A unified characterization of
generalized information and certainty measures

Velimir M. Ilić\textsuperscript{a}, Miomir S. Stanković\textsuperscript{b}

\textsuperscript{a}Mathematical Institute of the Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Beograd, Serbia
\textsuperscript{b}University of Niš, Faculty of Occupational Safety, Čarnojevića 10a, 18000 Niš, Serbia

Abstract

In this paper we consider the axiomatic characterization of information and certainty measures in a unified way. We present the general axiomatic system which captures the common properties of a large number of the measures previously considered by numerous authors. We provide the corresponding characterization theorems and define a new generalized measure called the Inforcer, which is the quasi-linear mean of the function associated to the event probability following the general composition law. In particular, we pay attention to the polynomial composition and the corresponding polynomially composable Inforcer measure. The most common measures appearing in literature can be obtained by specific choice of parameters appearing in our generic measures and they are listed in tables.

Keywords: information measure; entropy; inaccuracy; certainty; axiomatic characterization; pseudo-addition; polynomial addition

1. Introduction

In the past decades there was a plausible interest for definition and characterization of the measures of certainty and information associated to a probability distribution. The most commonly used ones are those which can be obtained as the average value of the information/certainty associated to the event.

Information measures determine the amount of uncertainty associated to a probability distribution. The basic one is the Shannon entropy [34], defined as a linear (trace-form) expectation of an additive decreasing function of an event probability called information content. Renyi [33], Varma [44] and Nath [24] considered the class of entropies which can be obtained as quasi-linear mean weighted by the random variable probability. Information entropies with more general weights were considered by Aczél and Daróczy [3], Kapur [20], Rathie [32], Khan and Autar [22] and Singh et. al [37]. Havrda and Charvát [16] and, subsequently, Daróczy [11] and Tsallis [41], considered the entropies which are the trace form of pseudo-additive information content. The trace form entropies based on the pseudo-additive content are also considered by Abe [1] and Kaniadakis [17, 18]. The class of entropies which are quasi-linear mean of the pseudo-additive information are considered by Sharma and Mittal [35], Frank and Daffertshofer [13], Arimoto [3], Boekee, Boxma and Van der Lubbe [9], [43] and Picard [31].

Inaccuracy measures are a generalization of information entropies, which deal with two distributions and reduce to the entropies if the distributions are identical. The firstly introduced one was the Kerridge inaccuracy [21], defined as the expected value of the information content of the first distribution, where the weights are are event probabilities with respect to the second distribution. Nath [25] considered two types

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\footnote{Corresponding author. Tel.: +38118224492; fax: +38118533014.}

\email{velimir.ilic@gmail.com (Velimir M. Ilic), miomir.stankovic@gmail.com (Miomir S. Stankovic)}
of the generalization - one to quasi-linear means and the other with pseudo-additive information content. The combination of these two approaches was considered by Gupta and Sharma [14].

Certainty measures are defined as the average value of a multiplicative increasing function of the event probability called certainty content. The certainty measures which can be represented as the trace form expectation of the certainty content are Onicesu’s information energy [28], also called Weaver’s expected probability called certainty content. The certainty measures which can be represented as the quasi-linear expectation were considered by Van der Lubbe [43] and Nath [6] and by Ebanks [12] for the case of trace form information entropies. Instead of this, we present the general axiomatic system, which characterizes in a unique manner the majority of the known information and certainty measures, and obtain the Inforcer measure as the unique one that satisfies it. By our axiomatic system, information and certainty measures are the particular cases of the Inforcer measure, which is the quasi-linear of the Inforcer content. The Inforcer content is defined as a monotonic function of event probability, having the information and certainty content as special cases.

In this paper we propose the general axiomatic system, which characterizes in a unique manner the majority of the known information and certainty measures, and obtain the Inforcer measure as the unique one that satisfies it. By our axiomatic system, information and certainty measures are the particular cases of the Inforcer measure, which is the quasi-linear of the Inforcer content. The Inforcer content is defined as a monotonic function of event probability, having the information and certainty content as special cases. The Inforcer measure and the Inforcer content follow the simple composition rule which asserts that the Inforcer measure and the Inforcer content of joint distributions can be obtained as the composition of the Inforcer measure and the Inforcer content of particular distributions.

In particular, we pay attention to the most common measures appearing in literature, generated by the polynomial composition operation. The polynomial composition has already been considered by Behara and Nath [6] and by Ebanks [12] for the case of trace form information entropies. Instead of this, we derive the more general form for information and certainty measures generated as a quasi-linear mean of the content which follows the polynomial composition law. It is shown that pseudo-addition [27] and real product represent the unique polynomial composition operations which preserve the decreasing information and increasing certainty content. Finally, we give the theorem which represents the interplay between the polynomial certainty and information measures generalizing the result from [43].

