Bosonization in four dimensions: The smooth way

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November 16, 1999

Abstract

I investigate bosonization in four dimensions, using the smooth bosonization scheme. I argue that generalized chiral “phases” of the fermion field corresponding to chiral phase rotations and “chiral Poincaré transformations” are the appropriate degrees of freedom for bosonization. Smooth bosonization is then applied to an Abelian fermion coupled to an external vector. The result is an exact rewriting of the theory, including the fermion, the bosonic fields, and ghosts. Exact bosonization is therefore not achieved since the fermion and the ghosts are not completely eliminated. The action for the bosons is given by the Jacobian of a change of variables in the path integral, and I calculate parts of this. The action describes a nonlinear field theory, and thus static, topologically stable solitons may exist in the bosonic sector of the theory, which become the fermions of the original theory after quantization.

PACS numbers: 11.15.Tk; 11.30.Rd; 11.10.Lm

Keywords: Bosonization; Chiral anomalies; Nonlinear field theories

1 Introduction

Bosonization is an operation which maps the description of a physical system in terms of fermionic fields into a description in terms of bosonic fields. Such operations have been known in relativistic field theory since the seminal paper by Coleman [1], and in condensed matter physics even for a longer time [2]. A more precise statement of bosonization is that there is a correspondence between a certain set of Green’s functions in the fermionic theory with another set in the bosonic theory. Reasons for wanting to bosonize a fermion are for example that the description of the system might be easier to handle mathematically, that it brings insight into the physics of the system, or that it is otherwise useful.

However, bosonization is understood only in two dimensions (2D); it is not yet completely understood how to generalize it to four. Some papers dealing with this problem in the path integral formalism are refs. [3, 4, 5, 6] and references therein. An investigation in the operator formalism is [7]; see also the references in [8]. The results from these papers do not coincide. In the case of the path integral formalism this is a reflection of the fact that a number can be

†Supported by Fonds zur Förderung der wissenschaftlichen Forschung, P11387-PHY

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1I thank J. Paaske and the referee for calling my attention to this reference.
represented by an integral in many different ways. Therefore, bosonization is not unique, and further requirements must be formulated in order to proceed.

What I will do in this paper is to investigate bosonization in the path integral in 4D from another point of view, different from those in the references above, but which perhaps follows more closely the ideas in 2D from Coleman’s paper [1]. Let us recall that in 2D the bosonic equivalent of a massive Dirac fermion is the sine–Gordon model (with coupling constant $\beta = \sqrt{4\pi}$), a non-linear field theory which has static, topologically stable solitons. One of the main messages of ref. [1] is that these solitons have properties which allow us to identify them with fermions in the quantum theory, and indeed Mandelstam [8] showed in the operator formalism that fermionic operators satisfying the Dirac equation (for the Thirring model) can be constructed from the bosonic operators satisfying the sine–Gordon equation.

Thus, we also want something like this in 4D: A nonlinear bosonized theory with static, topologically stable solitons that can be identified with the fermions in the original theory. For this to be at all possible in three space dimensions, it is necessary for topological reasons to have bosonic fields with at least three degrees of freedom. Furthermore, it is necessary to have a mechanism which prevents the solitons from collapsing according to Derrick’s theorem [9], like terms with four derivatives in the Lagrangian, which is the situation in the Skyrme model [10]. We also want a Skyrme-like picture to be possible for an Abelian Dirac fermion. This is different from the Skyrme model where there is a triplet of pseudoscalar fields which transforms under an internal symmetry group (the pion), and where the fermion (the nucleon) acquire a corresponding internal quantum number. I am, however, not aware of any reasons why the internal symmetry could not be spin, leading to a Skyrme soliton with a $U(1)$ quantum number.

Note that if we express the equivalence between the fermionic and bosonic descriptions in terms of Green’s functions, then only Green’s functions of bilinear operators of fermions could be considered, like in 2D. This may seem to imply that some information is lost by bosonization – namely the information in Green’s functions with external spinor lines. However, this is not necessarily the case. In 2D this information is encoded in the solitons of the bosonic theory, as is verified from Mandelstam’s construction of the Dirac field operators [3]. This is one way to understand our requirement of 4D bosonization that Skyrme-like fermionic solitons should appear in the theory: There is then a chance that spinor information is encoded in the bosonic theory.

Another thing we recall from 2D is that in the path integral investigations of bosonization the bosonic field equivalent to the Dirac fermion is essentially the chiral phase of the fermion. Moreover, the Lagrangian for the boson is in one way or another connected with the Jacobian of a chiral phase rotation of the fermion (see for instance ref. [11]), that is, if we write the Jacobian as $J = \exp(i \int d^2x \mathcal{L}_J)$, then the Lagrangian is connected with $\mathcal{L}_J$. The exact connection depends on the particular bosonization scheme, but $\mathcal{L}_J$ always contains at least the kinetic terms of the bosonic fields. Thus, in 4D we should be looking for something similar to the chiral phase of the fermion, which gives rise to a Jacobian when the fermion is transformed, and such that this Jacobian can be connected with the Lagrangian for the bosonic field.

To summarize our requirements, we want:

(i) “generalized chiral phases” of the fermion, which should be fields with at least three internal degrees of freedom, like a vector or a tensor,

(ii) that the “generalized chiral transformations” for the fermion give rise to a Jacobian when the variables are changed in the path integral, and when a suitable regularization scheme is applied, and

(iii) the Jacobian $J = \exp(i \int d^4x \mathcal{L}_J)$ must be such that the Lagrangian quantity $\mathcal{L}_J$ contains, at least, the kinetic terms of the chiral fields, and also other terms with four derivatives that can stabilize solitons in the bosonic theory.
I emphasize that these are not the only requirements one can make for 4D bosonization, but I think they are a reasonable starting point if we want to generalize from 2D the idea that solitons are fermions in the quantum theory.

As we shall see, we are able to meet these requirements when the generalized chiral phases are degrees of freedom (DOFs) connected with what may be called “chiral Poincaré transformations”. These are (global) transformations that act on the Dirac fermion in the same way as ordinary Lorentz transformations and translations, except that a $\gamma_5$ is multiplied onto the generators $J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} + (x_{\mu} i \partial_{\nu} - x_{\nu} i \partial_{\mu}) \equiv S_{\mu\nu} + L_{\mu\nu}$ and $P_\mu = i \partial_\mu$. The reason I consider exactly these DOFs is the following: The particle states in a relativistic quantum field theory are labelled by internal quantum numbers, spin, and mass, according to which representation of the internal symmetry group and Poincaré group they belong. Since these properties are connected with “phase rotations”, Lorentz transformations, and translations, and since the chiral phase is likely to take part in bosonization, then it is also likely that chiral Lorentz DOFs and chiral translation DOFs take part as well. Such transformations are not in general symmetries of the Lagrangian, but naively they are symmetries of the fermionic measure in the path integral. However, in ref. [12] it was shown that they give rise to a Jacobian when a change of fermionic variables is made in the regularized path integral. Moreover, the fields connected with these DOFs are an axial vector and a tensor, which may have nontrivial topology. Thus, we may regard the quantity $L_J$ for these chiral transformations as a candidate for the Lagrangian of a bosonized theory.

