Chiral dynamics in QED and QCD in a magnetic background and nonlocal noncommutative field theories

E.V. Gorbar‡ S. Homayouni† and V.A. Miransky‡
Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada
(Dated: November 4, 2018)

We study the connection of the chiral dynamics in QED and QCD in a strong magnetic field with noncommutative field theories (NCFT). It is shown that these dynamics determine complicated nonlocal NCFT. In particular, although the interaction vertices for electrically neutral composites in these gauge models can be represented in a space of noncommutative spatial coordinates, there is no background transformation that could put the vertices in the conventional form considered in the literature. It is unlike the Nambu-Jona-Lasinio (NJL) model in a magnetic field where such a field transformation can be found, with a cost of introducing an exponentially damping form factor in field propagators. The crucial distinction between these two types of models is in the characters of their interactions, being short-range in the NJL-like models and long-range in gauge theories. The relevance of the NCFT connected with the gauge models for the description of the quantum Hall effect in condensed matter systems with long-range interactions is briefly discussed.

PACS numbers: 11.10.Nx, 11.15.-q, 11.30.Rd, 11.30.Qc

1. INTRODUCTION

During the last few years, different aspects of noncommutative field theories (NCFT) have been intensively studied (for reviews, see Ref. [1]). In particular, it was revealed that NCFT are intimately related to the dynamics in quantum mechanical models in a strong magnetic field [2, 3], nonrelativistic field systems in a strong magnetic field [4, 5], nonrelativistic magnetohydrodynamical field theory [6], and, in the case of open strings attached to D-branes, to the dynamics in string theories in magnetic backgrounds [7, 8].

Recently, the connection between the dynamics in relativistic field theories in a strong homogeneous magnetic field and that in NCFT has been studied [9]. The main conclusion of that paper was that although relativistic field theories in the regime with the lowest Landau level (LLL) dominance indeed determine a class of NCFT, these NCFT are different from the conventional ones considered in the literature. In particular, the UV/IR mixing, taking place in the conventional NCFT [10], is absent in these theories. The reason of this is an inner structure (i.e., dynamical form factors) of electrically neutral composites in these theories. We emphasize that in order to establish the connection between dynamics in a homogeneous magnetic field and dynamics in NCFT, it is necessary to consider neutral fields. The point is that in homogeneous magnetic backgrounds, momentum is a good quantum number only for neutral states and therefore one can introduce asymptotic states and S-matrix only for them.

While studies of the origins of the noncommutativity in relativistic quantum field theories in a magnetic field are interesting in themselves, it is even more important that they lead to new physical results. In particular, as was shown in Ref. [9], the NCFT approach allows to derive interaction vertices for neutral composites. These vertices automatically include (in the form of the Moyal product [11]) all powers of transverse derivatives [i.e., the derivatives with respect to coordinates orthogonal to a magnetic field]. This result is quite noticeable because the dimension of the transverse subspace is two and it is very seldom that one can with a good accuracy calculate vertices for composites in quantum field models with spatial dimensions higher than one.

In the analysis in Ref. [3], the Nambu-Jona-Lasinio (NJL) model in a magnetic field was considered. It was shown that there exist two equivalent descriptions of its dynamics. In the first description, one uses the conventional composite operators \( \sigma(x) \sim \bar{\psi}(x)\gamma_5\psi(x) \) and \( \pi(x) \sim \bar{\psi}(x)\gamma_5\gamma_\alpha\gamma_\beta\sigma(x) \). In this case, besides the usual Moyal factor, the additional Gaussian-like (form-)factor \( e^{-(\sum_{i=1}^n \vec{k}_{i\perp}/4eB)} \) occurs in n-point interaction vertices of the fields \( \sigma(x) \) and \( \pi(x) \). Here \( \vec{k}_{i\perp} \) is a momentum of the \( i \)-th composite in a plane orthogonal to the magnetic field. These form factors reflect an inner structure of composites and play an important role in providing consistency of these NCFT. In particular, because of them, the UV/IR mixing is absent in these theories. In the second description, one considers...
other, “smeared”, fields $\Sigma(x)$ and $\Pi(x)$, connected with $\sigma(x)$ and $\pi(x)$ through a non-local transformation. Then, while the additional factors are removed in the vertices for the smeared fields, they appear in their propagators, again resulting in the UV/IR mixing removal.

The Gaussian form of the exponentially damping form factors reflects the Landau wave functions of fermions on the LLL. The form factors are intimately connected with the holomorphic representation in the problem of quantum oscillator (for a review of the holomorphic representation, see Ref. [11]). Indeed, in the problem of a free fermion in a magnetic field, the dynamics in a plane orthogonal to the magnetic field is an oscillator-like one. 1 And because weak short-range interactions between fermions in the NJL model in a magnetic field do not change this feature of the dynamics, the form factors in that model have the Gaussian form. But what happens in the case of more sophisticated dynamics, such as those with long-range interactions in gauge models? To find the answer to this question is the primary goal of the present paper.

In this work, we will extend the analysis of Ref. [2] to the more complicated cases of QED and QCD in a strong magnetic field. It will be shown that in these gauge models, the connection of the dynamics with NCFT is much more sophisticated. It is not just that the damping form-factors are not Gaussian in these models but there does not exist an analogue of the smeared fields at all. As a result, their interaction vertices cannot be transformed into the form of vertices in conventional NCFT. On the other hand, it is quite remarkable that, by using the Weyl symbols of the fields [1], their vertices can nevertheless be represented in the space with noncommutative spatial coordinates. The dynamics they describe correspond to complicated nonlocal NCFT. We will call these theories type II nonlocal NCFT. The name type I nonlocal NCFT will be reserved for models similar to the NJL model in a magnetic field, for which smeared fields exist. In both these cases, the term “nonlocal” reflects the point that, besides the Moyal factor, additional form factors are present in the theories.

The crucial distinction between these two types of models is in the characters of their interactions. While the interaction in the NJL-like models is local (short-range), it is long-range in gauge theories. This point is reflected in a much richer structure of neutral composites in the latter. We believe that both these types of nonlocal NCFT can be relevant not only for relativistic field theories but also for nonrelativistic systems in a magnetic field. In particular, while type I NCFT can be relevant for the description of the quantum Hall effect in condensed matter systems with short-range interactions [4, 5, 13], type II NCFT can be relevant in studies of this effect in condensed matter systems with long-range interactions (such as carbon materials).

The paper is organized as follows. In Sec. II a brief review of the dynamics of spontaneous chiral symmetry breaking in QED and QCD in a magnetic field is made. In Sec. III by using the formalism of bilocal composite fields, the effective action and interaction vertices for Nambu-Goldstone composites are derived in QED in a strong magnetic field. In Sec. IV these vertices are written in the space with noncommutative spatial coordinates and the concept of nonlocal type I and type II NCFT is introduced. In Sec. V the dynamics in QED with a large number of fermion flavors $N_f$ in a magnetic field is studied in a special dynamical regime with local (short-range) interactions. In this regime, the results obtained in the NJL model in a magnetic field [1] are reproduced in the formalism of bilocal composite fields. In Sec. VI it is shown that the chiral dynamics in QED with a weak coupling in a strong magnetic field is related to type II nonlocal NCFT. In Sec. VII it is shown that the chiral dynamics in QCD in a strong magnetic field also relates to such a type of nonlocal NCFT. In Section VIII we discuss the origins of the connection between relativistic field theories in magnetic backgrounds and NCFT. In Section IX we summarize the main results of the paper and discuss their possible applications. In the Appendix, some useful formulas and relations are derived.

II. CHIRAL SYMMETRY BREAKING IN QED AND QCD IN A MAGNETIC FIELD

The central dynamical phenomenon in relativistic field theories in a magnetic field is the phenomenon of the magnetic catalysis: a constant magnetic field is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions [14, 15]. The essence of this effect is the dimensional reduction in the dynamics of fermion pairing in a strong magnetic field, when the LLL dynamics dominates. In this section, we will indicate those features of this phenomenon in QED and QCD which are relevant for the present work.

The consistent theory of this phenomenon in QED was developed in Refs. [16] for earlier papers considering spontaneous chiral symmetry breaking in QED in a magnetic field, see Refs. [14, 17]. The crucial point of the analysis [16] was to recognize that there exists a special non-local (and non-covariant) gauge in which the so called improved rainbow (ladder) approximation is reliable in this problem, i.e., there is a consistent truncation of the system

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1 In particular, the holomorphic representation is widely used for the description of the quantum Hall effect [12].
of the Schwinger–Dyson equations. The full photon propagator in this gauge has the form

\[
D_{\mu\nu}(k) = i \frac{g_{\mu\nu}}{k^2 + k_\|^2 \Pi(k_\|^2, k_\perp^2)} + i \frac{g_{\mu\nu}k^2 - (k_\|^2 k_\|^2 + k_\perp^2 + k_\|^2 k_\perp^2)}{(k^2)^2},
\]

where \(\Pi(k_\|^2, k_\perp^2)\) is the polarization operator and the symbols \(\perp\) and \(\|\) in \(g_{\mu\nu}\) and \(k_\mu\) are related to the transverse \((1,2)\) and longitudinal \((0,3)\) space-time components, respectively (we consider a constant magnetic field \(B\) directed in the \(+x_3\) direction). Since the transverse degrees of freedom decouple from the LLL dynamics, only the first term in photon propagator \(D_{\mu\nu}\), proportional to \(g_{\mu\nu}\|\), is relevant. Therefore, as the full photon propagator in this special gauge, one can take the Feynman-like noncovariant propagator

\[
D_{\mu\nu}(k) = i \frac{g_{\mu\nu}}{k^2 + k_\|^2 \Pi(k_\|^2, k_\perp^2)}.
\]

It is important that this propagator does not lead to infrared mass singularities in loop corrections in a vertex that makes the improved rainbow approximation to be reliable in this gauge (because of mass singularities in covariant gauges, the loop corrections in the vertex are large \([15, 16]\)).

It was shown in Ref. \([16]\) that the kinematic region mostly responsible for generating the fermion mass is that with \(m_{\text{dyn}}^2 \ll |k_\|^2, k_\perp^2 \ll |eB|\). In that region, fermions can be treated as massless and, as a result, the polarization operator can be calculated in one-loop approximation. It is

\[
\Pi(k_\|^2, k_\perp^2) \simeq - \frac{2\tilde{\alpha}_b |eB|}{\pi k_\|^2},
\]

where \(\tilde{\alpha}_b = N_f\alpha_b = \frac{N_f e B}{4\pi}\). Here \(N_f\) is the number of fermion flavors and \(\alpha_b\) is the QED running coupling related to the magnetic scale \(\mu = \sqrt{|eB|}\).

