ON THE QUANTIZATION OF THE CHERN–SIMONS FIELD THEORY ON CURVED SPACE-TIMES: THE COULOMB GAUGE APPROACH

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ABSTRACT

We consider here the Chern-Simons field theory with gauge group SU(N) in the presence of a gravitational background that describes a two-dimensional expanding "universe". Two special cases are treated here in detail: the spatially flat Robertson-Walker space-time and the conformally static space-times having a general closed and orientable Riemann surface as spatial section. The propagator and the vertices are explicitly computed at the lowest order in perturbation theory imposing the Coulomb gauge fixing.

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1. INTRODUCTION

In this paper we address the problem of the perturbative quantization of the non-abelian Chern-Simons (C-S) field theory [1] [2] on a curved space-time. The space-time considered here has the form of a three dimensional manifold $M_3$ in which the metric is induced by the following length:

$$ds^2 = dt^2 + a(t)g_{ij}(x_1, x_2) [dx_i \otimes dx_j]$$  \hspace{1cm} (1.1)

where $g_{ij}(x_1, x_2) = h(x_1, x_2)\delta_{ij}$. $h(x_1, x_2)$ is assumed to be a conformally flat metric on a Riemann surface of genus $\Sigma_g$. The resulting three dimensional metric is static, conformally flat and with Euclidean signature. The variable $t$ takes its values in the real line $\mathbb{R}$. Along the $t$ axis the metric is flat. If the time dependent factor $a(t)$ is a constant, let say $a(t) = 1$, then we obtain the well known topological configuration $M_3 = \Sigma_g \otimes \mathbb{R}$. If instead $a(t) = e^{-2\rho(t)}$, $\rho(t)$ being for instance a decreasing function, then the above metric describes an expanding two dimensional “universe”.

On $M_3$ we consider the following C-S functional:

$$S_{CS} = \frac{s}{4\pi} \int_{M_3} d^3x \epsilon^{\mu\nu\rho}{\text{Tr}} \left( A_{\mu} \partial_{\nu} A_{\rho} + \frac{2}{3} A_{\mu} A_{\nu} A_{\rho} \right)$$ \hspace{1cm} (1.2)

where $A_{\mu} = A_{\mu}^a T^a$, the $T^a$ being the generators of the compact Lie group SU(N). To evaluate the trace in eq. (1.2), we exploit the conventions $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$ and $\text{Tr}[T^a T^b T^c] = \frac{1}{4}(d^{abc} + i f^{abc})$. Here $d^{abc}$ is a totally symmetric tensor in the indices $a, b, c$ while $f^{abc}$ are the usual structure constants of SU(N). Moreover, we have set $\mu, \nu, \rho = 0, 1, 2$, with $x_0 = t$. $x_1$ and $x_2$ are local coordinates on the Riemann surface $\Sigma_g$. The indices in the spatial coordinates will be denoted by the latin letters $i, j, k$ and so on. Finally we remember that $s$ should be an integer in order to preserve the gauge invariance of the theory. A simple calculation of the trace in eq. (1.2) yields the following result:

$$S_{CS} = \frac{s}{4\pi} \int_{M_3} d^3x \epsilon^{\mu\nu\rho}{\text{Tr}} \left( A_{\mu}^a \partial_{\nu} A_{\rho}^a - \frac{1}{3} f^{abc} A_{\mu}^a A_{\nu}^b A_{\rho}^c \right)$$ \hspace{1cm} (1.3)

This action is independent of the metric. The metric will be present however in the gauge fixing action and in the Faddeev popov term.
Despite of describing a topological field theory with no physical degrees of freedom, the C-S functional (1.3) can generate non trivial correlation functions even in the flat case, for example in the covariant gauge $\partial^\mu A_\mu^a = 0$, $\mu = 0, 1, 2$ [3], [4]. Moreover, it was shown in the above references that, computing the amplitudes of $n$ Wilson loops at any order in perturbation theory, one can extract informations on the HOMFLY polynomials [3] from the C-S field theory.

Unfortunately, it is not easy to perform analogous computations in the case of a non-flat space-time. For instance, in the covariant gauge $\partial^\mu A_\mu^a = 0$, the correlation functions of the C-S field theory defined on a curved metric background are in general not known. In the temporal gauge $A_0 = 0$, instead, one has to solve explicitly the Gauss constraint that fixes the residual gauge invariance [2], [7]. In the presence of Wilson loops this becomes a difficult task due to the zero modes [3] [8] and, in fact, we believe that this problem has not been solved until now. Finally, the light-cone gauge of ref. [10] is not compatible with the transition functions of a manifold like that described by the metric (1.1).

