TWISTED COHOMOLOGY OF THE HILBERT SCHEMES OF
POINTS ON SURFACES

MARC A. NIEPER-WISSKIRCHEN

Abstract. We calculate the cohomology spaces of the Hilbert schemes of
points on surfaces with values in locally constant systems. For that purpose,
we generalise I. Grojnowski’s and H. Nakajima’s description of the ordinary
cohomology in terms of a Fock space representation to the twisted case. We
further generalise M. Lehn’s work on the action of the Virasoro algebra to the
twisted case.

Building on work by M. Lehn and Ch. Sorger, we then give an explicit
description of the cup-product in the twisted case whenever the surface has a
numerically trivial canonical divisor.

We formulate our results in a way that they apply to the projective and
non-projective case in equal measure.

As an application of our methods, we give explicit models for the cohomol-
ygy rings of the generalised Kummer varieties and of a series of certain even
dimensional Calabi–Yau manifolds.

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1. Introduction and results

Let $X$ be a quasi-projective smooth surface over the complex numbers. We
denote by $X^{[n]}$ the Hilbert scheme of $n$ points on $X$, parametrising zero-dimensional
subschemes of $X$ of length $n$. It is a quasi-projective variety (Gro61) and smooth
of dimension $2n$ (Fog68). Recall that the Hilbert scheme $X^{[n]}$ can viewed as a
resolution of the $n$-th symmetric power $X^{(n)} := X^n/\mathfrak{S}_n$ of the surface $X$ by virtue
of the Hilbert–Chow morphism $\rho: X^{[n]} \to X^{(n)}$, which maps each zero-dimensional
subscheme $\xi$ of $X$ to its support $\text{supp}\xi$ counted with multiplicities.

Let $L$ be a locally constant system (always over the complex numbers and of
rank 1) over $X$. We can view it as a functor from the fundamental groupoid $\Pi$ of
$X$ to the category of one-dimensional complex vector spaces.

The fundamental groupoid $\Pi^{(n)}$ of $X^{(n)}$ is the quotient groupoid of $\Pi^n$ by the
natural $\mathfrak{S}_n$-action by Bro88 or (in terms of the fundamental group) Bea83. Thus
we can construct from \( L \) a locally constant system \( L^{(n)} \) on \( X^{(n)} \) by setting
\[
L^{(n)}(x_1, \ldots, x_n) := \bigotimes_i L(x_i),
\]
for each \((x_1, \ldots, x_n) \in X^{(n)}\) (for the notion of the tensor product over an unordered index set see, e.g., [LS03]). This induces the locally free system \( L^{[n]} := \rho^* L^{(n)} \) on \( X^{[n]} \). We are interested in the calculation of the cohomology space \( \bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}[2n]) \). Beside the natural grading given by the cohomological degree it carries a weighting given by the number of points \( n \). Likewise, the symmetric algebra \( S^*(\bigoplus_{\nu \geq 1} H^*(X, L^\nu[2])) \) carries a grading by cohomological degree and a weighting, which is defined such that \( H^*(X, L^\nu[2]) \) is of pure weight \( \nu \).

The first result of this paper is the following:

**Theorem 1.1.** There is a natural vector space isomorphism
\[
\bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}[2n]) \cong S^* \left( \bigoplus_{\nu \geq 1} H^*(X, L^\nu[2]) \right)
\]
that respects the grading and weighting.

For \( L = \mathbb{C} \), the trivial system, this result has already appeared in [Gro96] and [Nak97].

Theorem 1.1 is proven by defining a Heisenberg Lie algebra \( \mathfrak{h}_{X, L} \), whose underlying vector space is given by
\[
\bigoplus_{n \geq 0} H^*(X, L^n[2]) \oplus \bigoplus_{n \geq 0} H^*_c(X, L^{-n}[2]) \oplus \mathbb{C} c \oplus \mathbb{C} d
\]
and by showing that \( \bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}[2n]) \) is an irreducible lowest weight representation of this Lie algebra, as is done in [Nak97] for the untwisted case.

Let \( G \) be a finite subgroup of the group of locally constant systems on \( X \). Via the mapping \( L \mapsto L^{[n]} \), which is in fact an isomorphism between the groups of locally constant systems on \( X \) and \( X^{[n]} \), respectively, \( G \) becomes a subgroup of the group of locally constant systems on \( X^{[n]} \). Such a group naturally defines a Galois covering \( \eta: GX^{[n]} \to X^{[n]} \) of degree \( |G| \) with \( \eta_* \mathbb{C} = \bigoplus_{L \in G} L \). Let us call this covering the \( G \)-covering of \( X^{[n]} \). (In case that \( G \) is the group of all locally constant systems on \( X^{[n]} \), the \( G \)-covering is the universal one for \( n \geq 2 \).) Using the Leray spectral sequence for \( \eta \), which already degenerates at the \( E_2 \)-term, the cohomology of \( GX^{(n)} \) can be computed by Theorem 1.1.

**Corollary 1.2.** There is a natural vector space isomorphism
\[
\bigoplus_{n \geq 0} H^*(GX^{[n]}, \mathbb{C}[2n]) \cong \bigoplus_{L \in G} S^* \left( \bigoplus_{\nu \geq 1} H^*(X, L^\nu[2]) \right)
\]
that respects the grading and weighting.

We then proceed in the paper by defining a twisted version \( \mathfrak{v}_{X, L} \) of the Virasoro Lie algebra, whose underlying vector space will be given by
\[
\bigoplus_{n \geq 0} H^*(X, L^n) \oplus \bigoplus_{n \geq 0} H^*_c(X, L^{-n}) \oplus \mathbb{C} c \oplus \mathbb{C} d.
\]
(Note the different grading compared to \( \mathfrak{h}_{X, L} \).) We define an action of \( \mathfrak{v}_{X, L} \) on \( \bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}[2n]) \) by generalising results of [Leh99] to the twisted, not necessarily projective case. As in [Leh99], we calculate the commutators of the operators in \( \mathfrak{h}_{X, L} \) with the boundary operator \( \partial \) that is given by multiplying with \(-\frac{1}{2}\) of
the exceptional divisor class of the Hilbert–Chow morphism. It turns out that the same relations as in the untwisted, projective case hold.

The next main result of the paper is a description of the ring structure whenever $X$ has a numerically trivial divisor. Following ideas in [LS03], we introduce a family of explicitly described graded unital algebras $H^{[n]}$ associated to a $G$-weighted (non-counital) graded Frobenius algebra $H$ of degree $d$. For example, $H = \bigoplus_{L \in G} H^*(X, L[2])$ is such a Frobenius algebra of degree 2. The following holds for each $n \geq 0$:

**Theorem 1.3.** Assume that $X$ has a numerically trivial canonical divisor. Then there is a natural isomorphism

$$\bigoplus_{L \in G} H^*(X^[[n]], L^[[2n]]) \rightarrow \left( \bigoplus_{L \in G} H^*(X, L[2]) \right)^[[n]]$$

of $(G$-weighted) graded algebras of degree $2n$.

For $L = C$, and $X$ projective, this theorem is the main result in [LS03].

The idea of the proof of Theorem 1.3 is not to reinvent the wheel but to study how everything can already be deduced from the more special case considered in [LS03].

