. In particular, one of the main results is to analytically show that when the number of loads tends to infinity, the frequency deviations will be arbitrarily small for arbitrarily long time intervals.

The proposed scheme also ensures that load temperatures will not exceed their respective bounds, and hence that user comfort levels will not be affected. Furthermore, the fact that loads switch instantly at urgencies, leads to a fast response whereby randomized delays, often used in the literature to avoid synchronization, are avoided.

Our analytical results are verified with numerical simulations on the NPCC 140-bus network, where it is demonstrated that the proposed scheme offers reduced frequency overshoots in comparison with existing implementations.

**Paper structure:** In Section II we present some basic notation used in the paper and in Section III the considered power system. In Section IV we consider a conventional model for TCLs and study its properties in terms of the aggregate mean and variance. In Section V, we present our proposed scheme for frequency control using TCLs and state our main results regarding the performance of the power system. Numerical investigations of the results on the NPCC 140-bus system are provided in Section VI and conclusions are drawn in Section VII. The proofs of the main results are provided in the appendix.

## II. NOTATION

Real, natural and complex numbers are denoted by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{C}$ respectively, and the set of $n$-dimensional vectors with real entries is denoted by $\mathbb{R}^n$. Furthermore, we define the sets of integers and strictly positive rational and strictly positive real numbers by $\mathbb{Z}$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ respectively. The set of natural numbers including zero is denoted by $\mathbb{N}_0$.

The cardinality of a set $S$ is denoted by $|S|$. The average of a real valued time signal $x(t)$ with respect to time is defined as $\mathbb{E}(x(t)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt$ and its variance as $\mathbb{V}(x(t)) = \mathbb{E}(x(t)^2) - [\mathbb{E}(x(t))]^2$. For $c \in \mathbb{C}$ we denote its magnitude by $|c|$. The $1$-norm of a linear system with transfer function $G(s)$ is given by $\int_0^\infty |g(t)|dt$, where $g(t)$ is the inverse Laplace transformation of $G(s)$. We use $\omega_n$ to denote the $n \times 1$ vector with all elements equal to 0. We also say that a matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz if all its eigenvalues have strictly negative real part. Finally, a sequence $\{s_1, s_2, s_3, \ldots\}$ of real numbers is said to be uniformly bounded on an interval $[a, b]$ if for any subinterval $[c, d]$ of $[a, b]$ we have $\lim_{n \to \infty} \{s_1, s_2, s_3, \ldots\} \cap [c, d] = \frac{d-c}{b-a}$.

## III. POWER SYSTEM MODEL

We use the swing equation to describe the rate of change of the frequency of the power system (e.g. [20]). In particular, we consider the following assumptions on our studied model:

1. Bus voltage magnitudes satisfy $|V| = 1$ p.u. for all buses.
2. Lines are lossless and characterized by their susceptances.
3. Reactive power flows do not affect bus voltage phase angles and frequencies.
4. Frequencies between buses are synchronized.

The first three conditions have been widely used in the literature for the study of frequency control schemes in power networks [5], [11]. The fourth assumption is justified from the relatively small deviations between bus frequencies, which allows the study of power system characteristics using a single frequency (see also [16], [21], [22]). The latter follows from the fact that the dynamic behavior of TCLs is much slower than the frequency dynamics between buses, which justifies the assumption of small deviations among bus frequencies. Please note that a full complexity power network model, which includes multiple buses, voltage dynamics, line resistances and reactive power flows, is considered in the simulations presented in Section VI, which verify the main results of the paper. The above motivate the following system dynamics,

$$M \dot{\omega} = -p^L + p^M - D \dot{\omega} - \sum_{j=1}^N d_j,$$

(1)

In system (1) the time-dependent variables $p^M$, $d_j$ and $\omega$ represent, respectively, the aggregate mechanical power injection, the $j$th thermostatic load and the deviation from the nominal value of the frequency. Furthermore, we let $N := \{1, 2, \ldots, |N|\}$ be the set of TCLs. The constants $M > 0$ and $D > 0$ denote the generator inertia and damping coefficient respectively. Finally, the aggregate uncontrollable demand is denoted by $p^L$.

### A. Generation Dynamics

We consider a broad class of linear generation dynamics of the form

$$p^M = \hat{C} \dot{x} + \hat{D} \dot{\omega}, \quad \dot{x} = \hat{A} \dot{x} + \hat{B} \omega,$$

(2)

with input $\omega$, output $p^M$, state $\dot{x}$ that takes values in $\mathbb{R}^n$ and corresponding matrices $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^n$, $\hat{C} \in \mathbb{R}^{1 \times n}$ and $\hat{D} \in \mathbb{R}$. Note that linear systems are widely used in the literature to model generation dynamics (see e.g. [20, Section 11.1], [23, Section 11.1.7]). Such models are particularly relevant when small disturbances are considered.

