Stationary Scattering Theory: The N-Body Long-Range Case

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Abstract: Within the class of Dereziński–Enss pair-potentials which includes Coulomb potentials and for which asymptotic completeness is known (Dereziński in Ann Math 38:427–476, 1993), we show that all entries of the N-body quantum scattering matrix have a well-defined meaning at any given non-threshold energy. As a function of the energy parameter the scattering matrix is weakly continuous. This result generalizes a similar one obtained previously by Yafaev for systems of particles interacting by short-range potentials (Yafaev in Integr Equ Oper Theory 21:93–126, 1995). As for Yafaev’s paper we do not make any assumption on the decay of channel bound states. The main part of the proof consists in establishing a number of Kato-smoothness bounds needed for justifying a new formula for the scattering matrix. Similarly we construct and show strong continuity of channel wave matrices for all non-threshold energies. Away from a set of measure zero we show that the scattering and channel wave matrices constitute a well-defined ‘scattering theory’, in particular at such energies the scattering matrix is unitary, strongly continuous and characterized by asymptotics of generalized eigenfunctions of minimal growth.

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1. Introduction

Asymptotic completeness of systems of quantum particles interacting by long-range potentials (more precisely by pair-potentials decaying like $O(|x^\alpha|^{-\mu})$ with $\mu > \sqrt{3} - 1$) was proven by Dereziński [De] by entirely time-dependent methods. While asymptotic completeness of systems of quantum particles interacting by short-range potentials has been proven by a time-independent method, see [Ya1], there does not seem to appear any time-independent method for the long-range case in the literature. A notable virtue of the mentioned papers is their generality, thus the completeness results hold without implicit assumptions.

One deficiency of this situation is that we lack understanding of the scattering matrix in a general long-range setup. This is in contrast to Yafaev’s case where his method reveals some properties of the (short-range) scattering matrix. In particular Yafaev showed that this quantity is weakly continuous in the spectral parameter away from the threshold set, as deduced from a formula, [Ya3].

We show in this paper similar results for the long-range case. The continuity will be derived from a stationary formula, and similarly to Yafaev’s case our formula is justified in terms of various Kato-smoothness bounds. Only a few of them appears in Yafaev’s papers. In particular our paper is strongly dependent on certain time-independent bounds/observables which may be understood as modifications of Dereziński’s key ingredients. In addition to [De, Ya1] our paper relies on some resolvent bounds first proven in [GIS]
and then recently extended to a Besov space setting along with a (sharp) Sommerfeld uniqueness result in [AIIS].

For any channel \( \alpha = (a, \lambda^\alpha, u^\alpha) \) we consider the channel wave operators

\[
W^{\pm}_\alpha = \lim_{t \to \pm \infty} e^{itH} J_\alpha e^{-i(S^\pm_a(\xi_\alpha, t) + \lambda^\alpha t)}, \quad J_\alpha u^\alpha = u^\alpha \otimes f^\alpha,
\]

where \( H \) denotes the full \( N \)-body Schrödinger operator, \( S^\pm_a(\xi_\alpha, t) \) solves a certain Hamilton–Jacobi equation (it is a ‘distortion’ of the function \( t \xi^2_\alpha \)). For two channels, say incoming \( \alpha = (a, \lambda^\alpha, u^\alpha) \) and outgoing \( \beta = (b, \lambda^\beta, u^\beta) \), the corresponding entry \( \hat{S}_{\beta\alpha} = (W^{+}_\beta)^* W^{-}_\alpha \) of the scattering operator takes the form in a diagonalizing (momentum) space,

\[
\hat{S}_{\beta\alpha} = \int_{(\max|\lambda^\beta, \lambda^\alpha|, \infty)} S_{\beta\alpha}(\lambda) \, d\lambda.
\]

One can abstractly construct the fiber operators \( S_{\beta\alpha}(\cdot) \) as almost everywhere defined quantities, however such a construction does not seem to provide properties of scattering. On the contrary we prove (independently of any abstract construction) that there exist a (unique) weakly continuous candidate away from the threshold set. Another main result of the paper is that for fixed energy also the restriction of the wave operators and their adjoints (the latter are called wave matrices) exist. Given these results it makes sense to ask if there is a ‘scattering theory’ at a fixed energy? While this is not settled in this paper we do show that almost all non-threshold energies are ‘stationary complete’. In particular at any such energy \( S_{\beta\alpha}(\cdot) \) is strongly continuous and characterized by asymptotic properties of a class of generalized eigenfunctions.

The weak continuity property of \( S_{\beta\alpha}(\cdot) \) at all non-threshold energies will be read off from an explicit formula of the following form. Up to some trivial multiplicative factors in momentum space and localization away from ‘collision planes’

\[
S_{\beta\alpha}(\lambda) \approx \left( (H - \lambda) \Phi^+_b J_\beta \tilde{\gamma}^+_b (\lambda - \lambda^\beta)^* \right)^* \delta(H - \lambda) \left( (H - \lambda) \Phi^-_a J_\alpha \tilde{\gamma}^-_a (\lambda - \lambda^\alpha)^* \right) ^*.
\]

Here \( \delta(H - \lambda) = \pi^{-1/2} (H - \lambda - i0)^{-1} \) is the delta-function of \( H \) at \( \lambda \), \( \tilde{\gamma}^+_b (\lambda - \lambda^\beta)^* \) and \( \tilde{\gamma}^-_a (\lambda - \lambda^\alpha)^* \) are certain auxiliary ‘one-body wave matrices’ (constructed from inter-cluster potentials), while the stationary channel modifiers \( \Phi^-_a \) and \( \Phi^+_b \) should be considered as some appropriate incoming and outgoing phase-space ‘distortions of the identity’. The latter operators are introduced so that (looking only at the right-hand factor)

\[
(H - \lambda) \Phi^-_a J_\alpha \tilde{\gamma}^-_a (\lambda - \lambda^\alpha)^* \approx \sum_k Q^- (a, k)^* \left( B_- (a, k) Q^- (a, k) J_\alpha \tilde{\gamma}^-_a (\lambda - \lambda^\alpha)^* \right).
\]

Here the sum is finite, the operators \( B_- (a, k) \) are bounded and the only \( \lambda^- \)-dependence is sitting in the argument of the incoming one-body wave matrix. With a similar structure for the left-hand factor the weak continuity follows from three properties to be demonstrated in the paper:

(i) \( Q^+(b, l) \delta(H - \lambda) Q^- (a, k)^* \) is weakly continuous in \( \lambda \).
(ii) \( Q^- (a, k) J_\alpha \tilde{\gamma}^-_a (\lambda - \lambda^\alpha)^* \) is strongly continuous in \( \lambda \).
(iii) \( Q^+(b, l) J_\beta \tilde{\gamma}^+_b (\lambda - \lambda^\beta)^* \) is strongly continuous in \( \lambda \).
Note that uniform boundedness of the operator in (i) is very related to local smoothness of the $Q$-operators (relative to $H$) in the sense of Kato [Ka]. Apart from the limiting absorption principle bound the corresponding boundedness assertions for (i)–(iii) all arise from a well-known positive commutator technique, for example used extensively in the cited papers [De, Ya1, Ya3].

Although we take the existence of the wave operators $W^\pm_\alpha$ for granted in this paper, which is legitimate due to [De], we remark that it may be derived independently from the stationary setup of the present paper. The proof of the existence and weak continuity of the scattering matrix at all non-threshold energies may then also be considered as being independent of [De]. However for deducing unitarity and the strong continuity for almost all energies of the scattering matrix, as obtained in the paper, we do indeed use the completeness assertion of [De]. We leave it as a conjecture for $N$-body Schrödinger operators that those stronger properties are fulfilled for all non-threshold energies. This conjecture is partially motivated by related works by Isozaki [Is2, Is3, Is4] on the three-body stationary scattering problem. Although he needs strong decay assumptions on the pair-potentials, more precisely $O(|x^\alpha|^{-\rho})$ for $\rho > 5$, indeed Isozaki verifies the appropriate Parseval identity for an arbitrary non-threshold energy, see [Is4, Theorem 6.6]. The conjecture is recently resolved for the three-body problem with Dereziński–Enss pair-potentials [Sk3].

We remark that our method can be viewed as an extension of the one of [Ya3] for the short-range case (although involving several new ingredients and a different representation of the scattering matrix). In particular Yafaev’s result on the weak continuity of the short-range scattering matrix may be viewed as a consequence of our theory, see Remark 2.2 (v).

In Sect. 1.1 we elaborate on the consequences of our long-range theory (for the general class of Dereziński–Enss pair-potentials) applied to a well-known atomic physics model. The bulk of the paper is organized as follows. The general $N$-body model is introduced in Sect. 2, based on Condition 2.1 which will be imposed throughout the paper, along with various ingredients to be used in later sections. In particular we collect in Sect. 2.3 a number of results for the one-body case that will be essential for our treatment of the $N$-body problem. A main result of the paper (this is the weak continuity discussed above) is stated in Sect. 3 as Theorem 3.1. In the same section we outline the scheme for proving the theorem, in particular we introduce stationary channel modifiers. The ingredients from [Ya1] and [De] that we need (essentially vector field constructions) are given in Sects. 4 and 5, respectively. In addition the latter section contains a Sect. 5.1 with Mourre estimates, while Sect. 5.2 gives an account of a number of consequences of the Mourre estimates taken from [AIIS]. We need to do commutation with the stationary modifiers which are given by functional calculus. We devote Sect. 6 to studying a calculus facilitating this task. The commutation is then implemented in Sects. 7 and 8 in which the above assertions (i) and (ii)–(iii) are proven, respectively. We finalize the proof of Theorem 3.1 in Sect. 8.2. The exact expression for the scattering matrix, of the type indicated above, is derived in “Appendix C” with inputs from Sect. 5.2 and by using results from Sects. 7 and 8. We devote “Appendix A” to the proof of a few assertions from Sect. 3, given by using the stationary phase method. The content of “Appendix B” is an elementary commutator estimate of partial relevance for “Appendix C”, but of fundamental importance for Sects. 9.3 and 9.4. Our second main result, which concerns representation of channel states in terms of certain generalized eigenfunctions, is given in Sect. 9 as Theorem 9.1. The involved integrals are given in terms of channel wave matrices that we construct for all non-threshold energies. Another issue of Sect. 9 is the
construction of the *scattering matrix* for all non-threshold energies (done concretely), as well as its unitarity and strong continuity away from a null set of energies, see Definition 9.7 and Corollary 9.9. Such generic energies, introduced more precisely in Definition 9.2 by Parseval identities, we call ‘stationary complete’. We characterize for any stationary complete energy and for any incoming \(\alpha\)-channel the associated ‘minimum generalized eigenfunctions’ in terms of their explicit asymptotic properties, see Definition 9.10 and Theorems 9.11 and 9.13. The latter theorem may also be viewed as a characterization of the incoming \(\alpha\)-channel part of the scattering matrix at any such energy. Wave matrices have been studied in many contexts in the literature, for example abstractly in [KK] and concretely for \(N\)-body Schrödinger operators with short-range interactions in [Ya2] (our exposition is rather different from Yafaev’s). The developed theory for stationary complete energies is similar to the standard one for the one-body problem at any positive energy (in that case all positive energies are stationary complete), see Sect. 2.3. In fact we construct in Sect. 9.4 a weaker version of the ‘radial limit construction’ of restricted wave operators (appearing in Sect. 2.3 as (2.20)). This is then used in Sect. 9.5 to characterize stationary completeness in terms of top-order asymptotics of the (limiting) resolvent of \(H\).

As we already did in the above discussion, we ignore for simplicity in the bulk of the paper the possible existence of non-threshold embedded eigenvalues. It is a minor technical exercise to include non-threshold embedded eigenvalues for all results of the paper, see Remark 9.3.

1.1. A principal example, atomic and molecular \(N\)-body Hamiltonians. Consider a system of \(N\) charged particles of dimension \(n\) interacting by Coulomb forces. The Hamiltonian reads

\[
H = -\sum_{j=1}^{N} \frac{1}{2m_j} \Delta x_j + \sum_{1 \leq i < j \leq N} q_i q_j |x_i - x_j|^{-1}, \quad x_j \in \mathbb{R}^n, \quad n \geq 3, \quad (1.1)
\]

where \(x_j, m_j\) and \(q_j\) denote the position, mass and charge of the \(j\)’th particle, respectively.

The Hamiltonian \(H\) is regarded as a self-adjoint operator on \(L^2(X)\), where \(X\) is the \(n(N-1)\) dimensional real vector space \(X := \{\sum_{j=1}^{N} m_j x_j = 0\}\). Let \(\mathcal{A}\) denote the set of all cluster decompositions of the \(N\)-particle system. The notation \(d_{\text{max}}\) and \(d_{\text{min}}\) refers to the 1-cluster and \(N\)-cluster decompositions, respectively. Let for \(a \in \mathcal{A}\) the notation \(\#a\) denote the number of clusters in \(a\). For \(i, j \in \{1, \ldots, N\}, i < j\), we denote by \((ij)\) the \((N-1)\)-cluster decomposition given by letting \(C = \{i, j\}\) form a cluster and all other particles \(l \notin C\) form singletons. Write \((ij) \leq a\) if \(i\) and \(j\) belong to the same cluster in \(a\). More general, we write \(b \leq a\) if each cluster of \(b\) is a subset of a cluster of \(a\). If \(a\) is a \(k\)-cluster decomposition, \(a = (C_1, \ldots, C_k)\), we let

\[
X^a = \{x \in X \mid \sum_{l \in C_j} m_l x_l = 0, \quad j = 1, \ldots, k\} = X^{C_1} \oplus \cdots \oplus X^{C_k},
\]

and

\[
X_a = \{x \in X \mid x_i = x_j \text{ if } i, j \in C_m \text{ for some } m \in \{1, \ldots, k\}\}.
\]
Consequently any $x \in \mathbf{X}$ decomposes orthogonally as $x = x^a + x_a$ with $x^a = \pi^a x \in \mathbf{X}^a$ and $x_a = \pi_a x \in \mathbf{X}_a$.

With these notations, the many-body Schrödinger operator (1.1) takes the form $H = H_0 + V$, where $H_0 = p^2$ is (minus) the Laplace-Beltrami operator on the Euclidean space $(\mathbf{X}, q)$ and $V = V(x) = \sum_{b=(ij) \in A} V_b(x^b)$ with $V_b(x^b) = V_{ij}(x_i - x_j)$ for the $(N-1)$-cluster decomposition $b = (ij)$. Note for example that

$$x^{(12)} = (\frac{-m_2}{m_1+m_2}(x_1 - x_2), -\frac{m_1}{m_1+m_2}(x_1 - x_2), 0, \ldots, 0).$$

For any cluster decomposition $a \in \mathcal{A}$ we introduce the Hamiltonian $H^a$ as follows. For $a = a_{\text{min}}$ we define $H^{a_{\text{min}}} = 0$ on $\mathcal{H}^{a_{\text{min}}} := \mathbb{C}$. For $a \neq a_{\text{min}}$ we introduce the potential

$$V^a(x^a) = \sum_{b=(ij) \leq a} V_b(x^b); \quad x^a \in \mathbf{X}^a.$$

Then

$$H^a := -\Delta x^a + V^a(x^a) = (p^a)^2 + V^a \text{ on } \mathcal{H}^a = L^2(\mathbf{X}^a).$$

A channel $\alpha$ is by definition given as $\alpha = (a, \lambda^\alpha, u^\alpha)$, where $a \in \mathcal{A} = \mathcal{A} \setminus \{a_{\text{max}}\}$ and $u^\alpha \in \mathcal{H}^a$ obeys $\|u^\alpha\| = 1$ and $(H^a - \lambda^\alpha)u^\alpha = 0$ for a real number $\lambda^\alpha$, named a threshold. The set of thresholds is denoted $\mathcal{T}(H)$, and including eigenvalues of $H$ we introduce $\mathcal{T}_p(H) = \sigma_{pp}(H) \cup \mathcal{T}(H)$. For any $a \in \mathcal{A}$ the intercluster potential is by definition

$$I_a(x) = \sum_{b=(ij) \notin a} V_b(x^b).$$

Next we introduce atomic Dollard type channel wave operators. They read

$$W_{a, \text{atom}}^\pm = \operatorname{s-lim}_{t \to \pm \infty} e^{iHt} (u^\alpha \otimes e^{-i(D_{a, \text{atom}}^\pm (p_{a,t}) + \lambda^\alpha t)} \cdot), \quad (1.2)$$

where

$$D_{a, \text{atom}}^\pm (\xi_a, \pm |t|) = \pm D_{a, \text{atom}}^\pm (\pm \xi_a, |t|) \quad \text{and} \quad D_{a, \text{atom}}^\pm (\xi_a, t) = t \xi_a^2 + \int_1^t I_a(2s \xi_a) ds.$$

The well-definedness of $W_{a, \text{atom}}^\pm$ follows from the existence of $W_{a, \text{atom}}^\pm$, see Remarks 2.2 (i) and (iii). Let $I^a = (\lambda^\alpha, \infty)$ and $k_a = p_a^2 + \lambda^\alpha$. By the intertwining property $H W_{a, \text{atom}}^\pm \supseteq W_{a, \text{atom}}^\pm k_a$ and the fact that $k_a$ is diagonalized by the unitary map $F_{\alpha} : L^2(\mathbf{X}_a) \rightarrow L^2(I^\alpha, \mathcal{G}_a), \mathcal{G}_a = L^2(C_a), C_a = \mathbf{X}_a \cap \mathbb{S}^{n_a-1}$ with $n_a = \dim \mathbf{X}_a$, given by

$$(F_{\alpha} \varphi)(\lambda, \omega) = (2\pi)^{-n_a/2} 2^{-1/2} \lambda^\alpha (n_a-2)/4 \int e^{-i\lambda_a^{1/2} \omega \cdot x_a} \varphi(x_a) \, dx_a, \quad \lambda_a = \lambda - \lambda^\alpha,$$
we can for any given channels $\alpha$ and $\beta$ write
\[
\hat{S}_{\beta\alpha,\text{atom}} := F_\beta(W_{\beta,\text{atom}}^*)^*W_{\alpha,\text{atom}}F^{-1}_\alpha = \int_{I_{\beta\alpha}} S_{\beta\alpha,\text{atom}}(\lambda) \, d\lambda, \quad I_{\beta\alpha} = I^\beta \cap I^\alpha.
\]

The fiber operator $S_{\beta\alpha,\text{atom}}(\lambda) \in \mathcal{L}(\mathcal{G}_\alpha, \mathcal{G}_\beta)$ is from an abstract point of view (our point of view is different as discussed previously) a priori defined only for a.e. $\lambda \in I_{\beta\alpha}$. It is the $\beta\alpha$-entry of the atomic Dollard type scattering matrix $S_{\text{atom}}(\lambda) = (S_{\beta\alpha,\text{atom}}(\lambda))_{\beta\alpha}$.

Now a main result of this paper, specialized to the above atomic model, reads as follows (see Remark 2.2 (iv)).

**Theorem 1.1.** The operator-valued function $S_{\text{atom}}(\cdot)$ is a weakly continuous contraction away from $T_p(H)$. In particular its entries $S_{\beta\alpha,\text{atom}}(\cdot)$ are also weakly continuous away from $T_p(H)$.

A similar result is valid for Coulomb systems of charged particles with static (i.e. infinite mass) nuclei. It is a minor doable issue to include non-threshold embedded eigenvalues in the above result, cf. Remark 9.3. The behaviour at thresholds is much more intriguing and completely outside the scope of this paper. Strong continuity for all non-threshold energies (as conjectured before) is another difficult problem.

The second main result, again for convenience specialized to the above model, reads as follows. Recall the standard notation for weighted spaces
\[
L^2_s(X) = \langle x \rangle^{-s} L^2(X); \quad s \in \mathbb{R}, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.
\]

**Theorem 1.2.** Let $\alpha$ be a given channel $\alpha = (a, \lambda^a, u^a)$, $f : I^\alpha = (\lambda^a, \infty) \to \mathbb{C}$ be continuous and compactly supported away from $T_p(H)$, and let $s > 1/2$. For any $\varphi \in L^2(X_a)$
\[
W_{\alpha,\text{atom}}^\pm f(k_\alpha)\varphi = \int_{I^\alpha \setminus T_p(H)} f(\lambda)W_{\alpha,\text{atom}}^\pm(\lambda)(F_\alpha \varphi)(\lambda, \cdot) \, d\lambda \in L^2_{-s}(X), \quad (1.3)
\]
where the ‘wave matrices’ $W_{\alpha,\text{atom}}(\lambda) \in \mathcal{L}(\mathcal{G}_\alpha, L^2_s(X))$ with a strongly continuous dependence of $\lambda$. In particular for $\varphi \in L^2_s(X_b)$ the integrand is a continuous compactly supported $L^2_{-s}(X)$-valued function. In general the integral has the weak interpretation of an integral of a measurable $L^2_{-s}(X)$-valued function.

Of particular interest is the case $f(\lambda) = f_t(\lambda) = e^{-it\lambda}f_0(\lambda)$, $t \in \mathbb{R}$, in which case the formulas (1.3) represent exact Schrödinger wave packets of energy-localized states outgoing to or incoming from the channel $\alpha$. For a state $\varphi_\alpha = W_{\alpha,\text{atom}}f_0(k_\alpha)\varphi$ the asymptotics of the wave packet is ‘controlled’ as $t \to -\infty$. To ‘understand’ the behaviour as $t \to +\infty$ (and possibly compare with practical physics experiments) the right-hand side of (1.3) in principle provides a tool. The main contribution comes from the asymptotics of functions in the range of the outgoing resolvent $(H - (\lambda + i0))^{-1}$ (as confirmed by one of our formulas for the generalized eigenfunctions in the range of $W_{\alpha,\text{atom}}(\lambda)$). On the other hand this quantity is in general poorly understood for the $N$-body case. In the paper we derive several non-trivial estimates (more precisely mostly for the delta-function of $H$ only), however they are all of ‘weak type’. Strong resolvent bounds is yet another difficult problem with room for desirable improvements. So maybe the main virtue of Theorem 1.2 and its generalizations at this point is conceptual: The wave matrices $W_{\alpha}^\pm(\lambda)$ uniquely exist admitting representations like (1.3).
From a practical point of view it is more convenient to substitute the factor $W_{\alpha,\text{atom}}(\lambda)$ in (1.3) by

$$W_{\alpha,\text{atom}}(\lambda) = \sum_{\lambda^\beta < \lambda} W_{\beta,\text{atom}}(\lambda) S_{\beta,\alpha,\text{atom}}(\lambda).$$

(1.4)

Now each term contributes by a term which is ‘controlled’ as $t \to +\infty$, meaning that all relevant large time information of our ‘incoming $\alpha$-channel experiment’ is encoded in the components of the scattering matrix. In turn, although the resolvent is complicated, it appears in formulas only as the delta-function of $H$ sandwiched by some Kato-smooth operators (as discussed previously). We do not in this paper offer ‘finer analysis’ of the derived formulas on interesting (but difficult) issues like regularity or asymptotics of associated distributional kernels.

The delta-function of $H$ at any real $\lambda \notin \mathcal{T}_p(H)$, $\delta(H - \lambda) = \pi^{-1} \Im(H - \lambda - i0)^{-1}$, may be considered as a quadratic form on $L^2_{-\infty}(X) = \bigcap_s L^2_s(X)$. Correspondingly we introduce the following notion of completeness.

**Definition 1.3.** An energy $\lambda \in (\min \mathcal{T}(H), \infty) \setminus \mathcal{T}_p(H)$ is stationary complete if

$$\forall \psi \in L^2_{-\infty}(X) : \sum_{\lambda^\beta < \lambda} \| W_{\beta,\alpha,\text{atom}}(\lambda)^* \psi \|^2 = \langle \psi, \delta(H - \lambda)\psi \rangle.$$  

(1.5)

We show in this paper that asymptotic completeness (known to hold by time-dependent methods) implies that almost all energies in $(\min \mathcal{T}(H), \infty) \setminus \mathcal{T}_p(H)$ are stationary complete. (The other direction that stationary completeness almost everywhere implies asymptotic completeness obviously would follow by integration.) Parts of Theorems 1.1 and 1.2 extend as follows at stationary complete energies.

**Theorem 1.4.** At any stationary complete energy the scattering matrix $S_{\text{atom}}(\cdot)$ is a strongly continuous unitary operator determined uniquely by asymptotics of generalized eigenfunctions from the ranges of the wave matrices of Theorem 1.2.

Moreover at any such energy also (1.4) is valid as an identity in $\mathcal{L}(\mathcal{G}_a, L^2_{-s}(X))$ (summing in the weak sense and for arbitrary $s > 1/2$), and the ‘restricted wave operators’ $W_{\alpha,\text{atom}}(\cdot)^* \in \mathcal{L}(L^2_s(X), \mathcal{G}_a)$ are strongly continuous.

2. Preliminaries

We introduce the standard abstract $N$-body setup, our condition on the potentials, Condition 2.1, to be imposed throughout the paper and channel wave operators known to exist under Condition 2.1. For long-range potentials the channel wave operators are not uniquely defined and we discuss the relationships between various choices in Remarks 2.2. The issues we are studying in this paper do not depend on the choice of channel wave operators. We prefer to work with wave operators defined in terms of exact solutions to a Hamilton–Jacobi equation for a one-body problem as done in [Hő1, II], see Sect. 2.2. In Sect. 2.3 we collect a number of results for the one-body Schrödinger operators $\hat{h}_a$ of Sect. 2.2. All three subsections contain much notation to be used freely in later sections.
2.1. N-body Hamiltonians and limiting absorption principle. Let $X$ be a finite dimensional real inner product space, equipped with a finite family $\{X_a\}_{a \in \mathcal{A}}$ of subspaces closed under intersection: For any $a, b \in \mathcal{A}$ there exists $c \in \mathcal{A}$ such that

$$X_a \cap X_b = X_c.$$  

(2.1)

For the example of Sect. 1.1 the elements of $\mathcal{A}$ are cluster decompositions, but here $\mathcal{A}$ is an abstract index set. We order $\mathcal{A}$ by writing $a \leq b$ (or equivalently as $b \geq a$) if $X_a \supseteq X_b$. It is assumed that there exist $a_{\text{min}}, a_{\text{max}} \in \mathcal{A}$ such that

$$X_{a_{\text{min}}} = X, \quad X_{a_{\text{max}}} = \{0\}.$$  

Let $X^a \subseteq X$ be the orthogonal complement of $X_a \subseteq X$, and denote the associated orthogonal decomposition of $x \in X$ by

$$x = x^a \oplus x_a = \pi^a x \oplus \pi_a x \in X^a \oplus X_a.$$  

The vectors $x_a$ and $x^a$ may be called the inter-cluster and internal components of $x$, respectively. We note that the family $\{X^a\}_{a \in \mathcal{A}}$ is closed under addition: For any $a, b \in \mathcal{A}$ there exists $c \in \mathcal{A}$ such that

$$X^a + X^b = X^c,$$  

cf. (2.1).

A real-valued measurable function $V : X \to \mathbb{R}$ is a potential of many-body type if there exist real-valued measurable functions $V_a : X^a \to \mathbb{R}$ such that

$$V(x) = \sum_{a \in \mathcal{A}} V_a(x^a) \text{ for } x \in X.$$  

(2.2)

We take $V_{a_{\text{min}}} = 0$ (without loss of generality). We impose throughout the paper the following condition for $a \neq a_{\text{min}}$. By definition $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

**Condition 2.1.** There exists $\mu \in (\sqrt{3} - 1, 1)$ such that for all $a \in \mathcal{A}\{a_{\text{min}}\}$ the potential $V_a(x^a) = V_{\text{sr}}^a(x^a) + V_{\text{lr}}^a(x^a)$, where

1. $V_{\text{sr}}^a(-\Delta x^a + 1)^{-1}$ is compact and $|x^a|^{1+\mu} V_{\text{sr}}^a(-\Delta x^a + 1)^{-1}$ is bounded.
2. $V_{\text{lr}}^a \in C^\infty$ and for all $\beta \in \mathbb{N}_0^{\dim X^a}$

$$a^{\beta} V_{\text{lr}}^a(x^a) = O(|x^a|^{-\mu} |\beta|).$$

For any $a \in \mathcal{A}$ we associate a Hamiltonian $H^a$ as follows. For $a = a_{\text{min}}$ we define $H^{a_{\text{min}}} = 0$ on $\mathcal{H}^{a_{\text{min}}} = L^2(\{0\}) = \mathbb{C}$. For $a \neq a_{\text{min}}$ we introduce

$$V^a(x^a) = \sum_{b \leq a} V_b(x^b), \quad x^a \in X^a,$$

and then

$$H^a = -\Delta x^a + V^a \text{ on } \mathcal{H}^a = L^2(X^a).$$

We abbreviate

$$V_{a_{\text{max}}} = V, \quad H_{a_{\text{max}}} = H, \quad \mathcal{H}_{a_{\text{max}}} = \mathcal{H}.$$
The **thresholds** of $H$ are the eigenvalues of the sub-Hamiltonians $H^a$; $a \in \mathcal{A}' := \mathcal{A} \setminus \{a_{\text{max}}\}$. Equivalently stated the set of thresholds is

$$T(H) := \bigcup_{a \in \mathcal{A}'} \sigma_{pp}(H^a),$$

and including eigenvalues of $H$ we introduce $T_p(H) = \sigma_{pp}(H) \cup T(H)$. Note that for $a \in \mathcal{A} \setminus \{a_{\text{min}}, a_{\text{max}}\}$ the family $\{X_b \cap X^a\}_{b < a}$ forms a family of subspaces of many-body type in $X^a$. This self-similarity structure is useful for induction arguments involved in the proofs of various well-known properties: We recall (see for example [AIIS]) that under Condition 2.1 the set $T(H)$ is closed and countable. Moreover the set of non-threshold eigenvalues is discrete in $\mathbb{R} \setminus T(H)$, and it can only accumulate at points in $T(H)$ from below. The essential spectrum of $H$ is given by the formula $\sigma_{\text{ess}}(H) = [\min T(H), \infty)$. We refer to the minimum example $\mathcal{A} = \{a_{\text{min}}, a_{\text{max}}\}$ as the one-body model in which case $T(H) = \{0\}$.

Define the **Sobolev spaces** $\mathcal{H}^s$ of order $s \in \mathbb{R}$ associated with $H$ as

$$\mathcal{H}^s = (H - E)^{-s/2} \mathcal{H}; \quad E = \min \sigma(H) - 1. \quad (2.3)$$

We note that the form domain of $H$ is $\mathcal{H}^1 = H^1(X)$ and that the operator domain $\mathcal{D}(H) = \mathcal{H}^2 = H^2(X)$ (i.e. given by well-known Sobolev spaces). It is standard to consider $\mathcal{H}^{-1}$ and $\mathcal{H}^{-2}$ as the corresponding dual spaces. Denote $R(z) = (H - z)^{-1}$ for $z \notin \sigma(H)$.

Consider and fix $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 0 & \text{for } t \leq 4/3, \\ 1 & \text{for } t \geq 5/3, \end{cases} \quad \chi' \geq 0, \quad (2.4)$$

and such that the following properties are fulfilled:

$$\sqrt{\chi}, \sqrt{\chi'}, (1 - \chi^2)^{1/4}, \sqrt{-(1 - \chi^2)^{1/2}}' \in C^\infty.$$

We define correspondingly $\chi_+ = \chi$ and $\chi_- = (1 - \chi^2)^{1/2}$ and record that

$$\chi_+^2 + \chi_-^2 = 1 \quad \text{and} \quad \sqrt{\chi_+}, \sqrt{\chi_+'}, \sqrt{\chi_-}, \sqrt{-\chi_-'} \in C^\infty.$$

We shall use the standard notation $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in X$ (or more generally for $x$ in a normed space). If $T$ is an operator on a Hilbert space $\mathcal{G}$ and $\varphi \in \mathcal{G}$ then $\langle T \rangle_\varphi := \langle \varphi, T \varphi \rangle$. We denote the space of bounded operators from one (general) Banach space $X$ to another one $Y$ by $\mathcal{L}(X, Y)$ and abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. The dual space of $X$ is denoted by $X^*$.

To define **Besov spaces associated with the multiplication operator** $|x|$ on $\mathcal{H}$ let

$$F_0 = F(\{x \in X \mid |x| < 1\}),$$

$$F_m = F(\{x \in X \mid 2^{m-1} \leq |x| < 2^m\}) \quad \text{for } m = 1, 2, \ldots,$$

where $F(U) = F_U$ is the sharp characteristic function of any given subset $U \subseteq X$. The Besov spaces $B = B(X)$, $B^s = B(X)^s$ and $B^s_0 = B^s_0(X)$ are then given as

$$B = \{\psi \in L^2_{\text{loc}}(X) \mid \|\psi\|_B < \infty\}, \quad \|\psi\|_B = \sum_{m=0}^\infty 2^{m/2} \|F_m \psi\|_{\mathcal{H}}.$$
For any compact \( \Lambda_1 \subset \mathbb{R} \) any indication of the
Stationary Scattering Theory: The \( N \)
For notational convenience we take from this point and throughout the paper
\( \lambda \)
respectively. Denote the standard weighted \( L^2 \) spaces by
\[
L_s^2 = L_s^2(\mathbf{X}) = \langle x \rangle^{-s} L_s^2(\mathbf{X}) \quad \text{for } s \in \mathbb{R}, \quad L_{s-\infty}^2 = \bigcup_{s \in \mathbb{R}} L_s^2, \quad L_{-\infty}^2 = \bigcap_{s \in \mathbb{R}} L_s^2.
\]
Then for any \( s > 1/2 \)
\[
L_s^2 \subset L_{1/2}^2 \subset \mathcal{H} \subset L_{-1/2}^2 \subset B_0^* \subset B^* \subset L_{-s}^2.
\] (2.5)
Under Condition 2.1 (in fact under a weaker condition) the following limits exist locally uniformly in \( \lambda \notin T_p(H) \), see Sect. 5.2.
\[
R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon) \in \mathcal{L}(L_s^2, L_{-s}^2) \quad \text{for any } s > 1/2.
\] (2.6a)
Furthermore, in the strong weak*-topology (to be explained in Corollary 5.2),
\[
R(\lambda \pm i0) = \text{s-w}^* - \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon) \in \mathcal{L}(B, B^*)
\] with a locally uniform norm bound in \( \lambda \notin T_p(H) \). (2.6b)

2.2. A one-body effective potential and channel wave operators. For any \( a \in \mathcal{A} \) we introduce
\[
I^s_a = \sum b_{\not\subseteq a} V^b_{st}, I^r_a = \sum b_{\not\subseteq a} V^b_{tr}, I_a = I^s_a + I^r_a \quad \text{and}
\]
\[
\tilde{I}_a = \tilde{I}_{a,R} = I^s_a(x_a) = \chi_+(|x_a|/R) I^r_a(x_a) \prod_{b \not\subseteq a} \chi_+(|\pi^b x_a| \ln(x_a)/|x_a|); \quad R \geq 1.
\]
This ‘regularization’ \( \tilde{I}_a \) appears in [HS] for the free channel, i.e. for \( a = a_{\min} \). It is a one-body potential in the precise sense that for any \( \tilde{\mu} \in (\sqrt{3} - 1, \mu) \)
\[
\partial^\beta \tilde{I}_a(x_a) = \mathcal{O}(|x_a|^{-|\tilde{\mu}|-|\beta|}).
\] (2.7)
For notational convenience we take from this point and throughout the paper \( \tilde{\mu} = \mu \), i.e. more precisely we will assume (2.7) with \( \tilde{\mu} \) replaced by \( \mu \). We have introduced the auxiliary parameter \( R \geq 1 \) entailing a certain Mourre estimate at a given energy, see Sect. 5.1, however our main results can be stated in terms of the auxiliary one-body potential \( \tilde{I}_{a,R} \) with \( R = 1 \). We suppress the parameter \( R \) in notation and prefer to omit any indication of the \( R \)-dependence of quantities.

We let \( K_a(\cdot, \lambda), \lambda > 0 \), denote the corresponding approximate solution to the eikonal equation
\[
|\nabla_{x_a} K_a|^2 + \tilde{I}_a = \lambda \quad \text{as taken from [Is1, II]}. \quad \text{More precisely writing } K_a(x_a, \lambda) = \sqrt{\lambda} |x_a| - k_a(x_a, \lambda) \text{ the following properties are fulfilled with } \mathbb{R}_+ := (0, \infty) \text{ and } n_a := \dim X_a. \quad \text{The functions } k_a \in C^\infty(X_a \times \mathbb{R}_+) \text{ and:}
\]
1) For any compact \( \Lambda \subset \mathbb{R}_+ \) there exists \( \rho > 1 \) such that for all \( \lambda \in \Lambda \) and all \( x_a \in X_a \) with \( |x_a| > \rho \)
\[
2\sqrt{\lambda} \frac{\partial}{\partial |x_a|} k_a = \tilde{I}_a(x_a) + |\nabla_{x_a} k_a|^2.
\]
2) For all multiindices $\beta \in \mathbb{N}_0^n$, $m \in \mathbb{N}_0$ and compact $\Lambda \subset \mathbb{R}_+$

$$|\partial_{x_a}^\beta \partial_{\lambda}^m k_\alpha| \leq C (x_a)^{1-|\beta|-\mu} \text{ uniformly in } \lambda \in \Lambda.$$

The Legendre transform of the function $K_\alpha(\cdot, \lambda)$ leads to the following construction, citing here [II, Lemma 6.1]:

There exist an $X_\alpha$-valued function $x(\xi, t)$ and a positive function $\lambda(\xi, t)$ both in $C^\infty((X_\alpha \setminus \{0\}) \times \mathbb{R}_+)$ and satisfying the following requirements. For any compact set $B \subseteq X_\alpha \setminus \{0\}$, there exist $t_0, C > 0$ such that for $\xi \in B$ and $t > t_0$

$$\xi = \partial_{x_a} K_\alpha(x(\xi, t), \lambda(\xi, t)), \quad t = \partial_\lambda K_\alpha(x(\xi, t), \lambda(\xi, t)). \quad \text{(2.8a)}$$

$$|x(\xi, t) - 2t\xi| \leq C(t)^{1-\mu}, \quad |\lambda(\xi, t) - |\xi|^2| \leq C(t)^{-\mu}. \quad \text{(2.8b)}$$

Then we define

$$S_\alpha(\xi, t) = x(\xi, t) \cdot \lambda(\xi, t) - K_\alpha(x(\xi, t), \lambda(\xi, t)) \quad (\xi, t) \in (X_\alpha \setminus \{0\}) \times \mathbb{R}_+. \quad \text{(2.8c)}$$

Note that this function solves the Hamilton–Jacobi equation

$$\partial_t S_\alpha(\xi, t) = \xi^2 + \tilde{I}_a (\partial_\xi S_\alpha(\xi, t)); \quad t > t(\xi), \quad \xi \neq 0. \quad \text{(2.9)}$$

For any $\text{channel } \alpha = (a, \lambda^\alpha, u^\alpha), a \in A', u^\alpha \in \mathcal{H}_a, (H^a - \lambda^\alpha)u^\alpha = 0$ and $\|u^\alpha\| = 1$, we introduce, cf. [De], the channel wave operators

$$W^\pm_\alpha = \text{s-lim}_{t \to \pm \infty} e^{itH} J_\alpha e^{-i(S^\pm_\alpha(p_a, t) + \lambda^\alpha t)}, \quad \text{(2.10)}$$

where $J_\alpha f_a = u^\alpha \otimes f_a$ for $f_a \in \mathcal{H}_a = L^2(X_\alpha), p_a = -i \nabla_{x_a}$ and $S^\pm_\alpha(\xi_a, \pm |t|) = \pm S_\alpha(\xi_a, |t|)$.

