On the Existence of B-Root Subgroups
on Affine Spherical Varieties

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Abstract—Let X be an irreducible affine algebraic variety that is spherical with respect to an action of a connected reductive group G. In this paper, we provide sufficient conditions, formulated in terms of weight combinatorics, for the existence of one-parameter additive actions on X normalized by a Borel subgroup B ⊂ G. As an application, we prove that every G-stable prime divisor in X can be connected with an open G-orbit by means of a suitable B-normalized one-parameter additive action.

Keywords: additive group action, toric variety, spherical variety, Demazure root, locally nilpotent derivation, local structure theorem

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1. Let X be an irreducible algebraic variety over an algebraically closed field \( \mathbb{K} \) of characteristic zero equipped with an action of a connected reductive algebraic group G. Every nontrivial regular action on X of the additive group \( \mathbb{G}_a = (\mathbb{K}, +) \) induces an algebraic subgroup in the automorphism group \( \text{Aut}(X) \), which is called an \( \mathbb{G}_a \)-subgroup. For an arbitrary \( \mathbb{G}_a \)-subgroup \( H \) on X, every nonzero element of the one-dimensional Lie algebra \( \text{Lie}(H) \) naturally defines a locally nilpotent derivation (LND) \( \partial \) on the algebra of regular functions \( \mathbb{K}[X] \), and, in the case of quasi-affine X, the subgroup H can be recovered by taking the exponent of \( \partial \).

In this paper, we are interested in \( \mathbb{G}_a \)-subgroups on X normalized by the action of a Borel subgroup B ⊂ G. Following [1], we call such \( \mathbb{G}_a \)-subgroups B-root subgroups on X. For every B-root subgroup H on X, the adjoint action of B on \( \text{Lie}(H) \) reduces to multiplying by a character of B, which we denote by \( \chi_H \) and call the weight of H. If \( \partial \) is an LND on \( \mathbb{K}[X] \) corresponding to H, then \( \partial \) is also normalized by B with the same weight \( \chi_H \).

2. Throughout this paper, we assume that X is a spherical G-variety, that is, X is normal and possesses an open B-orbit. Let \( \mathfrak{D}^B \) (\( \mathfrak{D}^G \)) denote the finite set of all B-stable (respectively, G-stable) prime divisors in X. Elements of the set \( \mathfrak{D} = \mathfrak{D}^B \setminus \mathfrak{D}^G \) are traditionally called colors of X.

Now we recall the well-known division of colors in X into three types (see [2, 3]). Fix an arbitrary color D ∈ \( \mathfrak{D} \). Then one can always choose a minimal parabolic subgroup \( Q \supset B \) in G such that \( QD \neq D \). For every subgroup \( F \subset Q \), let \( \bar{F} \) denote its image in \( Q/Q_r \), where \( Q_r \) is the solvable radical of Q. Choose an arbitrary point \( z \) in the open B-orbit in X, and let \( Q_z \) be the stabilizer of \( z \) in Q. Note that \( Q_z \cap D \neq D \). Since \( Q_z \subset B \), the natural morphism

\[
Q_z \simeq Q/Q_r \rightarrow Q/(Q_zQ_r) \simeq \bar{Q}/\bar{Q}_z
\]

(1)
of factorizing by \( Q_z \) induces a codimension-preserving bijection between B-orbits in \( Qz \) and \( B \)-orbits in \( \bar{Q}/\bar{Q}_z \). In particular, \( \bar{Q}/\bar{Q}_z \) contains an open \( \bar{B} \)-orbit. Since \( \bar{Q} \) is isomorphic to either \( \text{SL}_2 \) or \( \text{PSL}_2 \), there are the following three possibilities for \( \bar{Q}_z \):

Type (U): \( \bar{Q}_z \) contains a maximal unipotent subgroup in \( \bar{Q} \). In this case, \( Qz\setminus Bz \) is a single B-orbit of codimension 1, which coincides with \( Qz \cap D \).