It is then necessary to show that this bosonic theory really is equivalent to the original fermionic one, or that it is in some sense a partial bosonization. In 2D an appealing method to do bosonization in the path integral is the “smooth bosonization scheme” of Damgaard, Nielsen and Sollacher [13, 14]. This is the scheme I will adopt here for 4D – hence the title of this paper. (This scheme has already been used to obtain partial bosonization in 4D for some special cases [4, 5].) Briefly stated, the method is to perform a change of variables in the path integral, using instead a fermion that is locally rotated with a chiral phase $\theta(x)$. This gives a Jacobian $J[\theta] = \exp(i \int d^2 x L_J[\theta])$, assuming an appropriate regularization is used. The field $\theta$ enters into the expression for the path integral, and the theory is enlarged by promoting it to a dynamical field, meaning that we path integrate over it. The theory now has a new local symmetry. This symmetry can be gauge-fixed, and by carefully choosing the gauge-fixing condition, we can get either the original fermionic theory or a new bosonic theory – the bosonized theory. The Lagrangian for this theory is $L_J[\theta]$. I emphasize that this bosonization scheme is based on exact manipulations of the path integral, and uses only familiar concepts from field theory – in particular gauge-fixing and chiral anomalies.

In this paper, I will investigate the bosonization of a massive Abelian Dirac fermion, coupled to an external vector field, along these lines. I will show how the new “generalized chiral phases” of the fermion, corresponding to chiral Poincaré transformations, can be introduced into the path integral. Then I will discuss how the new local symmetry of the path integral can be gauge-fixed, and I will use the gauge-fixing constraints for the chiral phase and chiral Poincaré symmetry, along with the conservation of the current, energy-momentum and angular momentum tensors, to try to eliminate the fermion from the theory. The resulting theory includes a Lagrangian $L_J$ for a pseudoscalar $\theta$, a tensor $\phi_{\mu\nu}$ and an axial vector $b_\mu$. These fields are the parameters of the local chiral transformations. Unfortunately, the ghosts do not decouple in the chosen gauge, and, furthermore, a “small” residual part of the fermion is not eliminated. Nevertheless, the resulting theory may still be useful. Perhaps it can serve as a starting point for obtaining a low energy effective description of the system.

The organization of the paper is the following. In sec. 2, I briefly review the smooth bosonization scheme [13]. The scheme was first applied to a massless fermion; the massive case was
considered in ref. [14]. However, the treatment of the massive case in ref. [14] was not as straightforward as for the massless case. In particular, the simple result that the bosonic Lagrangian equals $L_J$ was not found. I therefore reconsider this massive case, and find that one may indeed achieve that the bosonic Lagrangian is $L_J$. What makes this work can be understood as the effective vanishing of the axial current (and therefore also the current), which is a consequence of the combined effects of gauge-fixing and current conservation.

In sec. 3, I begin my investigations of a massive fermion coupled to a vector in 4D. First I introduce the chiral phase $\theta$ into the theory, inspired by the 2D case. The appropriate gauge-fixing for this DOF, together with energy-momentum and angular momentum conservation, is used to try to make the axial current effectively vanish. Then I introduce the chiral Lorentz and chiral translation “phases”, $\phi_{\mu\nu}$ and $b_\mu$, into the theory in an analogous way and use gauge-fixing of these DOFs, together with current conservation, to try to get the effective vanishing of the current. The idea is that the vanishing of both the current and the axial current should be enough to eliminate the fermion from the theory. The exact theory which results from this is given by a path integral with the original fermion, the bosonic fields (including Lagrange multipliers for the gauge-fixing delta functions), and ghosts.

In sec. 4, I discuss the calculation of the Jacobian of the chiral transformation, which would describe the bosonic theory in the best of all worlds where bosonization is exact. The complete Lagrangian $L_J$ is probably too hard to calculate exactly. However, the part containing only $\theta$ can be calculated exactly. I also calculate the Gaussian part (i.e. up to two orders in the field) of the spin part of the action for $\phi_{\mu\nu}$. Terms involving $L_{\mu\nu}$ and the Lagrangian for $b_\mu$ (which involves $P_\mu$) are not calculated. The terms that are left out are thereby small in the weak field, low energy limit. The lowest order coupling terms between $\theta$ and $\phi_{\mu\nu}$ are also calculated.

In sec. 5, I summarize and discuss the problems which prevented us from getting exact bosonization in sec. 3. I sketch how to generalize the results to the non-Abelian case and discuss some aspects of the physics of the bosonized theory.

\section{Smooth bosonization in 2D: The massive case revisited}

The smooth bosonization scheme has been used in 2D to bosonize a Dirac fermion coupled to a vector and axial vector [13], and a Dirac fermion coupled to scalar and pseudoscalar mass terms [14]. The bosonization procedures are completely different in these two papers. In the first paper the bosonic action is found from the Jacobian of the chiral rotation, while in the second paper it is the result of integrating out the fermion. I will demonstrate here that also mass term bosonization can be obtained from the Jacobian. This will also serve as a review of the smooth bosonization scheme. The point in performing this exercise is that it suggests that the identification of the quantity $L_J$ in the expression $J = \exp(i \int d^2 x L_J)$ for the Jacobian with the Lagrangian of the bosonic theory is a general feature of this scheme, and may be expected also in 4D.

The path integral with mass terms is

$$Z[m,m^\dagger] = \int D\psi D\bar{\psi} \exp i \int d^2 x \bar{\psi} [i\bar{\theta} - m P_+ - m^\dagger P_-] \psi, \quad (1)$$

where $P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5)$ and the chiral mass $m$ is defined from the scalar and pseudoscalar mass, $S$ and $P$, by $m(x) \equiv S(x) + iP(x)$. The chiral change of variables

$$\psi(x) \rightarrow e^{i\theta(x)\gamma_5} \bar{\psi}(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\theta(x)\gamma_5} \quad (2)$$
leads to the rotated path integral
\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left( i \int d^2x \left( \bar{\psi} [i\bar{\partial} - \bar{\theta}\gamma_5 - me^{2i\theta} P_+ - m\bar{e}^{-2i\theta} P_-] \psi \\
+ \frac{1}{2\pi} \partial_\mu \theta \partial^\mu \theta + \frac{1}{4\pi} \kappa_1 m(e^{2i\theta} - 1) + \frac{1}{4\pi} \kappa_1 m\bar{e}^{-2i\theta} - 1 + O(m^2) \right) \right).
\]
(3)

An appropriate regularization scheme is assumed here, one where the current is conserved\(^2\). This gives the second line in eq. (3) as the contribution from the Jacobian. \(\kappa_1\) is an arbitrary massive parameter. Note that the kinetic term and the terms proportional to \(m\) in the Jacobian is the desired result for the Lagrangian of the bosonic theory.

We now promote the local parameter \(\theta\) to a dynamical field and thus include the path integration over this field. The path integral does not depend on \(\theta\) (at this point), so this produces an irrelevant infinite numerical factor which is absorbed in the path integration measure. There is now a local symmetry
\[
\psi(x) \rightarrow e^{i\lambda(x)\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\lambda(x)\gamma_5}, \quad \theta(x) \rightarrow \theta(x) - \lambda(x)
\]
in the system, provided the transformation of the measure is included.

