ORBIFOLDS FROM A METRIC VIEWPOINT

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ABSTRACT. We establish the theory of orbifolds and their coverings from a metric point of view. In particular, we characterize coverings of Riemannian orbifolds in terms of submetries and show that the metric double of a Riemannian orbifold defines a two-fold orbifold covering. These metric results have applications to topological questions.

1. Introduction

The notion of an orbifold was introduced by Satake under the name of V-manifold [Sat56, Sat57]. In the seventies it was rediscovered by Thurston and the term “orbifold” was chosen after a vote in one of his seminars (see [Dav11] for a more detailed account). Thurston also defined the notion of coverings of orbifolds and showed that the usual theory of coverings and fundamental groups works in the setting of orbifolds [Thu79]. Later Haefliger generalized this theory to the setting of groupoids [Hae84, BH99]. Their relation to orbifolds is similar to the relation of general topological spaces to manifolds. Our purpose is to take a step back and to develop the theory of orbifolds and their coverings from a metric viewpoint. An advantage of this approach is that the definitions, and after their verification also certain proofs, become simpler and are more convenient to work with in certain situations.

The definition of a smooth orbifold in the sense of Thurston is recalled in Section 2. Every paracompact smooth orbifold $O$ can be endowed with a Riemannian metric [BH99, Ch. III.1]. In geometry such Riemannian orbifolds occur as quotients of isometric Lie group actions on Riemannian manifolds (see Lemma 2.2) and as collapsed limits of Riemannian manifolds that converge in the Gromov-Hausdorff topology [Fuk90]. The Riemannian metric induces a length metric on $O$. Recall that a length space is a metric space in which the distance between any pair of points can be realized as the infimum of the lengths of all rectifiable curves connecting these points [BBI01]. The following alternative definition of a Riemannian orbifold, which has been proposed to us by Alexander Lytchak, is based on the observation that the smooth orbifold structure of $O$ can be recovered from this length metric (see Section 2).

Definition 1.1. A Riemannian orbifold of dimension $n$ is a length space $O$ such that for each point $x \in O$ there exist an open neighborhood $U$ of $x$ in $O$ and a connected Riemannian manifold $M$ of dimension $n$ together with a finite group $G$ of isometries of $M$ such that $U$ and $M/G$ are isometric.

Here $M/G$ is endowed with the quotient metric, i.e. the distance between two points is defined as the distance between the respective orbits in $M$. More precisely, behind the above definition lies the fact that the isometric action of a finite group $G$ on a simply connected Riemannian manifold $M$ can be recovered from the metric quotient $M/G$, see Lemma 2.1 and Lemma 3.10.
To obtain a metric notion of orbifold coverings we follow the idea by Lytchak to describe submetries with discrete fibers as branched coverings \cite{Lyt02}. We denote the closed balls in a metric space $X$ as $B_r(x)$ and the open balls as $U_r(x)$. A map $p : X \to Y$ between metric spaces is called a submetry if $p(B_r(x)) = B_r(p(x))$ holds for all $x \in X$ and all $r \geq 0$. It is known that a submetry between Riemannian manifolds is a Riemannian submersion \cite{BG00}. In particular, a submetry between Riemannian manifolds of the same dimension is a local isometry. Using this notion we can characterize coverings of Riemannian orbifolds in metric terms as follows.

**Theorem 1.2.** For a map $p : O' \to O$ between $n$-dimensional Riemannian orbifolds $O'$ and $O$ the following conditions are equivalent.

(i) $p$ is a covering of Riemannian orbifolds in the sense of Thurston, cf. Definition 2.6.

(ii) $p$ is onto, locally 1-Lipschitz and each point $y \in O$ has a neighborhood $U$ such that the restriction of $p$ to $p^{-1}(U)$ is a submetry with respect to the restricted metrics.

Moreover, if $O'$ is complete, then conditions (i) and (ii) are satisfied if and only if $p$ is a submetry.

The additional characterization requires the completeness assumption. Indeed, in Example 2.8 we construct a covering of noncomplete Riemannian manifolds that is not a submetry.

The set of points in a Riemannian orbifold $O$ which have a neighborhood that is isometric to the quotient of a Riemannian manifold by an isometric reflection is called the codimension 1 stratum of $O$. Its closure coincides with the boundary of $O$ in the sense of Alexandrov geometry \cite{BGP92}. The double of $O$ along this closure admits a natural metric with respect to which the two copies are isometrically embedded (see Section 5). In Section 5 we prove the following statement that does not seem to appear in the literature.

**Proposition 1.3.** The metric double of a Riemannian orbifold along the closure of its codimension 1 stratum is a Riemannian orbifold and the natural projection to $O$ is a covering of Riemannian orbifolds.

In dimension 2 this result has been applied in \cite{La17a} and \cite{La17b}. Moreover, it can be conveniently applied in the solution of the topological question of when the quotient of $\mathbb{R}^n$ by a finite subgroup of the orthogonal group $O(n)$ is a topological manifold with boundary \cite{La17c}.

The proof of Proposition 1.3 relies on the covering space theory of (Riemannian) orbifolds. Establishing this theory essentially amounts to proving the existence of universal covering orbifolds. In Section 4 we explain two self-contained approaches to do this that use our Theorem 1.2. The two approaches are similar in the sense that both reduce the problem to the covering space theory of manifolds. One approach follows a suggestion by Thurston to work with coverings of the regular part of the orbifold (see Section 4.2). This approach a priori only works if the codimension one stratum is empty. The other approach is to work with coverings of the orthogonal frame bundle of the orbifold, and leads to formulations in terms of so-called classifying spaces (cf. Remark 4.10). To this end we give an alternative definition of the orbifold fundamental group that is more suitable for our purpose (see Definition 4.1 and Remark 4.2). As further applications in the Riemannian setting we show that the quotient of a Riemannian orbifold by a proper, isometric action of a discrete group is again a Riemannian orbifold with the projection being a covering of Riemannian orbifolds, and we give conditions...
for when isometries of the quotient lift to isometries of the total space (see Corollary 4.19 and Corollary 4.18).

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2. Basics on Riemannian orbifolds

An n-dimensional smooth orbifold in the sense of Thurston is defined in terms of a Hausdorff space X and a collection of data \( \{(U_i, G_i, U_i, \pi_i)\}_{i \in I} \), a so-called atlas, consisting of n-dimensional smooth manifolds \( \tilde{U}_i \) on which finite groups \( G_i \) act smoothly, an open covering \( \{U_i\}_{i \in I} \) of \( X \), and projections \( \pi_i : \tilde{U}_i \to U_i \) that induce homeomorphisms \( \pi_i : \tilde{U}_i/G_i \to U_i \). The charts \( (\tilde{U}_i, G_i, U_i, \pi_i) \) must satisfy the following compatibility condition. For two points \( x_i \in \tilde{U}_i \) and \( x_j \in \tilde{U}_j \) with \( \pi_i(x_i) = \pi_j(x_j) \) there should exist open neighborhoods \( \tilde{V}_i \) of \( x_i \) in \( \tilde{U}_i \) and \( \tilde{V}_j \) of \( x_j \) in \( \tilde{U}_j \) and a diffeomorphism \( \phi : \tilde{V}_i \to \tilde{V}_j \) with \( \pi_i = \pi_j \circ \phi \) on \( \tilde{V}_i \). In fact, Thurston demanded an equivariance condition on the transition map \( \phi \), which is however already implied by the condition that \( \pi_i = \pi_j \circ \phi \) holds on \( \tilde{V}_i \) as can be seen with the help of a compatible Riemannian metric (cf. Lemma 3.9).

Hence, in order to show that a Riemannian orbifold in the sense of Definition 1.1 has a natural structure of a smooth orbifold one only needs to prove the following lemma. Its proof is based on the fact that a finite subgroup \( G < O(n) \) is determined up to conjugation by the metric quotient \( S^{n-1}/G \) \cite{Swa02} Lem. 1. Note that in Section 3 we will also prove a global version of this lemma without referring to \cite{Swa02} Lem. 1, see Corollary 3.10. Recall that a ball in a Riemannian manifold is called normal, if its closure is contained in the diffeomorphic image of an open subset of the tangent space \( T_x M \) under the exponential map.

**Lemma 2.1.** Let \( U_r(x) \subset M \) and \( U_r(\bar{x}) \subset M \) be normal balls n-dimensional in Riemannian manifolds \( M \) and \( \bar{M} \). Suppose finite groups \( G \) and \( \bar{G} \) act isometrically and effectively on \( M \) and \( \bar{M} \) and fix the points \( x \) and \( \bar{x} \), respectively. Suppose further that the quotients \( U_r(x)/G \) and \( U_r(\bar{x})/\bar{G} \) are isometric. Then there exists an isometry \( \phi : U_r(x) \to U_r(\bar{x}) \) that conjugates the action of \( G \) on \( U_r(x) \) to the action of \( \bar{G} \) on \( U_r(\bar{x}) \).

**Proof.** The proof is by induction on the dimension \( n \). For \( n = 1 \) the claim follows readily. Let \( n > 1 \) and suppose that the claim is true in all dimensions smaller than \( n \). Since \( U_r(x)/G \) and \( U_r(\bar{x})/\bar{G} \) are isometric, so are the spaces of directions at \( x \) and \( \bar{x} \), i.e. the quotients of the unit spheres in the tangent spaces at \( x \) and \( \bar{x} \) by the linearized actions of \( G \) and \( \bar{G} \). Hence, by \cite{Swa02} Lem. 1 these linearized actions are conjugated by an isometry. Using the exponential map we obtain a diffeomorphism \( \phi : U_r(x) \to U_r(\bar{x}) \) that conjugates the actions of \( G \) and \( \bar{G} \), and that descends to an isometry between the respective quotient spaces.

Since the projections to the quotients are local isometries over the regular part, the diffeomorphism \( \phi \) is a Riemannian isometry there. Moreover, for any pair of points \( x_0, x_1 \in U_r(x) \) and any \( \varepsilon > 0 \) one can find a path \( \gamma : [0, 1] \to U_r(x) \) and a subdivision \( 0 = t_0 < t_1 < \cdots < t_k = 1 \) such that \( \gamma(0) = x_0, \gamma(1) = x_1 \) and \( \gamma(t_i, t_{i+1}), i = 0, \ldots, k-1 \), is a smooth curve in the regular part with \( L(\gamma) < d(x_0, x_1) + \varepsilon \). Since \( \phi \) is a Riemannian isometry on the regular
part, it follows that $\phi : U_r(x) \to U_r(\bar{x})$ is 1-Lipschitz. The same argument applied to $\phi^{-1}$ shows that $\phi$ is an isometry as claimed. \hfill \Box

Hence, a Riemannian orbifold in the sense of Definition \ref{def:orbifold} admits a smooth orbifold structure and a compatible Riemannian structure that in turn induces the metric structure. Conversely, recall that every paracompact smooth orbifold admits a compatible Riemannian structure that turns it into a Riemannian orbifold \cite[Ch. III.1]{BH99}.

As mentioned in the introduction, Riemannian orbifolds for instance arise in the following way.

Lemma 2.2. Let $G$ be a compact Lie group that acts isometrically and almost freely on a Riemannian manifold $(M, g)$. Then the metric quotient $M/G$ is a Riemannian orbifold.

Proof. We only sketch the argument and refer to \cite{Lyt10} for more details. By the slice theorem a tubular neighborhood of an orbit $G \cdot p$, $p \in M$, is equivariantly diffeomorphic to $G \times_{G_p} \nu_p(G \cdot p)$ where $\nu_p(G \cdot p)$ denotes the normal bundle of the orbit $G \cdot p$ at $p$ \cite{Bre72}. Let $S \subset M$ be the diffeomorphic image of $\nu_p(G \cdot p)$ through $p$ in $M$. For a point $x \in S$ and vectors $v, w \in T_xS$ we set $g_S(v, w) := g_{\exp_x}(v, w)$ where $\exp_x$ denotes the orthogonal projection from $T_xM$ onto $\nu_x(G \cdot x) = (T_xG \cdot x)^\perp$. Then $g_S$ defines a Riemannian metric on $S$ with respect to which the (finite) isotropy group $G_p$ acts isometrically. Moreover, a neighborhood of $G \cdot p$ in $M/G$ is isometric to a neighborhood of $p$ in $S/G_p$ with respect to the metric induced by $g_S$. \hfill \Box

Let us recall some basic facts about (Riemannian) orbifolds (in the following we will omit the term Riemannian when we want to emphasize that a property of a Riemannian orbifold is actually a property of the underlying smooth orbifold). For a point $x$ on a Riemannian orbifold $O$ choose a neighborhood isometric to $M/G$ as in Definition \ref{def:orbifold} such that $G$ fixed a preimage $\bar{x}$ of $x$ in $M$. It follows from Lemma 2.1 that the linearized orthonormal action of $G$ on $T_xM$ is uniquely determined up to conjugation by isometries. A representative of the corresponding conjugacy class in $O(n)$ is denoted as $G_x$ and is called local group of $O$ at $x$. The metric quotient $(T_xM)/G$ is the tangent space $T_xO$ of $O$ at $x$. It coincides with the tangent cone of $O$ at $x$ in the sense of metric geometry (cf. \cite{Lyt05}). The exponential map on $T_xM$ is $G$-equivariant and descends to an exponential map $\exp_x$ defined on an open neighborhood of the origin $x_o$ of $T_xO$. This exponential map is an infinitesimal isometry at $o$ in the sense of \cite[Def. 3.2]{Lyt05}, meaning that

$$|d(\exp_x(v_1), \exp_x(v_2)) - d(v_1, v_2)| \leq o(d(x_o, v_1) + d(x_o, v_2)) $$

holds for $v_1, v_2 \in T_xO$, and that $\exp_x$ is locally open at $x_o$. An orbifold $O$ can be stratified by manifolds as follows. We denote the union of all points $x \in O$ with $\text{codim}(\text{Fix}(G_x)) = i$ by $\Sigma_iO$ where $\text{Fix}(G_x)$ denotes the largest linear subspace of $\mathbb{R}^n$ that is pointwise fixed by $G_x$. The restriction of the exponential map to $\text{Fix}(G_x)$ for some $x \in \Sigma_iO$ induces a manifold chart for a neighborhood of $x$ in $\Sigma_iO$. Therefore $\Sigma_iO$ is a manifold. We call it the codimension $i$ stratum. It inherits a natural Riemannian metric from $O$ with respect to which it sits totally geodesic in $O$, i.e. geodesics on $\Sigma_i$ are orbifold geodesics in $O$ in the sense that they locally lift to ordinary geodesics on the Riemannian manifold charts (cf. \cite{La17a, La17b}). The stratum of codimension 0, i.e. the set of points with trivial local group, is called regular. The regular part is dense in $O$. All other strata are called singular. If $O$ is a global quotient $M/\Gamma$ of a
Riemannian manifold by a (possible infinite) group $\Gamma$, then we call points on $M$ regular or singular with respect to the projection $M \to M/\Gamma$ if they project to regular or singular points on $O$, respectively.