The system (1), (2) can be represented in the form

$$\begin{bmatrix} \dot{\omega} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}p^L + \sum_{j\in\mathbb{N}} d_j \\ \hat{D} \end{bmatrix},$$

(3)

where $A = \begin{bmatrix} (\hat{D} - D)/M & \hat{C}/M \\ B & \hat{A} \end{bmatrix}$, $B = \begin{bmatrix} -1/M \\ 0 \end{bmatrix}$.

We also denote $\dot{u} = p^L + \sum_{j\in\mathbb{N}} d_j$. The following assumption is made for (3).

**Assumption 1:** For system (3) the following hold

(i) $A$ is Hurwitz,

(ii) All equilibria of (3) with constant $\dot{u}$ satisfy $\omega^* = 0$.

Assumption 1(i) ensures that (3) is an asymptotically stable system. The latter is in line with current implementations where generation dynamics are designed such that the power

\[1\text{The nominal value is 50Hz or 60Hz.}\]
system is stable. Assumption 1(ii) is associated with the fact that secondary frequency control is implemented, where the objective is to recover the frequency to its nominal value at steady state.

IV. THERMOSTATICALLY CONTROLLED LOADS

In this section we consider a conventional model for cooling TCLs (e.g., refrigerators, air conditioner units) and study its properties. Note that the extension to heating TCLs, such as space heaters, is trivial and thus omitted. The analysis below enables to deduce important properties of TCL behavior, which are commonly described by (e.g., [16], [24])

\[
d_j^c = \bar{d}_j \sigma_j, \quad \sigma_j(t^+) = \begin{cases} 1, & T_j \geq T_j \text{ON} \\ 0, & T_j \leq T_j \text{OFF} \end{cases},
\]

where \( j \in N \) and \( t^+ = \lim_{\varepsilon \to 0} (t + \varepsilon) \). In (4), the time-dependent variables \( d_j^c \) and \( \sigma_j \in \{0, 1\} \) denote the demand and switch state of the \( j \)th load respectively. The time dependent variable \( T_j \) denotes the temperature of the \( j \)th load. The constants \( \bar{d}_j \) and \( T_j \) denote the load magnitude and lower and upper temperature thresholds for load \( j \) respectively and satisfy \( d_j, T_j \in \mathbb{R}_+ \) and \( T_j > T_j > 0, j \in N \). The hysteresis scheme in (4) is depicted in Figure 1.

Furthermore, the temperature dynamics satisfy

\[
\dot{T}_j = -k_j(T_j - \bar{T}_j + \lambda_j \bar{d}_j), j \in N,
\]

where constants \( k_j, \lambda_j \geq 0 \) denote the thermal insulation coefficient and coefficient of performance of load \( j \) respectively. Furthermore, \( \bar{T}_j \) denotes the ambient temperature of load \( j \) that is assumed to be constant. Moreover, it is assumed that \( \bar{T}_j > \bar{d}_j < \bar{T}_j \) and \( \bar{T}_j > \bar{T}_j, j \in N \), such that \( \hat{A} \); (4), (5), has no equilibria, as is the case in practice.

A. Periods and duty cycles of TCLs

The period \( \pi_j \) of thermal load \( j \), described by (4), (5), is defined as the time required for load \( j \) to switch twice, i.e.

\[
\pi_j = t_j, i + 1 - t_j, i, \text{ for } i \geq 1.
\]

It should be clear that for any \( j \in N \), it holds that \( t_j, i + 1 - t_j, i = t_j, k + 1 - t_j, k \), for all \( i, k \in N \). Note that, as follows from (4), (5), the time lengths that load \( j \) remains switched ON and OFF within each period are respectively given by

\[
\pi_j^{ON} = \frac{1}{k_j} \ln \left( \frac{T_j + \lambda_j \bar{d}_j - T_j}{T_j + \lambda_j \bar{d}_j - T_j} \right), j \in N, \quad (6a)
\]

\[
\pi_j^{OFF} = \frac{1}{k_j} \ln \left( \frac{T_j - T_j}{T_j - T_j} \right), j \in N, \quad (6b)
\]

and that it trivially follows that \( \pi_j = \pi_j^{ON} + \pi_j^{OFF} \). Furthermore, the duty cycle of each load is given by \( \alpha_j = \frac{\pi_j^{ON}}{\pi_j} \), i.e. the ratio of time the load is ON within each period.

B. Variance analysis

In this section we consider the aggregation of TCLs and analyze its mean and variance. In particular, we study how the latter is influenced when the number of loads tends to infinity, assuming a constant aggregate sum.

An important assumption in the following analysis is that period ratios lie in the set \( \mathbb{R}_+/Q_+ \). This is stated below.

Assumption 2: All loads \((i, j) \in E\) described by (4), (5), satisfy \( \rho_{ij} \in \mathbb{R}_+/Q_+ \).

