Eigenvalue spectrum of the spheroidal harmonics: a uniform asymptotic analysis

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The spheroidal harmonics $S_{lm}(\theta; c)$ have attracted the attention of both physicists and mathematicians over the years. These special functions play a central role in the mathematical description of diverse physical phenomena, including black-hole perturbation theory and wave scattering by nonspherical objects. The asymptotic eigenvalues $\{A_{lm}(c)\}$ of these functions have been determined by many authors. However, it should be emphasized that all previous asymptotic analyzes were restricted either to the regime $m \to \infty$ with a fixed value of $c$, or to the complementary regime $|c| \to \infty$ with a fixed value of $m$. A fuller understanding of the asymptotic behavior of the eigenvalue spectrum requires an analysis which is asymptotically uniform in both $m$ and $c$. In this paper we analyze the asymptotic eigenvalue spectrum of these important functions in the double limit $m \to \infty$ and $|c| \to \infty$ with a fixed $m/c$ ratio.

I. INTRODUCTION.

The spheroidal harmonic functions $S(\theta; c)$ appear in many branches of physics. These special functions are solutions of the angular differential equation 1

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left[ c^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + A \right] S = 0 \, , \quad (1)$$

where $\theta \in [0, \pi]$, $c \in \mathbb{Z}$, and the integer parameter $m$ is the azimuthal quantum number of the wave field 1, 8, 3.

These angular functions play a key role in the mathematical description of many physical phenomena, such as: perturbation theory of rotating Kerr black holes 2, 4, 6, electromagnetic wave scattering 7, quantum-mechanical description of molecules 8, 9, communication theory 10, and nuclear physics 11.

The characteristic angular equation (1) for the spheroidal harmonic functions is supplemented by a regularity requirement for the corresponding eigenfunctions $S(\theta; c)$ at the two boundaries $\theta = 0$ and $\theta = \pi$. These boundary conditions single out a discrete set of eigenvalues $\{A_{lm}\}$ which are labeled by the discrete spheroidal harmonic index $l$ (where $l - |m| = \{0, 1, 2, \ldots\}$). For the special case $c = 0$ the spheroidal harmonic functions $S(\theta; c)$ reduce to the spherical harmonic functions $Y(\theta)$, which are characterized by the familiar eigenvalue spectrum $A_{lm} = l(l + 1)$.

The various asymptotic spectrums of the spheroidal harmonics with $c^2 \in \mathbb{R}$ (when $c \in \mathbb{R}$ the corresponding eigenfunctions are called oblate, while for $ic \in \mathbb{R}$ the eigenfunctions are called prolate) were explored by many authors, see 1, 12, 17 and references therein. In particular, in the asymptotic regime $m^2 \gg |c|^2$ the eigenvalue spectrum is given by 12, 18:

$$A_{lm} = l(l + 1) - \frac{c^2}{2} \left[ 1 - \frac{m^2}{l(l+1)} \right] + O(1) \, , \quad (2)$$

while in the opposite limit, $|c|^2 \gg m^2$ with $ic \in \mathbb{R}$, the asymptotic spectrum is given by 1, 13, 11, 7

$$A_{lm} = [2(l - m) + 1]|c| + O(1) \, . \quad (3)$$

The asymptotic regime $c^2 \gg m^2$ (with $c \in \mathbb{R}$) was studied in 1, 13, 18, where it was found that the eigenvalues are given by:

$$A_{lm} = -c^2 + 2[l + 1 - \text{mod}(l - m, 2)]|c| + O(1) \, . \quad (4)$$

Note that the spectrum (4) is doubly degenerate.

It should be emphasized that all previous asymptotic analyzes of the eigenvalue spectrum were restricted either to the regime $m \to \infty$ with a fixed value of $c$ 12, 13, or to the complementary regime $|c| \to \infty$ with a fixed value of $m$ 1, 13, 16. A complete understanding of the asymptotic eigenvalue spectrum requires an analysis which is uniform in both $m$ and $c$ (that is, a uniform asymptotic analysis which is valid for a fixed (non-negligible) $m/c$ ratio as both $m$ and $c$ tend to infinity).
The main goal of the present paper is to present a uniform asymptotic analysis for the spheroidal harmonic eigenvalues in the double asymptotic limit

$$m \to \infty \quad \text{and} \quad |c| \to \infty$$

with a fixed $m/c$ ratio.

