TIED PSEUDO LINKS

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Abstract. Pseudo diagrams of knots were introduced by Hanaki as knot diagrams with some missing crossing information. They generalize the notion of knot shadows as well as the notion of knot diagrams in general. The knot theory that arises from considering equivalence classes of pseudo diagrams under equivalence relations generated by a natural set of Reidemeister moves is of particular interest and leads to the theory of pseudo knots. In this paper we introduce and study the theory of tied pseudo links. Tied links in $S^3$ were introduced by Aicardi and Juyumaya as standard links in $S^3$ equipped with some non-embedded arcs, called ties, joining some components of the link. Analogously, we consider pseudo links equipped with ties and we introduce the tied pseudo braid monoid, which is close related to the tied singular braid monoid.

We introduce $L$-moves for pseudo knots, with the use of which, we formulate a sharpened version of Markov’s theorem for pseudo knots. Finally, we prove Alexander’s & Markov’s theorems for tied pseudo knots.

0. Introduction

Pseudo diagrams of knots, links and spatial graphs were introduce by Hanaki in [H] as projections on the 2-sphere with over/under information at some of the double points. They comprise a relatively new and important model for DNA knots, since there exist cases of DNA knots that, after studying them by electron microscopes, it is hard to say a positive from a negative crossing. By considering equivalence classes of pseudo diagrams under equivalence relations generated by a specific set of Reidemeister moves, one obtains the theory of pseudo knots, that is, standard knots whose projections contain crossings with missing information (see [HJMR] for more). In [BJW], the pseudo braid monoid is introduced, which is related to the singular braid monoid, and with the use of which, the authors present Alexander’s and Markov’s theorems for pseudo knots. In this paper we introduce $L$-moves for pseudo knots and we obtain a sharpened version of Markov’s theorem for pseudo braid equivalence. Moreover, we introduce the notion of tied pseudo links and tied pseudo braids, that is, pseudo links/braids equipped with ties, which are non-embedded arcs joining some components of the pseudo link.

Tied links were introduced in [AJ1] as a generalization of links in $S^3$ forming a new class of knotted objects. They are classical links equipped with ties and they are obtained by considering the closure of tied braids, which appear from a diagrammatic interpretation of the defining generators of the algebra of braids and ties introduced in [Ju, AJ2]. Of particular interest are the tied singular links, introduced and studied in [AJ3], that is, singular links equipped with ties. They constitute a generalization of singular links in $S^3$ in the same way that tied links generalize standard links. In this paper we show how tied singular links are related to tied pseudo links and we define the tied pseudo braid monoid, which is fundamental for this paper and the basis for Alexander’s and Markov’s theorems for tied pseudo links. It is worth

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mentioning that the tied pseudo braid monoid is related to the **tied singular braid monoid**, the counterpart of the singular braid monoid which was introduced and studied in [AJ3].

The paper is organized as follows: In §1 we recall results concerning tied links and tied braids from [AJ1] and in §2 we recall results on pseudo knots and singular knots that play an important role for the rest of the paper. In §3 we introduce L-moves for pseudo knots, we present the braiding algorithm for pseudo knots and we present a sharpened version of Markov’s theorem. We then proceed by introducing the tied pseudo links (see §4) and the tied pseudo braid monoid. Finally, in §4.1 we prove the analogue of Alexander’s theorem for tied pseudo links and in §4.2 we present Markov’s theorem for this new class of knotted objects.

## 1. Tied links & tied braids

In this section we introduce tied links and we study the tied braid monoid introduced in [AJ1]. A tied link is a link whose set of components is subdivided into classes. More precisely:

**Definition 1.** A tied (oriented) link \(L(P)\) on \(n\) components is a set \(L\) of \(n\) disjoint smooth (oriented) closed curves embedded in \(S^3\), and a set \(P\) of ties, i.e., unordered pairs of points \((p_r, p_s)\) of such curves between which there is an arc called a tie. If \(P = \emptyset\), then \(L\) is a classical link in \(S^3\).

Note that ties are depicted as red springs or line segments connecting pairs of points lying on the curves and that ties are not embedded arcs, i.e., arcs can cross through ties. It is also worth mentioning that the set of ties on a tied link defines a partition of the set of components of the links. Moreover, a **tied link diagram** is defined as a diagram of a link provided with ties which are depicted as springs connecting pairs of points lying on the curves.

![Figure 1. A tied link.](attachment:image)

*Tie isotopy,* i.e., isotopy between tied links, is defined as ambient isotopy between links (ignoring the ties), taking also into consideration the set of ties in those links. More precisely:

**Definition 2.** Two oriented tied links \(L(P)\) and \(L'(P')\) are tie isotopic if:

- the links \(L\) and \(L'\) in \(S^3\) are ambient isotopic and
- the sets \(P\) and \(P'\) define the same partition of the set of components of \(L\) and \(L'\).

