Sub-Grid modelling based on vorticity-stretching

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Abstract. A non-isotropic vorticity dyadic tensor (VDT) sub-grid scale model is developed. Based on the direct numerical simulation database of channel turbulence, the sub-grid scale energy dissipation of the model in wall-bounded turbulence is studied, and a wall damping function for VDT model is developed. The sub-grid scale energy dissipation produced by the wall-corrected VDT model is more accurate than that of the Smagorinsky model with Van Driest damping and the dynamic Smagorinsky model.

1. Introduction

Current turbulence numerical simulation methods mainly include Direct Numerical Simulation (DNS), Large Eddy Simulation (LES) and Reynolds-Averaged Navier-Stokes equation method (RANS). Among them, DNS method completely solves all spatial and temporal scales of turbulence, which is the most accurate turbulence simulation method. However, current computer conditions can only support DNS simulations with simple configuration and low Reynolds numbers. The problems of complex configuration with high Reynolds numbers are unpractical, if not impossible, for DNS in the foreseeable future. The RANS method can obtain an averaged flow field with fewer costs. It is the most widely used turbulence simulation method in industrial flows. In LES, a Sub-Grid Scale (SGS) model is used to mimic the effects of small-scale flow structures, while the large-scale flow structures are directly solved by numerical methods. The LES method can obtain the unsteady flow information of large-scale flow field with much lower computational cost than the DNS method. This is an effective tool for predicting complex unsteady turbulent flows.

Ideally, LES should be applied to free turbulence with high Reynolds numbers. In these cases, turbulence has an inertia subrange [1, 2]. Therefore, under ideal conditions, LES cuts the flow into SGS motions and resolved motions in the inertia subrange. Due to the universality and scale self-similarity of the flow in the inertia subrange, there is a possibility to construct a general SGS model. Most SGS models are based on such ideal conditions. However, in wall turbulence, especially in the inner layer of the boundary layer, the high Reynolds number assumption is no longer valid, and the inertia sub-range becomes narrow or no longer exists, so the filtering cannot be performed in the inertia subrange. This makes it difficult to obtain ideal results for wall turbulence LES using SGS models developed based on ideal conditions. Therefore, many SGS models need special treatment close the wall to obtain the correct asymptotic near-wall behavior[3].

Classical wall correction methods usually modify the length scale of the SGS model according to certain rules close to the wall, thus avoiding excessive non-physical sub-grid stress in the near-
wall region [3]. The most commonly used wall modification method is the Van Driest damping function[2, 3], which makes the characteristic length of the SGS model increase exponentially according to the distance to the wall.

The Smagorinsky model [4] is the most widely used eddy-viscosity SGS model for industrial flow problems. The eddy-viscosity of Smagorinsky model is defined as,

$$\nu_{i}^{SMG} = (C_s \bar{\Delta})^2 |S|,$$

which is based on the magnitude of strain rate tensor $|S|$. The Smagorinsky model usually requires Van Driest damping in the wall turbulence. However, Smagorinsky model based on Van Driest damping is not completely ideal, and it is often necessary to adjust the model coefficients according to the problem and the specific numerical method.

The dynamic Smagorinsky model[5] is a milestone for SGS modeling. This model is based on the similarity of flows at different scales, and the Smagorinsky model coefficients are dynamically obtained by the Germano identity in the calculation process, which greatly improve the results. The dynamic process developed in the literature [5, 6] can be used not only for Smagorinsky model, but also for other SGS models.

Although the dynamic Smagorinsky model performs well in wall turbulence simulations, a test filtering of the resolved velocity fields is required, which is costly to perform[9]. Chester, et al[9] developed an approximate dynamic procedure, based on Taylor series expansions of the resolved velocity fields, which does not require test filtering. Qi et al [7] also developed a no-test-filtering dynamic procedure, by requiring the SGS energy dissipation of Smagorinsky model equal to the SGS energy dissipation of the tensor-diffusivity model [8]:

$$\tau_{ij} = \frac{1}{12} \bar{\Delta}^2 \partial \bar{u}_i \partial \bar{u}_j \partial \bar{u}_k \partial \bar{u}_l,$$

Smagorinsky model often leads to over dissipation in laminar and transitional boundary layers, since the magnitude of strain rate tensor $|S|$ is usually large in these regions. Deng, et al[10] developed a Vorticity-Stretching based eddy-Viscosity SGS model(VSV):

$$\nu_{i}^{VSV} = (C_{svs} \bar{\Delta})^2 |\tilde{\omega}_i \bar{S}_{ij} \tilde{\omega}_j|^{1/3},$$

which uses the vorticity stretching function $|\tilde{\omega}_i \bar{S}_{ij} \tilde{\omega}_j|^{1/3}$ instead of the magnitude of strain rate tensor $|S|$. In equation (3), $\tilde{\omega}_i$ is the resolved vorticity, and $\bar{S}_{ij}$ is the resolved train rate tensor. In laminar boundary layer, where there is no three-dimensional multi-scale energy cascade, vorticity stretching is vanishing, so the VSV model can automatically reduce the eddy viscosity into zero.

