Recollements arising from cotorsion pairs on extriangulated categories

Yonggang HU¹, Panyue ZHOU²

¹ Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
² College of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China

© Higher Education Press 2021

Abstract This paper is devoted to constructing some recollements of additive categories associated to concentric twin cotorsion pairs on an extriangulated category. As an application, this result generalizes the work by W. J. Chen, Z. K. Liu, and X. Y. Yang in a triangulated case [J. Algebra Appl., 2018, 17(5): 1–15]. Moreover, it highlights new phenomena when it applied to an exact category. Finally, we give some applications of our main results. In particular, we obtain Krause’s recollement whose proofs are both elementary and very general.

Keywords Extriangulated categories, recollements, cotorsion pairs, adjoint pairs

MSC2020 13D30, 18E05, 18G80, 18E10

1 Introduction

The recollement of triangulated categories was introduced in a geometric setting by Beilinson et al. [1]. Nowadays, it has become very powerful in understanding relationships among algebraic, geometric, or topological objects. A fundamental example of the recollement of abelian categories is due to MacPherson and Vilonen [10], in which it first appeared as an inductive step in the construction of perverse sheaves. Later, Wang and Lin [12] defined the notion of recollement of additive categories, which unifies the recollements of abelian categories and triangulated categories. Recently, Chen et al. [3] introduced localization sequences and colocalization sequences of additive categories which are similar to lower and upper recollements of triangulated categories, respectively. They considered recollements of additive categories by cotorsion pairs in triangulated categories. Based on this fact that a co-t-structure is a special cotorsion pair, they gave some recollements of additive categories associated to concentric twin
cotorsion pairs in a triangulated category by doing quotient.

Extriangulated categories were recently introduced by Nakaoka and Palu [11] by extracting those properties of \( \text{Ext}^1 \) on exact categories (which is itself a generalization of the concept of a module category and an abelian category) and on triangulated categories that seem relevant from the point of view of cotorsion pairs. In particular, exact categories and triangulated categories are extriangulated categories. There are a lot of examples of extriangulated categories which are neither exact categories nor triangulated categories, see [11,14]. Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Based on this idea, we extend Chen-Liu-Yang’s results to extriangulated categories.

Our main result is the following theorem.

**Theorem 1** Let \( \mathcal{C} \) be an extriangulated category with enough projectives and enough injectives, and \( (\mathcal{X}, \mathcal{T}), (\mathcal{U}, \mathcal{V}) \), and \( (\mathcal{X'}, \mathcal{Y}) \) be cotorsion pairs on \( \mathcal{C} \) with
\[
\mathcal{W} = \mathcal{X} \cap \mathcal{T} = \mathcal{U} \cap \mathcal{V} = \mathcal{X'} \cap \mathcal{Y}.
\]
If
\[
\mathcal{X'} \cap \mathcal{T} = \mathcal{V}, \quad \mathcal{Y} \subseteq \mathcal{T}, \quad \Sigma \mathcal{V} \subseteq \mathcal{V}, \quad \Sigma \mathcal{Y} \subseteq \mathcal{Y},
\]
then there are two recollements of additive categories as follows:

\[
\begin{array}{ccc}
\mathcal{V}/\mathcal{W} & F & \mathcal{T}/\mathcal{W} & G & (\mathcal{T} \cap \mathcal{U})/\mathcal{W} \\
& F_\delta & & G_\lambda \\
\mathcal{V}/\mathcal{W} & F & \mathcal{T}/\mathcal{W} & G' & \mathcal{Y}/\mathcal{W} \\
& F_\delta & & G'_\lambda \\
\end{array}
\]

(1.2)

where \( F, G_\lambda, \) and \( G_\delta \) are the full embeddings.

Our main result generalizes Chen-Liu-Yang’s results on a triangulated category and is new for an exact category case. In particular, applying our main results to complex categories, we reobtain Krause’s recollement. In fact, without considering homotopy categories, we reprove the existence of Krause’s recollement by our main results.

This article is organized as follows. In Section 2, we review some basic concepts and results concerning extriangulated categories. In Section 3, we show our main results. In Section 4, we give some applications to illustrate our main results.
2 Preliminaries

Let us briefly recall some definitions and basic properties of extriangulated categories from [11]. We omit some details here, but the reader can find them in [11].

Let $\mathcal{C}$ be an additive category equipped with an additive bifunctor $E: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}$, where Ab is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in E(C, A)$ is called an $E$-extension. Let $s$ be a correspondence which associates an equivalence class

$$s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$$

to any $E$-extension $\delta \in E(C, A)$. This $s$ is called a realization of $E$, if it makes the diagrams in [11, Definition 2.9] commutative. A triplet $(\mathcal{C}, E, s)$ is called an extriangulated category if it satisfies the following conditions:

1. $E: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}$ is an additive bifunctor;
2. $s$ is an additive realization of $E$;
3. $E$ and $s$ satisfy the compatibility conditions in [11, Definition 2.12].

Remark 1 Note that both exact categories and triangulated categories are extriangulated categories, see [11, Example 2.13] and extension closed subcategories of triangulated categories are again extriangulated, see [11, Remark 2.18]. Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories, see [11, Proposition 3.30] and [14, Example 4.14].

We will use the following terminology.

Definition 1 [11] Let $(\mathcal{C}, E, s)$ be an extriangulated category.

1. A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a conflation if it realizes some $E$-extension $\delta \in E(C, A)$. In this case, $x$ is called an inflation and $y$ is called a deflation.
2. If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in E(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $E$-triangle, and write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$$

We usually do not write this ‘$\delta$’ if it is not used in the argument.

3. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}$ be any pair of $E$-triangles. If a triplet $(a, b, c)$ realizes $(a, c): \delta \to \delta'$, then we write it as

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$$

$$\downarrow a \quad \downarrow b \quad \downarrow c$$

$$A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}$$
and call \((a, b, c)\) a morphism of \(\mathbb{E}\)-triangles.

(4) An object \(P \in \mathcal{C}\) is called projective if for any \(\mathbb{E}\)-triangle \(x \rightarrow B \rightarrow C \rightarrow \delta\) and any morphism \(c \in \mathcal{C}(P, C)\), there exists \(b \in \mathcal{C}(P, B)\) satisfying \(yb = c\). We denote the subcategory of projective objects by \(\mathcal{P} \subseteq \mathcal{C}\). Dually, the subcategory of injective objects is denoted by \(\mathcal{I} \subseteq \mathcal{C}\).

(5) We say that \(\mathcal{C}\) has enough projective objects if for any object \(C \in \mathcal{C}\), there exists an \(\mathbb{E}\)-triangle \(x \rightarrow P \rightarrow C \rightarrow \delta\) satisfying \(P \in \mathcal{P}\). We can define the notion of having enough injectives dually.

Assume that \((\mathcal{C}, \mathbb{E}, s)\) is an extriangulated category. By Yoneda’s lemma, any \(\mathbb{E}\)-extension \(\delta \in \mathbb{E}(C, A)\) induces natural transformations
\[
\delta_x^\#: \mathcal{C}(\_ , C) \Rightarrow \mathbb{E}(\_ , A), \quad \delta^\#: \mathcal{C}(A, \_ ) \Rightarrow \mathbb{E}(C, \_ )
\]
For any \(X \in \mathcal{C}\), these \((\delta_x^\#)_X\) and \(\delta^\#_X\) are given as follows:

1. \((\delta_x^\#)_X : \mathcal{C}(X, C) \rightarrow \mathbb{E}(X, A), f \mapsto f^*\delta;\)
2. \(\delta^\#_X : \mathcal{C}(A, X) \rightarrow \mathbb{E}(C, X), g \mapsto g_*\delta.\)

**Lemma 1** Let \((\mathcal{C}, \mathbb{E}, s)\) be an extriangulated category, and \(x \rightarrow B \rightarrow C \rightarrow \delta\) be an \(\mathbb{E}\)-triangle. Then we have the following long exact sequences:
\[
\mathcal{C}(-, A) \xrightarrow{\mathbb{E}(\_ , x)} \mathcal{C}(-, B) \xrightarrow{\mathbb{E}(\_ , y)} \mathcal{C}(-, C) \xrightarrow{\delta^\#} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(\_ , x)} \mathbb{E}(\_ , B) \xrightarrow{\mathbb{E}(\_ , y)} \mathbb{E}(\_ , C);
\]
\[
\mathbb{E}(C, \_ ) \xrightarrow{\mathbb{E}(y, \_ )} \mathbb{E}(B, \_ ) \xrightarrow{\mathbb{E}(x, \_ )} \mathbb{E}(A, \_ ) \xrightarrow{\delta^\#} \mathbb{E}(C, \_ ) \xrightarrow{\mathbb{E}(y, \_ )} \mathbb{E}(B, \_ ) \xrightarrow{\mathbb{E}(x, \_ )} \mathbb{E}(A, \_ )
\]

**Proof** It follows from [11, Propositions 3.3, 3.11].

Now, we recall higher extension groups from [9].

