Estimate of the Three-Loop $\overline{\text{MS}}$ Contribution to
\[ \sigma(W^+_L W^-_L \rightarrow Z_L Z_L) \]

F. A. Chishtie and V. Elias
Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7
Canada

Abstract
The three-loop contribution to the $\overline{\text{MS}}$ single-Higgs-doublet standard-model cross-section $\sigma(W^+_L W^-_L \rightarrow Z_L Z_L)$ at $s = (5M_H)^2$ is estimated via least-squares matching of the asymptotic Padé-approximant prediction of the next order term, a procedure that has been previously applied to QCD corrections to correlation functions and decay amplitudes. In contrast to these prior applications, the expansion parameter for the $W^+_L W^-_L \rightarrow Z_L Z_L$ process is the non-asymptotically-free quartic scalar-field coupling of the standard model, suggesting that the least-squares matching be performed over the “infrared” $\mu^2 \leq s$ region of the scale parameter. All three coefficients of logarithms within the three-loop term obtained by such matching are found to be within 6.6% relative error of their true values, as determined via renormalization-group methods. Surprisingly, almost identical results are obtained by performing the least squares matching over the $\mu^2 \geq s$ region.

Within standard-model single-Higgs-doublet electroweak physics, the cross-section for the scattering of two longitudinal W’s into two longitudinal Z’s at very high energy takes the form
\[ \sigma[s, L(\mu), g(\mu)] = \frac{8\pi^3}{9s} H[s, L(\mu), g(\mu)], \]  
where the scale-sensitive portion of the cross-section [1],
\[ H[s, L(\mu), g(\mu)] = g^2(\mu) \{1 + [-4L(\mu) - 10.0]g(\mu) + 12L^2(\mu) + 68.667L(\mu) + \left(93.553 + \frac{2}{3} \ln \left( \frac{s}{M^2_H} \right) \right) g^2(\mu) + \left[c_3L^3(\mu) + c_2L^2(\mu) + c_1L(\mu) + c_0 \right] g^3(\mu) + \ldots \}, \]  
depends on the renormalization scale $\mu$ explicitly through the logarithm
\[ L(\mu) \equiv \ln(\mu^2/s) \]  
and implicitly through the $\overline{\text{MS}}$ quartic scalar-field coupling
\[ g(\mu) = 6 \frac{\lambda_{\overline{\text{MS}}}(\mu)}{16\pi^2}. \]
The three-loop coefficients \( \{c_0, c_1, c_2, c_3\} \) in (2) are presently unknown. The factor of \( M_H \) appearing explicitly in the two-loop term calculated in [1] is a scale-independent pole mass; the coefficients \( \{c_0, c_1, c_2, c_3\} \) can also exhibit dependence on this mass without acquiring additional \( \mu \)-dependence.

In this note, we utilise asymptotic Padé-approximant methods to predict these four coefficients. Such methods have already been applied to predicting next-order terms within the renormalization group functions of QCD [2,3,4], supersymmetric QCD [2,5], and massive scalar field theory [3,4,6], as well as next-order QCD corrections to scalar and vector current correlation functions [4,7] and various decay processes [8,9,10]. In all of these applications, the Padé-estimation procedures are tested (with surprising success) against either those higher-order terms already known from explicit calculation, such as renormalization group functions in scalar field theories [11,12], QCD [13], and supersymmetric QCD [14], or against those coefficients of logarithms [such as \( \{c_1, c_2, c_3\} \) in (2)] which can be extracted via renormalization group methods [7,8,9,10]. Below, we shall apply the latter testing procedure to estimates of the next-order terms \( \{c_1, c_2, c_3\} \) in the \( \overline{\text{MS}} \) cross-section (1).

Given a perturbative series of the form 
\[
1 + R_1g + R_2g^2 + R_3g^3 + \ldots
\]
where \( R_3 \) is not known, as is the case in (2), asymptotic Padé-approximant methods can be employed to show that [4]
\[
R_3 \equiv \frac{2R_3^3}{R_1^3 + R_1R_2}. \tag{5}
\]
This result is, of course, contingent upon the field-theoretical series exhibiting appropriate asymptotic behaviour. Its derivation (explicitly presented in [8]) follows from the \( O(1/N) \) error anticipated from an \( [N]1 \) Padé-approximant prediction of \( R_{N+2} \) [15], a semi-empirical behaviour which is seen to characterise a number of field-theoretical applications even when \( N \) is small.

