We show that a virtual graviton has a $J = 0$ component, which serves to cancel the $J = 2, J_z = 0$ component when the graviton is on shell. In contrast, a massive graviton has no $J = 0$ component either on or off shell. This difference is responsible for the van Dam-Veltman-Zakharov discontinuity.

I. INTRODUCTION

It was recently noticed, in the consideration of $2 \rightarrow 2$ scattering amplitudes via graviton exchange, that a virtual graviton has a component with zero angular momentum ($J = 0$) [1, 2, 3]. While this is forbidden for a real graviton, which has $J = 2, J_z = \pm 2$, a $J = 0$ component of a virtual graviton does not violate any fundamental principle. Nevertheless, it is perhaps surprising, given that a virtual photon does not have a $J = 0$ component. Furthermore, a massive graviton does not have a $J = 0$ component either on or off shell [1], so it seems odd that a virtual massless graviton has a $J = 0$ component.

The fact that the graviton propagator has a $J = 0$ component was already shown in Ref. [4]. In this note we decompose the graviton propagator in terms of polarization tensors and explicitly display the $J = 0$ component. We then show that no such component is present in the massive graviton propagator. This is the reason for the van Dam-Veltman-Zakharov discontinuity between the massive and massless graviton propagators [5, 6].

We begin in Section II with a brief reminder about virtual photons. Virtual gravitons are dealt with in Section III, followed by massive gravitons in Section IV. We conclude with a discussion of a recent claim that angular momentum conservation is violated in quantum gravity [1, 2].

II. VIRTUAL PHOTONS

We begin by considering a virtual photon of four-momentum $q^\mu = (\omega, 0, 0, \kappa)$. The photon propagator in $R_\xi$ gauge is

$$D^{\mu\nu} = \left(-g^{\mu\nu} + (1 - \xi) \frac{q^\mu q^\nu}{q^2}\right) \frac{i}{q^2 + i\epsilon}. \quad (1)$$

The propagator connects two currents, both of which are conserved. The numerator of the amplitude is proportional to

$$J_{A\mu}D^{\mu\nu}J_{B\nu} \sim -J_A^0 J_B^0 + J_A^1 J_B^1 + J_A^2 J_B^2 + J_A^3 J_B^3 \quad (2)$$

where we have used current conservation, $q^\mu J_\mu = 0$, to eliminate the gauge-dependent part of the propagator. Alternatively, one could simply work in ‘t Hooft-Feynman gauge ($\xi = 1$).

Current conservation implies

$$\omega J^0 = \kappa J^3 \quad (3)$$

for both $J_A$ and $J_B$. For a real photon ($\omega = \kappa$), this gives $J^0 = J^3$, and Eq. (2) reduces to

$$J_{A\mu}D^{\mu\nu}J_{B\nu} \sim J_A^1 J_B^1 + J_A^2 J_B^2 \quad (4)$$
which shows that a real photon has only transverse polarizations. A virtual photon, however, also has a longitudinal component. The transverse and longitudinal polarization vectors are

\[
\epsilon_+^\mu = \frac{1}{\sqrt{2}}(0, -1, -i, 0) \\
\epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \\
\epsilon_0^\mu = \frac{1}{\sqrt{q^2}}(\kappa, 0, 0, \omega)
\]

which correspond to \( J = 1, J_z = +1, -1, 0 \), respectively. Through the use of current conservation, Eq. (3), Eq. (2) may be written in terms of these polarization vectors as

\[
J_{A \mu} D_{\mu \nu} J_{B \nu} \sim J_{A \mu} \sum_{\lambda = -1}^{1} \epsilon_\lambda^\mu \epsilon_\lambda^\nu J_{B \nu}
\]

where the sum runs over the three \( J = 1 \) polarization vectors given above [13]. In particular, there is no \( J = 0 \) component.

### III. VIRTUAL GRAVITONS

We now perform a similar analysis for the graviton propagator [7, 8],

\[
D^{\mu \nu \rho \sigma} = \frac{1}{2} \left( g^{\mu \rho} g^{\nu \sigma} + g^{\mu \sigma} g^{\nu \rho} - g^{\mu \nu} g^{\rho \sigma} \right) \frac{i}{q^2 + i\epsilon}.
\]

where we have specialized to harmonic (De Donder) gauge. The graviton propagator connects two conserved energy-momentum tensors,

\[
T_{A \mu \nu} T_{B \rho \sigma} \sim \frac{1}{2} \left( T_0^0 + T_1^1 + T_2^2 + T_3^3 \right)
\]

where we display only the numerator on the right-hand side. The symmetry \( T^{\mu \nu} = T^{\nu \mu} \) has been used to obtain this expression. In a more general gauge, terms proportional to the graviton four-momentum may be neglected due to conservation of the energy-momentum tensor, \( q^{\mu} T_{\mu \nu} = 0 \).

