GENERALIZATION OF KNUTH’S FORMULA FOR THE NUMBER OF SKEW TABLEAUX

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Abstract. We take an elementary approach to derive a generalization of Knuth’s formula using Lassalle’s explicit formula. In particular, we give a formula for the Kostka numbers of a shape \( \mu \vdash n \) and weight \( (m, 1^{n-m}) \) for \( m = 3, 4 \).

1. Introduction

Throughout this paper, \( n \) will denote a positive integer. We write \( \mu \vdash n \) if \( \mu \) is a partition of \( n \), that is, a non-increasing sequence \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) of positive integers such that \( |\mu| = \sum_{i=1}^{k} \mu_i = n \). We say that \( k \) is the height of \( \mu \) and denote it by \( h(\mu) \). We denote by \( D_\mu \) the Young diagram of \( \mu \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash m \) and \( D_\lambda \subset D_\mu \), then the skew shape \( \mu/\lambda \) is obtained by removing from \( D_\mu \) all the boxes belonging to \( D_\lambda \).

Let \( \mu, \lambda \vdash n \) and \( \nu \vdash m \leq n \). A semistandard Young tableau (SSYT) of shape \( \mu \) and weight \( \lambda \) is a filling of the Young diagram \( D_\mu \) with the numbers \( 1, 2, \ldots, h(\lambda) \) in such a way that

(i) \( i \) occupies \( \lambda_i \) boxes, for \( i = 1, 2, \ldots, h(\lambda) \),

(ii) the numbers are strictly increasing down the columns and weakly increasing along the rows.

The Kostka number \( K(\mu, \lambda) \) is the number of SSYT\'s of shape \( \mu \) and weight \( \lambda \). In particular, if \( \lambda = (1^n) \) then such a tableau is called a standard Young tableau (SYT) of shape \( \mu \), and for a skew shape \( \mu/\nu \) and weight \( (1^{n-m}) \) such a tableau is called a skew SYT of skew shape \( \mu/\nu \). We denote by \( f^{\mu/\nu} \) the number of skew SYT\'s of skew shape \( \mu/\nu \). Obviously, if \( \lambda = (m, 1^{n-m}) \vdash n \) and \( m \leq \mu_1 \), then for all SSYT\'s of shape \( \mu \) and weight \( \lambda \), a box \( (1, j) \in D_\mu \) is filled by 1 for \( 1 \leq j \leq m \), so \( K(\mu,(m, 1^{n-m})) = f^{\mu/(m)} \).

Naturally, if \( \nu = \emptyset \) then \( f^{\mu} \) is the number of SYT\'s of shape \( \mu \). We can easily compute \( f^{\mu} \) using the hook formula (see [4]), but the problem of computing Kostka numbers is in general difficult (see [3]). There is a recurrence formula for Kostka numbers (see [6] and [7]), but we have no explicit formula for Kostka numbers.

For \( z \in \mathbb{C} \), the falling factorial is defined by \( [z]_n = z(z-1) \cdots (z-n+1) = n!(\frac{z}{n}) \), and \( [z]_0 = 1 \). Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n \) and \( \mu' \) be the conjugate of \( \mu \). Knuth [5, p.67, Exercise 19] shows:

\[
\left( f^{\mu/(2)} \right) = \frac{f^\mu}{[n/2]} \left( \sum_{i=1}^{k} \left( \mu_i \over 2 \right) - \sum_{j \geq 1} \left( \mu_j' \over 2 \right) + \left( n \over 2 \right) \right).
\]
In fact, we can also compute \( f^{\mu/\lambda} \) using \([1, p.310]\), \([3, \text{Theorem}]\) and \([9, \text{Corollary 7.16.3}]\), but this requires evaluation of determinants and knowledge of Schur functions. If we compute \( \lambda = (2) \) using \([9, \text{Corollary 7.16.3}]\), then we get the following:

\[
(2) \quad f^{\mu/(2)} = \frac{f^{\mu}}{[n]_2} \left( \sum_{i=1}^{k} \left( \binom{\mu_i}{2} - \mu_i(i - 1) \right) \right) + \binom{n}{2}.
\]

