Primordial magnetic field and kinetic theory with Berry curvature

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We apply the modified kinetic theory to study generation of magnetic field due to anomaly in a primordial plasma of the standard model particles at temperature $T > 80$ TeV. It is known that a chiral imbalance in such plasma can lead to instabilities responsible for the origin of the magnetic fields. We have shown that inclusion of the Berry curvature term in the kinetic equation along with the collision term in the relaxation approximation can lead to the chiral vorticity effect. This effect was not considered in the earlier literature based on the heuristic application of the kinetic theory. But in the collisionless regime there may not be any vorticity generation. We also argued that the chiral imbalance in the collisionless regime remains subdominant for the primordial plasma compared to the case where the collisions are important.

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INTRODUCTION

There is a strong possibility that the observed magnetic fields in galaxies and in inter-galactic medium could be due to some process in the very early Universe. It should be noted here that there still exists a possibility that the fields may not be of primordial origin but might be created during the gravitational collapse of galaxies\cite{1,2}. Understanding the generation and dynamics of the primordial magnetic field is one of the most intriguing problems of the cosmology (see the recent reviews\cite{3,4}). At present there exists several models which discuss the origin of the primordial magnetic field in terms of cosmological defects\cite{6,8}; phase transitions\cite{9,10}; inflation\cite{11,12}; electroweak anomaly\cite{17,18}; string cosmology\cite{13,20}; temporary electric charge nonconservation\cite{21}; trace anomaly\cite{22} or breaking gauge invariance\cite{23}. In a recent work\cite{24}, it was shown that Biermann battery kind of mechanism can play a role in generating the primordial magnetic field just after the recombination era.

In recent time there has been a considerable interest in studying the role of quantum anomaly in generation of the primordial magnetic field\cite{25,26}. In Ref.\cite{17} (see also\cite{18}) it was argued that the chiral imbalance of the electrons can occur in the very early Universe, shortly before the electroweak scale that can give rise to a magnetic field \( B \sim 10^{22} \) G at temperature \( T \sim 100 \) GeV with a typical inhomogeneity scale \( \sim 10^6/T \). In this work the authors have studied the Maxwell equations with the Chern-Simon term and a kinetic equation consistent with the anomaly equation \( \partial_{\mu}J^\mu_R = -CF_{\mu\nu}\tilde{F}^{\mu\nu} \), here \( F^{\mu\nu} \) is the hypercharge field strength and \( C \) is a constant, to study the normal modes. It was found that the transverse modes can become unstable and give rise to the hypercharge magnetic field. In\cite{25} the authors have used magnetohydrodynamics in the presence of chiral asymmetry to study the evolution magnetic field. They have shown that the chiral-magnetic\cite{28,29} and chiral-vorticity effects\cite{30} can play a significant role in the generation and dynamics of primordial magnetic field. Further, it was demonstrated in Ref.\cite{27} that evolution of the primordial magnetic field is strongly influenced by the chiral anomaly even at a temperature as low as 10 MeV. It was shown that an isotropic and translationally invariant initial state of the standard model plasma in thermal equilibrium can become unstable in the presence of the global charges\cite{21}. The most general form of the polarization operator \( \Pi^{ij} \) can be written as:

\[
\Pi^{ij}(k) = (k^2\delta^{ij} - k^i k^j) \Pi_1(k^2) + i\epsilon^{ijk}k^i \Pi_2(k^2)
\]

where, \( k \) is a wave-vector and \( k^2 = |k|^2 \). This equation satisfy the transversality condition \( k_i \Pi^{ij} = 0 \). A non-zero value of \( \Pi_2 \) when \( k \rightarrow 0 \) imply the presence of Chern-Simon term in the expression for the free energy. Using the field theoretic framework in\cite{26} it was shown that for a sufficiently small \( k < \Pi_2(k^2)/\Pi_1(k^2) \), the polarization tensor \( \Pi^{ij} \) has a negative eigenvalue and the corresponding eigen mode gives the instability.

Recently there has been an interesting development in incorporating the parity-violating effects into a kinetic theory formalism. In Ref.\cite{28,31,32}, it was shown that the chiral anomaly can be involved in a kinetic theory frame work by including the Berry correction term in a classical action of a relativistic particle. This leads to the redefinition of the Poisson brackets which includes contribution from the Berry connection. The resultant “classical” kinetic equation can reproduce the triangle anomaly present in the underlying quantum-field theory and it can also incorporates the chiral-magnetic and chiral-vorticity effects. Further, it was shown in ref.\cite{33} that the parity-odd correlation obtained from the modified kinetic equation (with the Berry curvature term) is identical with the result obtained using perturbation theory in the hard dense loop approximation. The modified kinetic equation can also be derived from the Dirac Hamiltonian by performing semiclassical Foldy-Wouthuysen diagonalization\cite{34,35}. The modified kinetic equation can be applied to both the high density or high temperature regimes\cite{34}.

