ω-RECURRENT IN COCYCLES

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Abstract. After relating the notion of ω-recurrence in skew products to the range of values taken by partial ergodic sums and Lyapunov exponents, ergodic Z-valued cocycles over an irrational rotation are presented in detail. First, the generic situation is studied and shown to be $1/n$-recurrent. It is then shown that for any $ω(n) < n^{−e}$, where $e > 1/2$, there are uncountably many infinite staircases (a certain specific cocycle over a rotation) which are not ω-recurrent, and therefore have positive Lyapunov exponent. A further section makes brief remarks regarding cocycles over interval exchange transformations of periodic type.

1. Introduction

The notion of ω-recurrence in an ergodic system $\{X, µ, T\}$ as given by Krengel[10] is a way in measuring the rate at which points visit sets of positive measure. Let $ω : R^+ \to R^+$ be nonincreasing and regularly varying ($ω(kn) \sim ω(n)$ for all $k \in R^+$), and for a set $A$ of positive and finite measure, consider for each $x \in X$ the series

$$ζ(x) = \sum_{i=1}^{∞} χ_A(T^ix)ω(i).$$

Under the assumption that $\{X, µ, T\}$ is ergodic, $ζ(x)$ is seen to diverge either on a null set of $x$ or for almost-every $x$ (regardless of the choice of $A$); the notion is fruitful for $µ(X) = ∞$ and $ω(n) \to 0$. If the series diverges for almost ever $x$, the system is said to be ω-recurrent. Implications between rates of ω-recurrence and mixing properties were studied in [1, 2]; in those works towers over the dyadic adding machine were considered explicitly.

In this work we will concern ourselves with the following situation: let $\{Y, ν\}$ be a probability space, and $S : Y \to Y$ an ergodic transformation. Let $f : Y \to G$ be a function into a countable discrete group $G$, and denote the identity element of $G$ by $e$. Let $X = Y \times G$ and $µ$ be the product of $ν$ and the counting measure on $G$. Define the skew transformation

$$T(x, g) = (S(x), g + f(x)),$$

and assume that $\{X, µ, T\}$ is ergodic. Then the sequence of ergodic sums

$$e, f(x), f(x) + f(Tx), \ldots, \sum_{i=0}^{n-1} f(T^ix), \ldots$$

may be viewed as an analogue of a random walk on $G$.

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In §2 we will relate a function $\rho(n)$ which tracks the range of a typical random walk and the notion of $\omega$-recurrence to the set $Y \times \{e\}$ (Theorem 2.5). This machinery will be applied in §3 to cocycles over irrational rotations; let $Y = \mathbb{R}/\mathbb{Z}$ and $S$ be rotation by $\alpha \notin \mathbb{Q}$, and let $f : S^1 \to \mathbb{Z}$ be of bounded variation. In the particular case that 

$$f(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x),$$

the system is called the **infinite staircase**.

Compared to dyadic adding machines, the infinite staircase has a more natural geometric presentation as a continuous flow on an infinite-genus infinite-measure space (see [8]), and has been studied in depth; ergodicity of all irrational staircases was established in [7], and the situation of $\alpha$ a quadratic irrational was shown to be bounded rationally ergodic (a stronger notion than $\omega$-recurrence) in [3]. We will study in §3 the question of generic $\omega$-recurrence in $\mathbb{Z}$-valued cocycles over irrational rotations: for almost-every choice of $\alpha$, any such cocycle where $f$ is of bounded variation is $1/n$-recurrent (Theorem 3.2). Furthermore, using techniques developed in [11], we will show in §4 that for any $\omega(n) \in o(n^\epsilon)$ with $\epsilon < -1/2$ there is an uncountable set of $\alpha$ for which the infinite staircase is not $\omega$-recurrent (Theorem 4.4). In §5 we will consider a particular class of interval exchange transformations to derive $(\log m \cdot n/nz)$-recurrence (Theorem 5.2), where the powers $m > 1$ and $z \leq 1$ are explicit. Finally, the proofs of two technical lemmas are included in §6 in order to prevent them from interrupting their respective sections.

2. $\omega$-recurrence, ergodic sums and Lyapunov exponents

Let $\{Y, \nu\}$ be a probability space and $S : Y \to Y$ be an ergodic transformation. Let $f : Y \to G$ be a function into a countable discrete group $G$ with identity element $e$. Let $\{N_k\}$ be an increasing sequence of positive integers for $k = 1, 2, \ldots$, and for each $y \in Y$ let

$$r_k(y) = \# \left\{ g \in G : g = \sum_{i=0}^{n-1} f(S^i y), \ n \in \{1, \ldots, N_k\} \right\}$$

denote the number of values taken by the ergodic sums of $y$ through time $N_k$ (as the zeroth ergodic sum for any point is always the identity element, we ignore this value). Fixing some $\epsilon_1 \in (0, 1]$, let

$$\rho_k = \min \{ n \in \mathbb{N} : \nu\{y \in Y : r_k(y) \leq n\} \geq \epsilon_1 \},$$

and then denote

$$A_k = \{ y \in Y : r_k(y) \leq \rho_k \},$$

so that $\nu(A_k) \geq \epsilon_1$.

