Local Existence of Classical Solutions to Scalar Field Equation on Spatially Compact Spacetime as a Background

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Abstract. The goal of this paper is to prove the wellposedness of scalar field equation on spatially compact spacetime of Riemannian manifold. We construct the equation of motion from the Lagrangian of scalar field with non-minimal coupling, where the coupling interaction of the scalar field $\varphi$ is proportional to the scalar curvature of the spacetime. The equation of motion has the form like non-linear wave equation. The next step is to prove local existence of solutions. We have show that both the $k$\textsuperscript{th} linear energy and energy norm are bounded for the finite time with the initial data in $H^{k+1} \times H^k$. Finally, we prove the uniqueness and smoothness properties of the solution.

1. Introduction

The scalar field equation is a classical form of Klein-Gordon equation. It is one of fundamental equation describing dynamics of the spinless particle in our universe. Scalar field equation is one kind of partial differential equation. Study of analysis differential equation has many different purposes. The first as the fundamental goal is to find the exact solution of the equation. In addition, to solve the Klein-Gordon equation on the spacetime in four dimension with general couplings even at classical level is quite complicated\textsuperscript{1}. For that reason, we have to find the solvable models by specify both the spacetime and the coupling. Several simple models in four dimension have been studied for example in\textsuperscript{2–5}.

The second purpose of analysis differential equation as the simpler one is to prove the existence of solutions in both local and global cases. A local solution is a solution defined in the finite time, while a global solution is a solution defined in the interval $(-\infty, \infty)$. Local and global existence solution of scalar equation in higher dimensional spatially flat Friedmann-Robertson-Walker with non-minimal coupling has been studied in\textsuperscript{1}. In this paper we make a generalization of the previous work from Euclidean space to Riemannian manifold using Sobolev embedding theorem on manifold\textsuperscript{8}. We choose that the spacetime is spatially compact. We prove local existence of classical solution to the scalar field equation in higher dimensional compact Riemannian manifold with non-minimal coupling. This additional non-minimal coupling represents the coupling interaction of scalar field $\varphi$ that proportional to the scalar curvature of the spacetime. It is the simplest generalization of scalar field theory on curved spacetimes\textsuperscript{6,7}. 

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We organize the paper as follows. In Section 2, we give general setting to prove local solutions to scalar field equation. In Section 3 we study the uniqueness properties of solution. In Section 4 we give the proof of smoothness properties. Finally, we give some conclusions in Section 5 as the last section.

2. Local existence of solution

In this section we give general setting to prove local existence of solution to scalar field equation in manifold with spatially compact spacetime. Given the metric:

\[ ds^2 = -dt^2 + a^2(t)\hat{g}_{ij}dx^i dx^j \]  

(1)

\( \hat{g}_{ij} = g_{ij}(x^i, x^j) \) describes metric for spatially compact of \( D-1 \) dimensional manifold \( M \) where \( i, j = 1, 2, 3, ..., D-1 \). Let us define a new time coordinate \( \tau \) by

\[ \frac{d\tau}{dt} = \frac{1}{a(t)} \]  

(2)

Thus we have

\[ ds^2 = a^2(\tau) \left(-d\tau^2 + \hat{g}_{ij}dx^i dx^j\right) \]  

(3)

\( x^\mu = (x^0 = \tau, x^i) \), with \( \mu = 0, 1, ..., D-1 \) and \( i, j = 1, 2, ..., D-1 \).

To construct the equation of motion, we introduce the action of scalar field with non-minimal coupling and scalar potential \( V(\varphi) \) as follows,

\[ S = \int d\tau dx \sqrt{-g} \left( \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi + \frac{1}{2} \xi R \varphi^2 - V(\varphi) \right) \]  

(4)

where \( g \) and \( R \) describe the determinant and the scalar curvature from the metric (3). Non-minimal coupling is described by the second term of the R.H.S of (4), with \( \xi \) is a positive constant. Cosmological models which \( \xi \) positive is defined as canonical, hence for \( \xi \) negative is defined as phantom.\(^1\) Real function \( V(\varphi) \) that denotes the scalar potential is a smooth function and satisfies the condition \( V(0) = 0 \) and \( \partial_\varphi V(0) = 0 \).