For these reasons, we propose here a perturbative approach to the C-S field theory quantized in the Coulomb gauge. As a noncovariant gauge, the Coulomb gauge has not yet been investigated in connection with the C-S theory, as for example the temporal or light cone gauges, which have already been widely studied [11]. In this gauge, however, the quantization in a curved space-time endowed with the metric (1.1) will be drastically simplified as we will see.

The material of this paper is divided as follows. In Section 2 we quantize the C-S field theory in a spatially flat Robertson-Walker space-time. The explicit expressions of the propagators of the gauge fields and of the ghosts together with those of the vertices is given in the Coulomb gauge. In Section 3 these results are generalized to the case of a three dimensional manifold with conformal metric. The basic spatial section of the manifold is a closed and orientable Riemann surface of genus $g$. Finally, we discuss in the Conclusions some of the possible applications of the perturbative approach presented here.

2. THE CHERN-SIMONS FIELD THEORY IN THE COULOMB GAUGE: THE FLAT CASE

First of all we consider the quadratic part of the action (1.3) in order to compute the
propagators of the vector potentials $A_\mu$. We remember that throughout this Section we consider only the flat case, i.e. $g_{ij}(x_1, x_2) = \delta_{ij}$ in eq.(1.1). Firstly, we have to fix the gauge invariance of the C-S theory. A gauge transformation

$$A'_\mu = U A_\mu U^{-1} + \partial_\mu U U^{-1}$$  \hspace{1cm} (2.1)$$
is generated by the elements of SU(N):

$$U(x_1, x_2, t) = \exp \left[ \omega^a(z, \bar{z}, t) T^a \right]$$  \hspace{1cm} (2.2)$$
where the $\omega^a(x_1, x_2, t)$ represent functions on $M_3$. A convenient gauge fixing is the Coulomb gauge

$$\partial^i A_i^a = 0$$  \hspace{1cm} (2.3)$$
The coordinates $x_1$ and $x_2$ are in this case global coordinates on the two dimensional plane $\mathbb{R}^2$. In order to quantize the theory, we have to insert as usual a gauge-fixing term in the action (1.3) and the Faddeev popov ghosts:

$$S_q = S_{CS} + S_{gf} + S_{fp}$$  \hspace{1cm} (2.4)$$
where $S_{CS}$ was already defined above and

$$S_{gf} = \frac{s}{8\pi} \int_{M_3} d^3x \sqrt{g} \frac{1}{\lambda} (\partial^i A_i^a)^2$$  \hspace{1cm} (2.5)$$

$$S_{fp} = \int_{M_3} d^3x \sqrt{g} c^a \left[ \partial^i D_i^{ab}(A) \right] c^b$$  \hspace{1cm} (2.6)$$
In eq. (2.5) $\lambda$ represents an arbitrary real parameter. Moreover, $D_i^{ab}(A)$ in eq. (2.6) denotes the usual covariant derivative:

$$D_i^{ab}(A^c) = \partial_\mu \delta^{ab} - f^{abc} A^{c}_\mu$$  \hspace{1cm} (2.7)$$
and $\sqrt{g} = |det(g_{\mu\nu})|^\frac{1}{2}$. In the flat case we are treating, $\sqrt{g} = a(t)$. Finally, $c^a(z, \bar{z}, t)$ and $c^b(z, \bar{z}, t)$ represent the conjugate ghost fields. As we see from eqs. (2.5) and (2.6), the gauge fixing and the Faddeev-Popov Lagrangians depend on the metric. In our flat case, this means that there is a dependence on the function $a(t)$ defined in eq. (1.1). Of course, eq. (2.7) is only valid in a local sense, since in the case of a general manifold one needs to introduce also the Christoffel symbols in order to make it invariant under diffeomorphisms.
The expression of the propagator in the Coulomb gauge can be obtained solving the following equation:

$$\frac{s}{8\pi} \left[ \epsilon^{\mu\nu}\partial_{\nu} + (\lambda a_{x_0})^{-1} (\partial^\mu - \eta^\mu(\eta \cdot \partial)) (\partial^\nu - \eta^\nu(\eta \cdot \partial)) \right] G^{ab}_{\mu\kappa}(x-y) = \delta^a_b \delta^\mu_\kappa \delta(3)(x-y)$$

