SOME WEAK VERSIONS OF THE $M_1$-SPACES

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Abstract. We mainly introduce some weak versions of the $M_1$-spaces, and study some properties about these spaces. The main results are that: (1) If $X$ is a compact scattered space and $i(X) \leq 3$, then $X$ is an $s$-$m_1$-space; (2) If $X$ is a strongly monotonically normal space, then $X$ is an $s$-$m_2$-space; (3) If $X$ is a $\sigma$-$m_3$ space, then $t(X) \leq c(X)$, which extends a result of P.M. Gartside in [7]. Moreover, some questions are posed in the paper.

1. Introduction

All spaces are $T_1$ and regular unless stated otherwise, and all maps are continuous and onto. The letter $\mathbb{N}$ denotes the set of all positively natural numbers. Let $X$ be a topological space. Recalled that a family $\mathcal{P}$ of subsets of $X$ is called closure-preserving if, for any $\mathcal{P}' \subset \mathcal{P}$, we have $\bigcup \mathcal{P}' = \bigcup \{ \mathcal{P} : \mathcal{P} \in \mathcal{P}' \}$. Moreover, $X$ is called an $M_1$-space [3] if $X$ has a $\sigma$-closure-preserving base. It is still a famous open problem (usually called the $M_1 = M_3$ question, see [6]) whether each stratifiable space is $M_1$. M. Ito proved that every $M_3$-space with a closure-preserving local base at each point is $M_1$ [10], and T. Mizokami has just showed that every $M_1$-space has a closure-preserving local base at each point [12]. Therefore, to give a positive answer to $M_1 = M_3$, it is sufficient to prove that each stratifiable space has a closure-preserving local base at each point.

R.E. Buck first introduced and studied the $m_i$ (see Definition 2.4) properties in [1], where he gave some interesting and surprising results about $m_i$-spaces. Recently, A. Dow, R. Martínez and V.V. Tkachuk have also make some study on spaces with a closure-preserving local base at each point (In fact, they call such spaces for Japanese spaces in their paper.) [4]. In this paper we introduce some weak versions of $M_1$-spaces, and study some properties and relations on these spaces.

2. Preliminaries

Definition 2.1. Let $X$ be a topological space and $\mathcal{B}$ a family of subsets of $X$. $\mathcal{B}$ is called a quasi-base [6] for $X$ if, for each $x$ and open subset $U$ with $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in \text{int}(B) \subset B \subset U$.

Definition 2.2. Let $X$ be a topological space and $\mathcal{P}$ a pair-family for $X$, where, for any $P \in \mathcal{P}$, we denote $P = (P', P'')$. $\mathcal{P}$ is called a pairbase [6] if $\mathcal{P}$ satisfies the following conditions:

1. For any $(P', P'') \in \mathcal{P}$, $P' \subset P''$ and $P'$ is open subset of $X$;

2000 Mathematics Subject Classification. 54B10; 54C05; 54D30.

Key words and phrases. $m_i$-spaces; $s$-$m_i$-spaces; $s$-$\sigma$-$m_i$-spaces; closure-preserving; strongly monotonically normal; monotonically normal.

Supported by the NSFC (No. 10971185, No. 10971186) and the Educational Department of Fujian Province (No. JA09166) of China.
(2) For any \( x \in U \in \tau(X) \), there exists \((P', P'') \in \mathcal{P}\) such that \( x \in P' \subset P'' \subset U \).

Moreover, a pairbase \( \mathcal{P} \) is called a cushioned if, for each \( \mathcal{P}' \subset \mathcal{P} \), we have \( \bigcup\{P' : (P', P'') \in \mathcal{P}'\} \subset \bigcup\{P'' : (P', P'') \in \mathcal{P}'\} \).

**Definition 2.3.** Let \( \mathcal{P} \) be a collection of subsets of \( X \). \( \mathcal{P} \) is called closure-preserving if, for any \( \mathcal{P}' \subset \mathcal{P} \), we have \( \bigcup\mathcal{P}' = \bigcup\{P' : P' \in \mathcal{P}'\} \).

