Comparison of Results of Kinetic and Drift-Diffusion Models of Penning Discharge

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Abstract. Comparison of numerical models intended for the description of the processes in the Penning gas discharge plasma is performed in the study. Kinetic model is based on the particle-in-cell method. Modified drift-diffusion model belongs to the class of the hydrodynamic plasma models. Results of the numerical simulation based on the kinetic approach are obtained using author two-dimensional axisymmetric particle-in-cell computer code developed in VNIIA and proprietary computer code VORPAL (Tech-X Corporation). Results of the numerical simulation based on modified drift-diffusion model are obtained using author two-dimensional computer code developed in IPMech RAS. Results of numerical simulation are obtained at the following gas discharge parameters: working gas = H₂, p = 1.2 mTorr, V = 2.5 kV, spatial scale of discharge chamber ~ 1 cm, B = 0.1 T. Comparison of spatial distributions of Penning gas discharge plasma parameters obtained using mentioned approaches is performed.

1. Introduction
Penning gas discharge plasma is characterized by the following parameters: pressure p ~ 1 mTorr, anode voltage V ~ 1 kV, the typical spatial scale of the problem ~ 1 cm, magnetic field induction B ~ 0.1 T. Using these values one can estimate spatial and temporal parameters of the elementary processes occurring in considered type of discharge plasma, such as mean free path \( \lambda \), Debye length \( \lambda_{\text{De}} \), plasma frequency \( \omega_p \), cyclotron frequency and Larmor radius of electrons (\( \omega_{ge}, r_{ge} \)) and ions (\( \omega_{gi}, r_{gi} \)) [1]. The estimations are presented in table 1 (for presented estimations number density of electrons was assumed to be of the order of \( 10^{10} \text{ cm}^{-3} \)).

| \( r_{H_2} \), cm | \( T_e \), eV | \( \lambda_{\text{De}} \), cm | \( \lambda \), cm | \( r_{ge} \), cm | \( r_{gi} \), cm | \( 1/ \omega_p \), s | \( 1/ \omega_{ge} \), s | \( 1/ \omega_{gi} \), s |
|-----------------|-------------|-----------------|-------------|-------------|-------------|--------------|--------------|--------------|
| 1               | 7.42 \times 10^{-3} | 1.66 \times 10^{-2} | 180       | 0.11        | 6.4         | 1.77 \times 10^{-10} | 5.686 \times 10^{-11} | 2.072 \times 10^{-7} |
| 74 \times 10^{-10} | 5         | 3.43 \times 10^{-2} | 240       | 0.11        | 6.4         | 1.77 \times 10^{-10} | 5.686 \times 10^{-11} | 2.072 \times 10^{-7} |

Presented values indicate that \( Kn = \lambda/L > 1 \) which means that valid description of the system has to be carried out using kinetic approach.
The disadvantage of kinetic models is its demand for the computational resources. On the other hand, hydrodynamic models, for example, drift-diffusion model, more computationally efficient and can be used to perform series of simulations for example for the optimization purposes. However, external magnetic field imposed on the Penning discharge plasma is not accounted for in classical drift-diffusion model [1, 2].

In [2, 3] modifications of the classical drift-diffusion model described, which take into account presence of external magnetic field. Magnetic field is introduced in the model by means of the Hall parameter:

$$b_e = \mu_e B_e; \quad \mu_e p = 0.37 \cdot 10^6 \text{ cm}^2 \cdot \text{Torr} / \text{V} \cdot \text{s}$$

(1)

In its turn the $b_e$ parameter modifies coefficients in the model, in particular, a diffusion coefficient [2, 3]:

$$D_{e,\text{mod}} = \frac{D_e}{1 + b_e^2}; \quad D_e = \mu_e T_e;$$

(2)

Diffusion coefficient is connected with mean free path using the following formula [1]:

$$D = \frac{1}{3} \lambda \nu$$

(3)

Thus one can estimate the mean free path in the case when magnetic field imposed on the system. Also one can estimate the magnetic pressure:

$$p_B = \frac{B^2}{2\mu_0}$$

(4)