Here we would like to note that the authors in Ref.\cite{17} have used heuristically written kinetic theory compatible with the anomaly equation to study the generation of primordial magnetic field. In addition authors have used expression for the current by incorporating standard electric resistivity and the chiral magnetic effect. However the effect of chiral vorticity was not considered in this work by assuming that the primordial plasma has very small fluid velocity. Keeping this discussion in mind we believe that it is appropriate to consider the problem of generation of primordial magnetic field in presence of anomaly using the Berry curvature modified kinetic theory. We can derive the expression for the electric and magnetic resistivities within the kinetic frame work. Also the modified kinetic theory incorporates the chiral vorticity effect. Further the forms of the kinetic equation used in Ref.\cite{17} and the Berry curvature modified theory are very different and it would be interesting to see under what conditions they can have similar predictions. Further in\cite{36} normal modes of the chiral plasma were analyzed using the modified kinetic theory. In this work authors have found that in the collisionless limit transverse branch of the dispersion relation can become unstable with typical wave number \( k \sim \alpha \mu/\pi \), where \( \alpha \) is the coupling constant and \( \mu \) refers to the chiral chemical potential. It is surprising that similar length scales can also be obtained by the instability occurring in the regime where the collision term play an important role. This raises a natural question in the competition of these two instabilities which one will win. In this work we attempt to address these questions.
The manuscript is organized into three sections. In section 1 we briefly state the \((3+1)\) formalism of Thorne and Macdonald \[37\] and the kinetic theory with the Berry curvature. In section 2 we apply this formalism to study the primordial magnetic field generation in presence of the chiral asymmetry. We also calculated the vorticity generation in the plasma due to the chiral imbalance. Section 3, contains results and a brief discussion.

I. BASIC FORMALISM

Maxwell’s equations in the expanding universe

In the present work we intend to solve a coupled system of the kinetic and Maxwell’s equations in the expanding universe. Here we note that we ignore the fluctuations in the metric due to the matter perturbation. For this we need to write these equations in a general covariant form. Interestingly it is possible to write up the basic equations which incorporates the general relativistic effects (for the expanding background) in such a way that they resembles the equations of flat space-time\[38–41\]. In this formalism, the well developed intuitions and techniques of the flat space-time plasma physics can be exploited to study the problem at hand. This can be accomplished by choosing a particular set of fiducial observers (FIDO’s)\[38\] at each point of space-time. With respect to which the electric \(E\) and magnetic \(B\) fields and the other physical quantities are measured. The line element for the spatially flat background can be written using the Friedmann-Lemaître-Robertson-Walker metric as

\[
\text{where } ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{2}
\]

where \(x, y\) and \(z\) represent comoving coordinates. Here \(t\) is the proper time seen by observers at the fixed \(x, y\) \& \(z\) and \(a(t)\) is scale factor. One can introduce the conformal time \(\eta\) using the definition \(\eta = \int dt/a^2(t)\) to write this metric as:

\[
\text{ds}^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2). \tag{3}
\]

The (hyper)-electric \(E\), (hyper)-magnetic \(B\) and the current density \(J\) are related to the corresponding fiducial quantities by transformations: \(\mathcal{E} = a^2 E\), \(\mathcal{B} = a^2 B\), \(\mathcal{J} = a^2 J\). One can now write the Maxwell’s equations in the fiducial frame as:

\[
\frac{\partial \mathcal{B}}{\partial \eta} + \nabla \times \mathcal{E} = 0 \tag{4}
\]

\[
\nabla \cdot \mathcal{E} = 4\pi \rho_e \tag{5}
\]

\[
\nabla \cdot \mathcal{B} = 0 \tag{6}
\]

\[
\nabla \times \mathcal{B} = 4\pi \mathcal{J} + \frac{\partial \mathcal{E}}{\partial \eta} \tag{7}
\]

where \(\mathcal{B}, \mathcal{E}, \rho_e\) and \(\mathcal{J}\) are respectively magnetic field, electric field, charge density and current density seen by the fiducial observer.

Kinetic theory with Berry curvature

The charge \(\rho_e\) and current \(\mathcal{J}\) in the Maxwell’s equations \[44, 47\] can be calculated using the Berry curvature modified kinetic equation which is also consistent with the quantum anomaly equation. The modified kinetic equation is given by

\[
\frac{\partial f}{\partial \eta} + \frac{1}{1+ \epsilon_p \cdot \mathcal{B}} \left[ (e\mathcal{E} + e\tilde{v} \times \mathcal{B} + e^2 (\tilde{\mathcal{E}} \cdot \mathcal{B}) \Omega_p) \cdot \frac{\partial f}{\partial p} + (\tilde{v} + e\tilde{E} \times \mathcal{B} + e(\tilde{v} \cdot \mathcal{B}) \mathcal{B}) \cdot \frac{\partial f}{\partial \mathcal{B}} \right] = \left( \frac{\partial f}{\partial \eta} \right)_{\text{coll}}. \tag{8}
\]