To examine the structure of (2.10) let us for any $a \in A'$ introduce

$$\tilde{h}_a = p_a^2 + I_a, \quad \tilde{H}_a = H^a \otimes I + I \otimes \tilde{h}_a \text{ and } H_a = H^a \otimes I + I \otimes p_a^2. \quad \text{(2.11)}$$

Using (2.10) with $H$ replaced by $\tilde{H}_a$ we obtain

$$\tilde{W}^\pm_\alpha = \text{s-lim}_{t \to \pm \infty} e^{it\tilde{H}_a} J_\alpha e^{-i(S^\pm_\alpha(p_a, t) + \lambda^\alpha t)} = J_\alpha \tilde{W}^\pm_\alpha, \quad \tilde{W}^\pm_\alpha = \text{s-lim}_{t \to \pm \infty} e^{it\tilde{h}_a} e^{-iS^\pm_\alpha(p_a, t)}. \quad \text{(2.12)}$$

The operators $\tilde{W}^\pm_\alpha$ are one-body wave operators fulfilling certain ‘invariance properties’ to be used and elaborated on in Remarks 2.2 below. We are going to use many other properties of these operators, to be discussed in Sect. 2.3.

Let

$$k_\alpha = p_a^2 + \lambda^\alpha, \quad I^\alpha = (\lambda^\alpha, \infty),$$

$$C_a = X_a \cap S^{n_a-1}_a, \quad n_a = \text{dim } X_a \text{ and } C'_a = C_a \setminus \bigcup_b \bar{2}X_a X_b. \quad \text{(2.13a)}$$

By the intertwining property $HW^\pm_\alpha \supseteq W^\pm_\alpha k_\alpha$ and the fact that $k_\alpha$ is diagonalized by the unitary map $F_\alpha : L^2(X_a) \rightarrow L^2(I^\alpha; \mathcal{G}_a), \mathcal{G}_a = L^2(C_a)$, given by

$$(F_\alpha \varphi)(\lambda, \omega) = (2\pi)^{-n_a/2} 2^{-1/2} \lambda^{(n_a-2)/4} \int e^{-i\lambda^{1/2} \omega \cdot x_a} \varphi(x_a) \, dx_a, \quad \lambda_\alpha = \lambda - \lambda^\alpha, \quad \text{(2.13b)}$$
we can write
\[
\hat{S}_{\beta\alpha} := F_{\beta}(W_{\beta}^+) W_{\alpha} F_{\alpha}^{-1} = \int_{I_{\beta\alpha}} \beta(\lambda) \, d\lambda \quad \text{with } I_{\beta\alpha} = I_{\beta} \cap I_{\alpha}.
\]

The fiber operator \(S_{\beta\alpha}(\lambda) \in \mathcal{L}(\mathcal{G}_\alpha, \mathcal{G}_\beta)\) is a priori defined only for a.e. \(\lambda \in I_{\beta\alpha}\). It is the \(\beta\alpha\)-entry of the scattering matrix.

Remark 2.2. (i) One can alternatively use the Dollard-approximation (cf. [Do]) rather than the above exact solution \(S_a\) to the Hamilton–Jacobi equation to define channel wave operators. It reads
\[
S_{a, \text{dol}}(\xi, t) = t^2 + \int_1^t \tilde{I}_a(2s\xi) \, ds.
\]

Introducing \(S_{a, \text{dol}}^\pm(\xi, \pm |t|) = \pm S_a, \text{dol}(\pm \xi, |t|)\), it follows from the chain rule and the fact that \(W_a^\pm\) exist that
\[
W_{a, \text{dol}}^\pm := \text{s-lim}_{t \to \pm \infty} e^{itH} \int_a e^{-i(S_d^\pm(p_a,t) + t^2)}
= W_a^\pm \text{s-lim}_{t \to \pm \infty} e^{-i(S_d^\pm(p_a,t) - S_d^\pm(p_a,t))} = W_a^\pm e^{-i\theta_a^\pm(p_a)};
\]
\[
\theta_a^\pm \in C(X_a \setminus \{0\}).
\]

Note that the limits \(\theta_a^\pm\) exist thanks to (2.9) and the bounds
\[
\int_1^\infty |\tilde{I}_a(\partial_\xi S_a(\xi, s)) - \tilde{I}_a(2s\xi)| \, ds < \infty, \quad \xi \neq 0,
\]
which (since \(\mu > 1/2\)) are consequences of the bounds
\[
|\partial_\xi S_a(\xi, t) - 2t\xi| \leq C |t|^{1-\mu}; \quad C = C(\xi) < \infty.
\]

In turn (2.15) follows from the property (2.8b) combined with the fact that
\[
\partial_\xi S_a(\xi, t) = x(\xi, t) \text{ for } t > t(\xi).
\]

(ii) By the same arguments as in (i) we can track the \(R\)-dependence of the wave operators \(W^\pm_a\). Recall that \(\tilde{I}_a = \tilde{I}_{a, R}\) depends on \(R \geq 1\). So let us for given \(R_1, R_2 \geq 1\) introduce corresponding notation \(W^\pm_{a, R_1}\) and \(W^\pm_{a, R_2}\). We obtain as above the relationship
\[
W^\pm_{a, R_2} = W^\pm_{a, R_1} e^{-i\theta_a^\pm_{R_2, R_1}(p_a)};
\]
\[
\theta_a^\pm_{R_2, R_1} = \lim_{t \to \pm \infty} \left( S^\pm_{a, R_2}(\cdot, t) - S^\pm_{a, R_1}(\cdot, t) \right)
= S^\pm_{a, R_2}(\cdot, 1) - S^\pm_{a, R_1}(\cdot, 1)
+ \int_1^\infty (\partial_s S_{a, R_2}(\cdot, s) - \partial_s S_{a, R_1}(\cdot, s)) \, ds \in C(X_a \setminus \{0\}).
\]

Note then that
\[
W^\pm_{a, R_2} F^{-1}_{a} = W^\pm_{a, R_1} F^{-1}_{a} e^{-i\theta_a^\pm_{R_2, R_1}(\lambda^{1/2})},
\]
where the factors \(e^{-i\theta_a^\pm_{R_2, R_1}(\lambda^{1/2})}\) are strongly continuous \(\mathcal{L}(\mathcal{G}_a)\)-valued functions of \(\lambda\).
(iii) The construction of (i) applies to the example of Sect. 1.1 and for that case one can alternatively use the function $D_{\alpha, \text{atom}}(\xi, t)$ defined there. Note the property

$$\int_1^{\infty} |I_a(2s\xi) - \tilde{I}_a(2s\xi)| \, ds < \infty; \quad \xi \in X_a \setminus \bigcup_{c \leq a} X_c.$$ 

By the chain rule the existence of $W_{\alpha, \text{atom}}^\pm$ then follows from the existence of $W_{\alpha, \text{dol}}^\pm$. 

iv) It is also an elementary fact that the existence, along with continuity properties of the scattering matrix, to be stated in Theorem 3.1 imply similar assertions for the ‘Dollard’ and ‘atomic’ counterparts discussed above. For example, under the conditions of Theorem 3.1

$$S_{\beta\alpha, \text{atom}}(\lambda) = e^{i\theta^+_\beta(\lambda \alpha/2)} S_{\beta\alpha}(\lambda) e^{-i\theta^-_\beta(\lambda \alpha/2)},$$

where on the right-hand side the multiplication operators to the left and to the right are both strongly continuous factors (they are as indicated above given by explicit integrals). Whence Theorem 1.1 is an immediate consequence of Theorem 3.1. Similarly Theorems 1.2 and 1.4 are consequences of assertions from the bulk of the paper.

v) For short-range states, say given by Condition 2.1 with the additional requirement $V_w^\alpha(x^\alpha) = O(|x^\alpha|^{-1-\epsilon})$ for some $\epsilon > 0$, we may use $S_{a, \text{st}}(\xi, t) := t\xi^2$ rather than $S_a(\xi, t)$. Arguing as above we can then use the results of this paper (derived for the function $S_a$) to deduce analogous results for stationary short-range scattering theory (stated in terms of the function $S_{a, \text{st}}$), overlapping with [Ya2, Ya3].

2.3. Restricted wave operators and wave matrices; the one-body problem. In this section we collect various results for the one-body Schrödinger operator $\hat{h}_a = p^2_a + I_a$, $a \in A'$, with the potential $\tilde{I}_a = I_a(x_a)$ obeying (2.7) (taking here $\tilde{\mu} = \mu$). We remark that in fact, although it is not relevant for this paper, this theory holds under Condition 2.1 with $\mu \in (\sqrt{3} - 1, 1)$ replaced by $\mu \in (0, 1)$ (a part of the presented theory appears for $\mu \in (0, 1)$ in [Is1]). Given the functions $S_a^\pm(\xi_a, \pm|t|) = \pm S_a(\pm\xi_a, |t|)$ the corresponding wave operators read, cf. (2.12),

$$\tilde{w}_a^\pm = s\lim_{t \to \pm\infty} e^{i\tilde{h}_a} e^{-iS_a^\pm(p_a, t)}. \quad (2.17)$$

We consider for any vector $g$ in the Hilbert space $G_a = L^2(C_a)$ and any $\lambda > 0$ the one-body quasi-modes

$$\tilde{v}_{a, \lambda}^\pm[g](x_a) = \mp \frac{1}{2\pi} (c_a^\pm(\lambda))^{-1} \chi_+(|x_a|)|x_a|^{1-n_a/2} e^{\pm iK_a(x_a, \lambda)} g(\pm\hat{x}_a);$$

$$c_a^\pm(\lambda) = e^{\pm i\pi(n_a-3)/4} \pi^{-1/2} \lambda^{1/4}, \quad \lambda > 0, \quad \hat{x}_a = x_a/|x_a|. \quad (2.18)$$

This terminology is justified by the fact that for $g \in C^\infty(C_a) \subset G_a$

$$(\hat{h}_a - \lambda)\tilde{v}_{a, \lambda}^\pm[g] \in L^2_s(X_a) = \langle x_a \rangle^{-s} L^2(X_a) \text{ for any } s < \frac{1}{2} + \mu. \quad (2.19)$$

We also introduce the restricted wave operators (or alternatively named ‘restricted inverse wave operators’) $\tilde{\gamma}_a^\pm(\lambda)$ defined first by their action on any $\psi \in L^2_s(X_a), s > 1$, as the radial limits (of $G_a$-valued functions)

$$\tilde{\gamma}_a^+(\lambda)\psi = G_a \lim_{r \to \infty} c_a^+(\lambda)r^{(n_a-1)/2} e^{-iK_a(r, \cdot)} (\tilde{r}_a(\lambda + i0)\psi)(r)$$

$$\tilde{\gamma}_a^-(\lambda)\psi = R_a \left( G_a \lim_{r \to \infty} c_a^-(\lambda)r^{(n_a-1)/2} e^{iK_a(r, \cdot)} (\tilde{r}_a(\lambda - i0)\psi)(r) \right); \quad (2.20)$$
here $\lambda > 0$ and $\tilde{r}_a(\lambda \pm i0) = \lim_{\epsilon \to 0_+} (\lambda \mp i\epsilon)^{-1}$, cf. (2.6a) and (2.6b), and $R_a$ denotes the reflection operator on $G_a$, $(R_ag)(\omega) = g(-\omega)$. The existence of the limits (2.20) is an easy consequence of [IS2, Theorem 1.14], and by the same result

$$\gamma_a^\pm(\lambda) = 2\pi i^{-1}(\tilde{r}_a(\lambda + i0) - \tilde{r}_a(\lambda - i0)); \quad \lambda > 0. \quad (2.21)$$

Clearly (2.21) leads us to consider extensions, cf. [IS2, Proposition 1.15],

$$\gamma_a^\pm(\lambda) \in L(\mathcal{B}(X_a), G_a). \quad (2.22)$$

The restricted wave operators $\gamma_a^\pm(\cdot)$ are strongly continuous $L(\mathcal{B}(X_a), G_a)$-valued functions. We shall only use the latter property for the free case (i.e. with $\tilde{I}_a = 0$, see “Appendix C”), in which case we will use the notation $\gamma_a(\omega)$ for $\gamma_a^\pm(\lambda)$ (the two operators corresponding to different signs coincide in that case).

The wave matrices $\gamma_a^\pm(\lambda)^* \in L(G_a, \mathcal{B}(X_a)^*)$ are strongly weak*-continuous functions of $\lambda > 0$, and if considered as $L(G_a, L^2_{\ast\pi}(X_a))$-valued functions for any $s > 1/2$, the wave matrices are strongly continuous.

The above terminologies are justified by the following result relating $\gamma_a^\pm(\lambda)^*$ to the wave operators (2.17), see [Is1,II,HS].

**Lemma 2.3.** Let $a \in \mathcal{A}'$, $s > 1/2$, $f : \mathbb{R}_+ \to \mathbb{C}$ be bounded and continuous and let $J$ be a compact interval in $\mathbb{R}_+$. For any $\varphi \in L^2_s(X_a)$

$$\tilde{w}_a^\pm(f1_J)(p^2_a)\varphi = \int_J f(\lambda)\gamma_a^\pm(\lambda)^*\gamma_a,0(\lambda)\varphi \, d\lambda \in L^2_{\ast\pi}(X_a), \quad (2.23a)$$

More generally for any $\varphi \in 1_J(p^2_a)\mathcal{L}_{\pi}(X_b)$

$$\tilde{w}_a^\pm f(p^2_a)\varphi = \int_J f(\lambda)\gamma_a^\pm(\lambda)^*(F_a,0\varphi)(\lambda, \cdot) \, d\lambda \in L^2_{\ast\pi}(X_a); \quad (2.23b)$$

here $(F_a,0\varphi)(\lambda, \cdot)$ denotes the almost everywhere defined extensions of the restriction maps $\gamma_a,0(\lambda)\varphi$, cf. (2.13b) (in fact given explicitly by taking $\lambda^\varepsilon = 0$ in (2.13b)).

In (2.23a) the integrand is a bounded and continuous $L^2_{\ast\pi}(X_a)$-valued function. For (2.23b) the integral has the weak interpretation of the integral of a measurable $L^2_{\ast\pi}(X_a)$-valued function.

We record the following results, see [IS1,IS2] for similar formulas in a geometric context and see [GY,DS] for related results. For example (2.24b) may be viewed as a version of [IS2, (3.20)], while in turn (2.24c) is an immediate consequence of (2.19), (2.24b) and [IS1, Corollary 1.13].

**Lemma 2.4.** Let $a \in \mathcal{A}'$ and $\lambda > 0$. There exists an operator $\tilde{\gamma}_a(\lambda)$ on $G_a$ such that for any $g \in G_a$ the vector $u = \tilde{\gamma}_a^-(\lambda)^*g \in \mathcal{B}(X_a)^*$ has the asymptotic property

$$u - \tilde{v}_a,\lambda^-[g] - \tilde{v}_a,\lambda[\tilde{\gamma}_a(\lambda)g] \in \mathcal{B}_{\delta}(X_a). \quad (2.24a)$$

For $g \in C^\infty(C_a) \subset G_a$ the vectors $u^\pm = \tilde{\gamma}_a^\pm(\lambda)^*g$ are given by the formulas

$$u^\pm = \tilde{v}_a,\lambda^\pm[g] - (\tilde{h}_a - \lambda \pm i0)^{-1}(\tilde{h}_a - \lambda)\tilde{v}_a,\lambda^\pm[g], \quad (2.24b)$$

$$u^\pm = \pm 2\pi i\delta(\tilde{h}_a - \lambda)(\tilde{h}_a - \lambda)\tilde{v}_a,\lambda^\pm[g]. \quad (2.24c)$$
Remark. Let us for completeness of presentation note that (2.24a) uniquely determine the operator $\tilde{s}_a(\lambda)$, named the one-body scattering matrix. It is unitary, strongly continuous in $\lambda > 0$ and obeys the property (stated in terms of quantities of Lemma 2.3)

$$F_{a,0}(\tilde{\omega}_a^+)\tilde{\omega}_a^-F_{a,0}^{-1} = \int_{\mathbb{R}_+} \tilde{s}_a(\lambda) \, d\lambda.$$  

Yet another characterization is provided by the formula $\tilde{\gamma}_a^+(\lambda) = \tilde{s}_a(\lambda)\tilde{\gamma}_a^-(\lambda)$. I Sect. 9 we collect and examine analogous properties of the $N$-body scattering matrix. The interested reader may find justification for all of the mentioned one-body properties by combining the previously stated results with a proper extraction of the ‘one-body content’ of Sect. 9, although there are alternative approaches in this case [IS2,GY,DS].

Next we introduce a vector-valued local operator, somewhat formally as

$$G_a = \mathcal{H}_a(x_a) \cdot p_a,$$

where $\mathcal{H}_a(x_a) = \chi_+(4|x_a|)|x_a|^{-1/2}(I - |x_a|^{-2} |x_a\rangle \langle x_a|)$. (2.25)

A correct definition most conveniently uses local coordinates, however it might be useful to note that for any ‘nice function’, say $\phi \in C^\infty_c(X_a)$, $G_a\phi$ is the function on $X_a$ given at any point $x_a \in X_a$ by a vector in the (complexified) dual space by the action

$$\langle y, (G_a\phi)(x_a) \rangle = \chi_+(4|x_a|)|x_a|^{-1/2}(y \cdot \xi - |x_a|^{-2} y \cdot x_a \cdot \xi); \quad y \in X_a, \xi = (p_a\phi)(x_a).$$

Written in terms of any choice of coordinates, $G_a\phi \in L^2(X_a)^{n_a}$ in this case, however this property has a clean meaning also in the following example for which $G_a$ acts on smooth functions no longer being compactly supported.

**Lemma 2.5.** For any $a \in \mathcal{A}'$ the operators

$$G_a\tilde{\gamma}_a^\pm(\lambda)^* \in \mathcal{L}(\mathcal{G}_a, L^2(X_a)^{n_a})$$

with a locally uniform bound in $\lambda > 0$, (2.26a)

and in fact

$$G_a\tilde{\gamma}_a^\pm(\cdot)^* \in \mathcal{L}(\mathcal{G}_a, L^2(X_a)^{n_a})$$

are strongly continuous. (2.26b)

**Proof.** The assertion (2.26a) may be derived by a simplified version of ‘the $T^*T$ argument’ (with $T = G_a\tilde{\gamma}_a^\pm(\lambda)^*$) to be given in Step I of the proof of Lemma 8.3. Thus we introduce for any real $f \in C^\infty_c(\mathbb{R}_+)$

$$\Phi = f(\tilde{h}_a)Mf(\tilde{h}_a); \quad M = 2\chi_+(4|x_a|)^2\Im(|x_a|^{-1} x_a \cdot p_a),$$

and compute

$$i[\tilde{h}_a, \Phi] = 4f(\tilde{h}_a)\left((G_a)^*G_a + \mathcal{O}(\langle x_a \rangle^{-\mu - 1})\right)f(\tilde{h}_a).$$

(2.27)

Letting $\chi_\rho = \chi_-(r/\rho)$ for $\rho > 1$ and $\phi = \tilde{\gamma}_a^\pm(\lambda)^* g$ for $g \in \mathcal{G}_a$, we next compute

$$0 = (i[\tilde{h}_a, - \chi_\rho \Phi \chi_\rho])\phi$$

$$= (\chi_\rho i[\tilde{h}_a, \Phi] \chi_\rho)\phi + 2\Im(i[\tilde{h}_a, \chi_\rho] \Phi \chi_\rho)\phi.$$  

For the last term to the right we compute and insert the expression

$$i[\tilde{h}_a, \chi_\rho] = -2\rho^{-1} \sqrt{-\chi'_- (|x_a|/\rho)} \Im(|x_a|^{-1} x_a \cdot p_a) \sqrt{-\chi'_- (r/\rho)},$$
commute (cf. Sect. 6) and conclude that this term is bounded by $C \|g\|^2$ with a constant $C > 0$ being independent of $\rho > 1$ and $\lambda \in \text{supp } f$. For the first term we use (2.27), yielding after a commutation the final bound

$$\sup_{\lambda > 0, \rho > 1} f(\lambda)^2 \left\| \chi_{\rho} G_{a} \tilde{\gamma}_{a}^{\pm} (\lambda)^{*} g \right\|^2 \leq C \|g\|^2.$$ 

We conclude (2.26a) by taking $\rho \to \infty$ invoking Lebesgue’s monotone convergence theorem.

As for (2.26b) it suffices by (2.26a) to verify the continuity of $G_{a} \tilde{\gamma}_{a}^{\pm} (\cdot)^{*} g$ for $g \in C^\infty(C_{a})$. Using the representation (2.24b) the contribution from the second term to the right is ‘small’ thanks to (2.19) and [IS1, Theorem 1.10]. More precisely, with locally uniform bounds

$$G_{a}(\tilde{h}_{a} - \lambda \pm i0)^{-1}(\tilde{h}_{a} - \lambda)\tilde{v}_{a,\lambda}^{\pm}[g] \in L^{2}_{s}(X_{a})^{na} \text{ for any } s < \mu.$$ 

Whence for the second term it suffices to note that for any $\rho > 1$ the map

$$\mathbb{R}_{+} \ni \lambda \to \chi_{-}(|x_{a}|/\rho)G_{a}(\tilde{h}_{a} - \lambda \pm i0)^{-1}(\tilde{h}_{a} - \lambda)\tilde{v}_{a,\lambda}^{\pm}[g] \in L^{2}(X_{a})^{na}$$

is continuous.

The latter property is easily checked using a direct computation of $(\tilde{h}_{a} - \lambda)\tilde{v}_{a,\lambda}^{\pm}[g]$ (here omitted) and a well-known regularity property of $(\tilde{h}_{a} - \lambda \pm i0)^{-1}$, cf. Remark 2.6 (iii) given below.

For the first term in (2.24b) we check by direct computation (here omitted) that the map

$$\mathbb{R}_{+} \ni \lambda \to G_{a}\tilde{v}_{a,\lambda}^{\pm}[g] \in L^{2}(X_{a})^{na}$$

is continuous; hence (2.26b) is proven. \qed

Remark 2.6. (i) The assertion (2.26a) follows alternatively by (2.21) and ‘the TT’ argument’ (again with $T = G_{a} \tilde{\gamma}_{a}^{\pm} (\lambda)^{*}$) in combination with the resolvent bound [Ya3, Theorem 3.9]. We prefer the given proof primarily since it may be seen as a ‘warm-up’ for the somewhat technical proof of Lemma 8.3. As for (2.26b) we remark that for the free Laplacian the assertion appears in [Ya3] with a proof using a scaling argument.

(ii) We give in Lemma 8.3 an N-body version of Lemma 2.5 (extended to include several other locally Kato-smooth operators). A possible yet alternative scheme for proving (2.26b) would be to use the one appearing in Steps IV and V of the proof of Lemma 8.3 (the above proof of (2.26b) is easier though). The idea of this general scheme is first to use a commutator argument to derive a uniform bound (above (2.26a)) and then in turn to deduce the weak continuity of a Hilbert space valued function $f$ (above $f(\cdot) = G_{a} \tilde{\gamma}_{a}^{\pm} (\cdot)^{*} g$ for $g \in C^{\infty}(C_{a})$). Secondly, one extracts from the used commutator argument the continuity of $\|f(\cdot)\|^2$ and finally concludes the strong continuity by invoking the familiar functional analysis result [Yo, Theorem 8, p. 124].

(iii) In addition to Lemmas 2.3–2.5 we shall need one-body versions of assertions from Sect. 5.2 (in particular in the proof of Lemma 8.3), more precisely for the operators $\tilde{h}_{a}$ rather than for the N-body Hamiltonian $H$ appearing there. The assertions follow by taking $H$ in Sect. 5.2 to be the one-body Hamiltonian with a potential fulfilling Condition 2.1.
3. Stationary Channel Modifiers and a Main Result

We will in the present section explain our overall scheme for proving weak continuity of any entry of the scattering matrix. This result is stated as Theorem 3.1. Most of the needed notation for its proof will be finally fixed in this section.

We shall consider a ‘sufficiently small’ open interval \( \Lambda_1 \) containing any arbitrarily fixed \( \lambda_0 \) obeying

\[
\lambda_0 \in \Lambda \subseteq \mathcal{E} := (\min T(H), \infty) \setminus T_p(H).
\]

Only the scattering matrix in an open interval \( I_0 \ni \lambda_0, \bar{I}_0 \subseteq \Lambda \), needs consideration (by partitioning of unity). The smallness of \( \Lambda \) will depend on Mourre estimates at \( \lambda_0 \), see Sect. 5.1.

For \( f_1, f_2 \in C_c^\infty(\mathbb{R}) \) taking values in \([0, 1]\) we write \( f_1 \prec f_2 \) if \( f_2 = 1 \) in a neighbourhood of \( \text{supp } f_1 \). We consider real \( f_1, f_2 \in C_c^\infty(\Lambda) \) with \( f_1 \prec f_2 \) and such that \( f_1 = 1 \) in \( \bar{I}_0 \).

For a given channel \( \alpha = (a, \lambda^\alpha, u^\alpha) \) let

\[
\check{J}^\pm_\alpha = N^a_\pm (I \otimes \check{w}^\pm_\alpha) J_\alpha \quad \text{and} \quad \tilde{J}^\pm_\alpha = f_2(H) M_\alpha N^a_\pm M_\alpha (I \otimes \check{w}^\pm_\alpha) J_\alpha,
\]

where \( \check{w}^\pm_\alpha \) are given by (2.17), and \( N^a_\pm \) and \( M_\alpha \) are symmetric operators specified below. The latter operator \( M_\alpha \) is given as in [Ya3, (5.1)] (with changed notation \( M^{(a)} \to M_\alpha \), and recalled in Sect. 4), in particular \( M_\alpha \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}) \), and \( N^a_\pm \in \mathcal{L}(\mathcal{H}) \) will be constructed such that (recalling \( k_\alpha = p_\alpha^2 + \lambda^\alpha \))

\[
W^\pm_\alpha f_1(k_\alpha) = \text{s-lim}_{t \to \pm \infty} e^{itH} J_\alpha \check{w}^\pm_\alpha e^{-ikt_\alpha} f_1(k_\alpha) = \text{s-lim}_{t \to \pm \infty} e^{itH} \check{J}^\pm_\alpha e^{-ikt_\alpha} f_1(k_\alpha),
\]

By the form of \( \check{J}^\pm_\alpha \) this amounts to the property

\[
\text{s-lim}_{t \to \pm \infty} (I - N^a_\pm) J_\alpha \check{w}^\pm_\alpha e^{-ikt_\alpha} f_1(k_\alpha) = 0.
\]

Recalling the operator \( \tilde{H}_\alpha \) from (2.11) we introduce stationary channel modifiers

\[
\Phi^\pm_\alpha = f_2(H) M_\alpha N^a_\pm M_\alpha f_2(\tilde{H}_\alpha).
\]

Provided the two limits

\[
\Omega^\pm_\alpha = \text{s-lim}_{t \to \pm \infty} e^{itH} \Phi^\pm_\alpha e^{-it\tilde{H}_\alpha} \text{ exist},
\]

the limits

\[
\text{s-lim}_{t \to \pm \infty} e^{itH} \check{J}^\pm_\alpha e^{-ikt_\alpha} f_1(k_\alpha) = \text{s-lim}_{t \to \pm \infty} e^{itH} \Phi^\pm_\alpha e^{-it\tilde{H}_\alpha} J_\alpha \check{w}^\pm_\alpha f_1(k_\alpha) \text{ exist},
\]

and denoting them by \( \tilde{W}^\pm_\alpha \), respectively, obviously

\[
\tilde{W}^\pm_\alpha = \Omega^\pm_\alpha J_\alpha \check{w}^\pm_\alpha f_1(k_\alpha).
\]
For \( a \neq a_{\min} \) the operators \( N_{\pm}^a \) are of the form (one for the upper sign plus and one for the lower sign minus)

\[
N_{\pm}^a = A_1 A_2^a (A_3^a)^2 A_4^a A_1;
\]

\[
A_1 = A_{1\pm} = \chi_+(\pm B/\epsilon_0),
\]

\[
A_2^a = \chi_-(r^{\rho_2 - 1} r_0^a),
\]

\[
A_3^a = A_{3\pm}^a = \chi_-(\pm B_{\delta, \rho_1}^a), \quad B_{\delta, \rho_1}^a = r^{\rho_1/2} B_{\delta, \rho_1} r^{\rho_1/2},
\]

where \( \epsilon_0 > 0 \) is sufficiently small (determined by the Mourre estimates at \( \lambda_0 \) from Sect. 5.1),

\[
1 - \mu < \rho_2 < \rho_1 < 1 - \delta \quad \text{with} \quad \delta \in [2/(2 + \mu), \mu), \tag{3.6}
\]

and \( r, B, r_0^a \) and \( B_{\delta}^a \) are operators constructed by quantities from [De] (\( r \) and \( r_0^a \) are multiplication operators while \( B \) and \( B_{\delta}^a \) are corresponding Graf vector field type constructions). For \( a = a_{\min} \) we take \( N_{\pm}^a = A_1^a = A_2^a \).

More precisely \( r \) is a function on \( X \), which apart from a trivial rescaling (to assure Mourre estimates for the Graf vector field \( \nabla r^2 / 2 \) is given by the function \( r \) from [De]. It partly plays the role as a ‘stationary time variable’ compared to the usage of the real time parameter in [De].) (It should not be mixed up with the function \(|x|\).) The operator \( B := 2\mathfrak{h}(p \cdot \nabla r) \). Let \( r^a \) be the same function now constructed on \( X^a \) rather than on \( X \). Let then \( r_0^a = r_0^\delta r(x^a / r^\delta) \) and \( B_\delta^a = 2\mathfrak{h}(p^a \cdot (\nabla r^a)(x^a / r^\delta)) \). We further elaborate on the above quantities in Sect. 5.

We verify (3.3) and (3.2b) in Sect. 7.2 and “Appendix A”, respectively. Moreover by (3.2b) and stationary phase analysis we may rewrite (3.4a) and (3.4b) as

\[
\tilde{w}_a^\pm = \text{s-lim}_{t \to \pm \infty} e^{itH} e^{-itH_\alpha} J_\alpha \tilde{w}_a^\pm f_1(k_\alpha)(m_\alpha^+)^2 = W_\alpha^f f_1(k_\alpha)(m_\alpha^+)^2, \tag{3.7a}
\]

where \( m_\alpha^\pm \) (after conjugation by the Fourier transform \( F_\alpha \), see (2.13b)) act as multiplication operators on \( L^2(I^a; \mathcal{G}_\alpha) \) (with multiplicative fiber operators). More precisely

\[
F_\alpha m_\alpha^\pm F_\alpha^{-1} = \int_{I_\alpha} e^{\pm 2 i \chi_{\alpha}^a m_\alpha(\pm \xi_\alpha)} d\lambda = \pm 2 e^{i/2} m_\alpha(\pm \xi_\alpha), \quad \xi_\alpha = \xi_\alpha/|\xi_\alpha| \in C_\alpha, \tag{3.7b}
\]

and \( m_\alpha \) is the function used in (4.1) specifying the operator \( M_\alpha \). The verification of (3.7a) and (3.7b) is given in “Appendix A”.

We conclude the following property of any scattering matrix entry \( S_{\beta \alpha}(\lambda) \) (assuming \( \lambda^\alpha, \lambda^\beta < \lambda_0 \)),

\[
16 \lambda_{\beta \alpha} f_1^2(\lambda)m_b(\xi_\beta)^2 S_{\beta \alpha}(\lambda)m_a(\xi_\alpha)^2 = \tilde{S}_{\beta \alpha}(\lambda), \tag{3.8a}
\]

where (recalling \( I_{\beta \alpha} = I^\beta \cap I^\alpha \))

\[
F_\beta \tilde{S}_{\beta \alpha} F_\alpha^{-1} = \int_{I_{\beta \alpha}} \tilde{S}_{\beta \alpha}(\lambda) d\lambda \quad \text{with} \quad \tilde{S}_{\beta \alpha} = (\tilde{W}_\beta^+)^* \tilde{W}_\alpha^-. \tag{3.8b}
\]

We shall verify (3.8b) for a weakly continuous operator-valued function \( \tilde{S}_{\beta \alpha}(\cdot) \). By (3.8a) the operator-valued function \( S_{\beta \alpha}(\cdot) \) will then also be weakly continuous on \( I_0 \ni \lambda_0 \) thanks to a freedom in choosing the functions \( m_b \) and \( m_a \), see a paragraph in the beginning.
of Sect. 4. In the beginning of Sect. 9 these functions are taken explicitly (of course conforming with Sect. 4).

Since \( \lambda_0 \in \mathcal{E} \) is arbitrary a partition of unity allows us to conclude a main result of the paper, stated as follows. Let \( \mathcal{E}_{\beta \alpha} := I_{\beta \alpha} \setminus \mathcal{T}_p(H) \).

**Theorem 3.1.** For any incoming channel \( \alpha \) and any outgoing channel \( \beta \) the map

\[
\mathcal{E}_{\beta \alpha} \ni \lambda \rightarrow S_{\beta \alpha}(\lambda) \in \mathcal{L}(G_a, G_b)
\]

is a well-defined weakly continuous map. Within the class of such maps it is uniquely determined by the identity

\[
F_\beta S_{\beta \alpha} F_{\alpha}^{-1} = F_\beta (W_\beta^+) W_\alpha^- F_{\alpha}^{-1} = \int_{\mathcal{E}_{\beta \alpha}} S_{\beta \alpha}(\lambda) \, d\lambda.
\]

Motivated by (3.3) and (3.4b) it is convenient to introduce the notation

\[
T_\alpha^\pm = i(H\Phi_a^\pm - \Phi_a^\pm \hat{H}_a) = i f_2(H)(HM_a N_a^a M_a - M_a N_a^a M_a \hat{H}_a) f_2(\hat{H}_a).
\]

In Sect. 7 we shall compute it in detail, using various preliminary calculus considerations from Sect. 6, and then show the existence of the operators \( \Omega_{\beta \alpha}^\pm \) of (3.3) using a variety of ‘weak propagation estimates’. These estimates will then be applied to derive a representation formula of \( \tilde{S}_{\beta \alpha}(\lambda) \) in Sect. 8 (with support from “Appendix C”), and from this formula we finally deduce the weak continuity of Theorem 3.1.

We devote Sect. 9 to further results that come out as byproducts of our proof of Theorem 3.1. In particular we show that \( S_{\beta \alpha}(-) \) is strongly continuous for almost all energies in \( \mathcal{E}_{\beta \alpha} \), and we introduce and show strong continuity of channel wave matrices for all energies in \( \mathcal{E}_{\beta \alpha} \) (again with support from “Appendix C”).

### 3.1. Extended lattice structure of subspaces.

For the constructions of \( M_a, r, B, r_a^d, B_a^d \) we assume that the family of subspaces \( \{X^c\}, c \in \mathcal{A}, \) is stable under addition (the standard assumption) and includes all sets of the form \( X_c \) (a non-standard assumption). This can be done without loss of generality by adding to the collection \( \{X^c\} \cup \{X_c\} \) all sums of subspaces from the collection, and we can then consider \( H \) as an \( N \)-body operator with this new lattice structure, say in this subsection indexed by \( \mathcal{A}^{\text{new}} \), simply by taking \( V_d = 0 \) on all added subspaces \( X^d \). Obviously the notions of eigenvalues and thresholds of \( H \) are the same as with the original (i.e. ‘old’) lattice structure, say in this subsection indexed by \( \mathcal{A}^{\text{old}} \).

However with \( \mathcal{A}^{\text{new}} \) we can also consider the operators \( \tilde{H}_c \) and \( H_c \) from (2.11) as ‘full’ \( N \)-body operators. Whence for \( \tilde{H}_c, c \in \mathcal{A}^{\text{old}} \setminus \{a_{\text{max}}\} \), we define \( V_c = \tilde{I}_c \) on \( X_c \), and for all added subspaces \( X^d \neq X_c \) we take \( V_d = 0 \). If \( X^d \subseteq X_c \) is an old subspace the potential \( V_d \) is the same as for the old subspace, and for the remaining case (including \( X^d = X \) if \( c \neq a_{\text{min}} \)) we take \( V_d = 0 \).

With the extended constructions (including in particular the constructions \( r \) and \( B \)) the corresponding Mourre estimates (5.4a) and (5.4b) (stated in Sect. 5.1) are covered by the same (well-known) theory. Moreover Sect. 6.4 will work the same way with \( H \) replaced with \( \tilde{H}_a \) (by the extended construction of \( M_a \)). For simplicity we allow ourselves henceforth the freedom to omit the superscript ‘old’ in \( \mathcal{A}^{\text{old}} \) and write \( a \in \mathcal{A}' \) in any context where \( \tilde{H}_a, \Phi_a^\pm \) and \( T_a^\pm \) will appear, although the meaning in such cases is \( a \in \mathcal{A}^{\text{old}} \setminus \{a_{\text{max}}\} \).
4. Yafaev’s Constructions

Let for \( \epsilon \in (0, 1) \) (below taken sufficiently small) and for \( a \in \mathcal{A}' = \mathcal{A} \setminus \{a_{\text{max}}\} \)

\[
X_a(\epsilon) = \{|x_a| > (1 - \epsilon)|x|\}.
\]

Note that \( X_a \setminus \{0\} \subseteq X_a(\epsilon) \) and in fact that \( X_a \setminus \{0\} = \cap_{\epsilon \in (0,1)} X_a(\epsilon) \).