Assume that \((\mathcal{C}, \mathbb{E}, s)\) is an extriangulated category with enough projectives and enough injectives. For two subcategories \(\mathcal{X}, \mathcal{Y}\) of \(\mathcal{C}\), \(\text{Cone}(\mathcal{X}, \mathcal{Y})\) is defined to be the subcategory of \(\mathcal{C}\), consisting of objects \(C\) which admits an \(\mathbb{E}\)-triangle \(x \rightarrow Y \rightarrow C \rightarrow \delta\), where \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\). Dually, we can define \(\text{CoCone}(\mathcal{X}, \mathcal{Y})\).

For a subcategory \(\mathcal{C}_1 \subseteq \mathcal{C}\), put \(\Omega^0\mathcal{C}_1 = \mathcal{C}_1\), and for \(i > 0\), define \(\Omega^i\mathcal{C}_1\) inductively by
\[
\Omega^i\mathcal{C}_1 = \Omega(\Omega^{i-1}\mathcal{C}_1) = \text{CoCone}(\mathcal{P}, \Omega^{i-1}\mathcal{C}_1).
\]
We call \(\Omega^i\mathcal{C}_1\) the \(i\)-th syzygy of \(\mathcal{C}_1\). Dually, we define the \(i\)-th cosyzygy \(\Sigma^i\mathcal{C}_1\) by
\[
\Sigma^0\mathcal{C}_1 = \mathcal{C}_1, \quad \Sigma^i\mathcal{C}_1 = \text{Cone}(\Sigma^{i-1}\mathcal{C}_1, \mathcal{X}), \quad i > 0.
\]

Let \(X\) be any object in \(\mathcal{C}\). It admits an \(\mathbb{E}\)-triangle
\[
X \rightarrow I^0 \rightarrow \Sigma X \rightarrow \delta \quad (\text{resp., } \Omega X \rightarrow P_0 \rightarrow X \rightarrow \delta),
\]
Recollements arising from cotorsion pairs on extriangulated categories

where $I^0 \in \mathcal{I}$ (resp., $P_0 \in \mathcal{P}$). We can obtain $E$-triangles

$$\Sigma^i X \to I^i \to \Sigma^{i+1} X \longrightarrow \text{(resp., } \Omega^{i+1} \to P_i \to \Omega^i X \longrightarrow)$$

for $i > 0$ recursively.

Liu and Nakaoka [9] defined higher extension groups as

$$E^{i+1}(X, Y) := E(X, \Sigma^i Y) \cong E(\Omega^i X, Y)$$

for $i \geq 0$, and proved the following result.

**Lemma 2** [9, Proposition 5.2]  Let $(\mathcal{C}, E, \mathcal{s})$ be an extriangulated category with enough projectives and enough injectives, and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\delta}$ be an $E$-triangle. Then, for $i \geq 1$, we have the following long exact sequences:

$$\cdots \longrightarrow E^i(-, A) \longrightarrow E^i(-, B) \longrightarrow E^i(-, C) \longrightarrow \cdots$$

$$\cdots \longrightarrow E^{i+1}(-, A) \longrightarrow E^{i+1}(-, B) \longrightarrow E^{i+1}(-, C) \longrightarrow \cdots,$n

$$\cdots \longrightarrow E^i(C, -) \longrightarrow E^i(B, -) \longrightarrow E^i(A, -) \longrightarrow \cdots$$

$$\cdots \longrightarrow E^{i+1}(C, -) \longrightarrow E^{i+1}(B, -) \longrightarrow E^{i+1}(A, -) \longrightarrow \cdots.$$n

**Remark 2** [15, Lemma 2.14] Let $(\mathcal{C}, E, \mathcal{s})$ be an extriangulated category with enough projectives and enough injectives. Then

(a) $P$ is a projective object in $\mathcal{C}$ if and only if $E^i(P, C) = 0$ for any $C \in \mathcal{C}$ and $i \geq 1$;

(b) $I$ is an injective object in $\mathcal{C}$ if and only if $E^i(C, I) = 0$ for any $C \in \mathcal{C}$ and $i \geq 1$.

Let $\mathcal{C}$ be an additive category. For two objects $A, B$ in $\mathcal{X}$, denote by $\mathcal{X}(A, B)$ the subgroup of $\text{Hom}_\mathcal{C}(A, B)$ consisting of those morphisms which factor through an object in $\mathcal{X}$. Denote by $\mathcal{C}/\mathcal{X}$ the quotient category of $\mathcal{C}$ modulo $\mathcal{X}$: the objects are the same as the ones in $\mathcal{C}$, for the Hom space of objects $A$ and $B$ is given by the quotient group $\text{Hom}_\mathcal{C}(A, B)/\mathcal{X}(A, B)$. Note that the quotient category $\mathcal{C}/\mathcal{X}$ is an additive category. We denote $\bar{f}$ the image of $f: A \to B$ of $\mathcal{C}$ in $\mathcal{C}/\mathcal{X}$.

**Remark 3** If $\mathcal{X}$ is closed under direct summands, then, for any $C \in \mathcal{C}$, we have $C \cong 0$ in $\mathcal{C}/\mathcal{X}$ if and only if $C \in \mathcal{X}$.

**Definition 2** [11, Definition 4.1] Assume that $(\mathcal{C}, E, \mathcal{s})$ is an extriangulated category. Let $\mathcal{U}$ and $\mathcal{V}$ be two subcategories of $\mathcal{C}$. We call $(\mathcal{U}, \mathcal{V})$ a cotorsion pair if it satisfies the following conditions:

(a) $E(\mathcal{U}, \mathcal{V}) = 0$;
(b) for any object \( C \in \mathcal{C} \), there are two \( \mathcal{E} \)-triangles
\[
V_C \rightarrow U_C \rightarrow C \rightarrow \cdots, \quad C \rightarrow V^C \rightarrow U^C \rightarrow \cdots,
\]
satisfying \( U_C, U^C \in \mathcal{U} \) and \( V_C, V^C \in \mathcal{V} \).

By definition of the cotorsion pair, we can immediately conclude the following remark.

**Remark 4** Assume that \((\mathcal{C},\mathcal{E},\mathfrak{s})\) is an extriangulated category. Let \((\mathcal{U},\mathcal{V})\) be a cotorsion pair on \( \mathcal{C} \). Then

(a) \( C \) belongs to \( \mathcal{U} \) if and only if \( \mathcal{E}(C,\mathcal{V}) = 0 \);
(b) \( C \) belongs to \( \mathcal{V} \) if and only if \( \mathcal{E}(\mathcal{U},C) = 0 \);
(c) \( \mathcal{U} \) and \( \mathcal{V} \) are closed under direct sums, direct summands and extensions.

**Definition 3** \([11, \text{Definition 4.12}]\) Assume that \((\mathcal{C},\mathcal{E},\mathfrak{s})\) is an extriangulated category. Let \((\mathcal{S},\mathcal{T})\) and \((\mathcal{U},\mathcal{V})\) be cotorsion pairs on \( \mathcal{C} \). Then the pair \( \text{TCP} := ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) \) is called a *twin cotorsion pair* if it satisfies \( \mathcal{E}(\mathcal{S},\mathcal{V}) = 0 \). Note that this condition is equivalent to \( \mathcal{S} \subseteq \mathcal{U} \), and also to \( \mathcal{V} \subseteq \mathcal{T} \). If, moreover, it satisfies \( \mathcal{S} \cap \mathcal{T} = \mathcal{U} \cap \mathcal{V} \), then \( \text{TCP} \) is called a *concentric twin cotorsion pair*.