where \(\tilde{v} = \partial \epsilon_p / \partial p = v\) and \(e\tilde{E} = e\mathcal{E} - \partial \epsilon_p / \partial \mathcal{B} \mathcal{B}\). \(\Omega_p = \pm p/(2p^3)\) is Berry curvature. \(\epsilon_p\) is defined as \(\epsilon_p = p(1 - e \mathcal{B} \cdot \Omega_p)\) with \(p = |\mathcal{P}|\). The positive sign corresponds to right-handed fermions where as the negative sign is for left-handed ones. In the absence of Berry correction i.e. \(\Omega_p = 0\), above equation reduces to the Vlasov equation when the collision term on the right hand side of equation\[8\] is absent. Here we would like to note that the collision term in equation \[8\] refers to parity or chirality conserving process. In this work we have not considered the effect of chirality flipping rate \(\Gamma_R = (\Gamma_R/M_0) T\), in the plasma. As above \(T > T_R(\sim 80 \text{ TeV})\), the chirality-flipping rate can be ignored in comparison.
with the expansion rate of the Universe. Here $M_0 \sim 7 \times 10^{17}$ GeV. The modified particle number density is defined as:

$$N = \int \frac{d^3p}{(2\pi)^3} (1 + eB \cdot \Omega_p) f$$

(9)

Above equation can be converted to following form by multiplying by $(1 + eB \cdot \Omega_p)$ and integrating over $p$

$$\frac{\partial N}{\partial \eta} + \nabla \cdot \mathcal{J} = -e^2 \int \frac{d^3p}{(2\pi)^3} \left( \Omega_p \cdot \frac{\partial f}{\partial p} \right) (\mathcal{E} \cdot B),$$

(10)

In equation $\mathcal{J}$ is total current and is defined as $\mathcal{J} = \Sigma_\alpha \mathcal{J}_\alpha$. Here index $\alpha$ denotes current contribution from different species of the fermion e.g. right-left particles and there antiparticles. $\mathcal{J}_\alpha$ is defined as:

$$\mathcal{J}_\alpha = -e^\alpha \int \frac{d^3p}{(2\pi)^3} \left[ e^\alpha \left( \frac{\partial f_\alpha}{\partial p} + e^\alpha (\Omega_p \cdot \frac{\partial f_\alpha}{\partial p}) \right) \mathcal{E} + \epsilon^\alpha (\mathcal{E} \times \sigma^\alpha) \right]$$

(11)

where $\sigma^\alpha = \int d^3p/(2\pi)^3 \Omega_p f_\alpha$ and $\epsilon^\alpha = p(1 - e^\alpha B \cdot \Omega_p)$ with $p = |p|$. Depending on the species, charge $e$, energy of the particles $\epsilon_p$, Berry curvature $\Omega_p$ and form of the distribution function $f$ changes. For the right-handed particle $\alpha = R$ and hyper-charge $e$ and for right-handed anti-particle $\alpha = \bar{R}$ and for charge this case is $-e$ etc. Now one can obtain the total current $\mathcal{J}$ by adding the contribution from all the species of the particle $\mathcal{J}$. It is clear from equation $\mathcal{J}$ that in presence of external electric and magnetic fields the chiral current no longer conserved. The first term in equation $\partial_\mu \bar{\mathcal{J}}^\mu = 0$ is usual current equivalent to the kinetic theory and remaining second and third terms are the current contribution by Berry correction. The last term is due to the anomalous Hall effect and it vanishes for a spherically symmetric distribution function. One can write equation $\partial_\mu \bar{\mathcal{J}}^\mu = 0$ as:

$$\partial_\mu \bar{\mathcal{J}}^\mu = -\frac{g\gamma_2\gamma_3}{64\pi^2} F_{\mu
u} F^{\mu\nu},$$

(12)

where, $F$ and $\bar{F}$ are respectively, the $U_Y(1)$ hypercharge field strengths and their duals, $g'$ is the associated gauge coupling, and $\gamma_R = -2$ is the hypercharge of the right handed electron. If we follow the power counting scheme as in [28] $A_\mu = O(\epsilon)$, $\partial_\mu = O(\delta)$ (where $A_\mu$ represent gauge field) and considering only terms of the order of $O(\epsilon\delta)$ in equation

$$\left( \frac{\partial}{\partial \eta} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right) f_\alpha + \left( e^\alpha \mathcal{E} + e^\alpha (\mathbf{v} \times B) - \frac{\partial e^\alpha_\mu}{\partial \mathbf{r}} \right) \frac{\partial f_\alpha}{\partial p} = \left( \frac{\partial f_\alpha}{\partial \eta} \right) \text{coll.},$$

(13)

where, we have taken $\mathbf{v} = \mathbf{p}/p$. In the subsequent discussion we shall apply this equation to study the evolution of the primordial magnetic field.