Assumption 2 is a technical condition that enables to deduce Theorem 1 below which shows that when the number of TCLs tends to infinity, then the variance of their aggregation is zero for any initial condition. In particular, when Assumption 2 holds, then \( d^* \) is aperiodic signal that exhibits variability in the time instances the individual loads switch on and off, thus leading to Theorem 1. Assumption 2 excludes cases where two loads have identical periods, which makes the aggregation of any two loads periodic. The latter is true for all cases where \( \rho_{ij} \in Q_+ \), which are hence excluded. Note that \( Q_+ \) is a set of measure zero and hence the condition \( \rho_{ij} \in \mathbb{R}_+/Q_+ \) is unlikely to be violated in practice.

The following theorem states that the variance of the aggregation of TCLs tends to zero as their number tends to infinity. Its proof can be found in [25].

**Theorem 1:** Consider thermostatic loads described by (4), (5), with \( \bar{d}_j = \frac{1}{|N|} \) and let Assumption 2 hold. Then, \( \forall_{d^*} < \frac{\pi^2}{|N|} \) and hence \( \lim_{|N| \to \infty} \mathbb{V}(d^*) = 0 \).
demand should be expected, a desired feature to avoid large oscillations in the frequency response. Note that Theorem 1, as well as many of the results that follow, are stated for the case where \( \bar{d}_j = \frac{1}{|N|}, j \in N \), which suggests a constant aggregate sum \( \Gamma \) and loads of identical magnitude. The assumption that all load magnitudes are identical is made for simplicity and could potentially be relaxed, as part of future work.

Remark 1: A result analogous to Theorem 1 could be obtained by adopting a stochastic description for TCLs, where these are modeled as independent random processes. Theorem 1 is stated based on the presented deterministic setting, described by (4)-(5), since it is used to prove the main results of the paper, which also consider deterministic dynamics.

V. FREQUENCY CONTROL OF THERMOSTATIC LOADS

In this section we present a frequency control scheme for TCLs and propose appropriate conditions for its design. For the proposed scheme, we show that, as the number of loads tends to infinity, then no synchronization phenomena occur and that there exist arbitrarily long time intervals where frequency deviations are arbitrarily small.

A. Frequency control scheme for thermostatic loads

We introduce in this subsection the frequency control policy for the TCLs, which is a scheme that provides an ancillary service at urgencies, i.e. when frequency deviations exceed particular thresholds. The scheme, depicted in Figure 2, is described below

\[
\begin{align*}
\sigma_j(t^+) &= \begin{cases} 
1, & \{T_j \geq T_j, \\
0, & \{T_j < T_j, \\
\text{otherwise}, & \{T_j = T_j, \\
\end{cases}
\end{align*}
\]

For the rest of the manuscript, we let \( S(\bar{\omega}) = \{ j \in N : \omega_j \leq \bar{\omega} \} \) be the set of loads with respective frequency thresholds below \( \bar{\omega} \). Moreover, for any set \( S \subseteq N \), we let \( \omega_m(S) = \min_{j \in S} \omega_j, d_S^*(t) = \sum_{j \in S} d_j^*(t) \) and \( d_S^* = \sum_{j \in S} d_j \).

B. Hybrid system description

The behavior of system (1), (2), (5), (8), can be described by the states \( z = (\sigma, \dot{x}) \), where \( \sigma = (\omega, \dot{x}, T) \in \mathbb{R}^m, m = |N| + n + 1 \), is the continuous state, and \( \sigma \in P^{|N|} \) the discrete state, where \( P = \{0, 1\} \). Moreover, we let \( \Lambda = \mathbb{R}^m \times P^{|N|} \) be the space where the system states evolve. The continuous dynamics of the system (1), (2), (5), (8), are described by

\[
M \dot{\omega} = -p^L + p^H - D \dot{\omega} - \sum_{j \in N} \bar{d}_j \sigma_j, 
\]

\[
p^M = \hat{C} \dot{x} + \hat{D} \dot{\omega}, \quad \dot{x} = A \dot{x} + \hat{B} \omega, 
\]

\[
\dot{T}_j = -k_j(T_j - T_j + \lambda_j d_j \sigma_j), j \in N, \quad \sigma_j = 0, j \in N,
\]

\[
\sum_{j \in S} \alpha_j d_j, \quad \text{Furthermote, we let } \tilde{L} \text{ be the 1-norm of the system with input } d^* \text{ and output } \omega, \text{ described by (3), which is given by}
\]

\[
\tilde{L} = \int_0^\infty |C e^{At} B| dt, \quad (9)
\]

where \( C = [1 \ 0]^T \), noting that its boundedness follows from Assumption 1(i).

The following condition is imposed for the design of frequency thresholds. Within it, we let \( \zeta_j = \max(\alpha_j, 1 - \alpha_j) \), noting that \( \zeta_j \in [0.5, 1] \) since \( \alpha_j \in (0, 1) \).

