ERGODICITY CRITERIA FOR NON-EXPANDING
TRANSFORMATIONS OF 2-ADIC SPHERES

VLADIMIR ANASHIN, ANDREI KHRENNIKOV, AND EKATERINA YUROVA

Abstract. In the paper, we obtain necessary and sufficient conditions for ergodicity (with respect to the normalized Haar measure) of discrete dynamical systems \( (f, S_{2^{-r}}(a)) \) on 2-adic spheres \( S_{2^{-r}}(a) \) of radius \( 2^{-r} \), \( r \geq 1 \), centered at some point \( a \) from the ultrametric space of 2-adic integers \( \mathbb{Z}_2 \). The map \( f : \mathbb{Z}_2 \to \mathbb{Z}_2 \) is assumed to be non-expanding and measure-preserving; that is, \( f \) satisfies a Lipschitz condition with a constant 1 with respect to the 2-adic metric, and \( f \) preserves a natural probability measure on \( \mathbb{Z}_2 \), the Haar measure \( \mu_2 \) on \( \mathbb{Z}_2 \) which is normalized so that \( \mu_2(\mathbb{Z}_2) = 1 \).

1. Introduction

Algebraic and arithmetic dynamics are actively developed fields of general theory of dynamical systems, see [38] for extended bibliography, also monographs [35, 3]. Theory of dynamical systems in \( p \)-adic fields \( \mathbb{Q}_p \), where \( p \geq 2 \) is a prime number, is an important part of algebraic and arithmetic dynamics, see, e.g., [1, 2, 36, 37]; also [38, 24], and [3] for further references. As in general theory of dynamical systems, problems of measure preservation and ergodicity play fundamental role in the theory of \( p \)-adic dynamical systems, see e.g. [3, 6, 10, 11, 22, 23, 24, 17, 12, 13, 14, 15, 16, 20, 21, 30, 31, 32, 27, 33].

The case of non-expanding dynamics (the ones that satisfy a Lipschitz condition with a constant 1, a 1-Lipschitz for short) on the ring \( \mathbb{Z}_p \) of \( p \)-adic integers is sufficiently well studied [4, 5, 18, 19], see also [3] and references therein. However, it is not so much known about the dynamics in domains different from \( \mathbb{Z}_p \) although the later dynamics can be useful in applications to computer science (e.g. in computer simulations, numerical methods like Monte-Carlo, cryptography) and to mathematical physics, see [3], [7] and [24]. Dynamical systems on \( p \)-adic spheres are an interesting and nontrivial example of the dynamics. The first result in this direction, namely, the ergodicity criterion for monomial dynamical systems on \( p \)-adic spheres, was obtained in [22, 23]. It deserves a note that although these dynamical systems are a \( p \)-adic counterpart of a classical dynamical systems, circle rotations, in the \( p \)-adic case the dynamics exhibit quite another behavior than the classical one. Later the case of monomial dynamical systems on \( p \)-adic spheres was significantly extended: In [6], ergodicity criteria for locally

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analytic dynamical systems on $p$-adic spheres were obtained, for arbitrary prime $p$.

In the current paper, we consider an essentially wider class of dynamics than in [6], namely, the class of all 1-Lipschitz dynamical systems; however, on 2-adic spheres only: We find necessary and sufficient conditions for ergodicity of these dynamical systems, see further Theorem 3.1. Then with the use of the criterion we find necessary and sufficient conditions for ergodicity of perturbed monomial dynamical systems on 2-adic spheres around 1 in the case when perturbations are 1-Lipschitz and 2-adically small (Theorem 4.1). Earlier similar results were known only under additional restriction (however, for arbitrary prime $p$) that perturbations are smooth, cf. [6] and [3, Section 4.7]. In this connection it should be noticed that transition of results of the present paper for arbitrary prime $p$ seems to be a non-trivial task: It is well known that in $p$-adic analysis cases of even and odd primes differ essentially.

Our basic technique is van der Put series. The van der Put series were primarily known only as a tool to find antiderivatives (see [29, 28, 34]); recently by using the series the authors in [8, 9, 39] developed a new technique to determine whether a 1-Lipschitz transformation is measure-preserving and/or ergodic on $\mathbb{Z}_p$. Our approach seems to be fruitful: The analog of the techniques was successfully applied to determine ergodicity of 1-Lipschitz transformations on another complete non-Archimedean ring, the ring $\mathbb{F}_2[[X]]$ of formal power series over a two-element field, see recent paper [26].

We remark that as the mappings under consideration are in general not differentiable, it is impossible to apply the technique based on expansion into power series to the case under consideration as the said technique can be used for analytical and smooth dynamical systems only, see e.g. [6]. The van der Put basis is much better adapted to studies of non-smooth dynamics: A special collection of step-like functions, characteristic functions of balls, constitutes the basis. The van der Put basis reflects the ultrameric (non-Archimedean) structure of $p$-adic numbers, [28, 34]. We note that in the $p$-adic case the linear space consisting of linear combinations of step-like functions is a dense subspace of the space of continuous functions, [34].