**Lemma 3** Assume that \((\mathcal{C},\mathcal{E},\mathfrak{s})\) is an extriangulated category. Let \((\mathcal{U},\mathcal{V})\) be a cotorsion pair on \( \mathcal{C} \) and \( W := \mathcal{U} \cap \mathcal{V} \). Then \( \mathcal{C} / W \) is a cotorsion pair.

**Proof** Let \( f \in \mathcal{C}(U,V) \) be any morphism, where \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \). By the definition of a cotorsion pair, there exists an \( \mathcal{E} \)-triangle \( V' \xrightarrow{x} U' \xrightarrow{y} V \xrightarrow{\delta} \), where \( V' \in \mathcal{V} \) and \( U' \in \mathcal{U} \). Since \( \mathcal{V} \) is closed under extensions, we have \( U' \in \mathcal{V} \) and then \( U' \in \mathcal{U} \cap \mathcal{V} = W \). Applying the functor \( \mathcal{C}(U,-) \) to the \( \mathcal{E} \)-triangle \( V' \xrightarrow{x} U' \xrightarrow{y} V \xrightarrow{\delta} \), by Lemma 1, we have the exact sequence
\[
\mathcal{C}(U,U') \xrightarrow{C(U,y)} \mathcal{C}(U,V) \rightarrow \mathcal{E}(U,V') = 0.
\]
It follows that there exists a morphism \( a: U \rightarrow U' \) such that \( f = y \circ a \). Since \( U' \in W \), this means \( f = 0 \). \( \square \)

**Lemma 4** Assume that \((\mathcal{C},\mathcal{E},\mathfrak{s})\) is an extriangulated category. Let \((\mathcal{U},\mathcal{V})\) be a cotorsion pair on \( \mathcal{C} \) and \( W := \mathcal{U} \cap \mathcal{V} \), and let \( f: A \rightarrow B \) be any morphism in \( \mathcal{C} \).

(1) Let
\[
V_A \rightarrow U_A \xrightarrow{u_A} A \rightarrow \cdots, \quad V_B \rightarrow U_B \xrightarrow{u_B} B \rightarrow \cdots,
\]
be any two \( \mathcal{E} \)-triangles satisfying \( U_A, U_B \in \mathcal{U} \) and \( V_A, V_B \in \mathcal{V} \). Then there exists a morphism \( f_U \in \mathcal{C}(U_A, U_B) \) such that
\[
f \circ u_A = u_B \circ f_U,
\]
Moreover, $f_U$ with this property is unique in $(\mathcal{C}/\mathcal{W})(U_A, U_B)$.

(2) Dually, for any two $\mathcal{E}$-triangles

$$A \rightarrow V'_A \rightarrow U'_A \rightarrow, \quad B \rightarrow V'_B \rightarrow U'_B \rightarrow,$$

with $U'_A, U'_B \in \mathcal{U}$ and $V'_A, V'_B \in \mathcal{V}$, there exists a morphism $f'_V \in \mathcal{C}(V'_A, V'_B)$ compatible with $f$, uniquely up to $\mathcal{W}$.

Proof We only show (1). The existence immediately follows from $\mathcal{E}(U_A, V_B) = 0$. Moreover, if $f^1_U$ and $f^2_U$ in $\mathcal{C}(U_A, U_B)$ satisfy

$$u_B \circ f^1_U = f \circ u_A = u_B \circ f^2_U,$$

then, by $u_B \circ (f^1_U - f^2_U) = 0$, there exists $w \in \mathcal{C}(U_A, V_B)$ such that $f^1_U - f^2_U$ factors through $w$:

By Lemma 3, we have $\overline{w} = 0$, and then $\overline{f^1_U} = \overline{f^2_U}$. □

Lemma 5 Assume that $(\mathcal{C}, \mathcal{E}, \mathcal{s})$ is an extriangulated category. Let $(\mathcal{W}, \mathcal{V})$ be a cotorsion pair on $\mathcal{C}$, $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$, and $C$ be any object in $\mathcal{C}$.

(1) For any two $\mathcal{E}$-triangles

$$V \rightarrow U \xrightarrow{u} C \rightarrow, \quad V' \rightarrow U' \xrightarrow{u'} C \rightarrow,$$

satisfying $U, U' \in \mathcal{U}$ and $V, V' \in \mathcal{V}$, there exists a morphism $s \in \mathcal{C}(U, U')$ compatible with $u$ and $u'$, such that $\mathcal{s}$ is an isomorphism:

(2) Dually, those $V$ appearing in $\mathcal{E}$-triangles $C \rightarrow V \rightarrow U \rightarrow$, where $U \in \mathcal{U}, V \in \mathcal{V}$, are isomorphic in $\mathcal{C}/\mathcal{W}$.

Proof It immediately follows from Lemma 4. □
3 Recollements of additive categories

In this section, we will prove our main results. First, we need to do some preparations as follows.

**Definition 4** [12, Definition 2.1] and [3, Definition 3.1] Let \( \mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}'' \) be a sequence of additive functors between additive categories. We say that it is a *localization sequence* if the following conditions hold:

- (L1) the functor \( F \) is fully faithful and has a right adjoint \( F_\delta \);
- (L2) the functor \( G \) has a fully faithful right adjoint \( G_\delta \);
- (L3) there exists an equality of additive subcategories \( \text{Im } F = \text{Ker } G \), where 

\[
\text{Im } F = \{ A \in \mathcal{A} \mid A \cong F(X) \text{ for some } X \in \mathcal{A}' \},
\]
\[
\text{Ker } G = \{ A \in \mathcal{A} \mid G(A) = 0 \text{ in } \mathcal{A}'' \}.
\]

A *colocalization sequence* of additive categories is defined dually. A sequence of additive categories \( \mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}'' \) is called a *recollement* if it is both a localization sequence and a colocalization sequence.

From now on, we assume that \( (\mathcal{C}, \mathcal{E}, s) \) is an extriangulated category with enough projective and injective objects.

**Definition 5** [2, Definition 3.1.1] Let \( \mathcal{A} \) and \( \mathcal{B} \) be additive categories and \( F: \mathcal{A} \to \mathcal{B} \) a functor. For any \( B \in \mathcal{B} \), a *reflection* of \( B \) along \( F \) is a pair \((Q_B, \eta_B)\) of \( Q_B \in \mathcal{A} \) and \( \eta_B \in \mathcal{B}(B, F(Q_B)) \), satisfying the universality that for any \( A \in \mathcal{A} \) and any \( b \in \mathcal{B}(B, F(A)) \), there exists a unique morphism \( a \in \mathcal{A}(Q_B, A) \) such that \( F(a) \circ \eta_B = b \):

\[
\begin{array}{ccc}
B & \xrightarrow{\eta_B} & F(Q_B) \\
\downarrow{b} & & \downarrow{F(a)} \\
F(A) & & 
\end{array}
\]

A *coreflection* is defined dually.

**Lemma 6** Let \( ((\mathcal{I}, \mathcal{F}), (\mathcal{U}, \mathcal{V})) \) be a concentric twin cotorsion pair on \( \mathcal{C} \) with \( \mathcal{U} = \mathcal{I} \cap \mathcal{F} \). Then the inclusion \( \mathcal{V}/\mathcal{U} \hookrightarrow \mathcal{I}/\mathcal{U} \) admits an additive left adjoint \( F_\lambda: \mathcal{I}/\mathcal{U} \to \mathcal{V}/\mathcal{U} \). Indeed, for any \( T \in \mathcal{I} \) and \( \mathcal{E}-\text{triangle}, \)

\[
T \xrightarrow{v_T} V_T \to U_T \to 
\]

with \( U_T \in \mathcal{U} \) and \( V_T \in \mathcal{V} \), \( \overline{v_T}: T \to V_T \) gives a reflection of \( T \) along the inclusion \( \mathcal{V}/\mathcal{U} \hookrightarrow \mathcal{I}/\mathcal{U} \), where \( F_\lambda(T) = V_T \).