The corresponding estimations are presented in table 2.

| $\mu_e$, cm$^2$/(V*s) | $b_e$ | $T_e$, eV | $D_e$, cm$^2$/s | $\lambda$, cm | $p_B$, Torr | $\beta = p/p_B$ |
|---------------------|------|---------|----------------|----------|-----------|-------------|
| 1                   | 1    | 2.7·10$^1$ | 1.366·10$^-6$ |
| 3.7·10$^4$         | 3.7·10$^3$ | 5       | 1.35·10$^2$ | 3.054·10$^-6$ | 2.99·10$^1$ | 3.3·10$^-5$ |
| 10                  | 2.7·10$^2$ | 4.319·10$^-6$ |

$Kn$ parameter estimated using the mean free path shown in table 2 allows to conclude that drift-diffusion model which accounts for the presence of external magnetic field can be used for the simulation of Penning gas discharges plasma at parameters presented in the beginning of the section. Thus it is meaningful to compare results obtained using drift-diffusion and kinetic approaches.

The structure of the paper is as follows. In the second section kinetic model of the Penning gas discharge in hydrogen based on the particle-in-cell method is going to be presented. In the third section modified drift-diffusion model will be described. In the fourth section comparison of the results obtained using the modified drift-diffusion model, kinetic model based on the particle-in-cell method implemented in the computer code developed in VNIIA and kinetic model based on the particle-in-cell method implemented VORPAL computer code is going to be presented and analyzed.

2. Description of the kinetic model of Penning gas discharge in hydrogen

Kinetic model for the simulation of the Penning discharge plasma is based on the particle-in-cell method. Generally speaking particle-in-cell method is a way of solving of Vlasov-Maxwell system of equations [4]. In considered case two-dimensional axisymmetric electrostatic particle-in-cell method is
used. If particle-in-cell method is combined with Monte-Carlo collision technique then it allows performing kinetic simulation of plasma discharge parameters considering physical-chemical processes such as ionization, elastic scattering.

Particle-in-cell method uses a concept of macroparticles [5, 6]. Macroparticle is a computational particle that simulates behaviour of a large number of real plasma particles (electrons or different ion species). Macroparticles follow the trajectory of real particles in the plasma since the Lorentz force depends only on the charge-to-mass ratio [7]. The movement of macroparticles changes the distribution of electromagnetic field in the system. It means that the whole simulation process is self-consistent. Typical computational cycle of the particle-in-cell method consists of the following steps:

- solution of the Poisson equation for the calculation of electric field within the discharge chamber;
- interpolation of electric field from the mesh to the macroparticles position;
- calculation of the macroparticles displacement;
- simulating elementary processes on boundaries;
- simulating physical-chemical processes in the volume of discharge plasma using Monte-Carlo collision method;
- calculation of the charge density using information on the macroparticles weight and positions.

Algorithm presented above is implemented in the particle-in-cell code developed in VNIIA. In the study this code will be referred to as PIC--2D--UNSTR. Distinctive feature of the code is usage of unstructured grids. It allows to carry out simulation of the plasma devices with boundaries of complex shape. Detailed description of numerical algorithm implemented in PIC--2D--UNSTR is given in [8–10]. In this section we will focus our attention on the elements of the computed code specific to the model of Penning gas discharge in molecular hydrogen.

For the numerical simulation of the hydrogen plasma of Penning discharge using computer code PIC–2D–UNSTR two macroparticle species are considered: electrons and molecular hydrogen ions (H$_2^+$). Weight of each macroparticle is determined using initial number density of charged particles, initial number of macroparticles and initially volume occupied by them. This value is determined in the beginning of the simulation and remains constant in the process of computation.

