Existence of global weak solutions to the kinetic Peterlin model

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Abstract

We consider a class of kinetic models for polymeric fluids motivated by the Peterlin dumbbell theories for dilute polymer solutions with a nonlinear spring law for an infinitely extensible spring. The polymer molecules are suspended in an incompressible viscous Newtonian fluid confined to a bounded domain in two or three space dimensions. The unsteady motion of the solvent is described by the incompressible Navier-Stokes equations with the elastic extra stress tensor appearing as a forcing term in the momentum equation. The elastic stress tensor is defined by the Kramers expression through the probability density function that satisfies the corresponding Fokker-Planck equation. In this case, a coefficient depending on the average length of polymer molecules appears in the latter equation. Following the recent work of Barrett and Süli [5] we prove the existence of global-in-time weak solutions to the kinetic Peterlin model in two space dimensions.

1 Introduction

The Peterlin approximation is a nonlinear model falling into the category of Navier-Stokes-Fokker-Planck type systems. The nonlinearity of the model corresponds to the nonlinear spring potential for infinitely extensible molecular chains appearing in the Fokker-Planck equation. Among the nonlinear dumbbell models the most commonly studied one is the FENE model - finitely extensible nonlinear elastic model. Its advantage consists in a particular form of the spring potential, which forces that the system is considered in a bounded domain. Thus, even though in case of such a nonlinearity the macroscopic closure is not possible, but the methods developed in a series of papers, cf. [1,3,4] allowed for showing existence of global-in-time weak solutions. The case of spring potential in the Peterlin model does not provide finite extensibility of polymeric chains, thus the problem of unbounded domain (and integration by parts) has to be faced. However, the idea of averaging the coefficients (with respect to $\mathbf{R}$ - the vector corresponding to the length and orientation of polymers) gave that they depend on the macroscopic quantity only, namely the trace of the conformation tensor $\text{tr} \mathbf{C} = \langle |\mathbf{R}|^2 \rangle$, which is the average length of polymer molecules suspended in the solvent. This property apparently allows to prove a rigorous macroscopic closure of a kinetic equation and to use the results on existence and regularity of macroscopic quantities. This idea has very recently been used for a linear Hookean dumbbell model [5] by Barrett and Süli, who showed the existence of large-data global-in-time weak solutions. The Hookean model arises as a microscopic-macroscopic bead-spring model from the kinetic theory of dilute solutions of polymeric liquids with noninteracting polymer chains. The authors have also rigorously showed that the well-known Oldroyd-B model is a macroscopic closure of the Hookean dumbbell model in two space dimensions. It is worth to mention here that an attempt of mimicking the approach used for FENE models to linear Hookean case failed. Barrett and Süli in [2] firstly covered just the case of Hookean-type models,
meaning by that a slight modification of the spring potential to provide the uniform integrability of appropriate terms.

Motivated by their approach we study the kinetic Peterlin model representing a class of kinetic dumbbell-based models for dilute polymer solutions with a nonlinear spring force law. For the macroscopic closure of the corresponding kinetic equation, it is necessary to approximate the spring force. We consider the Peterlin approximation [12] that allows us to derive the so-called Peterlin viscoelastic model, which has been studied in our recent work [8,11] and these results will be essentially used in a current approach. See also [9,10] for our recent result on error analysis using the Lagrange-Galerkin method. As a consequence of the approximation of the force law, the Fokker-Planck equation contains additional coefficients, which depend on the trace of the (macroscopic) conformation tensor. As mentioned in [14], the Peterlin model can be therefore seen as the generalization of the upper-convected Maxwell model, in which the relaxation time and viscosity depend on a “structure parameter” $\text{tr}\ C$.

In Section 2 of the present paper we introduce the kinetic Peterlin model and its formal macroscopic closure, the so-called Peterlin viscoelastic model. In the next section we recall our recent results on uniqueness of regular weak solutions to the proposed macroscopic model. In Section 4 recalling the idea of Barrett and Süli [5], we show the existence of global-in-time weak solutions to the Fokker-Planck equation for some given fluid velocity $u_*$ and conformation tensor $C_*$. Let us mention that the difference between the kinetic Peterlin model analyzed in the present paper and the Hookean model studied in [5] is the dependence on the structure parameter $\text{tr}\ C$ appearing in the Fokker-Planck equation due to the Peterlin approximation of the nonlinear spring force law. Finally, in Section 5 we show the existence of global-in-time weak solutions to the kinetic Peterlin model in two space dimensions. We combine the result on uniqueness of solutions to the macroscopic model with the results presented in Section 4.

### 2 The kinetic Peterlin model

In the present paper we study the existence of global weak solutions to a kinetic dumbbell-based model for dilute polymer solutions. The polymer molecules are suspended in an incompressible viscous Newtonian fluid confined to an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The incompressible Navier-Stokes equations equipped with the no-slip boundary condition for the velocity are used to describe the unsteady motion of the solvent.

Let $T > 0$ be given, find $u : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}^d$ and $p : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ such that

$$
\frac{\partial u}{\partial t} + (u \cdot \nabla_x)u = \nu \Delta_x u + \text{div}_x T - \nabla_x p \quad \text{in } (0, T) \times \Omega, \quad (1a)
$$

$$
\text{div}_x u = 0 \quad \text{in } (0, T) \times \Omega, \quad (1b)
$$

$$
uu = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (1c)
$$

$$
uu(0) = u_0 \quad \text{in } \Omega. \quad (1d)
$$

The elastic extra stress tensor $T : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}^{d \times d}$, arising due to the random movement of polymers in the solvent, appears as the forcing term in equation (1a), and depends on the probability density function $\psi$. It is defined by the Kramers expression

$$
T(\psi) = n\gamma_3(\text{tr}\ C(\psi))C(\psi) - I, \quad (1e)
$$

where $n$ denotes the number density of polymer molecules, i.e., the number of polymer molecules per unit volume. Let us note that the above equations (1) are written in their non-dimensional

\[\]
form; \( U_0, L_0 \) denote in what follows the characteristic flow speed and the characteristic length-scale of the flow, respectively; viscosity \( \nu > 0 \) is the reciprocal of the Reynolds number.