Current and polarization tensor for chiral plasma

Here we assume that the plasma of the standard particles is in a state of ‘thermal-equilibrium’ with temperature $T > T_R$ and at these temperatures the masses of the plasma particles can be ignored. We also assume that there exist a left-right asymmetry and there is no large-scale (hyper-)electromagnetic field. Thus the equilibrium plasma considered to be in a homogeneous and isotropic state which similar to the assumptions made in Ref. [17, 26]. It has been found that in flat space, to terms of the first order in fine structure constant, the quantum and classical theories of the plasma give the same plasma frequencies for both spin-0 and spin-(1/2) particles. For a homogeneous and isotropic conducting plasma in thermal equilibrium, distribution function for different species are:

$$f_{0\alpha}(p) = \frac{1}{\exp\left(\frac{\epsilon^\alpha - \mu_\alpha}{T}\right) + 1}$$

(14)

If $\delta f_R$ and $\delta f_{\bar{R}}$ are fluctuations in the distribution functions of the right electron and right-antiparticles around there equilibrium distribution. Then we can write perturbed distribution functions as

$$f_R(r, p, \eta) = f_{0R}(p) + \delta f_R(r, p, \eta)$$

(15)

$$f_{\bar{R}}(r, p, \eta) = f_{0\bar{R}}(p) + \delta f_{\bar{R}}(r, p, \eta)$$

(16)
Subtracting equation for \( a = \tilde{R} \) from \( a = R \) using eq. (13) one can write:

\[
\left( \frac{\partial}{\partial \eta} + v \cdot \frac{\partial}{\partial r} \right) f(r, p, \eta) + (\mathbf{E}, \mathbf{v}) \frac{df_0}{dp} + e \mathbf{p} \cdot (\mathbf{B} \cdot \mathbf{\Omega}_p) \cdot \frac{\partial f_0}{\partial \eta} = \left[ \frac{\partial f(r, p, \eta)}{\partial \eta} \right]_{\text{coll.}}
\]

(17)

where, \( f(r, p, \eta) = (f_R - f_{\tilde{R}}) \) and \( f_0 = f_{0R} + f_{0\tilde{R}} \). Here we have used \( \frac{\partial f_0}{\partial \eta} = \mathbf{v} \frac{df_0}{dp} \). This equation relates the fluctuations of the distribution functions of the charged particles with the induced gauge field fluctuations. The gauge field fluctuations can be seen from the Maxwell’s electromagnetic equations. Under the relaxation time approximation, the collision term can be written as \( (\partial f_{su}/\partial \eta)_{\text{coll.}} \approx -\nu_c f_s \). Next, we take Fourier transform of the all perturbed quantities namely \( \mathbf{E}, \mathbf{B} \) and \( f(r, p, \eta) \) by considering the spatio-temporal variation of these quantities as \( exp[-i(\omega \eta - \mathbf{k} \cdot \mathbf{r})] \). Then using equation (17) one can get

\[
f_{k, \omega} = \frac{-e[(v \cdot \mathbf{E}_k) + \frac{1}{2p}(v \cdot \mathbf{B}_k)(k \cdot v)]}{i(k \cdot v - \omega - i\nu_c)} \frac{df_0}{dp}.
\]

(18)

So current contribution for the right handed particle and right handed antiparticles in terms of mode function can be obtained by adding contribution from both species using eq. (14), ignoring displacement current as

\[
\mathcal{J}_{kR} = e \int \frac{dp}{(2\pi)^d} \left\{ (v - \frac{i}{2p} (v \times k)) f_{k, \omega} - e \frac{2\omega}{2p^2} (\mathbf{B}_k - v (v \cdot \mathbf{B}_k)) f_0 + e \frac{p}{2p} \mathbf{B}_k \frac{df_0}{dp} \right\}
\]

(19)

In the similar way we can get current contribution from left handed particle and left handed antiparticle. So we can obtain total current by adding the contributions from both left-handed particles and anti particles as:

\[
\mathcal{J}_k = -m_D^2 \int \frac{dp}{(2\pi)^d} \left\{ \frac{v(v \cdot \mathbf{E}_k)}{k \cdot e \omega - \nu_c} - \frac{c_D^2}{2} \int \frac{dp}{(2\pi)^d} \left[ \frac{v(v \cdot \mathbf{B}_k)(k \cdot v - (v \cdot k)v)}{(k \cdot e \omega - \nu_c)} + \mathbf{B}_k \right] \right\} - \frac{i\gamma_D^2}{4} \int \frac{dp}{(2\pi)^d} \left[ \frac{(v \cdot k) \mathbf{E}_k}{(k \cdot e \omega - \nu_c)} \right] - \frac{\hbar^2}{4} \int \frac{dp}{(2\pi)^d} \left\{ \mathbf{B}_k - v (v \cdot \mathbf{B}_k) \right\}
\]

(20)

Here \( \Omega \) represent angular integrals. In eq. (20), we have defined \( m_D^2 = e^2 \int \frac{dp}{2\pi^d} \frac{df_0}{dp} \), \( c_D^2 = e^2 \int \frac{dp}{2\pi^d} \frac{df_0}{dp} \), \( g_D^2 = e^2 \int \frac{dp}{2\pi^d} \frac{df_0}{dp} \), \( \nu_c = \sum_a f_{oa} \) (a stands for different species \( R, \tilde{R}, L, \tilde{L} \)).

