Rigidly rotating dust in general relativity

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Abstract

A solution to the Einstein field equations that represents a rigidly rotating dust accompanied by a thin matter shell of the same type is found.

Key words: classical general relativity – exact solutions – rotating dust

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1 Introduction

Recently, Bonnor studied axially symmetric stationary solutions of Einstein field equations coupled with dust and showed the reasons why a density gradient parallel to the axis is allowed for General Relativity but not for Newtonian mechanics \(1\). He also considered an analytic solution which represents a rigidly rotating dust, and found that the solution is asymptotically flat in all the three spatial directions but with a total mass equal to zero. He attributed this to the center singularity that was believed to have an infinitely large negative mass, which just balances the positive infinite mass of the dust that fills all the spacetime. For the details, we refer the readers to \(1\). Motivated by this particular solution, Bonnor wondered whether or not a non-singular rotating dust exists. In the present article, we obtain a solution that generates an axially symmetric rigidly rotating dust, which is accompanied by a rigidly rotating thin disk, namely, a singular hypersurface perpendicular to the axis of the rigidly rotating dust.

It is worth mentioning that there is a paper by Georgiou \(2\), who studied rotating Einstein-Maxwell fields\(1\). Georgiou obtains exact exterior and matching interior stationary axially symmetric solutions for a rigidly rotating charged dust. His solution generates an infinitely long cylinder and a thin singular disk perpendicular to the axis of the cylinder. Later on we consider the differences and the similarities of our solution and that of Georgiou’s.

The present paper has also been motivated by a study by Opher, Santos and Wang (OSW) \(3\) concerning the origin of extragalactic jets. These authors argued that, under certain circumstances, the spacetime given by what they refer to as the van Stockum metric, which is associated with a dust cylinder (studied by van Stockum \(4\) and extensively analyzed by Bonnor \(5, 6\)), can account for the collimating effect present in extragalactic jets. OSW showed that this dust cylinder produces confinement for the geodesic motion of test particles for certain values of the radial energy and angular momentum. In fact, it was one of our aims to improve the OSW’s model by looking for a spacetime that could more realistically describe a jet.

It is worth mentioning, however, that van Stockum in fact rediscovered a solution which was first obtained by Lanczos \(7\). Hereafter, therefore, we refer to the metric related to the dust cylinder as Lanczos metric.

The extragalactic jets are ubiquitous in active galaxies, they are highly collimated and the matter which forms them is highly relativistic \(8\). It is

\(1\)It has been one of the referees who brought to our attention this paper by Georgiou.

\(2\)The referees brought to our attention that Lanczos’ paper preceded that by van Stockum.
worth mentioning that there is no consensus in the literature to explain why they are the way they are. Many authors assume that the jets propagate along a direction provided by, most probably, rapidly rotating Kerr black holes present at the centers of active galaxies (see, e.g., Begelman, Blandford and Rees [8] and also [9, 10]). This fact also suggest that putative general relativistic effects could be important (see, e.g., [12, 13, 14, 15, 16], among others).

The central engine that gives rise to the jets could be more complex than a simple super massive black hole, it could well occur that jets be driven by an axially symmetric structure present at the center of active galaxies. The Lanczos solution (referred to by OSW as van Stockum solution), however, has its weakness when applied to an actual physical situation: it represents an infinitely long cylinder. As a result, the spacetime is not asymptotically flat and has infinite mass.

The solution here studied generates an axially symmetric rigidly rotating dust accompanied by a surface layer, which does not satisfy the energy conditions (i.e., weak, dominant and strong [17]) in part of the hypersurface. We point out however, that other authors (see Refs. [18, 19, 20], and references cited therein) have investigated such structures and some of them, as in our case, do not satisfy any of the energy conditions. In some of these cases, the energy conditions may be satisfied by a suitable choice of parameters.

We argue that the present study may be by itself of interest, because it represents a new axially symmetric dust solution, which could also motivate other authors to find solutions physically satisfactory.

In section 2 a closed form of the solutions are given and the main properties of them are studied, while in section 3 our main conclusions and some discussions are presented.