(2.8)

where \( \int d^3x \delta(x)f(x) = f(0) \). Again, this equation is valid only in the flat case. In eq. (2.8) \( \eta_\mu \) denotes the vector \( \eta_\mu = (0, 0, 1) \) in the three dimensional space \( \mathbb{R}^3 \). Solving eq. (2.8) gives the following components of the propagator:

$$G^{ab}_{ij}(x-y) = 8\pi \lambda a(x_0) \delta^{ab} \delta(x_0 - y_0) \partial_i \partial_j \Delta_2(\overline{x} - \overline{y})$$

(2.9)

$$G^{ab}_{0i}(x-y) = \frac{8\pi}{s} \delta^{ab} \left[ \delta(x_0 - y_0) \epsilon_{ij} \partial^j \Delta_1(\overline{x} - \overline{y}) - \lambda \partial_{x_0} \left( \delta(x_0 - y_0) a(x_0) \right) \partial_i \Delta_2(\overline{x} - \overline{y}) \right]$$

(2.10)

$$G^{ab}_{i0}(x-y) = \frac{8\pi}{s} \delta^{ab} \left[ \delta(x_0 - y_0) \epsilon_{ij} \partial^j \Delta_1(\overline{x} - \overline{y}) - \lambda a(x_0) \partial_{x_0} \delta(x_0 - y_0) \partial_i \Delta_2(\overline{x} - \overline{y}) \right]$$

(2.11)

$$G^{ab}_{00}(x-y) = -\frac{8\pi}{s} \lambda \delta^{ab} \partial_{x_0} \left[ a(x_0) \partial_{x_0} \delta(x_0 - y_0) \right] \Delta_2(\overline{x} - \overline{y})$$

(2.12)

Here \( \epsilon_{\alpha\beta} \) is the two dimensional \( \epsilon \) tensor with \( \epsilon_{12} = 1 \) and \( \overline{x} = (x_1, x_2) \). \( \Delta_1(x-y) \) is the usual propagator of the massless scalar fields:

$$\Delta_1(\overline{x} - \overline{y}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\overline{k} \cdot (\overline{x} - \overline{y})}}{\overline{k}^2}$$

(2.13)

and

$$\Delta_2(\overline{x} - \overline{y}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\overline{k} \cdot (\overline{x} - \overline{y})}}{\overline{k}^4}$$

(2.14)

where \( \overline{k} = (k_1, k_2) \). It is important to stress here that the propagator \( \tilde{G}^{ab}_{ij}(x-y) \) is longitudinal in both indices \( i \) and \( j \). This descends from eq. (2.8) after setting \( \mu = 0 \) and \( \kappa = j \):

$$e^{0lj} \partial_l \tilde{G}^{ab}_{ij}(x-y) = 0$$

(2.15)

The propagator in the Coulomb gauge can be obtained performing the limit \( \lambda = 0 \). In this limit, it is easy to see that all the component of the propagator vanish but the transverse components of \( \tilde{G}^{ab}_{ij}(x-y) \) and \( \tilde{G}^{ab}_{i0}(x-y) \). Therefore, in the Coulomb gauge, the propagators are given by:

$$G^{ab}_{0i}(x-y) = \frac{8\pi}{s} \delta^{ab} \delta(x_0 - y_0) \epsilon_{ij} \partial^j \Delta_1(\overline{x} - \overline{y})$$

(2.16)
\[
G_{ab}^{\alpha}(x - y) = G_{ab}^{\beta}(x - y)
\]  
(2.17)

In the same way one can compute the propagator of the ghost fields \(G_{gh}^{ab}(x, y; t, t')\), which reads:

\[
G_{gh}^{ab}(x - y) = \delta^{ab} \Delta_1(x - y) \delta(x_0 - y_0)
\]  
(2.18)

Now we are ready to compute the vertices of the C-S field theory in the Coulomb gauge. As we see from the expressions of the propagators (2.16) and (2.17), the factor \(8\pi/s\) can be viewed as a coupling constant (see for example [3] on this point). With the settings \(x = (x, x_0)\), \(y = (y, y_0)\), \(z = (z, z_0)\) and \(w = (w, v)\) and at the first order in perturbation theory, the amplitude between three gauge fields \(A_{\mu}^a(x)\), \(A_{\nu}^b(y)\) and \(A_{\rho}^c(z)\) is:

\[
V_{\mu\nu\rho}^{abc}(x, y, z) = -\frac{s}{24\pi} \int d^3w f^{def} \epsilon^{\sigma\tau\eta} A_{\mu}^a(x) A_{\nu}^b(y) A_{\rho}^c(z) A_{\sigma}^d(w) A_{\tau}^e(w) A_{\eta}^f(w) \]  
(2.19)