Using variational principle of (4), we obtain scalar field equation

\[ \partial^2_\tau \varphi - \Delta \varphi = \mathcal{F}(\varphi, \partial_\tau \varphi) \]  

(5)

with the non-linear term is defined by,

\[ \mathcal{F}(\varphi, \partial_\tau \varphi) = -H(2D-1)\partial_\tau \varphi - \xi(D-1) \left( \dot{H} + H^2 \right) \varphi + a^2(\tau)\partial_\varphi V(\varphi) \]  

(6)

\( \Delta \) is defined as Laplace-Beltrami operator, and \( H \) denotes Hubble parameter with the scale factor \( a = a(\tau) \).

We claim that the solution of (5) must be bounded by some constant \( C \) for finite time \( T < \infty \) with the initial data in \( H^{k+\frac{1}{2}} \times H^k \),

\[ |\varphi(\tau, .)|_k := ||\varphi(\tau, .)||_{H^{k+\frac{1}{2}}(M)} + ||\partial_\tau \varphi(\tau, .)||_{H^k(M)} \]  

(7)

Thus we have Lemma as follows.
Lemma 2.1. Let $\varphi$ be a real function such that for $\tau \in [\tau_0, \tau_0 + T]$, $k \in \mathbb{N}$ and $p \geq 1$ real, we have the estimate as follows

$$|\varphi(\tau, .)|_k \leq C$$

(8)

Hence for $k > j \geq 0$ and $k > \frac{1}{2}(D - 1)$, where $D - 1$ denotes dimension of manifold $\mathcal{M}$, we have

$$\sum_{j=0}^{k} \left( \int_{\mathcal{M}} \left| \nabla^j F(\varphi, \partial_\tau \varphi) \right|^2 dv(g) \right)^{\frac{1}{2}} \leq C$$

(9)

The constant $C$ depends only on the initial data, $T$, $k$, and the bound of the scalar potential $V(\varphi)$.

Proof. Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold. For any integer $k$ and $\varphi : \mathcal{M} \to \mathbb{R}$ smooth, we can define $\nabla^j \varphi$ as $j^{th}$ covariant derivative of $\varphi$. We have,

$$\mathcal{C}_k^p(\mathcal{M}) = \left\{ \varphi \in C^\infty(\mathcal{M}) | \forall j = 0, \ldots, k, \int_{\mathcal{M}} |\nabla^j \varphi|^p dv(g) < +\infty \right\}$$

(10)

If $\mathcal{M}$ is a compact manifold, then for any $k$ and $p \geq 1$ we have $\mathcal{C}_k^p(\mathcal{M}) = C^\infty(\mathcal{M})$.

Let us consider the case for $|j| > 0$. We obtain

$$\nabla^j F(\varphi, \partial_\tau \varphi) = -H(2D - 1) \nabla^j (\partial_\tau \varphi) - \xi (D - 1) \left( \dot{H} + H^2 \right) \nabla^j \varphi + a^2(\tau) \nabla^j \partial_\varphi V(\varphi)$$

(11)

The first and the second term of (11) denote the linear term which bounded by some constant $C$ by the definition of (10). To prove the estimate of the third term we introduce

$$\nabla^j (\partial_\varphi V) = g^{j_1 \ldots j_k} (\nabla^k \partial_\varphi V(\varphi))_{i_1 \ldots i_k} \left( \nabla^{j_1} \partial_\varphi (V(\varphi)) \right)_{j_1 \ldots j_k} \equiv \partial_\varphi V(\varphi)$$

(12)

Since the scalar potential is a smooth function, by Sobolev embedding for the first part for any $\varphi \in H_k^2(\mathcal{M})$ there exists a constant $C$ such that

$$\| \partial_\varphi V(\varphi) \|_{H_n^2(\mathcal{M})} \leq C \| \varphi \|_{H_n^2(\mathcal{M})} \leq C$$

(13)

Thus we conclude that for $j > 0$, the non-linear term $F(\varphi, \partial_\varphi)$ is bounded by some constant $C$ locally for $\tau \in [\tau_0, \tau_0 + T]$.