**Tied braids,** that is, standard braids in \(S^3\) equipped with ties and whose standard closure gives rise to tied links, were also introduced in [AJ1]. They generalize the notion of standard braids and via the *monoid of tied braids,* i.e., the analogue of the braid group for tied links, we may extend Alexander’s and Markov’s theorems for tied links. In particular:

**Definition 3.** The tied braid monoid \(T\mathcal{B}_n\) is the monoid generated by \(\sigma_1, \ldots, \sigma_{n-1}\), the usual braid generators of \(\mathcal{B}_n\), and the generators \(\eta_1, \ldots, \eta_{n-1}\), called ties (see Figure 2), satisfying the braid relations of \(\mathcal{B}_n\) together with the following relations:
\[ \eta_i \eta_j = \eta_j \eta_i, \quad \text{for all } i, j \]

\[ \eta_i \sigma_i = \sigma_i \eta_i, \quad \text{for all } i \]

\[ \eta_i \sigma_j = \sigma_j \eta_i, \quad \text{for all } |i - j| > 1 \]

\[ \eta_i \sigma_j \sigma_i^\epsilon = \sigma_j \sigma_i^\epsilon \eta_j, \quad \text{for } |i - j| = 1 \text{ and where } \epsilon = \pm 1 \]

\[ \eta_i \eta_j \sigma_i = \eta_j \sigma_i \eta_j = \sigma_i \eta_i \eta_j, \quad \text{for } |i - j| = 1 \]

\[ \eta_i^2 = \eta_i, \quad \text{for all } i \]

**Figure 2.** Generators of \( TB_n \).

**Figure 3.** Relations in \( TB_n \), where the shaded crossing is either a positive or a negative crossing.

**Figure 4.** The relations \( \sigma_i \eta_j = \eta_j \sigma_i \).

We now introduce the generalized ties \( \eta_{i,j} \) and present their properties, with the use of which, we obtain the mobility property (see Proposition 1) for tied links, a useful property that allows us to prove Markov’s theorem for tied links. More precisely:
Figure 5. The relations $\eta_i \eta_j = \eta_j \eta_i$.

**Definition 4.** The generalized tie $\eta_{i,j}$, joining the $i^{th}$ with the $j^{th}$ strand is defined as:

\[
\eta_{i,j} := \sigma_i \ldots \sigma_{j-2} \eta_{j-1} \sigma_{j-2}^{-1} \ldots \cdot \sigma_i^{-1}
\]

Set $\eta_{i,i} := 1$ and define also the length of a generalized tie $\eta_{i,j}$ as $l(\eta_{i,j}) = |i - j|$ and define $l(\eta_i) = 1$.

Generalized ties $\eta_{i,j}$ are transparent with respect to all strands between the $i^{th}$ and $j^{th}$ strands, that is, they can be drawn no matter if in front or behind these strands. Moreover, the generalized ties are provided with elasticity, i.e. each generalized tie can be transformed by a second Reidemeister move in which the tie is stretched and represented as a spring. The generalized ties satisfy the following relations ([AJ1]):

\[
\begin{align*}
(i) & \quad \sigma_i \eta_{i,j} = \eta_{i+1} \sigma_i, \\
(ii) & \quad \sigma_j \eta_{i,j} = \eta_{i,j+1} \sigma_i, \\
(iii) & \quad \sigma_{i-1} \eta_{i,j} = \eta_{i-1,j} \sigma_{i-1}, \\
(iv) & \quad \sigma_{j-1} \eta_{i,j} = \eta_{i-1,j} \sigma_{j-1}, \\
v & \quad \eta_{k,k} \eta_{k,m} = \eta_{k,k} \eta_{k,m} = \eta_{k,m} \eta_{k,m}, \quad 1 \leq i, k, m \leq n
\end{align*}
\]

Using these properties and Equations 2 in [AJ1] it is shown that:

**Proposition 1.**

i. Mobility property: In any tied braid all ties can be moved to the bottom (or to the top), that is, $\alpha \sim \beta \gamma$, where $\alpha \in TB_n, \beta \in B_n$ and $\gamma$ the set of generalized ties.

ii. Any set of generalized ties in $TB_n$ defines an equivalence relation on the set of $n$ strands.

iii. Let $\alpha_1 = \beta_1 \gamma_1$, $\alpha_2 = \beta_2 \gamma_2$ be two tied braids in $TB_n$. Then, $\alpha_1 \sim \alpha_2$, if and only if $\beta_1 \sim \beta_2$ in $B_n$ and $\gamma_1, \gamma_2$ define the same partition of the set of the strands.