In [Ya1] various real functions \( m_a \in C^\infty(X), a \in \mathcal{A}' \), are constructed, fulfilling the following properties for any sufficiently small \( \epsilon > 0 \):

1. \( m_a(x) \) is homogeneous of degree 1 for \( |x| \geq 1 \) and \( m_a(x) = 0 \) for \( |x| \leq 1/2 \).
2. If \( x \in X_b(\epsilon) \) and \( |x| \geq 1 \), then \( m_a(x) = m_a(x_b) \) (i.e. \( m_a(x) \) does not depend on \( x^b \)).
3. If \( b \not\leq a \) and \( x \in X_b(\epsilon) \), then \( m_a(x) = 0 \).

Let for any such function \( w_a = \text{grad} m_a \) and

\[
M_a = 2\Re(w_a \cdot p) = -i \sum_{j \leq n} ((w_a)_j \partial x_j + \partial x_j (w_a)_j); \quad n := \dim X. \quad (4.1)
\]

Let for \( a \in \mathcal{A}' \)

\[
X'_a = X_a \setminus \cup_{c \geq a} X_c = X_a \setminus \cup_{c \not\leq a} X_c, \quad (4.2a)
\]

\[
X'_a(\delta) = X_a \setminus \cup_{c \geq a} \overline{X_c}(\delta); \quad \delta > 0.
\]

Here and below the overline means topological closure. Note that \( X'_a(\delta) \subseteq X'_a \).

We need the following additional information from [Ya1, Lemma 3.5] on the construction of the functions \( m_a \), cf. a discussion in Sect. 3: We can for any given point \( z_a \in X'_a \) choose \( m_a \) with the additional property \( m_a(\hat{z}_a) \neq 0 \); \( \hat{z}_a := z_a/|z_a| \). More generally, for any given \( \delta > 0 \) and for any sufficiently small \( \epsilon > 0 \) we can take \( m_a \) such that \( m_a(x) = |x_a| \) for \( x \in X'_a(\delta) \) with \( |x| \geq 1 \). We shall implement the latter property explicitly in the beginning of Sect. 9, however for our proof of weak continuity of the fiber operator \( \bar{S}_{\beta a}(\cdot) \) of (3.8b) it is not relevant. It is needed only for extracting the continuity of \( S_{\beta a}(\cdot) \) in terms of (3.8a).

To control the Hessian of the functions \( m_a \), \( \text{Hess} m_a = \nabla^2 m_a \), (or more precisely \( p \cdot \nabla^2 m_a \) \( p \) arising in commutator calculations with (4.1)) Yafaev used a family of similar convexity properties. Before recalling these functions we introduce more conical subsets of \( X \).

Let for \( a \in \mathcal{A}' \) and \( \epsilon \in (0, 1) \)

\[
\Gamma_a(\epsilon) = (X \setminus \{0\}) \setminus \cup_{b \not\leq a} X_b(\epsilon). \quad (4.2b)
\]

Note, as a motivation, that \( \{|x| \geq 1\} \cap \text{supp} m_a \subseteq \Gamma_a(\epsilon) \) for any small \( \epsilon > 0 \).

Next we note that

\[
\Gamma_a(\epsilon) \subseteq \cup_{d \leq a} X'_d. \quad (4.2c)
\]

In fact we can for any \( x \in \Gamma_a(\epsilon) \) introduce \( d(x) \leq a \) by \( X_{d(x)} = \cap_{d \leq a, x \in X_d} X_d \) and then check that \( x \in X'_{d(x)} \). If not, \( x \in X_c \cap X_{d(x)} \) for some \( c \) with \( d(x) \leq c \leq a \) and therefore \( X_c = X_c \cap X_{d(x)} = X_{d(x)} \), contradicting that \( c \neq d(x) \).

Introduce for \( \delta > 0 \) the open cone

\[
Y_d(\delta) = X_d(\delta) \setminus \cup_{c \geq d} \overline{X_c}(3\delta^{1/n}). \quad (4.2d)
\]
Thanks to (4.2c) we can for any \( \delta_0 > 0 \) write
\[
\Gamma_a(\epsilon) \subseteq \bigcup_{d \leq a} \bigcup_{\delta \in (0, \delta_0]} Y_d(\delta). \tag{4.2e}
\]
By compactness this leads, for any fixed \( \epsilon \in (0, 1) \), to the existence of \( J \in \mathbb{N} \), \( \delta_1, \ldots, \delta_J \in (0, \delta_0] \) and \( d_1, \ldots, d_J \leq a \) such that
\[
\Gamma_a(\epsilon) \subseteq \bigcup_{j \leq J} Y_{d_j}(\delta_j). \tag{4.2f}
\]

By [Ya1, p. 538] there exists \( \delta'_0 > 0 \) such that for all \( \delta \in (0, \delta'_0] \) there exists a real function \( m = m_{\text{max}} \in C^\infty(X) \) fulfilling the following properties:

(i) \( m(x) \) is homogeneous of degree 1 for \( |x| \geq 1 \) and \( m(x) = 0 \) for \( |x| \leq 1/2 \).
(ii) If \( x \in X_b(\delta) \) and \( |x| \geq 1 \), then \( m(x) = m(x_b) \) (i.e. \( m(x) \) does not depend on \( x^b \)).
(iii) \( m(x) \) is convex in \( |x| \geq 1 \).
(iv) For all \( d \in A' \) there exists \( \mu_d \geq 1 \) such that
\[
m(x) = \mu_d |x_d| \text{ for all } x \in Y_d(\delta) \text{ with } |x| \geq 1.
\]

Similar to (4.1) we introduce for any such function \( w = \text{grad } m \) and
\[
M = M_{\text{max}} = 2\Re(w \cdot p) = -i \sum_{j \leq n} (w_j \partial_{x_j} + \partial_{x_j} w_j). \tag{4.3}
\]

To relate and further specify the introduced functions \( m_a, a \in A' \) and \( a = a_{\text{max}} \), let us first fix the order of construction as follows: Fix \( \epsilon > 0 \) conforming with the construction of the family of functions \( m_a \) fulfilling the properties (1)–(3) (as well as the non-vanishing condition discussed above in a separate paragraph) and fix \( \delta'_0 > 0 \) conforming with the construction of the family of functions \( m \) fulfilling the properties (i)–(iv). Obviously we can assume \( \delta'_0 \leq \epsilon \). Take then \( \delta_0 = \delta'_0 \) in (4.2e) and (4.2f). Next for each function \( m_a, a \in A' \), we use (4.2f) and construct functions \( m_j \) fulfilling (i)–(iv) with \( \delta = \delta_j; j = 1 \ldots J \).

For each \( a \in A' \) we now choose a quadratic partition \( \xi_1, \ldots, \xi_J \in C^\infty(\mathbb{S}^{n-1}) \) (viz \( \sum_j \xi_j^2 = 1 \) subordinate to the covering (4.2f), and let \( \xi_j^+(x) = \xi_j(\hat{x}) \chi_+(|x|); \hat{x} := x/|x| \). Then we can write, using the support properties (1) and (3)
\[
m_a(x) = \sum_{j \leq J} m_{a,j}(x); \quad m_{a,j}(x) = \xi_j^+(x)^2 m_a(x),
\]
and using (2) and the properties \( Y_{d_j}(\delta_j) \subseteq X_{d_j}(\delta_j) \subseteq X_{d_j}(\epsilon) \)
\[
m_{a,j}(x) \chi_+(|x|) = \xi_j^+(x)^2 m_a(x_{d_j}) \chi_+(|x|).
\]
Similarly (using in addition the notation \( G_a \) of (2.25))
\[
p \cdot (\chi_+(|x|) \nabla^2 m_a(x)) p = \sum_{j \leq J} p \cdot (\chi_+(|x|) \xi_j^+(x)^2 \nabla^2 m_a(x_{d_j})) p,
\]
\[
= \sum_{j \leq J} G_{d_j}^* (\chi_+(|x|) \xi_j^+(x)^2 G_{d_j}) G_{d_j}; \quad G_j = G_j(x_{d_j}) \text{ bounded.} \tag{4.4a}
\]
In turn due to (iii) and (iv), the latter applied with \( d = d_j \) (and \( \delta = \delta_j \)),
\[
G_{d_j}^* (\chi_+(|x|) \xi_j^+(x)^2 G_{d_j}) \leq p \cdot (\chi_+(|x|) \nabla^2 m_j(x)) p. \tag{4.4b}
\]
Finally we introduce functions $\xi^+_a$ and $\tilde{\xi}^+_a$ as follows. First choose any $\xi_a \in C^\infty(\mathbb{S}^{n-1})$ such that $\xi_a = 1$ on $\mathbb{S}^{n-1} \cap \Gamma_a(\epsilon)$ and $\xi_a = 0$ on $\mathbb{S}^{n-1} \setminus \Gamma_a(\epsilon/2)$. Choose then any $\tilde{\xi}_a \in C^\infty(\mathbb{S}^{n-1})$ using this recipe with $\epsilon$ replaced by $\epsilon/2$. Finally let $\xi^+_a(x) = \xi_a(\hat{x})\chi_+(4|x|)$ and $\tilde{\xi}^+_a(x) = \xi_a(\hat{x})\chi_+(8|x|)$, and note that $\xi^+_a \xi^+_a = \xi^+_a$. Then by (1) and (3)

$$M_a = M_a \xi^+_a = \xi^+_a M_a = M_a \tilde{\xi}^+_a = \tilde{\xi}^+_a M_a,$$

which in applications will provide ‘free factors’ of $\xi^+_a$ and $\tilde{\xi}^+_a$ where convenient.

5. Dereziński Type Constructions

There exists a positive function $r \in C^\infty(X)$ fulfilling the following properties [De, AIIS]:

(i) For all $\beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$ there exists $C > 0$ such that

$$|\partial^\beta (x \cdot \nabla)^k (r(x) - \langle x \rangle)| \leq C\langle x \rangle^{-1}.$$

(ii) There exists $c > 0$ such that for all $a \in A$:

$$|x^a| \leq c \Rightarrow \nabla^a r(x) = 0$$

(i.e. if $|x^a| \leq c$, then $r(x)$ does not depend on $x^a$).

(iii) $r(x)$ is convex.

(iv) For any given non-threshold energy of $H$ there is a Mourre estimate for $H$ with $A := \Re(\nabla r^2 \cdot p)$, or more precisely with such $A$ for a suitably rescaled version of the function $r$ obeying (i)–(iii). (Here and later we slightly abusively use the same notation $r$ for the rescaled function $Rr(x/R)$ with $R \geq 1$ taken sufficiently large). See Sects. 5.1 and 6.3.2 for details.

(v) The family of functions $\{r^a | a \in A\}$, with $r^a$ being a function on $X^a$ (rather than on $X$, and with $r^a_{\text{min}}$ being a positive constant), fulfills the following self-similar structure.

$$\forall \epsilon > 0 \exists R' \geq 1 \forall x \in \Gamma_a(\epsilon) \cap \{|x| \geq R'\} : \quad r^2 = |x^a|^2 + (r^a)^2;$$

here we use the notation of (4.2b), $\Gamma_a(\epsilon) = (X \setminus \{0\}) \cup_{b \neq a} X_b(\epsilon)$.

It follows from (i) that

$$|\partial^\beta (r(x) - \langle x \rangle)| \leq C\langle x \rangle^{-1},$$

$$|\partial^\beta (x \cdot \nabla r(x) - r(x))| \leq C\langle x \rangle^{-1},$$

$$|\partial^\beta (x \cdot (\nabla^2 r)(x))| \leq C\langle x \rangle^{-1},$$

cf. [De].

For completeness of presentation we note that the bounds of (i), proven only for $k \leq 2$ in [De], are equivalent to

$$|\partial^\beta (x \cdot \nabla)^k (r(x)^2 - \langle x \rangle^2)| \leq C,$$

and the latter estimates can be checked using the construction of [De]. (They were verified explicitly in [Sk1, Appendix A] for the seminal construction of Graf [Gr].) As in [De] we shall need (5.2a)–(5.2c) rather than (i).
We also remark that a version of (5.1) appears as [Sk2, (5.10)] (proven in detail in the paper using the seminal construction of Graf). Again one can check the stated assertion using the construction of [De], however we have included (5.1) only for completeness of presentation. It will not be used in the paper.

Let \( \omega = \nabla r \) and
\[
B = 2\Re(\omega \cdot p) = -i \sum_{j \leq n} (\omega_j \partial_{x_j} + \partial_{x_j} \omega_j).
\]

(5.3a)

For the self-adjointness of \( B \) we refer to Sect. 6.3. This operator appears in the definition of the factor \( A_2 \) of the operators \( N_{a}^{\pm} \) of Sect. 3.

With \( \delta \) as in (3.6) we define for \( a \in \mathcal{A} \setminus \{ a_{\text{max}}, a_{\text{min}} \} \), and with \( r^a \) given by \( \nu \) and \( \omega^a = \nabla r^a \),
\begin{align*}
  r^a_\delta &= r^a_\delta(x) = r^\delta r^a(x^a / r^\delta), \\
  \omega^a_\delta &= \omega^a_\delta(x) = \omega^a(x^a / r^\delta), \\
  B^a_\delta &= 2\Re(\omega^a_\delta \cdot p^a).
\end{align*}

(5.3b) (5.3c) (5.3d)

The quantities of (5.3a)–(5.3d) are building blocks of the factors \( A^a_2 \) and \( A^a_3 \) of the operators \( N_{a}^{\pm} \) and fix them for any choice of parameters \( \rho_2, \rho_1 \) and \( \delta \) in (3.6). For the self-adjointness of \( B^a_\delta \) and \( B^a_{\delta, \rho_1} = r^{\rho_1/2} B^a_{\delta, \rho_1} \) we refer to Sect. 6.5. Recall that \( N_{a_{\text{min}}}^{a_{\text{min}}} = A_1^2 \).

5.1. Mourre estimates and limiting absorption principles. In addition to the function \( f_1 \prec f_2, f_1, f_2 \in C_c^\infty(\Lambda) \), introduced in Sect. 3 let \( f_3 \in C_c^\infty(\Lambda) \) with \( f_2 \prec f_3 \). Upon shrinking the support of \( f_3 \) (and therefore also the supports of \( f_1 \) and \( f_2 \)) the following Mourre estimate holds. For some \( \tilde{\epsilon} > 0 \)
\[
f_3(H)i[H, r^{1/2} B r^{1/2}] f_3(H) \geq \tilde{\epsilon} f_3(H)^2.
\]

(5.4a)

The commutator in (5.4a) is given by its formal expression, and a rescaled version \( r(x) \to R r(x / R) \), with \( R \geq 1 \) taken sufficiently big is used. See Sect. 6.3.2 for more details and for a convenient form of (5.4a) to be used in Sect. 7.

Strictly speaking the conjugate operator appearing in (5.4a) is defined in terms of the vector field \( \tilde{\omega} = \frac{1}{2} \nabla r^2 \) associated to the construction of [De] while the Mourre estimate of [AIIS] (originating from [Sk1]) is stated in terms of the vector field from the seminal paper [Gr]. However the difference of the two vector fields is minor, caused only by the fact that different regularization procedures are used. In particular one can indeed, based on the construction of [De], easily check that the same proof as the one referred to in [AIIS] works for the Mourre estimate (5.4a) (with the stated version of the Graf vector). A similar remark applies for (5.4b) stated below.

Recall that the scaling parameter \( R \geq 1 \) used above is needed but for convenience suppressed in the notation. For the potential \( \tilde{I}_a = \tilde{I}_a, a \in \mathcal{A} \), introduced in Sect. 2.2 there is a similarly suppressed parameter \( R \geq 1 \). For simplicity we have used the same notation and in fact they can taken equal. More precisely we can record that with the parameter \( R \geq 1 \) taken sufficiently large also
\[
f_3(\tilde{H}_a)i[\tilde{H}_a, r^{1/2} B r^{1/2}] f_3(\tilde{H}_a) \geq \tilde{\epsilon} f_3(\tilde{H}_a)^2,
\]

(5.4b)
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Note that although the operator \( \tilde{h}_a = \tilde{h}_{a,R} \) of (2.11) may have eigenvalues they are all located in an interval \([-\omega(R_0), 0] \), implying that \( \tilde{H}_a \) does not have eigenvalues or thresholds in the support of \( f_3 \) for \( R \geq 1 \) large.

We recall from [AIIS] that (5.4a) and (5.4b) imply the following bounds in some neighbourhood \( \mathcal{N} \) of \( \text{supp} f_2 \), cf. (2.6a), (2.6b) and Sect. 5.2 below.

\[
\forall s > 1/2 : \quad R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon) \in \mathcal{L}(L^2_s, L^2_{-s}) \quad \text{for} \quad \lambda \in \mathcal{N},
\]

\[
\forall s > 1/2 : \quad \tilde{R}_a(\lambda \pm i0) = \lim_{\epsilon \to 0^+} \tilde{R}_a(\lambda \pm i\epsilon) \in \mathcal{L}(L^2_s, L^2_{-s}) \quad \text{for} \quad \lambda \in \mathcal{N}. \tag{5.5a}
\]

In any such norm-topology the limits are taken uniformly on \( \mathcal{N} \), and for the (stronger) Besov space topology the limiting operators obey

\[
R(\lambda \pm i0), \quad \tilde{R}_a(\lambda \pm i0) \in \mathcal{L}(\mathcal{B}, \mathcal{B}^*) \quad \text{with uniform bounds in} \quad \lambda \in \mathcal{N}. \tag{5.5b}
\]

With (5.4a) and (5.4b) in place we can henceforth consider the scaling parameter \( R \geq 1 \) as fixed, see Sect. 6.3 for a more general discussion. In particular the above parameter \( \bar{\epsilon} \) can be chosen as a function of the distance to the biggest threshold of \( H \) below \( \lambda \), see Lemma 6.7. We could at this point fix the parameter \( \epsilon_0 > 0 \) in the factor \( A_1 \) of the operators \( N^a_{\pm} \) in (3.5) by the single requirement \( 4\bar{\epsilon}^2 < \bar{\epsilon} \), however its size will not play any role. Henceforth it is more convenient just to consider \( \epsilon_0 \) as a small positive parameter. (The freedom of possibly choosing \( \epsilon_0 \) smaller than indicated will come in conveniently, although not crucially, in Sect. 8.1.) In combination with the previous constructions (5.3a)–(5.3d) we have by now fixed \( N^a_{\pm} \) (for any given parameters \( \rho_2, \rho_1 \) and \( \delta \) as in (3.6)).

5.2. Strong resolvent estimates. We recall the following three results from [AIIS], to be used in Sect. 9 and “Appendix C”. (In addition the one-body analogues will be useful in Sect. 8.1, cf. Remark 2.6 (iii).) They are based on (5.4a) and (5.4b) (with \( \bar{\epsilon} > 0 \) given there), although for convenience we only consider \( H \) (for \( \tilde{H}_a \) the limiting absorption bounds, implicitly stated in (5.5a), will suffice). We let \( \mathcal{N} \) denote any open subset \( \mathcal{N} \) of \( \mathbb{R} \) containing \( \text{supp} f_2 \) such that \( f_3 = 1 \) in a neighbourhood of the closure \( \overline{\mathcal{N}} \). Let \( \mathcal{N}_{\pm} = \{ z \in \mathbb{C} : |\text{Re} z| \in \mathcal{N} \text{ and } 0 < |z| \leq 1 \} \).

Theorem 5.1.

(1) There exists \( C > 0 \) such that for all \( z \in \mathcal{N}_{\pm} \)

\[
\| R(z) \|_{\mathcal{L}(\mathcal{B}, \mathcal{B}^*)} \leq C. \tag{5.6}
\]

(2) For any \( \beta \in (0, \mu/2) \) and \( F \in C^\infty(\mathbb{R}) \) with

\[
\text{supp } F \subset (-\infty, \sqrt{\bar{\epsilon}}) \text{ and } F' \in C_c^\infty(\mathbb{R}),
\]

there exists \( C > 0 \), such that for all \( z \in \mathcal{N}_{\pm} \) and \( \psi \in L^2_{1/2+\beta} \)

\[
\| F(\pm B) R(z) \psi \|_{L^2_{-1/2+\beta}} \leq C \| \psi \|_{L^2_{1/2+\beta}}, \tag{5.7}
\]

respectively.
Corollary 5.2. For any $s > 1/2$ and $\alpha \in (0, \min\{\mu/2, s - 1/2\})$ there exists $C > 0$ such that for all $k \in \{0, 1\}$, $z \in \mathcal{N}_+$ and $z' \in \mathcal{N}_+$, respectively,
\[
\|p^k R(z) - p^k R(z')\|_{L(L^2_{\pm}, L^2_{\pm})} \leq C|z - z'|^\alpha.
\] (5.8)

In particular, for any $\lambda \in \mathcal{N}$ and $s > 1/2$ the following boundary values exist:
\[
p^k R(\lambda \pm i0) := \lim_{\epsilon \to 0+} p^k R(\lambda \pm i\epsilon) \quad \text{in} \quad L(L^2_{\pm}, L^2_{\pm}),
\]
respectively (here the limits are taken in the operator-norm topology). The same boundary values are realized (in an extended form) as
\[
p^k R(\lambda \pm i0) = s\text{-}w^*\text{-}\lim_{\epsilon \to 0+} p^k R(\lambda \pm i\epsilon) \quad \text{in} \quad L(B, B^*),
\]
respectively (here the right-hand side operators act on any $\psi \in B$ as the $\epsilon$-limits of $p^k R(\lambda \pm i\epsilon)\psi$ in the weak-star topology of $B^*$).

The microlocal Sommerfeld uniqueness result, characterizes the limiting resolvents $R(\lambda \pm i0)$ by the Helmholtz equation and ‘microlocal radiation conditions’. (It is an $N$-body version of [IS1, Corollary 1.13], that we used to prove (2.24c).) Given $\psi \in L^2_{\text{loc}}(X)$, we say a function $\phi \in H^2_{\text{loc}}(X)$ is a generalized solution to $(H - \lambda)\phi = \psi$, if it satisfies the equation in the distributional sense.

Corollary 5.3. Let $\lambda \in \mathcal{N}$, $\beta \in [0, \mu/2)$ and $\psi \in r^{-\beta}B$. Then $\phi^\pm = R(\lambda \pm i0)\psi \in B^*$ satisfy
\begin{enumerate}
  \item $\phi^\pm$ are generalized solutions to $(H - \lambda)\phi = \psi$,
  \item for all $F \in C^\infty(\mathbb{R})$ with
    \[
    \text{supp } F \subset (-\infty, \sigma) \quad \text{and} \quad F' \in C^\infty_c(\mathbb{R}), \quad \text{where} \quad \sigma = \sqrt{\epsilon},
    \]
    \[
    \text{the functions } F(\pm B)\phi^\pm \text{ belong to } r^{-\beta}B^*, \quad \text{respectively.}
    \]
  \end{enumerate}

Conversely, if $u \in L^2_{-\infty} \cap H^2_{\text{loc}}(X)$ satisfies
\begin{enumerate}
  \item $u$ is a generalized solution to $(H - \lambda)u = \psi$,
  \item there exists $\sigma > 0$ such that for all $F \in C^\infty(\mathbb{R})$ with
    \[
    \text{supp } F \subset (-\infty, \sigma) \quad \text{and} \quad F' \in C^\infty_c(\mathbb{R}),
    \]
    \[
    \text{the function } F(B)u \in B^*_0 \quad \text{(or } F(-B)u \in B^*_0),
    \]
    then $u = R(\lambda + i0)\psi$ (or $u = R(\lambda - i0)\psi$).
\end{enumerate}

6. Calculus Considerations

To facilitate our treatment of $T^\pm_a$ in Sect. 7 we recall from [AIIS, Sect. 2] a calculus in which the Mourre estimate (5.4a) can be implemented. This calculus fits well for computing $T^\pm_a$ by the usual product rule for commutation.

In Sect. 6.1 we introduce notation frequently used in the later arguments. Section 6.2 concerns the Helffer–Sjöstrand formula, and two applications on computing commutators are given. In Sect. 6.3.1 we provide the self-adjoint realization of the operator $B$ of (5.3a). We investigate the first commutator $i[H, B]$ in Sect. 6.3.2, and the second commutator $i[i[H, B], B]$ in Sect. 6.3.3. This will apply for commutation with the factor $A_1$. Commutation with the factor $A_3^a$ is similar. This will be treated in Sect. 6.5. Commutation with the factors $M_a$ and $A_2^\alpha$ are easier, to be treated in Sects. 6.4 and 6.6, respectively.
6.1. Notation. Let $T$ be a linear operator on $\mathcal{H} = L^2(X)$ such that $T, T^* : L^2_{\infty} \to L^2_{\infty}$, and let $t \in \mathbb{R}$. Then we say that $T$ is an operator of order $t$, if for each $s \in \mathbb{R}$ the restriction $T|_{L^2_s}$ extends to an operator $T_s \in L(L^2_s, L^2_{s-t})$. Alternatively stated,

$$\|r^{s-t} Tr^{-s} \psi\| \leq C_s \|\psi\| \text{ for all } \psi \in L^2_{\infty}. \quad (6.1)$$

We can (and will) use this with $r$ replaced by $r_R = r_R(x) := Rr(x/R)$ for a fixed sufficiently large $R \geq 1$, cf. Sect. 5.1. Slightly abusively we dont change the notation and use $r$ rather than $r_R$ throughout the section, although in the context of Sect. 6.3.2 the quantity $r_R$ is needed in our presentation (and meant). Note (for consistency) that $T_s$ extends the restriction $T|_{D(T) \cap L^2_s}$. If $T$ is of order $t$, we write

$$T = \mathcal{O}(r^t). \quad (6.2)$$

Note also that, if $T = \mathcal{O}(r^t)$ and $S = \mathcal{O}(r^s)$, then $T^* = \mathcal{O}(r^t)$ and $TS = \mathcal{O}(r^{t+s})$. If $T = \mathcal{O}(r^t)$ for all $t \in \mathbb{R}$, then we write $T = \mathcal{O}(r^{-\infty})$. If $T = \mathcal{O}(r^s)$ for some $s < t$, we write $T = \mathcal{O}(r^{t-})$. (This will in Sect. 7 be a desirable property with $t = -1$.)

6.2. Functional calculus. Here we present the Helffer–Sjöstrand formula to represent functions of self-adjoint operators, and its application to commutators.

For $t \in \mathbb{R}$ we denote by $\mathcal{F}^t$ the set of real $f \in C^{\infty}(\mathbb{R})$ obeying

$$|f^{(k)}(x)| \leq C_k \langle x \rangle^{-k} \text{ for any } k \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}.$$ 

It is known that for any $f \in \mathcal{F}^t$, $t \in \mathbb{R}$, there always exists a ‘good’ almost analytic extension $\tilde{f} \in C^{\infty}(\mathbb{C})$, meaning more precisely that such function $\tilde{f}$ obeys

$$\tilde{f}|_{\mathbb{R}} = f, \quad |\tilde{f}(z)| \leq C \langle z \rangle^t, \quad |(\partial \tilde{f})(z)| \leq C_k |\Im z|^k \langle z \rangle^{t-k-1} \text{ for any } k \in \mathbb{N}_0;$$

see [DG, Proposition C.2.2]. One can choose $\tilde{f} \in C^{\infty}_{\mathbb{C}}(\mathbb{C})$ if $f \in C^{\infty}_{\mathbb{R}}(\mathbb{R})$.

**Lemma 6.1.** Let $T$ be a self-adjoint operator on $\mathcal{H}$, and let $f \in \mathcal{F}^t$ with $t \in \mathbb{R}$. Take an almost analytic extension $\tilde{f} \in C^{\infty}(\mathbb{C})$ of $f$ as above. Then for any $k \in \mathbb{N}_0$ with $k > t$ the operator $f^{(k)}(T) \in \mathcal{L}(\mathcal{H})$ is expressed as

$$f^{(k)}(T) = (-1)^k k! \int_{\mathbb{C}} (T - z)^{-k-1} \, d\mu_f(z) \quad \text{with} \quad d\mu_f(z) = \pi^{-1}(\partial \tilde{f})(z) \, du \, dv, \quad z = u + iv. \quad (6.3)$$

The expression (6.3) for $k = 0$ is the well-known Helffer–Sjöstrand formula used extensively in the literature to compute and bound commutators. In general there are several variations of the definition of a commutator, and in this paper we do not fix a particular one. It will be clear from the context in what sense we will be considering a commutator. Typically, for symmetric operators $T$ and $S$, we first define $i[T, S]$ as the quadratic form

$$\langle i[T, S] \rangle_\psi = 2 \langle \Im(ST) \rangle_\psi = i \langle T\psi, S\psi \rangle - i \langle S\psi, T\psi \rangle \quad \text{for } \psi \in D(T) \cap D(S),$$

and then extend it to a larger space.

Let us provide an example of a commutator formula derived this way using (6.3).
Corollary 6.2. Let $T$ be a self-adjoint operator on $\mathcal{H}$, $S$ be a symmetric relatively $T$-bounded operator, and assume that there exists a bounded extension

$$(|T| + 1)^{-\varepsilon/2} (i[T, S]) (|T| + 1)^{-\varepsilon/2} \in \mathcal{L}(\mathcal{H}) \text{ for some } \varepsilon \in [0, 2].$$

Let $f \in \mathcal{F}^t$ with $t < 1 - \varepsilon$ be given, and let $d \mu_f$ be given as in (6.3). Then, as a quadratic form on $\mathcal{D}(f(T)) \cap \mathcal{D}(S)$,

$$i[f(T), S] = -\int_{\mathcal{C}} (T - z)^{-1} (i[T, S]) (T - z)^{-1} d\mu_f(z),$$

and it extends to a bounded self-adjoint operator on $\mathcal{H}$.

Another example is the commutator $[f(H), r^s]$ treated below, which is an example of commutators of functions of entries from the triple of operators $(H, r, B)$. Other such examples will be discussed in Lemmas 6.5 and 6.9. In Sect. 7 we will repeatedly use Lemmas 6.3, 6.5 and 6.9–6.11.

Lemma 6.3.

(1) For any $f \in \mathcal{F}^0$ the operator $f(H)$ is of order 0.

(2) Let any $f \in \mathcal{F}^t$ with $t < 1/2$ and $s \in \mathbb{R}$ be given. Then $i[f(H), r^s]$ has an expression, as a quadratic form on $\mathcal{D}(f(H)) \cap L^2_{\max(0,s)}$,

$$i[f(H), r^s] = -2s \int_{\mathcal{C}} (H - z)^{-1} \text{Re}(r^{s-1} \omega \cdot p) (H - z)^{-1} d\mu_f(z). \tag{6.4}$$

In particular $i[f(H), r^s]$ is of order $s - 1$.

6.3. The operator $B$, the Mourre estimate and commutation with $A_1$. We show that the operator $B$ is self-adjoint and examine commutators with functions of $B$, including the prime example $i[f(H), A_1]$, $f \in C^\infty_c(\mathbb{R})$ real and $A_1$ given by (3.5).

6.3.1. Self-adjoint realization We recall the self-adjoint realization of the operator $B$ from [AIIS] and accompanying properties related to the spaces (2.3).

Lemma 6.4. The operator $B$ defined as (5.3a) is essentially self-adjoint on $C^\infty_c(X)$, and the self-adjoint extension, denoted by $B$ again, satisfies that for some $C > 0$

$$\mathcal{D}(B) \supseteq \mathcal{H}^1, \quad \|B\psi\|_{\mathcal{H}} \leq C\|\psi\|_{\mathcal{H}^1} \text{ for } \psi \in \mathcal{H}^1. \tag{6.5a}$$

In addition, $e^{i t B}$ for each $t \in \mathbb{R}$ naturally restricts/extends as bounded operators $e^{i t B} : \mathcal{H}^{\pm k} \to \mathcal{H}^{\pm k}$, $k = 1, 2$, and they satisfy

$$\sup_{t \in [-1,1]} \|e^{i t B}\|_{\mathcal{L}(\mathcal{H}^{\pm k})} < \infty, \tag{6.5b}$$

respectively. Moreover, the restriction $e^{i t B} \in \mathcal{L}(\mathcal{H}^k)$, $k = 1, 2$, is strongly continuous in $t \in \mathbb{R}$.

Lemma 6.5. (1) For any $F \in \mathcal{F}^0$ the operator $F(B)$ is of order 0.

(2) Let $F \in \mathcal{F}^t$ with $t < 1$ and $s \in \mathbb{R}$ be given. Then $i[F(B), r^s]$ is represented, as a quadratic form on $\mathcal{D}(F(B)) \cap L^2_{\max(0,s)}$, by

$$i[F(B), r^s] = -2s \int_{\mathcal{C}} (B - z)^{-1} (\omega^2 r^{s-1}) (B - z)^{-1} d\mu_F(z).$$

In particular $i[F(B), r^s]$ is of order $s - 1$. 

6.3.2. First commutator with $B$ and the Mourre estimate  Here we are going to compute the commutator $i[H, B]$, and bound it from below. We define $i[H, B]$ first as a (bounded) quadratic form on $\mathcal{H}^2$:

$$(i[H, B])_\psi = 2(\text{Im}(BH))_\psi = i(H\psi, B\psi) - i(B\psi, H\psi) \quad \forall \psi \in \mathcal{H}^2.$$  (6.6)

We recall that $\omega := \text{grad } r$ and let

$$\tilde{\omega} = \frac{1}{2} \text{grad } r^2, \quad \tilde{h} = \frac{1}{2} \text{Hess } r^2, \quad h = \text{Hess } r.$$  

Then formal computations suggest that

$$B := i[H, r],$$

$$A := \frac{1}{2} [H, r^2] = r^{1/2} Br^{1/2},$$

$$DA := i[H, A] = 4p \cdot \tilde{h} r - \frac{1}{2} (\Delta^2 r^2) - 2\tilde{\omega} \cdot (\nabla V),$$

$$DB := i[H, B] = r^{-1/2}(i[H, A] - B^2)r^{-1/2} + r^{-2}\omega \cdot h\omega.$$  

Thus we could expect that $i[H, B]$ extends continuously to larger spaces, and this is justified in the following lemma, which in turn partly justifies the above computations.

**Lemma 6.6.** Denoting the extension of the quadratic form $DB = i[H, B]$ of (6.6) by the same notation, it is expressed as

$$DB = \text{s-lim}_{t \to 0} t^{-1}(He^{itB} - e^{itB}H) \quad \text{in } \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2})$$  (6.7a)

and, more explicitly, as

$$DB = r^{-1/2}(L - B^2)r^{-1/2},$$  (6.7b)

where $L \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2})$ is given by

$$L = 4p \cdot \tilde{h} r - \frac{1}{2} (\Delta^2 r^2) + r^{-1}\omega \cdot h\omega$$

$$+ 2 \sum_{a \in A} \left( -\tilde{\omega}^a \cdot (\nabla^a V_{\text{lr}}) + (V^a_{\text{st}} \tilde{\omega}^a) \cdot \nabla^a - \nabla^a \cdot (V^a_{\text{st}} \tilde{\omega}^a) + V^a_{\text{st}} \text{div } \tilde{\omega}^a \right).$$  (6.7c)

Here $\tilde{\omega}^a$ and $\nabla^a$ for any $a \in A$ denote the projection onto the internal components of $\tilde{\omega}$ and $\nabla$, respectively.

Note that formally

$$L = DA + r^{-1}\omega \cdot h\omega = DA + O(r^{-1}).$$

Whence we may use the following version of the Mourre estimate, cf. [AIIS]. For $\lambda \in \mathbb{R} \setminus T(H), \lambda > \lambda_c := \min \sigma_c(H) = \min \sigma_{\text{ess}}(H)$, the notation $d(\lambda)$ is used for the distance from $\lambda$ to the biggest threshold of $H$ below $\lambda$.

**Lemma 6.7.** For any $\lambda \in (\lambda_c, \infty) \setminus T(H)$ and $\epsilon > 0$ there exist a neighbourhood $\mathcal{U}$ of $\lambda$ and a compact operator $K$ on $\mathcal{H}$, such that for all real $f \in C_c^\infty(\mathcal{U})$

$$f(H)Lf(H) \geq f(H)(4d(\lambda) - \epsilon - K)f(H).$$

Here, strictly speaking, $r$ is meant as the rescaled version $r_R = r_R(x) := Rr(x/R)$ with $R = R(\lambda, \epsilon) \geq 1$ taken sufficiently large, and also $\mathcal{U}$ and $K$ may depend on $R$.  

We will in Sect. 7 implement Lemma 6.7 in combination with Lemma 6.6 in the following form, see also (6.13).

**Corollary 6.8.** For any \( \lambda \in (\lambda_c, \infty) \setminus \mathcal{T}(H) \) there exist \( \sigma > 0 \) and a neighbourhood \( \mathcal{U} \) of \( \lambda \): For any real \( f \in C_c^\infty(\mathcal{U}) \) there exists \( C > 0 \) such that (as quadratic forms on \( \mathcal{H} \))

\[
f(H)(DB)f(H) \geq f(H)r^{-1/2}(\sigma^2 - B^2)r^{-1/2}f(H) - Cr^{-2}. \tag{6.8}
\]

Here the strict meaning is the bound with \( r \) replaced with the rescaled version \( r_R \) with \( R \geq 1 \) taken sufficiently large (as in Lemma 6.7, and \( \mathcal{U} \) and \( C \) may depend on \( R \)).

Finally we compute and bound commutators of functions of \( H \) and \( B \).