The general axiomatic system for the Inforcer measure and the corresponding uniqueness theorem are presented in section 2. In section 3 we define the information and certainty measures as instances of Inforcer measure and derive relation between them. In section 4 we consider polynomial composition and derive the corresponding information and certainty measures. The majority of the information and certainty measures previously considered in literature are obtained by instantiation of the generalized average content and listed in Tables 1 and 2.

2. Axiomatic characterization of the Inforcer measure

Let $h : \mathbb{R} \to \mathbb{R}$ be a monotonic continuous (hence invertible) function such that $h(x) > 0$ for $x > 0$ and let the composition operation $\odot$ be defined as:

$$h(a + b) = h(a) \odot h(b); \quad a, b \in \mathbb{R}. \quad (1)$$

Let the set of all $n$-dimensional distributions and the set of all the positive ones be denoted with

$$\Delta_n \equiv \left\{ (p_1, \ldots, p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}; \quad \Delta_n^+ \equiv \left\{ (p_1, \ldots, p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}; \quad n > 1, \quad (2)$$

respectively. Let the direct product, $P \star Q \in \Delta_{nm}$, be defined as

$$P \star Q = (p_1 q_1, p_1 q_2, \ldots, p_n q_m), \quad (3)$$

for $P = (p_1, \ldots, p_n) \in \Delta_n$ and $Q = (q_1, \ldots, q_m) \in \Delta_m$ let and $\mathbb{R}^+$ denotes the set of positive real numbers.

The Inforcer measure is characterized with the following set of axioms.

[A1] The Inforcer content $\mathcal{EC} : (0, 1] \to \mathbb{R}^+$ is a continuous monotonic function, which is composable:

$$\mathcal{EC}(pq) = \mathcal{EC}(p) \odot \mathcal{EC}(q), \quad \text{for all} \quad p, q \in (0, 1]. \quad (4)$$
The \( n \)-dimensional Inforcer measure is a continuous function, \( \mathcal{G} : \Delta_n \times \Delta_n \rightarrow \mathbb{R}^+, n \in \mathbb{N} \), which to every pair of distributions \( U = (u_1, \ldots, u_n) \in \Delta_n \) and \( P = (p_1, \ldots, p_n) \in \Delta_n^+ \) assigns a quasi-linear mean of the content
\[
\mathcal{G}(U; P) = f^{-1}\left( \sum_{k=1}^{n} u_k \cdot f(\mathcal{E}(p_k)) \right),
\]  
(5)
where \( f \) is invertible and continuous.

\[ \mathcal{G} \] is composable:
\[
\mathcal{G}(U \star V; P \star Q) = \mathcal{G}(U; P) \odot \mathcal{G}(V; Q),
\]  
(6)
for all \( U, V, P, Q \in \Delta_n \).

The following theorem gives the unique class of functions to which the Inforcer measure belongs.

**Theorem 2.1.** Let the axioms [A1]-[A3] hold. Then, the Inforcer content and the Inforcer measure are uniquely determined with
\[
\mathcal{E}(p) = h(\tau \cdot \log_2 p), \quad \tau < 0
\]  
(7)
and
\[
\mathcal{G}(U; P) = \begin{cases} 
\mathcal{G}_\tau(U; P) = h\left( \sum_{k=1}^{n} u_k \log_2 p_k^\tau \right), & \lambda = 0 \\
\mathcal{G}_{\tau,\lambda}(U; P) = h\left( \frac{1}{\lambda} \log_2 \left( \sum_{k=1}^{n} u_k p_k^{\lambda \tau} \right) \right), & \lambda \neq 0
\end{cases}
\]  
(8)

**Proof.** Let \( \circ \) denote the composition of functions and let \( \mathcal{J} = h^{-1} \circ \mathcal{E} \). By applying isomorphism \( h^{-1} \) to (38), we get the Cauchy functional equation,
\[
\mathcal{J}(pq) = \mathcal{J}(p) + \mathcal{J}(p),
\]  
(9)
which has the unique solution
\[
\mathcal{J}(p) = \tau \cdot \log_2 p \quad \Leftrightarrow \quad \mathcal{E}(p) = h(\mathcal{J}(p)) = h(\tau \cdot \log_2 p),
\]  
(10)
which proves the equation (7), since the positivity of \( h \) implies that \( \tau < 0 \).