Again by the Leray spectral sequence, also Theorem 1.3 has a natural application to the cohomology ring of the $G$-coverings of $X^{[n]}$:

**Corollary 1.4.** There is a natural isomorphism

$$H^*(GX^[[n]], C[2n]) \rightarrow \left( \bigoplus_{L \in G} H^*(X, L[2]) \right)^[[n]]$$

of graded unital algebras of degree $2n$.

We want to point out at least two applications of our results. The first one is the computation of the cohomology ring of certain families of Calabi–Yau manifolds of even dimension: Let $X$ be an Enriques surface. Let $G$ be the group of all locally constant systems on $X$, i.e. $G \cong \mathbb{Z}/(2)$. We denote the non-trivial element in $G$ by $L$. The Hodge diamonds of $H^*(X, C[2])$ and $H^*(X, L[2])$ are given by

$$
\begin{array}{ccc}
1 & & 0 \\
0 & 10 & 0 \\
0 & 0 & 10 \\
1 & & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & & 0 \\
0 & 10 & 1 \\
0 & 0 & 0 \\
1 & & 0
\end{array}
$$

respectively.

Denote by $X^{[n]}$ the (two-fold) universal cover of $X^{[n]}$. By Remark 2.6, the isomorphism of Corollary 1.2 is in fact an isomorphism of Hodge structures. It follows that

$$H^{k,0}(X^{[n]}, C) = \begin{cases} 
\mathbb{C} & \text{for } k = 0 \text{ or } k = 2n, \\
0 & \text{for } 0 < k < 2n.
\end{cases}$$

In conjunction with Corollary 1.4, we have thus proven:

**Proposition 1.5.** For $n > 1$, the manifold $X^{[n]}$ is a Calabi–Yau manifold in the strict sense. Its cohomology ring $H^*(X^{[n]}, C[2n])$ is naturally isomorphic to $(H^*(X, C[2]) \oplus H^*(X, L[2]))[[n]]$.

Our second application is the calculation of the cohomology ring of the generalised Kummer varieties $X^{[[n]]}$ for an abelian surface $X$. (A slightly less explicit description of this ring has been obtained by more special methods in [Bri02].) Recall from [Ben83] that the generalised Kummer variety $X^{[[n]]}$ is defined as the fibre
over 0 of the morphism \( \sigma : X^{[n]} \to X \), which is the Hilbert–Chow morphism followed by the summation morphism \( X^{(n)} \to X \) of the abelian surface. The generalised Kummer surface is smooth and of dimension \( 2n - 2 \) \((\text{[Dea83]}).\)

As above, let \( H \) be a \( G \)-weighted graded Frobenius algebra of degree \( d \). Assume further that \( H \) is equipped with a compatible structure of a Hopf algebra of degree \( d \). For each \( n > 0 \), we associate to such an algebra an explicitly described graded unital algebra \( H[[n]] \) of degree \( n \).

In the following Theorem, we view \( H^*(X, \mathbb{C}[2]) \) as such an algebra (the Hopf algebra structure given by the group structure of \( X \)), where we give \( H^*(X, \mathbb{C}[2]) \) the trivial \( G \)-weighting for the group \( G := X[n]^\vee \), the character group of the group of \( n \)-torsion points on \( X \). We prove the following:

**Theorem 1.6.** There is a natural isomorphism

\[
H^*(X^{[n]}, \mathbb{C}[2n]) \to (H^*(X, \mathbb{C}[2]))^{[n]}
\]

of graded unital algebras of degree \( 2n \).

We should remark that most the “hard work” that is hidden behind the scenes has already been done by others (\([\text{Gro96]}, \text{Nak97}, \text{Leh99}, \text{LQW02}, \text{LS03}], \text{etc.}\)), and our own contribution is to see how the ideas and results in the cited papers can be applied an generalised to the twisted and to the non-projective case.

**Remark 1.7.** Let us finally mention that the restriction to algebraic, i.e. quasi-projected surfaces, is unnecessary. Our methods work equally well when we replace \( X \) by any complex surface. In this case, the Hilbert schemes become the Douady spaces (\([\text{Dou66]}\)).

2. **The Fock space description**

In this section, we prove Theorem 1.1 for a locally constant system \( L \) on \( X \) by the method that is used in \([\text{Nak97}]) \) for the untwisted case, i.e. by realising the cohomology space of the Hilbert schemes (with coefficients in a locally constant system) as an irreducible representation of a Heisenberg Lie algebra.

Let \( l \geq 0 \) and \( n \geq 1 \) be two natural numbers. Set

\[
X^{(l,n)} := \left\{ (\underline{x}', x, \underline{x}) \in X^{(n+l)} \times X \times X^{(l)} | \underline{x}' = \underline{x} + nx \right\}
\]

(we write the union of unordered tuples additively). The preimage in \( X^{(n+l)} \times X \times X^{(l)} \) of this closed subset under the Hilbert–Chow morphisms \( \rho \) is denoted by \( X^{[n,l]} \) (this incidence variety has already been considered in \([\text{Nak97}]) \).

We denote the projections of \( X^{(l+n)} \times X \times X^{(l)} \) onto its three factors by \( \tilde{p} \), \( \tilde{q} \) and \( \tilde{r} \), respectively. Likewise, we denote the three projections of \( X^{[l+n]} \times X \times X^{[l]} \) by \( p \), \( q \) and \( r \).

**Lemma 2.1.** It is \( q^*L^n \otimes r^*L^l \mid_{X^{[n,l]}} = p^*L^{[l+n]} \mid_{X^{[n,l]}} \).

**Proof.** It is \( \tilde{q}^*L^n \otimes \tilde{r}^*L^l \mid_{X^{(n,l)}} = \tilde{p}^*L^{(l+n)} \mid_{X^{(n,l)}} \). This follows from

\[
(q^*L^n \otimes r^*L^l)(\underline{x} + nx, x, \underline{x}) = L(x)^{\otimes n} \otimes \bigotimes_{x' \in \underline{x}} L(x') = \bigotimes_{x' \in \underline{x} + nx} L(x') = \tilde{p}^*L^{(l+n)}(\underline{x} + nx, x, \underline{x})
\]

for every \((\underline{x} + nx, x, \underline{x}) \in X^{(l,n)}\). By pulling back everything to the Hilbert schemes, the Lemma follows. \(\square\)
Due to Lemma 2.1 and the fact that $p|_{X^{[l,n]}}$ is proper (Nak97), the operator (a correspondence, see Nak97)

$$N: H^*(X, L^\nu[2]) \times H^*(X^{[l]}, L^{[l]}[2l]) \to H^*(X^{[l+n]}, L^{[l+n]}[2(l+n)]),$$

$$(\alpha, \beta) \mapsto \text{PD}^{-1} p_* (\langle q^* \alpha \sqcup r^* \beta \rangle \cap [X^{[l,n]}])$$

is well defined. Here,

$$\text{PD}: H^*(X^{[l+n]}, L^{[n+l]}[2(l+n)]) \to H^*_\text{BM}(X^{[l+n]}, L^{[n+l]}[-2(l+n)])$$

is the Poincaré-duality isomorphism between the cohomology and the Borel–Moore homology. (The degree shifts are chosen in a way that $N$ an operator of degree 0, see LS03.)