Now let us consider the case when $|j| = 0$. From (11) we have

$$\left( \int_{\mathcal{M}} \left| F(\varphi, \partial_\tau \varphi) \right|^2 dv(g) \right)^{\frac{1}{2}} \leq C \left[ \| \partial_\tau \varphi \|_{L^2(\mathcal{M})} + \| \varphi \|_{L^2(\mathcal{M})} + \| \partial_\varphi V(\varphi) \|_{L^2(\mathcal{M})} \right]$$

(14)

The first and the second term of (14) denotes the linear term bounded by some constant $C$. Using the condition $\partial_\varphi V(0) = 0$ we obtain

$$\| \partial_\varphi V(\varphi) \|_{L^2(\mathcal{M})} \leq C \| \varphi \|_{L^2(\mathcal{M})} \leq C$$

(15)

Thus the estimate (9) is true for $|j| = 0$. Moreover, the non-linear term $F(\varphi, \partial_\tau \varphi)$ is bounded by some constant $C$ locally for $\tau \in [\tau_0, \tau_0 + T]$ and the proof of Lemma 2.1 is finished.
Let us define,

$$E_k[\varphi] = \frac{1}{2} \sum_{j=0}^{k} \int_{M} \left[ (\nabla^j \partial_{\tau} \varphi)^2 + |\partial_i \nabla^j \varphi|^2 \right] dv(g)$$  \hspace{1cm} (16)

as the bounded linear energy satisfying the lemma as follows

**Lemma 2.2.** Given \( \varphi \) be a real function on \( D - 1 \) dimensional compact Riemannian manifold \((M, g)\). Hence for \( \tau \in [\tau_0, \tau_0 + T] \), \( k \in \mathbb{N} \) and \( k > \frac{1}{2}(D - 1) \) we have,

$$|\varphi(\tau,.)|_k \leq C$$  \hspace{1cm} (17)

such that,

$$E_k^2[\varphi](\tau) \leq E_k^2[\varphi](\tau_0) + \frac{1}{2} CT$$  \hspace{1cm} (18)

where some constant \( C \) depends only on the initial data, \( T, k \) and the bound of scalar potential.

**Proof.** From the definition of (16), we have

$$\frac{dE_k[\varphi]}{d\tau} = \sum_{j=0}^{k} \int_{M} (\nabla^j \partial_{\tau} \varphi) \nabla^j F(\varphi, \partial_{\tau} \varphi) dv(g) \leq CE_k^2[\varphi]$$  \hspace{1cm} (19)

Taking the integration of (19), we obtain

$$E_k^2[\varphi](\tau) \leq E_k^2[\varphi](\tau_0) + \frac{1}{2} CT$$

and the proof of Lemma 2.2 is finished. \(\square\)

In the next part of the proof, we will show the existence and uniqueness properties of \( \varphi \) satisfying Equation (5) with the initial data \( \varphi(\tau_0,.) = A \in H^{k+1}(M) \) and \( \partial_{\tau} \varphi(\tau_0,.) = B \in H^k(M) \) having compact support in \( M \).

Let us introduce the sequence \( \{\varphi_l\} \) such that

$$\partial^2_{\tau} \varphi_0 - \Delta \varphi_0 = 0$$  \hspace{1cm} (20)

with the initial data \( \varphi_0(\tau_0,.) = A_0 \) and \( \partial_{\tau} \varphi_0(\tau_0,.) = B_0 \). For \( l \geq 0 \), we have

$$\partial^2_{\tau} \varphi_{l+1} - \Delta \varphi_{l+1} = F(\varphi_l, \partial_{\tau} \varphi_l)$$  \hspace{1cm} (21)

with the initial data \( \varphi_l(\tau_0,.) = A_l \) and \( \partial_{\tau} \varphi_l(\tau_0,.) = B_l \).