II. A TRANSFORMATION INTO THE SCHRÖDINGER-TYPE WAVE EQUATION

For the analysis of the asymptotic eigenvalue spectrum, it is convenient to use the coordinate $x$ defined by

$$x \equiv \ln \left( \tan \left( \frac{\theta}{2} \right) \right),$$

in terms of which the angular equation (1) for the spheroidal harmonic eigenfunctions takes the form of a one-dimensional Schrödinger-like wave equation

$$\frac{d^2 S}{dx^2} - US = 0,$$

where the effective radial potential is given by

$$U(x(\theta)) = m^2 - \sin^2 \theta (c^2 \cos^2 \theta + A).$$

Note that the transformation (6) maps the interval $\theta \in [0, \pi]$ into $x \in [-\infty, \infty]$.

The effective potential $U(\theta)$ is invariant under the transformation $\theta \to \pi - \theta$. It is characterized by two qualitatively different spatial behaviors depending on the relative magnitudes of $A$ and $c^2$. We shall now study the asymptotic behaviors of the spheroidal eigenvalues in the two distinct cases: $A/c^2 > 1$ and $A/c^2 < 1$ [20].

III. THE ASYMPTOTIC EIGENVALUE SPECTRUM

A. The asymptotic regime $\{|c|, m\} \to \infty$ with $c^2 < m^2$.

If $A > c^2$ then the effective radial potential $U(x(\theta))$ is in the form of a symmetric potential well whose local minimum is located at

$$\theta_{\text{min}} = \frac{\pi}{2} \quad \text{with} \quad U(\theta_{\text{min}}) = -A + m^2.$$  

[Note that $\theta_{\text{min}} = \frac{\pi}{2}$ corresponds to $x_{\text{min}} = 0$.]

Spatial regions in which $U(x) < 0$ (the ‘classically allowed regions’) are characterized by an oscillatory behavior of the corresponding wave function $S$, whereas spatial regions in which $U(x) > 0$ are characterized by an exponentially decaying wave function (these are the ‘classically forbidden regions’). The effective radial potential $U(x)$ is characterized by two ‘classical turning points’ $\{x^-, x^+\}$ (or equivalently, $\{\theta^-, \theta^+\}$) for which $U(x) = 0$ [21].

The one-dimensional Schrödinger-like wave equation (7) is in a form that is amenable to a standard WKB analysis. In particular, a standard textbook second-order WKB approximation yields the well-known quantization condition [22–26]

$$\int_{x^-}^{x^+} dx \sqrt{-U(x)} = (N + \frac{1}{2})\pi \quad ; \quad N = \{0, 1, 2, \ldots\}$$

for the bound-state ‘energies’ (eigenvalues) of the Schrödinger-like wave equation (7), where $N$ is a non-negative integer. The characteristic WKB quantization condition (10) determines the eigenvalues $\{A\}$ of the spheroidal harmonic functions in the double limit $\{|c|, m\} \to \infty$. The relation so obtained between the angular eigenvalues and the parameters $m, c, \text{and} N$ is rather complex and involves elliptic integrals. However, if we restrict ourselves to the fundamental (low-lying) modes which have support in a small interval around the potential minimum $x_{\text{min}}$ [27], then we can use the expansion $U(x) \simeq U_{\text{min}} + \frac{1}{2} U''_{\text{min}} (x - x_{\text{min}})^2 + O((x - x_{\text{min}})^4)$ in (10) to obtain the WKB quantization condition [25]

$$\frac{|U_{\text{min}}|}{\sqrt{2U_{\text{min}}}} = N + \frac{1}{2} \quad ; \quad N = \{0, 1, 2, \ldots\}$$

