Semilinear Backward Doubly Stochastic Differential Equations and SPDEs Driven by Fractional Brownian Motion with Hurst Parameter in \((0, 1/2)\)

Shuai Jing\(^*\) and Jorge A. León\(^†\)

\(^*\) School of Mathematics, Shandong University, 250100, Jinan, China

\(^†\) Département de Mathématiques, Université de Bretagne Occidentale, 29285 Brest Cédex, France

Abstract

We study the existence of a unique solution to semilinear fractional backward doubly stochastic differential equation driven by a Brownian motion and a fractional Brownian motion with Hurst parameter less than 1/2. Here the stochastic integral with respect to the fractional Brownian motion is the extended divergence operator and the one with respect to Brownian motion is Itô’s backward integral. For this we use the technique developed by R. Buckdahn [3] to analyze stochastic differential equations on the Wiener space, which is based on the Girsanov theorem and the Malliavin calculus, and we reduce the backward doubly stochastic differential equation to a backward stochastic differential equation driven by the Brownian motion. We also prove that the solution of semilinear fractional backward doubly stochastic differential equation defines the unique stochastic viscosity solution of a semilinear stochastic partial differential equation driven by a fractional Brownian motion.

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1 Introduction

This paper investigates semilinear fractional backward doubly stochastic differential equations (BDS-DEs) and semilinear stochastic partial differential equations (SPDEs) driven by fractional Brownian motion. Fractional Brownian motions (fBms) and backward stochastic differential equations (BSDEs) have been extensively studied in recent twenty years. However, up to now there are only few works that combine both topics. Bender [1] considered a class of linear fractional BSDEs and gave their explicit solutions. There are two major obstacles depending on the properties of fBm: Firstly, the fBm is not a semimartingale except for the case of Brownian motion (Hurst parameter \(H = 1/2\)), hence the classical Itô calculus which is based on semimartingales cannot be transposed directly to

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\(^†\) Partially supported by the CONACyT grant 98998. E-mail: jleon@ctrl.cinvestav.mx
the fractional case. Secondly, there is no martingale representation theorem with respect to the fBm. However, such a martingale representation property with respect to the Brownian motion is the main tool in BSDE theory. Hu and Peng’s paper [9] overcame the second obstacle for the case of $H > 1/2$ by using the quasi-conditional expectation and by studying nonlinear fractional BSDEs in a special case only.

Nevertheless, there are many papers considering stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$ ([2], [15] and references therein) or $H < 1/2$ ([13]), or covering both cases ([10]). For the case $H < 1/2$, one of the main difficulties is how to properly define the stochastic integral with respect to the fBm. In the paper of Cheridito and Nualart [6], and then generalized by León and Nualart [12], the authors have defined the extended divergence operator which can be applied to the fBm for $H < 1/2$ as a special case. In this paper we will use such definition for the stochastic integration with respect to the fBm, and then apply the non-anticipating Girsanov transformation developed by Buckdahn [3] to transform the semilinear fractional doubly backward stochastic differential equation driven by the Brownian motion $W$ and the fractional Brownian motion $B$:

\begin{equation}
Y_t = \xi + \int_0^t f(s, Y_s, Z_s)ds - \int_0^t Z_s \circ dW_s + \int_0^t \gamma_s Y_s dB_s, \quad t \in [0, T],
\end{equation}

into a pathwise (in the sense of fBm) BSDE

\begin{equation}
\hat{Y}_t = \xi + \int_0^t f \left( s, \hat{Y}_s \? \varepsilon_s(T_s), \hat{Z}_s \? \varepsilon_s(T_s) \right) \varepsilon_s^{-1}(T_s)ds - \int_0^t \hat{Z}_s \? dW_s, \quad t \in [0, T].
\end{equation}

More precisely, the solutions $(Y, Z)$ and $(\hat{Y}, \hat{Z})$ are linked together by the following relations:

\[
\{(Y_t, Z_t), t \in [0, T]\} = \left\{ \left( \hat{Y}_t(A_t) \varepsilon_t, \hat{Z}_t(A_t) \varepsilon_t \right), t \in [0, T] \right\}
\]

and

\[
\left\{ \left( \hat{Y}_t, \hat{Z}_t \right), t \in [0, T] \right\} = \left\{ \left( Y_t(T_t) \varepsilon_t^{-1}(T_t), Z_t(T_t) \varepsilon_t^{-1}(T_t) \right), t \in [0, T] \right\},
\]

where $A_t$ and $T_t$ are Girsanov transformations. It is worth noting that such kind of method was also used by Jien and Ma [10] to deal with fractional stochastic differential equations.

It is well known that the solution of a BSDE can be regarded as a viscosity solution of an associated parabolic partial differential equation (PDE) (cf. [8] [18] and [23]), and the solution of BDSDE driven by two independent Brownian motions can be regarded as a stochastic viscosity solution of an SPDE (cf. [4] and [20]). So it is natural to consider the relationship between the solutions of our fractional BDSDE and the associated SPDE. We show that the solution of the above fractional BDSDE, which is a random field, is a stochastic viscosity solution of an SPDE driven by our fractional Brownian motion. To be more precise, the value function $u(t, x)$ defined by the solution of a fractional BSDE over the time interval $[0, t]$ instead of $[0, T]$, see (4.4), will be shown to possess a continuous version and to be the stochastic viscosity solution of the following semilinear SPDE

\[
\begin{cases}
\frac{du(t, x)}{dt} = [Lu(t, x) + f(t, x, u(t, x), \nabla_x u(t, x) \sigma(x))] dt + \gamma_t u(t, x) dB_t, \quad t \in [0, T], \\
u(0, x) = \Phi(x).
\end{cases}
\]

Taking a Brownian motion instead of the fBm, equation (1.1) becomes a classical BDSDE, which was first studied by Pardoux and Peng [20]. The associated stochastic viscosity solution of SPDE (1.3) (with $H = 1/2$) was studied by Buckdahn and Ma [4]. Let us point out that, unlike [4] considering the stochastic integral with respect to $B$ ($H = 1/2$) as the Stratonovich one and using a Doss-Sussman transformation as main tool, we have to do here with an extended divergence operator ($H < 1/2$), which condemns us to use the Girsanov transformation as main argument. However, this restricts us to semilinear equations. We will investigate the general case in a forthcoming paper, but with a different approach.
The paper is organized as follows: In Section 2 we recall some preliminaries which will be used in what follows: Malliavin calculus for fractional Brownian motion, the definition of extended divergence operator and the Girsanov transformation. In Section 3 we prove existence and uniqueness results for stochastic differential equations driven by a fractional Brownian motion and backward doubly stochastic differential equations driven by a Brownian motion as well as a fractional Brownian motion. The relationship between the stochastic viscosity solution of the stochastic partial differential equation (1.3) driven by fractional Brownian motion and that of an associated pathwise partial differential equation is given in Section 4.

2 Preliminaries

The purpose of this section is to describe the framework that will be used in this paper. Namely, we introduce briefly the transformations on the Wiener space, appearing in the construction of the solution to our equations, some preliminaries of the Malliavin calculus for the fBm, and the left and right-sided fractional derivatives, which are needed to understand the definition of the extension of the divergence operator with respect to the fBm. Although the most of results discussed in this section are known, we prefer to provide a self-contained exposition for the convenience of the reader.

2.1 Fractional calculus

For a detailed account on the fractional calculus theory, we refer, for instance, to Samko et al. [25].

Let $T > 0$ denote a positive time horizon, fixed throughout our paper, and let $f : [0, T] \to \mathbb{R}$ be an integrable function, and $\alpha \in (0, 1)$. The right-sided fractional integral of $f$ of order $\alpha$ is given by

$$I_{T-}^\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} \, du, \quad \text{for a.a. } x \in [0, T].$$

Note that $I_{T-}^\alpha(f)$ is well-defined because the Fubini theorem implies that it is a function in $L^p([0, T])$, $p \geq 1$, whenever $f \in L^p([0, T])$.

We denote by $I_{T-}^\alpha(L^p)$, $p \geq 1$, the family of all functions $f \in L^p([0, T])$ such that

$$f = I_{T-}^\alpha(\varphi), \quad (2.1)$$

for some $\varphi \in L^p([0, T])$. Samko et al. [25] (Theorem 13.2) provide a characterization of the space $I_{T-}^\alpha(L^p)$, $p > 1$. Namely, a measurable function $f$ belongs to $I_{T-}^\alpha(L^p)$ (i.e., it satisfies (2.1)) if and only if $f \in L^p((0, T])$ and the integral

$$\int_{s+\varepsilon}^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} \, du$$

converges in $L^p([0, T])$ as $\varepsilon \downarrow 0$. In this case a function $\varphi$ satisfying (2.1) coincides with the right–sided fractional derivative

$$(D_{T-}^\alpha f)(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(T-s)^{\alpha}} + \alpha \int_s^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} \, du \right), \quad (2.3)$$

where the integral is the $L^p([0, T])$–limit of (2.2). Moreover, it has also been shown in [25] (Lemma 2.5) that there is at most one solution $\varphi$ to the equation (2.1). Consequently, the inversion formulae

$$I_{T-}^\alpha(D_{T-}^\alpha f) = f, \quad \text{for all } f \in I_{T-}^\alpha(L^p),$$

and

$$D_{T-}^\alpha(I_{T-}^\alpha(f)) = f, \quad \text{for all } f \in L^1([0, T])$$
hold.

Similarly, the left-sided fractional integral and the derivative of $f$ of order $\alpha$, which are given, respectively, by

$$I_{0+}^\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(u)}{(x-u)^{1-\alpha}} du, \quad \text{for a.a. } x \in [0, T],$$

and

$$\left(D_{0+}^\alpha f\right)(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{s^\alpha} + \alpha \int_0^s \frac{f(s)-f(u)}{(s-u)^{1+\alpha}} du \right), \quad (2.4)$$

satisfy the inversion formulae

$$I_{0+}^\alpha \left(D_{0+}^\alpha f\right) = f, \quad \text{for all } f \in I_{0+}^\alpha(L^p),$$

and

$$D_{0+}^\alpha \left(I_{0+}^\alpha f\right) = f, \quad \text{for all } f \in L^1([0, T]).$$

### 2.2 Fractional Brownian motion

In this subsection we will recall some basic facts of the fBm. The reader can consult Mishura [15] and Nualart [16] and the references therein for a more complete presentation of this subject.

Henceforth $(\Omega, \mathcal{F}, P)$ and $W^0 = \{W^0_t : t \in [0, T]\}$ are the canonical Wiener space on the interval $[0, T]$ and the canonical Wiener process, respectively. This means, in particular, that $\Omega = C_0([0, T])$ is the space of all continuous functions $h : [0, T] \to \mathbb{R}$ with $h(0) = 0$. The coordinate process $W^0$ is the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$ and $\mathcal{F}$ is the completion of $\mathcal{B}(\Omega) = \sigma\{W^0_s, s \in [0, T]\}$ with respect to $P$. The noise under consideration is the process

$$B_t = \int_0^t K_H(t, s) dW^0_s, \quad t \in [0, T],$$

where $K_H$ is the kernel of the fBm with parameter $H \in (0, 1/2)$. That is,

$$K_H(t, s) = C_H \left[ \left( \frac{t}{s} \right)^{H-1/2} - (H-1/2) \frac{1}{s} \int_s^t u^{H-3/2} (u-s)^{H-1/2} du \right],$$

where $C_H = \sqrt{\frac{2H}{1-2H}(1-2H)^{1/2}}$. The process $B = \{B_t : t \in [0, T]\}$ is an fBm with Hurst parameter $H$, defined on $(\Omega, \mathcal{F}, P)$, i.e., $B$ is a Gaussian process with zero mean and covariance function

$$R_H(t, s) := \mathbb{E}[B_t B_s] = \int_0^{s\wedge t} K_H(t, r) K_H(s, r) dr = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad s, t \in [0, T].$$

Let $\mathcal{H}_H$ be the Hilbert space defined as the completion of the space $L^2(0, T)$ of step functions on $[0, T]$ with respect to the norm generated by the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}_H} = R_H(t, s) = \mathbb{E}[B_t B_s], \quad t, s \in [0, T].$$

From Pipiras and Taqqu [24] (see also [16]), it follows that $\mathcal{H}_H$ coincides with the Hilbert space

$$\Lambda_T^{1/2-H} := \left\{ f : [0, T] \to \mathbb{R} : \exists \varphi \in L^2(0, T) \text{ such that } f(u) = u^{1/2-H} I_{T-}^{1/2-H} \left( s^{H-1/2} \varphi(s) \right)(u) \right\}$$

equipped with scalar product

$$\langle f, g \rangle_{\Lambda_T^{1/2-n}} = C_H^2 \Gamma(H+1/2)^2 \langle \varphi, \varphi \rangle_{L^2(0, T)}.$$
So $1_{[0,t]} \mapsto B_t$ can be extended to an isometry of $\Lambda_T^{1/2-H}$ onto a Gaussian closed subspace of $L^2(\Omega, \mathcal{F}, P)$. This isometry is denoted by $\varphi \mapsto B(\varphi)$. Moreover, by the transfer principle (see Nualart [16]), the map $K : \Lambda_T^{1/2-H} \rightarrow L^2([0,T])$, defined by (see (2.3))
\[
(K\varphi)(s) = C_H \Gamma(H + 1/2) s^{1/2-H} \left( D_{T-s}^{1/2-H} u^{H-1/2} \varphi(u) \right)(s), \quad s \in [0,T],
\]
is an isometry such that
\[
B(\varphi) = \int_0^T (K\varphi)(s) dW_s^0 \quad \text{and} \quad K1_{[0,t]} = K_H(t, \cdot)1_{[0,t]}, \quad t \in [0,T].
\]

Using the properties of $K$, Cheridito and Nualart [6] have extended the domain of the divergence operator with respect to the fBm $B$. This extension of the divergence operator in the sense of Malliavin calculus holds also true for some suitable Gaussian processes (see León and Nualart [12]). For $B$, this extension is introduced as follows.

The following result identifies the adjoint of the operator $K$ (see [12]). It uses the left–sided fractional derivative $D_{0+}^q$ defined in (2.4).

**Proposition 2.1.** Let $g : [0, t] \rightarrow \mathbb{R}$ be a function such that $u \mapsto u^{1/2-H} g(u)$ belongs to $I_0^{1/2-H}(L^q([0,a]))$ for some $q > (1/2 - H)^{-1} \vee H^{-1}$. Then, $g \in \text{Dom } K^*$, and for all $u \in [0,T]$,
\[
(K^* g)(u) = C_H \Gamma(H + 1/2) u^{H-1/2} D_{0+}^{1/2-H} \left( s^{1/2-H} g(s) \right)(u).
\]

Let $S$ (resp. $S_K$) denote the class of smooth random variables of the form
\[
F = f(B(\varphi_1), \ldots, B(\varphi_n)), \quad (2.5)
\]
where $\varphi_1, \ldots, \varphi_n$ are in $\Lambda_T^{1/2-H}$ (resp. in the domain of the operator $K^* K$) and $f \in C_p^\infty(\mathbb{R}^n)$. Here, $C_p^\infty(\mathbb{R}^n)$ is the set of $C^\infty$ functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f$ and all its partial derivatives have polynomial growth.

The derivative of the smooth random variable $F$ given by (2.5) is the $\Lambda_T^{1/2-H}$–valued random variable $DF$ defined by
\[
DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n)) \varphi_i.
\]

Now we can introduce the stochastic integral that we use in this paper. It is an extended divergence operator with respect to $B$.

**Definition 2.2.** Let $u \in L^2(\Omega, \mathcal{F}, P; L^2([0,T]))$. We say that $u$ belongs to $\text{Dom } \delta$ if there exists $\delta(u) \in L^2(\Omega)$ such that
\[
E \left[ (K^* K) DF, u \right]_{L^2([0,T])} = E[F \delta(u)], \quad \text{for every } F \in S_K. \quad (2.6)
\]

In this case, the random variable $\delta(u)$ is called the extended divergence of $u$.