Thus, in this approximation, photon propagator \([2]\) becomes a propagator of a free massive boson with \(M_f^2 = 2\tilde{\alpha}_b |eB|\pi /\pi\):

\[
D_{\mu\nu}(x) = \frac{i}{(2\pi)^4} \int d^4 k e^{-ikx} \frac{g_{\mu\nu}}{k^2 - M_f^2}.
\]

As was shown in the first two papers of Ref. \([16]\), the improved rainbow approximation with the photon propagator \([2]\) is reliable when the parameter \(\tilde{\alpha}_b\) is small, i.e., \(\tilde{\alpha}_b \ll 1\). In the case of large \(N_f\), when one can use the \(1/N_f\) expansion, this approximation is reliable for arbitrary \(\tilde{\alpha}_b\) [see the last paper in Ref. \([16]\)].

In the weak coupling regime with \(\tilde{\alpha}_b = N_f\alpha_b \ll 1\), the dynamically generated mass of fermions is:

\[
m_{\text{dyn}} = C|eB|^{1/2} F(\tilde{\alpha}_b) \exp \left( - \frac{\pi N_f}{\tilde{\alpha}_b \ln(C_1/\tilde{\alpha}_b)} \right),
\]

where \(F(\tilde{\alpha}_b) \simeq (\tilde{\alpha}_b)^{1/3}\), \(C_1 \simeq 1.82\) and \(C\) is a numerical constant of order one. In the strong coupling limit for large \(N_f\), the dynamical mass takes the form

\[
m_{\text{dyn}} \simeq \sqrt{|eB|} \exp(-N_f).
\]

With appropriate modifications, this theory was extended to the case of QCD in a strong magnetic field in Ref. \([18]\) [for earlier consideration of QCD in a magnetic field, see Refs. \([19, 20]\)]. These modifications will be described in Sec. VII. The results of Refs. \([16, 17, 18]\) will be an essential ingredient in the analysis in this paper.

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2 We recall that in the improved rainbow (ladder) approximation the vertex \(\Gamma^\nu(x, y, z)\) is taken to be bare and the photon propagator is taken in the one-loop approximation.

3 This feature reflects the point that the spin of the LLL fermion (antifermion) states is polarized along (opposite to) the magnetic field.
III. QED IN A MAGNETIC FIELD: THE EFFECTIVE ACTION

In this Section, we analyze the dynamics in the chiral symmetric QED in a strong magnetic field. We will consider both the weakly coupling regime, with $\alpha_b \ll 1$, and (for large $N_f$) the strongly coupling regime with $\alpha_b \gtrsim 1$. In both these cases, one can use the results of the analysis of Ref. 11.

The chiral symmetry in this model is $SU(N_f)_L \times SU(N_f)_R$. The generation of a fermion mass breaks this symmetry down to $SU(N_f)_V$ and, as a result, $N_f^2 - 1$ neutral Nambu-Goldstone (NG) composites $\pi^A$, $A = 1, 2, ..., N_f^2 - 1$, occur (we do not consider here the anomalous $U(1)_A$). Our aim in this section is to derive the interaction vertices for $\pi^A$ in the regime with the LLL dominance and clarify whether their structure corresponds to a NCFT.

Integrating out the photon field $A_\mu$, we obtain the following non-local effective action for fermions in QED in a magnetic field:

\[ S = \int d^4x \bar{\psi} i\gamma^\mu D_\mu \psi - 2i\pi\alpha_b \int d^4x d^4y \bar{\psi}(x)\gamma^\mu \psi(x)D^{(0)}_{\mu\nu}(x-y)\bar{\psi}(y)\gamma^\nu\psi(y), \]

where, in the lowest order, the bare propagator corresponding to propagator \( \Box \)

\[ D^{(0)}_{\mu\nu}(x-y) = \frac{i}{(2\pi)^4} \int d^4k e^{-ik(x-y)} \frac{g_{\mu\nu}}{k^2}, \]

and the vector potential $A^{ext}_\mu$ in the covariant derivative $D_\mu = \partial_\mu - ieA^{ext}_\mu$ in \( \Box \) describes a constant magnetic field $B$ directed in the $+x^3$ direction. In the symmetric gauge, it is $A^{ext}_\mu = (0, Bx^2, -Bx^1, 0)$. Following the auxiliary field method developed for theories with nonlocal interaction in Refs. 21, 22, we add the term

\[ \Delta S = -2i\pi\alpha_b \int d^4x d^4y tr \{ \gamma^\mu [\varphi^b_a(x,y) - \psi_a(x)\bar{\psi}(y)] \gamma^\nu [\varphi^a_b(y,x) - \psi_b(y)\bar{\psi}(x)] \} D^{(0)}_{\mu\nu}(x-y) \]

in the action. Here $\varphi^b_a(x,y)$ is a bilocal auxiliary field with the indices $a$ and $b$ from the fundamental representation of $SU(N_f)$. Then we obtain the action

\[ S = \int d^4x \bar{\psi} i\gamma^\mu D_\mu \psi - 4i\pi\alpha_b \int d^4x d^4y \bar{\psi}(x)\gamma^\mu \varphi(x,y)\gamma^\nu\psi(y)D^{(0)}_{\mu\nu}(x-y) \]

\[ - 2i\pi\alpha_b \int d^4x d^4y tr [\gamma^\mu \varphi(x,y)\gamma^\nu \varphi(y,x)] D^{(0)}_{\mu\nu}(x-y) \]

(9)

[here, for clarity of the presentation, we omitted the $SU(N_f)$ indices]. Integrating over fermions, we find

\[ S(\varphi) = -iTrLn \left[ \gamma^\mu iD_\mu \delta^4(x-y) - 4i\pi\alpha_b\gamma^\mu \varphi(x,y)\gamma^\nu D^{(0)}_{\mu\nu}(x-y) \right] \]

\[ - 2i\pi\alpha_b \int d^4x d^4y tr [\gamma^\mu \varphi(x,y)\gamma^\nu \varphi(y,x)] D^{(0)}_{\mu\nu}(x-y), \]

(10)

where $Tr$ and $Ln$ are taken in the functional sense.

Following Ref. 22, we can expand $\varphi(x,y)$ as

\[ \varphi(x,y) = \varphi_0(x,y) + \tilde{\varphi}(x,y), \]

(11)

\[ \tilde{\varphi}(x,y) = \sum_n \int \frac{d^4P}{(2\pi)^4} \phi_n(P)\chi_n^{(0)}(x,y; P). \]

(12)

Here $\varphi_0(x,y)$ satisfies the equation

\[ \frac{\delta S}{\delta \varphi} = 0, \]

(13)
which is equivalent to the Schwinger–Dyson equation

\[
S^{-1}_{(1)}(x, y) = S_0^{-1}(x, y) - 4\pi\alpha_s\gamma^\mu S_{(1)}(x, y)\gamma^\nu D^{(0)}_{\mu\nu}(x - y),
\]

(14)

where \(S_0\) is the bare fermion propagator and \(S_{(1)} \equiv \varphi_0\) is the full fermion propagator in the rainbow (ladder) approximation. As to Eq. (12), \(\varphi_n(P)\) is a field operator describing a neutral composite \(|n, P\rangle\) and \(\chi^{(f)}(x, y; P)\) are solutions of the off-mass-shell Bethe–Salpeter (BS) equation in the ladder approximation,

\[
\chi^{(f)}(x, y; P) = 4\pi\alpha_s\lambda(P) \int d^4x_1 d^4y_1 S_{(1)}(x, x_1) \gamma^\mu \chi^{(f)}(x_1, y_1; P) \gamma^\nu S_{(1)}(y_1, y) D^{(0)}_{\mu\nu}(x_1 - y_1).
\]

(15)

The insertion of factor \(\lambda(P) \neq 1\) in this equation allows to consider off-mass-shell states with an arbitrary mass \(M^2 = P^2\). The on-mass-shell states correspond to \(\lambda(P) = 1\).

Using Eqs. (11) and (14), the action (10) can be rewritten as

\[
S(\tilde{\phi}) = -i \text{Tr} \ln \left[ S_{(1)}^{-1}(x, y) - 4\pi\alpha_b\gamma^\mu \tilde{\phi}(x, y)\gamma^\nu D^{(0)}_{\mu\nu}(x - y) \right]
\]

\[-2i\pi\alpha_b \int d^4x d^4y \text{tr}[\gamma^\mu(\varphi_0(x, y) + \tilde{\phi}(x, y))\gamma^\nu(\varphi_0(x, y) + \tilde{\phi}(y, x))]D^{(0)}_{\mu\nu}(x - y).\]

(16)

Expanding the action \(S(\tilde{\phi})\) in powers of \(\tilde{\phi}\) and ignoring its part that does not depend on \(\tilde{\phi}\), we obtain

\[
S(\tilde{\phi}) = \sum_{n=2}^{\infty} \frac{i}{n} \int d^4x_1 d^4y_1 \ldots d^4x_n d^4y_n \text{tr}[S_{(1)}(x_1, y_1)\varphi_D(y_1, y_2)S_{(1)}(y_2, y_3)\varphi_D(y_3, y_4)\ldots S_{(1)}(y_{n-1}, y_n)\varphi_D(y_n, x_1)]
\]

\[-2i\pi\alpha_b \int d^4x d^4y \text{tr}[\gamma^\mu\varphi_D(x, y)\gamma^\nu\varphi_D(y, x)]D^{(0)}_{\mu\nu}(x - y),\]

(17)

where

\[
\varphi_D(x, y) = 4\pi\alpha_b\gamma^\mu\tilde{\phi}(x, y)\gamma^\nu D^{(0)}_{\mu\nu}(x - y).
\]

Because \(\varphi_0\) satisfies the Schwinger–Dyson equation (13), the term linear in \(\tilde{\phi}\) is absent in (17).

