Research article

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Lie symmetry analysis and similarity solutions for the Jimbo – Miwa equation and generalisations

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Abstract: We study the Jimbo – Miwa equation and two of its extended forms, as proposed by Wazwaz et al., using Lie’s group approach. Interestingly, the travelling – wave solutions for all the three equations are similar. Moreover, we obtain certain new reductions which are completely different for each of the three equations. For example, for one of the extended forms of the Jimbo – Miwa equation, the subsequent reductions leads to a second – order equation with Hypergeometric solutions. In certain reductions, we obtain simpler first – order and linearisable second – order equations, which helps us to construct the analytic solution as a closed – form solution. The variation in the nonzero Lie brackets for each of the different forms of the Jimbo – Miwa also presents a different perspective. Finally, singularity analysis is applied in order to determine the integrability of the reduced equations and of the different forms of the Jimbo – Miwa equation.

Keywords: closed-form solution; similarity solutions; singularity analysis; symmetry analysis.

1991 Mathematics subject Classification: 34A05; 34A34; 34C14; 22E60; 35B06; 35C05; 35C07.

1 Introduction

The Jimbo – Miwa equation in 1 + 3 space dimensions is a partial differential equation (PDE) and along with its two different extended forms is our subject of study. The equation [1] was proposed in 1983 and has gained considerable attention in the past with respect to its study of integrability using various standard techniques. It is well known that this equation is a member of the KP hierarchy and possesses wider physical applications. Significant works were conducted by Dorizzi et al. [2], Wang et al. [3], Wazwaz et al. [4, 5], Cao et al. [6] and many others. Cao et al. [6] also provides a comprehensive list of most of the work done with respect to the algebraic structure of the equation.

The Jimbo – Miwa equation is defined to be

\[ u_{xxx} + 3u_xu_{xx} + 3u_uu_{xy} + 2u_{ty} - 3u_{tz} = 0, \]  \( (1.1) \)

while the extended forms given by Wazwaz et al. [4] are,

\[ u_{xxx} + 3u_xu_{xx} + 3u_uu_{xy} + 2u_{ty} - 3(u_{tz} + u_{xz} + u_{yz}) = 0, \]  \( (1.2) \)

and

\[ u_{xxx} + 3u_xu_{xx} + 3u_uu_{xy} + 2(u_{tx} + u_{ty} + u_{tz}) - 3u_{xz} = 0. \]  \( (1.3) \)

We study the determination of solutions for the three equations by using the method of Lie point symmetries. Lie symmetries are a powerful tools for the analysis of nonlinear differential equations. The main idea of Lie symmetries is to determine the transformations which leave the given differential equation invariant. Then, by using normal coordinates, the solution of the differential equation can be written in terms of the invariant functions for the Lie symmetry vector and in that way to reduce the number of independent variables in case of PDEs, or the order of a ordinary differential equation (ODE) [7–9]. Applications of Lie symmetries can be found for instance in [10–18] and references therein.

The application of Lie’s theory for equations (1.1)–(1.3) reveals that equation (1.1) admits six Lie point symmetries, equation (1.2) admits 10 Lie point symmetries while equation (1.3) is invariant under a six – dimensional group of point transformations.
The Lie symmetries are applied in order to determine similarity solutions for the equations under our consideration. What is more interesting is that the three equations of our study admit the same travelling-wave reduced equations with slight variation among each other. Reductions with other similarity variables provide different results for each equation. For example, with respect to equation (1.1), certain reductions lead to homogeneous equations with slight variation among each other. Reductions with other similarity variables provide different results obtained here are new and cannot be found in the literature. We took the aid of a symbolic manipulation code developed by Dimas et al. [19–21]. The paper also discusses the integrability of the three forms of Jimbo–Miwa using singularity analysis. It is shown that equations (1.1)–(1.3) satisfy the test.

The paper is arranged as follows: In Section 2 the preliminaries of Lie point symmetry analysis and singularity analysis are mentioned. In Section 3, and its subsequent subsections Lie’s point symmetry analysis of equation (1.1) is discussed in detail. In sections 4 and 5, analysis of equations (1.2) and (1.3) is discussed. Section 6, details the singularity analysis of the second–order equation obtained by the subsequent reductions of equation (1.1). In Section 7, the singularity analysis for the PDEs is presented. In the end, a brief conclusion and proper references are mentioned.

2 Preliminaries

In this Section, we briefly discuss the mathematical tools that we apply in this work to study the PDEs of our consideration. More specifically, we give the basic definitions for the theory of Lie symmetries and singularity analysis.

2.1 Lie symmetries

Let

\[ F(\mathbf{x}, u_a, u_{\beta}, u_{\delta}, u_{\alpha\beta}, \ldots) = 0, \]  

(2.1)

where \( \mathbf{x} = (x_a, x_\beta, x_\delta) \) are the set of independent variables, \( u_a = \partial u/\partial x_a \), \( u_{\beta} = \partial u/\partial x_\beta \), and \( u_{\alpha\beta} = \partial u/\partial x_\alpha x_\beta \), and the subsequent terms can be defined henceforth. Under an infinitesimal point transformation,

\[
\begin{align*}
\bar{x}_a &= x_a + \epsilon \xi^a(x_a, x_\beta, x_\delta, u) + O(\epsilon^2), \\
\bar{x}_\beta &= x_\beta + \epsilon \xi^\beta(x_a, x_\beta, x_\delta, u) + O(\epsilon^2), \\
\bar{x}_\delta &= x_\delta + \epsilon \xi^\delta(x_a, x_\beta, x_\delta, u) + O(\epsilon^2), \\
\bar{u} &= u + \epsilon \eta(x_a, x_\beta, x_\delta, u) + O(\epsilon^2),
\end{align*}
\]

(2.2)

equation (2.1) is said to be invariant if and only if,

\[ F(\mathbf{x}, u) = F(\mathbf{x}, \bar{u}). \]  

(2.3)

The transformations forms a symmetry group, say, \( G \), the generator of which can be defined as,

\[
\Gamma = \eta(x_a, x_\beta, x_\delta, u) \partial_u + \xi^a(x_a, x_\beta, x_\delta, u) \partial_{x_a},
\]

\[
+ \xi^\beta(x_a, x_\beta, x_\delta, u) \partial_{x_\beta} + \xi^\delta(x_a, x_\beta, x_\delta, u) \partial_{x_\delta},
\]

(2.4)

Therefore \( \Gamma \), which is the generator of the infinitesimal transformations, can be considered as a Lie point symmetry of equation (2.1). Now one can use equation (2.4) to discuss the reduction of the corresponding PDEs using the characteristic functions, obtained by solving the associated Lagrange’s system which is,

\[
\frac{dx_a}{\xi^a} = \frac{dx_\beta}{\xi^\beta} = \frac{dx_\delta}{\xi^\delta} = \frac{du}{\eta}.
\]

2.1.1 Singularity analysis

Our second method of investigation is known as singularity analysis which evolved at about the same time as symmetry analysis, i.e. towards the end of the nineteenth century. In essence, it is the determination of the existence of singularities in the dependent variable(s) in the complex plane of the independent variable. Indeed, it is interesting to note how analysis developed following the pioneering work of Cauchy in complex analysis. Equations which satisfy the analysis are said to possess the Painlevé Property. This is an interesting descriptor as the first known application of singularity analysis was due to Sophie Kowalevski who used the analysis to determine the third integrable case of the spinning top [22]. The treatment of the Painlevé Property in the classic text of E L Ince [23] is very clear. Later works by Ramani, Grammaticos and Bountis [24] and Michael Tabor [25] have kept the idea of proving integrability through singularity analysis before the public eye.

