Relativistic simultaneity and causality

V. J. Bolós\textsuperscript{1}, V. Liern\textsuperscript{2}, J. Olivert\textsuperscript{3}

\textsuperscript{1}Dpto. de Matemáticas, Facultad de Ciencias, Universidad de Extremadura.
Avda. de Elvas s/n. 06071–Badajoz, Spain.
e-mail: vjbolos@unex.es

\textsuperscript{2}Departament d'Economia Financera i Matemàtica, Universitat de València.
Avda. Tarongers s/n. 46071–Valencia, Spain.
e-mail: vicente.liern@uv.es

\textsuperscript{3}Departament d'Astronomia i Astrofísica, Universitat de València.
C/ Dr. Moliner s/n. 46100–Burjassot (Valencia), Spain.
e-mail: joaquin.olivert@uv.es

March 9, 2005

Abstract

We analyze two types of relativistic simultaneity associated to an observer: the spacelike simultaneity, given by Landau submanifolds, and the lightlike simultaneity (also known as observed simultaneity), given by past-pointing horismos submanifolds. We study some geometrical conditions to ensure that Landau submanifolds are spacelike and we prove that horismos submanifolds are always lightlike. Finally, we establish some conditions to guarantee the existence of foliations in the space-time whose leaves are these submanifolds of simultaneity generated by an observer.

1 Introduction

It is well known that some problems related with simultaneity have not been solved yet. A great number of works treat the local character of relativistic simultaneity accepting that Landau submanifolds\textsuperscript{4} generated by an observer are leaves of a spacelike foliation. However, the fulfillment of this property cannot be ensured on any neighborhood without assuming some additional geometrical conditions. Therefore, when working on a neighborhood where this property does not hold, some difficulties in setting a successful dynamical study arise. The study of some of these conditions is the main objective of this paper.
In this work, we consider two types of simultaneities: spacelike simultaneity, which describes those events that are simultaneous in the local inertial proper system of the observer; and lightlike (or observed) simultaneity, which describes those events which the observer observes as simultaneous although they are not simultaneous in its local inertial proper system. The sets of spacelike simultaneous events and lightlike simultaneous events determine the Landau submanifolds and the past-pointing horismos submanifolds respectively.

Our next concern is the causality related to these types of simultaneity, because we should be able to guarantee, for instance, that Landau submanifolds are spacelike in a given neighborhood. For this, we introduce a new concept, the tangential causality, more general than causality, and we prove that every Landau submanifold $L_{p,u}$ is $p$-tangentially spacelike, but it is not necessarily spacelike. On the other hand, we prove that every horismos submanifold $E_p$ is $p$-tangentially lightlike and lightlike.

In physics, it is usual to work with synchronizable timelike vector fields. In most cases, the leaves of the orthogonal foliation are considered “simultaneity submanifolds”. In this work it is proved that given an observer $\beta$ (i.e. a $C^\infty$ timelike curve), there exists, on a small enough tubular neighborhood of this observer, a synchronizable timelike vector field containing the 4-velocity of $\beta$. Moreover, this vector field is orthogonal to a Landau submanifolds foliation. But we can not assure the existence of these kind of vector fields on any convex normal neighborhood. On the other hand, it is also proved that given an observer, there exists a foliation whose leaves are past-pointing horismos submanifolds of events of the observer, and it is well defined on any convex normal neighborhood.

\section{Preliminary Concepts}

In what follows, $(\mathcal{M}, g)$ will be a 4-dimensional lorentzian space-time manifold. Given $v$ a vector, $v^\perp$ will denote the subspace orthogonal to $v$.

Let $p \in \mathcal{M}$. An open neighborhood $\mathcal{N}_0$ of the origin in $T_p\mathcal{M}$ is said to be normal if the following conditions hold:

(i) the mapping $\exp_p : \mathcal{N}_0 \to \mathcal{N}_p$ is a diffeomorphism, where $\mathcal{N}_p$ is an open neighborhood of $p$.

(ii) given $X \in \mathcal{N}_0$ and $t \in [0,1]$ we have that $tX \in \mathcal{N}_0$.

For a given event $p \in \mathcal{M}$, an open neighborhood $\mathcal{N}_p$ of $p$ is a normal neighborhood of $p$ if $\mathcal{N}_p = \exp_p \mathcal{N}_0$, where $\mathcal{N}_0$ is a normal neighborhood of the origin in $T_p\mathcal{M}$. Finally, an open set $V \neq \emptyset$ in $\mathcal{M}$, which is a normal neighborhood of each one of its points, is a convex normal neighborhood.

