ON POSITIVE LYAPUNOV EXPONENTS ALONG $E^{cu}$ AND NON-UNIFORMLY EXPANDING FOR PARTIALLY HYPERBOLIC SYSTEMS

REZA MOHAMMADPOUR

Abstract. In this paper we consider $C^1$ diffeomorphisms on compact Riemannian manifolds of any dimension that admit a dominated splitting $E^{cs} \oplus E^{cu}$. We prove that if the Lyapunov exponents along $E^{cu}$ are positive for Lebesgue almost every point, then a map $f$ is non-uniformly expanding along $E^{cu}$ under the assumption that the cocycle $Df^{-1}|_{E^{cu}(f)}$ has a dominated splitting with index 1 on the support of an ergodic Lyapunov maximizing observable measure. As a result, there exists a physical SRB measure for a $C^{1+\alpha}$ diffeomorphism map $f$ that admits a dominated splitting $E^{s} \oplus E^{cu}$ under assumptions that $f$ has non-zero Lyapunov exponents for Lebesgue almost every point and that the cocycle $Df^{-1}|_{E^{cu}(f)}$ has a dominated splitting with index 1 on the support of an ergodic Lyapunov maximizing observable measure.

1. Introduction

Let $f : M \to M$ be a $C^1$ diffeomorphism on a compact Riemannian manifold $M$ with a normalized Riemannian volume Lebesgue measure $\text{Leb}$. Given an $f$-invariant Borel probability measure $\mu$ in $M$, the basin of attraction $\mu$ is the set $B(\mu)$ of the points $x \in M$ such that the averages of Dirac measures along the orbit of $x$ converge to $\mu$ in the weak* sense:

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu$$

for any continuous $\varphi : M \to \mathbb{R}$. Then, we say that $\mu$ is a physical measure for $f$ if the basin of attraction $B(\mu)$ has the positive Lebesgue measure in $M$. We denote by Phy the set of all physical measures of a system.

A particular type of physical measures is called Sinai–Ruelle–Bowen (SRB) measure (see Section 6 for a precise definition). In the 1970s, Sinai, Ruelle, and Bowen introduced SRB measures for uniformly hyperbolic attractor sets $\Lambda$ under the assumption that $\Lambda$ has a dense set of periodic points or, equivalently, that $\Lambda$ is locally maximal or has a local product structure. These three properties of uniformly hyperbolic sets are equivalent [26]. For partially hyperbolic attractors, Pesin and Sinai [25] developed a technique called the
push-forward in 1982 to construct special measures known as \( u \)-measures. These measures share many geometric characteristics with the SRB measures. An ergodic SRB measure is physical, but not every physical measure is necessarily an SRB measure. The easiest counterexamples are the point masses on attractive fixed points and periodic orbits. Another interesting example is the figure-eight attractor that has a physical measure with a positive Lyapunov exponent that is not an SRB measure. Burns, Dolgopyat, Pesin, and Pollicott \cite{8} showed that a \( u \)-measure which has negative Lyapunov exponents for vectors in the central direction on a set of positive measure is a unique SRB measure under some transitivity assumptions.

There is a well-known conjecture by Viana that asks for the existence of an SRB measure under certain assumptions (see e.g., \cite{12}). Alves et al. \cite{1} considered the more difficult setting of a continuous splitting \( E^s \oplus E^{cu} \) with uniform contraction \( E^s \) and non-uniform expansion \( E^{cu} \). In this case, the construction of the SRB measure required a significantly more complex technique. In fact, they showed that \( f \) has finitely many ergodic physical SRB measures under the assumption that there exists a set \( H \) of positive Lebesgue measure on which \( f \) is non-uniformly expanding along \( E^{cu} \): There exists a \( \lambda > 0 \) such that for all \( x \in H \),

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| Df^{-1}_{|E^{cu}(f^j(x))} \right\| < -\lambda.
\]

In \cite{2}, they improved \cite[Theorem A]{1} by showing that the limsup condition (1.1) could be replaced by the liminf condition, and they also used Young towers to construct the SRB measure. Climenhaga, Dolgopyat and Pesin \cite{11} generalized the main result of the existence of SRB measures in \cite{1}. We recommend the survey \cite{12} for further references and results.

Apart from the one-dimensional setting, it has been difficult to work directly with Lyapunov exponents, therefore \cite{1,2} introduced stronger versions of non-uniform expansion. Moreover, \cite{1} asked whether their result is true if non-uniformly expanding condition (1.1) replaced by

\[
\limsup_{n \to \infty} \frac{1}{n} \log \left\| Df^{-n}_{|E^{cu}(f^n(x))} \right\| < 0
\]

for Leb almost every \( x \), which is the positivity of all Lyapunov exponents in the center unstable direction.

In this work, we deal with \( C^1 \) diffeomorphisms admitting partially hyperbolic invariant sets, where the tangent bundle over the set has an invariant dominated splitting into two subbundles, namely non-uniform expansion \( E^{cu} \) and non-uniform contraction \( E^{cs} \). In this paper, we show that equation (1.2) implies equation (1.1) under the assumption that the cocycle \( Df^{-1}_{|E^{cu}(f)} \) has a dominated splitting with index 1 on the support of an ergodic observable measure. Our key assumption is that the cocycle \( Df^{-1}_{|E^{cu}(f)} \) has a dominated splitting with index 1 on the support of an ergodic observable measure (the measure always exists; see Remark 4.2) and not on the entire manifold.
1.1. Precise setting. In this subsection, we fix \( T : X \to X \) as a homeomorphism on a compact manifold \( X \), and we let \( \text{Leb} \) be a normalized Lebesgue measure such that \( \text{Leb}(X) = 1 \). For simplicity, we denote that as \((X, T)\). Also, we denote by \( \mathcal{M}(X) \) and \( \mathcal{M}(X, T) \) the space of all Borel probability measure on \( X \), and the space of all \( T \)-invariant Borel probability measures on \( X \) with respect to weak* topology, respectively.

For any point \( x \in X \), we denote by \( V(x) \) the set of those Borel probabilities on \( X \) that are limits in the weak* topology of convergent subsequences of the sequence

\[
\gamma_{n,x} := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)},
\]

where \( \delta_y \) is the Dirac delta probability measure supported at \( y \in X \). In other words,

\[
V(x) = \left\{ \mu \in \mathcal{M}(X) : \exists n_j \to +\infty \text{ such that } \lim_{j \to \infty} \gamma_{n_j,x} = \mu \right\} \subset \mathcal{M}(X, T).
\]

The probabilities of the sequence (1.3) are called empirical probabilities of the orbit of \( x \). Note that empirical probabilities do not necessarily converge.

**Definition 1.1.** An invariant Borel probability measure \( \mu \in \mathcal{M}(X, T) \) is observable or physical-like if for any \( \varepsilon > 0 \) the set \( B_{\varepsilon}(\mu) = \{ x \in X : \text{dist}(V(x), \mu) < \varepsilon \} \) has positive Lebesgue measure. We call \( B_{\varepsilon}(\mu) \) the basin of \( \varepsilon \)-attraction of \( \mu \). We denote by \( \mathcal{O}_T \) the set of all observable measures.

Observable measures are introduced by Catsigeras and Enrich [10]. An observable measure is not necessarily physical (see Example 5.4). In general, for a system, physical measures may not exist; however, there always exists an observable measure (see [10, Theorem 1.3]).