1 The first two cases were due to Euler and Lagrange in the seventeenth century.
A major development is to be found in the works of Ablowitz, Ramani and Segur [26–28] who introduced a straightforward approach to determining whether a given differential equation possesses the Painlevé Property. The approach has become known as the ARS algorithm and is very simple in its concept although there are times when its application faces some serious challenges. There are three steps:

(1) Determine the existence or otherwise of a singularity by making a substitution of \( y = a(x - x_0)^p \) into the differential equation (for simplicity we consider a single dependent variable).

This is called the leading order term. The nature of the singularity is indicated by the value of \( p \) and its location is \( x_0 \). Originally, it was considered to be a negative integer, hence, the method of polelike expansions, but over the years it was accepted that \( p \) could equally be a fraction which could be positive or negative as differentiation of a positive fractional power eventually leads to a negative power and so a singularity. As a practical point, it must be borne in mind that a fraction requires the complex plane to be divided into segments by branch line cuts. A multitude of these is not good if numerical work has eventually to be undertaken. The terms which contribute to the evaluation of \( p \) are called the dominant terms.

(2) A differential equation of order greater than one needs additional constants of integration and the idea is to construct a Laurent expansion about the singularity.

The additional constants of integration enter the expansion at powers called resonances and these are identified by making the substitution \( y = a(x - x_0)^p + m(x - x_0)^{p+s} \) into the dominant terms.2 The result is a polynomial in \( m \).

A given coefficient enters the expansion when its coefficient is \( m \). The coefficient is arbitrary, i.e. to be determined by the initial conditions, if \( m \) is zero. The coefficient of \( m \) is a polynomial in \( s \) and the solution of polynomial equals zero gives the requisite values of \( s \). One of the resonances must take the value \(-1\). This value is associated with the location of the moveable singularity.

(3) The Laurent expansion is substituted into the complete equation with the coefficients of the resonant terms bring arbitrary and the other coefficients determined.

This can be a tedious process, but, if all goes well, the solution is inferred to be analytic albeit possibly restricted in applicability by branch cuts.

Parallel procedures for partial differential equations can be found in [29–35].

3 The symmetry analysis for equation (1.1)

For (1.1), we compute the Lie – Point symmetries,

\[
\begin{align*}
\Gamma_{1a} &= \partial_x, \\
\Gamma_{2a} &= t\partial_t - \frac{u}{3}\partial_u + \frac{x}{3}\partial_x + \frac{2v}{3}\partial_v, \\
\Gamma_{3a} &= \partial_x, \\
\Gamma_{4a} &= t\partial_x + \frac{2x}{3}\partial_u, \\
\Gamma_{5a} &= c_1(z)\partial_x + \frac{3tc_1(z)}{4}\partial_x + \left(\frac{2xc_1(z)}{4} - \frac{3tyc_1(z)}{4}\right)\partial_v, \\
\Gamma_{6a} &= E_0(t,z)\partial_u,
\end{align*}
\]

where \( c_1(z) \) and \( E_0(t,z) \) are arbitrary functions.

The nonzero Lie brackets are,

\[
\begin{align*}
[\Gamma_{1a}, \Gamma_{2a}] &= \Gamma_{1a}, \\
[\Gamma_{2a}, \Gamma_{3a}] &= -\frac{2\Gamma_{1a}}{3}, \\
[\Gamma_{1a}, \Gamma_{4a}] &= \partial_x, \\
[\Gamma_{1a}, \Gamma_{5a}] &= \frac{2\Gamma_{4a}}{3}.
\end{align*}
\]

Case 1. To study the travelling wave reductions, we follow a simple procedure, by which we consider \( m\Gamma_{3a} - c\Gamma_{1a} \), to reduce the equation (1.1) to a new PDE of dimension 1 + 2, where \( m \) is the wave number and \( c \) denotes the frequency. The similarity variables for this reduction are,

\[
\begin{align*}
w &= mz - ct, \\
u(t, x, y, z) &= v(x, y, w).
\end{align*}
\] (3.1)

The possible reduced PDE is

\[
\nu_{xxy} + 3v_{xy}v_{xy} + 3v_{yx}v_{xx} - 2c\nu_{yw} - 3mv_{ww} = 0.
\] (3.2)

The Lie – Point symmetries are,

\[
\begin{align*}
\Gamma_{1b} &= \partial_y, \\
\Gamma_{2b} &= sv\partial_v - 3w\partial_w - x\partial_x - y\partial_y, \\
\Gamma_{3b} &= \partial_x, \\
\Gamma_{ab} &= w\partial_x - \left(\frac{2cx}{3} + my\right)\partial_v, \\
\Gamma_{ib} &= \partial_y.
\end{align*}
\]

Case 1a. The translation, with respect to the variables \( x, w \) and \( y \) is obtained for the equation (3.2). Therefore, we use

2 Usually the letter \( r \) is used, but we use \( s \) in deference to the pioneering work of Kowalevskaya.
the operator \( k \Gamma_{3b} + \Gamma_{5b} + \Gamma_{1b} \), where \( k \) and \( l \) are the wave numbers, to obtain the reduced ode,

\[
k^3 l p'''(q) + 6k^2 l p''(q)p'(q) - (2cl + 3km)p'(q) = 0,
\]

where \( q = kx + ly + w \) and \( v(x, y, w) = p(q) \). This can be ascertained by the reader that the equation, (3.3), is the corresponding travelling - wave reduction for the PDE, equation (1.1). We compute the symmetries for equation (3.3). They are,

\[
\begin{align*}
\Gamma_{1c} &= \partial_q, \\
\Gamma_{2c} &= \partial_p, \\
\Gamma_{3c} &= q \partial_q + \left( \frac{mq}{kl} + \frac{2cq}{3k^2} - p \right) \partial_p.
\end{align*}
\]

**Case 1b.** We first look at the reduction using \( \Gamma_{3c} \), namely,

\[
g''(h) = \frac{10g'(h)g''(h)}{g(h)} \cdot \frac{15g^3}{g(h)^2} + \left( \frac{(2cl + 3km)g(h)^2}{k^l} - \frac{6g(h)}{k} \right) g'(h),
\]

where \( h = p(q) \) and \( g(h) = 1/p'(q) \). Equation (3.4) possesses a lone symmetry, \( \partial_h \). We use this to reduce (3.4) to a second – order equation.

\[
b''(a) = \frac{3b^2}{b(a)} + \frac{10b'(a)}{a} + \left( \frac{6a^2c - 3a^2m}{k^l} \right) b(a)^3 + \frac{15b(a)}{a^2},
\]

where \( a = g(h) \) and \( b(a) = 1/g'(h) \). Equation (3.5) is maximally symmetric. Therefore, the solution of (1.1), can be obtained in accordance to equation (3.5).