These neighborhoods are useful to obtain a dynamical study of simultaneity. Moreover, the Whitehead Lemma asserts that given $p \in \mathcal{M}$ and a neighborhood $\mathcal{U}$ of $p$, there always exists a convex normal neighborhood $V$ of $p$ such that $V \subset \mathcal{U}$.
Since we are not going to make a global study, we will consider the spacetime $\mathcal{M}$ as a convex normal neighborhood in order to simplify. So, given two events in $\mathcal{M}$, there exists a unique geodesic containing them.

We are going to introduce two static ways to analyze simultaneity: Landau and past-pointing horismos submanifolds. Given $u \in T_p \mathcal{M}$ the 4-velocity of an observer at $p$, and the metric tensor field $g$, we consider the submersion $\Phi : \mathcal{M} \to \mathbb{R}$ given by $\Phi (q) := g (\exp^{-1}_p q, u)$. The fiber

$$L_{p,u} := \Phi^{-1} (0)$$

is a regular 3-dimensional submanifold called Landau submanifold of $(p, u)$. In other words,

$$L_{p,u} = \exp_p u^\perp$$

(see Figure 1).

The next result is given in [4]:

**Theorem 1** Given $u \in T_p \mathcal{M}$ the 4-velocity of an observer at $p$, there exists a unique regular 3-dimensional submanifold $L_{p,u}$ such that $T_p L_{p,u} = u^\perp$ and whose points are simultaneous with $p$ in the local inertial proper system of $p$.

On the other hand, defining the submersion $\varphi : \mathcal{M} - \{p\} \to \mathbb{R}$ given by $\varphi (q) := g (\exp^{-1}_p q, \exp^{-1}_p q)$, the fiber

$$E_p := \varphi^{-1} (0)$$

(2)

is a regular 3-dimensional submanifold, called horismos submanifold of $p$, which has two connected components [5]. We will call past-pointing (respectively future-pointing) horismos submanifold of $p$, $E^-_p$ (resp. $E^+_p$), to the connected component of (2) in which, for each event $q \in \mathcal{M} - \{p\}$, the preimage $\exp^{-1}_p q$ is a past-pointing (respectively future-pointing) lightlike vector. In other words,

$$E^-_p = \exp_p C^-_p ; \quad E^+_p = \exp_p C^+_p ,$$

where $C^-_p$ and $C^+_p$ are the past-pointing and the future-pointing light cones of $T_p \mathcal{M}$ respectively (see Figure 2).
Figure 2: Construction of the horismos submanifolds $E^-_p$ and $E^+_p$ by means of the exponential map.

The events in a Landau submanifold $L_{p,u}$ are simultaneous with $p$ in the local inertial proper system of $p$ (i.e. they are synchronous with $p$), but they are not observed as simultaneous (in the sense that they are not observed at the same instant) by any observer. On the other hand, the events in a past-pointing horismos submanifold $E^-_p$ are observed as simultaneous by any observer at $p$, since they belong to light rays which arrive at $p$. So, a Landau submanifold $L_{p,u}$ defines an intrinsic simultaneity for an observer with 4-velocity $u$ at $p$, and a past-pointing horismos submanifold $E^-_p$ defines an observed simultaneity for any observer at $p$. In general, we will call both, Landau and past-pointing horismos submanifolds, \textit{simultaneity submanifolds}.

3 \textbf{Tangential causality}

Our aim is to study the causality of simultaneity submanifolds, specially in which cases a Landau submanifold is spacelike and in which cases a horismos submanifold is lightlike. But first, we are going to introduce a new concept called “tangential causality”, because some results on tangential causality will be useful to study causality in Section 4.

Let $V$ be a 4-dimensional vector space regarded as a $C^\infty$ manifold. It can be canonically identified with any of its tangent spaces: for each $v \in V$ there exists a unique isomorphism $\phi_v: T_vV \rightarrow V$ such that

$$\omega (\phi_v w) = w (\omega)$$

for all $w \in T_vV$ and for all $\omega \in V^*$.

