Principal Groupoid C*-Algebras
With Bounded Trace

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Abstract. Suppose G is a second countable, locally compact, Hausdorff, principal groupoid with a fixed left Haar system. We define a notion of integrability for groupoids and show G is integrable if and only if the groupoid C*-algebra C*(G) has bounded trace.

1. Introduction

Let H be a locally compact, Hausdorff group acting continuously on a locally compact, Hausdorff space X, so that (H, X) is a transformation group. A lovely theorem of Green says that if H acts freely on X, then the associated transformation-group C*-algebra C0(X) ⋊ H has continuous trace if and only if the action of H on X is proper [5, Theorem 17]. Muhly and Williams defined a notion of proper groupoid and proved that for principal groupoids G, the groupoid C*-algebra C*(G) has continuous trace if and only if the groupoid is proper [8, Theorem 2.3]. Of course, when G = H × X is the transformation-group groupoid, then G is proper if and only if the action of H on X is proper.

In [13] Rieffel introduced a notion of an integrable action of a group H on a C*-algebra A. This notion of integrability for A = C0(X) turned out to characterize when C0(X) ⋊ H, arising from a free action of H on X, has bounded trace [6, Theorem 4.8]. In this paper we define a notion of integrability for groupoids (see Definition 3.1) which, when G = H × X is the transformation-group groupoid, reduces to an integrable action of H on X (see Example 3.3). We then prove that for principal groupoids G, C*(G) has bounded trace if and only if the groupoid is integrable (see Theorem 4.4). This theorem is thus very much in the spirit of [8, Theorem 2.4], [4, Theorem 7.9], [4, Theorem 4.1] (see also [3, Corollary 5.9]) and [4, Theorem 5.3], which characterize when principal-groupoid C*-algebras are, respectively, continuous-trace, Fell, CCR and GCR C*-algebras. The key technical tools used to prove Theorem 4.4 are, first, a homeomorphism of the spectrum of C*(G) onto the orbit space [4, Proposition 5.1] and, second, a generalisation to groupoids of the notion of k-times convergence in the orbit space of a transformation group from [1].

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2. Preliminaries

Let $A$ be a $C^*$-algebra. An element $a$ of the positive cone $A^+$ of $A$ is called a bounded-trace element if the map $\pi \mapsto \text{tr}(\pi(a))$ is bounded on the spectrum $\hat{A}$ of $A$; the linear span of the bounded-trace elements is a two-sided $*$-ideal in $A$. We say $A$ has bounded trace if the ideal of (the span of) the bounded-trace elements is dense in $A$.

Throughout, $G$ is a locally compact, Hausdorff groupoid; in our main results $G$ is assumed to be second-countable and principal. We denote the unit space of $G$ by $G^0$, and the range and source maps $r, s : G \to G^0$ are $r(\gamma) = \gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, respectively. We let $\pi : G \to G^0 \times G^0$ be the map $\pi(\gamma) = (r(\gamma), s(\gamma))$; recall that $G$ is principal if $\pi$ is injective. In order to define the groupoid $C^*$-algebra, we also assume that $G$ is equipped with a fixed left Haar system: a set $\{\lambda^x : x \in G^0\}$ of non-negative Radon measures on $G$ such that

1. $\text{supp} \lambda^x = r^{-1}([x])$;
2. for $f \in C_c(G)$, the function $x \mapsto \int f \, d\lambda^x$ on $G^0$ is in $C_c(G^0)$; and
3. for $f \in C_c(G)$ and $\gamma \in G$, the following equation holds:

$$\int f(\gamma \alpha) \, d\lambda^{s(\gamma)}(\alpha) = \int f(\alpha) \, d\lambda^{s(\gamma)}(\alpha).$$

Condition (3) implies that $\lambda^{s(\gamma)}(\gamma^{-1} E) = \lambda^x(E)$ for measurable sets $E$. The collection $\{\lambda_x : x \in G^0\}$, where $\lambda_x(E) := \lambda^x(E^{-1})$, gives a right Haar system such that the measures are supported on $s^{-1}([x])$ and

$$\int f(\gamma \alpha) \, d\lambda_{r(\alpha)} = \int f(\gamma) \, d\lambda_{s(\alpha)}$$

for $f \in C_c(G)$ and $\gamma \in G$. We will move freely between these two Haar systems.

If $N \subseteq G^0$, then the saturation of $N$ is $r(s^{-1}(N)) = s(r^{-1}(N))$. In particular, we call the saturation of $\{x\}$ the orbit of $x \in G^0$ and denote it by $[x]$.

If $G$ is principal and all the orbits are locally closed, then by [4 Proposition 5.1] the orbit space $G^0/G = \{[x] : x \in G^0\}$ and the spectrum $C^*(G)^\wedge$ of the groupoid $C^*$-algebra $C^*(G)$ are homeomorphic. This homeomorphism is induced by the map $x \mapsto L^x : G^0 \to C^*(G)^\wedge$, where $L^x : C^*(G) \to B(L^2(G, \lambda_x))$ is given by

$$L^x(f)\xi(\gamma) = \int f(\gamma \alpha)\xi(\alpha^{-1})d\lambda^x(\alpha)$$

for $f \in C_c(G)$ and $\xi \in L^2(G, \lambda_x)$.

3. Integrable Groupoids and Convergence in the Orbit Space of a Groupoid

The following definition is motivated by the notion of an integrable action of a locally compact, Hausdorff group on a space from [3 Definition 3.2].