Furthermore, $q \times r|_{X \times X^{[l]}}$ is proper (Nak97). Thus we can also define an operator the other way round:

$$N^\dagger: H^*_c(X, L^{-\nu}[2]) \times H^*(X^{[n]}, L^{[l]}[2]) \to H^*_c(X^{[l]}, L^{[l]}[2l]),$$

$$(\alpha, \beta) \mapsto (-1)^n \text{PD}^{-1} r_* (\langle q^* \alpha \sqcup p^* \beta \rangle \cap [X^{[l,n]}])$$

As in Nak97, we will use these operators to define an action of a Heisenberg Lie algebra on

$$V_{X,L} := \bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}).$$

For this, let $A$ be a weighted, graded Frobenius algebra of degree $d$ (over the complex numbers), that is a weighted and graded vector space over $\mathbb{C}$ with a (graded) commutative and associative multiplication of degree $d$ and weight 0 and a unit element 1 (necessarily of degree $-d$ and weight 0) together with a linear form $f: A \to \mathbb{C}$ of degree $-d$ and weight 0 such that for each weight $\nu \in \mathbb{Z}$ the induced bilinear form $\langle \cdot, \cdot \rangle: A(\nu) \times A(-\nu) \to \mathbb{C}, (a, a') \mapsto \int_A aa'$ is non-degenerate (of degree 0). Here $A(\nu)$ denotes the weight space of weight $\nu$. In particular, all weight spaces are finite-dimensional. In the case of a trivial weighting, this notion of a graded Frobenius algebra has already appeared in LS03.

Example 2.2. The vector space

$$A_{X,L} := \bigoplus_{\nu \geq 0} H^*(X, L^\nu[2]) \oplus \bigoplus_{\nu \geq 0} H^*_c(X, L^{-\nu}[2])$$

is naturally a weighted, graded Frobenius algebra of degree 2 ($= \dim X$) as follows: The grading is given by the cohomology grading. The weighting is defined by defining $H^*(X, L^\nu[2])$ of pure weight $\nu$ for $\nu \geq 0$ and $H^*_c(X, L^{-\nu}[2])$ of pure weight $-\nu$. The multiplication is given by the cup-product, where we view the product of an ordinary cohomology class and of a cohomology class with compact support as an ordinary cohomology class whenever the resulting weight is strictly positive, and as a cohomology class with compact support otherwise. The linear form $f$ is given by evaluating a class with compact support on the fundamental class of $X$.

For such a weighted, graded Frobenius algebra $A$ we set

$$\mathfrak{h}_A := A \oplus \mathbb{C}c \oplus \mathbb{C}d.$$
For $A = A_{X,L}$, we set $\mathfrak{h}_{X,L} := \mathfrak{h}_A$. We define a linear map

$$q: \mathfrak{h}_{X,L} \to \text{End}(V_{X,L})$$

as follows: Let $l \geq 0$ and $\beta \in V_{X,L}(l) = H^*(X^{[l]}, L^[[2l]])$. We set $q(c)(\beta) := \beta$, and $q(d)(\beta) := \beta$. For $n \geq 0$, and $\alpha \in A_{X,L}(\nu) = H^*(X, L^[[2]])$, we set $q(\alpha)(\beta) := N(\alpha, \beta)$. For $\alpha \in A_{X,L}(-\nu) = H^*(X, L^{-\nu}[2])$, we set $q(\alpha)(\beta) := N^!(\alpha, \beta)$. Finally, we set $q(\alpha)(\beta) = 0$ for $\alpha \in A_{X,L}(0) = H^*(X, C) \oplus H^*_c(X, C)$.

**Proposition 2.4.** The map $q$ is a weighted, graded action of $\mathfrak{h}_{X,L}$ on $V_{X,L}$.

**Proof.** This Proposition is proven in [Nak97] for the untwisted case, i.e. for $L = C$. The proof there is based on calculating commutators on the level of cycles of the correspondences defined by the incidence schemes $X^{[l,n]}$. These commutators are independent of the locally constant system used. Thus the proof in [Nak97] also applies to this more general case. \hfill □

**Example 2.5.** Let $\alpha = \sum \alpha_{(1)} \otimes \cdots \otimes \alpha_{(n)} \in H^*(X^{(n)}, L^{(n)}[2n]) = S^n H^*(X, L[2])$ (we use the Sweedler notation to denote elements in tensor products). The pull-back of $\alpha$ by the Hilbert–Chow morphism $\rho: X^{[n]} \to X^{(n)}$ is then given by

$$\rho^* \alpha = \frac{1}{n!} \sum q(\alpha_{(1)}) \cdots q(\alpha_{(n)}) |0|,$$

where $|0|$ is the unit $1 \in H^*(X^{[0]}, C) = C$.

We will use Proposition 2.4 to prove our first Theorem.

**Proof of Theorem 1.1** The vector space $\hat{V}_{X,L} := S^*(\bigoplus_{\nu \geq 1} H^*(X, L^\nu[2]))$ carries a unique structure as an $\mathfrak{h}_{X,L}$-module such that $c$ acts as the identity, $d$ acts by multiplying with the weight, $\alpha \in H^*(X, L^n)$ for $n \geq 1$ acts by multiplying with $\alpha$, and $\alpha \in H^*(X, C) \oplus H^*_c(X, C)$ acts by zero. By the representation theory of the Lie algebras of Heisenberg-type, this is an irreducible lowest weight representation of $\mathfrak{h}_{X,L}$, which is generated by the lowest weight vector 1, which is of weight 0.

The $\mathfrak{h}_{V,L}$-module $V_{X,L}$ also has a vector of weight 0, namely $|0|$. Thus, there is a unique isomorphism $\Phi: \hat{V}_L \to V_L$ of $\mathfrak{h}_L$-modules that maps $1$ to $|0|$. This will be the inverse of the isomorphism mentioned in Theorem 1.1. It remains to show that $\Phi$ is bijective. The injectivity follows from the fact that $\hat{V}_{X,L}$ is irreducible as an $\mathfrak{h}_{X,L}$-module.

In order to prove the surjectivity, we will derive upper bounds on the dimensions of the weight spaces of the right hand side $V_{X,L}$ (see also [Leh04] about this proof method). By the Leray spectral sequence associated to the Hilbert–Chow morphism $\rho: X^{[n]} \to X^{(n)}$, such an upper bound is provided by the dimension of the spectral sequence’s $E_2$-term $H^*(X^{(n)}, R^* \rho_* L[2n])$. As shown in [GS93], it follows from the Beilinson–Bernstein–Deligne–Gabber decomposition theorem that

$$R^* \rho_! Q[2n] = \bigoplus_{\lambda \in P(n)} (i_\lambda)_* Q[2\ell(\lambda)].$$

Here, $P(n)$ is the set of all partitions of $n$, $\ell(\lambda) = r$ is the length of a partition $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^r)$, $X^{(\lambda)} := \{ \sum_{i=1}^r \lambda_i x_i \mid x_i \in X \} \subset X^{(n)}$, and $i_\lambda: X^{(\lambda)} \to X^{(n)}$ is the inclusion map.

Set $L^{(\lambda)} := i_\lambda^* L^{(n)}$. By the projection formula, it follows that $R^* \rho_* L[2n] = \bigoplus_{\lambda \in P(n)} (i_\lambda)_* L^{(\lambda)}[2\ell(\lambda)]$.

Thus, an upper bound on the dimension of $H^*(X^{[n]}, L^{[2n]}[2\ell(\lambda)])$ is provided by the dimension of $\bigoplus_{\lambda \in P(n)} H^*(X^{(\lambda)}, L^{(\lambda)}[2\ell(\lambda)])$. By [GS93], this can be seen to be
isomorphic to
\[
\bigoplus_{i \geq 1} S^i H^*(X, L[2]),
\]
where each \( \nu_i \geq 0 \).