It is easy to see that in the Coulomb gauge only those components of the vertex survive, for which two of the indices \(\mu, \nu\) and \(\rho\) are equal to zero, while the third index is a spatial index \(i\):

\[
V_{i00}^{abc}(x, y, z) = -\frac{s}{6\pi} \int d^3w d^3v f^{def} \epsilon^{ijk} G_{ab}^{ad}(x, w; x_0, v) G_{be}^{be}(y, w; y_0, v) G_{cf}^{cf}(z, w; z_0, v) \]  
(2.20)

In principle we had to use the Levi Civita tensor \([\epsilon]^{\mu\nu\rho} = a(x_0)\epsilon^{\mu\nu\rho}\) in eq. (2.20). However, the dependence on the metric \(a(x_0)\) cancels against the determinant of the metric present in the integration measure \(d^3w\), so that in the final form of the vertex the metric disappears. Let us also notice that, since each of the external propagator is proportional to the factor \(8\pi/s\), the effective coupling constant in front of the vertex (2.20) is going as \(\frac{1}{s^2}\). Finally, we have to compute the vertex of the interaction between ghosts and gauge fields. A straightforward calculation yields:

\[
V_{gh}^{abc}(x, y, z) = \int d^3w d^3v f^{def} G_{gh}^{ad}(x - w) \partial^i(w) G_{gh}^{be}(x - w) G_{gh}^{cf}(z - w) \]  
(2.21)

All the other components of this vertex vanish. As it is possible to see after summing over the index \(i\) using the expression of the metric (1.1) in the flat case, the ghost vertex does not depend on \(a(t)\).
3. THE CHERN-SIMONS FIELD THEORY IN THE COULOMB GAUGE: 
THE CASE OF CURVED SPACE-TIMES

Let us consider the most general manifold $M_3$ in which the metric is given by eq. (1.1). The spatial section $\Sigma_g$ is a Riemann surface of genus $g$. Covering $\Sigma_g$ with a system of open sets $\{U_i\}$, we define the local coordinates $z^{(i)}, \bar{z}^{(i)} : U_i \to \mathbb{C}$ as follows:

$$
\begin{align*}
    z^{(i)} &= x_1^{(i)} + i x_2^{(i)} \\
    \bar{z}^{(i)} &= x_1^{(i)} - i x_2^{(i)}
\end{align*}
$$

From now on, we will drop out the subscript $(i)$. The only nonvanishing components of the metric (1.1) are in these coordinates:

$$
g_{00} = 1 \quad g_{z\bar{z}}(z, \bar{z}, t) = g_{\bar{z}z}(z, \bar{z}, t) = a(t) h(z, \bar{z}) \quad g^{z\bar{z}} g_{z\bar{z}} = 1 \quad (3.2)
$$

The gauge fixed Chern-Simons action becomes in complex coordinates $S_{CS} = S_{\text{free}} + S_{\text{int}}$, where:

$$
S_{\text{free}} = \int_{M_3} d^2 z dt \left[ 2i (A_0^a \partial_z A_0^a + A_0^a \partial_0 A_\bar{z}^a + A_\bar{z}^a \partial_z A_0^a) - \text{c.c.} \right] + (a(t) \lambda)^{-1} g^{z\bar{z}} (\partial_z A_0^a + \partial_\bar{z} A_\bar{z}^a)^2 + 2 e^a \partial_z \partial_\bar{z} c^a \right] \quad (3.3)
$$

$$
S_{\text{int}} = \int_{M_3} d^2 z dt \left[ \varepsilon^{\mu\nu\rho} f^{abc} A_\mu^a A_\nu^b A_\rho^c - f^{abc} c^a (A_\mu^b \partial_z + A_\nu^b \partial_\bar{z}) c^c \right] \quad (3.4)
$$

and $d^2 z = \frac{1}{2i} dz \wedge d\bar{z}$. The factor $2i$ in eq. (3.3) comes from the form of the $\varepsilon^{\mu\nu\rho}$ tensor in complex coordinates. In fact, the Levi-Civita tensor $[\varepsilon]^{\mu\nu\rho} = g^{-\frac{1}{2}} \varepsilon^{\mu\nu\rho}$ becomes in these coordinates:

$$
[\varepsilon]^{0z\bar{z}} = -2ig^{z\bar{z}} a^{-1}(t) \quad (3.5)
$$

All the other components can be obtained from eq. (3.5) permuting the indices $0, z$ and $\bar{z}$ and changing the sign according to the order of the permutation. In this Section it will be useful to denote a sum over the complex indices with the first letters of the Greek alphabet $\alpha, \beta, \gamma$ and so on. For example, the gauge condition (2.3) becomes now $\partial^\alpha A_\alpha^a = 0$. Using the metric (1.1) to rise and lower the indices, this equation reads:

$$
\partial_z A_\bar{z}^a + \partial_\bar{z} A_z^a = 0 \quad (3.6)
$$
Eq. (3.6) does not contain the metric explicitly. This means that the Coulomb gauge condition is compatible with the transition functions at the intersections $U_i \cap U_j$ of two open sets $U_i$ and $U_j$ of the covering of $\Sigma_g$. Therefore eq. (3.6) is globally valid on $M_3$.

The gauge fields $(A^a_{\bar{z}}, A^a_{z}, A^a_0)$ are connections on the trivial principal bundle

$$P(M_3, SU(N)) = M_3 \otimes SU(N)$$

This bundle is trivial due to the fact that SU(N) is a simply connected Lie group. One can show as in the flat case that the Coulomb gauge (3.6) is a good gauge fixing without Gribov ambiguities [12], at least in the perturbative approach. A proof can be given as in [13], considering a gauge transformation (2.2) in which $\omega^a(z, \bar{z}, t)$ is taken to be an infinitesimal small function. The new fields, after this gauge transformation, are of the form $A'^a_\mu = A^a_\mu + (D_\mu \omega)^a$. The requirement that also $A'^a_\mu$ satisfies the gauge condition (3.6) yields the following equation:

$$\partial_z D_{\bar{z}}^{ab}(A)\omega^b(z, \bar{z}, t) + \partial_{\bar{z}} D_{z}^{ab}(A)\omega^b(z, \bar{z}, t) = 0 \quad (3.7)$$

If we suppose that $A^a_\mu$ is only a small perturbation around a classical solution, then we can expand the functions $\omega^a(z, \bar{z}, t)$ in powers of the effective coupling constant $\frac{1}{s}$

$$\omega^a(z, \bar{z}, t) = \sum_{n=0}^{\infty} \left( \frac{1}{s} \right)^n \omega^a_{(n)}(z, \bar{z}, t)$$

Solving eq. (3.7) in terms of the $\omega_{(n)}(z, \bar{z}, t)$, we get as a general solution on $M_3$:

$$\omega^a_{(n)}(z, \bar{z}, t) = \omega_{(n)}(t)$$

This is due to the fact that the spatial section of $M_3$ is a compact Riemann surface. Still we have to set the boundary conditions of the fields $A^a_{\bar{z}}(z, \bar{z}, t)$. Requiring that the fields $A^a_{\bar{z}}$ vanish at infinity, i.e. for large values of $|t|$, it is easy to see that the functions $\omega_{(n)}(t)$ should be constant. Therefore, the residual gauge invariance after the Coulomb gauge fixing amounts to the constant elements of the group SU(N) and it is not difficult to integrate it out in the path integral.

Having proven that the Coulomb gauge fixing is valid also in the case of the manifold $M_3$ with metric (1.1), we are ready to compute the propagators of the gauge fields. Let us put

$$G^{ab}_{\mu \nu}(z, w; t, t') = < A^a_{\mu}(z, \bar{z}, t) A^b_{\nu}(w, \bar{w}, t) >$$
where now $\mu, \nu = 0, z, \bar{z}$. Then the equations satisfied by the propagators of the $A$–fields become:

$$-4i\partial_z G^{ab}_{z0}(z, w; t, t') + 4i\partial_{\bar{z}} G^{ab}_{\bar{z}0}(z, w; t, t') = \frac{8\pi}{s} \delta^{ab} \delta^{(2)}_z(z, w) \delta(t - t')$$