**Definition 3.1.** A locally compact, Hausdorff groupoid $G$ is integrable if for every compact subset $N$ of $G^0$,

$$\sup_{x \in N} \{\lambda^x(s^{-1}(N))\} < \infty,$$

or, equivalently, \(\sup_{x \in N} \{\lambda_x(r^{-1}(N))\} < \infty\).
Remark 3.2. (1) Suppose that \( G \) is a principal groupoid. Then \( \lambda^x(s^{-1}(E)) = \lambda^y(s^{-1}(E)) \) for all \( x, y \in G^0 \) such that \( y \in [x] \). The map \( \lambda^x \mapsto s \ast \lambda^x \), where \( s \ast \lambda^x(E) = \lambda^x(s^{-1}(E)) \), gives a family of measures \( \{\alpha_{[x]} : [x] \in G^0/G\} \) such that \( \alpha_{[x]} \) is a measure on \([x]\) supported on \([x]\), and, for any \( f \in C_c(G) \), the function \[
x \mapsto \int_{y \in [x]} f(\pi^{-1}(x,y)) \, d\alpha_{[x]}(y)\]
is continuous. (Recall that \( \pi : \gamma \mapsto (r(\gamma), s(\gamma)) \) is injective by definition of princi-
pality.) In fact, the existence of the Haar system \( \{\lambda^x\} \) is equivalent to the existence of the family \( \{\alpha_{[x]}\} \) \cite[Examples 2.5(c)]{12}. Thus a principal groupoid \( G \) is integrable if and only if for every compact subset \( M \) of \( G^0/G \), the function \([x] \mapsto \alpha_{[x]}(M)\) is bounded.

(2) We could have taken the supremum in \((3.1)\) over the whole unit space, that is,
\[
\sup_{x \in G^0} \{\lambda^x(s^{-1}(N))\} = \sup_{x \in N} \{\lambda^x(s^{-1}(N))\}.
\]
To see this, first note that if \( y \) is not in the saturation \( r(s^{-1}(N)) = s(r^{-1}(N)) \) of \( N \), then \( s^{-1}(N) \cap r^{-1}(\{y\}) = \emptyset \), and hence \( \lambda^y(s^{-1}(N)) = 0 \). Second, if \( y \) is in the saturation of \( N \), then there exists a \( \gamma \in G \) such that \( s(\gamma) = y \) and \( r(\gamma) \in N \). Then
\[
\lambda^y(s^{-1}(N)) = \lambda^y(r^{-1}(\{y\}) \cap s^{-1}(N)) = \lambda^{r(\gamma)}(r^{-1}(\{r(\gamma)\}) \cap s^{-1}(N)) = \lambda^{s(\gamma)}(s^{-1}(N))\]
with \( r(\gamma) \in N \).

Example 3.3. Let \((H, X)\) be a locally compact, Hausdorff transformation group with \( H \) acting on the left of the space \( X \). Then \( G = H \times X \) with
\[
G^2 = \{(h, x), (k, y) \in G \times G : y = h^{-1} \cdot x\}
\]
and operations \((h, x)(k, h^{-1} \cdot x) = (hk, x) \) and \((h, x)^{-1} = (h^{-1}, h^{-1} \cdot x)\) is called the transformation-group groupoid. We identify the unit space \( \{e\} \times X \) with \( X \), and then the range and source maps \( r, s : G \to X \) are \( s(h, x) = h^{-1} \cdot x \) and \( r(h, x) = x \).

If \( \delta_x \) is the point-mass measure on \( X \) and \( \mu \) is a left Haar measure on \( H \), then \( \{\lambda^x := \mu \times \delta_x : x \in X\} \) is a left Haar system for \( G \). Now
\[
\lambda^x(s^{-1}(N)) = \mu(\{h \in H : h^{-1} \cdot x \in N\})
\]
and hence
\[
\sup_{x \in N} \{\lambda^x(s^{-1}(N))\} = \sup_{x \in N} \{\mu(\{h \in H : h^{-1} \cdot x \in N\})\};
\]
that is, Definition \ref{3.1} reduces to \cite{6} Definition 3.2.

Example 3.4. In \cite{5} pp. 95-96] Green describes an action as follows: the space \( X \) is a closed subset of \( \mathbb{R}^3 \) and consists of countably many orbits, with orbit representa-
tives \( x_0 = (0, 0, 0) \) and \( x_n = (2^{-2n}, 0, 0) \) for \( n = 1, 2, \ldots \). The action of the group \( H = \mathbb{R} \) on \( X \) is given by \( s \cdot x_0 = (0, s, 0) \) for all \( s \); and for \( n \geq 1 \),
\[
s \cdot x_n = \begin{cases} 
(2^{-2n}, s, 0) & \text{if } s \leq n; \\
(2^{-2n} - (\frac{2n}{\pi})2^{-2n-1}, n \cos(s-n), n \sin(s-n)) & \text{if } n < s < n + \pi; \\
(2^{-2n-1}, s - \pi - 2n, 0) & \text{if } s \geq n + \pi.
\end{cases}
\]
So the orbit of each \( x_n \) \( (n \geq 1) \) consists of two vertical lines joined by an arc of a helix situated on a cylinder of radius \( n \); the action moves \( x_n \) along the vertical lines at unit speed and along the arc at radial speed. This action is free, non-proper and integrable (see [13 Example 1.18] or [6 Example 3.3]). So the associated transformation-group groupoid \( G = H \times X \) is principal and integrable by Example 3.3.

The following characterization of integrability will be important later. In the case of a transformation-group groupoid, Lemma 3.5 reduces to a special case of [1 Lemma 3.5].