Conservation of the energy-momentum tensor implies

\[
\omega T^{0 \nu} = \kappa T^{3 \nu}.
\]

For a real graviton (\( \omega = \kappa \)), this gives \( T^{0 \nu} = T^{3 \nu} \), and Eq. (7) reduces to

\[
T_{A \mu \nu} D^{\mu \nu \rho \sigma} T_{B \rho \sigma} \sim \frac{1}{2} \left[ (T_1^1 - T_2^2)(T_1^1 - T_2^2) \right] + 2T_1^1 T_1^2
\]
which shows that a real graviton has only transverse components. A virtual graviton, however, has additional components. To identify them, we first rewrite Eq. (7) as

\[ T_{A \mu \nu} D^{\mu \nu \rho \sigma} T_{B \rho \sigma} \sim \frac{1}{2} [ (T_A^{11} - T_A^{22})(T_B^{11} - T_B^{22}) ] + 2T_A^{12}T_B^{12} + 2[ T_A^{13}T_B^{13} + T_A^{23}T_B^{23} - T_A^{31}T_B^{01} - T_A^{01}T_B^{31} ] + \frac{1}{6} [2(T_A^{00} - T_A^{33}) + T_A^{11} + T_A^{22}] [2(T_B^{00} - T_B^{33}) + T_B^{11} + T_B^{22}] - \frac{1}{6} [-T_A^{00} + T_A^{33} + T_A^{11} + T_A^{22}] [-T_B^{00} + T_B^{33} + T_B^{11} + T_B^{22}]. \] (10)

where current conservation, Eq. (8), has been used. We now proceed to identify each line above with a particular polarization.

An easy way to construct \( J = 2 \) polarization tensors is to take products of the \( J = 1 \) polarization vectors given in the previous section. The \( J = 2, J_z = \pm 2 \) polarization tensors are given by

\[ \epsilon_{2,\pm 2}^{\mu \nu} = \epsilon_{\pm 1}^{\mu} \epsilon_{\pm 1}^{\nu} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (11)

Using raising and lowering operators or, equivalently, Clebsch-Gordan coefficients \( [1] \), we derive the \( J = 2, J_z = \pm 1, 0 \) polarization tensors

\[ \epsilon_{2, \pm 1}^{\mu \nu} = \frac{1}{\sqrt{2}}(\epsilon_{\pm 3}^{\mu} \epsilon_{\mp 3}^{\nu} + \epsilon_{3}^{\mu} \epsilon_{-3}^{\nu}) = \frac{1}{2 \sqrt{q^2}} \begin{pmatrix} 0 & \mp \kappa & -i \kappa & 0 \\ \pm \kappa & 0 & 0 & \mp \omega \\ -i \kappa & 0 & 0 & -i \omega \\ 0 & \mp \omega & -i \omega & 0 \end{pmatrix}. \] (12)

\[ \epsilon_{2, 0}^{\mu \nu} = \frac{1}{\sqrt{6}}(\epsilon_{+}^{\mu} \epsilon_{-}^{\nu} + \epsilon_{-}^{\mu} \epsilon_{+}^{\nu} + 2\epsilon_{3}^{\mu} \epsilon_{3}^{\nu}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \frac{\kappa^2}{q^2} & 0 & 0 & 2 \frac{\imath \kappa \omega}{q^2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{2 \kappa \omega}{q^2} & 0 & 0 & 2 \frac{\omega^2}{q^2} \end{pmatrix}, \] (13)

and also the \( J = 0 \) polarization tensor

\[ \epsilon_{0, 0}^{\mu \nu} = \frac{1}{\sqrt{3}}(\epsilon_{+}^{\mu} \epsilon_{-}^{\nu} - \epsilon_{-}^{\mu} \epsilon_{+}^{\nu} - \epsilon_{3}^{\mu} \epsilon_{3}^{\nu}) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\frac{\kappa^2}{q^2} & 0 & 0 & -\frac{\kappa \omega}{q^2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{\kappa \omega}{q^2} & 0 & 0 & -\frac{\omega^2}{q^2} \end{pmatrix}. \] (14)

Using these polarization tensors, we can write Eq. (10) as

\[ T_{A \mu \nu} D^{\mu \nu \rho \sigma} T_{B \rho \sigma} \sim T_{A \mu \nu} \left( \sum_{\lambda = -2}^{2} \epsilon_{2, \lambda}^{\mu} \epsilon_{2, \lambda}^{\nu} - \frac{1}{2} \epsilon_{0, 0}^{\mu} \epsilon_{0, 0}^{\nu} \right) T_{B \rho \sigma}. \] (15)

The first line of Eq. (10) [which is identical to Eq. (9)] corresponds to the \( J = 2, J_z = \pm 2 \) contribution, which is present both on and off shell. Current conservation, Eq. (8), must be used to make this correspondence, as well as the correspondences below. The second line corresponds to the \( J = 2, J_z = \pm 1 \) contribution. It is evident that this vanishes on shell, where \( T_0^{\mu} = T_{3 \mu} \). The third and fourth lines correspond to the \( J = 2, J_z = 0 \) and \( J = 0 \) contributions, respectively. Although neither of these vanish on shell, it is evident that they cancel on shell (\( T_{00} = T_{33} \)). Thus the \( J = 0 \) component is necessary in order to ensure that a real graviton has no \( J = 2, J_z = 0 \) component. It is evident from Eq. (10) that the \( J = 0 \) component couples to the trace of the energy-momentum tensor.
The coefficient of the \( J = 0 \) contribution to Eq. (15) is unconventional in both its magnitude and its sign, but since this contribution does not contribute on shell, it does not violate any fundamental principles. In particular, the negative sign does not indicate a ghost, because there is no pole associated with the \( J = 0 \) contribution. Thus we conclude that a \( J = 0 \) component of the graviton propagator is present, and that it serves the purpose of canceling the \( J = 2, J_z = 0 \) contribution on shell [4].