**Case 1c.** Next, we use \( \Gamma_{2c} \) to study the reduction of equation (3.3).

The reduced third – order equation is,

\[
g''(h) = \left( \frac{2cl + 3km}{k^l} - \frac{6g(h)}{k} \right) g'(h),
\]

where \( h = q \) and \( g(h) = p'(q) \). The reduced third – order equation has two symmetries. They are \( \partial_h \) and \( h \partial_h + \frac{(2c - 6a^2k^l + 3km)}{3k^l} \partial_q \). The reduction with respect to \( \partial_h \) leads to the second – order equation

\[
b''(a) = \frac{3b^2}{b(a)} + \left( \frac{6ak^l - 2cl - 3km}{k^l} \right) b(a)^3,
\]

where \( a = g(h) \) and \( b(a) = 1/g'(h) \). Equation (3.7) possesses eight symmetries and hence is linearisable.

The reduction with respect to \( h \partial_h + \frac{(2c - 6a^2k^l + 3km)}{3k^l} \partial_q \) leads to the second – order equation,

\[
b''(a) = \frac{3b^2}{b(a)} + 9b'(a)b(a) + \left( \frac{(6a + 26k)b(a)}{k} - \frac{(12a^2 + 24ak)b(a)^3}{k} \right),
\]

where

\[
a = \frac{(6g(h)k^l - 2cl - 3km)}{6k^l} \quad \text{and} \quad b(a) = k^l \left( \frac{3k^l}{g(h)} - \frac{6g(h)}{k} \right).
\]

Equation (3.8) has zero Lie – Point symmetries. Singularity analysis of this equation is treated in Section 6. The reduction with respect to \( \Gamma_{3c} \) leads to a third – order equation with zero Lie point symmetries. We omit mentioning the equation here considering the high nonlinearity of the third – order equation. The equation is under study and we shall be discussing it in our subsequent work.

### 3.1 Further reductions for the Jimbo – Miwa equation

In this subsection, we study the reductions of equation (1.1) with respect to \( \Gamma_{2d} \). The similarity variables are,

\[
\begin{align*}
\omega_1 &= \frac{x}{t}, \\
\omega_2 &= \frac{z}{t}, \\
\omega &= \frac{v(y, \omega_1, \omega_2)}{t}.
\end{align*}
\]

The reduced PDE of \( 1 + 2 \) form is,

\[
9\nu_{w_{1}w_{1}} + (2 - 9\nu_{w_{1}})\nu_{y} + 4\nu_{w_{2}}\nu_{y} + 2\nu_{w_{1}}\nu_{y} - 9\nu_{w_{1}}\nu_{y} - 3\nu_{w_{1}w_{1}}w_{1} = 0.
\]

(3.9)

The symmetries of equation (3.9) are,

\[
\begin{align*}
\Gamma_{1d} &= y \partial_y + w_2 \partial_{w_2}, \\
\Gamma_{2d} &= y \partial_y, \\
\Gamma_{3d} &= \partial_{w_1} + \frac{2w_1}{y} \partial_y, \\
\Gamma_{4d} &= 2\sqrt{w_2} \partial_{w_2} - \frac{y}{\sqrt{w_2}} \partial_y, \\
\Gamma_{5d} &= c_2(w_2) \partial_y,
\end{align*}
\]

where \( c_2(w_2) \) is an arbitrary function with respect to \( w_2 \).

**Case 2.** We use \( \Gamma_{1d} \) for the reduction. The similarity variables are \( w_2 / y = w_3, v(y, w_1, w_2) = v_1(w_1, w_3) \). The reduced PDE in \( 1 + 1 \) dimensions is,

\[
0 = -4w_3^2\nu_{w_{1}w_{1}} + 9\nu_{w_{1}w_{1}} - 2w_1w_2\nu_{w_{1}w_{1}} + 9\nu_{w_{1}}\nu_{w_{1}} + 3w_3\nu_{w_{1}} - 2 + 3\nu_{w_{1}w_{1}} + 3w_3\nu_{w_{1}w_{1}w_{1}}.
\]
The Lie point symmetries are $\Gamma_{1\varepsilon} = \partial_v$ and $\Gamma_{2\varepsilon} = w_1 \partial_v + \frac{3}{2} \partial_w$. We use $\Gamma_{2\varepsilon}$ for reduction. The similarity variables are $v_1(w_1, w_2) = \frac{w_2}{9} + v_2(w_3)$. The reduced PDE is $v_2^2 + w_3 v_2^2 = 0$ which is maximally symmetric. Finally, the closed – form similarity solution of equation (1.1) is,

$$u(t, x, y, z) = \frac{f\left(\frac{x}{t}\right) + g\left(\frac{y}{t}\right)}{t^2}. \quad (3.10)$$

**Case 3.** Reduction with respect to $\Gamma_{3\varepsilon}$: The similarity variable is $v(y, w_1, w_2) = v_1(w_1, w_2)$. The reduced PDE is $v_3_{,w_2} = 0$. The solution of equation (1.1) can be given as,

$$u(t, x, y, z) = \frac{f\left(\frac{x}{t}\right) + g\left(\frac{y}{t}\right)}{t^2}. \quad (3.12)$$

**Case 4.** Reduction with respect to $\Gamma_{4\varepsilon}$: The similarity variable is,

$$v(y, w_1, w_2) = \frac{w_2}{9} + v_1(y, w_2).$$

The reduced PDE is $v_3_{,w_2} = 0$. Similarly, to the above case, the solution of the PDE can be easily determined. Therefore, the solution of the equation (1.1) can be given in terms of the solution of the reduced PDE.

**Case 5.** Reduction with respect to $\Gamma_{5\varepsilon}$: The similarity variables are $v(y, w_1, w_2) = \frac{-w_1}{w_2} + v_1(y, w_2)$. Hence, the reduced PDE is,

$$9y + 2v_1 + 4w_2 v_2_{,w_2} = 0. \quad (3.11)$$

The Lie – point symmetries for (3.11) are derived to be,

$$\Gamma_{y1} = c_1(y) \partial_y,$$

$$\Gamma_{y2} = c_2(w_2) \partial_{w_2} - \frac{v_1 c_2(w_2)}{2w_2} \partial_{v_1},$$

$$\Gamma_{y3} = v_0 \partial_{v_1},$$

$$\Gamma_{y4} = E_1(y, w_2) \partial_{v_1},$$

where $c_1(y)$, $c_2(w_2)$ and $E_1(y, w_2)$ are arbitrary functions. None of the above mentioned symmetries or linear combination of any of them leads to a satisfactory reduction. The Painlevé analysis of the equation (3.11) is under study to ascertain its integrability.