If, moreover, $(V, g)$ is a lorentzian vector space, we define a $(0, 2)$-tensor field $g$ on $V$ by

$$g (w, w') := g (\phi_v w, \phi_v w')$$

where $v \in V$ and $w, w' \in T_vV$. Then, $(V, g)$ is a lorentzian manifold [5].
Let $p$ be in $\mathcal{M}$, we can apply this to $T_p\mathcal{M}$ because $(T_p\mathcal{M}, g_p)$ is a lorentzian vector space, where $g_p := g|_{T_p\mathcal{M}}$. So, for each $v \in T_p\mathcal{M}$ we can define a canonical isomorphism $\phi_v$ of $T_v \mathcal{M}$ onto $T_p\mathcal{M}$ satisfying (3). If we define $g_p$ on $T_v (T_p\mathcal{M})$ from $g_p$ (according to (4)), then $(T_p\mathcal{M}, g_p)$ is a lorentzian manifold.

**Definition 2** Let $N$ be a regular submanifold of $\mathcal{M}$ and $p \in N$. The \textbf{p-tangential submanifold of} $N$ is

$$\exp_p^{-1} N$$

considered as a regular submanifold in $T_p\mathcal{M}$.

Given $q \in N$, we define the \textbf{p-tangential causality of} $N$ \textbf{at} $q$ as the causality of $\exp_p^{-1} N$ at $\exp_p^{-1} q$, using $g_p$. If this causality is the same at every point of $\exp_p^{-1} N$, then we define the \textbf{p-tangential causality of} $N$ as the causality of $\exp_p^{-1} N$ at any point.

The $p$-tangential causality can be interpreted as an “observed causality at $p$”, because an observer detects the events of the space-time through its tangent space.

It is easy to prove that

$$T_qN = \exp_{p\ast v} \left( T_v \left( \exp_p^{-1} N \right) \right),$$

where $N$ is a regular submanifold of $\mathcal{M}$, $p, q \in N$, and $v = \exp_p^{-1} q$. As a particular case, since $\exp_{p\ast 0} = \phi_0$ [1], we have

$$T_pN = \phi_0 \left( T_0 \left( \exp_p^{-1} N \right) \right).$$

Then, the causality of $N$ at $p$ coincides with the $p$-tangential causality of $N$ at $0$. However, given $v \in \exp_p^{-1} N$ such that $v \neq 0$, the causality of $N$ at $p$ is not necessarily the same as the $p$-tangential causality of $N$ at $v$.

Applying the tangential causality concept to simultaneity submanifolds, we obtain the next result:

**Proposition 3** Given $p \in \mathcal{M}$, and $u \in T_p\mathcal{M}$ the 4-velocity of an observer at $p$, we have that

(a) the $p$-tangential causality of $L_{p,u}$ is spacelike.

(b) the $p$-tangential causality of $E_p$ is lightlike.

**Proof.**

(a) Given $v \in \exp_p^{-1} L_{p,u}$ such that $v \neq 0$, we have that $g(u, v) = 0$. We define

$$g_u : T_p\mathcal{M} \to \mathbb{R} \quad v' \mapsto g_u (v') := g (u, v').$$

So, $\exp_p^{-1} L_{p,u} = g_u^{-1} (0)$. Therefore, given $w \in T_v (T_p\mathcal{M})$, we have that $w \in T_v \left( \exp_p^{-1} L_{p,u} \right)$ if and only if $w (g_u) = 0$, if and only if $g_u (\phi_v w) = 0$.
(by (4)), if and only if \( g(u, \phi_v w) = 0 \) (by (11)), if and only if \( g_p(\phi_v^{-1} u, w) = 0 \) (by (14)). Then
\[
T_v(\exp_p^{-1} L_{p,u}) = (\phi_v^{-1} u)^\perp. \tag{8}
\]
Moreover, \( \phi_v^{-1} u \) is timelike because \( g_p(\phi_v^{-1} u, \phi_v^{-1} u) = g(u, u) = -1 \). Thus, \( (8) \) is a spacelike subspace and hence \( \exp_p^{-1} L_{p,u} \) is a spacelike submanifold of \( (T_pM, g_p) \).