Type (T): \( \bar{Q}_z \) is a maximal torus in \( \bar{Q} \). In this case, \( Qz\setminus Bz \) contains two B-orbits of codimension 1 and one of them coincides with \( Qz \cap D \).
Type (N): $\tilde{Q}_z$ is the normalizer of a maximal torus in $\tilde{Q}$. In this case, $Q_z\cdot B_z$ is a single $B$-orbit of codimension 1, which coincides with $Q_z \cap D$.

It is well known that the above-defined type does not depend on the choice of a minimal parabolic subgroup $Q \supset B$ satisfying $QD \neq D$ (see [3, Proposition 1]). This makes the type of every color in $X$ well-defined.

**Remark 1.** The above-defined types $(U)$, $(T)$, and $(N)$ of colors in $X$ coincide with the types $b, a, a'$, respectively, in the notation of Luna (see [4, Sect. 2.7, 3.4] or [5, Sect. 30.10]).

3. It follows from [1, Proposition 1.6] that, for every $B$-root subgroup $H$ on $X$, there exists at most one divisor $D \in \mathcal{D}^+$ such that $HD \neq D$. The following result generalizes [1, Corollary 4.25], where the case of affine $X$ was considered.

**Proposition 1.** Suppose that a $B$-root subgroup $H$ on $X$ satisfies $HD \neq D$ for some divisor $D \in \mathcal{D}^+$, then $D$ is either $G$-stable or is a color of type $(T)$.

**Proof.** Assume that $D$ is a color of type $(U)$ or $(N)$ and choose a minimal parabolic subgroup $Q \supset B$ satisfying $QD \neq D$. Then, in the notation of Section 2, the orbit $Qz$ splits into two $B$-orbits, $O_z = Bz$ and $O_z' = O \cap D$. In this case, it follows from the discussion in [1, Sect. 3 1.5] that the set $Qz$ is $H$-stable; moreover, each $H$-orbit in $Qz$ is isomorphic to the affine line $\mathbb{A}^1$ and meets $O_z$ at exactly one point. For every subgroup $F \subset Q$, let $\tilde{F}$ denote its image in $Q/Qz$, where $Qz$ is the unipotent radical of $Q$. Similarly to (1), the morphism

$$\varphi: Qz \simeq Q/Q_z \to Q/(Qz_{Qz}) \simeq \tilde{Q}/\tilde{Q}_z \quad (2)$$

of factorizing by $Qz$ induces a codimension-preserving bijection between $B$-orbits in $Qz$ and $\tilde{B}$-orbits in $\tilde{Q}/\tilde{Q}_z$. Since the actions of $H$ and $Qz$ on $Qz$ commute, the former descends to a nontrivial action of $H$ on $\tilde{Q}/\tilde{Q}_z$ normalized by the action of $\tilde{B}$. In particular, there are exactly two $\tilde{B}$-orbits $\varphi(O_z)$ and $\varphi(O_z')$ in $\tilde{Q}/\tilde{Q}_z$, and every $H$-orbit is isomorphic to $\mathbb{A}^1$ and meets $\varphi(O_z')$ at exactly one point. Next, we consider the cases of types $(U)$ and $(N)$ separately. The condition on each of these types is reformulated in view of the fact that $\tilde{Q}$ is the quotient of $\tilde{Q}$ by its connected center.

Type $(U)$: $\tilde{Q}_z$ contains a maximal unipotent subgroup in $\tilde{Q}$. Then $\tilde{Q}/\tilde{Q}_z$ has a point fixed by the unipotent radical $\tilde{B}_u$ of $\tilde{B}$. Since $\tilde{B}_u$ is normalized by $\tilde{B}$ and commutes with the action of $H$, the set of $\tilde{B}_u$-fixed points in $\tilde{Q}/\tilde{Q}_z$ is stable with respect to both $\tilde{B}$ and $H$ and, hence, coincides with the whole $\tilde{Q}/\tilde{Q}_z$. Therefore, $\tilde{B}_u$ acts trivially on $\tilde{Q}/\tilde{Q}_z$, a contradiction.