**Proof** It is enough to prove that for any \( T \in \mathcal{I}, V \in \mathcal{V}, \) and \( f \in \mathcal{C}(T, V) \), there exists a morphism \( g \in \mathcal{C}(V_T, V) \) such that \( f = g \circ v_T \), uniquely in \( (\mathcal{C}/\mathcal{U})(V_T, V) \).
Applying the functor \( \mathcal{C}(-, V) \) to the \( \mathbb{E} \)-triangle \( T \overset{v_T}{\rightarrow} V_T \rightarrow U_T \rightarrow \), we have the following exact sequence:

\[
\mathcal{C}(V_T, V) \xrightarrow{\mathcal{C}(v_T, V)} \mathcal{C}(T, V) \rightarrow \mathbb{E}(U_T, V) = 0.
\]

Thus, there exists a morphism \( g \in \mathcal{C}(V_T, V) \) such that \( f = g \circ v_T \) and then \( \overline{f} = \overline{g} \circ \overline{v_T} \).

To prove the uniqueness, suppose that there exists a morphism \( g \in \mathcal{C}(V_T, V) \) such that \( (g - g') \circ v_T = 0 \), that is, \((g - g') \circ v_T\) factors through some \( W_0 \in \mathcal{W} \). Let \((g - g') \circ v_T = b \circ a\) with \( a: T \rightarrow W_0 \) and \( b: W_0 \rightarrow V \).

Applying the functor \( \mathcal{C}(-, W_0) \) to the \( \mathbb{E} \)-triangle \( T \overset{v_T}{\rightarrow} V_T \rightarrow U_T \rightarrow \), we have the following exact sequence:

\[
\mathcal{C}(V_T, W_0) \xrightarrow{\mathcal{C}(v_T, W_0)} \mathcal{C}(T, W_0) \rightarrow \mathbb{E}(U_T, W_0) = 0.
\]

Thus, there exists a morphism \( c \in \mathcal{C}(V_T, W_0) \) such that \( a = c \circ v_T \) and then \((g - g' - b \circ c) \circ v_T = 0\). By Lemma 1, there exists a morphism \( d \in \mathcal{C}(U_T, V) \) such that \((g - g') - b \circ c = d \circ e\). It follows that \( g - g' = d \circ e \). By Lemma 3, we have \((\mathcal{C}/\mathcal{W})(U_T, V) = 0\). It implies \( g = g' \). Therefore, \( v_T: T \rightarrow V_T \) gives a reflection of \( T \) along the inclusion \( \mathcal{V}/\mathcal{W} \hookrightarrow \mathcal{I}/\mathcal{W} \). Namely, \( F_\lambda: \mathcal{I}/\mathcal{W} \rightarrow \mathcal{V}/\mathcal{W} \) is left adjoint to the inclusion \( \mathcal{V}/\mathcal{W} \rightarrow \mathcal{I}/\mathcal{W} \).

**Lemma 7** Let \(((\mathcal{I}, \mathcal{F}), (\mathcal{W}, \mathcal{V}))\) be a concentric twin cotorsion pair on \( \mathcal{C} \) with \( \mathcal{W} = \mathcal{I} \cap \mathcal{F} \). If \( \Sigma \mathcal{V} \subseteq \mathcal{V} \), then there exists the following colocalization sequence of additive categories:

\[
\mathcal{V}/\mathcal{W} \xrightarrow{F_\lambda} \mathcal{I}/\mathcal{W} \xrightarrow{G_\lambda} (\mathcal{I} \cap \mathcal{W})/\mathcal{W},
\]

where \( F \) and \( G_\lambda \) are the full embeddings.

**Proof** Assume that \( F: \mathcal{V}/\mathcal{W} \rightarrow \mathcal{I}/\mathcal{W} \) is the full embedding. We define \( G: \mathcal{I}/\mathcal{W} \rightarrow (\mathcal{I} \cap \mathcal{W})/\mathcal{W} \) in the following way. For any \( T \in \mathcal{I} \), \( G(T) = Z_T \) appearing in an \( \mathbb{E} \)-triangle \( V_T \rightarrow Z_T \rightarrow T \rightarrow \) with \( Z_T \in \mathcal{I} \cap \mathcal{W} \) and \( V_T \in \mathcal{V} \). By [13, Definition 3.13], we know that \( G \) is an additive functor and a right adjoint of the inclusion \((\mathcal{I} \cap \mathcal{W})/\mathcal{W} \rightarrow \mathcal{I}/\mathcal{W} \), denoted by \( G_\lambda \). We claim that \( \text{Im} F = \text{Ker} G \). It is easy to see that \( \text{Im} F = \mathcal{V} \) since \( F \) is the full embedding. For any \( V \in \mathcal{V} \), there exists an \( \mathbb{E} \)-triangle \( V_0 \rightarrow W_0 \rightarrow V \rightarrow \) with \( W_0 \in \mathcal{W} \) and \( V_0 \in \mathcal{V} \). Hence, \( G(V) = W_0 \) in \((\mathcal{I} \cap \mathcal{W})/\mathcal{W} \). So we know that \( V \in \text{Ker} G \) by Remark 3. Then \( \text{Im} F \subseteq \text{Ker} G \). On the other hand, for any \( T \in \text{Ker} G \), there exists an \( \mathbb{E} \)-triangle \( V_T \rightarrow Z_T \rightarrow T \rightarrow \) with \( Z_T \in \mathcal{W} \) and \( V_T \in \mathcal{V} \).

Since \( \Sigma \mathcal{V} \subseteq \mathcal{V} \), we have \( \mathbb{E}(U, \Sigma V_T) = 0 \). Applying the functor \( \mathcal{C}(\mathcal{W}, -) \) to the \( \mathbb{E} \)-triangle \( T \overset{v_T}{\rightarrow} V_T \rightarrow U_T \rightarrow \), by Lemma 2, we obtain the following exact sequence:

\[
\mathbb{E}(\mathcal{W}, Z_T) = 0 \rightarrow \mathbb{E}(\mathcal{W}, T) \rightarrow \mathbb{E}(\mathcal{W}, \Sigma V_T) = 0.
\]
It follows that $E(\mathcal{U}, T) = 0$ and then $T \in \mathcal{V}$. Thus, $\text{Im} F = \text{Ker} G$. Let $F_\lambda$ be as in Lemma 6. Then $(F_\lambda, F)$ is an adjoint pair. This completes the proof. □

**Lemma 8** Let $(\mathcal{V}, \mathcal{V})$ be a cotorsion pair and $((\mathcal{I}, \mathcal{F}), (\mathcal{X}, \mathcal{Y}))$ a concentric twin cotorsion pair on $C$ with $\mathcal{W} = \mathcal{I} \cap \mathcal{F}$. If $\mathcal{I} \cap \mathcal{X} = \mathcal{V}$, then the inclusion $\mathcal{V}/\mathcal{W} \to \mathcal{F}/\mathcal{W}$ admits an additive right adjoint functor $F_\delta: \mathcal{F}/\mathcal{W} \to \mathcal{V}/\mathcal{W}$. Indeed, for any $T \in \mathcal{F}$ and $E$-triangle

$$Y_T \to V_T \xrightarrow{\pi_T} T \rightarrow$$

with $Y_T \in \mathcal{Y}$ and $V_T \in \mathcal{V}$, $\pi_T: V_T \to T$ gives a coreflection of $T$ along the inclusion $\mathcal{V}/\mathcal{W} \hookrightarrow \mathcal{F}/\mathcal{W}$, where $F_\delta(T) = V_T$.