**Lemma 6.9.** Let \( f \in \mathcal{F}^t \) and \( F \in \mathcal{F}^{t'} \) with \( t < -1/2 \) and \( t' < 1 \), respectively. Then the commutators \( i[f(H), B] \) and \( i[f(H), F(B)] \) extend from \( \mathcal{D}(B) \) and \( \mathcal{D}(F(B)) \) to bounded quadratic forms on \( \mathcal{H} \), represented as

\[
i[f(H), B] = -\int_C (H - z)^{-1}(DB)(H - z)^{-1}d\mu_f(z) \quad \text{and} \quad i[f(H), F(B)] = -\int_C (B - z)^{-1}(i[f(H), B])(B - z)^{-1}d\mu_{F}(z),
\]

respectively. Moreover, with the notation (6.2)

\[
i[f(H), B] = \mathcal{O}(r^{-1}) \quad \text{and} \quad i[f(H), F(B)] = \mathcal{O}(r^{-1}). \tag{6.9}
\]

6.3.3. **Second commutator \( \text{ad}^2_{iB}(\cdot) \)** Here we provide a realization of the second commutator \( i[DB, B] \), and bound it in some operator space. Although one can explicitly compute this second commutator, this will not be needed.

Note that by Lemmas 6.4 and 6.6 we may consider

\[
(DB)e^{itB} - e^{itB}(DB) \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2}).
\]

**Lemma 6.10.** There exists the strong limit

\[
\text{ad}^2_{iB}(H) = i[DB, B] := s-lim_{t \to 0} t^{-1}(DB)e^{itB} - e^{itB}(DB)) \quad \text{in} \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2}).
\]

Moreover

\[
(H - i)^{-1}\text{ad}^2_{iB}(H)(H + i)^{-1} = \mathcal{O}(r^{-1-\mu}). \tag{6.10}
\]

Finally, as a continuation of Lemmas 6.9 and 6.10, we consider a second commutator of \( B \) and a function of \( H \).

**Lemma 6.11.** For any \( f \in \mathcal{F}^t \) with \( t < -1 \) the second commutator

\[
\text{ad}^2_{iB}(f(H)) = i[i[f(H), B], B]
\]

extends from \( \mathcal{D}(B) \) to a bounded quadratic form on \( \mathcal{H} \), represented as

\[
\text{ad}^2_{iB}(f(H)) = -\int_C (H - z)^{-1}\text{ad}^2_{iB}(H)(H - z)^{-1}d\mu_f(z)
+ 2\int_C (H - z)^{-1}(DB)(H - z)^{-1}(DB)(H - z)^{-1}d\mu_f(z).
\]

In particular

\[
\text{ad}^2_{iB}(f(H)) = \mathcal{O}(r^{-1-\mu}). \tag{6.11}
\]
6.3.4. Commutation with $A$  By combining Lemmas 6.9–6.11 we obtain the following result.

**Lemma 6.12.** Let $f \in \mathcal{F}^t$ and $F \in \mathcal{F}^{t'}$ with $t < -1$ and $t' < 1$, respectively. Suppose also that $F' \geq 0$ with $\sqrt{F'} \in \mathcal{F}^0$. Then

\[
\mathbf{i}[f(H), F(B)] = \sqrt{F'}(B)[\mathbf{i}[f(H), B]]\sqrt{F'}(B) + O(r^{-1}).
\]

In particular for any real functions $f, g$ and $F$ with $F' \geq 0$, $f, g, \sqrt{F'} \in C_c^\infty(\mathbb{R})$ and given such that $g(\lambda)f(\lambda) = \lambda f(\lambda)$, 

\[
f(H)\mathbf{i}[g(H), F(B)]f(H) = \sqrt{F'}(B)f(H)(DB)f(H)\sqrt{F'}(B) + O(r^{-1}).
\]  

(6.12)

We shall in Sect. 7 use (6.12) in combination with Corollary 6.8. Note that with (6.8) and with (6.12) for $F' \in C_c^\infty(\mathbb{R})$ supported close to 0 we can infer the bound

\[
\exists \epsilon > 0 : f(H)\mathbf{i}[g(H), F(B)]f(H) \geq \epsilon f(\lambda)\sqrt{F'}(B)r^{-1}\sqrt{F'}(B)f(H) + O(r^{-1}).
\]  

(6.13)

6.4. Commutation with $M_a$. The operators $B, M = M_{a_{\text{max}}}$ and $M_a, a \in \mathcal{A}'$, are all of first order given in terms of smooth bounded globally Lipschitz vector fields. Consequently the conclusions of Lemma 6.4 also hold for $M_a, a \in \mathcal{A}$. However we shall actually not need the self-adjointness of the latter (and use their propagators for example); symmetry will suffice. However we will need to pass functions of $H$ through $M_a$. Recalling $w_a := \text{grad } m_a$ we let $w_a^b = (w_a)^b$ for any $a, b \in \mathcal{A}$.

**Lemma 6.13.** Let $f \in \mathcal{F}^t$ with $t < -1/2$ and $a \in \mathcal{A}$. Then the product $M_a f(H) = O(r^0)$, and the commutator $\mathbf{i}[f(H), M_a]$ extends from $\mathcal{H}^1$ to a bounded quadratic form on $\mathcal{H}$, represented as

\[
\mathbf{i}[f(H), M_a] = -\int_C (H - z)^{-1}\mathbf{i}[H, M_a](H - z)^{-1} d\mu_f(z);
\]

\[
\mathbf{i}[H, M_a] = 4p \cdot (\text{Hess } m_a)p - (\Delta^2 m_a)
\]

\[
+ 2 \sum_{b \in \mathcal{A}} \left( -w_a^b \cdot (\nabla^b V_{1r}^b) + (V^b_{sr}w_a^b) \cdot \nabla^b - V^b_{sr}w_a^b + V^b_{sr} \text{div } w_a^b \right).
\]

In particular (with the notation (6.2))

\[
\mathbf{i}[f(H), M_a] = O(r^{-1}),
\]  

(6.14a)

and

\[
\mathbf{i}[f(H), M_a] - 4f'(H)p \cdot (\text{Hess } m_a)p
\]

\[
= \mathbf{i}[f(H), M_a] + 4\int_C (H - z)^{-1}p \cdot (\text{Hess } m_a)p(H - z)^{-1} d\mu_f(z)
\]

\[
+ O(r^{-1-\mu}) = O(r^{-1-\mu}).
\]  

(6.14b)

In applications the second term to the left in (6.14b) for $a \in \mathcal{A}'$ will be controlled by (4.4a) and (4.4b), i.e. by such term for $a = a_{\text{max}}$. 


6.5. Commutation with $A^a_3$, $a \neq a_{\text{min}}$. Here we examine the operator $A^a_3$ appearing as a factor in $N^a_{\pm}$. For convenience we only look at the ‘plus’ case, i.e. we study the factor $A^a_3$ appearing as a factor in $N^a_{+}$. The ‘minus’ case may be treated similarly.

Clearly the operators

$$B_1 := B^a_3,$$

$$B_2 := B^a_{\delta, \rho_1} = r^{\rho_1/2}B^a_\delta r^{\rho_1/2} = 2\Im \left( r^{\rho_1} \omega^a_\delta \cdot p^a \right) \tag{6.15}$$

are of first order given with smooth globally Lipschitz vector fields. As for $B_1$ the vector field is bounded as well, and consequently all of the conclusions of Lemma 6.4 hold upon replacing $B$ by $B_1$. The operators $B_1$ and $B_2$ are essentially self-adjoint on $C^\infty_c(X)$ and the analogue of (6.5b) is fulfilled for both of them.

Now $A^a_3 = \chi_-(B_2)$ is well-defined and we can record that $A^a_3 \in \mathcal{O}(r^0)$, cf. Lemma 6.5. From a formula similar to the one of Lemma 6.5 2) we see that $[A^a_3, r^s]$ is of order $s - 1 + \rho_1 < s$, exemplifying good commutation properties when commuting with functions of $r$.

As for commutation with functions of $H$ we revisit Sects. 6.3.2–6.3.4 except for the part concerning the Mourre estimate. This is technically the most demanding part of Sect. 6.

Lemma 6.14. Introducing the quadratic form $D_a B_2 = i[H_a, B_2]$ as

$$D_a B_2 = \lim_{r \to 0} r^{-1} \left( H_a e^{i B_2} - e^{i B_2} H_a \right) \text{ in } \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2}), \tag{6.16a}$$

it is given by

$$D_a B_2 = r^{\rho_1/2} (T_1 + \cdots + T_5) r^{\rho_1/2};$$

$$T_1 = 4 p^a \cdot r^{-\delta} \left( \text{Hess } r^a \right) (x^a / r^\delta) p^a,$$

$$T_2 = -2 \Im \left( p^a \cdot r^{-\delta} \left( \text{Hess } r^a \right) (x^a / r^\delta) x^a r^{-1} B \right),$$

$$T_3 = \rho_1 \Im \left( B r^{-1} B^a_\delta \right),$$

$$T_4 = 2 \sum_{b \leq a} \left( \left( \omega^b_\delta \right)^a (\nabla^b V^b) + (V^b_{\text{st}} (\omega^b_\delta)^a) \cdot \nabla^b - \nabla^b \cdot (V^b_{\text{st}} (\omega^b_\delta)^a) + V^b_{\text{st}} \text{div} (\omega^b_\delta)^a \right),$$

$$T_5 = T_5(x) = \mathcal{O}(r^{-3\delta}) \text{ explicitly given by (summing over repeated indices)}:$$

$$\Delta^a_j \partial^a_i \left( r^{-\delta} \left( \text{Hess } r^a \right)_{ij} (x^a / r^\delta) \right) - \delta(\Delta r) \text{div}^a \left( r^{-1} F^a \right) - \frac{1}{2} \omega^a_j (x^a / r^\delta) \partial^a_j \{ |\omega|^2 r^{-2} \}$$

$$+ \partial^a_i \partial^a_j M_{ij} - \delta r^{-1} \partial^a_j \left( \Delta r \right) F^a_j;$$

$$F^a = F^a(x) = \{ (y^a \cdot \nabla y^a \omega^a)(y^a) \}_{y^a = x^a / r^\delta}, \quad M_{ij} = M_{ij}(x) = \delta r^{-1} \omega_i F^a_j.$$

Remarks. Although the stated exact expression for $T_5$ will not be relevant for us, we remark that the fifth term of the expression is on the form $\mathcal{O}(r^{-2})$ (and no better), however it cancels with a term from expanding the fourth term using the Leibniz rule for differentiation. The first term has the stated order $\mathcal{O}(r^{-3\delta})$, but the sum of all the others has a better order. The stated order $\mathcal{O}(r^{-3\delta})$ will by far be sufficient for our treatment of $T_5$. 

Lemma 6.15. Let \( f \in \mathcal{F}^t \) and \( F \in \mathcal{F}^{t'} \) with \( t < -1/2 \) and \( t' < 1 \), respectively. Then the commutators \( \text{i}[f(H_a), B_2] \) and \( \text{i}[f(H_a), F(B_2)] \) extend from \( \mathcal{D}(B_2) \) and \( \mathcal{D}(F(B_2)) \) to bounded quadratic forms on \( \mathcal{H} \), represented as

\[
\text{i}[f(H_a), B_2] = -\int_C (H_a - z)^{-1}(D_a B_2)(H_a - z)^{-1} \, d\mu_f(z) \quad \text{and} \quad (6.17a)
\]

\[
\text{i}[f(H_a), F(B_2)] = -\int_C (B_2 - z)^{-1}(\text{i}[f(H_a), B_2])(B_2 - z)^{-1} \, d\mu_F(z), \quad (6.17b)
\]

respectively. Moreover

\[
\text{i}[f(H_a), B_2] = \mathcal{O}(r^{\rho_1 - \delta}) \quad \text{and} \quad \text{i}[f(H_a), F(B_2)] = \mathcal{O}(r^{\rho_1 - \delta}). \quad (6.18)
\]

Ideally we would like the right-hand sides of (6.18) to be on the form \( \mathcal{O}(r^{-(1-\delta)}) \), which is not doable. However there is the following partial result of this type: Let us split

\[
D_a B_2 = D_1 + D_2;
\]

\[
D_1 = r^{\rho_1/2}(T_1 + T_2 + T_3)r^{\rho_1/2},
\]

\[
D_2 = r^{\rho_1/2}(T_4 + T_5)r^{\rho_1/2}.
\]

From the the property (ii) of Sect. 5 (with \( r \) replaced by \( r^\alpha \)) and the relations \( (1 + \mu)\delta - \rho_1 > 1 \) and \( 3\delta - \rho_1 > 1 \) it follows that

\[
-\int_C (H_a - z)^{-1} D_2 (H_a - z)^{-1} \, d\mu_f(z) = \mathcal{O}(r^{-(1-\delta)}),
\]

and therefore we conclude by plugging into (6.17a) and (6.17b) that the contribution from \( D_2 \) to \( \text{i}[f(H_a), F(B_2)] \) is on the desired form \( \mathcal{O}(r^{-(1-\delta)}) \). As for the contribution from \( D_1 \) to \( \text{i}[f(H_a), F(B_2)] \) we write

\[
\text{i}[f(H_a), F(B_2)] - \mathcal{O}(r^{-(1-\delta)})
\]

\[
= \text{Re}(T^{(1)} F'(B_2)) + \frac{1}{2} \int_C (B_2 - z)^{-2} \text{ad}_{B_2}^2(T^{(1)})(B_2 - z)^{-2} \, d\mu_F(z);
\]

\[
T^{(1)} := -\int_C (H_a - z)^{-1} D_1 (H_a - z)^{-1} \, d\mu_f(z) = \mathcal{O}(r^{\rho_1 - \delta}).
\]

The second term to the right is \( \mathcal{O}(r^{-(1-\delta)}) \), which follows from the following more complicated version of Lemma 6.11 involving

\[
T^{(2)} := -\int_C (H_a - z)^{-1} \text{ad}_{B_2}(D_1)(H_a - z)^{-1} \, d\mu_f(z)
\]

\[
+ 2 \int_C (H_a - z)^{-1} D_1(H_a - z)^{-1} D_1(H_a - z)^{-1} \, d\mu_f(z) = \mathcal{O}(r^{2(\rho_1 - \delta)}).
\]

For convenience we write in the remaining part of the section \( T \approx T' \) for any given operators \( T, T' \in \mathcal{L}(\mathcal{H}) \), if \( T - T' = \mathcal{O}(r^{-(1-\delta)}) \).
Lemma 6.16. Let $T^{(1)}$ and $T^{(2)}$ be given as above for $f \in \mathcal{F}^t$ with $t < -1/2$. Then there exists the strong limits

$$\text{ad}_{iB_2}(T^{(j)}) = i[T^{(j)}, B_2] := \lim_{t \to 0} t^{-1}(T^{(j)} e^{itB_2} - e^{itB_2} T^{(j)}) \text{ in } \mathcal{L}(\mathcal{H}); \quad j = 1, 2.$$ 

Moreover

$$\text{ad}_{iB_2}(T^{(1)}) = T^{(2)} + \mathcal{O}(r^{(-1)^-}),$$

and for the above second term we can substitute

$$\text{ad}_{iB_2}^2(T^{(1)}) = \text{ad}_{iB_2}(T^{(2)}) + i(\text{ad}_{iB_2}(T^{(1)}) - T^{(2)}) B_2 - iB_2(\text{ad}_{iB_2}(T^{(1)}) - T^{(2)})$$

$$\simeq \mathcal{O}(r^{(-1)^-}) + B_2 \mathcal{O}(r^{(-1)^-}).$$

Proof. We compute (for the very last assertion)

$$\text{ad}_{iB_2}(T^{(2)}) = -\int_{\mathcal{C}} (H_a - z)^{-1} \text{ad}_{iB_2}^2(D_1)(H_a - z)^{-1} \mu_f(z)$$

$$+ 3 \int_{\mathcal{C}} (H_a - z)^{-1} \text{ad}_{iB_2}(D_1)(H_a - z)^{-1} D_1(H_a - z)^{-1} \mu_f(z)$$

$$+ 3 \int_{\mathcal{C}} (H_a - z)^{-1} D_1(H_a - z)^{-1} \text{ad}_{iB_2}(D_1)(H_a - z)^{-1} \mu_f(z)$$

$$- 6 \int_{\mathcal{C}} (H_a - z)^{-1} D_1(H_a - z)^{-1} D_1(H_a - z)^{-1}$$

$$D_1(H_a - z)^{-1} \mu_f(z) + \mathcal{O}(r^{(-1)^-})$$

$$\simeq \mathcal{O}(r^{3(\rho_1 - \delta)}) \simeq 0.$$ 

By commutation – of the same sort, continuing the above calculations – we conclude the following result.

Lemma 6.17. Let $f \in \mathcal{F}^t$ and $F \in \mathcal{F}^{t'}$ with $t < -1/2$ and $t' < 1$, respectively. Suppose also that $F' \leq 0$ with $\sqrt{-F'} \in \mathcal{F}^0$. Then

$$i[f(H_a), F(B_2)] \simeq -\sqrt{-F'}(B_2)[i[f(H_a), B_2]] \sqrt{-F'}(B_2).$$

In particular for any real functions $f, g$ and $F$ with $F' \leq 0$, $f, g, \sqrt{-F'} \in C^\infty_c(\mathbb{R})$ and given such that $g(\lambda)f(\lambda) = \lambda f(\lambda)$,

$$f(H_a)i[g(H_a), F(B_2)]f(H_a)$$

$$\simeq -\sqrt{-F'}(B_2)f(H_a)[i[g(H_a), B_2]]f(H_a)\sqrt{-F'}(B_2)$$

$$\simeq -\sqrt{-F'}(B_2)f(H_a)D_1f(H_a)\sqrt{-F'}(B_2). \quad (6.19a)$$
Effectively the right-hand side of (6.19a) will contribute by a negative term. To see this we first write
\[
T_1 + T_2 = 4\left(p^a - \frac{\delta}{4} B \frac{x^a}{r}\right) \cdot r^{-\delta} \left(\text{Hess } r^a\right) \left(p^a - \frac{\delta}{4} \frac{x^a}{r} B\right) - \frac{\delta^2}{4} B \frac{x^a}{r} \cdot r^{-\delta} \left(\text{Hess } r^a\right) \left(p^a - \frac{\delta}{4} \frac{x^a}{r} B\right) \frac{x^a}{r} B.
\]

Using (5.2c) this leads to
\[
f(H_a)r^{\rho_1/2}(T_1 + T_2)r^{\rho_1/2} f(H_a) = f(H_a)r^{\rho_1/2} \left(4\left(p^a - \frac{\delta}{4} B \frac{x^a}{r}\right) \cdot r^{-\delta} \left(\text{Hess } r^a\right) \left(p^a - \frac{\delta}{4} \frac{x^a}{r} B\right)\right) f(H_a) + O\left(r^{\rho_1+\delta-2}\right).
\]

Now the first term to the right is positive, and the last term $O\left(r^{\rho_1+\delta-2}\right) = O\left(r^{(-1)-}\right)$.

We are only interested in $f \in C_{c}^\infty(\mathbb{R})$ taking values in [0, 1]. For convenience let us henceforth impose this condition. The contribution from $T_3$ is of order $O\left(r^{-1}\right)$ (here we use that $F'$ is compactly supported), but with a localization to $B > 0$ and provided $\sqrt{-F'} \in C_{c}^\infty(\mathbb{R}_+)$ we can again find a better lower bound. More precisely we have the following result for which it remains to be noted that the right-hand side of (6.19c) is positive.

**Lemma 6.18.** Suppose $\zeta := \sqrt{-F'} \in C_{c}^\infty(\mathbb{R}_+)$. Then for any $\epsilon > 0$ and with $\eta(b) := \sqrt{b} \chi_+(b/\epsilon)$ and $\zeta_1(b) := \sqrt{b} \zeta(b)$,

\[
\chi+(B/\epsilon)\zeta(B_2) f(H_a)r^{\rho_1/2} T_3 r^{\rho_1/2} f(H_a)\zeta(B_2) \chi+(B/\epsilon) \simeq \rho_1 f(H_a)\eta(B)\zeta_1(B_2) r^{-1} \zeta_1(B_2) \eta(B) f(H_a).
\]

**Proof.** Pick $\tilde{f} \in C_{c}^\infty(\mathbb{R})$ such that $f < \tilde{f}$, and let $\tilde{B}_a = \tilde{f}(H_a)B \tilde{f}(H_a)$. Then by commutation

\[
T := \chi+(B/\epsilon)\zeta(B_2) f(H_a)r^{\rho_1/2} \eta(B^-r^{-1} B^a) f(H_a)\zeta(B_2) \chi+(B/\epsilon) \simeq f(H_a)\chi+(B/\epsilon) \eta(\zeta(B_2)\tilde{B}_a r^{-1} B_2 \zeta(B_2)) \chi+(B/\epsilon) f(H_a).
\]

It follows from the first bound of (6.18) and the bound $\tilde{f}(H_a)[B_2, B] \tilde{f}(H_a) = O\left(r^{\rho_1-\delta}\right)$ that

\[
[\zeta(B_2), \tilde{B}_a] = - \int_{C} (B_2 - z)^{-1}[B_2, \tilde{B}_a](B_2 - z)^{-1} d\mu_{\zeta}(z) = O\left(r^{\rho_1-\delta}\right).
\]

Whence

\[
\eta(\zeta(B_2)\tilde{B}_a r^{-1} B_2 \zeta(B_2)) = \eta(\tilde{B}_a \zeta(B_2) r^{-1} B_2 \zeta(B_2)) + \eta(\zeta(B_2), \tilde{B}_a) r^{-1} B_2 \zeta(B_2)) = \eta(\tilde{B}_a r^{-1} \zeta^2(B_2)) + O\left(r^{\rho_1-\delta-1}\right),
\]

and we infer that

\[
T \simeq f(H_a)\chi+(B/\epsilon) \eta(\tilde{B}_a r^{-1} \zeta^2(B_2)) \chi+(B/\epsilon) f(H_a) \simeq f(H_a) \eta(\chi+(B/\epsilon) Br^{-1} \zeta^2(B_2) \chi+(B/\epsilon)) f(H_a).
\]
Next we write
\[
\mathcal{H}(\chi_+(B/\epsilon)Br^{-1}\zeta_1^2(B_2)\chi_+(B/\epsilon)) = \mathcal{H}(\eta^2(B)r^{-1}\zeta_1^2(B_2)) + \mathcal{H}(\chi_+(B/\epsilon)B[r^{-1}\zeta_1^2(B_2), \chi_+(B/\epsilon)]),
\]
and (for the second term)
\[
[r^{-1}\zeta_1^2(B_2), \chi_+(B/\epsilon)]
\]
\[
= [r^{-1}, \chi_+(B/\epsilon)]\zeta_1^2(B_2) + r^{-1}[\zeta_1^2(B_2), \chi_+(B/\epsilon)]
\]
\[
= \mathcal{O}(r^{-2}\zeta_1^2(B_2) + r^{-1}[\zeta_1^2(B_2), \chi_+(B/\epsilon)]).
\]

Here the first term contributes by a term on the form $\mathcal{O}(r^{(-1)-})$, so we are left with
\[
T \simeq f(H_a)\mathcal{H}(\eta^2(B)r^{-1}\zeta_1^2(B_2))f(H_a)
\]
\[
- f(H_a)\mathcal{H}\left(\chi_+(B/\epsilon)Br^{-1}[\chi_+(B/\epsilon), \zeta_1^2(B_2)]\right)f(H_a).
\]

The first term
\[
f(H_a)\mathcal{H}(\eta^2(B)r^{-1}\zeta_1^2(B_2))f(H_a)
\]
\[
\simeq f(H_a)\mathcal{H}(\eta(B)\zeta_1^2(B_2)r^{-1}\zeta_1(B_2)\eta(B))f(H_a)
\]
\[
+ f(H_a)\mathcal{H}(\eta(B)r^{-1}[\eta(B), \zeta_1^2(B_2)])f(H_a),
\]
and here in turn the first term is what we need for (6.19c).

It remains to show that
\[
- f(H_a)\mathcal{H}\left(\chi_+(B/\epsilon)Br^{-1}[\chi_+(B/\epsilon), \zeta_1^2(B_2)]\right)f(H_a)
\]
\[
+ f(H_a)\mathcal{H}(\eta(B)r^{-1}[\eta(B), \zeta_1^2(B_2)])f(H_a) \simeq 0.
\]

To deal with these commutators we let $\tilde{Z}_a = \tilde{f}(H_a)Z\tilde{f}(H_a)$, $Z = \zeta_1^2(B_2)$. By (6.9) it suffices to show that
\[
[\chi_+(B/\epsilon), \tilde{Z}_a], [\eta(B), \tilde{Z}_a] = \mathcal{O}(r^{0-}).
\]  

(6.20b)

Let us only do the second bound. We use (6.9) and (6.18) computing in the last step exactly as in (6.20a)
\[
[\eta(B), \tilde{Z}_a] = -\int_{\mathbb{C}} (B - z)^{-1}[B, \tilde{Z}_a](B - z)^{-1}d\mu_{\eta}(z)
\]
\[
= -\int_{\mathbb{C}} (B - z)^{-1}[\tilde{B}_a, \zeta_1^2(B_2)](B - z)^{-1}d\mu_{\eta}(z) + \mathcal{O}(r^{\rho_1-\delta})
\]
\[
= \mathcal{O}(r^{\rho_1-\delta}) = \mathcal{O}(r^{0-}).
\]
Remark 6.19. The procedure in the proof of Lemma 6.18 of regularizing by replacing an operator \( T \) by \( \tilde{T}_a = \tilde{f}(H_a) T \tilde{f}(H_a) \) before computing a commutator with \( T \) by the Helffer–Sjöstrand formula will be used frequently in the remaining part of the paper. We are dealing with various examples of first order operators, most typically \( B, B_{a,\delta,\rho_1}^a \) and \( \xi_j^+(x) G_{d,j} \), that along with functions of \( B \) and \( B_{a,\delta,\rho_1}^a \) at various points need to be commuted. The technical problem one should avoid (and possibly would encounter upon ‘blindly’ applying the Helffer–Sjöstrand formula) is the appearance of certain (presumably unbounded) commutators without a proper weight. Thus for example the formal commutator \([B, B_{a,\delta,\rho_1}^a]\) is of first order but does not seem relatively bounded neither to \( B \) nor to \( B_{a,\delta,\rho_1}^a \). In the proof of Lemma 6.18 the \( \tilde{T}_a \)-construction ‘cured’ this problem providing \( H_a \)-weights. We used ‘free’ factors of \( \tilde{f}(H_a) \) at our disposal and the feature that commutation with those was relatively harmless. We will proceed similarly later (mostly with cases where \( H_a \) is replaced by \( H \)), although typically without giving the details.

We have computed the commutator (6.19a) as a sum of two negative operators plus an operator on the form \( O(r^{(1)-}) \). This is with a localization to \( B > 0 \) and for \( \sqrt{-F^r} \in C^\infty_C(\mathbb{R}_+) \), which will be conditions met in our treatment of \( A_{a,\delta}^a \) in the ‘plus’ case. We shall in applications need commutation with \( A_{a,\delta}^a \), however with \( H_a \) replaced by \( H \). The presence of factors of \( M_a \) enables us to reduce to the case studied above. Note that thanks to (4.5) for any real \( f \in C^\infty_C(\mathbb{R}) \)

\[
(f(H_a) - f(H))\bar{\xi}_a^+ = O(r^{-\mu}),
\]

and note that \( O(r^{-\mu})O(r^{\rho_1-\delta}) = O(r^{(1)-}) \) (the second factor appears in (6.18)). Alternatively, one can essentially just mimic the above arguments with \( H_a \) replaced by \( H \); the additional complication is minor thanks to (4.5).

6.6. Commutation with \( A_{a,\delta}^a \), \( a \neq a_{\text{min}} \). Recalling

\[
A_{a,\delta}^a = \chi_-(r^\rho - 1) r_\delta^a \quad \text{with} \quad r_\delta^a = r^\delta r^a(r^\delta),
\]

we compute for real \( f \in C^\infty_C(\mathbb{R}) \)

\[
i[f(H), A_{a,\delta}^a] = -2 \int_C (H - z)^{-1} \Re \left( \chi'_-(\cdot) \nabla (r^\rho - 1) r_\delta^a \cdot p \right)(H - z)^{-1} \, d\mu_f(z)
= - \int_C (H - z)^{-1} (S_1 + S_2 + S_3) (H - z)^{-1} \, d\mu_f(z);
\]

where

\[
S_1 = \Re \left( \chi'_-(\cdot) r^{(\rho - 1)} B^a_\delta \right),
S_2 = (\rho - 1) \Re \left( \chi'_-(\cdot) r^{\rho - 2} r_\delta^a B \right),
S_3 = \delta \Re \left( \chi'_-(\cdot) r^{\rho + \delta - 2} (r^{-\delta} r_\delta^a - \omega_r^a \cdot r^{-\delta}) B \right).
\]

Due to (5.2b) we can write

\[
S_3 = \Re \left( O(r^{\rho + \delta - 2}) B \right) = \Re \left( O(r^{(1)-}) B \right),
\]

and consequently the contribution from \( S_3 \) to the commutator is of order \( O(r^{(1)-}) \). On the other hand the contributions from \( S_1 \) and \( S_2 \) are of order \( O(r^{\rho - 1}) \) and \( O(r^{-1}) \), respectively, and no better.
Applied to real $f, g \in C^\infty_c(\mathbb{R})$ with $g(\lambda) f(\lambda) = \lambda f(\lambda)$ (as we used for (6.12) and (6.19a)) we estimate with $\eta(b) = \sqrt{b} \chi_+(b/\epsilon)$ and for any $\epsilon > 0$, $\zeta(b) = -b \chi_+(-b)$, $\chi_1(s) = - (\chi^2)'(s) = 2 \chi_+(s) \chi'_+(s) = \chi_2(s) = - s (\chi^2)'(s) = 2 s \chi_+(s) \chi'_+(s)$

$$2 \chi_+(B/\epsilon) \Re \left[ \left( f(H)i[g(H), A^a_2] f(H) \chi_-(B_2) A^{\overline{a}}_2 \right) \chi_+(B/\epsilon) \right]$$

$$\simeq 2 \chi_+(B/\epsilon) \Re \left[ \left( f(H)(S_1 + S_2) f(H) \chi_-(B_2) A^{\overline{a}}_2 \right) \chi_+(B/\epsilon) \right]$$

$$\simeq - \chi_+(B/\epsilon) f(H) r^{(\rho_2 - \rho_1 - 1)/2} \sqrt{\chi_1(\cdot) B_2 \chi_-(B_2)} \sqrt{\chi_1(\cdot) B_2} f(H) \chi_+(B/\epsilon)$$

$$+ (1 - \rho_2) \eta(B) f(H) r^{(\rho_2 - \rho_1 - 1)/2} \sqrt{\chi_1(\cdot) r^{\rho_2 - 2} B_2} \chi_1 r^{\rho_1 - 1} r^a f(H) \chi_+(B/\epsilon)$$

$$\simeq \chi_+(B/\epsilon) f(H) r^{(\rho_2 - \rho_1 - 1)/2} \sqrt{\chi_1(\cdot) B_2} \chi_1 r^{\rho_1 - 1} r^a f(H) \chi_+(B/\epsilon)$$

We commuted repeatedly functions of $B_2$ and $B$ with multiplication operators (for example $A^a_2$). Using the fact that $\rho_1 < 1/3$ this is rather harmless (we do not give the details). Note also that the above treatment of $S_2$ used commutation of functions of $B$ and $\chi_-(B_2)$. This is implemented as explained in Remark 6.19 (with $H_a$ replaced by $H$).

To the right of the computation (6.22) both terms are positive.

7. Computation of $T^\pm_a$ and Existence of $\Omega^\pm_a$

Throughout the rest of the paper we fix any $f_4 \in C^\infty_c(\Lambda)$ such that in comparison with the functions $f_1 < f_2 < f_3$ from Sect. 5.1 (also henceforth fixed), $f_3 < f_4$. Let then $g(\lambda) = \lambda f_4(\lambda)$. This allows us to write

$$T^\pm_a = i f_2(H) \left( g(H) M_a N^a_\pm M_a - M_a N^a_\pm M_a g(\tilde{H}_a) \right) f_2(\tilde{H}_a),$$

and with this representation we can show that $T^\pm_a = O(r^{\rho_1 - \delta})$ in the sense of Sect. 6.1. Furthermore we can show that all terms in an expansion that are not on the form $O(r^{-1 - 2\epsilon})$ for some $\epsilon > 0$ tend to have a sign allowing us to prove non-trivial smoothness estimates for concrete expressions that fail to be on the favourite form $O(r^{-1/2 - \epsilon})$. Note that for any $\epsilon > 0$ the operators $r^{-1/2 - \epsilon} f_2(\tilde{H}_a)$ and $r^{-1/2 - \epsilon} f_2(H)$ are $\tilde{H}_a$- and $H$-smooth in the sense of Kato (see [Ka]), cf. (5.5a).

7.1. Propagation estimates. For any $\psi \in \mathcal{H}$ and any self-adjoint operator $T$ on $\mathcal{H}$ we let $\psi(t) = e^{-itH}\psi$ and denote $(T)_t = \langle \psi(t), T \psi(t) \rangle$, $t \in \mathbb{R}$. In the proof of the following lemma we calculate modulo terms on the form $f_2(H)O(r^{(\rho_1 - 1)\cdot}) f_2(H)$. We will in this subsection write $T_1 \simeq T_2$ if $T_1 - T_2$ has this form (which is slightly different from the relation used in Sects. 6.5–6.6). Recall the smoothness bounds

$$\int_{-\infty}^{\infty} \| Q_s f_2(H) \psi(t) \|^2 dt \leq C_s \| \psi \|^2; \quad Q_s = r^{-s}, \ s \in (1/2, 1). \quad (7.1)$$

Recall that the factors $A^a_3 = A^a_3\pm$ of $N^a_\pm$ are given in terms of $B^a_{\delta, \rho_1} = r^{\rho_1/2} B^a_{\delta} r^{\rho_1/2}$ with parameters given as in (3.6). In the list below (1)–(8) there appear respective operators $Q_1, \ldots, Q_8$ (equipped with additional indices). For $a = a_{\min}$ only $Q_1$ and $Q_2$ enter. The
‘\(Q\)-operators’ all have the feature of being \(H\)-bounded, and whence (when convenient) we can make them bounded for example by looking instead at \(Q_1 f_4(H), \ldots, Q_8 f_4(H)\), respectively. None of the latter operators appears to be on the form \(O(r^{(1-1/2)})\), so the corresponding bounds in the list do not conform with (7.1). On the other hand some are on the form \(O(r^{-1/2})\), just missing the applicability of (7.1), and the ‘worst’ appearing order is \(Q f_4(H) = O(r^{(\rho_1 - \delta)/2})\).

**Lemma 7.1.** Let \(a \in \mathcal{A}\). The following bounds hold for a constant \(C > 0\) being independent of \(\psi \in \mathcal{H}\).

1. For \(j \leq J = J(a)\) (as in (4.4a))
   \[
   \int_{-\infty}^{\infty} \| Q_1 f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2; \quad Q_1 = Q_1(a, j) = \xi_j^+ (x) G_{d_j}.
   \]
2. \[
   \int_{-\infty}^{\infty} \| Q_2 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2; \quad Q_2^\pm = r^{-1/2} \sqrt{(\chi_+^2)'(\mp B/\epsilon_0)}.
   \]
3. Let \(T_1^a = \sqrt{(\chi_+^2)'(\pm B_{\delta, \rho_1}^a) \chi_+(\pm B/\epsilon_0)}\) and \(\mathcal{H}^a = (\text{Hess } r^a)(x^a / r^\delta)\). Then
   \[
   \int_{-\infty}^{\infty} \| Q_3 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2;
   \quad Q_3^\pm = Q_3^\pm(a) = 2(\mathcal{H}^a)^{1/2}(p^a - \frac{\delta x^a}{4} B) f_3(H_a) r^{(\rho_1 - \delta)/2} T_1^a M_a.
   \]
4. Let \(\eta(b) = \sqrt{b_x} \chi_+(b/\epsilon_0)\), \(\zeta_1(b) = b(\chi_+^2)'(b)\) and \(T_2^a = \zeta_1(\pm B_{\delta, \rho_1}^a) \eta(\pm B)\). Then
   \[
   \int_{-\infty}^{\infty} \| Q_4 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2;
   \quad Q_4^\pm = Q_4^\pm(a) = \rho_{1/2} r^{-1/2} T_2^a M_a.
   \]
5. Let \(T_{1+}^a = \sqrt{(\chi_+^2)'(\pm B_{\delta, \rho_1}^a) \chi_+(r_{\rho_2-1}^a r_\delta^a) \chi_+(\pm B/\epsilon_0)}\) and \(\mathcal{H}^a\) be given as in 3). Then
   \[
   \int_{-\infty}^{\infty} \| Q_5 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2;
   \quad Q_5^\pm = Q_5^\pm(a) = 2(\mathcal{H}^a)^{1/2}(p^a - \frac{\delta x^a}{4} B) f_3(H_a) r^{(\rho_1 - \delta)/2} T_{1+}^a M_a.
   \]
6. Let \(\eta(b) = \sqrt{b_x} \chi_+(b/\epsilon_0)\), \(\zeta_1(b) = b(\chi_+^2)'(b)\) (as in 4)) and \(T_{2+}^a = \zeta_1(\pm B_{\delta, \rho_1}^a) \chi_+(r_{\rho_2-1}^a r_\delta^a) \eta(\pm B)\). Then
   \[
   \int_{-\infty}^{\infty} \| Q_6 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2;
   \quad Q_6^\pm = Q_6^\pm(a) = \rho_{1/2} r^{-1/2} T_{2+}^a M_a.
   \]
7. Let \(\xi_2(b) = \sqrt{- \chi_+(-b)}\), \(T_3^a = \xi_2(\pm B_{\delta, \rho_1}^a) f_3(H_a) \chi_+(\pm B/\epsilon_0)\) and \(\chi_{1+}(s) = 2 \chi_+(s) \chi_4'(s)\). Then
   \[
   \int_{-\infty}^{\infty} \| Q_7 \pm f_2(H) \psi(t) \|^2 \, dt \leq C \| \psi \|^2;
   \quad Q_7^\pm = Q_7^\pm(a) = r^{(\rho_2-\rho_1-1)/2} \sqrt{\chi_{1+}(r_{\rho_2-1}^a r_\delta^a)} T_3^a M_a.
   \]
(8) Let \( \eta(b) = \sqrt{b}\chi_+(b/\epsilon_0) \) (as in 4)), \( T_4^a = \chi_-\left( \pm B_{\delta,\rho_1}\right) \eta(\pm B) \) and \( \chi_2(s) = 2s\chi(s)\chi'_+(s) \). Then

\[
\int_{-\infty}^{\infty} \| Q_{\delta}^\pm f_2(H)\psi(t) \|^2 dt \leq C \| \psi \|^2;
\]

\[
Q_{\delta}^\pm = Q_\delta^\pm(a) = (1 - \rho_2)^{1/2}r^{-1/2}\sqrt{\chi_2(s)}(r^{\rho_2-1}r_\delta^a)T_4^aM_a.
\]

**Proof.** We only consider the upper sign case. The lower sign case assertions can be demonstrated similarly. We shall use a well-known positive commutator technique, based on 'propagation observables' denoted generically by \( \Psi \).