A family \( \mathcal{A} \) of open subsets of a space \( X \) is called a base of \( X \) at a set \( A \) if \( A = \bigcap \mathcal{A} \) and for any neighborhood \( U \) of \( A \), there is a \( V \in \mathcal{A} \) such that \( A \subset V \subset U \).

A family \( \mathcal{A} \) of subsets of a space \( X \) is called a quasi-base of \( X \) at a set \( A \) if \( A = \bigcap \mathcal{A} \) and for any neighborhood \( U \) of \( A \), there is a \( V \in \mathcal{A} \) such that \( A \subset \text{int}(V) \subset V \subset U \).

**Definition 2.4.** Let \( X \) be a space, \( x \in X \) and \( F \) a closed subset of \( X \). Then

1. \( X \) is \( m_1' (\sigma-m_1) \) at the point \( x \) if \( X \) has a closure-preserving (\( \sigma \)-closure-preserving) local base at the point \( x \). \( X \) is called an \( m_1' \)-space (\( \sigma \)-\( m_1' \)-space) if every point of \( X \) is \( m_1' (\sigma-m_1) \);
2. \( X \) is \( s-m_1' (\sigma-m_1') \) at \( F \) if \( X \) has a closure-preserving (\( \sigma \)-closure-preserving) local base at \( F \). \( X \) is called an \( s-m_1' \)-space (\( \sigma-m_1' \)-space) if every closed subset of \( X \) is \( m_1' (\sigma-m_1) \);
3. \( X \) is \( m_2' (\sigma-m_2) \) at the point \( x \) if \( X \) has a closure-preserving (\( \sigma \)-closure-preserving) local quasi-base at the point \( x \). \( X \) is called an \( m_2' \)-space (\( \sigma-m_2' \)-space) if every point of \( X \) is \( m_2' (\sigma-m_2) \);
4. \( X \) is \( s-m_2' (\sigma-m_2') \) at \( F \) if \( X \) has a closure-preserving (\( \sigma \)-closure-preserving) local quasi-base at \( F \). \( X \) is called an \( s-m_2' \)-space (\( \sigma-m_2' \)-space) if every closed subset of \( X \) is \( m_2' (\sigma-m_2) \);
5. \( X \) is \( m_3' (\sigma-m_3) \) at the point \( x \) if \( X \) has a cushioned local pairbase (\( \sigma \)-cushioned local pairbase) at the point \( x \). \( X \) is called an \( m_3' \)-space (\( \sigma-m_3' \)-space) if every point of \( X \) is \( m_3' (\sigma-m_3) \);
6. \( X \) is \( s-m_3' (\sigma-m_3') \) at \( F \) if \( X \) has a cushioned local pairbase (\( \sigma \)-cushioned local pairbase) at \( F \). \( X \) is called an \( s-m_3' \)-space (\( \sigma-m_3' \)-space) if every closed subset of \( X \) is \( m_3' (\sigma-m_3) \).

It is easy to see that

![Diagram](attachment:image.png)
A space $X$ is called a stratifiable or $M_3$-space if it has a $\sigma$-cushioned pairbase. By [12], we have that $X$ is $M_1$-space iff $X$ is $M_3$ and $m_1$ iff $X$ is $M_3$ and $s$-$m_1$. Moreover, if we let $X$ be a regular stratifiable space, then all the spaces on the above are equivalent, see [1][10][12].

For each space $X$, we let $I(X)$ be the set of all isolated points of $X$. If $X$ is scattered, then let $X_0 = X$; proceeding inductively assume that $\alpha$ is an ordinal and we constructed $X_\beta$ for all $\beta < \alpha$. If $\alpha = \beta + 1$ for some $\beta$, then we let $X_\alpha = X_\beta \setminus I(X_\beta)$. If $\alpha$ is a limit ordinal, then we let $X_\alpha = \bigcap \{X_\beta : \beta < \alpha\}$. The first ordinal $\alpha$ such that $X_\alpha = \emptyset$ is called the dispersion index of $X$ and is denoted by $i(X)$, see [4].

