Maxwell Optics: II. An Exact Formalism

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Abstract
We present a formalism for light optics starting with the Maxwell equations and casting them into an exact matrix form taking into account the spatial and temporal variations of the permittivity and permeability. This $8 \times 8$ matrix representation is used to construct the optical Hamiltonian. This has a close analogy with the algebraic structure of the Dirac equation, enabling the use of the rich machinery of the Dirac electron theory. We get interesting wavelength-dependent contributions which can not be obtained in any of the traditional approaches.

1 Introduction
The traditional scalar wave theory of optics (including aberrations to all orders) is based on the beam-optical Hamiltonian derived using the Fermat’s principle. This approach is purely geometrical and works adequately in the scalar regime. The other approach is based on the Helmholtz equation which is derived from the Maxwell equations. Then one makes the \textit{square-root} of the Helmholtz operator followed by an expansion of the radical \cite{1, 2}. This approach works to all orders and the resulting expansion is no different from the one obtained using the geometrical approach of the Fermat’s principle.

Another way of obtaining the aberration expansion is based on the algebraic similarities between the Helmholtz equation and the Klein-Gordon equation. Exploiting this algebraic similarity the Helmholtz equation is linearized in a procedure very similar to the one due to Feschbach-Villars, for linearizing the Klein-Gordon equation. This brings the Helmholtz equation...
to a Dirac-like form and then follows the procedure of the Foldy-Wouthuysen expansion used in the Dirac electron theory. This approach, which uses the algebraic machinery of quantum mechanics, was developed recently [3], providing an alternative to the traditional square-root procedure. This scalar formalism gives rise to wavelength-dependent contributions modifying the aberration coefficients [4]. The algebraic machinery of this formalism is very similar to the one used in the quantum theory of charged-particle beam optics, based on the Dirac [3] and the Klein-Gordon [6] equations respectively. The detailed account for both of these is available in [7]. A treatment of beam optics taking into account the anomalous magnetic moment is available in [8].

As for the polarization: A systematic procedure for the passage from scalar to vector wave optics to handle paraxial beam propagation problems, completely taking into account the way in which the Maxwell equations couple the spatial variation and polarization of light waves, has been formulated by analysing the basic Poincaré invariance of the system, and this procedure has been successfully used to clarify several issues in Maxwell optics [4, 10, 11].

In all the above approaches, the beam-optics and the polarization are studied separately, using very different machineries. The derivation of the Helmholtz equation from the Maxwell equations is an approximation as one neglects the spatial and temporal derivatives of the permittivity and permeability of the medium. Any prescription based on the Helmholtz equation is bound to be an approximation, irrespective of how good it may be in certain situations. It is very natural to look for a prescription based fully on the Maxwell equations. Such a prescription is sure to provide a deeper understanding of beam-optics and polarization in a unified manner. With this as the chief motivation we construct a formalism starting with the Maxwell equations in a matrix form: a single entity containing all the four Maxwell equations.

In our approach we require an exact matrix representation of the Maxwell equations in a medium taking into account the spatial and temporal variations of the permittivity and permeability. It is necessary and sufficient to use $8 \times 8$ matrices for such an exact representation. The derivation of the required matrix representation, and how it differs from the numerous other ones is presented in Part-I [12].

In the present Part (Part-II) we proceed with the exact matrix representation of the Maxwell equations derived in Part-I, and construct a general
formalism. The derived representation has a very close algebraic correspondence with the Dirac equation. This enables us to apply the machinery of the Foldy-Wouthuysen expansion used in the Dirac electron theory. The Foldy-Wouthuysen transformation technique is outlined in Appendix-A. General expressions for the Hamiltonians are derived without assuming any specific form for the refractive index. These Hamiltonians are shown to contain the extra wavelength-dependent contributions which arise very naturally in our approach. In Part-III we apply the general formalism to the specific examples: A. Medium with Constant Refractive Index. This example is essentially for illustrating some of the details of the machinery used.

The other application, B. Axially Symmetric Graded Index Medium is used to demonstrate the power of the formalism. Two points are worth mentioning, Image Rotation: Our formalism gives rise to the image rotation (proportional to the wavelength) and we have derived an explicit relationship for the angle of the image rotation. The other pertains to the aberrations: In our formalism we get all the nine aberrations permitted by the axial symmetry. The traditional approaches give six aberrations. Our formalism modifies these six aberration coefficients by wavelength-dependent contributions and also gives rise to the remaining three permitted by the axial symmetry. The existence of the nine aberrations and image rotation are well-known in axially symmetric magnetic lenses, even when treated classically. The quantum treatment of the same system leads to the wavelength-dependent modifications. The alternate procedure for the Helmholtz optics in gives the usual six aberrations (though modified by the wavelength-dependent contributions) and does not give any image rotation. These extra aberrations and the image rotation are the exclusive outcome of the fact that the formalism is based on the Maxwell equations, and done exactly.

The traditional beam-optics is completely obtained from our approach in the limit wavelength, $\lambda \to 0$, which we call as the traditional limit of our formalism. This is analogous to the classical limit obtained by taking $\hbar \to 0$ in the quantum prescriptions. The scheme of using the Foldy-Wouthuysen machinery in this formalism is very similar to the one used in the quantum theory of charged-particle beam optics. There too one recovers the classical prescriptions in the limit $\lambda_0 \to 0$ where $\lambda_0 = \hbar/p_0$ is the de Broglie wavelength and $p_0$ is the design momentum of the system under study.

The studies on the polarization are in progress. Some of the results in have been obtained as the lowest order approximation of the more general
framework presented here. These will be presented in Part-IV soon [14].

2 An exact matrix representation of the Maxwell equations in a medium

Matrix representations of the Maxwell equations are very well-known [15-16]. However, all these representations lack an exactness or/and are given in terms of a pair of matrix equations. A treatment expressing the Maxwell equations in a single matrix equation instead of a pair of matrix equations was obtained recently [12]. This representation contains all the four Maxwell equations in presence of sources taking into account the spatial and temporal variations of the permittivity $\epsilon(r,t)$ and the permeability $\mu(r,t)$.