Design condition 1: The frequency thresholds \( \omega_j \) are chosen such that for all \( \bar{\omega} \in \mathbb{R}_+ \) and some \( \delta > 0 \), \( \sum_{j \in S(\bar{\omega})} \zeta_j d_j \leq \max((\bar{\omega} - \delta)/\tilde{L}, 0) \), where \( \tilde{L} \) is given by (9).

Design condition 1 restricts the coupling of frequency and TCL dynamics by bounding the aggregate demand that actively contributes to frequency regulation. The condition allows to deduce that no synchronization occurs between TCLs when the scheme (8) is implemented. Note also that \( \delta \) in Design condition 1 satisfies \( \delta \in (0, \omega_m(N)) \) by definition, since \( \omega_j \) is such that for some \( j \in N \) would imply that Design condition 1 does not hold. To implement Design condition 1, the values of \( \omega_j \) for the TCL population should be selected such that the presented bound is satisfied at all values of \( \bar{\omega} \), i.e. given \( \bar{\omega} \), the condition restricts the aggregate demand of loads that may switch due to that particular frequency deviation.

Fig. 2: TCL scheme described by (8). In the green and red areas the switching state is ON and OFF respectively. In the orange area the switching state can be either ON or OFF.
which is valid when \( z \) belongs to the set \( F \) given by
\[
F = \{ z \in \Lambda : \sigma_j \in I_j(T_j), \forall j \in N \}.
\]
where
\[
I_j(T_j, \omega) = \begin{cases} 
1, & T_j > T_j, \\
0, & \omega > \omega_j^j \text{ and } T_j > T_j + \epsilon_j, \\
0, & T_j < T_j, \\
0, & \omega < -\omega_j^j \text{ and } T_j < T_j - \epsilon_j, \\
1, & \{ \omega \leq \omega_j^j \text{ and } T_j \leq T_j \}, \\
0, & \omega \leq -\omega_j^j \text{ and } T_j \in [T_j - \epsilon_j, T_j], \\
0, & \omega \geq \omega_j^j \text{ and } T_j \in [T_j, T_j + \epsilon_j].
\end{cases}
\]
Alternatively, when \( z \) belongs to the set \( G = (\Lambda \setminus F) \cup G \)
where \( G = \{ z \in \Lambda : \sigma_j \in I_j^D(T_j, \omega), \forall j \in N \} \),
\[
I_j^D(T_j, \omega) = \begin{cases} 
\omega \geq -\omega_j^j \text{ and } T_j = T_j, \\
\omega = -\omega_j^j \text{ and } T_j \in [T_j, T_j - \epsilon_j], \\
\omega \leq -\omega_j^j \text{ and } T_j = T_j - \epsilon_j, \\
0, \quad \omega \leq -\omega_j^j \text{ and } T_j \in [T_j - \epsilon_j, T_j], \\
0, \quad \omega \geq \omega_j^j \text{ and } T_j \in [T_j, T_j + \epsilon_j],
\end{cases}
\]
then its components follow the discrete update described below
\[
\pi^+ = \pi(t), \sigma_j(t^+) = \begin{cases} 
1, & T_j \geq T_j, \\
\omega \geq \omega_j^j \text{ and } T_j \in [T_j + \epsilon_j, T_j], \\
0, & T_j \leq T_j, \\
0, & \omega \leq -\omega_j^j \text{ and } T_j \in [T_j, T_j - \epsilon_j],
\end{cases}
\]
where \( \pi^+ = \lim_{\epsilon \to 0} \pi(t + \epsilon) \).

We can now provide the following compact representation for the hybrid system (1), (2), (5), (8),
\[
\dot{z} = f(z), z \in F, \quad z^+ = g(z), z \in G,
\]
where \( f(z) : F \to \Lambda \) and \( g(z) : G \to F \) are described by
(10) and (12) respectively. Note that \( z^+ = g(z) \) represents a discrete dynamical system where \( z^+ \) indicates that the next value of the state \( z \) is given as a function of its current value through \( g(z) \). Moreover, notice that \( F \cup G = \Lambda \).

C. Analysis of solutions

In this section we consider the solutions of (13) and show their existence and that no Zeno behavior occurs. We will be using the definitions of a hybrid system, hybrid time domain, hybrid solution and complete and maximal solutions for systems described by (13) as in [26, Ch. 2] (see also Definition 2 in [25]).

The following lemma shows the existence of complete solutions to (13). Furthermore, it demonstrates the boundedness of solutions to (13) and provides a lower bound on the time between consecutive switches, which suffices to show that no Zeno behavior occurs. Finally, it states that all maximal solutions to (13) are complete. Its proof can be found in [25].

We remind that \( t_{j,i} \) is the time of the \( i \)th switch of load \( j \).