(11)
where a prime denotes differentiation with respect to $x$. The subscript “min” means that the quantity is evaluated at the minimum $x_{\text{min}}$ of $U(x(\theta))$. Substituting (3) with $x_{\text{min}} = 0$ into the WKB quantization condition (11), one finds the asymptotic eigenvalue spectrum

$$A(c, m, N) = m^2 + (2N + 1) \sqrt{m^2 - c^2} + O(1) \quad N = \{0, 1, 2, \ldots\}$$

in the $N \ll \sqrt{m^2 - c^2}$ regime [27]. The resonance parameter $N = \{0, 1, 2, \ldots\}$ corresponds to $l - |m| = \{0, 1, 2, \ldots\}$, where $l$ is known as the spheroidal harmonic index.

It is worth noting that the eigenvalue spectrum (12), which was derived in the double asymptotic limit $\{|c|, m\} \to \infty$, reduces to (2) in the special case $m \gg |c|$ and reduces to (3) in the opposite special case $|c| \gg m$ with $ic \in \mathbb{R}$. The fact that our uniform eigenvalue spectrum (12) reduces to (2) and (3) in the appropriate special limits provides a consistency check for our analysis [28].

B. The asymptotic regime $\{c, m\} \to \infty$ with $c^2 > m^2$.

If $A < c^2$ then the effective radial potential $U(x(\theta))$ is in the form of a symmetric double-well potential: it has a local maximum at

$$\theta_{\text{max}} = \frac{\pi}{2} \quad \text{with} \quad U(\theta_{\text{max}}) = -A + m^2 ,$$

and two local minima at [29]

$$\theta_{\text{min}}^{\pm} = \frac{1}{2} \arccos(-A/c^2) \quad (14)$$

with

$$U(\theta_{\text{min}}^{\pm}) = -\frac{1}{4} c^2 [1 - (A/c^2)^2] - \frac{1}{2} A [1 + (A/c^2)] + m^2 .$$

Thus, the two potential wells are separated by a large potential-barrier of height

$$\Delta U \equiv U(\theta_{\text{max}}) - U(\theta_{\text{min}}^{\pm}) = \frac{1}{4} c^2 [1 - (A/c^2)^2] - \frac{1}{2} A [1 - (A/c^2)] \to \infty \quad \text{as} \quad c \to \infty .$$

The fact that the two potential wells are separated by an infinite potential-barrier in the $c \to \infty$ limit (with $c^2 > m^2$) [30] implies that the coupling between the wells (the ‘quantum tunneling’ through the potential barrier) is negligible in the $c \to \infty$ limit. The two potential wells can therefore be treated as independent of each other in the $c \to \infty$ limit [22, 31]. Thus, the two spectra of eigenvalues (which correspond to the two identical potential wells) are degenerate in the $c \to \infty$ limit [32].

Substituting (3) with $\theta_{\text{min}} = \frac{1}{2} \arccos(-A/c^2)$ into the WKB quantization condition (11), one finds the asymptotic eigenvalue spectrum

$$A(c, m, N) = -c^2 + 2\left[m + (2N + 1) \sqrt{1 - m/c} \right] c + O(1) \quad N = \{0, 1, 2, \ldots\}$$

in the $N \ll m \sqrt{1 - m/c}$ regime [33]. We recall that the spectrum (17) is doubly degenerate in the $c \to \infty$ regime [34]; each value of $N$ corresponds to two adjacent values of the spheroidal harmonic index $l$: $N = \frac{1}{2}[l - m - \text{mod}(l - m, 2)]$ [35].

It is worth noting that the eigenvalue spectrum (17), which was derived in the double asymptotic limit $\{|c|, m\} \to \infty$, reduces to (4) in the special case $c^2 \gg m^2$. The fact that our uniform eigenvalue spectrum (17) reduces to (4) in the appropriate special limit provides a consistency check for our analysis [36].

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[1] C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, 1957).
Note that in the quantum-mechanical terminology the dimensionless parameter $c$ stands for $\alpha\omega$, where $\alpha$ is the specific angular momentum (angular momentum per unit mass) of the spinning black hole and $\omega$ is the conserved frequency of a scalar perturbation mode $\xi$.

