Intransitivity in multiple solutions of Kemeny Ranking Problem

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Abstract – Kemeny rule is one of deeply justified ways to solve the problem allowing to find such a linear order (Kemeny ranking) of alternatives that a distance from it to the initial rankings (input preference profile) is minimal. The approach can give considerably more than one optimal solutions. The multiple solutions (output profile) can involve intransitivity of the input profile. Favorable obstacle in dealing with intransitive output profile is that the intransitive cycles are lexicographically ordered what can help when algorithmically revealing them.

1. Introduction
In this paper we deal with multidimensional ordinal measurements. Their results are consensus preference relations being discovered for given m rankings, possibly including ties, of n alternatives. A classical problem of single consensus ranking determination for m voters of n candidates has been intensively investigated as a Voting Problem in the framework of Social Choice Theory, see, e.g. [1-3]. Modern control and measurement systems frequently investigate complicated objects characterized by multiple heterogeneous properties using multisensors and distributed networking. This is just the domain where the problem could be affectively applied; see, for example [4-6]. Also promising is a development of measurement systems based on mobile agents, collective purposeful work of which could be organized using consensus ranking determination. We use here a widely known and deeply justified distance-based approach to determination of the consensus relation, so called Kemeny rule. However, it may result in not single and considerably greater amount of consensus relations. A set of the relations may include intransitivities. Although (in)transitivity of preferences is a subject of many papers (e.g. [7,8]), non-transitive Kemeny rankings were ignored by researchers. Aim of this paper is to demonstrate examples of this phenomenon and draw up some ways how to deal with it.

2. Definitions
To find the consensus ranking we use the Kemeny rule [9] that is shortly described as follows. Let us have m rankings on set \( A = \{a_1, a_2, ..., a_n\} \) of n alternatives and the relation set \( \Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\} \), where each of m rankings (also called preference relations or weak orders) \( \lambda = \{a_1 \succ a_2 \succ ... \succ a_r \sim a_t \succ ... \succ a_n\} \) may include \( \succ \), a strict preference relation \( \pi \), and \( \sim \), an equivalence relation (or tie) \( \nu \), so that \( \lambda = \pi \cup \nu \). The relation set \( \Lambda \) we will call an input preference profile for the given m rankings. Let \( \Pi \) be a set of all n! linear (strict) order relations \( \succ \) on \( A \). Each linear order corresponds to one of permutations of first n natural numbers \( N_n \).

Kemeny rule allows to find a consensus relation as such a linear order (Kemeny ranking) \( \beta \in \Pi \) of alternatives that a distance \( D(\beta, \Lambda) \) from \( \beta \) to the profile \( \Lambda \) is minimal, that is

\[
\beta = \arg \min_{\lambda \in \Pi} D(\lambda, \Lambda).
\]
The problem (1) is called Kemeny Ranking Problem (KRP).

The distance \( D(\lambda, \Lambda) \) between arbitrary ranking \( \lambda \) and profile \( \Lambda \) is defined as

\[
D(\lambda, \Lambda) = \sum_{i<j} \sum_{k=1}^{m} d_{ij}^k,
\]

where

\[
d_{ij}^k = \begin{cases}
0 & \text{if } a_i^k > a_j^k \\
1 & \text{if } a_i^k = a_j^k, \quad i, j = 1, \ldots, n. \\
2 & \text{if } a_i^k < a_j^k
\end{cases}
\]

(2)

To describe all rankings of a preference profile we use the profile matrix \( P \), rows and columns of which are labeled by the alternatives' numbers, where

\[
p_{ij} = \sum_{k=1}^{m} d_{ij}^k, \quad i, j = 1, \ldots, n.
\]

Then the KRP statement (1) is changed to

\[
\beta = \arg \min_{i<j} \sum_{i<j} p_{ij},
\]

(4)

that means the determination of such a transposition of profile matrix rows and columns that the sum of elements of its upper triangle submatrix is minimal.