Lemma 1: For any initial condition \( z(0, 0) \in \Lambda \) there exists a complete solution to (13). Furthermore, all maximal solutions to (13) are complete. Moreover, if Assumption 1 holds then the following hold:

(i) For each initial condition \( z(0, 0) \in \Lambda \), solutions to (13) are bounded.

(ii) For any solution to (13), there exists \( \tau_d > 0 \) such that \( \min_{i \geq 1}(t_{j,i+1} - t_{j,i}) \geq \tau_d \) for any \( j \in N \).

The boundedness of solutions to (13), demonstrated in the above lemma, follows also intuitively by noting that (13) consists of the asymptotically stable linear system (1), (2), with input \( d^a \) and output \( \omega \) in feedback with the hybrid system (5), (8) and that the magnitude of \( d^a \), which can be regarded as the output of (5), (8), is bounded. Furthermore, the boundedness of \( T_j, j \in N \) follows directly from the structure of (5), (8).

D. Performance analysis

In this section we state one of the main results of this paper, associated with the performance of solutions to (13). The following theorem, proven in the appendix, demonstrates that as the number of loads tends to infinity, then for all initial conditions there exist arbitrarily long time intervals where frequency deviations are arbitrarily small.

Theorem 2: Consider the system described by (13) and let Assumptions 1–2 and Design condition 1 hold. Furthermore, assume that the thermostatic loads described by (5), (8) satisfy \( \overline{T}_j = \frac{1}{|N|} \). Then, as \( |N| \to \infty \), for any \( z(0, 0) \in \mathbb{R}^m \times \mathbb{P}^{|N|} \), any maximal solution of (13) and any \( \epsilon > 0 \), \( \tau \in \mathbb{R}_+ \) such that \( |\omega(t, j)| \leq \epsilon \) for \( t \in [\tau, \tau + \hat{\tau}] \).

The importance of Theorem 2 is that it shows, for all initial conditions, that frequency trajectories become arbitrarily small for an arbitrarily long amount of time. Also, as shown in Lemma 1 the scheme in (8) avoids Zeno behavior. Furthermore, being deterministic, it allows the instant response to frequency deviations, thus providing improved ancillary services to the power system. The latter, is also numerically demonstrated in the following section.

Remark 2: Theorem 2 does not provide an analytical expression for \( \tau \). However, it is intuitive to note that its value in a real setting depends on: (i) the values of \( \epsilon \) and \( \hat{\tau} \), which are associated with its definition, (ii) the initial conditions and the speed of generation dynamics, which determine how long it takes for generation to match a potential disturbance, and (iii) the distribution of load periods.

VI. SIMULATION ON THE NPCC 140-BUS SYSTEM

In this section we verify our analytic results with a numerical simulation on the Northeast Power Coordinating Council (NPCC) 140-bus interconnection system, using the Power System Toolbox [27]. This model is more detailed and realistic than our analytical one, including line resistances, a DC12 exciter model, a transient reactance generator model, and turbine governor dynamics.

The test system consists of 93 load buses serving different types of loads including constant active and reactive loads and 47 generation buses. The overall system has a total real power of 28.55 GW. For our simulation, we added five loads on buses 2, 8, 9, 16 and 17, each having a step increase of magnitude 2 p.u. (base 100MVA) at \( t = 1 \) second.

Controllable loads were considered within the simulations at load buses 1 − 20, with loads controlled every 10ms. In
particular, we considered 500 refrigerators of equal magnitude at each of the 20 selected load buses with aggregate power of 3.25 GW. For comparison, we considered the system response when the following four schemes for TCLs were implemented:

(i) Conventional TCLs that do not contribute to frequency control, i.e., loads with dynamics as in (4), (5).
(ii) Frequency dependent TCLs with a deterministic control policy, i.e., loads with dynamics described by (5), (8).
(iii) Frequency dependent TCLs with a randomized control policy, as in [16], [17].
(iv) The scheme (iii) with larger feedback gains, aiming for a faster response.

The above cases will be referred to as case (i), (ii), (iii) and (iv) respectively. The values of the control parameters were randomly selected from uniform distributions with bounds provided in Table I in [25]. Furthermore, initial conditions were randomly selected in a similar manner. To ensure that incorporating the loads would not disturb the balance of the network, for each thermostatic load incorporated at a bus some constant demand equal to its average value was removed from the same bus. Moreover, frequency thresholds in case (ii) were selected in accordance with Design condition 1. In particular, following the approach described in [22], an equivalent single bus model of the power network, where generation was described with high order dynamics, was derived. The latter enabled to obtain \( \hat{L} \) (i.e. the 1-norm of the system (3) with input the aggregate demand and output \( \omega \)) and implement Design condition 1. To ensure that Design condition 1 was satisfied, we verified that the selected values of \( \omega \) satisfied

\[
\sum_{j \in S(\omega)} \omega_j \leq \max(\{\omega - \delta \}/\hat{L}, 0),
\]

letting \( \delta = 0.001 \text{Hz} \), for all \( \omega \in \mathbb{R}_+ \). For additional safety, frequency thresholds were designed with a 20% margin from the obtained upper bound. For case (iii), the implemented algorithm involved randomized transitions between the on/off states with controlled rates as described in cases (i) and (ii) is depicted in Figure 5. It should be noted that the proposed deterministic scheme.