In this paper, a non-eddy-viscosity SGS model based on vorticity stretching is developed: the non-isotropic Vorticity Dyadic Tensor model (VDT). The grid/filtering non-isotropic effect in near-wall region is considered in the model. Based on the DNS database of the channel flow of the Technical University of Madrid [11, 12, 13], the difference between the SGS energy dissipation of the VDT and the energy dissipation of real SGS stress is studied. A wall damping function, which makes the VDT model produce consistent SGS energy dissipation with the real SGS stress, is developed. VDT model with the new wall damping function produces better SGS dissipation than dynamic Smagorinsky model in boundary layer. The VDT model developed in this paper can be used to calibrate the coefficient of any eddy-viscosity SGS model.

2. Governing equations for LES

In LES, a low-pass filtering operation,

$$\bar{u}(\bar{x}, t) = \int G(\bar{x}', \bar{x}) \bar{u} (\bar{x}, t) d\bar{x}'$$

is used to separate the scales into resolved scales and sub-grid scales[2, 3]. The resulting resolved scales can be solved on a relatively coarse grid:

\[
\frac{\partial \bar{u}_i}{\partial t} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(2\nu \bar{S}_{ij}) - \frac{\partial \tau_{ij}}{\partial x_j},
\]

(5)

\[
\tau_{ij} = \bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j,
\]

(6)

\[
\bar{S}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right).
\]

(7)

The SGS stress tensor \( \tau_{ij} \) is not known, and should be replaced by a SGS model. If an eddy-viscosity model is used, the SGS stress tensor \( \tau_{ij} \) can be written as

\[
\tau_{ij} = 2\nu_t \bar{S}_{ij}.
\]

(8)

Where, \( \nu_t \) is the eddy-viscosity of SGS motions.

### 3. The DNS database and a-priori test

The DNS database [11, 12, 13] of channel turbulence at the Technical University of Madrid contains data sets of six Reynolds numbers: \( Re_\tau = 180, 350, 550, 934, 2009, \) and 4146. Among them, the three data sets of \( Re_\tau = 934, 2009, \) and 4146 have three velocity field components in physical space.

**Table 1.** Data sets used for a-priori tests.

| \( Re_\tau \) | \( L_x \) | \( L_y \) | \( N_x \) | \( N_y \) | \( N_z \) |
|---|---|---|---|---|---|
| 934 | \( \pi h \) | \( \pi h/2 \) | 256 | 256 | 385 |
| 2009 | 2\( \pi h \) | \( \pi h \) | 1024 | 1024 | 633 |
| 4164 | 2\( \pi h \) | \( \pi h \) | 2048 | 2048 | 1081 |

The a-priori test[14] of SGS energy dissipation is performed based on the velocity fields of three data sets. Table 1 shows the details of these three data sets[12, 13].

By performing Fourier filtering on the velocity fields in the streamwise direction \((x)\) and the spanwise direction \((y)\), the resolved velocity fields \( \bar{u}_i \) are obtained and the real SGS stresses are constructed:

\[
\tau_{ij}^\Delta = \bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j.
\]

(9)

Then SGS energy dissipation rate can be evaluated:

\[
\Pi_{SGS} = -<\tau_{ij}^\Delta \bar{S}_{ij}>.
\]

(10)

Where, \(<->\) represents the ensemble average, which can be approximated by performing spatial averaging along the streamwise direction and spanwise direction in the channel flow.