**Lemma 3.5.** Let \( G \) be a locally compact, Hausdorff groupoid. Then \( G \) is integrable if and only if, for each \( z \in G^0 \), there exists an open neighborhood \( U \) of \( z \) in \( G^0 \) such that

\[
\sup_{x \in U} \{ \lambda^x(s^{-1}(U)) \} < \infty.
\]

**Proof.** The proof is exactly the same as the proof of [1 Lemma 3.5]. \( \square \)

If a groupoid fails to be integrable, there exists a \( z \in G^0 \) such that

\[
\sup_{x \in U} \{ \lambda^x(s^{-1}(U)) \} = \infty
\]

for every open neighborhood \( U \) of \( z \); we then say that the groupoid fails to be integrable at \( z \).

It is evident from [1 2] that integrability and \( k \)-times convergence in the orbit space of a transformation group are closely related. Moreover, Lemma 2.6 of [8] says that, if a principal groupoid fails to be proper and the orbit space \( G^0/G \) is Hausdorff, then there exists a sequence that converges \( 2 \)-times in \( G^0/G \) in the sense of Definition 3.6.

**Definition 3.6.** A sequence \( \{x_n\} \) in the unit space of a groupoid \( G \) converges \( k \)-times in \( G^0/G \) to \( z \in G^0 \) if there exist \( k \) sequences

\[
\{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(k)}\} \subseteq G
\]

such that

1. \( r(\gamma_n^{(i)}) \to z \) as \( n \to \infty \) for \( 1 \leq i \leq k \);
2. \( s(\gamma_n^{(i)}) = x_n \) for \( 1 \leq i \leq k \);
3. if \( 1 \leq i < j \leq k \), then \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty \) as \( n \to \infty \), in the sense that \( \{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\} \) admits no convergent subsequence.

**Remarks 3.7.** (a) Condition (2) in Definition 3.6 is needed so that the composition in (3) makes sense.
(b) Definition 3.6 does not require that \( x_n \to z \), but as in the transformation-group case ([2 Definition 2.2]), this can be arranged by changing the sequence which converges \( k \)-times: replace \( x_n \) by \( r(\gamma_n^{(1)}) \) and replace \( \gamma_n^{(j)} \) by \( \gamma_n^{(j)}(\gamma_n^{(1)})^{-1} \).
(c) Part (3) of Definition 3.6 means \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \) is eventually outside every compact set. In particular, if \( LL^{-1} \) is compact, \( L\gamma_n^{(i)} \cap L\gamma_n^{(j)} = \emptyset \) eventually.

**Example 3.8.** Let \( G = H \times X \) be a transformation-group groupoid (see Example 3.3) and suppose that \( \{x_n\} \subseteq G^0 \) is a sequence converging \( 2 \)-times in \( G^0/G \) to \( z \in G^0 \). Then there exist two sequences

\[
\{\gamma_n^{(1)}\} = \{(s_n, y_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, z_n)\}
\]
in $G$ such that (1) $y_n \to z$ and $z_n \to z$; (2) $s_n^{-1} \cdot y_n = x_n$ and $t_n^{-1} \cdot z_n = x_n$; and (3) $(t_n s_n^{-1}, z_n) \to \infty$ as $n \to \infty$. To see that the sequence $\{x_n\}$ converges 2-times in $X/H$ to $z$ in the sense of [2, §4], consider the two sequences $\{s_n\}$ and $\{t_n\}$ in $H$. We have $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ using (1) and (2). Also, since $z_n \to z$ by (1), (3) implies that $t_n s_n^{-1} \to \infty$ in $H$.

Conversely, if $\{x_n\} \subseteq X$ converges 2-times in $X/H$ to $z$, then there exist two sequences $\{s_n\}$, $\{t_n\}$ in $H$ such that (1) $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ and (2) $t_n s_n^{-1} \to \infty$. It is easy to check that

$$\{\gamma_n^{(1)}\} = \{(s_n, s_n \cdot x_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, t_n \cdot x_n)\}$$

witness the 2-times convergence in $G^0/G$ of $\{x_n\} \subseteq G^0$ to $z \in G^0$.

In the transformation-group groupoid of Example 3.4, the sequence $\{x_n = (2^{-2n}, 0, 0)\}$ converges 2-times in $G^0/G$ to $z_0 = (0, 0, 0)$; to see this, just take $s_n = e$ and $t_n = 2n + \pi$ for each $n$.

In [4] we will prove that a principal groupoid $G$ is integrable if and only if $C^*(G)$ has bounded trace. For the “only if” direction we will need to know that the orbits are locally closed so that [4, Proposition 5.1] applies and $x \mapsto L^x$ induces a homeomorphism of $G^0/G$ onto $C^*(G)^\lor$: Lemma 3.9 below establishes that if $G$ is integrable, then the orbits are in fact closed, hence locally closed. We will prove the contrapositive of the “if” direction, and a key observation for the proof is Proposition 3.11 if a groupoid fails to be integrable at some $z$, then there is a non-trivial sequence $\{x_n\}$ which converges $k$-times in $G^0/G$ to $z$, for every $k \in N \setminus \{0\}$.

We thank an anonymous referee for providing the proof of Lemma 3.9.

**Lemma 3.9.** Let $G$ be a second countable, locally compact, Hausdorff, principal groupoid. If $G$ is integrable, then all orbits are closed.