By considering now $TB_n \subset TB_{n+1}$, we can consider the inductive limit $TB_{\infty}$. Using the generalized ties and the fact that ties can be transferred freely, in [AJ1] it is shown that the braiding algorithm in [LR1] can also be applied for tied links. In particular we have the following result:

**Theorem 1** (Alexander’s theorem for tied links in $S^3$). Every oriented tied link can be obtained by closing a tied braid.

In [AJ1] it is also proved the analogue of Markov’s theorem for tied braids using the mobility property and Equations 2. More precisely, we have the following theorem:
Theorem 2 (Markov’s Theorem for tied links). Two tied braids have tie isotopic closures if and only if one can obtained from the other by a finite sequence of the following moves:

1′. Markov Conjugation :  $\alpha \beta \sim \beta \alpha$, for all $\alpha, \beta \in TB_n$,

2′. Markov Stabilization : $\alpha \sim \alpha \sigma_n^{\pm 1}$, for all $\alpha \in TB_n$,

3′. Ties : $\alpha \sim \alpha \eta_{i,j}$, for all $\alpha \in TB_n$ such that $s_\alpha(i) = j$, where $s_\alpha$ denotes the permutation associated to the braid obtained from $\alpha$ by forgetting its ties.

Remark 1. It is also worth mentioning that in [AJ2] the authors prove that the bt-algebra (the algebraic counterpart of the braid group for tied links) supports a Markov trace. Moreover, in [AJ3], the authors provide a purely algebraic and combinatoric version of tied links, with the use of which, they prove first that the tied braid monoid has a decomposition like a semi-direct group product and present new proofs (different from the proofs presented in [AJ1]) of Alexander’s and Markov’s theorems for tied links.

2. Pseudo knots & Singular knots

In this section we introduce pseudo links and we study the pseudo braid monoid introduced in [BJW]. A pseudo diagram of a knot consists of a regular knot diagram where some crossing information may be missing, that is, it is unknown which strand passes over and which strand passes under the other. These undetermined crossings are called pre-crossings (for an illustration see Figure 6).

![Figure 6. A pseudo knot.](Figure6.png)

Definition 5. Pseudo knots are defined as equivalence classes of pseudo diagrams under an appropriate choice of Reidemeister moves that are illustrated in Figure 7.

As explained in [BJW], pseudo knots are closed related to singular knots, that is, knots that contain a finite number of self-intersections. In particular, there exists a bijection $f$ from the singular knot diagrams to the set of pseudo knot diagrams where singular crossings are mapped to pre-crossings. In that way we may also recover all of the pseudo knot Reidemeister moves, with the exception of the pseudo-Reidemeister I (PR1) move (see Figure 7). Moreover, $f$ induces an onto map from singular knots to pseudo knots, since the image of two isotopic singular knot diagrams are also isotopic pseudo knot diagrams with exactly the same sequence of Reidemeister moves.

We now introduce the pseudobraid monoid, $PM_n$, following [BJW].
**Definition 6.** The monoid of pseudo braids, $PM_n$, is the monoid generated by $\sigma_i^{\pm 1}, p_i, i = 1, \ldots, n - 1$, illustrated in Figure 8, where $\sigma_i^{\pm 1}$ generate the braid group $B_n$ and $p_i$ satisfy the following relations:

i. $p_i p_j = p_j p_i$, if $|i - j| \geq 2$

ii. $p_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} p_i$, if $|i - j| \geq 2$

iii. $p_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} p_i$, $i = 1, \ldots, n - 1$

iv. $\sigma_i \sigma_{i+1} p_i = p_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, \ldots, n - 2$

v. $\sigma_{i+1} \sigma_i p_{i+1} = p_i \sigma_{i+1} \sigma_i$, $i = 1, \ldots, n - 2$

![Figure 7. Reidemeister moves for pseudo knots.](image)

![Figure 8. The pseudo braid monoid generators.](image)

Note that $p_i$ corresponds to a standard crossing and not to a singular crossing. We denote a singular crossing by $\tau_i$ and we have that if we replace the pre-crossings, $p_i$, of Definition 6 by singular crossings $\tau_i$, then we obtain the singular braid monoid, $SM_n$, defined in [Ba, Bi]. Thus, we obtain the following result:

**Proposition 2** (Proposition 2.3 [BJW]). The monoid of pseudo braids is isomorphic to the singular braid monoid, $SM_n$.  

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Remark 2. In [FKR] it is shown that $SM_n$, the singular braid monoid, embeds in a group, the singular braid group $SB_n$. It follows that $PM_n$ embeds in a group also, the pseudo braid group $PB_n$, generated by $\sigma_i$ and $p_i$, $i = 1, \ldots, n - 1$, satisfying the same relations as $PM_n$. Obviously, $SB_n$ is isomorphic to $PB_n$ (Proposition 2.3 [BJW]).