Maxwell equations [17, 18] in an inhomogeneous medium with sources are

$$\nabla \cdot D(r,t) = \rho,$$
$$\nabla \times H(r,t) - \frac{\partial}{\partial t}D(r,t) = J,$$
$$\nabla \times E(r,t) + \frac{\partial}{\partial t}B(r,t) = 0,$$
$$\nabla \cdot B(r,t) = 0.$$  \hspace{1cm} (1)

We assume the media to be linear, that is $D = \epsilon(r,t)E$, and $B = \mu(r,t)H$, where $\epsilon$ is the permittivity of the medium and $\mu$ is the permeability of the medium. The magnitude of the velocity of light in the medium is given by $v(r,t) = |v(r,t)| = 1/\sqrt{\epsilon(r,t)\mu(r,t)}$. In vacuum we have, $\epsilon_0 = 8.85 \times 10^{-12} C^2/N.m^2$ and $\mu_0 = 4\pi \times 10^{-7} N/A^2$. Following the notation in [17, 12] we use the Riemann-Silberstein vector given by

$$F^\pm (r,t) = \frac{1}{\sqrt{2}} \left( \sqrt{\epsilon(r,t)}E(r,t) \pm i \frac{1}{\sqrt{\mu(r,t)}}B(r,t) \right).$$  \hspace{1cm} (2)

We further define,

$$\Psi^\pm (r,t) = \begin{bmatrix} -F^+_x \pm iF^+_y \\ F^+_z \\ F^+_x \pm iF^+_y \end{bmatrix}, \quad W^\pm = \left( \frac{1}{\sqrt{2\epsilon}} \right)^3 \begin{bmatrix} -J_x \pm iJ_y \\ J_z - v\rho \\ J_z + v\rho \end{bmatrix}. \hspace{1cm} (3)$$
where $W^\pm$ are the vectors for the sources. Following the notation in [12] the exact matrix representation of the Maxwell equations is

$$\frac{\partial}{\partial t} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix} - \frac{\dot{v}(r,t)}{2v(r,t)} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}$$

$$+ \frac{\dot{h}(r,t)}{2h(r,t)} \begin{bmatrix} 0 & i\beta \alpha_y \\ i\beta \alpha_y & 0 \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}$$

$$= -v(r,t) \begin{bmatrix} \{M \cdot \nabla + \Sigma \cdot u\} & -i\beta (\Sigma \cdot w) \alpha_y \\ -i\beta (\Sigma^* \cdot w) \alpha_y & \{M^* \cdot \nabla + \Sigma^* \cdot u\} \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}$$

$$- \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \end{bmatrix},$$

(4)

where $\cdot^*$ denotes complex-conjugation, $\dot{v} = \frac{\partial v}{\partial t}$ and $\dot{h} = \frac{\partial h}{\partial t}$. The various matrices are

$$M_x = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad M_y = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix}, \quad M_z = \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(5)

and $I$ is the $2 \times 2$ unit matrix. The triplet of the Pauli matrices, $\sigma$ is

$$\sigma = \begin{bmatrix} \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix},$$

(6)

and

$$u(r,t) = \frac{1}{2v(r,t)} \nabla v(r,t) = \frac{1}{2} \nabla \{\ln v(r,t)\} = -\frac{1}{2} \nabla \{\ln n(r,t)\}$$

$$w(r,t) = \frac{1}{2h(r,t)} \nabla h(r,t) = \frac{1}{2} \nabla \{\ln h(r,t)\}.$$ 

(7)

Lastly,

Velocity Function: $v(r,t) = \frac{1}{\sqrt{\epsilon(r,t)\mu(r,t)}}$

Resistance Function: $h(r,t) = \frac{\mu(r,t)}{\sqrt{\epsilon(r,t)}}.$

(8)
As we shall see soon, it is advantageous to use the above derived functions instead of the permittivity, $\varepsilon(r, t)$ and the permeability, $\mu(r, t)$. The functions, $v(r, t)$ and $h(r, t)$ have the dimensions of velocity and resistance respectively.

Let us consider the case without any sources ($W^\pm = 0$). We further assume,

$$\Psi^\pm(r, t) = \psi^\pm(r) e^{-i\omega t}, \quad \omega > 0, \quad (9)$$

with $\dot{v}(r, t) = 0$ and $\hat{h}(r, t) = 0$. Then,

$$\begin{align*}
\begin{bmatrix} M_z & 0 \\ 0 & M_z \end{bmatrix} \frac{\partial}{\partial z} & \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} \\
\frac{i}{v(r)} \frac{\omega}{\psi^+} & -v(r) \begin{bmatrix} \{M_\perp \cdot \nabla_\perp + \Sigma \cdot u\} & -i\beta \left(\Sigma \cdot w\right) \alpha_y \\ -i\beta \left(\Sigma^* \cdot w\right) \alpha_y & -\{M^*_\perp \cdot \nabla_\perp + \Sigma^* \cdot u\} \end{bmatrix} \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix}.
\end{align*} \quad (10)$$

At this stage we introduce the process of wavization, through the familiar Schrödinger replacement

$$-i\tilde{\lambda} \nabla_\perp \rightarrow \hat{p}_\perp, \quad -i\tilde{\lambda} \frac{\partial}{\partial z} \rightarrow p_z, \quad (11)$$

where $\tilde{\lambda} = \lambda/2\pi$ is the reduced wavelength, $c = \tilde{\lambda} \omega$ and $n(r) = c/v(r)$ is the refractive index of the medium. Noting, that $(pq - qp) = -i\tilde{\lambda}$, which is very similar to the commutation relation, $(pq - qp) = -i\hbar$, in quantum mechanics. In our formalism, ‘$\tilde{\lambda}$’ plays the same role which is played by the Planck constant, ‘$\hbar$’ in quantum mechanics. The traditional beam-optics is completely obtained from our formalism in the limit $\tilde{\lambda} \rightarrow 0$.

Noting, that $M_z^{-1} = M_z = \beta$, we multiply both sides of equation (10) by

$$\begin{bmatrix} M_z & 0 \\ 0 & M_z \end{bmatrix}^{-1} = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \quad (12)$$

and $(i\tilde{\lambda})$, then, we obtain

$$i\tilde{\lambda} \frac{\partial}{\partial z} \begin{bmatrix} \psi^+(r_\perp, z) \\ \psi^-(r_\perp, z) \end{bmatrix} = \hat{H}_g \begin{bmatrix} \psi^+(r_\perp, z) \\ \psi^-(r_\perp, z) \end{bmatrix}. \quad (13)$$
This is the basic optical equation, where

\[
\hat{H}_g = -n_0 \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix} + \hat{E}_g + \hat{O}_g
\]

\[
\hat{E}_g = -\left(n(r) - n_0\right) \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \beta_g
\]

\[
\hat{O}_g = \begin{bmatrix} 0 & -\bar{\lambda}(\Sigma \cdot w) \alpha_y \\ -\bar{\lambda}(\Sigma^* \cdot w) \alpha_y & 0 \end{bmatrix}
\]

(14)

where ‘g’ stands for grand, signifying the eight dimensions and

\[
\beta_g = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.
\]

(15)