**(1):** We choose \( m = m_j \) conforming with (4.4b) and let \( M = M_j \) be the corresponding operator defined in (4.3). Applying (6.14b) and (4.4b) we can then estimate

\[
if_2(H)[g(H), M]f_2(H)
\]

\[
\simeq 4 f_2(H)p \cdot (\chi_+(|x|) \text{Hess} m)p f_2(H)
\]

\[
\geq 4 f_2(H)G_{d_j}^\ast \chi_+(|x|)\xi_j^+(x)^2G_{d_j}f_2(H).
\]

Let

\[
\Psi_{1,j} = f_2(H)M_jf_2(H)
\]

Since \( \Psi_1 = \Psi_{1,j} \) is bounded, (1) follows from integration of \( \frac{d}{dt}\langle \Psi_1 \rangle_t = \langle i[g(H), \Psi_1] \rangle_t \) using the above estimation and (7.1).

**(2):** Let

\[
\Psi_2 = f_2(H)\chi_+^2(B/\epsilon_0)f_2(H).
\]

We obtain (2) by integrating \( \frac{d}{dt}\langle \Psi_2 \rangle_t = \langle i[g(H), \Psi_2] \rangle_t \) using (6.13), (7.1) and the boundedness of \( \Psi_2 \). Here (6.13) is used with \( f = f_3 \). Upon multiplying both sides of that estimate from the left and from the right by a factor \( f_2(H) \) the second term to the right gets the good form \( \simeq_1 0 \).

**(3), (4):** Let \( A_1 = \chi_+(B/\epsilon_0), A_2^a = \chi_-\left( B_{\delta,\rho_1}\right) \) (as in (3.5) for the ‘plus case’) and

\[
\Psi_3 = f_2(H)M_a A_1(A_2^a)^2 A_1 M_a f_2(H).
\]

We obtain (3) and (4) by integrating \( \frac{d}{dt}\langle \Psi_3 \rangle_t \) using (1), (2), (7.1), the boundedness of \( \Psi_3 \) and by using (6.19a)–(6.21) (as well as the arguments for the latter assertions and the accompanying comments). By the product rule for commutation there are several terms. Let us first treat the contribution from \( i[g(H), M_a] \): We write using (6.14b), (4.4a) and Remark 6.19

\[
2 f_2(H)\Re\langle [i[g(H), M_a]A_1(A_2^a)^2 A_1 M_a]f_2(H) \rangle
\]

\[
\simeq_1 8 f_2(H)\Re\langle p \cdot (\text{Hess} m_a)p A_1(A_2^a)^2 A_1 M_a \rangle f_2(H)
\]

\[
\simeq_1 8 \sum_{j \leq J} f_2(H)\Re\langle G_{d_j}^\ast \xi_j^+G_j\left(f_3(H)\xi_j^+G_{d_j}f_3(H)\right)A_1(A_2^a)^2 A_1 M_a \rangle f_2(H)
\]

\[
\simeq_1 \sum_{j \leq J} T_j; \quad T_j = f_2(H) G_{d_j}^\ast \xi_j^+G_j^\prime \xi_j^+G_{d_j}f_2(H) \text{ with } G_j^\prime \text{ bounded.}
\]

Indeed the last expression arises by commuting the factor \( f_3(H)\xi_j^+G_{d_j}f_3(H) \) to the right producing an error on the form \( \mathcal{O}(r^{\rho_1-\delta-1}) = \mathcal{O}(r^{(-1)-}) \) along the pattern of
Remark 6.19. Now we apply the Cauchy–Schwarz inequality and (1) to each term $T_j$ on the right-hand side.

As for the contribution from $i[g(H), A_1]$ we use (6.12), writing
\[ f_2(H) g_1(H) A_1(A_3^q A_1 A_1) f_2(H) \]
\[ \simeq 2 f_2(H) g_1(H) A_1(f_3(H) A_3^q A_1) f_2(H) \]
\[ \simeq 2 f_2(H) g_1(H) A_1 f_3(H) A_3^q A_1 f_2(H) \]
\[ \simeq 2 f_2(H) \sqrt{\left( \frac{B}{\epsilon_0} \right)} \bigg( \pm B/\epsilon_0 \bigg) O(r^{-1}) \sqrt{\left( \frac{B}{\epsilon_0} \right)} f_2(H). \]
The right-hand side is treated by 2).

Next let us elaborate on how (6.19a)–(6.21) are used for treating the contribution from $i[g(H), A_3^q]$. We need to compute
\[ f_2(H) A_1 i[g(H), A_3^q A_1] f_2(H). \]

Modulo a term in $O(r^{-1})$ this quantity is given by
\[ f_2(H) g_1(H) A_1 i[g(H), A_3^q A_1] f_2(H) \]
\[ \simeq 1 f_2(H) A_1 f_3(H) i[g(H), A_3^q A_1] f_2(H). \]

Note that any derivative of $\tilde{g}^a_3$ will contribute by an error on the form $O(r^{-\infty})$ due to (4.5). Then we apply the computations of Sect. 6.5 (keeping for (3) the factors of $f_3(H_a)$).

(5)–(8): With $A_1$ and $A_3^q$ as above we consider
\[ \Psi_4 = f_2(H) A_1 \chi_+ \left( r^{-r^2-1} r^3 \right) A_1 A_3^q A_1 f_2(H). \]
We compute and integrate $\frac{d}{dr}$ $\langle \Psi_4 \rangle$, using again (1), (2) and (7.1) as well as the boundedness of $\Psi_4$. We got from the above computation of $i[g(H), A_3^q]$ effectively negative main terms. There is an additional contribution, more precisely from the two appearances of $i[g(H), A_3^q]$ and this is also effectively negative (contributing by two additional negative terms), cf. the treatment of $i[g(H), A_3^q]$ in Sect. 6.6. Since the four non-trivial contributing terms come out with the same sign we conclude (5)–(8). \]

Remark 7.2. (i) Due to (6.21) the appearing factor of $f_3(H_a)$ in (3), (5) and (7) can be replaced by $f_3(H_a)$ (or by $f_3(H_a)$, cf. (ii) below). (This was actually implicitly used in the above proof.) The factor serves as a regularization indeed making $Q_5^\pm(a), Q_5^\pm(a)$ and $Q_7^\pm(a)$ bounded relatively to $H$ and can (presumably) be removed. As we already discussed we can make all of the ‘$Q$-operators’ bounded (when convenient) by the energy localization procedure.

(ii) There are very similar bounds for $\tilde{H}_a$. Denoting $\varphi(t) = e^{-ir \tilde{H}_a} \varphi$ for $\varphi \in \mathcal{H}$ we can replace $f_2(H) \psi(t)$ in (1)–(8) by $f_2(\tilde{H}_a) \varphi(t)$ and the expression $\| \psi(t) \|^2$ to right of the estimates by $\| \varphi \|^2$. The proof of these modifications of (1)–(8) is essentially the same. Note that the analogue of (7.1) reads, cf. (5.5a),
\[ \int_{-\infty}^{\infty} \| Q_s f_2(\tilde{H}_a) \varphi(t) \|^2 dt \leq C_s \| \varphi \|^2; \quad Q_s = r^{-s}, \ s \in (1/2, 1). \] (7.2)

The proof is based on modifying the above propagation observables $\Psi_{1,j}, j \leq J, \Psi_2, \Psi_3$ and $\Psi_4$ by replacing the factors of $f_2(H)$ by factors of $f_2(\tilde{H}_a)$. These modified propagation observables, say denoted by $\Phi_{1,j}, j \leq J, \Phi_2, \Phi_3$ and $\Phi_4$ respectively, will also be useful in the proof of Lemma 8.3.
(iii) The structure of the bounds (1)–(8) and the analogous ones mentioned in (ii) is given as
\[
\int_{-\infty}^{\infty} \left\| Q^\pm f_2(H)\psi(t) \right\|^2 \, dt \leq C \left\| \psi \right\|^2,
\]
\[
\int_{-\infty}^{\infty} \left\| Q^\pm f_2(\tilde{\mathcal{H}}_a)\psi(t) \right\|^2 \, dt \leq C \left\| \psi \right\|^2,
\]
respectively, where \( Q^\pm \) may (or may not) depend on \( a \) and \( j \leq J = J(a) \). In addition we also have such bounds with \( Q = \mathcal{O}(r^{(-1/2)-}) \), cf. (7.1) and (7.2).

We give the mentioned \( Q \)'s (including \( Q = \mathcal{O}(r^{(-1/2)-}) \)) some uniform index to distinguish them, say \( Q^\pm = Q^\pm(a, k) \) (not specifying the index \( k \)) or alternatively \( Q^\pm = Q^\pm(b, l) \) (with unspecified \( l \)). By the Kato-smoothness theory [Ka] and the Cauchy–Schwarz inequality a consequence of the above bounds are the following resolvent bounds, valid for any combination of the indices,
\[
\sup_{\lambda \in \mathbb{R}} \left\| (2\pi)^{-1} Q^+ (b, l) f_2(H)(R(z) - R(\tilde{z})) f_2(H)Q^- (a, k)^* \right\| < \infty. \tag{7.3a}
\]

In particular, recalling (2.6a), (5.5a) and the surrounding discussion, there exist weak limits as \( z \to \lambda \in \mathbb{R} \) from above and these limits define a weakly continuous function of \( \lambda \). Whence we can record that the limiting operators
\[
Q^+(b, l) f_2(H)\delta(H - \lambda) f_2(H)Q^- (a, k)^* \in \mathcal{L}(\mathcal{H})
\]
exist with a weakly continuous dependence of \( \lambda \in \mathbb{R} \). \tag{7.3b}

(iv) We do not show the existence of
\[
\lim_{\epsilon \to 0^+} Q^+(b, l) f_2(H)R(\lambda + i\epsilon) f_2(H)Q^- (a, k)^*,
\]
although it is likely doable by mimicking [Ya3, Sect. 3]. Up to Sect. 9 it is more convenient to deal with the delta function only. However in Sects. 9.3–9.5 we need the resolvent bounds
\[
\sup_{\lambda \neq 0} \left\| Q^\pm (b, l) f_2(H)R(z) \right\|_{\mathcal{L}(\mathcal{B}, \mathcal{H})} < \infty, \tag{7.3c}
\]

which follow by combining our commutator bounds and the Besov space bound (5.6), see “Appendix B”.

7.2. Integrability of \( T^\pm_a \). We shall show the assertion (3.3) in a standard fashion verifying ‘integrability of the time-derivative’.

**Lemma 7.3.** **The operators** \( \Omega^\pm_a \) **are well-defined.**

**Proof.** For convenience we treat the ‘plus case’, showing only the existence of \( \Omega^+_a \). The existence of the time-limit will be verified by means of Sect. 6, (7.1), Lemma 7.1 and Remark 7.2 (ii), computing and estimating the quantity
\[
T^+_a = i f_2(H)g(H)M_a \bar{N}^a_a M_a f_2(\tilde{\mathcal{H}}_a) - i f_2(H)M_a \bar{N}^a_a M_a g(\tilde{\mathcal{H}}_a) f_2(\tilde{\mathcal{H}}_a)
\]
\[
= f_2(H)il[g(H), M_a \bar{N}^a_a M_a]f_2(\tilde{\mathcal{H}}_a)
\]
\[
+ i f_2(H)M_a \bar{N}^a_a M_a \tilde{e}^+_a \left( g(H) - g(\tilde{\mathcal{H}}_a) \right) f_2(\tilde{\mathcal{H}}_a) =: T_1 + T_2.
\]
The calculation is modulo terms on the form \( T = f_2(H)\mathcal{O}(r^{(-1)_-})f_2(\tilde{H}_a) \). Let us abbreviate this property as \( T \simeq_2 0 \) (which is a different relation than the one used in Sect. 7.1). Any such ‘error’ can be treated by the smoothness bounds (7.1) and (7.2).

We aim at verifying integrability of \( T^+_a \) in the precise sense stated below as (7.6). Most of the work is essentially already done in the proof of Lemma 7.1. However the computation and estimation of the term \( T_2 \) to the right is different. We claim that this term \( \simeq_2 0 \). To see this we note that on the support of the factors of \( A^a_2 \) in the definition of \( N^a_+ \)

\[
\langle x^a \rangle \leq C r^{1-\rho_2}. \tag{7.4}
\]

We write and estimate for any \( b \not\leq a \), on the support of \( \tilde{\xi}^+_a \), with (7.4) and for \( r \) large

\[
|V^b_{\text{tr}}(x) - V^b_{\text{tr}}(x_a)| = \left| \int_0^1 \pi^b x^a \cdot (\nabla V^b_{\text{tr}})(\pi^b(x - tx^a)) \, dt \right| \\ \leq C_1 \langle x^a \rangle (|x^b| - C_2 |x^a|)^{(1+\mu)} \\ \leq C_3 r^{1-\rho_2} (|x| - C_4 r^{1-\rho_2})^{(1+\mu)} \\ \leq C_5 r^{-\rho_2 - \mu}.
\]

Similarly, under the same conditions as above,

\[
|\pi^b x_a| \geq |\pi^b x| - |\pi^b x^a| \geq c_1 r \geq c_2 |x_a|; \quad c_1 > c_2 > 0.
\]

For the case where \( I^a_{\text{tr}} = 0 \) we can use the above bounds and the fact that \( \rho_2 + \mu > 1 \) to argue, abbreviating \( \chi_-(\langle x^a \rangle r^{\rho_2 - 1} / C) = \chi_-(\cdot) \),

\[
T_2 \simeq_2 f_2(H)\mathcal{O}(r^0)\chi_-(\cdot)\tilde{\xi}^+_a(g(H) - g(\tilde{H}_a))f_2(\tilde{H}_a) \\ \simeq_2 \int \mathcal{C} f_2(H)\mathcal{O}(r^0)(H - z)^{-1}\chi_-(\cdot)\tilde{\xi}^+_a(I_1 - \tilde{I}_a)(\tilde{H}_a - z)^{-1}f_2(\tilde{H}_a) \, d\mu_g(z) \\ = f_2(H)\mathcal{O}(r^{-\rho_2 - \mu})f_2(\tilde{H}_a) \simeq_2 0.
\]

If \( I^a_{\text{tr}} \neq 0 \) there is an extra term on the form \( \mathcal{O}(r^{-1-\mu}) \) in the above integral, which obviously yields the same conclusion.

As for the term \( T_1 \) we write

\[
T_1 \simeq_2 f_2(H) f_3(H) i[g(H), M_a N^a_{\pm} M_a] f_3(H) f_2(\tilde{H}_a). \tag{7.5}
\]

Applying the product rule for commutation several terms arise, essentially all treated in Sect. 6 (see also the proof of Lemma 7.1). Let us show how to treat the contribution from the factors of \( i[g(H), A_1] \). (The contribution from the factors of \( i[g(H), M_a] \) may be treated similarly, cf. the proof of Lemma 7.1.)

By (6.12) and Remark 6.19 we can write it as

\[
2 f_2(H) f_3(H) M_a \mathcal{M}(i[g(H), A_1] A^a_2 (A^a_2)^2 A^a_2 A_1) M_a f_3(H) f_2(\tilde{H}_a) \\ \simeq_2 f_2(H) (Q^+_2)^* \mathcal{O}(r^{-0}) Q^+_2 f_2(\tilde{H}_a); \quad Q^+_2 = r^{-1/2} \sqrt{\langle \chi^+_2 \rangle} \left( \pm B / \epsilon_0 \right).
\]
Next we apply the Cauchy–Schwarz inequality estimating
\[
\int_{-\infty}^{\infty} |\langle f_2(H)\psi(t), (Q_2^\alpha)^* \mathcal{O}(r^{-0}) Q_2^\alpha f_2(\tilde{H}_a)\phi(t)\rangle| \, dt \\
\leq C \int_{-\infty}^{\infty} \|Q_2^\alpha f_2(H)\psi(t)\| \|Q_2^\alpha f_2(\tilde{H}_a)\phi(t)\| \, dt \\
\leq C \left( \int_{-\infty}^{\infty} \|Q_2^\alpha f_2(H)\psi(t)\|^2 \, dt \right)^{1/2} \left( \int_{-\infty}^{\infty} \|Q_2^\alpha f_2(\tilde{H}_a)\phi(t)\|^2 \, dt \right)^{1/2}.
\]
The first factor is finite due to Lemma 7.1 (2), in turn with a bound proportional to \(\|\psi\|\). Similarly, thanks to Remark 7.2 (ii), the second factor is finite too (although not needed, with a bound proportional with \(\|\phi\|\)).

We may argue similarly for the other commutators arising from expanding the commutator in (7.5). More precisely we first compute
\[
f_2(H)M_a A_1 i[g(H), A_2^\alpha (A_3^\alpha)^2 A_2^\alpha] A_1 M_a f_2(\tilde{H}_a) \\
\approx_2 f_2(H)M_a A_1 i[g(H), (\lambda^2 + \chi_+ + \chi_+ (r^{\mu_1} - r^{\mu_1} r^{\mu_2})^2 + \chi_+ (r^{\mu_2} - r^{\mu_2} r^{\mu_2})^2)] A_1 M_a f_2(\tilde{H}_a) \\
\approx_2 f_2(H) \left( - |Q_3^\alpha|^2 - |Q_4^\alpha|^2 + |Q_5^\alpha|^2 + |Q_6^\alpha|^2 + |Q_7^\alpha|^2 \right) f_2(\tilde{H}_a).
\]
We integrate, estimate by the Cauchy–Schwarz inequality and invoke Lemma 7.1 and Remarks 7.2 (ii) and (iii).

These estimates lead to the conclusion that
\[
\int_{-\infty}^{\infty} \|\langle \psi(t), T_a^+(\phi(t))\rangle\| \, dt \leq C_{\psi} \|\psi\|,
\]
and by the same arguments, that
\[
\forall \epsilon > 0 \exists \epsilon > 0 : \int_{t_\epsilon}^{\infty} \|\langle \psi(t), T_a^+(\phi(t))\rangle\| \, dt \leq \epsilon \|\psi\|. \tag{7.6}
\]
Clearly the existence of \(\Omega_a^+\) follows from (7.6).

\[\square\]

8. Formula for \(\tilde{S}_{\beta\alpha}(\lambda)\)

Recall from Sect. 3,
\[
\tilde{W}_a^{\pm} = \Omega_a^{\pm} J_a \tilde{w}_a^{\pm} f_1(k_\alpha) \quad \text{and} \quad \tilde{S}_{\beta\alpha} = \left( \Omega_b^{\pm} J_\beta \tilde{w}_b^{\pm} f_1(k_\beta) \right)^* \Omega_a^{-} J_a \tilde{w}_a^{-} f_1(k_\alpha). \tag{8.1}
\]

We show in “Appendix C” the following representation of \(\tilde{S}_{\beta\alpha}(\lambda)\), formally given as
\[
\tilde{S}_{\beta\alpha}(\lambda) = (2\pi i)^2 f_1^2(\lambda) \gamma_b^{(\lambda_\beta)} J_\beta^*(T_a^+) \delta(H - \lambda) T_a^- J_\alpha \gamma_a^{-(\lambda_\alpha)}; \quad \lambda_\alpha := \lambda - \lambda^\alpha. \tag{8.2}
\]

Parallel to Sect. 2.3 we call the appearing \(\lambda\)-depending operators \(J_\beta \gamma_b^{(\lambda_\beta)}\) and \(J_\alpha \gamma_a^{-(\lambda_\alpha)}\) auxiliary outgoing and incoming channel wave matrices, respectively. In Sects. 8.1 and 8.2 we examine the well-definedness of the right-hand side of (8.2) upon substituting expressions for \(T_a^+\) and \(T_a^-\) derived in the proof of Lemma 7.3. With the resulting final representation we then derive the weak continuity of \(\tilde{S}_{\beta\alpha}(\cdot)\), from which Theorem 3.1 follows.
8.1. Phase space estimates of auxiliary channel wave matrices. Recalling the index convention of Remark 7.2 (iii), we aim at bounding the operators

\[ Q^\pm(a, k) f_2(\tilde{H}_a) J_a \tilde{\gamma}_a^\pm(\lambda_\alpha)^* = f_2(\lambda) Q^\pm(a, k) J_a \tilde{\gamma}_a^\pm(\lambda_\alpha)^* \in \mathcal{L}(\mathcal{G}_a, \mathcal{H}), \quad \lambda_\alpha = \lambda - \lambda^\alpha, \]

as well as showing a continuity property in \( \lambda \in \Lambda \). This will along with (7.3b) be crucial for our usage of (8.2).

To be used frequently in this subsection, we let

\[ B_a = 2\mathfrak{M}(x_a/r \cdot p_a), \quad a \in \mathcal{A}', \quad (8.3) \]

and we recall the notation \( \mathcal{O}(r^t) \) of (6.2).

Lemma 8.1. For all \( a \in \mathcal{A}' \) and \( \epsilon > 0 \) taken small enough

\[ \sqrt{\chi_-(\pm B/\epsilon\epsilon_0)} \chi_-(|x^a|/(\epsilon \epsilon_0)) \chi_+(\pm B_a/(8\epsilon_0)) f_3(\tilde{H}_a) = \mathcal{O}(r^{-1/2}). \quad (8.4) \]

Proof. By the time-reversal property it suffices to consider the ‘plus’ case. Introducing

\[ S_{a, \epsilon}^+ = f_4(\tilde{H}_a) \sqrt{\chi_-(B/\epsilon\epsilon_0)} \chi_- (|x^a|/(\epsilon \epsilon_0)) r^{1/2} \chi_+(B_a/(8\epsilon_0)) f_3(\tilde{H}_a), \]

it suffices to show that \( S_{a, \epsilon}^+ = \mathcal{O}(r^0) \). Writing

\[ B = 2\mathfrak{M}(x/r \cdot p + \mathcal{O}(r^{-1}) \cdot p) = B_a + 2\mathfrak{M}(x^a/r \cdot p^a + \mathcal{O}(r^{-1}) \cdot p), \quad (8.5) \]

and using commutation (cf. Lemma 6.9 and Remark 6.19), we estimate \( \psi_\epsilon = S_{a, \epsilon}^+ \psi \) for any \( \psi \in L^2_{\infty} \subseteq \mathcal{H} \) as follows (recall the notation \( \langle T \rangle_{\psi} = \langle \psi, T\psi \rangle \)):

\[
\begin{align*}
\frac{1}{2} \| \psi_\epsilon \|^2 & \leq \langle I - B/(4\epsilon_0) \rangle \psi_\epsilon + C_1 \| \psi \|^2 \\
& \leq \langle (1 + C\epsilon) I - B_a/(4\epsilon_0) \rangle \psi_\epsilon + C_2 \| \psi \|^2 \\
& \leq \langle (C\epsilon - 1) I \rangle \psi_\epsilon + C_3 \| \psi \|^2 \\
& \leq C_3 \| \psi \|^2 \quad \text{for (for \( C \epsilon \leq 1 \).)}
\end{align*}
\]

For \( s \in \mathbb{R} \setminus \{0\} \) we can repeat the estimation (8.6) with \( \psi_\epsilon \) replaced by \( r^s S_{a, \epsilon}^+ r^{-s} \psi \) yielding the desired estimate (6.1) with \( t = 0 \). Note that the same constant \( C \) works in this case. \( \square \)

This result yields estimates for the \textit{outgoing and incoming N-body quasi-modes}

\[ J_a \tilde{v}_{a, \lambda_\alpha}^\pm[g] = u^a \otimes \tilde{v}_{a, \lambda_\alpha}^\pm[g] \in \mathcal{B}^s \quad \text{(with + and −, respectively), to be useful in Sect. 9.3. Here the functions \( \tilde{v}_{a, \lambda_\alpha}^\pm[g] \) are given by (2.18). Recall from the discussion at the end of Subsect 5.1 that we allow ourselves the freedom of considering only (sufficiently) small values of \( \epsilon_0 \).}

Corollary 8.2. Suppose \( \epsilon_0 < (64)^{-1} \min_{\lambda, \epsilon I_0} \sqrt{\lambda_\alpha} \). Then

\[
\forall \lambda \in I_0 \forall g \in \mathcal{G}_a : \quad \chi_- (\pm B/\epsilon_0)(u^a \otimes \tilde{v}_{a, \lambda_\alpha}^\pm[g]) \in \mathcal{B}^s_0. \quad (8.7)
\]
Proof. We will tacitly use the fact that any operator $T = \mathcal{O}(r^0)$ is bounded on $\mathcal{B}, \mathcal{B}^*$ as well as on $\mathcal{B}^*_0$, cf. [Hö3, Theorem 14.1.4]. By density we can assume that $g \in C^\infty(C_a)$. Like for (2.24b) and (2.24c) we first represent
\[
\tilde{v}_{a,\lambda}^\pm [g] = (\tilde{h}_a - \lambda \alpha \mp \mathrm{i}0)^{-1}(\tilde{h}_a - \lambda \alpha)\tilde{v}_{a,\lambda}^\pm [g].
\] (8.8)
Let $\mathcal{B}_{a}^1 = 2\mathbb{N}(x_a, p_a) \cdot \mathcal{B}$ and $\epsilon_1 = \sqrt{\lambda \alpha}$. By using the one-body version of Theorem 5.1 (cf. Remark 2.6 (iii)) we then deduce, abbreviating the function $\mu^\alpha \otimes \tilde{v}_{a,\lambda}^\pm [g] =: \phi \in \mathcal{B}^*$ and calculating modulo vectors in $\mathcal{B}_{a}^* (\text{indicated by the notation } \equiv)$, that for any sufficiently small $\epsilon > 0$
\[
\chi_-(|x^a|/(\epsilon r)) \chi_+(B_a/(8\epsilon_0)) f_3(\tilde{h}_a)\phi \\
= \chi_-(|x^a|/(\epsilon r)) \chi_+(B_a/(8\epsilon_0)) f_3(\tilde{h}_a + \lambda^\alpha)\phi \\
\simeq \chi_-(|x^a|/(\epsilon r)) \chi_+(B_a/(8\epsilon_0)) \phi \\
\simeq \chi_-(|x^a|/(\epsilon r)) \chi_+(B_a/(8\epsilon_0)) \chi_+(B_a^1/\epsilon_1) \phi \quad \text{(by (8.8)), cf. Theorem 5.1)} \\
\simeq \chi_-(|x^a|/(\epsilon r)) \phi \quad \text{(by a simplified version of (8.6))} \\
\simeq \phi.
\]
We now conclude (8.7) for the outgoing quasi-modes by Lemma 8.1. By the time-reversal property (8.7) for the incoming quasi-modes then holds as well. \hfill \Box

Lemma 8.3. For all channels $\alpha = (a, \lambda^\alpha, u^\alpha)$ and indices $k$
\[
\sup_{\lambda \in \Lambda} \left\| Q_{\pm}(a, k) f_2(\tilde{H}_a)J_a \tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^* \right\| < \infty,
\] (8.9)
and the operator-valued functions
\[
|Q_{\pm}(a, k) f_2(\tilde{H}_a)|J_a \tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^* \in \mathcal{L}(G_a, \mathcal{H})
\]
are strongly continuous in $\lambda \in \Lambda$. (In particular also $Q_{\pm}(a, k) f_2(\tilde{H}_a)J_a \tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^*$ is strongly continuous in $\lambda \in \Lambda$.)

Proof. It suffices to consider the cases $Q_{\pm}(a, k) \neq Q_s$ since for $Q_s$ (as given by (7.1)) the assertions reduce to properties of $\tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^*$ stated in Sect. 2.3. Whence from this point we assume $Q_{\pm}(a, k) \neq Q_s$. The uniform bounds (8.9) will be proved in Step I below. The stated strong continuity assertions will then be shown in Steps II–V. Note that weak continuity is an immediate consequence of (8.9) and the strong continuity for $Q_{\pm}(a, k) = Q_s$ (the weak continuity will be used in Steps IV and V).

I. By ‘the $T^*T$ argument’ (cf. the proof of Lemma 2.5) it suffices for (8.9) to show that
\[
\sup_{\lambda \in \Lambda} \left\| \tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^* J_a^* f_2(\tilde{H}_a) Q_{\pm}(a, k)^* Q_{\pm}(a, k) f_2(\tilde{H}_a)J_a \tilde{y}_{a,\lambda}^\pm (\lambda \alpha)^* \right\| < \infty.
\]
We pick, using quantities from Remark 7.2 (ii),
\[
\Phi \in \text{span}_\mathbb{R}\{\Phi_1, j, \Phi_2, \Phi_3, \Phi_4 | j \leq J\}: \\
i[\tilde{H}_a, \Phi] \geq f_2(\tilde{H}_a) Q_{\pm}(a, k)^* Q_{\pm}(a, k) f_2(\tilde{H}_a) \\
+ f_2(\tilde{H}_a)\mathcal{O}(r^{(1-\)} f_2(\tilde{H}_a).
\] (8.10)
Letting $\chi_\rho = \chi_-(r/\rho)$ for $\rho > 1$ and $\phi = J_a \hat{\gamma}_a^\pm(\lambda_\alpha)^* g$ for $g \in G_a$, we compute

$$0 = \langle i[\hat{H}_a - \lambda, \chi_\rho \Phi \chi_\rho] \rangle_{\phi} = \langle \chi_\rho i[\hat{H}_a, \Phi] \chi_\rho \rangle_{\phi} + 2\Re \langle i[\hat{H}_a, \chi_\rho] \Phi \chi_\rho \rangle_{\phi}.$$  

For the last term to the right we compute and insert the expression

$$i[\hat{H}_a, \chi_\rho] = -\rho^{-1} \sqrt{-\chi_-(r/\rho)} B \sqrt{-\chi'_-(r/\rho)},$$

commute and conclude that this term is bounded by $C \|g\|^2$ with a constant $C > 0$ being independent of $\rho > 1$ and $\lambda \in \Lambda$. For the first term we use (8.10), yielding after a commutation the bound

$$\sup_{\lambda \in \Lambda, \rho > 1} \left\| \chi_\rho Q^\pm(a, k) f_2(\hat{H}_a) J_a \hat{\gamma}_a^\pm(\lambda_\alpha)^* \right\|^2 < \infty. \quad (8.11)$$

Clearly (8.9) follows from (8.11) and Lebesgue’s monotone convergence theorem (by taking $\rho \to \infty$).

II. For $Q_1(a, j) = \xi_j^+ G_{d_j}$ (as in Lemma 7.1 1)) the strong continuity assertion follows from (2.26b) if $d_j = a$ due to the representation

$$\xi_j^+ G_{d_j} f_2(\hat{H}_a) J_a \hat{\gamma}_a^\pm(\lambda_\alpha)^* = f_2(\lambda) \xi_j^+ J_a G_{d_j} \hat{\gamma}_a^\pm(\lambda_\alpha)^*.$$  

For $d_j \neq a$ we have $d_j \leq a$ (since $d_j \leq a$). The fact that the factor $\xi_j^+$ is supported in $Y_{d_j}(\delta_j)$ then yields the estimate $|x^a| \geq \epsilon |x|$ on supp $\xi_j^+$ for some $\epsilon > 0$. We decompose for (large) $\rho > 1$

$$\xi_j^+ G_{d_j} = \chi_1 \xi_j^+ G_{d_j} + \chi_2 \xi_j^+ G_{d_j}, \quad \chi_1 = \chi_\alpha^2(|x^a|/\rho), \quad \chi_2 = \chi_\alpha^2(|x^a|/\rho),$$

and write correspondingly, again with $\phi = J_a \hat{\gamma}_a^\pm(\lambda_\alpha)^* g$ for $g \in G_a$,

$$\xi_j^+ G_{d_j} \phi = \chi_1 \xi_j^+ G_{d_j} \phi + \chi_2 \xi_j^+ G_{d_j} \phi.$$  

For the first term we estimate using (2.22), cf. the proof of [Ya3, Lemma 4.5],

$$\left\| \chi_1 \xi_j^+ G_{d_j} \phi \right\|^2 \leq C_1 \int_{|x^a| \geq \rho} dx^a \left( |u^a|^2 + |p^a u^a|^2 \right)$$

$$\left| x^a \right|^{-1} \int_{|x^a| \leq \epsilon^{-1} |x^a|} \left( |\hat{\gamma}_a^\pm(\lambda_\alpha)^* g|^2 + |p_a \hat{\gamma}_a^\pm(\lambda_\alpha)^* g|^2 \right) dx_a$$

$$\leq C_2 \int_{|x^a| \geq \rho} \left( |u^a|^2 + |p^a u^a|^2 \right) dx^a \|g\|^2.$$  

Here the constant $C_2$ can be chosen independently of $\lambda \in \Lambda$. Consequently the right-hand side can be taken arbitrarily small uniformly in $\lambda$ by taken $\rho > 1$ big enough (henceforth considered fixed).

For the second term note that $|x| \leq 2\epsilon^{-1} \rho$ on the support of $\chi_2$. Hence this term is continuous in $\lambda \in \Lambda$ thanks to the established continuity for $Q^\pm(a, k) = Q_2$. Whence we have treated the case $Q_1(a, j) = \xi_j^+ G_{d_j}$ for any $j \leq J$.  

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III. For $Q_3^\pm$ (as in Lemma 7.1.2), leaving out a similar discussion for $Q_2^-$ it suffices by (8.9) to show continuity of the map

$$\Lambda \ni \lambda \to Q_3^\pm f_2(\tilde{H}_a)J_a\tilde{\gamma}_a^\pm(\lambda,\alpha)^*g \in \mathcal{H}; \quad g \in C^\infty(C_a) \text{ fixed.}$$

For $\lambda$ close to a fixed $\lambda' \in \Lambda$ we have correspondingly that $\lambda_\alpha = \lambda - \lambda'^\alpha$ is close to $\lambda'_\alpha = \lambda' - \lambda'^\alpha$, and according to an assertion in Sect. 2.3 we can record that

$$\forall s > 1/2: \quad \| (x_\alpha)^{-s}\phi(\lambda)\| \to 0 \text{ for } \lambda \to \lambda';$$

$$\phi(\lambda) = J_\alpha\tilde{\gamma}_a^+(\lambda,\alpha)^*g - J_\alpha\tilde{\gamma}_a^+(\lambda',\alpha^\prime)^*g.$$  \hfill (8.12)

By the arguments from the last part of Step II, it suffices to show that for some $\epsilon > 0$

$$\left\| Q_3^\pm \chi_-(|x_\alpha|/(\epsilon r))f_2(\tilde{H}_a)\phi(\lambda) \right\| \to 0 \text{ for } \lambda \to \lambda'.$$

In turn, easily seen by using (2.24c) and the one-body version of Theorem 5.1 (cf. the proof of Corollary 8.2), it suffices to show that $\| \varphi_\pm(\lambda) \| = o(|\lambda - \lambda'|^0)$, where for $\epsilon, \epsilon_0 > 0$ take small

$$\varphi_\pm(\lambda) = \sqrt{(\chi_\pm^2)(B/\epsilon_0)r^{-1/2}T_\epsilon^\pm f_2(\tilde{H}_a)\phi(\lambda); \quad T_\epsilon^\pm = \chi_-(|x_\alpha|/(\epsilon r))\chi_+(\pm B_a/(8\epsilon_0)).$$

Thanks to Lemma 8.1 we only need (by a quick check of both cases, $T_\epsilon^+$ and $T_\epsilon^-$) to show that $\| r^{-1}\phi(\lambda) \| = o(|\lambda - \lambda'|^0)$, and the latter is obviously fulfilled by (8.12).

IV. As for $Q_3^\pm(a)$ and $Q_4^\pm(a)$ we introduce

$$Q^\pm = \sqrt{|Q_3^\pm(a) f_4(\tilde{H}_a)|^2 + |Q_4^\pm(a) f_4(\tilde{H}_a)|^2}.$$  \hfill (8.13)

It suffices (thanks to (8.9) and the weak continuity) to show that

$$\left\| Q^\pm f_2(\tilde{H}_a)J_a\tilde{\gamma}_a^\pm(\lambda,\alpha)^*g \right\|^2 \to \left\| Q^\pm f_2(\tilde{H}_a)J_a\tilde{\gamma}_a^\pm(\lambda',\alpha)^*g \right\|^2 \text{ for } \lambda \to \lambda', \quad g \in C^\infty(C_a) \text{ fixed,}$$

cf. Remark 2.6 (ii) (the Hilbert space for this abstract scheme of proof is here concretely taken to be $\mathcal{H} \oplus \mathcal{K}$).

Letting $\chi_\rho = \chi_-(|x_\alpha|^\rho/\rho)$ for $\rho > 1$ and $\phi^\pm(\lambda) = J_\alpha\tilde{\gamma}_a^\pm(\lambda,\alpha)^*g$, we compute as in Step I (now with $\Phi_3^\pm = \Phi_3^\pm(a)$)

$$0 = \langle i[\tilde{H}_a, \chi_\rho \Phi_3^\pm \chi_\rho] \rangle \phi^\pm(\lambda),$$

$$= \langle i[\tilde{H}_a, \Phi_3^\pm] \rangle \chi_\rho \phi^\pm(\lambda) + 2\Re \langle i[\tilde{H}_a, \chi_\rho] \Phi_3^\pm \chi_\rho \rangle \phi^\pm(\lambda).$$

We take the limit $\rho \to \infty$. For the first term we get in the limit $\| f_2(\lambda) Q^\pm \chi_\rho \phi^\pm(\lambda) \|^2$ plus terms that we know are continuous in $\lambda$ by the proof of Lemma 7.1 and Steps II and III. Here we used commutation to replace $\| f_2(\lambda) Q^\pm \chi_\rho \phi^\pm(\lambda) \|^2$ by

$$\langle Q_3^\pm(a)^* \chi_\rho^2 Q_3^\pm(a) \rangle f_2(\tilde{H}_a) \phi^\pm(\lambda) + \langle Q_4^\pm(a)^* \chi_\rho Q_4^\pm(a) \rangle f_2(\tilde{H}_a) \phi^\pm(\lambda).$$
before taking the limit.