(3.8)

$$-4i\partial_z G_{0w}^{ab}(z, w; t, t') + 4i\partial_t G^{ab}_{zw}(z, w; t, t') - 2\frac{a^{-1}(t)}{\lambda} \partial_t \left[ g^{zz} \partial_z G^{ab}_{zw}(z, w; t, t') + g^{\bar{z}z} \partial_{\bar{z}} G^{ab}_{\bar{z}w}(z, w; t, t') \right] = \frac{8\pi}{s} \delta^{ab} \delta^{(2)}_z(z, w) \delta(t - t')$$

(3.9)

Another equation can be obtained from (3.9) permuting the indices $z$ and $\bar{z}$ and substituting the index $w$ with $\bar{w}$. There are still other relations relating the various components of the propagators together:

$$\partial_z G^{ab}_{z\alpha}(z, w; t, t') - \partial_{\bar{z}} G^{ab}_{\bar{z}\alpha}(z, w; t, t') = 0$$

(3.10)

$$-4i\partial_\alpha G^{ab}_{00}(z, w; t, t') + 4i\partial_0 G^{ab}_{\alpha0}(z, w; t, t') -$$

$$-\frac{a^{-1}(t)}{\lambda} \partial_\alpha \left[ g^{zz} \partial_z G^{ab}_{z0}(z, w; t, t') + g^{\bar{z}z} \partial_{\bar{z}} G^{ab}_{\bar{z}0}(z, w; t, t') \right] = 0$$

(3.11)

where $\alpha = w, \bar{w}$. Eq. (3.10) implies that the propagators $G^{ab}_{z\bar{z}}(z, w; t, t')$ and $G^{ab}_{\bar{z}z}(z, w; t, t')$ do not have transverse components. This equation is the equivalent in complex coordinates of the relation (2.15). Finally we have:

$$-4i\partial_z G^{ab}_{0\bar{w}}(z, w; t, t') + 4i\partial_{\bar{w}} G^{ab}_{z\bar{w}}(z, w; t, t') -$$

$$-\frac{a^{-1}(t)}{\lambda} \partial_z \left[ g^{zz} \partial_z G^{ab}_{z\bar{w}}(z, w; t, t') + g^{\bar{z}\bar{z}} \partial_{\bar{z}} G^{ab}_{\bar{z}\bar{w}}(z, w; t, t') \right] = 0$$

(3.12)

Again it is possible to get another independent relation from eq. (3.12) interchanging the two indices $z$ and $\bar{z}$ and substituting $\bar{w}$ with $w$. Eqs. (3.8)–(3.12) are equivalent to the system (2.8) given in the flat case. It is very difficult to solve these equations when the metric is the general metric given in eq. (1.1). Moreover, we should remember that due to a theorem stating that the total charge on a Riemann surface (like in any other two dimensional compact manifold) is always zero, an isolated $\delta$ function $\delta^{(2)}(z, w)$ is not allowed. Therefore, in the right hand side of eqs. (3.8)–(3.9) there must be also zero modes. The form of these zero modes will be uniquely determined below. Despite of all these difficulties, in the Coulomb gauge, i.e. in the limit in which $\lambda \to 0$, there are drastic simplifications, so that the above equations are reduced to the following two relations:

$$\partial_z G^{ab}_{z\bar{0}}(z, w; t, t') - \partial_{\bar{z}} G^{ab}_{\bar{z}0}(z, w; t, t') = \frac{4\pi i}{s} \delta^{ab} \delta^{(2)}(z - w) \delta(t - t') + \text{zero modes}$$

(3.13)
\[ \partial \bar{z} G^{ab}_{z0}(z, w; t, t') + \partial z G^{\bar{a}b}_{z0}(z, w; t, t') = 0 \quad (3.14) \]

These equations describe exactly the main requirement of the Coulomb gauge, i.e. the fact that only the transverse fields in the two dimensional spatial section \( \Sigma_g \) of \( M_3 \) propagate. The transverse fields in complex coordinates are in fact described by the following condition:

\[ A^a_z = \overline{(A^a_{\bar{z}})} = -A^a_{\bar{z}}. \]

