Two-component nonlinear wave of the Born-Infeld equation

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The generalized perturbative reduction method is used to find the two-component vector breather solution of the Born-Infeld equation
\[ U_{tt} - CU_{zz} = -AU_t^2U_{zz} - \sigma U_z^2U_{tt} + BU_tU_zU_{zt}. \]
It is shown that the solution of the two-component nonlinear wave oscillates with the sum and difference of frequencies and wave numbers.

Keywords: Generalized perturbative reduction method, Two-component nonlinear waves, Born-Infeld equation.

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I. INTRODUCTION

The nonlinear solitary waves plays a fundamental role in the study of nonlinear phenomena in completely different fields of physics and applied mathematics. The solitary wave behavior can mathematically be described in nonlinear partial differential equations. Some of the equations are the Boussinesq, Benjamin-Bona-Mahony, Maxwell-Bloch, Hirota and many others. The nonlinear solitary waves can be divided into two main types: single-component (scalar) and two-component (vector) solitary waves. This is one of the most interesting topics for study in various fields of physics, various fields of physics such as optics, hydrodynamics, acoustics, plasma and others[1-5].

Among other nonlinear partial differential equations, there is a nonlinear modification of the Maxwell wave equation, which includes the so-called Born-Infeld nonlinearity. As a result, the Born-Infeld equation is obtained, which describes the properties of particles and has the form [1,6]
\[ \frac{\partial^2 U}{\partial t^2} - C \frac{\partial^2 U}{\partial z^2} = -A(\frac{\partial U}{\partial t})^2 \frac{\partial^2 U}{\partial z^2} - \sigma (\frac{\partial U}{\partial z})^2 \frac{\partial^2 U}{\partial t^2} + B \frac{\partial U}{\partial t} \frac{\partial U}{\partial z} \frac{\partial^2 U}{\partial t \partial z}, \quad (1) \]
or in the dimensionless form
\[ \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial z^2} = -(\frac{\partial U}{\partial t})^2 \frac{\partial^2 U}{\partial z^2} - (\frac{\partial U}{\partial z})^2 \frac{\partial^2 U}{\partial t^2} + 2 \frac{\partial U}{\partial t} \frac{\partial U}{\partial z} \frac{\partial^2 U}{\partial t \partial z}, \quad (2) \]
where \( U(z,t) \) is a real function of space coordinate \( z \) and time \( t \) and represents the wave profile, while \( A, B, C \) and \( \sigma \) are the real constants. Eq.(2) has solutions in the form of the solitary waves \( U = \Phi(x + t) \) and \( U = \Phi(x - t) \), where \( \Phi \) is arbitrary function[1,6,7].

Sometimes 2+1 (two space coordinates and time) dimensional version of the Born-Infeld equation also have been investigated[8].

We shall consider a pulse with width \( T \), the carrier frequency \( \omega \) and wave number \( k \), propagating along the positive \( z \) axis. We are interested in the case where the pulse duration is much longer than the inverse frequency of the carrier wave, i.e. \( T >> 1/\omega \). Following the standard procedure of the slowly varying envelope approximation, we will transform the Born-Infeld equation (11) for real function \( U \) into the slowly varying envelope functions, using the expansion [9,10]
\[ U(z,t) = \sum_{l=\pm 1} \hat{u}_l(z,t)Z_l, \quad (3) \]
where \( Z_l = e^{i(kz - \omega t)} \) is the fast oscillating function, \( \hat{u}_l \) are the slowly varying complex envelope functions, which satisfied inequalities
\[ \left| \frac{\partial \hat{u}_l}{\partial t} \right| \ll \omega |\hat{u}_l|, \quad \left| \frac{\partial \hat{u}_l}{\partial z} \right| \ll k|\hat{u}_l|. \quad (4) \]

For the reality of \( U \), we set: \( \hat{u}_{+1} = \hat{u}_{-1}^* \).

The purpose of the present work is to consider the two-component vector breather solution of the Born-Infeld equation (1) using the generalized perturbative reduction method [3,4,11-16].
The rest of this paper is organized as follows: Section II is devoted to the linear part of the Born-Infeld equation for slowly varying complex envelope functions. In Section III, using the generalized perturbative reduction method, we will transform Eq.(1) to the coupled nonlinear Schrödinger equations for auxiliary functions. In Section IV, will be presented the solution of the two-component nonlinear pulse. Finally, in Section V, we will discuss the obtained results.