IV. MASSIVE GRAVITONS

The propagator of a massive graviton of mass \( M \) is [2]

\[
D_M^{\mu\nu\rho\sigma} = \frac{1}{2} \left( g^{\mu\rho} \bar{g}^{\nu\sigma} + g^{\mu\sigma} \bar{g}^{\nu\rho} - \frac{2}{3} \bar{g}^{\mu\nu} g^{\rho\sigma} \right) \frac{i}{q^2 - M^2 + i\epsilon}
\]

where

\[
\bar{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^2}{M^2} g^{\mu\nu}.
\]

When sandwiched between two conserved energy-momentum tensors, terms proportional to the graviton four-momentum vanish, and \( \bar{g}^{\mu\nu} \) may be replaced by \( g^{\mu\nu} \). The numerator of the massive graviton propagator is then identical to that of Eq. (15) for a massless graviton, with the exception of the coefficient of the last term. This gives rise to the van Dam-Veltman-Zakharov discontinuity [5, 6, 10, 11, 12].

The numerator of the massive graviton propagator may be written in terms of massive \( J = 2 \) polarization tensors as

\[
D_M^{\mu\nu\rho\sigma} \sim \sum_{\lambda = -2}^{2} \varepsilon_{\lambda}^{\mu\nu} \varepsilon_{\lambda}^{\rho\sigma}
\]

where the massive polarization tensors are identical to those of the massless graviton, Eqs. (11) – (13), but with \( q^2 \) replaced by \( M^2 \) throughout. We see that, unlike the massless graviton propagator, Eq. (15), there is no \( J = 0 \) contribution. In the massless case, the \( J = 0 \) contribution was necessary to cancel the \( J = 2, J_z = 0 \) contribution on shell. In the massive case no such cancellation is necessary, as an on-shell massive graviton has a \( J = 2, J_z = 0 \) component.

V. DISCUSSION

Recently a pair of papers have appeared that argue that the presence of a \( J = 0 \) contribution in the graviton propagator implies a violation of angular momentum conservation [1, 2]. In the latter paper, it is argued that although a partial-wave analysis reveals a \( J = 0 \) component in certain \( 2 \rightarrow 2 \) amplitudes, an explicit decomposition of that amplitude in terms of polarization tensors shows that only \( J = 2, J_z = \pm 2 \) components are present [12]. However, while the partial-wave analysis refers to the angular momentum along the collision axis, \( z \), the decomposition is performed in terms of polarization tensors along an axis orthogonal to the scattering plane (\( J_3 \neq J_z \)). We will show that when the amplitude is decomposed along the \( z \) axis, a \( J = 0 \) component is indeed present, consistent with the partial-wave analysis. Thus there is no contradiction, and hence no evidence for violation of angular momentum conservation.

The amplitudes studied in Ref. [2] involve same-helicity massive fermion-antifermion or massive vector bosons. It is shown that in both cases the energy-momentum tensor with a virtual graviton, in the center-of-momentum frame, is proportional to

\[
\hat{T}^{ij} [f_\pm, \bar{f}_\pm] \sim \hat{T}^{ij} [V_\pm, V_\pm] \sim k^i k^j
\]
where \( k \) is the three-momentum of one of the external particles. If we decompose the angular momentum along the collision axis, \( z \), one obtains

\[
k'k' = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} = \frac{1}{\sqrt{3}} (\sqrt{2} \epsilon_{ij}^2 - \epsilon_{ij}^0)
\]  

(20)

where \( \epsilon_{i,j} \) are the spatial components of the polarization tensors of Eqs. [13] and [14] in the center-of-momentum frame (\( \kappa = 0 \)). This agrees with the partial-wave analysis of Ref. [2], which reveals a linear combination of \( J = 2, J_z = 0 \) and \( J = 0 \) contributions.

In this paper we have shown that a virtual graviton has a \( J = 0 \) component, and that it serves the purpose of canceling the \( J = 2, J_z = 0 \) component when the graviton is on shell. In contrast, a massive graviton has no \( J = 0 \) component either on or off shell. This difference gives rise to the van Dam-Veltman-Zakharov discontinuity.

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[13] In this expression, one should treat \( \sqrt{q^2} \) formally as a real number, even when the photon is spacelike (\( q^2 < 0 \)).
[14] See the previous footnote.
[15] Actually, \( J = 2, J_3 = 0 \) and \( J = 0 \) components are also present, but happen to cancel for these particular amplitudes.