According to the smooth bosonization scheme, this local symmetry is viewed as an ordinary gauge symmetry which can be gauge-fixed. The gauge fixing procedure is then to choose a gauge-fixing function \(\Phi(x)\) and to insert a functional delta function
\[
\delta(\Phi) = \int \mathcal{D}\beta e^{i \int d^2x \beta \Phi} \equiv \int \mathcal{D}\beta e^{i \int d^2x L_{gf}}
\]
and a Faddeev–Popov determinant
\[
\text{Det} \left( \frac{\delta \Phi}{\delta \lambda} \right) = \int \mathcal{D}c \mathcal{D}\bar{c} e^{i \int d^2x (\bar{c} \frac{\delta \Phi}{\delta \lambda})c} \equiv \int \mathcal{D}c \mathcal{D}\bar{c} e^{i \int d^2x L_{\text{ghosts}}}
\]
into the path integral. It is necessary to include the anomalous variation of the gauge-fixing function in order to get the correct Faddeev–Popov determinant. In other words, it is necessary to include the contribution from the Jacobian for the BRST variation
\[
\delta(-c\Phi) = \beta \Phi + \bar{c} \left( \frac{\delta \Phi}{\delta \lambda} \right) c
\]
\[
= L_{gf} + L_{\text{ghosts}}
\]
that is added to the Lagrangian, see the discussion in ref. [13].

One can now consider a general gauge-fixing function \(\Phi\) which interpolates between fermionic and bosonic variables. In this paper I am only interested in gauges which give bosonization, thus \(\Phi\) should depend on the fermion fields only. The choice I make is
\[
\Phi = \partial_\mu j_5^\mu, \quad j_5^\mu \equiv \bar{\psi} \gamma_\mu \gamma_5 \psi.
\]
(8)

This is the preliminary gauge-fixing chosen in ref. [13] (eq. (20) in that paper with \(\Delta = 1\)). It is one of the “endpoints” of a smooth gauge; the other one is \(\Phi = \Box \theta/\pi\) (\(\Delta = 0\) in eq. (20) of ref. [13]) which returns the original fermionic theory. Eventually, another gauge-fixing condition was

\(^2\)The requirement of current conservation alone does not completely fix the regularization scheme. The ambiguities can be resolved by imposing Bose symmetry in triangle diagrams instead \[13\]. The conservation of the current then follows.
adopted in ref. [13], which was a formal integration of eq. (8), in order to take care of certain zero modes. However, in this paper I will assume that appropriate boundary conditions have been imposed on the fields of the theory in order to make all free fields vanish. Then there are only trivial zero modes in the theory. When we set $\Phi = 0$, it gives $\partial_\mu j_5^\mu = 0$, and this will then decouple the fermion from the bosonic field $\theta$ – except for the mass terms – as seen from the path integral (3). Furthermore, in this gauge the ghosts decouple.

Thus, the problem is to decouple also the fermionic mass terms from $\theta$. In the treatment of this problem in ref. [5] the massive parameter $\kappa_1$ in the Jacobian is chosen to be zero. Instead the required terms – linear in $m$ and $m^\dagger$ – are generated by rewriting parts of the theory into the Schwinger model with a “perturbation” and evaluating a few relevant expectation values. However, we can solve this problem in another way if we make use of Coleman’s results [1]. The point is that decoupling will occur if the expression

$$
\int D\beta D\psi D\bar{\psi} \exp i \int d^2 x \bar{\psi} [i\partial \bar{\psi} - \partial_\beta \gamma_5 - mP_+ e^{2i\theta} - m^\dagger P_- e^{-2i\theta}] \psi
$$

(9)
is unity. By using Coleman’s results, we can bosonize this path integral. We get

$$
\int D\beta D\phi D\bar{\psi} \exp i \int d^2 x \left( \frac{1}{2\pi} \partial_\mu \phi \partial^\mu \phi + \partial_\mu \frac{1}{\pi} \bar{\phi} \right)

\left( \frac{1}{4\pi} \kappa me^{2i\theta} (e^{2i\phi} - 1) + \frac{1}{4\pi} \kappa m^\dagger e^{-2i\theta} (e^{-2i\phi} - 1) + O(m^2) \right).
$$

(10)

Here, the massive parameter $\kappa$ is arbitrary. It is clear from consistency reasons that it must be equal to $\kappa_1$, and also that the terms of $O(m^2)$ should be present, but this is not needed for the argument. (The terms of $O(m^2)$ do not appear in Coleman’s expressions, probably as a consequence of his definition of the composite operators $\sigma_\pm(x)$ in terms of point-splitting [1].) We should also recall that the replacement of eq. (9) with eq. (10) holds in the sense of a perturbative expansion in $m$ and $m^\dagger$. This is what Coleman showed, and is a result that to our knowledge has not been improved upon. Now it is seen that the path integration over $\beta$ gives a delta function for $\phi$ to be a free field, hence to vanish, and so the expression is unity.

This demonstrates that $\mathcal{L}_J$ is the Lagrangian of the bosonized theory within the smooth bosonization scheme, even for massive fermions. Unfortunately, we needed Coleman’s results to prove it. Therefore, what we just did amounts to a consistency check, and not an independent derivation of bosonization. Similar results are not available in 4D because it involves statements about the explicit forms of nontrivial Green’s functions. However, this has not necessarily been a useless exercise if we can understand why the condition for the divergence of the axial current to vanish was strong enough to decouple the mass terms. Let us now consider this point.

First of all, let us observe that there is phase rotation invariance in the theory, which implies a conservation equation for the current:

$$
\partial_\mu j^\mu = 0, \quad j_\mu \equiv \bar{\psi} \gamma_\mu \psi.
$$

(11)

Note the similarity between this conservation equation and the gauge-fixing condition. Indeed,

$$
\Psi \equiv \partial_\mu j^\mu
$$

(12)
is itself a gauge-fixing function if we carry out the same procedure for the phase rotations as we have just done for the chiral phase rotations. The new “dynamical” field $\alpha$ then couples to the fermion through

$$
\int D\alpha \exp i \int d^2 x \alpha \partial_\mu j^\mu = \int D\alpha \exp i \int d^2 x \Psi = \delta(\Psi),
$$

(13)
and is therefore the field that implements the delta function for \( \Psi \). Strictly speaking, we have not really done any gauge-fixing, in the sense of inserting a further delta function and an associated Faddeev–Popov determinant, and therefore there are no ghosts.

The point is now this: Let us consider the axial current and make the Hodge decomposition

\[
\bar{j}_5^\mu = \partial_\mu \xi + \epsilon_{\mu\nu} \partial_\nu \eta. \tag{14}
\]

(I work in a spacetime with trivial topology in this paper, so there is no harmonic form in the decomposition.) The gauge-fixing condition now implies

\[
\Box \xi = 0. \tag{15}
\]

Conversely, due to the property

\[
\gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu \gamma_5 \tag{16}
\]

of the gamma matrices in two dimensions, current conservation implies

\[
- \epsilon_{\mu\nu} \partial_\nu \epsilon^{\nu\rho} \partial_\rho \eta = \Box \eta = 0. \tag{17}
\]

This means that effectively

\[
\Box \bar{j}_5^\mu = 0 \tag{18}
\]

in the path integral, i.e. in the sense of a delta function constraint in the path integral. In other words, the axial vector vanishes.

We are therefore dealing with a theory where the axial current, hence also the current, vanishes. Could this be the “real” reason for the decoupling of \( \theta \) from the mass terms? The following heuristic argument suggests that it is: Since the axial current vanishes, we have a local condition (it applies pointwise) which effectively puts two DOFs of the fermion to zero. However, a physical Dirac fermion in two dimensions only have two DOFs, which means that there were not any DOFs “left over” for the mass terms.