The 2-adic spheres are a special case of $p$-adic spheres; as a matter of fact, 2-adic spheres are 2-adic balls: Denote the functions is a dense subspace of the space of continuous functions, $\mathbb{Z}_p$.

Indeed, if $f$ is a 1-Lipschitz transformation such that $f(a + p^k\mathbb{Z}_p) \subset a + p^k\mathbb{Z}_p$, then necessarily $f(a) = a + p^k y$ for a suitable $y \in \mathbb{Z}_p$. Thus, $f(a + p^k z) = f(a) + p^k \cdot u(z)$ for any $z \in \mathbb{Z}_p$; so we can relate to $f$ the following 1-Lipschitz transformation on $\mathbb{Z}_p$:

$$u: z \mapsto u(z) = \frac{1}{p^k}(f(a + p^k z) - a - p^k y); \quad z \in \mathbb{Z}_p.$$
It can be shown that the transformation \( f \) is ergodic on the ball \( B_{p^{-r}}(a) \) if and only if the transformation \( u \) is ergodic on \( \mathbb{Z}_p \). To determine ergodicity of a transformation on the space \( \mathbb{Z}_p \) various techniques may be used, see [3] for details. In the paper, we exploit a version of the idea described above to reduce the case of ergodicity on 2-adic spheres to the case of ergodicity on the whole space \( \mathbb{Z}_2 \) (cf. further Proposition 3), and we use van der Put series for the latter study since the series turned out to be the most effective technique in the case when \( p = 2 \), cf. [8, 9, 26, 39].

2. PRELIMINARIES

We remind that \( p \)-adic absolute value satisfies strong triangle inequality:

\[ |x + y|_p \leq \max\{|x|_p, |y|_p\}. \]

The \( p \)-adic absolute value induces \((p \text{-adic})\) metric on \( \mathbb{Z}_p \) in a standard way: given \( a, b \in \mathbb{Z}_p \), the \( p \)-adic distance between \( a \) and \( b \) is \( |a - b|_p \). Absolute values (and also metrics induces by these absolute values) that satisfy strong triangle inequality are called non-Archimedean. Although the strong triangle inequality is the only difference of the \( p \)-adic metric from real or complex metrics it results in dramatic differences in behaviour of \( p \)-adic dynamical systems compared to that of real or complex counterparts.

The space \( \mathbb{Z}_p \) is equipped with a natural probability measure, namely, the Haar measure \( \mu_p \) normalized so that \( \mu_p(\mathbb{Z}_p) = 1 \): Balls \( B_{p^{-r}}(a) \) of non-zero radii constitute the base of the corresponding \( \sigma \)-algebra of measurable subsets, \( \mu_p(B_{p^{-r}}(a)) = p^{-r} \). The measure \( \mu_p \) is a regular Borel measure, so all continuous transformations \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) are measurable with respect to \( \mu_p \). As usual, a measurable mapping \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) is called measure-preserving if \( \mu_p(f^{-1}(S)) = \mu(S) \) for each measurable subset \( S \subset \mathbb{Z}_p \). A measure-preserving mapping \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) is called ergodic if \( f^{-1}(S) = S \) implies either \( \mu_p(S) = 0 \) or \( \mu_p(S) = 1 \) (in the paper, speaking of ergodic mapping we mean that the mappings are also measure-preserving).

Let a transformation \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) be non-expanding with respect to the \( p \)-adic metric; that is, let \( f \) be a \( 1 \)-Lipschitz with respect to the \( p \)-adic metric:

\[ |f(x) - f(y)|_p \leq |x - y|_p \]

for all \( x, y \in \mathbb{Z}_p \). The \( 1 \)-Lipschitz property may be re-stated in terms of congruences rather than in term of inequalities, in the following way.

Given \( a, b \in \mathbb{Z}_p \) and \( k \in \mathbb{N} = \{1, 2, 3, \ldots, \} \), the congruence \( a \equiv b \) (mod \( p^k \)) is well defined: the congruence just means that images of \( a \) of \( b \) under action of the ring epimorphism mod\( p^k \): \( \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z} \) of the ring \( \mathbb{Z}_p \) onto the residue ring \( \mathbb{Z}/p^k\mathbb{Z} \) modulo \( p^k \) coincide. Remind that by the definition the epimorphism mod\( p^k \) sends a \( p \)-adic integer that has a canonical representation \( \sum_{i=0}^{\infty} \alpha_i p^i, \alpha_i \in \{0, 1, \ldots, p - 1\}, i = 0, 1, 2, \ldots, , \) to \( \sum_{i=0}^{k-1} \alpha_i p^i \in \mathbb{Z}/p^k\mathbb{Z} \). Note also that we treat if necessary elements from \( \mathbb{Z}/p^k\mathbb{Z} \) as numbers from \( \{0, 1, \ldots, p^k - 1\} \).