Let \((M, g)\) be a smooth compact Riemannian manifold. A set of smooth functions with compact support in \( M \) denoted by \( D(M) \) is dense in \( H^p(M) \) for any \( p \geq 1 \) thus Sobolev embedding theorem can be applied. We can choose the sequences \( \{A_l\} \) and \( \{B_l\} \) such that \( A_l, B_l \in D(M) \) where \( A_l \rightarrow A \) in \( H^{k+1}(M) \) and \( B_l \rightarrow B \) in \( H^k(M) \). Given \( A_l \) dan \( B_l \) satisfying the condition as follows

$$\begin{cases} 
\|A_l\|_{H^{k+1}(M)} \leq C\|A\|_{H^{k+1}(M)} \\
\|B_l\|_{H^k(M)} \leq C\|B\|_{H^k(M)} 
\end{cases}$$  \hspace{1cm} (22)

where the constant \( C \) depends only on \( k \).

Next we prove that the linear energy \( E_k \) of \( \varphi_l \) is bounded by some constant as follows.
Lemma 2.3. Let \( \{ \varphi_l \} \) be a solution of the Equation (21) on the compact Riemannian manifold \((\mathcal{M}, g)\) and \( \varphi_l \) is a real function. Then for \( \tau \in [\tau_0, \tau_0 + T] \) we have
\[
\mathcal{E}_k[\varphi_l](\tau) \leq C
\]
where the constant \( C \) depends on the initial data, \( k \), and the bound of the scalar potential.

Proof. First we define \( \varphi_0 \) as the solution of linear Equation (20) such that \( \mathcal{E}_k \) is conserved
\[
\mathcal{E}_k[\varphi_0](\tau_0) = \mathcal{E}_k[\varphi_0](\tau) \leq \tilde{C}
\]
where the bounded constant \( \tilde{C} \) depends only on the initial data such for every \( \varphi_l \) satisfying Equation (21) we have
\[
\mathcal{E}_k[\varphi_l](\tau_0) \leq \tilde{C}
\]
Thus we conclude that inequality (23) is true for \( l = 0 \). Next we will show that inequality (23) also true for \( l = n > 0 \). We have the estimate
\[
\frac{\partial}{\partial \tau} \int_{\mathcal{M}} |\varphi_n|^2 dv(g) \leq C \|\varphi_n\|_{L^2(\mathcal{M})} \mathcal{E}_k^{\frac{1}{2}}(\varphi_n)
\]
Integrating the inequality (26) we obtain
\[
\|\varphi_n(\tau, .)\|_{L^2(\mathcal{M})} \leq C \left[ \|A\|_{L^2(\mathcal{M})} + T \right]
\]
For \( T \leq 1 \), we have \( \|\varphi_n(\tau, .)\|_{L^2(\mathcal{M})} \) bounded for \( \tau \in [\tau_0, \tau_0 + T] \) where the constant \( C \) depends only on the initial data. We conclude that the inequality (23) also true for \( l = n > 0 \). Finally, we will prove that the inequality (23) is true for \( l = n + 1 \). Let us consider the estimate
\[
\frac{d\mathcal{E}_k[\varphi_{n+1}]}{dT} \leq \left( \sum_{j=0}^{k} \int_{\mathcal{M}} \left[ \nabla^j F(\varphi_{n+1}, \partial_\tau \varphi_{n+1}) \right]^2 dv(g) \right)^{\frac{1}{2}} \left( \sum_{j=0}^{k} \int_{\mathcal{M}} \left( \nabla^j \partial_\tau \varphi_{n+1} \right)^2 dv(g) \right)^{\frac{1}{2}}
\]
After some calculation we obtain
\[
\mathcal{E}_k[\varphi_{n+1}](\tau) \leq \mathcal{E}_k[\varphi_{n+1}](\tau_0) + \frac{1}{2} CT
\]
by the assumption \( T \leq 1 \) we have show that \( \mathcal{E}_k[\varphi_{n+1}](\tau) \) bounded for \( \tau \in (\tau_0, \tau_0 + T) \). This is the end of the proof of Lemma 2.3. \( \square \)