As is clear from the discussion in the previous section, one should use the improved rainbow (ladder) approximation in the present problem. The Schwinger–Dyson equation for the fermion propagator in this approximation takes the form

\[
S^{-1}_{(1)}(x, y) = S_0^{-1}(x, y) - 4\pi\alpha_s\gamma^\mu S(x, y)\gamma^\nu D_{\mu\nu}(x - y),
\]

(18)

where the photon propagator \(D_{\mu\nu}(x)\) is given in Eq. (4). The off-mass-shell BS equation in the improved ladder approximation is given by

\[
\chi(x, y; P) = 4\pi\alpha_s\lambda(P) \int d^4x_1 d^4y_1 S(x, x_1) \gamma^\mu \chi(x_1, y_1; P) \gamma^\nu S(y_1, y) D_{\mu\nu}(x_1 - y_1).
\]

(19)

The comparison of Eqs. (18), (19) with Eqs. (12), (14) suggests that in the improved rainbow (ladder) approximation the effective action (17) should be replaced by the following one:

\[
S(\tilde{\phi}) = \sum_{n=2}^{\infty} \frac{i}{n} \int d^4x_1 d^4y_1 \ldots d^4x_n d^4y_n \text{tr}[S(x_1, y_1)\varphi_D(y_1, y_2)S(x_2, y_2)\varphi_D(y_2, y_3)\ldots S(x_{n-1}, y_{n-1})\varphi_D(y_{n-1}, x_1)]
\]

\[-2i\pi\alpha_b \int d^4x d^4y \text{tr}[\gamma^\mu\varphi_D(x, y)\gamma^\nu\varphi_D(y, x)]D_{\mu\nu}(x - y),\]

(20)

where

\[
\varphi_D(x, y) = 4\pi\alpha_b\gamma^\mu\tilde{\phi}(x, y)\gamma^\nu D_{\mu\nu}(x - y).
\]
and
\[ \hat{\varphi}(x, y) = \sum_n \int \frac{d^4 P}{(2\pi)^4} \phi_n(P) \chi_n(x, y; P) \]  
(21)

(compare with Eq. (22)). Here \( \chi_n(x, y; P) \) are solutions of the off-mass-shell BS equation (19).

As in the case of the NJL model \[24\], in the improved rainbow approximation in QED, the LLL fermion propagator factorizes into two parts: the part depending on the transverse coordinates \( x_\perp = (x^1, x^2) \) and that depending on the longitudinal coordinates \( x_\parallel = (x^0, x^3) \),
\[ S_{LLL}(x, y) = P(x_\perp, y_\perp) S_{\parallel}(x_\parallel - y_\parallel). \]  
(22)

Here \( P(x_\perp, y_\perp) \) is the projection operator on the LLL states \[10\] which in the symmetric gauge is
\[ P(x_\perp, y_\perp) = \frac{|eB|}{2\pi} e^{\frac{\mu}{\pi} x_\perp y_\perp} e^{-\frac{\mu}{\pi} (x_\perp - y_\perp)^2}. \]  
(23)

The first exponential factor in \( P(x_\perp, y_\perp) \) is the Schwinger phase \[23\]. Its presence is dictated by the group of magnetic translations in this problem (for more details, see Sec. VIII below). As was shown in Ref. \[9\], it is the Schwinger phase that is responsible for producing the Moyal factor (a signature of NCFT \[1\]) in interaction vertices.

As to the longitudinal part, in the improved rainbow approximation it has the form \[10\]
\[ S_{\parallel}(x_\parallel - y_\parallel) = \int \frac{d^2 k_\parallel}{(2\pi)^2} e^{ik_\parallel(x_\parallel - y_\parallel)} \frac{i}{k_\parallel \gamma_\parallel} \frac{1 - i\gamma_\parallel^2}{2}, \]  
(24)
i.e., it has the form of a fermion propagator in 1+1 dimensions. 4 The dynamical mass function \( m(k_\parallel^2) \) is essentially constant for \( k_\parallel^2 \lesssim |eB| \) and rapidly decreases for \( k_\parallel^2 > |eB| \[10\]. Therefore, a simple and reliable approximation for \( m(k_\parallel^2) \) is
\[ m(k_\parallel^2) = \theta(|eB| - k_\parallel^2) m_{dyn}, \]  
(25)
where \( \theta(x) \) is the step function and \( m_{dyn} \) is the fermion pole mass \[5\] or \[6\] [this conclusion was later confirmed in papers \[24\].

The operators \( \phi_n(P) \) in equation \[24\] describe all possible neutral fermion-antifermion composites. The description of their interaction vertices in QED in a magnetic field is quite a formidable problem. Henceforth we limit ourselves to considering only the interaction vertices for the NG boson states \(|A; P >\) and their operators \( \phi^A(P) \) (for a discussion concerning other states, see Sec. VIII \[5\]).

In a magnetic field, the wave function of the states \(|A; P >\) satisfying BS equation \[19\] has the following form \[15\]:
\[ \chi^A(x, y; P) = \langle 0| T \psi(x) \bar{\psi}(y)|A; P > = e^{-iP \cdot X} e^{i\epsilon r \cdot A^a_{\mu} r}(X) \chi^A(r; P), \]  
(26)
where \( r = x - y, X = \frac{eB}{2} \) and, that is very important, the function \( \chi^A(r; P) \) is independent of the center of mass coordinate \( X \). This fact reflects the existence of the group of magnetic translations in the present problem. As in the case of the fermion propagator, the presence of the Schwinger factor \( e^{i\epsilon r \cdot A^a_{\mu} r}(X) \) in expression \[20\] is dictated by this symmetry (see Sec. VIII \[12\] below). As to the \( SU(N_f) \) structure of \( \chi^A(r; P) \), it is:
\[ \hat{\chi}^A(r; P) = \frac{\lambda^A}{2} \hat{\chi}(r; P), \]  
(27)
where \( \lambda^A \) are \( N_f^2 - 1 \) matrices in the fundamental representation of \( SU(N_f) \).

Now, transforming BS equation \[19\] into momentum space, we get:
\[ \hat{\chi}^A(p; P) = \frac{16\pi\alpha_b \lambda^A(P)}{|eB|^2} \int \frac{d^2 q_\perp d^2 k_\perp k_\parallel d^2 k_{\parallel}}{(2\pi)^6} e^{i(p_\perp - q_\perp) \times (A_\perp - p_\perp)} e^{-\frac{(k_\parallel^2 + k_\perp^2)}{2m_{dyn}}}, \]  

4 In particular, the matrix \( (1 - i\gamma^4 \gamma^2)/2 \) is the projection operator on the fermion (antifermion) states with the spin polarized along (opposite to) the magnetic field, and therefore it projects on two states of the four ones, as should be in 1+1 dimensions.

5 As is well known, in some approximations, the BS equation is plagued by the appearance of spurious solutions. Because we restrict ourselves to calculating the vertices for the NG bosons, which are manifestly physical, no such problem occurs in this study.
work, we immediately find that to the BS equation for NG bosons with of the factorization of the LLL propagator. Therefore we can explicitly integrate over $\chi$ space depends only on the coordinates $p$. Only with the transverse coordinates, this dependence is the most relevant for our purposes. The case with nonzero $\chi$ is quite formidable. In fact, as will become clear below (see also Appendix), it is the very special case with nonzero $A_\perp = 0$, i.e., for $||$ $\phi(x, y)$ in Eq. (21) is

$$\phi(x, y) = \int \frac{d^2 p_\perp}{(2\pi)^2} \phi^A(P_\perp) \chi^A(x; y; P_\perp)$$

with the BS wave function $\chi^A$ depending only on $P_\perp$. For $P_\perp = 0$, the Fourier transform of the fields $\phi^A$ in coordinate space depends only on the coordinates $X_\perp$. Because in this problem the noncommutative geometry is connected only with the transverse coordinates, this dependence is the most relevant for our purposes. The case with nonzero longitudinal momenta $P_\parallel$ is quite cumbersome and is considered in the Appendix.

We would like to emphasize that in the case of nonlocal interactions, such as those in QED, the bound state problem with nonzero $P$ is quite formidable. In fact, as will become clear below (see also Appendix), it is the very special property of the factorization of the longitudinal and transverse dynamics in the LLL approximation that will allow us to succeed in the derivation of explicit expressions for interaction vertices for NG fields $\phi^A$ for nonzero $P$.

When $P_\parallel = 0$, one can check that, up to the factors $\lambda(P_\perp)$ and $e^{iP_\perp \times u_\perp}$, the structure of equation (31) is similar to the BS equation for NG bosons with $P_\perp = P_\parallel = 0$ considered in Ref. [15]. By using the analysis performed in that work, we immediately find that

$$f^A(p_\parallel; P_\perp) = S_\parallel(p_\parallel) F^A(p_\parallel; P_\perp) \gamma^5 \frac{1 - i\gamma^1 \gamma^2}{2} S_\parallel(p_\parallel),$$

where $F^A(p_\parallel; P_\perp)$ is a scalar function. It satisfies the following equation in Euclidean space:

$$F^A(p_\parallel; P_\perp) = 8\pi\alpha_b\lambda(P_\perp) \int \frac{d^2 u_\perp d^2 k_\perp}{(2\pi)^4} \frac{F^A(k_\perp; P_\perp)}{k_\perp^2 + m^2} \frac{e^{iP_\perp \times u_\perp} e^{-\frac{q^2}{2m^2}}}{(k_\perp - p_\parallel)^2 + u_\perp^2 + M^2_{\perp}}.$$  

In the derivation of this equation, Eq. (11) was used. Now, taking into account Eqs. (26), (29), and (33), and integrating explicitly over $p_\perp$, we find that the BS wave function in coordinate space is

$$\chi(x, y; P_\perp) = P(x_\parallel, y_\parallel) \int \frac{d^2 p_\parallel}{2(2\pi)^2} e^{iP_\perp \times u_\perp} e^{-i p_\parallel (x_\parallel - y_\parallel)} e^{-\frac{q^2}{2m^2}} e^{\epsilon_b p_\parallel \times (\epsilon^A - \phi^A) \text{sign} u_\parallel}$$

$$\times S_\parallel(p_\parallel) F^A(p_\parallel; P_\perp) \gamma^5 \frac{1 - i\gamma^1 \gamma^2}{2} S_\parallel(p_\parallel).$$

(35)
Then, inserting the bilocal field $\tilde{\varphi}(x, y)$ from Eq. (32) into action (20) and using the BS equation (19), we obtain the following explicit form for the effective action:

$$S(\hat{\varphi}) = \sum_{n=2}^{\infty} \frac{i}{n} \int d^4x_1 d^4y_1 \ldots d^4x_n d^4y_n \int \frac{d^2P_1^1 \ldots d^2P_n^1}{(2\pi)^{2n}} \phi^{A_1}(P_1^+) \ldots \phi^{A_n}(P_n^+)$$

$$\times \frac{\text{tr}[S_{LLL}^{-1}(x_1, y_1)\chi^{A_1}(y_1, x_2; P_1^+) \ldots S_{LLL}^{-1}(x_{n-1}, y_{n-1})\chi^{A_n}(y_n, x_1; P_n^+)]}{\Pi_{i=1}^{n}\lambda(P_i^+)} - \frac{i}{2} \int d^4x_1 d^4y_1 d^4x_2 d^4y_2 \int \frac{d^2P_1^1 d^2P_2^2}{(2\pi)^4} \phi^{A_1}(P_1^+) \phi^{A_2}(P_2^+) \text{tr}[\chi^{A_1}(x_1, y_1; P_1^+) S_{LLL}^{-1}(y_1, y_2)\chi^{A_2}(y_2, x_2; P_2^+) S_{LLL}^{-1}(x_2, x_1)]$$

(36)