We are considering following physical chemical processes occurring in the bulk of hydrogen plasma of Penning gas discharge:

- $e + H_2 \rightarrow e + H_2$ (elastic scattering of electrons on molecules H$_2$);
- $e + H_2 \rightarrow e + e + H_2^+$ (ionization of H$_2$ molecules by electron impact, ionization threshold is 15.34 eV);

These processes are simulated using Monte-Carlo collision technique [11–13]. In PIC–2D–UNSTR computer code this method is used in combination with null collision method [11, 12]. Null collision method allows speeding up calculation of collision processes by introducing additional fictitious process which makes collision frequency constant. This constant collision frequency allows determining in advance total number of macroparticles that might experience interaction on the current time step. This number generally is much less (fraction of the order of 10$^{-2}$) than total number of macroparticles of given specie. Thus computational efficiency is provided by reduction of number of arithmetic operations (kinetic energy calculation, evaluation of collision frequency) due to reduction of number of macroparticles to which these operations are applied.

Another option for speeding up the overall simulation process in the current version of PIC–2D–UNSTR code is usage of different time steps for the solution of charged macroparticle motion equations and for the simulation of collision processes by means Monte-Carlo method. It means that collision processes are not evaluated every PIC cycle (PIC cycle is connected with the solution of macroparticle motion equations).

Addition to the modeling process of macroparticles created during collision processes or altering of state of existing macroparticles is carried out using models of corresponding elementary processes.
[12, 13]. Cross-sections for the processes considered in the model were taken from [14] and presented in figures 1 and 2.

In the considered model of Penning gas discharge in hydrogen process of ion-electron emission from the cathode was considered. The data on the dependence of the electron yield per ion versus energy was taken from [15].

**Figure 1.** Cross-section of elastic scattering of electrons on the neutral hydrogen molecules used in the kinetic model: solid line – PIC–2D–UNSTR, dashed line - VORPAL.

**Figure 2.** Cross-section of electron impact ionization of hydrogen molecules used in the kinetic model: solid line – PIC-2D-UNSTR, dashed line - VORPAL.
In order to obtain physically valid results by means of the PIC–MCC method several restrictions have to be satisfied [16]:

- maximum size of computational grid elements has to be less than Debye length: \( \Delta x_{\text{max}} \leq \lambda_D \);
- the time step has to resolve the highest possible frequency \( \omega \) in the system: \( \omega \Delta t \leq 0.2 \);
- during one time step distance travelled by the macroparticle should not exceed the minimum size of the element (Courant condition): \( \max_{i \in \{1..N_p\}} \left( \left| u_i \right| \right) \cdot \Delta t_{\text{motion}} < \Delta x_{\text{min}} \);
- in order to reduce numerical noise inherent to PIC method number of macroparticles within the Debye sphere has to be reasonably high: \( N_D \gg 1 \);
- in order to minimize the probability of more than one collision per collision time step for the given particle \( \Delta t_{\text{collision}} \) should be chosen so that: \( \Delta t_{\text{collision}} \cdot \sigma_{\text{T}}(\epsilon_i) \cdot n_i(\vec{r}) < 0.1 \).

Alternative to the PIC–2D–UNSTR code for the kinetic modeling of Penning discharge plasma available in VNIIA is proprietary computer code VSim (Vorpal) developed in Tech–X Corporation [17]. VSim is a flexible, multiplatform, multiphysics software tool for running computationally intensive electromagnetic, electrostatic, magnetostatic and plasma simulations in the presence of complex dielectric, magnetic and metallic shapes. It allows to solve 1D, 2D and 3D problems. VSim can be used to simulate a variety of problems on regular, structured, orthogonal meshes with embedded boundaries (cut cells) for complex geometries. Self-consistent solutions in the presence of charged particles can be carried out using particle-in-cell method and/or charge fluid methods. In the framework of VSim particles collision processes can be simulated. Model of the hydrogen Penning discharge plasma described above was also implemented using VSim software for the sake of comparison. Cross-sections for the elementary processes used in VSim simulation are presented in figures 1 and 2 and compared against cross-sections used in the PIC–2D–UNSTR code.

3. Description of the modified drift-diffusion model of Penning gas discharge in hydrogen [3]

Detailed description of the modified drift diffusion model including derivation of its equations is given in [3]. In this section equations will be shown as well as closing relations. Considered model is two-dimensional axisymmetric. It includes continuity equations for electrons number density \( n_e \), ions number density \( n_i \) and Poisson equations for electric potential \( \varphi \). Axial magnetic field is accounted for in the model. Also the model was modified in order to take into account large reduced fields which can be observed in the Penning gas discharge.