Define now the closure of a pseudo braid as in the standard case (see Figure 13 by ignoring the ties). By considering $PM_n \subset PM_{n+1}$, we can consider the inductive limit $PM_\infty$. Using Alexander’s theorem for singular knots ([Bi]), in [BJW] the analogue of Alexander’s theorem for pseudo links is presented. In particular:

**Theorem 3 (Alexander’s theorem for pseudo links).** Every pseudo link can be obtained by closing a pseudo braid.

Finally, we have the analogue of Markov’s theorem for pseudo braids.

**Theorem 4 (Markov’s Theorem for pseudo braids).** Two pseudo braids have isotopic closures if and only if one can obtained from the other by a finite sequence of the following moves:

1. Markov Conjugation: $\alpha \sim \beta \pm 1 \alpha \beta \mp 1$, for $\alpha \in PM_n \& \beta \in B_n$,
2. Markov Commuting: $\alpha \beta \sim \beta \alpha$, for $\alpha, \beta \in PM_n$,
3. Markov Stabilization: $\alpha \sim \alpha \sigma_n \pm 1$, for $\alpha \in PM_n$,
4. Markov Pseudo – Stabilization: $\alpha \sim \alpha p_n$, $\in PM_{n+1}$.

In the proof of Theorem 4, the fact that singular knots are close related to pseudo knots is used, and the only interesting case is the case where two pseudo knots differ by a pseudo-Reidemeister move 1 (see left-most and right-most illustrations in Figure 9). More precisely, let $L, L'$ be two pseudo knots that differ by a PR1 move and let $l, l'$ be the images of $L, L'$ respectively under the isomorphism between singulars and pseudo knots. By Alexander’s theorem for singulars there are singular braids $\alpha, \alpha'$, whose closure are $l, l'$ respectively. The images of these singular braids under the natural map between singular braids and pseudo braids are the pseudo braids $\beta, \beta'$, whose closures are $L$ and $L'$ respectively. In [BJW] it is shown then that $\alpha$ differs from $\alpha'$ by the Relation 4 of Theorem 4 and the proof is concluded. The above are summarized in the following diagram:

3. $L$-moves & pseudo knots

In this section we introduce the notion of $L$-moves for pseudo knots. $L$-moves make up an important tool for braid equivalence in any topological setting and they allow us to formulate sharpened versions of Markov’s theorems.

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Definition 7. An $L$-move on a braid $\beta$, consists in cutting an arc of $\beta$ open and pulling the upper cutpoint downward and the lower upward, so as to create a new pair of braid strands with corresponding endpoints (on the vertical line of the cutpoint), and such that both strands cross entirely over or under with the rest of the braid. Stretching the new strands over will give rise to an $L_o$-move and under to an $L_u$-move as shown in Figure 10 by ignoring all pre-crossings.

In [La], $L$-moves and Markov theorems are presented for different knot theories. We only recall the case of singular knots. More precisely, $L$-moves for singular braid equivalence are defined as in the classical case (Definition 7), that is, the two strands that appear after the performance of an $L$-move should cross the rest of the braid only with real crossings (all over in the case of an $L_o$-move or all under in the case of an $L_u$-move). Finally, with the use of $L$-moves, in [La], a sharpened version of Markov’s theorem (compared to [G]) for singular braids is presented:

Theorem 5 (Markov’s theorem for singular links). Two oriented singular links are isotopic if and only if any two corresponding singular braids differ by braid relations in $SB_\infty$ and a finite sequence of the following moves:

i. Singular commuting : $\tau_i \alpha \sim \alpha \tau_i$

ii. $L$ - moves

where $\alpha \in SB_n$, the singular braid group, and $\tau_i$ a singular crossing.

Our intention is to present an $L$-move Markov theorem for pseudo knots. We first need to define $L$-moves for pseudo braids. $L$-moves for pseudo braids are defined in the same way as in the singular case (for an illustration see Figure 10).

$L$-moves on pseudo knots can be used in order to obtain Alexander’s theorem for pseudo knots following the braiding algorithm in [Bi], [A] and [LR1]. The main idea of this algorithm is to keep the arcs of the oriented links diagram that go downwards with respect to the height.
function unaffected and replace arcs that go upwards with braid strands. These arcs are called opposite arcs. As noted in [B], before we apply the braiding algorithm we have to isotope the pseudo link in such a way that, roughly speaking, the pre-crossings are in a strictly decreasing order with respect to the height function, so that the braiding algorithm will not affect them. A detailed proof can be found in [PR] Theorem 2.3. Then we may apply the braiding algorithm of [LR1] for the knot (ignoring the pre-crossings).