We will also need the following terminology. Recall that a ball $B_r(x)$ in a Riemannian manifold is **totally convex** if it is normal and every pair of points in $B_r(x)$ can be connected by a unique minimizing geodesic in $M$ that is contained in $B_r(x)$. Totally convex balls always exist [DoC92, Prop. 4.2]. We call balls $B_r(\bar{x})$ and $B_r(\bar{x})$ in a Riemannian orbifold $O$ **normal** or **totally convex**, respectively, if $U_{4r}(x)$ is isometric to $M/G$ as in Definition 1.1 where $G$ fixes a preimage $x$ of $\bar{x}$ in $M$, and the ball $B_r(x)$ is normal or totally convex, respectively. Note that if $B_r(\bar{x})$ is normal then any orbifold geodesic starting at $\bar{x}$ exists and is distance minimizing up to length $r$. Also note that if $B_r(\bar{x})$ is totally convex in the above sense, then $B_r(\bar{x})$ is also totally convex in the sense that any pair of points in $B_r(\bar{x})$ can be connected by a distance minimizing geodesic and that any such distance minimizing geodesic in $O$ is contained in $B_r(\bar{x})$.

In the following all orbifolds are assumed to be **connected**. In this case the regular stratum is connected as well.

**Lemma 2.3.** The regular part of an orbifold $O$ is connected.

*Proof.* Let us first prove the statement locally. So we assume that $O$ is a quotient $M/G$ as in Definition 1.1 where $M$ is an open normal ball with center $x$ and $G$ fixes the point $x$. By using the exponential map we can also assume that $M = \mathbb{R}^n$ and that $G$ is a finite subgroup of $O(n)$. Showing the claim is equivalent to showing that the components of the regular part in $\mathbb{R}^n$ are permuted transitively by the group $G$. The singular set in $\mathbb{R}^n$ is the union of all fixed-point subspaces of elements in $G$. Since only hyperplanes create new connected components in $\mathbb{R}^n$ we can assume that $G$ is generated by reflections. In this case the connected components in $\mathbb{R}^n$ are also known as Weyl chambers and the claim is e.g. proven in [Hum90].

Now let $O$ be any orbifold and suppose its regular part is the union of two disjoint open subsets $U$ and $V$. Since the regular part is dense in $O$ the closures $\overline{U}$ and $\overline{V}$ cover $O$. Since $O$ is connected by assumption, their intersection is nontrivial, i.e. there exists some singular point $x \in \overline{U} \cap \overline{V}$. But this contradicts the fact that the regular part of $O$ is locally connected as shown above. This completes the proof of the lemma. \hfill \Box

As a consequence we can prove the following lemma.

**Lemma 2.4.** An isometry of a Riemannian orbifold $O$ restricts to a Riemannian isometry of the regular part of $O$. Moreover, it is uniquely determined by its value and differential at a regular point.

*Proof.* By Lemma 2.1 an isometry of $O$ preserves the strata. In particular, it restricts to an isometry of the regular part of $O$ which is a Riemannian manifold. By [Hel01, Thm. 11.1] a metric isometry between Riemannian manifolds is also a Riemannian isometry. This shows the first claim. Since the regular part of $O$ is connected by Lemma 2.3, the restriction to the regular part is determined by its value and differential at a point, see [DoC92, Lem. 4.2]. Since the regular part of $O$ is dense in $O$, the isometry we started with is the metric completion of its restriction to the regular part. This completes the proof of the lemma. \hfill \Box

Among the singular strata, we will be in particular concerned with the codimension 1 stratum. It and its closure can be characterized as follows.
Lemma 2.5. A point \( x \) in an orbifold \( O \) belongs to the codimension 1 stratum \( \Sigma_1 O \) of \( O \) if and only if its local group \( G_x \) is generated by one reflection. The point \( x \) belongs to the closure of \( \Sigma_1 O \) if and only if its local group \( G_x \) contains a reflection.

Proof. As in Lemma 2.3 the claim can be reduced to the case in which \( O = \mathbb{R}^n/G_x \). In this case the first claim is immediate from the definition. Now the second follows from continuity and dimension reasons. \( \square \)

The concept of a covering orbifold has been introduced by Thurston [Thu79, Def. 13.2.2]. In the Riemannian setting his definition reads as follows.

Definition 2.6 (Thurston). A covering orbifold of a Riemannian orbifold \( O \) is a Riemannian orbifold \( \hat{O} \) together with a surjective map \( p : \hat{O} \to O \) that satisfies the following property. Each point \( x \in O \) has a neighborhood \( U \) isometric to some \( M/G \), as in Definition 1.1, for which each connected component \( U_i \) of \( p^{-1}(U) \) is isometric to \( M/G_i \) for some subgroup \( G_i < G \) such that the following diagram commutes

\[
\begin{array}{ccc}
U_i & \sim \rightarrow & M/G_i \\
\downarrow p & & \downarrow \\
U & \sim \rightarrow & M/G
\end{array}
\]

We refer to the map \( p : \hat{O} \to O \) in the definition as an orbifold covering or a covering of Riemannian orbifolds.

Remark 2.7. The \( G_i \) appearing in Definition 2.6 do not have to be isomorphic. For instance, consider the 4-fold orbifold covering sketched in Figure 2. The singular points with local group \( \mathbb{Z}_2 \) on the base orbifold have three preimages. Two of them are singular with local group \( \mathbb{Z}_2 \) and the third one is regular.

Note that if \( p : \hat{O} \to O \) is a covering of smooth orbifolds in the sense of [Thu79, Def. 13.2.2], then any Riemannian orbifold metric on \( O \) lifts to a Riemannian orbifold metric on \( \hat{O} \) with respect to which \( p \) becomes a covering of Riemannian orbifolds.

As pointed out in the introduction, in the noncomplete case a covering in the sense of Thurston does not need to be a submetry. In fact, this property already fails for ordinary coverings of Riemannian manifolds as the following example illustrates.

Example 2.8. Consider the subset \( X := (\mathbb{R}^2 - \{0\} \times \mathbb{R}) \cup \{0\} \times \bigcup_{n=1}^{\infty} (1/10^n, 2/10^n) \) of \( \mathbb{R}^2 \) with its restricted Riemannian metric. Let \( \hat{X} \) be the universal cover of \( X \) with the lifted Riemannian metric for which the covering \( p : \hat{X} \to X \) becomes a local isometry. Endow \( X \) with the usual topology and consider the projection \( \pi : \hat{X} \to X \) as the universal cover of \( X \) with the lifted Riemannian metric for which the covering \( p : \hat{X} \to X \) becomes a local isometry. Endow \( X \) with the usual topology and consider the projection \( \pi : \hat{X} \to X \) as the universal cover of \( X \) with the lifted Riemannian metric for which the covering \( p : \hat{X} \to X \) becomes a local isometry.
and $\tilde{X}$ with the induced length metrics. With respect to this metric the distance between the points $x = (-1,0)$ and $y = (1,0)$ in $X$ is 2. Now let $\tilde{x}$ and $\tilde{y}$ be lifts of $x$ and $y$ in $\tilde{X}$. The point $\tilde{y}$ is the endpoint of a lift $\tilde{\gamma}$ of a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Moreover, any path between $\tilde{x}$ and $\tilde{y}$ projects to a path on $X$ that is homotopic to $\gamma$ with its endpoints fixed. Since the distance between $\tilde{x}$ and $\tilde{y}$ is the infimum of the length of all curves between $\tilde{x}$ and $\tilde{y}$, it follows that this distance is strictly larger than 2. In particular, the point $y$ is not contained in the ball $p(B_2(\tilde{x}))$ and so $p$ is not a submetry.

However, using the fact that covering maps have the curve lifting property, one can show that the covering constructed in Example 2.8 is a weak submetry. Here a map $f : X \to Y$ between metric spaces $X$ and $Y$ is called weak submetry if $p(U_r(x)) = U_r(p(x))$ holds for all $x \in X$ and all $r \geq 0$. A submetry is always a weak submetry. The converse is true if the space $X$ is proper, i.e. its bounded closed balls are compact. More generally, in Lemma 3.13 we show that a covering of Riemannian orbifolds is a weak submetry. We could not decide yet whether the converse is always true or not, even in the manifold case.

**Question 1.** Is a weak submetry between Riemannian manifolds (orbifolds) of the same dimension always a covering map?

### 3. Metric definition of orbifold coverings

Our aim in this section is to prove Theorem 1.2 that characterizes coverings $p : O' \to O$ of Riemannian orbifolds $O'$ and $O$ in terms of submetries. Recall that a map $p : X \to Y$ between metric spaces $X$ and $Y$ is called a submetry if $p(U_r(x)) = B_r(p(x))$ holds for all $x \in X$ and all $r \geq 0$. A submetry is always a weak submetry. The converse is true if the space $X$ is proper, i.e. its bounded closed balls are compact. More generally, in Lemma 3.13 we show that a covering of Riemannian orbifolds is a weak submetry. We could not decide yet whether the converse is always true or not, even in the manifold case.

In the following suppose that $p : O' \to O$ is a map between $n$-dimensional Riemannian orbifolds $O'$ and $O$. From what we have just said it is clear that the map $p$ satisfies condition (ii) of Theorem 1.2 if it is a submetry. Suppose that $p$ is a covering of orbifolds in the sense of Definition 2.6. In this case $p$ is onto and locally 1-Lipschitz. For a point $x \in O$ we can choose a neighborhood $U$ of $x$ as in Definition 2.6 that is isometric to a quotient $M/G$ where $G$ fixes a preimage of $x$ in $M$ and that is a totally convex ball around $x$. Then also the connected components $U_i$ of the preimage $p^{-1}(U)$ have these properties. In this case the restriction of $p$ to each of these connected components is a submetry. Moreover, if we decrease the radius of the ball $U$ further until the distance of different connected components of $p^{-1}(U)$ is much larger than the diameter of $U$, then it follows that also the restriction of $p$ to $p^{-1}(U)$ is a submetry. Hence, a covering of orbifolds in the sense of Definition 2.6 satisfies condition (ii) of Theorem 1.2. In the following we are going to show that the converse is also true. Moreover, at the end of this section we show that both conditions imply that the map $p$ is a weak submetry (see Lemma 3.13) and thus a submetry if $O'$ is complete. This will complete the proof of Theorem 1.2.

To prove the remaining implications we will draw on ideas and results from the thesis [Lyt02] by Alexander Lytchak on general properties of submetries. The relevant results in [Lyt02] are proven for submetries between Alexandrov spaces with curvature bounded from below in the sense of Alexandrov. For the precise definition we refer the reader to [BGP92].
Here we just mention that a (noncompact) Riemannian orbifold, although it is in general neither complete nor has a lower curvature bound, is locally an Alexandrov space. More precisely, a compact, totally convex ball in a Riemannian orbifolds has this property. Therefore, we will be able to apply ideas from \[\text{Lyt02}\] in the present situation. Notice, however, that the restriction of a submetry to a subspace is in general not a submetry anymore.