Now it is obvious that the congruence \( a \equiv b \) (mod \( p^k \)) is equivalent to the inequality \( |a - b|_p \leq p^{-k} \). Therefore the transformation \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) is 1-Lipschitz if and only if it is compatible; that is,

\[ f(a) \equiv f(b) \pmod{p^k} \text{ once } a \equiv b \pmod{p^k}. \]
The compatibility property implies that given a 1-Lipschitz transformation \( f : \mathbb{Z}_p \to \mathbb{Z}_p \), the reduced mapping modulo \( p^k \)

\[
f \mod p^k : z \mod p^k \mapsto f(z) \mod p^k
\]
is a well defined mapping \( f \mod p^k \) of residue ring \( \mathbb{Z}/p^k\mathbb{Z} \) into itself: The mapping \( f \mod p^k \) does not depend on the choice of representative \( z \) in the ball \( \mathbb{Z} + p^k\mathbb{Z} \) (the latter ball is a coset with respect to the epimorphism \( \mod p^k \)); that is, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}_p & \xrightarrow{f} & \mathbb{Z}_p \\
\mod p^k & & \mod p^k \\
\mathbb{Z}/p^k\mathbb{Z} & \xrightarrow{f \mod p^k} & \mathbb{Z}/p^k\mathbb{Z}
\end{array}
\]

A 1-Lipschitz transformation \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) is called bijective modulo \( p^k \) if the reduced mapping \( f \mod p^k \) is a permutation on \( \mathbb{Z}/p^k\mathbb{Z} \); and \( f \) is called transitive modulo \( p^k \) if \( f \mod p^k \) is a permutation that is cycle of length \( p^k \).

Main ergodic theorem for 1-Lipschitz transformations on \( \mathbb{Z}_p \) \([3, \text{Theorem } 4.23]\) yields:

**Theorem 2.1 (Main ergodic theorem).** A 1-Lipschitz transformation \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) is measure-preserving if and only if it is bijective modulo \( p^k \) for all \( k = 1, 2, 3, \ldots \); and \( f \) is ergodic if and only if \( f \) is transitive modulo \( p^k \) for all \( k = 1, 2, 3, \ldots \).

Now we remind definition and basic properties of van der Put series following \([28]\). Given a continuous \( p \)-adic function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) defined on \( \mathbb{Z}_p \) and valuated in \( \mathbb{Z}_p \), there exists a unique sequence \( B_0, B_1, B_2, \ldots \) of \( p \)-adic integers such that

\[
f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x)
\]

for all \( x \in \mathbb{Z}_p \), where

\[
\chi(m, x) = \begin{cases} 
1, & \text{if } |x - m|_p \leq p^{-n} \\
0, & \text{otherwise}
\end{cases}
\]

and \( n = 1 \) if \( m = 0 \); \( n \) is uniquely defined by the inequality \( p^{n-1} \leq m \leq p^n - 1 \) otherwise. The right side series in (2.2) is called the van der Put series of the function \( f \). Note that the sequence \( B_0, B_1, \ldots, B_m, \ldots \) of van der Put coefficients of the function \( f \) tends \( p \)-adically to 0 as \( m \to \infty \), and the series converges uniformly on \( \mathbb{Z}_p \). Vice versa, if a sequence \( B_0, B_1, \ldots, B_m, \ldots \) of \( p \)-adic integers tends \( p \)-adically to 0 as \( m \to \infty \), then the series in the right-hand side of (2.2) converges uniformly on \( \mathbb{Z}_p \) and thus defines a continuous function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \).

The number \( n \) in the definition of \( \chi(m, x) \) has a very natural meaning: As

\[
|\log_p m| = (\text{the number of digits in a base-} p \text{ expansion for } m) - 1,
\]
therefore \( n = \lfloor \log_p m \rfloor + 1 \) for all \( m \in \mathbb{N}_0 \); we put \( \lfloor \log_0 \rfloor = 0 \) by this reason. Recall that \( \lfloor \alpha \rfloor \) for a real \( \alpha \) denotes the integral part of \( \alpha \), that is, the nearest to \( \alpha \) rational integer which does not exceed \( \alpha \).

Coefficients \( B_m \) are related to values of the function \( f \) in the following way: Let \( m = m_0 + \ldots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1} \) be a base-\( p \) expansion for \( m \), i.e., \( m_j \in \{0, \ldots, p-1\} \), \( j = 0, 1, \ldots, n-1 \) and \( m_{n-1} \neq 0 \), then

\[
B_m = \begin{cases}
    f(m) - f(m - m_{n-1}p^{n-1}), & \text{if } m \geq p; \\
    f(m), & \text{if otherwise}.
\end{cases}
\]

(2.4)

It is worth noticing also that \( \chi(m, x) \) is merely a characteristic function of the ball \( B_{p^{-\lfloor \log_p m \rfloor}}(m) = m + p^{\lfloor \log_p m \rfloor} - 1 \mathbb{Z}_p \) of radius \( p^{-\lfloor \log_p m \rfloor} - 1 \) centered at \( m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \):

(2.5) \[
\chi(m, x) = \begin{cases}
    1, & \text{if } x \equiv m \pmod{p^{\lfloor \log_p m \rfloor} + 1}; \\
    0, & \text{if otherwise}
\end{cases}
\]

\[
\begin{cases}
    1, & \text{if } x \in B_{p^{-\lfloor \log_p m \rfloor}}(m); \\
    0, & \text{if otherwise}
\end{cases}
\]

The following theorem that characterizes 1-Lipschitz functions in terms of van der Put basis was proved in [8]:

**Theorem 2.2.** Let a function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) be represented via van der Put series (2.2); then \( f \) is compatible (that is, satisfies the \( p \)-adic Lipschitz condition with a constant 1) if and only if \( |B_m|_p \leq p^{-\lfloor \log_p m \rfloor} \) for all \( m = 0, 1, 2, \ldots \).