**Remark 2.3.** i) In [12] it is shown that the domain of $K^* K$ is a dense subset of $\Lambda_T^{1/2-H}$. Therefore, there is at most one square integrable random variable $\delta(u)$ such that (2.6) holds.

(ii) In [6] and [12], it is proven that the domain of $\delta$ is bigger than that of the classical divergence operator, which is defined by the chaos decomposition approach (see Nualart [17]).

(iii) In Section 3 and Section 4, we use the convention
\[
\int_0^t u_s dB_s = \delta(u1_{[0,t]}),
\]
whenever $u1_{[0,t]} \in \text{Dom } \delta$. 

5
2.3 Girsanov transformations

In this section we introduce the Girsanov transformations on $\Omega$, which we consider in this paper.

In all which follows, we assume that $\gamma$ is a square-integrable function satisfying the following hypothesis:

\[(H1) \quad \gamma_{[0,t]} \text{ belongs to } H^{1/2-H}_p, \text{ for every } t \in [0,T].\]

We emphasize that León and San Martín [13] (Lemma 2.3) have shown the existence of square-integrable functions satisfying the above hypothesis.

Now, for $t \in [0, T]$, we consider the transformations on $\Omega$ of the form

\[T_t(\omega) = \omega + \int_0^t (\mathcal{K}\gamma_{[0,t]})(r)dr\]

and

\[A_t(\omega) = \omega - \int_0^t (\mathcal{K}\gamma_{[0,t]})(r)dr.\]

Notice that $A_tT_t$ and $T_tA_t$ are the identity operator of $\Omega$, and that the Girsanov theorem leads to write

\[B(\varphi)T_t = B(\varphi) + \int_0^t (\mathcal{K}\gamma_{[0,t]})(r)(\mathcal{K}\varphi)(r)dr = B(\varphi) + \int_0^t \gamma_r(\mathcal{K}\mathcal{K}\varphi)(r)dr,\]

for all $\varphi \in \text{Dom}(\mathcal{K}^*\mathcal{K})$, and

\[E[F] = E[F(A_t)\varepsilon_t],\]

with

\[\varepsilon_t = \exp\left(\int_0^t \gamma_r dB_r - \frac{1}{2} \int_0^t (\mathcal{K}\gamma_{[0,t]})(r)^2dr\right).\]

We will need the following estimate of the above exponential of the integral with respect to the fractional Brownian motion:

\[\text{Lemma 2.4. Let } \gamma : [0, T] \mapsto \mathbb{R}, \gamma \in L_p[0,T] \cap D^p_p[0,T], \text{ for some } p > 1/H, \text{ where } D^p_p[0,T] = \{ \gamma : [0,T] \mapsto \mathbb{R} | \int_0^T (\int_0^T \varphi(x,t)dt)^pdx < \infty \} \text{ and set } \varphi(x, t) = \frac{\gamma(t) - \gamma(x)}{(t-x)^{\nu/2}}1_{\{0 < x < t \leq T\}}. \text{ Then there exists a constant } C(H, p) \text{ only depending on } H \text{ and } p, \text{ such that}\]

\[E\left[\exp\left(\sup_{0 \leq t \leq T} \int_0^t \gamma_s dB_s\right)\right] \leq 2 \exp\left\{1/2 \left(C(H, p)G_p(0,T,\gamma) + 4\sqrt{\gamma}\right)^2\right\}, \quad (2.7)\]

where $G_p(0,T,\gamma) := \|\gamma\|_{L_p[0,T]} \cdot T^{H-1/p} + T^{1/2-1/p} \left(\int_0^T (\int_0^T \varphi(x,t)dt)^p\right)^{1/p}.$

\[\text{Proof: Let } I_T = \sup_{0 \leq t \leq T} |\int_0^t \gamma_s dB_s|. \text{ According to Lifshitz[14] Theorem 1, P.141, and its corollary, for all } r > 4\sqrt{\gamma}D(T,\lambda/2), \text{ we have the inequality}\]

\[P\{I_T > r\} \leq 2 \left(1 - \Phi\left(\frac{r - 4\sqrt{\gamma}D(T,\lambda/2)}{\lambda}\right)\right), \quad (2.8)\]

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\{-y^2/2\}dy$, $D(T,\lambda/2)$ is the Dudley integral (for more details, we refer to Mishura[15]), and $\lambda^2 = \sup_{t \in [0,T]} E\left(\left|\int_0^t \gamma_s dB_s\right|^2\right).$
Since $E[\exp\{I_T^r\}] = 1 + \int_0^{+\infty} \exp\{x\} P(I_T^r > x)dx$, by using the estimate of (2.8) we have

\[
E[\exp\{I_T^r\}] = 1 + \int_0^{4\sqrt{2}D(T,\lambda/2)} \exp\{x\} P(I_T^r > x)dx + \int_{4\sqrt{2}D(T,\lambda/2)}^{+\infty} \exp\{x\} P(I_T^r > x)dx
\]

\[
\leq \exp\left\{4\sqrt{2}D(T,\lambda/2)\right\} + 2 \int_{4\sqrt{2}D(T,\lambda/2)}^{+\infty} \exp\{x\} \left(1 - \Phi\left(\frac{x - 4\sqrt{2}D(T,\lambda/2)}{\lambda}\right)\right)dx
\]

\[
= \exp\left\{4\sqrt{2}D(T,\lambda/2)\right\} + 2 \int_{0}^{+\infty} \exp\left\{4\sqrt{2}D(T,\lambda/2) + x\right\} \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\lambda} \exp\{-y^2/2\}dy\right)dx
\]

\[
\leq 2 \exp\left\{\lambda^2/2 + 4\sqrt{2}D(T,\lambda/2)\right\}.
\]

Moreover, from Theorem 1.10.6 of Mishura[15] and its proof we know that

\[\lambda \leq C_1(H,p)G_p(0,T,\gamma)\]

and

\[D(T,\lambda/2) \leq C_2(p)G_p(0,T,\gamma).
\]

By substituting them to the former inequality, we easily get the wished result.

\[\square\]

### 3 Semilinear fractional SDEs and fractional backward doubly SDEs

#### 3.1 Fractional anticipating semilinear equations

In this subsection we discuss the existence and uniqueness of solutions to anticipating semilinear equations driven by a fractional Brownian motion $B$ with Hurst parameter $H \in (0, 1/2)$. This type of equation was studied by Jien and Ma [10], and since it motivates the approach in our work, we give it in details.

We consider the fractional anticipating equation

\[
X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \gamma_s X_s dB_s, \quad t \in [0,T].
\]  

(3.1)

Here $\xi \in L^p(\Omega)$, $p > 2$, and $b : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that:

**(H2)** There exist $\nu \in L^1([0,T])$, $\nu \geq 0$, and a positive constant $L$ such that

\[
|b(\omega, t, x) - b(\omega, t, y)| \leq \nu_t|x - y|, \quad \int_0^T \nu_t dt \leq L,
\]

\[
|b(\omega, t, 0)| \leq L,
\]

for all $x, y \in \mathbb{R}$ and almost all $\omega \in \Omega$.

We observe that the above assumption guarantees that the pathwise equation

\[
\zeta_t(x) = x + \int_0^t \varepsilon_s^{-1}(T_s)b(T_s, s, \varepsilon_s(T_s)\zeta_s(x))ds, \quad t \in [0,T],
\]

has a unique solution. Henceforth, we denote it by $\zeta$.

Now we can state the existence of a unique solution equation (3.1):
Theorem 3.1. Under Hypotheses (H1) and (H2), the process
\[ X_t = \varepsilon_t \zeta_t(A_t, \xi(A_t)) \] (3.2)
is the unique solution in \( L^2(\Omega \times [0, T]) \) of the equation (3.1), such that \( \gamma X_{1[0,t]} \in \text{Dom } \delta \), for all \( t \in [0, T] \).

Proof: We first show that the process \( X \) given in (3.2) is a solution of equation (3.1). For this we first observe that \( X \) belongs to \( L^2(\Omega \times [0, T]) \) and we let \( F \in \mathcal{S}_K \). Then, by the integration by parts formula and the Girsanov theorem, together with the fact that \( \frac{dF(T_t)}{dt} = \gamma_t(K^*KDF)(T_t, t) \), we have
\[
E[FX_t - F\xi] = E[F(T_t)\zeta_t(\xi) - F\zeta_0(\xi)] = E\left[\int_0^t \frac{d}{ds}(F(T_s)\zeta_s(\xi))ds\right]
\]
\[
= \int_0^t \gamma_s E[(K^*KDF)(T_s, s)\zeta_s(\xi)]ds + \int_0^t E[F(T_s)\varepsilon_s^{-1}(T_s)b(T_s, s, \varepsilon_s(T_s)\zeta_s(\xi))]ds
\]
\[
= \int_0^t (\gamma_s E[(K^*KDF)(s)X_s] + E[Fb(s, X_s)])ds.
\]
Hence, since \( \gamma X_{1[0,t]} \) is square integrable, Definition 2.2 implies that \( \gamma X_{1[0,t]} \) belongs to \( \text{Dom } \delta \) and the equality in (3.1) holds, for all \( t \in [0, T] \).

Now we deal with the uniqueness of equation (3.1). For this end, let \( Y \) be another solution of equation (3.1), \( F \in \mathcal{S}_K \) and \( t \in [0, T] \). Then,
\[
E[Y_t F(A_t)] = E[\xi F(A_t)] + E\left[\int_0^t F(A_t)b(s, Y_s)ds\right] + E\left[\int_0^t \gamma_s Y_s(K^*KDF(A_t))(s)ds\right].
\]
Therefore, the integration by parts formula, Fubini’s theorem as well as the fact that \( \frac{dF(A_s)}{ds} = -\gamma_s(K^*KDF(A_s))(s) \) yield
\[
E[Y_t F] = E[\xi F(A_t)] + E\left[\int_0^t \gamma_s (K^*KDF(A_s))(s)ds\right]
\]
\[
+ E\left[\int_0^t F(A_s)b(s, Y_s)ds\right] - E\left[\int_0^t \gamma_r (K^*KDF(A_r))(r) \int_0^r b(s, Y_s)dsdr\right]
\]
\[
+ E\left[\int_0^t \gamma_s Y_s(K^*KDF(A_s))(s)ds\right] - E\left[\int_0^t \int_0^r \gamma_r (K^*KDF((K^*KDF(A_r))(s))(r) \gamma_s Y_s dsdr\right].
\]
Hence, by using that \( Y \) is a solution of (3.1), Definition 2.2 and the relation
\[
(K^*KDF((K^*KDF(A_r))(s))(r) = (K^*KDF((K^*KDF(A_r))(r)))(s),
\]
we obtain
\[
E[Y_t F(A_t)] = E[\xi F] + E\left[\int_0^t F(A_s)b(s, Y_s)ds\right].
\]
Consequently, by using the Girsanov theorem again, we get
\[
Y_t(T_t)\varepsilon_t^{-1}(T_t) = \xi + \int_0^t b(T_s, s)Y_s(T_s)\varepsilon_s^{-1}(T_s)ds,
\]
which implies that \( Y_t(T_t)\varepsilon_t^{-1}(T_t) = \zeta(\xi) \). That is, \( Y_t = \varepsilon_t \zeta_t(A_t, \xi(A_t)) \), and therefore the proof is complete. \( \blacksquare \)
3.2 Fractional backward doubly stochastic differential equations

In this section we state some of the main results of this paper. Namely, the existence and uniqueness of backward doubly stochastic differential equations driven by both a fractional Brownian motion \( B \) and a standard Brownian motion \( W \).

Let \( \{ B_t : 0 \leq t \leq T \} \) be a one-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \), defined on the classical Wiener space \((\Omega, \mathcal{F}^B, P^B)\) with \( \mathcal{F}^B = C_0([0, T]; \mathbb{R}) \), and \( \{ W_t = (W^1_t, W^2_t, \ldots, W^d_t) : 0 \leq t \leq T \} \) a \( d \)-dimensional canonical Brownian motion defined on the classical Wiener space \((\Omega^W, \mathcal{F}^W, P^W)\) with \( \mathcal{F}^W = C_0([0, T]; \mathbb{R}^d) \). We put \((\Omega, \mathcal{F}, P) = (\Omega^B, \mathcal{F}^B, P^B) \otimes (\Omega^W, \mathcal{F}^W, P^W)\) and let \( \mathcal{F} = \mathcal{F}^B \vee \mathcal{F}^W \), where \( \mathcal{F} \) is the class of the \( P \)-null sets. We denote again by \( B \) and \( W \) their canonical extension from \( \Omega^B \) and \( \Omega^W \), respectively, to \( \Omega \).

We let \( \mathcal{F}_{t,T}^W = \sigma \{ W_T - W_s, s \leq t \} \vee \mathcal{N} \), \( \mathcal{F}_t^B = \sigma \{ B_s, 0 \leq s \leq t \} \vee \mathcal{N} \), and \( \mathcal{G}_t = \mathcal{F}_{t,T}^W \vee \mathcal{F}_t^B, t \in [0, T] \). Let us point out that \( \mathcal{F}_{t,T}^W \) is decreasing and \( \mathcal{F}_t^B \) is increasing in \( t \), but \( \mathcal{G}_t \) is neither decreasing nor increasing. We denote the family of \( \sigma \)-fields \( \{ \mathcal{G}_t \}_{0 \leq t \leq T} \) by \( \mathcal{G} \). Moreover, we shall also introduce the backward filtrations \( \mathcal{H}_t = \mathcal{F}_{t,T}^W \vee \mathcal{F}_t^B, t \in [0, T] \) and \( \mathcal{F}^W = \{ \mathcal{F}_{t,T}^W \}_{t \in [0, T]} \).

Let \( S'_K \) denote the class of smooth random variables of the form

\[
F = f(B(\varphi_1), \ldots, B(\varphi_n), W(\psi_1), \ldots, W(\psi_m)),
\]

where \( \varphi_1, \ldots, \varphi_n \) are elements of the domain of the operator \( \mathcal{K}^*\mathcal{K} \), \( \psi_1, \ldots, \psi_m \in C([0, T], \mathbb{R}^d) \), \( f \in C^\infty_p(\mathbb{R}^{n+m}) \) and \( n, m \geq 1 \). Here, \( C^\infty_p(\mathbb{R}^{n+m}) \) is the set of all \( C^\infty \) functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( f \) and all its partial derivatives have polynomial growth.

The Malliavin derivative of the smooth random variable \( F \) w.r.t. \( B \) is the \( \Lambda^{1/2-H} \)-valued random variable \( D^B F \) defined by

\[
D^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (B(\varphi_1), \ldots, B(\varphi_n), W(\psi_1), \ldots, W(\psi_m)) \varphi_i,
\]

and the Malliavin derivative \( D^W F \) of the smooth random variable \( F \) w.r.t. \( W \) is given by

\[
D^W F = \sum_{i=1}^m \frac{\partial f}{\partial x_{n+i}} (B(\varphi_1), \ldots, B(\varphi_n), W(\psi_1), \ldots, W(\psi_m)) \psi_i.
\]

**Definition 3.2.** (Skorohod integral w.r.t. \( B \). Extension of Definition 2.2.) We say that \( u \in L^2(\Omega \times [0, T]) \) belongs to \( \text{Dom } \delta^B \) if there exists a random variable \( \delta^B(u) \in L^2(\Omega) \) such that

\[
E \left[ (\mathcal{K}^*\mathcal{K} D^B F, u)_{L^2([0, T])} \right] = E \left[ F \delta^B(u) \right], \quad \text{for all } F \in S'_K.
\]

We call \( \delta^B(u) \) the Skorohod integral with respect to \( B \).

**Definition 3.3.** (Skorohod integral w.r.t. \( W \).) We say that \( u \in L^2(\Omega \times [0, T]) \) belongs to \( \text{Dom } \delta^W \) if there exists a random variable \( \delta^W(u) \in L^2(\Omega) \) such that

\[
E \left[ \int_0^T (D^W F) u_s ds \right] = E \left[ F \delta^W(u) \right], \quad \text{for all } F \in S'_K.
\]

We call \( \delta^W(u) \) the Skorohod integral with respect to \( W \).