From the effective action (36), one can obtain the n-point vertices of NG bosons. In fact, using Eq. (34), the factorized form of the fermion propagator (22), and the fact that $P(x_\perp, y_\perp)$ is a projection operator, we can integrate over all coordinates in the effective action (36), similarly as it was done in the case of the NJL model in Ref. [9]. Then, we get the following expression for the effective action:

$$S(\phi) = \sum_{n=2}^{\infty} \Gamma_n,$$

where the interaction vertices $\Gamma_n$, $n > 2$, are

$$\Gamma_n = \frac{\pi |eB|}{2^{n-1}n} \int d^2X || \int d^2k_1 \lambda(P_1^+) \ldots d^2P_n^1 \delta^n \left( \sum_{i=1}^{n} P_i^+ \right) \phi^{A_1}(P_1^+) \ldots \phi^{A_n}(P_n^+)$$

$$\times \frac{\text{tr}[S_{||}(k_1)F^{A_1}(k_1; P_1^+)\gamma^5 \frac{1 - i\gamma^1\gamma^2}{2} \ldots S_{||}(k_n)F^{A_n}(k_n; P_n^+)\gamma^5 \frac{1 - i\gamma^1\gamma^2}{2}]}{\Pi_{i=1}^{n}\lambda(P_i^+)} e^{-\frac{i}{\hbar} \sum_{i<j} P_i^+ \times P_j^+},$$

(37)

and the quadratic part of the action is

$$\Gamma_2 = -\frac{|eB|}{16\pi} \int d^2X || \int d^2k_1 \lambda(P_1^+) \lambda(P_2^+) \phi^{A_1}(P_1^+)$$

$$\times \frac{\text{tr}[S_{||}(k_1)F^{A_1}(k_1; P_1^+)\gamma^5 \frac{1 - i\gamma^1\gamma^2}{2} S_{||}(k_2)F^{A_2}(k_2; P_2^+)\gamma^5 \frac{1 - i\gamma^1\gamma^2}{2}]}{\lambda(P_1^+) \lambda(P_2^+)} \phi^{A_2}(-P_2^+).$$

(38)

For the expressions of the vertices in the case of nonzero longitudinal momenta, see the Appendix.

In the next section, we will discuss the connection of the structure of vertices (37) with vertices in NCFT.

### IV. TYPE I AND TYPE II NONLOCAL NCFT

The derivation of the expressions for vertices (37) and quadratic part of the action (38) is one of the main results of this work (the generalization of these expressions for the case of nonzero longitudinal momenta are given in Eqs. (35) and (38) in the Appendix). Let us discuss the connection of the structure of these vertices with vertices in NCFT. According to [1], an $n$-point vertex in a noncommutative theory in momentum space has the following canonical form:

$$\int \frac{d^Dk_1}{(2\pi)^D} \ldots \frac{d^Dk_n}{(2\pi)^D} \phi(k_1) \ldots \phi(k_n) \delta^D \left( \sum_i k_i \right) e^{-\frac{i}{\hbar} \sum_{i<j} k_i \times k_j},$$

(39)

where here $\phi$ denotes a generic field and the exponent $e^{-\frac{i}{\hbar} \sum_{i<j} k_i \times k_j} \equiv e^{-\frac{i}{\hbar} \sum_{i<j} \theta^{ab} k_i \theta k_j}$ is the Moyal exponent factor. Here the antisymmetric matrix $\theta^{ab}$ determines the commutator of spatial coordinates:

$$[\hat{x}^a, \hat{x}^b] = i\theta^{ab}.$$

(40)
When can the vertex (37) be transformed into the conventional form (39)? In order to answer this question, it will be convenient to ignore for a moment the fact that $F^A(p_i; P_{\perp})$ in (37) is a solution of equation (44) and consider it as an arbitrary function of the momenta $p_i$ and $P_{\perp}$. Then, comparing expressions (37) and (39), it is not difficult to figure out that if the function $F(p_i; P_{\perp})$, defined as

$$F^A(p_i; P_{\perp}) = \left(\lambda^A/2\right) F(p_i; P_{\perp}),$$

has the factorized form

$$F(p_i; P_{\perp}) = F_i(p_i) F(P_{\perp}),$$

then there exists a map of the fields $\phi^A(P_{\perp})$ into new fields in terms of which vertices $\Gamma_n$ take the conventional form (39). Indeed, let us introduce new fields

$$\Phi^A(P_{\perp}) = \frac{F_i(P_{\perp})}{\lambda(P_{\perp})} \phi^A(P_{\perp}).$$

Then, after integrating over $k_i$ and taking trace over Dirac matrices, we get the conventional form for $\Gamma_n$:

$$\Gamma_n = C_n |eB| \int d^2 X_\| \int d^2 P_{\perp} \left(\frac{2\pi}{2}\right)^2 \delta^2 \left(\sum_{i=1}^n \hat{P}_{\perp}^i\right)$$

$$\times \text{tr} \left[ \hat{\Phi}(P_{\perp}^1) \ldots \hat{\Phi}(P_{\perp}^n) \right] e^{-\frac{i}{\hbar} \sum_{i<j} P^i_{\perp} \times P^j_{\perp}}, \quad \hat{\Phi} \equiv (\lambda^A/2) \Phi^A,$$

where $C_n$ is some constant. The propagator of these fields is determined from the quadratic part (43) of the action:

$$\Gamma_2 = C_2 |eB| \int d^2 X_\| \int d^2 P_{\perp} \left(\frac{2\pi}{2}\right)^2 \delta^2 \left(\lambda(P_{\perp}) - 1\right) \text{tr} \left[ \hat{\Phi}(P_{\perp}) \hat{\Phi}(-P_{\perp}) \right].$$

Thus, in terms of the new fields $\Phi^A(P_{\perp})$, vertices (37) can be transformed into the canonical form. As will be shown in the next section, there exists a special dynamical regime in QED with a large number of fermion flavors $N_f$ and $M_2^2 \gg |eB|$ in which the constraint (42) can be fulfilled. In fact, this dynamical regime is essentially the same as that in the NJL model in a strong magnetic field $|eB|$. In this case, function $F$ is $p_\| \perp$ independent. Following Ref. [9], the fields $\Phi^A(P_{\perp})$ with a built-in form factor will be called smeared fields.

In coordinate space, the interaction vertices (44) of the smeared fields take the form:

$$\Gamma_n = \frac{C_n}{4\pi^2} |eB| \int d^2 X_\| \int d^2 X_{\perp} \text{tr} \left[ \hat{\Phi}(X_{\perp}) \ast \ldots \ast \hat{\Phi}(X_{\perp}) \right],$$

where the symbol $\ast$ is the conventional star product (11) relating to the transverse coordinates. In the space with noncommutative transverse coordinates $X^a_{\perp}, \ a = 1, 2$, these vertices can be represented as

$$\Gamma_n = \frac{C_n}{4\pi^2} |eB| \int d^2 X_\| \text{Tr} \left[ \hat{\Phi}(X_{\perp}) \ldots \hat{\Phi}(X_{\perp}) \right],$$

where $\hat{\Phi}(X)$ is the Weyl symbol of the field $\Phi(X)$, the operation $\text{Tr}$ is defined as in (11), and

$$[\hat{X}^a_{\perp}, \hat{X}^b_{\perp}] = \frac{i}{e\hbar} \epsilon^{ab} \equiv i\theta^{ab}, \ a, b = 1, 2.$$

We will refer to theories with a factorized function $F(p_i; P_{\perp})$ as type I nonlocal NCFT.

In the case when the function $F(p_i; P_{\perp})$ in Eq. (41) is not factorized, one cannot represent interaction vertices (37) as a nonlocal vertex in the noncommutative space. Indeed, we can rewrite (37) as

$$\Gamma_n = \frac{\beta}{2\pi |eB|} \int d^4 X \left[ V^A_1 \ldots V^A_n - i \nabla^A_1 \ldots - i \nabla^A_n \right] \phi^A_1(X^1) \ast \ldots \ast \phi^A_n(X^n) \bigg|_{x^1 = x^2 = \ldots = x^n},$$

where $\phi^A_n(X^i)$ is the conventional star product [1] relating to the transverse coordinates. In the space with noncommutative transverse coordinates $X^a_{\perp}, \ a = 1, 2$, these vertices can be represented as

$$\Gamma_n = \frac{C_n}{4\pi^2} |eB| \int d^2 X_\| \text{Tr} \left[ \hat{\Phi}(X_{\perp}) \ldots \hat{\Phi}(X_{\perp}) \right],$$

where $\hat{\Phi}(X)$ is the Weyl symbol of the field $\Phi(X)$, the operation $\text{Tr}$ is defined as in (11), and

$$[\hat{X}^a_{\perp}, \hat{X}^b_{\perp}] = \frac{i}{e\hbar} \epsilon^{ab} \equiv i\theta^{ab}, \ a, b = 1, 2.$$

We will refer to theories with a factorized function $F(p_i; P_{\perp})$ as type I nonlocal NCFT.
where the coincidence limit $X_1 = X_2 = \ldots = X$ is taken after the action of a nonlocal operator $V_n^{A_1 \ldots A_n}$ on the fields $\phi^{A_i}$. In momentum space, the operator $V_n^{A_1 \ldots A_n}$ is

$$V_n^{A_1 \ldots A_n}(P_1, \ldots, P_n) = \int \frac{d^2 k_1}{(2\pi)^2} \text{tr} \left[ S_{||}(k_1) F^{A_1}(k_1; P_1) \gamma_5 \frac{1 - i \gamma_4 \gamma_2}{2} \ldots S_{||}(k_1) F^{A_n}(k_1; P_n) \gamma_5 \frac{1 - i \gamma_4 \gamma_2}{2} \right] \times \frac{1}{\Pi_{i=1}^n \lambda(P_i^\perp)}. \quad (50)$$

By using the fact that the Weyl symbol of the derivative in noncommutative space is given by the operator $\hat{\nabla}_a$, acting as $\hat{\nabla}_a$,

$$\hat{\nabla}_a \phi(X) = -i[(\theta^{-1})_{ab} \hat{X}^b_\perp, \phi(X)], \quad (51)$$

we obtain the following form for the interaction vertices in the noncommutative space:

$$\Gamma_n = \frac{i|eB|}{2n+1\pi n} \int d^2 X_1 \text{Tr} \left\{ V_n^{A_1 \ldots A_n}(-i\hat{\nabla}_1, \ldots, -i\hat{\nabla}_n) \phi^{A_1}(\hat{X}_1) \ldots \phi^{A_n}(\hat{X}_n) \right\} \hat{X}_1 \hat{X}_2 \ldots \hat{X}_n. \quad (52)$$

It is clear that NCFT with such vertices are much more complicated than type I nonlocal NCFT discussed above. We will call them type II nonlocal NCFT. As will be shown in Secs. VI and VII, QED and QCD in a strong magnetic field yield examples of such theories. For the case of nonzero longitudinal momentum $P_1$, the counterparts of expressions (49) and (52) are derived in the Appendix (see Eqs. (57) and (59) there).