If we set \( g = f \circ h \) or, equivalently, \( f = g \circ h^{-1} \), (5) can be transformed to
\[
\mathcal{G}(U; P) = h\left( g^{-1}\left( \sum_{k=1}^{n} u_k g(\mathcal{J}(p_k)) \right) \right) = h\left( g^{-1}\left( \sum_{k=1}^{n} u_k g(\tau \cdot \log_2 p_k) \right) \right).
\]  
(11)

The function \( g \) can be determined using the pseudo-additivity of entropy (6) which has the form
\[
h\left( g^{-1}\left( \sum_{k=1}^{n} \sum_{l=1}^{m} u_k v_l g(\mathcal{J}(p_{kl})) \right) \right) = h\left( g^{-1}\left( \sum_{k=1}^{n} u_k g(\mathcal{J}(p_k)) \right) \right) \circ h\left( g^{-1}\left( \sum_{l=1}^{m} v_l g(\mathcal{J}(q_l)) \right) \right).
\]  
(12)

Let \( Q = (1/m, \ldots, 1/m) \) be uniform and \( \mathcal{J}(1/m) = \mathcal{J} \). By applying \( h^{-1} \) and after using \( \mathcal{J}(pq) = \mathcal{J}(p) + \mathcal{J}(q) \), we get
\[
g^{-1}\left( \sum_{k=1}^{n} u_k g(\mathcal{J}(p_k) + \mathcal{J}) \right) = g^{-1}\left( \sum_{k=1}^{n} u_k g(\mathcal{J}(p_k)) \right) + \mathcal{J}
\]  
(13)
or, if we set \( g_J(z) = g(z + J) \),

\[
g_J^{-1}\left(\sum_{k=1}^{n} u_k g_J(J(p_k))\right) = g^{-1}\left(\sum_{k=1}^{n} u_k g(J(p_k))\right).
\]

Since \( g_J \) and \( g \) generate the same mean, \( g_J \) is the linear function of \( g \) and we get the equation

\[
g_J(z) = a(J) \cdot g(z) + b(J).
\]

The functional equation (15) can be solved as in \([15]\). Since \( g(z + J) = g(J + z) \), we can write \( a(J)g(z) + g(J) = a(J)\gamma g(J) + g(J) \) or \( (a(J) - 1)/g(J) = (\gamma - 1)/g(z) = \gamma \). Inserting this into equation (15) leads to the following functional equations

\[
g(J + z) = g(J) + g(z) \text{ for } \gamma = 0
\]

\[
a(J + z) = a(J) \cdot a(z) \text{ for } \gamma \neq 0.
\]

The first case leads to \( g(z) = cz \) whereas the second one imposes to the function \( a(z) = 2^{|z|} \). Then, we can write \( g(z) = (2^{|z|} - 1)/\gamma \), where \( \lambda \neq 0 \), in this case since \( g \) must be invertible. The theorem is proven by substitution of the solution for \( g \) in \([11]\). \( \square \)

If \( U \equiv P \) and the normalization condition \( \mathcal{G}(\frac{1}{\lambda}; X ; U) = h(1) \) is additionally satisfied, then \( \tau = -1 \), and the class of Inforcer measures \( \mathcal{G} \) reduces to the class derived and characterized in \([4]\).

\[
\mathcal{G}(U; P) = \begin{cases} 
    h\left( -\sum_{k=1}^{n} p_k \log_2 p_k \right), & \lambda = 0 \\
    h\left( \frac{1}{\lambda} \log_2 \sum_{k=1}^{n} p_k^{1-\lambda} \right), & \lambda \neq 0.
\end{cases}
\]

Note that, although not explicitly mentioned in \([4]\), the proof from \([4]\) requires continuity of the event and the average content, the continuity of \( h^{-1} \), as well as the normalization condition (compare section 2 from \([4]\) with the proof of theorem \([2.1]\)).

**Remark 2.2.** The structure \((\mathbb{R}, \circ)\) is a commutative topological group isomorphic to \((\mathbb{R}, +)\) and \( h \) is an isomorphism from \((\mathbb{R}, +)\) to \((\mathbb{R}, \circ)\), i.e. \((\mathbb{R}, \circ)\) is Abelian, \( \mathbb{R}^2 \to \mathbb{R} : (a, b) \to a \circ b \), and \( \mathbb{R} \to \mathbb{R} : a \to a \circ \alpha \) are continuous. Here, for simplicity, we consider topological groups defined over \( \mathbb{R} \). However, the results from theorem \([2.1]\) are valid for an arbitrary topological group.