For the the Skorohod integral with respect to \( W \) we have, in particular, the following well known result:

**Proposition 3.4.** Let \( u \in L^2(\Omega \times [0, T]) \) be \( \mathcal{H} \)-adapted. Then the Itô backward integral \( \int_0^T u_s \downarrow dW_s \) coincides with the Skorohod integral with respect to \( W \):

\[
\int_0^T u_s \downarrow dW_s = \delta^W(u).
\]
The extension to Ω of the operators $T_t$ and $A_t$ introduced in subsection 2.3 as acting over $\Omega'$, is done in a canonical way:

$$T_t(\omega', \omega'') := (T_t \omega', \omega''), \quad A_t(\omega', \omega'') := (A_t \omega', \omega''), \quad \text{for} \ (\omega', \omega'') \in \Omega = \Omega' \times \Omega''.$$

We denote by $L^2_G(0, T; \mathbb{R}^n)$ (resp., $L^2_H(0, T; \mathbb{R}^n)$) the set of $n$-dimensional measurable random processes $\{\varphi_t, t \in [0, T]\}$ which satisfy:

i) $E \left[ \int_0^T |\varphi_t|^2 dt \right] < +\infty$,

ii) $\varphi_t$ is $\mathcal{G}_t$- (resp., $\mathcal{H}_t$-) measurable, for a.e. $t \in [0, T]$.

We shall introduce a subspace of $L^2_G(0, T; \mathbb{R}^n)$, which stems its importance from its invariance with respect to a class of Girsanov transformations. Recalling the notation 

$$I_T^* := \sup_{t \in [0, T]} \left| \int_0^t \gamma_s dB_s \right|$$

from subsection 2.3, we define $L^2_G^{2, *}(0, T; \mathbb{R} \times \mathbb{R}^d)$ to be the space of all $\mathcal{G}$-adapted processes $(Y, Z)$ which are such that

$$E \left[ \exp\{p I_T^* \} \int_0^T (|Y_t|^2 + |Z_t|^2) \ dt \right] < \infty, \text{ for all } p \geq 1.$$

For the space $L^2_G^{2, *}(0, T; \mathbb{R} \times \mathbb{R}^d)$ we have the following invariance property:

**Proposition 3.5.** For all processes $(Y, Z) \in L^2_G^{2, *}(0, T; \mathbb{R} \times \mathbb{R}^d)$ we have:

i) $(\tilde{Y}_t, \tilde{Z}_t) := (Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t)) \in L^2_G^{2, *}(0, T; \mathbb{R} \times \mathbb{R}^d)$ and

ii) $(\bar{Y}_t, \bar{Z}_t) := (Y_t(A_t)\varepsilon_t, Z_t(A_t)\varepsilon_t) \in L^2_G^{2, *}(0, T; \mathbb{R} \times \mathbb{R}^d)$.

**Proof:** Since the proofs of i) and ii) are similar, we only prove i): For the case of $(\bar{Y}, \bar{Z})$, from the Girsanov transformation and Lemma 2.4 we have

$$E \left[ \exp\{p I_T^* \} \int_0^T \left( |\tilde{Y}_t|^2 + |\tilde{Z}_t|^2 \right) \ dt \right]$$

$$= \int_0^T E \left[ \exp\{p I_T^* \} (|Y_t(T_t)|^2 + |Z_t(T_t)|^2)\varepsilon_t^{-2}(T_t) \right] \ dt$$

$$= \int_0^T E \left[ \exp\{p I_T^* \} (|Y_t|^2 + |Z_t|^2)\varepsilon_t^{-1} \right] \ dt$$

$$\leq E \left[ \exp \left\{ p \sup_{0 \leq \tau \leq T} \left| \int_0^\tau \gamma_s dB_s \right| + \sup_{0 \leq \tau \leq T} \int_0^T (K\gamma_1[0, t](s)(K\gamma_1[0, t])(s)) ds \right\} \int_0^T (|Y_t|^2 + |Z_t|^2)\varepsilon_t^{-1} \ dt \right]$$

$$\leq CE \left[ \exp\{(p + 1) I_T^* \} \int_0^T (|Y_t|^2 + |Z_t|^2) \ dt \right]$$

$$< +\infty, \text{ for all } p \geq 1.$$

Hence, the proof is complete. \hfill \blacksquare

We now consider the following type of backward doubly stochastic differential equation driven by the Brownian motion $W$ and the fractional Brownian motion $B$:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\ ds - \int_0^t Z_s \ dW_s + \int_0^t \gamma_s Y_s dB_s, \quad t \in [0, T]. \quad (3.4)$$

Here $\xi \in L^2(\Omega, \mathcal{F}_0^W, P)$ and $f : \Omega'' \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function such that
(H3) i). $f(\cdot, t, y, z)$ is $\mathcal{F}_{t,T}^W$-measurable, for all $t \in [0, T]$, and for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$;

(H3) ii). $f(\cdot, 0, 0) \in L^2(\Omega \times [0, T])$;

(H3) iii). There exists a constant $C \in \mathbb{R}^+$ such that for all $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \text{ a.e., a.s.}$$

Remark 3.6. Let us refer to some special cases of the above BDSDE:

i) If $\gamma = 0$, equation (3.4) becomes a classical BSDE (Pardoux and Peng [19]) with a unique solution $(Y, Z) \in L^2_{\mathbb{P}}(0, T; \mathbb{R} \times \mathbb{R}^d)$;

ii) If $\xi \in \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are deterministic, we can choose $Z = 0$ and $Y \in L^2(\mathbb{P} \times [0, T])$ with $\gamma Y \mathbb{1}_{[0,t]} \in \text{Dom}(\delta^B)$, $t \in [0, T]$, as the unique solution of the fractional SDE

$$Y_t = \xi + \int_0^t f(s, Y_s, 0)ds + \int_0^t \gamma sY_s dB_s, \ t \in [0, T],$$

which can be solved due to subsection 3.1.

iii) Pardoux and Peng [20] considered backward doubly SDEs in the case that $B$ and $W$ are two independent Brownian motions and in a nonlinear framework.

We let $\tilde{\Omega} := \{\omega' \in \Omega' | I_T'(\omega') < \infty\}$, which satisfies $P^B(\tilde{\Omega}) = 1$ from Lemma 2.4. We first establish the following theorem:

**Theorem 3.7.** For all $\omega' \in \tilde{\Omega}$, the backward stochastic differential equation

$$\hat{Y}_t(\omega', \cdot) = \xi + \int_0^t f(s, \hat{Y}_s(\omega', \cdot), \hat{Z}_s(\omega', \cdot)) ds - \int_0^t \hat{Z}_s(\omega', \cdot) dW_s, \ t \in [0, T],$$

$t \in [0, T]$, has a unique solution $(\hat{Y}(\omega', \cdot), \hat{Z}(\omega', \cdot)) \in L^2_{\mathbb{P}}(0, T; \mathbb{R} \times \mathbb{R}^d)$.

Moreover, putting $(\hat{Y}_t(\omega', \cdot), \hat{Z}_t(\omega', \cdot)) := (0, 0)$, for $\omega' \in \tilde{\Omega}$, the random variable $(\hat{Y}_t(\omega', \omega''), \hat{Z}_t(\omega', \omega''))$ is jointly measurable in $(\omega', \omega'')$, and $(\hat{Y}, \hat{Z}) \in L^2_{\mathbb{P}}(0, T; \mathbb{R} \times \mathbb{R}^d)$.

Furthermore, there exists a positive constant $C$ (only depending on the $L^2$-norm of $\xi$ and $K\gamma \mathbb{1}_{[0,t]}$, $L^2$-bound of $f(\cdot, 0, 0)$ and the Lipschitz constant of $f$) such that, for all $\omega' \in \tilde{\Omega}$:

$$E^W \left[ \sup_{t \in [0, T]} |\hat{Y}_t(\omega', \cdot)|^2 + \int_0^T |\hat{Z}_t(\omega', \cdot)|^2 dt \right] \leq C \exp\{2I_T'(\omega')\}. \quad (3.6)$$

**Proof:** We put $F_s(\omega', y, z) = f(s, y \mathbb{1}_{(T_s, \omega')}(T_s, \omega'), z \mathbb{1}_{(T_s, \omega')}(T_s, \omega')) \mathbb{1}_{(T_s, \omega')}(s, \omega')$, $s \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d, \omega' \in \tilde{\Omega}$. Obviously,

i) $F_s(\cdot, y, z)$ is $\mathcal{G}_s$-measurable and $F_s(\omega', \cdot, y, z)$ is $\mathcal{F}_{s,T}^W$-measurable, $\omega' \in \tilde{\Omega}$;

ii) $F_s(\omega', \omega'', y, z)$ is Lipschitz in $(y, z)$, uniformly with respect to $(s, \omega', \omega'')$;

iii) $|F_s(\omega', \omega', 0, 0)| \leq C|f(s, 0, 0)| \exp\{I_T'(\omega')\}$.

Using $F_s$, equation (3.5) can be rewritten as follows:

$$\hat{Y}_t(\omega') = \xi + \int_0^t F_s(\omega', \hat{Y}_s(\omega', \cdot), \hat{Z}_s(\omega', \cdot)) ds - \int_0^t \hat{Z}_s(\omega', \cdot) dW_s, \ t \in [0, T], \omega' \in \tilde{\Omega}. \quad (3.7)$$

**Step 1:** We begin by proving the existence: From the conditions i)-iii) and standard BSDE arguments (see: Pardoux and Peng [19]) we know that, for all $\omega' \in \tilde{\Omega}$, there is a unique solution $(\hat{Y}(\omega', \cdot), \hat{Z}(\omega', \cdot)) \in L^2_{\mathbb{P}}(0, T; \mathbb{R} \times \mathbb{R}^d)$. On the other hand, the joint measurability of $F_s$ with respect
to \((\omega', \omega'')\) allows to show that, extended to \(\Omega' \times \Omega''\) by putting \((\bar{Y}_t(\omega', \cdot), \bar{Z}_t(\omega', \cdot)) := (0, 0), \ \omega' \in \bar{\Omega}'\), the process \((\bar{Y}, \bar{Z})\) is \(\mathbb{H}\)-adapted. Let us show that \((\bar{Y}, \bar{Z}) \in L^2_{\mathbb{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\). For this end, it suffices to prove (3.6).

Let \(\omega' \in \bar{\Omega}'\) be arbitrarily fixed. By applying Itô’s formula to \(|\bar{Y}_t|^2\) we have at \(\omega', P^{W\cdot}\)-a.s.,

\[
d |\bar{Y}_t|^2 = 2 \bar{Y}_t \left( F_t \left( \bar{Y}_t, \bar{Z}_t \right) dt - \bar{Z}_t \downarrow dW_t \right) - |\bar{Z}_t|^2 dt.
\]

It follows that at \(\omega', P^{W\cdot}\)-a.s.,

\[
|\bar{Y}_t|^2 + \int_0^t |\bar{Z}_s|^2 ds = \xi^2 + 2 \int_0^t \bar{Y}_s F_s \left( \bar{Y}_s, \bar{Z}_s \right) ds - 2 \int_0^t \bar{Y}_s \bar{Z}_s \downarrow dW_s \leq \xi^2 + \int_0^t \left( C_1 |\bar{Y}_s|^2 + \frac{1}{2} |\bar{Z}_s|^2 + C_2 |F_s(0, 0)|^2 \right) ds - 2 \int_0^t \bar{Y}_s \bar{Z}_s \downarrow dW_s.
\]

Hence, at \(\omega', P^{W\cdot}\)-a.s.,

\[
|\bar{Y}_t|^2 + \frac{1}{2} \int_0^t |\bar{Z}_s|^2 ds \leq \xi^2 + \int_0^t \left( C_1 |\bar{Y}_s|^2 + C_3 \varepsilon_s^{-2} \right) ds - 2 \int_0^t \bar{Y}_s \bar{Z}_s \downarrow dW_s.
\]

Taking the expectation with respect to \(P^{W}\), we notice that

\[
E^{W} \left[ \left( \int_0^T |\bar{Y}_t(\omega', \cdot)\bar{Z}_t(\omega', \cdot)|^2 dt \right)^{1/2} \right] \leq \left( E^{W} \left[ \sup_{t \in [0, T]} |\bar{Y}_t(\omega', \cdot)|^2 \right] \right)^{1/2} \left( E^{W} \left[ \int_0^T |\bar{Z}_t(\omega', \cdot)|^2 dt \right] \right)^{1/2} < +\infty.
\]

Consequently, \(E^{W} \left[ \int_0^t |\bar{Y}_s(\omega', \cdot)|^2 ds \right] = 0\), and by taking the conditional expectation with respect to \(\mathcal{F}_{\frac{1}{2}}\), we obtain

\[
E^{W} \left[ \left( \bar{Y}_t(\omega', \cdot) \right)^2 + \frac{1}{2} \int_0^t |\bar{Z}_s(\omega', \cdot)|^2 ds \right] \leq E^{W} \left[ \xi^2 \right] + \int_0^t C_1 E^{W} \left[ |\bar{Y}_s(\omega', \cdot)|^2 \right] ds + C_4 \exp\{2I_T(\omega')\}.
\]

Thus, from Gronwall’s inequality, we have

\[
E^{W} \left[ |\bar{Y}_t(\omega', \cdot)|^2 \right] \leq \left( E^{W} \left[ \xi^2 \right] + C_4 \exp\{2I_T(\omega')\} \right) \exp\{C_1 t\}, t \in [0, T],
\]

which, combined with the previous estimate, yields

\[
E^{W} \left[ \int_0^T \left( |\bar{Y}_t(\omega', \cdot)|^2 + |\bar{Z}_t(\omega', \cdot)|^2 \right) dt \right] \leq C \exp\{2I_T(\omega')\}.
\]

In order to get the estimate (3.6), it suffices now to estimate \(E^{W} \left[ \sup_{t \in [0, T]} |\bar{Y}_t(\omega', \cdot)|^2 \right]\) by using equation (3.7), Burkholer-Davis-Gundy’s inequality and the above estimate.

To prove that \((\bar{Y}, \bar{Z})\) belongs even to \(L^2_{\mathbb{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\), we have to prove the uniqueness in \(L^2_{\mathbb{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\).
Step 2: We suppose that \((\tilde{Y}^1, \tilde{Z}^1)\) and \((\tilde{Y}^2, \tilde{Z}^2)\) are two solutions of equation (3.7) belonging to \(L^2_{\mathbb{H}}(0, T; \mathbb{R} \times \mathbb{R}^d)\) (Notice that \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d) \subset L^2_{\mathbb{H}}(0, T; \mathbb{R} \times \mathbb{R}^d)\)). Putting \(\Delta \tilde{Y}_t = \tilde{Y}^1_t - \tilde{Y}^2_t\) and \(\Delta \tilde{Z}_t = \tilde{Z}^1_t - \tilde{Z}^2_t\), we have

\[
\Delta \tilde{Y}_t = \int_0^t \left[ F_s \left( \tilde{Y}^1_s, \tilde{Z}^1_s \right) - F_s \left( \tilde{Y}^2_s, \tilde{Z}^2_s \right) \right] ds - \int_0^t \Delta \tilde{Z}_s \downarrow dW_s, \quad t \in [0, T].
\]

By applying Itô’s formula to \(|\Delta \tilde{Y}_t|^2\), we get that

\[
E|\Delta \tilde{Y}_t|^2 + E \left[ \int_0^t |\Delta \tilde{Z}_s|^2 \, ds \right] = 2E \left[ \int_0^t \Delta \tilde{Y}_s \left[ F_s \left( \tilde{Y}^1_s, \tilde{Z}^1_s \right) - F_s \left( \tilde{Y}^2_s, \tilde{Z}^2_s \right) \right] ds \right]
\leq E \left[ \int_0^t \left( 2C_0 + C_0^2 \right) |\Delta \tilde{Y}_s|^2 + \frac{1}{2} |\Delta \tilde{Z}_s|^2 \right] ds.
\]

and finally from Gronwall’s lemma, we conclude that \(\Delta \tilde{Y}_t = 0, \Delta \tilde{Z}_t = 0\), a.s., a.e.