V. QED WITH LARGE $N_f$ IN A STRONG MAGNETIC FIELD AND NCFT: DYNAMICAL REGIME WITH LOCAL INTERACTIONS

Let us now consider the dynamical regime with such large $N_f$ that the $1/N_f$ expansion is reliable, and the coupling $\alpha_b$ is so strong that $M_\gamma^2 = 2\alpha_b |eB|/\pi$ in (11) is of order $|eB|$ or larger. In this case, we have a NJL model with the current-current local interaction, in which the coupling constant $G = 4\pi\alpha_b/M_\gamma^2 = 2\pi^2/N_f|eB|$ and the ultraviolet cutoff $\Lambda_{||}$ connected with longitudinal momenta is $\Lambda_{||} = |eB|$ (see the last paper in Ref. [11]). The dynamical fermion mass is now given by expression (51) and the mass function $m(k^2)$ is a constant, $m(k^2) = m_{\text{dyn}}$. In this regime, Eq. (52) takes the form

$$F(p_1; P_\perp) = \frac{8\pi\alpha_b \lambda_L(P_\perp)}{M_\gamma^2} \int \frac{d^2 u \cdot d^2 k_{||}}{(2\pi)^4} \frac{e^{iP_\perp \cdot u_{||}} e^{-\frac{p_1^2}{2\pi m_{\text{dyn}}}}}{k_{||}^2 + m_{\text{dyn}}^2}, \quad (53)$$

where the function $F(p_1; P_\perp)$ is defined in Eq. (11). The subscript $L$ in $\lambda_L(P_\perp)$ reflects the consideration of the limit with local interactions here. Since the right-hand side of equation (53) does not depend on $p_1$, the function $F(p_1; P_\perp)$ is $p_1$-independent, $F(p_1; P_\perp) = F(P_\perp)$, i.e., this dynamics relates to the type I nonlocal NCFT considered in the previous section. Then, we immediately find from (53) that

$$\lambda_L(P_\perp) = \frac{\pi M_\gamma^2 e^{\frac{p_1^2}{2\pi m_{\text{dyn}}}}}{2N_f e^{\frac{p_1^2}{2\pi m_{\text{dyn}}}}} \frac{\alpha_b |eB|}{\ln \frac{|eB|}{m_{\text{dyn}}}}, \quad (54)$$

where we used $M_\gamma^2 = 2\alpha_b |eB|/\pi \gg m_{\text{dyn}}^2$. Using now the on-mass-shell condition $P_\perp \to 0$, $\lambda_L(P_\perp) \to 1$ for the NG bosons in equation (54), we arrive at the following gap equation for $m_{\text{dyn}}$:

$$\frac{2N_f}{\ln \frac{|eB|}{m_{\text{dyn}}}} = 1. \quad (55)$$

This gap equation yields expression (16) for the mass. It also implies that $\lambda_L$ (54) can be rewritten in a very simple form:

$$\lambda_L(P_\perp) = e^{\frac{p_1^2}{2\pi m_{\text{dyn}}}}, \quad (56)$$
The choice of the off-mass-shell operators $\phi^A$ in expansion (62) is not unique. They are determined by the choice of their propagator. If one chooses the conventional composite NG fields $\pi^A = (G/2)\bar{\psi}\gamma_5\lambda^A\psi$ as $\phi^A$, their propagator is

$$D^{AB}_\pi = \frac{8\pi^2 \delta_{AB}}{|eB| \ln \frac{|eB|}{m_{dyn}^2} (1 - e^{-\frac{P^2}{m_{dyn}^2}})} = \frac{8\pi^2 \delta_{AB}}{|eB| \ln \frac{|eB|}{m_{dyn}^2} (1 - \lambda^A_L(P_\perp))} = \frac{4\pi^2 \delta_{AB}}{N_f |eB| (1 - \lambda^A_L(P_\perp))}. \quad (57)$$

Up to the factor 2, this propagator coincides with that in the NJL model in a magnetic field. From propagator (57) and Eq. (35) we find the function $F(P_\perp)$:

$$F(P_\perp) = 2\lambda^A_L(P_\perp) = 2e^{-\frac{P^2}{m_{dyn}^2}}. \quad (58)$$

Interaction vertices (37) are nonzero only for even $n$ and they take now the form

$$\Gamma_n = -\frac{2|eB|(-1)^{\frac{n}{2}}}{n(n-2)m_{dyn}^2} \int d^2X || \int \frac{d^2 P_1^\perp}{(2\pi)^2} ... \frac{d^2 P_n^\perp}{(2\pi)^2} \delta^2(\sum_{i=1}^n \vec{P}_i^\perp) \times \text{tr} [\bar{\pi}(P_1^\perp) ... \bar{\pi}(P_n^\perp)] e^{-\frac{1}{2} \sum_{i\neq j} P_i^\perp \cdot P_j^\perp} \frac{\sum_{i<j}P_i^\perp \cdot P_j^\perp}{(2\pi)^2}, \quad \pi \equiv (\lambda^A/2)\pi^A. \quad (59)$$

These expressions for the vertices agree with those found in Ref. 3 (see footnote 6).

Apart from the exponentially damping factor $e^{-\frac{1}{2} \sum_{i\neq j} P_i^\perp \cdot P_j^\perp}$, the form of these vertices coincide with that in NCFT with noncommutative space transverse coordinates $X^\perp$ satisfying commutation relation (13). The appearance of the additional Gaussian (form-) factors in vertices reflects an inner structure of composites $\pi^A$ in the LLL dynamics. These form factors, reflecting the Landau wave functions on the LLL, are intimately connected with the holomorphic representation in the problem of a free fermion in a strong magnetic field [11, 12]. The short-range interactions between fermions in this dynamical regime do not change their Gaussian form. As was shown in Ref. 3, because of these form factors, the UV/IR mixing is absent in the model.

As we discussed in the previous section, in order to take properly into account these form factors, it is convenient to introduce new, smeared, fields:

$$\Pi(X) = e^{\frac{\nabla^2_\perp}{2eB}} \bar{\pi}(X), \quad (60)$$

where $\nabla^2_\perp$ is the transverse Laplacian. Then, in terms of the smeared fields, the vertices can be rewritten in the standard form with the Moyal exponent factor only:

$$\Gamma_n = -\frac{2|eB|(-1)^{\frac{n}{2}}}{n(n-2)m_{dyn}^2} \int d^2X || \int \frac{d^2 P_1^\perp}{(2\pi)^2} ... \frac{d^2 P_n^\perp}{(2\pi)^2} \delta^2(\sum_{i=1}^n \vec{P}_i^\perp) \times \text{tr} [\bar{\Pi}(P_1^\perp) ... \bar{\Pi}(P_n^\perp)] e^{-\frac{1}{2} \sum_{i\neq j} P_i^\perp \cdot P_j^\perp}. \quad (61)$$

But now the form factor occurs in the propagator of the smeared fields:

$$D^{AB}_\Pi(P_\perp) = e^{-\frac{\nabla^2_\perp}{2eB}} D^{AB}_\pi(P_\perp). \quad (62)$$

In this case, it is again the form factor $e^{-\frac{\nabla^2_\perp}{2eB}}$, now built in the propagator $D^{AB}_\Pi(P_\perp)$, that is responsible for the absence of the UV/IR mixing.

The extension of the present analysis to the case with nonzero longitudinal momenta $P_1$ is straightforward for this NJL-like dynamical regime (see the Appendix). The results coincide with those obtained in Ref. 3.

Therefore the dynamics in this regime relates to type I nonlocal NCFT. As was discussed in the previous section (see also Ref. 3), n-point vertex (61) can be rewritten in the coordinate space in the standard NCFT form with the star product. Moreover, as was shown in 3, in the space with noncommutative transverse coordinates, one can derive the effective action for the composite fields in this model. Thus, here we reproduced the results of Ref. 3 by using the method of bilocal auxiliary fields.

---

6 In 3, the conventional NJL model with the chiral group $U(1)_L \times U(1)_R$ and with the number of fermion colors $N_c$ was considered. Therefore, comparing these two models, one should take $N_c = 1$ and replace the flavor matrices $\lambda^A$ by 1. Since $\text{tr}(\lambda^A/2)^2 = 1/2$, there is an additional factor 2 in the propagator of $\pi^A$ in the present model.
VI. QED WITH WEAK COUPLING IN A STRONG MAGNETIC FIELD AND TYPE II NONLOCAL NCFT

In this section, we will consider the dynamics of QED in a magnetic field in a weak coupling regime, when the coupling $\alpha_b$ is small (the number of flavors $N_f$ can now be arbitrary). As will become clear in a moment, this dynamics yields an example of a type II nonlocal NCFT.

According to the analysis in Section II, the integral equation for $F(p_\parallel; P_\perp)$ in this regime is

$$ F(p_\parallel; P_\perp) = 8\pi\alpha_b\lambda_W(P_\perp) \int \frac{d^2u_\perp d^2k_\parallel F(k_\parallel; P_\perp)}{(2\pi)^4} \frac{e^{iP_\perp \times u_\perp} e^{-\frac{q_\perp^2}{2\gamma}}}{k_\parallel^2 + m_{dyn}^2 (k_\parallel - p_\parallel)^2 + \bar{u}_\perp^2 + M_\gamma^2}, $$

where $m_{dyn}$ is given in Eq. (48) and the subscript $W$ in $\lambda_W$ reflects the consideration of the weak coupling regime. Unlike the integral equation (48), the kernel of this equation does not have a separable form. Therefore, the function $F(p_\parallel; P_\perp)$ is not factorized in this case and the present dynamics relates to type II nonlocal NCFT. According to the analysis in Sec. IV, its n-point vertices can be written either through the star product in the form (49) in coordinate space or in the form (50) in the noncommutative space.

In order to illustrate the difference of this dynamics from that considered in the previous section, it will be instructive to analyze it in a special limit with $P_\perp^2 \gg |eB|$. We will that in this limit the approximation with $F(p_\parallel; P_\perp)$ being independent of $p_\parallel$ is quite good and, therefore, the dynamics in this limit can be considered as approximately relating to type I nonlocal NCFT. However, as will be shown below, the form factor in this dynamics is very different from the Gaussian form factor that occurs in the NJL-like dynamics. This point reflects a long-range character of the QED interactions.