Since the following equation is well known (see \([7, (1.6)]\), also see Proposition \([6]\) for a generalization):

\[
(3) \quad \sum_{i=1}^{k} \mu_i(i - 1) = \sum_{j \geq 1} \binom{\mu_j}{2},
\]

we have (11). As previously stated, since \( K(\mu, (m, 1^{n-m})) = f^{\mu/(m)} \), we know the value of \( K(\mu, (2, 1^{n-2})) \) from (11), so we are interested in the extent to which (11) can be generalized to an arbitrary positive integer \( m \). In fact, if \( \lambda = (3) \) then we get the following using \([9, \text{Corollary 7.16.3}]\):

\[
(4) \quad f^{\mu/(3)} = \frac{f^{\mu}}{[n]_3} \left( \sum_{i=1}^{k} \left( \mu_i(i - 1) + \binom{\mu_i}{2} \right) \right) + \binom{n-2}{2} \sum_{i=1}^{k} \left( \binom{\mu_i}{2} - \mu_i(i - 1) \right)
\]

\[
+ \frac{f^{\mu}}{[n]_3} \left( 2 \sum_{i=1}^{k} \left( \mu_i(i - 1) + \binom{\mu_i}{3} \right) - 2 \sum_{i=1}^{k} \binom{\mu_i}{2} \right)
\]

The proof of (4) using Lassalle’s explicit formula for characters will be given in Section 4.

Let \( l \) be a nonnegative integer. Let \( C(\mu) = \{ j - i \mid (i, j) \in D_\mu \} \) be the multiset of contents of the partition \( \mu \), and \( p_l[C(\mu)] = \sum_{(i,j) \in D_\mu} (j - i)^l \) be the \( l \)th power sum symmetric function evaluated at the contents of \( \mu \). In this paper, we take an elementary approach to derive a formula for \( f^{\mu/(m)} \) using \([2, \text{Section 5.3}]\) and \( p_l[C(\mu)] \).

This paper is organized as follows. After giving preliminaries in Section 2, we prove that \( p_l[C(\mu)] \) can be written as a linear combination of \( q_{r,t}^\pm \) in Section 3. We give an expression for \( f^{\mu/(m)} \) in terms of \( q_{r,t}^\pm \) for \( m \leq 4 \) in Section 4. Finally, we prove a generalization of (3) in Section 5.

2. Preliminaries

Throughout this section, \( h, \ l, \ r \) and \( t \) be nonnegative integers. We denote by \( S(n, k) \) the Stirling numbers of the second kind. First of all, we define

\[
C(r, t) = t!S(r + 1, t + 1).
\]

Then

\[
C(r, t) = t!S(r + 1, t + 1)
\]

\[
= t!(S(r, t) + (t + 1)S(r, t + 1))
\]
(5) \[ tC(r - 1, t - 1) + (t + 1)C(r - 1, t), \]
since \( S(r + 1, t + 1) = S(r, t) + (t + 1)S(r, t + 1). \)

Set

(6) \[ \varphi_l(h, r, t) = \binom{l}{h}C(h, r)C(l - h, t). \]

Clearly,

(7) \[ \varphi_l(h, r, t) = \binom{l - h}{t}C(l - h, t)C(h, r) \]

We define

\[ R_l(t) = \sum_{i=1}^{t} i^l. \]

Lemma 1. We have

\[ R_{l+1}(t) = (t + 1)R_l(t) - \sum_{i=1}^{t} R_i(i). \]

Proof. We have

\[
(t + 1)R_l(t) = (t + 1) \sum_{i=1}^{t} i^l \\
= \sum_{i=1}^{t} i^{l+1} + \sum_{i=1}^{t} \sum_{j=1}^{i} j^l \\
= R_{l+1}(t) + \sum_{i=1}^{t} R_i(i).
\]

Lemma 2. We have

\[ R_l(t) = \sum_{i=0}^{t} C(l, i) \binom{t}{i+1}. \]

Proof. Setting \( n = q = 0 \) in [2, Proposition 5.1.2]. We have

(8) \[ \sum_{k=0}^{l} \binom{k}{m} = \binom{l + 1}{m + 1}. \]