From the definition (7) and (16) above, let us introduce functional energy
\[
\Theta_{l,k} = \sup_{\tau \in [\tau_0, \tau_0 + T]} \left[ \mathcal{E}_k^{\frac{1}{2}}[\varphi_l - \varphi_{l-1}](\tau) + \|\varphi_l - \varphi_{l-1}\|_{L^2(\mathcal{M})} \right]
\]

Lemma 2.4. Let \( \{ \varphi_l \} \) be a solution of (20) and (21) on compact Riemannian manifold \((\mathcal{M}, g)\) and \( \varphi_l \) is a real function. For \( \tau \in [\tau_0, \tau_0 + T] \), we have the estimate of \( \Theta_{l,k} \) satisfying the following condition
\[
\mathcal{E}_k^{\frac{1}{2}}[\varphi_l - \varphi_{l-1}](\tau) \leq \mathcal{E}_k^{\frac{1}{2}}[\varphi_l - \varphi_{l-1}](\tau_0) + \frac{1}{2} C \Theta_{l,k} T
\]
Proof. Using the similar method to the proof of Lemma 2.3, we have
\[ \frac{\partial}{\partial \tau} \mathcal{E}_k[\varphi_l - \varphi_{l-1}] \leq C \Theta_{l,k} \mathcal{E}_k^\frac{1}{2}[\varphi_l - \varphi_{l-1}] \] (32)

Integrating the above inequality we obtain
\[ \mathcal{E}_k^\frac{1}{2}[\varphi_l - \varphi_{l-1}](\tau) \leq \mathcal{E}_k^\frac{1}{2}[\varphi_l - \varphi_{l-1}](\tau_0) + \frac{1}{2} C \Theta_{l,k} T \]

and the proof of Lemma 2.4 is finished.

Lemma 2.5. Let \( \{\varphi_l\} \) be a solution of (20) and (21) on compact Riemannian manifold \((M, g)\) and \( \varphi_l \) is a real function. \( \Theta_{l,k} \) is defined as in Equation (30). There exists some constant \( C_0 > 1 \) and \( T \) such that,
\[ \Theta_{l,k} \leq \frac{C_0}{2^l} \] (33)

for all \( l \).

Proof. Lemma 2.5 is true for \( l = 1 \) by assuming that the constant \( C_0 \) is big enough. Let us consider
\[ \frac{\partial}{\partial \tau} \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}^2 \leq 2 \|\varphi_{l+1} - \varphi_l\|_{L^2(M)} \|\dot{\varphi}_{l+1} - \dot{\varphi}_l\|_{L^2(M)} \] (34)

Integrating the inequality (34) we obtain,
\[ \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau) - \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau_0) \leq \frac{1}{2} C \Theta_{l+1,k} T \] (35)

Equation (30) for \( l + 1 \) can be rewrite as
\[ \Theta_{l+1,k} = \sup_{\tau \in [\tau_0, \tau_0 + T]} \left[ \mathcal{E}_k^\frac{1}{2}[\varphi_{l+1} - \varphi_l](\tau) + \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau) \right] \] (36)

such that
\[ \Theta_{l+1,k} \leq 2 \mathcal{E}_k^\frac{1}{2}[\varphi_{l+1} - \varphi_l](\tau_0) + \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau_0) + C \Theta_{l,k} T \] (37)

The first and second term of R.H.S depend only on the initial data. Moreover, we can choose them as such that
\[ 2 \mathcal{E}_k^\frac{1}{2}[\varphi_{l+1} - \varphi_l](\tau_0) + \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau_0) \leq \frac{1}{2^{l+2}} \] (38)

Then by assuming \( CT < \frac{1}{4} \), we obtain
\[ \Theta_{l+1,k} \leq \frac{C_0}{2^{l+1}} \] (39)