Type (N): $\tilde{Q}$ contains a subgroup $\tilde{Q}_z$ of index 2 that is the preimage of the connected component of the identity in $\tilde{Q}_z$. Then the natural morphism $\psi: \tilde{Q}/\tilde{Q}_z \to \tilde{Q}/\tilde{Q}_z$ is an unramified two-fold covering. Moreover, the set $\psi^{-1}(\varphi(O_z))$ is an open $\tilde{B}$-orbit in $\tilde{Q}/\tilde{Q}_z$, and the set $\psi^{-1}(\varphi(O_z'))$ splits into two $\tilde{B}$-orbits of codimension 1, which we denote by $D_1$ and $D_2$. Now let $y = \varphi(z)$ and $\psi^{-1}(y) = \{y_1, y_2\}$. Since $Hy \simeq \mathbb{A}^1$, the set $\psi^{-1}(Hy)$ is a disjoint union of two components $Y_1$ and $Y_2$, each of which maps isomorphically onto $Hy$. Without loss of generality, we assume that $y_i \in Y_j$ for $i = 1, 2$. Let $b \in B$ be such that $hy_i = y_j$. Then $b \in \tilde{B}_z$ and, hence, $hy_2 = y_1$. Since $Hy$ is $\tilde{B}$-stable, the action of $b$ interchanges $Y_1$ and $Y_2$. On the other hand, the set $\psi^{-1}(Hy \cap \varphi(O_z))$ consists of two points belonging to different $\tilde{B}$-orbits, a contradiction.

In the terminology of [1, Sect. 3 4.2], a $B$-root subgroup $H$ on $X$ is called vertical if it preserves an open $B$-orbit and horizontal otherwise. If $H$ is horizontal and $HD \neq D$ for some $D \in \mathcal{D}^+$, then we say that $H$ moves $D$. According to Proposition 1, horizontal $B$-root subgroups can be divided into two types.

**Definition 1.** Let $H$ be a horizontal $B$-root subgroup on $X$, and let a divisor $D \in \mathcal{D}^+$ be such that $HD \neq D$. If $D \in \mathcal{D}^G$, then we call $H$ toroidal. If $D$ is a color of type $(T)$, then we call $H$ blurring.

4. For every subset $\mathcal{F} \subset \mathcal{D}$, put $D_\mathcal{F} = \bigcup_{D \in \mathcal{D}^\mathcal{F}} D$, $X_\mathcal{F} = X \setminus D_\mathcal{F}$, and let $P_\mathcal{F}$ denote the stabilizer of the set $X_\mathcal{F}$ in $G$. Then $P_\mathcal{F}$ is a parabolic subgroup of $G$ containing $B$. In our subsequent consideration, a key role is played by the local structure theorem (see [7, Theorem 2.3, Proposition 2.4; 8, Theorem 1.4]), which in our situation can be stated as follows.

**Theorem 1.** Suppose that $\mathcal{F} \subset \mathcal{D}$ is an arbitrary subset and $P = L \times P_u$ is a Levi decomposition of the group $P = P_\mathcal{F}$. Then there exists a closed $L$-stable subvariety $Z \subset X_\mathcal{F}$ such that the map $P \times Z \to X_\mathcal{F}$ given by the formula $(p, z) \mapsto px$ is a $P$-equivariant isomorphism, where the action of $P$ on $P \times Z$ is defined as $lu(p, z) = (lul^{-1}, lz)$ for all $l \in L$, $u, p \in P_u$, $z \in Z$. Moreover, if $P$ coincides with the stabilizer of the open $B$-orbit in $X$, then the derived subgroup of $L$ acts trivially on $Z$.

Below we will need the following observation.
Proposition 2. Suppose that $\mathcal{F} = \mathcal{D}$ or $\mathcal{F} = \mathcal{D}\{D_0\}$, where $D_0$ is a color of type $(T)$. Then the group $P_\rho$ coincides with the stabilizer of the open $B$-orbit in $X$.