II. THE LINEAR PART OF THE BORN-INFELD EQUATION

The linear part of the Born-Infeld equation (1) is coincide with the linear Maxwell wave equation and is given by

\[ \frac{\partial^2 U}{\partial t^2} - C \frac{\partial^2 U}{\partial z^2} = 0. \]  

(5)

We consider a pulse whose duration satisfies the condition \( T >> \omega^{-1} \). Substituting Eq.(3) into (5) we obtain the dispersion relation

\[ \omega^2 = C k^2 \]  

(6)

and the rest part of the linear equation (5) in the form

\[ \sum_{l=\pm 1} Z_l [-2il\omega \frac{\partial \hat{u}_l}{\partial t} - 2ilkC \frac{\partial \hat{u}_l}{\partial z} + \frac{\partial^2 \hat{u}_l}{\partial t^2} - C \frac{\partial^2 \hat{u}_l}{\partial z^2}] = 0. \]  

(7)

III. THE GENERALIZED PERTURBATIVE REDUCTION METHOD

In order to consider the two-component vector breather solution of the Eq.(1), we use the generalized perturbative reduction method \([3,4,11-16]\) which makes it possible to transform the Born-Infeld equation for the functions \( \hat{u}_l \) to the coupled nonlinear Schrödinger equations for auxiliary functions \( f^{(\alpha)}_{l,n} \). As a result, we obtain a two-component nonlinear pulse oscillating with the difference and sum of the frequencies and wave numbers. In the frame of this method, the complex envelope function \( \hat{u}_l \) can be represented as

\[ \hat{u}_l(z,t) = \sum_{\alpha=1}^{\infty} \sum_{n=-\infty}^{+\infty} \varepsilon^\alpha Y_{l,n} f^{(\alpha)}_{l,n}(\zeta_{l,n}, \tau), \]  

(8)

where \( \varepsilon \) is a small parameter,

\[ Y_{l,n} = e^{in(Q_{l,n} z - \Omega_{l,n} t)}, \quad \zeta_{l,n} = \varepsilon Q_{l,n} (z - v_{g_{l,n}} t), \]

\[ \tau = \varepsilon^2 t, \quad v_{g_{l,n}} = \frac{\partial \Omega_{l,n}}{\partial Q_{l,n}}. \]

It is assumed that the quantities \( \Omega_{l,n}, Q_{l,n} \) and \( f^{(\alpha)}_{l,n} \) satisfies the inequalities for any \( l \) and \( n \):

\[ \omega \gg \Omega_{l,n}, \quad k \gg Q_{l,n}, \]

\[ \left| \frac{\partial f^{(\alpha)}_{l,n}}{\partial \tau} \right| \ll \Omega_{l,n} \left| f^{(\alpha)}_{l,n} \right|, \quad \left| \frac{\partial f^{(\alpha)}_{l,n}}{\partial \eta} \right| \ll Q_{l,n} \left| f^{(\alpha)}_{l,n} \right|. \]

Substituting Eq.(8) into (7), for the linear part of the Born-Infeld equation (5) we obtain

\[ \sum_{l=\pm 1} \sum_{\alpha=1}^{\infty} \sum_{n=\pm 1} \varepsilon^\alpha Z_l Y_{l,n} [W_{l,n} + \varepsilon J_{l,n} - \varepsilon^2 ilh_{l,n} \frac{\partial}{\partial \tau} - \varepsilon^2 Q^2 H_{l,n} \frac{\partial^2}{\partial \zeta^2} + O(\varepsilon^3)] f^{(\alpha)}_{l,n} = 0, \]  

(9)
where

\[ W_{l,n} = -2n\omega\Omega_{l,n} + 2nk\Omega_{l,n}C - \Omega_{l,n}^2 + CQ_{l,n}^2, \]

\[ J_{l,n} = 2i\Omega_{l,n} [\omega v_{g_{l,n}} - \omega v_{g_{l,n}}] - \Omega_{l,n} v_{g_{l,n}} - C\Omega_{l,n}], \]

\[ h_{l,n} = 2(\omega + ln\Omega_{l,n}), \]

\[ H_{l,n} = C - v_{g_{l,n}}^2. \]