2 The Rotating Dust Metric

Our starting point is the Lanczos metric given by [4]

$$ds^2 = dt^2 - 2kdt d\varphi - l d\varphi^2 - e^\mu (dr^2 + dz^2),$$

where

$$k = \frac{\alpha \eta}{r}, \quad \eta = r \xi, \quad l = r^2 - \alpha^2 \eta^2,$$

$$\mu_r = \frac{\alpha^2}{2r} (\eta_z^2 - \eta_r^2), \quad \mu_z = -\frac{\alpha^2}{2r} \eta_r \eta_z,$$
and the function $\xi(r, z)$ satisfies the Laplacian equation $\nabla^2 \xi = 0$, with $\nabla^2$ being the Laplacian operator in Euclidean three-space. The symbol $(.)_x$ denotes partial derivative with respect to the argument $x$, and $\{x^\mu\} \equiv \{t, r, z, \varphi\}$, $(\mu = 0, 1, 2, 3)$ are the usual axisymmetric coordinates. One can show that the above solutions satisfy the Einstein field equations: $R_{\mu\nu} - g_{\mu\nu}R/2 = -8\pi\rho u_\mu u_\nu$ with the energy density and four-velocity of the dust being given, respectively, by

$$\rho = \frac{e^{-\mu}}{8\pi r^2} (\eta_z^2 + \eta_r^2), \quad u^\mu = \delta^\mu_0.$$  

The above solutions represent rigidly rotating dust. This can be seen, for example, by calculating the shear in this non-expanding spacetime, $q_{\mu\nu} \equiv (u^\mu_{;\nu} + u^\nu_{;\mu})/2$, which is identically zero for the solutions given by Eqs.(1) and (2). However, the angular velocity of the dust, which is given by $w_{\mu\nu} \equiv (u^\mu_{;\nu} - u^\nu_{;\mu})/2$, does not vanish.

The specific solution considered by Bonnor [1], for example, is $\xi = 2h/\sqrt{r^2 + z^2}$ with $h$ being a constant. As shown in [1] this spacetime is free of any spacetime singularities, except for that located at the origin of the coordinate system, namely, $r = z = 0$. This singularity is a curvature singularity with an infinitely large negative mass.

In this paper, we consider the solution with $\xi = J_0(r)e^{-z}$, where $J_0(r)$ denotes the zero-order Bessel function. Then, substituting it into Eqs.(2) and (3) we find

$$k = \alpha r J_1(r)e^{-z}, \quad l = r^2 \left[1 - \alpha^2 J_1^2(r)e^{-2z}\right],$$

$$\mu = -\frac{\alpha^2 r}{2} J_0(r)J_1(r)e^{-2z},$$

$$\rho = \frac{\alpha^2 e^{-2z}}{8\pi} \left[J_0^2(r) + J_1^2(r)\right] e^{-\mu}. $$

From the above equations we can see that the spacetime is singular when $z \to -\infty$. To remedy this undesirable feature we can replace $z$ by $|z|$ in Eq.(4), i.e.,

$$k = \alpha r J_1(r)e^{-|z|}, \quad l = r^2 \left[1 - \alpha^2 J_1^2(r)e^{-2|z|}\right],$$

$$\mu = -\frac{\alpha^2 r}{2} J_0(r)J_1(r)e^{-2|z|}. $$

Before proceeding it is worth mentioning that in a paper by Georgiou [3], exact exterior and matching interior solutions are found, where the

In this paper we choose units such that $G = 1 = c$, where $G$ is the gravitational constant, and $c$ the speed of light.
interior solution is similar to the solution present here. As shown in [2] the spacetime refers to solutions of the Einstein-Maxwell field equations for a rigidly rotating charged dust with vanishing Lorentz force. The solutions generate an infinitely long cylinder of charged dust rigidly rotating about its axis and a 4-current located on a singular hypersurface perpendicular to the axis of the cylinder at the origin of the coordinate system.

Due to the fact that Georgiou’s interior solutions present a non null electromagnetic 4-potential, namely, \( A_\mu = (0, 0, 0, A_3) \), his equations related to the \( \mu \) function involves the \( A_3 \) function (see Eqs.(4.2) and (4.3) in [2]). In such a way he could set \( F = e^\mu = 1 \). On the other hand, we have \( F = 1 \) and \( \mu \) is given by Eq.(4), as a result our solutions are different. Apart from the constants our mass density contains in addition the term \( e^{-\mu} \), as a result a stronger dependence on the \( z \) coordinate occurs as compared to the Georgiou’s mass density. Our solution describes rigidly rotating neutral dust and Georgiou’s solution, rigidly rotating charged dust. Consequently the resulting spacetimes are different.

One can show that such resulted spacetime present here is asymptotically flat in the \( z \) direction, since as \( |z| \rightarrow \infty \) for any particular finite value of \( r \), \( k \rightarrow 0 \), \( l \rightarrow r^2 \) and \( e^\mu \rightarrow 1 \). On the other hand, the behaviour of the solution as \( r \rightarrow \infty \), for any particular finite value of \( z \), shows that, for example, \( k \) oscillates infinitely between \(-\infty \) and \(+\infty \). This would indicate that the spacetime is not asymptotically flat.