Substitute (39) to (35) such that
\[ \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau) - \|\varphi_{l+1} - \varphi_l\|_{L^2(M)}(\tau_0) \leq \frac{C_0}{2^{l+1}} \] (40)

Then we obtain that the solution \( \{\varphi_l\} \) is bounded by some constant \( C_0 \) for all \( l \). This is the end of the proof of Lemma 2.5.
Now, let us consider for all $|j| \leq k + 1$. From the definition of (16) we have the estimate
\[
\| \partial_t \nabla^j (\varphi_l - \varphi_{l-1})(\tau, \cdot) \|_{L^2(M)} \leq \sup_{\tau \in [\tau_0, \tau_0 + T]} \left[ \epsilon_k^j \| \varphi_l - \varphi_{l-1}(\tau) \| + \| \phi_l - \phi_{l-1}(\tau, \cdot) \|_{L^2(M)} \right]
\] (41)

In addition, we have $\varphi_l - \varphi_{l-1} \in H^{k+1}(M)$. From Lemma 2.5 we obtain,
\[
\Theta_{l,k} \leq \sup_{\tau \in [\tau_0, \tau_0 + T]} \| \varphi_l - \varphi_{l-1} \|_{H^{k+1}(M)} \leq \frac{C_0}{2^l}
\] (42)

which show that $\{ \varphi_l \}$ is a Cauchy sequence on $C([\tau_0, \tau_0 + T], H^{k+1}(M))$. Using the similar method we can show that $\{ \partial_t \varphi_l \}$ also a Cauchy sequence on $C([\tau_0, \tau_0 + T], H^k(M))$. Finally we have prove the local existence solution of Equation (5) such that
\[
\varphi \in C \left([\tau_0, \tau_0 + T], H^{k+1}(M) \right) \cap C^1 \left([\tau_0, \tau_0 + T], H^k(M) \right)
\] (43)

3. Uniqueness properties
In this section, we consider the uniqueness properties of the solutions. Let us define $\varphi$ and $\varphi'$ be the solutions of (5) with the same initial data. By the similar method of the proof of Lemma 2.4, we can define the estimate
\[
\frac{d}{d\tau} \epsilon_k[\varphi' - \varphi] \leq C \| F(\varphi', \partial_t \varphi') - F(\varphi, \partial_t \varphi) \|_{H^k(M)} \| \partial_t \varphi' - \partial_t \varphi \|_{H^k(M)} \leq C \epsilon_k[\varphi' - \varphi]
\] (44)

Using Gronwall lemma where $\varphi$ and $\varphi'$ have the same initial data, then for $\tau \in [\tau_0, \tau_0 + T]$ we obtain $\varphi'(\tau) = \varphi(\tau)$. Finally we conclude that the solution of (5) is said to be unique.

4. Smoothness properties
From Lemma 2.5 we can get that $\{ \partial_t \partial_t \varphi_l \}$ and $\{ \partial_t \partial_t \varphi \}$ is a Cauchy sequence on $C([\tau_0, \tau_0 + T], H^{k-1}(M))$. From Equation (5), we can obtain the estimate
\[
\| \partial^2_t (\varphi_{l+1} - \varphi_l) \|_{H^{k-1}(M)} \leq \| F(\varphi_{l+1}, \partial_t \varphi_{l+1}) - F(\varphi_l, \partial_t \varphi_l) \|_{H^{k-1}(M)} + \| \partial_t \partial_t (\varphi_{l+1} - \varphi_l) \|_{H^{k-1}(M)}
\] (45)

The first term of the R.H.S is bounded by some constant $C \Theta_{l+1,k}$. While the second term is bounded by some constant since it is a Cauchy sequence on $H^{k-1}(M)$. From Lemma 2.5 we can conclude that $\{ \partial^2_t \varphi \}$ is also a Cauchy sequence on $H^{k-1}(M)$. Thus for $k > \frac{1}{2}(D - 1)$ there exists a real function $\varphi$ defined on compact Riemannian manifold $(M, g)$ such that,
\[
\sup_{\tau \in [\tau_0, \tau_0 + T]} \| \partial^2_t \varphi \|_{H^{k-1}(M)} \leq C
\] (46)

where $\partial^2_t \varphi \in C([\tau_0, \tau_0 + T], H^{k-1}(M))$.