**Proof** For any $T \in \mathcal{F}$, there exists an $E$-triangle (3.1) with $Y_T \in \mathcal{Y}$ and $V_T \in \mathcal{X}$. Since $\mathcal{Y} \subseteq \mathcal{F}$ and $\mathcal{F}$ is closed under extensions, we have $V_T \in \mathcal{F}$. Thus, $V_T \in \mathcal{X} \cap \mathcal{F} = \mathcal{V}$. The remaining proof is similar to that of Lemma 6. □

**Lemma 9** Let $((\mathcal{I}, \mathcal{F}), (\mathcal{X}, \mathcal{Y}))$ be a concentric twin cotorsion pair on $C$ with $\mathcal{W} = \mathcal{I} \cap \mathcal{F}$. Then the inclusion $\mathcal{Y}/\mathcal{W} \hookrightarrow \mathcal{F}/\mathcal{W}$ admits an additive left adjoint $G': \mathcal{F}/\mathcal{W} \to \mathcal{Y}/\mathcal{W}$. Indeed, for any $T \in \mathcal{F}$ and $E$-triangle

$$T \xrightarrow{\gamma_T} Y_T \to X_T \rightarrow$$

with $Y_T \in \mathcal{Y}$ and $X_T \in \mathcal{X}$, $\gamma_T: T \to Y_T$ gives a reflection of $T$ along the inclusion $\mathcal{Y}/\mathcal{W} \hookrightarrow \mathcal{F}/\mathcal{W}$, where $G'(T) = Y_T$.

**Proof** The proof is similar to that of Lemma 6. □

**Lemma 10** Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair and $((\mathcal{I}, \mathcal{F}), (\mathcal{X}, \mathcal{Y}))$ a concentric twin cotorsion pair on $C$ with $\mathcal{W} = \mathcal{I} \cap \mathcal{F}$. If $\mathcal{I} \cap \mathcal{X} = \mathcal{V}$ and $\Sigma\mathcal{W} \subseteq \mathcal{Y}$, then there exists the following localization sequence of additive categories:

$$\begin{array}{ccc}
\mathcal{V}/\mathcal{W} & \xrightarrow{F} & \mathcal{F}/\mathcal{W} \\
\downarrow & & \downarrow \\
\mathcal{Y}/\mathcal{W} & \xrightarrow{G'} & \mathcal{X}/\mathcal{W} \\
\end{array}$$

where $F$ and $G'$ are the full embeddings.

**Proof** Suppose that $F: \mathcal{V}/\mathcal{W} \to \mathcal{F}/\mathcal{W}$ and $G': \mathcal{Y}/\mathcal{W} \to \mathcal{X}/\mathcal{W}$ are the full embeddings. Let $F_\delta$ and $G'$ be as in Lemmas 8 and 9, respectively. Then $(F, F_\delta)$ and $(G', G_\delta')$ are adjoint pairs. Next, we claim that $\text{Im} F = \text{Ker} G'$. Obviously, $\text{Im} F = \mathcal{V}$. For any $V \in \mathcal{V}$, there exists an $E$-triangle

$$V \xrightarrow{Y_V} Y_V \to X_V \rightarrow$$

with $Y_V \in \mathcal{Y}$ and $X_V \in \mathcal{X}$. Since $\mathcal{I} \cap \mathcal{X} = \mathcal{V}$ and $\mathcal{X}$ is closed under extensions, $Y_V \in \mathcal{Y} \cap \mathcal{X} = \mathcal{W}$. Since $G'(V) = Y_V$, $V \in \text{Ker} G'$ by Remark 3. Then $\text{Im} F \subseteq \text{Ker} G'$. On the other hand, for any $T \in \text{Ker} G'$, $G'(T) = Y_T$ appearing in an $E$-triangle

$$T \to Y_T \to X_T \rightarrow$$
Recollements arising from cotorsion pairs on extriangulated categories

with \( Y_T \in \mathcal{Y} \) and \( X_T \in \mathcal{X} \). Then \( Y_T \in \mathcal{W} \).

Applying the functor \( \mathcal{E}(-, \mathcal{Y}) \) to the \( \mathcal{E} \)-triangle \( T \to Y_T \to X_T \to \), by Lemma 2, we have the following exact sequence:

\[
\mathbb{E}(Y_T, \mathcal{Y}) = 0 \to \mathbb{E}(T, \mathcal{Y}) \to \mathbb{E}(X_T, \Sigma \mathcal{Y}) = 0.
\]

It follows that \( \mathbb{E}(T, \mathcal{Y}) = 0 \) and then \( T \in \mathcal{X} \). Thus, \( T \in \mathcal{T} \cap \mathcal{X} = \mathcal{Y} \), namely, \( T \in \text{Im} \mathcal{F} \). So \( \text{Im} \mathcal{F} = \text{Ker} \mathcal{G}' \). This completes the proof. \( \square \)

**Proof of Theorem 1** Define

\[ J: (\mathcal{T} \cap \mathcal{U})/\mathcal{W} \to \mathcal{Y}/\mathcal{W} \]

in the following way. For any \( M \in \mathcal{T} \cap \mathcal{U} \), \( J(M) = Y_M \) appears in an \( \mathcal{E} \)-triangle

\[ M \xrightarrow{y_M} Y_M \to X_M \to \]

with \( Y_M \in \mathcal{Y} \) and \( X_M \in \mathcal{X} \). Applying the functor \( \mathcal{E}(-, Y_{M'}) \) to the above \( \mathcal{E} \)-triangle, we have the following exact sequence:

\[ \mathcal{E}(Y_M, Y_{M'}) \xrightarrow{\mathcal{E}(y_M, W_0)} \mathcal{E}(M, Y_{M'}) \to \mathbb{E}(X_M, Y_{M'}) = 0. \]

Thus, for any \( f \in (\mathcal{T} \cap \mathcal{U})/\mathcal{W})(M, M') \), there is a morphism \( f_M \in \mathcal{E}(Y_M, Y_{M'}) \) such that \( y_{M'} \circ f = f_M \circ y_M \). Hence, we have the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{y_M} & Y_M \\
\downarrow{f} & & \downarrow{f_M} \\
M' & \xrightarrow{y_{M'}} & Y_{M'}
\end{array}
\]

of \( \mathcal{E} \)-triangles. Define \( J(f) = f_M \). By Lemmas 4 and 5, \( J \) is well defined and is an additive functor.

Define

\[ K: \mathcal{Y}/\mathcal{W} \to (\mathcal{T} \cap \mathcal{U})/\mathcal{W} \]

in the following way. For any \( Y \in \mathcal{Y} \), \( K(Y) = U_Y \) appears in an \( \mathcal{E} \)-triangle

\[ V_Y \to U_Y \xrightarrow{u_Y} Y \to \]

with \( V_Y \in \mathcal{V} \) and \( U_Y \in \mathcal{U} \). Since \( \mathcal{V} \subseteq \mathcal{T} \), \( \mathcal{V} \subseteq \mathcal{T} \), and \( \mathcal{T} \) is closed under extensions, \( U_Y \in \mathcal{T} \cap \mathcal{V} \). For any \( g \in (\mathcal{V}/\mathcal{W})(Y, Y') \), since \( \mathbb{E}(U_Y, V_{Y'}) = 0 \), there exists a morphism \( g_Y \in \mathcal{E}(U_Y, U_{Y'}) \) such that \( u_{Y'} \circ g_Y = g \circ u_Y \). Hence, we have the commutative diagram

\[
\begin{array}{ccc}
V_Y & \to & U_Y \\
\downarrow{g_Y} & & \downarrow{g} \\
V_{Y'} & \to & U_{Y'}
\end{array}
\]
of $E$-triangles. Define $K(g) = g_Y$. By Lemmas 4 and 5, $K$ is well defined and is an additive functor. Now, we have the following diagram:

$$
\begin{array}{cccccc}
V/W & F & G & (T \cap U)/W \\
\downarrow{id} & \downarrow{id} & \downarrow{id} & \downarrow{J} & \downarrow{K} \\
V/W & F & G & (T \cap U)/W \\
\end{array}
$$

For any $Y \in \mathcal{Y}$, since $0 \rightarrow Y \rightarrow Y \rightarrow Y$ is an $E$-triangle, we have $G'(Y) = Y$. So $G'$ is dense. Then there exists a natural isomorphism from the composite functor

$$
\begin{array}{c}
\mathcal{Y}/W \xrightarrow{G'} \mathcal{Y}/W \xrightarrow{K} (T \cap U)/W
\end{array}
$$

to the functor

$$
\begin{array}{c}
\mathcal{Y}/W \xrightarrow{K} (T \cap U)/W.
\end{array}
$$

Consequently, we know that the composite functor

$$
\begin{array}{c}
\mathcal{Y}/W \xrightarrow{K} (T \cap U)/W \xrightarrow{G'} T/W
\end{array}
$$

is indeed left adjoint to $\mathcal{T}/W \xrightarrow{G'} \mathcal{Y}/W$. Put $G'_\lambda = G\lambda K$. Then we have (1.3).