The system of equations is presented below:

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} \left( \mu_n n_e E_z - D_e \frac{\partial n_e}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n_e}{\partial r} \right) = \alpha(E)\varphi - \beta n_e n_i \tag{5}
\]

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z} \left( \mu_n n_i E_z - D_e \frac{\partial n_i}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n_i}{\partial r} \right) = \alpha(E)\varphi - \beta n_e n_i \tag{6}
\]

\[
\frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) = 4\pi e(n_e - n_i) \tag{7}
\]

\[
\Gamma_{e,z} = n_e u_{e,z} = \mu_e n_e E_z - D_e \frac{\partial n_e}{\partial z} \tag{8}
\]

\[
\Gamma_{i,z} = n_i u_{i,z} = \frac{\mu_i}{1 + b_e} n_i E_z - \frac{D_e}{1 + b_e} \frac{\partial n_i}{\partial z} \tag{9}
\]

\[
\Gamma_{t,z} = n_e u_{e,z} = \mu_e n_e E_z - D_e \frac{\partial n_e}{\partial z} \tag{10}
\]
\[ \hat{\Gamma}_e = n_n u_n = \frac{\mu_e}{1 + b_e^2} n_e - \frac{D_e}{1 + b_e^2} \frac{\partial n_e}{\partial r} \]  
(11)

\[ \hat{\Gamma}_{ee} = n_u u_{ee} = -b_e \hat{\Gamma}_e; \quad \hat{\Gamma}_{ii} = n_u u_{ii} = -b_i \hat{\Gamma}_i \]  
(12)

Here \( \alpha(\bar{E}) \) is ionization coefficient, \( \hat{\Gamma}_e, \hat{\Gamma}_i \) are electron and ion flux densities, \( \mu_e, \mu_i \) are electron and ion mobilities, \( D_e, D_i \) are electron and ion diffusion coefficients, \( b_e, b_i \) is the electron and ion Hall parameter, \( \bar{E} = -\nabla \varphi \) is the electric field, \( \bar{E} = \sqrt{\Gamma_{ee} + \Gamma_{ei} + \Gamma_{ii}} \), \( \beta = 2 \cdot 10^{-7} \text{ cm}^3/\text{s} \) is the recombination coefficient (assumed constant in the model).

For the hydrogen plasma, closing relations are [3]:

\[ (\mu_e p) = 3.7 \cdot 10^4 \text{ cm}^3 \cdot \text{Torr} / \text{V} \cdot \text{s}; \quad (\mu_i p) = 6.55 \cdot 10^3 \text{ cm}^3 \cdot \text{Torr} / \text{V} \cdot \text{s} \]  
(13)

\[ \mu_e = \min \left( \frac{(\mu_e p)}{p}, \frac{2.4 \cdot 10^{10}}{\sqrt{p |\bar{E}|}} \right), \quad \mu_i = \min \left( \frac{(\mu_i p)}{p}, \frac{9.2 \cdot 10^7}{\sqrt{p |\bar{E}|}} \right) \]  
(14)

\[ D_e = \mu_e T_e; \quad D_i = \mu_i T_i \]  
(15)

\[ b_e = \mu_e B; \quad b_i = \mu_i B \]  
(16)

Here \( p \) is the pressure of the background gas, \( B \) is the induction of the magnetic field, \( T_e = 11610 \text{ K} \) is the temperature of electrons. The closing relation (14) takes into account that in Penning gas discharge large reduced fields are possible.

