Diffraction at a Time Grating in the Dynamical Sauter-Schwinger Process

K Krajewska¹, W Gac¹, M Twardy² and J Z Kamiński¹

¹ Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
² Faculty of Electrical Engineering, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland
E-mail: Katarzyna.Krajewska@fuw.edu.pl

Abstract. The Sauter-Schwinger pair creation from the quantum vacuum driven by a sequence of identical time-dependent electric-field pulses is considered. We show the appearance of intra- and inter-pulse interference structures in momentum distributions of produced pairs. These structures are interpreted in terms of the eigenvalues of the time-evolution operator for an arbitrary eigenmode of the fermionic field. We demonstrate a nearly perfect coherence of inter-pulse structures, which is due to nonadiabatic transitions at avoided crossings of the phases defining the time evolution operator. As we also show, this nearly perfect coherent enhancement can be accidentally lost.

1. Introduction
The vacuum instability in the presence of a static electric field, which results in electron-positron \((e^-e^+)\) pair creation, has been predicted long time ago [1, 2, 3]. Breaking the vacuum requires an enormous electric field strength, \(\mathcal{E}_S = m_e^2 c^3 /|e| = 1.32 \times 10^{18} \text{ V/m}\), where \(m_e\) is the electron rest mass and \(e = -|e| < 0\) is its charge (here and in what follows, we keep \(\hbar = 1\)). \(\mathcal{E}_S\) is referred to as the Sauter-Schwinger critical field. Since such electric field cannot be achieved in laboratory, the Sauter-Schwinger pair production has not been verified experimentally yet. Another disadvantage is that the process is very weak. For this reason, various ideas have been proposed to enhance the signal of \(e^-e^+\) pairs, with the dynamically-assisted mechanism [4] or the diffraction-based one [5, 6] among others.

This paper is closely related to Ref. [6] where we have considered the pair creation from the vacuum by a train of alternating-sign Sauter pulses. In contrast, in the current paper we consider a sequence of \(N_{\text{rep}}\) identical pulses with an envelope and a carrier wave. While essentially we observe similar nearly coherent enhancement in the momentum distributions of created particles for \(N_{\text{rep}} > 1\), additional accidental loss of coherence is also observed. Moreover, we demonstrate that the same interpretation of the diffraction patterns in the momentum distributions of created particles as in [6] holds.

2. Theoretical formulation
Consider the electron-positron pair creation from the vacuum by a homogeneous in-space, time-dependent electric field which oscillates in time along the \(z\)-axis. In this case, the electromagnetic...
field tensor is $c\mathcal{F} \mu(x) = (0, \mathcal{E}(t)) = (0, 0, 0, \mathcal{E}(t))$. Moreover, we assume that the electric field $\mathcal{E}(t)$ is generated by a laser, meaning that [6]

$$\int_{-\infty}^{+\infty} dt \mathcal{E}(t) = 0. \quad (1)$$

The corresponding four-vector potential equals $A^\mu(x) = (0, A(t)) \equiv (0, 0, 0, A(t))$, with $A(t)$ such that $\mathcal{E}(t) = -\mathbf{\dot{A}}(t)$, and

$$\lim_{t \to +\infty} A(t) = \lim_{t \to -\infty} A(t). \quad (2)$$

As it was demonstrated in [5, 6], in order to calculate the probability distribution of pair creation one has to solve the system of equations

$$i \frac{d}{dt} \begin{bmatrix} c_p^{(1)}(t) \\ c_p^{(2)}(t) \end{bmatrix} = \begin{bmatrix} \omega_p(t) & i\Omega_p(t) \\ -i\Omega_p(t) & -\omega_p(t) \end{bmatrix} \begin{bmatrix} c_p^{(1)}(t) \\ c_p^{(2)}(t) \end{bmatrix}. \quad (3)$$