On the other hand, the Kretschmann scalar is given by

\[
\mathcal{R} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = -\frac{\alpha^2 e^{-2(|z|+\mu)}}{4r^2} \left\{ 16 \left[ J_1^2(r) - rJ_0(r)J_1(r) \right] + r^2 \left( J_0^2(r) + J_1^2(r) \right) \right\} - 4\alpha^2 r^2 e^{-2|z|} \left[ 2J_0^4(r) + J_1^4(r) \right] + 7J_0^2(r)J_1^2(r) - 4rJ_0(r)J_1(r) \left( J_0^2(r) + J_1^2(r) \right) \right\} + \mathcal{R}_0 \delta(z),
\]

where \( \mathcal{R}_0(r) \) is a bounded function of \( r \) (see the discussions following Eq.(14) below), and \( \delta(z) \) denotes the Dirac delta function. Using the relations

\[
J_n(x) \approx \begin{cases} 
\frac{x^n}{2^{2n}n!}, & x \rightarrow 0, \\
\sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{2n+1}{4} \pi \right), & x \rightarrow +\infty,
\end{cases}
\]

we can see from Eq.(7) that \( \mathcal{R} \rightarrow \) finite, as \( r \rightarrow 0 \), and that \( \mathcal{R} \rightarrow 0 \), as \( |z| \) or \( r \rightarrow +\infty \) indicating that the spacetime is asymptotically flat. Although we
have considered here some discussion on flatness, it is worth bearing in mind that some authors argue that this concept is not well defined (see, e.g., [21]).

Also, from Eqs. (6) and (8) we have

\[ X \equiv \left| \partial_\varphi \right|^2 = \left| g_{\varphi\varphi} \right| \to O(r^2), \]
\[ \frac{X_\alpha X^\alpha}{4X} \to 1, \] (9)

as \( r \to 0 \). Hence, the axis \( (r = 0) \) of the spacetime is well defined and locally flat. From the above equations we can also see that, by properly choosing the constant \( \alpha \), we may have \( g_{\varphi\varphi} < 0 \) for any \( r \). That is, the spacetime may be free of any closed time-like curves. Therefore, the solution given by Eq. (6) represents an axially symmetric and rigidly rotating dust spacetime.

It should be noted that the replacement of \( z \) by \( \left| z \right| \) gives rise to a thin matter shell. As a matter of fact, this replacement mathematically is equivalent first to cut the original spacetime given by Eq. (4) into two parts, \( z > 0 \) and \( z < 0 \), and then join the part \( z > 0 \) with a copy of it along the hypersurface \( z = 0 \), so that the resulted spacetime has a reflection symmetry with respect to the surface. After this cut-paste operation, the spacetime is no longer analytic across the surface \( z = 0 \). Actually, the metric coefficients are continuous, but their first derivatives with respect to \( z \) are not. Then, according to Taub’s theory [18, 22], a thin matter shell appears on the hypersurface. Introducing the quantity \( b_{\mu\nu} \) via the relation

\[ b_{\mu\nu} \equiv g^+_{\mu\nu,z}\big|_{z=0+} - g^-_{\mu\nu,z}\big|_{z=0-}, \] (10)

where \( g^+_{\mu\nu} \) (\( g^-_{\mu\nu} \)) are quantities defined in the region \( z > 0 \) (\( z < 0 \)), we find that the non-vanishing components of \( b_{\mu\nu} \) are given by

\[ b_{11} = b_{22} = -2\alpha^2 r J_0(r) J_1(r) e^{\mu_0}, \]
\[ b_{03} = b_{30} = 2\alpha r J_1(r), \]
\[ b_{33} = -4\alpha^2 r^2 J^2_1(r), \] (11)

where \( \mu_0 \equiv \mu(r, z)|_{z=0} \). Then, the surface energy-momentum tensor \( \tau_{\mu\nu} \) is given by [18]

\[ \tau_{\mu\nu} = \frac{1}{16\pi} \left\{ b(n g_{\mu\nu} - n_\mu n_\nu) + n_\lambda (n_\mu b^\lambda_\nu + n_\nu b^\lambda_\mu) - (n b_{\mu\nu} + n_\lambda n_\delta b^{\lambda\delta} g_{\mu\nu}) \right\}, \] (12)

where \( n_\mu \) is the normal vector to the hypersurface \( z = 0 \), given by \( n_\mu = \delta^2_\mu \), with \( n \equiv n_\lambda n^\lambda \) and \( b \equiv b^\lambda_\lambda \). Substituting Eq. (11) into Eq. (12), we find that \( \tau_{\mu\nu} \) can be written in the form

\[ 8\pi \tau_{\mu\nu} = \sigma u_\mu u_\nu + px_\mu x_\nu + q(u_\mu x_\nu + u_\nu x_\mu), \] (13)
\[
\sigma = -p = -\alpha^2 r J_0(r) J_1(r) e^{-\mu_0}, \\
q = \alpha J_1(r) e^{-\mu_0},
\]