The last important part of the model to be discussed is the ionization coefficient. In the modified drift-diffusion model possibility for large reduced fields is accounted for in the method intended for calculation of ionization coefficient. For the ionization coefficient in relatively law reduced field Townsend formula is used with approximation parameters relevant for molecular hydrogen [1, 3]:

\[ \alpha = \frac{A \exp \left( -\frac{B}{(|\bar{E}| / p) \right)}}, \quad A = 5 \text{ cm} \cdot \text{Torr}^{-1}, \quad B = 130 \text{ V} \text{ cm} \cdot \text{Torr}^{-1} \]  
(17)

This approximation is valid for \( E / p = 150 \div 600 \text{ V/(cm Torr)} \). At large reduced fields \([1, 3]\):

\[ \alpha = N_0 \sigma_i \]  
(18)

where \( \sigma_i \) is determined using the Thompson formula:

\[ \sigma_i = 4 \pi \alpha_i \left( \frac{I_H}{\epsilon} \right) \left( \frac{\epsilon - I}{I} \right) \]  
(19)

Here \( I \), \( I_H \) are the potentials of ionization of hydrogen molecule and hydrogen atom, \( \epsilon \) is the kinetic energy of electron, \( \alpha_i \) is the Bohr radius. Kinetic energy of electrons and heavy particles can be determined using the following formulas:

\[ \epsilon_e [\text{eV}] = 2.84 \cdot 10^{-14} V_e^2 [\text{cm/s}] \]  
(20)

\[ \epsilon_A [\text{eV}] = 1.05 \cdot 10^{-12} V_A^2 [\text{cm/s}] \]  
(21)

Equations (5) – (7) are solved in combination with the equation for the external circuit:

\[ \epsilon_{\text{emf}} = V + IR \]  
(22)
Here \( \varepsilon \) is the EMF of power supply, \( V \) is voltage drop on the electrodes, \( R \) is the external resistance and \( I \) is the discharge current. The solution of the equations is performed on structured grids.

Computer code that implements this model was developed in Laboratory of Radiative Gas Dynamics, Institute for Problems in Mechanics RAS by Surzhikov S.T. We will refer it as MDDM code throughout the paper.

4. Comparison of results of kinetic and modified drift-diffusion simulation of Penning discharge

In order to compare results obtained using kinetic and hydrodynamic approach the following problem will be considered.

Cylindrical anode with inner radius \( r_a = 0.55 \) cm is placed coaxially between circular cathode \( r_c = 0.55 \) cm and circular anticathode \( r_{ac} = 0.55 \) cm with orifice \( r_{orf} = 0.4 \) cm. Distance between cathode and anticathode is 1.1 cm. Length of anode is 0.5 cm. It is assumed that orifice is closed with the mesh. This feature is modeled by assuming corresponding boundary condition: \( \varphi_{orf} = 0 \). Schematic view of discharge chamber is shown in figure 3 as well as unstructured mesh \( (N_{el} = 6381, N_{pnt} = 3293, \text{grid nodes were condensed near anode}) \) and boundary conditions for Poisson equations used in simulation performed by means of PIC-2D-UNSTR code. The rest of the conditions for the simulation are as follows: buffer gas – \( \text{H}_2 \), buffer gas pressure \( p = 1.2 \) mTorr, buffer gas temperature \( T = 300 \) K, anode voltage \( V_a = 2500 \) V, cathode, anticathode and orifice are grounded, induction of axial magnetic field \( -B_z = 0.1 \) T.

In PIC-2D-UNSTR simulation initially 60000 of electrons and ions (quasi-neutral plasma) were placed in the cylindrical region (height and radius 0.5 cm) in the central part of computational area with initial number density of \( 1.5 \times 10^9 \) cm\(^{-3} \). Corresponding particle weight is \( \sim 10^5 \). Computational time step used in these calculation is 5.0 \( \times 10^{-12} \) s.

![Figure 3](attachment:image.png)

**Figure 3.** Geometry of the computational area and mesh used in PIC-2D-UNSTR code.

Another kinetic model is implemented using VSim computer code. Simulation using VSim is carried out on structured grid. Number of grid points used for the modeling is 86x86. Size of
structured mesh elements: $h_z = 1.28 \times 10^{-4}$ m, $h_r = 8.14 \times 10^{-5}$ m. Initially in computations using VSim code 100 electrons were placed in each cell. Initial electron number density is $10^8$ cm$^{-3}$. Time step chosen for simulation is $1.358 \times 10^{-12}$ s.