The polymers are modelled as two beads connected by a spring and are assumed not to interact with each other. The spring connecting the beads exerts a spring force \( \mathbf{F}(\mathbf{R}) \) with \( \mathbf{R} \) being the vector connecting the beads. We consider the spring force to be nonlinear, i.e. \( \mathbf{F}(\mathbf{R}) = \gamma_1(\|\mathbf{R}\|^2)\mathbf{R} \). On each of the beads there is a balance between the spring force, a friction force exerted by the surrounding fluid and a stochastic force due to Brownian motion. Let \( \zeta > 0 \) be a friction coefficient, \( k\tau \) be the magnitude of stochastic forces with \( k \) being the Boltzmann constant and \( \tau \) being the absolute temperature. Then the probability density \( \psi : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}_0^+ \) satisfies the following Fokker-Planck equation

\[
\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi + \text{div}_R [\nabla_x \mathbf{u} \cdot \mathbf{R} \psi] = \frac{2k\tau}{\zeta} \gamma_2(\|\mathbf{R}\|^2) \Delta_R \psi + \frac{2}{\zeta} \text{div}_R [\mathbf{F}(\mathbf{R})\psi] + \frac{k\tau}{2\zeta} \Delta_x \psi \tag{2}
\]

with the center-of-mass diffusion coefficient \( (k\tau)/(2\zeta) > 0 \). The constitutive functions \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) defined on \( \mathbb{R} \) are from now on assumed to be continuous and positive-valued. If they are constant, then we obtain the Hookean dumbbell model whose closure is the well-known Oldroyd-B model. In order to derive an analogous closed system of equations for the conformation tensor we employ the Peterlin approximation of the spring force, which replaces the length of the spring \( \|\mathbf{R}\|^2 \) in the spring constant \( \gamma_1 \) by the average length of the spring \( \langle|\mathbf{R}|^2\rangle \). The force law thus reads \( \mathbf{F}(\mathbf{R}) = \gamma_1(\langle|\mathbf{R}|^2\rangle)\mathbf{R} = \gamma_1(\text{tr} \mathbf{C}(\psi))\mathbf{R} \). We note that for the trace of the macroscopic conformation tensor \( \mathbf{C}(\psi) := \langle \mathbf{R} \otimes \mathbf{R} \rangle \) it holds that \( \text{tr} \mathbf{C}(\psi) = \langle|\mathbf{R}|^2\rangle \). Here \( \otimes \) denotes the dyadic product and

\[
\langle f \rangle := \int_{\mathbb{R}^d} f(\mathbf{R})\psi(t, \mathbf{x}, \mathbf{R}) \, d\mathbf{R}.
\]

For more details on deriving equation (2) we refer the reader to e.g., [5, 6, 12, 13, 15] and the references therein.

**Definition 1. (normalized Maxwellian)**

We define the equilibrium distribution of the probability density function by

\[
M := b \exp \left\{ -\frac{\|\mathbf{R}\|^2}{2a} \right\} \quad \text{with} \quad a := \frac{k\tau \gamma_{2,\text{M}}}{\gamma_{1,\text{M}}}, \quad b := (2\pi a)^{-d/2}. \tag{3a}
\]

Here \( \gamma_{i,\text{M}} := \gamma_i(\text{tr} \mathbf{C}_\text{M}) > 0, \ i = 1, 2, \) denote the values of the functions \( \gamma_1 \) and \( \gamma_2 \) at the equilibrium. We note that \( \text{tr} \mathbf{C}_\text{M} := \text{tr} \mathbf{C}(M) = d \).

The next lemma provides the non-dimensional form of the Fokker-Planck equation (2) rewritten using the Maxwellian \( M \) defined above.

**Lemma 2.**

Let the functions \( \gamma_1, \gamma_2 \) be such that the identity

\[
\frac{\gamma_{1,\text{M}}}{\gamma_{2,\text{M}}} = \frac{\gamma_1(\text{tr} \mathbf{C})}{\gamma_2(\text{tr} \mathbf{C})} = k\tau := \gamma_{\text{M}} \tag{3b}
\]

is satisfied for a.e. \( (t, x) \in (0, T) \times \Omega \). Then the Fokker-Planck equation (2) can be rewritten in its non-dimensional form as

\[
\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi + \text{div}_R [\nabla_x \mathbf{u} \cdot \mathbf{R} \psi] = \Gamma(\text{tr} \mathbf{C}) \nabla_R \cdot \left( M \nabla_R \left( \frac{\psi}{M} \right) \right) + \varepsilon \Delta_x \psi, \tag{3c}
\]
where the coefficient $\Gamma(\text{tr } C) > 0$ and the center-of-mass diffusion coefficient $\varepsilon > 0$ are given by

$$\Gamma(\text{tr } C) := \frac{\gamma^2(\text{tr } C)}{2\lambda}, \quad \varepsilon := \left(\frac{l_0}{L_0}\right)^2 \frac{1}{8\lambda},$$

(3d)

respectively. The coefficient $\lambda := (\zeta/4\gamma_M)(U_0/L_0)$, usually called the Deborah number, characterises the elastic relaxation property of the fluid, and $l_0 := \sqrt{\text{tr } C_M/d}$ denotes the characteristic microscopic length-scale.