### 3. Information and certainty measures

Using the Inforcer content and the Inforcer measure, entropy and certainty measures can be defined as follows. Let \( u : \Delta_n \to \Delta_n \) be a continuous function such that \( u(P) = (u_1(P), \ldots, u_n(P)) \in \Delta_n \).

If \( h : \mathbb{R} \to \mathbb{R} \) is increasing and \( h(0) = 0 \), the Inforcer content is a decreasing function \( EC : (0, 1] \to \mathbb{R}^+ \)

and is called the *information content*. The corresponding Inforcer measure, \( I : \Delta_n \times \Delta_n \to \mathbb{R}^+ \), is called the *generalized inaccuracy measure*. The function \( \mathcal{H} : \Delta_n \to \mathbb{R}^+ ; \mathcal{H}(P) = \mathcal{G}(u(P); P) \) is called the *generalized entropy*. The inaccuracy and the entropy are both referred to as the *information measures*.

If \( h : \mathbb{R} \to \mathbb{R}^+ \) is decreasing and \( h(+\infty) = 0 \), the Inforcer content is an increasing function \( EC : (0, 1] \to (0, 1] \)

and is called the *certainty content*. The corresponding Inforcer measure, \( C : \Delta_n \times \Delta_n \to \mathbb{R}^+ \), is called the *generalized certainty measure*.

Previously, the inaccuracy measures have been considered in \([21]\), \([25]\), \([14]\), entropies in \([34]\), \([33]\), \([44]\), \([24]\), \([8]\), \([20]\), \([32]\), \([22]\), \([37]\), \([16]\), \([11]\), \([41]\), \([1]\), \([17]\), \([18]\), \([35]\), \([13]\), \([5]\), \([9]\), \([43]\), \([31]\), and certainty measures in \([28]\), \([45]\), \([31]\), \([43]\), \([8]\). Most of these measures follow the polynomial composition law, which will be discussed in section \([4]\).
Theorem 3.1. Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $Q = (q_1, \ldots, q_m) \in \Delta_m$, $n, m \in \mathbb{N}$ and let $C : \Delta_n \times \Delta_n \to \mathbb{R}^+$ be a certainty measure and a function $I : \Delta_n \times \Delta_n \to \mathbb{R}^+$ have the following properties:

- **[P1]** $I$ is the pseudo-additive measure
  \[ I(P \star Q; U \star V) = I(U; P) \oplus I(Q; V); \quad (19) \]
  where $\oplus$ is defined with $h_1(a + b) = h_1(a) \oplus h_1(b)$ for all $a, b \in \mathbb{R}$ and $h_1 : \mathbb{R} \to \mathbb{R}^+$ is an increasing function, such that $h(0) = 0$.

- **[P2]** $I$ is a continuous and strictly monotonic function $g : \mathbb{R} \to \mathbb{R}$ of the certainty measure $C$, so that
  \[ I(P; U) = g(C(P; U)). \quad (20) \]

Thus, $I$ is the generalized inaccuracy measure:

- \[ I(U; P) = \begin{cases} h_1 \left( \frac{1}{n} \sum_{k=1}^{n} u_k \log_2 p_k^2 \right), & \lambda = 0 \\ h_1 \left( \frac{1}{n} \log_2 \left( \frac{1}{n} \sum_{k=1}^{n} u_k p_k^{2\lambda} \right) \right), & \lambda \neq 0 \end{cases} \quad (21) \]

**Proof:** By the definition, a certainty measure is an Inforcer measure if the function $h \equiv h_C : \mathbb{R} \to \mathbb{R}^+$ is a decreasing function and $h_C(+\infty) = 0$, so that we have

- \[ C(U; P) = \begin{cases} h_C \left( \frac{1}{n} \sum_{k=1}^{n} u_k \log_2 p_k^2 \right), & \lambda = 0 \\ h_C \left( \frac{1}{n} \log_2 \left( \frac{1}{n} \sum_{k=1}^{n} u_k p_k^{2\lambda} \right) \right), & \lambda \neq 0 \end{cases} \quad (22) \]

We will show that $g(y) = h_1(k \cdot h_C^{-1}(y))$ and the result follows from **[P2]**. Note that $k > 0$ since $I$ is by assumption positive, $h_1$ is increasing, and $h_C$ is decreasing, so we can fix the value of the $k$ to 1 since the $\tau$ and $\lambda$ in (22) can be arbitrarily chosen.