We prove the statement by induction on \( l \). If \( l = 0 \), then the statement holds since \( C(0, 0) = 1 \). Assume that the statement holds for \( l - 1 \). Then

\[
R_l(t) = (t + 1)R_{l-1}(t) - \sum_{j=1}^{t} R_{l-1}(j)(\text{by Lemma 1}) \\
= (t + 1) \sum_{i=0}^{l-1} C(l - 1, i) \binom{t}{i+1} - \sum_{j=1}^{t} \sum_{i=0}^{l-1} C(l - 1, i) \binom{j}{i+1}
\]
\[
\begin{align*}
    &= \sum_{i=0}^{l-1} (i+2)C(l-1,i) \binom{t+1}{i+2} - \sum_{i=0}^{l-1} C(l-1,i) \binom{t+1}{i+2} \quad \text{(by (8))} \\
    &= \sum_{i=0}^{l-1} (i+1)C(l-1,i) \binom{t+1}{i+2} \\
    &= \sum_{i=0}^{l-1} (i+1)C(l-1,i) \binom{t}{i+2} + \sum_{i=0}^{l-1} (i+1)C(l-1,i) \binom{t}{i+1} \\
    &= \sum_{i=0}^{l} iC(l-1,i-1) \binom{t}{i+1} + \sum_{i=0}^{l-1} (i+1)C(l-1,i) \binom{t}{i+1} \\
    &= \sum_{i=0}^{l} (iC(l-1,i-1) + (i+1)C(l-1,i)) \binom{t}{i+1} \\
    &= \sum_{i=0}^{l} C(l,i) \binom{t}{i+1} \quad \text{(by (5))}. 
\end{align*}
\]

**Lemma 3.** For \( z \in \mathbb{C} \), we have

\[
z^l = \sum_{i=0}^{l} C(l,i) \binom{z-1}{i}.
\]

**Proof.** From [2, p.211, (4.65)], we have

\[
z^l = \sum_{i=0}^{l} S(l,i)[z]_i,
\]

so

\[
z^l = \sum_{i=0}^{l} S(l,i)[z]_i \\
= \sum_{i=0}^{l} S(l,i)z[z-1]_{i-1} \\
= \sum_{i=0}^{l} S(l,i)[z-1]_{i-1}(z-i+i) \\
= \sum_{i=0}^{l} S(l,i)[z-1]_{i} + \sum_{i=1}^{l} iS(l,i)[z-1]_{i-1} \\
= \sum_{i=0}^{l} S(l,i)[z-1]_{i} + \sum_{i=0}^{l-1} (i+1)S(l,i+1)[z-1]_{i} \\
= \sum_{i=0}^{l} (S(l,i) + (i+1)S(l,i+1)) [z-1]_{i}
\]
\[= \sum_{i=0}^{l} S(l + 1, i + 1)[z - 1]^i\]
\[= \sum_{i=0}^{l} i!S(l + 1, i + 1)\binom{z - 1}{i}\]
\[= \sum_{i=0}^{l} C(l, i)\binom{z - 1}{i}.\]

\[\square\]

Let \(\mu, \lambda \vdash n\). We denote by \(\chi_{\mu}(\lambda)\) the value of the character of the Specht module \(S_{\mu}\) evaluated at a permutation \(\pi\) belonging to the conjugacy class of type \(\lambda\). From [2, Example 5.3.3], we have
\[\chi_{\mu}(2, 1^{n-2}) = \frac{f_{\mu}}{[n]_2} 2p_1[C(\mu)],\]
\[\chi_{\mu}(3, 1^{n-3}) = \frac{f_{\mu}}{[n]_3} 3\left(p_2[C(\mu)] - \binom{n}{2}\right),\]
\[\chi_{\mu}(4, 1^{n-4}) = \frac{f_{\mu}}{[n]_4} 4\left(p_3[C(\mu)] - (2n - 3)p_1[C(\mu)]\right),\]
\[\chi_{\mu}(5, 1^{n-5}) = \frac{f_{\mu}}{[n]_5} 5\left(p_4[C(\mu)] - (3n - 10)p_2[C(\mu)] - 2p_1[C(\mu)]^2 + 5\binom{n}{3} - 3\binom{n}{2}\right),\]
\[\chi_{\mu}(6, 1^{n-6}) = \frac{f_{\mu}}{[n]_6} 6\left(p_5[C(\mu)] + (25 - 4n)p_3[C(\mu)] + 2(3n - 4)(n - 5)p_1[C(\mu)]\right)\]
\[= \frac{f_{\mu}}{[n]_6} 36p_1[C(\mu)]p_2[C(\mu)].\]