The above optical Hamiltonian is exact (as exact as the Maxwell equations in a time-independent linear media). The approximations are made only at the time of doing specific calculations. Apart from the exactness, the optical Hamiltonian is in complete algebraic analogy with the Dirac equation with appropriate physical interpretations. The relevant point is:

\[
\beta_g \hat{E}_g = \hat{E}_g \beta_g, \quad \beta_g \hat{O}_g = -\hat{O}_g \beta_g.
\]

(16)

We note that the upper component (\( \Psi^+ \)) is coupled to the lower component (\( \Psi^- \)) through the logarithmic divergence of the resistance function. If this coupling function, \( w = 0 \) or is approximated to be zero, then the equations for (\( \Psi^+ \)) and (\( \Psi^- \)) get completely decoupled, leading to two independent equations. Each of these two equations is equivalent to the other. These are the leading equations for our studies of beam-optics and polarization. In the optics context any contribution from the gradient of the resistance function can be assumed to be negligible. With this reasonable assumption we can decouple the equations and reduce the problem from eight dimensions to four dimensions. In the following sections we shall present a formalism with the approximation \( w \approx 0 \). After constructing the formalism in four dimensions we shall also address the question of dealing with the contributions coming from the gradient of the resistance function. This will require the application
of the Foldy-Wouthuysen transformation technique in *cascade* as we shall see. This justifies the usage of the two derived laboratory functions in place of permittivity and permeability respectively.

### 3 The Beam-Optical Formalism

In the previous section, starting with the Maxwell equations we presented the exact representation of the Maxwell equations using $8 \times 8$ matrices. From this representation we constructed the optical Hamiltonian having $8 \times 8$ matrices. The coupling of the upper and lower components of the corresponding eight-vector was neatly expressed through the logarithmic divergence of the laboratory function, the resistance. We reason that in the optical context we can safely ignore this term and reduce the problem from eight to four dimensions without any loss of physical content.

We drop the '$+$' throughout and then the beam-optical Hamiltonian is

$$i\bar{\lambda} \frac{\partial}{\partial z} \psi(\mathbf{r}) = \hat{H}\psi(\mathbf{r})$$

$$\hat{H} = -n_0\beta + \hat{\mathcal{E}} + \hat{\mathcal{O}}$$

$$\hat{\mathcal{E}} = -(n(\mathbf{r}) - n_0)\beta - i\bar{\lambda}\beta\mathbf{\Sigma} \cdot \mathbf{u}$$

$$\hat{\mathcal{O}} = i(M_yp_x - M_xp_y)$$

$$= \beta(M_{\perp} \cdot \mathbf{p}_{\perp})$$

(17)

If we were to neglect the derivatives of the permittivity and permeability, we would have missed the term, $(-i\bar{\lambda}\beta\mathbf{\Sigma} \cdot \mathbf{u})$. This is an outcome of the exact treatment.

Proceeding with our analogy with the Dirac equation: this extra term is analogous to the anomalous magnetic/electric moment term coupled to the magnetic/electric field respectively in the Dirac equation. The term we dropped (while going from the exact to the almost-exact) is analogous to the anomalous magnetic/electric moment term coupled to the electric/magnetic fields respectively. However it should be born in mind that in our exact treatment, both the terms were derived from the Maxwell equations, where as in the Dirac theory the anomalous terms are added based on experimental results and certain arguments of invariances. Besides, these are the only two terms one gets. The term, $(-i\bar{\lambda}\beta\mathbf{\Sigma} \cdot \mathbf{u})$ is related to the polarization and we shall call it as the *polarization term*.
One of the other similarities worth noting, relates to the square of the optical Hamiltonian.

\[
\hat{H}^2 = \left\{ n^2(r) - \hat{p}_\perp^2 \right\} - \bar{\lambda}^2 u^2 + [M_\perp \cdot \hat{p}_\perp, n(r)] \\
+ 2i\bar{\lambda} n(r) \Sigma \cdot u + i\bar{\lambda} [M_\perp \cdot \hat{p}_\perp, \Sigma \cdot u] \\
= \left\{ n(r) + i\bar{\lambda} \Sigma \cdot u \right\}^2 - \hat{p}_\perp^2 \\
+ [M_\perp \cdot \hat{p}_\perp, \left\{ n(r) + i\bar{\lambda} \Sigma \cdot u \right\}] \\
\]

(18)

It is to be noted that the square of the Hamiltonian in our formalism differs from the square of the Hamiltonian in the square-root approaches [1, 2] and the scalar approach in [3, 4]. This is essentially the same type of difference which exists in the Dirac case. There too, the square of the Dirac Hamiltonian gives rise to extra pieces (such as, \(-\hbar q \Sigma \cdot B\), the Pauli term which couples the spin to the magnetic field) which is absent in the Schrödinger and the Klein-Gordon descriptions. It is this difference in the square of the Hamiltonians which give rise to the various extra wavelength-dependent contributions in our formalism. These differences persist even in the approximation when the polarization term is neglected.

Recalling, that in the traditional scalar wave theory for treating monochromatic quasiparaxial light beam propagating along the positive \(z\)-axis, the \(z\)-evolution of the optical wave function \(\psi(r)\) is taken to obey the Schrödinger-like equation

\[
i\bar{\lambda} \frac{\partial}{\partial z} \psi(r) = \hat{H} \psi(r),
\]

(19)

where the optical Hamiltonian \(\hat{H}\) is formally given by the radical

\[
\hat{H} = - \left( n^2(r) - \hat{p}_\perp^2 \right)^{1/2},
\]

(20)

and \(n(r) = n(x, y, z)\). In beam optics the rays are assumed to propagate almost parallel to the optic-axis, chosen to be \(z\)-axis, here. That is, \(|\hat{p}_\perp| \ll 1\). The refractive index is the order of unity. For a medium with uniform refractive index, \(n(r) = n_0\) and the Taylor expansion of the radical is

\[
\left( n^2(r) - \hat{p}_\perp^2 \right)^{1/2} = n_0 \left\{ 1 - \frac{1}{n_0^2} \hat{p}_\perp^2 \right\}^{1/2}
\]
\[ n_0 \left\{ 1 - \frac{1}{2n_0^2} \hat{p}_\perp^2 - \frac{1}{8n_0^4} \hat{p}_\perp^4 - \frac{1}{16n_0^6} \hat{p}_\perp^6 \right. \\
\left. \quad - \frac{5}{128n_0^8} \hat{p}_\perp^8 - \frac{7}{256n_0^{10}} \hat{p}_\perp^{10} - \cdots \right\} \right. \]  

(21)

In the above expansion one retains terms to any desired degree of accuracy in powers of \( \left( \frac{1}{n_0^2} \hat{p}_\perp^2 \right) \). In general the refractive index is not a constant and varies. The variation of the refractive index \( n(r) \), is expressed as a Taylor expansion in the spatial variables \( x, y \) with \( z \)-dependent coefficients. To get the beam optical Hamiltonian one makes the expansion of the radical as before, and retains terms to the desired order of accuracy in \( \left( \frac{1}{n_0^2} \hat{p}_\perp^2 \right) \) along with all the other terms (coming from the expansion of the refractive index \( n(r) \)) in the phase-space components up to the same order. In this expansion procedure the problem is partitioned into paraxial behaviour + aberrations, order-by-order.

