PRODUCTS OF CYCLOTOMIC POLYNOMIALS ON UNIT CIRCLE

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ABSTRACT. We present a method to deal with the values of polynomials of type \( P(z) = \prod_{d \in D} (1 - z^d)^{j_d} \) on the unit circle. We use it to improve the known bounds on various measures of coefficients of cyclotomic and similar polynomials.

1. Introduction and earlier results

Let \( \Phi_n(z) = \sum_m a_n(m) z^m \) be the \( n \)th cyclotomic polynomial, where we assume that \( n \) is odd and square free. Then the order of \( \Phi_n \) is the number \( \omega(n) \) of prime divisors of \( n \). Put
\[
A_n = \max_m |a_n(m)|, \quad S_n = \sum_m |a_n(m)|, \quad Q_n = \sum_m a_n(m)^2, \quad L_n = \max_{|z|=1} |\Phi_n(z)|.
\]

We have \( \deg \Phi_n = \varphi(n) \leq n \), so these quantities are bounded by the inequalities
\[
L_n/n \leq S_n/n \leq \sqrt{Q_n/n} \leq A_n.
\]

The author [2, 4] proved that for \( n = p_1 p_2 \ldots p_k \), \( p_1 < p_2 < \ldots < p_k \) we have
\[
(c + \epsilon_k)^{2k} \leq \sup_{\omega(n) = k} \frac{L_n/n}{M_n} \leq \frac{A_n}{M_n} \leq (C + \epsilon_k)^{2k},
\]
where \( c \approx 0.71 \), \( C \approx 0.95 \), \( M_n = \prod_{j=1}^{k-2} p_j^{2k-j-1} \) and \( \epsilon_k \to 0 \) for \( k \to \infty \). Moreover \( M_n \) gives the optimal order, i.e. it cannot be replaced by any smaller (in lexicographical sense) product of powers of prime factors of \( n \).

Below we give a review of results for cyclotomic polynomials of order a most 3, where we assume \( p < q < r \).

For a cyclotomic polynomial of order 1 obviously \( A_p = 1 \) and \( S_p = L_p = Q_p = p \).

For binary cyclotomic polynomials A.Migotti [10] proved that \( A_{pq} = 1 \). L.Carlitz [6] obtained \( S_{pq} = Q_{pq} = 2p^*q^* - 1 < \frac{1}{2} pq \), where \( p^* \in \{1, 2, \ldots, q-1\} \) and \( q^* \in \{1, 2, \ldots, p-1\} \).

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1) is the inverse of \( p \) modulo \( q \) and \( q^* \) is defined similarly. The author [4] found such \( p \) and \( q \) that \( L_{pq} \geq \left( \frac{4}{\pi^2} - \epsilon \right) pq \), but no upper bound for \( L_{pq} \) has been known so far.

For ternary cyclotomic polynomials G.Bachman [1] proved that \( A_{pqr} \leq \frac{3}{4} p \). There is a conjecture of Y.Gallot and P.Moree [8] that \( A_{pqr} \leq \frac{3}{4} p \) and it is known that the constant \( \frac{2}{3} \) cannot be smaller. The author [2] proved that \( S_{pqr} \leq \frac{16}{32} p^2 q r \) and if the conjecture \( A_{pqr} \leq \frac{3}{4} p \) is true, then \( S_{pqr} \leq \frac{4}{9} p^2 q r \). Furthermore the author [4] proved that \( L_{pqr} \geq \left( \frac{8}{3} \pi^2 - \epsilon \right) p^2 q r \). No upper bound for \( L_{pqr} \) and almost nothing about \( Q_{pqr} \) has been known so far.

2. Main results

As we mentioned in the previous section, \( M_n \) gives the optimal order for \( A_n, S_n/n, \sqrt{Q(n)/n} \) and \( L_n/n \). We define the following constants:

\[
B_k = \limsup_{p_1 \to \infty} \frac{A_n}{M_n}, \quad B_k^\Sigma = \limsup_{p_1 \to \infty} \frac{S_n/n}{M_n},
\]

\[
B_k^\square = \limsup_{p_1 \to \infty} \frac{\sqrt{Q(n)/n}}{M_n}, \quad B_k^\circ = \limsup_{p_1 \to \infty} \frac{L_n/n}{M_n},
\]

where \( \limsup \) is taken over all \( n = p_1 p_2 \ldots p_k \) with \( p_1 < p_2 < \ldots < p_k \). Clearly, \( B_k^\circ \leq B_k^\Sigma \leq B_k^\square \leq B_k \) for \( k > 0 \). We prove the following theorems.

**Theorem 1.** For binary cyclotomic polynomials \( B_2^\circ = \frac{4}{\pi^2} \).

**Theorem 2.** For ternary cyclotomic polynomials:

(i) \( B_3^\circ = \frac{1}{\pi^2} \),

(ii) \( \sqrt{3/2}/\pi^2 \leq B_3^\Sigma \leq \sqrt{1/12} \).

(iii) \( B_3^\Sigma < 0.2731 \).

An upper bound on \( Q_{pqr} \) is given later by Theorem 13.

Summarizing, the known results for cyclotomic polynomials of order 2 are:

\[
B_2^\circ = \frac{4}{\pi^2}, \quad B_2^\Sigma = \frac{1}{2}, \quad B_2^\square = \frac{\sqrt{2}}{2}, \quad B_2 = 1,
\]

and for order 3 we have:

\[
B_3^\circ = \frac{1}{\pi^2} \leq B_3^\Sigma \leq 0.2731, \quad \sqrt{3/2}/\pi^2 \leq B_3^\square \leq \sqrt{1/12} < \frac{2}{3} \leq B_3 \leq \frac{3}{4}.
\]

These theorems allow us to improve the bound on \( B_k \) from [2].