For the case of the series (2), however, \( R_1 \) is linear in \( L \) and \( R_2 \) is quadratic in \( L \). Consequently, the prediction (5) for \( R_3 \) corresponds to a rational function of \( L \) incompatible with the degree-3 polynomial in \( L \) anticipated from (2). Clearly, a procedure is required by which predictions for the polynomial coefficients \( \{c_0, c_1, c_2, c_3\} \) in (2) can be extracted from (5). In past applications where the same problem arises [7,8,9,10], one method employed is a least-squares matching of (5) to the form 
\[
R_3 = c_0 + c_1L(\mu) + c_2L^2(\mu) + c_3L^3(\mu)
\]
over the full perturbative domain of \( \mu \). For QCD calculations this domain is ultraviolet; e.g. in estimating three-loop QCD corrections to \( B \rightarrow X_u\bar{\nu} \) the matching is over the ultraviolet domain \( \mu \geq m_b(\mu) \) [8]. For the expression (2), in which the perturbative expansion parameter is the non-asymptotically-free quartic scalar coupling \( \lambda_{\overline{\text{MS}}} \), the appropriate domain for such a least-squares matching is infrared.

Thus, to obtain predicted values for \( \{c_0, c_1, c_2, c_3\} \) for a given choice of \( M_H \), we choose a least-squares matching over the region \( \mu^2 \leq s \), or alternatively \( 0 < w \leq 1 \), where \( w \equiv \mu^2/s \) is the argument of the logarithm (3). From (2) and (5), this matching is achieved by minimizing the function
\[ \chi^2(c_0, c_1, c_2, c_3) \equiv \int_{w_{\text{min}}}^{1} dw \left[ \frac{2R_2^3(w)}{R_1^3(w) + R_1(w)R_2(w)} - c_0 - c_1 \ln(w) - c_2 \ln^2(w) - c_3 \ln^3(w) \right]^2 \] (6)

with respect to \( c_0, c_1, c_2, c_3 \), where \( R_1(w) \) and \( R_2(w) \) are explicitly given in (2):

\[ R_1(w) = -4 \ln(w) - 10.0, \] (7)

\[ R_2(w) = 12 \ln^2(w) + 68.667 \ln(w) + \left( 93.553 + \frac{2}{3} \ln \left( \frac{s}{M_H^2} \right) \right). \] (8)

The lower bound of integration \( w_{\text{min}} \) in (6) would ordinarily be zero to encompass the full \( \mu^2 \leq s \) range. However, we are compelled to consider a nonzero value of \( w_{\text{min}} \) in order to avoid any integrand poles, as discussed below. The expressions (1) and (2) are stated in ref. [1] to be accurate (within single-digit percent errors) only in the high-energy limit \( \sqrt{s} \gtrsim 5M_H \). Although the projected linear and quadratic dependence of \( c_1 \) and \( c_0 \) on \( \ln(s/M_H^2) \) could, in principle, be extracted via Padé methods\footnote{In ref. [10], the polynomial dependence of three-loop order terms in \( H \to gg \) on the logarithm of the pole-mass ratio \( M_H/M_1 \) is similarly extracted.}, the relatively small coefficient of this logarithm in the previous-order term (8) necessarily implies a similar insensitivity to this logarithm in Padé estimates of next-order terms. Consequently, we restrict our analysis here to the \( s = (5M_H)^2 \) kinematic boundary of applicability for (1) and (2). With this choice, the integrand of (6) acquires singularities at 0.0552, 0.0821, and 0.0896. Consequently, we choose \( w_{\text{min}} = 0.09 \) to include virtually all of the integrable infrared region, and we find that

\[ \chi^2(c_0, c_1, c_2, c_3) \]

\[ = 248033 + 813.779c_0 - 332.772c_1 + 244.574c_2 - 242.664c_3 + 0.91c_0^2 - 1.38657c_0c_1 + 1.72946c_0c_2 - 2.67527c_0c_3 + 0.86473c_1^2 - 2.67527c_1c_2 + 4.64965c_1c_3 + 2.32482c_2^2 - 8.67669c_2c_3 + 8.48631c_3^2 \] (9)

By then optimizing (9) with respect to \( c_0, c_1, c_2, c_3 \), we obtain the following Padé predictions for these coefficients:

\[ c_0^{P_{\text{Pad}}} = -896, \quad c_1^{P_{\text{Pad}}} = -889, \quad c_2^{P_{\text{Pad}}} = -288, \quad c_3^{P_{\text{Pad}}} = -30.5. \] (10)

As noted earlier, the true values of the coefficients \( c_1, c_2, c_3 \) can be extracted via the scale-invariance of the physical cross-section (1):

\[ O = \mu^2 \frac{dH}{d\mu^2}(s, L(\mu), g(\mu)) = \frac{\partial H}{\partial L} + \beta(g) \frac{\partial H}{\partial g}, \] (11)

where [11]

\[ \beta(g) = 2g^2 - \frac{13}{3} g^3 + 27.803g^4 + \ldots . \] (12)
One can verify that the known terms in (2) satisfy the renormalization-group equation (11) to $O(g^3)$ and $O(g^4)$, as is evident from the series expansions

$$\frac{\partial H}{\partial L} = -4g^3 + (24L + 68.667)g^4 + (3c_3L^2 + 2c_2L + c_1)g^5 + ... ,$$  \hspace{1cm} (13)

$$\beta(g) \frac{\partial H}{\partial g} = 4g^3 + (-24L - 68.667)g^4 + (96L^2 + 601.333L + 934.028 \ldots ,$$  \hspace{1cm} (14)

We find upon incorporating $O(g^5)$ terms of (13) and (14) into the right-hand side of (11) that

$$c_1 = -934.028 - 5.333ln(s/M_H^2) \quad \text{approaches} \quad -951.2,$$

$$c_2 = -300.67, \quad c_3 = -32.$$  \hspace{1cm} (15)

The Padé predictions (10) for $c_1, c_2, c_3$ are respectively seen to be within relative errors of 6.6%, 4.3%, and 4.7% of their true values (15).