Step 3: Let us now show that \((\tilde{Y}, \tilde{Z})\) is not only in \(L^2_{\mathbb{H}}(0, T; \mathbb{R} \times \mathbb{R}^d)\) but even in \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\). For this we consider for an arbitrarily given \(\tau \in [0, T]\) equation (3.7) over the time interval \([0, \tau]\):

\[
\tilde{Y}^\tau_t = \xi + \int_0^t F_s \left( \tilde{Y}^\tau_s, \tilde{Z}^\tau_s \right) ds - \int_0^t \tilde{Z}^\tau_s \downarrow dW_s, \quad t \in [0, \tau].
\]

Let \(\mathcal{H}^\tau_t := \mathcal{F}^{W,T} \vee \mathcal{F}^B_t, \ t \in [0, \tau]\). Then \(\mathcal{H}^\tau = \{\mathcal{H}^\tau_t\}_{t \in [0, \tau]}\) is a filtration with respect to which \(W\) has the martingale representation property. Since \(F_t(y, z)\) is \(\mathcal{G}_t\)- and hence also \(\mathcal{H}^\tau_t\)-measurable, \(dt\) a.e. on \([0, \tau]\), it follows from the classical BSDE theory that BSDE (3.8) admits a solution \((\tilde{Y}^\tau, \tilde{Z}^\tau)\) belongs to \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\). Hence, \((\tilde{Y}^\tau_t, \tilde{Z}^\tau_t) = (\tilde{Y}^\tau_t, \tilde{Z}^\tau_t), dt\) a.e., for \(t < \tau\). Consequently, \((\tilde{Y}^\tau, \tilde{Z}^\tau)\) is \(\mathcal{H}^\tau_t\)- measurable, \(dt\) a.e., for \(t < \tau\). Therefore, letting \(\tau \downarrow t\) we can deduce from the right continuity of the filtration \(\mathcal{F}^B\) that \((\tilde{Y}, \tilde{Z})\) is \(\mathcal{G}\)-adapted.

It still remains to prove that \((\tilde{Y}, \tilde{Z})\) belongs to \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\).

Step 4: For the proof that \((\tilde{Y}, \tilde{Z})\) belongs to \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\) we notice that, by the above estimates and Lemma 2.4,

\[
E \left[ \left| \exp \left\{ pI^*_{\tau_T} \right\} \int_0^T \left( |\tilde{Y}_t|^2 + |\tilde{Z}_t|^2 \right) \, dt \right| \right] \leq E \left[ C \exp \{ (2 + p)I^*_{\tau_T} \} \right] < \infty, \quad \text{for all } p \geq 1.
\]

Hence, the proof is complete.

Corollary 3.8. The process \((\tilde{Y}, \tilde{Z})\) given by Theorem 3.7 is the unique solution of equation (3.5) in \(L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)\).

Now we state the main result of this subsection:

Theorem 3.9. 1) Let \((\tilde{Y}, \tilde{Z})\) be a solution of BSDE (3.5). Then

\[
\{(Y_t, Z_t), t \in [0, T]\} = \left\{ \left( \tilde{Y}_t(A_t)\xi_t, \tilde{Z}_t(A_t)\xi_t \right), t \in [0, T] \right\} \in L^2_{\mathcal{G}}(0, T; \mathbb{R} \times \mathbb{R}^d)
\]

is a solution of equation (3.4) with \(\gamma Y_{1|0,t} \in \operatorname{Dom} \delta^B\), for all \(t \in [0, T]\).
2) Conversely, given an arbitrary solution \((Y,Z) \in L^2_d(0,T;\mathbb{R} \times \mathbb{R}^d)\) of the equation (3.4) with \(\gamma Y_{[0,t]} \in Dom \, \delta^B\), for all \(t \in [0,T]\), the process
\[
\left\{ \left( \tilde{Y}_t, \tilde{Z}_t \right), t \in [0,T] \right\} = \{(Y_t(T_t), Z_t(T_t) \gamma^{-1}(T_t), t \in [0,T]\} \in L^2_d(0,T;\mathbb{R} \times \mathbb{R}^d)
\]
is a solution of BSDE (3.5).

From the Theorems 3.7 and 3.9 we can immediately conclude the following

**Corollary 3.10.** The solution of equation (3.4) in \(L^2_d(0,T;\mathbb{R} \times \mathbb{R}^d)\) exists and is unique.

**Proof (of Theorem 3.9):** We first prove that, given the solution \((\tilde{Y}, \tilde{Z})\) of equation (3.5), \((Y,Z)\) defined in the theorem solves (3.4). For this we notice that for \(F\) being an arbitrary but fixed element of \(\mathcal{S}_\mathcal{K}\), from Girsanov transformation and from the equation (3.5), it follows
\[
E[FY_t - F\xi] = E[F(T_t)\tilde{Y}_t - F\tilde{Y}_0] = E[F(T_t)\tilde{Y}_t + F(T_t)\int_0^t F_s(\tilde{Y}_s, \tilde{Z}_s) \, ds - F(T_t)\int_0^t \tilde{Z}_s \downarrow dW_s - F\tilde{Y}_0].
\]
We recall that \(E[F(T_t)\int_0^t \tilde{Z}_s \downarrow dW_s] = E[\int_0^t D^W_s F(T_t)\tilde{Z}_s \, ds]\). Thus, from the fact that \(\frac{d}{dt}F(T_t) = \gamma_t(K^s KD^B F)(t,T_t)\), we have
\[
E[FY_t - F\xi] = E\left[\int_0^t \gamma_s(K^s KD^B F)(s,T_s) \, ds + \int_0^t \gamma_r(K^s KD^B F)(r,T_r)dr F_s(\tilde{Y}_s, \tilde{Z}_s) \, ds - \int_0^t D^W_s \gamma_r(K^s KD^B F)(r,T_r)dr \tilde{Z}_s \, ds\right].
\]
Moreover, from Fubini’s theorem, the definition of the Skorohod integral with respect to \(W\), and from Proposition 3.4, we obtain that
\[
E\left[\int_0^t \gamma_r(K^s KD^B F)(r,T_r)dr \tilde{Z}_s \, ds\right] = \int_0^t \gamma_r E\left[\int_0^r D^W_s (K^s KD^B F)(r,T_r)\tilde{Z}_s \, ds\right]dr
\]
\[
= \int_0^t \gamma_r E\left[\int_0^r D^W_s (K^s KD^B F)(r,T_r)\tilde{Z}_s \, ds\right]dr.
\]
Thus, by applying the inverse Girsanov transformation as well as Fubini’s theorem, we obtain
\[
E[FY_t - F\xi] = E\left[\int_0^t F(T_s)F_s(\tilde{Y}_s, \tilde{Z}_s) \, ds - \int_0^t D^W_s F(T_s)\tilde{Z}_s \, ds\right]
\]
\[
+ E\left[\int_0^t \gamma_s(K^s KD^B F)(s,T_s)(\tilde{Y}_0 + \int_0^s F_r(\tilde{Y}_r, \tilde{Z}_r)dr - \int_0^s \tilde{Z}_r \downarrow dW_r) \, ds\right]
\]
\[
= E\left[\int_0^t Ff(s,Y_s, Z_s)ds\right] - E\left[\int_0^t D^W_s FZ_s \, ds\right] + E\left[\int_0^t \gamma_s(K^s KD^B F)(s)Y_s \, ds\right],
\]
where, for the latter expression, we have used that \(\tilde{Y}_s\) is a solution of (3.5). Since \(Z \in L^2_d(0,T;\mathbb{R} \times \mathbb{R}^d) \subset L^2_d(0,T;\mathbb{R} \times \mathbb{R}^d)\), it holds
\[
E\left[\int_0^t D^W_s FZ_s \, ds\right] = E\left[F \int_0^t Z_s \downarrow dW_s\right].
\]
Consequently, \( E \left[ \int_0^t (K^* KD^B F)(s) Y_s ds \right] = E \left[ \int_0^t \left( Y_t - \xi - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s \downarrow dW_s \right) \right] \) holds for all \( F \in \mathcal{S}_K \). From Proposition 3.5 we know that both, \( \left( \tilde{Y}, \tilde{Z} \right) \) and \((Y, Z)\), belong to \( L^2_{2*}(0, T; \mathbb{R} \times \mathbb{R}^d) \). Consequently, \( Y_t - \xi - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s \downarrow dW_s \in L^2(\Omega, F, P) \), for all \( t \in [0, T] \). Moreover, using (3.6),

\[
E \left[ \int_0^T |\gamma_{1[0,t]}(r) Y_r|^2 dr \right] = E \left[ \int_0^T |\gamma_{1[0,t]}(r) \tilde{Y}_r(\xi_r)|^2 \varepsilon_r^2 dr \right]
\]

\[
= E \left[ \int_0^T |\gamma_{1[0,t]}(r) \tilde{Y}_r|^2 \varepsilon_r(I_T^*) dr \right] \leq CE \left[ \int_0^T |\gamma_{1[0,t]}(r) \tilde{Y}_r|^2 \exp\{I_T^*\} dr \right]
\]

\[
\leq CE \left[ \int_0^T |\gamma_{1[0,t]}(r)|^2 \sup_{r \in [0,T]} |\tilde{Y}_r|^2 \exp\{I_T^*\} dr \right] \leq C \int_0^T |\gamma_{1[0,t]}(r)|^2 E \left[ \exp\{C_I^*\} \right] dr
\]

\[
\leq C \int_0^T |\gamma_{1[0,t]}(r)|^2 dr < \infty.
\]

Thus, according to the definition of the Skorohod integral with respect to \( B \) we then conclude \( \gamma Y 1_{[0,t]} \in Dom \delta^B \) and

\[
\int_0^t \gamma_s Y_s dB_s = Y_t - \left( \xi + \int_0^t f(s, Y_s, Z_s) ds - \int_0^t Z_s \downarrow dW_s \right), \ t \in [0, T].
\]

Hence \( Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds - \int_0^t Z_s \downarrow dW_s + \int_0^t \gamma_s Y_s dB_s, t \in [0, T]. \)

Let us prove now the second assertion of the Theorem. For this end we let \( Y, Z) \in L^2_{2*}(0, T; \mathbb{R} \times \mathbb{R}^d) \) be an arbitrary solution of equation (3.4) and \( F \) be an arbitrary but fixed element of \( \mathcal{S}_K \). Then we have

\[
E[\gamma Y_t F(A_t)] = E[\gamma Y_0 F(A_0)] + E\left[ \int_0^t F(A_s) f(s, Y_s, Z_s) ds \right] - E \left[ F(A_t) \int_0^t Z_s \downarrow dW_s \right] + E \left[ F(A_t) \int_0^t \gamma_s Y_s dB_s \right]
\]

\[
= I_1 + I_2 - I_3 + I_4,
\]

where, using the fact that \( \frac{d}{dt} F(A_t) = -\gamma_t (K^* KD^B F(A_t))(t) \) and Fubini's theorem,

\[
I_1 = E[\gamma Y_0 F(A_0)] - E \left[ \xi \int_0^t \gamma_s (K^* KD^B F(A_s))(s) ds \right],
\]

\[
I_2 = E \left[ \int_0^t F(A_s) f(s, Y_s, Z_s) ds \right] - E \left[ \int_0^t \gamma_t (K^* KD^B F(A_t))(r) \int_0^r f(s, Y_s, Z_s) ds dr \right].
\]

From the duality between the Itô backward integral and the Malliavin derivative \( D^W \) (recall that \( Z \) is square integrable), we get

\[
I_3 = E \left[ \int_0^t Z_s D^W f(A_s) ds \right] - E \left[ \int_0^t Z_s D^W \int_0^t \gamma_r (K^* KD^B F(A_r))(r) dr ds \right]
\]

\[
= E \left[ \int_0^t Z_s D^W f(A_s) ds \right] - E \left[ \int_0^t \int_0^r Z_s \gamma_r D^W (K^* KD^B F(A_r))(r) ds dr \right]
\]

\[
= E \left[ \int_0^t Z_s D^W f(A_s) ds \right] - E \left[ \int_0^t \gamma_s (K^* KD^B F(A_s))(s) \int_0^s Z_r \downarrow dW_r ds \right].
\]
Moreover, from the duality between the integral w.r.t. \( B \) and \( D^B \) (observe that \( \gamma Y_{[0,t]} \in \text{Dom } \delta^B \)) we obtain
\[
I_4 = E \left[ \int_0^t \gamma_s Y_s (K^* K D^B F(A_s))(s) \, ds \right] - E \left[ \int_0^t \int_0^r \gamma_r (K^* K D^B (K^* K D^B F(A_r))(r))(s) \gamma_s Y_s \, ds \, dr \right] 
= E \left[ \int_0^t \gamma_s Y_s (K^* K D^B F(A_s))(s) \, ds \right] - E \left[ \int_0^t \gamma_s (K^* K D^B F(A_s))(s) \int_0^s \gamma_r Y_r \, dB_r \, ds \right].
\]
Consequently, using the fact that \( Y, Z \) is a solution of equation (3.4), i.e.,
\[
Y_t = \xi + \int_0^t f(s, Y_s, Z_s) \, ds - \int_0^t Z_s \downarrow dW_s + \int_0^t \gamma_s Y_s \, dB_s, \quad t \in [0, T],
\]
we obtain that \( E[Y_t F(A_t)] = E[\xi F] + E \left[ \int_0^t F(A_s) f(s, Y_s, Z_s) \, ds \right] - E \left[ \int_0^t Z_s D^W_s F(A_s) \, ds \right]. \)
Therefore, by applying Girsanov transformation again and taking into account the arbitrariness of \( F \in S^*_K \), it follows that for all \( F \in S^*_K \),
\[
E \left[ \int_0^t Z_s(T_s) \xi^{-1}_s(T_s) D^W_s F \, ds \right] = E \left[ F \left\{ \xi - Y_t(T_t) \xi^{-1}_t(T_t) + \int_0^t f(s, Y_s(T_s), Z_s(T_s)) \xi^{-1}_s(T_s) \, ds \right\} \right].
\]
Now, since according to Proposition 3.5, \( Y_t(T_t) \xi^{-1}_t(T_t), Z_t(T_t) \xi^{-1}_t(T_t) \in L^2_{\alpha^*}(0, T; \mathbb{R} \times \mathbb{R}^d) \), we have \( Y_t(T_t) \xi^{-1}_t(T_t) - \xi - \int_0^t f(s, Y_s(T_s), Z_s(T_s)) \xi^{-1}_s(T_s) \, ds \in L^2(\Omega, \mathcal{F}, P) \). Therefore,
\[
Y_t(T_t) \xi^{-1}_t(T_t) = \xi + \int_0^t f(s, Y_s(T_s), Z_s(T_s)) \xi^{-1}_s(T_s) \, ds - \int_0^t Z_s(T_s) \xi^{-1}_s(T_s) \downarrow dW_s, a.s., \quad (3.9)
\]
for all \( t \in [0, T] \), which means that \( (\hat{Y}, \hat{Z}) := \{ Y_t(T_t) \xi^{-1}_t(T_t), Z_t(T_t) \xi^{-1}_t(T_t), t \in [0, T] \} \) is a solution of equation (3.5). Hence, the proof is complete.

4 The associated stochastic partial differential equations

In this section we will use the following standard assumptions:

(H4) i). The functions \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}, b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \Phi : \mathbb{R}^d \to \mathbb{R} \) are Lipschitz.

(H4) ii). The function \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is continuous, \( f(t, \cdot, \cdot, \cdot) \) is Lipschitz, uniformly with respect to \( t \) and \( f(\cdot, 0, 0, 0) \in L^2(\Omega \times [0, T]) \).