We start the analysis of integral equation (63) for $P_\perp^2 \gg |eB|$ by considering the integral

$$ I = \int \frac{d^2u_\perp e^{iP_\perp \times u_\perp} e^{-\frac{q_\perp^2}{2\gamma}}}{(2\pi)^2} \frac{1}{q_\parallel^2 + u_\perp^2 + M_\gamma^2} = \int \frac{d^2u_\perp e^{i\Delta_\perp \cdot u_\perp} e^{-\frac{\Delta_\perp^2}{2\gamma}}}{(2\pi)^2} \frac{1}{q_\parallel^2 + \bar{u}_\perp^2 + M_\gamma^2}, $$

where $q_\parallel = k_\parallel - p_\parallel$, $\Delta_\perp = \frac{eB}{2\gamma}$ and, for convenience, we made the change of variable $u_\perp^1 \to -u_\perp^1$, $u_\perp^2 \to u_\perp^1$ in the last equality. By representing $e^{-\Delta_\perp^2/2|eB|}$ and $(q_\parallel^2 + \bar{u}_\perp^2 + M_\gamma^2)^{-1}$ through their Fourier transforms, we obtain

$$ I = \int d^2\Delta_\perp \frac{|eB| e^{-\frac{|eB|^2}{2\gamma^2}}}{2\pi} |K_0(\Delta_\perp^{-2} / \sqrt{q_\parallel^2 + M_\gamma^2})|, $$

where $K_0(z)$ is the Bessel function of imaginary argument. For $P_\perp^2 \gg |eB|$, when $|\Delta_\perp| \gg 1$, one can neglect the dependence of $K_0$ on $\Delta_\perp$. Then, using the asymptotics of $K_0(z)$ at $z \to +\infty$, we find that

$$ I \approx \frac{1}{2} \left( \frac{|eB|}{2\pi |P_\perp| (q_\parallel^2 + M_\gamma^2)^{1/2}} \right)^{1/2} e^{-\frac{|P_\perp| (q_\parallel^2 + M_\gamma^2)^{1/2}}{|eB|}}, $$

where $|P_\perp| \equiv \sqrt{P_\perp^2}$. Since this $I$ as a function of $q_\parallel$ exponentially decreases starting from $M_\gamma$, it is sufficient to take into account only the region with $q_\parallel \lesssim M_\gamma$ and approximate $I$ there by

$$ I \approx \frac{1}{2} \left( \frac{|eB|}{2\pi |P_\perp| M_\gamma} \right)^{1/2} e^{-\frac{|P_\perp| M_\gamma}{|eB|}}, $$

Thus, we conclude that for $P_\perp^2 \gg |eB|$, a good approximation for equation (63) is to integrate over $q_\parallel^2$ up to $M_\gamma^2$ and to use expression (67) for $I$. This implies that for large $P_\perp^2$ the function $F(p_\parallel; P_\perp)$ can be taken independent of $p_\parallel$. Then we find from Eq. (58) that

$$ \lambda_W(P_\perp) \simeq \frac{1}{\alpha_b \ln \frac{M_\gamma^2}{m_{dyn}}} \left( \frac{2\pi |P_\perp| M_\gamma}{|eB|} \right)^{1/2} e^{-\frac{|P_\perp| M_\gamma}{|eB|}}. $$

7 While in the dynamical regime with the LLL dominance longitudinal momenta should satisfy the inequality $P_\parallel \ll \sqrt{|eB|}$, transverse momenta can be large.
By choosing \( F(P_\perp) = 2\lambda_{\text{W}}^{1/2}(P_\perp) \), we obtain the following pion propagator from Eq. (68) (compare with Eq. (67)):

\[
D^{AB}(P_\perp) = \frac{8\pi^2\delta_{AB}}{|eB| \ln \frac{M_\perp^2}{m_\text{dyn}} (1 - \lambda_{\text{W}}^{-1}(P_\perp))}.
\]

(69)

And, using Eq. (67), we find the corresponding vertices:

\[
\Gamma_n = \frac{2\pi |eB|}{n} \int d^2x_1 \int d^2k_1 \frac{d^2P_1}{(2\pi)^2} \int d^2P_2 \frac{d^2P_2}{(2\pi)^2} \cdots \frac{d^2P_n}{(2\pi)^2} \delta^2(\sum_{i=1}^n \vec{k}_i) \\
\times \text{tr} \left[ S_{||}(k_1) \gamma^5 \bar{\pi}(P_1) \frac{1 - i\gamma^1\gamma^2}{2} \cdots S_{||}(k_n) \bar{\pi}(P_n) \right] \Pi_{i=1}^n \lambda_{\text{W}}^{-1/2}(P_1) e^{-\sum_i \vec{k}_i \times \vec{P}_i}.
\]

(70)

Now, let us compare expressions (68) and (69) with their counterparts (67) and (66) in the case of local interactions. While the behaviors of propagators (67) and (66) are similar for large \( P_\perp^2 \gg |eB| \) (they both approach a constant as \( P_\perp^2 \to \infty \)), the behaviors of vertices (66) and (70) are quite different. While the form factor in vertices (66) has the Gaussian form \( e^{-\frac{2\pi^2}{|eB|}} \), the form factor in vertices (70) is proportional to \( (P_1|M_\perp|/|eB|)^{-1/4} e^{-\frac{1}{2}(P_1^2|/|eB|)} \) and, therefore, decreases much slower for large \( P_\perp^2 \). The reason of that is a non-local character of the interactions in QED in a weak coupling limit. To see this more clearly, let us compare integral equations (65) and (66). The transition to the local interactions corresponds to the replacement of the propagator \( [(k_1 - p_1)^2 + \vec{u}_1 \cdot \vec{M}_1^2]^{-1} \) in Eq. (65) by \( M_\perp^{-2} \). This in turn leads to the replacement of the Bessel function \( K_0(\sqrt{\frac{k_1^2}{|eB|} + M_\perp^2}) \) in Eq. (66) by the delta function \( 2\pi / |k_1^2| \delta^2(\vec{k}_1 - \vec{\Delta}_1) \). The substitution of the delta function in \( I(65) \) leads to the Gaussian form factor, which, therefore, is a signature of short-range interactions.

VII. CHIRAL DYNAMICS IN QCD IN A MAGNETIC FIELD AND TYPE II NONLOCAL NCFT

In this section, we will show that the chiral dynamics in QCD in a strong magnetic field relates to type II nonlocal NCFT. Here under strong magnetic fields, we understand the fields satisfying \( |eB| \gg A_{QCD}^2 \), where \( A_{QCD} \) is the QCD confinement scale.

A crucial difference between the dynamics in QED and QCD in strong magnetic backgrounds is of course the property of asymptotic freedom and confinement in QCD. The infrared dynamics in quantum chromodynamics is much richer and more sophisticated. As was shown in Ref. 13, the confinement scale \( \lambda_{QCD}(B) \) in QCD in strong magnetic field can be much less than the confinement scale \( \lambda_{QCD} \) in the vacuum. As a result, an anisotropic dynamics of confinement is realized with a rich and unusual spectrum of very light glueballs.

On the other hand, the chiral dynamics in QED and QCD in strong magnetic backgrounds have a lot in common. If the magnetic field is so strong that the dynamical fermion mass \( m_\text{dyn} \) is much larger than the confinement scale \( \lambda_{QCD}(B) \), the running coupling \( \alpha_s \) is small for such momenta. As a result, the dynamics in that region is essentially Abelian. Indeed, while the contribution of (electrically neutral) gluons and ghosts in the polarization operator is proportional to \( k^2 \), the fermion contribution is proportional to \( |eB| \), similarly to the case of QED in a magnetic field (see Eq. (6) and Eq. VII.1 below). As a result, the fermion contribution dominates in the relevant region with \( |k^2| \ll |eB| \).

Because of the Abelian like structure of the dynamics in this problem, one can use the results of the analysis in QED in a magnetic field 10, by introducing appropriate modifications. One of the modifications is that the chiral symmetry in QCD in a magnetic field is different from that in QED. Indeed, since the background magnetic field breaks explicitly the global chiral symmetry that interchanges the up and down quark flavors (having different electric charges), the chiral symmetry in this problem is \( SU(N_u)_L \times SU(N_d)_R \times SU(N_d)_L \times SU(N_d)_R \times U^{(-)}(1)_A \), where \( N_u \) and \( N_d \) are the numbers of up and down quarks, respectively (the total number of quark flavors is \( N_f = N_u + N_d \)). The \( U^{(-)}(1)_A \) is connected with the current which is an anomaly free linear combination of the \( U^{(d)}(1)_A \) and \( U^{(u)}(1)_A \) currents. [The \( U^{(-)}(1)_A \) symmetry is of course absent if either \( N_d \) or \( N_u \) is equal to zero]. The generation of quark masses breaks this symmetry spontaneously down to \( SU(N_u)_V \times SU(N_d)_V \) and, as a result, \( N_u^2 + N_d^2 - 1 \) neutral NG bosons occur.

Another modification is connected with the presence of a new quantum number, the color. As was shown in Ref. 15, there exists a threshold value of the number of colors \( N_c^{\text{thr}} \) dividing the theories with essentially different
For the number of colors $N_c \ll N_c^{thr}$, an anisotropic dynamics of confinement with the confinement scale $\Lambda_{QCD}(B)$ much less than $\Lambda_{QCD}$ and a rich spectrum of light glueballs is realized. For $N_c$ of order $N_c^{thr}$ or larger, a conventional confinement dynamics with $\Lambda_{QCD}(B) \approx \Lambda_{QCD}$ takes place. The threshold value $N_c^{thr}$ grows rapidly with the magnetic field. For example, for $\Lambda_{QCD} = 250$ MeV and $N_c = 1$, $N_d = 2$, the threshold value is $N_c^{thr} \gtrsim 100$ for $|eB| \gtrsim (1 \text{ GeV})^2$. We will consider both the case with $N_c \ll N_c^{thr}$ and that with $N_c \gtrsim N_c^{thr}$.

For $N_c \ll N_c^{thr}$, the dynamical mass $m_{dyn}^{(q)}$ of a $q$-th quark is defined as

$$m_{dyn}^{(q)} \approx \sqrt{|e_qB| (c_q\alpha_s)^{1/3}} \exp \left[ -\frac{2N_c\pi}{\alpha_s(N_c^2 - 1)\ln(C_1/c_q\alpha_s)} \right]$$

(compare with expression (13) for the dynamical mass in QED). Here $e_q$ is the electric charge of the $q$-th quark, the numerical factors $C$ and $C_1$ are of order one and the constant $c_q$ is defined as

$$c_q = \frac{1}{6\pi(2N_u + N_d)} \left| \frac{e}{e_q} \right|.$$  

The strong coupling $\alpha_s$ in Eq. (71) is related to the scale $\sqrt{|eB|}$, i.e.,

$$\frac{1}{\alpha_s} \approx b \ln \frac{|eB|}{\Lambda_{QCD}^2}, \quad b = \frac{11N_c - 2N_f}{12\pi}.$$  

In QCD, there are two sets of NG bosons related to the $SU(N_u)_V$ and $SU(N_d)_V$ symmetries. Their BS wave functions are defined as in Eq. (20) with the superscript $A$ replaced by $A_u$ ($A_d$) for the set connected with the $SU(N_u)_V$ ($SU(N_d)_V$). The corresponding BS equations have the form of equation (30) with the coupling $\alpha_s$ replaced by the strong coupling $(N_u^2 - 1)\alpha_s/2N_c$, where $(N_u^2 - 1)/2N_c$ is the quadratic Casimir invariant in the fundamental representation of $SU(N_c)$.