Simulation based on the modified drift-diffusion model of Penning gas discharge was performed using structured grid. Number of nodes is ~151x101. Condensation of the grid nodes in the vicinity of electrodes was performed. Boundary conditions used in the simulation performed using modified drift-diffusion model are as follows:

$$z = 0: \frac{\partial n_e}{\partial z} = 0, \Gamma_x = \Phi_x, \varphi = 0$$

$$z = 1.1 \text{ cm}, r > 0.4 \text{ cm}: \frac{\partial n_e}{\partial z} = 0, \Gamma_x = \Phi_x, \varphi = 0$$

$$r = 0: \frac{\partial n_e}{\partial r} = \frac{\partial n_i}{\partial r} = \frac{\partial \varphi}{\partial r} = 0$$

$$r = 0.55 \text{ cm}, z < 0.3 \text{ cm}, z > 0.8 \text{ cm}: \frac{\partial n_e}{\partial r} = \frac{\partial n_i}{\partial r} = 0, n_i = 0, \varphi = V_a$$

$$r = 0.55 \text{ cm}, 0.3 \text{ cm} < z < 0.8 \text{ cm}: \frac{\partial n_e}{\partial r} = 0, n_i = 0, \varphi = V_a$$

$$z = 1.1 \text{ cm}, r < 0.4 \text{ cm}: \frac{\partial n_e}{\partial z} = \frac{\partial n_i}{\partial z} = 0, \varphi = 0$$

A quasi-neutral plasma cloud of a spherical shape in the central region of computational area was used as the initial conditions.

In the figure 4 temporal evolution of number of macroelectrons and macroins is the system calculated using PIC–2D–UNSTR computer code is presented.

![Figure 4. Temporal dependence of number of macroparticles in the system calculated using PIC–2D–UNSTR.](image-url)
One can observe that hydrogen plasma of the Penning gas discharge tends to steady state. In the steady state number of macroions is about $5.26 \cdot 10^5$ and number of macroelectrons is $1.96 \cdot 10^6$.

Comparison of results of numerical simulation obtained using three computer codes is presented below. In the figure 5 two-dimensional distributions of electric potential $\phi$ [V] obtained using kinetic codes and modified drift-diffusion code are given.

**Figure 5.** Distribution of the potential $\phi$ [V] in the computational area: a – results obtained using VSim computer code; b – results obtained using PIC–2D–UNSTR, c – results obtained using MDDM code.
Qualitative picture of potential distribution in the discharge chamber obtained using kinetic and hydrodynamic codes is the same: potential decreases in the direction from anode to the axis of symmetry. However in the case of VSim computer code, one can observe slight increase of the potential towards the axis of symmetry, i.e. in this case distribution of the potential is nonmonotonically decreasing.

One can notice that distributions obtained using kinetic codes shows faster decrease of the potential (on the Δr ~ 0.2 cm from anode potential drops from 2400 V to 200V) than in the case of MDDM – code (on the Δr ~ 0.45 cm from anode potential drops from 2400 V to 200V).

In the figure 6 two-dimensional distribution of the radial component $E_r$ [V/cm] of electric field obtained using kinetic codes and modified drift-diffusion code are given.

In the vicinity of anode one can notice qualitative discrepancy between results obtained using kinetic codes and MDDM code: in the case of MDDM-code in the vicinity of anode $E_r \rightarrow 0$, while in the case of kinetic codes $E_r \rightarrow \sim 10000$ V/cm.

In the figure 7 two-dimensional distribution of the axial component $E_z$ [V/cm] of electric field obtained using kinetic codes and modified drift-diffusion code are given. Qualitative picture of distribution of axial component of electric field predicted using kinetic and hydrodynamic codes is the same. However, absolute value of $E_z$ predicted using kinetic codes is higher, than the absolute value predicted using the MDDM code.