The algorithm is summarized as follows:

- We first isotope the diagram of the pseudo link as described above. Then, we apply the following braiding algorithm for the knot (ignoring the pre-crossings):
- We chose a base-point and we run along the diagram of the (pseudo) link according to its orientation.
- When/If we run along an opposite arc, we subdivide it into smaller arcs, each containing crossings of one type only as shown in Figure 11. These arcs are called up-arcs.

![](Figure 11. Up-arcs.)

- We now label every up-arc with an “o” or a “u”, according to the crossings it contains. If it contains no crossings, then the choice is arbitrary.
- We perform an $L_o$-move on all up-arcs which were labeled with an “o” and an $L_u$-move on all up-arcs which were labeled with an “u” (see Figure 12).

![](Figure 12. Moves involved in the braiding algorithm.)

- The result is a pseudo braid whose closure is isotopic to the initial pseudo link.

As shown in [La], $L$ - moves capture real conjugation and real stabilization moves. Thus, we may replace relations 2 and 3 in Theorem 4, with $L$-moves and we obtain the following straightforward result:

**Theorem 6 (L-move Markov’s Theorem for pseudo braids).** Two pseudo braids have isotopic closures if and only if one can obtained from the other by a finite sequence of the
following moves:

1'. $L$ - moves

2'. Markov Commuting: $\alpha \beta \sim \beta \alpha$, for $\alpha, \beta \in PM_n$.

3'. Markov Pseudo - Stabilization: $\alpha \sim \alpha p_n$, $\in PM_{n+1}$.

4. Tied Pseudo links & Tied Singular links

In this section we introduce and study tied pseudo links and tied pseudo braids with the use of tied singular links and tied singular braids, introduced and studied in [AJ3]. We also introduce the tied pseudo braid monoid and we conclude by proving the Alexander’s and Markov’s theorems for tied pseudo links.

Definition 8. A tied singular link is a singular link with ties, while a tied pseudo knot is a pseudo knot equipped with ties.

Definition 9. The tied singular braid monoid $TSM_n$ is defined as the monoid generated by the standard braid generators $\sigma_i^{\pm 1}$’s, the singular braiding generators $\tau_i$’s of $SB_n$ and the ties generators $\eta_i$’s of $TB_n$ satisfying the defining relations of $SB_n$, the defining relations of $TB_n$ together with the following relations:

\[
\tau_i \eta_i = \eta_i \tau_i, \quad \text{for all } i, \\
\tau_i \eta_j = \eta_j \tau_i, \quad \text{for } |i - j| \geq 2, \\
\eta_i \tau_j \tau_i = \tau_j \tau_i \eta_j, \quad \text{for } |i - j| = 1, \\
\eta_i \tau_j \eta_i = \eta_j \tau_i \eta_j \tau_i \eta_j, \quad \text{for } |i - j| = 1, \\
\eta_i \tau_j \sigma_i = \tau_j \sigma_i \eta_j, \quad \text{for } |i - j| = 1, \\
\eta_i \sigma_j \tau_i = \sigma_j \tau_i \eta_j, \quad \text{for } |i - j| = 1, \\
\tau_i \eta_j = \sigma_i \eta_j \sigma_i^{-1} \tau_i, \quad \text{for } |i - j| = 1.
\]

To define the monoid of tied pseudo braids $PM_n$, we consider the generators $\sigma_i^{\pm 1}$ of the braid group $B_n$, the ties $\eta_i$ of $TB_n$ and we add the generators $p_i$, replacing the $\tau_i$’s. We need to find the defining relations of this monoid and for this reason we analyze the pseudo-Reidemeister moves and obtain the pseudo braid monoid (see [BJW]) and we also analyze relations concerning ties and pre-crossings. Straightforward calculations lead to the following definition:

Definition 10. The tied pseudo braid monoid $TPM_n$ is defined as the monoid generated by the standard braid generators $\sigma_i^{\pm 1}$’s, the pseudo-generators $p_i$’s of $SM_n$ and the ties generators $\eta_i$’s of $TB_n$ satisfying the defining relations of $PB_n$, the defining relations of $TB_n$ together with the following relations:
\[ p_i \eta_i = \eta_i p_i, \quad \text{for all } i, \]
\[ p_i \eta_j = \eta_j p_i, \quad \text{for } |i - j| \geq 2, \]
\[ \eta_i p_j \eta_i = p_j \eta_i \eta_j, \quad \text{for } |i - j| = 1, \]
\[ \eta_i \eta_j p_i = \eta_j \eta_i \eta_j = p_i \eta_i \eta_j, \quad \text{for } |i - j| = 1, \]
\[ \eta_i p_j \sigma_i = p_j \sigma_i \eta_j, \quad \text{for } |i - j| = 1, \]
\[ \eta_i \sigma_j p_i = \sigma_j p_i \eta_j, \quad \text{for } |i - j| = 1, \]
\[ p_i \eta_j = \sigma_i \eta_j \sigma_i^{-1} p_i, \quad \text{for } |i - j| = 1. \]