**Proof.** Let $\{\alpha_x : [x] \in G^0/G\}$ be the family of measures from Remark 3.2(1). We claim that, for fixed $h \in C_c(G^0/G)$, the function $[x] \mapsto \int y \in [x] h(y) \, d\alpha_x(y)$ is continuous. To see this, choose $g_n \in C_c(G^0 \times G^0)$ such that, for all $u \in G^0$, the function $g_n(u, \cdot)$ increases to the function $v \mapsto 1$. Then

$$\int y \in [x] h(y) \, d\alpha_x(y) = \lim_n \int y \in [x] g_n(x, y) h(y) \, d\alpha_x(y) = \lim_n \int x \in G f_n(\gamma) \, d\lambda^x(\gamma),$$

where $f_n(\gamma) = g_n(x(\gamma), h(s(\gamma)))$. Since $f_n \in C_c(G)$, the function

$$x \mapsto \int x \in G f_n(\gamma) \, d\lambda^x(\gamma)$$

is continuous for each $n$. Note that $x \mapsto \int y \in [x] g_n(x, y) h(y) \, d\alpha_x(y)$ is compactly supported for each $n$. Since limits of uniformly continuous functions are continuous, $x \mapsto \int y \in [x] h(y) \, d\alpha_x(y)$ is continuous; this function is constant on orbits, which proves the claim.

Fix $x_0 \in G^0$ and suppose that $G$ is integrable. Since $G$ is principal, for each compact subset $M$ of $G^0/G$, the function $[x] \mapsto \alpha_x(M)$ is bounded. In particular, for each $h \in C_c(G^0/G)^+$, $\int h \, d\alpha_{[x_0]} \in \mathbb{R}$. Since the the support of $\alpha_{[x]}$ is $[x]$, we have

$$\{x_0\} = \bigcap_{h \in C_c(G^0/G)^+} \{x : \int h \, d\alpha_{[x]} \leq \int h \, d\alpha_{[x_0]}\}. $$

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But the function \( [x] \rightarrow \int_{y \in [x]} h(y) \, d\alpha_{[x]}(y) \) is continuous, hence lower semi-continuous, so the left-hand side of (3.2) is an intersection of closed sets. Thus \( \{x_0\} \) is closed in \( G^0/G \), and hence \( [x_0] \) is closed in \( G^0 \). □

The transformation group of [13] Example 1.18 provides an example of a non-integrable free action with closed orbits (by choosing repetition numbers with infinite supremum). Thus there are non-integrable principal groupoids with closed orbits.

Recall that a neighborhood \( W \) of \( G^0 \) is called conditionally compact if the sets \( WV \) and \( VW \) are relatively compact for every compact set \( V \) in \( G \). The following lemma will be used repeatedly.

**Lemma 3.10.** Let \( G \) be a second countable, locally compact, Hausdorff groupoid.

1. Let \( z \in G^0 \) and let \( K \) be a relatively compact neighborhood of \( z \) in \( G \). There exist \( a \in \mathbb{R} \) and a neighborhood \( U \) of \( z \) in \( G^0 \) such that \( 0 < a \leq \lambda_x(K) \) for all \( x \in U \).
2. Let \( Q \) be a conditionally compact neighborhood in \( G \). Given any relatively compact neighborhood \( V \) in \( G^0 \) such that \( QV \neq \emptyset \), there exists \( c \in \mathbb{R} \) such that \( c > 0 \) and \( \lambda_x(Q) \leq c \) for all \( x \in V \).

**Proof.**

(1) Suppose not. Let \( \{U_i\} \) be a decreasing sequence of open neighborhoods of \( z \) in \( G^0 \). There exists an increasing sequence \( i_1 < i_2 < \cdots < i_n < \cdots \) and \( x_n \in U_{i_n} \) such that \( \lambda_{x_n}(K) < 1/n \) for each \( n \geq 1 \). Note that \( x_n \to z \).

Let \( f \in C_c(G) \) such that \( 0 \leq f \leq 1 \), \( f(z) = 1 \) and \( \text{supp } f \subseteq K \); note that \( \int f(\gamma) \, d\lambda_x(\gamma) > 0 \). By the continuity of the Haar system,

\[
\frac{1}{n} > \frac{1}{n} \lambda_{x_n}(K) \geq \int f(\gamma) \, d\lambda_{x_n}(\gamma) = \int f(\gamma) \, d\lambda_x(\gamma) \quad \text{as } n \to \infty,
\]

which is impossible since the left-hand side converges to 0 and \( \int f(\gamma) \, d\lambda_x(\gamma) > 0 \).

(2) Let \( V \) be any relatively compact neighborhood in \( G^0 \) such that \( QV \neq \emptyset \). Let \( f \in C_c(G) \) such that \( 0 \leq f \leq 1 \) and \( f \) is identically one on the relatively compact subset \( QV \). The function \( w \mapsto \int f(\gamma) \, d\lambda_w(\gamma) \) is in \( C_c(G^0) \), so it achieves a maximum \( c > 0 \). Then, for \( x \in V \),

\[
\lambda_x(Q) = \lambda_x(Qx) \leq \int f(\gamma) \, d\lambda_x(\gamma) \leq c. \quad \Box
\]

**Proposition 3.11.** Let \( G \) be a locally compact, Hausdorff groupoid. Let \( z \in G^0 \) and suppose that \( G \) fails to be integrable at \( z \). Then there exists a sequence \( \{x_n\} \) in \( G^0 \) such that \( x_n \to z \), and \( \{x_n\} \) converges \( k \)-times in \( G^0/G \) to \( z \), for every \( k \in \mathbb{N} \setminus \{0\} \). In addition, if \( G \) is second countable, principal and the orbits are locally closed, then \( x_n \neq z \) eventually.