Reader may refer to [5][6] for notations and terminology not explicitly given here.

3. $s,m_1$ and $s,\sigma,m_1$ spaces

In [1], A. Dow, R. Martínez and V.V. Tkachuk proved that each space with a finite number of non-isolated points is an $m_1$-space, and each compact scattered space $X$ and $i(X) \leq 3$ is an $m_1$-space. However, we shall see that there exists an $m_1$-space $X$ such that $X$ is non-$s$-$m_1$, see Example 3.1. But we have the follow Theorems [3.1] and [3.2] which extend the results of the above.

**Lemma 3.1.** Let $\mathcal{P}$ be a topological property such that (a) if space $X$ has $\mathcal{P}$ then $X$ has $m_i$ and (b) if $X$ has $\mathcal{P}$ and $A$ is a closed subspace then $X/A$ has $\mathcal{P}$. Then every space $X$ with property $\mathcal{P}$ has the $s$-variant property $s$-$m_i$.

**Proof.** Take any space $X$ with property $\mathcal{P}$, and take any closed subset $A$ of $X$. Then, by (b), $X/A$ has property $\mathcal{P}$, and so is $m_i$ (by (a)). In particular, the point $A$ in $X/A$ is an $m_i$ point, and so the set $A$ has a ‘nice’ outer base in $X$. From which it follows that $X$ has $s$-$m_i$. $\square$

**Theorem 3.1.** Each space with a finite number of non-isolated points is an $s$-$m_1$-space.

**Proof.** Let $X$ be a space with a finite number of non-isolated points, and let $A$ be a closed subspace of $X$. It is easy to see that $X/A$ has finite number of non-isolated points. By [3] Proposition 2.9, a space with a finite number of non-isolated points is $m_i$, and thus $X$ is an $s$-$m_1$-space by Lemma 3.1. $\square$

**Theorem 3.2.** If $X$ is a compact scattered space and $i(X) \leq 3$, then $X$ is an $s$-$m_1$-space.

**Proof.** Let $X$ be a compact scattered space and $i(X) \leq 3$, and let $A$ be a closed subspace of $X$. It is easy to see that $X/A$ is also a compact scattered space and $i(X) \leq 3$. By [3] Theorem 3.1, a compact scattered space and $i(X) \leq 3$ is $m_i$, and thus $X$ is an $s$-$m_1$-space by Lemma 3.1. $\square$

The proofs of the following Propositions 3.1 and 3.2 are easy, and so we omit them.

**Proposition 3.1.** If $X$ has a clopen closure-preserving neighborhood base at any closed set then $X$ is hereditarily $s$-$m_1$. In particular, any extremally disconnected $s$-$m_1$ space is hereditarily $s$-$m_1$. 


Proposition 3.2. Suppose that $X$ is a space and a closed set $F \subset X$ has an open neighborhood base in $X$ which is well-ordered by the reverse inclusion. Then $X$ is $s$-$m_1$ at $F$.

Proposition 3.3. If $X$ is an $s$-$m_1$-space, and $D$ is dense in $X$. Then $D$ is also an $s$-$m_1$-subspace.

A map $f : X \to Y$ is called quasi-open if, for each non-empty open subset $U$ of $X$, the interior of $f(U)$ is non-empty. $f$ is called an irreducible map if, for each proper closed subset $F$ of $X$, we have $f(F) \neq Y$.

Lemma 3.2. Let $f : X \to Y$ be a quasi-open closed map. If $\mathcal{B}$ is a closure-preserving open family of $X$, then $\varphi = \{\text{int}(f(B)) : B \in \mathcal{B}\}$ is a closure-preserving open family of $Y$.

Theorem 3.3. Let $f : X \to Y$ be a quasi-open closed map. If $X$ is an $s$-$m_1$-space, then $Y$ is also an $s$-$m_1$ space.