Fixing some choice of $k$, let $y \in A_k$ be arbitrary. The different elements of $G$ which are realized as ergodic sums of $y$ through time $N_k$ we will refer to as **bins**, and the individual ergodic sums

$$\sum_{i=0}^{n-1} f(S^i y)$$

for $n \leq N_k$ will be called **balls**; we have $N_k$ balls distributed amongst no more than $\rho_k$ different bins for each $y \in A_k$. Choose some $\epsilon_2 \in (0, 1)$. Then having chosen
some \( y \in A_k \), a particular bin \( g \) will be called \( \epsilon_2 \)-crowded if
\[
\sum_{n=1}^{N_k} \chi_g \left( \sum_{i=0}^{n-1} f(S^i y) \right) \geq \epsilon_2 \frac{N_k}{\rho_k}.
\]
An \( \epsilon_2 \)-crowded bin represents a particular ergodic sum which is achieved as often
(up to scaling by \( \epsilon_2 \)) as would be expected if the balls were distributed uniformly
across the available bins.

**Lemma 2.1.** For each \( y \in A_k \), at least \((1 - \epsilon_2)N_k\) balls belong to an \( \epsilon_2 \)-crowded bin.

**Proof.** A simple pigeonholing argument suffices, as every bin containing fewer than
\( \epsilon_2 N_k/\rho_k \) balls would account for no more than \( \epsilon_2 N_k \) of the balls, leaving \((1 - \epsilon_2)N_k\) balls in crowded bins. \( \square \)

Picking \( \epsilon_3 \in (0, 1) \), a ball \( m \in \{1, \ldots, N_k\} \) will be called \( \epsilon_3 \)-predicting if
\[
\# \left\{ n \in \{1, \ldots, N_k\} : \sum_{i=0}^{m+n} f(S^i y) = \sum_{i=0}^{m} f(S^i y) \right\} \geq \epsilon_3 \epsilon_2 \frac{N_k}{\rho_k}.
\]
An \( \epsilon_3 \)-predicting ball is a particular time whose ergodic sum is achieved many times
within the next \( N_k \) sums. As every point \( y \in A_k \) generates \((1 - \epsilon_2)N_k\) balls which
are in \( \epsilon_2 \)-crowded bins, we find

**Lemma 2.2.** For each \( y \in A_k \), at least
\[
(1 - \epsilon_2)(1 - \epsilon_3)N_k - \rho_k
\]
balls are \( \epsilon_3 \)-predicting.

**Proof.** Fix \( y \) and enumerate the \( \epsilon_2 \)-crowded bins by \( g_1, g_2, \ldots, g_m \), where \( m \leq \rho_k \). Let \( j_i \) be the number of balls in bin \( g_i \), so that by Lemma 2.1 we have
\[
\sum_{i=1}^{m} j_i \geq (1 - \epsilon_2)N_k.
\]
In bin \( i \), then,
\[
[(1 - \epsilon_3)j_i] = (1 - \epsilon_3)j_i - \{(1 - \epsilon_3)j_i\}
\]
balls are \( \epsilon_3 \)-predicting, where \( \lfloor x \rfloor \) denotes the integer part of \( x \) and \( \{x\} = x - \lfloor x \rfloor = x \text{ mod } 1 \). Then the number of \( \epsilon_3 \)-crowded balls is at least
\[
\sum_{i=1}^{m} [(1 - \epsilon_3)j_i] \geq (1 - \epsilon_3)(1 - \epsilon_2)N_k - \sum_{i=1}^{m} \{(1 - \epsilon_3)j_i\} \geq (1 - \epsilon_3)(1 - \epsilon_2)N_k - \rho_k. \quad \square
\]

Finally, denote
\[
B_k = \left\{ y \in Y : \sum_{m=1}^{N_k} \chi_0 \left( \sum_{i=0}^{m-1} f(S^i y) \right) \geq \epsilon_2 \epsilon_3 \frac{N_k}{\rho_k} \right\},
\]
the set of points which have at least \( \epsilon_2 \epsilon_3 N_k/\rho_k \) ergodic sums equal to zero in the
first \( N_k \) times. As \( \epsilon_3 \)-predicting balls see many ergodic sums equal to zero and
there are many \( \epsilon_3 \)-predicting balls, by combining all the results of this section with
the fact that the sequence of ergodic sums form an additive cocycle we have
Proposition 2.3. \[
\nu(B_k) \geq \frac{\epsilon_1 ((1 - \epsilon_2)(1 - \epsilon_3)N_k - \rho_k)}{N_k} > \epsilon_1 (1 - \epsilon_2)(1 - \epsilon_3) - \frac{\rho_k}{N_k}.
\]

Proof. First denote the sets
\[ C_i = S^i(A_k), \quad \bar{C}_i = (C_i \cap B_k), \]
for \( i = 0, 1, \ldots, N_k - 1 \). As each ball within \( C_0 = A_k \) has \((1 - \epsilon_2)(1 - \epsilon_3)N_k - \rho_k\) such balls as forward images, and \( \nu(C_0) \geq \epsilon_1 \), we have
\[
\sum_{j=0}^{N_k-1} \nu(\bar{C}_j) \geq \epsilon_1 ((1 - \epsilon_2)(1 - \epsilon_3)N_k - \rho_k),
\]
so the measure of the union of the \( \bar{C}_j \) (which is contained within \( B_k \)) is at least this quantity divided by the number of sets, \( N_k \). \qed

We state without proof the following elementary fact:

Lemma 2.4. Suppose that \( \{Y, \nu\} \) is a probability space, and \( f : Y \to \mathbb{R}^+ \cup \{\infty\} \). If \( f(y) < \infty \) almost everywhere, then for any \( \epsilon > 0 \), there exists a set \( S \) of \( \nu \)-measure \( 1 - \epsilon \) such that
\[
\int_S f(y) d\nu < \infty.
\]

Theorem 2.5. Suppose that \( \rho_k \in o(N_k) \), and that there is a \( \delta > 0 \) such that the inequality
\[
(2) \quad \frac{N_k}{\rho_k} \geq \frac{\delta}{1 - \delta} \sum_{i=1}^{k-1} \frac{N_i}{\rho_i}
\]
holds for sufficiently large \( k \). Then the system is \( \omega \)-recurrent for all \( \omega \) such that
\[
\sum_{k=1}^{\infty} \omega(N_k) \frac{N_k}{\rho_k} = \infty.
\]

Proof. Suppose that \( k \) is such that \( (2) \) holds. Let \( \epsilon \) be such that
\[
0 < \epsilon < (1 - \epsilon_2)(1 - \epsilon_3).
\]
Then we have
\[
\frac{\epsilon N_k}{\rho_k} - \delta \sum_{i=1}^{k-1} \left( \frac{N_i}{\rho_i} \right) \geq \delta \frac{N_k}{\rho_k}.
\]
So for some \( y \in B_k \), when considering the contribution of terms through time \( N_k \) to the series \( 1 \), as \( \rho_k \in o(N_k) \), we know that for large enough \( k \) at least \( \epsilon N_k/\rho_k \) terms appear. To determine the contribution of those terms between \( N_{i-1} \) and \( N_i \), we assume without loss of generality that for each \( i < k \), a total of \( \delta \epsilon N_i/\rho_i \) terms were ‘used up’ in computing the contribution of the terms between \( N_{i-1} \) and \( N_i \); all remaining terms are then replaced with \( \omega(N_i) \) (recall that \( \omega \) is monotone). Our inequality ensures, then, that at least \( \delta \epsilon N_k/\rho_k \) terms remain. Let \( S \subset Y \) be an arbitrary set of measure
\[
\nu(S) > 1 - \liminf_{k \to \infty} \mu(B_k).
\]
By Proposition 2.3 and the assumption that \( \rho_k \in o(N_k) \), there is some \( \epsilon' > 0 \) such that for large \( k \)

\[ \nu(S \cap B_k) \geq \epsilon'. \]

Consider, then, the integral

\[ \int_S \zeta(x) d\nu. \]

Let \( K \) be such that for all \( k > K \) the set \( S \) contains a subset of \( B_k \) of measure \( \epsilon' \) and (2) holds. So, between times \( N_{k-1} \) and \( N_k \), many ergodic sums are zero on a set of nontrivial measure:

\[ \int_S \zeta(x) d\nu \geq \epsilon' \delta \sum_{k=K}^{\infty} \omega(N_k) \frac{N_k}{\rho_k} = \infty. \]

We have already remarked that \( \zeta(x) \) is either finite almost everywhere or infinite almost everywhere; the former case is now excluded by Lemma 2.4.

\[ \square \]

In summary, the term \( \epsilon_1 \) is useful only in finding some set on which \( \rho_k \) can be controlled for some sequence \( N_k \). The terms \( \epsilon_2 \) and \( \epsilon_3 \) are simple combinatorial ‘placeholder’ constants. The condition that \( \rho_k \in o(N_k) \) can be thought of as requiring any nontrivial \( \rho_k \), leaving only (2) as the only significant impediment to concluding \( \omega \)-recurrence.

Suppose that \( G \) is a discrete metric group and at most \( d \)-dimensional: there is a constant \( C \) such that the number of points \( g(r) \) in a ball of radius \( r \) centered at the identity is bounded by

\[ g(r) \leq Cr^d. \]

**Corollary 2.6.** Assume that \( \{Y, \nu, S\} \) is ergodic, with \( f : Y \to G \), where \( G \) is discrete and of dimension no larger than \( d \). Denote

\[ \lambda = \limsup_{n \to \infty} \frac{\log \left( \text{ess sup}_{x \in S^1} \| \sum_{i=0}^{n-1} f(S^i x) \| \right)}{\log n}, \]

the principal Lyapunov exponent. If \( \lambda < 1/d \) and the skew product is ergodic, then for any \( \lambda' > \lambda \), the skew product is \( \omega \)-recurrent, where

\[ \omega(n) = \frac{1}{n^{(1-d\lambda')}}. \]

**Proof.** Between the dimension \( d \) and the principal Lyapunov exponent \( \lambda \), for any \( \lambda' > \lambda \) we may set \( N_k = \gamma^k \) for some \( \gamma > 1 \) and (for sufficiently large \( k \))

\[ \rho_k = \gamma^{kd\lambda'}. \]

We may assume that \( \lambda' \) also satisfies \( d\lambda' < 1 \) so that \( \rho_k \in o(N_k) \), and we apply Theorem 2.5 the verification of (2) is direct in this situation and is left to the reader.

\[ \square \]
3. \(\omega\)-recurrence in \(\mathbb{Z}\)-valued cocycles over rotations