1.2. Main result. Our goal is to show that for partially hyperbolic systems, the positive Lyapunov exponents in the center-unstable direction imply non-uniform expansion under a mild assumption. It is clear that the two formulations (1.5) and (1.6) are equivalent when the non-uniformly expanding direction is 1-dimensional, that is, when \( E^{cu} \) can be split as \( E^c \oplus E^u \) with \( E^c \) having dimension 1 and \( E^u \) being uniformly expanding. In the case where \( E^c \) has dimension 1, there are some results (see e.g., [15, 18, 7, 9]) that studied SRB measures. A distinction in our result from similar results is that we only require the dominated splitting property on the support of an ergodic observable measure (the measure always exists, see Remark 4.2), not on the entire manifold or the attractor.

Let \( f : M \to M \) be a \( C^1 \) diffeomorphism on a compact Riemannian manifold that admits a dominated splitting \( TM = E^{cs} \oplus E^{cu} \) (See Subsection 3.3 for more information). Any diffeomorphism \( C^1 \) away from the set of diffeomorphisms exhibiting a homoclinic tangency may be approximated by partially hyperbolic diffeomorphisms of this form (see [14]). Let \( \mu \) be the invariant probability measure. We denote by

\[
\lambda^-(\mu, Df^{-1}_{|E^{cu}(f)}) := \lim_{n \to \infty} \frac{1}{n} \int \log \| Df^{-n}_{|E^{cu}(f_n(x))} \| \, d\mu(x).
\]
\( \lambda^- (\mu, Df^{-1}_{E^{cu}(f)}) \) is the minimal Lyapunov exponent in the center-unstable direction. For simplicity, we denote by \( \lambda^- (\mu, f) := \lambda^- (\mu, Df^{-1}_{E^{cu}(f)}) \). Let \( Y \subset M \) be a non-empty \( f \)-invariant compact set. We say that \( Df^{-1}_{E^{cu}(f)} \) has a dominated splitting with index 1 on \( Y \) if there exists a 1-dimensional invariant cone fields \((C_x)_{x \in Y}\) on \( Y \) for \( Df^{-1}_{E^{cu}(f)} \) (see Subsection 3.3 for further details).

The main result of this paper is the following theorem.

**Theorem 1.2.** Let \( f : M \to M \) be a \( C^1 \) diffeomorphism on a compact Riemannian manifold that admits a dominated splitting \( T_M = E^{cs} \oplus E^{cu} \). Assume that \( \nu \) is an ergodic observable measure such that

\[
\lambda^- (\nu, f) = \sup_{\mu \in \mathcal{O}_f} \lambda^- (\mu, f) = \max_{\mu \in \mathcal{O}_f} \lambda^- (\mu, f),
\]

and that \( Df^{-1}_{E^{cu}(f)} \) has a dominated splitting with index 1 on the support of \( \nu \). If Lebesgue a.e. \( x \) satisfies

\[
\limsup_{n \to \infty} \frac{1}{n} \log \left\| Df^{-n}_{E^{cu}(f^n(x))} \right\| < 0,
\]

then there are \( \lambda > 0 \) and \( K \in \mathbb{N} \) such that Lebesgue a.e. \( x \) satisfies

\[
\limsup_{n \to +\infty} \frac{1}{nK} \sum_{i=1}^{n} \log \left\| Df^{-K}_{E^{cu}(f^K(x))} \right\| \leq -\lambda < 0.
\]

Note that the supremum (1.4) is always attained because the Lyapunov exponent is upper semi-continuous and \( \mathcal{O}_f \) is compact with respect to the weak* topology (see Theorem 2.2). Therefore, the invariant measure \( \nu \) in Theorem 1.2 always exists. Without loss of generality, we assume that \( \nu \) is ergodic (see [17, Proposition A.1]).

Section 5 provides several examples of observable measures and some systems where there is a dominated splitting on the support of an ergodic observable measure.

**Remark 1.3.** Theorem A can also be localized as asked by [1]: if (1.5) holds on a set of positive Lebesgue measure, then (1.6) holds on some set of positive Lebesgue measure. In order to state it, we need to define an observable measure relative to some \( C^1 \)-disk. Let \( D \) be an embedded compact \( C^1 \)-disk in \( M \). Then \( \mu \) is called an observable measure relative to \( D \), if for any \( \eta > 0 \), one has

\[
\text{Leb}_D(\{x \in D : \text{dist}(V(x), \mu) < \eta\}) > 0,
\]

where \( \text{Leb}_D \) is the normalized Lebesgue measure on a disc \( D \).

Crovisier, Yang and Zhang [15] generalized Theorem 2.2 to the set of all observable measures relative to some \( C^1 \)-disk \( D \), which is denoted by \( \mathcal{O}_D \). Moreover, for a set of positive \( \text{Leb}_D \) measure \( B \subset D \), we say that \( \mu \) is called a observable measure relative to \( B \subset D \), if for any \( \eta > 0 \),

\[
\text{Leb}_D(\{x \in B : \text{dist}(V(x), \mu) < \eta\}) > 0.
\]
By replacing $O_f$ with $O_D$ in Definition 4.1, we can state Theorem A as follows, if (1.5) holds on a set of positive Lebesgue measure $B \subset D$, then (1.6) holds on some set of positive Lebesgue measure.

Thanks to Theorem 1.2, we can show the existence of an SRB measure for partially hyperbolic systems.

**Theorem 1.4.** Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold that admits a dominated splitting $T\mathcal{M} = E^s \oplus E^{cu}$, where $E^s$ is uniformly contracting. Assume that $\nu$ is an ergodic observable measure such that

$$\lambda^-(\nu, Df^{-1}_{|E^{cu}}) = \sup_{\mu \in O_f} \lambda^-(\mu, Df^{-1}_{|E^{cu}}) = \max_{\mu \in O_f} \lambda^-(\mu, Df^{-1}_{|E^{cu}}),$$

and that $Df^{-1}_{|E^{cu}(f^n(x))}$ has a dominated splitting with index 1 on the support of $\nu$. If Lebesgue a.e. $x$ satisfies

$$\limsup_{n \to \infty} \frac{1}{n} \log \left\| Df^{-n}_{|E^{cu}(f^n(x))} \right\| < 0,$$

then $f$ has finitely many ergodic physical SRB measures and the union of their basins has full Lebesgue measure.

**Proof.** It follows from Theorems 1.2 and [2, Theorem A].

The assumption of uniform contraction in Theorem 1.4 can be somewhat relaxed, as explained in Section 6 of their paper [1]. For systems with partial hyperbolicity and non-uniform expansion in the $E^{cu}$ direction, [1] proved the existence of invariant probability measures with absolutely continuous conditional measures along the center-unstable direction with respect to Lebesgue measure. These measures are referred to as Gibbs $cu$-states (for a precise definition and properties, see [16]).

Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^1$ diffeomorphism on a compact Riemannian manifold. For two $Df$-invariant subbundles $E, F \subset T\mathcal{M}$, the bundle $E$ is dominated by $F$, denoted as $E \oplus F$, when there are $C > 0$ and $0 < \lambda < 1$ such that $\left\| Df^n_{|E(x)} \right\| \left\| Df^{-n}_{|F(f^n(x))} \right\| \leq C\lambda^n$ for any $x \in \mathcal{M}$ and any $n \in \mathbb{N}$. Let $\mu$ be an $f$-invariant Borel probability measure. We define the *Lyapunov exponent* of $Df^{-1}|_E$ with respect to $\mu$ and $f$ as

$$\lambda(\mu, Df^{-1}|_E) := \lim_{n \to \infty} \frac{1}{n} \int \log \left\| Df^{-n}_{|E(f^n(x))} \right\| d\mu(x).$$

Working with the Lyapunov exponent (1.2) can be challenging due to the lack of invariance of the Lebesgue measure. However, in the following theorem, we establish the existence of an invariant measure whose corresponding Lyapunov exponent is equal to (1.2).

**Theorem 1.5.** Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^1$ diffeomorphism on a compact Riemannian manifold that admits a dominated splitting $T\mathcal{M} = E \oplus F$. Assume that $\nu$ is an ergodic observable measure such that

$$\lambda(\nu, Df^{-1}_{|F}) = \sup_{\mu \in O_f} \lambda(\mu, Df^{-1}_{|F}) = \max_{\mu \in O_f} \lambda(\mu, Df^{-1}_{|F}),$$

then $f$ has finitely many ergodic physical SRB measures and the union of their basins has full Lebesgue measure.
and that $Df^{-1}|_F$ has a dominated splitting with index 1 on the support of $\nu$. Then,

$$\text{ess sup lim sup } \frac{1}{n} \log \|Df^{-n}|_{F(f^n,x)}\| = \sup_{\mu \in \mathcal{O}_f} \lambda(\mu, Df^{-1}|_F) = \max_{\mu \in \mathcal{O}_f} \lambda(\mu, Df^{-1}|_F),$$

where the ess sup denotes the essential supremum taken against the Lebesgue measure $\text{Leb}$.

Remark 1.6. All results of the present paper also work for diffeomorphisms with partially hyperbolic attractors (see Subsection 3.3).

In Section 2 we provide properties of observable measures and related concepts. In Section 3 we study Lyapunov exponents, dominated splitting and related concepts. In Section 4 we discuss the maximal Lyapunov exponents over the set of observable measures and we prove Theorem 1.5. In Section 5 we give some examples about observable measures. The proof of Theorem 1.2 is given in Section 6.

Acknowledgments. The author is grateful to José Alves, David Burguet, Snir Ben Ovadia and Kiho Park for the useful discussion. He also thanks Davi Obata and Paulo Varandas for the influential conversation. This work was supported by the Knut and Alice Wallenberg Foundation.

2. Observable Measures

Let $T : X \to X$ be a homeomorphism on a compact manifold $X$ and let $\text{Leb}$ be a Lebesgue measure normalized so that $\text{Leb}(X) = 1$. We recall some results and facts related with observable measures.

We recall that an invariant Borel probability measure $\mu \in \mathcal{M}(X, T)$ is observable or physical-like if for any $\varepsilon > 0$ the set $B_\varepsilon(\mu) = \{x \in X : \text{dist}(V(x), \mu) < \varepsilon\}$ has positive Lebesgue measure. We call $B_\varepsilon(\mu)$ the basin of $\varepsilon$-attraction of $\mu$. We denote by $\mathcal{O}_T$ the set of all observable measures. When the context is clear, we denote by $\mathcal{O} := \mathcal{O}_T$ the set of all observable measures.

Remark 2.1. By the definition of observable measures, $\varepsilon$-approximation in the space $\mathcal{M}(X, T)$ can be transferred to an $\varepsilon$-approximation (in time-mean) towards an attractor in the ambient manifold $X$. In other words, if $\mu$ is observable and $x \in B_\varepsilon(\mu)$, then the iterates $T^n(x)$ will $\varepsilon$-approach the support of $\mu$ with a frequency which is asymptotically bounded away from zero.

The set $B(\mu) = \{x \in X : V(x) = \{\mu\}\}$ is the basin of attraction of a physical-measure $\mu$. Inspired by the latter definition, we define the basin of attraction $\mathcal{O}$:

\begin{equation}
\mathcal{G}(\mathcal{O}) = \{x \in X : V(x) \subset \mathcal{O}\}.
\end{equation}

For simplicity, let us denote $\mathcal{G} := \mathcal{G}(\mathcal{O})$. Catsigeras and Enrich [10, Theorems 1.3] have shown that $\mathcal{O}$ is non-empty. They have also proved the following results:

Theorem 2.2 ([10, Theorem 1.5]). $\mathcal{O}$ is the smallest weak* compact set that contains, for Lebesgue almost all initial states, the limits of all convergent subsequences of (1.3).
In other words, \( \mathcal{O} = \bigcap_{K \in \mathcal{W}} K \), where
\[
W := \{ K \subset \mathcal{M}(X) : K \text{ is compact and } \text{Leb}(\mathcal{G}(K)) = 1 \}.
\]
By Theorem 2.2, \( \text{Leb}(\mathcal{G}) = 1 \).

**Theorem 2.3** ([10, Theorem 1.6]). There are finitely many physical measures such that the union of their basins of attraction covers \( X \) Lebesgue almost everywhere if and only if \( \mathcal{O} \) is finite. In this case, \( \mathcal{O} = \text{Phy} \).

**Theorem 2.4** ([10, Theorem 1.7]). If \( \mathcal{O} \) is countably infinite, then there are countably infinitely many physical measures such that their basins of attraction cover \( X \) Lebesgue almost everywhere. In this case \( \mathcal{O} \) is the weak* closure of the set of physical measures.

3. **Lyaunov exponents and dominated systems**

3.1. **Subadditive potentials.** A topological dynamical system \((X, T)\) is a continuous map \( T : X \to X \) on a compact metric space \( X \). We say that \( \Phi := \{ \log \phi_n \}_{n=1}^{\infty} \) is a subadditive potential over \((X, T)\) if each \( \phi_n \) is a continuous positive-valued function on \( X \) such that
\[
0 < \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \quad \forall x \in X, m, n \in \mathbb{N}.
\]
Moreover, we say that \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) is an almost additive potential over \((X, T)\) if there exists a constant \( C > 0 \) such that for any \( m, n \in \mathbb{N}, x \in X \), we have
\[
C^{-1} \phi_n(x)\phi_m(T^n(x)) \leq \phi_{n+m}(x) \leq C\phi_n(x)\phi_m(T^n(x)).
\]

Let \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) be a subadditive potential over \((X, T)\). For an \( T \)-invariant measure \( \mu \), we define the Lyapunov exponent of \( \Phi \) with respect to \( \mu \) and \( T \) as
\[
\chi(\mu, \Phi) := \lim_{n \to \infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x),
\]
where the limit exists by Kingman’s subadditive theorem.