In other words, \( f \) is compatible if and only if it can be represented as

(2.6) \[
f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x),
\]

for suitable \( b_m \in \mathbb{Z}_p; \ m = 0, 1, 2, \ldots \).

To study ergodicity on \( p \)-adic spheres, the following lemma is useful (further \( f^t \) stands for the \( t \)-th iterate of \( f \)):

**Lemma 1** ([3, Lemma 4.76]). A 1-Lipschitz transformation \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) is ergodic on the sphere \( S_{p^{-r}}(y) \) if and only if the following two conditions hold simultaneously:

1. The mapping \( z \mapsto f(z) \mod {p^{r+1}} \) is transitive on the set \( S_{p^{-r}}(y) \mod {p^{r+1}} = \{y + p^r s : s = 1, 2, \ldots, p - 1\} \subset \mathbb{Z}/p^{r+1}\mathbb{Z}; \)

2. The mapping \( z \mapsto f^{p^{-1}}(z) \mod {p^{r+1}} \) is transitive on the set \( B_{p^{-r+1}}(y + p^r s) \mod {p^{r+1}} = \{y + p^r s + p^{r+1} S : S = 0, 1, 2, \ldots, p^t - 1\} \),

for all \( t = 1, 2, \ldots \) and for some (equivalently, for all) \( s \in \{1, 2, \ldots, p - 1\} \).

Condition 2 holds if and only if \( f^{p^{-1}} \) is an ergodic transformation on the ball \( B_{p^{-r+1}}(y + p^r s) = y + p^r s + p^{r+1}\mathbb{Z}_p \) of radius \( p^{-r+1} \) centered at \( y + p^r s \), for some (equivalently, for all) \( s \in \{1, 2, \ldots, p - 1\} \).
The ergodic 1-Lipschitz transformations of $\mathbb{Z}_2$ are completely characterized by the following theorem, see [8]:

**Theorem 2.3.** A 1-Lipschitz transformation $f: \mathbb{Z}_2 \to \mathbb{Z}_2$ is ergodic if and only if it can be represented as

$$f(x) = b_0 \chi(0, x) + b_1 \chi(1, x) + \sum_{m=2}^{\infty} 2^{[\log_2 m]} b_m \chi(m, x)$$

for suitable $b_m \in \mathbb{Z}_2$ that satisfy the following conditions:

1. $b_0 \equiv 1 \pmod{2}$;
2. $b_0 + b_1 \equiv 3 \pmod{4}$;
3. $|b_m|_2 = 1$, $m \geq 2$;
4. $b_0 + b_3 \equiv 2 \pmod{4}$;
5. $\sum_{m=2^{n-1}}^{2^n-1} b_m \equiv 0 \pmod{4}$, $n \geq 3$.

**Remark 2.** It is an elementary exercise to show that condition 2 in the statement of Theorem 2.3 can be replaced by the condition $b_1 - b_0 \equiv 1 \pmod{4}$.

3. Ergodicity of 1-Lipschitz dynamical systems on 2-adic spheres

In this section we prove ergodicity criterion for 1-Lipschitz dynamics on 2-adic spheres, so further $p = 2$ and $f: \mathbb{Z}_2 \to \mathbb{Z}_2$ is a 1-Lipschitz function. Let $S_{2^{-r}}(a)$ be a sphere of radius $2^{-r}$ with a center at the point $a \in \{0, \ldots, 2^r - 1\}$, and let the sphere $S_{2^{-r}}(a)$ be invariant under action of $f$; that is, $f(S_{2^{-r}}(a)) \subset S_{2^{-r}}(a)$. As $p = 2$, the sphere $S_{2^{-r}}(a)$ coincides with the ball $B_{2^{1-r}}(a + 2^r)$ of radius $2^{1-r}$ centered at the point $a + 2^r$: $S_{2^{-r}}(a) = \{a + 2^r + 2^r x : x \in \mathbb{Z}_2\} = B_{2^{1-r}}(a + 2^r)$. Therefore the sphere $S_{2^{-r}}(a)$ is $f$-invariant if and only if $f(a + 2^r + 2^r + 1 \mathbb{Z}_p) \subset a + 2^r + 2^r + 1 \mathbb{Z}_p$; that is, if and only if

$$(3.1) \quad f(a + 2^r) \equiv a + 2^r \pmod{2^r + 1}$$

as a 1-Lipschitz function maps a ball of radius $2^{-\ell}$ into a ball of radius $2^{-\ell}$.