It follows that the upper bound given by the \( E_2 \)-term is exactly the dimension of the \( n \)-th weight space of \( \tilde{V}_{X, L} \).

Thus the dimension of the weight spaces of \( V_{X, L} \) cannot be greater than the dimensions of the weight spaces of \( \tilde{V}_{X, L} \).

Thus the Theorem is proven.  \( \square \)

**Remark 2.6.** Assume that \( X \) is projective. In this case, the (twisted) cohomology spaces of \( X \) and its Hilbert schemes \( X^{[n]} \) carry pure Hodge structures. As the isomorphism of Theorem 1.1 is defined by algebraic correspondences (i.e. by correspondences of Hodge type \((p, p)\)), it follows that the isomorphism in Theorem 1.1 is compatible with the natural Hodge structures on both sides.

In terms of Hodge numbers, the following equation encodes our result:

\[
\sum_{n \geq 0} \prod_{i, j} h^{i, j}(X^{[n]}, L^{[n]}[2n]) p^i q^j z^n = \prod_{m \geq 1} \prod_{i, j} (1 - (-1)^{i+j} p^i q^j z^m)^{(-1)^{i+j} h^{i, j}(X, L^m[2])}
\]

3. The Virasoro algebra in the twisted case

To each weighted, graded Frobenius algebra \( A \) of degree \( d \), we associate a skew-symmetric form \( \epsilon : A \times A \to C \) of degree \( d \) as follows:

Let \( n \in Z \).

We note that \( A(n) \) and \( A(-n) \) are dual to each other via the linear form \( f \).

Thus we can consider the linear map \( \Delta(n) : C \to A(n) \otimes A(-n) \) dual to the bilinear form \( \langle \cdot, \cdot \rangle : A(n) \otimes A(-n) \to C \).

Write \( \Delta(n)1 = \sum e_{(1)}(n) \otimes e_{(2)}(n) \) in Sweedler notation.

Then we define \( \epsilon \) by setting

\[
\epsilon(\alpha, \beta) := \sum_{\nu=0}^{n} \frac{\nu(n - \nu)}{2} \int \sum e_{(1)}(\nu) e_{(2)}(\nu) \alpha \beta
\]

for all \( \alpha \in A(n) \) whenever \( n \geq 0 \). We shall call this form the Euler form of \( A \).

**Example 3.1.** Assume that \( A(n) \equiv A(0) \) for all \( n \in Z \).

In this case, we have

\[
\epsilon(\alpha, \beta) = \frac{n^3 - n}{12} \int e \alpha \beta
\]

for \( \alpha \in A(n) \) with \( e := \int \sum e_{(1)}(0) e_{(2)}(0) \) (Leh99).

We use the Euler form to define another Lie algebra associated to \( A \). We set

\[ v_A := A[-2] \oplus Cc \oplus Cd. \]

We define the structure of a weighted, graded Lie algebra on \( v_A \) by defining \( c \) to be a central element or weight 0 and degree 0, \( d \) an element of weight 0 and degree 0 and by introducing the following commutator relations:

\[ [d, a] := n \cdot a \text{ for each element } a \in A[-2] \text{ of weight } n, \text{ and } [a, a'] := (da)a' - a(da') - c(a, a') \text{ for elements } a, a' \in A. \]

**Definition 3.2.** The Lie algebra \( v_A \) is the Virasoro algebra associated to \( A \).

For \( A = A_{X, L} \), we set \( v_{X, L} := v_A \). The whole construction is a generalisation to the twisted case of the Virasoro algebra found in [Leh99].
We now define a linear map $L: \mathfrak{B}_{X,L} \to \text{End}(V_{X,L})$ as follows: We define $L(e)$ to be the identity, $L(d)$ to be multiplication with the weight, and for $\alpha \in A[-2]$ we set

$$L(\alpha) := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} \sum_{\nu} q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha):,$$

where the normal ordered product $:aa:$ of two operators is defined to be $aa'$ if the weight of $a$ is greater or equal to the weight of $a'$ and is defined to be $a'a$ if the weight of $a'$ is greater than the weight of $a$.

The following Lemma is proven for the untwisted case in [Leh99].

**Lemma 3.3.** For $\alpha \in A_{V,L}[-2]$ and $\beta \in A_{V,L}$, we have

$$[L(\alpha), q(\beta)] = -q(\alpha[d, \beta]).$$

**Proof.** Let $\alpha \in A_{V,L}[2](n)$ and $\beta \in A_{V,L}(m)$ with $n, m \in \mathbb{Z}$. In the following calculations we omit all Koszul signs arising from commuting the graded elements $\alpha$ and $\beta$. By definition, we have $[L(\alpha), q(\beta)] = \frac{1}{2} \sum_{\nu \in \mathbb{Z}} q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha):, q(\beta)],$ where $\nu$ runs through all integers. As the commutator of two operators in $\mathfrak{B}_{V,L}$ is central, we do not have to pay attention to the order of the factors of the normally ordered product when calculating the commutator:

$$[q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha), q(\beta)] = \nu q(e_{(1)}(\nu), \beta)q(e_{(2)}(\nu)\alpha) + (n - \nu)(e_{(2)}(\nu)\alpha, \beta)q(e_{(1)}(\nu)).$$

As $(\cdot, \cdot)$ is of weight zero, the first summand is only non-zero for $\nu = -m$, while the second summand is only non-zero for $\nu = n + m$. Thus we have

$$[L(\alpha), q(\beta)] = -\frac{m}{2} \sum \left( (e_{(1)}(-m), \beta)q(e_{(2)}(-m)\alpha) + (e_{(2)}(n + m)\alpha, \beta)q(e_{(1)}(n + m)) \right).$$

As $e_{(1)}(\cdot)$ is the dual basis to $e_{(2)}(\cdot)$, the right hand side simplifies to $-mq(\alpha, \beta)$, which proves the Lemma. \qed

We use Lemma 3.3 to prove the following Proposition, which has already appeared in [Leh99] for the untwisted, projective case:

**Proposition 3.4.** The map $L$ is a weighted, graded action of the Virasoro algebra $\mathfrak{B}_{X,L}$ on $V_{X,L}$.

**Proof.** Let $\alpha \in A[-2](m)$ and $\beta \in A[-2](n)$ with $m, n \in \mathbb{Z}$. We have to prove that $[L(\alpha), L(\beta)] = (m - n)L(\alpha, \beta) - e(\alpha, \beta)$. We follow ideas in [FLMSS]. In all summations below, $\nu$ runs through all integers if not specified otherwise.