We now set \(Y = S^1\), \(\nu\) be Lebesgue measure, \(S(y) = y + \alpha \mod 1\) for some \(\alpha \in (0, 1) \setminus \mathbb{Q}\), and \(f : S^1 \to \mathbb{Z}\) to be a step function with finitely many discontinuities (i.e. of bounded variation). We use standard continued fraction notation, where:

\[
\alpha = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}
\]

Denote \(p_n/q_n\) the \(n\)-th convergent to \(\alpha\), and let

\[
a_n = \sum_{i=1}^{n} a_i.
\]

It follows from the Dejoy-Koksma inequality and known metric results in Diophantine approximation (which we do not include herein; see for instance \([5, \S 2]\)) that the principal Lyapunov exponent is, for almost every \(\alpha\), equal to zero. Corollary 2.6 allows us to conclude that every bounded variation \(\mathbb{Z}\)-valued cocycle over almost-every rotation is \(1/n\)-recurrent for all \(\epsilon > 0\), but we will show something stronger. For every \(\alpha\), if we let (using the notation and terminology of \([2]\) \(N_k = q_k\), then we may set \(\epsilon_1 = 1\) and \(\rho_k = C a_k\), where \(C\) is the (finite) variation of \(f\) (this is the content of the Denjoy-Koksma inequality). We will always assume our cocycles are ergodic, by which we mean ergodic with respect to Lebesgue measure.

The proof of the following Lemma is postponed until \([6]\)

**Lemma 3.1.** For almost every \(\alpha\), there is a \(\delta > 0\) such that for all \(k\)

\[
\delta \sum_{i=1}^{k-1} \frac{q_i}{a_i} \leq \frac{q_k}{a_k}
\]

**Theorem 3.2.** For almost every \(\alpha\), any ergodic bounded-variation \(\mathbb{Z}\)-valued cocycle over rotation by \(\alpha\) is \(1/n\) recurrent.

**Proof.** It is known (see e.g. the final Corollary in \([9]\)) that for almost-every \(\alpha\) we have

\[
\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty.
\]

Set \(N_k = q_k\), \(\rho_k = a_k\), and \(\omega(n) = 1/n\). Apply Theorem 2.5, justified by Lemma 3.1

**Remark.** Using the fact that the sequence \(a_i/(i \log i)\) converges to \(1/\log 2\) in measure (\([9, \S 4]\)) for generic \(\alpha\) it is possible to strengthen this recurrence to any \(\omega\) of the form

\[
\omega(n) = \frac{1}{n \cdot \log^{(3)} n \cdot \log^{(4)} n \cdots \log^{(j)} n},
\]

where \(\log^{(i)} n\) is the \(i\)-th iterated logarithm and \(j\) is arbitrary; approximating \(a_i > i \log i\) for all but a zero-denisty sequence of \(i\) and \(q_i > (K - \epsilon)^i\) (where \(K\) is the
for all sufficiently large $i$ and $\epsilon > 0$ arbitrary, we see that
\[
\omega(q_k) \frac{q_k}{a_k} \sim \frac{1}{k \cdot \log k \cdot \log^2 k \cdots \log^{(j-1)} k}
\]
for all but a zero-density subsequence of $k$. Details of this extension are omitted in favor of the straightforward and simply-stated Theorem 3.2.

4. **Different rates of $\omega$-recurrence in the infinite staircase**

We now turn our attention to the problem of specific $\alpha$, and the rates of $\omega$-recurrence in the associated infinite staircase. We will restrict the form of $\alpha$:

$$(3) \quad \alpha = [2r_1, s_1, 2r_2, s_2, \ldots].$$

This particular form of $\alpha$ has many nice Diophantine properties related to the infinite staircase and first appeared in [4]. It was shown in [11] that the orbit of the origin may be realized through a sequence of substitutions on a three-letter alphabet $\{A, B, C\}$. We will not present any background in substitutions; the following results are presented in detail, with all terminology defined, in [11]. Define the substitutions

$$\sigma: \begin{cases} 
A \rightarrow A(A^r B^{r-1} C)(A^{r_1} B^{r_1} C)^{s_1-1} \\
B \rightarrow A(A^{r_1} B^{r_1} C)(A^{r_2} B^{r_2} C)^{s_2-1} \\
C \rightarrow A(A^{r_2} B^{r_2} C)(A^{r_2} B^{r_2} C)^{s_2}
\end{cases}$$

For convenience denote $\sigma^{(n)} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$.

**Proposition 4.1.** The symbolic coding of the orbit of the origin with respect to the labeling

$$A = \left[0, \frac{1}{2}\right], \quad B = \left[\frac{1}{2}, 1 - \alpha\right], \quad C = [1 - \alpha, 1],$$

is given by the limiting word

$$W = \lim_{n \to \infty} \sigma^{(n)}(A).$$

Furthermore, the encoding of any point $x \in [0, 1)$ may be presented as beginning with the concatenated word $W_1(y)W_2(y)$, where $W_2(y) \in \{\sigma^{(n)}(A), \sigma^{(n)}(B), \sigma^{(n)}(C)\}$, and $W_1(y)$ is a proper right factor (possibly empty) of one of these words. Finally, the word $\sigma^{(n)}(A)$ is of length $q_{2n}$, and both $\sigma^{(n)}(B)$ and $\sigma^{(n)}(C)$ are of length $q_{2n} + q_{2n-1}$.