Four low-pass filters with different filter-widths are used. Table 2 shows the filter-resolutions of the resolved flow fields, where resolution \( g_0 \) corresponds to the original grid-resolution of the DNS database. In Table 2 the filter-width(or grid-space) are non-dimensionalized by wall-unit: \( \Delta_x^+ = u_\tau \Delta_x / \nu \). From \( g_0 \) to \( g_4 \), the filter-widths are doubled successively in both the streamwise and spanwise directions. That is, \( \Delta_x^+ \) and \( \Delta_y^+ \) at a resolution \( g_1 \) are twice as large as \( \Delta_x^+ \) and \( \Delta_y^+ \) at a resolution \( g_0 \), and so on.
Whenever differentiation is needed, the fourier spectral method is used to obtain the derivatives in streamwise and spanwise directions, while a fourth-order central difference compact scheme similar to those developed in [15] is used to get the derivatives in the wall-normal direction.

4. The non-isotropic vorticity dyadic tensor model

For convenience of description, in this paper, the viscous sublayer and the buffer layer are referred to as the viscous wall zone, which is resided in \( z^+ < 30 \), and the part of boundary layer outside the buffer layer is referred to as the turbulent zone, which is the region with \( z^+ > 30 \).

As mentioned in the foregoing section, Deng et al. developed a vorticity-stretching eddy-viscosity SGS model (VSV)[10]. Contrary to the magnitude of the strain rate tensor \( |S| \), the vorticity-stretching function 
\[
\bar{\omega}_i \bar{S}_{ij} \bar{\omega}_j \]

decreases to zero as it approaches the wall. However, replacing \( |S| \) using 
\[
|\bar{\omega}_i \bar{S}_{ij} \bar{\omega}_j|^{1/3}
\]

in the eddy-viscosity model does not ensure satisfactory asymptotic near-wall behaviour of the SGS model. In wall turbulence simulations, the VSV model still requires wall damping.

In the process of constructing a SGS model with proper wall asymptotic behaviour using the concept of vorticity stretching, we found that the following filter-width-normalized Vorticity Dyadic Tensor (VDT) model behaves fairly well in the turbulent boundary layer:

\[
\tau^D_{ij} = -\bar{\omega}_i \bar{\omega}_j .
\]

Where,

\[
\bar{\omega}_i = \epsilon_{ijk} \frac{\partial \bar{u}_j}{\partial x_k}
\]

is the filter-width-normalized vorticity. With \( \frac{\partial \bar{u}_j}{\partial x_k} \) defined by:

\[
\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k} \Delta_k \text{ (no summation)},
\]

\[
\epsilon_{ijk} = \begin{cases} 
1, & \text{if (i, j, k) are cyclic} \\
-1, & \text{if (i, j, k) are anticyclic} \\
0, & \text{otherwise}
\end{cases}
\]

It is worth noting that the SGS energy dissipation produced by the VDT model is exactly the vorticity-stretching function based on resolved velocity:

\[
\Pi^D_{SGS} = -< \tau^D_{ij} \bar{S}_{ij} > = < \bar{\omega}_i \bar{S}_{ij} \bar{\omega}_j > .
\]
Figure 1. Comparison between SGS dissipation of VDT and that of the real stress ($Re_\tau = 934$).

Fig.1 shows the comparison of the real SGS dissipation obtained by DNS in the turbulent channel at $Re_\tau = 934$ and the SGS dissipation obtained by the VDT model (11). The VDT model produces correct exponential behaviour in both the viscous underlayer ($z^+ < 5$) and the turbulent zone. However, the SGS energy dissipation given by the VDT model is larger than the real SGS dissipation. In addition, the variation of the SGS dissipation with the filter-width is also inconsistent with the real SGS dissipation.

5. Difference between the VDT model and the real SGS stress

In this section, the reason why the SGS energy dissipation generated by the VDT model in wall turbulence is inconsistent with the real SGS stress is analysed.

It can be seen from Fig. 1 that the VDT model is different from the real SGS stress in both the viscous zone and the turbulent zone, but the deviation behavior is similar in the two regions. As shown in Fig.1, in the turbulent region, the SGS energy dissipation given by the VDT model does not change with the filter-width, but the real SGS stress energy dissipation is positively related to the filter-width. In Fig.2, the real SGS dissipation is normalized with the square of the filter-width $\bar{\Delta}$,

$$\Pi_{SGS}^{real} = - < \tau_{ij} \bar{S}_{ij} > \Rightarrow - < \tau_{ij} \bar{S}_{ij} > / (\bar{\Delta}/\eta)^2. \quad (16)$$

Where, the Kolmogorov scale $\eta$ can be obtained by [2]:

$$\eta = (\nu^3/\epsilon)^{1/4}, \quad (17)$$

and the rate of dissipation of turbulent kinetic energy $\epsilon$ can be obtained from the DNS database.