Proof. If $\mathcal{F} = \mathcal{D}$, then the assertion is obvious, so in what follows we assume that $\mathcal{F} = \mathcal{D}\{D_0\}$ for a color $D_0$ of type $(T)$. Let $Q \supset B$ be an arbitrary minimal parabolic subgroup of $G$. Then the condition $QD_0 \neq D_0$ can hold only if $QD' \neq D'$ for some color $D' \in \mathcal{D}\{D_0\}$. Therefore, if $Q \subset P_\rho$, then $QD_0 = D_0$. Since $P_\rho$ is generated as a group by all minimal parabolic subgroups contained in it, we obtain $P_\rho D_0 = D_0$, which yields the required result.

5. Throughout the rest of this paper, we assume that $X$ is an affine spherical $(T)$-variety. Now we introduce some notation.

Fix a maximal torus $T \subset B$, and let $\chi(T)$ denote its character lattice. Let $\Delta \subset \chi(T)$ be the root system of $G$ with respect to $T$, and let $\Lambda^+ \subset \chi(T)$ be the monoid of dominant weights with respect to $B$.

Let $M(\Delta)$ be the lattice (respectively, the monoid) of weights of $B$-semi-invariant rational (respectively, regular) functions on $X$. Since $X$ is affine, we have $M = \mathbb{Z}\Delta$ (see, e.g., [5, Proposition 5.14]). Consider the dual lattice $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and the corresponding rational vector space $N_Q = N \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural pairing $N \times M \to \mathbb{Z}$ is denoted by $\langle \cdot , \cdot \rangle$.

Since $X$ contains an open $B$-orbit, for every $\lambda \in M$, there exists a unique, up to proportionality, $B$-semi-invariant rational function $f_\lambda$ on $X$ of weight $\lambda$. Requiring all such functions to take the value 1 at a fixed point of the open $B$-orbit, we assume that $f_\lambda f_\mu = f_{\lambda+\mu}$ for all $\lambda, \mu \in M$. Every divisor $D \in \mathcal{D}^B$ defines an element $\alpha(D) \in N$ by the formula $\langle \alpha(D), \lambda \rangle = \text{ord}_D(f_\lambda)$ for all $\lambda \in M$. Since $X$ is normal, we have

$$\Gamma = \{ \lambda \in M \mid \langle \alpha(D), \lambda \rangle \geq 0 \text{ for all } D \in \mathcal{D}^B \}.$$  

(3)

In particular, the set $\{ \langle \alpha(D), \lambda \rangle \mid D \in \mathcal{D}^B \}$ generates a strictly convex cone in $N_Q$.

For each strictly convex finitely generated cone $\mathcal{C} \subset N_Q$, let $\mathcal{C}_1$ denote the set of primitive elements $\rho$ of the lattice $N$ such that the ray $Q_{\rho} = \mathbb{R}_{\rho}$ is a face of $\mathcal{C}$. For every $\rho \in \mathcal{C}_1$, define the set

$$\mathcal{R}_\rho(\mathcal{C}) = \{ \mu \in M \mid \langle \rho, \mu \rangle = -1 \};$$

$$\langle \rho', \mu \rangle \geq 0 \text{ for all } \rho' \in \mathcal{C}_1, \rho \in \mathcal{C}_1.$$

(4)

Elements of the set $\mathcal{R}(\mathcal{C}) = \bigcup_{\rho \in \mathcal{C}_1} \mathcal{R}_\rho(\mathcal{C})$ are called Demazure roots of the cone $\mathcal{C}$. Put

$$\Gamma(\mathcal{C}) = \{ \lambda \in M \mid \langle \lambda, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathcal{C} \}$$

and consider the algebra $A(\mathcal{C}) = \bigoplus_{\lambda \in \Gamma(\mathcal{C})} \mathbb{K}_{f_\lambda}$. Below, we will need the following well-known result (see [9, Theorem 2.7]), which provides a description of all $T$-normalized LNDs on $A(\mathcal{C})$.