As for the second term we compute as follows, using again conveniently Lemma 2.4 and the one-body version of Theorem 5.1, and using in the second step also (1) and (2) from Sect. 4. A further elaboration is needed, to be given after the computation. We compute for any \( \epsilon, \epsilon_0 > 0 \) taken small

\[
\lim_{\rho \to \infty} 2 \Im(i [\tilde{H}_a, \chi_\rho]) \Phi_{\frac{1}{3}}^\pm \chi_\rho \phi^\pm(\lambda)
\]

\[= \pm 2\sqrt{\lambda_\alpha} f_2^2(\lambda) \lim_{\rho \to \infty} \rho^{-1} \Im(\langle \chi_2^- (\frac{|z|}{\rho}) \chi_{\pm}(\frac{|x|}{\rho}) M_a^2 f_3(\tilde{H}_a) \chi_{\pm}(\pm B_a/(8\epsilon_0)) \phi^\pm(\lambda) \rangle)
\]

\[= \pm \left(2\sqrt{\lambda_\alpha}\right)^3 f_2^2(\lambda) \lim_{\rho \to \infty} \rho^{-1} \langle \chi_2^- (\frac{|x|}{\rho}) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \phi^\pm(\lambda) \rangle
\]

\[\Rightarrow \mp \frac{2\sqrt{\lambda_\alpha}}{\pi} f_2^2(\lambda) \| m_{\mathfrak{g}} \mathfrak{g}(\pm) \|_{\tilde{H}_a}^2.
\]

Note that in the first step of (8.14) we used that effectively the factor \( \frac{1}{\lambda_\alpha} \) can be removed. This is essentially due to the fact that \( B^a_{\delta, \rho_1} = 0 \in \{ |x| < cr_\delta \} \), cf. (ii) of Sect. 5, since for the complement we have

\[2 \lim_{\rho \to \infty} \rho^{-1} \Im(\langle p_a \cdot \hat{\chi}_a(\chi_2^- (\frac{|x|}{\rho}) \chi_{\pm}(\frac{|x|}{\rho}) M_a^2 \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \phi^\pm(\lambda) \rangle) = 0.
\]

In fact (using commutation and Remark 6.19)

\[2 \lim_{\rho \to \infty} \rho^{-1} \Im(\langle p_a \cdot \hat{\chi}_a(\chi_2^- (\frac{|x|}{\rho}) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \phi^\pm(\lambda) \rangle)
\]

\[= f_2^2(\lambda) \lim_{\rho \to \infty} \frac{2}{\rho} \Im(\langle p_a \cdot \hat{\chi}_a(\chi_2^- (\frac{|x|}{\rho}) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0)) \phi^\pm(\lambda) \rangle)
\]

and

\[\chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I = \pm \int \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 d\mu_{\chi_{\mathfrak{r}}}(z) = 0.
\]

(8.15)

Once \( (A_3^a)^2 \) is removed we can also remove the remaining factor \( \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 \). Similarly in the first step we also replaced the factors of \( A_1 = A_1 \pm \) by factors of \( \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 \).

Using (2.24c) and the one-body version of Theorem 5.1 as in Step III this is justified by the assertions

\[2 f_2^2(\lambda) \lim_{\rho \to \infty} \rho^{-1} \Im(\langle [\tilde{H}_a] p_a \cdot \hat{\chi}_a(\chi_2^- (\frac{|x|}{\rho}) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I) f_3(\tilde{H}_a) M_a \phi_{\mathfrak{r}}^\pm(\lambda) \rangle) = 0.
\]

\[2 f_2^2(\lambda) \lim_{\rho \to \infty} \rho^{-1} \Im(\langle [\tilde{H}_a] p_a \cdot \hat{\chi}_a(\chi_2^- (\frac{|x|}{\rho}) \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I) f_3(\tilde{H}_a) M_a \phi_{\mathfrak{r}}^\pm(\lambda) \rangle) = 0;
\]

\[\phi_{\mathfrak{r}}^\pm(\lambda) = \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 \phi_{\mathfrak{r}}^\pm(\lambda), \quad \phi_{\mathfrak{r}}^\pm(\lambda) = \chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 \phi_{\mathfrak{r}}^\pm(\lambda).
\]

(Defining \( \phi_{\mathfrak{r}}^\pm(\lambda) \) by the identity \( \phi_{\mathfrak{r}}^\pm(\lambda) = \phi_{\mathfrak{r}}^\pm + \phi_{\mathfrak{r}}^\pm + \phi_{\mathfrak{r}}^\pm \), substituting and expanding the inner product into a sum of nine terms, indeed seven of those vanish in the limit.) In turn these assertions follow by commutation and Lemma 8.1. Note that thanks to the latter result

\[\chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I) f_3(\tilde{H}_a) = O(r^{-1/2}) = O(r^{0-}), \quad (8.16a)
\]

and

\[\chi_{\mathfrak{r}}(\pm B_a/(8\epsilon_0))^2 (A_3^a)^2 - I) f_3(\tilde{H}_a) = O(r^{-1/2}) = O(r^{0-}). \quad (8.16b)
\]
We conclude that indeed the factors of $A_1 = A_{1\pm}$ can be replaced by the factors of $\chi_+((\pm B_a / (8\epsilon_0)))$, as claimed.

The last part of the first step of the computation (8.14), as well as the remaining steps, use asymptotics of $\chi_+((\pm B_a / (8\epsilon_0))\phi_\pm(\lambda)$ in $B^*$. More precisely, cf. Lemma 2.4, (8.8) and the one-body version of Theorem 5.1, we can record

$$
\chi_+((\pm B_a / (8\epsilon_0))\phi_\pm(\lambda) - u^\alpha \otimes \tilde{v}^\pm_{a,\lambda_a} [g] \in B_0^*.
$$

(8.17)

The rest of the computation (8.14) amounts to replacing the operators $p_a \cdot \hat{x}_a$ and the factors of $M_a$, acting effectively on the quasi-modes $u^\alpha \otimes \tilde{v}^\pm_{a,\lambda_a} [g]$, in agreement with the asymptotics (8.17).

Obviously the far right-hand side of (8.14) is continuous in $\lambda$. We conclude that the function

$$
\Lambda \ni \lambda \rightarrow \|Q^\pm f_2(\hat{H}_a)\phi_\pm(\lambda)\|^2 \in \mathbb{R}
$$

is continuous, as wanted.

V. We mimic Step IV using now

$$
Q^\pm := \sqrt{|Q^\pm_5(a) f_4(\hat{H}_a)|^2 + |Q^\pm_6(a) f_4(\hat{H}_a)|^2 + |Q^\pm_7(a) f_4(\hat{H}_a)|^2 + |Q^\pm_8(a) f_4(\hat{H}_a)|^2},
$$

$\Phi^\pm_4 = \Phi^\pm_4(a)$ (replacing $\Phi^\pm_3(a)$) and using again Steps II and III, yielding similarly the strong continuity for $Q^\pm_5(a), Q^\pm_6(a), Q^\pm_7(a)$ and $Q^\pm_8(a)$. In fact this case is simpler in that one readily shows the following analogue of (8.14),

$$
\lim_{\rho \to \infty} 2\mathfrak{M}(i[H, \chi_\rho] \Phi^\pm_4 \chi_\rho)_{\phi_\pm(\lambda)} = 0,
$$

and a constant function is obviously continuous. 

8.2. Conclusion of argument, the weak continuity. The expression to the right in (8.2) can (and should) be written, cf. the proof of Lemma 7.3, as a sum of terms on the form

$$
f_1^2(\lambda)(\tilde{y}_b(\lambda_\beta) J^*_b Q^+(b, l) \ast B^*_+ )

\left(Q^+(b, l) f_2(H) \delta(H - \lambda) f_2(H) Q^-(a, k)^* \right)(B_- Q^-(a, k) J_a \tilde{y}_a^- (\lambda_\alpha)^*).
$$

where $B_+ = B_+(b, l)$ and $B_- = B_-(a, k)$ are bounded operators. The middle factor is weakly continuous by Remark 7.2 (iii). The factor to the right is strongly continuous by Lemma 8.3. The adjoint of the factor to the left is strongly continuous by the same result. Consequently the entire product is weakly continuous, and Theorem 3.1 follows (cf. the discussion given before the statement).
9. Channel Wave Matrices and Scattering at Fixed Energy

In this concluding section we derive various consequences of our proof of Theorem 3.1. Whence we concretely construct the open channel wave matrices and the scattering matrix for all energies in the small neighbourhood $I_0 \ni \lambda_0$ fixed in Sect. 3. Away from a null set of energies, more precisely at any stationary scattering energy in $I_0$, the scattering matrix is unitary and strongly continuous and it is characterized by asymptotic properties of generalized eigenfunctions of minimal growth. The discussion relies heavily on “Appendix C”.

For all $b \in A'$ we pick an arbitrary increasing sequence $(C_{b,k})_k$ of open reflection symmetric subsets of $C'_b$ (recall the notation (2.13a)) with closure $\overline{C}_{b,k} \subset C'_b$ and union $\cup_k C_{b,k} = C'_b$. We pick for each such set a function $m_b = m_{b,k}$ as in Sect. 4 such that $m_b(\xi) = 1$ for all $\xi \in C_{b,k}$. Let for any channel $\beta = (b, \lambda^\beta, u^\beta)$

$$\chi_{\beta,k} = \chi_{\beta,k}(p_b) = F^{-1}_{\beta}1_{C_{b,k}}(\xi_b)F_{\beta},$$

$$\tilde{\chi}_{\beta,k} = \tilde{\chi}_{\beta,k}(p_b) = F^{-1}_{\beta}(4\lambda^\beta)^{-1}1_{C_{b,k}}(\xi_b)F_{\beta}.$$

By combining (3.7a), (3.7b), (C.5a) and (C.11a) we then obtain (by an elementary approximation argument)

$$W^\pm_{\beta}(f_1g)(k\beta)\chi_{\beta,k}G = \tilde{W}^\pm_{\beta}g(k\beta)\tilde{\chi}_{\beta,k}G = \int \pm \frac{\pi}{2\lambda^\beta}(f_1g)(\lambda)(f_2(H)\delta(H - \lambda)T^\pm_{b,k}J_{\beta}\tilde{\gamma}^\pm(\lambda^\beta)^*)1_{C_{b,k}}g_{\beta,0}(\lambda^\beta)\varphi \, d\lambda; \quad (9.1)$$

for any $\varphi \in L^2_{\gamma}(X_b)$ with $s > 1/2$.

Here $g$ is any complex continuous function on $\mathbb{R}$ vanishing at infinity, and we note that indeed $T^\pm_{b,k} = T^\pm_{b,k}(b)$ depends on $k$.

We are lead to introduce the approximate channel wave matrices

$$\Gamma^\pm_{\beta,k}(\lambda)^* = \pm \frac{\pi}{2\lambda^\beta} f_2(H)\delta(H - \lambda)T^\pm_{b,k}J_{\beta}\tilde{\gamma}^\pm(\lambda^\beta)^*1_{C_{b,k}}; \quad k \in \mathbb{N}, \lambda \in I_0 (\ni \lambda_0). \quad (9.2a)$$

Equivalently, cf. (C.5b),

$$\Gamma^\pm_{\beta,k}(\lambda)^* = \frac{1}{4\lambda^\beta} f_2(H)(\Phi^\pm_{b,k} + iR(\lambda \mp i0)T^\pm_{b,k})J_{\beta}\tilde{\gamma}^\pm(\lambda^\beta)^*1_{C_{b,k}}. \quad (9.2b)$$

9.1. Restricted channel wave operators and channel wave matrices. By (9.1) the right-hand side of (9.2a) is independent of details of the construction of $T^\pm_{b,k}$. Considered as an $\mathcal{L}(G_b, L^2_{\gamma,s}(X))$-valued function of $\lambda \in I_0$ for any $s > 1/2$, it is strongly continuous. Clearly

$$1_{C_{b,k}}\Gamma^\pm_{\beta,k+1}(\lambda) = \Gamma^\pm_{\beta,k}(\lambda) \in \mathcal{L}(L^2_{\gamma}(X), G_b).$$

We can then for any $\lambda \in I_0$ and $\psi \in L^2_{\gamma}(X)$ with $s > 1/2$ introduce a measurable function $\Gamma^\pm_{\beta}(\lambda)\psi$ on $C_b$ by

$$1_{C_{b,k}}\Gamma^\pm_{\beta}(\lambda)\psi = \Gamma^\pm_{\beta,k}(\lambda)\psi; \quad k \in \mathbb{N}. \quad (9.3)$$
We argue that $\Gamma^{\pm}_\beta(\lambda)\psi \in \mathcal{G}_b$ as follows. With (9.1) it follows from Stone’s formula that
\[
\int_{\Delta} \left\| \Gamma^{\pm}_{\beta,k}(\lambda)\psi \right\|^2 d\lambda \leq \int_{\Delta} \langle \delta(H - \lambda) \rangle \psi \, d\lambda \quad \text{for any interval } \Delta \subseteq I_0,
\]
leading to the conclusion (thanks to the regularity of finite Borel measures [Ru2, Theorem 2.18]) that
\[
\left\| \Gamma^{\pm}_{\beta,k}(\lambda)\psi \right\|^2 \leq \langle \delta(H - \lambda) \rangle \psi \quad \text{for a.e. } \lambda \in I_0.
\]
Since $\Gamma^{\pm}_{\beta,k}(\cdot)\psi \in \mathcal{G}_b$ is weakly continuous and the functional $\| \cdot \|^2$ is weakly lower semi-continuous we conclude that
\[
\left\| \Gamma^{\pm}_{\beta,k}(\lambda)\psi \right\|^2 \leq \langle \delta(H - \lambda) \rangle \psi \quad \text{for all } \lambda \in I_0.
\]
(9.4)
In particular also
\[
\left\| \Gamma^{\pm}_{\beta}(\lambda)\psi \right\|^2 = \lim_{k \to \infty} \left\| \Gamma^{\pm}_{\beta,k}(\lambda)\psi \right\|^2 \leq \langle \delta(H - \lambda) \rangle \psi,
\]
showing that indeed $\Gamma^{\pm}_{\beta}(\lambda)\psi \in \mathcal{G}_b$.

We can now record, using Lemma 8.3 and (7.3b), that for any $s > 1/2$ the restricted channel wave operators
\[
\Gamma^{\pm}_\beta(\lambda) \in \mathcal{L}(L^2_s(X), \mathcal{G}_b)
\]
with a weakly continuous dependence of $\lambda \in I_0$,
and in fact that the channel wave matrices
\[
\Gamma^{\pm}_\beta(\lambda)^* \in \mathcal{L}(\mathcal{G}_b, L^2_{-s}(X))
\]
depend strongly continuously of $\lambda \in I_0$.

By taking $k \to \infty$ in (9.1) we conclude that for any $\psi \in L^2_s(X)$, $s > 1/2$,
\[
(F^{\beta}_\psi(W^{\pm}_\beta f)(\lambda) = \Gamma^{\pm}_\beta(\lambda)\psi \quad \text{for a.e. } \lambda \in I_0.
\]

We summarize as follows.

**Theorem 9.1.** Let $\beta$ be a given channel $\beta = (\lambda^\beta, u^\beta)$, $f : I^\beta = (\lambda^\beta, \infty) \to \mathbb{C}$ be bounded and continuous, and let $s > 1/2$. For any $\varphi \in L^2_s(X_b)$
\[
W^\pm_{\beta}(f 1_{I_0}(k_{\beta})\varphi = \int_{I_0} f(\lambda)\Gamma^\pm_{\beta}(\lambda)^* \gamma_{b,0}(\lambda, \beta)\varphi \, d\lambda \in L^2_s(X), \quad (9.5a)
\]
More generally for any $\varphi \in 1_{I_0}(k_{\beta})L^2_s(X_b)$
\[
W^\pm_{\beta} f(k_{\beta})\varphi = \int_{I_0} f(\lambda)\Gamma^\pm_{\beta}(\lambda)^* (F^{\beta}\varphi)(\lambda, \cdot) \, d\lambda \in L^2_{-s}(X). \quad (9.5b)
\]

In (9.5a) the integrand is a bounded and continuous $L^2_{-s}(X)$-valued function. For (9.5b) the integral has the weak interpretation of an integral of a measurable $L^2_{-s}(X)$-valued function.

For $f(\lambda) = e^{-it\lambda}$, $t \in \mathbb{R}$, the formulas (9.5a) and (9.5b) represent Schrödinger wave packets of energy-localized states outgoing to or incoming from the channel $\beta$, cf. a discussion in Sect. 1.1.
9.2. Parseval identities and construction of the scattering matrix. Let \( Q = Q_s \) be given as in (7.1) (for an arbitrary \( s \in (1/2, 1) \)). By the orthogonality and completeness of channel wave operators we deduce the Parseval formulas

\[
\int_{\Delta} \sum_{\lambda^+ < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) Q^*_s \varphi \right\|^2 d\lambda = \int_{\Delta} \langle \delta (H - \lambda) \rangle_{Q^*_s} \varphi \ d\lambda;
\]

(9.6)

for all intervals \( \Delta \subseteq I_0 \) and \( \varphi \in \mathcal{H} \).

Moreover, arguing essentially as before, we can deduce from (9.6) that

\[
\sum_{\lambda < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) Q^*_s \varphi \right\|^2 \leq \langle \delta (H - \lambda) \rangle_{Q^*_s} \varphi \text{ for all } \lambda \in I_0,
\]

(9.7a)

\[
\sum_{\lambda < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) Q^*_s \varphi \right\|^2 = \langle \delta (H - \lambda) \rangle_{Q^*_s} \varphi \text{ a.e. in } I_0, \text{ say for } \lambda \in I_0 \setminus N_0.
\]

(9.7b)

By first taking \( \varphi \) from any dense countable subset \( C \subseteq \mathcal{H} \) we then obtain (9.7a) and (9.7b) for all vectors from \( C \) with the null set \( N_0 \) in (9.7b) being independent of \( \varphi \in C \) (using the countability). Secondly we extend (9.7a) by continuity (using the density) and then in turn conclude that (9.7b) holds for all \( \varphi \in \mathcal{H} \) with the same null set \( N_0 \).

In particular we obtain that for some null subset \( N_0 \subseteq I_0 \)

\[
\forall \psi \in L^2_\infty : \sum_{\lambda < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) \psi \right\|^2 \leq \langle \delta (H - \lambda) \rangle_{\psi} \text{ for all } \lambda \in I_0.
\]

(9.8a)

\[
\forall \psi \in L^2_\infty : \sum_{\lambda < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) \psi \right\|^2 = \langle \delta (H - \lambda) \rangle_{\psi} \text{ for all } \lambda \in I_0 \setminus N_0.
\]

(9.8b)

We are lead to introduce the following concept.

**Definition 9.2.** An energy \( \lambda \in I_0 \) is stationary complete if

\[
\forall \psi \in L^2_\infty : \sum_{\lambda < \lambda_0} \left\| \Gamma^\pm_\beta (\lambda) \psi \right\|^2 = \langle \delta (H - \lambda) \rangle_{\psi}.
\]

(9.9)

We have shown that almost all energies in \( I_0 \) are stationary complete. The property missing for possibly having this conclusion for all energies in \( I_0 \) is clearly the continuity in \( \lambda \) of the sum to the left in (9.9). In Sect. 9.5 we characterize the concept of stationary completeness in terms of asymptotics of the (limiting) resolvent of \( H \).

**Remark 9.3.** The right-hand side of the identity (9.9) makes sense only for \( \lambda \notin \mathcal{T}_p(H) \) (which holds for \( \lambda \in I_0 \)). However the essential property for the concept of ‘on-shell stationary completeness’ would actually only need \( \lambda \notin \mathcal{T}(H) \). Indeed if \( \lambda \notin \mathcal{T}(H) \) is an eigenvalue of \( H \) the corresponding eigenprojection, say denoted \( P_\lambda \), maps to \( L^2_\infty \) and the expression \((H - P_\lambda - \lambda - i0)^{-1}\) (along with its imaginary part) does have an interpretation, cf. [AHS]. On the other hand the scattering theories for \( H \) and \( H - P_\lambda \) coincide, in particular the restricted generalized Fourier transform is the same. Consequently a more
general notion of completeness for $\lambda \in (\min T(H), \infty) \setminus T(H)$ would be (with $P_\lambda = 0$ if $\lambda \notin \sigma_{pp}(H)$)

$$\forall \psi \in L^2_\infty : \sum_{\lambda^\beta < \lambda_0} \| \Gamma_{\beta}^\pm (\lambda) \psi \|^2 = \langle \delta(H - P_\lambda - \lambda) \rangle \psi.$$ 

For simplicity we only consider $\lambda \in (\min T(H), \infty) \setminus T_p(H)$ in this paper (in fact localized by the requirement $\lambda \in I_0$).

Recall that the above operator $Q_s$ is only one out of several ‘$Q$-operators’ from our procedure. More precisely for each $k$ the expansions of $T^\pm_b$ used in Sects. 7 and 8 involved several (but finitely many) ‘$Q$-operators’ either on the above form $Q_s$ or being one of those from Lemma 7.1. So when varying $k$ we obtain in total an infinite family of ‘$Q$-operators’, say denoted by $Q$.

For any $Q \in Q$ and $\varphi \in L^2_\infty$ we record that $f_3(H)Q^*\varphi \in L^2_\infty$, cf. Remark 7.2 (i), and

$$\Gamma_{\beta}^\pm (\lambda)Q^*\varphi = \lim_{k \to \infty} \Gamma_{\beta,k}^\pm (\lambda) f_3(H)Q^*\varphi \in G_b; \ \lambda \in I_0.$$ 

However (more generally) for any finite sum $\psi = \sum_j Q^*_j \varphi_j$, where now $\varphi_j$ is arbitrary from $\mathcal{H}$, the quantity $\Gamma_{\beta,k}^\pm (\lambda)\varphi = \Gamma_{\beta,k}^\pm (\lambda) f_3(H)\psi$ is well-defined. Thanks to (9.1) it is unambiguously given as

$$\Gamma_{\beta,k}^\pm (\lambda)\varphi = \sum_j 1_{C_{b,k}} \Gamma_{\beta,k}^\pm (\lambda) Q^*_j \varphi_j \in G_b,$$

and as used before $1_{C_{b,k}} \Gamma_{\beta,k}^\pm (\lambda) = 1_{C_{b,k}} \Gamma_{\beta,k+1}^\pm (\lambda)$, allowing us to compute

$$\Gamma_{\beta}^\pm (\lambda)\varphi = \lim_{k \to \infty} \Gamma_{\beta,k}^\pm (\lambda)\varphi = \sum_j \lim_{k \to \infty} 1_{C_{b,k}} \Gamma_{\beta,k}^\pm (\lambda) Q^*_j \varphi_j$$

$$= \sum_j \lim_{k \to \infty} 1_{C_{b,k}} \Gamma_{\beta}^\pm (\lambda) Q^*_j \varphi_j$$

$$= \sum_j \Gamma_{\beta}^\pm (\lambda) Q^*_j \varphi_j \in G_b.$$ 

We used the following analogue of (9.4),

$$\| \Gamma_{\beta,k}^\pm (\lambda) Q^*_j \varphi_j \|^2 \leq \langle \delta(H - \lambda) \rangle f_2(H)Q^*_j \varphi_j \text{ for all } \lambda \in I_0.$$ 

(9.10)

This reasoning yields that

$$\forall Q \in Q : \Gamma_{\beta}^\pm (\lambda)Q^* \in \mathcal{L}(\mathcal{H}, G_b),$$

in fact with a weakly continuous dependence of $\lambda \in I_0$.

We are lead to introduce the subspace $\mathcal{V} \subseteq \mathcal{H}^{-2}$ of vectors $\psi$ as above, i.e. vectors being given as a finite sum of terms on the form $Q^*\varphi$, where $Q \in Q$ and $\varphi \in \mathcal{H}$. In the
following we consider \( V \) just as a vector space, it will not be equipped with any topology. Mimicking the proof of (9.7a) we conclude the following extension of (9.10),

\[
\forall \psi \in V : \sum_{\lambda^\beta < \lambda} \| \Gamma^\pm_\beta (\lambda) \psi \|^2 \leq (\delta(H - \lambda)) f_\pm(H) \psi \text{ for all } \lambda \in I_0.
\] (9.11)

We are lead to define

\[
\Gamma^\pm(\lambda) : V \to G := \sum_{\lambda^\beta < \lambda} G_b, \quad \Gamma^\pm(\lambda) = (\Gamma^\pm_\beta(\lambda))_{\lambda^\beta < \lambda}.
\]

By (9.11) and the above computation, \( \Gamma^\pm(\lambda) = \Gamma^\pm(\lambda) f_\pm(H) \) are well-defined linear maps. Using the natural Hilbert space structure on \( G \), (9.11) may be phrased as

\[
\forall \psi \in V : \| \Gamma^\pm(\lambda) \psi \|^2 \leq (\delta(H - \lambda)) f_\pm(H) \psi \text{ for all } \lambda \in I_0.
\] (9.12a)

The expression \( (\delta(H - \lambda)) f_\pm(H) \psi \) appearing to the right in (9.12a) is continuous in \( \lambda \in I_0 \), cf. Remark 7.2 (iii). The quantities on the left-hand side might not have this property, however we can record that \( \Gamma^\pm(\cdot) \psi \) are weakly continuous and note the following version of (9.9),

\[
\forall \psi \in V, \forall \text{ intervals } \Delta \subseteq I_0 : \int_\Delta \| \Gamma^\pm(\lambda) \psi \|^2 d\lambda = \int_\Delta (\delta(H - \lambda)) f_\pm(H) \psi \, d\lambda.
\] (9.12b)

Since almost all energies in \( I_0 \) are stationary complete the Parseval formulas (9.12b) may (independently) be seen as a consequence of the following extension of (9.9).

**Lemma 9.4.** Suppose \( \lambda \in I_0 \) is stationary complete. Then

\[
\forall \psi \in V : \| \Gamma^\pm(\lambda) \psi \|^2 = \sum_{\lambda^\beta < \lambda} \| \Gamma^\pm_\beta(\lambda) \psi \|^2 = (\delta(H - \lambda)) f_\pm(H) \psi.
\] (9.13)

**Proof.** Fix an arbitrary \( \psi = \sum_j Q_j^\dagger \varphi_j \in V \), let \( \chi_m = \chi_{-(r/m)} \) for \( m \in \mathbb{N} \) and then \( \psi_m = \sum_j Q_j^\dagger \chi_m \varphi_j \). Since \( \chi_m \varphi_j \in L^2_\infty \) and \( f_\pm(H) Q_j^* = O(r^0) \) (cf. Remark 7.2 (i)), also \( f_\pm(H) \psi_m \in L^2_\infty \), and for such vector we know the identities (9.9). By (9.12a) the sequences \( \Gamma^\pm(\lambda) f_\pm(H) \psi_m = \Gamma^\pm(\lambda) \psi_m \to \Gamma^\pm(\lambda) \psi \) in \( G \). Whence also \( \| \Gamma^\pm(\lambda) f_\pm(H) \psi_m \|^2 \to \| \Gamma^\pm(\lambda) \psi \|^2 \). Similarly \( (\delta(H - \lambda)) f_\pm(H) \psi_m \to (\delta(H - \lambda)) f_\pm(H) \psi \). Since the identities in (9.9) hold for all elements of two sequences we conclude the limiting identities as expressed by (9.13). \( \square \)

**Lemma 9.5.** Let \( \lambda \in I_0 \). Then the subspaces

\[
G^\pm_\gamma = (\Gamma^\pm(\lambda) \psi)_{\lambda^\beta < \lambda} \subseteq G \mid \psi \in L^2_\infty \text{ are dense in } G.
\]

**Proof.** It suffices to prove density of the subspaces \( G^\pm_\gamma = \{ \Gamma^\pm(\lambda) \psi \in G \mid \psi \in V \} \), cf. the proof of Lemma 9.4. Let us only consider the ‘minus’ case. For any ‘open channel’ \( \alpha, k \in \mathbb{N} \) and \( g_a \in G_a \) we consider

\[
\psi_a = \psi_{\alpha,k,g_a} = T_{\alpha,k}^a J_{\alpha}^a \gamma_a^\dagger (\lambda) 1_{C_{a,k}}(\xi_a) g_a.
\] (9.14a)

We can write \( \psi_a \) as a finite sum of vectors on the form \( Q^* \varphi \) where \( Q \in Q \) and \( \varphi \in \mathcal{H} \). Whence \( \psi_a \in V \) and \( \Gamma^\pm_\beta(\lambda) \psi_a \) is a well-defined vector in \( G_b \). We compute for
any \( l \in \mathbb{N} \) the vector \( 1_{C_{b,l}}(\hat{\xi}_b)\Gamma^\beta_\beta(\lambda)\psi_\alpha \) by using (9.2a) and (c.11b), concluding that
\[
1_{C_{b,l}}(\hat{\xi}_b)\Gamma^\beta_\beta(\lambda)\psi_\alpha = 0 \quad \text{if} \quad \beta \neq \alpha, \quad \text{while for} \quad \beta = \alpha \\
1_{C_{b,l}}(\hat{\xi}_b)\Gamma^\beta_\beta(\lambda)\psi_\alpha = 1_{C_{a,l}}(\hat{\xi}_a)\Gamma^-_\beta_\alpha(\lambda)\psi_\alpha = -\frac{2\lambda}{\pi}1_{C_{a,l}}\cap C_{a,k}(\hat{\xi}_a)g_a.
\]
By taking \( l \to \infty \) it follows that
\[
G^\gamma_\gamma \ni \Gamma^-_\beta(\lambda)\psi_\alpha = (\delta_{\beta\alpha} - \frac{2\lambda}{\pi}1_{C_{a,k}}g_a)\lambda_\beta < \lambda_0.
\] (9.14b)
By varying over \( \alpha \) (with \( \lambda^\alpha < \lambda_0 \)), \( k \in \mathbb{N} \) and \( g_a \in G_a \) the corresponding vectors to the right in (9.14b) span a dense subspace of \( G \). We are done since this subspace is contained in \( G^\gamma_\gamma \).

**Definition 9.6.** For \( \lambda \in I_0 \), assumed stationary complete, we let \( U(\lambda) \in \mathcal{L}(G) \) be the uniquely defined unitary operator \( U(\lambda) \) on \( G \) such that
\[
U(\lambda)\Gamma^-_\beta(\lambda)\psi = \Gamma^+_\beta(\lambda)\psi; \quad \psi \in \mathcal{V}.
\]
(Note that Lemmas 9.4 and 9.5 are used.)

**Definition 9.7.** For given \( \lambda \in I_0 \) the scattering matrix \( S(\lambda) \) is the unique bounded operator on \( G \) with each \( \beta\alpha \)-entry given by \( S_{\beta\alpha}(\lambda) \) as specified in Theorem 3.1. (Note that \( \|S(\lambda)\| \leq 1 \) by the orthogonality of channels and the weak continuity.)

**Theorem 9.8.** For any \( \lambda \in I_0 \) assumed stationary complete the scattering matrix \( S(\lambda) \) is unitary. Moreover the \( \mathcal{L}(G) \)-valued function \( S(\cdot) \) is strongly continuous at \( \lambda \), for any \( \psi \in \mathcal{V} \) the \( G \)-valued functions \( \Gamma^\pm(\lambda)\psi = (\Gamma^\pm_\alpha(\cdot)\psi)_{\lambda^\alpha < \lambda_0} \) are norm-continuous at \( \lambda \) and
\[
S(\lambda)\Gamma^-_\beta(\lambda)\psi = \Gamma^+_\beta(\lambda)\psi.
\] (9.15)

**Proof.** It suffices to check that \( S_{\beta\alpha}(\lambda)g_a = U_{\beta\alpha}(\lambda)g_a \). By (9.14a), (9.14b), (9.2a), (8.2) and (3.8a)
\[
1_{C_{b,l}}(\hat{\xi}_b)U_{\beta\alpha}(\lambda)1_{C_{a,k}}g_a = -\frac{\pi}{2\lambda_\alpha}\Gamma^+_\beta_\alpha(\lambda)T^-_{\beta,k}J^-_{\beta}J^+_{\beta}((T^+_l)^*)_\beta\delta(H - \lambda)T^-_{\alpha,k}J^-_{\alpha}J^+_{\alpha}((T^+_l)^*)_\alpha1_{C_{a,k}}(\hat{\xi}_a)g_a = \frac{\lambda_\alpha}{160\lambda_\lambda_\alpha}1_{C_{b,l}}(\hat{\xi}_b)\hat{S}_{\beta\alpha}(\lambda)1_{C_{a,k}}(\hat{\xi}_a)g_a = 1_{C_{b,l}}(\hat{\xi}_b)\hat{S}_{\beta\alpha}(\lambda)1_{C_{a,k}}(\hat{\xi}_a)g_a.
\]
We take \( k, l \to \infty \) and conclude that indeed \( U_{\beta\alpha}(\lambda)g_a = S_{\beta\alpha}(\lambda)g_a \) for any \( g_a \in G_a \). \( \Box \)

The free energy \( K := \text{diag}(k_a) \) is diagonalised on the neighbourhood \( I_0 \ni \lambda_0 \) by
\[
F_{I_0} : \Sigma^\circ_{\lambda^\alpha < \lambda_0} 1_{I_0}(k_a)L^2(X_a) \to \Sigma^\circ_{\lambda^\alpha < \lambda_0} L^2(I_0, G_a) = \int_{I_0} G \, d\lambda, \\
F_{I_0} = \text{diag}(F_a).
\]
Recall from Theorem 3.1 the notation \( S_{\beta\alpha} = (W^+_\beta)^{-1}W^-_{\alpha} \).
Corollary 9.9. The scattering matrix considered as a map

\[ I_0 \ni \lambda \rightarrow S(\lambda) \in \mathcal{L}(\mathcal{G}) \] is weakly continuous.

Within the class of such maps it is uniquely determined by the identity

\[ F_{I_0} S F_{I_0}^{-1} = F_{I_0} (S_{\beta \alpha})_{\lambda^\beta, \lambda^\alpha < \lambda_0} F_{I_0}^{-1} = \int_{I_0} \oplus S(\lambda) \, d\lambda. \] (9.16)

For all stationary complete energies in \( I_0 \) (in particular for almost all energies in \( I_0 \)) the scattering matrix is unitary and strongly continuous.

Remark. It is stated in [Ya3, Theorem 6.7] that for short-range systems the scattering matrix is strongly continuous away from thresholds and eigenvalues. The proof appears incorrect to the author (however the assertion of weak continuity is correctly derived). Similarly the same assertion is incorrectly stated in [As, Corollary 1].

9.3. Besov space setting and minimum generalized eigenfunctions. We considered in Lemma 9.4 the operators \( \Gamma^{\pm}(\lambda) = \left( \Gamma^{\pm}(\lambda) \right)_{\lambda^\beta < \lambda} \) as mappings \( \Gamma^{\pm}(\lambda) : \mathcal{V} \rightarrow \mathcal{G}; \lambda \in I_0 \). The extended space \( \mathcal{V} \supseteq L^2_\infty \) was convenient for our constructions, however there is a different extension which appears 'more natural', namely the Besov space \( B \) in which \( L^2_\infty \) is densely embedded. Using (2.6b), (9.8a) and extension by continuity it follows that \( \Gamma^{\pm}(\lambda) \in \mathcal{L}(\mathcal{B}, \mathcal{G}) \) with a uniform bound in \( \lambda \in I_0 \). Clearly \( \Gamma^{\pm}(\cdot) \in \mathcal{L}(\mathcal{B}, \mathcal{G}) \) are strongly continuous at any stationary complete \( \lambda \in I_0 \) in which case (9.15) also holds for \( \psi \in B \). Due to Theorem 9.11 (stated below) the vector \( \Gamma^{-}(\lambda)\psi \) with \( \psi \in B \) (appearing to the left in (9.15)) in this case is the generic form of vectors in \( G \), indeed explaining why \( B \) is 'natural'.

Definition 9.10. For any \( \lambda \in I_0 \) we let

\[ \mathcal{E}_\lambda = \{ \phi \in B^* \cap H^2_{\text{loc}}(X) \mid (H - \lambda)\phi = 0 \} \]

and call its elements minimum generalized eigenfunctions.

This terminology is motivated by Corollary 5.3. Note also that \( \delta(H - \lambda)\psi \in \mathcal{E}_\lambda \) for \( \psi \in B \) (as well as for \( \psi \in f_2(H)\mathcal{V} \)). However if \( \lambda \) is stationary complete, in fact any \( \phi \in \mathcal{E}_\lambda \) has this form. This is a consequence of the following more general result. We remark that there are analogous assertions for various one-body type Hamiltonians in the literature, see for example [GY,DS,IS2].

Theorem 9.11. For any \( \lambda \in I_0 \) assumed stationary complete the operators \( \Gamma^{\pm}(\lambda) : B \rightarrow G, \Gamma^{\pm}(\lambda)^* : G \rightarrow \mathcal{E}_\lambda \) and \( \delta(H - \lambda) : B \rightarrow \mathcal{E}_\lambda \) all map onto. Moreover the wave matrices \( \Gamma^{\pm}(\lambda)^* : G \rightarrow \mathcal{E}_\lambda(\subseteq B^*) \) are bi-continuous linear isomorphisms.

Proof. The ranges of \( \Gamma^{\pm}(\lambda) \) are dense due to Lemma 9.5, and clearly \( \mathcal{E}_\lambda \) is closed in \( B^* \). Thanks to the open mapping and closed range theorems (see for example see [Yo]) it then suffices to show that \( \Gamma^{\pm}(\lambda)^* \) map onto \( \mathcal{E}_\lambda \). (Here and below we use the identities \( \Gamma^{\pm}(\lambda)^*\Gamma^{\pm}(\lambda) = \delta(H - \lambda) \) as quadratic forms on \( B \).)