We denote by \( (X^{t,x}_s)_{0 \leq s \leq t} \) the unique solution of the following stochastic differential equation:
\[
\begin{cases}
  dX^{t,x}_s = -b(X^{t,x}_s) \, ds - \sigma(X^{t,x}_s) \downarrow dW_s, \quad s \in [0, t], \\
  X^{t,x}_0 = x \in \mathbb{R}^d.
\end{cases}
\]

(4.1)

We note that this equation looks like a backward stochastic differential equation, but due to the backward Itô integral, the SDE (4.1) is indeed a classical forward SDE. Under our standard assumptions on \( \sigma \) and \( b \), it has a unique strong solution which is \( \{ F^{W}_{s,t} \}_{s \leq t} \)-adapted. The following result provides some standard estimates for the solution of equation (4.1) (cf. [21] Lemma 2.7).

Lemma 4.1. Let \( X^{t,x} = \{ X^{t,x}_s, s \in [0, t] \} \) be the solution of the SDE (4.1). Then

(i) There exists a continuous version of \( X^{t,x} \) such that \( (s, x) \mapsto X^{t,x}_s \) is locally Hölder \( (C^{\alpha,2\alpha}, \text{for all } \alpha \in (0, 1/2)) \);

(ii) For all \( q \geq 1 \), there exists \( C_q > 0 \) such that, for \( t, t' \in [0, T] \) and \( x, x' \in \mathbb{R}^d \),
\[
E \left[ \sup_{0 \leq s \leq t} |X^{t,x}_s|^q \right] \leq C_q (1 + |x|^q), \quad (4.2)
\]
\[
E \left[ \sup_{0 \leq s \leq T} \left| X^{t,x}_{s\wedge t} - X^{t',x'}_{s\wedge t'} \right|^q \right] \leq C_q \left[ (1 + |x|^q + |x'|^q)(t - t'|q/2 + |x - x'|^q) \right].
\] (4.3)

With the forward SDE we associate a BDSDE with driving coefficient \( f \):
\[
Y^{t,x}_s = \Phi(X^{t,x}_0) + \int_0^s f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) \, dr - \int_0^s Z^{t,x}_r \, dW_r + \int_0^s \gamma_r Y^{t,x}_r \, dB_r, \quad s \in [0, t].
\] (4.4)

By Theorem 3.7 and Theorem 3.9, the above BDSDE has a unique solution \((Y^{t,x}_s, Z^{t,x}_s)\) given by
\[
(Y^{t,x}_s, Z^{t,x}_s) = \left( \tilde{Y}^{t,x}_s(A_s) \varepsilon_s, \tilde{Z}^{t,x}_s(A_s) \varepsilon_s \right), s \in [0, t],
\]
where, for all \( \omega' \in \tilde{\Omega}' \), \( P^W \)-a.s.,
\[
\tilde{Y}^{t,x}_s(\omega', \cdot) = \Phi(X^{t,x}_0) + \int_0^s f(r, X^{t,x}_r, \tilde{Y}^{t,x}_r(\omega', \cdot) \varepsilon_r(T_r, \omega'), \tilde{Z}^{t,x}_r(\omega', \cdot) \varepsilon_r(T_r, \omega')) \varepsilon_r^{-1}(T_r, \omega') \, dr
- \int_0^s \tilde{Z}^{t,x}_r(\omega', \cdot) \, dW_r, \quad s \in [0, t].
\] (4.5)

Pardoux and Peng [19] [20] have studied BSDEs in a Markovian context in which the driver \( F_r(x, y, z) \) is deterministic; here, in our framework the driver
\[
F_r(\omega', x, y, z) := f(r, x, y \varepsilon_r(T_r, \omega'), z \varepsilon_r(T_r, \omega')) \varepsilon_r^{-1}(T_r, \omega'), \quad (r, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d,
\]
is random but it depends only on \( B \) and is independent of the driving Brownian motion \( W \). In the following we shall define \( X^{t,x}_s, \tilde{Y}^{t,x}_s \) and \( Z^{t,x}_s \) for all \((s, t) \in [0, T]^2\) by setting \( X^{t,x}_s = x, \tilde{Y}^{t,x}_s = \tilde{Y}^{t,x}_t \) and \( Z^{t,x}_s = 0 \), for \( t < s \leq T \).

We have the following standard estimates for the solution:

**Lemma 4.2.** For all \( p \geq 1 \), there exists a constant \( C_p \in \mathbb{R}_+ \) such that for all \((t, x), (t', x') \in [0, T] \times \mathbb{R}^d, \omega' \in \tilde{\Omega}' \), \( P^W \)-a.s.,
\[
E^W \left[ \sup_{0 \leq s \leq t} \left| \tilde{Y}^{t,x}_s(\omega', \cdot) \right|^p \right]^{1/p} + \left( \int_0^t \left| \tilde{Z}^{t,x}_r(\omega', \cdot) \right|^2 \, dr \right)^{1/2} \leq C_p \left( 1 + |x| \right) \exp \left( p I^W_T(\omega') \right); \] (4.6)
\[
E^W \left[ \sup_{0 \leq s \leq t \wedge T'} \left| \tilde{Y}^{t,x}_s(\omega', \cdot) - \tilde{Y}^{t',x'}_s(\omega', \cdot) \right|^p + \left( \int_0^{t \wedge T'} \left| \tilde{Z}^{t,x}_s(\omega', \cdot) - \tilde{Z}^{t',x'}_s(\omega', \cdot) \right|^2 \, ds \right)^{p/2} \right] \leq C_p \exp \left( p I^W_T(\omega') \right) \left( |x - x'|^p + (1 + |x|^p + |x'|^p)(t - t'|p/2) \right). \] (4.7)

**Proof:** Let us fix any \( \omega' \in \tilde{\Omega}' \). For \( p \geq 2 \), by applying Itô’s formula to \( |\tilde{Y}^{t,x}_s(\omega', \cdot)|^p \), it follows that, \( P^W \)-a.s., for \( s \in [0, t],
\[
|\tilde{Y}^{t,x}_s(\omega', \cdot)|^p + \frac{p(p - 1)}{2} \int_0^s |\tilde{Y}^{t,x}_r(\omega', \cdot)|^{p-2} \left| \tilde{Z}^{t,x}_r(\omega', \cdot) \right|^2 \, dr
= |\Phi(X^{t,x}_0)|^p
+ p \int_0^s \tilde{Y}^{t,x}_r(\omega', \cdot) \left( F_r(\omega', X^{t,x}_r, \tilde{Y}^{t,x}_r(\omega', \cdot), \tilde{Z}^{t,x}_r(\omega', \cdot)) \right) \, dr + \tilde{Z}^{t,x}_s(\omega', \cdot) \, dW_r.
\]
Let $0 \leq s \leq t' \leq t \leq T$. We take the conditional expectation with respect to $\mathcal{F}_{t',t}$ on both sides of the above equality, and we obtain

$$E^{W} \left[ \left| \hat{Y}_{s}^{t,x} (\omega', \cdot) \right|^p \right]_{\mathcal{F}_{t',t}} + E^{W} \left[ \frac{p(p-1)}{2} \int_{0}^{s} \left| \hat{Y}_{r}^{t,x} (\omega', \cdot) \right|^{p-2} \left| \hat{Z}_{r}^{t,x} (\omega', \cdot) \right|^2 \, dr \right]_{\mathcal{F}_{t',t}}$$

$$= E^{W} \left[ \Phi \left( X_{0}^{t,x} \right)^p \right]_{\mathcal{F}_{t',t}} + p E^{W} \left[ \int_{0}^{s} \left| \hat{Y}_{r}^{t,x} (\omega', \cdot) \right|^{p-2} \left| \hat{Z}_{r}^{t,x} (\omega', \cdot) \right|^2 \, dr \right]_{\mathcal{F}_{t',t}}$$

$$\leq E^{W} \left[ \Phi \left( X_{0}^{t,x} \right)^p \right]_{\mathcal{F}_{t',t}} + p E^{W} \left[ \int_{0}^{s} \left( C_p \left| \hat{Y}_{r}^{t,x} (\omega', \cdot) \right|^{p} + C_p \left| X_{r}^{t,x} \right|^p \right) + \frac{p(p-1)}{4} \left| \hat{Z}_{r}^{t,x} (\omega', \cdot) \right|^{p-2} \left| \hat{Z}_{r}^{t,x} (\omega', \cdot) \right|^2 \, dr \right]_{\mathcal{F}_{t',t}}$$

Thus, from Gronwall’s inequality and (H4) we have

$$\left| \hat{Y}_{s}^{t,x} (\omega', \cdot) \right|^{p} \leq C_p \left( E^{W} \left[ \sup_{0 \leq r \leq s} \left| X_{r}^{t,x} \right|^p \right]_{\mathcal{F}_{s,t}} + \exp \left\{ p I^*_T (\omega') \right\} \right) \leq C_p \left( 1 + \left| X_{s}^{t,x} \right|^p \right) \exp \left\{ p I^*_T (\omega') \right\}$$

Consequently, by using Doob’s inequality, we get from the arbitrariness of $p \geq 1$:

$$E^{W} \left[ \sup_{0 \leq s \leq t'} \left| \hat{Y}_{s}^{t,x} (\omega', \cdot) \right|^p \right] \leq C_p \left( 1 + \left| x \right|^p \right) \exp \left\{ p I^*_T (\omega') \right\}$$

The first result follows from Burkholder-Davis-Gundy inequality applied to $\int_{0}^{t'} \left( \hat{Z}_{r}^{t,x} (\omega', \cdot) \right) \, dW_{r}$ (see, e.g., [21]).

Concerning the second assertion, without loss of generality, we can suppose $t \geq t'$. Let $0 \leq s \leq t'' \leq t'$.

Using an argument similar to that developed by Pardoux and Peng [21], we see that, for some constants $\theta > 0$ and $C > 0$,

$$E^{W} \left[ \left| \hat{Y}_{s}^{t,x} (\omega', \cdot) - \hat{Y}_{s}^{t',x'} (\omega', \cdot) \right|^p \right]_{\mathcal{F}_{t',t}}$$

$$\leq CE^{W} \left[ \Phi \left( X_{0}^{t,x} \right)^p - \Phi \left( X_{0}^{t',x'} \right)^p \right] 
\leq \int_{0}^{s} \left| \hat{Y}_{r}^{t,x} (\omega', \cdot) - \hat{Y}_{r}^{t',x'} (\omega', \cdot) \right|^p \, dr \right]_{\mathcal{F}_{t',t}}$$

$$\leq CE^{W} \left[ \Phi \left( X_{0}^{t,x} \right)^p - \Phi \left( X_{0}^{t',x'} \right)^p \right]_{\mathcal{F}_{t',t}} + \exp \left\{ p I^*_T (\omega') \right\} \left( E \left[ \int_{0}^{s} \left| X_{r}^{t,x} - X_{r}^{t',x'} \right|^2 \, dr \right]_{\mathcal{F}_{t',t}} \right)^{1/2}$$

Consequently, from Gronwall’s lemma and according to Lemma 4.1,

$$\left| \hat{Y}_{s}^{t,x} (\omega', \cdot) - \hat{Y}_{s}^{t',x'} (\omega', \cdot) \right|^{p} \leq C_p \left| X_{s}^{t,x} - X_{s}^{t',x'} \right|^{p} \exp \left\{ p I^*_T (\omega') \right\}, 0 \leq s \leq t' \leq t,$n

and

$$E^{W} \left[ \sup_{0 \leq s \leq t'} \left| \hat{Y}_{s}^{t,x} (\omega', \cdot) - \hat{Y}_{s}^{t',x'} (\omega', \cdot) \right|^p \right] \leq C_p \exp \left\{ p I^*_T (\omega') \right\} \left( 1 + \left| x \right|^p + \left| x' \right|^p \right) |t - t'|^{p/2} + \left| x - x' \right|^p \right)$$

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Finally, with the help of the Burkholder-Davis-Gundy inequality together with Lemma 2.4, we deduce that for all \( p \geq 2 \), there exists \( C_p \) such that

\[
E^W \left[ \sup_{0 \leq s \leq t'} \left| \hat{Y}_s^{t,x}(\omega', \cdot) - \hat{Y}_{s'}^{t',x'}(\omega', \cdot) \right|^p + \left( \int_0^{t'} \left| \hat{Z}_s^{t,x}(\omega', \cdot) - \hat{Z}_{s'}^{t',x'}(\omega', \cdot) \right|^2 ds \right]^{p/2} \right] 
\leq C_p \exp \left\{ p \left( \frac{1}{2} \right) \right\} \left( |x - x'|^p + (1 + |x|^p + |x'|^p)|t - t'|^{p/2} \right).
\]

The case \( p \geq 1 \) follows easily from the case \( p \geq 2 \). This completes the proof of the lemma.

We now introduce the random field: \( \tilde{u}(t, x) = \hat{Y}_t^{t,x} \), \( (t, x) \in [0, T] \times \mathbb{R}^d \), which has the following regularity properties:

**Proposition 4.3.** The random field \( \tilde{u}(t, x) \) is \( \mathcal{F}_t^B \)-measurable and we have \( \hat{Y}_s^{t,x}(\omega', \cdot) = \tilde{u}(\omega', s, X_s^{t,x}) \), \( P^W \)-a.s., \( 0 \leq s \leq t \leq T, \omega' \in \Omega' \).

**Proof:** From Theorem 3.7 with terminal time \( t \), we know that \( \hat{Y}_s^{t,x} \) is \( \mathcal{F}_s^W \cup \mathcal{F}^B_s \)-measurable. Hence \( \tilde{u}(t, x) = \hat{Y}_t^{t,x} \) is \( \mathcal{F}_t^W \cup \mathcal{F}^B_t \)-measurable. By applying Blumenthal zero-one law we deduce that \( \tilde{u} \) is \( \mathcal{F}_t^B \)-measurable and independent of \( W \). The second assertion is a direct result from the uniqueness property of the solutions of (4.1) and (4.5) (cf. [8]).

**Lemma 4.4.** The process \( \{ \hat{Y}_s^{t,x}; s, t \in [0, T], x \in \mathbb{R}^d \} \) possesses a continuous version. Moreover, \( |\tilde{u}(t, x)| \leq C \exp \{ pI_T^* \}(1 + |x|), P\text{-a.s.} \)

**Proof:** Recall that, for \( s \in [0, t] \),

\[
\hat{Y}_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f \left( r, X_r^{t,x}, \hat{Y}_r^{t,x} \varepsilon_r(T_s), \hat{Z}_r^{t,x} \varepsilon_r(T_s) \right) \varepsilon_r^{-1}(T_s) dr - \int_0^s \hat{Z}_r^{t,x} dW_r.
\]

Let \( 0 \leq t \leq t' \leq T, \ x, x' \in \mathbb{R}^d \). Then by Proposition 4.3, we have,

\[
\tilde{u}(t, x) - \tilde{u}(t', x') = E \left[ \hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'} \mid \mathcal{F}_T^B \right]
\]

and, thus,

\[
|\tilde{u}(t, x) - \tilde{u}(t', x')|^p = E \left[ |\hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'}| \mid \mathcal{F}_T^B \right]^p 
\leq C E \left[ |\hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'}|^2 \mid \mathcal{F}_T^B \right] + C E \left[ \left| \hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'} \mid \mathcal{F}_T^B \right|^p, \right.
\]

where, \( P\text{-a.s.}, \) by Lemma 4.4,

\[
E \left[ |\hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'}|^p \mid \mathcal{F}_T^B \right] \leq C \exp \{ pI_T^* \} \left( (1 + |x|^p + |x'|^p)|t - t'|^{p/2} + |x - x'|^p \right),
\]

and

\[
E \left[ |\hat{Y}_t^{t,x} - \hat{Y}_{t'}^{t',x'}|^p \mid \mathcal{F}_T^B \right] 
\leq C \left( (1 + |x'|^p)|t - t'|^p \exp \{ pI_T^* \} + C|t - t'|^{p/2} \left( E \left[ \left( \int_t^{t'} \left( |\hat{Y}_s^{t,x}|^2 + |\hat{Z}_s^{t,x}|^2 \right) ds \right)^{p/2} \right] \right) \right.
\]

\[
\leq C(1 + |x'|^p) \exp \{ pI_T^* \}|t - t'|^{p/2}.
\]
Consequently, for all \((t, x), (t', x') \in [0, T] \times \mathbb{R}^d\), \(p \geq 1\), \(\mathbb{P}\)-a.s.,

\[
|\tilde{u}(t, x) - \tilde{u}(t', x')| \leq C \exp\{I_T\} \left( (1 + |x| + |x'|)(t - t')^{1/2} + |x - x'| \right).
\]

Hence,

\[
E(|\tilde{u}(t, x) - \tilde{u}(t', x')|^p) \leq C_p \left( (1 + |x| + |x'|)^p (t - t')^{p/2} + |x - x'|^p \right),
\]

and Kolmogorov’s continuity criterion gives the existence of a continuous version of \(\tilde{u}\).