2. Profile intransitivity

Though each ranking of the preference profile is transitive, the corresponding matrix \( P \) (and, hence, the preference profile) may be both transitive, i.e. \( p_{i\beta} \leq p_{j\beta} \) if \( p_{ij} \leq p_{j\beta} \) and \( p_{ij} \leq p_{i\beta} \), for all \( i \neq j \neq k = 1, \ldots, n \), and non-transitive if the condition fails. Well known Condorcet paradox (Condorcet rule: if some alternative obtains a majority of votes in pair-wise contests against every other alternative, the alternative is chosen as the winner in the consensus ranking \( \beta \); however, for some \( \beta \) and \( a_i, a_j, a_k \in \beta \), it can be that \( a_i \succ a_j \) and \( a_j \succ a_k \) while \( a_k \succ a_i \) just takes place if the given preference profile is non-transitive.

The following profile

\[
\lambda_1: \ a_1 \succ a_2 \succ a_3
\]

\[
\lambda_2: \ a_2 \succ a_1 \succ a_3
\]

\[
\lambda_3: \ a_3 \succ a_1 \succ a_2
\]

includes an evident circular ambiguity (shortly, cycle) that leads to the Condorcet paradox if we would apply the Condorcet rule to determine a consensus ranking. In this case a Condorcet winner does not exist. The Kemeny rule being applied in this case gives three optimal solutions (with \( D(\beta, \Lambda) = 8 \)) coinciding with rankings of the initial profile.

Thus, in this case, the initial profile intransitivity seems to produce multiple solutions in the Kemeny rules. Reasonable way to resolve the situation is to declare all three alternatives to be equivalent to each other, that is the final solution is \( a_1 \sim a_2 \sim a_3 \). In fact, this weak order is in minimal feasible distance, equal to 9, from all the rankings (5).

Matrix \( P \) can be characterized by a least distance \( D_{\text{least}} \) from its preference profile \( \Lambda \) to some linear order \([5,10]\). To find the distance a lesser element of each pair \( (p_{ij}, p_{ji}) \) should be included into, that is

\[
D_{\text{least}} = \sum_{i<j} \min(p_{ij}, p_{ji}), \quad i, j = 1, \ldots, n.
\]

(6)

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It can be easily shown that if matrix $P$ is transitive (this means all initial rankings are consistent), then $D_{\text{least}} = D(\beta, \Lambda)$ and $D_{\text{least}}$ is an accessible value. An inverse proposition is also valid. Thus, for any input profile, after determination of $\beta$ by Kemeny rule (4), its transitivity can be determined using the following criterion:

$$ \text{matrix } P \text{ is } \begin{cases} \text{transitive if } D_{\text{least}} = D(\beta, \Lambda) \\ \text{intransitive if } D_{\text{least}} < D(\beta, \Lambda) \end{cases} \quad (7) $$

3. Monte-Carlo investigation of input profile intransitivity

We have experimentally investigated intransitivity of input profiles using our branch-and-bound (B&B) algorithm described in [5] to solve the Kemeny Ranking Problem. We used Monte-Carlo approach in order to simulate profile matrices calculated for weak orders that were obtained by uniting pseudo-random strict orders and ties generated separately. Strict orders were generated using the uniform distribution of integers in a specified range $1, \ldots, n$.

For each combination $(m, n)$, where number of rankings $m = 4, 5, 6, \text{ and } 15$ and number of ranking elements $n = 10, 15 \text{ and } 20$, 100 individual profiles (problems) were generated (solved). Thus, totally 1500 profiles were generated each served as input of the B&B algorithm.

![Figure 1. Distances $D_{\text{least}}$ and $D(\beta, \Lambda)$ (vertical axis) for each of 100 individual problems (horizontal axis) for $m = 4, 5, 6 \text{ and } 15$ and candidate numbers $n = 10, 15 \text{ and } 20$; $D(\beta, \Lambda)$ curves are shown with boxes, $D_{\text{least}}$ with diamonds.](image-url)
Figure 1 shows curves of distances $D_{\text{least}}$ and $D(\beta, \Lambda)$ plotted by their values obtained for each of 100 individual problems and sorted by distance $D(\beta, \Lambda)$ in descending order.

One can see from figure 1 that, for the given $n$ and $m$, profiles were frequently transitive (that is the condition $D_{\text{least}} = D(\beta, \Lambda)$ was valid) only for even values of $m$. Odd $m$ have practically always resulted in intransitive input profiles. In all cases amount of transitive profiles decreases as the number $n$ of alternatives increases.