(b) Analogously,
\[
T_v(\exp_p^{-1} E_p) = (\phi_v^{-1} v)^\perp. \tag{9}
\]
Since \( \phi_v^{-1} v \) is lightlike, we have that \( \exp_p^{-1} E_p \) is a lightlike submanifold of \( (T_pM, g_p) \).

\[\blacksquare\]

4 Causality of simultaneity submanifolds

Definition 4 Given \( p \in \mathcal{M} \) and \( v \in T_p\mathcal{M} \), we define
\[
v_q^* := \tau_{pq}v, \tag{10}
\]
where \( q \in \mathcal{M} \) and \( \tau_{pq} \) is the parallel transport along the unique geodesic segment from \( p \) to \( q \) (note that we are considering \( \mathcal{M} \) as a convex normal neighborhood, and so there exists a unique geodesic containing \( p \) and \( q \)). It is clear that \( (10) \) depends differentiably on \( q \) and thus \( v^* \) is a vector field, named vector field adapted to \( v \).

The vector field \( v^* \) adapted to \( v \) has the same causal character as \( v \), since parallel transport keeps orthogonality.

Definition 5 Let \( U \) be a timelike vector field on an open neighborhood \( \mathcal{U} \). \( U \) is synchronizable on \( \mathcal{U} \) if and only if its orthogonal \( 3 \)-distribution \( U^\perp \) is a foliation on \( \mathcal{U} \). Then, \( U^\perp \) is called the physical spaces \( 3 \)-foliation of \( U \) and it is also denoted by \( S_U \).

Theorem 6 Given \( p \in \mathcal{M} \), and \( u \in T_p\mathcal{M} \) the 4-velocity of an observer at \( p \), we have that

(a) if \( u^* \) is synchronizable on an open neighborhood of \( q \in L_{p,u} \), then \( L_{p,u} \) is spacelike at \( q \).

(b) \( E_p \) is always lightlike.

Proof.
(a) Let us call $v := \exp_p^{-1} q$. By (9) and (10) we have

$$T_q L_{p,u} = \exp_{p* v} \left( T_v \left( \exp_p^{-1} L_{p,u} \right) \right) = \exp_{p* v} \left( \left( \phi^{-1}_v u \right) \right). \quad (11)$$

Let $w$ be in $\left( \phi^{-1}_v u \right)$. Then $g(u, \phi_v w) = 0$, and hence $g(u, (\phi_v w)^*) = 0$ because parallel transport keeps orthogonality. So, $(\phi_v w)^*_q$ and $v_q^*$ are in $(u_q^*)^\perp$.

Since $u^*$ is synchronizable on an open neighborhood of $q$, $(u^*)^\perp$ is a foliation in this open neighborhood (that is, it is involutive) and hence $[(v^*, \phi_v w)^*]_q \in (u_q^*)^\perp$. Let us denote $\theta(X)(Y)$ the Lie bracket $[X,Y]$ where $X, Y$ are vector fields. Then $\theta(v^*) ((\phi_v w)^*)_q \in (u_q^*)^\perp$. Using induction over $n \in \mathbb{N}$

$$\theta(v^*)^n (\phi_v w)^*)_q \in (u_q^*)^\perp. \quad (12)$$

Since

$$\exp_{p* v} w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \theta(v^*) ((\phi_v w)^*)_q \quad (13)$$

(see [3], applying [12] in [13], we have $\exp_{p* v} w \in (u_q^*)^\perp$. So, by (11), we have

$$T_q L_{p,u} = (u_q^*)^\perp, \quad (14)$$

because they have the same dimension. Concluding, $L_{p,u}$ is spacelike since $u_q^*$ is timelike.

(b) Given $q \in E_p$ and $v := \exp_p^{-1} q$, by (9) and (10) we have

$$T_q E_p = \exp_{p* v} \left( T_v \left( \exp_p^{-1} E_p \right) \right) = \exp_{p* v} \left( \left( \phi^{-1}_v v \right) \right). \quad (15)$$

Let $w$ be in $\left( \phi^{-1}_v v \right)$. Then $g(v, \phi_v w) = 0$, and hence $g(v^*, (\phi_v w)^*) = 0$ because parallel transport keeps orthogonality. So, $(\phi_v w)^*_q$ and $v_q^*$ are in $(v_q^*)^\perp$.