**Case 6.** Reduction with respect to $\Gamma_{6\varepsilon}$: Next, we study the reduction with respect to $\Gamma_{6\varepsilon}$ for equation (1.1). The similarity variable is $u(t, x, y, z) = \frac{x^3}{9} + v(t, y, z)$. The reduced PDE of dimension $1 + 2$ is,

$$v_y + tv_{ty} = 0. \quad (3.12)$$

The reduced PDE possesses the following Lie point symmetries,

$$\Gamma_{y1} = E_2(t, z) \partial_{y_1},$$

$$\Gamma_{y2} = E_3(y, z) \partial_{y_2},$$

$$\Gamma_{y3} = E_4(z) \partial_{y_3} + v E_5(z) \partial_{y_1},$$

$$\Gamma_{y4} = E_6(t, y, z) \partial_{y_3},$$

where $E_i$’s are arbitrary functions of the variables mentioned against them. Equation (3.12) can easily be integrated $v + tv + f(t) = 0$, from which we find $v(t) = \frac{v_0}{t} - \frac{f(t)e^t}{t}$.

### 3.2 Reductions for the Jimbo – Miwa equation

In this subsection, we study subsequent reductions for equations (3.2) and (1.1). We start with $\Gamma_{2b}$, for which we consider different possibilities for the similarity variables and study the subsequent reductions. We consider firstly the similarity variables,

$$\frac{x^3}{w} = w_1, \frac{x}{y} = w_2,$$

$$v(x, y, w) = \frac{v_1(w_1, w_2)}{x}.$$

The reduced PDE is,

$$0 = -3w_2^2 w_1 (w_1, w_2) \left(3w_1 v_1_{,w_2} - 2w_1 v_2_{,w_2} + 2w_2 v_1_{,w_2}\right) + 2w_2^2 w_1^2 v_1_{,w_2} - 9w_1 w_2 v_1_{,w_2} + 3w_1 w_2 v_2_{,w_2} - 6w_2 v_2_{,w_2} + 27w_1^2 w_2 v_1_{,w_2} + 27w_1 w_2^2 v_1_{,w_2} + 27w_1 w_2 v_1_{,w_2} + 54w_1^2 w_2 v_1_{,w_2} + 9w_1 w_2^2 v_1_{,w_2} + 9w_1 w_2 v_1_{,w_2} + 18w_1 w_2 v_1_{,w_2} + 6w_1 w_2 v_1_{,w_2} + 3w_2^2 v_{,w_2} \left(9w_1 (w_2 v_1_{,w_2} + w_1 v_1_{,w_2}) + 2w_2 v_{,w_2}\right) + 2w_2^2 v_{,w_2} + w_2^2 v_{,w_2} = 6w_2 v_{,w_2}.$$
= p_1(q_1, q_2), where w_1 = q_1 and w_2 = q_2. We assume that
p_1(q_1, q_2) = p_{1a}(q_1)p_{2a}(q_2), where p_{1a} and p_{2a} are arbitrary
functions of q_1 and q_2 respectively. The computation becomes less tedious
for two particular cases, firstly when p_{2a}(q_2) = F_0, where F_0 is an
arbitrary constant. The reduced second – order ode which is maximally
symmetric is,
\[ 2p_{1a}^2 + 3q_1(p_{1a})^2 = 0. \]  
(3.13)

Therefore, the solution of equation (1.1) can be given in
terms of equation (3.13).

Similarly, when p_{1a}(q_1) = F_t, where F_t is arbitrary, we
obtain a fourth – order equation, namely,
\[ 6p_{2a}(q_2)p_{2a}^2(q_2) - 6q_2p_{2a}^2(q_2)^2 - 3q_2p_{2a}(q_2)p_{2a}^\prime(q_2) 
+ 6q_2^2p_{2a}^2(q_2)p_{2a}(q_2) + 4q_2^3p_{2a}^\prime(q_2) = 0. \]  
(3.14)

This equation has a single symmetry, \( \partial_{q_1} \), which
reduces equation (3.14) to a third – order equation with zero
point symmetries. The equation is,
\[
p_{2a}^\prime(q_2) = \left[ \frac{10p_{2a}(q_2)}{p_{2a}(q_2)} + 9p_{2a}(q_2) \right] \frac{p_{2a}^\prime(q_2)}{p_{2a}(q_2)} - 15p_{2a}^\prime(q_2) 
+ 6q_2p_{2a}^\prime(q_2)p_{2a}(q_2) - 12p_{2a}(q_2) + (9p_{2a}^\prime(q_2))^2,
\]  
(3.15)

where \( q_1 = p_{2a}(q_2) \) and \( p_{3a}(q_1) = 1/q_2p_{2a}^\prime(q_2) \). Singularity
analysis of the equation (3.15) is under study.

**Case 8.** The similarity variable considered using \( \Gamma_{4b} \) is
\( v(x, y, w) = \frac{c_3w^2}{3w} + v_1(y, w) \), where y and w are the new
independent variables. The reduced PDE is,
\[ v_{1w} + wv_{1w} = 0. \]  
(3.16)

The reduced PDE (3.16) is similar to equation (3.12)
which is also a reduced PDE of 1 + 1 dimensions. Also,
it can be easily observed that equation(3.16) is a
variant of equation (3.11). The Lie point symmetries
are almost in similar nature, hence we omit mentioning them here.

## 4 The symmetry analysis for
equation (1.2)

The Lie-point symmetries are,
\[ \Gamma_{u} = \partial_x, \]
\[ \Gamma_2 = \partial_y, \]
\[ \Gamma_{ui} = \frac{t^2}{2}x + \left( \frac{t}{3} + \frac{z}{3} \right) \partial_y + t \partial_z, \]
\[ \Gamma_{ui} = \frac{1}{2} \partial_x + \partial_z, \]
\[ \Gamma_u = \frac{t^2}{2}x + \left( \frac{t}{3} + \frac{z}{3} \right) \partial_y + t \partial_z, \]
\[ \Gamma_{ui} = \frac{1}{2} \partial_x + \partial_z, \]
\[ \Gamma_{ui} = \frac{3c_3(t)}{2} \partial_x + x \partial_y(t) \partial_{u}, \]
\[ \Gamma_{ui} = \frac{3c_3(t)}{2} \partial_x + (c_3(t) - x c'_3(t)) \partial_{u}, \]
\[ \Gamma_{ui} = \frac{3c_3(t)}{2} \partial_x + \frac{3c_3(t)}{2} \partial_x + x c'_3(t) \partial_{u}, \]

where \( c_3, c_4, \) and \( c_7 \) are arbitrary functions of \( t \). The nonzero
Lie brackets are,
\[ [\Gamma_{u}, \Gamma_{ui}] = \frac{\Gamma_{ui}}{3}, \]
\[ [\Gamma_{2}, \Gamma_{ui}] = \frac{\partial_x}{12}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{\partial_x + 2y}{2}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{\Gamma_{ui}}{3}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{\partial_x}{4} + \frac{\partial_y}{12} + \frac{\partial_z}{2}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{\partial_x}{4} + \partial_y + \frac{\partial_z}{2}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = -\partial_x + \frac{\partial_y}{12} - \frac{\partial_z}{2} - \frac{3c_3}{8}, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{3c_3(t)}{2} \partial_u, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{3c_3(t)}{2} \partial_u, \]
\[ [\Gamma_{ui}, \Gamma_{ui}] = \frac{3c_3(t)}{2} \partial_u, \]