In \[\text{Lyt02}\], a notion of differentiability is introduced for Lipschitz maps between metric spaces. In the present situation this notion reduces to the following. A Lipschitz map \(f : O' \to O\) is called differentiable at a point \(x \in O'\) if there exists a homogenous Lipschitz map \(D_x f : T_x O' \to T_{f(x)} O\) such that

\[
d(f(\exp_x(v)), \exp_{f(x)}(D_x f(v))) = o(||v||^2)
\]

holds. In this case \(D_x f\) is called the differential of \(f\) at \(x\). It is clear from this definition that differentiability at some point is a local property. By \[\text{Lyt02}\text{, Lem. 3.6}\] the map \(f\) is differentiable at \(x\) if and only if for each \(y \neq f(x)\) in a neighborhood of \(f(x)\), the composition \(d_y \circ f\) \(d_y\) being the distance function to \(y\), is differentiable at \(x\). Moreover, if \(f\) is differentiable at \(x\) and \(\gamma : [0,1) \to O\) is a path with \(\gamma(0) = x\) that is differentiable at \(0\), then also \(f \circ \gamma\) is differentiable at \(0\) and \(D_x f(\gamma'(0)) = (f \circ \gamma)'(0)\) holds. Here we have set \(\gamma'(0) := (D_0 \gamma)(1)\) where we identify the tangent space of \([0,1)\) at \(0\) with \([0,\infty)\).

In the following we always assume that the map \(p : O' \to O\) has the properties stated in Theorem 1.2. (\(ii\)). In particular, in this case \(p\) is 1-Lipschitz as a locally 1-Lipschitz map between length spaces. The analogue of the following statement for submetries between Alexandrov is proven in \[\text{Lyt02}\text{, Prop. 5.1}\].

**Lemma 3.1.** The map \(p\) is differentiable everywhere and its differentials are submetries.

**Proof.** Let \(x \in O'\) and choose \(r\) such that the balls \(B_r(x)\) and \(B_r(p(x))\) are totally convex and such that the restriction \(p_X\) of \(p\) to \(X := p^{-1}(B_r(p(x))) \supseteq B_r(x)\) is a submetry. For the first claim it suffices to show that \(d_y \circ p\) is differentiable for every \(y \in B_r(p(x))\). Since \(p_X\) is a submetry we have that \(d_y \circ p_X = d_{p^{-1}(y)}(\text{cf.} \text{Lyt02\text{, Lem. 4.3}})\) and \(d(p^{-1}(y), x) < r\) for \(y \in B_r(p(x))\). Hence, we have \(d_y \circ p_X = d_{p^{-1}(y) \cap B_r(x)}(\text{for } y \in B_r(p(x)))\). The totally convex balls \(B_r(x)\) and \(B_r(p(x))\) are Alexandrov spaces with curvature bounded from below. Since the distance function from a point to a disjoint closed subspace of such an Alexandrov space is differentiable \[\text{BGP92}\text{, p. 44}\] at this point, the submetry \(p\) is differentiable at \(x\) as claimed.

Now it follows from equations (1) and (2), and the homogeneity of the differential that the differentials of \(p\) are 1-Lipschitz. Moreover, using (1), (2) and the fact that the tangent spaces are proper it follows that the differentials are weak submetries, and thus submetries since all tangent spaces are proper. This completes the proof of the lemma.

Next we verify an analogue of a part of \[\text{Lyt02}\text{, Prop. 9.1}\] in case of orbifolds. We call \(p\) discrete if its fibers are discrete. Moreover, we call the distance of an element \(v\) in a tangent space to the origin its norm and denote it as \(||v||\).

**Lemma 3.2.** The map \(p\) is discrete and its differentials are norm-preserving.

**Proof.** Let \(x\) be an element in the fiber of a point \(y \in O\). By Lemma 3.1, the differential \(D_x p : T_x O' \to T_{p(x)} O\) is a submetry. The tangent space at a point in a Riemannian orbifold is isometric to the quotient of \(\mathbb{R}^n\) by a finite subgroup of \(O(n)\), and as such an Alexandrov space.
Since \( T_x\mathcal{O}' \) and \( T_{p(x)}\mathcal{O} \) have the same dimension and the differential \( D_xp \) is homogenous, the so-called vertical subspace \( (D_xp)^{-1}(0) \) of \( T_x\mathcal{O}' \) is trivial by [Lyt02] Prop. 9.1. Moreover, by [Lyt02] Prop. 6.4 all directions at \( x \) are horizontal in this case, i.e. \( \|D_xp(v)\| = \|v\| \) holds for all \( v \in T_x\mathcal{O}' \). Now it follows from the definition of differentiability, equation (2), that \( x \) is isolated in the fiber \( p^{-1}(y) \). This completes the proof of the lemma.

In the case of Alexandrov spaces the following lemma is proven in [Lyt02] Lem. 5.4.

**Lemma 3.3.** Let \( x \in \mathcal{O}' \) be a point with image \( y \in \mathcal{O} \). There exists some \( r > 0 \) for which any distance minimizing geodesic \( \tilde{\gamma} : [0, r] \to \mathcal{O} \) with \( \tilde{\gamma}(0) = y \) can be lifted to a distance minimizing geodesic \( \gamma \) on \( \mathcal{O}' \) with \( \gamma(0) = x \). Moreover, given any \( v \in T_x\mathcal{O}' \) with \( D_xp(v) = \tilde{\gamma}'(0) \) the lift \( \gamma \) can be chosen such that \( \gamma'(0) = v \).

**Proof.** We choose \( r > 0 \) such that the balls \( B_r(x) \) and \( B_r(p(x)) \) are totally convex, such that the restriction of \( p \) to \( X := p^{-1}(B_{4r}(p(x))) \supseteq B_{4r}(x) \) is a submetry, and such that the distance from \( x \) to other points in the fiber of \( y = p(x) \) is at least \( 8r \), which is possible by Lemma 3.2.

Then we have \( p^{-1}(B_r(y)) \cap B_{4r}(x) = B_{2r}(x) \). It follows that the restriction of \( p \) to \( B_r(x) \) is a submetry onto \( B_r(y) \). Moreover, the totally convex balls \( B_r(x) \) and \( B_r(y) \) are Alexandrov spaces and so we can apply the results from [Lyt02] to the submetry \( p : B_r(x) \to B_r(y) \). More specifically, for any geodesic \( \tilde{\gamma} : [0, r] \to \mathcal{O} \) with \( \tilde{\gamma}(0) = y \), and any \( v \in T_x\mathcal{O}' \) with \( D_xp(v) = \tilde{\gamma}'(0) \), there exists a lift \( \gamma \) of \( \tilde{\gamma} \) to \( \mathcal{O}' \) with \( \gamma'(0) = v \) by [Lyt02] Lem. 5.4. By the submetry property also the lift \( \gamma \) of \( \tilde{\gamma} \) is distance minimizing as claimed. \( \square \)

Next we prove the following completeness property.

**Lemma 3.4.** Suppose that the restriction of a distance minimizing geodesic \( \tilde{\gamma} : [0, s] \to \mathcal{O} \) to \([0, s) \) lifts to a distance minimizing geodesic \( \gamma : [0, s) \to \mathcal{O}' \). Then \( \gamma \) can be extended to a lift of \( \tilde{\gamma} \) on \([0, s] \).

**Proof.** Let \( r > 0 \) such that the restriction \( p_X \) of \( p \) to \( X := p^{-1}(B_r(\tilde{\gamma}(s))) \) is a submetry. We can assume that \( \gamma \) is contained in \( X \). Set \( x = \gamma(0) \) and \( y = p(x) \). Since \( p_X \) is a submetry and \( \tilde{\gamma} \) is distance minimizing, there exists some \( z \in X \) with \( d(\gamma(s/2), z) = s/2 \) and \( p(z) = \tilde{\gamma}(s) \). By the same reason we have \( d(x, z) = s \). We claim that \( z \) is the limit of \( \gamma(t) \) as \( t \) tends to \( s \). Suppose this is not the case. Then there exists some \( s' \in [s/2, s) \) maximal with the property that \( d(\gamma(s/2), z) = d(\gamma(s/2), \gamma(s')) + d(\gamma(s'), z) \). Since \( \mathcal{O} \) is a length space, for some small normal ball \( B_{\varepsilon}(\gamma(s')) \subseteq X \), \( \varepsilon < s - s' \), we have that \( d(\gamma(s'), z) = \inf_{u \in S_{\varepsilon}(\gamma(s'))}(d(\gamma(s'), u) + d(u, z)) \) where \( S_{\varepsilon}(\gamma(s')) \) is the distance \( \varepsilon \)-sphere at \( \gamma(s') \). By compactness the infimum is attained, say at \( u_0 \in S_{\varepsilon}(\gamma(s')) \). By maximality of \( s' \) the point \( u_0 \) does not lie on \( \gamma \). But this implies \( d(x, z) < s \) since we can short-cut at \( \gamma(s') \), a contradiction. It follows that \( \gamma \) can be extended to a lift of \( \tilde{\gamma} \). \( \square \)

Now we can strengthen the statement of Lemma 3.3

**Lemma 3.5.** Suppose we are in the situation of Lemma 3.3. Suppose in addition that we have fixed some \( r > 0 \) for which the restriction of \( p \) to \( p^{-1}(B_r(y)) \) is a submetry. Then the conclusion of Lemma 3.3 holds for this \( r \).

**Proof.** Let \( \tilde{\gamma} : [0, r] \to \mathcal{O}' \) be a distance minimizing geodesic with \( \tilde{\gamma}(0) = y \), and let \( v \in T_x\mathcal{O}' \) with \( D_xp(v) = \tilde{\gamma}'(0) \). The set of all \( s \in [0, 1] \) for which there exists a lift \( \gamma : [0, s] \to \mathcal{O}' \)
of $\tilde{\gamma}_{|[0,r]}$ with $\gamma(0) = x$ and $\gamma'(x) = v$ is closed by Lemma 3.4 and open in $[0,r]$ by Lemma 3.3. Hence, the lift $\gamma$ exists on $[0,r]$. Since the restriction of $p$ to $p^{-1}(B_r(y))$ is a submetry and $\tilde{\gamma}$ is distance minimizing, the path $\gamma$ is distance minimizing, too. □

As a consequence, we can prove the following lemma.

**Lemma 3.6.** Suppose in addition that $B_r(y)$ is a normal ball and that the restriction of $p$ to $p^{-1}(B_{2r}(y))$ is a submetry. Then for any $v \in T_xO'$ there exists a distance minimizing geodesic of length $r$ starting at $x$ in the direction of $v$, and it projects to a geodesic on $O$. Moreover, the $r$-ball $B_r(x)$ is compact, and for $s \leq r$ the map $p$ maps the distance $s$-sphere $S_s(x)$ onto the distance $s$-sphere $S_s(y)$.

**Proof.** Since the ball $B_r(y)$ is normal by assumption, for any $v \in T_xO$ there exists a minimizing geodesic $\tilde{\gamma} : [0,r] \to O$ with $\gamma(0) = y$ and $\gamma'(0) = D_sp(v)$. By Lemma 3.5 this geodesic can be lifted to a minimizing geodesic $\gamma : [0,r] \to O$ with $\gamma(0) = x$ and $\gamma(0) = v$. Now it follows from Lemma 3.4 and the proof of the Hopf-Rinow theorem, see [BBI01, p. 52], that the ball $B_r(x)$ is compact. Let $s \leq r$. Our assumption that $p$ is a submetry on $p^{-1}(B_{2r}(y))$ implies that $S_s(y) \subseteq p(S_s(x))$. Since a ball slightly bigger than $B_r(y)$ is still normal, the same argument as above shows that a ball slightly bigger than $B_r(x)$ is still compact. Therefore, by compactness and our assumption that $O$ is a length space, any point on $S_s(x)$ can be connected by a minimizing geodesic with $x$ [BBI01, Prop. 2.5.19]. Since this geodesic projects to a minimizing geodesic on $O$, it follows that $S_s(x)$ is mapped onto $S_s(y)$. This completes the proof of the lemma. □

The next step is to prove the following statement.

**Lemma 3.7.** Let $M$ and $\tilde{M}$ be $n$-dimensional, simply connected Riemannian manifolds. Suppose that a finite group $G$ acts isometrically on $M$ and that there exists a submetry $\tilde{p} : \tilde{M} \to M/G$. Then there exists an isometry $\Phi : M \to \tilde{M}$ for which the following diagram commutes.

$$
\begin{array}{ccc}
M & \xrightarrow{p} & M/G \\
\Phi \downarrow & & \downarrow & \Phi \\
\tilde{M} & \xrightarrow{\tilde{p}} & \tilde{M}
\end{array}
$$

Before we prove Lemma 3.7, we prove the following local version of it.

**Lemma 3.8.** Suppose we are in the situation of Lemma 3.7 and that the statement of Lemma 3.7 holds true up to dimension $n-1$. Then for each pair of points $x \in M$ and $\tilde{x} \in \tilde{M}$ with $p(x) = \tilde{p}(\tilde{x})$, there exists some $r > 0$ and an isometry $\phi : B_r(x) \to B_r(\tilde{x})$ with $p = \tilde{p} \circ \phi$.