**Theorem 3.1.** Suppose \( \{ \nu_n \}_{n=1}^{\infty} \) is a sequence in \( \mathcal{M}(X) \) and \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) is a subadditive potential over a topological dynamical system \((X, T)\). We form the new sequence \( \{ \mu_n \}_{n=1}^{\infty} \) by \( \mu_n = \frac{1}{n} \sum_{i=0}^{n-1} T_i^* \nu_n \). Assume that \( \mu_{n_i} \) converges to \( \mu \) in \( \mathcal{M}(X) \) for some subsequence \( \{ n_i \} \) of natural numbers. Then \( \mu \) is an \( T \)-invariant measure and
\[
\limsup_{i \to \infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\nu_{n_i}(x) \leq \chi(\mu, \Phi).
\]
Moreover, if \( \Phi \) is an almost additive potential over a topological dynamical system \((X, T)\), then
\[
\lim_{i \to \infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\nu_{n_i}(x) = \chi(\mu, \Phi).
\]

**Proof.** It follows from [17, Lemmas A2 and A4].

Feng and Huang [17, Lemma A.4] proved the continuity of Lyapunov exponents for almost additive potentials.
Lemma 3.2. Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be an almost additive potential over a topological dynamical system $(X, T)$. For every $\epsilon > 0$, there is $\delta > 0$ such that if $\mu, \nu \in M(X, T)$, and $d(\mu, \nu) < \delta$, then $|\lambda(\mu, \Phi) - \lambda(\nu, \Phi)| < \epsilon$.

3.2. Matrix cocycles. A matrix cocycle $A$ over a topological dynamical system $(X, T)$ is a continuous map $A : X \to GL_d(\mathbb{R})$. For $n \in \mathbb{N}$ and $x \in X$, we define the product of $A$ along the length $n$ orbit of $x$ as

$$A^n(x) = A(T^{n-1}(x))A(T^{n-2}(x)) \cdots A(x).$$

When the context is clear, we say that $A$ is a cocycle. Also, it induces a skew-product $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ by $F(x, v) = (T(x), A(x)v)$. Observe that $F^n(x, v) = (T^n(x), A^n(x)v)$ for each $n \geq 1$.

Note that $\|A\|$, the Euclidean operator norm of a matrix $A$, is submultiplicative i.e.,

$$0 < \|A^{n+m}(x)\| \leq \|A^n(x)\| \cdot \|A^m(T^n(x))\| \quad \forall x \in X, m, n \in \mathbb{N},$$

therefore the potential $\Phi_A := \{\log \|A^n\|\}_{n=1}^{\infty}$ is subadditive for a matrix cocycle $A$.

An important class of matrix cocycles are derivative cocycles, which are defined as follows. Let $f : M \to M$ be a local diffeomorphism on a manifold $M$. Assume that the tangent bundle $TM$ is trivial. We then consider $A(x) = Df_x$ at each point as a matrix. Thus the tangent map

$$Df : TM \to TM, \quad Df(x, v) = (f(x), A(x)v)$$

is a skew product that generated by a matrix cocycle $(f, A)$, in the previous sense.

We denote by $Gr(d)$ the Grassmanian of $\mathbb{R}^d$.

Theorem 3.3 (Oseledets). Let $A : X \to GL_d(\mathbb{R})$ be a matrix cocycle over a homeomorphism $(X, T)$. Let $\mu$ be an $T$-invariant measure. For $\mu$ almost every $x \in X$, there are functions $k = k(x)$, $\lambda_1(x, A) > \lambda_2(x, A) > \ldots > \lambda_k(x, A)$ and a direct sum decompositions $\mathbb{R}^d = E^1(x) \oplus E^2(x) \oplus \ldots \oplus E^k(x)$ such that, for every $i = 1, \ldots, k$:

1) The maps $x \to k(x)$, $x \to \lambda(x, A)$ and $x \to E^i(x)$ (with values $\mathbb{N}$, $\mathbb{R}$ and $Gr(d)$, respectively) are measurable.

2) $k(T(x)) = k(x)$, $\lambda_i(T(x), A) = \lambda_i(x, A)$ and $A(x)E^i(x) = E^i(T(x))$.

3) For every non-null $v \in E^i(x)$,

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x, A).$$

4) For any disjoint sum of the spaces within the splitting

$$\lim_{n \to \pm \infty} \frac{1}{n} \log |\sin \angle(\oplus_{i \in I} E^i(T^n(x)), \oplus_{i \in J} E^i(T^n(x))| = 0$$

whenever $I \cap J = \emptyset$.

The splitting $\mathbb{R}^d = E^1(x) \oplus \ldots E^k(x)$ is called Oseledets splitting.
We denote by $\chi_1(x, A) \geq \chi_2(x, A) > \ldots \geq \chi_d(x, A)$ the Lyapunov exponents with respect to $\mu$, counted with multiplicity. This is the Lyapunov spectrum of the cocycle with respect to some invariant measure $\mu$.

Let $\chi(\mu, \Phi_A) := \int \chi_1(., A) \, d\mu$. If the measure $\mu$ is ergodic, then $\chi_1(x, A) = \chi(\mu, \Phi_A)$ for $\mu$-almost every $x \in X$.

### 3.3. Dominated splitting

**Partial hyperbolicity** is a relaxed form of uniform hyperbolicity that plans to address larger families of dynamics. A main aim of their study consists in understanding how the properties of uniformly hyperbolic systems extends.

We consider $f$ in the space $\text{Diff}^r(M)$ of $C^r$-diffeomorphisms endowed with the $C^r$-topology, where $r \geq 1$ and $M$ is a closed connected $d$-dimensional Riemannian manifold. For $K \subset M$, a splitting $T_K M = E^1 \oplus E^2 \oplus \ldots \oplus E^\ell$ is a linear decomposition $T_x M = E^1(x) \oplus E^2(x) \oplus \ldots \oplus E^\ell(x)$ at each $x \in K$ such that $\dim E^i(x)$ does not depend on $x$ for any $1 \leq i \leq \ell$.

If $K \subset M$ is an $f$-invariant set (i.e., $f(K) = K$), then we say that a splitting $T_K M = E^1 \oplus E^2 \oplus \ldots \oplus E^\ell$ is a dominated splitting for the cocycle $Df$ if and only if the following conditions hold:

- **Invariance**: The bundles are $Df$-invariant. This means that for every $x \in K$ and $1 \leq i \leq \ell$, we have $Df_x(E^i(x)) = E^i(f(x))$.
- **Domination**: There exist constants $c > 0$ and $\lambda \in (0, 1)$ such that for every $1 \leq i \leq \ell - 1$, every $x \in K$, and every nonzero vectors $u \in E^i(x)$ and $v \in E^{i+1}(x)$, the following inequality holds:

  \[
  \frac{\|Df_x^n u\|}{\|u\|} \leq c \lambda^n \frac{\|Df_x^n v\|}{\|v\|} \quad \text{for all } n \geq 0.
  \]

Bochi and Gourmelon [4] showed that a cocycle has dominated splitting if and only if there is a uniform exponential gap between singular values of its iterates. Indeed, assuming that $X$ is a compact metric space, and letting $A : X \to GL(d, \mathbb{R})$ be a matrix cocycle over a homeomorphism $(X, T)$. Let $Y \subset X$ be a non-empty $T$-invariant compact set. We say that the cocycle $A$ has dominated splitting with index $i$ on $Y$ if there are constants $C > 0$ and $0 < \tau < 1$ such that

\[
\frac{\sigma_{i+1}(A^n(x))}{\sigma_i(A^n(x))} \leq C \tau^n \quad \forall x \in Y, \forall n \in \mathbb{N},
\]

where $\sigma_i$’s are singular values.