Comparing Definitions 9 & 10 we obtain the following result:

**Theorem 7.** There exists an isomorphism \( \mu \) from the tied singular braid monoid to the tied pseudo braid monoid, defined as follows:

\[
\mu : TSM_n \to TPM_n
\]

\[
\sigma_i^{\pm 1} \mapsto \sigma_i^{\pm 1}, \quad \tau_i \mapsto p_i
\]

Moreover, in Theorem 9 [AJ3] it is shown that the monoid \( TSM_n \) is isomorphic to \( P_n \times SB_n \), where \( P_n \) is the monoid generated by the set of partitions generated by the ties. This separates the ties from the rest of the singular braid, i.e. in a tied singular braid one can bring the ties at the top or bottom of the tied singular braid. Since now \( TPM_n \) is isomorphic to \( TSM_n \) and \( TB_n \) is isomorphic to \( SB_n \), it is natural to consider \( TPM_n \) to be isomorphic to \( P_n \times TB_n \). A proof based on straightforward calculations is obtained by recalling that generalized ties \( \eta_{i,j} \) are transparent with respect to all strands between the \( i^{th} \) and \( j^{th} \) strands, and that they are provided with elasticity. Moreover, the generalized ties satisfy the following relations (recall Eq. 2):

\[
(i) \quad p_i \eta_{i,j} = \eta_{i+1} p_i, \quad (ii) \quad p_j \eta_{i,j} = \eta_{i,j+1} p_i, \\
(iii) \quad p_{i-1} \eta_{i,j} = \eta_{i-1,j} p_{i-1}, \quad (iv) \quad p_{j-1} \eta_{i,j} = \eta_{i,j-1} p_{j-1},
\]

Using the transparent property, elasticity of ties and Equations 4, we obtain the following results:

**Theorem 8.** Let \( \alpha \) be a tied pseudo braid in \( TPM_n \). Then, \( \alpha \) can be written as \( \alpha = \gamma \beta \) (or \( \alpha = \beta \gamma \)), where \( \gamma \in PM_n \) and \( \beta \in TPM_n^\sim \), where \( TPM_n^\sim \) denotes the subset of \( TPM_n \) generated by the ties.

Moreover:

**Proposition 3.** Let \( \alpha_1 = \beta_1 \gamma_1, \ \alpha_2 = \beta_2 \gamma_2 \) be two tied pseudo braids in \( TPM_n \). Then, \( \alpha_1 \sim \alpha_2 \), if and only if \( \beta_1 \sim \beta_2 \) in \( TB_n \) and \( \gamma_1, \gamma_2 \) define the same partition of the set of the strands.
4.1. Alexander’s theorem for tied pseudo braids.

**Definition 11.** The closure of a tied pseudo braid is defined in the same way as the closure of a tied braid. For an illustration see Figure 13.

![Figure 13. The closure operation for tied pseudo braids.](image)

We are now in position to prove Alexander’s theorem for tied pseudo braids.

**Theorem 9 (Alexander’s theorem for tied pseudo braids).** Every tied pseudo link is equivalent to the closure of a tied pseudo braid.

**Proof.** The proof is similar to that of Theorem 3.1 in [BJW]. More precisely, let $L$ be a tied pseudo link and let $f$ be the natural map from the tied singular links to tied pseudo links. Consider the tied singular link $L'$ in $f^{-1}(L)$. From Theorem 10 in [AJ3] we have that there exists a tied singular braid $\beta'$, whose closure is equivalent to $L'$. From Theorem 7 and Eq. 8 there exist a unique tied pseudo braid $\beta$ whose closure is exactly the tied pseudo link $L$. □

**Remark 3.** An alternative proof follows by using the braiding algorithm of § 3 and the fact that ties can freely move along strands of the pseudo links (see also [D] for Alexander’s theorem for tied links in some c.c.o. 3-manifolds).

4.2. Markov’s theorem for tied pseudo braids. We begin this subsection by recalling Markov’s theorem for tied singular braids from [AJ3].