**Proof.** We choose $r > 0$ such that $B_{8r}(x)$ is a normal ball. By Lemma 3.6 for $s \leq 8r$ the submetry $\tilde{p}$ maps the distance $s$-sphere $S_s(\tilde{x})$ onto the distance $s$-sphere $S_s(p(x))$. The restriction of $\tilde{p}$ to $\tilde{p}^{-1}(S_r(p(x)))$ is a submetry onto $S_r(p(x))$ with respect to the restricted metrics. Lemma 3.6 implies that $\tilde{p}^{-1}(S_r(p(x))) \cap B_{2r}(\tilde{x}) = S_r(\tilde{x})$, and hence also the restriction of $\tilde{p}$ to $S_r(\tilde{x})$ is a submetry onto $S_r(p(x))$ with respect to the restricted metrics. Since $S_r(\tilde{x})$ is compact by Lemma 3.6 it follows that $\tilde{p} : S_r(\tilde{x}) \to S_r(p(x))$ is also a submetry with respect to the intrinsic metrics (cf. [Lyt06, Lem. 4.4]).
We claim that there exists an isometry $\Phi : S_r(x) \to S_r(\bar{x})$ with $p = \bar{p} \circ \Phi$. For $n > 2$ this follows from Lemma 3.6 by our induction assumption since $S_r(x)$ and $S_r(\bar{x})$ are simply connected in this case. For $n = 1$ the claim is immediate. For $n = 2$ it follows easily that $\bar{p}$ is the quotient map for an isometric action of a cyclic or dihedral group $G_x$. By continuity the isomorphism type of $G_x$ is independent of $r$. The group $G_x$ is cyclic if and only if $S_r(p(x))$ is a circle, and hence, if and only if the isotropy group $G_x$ is cyclic. Moreover, since the length of $S_r(\bar{x})$ is asymptotic to $2\pi r$ for $r$ tending to zero, $G_x$ and $G_{\bar{x}}$ must have the same order and are thus isomorphic. In particular, $S_r(x)$ and $S_r(\bar{x})$ are isometric and an isometry $\Phi$ can be chosen such that it conjugates the group actions so that $p = \bar{p} \circ \Phi$ holds.

Given Lemma 3.6 we can extend this isometry radially to a homeomorphism $\phi : B_r(x) \to B_r(\bar{x})$. We claim that this homeomorphism has the desired properties. Since the minimizing geodesic between $x, \bar{x}, p(x)$ and a point on $S_r(x), S_r(\bar{x}), S_r(p(x))$, respectively, is unique, we have $p = \bar{p} \circ \phi$. Moreover, on the regular parts $p$ and $\bar{p}$ are local isometries and so $\phi$ is a Riemannian isometry there. Because of $p = \bar{p} \circ \phi$, the regular part of $B_r(x)$ with respect to $p$ is the radial extension of the regular part of $S_r(x)$ with respect to $p$, which in turn is isometric via $\Phi$ to the regular part of $S_r(\bar{x})$ with respect to $\bar{p}$. Therefore, for any pair of points $x_0, x_1 \in B_r(x)$ and any $\varepsilon > 0$ one can find a path $\gamma : [0,1] \to B_r(x)$ and a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\gamma(0) = x_0, \gamma(1) = x_1$ and $\gamma(t_i, t_{i+1})$, $i = 0, \ldots, k - 1$, is a smooth curve in the regular part with $L(\gamma) < d(x_0, x_1) + \varepsilon$. Since $\phi$ is a Riemannian isometry on the regular part, it follows that $\phi : B_r(x) \to B_r(\bar{x})$ is 1-Lipschitz. The same argument applied to $\phi^{-1}$ shows that $\phi$ is an isometry as claimed.

To prove the global statement of Lemma 3.8 we need the following technique (cf. [CH75, Ch. 1.13]). Let $M$ and $\bar{M}$ be Riemannian manifolds, fix points $x \in M, \bar{x} \in \bar{M}$ and choose a linear isometry $I : T_xM \to T_{\bar{x}}\bar{M}$. A broken geodesic is a continuous curve $\gamma : [0, l] \to M$ such that there exist $0 = t_0 < t_1 \cdots < t_{n-1} < t_n = l$ for which the restrictions $\gamma([t_i, t_{i+1}])$, $i = 0, \ldots, n - 1$, are smooth geodesics. Set

$$\gamma_i = \gamma|[0,t_i]$$

and define $v_i \in T_{\gamma(t_i)}M$ by

$$\gamma|[t_i,t_{i+1}] = t \mapsto \exp_{\gamma(t_i)}((t-t_i)v_i).$$

A correspondence between broken geodesics $\gamma$ and $\bar{\gamma}$ emanating from $x$ and $\bar{x}$ can be defined as follows. Let $\gamma$ be a broken geodesic in $M$ starting at $x$. We set $\bar{\gamma}(0) = \bar{x}$ and $\bar{\gamma}_0 = \bar{\gamma}|[0,t_0]$. Assume that $\bar{\gamma}_i$ is already defined. Then we set

$$\bar{\gamma}|[t_i, t_{i+1}] = \exp_{\bar{\gamma}(t_i)}((t-t_i)(P_{\bar{\gamma}} \circ I \circ P_{\bar{\gamma}}^{-1})(v_i))$$

and $\bar{\gamma}_{i+1} = \bar{\gamma}|[0,t_i]$ if the exponential map is defined where $P_{\gamma}$ denotes the parallel transport along $\gamma$. If the exponential map is defined in each step, this construction yields a broken geodesic $\bar{\gamma} = \bar{\gamma}_n : [0,1] \to \bar{M}$. In this case we set $I_{\gamma(t)} = P_{\gamma|[0,t]} \circ I \circ P_{\gamma|[0,t]}^{-1} : T_{\gamma(t)}M \to T_{\gamma(t)}\bar{M}$. Note that if $\gamma$ is contained in an open subset $U \subset M$ with $x \in U$ on which an isometry $\phi : U \to V \subset \bar{M}$ is defined with $D_x\phi = I$, then $I_{\gamma(t)} = D_{\gamma(t)}\phi$ holds for all $t \in [0, l]$.

Moreover, we need the following statement from Riemannian geometry.

**Lemma 3.9.** Let $M$ be a Riemannian manifold on which a finite group $G$ acts isometrically and let $p : M \to M/G$ be the quotient map. Then any isometry $\phi : U \to V$ between open
connected subset of $M$ with $p = p \circ \phi$ on $U$ is the restriction of the action of some element $g \in G$ on $M$.

Proof. Let $x \in U$ be a regular point for the covering $p : M \to M/G$. Since $p$ is a local isometry in a neighborhood of $x$ there exists some $g \in G$ such that the action of $g$ and $\phi$ coincide in this neighborhood. Now the claim follows from the fact that a Riemannian isometry of a connected space is determined by its local behaviour \cite[Lem. 4.2]{DoC92}.

Let us now prove Lemma 3.7.

Proof of Lemma 3.7. The proof is by induction on $n$. For $n = 1$ the claim follows easily. Suppose that $n > 1$ and that the statement holds in all dimensions lower than $n$. Let $x \in M$ and $\bar{x} \in \bar{M}$ be points with $p(x) = \bar{p}(\bar{x})$. By Lemma 3.8 there exists some $r > 0$ and an isometry $\phi_x : B_r(x) \to B_r(\bar{x})$ with $p = \bar{p} \circ \phi_x$. We set $I := D_x \phi : T_x M \to T_{\bar{x}} \bar{M}$.

We are going to show that for every broken geodesic $\gamma : [0, l] \to M$ with $\gamma(0) = x$ the broken geodesic $\tilde{\gamma} : [0, l] \to M$ can be defined as above and vice versa, that we have $p(\gamma(t)) = \bar{p}(\tilde{\gamma}(t))$ for all $t \in [0, l]$ and that there exists a locally defined isometry $\phi : B_r(\gamma(l)) \to B_r(\tilde{\gamma}(l))$ with $p = \bar{p} \circ \phi$, and $D_{\gamma(l)} \phi = I_{\tilde{\gamma}(l)}$. Then the proof of the Cartan-Ambrose-Hicks Theorem (see \cite[Thm. 1.42, p. 32]{CH75}) implies that for all broken geodesics $\gamma_0$ and $\gamma_1$ starting at $x$ such that $\gamma_0(0) = \gamma_1(0)$ holds, we have $\gamma_0(1) = \gamma_1(1)$ and that the map $\Phi : M \to \bar{M}$ defined by $\gamma(0) \mapsto \gamma(t)$ is a local isometry, and hence an isometry since the argument works in both directions.

So let $\gamma : [0, l] \to M$ be a broken geodesic with $\gamma(0) = x$ and let $l' \in [0, l]$ be the supremum of all $t \in [0, l]$ for which $\tilde{\gamma}$ can be defined on $[0, l']$ such that $p(\gamma(s)) = \bar{p}(\tilde{\gamma}(s))$ holds for all $s \in [0, l']$ and such that there exists a locally defined isometry $\phi_\gamma(t) : B_r(\gamma(t)) \to B_r(\tilde{\gamma}(t))$ with $p = \bar{p} \circ \phi_\gamma(t)$ and $D_{\gamma(t)} \phi_\gamma(t) = I_{\tilde{\gamma}(t)}$. Since we have already constructed the isometry $\phi_x$, the supremum $l'$ exists and is larger than 0. Since $l'$ is a supremum of an open set, it suffices to prove that the supremum is attained in order to prove $l' = l$.

To this end we can assume that $p \circ \gamma$ is a minimizing geodesic on $M/G$. In this case, by Lemma 3.8 $\tilde{\gamma}$ can be extended to $l'$ such that $p(\gamma(l')) = \bar{p}(\tilde{\gamma}(l'))$ holds. By Lemma 3.8 there exists some $r'$ and a local isometry $\phi : B_{r'}(\gamma(l')) \to B_{r'}(\tilde{\gamma}(l'))$ with $p = \bar{p} \circ \phi$. We choose $r'$ so small that the minimal distance between $\gamma(l')$ and other points in the $G$-orbit of $\gamma(l')$ is at least $8r'$. Let $s \in [0, l']$ be such that $\gamma(t) \in B_{r'/2}(\gamma(l'))$ for all $t \in [s, l']$. By Lemma 3.9 there is some isometry $g \in G$ such that $\phi_\gamma(t) := \phi \circ g$ coincides with $\phi_\gamma(s)$ in a neighborhood of $\gamma(s)$. In particular, we have $D_{\gamma(t)} \phi_\gamma(t) = I_{\tilde{\gamma}(t)}$. Our choice of $r'$ implies that $g(\gamma(l')) = \gamma(l')$. Therefore, $\gamma(t)$ lies in the domain of $\phi_\gamma(t)$, which is $B_{r'}(\gamma(l'))$, for all $t \in [s, l']$. Hence we have $D_{\gamma(t)} \phi_\gamma(t) = I_{\tilde{\gamma}(t)}$ and so $l' = l$. Note that by the same argument we can construct $\gamma$ from a given $\tilde{\gamma}$. It follows as described above that $\Phi : M \to \bar{M}$ is an isometry with $p = \bar{p} \circ \Phi$. Now Lemma 3.7 follows by induction.

As a corollary of Lemma 3.7 and Lemma 3.9 we record the following statement.

Corollary 3.10. Let $M$ and $\bar{M}$ be simply connected Riemannian manifolds on which finite groups $G$ and $\bar{G}$, respectively, act isometrically and effectively. Suppose that $M/G$ and $\bar{M}/\bar{G}$ are isometric. Then there exists an isomorphism $\phi : G \to \bar{G}$ and a $\phi$-equivariant isometry $\Phi : M \to \bar{M}$.

We can apply Lemma 3.7 to submetries between Riemannian orbifolds as follows.
Lemma 3.11. Suppose that a ball \( U_r(y) \subset \mathcal{O} \) is normal and isometric to \( M/G \) as in Definition 1.1 where \( G \) fixes a preimage \( z \) of \( y \) in \( M \). Then for \( x \in p^{-1}(y) \) the projection \( q : M \to M/G = U_r(y) \) lifts to a map \( q' : M \to U_r(x) \) which is the metric quotient map for the action of a finite subgroup \( H \) of \( G \) on \( M \). In particular, the ball \( B_{r/4}(x) \subset \mathcal{O}' \) is normal.

**Proof.** For some \( s \leq r \) the ball \( U_s(x) \) is normal and isometric to \( M/G \) where \( G \) fixes a preimage \( \bar{z} \) of \( x \) in \( M \), and the restriction \( p : U_s(x) \to U_s(y) \) is a submetry (cf. proof of Lemma 3.3). By Lemma 3.7 there exists an isometry \( \Phi : U_s(\bar{z}) \to U_s(z) \subset M \) for which the following diagram commutes

\[
\begin{array}{ccc}
U_s(\bar{z}) & \xrightarrow{\Phi} & U_s(z) \\
\downarrow{q'} & & \downarrow{q} \\
U_s(x) & \xrightarrow{p} & U_s(y)
\end{array}
\]

By Lemma 3.9 the group \( G \) conjugated by \( \Phi \) is a subgroup of \( G \). By Lemma 3.6 we can radially extend the map \( q' \) to a continuous map \( q : U_r(z) \to U_r(x) \) for which the lower triangle in the diagram still commutes. Let \( H \) be the subgroup of \( G \) obtained from conjugating \( G \) by \( \Phi \). Using Lemma 3.6 one shows that the induced map \( U_r(z)/H \to U_r(x) \) is a homeomorphism. Moreover, since all maps involved are local isometries, the induced map \( U_r(z)/H \to U_r(x) \) is a Riemannian isometry on the regular part. It follows as in the proof of Lemma 3.8 that this induced map is in fact an isometry. This completes the proof of the lemma. \( \square \)

Now we can complete the proof of the equivalence between condition (i) and (ii) in Theorem 1.2.