This argument works in 4D as well if we can find similar local constraints. The idea is to count the DOFs affected by these constraints, and conclude that there is no room left for mass terms or other bilinear combinations of the fermion field. This leads to our strategy for smooth bosonization in 4D. The local constraints that I will consider in 4D are to have both the vector and the axial vector current vanish. Thus, my choice of gauge-fixing functions will be guided by this, rather than trying to find gauges where the ghosts decouple (which may not even exist).

3 Smooth bosonization of an Abelian fermion in 4D

I consider an Abelian fermion of mass \( m \) coupled to an external vector \( A_\mu(x) \). The theory is described by the path integral

\[
Z[A] = \int \mathcal{D}\psi \mathcal{D} \bar{\psi} e^{i \int d^4x \mathcal{L}},
\]

\[
\mathcal{L} = \bar{\psi} \left[ i \partial - A - m \right] \psi. \tag{19}
\]

I will briefly consider generalizations to non-Abelian fermions later.

First let us try the chiral phase of the fermion and see how far we can get with bosonization in this case. Of course, our requirements for the bosonized theory can not be fulfilled with only the chiral phase, because topologically stable configurations can not be achieved with just one
pseudoscalar field, but we will get an idea of what is going on. Thus, we perform the change of variables

$$\psi \to e^{i\theta\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\theta\gamma_5}$$  \hspace{1cm} (20)

in the path integral, which then becomes

$$Z[A] = J[\theta] \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{i \int d^4x \mathcal{L}'}$$

where

$$\mathcal{L}' = \bar{\psi}[-\partial_\theta\gamma_5 - A - me^{2i\theta\gamma_5}]\psi,$$

and $J[\theta] = \exp(iS_J[\theta])$ is the Jacobian of the transformation. According to our previous considerations, $S_J[\theta]$ is an action for the field $\theta$ and a part of the final action for the bosonized theory. I will consider the difficult problem of how to calculate this in the next section, but we do not need the explicit form here.

By the smooth bosonization scheme we now path integrate over $\theta$ and insert a gauge-fixing delta function and Faddeev–Popov determinant. An obvious choice of gauge-fixing function is

$$\Phi = \partial_\mu j^5_\mu.$$  \hspace{1cm} (22)

This decouples the axial current from $\theta$, leaving only the mass term to couple $\theta$ and the fermion.

The Faddeev–Popov determinant can now be found by adding $\beta \Phi$ to the Lagrangian and performing a gauge transformation. There is actually a subtlety connected with this, since the new terms in the Lagrangian is of the form

$$\left(\beta F_1(\theta) + \beta^2 F_2(\theta) + \beta^3 F_3(\theta)\right) \delta \lambda,$$  \hspace{1cm} (23)

where $\delta \lambda$ is the parameter of the gauge transformation. This can be found from a calculation similar to that described in the next section. Thus it appears that the path integration over $\beta$ no longer gives a delta function. It turns out, however, that only $F_1$ contributes to the Faddeev–Popov determinant:

$$\frac{\delta \Phi}{\delta \lambda} = F_1(\theta).$$  \hspace{1cm} (24)

This gives the BRST invariant result, and can be verified by the alternative procedure of adding $\delta(-\bar{\psi})$ to the Lagrangian. $F_1$ is a derivative operator depending on $\theta$. I will not give the explicit form here, mainly because it is tedious to calculate, but also because the inclusion of further DOFs will modify it, and because we will not need the explicit form. Thus the ghosts do not decouple in this gauge. However, the meaning of the path integral over $\beta$ as a delta function is still intact.

Let us see if the gauge-fixing condition leads to the vanishing of the axial current $j^5_\mu$ itself. To find out we perform a Hodge decomposition like we did in 2D. It is

$$j^5_\mu = \partial_\mu \xi + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \eta^\rho. \hspace{1cm} (25)$$

The gauge-fixing condition leads immediately to $\Box \xi = 0$. $j^5_\mu$ will then be a free field – hence vanish – if $\eta_{\mu\nu}$ is a free field. This happens if

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho j^5_\sigma = 0.$$  \hspace{1cm} (26)

Using the property

$$\frac{1}{2} \{\sigma_{\mu\nu}, \gamma_\rho\} = -\epsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma_5,$$  \hspace{1cm} (27)
of the gamma matrices, eq. (24) is equivalent to
\[ \partial_\mu \left( \frac{i}{2} \bar{\psi} \left\{ \frac{1}{2} \sigma^{\mu\nu}, \gamma^\nu \right\} \psi \right) = 0. \] (28)

We can compare this to the equation for the conservation of the angular momentum current:
\[ \partial_\mu j_\mu^{\alpha\beta} \equiv \partial_\mu \left( \frac{1}{2} \bar{\psi} \left\{ \gamma^\mu, \frac{1}{2} \sigma^{\alpha\beta} \right\} \psi + \bar{\psi} \gamma^\mu L^{\alpha\beta} \psi \right) = 0, \] (29)

where \( L_{\mu\nu} = x_\mu i \partial_\nu - x_\nu i \partial_\mu \) is the orbital angular momentum operator. Thus, the spin part of this conservation equation, together with the gauge-fixing condition, would make the axial current a free field.

The orbital part of the conservation equation can also be written
\[ \partial_\mu (x^\alpha \Theta_{\mu\beta} - x^\beta \Theta_{\mu\alpha}), \] (30)
where
\[ \Theta_{\mu\nu} \equiv \bar{\psi} \gamma_\mu i \partial_\nu \psi \] (31)
is the energy-momentum tensor. But translation invariance implies
\[ \partial_\mu \Theta_{\mu\nu} = 0, \] (32)
and the orbital part, eq. (30), becomes
\[ \partial_\mu (x^\alpha \Theta_{\mu\beta} - x^\beta \Theta_{\mu\alpha}) = \Theta^{\alpha\beta} - \Theta^{\beta\alpha}. \] (33)

Thus, gauge-fixing and Poincaré invariance does not lead to the vanishing of the axial current as a local constraint because the energy-momentum tensor of a Dirac fermion is not symmetric.

Let us now consider the chiral Poincaré transformations. First, we concentrate on the chiral Lorentz transformations, acting on the Dirac fermion by
\[ \psi \rightarrow e^{i \frac{1}{2} \phi_{\mu\nu} J^{\mu\nu} \gamma_5} \psi \quad \bar{\psi} \rightarrow \bar{\psi} e^{i \frac{1}{2} \phi_{\mu\nu} J^{\mu\nu} \gamma_5}. \] (34)

This is a global transformation, and a symmetry of the kinetic part of the Lagrangian. Couplings to vectors and mass terms (and tensors) break the symmetry explicitly. In addition there are anomalies, which are crucial to our discussion since they are responsible for the Jacobian.