Proof. Let $F$ be any closed set of $Y$ and $F \neq \emptyset$. Then $f^{-1}(F)$ is closed in $X$. Since $X$ is $s$-$m_1$, $f^{-1}(F)$ has a closure-preserving open neighborhood base $\mathcal{B}$ at $f^{-1}(F)$. Since $f$ is a quasi-open closed map, the family $\varphi = \{\text{int}(f(B)) : B \in \mathcal{B}\}$ is a closure-preserving open family of $Y$ by Lemma 3.2. Moreover, because $f$ is a closed map, we have $F \subset \text{int}(f(B))$ for each $B \in \mathcal{B}$. It is easy to see that $\varphi$ is an open neighborhood base at $F$ in $Y$. \hfill $\Box$

Corollary 3.1. Closed and irreducible maps preserve $s$-$m_1$-spaces.

Proof. Since closed and irreducible maps are quasi-open maps, closed and irreducible maps preserve $s$-$m_1$ property by Theorem 3.3. \hfill $\Box$

Next, we shall give an example to show that there exists an $m_1$-space $X$ which is non-$s$-$m_1$. Firstly, we prove the following Theorem 3.4.

Let $A$ be a subset of a space $X$. We call a family $N$ of open subsets of $X$ an outer base of $A$ in $X$ if for any $x \in A$ and open subset $U$ with $x \in U$ there is a $V \in N$ such that $x \in V \subset U$.

Theorem 3.4. If $X$ is Eberlein compact then $X$ is an $s$-$\sigma$-$m_1$-space.

Proof. Let $F$ be any closed subset of $X$. Since $X$ is Eberlein compact, it follows from [1] Theorem 3.13] that $F$ has a $\sigma$-closure-preserving outer base $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ in $X$, where, for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is closure-preserving and $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. For each $n \in \mathbb{N}$, let

$$\mathcal{P}_n = \{\bigcup \mathcal{B} : \mathcal{B} \text{ is a finite subfamily of } \mathcal{B}_n \text{ and } F \subset \bigcup \mathcal{B}\}.$$

It is easy to see that $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ is a $\sigma$-closure-preserving local base at the set $F$ in $X$, where, for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is closure-preserving. \hfill $\Box$

Recalled that a closed map $f : X \to Y$ which is perfect if, for each $y \in Y$, $f^{-1}(y)$ is compact.

Example 3.1. There exists an $m_1$-space $X$ such that the following conditions are satisfied:

1. $X$ is an $s$-$\sigma$-$m_1$-space, and non-$s$-$m_1$-space;
2. The image of $X$ under some perfect and irreducible map is not an $m_1$-space.
Proof. Let $X$ be the Alexandroff double $D$ of the Cantor set $C$. Then $X$ is first countable Elberlein compact space [4]. Hence $X$ is $s$-$m_1$ by Theorem 3.3. Let $f : X \to Y$ be the quotient map by identifying the non-isolated point of $X$ to one point. Then $f$ is an irreducible and perfect map. However, $Y$ is not an $m_1$-space by [4, Corollary 3.18], and hence $X$ is not an $s$-$m_1$-space by Corollary 3.1. Moreover, it is easy to see that first-countable spaces are $m_1$. However, $X$ is non-$s$-$m_1$-space. Therefore, compact first-countable is not need to be an $s$-$m_1$-space. □

In [4], the authors prove that each $GO$ space is $m_1$. However, we don’t know whether each $GO$ space is $s$-$m_1$, and so we have the following question.

Question 3.1. Let $X$ be a $GO$ space. Is $X$ $s$-$m_1$?

4. $s$-$m_2$ AND $s$-$\sigma$-$m_2$ SPACES

Since closed maps preserve a closure-preserving family, we have the following theorem.

Theorem 4.1. Closed maps preserve $s$-$m_2$ and $s$-$\sigma$-$m_2$ spaces, respectively.