Equating to zero, the terms with the same powers of \( \varepsilon \), from the Eq.(9) we obtain a series of equations. In the first order of \( \varepsilon \), we have a connection between of the parameters \( \Omega_{l,n} \) and \( Q_{l,n} \). When

\[ 2(C\Omega_{l,n} - \omega\Omega_{l,n} + \Omega_{l,n}^2 + CQ_{l,n}^2 = 0, \]

then \( f^{(1)}_{l,n} \neq 0 \) and when

\[ 2(C\Omega_{l,n} - \omega\Omega_{l,n} + \Omega_{l,n}^2 - CQ_{l,n}^2 = 0, \]

than \( f^{(1)}_{l,n} \neq 0 \).

From Eq.(10), in the second order of \( \varepsilon \), we obtain the equation

\[ J_{l,n} = J_{l,n} = 0 \]

and the expression

\[ v_{g_{l,n}} = C\frac{\lambda + ln\Omega_{l,n}}{\omega + ln\Omega_{l,n}}. \]

In the third order of \( \varepsilon \), the linear part of the Born-Infeld equation (1), is given by

\[ \sum_{l=\pm 1} \sum_{n=\pm 1} \varepsilon^3 Z_l Y_l_n [-i\hbar h_{l,n} \frac{\partial}{\partial r} - Q_{l,n} H_{l,n} \frac{\partial^2}{\partial \xi^2}]. \]

Next we consider the nonlinear term of the Born-Infeld equation (1)

\[ A(\frac{\partial U}{\partial r})^2 \frac{\partial^2 U}{\partial z^2} - A(\frac{\partial^2 U}{\partial z})^2 \frac{\partial^2 U}{\partial r^2} + B \frac{\partial U}{\partial r} \frac{\partial U}{\partial z} \frac{\partial^2 U}{\partial r \partial z}. \]

Substituting Eqs.(3) and (8) into Eq.(15), for the nonlinear part of the Born-Infeld equation (1) we obtain

\[ \varepsilon^3 Z_{l,n}[\tilde{q}_l + |f^{(1)}_{l,n}|^2 + \tilde{r}_l + |f^{(1)}_{l,-n}|^2]f^{(1)}_{l+1,l} Y_{l+1,l+1} + (\tilde{q}_l - |f^{(1)}_{l,n}|^2 + \tilde{r}_l - |f^{(1)}_{l,n}|^2) Y_{l+1,l-1} \]

and plus terms proportional to \( Z_{l,n} \). Here

\[ \tilde{q}_l = (A + \sigma - B)(\omega \pm \Omega_{l,n})^2 (k \pm Q_{l,n}), \]

\[ \tilde{r}_l = 2[A(k \pm Q_{l,n})(\omega \mp \Omega_{l,n})^2 + \sigma(\omega \pm \Omega_{l,n})^2 (k \mp Q_{l,n}) - B(\omega + \Omega_{l,n})(\omega - \Omega_{l,n})(k + Q_{l,n})(k - Q_{l,n})]. \]

In the third order of \( \varepsilon \) for the Born-Infeld equation (1), from Eqs.(14) and (16) we obtain the system of nonlinear equations

\[ i\frac{\partial f^{(1)}_{l,n}}{\partial \tau} + Q_{l+1,l} \frac{\partial^2 f^{(1)}_{l,n}}{\partial \xi^2} + \frac{\tilde{q}_l}{h_{l+1,l+1}} |f^{(1)}_{l+1,l+1}|^2 + \tilde{r}_l |f^{(1)}_{l+1,l+1}|^2 f^{(1)}_{l+1,l+1} = 0, \]

\[ i\frac{\partial f^{(1)}_{l,n}}{\partial \tau} + Q_{l+1,l} \frac{\partial^2 f^{(1)}_{l,n}}{\partial \xi^2} + \frac{\tilde{q}_l}{h_{l+1,l+1}} |f^{(1)}_{l+1,l+1}|^2 + \tilde{r}_l |f^{(1)}_{l+1,l+1}|^2 f^{(1)}_{l+1,l+1} = 0. \]
IV. THE TWO-COMPONENT VECTOR BREATHER OF THE BORN-INFELD EQUATION