For convenience we only show that \( \Gamma^+(\lambda)^* \) maps onto \( \mathcal{E}_\lambda \). Let \( B \) be given by (5.3a) (possibly rescaled), and let \( h : \mathbb{R} \rightarrow \mathbb{R} \) be given by \( h(e) = e f_4(e) \) (this function is
denoted by \( g \) in Sect. 7) and note that \( h(H) - \lambda \) annihilates the spaces \( \Gamma^+(\lambda)^* \subseteq \mathcal{E}_\lambda \). Thanks to Corollary 5.3 2) there exists \( \epsilon > 0 \) such that

\[
\chi_-(\pm B/\epsilon)R(\lambda \pm i0)\psi \in \mathcal{B}_0^* \quad \text{for all} \quad \psi \in \mathcal{B}. 
\]  

(Recall the fact that any operator \( T = O(r^0) \) is bounded on \( \mathcal{B}^* \).) For any given \( \phi \in \mathcal{E}_\lambda \) we introduce for \( \epsilon > 0 \) taken small enough

\[
\phi_+ = \chi_+^2(B/\epsilon)\phi, \quad \phi_- = \phi - \phi_+, \quad g_\rho = \Gamma^+(\lambda)\chi_\rho(h(H) - \lambda)\phi_+;
\]

here \( \chi_\rho = \chi_-(r/\rho), \rho > 1 \). By commuting the factor \( h(H) - \lambda \) through the factor \( \chi_\rho \) we deduce that \( \sup_{\rho > 1} \| g_\rho \| < \infty \) (note that only the commutator contributes). Let \( g_\infty = \text{w-lim}_{n \to \infty} g_{\rho_n} \in \mathcal{G} \), the limit taken along a suitable sequence \( \rho_n \to \infty \), and let \( \phi' = 2\pi i\Gamma^+(\lambda)^*g_\infty. \) We can now mimic the proof of [IS2, Lemma 3.12] as follows concluding that \( \phi = \phi' \). The needed ingredients are the identity \( \Gamma^+(\lambda)^*\Gamma^+(\lambda) = \delta(H - \lambda) \), the decomposition \( \phi_+ = \phi - \phi_- \) and the bounds (9.17) (in the last step below implemented after commuting using parts of Sects. 6.2 and 6.3). We compute the weak*-limits

\[
\phi' = \text{w*-lim}_{n \to \infty} \left( R(\lambda + i0) - R(\lambda - i0) \right) \chi_{\rho_n}(h(H) - \lambda) \chi_+^2(B/\epsilon)\phi \\
= \text{w*-lim}_{n \to \infty} \left( R(\lambda + i0)\chi_{\rho_n}, h(H) - \lambda \right) \chi_+^2(B/\epsilon)\phi + \tilde{h}(H)\chi_+^2(B/\epsilon)\phi \\
+ \text{w*-lim}_{n \to \infty} R(\lambda - i0)\chi_{\rho_n}(h(H) - \lambda) \chi_+^2(B/\epsilon)\phi \\
= \text{w*-lim}_{n \to \infty} \left( R(\lambda + i0)\chi_{\rho_n}, h(H) \right) \chi_+^2(B/\epsilon)\phi + \tilde{h}(H)\chi_+^2(B/\epsilon)\phi \\
+ \text{w*-lim}_{n \to \infty} R(\lambda - i0)\chi_{\rho_n}, h(H) \chi_+^2(B/\epsilon)\phi + \tilde{h}(H)\chi_+^2(B/\epsilon)\phi \\
= \tilde{h}(H)(\chi_+^2(B/\epsilon)\phi + \chi_-^2(B/\epsilon)\phi) = \phi,
\]

showing that indeed \( \phi = \phi' \), and therefore in particular that \( \phi \in \text{ran} \Gamma^+(\lambda)^* \) as wanted.

For any channel \( \beta = (b, \lambda^\beta, u^\beta) \) and \( g \in \mathcal{G}_b \) we introduce functions on \( X_b \) as

\[
v^\pm_{\beta,\lambda}[g](x_b) = \pm \frac{1}{2\pi} \left( c^\pm_{\beta}(\lambda) \right)^{-1} \chi_+(|x_b|) |x_b|^{(1-n_b)/2} e^{\pm iK_b(x_b, \lambda^\beta)} \varphi(\pm x_b) = \frac{1}{\sqrt{2\pi}} e^{\pm i\pi \lambda^- b / 2} e^{\pm i\pi \lambda^+ b / 2} \chi_+^2(B/\epsilon)\phi + \tilde{h}(H)\chi_+^2(B/\epsilon)\phi.
\]  

(9.18)

Note that in terms of the notation (2.18), \( v^\pm_{\beta,\lambda}[g] = \check{v}^\pm_{\beta,\lambda}[g] \).

**Lemma 9.12.** Let \( \beta = (b, \lambda^\beta, u^\beta) \) be any channel with \( \lambda^\beta < \lambda_0 \). For \( \epsilon > 0 \) taken small enough the following bounds hold for any \( \lambda \in I_0 \) and any \( g \in \mathcal{G}_b \).

\[
\chi_-^2(\mp B/\epsilon)\Gamma^\pm_{\beta}(\lambda)^*g - J_{\beta}v^\pm_{\beta,\lambda}[g] \in B_0^*.
\]

**Proof.** We shall use (9.2b). From its derivation, see (C.5b), the second term contributes by a term in agreement with the following interpretation given by taking limits in \( \mathcal{L}(\mathcal{G}_b, L^2_{-1}(X)) \).

\[
f_2(H)R(\lambda \mp i0)T_{b,k}^\pm J_{\beta}v^\pm_{\beta,\lambda}[g] \in B_0^*.
\]  

For any channel \( \beta = (b, \lambda^\beta, u^\beta) \) and any \( g \in \mathcal{G}_b \).
Conversely, if \( g_k \) for any \( g_k \) \( B \)

By combining (7.3c) and Lemma 8.3 we obtain that

\[
\sup_{\epsilon > 0} \left\| R(\lambda \mp i\epsilon) T_{b,k}^\pm J_\beta \tilde{\gamma}_b^\pm(\lambda_\beta)^*1_{C_{b,k}} \right\|_{\mathcal{L}(G_b, B^*)} < \infty.
\]

Whence in fact

\[
S^\mp := f_2(H) R(\lambda \mp i0) T_{b,k}^\pm J_\beta \tilde{\gamma}_b^\pm(\lambda_\beta)^*1_{C_{b,k}} \in \mathcal{L}(G_b, B^*).
\]

Next we expand for each sign \( T_{b,k}^\pm \) in terms of the \( Q \)-operators and consider the approximation of \( S^\mp \) given by replacing \( R(\lambda \mp i0) f_2(H) Q_j^* \) by \( R(\lambda \mp i0) f_2(H) Q_j^* \chi_\rho \), say denoted by \( S^\mp_\rho \) (as in the proof of Theorem 9.11, here \( \chi_\rho = \chi_-(r/\rho) \) with \( \rho \) considered large). Now \( S^\mp_\rho \to S^\mp \) strongly in \( \mathcal{L}(G_b, B^*) \), and by (9.17) \( \chi^2(\mp B/\epsilon) S^\mp \in \mathcal{L}(G_b, B^*_0) \).

Whence also \( \chi^2(\mp B/\epsilon) S^\mp \in \mathcal{L}(G_b, B^*_0) \).

By density and continuity we are left with showing

\[
\frac{1}{4\lambda_\beta} \chi^2(\mp B/\epsilon) f_2(H) \Phi_{b,k}^\pm J_\beta \tilde{\gamma}_b^\pm(\lambda_\beta)^*1_{C_{b,k}} g_k - J_\beta \tilde{\nu}_{b,\beta,\lambda}[g_k] \in B^*_0
\]

for any \( g_k \in 1_{C_{b,k}} g \in C^\infty(C_b), k \in \mathbb{N} \). In turn, thanks to the form of \( \Phi_{b,k}^\pm \) (and commutation), it remains to show that

\[
\frac{1}{4\lambda_\beta} f_2(H) \Phi_{b,k}^\pm J_\beta \tilde{\gamma}_b^\pm(\lambda_\beta)^*1_{C_{b,k}} g_k - J_\beta \tilde{\nu}_{b,\beta,\lambda}[g_k] \in B^*_0
\]

for any such \( g_k \).

By using the analysis of Step IV in the proof of Lemma 8.3 (not to be repeated here) the first term simplifies exactly as the subtracted second term modulo a term in \( B^*_0 \). \( \square \)

We can now give a ‘geometric interpretation’ of the operator \( \Gamma_\alpha^- (\lambda)^* \) needed for a wave packet description of an incoming \( \alpha \)-channel experiment, energy-localized in \( I_0 \), cf. a discussion in Sect. 1.1. More precisely we characterize for any stationary scattering energy the space of generalized eigenfunctions in the range of \( \Gamma_\alpha^- (\lambda)^* \) in terms of asymptotics expressed by the quasi-modes \( J_\beta \tilde{\nu}_{b,\beta,\lambda}[-] \) of Lemma 9.12. The result may also be viewed as a characterization of the incoming \( \alpha \)-channel part of the scattering matrix. We equip the space \( B^* / B^*_0 \) with the quotient-norm, thus making it a Banach space. With \( F_\rho := F([\lambda \in X \mid |\lambda| < \rho]) \), \( \rho > 1 \), the expression

\[
\|u\|_{\text{quo}} = \limsup_{\rho \to \infty} \rho^{-1/2} \| F_\rho u \| \quad (9.19)
\]

is a norm on \( B^* / B^*_0 \) equivalent with the quotient-norm.

**Theorem 9.13.** Let \( \lambda \in I_0 \) be stationary complete and \( \alpha = (a, \lambda_\alpha, u_\alpha) \) be any channel with \( \lambda_\alpha < \lambda_0 \). Then the following existence and uniqueness results hold for any \( g \in \mathcal{G}_\alpha \).

1. Let \( u = \Gamma_\alpha^- (\lambda)^* g \), and let \( (g_\beta)_{\lambda_\beta < \lambda_0} \in \mathcal{G} \) be given by \( g_\beta = S_{\beta\alpha}(\lambda)^* g \). Then, as an identity in \( B^* / B^*_0 \),

\[
u = J_\alpha \tilde{\nu}_{\alpha,\beta} [g] + \sum_{\lambda_\beta < \lambda_0} J_\beta \tilde{\nu}_{\beta,\lambda} [g_\beta].
\] \( (9.20) \)

2. Conversely, if (9.20) is fulfilled for some \( u \in \mathcal{E}_\lambda \) and some \( (g_\beta)_{\lambda_\beta < \lambda_0} \in \mathcal{G} \), then \( u = \Gamma_\alpha^- (\lambda)^* g \) and \( g_\beta = S_{\beta\alpha}(\lambda)^* g \) for all \( \lambda_\beta < \lambda_0 \).
Proof. I. Using the norm (9.19) the Cauchy criterion assures that for any \((g_\beta)_{\lambda_0^\beta<\lambda_0} \in \mathcal{G}\) the right-hand side of (9.20) is well-defined. The cross-terms do not contribute for the following reason: For given different open channels \(\beta = (b, \lambda_0^\beta, u_0^\beta) \) and \(\beta' = (b', \lambda_0^\beta', u_0^\beta')\) with \(b \neq b'\), Fubini's theorem and the fact that \(<u_0^\beta, u_0^\beta'>_{\mathcal{H}^b} = 0\) clearly imply that in the \(\rho \to \infty\) limit
\[
\langle F_\rho J_\beta \tilde{v}_\beta^+, [g_\beta], F_\rho J_{\beta'} \tilde{v}_{\beta'}^+, [g_{\beta'}] \rangle_{\mathcal{H}} = \int_X F_\rho u_0^{\beta'} \tilde{v}_{\beta'}^+, [g_{\beta'}] F_\rho u_0^\beta \tilde{v}_\beta^+, [g_\beta] \, dx \to 0.
\]
(9.21)

If \(b \neq b'\) we let \(c \in \mathcal{A}\) be given by \(X^c = X^b + X^{b'}\) and \(n_c = \dim X_c\), and note that in this case \(n_c \leq (n_b + n_{b'})/2\). By approximation, and by using Fubini's theorem and the Cauchy–Schwarz inequality, we can assume that the functions \(g_\beta, g_{\beta'}, u_0^\beta\) and \(u_0^{\beta'}\) are all bounded with compact support. With this assumption we claim that the integral in (9.21) is \(\mathcal{O}(\rho^{1/2})\) (and hence in particular \(o(\rho)\) as wanted). On the support of the integrand \(|x^c| \leq C_1 (|x^b| + |x^{b'}|) \leq C_2\). Whence for \(n_c = 0\) it follows that \(|x|\) is bounded, and consequently the integral is \(\mathcal{O}(\rho^0) = \mathcal{O}(\rho^{1/2})\). More generally we can further assume that the integrand is supported in \(|x| \geq C_3\) for a (big) constant \(C_3 \geq 1\), allowing us to estimate factors of the integrand as
\[
|x_b|^{(1-n_b)/2}|x_{b'}|^{(1-n_{b'})/2} \leq C_4 |x|^{1-(n_b+n_{b'})/2} \leq C_4 |x|^{(1-n_c)|x|^{-1/2}}.
\]

For \(n_c \geq 1\) we estimate \(|x|^{(1-n_c)|x|^{-1/2}} \leq |x_c|^{(1-n_c)|x_c|^{-1/2}}\) and use spherical coordinates in \(X_c\) and the fact that \(|x_c| \leq |x| < \rho\), indeed yielding the desired bound \(\mathcal{O}(\rho^{1/2})\).

We conclude that indeed the right-hand side limit in (9.20) exists and that
\[
\left\| \sum_{\lambda_0^\beta<\lambda_0} J_\beta \tilde{v}_\beta^+, [g_\beta] \right\|_{quo}^2 = \sum_{\lambda_0^\beta<\lambda_0} (4\pi \lambda_0^{1/2})^{-1} \|g_\beta\|^2.
\]
(9.22)

II. For 1) we decompose \(u = \Gamma^-_\alpha (\lambda)^* g\) as
\[
u = \chi_+^2 (B/\epsilon) u + \chi_+^2 (B/\epsilon) u.
\]
The first term corresponds to the first term to the right in (9.20) thanks to Lemma 9.12. For the second term we substitute, cf. (1.4) and (9.15),
\[
\Gamma^-_\alpha (\lambda)^* g = \sum_{\lambda_0^\beta<\lambda} \Gamma_\beta^+(\lambda)^* g_\beta; \quad g_\beta = S_{\beta\alpha}(\lambda) g.
\]
The series converges in \(\mathcal{B}^*\). By continuity we can take the factor \(\chi_+^2 (B/\epsilon)\) inside the summation. Then we obtain (1) by applying Lemma 9.12 and Corollary 8.2 to the terms of the (convergent) series \(\sum_{\lambda_0^\beta<\lambda} \chi_+^2 (B/\epsilon) \Gamma_\beta^+(\lambda)^* g_\beta\).

III. For (2) we note that \(u' := u - \Gamma^-_\alpha (\lambda)^* g\) is in \(\mathcal{E}_\lambda\) and obeys
\[
\chi_-(B/\epsilon) u' = 0 \text{ in } \mathcal{B}^*/\mathcal{B}_0^*.
\]
Here we first used (9.20) to \(\Gamma^-_\alpha (\lambda)^* g\) to obtain a similar representation of \(u' \in \mathcal{B}^*/\mathcal{B}_0^*\), then we multiplied by \(\chi_-(B/\epsilon)\) using the fact that this operator is continuous in \(\mathcal{B}^*/\mathcal{B}_0^*\) and finally we used again Corollary 8.2 to the terms of the resulting series. This means that \(\chi_-(B/\epsilon) u' \in \mathcal{B}^*_0\) and then in turn, thanks to the second part of Corollary 5.3, that \(u' = 0\). Finally (9.22) applied to the mentioned representation of \(u'\) implies that also \(g_\beta - S_{\beta\alpha}(\lambda) g = 0\), as wanted.

\(\square\)
9.4. Restricted channel wave operators as radial limits. We represent the restricted wave operators \( \Gamma^\pm_\alpha(\lambda) \) in Theorem 9.1 as radial limits (taken in a weak sense). We denote \( F_\rho = F([x \in \mathbf{X} \mid |x| < \rho]) \) for \( \rho > 1 \).

**Proposition 9.14.** For any channel \( \alpha = (\alpha, \lambda^\alpha, u^\alpha), \lambda \in I_0, \psi \in \mathcal{B}(\mathbf{X}) \) and \( g \in \mathcal{G}_a \)

\[
(\Gamma^\pm_\alpha(\lambda), \psi, g) = \lim_{\rho \to \infty} c^\pm_\alpha(\lambda) \rho^{-1} (F_\rho R(\lambda \pm i0) \psi, F_\rho \left( u^\alpha \otimes |x_a|^{(1-n_a)/2} e^{\pm iK_a(|x_a| \cdot \lambda_a)} g(\pm \cdot) \right)).
\]

**(9.23)**

**Proof.** Since \( r(x) - |x| \) is a bounded function on \( \mathbf{X} \) we can replace \( F_\rho \) by \( F([x \in \mathbf{X} \mid r(x) < \rho]) \). Moreover we can assume that \( g \) is in the dense subspace \( C_\infty^r(C'_a) \) of \( \mathcal{G}_a \).

Fixing any such \( g \) we verify (9.23), noting that \( g = 1_{C_2 \cdot k} \) for \( k \) large enough.

We approximate by a decreasing smooth cutoff function \( \chi_\epsilon \in C_\infty(\mathbb{R}) \) (for \( \epsilon \in (0, 1/3) \)) such that

\[
\chi_\epsilon(t) = \begin{cases} 
1 & \text{for } t \leq \epsilon, \\
1 + \frac{3}{2} \epsilon - t & \text{for } 3\epsilon \leq t \leq 1, \\
0 & \text{for } t \geq 1 + 2\epsilon,
\end{cases}
\]

and \( \chi'_\epsilon \geq -1 \). Letting \( \chi_{\epsilon, \rho} = \chi_\epsilon(r/\rho), \rho > 1, \) and \( \psi_2 = f_2(H)\psi \) we compute using (9.2b)

\[
\langle \Gamma^\pm_\alpha(\lambda), \psi, g \rangle = \langle \Gamma^\pm_{\alpha,k}(\lambda), \psi, g \rangle = (4\lambda_\alpha)^{-1} \left( \langle \psi_2, \Phi^\pm_{\alpha,k} J_a \gamma_\alpha^\pm(\lambda_\alpha)^* g \rangle - \lim_{\rho \to \infty} \langle R(\lambda \pm i0) \psi_2, \chi_{\epsilon, \rho} (H - \lambda) \Phi^\pm_{\alpha,k} J_a \gamma_\alpha^\pm(\lambda_\alpha)^* g \rangle \right)
\]

\[
= -i(4\lambda_\alpha)^{-1} \lim_{\rho \to \infty} \rho^{-1} \langle R(\lambda \pm i0) \psi_2, B \chi'_\epsilon(r/\rho) \Phi^\pm_{\alpha,k} J_a \gamma_\alpha^\pm(\lambda_\alpha)^* g \rangle
\]

\[
= \mp i2\sqrt{\lambda_\alpha} \lim_{\rho \to \infty} \rho^{-1} \langle R(\lambda \pm i0) \psi_2, \chi'_\epsilon(r/\rho) J_a \gamma_\alpha^\pm[g] \rangle
\]

\[
= \mp i2\sqrt{\lambda_\alpha} \left( \pm 2^{-1} i\lambda_\alpha^{-1/2} c^\pm_\alpha(\lambda) \right) \lim_{\rho \to \infty} \rho^{-1} \langle R(\lambda \pm i0) \psi, \chi'_\epsilon(r/\rho) \left( u^\alpha \otimes |x_a|^{(1-n_a)/2} e^{\pm iK_a(|x_a| \cdot \lambda_a)} g(\pm \cdot) \right) \rangle
\]

\[
= -c^\pm_\alpha(\lambda) \lim_{\rho \to \infty} \rho^{-1} \langle R(\lambda \pm i0) \psi, \chi'_\epsilon(r/\rho) \left( u^\alpha \otimes |x_a|^{(1-n_a)/2} e^{\pm iK_a(|x_a| \cdot \lambda_a)} g(\pm \cdot) \right) \rangle.
\]

In the fourth step of the computation we used the analysis of Step IV in the proof of Lemma 8.3. We can now take \( \epsilon \to 0 \) swapping limits, using that the error is \( \mathcal{O}(\sqrt{\epsilon}) \) uniformly in \( \rho > 1 \) and [Ru1, Theorem 7.11]).

**Remark 9.15.** (1) The formula extends (by continuity) to \( \psi \) being a sum of vectors of the form \( Q^* \varphi, \varphi \in \mathcal{H} = L^2(\mathbf{X}) \) and operators \( Q \in \mathcal{Q} \) as specified in Sect. 9.2, i.e. with the notation from there to any vector \( \psi \in \mathcal{Y} \).

(2) The right-hand side of (9.23) may be viewed as weak radial limits of states of the form \( R(\lambda \pm i0) \psi \). A stronger concept of radial limits (conforming with the weak one) appears in several papers, see [GY, HS, II, Is1, IS2], and is actually needed for the conclusions on the one-body problem in Sect. 2.3. It requires radiation condition
bounds, which presently are only fully understood for the one-body problem. Nevertheless a formula like \((9.23)\) may be viewed as an independent ‘geometric’ definition of stationary scattering theory, and it is possible to show that such theory (essentially) coincides with the time-dependent one \([Va]\), see also the recent work \([As]\) for a similar accomplishment. However, as far as we know, formulas like \((9.23)\) were proven previously only under an additional decay condition on the channel bound states.

9.5. Stationary completeness, a characterization. Recall that from Definition 9.2 that \(\lambda \in I_0\) is stationary complete if

\[
\forall \psi \in L^2_\infty : \sum_{\lambda^\beta < \lambda_0} \left\| \Gamma^\pm_{\beta}(\lambda) \psi \right\|^2 = \langle \delta(H - \lambda) \rangle \psi \quad \text{(for both signs)}.
\]

Proposition 9.16. Let \(\lambda \in I_0\). Then the following four assertions are equivalent:

1. \(\lambda \in I_0\) is stationary complete.
2. For all \(\psi \in L^2_\infty\)

\[
\sum_{\lambda^\beta < \lambda_0} \left\| \Gamma^\pm_{\beta}(\lambda) \psi \right\|^2 = \langle \delta(H - \lambda) \rangle \psi.
\]

3. For all \(\psi \in L^2_\infty\), as an identity in \(B^*/B^*_0\):

\[
R(\lambda + i0) \psi = 2\pi i \sum_{\lambda^\beta < \lambda_0} J_{\beta} \tilde{v}^*_{\beta,\lambda}[g_{\beta}]; \quad g_{\beta} = \Gamma^+_\beta(\lambda) \psi.
\]

4. For all \(\psi \in L^2_\infty\) there exists \((g_{\beta})_{\lambda^\beta < \lambda_0} \in \mathcal{G}\) such that, as an identity in \(B^*/B^*_0\):

\[
R(\lambda + i0) \psi = 2\pi i \sum_{\lambda^\beta < \lambda_0} J_{\beta} \tilde{v}^*_{\beta,\lambda}[g_{\beta}].
\]

Proof. The equivalence of (1) and (2) follows from the time-reversal property, i.e. by properties of conjugation by the operator of complex conjugation (seen for example by using Proposition 9.14).

Let for any \(\psi \in L^2_\infty\)

\[
u = \delta(H - \lambda) \psi - \sum_{\lambda^\beta < \lambda_0} \Gamma^+_\beta(\lambda)^* g_{\beta}; \quad g_{\beta} = \Gamma^+_\beta(\lambda) \psi.
\]

Note that \(\nu\) is a well-defined element of \(E_\lambda\), cf. the Besov space version of \((9.8a)\). Applying \((9.17)\) and Lemma 9.12 we obtain, as an identity in \(B^*/B^*_0\),

\[2\pi \chi_-(\frac{-B}{\epsilon}) u = R(\lambda + i0) \psi - 2\pi i \sum_{\lambda^\beta < \lambda_0} J_{\beta} \tilde{v}^*_{\beta,\lambda}[g_{\beta}].\]

Using the fact that (2) implies that \(u = 0\) (thanks to \((9.8a)\) and the Cauchy–Schwarz inequality), clearly (3) is a consequence of (2). Conversely (3) implies (2) by combining the above computation with Corollary 5.3, cf. Step III of the proof of Theorem 9.13.

The assertion (3) is stronger than (4). Conversely if \((9.26)\) holds for \(\psi \in L^2_\infty\) for some \((g_{\beta})_{\lambda^\beta < \lambda_0} \in \mathcal{G}\), then the recipe \((9.23)\) and Step I of the proof of Theorem 9.13 allow us to conclude that indeed \(g_{\beta} = \Gamma^+_\beta(\lambda) \psi\). (Note that the expression \((9.23)\) fits well with the norm-topology on \(B^*\), like also \((9.19)\) does.)
Remark. It is not known for the general many-body problem (considered in this paper) if all energies in $\mathcal{E} = (\min T(H), \infty) \setminus T_0(H)$ are stationary complete. However this property is proven for the three-body problem in [Sk3] (under slightly stronger conditions than Condition 2.1), done by verifying Proposition 9.16 4) in this case.

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Appendix A: Proof of (3.2b), (3.7a) and (3.7b)

For convenience we consider only the assertions (3.2b), (3.7a) and (3.7b) for $t \to +\infty$. Our procedure of proof relies on standard stationary phase analysis on which we omit details, see for example [Hö1, Hö2, II] for elaborate accounts. In addition we will use Lemma 8.1 (proved in Sect. 8 independently of the mentioned assertions). Recall that Lemma 8.1 is stated in terms of the auxiliary operator $B_a = 2\Re((x_a/r) \cdot p_a)$ introduced in (8.3).

(3.2b): We need to show that

$$s\text{-}\lim_{t \to +\infty} (I - N^a_+) J_a \hat{\varphi} e^{-ixka} f_1(k_a) = 0,$$

which by the boundedness of $N^a_+ = A_1 A^q_2 (A^q_3)^2 A^q_2 A_1$ (with $A_1 = A_{1+}$ and $A^q_3 = A^q_{3+}$) amounts to showing that

$$\forall \varphi \in L^2(X_a) : \quad \lim_{t \to +\infty} \| (I - A_1 A^q_2 (A^q_3)^2 A^q_2 A_1)(u^a \otimes \varphi(t)) \| = 0; \quad \varphi(t) = e^{-iS_a(p_a,t)} \varphi_1, \quad \varphi_1 = f_1(k_a) \varphi. \tag{A.1}$$

We can assume that $\varphi$ is smooth in momentum space, i.e. that its Fourier transform $\hat{\varphi} \in C^\infty(X_a)$. Then $\hat{\varphi}_1 \in C^\infty_c(X_a \setminus \{0\}),$ and

$$\varphi(t)(x_a) = (2\pi)^{-n/2} \int e^{i\xi_a \cdot x_a} e^{-iS_a(\xi_a,t)} \hat{\varphi}_1(\xi_a) d\xi_a. \tag{A.2}$$

By assumption $B = \sup \hat{\varphi}_1 \subseteq X_a \setminus \{0\}$ is compact. Uniformly in $\xi \in B$ for any such $B$ the function $S_a$ obeys the bounds (as $t \to \infty$)

$$\partial_{\xi}^r \partial_{\xi}^k (S_a(\xi,t) - t\xi^2) = \begin{cases} \mathcal{O}(t^{1-k-\mu}) & \text{for } |\gamma| + k \leq 2, \\ \mathcal{O}(t^{1-k}) & \text{for } |\gamma| + k \geq 3. \end{cases} \tag{A.3}$$

The stationary phase method then applies, see for example [Hö2, Theorem 7.7.6]. We note that the part of (A.3) given by the cases $|\gamma| + k \leq 1$ is a consequence of (2.8a)–(2.8c) (in turn appearing as [II, Lemma 6.1]). We give at the end of the “Appendix” a proof of (A.3) for the cases $|\gamma| + k \geq 2$. Taking for the moment the bounds for granted the stationary phase method yields the effective localization $|x_a| \geq \varepsilon \gamma$ for some $\varepsilon > 0$, which means that we can freely insert the factor $\chi_+ (|x_a|/(\varepsilon \gamma))$ for some $\varepsilon > 0$ in front
of \( \varphi(t) \) in the tensor product in (A.1). (Recalling that \( f_1 \) is supported near \( \lambda_0 \) we can for example use this factor with \( \epsilon' = \sqrt{\lambda_0 - \lambda^a} \).)

Similarly we can record that

\[
\left\| \left( B_a - 2k_a^2 \right) \left( u^a \otimes \varphi(t) \right) \right\| \to 0 \text{ for } t \to \infty, \tag{A.4}
\]

which in addition to (A.2) and the stationary phase method is based on (5.2a) and the fact that \( x^a \) is a localized variable due to the presence of the channel bound state \( u^a \).

More precisely the localization can be implemented in terms of an insertion of a factor of \( \chi_- (r^{-\delta'} |x^a|) \) for any \( \delta' \in (0, 1) \) in front of the tensor product (which is harmless since effectively \( |x_a| \) grows linearly in time). For the same reason we can freely insert the factor \( \chi_- (r^{-\delta'} |x^a|) \) in front of the tensor product in (A.1). It turns out to be useful to do this for some \( \delta' \in (0, \delta) \) (where \( \delta \) is fixed in (3.6)).

With these conventions it remains to argue that for the expression

\[
A_1 A_2^a (A_2^a)^2 A_2^a A_1 \psi(t) \text{ with } \psi(t) = \chi_- (r^{-\delta'} |x^a|) \left( u^a \otimes \left( \chi_+ (|x_a|/(\epsilon t)) \varphi(t) \right) \right), \tag{A.5}
\]

all factors of \( A_j \) (omitting for convenience the superscript \( a \) for \( j = 2, 3 \)) can be replaced by \( I \) in the large time limit. This will be accomplished by showing that

\[
\left\| (A_j - I) \psi(t) \right\| \to 0 \text{ for } t \to \infty; \quad j = 1, 2, 3. \tag{A.6}
\]

For the case of \( j = 1 \) we can freely insert the factor \( \chi_+ (B_a/(8\epsilon_0)) \) in front of \( \psi(t) = u^a \otimes \varphi(t) + o(t^0) \), which may be seen as follows, using in the second step (A.4) and a commutation,

\[
6^{-1} \left\| \chi_- (B_a/(8\epsilon_0)) (u^a \otimes \varphi(t)) \right\|^2 \\
\leq \langle \chi_- (B_a/(8\epsilon_0))^2 (2 - B_a/(8\epsilon_0)) \rangle u^a \otimes \varphi(t) \\
= -2 \langle (p_a^2/(8\epsilon_0) - 1) f_2(k_a) \chi_- (B_a/(8\epsilon_0)) (u^a \otimes \varphi(t)) \rangle + o(t^0) \\
\leq -2 \langle (\lambda_0 - \lambda^a)/9 \epsilon_0 - 1 \rangle f_2(k_a) \chi_- (B_a/(8\epsilon_0)) (u^a \otimes \varphi(t)) + o(t^0) \\
\leq o(t^0).
\]

Along with additional ‘free factors’ of \( f_3(\tilde{H}_a) \) and \( \chi_- (|x^a|/(\epsilon r)) \) we then write, using in the last step Lemma 8.1,

\[
(I - A_1^2) \psi(t) = (I - A_1^2) \chi_+ (B_a/(8\epsilon_0)) \psi(t) + o(t^0) \\
= \chi_- (B/(8\epsilon_0))^2 \chi_- (|x^a|/(\epsilon r)) \chi_+ (B_a/(8\epsilon_0)) f_3(\tilde{H}_a) \psi(t) + o(t^0) \\
= o(t^0).
\]

Since \( A_1 \geq 0 \), the assertion (A.6) for \( j = 1 \) follows.

To show (A.6) for \( j = 2, 3 \) we need the facts that in \( |x^a| < cr^a \) with \( c > 0 \) given in the property (ii) of Sect. 5 (with ii) used for \( r^a \) rather than for \( r \), \( r^{a_2}_\delta = r^{a_1}_\delta r^a(0) \) and \( B^{a_2}_{\delta, r^a_1} = r^{a_1}_\delta B^{a}_\delta r^{a_1}/2 = 0 \). For \( A_2 = A_2^a \) we abbreviate \( \chi_{\delta'} = \chi_- (r^{-\delta'} |x^a|) \) and recall that we took it with \( \delta' \in (0, \delta) \). Using then that for all large enough \( r \),

\[
A_2 \chi_{\delta'} = \chi_- (r^{\delta - 1 + \delta} r^a(0)) \chi_{\delta'} = \chi_{\delta'},
\]
cf. (3.6), combined with the ‘free factor’ of \( \chi_+(|x_\alpha|/(e't)) \), indeed (A.6) for \( j = 2 \) follows. Similarly we can replace \( A_3 = \chi_-(B^a_δ,\rho_1) \) by arguing that for all large enough \( \rho > 1 \),

\[
(A_3 - I) \chi_+(r/\rho) = \int_{\mathbb{C}} (B^a_δ,\rho_1 - z)^{-1} (B^a_δ,\rho_1 \chi_+(r/\rho)) z^{-1} d\mu_\chi_-(z) = 0,
\]

cf. (8.15). This proves (A.6) holds for \( j = 3 \).

(3.7a) and (3.7b): We take (3.7b), i.e.

\[
F_a m_a^+ F_a^{-1} = \int_{s-lim} 2\lambda^{1/2} m_a(\xi_a) d\lambda = 2\lambda^{1/2} m_a(\hat{\xi}_a), \quad \hat{\xi}_a = \xi_a/|\xi_a| \in C_a,
\]
as a definition of \( m_a^+ \), and it remains (for (3.7a)) to show

\[
\left( s-lim e^{itH} \Phi^+_a e^{-it\tilde{H}_a} \right) J_a \tilde{w}_a^+ f_1(k_a) = s-lim e^{itH} e^{-it\tilde{H}_a} J_a \tilde{w}_a^\pm f_1(k_a)(m_a^+)^2;
\]

\[
\Phi^+_a = f_2(H)M_a N^a_+ M_a f_2(\tilde{H}_a).
\]

Due to (A.1) it suffices to show (recalling the notation (4.2a)) that for all \( \varphi \in L^2(X_a) \) with \( \tilde{\varphi} \in C^\infty_c(X'_a) \):

\[
\lim_{t \to \infty} \left\| M_a \left( u^a \otimes \varphi(t) \right) - u^a \otimes \left( m_a^+ \varphi(t) \right) \right\| = 0,
\]

\[
\lim_{t \to \infty} \left\| \left( f_2(H) - f_2(\tilde{H}_a) \right) \left( u^a \otimes \left( (m_a^+)^2 \varphi(t) \right) \right) \right\| = 0; \quad (A.7)
\]

\[
\varphi(t) = e^{-iS_a(\rho_a^{-1})} \varphi_1, \quad \varphi_1 = f_1(k_a) \varphi.
\]

As for the first assertion of (A.7) we can (as above) replace the tensor product \( u^a \otimes \varphi(t) \) with its modification \( \psi(t) \) from (A.5). We can then use stationary phase analysis in combination with the properties (1) and (2) from Sect. 4 (here 2) is used with \( b = a \). The second part also follows from standard stationary phase analysis noting that the support property of \( \tilde{\varphi} \) yields ‘effective decay’ of \( u^a \otimes (m_a^+)^2 \varphi(t) \) near the ‘collision planes’ \( X_c \) with \( c \not\leq a \) (where the intercluster potential \( I_a \) lacks decay), yielding the same conclusion for \( I_a - \tilde{I}_a \) when first representing

\[
f_2(H) - f_2(\tilde{H}_a) = - \int_{\mathbb{C}} (H - z)^{-1} (I_a - \tilde{I}_a)(\tilde{H}_a - z)^{-1} d\mu f(z),
\]

and then using a commutation. We skip the detailed arguments.

**Remark A.1.** To be used in “Appendix C” let us note the related result (with \( \tilde{I}_a^{-1} \) given by (3.1))

\[
\begin{aligned}
&\lim_{t \to +\infty} e^{itH} \tilde{J}_a^- e^{-itk_a} f_1(k_a) = 0, \quad (A.8)
\end{aligned}
\]

which follows by the above arguments.

**Proof of A.3 for the cases \(|\gamma| + k \geq 2 \)** Let \( B \subseteq X_a \setminus \{0\} \) be compact. All bounds below are uniform in \( \xi \in B \). Using the functions \( x = x(\xi, t) \) and \( \lambda = \lambda(\xi, t) \) of (2.8a) and (2.8b) for \( \xi \in B \) and \( t \) large enough it follows (using also (2.8c) and representing as column vectors) that

\[
\nabla_{(x, \lambda)} K_a = (\xi^{ir}, t)^{ir} \quad \text{and} \quad \nabla_{(\xi, t)} S_a = (\lambda^{ir}, \lambda)^{ir}. \quad (A.9)
\]
Motivated by (2.8b) we ‘renormalize’ the variables in terms of the ‘time’ $\tau := \sqrt{t}$ as

$$\tilde{t} = t/\tau \quad (= \sqrt{t} = \tau), \quad \tilde{\xi} = \tau \xi, \quad \tilde{x} = x/\tau, \quad \tilde{\lambda} = \tau \lambda.$$  

We need to show the bounds

$$\partial^{\alpha}_{(\tilde{\xi}, \tilde{t})} (\tilde{x} - 2\tilde{\xi}) = O(\tau^{1-|\alpha|}2^\mu); \quad |\alpha| \leq 1, \quad (A.10a)$$

$$\partial^{\alpha}_{(\tilde{\xi}, \tilde{t})} (\tilde{\lambda} - |\tilde{\xi}|^2/\tilde{t}) = O(\tau^{1-|\alpha|}2^\mu); \quad |\alpha| \leq 1, \quad (A.10b)$$

$$\partial^{\alpha}_{(\tilde{\xi}, \tilde{t})} z = O(\tau^{1-|\alpha|}); \quad z = \tilde{x}, \quad z = \tilde{\lambda}, \quad z = \tilde{\xi} \text{ or } z = \tilde{t}. \quad (A.10c)$$

Upon substituting the expressions for the dependent variables $\tilde{x}$ and $\tilde{\lambda}$, and using for the independent variables,

$$\partial_{\tilde{\xi}} = \tilde{t} \partial_{\xi} \text{ and } \partial_{\tilde{t}} = (2\tilde{t})^{-1} \left( \partial_{t} + (\tilde{t})^{-1} \tilde{\xi} \cdot \partial_{\xi} \right),$$

it is an elementary check (here omitted) that (A.3) follows from (2.8a)--(2.8c) and (A.10a)--(A.10c).