As shown in Fig.2, the real SGS dissipation collapsed into each other after normalization. This suggests that the real SGS dissipation in the turbulent region is proportional to $\bar{\Delta}^2$:

$$\Pi_{SGS}^{real} \sim (\bar{\Delta}/\eta)^2. \quad (18)$$
Equation (18) obviously violates the assumption that the SGS dissipation in the inertia subrange is independent of the filter-width:
\[ \Pi_{SGS}^{real} \sim \left( \bar{\Delta}/\eta \right)^0. \] (19)

In the LES of the boundary flow, the filter-width is required to be within the following range [16, 17, 18]:
\[ \bar{\Delta}_x^+ \leq 150, \quad \bar{\Delta}_y^+ \leq 40, \quad \bar{\Delta}_{wall}^+ \leq 1. \] (20)
Where \( \bar{\Delta}_x^+ \) and \( \bar{\Delta}_y^+ \) are the filter widths in the streamwise and spanwise directions respectively (wall unit), \( \bar{\Delta}_{wall}^+ \) is the wall normal filter-width to the first layer of grid. Since equation (20) gives a strict limit on the filter-width, the reason why the real SGS stress does not follow the ideal behavior is likely due to that the filter-width used is smaller than the lower limit of the inertia subrange and falls into dissipation range. According to Kolmogorov’s K41 law [19, 20, 2], we have:
\[ \delta \bar{u} \sim \bar{\Delta}^{1/3}, \quad \tau_{ij} \sim \bar{\Delta}^{2/3}, \quad \bar{S}_{ij} \sim \bar{\Delta}^{-2/3}, \quad \Pi_{SGS} \sim \bar{\Delta}^0, \] (21)
for filter-width \( \bar{\Delta} \) in the inertia subregion. While for \( \bar{\Delta} \) in dissipation range these variables scale like:
\[ \delta \bar{u} \sim \bar{\Delta}^1, \quad \tau_{ij} \sim \bar{\Delta}^2, \quad \bar{S}_{ij} \sim \bar{\Delta}^0, \quad \Pi_{SGS} \sim \bar{\Delta}^2. \] (22)
Note that the relation between the SGS dissipation and the filter-width given by equation (22) agrees with the DNS data shown in Fig.2.

Fig.3 shows the variation of the ratio \( \bar{\Delta}/\eta \) with the normal distance of the wall at different Reynolds numbers and filter-widths. For all cases shown in Fig.3, the \( \bar{\Delta}/\eta \) ratios are far below the lower limit of the inertia subrange, which is [2]:
\[ l_{DI} = 60\eta. \] (23)
Figure 3. Comparison between $\bar{\Delta}/\eta$ and the lower limit of the inertia subrange.

Note that in Fig.3, the largest filter-width in the streamwise direction is close to the upper limit given by equation (20), while the largest filter-width in the spanwise direction is more than twice the upper limit. This confirms that in wall turbulence, the grid-resolution requirement of LES will limit the filter-width, so that the filter-width falls into the dissipation region. This is the reason why the real SGS dissipation is rather proportional to the square of the filter-width than filter-width-independent.

6. Wall damping for VDT model

The interaction between the eddy-viscosity SGS model and the resolved scales is essentially an energetic interaction, and the balance of the energy transfers alone between the two scale ranges is enough to represent the action of the sub-grid scales[3].

According to the difference between the SGS dissipation of VDT model and the real SGS stress in the turbulent region, the VDT model (11) can be modified to produce the same amount SGS dissipation as the real SGS stress.

We want to make the following corrections to the filter-width $\bar{\Delta}$:

$$\bar{\Delta}_w = f_w \bar{\Delta},$$

so that the SGS dissipation given by equation (11) is equal to the real SGS dissipation. From Fig. (1) and Fig. (2), $f_w$ should scale like,

$$f_w^2 \sim (\bar{\Delta}/\eta)^2.$$  \hspace{1cm} (25)

Using the DNS database to compare the VDT model dissipation with the real SGS dissipation, we get:

$$f_w^2 = 1 - exp\left[- \left(\frac{\bar{\Delta}}{19\eta}\right)^2\right].$$  \hspace{1cm} (26)
Fig. 4 shows the variation of $f_w^2$ with $\bar{\Delta}/\eta$. It can be seen that, when $\bar{\Delta}/\eta$ approaches infinity, $f_w^2$ approaches 1:

$$f_w \to 1, \text{ if } \bar{\Delta}/\eta \to 60.$$  

(27)

According to equation (27), when the filter-width enters the inertia subrange from the dissipation region, the damping function (26) is automatically reduced to 1, which ensures the validity of using it in turbulence away from the wall. Substituting (26) into (24), we get the modified filter width $\bar{\Delta}_w$. By replacing the filter width $\bar{\Delta}$ with $\bar{\Delta}_w$ in equation (13), we obtained the wall-corrected VDT model.