**Remark 4.** In [2, Example 5.3.3], the coefficient of \(d_3(\lambda)\) (in this paper, we denote by \(p_3[C(\mu)]\)) in the character value \(\hat{\chi}_{6,1^{n-6}}^\lambda\) is 24(7 - \(n\)). Since \(c_6^1\) and \(c_7^2\) are incorrect in [2, p.251], the value of the character \(\hat{\chi}_{6,1^{n-6}}^\lambda\) is also incorrect. In fact, the coefficient of \(d_3(\lambda)\) in the character value \(\hat{\chi}_{6,1^{n-6}}^\lambda\) is 6(25 - 4\(n\)), as given in (9).

We obtain [2, Example 5.3.8]:
\[\chi_{\mu}(2, 2, 1^{n-4}) = \frac{f_{\mu}}{[n]_4} 4\left(p_1[C(\mu)]^2 - 3p_2[C(\mu)] + 2\binom{n}{2}\right).\]

In general, for \(\mu \vdash n\) and \(\lambda \vdash m \leq n\), the character \(\chi_{\mu}(\lambda, 1^{n-m})\) can be expressed as a polynomial of \(c_\mu^t(t)\) using Lassalle’s explicit formula [2, Theorem 5.3.11].

3. \(p_1[C(\mu)]\) AND \(q_{r,t}^\pm\)

Let \(\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n\), and let \(r, t\) be nonnegative integers. We define
\[q_{r,t}^\pm = \sum_{i=1}^{k} \left(\binom{\mu_i}{r + 1}\binom{i - 1}{t} \pm \binom{\mu_i}{t + 1}\binom{i - 1}{r}\right).\]
Observe that if $r = t$ then
\begin{equation}
q_{r,r}^- = 0,
\end{equation}
and
\begin{equation}
q_{r,t}^+ = q_{t,r}^+,
\end{equation}
\begin{equation}
q_{r,t}^- = -q_{t,r}^-.
\end{equation}

**Proposition 5.** Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and $l$ be a nonnegative integer. Then
\begin{align*}
p_{2l+1}[C(\mu)] &= \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} (j - i)^l, \\
&= \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} \sum_{h=0}^{l} \sum_{r=0}^{2l+1-h} (-1)^h \varphi_{2l+1}(h, r, t) q_{t,r}^-,
\end{align*}
\begin{align*}
p_{2l}[C(\mu)] &= \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} \sum_{h=0}^{l} \sum_{r=0}^{2l-h} (-1)^h \varphi_{2l}(h, r, t) q_{r,t}^+ + \frac{1}{2} (-1)^l \sum_{r=0}^{l} \varphi_{2l}(l, r, t) q_{r,t}^+.
\end{align*}