Let us denote \( \bar{I} = h_1^{-1} \circ I, \bar{C} = h_C^{-1} \circ C, \) and \( \bar{g} = h_1^{-1} \circ g \circ h_C \). Since $I = g \circ C$, we have \( \bar{I} = \bar{g} \circ \bar{C} \) or

\[ \bar{g}(\bar{C}(P \star Q; U \star V)) = \bar{I}(P \star Q; U \star V) = \bar{I}(U; P) + \bar{I}(Q; V), \quad (23) \]

where the right-hand side equality follows from (19). As the special case of the Inforcer measure, $C$ satisfies the composability axiom \( [A3] \)

\[ C(P \star Q; U \star V) = C(U; P) \otimes C(Q; V), \quad (24) \]

where $\otimes$ is defined with $h_C(a + b) = h_C(a) \otimes h_C(b)$ for all $a, b \in \mathbb{R}$, and we have

\[ \bar{C}(P \star Q; U \star V) = \bar{C}(U; P) + \bar{C}(Q; V). \quad (25) \]

By combining the equations (23) and (25) we get

\[ \bar{g}(\bar{C}(U; P) + \bar{C}(Q; V)) = \bar{I}(U; P) + \bar{I}(Q; V). \quad (26) \]

By using \( \bar{I} = \bar{g} \circ \bar{C} \) and setting $a = \bar{C}(U; P), b = \bar{C}(Q; V)$, we get the Cauchy functional equation $\bar{g}(a) + \bar{g}(b) = \bar{g}(a + b)$, which has the unique solution $\bar{g}(x) = k \cdot x, k \in \mathbb{R}$. Accordingly, $\bar{g}(x) = h_1^{-1}(g(h_C(x))) = kx$ or equivalently, $g(h_C(x)) = h_1(kx)$. Finally, if we set $y = h_C(x)$, we get $g(y) = h_1(k \cdot h_C^{-1}(y))$ and the theorem is proven. □
4. The polynomial composable Inforcer measures

In this section we consider the Inforcer measures which follow the polynomial composition law. More specifically, we consider the case when the operation $\odot$ can be represented as

$$h(x + y) = h(x) \oplus h(y) = F(h(x), h(y))$$  \hspace{1cm} (27)

and the function $F : \mathbb{R}^2 \to \mathbb{R}$ is a two-variable polynomial.

The equation of the type (27) is called a polynomial addition theorem. In [2], it is shown that the most general functions with a polynomial addition theorem are

$$h(x) = a \cdot x + b, \quad \text{and} \quad h(x) = \frac{2^e - d}{e},$$  \hspace{1cm} (28)

where $e \neq 0, a, b, c, d$ are arbitrary constants. The respective polynomials $F$ in (27) are

$$F(u, v) = u + v + b \quad \text{and} \quad F(u, v) = euv + du + dv + \frac{d^2 - d}{e}.$$  \hspace{1cm} (29)

Note that the first formula reduces to real addition for $b = 0$ and the second one reduces to multiplication for $e = 1$ and $d = 0$.

The formulas (28) and (29) determine the most general form of polynomially composable Inforcer measure. The values of the parameters are further restricted if the Inforcer is considered as an information or a certainty measure. Previously, the polynomial composition has been considered by Behara and Nath [6] and by Ebanks [12], for the case of trace form information measures. In the following subsection, we derive the more general form for information measures generated as a quasi-linear mean of the content. Subsequently, we consider the polynomially composable certainty measures.

4.1. Polynomially composable information measures

In the case of information measures, by definition, $h : \mathbb{R} \to \mathbb{R}$ is an increasing function such that $h(0) = 0$, which implies $b = 0$, $a > 0$ and $d = 1$, $c \cdot e > 0$ in (28), so we have

$$h(x) = \begin{cases} \frac{ax}{e}, & a > 0, \quad \text{for } e = 0 \\ \frac{2^e - 1}{e}, & c \cdot e > 0, \quad \text{for } e \neq 0. \end{cases}$$ \hspace{1cm} (30)

The corresponding composition operation is defined with

$$F(u, v) = u \oplus v = u + v \quad \text{and} \quad F(u, v) = u \oplus v = u + v + euv,$$ \hspace{1cm} (31)

which is the pseudo-addition defined in [27], [10].

Using the pseudo-addition (31) and the form of function $h$ given by (30), polynomially additive information measures can be characterized with the following instance of the axiomatic system [A1]-[A3].