**Proof.** That the coding of the origin takes the form given is [11] Thm. 1.1, Prop. 4.3]. That the orbit of any $y$ may be realized through the concatenation given follows from [11] Prop. 4.1; it is also stated in [12] §2. Finally, the lengths of all the words are computed in [11] Lem. 5.4; we let

$$M_n = \begin{bmatrix} (2r_n - 1)s_n + 1 & s_n \\
(2r_n - 1)s_n + r_n & s_n + 1
\end{bmatrix},$$

and then we have

$$M_n \cdot M_{n-1} \cdots M_1 \begin{bmatrix} 1 \\
1
\end{bmatrix} = \begin{bmatrix} |\sigma^{(n)}(A)| \\
|\sigma^{(n)}(B)| = |\sigma^{(n)}(C)|
\end{bmatrix},$$

and the computation of the lengths is then a direct inductive exercise which we leave to the reader. \(\square\)
Given our labeling of the intervals $A$, $B$ and $C$, given a word $P = p_1p_2 \ldots p_n$ of length $n$ in this alphabet, we define
\[
g(P) = \# \{i \leq n : p_i = A \} - \# \{i \leq n : p_i = B \} - \# \{i \leq n : p_i = C \},
\]
where $\#S$ denotes the cardinality of a set $S$. If $P$ is the coding of the length $(n-1)$ orbit of some point $x$, then $g(P)$ is the $n$-th ergodic sum of $x$. Further denote for $k \in \mathbb{Z}$
\[
g(P,k) = \# \{i \leq n : g(p_1p_2 \ldots p_i) = k \},
\]
the number of times that the intermediate ergodic sums are equal to $k$.

**Lemma 4.2.** Suppose $\alpha$ is of the form given by $\mathbb{A}$ and the $s_i$ are bounded. Then there is a constant $\tau$ such that for all $n$ we have
\[
\max \left\{ g(P,k) : k \in \mathbb{Z}, P \in \{ \sigma^{(n)}(A), \sigma^{(n)}(B), \sigma^{(n)}(C) \} \right\} \leq \tau q_{2n-2}.
\]

*Proof.* As the substitutions $\sigma_i$ are homomorphisms, we consider the example of
\[
\sigma^{(n)}(A) = \sigma^{(n-1)}(\sigma_n A)
\]
\[
= \sigma^{(n-1)}(A(A^n B^{r_{n-1}} C)^n)
\]
\[
= \sigma^{(n-1)}(A) \left[ \left( \sigma^{(n-1)}(A) \right)^{r_n} \left( \sigma^{(n-1)}(B) \right)^{r_{n-1}} \left( \sigma^{(n-1)}(C) \right) \right]^n.
\]
It is direct to show inductively that for our substitutions $\sigma_i$, we have for all $n$
\[
g(\sigma^{(n)}(A)) = 1, \quad g(\sigma^{(n)}(B)) = g(\sigma^{(n)}(C)) = -1;
\]
see also [11, Prop. 5.1]. Consider as an example the word $P = \sigma^{(n)}(AB) = \sigma^{(n)}A \cdot \sigma^{(n)}B$.

As $g(\sigma^{(n)}(A)) = 1$ and the sums are an additive cocycle, we have
\[
g(P,k) = g(\sigma^{(n)}(A), k) + g(\sigma^{(n)}(B), k - 1).
\]
In this manner we have
\[
g(\sigma^{(n)}(A), k) = g(\sigma^{(n-1)}(A), k) + s_n \left( \sum_{j=1}^{r_n} g(\sigma^{(n-1)}(A), k - j) + \sum_{j=2}^{r_n+1} g(\sigma^{(n-1)}(B), k - j) + g(\sigma^{(n-1)}(C), k - 1) \right).
\]
Regardless of how large we choose to make $r_n$, we must have
\[
\sum_{j=1}^{r_n} g(\sigma^{(n-1)}(A), k - j) \leq q_{2n-2},
\]
as $q_{2n-2}$ is the length of the word $\sigma^{(n-1)}(A)$ (part of Prop. 11). Similarly we have
\[
\sum_{j=2}^{r_n+1} g(\sigma^{(n-1)}(B), k - j) \leq q_{2n-2} + q_{2n-3}, \quad g(\sigma^{(n-1)}(C), k - 1) \leq q_{2n-2} + q_{2n-3}.
\]
If we take $M$ to be such that $s_n \leq M$, then
\[
g(\sigma^{(n)}(A), k) \leq q_{2n-2} + M(q_{2n-2} + 2(q_{2n-2} + q_{2n-3})) \leq \tau q_{2n-2},
\]
where the constant \( \tau \) does not depend on the index \( n \). Similar arguments apply to \( \sigma^{(n)}(B) \) and \( \sigma^{(n)}(C) \).