Proof. Let us consider the non-dimensional variables denoted by $\sim$, e.g., $\tilde{\psi} := \psi/d_0$, where $d_0$ is the characteristic probability density. We insert these variables into (2) and multiply the resulting equation by $\frac{L_0}{U_0d_0}$. On noting $T_0 = U_0/L_0$, equation (2) becomes

$$\frac{\partial \tilde{\psi}}{\partial t} + (\bar{u} \cdot \nabla) \tilde{\psi} + \text{div}_R \left[ \nabla_x \bar{u} \cdot \hat{R} \tilde{\psi} \right] = \frac{2k\tau \gamma^2(\text{tr } C)}{\zeta U_0(l_0)^2} \Delta_R \tilde{\psi} + \frac{2L_0}{\zeta U_0 \text{div}_R \left[ \gamma_1(\text{tr } C) \hat{R} \tilde{\psi} \right]} + \frac{k\tau}{2\zeta L_0 U_0} \Delta_x \tilde{\psi}.$$  

(4)

The direct calculation, taking into account (3a), yields

$$\nabla_R \cdot \left( M \nabla_R \left( \frac{\tilde{\psi}}{M} \right) \right) = \Delta_R \tilde{\psi} + \frac{1}{a} \text{div}_R \left[ \hat{R} \tilde{\psi} \right].$$

We note that $a = l_0 = 1$. Thus, it holds that

$$\frac{2k\tau \gamma^2(\text{tr } C)}{\zeta U_0(l_0)^2} \Delta_R \tilde{\psi} + \frac{2L_0}{\zeta U_0 \text{div}_R \left[ \gamma_1(\text{tr } C) \hat{R} \tilde{\psi} \right]} = \frac{2k\tau \gamma^2(\text{tr } C)}{\zeta U_0(l_0)^2} \nabla_R \cdot \left( M \nabla_R \left( \frac{\tilde{\psi}}{M} \right) \right).$$

By the definition of $\lambda$ and $\gamma_M$ it holds that

$$\frac{2k\tau \gamma^2(\text{tr } C)}{\zeta U_0(l_0)^2} = \frac{\gamma^2(\text{tr } C)}{2\lambda} = \Gamma(\text{tr } C), \quad \varepsilon = \frac{k\tau}{2\zeta L_0 U_0}.$$  

Omitting the $\sim$-notation we get equation (3c).

Let us point out that the $\sim$-notation of the non-dimensional variables has been used only in the proof of Lemma 2. In what follows all the equations are non-dimensional.

Finally, we impose the following decay/boundary and initial conditions on $\psi$:

$$\left| M \left( \Gamma(\text{tr } C) \nabla_R \left( \frac{\psi}{M} \right) - (\nabla_x u) \hat{R} \frac{\psi}{M} \right) \right| \to 0 \quad \text{as} \quad |R| \to \infty \quad \text{on} \quad (0, T) \times \Omega, \quad (5a)$$

$$\varepsilon \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega \times \mathbb{R}^d, \quad (5b)$$

$$\psi(0) = \psi_0 \quad \text{on} \quad \Omega \times \mathbb{R}^d, \quad (5c)$$

where $n$ is the unit outward normal vector on $\partial \Omega$ and $\psi_0$ is a given non-negative function defined on $\Omega \times \mathbb{R}^d$ with $\int_{\mathbb{R}^d} \psi_0(x, R) \, dR = 1$ for a.e. $x \in \Omega$.

Definition 3. Throughout the paper we refer to the system of equations and conditions (1), (3), (5) as the kinetic Peterlin model (KP).
In order to obtain a formal macroscopic closure of the above introduced kinetic model we multiply
the non-dimensional form of the Fokker-Planck equation (4) by \( R \otimes R \) and integrate by parts
over \( \mathbb{R}^d \) to get that the conformation tensor \( C : [0, T] \times \Omega \rightarrow \mathbb{R}^{d \times d} \) satisfies the following equation
\[
\frac{\partial C}{\partial t} + (u \cdot \nabla)C - (\nabla u)C - C(\nabla u)^T = \frac{\gamma_2 (\text{tr } C)}{\lambda} I - \frac{\gamma_1 (\text{tr } C)}{\lambda \gamma_M} C + \varepsilon \Delta C \quad \text{in } (0, T) \times \Omega \tag{6a}
\]
subject to the boundary and initial conditions
\[
\varepsilon \frac{\partial C}{\partial n} = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad C(0) = C_0 \quad \text{in } \Omega. \tag{6b}
\]

**Definition 4.** Throughout the paper we refer to the system of equations and conditions (1), (6) as the (macroscopic) Peterlin model (MP).

### 2.1 Notation and preliminaries

Let \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), be a bounded domain with smooth boundary \( \partial \Omega \). We define the following
functional spaces
\[
V := \{ v \in H_0^1(\Omega)^d : \text{div}_x v = 0 \}, \quad H := \{ v \in L^2(\Omega)^d : \text{div}_x v = 0, v \cdot n = 0 \text{ on } \partial \Omega \},
\]
where the divergence is understood in the sense of distributions. We shall use the notation
\[
\hat{\psi} := \frac{\psi}{M}
\]
and the Maxwellian-weighted \( L^p \) space over \( \Omega \times \mathbb{R}^d \) denoted by \( L^p_M(\Omega \times \mathbb{R}^d), p \in [1, \infty) \), equipped with the norm
\[
\| \hat{\psi} \|_{L^p_M(\Omega \times \mathbb{R}^d)} := \left( \int_{\Omega \times \mathbb{R}^d} M|\hat{\psi}|^p \, dR \, dx \right)^{1/p}.
\]
Analogously, we define the space \( \hat{X} \equiv H^1_M(\Omega \times \mathbb{R}^d) := \{ \hat{\psi} \in L^1_{\text{loc}}(\Omega \times \mathbb{R}^d) : \| \hat{\psi} \|_{H^1_M(\Omega \times \mathbb{R}^d)} < \infty \} \) with the norm
\[
\| \hat{\psi} \|_{H^1_M(\Omega \times \mathbb{R}^d)} := \left( \int_{\Omega \times \mathbb{R}^d} M \left[ |\hat{\psi}|^2 + |\nabla_x \hat{\psi}|^2 + |\nabla_R \hat{\psi}|^2 \right] \, dR \, dx \right)^{1/2}.
\]
Finally, let
\[
\tilde{Z}_2 := \left\{ \hat{\psi} \in L^2_M(\Omega \times \mathbb{R}^d) : \hat{\psi} \geq 0 \text{ a.e. on } \Omega \times \mathbb{R}^d; \int_{\mathbb{R}^d} M(R) \hat{\psi}(x, R) \, dR \leq 1 \text{ for a.e. } x \in \Omega \right\}.
\]

The proof of existence of weak solutions to the Fokker-Planck equation is based on the compactness theorem due to Dubinskii [7], that is a generalization of the Lions-Aubin compactness theorem. We refer to [5, 7] and the references therein for more details.