**Proof.** By the definition of $p_l[C(\mu)]$, we get the following:
\begin{align*}
p_l[C(\mu)] &= \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} (j - i)^l \\
&= \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} \sum_{h=0}^{l} \sum_{r=0}^{l-h} (-1)^{l-h} \binom{l}{h} j^h i^{l-h} \\
&= \sum_{i=1}^{k} \sum_{h=0}^{l} \sum_{r=0}^{l-h} (-1)^{l-h} \binom{l}{h} i^{l-h} R_h(\mu_i) \\
&= \sum_{i=1}^{k} \sum_{h=0}^{l} \sum_{r=0}^{l-h} (-1)^{l-h} \binom{l}{h} C(h, r) C(l - h, t) \binom{\mu_i}{r+1} \binom{i-1}{t} \\
&= \sum_{i=1}^{k} \sum_{h=0}^{l} \sum_{r=0}^{l-h} (-1)^{l-h} \varphi_l(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} 
\text{(by (14))},
\end{align*}
where the fourth equality follows from Lemma 2 and Lemma 3. Thus
\begin{align*}
p_{2l+1}[C(\mu)] &= \sum_{i=1}^{k} \sum_{h=0}^{2l+1} \sum_{r=0}^{2l+1-h} (-1)^{2l+1-h} \varphi_{2l+1}(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} \\
&= \sum_{i=1}^{k} \sum_{h=0}^{l} \sum_{r=0}^{2l+1-h} (-1)^{2l+1-h} \varphi_{2l+1}(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} \\
&\quad + \sum_{i=1}^{k} \sum_{h=0}^{2l+1} \sum_{r=0}^{2l+1-h} (-1)^{2l+1-h} \varphi_{2l+1}(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} \\
&= \sum_{i=1}^{k} \sum_{h=0}^{l} \sum_{r=0}^{2l+1-h} (-1)^{h-1} \varphi_{2l+1}(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} \\
&\quad + \sum_{i=1}^{k} \sum_{h=0}^{2l+1} \sum_{r=0}^{2l+1-h} (-1)^{h} \varphi_{2l+1}(h, r, t) \binom{\mu_i}{r+1} \binom{i-1}{t} 
\text{(by (14))}.
\end{align*}
where the third equality can be shown as follows:

\[
\sum_{h=t+1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{2l-h} (-1)^h \varphi_{2l+1}(h, r, t) \left( \frac{\mu_i}{r+1} \right) \left( \frac{r}{t} \right) = \sum_{h=0}^{l} \sum_{r=0}^{2l-1-h} \sum_{t=0}^{l} (-1)^h \varphi_{2l+1}(h, r, t)q_{r,t}^+,
\]

where the second equality can be shown as follows:

\[
\sum_{h=t+1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{2l-h} (-1)^h \varphi_{2l+1}(h, r, t) \left( \frac{\mu_i}{r+1} \right) \left( \frac{r}{t} \right) = \sum_{h=0}^{l} \sum_{r=0}^{2l-1-h} \sum_{t=0}^{l} (-1)^h \varphi_{2l+1}(h, r, t)q_{r,t}^+.
\]

Similarly, we have

\[
p_{2l}[C(\mu)] = \sum_{h=0}^{l-1} \sum_{r=0}^{l} \sum_{t=0}^{2l-h} (-1)^h \varphi_{2l}(h, r, t)q_{r,t}^+ + \sum_{i=1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{2l-h} (-1)^i \varphi_{2l}(l, r, t) \left( \frac{\mu_i}{r+1} \right) \left( \frac{r}{t} \right) = \sum_{h=0}^{l} \sum_{r=0}^{2l-1-h} \sum_{t=0}^{l} (-1)^h \varphi_{2l}(h, r, t)q_{r,t}^+ + \frac{1}{2} (-1)^l \sum_{r=0}^{l} \sum_{t=0}^{l} \varphi_{2l}(l, r, t)q_{r,t}^+,
\]

where the second equality can be shown as follows:

\[
\sum_{i=1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{l} (-1)^i \varphi_{2l}(l, r, t) \left( \frac{\mu_i}{r+1} \right) \left( \frac{r}{t} \right) = \frac{1}{2} (-1)^l \sum_{i=1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{l} \varphi_{2l}(l, r, t) \left( \frac{\mu_i}{r+1} \right) \left( \frac{r}{t} \right) + \frac{1}{2} (-1)^l \sum_{i=1}^{k} \sum_{r=0}^{l} \sum_{t=0}^{l} \varphi_{2l}(l, r, t) \left( \frac{\mu_i}{t+1} \right) \left( \frac{r}{t} \right) = \frac{1}{2} (-1)^l \sum_{r=0}^{l} \sum_{t=0}^{l} \varphi_{2l}(l, r, t)q_{r,t}^+.
\]
By Proposition 5, we have