Curiously, the accuracy of these Padé results does not appear to be contingent upon the matching being performed over the “infrared” $\mu^2 < s = 25 M_H^2$ range, as motivated by the non-asymptotically free character of the scalar field coupling $g(\mu)$. Indeed, a potential drawback of fitting over the $\mu^2 \leq s$ (hence, $w \leq 1$) region, as in (6), is the negativity of $ln(w)$ over the entire range of integration. Since the $c_i$ ultimately obtained in (10) are all same-sign (negative), cancellations necessarily occur between successive $c_kln^k(w)$ terms in the integrand of (6) in the best-fit region of $c_k$ parameter-space. To address this issue, we have also performed a fit of the Padé-prediction (5) to the third-subleading order of (2) over the entire $\mu^2 \geq s$ (i.e. $w \geq 1$) region in which $ln(w)$ is positive. This entails integration of the integrand of (6) with appropriately modified bounds of integration to encompass the ultraviolet region:

$$\int_{w_{\text{min}}}^{1} dw[...]^2 \rightarrow \int_{1}^{\infty} dw[...]^2.$$

We then find that

$$\chi^2(c_0, c_1, c_2, c_3) = 1.45665 \cdot 10^7 + 5095.28c_0 + 10294.6c_1 + 35519.6c_2 + 167166c_3 + c_0^2 + 2c_0c_1 + 4c_0c_2 + 12c_0c_3 + 2c_1^2 + 12c_1c_2 + 48c_1c_3 + 24c_2^2 + 240c_2c_3 + 720c_3^2,$$

which upon optimization, yields values for $c_k$ [$c_0 = -896, \quad c_1 = -889, \quad c_2 = -289, \quad c_3 = -30.9$] that are virtually the same as those listed in (10). The
small relative errors characterising our Padé estimates of $c_1, c_2, c_3$ are comparable to those characterising Padé estimates of renormalization-group accessible coefficients within next-order QCD corrections to other processes [7,8,9,10], and suggest similar accuracy in the estimated value of the renormalization-group inaccessible three-loop coefficient $c_0$ in (10).

We therefore conclude that Padé-approximant predictions of the next order contribution to $WW \rightarrow ZZ$ at very high energies appear to be consistent and reliable. It should also be noted that higher order $\beta$-function terms associated with the evolution of the quartic scalar-field coupling constant (4) are themselves accurately predicted by the same asymptotic Padé-approximant methods employed above for $\sigma(WW \rightarrow ZZ)$. If we express the $\beta$-function (12) in the form

$$\beta(g) = 2g^2(1 + R_1 g + R_2 g^2 + R_3 g^3 + \ldots); \quad R_1 = -13/6, \quad R_2 = 13.915,$$

we predict via (5) that $R_3 = -133.6$, or alternatively, that the predicted next term in the series (12) is $-267.2g^3$. This is quite close to $-266.495g^3$, the true calculated value [11] of the next-order $\beta$-function contribution. Similarly close agreement between the somewhat more complicated asymptotic Padé-approximant prediction and the explicit calculation of the $O(g^6)$ contribution to this $\beta$-function is demonstrated in ref. [6].

Thus, the results presented above are an example of how Padé estimation procedures can anticipate next-order contributions whose exact values are obtainable only by lengthy calculation. For the particular process in question, the distinction between two- and three-loop order results is seen to be unimportant unless the mass of the (Salam-Weinberg) Higgs field mediating the scattering process is very large. This is illustrated in Figures 1-3, which compare two loop and three loop expressions for the scale sensitive portion (2) of the $WW \rightarrow ZZ$ cross section (1) for Higgs-field masses of 200, 400, and 600 GeV, respectively. Only for the largest of these three choices is an appreciable difference anticipated between two- and three-loop order predictions for the cross section.
Figure 1: The two-loop (bottom curve) and predicted three-loop (top curve) expressions for the scale-sensitive portion $H[s = (5M_H)^2, L(\mu), g(\mu)]$ of the cross-section (1) are plotted for Higgs mass $M_H = 200$ GeV.
Figure 2: Comparison of two-loop (2L) and three-loop (3L) expressions, as in Figure 1, but with $M_H = 400$ GeV.
Figure 3: Comparison of two-loop (2L) and three-loop (3L) expressions, as in Figure 1, but with $M_H = 600$ GeV.
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