In relativistic quantum mechanics too, one has the problem of understanding the behaviour in terms of nonrelativistic limit + relativistic corrections, order-by-order. In the Dirac theory of the electron this is done most conveniently through the Foldy-Wouthuysen transformation \([19, 20]\). The Hamiltonian derived in (17) has a very close algebraic resemblance with the Dirac case, accompanied by the analogous physical interpretations. The details of the analogy and the Foldy-Wouthuysen transformation are given in Appendix-A.

To the leading order, that is to order, \( \left( \frac{1}{n_0^2} \hat{p}_\perp^2 \right) \) the beam-optical Hamiltonian in terms of \( \hat{E} \) and \( \hat{O} \) is formally given by

\[
\hat{H}^{(2)} = -n_0\beta + \hat{E} - \frac{1}{2n_0} \beta \hat{O}^2.
\]

(22)

Note that \( \hat{O}^2 = -\hat{p}_\perp^2 \) and \( \hat{E} = -(n(r) - n_0) \beta - i\lambda \beta \Sigma \cdot \mathbf{u} \). Since, we are primarily interested in the forward propagation, we drop the \( \beta \) from the non-matrix parts of the Hamiltonian. The matrix terms are related to the polarization. The formal Hamiltonian in (22), expressed in terms of the phase-space variables is:

\[
\hat{H}^{(2)} = - \left\{ n(r) - \frac{1}{2n_0} \hat{p}_\perp^2 \right\} - i\lambda \beta \Sigma \cdot \mathbf{u}.
\]

(23)
Note that one retains terms up to quadratic in the Taylor expansion of the refractive index $n(r)$ to be consistent with the order of $\left(\frac{1}{n_0^2}p_\perp^2\right)$. This is the paraxial Hamiltonian which also contains an extra matrix dependent term, which we call as the polarization term. Rest of it is similar to the one obtained in the traditional approaches.

To go beyond the paraxial approximation one goes a step further in the Foldy-Wouthuysen iterative procedure. Note that, $\hat{O}$ is the order of $\hat{p}_\perp$. To order $\left(\frac{1}{n_0^2}p_\perp^2\right)^2$, the beam-optical Hamiltonian in terms of $\hat{E}$ and $\hat{O}$ is formally given by

$$
i\bar{\lambda} \frac{\partial}{\partial z} |\psi\rangle = \hat{H}^{(4)} |\psi\rangle,$$

$$\hat{H}^{(4)} = -n_0\beta + \hat{E} - \frac{1}{2n_0}\hat{O}^2$$

$$- \frac{1}{8n_0^2} \left[ \hat{O}, \left( [\hat{O}, \hat{E}] + i\bar{\lambda} \frac{\partial}{\partial z} \hat{O} \right) \right]$$

$$+ \frac{1}{8n_0^3}\beta \left\{ \hat{O}^4 + \left( [\hat{O}, \hat{E}] + i\bar{\lambda} \frac{\partial}{\partial z} \hat{O} \right)^2 \right\}.$$  \hspace{1cm} (24)

Note that $\hat{O}^4 = \hat{p}_\perp^4$, and $\frac{\partial}{\partial z} \hat{O} = 0$. The formal Hamiltonian in (24) when expressed in terms of the phase-space variables is

$$\hat{H}^{(4)} = -\left\{ n(r) - \frac{1}{2n_0} \hat{p}_\perp^2 - \frac{1}{8n_0^3} \hat{p}_\perp^4 \right\}$$

$$- \frac{1}{8n_0^2} \left\{ [\hat{p}_\perp^2, (n(r) - n_0)]_+ + 2\left(p_x (n(r) - n_0) p_x + p_y (n(r) - n_0) p_y\right) \right\}$$

$$- \frac{i}{8n_0^2} \left\{ [p_x, [p_y, (n(r) - n_0)]_+]_+ - [p_y, [p_x, (n(r) - n_0)]_+]_+ \right\}$$

$$+ \frac{1}{8n_0^3} \left\{ [p_x, (n(r) - n_0)]_+^2 + [p_y, (n(r) - n_0)]_+^2 \right\}$$

$$+ \frac{i}{8n_0^3} \left\{ [[p_x, (n(r) - n_0)]_+, [p_y, (n(r) - n_0)]_+]_+ \right\}$$

$$\cdots$$  \hspace{1cm} (25)

where $[A, B]_+ = (AB + BA)$ and ‘\ldots’ are the contributions arising from the
presence of the polarization term. Any further simplification would require information about the refractive index $n(r)$.

Note that, the paraxial Hamiltonian (23) and the leading order aberration Hamiltonian (25) differs from the ones derived in the traditional approaches. These differences arise by the presence of the wavelength-dependent contributions which occur in two guises. One set occurs totally independent of the polarization term in the basic Hamiltonian. This set is a multiple of the unit matrix or at most the matrix $\beta$. The other set involves the contributions coming from the polarization term in the starting optical Hamiltonian. This gives rise to both matrix contributions and the non-matrix contributions, as the squares of the polarization matrices is unity. We shall discuss the contributions of the polarization to the beam optics elsewhere. Here, it suffices to note existence of the the wavelength-dependent contributions in two distinguishable guises, which are not present in the traditional prescriptions.

4 When $w \neq 0$

In the previous sections we assumed, $w = 0$ and this enabled us to develop a formalism using $4 \times 4$ matrices via the Foldy-Wouthuysen machinery. The Foldy-Wouthuysen transformation enables us to eliminate the odd part in the $4 \times 4$ matrices, to any desired order of accuracy. Here too we have the identical problem, but a step higher in dimensions. So, we need to apply the Foldy-Wouthuysen to reduce the strength of the odd part in eight dimensions. This will reduce the problem from eight to four dimensions.