Similarly, for any $M \in (T \cap U)$, $G(M) = M$ since $0 \rightarrow W \rightarrow W \rightarrow W$ is an $E$-triangle. So $G$ is dense. Then there exists a natural isomorphism from the composite functor

$$
\begin{array}{c}
\mathcal{T}/W \xrightarrow{G} (T \cap U)/W \xrightarrow{J} \mathcal{Y}/W
\end{array}
$$

to the functor

$$
\begin{array}{c}
(T \cap U)/W \xrightarrow{J} \mathcal{Y}/W.
\end{array}
$$

Thus,

$$
\begin{array}{c}
(T \cap U)/W \xrightarrow{J} \mathcal{Y}/W \xrightarrow{G'_\delta} \mathcal{T}/W
\end{array}
$$

is indeed right adjoint to $\mathcal{T}/W \xrightarrow{G} (T \cap U)/W$. Put $G_\delta = G'_\delta J$. Then we obtain (1.2).

□

We give the dual result of Theorem 1, but omit its proof.

**Theorem 2** Let $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$, and $(\mathcal{X}, \mathcal{Y})$ be cotorsion pairs on $C$ with (1.1). If

$$
\begin{array}{c}
\mathcal{Y} \cap \mathcal{S} = \mathcal{U}, \quad \mathcal{X} \subseteq \mathcal{S}, \quad \Sigma \mathcal{V} \subseteq \mathcal{V}, \quad \Sigma \mathcal{Y} \subseteq \mathcal{Y},
\end{array}
$$

then $\mathcal{Y}$ is the cotorsion class of $\mathcal{X}$.
then there are two recollements of additive categories as follows:

\[
\begin{array}{ccc}
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{I_\lambda} & \mathcal{I} \setminus \mathcal{W} \\
\angle & & \angle \\
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{I_\delta} & \mathcal{I} \setminus \mathcal{W} \\
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{J_\lambda} & (\mathcal{I} \cap \mathcal{W}) \setminus \mathcal{W} \\
\angle & & \angle \\
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{I_\delta} & \mathcal{I} \setminus \mathcal{W} \\
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{J_\delta} & \mathcal{J} \setminus \mathcal{W} \\
\mathcal{U} \setminus \mathcal{W} & \xrightarrow{J'_\delta} & \mathcal{X} \setminus \mathcal{W} ,
\end{array}
\]

where \( I, J_\delta, \) and \( J'_\delta \) are the full embeddings.

Note that a triangulated category can be viewed as an extriangulated category with enough projective and injective objects.

**Remark 5** When we apply Theorem 1 to a triangulated category, it is just [3, Theorem 3.2]. When we apply Theorem 2 to a triangulated category, it is just [3, Theorem 3.9].

### 4 Applications

We assume that \( R \) is a ring with unit. We denote by \( \text{Mod} \, R \) the category of left \( R \)-modules. Unless otherwise stated, all modules are left modules. In this section, we will give some applications. Meanwhile, we will see that [3, Propositions 4.3, 4.6] can be obtained directly from the cotorsion pairs on the categories of the complexes by our main results.

Let \( \mathcal{A} \) be an abelian category with enough projective and injective objects. For a subclass \( \mathcal{C} \) of \( \mathcal{A} \), we set

\[
\begin{align*}
\mathcal{C}^\perp &= \{ M \in \mathcal{A} \mid \text{Ext}^1_R(M, N) = 0, \forall N \in \mathcal{C} \}, \\
\mathcal{C}^{\perp} &= \{ N \in \mathcal{A} \mid \text{Ext}^1_R(M, N) = 0, \forall M \in \mathcal{C} \}.
\end{align*}
\]

A pair \((\mathcal{X}, \mathcal{Y})\) of \( \mathcal{A} \) is said to be a cotorsion pair if \( \mathcal{X} = \mathcal{Y}^\perp \) and \( \mathcal{Y} = \mathcal{X}^{\perp} \). The cotorsion pair is called complete if for any \( M \in \mathcal{A} \), there exist two short exact sequences

\[
\begin{align*}
0 &\rightarrow Y \rightarrow X \rightarrow M \rightarrow 0, \\
0 &\rightarrow M \rightarrow Y' \rightarrow X' \rightarrow 0,
\end{align*}
\]

with \( X, X' \in \mathcal{X} \) and \( Y, Y' \in \mathcal{Y} \). A pair of cotorsion pairs \((\mathcal{I}, \mathcal{F}), (\mathcal{U}, \mathcal{V})\) is said to be a twin cotorsion pair if \( \mathcal{I} \subseteq \mathcal{U} \), or equivalently, \( \mathcal{V} \subseteq \mathcal{F} \). A twin cotorsion pair is called concentric if \( \mathcal{I} \cap \mathcal{F} = \mathcal{U} \cap \mathcal{V} \).

Let \( \mathcal{P} \) be the full subcategory of \( \text{Mod} \, R \) consisting of all projective modules. It is well known that the canonical cotorsion pair \((\mathcal{P}, \text{Mod} \, R)\) is completed by [8, Theorem 3.2.1] since it is generated by \( R \).
Recall that $R$ is said to be a **Gorenstein ring** if it is a left and right noetherian ring with finite injective dimension on either side. An $R$-module $N$ is called **Gorenstein projective** if there is an exact sequence of projective $R$-modules

$$P : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

with $N = \text{Ker}(P_0 \rightarrow P^0)$ such that $\text{Hom}(P, Q)$ is exact for any projective module $Q$.

Let $\mathcal{GP}(R)$ be the full subcategory of $\text{Mod} R$ consisting of all the Gorenstein projective modules. We denote by $\mathcal{P}^{\infty}$ the class of modules admitting finite projective dimension. If $R$ is a Gorenstein ring, then $(\mathcal{GP}(R), \mathcal{P}^{\infty})$ is a complete cotorsion pair, see [8, Example 4.1.14].

Let $\mathcal{A}$ be an additive category. The subcategory $\mathcal{B}$ of $\mathcal{A}$ is called coreflective if the inclusion functor admits a right adjoint functor.

**Proposition 1**  Let $R$ be a Gorenstein ring. Then the stable category $\mathcal{GP}(R)$ is a coreflection subcategory of $\text{Mod} R$.

*Proof*  It is easy to see that $((\mathcal{P}, \text{Mod} R), (\mathcal{GP}(R), \mathcal{P}^{\infty}))$ is a concentric twin cotorsion pair and $\Sigma \mathcal{P}^{\infty} \subseteq \mathcal{P}^{\infty}$ since $R$ is a Gorenstein ring. By Lemma 7, we obtain a colocalization sequence of additive categories:

$$\xymatrix{ \mathcal{P}^{\infty} \ar[r]^F \ar[rrd]^{G} \ar[rd]^{F_{\lambda}} & \text{Mod} R \ar[r]^{G_{\lambda}} & \mathcal{GP}(R) \ar[ld]_{G}.}$$

Thus, the inclusion functor $G_{\lambda}$ admits a right adjoint functor. That is to say, $\mathcal{GP}(R)$ is a coreflection subcategory of $\text{Mod} R$. \qed

We denote the class of projective modules by $\mathcal{P}$, the class of injective modules by $\mathcal{I}$, the category of a complex of left $R$-modules by $C(\text{Mod} R)$, and the class of exact complexes by $\mathcal{E}$.