**Theorem 3.** For \( n = p_1 p_2 \ldots p_k \) we have

\[
B_k \leq (C + \epsilon_k)^{2k} M_n,
\]

where \( C < 0.859125 \) and \( \epsilon_k \to 0 \) with \( k \to \infty \).
Additionally we use our methods to estimate the number

$$J_{pqr} = \sum_k |a_{pqr}(k) - a_{pqr}(k - 1)|$$

of jumps of ternary cyclotomic coefficients studied by the author \[3\] and Camburu, Ciolan, Luca, Moree and Shparlinski \[5\].

**Theorem 4.** For some constant $c$ we have $J_{pqr} \leq cpqrU^2$, where $U = \max\{u_{qr}, u_{rp}, u_{pq}\}$, $u_{pq} = u_{qp} = 1$ and $q^*$ is the inverse of $q$ modulo $p$, etc.

At the end we prove the following result for the so-called relatives of cyclotomic polynomials, i.e. polynomials of form

$$P_n(z) = \prod_{d \in D} (1 - z^d)^{j_d},$$

where $n = p_1 p_2 \ldots p_k$, introduced in \[9\] by Liu.

**Theorem 5.** Let

$$L(k) = \sup_{\omega(n) = k} \max_{|z| = 1} \frac{|P_n(z)|}{n}.$$  

Then $\log L(k) \geq \frac{\log 2}{2} k^2 + O(k \log k)$.

In \[9\] was proved that $\log M(k) = \frac{\log 2}{2} k^2 + O(k \log k)$, where $M(k)$ is maximal absolute value of a coefficient of $P_n$ with $\omega(n) = k$, so in fact in Theorem 5 we have equality. Theorem 5 gives a constructive and a bit simpler proof of the lower bound $\log M(k) \geq \frac{\log 2}{2} k^2 + O(k \log k)$.

3. Preliminaries

In this section we introduce additional notation used in the paper and we present the main ideas of the proofs.

Let $P(z) = \prod_{d \in D} (1 - z^d)^{j_d}$ be a polynomial with integers $j_d$ (not necessarily positive) and $\text{lcm}(D) = n = p_1 p_2 \ldots p_k$, where $2 < p_1 < p_2 < \ldots < p_k$ are primes. We are interested in the values of $|P(z)|$ for $|z| = 1$, so let $z = e^{2\pi ix}$. We have $|1 - z^d| = 2s(dx)$, where $s(x) = |\sin(\pi x)|$. Therefore

$$F(x) := |P(z)| = 2^{\Sigma_d j_d} \prod_d s(dx)^{j_d}.$$  

We can only consider $x \in [-1/2, 1/2)$, because $F(x) = F(x + 1)$. Let $x = \frac{N + t}{n}$, where $|N| < n/2$ is an integer and $t \in [-1/2, 1/2)$. by the Chinese remainder theorem we can uniquely write $N = N_{a_1, a_2, \ldots, a_k}$, where $N \equiv a_i \pmod{p_i}$ and $|a_i| < p_i/2$ for $i = 1, 2, \ldots, k$. This substitution will allow us to deal with some expressions of form $s(dx)$ quite easily.
In order to estimate the sum \( Q \) of squares of coefficients of \( P \) we use the Parseval identity

\[
Q = \int_{-1/2}^{1/2} |P(e^{2\pi i x})|^2 dx = \int_{-1/2}^{1/2} F(x)^2 dx = \frac{1}{n} \sum_{a_1, a_2, \ldots, a_k} I_{a_1, a_2, \ldots, a_k},
\]

where

\[
I_{a_1, a_2, \ldots, a_k} = \int_{-1/2}^{1/2} F \left( \frac{N_{a_1, a_2, \ldots, a_k} + t}{n} \right)^2 dt
\]

and the sum is over all \(|a_i| < p_i/2, i = 1, 2, \ldots, k\).

Throughout the paper we use the notation

\[
s(x) = |\sin(\pi x)|, \quad s_d(x) = s(x/d).
\]

We also use the following asymptotic notation: \( f(n) \ll g(n) \) if there exists an absolute constant \( c \) such that \( f(n) < cg(n) \) and \( f(n) \lesssim g(n) \) if \( f(n) \leq (1 + o(1))g(n) \).

The following lemmas are crucial in the proofs in the next sections.

**Lemma 6.** Let \( n = p_1 p_2 \ldots p_k, N = N_{a_1, a_2, \ldots, a_k} \) and \( x = \frac{N + t}{n} \). Then

(i) \( s(nx) = s(t) \),

(ii) \( s((n/p_i)x) = s_{p_i}(a_i + t) \),

(iii) \( s\left((n/p_i p_j)x\right) = s_{p_i p_j}(a_j a_i p_i^* + a_i + t) = s_{p_i}(a_i - a_j) p_j p_i^* + a_j + t \),

where \( p_i^* \) is the inverse of \( p_i \) modulo \( p_j \) and \( p_j^* \) is the inverse of \( p_j \) modulo \( p_i \).

**Proof.** Parts (i) and (ii) are trivial. For (iii) note that

\[ N \equiv a_i p_j p_i^* + a_j p_i p_i^* \equiv a_i (1 - p_i p_i^*) + a_j p_i p_i^* = (a_i - a_j) p_i p_i^* + a_i \pmod{p_i p_j}. \]

\[ \square \]

**Lemma 7.** Let \( N, n \) and \( x \) be the numbers defined in Lemma 6. Let \( p_1 = \min\{p_1, p_2, \ldots, p_k\} \). If \( p_1 \to \infty \), then the following holds.

(i) \[ \frac{|a_i + t|}{p_i} \ll s((n/p_i)x) \ll \frac{|a_i + t|}{p_i}. \]

(ii) If \(|a_i| < p_i^{1-\varepsilon}\) then \( s((n/p_i)x) \sim \pi \frac{|a_i + t|}{p_i} \).