In the figure 8 two-dimensional distributions of the reduced electric field $\frac{E}{p}$ [V/(cm·Torr)] obtained using kinetic and MDDM codes are shown. Qualitative picture of distribution of reduced electric field calculated using kinetic codes is the same. The main discrepancy between results obtained using kinetic and hydrodynamic codes is the distribution of reduced electric field in the vicinity of the anode. MDDM code produces lower values of reduced electric field in the vicinity of the anode than kinetic codes PIC-2D-UNSTR and VSim. On the other hand kinetic codes predict lower values of the reduced electric field below $r \sim 0.3$ cm than MDDM code.

Detailed comparison of potential distribution and radial component of electric field distribution obtained using kinetic and hydrodynamic codes is presented in figures 9, 10 correspondingly. It may be noted that quantitative agreement of results obtained using two completely different kinetic computer codes is satisfactory however not perfect. Electric potential in the vicinity of the axis of symmetry is about 350 V in the case of VSim code, ~ 25 V in the case of PIC-2D-UNSTR code. Results obtained using MDDM code are in qualitative agreement with results of kinetic codes except for the distribution of radial component of electric field in the vicinity of anode in the range of $r = 0.45 - 0.55$ cm (as it was already stated above).

In the figure 11 and 12 two-dimensional distributions of number density of electrons and $\text{H}_2^+$ ions obtained using kinetic codes and MDDM code are presented. Qualitative picture of electrons number density $n_e$ [cm$^{-3}$] distribution in the discharge chamber obtained using kinetic codes is the same. Discrepancy between kinetic codes and hydrodynamic code is observed in the vicinity of anode ($r \sim 0.35$ cm). In this region kinetic codes predict local maximum in the distribution of electron number density, while hydrodynamic code predicts monotonic decrease in electron number density from axis of symmetry to the anode. Both kinetic and hydrodynamic codes predict global maximum in the distribution of electron number density in the vicinity of axis of symmetry.

Qualitative picture of $\text{H}_2^+$ ions number density $n_i$ [cm$^{-3}$] distribution in the discharge chamber obtained using kinetic and hydrodynamic codes is the same. Maximum value of ions number density is reached in the vicinity of axis of symmetry and monotonically decreases towards the anode.

In the figures 13 and 14 distributions of the electron and ion number densities along the line $z = 0.55$ cm obtained using kinetic and hydrodynamic codes are shown.
Figure 6. Distribution of the radial component $E_r$ [V/cm] of electric field in the computational area: a – results obtained using VSim computer code; b – results obtained using PIC-2D-UNSTR; c – results obtained using MDDM code.
Figure 7. Distribution of the axial component $E_z$ [V/cm] of electric field in the computational area: a – results obtained using VSim computer code; b – results obtained using PIC-2D-UNSTR; c – results obtained using MDDM code.
Figure 8. Distribution of the reduced electric field $|\mathbf{E}|/\rho$ [V/(cm·Torr)] in the computational area: 
a – results obtained using VSim computer code; b – results obtained using PIC–2D–UNSTR; 
c – results obtained using MDDM code.
Figure 9. Distribution of the potential $\varphi$ [V] along the line $z = 0.55$ cm: solid line – result obtained using PIC–2D–UNSTR, dashed line – results obtained using VSim.

Figure 10. Distribution of the radial component $E_r$ [V/cm] of electric field along the line $z = 0.55$ cm: solid line – result obtained using PIC–2D–UNSTR, dashed line – results obtained using VSim.
Figure 11. Distribution of the electrons number density $n_e \text{[cm}^{-3}]$ in the computational area: a – results obtained using VSim computer code; b – results obtained using PIC–2D–UNSTR; c – results obtained using MDDM code.
Figure 12. Distribution of the ions $\text{H}_2^+$ number density $n_i$ [cm$^{-3}$] in the computational area: a – results obtained using VSim computer code; b – results obtained using PIC–2D–UNSTR; c – results obtained using MDDM code.
Figure 13. Distribution of the electrons number density $n_e$ [cm$^{-3}$] along the line $z = 0.55$ cm: solid line – result obtained using PIC–2D–UNSTR, dashed line – results obtained using VSim.