Henceforth we denote by \(\mathcal{L}\) the second-order differential operator:

\[
\mathcal{L} := \frac{1}{2} \text{tr}(\sigma \sigma^*(x) D^2 u_x) + b(x) \nabla u,
\]

and we consider the following stochastic partial differential equations:

\[
\left\{ \begin{array}{ll}
\displaystyle d\tilde{u}(t, x) = \left[ \mathcal{L}\tilde{u}(t, x) + f(t, x, \tilde{u}(t, x) \varepsilon_t(T_t), \nabla_x \tilde{u}(t, x) \sigma(x) \varepsilon_t(T_t) \varepsilon_t^{-1}(T_t) \right] dt, & t \in [0, T]; \\
\tilde{u}(0, x) = \Phi(x).
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{ll}
\displaystyle du(t, x) = \left[ \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla_x u(t, x) \sigma(x)) dt + \gamma_t u(t, x) dB_t, & t \in [0, T]; \\
u(0, x) = \Phi(x).
\end{array} \right.
\]

Our objective is to characterize \(\tilde{u}(t, x) = \hat{Y}^{t,x}_{\cdot}^1\) and \(u(t, x) = \hat{u}(A_t, t, x) \varepsilon_t = Y^{t,x}_{\cdot}^1\) as the viscosity solutions of the above stochastic partial differential equations (4.8) and (4.9), respectively.

**Remark 4.5.** In fact, equation (4.8) is a partial differential equation with random coefficients which can be solved pathwisely, and the equation (4.9) is a stochastic partial differential equation driven by the fractional Brownian motion \(B\).

First we give the definition of a pathwise viscosity solution of SPDE (4.8).

**Definition 4.6.** A real valued continuous random field \(\hat{u}: \Omega' \times [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}\) is called a pathwise viscosity solution of equation (4.8) if there exists a subset \(\tilde{\Omega}'\) of \(\Omega'\) with \(P'(\Omega') = 1\), such that for all \(\omega' \in \tilde{\Omega}'\), \(\hat{u}(\omega', \cdot, \cdot)\) is a viscosity solution for the PDE (4.8) at \(\omega'\).

For the definition of the viscosity solution, which is a well-known concept by now, we refer to the User’s Guide by Crandall et al. [7].

For the proof that \(\hat{u}(t, x)\) is the pathwise viscosity solution of equation (4.8), we need the following two auxiliary results.

**Lemma 4.7.** (Comparison result.) Let (H3) hold. Let \((\hat{Y}^1(\omega', \cdot), \hat{Z}^1(\omega', \cdot))\) and \((\hat{Y}^2(\omega', \cdot), \hat{Z}^2(\omega', \cdot))\) be the solutions of BSDE (3.5) with coefficients \((\xi_1, f_1)\) and \((\xi_2, f_2)\), respectively. Then, if \(\xi_1 \leq \xi_2\) and \(f_1 \leq f_2\), it holds that \(\hat{Y}_t^1(\omega', \cdot) \leq \hat{Y}_t^2(\omega', \cdot), t \in [0, T], P^W\)-a.s., for all \(\omega' \in \tilde{\Omega}'\).

**Proof:** For the proof the reader is referred to the comparison for BSDEs by Peng [22], or also to Buckdahn and Ma [3].

**Lemma 4.8.** (A priori estimate.) Let \(\omega' \in \tilde{\Omega}'\) and let \((\hat{Y}^1, \hat{Z}^1)\) and \((\hat{Y}^2, \hat{Z}^2)\) be the solutions of BSDE (3.5) with coefficients \((\xi_1, f_1)\) and \((\xi_2, f_2)\), respectively, and put \(\delta \hat{Y}(\omega', \cdot) = \hat{Y}^1(\omega', \cdot) - \hat{Y}^2(\omega', \cdot), \delta \xi = \xi_1 - \xi_2\) and \(\delta f_s(\omega', \cdot) = (f_1 - f_2) (s, \hat{Y}^2_s(\omega', \cdot) \varepsilon_s(T_s, \omega'), \hat{Z}^2_s(\omega', \cdot) \varepsilon_s(T_s, \omega')) \varepsilon_s^{-1}(T_s, \omega'). \) Moreover,
let $C$ be the Lipschitz constant of $f_1$. Then there exist $\beta \geq C(2 + \nu^2) + \mu^2, \nu > 0, \mu > 0$, such that $P^W$-a.s., for $0 \leq s \leq T$, and for all $\omega' \in \tilde{\Omega}'$,

$$E^W \left[ \exp\{\beta(T-s)\} \left| \delta\tilde{Y}_s(\omega', \cdot) \right|^2 \right] \leq E^W \left[ \exp\{\beta(T-s)\} \left| \beta T \right| \delta\xi^2 + \frac{1}{\mu^2} \int_0^s \exp\{\beta(T-r)\} \left| \delta f_r(\omega', \cdot) \right|^2 dr \right],$$

(4.10)

$$E^W \left[ \sup_{0 \leq r \leq T} \left| \delta\tilde{Y}_r(\omega', \cdot) \right|^2 + \int_0^T \left| \delta\tilde{Z}_r(\omega', \cdot) \right|^2 dr \right] \leq CE^W \left[ \left| \delta\xi^2 + \int_0^T \left| \delta f_r(\omega', \cdot) \right|^2 dr \right] .$$

(4.11)

Proof: Let $\omega' \in \tilde{\Omega}'$. By applying Itô’s formula to $\left| \delta\tilde{Y}_s(\omega', \cdot) \right|^2 \exp\{\beta(T-s)\}$, we obtain

$$\left| \delta\tilde{Y}_s(\omega', \cdot) \right|^2 \exp\{\beta(T-s)\} + \int_0^s \exp\{\beta(T-r)\} \left( \beta \left| \delta\tilde{Y}_r(\omega', \cdot) \right|^2 + \left| \delta\tilde{Z}_r(\omega', \cdot) \right|^2 \right) dr$$

$$+ 2 \int_0^s \exp\{\beta(T-r)\} \delta\tilde{Y}_r(\omega', \cdot) \delta\tilde{Z}_r(\omega', \cdot) \downarrow dW_r$$

$$= \left| \delta\xi^2 \exp\{\beta T\} + 2 \int_0^s \delta\tilde{Y}_r(\omega', \cdot) \exp\{\beta(T-r)\} \left[ f_1 \left( r, \tilde{Y}_r(\omega', \cdot) \varepsilon_r(T_r, \omega'), \tilde{Z}_r(\omega', \cdot) \varepsilon_r(T_r, \omega') \right) \right] \varepsilon_r^{-1}(T_r, \omega')dr, $$

and by taking the expectation with respect to $P^W$ on both sides we get, for $\nu > 0, \mu > 0$, $P$-a.s.,

$$E^W \left[ \left| \delta\tilde{Y}_s(\omega', \cdot) \right|^2 + \int_0^s \exp\{\beta(T-r)\} \left( \beta \left| \delta\tilde{Y}_r(\omega', \cdot) \right|^2 + \left| \delta\tilde{Z}_r(\omega', \cdot) \right|^2 \right) dr \right]$$

$$\leq E^W \left[ \left| \delta\xi^2 \exp\{\beta T\} \right| + \frac{\nu^2}{\mu^2} \right]$$

$$+ E^W \left[ \int_0^s \exp\{\beta(T-r)\} \left( (2 + \nu^2) \bar{C} + \mu^2 \right) \left| \delta\tilde{Y}_r(\omega', \cdot) \right|^2 + \frac{\bar{C}}{\nu^2} \left| \delta\tilde{Z}_r(\omega', \cdot) \right|^2 + \frac{\left| \delta f_r(\omega', \cdot) \right|^2}{\mu^2} \right] dr \right].$$

Finally, by choosing $\beta \geq \bar{C}(2 + \nu^2) + \mu^2$, with $\nu^2 > \bar{C}$, we obtain that $P$-a.s.,

$$E^W \left[ \left| \delta\tilde{Y}_s(\omega', \cdot) \right|^2 \right] \leq E^W \left[ \exp\{\beta T\} \left| \delta\xi^2 + \int_0^s \exp\{\beta(T-r)\} \frac{1}{\mu^2} \left| \delta f_r(\omega', \cdot) \right|^2 dr \right] ,$$

which is exactly (4.10).

Estimate (4.11) can be proven by the arguments developed for (4.7).

Let us now turn to the solutions of our SPDEs. The next theorem is one of the main results of this section. For this let $\tilde{\Omega}' = \left\{ \omega' \in \tilde{\Omega}' | \tilde{u}(\omega', \cdot, \cdot) \right\}$ continuous and notice that in light of Lemma 4.4, $P^B(\tilde{\Omega}') = 1$.

**Theorem 4.9.** The random field $\tilde{u}$ defined by $\tilde{u}(\omega', t, x) = \tilde{Y}^{t, x}_r(\omega')$ for all $\omega' \in \tilde{\Omega}'$ is a pathwise viscosity solution of equation (4.8), where $\tilde{Y}^{t, x}_r(\omega', \cdot)$ is the solution of equation (4.5). Furthermore, this solution $\tilde{u}(\omega', t, x)$ is unique in the class of continuous stochastic fields $\tilde{u} : \Omega' \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that, for some random variable $\eta \in L^1(F^H_B)$,

$$|\tilde{u}(\omega', t, x)| \leq \eta(\omega')(1 + |x|), (t, x) \in [0, T] \times \mathbb{R}^d, P^B(\text{d}\omega') \text{-a.s.}$$

**Remark 4.10.** The uniqueness of the solution is to be understood as a $P$-almost sure one: let $\tilde{u}_i(i = 1, 2)$ be such that $\tilde{u}_i(\omega')$ is a viscosity solution of PDE (4.8) at $\omega'$, for all $\omega' \in \tilde{\Omega}'$. Then, by the uniqueness result of viscosity solution of deterministic PDEs (see: Pardoux [18], Theorem 6.14) we know that $\tilde{u}_1(\omega', \cdot) = \tilde{u}_2(\omega', \cdot)$, for all $\omega' \in \tilde{\Omega}' \cap \tilde{\Omega}'_2$. In particular, $\tilde{u}_1 = \tilde{u}_2, P$-a.s.
Proof of Theorem 4.9: We adapt the method used in the paper of El Karoui et al.[8] to show that \( \hat{u} \) is a pathwise viscosity subsolution of equation \((4.8)\). Recall that the set \( \mathcal{O} := \{ \omega' \in \Omega'|I_T^\mathcal{P}(\omega') < +\infty \} \) satisfies \( P^\mathcal{B}(\mathcal{O}) = 1 \). We work here on the set \( \mathcal{O} := \{ \omega' \in \Omega'|\hat{u}(\omega', \cdot) \text{ is continuous} \} \) which satisfies \( P^\mathcal{B}(\mathcal{O}) = 1 \) in light of Lemma 4.4.

Now, according to the definition of the viscosity solution, for an arbitrarily chosen \( \omega' \in \mathcal{O} \), we fix arbitrarily a point \((t, x) \in [0, T] \times \mathbb{R}^d \) and a test function \( \varphi \in C^\infty_b \) such that \( \varphi(t, x) = \hat{u}(\omega', t, x) \) and \( \varphi \geq \hat{u}(\omega', \cdot) \).

For \( t \in [0, T] \) and \( h \geq 0 \), we have, thanks to the Proposition 4.3 and equation \((4.5)\),

\[
\hat{u}(\omega', t, x) = \hat{u}(\omega', t - h, X^x_{t-h}) \rightleftharpoons \int_{t-h}^t F_r \left( \omega', X^x_r, Y^x_r(\omega', \cdot), Z^x_r(\omega', \cdot) \right) dr - \int_{t-h}^t \hat{Z}^x_r(\omega', \cdot) \downarrow dW_r.
\]

We emphasize that for fixed \( \omega' \), this BSDE can be viewed as a classical BSDE with respect to \( W \) and we recall that \( \hat{Y}^x_{t-h}(\omega', \cdot) = \hat{u}(\omega', t - h, X^x_{t-h}) \). Now for the fixed \( \omega' \in \mathcal{O} \), it holds that

\[
\varphi(t, x) \leq \varphi(t - h, X^x_{t-h}) + \int_{t-h}^t F_r \left( \omega', X^x_r, \hat{Y}^x_r(\omega', \cdot), \hat{Z}^x_r(\omega', \cdot) \right) dr - \int_{t-h}^t \hat{Z}^x_r(\omega', \cdot) \downarrow dW_r.
\]

Let \( (\hat{Y}^x, \hat{Z}^x) ∈ L^2_0(t - h, t; \mathbb{R}^d) \) be the solution of the following equation evaluated at \( \omega' \): for \( s \in [t - h, t] \),

\[
\hat{Y}^x_s(\omega', \cdot) = \varphi(t - h, X^x_{t-h}) + \int_{t-h}^s F_r \left( \omega', X^x_r, \hat{Y}^x_r(\omega', \cdot), \hat{Z}^x_r(\omega', \cdot) \right) dr - \int_{t-h}^s \hat{Z}^x_r(\omega', \cdot) \downarrow dW_r.
\]  

(4.12)

From Lemma 4.7, it follows that \( \hat{Y}^x(\omega', \cdot) ≥ \varphi(t, x) = \hat{u}(\omega', t, x) \). Now we put

\[
\hat{Y}^x(\omega', \cdot) = \hat{Y}^x(\omega', \cdot) - \varphi(s, X^x_s) - \int_{t-h}^s G(\omega', r, x) dr
\]  

(4.13)

and

\[
\hat{Z}^x(\omega', \cdot) = \hat{Z}^x(\omega', \cdot) - (\nabla x \varphi \sigma)(s, X^x_s),
\]

where \( G(\omega', s, x) = \partial_r \varphi(s, x) - LP(\omega', x, \varphi(s, x), \nabla x \varphi(s, x)(x)) \). From the equations \((4.12)\), \((4.13)\) and Itô’s formula we have

\[
\hat{Y}^x_s(\omega', \cdot) = \int_{t-h}^s \left[ F_r \left( \omega', X^x_r, \varphi(r, X^x_s) + \hat{Y}^x_r(\omega', \cdot) + \int_{t-h}^r G(\omega', s, x) ds, (\nabla x \varphi \sigma)(r, X^x_s) + \hat{Z}^x_r(\omega', \cdot) \right) \right] dr - \int_{t-h}^s \hat{Z}^x_r(\omega', \cdot) \downarrow dW_r.
\]

Puting

\[
\delta(\omega', r, h) = F_r \left( \omega', X^x_r, \varphi(r, X^x_s) + \int_{t-h}^r G(\omega', s, x) ds, (\nabla x \varphi \sigma)(r, X^x_s) \right) - (\partial_r \varphi - LP(\omega', x, \varphi(s, x), \nabla x \varphi(s, x)(x)) \right)
\]

we have \( |\delta(\omega', r, h)| ≤ \kappa(\omega') |X^x_r - x|, r \in [0, t] \), for some \( F^B \)-measurable and \( P^B \)-integrable \( \kappa : \Omega' \rightarrow \mathbb{R}^+ \). From the a priori estimate \((4.10)\), it follows that

\[
E^W \left[ \sup_{t-h ≤ s ≤ t} |\hat{Y}^x_s(\omega', \cdot)|^2 + \int_{t-h}^t |\hat{Z}^x_r(\omega', \cdot)|^2 dr \right] ≤ CE^W \left[ \int_{t-h}^t |\delta(\omega', r, h)|^2 dr \right] = h \rho(\omega', h).
\]  

(4.14)
where \( \rho(\omega', h) \) tends to 0 as \( h \to 0 \). Consequently, it yields

\[
E^W \left[ \int_{t-h}^{t} \left( |\tilde{Y}^{t,x}_{r}(\omega', \cdot)| + |\tilde{Z}^{t,x}_{r}(\omega', \cdot)| \right) dr \right] = h \sqrt{\rho(\omega', h)}. \tag{4.15}
\]

Furthermore, we have

\[
\tilde{Y}^{t,x}_{r}(\omega') = E^W \left[ \tilde{Y}^{t,x}_{r}(\omega', \cdot) \right] = E^W \left[ \int_{t-h}^{t} \tilde{\delta}(\omega', r, h) dr \right],
\]

where

\[
\tilde{\delta}(\omega', r, h) = - (\partial_s \varphi - \bar{\mathcal{L}} \varphi)(r, X^{t,x}_r) + G(\omega', r, x)
+ F_r \left( \omega', X^{t,x}_r, \varphi(r, X^{t,x}_r) + \tilde{Y}^{t,x}_{r}(\omega', \cdot) + \int_{t-h}^{r} G(\omega', s, x) ds, (\nabla_x \varphi)(r, X^{t,x}_r) + \tilde{Z}^{t,x}_{r}(\omega', \cdot) \right).
\]

From the fact that \( f \) is Lipschitz and the estimates (4.14) and (4.15), we get

\[
E^W \left[ \int_{t-h}^{t} \left( |\tilde{\delta}(\omega', r, h)| + |\tilde{Y}^{t,x}_{r}(\omega', \cdot)| + |\tilde{Z}^{t,x}_{r}(\omega', \cdot)| \right) dr \right] = h \rho(\omega', h).
\]

Thus, from (4.13) (for \( s = t \)) and since \( \tilde{Y}^{t,x}_{r}(\omega', \cdot) \geq \varphi(t, x) \), we obtain \( \int_{t-h}^{t} G(\omega', r, x) dr \geq -h \rho(\omega', h) \).