(iii) For \( a_i = a_j \) we have \( s((n/p_i p_j)x) \ll \frac{1}{\max\{p_i, p_j\}} \).

(iv) If \(|a_i|, |a_j| < p_i^{1-\varepsilon}\) and \( a_i \neq a_j \) then \( s((n/p_i p_j)x) \sim s_{p_i}(a_j - a_i) p_j p_i^* \).

**Proof.** Parts (i) and (ii) follow easily from the properties of the sine function.

To prove (iii), note that by Lemma 6 we have

\[ s((n/p_i p_j)x) = s_{p_i p_j}(a_i + t) \ll \min\{p_i, p_j\} = \frac{1}{p_i p_j} = \frac{1}{\max\{p_i, p_j\}}. \]

For the last part note that \( s_{p_i p_j}(a_i - a_j) p_j p_i^* \gg \frac{1}{p_i} \) and \( \frac{a_i + t}{p_i p_j} \ll \frac{1}{p_i^{1+\varepsilon}}. \]

\[ \square \]
The following fact is easy and we omit its proof.

**Proposition 8.** We have $s(px)/s(x) \leq p$ and $s(px)s(qx)/s(x) \leq \min\{p, q\}$.

Let

$$F_n(x) = |\Phi_n(e^{2\pi ix})|.$$  

To work with $\Phi_n$ on the unit circle we use the following formula obtained in [4].

**Proposition 9.** For $n > 1$ we have $F_n(x) = \prod_{d|n} s(dx)^{\mu(n/d)}$.

4. Warm up: binary polynomials

**Proof of Theorem 1.** By the results from [4] we already know that for all $\epsilon > 0$ there exist $p$ and $q$ such that $L_{pq} \geq (\frac{4}{\pi^2} - \epsilon)pq$. Therefore in order to prove Theorem 1 it is enough to show that $L_{pq} \lessapprox \frac{4}{\pi^2}$ with $p = \min\{p, q\} \to \infty$.

Let $2 < p < q$ and $F = F_{pq}$. Like explained in the previous section, every $x \in [-1/2, 1/2)$ can be uniquely expressed as $x = \frac{N_{a,b} + t}{pq}$ with $|a| < p/2$ and $|b| < q/2$. Let $N = N_{a,b}$.

By Proposition 9 and Lemma 6 we have

$$F(x) = s(t)s_{pq}(N + t) = \frac{s(t) s_{pq}(N + t)}{s_p(a + t)s_q(b + t)}.$$  

Now we use Lemma 7 and Proposition 8 to estimate $F(x)$. We consider three cases.

**Case 1.** $a = b = 0$. Then $N = 0$ and $F(x) \ll 1$.

**Case 2.** $a, b \neq 0$. Then

$$F(x) \lessapprox \frac{s(t)}{s_p(a + t)s_q(b + t)} \ll \frac{pq}{|ab|}.$$  

If $|a| > p^{1-\epsilon}$ or $|b| > p^{1-\epsilon}$ then $F(x) \ll pq$. Otherwise

$$F(x) \lessapprox \frac{pq}{\pi^2} \cdot s(t) \frac{1}{|a + t| \cdot |b + t|} \leq \frac{4}{\pi^2} pq.$$  

**Case 3.** $a \neq 0$ and $b = 0$ (or reverse, which is analogous). For $|a| > p^{1-\epsilon}$ we have $F(x) \lessapprox \frac{pq}{|a|} < pq$. If $|a| \leq p^{1-\epsilon}$, then

$$F(x) \lessapprox \frac{pq}{\pi^2} \cdot \frac{s(t)}{|t| \cdot |a + t|} \leq \frac{pq}{\pi^2} \cdot \frac{s(t)}{|t| \cdot (1 - |t|)} \leq \frac{4}{\pi^2} pq$$  

by elementary computations.

Recall that $L_{pq} = \max_x F(x)$, so the proof is done. \qed
Let us add that the inequality $L_{pq} \leq \frac{4}{\pi}pq$ is not true in general. For example, let us fix $p$, choose $q \equiv -2 \pmod{p}$ and $x = \frac{pq - q - 1}{2pq}$. Then by using the expansion of $\sin x$ we obtain

$$\frac{F(x)}{pq} \to \frac{4}{\pi^2} + \frac{2\pi^2 - 3}{6\pi^2} p^{-2} + O(p^{-4})$$

with $q \to \infty$.

It justifies the assumption $p_1 \to \infty$ in the definition of $B_k^0$.

5. Ternary polynomials: maximum on circle

In this section we prove part (i) of Theorem 2. It is an instant consequence of Lemmas 10 and 11 below.

Let $2 < p < q < r$, $F = F_{pqr}$ and $N = N_{a,b,c}$, $|a| < p/2$, $|b| < q/2$, $|c| < r/2$.

**Lemma 10.** Let $x = \frac{N_{a,b,c} + t}{pq}$.

(i) If $a = b = c = 0$ then $F_{pqr}(x) \ll 1$.

(ii) If $a = b = 0$ and $c \neq 0$ then

$$F_{pqr}(x) \ll \begin{cases} 1, & \text{for } |dpq| < r/2, \\ r^2/|d| \ll pqr, & \text{for } |dpq| > r/2, \end{cases}$$

where $|d| < r/2$ and $dpq \equiv c \pmod{r}$.

The analogous bound holds for any permutation of $(p,q,r)$ and the appropriate permutation of $(a,b,c)$.

(iii) For $b = c \neq 0$ we have

$$F_{pqr}(x) \ll \begin{cases} pqr/ab^2, & \text{for } a = 0, \\ pqr/|ab^2|, & \text{for } a \neq 0, \end{cases}$$

As in the previous case, this bound has its symmetric versions.

