ON COEFFICIENT FUNCTIONALS ASSOCIATED WITH THE ZALCMAN CONJECTURE

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Abstract. For a function \( f \) which is analytic and univalent in the unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) having the power series expansion of the normalized form \( z + \sum_{n=2}^{\infty} a_n z^n \), Zalcman conjectured that \( |a_n^2 - a_{2n-1}| \leq (n-1)^2 \), \( n = 2, 3, \ldots \). In this article, we obtain the sharp estimate for the classical Zalcman coefficient functional \( a_n^2 - a_{2n-1} \) for the above class of functions with the restriction that the \( n \)-th coefficient, \( a_n \), has certain integral representation associated with probability measure. Moreover, we also study a similar problem for the classes of functions of the above form whose coefficients satisfy certain inequalities.

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REFERENCES

[1] Y. ABU MUHANNA, L. LI, AND S. PONNUSAMY, Extremal problems on the class of convex functions of order \(-1/2\), Arch. Math. (Basel) 103 (6) (2014), 461–471.
[2] O. P. AHUJA AND H. SILVERMAN, A survey on spiral-like and related function classes, Math. Chronicle 20 (1991), 39–66.
[3] L. DE BRANGES, A proof of the Bieberbach conjecture, Acta Math. 154 (1–2) (1985), 137–152.
[4] J. E. BROWN AND A. TSAO, On the Zalcman conjecture for starlike and typically real functions, Math. Z. 191 (3) (1986), 467–474.
[5] P. L. DUREN, Univalent Functions, Springer-Verlag, New York, 1983.
[6] I. EFRAIMIDIS AND D. YUKOTIĆ, Applications of Livingston-type inequalities to the generalized Zalcman functional, Math. Nachr., doi:10.1002/mana.201700022, 1–12.
[7] A. W. GOODMAN, On uniformly starlike functions, J. Math. Anal. Appl. 155 (1991), 364–370.
[8] A. W. GOODMAN, On uniformly convex functions, Ann. Polon. Math. 56 (1) (1991), 87–92.
[9] J. A. KIM AND N. E. CHO, Properties of convolutions for hypergeometric series with univalent functions, Adv. Difference Equ. 2013, 2013:101, 1–11.
[10] S. L. KRUSHKAL, Univalent functions and holomorphic motions, J. Anal. Math. 66 (1995), 253–275.
[11] S. L. KRUSHKAL, Proof of the Zalcman conjecture for initial coefficients, Georgian Math. J. 17 (4) (2010), 663–681. (Erratum in Georgian Math. J., 19 (4) (2012), 777.)
[12] O. S. KWON AND S. OWA, The subordination theorem for \( \lambda \)-spirallike functions of order \( \alpha \), Int. J. Appl. Math. 11(2) (2002), 113–119.
[13] L. LI AND S. PONNUSAMY, Generalized Zalcman conjecture for convex functions of order \(-1/2\), J. Analysis 22 (2014), 77–87.
[14] L. LI AND S. PONNUSAMY, On the generalized Zalcman functional \( \lambda a_n^2 - a_{2n-1} \) in the close-to-convex family, Proc. Amer. Math. Soc. 145 (2017), 833–846.
[15] L. LI AND S. PONNUSAMY AND J. QIAO, Generalized Zalcman conjecture for convex functions of order \( \alpha \), Acta. Math. Hungar. 150 (1) (2016), 234–246.
[16] R. J. LIBERA, Univalent \( \alpha \)-spiral functions, Canad. J. Math. 19 (1967), 449–456.
[17] W. MA, The Zalcman conjecture for close-to-convex functions, Proc. Amer. Math. Soc. 104 (3) (1988), 741–744.
[18] W. Ma, *Generalized Zalcman conjecture for starlike and typically real functions*, J. Math. Anal. Appl. **234** (1) (1999), 328–339.

[19] M. L. Morga and O. P. Ahuja, *On spiral-like functions of order \( \alpha \) and type \( \beta \)*, Yokohama Math. J. **29** (2) (1981), 145–156.

[20] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[21] S. Ponnusamy and K.-J. Wirths, *On the problem of Gromova and Vasil’ev on integral means, and Yamashita’s conjecture for spirallike functions*, Ann. Acad. Sci. Fenn. Math. **39** (2014), 721–731.

[22] V. Ravichandran and S. Verma, *Generalized Zalcman conjecture for some classes of analytic functions*, J. Math. Anal. Appl. **450** (2017), 592–605.

[23] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. **209** (1997), 221–227.