\[ p_0[C(\mu)] = \frac{1}{2}q_{0,0} = n, \]
\[ p_1[C(\mu)] = q^-_{0,0} + q^-_{1,0} \]
\[ = q^-_{1,0}, \quad \text{(by (12))} \]
\[ p_2[C(\mu)] = 2q^+_{0,1} + 2q^+_{0,2} - q^+_{1,0} - q^+_{1,1} \]
\[ = q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1}, \quad \text{(by (13))} \]
\[ p_3[C(\mu)] = -2q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 3q^-_{0,1} - 9q^-_{1,1} - 6q^-_{2,1} \]
\[ = q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1} \quad \text{(by (12) and (14))}.} \]

4. Main results

For any \( i \geq 1, m_i(\mu) = |\{j \mid \mu_j = i\}| \) is the multiplicity of \( i \) in \( \mu \). Set

\[ z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!. \]

Let \( \mu \vdash n \) and \( \lambda \vdash m \leq n \). From [10, Theorem 3.1], we have

\[ f^{\mu/\lambda} = \sum_{\nu \vdash m} z^{-1}_\nu \chi^\mu(\nu, 1^{n-m}) \chi^\lambda(\nu). \]

If \( \lambda = (m) \), then

\[ f^{\mu/(m)} = \sum_{\nu \vdash m} z^{-1}_\nu \chi^\mu(\nu, 1^{n-m}) \chi^{(m)}(\nu) \]
\[ = \sum_{\nu \vdash m} z^{-1}_\nu \chi^\mu(\nu, 1^{n-m}). \] (16)

We already proved that \( p_i[C(\mu)] \) can be expressed as a linear combination of \( q^\pm_{r,t} \) (Proposition 5), so the character value \( \chi^\mu(\lambda, 1^{n-m}) \) can be written as a polynomial in \( q^\pm_{r,t} \) using Lassalle’s explicit formula [2, Theorem 5.3.11]. We compute \( \chi^\mu(m, 1^{n-m}) \) for \( 2 \leq m \leq 4 \) and \( \chi^\mu(2, 2, 1^{n-4}) \) using (9), (10) and (15).

\[ \chi^\mu(2, 1^{n-2}) = \frac{f^\mu}{[n]_2} 2p_1[C(\mu)] \]
\[ = \frac{f^\mu}{[n]_2} 2q^-_{1,0}, \]
\[ \chi^\mu(3, 1^{n-3}) = \frac{f^\mu}{[n]_3} 3 \left( p_2[C(\mu)] - \binom{n}{2} \right) \]
\[ = \frac{f^\mu}{[n]_3} 3 \left( q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2} \right), \]
\[ \chi^\mu(4, 1^{n-4}) = \frac{f^\mu}{[n]_4} 4 (p_3[C(\mu)] - (2n - 3)p_1[C(\mu)]) \]
Substituting (17) into (16), we find

\[ \chi^{\mu}(2, 2, 1^{n-4}) = \frac{f^\mu}{[n]_4} 4 \left( (4 - 2n)q_{1,0}^- + 6q_{2,0}^- + 6q_{3,0}^- - 6q_{2,1}^- \right), \]

\[ \chi^{\mu}(2, 2, 1^{n-4}) = \frac{f^\mu}{[n]_4} 4 \left( p_1 [C(\mu)]^2 - 3p_2 [C(\mu)] + 2 \left( \frac{n}{2} \right) \right) \]

(17)

Substituting (17) into (16), we find

\[ f^{\mu/(2)} = \frac{1}{z(2)} \chi^{\mu}(2, 1^{n-2}) + \frac{1}{z(1,1)} \chi^{\mu}(1^n) \]

\[ = \frac{1}{2} \cdot \frac{f^\mu}{[n]_2} \cdot 2q_{1,0}^- + \frac{1}{2} f^\mu \]