**Corollary 4.3.** For any \( x \), if we denote \( R_n(x) \) to be the symbolic encoding of the \( q_{2n} \)-length orbit of \( x \), then there is a constant \( \tau \) (independent of both \( x \) and \( n \)) such that
\[
\max\{g(R_n(x), k) : k \in \mathbb{Z}\} \leq \tau q_{2n}^{-2}.
\]

**Proof.** Recall that the orbit of any point may be taken to begin with the concatenation of two words \( W_1(x) \) and \( W_2(x) \), where \( W_2(x) \) is of length at least \( q_{2n} \). Considering the words \( W_0(x) \) and \( W_1(x) \) independently, we need only double the constant \( \tau \) from Lemma 4.2. \( \square \)

Note that if the \( s_n \) are bounded, then for some \( \tau' \) we have (independent of \( n \))
\[
\tag{4}
2r_{n+1}q_{2n} < q_{2n+2} = (2r_{n+1}s_{n+1} + 1)q_{2n} + 2r_{n+1}q_{2n-1} < \tau'r_{n+1}q_{2n}.
\]

**Theorem 4.4.** Let \( \omega(n) \in o(1/n^\epsilon) \) for some \( \epsilon > 1/2 \) (and be monotone decreasing and regularly varying). Then there is an uncountable set of \( \alpha \) such that the infinite staircase is not \( \omega \)-recurrent. In fact, there is an uncountable set of \( \alpha \) for which there is a constant \( K = K(\alpha) \) such that for all \( x \in S^1 \)
\[
\sum_{i=1}^\infty \omega(i) \cdot \chi_0 \left( \sum_{j=1}^{i-1} f(x + j\alpha) \right) < K.
\]

**Proof.** For any fixed \( x \), denote
\[
\phi_n = \sum_{i=1}^{q_{2n}} \omega(i) \cdot \chi_0 \left( \sum_{j=1}^{i-1} f(x + j\alpha) \right).
\]

Then we have
\[
\phi_{n+1} \leq \phi_n + \sum_{i=1}^{2r_{n+1}s_{n+1}+1} \left( \sum_{j=1}^{q_{2n}} \omega(lq_{2n} + j) \cdot \chi_0 \left( \sum_{t=1}^{j} f(x + (l - 1)q_{2n}\alpha + t\alpha) \right) \right).
\]

Each index \( l \) represents a \( q_{2n} \)-length portion of the sum (between times \( (l - 1)q_{2n} \) and \( lq_{2n} \)), so by Corollary 4.3 (and monotonicity of \( \omega \)) we have that each of these segments contributes no more than
\[
\gamma q_{2n-2}\omega(lq_{2n})
\]
to the series \( \| \). Furthermore, via \( \| \) and Corollary 4.3 we know that only so many of these \( q_{2n} \)-portions can contribute their maximal allowed contribution to the sum. So (recall again that \( \omega \) is monotone):
\[
\phi_{n+1} \leq \phi_n + \sum_{l=1}^{\tau'r_{n}} \tau q_{2n-2}\omega(lq_{2n}).
\]
Now, by the assumption that $\omega(N) \in o(1/n^\epsilon)$ and by (4), we have that
\[
\sum_{l=1}^{r_n} \tau q_{2n-2} \omega(lq_{2n}) \leq \sum_{l=1}^{r_n} \frac{\tau q_{2n-2}}{(2r_n q_{2n})^\epsilon} 
< \frac{\tau q_{2n-2}^{1-\epsilon}}{(2r_n)^\epsilon} \sum_{l=1}^{r_n} \frac{1}{l^\epsilon} 
< \frac{\tau q_{2n-2}^{1-\epsilon}}{(2r_n)^\epsilon} (r_n + 1)^{1-\epsilon}
\]
As $q_{2n-2}$ does not depend on $r_n$, and $\epsilon > 1/2$, with $r_1, s_1, \ldots, r_{n-1}, s_{n-1}$ fixed, we are free to set $r_n$ large enough so that
\[
\frac{\tau q_{2n-2}^{1-\epsilon}}{(2r_n)^\epsilon} (r_n + 1)^{1-\epsilon} < b_n,
\]
where $b_n$ is any fixed series such that $\sum b_n < \infty$; it follows that (1), the series $\zeta(x)$, is bounded by this sum for every $x$ (the only term we can not control is the first, and $\phi_0 \leq \omega(1)$ for all $x$). As the choice of $r_n$ is not specific (just some lower bound depending on prior partial quotients), the set of such $\alpha$ that we may construct in this manner is uncountable. □

**Corollary 4.5.** For each $\lambda \in (0, 1/2)$, there is an uncountable collection of $\alpha$ such that the infinite staircase in the direction $\alpha$ has Lyapunov exponent at least $\lambda$.

**Proof.** Compare the conclusions of Theorem 4.4 and Corollary 2.6. Note that as the Lyapunov spectrum of a cocycle into $\mathbb{Z}$ is a single number, these are ergodic skew products with no zero Lyapounov exponents. □

5. Interval Exchanges of Periodic Type

We will not present any background in interval exchange transformations (IETs); an excellent survey on the subject is [13]. We note, however, that finding $\rho_k$, appropriate bounds on the growth of ergodic sums along some sequence of times $N_k$, is much more difficult than in the case of rotations. In the case that the IET is of periodic type (eventually periodic sequence of Rauzy classes), we have for all $n \in \mathbb{N}$ [9] Theorem 2.2:
\[
\sup_{x \in [0, 1]} \left| \sum_{i=0}^{n-1} \phi \circ T^i(x) \right| \leq C \cdot (\log n)^{M+1} \cdot n^{\theta_2 / \theta_1} \cdot V(\phi),
\]
where $\phi : S^1 \to \mathbb{R}$, $V(\phi)$ is the variation of $\phi$, $0 \leq \theta_2 < \theta_1$ are the two largest Lyapunov exponents, and $M$ is the size of the largest Jordan block in what is called the periodic matrix of $T$, and $C$ is a constant not dependent on $n$; $M$ is no larger than the number of exchanged intervals.
Assume, then, that $T$ is an interval exchange of periodic type on $M$ intervals, ergodic with respect to Lebesgue measure, that $\phi : S^1 \to \mathbb{Z}$ is of bounded variation (equivalently, piecewise constant over finitely many intervals) and define for some fixed $\gamma > 1$

$$N_k = \gamma^k, \quad \rho_k = Ck^{M+1}\gamma^{\frac{M}{M+1}},$$

where the constant $C$ does not depend on $k$. The proof of the following is postponed until §6.