**Case 9.** We study the reduction firstly with the travelling
-wave simplification. It can be easily verified that \( \Gamma_{4i} - \frac{\Gamma_2}{2} \)
and \( \Gamma_{ij} - \frac{3i}{8} \) are symmetries. Therefore, we use linear combination of \( \Gamma_{ij}, \Gamma_{kij}, \Gamma_{li} - \frac{l^2}{2} \) and \( \Gamma_{ii} - \frac{4i^2}{3} \), i.e.
\[
k\Gamma_{ij} + l\Gamma_{kl} + m\left(\Gamma_{il} - \frac{l^2}{2}\right) - c\left(\Gamma_{ii} - \frac{4i^2}{3}\right),
\]
where \( k, l, m \) are wave numbers and \( c \) is the frequency, to reduce the equation to a fourth-order ode,
\[
k^3p^{\prime\prime}(q) + 6k^3p^\prime q^{\prime\prime}(q) - (2cl + 3km + 3lm + 3m^2)p^\prime(q) = 0,
\]
(4.1)
where \( q = kx + ly + mz - ct \) and \( p(q) = u(t, x, y, z) \). It is to be noted here that equation (4.1) is similar to the equation (3.3). The only difference can be found in the coefficient of second derivative of \( p \) with respect to \( q \). Also, the subsequent reductions are similar to the previous section.

### 4.1 Further reductions of equation (1.2)

**Case 10.** We reduce equation (1.2) using \( \Gamma_{ij} \). It is to be mentioned here that other symmetries such as, \( \Gamma_{ij} \) and \( \Gamma_{ki} \) do not provide favourable reductions.

The similarity variables for \( \Gamma_{ij} \),
\[
\begin{align*}
-3t + 8y - 4z &= w_t, \\
\frac{z}{\sqrt{t}} &= w_t, \\
4x - 2x - 3t &= w_{t}, \\
\frac{t^4}{4} &= w_{t},
\end{align*}
\]
\[
u(t, x, y, z) = \frac{45t^2 - 160ty + 48t^2z + 640v_1(w_t, w_{t}, w_{y})}{640t^4}.
\]

The reduced PDE of dimension 1 + 2 is
\[
9v_1_{w_{t}w_{t}} + \left(2 - 9v_1_{w_{t}w_{t}}\right)v_{w_{t}} - 2w_{t}v_{w_{t}w_{t}} - 9v_1v_{w_{t}w_{t}} - 3w_{t}v_{w_{t}w_{t}} + 3w_{t}v_{w_{t}w_{t}} = 0.
\]

The Lie point symmetries are,
\[
\begin{align*}
\Gamma_{ij} &= 2w_1\partial_{w_1} + w_2\partial_{w_2}, \\
\Gamma_{kij} &= \partial_{w_k}, \\
\Gamma_{lij} &= \partial_{w_l}, \\
\Gamma_{jij} &= \frac{9}{2}\partial_{w_1} - w_3\partial_{w_1}, \\
\Gamma_{jij} &= w_3\partial_{w_3}.
\end{align*}
\]

**Case 11.** We study the reduction with respect to \( \Gamma_{ij} \). The similarity variables are \( w_{t} = \frac{w_{t}}{\sqrt{t}}, \ v_{1}(w_{t}, w_{t}, w_{t}) = v_{2}(w_{t}, w_{t}) \). The reduced PDE of 1 + 1 dimensions is,
\[
3\left(-6 + w_4^2v_{2n_4} + w_4v_{2n} + 5 - 9v_{2n_4}\right)
\]
\[
- w_4\left(2w_3 + 9v_{2n_4}\right)v_{2n_4} + 3v_{2n_4} = 0.
\]

The Lie point symmetries are \( \Gamma_{ij} = \partial_{w_1} - \frac{2w_3}{9}\partial_{w_3}, \ \Gamma_{kij} = \partial_{w_k} \).

**Case 12.** We study the reduction with \( \Gamma_{1k} \). The similarity variable is,
\[
v_2(w_{t}, w_{t}) = \frac{w_3}{9} + v_3(w_{t}).
\]

The reduced ode is,
\[
7w_4v_3'(w_{t}) - 18v_3'(w_{t}) + 3w_4^2v_3'(w_{t}) = 0.
\]

Equation (4.2) is maximally symmetric. Also, one interesting observation regarding (4.2) is that the solution of the equation is in terms of Hypergeometric function. Therefore, solution of (1.2) can be given in terms of equation (4.2).

**Case 13.** We study the reduction using \( \Gamma_{ij} + \Gamma_{ij} \), the similarity variables \( w_2, w_3 \), are the newly defined independent variable and \( v_{1} (w_{t}, w_{t}, w_{t}) = w_{t} + v_{2} (w_{t}, w_{t}) \). The reduced PDE is,
\[
2 - 9v_{2n_4} + 9v_{2n_4} = 0.
\]

The symmetries of equation (4.3) are,
\[
\begin{align*}
\Gamma_{ij} &= v_{2}\partial_{v_2}, \\
\Gamma_{kij} &= \partial_{v_k}, \\
\Gamma_{lij} &= (\ -c_6(w_{t} - w_{t}) + c_6(w_{t} + w_{t})\ )\partial_{v_2} \\
&\ + (\ -c_6(w_{t} - w_{t}) + c_6(w_{t} + w_{t})\ )\partial_{v_1}, \\
\Gamma_{jij} &= E_6(w_{t}, w_{t})\partial_{v_1},
\end{align*}
\]
where \( c_6 (w_{t} - w_{t}), c_6 (w_{t} + w_{t}) \) and \( E_6 (w_{t}, w_{t}) \) are arbitrary functions.

**Case 14.** We use \( \Gamma_{ij} \) for reduction. The similarity variable is \( v_{2} (w_{t}, w_{t}) = v_3 (w_{t}) \). The reduced ode is,
\[
2 + 9v_3'(w_{t}) = 0.
\]

Equation (4.4) is linearisable. Therefore, the solution of (1.2) can be given in terms of equation (4.4).

**Case 15.** We study the reduction with respect to \( \Gamma_{ij} \). The similarity variables are \( v_{1} (w_{t}, w_{t}, w_{t}) = \frac{w_{t}}{9} + v_{2} (w_{t}, w_{t}) \).
The new independent variables are \( w_1 \) and \( w_2 \). The reduced PDE is, 
\[ 9v_{2,ww} + 4v_{2,ww} + 3w_2v_{2,ww} = 0. \tag{4.5} \]
The Lie – point symmetries are,
\[ \begin{align*}
\Gamma_{1m} &= \partial_{w_1}, \\
\Gamma_{2m} &= v_2\partial_{w_1}, \\
\Gamma_{3m} &= E_1(w_1, w_2)\partial_{w_2}.
\end{align*} \]

**Case 16.** The reduction using \( \Gamma_{1m} \), leads to the well – known linearisable second – order equation, which is \( v'_2(w_2) = 0 \), where the similarity variable is \( v_2(w_1, w_2) = v_3(w_2) \). Next, using \( \Gamma_{1m} + \Gamma_{2m} \), which is a symmetry, the corresponding reduced ode is, 
\[ 4v_1(w_2) + 3w_2v'_1(w_2) + 9v'_2(w_2) = 0. \tag{4.6} \]
where the similarity variables are \( v_2(w_1, w_2) = e^{\theta_1}v_3(w_2) \).