[11] The information content $\mathcal{E}I : (0, 1] \to \mathbb{R}^+$ is a continuous decreasing function, which is pseudo-additive:

$$\mathcal{E}I(pq) = \mathcal{E}I(p) \oplus \mathcal{E}I(q), \quad \text{for all } p, q \in (0, 1].$$ \hspace{1cm} (32)

[12] The $n$-dimensional inaccuracy measure is a continuous function, $I : \Delta_n \times \Delta_n \to \mathbb{R}^+$, $n \in \mathbb{N}$, which to every pair of distributions $U = (u_1, \ldots, u_n) \in \Delta_n$, $P = (p_1, \ldots, p_n) \in \Delta_n^+$ assigns a quasi-linear mean of the information content

$$I(U; P) = f^{-1}\left( \sum_{k=1}^n u_k \cdot f(\mathcal{E}I(p_k)) \right),$$ \hspace{1cm} (33)

where $f$ is invertible and continuous.
The measure can be characterized with the following instance of the axiomatic system \( I(U; P) = I(U) \cap I(P) \), for all \( U, V, P, Q \in \Delta_n \).

According to formulas (38) and (39), the generalized inaccuracy measures satisfying [I1]-[I3] are uniquely determined with the class

\[
I(U; P) = \left\{
\begin{aligned}
I_1(U; P) &= \sum_{k=1}^{n} u_k \log_2 p_k^7, & \lambda = 0, & e = 0 \\
I_{x,\lambda}(U; P) &= \frac{1}{\lambda} \log_2 \left( \sum_{k=1}^{n} u_k p_k^{x\lambda} \right), & \lambda \neq 0, & e = 0 \\
I_{x}^{(x)}(U; P) &= \frac{1}{e} \left( \exp \left( \sum_{k=1}^{n} \tau \cdot c \cdot u_k \log p_k \right) - 1 \right), & \lambda = 0, & e \neq 0 \\
I_{x,\lambda}^{(x)}(U; P) &= \frac{1}{e} \left( \left( \sum_{k=1}^{n} u_k p_k^{x\lambda} \right)^e - 1 \right), & \lambda \neq 0, & e \neq 0
\end{aligned}\right.
\]

where \( \tau < 0 \) and \( c \cdot e > 0 \). Note that, in the case of \( e = 0 \), we can fix the value of the parameter \( a \) appearing in the function (30) to \( a = 1 \), since \( a \) is a multiplicative term and \( \tau \) and \( \lambda \) in (35) can be arbitrarily chosen.

If the normalization condition \( I(U; P) = 1 \) is additionally satisfied, then \( \tau = -1 \) and \( e = 2^\tau - 1 \) and the class of inaccuracy measures (35) reduce to the class derived and characterized in [14]. The majority of previously considered information measures obtainable by a specific choice of the parameters in (35) are listed in the Table [1].

**4.2. Polynomially composable certainty measures**

In the case of certainty measures, by definition, \( h : \mathbb{R} \to \mathbb{R}^+ \) is a positive decreasing function such that \( h(+\infty) = 0 \). The linear function case is not possible since a linear decreasing function cannot be positive for all \( x \in \mathbb{R}^+ \). In the case of the exponential function, \( d = 0 \) since \( h(+\infty) = 0 \). Accordingly, for certainty measures, (28) has the following form

\[
h(x) = \frac{2^{-e\cdot x}}{e}; \quad c \cdot e > 0. \tag{36}
\]

The corresponding composition operation is defined with

\[
x \odot y = e \cdot x \od y. \tag{37}
\]

Using the composition operation (27) and the corresponding isomorphism (56) the polynomial certainty measure can be characterized with the following instance of the axiomatic system [A1]-[A3].

**[C1]** The certainty content \( CC : (0, 1] \to (0, 1] \) is a continuous increasing function, which is multiplicative:

\[
CC(pq) = c \cdot CC(p) \cdot CC(q), \quad \text{for all } p, q \in (0, 1]. \tag{38}
\]

**[C2]** The \( n \)-dimensional certainty measure is a continuous function, \( C : \Delta_n \times \Delta_n \to \mathbb{R}^+, n \in \mathbb{N}, \) which to every pair of distributions \( U = (u_1, \ldots, u_n) \in \Delta_n, P = (p_1, \ldots, p_n) \in \Delta_n^+ \) assigns a quasi-linear mean of the certainty content

\[
C(U; P) = f^{-1} \left( \sum_{k=1}^{n} u_k \cdot f(CC(p_k)) \right), \tag{39}
\]

where \( f \) isinvertible and continuous.
[C3] \( C \) is multiplicative:
\[
C(U \ast V; P \ast Q) = C(U; P) \odot C(V; Q),
\]
for all \( U, V, P, Q \in \Delta_n \).