Consequently, \( \frac{1}{h} \int_{t-h}^{t} G(\omega', r, x) dr \geq - \rho(\omega', h) \). By letting \( h \) tend to 0, we finally get for \( \omega' \in \bar{\Omega}' \),

\[
G(\omega', t, x) = \partial_t \varphi(t, x) - \bar{\mathcal{L}} \varphi(t, x) - F_t \left( \omega', x, \varphi(t, x), \nabla_x \varphi(t, x) \sigma(x) \right) \geq 0.
\]

Hence \( \hat{u}(\omega', t, x) \) is a pathwise viscosity subsolution of (4.8). The proof of \( \hat{u} \) being a pathwise viscosity supersolution is similar.

The proof of uniqueness becomes clear from Remark 4.10.

In analogy to the relation between the solutions \((Y, Z)\) and \((\hat{Y}, \hat{Z})\) of the associated BDSDE and the BSDE, respectively, we shall expect that \( u(t, x) = Y^{t,x}_t = \hat{Y}^{t,x}_t(A_t)x_t, (t, x) \in [0, T) \times \mathbb{R}^d \), is a solution of SPDE (4.9). This claim is confirmed by

**Proposition 4.11.** Suppose that \( u, \hat{u} \) are \( C^{0,2}\)-stochastic fields over \( \Omega' \times [0, T] \times \mathbb{R}^d \) such that there exist \( \delta > 0 \) and a constant \( C_{\delta,x} > 0 \) (only depending on \( \delta \) and \( x \)) with:

\[
E \left[ |w(t, x)|^{2+\delta} + \int_0^t \left( |\nabla_x w(s, x)|^{2+\delta} + |D^2_{xx} w(s, x)|^{2+\delta} \right) ds \right] \leq C_{\delta,x}, t \in [0, T], \text{ for } w = u, \hat{u}. \tag{4.16}
\]

Then \( \hat{u}(t, x) \) is a classical pathwise solution of equation (4.8) if and only if \( u(t, x) \) is a classical solution of SPDE (4.9).

**Proof:** We restrict ourselves to show \( u(t, x) \) solves equation (4.9) whenever \( \hat{u} \) solves (4.8). For this, we proceed in analogy to the proof of Theorem 3.9. Let \( F \) be an arbitrary but fixed element of \( \mathcal{S}_K \).

By the Girsanov transformation we have

\[
E \left[ |u(t, x)F - u(0, x)F| \right] = E \left[ \hat{u}(A_t, t, x)x_t F - \hat{u}(0, x)F \right] = E \left[ F(T_t)\hat{u}(0, x) - F\hat{u}(0, x) \right]
= E \left[ F(T_t)\int_0^t \left( \nabla_x \hat{u}(s, x) \sigma(s, T_s) \nabla_x \hat{u}(s, x) x_t \right) ds \right].
\]

As in the proof of Theorem 3.9 we use the fact that \( \frac{d}{dt} F(T_t) = \gamma_t(K^*KD^BF)(t, T_t) \) to deduce the following

\[
E \left[ |u(t, x)F - u(0, x)F| \right]
= E \left[ \hat{u}(0, x) \int_0^t \gamma_s(K^*KD^BF(T_s))(s) ds \right] + E \left[ \int_0^t \int_0^s \gamma_r(K^*KD^BF(T_r))(r) dr ds \right]
+ E \left[ \int_0^t \int_0^s \gamma_r(K^*KD^BF(T_r))(r) dr ds \right] + E \left[ \int_0^t \int_0^s \gamma_r(K^*KD^BF(T_r))(r) dr ds \right].
\]
Thanks to the assumption that $\hat{u}(t, x)$ is a pathwise classical solution of equation (4.8), we obtain
\[
E [u(t, x)F - u(0, x)F] = E \left[ \int_0^t F(T_s) \left( \mathcal{L}\hat{u}(s, x) + f(s, x, \hat{u}(s, x)\varepsilon_s(T_s), \nabla_x \hat{u}(s, x)\sigma(x)\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s) \right) ds \right] \\
+ E \left[ \int_0^t \gamma_s(K^* \mathcal{K}D^BF(T_s))(s)\hat{u}(s, x)ds \right] 
= E \left[ \int_0^t F \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla_x u(s, x)\sigma(x)) ds \right] \\
+ E \left[ \int_0^t \gamma_s(K^* \mathcal{K}D^BF)(s)u(s, x)ds \right].
\]

Consequently,
\[
E \left[ \int_0^t \gamma_s(K^* \mathcal{K}D^BF)(s)u(s, x)ds \right] = E \left[ F \left( u(t, x) - u(0, x) - \int_0^t \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla_x u(s, x)\sigma(x)) \right) ds \right].
\]

From the integrability condition (4.16) we know that
\[
0 \leq u(t, x) - u(0, x) - \int_0^t (\mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla_x u(s, x)\sigma(x))) ds \in L^2(\Omega, \mathcal{F}, \mathbb{P}).
\]

Moreover, $\gamma_{1|0, t}(u) \in L^2([0, T] \times \Omega)$. Indeed,
\[
E \left[ \int_0^T |\gamma_{1|0, t}(r)|^2 |u(r, x)|^2 dr \right] = \int_0^T |\gamma_{1|0, t}(r)|^2 E|u(r, x)|^2 dr \leq C_{\delta, x} \int_0^T |\gamma_{1|0, t}(r)|^2 dr < \infty.
\]

By Definition 3.2 we get
\[
E \left[ F \int_0^t \gamma_s u(s, x) d\mathbb{B}_s \right] = E \left[ F \left( u(t, x) - u(0, x) - \int_0^t \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla_x u(s, x)\sigma(x)) \right) ds \right].
\]

It then follows from the arbitrariness of $F \in \mathcal{S}_K$ that
\[
u(t, x) = \Phi(x) + \int_0^t \left[ \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla_x u(s, x)\sigma(x)) \right] ds + \int_0^t \gamma_s u(s, x) d\mathbb{B}_s.
\]

The proof is complete now.

**Remark 4.12.** The regularity of $\hat{u}$ in the above proposition is difficult to get under not too restrictive assumptions (like coefficients of class $C^{3}_{1,b}$, linearity of $f$ in $x$).

**Remark 4.13.** Notice that generally speaking, a continuous random field after Girsanov transformation $A_t$ is not necessarily continuous in $t$ any more. We give a simple counterexample: Let $0 < s < T$ be fixed and
\[
\hat{u}(t, x) = \begin{cases} 
(t - s)B_t, & 0 \leq t \leq s; \\
(t - s)\text{sgn}(B_s), & s < t \leq T.
\end{cases}
\]

It is obvious that $\hat{u}(t, x)$ is $\mathcal{F}_t^B$-measurable and continuous in $t$. But after Girsanov transformation $A_t$, it becomes
\[
u(t, x) = \hat{u}(A_t, t, x)\varepsilon_t = \begin{cases} 
(t - s) \left( B_t - \int_0^t (K\gamma 1_{[0, t]})(r) dr \right) \varepsilon_t, & 0 \leq t \leq s; \\
(t - s)\text{sgn} \left( B_s - \int_0^t (K\gamma 1_{[0, t]})(r) dr \right) \varepsilon_t, & s < t \leq T.
\end{cases}
\]
which is not continuous in \( t \) on

\[
\left\{ \omega' : \inf_{t \in [x,T]} \left( B_s - \int_0^s (K \gamma_1_{[0,t]})(r) \, dr \right) < 0 < \sup_{t \in [x,T]} \left( B_s - \int_0^s (K \gamma_1_{[0,t]})(r) \, dr \right) \right\}.
\]

However, as we state below, the random field \( u \) has a continuous version in our case. To this end we need the following technical result:

**Lemma 4.14.** Let \( \gamma \) be such that (H1) holds. Then there exist positive constants \( C \) and \( q \) such that for all \( r, v, v' \in [0, T] \), \( v \leq v' \),

\[
\left| \int_0^T (K \gamma_1_{[v',v]})(s)(K \gamma_1_{[0,r]})(s) \, ds \right| \leq C|v - v'|^q.
\]

**Proof:** We have

\[
\left| \int_0^T (K \gamma_1_{[v',v]})(s)(K \gamma_1_{[0,r]})(s) \, ds \right| \leq \left( \int_0^T (K \gamma_1_{[v',v]})(s)^2 \, ds \right)^{1/2} \left( \int_0^T (K \gamma_1_{[0,r]})(s)^2 \, ds \right)^{1/2}
\]

\[
\leq C \left( \int_0^T (K \gamma_1_{[v',v]})(s)^2 \, ds \right)^{1/2},
\]

where the last inequality follows from [13] (Lemma 2.3). Also, by the proof of Lemma 2.3 in [13] and using the notation \( \alpha = 1/2 - H \), we have

\[
(K \gamma_1_{[v',v]})(s) = 1_{[v',v]}(s) \left( \phi_\gamma(s) + \frac{\alpha s^\alpha}{(1 - \alpha)} \int_0^s \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}} \, dr \right) - 1_{[0,v]}(s) \frac{\alpha s^\alpha}{(1 - \alpha)} \int_v^s \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}} \, dr
\]

\[
= I_1(s) + I_2(s).
\]

Now, applying [13] (Lemma 2.3) again, we obtain

\[
\int_0^T I_1(s)^2 \, ds = \int_0^T 1_{[v',v]}(s) \left( 1_{[0,v]}(s) \left[ \phi_\gamma(s) + \frac{\alpha s^\alpha}{(1 - \alpha)} \int_v^s \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}} \, dr \right] \right)^2 \, ds
\]

\[
\leq |v - v'|^{(p'-2)/p'} |\phi_\gamma_{[v',v]}|^2_{L_{p'}} \leq C|v - v'|^{(p'-2)/p'}.
\]

On the other hand, for \( m = 1 + \eta \) and \( q_m = m/\eta \), with \( \eta \) small enough, we can write

\[
\int_v^{v'} \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}} \, dr
\]

\[
\leq |v - v'|^{1/q_m} \left[ \int_v^{v'} \frac{r^{-m\alpha}\gamma_r^m}{(r-s)^{m(1+\alpha)}} \, dr \right]^{1/m}
\]

\[
= \frac{1}{\Gamma(\alpha)} |v - v'|^{1/q_m} \left[ \int_v^{v'} \frac{\int_0^T \phi_\gamma(\theta) \theta^{-\alpha}(\theta - r)^{\alpha-1} \, d\theta}{(r-s)^{m(1+\alpha)}} \, dr \right]^{1/m}
\]

\[
\leq C|v - v'|^{1/q_m} \left[ \int_v^{v'} |\phi_\gamma(\theta)|^m \theta^{-\eta} \int_0^\theta (r-s)^{-m(1+\alpha)} \theta^{-m\alpha + \eta}(\theta - r)^{m(\alpha-1)} \, d\theta \, dr \right]^{1/m}
\]

\[
\leq C|v - v'|^{1/q_m} (v' - s)^{-2\eta/m} \left[ \int_v^{v'} |\phi_\gamma(\theta)|^m \theta^{-\eta} \int_0^\theta (r-s)^{-m\alpha + \eta - 1} \theta^{-m\alpha + \eta}(\theta - r)^{m\alpha - \eta - 1} \, d\theta \, dr \right]^{1/m}.
\]
Hence, [13] (Lemma 2.2) gives
\[
\int_0^T \frac{r^{-\alpha} \gamma_r}{(r-s)^{1+\alpha}} dr
\leq C |v - v'|^{1/q_m} (v' - s)^{-2\eta/m} \left[ \int_{v'}^T |\phi_\gamma(\theta)|^m \theta^{-\eta} v'^{-m\alpha+\eta} (v' - s)^{-\alpha+\eta} (\theta - s)^{-1} (\theta - v')^{m\alpha-\eta} d\theta \right]^{1/m}
\]
\[
\leq C |v - v'|^{1/q_m} (v' - s)^{-\alpha-\eta/m} \left[ \int_{v'}^T |\phi_\gamma(\theta)|^m \theta^{-\eta} v'^{-m\alpha+\eta} (\theta - s)^{-1+m\alpha-\eta} d\theta \right]^{1/m},
\]
which, together with (4.17), implies, for \( p < 1/\alpha, p' = \frac{pm}{2(1-(m\alpha-\eta)p)} \) and \( 1/p' + 1/q' = 1 \),
\[
\int_0^T I_2(s)^2 ds
\leq C |v - v'|^{\frac{2}{q_m}} \left( \int_0^\infty (v' - s)^{2q'(-\alpha-\eta/m)} ds \right)^\frac{1}{q'} \left( \int_0^T \left[ \int_{v'}^T |\phi_\gamma(\theta)|^m \theta^{-\eta} (\theta - s)^{-1+m\alpha-\eta} d\theta \right]^{\frac{2p'}{p+m}} ds \right)^{\frac{1}{p'}}
\]
\[
\leq C |v - v'|^{2/q_m}.
\]
Thus, we get the wished result. 

**Lemma 4.15.** The random field \( u(t, x) := \hat{u}(A_t, T, A_x, \varepsilon_t, (t, x) \in [0, T] \times \mathbb{R}^d \) has a continuous version.