Since torsion vanishes,

$$[v^*, (\phi_v w)^*]_q = (\nabla_{v^*} (\phi_v w)^*)_q = 0 \quad (16)$$

but

$$\left( \nabla_{v^*} (\phi_v w)^* \right)_q \quad (17)$$

because $(\phi_v w)^*$ is parallely transported along the geodesic that joins $p$ and $q$ (i.e., the integral curve of $v^*$ passing through $q$). Moreover, given $X, Y, Z$ three vector fields, we have $Zg(X, Y) = g(\nabla_Z X, Y) + g(\nabla_Z Y, X)$ (see [3]). Taking $X, Y = v^*$ and $Z = (\phi_v w)^*$, we obtain

$$g \left( \nabla_{(\phi_v w)^* v^*}, v^* \right) = 0. \quad (18)$$
Applying (17) and (18) in (16), and using the previous notation for the Lie bracket, we have
\[ \theta (v^*) ((\phi, w)^* )_q \in (u^*_q) \perp. \] Using induction over \( n \in \mathbb{N} \)
\[ \theta (v^*)^n ((\phi, w)^* )_q \in (v^*_q) \perp. \quad (19) \]

Applying (19) in (13), we have \( \exp_p v^* w \in (v^*_q) \perp \). So, by (15), we have
\[ T_q E_p = (v^*_q) \perp, \quad (20) \]
because they have the same dimension. Concluding, \( E_p \) is lightlike since \( v^*_q \) is lightlike.

Under the hypotheses of Theorem 6, by (14) and (20), we have
\[ T_q L_{p,u} = \tau_{pq} u^\perp; \quad T_q E_p = \tau_{pq} v^\perp, \]
since \( (u^*_q) \perp = \tau_{pq} u^\perp \) and \( (v^*_q) \perp = \tau_{pq} v^\perp \).

It is important to remark that if the adapted vector field \( u^* \) is not synchronizable in any open neighborhood of \( q \), then we can not assure that \( L_{p,u} \) is spacelike at \( q \), but we can always assure that the \( p \)-tangential causality of \( L_{p,u} \) at \( q \) is spacelike, by Proposition 3.

Remark 7 Since \( L_{p,u} \) is spacelike at \( p \), we can always assure that there exists a small enough open neighborhood of \( p \) in which \( L_{p,u} \) is spacelike.

5 Simultaneity foliations

Since we are supposing that the space-time \( \mathcal{M} \) is a convex normal neighborhood, we can define Landau and horismos submanifolds at any event \( p \). In particular, given \( \beta : I \rightarrow \mathcal{M} \) an observer (where \( I \) is an open real interval and \( \beta \) is parameterized by its proper time), we can define the sets of Landau and horismos submanifolds
\[
\{ L_{\beta(t)} \}_{t \in I} : \quad \{ E_{\beta(t)}^- \}_{t \in I} : \quad \{ E_{\beta(t)}^+ \}_{t \in I}
\]
where \( L_{\beta(t)} \) denotes \( L_{\beta(t), \dot{\beta}(t)} \). Our aim in this section is to study these sets of Landau and horismos submanifolds as leaves of a foliation.

Theorem 8 Let \( \beta : I \rightarrow \mathcal{M} \) be an observer and \( p \in \beta I \). There exists a convex normal neighborhood \( \mathcal{V} \) of \( p \) where a foliation \( \mathcal{L}_\beta \) (called Landau foliation generated by \( \beta \)) is defined, and whose leaves are \( \{ L_{\beta(t)} \cap \mathcal{V} \}_{t \in I} \) (see Figure 3).

Proof. Let us introduce some definitions that can be found in [6]. Let \( N \) be a submanifold of \( \mathcal{M} \), we define the normal tangent fiber bundle \( TN^\perp \) as the fiber bundle composed by the subspaces \( T_p N^\perp \). On an open neighborhood of
the zero section $O(TN^\perp) = \{0_p \in T_pN^\perp : p \in N\}$ we can define the normal exponential map of $N$ as follows:

$$
\exp^\perp : T N^\perp \rightarrow M, \quad v \in T_pN^\perp \mapsto \exp_p v.
$$

Considering $\beta I$ as a submanifold of $M$, we can define the normal exponential map of $\beta I$ on an open neighborhood of the zero section of $T\beta I^\perp$. Using an analogous reasoning given in [6], it is proved that for all $p \in \beta I$ there exists an small enough open neighborhood $V$ of $p$ (that we can assume to be convex normal neighborhood) in which the normal exponential map of $\beta I$ is a diffeomorphism. So, given $q \in V$, it belongs to one and only one Landau submanifold of the family $\{L_{\beta(t)} \cap V\}_{t \in I}$. Since these submanifolds are regular, they are the leaves of a foliation on $V$.

According to Theorem 8 and Remark 7 there exists a small enough tubular neighborhood of $\beta I$ where the Landau foliation generated by $\beta$ is well defined and spacelike. So, there exists a synchronizable future-pointing timelike unit vector field (defined on this tubular neighborhood) orthogonal to the Landau foliation $L_\beta$. The integral curves of this vector field are a congruence of observers orthogonal to $L_\beta$.