Dominated splitting can be characterized in terms of existence of invariant cone fields (see [13]). We say that a family of convex cones $C := (C_x)_{x \in Y}$ is a $i$-dimensional invariant cone field on $Y$ if

\[
A(x)C_x \subset C^0_{T(x)} \quad \forall x \in Y,
\]

where $i$ is the dimension $C$. $C$ is also called unstable invariant cone field (see [13] for more information).

---

1The multiplicity of $\lambda_i(x, A)$ is the same as the dimension of $E^i(x)$. 
For a compact invariant set $K$, a $Df$-invariant bundle $F$ is uniformly contracted (by $Df$) if there are constants $C > 0$ and $\lambda \in (0,1)$ such that for any point $x \in K$, we have $\|Df^n|_{F(x)}\| \leq C\lambda^n$; a $Df$-invariant bundle $F$ is uniformly expanded (by $Df$) if it is uniformly contracted by $Df^{-1}$. We say a compact invariant set $K$ is partially hyperbolic if there is a $Df$-invariant splitting $T_KM = E^u \oplus E^c_1 \oplus E^c_2 \oplus E^s$ such that

- $E^u$ is uniformly expanded and $E^s$ is uniformly contracted.
- $E^u$ or $E^s$ may be trivial, but cannot be trivial simultaneously.

The bundles $E^c_1, E^c_2$ are not uniformly contracted or expanded, and are called center bundles. The bundles $E^{cs} := E^s \oplus E^c$ and $E^{cu} := E^u \oplus E^c$ are the center-stable and center-unstable subbundles, which are mostly contracting and expanding, respectively.

In this paper, we are interested in a $C^1$ diffeomorphism $f : M \to M$ on a compact Riemannian manifold that admits a dominated splitting $TM = E^{cs} \oplus E^{cu}$.

In [23], the author showed that if a cocycle has dominated splitting with index 1 on some compact metric space, which can be characterized in terms of existence of invariant cone fields (or multicones), then the potential $\Phi_A = \{\log \|A^n\|\}_{n=1}^\infty$ is almost additive.

**Proposition 3.4** ([23, Proposition 5.8]). Let $X$ be a compact metric space, and let $A : X \to GL(d,\mathbb{R})$ be a matrix cocycle over a homeomorphism $(X,T)$. Assume that $(C_x)_{x \in X}$ is a 1-dimensional invariant cone field on $X$. Then, there exists $\kappa > 0$ such that for every $m,n > 0$ and for every $x \in X$ we have

$$\|A^{m+n}(x)\| \geq \kappa\|A^m(x)\| \cdot \|A^n(T^m(x))\|.$$  

### 4. Proof of Theorem 1.5

In this section, we fix $T : X \to X$ as a homeomorphism on a compact manifold $X$ and let $\text{Leb}$ be a Lebesgue measure normalized so that $\text{Leb}(X) = 1$. Let $\Phi = \{\log \phi_n\}_{n=1}^\infty$ be a subadditive potential over $(X,T)$. Ergodic optimization of Lyapunov exponents concerns the supremum of the Lyapunov exponents of measures over invariant measures:

$$\beta(\Phi) := \sup_{\mu \in \mathcal{M}(X,T)} \chi(\mu, \Phi).$$

In (4.1), the supremum is always attained by an ergodic measure; such measures will be called Lyapunov maximizing measures. This follows from the fact that $\mathcal{M}(X,T)$ is a compact convex set whose extreme points are exactly the ergodic measure, and $\chi(\cdot, \Phi)$ is upper semi continuous with respect to the weak* topology (see [20, 3, ?]).

Let us stress that the maximal Lyapunov exponent can also be characterized as follows:

$$\beta(\Phi) = \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log \phi_n(x) = \sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} \log \phi_n(x).$$

**Definition 4.1.** An observable measure $\mu$ is called a Lyapunov maximizing observable measure for a subadditive potential $\Phi = \{\log \phi_n\}_{n=1}^\infty$ over $(X,T)$ if

$$\chi(\mu, \Phi) = \sup_{\nu \in \mathcal{O}} \chi(\nu, \Phi)$$
Similarly, one can define a Birkhoff maximizing observable measure (see [30]).

Remark 4.2. Since $\chi(\cdot, \Phi)$ is upper semi-continuous with respect to weak* topology and $O$ is weak* compact by Theorem 2.2, the supremum is always attained in (4.3). Therefore, a Lyapunov maximizing observable measure always exists.

Similar to (4.2), we obtain the following theorem for observable measures.

**Theorem 4.3.** Let $A : X \to GL(d, \mathbb{R})$ be a matrix cocycle over a homeomorphism $(X, T)$. Let $\mu$ be an ergodic Lyapunov maximizing observable measure for the subadditive potential $\Phi_A$. Suppose that the cocycle $A$ has a dominated splitting with index 1 on the support $\mu$. Then

$$\text{ess sup lim sup } \frac{1}{n} \log \| A^n(x) \| = \sup \{ \chi(\mu, \Phi_A) \mid \mu \in O \}$$

where the ess sup denotes the essential supremum taken against the Lebesgue measure $\text{Leb}$.

Remark 4.4. The inequality in the formula (4.5) could be strict (see Example 5.3).

**Proof of Theorem 1.5.** It follows from Theorem 4.3. □

Now, we prove Theorem 4.3.

**Lemma 4.5.** Let $\Phi = \{ \log \phi_n \}_{n=1}^{\infty}$ be a subadditive potential over a homeomorphism $(X, T)$. Then

$$\text{ess sup lim sup } \frac{1}{n} \log \phi_n(x) \leq \sup \{ \chi(\mu, \Phi) \mid \mu \in O \}$$

where the essential supremum taken against $\text{Leb}$.

**Proof.** Choose a point $x \in G$ (see (2.1)), and then a subsequence of integers $\{ n_i \}$ such that $\limsup_{n \to \infty} \frac{1}{n} \log \phi_n(x) = \lim_{i \to \infty} \frac{1}{n_i} \log \phi_{n_i}(x)$. Since $x \in G$, there is a subsequence $\gamma_{n_i, x}$ of $\gamma_{n, x}$ (see (1.3)) such that converges in weak* topology to an observable measure $\mu$ when $k \to \infty$ by definition. Without loss of generality, we can suppose that $\gamma_{n_i, x}$ converges to $\mu$ as $i \to \infty$. Then, by Theorem 3.1

$$\lim_{i \to \infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\delta_x \leq \chi(\mu, \Phi) \leq \sup \{ \chi(\mu, \Phi) \mid \mu \in O \}.$$  

On the other hand,

$$\lim_{i \to \infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\delta_x = \limsup_{n \to \infty} \frac{1}{n} \log \phi_n(x).$$

Combination (4.6) and (4.7) and the arbitrariness of $x \in G$ imply

$$\text{ess sup lim sup } \frac{1}{n} \log \phi_n(x) \leq \sup \{ \chi(\mu, \Phi) \mid \mu \in O \}.$$
We consider any subset $U$ with full Lebesgue measure. We know that each subset of full Lebesgue measure is dense, hence
\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{x \in U} \log \phi_n(x) = \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log \phi_n(x) = \beta(\Phi).
\]

By (4.2) and (4.1),
\[
\text{ess sup } \limsup_{n \to \infty} \frac{1}{n} \log \phi_n(x) \leq \sup \{ \chi(\mu, \Phi) \mid \mu \in \mathcal{O} \} \leq \limsup_{n \to \infty} \frac{1}{n} \text{ess sup } \log \phi_n(x).
\]

□

Let $f : M \to M$ be a $C^\infty$ map on a compact smooth manifold $M$. Kozlovski [21] showed the following integral formula for the topological entropy $C^\infty$ smooth systems:
\[
h_{\text{top}}(f) = \lim_{n \to \infty} \frac{1}{n} \log \int \max_k \| \Lambda_k Df^n_x \| \, d\text{Leb}(x),
\]
where $\Lambda^k Df^n_x$ is a mapping between $k$-exterior algebras of the tangent spaces.