For $|\alpha| = 0$ the bounds (A.10a)--(A.10c) are already known by (2.8b). The proofs for $|\alpha| = 1$ will be based on the representation of the Hessian of

$$K_0 = K_0(x, \lambda) := \sqrt{\lambda} r, \quad r = |x|, \quad x = x_a,$$

as the square block-matrix

$$\nabla^2 K_0 = \begin{pmatrix} \sqrt{\lambda} r^{-1} P_\perp & (2\sqrt{\lambda})^{-1} \hat{x} \\ (2\sqrt{\lambda})^{-1} \hat{x}^{\text{tr}} & -4^{-1} \hat{x}^{-3/2} \end{pmatrix}; \quad \hat{x} = x/r, \quad P_\perp = I - |\hat{x}>(\hat{x}|.$$

We introduce

$$A_0 = \begin{pmatrix} \tau^2 r^{-1} \sqrt{\lambda} P_\perp & (2\sqrt{\lambda})^{-1} \hat{x} \\ (2\sqrt{\lambda})^{-1} \hat{x}^{\text{tr}} & -\tau^{-2} r^{-1} \lambda^{-3/2} \end{pmatrix} = R_\tau(\nabla^2 K_0) R_\tau; \quad R_\tau = \text{diag}(\tau, \tau^{-1}),$$

and compute

$$B_0 := A_0^{-1} = \begin{pmatrix} \tau^{-2} r \lambda^{-1/2} I & 2\sqrt{\lambda} \hat{x} \\ 2\sqrt{\lambda} \hat{x}^{\text{tr}} & 0 \end{pmatrix}.$$  

We record that the matrices

$$A_0 \text{ and } A := R_\tau(\nabla^2 K_a) R_\tau = A_0 + O(\tau^{-2}) \quad (A.11)$$

$$B_0 \text{ and } B := A^{-1} = R_\tau^{-1}(\nabla^2 K_a)^{-1} R_\tau^{-1} = B_0 + O(\tau^{-2})$$

are all bounded (with uniform bounds in $\xi \in B$). Note also the square matrix identity (obtained by differentiating (A.9)),

$$D_{(\xi, t)} (\xi^{\text{tr}}, t)^{\text{tr}} = \nabla^2 K_a(x, \lambda) D_{(\xi, t)} (x^{\text{tr}}, \lambda)^{\text{tr}}, \quad (A.12)$$

where $D_z F$ is used as notation for the derivative of a vector field $F = F(z)$. Next we write (A.12) as

$$R_\tau^{-1} C_1 = \nabla^2 K_a(x, \lambda) R_\tau (D_{(\xi, t)} (\xi^{\text{tr}}, \lambda)^{\text{tr}} + C_2),$$
and therefore in turn as
\[ C_1 = A(D_{\xi, \bar{\lambda}}(\bar{x}^{\text{tr}}, \bar{\lambda})^{\text{tr}} + C_2), \]
where
\[ C_1 = \begin{pmatrix} I - \bar{\xi}/\tau \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 \\ \bar{x}/\tau \\ 0 - \bar{\lambda}/\tau \end{pmatrix}. \]

**I.** We prove (A.10c) for \(|\alpha| = 1\). The result follows from writing (A.13) as
\[ D_{\xi, \bar{\lambda}}(\bar{x}^{\text{tr}}, \bar{\lambda})^{\text{tr}} = BC_1 - C_2, \]
since indeed (by the previous remarks) the right-hand side is bounded uniformly in \(\xi \in B\) for all large \(\tau\).

**II.** We prove (A.10a) and (A.10b) for \(|\alpha| = 1\) (yielding (A.3) for the cases \(|\gamma| + k \leq 2\)). Introducing the column vector
\[ \bar{\gamma} = \bar{y}(\tau) = (\bar{x}^{\text{tr}} - 2\bar{\xi}^{\text{tr}}, \bar{\lambda} - |\bar{\xi}|^2/\bar{\tau}), \]
we record that \(\bar{\gamma} = O(\tau^{1-2\mu})\) and compute using (A.14)
\[ D_{\xi, \bar{\lambda}}(\bar{y}) = BC_1 - C_2 - C_3; \quad C_3 = \begin{pmatrix} 2I \\ 2\bar{x}^{\text{tr}}/\tau \end{pmatrix} - \left(\frac{\bar{\xi}}{\bar{\tau}}\right)^2. \]
We need to check that the right-hand side is \(O(\tau^{-2\mu})\). Thanks to (A.11) it suffices to check that
\[ B_0C_1 - C_2 - C_3 = O(\tau^{-2\mu}), \]
but an elementary inspection yields the matrix bound
\[ \|B_0C_1 - C_2 - C_3\| \leq \text{Const.} \tau^{-1} |\bar{y}| \quad \text{for large } \tau, \]
showing the desired bound \(O(\tau^{-2\mu})\).

**III.** We prove (A.10c) for \(|\alpha| \geq 2\) (yielding (A.3) for the cases \(|\gamma| + k \geq 3\)). Assuming by induction that the bounds hold for \(|\alpha| \leq m - 1\) for some \(m \geq 2\) we obtain, after \(m - 1\) differentiations of (A.13) (more precisely by applying \(D^\beta_{\xi, \bar{\lambda}}\) with \(|\beta| = m - 1\) to (A.13)) computing by the chain rule, an expression for
\[ AD^\beta_{\xi, \bar{\lambda}}D_{\xi, \bar{\lambda}}(\bar{x}^{\text{tr}}, \bar{\lambda})^{\text{tr}}, \]
in terms of derivatives of lower order. The resulting formula is an example of Faa di Bruno’s formula, see [Hö1, Lemma 3.6] for a similar problem. The induction scheme works thanks to proper control of derivatives of the ‘outer function’. More precisely we consider above the matrix \(A = A(\tau, x, \lambda)\) as a function of \(z := (\bar{\tau}, \bar{x}, \bar{\lambda})\), introducing
\[ \tilde{A}(z) = \tilde{A}(\bar{\tau}, \bar{x}, \bar{\lambda}) = \tilde{A}(\bar{\tau}, \bar{x}, \bar{\lambda}/\bar{\tau}). \]
Then an elementary check shows that \(\partial^\gamma \tilde{A}(z) = O(\tau^{-|\gamma|})\), and with these bounds the right-hand side of Faa di Bruno’s formula can be dealt with by the induction hypothesis, implementing the induction step. \(\square\)
Appendix B: Proof of (7.3c)

We shall prove (7.3c), i.e. the bound

$$\sup_{\Im z \neq 0} \| Q^\pm(a, k) f_2(H) R(z) \|_{L(B, H)} < \infty.$$ 

For that we invoke the following (standard) scheme of stationary scattering theory (see for example [Ya3,IS1]), proceeding partially abstractly and proving a more general version than needed for (7.3c) (useful in [Sk3] as well).

Pick any $f \in C^\infty_c(\Lambda)$ such that $f_2 < f < f_3$ (implying that the bound (5.6) of Theorem 5.1 is at disposal provided $\forall \lambda \in \Lambda$ is in some neighbourhood of $\sup f$).

**Lemma B.1.** Suppose $\Psi = f_2(H) P f_2(H) \in L(H) \cap L(B)$ is self-adjoint (on $H$), and suppose

$$i[H, \Psi] \geq f_2(H)\left(Q^* Q - T^* T\right)f_2(H) \tag{B.1}$$

for $H$-bounded operators $Q$ and $T$. Then (with $f \in C^\infty_c(\Lambda)$ given as above) the following estimates hold for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all $\psi \in B$.

$$\| Q R(z) f_2(H) \psi \|^2 \leq \| T R(z) f_2(H) \psi \|^2 + 2\left( \| \psi \|_{L(H)} + \| \psi \|_{L(B)} \right) \| R(z) f(H) \|_{L(B, B^*)} \| \psi \|^2_B \tag{B.2}$$

In particular,

If $\sup_{\Im z \neq 0} \| T f_2(H) R(z) \|_{L(B, H)} < \infty$, then also $\sup_{\Im z \neq 0} \| Q f_2(H) R(z) \|_{L(B, H)} < \infty$.

**Proof.** Letting $\psi_2 = f_2(H) \psi$ and $\phi = R(z) \psi_2$ the estimate (B.1) leads to

$$\| Q \phi \|^2 - \| T \phi \|^2 \leq 2(\langle \Im z \rangle R(z) \psi, \Psi R(z) \psi) - \Im(\psi, \Psi R(z) \psi))$$

$$\leq 2\left( \| \psi \|_{L(H)} \| R(z) f(H) \psi \| + \| R(z) f(H) \psi \| \right)$$

$$\leq 2\left( \| \psi \|_{L(H)} + \| \psi \|_{L(B)} \right) \| R(z) f(H) \|_{L(B, B^*)} \| \psi \|^2_B.$$

$$\Box$$

(7.3c): We repeatedly apply Lemma B.1 and the ‘propagation observables’ $\Psi_1, \ldots, \Psi_4 \in L(H) \cap L(B)$ of the proof of Lemma 7.1 (or alternatively the version of (8.10) with $\tilde{H}_a$ replaced with $H$). Note for example that $Q_1 = Q_1(a, j) = \xi_j^+(x) G_{j, a}$ in Lemma 7.1 (1) may be treated by a single application of Lemma B.1, which is obvious from the proof of Lemma 7.1 (1).

Appendix C: Formulas for Scattering and Channel Wave Matrices

A main goal of this “Appendix” is to derive the formula (8.2), i.e.

$$\tilde{S}_{\beta a}(\lambda) = (2\pi i)^2 f_1^\pm(\lambda) \tilde{y}_b^+ (\lambda_\beta) J_\beta^a (T_b^+) \delta(H - \lambda) T_a^- J_a \tilde{y}_a^- (\lambda_\alpha)^*,$$

$$\lambda_a = \lambda - \lambda^\alpha,$$

which is interpreted correctly in Sect. 8.2. As in Sect. 3 here $\lambda \in \Lambda$, where $\Lambda$ is a small open interval containing a fixed $\lambda_0 \notin \mathcal{F}(H).$ The derivation will be given using smoothness bounds from Sect. 7, arguments from Sects. 8.1 and 8.2, (A.8) and finally
bounds from Sect. 5.2. We follow essentially the scheme of [DS, Appendix A] (for a similar issue, see for example the proof of [Ya4, Proposition 7.2]). Let for \( \epsilon > 0 \) and \( \lambda \in \Lambda \)

\[
\delta_{\epsilon, \beta}(\lambda) = (2i\pi)^{-1}((k_{\beta} - \lambda - i\epsilon)^{-1} - (k_{\beta} - \lambda + i\epsilon)^{-1}) \\
= \frac{\epsilon}{\pi}((k_{\beta} - \lambda + i\epsilon)^{-1}(k_{\beta} - \lambda - i\epsilon)^{-1}); \quad k_{\beta} = p_{b}^{2} + \lambda^{\beta}.
\]

The outset for our analysis is the following two formulas which can be derived as in [DS, Appendix A]. In the first formula \( g \) is any complex continuous function on \( \mathbb{R} \) vanishing at infinity.

\[
\tilde{W}_{\beta}^{\pm}(g1_{\Lambda})(k_{\beta})\varphi \\
= \lim_{\epsilon \to 0^{+}} \frac{1}{\epsilon} \int_{\Lambda} g(\lambda) f_{2}(H)(\Phi_{b}^{+} + iR(\lambda - i\epsilon)T_{b}^{+})J_{\beta} \tilde{w}_{b} f_{1}(k_{\beta})\delta_{\epsilon, \beta}(\lambda)\varphi d\lambda, \tag{C.1}
\]

\[
\langle \varphi_{b}, \tilde{\varphi}_{\beta b_{\alpha}} \rangle \\
= -2\pi \lim_{\epsilon \to 0^{+}} \int_{-\infty}^{\infty} \langle \varphi_{b}, \delta_{\epsilon, \beta}(\lambda') (\tilde{W}_{\beta}^{+})^{*}T_{\alpha}^{-} J_{\alpha} \tilde{w}_{\alpha}^{-} f_{1}(k_{\alpha})\delta_{\epsilon, \alpha}(\lambda')\varphi_{\alpha} \rangle d\lambda'. \tag{C.2}
\]

To make the exposition self-contained we prove these formulas, which only (compared to Sect. 3) require the properties \( f_{1} \prec f_{2} \) for some \( f_{1}, f_{2} \in \mathbb{C}_{c}^{\infty}(\Lambda) \).

**Proof of C.1 and C.2.** Recall from Sect. 3 that

\[
\tilde{W}_{\beta}^{\pm} = \text{s-lim}_{t \to \pm \infty} e^{itH} J_{\beta}^{+} e^{-itk_{\beta}} f_{1}(k_{\beta}) = \text{s-lim}_{t \to \pm \infty} e^{itH} \Phi_{b}^{+} e^{-it\tilde{H}_{b}} J_{\beta} \tilde{w}_{b}^{\pm} f_{1}(k_{\beta}).
\]

By the intertwining property

\[
\tilde{W}_{\beta}^{\pm} = \tilde{W}_{\beta}^{\pm} f_{2}(k_{\beta}) = f_{2}(H) \tilde{W}_{\beta}^{\pm}. \tag{C.3}
\]

For any interval \( I \) and any \( \varphi \in L^{2}(X_{b}) \) we compute using the vector-valued Plancherel formula

\[
\lim_{t \to \pm \infty} e^{itH} J_{\beta}^{+} e^{-itk_{\beta}} f_{1}(k_{\beta})1_{I}(k_{\beta})\varphi = \lim_{\epsilon \to 0^{+}} 2\epsilon \int_{0}^{\infty} e^{-2\epsilon t} e^{itH} J_{\beta}^{+} e^{-itk_{\beta}} (1_{I} f_{1})(k_{\beta})\varphi dt
\]

\[
= \lim_{\epsilon \to 0^{+}} \frac{\epsilon}{\pi} \int_{\mathbb{R}} R(\lambda - i\epsilon) J_{\beta}^{+} (k_{\beta} - \lambda - i\epsilon)^{-1} (1_{I} f_{1})(k_{\beta})\varphi d\lambda
\]

\[
= \lim_{\epsilon \to 0^{+}} \frac{\epsilon}{\pi} \int_{I} R(\lambda - i\epsilon) J_{\beta}^{+} (k_{\beta} - \lambda - i\epsilon)^{-1} f_{1}(k_{\beta})\varphi d\lambda
\]

\[
- \lim_{\epsilon \to 0^{+}} \frac{\epsilon}{\pi} \int_{I} R(\lambda - i\epsilon) J_{\beta}^{+} (k_{\beta} - \lambda - i\epsilon)^{-1} (1_{I^{c}} f_{1})(k_{\beta})\varphi d\lambda
\]

\[
+ \lim_{\epsilon \to 0^{+}} \frac{\epsilon}{\pi} \int_{I^{c}} R(\lambda - i\epsilon) J_{\beta}^{+} (k_{\beta} - \lambda - i\epsilon)^{-1} (1_{I} f_{1})(k_{\beta})\varphi d\lambda.
\]

The two last terms vanish thanks to the Cauchy–Schwarz inequality and the fact that \( J_{\beta}^{+} f_{2}(k_{\beta}) \) is bounded, using as well the (general) features

\[
\left\| \frac{\epsilon}{\pi} \int_{I} R(\lambda - i\epsilon) R(\lambda + i\epsilon)\psi d\lambda \right\| \leq \| \psi \|; \quad \psi \in \mathcal{H}, \tag{C.4a}
\]
Thanks to (C.4a) and (C.4b) it follows that uniformly in \( \varepsilon \in \mathbb{R} \) it holds

\[
\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi} \int_{1^c} (k_\beta - \lambda + i\varepsilon)^{-1}(k_\beta - \lambda - i\varepsilon)^{-1} \tilde{\varphi} \, d\lambda = 1_{1^c}(k_\beta)\tilde{\varphi}; \quad \tilde{\varphi} \in L^2(X_\beta). \tag{C.4b}
\]

Note that in our application \( 1_{1^c}(k_\beta)\tilde{\varphi} = 1_{1^c}(k_\beta)(1_{1^c}f_1)(k_\beta)\varphi = 0 \).

Any continuous function \( g \) vanishing at infinity can be uniformly approximated by \( g_m \), finite linear combinations of characteristic functions of intervals, and for each such function \( g_m \) we can record (using (C.3)) that

\[
\tilde{W}_\beta^+ g_m(k_\beta) \varphi = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi} \int g_m(\lambda)f_2(H)R(\lambda - i\varepsilon)\tilde{J}_\beta^+(k_\beta - \lambda - i\varepsilon)^{-1}f_1(k_\beta)\varphi \, d\lambda.
\]

Thanks to (C.4a) and (C.4b) it follows that uniformly in \( \varepsilon > 0 \)

\[
\frac{\varepsilon}{\pi} \int g_m(\lambda)f_2(H)R(\lambda - i\varepsilon)\tilde{J}_\beta^+(k_\beta - \lambda - i\varepsilon)^{-1}f_1(k_\beta)\varphi \, d\lambda \to \frac{\varepsilon}{\pi} \int g(\lambda)f_2(H)R(\lambda - i\varepsilon)\tilde{J}_\beta^+(k_\beta - \lambda - i\varepsilon)^{-1}f_1(k_\beta)\varphi \, d\lambda.
\]

Hence we can interchange limits (cf. [Ru1, Theorem 7.11]) obtaining that

\[
\tilde{W}_\beta^+ g(k_\beta) \varphi = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi} \int g(\lambda)f_2(H)R(\lambda - i\varepsilon)\tilde{J}_\beta^+(k_\beta - \lambda - i\varepsilon)^{-1}f_1(k_\beta)\varphi \, d\lambda,
\]

from which we conclude (C.1) by substituting

\[
R(\lambda - i\varepsilon)\tilde{J}_\beta^+ f_2(k_\beta) = R(\lambda - i\varepsilon)\Phi_b^+ J_\beta \tilde{w}_b^+ = (\Phi_b^+ + iR(\lambda - i\varepsilon)T_b^+)J_\beta \tilde{w}_b^+(k_\beta - \lambda + i\varepsilon)^{-1}.
\]

As for (C.2) we compute using (A.8)

\[
\tilde{W}_\alpha^- = \lim_{t \to -\infty} e^{itH}\tilde{J}_\alpha^- e^{-itk_\alpha} f_1(k_\alpha) - \lim_{t \to +\infty} e^{itH}\tilde{J}_\alpha^- e^{-itk_\alpha} f_1(k_\alpha)
\]

\[
= -\lim_{t \to +\infty} \left( e^{itH}\tilde{J}_\alpha^- e^{-itk_\alpha} - e^{-itH}\tilde{J}_\alpha^- e^{itk_\alpha} \right) f_1(k_\alpha)
\]

\[
= -\lim_{t \to +\infty} \int_{-t}^{t} e^{isH}T_a^- J_\alpha \tilde{w}_a^- e^{-isk_\alpha} f_1(k_\alpha) \, ds
\]

\[
= -\lim_{t \to +\infty} \int_{0}^{\infty} e^{-e^{-t}s} \left( \int_{-t}^{t} e^{isH}T_a^- J_\alpha \tilde{w}_a^- e^{-isk_\alpha} f_1(k_\alpha) \, ds \right) \, dt
\]

\[
= -\lim_{t \to +\infty} \int_{\mathbb{R}} e^{-|s|} e^{isH}T_a^- J_\alpha \tilde{w}_a^- e^{-isk_\alpha} f_1(k_\alpha) \, ds.
\]

Then we multiply by \( \left( \tilde{W}_\beta^+ \right)^* \) (in agreement with the definition of \( \tilde{S}_{\beta\alpha} \)) and invoke again the intertwining property and the vector-valued Plancherel theorem, calculating for any \( \varphi \in L^2(X_\alpha) \)

\[
\tilde{S}_{\beta\alpha} \varphi = -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} e^{-2\varepsilon|s|} e^{isk_\beta} \left( \tilde{W}_\beta^+ \right)^* T_a^- J_\alpha \tilde{w}_a^- e^{-isk_\alpha} f_1(k_\alpha) \varphi \, ds
\]

\[
= -2\pi \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \delta_{\varepsilon,\beta}(\lambda) \left( \tilde{W}_\beta^+ \right)^* T_a^- J_\alpha \tilde{w}_a^- f_1(k_\alpha) \delta_{\varepsilon,\alpha}(\lambda) \varphi \, d\lambda.
\]

Whence (C.2) is proven. \( \square \)
Appendix C.1.: Taking $\epsilon \to 0$ in (C.1) and (C.2). Although (C.1) is valid for any $\varphi \in L^2(X_b)$ we take below $\varphi = \varphi_b \in L^2_s(X_b)$ for $s > 1/2$, and compute by taking $\epsilon \to 0$ using Sect. 5.2, here first done formally,

$$\tilde{W}_\beta g(k\beta)\varphi = 2\pi \int \Lambda g(\lambda)f_2(H)\delta(H - \lambda)T^+_\beta J_\beta \tilde{w}_\beta f_1(k\beta)\delta_{0,\beta}(\lambda)\varphi d\lambda;$$

$$\delta_{0,\beta}(\lambda) = (2i\pi)^{-1}((k_\beta - \lambda - i0)^{-1} - (k_\beta - \lambda + i0)^{-1}) = \gamma_{b,0}(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta).$$

(We prove in Step I below the two resulting formulas (C.5a) and (C.5b).) See Sect. 2.3 for notation, and note that the above formula has the following precise meaning. We substitute, referring again to Sect. 2.3,

$$\tilde{w}_\beta f_1(k\beta)\delta_{0,\beta}(\lambda) = f_1(\lambda)\tilde{\gamma}_\beta^+(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta).$$

Then by combining the proof of Lemma 7.3, Remark 7.2 (iii) and Lemma 8.3 we conclude that the integral to the right is well-defined,

$$(C.5a)$$

$$(\tilde{W}_\beta g(k\beta)\varphi = 2\pi \int (g f_1)(\lambda)(f_2(H)\delta(H - \lambda)T^+_\beta J_\beta \tilde{\gamma}_\beta^+(\lambda_\beta)^*) \gamma_{b,0}(\lambda_\beta)\varphi d\lambda, \quad (C.5a)$$

where the correct interpretation of the product in parentheses involves the ‘$Q$-operators’ as in Sect. 8.2 (more precisely obtained by expanding $T_b^+$ into terms on the form (C.8) given below). We call (C.5a) a ‘channel wave matrix’ representation (examined closer with exceptions in Sects. 9.3–9.5). The alternative formula reads

$$(C.5b)$$

$$(\tilde{W}_\beta g(k\beta)\varphi = \int (g f_1)(\lambda)(f_2(H)(\Phi^+_b + iR(\lambda - i0)T^+_b) J_\beta \tilde{\gamma}_\beta^+(\lambda_\beta)^*) \gamma_{b,0}(\lambda_\beta)\varphi d\lambda. \quad (C.5b)$$

Although it is not relevant for (8.2) we remark that the integrands in (C.5a) and (C.5b) take value in $\mathcal{B}(X)^*$. This is thanks to Remark 7.2 iv) and Lemma 8.3, see the proof of Lemma 9.12 for an elaboration. However the below procedure is somewhat softer. As the reader will see we shall consider the integrands in (C.5a) and (C.5b) as taking value in $L^2_{-1}(X)$ and, writing $\delta(H - \lambda) = (2i\pi)^{-1}(R(\lambda + i0) - R(\lambda - i0))$, interpreting the integrands in agreement with taking limits in $\mathcal{L}(\mathcal{G}_b, L^2_{-1}(X))$ as

$$f_2(H)R(\lambda \pm i0)T^+_b J_\beta \tilde{\gamma}_\beta^+(\lambda_\beta)^* = s\text{-w-limit} f_2(H)R(\lambda \pm i\epsilon)T^+_b J_\beta \tilde{\gamma}_\beta^+(\lambda_\beta)^* \quad \epsilon \to 0_+.$$

With this interpretation the integrand in (C.5a) is better. Thanks to Remark 7.2 (iii) and Lemma 8.3 it obviously takes value in $L^2_{-1}(X)$, not only for $s = 1$, but for any $s > 1/2$. I. We prove (C.5a) and (C.5b). For (C.5a) we compute the ‘$\epsilon \to 0$’–limit in (C.1) in the precise meaning of taking limit in the weak topology of $L^2_{-1}(X)$. There are two assertions that need to be checked:

$$i \lim_{\epsilon \to 0} \int \Lambda g(\lambda)f_2(H)(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))T^+_b J_\beta \tilde{w}_\beta f_1(k\beta)\delta_{\epsilon,\beta}(\lambda)\varphi d\lambda = i \lim_{\epsilon \to 0} \int \Lambda g(\lambda)f_2(H)(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))T^+_b J_\beta f_1(\lambda)\tilde{\gamma}_\beta^+(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta)\varphi d\lambda.$$
\[ 2\pi \int g(\lambda) f_2(H) \delta(H - \lambda) T^+_b J_\beta f_1(\lambda) \bar{\gamma}^+_b(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta) \varphi \, d\lambda, \quad (C.6a) \]

and

\[
\lim_{\epsilon \to 0} \int g(\lambda) f_2(H) \left( \Phi^+_b + i R(\lambda + i \epsilon) T^+_b \right) J_\beta f_1(k_\beta) \delta_{\epsilon,\beta}(\lambda) \varphi \, d\lambda
\]

\[= \lim_{\epsilon \to 0} \int g(\lambda) f_2(H) \left( \Phi^+_b + i R(\lambda + i \epsilon) T^+_b \right) J_\beta f_1(\lambda) \bar{\gamma}^+_b(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta) \varphi \, d\lambda
\]

\[= \int g(\lambda) f_2(H) \left( \Phi^+_b + i R(\lambda + i 0) T^+_b \right) J_\beta f_1(\lambda) \bar{\gamma}^+_b(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta) \varphi \, d\lambda
\]

\[= 0. \quad (C.6b) \]

For \(C.6a\) we write with \(\delta_{\epsilon,\lambda}(\lambda') := \pi^{-1} \epsilon/(\lambda - \lambda' + \epsilon^2)\) for any fixed \(\lambda \in \mathbb{R}\) and \(\epsilon > 0\), and by using Lemma 2.3,

\[\tilde{w}^+_b f_1(k_\beta) \delta_{\epsilon,\beta}(\lambda) \varphi = \int_{\lambda_\beta}^{\infty} f_1(\lambda') \delta_{\epsilon,\lambda}(\lambda') \bar{\gamma}^+_b(\lambda' - \lambda_\beta)^* \gamma_{b,0}(\lambda' - \lambda_\beta) \varphi \, d\lambda', \quad (C.7)\]

substitute and replace by the limit (when taking \(\epsilon \to 0\)). This is doable thanks to Remark 7.2 (iii), Lemma 8.3 and the continuity of \(\gamma_{b,0}(\lambda_\beta)\varphi\) justifying the first equality, and the second equality is a consequence of Remark 7.2 (iii) and Lemma 8.3 too. In addition we used the fact that \(T^+_b\) is a finite sum of terms expressed as

\[f_2(H) Q^+(b, l)^* B_+ Q^+(b, l) f_2(\tilde{H}_b)\text{ with } B_+ \text{ bounded}, \quad (C.8)\]

cf. Sect. 8.2. We shall use that \(T^+_b\) is expanded this way again below.

For \(C.6b\) we will invoke Theorem 5.1 and Corollary 5.2. Indeed thanks to these results and the presence of the factors \(A_{1+}\) in \(\Phi^+_b\) we can compute the left-hand side to be equal

\[\lim_{\epsilon \to 0} \int g(\lambda) f_2(H) \left( \Phi^+_b + i R(\lambda + i \epsilon) T^+_b \right) J_\beta f_1(\lambda) \bar{\gamma}^+_b(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta) \varphi \, d\lambda
\]

\[= \int g(\lambda) f_2(H) \left( \Phi^+_b + i R(\lambda + i 0) T^+_b \right) J_\beta f_1(\lambda) \bar{\gamma}^+_b(\lambda_\beta)^* \gamma_{b,0}(\lambda_\beta) \varphi \, d\lambda. \quad (C.9)\]

To see this, considering only the ‘difficult term’

\[\int g(\lambda) f_2(H) R(\lambda + i \epsilon) T^+_b J_\beta \tilde{w}^+_b f_1(k_\beta) \delta_{\epsilon,\beta}(\lambda) \varphi \, d\lambda, \]

we insert \(I = \chi^2_+(2B/\epsilon_0) + \chi^2_-(2B/\epsilon_0)\) to the right of the factor \(R(\lambda + i \epsilon)\). Using the factors of \(A_{1+}\) and commutation the contribution from the second term \(\chi^2_-(2B/\epsilon_0)\) allows the insertion of the weight \(\langle x \rangle^{-s}\) for some \(s > 1/2\) to the right of \(R(\lambda + i \epsilon)\) and we can use \((5.5a)\) and \((C.7)\) to compute the ‘\(\epsilon \to 0\)’-limit for this term in agreement with \((C.9)\). Note for this argument that \(\langle x \rangle^s \chi^2_+(2B/\epsilon_0) f_2(H) Q^+(b, l)^* B_+\) is bounded. As for the contribution from the first term \(\chi^2_+(2B/\epsilon_0)\) we write

\[R(\lambda + i \epsilon) \chi^2_+(2B/\epsilon_0) = R(\lambda + i \epsilon) \chi^2_+(2B/\epsilon_0) \langle x \rangle^{-s} \langle x \rangle^s; \quad s > 1/2 - \mu/2.\]

Then by Theorem 5.1 and Corollary 5.2

\[\langle x \rangle^{-1} f_2(H) R(\lambda + i \epsilon) \chi^2_+(2B/\epsilon_0) \langle x \rangle^{-s}\] is uniformly bounded,
and there exists
\[ \Lambda \ni \lambda \rightarrow \langle x \rangle^{-1} f_2(H) R(\lambda + \imath 0) \chi_+^2(2B/\epsilon_0) (x)^{-s} = \lim_{\epsilon \to 0} \langle x \rangle^{-1} f_2(H) R(\lambda + \imath \epsilon) \chi_+^2(2B/\epsilon_0) (x)^{-s}, \]
constituting a continuous \( \mathcal{L}(\mathcal{H}) \)-valued function (the limit is taken uniformly in \( \lambda \)). Consequently, using again (C.7), it suffices to check the existence and continuity of the \( \mathcal{L}(\mathcal{G}_b, \mathcal{H}) \)-valued function
\[ R \ni \lambda \rightarrow f_1(\lambda) (x)^s T_b^+ J_\beta \gamma_b^+ (\lambda, \beta)^*. \]

Expanding \( T_b^+ \) as above, we can use Lemma 8.3 to combine for each term the factor \( Q^+ (b, l) \) with the factor \( J_\beta \gamma_b^+ (\lambda, \beta)^* \) (as done above). Each remaining factor of a ‘\( Q \)-operator’ (more precisely \( Q^+(b, l)^* \) appearing to the left) contributes by a factor \( O(\rho_i^{1/2 - \delta/2}), \) cf. Remark 7.2 (i). Whence we are left with checking that the right-hand side of (C.9) vanishes. For that we note that the ‘\( \epsilon \rightarrow 0 \)’-limit in the modified versions of (C.9) is taken uniformly in \( \rho > 1 \). Whence the integrand to the right can be computed (in the weak topology of \( L_{-1}^2(X) \)) as
\[ g(\lambda) f_2(H) (\Phi^+ - R(\lambda + \imath 0)(H - \lambda) \Phi^+ J_\beta f_1(\lambda) \gamma_b^+ (\lambda, \beta)^* \gamma_{b,0}(\lambda, \beta) \varphi \]
\[ = \lim_{\rho \to \infty} \int_{\mathcal{G}_b} g(\lambda) f_2(H) (\Phi^+ - R(\lambda + \imath \rho)(H - \lambda) \Phi^+ J_\beta f_1(\lambda) \gamma_b^+ (\lambda, \beta)^* \gamma_{b,0}(\lambda, \beta) \varphi \]
\[ = \lim_{\rho \to \infty} \int_{\mathcal{G}_b} g(\lambda) f_2(H) [H, \chi_\rho(\lambda) \Phi^+ J_\beta f_1(\lambda) \gamma_b^+ (\lambda, \beta)^* \gamma_{b,0}(\lambda, \beta) \varphi \]
\[ = 0 \]
proving (C.6b). We have shown (C.5a).

The second formula (C.5b) (interpreted as an identity in \( L_{-1}^2(X) \)) is a consequence of the above proof of (C.5a).

II. We prove (8.2). It is tempting to substitute the adjoint expression of (C.5a) with \( g(\lambda) = \delta_{\epsilon, \lambda}^\prime(\lambda) \) into (C.2) and then interchange the order of the two integrations. This is doable but requires of course some modification since the meaning of the right-hand side of (C.5a) is a vector in \( L_{-s}^2(X) \) for \( s = 1 \), or in fact for any \( s > 1/2 \) (but unlikely any smaller), and there is no obvious way of controlling the vector \( T_a^{-} J_a \tilde{\varphi}_{a} f_1(k_a) \delta_{\epsilon, \alpha}(\lambda^\prime) \varphi_{a} \)
uniformly in \( \epsilon > 0 \) in \( L_{-s}^{2}(X) \) for some \( s > 1/2 \). However, writing \( T_a^{-} \) as a finite sum of terms on the form
\[ f_2(H) Q^-(a, k)^* B^- Q^-(a, k) f_2(\tilde{H}_a) \text{ with } B^- \text{ bounded}, \]
we can introduce the modification, say denoted by \( T_{a,-}^{-, \rho} \), given by inserting for each such term the above factor \( \chi_\rho \) to the right of \( Q^-(a, k)^* \). Using Fubini’s theorem and the computation
\[ \int_{-\infty}^{\infty} \delta_{\epsilon, \lambda^\prime}(\lambda) \delta_{\epsilon, \alpha}(\lambda^\prime) \, d\lambda^\prime = \delta_{2\epsilon, \alpha}(\lambda), \]
we then obtain
\[
(2\pi i)^{-2} \langle \varphi_b, \tilde{S}_{\beta\alpha} \varphi_a \rangle = \lim_{\epsilon \to 0} \lim_{\rho \to \infty} \int_{\Lambda} f_1(\lambda) \langle \gamma_{b,0}(\lambda) \varphi_b, \tilde{\gamma}^+_\beta(\lambda)J^+_\beta(T^+_b)^* \delta(H - \lambda)T^-_{a,\rho}J_a \tilde{w}_a f_1(k_\alpha)\delta_{2\epsilon,\alpha}(\lambda) \varphi_a \rangle d\lambda,
\]
valid for any $\varphi_a \in L^2_\alpha(X_a)$ and $\varphi_b \in L^2_\beta(X_b)$ for $s > 1/2$.

With an analogue of (C.7) (with $\alpha$ rather than $\beta$) we compute
\[
\lim_{\epsilon \to 0} \int_{\Lambda} f_1(\lambda) \langle \gamma_{b,0}(\lambda) \varphi_b, \tilde{\gamma}^+_\beta(\lambda)J^+_\beta(T^+_b)^* \delta(H - \lambda)T^-_{a,\rho}J_a \tilde{w}_a f_1(k_\alpha)\delta_{2\epsilon,\alpha}(\lambda) \varphi_a \rangle d\lambda = \int_{\Lambda} f_1(\lambda) \langle \gamma_{b,0}(\lambda) \varphi_b, \tilde{\gamma}^+_\beta(\lambda)J^+_\beta(T^+_b)^* \delta(H - \lambda)T^-_{a,\rho}J_a f_1(\lambda)\gamma^-_{a,0}(\lambda) \varphi_a \rangle d\lambda.
\]
Here the limit is taken uniformly in $\rho > 1$. Due to these features we can interchange limits above and conclude that
\[
(2\pi i)^{-2} \langle \varphi_b, \tilde{S}_{\beta\alpha} \varphi_a \rangle = \lim_{\rho \to \infty} \left( \lim_{\epsilon \to 0} \int_{\Lambda} f_1(\lambda) \langle \gamma_{b,0}(\lambda) \varphi_b, \tilde{\gamma}^+_\beta(\lambda)J^+_\beta(T^+_b)^* \delta(H - \lambda)T^-_{a,\rho}J_a f_1(\lambda)\gamma^-_{a,0}(\lambda) \varphi_a \rangle d\lambda \right).
\]
showing (8.2). Note that indeed the right interpretation of the formula requires the expansion into a (finite) sum of expressions arising upon substituting for $T^+_b$ and $T^-_{a,\rho}$ sums of operators on the form (C.8) and (C.10), respectively.

\[\square\]

**Remark C.1.** Note the following analogues of (C.5a) and (8.2), cf. (3.8a) and (3.8b),
\[
\tilde{W}_\beta g(k_\beta)\varphi = -2\pi \int (g f_1)(\lambda) (f_2(H)\delta(H - \lambda)T^-_b J_b^+(\lambda) \varphi) \gamma_{b,0}(\lambda) \varphi d\lambda, \quad (C.11a)
\]
\[
16\lambda_\beta \lambda_\alpha f_1^2(\lambda) m_b(\pm \xi_b)^2 m_b(\pm \xi_a)^2 \delta_{\beta\alpha} \frac{\epsilon,\alpha(\lambda)\varphi}{\lambda} = -(2\pi i)^2 f_1^2(\lambda) \tilde{\gamma}^+_\beta(\lambda)J^+_\beta(T^+_b)^* \delta(H - \lambda)T^+_a J_a^\pm \gamma^\pm_{a,0}(\lambda) \varphi, \quad (C.11b)
\]
The quantity to the left in (c.11b) (for each sign) is meant to be an operator in $\mathcal{L}(\mathcal{G}_a, \mathcal{G}_b)$; the use of the Kronecker symbol $\delta_{\beta\alpha}$ specifies that it vanishes unless $\beta = \alpha$. The operator can be considered as the fiber of $F_\beta(\tilde{W}^\pm_\beta) \tilde{W}^\pm_\alpha F_\alpha^{-1}$ at energy $\lambda$, invoking the orthogonality of channels. Whence the formula (c.11b) results by mimicking the above procedure for showing (8.2). As before its correct interpretation requires the expansion into a sum of expressions arising upon substituting for $T^\pm_b$ and $T^\pm_{a,\rho}$ sums of operators on the form (C.8) or (C.10).

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