We start with the grand optical equation in (13) and proceed with the Foldy-Wouthuysen transformations as before, but with each quantity in double the number of dimensions. Symbolically this means:

$$
\hat{H} \rightarrow \hat{H}_g, \quad \psi \rightarrow \psi_g = \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix}, \\
\hat{E} \rightarrow \hat{E}_g, \quad \hat{O} \rightarrow \hat{O}_g, \\
n_0 \rightarrow n_g = n_0 \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}. \tag{26}
$$
The first Foldy-Wouthuysen iteration gives
\[ \hat{H}^{(2)}_g = -n_0 \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix} + \hat{\varepsilon}_g - \frac{1}{2n_0} \beta_g \hat{O}^2_g \]
\[ = -n_0 \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix} \beta_g + \hat{\varepsilon}_g + \frac{1}{2n_0} \bar{\lambda}^2 w \cdot w \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix} \beta_g. \] (27)

We drop the \( \beta_g \) as before and then get the following
\[ i\bar{\lambda} \frac{\partial}{\partial z} \psi (r) = \hat{H} \psi (r) \]
\[ \hat{H} = -n_0 \beta + \hat{\varepsilon} + \hat{O} \]
\[ \hat{\varepsilon} = -(n (r) - n_0) \beta - i\bar{\lambda} \beta \Sigma \cdot u + \frac{1}{2n_0} \bar{\lambda}^2 w^2 \beta \]
\[ \hat{O} = i (M_y p_x - M_x p_y) \]
\[ = \beta (M \perp \cdot \hat{p} \perp), \] (28)

where, \( w^2 = w \cdot w \), the square of the logarithmic gradient of the resistance function. This is how the basic optical Hamiltonian (17) gets modified. The next degree of accuracy is achieved by going a step further in the Foldy-Wouthuysen iteration and obtaining the \( \hat{H}^{(4)}_g \). Then, this would be the higher refined starting optical Hamiltonian, further modifying the basic optical Hamiltonian (17). This way we can apply the Foldy-Wouthuysen in \textit{cascade} to obtain the higher order contributions coming from the logarithmic gradient of the resistance function, to any desired degree of accuracy. We are very unlikely to need any of these contributions, but it is possible to keep track of them.

5 Concluding Remarks

We start with the Maxwell equations and express them in a matrix form in a medium with varying permittivity and permeability in presence of sources using \( 8 \times 8 \) matrices. From this exact matrix representation we construct the exact optical Hamiltonian for a monochromatic quasiparaxial light beam. The optical Hamiltonian has a very close algebraic similarity with the Dirac equation. We exploit this similarity to adopt the standard machinery, namely
the Foldy-Wouthuysen transformation technique of the Dirac theory. This enabled us to obtain the beam-optical Hamiltonian to any desired degree of accuracy. We further get the wavelength-dependent contributions to at each order, starting with the lowest-order paraxial paraxial Hamiltonian.

The beam-optical Hamiltonians also have the wavelength-dependent matrix terms which are associated with the polarization. In this approach we have been able to derive a Hamiltonian which contains both the beam-optics and the polarization. In Part-III [13] we shall apply the formalism to the specific examples and see how the beam-optics (paraxial behaviour and the aberrations) gets modified by the wavelength-dependent contributions. In Part-IV [14] we shall examine the polarization component of the formalism presented here.

Appendix-FW

Foldy-Wouthuysen Transformation

In the traditional scheme the purpose of expanding the light optics Hamiltonian $\hat{H} = -\left( n^2(\mathbf{r}) - \hat{\mathbf{p}}^2_\perp \right)^{1/2}$ in a series using $\left( \frac{\hat{\mathbf{p}}^2_\perp}{n^2} \right)$ as the expansion parameter is to understand the propagation of the quasiparaxial beam in terms of a series of approximations (paraxial + nonparaxial). Similar is the situation in the case of the charged-particle optics. Let us recall that in relativistic quantum mechanics too one has a similar problem of understanding the relativistic wave equations as the nonrelativistic approximation plus the relativistic correction terms in the quasirelativistic regime. For the Dirac equation (which is first order in time) this is done most conveniently using the Foldy-Wouthuysen transformation leading to an iterative diagonalization technique.

The main framework of the formalism of optics, used here (and in the charged-particle optics) is based on the transformation technique of the Foldy-Wouthuysen theory which casts the Dirac equation in a form displaying the different interaction terms between the Dirac particle and an applied electromagnetic field in a nonrelativistic and easily interpretable form (see, [19]-[23], for a general discussion of the role of the Foldy-Wouthuysen-type transformations in particle interpretation of relativistic wave equations). In the Foldy-Wouthuysen theory the Dirac equation is decoupled through a canonical transformation into two two-component equations: one reduces to
the Pauli equation in the nonrelativistic limit and the other describes the negative-energy states.

Let us describe here briefly the standard Foldy-Wouthuysen theory so that the way it has been adopted for the purposes of the above studies in optics will be clear. Let us consider a charged-particle of rest-mass \( m_0 \), charge \( q \) in the presence of an electromagnetic field characterized by \( E = -\nabla \phi - \frac{\partial}{\partial t} A \) and \( B = \nabla \times A \). Then the Dirac equation is

\[
\begin{align*}
\frac{i\hbar}{\partial t} \Psi(r, t) & = \hat{H}_D \Psi(r, t) \\
\hat{H}_D & = m_0 c^2 \beta + q\phi + c\alpha \cdot \hat{\pi} \\
\hat{\pi} & = \hat{p} - qA, \quad \hat{p} = -i\hbar \nabla, \quad \text{and} \quad \hat{\pi}^2 = (\hat{\pi}_x^2 + \hat{\pi}_y^2 + \hat{\pi}_z^2) \\
\end{align*}
\]

where

\[
\begin{align*}
\alpha &= \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\sigma &= \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
\end{align*}
\]

with \( \hat{\pi} = \hat{p} - qA, \hat{p} = -i\hbar \nabla \), and \( \hat{\pi}^2 = (\hat{\pi}_x^2 + \hat{\pi}_y^2 + \hat{\pi}_z^2) \).

In the nonrelativistic situation the upper pair of components of the Dirac Spinor \( \Psi \) are large compared to the lower pair of components. The operator \( \hat{\mathcal{E}} \) which does not couple the large and small components of \( \Psi \) is called ‘even’ and \( \hat{\mathcal{O}} \) is called an ‘odd’ operator which couples the large to the small components. Note that

\[
\beta \hat{\mathcal{O}} = -\hat{\mathcal{O}} \beta, \quad \beta \hat{\mathcal{E}} = \hat{\mathcal{E}} \beta.
\]

Now, the search is for a unitary transformation, \( \Psi' = \Psi \rightarrow \hat{U} \Psi \), such that the equation for \( \Psi' \) does not contain any odd operator.