We begin with the case $n \neq 0$ and $m + n \neq 0$. In this case, by Lemma 3.3 it is

$$[L(\alpha), L(\beta)] = \frac{1}{2} \sum_{\nu \in \mathbb{Z}} q(e_{(1)}(\nu))q(e_{(2)}(\nu)\beta)$$

$$= \frac{1}{2} \left( \sum_{\nu} (-\nu)q(e_{(1)}(\nu)\alpha)q(e_{(2)}(\nu)\beta) + \sum_{\nu}(\nu - n)q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha\beta) \right).$$

As $\sum q(e_{(1)}(\nu)\alpha)q(e_{(2)}(\nu)\beta) = q(e_{(1)}(\nu + m))q(e_{(2)}(\nu - m)\alpha\beta)$, the right hand side is equal to

$$\frac{1}{2} \sum_{\nu} ((-\nu)q(e_{(1)}(\nu + m))q(e_{(2)}(\nu + m)\alpha\beta) + (\nu - n)q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha\beta)), $$

which is nothing else than $(m - n)L(\alpha, \beta)$. Note that $e(\alpha, \beta) = 0$ in this case.
The next case we study is \( m > 0 \) and \( n = -m \). In order to ensure convergence in the following calculations we have to split up \( L(\beta) \) as follows:

\[
L(\beta) = \sum_{\nu \geq m} q(e(1)(\nu)\beta)q(e(2)(\nu)) + \sum_{\nu < m} q(e(2)(\nu))q(e(1)(\nu)\beta)
\]

Calculating the commutator \([L(\alpha), L(\beta)]\) thus yields the four terms:

\[
\frac{1}{2} \left( \sum m - \nu q(e(1)(\nu)\alpha\beta)q(e(2)(\nu)) + \frac{1}{2} \sum \nu q(e(1)(\nu)\beta)q(e(1)(\nu)\alpha) + \frac{1}{2} \sum \nu q(e(2)(\nu)\alpha)q(e(1)(\nu)\beta) + \frac{1}{2} \sum (m - \nu)q(e(2)(\nu))q(e(1)(\nu)\alpha\beta). \right.
\]

As in the first case, we now move \( \alpha \) and \( \beta \) rightwards. Then we can split off an infinite part given by a multiple of \( L(\alpha\beta) \) and are left over with the finite sum

\[
[L(\alpha), L(\beta)] - 2mL(\alpha\beta)
\]

\[
= \frac{1}{2} \sum_{\nu=0}^m (m - \nu) \left( q(e(2)(\nu))q(e(1)(\nu)\alpha\beta) - q(e(1)(\nu))q(e(2)(\nu)\alpha\beta) \right).
\]

The right side is exactly \( c(\alpha, \beta) \).

The remaining cases either follow from the above by exchanging \( n \) and \( m \) or are trivial (\( n = m = 0 \)).

4. THE BOUNDARY OPERATOR

We proceed as in [Leh99] by introducing a boundary operator on \( V_{X,L} \). Recall the definition of the tautological classes of the Hilbert scheme [LQW02]: Let \( \Xi^n \) be the universal family over \( X^{[n]} \), which is a subscheme of \( X^{[n]} \times X \). We denote the projections of \( X^{[n]} \times X \) onto its factors by \( p \) and \( q \). To each \( \alpha \in H^*(X,\mathbb{C}) \) we associate the tautological classes

\[
\alpha^{[n]} := p_*(\text{ch}(\mathcal{O}_\Xi^n) \cup q^*(\text{td}(X) \cup \alpha))
\]

in \( H^*(X^{[n]},\mathbb{C}) \).

Remark 4.1. Note that the tautological classes live in the cohomology with untwisted coefficients, and we have not generalised this concept to the twisted case.

Each \( \alpha \in H^*(X,\mathbb{C}) \) defines an operator \( m(\alpha) \in \text{End}(V_{X,L}) \), which is given by \( m(\alpha)(\beta) := \alpha^{[n]} \cup \beta \) for all \( \beta \in H^*(X^{[n]},L^{[n]}) \). It is an operator of weight zero. As it does not respect the grading, we split it up into its homogeneous components \( m(\alpha) = \sum m^*(\alpha) \) with respect to the grading. Following [Leh99], we set \( d := m^2(1) \) and call it the boundary operator. It is an operator of weight 0 and degree 2. For each operator \( p \in \text{End}(V_{X,L}) \), we set \( p' := [\partial, p] \) and call it the derivative of \( p \).

The main theorem in [Leh99] is the calculation of the derivatives of the Heisenberg operators in the untwisted, projective case. In the sequel, we will do this in our more general case:

Let \( K \) be the canonical divisor of \( X \). We make it into an operator \( K : A_{X,L} \to A_{X,L}[-2] \) of weight zero by setting

\[
K(\alpha) := \frac{|n|-1}{2} K \alpha
\]

for \( \alpha \in H^*(X^{[n]},L^{[n]}) \).

Proposition 4.2. For all \( \alpha, \beta \in A_{X,L} \) the following holds:

\[
[q'(\alpha), q(\beta)] = -q([d, \alpha][d, \beta]) - \int K([d, \alpha])|d, \beta|.
\]
Proof. Let us first consider the case of \( \alpha \in A(m) \) and \( \beta \in A(n) \) with \( n + m \neq 0 \). We have to show that \( \{q'(\alpha), q(\beta)\} = -nmq(\alpha \beta) \). This is proven in [Leh99] for the projective, untwisted case. The proof in [Leh99] is based on calculating the commutator on the level of cycles. As these calculations are local in \( X \), the result remains true for non-projective \( X \). Furthermore, the proof literally works in the twisted case.

The case \( n + m = 0 \) remains. Here we have to show that \( \{q'(\alpha), q(\beta)\} = m^2 \frac{\chi}{2} K \alpha \beta \). In [Leh99] the following intermediate result is formulated for the projective, untwisted case: For all \( m \in \mathbb{Z} \), there exists a class \( K_m \in H^*(X, \mathbb{C}) \) such that \( \{q'(\alpha), q(\beta)\} = m^2 \chi K_m \alpha \beta \). As above the proof for this intermediate result that is given in [Leh99] also works in the twisted and non-projective case. The classes \( K_m \) do not depend on the choice of \( L \), i.e. are universal for the surface. In [Leh99], the classes \( K_m \) are computed for the projective case, namely \( K_m = \frac{\chi}{2} K \), where \( K \) is the class of the canonical divisor. All that remains is to calculate the classes \( K_m \) for the non-projective (untwisted) case. As \( \{q'(\alpha), q(\beta)\} = \{q'(\beta), q(\alpha)\} \) (up to Koszul signs), it is enough to calculate \( K_m \) for \( m > 0 \):

Let \( \beta \in A_X(\mathbb{C}; -m) = H^*_c(X, \mathbb{C}[2]) \). Consider an open embedding \( j : X \to \tilde{X} \) of \( X \) into a smooth, projective surface \( \tilde{X} \). We denote the correspondings embeddings \( X^{[m]} \to \tilde{X}^{[m]} \) also by the letter \( j \). Denote the 1 in \( \tilde{A}_\chi(m) = H^*(X, \mathbb{C}[2]) \) by \( 1(m) \). As all constructions considered so far are functorial (in the appropriate senses) with respect to open embeddings, we have

\[
j^*(\{q'(1(m)), q(j_\* \beta)\}) = \{q'(j^* 1(m)), q(\beta)\}.
\]

The right hand side is given by \( m^2 \chi K \beta \), where \( K \) is the class corresponding to \( X \). By the calculations in [Leh99], the left hand side is given by \( m^2 \chi K \beta \), where \( K \beta \) is the canonical divisor class of \( \tilde{X} \). As \( j^* K \beta = K \beta \), we see that \( K_m = \frac{\chi}{2} K \) also holds in the non-projective case, which proves the Proposition.

\[\square\]

Corollary 4.3. For all \( \alpha \in A_X, L \), the following holds:

\[q'(\alpha) = \text{L}(\langle 1, d\alpha \rangle) + q(K(\langle 1, d\alpha \rangle)).\]

Proof. This can be deduced from 4.2 as the respective statement for the untwisted, projective case in [Leh99] is proven.

\[\square\]

5. THE RING STRUCTURE

From now on, we assume that the canonical divisor of \( X \) is numerically trivial.