Taking into account Eqs.(8) and (13), after transformation back to the space coordinate $z$ and time $t$, from the system of equations (17) we obtain the coupled nonlinear Schrödinger equations for the auxiliary functions $\Lambda_{\pm} = \varepsilon f_{i+1,\pm 1}^{(1)}$ in the following form

$$i(\frac{\partial \Lambda_{\pm}}{\partial t} + v_{\pm} \frac{\partial \Lambda_{\pm}}{\partial z}) + p_{\pm} \frac{\partial^2 \Lambda_{\pm}}{\partial z^2} + q_{\pm} |\Lambda_{\pm}|^2 \Lambda_{\pm} + r_{\pm} |\Lambda_{\mp}|^2 \Lambda_{\pm} = 0,$$

where

$$p_{\pm} = \frac{C - v_{\pm}^2}{2(\omega \pm \Omega_{\pm})},$$

$$q_{\pm} = \frac{\tilde{q}_{\pm}}{2(\omega \pm \Omega_{\pm})},$$

$$r_{\pm} = \frac{\tilde{r}_{\pm}}{2(\omega \pm \Omega_{\pm})},$$

$$v_{\pm} = v_{g;+1,\pm 1} = C \frac{k_{\pm} Q_{\pm}}{\omega \mp \Omega_{\pm}},$$

$$\Omega_{+} = \Omega_{+1,+1} = \Omega_{-1,-1}, \quad \Omega_{-} = \Omega_{+1,-1} = \Omega_{-1,+1},$$

$$Q_{+} = Q_{+1,+1} = Q_{-1,-1}, \quad Q_{-} = Q_{+1,-1} = Q_{-1,+1}.$$

The solution of Eq.(18) is given by [3,11-14]

$$\Lambda_{\pm} = \frac{A_{\pm}}{T} \text{Sech}\left(\frac{t - \frac{v_{\pm}}{T}}{\frac{T}{2}}\right)e^{i(k_{\pm} z - \omega_{\pm} t)},$$

where $A_{\pm}$, $k_{\pm}$ and $\omega_{\pm}$ are the real constants, $V_0$ is the velocity of the nonlinear wave. We assume that $k_{\pm} << Q_{\pm}$ and $\omega_{\pm} << \Omega_{\pm}$.

Combining Eqs.(3), (8) and (20), we obtain the two-component vector breather solution of the Born-Infeld equation (1) in the following form:

$$U(z, t) = \mathfrak{A} \text{Sech}\left(\frac{t - \frac{v_{\pm}}{T}}{\frac{T}{2}}\right)\left\{\cos[(k_{+} + Q_{+}) z - (\omega_{+} + \Omega_{+}) t]ight\}$$

$$+ \left(\frac{p_{-} - p_{+} r_{-}}{p_{+} q_{-} - p_{-} r_{+}}\right) \frac{v_{\pm}}{T} \cos[(k_{-} + Q_{-}) z - (\omega_{-} + \Omega_{-}) t],$$

where $\mathfrak{A}$ is amplitude of the nonlinear pulse. The expressions for the parameters $k_{\pm}$ and $\omega_{\pm}$ are given by

$$k_{\pm} = \frac{V_0 - v_{\pm}}{2p_{\pm}}, \quad \omega_{+} = \frac{p_{+} \omega_{+}}{p_{-} \omega_{-} + \frac{V_0^2 (p_{-}^2 - p_{+}^2) + v_{\pm}^2 p_{+}^2 - v_{\pm}^2 p_{-}^2}{4p_{+} p_{-}^2}}.$$

V. CONCLUSION

We investigate the two-component vector breather solution of the Born-Infeld equation (1) in case when the slowly varying envelope approximation Eq.(4) is valid. The nonlinear pulse with the width $T >> \Omega_{\pm}^{-1} >> \omega^{-1}$ is considered. Using the generalized perturbative reduction method Eq.(8), the Eq.(1) is transformed to the coupled nonlinear Schrödinger equations (18) for the functions $\Lambda_{\pm 1}$. As a result, the two-component nonlinear pulse oscillating with
the sum and difference of the frequencies and wave numbers Eq. (21), is obtained. The dispersion relation and the connection between parameters $\Omega_\pm$ and $Q_\pm$ are determined from Eqs. (6), (11) and (12). The parameters of the pulse from Eqs. (19) and (22) are determined.

We have to note that the two-component vector breathers can propagate also in the other physical systems [4, 14–16].

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