(iv) For distinct $a, b, c$ we have $F_{pqr}(x) \lesssim \frac{1}{p^2} p^2 qr$ with $p = \min\{p, q, r\} \to \infty$.

**Proof.** By Proposition 9 and Lemma 6 we have

$$F_{pqr}(x) = \frac{s(t)s_{qr}(N + t)s_{rp}(N + t)s_{pq}(N + t)}{s_{pqr}(N + t)s_{p}(a + t)s_{q}(b + t)s_{r}(c + t)}.$$

We deal with all parts separately. As for the binary case, we write $F$ instead of $F_{pqr}$. Part (i) is trivial. In the remaining parts we often use Lemmas 6 and 7 and Proposition 8.

**Part (ii).** Let $p'$ and $q'$ be the inverses of $p$ and $q$ modulo $r$. Obviously $d \neq 0$.

Then

$$F(x) = \frac{s(t)s_{pq}(t)}{s_{p}(t)s_{q}(t)} \cdot \frac{s_{qr}(cpp' + t)s_{rp}(cpp' + t)s_{pq}(cppq'q' + t)s_{r}(c + t)}{s_{pqr}(cppq'q' + t)s_{r}(c + t)} \ll \frac{s_{r}(dp)s_{r}(dq)}{s_{r}(dpq)s_{r}(d')}.$$
Now it is clear that if $|dpq| < r/2$ then $F_{pqr}(x) \ll 1$. For all $d$ we have

$$F(x) \ll \frac{1}{(1/r)(|d|/r)} = r^2/|d|.$$ 

Similarly we deal with cases symmetric to this one.

**Part (iii).** For $a = 0$ we have

$$F(x) \ll pqr \frac{\min\{q, r\}}{|bc| \max\{q, r\}} < \frac{pqr}{|bc|}.$$ 

If $a \neq 0$, then we use the bound $s(t)/s_p(a+t) \ll p/|a|$ instead of $s(t)/s_p(0+t) \leq p$.

**Part (iv).** We have

$$F(x) \leq p \cdot \frac{s(t)}{s_p(a+t)s_q(b+t)s_r(c+t)}.$$ 

Because at most one of $a, b, c$ equals 0, the quotient above is well defined, as we may replace $t = 0$ by $t \to 0$ if necessary. If $\max\{|a|, |b|, |c|\} > p^{1-\epsilon}$, then $F(x) \ll p^{2+\epsilon}qr$, so let $|a|, |b|, |c| \leq p^{1-\epsilon}$. In this case

$$F(x) \leq \frac{p^2qr}{\pi^3} \cdot \frac{s(t)}{|a+t| \cdot |b+t| \cdot |c+t|}.$$ 

By elementary computations, the last quotient is maximal for $\{a, b, c\} = \{-1, 0, 1\}$ and $t \to 0$. The limit equals $\pi$, which completes the proof. 

The following Lemma gives an explicit example of $p, q, r$ for which $L_{pqr}$ is large.

**Lemma 11.** Let $q \equiv r \equiv 2 \pmod{p}$, $r \equiv \frac{-4}{p-1} \pmod{q}$, $N = r \cdot \frac{p-1}{2} + 1$ and $x = \frac{N}{pqr}$. Then $F_{pqr}(x) \sim \frac{1}{\pi^2} p^2 qr$ with $p \to \infty$ and $\frac{q}{p} \to \infty$.

**Proof.** We have $N \equiv 0 \pmod{p}$, $N \equiv -1 \pmod{q}$ and $N \equiv 1 \pmod{r}$, so

$$F(x) \sim \frac{pqr}{\pi^2} \cdot \frac{s_{qr}(N)}{s_{pqr}(N)} \cdot \frac{s_{pq}(N)}{s_{pqr}(N)}.$$ 

Taking $q/p \to \infty$ we obtain $s_{pr}(N) \to 1$ and by Lemma 7, $s_{pq}(N) \sim s_{pq}(qq^*) \sim 1$. Finally

$$s_{qr}(N)/s_{pqr}(N) \sim s(p/2q)/s(1/2q) \sim p$$

with $q/p \to \infty$. 

□
6. TERNARY POLYNOMIALS: SUM OF SQUARES

In this section we derive an upper bound on \( Q_{pqr} \) and use it to prove the second part of Theorem 2. As mentioned in Preliminaries,

\[
Q_{pqr} = \frac{1}{pqr} \sum_{|a|<p/q; |b|<q/2; |c|<r/2} I_{a,b,c},
\]

where

\[
I_{a,b,c} = \int_{-1/2}^{1/2} F \left( \frac{N_{a,b,c} + t}{pqr} \right)^2 dt.
\]

First we deal with some specific triples \((a, b, c)\).

**Lemma 12.** We have

(i) \( I_{0,0,0} \ll 1 \),

(ii) \( \sum_{c \neq 0} I_{0,0,c} \ll (pqr)^2 \), similarly for \( I_{0,b,0} \) and \( I_{a,0,0} \),

(iii) \( \sum_{b \neq 0} I_{a,b,b} \ll (pqr)^2 \), similarly for \( I_{a,b,a} \) and \( I_{a,a,c} \),

(iv) \( \sum_{\max\{a,b,c\} > p^{1-\epsilon}} I_{a,b,c} \ll p^3 q^2 r^2 \).

**Proof.** For (i) – (iii) we use the analogous parts of Lemma 10. Part (i) is again trivial.

**Part (ii).** By Lemma 10

\[
\sum_{c \neq 0} I_{0,0,c} \ll \frac{r^2}{p^2 q^2} + r^4 \sum_{d > r/pq} \frac{1}{d^2} \ll (pqr)^2,
\]

where we used the known fact that \( \sum_{k \geq n} \frac{1}{k^2} \ll \frac{1}{n} \).