**Proof.** Suppose the groupoid fails to be integrable at \( z \). Fix \( k \in \mathbb{N} \setminus \{0\} \). Let \( \{U_n\} \) be a decreasing sequence of open relatively compact neighborhoods of \( z \) in \( G^0 \). By Lemma 3.3,

\[
\sup_{y \in U_n} \{\lambda^y(s^{-1}(U_n))\} = \infty
\]

for each \( n \). So we can choose a sequence \( \{x_n\} \) such that \( x_n \in U_n \) and \( \lambda^{x_n}(s^{-1}(U_n)) > n \). Note that \( x_n \to z \) as \( n \to \infty \).

Let \( Q \) be an open symmetric conditionally compact neighborhood of \( z \) in \( G \) and let \( V \) be an open relatively compact neighborhood of \( z \) in \( G^0 \). By Lemma 3.10(2)
there exists $c > 0$ such that $\lambda_v(Q^2) \leq c$ whenever $v \in V$. Choose $n_0$ such that $n_0 > (k-1)c$ and $U_{n_0} \subseteq V$. Temporarily fix $n > n_0$. Set $\gamma_n^{(1)} = x_n$. For $k \geq 2$ choose $k-1$ elements $\gamma_n^{(2)}, \ldots, \gamma_n^{(k)}$ as follows. Note that since $x_n = r(\gamma_n^{(1)}) \in V$, we have

$$
\lambda_{x_n}(r^{-1}(U_n) \setminus Q^2\gamma_n^{(1)}) \geq \lambda_{x_n}(r^{-1}(U_n)) - \lambda_{x_n}(Q^2\gamma_n^{(1)})
= \lambda_{x_n}(r^{-1}(U_n) \setminus s^{-1}(\{x_n\})) - \lambda_{r(\gamma_n^{(1)})}(Q^2)
> (k-1)c - c = (k-2)c \geq 0.
$$

So there exists

$$
\gamma_n^{(2)} \in (r^{-1}(U_n) \setminus s^{-1}(\{x_n\})) \setminus Q^2\gamma_n^{(1)};
$$

note that $r(\gamma_n^{(2)}) \in U_n \subset V$ and $s(\gamma_n^{(2)}) = x_n$. Next,

$$
\lambda_{x_n}(r^{-1}(U_n) \setminus (Q^2\gamma_n^{(1)} \cup Q^2\gamma_n^{(2)}))
\geq \lambda_{x_n}(r^{-1}(U_n)) - \lambda_{x_n}(Q^2\gamma_n^{(1)}) - \lambda_{x_n}(Q^2\gamma_n^{(2)})
\geq \lambda_{x_n}(r^{-1}(U_n) \setminus s^{-1}(\{x_n\})) - \lambda_{r(\gamma_n^{(1)})}(Q^2) - \lambda_{r(\gamma_n^{(2)})}(Q^2)
> (k-3)c \geq 0.
$$

Continue until $\gamma_n^{(1)}, \ldots, \gamma_n^{(k)}$ have been chosen in this way.

If $n > n_0$, then by construction $s(\gamma_n^{(i)}) = x_n$ and $r(\gamma_n^{(i)}) \in U_n$ for each $n$; so $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leq i \leq k$. Moreover $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin Q^2$ for $1 \leq i < j \leq k$ and $n > n_0$. To see that $\{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\}$ tends to infinity, suppose that it doesn’t. Then, $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \gamma$ by passing to a subsequence and relabelling. But then $s(\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}) = r(\gamma_n^{(i)}) \to z$ and $r(\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}) = r(\gamma_n^{(j)}) \to z$ implies $\gamma = z$, which is impossible because $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin Q^2$ and $Q$ contains $G^0$. Hence $\{x_n\}$ converges $k$-times in $G^0/G$ to $z$.

We claim that if $G$ is second countable and principal, then $x_n \neq z$ eventually. To see this, suppose $x_n = z$ frequently. Then $\lambda^z(s^{-1}(U_n)) > n$ frequently, and hence

$$
\lambda^z(s^{-1}(U_1)) = \infty.
$$

The orbits are locally closed and $G$ is second countable and principal, so the source map restricts to a homeomorphism $s : r^{-1}(\{z\}) \to [z]$. Since $U_1$ is relatively compact, $s^{-1}(\{z\} \cap U_1)$ is relatively compact in $r^{-1}(\{z\})$ because $s : r^{-1}(\{z\}) \to [z]$ is a homeomorphism. But now $\lambda^z(s^{-1}(\{z\} \cap U_1)) = \lambda^z(s^{-1}(U_1)) < \infty$, contradicting (3.3).

4. Integrability of $G$ and Trace Properties of $C^*(G)$

**Proposition 4.1.** Let $G$ be a second-countable, locally compact, Hausdorff, principal groupoid. If $C^*(G)$ has bounded trace, then $G$ is integrable.

The proof of Proposition 4.1 is based on that of [8] Theorem 2.3. There, Muhly and Williams choose a sequence $\{x_n\} \subseteq G^0$ with $x_n \to z$ which witnesses the failure of the groupoid to be proper. They then carefully construct a function $f \in C_c(G)$ to obtain an element $d$ of the Pedersen ideal of $C^*(G)$ such that $\text{tr}(L^2(d))$ does not converge to $\text{tr}(L^2(z))$. Since the Pedersen ideal is the minimal dense ideal [9] Theorem 5.6.1, the ideal of continuous-trace elements cannot be dense, so $C^*(G)$ does not have continuous trace. We adopt the same strategy, use exactly the
same function $f$, but adapt the proof of [8, Theorem 2.3] using ideas from [5, Proposition 3.5].