Further, as $f$ is 1-Lipschitz, we can represent $f: \mathbb{Z}_2 \to \mathbb{Z}_2$ as

$$(3.2) \quad f(a + 2^r + 2^r x) = f(a + 2^r) + 2^r g(x);$$

then $g: \mathbb{Z}_2 \to \mathbb{Z}_2$ is a 1-Lipschitz function. The following proposition holds:

**Proposition 3.** The function $f$ is ergodic on the sphere $S_{2^{-r}}(a)$ if and only if $f(a + 2^r) \equiv a + 2^r \pmod{2^r + 1}$ and the function $g$ defined as

$$(3.3) \quad g(x) = \frac{f(a + 2^r + 2^r x) - (a + 2^r)}{2^r + 1}$$

is a 1-Lipschitz ergodic transformation on $\mathbb{Z}_2$.

**Proof.** The first of conditions of the proposition just means that the sphere $S_{2^{-r}}(a)$ is $f$-invariant, see (3.1). In view of that condition, Condition 2 of Lemma 1 holds then if and only if the function $g$ is transitive modulo $2^r$ for all $t = 1, 2, 3, \ldots$. However, the latter condition is equivalent to the ergodicity of the function $g$ on $\mathbb{Z}_2$ by Theorem 2.1. $\square$
Now given a 1-Lipschitz function \( f: \mathbb{Z}_2 \to \mathbb{Z}_2 \), in view of Theorem 2.2 and (2.6), \( f \) has a unique representation via van der Put series:

\[
(3.4) \quad f(x) = \sum_{m=0}^{\infty} B_f(m) \chi(m, x) = \sum_{m=0}^{\infty} 2^{[\log_2 m]} b_f(m) \chi(m, x),
\]

where \( b_f(m) \in \mathbb{Z}_2; \) so \( B_f(m) = 2^{[\log_2 m]} b_f(m) \) for all \( m = 0, 1, 2, \ldots \).

**Theorem 3.1.** The function \( f \) represented by van der Put series (3.4) is ergodic on the sphere \( S_{2^r}(a) \) if and only if the following conditions hold simultaneously:

1. \( f(a + 2^r) \equiv a + 2^r + 2^{r+1} \pmod{2^{r+2}}; \)
2. \( b_f(a + 2^r + m \cdot 2^{r+1}) \equiv 1, \) for \( m \geq 1; \)
3. \( b_f(a + 2^r + 2^{r+1}) \equiv 1 \pmod{4}; \)
4. \( b_f(a + 2^r + 2^{r+2}) + b_f(a + 2^r + 3 \cdot 2^{r+1}) \equiv 2 \pmod{4}; \)
5. \( \sum_{m=2^{n-1}}^{2^n-1} b_f(a + 2^r + m \cdot 2^{r+1}) \equiv 0 \pmod{4}, \) for \( n \geq 3. \)

**Proof.** Represent \( f \) by (3.2) and \( g \) by (3.3), then calculate van der Put coefficients \( B_m \) of the function \( g(x) = \sum_{m=0}^{\infty} B_m \chi(m, x) \) as follows. Given \( m \in \{2, 3, \ldots\} \), denote \( \hat{m} = m - 2^{[\log_2 m]} \); then using formula (2.4) for van der Put coefficients \( B_m \) and \( B_f(m) \) we find that

\[
(3.5) \quad B_0 = g(0) = \frac{f(a + 2^r) - (a + 2^r)}{2^{r+1}};
\]

\[
(3.6) \quad B_1 = g(1) = \frac{f(a + 2^r + 2^{r+1}) - (a + 2^r)}{2^{r+1}};
\]

\[
B_m = B_{\hat{m} + 2^n} = g(\hat{m} + 2^n) - g(\hat{m}) = \frac{f(a + 2^r + 2^{r+1} \hat{m} + 2^n + r + 1) - (a + 2^r)}{2^{r+1}} - \frac{f(a + 2^r + 2^{r+1} \hat{m}) - (a + 2^r)}{2^{r+1}} = \frac{f(a + 2^r + 2^{r+1} \hat{m} + 2^n + r + 1) - f(a + 2^r + 2^{r+1} \hat{m})}{2^{r+1}}, \quad (m \geq 2, n = [\log_2 m]).
\]

As \( g \) is 1-Lipschitz, Theorem 2.2 yields that for every \( k = 0, 1, 2, \ldots \) there exists \( b_k \in \mathbb{Z}_2 \) such that \( B_k = b_k 2^{[\log_2 k]} \). So from (3.7) it follows that if \( m \geq 2 \) and \( n = [\log_2 m] \) then

\[
(3.8) \quad b_m = \frac{B_m}{2^n} = \frac{B_f(a + 2^r + 2^{r+1} \hat{m} + 2^n + r + 1)}{2^{n+r+1}} = b_f(a + 2^r + m \cdot 2^{r+1}).
\]

Now apply Theorem 2.3. In view of (3.8), condition 3 (for \( m \geq 2 \)) of Theorem 2.3 is equivalent to condition 2 of the theorem under proof. By the same reason, conditions 4 and 5 of Theorem 2.3 accordingly are equivalent to conditions 4 and 5 of the theorem under proof. Combining (3.3) with (2.4), we see that condition 1 of Theorem 2.3 is equivalent to condition 1 of the theorem under proof.
Finally, combining (3.5) and (3.6) with (2.4) we see that

\[(3.9) \quad b_1 - b_0 = g(1) - g(0) = \frac{f(a + 2^r + 2^{r+1}) - f(a + 2^r)}{2^{r+1}} = \\
\frac{B_f(a + 2^r + 2^{r+1})}{2^{r+1}} = b_f(a + 2^r + 2^{r+1})\]