**Part (iii).** Using the bounds from (iii) of Lemma 10 we obtain

\[
\sum_{b \neq 0} I_{a,b,b} \ll (pqr)^2 \sum_{b \neq 0} \frac{1}{b^4} + (pqr)^2 \sum_{a,b \neq 0} \frac{1}{a^2 b^4} \ll (pqr)^2.
\]

**Part (iv).** By the previous cases, we may consider only distinct \( a, b, c \). For such triples we have

\[
F(x) \ll \frac{p^2 qr \cdot s(t)}{|a+t| \cdot |b+t| \cdot |c+t|}
\]

which yields

\[
\sum_{c > p^{1-\epsilon}} I_{a,b,c} \ll p^4 q^2 r^2 \sum_{c > p^{1-\epsilon}} \frac{1}{c^2} \sum_{a,b \neq (0,0)} \int_{-1/2}^{1/2} \left( \frac{s(t)}{|a+t| \cdot |b+t|} \right)^2 dt
\]

\[
\ll p^4 q^2 r^2 \sum_{c > p^{1-\epsilon}} \frac{1}{c^2} \ll p^{3+\epsilon} q^2 r^2.
\]

It completes the proof of the last part. \( \square \)

Now we are ready to prove the following theorem.
**Theorem 13.** Let \( x = \frac{1}{p} \min\{q', p - q'\} \) and \( y = \frac{1}{p} \min\{r', p - r'\} \), where \( q' \) and \( r' \) are the inverses of \( q \) and \( r \) modulo \( p \). Without loss of generality we assume that \( x \leq y \). Then

\[
\frac{Q_{pqr}}{p^3qr} \lesssim \frac{1}{6} P(x, y) + \frac{1}{12} f(x, y),
\]

where

\[
P(x, y) = 2x - 11x^2 + 26x^3 - 17x^4 - 5y^2 + 18y^3 - 17y^4 \\
+ 12xy - 24x^2 y - 12xy^2 + 24x^2 y^2, \\
f(x, y) = \{2x + y\}^2(1 - \{2x + y\})^2 + \{2x - y\}^2(1 - \{2x - y\})^2 \\
+ \{2y + x\}^2(1 - \{2y + x\})^2 + \{2y - x\}^2(1 - \{2y - x\})^2
\]

and \( \{\cdot\} \) denotes the fractional part of given real number.

**Proof.** By Lemma 12 we may focus on distinct \( a, b, c \leq p^{1-\epsilon} \). By Lemma 6 and 7

\[
F\left(\frac{N + t}{pqr}\right) \lesssim \frac{p^2 qr}{\pi^3} \frac{s(t)s((a-b)x)s((a-c)y)}{|a + t| \cdot |b + t| \cdot |c + t|}.
\]

By Lemma 12 and then by the substitution \( m = a - b, n = a - c \) and \( u = a + t \), we obtain

\[
\frac{Q_{pqr}}{p^3qr} \lesssim \pi^{-6} \sum_{a, b, c} s((a - b)x)^2 s((a - c)y)^2 \int_{-1/2}^{1/2} \left( \frac{s(t)}{(a + t)(b + t)(c + t)} \right)^2 dt
\]

\[
\sim \pi^{-6} \sum_{m, n \neq 0; m \neq n} s(mx)^2 s(ny)^2 \int_{-\infty}^{+\infty} \left( \frac{s(u)}{u(u - m)(u - n)} \right)^2 du.
\]

Computing of the integral is a routine; it equals

\[
\pi^2 \left( \frac{1}{m^2 n^2} + \frac{1}{m^2 (m - n)^2} + \frac{1}{n^2 (m - n)^2} \right).
\]

After reorganizing the variables \( m \) and \( n \) we arrive at the following asymptotic bound on \( Q_{pqr}/p^3qr \):

\[
\pi^{-4} \sum_{m, n \neq 0; m \neq n} \frac{s(mx)^2 s(ny)^2 + s(mx)^2 s((m + n)x)^2 + s((m + n)x)^2 s(ny)^2}{m^2 n^2}.
\]

We may express this bound as \( S_1 - S_2 \), where \( S_1 \) runs over all \( m, n \neq 0 \) and \( S_2 \) runs over \( m = n \neq 0 \), i.e.

\[
S_1 = \pi^{-4} \sum_{m, n \neq 0} \frac{(\ldots)}{m^2 n^2}, \quad S_2 = \pi^{-4} \sum_{n \neq 0} \frac{(\ldots)}{n^4}.
\]

Recall that the \( k \)th Bernoulli polynomial is

\[
B_k(x) = \sum_{j=0}^{k} \binom{k}{j} b_{k-j} x^j,
\]
where \( k > 0 \), and \( b_j \) are the Bernoulli numbers. Particularly

\[
B_2(x) = x^2 - x + \frac{1}{6}, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.
\]

The Fourier series of \( B_k(x) \) is given by

\[
B_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{j \neq 0} e^{2\pi ijx} j^k
\]

for \( 0 \leq x \leq 1 \) and \( k \geq 2 \).