(18)

\[ f^{\mu/(3)} = \frac{1}{z(3)} \chi^{\mu}(3, 1^{n-3}) + \frac{1}{z(2,1)} \chi^{\mu}(2, 1^{n-2}) + \frac{1}{z(1,1,1)} \chi^{\mu}(1^n) \]

\[ = \frac{1}{3} \cdot \frac{f^\mu}{[n]_3} \cdot 3 \left( q_{0,1}^+ + 2q_{0,2}^+ - q_{1,1}^- - \left( \frac{n}{2} \right) \right) + \frac{1}{2} \cdot \frac{f^\mu}{[n]_2} \cdot 2q_{1,0}^- + \frac{1}{6} f^\mu \]

(19)

and

\[ f^{\mu/(4)} = \frac{1}{z(4)} \chi^{\mu}(4, 1^{n-4}) + \frac{1}{z(3,1)} \chi^{\mu}(3, 1^{n-3}) + \frac{1}{z(2,2)} \chi^{\mu}(2, 2, 1^{n-4}) + \frac{1}{z(2,1,1)} \chi^{\mu}(1^n) \]

\[ = \frac{1}{4} \cdot \frac{f^\mu}{[n]_4} \cdot 4 \left( (4 - 2n)q_{1,0}^- + 6q_{2,0}^- + 6q_{3,0}^- - 6q_{2,1}^- \right) \]

\[ + \frac{1}{3} \cdot \frac{f^\mu}{[n]_3} \cdot 3 \left( q_{0,1}^+ + 2q_{0,2}^+ - q_{1,1}^- - \left( \frac{n}{2} \right) \right) \]

\[ + \frac{1}{8} \cdot \frac{f^\mu}{[n]_4} \cdot 4 \left( (q_{1,0}^-)^2 - 3q_{0,1}^- - 6q_{0,2}^+ + 3q_{1,1}^+ + 2 \left( \frac{n}{2} \right) \right) \]

\[ + \frac{1}{4} \cdot \frac{f^\mu}{[n]_2} \cdot 2q_{1,0}^- + \frac{1}{24} f^\mu \]

\[ = \frac{f^\mu}{[n]_4} \left( \frac{1}{2} (n - 2)(n - 7)q_{1,0}^- + 6q_{2,0}^- + 6q_{3,0}^- - 6q_{2,1}^- + \frac{1}{2} (q_{1,0}^-)^2 \right) \]

\[ + \frac{f^\mu}{[n]_4} \left( (n - \frac{9}{2})q_{0,1}^+ + (2n - 9)q_{0,2}^+ - (n - \frac{9}{2})q_{1,1}^+ \right) \]

\[ + \frac{f^\mu}{[n]_4} \left( \left( \frac{n}{4} \right) - 3 \left( \frac{n}{3} \right) + 2 \left( \frac{n}{2} \right) \right). \]

We get (2) and (4) by substituting (11) into (18) and (19), respectively.
5. A generalization of a polynomial identity for a partition and its conjugate

**Proposition 6.** Let $\mu$ be a partition of an integer. Then $\mu'$ is the conjugate of $\mu$ if and only if

$$
\sum_{i=1}^{k} \binom{\mu_i}{t+1} \binom{i-1}{r} = \sum_{j \geq 1} \binom{\mu'_j}{r+1} \binom{j-1}{t},
$$

for all nonnegative integers $r$ and $t$.

**Proof.** First, we show the “only if” part. Then

$$
\sum_{j \geq 1} \binom{\mu'_j}{r+1} \binom{j-1}{t} = \sum_{j \geq t+1} \sum_{\begin{subarray}{l} J \subseteq \{1, 2, \ldots, \mu_1\} \\ |J| = t+1, \max I = j \end{subarray}} |\{I \mid I \times J \subseteq D_\mu, |I| = r + 1\}|
$$

$$
= \sum_{i=r+1}^{k} \sum_{\begin{subarray}{l} J \subseteq \{1, 2, \ldots, \mu\} \\ |J| = r+1, \max I \leq \mu_i \end{subarray}} |\{J \mid J \subseteq \{1, 2, \ldots, \mu_i\}, |J| = t+1\}|
$$

$$
= \sum_{i=r+1}^{k} \sum_{\begin{subarray}{l} J \subseteq \{1, 2, \ldots, \mu\} \\ |J| = r+1, \max I \leq \mu_i \end{subarray}} \binom{\mu_i}{t+1}
$$