**Theorem 10 (Markov’s Theorem for tied singular braids).** Two tied singular braids have equivalent closures if and only if one can obtained from the other by a finite sequence of the following moves:

1'. Conjugation : $\alpha \beta \sim \beta \alpha$, for all $\alpha, \beta \in TSB_n$,

2'. Stabilization : $\alpha \sim \alpha \sigma_n^{\pm 1}$, for all $\alpha \in TSB_n$,

3'. $t$ – Stabilization : $\alpha \sim \alpha \eta_{i,j}$, for all $\alpha \in TSB_n$ such that $s_\alpha(i) = j$, where $s_\alpha$ denotes the permutation associated to the braid obtained from $\alpha$ by forgetting its ties.
Remark 4. It is worth mentioning that Theorem 10 is stated and proved in [AJ3] in the context of combinatoric tied singular links.

We may consider the theory of tied pseudo links as the quotient of the theory of tied singular links by the pseudo-Reidemeister move 1 (see Figure 7). Thus, in order to obtain Markov’s theorem for tied pseudo braids, we need to consider moves of Theorem 10 together with the pseudo-stabilization moves of Theorem 6. Recall also the moves that were used in Markov’s theorem for pseudo braids, that is, (real) conjugation, (real) stabilization, Markov-commuting and pseudo-stabilization (or equivalently $L$-moves, Markov-commuting and pseudo-stabilization). We have the following result:

Theorem 11 (Markov’s Theorem for tied pseudo braids). Two tied pseudo braids have equivalent closures if and only if one can obtained from the other by a finite sequence of the following moves:

1$. Commuting : \( \alpha \beta \sim \beta \alpha \), for all \( \alpha, \beta \in TPM_n \),

2$. Markov – Conjugation : \( \beta \sim \alpha^{\pm 1} \beta \alpha^{\mp 1} \) for all \( \beta \in TPM_n \) & \( \alpha \in B_n \),

3$. Real – Stabilization : \( \alpha \sim \alpha \sigma_n^{\mp 1} \), for all \( \alpha \in TPM_n \),

4$. Pseudo – Stabilization : \( \alpha \sim \alpha p_n \), for all \( \alpha \in TPM_n \).

5$. t – Stabilizations : \( \alpha \sim \alpha \eta_{i,j} \), for all \( \alpha \in TPM_n \) such that \( s_\alpha(i) = j \), where \( s_\alpha \) denotes the permutation associated to the braid obtained from \( \alpha \) by forgetting its ties.

Remark 5. Note that from the discussion above it follows that relations 2" and 3" can be replaced by the $L$-moves.

We now proceed with the proof of Theorem 11.

Proof. Obviously, if two tied pseudo braids differ by moves of Theorem 11, then their closures are equivalent. For the converse, let \( L, L' \) be two tied pseudo knots and let \( l, l' \) be the corresponding tied singular links obtained from \( L \) and \( L' \) respectively, by changing pre-crossings to singular crossings. From Alexander’s theorem for tied singular braids, we may braid \( l \) and \( l' \) to \( \alpha \) and \( \alpha' \) respectively. By Theorem 7 we may assume that \( \alpha = \mu(\beta) \) and \( \alpha' = \mu(\beta') \), where \( \mu \) is the isomorphism between \( TSM_n \) and \( TPM_n \). If now \( L \) differs from \( L' \) by any isotopy move except from pseudo Reidemeister moves 1, then by Theorem 10 the tie singular braids \( \alpha, \alpha' \) differ by moves 1', 2', 3' and thus, \( \beta, \beta' \) differ by moves 1", 2", 3" and 5".

If now \( L, L' \) differ by a PR1 move, we follow the proof of Theorem 4.2 in [BJW] keeping in mind that the endpoints of the ties can freely move along strands. In that way, we may always reduce the complexity that ties may create in the diagrams. Without loss of generality, we place now \( L \) and \( L' \) in \( S^2 \) in such a way that their diagrams are identical except for a region as shown in top part of Figure 13. We now perform the braiding algorithm from §3 on all identical up-arcs of \( L \) and \( L' \), leaving the region that the tied pseudo knots differ at, in the end. Obviously, the tied pseudo braids \( \beta \) and \( \beta' \) differ by a 4" move.

\[ \square \]