Generally, the Kemeny rule produces a number $N_{\text{kem}}$ of optimal solutions which shape output profile so that there can be $N_{\text{kem}} \gg m$. Multiple solutions of the KRP is not a seldom phenomenon. Computing experiments [5,10-12] show that a probability of single KRP solution is in close agreement with the probability of Condorcet winner existence [13-15]. The latter decreases as the values of $n$ increase. The same can be said of $m$, however, evenness of $m$ seriously decreases chances of Condorcet winner comparatively to odd values of $m$ [13].

Using the obtained experimental data presented in figure 1 it is possible to estimate probabilities of intransitive input profiles $P_{\text{intr}}(m, n)$ for fixed $m$ and $n$, dividing a number of cases where $D_{\text{least}} \neq D(\beta, \Lambda)$ by 100. In this way the data in table 1 were calculated; they, in turn, were used to the curves in figure 2.

### Table 1
Experimental estimations of probability $P_{\text{intr}}(m, n)$ of intransitive profile.

| $m$ | $n$  | $m$  | $n$  | $m$  | $n$  |
|-----|------|------|------|------|------|
|     | 10   | 15   | 20   |
| 4   | 0.06 | 0.21 | 0.53 |
| 5   | 0.96 | 1    | 1    |
| 6   | 0.33 | 0.7  | 0.94 |
| 15  | 0.97 | 1    | 1    |

Figure 2. Experimental curves of probability $P_{\text{intr}}(m, n)$ of intransitive input profile.
One can see from figure 2 that probability $P_{\text{in}}(m, n)$ of intransitive input profile is close to one for odd values of $m$ in the range $10 \leq n \leq 20$. If $m$ is even, the input profile can rather frequently be transitive (or intransitive) in dependence on particular values of $m$ and $n$. On the whole, the probability increases as both the number $n$ of alternatives and the number $m$ of rankings increase. This observation is in satisfactory agreement with findings presented in papers [13-15].

4. Multiple solutions and their intransitivity

Necessary condition of intransitivity presence in the KRP multiple solutions is $N_{\text{kem}} \geq 3$. Also, if, in accordance with criterion (7), an initial profile is intransitive, an output profile is intransitive too but the cycles are lexicographically ordered and can be algorithmically revealed. To illustrate this let us consider two short examples.

In table 2 one can see that the initial profile is intransitive, the output profile consisting of three solutions has one cycle (shown by bold frame) attesting intransitivity. Following to approach described in Section 2 we have the final solution $a_1 \succ a_2 \sim a_4 \sim a_5 \succ a_3$.

| Input profile | Output profile |
|---------------|---------------|
| 1 4 5 2 ~3    | 1 2 4 5 3     |
| 3 2 1 4 5     | 1 4 5 2 3     |
| 1 5 4 2 3     | 1 5 2 4 3     |
| 1 ~5 2 3 4    |               |
| 2 4 1 5 3     |               |

$D_{\text{least}} = 30 \quad D(\beta, \Lambda) = 32 \quad N_{\text{kem}} = 3$

An example of output profile having four cycles reduced in table 3. In this case the final optimal solution could be $a_1 \sim a_6 \sim a_9 \succ a_2 \sim a_{10} \succ a_8 \succ a_4 \succ a_5 \sim a_7 \sim a_3$.

| Cycle 1 | Cycle 2 | Cycle 1 | Cycle 2 | Cycle 1 |
|---------|---------|---------|---------|---------|
| 1 6 9 2 10 8 4 3 7 5 | 1 9 2 10 8 4 5 3 7 | 1 6 9 10 2 8 4 3 7 5 | 9 6 1 10 2 8 4 5 3 7 | 9 1 6 10 2 8 4 7 5 3 |

$D_{\text{least}} = 129 \quad D(\beta, \Lambda) = 131 \quad N_{\text{kem}} = 6$
5. Conclusion
Output profile of the Kemeny Ranking Problem can involve intransitivity of an input preference profile. Computing experiment had shown that most likely intransitivities of input profiles are in case of odd values of $m$ in the range $10 \leq n \leq 20$. Favorable obstacle in dealing with intransitive output profile is that the intransitive cycles are lexicographically ordered and can be algorithmically revealed.

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