The percentage of TCLs that are ON for the TCL schemes described in cases (i) and (ii) is depicted in Figure 5. It should be noted that the almost flat response in case (i) validates Theorem 1.

\[\text{Fig. 3: Frequency at bus 27 with TCL dynamics in the following cases: i) Conventional TCLs, ii) Deterministic frequency dependent TCLs, iii) Frequency dependent TCLs with randomized control policy, iv) As in (iii) but feedback gains are ten times larger.}\]

\[\text{Fig. 4: Largest frequency overshoot for buses 1 \text{–} 40 for the four cases described in the caption of Figure 3.}\]

\[\text{Fig. 5: Percentage of TCLs switched ON for the following cases: i) Conventional TCLs, ii) Deterministic frequency dependent TCLs, increasing only one of the transition rates (as is the effect of increasing \( K_\pi \)) will maintain a slower recovery of the TCLs. The speed of response of schemes with randomization can potentially be improved by combining them with deterministic schemes, which is an interesting direction for further theoretical analysis. Figures 3, 4 also demonstrate that no Zeno behavior or load synchronization are experienced with the proposed deterministic scheme.}\]

The percentage of TCLs that are ON for the TCL schemes described in cases (i) and (ii) is depicted in Figure 5. It should be noted that the almost flat response in case (i) validates Theorem 1.

**VII. Conclusion**

We have studied the problem of controlling thermostatic loads to provide ancillary services to the power network at emergencies. We first considered conventional TCLs and showed
that their aggregation has zero variance when their number tends to infinity and a mild condition on their period ratios holds. Then, we proposed a deterministic control scheme for TCLs which induces switching when frequency deviations exceed particular frequency thresholds. For the considered scheme, we explain how frequency thresholds could be designed such that the coupling between load and frequency dynamics does not cause load synchronization. In particular, when the number of loads tends to infinity, we showed that frequency deviations are arbitrarily small for arbitrarily large periods of time. Our analytic results have been numerically verified with simulations on the NPCC 140-bus system, which demonstrate improved frequency response when frequency dependent TCLs are incorporated compared to when conventional implementations are considered.

Future extensions of this work could consider more involved dynamics, including a network model of the power grid. In addition, future studies could consider more advanced control designs for TCLs, taking into account elements such as their economic performance and the rate at which they desynchronize, which may yield improved response.

**APPENDIX**

The following results will be used in the proof of Theorem 2.

**Corollary 1:** Let Assumption 2 hold and consider TCLs described by (5), (8) with $\bar{d}_j = \frac{1}{|S|}$ and any set $S \subseteq N$. Then, if there exists $\tau \geq 0$ such that $|\omega(t)| < \omega_m(S)$ for all $t \geq \tau$ then $\lim_{\tau \to \infty} \|d_S^{\tau}\| = 0$.

**Proof of Corollary 1:** When for some finite $\tau$ it holds that $|\omega(t)| < \omega_1(t)$, $t \geq \tau$, $j \in S$, the scheme in (8) reduces to (4) for $j \in S$. If $|S| \to \infty$ as $|N| \to \infty$, the proof follows directly from Theorem 1 and the boundedness of $d_S^{\tau}$. Alternatively, if $|S| < \infty$ as $|N| \to \infty$, then the proof follows trivially by noting that $\lim_{|N| \to \infty} \sum_{j \in S} d_{j}^{\tau} \leq \lim_{|N| \to \infty} |S| \frac{1}{|N|} = 0$.

**Lemma 3:** Consider TCLs described by (5), (8), with $\bar{d}_j = \frac{1}{|S|}$, any set $S \subseteq N$ and let Assumption 2 hold. Then, when $|N| \to \infty$, for any initial condition $(T(0), \sigma(0)) \in \mathbb{R}^{|N|} \times P^{|N|}$ and any $0 < \delta_1, \delta_0 \in \mathbb{R}_{++}$, there exist $\tau, \tau_1, \tau_0 \in \mathbb{R}_{++}, \tau \leq \tau_1 + \tau_0$ such that if $|\omega| < \omega_m(S)$ for $t \in [\tau, \tau_0]$, then $\int_{\tau_1}^{\tau_1 + \tau_2} (d_S^{\tau}(t) - d_S^{\ast})^2 dt \leq \epsilon$.