Expression for the polarization tensor \( \Pi^{ij} \) can be obtained from eq. (20) by writing total current in the following form \( \mathcal{J}_k^i = \Pi^{ij}(k)A_j(k) \) using \( \mathcal{E}_k = -i\omega \mathbf{A}_k \) and \( \mathbf{B}_k = k \times \mathbf{A}_k \). One can express \( \Pi^{ij} \) in terms of longitudinal \( P_L^{ij} = k^i k^j / k^2 \), transverse \( P_T^{ij} = (\delta^{ij} - k^i k^j / k^2) \) and the axial \( P_A^{ij} = i\epsilon^{ijk}k^k \) projection operators as \( \Pi^{ik} = \Pi_L P_L^{ik} + \Pi_T P_T^{ik} + \Pi_A P_A^{ik} \). After performing the angular integrations in eq. (20) one obtains \( \Pi_L, \Pi_T \) and \( \Pi_A \) as given below

\[
\Pi_L = -m_D^2 \frac{k \omega}{k^2} \left[ 1 - \omega L(k) \right],
\]

(21)

\[
\Pi_T = m_D^2 \frac{k \omega}{k^2} \left[ 1 + \frac{k^2 - \omega^2}{\omega^2} L(k) \right],
\]

(22)

\[
\Pi_A = -\frac{\hbar_D}{2} \left[ 1 - \omega \left( 1 - \frac{\omega^2}{k^2} \right) L(k) - \frac{\omega^2}{k^2} \right]
\]

(23)

where, \( L(k) = \frac{1}{2k} \ln \left( \omega + \frac{\omega^2}{k^2} \right) \) and \( \omega' = \omega + i\nu_c \). Also \( m_D^2 = 4\pi \alpha \left( \frac{e^2}{3} + \frac{\hbar_D^2 + \mu^2}{2e^2} \right) \) and \( \hbar_D^2 = \frac{2\alpha \Delta \mu}{\pi} \).

First, consider case when \( \omega \to 0 \) and \( \nu_c \). In this limit \( \Pi_L \) and \( \Pi_T \) at the lowest order will vanish and the parity odd part of the polarization \( \Pi_A = \Delta \Pi / 2 \approx \alpha \Delta \mu / \pi \). Here it should be noted that \( \Pi \) does not get thermal correction. This could be due to the fact that origin of \( \Pi_A \) term is related with the axial anomaly and it is well known that anomaly does not receive any thermal correction. This form of \( \Pi_A \) is similar to the result obtained in using quantum field theoretic arguments at \( T \leq 40 \text{ GeV} \). But in the kinetic theory approach presented here such no assumption is made. Normal modes for the plasma can be obtained by using expressions for \( \Pi_L, \Pi_T \) and \( \Pi_A \). Using equation \( \partial_t F^{\mu \nu} = -4\pi \mathcal{J}^\mu \), we can write to the following relation

\[
[M^{-1}]^{ij} A_j(k) = -4\pi \mathcal{J}_k^i,
\]

(24)

where, \( [M^{-1}]^{ik} = [(k^2 - \omega^2)\delta^{ik} - k^i k^j + \Pi^{ik}] \). Dispersion relations can be obtained from the poles of \( [M^{-1}]^{ik} \), which are as given below

\[
\omega^2 = \Pi_L,
\]

\[
\omega^2 = k^2 + \Pi_T(k) \pm k \Pi_A.
\]
One can study the normal modes of the chiral plasma and instabilities using these dispersion relations. However, it is more instructive to study dynamical evolution of magnetic field by explicitly writing time dependent Maxwell equations.

II. GENERATION OF THE PRIMORDIAL MAGNETIC FIELD AND VORTICITY

Plasma with chirality imbalance are known to have instabilities that can generate magnetic fields in two different regimes: (i) for the case when \( k << \omega << \nu_c \) and (ii) in the quasi-static limit i.e. \( \omega << k \) and \( \nu_c = 0 \). In this section we analyze how the magnetic fields evolve in the plasma due to these instabilities, within the modified kinetic theory framework. Expression for the total current described by Eq. (28) can be written as \( J_k^i = \sigma_{ij}^E E_k^j + \sigma_{ij}^B B_k^j \), where \( \sigma_{ij}^E \) and \( \sigma_{ij}^B \) are electrical and magnetic conductivities. The integrals involved in Eq. (28) are rather easy to evaluate in the limit \( k << \omega << \nu_c \) and one can write the expression for \( \sigma_{ij}^E \) and \( \sigma_{ij}^B \) as:

\[
\sigma_{ij}^E \approx \left( \frac{m_D^2}{3\nu_c} \right) + \frac{i}{3\nu_c} \frac{\alpha \Delta \mu}{\pi} \delta^{ij} k^4 \]  
(25)

\[
\sigma_{ij}^B \approx -\frac{4}{3} \frac{\alpha \Delta \mu}{\pi} \]  
(26)

Here, we would like to note that the Berry curvature correction in the kinetic equation gives us an additional contribution in the expression for \( \sigma_{ij}^B \) which was not incorporated in Ref. [17].