Theorem 2. Let $\mathcal{C} \subset N_Q$ be an arbitrary strictly convex finitely generated cone.

(a) The set of weights of all $T$-normalized LNDs on $A(\mathcal{C})$ equals $\mathcal{R}(\mathcal{C})$.

(b) For each $\rho \in \mathcal{C}_1$ and each $\mu \in \mathcal{R}_\rho(\mathcal{C})$, there exists a unique, up to proportionality, $T$-normalized LND $\partial_\mu$ on $A(\mathcal{C})$ of weight $\mu$, which is defined by the formula

$$\partial_\mu(f_\lambda) = \langle \rho, \lambda \rangle f_\lambda f_\mu$$

(6)

for all $\lambda \in \Gamma(\mathcal{C})$.

6. Suppose that $H$ is a $B$-root subgroup on $X$ and $\mathcal{F}_H = \{ D \in \mathcal{D} \mid HD = D \}$. Then $H$ preserves the open subset $X_{\mathcal{F}_H} \subset X$, thus defining a $B$-normalized LND on the algebra $\mathbb{K}[X_{\mathcal{F}_H}]$. Note that $\mathcal{F}_H = \mathcal{D}$ in the case of vertical or toroidal $H$ and $\mathcal{F}_H = \mathcal{D}\{D_0\}$ in the case of blurring $H$ moving a color $D_0$ of type $(T)$.

Now let $\mathcal{F} = \mathcal{D}$ or $\mathcal{F} = \mathcal{D}\{D_0\}$, where $D_0 \in \mathcal{D}$ is a color of type $(T)$. Our goal in this section is to describe all $B$-normalized LNDs on the algebra $\mathbb{K}[X_{\mathcal{F}}]$.

Applying Theorem 1 and retaining the notation $Z$, $L$, $P_\rho$ used in this theorem. Then there is a $P$-equivariant isomorphism $X_{\mathcal{F}} \simeq P_\rho \times Z$, via which we will identify these two varieties below. Without loss of generality, we assume that $L \supset T$. By Proposition 2, the derived subgroup of $L$ acts trivially on $Z$. Since $X_\rho$ contains an open $B$-orbit, the variety $Z$ contains an open $T$-orbit, which will be denoted by $Z_\rho$. Fix also an arbitrary point $z_0 \in Z_\rho$. Let $L_\rho$ denote the kernel of the action of $L$ on $Z$ and put $T_\rho = T \cap L_\rho$. Note that $M$ consists of exactly those characters of $T$ that restrict trivially to $T_\rho$.

For every $\lambda \in M$, the restriction of $f_\lambda$ to the subvariety $Z$ is a $T$-semi-invariant rational function, which will still be denoted by $f_\lambda$. The $\mathbb{K}[Z] = \bigoplus_{\lambda \in \Gamma_{T_\rho}} \mathbb{K}_{f_\lambda}$, where

$$\Gamma_{T_\rho} = \{ \lambda \in M \mid \langle \alpha(D), \lambda \rangle \geq 0 \text{ for all } D \in \mathcal{D}^B \{ \mathcal{F} \} \}.$$  

(7)

Without loss of generality, we assume that $f_\lambda(z_0) = 1$ for all $\lambda \in M$.

Consider the adjoint representation of the group $L$ on the space $p_\rho = \text{Lie}P_\rho$, and decompose $p_\rho$ into a direct sum of irreducible $L$-invariant subspaces. It is well known (see [6, Theorem 0.1]) that all summants in this decomposition are pairwise nonisomorphic as
\[ \begin{align*}
\Omega_\mu &= \{ \alpha \in \Omega \mid \mu|_{t_\alpha} = \alpha|_{t_\alpha} \}; \\
\Omega_\mu^0 &= \{ \alpha \in \Omega_\mu \mid \mu - \alpha \in \Gamma_Z \}. 
\end{align*} \]

Note that the condition \( \mu|_{t_\alpha} = \alpha|_{t_\alpha} \) is equivalent to \( \mu - \alpha \in M \).