**Corollary 4.6.** Assume that $f : M \to M$ is a $C^\infty$ map on a compact smooth manifold $M$. Then
\[
h_{\text{top}}(f) \geq \text{ess sup } \limsup_{n \to \infty} \frac{1}{n} \log \max_k \| \Lambda^k Df^n_x \|.
\]

**Proof.** We consider $\phi_n = \max_k \| \Lambda^k Df^n_x \|$ in Lemma 4.5, then
\[
\text{ess sup } \limsup_{n \to \infty} \frac{1}{n} \log \max_k \| \Lambda^k Df^n_x \| \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{ess sup } \max_k \| \Lambda^k Df^n_x \|
\leq h_{\text{top}}(f).
\]

□

This formula was demonstrated using a different method by Burguet [6] (see (1.3) in [6]).

**Proof of Theorem 4.3.** Lemma 4.5 implies that one side of the equality (4.4) and the inequality (4.5) hold. Now, we explain how one can tackle the other side of the equality (4.4) in Theorem 4.3.

Let $\mathcal{G} \subset X$ such that $V(x) \subset \mathcal{O}$ for any $x \in \mathcal{G}$. By Theorem 2.2, $\text{Leb}(\mathcal{G}) = 1$. We fix a subset $L \subset X$ with full Lebesgue measure.

The set of observable measures $\mathcal{O}$ is weak* compact (see Theorem 2.2), hence there is a Lyapunov maximizing observable measure $\mu$ for the subadditive potential $\Phi_A$ (see Remark 4.2). We assume that $\mu$ is ergodic. Moreover, we recall that we assume the cocycle $A$ has a dominated splitting with index 1 on $Y := \text{supp}(\mu)$ (the support of $\mu$). By the robustness of dominated splittings (see [13, Corollary 2.8]), there exists a closed neighborhood $U$ of $\text{supp}(\mu)$ such that if $Z := \bigcap_{k \in \mathbb{Z}} T^{-k}(U) \supseteq \text{supp}(\mu)$ is the maximal invariant set in this neighborhood. Then, the restricted bundle $E_Z$ over the compact invariant set $Z$ admits a dominated splitting $\mathbb{F} \oplus \mathcal{G}$ such that $\dim(\mathbb{F}) = 1$, extending the previously found dominated
LYAPUNOV EXPONENTS AND NON-UNIFORMLY EXPANDING

splitting on $E_Y$. Recall that the bundles of a dominated splitting are Hölder continuous. Therefore, the cocycle $A$ has dominated splitting with index 1 on $Z$.

We choose $\delta$ small enough such that $B_\delta(\mu) \subset Z$ (see Remark 2.1). Note that the basin of $\delta$-attraction of $\mu$, $B_\delta(\mu)$, has positive Lebesgue measure by the definition of the observable measure.

Dominated splitting can be characterized in terms of existence of invariant cone fields. Therefore, there is a family of cone fields $(C_x)_{x \in Z}$ such that $A(x)(C_x) \subset C^0_{T(x)}$.

By Proposition 3.4, the potential $\Phi_A$ is almost additive on $Z$. Hence, $\mu \to \chi(\mu, \Phi_A)$ is continuous on $\mathcal{M}(Z, T|Z)$ by Lemma 3.2.

Let $y \in B_\delta(\mu) \subset Z \cap L \cap G$. There is a subsequence of integers $\{n_i\}$ such that $\frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{T^j(y)}$ converges to a measure $\nu$ with $d(\mu, \nu) < \delta$ (distance in weak* topology) by definition.

Now, we prove the other side of the equality (4.4).

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A^n(y)\| \geq \limsup_{i \to \infty} \frac{1}{n_i} \log \|A^{n_i}(y)\| = \limsup_{i \to \infty} \frac{1}{n_i} \int \log \|A^{n_i}(y)\| d\delta_y. \tag{1}
\]

Since $\frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{T^j(y)} \to \nu$ (in weak* topology) and $\Phi_A$ is almost additive on $Z$, by Theorem 3.1,

\[
\tag{2} \chi(\nu, \Phi_A) = \chi(\mu, \Phi_A). \tag{2}
\]

By continuity (Lemma 3.2), for given $\epsilon > 0$,

\[
\tag{2} \geq \chi(\mu, \Phi_A) - \epsilon.
\]

That implies

\[
\sup_{x \in L} \limsup_{n \to \infty} \frac{1}{n} \log \|A^n(x)\| \geq \chi(\mu, \Phi_A) - \epsilon.
\]

As we consider arbitrary $L$ and $\epsilon$,

\[
\text{ess sup lim sup} \frac{1}{n} \log \|A^n(x)\| \geq \sup \{\chi(\mu, \Phi_A) | \mu \in \mathcal{O}\}. \tag{4.8}
\]

Combining (4.8) and Lemma 4.5 completes the proof.

\[\square\]

**Corollary 4.7.** Under conditions of Theorem 4.3, we assume that the set of observable measures $\mathcal{O}$ is at most countable, then we have

\[
\text{ess sup lim sup} \frac{1}{n} \log \|A^n(x)\| = \sup \{\chi(\mu, \Phi_A) | \mu \in \mathcal{O}\} = \sup \{\chi(\mu, \Phi_A) | \mu \in \text{Phy}\}.
\]
Proof. It is enough to prove that
\[ \sup \{ \chi(\mu, \Phi_A) \mid \mu \in \mathcal{O} \} = \sup \{ \chi(\mu, \Phi_A) \mid \mu \in \text{Phy} \}, \]
by Theorem 4.3.

If the set of observable measures \( \mathcal{O} \) is finite, then \( \mathcal{O} = \text{Phy} \) by Theorem 2.3, and if \( \mathcal{O} \) is countable infinite, then \( \mathcal{O} \) is the weak*closure of the set of Phy measures by Theorem 2.4. In both cases, the equality is clear. \( \square \)

Remark 4.8. The physical measures may not exist, if the set \( \mathcal{O} \) is uncountable. For instance, identity map on a manifold has observable measures but has no physical measures. Hence, the above result fails in general when the set \( \mathcal{O} \) is uncountable.

5. Examples

In the following section, we will present simple examples that highlight the concept of observable measures.