Theorem 4.2. If $X$ is an $s$-$m_2$-space and $Y \subset X$ then $Y$ is $s$-$m_2$.

A space $X$ is monotonically normal if there is a function $G$ which assigns to each closed ordered pair $(H, K)$ of disjoint closed subsets of $X$ an open subset $G(H, K) \subset X$ such that:

1. $H \subset G(H, K) \subset G(H, K) \subset X \setminus K$;
2. $G(H, K) \subset G(H', K')$ for disjoint closed subsets $H'$ and $K'$ with $H \subset H'$ and $K \supset K'$;
3. $G(H, K) \cap G(K, H) = \emptyset$.

Moreover, if $X$ also satisfies the following condition
4. if $H' \subset G(H, K)$ with $H'$ closed in $X$, then $G(H', K) \subset G(H, K)$.

Then $X$ is called strongly monotonically normal [8, 9].

In [4], the authors pose the following question.

Question 4.1. [4] Must every monotonically normal space be an $m_1$-space?

Next, we shall give some partial answer for this Question 4.1 see Theorem 4.1.

Theorem 4.3. Let $X$ be a strongly monotonically normal space. Then $X$ is an $s$-$m_2$-space.

Proof. Let $A$ be a closed subspace of $X$. In [2], R.E. Buck, R.W. Heath and P.L. Zenora showed that a closed image of a strongly monotonically normal space is again strongly monotonically normal, and hence $X/A$ is strongly monotonically normal. By [1, Theorem 3.13], a strongly monotonically normal space is $m_2$, and thus $X$ is an $s$-$m_2$-space by Lemma 3.1. □

Question 4.2. Let $X$ be a strongly monotonically normal space. Is $X$ an $s$-$m_1$-space or an $m_1$-space?
5. $s_{m_3}$ AND $\sigma$-$m_3$ SPACES

The proofs of the following two theorems are obvious, and so we omit them.

**Theorem 5.1.** Closed maps preserve $s_{m_3}$ and $s$-$\sigma$-$m_3$-spaces, respectively.

**Theorem 5.2.** If $X$ is an $s$-$m_3$-space and $Y \subset X$ then $Y$ is $s$-$m_3$.

The following theorem is also a partial answer for the Question 4.1.

**Theorem 5.3.** Let $X$ be a monotonically normal space. Then $X$ is an $s$-$m_3$-space.

**Proof.** Let $A$ be a closed subspace of $X$. It is well known that monotonically normal spaces are preserved by closed images, and implies $m_3$. Hence $X$ is $m_3$ and $X/A$ is monotonically normal, which follows that $X$ is an $s$-$m_3$-space by Lemma 3.1. \qed

Let $X$ be a space and $\kappa$ an infinite cardinal. For each $x \in X$, we denote $t(x, X)$ means that for any $A \subset X$ with $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$; moreover, $t(x, X) \leq \kappa$ iff $t(x, X) \leq \kappa$ for each $x \in X$. The space $X$ with $t(x, X) \leq \kappa$ are said to have tightness $\leq \kappa$.

A pairwise disjoint collection of non-empty open subsets in $X$ is called a cellular family. The cellularity of $X$, defined as follows:

$$c(X) = \sup\{|U| : U \text{ is a cellular family in } X\} + \omega.$$ 

In [5], P.M. Gartside proved that for each monotonically normal space $X$, we have $t(X) \leq c(X)$. We shall extend this result of P.M. Gartside, and prove that, for each $\sigma$-$m_3$-space $X$, we have $t(X) \leq c(X)$.

**Theorem 5.4.** Suppose that a space $X$ is $\sigma$-$m_3$ at some point $x \in X$ and $\kappa$ is an infinite cardinal such that $c(U) \leq \kappa$ for some open neighborhood $U$ of the point $x$. Then $t(x, X) \leq \kappa$. In particular, if $X$ is an $\sigma$-$m_3$-space, then $t(X) \leq c(X)$.