Figure 14. Distribution of the H$_2^+$ ions number density $n_i$ [cm$^{-3}$] along the line $z = 0.55$ cm: solid line – result obtained using PIC–2D–UNSTR, dashed line – results obtained using VSim.
It is worth noting that results obtained using kinetic codes are quite similar to each other. MDDM code predicts monotonic decrease of electron number densities form axis of symmetry to the anode, while kinetic code predict nonmonotonic behavior of electron number density in this area.

Maximum number density of electrons predicted using MDDM code is one order of magnitude higher than the corresponding value predicted using kinetic codes. One can observe sharp decrease in the electron number density distribution obtained using MDDM code in the range of \( r = 0 \pm 0.025 \) cm. After the sharp decrease \( n_e \) calculated using MDDM code reaches the value \( \sim 2 \times 10^{10} \) cm\(^{-3}\). This value slightly varies in the range of \( r = 0.025 \pm 0.5 \) cm and is in satisfactory agreement with the typical value of the \( n_e \) in the bulk of the gas discharge plasma obtained using kinetic codes.

Maximum number density of ions predicted using MDDM code is one order of magnitude higher than the corresponding value predicted using kinetic codes. Results obtained using kinetic and hydrodynamic code show monotonic decrease of \( n_i \) from axis of symmetry towards the anode as it has been already noted. However results obtained using MDDM code exhibit sharp decrease of \( n_i \) in the range of \( r = 0 \pm 0.025 \) cm as well as it was observed for the \( n_e \). In the bulk of the plasma (in the range of \( r = 0.025 \pm 0.5 \) cm) \( n_i \) predicted by MDDM code approximately two orders of magnitude lower than the corresponding value obtained using kinetic codes.

Based on the plots presented above one can conclude that results obtained using modified drift-diffusion model are in qualitative agreement with the results of the kinetic codes. Trend directions are generally assumed by the authors of the paper when comparing results qualitatively to each other. On the other hand quantitative discrepancy may reach orders of magnitude. It should be stressed that modified drift-diffusion model is a phenomenological (semiempirical) model. It means that there are parameters in this model such as electron temperature, mobility of electrons and ions, diffusion coefficients of electrons and ions, ionization and recombination coefficients that has to be somehow estimated (calculated) or taken from experiments. Improvement of the results obtained using modified drift-diffusion model can be expected if estimations of these phenomenological parameters are correct. Kinetic codes can provide reasonable estimations of these parameters.

For example in the figure 15 distribution of the electron temperature \( T_e [\text{eV}] \) predicted using PIC–2D–UNSTR code is shown. It can be easily noted that electron temperature in the Penning gas discharge considered in the paper is not lower than 5 eV. In the vicinity of anode electron temperature reaches the value of 95 eV. In the drift-diffusion model used in this work electron temperature was assumed constant in the discharge chamber and equal to 1 eV.

![Figure 15. Distribution of the electron temperature \( T_e [\text{eV}] \) obtained using PIC–2D–UNSTR computer code.](image-url)
Also kinetic codes can provide estimations of energy distribution functions of electrons and ions. In the figure 16 electrons and $\text{H}_2^+$ ions energy distribution function computed using PIC–2D–UNSTR code for discharge parameters considered in the paper are shown. These functions can be used for the estimations of mobility of charged particles and diffusion coefficients.

![Figure 16](image)

**Figure 16.** Electrons and $\text{H}_2^+$ ions energy distribution functions calculated using the PIC–2D–UNSTR code for the parameters of the discharge considered in the study.

5. Conclusion

In the study attempt was made to compare results of simulation of plasma of Penning discharge obtained using kinetic and hydrodynamic models. Estimations of mean free path of electrons in plasma based on diffusion coefficient modified by considering the external magnetic field justify the usage of hydrodynamic model under the typical conditions of existence of Penning discharge. Results obtained using two PIC–MCC codes (PIC–2D–UNSTR developed in VNIAA and proprietary code VSim) are in satisfactory agreement with each other. Some discrepancies are observed when results obtained using modified drift-diffusion model are compared with results of kinetic simulations. Assumption was made that the results of modified drift-diffusion model may be improved using the data obtained by means of the kinetic codes (e.g. temperature, mobility and diffusion coefficients of charged particles).

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