**Proof of Proposition 4.1** Fix $M \in \mathbb{N} \setminus \{0\}$. We will show that there is an element $d$ of the Pedersen ideal of $C^*(G)$, a sequence of representations $\{L^\ast_n\}$ and $n_0 > 0$ such that $\text{tr}(L^\ast_n(d)) > M$ whenever $n > n_0$. Since $M$ is arbitrary, $C^*(G)$ cannot have bounded trace.

If $G$ is not integrable, then the integrability fails at some $z \in G^0$ by Lemma 3.5. If the orbits are not closed, then $C^*(G)$ cannot be CCR by [3, Theorem 4.1] and hence cannot have bounded trace. So from now on we may assume that the orbits are closed. By Proposition 3.11, there exists a sequence $\{x_n\}$ such that $x_n \neq z$, $x_n \to z$, and $\{x_n\}$ converges $k$-times in $G^0/G$ to $z$, for every $k \in \mathbb{N} \setminus \{0\}$.

Since we will use exactly the same function $f$ that was used in the proof of [8, Theorem 2.3], our first task is to briefly outline its construction. Fix a function $g \in C_0(G^0)$ such that $0 \leq g \leq 1$ and $g$ is identically one on a neighborhood $U$ of $z$. Let $N = \text{supp } g$ and let

$$F^N_z := s^{-1}(\{z\}) \cap r^{-1}(\{z\} \cap N) = s^{-1}(\{z\}) \cap r^{-1}(N),$$
$$F^N := r^{-1}(\{z\}) \cap s^{-1}(\{z\} \cap N) = r^{-1}(\{z\}) \cap s^{-1}(N).$$

There exist symmetric, open, conditionally compact neighborhoods $W_0$ and $W_1$ in $G$ such that

$$G^0 \subseteq W_0 \subseteq \overline{W_0} \subseteq W_1 \text{ and } F^N \cup F^N_z \subseteq W_0.$$

Thus $\overline{W_1^7} z \setminus W_0 z \subseteq r^{-1}(G^0 \setminus N)$. (The reason for using $\overline{W_1^7}$ becomes clear at (4.4) below.) By a compactness argument, there exist open, symmetric, relatively compact neighborhoods $V_0 \subseteq G^0$ and $V_1$ of $z$ in $G$ such that $V_0 \subseteq V_1$ and

$$\overline{V_1^7} z \setminus W_0 z \subseteq r^{-1}(G^0 \setminus N).$$

Now note that if $\gamma \in \overline{W_1^7} z \setminus W_0 V_0 W_0$, then $r(\gamma) \in r(\overline{W_1^7} z \setminus W_0 V_0) \subseteq G^0 \setminus N$. It follows that the function $g^{(1)} : G \to [0,1]$ defined by

$$g^{(1)}(\gamma) = \begin{cases} 
 g(r(\gamma)) & \text{if } \gamma \in \overline{W_1^7} z \setminus W_0 V_0, \\
 0 & \text{if } \gamma \notin W_0 V_0 W_0
\end{cases}$$

is well-defined and continuous with compact support in $G$. By construction

$$(W_0 V_0 W_0)^2 = W_0 V_0 W_0^2 V_0 W_0 \subseteq W_0^4 V_0 W_0^4 \subseteq \overline{W_0^4} V_0 \overline{W_0^4} \subseteq W_1^4 V_1 W_1^4 \subseteq \overline{W_1^4} V_1 W_1^4.$$

So there exists a function $b \in C_0(G)$ such that $0 \leq b \leq 1$, $b$ is identically one on $W_0 V_0 W_0^2 V_0 W_0$ and it is identically zero on the complement of $\overline{W_1^7} z \setminus W_0 V_0$. Further, we can replace $b$ with $(b + b^\ast)/2$ to ensure that $b$ is self-adjoint. Set

$$f(\gamma) = g(r(\gamma)) g(s(\gamma)) b(\gamma);$$

note that $f \in C_c(G)$ is self-adjoint.
For \( \xi \in L^2(G, \lambda_a) \) and \( \gamma \in G \) we have

\[
L^u(f)(\gamma) = \int f(\gamma \alpha) \xi(\alpha^{-1}) \, d\lambda^u(\alpha) = \int g(r(\gamma)) g(s(\alpha)) b(\gamma \alpha) \xi(\alpha^{-1}) \, d\lambda^u(\alpha) = g(r(\gamma)) \int g(s(\alpha)) b(\gamma \alpha) \xi(\alpha^{-1}) \, d\lambda^u(\alpha)
\]

(4.2)

By \cite{S} Lemma 2.8, \( g^{(1)} \) is an eigenvector for \( L^x(f) \) with eigenvalue

\[
\mu_x^{(1)} = \int g(r(\alpha)) g^{(1)}(\alpha) \, d\lambda_x(\alpha) = \int_{W_0 V_0 W_0} g(r(\alpha))^2 \, d\lambda_x(\alpha).
\]

By \cite{S} Lemma 2.9, there exist an open \( V_2 \subseteq V_0 \) and a conditionally compact neighborhood \( Y \) of \( G^0 \) so that \( Y \subseteq W_0 \) and if \( v \in V_2 \), then \( r(Y v) \subseteq U \). Notice that \( Y V_2 Y \) is a relatively compact subset of \( W_0 V_2 W_0 \). By Lemma \cite{S} 3.10(1) there exist an open neighborhood \( V_3 \) of \( z \) and \( a > 0 \) such that

\[
\lambda_v(Y V_2 Y) \geq a \quad \text{whenever } v \in V_3.
\]

(4.3)

Now, if \( \alpha \in Y V_2 Y \), then \( r(\alpha) \in U \) and hence \( g(r(\alpha)) = 1 \); it follows that

\[
\mu_x^{(1)} \geq \int_{Y V_2 Y} g(r(\alpha))^2 \, d\lambda_x(\alpha) = \lambda_x(Y V_2 Y) \geq a > 0
\]

whenever \( x_n \in V_3 \).