Example 5.1. We consider derivative cocycles as follows. Let \( f : M \to M \) be a \( C^1 \)-diffeomorphism on a smooth compact Riemannian manifold \( M \) that admits a dominated splitting \( TM = E^{cs} \oplus E^{cu} \). Let \( \mu \) be an ergodic Lyapunov maximal observable measures for a subadditive potential \( \{ \log \| Df^{-n}_{|E^{cu}(f^n)} \| \}_{n=1}^{\infty} \). Assume that the cocycle \( Df^{-1}_{|E^{cu}(f)} \) satisfies the uniform 1-gap property with respect to \( \mu \), i.e., there are a \( C^0 \)-neighborhood \( U_{Df^{-1}_{|E^{cu}(f)}} \) of \( Df^{-1}_{|E^{cu}(f)} \) \(^2\) and \( \beta > 0 \) such that for every \( B \in U_{Df^{-1}_{|E^{cu}(f)}} \) we have
\[
\int (\chi_1(x, B) - \chi_2(x, B)) \, d\mu(x) > \beta.
\]

Then, by the main result of [28], the cocycle \( Df^{-1}_{|E^{cu}(f)} \) has a dominated splitting with index 1 on the support of the ergodic Lyapunov maximal observable measures \( \mu \).

Example 5.2. By Theorem 2.3, the unique SRB measure \( \mu \) is the unique observable measure for any transitive \( C^{1+\alpha} \) Anosov diffeomorphism.

Example 5.3. We are going to show that the inequality in the formula (4.4) could be strict. Consider Example 5.2, there are infinitely many other ergodic measures, which are not observable (e.g., those supported on the periodic orbits). Hence, by [19, Theorem 1], one can find a continuous function whose integral with respect to some ergodic measure is larger than its integral with respect to any observable measure \( \mu \).

Example 5.4. Let \( g : T^2 \to T^2 \) be a transitive \( C^2 \) Anosov diffeomorphism. By Sinai’s theorem, there exists a \( g \)-ergodic physical measure \( \mu \) on the two-torus, which is a SRB measure for \( g \). Denote by \( f : T^3 \to T^3 \) the \( C^2 \) map on the three-dimensional torus \( T^3 \), defined by \( f(x, y, z) = (x, g(y, z)) \). For Lebesgue almost all initial states \( (x, y, z) \in T^3 \), the sequence (1.3) of the empirical probabilities converges to a measure \( \mu_x = \delta_x \times \mu \), which

\(^2\)Note that in the preceding definition, the base \( f \) and the measure \( \mu \) remain unaffected by the choice of \( B \in U_{Df^{-1}_{|E^{cu}(f)}} \).
is supported on a 1-dimensional unstable manifold injectively immersed in the two-torus \( \{x\} \times \mathbb{T}^2 \). Measures \( \mu_x \) are mutually singular for different values of \( x \in S^1 \) as they are supported on disjoint compact two-tori embedded on \( \mathbb{T}^2 \). The basin of attraction \( B(\mu_x) \) has zero Lebesgue measure in the ambient manifold, therefore, none of the probabilities \( \mu_x \) is physical for \( f \). Moreover, the set of all those measures \( \mu_x \), which is weak* compact, coincides with the set \( \mathcal{O} \) of observable measures for \( f \) by Theorem 2.2.

6. Proof of Theorem 1.2

Tian [27] showed that the negative Lyapunov exponents of all observable measures imply non-uniform expansion.

**Theorem 6.1** ([27, Proposition A]). Let \( T : X \to X \) be a continuous map on a compact manifold \( X \). Assume that \( a_n : X \to \mathbb{R} \) and \( b_n : X \to \mathbb{R}, \ n \geq 0 \), are two sequences of continuous functions such that for every \( x \in X, n, k \geq 0 \),
\[
(6.1) \quad a_{n+k}(x) \leq a_n(T^k(x)) + a_k(x), \quad b_{n+k}(x) \leq b_n(T^k(x)) + b_k(x);
\]
and
\[
(6.2) \quad a_n(x) \leq a_{n+k}(x) + b_k(T^n(x)).
\]
If
\[
\lim_{n \to \infty} \frac{1}{n} \int a_n(x) d\mu(x) < 0
\]
for every \( \mu \in \mathcal{O} \), then there are \( \lambda > 0 \) and \( K \in \mathbb{N} \) such that Lebesgue almost every \( x \) satisfies
\[
\limsup_{n \to \infty} \frac{1}{nK} \sum_{i=0}^{n-1} a_K(T^{iK}(x)) \leq -\lambda < 0.
\]

**Proof of Theorem 1.2.** Note that condition (1.5) of Theorem 1.2 is equivalent to
\[
(6.3) \quad \text{ess sup} \limsup_{n \to \infty} \frac{1}{n} \log \| Df^{-n}_{|E_{cu}(f^n(x))} \| < 0.
\]
On the other hand,
\[
\text{ess sup} \limsup_{n \to \infty} \frac{1}{n} \log \| Df^{-n}_{|E_{cu}(f^n(x))} \| = \max\{\lambda^-(\mu, f) \mid \mu \in \mathcal{O}\},
\]
by Theorem 4.3. Therefore, by (6.3),
\[
(6.4) \quad \lambda^-(\mu, f) < 0
\]
for any \( \mu \in \mathcal{O} \).

We denote \( a_n(x) := \log \| Df^{-n}_{|E_{cu}(f^n(x))} \| \) and \( b_n(x) := \log \| Df^n_{|E_{cu}(x)} \| \). They satisfy conditions (6.1) and (6.2) of Theorem 6.1. Moreover, Lyapunov exponents of all observable measures are negative (see (6.4)). Then there are \( \lambda > 0 \) and \( K \in \mathbb{N} \) such that Lebesgue a.e. \( x \) satisfies
\[
\limsup_{n \to +\infty} \frac{1}{nK} \sum_{i=1}^{n} \log \| Df^{-K}_{|E^n(f^{iK}(x))} \| \leq -\lambda < 0.
\]
Let \( f : M \to M \) be a \( C^1 \) diffeomorphism on a compact Riemannian manifold \( M \). Let \( A := Df \). We say that a Borel invariant measure \( \mu \) is hyperbolic if for \( \mu \) a.e. \( x \), \( \chi_i(x, A) \neq 0 \) and \( \chi_i(x, A) < \chi_{k(x)}(x, A) \), which \( k(x) \leq \dim M \), that is
\[
\chi_{k(x)}(x, A) \leq \ldots \leq \chi_{t+1}(A) < 0 < \chi_t(x, A) \leq \ldots \leq \chi_1(x, A).
\]

We say that a Borel invariant measure \( \mu \) is SRB (Sinai–Ruelle–Bowen) measure if \( \mu \) is hyperbolic and admitting a system of conditional measures such that the conditional measures on unstable manifolds are absolutely continuous with respect to the Lebesgue measures \( \text{Leb} \) on these manifolds induced by the restriction of the Riemannian structure (see [16, 24, 29]). We denote by \( \text{SRB} \) the set of SRB measures. Recently, [22] proved that a partially hyperbolic attractor for a \( C^1 \) vector field with two dimensional center supports an SRB measure.

**Corollary 6.2.** Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism on a compact Riemannian manifold that admits a dominated splitting \( TM = E^s \oplus E^{cu} \), where \( E^s \) is uniformly contracting. Assume that \( \nu \) is an ergodic observable measure such that
\[
(6.5) \quad \lambda^{-}(\nu, f) = \sup_{\mu \in \mathcal{O}_f} \lambda^{-}(\mu, f) = \max_{\mu \in \mathcal{O}_f} \lambda^{-}(\mu, f),
\]
and that \( Df^{-1}_{|E^{cu}(f^n(x))} \) has a dominated splitting with index 1 on the support of \( \nu \). Then
\[
\text{ess sup} \lim sup_{n \to \infty} \frac{1}{n} \log \left\| Df^{-n}_{|E^{cu}(f^n(x))} \right\| = \sup_{\mu \in \mathcal{O}_f} \lambda^{-}(\mu, f) = \sup_{\mu \in \text{Phy}} \lambda^{-}(\mu, f) = \sup_{\mu \in \text{SRB}} \lambda^{-}(\mu, f),
\]
where the essential supremum taken against \( \text{Leb} \).