In order to bosonize these DOFs we proceed as for the chiral phase: A local change of variables in the path integral, with now the field \( \phi_{\mu\nu}(x) \) as parameter of the transformation, is performed. This gives rise to a Jacobian which depends on this field, and to which we will return in the next section. The fermionic part of the new transformed Lagrangian is
\[ \mathcal{L}' = \bar{\psi} i \partial_\mu - \frac{1}{2} \partial_\mu \phi_{\alpha\beta} \left( \frac{1}{2} \{ \gamma^\mu, \frac{1}{2} \sigma^{\alpha\beta} \} \gamma_5 + \gamma^\mu \gamma_5 L^{\alpha\beta} \right) - \mathcal{A} - \phi_{\mu\nu} \left( A^\nu \gamma^\mu \gamma_5 + x^\mu \partial^\nu A^\gamma_5 \right) - m - im \phi_{\mu\nu} J^{\mu\nu} \gamma_5 \psi + \cdots. \] (35)

I have only given the terms of lowest order in \( \phi_{\mu\nu} \); the dots refer to higher orders. It is understood that a symmetric product of \( \phi_{\mu\nu} \) and \( J_{\mu\nu} \) is used, and derivative operators are symmetrized to act both to the right and to the left, all in order to get a real expression. From the last term in eq. (35), the coupling term involving the mass \( m \), it is seen that \( \phi_{\mu\nu} \) must have the same properties under discrete symmetries as the bilinear combination \( \bar{\psi} \sigma_{\mu\nu} \gamma_5 \psi \) in order not to come in conflict with these discrete symmetries. This means that the dual tensor \( \tilde{\phi}_{\mu\nu} \) transforms in the same way as the electromagnetic tensor \( F_{\mu\nu} \).
The bosonic field $\phi_{\mu\nu}$ couples in a complicated way to the fermion. But a choice for a gauge-fixing function which almost suggests itself (by analogy to the previous coupling to $\theta$) is

$$\Phi^{\alpha\beta} = \partial_\mu \left( \frac{1}{2} \bar{\psi} \left\{ \gamma^\mu, \frac{1}{2} \sigma^{\alpha\beta} \right\} \gamma_5 \psi + \bar{\psi} \gamma^\mu \gamma_5 L^{\alpha\beta} \psi \right) \equiv \partial_\mu j_5^\mu{}^{\mu,\alpha\beta}. \quad (36)$$

The gauge-fixing condition leads to a manifest decoupling only between $\psi$ and $\phi_{\mu\nu}$ in the lowest order term with $\partial_\mu \phi_{\alpha\beta}$ in $L'$, but the sense of the decoupling is implicitly stronger.

The remarks about the gauge-fixing procedure for the chiral phase apply here as well. The delta function leads to the term

$$\mathcal{L}_{\text{gf}} = \frac{1}{2} \beta_{\mu\nu} \Phi^{\mu\nu} \quad (37)$$

in the Lagrangian, where $\beta_{\mu\nu}$ is the Lagrange multiplier field. The Faddeev–Popov determinant gives

$$\mathcal{L}_{\text{ghosts}} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \left( \delta \Phi^{\mu\nu} / \delta \lambda^{\rho\sigma} \right) e^{\rho\sigma}, \quad (38)$$

where $\delta \Phi^{\mu\nu} / \delta \lambda^{\rho\sigma}$ is a complicated derivative operator depending on $\phi_{\mu\nu}$.

Let us investigate the spin part of the gauge-fixing condition. It is

$$\partial_\mu \left( \frac{1}{2} \bar{\psi} \left\{ \gamma^\mu, \frac{1}{2} \sigma^{\alpha\beta} \right\} \gamma_5 \psi \right) = 0, \quad (39)$$

which is equivalent to

$$\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \partial_\mu j_\nu = 0 \quad (40)$$

due to the property (27). The meaning of this equation is revealed if we Hodge decompose the vector $j_\mu$:

$$j_\mu = \partial_\mu \xi + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial_\nu \eta^{\rho\sigma}. \quad (41)$$

From this we can deduce that eq. (40) implies $\Box \eta_{\mu\nu} = 0$. Moreover, from phase rotation invariance, we have the current conservation equation

$$\partial_\mu j^\mu = 0, \quad (42)$$

from which also $\Box \xi = 0$ follows. Thus, modulo the orbital part of the gauge-fixing condition, we have effectively a delta function in the path integral which ensures that the current $j_\mu$ is a free field.

Then what about the orbital part? This reads

$$\partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 L^{\alpha\beta} \psi) = \partial_\mu (x^\alpha \Theta_5^{\mu\beta} - x^\beta \Theta_5^{\mu\alpha}), \quad (43)$$

where I have introduced the “axial energy-momentum tensor”

$$\Theta_5^{\mu\nu} \equiv \bar{\psi} \gamma^\mu \gamma_5 i \partial^\nu \psi. \quad (44)$$

At this point we introduce further DOFs in the theory, corresponding to chiral translations. Thus we make the change of variables

$$\psi \rightarrow e^{ib_\mu P^\mu} \gamma_5 \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{ib_\mu P^\mu} \gamma_5 \quad (45)$$
in the path integral. This leads to a Jacobian and to the new Lagrangian
\[
\mathcal{L}' = \bar{\psi} [i\partial - b_\mu \gamma^\mu \gamma_5 \partial^\nu - A - b_\mu \partial^\mu A \gamma_5 - m - 2im b_\mu P^\mu \gamma_5] \psi + \cdots.
\] (46)
The field \(b_\mu\) is an axial vector.

To fix the new gauge symmetry we choose the condition
\[
\Phi^\nu = \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 i \partial^\nu \psi) = \partial_\mu \Theta^{\mu \nu}_5 = 0.
\] (47)
The new terms in the Lagrangian from the gauge-fixing procedure are
\[
\mathcal{L}_{gf} = \beta_\mu \Phi^\mu
\] (48)
and
\[
\mathcal{L}_{\text{ghosts}} = \bar{c}_\mu \left( \frac{\delta \Phi^\mu}{\delta \lambda^\nu} \right) c^\nu.
\] (49)

In the presence of the delta function for the gauge-fixing condition, the orbital part of the chiral Lorentz gauge-fixing function, eq. (43), effectively becomes
\[
\Theta_5^{\mu \nu} - \Theta_5^{\nu \mu}.
\] (50)
Thus, an obstacle for having effectively a vanishing current \(j_\mu\) is the antisymmetric part of \(\Theta_5^{\mu \nu}\).

We have now included all the chiral DOFs advertised in the Introduction in the path integral. Putting all together, we come to the following result:
\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} [a] \mathcal{D}[b] \mathcal{D}[c] \mathcal{D}[\bar{c}] e^{i \int d^4x \mathcal{L}_{\text{eff}}}
\] (51)
where
\[
\mathcal{D}[a] = \mathcal{D}\theta \mathcal{D}\phi_{\mu \nu} \mathcal{D}b_\mu,
\]
\[
\mathcal{D}[b] = \mathcal{D}\beta \mathcal{D}\beta_{\mu \nu} \mathcal{D}\beta_\mu,
\]
\[
\mathcal{D}[c] \mathcal{D}[\bar{c}] = \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}c_{\mu \nu} \mathcal{D}\bar{c}_{\mu \nu} \mathcal{D}c_\mu \mathcal{D}\bar{c}_\mu,
\] (52)
and the effective Lagrangian is
\[
\mathcal{L}_{\text{eff}} = \mathcal{L}' + \mathcal{L}_{gf} + \mathcal{L}_{\text{ghosts}} + \mathcal{L}_J,
\] (53)
where
\[
\mathcal{L}' = \bar{\psi} e^{iB \gamma_5 [i\partial - A - m]} e^{iB \gamma_5} \psi, \quad B \equiv \theta + \frac{i}{2} \phi_{\mu \nu} J^{\mu \nu} + b_\mu P^\mu,
\]
\[
\mathcal{L}_{gf} = \beta \Phi + \frac{1}{2} \beta_{\mu \nu} \Phi_{\mu \nu} + \beta_\mu \Phi^\mu,
\]
\[
\mathcal{L}_{\text{ghosts}} = \bar{c} \left( \frac{\delta \Phi}{\delta \lambda} \right) c + \frac{1}{2} \bar{c}_{\mu \nu} \left( \frac{\delta \Phi_{\mu \nu}}{\delta \lambda^{\sigma \nu}} \right) c^{\sigma \nu} + \bar{c}_\mu \left( \frac{\delta \Phi^\mu}{\delta \lambda^\nu} \right) c^\nu,
\]
\[
\int d^4x \mathcal{L}_J = -i \ln J[B].
\] (54)