So far our set-up is the one from \cite{S}. Now choose \( l \in \mathbb{N} \setminus \{0\} \) such that \( la^2 > M \). (Note that \( a \) is independent of \( l \)) The sequence \( \{x_n\} \) converges \( k \)-times in \( G/G^0 \) to \( z \) for every \( k \in \mathbb{N} \setminus \{0\} \), so it certainly converges \( l \) times. So there exist \( l \) sequences

\[
\{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(l)}\} \subseteq G
\]

such that

1. \( r(\gamma_n^{(i)}) \to z \) as \( n \to \infty \) for \( 1 \leq i \leq l \);
2. \( s(\gamma_n^{(i)}) = x_n \) for \( 1 \leq i \leq k \);
3. if \( 1 \leq i < j \leq l \), then \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty \).

Moreover, by construction (see Proposition \cite{S} 3.11), we may take \( \gamma_n^{(1)} = x_n \). Temporarily fix \( n \). Set \( g_n^{(1)} := g^{(1)} \), and for \( 2 \leq j \leq l \) set

\[
g_n^{(j)}(\gamma) :=
\begin{cases} 
  g^{(1)}(\gamma(\gamma_n^{(j)})^{-1}), & \text{if } s(\gamma) = s(\gamma_n^{(j)}); \\
  0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
  g(r(\gamma)), & \text{if } \gamma \in W_1 V_1 W_1^{-1} \gamma_n^{(j)}; \\
  0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
  g(r(\gamma)), & \text{if } \gamma \in W_1 V_1 W_1^{-1} \gamma_n^{(j)}; \\
  0, & \text{if } \gamma \notin W_0 V_0 W_0 \gamma_n^{(j)}
\end{cases}
\]

Each \( g_n^{(j)} \) \( (1 \leq j \leq l) \) is a well-defined function in \( C_c(G) \) with support contained in \( W_0 V_0 W_0 \gamma_n^{(j)} \). For \( 1 \leq i < j \leq l \), \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin (W_0 V_0 W_0)^2 \) eventually, so there
exists \( n_0 > 0 \) such that, for every \( 0 \leq i, j \leq l, i \neq j \),
\[
W_0V_0W_0\gamma_n^{(j)} \cap W_0V_0W_0\gamma_n^{(i)} = \emptyset
\]
whenever \( n > n_0 \).

We now prove a generalization of [8, Lemma 2.8] which, together with (4.2),
immediately implies that each \( \gamma_n^{(j)} \) is an eigenvector of \( L^x_n(f) \) for \( 1 \leq j \leq l \).

**Lemma 4.2.** With the choices made above, for all \( \alpha, \gamma \in G \) and \( 1 \leq j \leq l \),
\[
g(r(\gamma))g(r(\alpha))b(\gamma \alpha^{-1})g_n^{(j)}(\alpha) = g_n^{(j)}(\gamma)g(r(\alpha))g_n^{(j)}(\alpha).
\]

**Proof.** If \( \alpha \notin W_0V_0W_0\gamma_n^{(j)} \), then both sides are zero. So we may assume throughout
that \( \alpha \in W_0V_0W_0\gamma_n^{(j)} \).

If \( \gamma \in W_0V_0W_0\gamma_n^{(j)} \), then \( g_n^{(j)}(\gamma) = g(r(\gamma)) \) and \( \gamma \alpha^{-1} \in W_0V_0W_0^2V_0W_0 \), so
\( b(\gamma \alpha^{-1}) = 1 \) and both sides agree.

If \( \gamma \in \overline{W_1V_1W_1\gamma_n^{(j)}} \setminus W_0V_0W_0\gamma_n^{(j)} \), then \( g(r(\gamma)) = 0 = g_n^{(j)}(\gamma) \), so both sides are zero.

Finally, if \( \gamma \notin \overline{W_1V_1W_1\gamma_n^{(j)}} \), then \( g_n^{(j)}(\gamma) = 0 \), so the right-hand side is zero. On
the other hand, if \( \gamma \alpha^{-1} \in \overline{W_1V_1W_1(= \text{supp} b)} \), then
\[
(4.4) \quad \gamma \in \overline{W_1V_1W_1\gamma_n^{(j)}} \subseteq \overline{W_1V_1W_1\gamma_n^{(j)}}.
\]
So \( \gamma \notin \overline{W_1V_1W_1\gamma_n^{(j)}} \) implies \( \gamma \alpha^{-1} \notin \text{supp} b \), so the left-hand side is zero as well. \( \square \)