Equation (4.6) is a second – order linear equation and it is maximally symmetric.

**5 The symmetry analysis for equation (1.3)**

Similarly, for equation (1.3), we only mention the Lie – point symmetries here. The results with respect to the travelling – wave reduction are similar and hence we omit it here.

\[ \begin{align*}
\Gamma_{10} &= \partial_{x}, \\
\Gamma_{20} &= \partial_{y}, \\
\Gamma_{30} &= \partial_{u}, \\
\Gamma_{40} &= \partial_{z}, \\
\Gamma_{50} &= f_4(t)\partial_{u}, \\
\Gamma_{60} &= g_4(z)\partial_{u}.
\end{align*} \]

where \( f_4(t) \) and \( g_4(z) \) are the arbitrary functions. It is clear that the Lie brackets of the point symmetries, from \( \Gamma_{10} \) to \( \Gamma_{60} \) of equation (1.3) do not possess any nonzero output and so the algebra is abelian.

**5.1 Further reduction for equation (1.3)**

**Case 17.** We study the reduction with respect to \( \Gamma_{50} + \Gamma_{10} \). The similarity variable is \( u(t, x, y, z) = f_1(t) + v_1(x, y, z) \), where \( f_1(t) = \int f_4(t)dt \). The reduced PDE in \(+2 \) dimensions is, 
\[ -3v_{1,x} + 3v_{1,y} + 3v_{1,z} + v_{1,xx} = 0. \tag{5.1} \]
The Lie – point symmetries are,
\[ \begin{align*}
\Gamma_{1p} &= x\partial_{v_1} - z\partial_{v_y}, \\
\Gamma_{2p} &= \partial_{v_y}, \\
\Gamma_{3p} &= \partial_{v_z}, \\
\Gamma_{4p} &= h_1(w_1)\partial_{v_1}, \\
\Gamma_{5p} &= h_2(z)\partial_{v_1},
\end{align*} \]
where \( h_1(w_1) \) and \( h_2(z) \) are arbitrary functions with respect to \( z \).

**Case 18.** The similarity variable with respect to \( \Gamma_{1p} \) is \( v_1(x, y, z) = -\frac{y}{x} + v_2(x, z) \). The reduced PDE in \( 1+1 \) dimensions is, 
\[ v_{2,x} + zv_{2,y} + xv_{2,zz} = 0. \tag{5.2} \]
The Lie – point symmetries are,
\[ \begin{align*}
\Gamma_{1q} &= E_1(x, z)\partial_{x} + E_9(x, z)\partial_{y}, \\
\Gamma_{2q} &= E_3(z)\partial_{z}, \\
\Gamma_{3q} &= v_2E_{10}(z)\partial_{z},
\end{align*} \]
where \( E_1(x, z), E_3(z) \) and \( E_{10}(z) \) are the arbitrary functions.

The similarity variable for \( \Gamma_{4q} \) is \( v_2(x, z) = xv_3(z) \) and the reduced ode is of Euler type, namely,
\[ v_1(z) + zv'_2(z) = 0. \tag{5.3} \]

Next, the similarity variable using \( \Gamma_{2q} + \Gamma_{4q} \), is of the form \( v_2(x, z) = e^{E_1(z)} + v_3(x) \), where \( E_1(z) = \int E_{10}(z)dz \). The reduced ode is,
\[ v_3(x) + xv'_2(x) = 0. \tag{5.4} \]

Therefore, the solution of equation (1.3) can be given in terms of equation (5.3) and (5.4).

**Case 19.** We use \( \Gamma_{3p} + \Gamma_{4p} \) for the reduction. The similarity variables are \( w_1 = z + y, \quad v_1(x, y, z) = v_2(x, z + y) \). The reduced PDE is,
\[ 3v_{2,xx} + 3v_{2,yy} + 3v_{2,zz} + v_{2,xxx} = 0. \tag{5.5} \]

The Lie – point symmetries of equation (5.5) are,
\[ \begin{align*}
\Gamma_{1r} &= x\partial_{x} - (2x + v_2)\partial_{y}, \\
\Gamma_{2r} &= \partial_{y}, \\
\Gamma_{3r} &= \partial_{z}, \\
\Gamma_{4r} &= h_1(w_1)\partial_{w_1},
\end{align*} \]
where \( h_1(w_1) \) is an arbitrary function of \( w_1 \).

We use \( \Gamma_{1r} \) for reduction. The similarity variable, \( v_2(x, w_1) = -x + \frac{v_1(w_1)}{2} \), leads to the reduced first – order ode,
\[ ( -2 + 3v_3(w_1))v'_3(w_1) = 0, \tag{5.6} \]
which can be solved easily.
**Case 20.** We now study the reduction with respect to \( \Gamma_{60} + \Gamma_{40} \), for which the similarity variables are \( u(t, x, y, z) = g_3(z) + v_3(t, x, y) \), and \( g_3(z) = \int \frac{dz}{7 \varphi(z)} \). The reduced PDE of dimension 1 + 1 is,
\[
3v_2 v_{2x} + 3v_3 v_{2z} + v_{xxxy} + 2v_{2y} + 2v_{2z} = 0. 
\]
(5.7)

The Lie point symmetries are,
\[
\begin{align*}
\Gamma_{1t} &= \partial_t, \\
\Gamma_{2t} &= t \partial_t + \frac{x}{3} \partial_x + \frac{y}{3} \partial_y - \frac{v_2}{3} \partial_x, \\
\Gamma_{3t} &= \partial_z, \\
\Gamma_{1x} &= t \partial_x + \left(\frac{2x}{3} + \frac{2y}{3}\right) \partial_y, \\
\Gamma_{2x} &= x \partial_x, \\
\Gamma_{3x} &= h(t) \partial_x, \\
\end{align*}
\]
where \( h(t) \) is an arbitrary function of \( t \). We use \( \Gamma_{1t} + \Gamma_{2t} + \Gamma_{3t} \) to study the further reduction.

The similarity variables are,
\[
\begin{align*}
x - t &= w_1, \\
y - t &= w_2, \\
v_2(t, x, y) &= v_3(w_1, w_2).
\end{align*}
\]

The reduced PDE of dimension 1 + 1 is,
\[
-2v_{3w_1w_2} - 4v_{3w_1} + 3v_{3w_2} + 2v_{3w_1} + 3v_{3w_2} + v_{3w_1w_2} = 0. 
\]
(5.8)

The Lie point symmetries are,
\[
\begin{align*}
\Gamma_{1t} &= \partial_{w_1}, \\
\Gamma_{2t} &= \partial_{w_2}, \\
\Gamma_{3t} &= \partial_v.
\end{align*}
\]

The similarity variables with respect to \( \Gamma_{1t} + \Gamma_{2t} + \Gamma_{3t} \) are,
\[
\begin{align*}
w_2 &- w_1 = w_1, \\
v_3(w_1, w_2) &= w_1 + v_3(w_1).
\end{align*}
\]

The reduced ode is,
\[
\frac{v''_3(w_3) - 6v'_3(w_3)v''_3(w_3) + 3v''_3(w_3)}{v_3''(w_3)} = 0. 
\]
(5.9)