We write a complex $X^\bullet \in C(\text{Mod} R)$ as

$$\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots.$$ 

Given $X^\bullet \in C(\text{Mod} R)$, the suspension of $X^\bullet$, denoted by $S(X^\bullet)$, is the complex given by $S(X^\bullet)_n = X_{n-1}$ and $d_{S(X^\bullet)} = -d_n$. Inductively, one can define $S^{n+1}(X^\bullet) = S(S^n(X^\bullet))$ for all $n \in \mathbb{Z}$. Given an $R$-module $X$, we denote by $\overline{X}$ the complex

$$\cdots \rightarrow 0 \rightarrow X \xrightarrow{1} X \rightarrow 0 \rightarrow \cdots,$$

where the two $X$'s are in the 0th and -1st place.

Recall that a complex $P^\bullet \in C(\text{Mod} R)$ is said to be **projective** if for any morphism $P^\bullet \rightarrow D^\bullet$ and any epimorphism $C^\bullet \rightarrow D^\bullet$, the diagram

$$\xymatrix{ P^\bullet & \ar[l] C^\bullet \ar[r] & D^\bullet}$$

is a pullback.
Recollements arising from cotorsion pairs on extriangulated categories can be completed to a commutative diagram by a morphism $P^\bullet \to C^\bullet$. Dually, one can define the injective complex $I^\bullet$. It is well known that each projective complex $P^\bullet$ and injective complex $I^\bullet$ are exact complexes. Moreover, the components $P_n$ and $I_n$ of projective complex $P^\bullet$ and injective complex $I^\bullet$ are projective and injective modules in Mod $R$, respectively. However, conversely, it may not true, see [4, Example 1.4.5].

From [4, Section 1.4], we know that C(Mod $R$) admits enough projective and injective complexes. Then, for any complex $C^\bullet \in$ C(Mod $R$), there exist a projective resolution and an injective resolution of $C^\bullet$. It means that there exist two exact sequences of complexes

$$\cdots \to P_n^\bullet \to \cdots \to P_1^\bullet \to P_0^\bullet \to C^\bullet \to 0,$$

$$0 \to C^\bullet \to I_0^\bullet \to I_{-1}^\bullet \to \cdots \to I_{-n}^\bullet \to \cdots,$$

where each $P_n^\bullet$ and $I_{-n}^\bullet$ is projective and injective, respectively.

Now, we can define the groups $\text{Ext}^n_{\text{C}(\text{Mod} \ R)}(C^\bullet, D^\bullet)$ (simply, $\text{Ext}^n(C^\bullet, D^\bullet)$) for any complexes $C^\bullet$ and $D^\bullet$. If exact sequence of complexes (4.1) is the projective resolution of $C^\bullet$, then $\text{Ext}^n(C^\bullet, D^\bullet)$ is defined to be the $n$-cohomology group of the complex

$$0 \to \text{Hom}(P_0^\bullet, D^\bullet) \to \text{Hom}(P_1^\bullet, D^\bullet) \to \cdots.$$

It also can be computed by the injective resolution of $D^\bullet$. Especially, the $\xi \in \text{Ext}^1(C^\bullet, D^\bullet)$ can be put in bijective correspondence with the equivalence classes of short exact sequence

$$0 \to D^\bullet \to E^\bullet \to C^\bullet \to 0$$

in $C(\text{Mod} \ R)$.

We denote by $K(\text{Mod} \ R)$ and $D(R)$ the corresponding homotopy category and derived category of Mod $R$, respectively.

**Definition 6** [5,6] Given a class of $R$-modules $\mathcal{C}$, we define the following classes of chain complexes in $C(\text{Mod} \ R)$.

1. $\text{dwC}$ is the class of all chain complexes $C^\bullet$ with $C_n \in \mathcal{C}$ for any $n \in \mathbb{Z}$.
2. $\text{exC}$ is the class of all exact chain complexes $C^\bullet$ with $C_n \in \mathcal{C}$ for any $n \in \mathbb{Z}$.
3. $\widetilde{\mathcal{C}}$ is the class of all exact chain complexes $C^\bullet$ with cycles $Z_n(C^\bullet) \in \mathcal{C}$.
4. Given any cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod $R$, $dg\mathcal{X}$ is the class of all complexes of $R$-modules satisfying that each $X_n \in \mathcal{X}$ and $\text{Hom}_{K(\text{Mod} \ R)}(X^\bullet, Y^\bullet) = 0$ for any $X^\bullet \in dg\mathcal{X}$ and any $Y^\bullet \in \widetilde{\mathcal{Y}}$. Similarly, one can define $dg\mathcal{Y}$.

We denote by $K_{ex}(\mathcal{C})$ and $K(\mathcal{C})$ the corresponding homotopy categories consisting of complexes in $ex\mathcal{C}$ and $dw\mathcal{C}$, respectively.

**Lemma 11** [3,7] For the canonical cotorsion pair $(\mathcal{P}, \text{Mod} \ R)$,

$$\text{dwP}, \text{dwP}^\perp, \text{exP}, \text{exP}^\perp, \text{dgP}, \mathcal{C},$$
are the complete cotorsion pairs in \( C(\text{Mod } R) \). Moreover, they satisfy the following conditions:

1. \( \widetilde{\mathcal{P}} = \text{dw} \mathcal{P} \cap \text{dw} \mathcal{P} \perp = \text{ex} \mathcal{P} \cap \text{ex} \mathcal{P} \perp = \text{dg} \mathcal{P} \cap \mathcal{E} \); 
2. \( \mathcal{E} \cap \text{dw} \mathcal{P} = \text{ex} \mathcal{P}, \text{dg} \mathcal{P} \subseteq \text{dw} \mathcal{P} \).

Lemma 12

The classes \( \text{ex} \mathcal{P} \perp \) and \( \mathcal{E} \) are closed under cosyzygy \( \Sigma \).

Proof

Let \( M^\bullet \in \mathcal{E} \) and \( N^\bullet \in \text{ex} \mathcal{P} \perp \). Since \( C(\text{Mod } R) \) admits enough injective objects, there exist two short exact sequences

\[
0 \to M^\bullet \to E^\bullet(M^\bullet) \to \Sigma M^\bullet \to 0, \\
0 \to N^\bullet \to E^\bullet(N^\bullet) \to \Sigma N^\bullet \to 0,
\]

with \( E^\bullet(M^\bullet) \) and \( E^\bullet(N^\bullet) \) are injective complexes. By the long exact sequence theorem and since the injective complexes are exact, it is easy to see that \( \Sigma M^\bullet \) is exact. Hence, \( \mathcal{E} \) is closed under cosyzygy.

Now, it remains to show that \( \Sigma N^\bullet \in \text{ex} \mathcal{P} \perp \). Let \( Q^\bullet \) be a complex of \( \text{ex} \mathcal{P} \). Applying the functor \( \text{Hom}(Q^\bullet, -) \) to the exact sequence (4.2), we obtain a long exact sequence

\[
\text{Ext}^1(Q^\bullet, N^\bullet) \to \text{Ext}^1(Q^\bullet, E^\bullet(N^\bullet)) \to \text{Ext}^1(Q^\bullet, \Sigma N^\bullet) \to \text{Ext}^2(Q^\bullet, N^\bullet).
\]

Note that there is a short exact sequence

\[
0 \to \Omega Q^\bullet \to P^\bullet \to Q^\bullet \to 0.
\]

Thus, we know that

\[
\text{Ext}^2(Q^\bullet, N^\bullet) \cong \text{Ext}^1(\Omega Q^\bullet, N^\bullet).
\]

We claim that \( \Omega Q^\bullet \in \text{ex} \mathcal{P} \). Indeed, it is easy to see that \( \Omega Q^\bullet \) is exact. Moreover, sequence (4.3) is degree-wise split. Hence, each \( Q_n \) is a projective module and so, the desired result comes. In this case,

\[
\text{Ext}^2(Q^\bullet, N^\bullet) = \text{Ext}^1(Q^\bullet, E^\bullet(N^\bullet)) = 0.
\]

Therefore, \( \text{Ext}^1(Q^\bullet, \Sigma N^\bullet) = 0 \). This completes the proof. \( \Box \)

Now, we can apply one of our main results to reprove the existence of the following recollement.