Let \( e(x) = e^{2\pi ix} \). After some straightforward computations we obtain

\[
S_1 = \frac{1}{16\pi^4} \sum_{m,n \neq 0} \frac{1}{m^2 n^2} \left( 12 - 8e(mx) - 8e(ny) \\
+ 4e(mx + ny) - 4e(ny + mx) - 4e(mx + ny) \\
+ 2e(m(y + x) + ny) + 2e(m(y - x) + ny) \\
+ 2e(mx + n(y + x)) + 2e(mx + n(y - x)) \right).
\]

Now we use the equality

\[
\sum_{m,n \neq 0} \frac{e(mu + nv)}{m^2 n^2} = 4\pi^4 B_2(\{u\}) B_2(\{v\}).
\]

Notice that \( 0 < x \leq y < \frac{1}{2} \), so we obtain

\[
S_1 = 3B_2(0)^2 - 2B_2(x)B_2(0) - 2B_2(y)B_2(0) + B_2(x)B_2(y) \\
- B_2(x)^2 - B_2(y)^2 + \frac{1}{2} B_2(y + x)B_2(y) + \frac{1}{2} B_2(y - x)B_2(y) \\
+ \frac{1}{2} B_2(x)B_2(y + x) + \frac{1}{2} B_2(x)B_2(y - x).
\]

After some elementary calculations we have

\[
S_1 = \frac{1}{3} x + 2xy - x^2 + x^3 - 2xy^2 - 3x^2y + 3x^2y^2.
\]

It remains to simplify

\[
S_2 = \frac{1}{16\pi^4} \sum_{n \neq 0} \frac{1}{n^4} \left( 12 - 8e(nx) - 8e(ny) + 2e(n(y + x)) + 2e(n(y - x)) \\
- 4e(2nx) - 4e(2ny) + 2e(n(2y - x)) \\
+ 2e(n(2y + x)) + 2e(n(2x - y)) + 2e(n(2x + y)) \right).
\]

We use the equality

\[
\sum_{n \neq 0} \frac{e(nu)}{n^4} = -\frac{2}{3} \pi^4 B_4(\{u\}).
\]
By $0 < x \leq y < \frac{1}{2}$ we have

$$
S_2 = -\frac{1}{2}B_4(0) + \frac{1}{3}B_4(x) + \frac{1}{3}B_4(y) + \frac{1}{6}B_4(2x) + \frac{1}{6}B_4(2y)
- \frac{1}{12}B_4(y + x) - \frac{1}{12}B_4(y - x)
- \frac{1}{12}(B_4(2y - x) + B_4(2y + x)) + B_4(2x - y) + B_4(2x + y))
.$$ 

By elementary computations

$$
S_2 = \frac{17}{6}x^4 + \frac{17}{6}y^4 - \frac{10}{3}x^3 - 3y^3 + \frac{5}{6}x^2 - \frac{5}{6}y^2 - x^2y^2 + x^2y - \frac{1}{12}f(x, y).
$$

Since $\frac{Q_{pqr}}{p^{3qr}} \preceq S_1 - S_2$, we obtain the assertion of Theorem 13 by verifying the value of $S_1 - S_2$. □

**Proof of Theorem 2(ii).** By Theorem 13, for the upper bound it is enough to prove that $f(x, y) \leq \frac{1}{4}$ and $P(x, y) \leq \frac{3}{8}$. The first inequality is obvious.

As for the second one, note that

$$
\frac{\partial P(x, y)}{\partial x} = (2 + 12y) + (-22 - 48y^2)x + 78x^2 - 68x^3
\geq 2 + 12x - 34x + 78x^2 - 68x^3 = 2 - 22x + 78x^2 - 68x^3 \geq 0
$$

for $0 < x \leq y < \frac{1}{2}$. Furthermore

$$
\frac{\partial P(y, y)}{\partial y} = 2 - 8y + 24y^2 - 30y^3 \geq 0
$$

for $0 < y < \frac{1}{2}$. Thus

$$
P(x, y) \leq P(y, y) \leq P(1/2, 1/2) = 3/8.
$$

In order to prove the lower bound, we again use the polynomial $\Phi_{pqr}$ with $q \equiv r \equiv 2 \pmod{p}$ and $r \equiv -4 \pmod{q}$. We have $N_{a,a-1,a+1} = r \cdot \frac{p-1}{2} + 1 + a$ for $|a| < p/2$ and

$$
q_{pqr}(N_{a,a-1,a+1} + t)/s_{qr}(N_{a,a-1,a+1} + t) \sim p
$$

with $q/p \to \infty$. Moreover,

$$
s_{rp}(N_{a,a-1,a+1} + t) \sim 1, \quad s_{pq}(N_{a,a-1,a+1} + t) \sim 1
$$

with $p \to \infty$. Therefore for $|a| < p^{1-\epsilon}$ we have

$$
I_{a,a-1,a+1} \sim \frac{p^3qr}{\pi^6} \int_{-1/2}^{1/2} \left( \frac{s(t)}{(a + t)(a - 1 + t)(a + 1 + t)} \right)^2 dt.
$$

Thus

$$
\frac{Q_{pqr}}{p^3qr} \gtrsim \frac{1}{\pi^6} \int_{-\infty}^{+\infty} \left( \frac{s(x)}{x(x-1)(x+1)} \right)^2 dx = \frac{3}{2\pi^4},
$$

which completes the proof. □
7. Ternary polynomials: sum of absolute values

Proof of Theorem 2(iii). Let $\Phi_{pqr}(x) = \sum_{n} a(n)x^n$. Put $a = \frac{S_{pqr}}{p^qr}$, $m = \frac{A_{pqr}}{p}$ and $|a(n)|/p = a + r_n$. Clearly, $\sum_{n} r_n = 0$. Then

$$\frac{1}{12} \geq \frac{Q_{pqr}}{p^qr} = \frac{1}{pq} \sum_{n} (a + r_n)^2 = a^2 + \frac{1}{pqr} \sum_{n} r_n^2.$$ We will estimate the sum $R = \sum_{n} r_n^2$ from below. Note that $|a(n) - a(n - qr)| \leq 1$ and $|a(n) - a(n - qr)| \leq 2$, because

$$(1 - x^{qr})\Phi_{pqr}(x) = (1 - x^r)(1 + x + \ldots + x^{q-1})\Phi_{qr}(x^p).$$