Equation (5.8) has three symmetries, which are,
\[
\begin{align*}
\Gamma_{1t} &= \partial_{w_1}, \\
\Gamma_{2t} &= \partial_{w_2}, \\
\Gamma_{3t} &= w_2 \partial_{w_3} + (w_3 - v_3) \partial_{v_3}.
\end{align*}
\]

**Case 21.** The reduction with respect to \( \Gamma_{1u} \), leads to a third order ode,
\[
\frac{v''_3(w_3) - 6v'_3(w_3)v''_3(w_3) + 3v''_3(w_3)}{v_3''(w_3)} = 0. 
\]

where \( v_3(w_3) = v_3(w_3) \) and \( v_5(w_3) = \frac{1}{v_3(w_3)} \). The equation (5.9) has a sole symmetry, \( \partial_{w_3} \). The reduced third order ode is further reduced to the second order equation, which is,
\[
\frac{v''_6(w_5)}{v_6(w_5)} = \frac{3v''_6(w_5)^2}{v_6(w_5)} + \frac{10v'_6(w_5)}{v_6(w_5)} + \frac{(3w_5^2 - 6w_5^2)'}{v_6(w_5)} + \frac{15v_6(w_5)}{v_6(w_5)}, 
\]
(5.10)

where \( w_5 = v_6(w_3) \) and \( v_6(w_3) = v_3(w_3) \). The equation (5.10) is maximally symmetric.

**Case 22.** The reduction with respect to \( \Gamma_{2u} \), leads to the third order equation,
\[
\frac{v''_5(w_5)}{v_5(w_5)} = \frac{3v''_6(w_5)^2}{v_6(w_5)} + \frac{6v'_6(w_5)}{v_6(w_5)}, 
\]
(5.11)

where \( w_5 = v_5(w_3) \) and \( v_5(w_3) = \frac{1}{v_3(w_3)} \). The equation (5.11) has two symmetries, which are \( \partial_{w_3} \) and \( w_3 \partial_{w_3} + (1 - 2v_3) \partial_{v_3} \). The reduction with respect to \( \partial_{w_3} \) leads to the second order equation,
\[
\frac{v''_6(w_5)}{v_6(w_5)} = \frac{3v''_6(w_5)^2}{v_6(w_5)} + \frac{6v'_6(w_5)}{v_6(w_5)} + \frac{26 - 6w_3}{w_5}v_6(w_5), 
\]
(5.12)

where \( w_5 = v_6(w_3) \) and \( v_6(w_3) = \frac{1}{v_3(w_3)} \). The equation (5.12) possess eight symmetries and hence is maximally symmetric. Next, we use \( w_3 \partial_{w_3} + (1 - 2v_3) \partial_{v_3} \) for reduction. The reduced second order ode is,
\[
\frac{v''_6(w_5)}{v_6(w_5)} = \frac{3v''_6(w_5)^2}{v_6(w_5)} + \frac{9v_6(w_5)}{v_6(w_5)}v_6'(w_5) + (26 - 6w_3)v_6(w_5)^3 
\]
\[
+ (12w_5^2 - 24w_3) v_6(w_5)^4, 
\]
(5.13)

where \( w_5 = \frac{2w_3(m_3 - 1)}{2} \) and \( v_6(w_3) = \frac{1}{v_3(w_3)} \). Equation (5.13) possesses zero Lie point symmetries and it is similar to equation (3.8). The singularity analysis fails to infer any specific conclusion for equation (5.13). Therefore, the solution of equation (1.3) can be given in terms of equation (5.10) and (5.12).

**Case 23.** The reduction with respect to \( \Gamma_{1u} \), leads to a third order equation with zero Lie point symmetries. The equation is,
\[
\frac{v''_5(w_5)}{v_5(w_5)} = \frac{10v'_5(w_5)}{v_5(w_5)} + 10v_5(w_5) + \frac{15v''_5(w_5)}{v_5(w_5)} - 30v'_5(w_5)^3 
\]
\[
+ (12w_5^2 + 26w_3) v_5(w_3)^4, 
\]
(5.9)

\[
\]
where \( w_6 = \frac{(2w_1 w_5) - w_2 w_3}{2} \) and \( v_5 (w_6) = \frac{1}{w_5 (w_5 w_7 + w_4) (w_5 w_7 - w_4)} \).

Singularity analysis of this equation is under study.

Case 24. The reduction with respect to \( \Gamma_{2v} \) leads to a PDE of dimension 1 + 1, with similarity variable,

\[
v_2 (t, x, y) = \frac{v_5 \left( \frac{x}{p}, \frac{y}{p} \right)}{p},
\]

\[
0 = 2w_1 v_{3y_1} + 4v_{3x_1} + 2w_2 v_{3y_2} + 2w_3 v_{3y_2} - 9v_{3x_1} v_{3y_2} + v_{3x_2} \left( 4 - 9v_{3y_2} \right) + 2w_1 v_{3x_1} - 3v_{3x_1 y_1 y_2}.
\]

(5.14)

Equation (5.14) has a sole point symmetry, \( \partial_{\nu_5} \). The subsequent reductions do not yield any favourable reductions.

Case 25. The reductions with respect to \( \Gamma_{4s} \) leads to a PDE of 1 + 1 dimension. Similar equations are obtained in section 3 equation (3.12), where we have discussed it in detail. The equation is,

\[
v_{3y} + tv_{3y} = 0,
\]

(5.15)

where the similarity variable is \( v_2 (t, x, y) = \frac{\left( \frac{x}{p} \right)}{p} + v_3 (t, y) \).

6 Singularity analysis

We study the singularity analysis of equation (3.8). We suggest that readers refer \([3, 5, 36-40]\) to understand the preliminaries. Firstly, we write it in a convenient format,

\[
-2(13k + 3x)y(x)^4 + 12x(2k + x)y(x)^3
-9ky(x)^3y(x) - 3ky(x)^2 + kyx(x) = 0,
\]

(6.1)

where we mention \( b(a) = y(x) \). We substitute \( y \rightarrow ax^p \) in equation (6.1) and look for the possible values of the exponent \( p \). The substitution leads to,

\[
-26a^2 k x^{6p} - a^2 k p x^{2+2p} = 2a^2 k p x^{2+2p} - 9a^2 k p x^{-1+3p}
-6a^2 x^{1+6p} + 24a^2 k x^{1+6p} + 12a^2 x^{2+6p} = 0
\]

(6.2)

One of the possible values obtained from the dominant terms is \(-1\). For \( p = -1 \), the possible values of the leading-order coefficient \( a \) are \( 0, \frac{1}{2}, \frac{1}{2} \). Next we substitute \( y \rightarrow ax^{-1} + mx^{-1+6} \) into (6.1) to compute the resonances (\( s \) denotes the resonance). The substitution leads to,

\[
0 = -a^2 k^2 + 9a^2 k x^{2} - 26a^2 k x^{3} - 26a^2 k x^{4} + 26a^2 k x^{5} - 12a^5
-2akm x^{2+5s} + 27akmx^{2+5s} - 104akmx^{2+5s} + 120a^2 km x^{2+5s} + 3akms x^{2+5s} + akms x^{2+5s} + 24a^2 mx^{2+5s}
+60a^2 mx^{2+5s} + 156a^2 km x^{2+5s} + 240a^2 km x^{2+5s} + 3km^2 x^{2+5s} - 18akm^2 x^{2+5s}
-2km^2 x^{2+5s} - 36a^2 m x^{3+1x} + 120a^2 m x^{3+1x} + 9km^2 x^{3+1x}
-104akm^2 x^{3+1x} + 240a^2 km x^{3+1x} - 24akm^2 x^{3+1x} + 24am^2 x^{3+1x}
+120a^2 m x^{3+1x} - 26km^2 x^{4+1x} + 120akm^2 x^{4+1x} - 6m^4 x^{3+1x} + 60am^4 x^{3+1x} + 24km^4 x^{4+1x} + 12m^4 x^{3+1x}.
\]