**Proof of Lemma 3:** From Theorem 1 it follows that when Assumption 2 holds for TCLs described by (4), (5), with $\bar{d}_j = \frac{1}{|S|}$ then $\lim_{|N| \to \infty} \|d^\tau\| = 0$. Corollary 1 extends this result to any subset $S \subseteq N$, i.e., $\lim_{|N| \to \infty} \|d_S^\tau\| = 0$. The latter suggests that $\lim_{|N| \to \infty} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (d_S^{\tau}(t) - d_S^{\ast})^2 dt = 0$ which follows since

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (d_S^{\tau}(t) - d_S^{\ast})^2 dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (d_S^{\ast}(t))^2 dt - (d_S^{\ast})^2 dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (d_S^{\ast}(t))^2 dt - (d_S^{\ast})^2 dt = 0$$

which is the first step follows by expanding the squared term and using the definition of $d_S^{\ast}$.

Now consider the condition on the lemma statement, and temporarily assume that $|\omega(t)| < \omega_m(S)$ for all $t \geq \tau$. Therefore, (5), (8) reduces to (4), (5) for $t \geq \tau$.

Next, the proof follows by contradiction. In particular, assume there exist $\epsilon > 0$ and $\tau_1, \tau_0 \in \mathbb{R}_{++}$ such that $\int_{\tau_1}^{\tau_1 + \tau_2} (d_S^{\tau}(t) - d_S^{\ast})^2 dt \geq \epsilon$ for all $S \in \{\tau_1 + k\tau : k \in \mathbb{N}_0\}$. Then, $V(d_S^\tau) \geq \frac{\epsilon}{\tau}$, which contradicts the result of Theorem 1. Hence, if $|\omega(t)| < \omega_m(S)$ for all $t \geq \tau$, then for any $\epsilon > 0$, $\tau_1 \in \mathbb{R}_{++}$, there exists finite $\tau_2$ such that $\int_{\tau_1}^{\tau_1 + \tau_2} (d_S^{\tau}(t) - d_S^{\ast})^2 dt \leq \epsilon$.

To conclude the proof, note that the trajectory of $d_S^{\tau}(t)$ depends only on the initial conditions and the trajectory of $\omega(t)$. Hence, the trajectory of $\omega(t)$ for $t \geq \tau_1 + \tau_0$, does not affect the result that $\int_{\tau_1}^{\tau_1 + \tau_2} (d_S^{\tau}(t) - d_S^{\ast})^2 dt \leq \epsilon$. Therefore, the condition on $\omega(t)$ reduces to $|\omega(t)| < \omega_m(S)$ for $t \in [\tau_1, \tau_0]$, for any $\tau_0 \geq \tau_1 + \tau_2$.

Before continuing with the rest of the results, it will be convenient to note that system (13) consists of the linear system (3) in feedback with the hybrid system (5), (8). Let $\hat{x}$ be the equilibrium value of $x$ in (3) when $d^x = d^{x^*} = \sum_{j \in N} \alpha_j d_j$. System (3), can be equivalently written in terms of deviations from these equilibrium values as follows:

$$y = Cx, \quad \dot{x} = Ax + Bu, \quad (15)$$

where $x = \begin{bmatrix} \omega \\ \hat{x} - \bar{x} \end{bmatrix}$, $u = [d^x - d^{x^*}]$, $y = \omega$, $C = [1 \ 0]$, and $A$ and $B$ as given in the description immediately after (3). Furthermore, note that $A$ is Hurwitz from Assumption 1(i).

**Lemma 3:** Consider the system (15). Let $|\omega(t)|$ be uniformly bounded for $t \geq 0$ and satisfy the property that for any $\epsilon > 0$, $\tau_1 \in \mathbb{R}_{++}$, there exists $\tau_1 \in \mathbb{R}_{++}$ such that $\int_{\tau_1}^{\tau_1 + \tau_2} (d_S^{\tau}(t) - d_S^{\ast})^2 dt \leq \epsilon$.

Then, for any $x(0) \in \mathbb{R}^{n+1}$ and any $\epsilon > 0$, $\tau_1 \in \mathbb{R}_{++}$, there exists $\tau_1 \in \mathbb{R}_{++}$ such that $|y(t)| \leq \epsilon$ for all $t \in [\tau_1, \tau_1 + \tau_1]$.

**Proof of Lemma 3:** The proof can be found in [25].

**Proof of Theorem 2:** The trajectories $z(t,j)$ of system (13) are in general non-unique. However, it can be trivially shown that for each trajectory of $d_S^{\tau}(t,j)$, there exists a unique trajectory for $\omega(t,j)$, since $\omega$ is the output of linear system (15) with input $d^x$. The analysis below concerns $\omega(t,j)$ given any trajectory $d_S^{\tau}(t,j)$ that is compatible with (13) such that the conditions of Theorem 2 hold. For simplicity, in the analysis below we drop the element $j$ from the argument of the solutions, i.e. denoting $x(t,j)$ and $d_S^{\tau}(t,j), i \in N$, by simply $x(t)$ and $d_S^{\tau}(t), i \in N$ respectively.