**Proof.** We are going to show that \( \mathcal{O} = \text{Phy} = \text{SRB} \). By Theorem 1.4, there exist finite ergodic SRB measures and the union of their basins forms a set with Lebesgue full measure; denote them by \( \mu_1, \ldots, \mu_k \). We define define \( \nu_i (B) = \frac{1}{K} (\mu_i (B) + \mu_i (f^{-1}(B)) + \ldots + \mu_i (f^{-K-1}(B)) \) for any Borel measurable set \( B \), then \( \nu_i \) is ergodic, and SRB for \( f \) that form a compact subset of \( \mathcal{O} \) and the union of the basins of \( \nu_i \) forms a set with Lebesgue full measure. \( \mathcal{O} \) coincides with these finite \( \nu_i \), by Theorem 2.2. Then the result follows from Corollary 4.7.

We consider the largest Lyapunov exponent in the \( E^{cs} \)-direction
\[
\lambda^{cs}(x) := \lim sup_{n \to \infty} \frac{1}{n} \log \| Df^n_{|E^{cs}(x)}(x) \|.
\]

\(^3K \) appears in Theorem A.
We say that $E^{cs}$ is mostly contracting if for any local unstable manifold $\gamma^u$, then we have $\lambda^{cs}(x) < 0$ for a positive Leb$^u$ measure set of points $x \in \gamma^u$.

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold that admits a dominated splitting $TM = E^{cs} \oplus E^{u}$ with $E^{cs}$ mostly contracting. Bonatti and Viana [5] showed that $f$ has finitely many ergodic physical SRB measures and the union of their basins has full Lebesgue measure. Therefore, one can use their result to prove Corollary 6.2 for the above case.

References

[1] Alves, J., Bonatti, C., and Viana, M. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones Mathematicae* 140, 2 (2000), 351–398.

[2] Alves, J., Dias, L., Luzzatto, S., and Pinheiro, V. SRB measures for partially hyperbolic systems whose central direction is weakly expanding. *Journal of the European Mathematical Society* 19, 10 (2017), 2911–2946.

[3] Bochi, J. Ergodic optimization of Birkhoff averages and Lyapunov exponents. *Proceedings of the ICM, 2018* 2 (2018), 1821–1842.

[4] Bochi, J., and Gourmelon, N. Some characterizations of domination. *Mathematische Zeitschrift* 263, 1 (2009), 221–231.

[5] Bonatti, C., and Viana, M. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel Journal of Mathematics* 115 (2000), 157–193.

[6] Burguet, D. Entropy of physical measures for $C^\infty$ dynamical systems. *Communications in Mathematical Physics* 2, 357 (2020), 1201–1222.

[7] Burguet, D. SRB measures for partially hyperbolic systems with one-dimensional center subbundles. *arxiv.org/abs/2403.11654* (2024).

[8] Burns, K., Dolgopyat, D., Pesin, Y., and Pollicott, M. Stable ergodicity for partially hyperbolic attractors with negative central exponents. *Journal of Modern Dynamics* 1, 2 (2008), 63–81.

[9] Cao, Y., Mi, Z., and Yang, D. On the abundance of Sinai-Ruelle-Bowen measures. *Communications in Mathematical Physics* 391, 3 (2022), 1271–1306.

[10] Catsigeras, E., and Enrich, H. SRB-like measures for $C^0$ dynamics. *Bulletin of the Polish Academy of Sciences* 59 (2011), 151–164.

[11] Climenhaga, V., Dolgopyat, D., and Pesin, Y. Stationary non-uniform hyperbolicity; SRB measures for dissipative maps. *Communications in Mathematical Physics* 346 (2016), 553–602.

[12] Climenhaga, V., Luzzatto, S., and Pesin, Y. The geometric approach for constructing Sinai–Ruelle–Bowen measures. *Journal of Statistical Physics* 166 (2017), 467–493.

[13] Crovisier, S., and Potrie, R. Introduction to partially hyperbolic dynamics. Notes. *ICTP* (2015).

[14] Crovisier, S., Sambarino, M., and Yang, D. Partial hyperbolicity and homoclinic tangencies. *Journal of the European Mathematical Society* 017, 1 (2015), 1–49.

[15] Crovisier, S., Yang, D., and Zhang, J. Empirical measures of partially hyperbolic attractors. *Communications in Mathematical Physics* 375 (2020), 725–764.

[16] Diaz, C. B. L., and Viana, M. Dynamics beyond uniform hyperbolicity. *Springer, Berlin* (2005).

[17] Feng, D., and Huang, W. Lyapunov spectrum of asymptotically sub-additive potentials. *Communications in Mathematical Physics* 297, 1 (2010), 1–43.

[18] Hua, Y., Yang, F., and Yang, J. A new criterion of physical measures for partially hyperbolic diffeomorphisms. *Transactions of the American Mathematical Society* 373, 1 (2020), 385–417.

[19] Jenkinson, O. Every ergodic measure is uniquely maximizing. *Discrete and Continuous Dynamical Systems* 16, 2 (2006), 383–392.

[20] Jenkinson, O. Ergodic optimization in dynamical systems. *Ergodic Theory and Dynamical Systems* 39, 10 (2019), 2593–2618.
21 Kozlovski, O. An integral formula for topological entropy of $C^\infty$ maps. *Ergodic Theory and Dynamical Systems* 18, 2 (1998), 405–424.
22 Mi, Z., You, B., and Zang, Y. SRB measures for partially hyperbolic flows with mostly expanding center. *Journal of Dynamics and Differential Equations* (2022).
23 Mohammadpour, R. Lyapunov spectrum properties and continuity of the lower joint spectral radius. *Journal of Statistical Physics* 187, 3 (2022), Paper No. 23, 29.
24 Pesin, Y. Characteristic Lyapunov exponents, and smooth ergodic theory. *Russian Mathematical Surveys* 32, 4 (1977), 55–112.
25 Pesin, Y., and Sinai, Y. G. Gibbs measures for partially hyperbolic attractors. *Ergodic Theory and Dynamical Systems* 2, 3-4 (1980), 417–438.
26 Shub, M. Global Stability of Dynamical Systems. *Springer, Berlin* (1987).
27 Tian, X. When a physical-like measure is physical or SRB? *Journal of Differential Equations* 256, 8 (2018), 3567–3586.
28 Yemini, B. Uniform gap in Lyapunov exponents and dominated splitting for linear cocycles. https://arxiv.org/abs/2110.11301 (2022).
29 Young, L.-S. What are SRB measures, and which dynamical systems have them? *Journal of Statistical Physics* 108 (2002), 733–754.
30 Zhao, Y. Maximal integral over observable measures. *Acta Mathematica Sinica* 32 (2016), 571–578.

Department of Mathematics, Uppsala University, Box 480, SE-75106, Uppsala, Sweden.
Email address: reza.mohammadpour@math.uu.se