**Theorem 3.** Every \( B \)-normalized LND of weight \( \mu \) on the algebra \( \mathbb{K}[P_u \times Z] \) has the form
\[ \sum_{\alpha \in \Omega_\mu} c_\alpha f_{\mu-\alpha} \delta_\alpha + \partial_Z, \tag{9} \]
where \( c_\alpha \in \mathbb{K} \) and \( \partial_Z \) is a \( T \)-normalized LND of weight \( \mu \) on \( \mathbb{K}[Z] \) extended trivially to \( \mathbb{K}[P_u] \). Conversely, every derivation on \( \mathbb{K}[P_u \times Z] \) of the above form is \( B \)-normalized of weight \( \mu \) and locally nilpotent.

**Proof.** Suppose that \( \partial \) is a \( B \)-normalized LND of weight \( \mu \) on \( \mathbb{K}[P_u \times Z] \), and let \( \partial_Z \) be the restriction of \( \partial \) to the subalgebra \( \mathbb{K}[Z] \). In what follows, we regard \( \partial_Z \) as a derivation on the whole algebra \( \mathbb{K}[P_u \times Z] \) by setting \( \partial_Z(\mathbb{K}[P_u]) = 0 \). The extension of the derivation \( \partial - \partial_Z \) to the algebra \( \mathbb{K}[P_u \times Z_0] \) determines a \( B \)-semi-invariant vector field \( \xi \) of weight \( \mu \) on the smooth variety \( P_u \times Z_0 \). Since \( B \) acts transitively on \( P_u \times Z_0 \), \( \xi \) is uniquely determined by its value \( v \) at the point \( (e, z_0) \), where \( e \in P_u \) is the identity element. Since \( \partial - \partial_Z \) acts trivially on \( \mathbb{K}[Z_0] \), it follows that \( v \) is a \( B \cap L_0 \)-semi-invariant vector in \( P_u \) of weight \( \mu|_{t_\alpha} \); therefore,
\[ v = \sum_{\alpha \in \Omega_\mu} c_\alpha e_\alpha \]
for some \( c_\alpha \in \mathbb{K} \). On the other hand, we observe that the derivation \( \sum_{\alpha \in \Omega_\mu} c_\alpha f_{\mu-\alpha} \delta_\alpha \) on \( \mathbb{K}[Z_0] \) is also \( B \)-semi-invariant of weight \( \mu \) and corresponds to the same tangent vector at \( (e, z_0) \); hence, it coincides with \( \partial - \partial_Z \). Since this derivation preserves the algebra \( \mathbb{K}[P_u \times Z] \), the condition \( c_\alpha = 0 \) should hold for all \( \alpha \in \Omega_\mu \) with \( \mu - \alpha \in \Gamma_Z \), which proves the first claim.

Now suppose that \( \partial \) is a derivation on \( \mathbb{K}[P_u \times Z] \) of the form (9). Then \( \partial \) is automatically \( B \)-normalized of weight \( \mu \), and it remains to prove that \( \partial \) is locally nilpotent. Since \( \mathbb{K}[P_u] \) is a rational \( P_u \)-module (with respect to the action on the right), it suffices to check that \( \partial \) is locally nilpotent on an arbitrary subspace of the form \( V \otimes_{\mathbb{K}} \mathbb{K}[Z] \), where \( V \subset \mathbb{K}[P_u] \) is a finite-dimensional \( P_u \)-invariant subspace. Since the image of the algebra \( P_u \) in \( gl(V) \) nilpotent, in \( V \) there exists a flag of subspaces
\[ 0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_s = V \tag{10} \]
with the property \( p_u V_i \subset V_{i-1} \) for all \( i = 1, \ldots, s \). It follows that, for all \( i = 1, \ldots, s, g \in V_i \), and \( f \in \mathbb{K}[Z] \), we have
\[ \partial(gf) = \sum_{\alpha \in \Omega_\mu} c_\alpha \delta_\alpha(g)f_{\mu-\alpha}f + g\partial_Z(f) \tag{11} \]
in \( g\partial_Z(f) + V_{i-1} \otimes_{\mathbb{K}} \mathbb{K}[Z] \), because \( \delta_\alpha(g) = e_\alpha g \in V_{i-1} \) for all \( \alpha \in \Omega \). Since \( \partial_Z \) is an LND on \( \mathbb{K}[Z] \), we obtain \( \partial^k \) \( (gf) \in V_{i-1} \otimes_{\mathbb{K}} \mathbb{K}[Z] \) for some \( k > 0 \). The proof is completed by induction on \( i \).