In addition, the three conservation equations (current, angular momentum and energy-momentum) implicitly hold. They can be included explicitly by introducing the Lagrange multiplier \(\alpha\), \(\alpha_{\mu \nu}\) and \(\alpha_\mu\), and adding
\[
\alpha \partial_\mu j^\mu + \frac{1}{2} \alpha_{\mu \nu} \partial_\mu j^{\mu \nu} + \alpha_\mu \partial_\nu \Theta^\nu\mu
\] (55)
to the effective Lagrangian.

It is now clear that exact bosonization fails, at least with only the chiral DOFs that we have considered. The currents do not vanish, and the ghosts do not decouple. The meaning of this result, and the question of how far we are from exact bosonization, will be discussed in sec. 5.
4 The bosonic action

I now turn to the problem of calculating the quantity $\mathcal{L}_J$ from the Jacobian $J = \exp(i \int d^4x \mathcal{L}_J)$ of the chiral transformations. This is the Lagrangian for the would-be bosonized theory.

Technically, the full change of variables
\[ \psi \to e^{iB\gamma_5} \psi, \quad \bar{\psi} \to \bar{\psi} e^{iB\gamma_5}, \]
(56)
with $B$ as in eq. (54), is difficult to implement. We may anticipate that the action $S_J$ is a complicated nonlinear functional of the bosonic fields, containing infinitely many derivatives and angular momentum operators. Approximations are therefore necessary. Observe that the latter features come from the derivatives in the infinitesimal generators, that is, their spacetime parts. The first approximation I will adopt for my calculations is to ignore these parts of the generators, that is, I will ignore the orbital part of $J_{\mu\nu}$ in the term with $\phi_{\mu\nu}$, and the field $b_{\mu}$ altogether. Intuitively this is a low energy approximation.

Still further simplifications are necessary. I will consider the action for $\theta$ and $\phi_{\mu\nu}$ separately to begin with; interactions between the fields will be investigated later.

The Jacobian $J[\theta] = \exp(iS_J[\theta])$ can be calculated exactly provided the regularization scheme is carefully chosen. There are two things we must pay special attention to. The first one is that phase and Poincaré invariance must be respected. We can achieve this by using the proper time regularization scheme described in ref. [12]. This is not an unimportant point, because an axial vector will appear in the Dirac operator in intermediate calculations, and this may destroy Poincaré invariance for certain schemes, see ref. [12]. But other schemes should also be possible. The second thing is that proper time regularization by itself is quadratically divergent. For our calculations we have removed the quadratic divergences by extending the regularized proper time integral into a Pauli–Villars sum [16]. The procedure is described in ref. [17], where it is used for the calculation of chiral anomalies in the path integral.

The calculation proceeds by integrating up a sequence of infinitesimal chiral rotations, and the result is
\[
\mathcal{L}_\theta = \frac{1}{24\pi^2} \theta \Box^2 \theta + \frac{1}{12\pi^2} (\partial_\mu \theta \partial^\mu \theta)^2
\]
\[
- \frac{m^2}{12\pi^2} \partial_\mu \theta \partial^\mu \theta - \frac{m^2}{6\pi^2} \left(e^{4i\theta} + e^{-4i\theta}\right) \partial_\mu \theta \partial^\mu \theta
\]
\[
- \frac{m^4}{24\pi^2} \left(e^{4i\theta} + e^{-4i\theta} - 2\right) - \frac{m^4}{192\pi^2} \left(e^{8i\theta} + e^{-8i\theta} - 2\right)
\]
\[
+ \frac{1}{8\pi^2} \theta F \tilde{F},
\]
(57)
with $S_J = \int d^4x \mathcal{L}_\theta$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is a kind of generalized sine–Gordon Lagrangian; the last term is the familiar Adler–Bell–Jackiw (ABJ) anomaly. I repeat that this is an exact result. There is no hidden dependence on the regularization scheme in terms of carefully chosen parameters, and the cutoff has been taken to infinity. Furthermore, there are no higher powers of derivatives than four.

I will return to the physics of $\mathcal{L}_\theta$ in the next section, but let us here expand it in powers of $\theta$ and keep only the Gaussian part – a weak field approximation. We also ignore the coupling to the vector field. We get
\[
\mathcal{L}_\theta = \frac{1}{24\pi^2} \theta \Box^2 \theta + \frac{5m^2}{12\pi^2} \theta \Box \theta + \frac{m^4}{\pi^2} \theta^2 + \cdots.
\]
(58)
It is more convenient to rewrite this as
\[ \mathcal{L}_\theta = \frac{1}{24\pi^2} \theta \left( \Box + 10m^2 \Box + 24m^4 \right) \theta + \cdots = \frac{1}{24\pi^2} \theta (\Box + 6m^2)(\Box + 4m^2) \theta + \cdots. \] (59)

Thus \( \theta \) has a negative metric, assuming the sign of the metric is defined from the Gaussian part of the action. Let us now try to insert an \( i \epsilon \) according to the usual prescription \( m \to m - i \epsilon \). This appears to lead to a negative imaginary part for the Lagrangian, hence jeopardizing the convergence of the path integral. However, we should really consider the full Lagrangian (57), which contains several oscillating terms. These are harmless with respect to divergence, and the only potentially dangerous term is the third one,
\[ -\frac{m^2 - i \epsilon}{12\pi^2} \partial_\mu \theta \partial^\mu \theta. \] (60)

But if we take a plane wave \( \theta(x) = e^{ikx} \) for \( \theta \), then this term becomes
\[ -\frac{m^2 - i \epsilon}{12\pi^2} k^2, \] (61)
which leads to convergence for \( k_\mu \) time-like, as is expected on-shell. Thus the theory is well defined despite the negative metric of \( \theta \), although this may severely restrict the physical role of this field.

We can in principle perform a similar analysis for the field \( \phi_{\mu\nu} \) when the orbital part of \( J_{\mu\nu} \) is ignored, but in practice our level of technology limits what we can do. A full nonlinear Lagrangian like (57) would be very hard to calculate. It is, however, possible to find the Gaussian part. A procedure for this is first to perform one infinitesimal chiral Lorentz transformation and calculate the Jacobian, then perform a second transformation and calculate the Jacobian for that with the new Dirac operator, this time keeping two orders of \( \phi_{\mu\nu} \). The correct, “total” Jacobian is the sum of these, but with the terms proportional to \( \phi^2 \) divided by two. This procedure can be proved by considering the total Jacobian as an integral over a sequence of infinitesimal contributions. The Gaussian approximation for the \( \phi_{\mu\nu} \) Lagrangian is
\[ \mathcal{L}_\phi = \frac{1}{192\pi^2} \phi_{\mu\nu}(\Box + 6m^2)(\Box + 4m^2)\phi^{\mu\nu} + \frac{1}{48\pi^2} \phi_{\mu\nu}(\Box + 6m^2)\tilde{F}^{\mu\nu}. \] (62)

The last term is the spin part of the anomaly from ref. [12].