**Proposition 3.12.** Suppose \( p : \mathcal{O}' \to \mathcal{O} \) has the properties stated in Theorem 1.2 (ii). Then \( p \) is a covering of Riemannian orbifolds in the sense of Definition 2.6.

**Proof.** For a point \( x \in \mathcal{O} \) we choose \( r > 0 \) such that \( B_{16r}(x) \) is a normal ball isometric to \( M/G \) where \( G \) fixes a preimage \( z \) of \( x \) in \( M \), and such that the restriction of \( p \) to \( p^{-1}(B_{16r}(x)) \) is a submetry. Let \( \{x_i\}_{i \in I} \) be the preimages of \( x \) in \( \mathcal{O}' \). By Lemma 3.11 the balls \( B_{4r}(x_i) \), \( i \in I \), are normal. It follows that the balls \( B_r(x_i) \), \( i \in I \), are pairwise disjoint and we have \( p^{-1}(B_r(x)) = \bigcup_{i \in I} B_r(x_i) \). Moreover, by Lemma 3.11 the balls \( U_r(x_i) \) are isometric to \( U_r(z)/G_i \) for some subgroup \( G_i \) of \( G \) and the diagrams in Definition 2.6 commute for \( U = U_r(x) \) and \( U_i = U_r(x_i) \). \( \square \)

Recall from the beginning of this section that condition (ii) in Theorem 1.2 is trivially holds if \( p \) is a submetry. Hence, in order to complete the proof of Theorem 1.2 it remains to prove the following lemma.

**Lemma 3.13.** A map \( p : \mathcal{O}' \to \mathcal{O} \) that satisfies conditions (i) and (ii) in Theorem 1.2 is a weak submetry. Moreover, it is a submetry if \( \mathcal{O}' \) is complete.

**Proof.** We have already observed that the map \( p \) is 1-Lipschitz as a locally 1-Lipschitz map between length spaces. To recognize it as a weak submetry we have to show that for any points \( x' \in \mathcal{O}' \) and \( x, y \in \mathcal{O} \) with \( p(x') = x \) and \( d(x, y) = r \), and any \( \varepsilon > 0 \) there exists a point \( y' \in \mathcal{O} \) with \( p(y') = y \) and \( d(x', y') < r + \varepsilon \).

So let \( \varepsilon > 0 \) and let \( \tilde{\gamma} : [0, r'] \to \mathcal{O} \) be a 1-Lipschitz path with \( \tilde{\gamma}(0) = x \), \( \tilde{\gamma}(r') = y \) and length \( L(\tilde{\gamma}) < d(x, y) + \varepsilon \). By compactness there exists a finite subdivision \( 0 = t_0 < t_1 < \cdots < t_k = r' \)
and radii \( r_i, i = 0, \ldots, k \), such that each ball \( B_{r_i}(\tilde{\gamma}(t_i)) \) is totally convex and satisfies the condition in Definition 2.6, and such that \( \tilde{\gamma}([0, t_1]) \subseteq B_{r_0}(\tilde{\gamma}(0)), \tilde{\gamma}([t_{i-1}, t_{i+1}]) \subseteq B_{r_i}(\tilde{\gamma}(t_i)), \) \( i = 1, \ldots, k - 1 \), and \( \tilde{\gamma}([t_{k-1}, r']) \subseteq B_{r_k}(\tilde{\gamma}(r')) \) holds. Therefore, we can assume that the restriction of \( \tilde{\gamma} \) to each interval \([t_i, t_{i+1}]\), \( i = 0, \ldots, k - 1 \), is distance minimizing. By Lemma 3.5 we can lift \( \tilde{\gamma} \) to a 1-Lipschitz path \( \gamma : [0, r'] \rightarrow O \) with the same property and length \( \leq d(x, y) + \varepsilon \). The point \( y' = \gamma(r') \) has the desired properties and so the first claim follows.

If \( O' \) is complete, then its closed, bounded balls are compact by the Hopf-Rinow theorem for length spaces [BB01, Thm. 2.5.28]. In this case a subsequence of the points \( y'(\varepsilon) \) in the argument above converge to a point \( y' \) with \( p(y') = y \) and \( d(x', y') = r \). Hence, \( p \) is a submetry in this case as claimed. \( \Box \)

4. Orbifold coverings and fundamental groups

In the following we assume familiarity with the covering space theory of topological spaces (see e.g. [Hat02]). An orbifold covering \( p : \hat{O} \rightarrow O \) is called universal if, given a choice of points \( \hat{x}_0 \in \hat{O} \) and \( x_0 \in O \) with \( \varphi(\hat{x}_0) = x_0 \), for any orbifold covering \( p' : O' \rightarrow O \) and a base point \( x'_0 \) with \( \varphi'(x'_0) = x_0 \), there exists an orbifold covering \( q : \hat{O} \rightarrow O' \) with \( q(\hat{x}_0) = x'_0 \) and \( p = p' \circ q \). The existence of universal orbifold coverings is well-known and there are several ways of proving it. For instance, in [Thu79] Thurston describes a fiber product for orbifolds and uses it to obtain a universal covering orbifold as an inverse limit. In [BMP03] a proof using the notion of “orbifold loops” similarly as ordinary loops in case of topological spaces is sketched. The problem can also be reduced to the existence of universal covering of sufficiently well-behaved topological spaces (cf. Remark 4.10 and [ALR07]). For orbifolds without codimension 1 stratum a proof along such lines was already suggested by Thurston [Thu79].

In this section we give two proofs for the existence of universal orbifold coverings that make use of Theorem 1.2 and record some of its corollaries. These results are needed in the proof of Proposition 1.3 (cf. Section 5). The proofs rely on the ordinary covering space theory of manifolds but are otherwise self-contained. The first proof is similar in spirit to Thurston’s suggestion and leads to a formulation in terms of coverings between classifying spaces (cf. Remark 4.10). In the second approach we explain Thurston’s suggestion. Both approaches require the following notion.

To a connected manifold \( M \) one can associate the orthonormal frame bundle \( FM \) on which the orthogonal group \( O(n) \) acts freely with quotient space \( M \). The frame bundle \( FM \) is disconnected if and only if \( M \) is orientable. Moreover, if \( M \) is Riemannian, then \( FM \) can be endowed with a canonical Riemannian metric with respect to which \( O(n) \) acts isometrically with metric quotient \( M \). This construction was first carried out by O’Neill [O’Ne66] and independently by Mok [Mok78], and it is related to Sasaki’s Riemannian metric on the tangent bundle of \( M \) [Sas58, Sas62]. Therefore, we refer to it as the Sasaki-Mok-O’Neill metric. Since the \( O(n) \)-action on \( FM \) commutes with the action of isometries induced by isometries of \( M \), this construction can be generalized to orbifolds [ALR07]. More precisely, for an orbifold of the form \( M/G \) as in Definition 1.1 the orthonormal frame bundle is \((FM)/G\). For a general orbifold the orthonormal frame bundle can be patched together from the frame bundles of charts of the form \( M/G \), cf. [ALR07, Def. 1.22]. The resulting frame bundle \( FO \) is a manifold
on which \( O(n) \) acts isometrically with quotient \( O \). If the metric on \( O \) is complete, then so is the induced metric on \( FO \).

Next we introduce orbifold fundamental groups. Let \( \pi : FO \to O \) be the orthonormal frame bundle of a Riemannian orbifold \( O \) and let \( x_0 \in O \) be a regular point. Consider the set \( P_{x_0}FO := \{ \gamma \in C^0([0, 1], FO) \mid p(\gamma(0)) = x_0 = p(\gamma(1)) \} \) of all continuous paths in \( FO \) that start and end at the fiber \( p^{-1}(x_0) \) over \( x_0 \). The action of \( O(n) \) on \( FO \) induces an action of \( O(n) \) on \( P_{x_0}FO \). One can also think of an element in \( P_{x_0}FO \) as a path in \( O \) together with a moving orthogonal frame.

**Definition 4.1.** The (orbifold) fundamental group \( \pi_1^{orb}(O, x_0) \) of \( O \) based at \( x_0 \) is defined as the quotient of \( P_{x_0}FO \) by the equivalence relation generated by the action of \( O(n) \) on \( FO \) and by homotopies through paths in \( P_{x_0}FO \). The group multiplication is defined as the usual concatenation of paths.

Similarly as for ordinary fundamental groups one checks that \( \pi_1^{orb}(O, x_0) \) is well-defined. If \( \gamma \) represents the trivial class in \( \pi_1^{orb}(O, x_0) \), then it is homotopic through paths in \( P_{x_0}FO \) to a trivial path. A path that connects two distinct components of \( \pi^{-1}(x_0) \) always represents a non-trivial class in \( \pi_1^{orb}(O, x_0) \). Such a path exists if and only if \( O \) is not orientable.

**Remark 4.2.** It is not hard to show that the orbifold fundamental group of an orbifold \( O \) in the sense of Definition 4.1 is isomorphic to the ordinary fundamental group of the so-called classifying space \( BO \) of \( O \). This classifying space is defined as the Borel-construction of the action of \( O(n) \) on \( FO \). More precisely, let \( EO(n) \to BO(n) \) be a classifying bundle of \( O(n) \) \( \text{[AM94], Thm. 1.1, e.g.} \) the infinite Stiefel manifold of \( n \)-frames over the infinite Grassmannian of \( n \)-planes \( \text{[MS74] \S 5} \), then \( BO \) is the quotient of \( FO \times EO(n) \) by the diagonal action of \( O(n) \). Moreover, one can then define higher-dimensional orbifold homotopy groups and (co)homology groups of \( O \) as the respective invariants of \( BO \) (cf. \[ALR07\]).

**Remark 4.3.** Still other definitions of the orbifold fundamental group are discussed in \[Dav11\], p. 7. For instance, \( \pi_1^{orb}(O, x_0) \) may be described as “homotopy classes” of loops \( \gamma : [0, 1] \to O \) in a suitable sense (cf. \[BMP03\], p. 35) or in terms of generators and relations that only depend on the topology of \( O \) and the structure of its codimension 1- and 2-stratum \[Dav11\].

For another point \( x_1 \) in the regular part of \( O \) we can choose a path \( \gamma \) in \( FO \) that connects the fiber over \( x_0 \) with the fiber over \( x_1 \) and projects to the regular part of \( O \) by Lemma 2.3. For a loop \( \gamma_{x_0} \in P_{x_0}FO \) there exist paths \( \gamma_1 \in O(n)\gamma \) and \( \gamma_2 \in O(n)\gamma^{-1} \) for which the concatenation \( \gamma_{x_1} := \gamma_2 \star \gamma_{x_0} \star \gamma_1 \) defines a path in \( P_{x_1}FO \). The assignment \( \gamma_{x_0} \mapsto \gamma_{x_1} \) induces an isomorphism \( \pi_1^{orb}(O, x_0) \to \pi_1^{orb}(O, x_1) \). Indeed, paths homotopic in \( P_{x_0}FO \) are assigned to homotopic paths in \( P_{x_1}FO \), and the induced homomorphism is inverse to the analogous homomorphism \( \pi_1^{orb}(O, x_1) \to \pi_1^{orb}(O, x_0) \) induced by the inverse path \( \gamma^{-1} \).

For a point \( z_0 \in p^{-1}(x_0) \) there is a natural map

\[ j : \pi_1(FO, z_0) \to \pi_1^{orb}(O, x_0) \]

given by inclusions of equivalence classes and this map commutes with the change of base point isomorphism discussed above.
Lemma 4.4. The image of \( j : \pi_1(\mathcal{O}, z_0) \to \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) has index 1 or 2 in \( \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) depending on whether \( \mathcal{O} \) is orientable or not. Its kernel is given by \( i_*(\pi_1(-1(x_0), z_0)) \) where \( i : SO(n) \cong \pi^{-1}(x_0) \to FO \) is the inclusion.

Proof. A class in \( \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) lies in the image of \( j \) if and only if it is represented by a path whose end points lie on the same connected component of a fiber of the projection \( FO \to \mathcal{O} \). In particular, since the fiber of a regular point has two connected component, the square of any class in \( \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) lies in the image of \( j \). Now the first claim follows from the fact that different components of the same fiber lie in the same component of \( FO \) if and only if \( \mathcal{O} \) is non-orientable. A loop representing an element in the kernel of \( j \) is homotopic relative its endpoints to a loop in the fiber of \( x_0 \). Hence its homotopy class lies in the image of \( i_* \). Conversely, loops that lie in a fiber represent a trivial class in \( \pi_1^{\text{orb}}(\mathcal{O}, x_0) \), and so the second claim holds true, too. \( \square \)

The projection \( \pi : FO \to \mathcal{O} \) induces a homomorphism \( \pi_* : \pi_1^{\text{orb}}(\mathcal{O}, x_0) \to \pi_1(\mathcal{O}, x_0) \).

Lemma 4.5. For a manifold \( M \) the homomorphism \( \pi_* : \pi_1^{\text{orb}}(M, x_0) \to \pi_1(M, x_0) \) is an isomorphism. In particular, for the regular part \( \mathcal{O}_{\text{reg}} \) of an orbifold \( \mathcal{O} \) we have an isomorphism \( \pi_* : \pi_1^{\text{orb}}(\mathcal{O}_{\text{reg}}, x_0) \to \pi_1(\mathcal{O}_{\text{reg}}, x_0) \) (cf. Lemma 2.3).