By Remark 2 to Theorem 2.3, (3.9) proves condition 3 (as well as condition 1 for \(m = 1\)) of the theorem under proof. \(\square\)

4. Ergodicity of Perturbed Monomial Systems on 2-adic Spheres Around 1

In this section, we study ergodicity of a function of the form \(f(x) = x^s + 2^{r+1}u(x)\) on the sphere \(S_{2-r}(1) = \{1 + 2^r + 2^{r+1}x : x \in \mathbb{Z}_2\}\), \(r \geq 1\), \(s \in \mathbb{N}\). The perturbation function \(u\) is assumed to be 1-Lipschitz.

**Theorem 4.1.** Let \(u : \mathbb{Z}_2 \to \mathbb{Z}_2\) be an arbitrary 1-Lipschitz function, let \(s, r \in \mathbb{N}\). The function \(f(x) = x^s + 2^{r+1}u(x)\) is ergodic on the sphere \(S_{2-r}(1) = \{1 + 2^r + 2^{r+1}x : x \in \mathbb{Z}_2\}\) if and only if \(s \equiv 1\) (mod 4) and \(u(1) \equiv 1\) (mod 2).

**Proof.** As \(u\) is 1-Lipschitz, \(u(1 + 2^r + 2^{r+1}) = u(1 + 2^r) + 2^{r+1}\xi(x)\) for a suitable map \(\xi : \mathbb{Z}_2 \to \mathbb{Z}_2\); it is an exercise to prove that \(\xi\) is also 1-Lipschitz. Combining (3.3), (3.2) and Newton’s binomial we get

\[(4.1) \quad g(x) = \frac{f(1 + 2^r + 2^{r+1}) - (1 + 2^r)}{2^{r+1}} = \\
u(1 + 2^r + 2^{r+1}) + \frac{1}{2^{r+1}}((1 + 2^r + 2^{r+1}x)^s - (1 + 2^r)) = \\
2^{r+1}\xi(x) + u(1 + 2^r) + xs + 2^{r-1}(1 + 2x)^2\left(\frac{s}{2}\right) + 2^{2r-1}(1 + 2x)^3\left(\frac{s}{3}\right) + \cdots\]

By Proposition 3 the function \(f\) is ergodic on the sphere \(S_{2-r}(1)\) if and only the function \(g\) is ergodic on \(\mathbb{Z}_p\). As \(r \geq 1\) and \(\xi\) is 1-Lipschitz, the right-hand side of (4.1) is ergodic on \(\mathbb{Z}_p\) if and only if the polynomial

\[v(x) = u(1 + 2^r) + xs + 2^{r-1}(1 + 2x)^2\left(\frac{s}{2}\right) + 2^{2r-1}(1 + 2x)^3\left(\frac{s}{3}\right) + \cdots\]

in variable \(x\) is ergodic on \(\mathbb{Z}_p\): This follows from [3, Proposition 9.29] where it is shown in particular that given 1-Lipschitz functions \(t, w : \mathbb{Z}_2 \to \mathbb{Z}_2\), the function \(t + 4w\) is ergodic on \(\mathbb{Z}_2\) if and only if the function \(t\) is ergodic on \(\mathbb{Z}_2\). Nonetheless, we would better deduce the same claim from Theorem 3.1 to obtain a formula that will be used later in the current proof.

We first note that conditions 2-5 of Theorem 3.1 are all “modulo 4”, meaning the conditions do not depend on terms of order greater than 1 of canonic 2-adic representations of van der Put coefficients of the function. Then, condition 1 of Theorem 3.1 holds if and only if the following congruence holds:

\[f(1 + 2^r) = (1 + 2^r)^s + 2^{r+1}u(1 + 2^r) \equiv 1 + 2^r + 2^{r+1} \pmod{2^{r+2}}.\]
As $u$ is 1-Lipschitz, the latter congruence is equivalent to the congruence 
\[1 + 2^r s + 2^{2r} \binom{s}{2} + 2^{r+1}u(1) \equiv 1 + 2^r + 2^r+1 \mod 2^{r+2}\] 
which by Newton’s binomial is equivalent to the congruence
\[(4.2) \quad s + 2^r \binom{s}{2} + 2u(1) \equiv 3 \mod 4\]
and therefore $s \equiv 1 \mod 2$: $s = 2\tilde{s} + 1$ for suitable $\tilde{s} \in \mathbb{N}_0$. So we conclude that the summand $2^{r+1}\xi(x)$ has no effect on ergodicity of the function $g$ on $\mathbb{Z}_2$: The latter function is ergodic if and only if the polynomial $v$ is ergodic on $\mathbb{Z}_2$.