**Proof:** In the following, for simplicity of notations, we put
\[
\Theta_t^{t, x, v, v'} = \left( r, X_r^{t, x}, \hat{Y}_r^{t, x} (A_v) \varepsilon_r (T_r A_v), \hat{Z}_r^{t, x} (A_v) \varepsilon_r (T_r A_v) \right).
\]

For \( 0 \leq v' \leq v \leq T \), we notice that \( \left( \hat{Y}_r^{t, x} (A_v) \right) \) is the solution of the BSDE
\[
\hat{Y}_s^{t, x} (A_v) = \Phi (X_0^{t, x}) + \int_0^s f \left( \Theta_r^{t, x, v, v'} \right) \varepsilon_r^{-1} (T_r A_v) dr - \int_0^s \hat{Z}_r^{t, x} (A_v) \downarrow dW_r, \tag{4.19}
\]
while \( \left( \hat{Y}_r^{t, x} (A_v') \right) \) is the solution of the BSDE
\[
\hat{Y}_s^{t, x} (A_v') = \Phi (X_0^{t, x}) + \int_0^s f \left( \Theta_r^{t, x, v, v'} \right) \varepsilon_r^{-1} (T_r A_v) dr - \int_0^s \hat{Z}_r^{t, x} (A_v) \downarrow dW_r.
\]
We set \( J_t^v = \exp \left\{ \int_0^T (K_\gamma 1_{[0, v]} (s) (K_\gamma 1_{[0, v]}) (s) ds \right\} \), and we observe that \( \varepsilon_r^{-1} (T_r A_v) = \varepsilon_r^{-1} (T_r) J_t^v \). Moreover, equation (4.19) can be written as follows:
\[
\hat{Y}_s^{t, x} (A_v) = \Phi (X_0^{t, x}) + \int_0^s f \left( r, X_r^{t, x}, \hat{Y}_r^{t, x} (A_v), \hat{Z}_r^{t, x} (A_v) \right) \varepsilon_r (T_r) (J_t^v)^{-1} \varepsilon_r^{-1} (T_r) J_t^v dr
\]
\[
- \int_0^s \hat{Z}_r^{t, x} (A_v) \downarrow dW_r, \quad s \in [0, t],
\]
26
and a comparison with (4.5) suggests the similarity of arguments which can be applied. So, by a standard BSDE estimate, see, for instance, the proof of Lemma 4.2, we have
\[
E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t,x}(A_v) - \hat{Y}_s^{t,x}(A_{v'}) \right|^p \right] 
\leq CE \left[ \int_0^T \left| f(\Theta_t^{t,x,v,v}) - f(\Theta_t^{t,x,v,v'}) \right|^p \, dr \right]
\leq CE \left[ \int_0^T \left[ \left| f(\Theta_t^{t,x,v}) \right| - \left| f(\Theta_t^{t,x,v'}) \right| \right] \, dr \right].
\]
(4.20)
Recalling that \( \varepsilon_r^{-1}(T, A_v) = \varepsilon_r^{-1}(T, A_v') \) and applying Lemma 4.14, we get
\[
\left| \varepsilon_r^{-1}(T, A_v) - \varepsilon_r^{-1}(T, A_v') \right|^p = \varepsilon_r^{-p}(T_r) \left| J_p - J_p' \right|^p 
\leq C\varepsilon_r^{-p}(T_r) \int_0^T (K\gamma_{1|v,v'}) \, ds - \int_0^T (K\gamma_{1|v,v'}) \, ds \right|^p 
\leq C\varepsilon_r^{-p}(T_r) \int_0^T (K\gamma_{1|v,v'}) \, ds \right|^p \leq C\varepsilon_r^{-p}(T_r) |v - v'|^{pq}.
\]
(4.21)
On the other hand,
\[
\left| f(\Theta_t^{t,x,v,v}) - f(\Theta_t^{t,x,v,v'}) \right|^p \leq C \left( \left| \hat{Y}_r^{t,x}(A_v) \right|^p + \left| \hat{Z}_r^{t,x}(A_v) \right|^p \right) \left| \varepsilon_r(T, A_v) - \varepsilon_r(T, A_v') \right|^p 
\leq C \left( \left| \hat{Y}_r^{t,x}(A_v) \right|^p + \left| \hat{Z}_r^{t,x}(A_v) \right|^p \right) \left| \varepsilon_r(T, A_v) - \varepsilon_r(T, A_v') \right|^p 
\leq C \left( \left| \hat{Y}_r^{t,x}(A_v) \right|^p + \left| \hat{Z}_r^{t,x}(A_v) \right|^p \right) |v - v'|^{pq}.
\]
(4.22)
Plugging estimates (4.21) and (4.22) into the equation (4.20), Lemma 4.2 yields that
\[
E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t,x}(A_v) - \hat{Y}_s^{t,x}(A_{v'}) \right|^p \right] 
\leq CE \left[ \int_0^T \left[ \left| f(\Theta_t^{t,x,v,v}) \right|^p + \left| \hat{Y}_r^{t,x}(A_v) \right|^p + \left| \hat{Z}_r^{t,x}(A_v) \right|^p \right] \, ds \right] |v - v'|^{pq} 
\leq C(1 + |x|^p) |v - v'|^{pq}.
\]
For the latter inequality we have used the following estimate of \( \hat{Z} \), which proof will be postponed until the end of the current proof.

**Lemma 4.16.** There exists a constant \( C_p \) such that \( E \left[ \sup_{0 \leq s \leq T} \left| \hat{Z}_s^{t,x} \right|^p \right] \leq C_p (1 + |x|^p) \).

We continue our proof of Lemma 4.15: According to the proof of Lemma 4.4 we have
\[
E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t',x'}(A_v) - \hat{Y}_s^{t,x}(A_v) \right|^p \right] = E \left[ \sup_{s \in [0,T]} \left| \hat{Y}_s^{t',x'} - \hat{Y}_s^{t,x} \right|^p |F_T^p \circ A_v \right] 
\leq \left( E \left[ \sup_{s \in [0,T]} \left| \hat{Y}_s^{t',x'} - \hat{Y}_s^{t,x} \right|^2 \right] \right)^{1/2} \left( E \left[ \varepsilon_{t}^{-2}(T_v) \right] \right)^{1/2} \leq C \left( (1 + |x|^p + |x'|^p)|t - t'|^{p/2} + |x - x'|^p \right).
\]
Hence, by combining the above estimates we obtain

$$E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t,x}(A_v) - \hat{Y}_s^{t,x}(A_v') \right|^p \right]$$

$$\leq C \left( E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t,x}(A_v) - \hat{Y}_s^{t,x'}(A_v) \right|^p \right] + E \left[ \sup_{0 \leq s \leq T} \left| \hat{Y}_s^{t,x'}(A_v) - \hat{Y}_s^{t,x}(A_v) \right|^p \right] \right)$$

$$\leq C \left( (1 + |x|^p + |x'|^p)(|t-t'|^{p/2} + |v-v'|^p) + |x-x'|^p \right).$$

Consequently, from the Kolmogorov continuity criterion we know the process \( \{\hat{Y}_s^{t,x}(A_v); s, t, v \in [0, T], x \in \mathbb{R}^d\} \) has an a.s. continuous version. From Lemma 4.14 we have that \( \varepsilon_t \) is continuous in \( \varepsilon_t \). It then follows that \( u(t, x) = \hat{Y}_t^{t,x}(A_t) \) has a version which is jointly continuous in \( t \) and \( x \). □

**Proof of Lemma 4.16:** For simplicity we suppose all the functions \( \Phi, f, \sigma, b \) are smooth. For the proof of the general case, the Lipschitz functions have to be approximated by smooth functions with the same Lipschitz constants. We define \( \left( \overline{Y}_s^{t,x}, \overline{Z}_s^{t,x} \right) \) to be the solution of the equation

$$\overline{Y}_s^{t,x} = \Phi'\left(X_0^{t,x}\right) \nabla_x X_0^{t,x} + \int_0^s \left[ f_x'\left(\Xi_t\right) \varepsilon_r^{-1}(T_r) \nabla_x X_r^{t,x} + f_y'\left(\Xi_t\right) Y_r^{t,x} + f_z'\left(\Xi_t\right) Z_r^{t,x}\right] dr - \int_0^s \overline{Z}_r^{t,x} \downarrow dW_r,$$

where \( \Xi_t = \left(r, X_r^{t,x}, \overline{Y}_r^{t,x}(T_r), \overline{Z}_r^{t,x}(T_r)\right) \) and

$$\nabla_x X_r^{t,x} = I - \int_r^t \nabla_x X_s^{t,x} b_x\left(X_s^{t,x}\right) ds - \int_r^t \nabla_x X_s^{t,x} \sigma_x\left(X_s^{t,x}\right) ds\downarrow dW_s, \ r \in [0, t].$$

With the arguments used in the proof of Lemma 4.2, we get

$$E \left[ \sup_{0 \leq s \leq T} \left| \overline{Y}_s^{t,x} \right|^p + \left( \int_0^T |\overline{Z}_s^{t,x}|^2 ds \right)^{p/2} \right] \leq C_p (1 + |x|^p).$$

By arguments which by now are standard, it can be seen that the processes \( X_s^{t,x}, \hat{Y}_s^{t,x} \) and \( \hat{Z}_s^{t,x} \) are Malliavin differentiable with respect to \( W \), and thus, for \( s \leq u \leq t \leq T \),

$$D_u W \hat{Y}_s^{t,x} = \Phi' \left(X_0^{t,x}\right) D_u W X_0^{t,x} - \int_0^s D_u W \hat{Z}_r^{t,x} \downarrow dW_r$$

$$+ \int_0^s \left[ f_x'\left(\Xi_t\right) \varepsilon_r^{-1}(T_r) D_u W X_r^{t,x} + f_y'\left(\Xi_t\right) D_u W \hat{Y}_r^{t,x} + f_z'\left(\Xi_t\right) D_u W \hat{Z}_r^{t,x}\right] dr.$$  (4.26)

On the other hand, from

$$\hat{Y}_s^{t,x} = \hat{Y}_s^{t,x} + \int_0^s f \left(r, X_r^{t,x}, \hat{Y}_r^{t,x}(T_r), \hat{Z}_r^{t,x}(T_r)\right) \varepsilon_r^{-1}(T_r) dr - \int_0^s \hat{Z}_r^{t,x} \downarrow dW_r, \theta < s \leq u \leq T,$$

we get

$$D_u W \hat{Y}_s^{t,x} = - \hat{Z}_u^{t,x} + \int_s^u D_u W \hat{Z}_r^{t,x} \downarrow dW_r$$

$$+ \int_s^u \left[ f_x'\left(\Xi_t\right) \varepsilon_r^{-1}(T_r) D_u W X_r^{t,x} + f_y'\left(\Xi_t\right) D_u W \hat{Y}_r^{t,x} + f_z'\left(\Xi_t\right) D_u W \hat{Z}_r^{t,x}\right] dr.$$  (4.27)

From the above three equations (4.23), (4.26) and (4.27) and the relation

$$D_u W \left(X_s^{t,x}\right) = \nabla_x X_s^{t,x} \left(\nabla_x X_s^{t,x}\right)^{-1} \sigma_x\left(X_s^{t,x}\right)$$

(4.28)
Finally, from standard estimates for (4.24) and from the estimate (4.25), we get
\[
E \left[ \sup_{0 \leq s \leq T} \left| \tilde{Z}^{t,x}_s \right|^p \right] \leq E \left[ \sup_{0 \leq s \leq T} \left| \nabla_{x} X^{t,x}_s \right|^{-p} \sup_{0 \leq s \leq T} \left| \sigma(X^{t,x}_s) \right|^p \right] \\
\leq E \left[ \sup_{0 \leq s \leq T} \left| \nabla_{x} X^{t,x}_s \right|^{-p} \right]^{1/q_1} E \left[ \sup_{0 \leq s \leq T} \left| \sigma(X^{t,x}_s) \right|^p \right]^{1/q_3} \\
\leq C_p (1 + |x|^p),
\]
where \(1/q_1 + 1/q_2 + 1/q_3 = 1\), with \(q_1, q_2, q_3 > 1\). The proof is complete.

The above proposition motivates the following definition:

**Definition 4.17.** A continuous random field \(u : [0, T] \times \mathbb{R}^d \times \Omega' \rightarrow \mathbb{R}\) is a (stochastic) viscosity solution of equation (4.9) if and only if \(\tilde{u}(t, x) = u(T, t, x)\epsilon_t^{-1}(T_t), (t, x) \in [0, T] \times \mathbb{R}^d\) is a pathwise viscosity solution of equation (4.8).

As a consequence of our preceding discussion, we can formulate the following statement:

**Theorem 4.18.** The continuous stochastic field \(u(t, x) := \tilde{u}(A_t, t, x)\epsilon_t = \tilde{Y}^{t,x}_t(A_t)\epsilon_t = Y^{t,x}_t\) is a stochastic viscosity solution of the semilinear SPDE (4.9). This solution is unique inside the class \(C_p^B\) of continuous stochastic field \(\tilde{u} : \Omega' \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that,
\[
|\tilde{u}(t, x)| \leq C \exp\{I_T\} (1 + |x|), (t, x) \in [0, T] \times \mathbb{R}^d, P \text{-a.s.,}
\]
for some constant \(C\) only depending on \(\tilde{u}\).

**Remark 4.19.** 1) It can be easily checked that \(\tilde{u} \in C_p^B\) if and only if \((\tilde{u}(A_t, t, x)\epsilon_t) \in C_p^B\) if and only if \((\tilde{u}(T, t, x)\epsilon_t^{-1}(T_t)) \in C_p^B\).

2) As a consequence of the preceding theorem we have that \(u(t, x) = Y^{t,x}_t\) is the unique (in \(C_p^B\)) stochastic viscosity solution of SPDE (4.9). This extends the Feynman-Kac formula to SPDEs driven by a fractional Brownian motion.

We conclude the main theorems of Section 4 with the following relation diagram, which shows the mutual relationship between fractional backward SDEs and SPDEs:

\[
\begin{array}{ccc}
\tilde{u}(t, x) \text{ is the viscosity solution of (4.8)} & \overset{GT}{\leftrightarrow} & u(t, x) \text{ is the viscosity solution of (4.9)} \\
\downarrow & & \downarrow \\
(\tilde{Y}, \tilde{Z}) \text{ is the solution of BSDE (4.5)} & \overset{GT}{\leftrightarrow} & (Y, Z) \text{ is the solution of BDSDE (4.4)}
\end{array}
\]

where 'GT' stands for 'Girsanov transformation'.

Finally, in order to illustrate how our method works, we give the example of a linear fractional backward doubly stochastic differential equation.

**Example 4.20.** We let \(d = 1\) and \(f(s, x, y, z) = f^1_s x + f^2_s y + f^3_s z\), where the coefficients \(f^1_s, f^2_s\) and \(f^3_s\) are bounded and deterministic functions. The associated fractional backward doubly stochastic differential equation is linear and writes:
\[
Y^{t,x}_s = \Phi(X^{t,x}_0) + \int^s_0 \left( f^1_r X^{t,x}_r + f^2_r Y^{t,x}_r + f^3_r Z^{t,x}_r \right) dr - \int^s_0 Z^{t,x}_r dW_r + \int^s_0 \gamma_r Y^{t,x}_r dB_r. \tag{4.29}
\]
After Girsanov transformation, it becomes

\[
\hat{Y}_t^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \left( f_1^{t,x} X_r^{t,x} \varepsilon_r^{-1}(T_r) + f_2^{t,x} \hat{Y}_r^{t,x} + f_3^{t,x} \hat{Z}_r^{t,x} \right) dr - \int_0^s \hat{Z}_r^{t,x} \downarrow dW_r.
\]

and has the following solution:

\[
\hat{Y}_s^{t,x} = \mathbb{E}_Q \left[ \int_0^s \left( f_1^{t,x} X_r^{t,x} \varepsilon_r^{-1}(T_r) \exp \left\{ \int_r^s f_2^{t,x} du \right\} \right) dr + \exp \left\{ \int_0^s f_2^{t,x} dr \right\} \Phi(X_0^{t,x}) \bigg| \mathcal{F}_{s,t} \cup \mathcal{F}_t \right],
\]

where \( \mathbb{E}_Q \) is the expectation with respect to \( Q = \exp \left\{ \int_0^t f_3^{t,x} dW_r - \frac{1}{2} \int_0^t (f_3^{t,x})^2 dr \right\} \mathbb{P} \). According to Theorem 3.9, the solution of (4.29) is then

\[
Y_s^{t,x} = \mathbb{E}_Q \left[ \varepsilon_s \int_0^s \left( f_1^{t,x} X_r^{t,x} \varepsilon_r^{-1}(T_r, A_s) \exp \left\{ \int_r^s f_2^{t,x} du \right\} \right) dr + \varepsilon_s \exp \left\{ \int_0^s f_2^{t,x} dr \right\} \Phi(X_0^{t,x}) \bigg| \mathcal{F}_{s,t} \cup \mathcal{F}_t \right].
\]

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