Proof. For a manifold \( M \) the orthonormal frame bundle \( \pi : FO \to O \) is a fiber bundle. By the homotopy lifting property of fiber bundles any loop in \( M \) based at \( x_0 \) can be lifted to a path in \( P_{x_0}FM \) and so \( \pi_* \) is onto. Moreover, any homotopy between such loops can be lifted to a homotopy in \( P_{x_0}FM \) and so \( \pi_* \) is an isomorphism as claimed. \( \square \)

4.1. Existence of universal orbifold coverings. The idea to prove the existence of universal covering orbifolds is to translate the problem into a question about ordinary coverings of manifolds. In the following we assume that our orbifold \( \mathcal{O} \) is endowed with an auxiliary complete (Riemannian) metric. Essentially the same proof as in the manifold case in \([NO61]\) shows that any Riemannian metric on \( \mathcal{O} \) is conformally equivalent to a complete metric. Note that once a universal orbifold covering \( p : \hat{\mathcal{O}} \to \mathcal{O} \) has been constructed any Riemannian metric on \( \mathcal{O} \) lifts to a Riemannian metric on \( \hat{\mathcal{O}} \) such that \( p \) becomes a covering of Riemannian orbifolds with respect to the initial metric on \( \mathcal{O} \) and the lifted metric on \( \hat{\mathcal{O}} \).

The following lemma is the first step.

Lemma 4.6. An orbifold covering \( p : \hat{\mathcal{O}} \to \mathcal{O} \) induces an \( O(n) \)-equivariant covering map \( dp : F\hat{\mathcal{O}} \to FO \) on the level of orthonormal frame bundles and a homomorphism \( p_* : \pi_1^{\text{orb}}(\hat{\mathcal{O}}, \hat{x}_0) \to \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) such that the following diagrams commute

\[
\begin{array}{ccc}
(F\hat{\mathcal{O}}, \hat{z}_0) & \xrightarrow{\pi} & (\hat{\mathcal{O}}, \hat{x}_0) \\
\downarrow{dp} & & \downarrow{p} \\
(F\mathcal{O}, z_0) & \xrightarrow{\pi} & (\mathcal{O}, x_0)
\end{array}
\quad
\begin{array}{ccc}
\pi_1(F\hat{\mathcal{O}}, \hat{z}_0) & \xrightarrow{j} & \pi_1^{\text{orb}}(\hat{\mathcal{O}}, \hat{x}_0) \\
\downarrow{dp_*} & & \downarrow{p_*} \\
\pi_1(F\mathcal{O}, z_0) & \xrightarrow{j} & \pi_1^{\text{orb}}(\mathcal{O}, x_0)
\end{array}
\]

Moreover, we have

\[\ker(j) = dp_*(\ker(j)).\]

In particular, the homomorphism \( p_* : \pi_1^{\text{orb}}(\hat{\mathcal{O}}, \hat{x}_0) \to \pi_1^{\text{orb}}(\mathcal{O}, x_0) \) is injective.
Proof. An orbifold covering of the form $M/G_i \to M/G$ for a subgroup $G_i$ of $G$ as in Definition 2.6 induces an $O(n)$-equivariant covering $FM/G_i \to FM/G$ since the action of $G$ on $FM$ is free and commutes with the action of $O(n)$. In this way we obtain an $O(n)$-equivariant covering $dp : F\hat{O} \to F\hat{O}$ for which the left diagram commutes. This and the fact that homotopies can be lifted and projected implies that there is a induced homomorphism $p_* : \pi_1^{orb}(\hat{O},\hat{x}_0) \to \pi_1^{orb}(O,x_0)$ for which the right diagram commutes.

Suppose a loop $\gamma$ in $F\hat{O}$ based at $z_0$ represents a class in the kernel of $j$. By Lemma 4.4 we can assume that it is contained in a connected component of the fiber over $x_0$. Since the point $x_0$ is regular, the restriction $dp : \pi^{-1}(\hat{x}_0) \to \pi^{-1}(x_0)$ is an isometry. Hence, we can lift $\gamma$ to a loop $\hat{\gamma}$ based at $\hat{z}_0$ that is contained in the fiber $\pi^{-1}(\hat{x}_0)$ and whose class thus lies in the connected component of $F\hat{O}$ by Lemma 4.4. This shows (3) and completes the proof of the lemma.

An orbifold $O$ is called orientable if $F\hat{O}$ is disconnected and non-orientable otherwise. Recall that an orbifold can be written as a quotient $O = \hat{F}\hat{O}/O(n)$. The subgroup $SO(n)$ of $O(n)$ leaves the connected components of $F\hat{O}$ invariant while the full group $O(n)$ interchanges them in the orientable case. In particular, an orientable orbifold $O$ can also be written as $F_0\hat{O}/SO(n)$ where $F_0\hat{O}$ denotes a connected component of $F\hat{O}$.

Lemma 4.7. Let $O$ be an orientable Riemannian orbifold and let $\varphi : X \to F_0\hat{O}$ be a covering map from a connected Riemannian manifold $X$ onto a connected component $F_0\hat{O}$ of $F\hat{O}$. Then the metric quotient $X/\sim$ of $X$ obtained by collapsing the connected components of the fibers of the composition $\pi \circ \varphi$ is a Riemannian orbifold and the induced map $\varpi : X/\sim \to O$ is a covering of Riemannian orbifolds.

Proof. Possibly after dividing out the action of $\ker(j)$ on $X$ which leaves each connected component of the fibers of $\pi \circ \varphi$ invariant, we may assume that the condition

\[ \ker(j) \subseteq \text{Im}(\varphi_1 : \pi_1(X,\hat{z}_0) \to \pi_1(F_0\hat{O},z_0)) \]

is satisfied. In this case, by Lemma 4.1 and condition 4, the action of $SO(n)$ on $F_0\hat{O}$ lifts to an almost free action on $X$ with quotient $X/\sim$. Therefore $X/\sim$ is a Riemannian orbifold by Lemma 2.2 (alternatively, condition 4 implies that the charts of $O$ can be lifted to charts of $X/\sim$). Moreover, by Lemma 3.13 the covering $\varphi : X \to F_0\hat{O}$ between the complete spaces $X$ and $F_0\hat{O}$ is a submetry. Therefore, by compactness of $SO(n)$, also the induced map $\varpi : X/\sim \to O$ is a submetry and thus a covering of Riemannian orbifolds by Theorem 1.2.

We treat non-orientable orbifolds separately in order to avoid technicalities with non-connected coverings (but see Remark 4.10). To this end, similarly as in the manifold case, we associate an orientable orbifold to each non-orientable orbifold in a canonical way, its so-called orientable double cover.

Lemma 4.8. Let $O$ be a Riemannian orbifold. The quotient $O^+ := F_0\hat{O}/SO(n)$ is an orientable Riemannian orbifold and the natural map $p : F_0\hat{O}/SO(n) \to F\hat{O}/O(n)$ is an orbifold covering. Moreover, if $q : \hat{O} \to O$ is a covering of Riemannian orbifolds, then $q$ lifts to a covering $q^+ : \hat{O}^+ \to O^+$.

Proof. The fact that $p : F_0\hat{O}/SO(n) \to F\hat{O}/O(n)$ is a covering of Riemannian orbifolds follows as in the proof of Lemma 4.7. By construction the orthonormal frame bundle of $O^+$ consists of two copies of $F_0\hat{O}$ and thus $O^+$ is orientable.
By Lemma 4.6, the covering \( \varphi : \hat{O} \to O \) gives rise to an \( O(n) \)-equivariant covering map \( dp : F\hat{O} \to FO \) which restricts to an \( SO(n) \)-equivariant covering map \( dq : F_0\hat{O} = F_0\hat{O}^+ \to F_0O = F_0O^+ \) (cf. Lemma 4.8). Dividing out the \( SO(n) \)-action yields a lift \( q^+ : \hat{O}^+ \to O^+ \) of \( q \) which is a covering of Riemannian orbifolds by Lemma 4.7. \( \square \)

Now we can prove the existence of universal covering orbifolds.

**Proposition 4.9.** For every Riemannian orbifold \( O \) there exists a universal orbifold covering \( p : \hat{O} \to O \). Moreover, the orbifold fundamental group of \( \hat{O} \) is trivial.

*Proof.* By Lemma 4.8 we can assume that \( O \) is orientable. By the theory of ordinary covering spaces there exists a covering \( \varphi : X \to F_0O \) as in Lemma 4.7 with

\[
(\text{5}) \quad \ker(j) = \text{Im}(\varphi_* : \pi_1(X, \hat{z}_0) \to \pi_1(FO, z_0)).
\]

Set \( \hat{O} = X/\sim \) and \( p = \varphi \) as in Lemma 4.7. By Lemma 4.7, the map \( p : \hat{O} \to O \) is an orbifold covering. We have to show that for any orbifold covering \( q : O' \to O \), there exists a lift \( q' : \hat{O} \to O' \) with \( p = q \circ q' \). By Lemma 4.6 and (5) we have that

\[
\varphi'_*(\pi_1(X, z_0)) \subseteq dq_*(\pi_1(O', \hat{z}_0))
\]

for some base point \( \hat{z}_0 \in (dq)^{-1}(z_0) \). Therefore the covering \( \varphi \) lifts to a covering \( \varphi' : X \to F_0O' \) with \( \varphi = dq \circ \varphi' \) and \( \varphi'(\hat{z}_0) = \hat{z}_0' \). By Lemma 4.7, the map \( \varphi' \) induces an orbifold covering \( q' : \hat{O} \to O' \) with the desired properties.

It remains to show that \( \hat{O} \) is simply connected as an orbifold. First note that \( \hat{O} \) is orientable as a covering of the orientable orbifold \( O \). By Lemma 4.6, we have \( j \circ \varphi_* = p_* \circ j \) on \( \pi_1(X, \hat{x}_0) \) and so, by (5), these compositions are trivial. Hence, the fact that \( j \) is onto by Lemma 4.4 and \( p_* \) is injective by Lemma 4.6 implies that \( \pi^\text{orb}_1(\hat{O}, \hat{x}_0) \) is trivial as claimed. \( \square \)

**Remark 4.10.** In Lemma 4.6, we have seen that an orbifold covering \( p : \hat{O} \to O \) gives rise to an \( O(n) \)-equivariant covering map \( dp : F\hat{O} \to FO \). This covering induces a covering \( \varphi : B\hat{O} \to BO \) between the corresponding classifying space (see Remark 4.2). In fact, one can show that the orbifold coverings of \( O \) are in one-to-one correspondence to the ordinary coverings of the classifying space \( BO \) (cf. [ALR07, Prop. 2.17]).

4.2. **Thurston’s suggestion.** Now we explain Thurston’s suggestion to prove the existence of universal covering orbifolds of an orbifold \( O \) with empty codimension 1 stratum. We still endow \( O \) with an auxiliary Riemannian metric (cf. the discussion in Section 4.1). Since the codimension 1 stratum of an orientable orbifold is empty, we have already seen in the preceding section how the case of a non-empty codimension 1 stratum can be handled (see Lemma 4.8).

Let \( x_0 \in O \) be a regular point. According to Thurston a covering map \( p : (Y, y_0) \to (O\text{reg}, x_0), \ Y \) connected, is the restriction of a covering of \( p : \hat{O} \to O \) to the regular parts, if \( p_* (\pi_1(Y, y_0)) \) contains a “certain obvious normal subgroup of \( \pi_1(O\text{reg}, x_0) \)” [Thu79, p. 305]. Thurston’s assertion can be explained as follows. The inclusion \( \iota : O\text{reg} \subseteq O \) induces a map \( i_* : \pi_1(O\text{reg}, x_0) \cong \pi^\text{orb}_1(O\text{reg}, x_0) \to \pi^\text{orb}_1(O, x_0) \) where we have applied the isomorphism from Lemma 4.5. For a covering \( p : (\hat{O}, y_0) \to (O, x_0) \) it follows from the homotopy lifting property applied to the covering \( dp : F_0\hat{O} \to F_0O \) (cf. Lemma 4.6) that the condition

\[
(\text{6}) \quad \text{Ker}(i_* : \pi_1(O\text{reg}, x_0) \cong \pi^\text{orb}_1(O\text{reg}, x_0) \to \pi^\text{orb}_1(O, x_0)) \subseteq p_* (\pi_1(Y, y_0))
\]
is satisfied, where we have set \( Y := p^{-1}(\mathcal{O}_{\text{reg}}) \). Conversely, we call a covering \( p : (Y, y_0) \to (\mathcal{O}_{\text{reg}}, x_0) \) admissible if condition (6) is satisfied. Using the compatibility with the change of base point isomorphisms, one checks that the validity of this condition is independent of the specific base points. The kernel occurring in this condition is the “obvious normal subgroup” alluded to by Thurston. Thurston’s remark can be justified by the following two lemmas.

**Lemma 4.11.** Let \( p : (Y, y_0) \to (\mathcal{O}_{\text{reg}}, x_0) \) be a covering map, let \( U \) be a connected neighborhood of \( x_0 \) in \( \mathcal{O} \) and let \( \hat{U} \) be the connected component of \( p^{-1}(U) \) that contains \( y_0 \). If \( p \) is an admissible covering, then so is its restriction \( p : (\hat{U}, y_0) \to (U \cap \mathcal{O}_{\text{reg}}, x_0) \).