Let \( H \) be a non-counital graded Frobenius algebra of degree \( d \) (over the complex numbers), that is a graded vector space over \( \mathbb{C} \) with a (graded) commutative and associative multiplication of degree \( d \) and a unit element \( 1 \) (of degree \( -d \)) together with a coassociative and cocommutative \( H \)-module homomorphism \( \Delta : H \to H \otimes H \) of degree \( d \). (We regard \( H \otimes H \) as an \( H \)-algebra by multiplying on the left factor.) The map \( \Delta \) is called the diagonal.

Example 5.1. Let \( H \) be a graded Frobenius algebra of degree \( d \). The dual \( \Delta \) to the multiplication map \( H \otimes H \to H \) with respect to \( \langle \cdot, \cdot \rangle \) makes \( H \) a non-counital, graded Frobenius algebra of degree \( d \). In this context, the integral \( \int \) of the Frobenius algebra is the counit of \( H \).

Let \( G \) be a finite abelian group, which will be written additively in the sequel. A \( G \)-weighting on \( H \) is an action of the character group \( G^\vee \) of \( G \) on \( H \). In other words, \( H \) comes together with a weight decomposition of the form \( \bigoplus_{L \in G} H(L) \), where each \( \chi \in G^\vee \) acts on \( H(L) \) by multiplication with \( \chi(L) \).
**Example 5.2.** Let $G$ be a finite subgroup of the group of locally constant systems on $X$, written additively. The $G$-weighted vector space

$$H_{X,G} := \bigoplus_{L \in G} H^*(X, L[2])$$

is naturally a non-comital $G$-weighted, graded Frobenius algebra of degree 2 as follows: the grading is given by the cohomological grading. The multiplication is given by the cup product. The diagonal is given by the proper push-forward $\delta_* : H_{X,G} \to H_{X,G} \otimes H_{X,G}$ that is induced by the diagonal map $\delta : X \times X \to X$.

By iterated application, $\Delta$ induce maps $\Delta : H \to H^{\otimes n}$ with $n \geq 1$. We denote the restriction of $\Delta : H \to H^{\otimes n}$ to $H(nL)$, $L \in G$, followed by the projection onto $H(L)^{\otimes n}$ by $\Delta(L) : H(nL) \to H(L)^{\otimes n}$. The element $e := (\nabla \circ \Delta(0))(1) \in H$ is called the Euler class of $H$, where $\nabla : H \otimes H \to H$ is the multiplication map.

There is a construction given in [LS03] that associates to each graded Frobenius algebra $H$ of degree of $d$ a sequence of graded Frobenius algebras $H^n$ (whose degrees are given by $nd$). We extend this construction to $G$-weighted not necessarily comital Frobenius algebras as follows: For each $L \in G$, set

$$H_n(L) := \bigoplus_{\sigma \in \mathfrak{S}_n} \left( \bigotimes_{B \in \sigma \setminus \{n\}} H(|B| L) \right) \sigma$$

and $H_n := \bigoplus_{L \in G} H_n(L)$

where $[n] := \{1, \ldots, n\}$ and $\sigma \setminus [n]$ is the set of orbits of the action of the cyclic group generated by $\sigma$ on the set $[n]$. (Note that $H_n(0) = H(0)\{\mathfrak{S}_n\}$ in the terminology of [LS03].) The symmetric group $\mathfrak{S}_n$ acts on $H_n$. The graded vector space of invariants, $H_n^{\mathfrak{S}_n}$, is denoted by $H^n$.

Let $f : I \to J$ a surjection of finite sets and $(n_i)_{i \in I}$ a tuple of integers. Fibre-wise multiplication yields ring homomorphisms

$$\nabla^{I,J} := \nabla^f : \bigotimes_{i \in I} H(n_i L) \to \bigotimes_{j \in J} H \left( \sum_{f(i) = j} n_i L \right)$$

degree $d(|I| - |J|)$. (These correspond to the ring homomorphism $f^{I,J}$ in [LS03]!) Dually, by using the diagonal morphisms $\Delta(L)$ and relying on their coassociativity and cocommutativity, we can define $\nabla^{I,J}$-module homomorphisms

$$\Delta_{I,J} := \Delta^f : \bigotimes_{j \in J} H \left( \sum_{f(i) = j} n_i L \right) \to \bigotimes_{i \in I} H(n_i L),$$

which are also of degree $d(|I| - |J|)$. (These correspond to the module homomorphisms $f_{I,J}$ in [LS03].)

Let $\sigma, \tau \in \mathfrak{S}_n$ be two permutations. By $\langle \sigma, \tau \rangle$ we denote the subgroup of $\mathfrak{S}_n$ generated by the two permutations. Note that there are natural surjections $\sigma \setminus [n] \to \langle \sigma, \tau \rangle \setminus [n]$, $\tau \setminus [n] \to \langle \sigma, \tau \rangle \setminus [n]$, and $(\sigma \tau) \setminus [n] \to \langle \sigma, \tau \rangle \setminus [n]$. The corresponding ring and module homomorphisms are denoted by $\nabla^{\sigma, \langle \sigma, \tau \rangle}$, etc., and $\Delta_{\langle \sigma, \tau \rangle, \sigma}$, etc.

Let $L, M \in G$. We define a linear map

$$m_{\sigma, \tau} : \bigotimes_{B \in \sigma \setminus [n]} H(|B| L) \otimes \bigotimes_{B \in \tau \setminus [n]} H(|B| M) \to \bigotimes_{B \in (\sigma \tau) \setminus [n]} H(|B| (L + M))$$

by

$$m_{\sigma, \tau}(\alpha \otimes \beta) = \Delta_{\langle \sigma, \tau \rangle, \sigma \tau}(\nabla^{\sigma, \langle \sigma, \tau \rangle}(\alpha) \nabla^{\tau, \langle \sigma, \tau \rangle}(\beta) e^{\gamma(\sigma, \tau)}),$$

where the expression $e^{\gamma(\sigma, \tau)}$ is defined as in [LS03] (we have to use our Euler class $e$, which is defined above). This defines a product $H_n \otimes H_n \to H_n$ which is given
by
\[(\alpha \sigma) \cdot (\beta \sigma) := m_{\sigma, \tau}(\alpha, \beta) \sigma \tau\]
for \(\alpha \sigma \in H_n(L)\) and \(\beta \sigma \in H_n(M)\). This product is associative, \(\mathcal{S}_n\)-equivariant, and of degree \(nd\), which can be proven exactly as the corresponding statements about the product of the rings \(H\{\mathcal{S}_n\}\), which are defined in [LS03]. The product becomes (graded) commutative when restricted to \(H[^n]\). Thus we have made \(H[^n]\) a graded commutative, unital algebra of degree \(nd\).

**Definition 5.3.** The algebra \(H[^n]\) is the \(n\)-th Hilbert algebra of \(H\).

In case \(G\) is trivial, the \(n\)-Hilbert algebra of \(H\) defined here is exactly the algebra \(H[^n]\) of [LS03]. For non-trivial \(G\), this is no longer true.