In the free particle case (with \( \phi = 0 \) and \( \hat{\pi} = \hat{p} \)) such a Foldy-Wouthuysen transformation is given by

\[
\begin{align*}
\Psi & \rightarrow \Psi' = \hat{U}_F \Psi \\
\hat{U}_F & = e^{i\hat{S}} = e^{i\beta \alpha \cdot \hat{p}} \\
\tan 2|\hat{p}|\theta & = \frac{|\hat{p}|}{m_0 c}.
\end{align*}
\]
This transformation eliminates the odd part completely from the free particle Dirac Hamiltonian reducing it to the diagonal form:

\[ i\hbar \frac{\partial}{\partial t} \Psi' = e^{i\hat{S}} \left( m_0 c^2 \beta + c\alpha \cdot \hat{p} \right) e^{-i\hat{S}} \Psi' \]

\[ = \left( \cos |\hat{p}|\theta + \frac{\beta \alpha \cdot \hat{p}}{|\hat{p}|} \sin |\hat{p}|\theta \right) \left( m_0 c^2 \beta + c\alpha \cdot \hat{p} \right) \]

\[ \times \left( \cos |\hat{p}|\theta - \frac{\beta \alpha \cdot \hat{p}}{|\hat{p}|} \sin |\hat{p}|\theta \right) \Psi' \]

\[ = \left( m_0 c^2 \cos 2|\hat{p}|\theta + c|\hat{p}| \sin 2|\hat{p}|\theta \right) \beta \Psi' \]

\[ = \left( \sqrt{m_0^2 c^4 + c^2 \hat{p}^2} \right) \beta \Psi' . \]  

(A.6)

In the general case, when the electron is in a time-dependent electromagnetic field it is not possible to construct an \( \exp(i\hat{S}) \) which removes the odd operators from the transformed Hamiltonian completely. Therefore, one has to be content with a nonrelativistic expansion of the transformed Hamiltonian in a power series in \( 1/m_0 c^2 \) keeping through any desired order. Note that in the nonrelativistic case, when \( |\hat{p}| \ll m_0 c \), the transformation operator \( \hat{U}_F = \exp(i\hat{S}) \) with \( \hat{S} \approx -i\beta \hat{O}/2m_0 c^2 \), where \( \hat{O} = c\alpha \cdot \hat{p} \) is the odd part of the free Hamiltonian. So, in the general case we can start with the transformation

\[ \Psi^{(1)} = e^{i\hat{S}_1} \Psi, \quad \hat{S}_1 = -\frac{i\beta \hat{O}}{2m_0 c^2} = -\frac{i\beta \alpha \cdot \hat{p}}{2m_0 c} . \]  

(A.7)

Then, the equation for \( \Psi^{(1)} \) is

\[ i\hbar \frac{\partial}{\partial t} \Psi^{(1)} = i\hbar \frac{\partial}{\partial t} \left( e^{i\hat{S}_1} \Psi \right) = i\hbar \frac{\partial}{\partial t} \left( e^{i\hat{S}_1} \right) \Psi + e^{i\hat{S}_1} \left( i\hbar \frac{\partial}{\partial t} \Psi \right) \]

\[ = \left[ i\hbar \frac{\partial}{\partial t} \left( e^{i\hat{S}_1} \right) + e^{i\hat{S}_1} \hat{H}_D \right] \Psi \]

\[ = \left[ i\hbar \frac{\partial}{\partial t} \left( e^{i\hat{S}_1} \right) e^{-i\hat{S}_1} + e^{i\hat{S}_1} \hat{H}_D e^{-i\hat{S}_1} \right] \Psi^{(1)} \]

\[ = \left[ e^{i\hat{S}_1} \hat{H}_D e^{-i\hat{S}_1} - i\hbar e^{i\hat{S}_1} \frac{\partial}{\partial t} \left( e^{-i\hat{S}_1} \right) \right] \Psi^{(1)} \]

\[ = \hat{H}^{(1)} \Psi^{(1)} . \]  

(A.8)
where we have used the identity $$\frac{\partial}{\partial t} (e^{A}) e^{-A} + e^{A} \frac{\partial}{\partial t} (e^{-A}) = \frac{\partial}{\partial t} \hat{I} = 0.$$ Now, using the identities

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, \{ \hat{A}, \hat{B} \}] + \frac{1}{3!} [\hat{A}, \{ [\hat{A}, [\hat{A}, \hat{B}]] \} + \ldots$$

$$= \left( 1 + \hat{A} + \frac{1}{2!} \hat{A}^2 + \frac{1}{3!} \hat{A}^3 \ldots \right)$$

$$\times \frac{\partial}{\partial t} \left( 1 - \hat{A} + \frac{1}{2!} \hat{A}^2 - \frac{1}{3!} \hat{A}^3 \ldots \right)$$

$$= \left( 1 + \frac{\partial}{\partial t} \hat{A} + \frac{1}{2!} \hat{A}^2 + \frac{1}{3!} \hat{A}^3 \ldots \right)$$

$$\times \left( -\frac{\partial}{\partial t} \hat{A} + \frac{1}{2!} \left\{ \frac{\partial^2}{\partial t^2} \hat{A} + \hat{A} \frac{\partial}{\partial t} \hat{A} \right\} \right)$$

$$- \frac{1}{3!} \left\{ \frac{\partial^2}{\partial t^2} \hat{A}^2 + \hat{A} \frac{\partial}{\partial t} \hat{A} \right\} + \hat{A} \left( \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} \hat{A} \right\} \ldots$$

$$\approx -\frac{\partial}{\partial t} \hat{A} - \frac{1}{2!} \left[ \hat{A}, \frac{\partial}{\partial t} \hat{A} \right]$$

$$- \frac{1}{3!} \left[ \hat{A}, \left[ \hat{A}, \frac{\partial}{\partial t} \hat{A} \right] \right]$$

$$- \frac{1}{4!} \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \frac{\partial}{\partial t} \hat{A} \right] \right] \right] , \quad (A.9)$$

with $$\hat{A} = i \hat{S}_1$$, we find

$$\hat{H}_D^{(1)} \approx \hat{H}_D - \frac{i}{\hbar} \frac{\partial}{\partial t} \hat{S}_1 + \frac{1}{2!} \left[ \hat{S}_1, \hat{H}_D - \frac{\hbar}{2} \frac{\partial}{\partial t} \hat{S}_1 \right]$$

$$- \frac{1}{3!} \left[ \hat{S}_1, \left[ \hat{S}_1, \hat{H}_D - \frac{\hbar}{3} \frac{\partial}{\partial t} \hat{S}_1 \right] \right]$$