Let us consider a fixed point $(\tau, x) \in M$. We can define a sequence $(t_l, x_l) \rightarrow (\tau, x)$ where $\tau_0 \leq t_l, \tau_0 + T$. Then we have the estimate
\[
| \partial^2_t \varphi(\tau, x) - \partial^2_t \varphi(\tau_0, x_0) | \leq | \partial^2_t \varphi(\tau, x) - \partial^2_t \varphi(\tau_0, x_0) | + | \partial^2_t \varphi(\tau_0, x_0) | (47)
\]

By Sobolev embedding theorem, we can define $\partial^2_t \varphi$ as a continuous function. Thus the first term of R.H.S vanishes as $l \rightarrow \infty$. Since $\partial^2_t \varphi \subset C([\tau_0, \tau_0 + T], H^{k-1}(M))$ we can estimates the
second term by some constant times \(\|\partial^2_r \varphi(\tau, \cdot) - \partial^2_r \varphi(\tau_1, \cdot)\|_{H^{k-1}(\mathcal{M})}\) such that it also vanishes as \(l \to \infty\). Finally we conclude that \(\partial^2_r \varphi \in C((\tau_0, \tau_0 + T) \times \mathcal{M})\) such that,
\[
\varphi \in C^2((\tau_0, \tau_0 + T) \times \mathcal{M})
\] (48)

To prove the smoothness properties, we claim that the solution \(\varphi\) is \(C^{k-1}((\tau_0, \tau_0 + T) \times \mathcal{M})\) with \(k \geq 3\). For \(k = 3\) the claim is true as defined in (48). For \(k - n\) we have
\[
\partial^r g^{i_1 i_1} \cdots g^{i_k i_{k-1}} g^{j_k \cdots j_{k-1}} \left( \nabla^{k-r-1} \varphi \right)_{i_1 \cdots i_{k-r-1}} \left( \nabla^{k-r-1} \varphi \right)_{j_1 \cdots j_{k-r-1}} \in C((\tau_0, \tau_0 + T) \times \mathcal{M})
\] (49)

with \(r = 0, 1, \ldots n - 1\). For \(k = n + 1\) the limit of sequence defined by (20) and (21) satisfies
\[
g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} \varphi \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} \varphi \right)_{j_1 \cdots j_k-p} \partial^r_p \varphi \in C((\tau_0, \tau_0 + T) \times \mathcal{M})
\] (50)

with \(p = 0, 1, \ldots, n - 1\). From (21) we obtain
\[
\partial^p g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} (\varphi_l - \varphi_{l-1}) \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} (\varphi_l - \varphi_{l-1}) \right)_{j_1 \cdots j_k-p} =
\partial^{p-2} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} \varphi \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} \varphi \right)_{j_1 \cdots j_k-p} +
\partial^{p-2} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} \partial_k \partial^k (\varphi_l - \varphi_{l-1}) \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} \partial_k \partial^k (\varphi_l - \varphi_{l-1}) \right)_{j_1 \cdots j_k-p}
\] (51)

where
\[
\hat{\varphi}_l = \mathcal{F}(\varphi_l, \partial_\tau \varphi_l) - \mathcal{F}(\varphi_{l-1}, \partial_\tau \varphi_{l-1})
\] (52)