The underlying graded vector space of \(\sum_{n \geq 0} H[^n](L)\) is naturally isomorphic to \(S(L) := S^*\bigoplus_{n \geq 1} H(nL)\), namely as follows: Firstly, we introduce linear maps \(H_n(L) \to S(L)\), which are defined by mapping an element of the form \(\sum_{\sigma \in \mathcal{S}_n} \bigotimes_{B \in \sigma \setminus [n]} \alpha_{\sigma, B}\) to \(\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \prod_{B \in \sigma \setminus [n]} \alpha_{\sigma, B}\). The restrictions of these morphisms to the \(\mathcal{S}_n\)-invariant parts define a linear map \(\bigoplus_{n \geq 0} H[^n](L) \to S(L)\). This map is an isomorphism, which can be proven exactly as in [LS03] for trivial \(G\).

Recall that \(H^*(X, C[2])\) is a (trivially weighted) graded non-counital Frobenius algebra of degree \(d\).

**Lemma 5.4.** There is a natural isomorphism \(H^*(X, C[2])[^n] \to H^*(X[^n], C[2n])\) of graded unital algebras of degree \(nd\).

**Proof.** Recall the just defined isomorphism between the spaces \(\bigoplus_{n \geq 0} H^*(X, C[2])[^n]\) and \(S^*(\bigoplus_{n \geq 0} H^*(X, C[2]))\) (for \(L = C\)). The composition of this isomorphism with isomorphism between \(S^*(\bigoplus_{n \geq 0} H^*(X, C[2]))\) and \(\bigoplus_{n \geq 0} H^*(X[^n], C[2n])\) of Theorem 1.4 induces by restriction the claimed isomorphism of the Lemma on the level of graded vector spaces.

That this isomorphism is in fact an isomorphism of unital algebras, is proven in [LS03] for \(X\) being projective. The proof there does not use the fact that \(H^*(X, C[2])\) has a counit, in fact it only uses its diagonal map. It relies on the earlier work in [Leh99], which has been extended to the non-projective case above, and [LQW02], which can similarly be extended. Thus the proof in [LS03] also works in the non-projective case, when we replace the notion of a Frobenius algebra by the notion of a non-counital Frobenius algebra. \(\square\)

We will now deduce Theorem 1.3 from Lemma 5.4.

**Proof of Theorem 1.3.** Let \(L, M \in G\).

Let \(\lambda = (\lambda_1, \ldots, \lambda_i)\) be a partition of \(n\). Let \(\nu_i\) the multiplicity of \(i\) in \(\lambda\), i.e. \(\lambda = \sum \nu_i \cdot i\). Set \(X(\lambda) := \prod X(\nu_i)\), and \(L(\lambda) := \prod pr_i^* L(\nu_i)\), where the \(pr_i\) denote the projections onto the factors \(X(\nu_i)\). Let \(\alpha = \sum \alpha(1) \cdots \alpha(\mu) \in H^*(X(\lambda), L(\lambda)[2l]) = \bigotimes S^\nu H^*(X,L[2])\).

We set
\[|\alpha| := \sum q(\alpha(1)) \cdots q(\alpha(\mu))[0].\]

By Theorem 1.3 the cohomology space \(H^*(X[^n], L[^n][2n])\) is linearly spanned by classes of the form \(|\alpha|\).

Let \(\mu = (\mu_1, \ldots, \mu_m)\) be another partition of \(n\) and \(\beta \in H^*(X[^n], M[^n][2m])\). In order to describe the ring structure of \(H^*(X[^n], L[^n][2n])\), we have to calculate the classes \(|\alpha \cup \beta| := |\alpha| \cup |\beta|\) in terms of the vector space description given by Theorem 1.3.

This means that we have to calculate the numbers
\[\langle \gamma | \alpha \cup \beta \rangle := q(\gamma)|\alpha \cup \beta| \in H^*(X[^0], C) = C\]
for all $\gamma \in H^2_c(X(\kappa), ((LM)^{-1}1(\kappa))[2k])$ for all partitions $\kappa = (\kappa_1, \ldots, \kappa_k)$ of $n$, and we have to show that they are equal to the numbers that would come out if we calculated the product of $\alpha$ and $\beta$ by the right hand side of the claimed isomorphism of the Theorem.

The class $|\alpha\rangle$ is given by applying a sequence of correspondences to the vacuum vector: Recall from [Nak07] how to compose correspondences. It follows that $|\alpha\rangle$ is given by

$$\text{PD}^{-1}(\text{pr}_1)_*(\text{pr}_2^*\alpha \cap \zeta_\lambda),$$

where the symbols have the following meaning: The maps $\text{pr}_1$ and $\text{pr}_2$ are the projections of $X[n] \times X(\lambda)$ onto its factors $X[n]$ and $X(\lambda)$. Further, $\zeta_\lambda$ is a certain class in $H^3_{BM}(Z_\lambda)$, where $Z_\lambda$ is the incidence variety

$$Z_\lambda := \left\{ (\xi, (x_1, x_2, \ldots)) \in X[n] \times X(\lambda) \mid \supp \xi = \sum_i ix_i \right\}$$
in $X[n] \times X(\lambda)$. (Note that $\text{pr}_1^*L[n]|_{Z_\lambda} = \text{pr}_2^*L(\lambda)|_{Z_\lambda}$, and that $p|_{Z_1}$ is proper.)

For $|\beta\rangle$ and $|\gamma\rangle$ we get similar expressions. By definition of the cup-product (pull-back along the diagonal), it follows that $|\gamma\rangle \cup |\alpha \cup \beta\rangle = (r^*\gamma \cup p^*\alpha \cup q^*\beta, \zeta_{\lambda,\mu,\kappa})$, where $p$, $q$, and $r$ are the projections from $X(\lambda) \times X(\mu) \times X(\kappa)$ onto its three factors, and $\zeta_{\lambda,\mu,\kappa}$ is a certain class in $H^3_{BM}(Z_{\lambda,\mu,\kappa})$ with

$$Z_{\lambda,\mu,\kappa} := \left\{ (\xi, (x_1, x_2, \ldots), (y_1, y_2, \ldots), (z_1, z_2, \ldots)) \mid \sum_i ix_i = \sum_j jy_j = \sum_k kz_k \right\}.$$

(The incidence variety is proper over any of the three factors, so everything is well-defined.) The main point is now that the incidence variety $Z_{\lambda,\mu,\kappa}$ and the homology class $\zeta_{\lambda,\mu,\kappa}$ are independent of the locally constant systems $L$ and $M$. In particular, we can calculate $\zeta_{\lambda,\mu,\kappa}$ once we know the cup-product in the case $L = M = \mathbb{C}$. But this is the case that is described in Lemma 5.4, which we will analyse now.

First of all, the incidence variety is given by

$$Z_{\lambda,\mu,\kappa} = \sum_{\sigma, \tau} Z_{\sigma, \tau}$$

where $\sigma$ and $\tau$ run through all permutations with cycle type $\lambda$ and $\mu$, respectively, such that $\rho := \sigma \tau$ has cycle type $\kappa$. The varieties $Z_{\sigma, \tau}$ are defined as follows:

As the orbits of the group action of $\langle \sigma \rangle$ on $[n]$ correspond to the entries of the partition $\lambda$, there exists a natural map $X^\sigma[n] \rightarrow X(\lambda)$, which is given by symmetrising. Furthermore the natural surjection $\sigma|n[| \rightarrow \langle \sigma, \tau\rangle |n]$ induces a diagonal embedding $X^\sigma([n]) \rightarrow X^\sigma([n])$. Composing both maps, we get a natural map $X^\sigma([n]) \rightarrow X(\lambda)$. Analogously, we get maps from $X^\langle \sigma, \tau\rangle |n]$ to $X(\mu)$ and $X^\langle \kappa\rangle$. Together, these maps define a diagonal embedding

$$i_{\tau, \sigma}: X^\langle \sigma, \tau\rangle |n| \rightarrow X^\langle \kappa\rangle \times X^\langle \lambda\rangle \times X^\langle \mu\rangle.$$

We define $Z_{\sigma, \tau}$ to be the image of this map.