Besides these NG bosons, there is one NG boson connected with the anomaly free $U^-(1)_A$ discussed above. Its BS wave function is defined as in Eqs. (20) and (27) but with the matrix $\lambda^A$ replaced by the traceless matrix $\lambda^0/2 \equiv (\sqrt{N_d/N_f}\lambda^0_u - \sqrt{N_u/N_f}\lambda^0_d)/2$ [13]. Here $\lambda^0_u$ and $\lambda^0_d$ are proportional to the unit matrices in the up and down flavor sectors, respectively. They are normalized as the $\lambda^A$ matrices: $\text{tr}[(\lambda^0_u)^2] = \text{tr}[(\lambda^0_d)^2] = 2$.

The polarization operator in the propagator of gluons has the form similar to that for photons [18]:

$$\Pi \approx -\frac{\alpha_s}{\pi} \sum_{q=1}^{N_f} \frac{|e_qB|}{k^2}.$$  

(compare with Eq. (3)). This expression implies that the gluon mass is

$$M_g^2 = \sum_{q=1}^{N_f} \frac{\alpha_s}{\pi} |e_qB| = (2N_u + N_d)\frac{\alpha_s}{3\pi} |eB|.$$  

It is clear that the chiral dynamics in this case is similar to that in QED in a weak coupling regime considered in Sec. 11 and it relates to the type II nonlocal NCFT. In the noncommutative space, the expressions for the vertices of each of the two sets of NG bosons have the form (32) (34), in the Appendix) for the case when their fields are independent of (depend on) longitudinal coordinates. Notice that in the leading approximation, the vertices do not mix NG fields from the different $SU(N_u)_V$ and $SU(N_d)_V$ sets: the mixing is suppressed by powers of the small coupling $\alpha_s$ (an analogue of the Zweig-Okubo rule). On the other hand, the NG boson related to the $U^-(1)_A$ interacts with NG bosons from both these sets without any suppression.

Let us now turn to the case with large $N_c$, in particular, to the 't Hooft limit $N_c \to \infty$. Just a look at expression (75) for the gluon mass is enough to recognize that the dynamics in this limit is very different from that considered above. Indeed, as is well known, the strong coupling constant $\alpha_s$ is proportional to $1/N_c$ in this limit. More precisely, it rescales as

$$\alpha_s = \frac{\tilde{\alpha}_s}{N_c},$$

where the new coupling constant $\tilde{\alpha}_s$ remains finite as $N_c \to \infty$. Then, expression (76) implies that the gluon mass goes to zero in this limit. This in turn implies that the appropriate approximation is now not the improved rainbow (ladder) approximation but the rainbow (ladder) approximation itself, when the gluon propagator in the gap
(Schwinger-Dyson) equation and in the BS equation is taken to be bare with $\Pi = 0$. In other words, gluons are massless and genuine long range interactions take place in this regime. The dynamical mass of quarks now is

$$m_{dyn} = C \sqrt{|e_B|} \exp \left[ -\pi \left( \frac{\pi}{4 \alpha_s} \right)^{1/2} \right],$$

where the constant $C$ is of order one. As was shown in Ref. [18], this expression is a good approximation for the quark mass when $N_c$ is of order $N_c^{thr}$ or larger. As to the BS equation, repeating the analysis of Sec. [17] it is easy to show that, unlike the QED case, the amplitude $F(p, p_1)$, defined in Sec. [17] is not factorized in this dynamical regime even for large transverse momenta $P_T^2 \gg |e_B|$. Therefore, the dynamics with large $N_c$ yields even a more striking example of the type II nonlocal NCFT than the previous dynamical regime with $N_c \ll N_c^{thr}$.

VIII. MORE ABOUT THE CONNECTION BETWEEN FIELD THEORIES IN A MAGNETIC FIELD AND NCFT

What are the origins of the connection between field theories in a magnetic field and NCFT? As was emphasized in Ref. [14] and in Sec. [13] of this paper, it is the Schwinger phase in the LLL fermion propagator and BS wave functions of neutral composites that leads to the Moyal factor (a signature of NCFT) in interaction vertices of neutral composites. But what is the origin of the Schwinger phase itself? Let us show that it reflects the existence of the group of magnetic translations in an external magnetic field [13,20]. The generators of this group in the symmetric gauge, used in this paper, are:

$$\hat{P}_1 = \frac{1}{i} \frac{\partial}{\partial x_1} - \frac{\hat{Q}}{2} B x_2, \quad \hat{P}_2 = \frac{1}{i} \frac{\partial}{\partial x_2} + \frac{\hat{Q}}{2} B x_1, \quad \hat{P}_3 = \frac{1}{i} \frac{\partial}{\partial x_3},$$

(78)

where $\hat{Q}$ is the charge operator. The commutators of these generators are:

$$[\hat{P}_1, \hat{P}_2] = \frac{1}{i} \hat{Q} B, \quad [\hat{P}_1, \hat{P}_3] = [\hat{P}_2, \hat{P}_3] = 0.$$

(79)

Therefore all the commutators equal zero for neutral states, and the momentum $\tilde{P} = (P_1, P_2, P_3)$ of their center of mass is a good quantum number.

It is easy to check that the structure of the generators (78) implies the presence of the Schwinger phase in the matrix elements of the time ordered bilocal operator $T(\psi(x)\bar{\psi}(y))$ taken between two arbitrary neutral states, $|a; P_a >$ and $|b; P_b >$. More precisely,

$$M_{ba}(x, y; P_b, P_a) = < P_b; b|T\psi(x)\bar{\psi}(y)|a; P_a > = e^{-i(P_a - P_b)X} e^{i\epsilon r \nu A_{\mu}^{ext}(X)} \tilde{M}_{ba}(r; P_b, P_a),$$

(80)

where $r = x - y, \ X = \frac{r}{\sqrt{2}}$ (compare with Eq. [20]). The second exponent factor on the right hand side of this equation is the Schwinger phase. Taking $|a; P_a >$ and $|b; P_b >$ to be the vacuum state $|0 >$, we get the fermion propagator. And taking $|b; P_b >= |0 >$ and $|a; P_a >$ to be a state of some neutral composite, we get the BS wave function of this composite. Thus, the group of magnetic translations is in the heart of the connection between the field dynamics in a magnetic field and NCFT. In particular, one of the consequences of this consideration is that although the treatment of neutral composites other than NG bosons is much more involved, one can be sure that, in a strong magnetic field, the noncommutative structure of interaction vertices which include those composites is similar to that of the vertices for NG bosons that was derived in Secs. [14,15] and in the Appendix.

Now, what is the form of interaction vertices which include such elementary electrically neutral fields as photon and gluon fields in QED and QCD, respectively. This problem in QED has been recently considered in Ref. [27]. As was shown there, in the LLL approximation, fermion loops infect a noncommutative structure for $n$-point photon vertices and, as a result, the Moyal factor occurs there. The situation in QCD is more subtle. In that case, besides induced gluon vertices generated by quark loops, there are also triple and quartic gluon vertices in the initial QCD action. There is of course no Moyal factor in those vertices. Still, since the Abelian approximation is reliable in the description of the chiral dynamics in a strong magnetic field in QCD (see the discussion in the previous section), the situation for this dynamics in QCD is similar to that in QED. However, the description of the confinement dynamics in QCD, relating to the deep infrared region, is much more involved, although the influence of the magnetic field on that dynamics is quite essential [18].

The last point we want to address in this section is the reliability of the LLL approximation in a strong magnetic field. As to the chiral dynamics, there are solid arguments that it is a reliable approximation for it [14,15,16]. In particular,
its reliability was shown explicitly in the NJL model in a magnetic field in the leading order in $1/N_c$ expansion \[15\]. On the other hand, as has been recently shown in Ref. \[27\], the cumulative effect of higher Landau levels can be important for n-point photon vertices, at least in some kinematic regions (a nondecoupling phenomenon). It is however noticeable that in the kinematic region with momenta $k^2_i \gg |k_{ij}^2|$, which provides the dominant contribution in the chiral dynamics \[16\], the LLL contribution is dominant \[27\]. Thus, although this question deserves further study, the assumption about the LLL dominance in the chiral dynamics in relativistic field theories seems to be well justified.

IX. CONCLUSION

The main result of this paper is that the chiral dynamics in QED and QCD in a strong magnetic field determine complicated nonlocal NCFT (type II nonlocal NCFT). These NCFT are quite different both from the NCFT considered in the literature and the NCFT corresponding to the NJL model in a magnetic field (type I nonlocal NCFT) \[9\]. While in type I NCFT there exists a field transformation that puts interaction vertices in the conventional form (with a cost of introducing an exponentially damping form factor in field propagators), no such a transformation exists for type II nonlocal NCFT.

The reason of this distinction between the two types of models is in the characters of their interactions, being short-range in the NJL-like models and long-range in gauge theories. While the influence of the short-range interactions on the LLL dynamics is quite minor, the long-range interactions change essentially that dynamics. As a result, the structure of neutral composites and manifestations of nonlocality in gauge theories in strong magnetic backgrounds are much richer.

We believe that both these types of nonlocal NCFT can be relevant not only for relativistic field theories but also for nonrelativistic systems in a magnetic field. In particular, while type I NCFT can be relevant for the description of the quantum Hall effect in condensed matter systems with short-range interactions \[2, 7, 12\], type II NCFT can be relevant in studies of this effect in condensed matter systems with long-range interactions. Concrete examples of such systems are provided by carbon materials, in particular, by highly oriented pyrolytic graphite (HOPG), where Coulomb-like interactions take place \[28, 29, 30\]. The effective theory of this system is QED$_{2+1}$. Recent experiments in HOPG in strong magnetic fields \[31\] suggest the existence of the quantum Hall effect in this system. It would be interesting to clarify the dynamics of this effect in the framework of the QED$_{2+1}$ effective theory.