Note the similarity between the two Gaussian Lagrangians for \( \theta \) and \( \phi_{\mu\nu} \), apart from the coupling to the vector field \( A_\mu \). In fact, we can get the latter Lagrangian by making the replacement \( \theta \to \frac{1}{2} \phi_{\mu\nu} \frac{1}{2} \sigma^{\mu\nu} \) and taking the Dirac trace: \( \frac{1}{2} \text{tr}(\cdots) \). This suggests that writing all the terms in the Lagrangian (57) in terms of exponentials (i.e. also the kinetic terms), making the replacement
\[ \theta \to \theta + \frac{1}{2} \phi_{\mu\nu} J^{\mu\nu} + b_\mu P^\mu, \] (63)
and taking the Dirac trace divided by four leads to a quantity that is a part of the full bosonic Lagrangian. Additional terms are expected to appear in a “real” calculation from the derivative operators \( L_{\mu\nu} \) and \( P_\mu \), and various terms containing the vector field \( A_\mu \). Furthermore, there may be a term similar to the Wess–Zumino term in the chiral Lagrangian of the strong interactions, because the \( \frac{1}{2} \sigma_{\mu\nu} \) generate a non-Abelian group, like the Gell-Mann matrices \( \frac{1}{2} \lambda^a \) of SU(3) in the usual case.
Finally, we can calculate the interaction terms between $\theta$ and $\phi_{\mu \nu}$. There are several terms which couples one $\theta$ with one $\phi_{\mu \nu}$. These can be computed in the same way as the Gaussian Lagrangian for $\phi_{\mu \nu}$ above, and are therefore the only ones I will consider. It has the complicated form

$$L_{\theta \phi} = \frac{1}{24 \pi^2} \left( \frac{1}{\Box} \partial^{\rho} F_{\rho \mu} \phi^{\nu \mu} \left( \Box + 6m^2 \right) \partial_{\mu} \theta + 3 F_{\rho \mu} \partial^{\rho} \partial_{\mu} \phi^{\nu \mu} + \frac{3}{2} F_{\mu \nu} \theta \Box \phi^{\mu \nu} - 2 F_{\rho \nu} \partial^{\rho} \partial_{\mu} \phi^{\nu \mu} + 2 F_{\rho \nu} \partial_{\mu} \partial^{\rho} \phi^{\nu \mu} \right) + F_{\mu \nu} \Box \theta \phi^{\mu \nu} + 3 F_{\rho \nu} \phi_{\sigma}^{\rho \mu} + \frac{3}{2} F_{\mu \nu} \Box \theta \phi^{\mu \nu} + 12 m^2 F_{\mu \nu} \theta \phi^{\mu \nu} \right) \quad (64)$$

To obtain this result I have made the replacement

$$A_{\mu} \rightarrow \Pi_{\mu \nu} A_{\nu} \equiv \left( g_{\mu \nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) A_{\nu} = \frac{1}{\Box} \partial^{\nu} F_{\nu \mu}, \quad (65)$$

where the projection operator $\Pi_{\mu \nu}$ removes the gradient part of $A_{\mu}$ and thus renders the expression gauge invariant. This can be done without loss of generality, since the gradient part of $A_{\mu}$ does not couple to the fermion. The presence of an $F_{\mu \nu}$ could be anticipated since a nonvanishing expression could not be obtained from one $\theta$, one $\phi_{\mu \nu}$, and derivatives alone.

This completes my calculation of the bosonic action. Two more terms are actually known: A coupling term between one $\phi_{\mu \nu}$ and three $A_{\mu}$’s, and a coupling term between one $b_{\mu}$ and three $A_{\mu}$’s. These are part of the chiral Poincaré anomalies, and are found in ref. [12]. They are not given here, because they contain an $L_{\mu \nu}$ and a $P_{\mu}$, respectively.

It should also be possible to calculate the Faddeev–Popov determinants in $L_{\text{ghosts}}$ to the same precision. This would require only a Jacobian for an infinitesimal transformation, but the change (64) in the Lagrangian, which clearly complicates the calculation.

5 Discussion

5.1 Summary

Let us first recall how our initial requirements are met:

(i) the generalized chiral phases are the antisymmetrical tensor field $\phi_{\mu \nu}$ from the chiral Lorentz transformations, and an axial vector $b_{\mu}$ from the chiral translations; both of these can be used for making configurations of nontrivial topology,

(ii) the chiral phase rotation and chiral Poincaré transformations give rise to a Jacobian when a change of variables are made in the path integral, and

(iii) the quantity $L_{ij}$ is at least a part of the Lagrangian for the bosonic theory – if bosonization had been exact, it would be the complete Lagrangian; it is nonlinear, hence possibly have soliton solutions, and contains fourth order derivative terms, which can stabilize these solitons.

The question of whether or not topologically stable solitons can be formed in the bosonic theory is interesting by itself, and deserves further investigation. I favor the possibility that the tensor $\phi_{\mu \nu}$ is responsible for these configurations, rather than the axial vector $b_{\mu}$. The reason is that $b_{\mu}$ is the parameter of a pure derivative operator, $P_{\mu} \gamma_5$, while $\phi_{\mu \nu}$ is the parameter of $J_{\mu \nu} \gamma_5$ which at least has an internal part, $\frac{1}{2} \sigma_{\mu \nu} \gamma_5$. Hence, if part of the Lagrangian for $\phi_{\mu \nu}$ is indeed given by the replacement (63) in eq. (57), then that part bears a resemblance to the Skyrme model where the matrices $\sigma_{\mu \nu}$ play the role of the Pauli or Gell-Mann matrices.

Guided by the observation that the axial vector current (and vector current) vanishes in a certain gauge of the smooth bosonization scheme, I tried in this paper to find the gauges which
would implement the vanishing of the vector and axial vector currents in 4D. However, this program was not completely successful due to the fact that the energy-momentum tensor for a Dirac fermion, and its axial counterpart, is not symmetrical. Furthermore, in the chosen gauge the ghosts did not decouple. Nevertheless, this is an exact rewriting of the theory, exhibiting some DOFs of the fermion in an unusual way.

5.2 Problems with the approach

If what we desire is exact bosonization, the most serious problem is that the ghosts do not decouple from the bosonic fields. Even if we did get rid of the fermion, the ghosts would be there as an obstacle for this. Conversely, if we could find a ghost-free gauge – and it is far from clear that this exists – then the fermion would probably not be decoupled. Thus exact bosonization must fail, at least for the model and DOFs I have considered in this paper.

Another problem is the lack of symmetry of the energy-momentum tensors. However, in applications the energy-momentum tensor is frequently replaced by a symmetrical improved tensor. This suggests that the antisymmetrical part is somehow unimportant. If we were allowed to replace the two energy-momentum tensors with their symmetrized versions, this would mean that the current and axial current would effectively be free fields, hence vanish, and the fermion would be completely eliminated from the theory, according to our previous arguments. In this case the resulting theory would be a theory of “only” bosons and ghosts.