**Lemma 5.1.** For any $\gamma > 1$, we have

$$\liminf_{k \to \infty} \frac{\gamma^k}{\rho_k \sum_{i=0}^{k-1} \frac{\gamma^i}{\rho_i}} > 0.$$  

Then we may apply Theorem 2.5 to obtain a result stronger than that implied solely by Corollary 2.6.

**Theorem 5.2.** If $T$ is an ergodic IET of periodic type defined on $M$ intervals and $\phi : S^1 \to \mathbb{Z}$ is of bounded variation, then the cocycle is $\omega$-recurrent, where

$$\omega(n) = \frac{(\log n)^M}{\frac{e_1 - e_2}{n} \cdot \log^2 n \cdot \log^3 n \cdot \cdots \cdot \log^j n},$$

where $\log^i n$ again represents the $i$-th iterated logarithm and $j$ is arbitrary.

**Proof.** We have

$$\frac{\gamma^k \omega(\gamma^k)}{\rho_k} > C' \frac{\gamma^k k^M}{k^{M+1}\gamma^{\frac{M}{M+1}} \cdot \log k \cdot \log^2 k \cdot \cdots \cdot \log^j k},$$

and the proof is completed with an appeal to Theorem 2.5.

Note that the recurrence of such cocycles is strongest for $\theta_2 = 0$ (the denominator of $\omega(n)$ contains a full power of $n$, as in Theorem 3.2). The ergodicity of cocycles over interval exchanges is increasingly seen to be interwoven with zero values for non-principal Lyapunov exponents, and this distinction makes an cameo appearance here in $\omega$-recurrence as well.

6. **Proof of Lemmas 3.1 and 5.1**

**Proof of Lemma 3.1.** We first wish to show that for almost every $\alpha$, there is some $\delta > 0$ such that

$$\delta \sum_{i=1}^{k-1} \frac{q_i}{a_i} \leq \frac{q_k}{a_k}.$$  

Equivalently, we will show that there is an $M$ such that

$$\frac{a_k}{q_k} \sum_{i=1}^{k-1} \frac{q_i}{a_i} \leq M.$$

Denote

$$C_k = \frac{a_k}{q_k} \sum_{i=1}^{k-1} \frac{q_i}{a_i}.$$
Then we have for $k > 0$

\[ C_{k+1} = \frac{a_{k+1}}{q_{k+1}} \sum_{i=1}^{k} \frac{q_i}{a_i} \]

\[ = \frac{a_k + a_{k+1}}{a_{k+1}q_k + q_{k-1}} \left( \frac{q_k}{a_k} + \sum_{i=1}^{k-1} \frac{q_i}{a_i} \right) \]

\[ < \frac{a_k + a_{k+1}}{a_{k+1}q_k} \left( \frac{q_k}{a_k} + \sum_{i=1}^{k-1} \frac{q_i}{a_i} \right) \]

\[ = \left( \frac{1}{a_{k+1}} + \frac{1}{a_k} \right) + \frac{C_k}{a_{k+1}} + \frac{C_k}{a_k} \]

\[ = \left( \frac{1}{a_k} + \frac{1}{a_{k+1}} \right) \frac{1}{a_k} (C_k + 1) \]

For any $\alpha$ we have $a_k \to \infty$, so for arbitrary $\tau > 0$, for all $a_{k+1} \geq 2$ we eventually have

\[ C_{k+1} \leq \left( \frac{1}{2} + \tau \right) (C_k + 1) \]

If $C_k \geq 4$ and $a_{k+1} \geq 2$, then, we have

\[ C_{k+1} < \frac{2}{3} C_k, \]

while for $C_k \leq 4$ and $a_{k+1} \geq 2$ we have $C_{k+1} < 3$. Suppose, then, that $a_{n+i} = 1$ for $i = 1, 2, \ldots, m$. In this case we have

\[ a_{n+i} = a_n + i, \quad q_{n+i} = \varphi_i q_n + \varphi_{i-1} q_{n-1}, \]

where $\varphi_i$ is the $i$-th Fibonacci number (beginning with $\varphi_0 = \varphi_1 = 1$ and $\varphi_{-1} = 0$ by convention). Then we have for $m, n > 0$

\[ \frac{a_n + m}{a_n \varphi_m q_n} \sum_{i=1}^{n} \frac{q_i}{a_i} < \frac{a_n + m}{\varphi_m q_n} \left( \sum_{i=1}^{n} \frac{q_i}{a_i} + \sum_{i=1}^{m} \frac{\varphi_i q_n + \varphi_{i-1} q_{n-1}}{a_n + i} \right). \]

As $q_{n-1} < q_n$ and $\varphi_{n-1} + \varphi_n = \varphi_{n+1}$, we have

\[ C_{n+m} < \frac{a_n + m}{a_n \varphi_m} C_n + \frac{a_n + m}{a_n \varphi_m} \sum_{i=1}^{m} \frac{\varphi_{i+1}}{\varphi_m}. \]