As is well known, dynamics and symmetry structures of field theories in two spatial dimensions can significantly differ from those in three spatial dimensions. These differences can be especially important in gauge theories. The reason of that is a possibility of generating some unique 2+1-dimensional terms, like the Chern-Simons term, in effective actions of gauge theories. This makes the studies of gauge theories in a magnetic field in 2+1 dimensions to be interesting not only for applications in condensed matter physics but also from the theoretical point of view. We are planning to study 2+1 dimensional gauge theories in a magnetic field by using the NCFT approach elsewhere. [In the NCFT approach, some aspects of dynamics in the 2+1 dimensional NJL model in a magnetic field were considered in Ref. \[9\]].

Another potentially interesting problem would be the examination of the existence of type II nonlocal NCFT in string theories in magnetic backgrounds with broken supersymmetries.

Acknowledgments

We thank Michio Hashimoto and Igor Shovkovy for useful remarks. The work was supported by the Natural Sciences and Engineering Research Council of Canada.

Appendix

In the main body of the paper we restricted our analysis to the case when fields $\phi^A(X)$ depend only on the transverse coordinates $X_\perp$. In this Appendix, we consider the general case of fields $\phi^A(X)$ depending on both transverse and longitudinal coordinates.

Instead expression \[32\], now we have the following representation for the bilocal field $\tilde{\varphi}(x, y)$:

$$\tilde{\varphi}(x, y) = \int \frac{d_4 P}{(2\pi)^3} \phi^A(P) \chi^A(x, y; P), \quad (81)$$
where the structure of the BS wave function $\chi^A(x, y; P)$ is described in Eqs. (26), (29) and (31). While for the case $P_\parallel = 0$ the effective action was given in expression (36), it now takes the form

$$S(\check{\varphi}) = \sum_{n=2}^{\infty} \frac{i}{n} \int d^4x_1 d^4y_1 ... d^4x_n d^4y_n \int \frac{d^4P_1 ... d^4P_n}{(2\pi)^{4n}} \phi^A_1(P_1) ... \phi^A_n(P_n)$$

$$\times \text{tr} \left[ S_{LLL}^{-1}(x_1, y_1) \chi^A_1(y_1, x_2; P_1) ... S_{LLL}^{-1}(x_{n-1}, y_{n-1}) \chi^A_{n-1}(y_{n-1}, x_n; P_n) \right]$$

$$\Pi_{i=1}^n \lambda(P_i)$$

Then using (26) and (29) and integrating over $P_\parallel$, we get

$$\chi^A(x, y; P) = P(x, y) \int \frac{d^2P_\parallel}{(2\pi)^2} e^{-iP_\parallel \cdot (x_\parallel - y_\parallel)} e^{-\mu P_\parallel} e^{i\mu P_\parallel} \text{sign}(|\mu|)$$

$$\times S_\parallel\chi^A_1(y_1, x_2; P_1) S_\parallel\chi^A_2(y_2, x_3; P_2) S_\parallel\chi^A_3(y_3, x_4; P_3)$$

(82)

It is convenient to represent $f^A(p_\parallel; P)$, defined in Eq. (31), as

$$f^A(p_\parallel; P) = S_\parallel(p_\parallel + \frac{P_\parallel}{2}) G^A(p_\parallel; P) S_\parallel(p_\parallel - \frac{P_\parallel}{2}).$$

(83)

[Note that a $\gamma$-matrix structure of $G^A(p_\parallel; P)$ is determined from the corresponding BS equation and it can be different from that in Eq. (31).] Then using (26) and (29) and integrating over $p_\parallel$, we get

$$\chi^A(x, y; P) = P(x, y) \int \frac{d^2P_\parallel}{(2\pi)^2} e^{-iP_\parallel \cdot (x_\parallel - y_\parallel)} e^{-\mu P_\parallel} e^{i\mu P_\parallel} \text{sign}(|\mu|)$$

$$\times S_\parallel(p_\parallel + \frac{P_\parallel}{2}) G^A(p_\parallel; P) S_\parallel(p_\parallel - \frac{P_\parallel}{2})$$

(84)

(compare with Eq. (55)). Substituting $\chi^A(x, y; P)$ in Eq. (22), we obtain the effective action in momentum space:

$$S(\varphi) = \sum_{n=2}^{\infty} \Gamma_n,$$

where the interaction vertices $\Gamma_n$, $n > 2$, are

$$\Gamma_n = \frac{2^{3-n} \pi \sqrt{|eB|}}{n} \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d^4P_1}{(2\pi)^4} ... \frac{d^4P_n}{(2\pi)^4} \delta^4 \left( \sum_{i=1}^n P_i \right) \phi^A_1(P_1) ... \phi^A_n(P_n)$$

$$\times \text{tr} \left[ G^A(k_\parallel - \frac{P_\parallel}{2}; P_1) S_\parallel(k_\parallel - P_\parallel) G^A(k_\parallel - \frac{P_\parallel}{2}; P_2) S_\parallel(k_\parallel - P_\parallel) \right]$$

$$... G^A(k_\parallel - \sum_{i=1}^{n-1} \frac{P_\parallel}{2}; P_n) S_\parallel(k_\parallel - \sum_{i=1}^n P_\parallel) \right] e^{-\frac{1}{2} \sum_{i<j} \frac{P_\parallel}{2} \times P_\parallel}$$

(85)

(compare with expression (57), and the quadratic part of the action is

$$\Gamma_2 = -\frac{\pi |eB|}{4} \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d^4P}{(2\pi)^4} \lambda(P) \phi^A_1(P)$$

$$\times \text{tr} \left[ G^A(k_\parallel - \frac{P_\parallel}{2}; P) S_\parallel(k_\parallel - P_\parallel) G^A(k_\parallel - \frac{P_\parallel}{2}; -P) S_\parallel(k_\parallel) \right] \phi^A_2(-P)$$

(86)

(compare with expression (58)).
Further, in coordinate space, the interaction vertices take the form similar to that in Eq. (19):

\[ \Gamma_n = \frac{i|eB|}{2^{n+1} \pi n} \int d^4 X \left[ \sum_{n=1}^{\infty} \left( -i \nabla_1, ..., -i \nabla_n \right) \phi^{A_1}(X_1) \ast ... \ast \phi^{A_n}(X_n) \right] |X_1=\ldots=X_n=0, \]

where the nonlocal operator \( V^{A_1\ldots A_n}_1 \ldots n \) now depends both on transverse \( \nabla_\perp \) and longitudinal \( \nabla_\parallel \). In momentum space, this operator is

\[ V^{A_1\ldots A_n}_1\ldots n(P_1, ..., P_n) = \int \frac{d^2 k||}{(2\pi)^2} \text{Tr} \left[ G^{A_1}(k_1) - \frac{P_1||}{2}; P_1 \right] S||_1(k|| - P_1||) G^{A_2}(k|| - P_1|| - \frac{P_2||}{2}; P_2) S||_2(k|| - P_1|| - P_2||) \]

\[ \ldots G^{A_n}(k|| - \sum_{i=1}^{n-1} P_i|| - \frac{P_n||}{2}; P_n) S||_n(k|| - \sum_{i=1}^{n} P_i||) \frac{1}{\Pi_{i=1}^{n} \lambda(P_i)} \]

(compare with Eq. (50)). Finally, the interaction vertices are given by the following expression in noncommutative space:

\[ \Gamma_n = \frac{i|eB|}{2^{n+1} \pi n} \int d^2 x_\parallel \text{Tr} \left[ \left\{ V^{A_1\ldots A_n}_1\ldots n(-i \nabla_\parallel, -i \nabla_\perp_1, ..., -i \nabla_\perp_n) \phi^{A_1}(X_1^||, \vec{X}_1^\perp) \right\} \chi^{A_n}(x_\parallel, \vec{X}_n^\perp) \right] \]

\[ \ldots \phi^{A_n}(x_\parallel, \vec{X}_n^\perp) \} \chi^{A_1}(x_\perp) \equiv \hat{\chi}^{A}(p; P), \]

where here the subscript \( i \) runs from 1 to \( n \) (compare with expression (92)).

What is the connection between the forms of the function \( f^A(p||; P) \) in the cases with zero and nonzero longitudinal momentum \( P|| \)? In regard to this question, it is appropriate to recall the Pagels-Stokar (PS) approximation [32] for a BS wave function \( \chi^A(p; P) \), which is often used in Lorentz invariant field theories. It is assumed in this approximation that the amputated BS wave function, defined as

\[ \tilde{\chi}^A(p; P) \equiv S^{-1}(p + \frac{P}{2}) \chi^A(p; P) S^{-1}(p - \frac{P}{2}), \]

is approximately the same for the cases with zero and nonzero \( P || \), i.e., \( \tilde{\chi}^A(p; P) \simeq \hat{\chi}^A(p) \) where \( \hat{\chi}^A(p) \equiv \chi^A(p; P)|_{P=0} \).

Then, in this approximation, the BS wave function \( \chi^A(p; P) \) is

\[ \chi^A(p; P) = S(p + \frac{P}{2}) \tilde{\chi}^A(p) S(p - \frac{P}{2}), \]

i.e., the whole dependence of the BS wave function on the momentum \( P \) comes from the fermion propagator. It is known (for a review, see Ref. [33]) that the PS approximation can be justified both for weak coupling dynamics and, in the case of the NJL model, in the regime with large \( N_c \).

Here we would like to suggest an anisotropic version of the PS approximation for dynamics in a magnetic field. The main assumption we make is that the function \( G^A(p||; P) \), defined in Eq. (39), is approximately \( P|| \) independent, i.e.,

\[ G^A(p||; P) \simeq F^A(p||; P) \gamma^5 \frac{1 - i\gamma^5 \gamma^2}{2}, \]

where the function \( F^A(p||; P) \) is defined in Eq. (33). Then, the whole dependence of the function \( f^A(p||; P) \) on the longitudinal momentum \( P|| \) comes from the fermion propagator:

\[ f^A(p||; P) = S||_1(p|| + \frac{P}{2}) F^A(p||; P) \gamma^5 \frac{1 - i\gamma^5 \gamma^2}{2} S||_1(p|| - \frac{P}{2}), \]

(compare with Eq. (33)).

We utilized expression (92) in QED in a magnetic field in the dynamical regime with the local interaction considered in Sec. V. In that case, the function \( F^A(p||; P) \) is a constant. Then, using expression (33) for \( \Gamma_n \) with \( G^A \) in Eq. (92), it is not difficult to derive the n-point vertices for fields \( \phi^A(P) \) for a general momentum \( P \). The result coincides with that obtained in Ref. [4] in the NJL model.
Expressions (12) and (13) can also be useful for the analysis of the dynamics in QED and QCD in a magnetic field in the weak coupling regime. As was shown in Secs. VI and VII, in those cases the function $F^\perp(p_\perp; P_\perp)$ depends on both momenta $p_\perp$ and $P_\perp$, (the dynamics in this regime relate to type II NCFT). The determination of this dependence is a nontrivial problem and is outside the scope of this paper.

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