Further, according to [25] (see also [3, Corollary 4.71]) a polynomial over $\mathbb{Z}_2$ is ergodic on $\mathbb{Z}_2$ if and only if it is transitive modulo 8. This already proves the theorem if $r \geq 3$ since in the latter case $v(x) \equiv u(1 + 2^r) + xs \mod 8$; however, in force of a well know criterion of ergodicity of affine maps (see e.g. [3, Theorem 4.36]), the affine map $x \mapsto u(1 + 2^r) + xs$ is ergodic on $\mathbb{Z}_2$ if and only if $u(1 + 2^r) \equiv 1 \mod 2$ (so $u(1) \equiv 1 \mod 2$ as $u$ is 1-Lipschitz) and $s \equiv 1 \mod 4$.

To complete the proof of the theorem, only cases $r = 1$ and $r = 2$ are to be considered. If $r = 2$ then $v(x) \equiv u(1 + 2^2) + xs + 2\binom{s}{2} + 4\binom{s}{4}x \mod 8$ and we conclude the proof as in the case $r \geq 2$. In the remaining case $r = 1$ we have that 
\[v(x) \equiv u(1 + 2) + xs + \binom{s}{2} + 2\binom{s}{2}x + 4\binom{s}{2}x^2 + 2\binom{s}{3} + 6\binom{s}{3}x + 4\binom{s}{4} \mod 8.\]

However, if $r = 1$ then congruence (4.2) implies the congruence $1 + 2\tilde{s} + 2(2\tilde{s} + 1)\tilde{s} + 2u(1) \equiv 3 \mod 4$ (remind that $s = 2\tilde{s} + 1$) which is equivalent to the congruence $u(1) \equiv 1 \mod 2$. Yet the latter congruence implies that necessarily $\tilde{s} \equiv 0 \mod 2$: If otherwise, then from (4.3) it follows that $v(x) \equiv u(1) + xs + 1 \equiv x \mod 2$ which means that $v(x)$ is not transitive modulo 2 and thus the polynomial $v(x)$ can not be ergodic on $\mathbb{Z}_2$ (cf. Theorem 2.1). On the other hand, if $u(1) \equiv 1 \mod 2$ and $s \equiv 1 \mod 4$ (i.e., $s = 4\tilde{s} + 1$ for some $\tilde{s} \in \mathbb{N}_0$) then the polynomial in the right-hand side of (4.3) is congruent modulo 8 to the polynomial $u(3) + x + 6\tilde{s}$ which induces an ergodic affine transformation on $\mathbb{Z}_2$ (cf. [3, Theorem 4.36]) since $u(3) \equiv u(1) \equiv 1 \mod 2$ (recall that $u$ is 1-Lipschitz). Thus, the polynomial $v(x)$ (and whence the function $g$) are also ergodic on $\mathbb{Z}_2$. This completes the proof. □

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References

[1] S. Albeverio, A. Khrennikov, and P. E. Kloeden. Memory retrieval as a $p$-adic dynamical system. BioSystems. 49 (1999), 105–115.

[2] S. Albeverio, A. Khrennikov, B. Tirozzi, and D. De Smedt. p-adic dynamical systems. Theor. Math. Phys. 114 (1998), 276–287.
[3] V. Anashin and A. Khrennikov. *Applied Algebraic Dynamics* (de Gruyter Expositions in Mathematics, vol. 49). Walter de Gruyter, Berlin–New York, 2009.

[4] V. Anashin. Uniformly distributed sequences of p-adic integers. *Math. Notes* **55** (1994), 109–133.

[5] V. Anashin. Uniformly distributed sequences of p-adic integers. *Discrete Math. Appl.* **12**(6) (2009), 527–590.

[6] V. Anashin. Ergodic transformations in the space of p-adic integers. *p-adic Mathematical Physics. 2nd Int'l Conference* (Belgrade, Serbia and Montenegro 15–21 September 2005) (AIP Conference Proceedings, 826), American Institute of Physics, Melville, New York, 2006, pp. 3–24.

[7] V. Anashin. Non-Archimedean theory of T-functions. *Proc. Advanced Study Institute Boolean Functions in Cryptology and Information Security* (NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., 18), IOS Press, Amsterdam, 2008, pp. 33–57.

[8] V. S. Anashin, A. Yu. Khrennikov and E. I. Yurova. Characterization of ergodicity of p-adic dynamical systems by using the van der Put basis. *Doklady Akademii Nauk*, **438**(2) (2011), 151–153.

[9] V. Anashin, A. Khrennikov and E. Yurova. Using van der Put basis to determine if a 2-adic function is measure-preserving or ergodic w.r.t. Haar measure. *Advances in Non-Archimedean Analysis* (Contemporary Mathematics, 551), American Mathematical Society, Providence, RI, 2011, pp. 35–38.

[10] D. K. Arrowsmith and F. Vivaldi. Some p-adic representations of the Smale horseshoe. *Phys. Lett. A*, **176** (1993), 292–294.

[11] D. K. Arrowsmith and F. Vivaldi. Geometry of p-adic Siegel discs. *Physica D*, **71** (1994), 222–236.

[12] R. Benedetto. p-adic dynamics and Sullivan’s no wandering domain theorem. *Compos. Math.* **122** (2000), 281–298.