The solution of eqs. (3.13) and (3.14) is provided by the following Green functions:

\[ \langle A^a_z(z, t) A^b_0(w, t') \rangle = \frac{2 \pi i}{s} \delta^{ab} \partial \bar{z} K(z, w) \delta(t - t') \quad (3.15) \]

\[ \langle A^a_{\bar{z}}(z, t) A^b_0(w, t') \rangle = -\frac{2 \pi i}{s} \delta^{ab} \partial z K(z, w) \delta(t - t') \quad (3.16) \]

where \( K(z, w) \) is the usual propagator of the scalar fields on a Riemann surface (see ref. [14] for more details):

\[ K(z, w) = \delta^{(2)}_{zz}(z, w) + \frac{g_{z \bar{z}}}{\int_{\Sigma_g} d^2 u g_{u \bar{u}}} \quad (3.17) \]

\[ \partial z \partial w K(w, z) = -\delta^{(2)}_{z \bar{w}}(z, w) + \tilde{\omega}_i(z) [\text{Im } \Omega]^{-1}_{ij} \omega_j(w) \quad (3.18) \]

\[ \int_{\Sigma_g} d^2 z g_{z \bar{z}} K(z, w) = 0 \quad (3.19) \]

In eq. (3.18) the \( \omega_i(z) dz, i = 1, \ldots, g \), denote the usual holomorphic differentials and \( \Omega_{ij} \) represents the period matrix. It is important to stress here that \( K(z, w) \) is a singlevalued function on \( \Sigma_g \). Using the propagators (3.15) and (3.16) it is easy to see that eq. (3.14) is trivially satisfied. Therefore, the Coulomb gauge requirement (3.6) is fulfilled and the above defined propagators describe exactly the transverse components of the gauge fields. Still there is an ambiguity in the solutions (3.13) and (3.14) due to the zero mode sector of the fields \( A^a_z \) and \( A^a_{\bar{z}} \). In order to remove this ambiguity, we have to require that the above propagators are singlevalued along the nontrivial homology cycles of the Riemann surface. Otherwise, the propagators are not well defined on \( M_3 \), but in one of its coverings. Therefore, the propagators should obey the following relations:

\[ \oint_{\gamma} dz < A^a_z(z, t) A^b_0(w, t') >= \oint_{\gamma} d\bar{z} < A^a_{\bar{z}}(z, t) A^b_0(w, t') >= 0 \quad (3.20) \]

along any nontrivial homology cycles \( \gamma \). Due to the properties of singlevaluedness of the Green function \( K(z, w) \), eq. (3.20) is trivially satisfied by the propagators given in eqs. (3.15) and (3.16). In this way these two propagators are well defined and also the freedom in the zero mode sector is removed. Now we insert their expressions in eq. (3.13) in order
to get the exact form of the zero mode terms appearing in the right hand side of this equation:

\[ \partial_z G^{ab}(z, w; t, t') - \partial_{\bar{z}} G^{ab}(z, w; t, t') = \frac{4\pi i}{s} \delta^{ab} \delta^{(2)}(z, w) \delta(t - t') + 4\pi is \frac{g_{zz}}{\int_{\Sigma} d^2 u g_{uu}} \delta(t - t') \]

(3.21)

The fact that the propagators in the Coulomb gauge must obey eq. (3.20) can be understood also decomposing the fields by means of the Hodge decomposition of the gauge fields in a coexact, exact and harmonic part:

\[ A^a_z = i \partial_z \phi^a + \partial_z \rho^a + A^a_{\text{har}} \]

(3.22)

\[ A^a_{\bar{z}} = i \partial_{\bar{z}} \phi^a + \partial_{\bar{z}} \rho^a + A^a_{\text{har}} \]

(3.23)

\( \phi^a \) and \( \rho^a \) represent two real scalar fields. The above decomposition is allowed since the gauge invariance has been completely fixed by the choice of the Coulomb gauge, at least in the perturbative approach, and the \( G \)-bundle \( P(M_3, SU(N)) \) is trivial as we previously remarked. In the Coulomb gauge, the only components of the gauge fields which are allowed to propagate are the coexact differentials, i.e. the 1–forms obtained differentiating the scalar fields \( \phi^a \) in eqs. (3.22) and (3.23). Therefore, the requirement (3.20) is a pure consequence of the fact that the coexact forms have vanishing holonomies around the nontrivial homology cycles.

Let us notice that the zero mode term appearing in the right hand side of eq. (3.21) is totally irrelevant. To eliminate it it is sufficient to introduce new gauge fields, let say \( \tilde{A}_z \), \( \tilde{A}_{\bar{z}} \), differing from the old ones by the fact that they are normalized to zero at a point \((0, 0)\) of the Riemann surface:

\[ \tilde{A}^a_z(z, \bar{z}, t) = A^a_z(z, \bar{z}, t) - A^a_z(0, 0, t) \]

(3.24)

\[ \tilde{A}^a_{\bar{z}}(z, \bar{z}, t) = A^a_{\bar{z}}(z, \bar{z}, t) - A^a_{\bar{z}}(0, 0, t) \]

(3.25)

Using the above new fields it is easy to check that the second term in the right hand side of eq. (3.24), which is a zero mode contribution, cancels out. Finally, we notice that in the flat case discussed in Section 2, the propagators (3.15) and (3.16) are in agreement with the propagators given in eqs. (2.16) and (2.17).