(14)

where \(u_\mu\) is the four-velocity of the dust restricted to the surface \(z = 0\), and \(x_\mu\) is a space-like unit vector on the surface, given by \(x_\mu = r \delta^3_\mu\), and has the properties: \(x_\lambda x^\lambda = -1\) and \(x_\lambda u^\lambda = 0\). The non null components of \(\tau^\mu_\nu\) are explicit shown in the appendix, where we also obtain them using the alternative technique derived by Israel [23].

Eq.(13) shows that \(\sigma\) represents the surface energy density of the shell measured by observers who are comoving with the dust fluid, and \(p\) is the pressure in the direction perpendicular to the observers’ world lines, while \(q\) represents the heat flow. On the other hand, Eqs.(14) show that all these quantities are finite for any \(r\). One can also see that \(\sigma\) oscillates between positive and negative values; when \(\sigma < 0\) the energy conditions [17] are violated. Note that the function \(R_0(r)\) appearing in Eq.(7) is a combination of them and is finite, too.

Note that the quantities \(q_{\mu\nu}\) and \(w_{\mu\nu}\) defined above contain only first derivatives of the metric coefficients, as a result they can be written generally in the form

\[
Y_{\mu\nu} = Y^+_{\mu\nu} H(z) + Y^-_{\mu\nu} H(-z),
\]

where \(Y^\pm_{\mu\nu}(\equiv \{q^\pm_{\mu\nu}, w^\pm_{\mu\nu}\})\) are quantities calculated respectively in the regions \(z > 0\) and \(z < 0\), and \(H(z)\) is the Heaviside function, which is 1 for \(z > 0\), 1/2 for \(z = 0\), and 0 for \(z < 0\). Since \(q^\pm_{\mu\nu}\) are all equal zero, we can see that \(q_{\mu\nu}\) is zero even on the surface \(z = 0\). That is, the matter shell is also rigidly rotating.

On the other hand, in terms of \(\tau_{\mu\nu}\), the Einstein field equations for the solution (6) takes the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi [\rho u_\mu u_\nu + \tau_{\mu\nu} \delta(z)],
\]

(15)

where \(\rho\) is given by Eq.(5) but with \(z\) being replaced by \(|z|\). Thus, the solution given by Eq.(6) represents a rigidly rotating dust accompanied by a matter shell of the same type.

To further study the spacetime of this solution, let us consider the total mass of the spacetime. Using the formula of Tolman [24], we find that the mass inside the cylinder \(r = R\) is given by

\[
M(R) = \alpha^2 R \left\{ 2 J_0(R) J_1(R) + R \left[ J_2^2(R) - J_1^2(R) \right] \right\},
\]

(16)
where $R$ is a constant, and $t^\nu_\mu$ is the so-called gravitational energy-momentum pseudo-tensor \[24\]. The combination of Eq.(8) and Eq.(16) shows that $M(R)$ oscillates infinitely between $-\infty$ and $+\infty$ as $R \rightarrow +\infty$. Thus, in the present case the total mass of the spacetime is not well defined. Were $\sigma$ always negative it could occur that the total mass of the dust structure were zero. This behaviour of $M(R)$ as $R \rightarrow \infty$ is related to the fact that $\sigma$ oscillates between positive and negative values. To remedy this problem, one might cut the spacetime and smoothly match the dust structure where $\sigma > 0$ to vacuum spacetimes. As result one would have a hollow rigidly rotating dust structure with two vacuums, one inside and other outside it. It could occur however that this procedure generated unphysical surface layers. Due to the complexity of these issues, we have not yet been successful in this direction.

3 Discussion and Conclusions

Motivated by the article by Bonnor \[1\] we have studied a particular axially symmetric rigidly rotating dust solution, and found that it is accompanied by a thin disk located on an hypersurface perpendicular to the symmetry axis. The undesirable feature is that the thin disk has negative energy density in part of the hypersurface.

The solution we have found is in some sense similar to a solution found by Georgiou \[2\]. This author obtains exact interior and matching exterior axially symmetric solutions of the Einstein-Maxwell fields equations, for rigidly rotating charged dust. The fact that our metric function $\mu$ be given by Eq.(4), instead of being $\mu = 1$ as in \[2\], turns the spacetimes different. Therefore, the fact that the dust be charged modify significantly the spacetime generated.