Fig. 5 shows a comparison of the SGS dissipation of the wall-corrected VDT model with the real SGS dissipation. For all the given Reynolds numbers and filter-widths, the SGS dissipation produced by the modified VDT model agrees well with the real value.

Fig. 6 shows a comparison of the SGS dissipation produced by the wall-corrected VDT model and the SGS dissipation produced by the Smagorinsky model (with Van Driest damping) for the case of $Re_\tau = 934$. In the viscous sublayer, the SGS dissipation produced by Smagorinsky model with Van Driest damping has the asymptotic behaviour of $\Pi_{SGS} \sim y^2$, while the SGS dissipation produced by wall-corrected VDT model and the real SGS stress both scale as $\Pi_{SGS} \sim y^3$. In the turbulent region, the SGS dissipation of the wall-corrected VDT model agrees well with the real SGS dissipation for all filter widths and Reynolds numbers, while the Smagorinsky model with Van Driest damping does not produce correct filter-width-dependent behavior.

Fig. 7 shows a comparison of the SGS dissipation produced by the wall-corrected VDT model with the SGS dissipation produced by the dynamic Smagorinsky model for the case of $Re_\tau = 934$. In the viscous sublayer, both models produce the correct asymptotic behavior: $\Pi_{SGS} \sim y^3$. In the turbulence region, although the results of the dynamic Smagorinsky model are more accurate than those of the Smagorinsky model, their accuracy is not as good as that of the wall-corrected VDT model.
(a) $Re_\tau = 934$

(b) $Re_\tau = 2009$
7. A dynamic procedure without test filtering
Since the wall-corrected VDT model produces very accurate SGS dissipation for wall turbulence, we can use it as an agent model for real SGS dissipation. For any eddy viscosity SGS model, the wall-corrected VDT model can be used to calibrate its SGS dissipation. In this way, we propose a dynamic procedure that does not require test filtering.

Take the Smagorinsky model\[4\] as an example:

\[ \tau_{ij}^{\text{smag}} = -2c\bar{\Delta}^2|\bar{S}|\bar{S}_{ij}. \]  
\hspace{3.5cm} (28)

We require,

\[ -<\tau_{ij}^{\text{VDT}}\bar{S}_{ij}> = -<\tau_{ij}^{\text{smag}}\bar{S}_{ij}> = <c\bar{\Delta}^2|\bar{S}|^3>. \]  
\hspace{3.5cm} (29)

Assuming the model coefficient \(c\) is constant in the process of averaging in equation (29), \(c\) can be uniquely determined by equation (29). In this way, the model coefficient \(c\) can be calculated dynamically without test filtering. Obviously, the model coefficient \(c\) thus obtained can make the SGS dissipation of the Smagorinsky model equal to the wall-corrected VDT model.

This dynamic procedure is similar to the method proposed in [7], where the tensor-diffusivity model [8](2) is used instead of the wall-corrected VDT model.

8. Conclusions
The vorticity dyadic tensor model (VDT) was developed based on the concept of resolved scale vorticity stretching. Using the DNS database of channel turbulence at the Technical University of Madrid, the performance of VDT model in LES of wall turbulence is studied. Based on this, a wall correction of VDT model based on filter-width is developed. The wall-corrected VDT model can give more accurate SGS dissipation than the dynamic Smagorinsky model, and thus
**Figure 6.** Comparison of wall-corrected VDT model with Smagorinsky model plus Van Driest damping (SMG). $Re_\tau = 934$.

**Figure 7.** Comparison of wall-corrected VDT model with dynamic Smagorinsky model (DSM). $Re_\tau = 934$. 
is a good agent model for real SGS dissipation. A new dynamic procedure without test filtering is proposed based on wall-corrected VDT model.

Acknowledgements
This research was supported by the National Key Research and Development Program of China (2016YFA0401203) and the National Natural Science Foundation of China (11372343). This work was also funded in part by the Coturb program of the European Research Council. We thank Dr. Oriol Lehmkuhl for reading the manuscript and for his valuable suggestions.

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