\(^2\) On \( M_3 \) this implies that the new fields are normalized to zero along the whole line of the time. This is possible to do since the three dimensional manifold is flat in the time direction.
We finish this Section providing the explicit form of the other correlation functions of Chern-Simons field theory. The propagator of the ghost fields becomes:

\[ G_{gh}^{ab}(z, w; t, t') = \delta^{ab} K(z, w) \delta(t - t') \] (3.26)

The vertex coming from the cubic interaction between the gauge fields reads instead:

\[ V_{abc}^{z_100}(z_1, z_2, z_3; t, t', t'') = \frac{2\pi i s}{3} \int_{\Sigma_g} d^2 z f^{abc} \partial_{z_1} K(z_1, z) \left[ \partial_z K(z_2, z) \partial_{\bar{z}} K(z_3, z) - \partial_{\bar{z}} K(z_2, z) \partial_z K(z_3, z) \right] \delta(t - t') \delta(t' - t'') \] (3.27)

The simple integration in the variable \( t \) has been already carried out in the above expression of the vertex. The component \( V_{abc}^{z_100}(z_1, z_2, z_3; t, t', t'') \) of the vertex can be simply obtained replacing the derivative \( \partial_{z_1} \) in eq. (3.27) with its complex conjugate. The vertex describing the interaction between ghost and gauge fields has only one component which is given by:

\[ V_{0 gh}^{abc}(z_1, z_2, z_3; t, t', t'') = \frac{2\pi i s}{3} \int_{M_3} d^2 z f^{abc} K(z_1, z) \left[ \partial_z K(z_2, z) \partial_{\bar{z}} K(z_3, z) \right] \delta(t - t') \delta(t' - t'') \] (3.28)

It is easy to check that the above expressions of the vertices (3.27) and (3.28) are real as it should be.

4. CONCLUSIONS

Concluding, we would like to outline some of the possible applications of the perturbative approach presented here. On one side, the introduction of the C-S functional has been proposed in condensed matter physics as a possible explanation of some physical phenomena, like for example the fractional quantum Hall effect [13]. We expect therefore that the presence of a background can be revealed by some new and observable effects. For instance, it was proven in [13] that a curved space-time manifests itself in a two dimensional scalar field theory through the appearance in the amplitudes of induced vertex operators satisfying a nonabelian braid group statistics. Unfortunately this example, unlike the C-S field theory, is unphysical, apart from its consequences in string theory.
In the C-S field theory we suppose that the mechanism through which topological effects become evident, is provided by the edge states of ref. [17]. In fact, one can always choose the metric (1.1) in such a way that the curvature of the Riemann surface $\Sigma_g$ has point-like singularities at some points $a_1, \ldots, a_M \in \Sigma_g$ (see ref. [16] for more details). These points play the role of punctures and, therefore, they generate edge states with nontrivial statistics in the presence of the C-S fields. Preliminary calculations, done for the abelian C-S theory, show indeed that edge states of this kind are induced by the punctures $a_1, \ldots, a_M$ as the vertex operators are induced in the two dimensional scalar model mentioned above.

On the other side, the nonabelian C-S field theory is currently being studied on curved space-times due to its applications to knot theory [2], [18] and braid group statistics [10], [17], [19]. For instance, the perturbative approach developed here makes it possible to derive the multiparameter link invariants of refs. [20] and [21] from a nonabelian C-S field theory defined on a 3-manifold $\Sigma_g \otimes \mathbb{R}$. This is a very interesting application since these link invariants were obtained until now only within the framework of the quasi-triangular Hopf algebras, but a field theoretic approach is still missing. Finally, we notice that the manifold $\Sigma_g \otimes \mathbb{R}$ is just a subcase of the manifolds $M_3$ treated here, which can be obtained setting $a(t) = 1$ in eq. (1.1). However, as we have seen here, the dilation factor $a(t)$ does not appear in the correlation functions of the gauge fixed C-S theory, at least in the Coulomb gauge. Thus, we do not expect that this factor can play an important role in the computation of the link invariants.

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