Taking vector product of \( k \) with the Maxwell equation \( i(k \times B_k)^i = 4\pi[\sigma_{ij}^E E_k^j + \sigma_{ij}^B B_k^j] \), ignoring the displacement current term, one can write:

\[
\frac{\partial B_k}{\partial \eta} + \left( \frac{3\nu_c}{4\pi m_D^2} \right) k^2 B_k - \left( \frac{\alpha \Delta \mu}{\pi m_D^2} \right) (k \times (k \times E_k)) + \frac{i}{\pi m_D^2} \frac{4\alpha \nu_c \Delta \mu}{\pi m_D^2} (k \times B_k) = 0. \]  
(27)

This is the magnetic diffusivity equation for the chiral plasma. By replacing \((k \times E_k)\) by \(-\frac{1}{i} \frac{\partial B_k}{\partial \eta} \) in eq. (27), we can solve this equation without a loss of generality by considering the propagation vector \( k \) in \( z \)-direction and the magnetic field having components perpendicular to \( z \)-axis. After defining two new variables: \( \tilde{B}_k = (B_k^1 + iB_k^2) \) and \( \tilde{B}_k' = (B_k^1 - iB_k^2) \) one can rewrite eq. (27) as,

\[
\frac{\partial \tilde{B}_k}{\partial \eta} + \left( \frac{3\nu_c}{4\pi m_D^2} \right) k^2 \tilde{B}_k - \left( \frac{\alpha \Delta \mu}{\pi m_D^2} \right) \left( 1 + \frac{4\alpha \nu_c \Delta \mu}{\pi m_D^2} \right) \tilde{B}_k = 0, \]  
(28)

\[
\frac{\partial \tilde{B}_k'}{\partial \eta} + \left( \frac{3\nu_c}{4\pi m_D^2} \right) k^2 \tilde{B}_k' - \left( \frac{\alpha \Delta \mu}{\pi m_D^2} \right) \left( 1 - \frac{4\alpha \nu_c \Delta \mu}{\pi m_D^2} \right) \tilde{B}_k' = 0. \]  
(29)

It should be noted here that if \( \frac{\alpha \Delta \mu}{\pi m_D^2} k^2 \ll 1 \) Eq. (29) is similar to the magnetic field evolution equation considered in Ref. [17]. In this limit Eq. (29) will give a purely damping mode.

Another regime where chiral imbalance instability can occur is in the quasi-static limit when \( \omega << k \) and \( \nu_c = 0 \). In this limit one can 'define' the electric conductivity as \( \sigma_{ij}^E \approx \pi(m_D^2/2k)\delta^{ij} \) and magnetic conductivity \( \sigma_{ij}^B \approx (\hbar^2/2\pi\delta^{ij}) \). Here it should be noted that the above conductivities do not depend upon the collision frequency. Similar to the previous case one can take the propagation vector in \( z \)-direction and consider the components of the magnetic field in the direction perpendicular directions. One can write the set of decoupled equations describing the evolution of magnetic field using the variables \( \tilde{B}_k \) and \( \tilde{B}_k' \) as:

\[
\frac{\partial \tilde{B}_k}{\partial \eta} + \left[ \frac{k^2 - \frac{4\alpha \nu_c \Delta \mu}{\pi m_D^2} k}{\frac{2k}{4\pi m_D^2}} \right] \tilde{B}_k = 0, \]  
(30)

\[
\frac{\partial \tilde{B}_k'}{\partial \eta} + \left[ \frac{k^2 + \frac{4\alpha \nu_c \Delta \mu}{\pi m_D^2} k}{\frac{2k}{4\pi m_D^2}} \right] \tilde{B}_k' = 0. \]  
(31)

Here we note that if one replaces \( \partial/\partial \eta \) by \(-i\omega \) equations (30) and (31) gives the same dispersion relation for the instability as discussed in Ref. [36].
Vorticity generated from chiral imbalance in the plasma

It would be interesting to see if the instabilities arising due to chiral-imbalance can lead to vorticity generation in the plasma. In order to study vorticity of the plasma, we define the average velocity as:

\[
\langle \mathbf{v} \rangle = \frac{\int \frac{d^3p}{(2\pi)^3} p (\mathbf{v} \cdot \mathbf{f}_R + \mathbf{v} \cdot \mathbf{f}_L + \delta f_R + \delta f_L)}{\int \frac{d^3p}{(2\pi)^3} (\mathbf{f}_0R + \mathbf{f}_0L + \delta f_0R + \delta f_0L)} \tag{32}
\]

Here we have used the perturbed distribution function in the numerator of the above equation which is due to the fact that the equilibrium distribution function is assumed to homogeneous and isotropic and therefore will not contribute to vorticity dynamics. The denominator is the usual normalizing factor used in defining mean quantities using the distribution function which contains the equilibrium distribution function and gives on the leading order 6T^3\zeta(3)/2\pi^2.