Let \( \mu_n^{(j)} \) be the eigenvalue corresponding to the eigenvector \( g_n^{(j)} \). Using (4.3),
\[
\mu_n^{(j)} = \int_{W_0V_0W_0\gamma_n^{(j)}} g(r(\alpha))^2 \, d\lambda_{x_n}(\alpha) \geq \lambda_{x_n}(YV_2Y\gamma_n^{(j)}) = \lambda_{r(\gamma_n^{(j)})(YV_2Y)} \geq a
\]
whenever \( r(\gamma_n^{(j)}) \in V_3 \). Choose \( n_1 > n_0 \) such that \( n > n_1 \) implies \( x_n \in V_3 \) and
\( r(\gamma_n^{(j)}) \in V_3 \) for \( 1 \leq j \leq l \). Then \( L^x_n(f \ast f) \) is a positive compact operator with
\( l \) eigenvalues \( \mu_n^{(j)} \geq a^2 \) for \( 1 \leq j \leq l \). To push \( f \ast f \) into the Pedersen ideal, let \( r \in C_c(0, \infty) \)
be any function satisfying
\[
r(t) = \begin{cases} 0, & \text{if } t < \frac{a^2}{3}; \\ 2t - \frac{2a^2}{3}, & \text{if } \frac{a^2}{3} \leq t < \frac{2a^2}{3}; \\ t, & \text{if } \frac{2a^2}{3} \leq t \leq \|f \ast f\|. \end{cases}
\]
Set \( d := r(f \ast f) \). Now \( d \) is a positive element of the Pedersen ideal of \( C^*(G) \) with
\( \text{tr}(L^x_n(d)) \geq ta^2 > M \) whenever \( n > n_1 \). Since \( M \) was arbitrary, \( L^x \mapsto \text{tr}(L^x(d)) \) is
unbounded on \( C^*(G) \). Thus \( C^*(G) \) does not have bounded trace. \( \square \)

**Proposition 4.3.** Suppose \( G \) is a second countable, locally compact, Hausdorff, principal groupoid. If \( G \) is integrable, then \( C^*(G) \) has bounded trace.

**Proof.** Since \( G \) is principal and integrable, the orbits are closed by Lemma [8, Lemma 2.8] and
\( x \mapsto L^x \) induces a homeomorphism of \( G^0/G \) onto \( C^*(G)^\wedge \) by [3] Proposition 5.1. To show that \( C^*(G) \) has bounded trace, it suffices to see that for a fixed \( u \in G^0 \) and all \( f \in C_c(G) \), \( \text{tr}(L^u(f \ast f)) \) is bounded independent of \( u \).

Fix \( u \in G^0 \) and let \( \xi \in L^2(G, \lambda_u) \). Since
\[
L^u(f)\xi(\gamma) = \int f(\gamma \alpha^{-1})\xi(\alpha) \, d\lambda_u(\alpha),
\]
$L^u(f)$ is a kernel operator on $L^2(G, \lambda_u)$ with kernel $k_f$ given by $k_f(\gamma, \alpha) = f(\gamma \alpha^{-1})$. We will show that $k_f \in L^2(G \times G, \lambda_u \times \lambda_u)$ and we will find a bound on $k_f$ independent of $u$. This will complete the proof since $\text{tr}(L^u(f^* f)) = \|k_f\|^2$ by, for example, [10] Theorem 3.4.16.

Notice that

$$\|k_f\|^2 = \int_{G \times G} |k_f(\gamma, \alpha)|^2 \, d(\lambda_u \times \lambda_u)(\gamma, \alpha) = \int_G \int_G |f(\gamma \alpha^{-1})|^2 \, d\lambda_u(\gamma) \, d\lambda_u(\alpha)$$

by Tonelli’s Theorem and right invariance. For a fixed $\alpha$, the inner integral

$$\int_G |f(\gamma)|^2 \, d\lambda_{\alpha}(\gamma) \leq \|f\|^2_{\infty} \lambda_{\alpha}(\text{supp } f)$$

and is zero unless $r(\alpha) \in s(\text{supp } f)$. The outer integral is zero unless $s(\alpha) = u$. Let

$$K = r^{-1}(s(\text{supp } f)) \cap s^{-1}\{u\}.$$

So

$$\|k_f\|^2 \leq \int_K \|f\|^2_{\infty} \lambda_{\alpha}(\text{supp } f) \, d\lambda_u(\alpha)$$

$$\leq \|f\|^2_{\infty} \sup \{\lambda_{\alpha}(\text{supp } f) : r(\alpha) \in s(\text{supp } f)\} \lambda_u(K)$$

$$\leq \|f\|^2_{\infty} \sup \{\lambda_x(\text{supp } f) : x \in s(\text{supp } f)\} \sup \{\lambda_x(r^{-1}(s(\text{supp } f)) : x \in G^0)\}.$$

Since $s(\text{supp } f)$ is a compact subset of $G^0$, by integrability there exists $N > 0$ such that

$$\sup \{\lambda_x(r^{-1}(s(\text{supp } f)) : x \in G^0)\} < N$$

(see also Remark 3.2). Note that $N$ does not depend on $u$. By Lemma 3.10(2), applied to the conditionally compact neighborhood $\text{supp } f$ and the relatively compact neighborhood $s(\text{supp } f)$, there exists $M > 0$ such that $\lambda_x(s(\text{supp } f)) < M$ for all $x \in s(\text{supp } f)$; that is,

$$\sup \{\lambda_x(\text{supp } f) : x \in s(\text{supp } f)\} < M.$$

Note that $M$ does not depend on $u$.

Thus $\|k_f\|^2 < \|f\|^2_{\infty} MN$, so $k_f \in L^2(G \times G, \lambda_u \times \lambda_u)$ as claimed, and

$$\text{tr}(L^u(f^* f)) = \|k_f\|^2 < \|f\|^2_{\infty} MN,$$

which is a bound on $\text{tr}(L^u(f^* f))$ independent of $u$. □

Combining Propositions 4.1 and 4.3 we have

**Theorem 4.4.** Suppose $G$ is a second countable, locally compact, Hausdorff principal groupoid. Then $G$ is integrable if and only if $C^*(G)$ has bounded trace.

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