In the context of black-hole perturbation theory, the dimensionless parameter $c$ is composed of two identical potential wells which, in the limit with $c \rightarrow \infty$, are separated by an energy gap that is negligible in the $c \rightarrow \infty$ limit.

We shall assume without loss of generality that $Re \geq 0$, $3c \geq 0$, and $m \geq 0$. Note that the angular differential equation (I) is invariant under the transformations $c \rightarrow -c$ and $m \rightarrow -m$. Thus, the eigenvalues are also invariant under these transformations.

Note that in the quantum-mechanical terminology $-U$ stands for $2\omega (E - V)$, where $E, V,$ and $m$ are the total energy, potential energy, and mass of the of the particle, respectively.

Below we shall show that these two cases correspond to $c^2 < m^2$ and $c^2 > m^2$, respectively.

Note that these turning points are characterized by the relation $\theta^* < \theta_{\text{min}} < \theta^+$.

Substituting our final formula [see Eq. (12) below] into the effective potential (8), one finds that the potential barrier (16) is given by $\Delta U = (c - m)^2 + O(c)$. Thus, $\Delta U \rightarrow \infty$ in the $c \rightarrow \infty$ limit with $c^2 > m^2$.

More precisely, the coupling between the two potential wells (due to the weak ‘quantum tunneling’ through the large potential barrier) introduces a small correction term of order $\exp[-\int_{\beta_2^-}^{\beta_2^+} d\theta \sqrt{\Delta U(\theta)}]$ to the r.h.s of the WKB quantization condition (10) [22, 21], where $\theta_2^-$ and $\theta_2^+$ are the inner turning points of the effective potential barrier. This term is of the order of $e^{-\sqrt{\Delta U}} \sim e^{-(c - m)^2 + O(c)}$, which is negligible in the $c \rightarrow \infty$ limit with $c > m$ [22, 31].

Substituting our final formula (17) into the effective potential (8), one finds that the turning points are located at $x^\pm - x_{\text{min}} \simeq \pm \sqrt{\frac{N + 1}{m \sqrt{1 - \frac{c^2}{m^2}}} + c^2}$. Thus, the assumption $|x^\pm - x_{\text{min}}| \ll 1$ is valid in the $N \ll \sqrt{m^2 - c^2}$ regime.

It is worth noting that our analytical formula (12) agrees with the numerical results of (8) for the case $l = m = 100$ with $c = 100i$ with a remarkable accuracy of $3.68 \times 10^{-3} \%$. (Note that $c \rightarrow ic$ in the notations of (1)).

Note the symmetry relation $\theta_{\text{min}} = \pi - \theta_{\text{min}}$.

Note that our final formula [see Eq. (17) below] into the effective potential (8), one finds that the potential barrier (10) is given by $\Delta U = (c - m)^2 + O(c)$. Thus, $\Delta U \rightarrow \infty$ in the $c \rightarrow \infty$ limit with $c^2 > m^2$.

C. S. Park, M. G. Jeong, S. K. Yoo, and D. K. Park, Phys. Rev. A 58, 3443 (1998); Z. Cao, Q. Liu, Q. Shen, X. Dou, Y. Chen, and Y. Ozaki, Phys. Rev. A 63, 054103 (2001); F. Zhou, Z. Cao, and Q. Shen, Phys. Rev. A 67, 062112 (2003).

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As discussed above, this double degeneracy of the asymptotic eigenvalue spectrum reflects the fact that the effective potential (8) with $c^2 > m^2$ is composed of two identical potential wells which, in the $c \rightarrow \infty$ limit, are separated by an infinite potential-barrier.
Thus, $l - |m| = \{0, 1, 2, 3, 4, 5, \ldots \}$ correspond to $N = \{0, 0, 1, 1, 2, 2, \ldots \}$.

It is worth noting that our analytical formula (17) agrees with the numerical results of [9] for the case $l = m = 100$ with $c = 100$ with a remarkable accuracy of $0.22\%$. (Note that $c \rightarrow ic$ in the notations of [9].)