But, given a convex normal neighborhood, we can not assure that the Landau foliation generated by $\beta$ is well defined on it, because the leaves can intersect themselves. In fact, it is usual that $L_{\beta(t_1)} \cap L_{\beta(t_2)} \neq \emptyset$ for $t_1, t_2 \in I$, $t_1 \neq t_2$; for instance, in Minkowski space-time if $\beta$ is not geodesic (see Figure 4).

Unlike the Landau foliations, the horismos foliations (past-pointing and future-pointing) are well defined on any convex normal neighborhood, because
Figure 4: Intersection of two Landau submanifolds $L_{p,u}$ and $L_{p',u'}$ in Minkowski space-time, where $p, p'$ are events of a not geodesic observer $\beta$ and $u, u'$ are the 4-velocities of this observer at $p, p'$ respectively.

their leaves do not intersect in any case:

**Theorem 9** Let $\beta : I \to \mathcal{M}$ be an observer. There exists a foliation $E^{-}_{\beta}$ (called past-pointing horismos foliation generated by $\beta$) defined on $\bigcup_{t \in I} E^{-}_{\beta(t)}$ and whose leaves are $\{E^{-}_{\beta(t)}\}_{t \in I}$ (see Figure 5).

Analogous for future-pointing horismos.

**Proof.** It is proved in [5] that, in a convex normal neighborhood, past-pointing (or future-pointing) horismos submanifolds of different events of a given observer do not intersect. Since they are regular submanifolds, it is clear that they form a foliation.

According to Theorems 6(b) and 9, the past-pointing (and future-pointing) horismos foliation generated by an observer is always well defined (on any convex normal neighborhood) and lightlike.

### 6 Discussion and open problems

If we work in the framework of *spacelike simultaneity* (i.e., simultaneity in the local inertial proper system of the observer, see [4]) there appear several serious mathematical problems related with Landau submanifolds: it can not be assured that they were spacelike at any point (see Theorem 3) and the leaves of a “Landau foliation” associated to a given observer can intersect themselves, even working in a convex normal neighborhood (see Figure 4). On the other hand, all these problems disappear if we work in the framework of *lightlike* (or observed) *simultaneity*, i.e., working with past-pointing horismos submanifolds instead of Landau submanifolds.

Moreover, given an observer at an event $p$ with 4-velocity $u$, the events of its Landau submanifold $L_{p,u}$ do not affect the observer at $p$ in any way, since
both electromagnetic and gravitational waves travel at the speed of light. On the other hand, the events of its past-pointing horismos submanifold $E_p^-$ are precisely the events that affect and are observed by the observer at $p$, i.e. the events that exist for the observer at $p$. So, we can say “what you see is what happens”. More arguments in favor of lightlike simultaneity against spacelike simultaneity are discussed in [2].

To conclude, we are going to present the main open problems, that are related with Landau submanifolds:

- A Landau submanifold $L_{p,u}$ is spacelike at any point of any convex normal neighborhood of $p$.

There is not any satisfactory proof of this fact, neither a counterexample showing that it is false.

- Given $\beta : I \rightarrow M$ an observer, we have that $L_{\beta(t_1)} \cap L_{\beta(t_2)} = \emptyset$ for any $t_1, t_2 \in I$, $t_1 \neq t_2$, if and only if... ? (Supposing that $M$ is a convex normal neighborhood, of course.)

In Minkowski space-time the answer is “if and only if $\beta$ is geodesic”, but this property has not been yet generalized to General Relativity. It would be useful to characterize when an observer has focal points (see [3]).

References

[1] J. K. Beem, P. E. Ehrlich. Global Lorentzian Geometry (Marcel Dekker: New York 1981).
[2] V. J. Bolós. Lightlike simultaneity, comoving observers and distances in general relativity. gr-qc/0501085.

[3] S. Helgason. Differential geometry and symmetric spaces (Academic Press: London 1962).

[4] J. Olivert. On the local simultaneity in general relativity. J. Math. Phys. 21 (1980), 1783.

[5] R. K. Sachs, H. Wu. Relativity for Mathematicians (Springer Verlag: Berlin, Heidleberg, New York 1977).

[6] T. Sakai. Riemannian Geometry (American Mathematical Society: New York 1996).