The underbraced polynomial is flat because every two nonzero consecutive coefficients of $\Phi_{qr}$ are ±1 and ±1. Therefore we may consider a continuous function $f : [0, 1] \to [0, m]$ satisfying $a + r_n = f(n/pqr)$. Moreover, $f(0) = f(1) = 0$ and $|f(x) - f(y)| \leq 2|x - y|$ for $x, y \in [0, 1]$. For $p \to \infty$ we have

$$\int_{0}^{1} f(x)dx \sim a, \quad \int_{0}^{1} (f(x) - a)^2dx \sim R.$$ The value of $R$ is minimal for the function

$$f(x) = \begin{cases} 2x, & \text{for } 0 \leq x \leq m/2, \\ m, & \text{for } m/2 \leq x \leq 1 - m/2, \\ 2(1 - x), & \text{for } 1 - m/2 \leq x \leq 1, \end{cases}$$

where the optimal $m$ equals $1 - \sqrt{1 - 2a}$ (note that we already know that $a < 1/2$). So we have

$$\int_{0}^{1} (f(x) - a)^2dx = \frac{2}{3}m^3 + (m - a)^2 + am^2.$$ We need to solve the system

$$\begin{cases} m = 1 - \sqrt{1 - 2a}, \\ \frac{1}{12} \geq a^2 - \frac{2}{3}m^3 + (m - a)^2 + am^2. \end{cases}$$

By numerical computations, the solution is

$$a \leq 0.273099 \ldots < 0.2731$$

and the proof is done. \hfill \square

8. General case

Let us recall the following lemma from [2].

Lemma 14. Let $p_1 < p_2 < \ldots < p_k$ be primes and $n = p_1p_2\ldots p_k$. Then

$$\Phi_n(x) = f_n(x) \cdot \prod_{j=1}^{k-2} \prod_{i=j+2}^{k} \Phi_{p_1\ldots p_j}(x^{p_j+2\ldots p_k/p_1}).$$
where \( f_n \) is a formal power series satisfying \( f_n(x) = (1 - x^n) \prod_{i=1}^{k} (1 - x^{n/p_i}) / \prod_{i=1}^{k} (1 - x^{n/p_i}) \).

Let \( f^*_n \) be the polynomial of degree smaller than \( n \), satisfying \( f^*_n(x) \equiv f_n(x) \pmod{x^n} \). In the same paper the author proved that for \( k \geq 2 \) the height of \( f^*_n \) does not exceed \( \left( \frac{k-2}{|k/2|-1} \right) \). Here we prove the following bound.

**Lemma 15.** The sum of absolute values of coefficients of \( f^*_n \) is \( \lesssim \frac{2^{k-1}n}{k!} \) with \( p_1 = \min\{p_1, p_2, \ldots, p_k\} \to \infty \).

**Proof.** We have

\[
\begin{align*}
f^*_n(x) &\equiv \prod_{i=2}^{k} (1 - x^{n/p_i}) \sum_{\alpha_1, \ldots, \alpha_k \geq 0} x^{\alpha_1 n/p_1 + \cdots + \alpha_k n/p_k} \pmod{x^n}.
\end{align*}
\]

The product has the sum of absolute values of coefficients not greater than \( 2^{k-1} \). Let \( p_1 \to \infty \). Then the sum of coefficients of \( \sum x^{\alpha_1 n/p_1 + \cdots} \) with exponents smaller than \( n \) equals asymptotically the volume of simplex on vertices

\[
(p_1, 0, \ldots, 0), (0, p_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, p_k).
\]

It equals \( \frac{n}{k!} \). \( \square \)

**Proof of Theorem** By Lemma \([14]\) we have

\[
S_{p_1 \ldots p_k} \leq \frac{2^{k-1}p_1 \cdots p_k}{k!} \prod_{j=1}^{k-2} S_{p_1 \ldots p_j}^{k-j-1}.
\]

By the inequality \( S_n \leq nM_n \mathcal{B}(\omega(n)) \) with \( \min\{p : p \mid n\} \to \infty \), after some calculations we obtain

\[
\mathcal{B}_k^{\Sigma} \leq \frac{2^{k-1}}{k!} \prod_{j=1}^{k-2} (\mathcal{B}_j^{\Sigma})^{k-j-1}
\]

for \( k \geq 3 \). Put \( b_k = \mathcal{B}_k^{\Sigma} \) for \( k = 1, 2, 3 \) and \( b_k = \frac{2^{k-1}}{k!} \prod_{j=1}^{k-2} b_j^{k-j-1} \) for \( k > 3 \). Then clearly \( \mathcal{B}_k^{\Sigma} \leq b_k \). For \( k \geq 6 \) we have

\[
\frac{b_k / b_{k-1}}{b_{k-1} / b_{k-2}} = \frac{k - 1}{k} b_{k-2},
\]

so \( b_k = \frac{k-1}{k} b_{k-1}^2 \). Furthermore,

\[
C := \lim_{k \to \infty} b_k^{2^{-k}} = b_5^{1/32} \prod_{k=6}^{\infty} \left( \frac{k - 1}{k} \right)^{2^{-k}},
\]

where \( b_5 = \frac{3^4}{30} \cdot b_3 \cdot b_2^2 \cdot b_1^3 = b_3/30 = \mathcal{B}_5^{\Sigma} / 30 \). Based on Theorem \([2]\) and some numerical computations we conclude that \( C < 0.859125 \).
Now by Lemma 14 and the bound \((k-2)\binom{k-2}{\lfloor k/2 \rfloor - 1}\) on the height of \(f_n^*\) we have
\[
\mathcal{B}_k \leq \left( \frac{k-2}{\lfloor k/2 \rfloor - 1} \right) \prod_{j=1}^{k-2} (\mathcal{B}_j^\Sigma)^{k-j-1} < 2^{k-1} \prod_{j=1}^{k-2} b_j^{k-j-1} = k!b_k,
\]
so by verifying that \(\lim_{k \to \infty} (k!)^{-k} = 1\) we complete the proof. \(\square\)

9. JUMPS OF TERNARY CYCLOMOTIC COEFFICIENTS

Proof of Theorem 4. In the proof we do not assume that \(p < q < r\). Let
\[
F(x) = \left| (1 - z) \Phi_{pqr}(z) \right|,
\]
where \(z = e^{2\pi ix}\). We have
\[
J = \frac{1}{pqr} \sum_{|a| < p/2; |b| < q/2; |c| < r/2} I_{a,b,c},
\]
where \(I_{a,b,c} = \int_{-1/2}^{1/2} F \left( \frac{N_{a,b,c} + t}{pqr} \right)^2 dt\). Now we need to consider some cases. We omit details, as the computations are similar to those in the proofs of Lemma 12 and Theorem 13.