According to formulas (8) and (30) the generalized certainty measures satisfying [C1]-[C3] are uniquely determined with the class
\[
C(\tau; P) = \begin{cases} 
\frac{1}{\tau} \cdot \exp \left\{ -\frac{n}{\tau} \cdot c \cdot u_k \log p_k \right\}, & \tau > 0, \gamma \neq 0 \\
\frac{1}{\tau} \cdot \left( \sum_{k=1}^{n} u_k p_k^{\tau \lambda} \right)^{-\frac{1}{\tau}}, & \lambda \neq 0, \gamma \neq 0
\end{cases}
\]
where \( \tau < 0 \) and \( c \cdot e > 0 \).

If \( e = 1 \) and \( U \equiv P \), the axiomatic system [C1]-[C3] and the class of certainty measures (41) reduces to the one considered in [43], while the case of \( U = (p^{(1)}_1, \ldots, p^{(n)}_n) \) is considered in [8]. The majority of previously considered certainty measures which can be obtained by a specific choice of the parameters in (41) are listed in Table 2.

5. Conclusion and further work

In this paper we considered the axiomatic characterization of information and certainty measures and derived the Inforcer measure, which generalizes all of them. The definition of the Inforcer measure paves the way for unification of generalized divergence measures [19], [23], [7], [39], [38], which should be explored further.

According to the axiomatic system the Inforcer is a composable measure which can be represented as quasi-linear mean-value of composable Inforcer content. The composition operation was defined using the monotonic function \( h : \mathbb{R} \rightarrow \mathbb{R} \), which is increasing for the information and decreasing for the certainty measures. As the simplest case, we defined the class of composition operations under the assumption that the operation should have polynomial representation by use of the polynomial addition theorem [2]. The discussion can be further generalized by assuming the rational or algebraic functions instead of the polynomial, in which case the rational and algebraic addition theorems [2] should be used.

In addition, it seems important to further generalize the Inforcer axiomatic system and to derive the generalization of the Inforcer measure which covers Abe [1], Kaniadakis [17], [18] and Sharma-Mittal-Taneja [36], [23] entropies, which were not covered by our framework.
Table 1: Generalized information measures \(^{(35)}\), with \(u_k = \frac{\phi_k(p_t)}{\sum_{i=1}^{n} p_i \phi_k}\)

| Measure | type | \(\phi_k(p)\) | \(c\) | \(e\) | \(\tau\) | \(\lambda\) |
|---------|------|-------------|------|------|------|------|
| Shannon \(^{(24)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | – |
| Rényi \(^{(33)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(1 - \alpha\) |
| Varma \(^{(44)}\), Nath \(^{(26)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(\mu - \alpha\) |
| Nath \(^{(24)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(1 - \frac{\alpha}{\mu}\) |
| Aczél and Daróczy \(^{(3)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(1 - \alpha\) |
| Kapur \(^{(20)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(1 - \alpha\) |
| Rathie \(^{(32)}\) | \(I_{t,\lambda}\) | \(p\) | – | – | –1 | \(1 - \alpha\) |
| Measure | type | $\phi_c(p)$ | $c$ | $e$ | $\tau$ | $\lambda$ |
|---------|------|-------------|-----|-----|-------|---------|
| Khan and Autar [22] | $I_{\tau,\lambda}$ | $p^{\beta}v_i$ | – | – | $-1$ | $1-\alpha$ |
| Singh et al. [37] | $I_{\tau,\lambda}$ | $p^{\beta}v_i$ | – | – | $-\beta$ | $1-\alpha$ |
| Havrda and Charvát [16], Daróczy [11] | $I_{c, e, \tau, \lambda}$ | $p$ | $1-\gamma$ | $2^{1-\gamma}-1$ | $-1$ | $1-\gamma$ |
| Sharma and Mittal [35] | $I_{c, e, \tau, \lambda}$ | $p$ | $1-\gamma$ | $2^{1-\gamma}-1$ | $-1$ | – |
| Tsallis [41] | $I_{c, e, \tau, \lambda}$ | $p$ | $1-\gamma$ | $2^{1-\gamma}-1$ | $1-\alpha$ | $-1$ |
| Frank and Daffertshofer [13] | $I_{c, e, \tau, \lambda}$ | $p$ | $1-\gamma$ | $1-\gamma$ | $-1$ | $1-\gamma$ |
| Arimoto [5] | $I_{c, e, \tau, \lambda}$ | $p$ | $1-\gamma$ | $1-\gamma$ | $-1$ | $1-\alpha$ |
| Measure | type | $\phi_t(p)$ | $c$ | $e$ | $\tau$ | $\lambda$ |
|---------|------|-------------|-----|-----|-------|-------|