We consider the case when \( k << \omega << \nu_c \), in this case the perturbed distribution function say for the right-handed particles can be written as:

\[
\delta f_{k,\omega R} = -\frac{e}{\nu_c} \left( (\mathbf{v} \cdot \mathbf{E}_k) + i \frac{1}{2p} (\mathbf{v} \cdot \mathbf{B}_k)(\mathbf{k} \cdot \mathbf{v}) \right) \frac{df_{0R}}{dp} \tag{33}
\]

If we add the contribution for all the particles species and their anti-particles one can write the numerator in equation (32) as:

\[
\frac{4 \log 2}{3\nu_c} \sqrt{\frac{\alpha}{\pi^3}} (\Delta \mu T) \mathbf{E} \tag{34}
\]

And one can write average velocity as given below

\[
\langle \mathbf{v} \rangle = \frac{4 \log 2}{9\zeta(3)} \frac{\sqrt{\pi \alpha}}{\nu_c T} \mathbf{E} \tag{35}
\]

where, \( \delta = \Delta \mu / T \). We note here that the forms of some integrals in equation (32) are quite complicated and we have evaluated them approximately by assuming the chiral-chemical potential is smaller than the temperature i.e. \( \delta << 1 \).

Now vorticity can be obtained by taking curl of the eq.(34) and assuming that the chemical potentials and temperature are constant in space and time:

\[
\langle \mathbf{\omega} \rangle = i \frac{4 \log 2}{9\zeta(3)} \frac{\sqrt{\pi \alpha}}{\nu_c T} \delta (\mathbf{k} \times \mathbf{E}_k) \tag{35}
\]

It should be noted that this vorticity is generated essentially due to the finite chiral chemical potential.

III. RESULTS AND DISCUSSION

In the previous section we have applied modified kinetic theory to study generation of primordial magnetic field in the presence of chirality imbalance. Equation (21) is generalization of the results of Ref[50] in presence of collision frequency. We have also shown that this generalization is necessary in order to derive results of Ref. [17]. We note here that the axial part of the polarization tensor matches with the expression given in Ref.[27] when the limit \( \omega \to 0 \) and \( \nu_c = 0 \) is considered.

In the previous section we have also calculated vorticity as given by eq.(35) in the regime \( k << \omega << \nu_c \). Expression for the electric conductivity in eq.(25) contains two terms: first term \( m_D^2/3\nu_c \) can be identified with the usual resistivity \( \sigma \). The another term arises due to the Berry curvature correction in the kinetic equation. Using eq.(35) we can find out contribution of the vorticity in the total current. From relation \( \mathcal{J}_k = \sigma_{\nu \nu}^k \mathcal{E} \times \mathbf{B}_k^e \) and eliminating \( \mathbf{k} \times \mathbf{E}_k \) by using eq.(35) one can separate out the current due to vorticity as:

\[
\mathcal{J}_{k(\omega)} = -\frac{\alpha}{\pi} \left( \frac{T^2}{3\pi} \right) \mathbf{\omega}_k \tag{36}
\]

The expression for current contribution from vorticity has been derived from the modified kinetic theory and which is similar to the expression given in Ref.[30] at the leading order. Further using eq.(35), one can eliminating \( \mathbf{k} \times \mathbf{E}_k \) in eq.(27) one can obtain:

\[
\frac{\partial \mathbf{B}_k}{\partial \eta} + \left( \frac{3\nu_c}{4\pi m_d} \right) k^2 \mathbf{B}_k + i \frac{\alpha}{\pi} \left( \frac{\nu_c T^2}{\pi m_D^2} \right) \mathbf{k} \times \mathbf{\omega}_k + i \left( \frac{2\nu_c T \delta}{m_D^2} \right) \mathbf{k} \times \mathbf{B}_k = 0. \tag{37}
\]
In this equation, second term is the usual diffusivity term while the third and fourth terms are respectively denotes the contributions from chiral-vorticity and chiral-magnetic effects.

Next, we analyze solutions of eqs. (28, 31). To estimate \( \nu_c \) one can use the definition of the mean collision time \( \nu_c \sim \alpha^2 T \) in Ref. [40]. With this one can also estimate the resistivity calculated using the kinetic approach to be

\[
\sigma = \frac{m_0^2}{4\nu_c} \sim 100T.
\]

Eq. (28) clearly gives unstable modes for \( k < (16/3)\alpha \Delta \mu \). Eq. (29) gives purely a damping mode, if the condition \( \left( \frac{\alpha^2 T}{\pi m_0^2} \right) k < 1 \) is satisfied. One can rewrite this inequality as \( \left( \frac{\alpha T}{\Delta \mu} \frac{\delta}{\sigma} \frac{\nu_c}{\nu} \right) k < 1 \) and which can easily be satisfied for \( k < \nu_c \) and \( \delta < 1 \). If one replaces \( \frac{\sigma}{\nu} \) by \( -i\omega \) in eq. (28) one obtains the dispersion relation for the instability. \( \tilde{B}_k \) grows fastest for \( \nu_{max} \sim \frac{8\alpha T \delta}{3} \) and the maximum growth rate is \( \gamma_1 \sim \frac{16}{3} \pi^2 \alpha^2 T \nu_c \). Here we have assumed that \( \frac{\alpha^2 T}{\pi m_0^2} k \ll 1 \). This expression for \( \nu_{max} \) agrees with the numerical results given in Ref. [24] for the peak in the magnetic energy. However our \( \nu_{max} \) differs from that given in the above work by a numerical factor. The magnetic field will continue to grow at the cost of the chiral charge. This can be seen from the anomaly equation which at \( T > 80 \) TeV gives \( n_L - n_R + 2\alpha \mathcal{H} = \) constant, where, \( n_{L,R} = \frac{\nu L_R T}{6} \) and \( \mathcal{H} \) is the magnetic helicity which is defined as below