**Theorem 5.** (*Dubinskii*)

Suppose that \( A_0 \) and \( A_1 \) are Banach spaces, \( A_0 \hookrightarrow A_1 \), and \( M \) is a semi-normed subset of \( A_0 \) with the compact embedding \( M \hookrightarrow A_0 \). Then, for \( \alpha_i > 1, i = 0, 1 \), the embedding
\[
\left\{ \eta \in L^{\alpha_0}(0, T; M) : \frac{\partial \eta}{\partial t} \in L^{\alpha_1}(0, T; A_1) \right\} \hookrightarrow L^{\alpha_0}(0, T; A_0)
\]
is compact.
3 Uniqueness result for the macroscopic Peterlin model

One of the crucial parts of the proof of existence of global weak solutions to the kinetic Peterlin model, presented in Section 5 below, is uniqueness of regular weak solutions to (MP). In this section we list the available results.

The couple \((u, C)\) with
\[
 u \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad C \in L^\infty(0, T; L^2(\Omega)^{d \times d}) \cap L^2(0, T; H^1(\Omega)^{d \times d})
\]
is called a weak solution to the Peterlin model (MP) if it satisfies, for any \((v, D) \in V \times H^1(\Omega)^{d \times d},\)
\[
 \int_\Omega \frac{\partial u}{\partial t} \cdot v \, dx + \int_\Omega (u \cdot \nabla) u \cdot v \, dx + \nu \int_\Omega \nabla u : \nabla v \, dx + \int_\Omega \gamma_3(\text{tr} C) C : \nabla v \, dx = 0
\]
\[
 \int_\Omega \frac{\partial C}{\partial t} : D \, dx + \int_\Omega (u \cdot \nabla) C : D \, dx - 2 \int_\Omega (\nabla u) C : D \, dx + \varepsilon \int_\Omega \nabla C : \nabla D \, dx = \frac{1}{\lambda} \int_\Omega \gamma_2(\text{tr} C) I : D \, dx - \frac{1}{\lambda \gamma_M} \int_\Omega \gamma_1(\text{tr} C) C : D \, dx,
\]
and if \((u(0), C(0)) = (u_0, C_0),\) for a given initial data \((u_0, C_0) \in H \times L^2(\Omega)^{d \times d}.\)

Assumptions on the constitutive functions

Let us assume that \(\gamma_1, \gamma_2\) and \(\gamma_3\) are smooth positive functions defined on \(\mathbb{R}\) and \(\gamma_3\) is moreover non-decreasing. Further, we suppose that for some positive constants \(A_i, B_i, C_i, i = 1, 2,\) the following polynomial growth conditions are satisfied for large \(s:\)
\[
 A_1 s^\alpha \leq \gamma_1(s) \leq A_2 s^\alpha, \quad C_1 s^\beta \leq \gamma_2(s) \leq C_2 s^\beta, \quad B_1 s^\gamma \leq \gamma_3(s) \leq B_2 s^\gamma.
\]

In [8] we have studied the existence and uniqueness of global weak and classical solutions to the Peterlin viscoelastic model with \(\lambda = \gamma_M = 1.\) We have shown the existence of global-in-time weak solutions in both two and three space dimensions with only \(C \in L^p((0, T) \times \Omega)^{d \times d} \cap L^{1+\delta}(0, T; W^{1,1+\delta})\) for \(p > 2\) and \(0 < \delta << 1.\) Moreover, for the two-dimensional case we have proven the existence and uniqueness of classical solutions to model (MP), which is of interest for our further needs; see Theorem 3 in [8].

Theorem 6. (unique classical solution to (MP))

Let the assumptions [8] on \(\gamma_1, \gamma_2\) and \(\gamma_3\) be satisfied with
\[
 \alpha + \beta + 1 > 2, \quad \alpha > \beta + 1, \quad \beta \geq 0, \quad \text{and} \quad \gamma < \alpha + 1 \text{ or } \gamma = \alpha + 1 \text{ with } dB_2 C_2 < A_1 B_1.
\]

In addition, let \(|\psi'(s)| \leq K s^{\beta-1}\) for large \(s.\) Then there exists a global classical solution to the Peterlin model (MP) for \(d = 2.\)

In [11] we studied a particular case of (MP) in which \(\gamma_3(s) = s,\) and the two functions \(\gamma_1, \gamma_2\) were taken as in [8]. For the two-dimensional case we showed the existence and uniqueness of regular global-in-time weak solutions as defined in [7]. Another technique of the proof allowed us to cover different choices of the constitutive functions than in Theorem 6 above. For completeness, we recall Theorem 5.3 from [11].

Theorem 7. (unique regular weak solution to (MP))

Let \(\Omega \subset \mathbb{R}^2\) be of class \(C^2\) and the initial data \((u_0, C_0) \in [H^2(\Omega)^2 \cap V] \times H^2(\Omega)^{2 \times 2}.\) Let the assumptions [8] and one of the following conditions be satisfied
\[
 0 < \alpha \leq 2, \quad 1 \leq \gamma < \alpha + 1, \quad \text{or} \quad \gamma = \alpha + 1 \text{ with } dB_2 C_2 < A_1 B_1,
\]
\[
 \text{or } \alpha = 0 \text{ and } \gamma = 1.
\]
Then the weak solution to the Peterlin model (MP) with $\gamma_3(s) = s$ satisfies

$$u \in L^\infty(0, T; H^2(\Omega)^2), \quad C \in L^\infty(0, T; H^2(\Omega)^2 \times 2)$$

and it is unique.