For system (15), from any initial condition $x(0) \in \mathbb{R}^{n+1}$, $\omega(t)$ is given by

$$\omega(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(t-\tau)d\tau, \quad (16)$$

which suggests that the magnitude of $\omega(t)$ satisfies

$$|\omega(t)| \leq |Ce^{At}x(0)| + \int_0^t Ce^{A(t-\tau)}Bu(t-\tau)d\tau. \quad (17)$$

Since $A$ is Hurwitz, it follows that for any $\hat{\epsilon} > 0$ there exists $\tau \in \mathbb{R}_{++}$ such that $|Ce^{At}x(0)| \leq \hat{\epsilon}$ for all $t \geq \tau$. Furthermore, for the integral part of (17), it holds that

$$\int_0^\tau |Ce^{At}Bu(t-\tau)d\tau| \leq \int_0^\tau |Ce^{At}B||u||_\infty \leq \int_0^\infty |Ce^{At}B||u||_\infty = \hat{L}||u||_\infty,$$}

noting that $\hat{L}$ is bounded from Assumption 1(i). Hence, for any $x(0) \in \mathbb{R}^{n+1}$ and any $\epsilon > 0$, there exists $\tau \in \mathbb{R}_{++}$ such that $|\omega(t)| \leq \epsilon + \hat{L}||u||_\infty = \hat{\omega}$ for all $t \geq \tau$.

Now for given $\hat{\omega}$ consider the sets $S(\hat{\omega}) = \{j : \omega_j \leq \hat{\omega}\}$ and $S(\hat{\omega}) = N \setminus S(\hat{\omega})$, which should be interpreted as the sets of loads with and without active frequency feedback. In particular, since $|\omega(t)| \leq \hat{\omega}, t \geq \tau$, the dynamics of $d_S^{\tau}$
reduce from (5), (8), to (4), (5), for \( j \in \tilde{S}(\hat{\omega}) \). Furthermore, note that Corollary 1 applies to the set \( \tilde{S}(\hat{\omega}) \), suggesting that 
\[
\lim_{|N| \to \infty} V(d^*_S(\omega)) = 0.
\]

In the arguments below the variables \( \tau_1, \tau_1, \tilde{\tau}_1, \) and \( \tau_1, \tau_1 \) are used as in Lemmas 2 and 3 respectively. From Lemma 2, it follows that as \( |N| \to \infty \), then for any \( \epsilon_1 > 0, \tau_1 \in \mathbb{R}_+ \) there exists \( \tau_1 \in \mathbb{R}_+ \) such that 
\[
\int_0^{\tau_1} \left( d^*_S(\hat{\omega})(t) - d^*_S(\hat{\omega}) \right)^2 dt \leq \epsilon_1.
\]
Note that the value of \( \tau_1 \) depends on \( \tau_1, \tau_1, \) and the initial conditions. It then follows by applying Lemma 3 with \( u = (d^*_S(\hat{\omega})(t) - d^*_S(\hat{\omega})) \), that for any \( \epsilon > 0, \tau_1 \in \mathbb{R}_+ \), there exists \( \tau_1 \in \mathbb{R}_+ \) such that 
\[
\omega(t) \leq \left| C e^{A_t} x(0) \right| + \int_0^t C e^{A_t} B u(t - \tau) d\tau = |C e^{A_t} x(0)| + \int_0^t C e^{A_t} B (d^*_S(\hat{\omega})(t - \tau) - d^*_S(\hat{\omega})) d\tau + \int_0^t C e^{A_t} B (d^*_S(\hat{\omega})(t - \tau) - d^*_S(\hat{\omega})) d\tau \leq \epsilon + \hat{l} \left| d^*_S(\hat{\omega}) - d^*_S(\hat{\omega}) \right| \leq \epsilon
\]
for all \( t \in [\tilde{\tau}_1, \tilde{\tau}_1 + \tau_1] \). Note that, as follows from the arguments in the proof of Lemma 3, it holds that \( \tilde{\tau}_1 \geq \tau_1 \).

Part 1: If \( S(\hat{\omega}) = \emptyset \) then the proof is complete from the above arguments.

Part 2: If \( S(\hat{\omega}) \neq \emptyset \), then from Design Condition 1 it holds that 
\[
\left| d^*_S(\hat{\omega}) - d^*_S(\hat{\omega}) \right| \leq \sum_{i \in S(\hat{\omega})} \xi_i d^*_S(\hat{\omega}) \leq \max(L^* - L) \cdot (\hat{\omega} - \hat{\omega}_0).\]

Part 2: If \( S(\hat{\omega}) \neq \emptyset \), then from Design Condition 1 it holds that 
\[
\left| d^*_S(\hat{\omega}) - d^*_S(\hat{\omega}) \right| \leq \sum_{i \in S(\hat{\omega})} \xi_i d^*_S(\hat{\omega}) \leq \max(L^* - L) \cdot (\hat{\omega} - \hat{\omega}_0).
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\]