7. Retain the assumptions and notation used in Section 6. Now we study when a \( B \)-normalized LND on the algebra \( \mathbb{K}[P_u \times Z] \) preserves the subalgebra \( \mathbb{K}[X] \) and thus defines a \( B \)-root subgroup on the whole variety \( X \). Let \( \lambda \mapsto \overline{\lambda} \) be an arbitrary projection operator from \( \mathcal{X}(T) \otimes_{\mathbb{Q}} \mathbb{Q} \) onto the subspace \( M \otimes_{\mathbb{Q}} \mathbb{Q} \). Let \( \mathcal{E}_z \subset N_{\mathbb{Q}} \) be the cone generated by the set \( \{ \alpha(D) \mid D \in \mathcal{D}_z \} \), so that \( \Gamma_z = \Gamma(\mathcal{E}_z) \) in view of (7) and \( \mathbb{K}[Z] = A(\mathcal{E}_z) \) (see the notation in Section 5).

**Theorem 4.** There exists a collection of constant \( \{ C_D \mid D \in \mathcal{F} \} \) with the following property: if \( \mu \in \mathcal{X}(T) \) and \( \langle \nu_{\mu}, \overline{\nu} \rangle \geq C_D \) for all \( D \in \mathcal{F} \), then every \( B \)-normalized LND on \( \mathbb{K}[P_u \times Z] \) of weight \( \mu \) preserves \( \mathbb{K}[X] \).

**Proof.** Fix a generating system \( F_1, \ldots, F_k \) of the algebra \( \mathbb{K}[X] \). For every \( i = 1, \ldots, k \), we have
\[ F_i = \sum_{j=1}^n g_{ij} f_{\lambda_j} \]
for some functions \( g_{ij} \in \mathbb{K}[P_u] \) and weights \( \lambda_j \in \Gamma_Z \). If \( \partial \) is an arbitrary derivation of the algebra \( \mathbb{K}[P_u \times Z] \), then \( \partial \) preserves \( \mathbb{K}[X] \) if and only if \( \text{ord}_P(\partial(F_i)) \geq 0 \) for all \( D \in \mathcal{F} \) and \( i = 1, \ldots, k \).

By Theorem 3, to find the required collection of constants, for each weight \( \mu \in \mathcal{X}(T) \), it suffices to require that each summand in (9) preserve \( \mathbb{K}[X] \).
ON THE EXISTENCE OF B-ROOT SUBGROUPS

If \( \partial_Z \neq 0 \), then, by Theorem 2, we have \( \mu \in \mathcal{R}_\mu(\mathfrak{g}_Z) \) for some \( \rho \in \mathfrak{g}_Z^1 \) (in particular, \( \mu \in M \) and \( \mu = \mu \)) and there exists a nonzero constant \( c \in \mathbb{K} \) such that \( \partial_Z(f_\lambda) = c(\rho, \lambda)f_\lambda f_\mu \) for all \( \lambda \in \Gamma_Z \). Then, for all \( D \in \mathcal{D} \) and \( i = 1, \ldots, k \), with \( \partial_Z(F_i) \neq 0 \), we have

\[
\text{ord}_D(\partial_Z(F_i)) = \text{ord}_D \left( \sum_{j=1}^n \langle \rho, \lambda_{ij} \rangle g_i f_\lambda f_\mu \right) = \langle \nu_D, \overline{\lambda} \rangle + \text{ord}_D \left( \sum_{j=1}^n \langle \rho, \lambda_{ij} \rangle g_i f_\lambda f_\mu \right) \geq \langle \nu_D, \overline{\lambda} \rangle + \min \{ \text{ord}_D(g_i f_\lambda) \mid j = 1, \ldots, n \}.
\]

It remains to be noted that all expressions in (12) and (13) are nonnegative for a suitable choice of the required constants.