The proof of higher regularity is based on the regularity results for the Stokes and the Laplace operators. Analogously as in [8, 11, 16], assuming enough regular data, conditions (8) and (10), we can repeat the argument several times to obtain arbitrarily regular solution to (1). For our further needs it is sufficient to have $u \in L^\infty(0, T; H^3(\Omega)^2)$ and $C \in L^\infty(0, T; H^2(\Omega)^2 \times 2)$.

**Corollary 8.**

Let $\Omega$ be of class $C^3$ and the initial data $(u_0, C_0) \in [H^3(\Omega)^2 \cap V] \times H^3(\Omega)^2 \times 2$. Let the assumptions (8) and (10) be satisfied. Then the weak solution to the Peterlin model (MP) with $\gamma_3(s) = s$ satisfies

$$u \in L^\infty(0, T; H^3(\Omega)^2), \quad C \in L^\infty(0, T; H^2(\Omega)^2 \times 2).$$

Let us conclude the above two results for further reference. Let the assumptions of Theorem 6 and Corollary 8 be satisfied. Then there exists a unique regular weak solution to the Peterlin model (MP), i.e., a couple $(u, C)$ satisfying (11) - (12) such that

$$u \in L^2(0, T; V) \cap L^\infty(0, T; H^3(\Omega)^2), \quad C \in L^2(0, T; H^1(\Omega)^2 \times 2) \cap L^\infty(0, T; H^2(\Omega)^2 \times 2).$$

(11)

**4 The Fokker-Planck equation**

In this section we want to prove the existence of the weak solution $\psi = \psi_* = M \hat{\psi}_*$ to the Fokker-Planck equation (34) for a given couple $(u_*, C_*)$.

We set $(u, C) = (u_*, C_*)$, where

$$u_* \in L^2(0, T; V) \cap L^\infty(0, T; H^3(\Omega)^2), \quad C_* \in L^\infty(0, T; H^2(\Omega)^d \times d)$$

(12)

and we seek the solution $\hat{\psi}_*(t, x, R) = \psi(t, x, R)/M(R)$ such that

$$\frac{\partial \hat{\psi}_*}{\partial t} + (u_* \cdot \nabla) \hat{\psi}_* + \text{div}_R \left[ \nabla_x u_* \cdot R \hat{\psi}_* \right] = \Gamma(\text{tr C}_*) \nabla_R \cdot \left( M \nabla_R \left( \frac{\hat{\psi}_*}{M} \right) \right) + \varepsilon \Delta_x \hat{\psi}_*$$

(13a)

subject to the following decay/boundary and initial conditions

$$\left| M \left( \Gamma(\text{tr C}_*) \nabla_R \hat{\psi}_* - (\nabla_x u_*)R \hat{\psi}_* \right) \right| \rightarrow 0 \quad \text{as} \quad |R| \rightarrow \infty \quad \text{on} \quad (0, T) \times \Omega,$$

(13b)

$$\varepsilon \frac{\partial \hat{\psi}_*}{\partial n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega \times \mathbb{R}^d,$$

(13c)

$$\hat{\psi}_*(0) = \hat{\psi}_0 \quad \text{on} \quad \Omega \times \mathbb{R}^d.$$  

(13d)

Further we assume that

$$\hat{\psi}_0 \in L^2_M(\Omega \times \mathbb{R}^d) \quad \text{with} \quad \hat{\psi}_0 \geq 0 \quad \text{a.e. on} \quad \Omega \times \mathbb{R}^d, \quad \int_{\mathbb{R}^d} M(R) \hat{\psi}_0(x, R) \, dR = 1 \quad \text{a.e.} \quad x \in \Omega.$$  

(13e)
Definition 9. We refer to the system of equations and conditions (12), (13) as problem (FP).

Let us point out that the difference between equation (13a) and the Fokker-Planck equation of the Hookean dumbbell model studied in [5] is the coefficient $\Gamma(\text{tr } C_s)$. Under the assumptions (3d), (5) and (12) it holds that $\|\Gamma(\text{tr } C_s)\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^d))} \leq c$. Thus, the whole proof of the existence result for the Fokker-Planck equation from Section 4 in [5] can be repeated for problem (FP) defined above. In what follows we recall the key steps of the proof to recall its main idea.

4.1 Semi-discrete approximation of a regularized problem

Firstly, we consider a regularization (FP$_L$) of problem (FP) that is based on the parameter $L > 1$. The term involving $\nabla_x u_s$ in (13a) and the corresponding term in (13b) are modified using the cut-off function $\beta^L \in C(\mathbb{R})$ defined as

$$\beta^L(s) = \min(s, L) = \begin{cases} s, & s \leq L \\ L, & s \geq L. \end{cases}$$

(14)

We seek a solution $\hat{\psi}_{s,L} \in L^\infty(0,T;L^2_M(\Omega \times \mathbb{R}^d)) \cap L^2(0,T;\hat{X})$ that, for any $\phi \in W^{1,1}(0,T;\hat{X})$ with $\hat{\phi}(T, \cdot, \cdot) = 0$, satisfies

$$- \int_0^T \int_{\Omega \times \mathbb{R}^d} M \hat{\psi}_{s,L} \frac{\partial \hat{\phi}}{\partial t} d\mathbf{R} \, dx \, dt + \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \frac{\partial}{\partial x} \hat{\psi}_{s,L} - u_s \hat{\psi}_{s,L} \right] \cdot \nabla_x \hat{\phi} \, d\mathbf{R} \, dx \, dt +$$

$$+ \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \Gamma(\text{tr } C_s) \nabla_R \hat{\psi}_{s,L} - \left( (\nabla_x u_s) \mathbf{R} \right) \beta^L(\hat{\psi}_{s,L}) \right] \cdot \nabla_R \hat{\phi} \, d\mathbf{R} \, dx \, dt =$$

$$= \int_{\Omega \times \mathbb{R}^d} M \beta^L(\hat{\phi}_0) \hat{\phi} \, d\mathbf{R} \, dx.$$  

(15)

In order to prove the existence of weak solutions to (FP$_L$) we study a discrete-in-time approximation of (15). To this end we consider a regular mesh $\{t_1, \ldots, t_N\}$ on the time interval $[0,T]$. For $n = 1, \ldots, N$, we seek the values $\hat{\psi}^n_{s,L} \in \hat{Z}_2 \cap \hat{X}$ representing the approximate solution.