$$
= \sum_{i=r+1}^{k} \binom{\mu_i}{t+1} \binom{i-1}{r}.
$$

Next, let $\lambda$ be the conjugate of $\mu$. Set $h(\lambda) = h$. Then

$$
\sum_{j=1}^{h} \binom{\lambda_j}{r+1} \binom{j-1}{t} = \sum_{i=1}^{k} \binom{\mu_i}{t+1} \binom{i-1}{r}
$$

$$
= \sum_{j \geq 1} \binom{\mu'_j}{r+1} \binom{j-1}{t}.
$$

Setting $h(\mu') = l$ and $r = 0$ in (20), we have

$$
\sum_{j=1}^{h} \lambda_j \binom{j-1}{t} = \sum_{i=1}^{l} \mu'_i \binom{i-1}{t}.
$$
Suppose $h > l$ and set $t = h - 1$ in (21), then $\lambda_h = 0$. Similarly, suppose $h < l$ and set $t = l - 1$ in (21). Then $\mu'_l = 0$, and both cases are contradictions. Thus $h = l$.

We show that $\lambda_{h-i} = \mu'_{h-i}$ for all $i$ with $0 \leq i \leq h - 1$ by induction on $i$. If $i = 0$, setting $t = h - 1$ in (21), then $\lambda_h = \mu'_h$.

Assume that the assertion holds for some $i \in \{0, 1, \ldots, h - 2\}$. Let $t = h - (i + 2)$ in (21). By the inductive hypothesis, we have

$$\sum_{j=h-i}^{h} \lambda_j \left( \frac{j-1}{h-i-2} \right) = \sum_{j=h-i}^{h} \mu'_j \left( \frac{j-1}{h-i-2} \right).$$

Therefore, $\lambda_{h-1} = \mu'_{h-1}$ since $\left( \frac{j-1}{h-i-2} \right) = 0$ for all $j$ with $1 \leq j \leq h - j - 2$. Thus $\lambda = \mu'$ and $\mu'$ is the conjugate of $\mu$. \hfill \Box

From Proposition 6 we have

$$q_{r,t}^{\pm} = \sum_{i=1}^{k} \left( \frac{\mu_i}{r+1} \right) \left( \frac{i-1}{t} \right) \pm \sum_{j=1}^{k} \left( \frac{\mu'_j}{r+1} \right) \left( \frac{j-1}{t} \right).$$

By substituting (22) into (18) and (19), we get (11) and

$$f^{\mu/3} = \frac{f^\mu}{n^3} \left( q_{0,1}^+ + 2q_{0,2}^+ - q_{1,1}^+ + (n-2)q_{1,0}^- + \left( \frac{n}{3} \right) - \left( \frac{n}{2} \right) \right)$$

(by (13))

$$= \frac{f^\mu}{n^3} \left( q_{1,0}^+ + 2q_{2,0}^+ - q_{1,1}^+ + (n-2)q_{1,0}^- + \left( \frac{n}{3} \right) - \left( \frac{n}{2} \right) \right)$$

$$= \frac{f^\mu}{n^3} \left( \sum_{i=1}^{k} \left( \frac{\mu_i}{2} \right) + \sum_{j=1}^{k} \left( \frac{\mu'_j}{2} \right) + 2 \left( \frac{\sum_{i=1}^{k} \mu_i}{3} + \sum_{j=1}^{k} \frac{\mu'_j}{3} \right) \right)$$

$$+ \frac{f^\mu}{n^3} \left( \sum_{i=1}^{k} \left( \frac{\mu_i}{2} \right) (i-1) + \sum_{j=1}^{k} \frac{\mu'_j}{2} (j-1) \right)$$

$$+ \frac{f^\mu}{n^3} \left( (n-2) \left( \sum_{i=1}^{k} \frac{\mu_i}{2} - \sum_{j=1}^{k} \frac{\mu'_j}{2} \right) + \left( \frac{n}{3} \right) - \left( \frac{n}{2} \right) \right),$$

respectively.

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