8. Let us deduce several consequences of Theorems 3 and 4.

**Corollary 1.** All \( B \)-normalized LNDs on \( \mathbb{K}[X] \) of the same weight form a finite-dimensional vector space over \( \mathbb{K} \).

**Proof.** In view of [1, Proposition 4.22], any two horizontal \( B \)-root subgroups on \( X \) of the same weight \( \mu \) move the same divisor \( D \in \mathcal{D} \). Therefore, under the conditions of Section 6, one can choose a subset \( \mathcal{D} \subset \mathcal{D} \) such that all \( B \)-root subgroups on \( X \) of weight \( \mu \) move the same divisor \( D \). By Theorems 3 and 2, all \( B \)-normalized LNDs on \( \mathbb{K}[X] \) of weight \( \mu \) form a finite-dimensional vector space. The condition of preserving the subalgebra \( \mathbb{K}[X] \) determines a subspace in that vector space.

Let \( \mathcal{E} \subset N_0^+ \) be the cone generated by the set \( \{ a(D) \mid D \in \mathcal{D} \} \), so that \( \Gamma = \Gamma(\mathcal{E}) \) in view of (3).

**Corollary 2.** Let \( D \in \mathcal{D} \) be such that \( D \in \mathcal{D}^G \) or \( D \) is a color of type \( T \). Suppose that there exists an element \( \rho \in \mathcal{E}^1 \) such that \( a(D) \in \mathbb{Q}_{\geq 0} \rho \) and \( a(D') \notin \mathbb{Q}_{\geq 0} \rho \) for all \( D' \in \mathcal{D}\setminus \{D\} \). Then there exists a \( B \)-root subgroup on \( X \) that moves \( D \).

**Proof.** Put \( \mathcal{F} = \mathcal{D} \) for \( D \in \mathcal{D}^G \) and \( \mathcal{F} = \mathcal{D}\setminus \{D\} \) otherwise. Retain the notation of Sections 6 and 7. Since \( \rho \in \mathcal{C}_Z \) and \( \mathcal{C}_Z \subset \mathcal{E} \), we have \( \rho \in \mathcal{C}_Z \). Choose any element \( \mu \in \mathcal{R}_\mu(\mathcal{C}_Z) \) and consider the \( B \)-normalized LND \( \partial_\mu \) on \( \mathbb{K}[P] \times Z \) of weight \( \mu \) that acts trivially on \( \mathbb{K}[P] \) and by formula (6) on \( \mathbb{K}[Z] \). It follows from the hypothesis that there exists a weight \( \lambda \in \Gamma \) such that \( \langle \rho, \lambda \rangle = 0 \) and \( \langle \lambda(D'), \lambda \rangle > 0 \) for all \( D' \in \mathcal{F} \). Then, for all integers \( N > 0 \), we have \( N\lambda + \mu \in \mathcal{R}_\mu(\mathcal{C}_Z) \). By Theorem 4, there is a value \( N_0 \) such that, for all \( N \geq N_0 \), the LND \( \partial_{N\lambda+\mu} = f_{N\lambda+\mu} \partial_\mu \) preserves \( \mathbb{K}[X] \) and, therefore, defines a \( B \)-root subgroup on \( X \). This \( B \)-root subgroup moves \( D \) due to [1, Proposition 4.22].

In view of [1, Proposition 3.9], every divisor \( D \in \mathcal{D}^G \) automatically satisfies the conditions of Corollary 2. This implies the following result, which was stated as a conjecture in [1, Conjecture 4.29].

**Corollary 3.** For every \( D \in \mathcal{D}^G \), there exists a \( B \)-root subgroup on \( X \) that moves \( D \).

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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