By $\hat{\psi}^n_{s,L}(t, \cdot)$ we denote the piecewise linear approximation of $\hat{\psi}_{s,L}(t, \cdot)$. Further, we employ a collective symbol $\hat{\psi}^n_{s,L}(\pm)$ for $\hat{\psi}^n_{s,L}$ and the values $\hat{\psi}_{s,L}$, $\hat{\psi}^n_{s,L}$ at the end points of the interval $[t_{n-1}, t_n]$. For more details we refer to the works of Barret and Süli [12,5].

Now, we recall the most important uniform estimates. As first, it can be shown, cf., Lemma 4.3 in [5], that an arbitrary $r$-th moment of the approximate solution is uniformly bounded.

Lemma 10. (uniform bounds on the moments)
Let the assumptions (12) and (13e) be satisfied. Then we have, for any $r \in \mathbb{R}_0^+$, that

$$\int_{\Omega \times \mathbb{R}^d} M|\mathbf{R}|^r \hat{\psi}^n_{s,L} \, d\mathbf{R} \, dx \leq c, \quad n = 0, \ldots, N.$$

The next estimate implies that the solution $\hat{\psi}^n_{s,L}$ has finite Fisher information and finite relative entropy with respect to the Maxwellian $M$. We refer to Lemma 4.4 in [5].

Lemma 11. (finite Fisher information and relative entropy)
Under the assumptions of Lemma 10 it holds that

$$\text{ess sup}_{t \in [0,T]} \int_{\Omega \times \mathbb{R}^d} M \mathcal{F}(\hat{\psi}^n_{s,L}(\pm))(t) \, d\mathbf{R} \, dx + \frac{1}{\Delta t L} \int_{\Omega \times \mathbb{R}^d} M (\hat{\psi}^n_{s,L} - \hat{\psi}_{s,L})^2 \, d\mathbf{R} \, dx +$$

$$+ \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \left| \nabla_x \sqrt{\hat{\psi}^n_{s,L}} \right|^2 + \left| \nabla_R \sqrt{\hat{\psi}^n_{s,L}} \right|^2 \right] \, d\mathbf{R} \, dx \, dt \leq c.$$  

(16)
Moreover, we have that
\[
\left| \int_0^T \int_{\Omega \times \mathbb{R}^d} M \frac{\partial \hat{\psi}_{\Delta t}^L}{\partial t} \, d\mathbf{R} \, dx \, dt \right| \leq c \|\phi\|_{L^2(0,T;W^{1,\infty}(\Omega \times \mathbb{R}^d))} \quad \forall \phi \in L^2(0,T;W^{1,\infty}(\Omega \times \mathbb{R}^d)).
\]

The function $F \in C(\mathbb{R}^+) \) appearing in (16) is given by $F(s) := s \log s - 1) + 1$. As pointed out in [5], it is a non-negative, strictly convex function that can be considered to be defined on $[0, \infty)$ with $F(1) = 0$.

### 4.2 Existence of weak solutions to (FP)

Passage to the limit with $L \to \infty$ implies the existence of weak solution to (FP) as shown in Theorem 4.1 in [5].

**Theorem 12.** (existence of weak solution to (FP))

Let the assumptions (12), (13c) be satisfied, and let $\Delta t \leq (4L^2)^{-1}$ as $L \to \infty$. Then, there exists a subsequence of $\{\hat{\psi}^t_{\Delta t} \} \subseteq L^1$ and a function $\hat{\psi}_*$ such that

\[
\begin{align*}
|\mathbf{R}|^r \hat{\psi}_* \in L^\infty(0,T;L^1_M(\Omega \times \mathbb{R}^d)), & \quad \text{for any } r \in [0, \infty), \\
\hat{\psi}_* \in H^1(0,T;M^{-1}[H^s_M(\Omega \times \mathbb{R}^d)]'), & \quad \text{for any } s > d + 1,
\end{align*}
\]

with

\[
\hat{\psi}_* \geq 0 \text{ a.e. on } [0,T] \times \Omega \times \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d} M(\mathbf{R}) \hat{\psi}_*(t,x,\mathbf{R}) \, d\mathbf{R} \leq 1 \text{ for a.e. } (x,t) \in [0,T] \times \Omega,
\]

and finite relative entropy and Fisher information, with

\[
\mathcal{F}(\hat{\psi}_*) \in L^\infty(0,T;L^1_M(\Omega \times \mathbb{R}^d)) \quad \text{and} \quad \sqrt{\hat{\psi}_*} \in L^2(0,T;H^1_M(\Omega \times \mathbb{R}^d)),
\]

such that as $L \to \infty \, (\text{and } \Delta t \to 0)$

\[
\begin{align*}
M^{1/2} \nabla_x \sqrt{\hat{\psi}^\Delta t_{\psi_* L}} & \to M^{1/2} \nabla_x \sqrt{\hat{\psi}_*} \quad \text{weakly in } L^2(0,T;L^2(\Omega \times \mathbb{R}^d)), \\
M^{1/2} \nabla_R \sqrt{\hat{\psi}^\Delta t_{\psi_* L}} & \to M^{1/2} \nabla_R \sqrt{\hat{\psi}_*} \quad \text{weakly in } L^2(0,T;L^2(\Omega \times \mathbb{R}^d)), \\
M \frac{\partial \hat{\psi}^\Delta t_{\psi_* L}}{\partial t} & \to \frac{\partial \hat{\psi}_*}{\partial t} \quad \text{weakly in } L^2(0,T;[H^s(\Omega \times \mathbb{R}^d)]'), \\
|\mathbf{R}|^r \beta L(\psi^\Delta t_{\psi_* L}) & \to |\mathbf{R}|^r \beta \hat{\psi}_* \quad \text{strongly in } L^p(0,T;L^1_M(\Omega \times \mathbb{R}^d)),
\end{align*}
\]

for any $p \in [1, \infty)$.