By Lemma 5.4 the class $\zeta_{\lambda,\mu,\kappa}$ is given by $\sum_{\sigma, \tau}(i_{\sigma, \tau})_*\zeta_{\sigma, \tau}$, where each class $\zeta_{\sigma, \tau} \in H^3_{BM}(X^\langle \sigma, \tau\rangle |n]$ is Poincaré dual to $c_{\sigma, \tau} e^\gamma(\sigma, \tau)$. Here, $c_{\sigma, \tau}$ is a certain combinatorial factor (possibly depending on $\sigma$ and $\tau$), whose precise value is of no concern for us.

Having derived the value of $\zeta_{\lambda,\mu,\kappa}$ from Lemma 5.4 we have thus calculated the value $|\gamma\rangle \cup |\alpha \cup \beta\rangle$.

Now we have to compare this value with the one that is predicted by the description of the cup-product given by the right hand side of the claimed isomorphism of the Theorem. With the same analysis as above, we find this value is also given by a
comultiplication \( \delta \) endowed with a compatible structure of a cocommutative Hopf algebra of degree \( \lambda, \mu, \kappa \) with the class \( \sum_{\tau} (\iota_{\sigma, \tau}) e_{\sigma \tau} \) with the same combinatorial factors \( e_{\sigma \tau} \) as above. We thus find that the claimed ring structure yields the correct value of \( \langle \gamma | \alpha \cup \beta \rangle \). \( \square \)

Remark 5.5. One can also define a natural diagonal map for the Hilbert algebras \( H^{[n]} \) making them into graded, non-counital Frobenius algebras of degree \( nd \). The isomorphism of Theorem 1.3 then becomes an isomorphism of graded non-counital Frobenius algebras.

6. The generalised Kummer varieties

Finally, we want to use Theorem 1.3 to study the cohomology ring of the generalised Kummer varieties.

Let \( H \) be a non-counital graded Frobenius algebra of degree \( d \) that is moreover endowed with a compatible structure of a cocommutative Hopf algebra of degree \( d \). The comultiplication \( \delta \) of the Hopf algebra structure is of degree \( -d \). The counit of the Hopf algebra structure is denoted by \( \epsilon \) and is of degree \( d \). We further assume that \( H \) is also equipped with a \( G \)-weighting for a finite group \( G \).

Example 6.1. Let \( X \) be an abelian surface. The group structure on \( X \) induces naturally a graded Hopf algebra structure of degree 2 on the graded Frobenius algebra \( H^*(X, \mathbb{C}[2]) \). This algebra is also trivially \( X[n] \)-weighted, where \( X[n] \) is the character group of the group of \( n \)-torsion points on \( X \). (Trivially weighted means that the only non-trivial \( X[n] \)-weight space of \( H^*(X, \mathbb{C}[2]) \) is the one corresponding to the identity element 0.)

Let \( n \) be a positive integer. Recall the definition of the \((G\text{-}weighted)\) Hilbert algebra \( H^{[n]} \). Repeated application of the comultiplication \( \delta \) induces a map \( \delta: H \to H^\otimes_n = H^{id\setminus[n]} \), which is of degree \(-n-1\). Its image lies in the subspace of symmetric tensors. Thus we can define a map \( \phi: H \to H^{[n]} \) with \( \phi(\alpha) := \delta(\alpha)id \). One can easily check that this map is an algebra homomorphism of degree \(-n-1\), making \( H^{[n]} \) into an \( H \)-algebra.

Define
\[
H^{[n]} := H^{[n]} \otimes_H \mathbb{C},
\]
where we view \( \mathbb{C} \) as an \( H \)-algebra of degree \( d \) via the Hopf counit \( \epsilon \). It is \( H^{[n]} \) a \((G\text{-}weighted)\) graded Frobenius algebra of degree \( nd \).

Definition 6.2. The algebra \( H^{[n]} \) is the \( n \)-th Kummer algebra of \( H \).

The reason of this naming is of course Theorem 1.3.

Proof of Theorem 1.3. Let \( n: X \to X \) denote the morphism that maps \( x \) to \( n \cdot x \). There is a natural cartesian square
\[
\begin{array}{ccc}
X \times X^{[n]} & \xrightarrow{\nu} & X^{[n]} \\
\downarrow p & & \downarrow \sigma \\
X & \xrightarrow{n} & X,
\end{array}
\]
where \( p \) is the projection on the first factor and \( \nu \) maps a pair \((x, \xi)\) to \( x + \xi \), the subscheme that is given by translating \( \xi \) by \( x \) \((\text{Bea83})\). Let \( G \) be the character group of the Galois group of \( n \), i.e. \( G = X[n] \). Each element \( L \) of \( G \) corresponds to a locally constant system \( L \) on \( X \), and we have \( n_* \mathcal{C} = \bigoplus_{L \in G} L^{[n]} \). It follows that \( \nu \) is the \( G \)-covering of \( X^{[n]} \) with \( \nu_* \mathcal{C} = \bigoplus_{L \in G} L^{[n]} \).
Together with Theorem 4.3, this leads to the claimed description of the cohomology ring of $X^{[n]}$: Firstly, there is a natural isomorphism
\[ H^*(X^{[n]}, C[2n]) \rightarrow H^*(X \times X^{[n]}, C[2n]) \otimes H^*(X, C[2]) \]
of unital algebras (the tensor product is taken with respect to the map $p^*$ and the Hopf counit $H^*(X, C[2]) \rightarrow C$). By the Leray spectral sequence for $\nu$ and by (1), the right hand side is naturally isomorphic to
\[ H^*(X^{[n]}, \nu, C[2n]) \otimes H^*(X, C[2]) \]
(where the tensor product is taken with respect to the map $\sigma^*$ and the Hopf counit).

By Theorem 4.3 the algebra $\bigoplus_{L \in G} H^*(X^{[n]}, L[n][2n])$ is naturally isomorphic to $\bigoplus_{L \in G} H^*(X, L[2])^{[n]}$. Now $H^*(X, L[2]) = 0$ unless $L$ is the trivial bundle, which follows from the fact that all classes in $H^*(X, C)$ are invariant under the action of the Galois group of $n$, i.e., correspond to the trivial character. Thus there is a natural isomorphism
\[ \bigoplus_{L \in G} H^*(X^{[n]}, L[n][2n]) \rightarrow H^*(X, C[2])^{[n]}, \]
of $G$-weighted algebras, where we endow $H^*(X, C[2])$ with the trivial $G$-weighting. Under this isomorphism, the map $\sigma^*$ corresponds to the homomorphism $\phi$ defined above by Example 2.5. Thus we have proven the existence of a natural isomorphism
\[ H^*(X^{[n]}, C[2n]) \rightarrow H^*(X, C[2])^{[n]} \otimes H^*(X, C[2]) \]
of unital, graded algebras. But the right hand side is nothing but $H^*[n]$, thus the Theorem is proven. \qed

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Max-Planck-Institut for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: marc@nieper-wisskirchen.de