Of course, also a problem is the lack of a proof that vanishing currents imply a vanishing fermion. Indeed, if such a proof could be found, it is likely to be known first in 2D. However, no such proof is known in 2D.

These problems may imply the possibility that we have really been going the “wrong direction”. Perhaps it is some complicated nonlinear bosonic theory which is the fundamental thing, and that when “fermionized” gives a theory of a fermion coupled to bosonic fields. This is the situation with the Skyrme model [10]. If this possibility is correct, then it would explain why an arbitrary fermion could not be bosonized, and it would imply that certain theories of fermions coupled to bosons could.

There is also a potential problem that may occur at finite temperature. In the literature [7] there exists an argument that each fermion DOF has the same energy as 7/8 bosonic DOFs. This seems to restrict the construction of bosonic theories from fermionic ones. However, it is important to take into account that the “7/8-rule” only applies if the system is in a state of an approximately free gas of particles, both with respect to the fermionic description and the bosonic description. This is not the case for the bosonization procedure in this paper, because the bosonized theory is nonlinear, hence cannot describe a free gas. The same is of course true for other bosonization schemes that produces nonlinear bosonic theories.

5.3 Non-Abelian fermions

I now briefly consider the possibility of generalizing the results to a non-Abelian Dirac fermion. (See also refs. [5, 12].) The fermion will then transform in some representation of the non-Abelian group. If we for simplicity choose $U(N)$ and the fundamental representation, we have the “color” transformations

$$
\psi \rightarrow e^{i\omega^a t^a} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\omega^a t^a},
$$

where $a = 1, \ldots, N$ and $t^a$ are the generators. This is of course a generalization of the phase rotation of the Abelian phase, and as such should be considered together with the Poincaré

\footnote{I thank the referee for pointing out this argument.}
transformations and their chiral counterparts. We must therefore admit the possibility that also Poincaré transformations, chiral phase rotations and chiral Poincaré transformations can be generalized to color space. That is, each color of fermion can be transformed separately. Thus we have the generators

\[ t^a, \ t^a J_{\mu \nu}, \ t^a P_{\mu}, \ t^a \gamma_5, \ t^a J_{\mu \nu} \gamma_5 \text{ and } t^a P_{\mu} \gamma_5, \]  

(respectively, for these transformations. Note that they do not generate a group, since their algebra does not close. They are also not in general symmetries of the fermion Lagrangian. However, they have the property that they are naive symmetries of the fermionic measure while giving rise to a Jacobian in the regularized theory. It is for this reason that they are important for smooth bosonization.

We can now follow the same procedure as for the Abelian case, introducing the new DOFs in the path integral, gauge-fixing etc. The bosonic action will again be \( S_J \) from the Jacobian with bosonic fields \( \theta^a, \phi^a_{\mu \nu} \text{ and } b^a_\mu \). The action will then have further complications from a non-Abelian structure. The presence of colored matrices in the Dirac operator will break the color symmetries and may change the conservation equations needed for our previous arguments. However, the new conservation equations will probably be equally effective, and a generalization of the results of sec. 3 should be straightforward.

5.4 The physics of \( L_J \): A model theory

Even if bosonization is not exact, it may be possible to use the rewritten theory as a starting point for an approximate description of certain processes, perhaps at low energies. This may be relevant in particular when the “effective DOFs” in these processes are pseudoscalars, tensors or axial vectors. But I will now consider another approximation, motivated instead by simplicity. Namely, I will consider the theory described by the Lagrangian \( L_J \) of the bosonic theory in its own right. I can think of no reason why this theory cannot be a perfectly healthy quantum field theory. In a sense it is like a caricature of the true fermionic system. A heuristic argument for this is given in terms of a physical interpretation of the Jacobian \( J \) below.

First, let us assume that the vector field \( A_\mu \) is absent. In the weak field limit we can find the equation of motion for \( \theta \) from eq. (59):

\[ (\Box + 6m^2)(\Box + 4m^2)\theta = 0, \]  

(a “double” Klein–Gordon equation. This has plane wave solutions:

\[ \theta(x) = e^{ikx}, \quad k^2 = m_1^2 \equiv 4m^2 \text{ or } k^2 = m_2^2 \equiv 6m^2. \]  

Thus there are apparently two mass shells for the field, corresponding to two “branches” of propagation. The same is true for weak \( \phi_{\mu \nu} \)’s, as can be seen from eq. (62). Of course, these modes may turn out to be irrelevant when higher orders of the fields and interactions between them are taken into account.

If we now include the vector \( A_\mu \), and furthermore, assume that this field is a dynamical field in a larger theory, then the situation becomes more interesting. The \( \theta \) may then decay into two \( A_\mu \)’s through the ABJ anomaly, and the \( \phi_{\mu \nu} \)mixes with \( A_\mu \) (\( \phi_{\mu \nu} \text{ and } b_\mu \) decays also into three \( A_\mu \)’s through the anomaly terms we mentioned in the previous section). It also induces further couplings between \( \theta \) and \( \phi_{\mu \nu} \) (and \( b_\mu \)) by \( A_\mu \)-exchange.

Finally, I will discuss the physical interpretation for the use of the Jacobian of chiral transformations in the bosonic theory. We can define the effective action \( W \) for the fermionic theory by

\[ e^{iW} \equiv Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \bar{\psi} e^{i \int d^4x \bar{\psi} D\psi}, \]  

(70)
where $D$ is the Dirac operator. A change of variables will give

$$Z = J[B] \int D\psi D\bar{\psi} e^{i \int d^4x \bar{\psi} (e^{iB\gamma_5} D e^{iB\gamma_5}) \psi} = J[B] e^{iW[B]},$$

(71)

with $B$ defined in eq. (54) and $W[B]$ the rotated effective fermion action, so that the Jacobian can be written

$$J[B] = \frac{\int D\psi D\bar{\psi} e^{i \int d^4x \bar{\psi} D\psi}}{\int D\psi D\bar{\psi} e^{i \int d^4x \bar{\psi} e^{iB\gamma_5} D e^{iB\gamma_5} \psi}} = e^{i(W-W[B])}.$$  

(72)

The quantity $S_J[B]$ in $J = e^{iS_J[B]}$ is therefore the difference between the effective action of an unrotated fermion field and a rotated fermion field. In a sense, $S_J[B]$ measures the response of the fermion to external “forces” – it is the amount of action we must inject into the system to maintain the rotated configurations $e^{iB\gamma_5} \psi$, $\bar{\psi} e^{iB\gamma_5}$ compared to the unrotated ones $\psi$, $\bar{\psi}$. An analogy which comes to mind is the physics of an elastically deformable solid, where the fermion field is like the solid, and the chiral transformations are like compression, shear and tension deformations. The Lagrangian $L_J$ thus describes the theory of such “deformations” of the fermion field.

Ideas reminiscent of these have been used in ref. [3] to justify a derivation of the chiral Lagrangian [18] from QCD. In these papers, however, mainly the chiral flavor phase DOFs of the quarks were considered. It would be interesting to try to bosonize all DOFs of the quark field, including both color and chiral Poincaré phases. (For a related investigation see ref. [5].)

Acknowledgments I thank P.H. Damgaard for invaluable discussions and comments, and M. Faber, A.N. Ivanov, S.H. Hansen, and O. Borisenko for discussions.

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