Additionally, for $s > d + 1$, the function $\hat{\psi}_*$ satisfies

\[
\begin{align*}
& - \int_0^T \int_{\Omega \times \mathbb{R}^d} M \hat{\psi}_* \frac{\partial \hat{\psi}_*}{\partial t} \, d\mathbf{R} \, dx \, dt + \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \varepsilon \nabla_x \hat{\psi}_* - \mathbf{u}_x \hat{\psi}_* \right] \cdot \nabla_x \phi \, d\mathbf{R} \, dx \, dt + \\
& + \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \Gamma(\text{tr } C_+) \nabla_R \hat{\psi}_* - [\nabla_x \mathbf{u}_x] \hat{\psi}_* \right] \cdot \nabla_R \phi \, d\mathbf{R} \, dx \, dt = \\
& = \int_{\Omega \times \mathbb{R}^d} M \hat{\psi}_0(x,\mathbf{R}) \phi(0,x,\mathbf{R}) \, d\mathbf{R} \, dx, \quad \forall \phi \in W^{1,1}(0,T;\hat{X}) \text{ with } \phi(T,\cdot,\cdot) = 0.
\end{align*}
\]
4.3 Macroscopic closure

In what follows we shall discuss the rigorous macroscopic closure of the Fokker-Planck equation. It can be shown that under the assumptions of Theorem 12 the weak solution \( \hat{\psi}_s \) to problem (FP) is such that

\[
C(M\hat{\psi}_s) = C(M\hat{\psi}_s)^T \in L^\infty(0,T;L^2(\Omega)^{d \times d}) \cap L^2(0,T;H^1(\Omega)^{d \times d}),
\]

and it satisfies, for any \( D \in W^{1,1}(0,T;H^1(\Omega)^{d \times d}), \)

\[
- \int_0^T \int_\Omega C(M\hat{\psi}_s) : \frac{\partial D}{\partial t} \ dx \ dt + \int_0^T \int_\Omega \varepsilon \nabla_x C(M\hat{\psi}_s) : \nabla_x D - (u_s \cdot \nabla_x)D : C(M\hat{\psi}_s) \ dx \ dt - \\
- \int_0^T \int_\Omega (\nabla_x u_s)C(M\hat{\psi}_s) + C(M\hat{\psi}_s)(\nabla_x u_s)^T \ dx \ dt - \int_0^T \int_\Omega 2\Gamma(\text{tr } C_s) \left[ I - C(M\hat{\psi}_s) \right] : D \ dx \ dt = \\
= \int_{\Omega \times \mathbb{R}^d} C(M\hat{\psi}_0)(x) : D(0,x) \ dR \ dx.
\]

Let us note that by (3b) and (3d) it holds that all the terms in (18) except the term containing the gradient with \( \nabla_x \) and it satisfies, for any \( D \in H^1(\Omega)^{d \times d}, \)

\[
\int_0^T \int_\Omega \varepsilon \nabla_x C(M\hat{\psi}_s) : \nabla_x D - (u_s \cdot \nabla_x)D : C(M\hat{\psi}_s) \ dx \ dt - \\
\int_0^T \int_\Omega (\nabla_x u_s)C(M\hat{\psi}_s) + C(M\hat{\psi}_s)(\nabla_x u_s)^T \ dx \ dt - \int_0^T \int_\Omega 2\Gamma(\text{tr } C_s) \left[ I - C(M\hat{\psi}_s) \right] : D \ dx \ dt = \\
= \int_{\Omega \times \mathbb{R}^d} C(M\hat{\psi}_0)(x) : D(0,x) \ dR \ dx.
\]

For the careful derivation we refer to Lemmas 4.2, 4.5 and 4.6 in [5]. The main idea is to test the semi-discrete approximation of the Fokker-Planck equation with \( R \otimes R : D \in \hat{X} \) for \( D \in C^\infty(\Omega). \)

The definition of the conformation tensor along with some useful identities mentioned below yields all the terms in (18) except the term containing the gradient \( \nabla R \hat{\psi}_s(L). \) In the latter term we have to integrate by parts with respect to \( R, \) which requires the approximation of \( \hat{\psi}_s(L) \) by a sequence of smooth functions. The dense embedding of \( C^\infty(\Omega;C_0^\infty(\mathbb{R}^d)) \) in \( \hat{X}, \) cf., [2], then implies that the closure is indeed valid for \( \hat{\psi}_s(L) \in \hat{X}. \)

Here we only present formal macroscopic closure of the weak formulation (17). To this end let \( \hat{\phi} \) in (17) be \( R \otimes R : D \) with \( D \in W^{1,1}(0,T;C^\infty(\Omega)) \) such that \( D(T, \cdot) = 0. \) Taking into account the definition of the conformation tensor \( C(\hat{\psi}) := (R \otimes R) = \int_{\mathbb{R}^d} R \otimes R \psi(t,x,R) \ dR \)

we directly get

\[
- \int_0^T \int_\Omega C(M\hat{\psi}_s) : \frac{\partial D}{\partial t} \ dx \ dt + \int_0^T \int_\Omega \varepsilon \nabla_x C(M\hat{\psi}_s) : \nabla_x D - (u_s \cdot \nabla_x)D : C(M\hat{\psi}_s) \ dx \ dt + \\
+ \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \Gamma(\text{tr } C_s) \nabla R \hat{\psi}_s - [(\nabla_x u_s)R] \hat{\psi}_s \right] \cdot \nabla R (R \otimes R : D) \ dR \ dx \ dt = \\
= \int_{\Omega \times \mathbb{R}^d} C(M\hat{\psi}_0)(x) : D(0,x) \ dR \ dx.
\]