**Proof.** Fix any set $A \subset X \setminus \{x\}$ with $x \in \overline{A}$. Let $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a $\sigma$-cushioned pairwise at the point $x$, where $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$, and for each $P \in \mathcal{P}$, $P_2$ is closed in $X$. Without loss of generality, we may assume that $A \subset U$ and $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n \subset U$. For each $y \in X \setminus \{x\}$ and $n \in \mathbb{N}$, put

$$W_{ny} = X \setminus \bigcup\{P_1 : P \in \mathcal{P} \text{ and } y \notin P_2\}.$$ 

For each $n \in \mathbb{N}$, since $y \in X \setminus \bigcup\{P_2 : P \in \mathcal{P} \text{ and } y \notin P_2\} \subset W_{ny}$, $W_{ny}$ is an open neighborhood of $y$.

Claim 1: If $Q \subset X \setminus \{x\}$, then $x \in \overline{Q}$ if and only if, for each $n \in \mathbb{N}$, $x \in \bigcup\{W_{ny} : y \in Q\}$.

In fact, if $x \in \overline{Q}$, then, for each open neighborhood $V$ of point $x$, we have $V \cap Q \neq \emptyset$. Choose a point $y \in V \cap Q$. It follows that $V \cap W_{ny} \neq \emptyset$ for each $n \in \mathbb{N}$, and hence we have $x \in \bigcup\{W_{ny} : y \in Q\}$. For each $n \in \mathbb{N}$, let $x \in \bigcup\{W_{ny} : y \in Q\}$.

Suppose that $x \notin \overline{Q}$. Then there exist an $n \in \mathbb{N}$ and $P \in \mathcal{P}_n$ such that $x \in P_1$ and $P_2 \cap Q = \emptyset$. For each $y \in Q$, since $y \notin P_2$, we have $W_{ny} \cap P_1 = \emptyset$, and so $x \notin \bigcup\{W_{ny} : y \in Q\}$, which is a contradiction.

For each $n \in \mathbb{N}$ and $y \in A$, put $G_{ny} = W_{ny} \cap U$. It follows from $c(U) \leq \kappa$ that we can choose a set $D_n \subset A$ such that $|D_n| \leq \kappa$ and $G_n = \bigcup\{G_{ny} : y \in D_n\}$ is dense in $H_n = \bigcup\{G_{ny} : y \in A\}$. Let $D = \bigcup_{n \in \mathbb{N}} D_n$. Then $|D| \leq \kappa$. Obviously, we have that

$$x \in \overline{A} \subset \bigcap_{n \in \mathbb{N}} \overline{H_n} = \bigcap_{n \in \mathbb{N}} \overline{G_n},$$
which implies that $x \in \overline{G_n} \subset \bigcup \{W_{ny} : y \in D\}$ for each $n \in \mathbb{N}$. By the Claim 1, we have $x \in \overline{D}$. Therefore, $t(x, X) \leq \kappa$. \hfill \qed

**Corollary 5.1.** ([4] If $X$ is an $m_2$-space, then $t(X) \leq c(X)$.

**Corollary 5.2.** ([7] Theorem 10) If $X$ is a monotonically normal space, then $t(X) \leq c(X)$.

Recall that a family $\mathcal{U}$ of non-empty open sets of a space $X$ is called a $\pi$-base if for each non-empty open set $V$ of $X$, there exists an $U \in \mathcal{U}$ such that $V \subset U$. The $\pi$-character of $x$ in $X$ is defined by $\pi(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a local $\pi$-base at $x$ in $X\}$. The $\pi$-character of $X$ is defined by $\pi(X) = \sup\{\pi(x, X) : x \in X\}$.

In [4], A. Dow, R. Ramírez and V.V. Tkachuk proved that if $X$ is a separable $m_2$-space with the Baire property then $X$ has countable $\pi$-character. However, we find the proof has a gap. Next, we shall give out the correct proof. In fact, we have more general result, see Theorem 5.5.