[13] R. Benedetto. Hyperbolic maps in p-adic dynamics. *Ergod. Theory and Dyn. Sys.* **21** (2001), 1–11.

[14] R. Benedetto. Components and periodic points in non-Archimedean dynamics. *Proc. London Math. Soc.*, **84** (2002), 231–256.

[15] R. Benedetto. Heights and preperiodic points of polynomials over function fields. *Int. Math. Res. Notices*, **62** (2005), 3855–3866.

[16] J.-L. Chabert, A.-H. Fan, Y. Fares. Minimal dynamical systems on a discrete valuation domain. *Discrete and Continuous Dynamical Systems - Series A*, **25** (2009), 777–795.

[17] Z. Coelho and W. Parry. Ergodicity of p-adic multiplication and the distribution of Fibonacci numbers. *Topology, ergodic theory, real algebraic geometry* (Amer. Math. Soc. Transl. Ser., 202) American Mathematical Society, Providence, RI, 2001, pp. 51–70.

[18] A.-H. Fan, M.-T. Li, J.-Y. Yao, and D. Zhou. p-adic affine dynamical systems and applications. *C. R. Acad. Sci. Paris Ser. I*, **342** (2006), 129–134.

[19] A.-H. Fan, M.-T. Li, J.-Y. Yao, and D. Zhou. Strict ergodicity of affine p-adic dynamical systems. *Adv. Math.* **214** (2007), 666–700.

[20] A.-H. Fan, L. Liao, Y.-F. Wang, and D. Zhou. p-adic repellors in $\mathbb{Q}_p$ are subshifts of finite type. *C. R. Math. Acad. Sci. Paris*, **344** (2007), 219–224.

[21] C. Favre and J. Rivera-Letelier. Théorème de Ruelle-distribution de Brolin en dynamique p-adique. *C. R. Math. Acad. Sci. Paris*, **339** (2004), 271–276.

[22] M. Gundlach, A. Khrennikov, K.-O. Lindahl. On ergodic behaviour of p-adic dynamical systems. *Infinite Dimensional Analysis, Quantum Prob. and Related Fields*, **4** (2001), 569–577.

[23] M. Gundlach, A. Khrennikov, K.-O. Lindahl. Topological transitivity for p-adic dynamical systems. *p-adic functional analysis* (Lecture notes in pure and applied mathematics, 222), Dekker, New York, 2001, pp. 127–132.

[24] A. Khrennikov and M. Nilsson. *p-adic deterministic and random dynamics*. Kluwer, Dordrecht, 2004.
[25] M. V. Larin, Transitive polynomial transformations of residue class rings, *Discrete Math. Appl.* **12** (2002), 141–154.

[26] D.-D. Lin, T. Shi, and Z.-F. Yang. Ergodic theory over $\mathbb{F}_2[[X]]$, *Finite Fields Appl.* **18** (2012), 473–491.

[27] K-O. Lindhal. On Siegels linearization theorem for fields of prime characteristic. *Nonlinearity*, **17** (2004), 745–763.

[28] K. Mahler. *p-adic numbers and their functions*, Cambridge Univ. Press, Cambridge, 1981.

[29] M. van der Put, *Algèbres de fonctions continues p-adiques*, Universiteit Utrecht, 1967.

[30] J. Rivera-Letelier. *Dynamique des fonctions rationnelles sur des corps locaux*. (PhD thesis), Orsay, 2000.

[31] J. Rivera-Letelier. *Dynamique des fonctions rationnelles sur des corps locaux*. *Astérisque*, **147** (2003), 147–230.

[32] J. Rivera-Letelier. *Espace hyperbolique p-adique et dynamique des fonctions rationnelles*. *Compos. Math.*, **138** (2003), 199–231.

[33] A. De Smedt and A. Khrennikov. A p-adic behaviour of dynamical systems. *Rev. Mat. Complut.*, **12** (1999), 301–323.

[34] W. H. Schikhof *Ultrametric calculus. An introduction to p-adic analysis*. Cambridge University Press, Cambridge, 1984.

[35] J. H. Silverman. *The arithmetic of dynamical systems (Graduate Texts in Mathematics, 241)*, Springer, New York, 2007.

[36] F. Vivaldi. The arithmetic of discretized rotations. *p-adic Mathematical Physics. 2nd Int’l Conference (Belgrade, Serbia and Montenegro 15–21 September 2005)* (AIP Conference Proceedings, 826), American Institute of Physics, Melville, New York, 2006, pp. 162–173.

[37] F. Vivaldi and I. Vladimirov. Pseudo-randomness of round-off errors in discretized linear maps on the plane. *Int. J. of Bifurcations and Chaos*, **13** (2003), 3373–3393.

[38] F. Vivaldi, Algebraic and arithmetic dynamics bibliographical database, [http://www.maths.qmul.ac.uk/~fv/database/algdyn.pdf](http://www.maths.qmul.ac.uk/~fv/database/algdyn.pdf)

[39] E. I. Yurova. Van der Put basis and p-adic dynamics *p-Adic Numbers, Ultrametric Analysis and Applications*, **2**(2)(2010) 175–178.