(6.3)

We consider the linear terms with respect to \( m \) from equation (6.3),

\[
-2akm^{2+5s} + 27akmx^{2+5s} - 104akmx^{2+5s} + 120a^2 km x^{2+5s} + 3akms x^{2+5s} + akms x^{2+5s} + 24a^2 mx^{2+5s}
+60a^2 mx^{2+5s} + 156a^2 km x^{2+5s} + 240a^2 km x^{2+5s} + 3km^2 x^{2+5s} - 18akm^2 x^{2+5s}
-2km^2 x^{2+5s} - 36a^2 m x^{3+1x} + 120a^2 m x^{3+1x} + 9km^2 x^{3+1x}
-104akm^2 x^{3+1x} + 240a^2 km x^{3+1x} - 24akm^2 x^{3+1x} + 24am^2 x^{3+1x}
+120a^2 m x^{3+1x} - 26km^2 x^{4+1x} + 120akm^2 x^{4+1x} - 6m^4 x^{3+1x} + 60am^4 x^{3+1x} + 24km^4 x^{4+1x} + 12m^4 x^{3+1x}.
\]

(6.4)

We list the corresponding values of resonances for various nonzero values of the leading – order coefficient \( a \),

\[
\left( a = \frac{1}{2} \rightarrow s = \left( \frac{1}{2}, 1 \right) \right), \quad \left( a = \frac{1}{2} \rightarrow s = \left( -\frac{1}{2}, \frac{1}{2} \right) \right), \quad \left( a = \frac{1}{2} \rightarrow s = \left( -\frac{1}{2}, -\frac{1}{2} \right) \right).
\]

It is to be observed that the generic value of \( s \), which is \(-1\), is not obtained for any of the possible values of the leading – order coefficient \([36, 37]\) and hence we cannot infer about the integrability of equation (3.8).

7 Singularity analysis for the Jimbo–Miwa PDE

We apply the singularity analysis for the PDE (1.1). We do that by replacing \( u \rightarrow U \phi(t, x, y, z)^p \) \([41, 42]\), in equation (1.1).
Solving the dominant terms, we obtain the value of $p$ is $-1$. To obtain the coefficient of the leading order exponent we substitute the value of $p$ and collect the dominant terms, which gives,

$$-12U_0^2 \Phi_x \Phi_{\xi_x}^2 + 24U_0 \Phi_x \Phi_{\xi_x}^3 = 0.$$  

(7.2)

We solve equation (7.2) to obtain the values of $U_0$, which are,

$$U_0 \to 0, \quad U_0 \to 2\Phi_x.$$

For the nonzero $U_0$, we compute the resonances, for which we substitute,

$$U(t, x, y, z) = U_0 \Phi(t, x, y, z)^{-1} + m\Phi(t, x, y, z)^{-1}$$

into equation (1.1). The substitution leads to,

$$0 = 6m^2(-2 + S)(-1 + S)\Phi^2 \Phi_{\xi_x}^2 + 12U_0 \Phi_x (U_0 - 2\Phi_x) \Phi_x$$

$$+ m(-4 + S)(1 + S)\Phi^2 \Phi_{\xi_x}^2 + 6U_0 + (S - 3)(-2 + S)\Phi_x$$

$$+ 3m^2(-1 + S)\Phi^2 (\Phi_{\xi_x} + \Phi_{\xi_y})$$

$$+ 3m(1 + S)\Phi(1 + S)(-2U_0 + (S - 3)(-2 + S)\Phi_x)(\Phi_{\xi_x} + \Phi_{\xi_y})$$

$$- m(-2 + S)(-1 + S)\Phi^2 \Phi_{\xi_x}^2$$

$$+ 3(1 + S)(2U_0 + (S - 3)(-2 + S)\Phi_x)(\Phi_{\xi_x} + \Phi_{\xi_y})$$

$$+ 3(1 + S)(2U_0 + (S - 3)(-2 + S)\Phi_x)(\Phi_{\xi_x} + \Phi_{\xi_y})$$

$$- m(2 + S)(-2 + S)\Phi_x.$$

Next, we collect all the linear terms with respect to $m$, which gives,

$$(-4 + S)(-1 + S)\Phi^2 \Phi_{\xi_x}^2 - 6U_0 + (S - 3)(-2 + S)\Phi_x + 3(-1 + S)\Phi^2 \Phi_{\xi_x}^2$$

$$+ (2U_0 + (S - 3)(-2 + S)\Phi_x)(\Phi_{\xi_x} + \Phi_{\xi_y})$$

$$- (S - 2)(-1 + S)\Phi_{\xi_x} (\Phi_{\xi_x} + 2\Phi_y)$$

$$+ (S - 1)\Phi_{\xi_x} (\Phi_{\xi_x} + 2\Phi_y)$$

$$+ (S - 1)\Phi_{\xi_x} (\Phi_{\xi_x} + 2\Phi_y) - 2\Phi_x \Phi_{\xi_x} + 4\Phi_{\xi_x} \Phi_{\xi_y} + \Phi_{\xi_x} \Phi_{\xi_y} + 2\Phi_x \Phi_{\xi_y} + 2\Phi_x \Phi_{\xi_y}.$$

As, observed from the values of $U_2(t, x, y, z)$ and $U_3(t, x, y, z)$, arbitrary term is $U_1(t, x, y, z)$. Similar observations can be made from $U_4(t, x, y, z)$ to $U_6(t, x, y, z)$. The calculations are bit tedious and we omit mentioning them here. We summarize our analysis in the following theorem.

**Theorem:** The Jimbo-Miwa equation (1.1) and its generalizations (1.2) and (1.3) pass the singularity test, that is, have the Painlevé property and are integrable. Their solution is expressed by a Right Painlevé Series.

**8 Conclusion**

An elaborate study of the reductions of Jimbo–Miwa and its extended forms is discussed. As mentioned above, new reductions of the equations were obtained. The singularity analysis of certain reduced odes and all the three forms of Jimbo–Miwa equation are discussed. This paper forms the first part of our subsequent papers on Jimbo–Miwa equations. The conservation laws and the solutions from them are under study. Moreover, we have discussed reductions elaborately, but not exhaustively. Therefore, the remaining reductions will also be discussed in our future work.

Finally, by applying the singularity analysis for the PDEs of our consideration, without applying any similarity transformation, we found that the equations of our consideration have the Painlevé property and the analytic solution can be expressed in terms of a Right Painlevé Series.

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