$$- \frac{i}{4!} \left[ \hat{S}_1, \left[ \hat{S}_1, \left[ \hat{S}_1, \hat{H}_D - \frac{\hbar}{4} \frac{\partial}{\partial t} \hat{S}_1 \right] \right] \right] . \quad (A.10)$$
Substituting in (A.10), $\hat{H}_D = m_0 c^2 \beta + \hat{E} + \hat{O}$, simplifying the right hand side using the relations $\beta \hat{O} = -\hat{O} \beta$ and $\beta \hat{E} = \hat{E} \beta$ and collecting everything together, we have

\[
\hat{H}^{(1)}_D \approx m_0 c^2 \beta + \hat{E}_1 + \hat{O}_1
\]

with $\hat{E}_1$ and $\hat{O}_1$ obeying the relations $\beta \hat{O}_1 = -\hat{O}_1 \beta$ and $\beta \hat{E}_1 = \hat{E}_1 \beta$ exactly like $\hat{E}$ and $\hat{O}$. It is seen that while the term $\hat{O}$ in $\hat{H}_D$ is of order zero with respect to the expansion parameter $1/m_0 c^2$ (i.e., $\hat{O} = O \left((1/m_0 c^2)^0\right)$) the odd part of $\hat{H}^{(1)}_D$, namely $\hat{O}_1$, contains only terms of order $1/m_0 c^2$ and higher powers of $1/m_0 c^2$ (i.e., $\hat{O}_1 = O \left((1/m_0 c^2)^2\right)$).

To reduce the strength of the odd terms further in the transformed Hamiltonian a second Foldy-Wouthuysen transformation is applied with the same prescription:

\[
\Psi^{(2)} = e^{i\hat{S}_2 \Psi^{(1)}},
\hat{S}_2 = -\frac{i\beta \hat{O}_1}{2m_0 c^2} = -\frac{i\beta}{2m_0 c^2} \left( \frac{\beta}{2m_0 c^2} \left[ \hat{O}, \hat{E} \right] + i\hbar \frac{\partial \hat{O}}{\partial t} \right) - \frac{1}{3m_0^2 c^4} \hat{O}^3,
\]

(A.11)

After this transformation,

\[
\frac{i\hbar}{\partial t} \Psi^{(2)} = \hat{H}^{(2)}_D \Psi^{(2)}, \quad \hat{H}^{(2)}_D = m_0 c^2 \beta + \hat{E}_2 + \hat{O}_2
\]

\[
\hat{E}_2 \approx \hat{E}_1, \quad \hat{O}_2 \approx \frac{\beta}{2m_0 c^2} \left( \left[ \hat{O}_1, \hat{E}_1 \right] + i\hbar \frac{\partial \hat{O}_1}{\partial t} \right),
\]

(A.13)

where, now, $\hat{O}_2 = O \left((1/m_0 c^2)^2\right)$. After the third transformation

\[
\Psi^{(3)} = e^{i\hat{S}_3 \Psi^{(2)}}, \quad \hat{S}_3 = -\frac{i\beta \hat{O}_2}{2m_0 c^2}
\]

(A.14)
we have
\[ i\hbar \frac{\partial}{\partial t} \Psi^{(3)} = \hat{H}^{(3)}_D \Psi^{(3)}, \quad \hat{H}^{(3)}_D = m_0 c^2 \beta + \hat{\mathcal{E}}_3 + \hat{\mathcal{O}}_3 \]
\[ \hat{\mathcal{E}}_3 \approx \hat{\mathcal{E}}_2 \approx \hat{\mathcal{E}}_1, \quad \hat{\mathcal{O}}_3 \approx \frac{\beta}{2m_0 c^2} \left[ [\hat{\mathcal{O}}_2, \hat{\mathcal{E}}_2] + i\hbar \frac{\partial \hat{\mathcal{O}}_2}{\partial t} \right], \quad (A.15) \]
where \( \hat{\mathcal{O}}_3 = O \left( 1/m_0 c^2 \right) \). So, neglecting \( \hat{\mathcal{O}}_3 \),
\[ \hat{H}^{(3)}_D \approx m_0 c^2 \beta + \hat{\mathcal{E}} + \frac{1}{2m_0 c^2} \beta \hat{\mathcal{O}}^2 \]
\[ -\frac{1}{8m_0^2 c^4} \left[ \hat{\mathcal{O}}, \left[ \hat{\mathcal{O}}, \hat{\mathcal{E}} \right] + i\hbar \frac{\partial \hat{\mathcal{O}}}{\partial t} \right] \]
\[ -\frac{1}{8m_0^3 c^6} \beta \left\{ \hat{\mathcal{O}}^4 + \left[ \hat{\mathcal{O}}, \hat{\mathcal{E}} \right] + i\hbar \frac{\partial \hat{\mathcal{O}}}{\partial t} \right\}^2 \quad (A.16) \]

It may be noted that starting with the second transformation successive \((\hat{\mathcal{E}}, \hat{\mathcal{O}})\) pairs can be obtained recursively using the rule
\[ \hat{\mathcal{E}}_j = \hat{\mathcal{E}}_1 \left( \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}_{j-1}, \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}_{j-1} \right) \]
\[ \hat{\mathcal{O}}_j = \hat{\mathcal{O}}_1 \left( \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}_{j-1}, \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}_{j-1} \right), \quad j > 1, \quad (A.17) \]

and retaining only the relevant terms of desired order at each step.

With \( \hat{\mathcal{E}} = q\phi \) and \( \hat{\mathcal{O}} = c\mathbf{\alpha} \cdot \hat{\mathbf{\pi}} \), the final reduced Hamiltonian \((A.16)\) is, to the order calculated,
\[ \hat{H}^{(3)}_D = \beta \left( m_0 c^2 + \frac{\hat{\mathbf{\pi}}^2}{2m_0} - \frac{\hat{\mathbf{p}}^4}{8m_0^2 c^2} \right) + q\phi - \frac{q\hbar}{2m_0 c^2} \beta \Sigma \cdot \mathbf{B} \]
\[ -\frac{iq\hbar^2}{8m_0^2 c^2} \Sigma \cdot \text{curl} \mathbf{E} - \frac{q\hbar}{4m_0^2 c^2} \Sigma \cdot \mathbf{E} \times \hat{\mathbf{p}} \]
\[ -\frac{q\hbar^2}{8m_0^2 c^2} \text{div} \mathbf{E}, \quad (A.18) \]

with the individual terms having direct physical interpretations. The terms in the first parenthesis result from the expansion of \( \sqrt{m_0^2 c^4 + c^2 \hat{\mathbf{\pi}}^2} \) showing the effect of the relativistic mass increase. The second and third terms are
the electrostatic and magnetic dipole energies. The next two terms, taken together (for hermiticity), contain the spin-orbit interaction. The last term, the so-called Darwin term, is attributed to the zitterbewegung (trembling motion) of the Dirac particle: because of the rapid coordinate fluctuations over distances of the order of the Compton wavelength \(2\pi\hbar/m_0c\) the particle sees a somewhat smeared out electric potential.