Remark 6. An alternative proof of Theorem 11 can be obtained by using Theorem 8 in order to separate the ties from the rest of the braiding generators of a tied pseudo braid and then
follow the proof of Theorem 13 in [D] by using generalized ties and properties that generalized ties satisfy. Roughly, if $L$ and $L'$ are two isotopic tied pseudo links and $\beta, \beta'$ corresponding tied pseudo braids of theirs, then from Theorem 8 we have that $\beta = \gamma k$ and $\beta' = \gamma' k'$, where $\gamma, \gamma' \in PM_n$ and $k, k' \in TPM_n$. Ignoring the ties in the tied pseudo links $L$ and $L'$ results into two isotopic pseudo links. Thus, corresponding pseudo braids of theirs, $\alpha, \alpha'$, are equivalent and one may obtain $\alpha'$ from $\alpha$ via a sequence of moves in Theorem 4 or Theorem 6. Note that $\alpha$ and $\alpha'$ are obtained from $\beta, \beta'$ respectively, by ignoring the ties. Thus, $\gamma$ is equivalent to $\gamma'$ by a sequence of Theorem 4 moves. From Proposition 3, it suffices to show now that $k, k'$ define the same partition of components of the links. This is achieved in the same way as in [AJ1] for tied links in $S^3$, as in [F] for tied links in the Solid Torus and as in [D] for tied links in various 3-manifolds.

5. Conclusions

In this paper we introduced and studied tied pseudo links via the tied pseudo braid monoid that is related to the tied singular braid monoid. We believe that tied pseudo knots can be used to model biological objects related to DNA and our results set the bases for the construction of tied pseudo link invariants in the sense of [Jo]. This method has been successfully applied in the case of tied links in $S^3$ in [AJ4], for tied links in the Solid Torus in [F] and in [D], Alexander’s and Markov’s theorems for tied links in some 3-manifolds are presented and work toward the study of tied links in knot complements and other c.c.o. 3-manifolds is currently being done [D4]. Finally, it is worth mentioning that this braid approach in defining invariants for knots in various 3-manifolds has been implemented for the construction of HOMFLYPT type invariants for knots in the Solid Torus in [La1, DL2] and work toward the construction of such invariants for knots in lens spaces $L(p, 1)$ has been done in [DL3, DL4, DL6, DL1]. For the case of Kauffman bracket type invariants via braids the reader is referred to [D1] for knots in the Solid Torus and in [D2] for knots in the genus 2 handlebody. In a sequel paper we shall adopt this braid approach for the study of other knot theories.
knots, J. Knot Theory and Ramifications, (2019). doi:10.1016/j.jktorr.2019.06.014.

[DL3] I. Diamantis, S. Lambropoulou, The braid approach to the HOMFLYPT skein module of the lens spaces $L(p,1)$ via braids, J. Knot Theory and Ramifications, 28, No. 13, 1940020 (2019). doi:10.1142/S0218216519400200.

[DL4] I. Diamantis, S. Lambropoulou, An important step for the computation of the HOMFLYPT skein module of the lens spaces $L(p,1)$ via braids, J. Knot Theory and Ramifications, 28, No. 11, 1940007 (2019). doi:10.1142/S0218216519400078.

[DLP] I. Diamantis, S. Lambropoulou, J. H. Przytycki, Topological steps on the HOMFLYPT skein module of the lens spaces $L(p,1)$ via braids, J. Knot Theory and Ramifications, 25, No. 14, (2016). doi:10.1142/S021821651650084X.

[F] M. Flores, Tied Links in the Solid Torus, [arXiv:1910.10778]

[FKR] R. Fenn, E. Keyman & C.P. Rourke, The singular braid monoid embeds in a group, J. Knot Theory and Ramifications, 7, No. 07, 881–892 (1998).

[G] B. Gemein, Singular braids and Markov’s theorem, Osaka J. Math., J. Knot Theory and Ramifications, 6, No. 04, (1996) 441–454.

[H] R. Hanaki, Pseudo diagrams of links, links and spatial graphs, Osaka J. Math., 47, (2010) 863–883.

[HJMR] A. Henriques, R. Hoberg, S. Jablan, L. Johnson, E. Minten, L. Radovic, The theory of pseudo-knots, J. Knot Theory and Ramifications, 22, No. 07, (2013) 1350032.

[Jo] V. F. R. Jones Hecke algebra representations of braid groups and link polynomials, Ann. Math. 126, 335–388 (1987).

[Ju] J. Juyumaya, Another algebra from the Yokonuma-Hecke algebra, Preprint ICTP IC/1999/160, Trieste.

[La] S. Lambropoulou, L-Moves and Markov theorems, Journal of Knot Theory and Its Ramifications 16, Ni.10, (2007) 1459-1468.

[La1] S. Lambropoulou, Knot theory related to generalized and cyclotomic Hecke algebras of type $B$, J. Knot Theory and its Ramifications 8, No. 5, (1999) 621-658.

[LR1] S. Lambropoulou, C.P. Rourke (2006), Markov’s theorem in 3-manifolds, Topology and its Applications 178, (1997) 95-122.

[PR] L. Paris & L. Rabenda, Singular Hecke algebras, Markov traces, and HOMFLY-type invariants, Annales de l’Institut Fourier, 58 (2008) No. 7, pp. 2413-2443. doi : 10.5802/aif.2419.
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