\[
\mathcal{H} = \frac{1}{V} \int d^3 x A \cdot B
\]

For the saturated state of the instability one can study the steady state solution of the magnetic diffusivity equation [31] i.e. by setting \( \partial_t \mathbf{B}_k = 0 \). After taking a dot product of the eq.(37) with fluid velocity \( \mathbf{v}_k \) one obtains,

\[
\left( \omega_k - \frac{16\alpha T \delta}{3} \mathbf{v}_k \right) \cdot \mathbf{B}_k = 0.
\]

Therefore we can express the magnetic field,

\[
\mathbf{B}_k = g(k) k \times \left[ \omega_k - \frac{16\alpha T \delta}{3} \mathbf{v}_k \right]
\]

The scalar function \( g(k) \) can be determined by substituting the above expression for the magnetic field into equation [37]. In the very large length scale i.e. \( k \to 0 \):

\[
g(k) = \frac{3}{64} \sqrt{T} \left( \frac{1}{\alpha} \right)^2
\]

So for a very large length scale \( k \to 0 \), magnetic field in the steady state is:

\[
\mathbf{B}_k = -\frac{1}{4} \sqrt{T} \left( \frac{T}{\alpha} \right) \omega_k
\]

This equation relates the vorticity generated during the instability with the magnetic field in the steady state.

Interestingly in \( \omega \ll k \) and \( \nu_c = 0 \) regime, one can have an instability described by eq. (39) with typical scales \( k \sim \alpha \Delta \mu \) and \( |\omega| \sim \alpha^2 T \delta \) [31]. Using the expression for electric and magnetic conductivities for modes in this regime one can write the magnetic diffusivity equation as:

\[
\frac{\partial \mathbf{B}_k}{\partial \eta} + \frac{k^2}{4\pi \sigma_1} \mathbf{B}_k - \frac{i\alpha T \delta}{\pi \sigma_1} \left( \mathbf{k} \times \mathbf{B}_k \right) = 0
\]

where \( \sigma_1 = \frac{\pi m_D^2}{2k} \). Here it should be noted that unlike eq. (37), the above equation does not have a vorticity term. Thus one can not relate the steady state magnetic field with the vorticity and the characteristics of the steady state magnetic field can be different than the previous results where we have taken \( k \to 0 \) limit. The length scale of this instability is strikingly similar to \( \nu_{max} \) that we have obtained for modes where \( \nu_c \) play the most dominant role. This raises a question if both the instabilities operate on the similar scale then which one will win. For this it is necessary to compare the maximum growth rates of these instabilities. One can find that in the collisionless regime the growth of the perturbation described by eq. (30) is maximum for \( \nu_{max} \sim \frac{2\alpha T}{9} \) and the maximum growth rate can be found to be \( \gamma_2 \sim \frac{T}{\pi} \left( \frac{T \delta}{m_0^2} \right)^2 (T \delta) \). One can compare \( \gamma_1 \) for the collisional plasma with \( \gamma_2 \) for the collisionless limit: \( \frac{\gamma_1}{\gamma_2} \sim 10^5 \). Clearly for \( \delta \ll 1 \) it is the collisional plasma[17] that will grow much faster than instability in the \( k \gg \omega \) regime. However for the plasma where the chiral chemical potential is much larger than temperature the
situation can be reverse.

In conclusion we have applied the kinetic theory with the Berry curvature correction to study origin of the primordial magnetic field due to anomaly. We have incorporated the effect of collisions using the relaxation time approximation. Eqs. [21] generalize the expression for the polarization tensor given in Ref. [36] in presence of the collision term. We have derived the expressions for electric and magnetic conductivities using the modified kinetic theory. Our expression for magnetic conductivity (arising due to chiral magnetic effect) broadly agrees with that used in Ref. [17]. However our results on electric conductivity has an extra term compared to equation of conductivity found in Ref. [17]. We have further shown that this extra term in the conductivity is equivalent of the current due to vorticity. Our equation for the vorticity current is in agreement with the result given in Ref. [30]. Further we have shown that the instability that can be present in a collisionless chiral plasma may not grow as fast as one found in presence of collision in Ref. [17] when \( \delta << 1 \). We have also demonstrated that the instability in presence of the collision term can generate vorticity in the plasma while there may not be any significant vorticity generation in the collisionless limit.

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