It is clear that the Foldy-Wouthuysen transformation technique expands the Dirac Hamiltonian as a power series in the parameter \(1/m_0c^2\) enabling the use of a systematic approximation procedure for studying the deviations from the nonrelativistic situation. We note the analogy between the nonrelativistic particle dynamics and paraxial optics:

| The Analogy |  |
|-------------|-------------------|
| **Standard Dirac Equation** | **Beam Optical Form** |
| \(m_0c^2\beta + \hat{E}_D + \hat{O}_D\) | \(-n_0\beta + \hat{E} + \hat{O}\) |
| Positive Energy | Forward Propagation |
| Nonrelativistic, \(|\pi| \ll m_0c\) | Paraxial Beam, \(|\hat{p}_\perp| \ll n_0\) |
| Non relativistic Motion | Paraxial Behavior |
| + Relativistic Corrections | + Aberration Corrections |

Noting the above analogy, the idea of Foldy-Wouthuysen form of the Dirac theory has been adopted to study the paraxial optics and deviations from it by first casting the Maxwell equations in a spinor form resembling exactly the Dirac equation (A.1, A.2) in all respects: \(i.e.,\) a multicomponent \(\Psi\) having the upper half of its components large compared to the lower components and the Hamiltonian having an even part (\(\hat{E}\)), an odd part (\(\hat{O}\)), a suitable expansion parameter, \(|\hat{p}_\perp|/n_0 \ll 1\) characterizing the dominant forward propagation and a leading term with a \(\beta\) coefficient commuting with \(\hat{E}\) and anticommuting with \(\hat{O}\). The additional feature of our formalism is to return finally to the original representation after making an extra approximation, dropping \(\beta\) from the final reduced optical Hamiltonian, taking into account the fact that we are primarily interested only in the forward-propagating beam.

**References**
[1] Alex J. Dragt, Etienne Forest and Kurt Bernardo Wolf, **Foundations of a Lie algebraic theory of geometrical optics**, in *Lie Methods in Optics*, Lecture notes in physics No. 250 (Springer Verlag, 1986) pp. 105-157.

[2] Alex J. Dragt, *Lie algebraic method for ray and wave optics*, (University of Maryland Report in preparation, 1995).

[3] Sameen Ahmed Khan, Ramaswamy Jagannathan and Rajiah Simon, **Foldy-Wouthuysen transformation and a quasiparaxial approximation scheme for the scalar wave theory of light beams**, (communicated).

[4] Sameen Ahmed Khan, **An alternate way to obtain the aberration expansion in Helmholtz Optics**, (In preparation)

[5] R. Jagannathan, R. Simon, E. C. G. Sudarshan and N. Mukunda, **Quantum theory of magnetic electron lenses based on the Dirac equation**, *Phys. Lett. A* **134**, 457-464 (1989); R. Jagannathan, *Dirac equation and electron optics*, in *Dirac and Feynman: Pioneers in Quantum Mechanics*, Ed. R. Dutt and A. K. Ray (Wiley Eastern, New Delhi, 1993), pp. 75-82.

[6] S. A. Khan and R. Jagannathan, **On the quantum mechanics of charged particle beam transport through magnetic lenses**, *Phys. Rev. E* **51**, 2510–2515 (March 1995).

[7] R. Jagannathan and S. A. Khan, **Quantum theory of the optics of charged particles**, *Advances in Imaging and Electron Physics* Vol. **97**, Ed. P. W. Hawkes (Academic Press, San Diego, 1996) 257-358.

[8] M. Conte, R. Jagannathan, S. A. Khan and M. Pusterla, **Beam optics of the Dirac particle with anomalous magnetic moment**, *Particle Accelerators* **56** (1996) 99-126.

[9] N. Mukunda, R. Simon, and E. C. G. Sudarshan, **Paraxial-wave optics and relativistic front description. I. The scalar theory**, *Phys. Rev. A* **28** 2921-2932 (1983); N. Mukunda, R. Simon, and E. C. G. Sudarshan, **Paraxial-wave optics and relativistic front description. II. The vector theory**, *Phys. Rev. A* **28** 2933-2942 (1983);
N. Mukunda, R. Simon, and E. C. G. Sudarshan, *Fourier optics for the Maxwell field: formalism and applications*, *J. Opt. Soc. Am.* A 2(3) 416-426 (1985).

[10] R. Simon, E. C. G. Sudarshan and N. Mukunda, *Gaussian-Maxwell beams*, *J. Opt. Soc. Am.* A 3(4) 536-5?? (1986).

[11] R. Simon, E. C. G. Sudarshan and N. Mukunda, *Cross polarization in laser beams*, *Appl. Optics* 26(9), 1589-1593 (01 May 1987).

[12] Sameen Ahmed Khan, *Maxwell Optics: I. An exact matrix representation of the Maxwell equations in a medium*, e-print: physics/0205083.

[13] Sameen Ahmed Khan, *Maxwell Optics: III. Applications*, e-print: physics/0205085.

[14] R. Jagannathan *et al*, *Maxwell Optics: IV. Polarization*, (in preparation)

[15] E. Moses, *Solutions of Maxwell’s equations in terms of a spinor notation: the direct and inverse problems*, *Phys. Rev.*, 113 (6), 1670-1679 (15 March 1959).

[16] Bialynicki-Birula, *Photon wave function*, in *Progress in Optics*, Vol. XXXVI, Ed. E. Wolf, pp 248-294, (North-Holland 1996).

[17] J. D. Jackson, *Classical Electrodynamics*, (Third Edition, John Wiley & Sons, 1998).

[18] Wolfgang K. H. Pnifsky and Melba Phillips, *Classical Electricity and Magnetics*, (Addison-Wesley Publishing Company, 1962).

[19] L. L. Foldy and S. A. Wouthuysen, *On the Dirac Theory of Spin 1/2 Particles and its Non-Relativistic Limit*, *Phys. Rev.* 78, 29-36 (1950).

[20] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, San Francisco, 1964).
[21] M. H. L. Pryce, The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles, *Proc. Roy. Soc. Ser., A* 195, 62-81 (1948).

[22] S. Tani, Connection between particle models and field theories. I. The case spin $1/2$, *Prog. Theor. Phys.*, 6, 267-285 (1951).

[23] R. Acharya and E. C. G. Sudarshan, *J. Math. Phys.*, 1, 532-536 (1960).