Boekee and Van der Lubbe [9]

\[
\frac{\gamma}{1 - \gamma} \left( \left( \sum_{i=1}^{n} p_i^\gamma \right)^\frac{1}{\gamma} - 1 \right)
\]

\[
\frac{1}{\lambda} \log \left( \sum_{k=1}^{n} p_k^{\tau+1} \right)
\]

\[
\frac{1}{\gamma} \left( \exp \left\{ \sum_{i=1}^{n} \gamma \cdot c \cdot p_k \log p_k \right\} - 1 \right)
\]

Van der Lubbe et al. [43]

\[
\sum_{i=1}^{n} p_i \log p_i^\gamma
\]

\[
\frac{1}{\lambda} \log \left( \sum_{k=1}^{n} p_k^{\tau+1} \right)
\]

\[
\frac{1}{\gamma} \left( \left( \sum_{i=1}^{n} p_i^{\tau+1} \right)^\frac{1}{\gamma} - 1 \right)
\]

Kerridge [21]

\[
- \sum_{i=1}^{n} u_i \log p_i
\]

Nath [25]

\[
\frac{1}{2^{1 - \gamma} - 1} \left( \sum_{i=1}^{n} u_i p_i^{\gamma - 1} - 1 \right)
\]

Gupta and Sharma [14], Picard [31]

\[
\frac{1}{2^{1 - \gamma} - 1} \left( \exp \left\{ \sum_{k=1}^{n} (\gamma - 1) u_k \log p_k \right\} - 1 \right)
\]

\[
\frac{1}{2^{1 - \gamma} - 1} \left[ \left( \sum_{i=1}^{n} u_i p_i^{\tau+1} \right)^{\frac{1}{\gamma}} - 1 \right]
\]
Table 2: Generalized certainty measures (41), with $u_k = \frac{\phi(p_k)}{\sum_{i=1}^{n} p_i}$

| Measure | type | $\phi(p)$ | $c$ | $e$ | $\tau$ | $\lambda$ |
|---------|------|-----------|-----|-----|-------|--------|
| Onicescu [28], Weaver [45] | $\sum_{i=1}^{n} P_i^2$ | $C_{\tau,\lambda}$ | $p$ | 1 | 1 | $-1$ | $-1$ |
| Teodorescu [40] | $\frac{1}{\gamma - 1} \sum_{i=1}^{n} p_i^\gamma$ | $C_{\tau,\lambda}$ | $p$ | $\gamma - 1$ | $\gamma - 1$ | $-1$ | $1 - \gamma$ |
| Pardo and Taneja [29] | $\sum_{i=1}^{n} p_i^\gamma$ | $C_{\tau,\lambda}$ | $p$ | $\gamma - 1$ | 1 | $-1$ | $1 - \gamma$ |
| Pardo [30] | $\frac{1}{\gamma - 1} \sum_{i=1}^{n} u_ip_i^\gamma$ | $C_{\tau,\lambda}$ | $u_k \cdot p$ | $\gamma - 1$ | $\gamma - 1$ | $-1$ | $1 - \gamma$ |
| Tuteja et al. [42] | $\frac{1}{\gamma - 1} \left( \frac{\sum_{i=1}^{n} u_ip_i^\gamma}{\sum_{i=1}^{n} u_ip_i} \right)^\frac{\gamma - 1}{1 - \beta}$ | $C^{\gamma,\beta}_{\tau,\lambda}$ | $u_k \cdot p$ | $\gamma - 1$ | $\gamma - 1$ | $\frac{\gamma - 1}{1 - \beta}$ | $1 - \beta$ |
| Van der Lubbe et al. [43] | $\sum_{i=1}^{n} \exp \{ \tau \cdot p_k \log p_k \} \left( \sum_{k=1}^{n} P_k^{1+\tau} \right)^{1/\lambda}$ | $C_{\tau}$ | $p$ | $-1$ | 1 | $\tau$ | $-\lambda$ |
| Bhatia [8] | $\sum_{k=1}^{n} \exp \left\{ \tau \cdot p_k^\beta \log p_k \right\} \left( \sum_{k=1}^{n} P_k^\beta \right)^{1/\lambda}$ | $C_{\tau,\lambda}$ | $p^\beta$ | $-1$ | 1 | $\tau$ | $\lambda$ |

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