The first time of R.H.S of Equation (51) can be written as
\[
\partial^{p-2} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} \hat{\varphi}_l \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} \hat{\varphi}_l \right)_{j_1 \cdots j_k-p} =
\frac{14}{\tau} \partial^{p-1} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} (\varphi_l - \varphi_{k-1}) \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} (\varphi_l - \varphi_{k-1}) \right)_{j_1 \cdots j_k-p} -
\frac{\xi}{\tau^2} \partial^{p-2} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} (\varphi_l - \varphi_{k-1}) \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} (\varphi_l - \varphi_{k-1}) \right)_{j_1 \cdots j_k-p} -
\frac{\tau^4}{\tau^2} \partial^{p-2} g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left( \nabla^{k-p} (\partial_\varphi V(\varphi_l) - \partial_\varphi V(\varphi_{l-1})) \right)_{i_1 \cdots i_k-p} \left( \nabla^{k-p} (\partial_\varphi V(\varphi_l) - \partial_\varphi V(\varphi_{l-1})) \right)_{j_1 \cdots j_k-p}
\] (53)

The first and second term of R.H.S of (53) is \(C((\tau_0, \tau_0 + T \times \mathcal{M})\). Furthermore, the last term can be estimate as follows
\[
\partial^p g^{i_1 i_1} \cdots g^{i_k \cdots i_k} g^{j_k \cdots j_k} \left[ \nabla^{k-p} \left( G(\varphi_l, \varphi_{l-1}) \hat{\varphi}_{l-1} \right) \right]_{i_1 \cdots i_k-p} \left[ \nabla^{k-p} \left( G(\varphi_l, \varphi_{l-1}) \hat{\varphi}_{l-1} \right) \right]_{j_1 \cdots j_k-p} =
\sum_{c+d \leq p \atop c + s \leq p \atop s \leq p} \sum_{c+d \leq p \atop c \leq p \atop c \leq p} \left( \partial^p g^{i_1 i_1} \cdots g^{i_c \cdots i_c} (\nabla^c G)_{i_1 \cdots i_c} \left( \nabla^c G \right)_{j_1 \cdots j_c} \right) \left( \nabla^{c+d+1} G \right)_{i_{c+1} \cdots i_{c+d}} \left( \nabla^{c+d+1} G \right)_{j_{c+1} \cdots j_{c+d}}
\] (54)
for \( c \geq 1 \) and \( d \geq 0 \), where
\[
G(\varphi_t - \varphi_{t-1}) = \int_0^1 \partial_\sigma \partial_\tau V [\sigma \varphi_t + (1 - \sigma) \varphi_{t-1}] d\sigma
\] (55)

To the fact that \( V(\varphi) \) is a smooth function and by Lemma 2.3 we have \( \Theta_c[\varphi_t] \) is bounded for \( c \leq n - p \), thus \( (\partial^p g^{i_1 j_1}...g^{i_p j_p} (\nabla^c G)_{i_1...i_c}(\nabla^c G)_{j_1...j_c}) \) can be bounded by some constant. We conclude that
\[
(\partial^p g^{i_1 j_1}...g^{i_p j_p} (\nabla^c G)_{i_1...i_c}(\nabla^c G)_{j_1...j_c}) \in C([\tau_0, \tau_t + T] \times M)
\]
and
\[
(\partial^p g^{i_1 j_1}...g^{i_p j_p} (\nabla^c G)_{i_1...i_c}(\nabla^c G)_{j_1...j_c}) \in C([\tau_0, \tau_t + T] \times M) \quad \text{for} \quad p = 0, 1, ..., n - 1.
\]

5. Conclusion
Finally we have the conclusion of this work by the theorem as follows.

**Theorem 5.1.** Let \((M, g)\) be \( D - 1 \) dimensional compact Riemannian manifold. Assume that the scalar potential is a smooth function satisfying \( V(0) = 0 \) and \( \partial_\varphi V(0) = 0 \). For \( k > \frac{1}{2}(D - 1) \), there exists \( T > 0 \) and local classical solution \( \varphi \in C^{k-1}([\tau_0, \tau_0 + T] \times M) \) of Equation (5) such that
\[
\varphi \in C^0([\tau_0, \tau_0 + T], H^{k+1}(M)) \cap C^1([\tau_0, \tau_0 + T], H^k(M))
\] (56)

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