Here,

$$\omega_p^2(t) = c^2 p^2 + c^2 (p_0 - eA(t))^2 + (mc^2)^2 \quad (4)$$

is expressed in terms of the longitudinal $p_\parallel$ and the transverse $p_\perp$ components of the particle asymptotic momentum, which are defined as

$$p_\parallel = p \cdot e_z, \quad p_\perp = p - p_\parallel e_z, \quad (5)$$

whereas $\Omega_p(t) = \frac{c|e|\mathcal{E}(t)e_\perp}{2\omega_p^2(t)}$ with $e_\perp = \sqrt{(cp_\perp)^2 + (mc^2)^2}$. Eq. (3) will be solved for an electric field model consisting of a sequence of $N_{\text{rep}}$ pulses. In this case, the momentum distribution of pairs created from the vacuum in the eigenmode of the fermionic field defined by the momentum $p$ is

$$\mathcal{P}_{N_{\text{rep}}} = \lim_{t \to +\infty} |c_p^{(2)}(t)|^2, \quad (6)$$

where $|c_p^{(1)}(t)|^2 + |c_p^{(2)}(t)|^2 = 1$.

As it was mentioned above, the electric field consists of a sequence of $N_{\text{rep}}$ pulses,

$$\mathcal{E}(t) = \mathcal{E}_0 F(t), \quad (7)$$

where

$$F(t) = \sum_{N=1}^{N_{\text{rep}}} F_0 \left[ t + (2N - 1 - N_{\text{rep}})T/2 \right]. \quad (8)$$

Here, $F_0$ represents a single pulse shape function,

$$F_0(t) = \begin{cases} \frac{M}{2} \left[ 1 + \cos\left( \frac{\pi t}{\sigma} \right) \right] \sin\left( \frac{2\pi t}{\sigma} + \chi \right) & \text{for } |t| \leq \sigma, \\
0 & \text{for } |t| > \sigma, \end{cases} \quad (9)$$

whereas $T$ defines a time-delay between the subsequent pulses. For the pulses to be well-separated it must hold that $T \geq 2\sigma$, where $2\sigma$ denotes the individual pulse duration. This is illustrated in Fig. 1, where we plot the electric field shape function for a train of two pulses for either $T = 2\sigma$ (left panel) or $T = 4\sigma$ (right panel). The constant $M$ in Eq. (9) is chosen such that the maximum absolute value of $F_0(t)$ equals 1. Further, $N_{\text{osc}} = 1, 2, 3, \ldots$ is the number of
cycles in the pulse and $\chi$ is its carrier-envelope phase (CEP). Since the vector potential equals, in general,

$$A(t) = -\int_{-\infty}^{t} E(\tau) d\tau = \int_{t}^{\infty} E(\tau) d\tau,$$

the vector potential shape function for the single pulse is

$$f_0(t) = \int_{t}^{\infty} F_0(\tau) d\tau,$$

and for the train of pulses,

$$f(t) = \sum_{N=1}^{N_{\text{rep}}} f_0[t + (2N - 1 - N_{\text{rep}})T/2].$$

One can check that in our case,

$$f_0(t) = \frac{\sigma M}{2\pi 2N_{\text{osc}}} \left[ \cos\left(\frac{2\pi N_{\text{osc}}}{\sigma} t + \chi\right) - \cos \chi \right] + \frac{\sigma M}{4\pi 2N_{\text{osc}} + 1} \left[ \cos\left(\frac{\pi(2N_{\text{osc}} + 1)}{\sigma} t + \chi\right) + \cos \chi \right]$$

$$+ \frac{\sigma M}{4\pi 2N_{\text{osc}} - 1} \cos\left(\frac{\pi(2N_{\text{osc}} - 1)}{\sigma} t + \chi\right) + \cos \chi,$$

which is defined for $|t| < \sigma$ and is 0 otherwise.