**Theorem 5.5.** Suppose that a space $X$ is a separable space with the Baire property. If $X$ is a $\sigma$-$m_3$ at some point $x \in X$, then it has countable $\pi$-character at the point $x$.

**Proof.** Suppose that $x$ is a non-isolated point in $X$, and that $D = \{d_n : n \in \mathbb{N}\} \subset X \setminus \{x\}$ is a dense subset of $X$. Let $\mathcal{P} = \bigcup P_n$ be a local $\sigma$-cushioned pairwise base at the point $x$, where, for each $n \in \mathbb{N}$, $P_n$ is cushioned. For each $n \in \mathbb{N}$, put $D_n = D \cap \bigcup \{P_1 : P \in P_n\} = \{d_{n,m} : m \in \mathbb{N}\}$. For each $y \in X \setminus \{x\}$ and $n \in \mathbb{N}$, put

$$W_{ny} = X \setminus \bigcup \{P_1 : P \in P_n \text{ and } y \notin P_2\}.$$

Then $W_{ny}$ is an open neighborhood of $y$. For each $n, m \in \mathbb{N}$, put $B_{n,m} = \{y \in X \setminus \{x\} : d_{n,m} \in W_{ny}\}$. It is easy to see that $B_{n,m} \subset \bigcap \{P_2 : d_{n,m} \in P_1, P \in P_n\}$. Further, for each $n, m \in \mathbb{N}$, $B_{n,m}$ is closed in $X \setminus \{x\}$. In fact, let $y \in \overline{B_{n,m}} \setminus \{x\}$. Then $y \in W_{ny}$, and hence $W_{ny} \cap B_{n,m} \neq \emptyset$. For each $z \in W_{ny} \cap B_{n,m}$, we have $W_{nz} \subset W_{ny}$, and thus $d_{n,m} \in W_{nz} \subset W_{ny}$. So $y \in B_{n,m}$. Let $\mathcal{W} = \{\text{int}(B_{n,m}) : n, m \in \mathbb{N}\}$. Then $\mathcal{W}$ is a countable $\pi$-base at the point $x$.

In fact, let any open subset $U$ with $x \in U$. Since $\mathcal{P}$ is a local pairwise base at the point $x$, there exist $n \in \mathbb{N}$ and $P \in P_n$ such that $x \in P \subset P_2 \subset U$, where $P = (P_1, P_2)$. It is easy to see that, for each $m \in \mathbb{N}$, $d_{n,m} \in B_{n,m} \subset P_2$ whenever $d_{n,m} \in P_1$. Let $V = P \setminus \{x\}$. Then $V$ is a non-empty open subset and has the Baire property. If $y \in V$, then $D_n \cap W_{ny} \cap V \neq \emptyset$, and hence there exists an $m \in \mathbb{N}$ such that $d_{n,m} \in W_{ny} \cap V$, i.e., $y \in B_{n,m}$. Therefore, $V \subset \bigcup \{B_{n,m} : d_{n,m} \in P_1\}$. Since $V$ has the Baire property, there exists an $m \in \mathbb{N}$ such that $G = \text{int}(B_{n,m}) \neq \emptyset$. Hence $G \in \mathcal{W}$ and $G \subset B_{n,m} \subset P_2 \subset U$. Therefore, $\mathcal{W}$ is a countable $\pi$-base at the point $x$. \hfill \qed

**Corollary 5.3.** ([4] Suppose that a space $X$ is a separable space with the Baire property and $X$ is $m_2$ at some point $x \in X$. Then $X$ has countable $\pi$-character at the point $x$.

**Corollary 5.4.** If $G$ is a separable $\sigma$-$m_3$-group with the Baire property then $G$ is metrizable.

**Proof.** Since $w(G) = \pi w(G)$ for any topological group, $G$ is metrizable. \hfill \qed
Acknowledgements. We wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper. In particular, Lemma 3.1 is due to the reviewers, which gives an easy proofs for Theorems 3.1 3.2 3.3 and 5.3 in our original paper.

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