Case 1. \(a = b = c = 0\). We have \(I_{0,0,0} \ll (pqr)^{-2}\).

Case 2. \(b = c = 0\) and \(a \neq 0\). Then \(F(x) \ll p/a\), so \(\sum_{a \neq 0} I_{a,0,0} \ll p^2\). Similarly we deal with two symmetric cases.

Case 3. \(b \neq c \neq 0\). Then \(f(x) \ll \frac{p}{ab}\) and \(\sum_{a,b \neq 0} I_{a,b,b} \ll p^2\)

Case 4. \(a, b \neq 0\), \(a \neq b\) and \(c = 0\). Then
\[
f(x) \ll \frac{pqr}{ab} s((a-b)u_{pq}) s(a_{upq}) s(b_{upr}).
\]
Now, depending on \(a\) and \(b\), we use different bounds on \(s(au)\), \(s(bu)\) and \(s((a-b)u)\). The number \(n\) we determine later. For \(|a| \leq n\) we use \(s(au) \ll |a|u\) and for \(a > n\) we use \(s(au) \leq 1\). Similarly for \(b\) and \(a - b\). We have
\[
\sum_{0 < |a|, |b| \leq n} I_{a,b,0} \ll (pqr)^2 U^6 \sum_{0 < |a|, |b| \leq n} (a-b)^2 \ll (pqr)^2 U^6 n^4,
\]
\[
\sum_{0 < |a| \leq n < |b|} I_{a,b,0} \ll (pqr)^2 U^2 \sum_{0 < |a| \leq n < |b|} \frac{1}{b^2} \ll (pqr)^2 U^2,
\]
\[
\sum_{|a|, |b| > n} I_{a,b,0} \ll (pqr)^2 \sum_{|a|, |b| > n} \frac{1}{a^2 b^2} \ll \frac{(pqr)^2}{n^2}.
\]
The case of the sum \(\sum_{0 < |b| \leq n < |a|} I_{a,b,0}\) is analogous to the second sum above. The optimal choice of \(n\) is \(1/U\). Then we have
\[
\sum_{|a| < p/2; |b| < q/2} I_{a,b,0} \ll (pqr)^2 U^2.
\]
Case 5. It remains to consider distinct $a, b, c \neq 0$. We have

$$f(x) \ll \frac{pq}{abc} s((a - b)u_{pq}) s((b - c)u_{qr}) s((c - a)u_{rp}).$$

Each case is symmetric to one of the following.

$$\sum_{0 < |a|, |b|, |c| \leq n} I_{a,b,c} \ll (pqr)^2 U^6 \sum_{0 < |a|, |b|, |c| \leq n} \frac{(a - b)^2 (b - c)^2 (c - a)^2}{a^2 b^2 c^2},$$

$$\ll (pqr)^2 U^6 n^4,$$

$$\sum_{0 < |a|, |b| \leq n < |c|} I_{a,b,c} \ll (pqr)^2 U^2 \sum_{0 < |a|, |b| \leq n < |c|} \frac{(a - b)^2}{a^2 b^2 c^2} \ll (pqr)^2 U^2,$$

$$\sum_{|a|, |b| > n; |c| > 0} I_{a,b,c} \ll (pqr)^2 \sum_{|a|, |b| > n; |c| > 0} \frac{1}{a^2 b^2 c^2} \ll \frac{(pqr)^2}{n^2}.$$

Note that the obtained bounds are the same as in the previous case, so again we have

$$\sum_{a,b,c \neq 0} I_{a,b,c} \ll (pqr)^2 U^2$$

as desired.

To complete the proof, note that $\max \{p, q, r\}^2 = o(1)(pqr)^2 U^2.$ ∎

### 10. RELATIVES OF CYCLOTOMIC POLYNOMIALS

**Proof of Theorem**. We consider primes $p_1 < p_2 < \ldots < p_k$ with $p_1 \to \infty$, satisfying the congruences

$$p_j \equiv 2(j - i) \pmod{p_i} \quad \text{for all } 1 \leq i < j \leq k.$$

The existence of such primes is guaranteed by the Chinese remainder theorem and Dirichlet’s theorem on primes in arithmetic progressions. Put $N = N_1, N_2, \ldots, N_k$ and $x = \frac{N - 1}{n}$. By Lemma 6 we have

$$F(z) = |P_n(x)| = 2^{k-1} \frac{s(1/2)}{s(p_1)} \prod_{1 \leq i < j \leq k} s_{p_ip_j}(N - 1/2) \prod_{i=1}^k s_{p_i}(i - 1/2).$$

By using Lemma 7 we obtain $s_{p_i}(i - 1/2) \sim \frac{2^{i-1}}{p_i}$ and

$$s_{p_ip_j}(N - 1/2) \sim s_{p_ip_j}((i - j)p_j p_i^*),$$

where $p_j p_i^* = (p_j(p_i - 1)/2) = 1$. It gives

$$F(x) \sim \frac{2^{k-1} n}{\pi^k(2k - 1)!!} = n \cdot 2^{k^2/2 + O(k \log k)},$$

which completes the proof. ∎

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