3. Momentum distributions
In Fig. 2, we plot the momentum distributions of pairs created from the vacuum (6) by a single electric field pulse ($N_{\text{rep}} = 1$), with the shape function (9). The distributions denoted by the black, blue, red, and green lines correspond to different CEPs of the pulse: $\chi = 0$, $\chi = \pi/4$, $\chi = \pi/3$, and $\chi = \pi/2$, respectively. For the remaining parameters we keep $\sigma = 5\tau_C$, where $\tau_C$ is the Compton time $\tau_C = 1/(m_e c^2)$, and the field amplitude is $E_0 = -0.1E_S$. For the left (right) panel, the number of cycles within a pulse is $N_{\text{osc}} = 2$ ($N_{\text{osc}} = 5$). The spectra are plotted as
Figure 2. Longitudinal momentum distributions of $e^-e^+$ pairs created from the vacuum (6) by a single electric pulse ($N_{rep} = 1$) with $N_{osc} = 2$ (left panel) and $N_{osc} = 5$ (right panel) cycles and different values of CEP; namely, $\chi = 0$ (black line), $\chi = \pi/4$ (blue line), $\chi = \pi/3$ (red line), or $\chi = \pi/2$ (green line). The shape function of the driving electric field is defined by Eqs. (7), (8) and (9) with $\sigma = 5\tau_C$, and the amplitude of the field, $E_0 = -0.1E_S$.

functions of the longitudinal momentum $p_\parallel$, i.e., assuming that $p_\perp = 0$. Such a choice is justified as the particles are mostly generated in the direction of the electric field. As we can see from this figure, the influence of CEP on the spectra is negligible. This follows from the fact that the process is basically determined by the field amplitude, which is kept the same in each case. Thus, the momentum distributions of created pairs are very similar for different values of $\chi$. For this reason, we shall keep $\chi = 0$ in our further analysis. Similar to our previous work [6], we shall refer to the observed structures as intra-pulse interference patterns. This is to distinguish them from the peak structures observed in Figs. 3 and 4.

When applying a train of electric field pulses to the quantum vacuum additional peak structures in the spectrum appear (see, the results for two and three pulse repetitions in Fig. 3 and Fig. 4). Such structures are much finer than the intra-pulse modulations. Typically, they consist of maxima which appear at the same values of the longitudinal momenta $p_\parallel$, independently of $N_{rep}$. At these momenta, the distributions $P_{N_{rep}}$ approximately scale to the one resulting from the interaction of a single pulse with the vacuum (black envelope), with a typical scaling factor $N_{rep}^2$. This indicates the coherent enhancement of probability distributions for $N_{rep} > 1$. In addition, these peaks become more narrow with increasing the number of pulses in the train, i.e., with increasing $N_{rep}$. Note that the main peaks in the spectra are accompanied by secondary maxima. For a given $N_{rep}$, there is always $(N_{rep} - 2)$ such secondary peaks. While modulations of the peak structures originate from the intra-pulse interference, the peaks themselves occur only when a sequence of electric field pulses is applied. Hence, we conclude that those structures originate from inter-pulse interferences.

Note that the distinction between the inter- and intra-pulse interference effects follows from the fact that, while the inter-pulse interferences give rise to the pronounced and sharp peaks in the momentum distributions of created particles (with the separation of peaks entirely controlled by the time delay of electric field pulses comprising the train, not by their temporal shapes; see, Figs. 3 and 4), the intra-pulse interferences lead to the smooth modulations of those distributions. This has been already seen in [6], where a different pulse train was considered. While in [6] a sequence of alternating-sign Sauter pulses was studied, in this paper we perform calculations for pulses with the carrier wave. This increases the number of parameters defining the train and so it may lead to an accidental loss of coherence, as illustrated in the right column of Fig. 3.
Figure 3. Distributions of e−e+ pairs, $P_{N_{\text{rep}}}$, (upper panels) as functions of their longitudinal momentum $p_\parallel$ (for $p_\perp = 0$), when generated by a single pulse ($N_{\text{rep}} = 1$) (black line), or by a train of two ($N_{\text{rep}} = 2$) (blue line) or three pulses ($N_{\text{rep}} = 3$) (red line). The shape function of the driving electric field is defined by Eqs. (7), (8) and (9) with the parameters: $E_0 = -0.1E_S$, $\chi = 0$, $\sigma = 5\tau_C$, and $T = 10\tau_C$. Lower panels show the dependence of the phases $\vartheta_1 \in (-\pi, 0)$ (modulo 2$\pi$) (in blue) and $\vartheta_2 \in (0, \pi)$ (modulo 2$\pi$) (in red) on $p_\parallel$.