The Fibonacci sequence is exponentially growing, so as $m \to \infty$, the first term decreases to zero, and the sum in the second term approaches the limit of a convergent series. To complete the proof of the Lemma we need only address the term

\[ \frac{a_n + m}{a_n} = 1 + \frac{m}{a_n}. \]

If $\alpha$ is such that for sufficiently large $n$ we always have $a_n \geq 2$, the conclusion is clearly satisfied for such $\alpha$. Disregarding the null set of $\alpha$ for which all sufficiently large $a_n = 1$, then, we define for each $\alpha$ with infinitely many $a_n = 1$ and infinitely many $a_n \neq 1$ a sequence of pairs of positive integers $(n_i, m_i)$ as follows: for $j = 1, 2, \ldots, m_i$, each $a_{n_{i+j}} = 1$, and all partial quotients which are one appear in this manner. If we have

\[ \limsup_{i \to \infty} \frac{m_i}{n_i} \geq 1, \]
then it follows that in the sequence of partial quotients, the subsequence of \( a_i = 1 \) is of upper density at least one half. For generic \( \alpha \) this density is \( (\log 4 - \log 3)/\log 2 < 1/2 \), so for almost every \( \alpha \) we eventually have for those appearances of \( m \) successive \( a_{n+i} = 1 \)

\[
\frac{a_n + m}{a_n} \leq 1 + \frac{m}{n} < 2.
\]

\( \square \)

**Proof of Lemma 5.1.** Lemma 5.1 is very similar, except that \( N_k = \gamma^k \) for some \( \gamma > 0 \) does not depend on any dynamic or geometric data like partial quotients. Again, we will show that \( C_k \) are bounded, where

\[
C_k = \rho_k \gamma^k \sum_{i=1}^{k-1} \frac{\gamma^i}{\rho^i}.
\]

We first rearrange

\[
C_k = \sum_{i=1}^{k-1} \frac{C_k^{M+1, \gamma^i \rho^i}}{C_i^{M+1, \gamma^i \rho^i}} \gamma^k
\]

\[
= \sum_{i=1}^{k-1} \left( \frac{k}{i} \right)^{M+1} \frac{\gamma^{(k-i)\theta_2}}{\gamma^{k-i}} \gamma^j \theta_1
\]

\[
= \sum_{j=1}^{k-1} \left( \frac{k}{k-j} \right)^{M+1} \gamma^j \theta_1
\]

where we make the simple substitution \( j = k - i \) in the last line. We will break this sum into two parts \( C_k = S_1(k) + S_2(k) \), where

\[
T(k) = \left[ \frac{\theta_1(M+1) \log k}{(\theta_1 - \theta_2) \log \gamma} \right],
\]

\[
S_1(k) = \sum_{j=T(k)+1}^{k-1} \left( \frac{k}{k-j} \right)^{M+1} \gamma^j \theta_1
\]

\[
S_2(k) = \sum_{j=T(k)+1}^{k-1} \left( \frac{k}{k-j} \right)^{M+1} \gamma^j \theta_1
\]

The relevance of this cutoff \( T(k) \) is that for \( j > T(k) \) we have

\[
\gamma^{j(\theta_2 - \theta_1)/\theta_1} \leq \frac{1}{k^{M+1}},
\]
so we may estimate

\[ S_2 \leq \sum_{j=T(k)+1}^{k-1} \left( \frac{k}{k-j} \right)^{M+1} \frac{1}{k^{M+1}} \]

\[ \leq \sum_{j=1}^{k-1} \left( \frac{k}{k-j} \right)^{M+1} \frac{1}{k^{M+1}} \]

\[ \leq \sum_{i=1}^{k} \frac{k^{M+1}}{i^{M+1}} \]

\[ \leq \sum_{i=1}^{\infty} \frac{1}{i^{M+1}}. \]

so \( S_2 \) is bounded by the same constant regardless of \( k \). On the other hand, \( S_1 \) now involves only terms with comparatively small values of the index \( j \):

\[ S_1 = \sum_{j=1}^{T(k)} \left( \frac{k}{k-j} \right)^{M+1} \frac{\theta_2 - \theta_1}{\gamma^j} \gamma^j \frac{\theta_2 - \theta_1}{\gamma^j} \]

\[ \leq \left( \frac{k}{k-T(k)} \right)^{(M+1)T(k)} \sum_{j=1}^{\infty} \gamma^j \frac{\theta_2 - \theta_1}{\gamma^j} \].

The sum is convergent (letting \( k \to \infty \)) as \( \theta_2 < \theta_1 \) and \( \gamma > 1 \). On the other hand, as \( T(k) \in o(k) \), the fraction \( k/(k-T(k)) \) converges to one as \( k \to \infty \). As the power \( M+1 \) is not dependent on \( k \), then, \( S_1 \) is also bounded. 

\[ \square \]

7. Concluding Remarks

The Central Limit Theorem and Law of the Iterated Logarithm, if known in a certain skew product, allows for stronger choices of \( \rho_k \). The problem of establishing the CLT in the context of skew products over rotations and interval exchanges is the subject of active research by numerous mathematicians; any results have immediate implications in \( \omega \)-recurrence through Theorem 2.5.

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