**Proof.** We need to show that an equivalence class \( [\gamma] \in \text{Ker}(i_\ast : \pi_1(U \cap \mathcal{O}_{\text{reg}}, x_0) \to \pi_1^{\text{orb}}(U, x_0)) \) is contained in \( p_\ast(\pi_1(\hat{U}, y_0)) \). Clearly we have \( [\gamma] \in \text{Ker}(i_\ast : \pi_1(\mathcal{O}_{\text{reg}}, x_0) \to \pi_1^{\text{orb}}(\mathcal{O}, x_0)) \) and so \( [\gamma] \in p_\ast(\pi_1(Y, y_0)) \) by (6). Now the homotopy lifting property of coverings implies that \( [\gamma] \in p_\ast(\pi_1(\hat{U}, y_0)) \). \( \square \)

**Lemma 4.12.** Let \( p : (Y, y_0) \to (\mathcal{O}_{\text{reg}}, x_0) \) be an admissible covering map. Then the metric completion \( p : \bar{Y} \to \mathcal{O} \) is a covering of Riemannian orbifolds.

**Proof.** Let \( x \in \mathcal{O} \) be a point and let \( y \) be a preimage in the completion \( Y \). We can choose a neighborhood \( U \) of \( x \) isometric to \( M/G \) as in Definition 1.12 such that \( M \) is simply connected, \( G \) fixes a preimage \( z \) of \( x \) in \( M \) and \( M \) is complete in a neighborhood of \( z \). Let \( \hat{U} \) be the connected component of \( p^{-1}(U \cap \mathcal{O}_{\text{reg}}) \) that contains \( y_0 \). We can assume that \( x_0 \in U \) and that \( y_0 \in \hat{U} \). Let \( z_0 \) be a preimage of \( x_0 \) in \( M \). Since \( M \) is simply connected as an orbifold, it follows from the homotopy lifting property that

\[
\text{Ker}(i_\ast : \pi_1(U \cap \mathcal{O}_{\text{reg}}, x_0) \to \pi_1^{\text{orb}}(U, x_0)) = p_\ast(\pi_1(M_{\text{reg}, G}, z_0))
\]

where \( M_{\text{reg}, G} \) are the points in \( M \) that project to \( U \). By Lemma 4.11 and the lifting condition for ordinary coverings this implies that the covering \((M_{\text{reg}, G}, z_0) \to (U, x_0)\) can be lifted to a covering \((M_{\text{reg}, G}, z_0) \to (\hat{U}, y_0)\). It follows that a neighborhood of \( y \) in \( \bar{Y} \) is isometric to \( M/H \) for some subgroup \( H \) of \( G \). This shows that \( \bar{Y} \) is a complete orbifold. Moreover, the covering map \( p : (Y, y_0) \to (\mathcal{O}_{\text{reg}}, x_0) \) is a weak submetry by Lemma 3.13. Therefore also its metric completion \( p : \bar{Y} \to \mathcal{O} \) is a weak submetry, and hence a submetry since the complete length space \( \bar{Y} \) is proper by the Hopf-Rinow theorem \([BB01\text{ Thm. 2.5.28}]\). Hence, \( p : \bar{Y} \to \mathcal{O} \) is a covering of Riemannian orbifolds by Theorem 1.12 as claimed. \( \square \)

In fact, let \( p : (\hat{O}, \hat{y}_0) \to (\mathcal{O}, x_0) \) be the metric completion of the covering \( p : (\bar{Y}, \bar{y}_0) \to (\mathcal{O}_{\text{reg}}, x_0) \) corresponding to the subgroup \( \text{Ker}(i_\ast : \pi_1(\mathcal{O}_{\text{reg}}, x_0) \to \pi_1^{\text{orb}}(\mathcal{O}, x_0)) \) of \( \pi_1(\mathcal{O}_{\text{reg}}, x_0) \) and let \( q : (\hat{O}, \hat{y}_0) \to (\mathcal{O}, x_0) \) be any orbifold covering. Then it follows with condition (6) and the argument at the end of Lemma 4.12 that there exists a covering \( q' : (\hat{O}, \hat{y}_0) \to (\mathcal{O}, y_0) \) lifting \( p \). This shows that \( p : (\hat{O}, \hat{y}_0) \to (\mathcal{O}, x_0) \) is a universal orbifold covering.

4.3. **Consequences of the existence of universal orbifold coverings.** We record some consequences of the existence of universal orbifold coverings. Like regular coverings, coverings of Riemannian orbifolds come along with a deck transformation group.

**Definition 4.13.** The deck transformation group of a covering \( p : \hat{O} \to \mathcal{O} \) of Riemannian orbifolds is defined as the group of all isometries of \( \hat{O} \) that leave the fibers of \( p \) invariant. The
covering $p$ is called Galois if the deck transformation group acts transitively on the fibers of $p$.

It follows from Definition 2.6 that the deck transformation group $G$ of a covering $p: \hat{O} \to O$ of Riemannian orbifolds acts properly on $\hat{O}$, i.e. given any two points $x, y \in \hat{O}$ there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that $gU \cap V \neq \emptyset$ for only finitely many $g \in G$. Using the universal covering property the next statement follows as in the manifold case.

**Proposition 4.14.** A universal orbifold covering $p: \hat{O} \to O$ is Galois. In particular, any covering $q: O' \to O$ is of the form $q: \hat{O}/H \to \hat{O}/G$ for some subgroup $H$ of the deck transformation group $G$ of $p: \hat{O} \to O$. Moreover, the group of deck transformations of such a covering $q$ is given by the normalizer $N_G(H)$ of $H$ in $G$.

The deck transformation group of $p: \hat{O} \to O$ is related to the fundamental group of $O$ as follows.

**Proposition 4.15.** The deck transformation group $G$ of a universal orbifold covering $p: \hat{O} \to O = \hat{O}/G$ is isomorphic to $\pi_1^{orb}(\hat{O}, x_0)$ via a naturally isomorphism $\tau_G$ in the sense that for a subgroup $H$ of $G$ we have $q_* \tau_H = \tau_G$ on $H$ with $q: \hat{O}/H \to O = \hat{O}/G$.

**Proof.** For an element $g \in G$ choose a path $\gamma: [0, 1] \to \hat{F} \hat{O}$ connecting $\pi^{-1}(\hat{x}_0)$ and $\pi^{-1}(g \hat{x}_0)$. This path projects to a path $\tilde{\gamma}: [0, 1] \to \tilde{F} \hat{O}$ with $\tilde{\gamma}(0), \tilde{\gamma}(1) \in \pi^{-1}(x_0)$ and hence represents a class $[\tilde{\gamma}]$ in $\pi_1^{orb}(\hat{O}, x_0)$. Since $\hat{O}$ is simply connected as an orbifold by Proposition 4.9 this class does not depend on the choice of $\gamma$. This shows that $\tau_G(g) := [\tilde{\gamma}]$ is well-defined and implies that $\tau_G$ is in fact a homomorphism. Moreover, it follows from the homotopy lifting property and the regularity of the point $x_0$ that $\tau_G$ is an isomorphism. The naturality property is an immediate consequence of the definition of $\tau_G$ and $p_*$. \hfill $\square$

With this Proposition we can prove the following corollaries.

**Corollary 4.16.** An orbifold $O$ has the universal covering property if and only if its orbifold fundamental group is trivial.

**Proof.** This follows easily from Proposition 4.9 and Proposition 4.15. \hfill $\square$

**Corollary 4.17.** An orbifold covering $q: \hat{O} \to O$ is regular if and only if the image of $\pi_1^{orb}(\hat{O}, \hat{x}_0)$ under $p_*$ is a normal subgroup of $\pi_1^{orb}(O, x_0)$.

**Proof.** By Proposition 4.9 and Proposition 4.14 we can assume that the covering $q$ is of the form $q: \hat{O}/H \to \hat{O}/G$ as in the statement of Proposition 4.14. By Proposition 4.14 the covering $q: \hat{O} \to O$ is regular if and only if $H$ is a normal subgroup of $G$. Hence, the claim follows from Proposition 4.15. \hfill $\square$

**Corollary 4.18.** Let $p_1: (O_1, x_1) \to (O, x_0)$ and $p_2: (O_2, x_2) \to (O, x_0)$ be two orbifold coverings. Then $p_1$ lifts to a covering $q: (\hat{O}_1, x_1) \to (\hat{O}_2, x_2)$ if and only if $(p_1)_*(\pi_1^{orb}(O_1, x_1)) \subseteq (p_2)_*(\pi_1^{orb}(O_2, x_2))$.

**Proof.** As in Corollary 4.17 we can assume that the coverings $p_1$ and $p_2$ are of the form $p_1: \hat{O}/H_1 \to \hat{O}/G$ and $p_2: \hat{O}/H_2 \to \hat{O}/G$ for subgroups $H_1$ and $H_2$ of the deck transformation group $G$ of the universal covering $\hat{O} \to O$. Hence, the claim follows from Proposition 4.15. \hfill $\square$
**Corollary 4.19.** Let $O$ be a Riemannian orbifold and let $\Gamma$ be a discrete group that acts isometrically and properly on $O$. Then $O \to O/\Gamma$ is a covering of Riemannian orbifolds with deck transformation group $\Gamma$. In particular, if $O$ is a simply connected manifold, then any isometry of $O/\Gamma$ lifts to an isometry of $O$.

**Proof.** Since the group $\Gamma$ is discrete and its action on $O$ is proper, the first claim is local. More precisely, to prove it we can assume that $O$ is a quotient $M/H$ of a Riemannian ball $M$ by a finite group of isometries $H$ as in Definition 1.1 where $H$ fixes a point $x \in M$, and that $\Gamma$ is a finite group that fixes the coset $\varpi$ of $x$ in $O$. Since $M$ is simply connected as an orbifold it follows from Corollary 4.18 that the action of $\Gamma$ on $O$ lifts to an action of a group $G$ on $M$ with normal subgroup $H$. In particular, $O \to O/\Gamma$ is a covering of Riemannian orbifolds as claimed.

Clearly $\Gamma$ is contained in the deck transformation group of the covering $O \to O/\Gamma$. A deck transformation of the covering $O \to O/\Gamma$ restricts to a smooth isometry of the regular part of $O$ by Lemma 2.4. Let $g$ be a deck transformation of the covering $O \to O/\Gamma$ and let $x \in O$ be a regular point of this covering. It follows that there exists some $\gamma \in \Gamma$ whose differential at $x$ coincides with the differential of $g$. Now the second part of Lemma 2.4 implies that $g$ and $\gamma$ coincide everywhere. The last statement about the existence of lifts follows from Corollary 4.18. This completes the proof of the corollary. \qed

Finally, we mention an interesting characterization of good orbifold due to Thurston. Following Thurston an orbifold is called *good* if it is covered by a manifold, and otherwise it is called *bad*. Recall that a neighborhood $U$ of a point $x$ in a Riemannian orbifold $O$ is isometric to a quotient $M/G_x$ where $G_x$ fixes a preimage of $x$ in $M$ and that we refer to $G_x$ as the local group of $x$. For any regular point $x_0 \in U$ the local group $G_x$ is isomorphic to $\pi_1^\text{orb}(U,x_0)$ via the isomorphism described in Proposition 4.15. The composition with the homomorphism $\pi_1^\text{orb}(U,x_0) \to \pi_1^\text{orb}(O,x_0)$ induced by the inclusion yields a homomorphism $i_x : G_x \to \pi_1^\text{orb}(O,x_0)$. One readily checks that the condition if the map $i_x$ is injective does not depend on the choices made. At this point one easily proves the following characterization.

**Proposition 4.20** (Thurston). The orbifold $O$ is good if and only if for any $x \in O$ the homomorphism $i_x : G_x \to \pi_1^\text{orb}(O,x_0)$ is injective.

5. The metric double covering

The aim of this section is to prove Proposition 1.3. Let us first the required notions before we recall its statement.

The *metric double* of a metric space $X$ along a closed subspace $Y \subset X$ is defined as the topological double $2_Y X$ endowed with the unique maximal metric that is majorized by the metrics on the two copies $X_1$ and $X_2$ of $X$ in $2_Y X$ [BBI01 3.1.24]. Equivalently, this metric can be described as a *gluing metric* where the distance between two points $x, y \in 2_Y X$ is defined as the infimum of

$$\sum_{i=0}^k d(x_i,y_i)$$

over all sequences $x_i, y_i$, $i = 0, \ldots, k$ with $x = x_0$, $y = y_k$, $x_i, y_i \in X_i$, $i = 1, 2$, and $x_i = y_{i+1}$, $i = 0, \ldots, k - 1$ [BBI01 3.1.12, 3.1.27]. If $X$ is in addition a length space, then so is $2_Y X$ [BBI01 3.1.24].
We apply this construction in case of a Riemannian orbifold \( X = \mathcal{O} \) and \( Y \) being the closure of its codimension 1 stratum \( \Sigma_1 \mathcal{O} \). In the following we refer to this closure as the **boundary** of \( \mathcal{O} \) and denote it as \( \partial \mathcal{O} \) (since it coincides with the boundary of \( \mathcal{O} \) in the sense of Alexandrov geometry by Lemma 2.5, cf. [BGP92]). Recall the statement of Proposition 1.3. We want to prove that the metric double \( 2\partial \mathcal{O} \) is a Riemannian orbifold and that the natural projection \( 2\partial \mathcal{O} \to \mathcal{O} \) is a covering of Riemannian orbifolds.