Having analyzed the sensitivity of the resulting distributions to the external field parameters, we next demonstrate that the inter-pulse structures can be interpreted in terms of the eigenvalues of the time evolution operator which evolves in time a given eigenmode of the fermionic field.

4. Interpretation of the inter-pulse interference patterns

In order to interpret our numerical results we consider the time evolution matrix $\hat{U}(t, t')$, $t \geq t'$, that satisfies the equation,

$$i\frac{d}{dt} \hat{U}(t, t') = \left( \begin{array}{cc} \omega_p(t) & i\Omega_p(t) \\ -i\Omega_p(t) & -\omega_p(t) \end{array} \right) \hat{U}(t, t'),$$

with the initial condition $\hat{U}(t', t') = \hat{I}$. Note that for a train of $N_{\text{rep}}$ identical pulses driving the pair creation, the functions $\omega_p(t)$ and $\Omega_p(t)$ are periodic in the interval $N_{\text{rep}}T$. Thus, the system dynamics is determined by its evolution over time $T$, as indicated by

$$\hat{U}(t' + N_{\text{rep}}T, t') = [\hat{U}(T)]^{N_{\text{rep}}}. \quad (15)$$

Keeping this in mind, we introduce the eigenvalue problem for the operator $\hat{U}(T)$ (which is also called the monodromy matrix [7]),

$$\hat{U}(T)[j] = e^{-i\vartheta_j}[j], \quad j = 1, 2, \quad (16)$$
where the eigenvalues are $e^{-i\vartheta_j}$, and $|j\rangle$ denote their corresponding eigenstates. As discussed in [6], $\vartheta_j$ are defined modulo $2\pi$ and the eigenstates $|j\rangle$ can be parametrized as

$$
|1\rangle = e^{i\psi_1} \begin{pmatrix} e^{-i\beta/2} \cos(\gamma/2) \\
 e^{i\beta/2} \sin(\gamma/2) \end{pmatrix},
$$

$$
|2\rangle = e^{i\psi_2} \begin{pmatrix} -e^{-i\beta/2} \sin(\gamma/2) \\
 e^{i\beta/2} \cos(\gamma/2) \end{pmatrix}.
$$

Here, $0 \leq \gamma \leq \pi$, $0 \leq \beta < 2\pi$, and the global phases $\psi_j$ can be chosen arbitrary as they are irrelevant in our further analysis. Using these definitions, one can show that [6]

$$
[\hat{U}(T)]^{N_{\text{rep}}} = e^{-iN_{\text{rep}}\vartheta_0} \begin{pmatrix} \cos(N_{\text{rep}}\vartheta) + i\sin(N_{\text{rep}}\vartheta) \cos \gamma & ie^{-i\beta} \sin(N_{\text{rep}}\vartheta) \sin \gamma \\
 ie^{i\beta} \sin(N_{\text{rep}}\vartheta) \cos \gamma & \cos(N_{\text{rep}}\vartheta) - i\sin(N_{\text{rep}}\vartheta) \cos \gamma \end{pmatrix},
$$

where we have introduced: $\vartheta_0 = (\vartheta_2 + \vartheta_1)/2$ and $\vartheta = (\vartheta_2 - \vartheta_1)/2$. Thus, there are four real angles $0 \leq \vartheta_0, \vartheta < 2\pi$, $\beta$, and $\gamma$ which define the evolution of the system while it interacts with the pulsed electric field. Note that only two of them define the momentum distribution of created pairs [6]. More importantly, for a train of $N_{\text{rep}}$ pulses, the corresponding probability distribution $P_{N_{\text{rep}}}$ can be expressed in terms of the probability distribution for a single pulse $P_1$,

$$
P_{N_{\text{rep}}} = P_1 \left[ \frac{\sin(N_{\text{rep}}\vartheta)}{\sin \vartheta} \right]^2.
$$