A completely satisfactory solution for rigidly rotating dust fluids is yet to be derived and deserves to be investigated. Such a solution must be asymptotically flat, have finite mass, have non-singularities, and whether a thin shell appears it must satisfy the energy conditions.

To see whether or not our solution allow confinement, one needs to follow a similar procedure as in \[3\]. However, due to the fact that the metric functions now depend on both of the $r$ and $z$ coordinates, the study of geodesic motions become very complicated. We therefore leave such an issue for a future study.

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[22] It should be noted that the Taub method is equivalent to that of Israel [23] when the hypersurface is non-null. As a matter of fact, introducing the extrinsic curvature of the hypersurface by $K_{\mu\nu}^\pm = (n_{\alpha;\beta} - n_\beta \Gamma^\pm_{\alpha\beta})p_\mu^\alpha p_\nu^\beta$, where $p_\mu^\alpha$ is the first fundamental form of the hypersurface, defined as $p_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu/n_\alpha n^\alpha$, the Einstein field equations on the hypersurface take the form

$$[K_{\mu\nu}]^- - p_{\mu\nu} [K]^- = 8\pi \tau_{\mu\nu},$$

which are exactly the ones found by Israel [23], where $[K_{\mu\nu}]^- \equiv K_{\mu\nu}^+ - K_{\mu\nu}^-$, and $[K]^- \equiv [K_{\mu\nu}]^- g^{\mu\nu}$. In the appendix we show explicitly that both techniques are equivalent.

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A Alternative Calculation of $\tau_{\mu\nu}$

Here we show that Taub’s and Israel’s techniques to calculate the surface energy-momentum tensor, $\tau_{\mu\nu}$, are equivalent.

We start from

$$8\pi\tau_{\mu\nu} \equiv 8\pi \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} T_{\mu\nu} \, dz = -\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \left( R_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} R \right) \, dz$$

$$= -\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \left( \mathfrak{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \mathfrak{R} \right) \, dz$$

(17)

(see, e.g., [2, 23]).

Due to the presence of the thin disk, $R_{\mu\nu}$ and $\mathfrak{R}$ contain the terms $R_{\mu\nu}$ and $\mathfrak{R}$, respectively. These terms contain delta functions, and appear only with components of $R_{\mu\nu}$ whose expressions contain second order partial derivatives with respect to $z$. Following [2], it is easy to show that the non null components of $\mathfrak{R}_{\mu\nu}$ read

$$\mathfrak{R}^1_1 = \mathfrak{R}^2_2 = \frac{1}{2} e^{-\mu} \left( \mu^-_{z} - \mu^+_{z} \right) \delta(z)$$

$$-\mathfrak{R}^3_3 = \mathfrak{R}^0_0 = \frac{1}{2} e^{-\mu} \frac{k}{r^2} \left( k^-_{z} - k^+_{z} \right) \delta(z)$$

$$\mathfrak{R}^3_0 = \frac{e^{-\mu}}{2r^2} \left( k^-_{z} - k^+_{z} \right) \delta(z)$$

$$\mathfrak{R}^0_3 = 2k \mathfrak{R}^3_3 - t \mathfrak{R}^3_0 \,,$$

(18)

where the functions with superscript $+$ ($-$) stand for the functions defined in the region $z > 0$ ($z < 0$).

Substituting Eq.(18) into (17) it is straightforward to show that the non null components of $\tau_{\mu\nu}$ are given by:

$$8\pi\tau^0_0 = -\alpha^2[J^2_0(r) + r J_0(r) J_1(r)] e^{-\mu_0}$$

$$8\pi\tau^0_3 = \alpha r J_1(r) [1 + \alpha^2 J^2_1(r)] e^{-\mu_0}$$

$$8\pi\tau^3_0 = -\frac{\alpha}{r} J_1(r) e^{-\mu_0}$$

$$8\pi\tau^3_3 = \alpha^2[J^2_0(r) - r J_0(r) J_1(r)] e^{-\mu_0}$$

(19)

which agree with Eqs.(13), showing, therefore, that Taub’s and Israel’s techniques are equivalent.
Finally, it is worth mentioning that Georgiou’s definition for $\tau_{\mu\nu}$ is a little bit different from Israel’s. The former integrates $T_{\mu\nu}$ with respect to the proper distance measured perpendicularly through the thin disk. On the other hand, Eq.(17) is an integral of $T_{\mu\nu}$ with respect to the $z$ coordinate. If one followed Georgiou’s definition, instead of having the term $e^{-\mu_0}$ in Eq.(19) one would have $e^{-\mu_0/2}$. 