Moreover, for any \( a \in \mathbb{R}^d \) it holds that \( (a \cdot \nabla R)(R \otimes R) = aR^T + Ra^T. \) Thus, the latter identity with \( a = M[(\nabla_x u_s)R] \hat{\psi}_s \) yields

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ (\nabla_x u_s)R \right] \hat{\psi}_s \cdot \nabla R (R \otimes R : D) \ dR \ dx \ dt \\
= \int_0^T \int_\Omega (\nabla_x u_s)C(M\hat{\psi}_s) + C(M\hat{\psi}_s)(\nabla_x u_s)^T \ dx \ dt.
\]
Further, formal integration by parts, which is rigorously done by employing the density argument mentioned above, yields the term
\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} M \Gamma(\text{tr } C_s) \nabla_R \hat{\psi}_s \cdot \nabla_R (R \otimes R : D) \, dR \, dx \, dt = \\
= - \int_0^T \int_{\Omega \times \mathbb{R}^d} \Gamma(\text{tr } C_s) \hat{\psi}_s \left[ \nabla_R \cdot (M \nabla_R (R \otimes R)) \right] : D \, dR \, dx \, dt.
\]

By (3a) and identity \(\Delta_R (R \otimes R) = 2I\), we finally obtain
\[
- \int_0^T \int_{\Omega \times \mathbb{R}^d} \Gamma(\text{tr } C_s) \hat{\psi}_s \left[ \nabla_R \cdot (M \nabla_R (R \otimes R)) \right] : D \, dR \, dx \, dt = \\
= - \int_0^T \int_{\Omega} 2 \Gamma(\text{tr } C_s) \left[ I - C(M \hat{\psi}_s) \right] : D \, dx \, dt,
\]
and thus equation (18).

5 The existence result for the kinetic Peterlin model

In what follows we combine the results from the previous two sections to prove the existence of large-data global-in-time weak solutions to the kinetic Peterlin model (KP). Let us note that the weak solutions to the Fokker-Planck equation for a given pair \((u_0, C_0)\) exist in both two and three space dimensions, cf., Section 3. However, the main result is only valid in two space dimensions due to the uniqueness result for the macroscopic model from Section 3.

**Theorem 13. (existence of weak solution to (KP))**

Let \(d = 2\) and \(\Omega\) be of class \(C^3\). Let \(u_0 \in V \cap H^3(\Omega)^2\) and let \(\hat{\psi}_0\) satisfy (13e) with \(C_0 := C(M \hat{\psi}_0) \in H^3(\Omega)^{2 \times 2}\). It follows that there exists a couple \((u_\psi, C_\psi)\) satisfying (13d) and solving the weak formulation (11) of the Peterlin model.

In addition, there exists \(\hat{\psi}_\psi\) satisfying
\[
|R| \hat{\psi}_\psi \in L^\infty(0, T; L^1_M(\Omega \times \mathbb{R}^d)), \quad \text{for any } r \in [0, \infty),
\]
with
\[
\hat{\psi}_\psi \geq 0 \quad \text{a.e. on } [0, T] \times \Omega \times \mathbb{R}^d \quad \text{and} \quad \int_{\mathbb{R}^d} M(R) \hat{\psi}_\psi(t, x, R) \, dR = 1 \quad \text{for a.e. } (t, x) \in [0, T] \times \Omega,
\]
\[
\mathcal{F}(\hat{\psi}_\psi) \in L^\infty(0, T; L^1_M(\Omega \times \mathbb{R}^d)) \quad \text{and} \quad \sqrt{\hat{\psi}_\psi} \in L^2(0, T; H^1_M(\Omega \times \mathbb{R}^d)),
\]
and solving
\[
- \int_0^T \int_{\Omega \times \mathbb{R}^d} M \hat{\psi}_\psi \frac{\partial \hat{\psi}_\psi}{\partial t} \, dR \, dx \, dt + \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \hat{\psi}_\psi \hat{\psi}_\psi - u_\psi \hat{\psi}_\psi \right] \cdot \nabla_x \hat{\psi}_\psi \, dR \, dx \, dt + \]
\[
+ \int_0^T \int_{\Omega \times \mathbb{R}^d} M \left[ \Gamma(\text{tr } C_\psi) \nabla_R \hat{\psi}_\psi - \left[ (\nabla_x u_\psi) R \right] \hat{\psi}_\psi \right] \cdot \nabla_R \hat{\psi}_\psi \, dR \, dx \, dt = \\
= \int_{\Omega \times \mathbb{R}^d} M \hat{\psi}_\psi (0, x, R) \hat{\psi} (0, x, R) \, dR \, dx, \quad \forall \hat{\psi} \in W^{-1,1}(0, T; \hat{X}) \quad \text{with} \quad \hat{\psi}(T, \cdot, \cdot) = 0.\]

Moreover, we have that \(C_\psi = C(M \hat{\psi}_\psi)\).
Proof. The existence of \((u_p, C_p)\) satisfying (12) as a unique solution to (7) is a straightforward consequence of Theorem 6 and Corollary 8. Theorem 12 with \((u_*, C_*) = (u_p, C_p)\) yields the existence of \(\hat{\psi}_p\) satisfying (19). Comparing (7) with (18) and recalling uniqueness of the regular weak solution to \((MP)\) from Section 3, we can conclude that \(C_p = C(M\hat{\psi}_p)\). \(\square\)

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