Eq. (19) shows that $P_{N_{\text{rep}}}/P_1$ is equal to the so-called diffraction (interference) factor. Thus, it takes maximum values for small $\vartheta$ or, in other words, for $\vartheta \approx \vartheta_1 + 2\vartheta_{\text{G}}$ (modulo $2\pi$), where
$2\vartheta_G$ determines the difference between the phases of eigenvalues of the monodromy matrix (16). Specifically, one can show that for $N_{\text{rep}} > 1$,

$$\frac{P_{N_{\text{rep}}}}{P_1} \approx N_{\text{rep}}^2 \left[ 1 - \frac{4}{3} (N_{\text{rep}} - 1) \vartheta_G^2 \right]. \tag{20}$$

It follows from here that for as long as the phase difference is not too large (i.e., for $\vartheta_G \ll 1$ and $N_{\text{rep}} \vartheta_G \ll 1$), the $N_{\text{rep}}^2$-type enhancement should be observed. This is confirmed by Figs. 3 and 4, where in the lower panels we present the dependence of both phases $\vartheta_1$ and $\vartheta_2$ on the particles longitudinal momentum. For clarity, the phases have been defined such that $\vartheta_1 \in (-\pi, 0)$ (modulo $2\pi$) (in blue) whereas $\vartheta_2 \in (0, \pi)$ (modulo $2\pi$) (in red). Whenever these phases nearly intersect each other (at the so-called avoided crossings), we observe the peaks in the corresponding momentum distributions. If, however, the phase difference is too large, as it happens for $p_\parallel = 0$, the $N_{\text{rep}}^2$ scaling is lost. One should also note that this analysis does not apply to the results presented in the right column of Fig. 3 due to accidental loss of coherence, mentioned in Section 3. Basically, in this case the zeros of the momentum distribution for $N_{\text{rep}} = 1$ (black envelope) nearly coincide with the avoided crossings of the phases $\vartheta_1$ and $\vartheta_2$. Thus, instead of large peaks in the momentum distributions of created particles, the distributions almost vanish at the avoided crossings.

5. Conclusions

We have analyzed the electron-positron pair creation from vacuum by a sequence of $N_{\text{rep}}$ identical electric-field pulses. We have confirmed that the particle momentum distributions exhibit the intra- and inter-pulse interferences, similar to Ref. [6] where the Sauter pulses were considered. In the current paper, however, we have analyzed a different model of the pulsed electric field, with an envelope and a carrier wave. We have demonstrated, for instance, that the momentum distributions of created pairs marginally depend on the carrier envelope phase provided that we keep the peak electric field constant.

We have focused on inter-pulse interferences, demonstrating that they lead to a nearly perfect coherent enhancement of momentum distributions of created pairs. Namely, at certain particle momenta, the major inter-pulse peaks scale as $N_{\text{rep}}^2$ compared to the intra-pulse modulations. At those momenta, the corresponding phases of the time-evolution matrix $\vartheta_1$ and $\vartheta_2$ exhibit an avoided crossing. If, however, the phase difference is too large, the nearly perfect coherence is lost. Other detailed features of the momentum distributions have also been described using this interpretation. The numerical results have shown, for instance, that there exists the possibility of accidental loss of coherence. This happens when $P_1(p_\parallel, p_\perp = 0) = 0$ for those values of $p_\parallel$ for which the avoided crossings of phases $\vartheta_1$ and $\vartheta_2$ occur.

Acknowledgments

This work is supported by the National Science Centre (Poland) under Grant No. 2014/15/B/ST2/02203.

References

[1] Sauter F 1931 Zeitschrift für Physik 69 742–764
[2] Heisenberg W and Euler H 1936 Zeitschrift für Physik 98 714–732
[3] Schwinger J 1951 Phys. Rev. 82 664–679
[4] Schützhold R, Gies H and Dunne G 2008 Phys. Rev. Lett. 101 130404
[5] Akkermans E and Dunne G V 2012 Phys. Rev. Lett. 108 030401
[6] Kamiński J Z, Twardy M and Krajewska K 2018 Phys. Rev. D 98 056009
[7] Yakubovich V A and Starzhinskii V M 1975 Linear differential equations with periodic coefficients (New York: Wiley & Sons)