Proposition 2 [3]

There exists the following recollement of triangulated category:

\[
K_{ex}(\mathcal{P}) \xleftarrow{I_\lambda} K(\mathcal{P}) \xrightarrow{J_\lambda} D(R).
\]
Proof By Theorem 3, Lemmas 11 and 12, we have the following recollement:

\[
\begin{array}{c}
\text{ex} \mathcal{P}/ \tilde{\mathcal{P}} \xrightarrow{I} \text{dw} \mathcal{P}/ \tilde{\mathcal{P}} \xrightarrow{J} \text{dg} \mathcal{P}/ \tilde{\mathcal{P}}.
\end{array}
\]

It is well known that there exist triangulated equivalences

\[
\text{ex} \mathcal{P}/ \tilde{\mathcal{P}} \cong K_{\text{ex}}(\mathcal{P}), \quad \text{dw} \mathcal{P}/ \tilde{\mathcal{P}} \cong K(\mathcal{P}), \quad \text{dg} \mathcal{P}/ \tilde{\mathcal{P}} \cong D(R).
\]

Moreover, it is easy to see that the functor \( I \) and \( J_\lambda \) are triangulated functors. By the adjointness, we know that the remained four functors are triangulated functors.

Next, we hope to get Krause’s recollement from the cotorsion pair on the category of complexes. The following observation is very important. Let \( X^\bullet \) be a complex. Now, we construct a short exact sequence of complexes as follows:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & X_{n+1} & \xrightarrow{[d_{n+1}]} & X_{n+1} \oplus X_n & \xrightarrow{[\begin{smallmatrix} -d_{n+1} & 1 \end{smallmatrix}]} & X_n & \rightarrow & 0 \\
& & d_{n+1} & \downarrow & \downarrow & \downarrow & -d_n & & \\
0 & \rightarrow & X_n & \xrightarrow{[1 \ 0]} & X_n \oplus X_{n-1} & \xrightarrow{[-d_n \ 1]} & X_{n-1} & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & & \\
\end{array}
\]

That is, there exists an exact sequence

\[
0 \rightarrow X^\bullet \xrightarrow{[d]} \prod_{n \in \mathbb{Z}} X_n \xrightarrow{[\begin{smallmatrix} -d & 1 \end{smallmatrix}]} S(X) \rightarrow 0.
\]

From this observation, we have the following result.

Lemma 13 If \( X^\bullet \in \text{dg} \mathcal{I} \), then there exists an isomorphism of complexes \( \Sigma X^\bullet \cong S(X) \). In particularly, \( \text{dg} \mathcal{I} \) is closed under the cosyzygy \( \Sigma \).

Proof Since \( X^\bullet \in \text{dg} \mathcal{I} \), \( X_n \) are injective modules for all \( n \in \mathbb{Z} \). Note that \( \prod_{n \in \mathbb{Z}} X_n \) is an injective complex since \( X_n \) are injective complexes for all \( n \in \mathbb{Z} \). Then we obtain the desired result.

Finally, we are in proposition to reobtain Krause’s recollement, which also reprove [3, Proposition 4.3].
Proposition 3  For the canonical cotorsion pair $(\text{Mod } R, \mathcal{I})$,

$$(\perp dw \mathcal{I}, dw \mathcal{I}), \quad (\perp ex \mathcal{I}, ex \mathcal{I}), \quad (\mathcal{E}, dg \mathcal{I}),$$

are the completed cotorsion pairs in $C(\text{Mod } R)$. Moreover, they induce Krause’s recollement

$$K_{ex}(\mathcal{I}) \xleftarrow{F_{\lambda}} F \xrightarrow{G_{\lambda}} K(\mathcal{I}) \xrightarrow{G'} D(R).$$

Proof  Following [3,7], we know that these three cotorsion pairs are complete and they also satisfy the following conditions:

1. $\overline{\mathcal{I}} = \perp dw \mathcal{I} \cap dw \mathcal{I} = \perp ex \mathcal{I} \cap ex \mathcal{I} = \mathcal{E} \cap dg \mathcal{I};$

2. $\mathcal{E} \cap dw \mathcal{I} = ex \mathcal{I}, \ dg \mathcal{I} \subseteq dw \mathcal{I}.$

Now, we show that $ex \mathcal{I}$ is closed under cosyzygy. Indeed, there exists a short exact sequence

$$0 \to X^\bullet \to E^\bullet(X^\bullet) \to \Sigma X^\bullet \to 0$$

for each $X \in ex \mathcal{I}$. Clearly, $\Sigma X^\bullet$ is an exact complex. It is easy to see that this sequence is degree-wise split. Thus, each component $(\Sigma X^\bullet)_n$ of $\Sigma X^\bullet$ is injective and so, $\Sigma X^\bullet \in ex \mathcal{I}$.

By Lemma 13, $dg \mathcal{I}$ is also closed under the cosyzygy $\Sigma$. Therefore, by Theorem 1, we know that there exists a recollement of additive categories:

$$ex \mathcal{I}/\overline{\mathcal{I}} \xleftarrow{F_{\lambda}} F \xrightarrow{G_{\lambda}} dw \mathcal{I}/\overline{\mathcal{I}} \xrightarrow{G'} dg \mathcal{I}/\overline{\mathcal{I}}.$$

It is well known that there exist triangulated equivalences

$$ex \mathcal{I}/\overline{\mathcal{I}} \cong K_{ex}(\mathcal{I}), \quad dw \mathcal{I}/\overline{\mathcal{I}} \cong K(\mathcal{I}), \quad dg \mathcal{I}/\overline{\mathcal{I}} \cong D(R).$$

The remaining arguments are similar to that of Proposition 2.  

Acknowledgements  This work was supported by the National Natural Science Foundation of China (Grant Nos. 11901190, 11671126, 12071120) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19B239).
References

1. Beilinson A, Bernstein J, Deligne P. Faisceaux pervers. In: Analysis and Topology on Singular Spaces, I (Luminy, 1981). Paris: Soc Math France, 1982, 5–171
2. Borceux F. Handbook of Categorical Algebra 1, Basic Category Theory. Encyclopedia Math Appl, Vol 50. Cambridge: Cambridge Univ Press, 1994
3. Chen W J, Liu Z K, Yang X Y. Recollements associated to cotorsion pairs. J Algebra Appl, 2018, 17(5): 1–15
4. Enochs E E, Jenda O M G. Relative Homological Algebra, Vol 2. Berlin: De Gruyter, 2011
5. Gillespie J. The flat model structure on $\text{Ch}(R)$. Trans Amer Math Soc, 2004, 356(8): 3369–3390
6. Gillespie J. Cotorsion pairs and degreewise homological model structures. Homology Homotopy Appl, 2008, 10(1): 283–304
7. Gillespie J. Gorenstein complexes and recollements from cotorsion pairs. Adv Math, 2016, 291: 859–911
8. Göbel R, Trlifaj J. Approximations and Endomorphism Algebras of Modules. Berlin: Walter de Gruyter, 2006
9. Liu Y, Nakaoka H. Hearts of twin cotorsion pairs on extriangulated categories. J Algebra, 2019, 528: 96–149
10. MacPherson R, Vilonen K. Elementary construction of perverse sheaves. Invent Math, 1986, 84(2): 403–435
11. Nakaoka H, Palu Y. Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah Topol Géom Différ Catég, 2019, 60(2): 117–193
12. Wang M X, Lin Z Q. Recollement of additive quotient categories. arXiv: 1502.00479
13. Zheng Q L, Wei J Q. One-sided triangulated categories induced by concentric twin cotorsion pairs. J Algebra Appl, 2020, 19(8): 2050142
14. Zhou P Y, Zhu B. Triangulated quotient categories revisited. J Algebra, 2018, 502: 196–232
15. Zhu B, Zhuang X. Tilting subcategories in extriangulated categories. Front Math China, 2020, 15(1): 225–253