**Remark 5.1.** The metric double along the boundary and the orientable double cover of a Riemannian orbifold with nonempty codimension 1 stratum may differ as the following example illustrates. Consider a flat Möbius strip. Then its double along its boundary is a Klein bottle whereas its orientable double cover is a torus.

To prove Proposition 1.3 we first consider a connected metric space \( X \) with a closed subspace \( Y \) and observe some general facts about the metric double of \( X \) along \( Y \). In the following we denote the two copies of \( X \) as \( X_1 \) and \( X_2 \).

**Lemma 5.2.** Let \( \gamma : [0, 1] \to 2_Y X \) be a path connecting points \( \gamma(0) \in X_1 \) and \( \gamma(1) \in X_2 \). Then the set \( M = \{ t \in [0, 1] | \gamma(t) \in Y \} \subset [0, 1] \) is a nonempty union of closed intervals.

**Proof.** The subspaces \( X_1 \) and \( X_2 \) are closed in \( 2_Y X \) since \( Y \) is closed in \( X \). Moreover, we have \( 2_Y X = X_1 \cup X_2 \) and \( Y = X_1 \cap X_2 \). Hence, \( [0, 1] = \gamma^{-1}(X_1) \cup \gamma^{-1}(X_2) \) and so there is some \( t \in [0, 1] \) with \( \gamma(t) \in X_1 \cap X_2 = Y \) since \( [0, 1] \) is connected. This shows that \( M \) is non-empty. By continuity \( M \) is also closed. Every closed subset of \([0, 1]\) is a union of closed intervals. \( \square \)

There is a natural reflection \( s : 2_Y X \to 2_Y X \) that interchanges the two copies \( (X_1, d) \) and \( (X_2, d) \) of \( X \) in \( 2_Y X \) and fixes the subspace \( Y = X_1 \cap X_2 \subset 2_Y X \) pointwise. The fact that it identifies \( X_1 \) and \( X_2 \) isometrically by definition implies that it is an isometry.

**Lemma 5.3.** For two points \( x, x' \in X_1 \) we have \( d(x, x') = d_2(x, x') \leq d_2(x, s(x')) \). In other words, the embedding \( X_1 \hookrightarrow 2_Y X \) is isometric and the composition with the projection from \( 2_Y X \) to \( 2_Y X/s \) with the quotient metric is an isometry.

**Proof.** For any approximation \( \sum_{i=0}^{k-1} d(x_i, x_{i+1}) \) of \( d_2(x, x') \) or \( d_2(x, s(x')) \) we obtain the same approximation of \( d(x, x') \) by mapping all the \( x_i \) that lie in \( X_2 \) to \( X_1 \) via \( s \). This shows \( d_2(x, x') \leq d_2(x, s(x')) \), and \( d(x, x') \leq d_2(x, x') \) by the triangle inequality. On the other hand, we have \( d_2 \leq d \) by the characterization of \( d_2 \) as a maximal metric that is majorized by \( d \) and so the claim follows. \( \square \)

Let us now explain the proof of Proposition 1.3.

**Proof of Proposition 1.3.** The proof is by induction on the dimension. The claim is local and it is clear for points in \( 2\mathcal{O} = 2\partial \mathcal{O} \) that do not project to \( \partial \mathcal{O} \). Let \( x \in 2\mathcal{O} \) be a point that projects to \( \partial \mathcal{O} \). According to Definition 1.1 we can assume that \( \mathcal{O} \) is of the form \( M/G \), the quotient of a Riemannian manifold ball by a finite group of isometries \( G \) that fixes a preimage \( \hat{x} \) of \( x \) in \( M \). In particular, for \( n = 1 \) the underlying topological space of \( \mathcal{O} \) is \([0, 1]\) and in this case the claim is clear.

Fix some dimension \( n > 1 \) and suppose that the claim is true in all lower dimensions. Let \( S^2 \) be the unit sphere in the tangent space \( T_{\hat{x}} M \). Then \( S^2/G \) is a Riemannian orbifold with \( \mathbb{R}_{>0} \cdot (\partial(S^2/G)) = \partial(T_{\hat{x}} M/G) \). By induction assumption the metric double \( 2(S^2 x/G) \) is a
covering orbifold of $S_2/G$. Therefore, there exists an index 2 subgroup $H$ of $G$ such that the identity on $S_2/G$ lifts to an isometry $\theta : 2\langle S_2/G \rangle \to S_2/H$. For $n = 2$ this follows from the classification of finite subgroups of $O(2)$ and for $n > 2$ it follows from Proposition 4.14 and Corollary 4.16. In particular, $\theta$ is equivariant with respect to the natural reflection of $2\langle S_2/G \rangle$ and the action of $G/H \cong \mathbb{Z}_2$ on $S_2/H$.

Applying the exponential map yields an equivariant homeomorphism between small metric balls $\theta : 2\mathcal{O} \supset B_r(x) \to B_r(y) \subset M/H$ where $y$ is the coset of $\hat{x}$ in $M/H$. By construction this map has the property that $\theta(B_{r'}(x)) = B_{r'}(y)$ holds for all $r' \in [0, r]$ and it descends to the identity map on $B_r(x) \subset \mathcal{O}$ which is an isometry. Since all metrics involved are length metrics, this together with the preceding lemmas implies that the map $\theta$ restricts to an equivariant isometry $\theta : 2\mathcal{O} \supset B_{r/\theta}(x) \to B_{r/\theta}(y) \subset M/H$. For completeness and convenience of the reader we spell out the details of this implication in two subsequent, separate lemmas. This shows that the projection from $2\mathcal{O}$ to $\mathcal{O}$ is a covering of Riemannian orbifolds and so the claim follows by induction.

It remains to show that $\theta : 2\mathcal{O} \supset B_{r/\theta}(x) \to B_{r/\theta}(y) \subset M/H$ is an isometry. To this end we first characterize the metrics on $B_{r/\theta}(x)$ and $B_{r/\theta}(y)$ as follows. Let $s_G$ be the generator of $G/H$ acting on $M/H$. We denote the metrics on $2\mathcal{O}$ and on $M/H$ by $d_2$ and $d_9$, respectively. In the subsequent lemma $(Z, z, d_Z, s_Z, \phi)$ may either be $(2\mathcal{O}, x, d_2, s, \text{id})$ or $(M/H, y, d_9, s_G, \theta)$. Moreover, we continue to denote the two copies of $\mathcal{O}$ in $2\mathcal{O}$ as $X_1$ and $X_2$ and their intersection as $Y$.

**Lemma 5.4.** Let $(Z, d_Z)$ be a length space with an isometric involution $s_Z$. Suppose that there exists a homeomorphism $\phi : 2\mathcal{O} \supset B_r(x) \to B_r(z) \subset Z$ with $\phi(x) = Z_0 := \text{Fix}(s_Z)$ that is $Z_2$-equivariant with respect to the action of $s$ on $B_r(x)$ and the action of $s_Z$ on $B_r(0)$, and that is a radial isometry in the sense that $\phi(B_{r'}(x)) = B_{r'}(z)$ holds for all $r' \in [0, r]$. Then the following holds true.

(i) For $w, w' \in \phi(B_{r/\beta}(x) \cap X_1)$ we have
\[ d_Z(w, w') \leq d_Z(w, s_Z(w')). \]

(ii) For $w \in \phi(B_{r/\beta}(z) \cap X_1)$ and $w' \in \phi(B_{r/\beta}(z) \cap X_2)$ we have
\[ d_Z(w, w') = \inf_{y' \in Z_0} (d_Z(w, y') + d_Z(y', w')). \]

**Proof.** (i) Let $\gamma : [0, 1] \to Z$ be a path connecting $w$ and $s_Z(z)$ whose length approximates $d_Z(w, s_Z(z))$ up to some small $\varepsilon > 0$. Because of $w, s_Z(z) \in B_{r/\beta}(z)$, we can assume that $\gamma$ is completely contained in $B_r(z)$. By Lemma 5.2 applied to $\phi^{-1} \circ \gamma$ there is some $t_0 \in [0, 1]$ with $\gamma(t_0) \in \phi(Y \cap B_r(x) \subset \text{Fix}(s_Z)$. We define a new path $\tilde{\gamma} : [0, 1] \to Z$ by $\tilde{\gamma}(t) = \gamma(t)$ for $t \in [0, t_0]$ and $\tilde{\gamma}(t) = s_Z(\gamma(t))$ for $t \in [t_0, 1]$. The path $\tilde{\gamma}$ connects $w$ and $w'$ and has length $L(\tilde{\gamma}) = L(\gamma)$ since $s$ is an isometry. Now the claim follows since $Z$ is a length space.

(ii) By the triangle inequality we have $d_Z(w, w') \leq \inf_{y' \in Z_0} (d_Z(w, y') + d_Z(y', w'))$. On the other hand, let $\gamma$ be a path connecting $w$ and $w'$ whose length approximates $d_Z(w, w')$. As above we can assume that $\gamma$ is completely contained in $B_r(z)$. Similarly as in (i) we can use Lemma 5.2 to construct a path $\tilde{\gamma}$ connecting $w$ and $s_Z(z)$ that lies completely in $\theta(B_r(z) \cap X_1)$, intersects $Z_0$ and satisfies $L(\tilde{\gamma}) = L(\gamma)$. Since $\tilde{\gamma}$ intersects $Z_0$, we have
\[ L(\gamma) \geq \inf_{y' \in Z_0} (d(w, y') + d(y', w')). \] The fact that \( Z \) is a length space implies \( d(w, w') \geq \inf_{y' \in Z_0} (d(w, y') + d(y', w')) \) and hence the claim follows. \qed

**Lemma 5.5.** The map \( \theta : 2\mathcal{O} \supset B_{r/9}(x) \to B_{r/9}(y) \subset M/H \) is an isometry as claimed in the proof of Proposition 1.3.

Proof. Let \( z, z' \in B_{r/3}(x) \cap X_1 \). By Lemma 5.3 we have \( d(z, z') = d_2(z, z') = d(\overline{z}, \overline{z}') \) where \( \overline{z} \) and \( \overline{z}' \) are the cosets of \( z \) and \( z' \) in \( 2\mathcal{O}/s \cong \mathcal{O} \). Since \( \theta \) descends to an isometry on \( B_r(x) \subset \mathcal{O} \), the definition of the quotient metric on \( M/G \) implies
\[
d_2(z, z') = \min\{d_q(\theta(z), \theta(z')), d(\theta(z), s_G(\theta(z'))\}\}.
\]

Now Lemma 5.4 (i), shows that \( d_2(z, z') = d_q(\theta(z), \theta(z')) \). By the same reason this identity holds for points \( z, z' \in B_{r/3}(x) \cap X_2 \).

Now let \( z \in B_{r/9}(x) \cap X_1 \) and \( z' \in B_{r/9}(x) \cap X_2 \). Then we have \( d_2(z, z'), d_q(\theta(z), \theta(z')) \leq 2r/9 \). Applying the first paragraph and Lemma 5.4 (ii), twice yields
\[
d_2(z, z') = \inf_{y' \in Y} (d_2(z, y') + d_2(y', z'))
= \inf_{y' \in B_{r/3}(x) \cap Y} (d_2(z, y') + d_2(y', s(z')))
= \inf_{y' \in B_{r/3}(x) \cap Y} (d_q(\theta(z), \theta(y')) + d_q(\theta(y'), s(z')))
= \inf_{y' \in B_{r/3}(y) \cap \text{Fix}(s_G)} (d_q(\theta(z), y') + d_q(y', \theta(z')))
= d_q(\theta(z), \theta(z')).
\]

Hence, \( \theta : B_{r/9}(y) \to B_{r/9}(z) \) is an isometry as claimed. \qed

This completes the proof of Proposition 1.3 as explained above. \qed

**References**

[ALR07] A. Adem, J. Leida and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, 171, Cambridge Univ. Press, Cambridge, 2007.

[AM94] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der Mathematischen Wissenschaften, 309, Springer-Verlag, Berlin, 1994.

[BG00] V. N. Berestovskii and L. Guijarro, *A metric characterization of Riemannian submersions*, Ann. Global Anal. Geom. 18 (2000), no. 6, 577–588.

[Bre72] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[BH99] M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[BMP03] M. Boileau, S. Maillot and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses, 15, Société Mathématique de France, Paris, 2003.

[BGP92] Yu. Burago, M. Gromov and G. Perelman, Russian Math. Surveys 47 (1992), no. 2, 1–58; translated from Uspekhi Mat. Nauk 47 (1992), no. 2(284), 3–51, 222.

[BBI01] D. Burago, Yu. Burago and S. Ivanov, *A course in metric geometry*, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

[Dav11] M. W. Davis, *Lectures on orbifolds and reflection groups*, Transformation groups and moduli spaces of curves, Adv. Lect. Math. (ALM), vol. 16, Int. Press, Somerville, MA, 2011, pp. 63-93, MR2883685

[DoC92] M. P. do Carmo, *Riemannian geometry*, translated from the second Portuguese edition by Francis Flaherty, Mathematics: Theory & Applications, Birkhäuser Boston, Boston, MA, 1992.
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