Algebraic Geometry Approach in Gravity Theory and New Relations between the Parameters in Type I Low-Energy String Theory Action in Theories with Extra Dimensions

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On the base of the distinction between covariant and contravariant metric tensor components, a new (multi-variable) cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian has been derived and parametrized with complicated non-elliptic functions, depending on the (elliptic) Weierstrass function and its derivative. This is different from standard algebraic geometry, where only two-dimensional cubic equations are parametrized with elliptic functions and not multivariable ones.

Physical applications of the approach have been considered in reference to theories with extra dimensions. The so-called "length function" \( l(x) \) has been introduced and found as a solution of quasilinear differential equations in partial derivatives for two different cases of "compactification + rescaling" and "rescaling + compactification". New physically important relations (inequalities) between the parameters in the action are established, which cannot be derived in the case \( l = 1 \) of the standard gravitational theory, but should be fulfilled also for that case.

1 Introduction

Inhomogeneous cosmological models have been intensively studied in the past in reference to colliding gravitational waves [1] or singularity structure and generalizations of the Bondi - Tolman and Eardley-Liang-Sachs metrics [2, 3]. In these models the inhomogeneous metric is called the Szafron-Szekeres metric [4-7]. In [7], after an integration of one of the components - \( G_{0}^{1} \) of the Einstein’s equations, a solution in terms of an elliptic function is obtained. This is important since valuable cosmological characteristics for observational cosmology such as the Hubble’s constant \( H(t) = \frac{R(t)}{R'(t)} \) and the deceleration parameter \( q = -\frac{R'(t)R(t)}{R^{2}(t)} \) may be expressed in terms of the Jacobi’s theta function and of the Weierstrass elliptic function respectively [8]. Also in [7], the expression for the metric in the Szafron-Szekeres approach has been obtained in terms of the Weierstrass elliptic function after reducing the component \( G_{0}^{1} \) of the Einstein’s equations [7, 8] to the nonlinear differential equation \( \left( \frac{d\Phi}{dz} \right)^{2} = -K(z) + 2M(z)\Phi^{-1} + \frac{1}{4}\Lambda\Phi^{2} \). Then by introducing some notations this equation can be brought to the two-dimensional cubic algebraic equation \( y^{2} = 4x^{3} - g_{2}x - g_{3} \), which according to the standard algebraic geometry prescription (see [9] for a contemporary introduction into algebraic geometry) can be parametrized as \( x = \rho(z) \), \( y = \rho^{'}(z) \),

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where $\rho(z)$ is the well-known Weierstrass elliptic function

$$\rho(z) = \frac{1}{z^2} + \sum_{\omega} \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]$$

and the summation is over the poles in the complex plane. According to the standard definition of an elliptic curve [9], such a parametrization is possible if the functions $g_2$ and $g_3$ are equal to the s.c. "Eisenstein series" (define complex numbers) $g_2 = 60 \sum_{\omega \in \Gamma} \frac{1}{\omega^2}$, $g_3 = 140 \sum_{\omega \in \Gamma} \frac{1}{\omega^4}$.

The main goal of the present paper and of the preceding ones [10, 11] is to propose a new algebraic geometry approach for finding new solutions of the Einstein's equations by representing them in an algebraic form. The approach is based essentially on the s.c. gravitational theory with covariant and contravariant metrics and connections (GTCCMC) [12], which makes a clear distinction between covariant $g_{ij}$ and contravariant metric tensor components $\tilde{g}^{ij}$. This means that $\tilde{g}^{18}$ should not be considered to be the inverse ones to the covariant components $g_{ij}$, Consequently $\tilde{g}^{18} g_{im} \equiv f_n^a(x)$. In the special case when $f_n^a(x) = (x) \delta_{n}^{a}$, important new relations in the form of inequalities will be found between the parameters in the type I low energy string theory action - the string coupling constant $\lambda$, the string scale $m_s$ (which in these theories is identified with $m_{grav}$) and the electromagnetic coupling constant $g_4$.

## 2 New Algebraic Geometry Approach in Gravity Theory. Embedded Sequence of Cubic Algebraic Equations.

In the framework of the GTCCMC and the distinction between covariant and contravariant metric tensor components, we shall assume that the contravariant metric tensor can be represented in the form of the factorized product $\tilde{g}^{ij} = dX^i dX^j$, where the differentials $dX^i$ remain in the tangent space $T_X$ of the defined on the initially given manifold generalized coordinates $X^i = X^i(x_1, x_2, ..., x_n)$. The existence of different from $g^{ij}$ contravariant metric tensor components $\tilde{g}^{ij}$ means that another connection $\tilde{\Gamma}_k_{il}^{m} = \tilde{g}^{im} \Gamma^m_{kl} = \frac{1}{2} \delta^{im} (g_{kl,1} + g_{lk,1} - g_{kl,1})$, not consistent with the initial metric $g_{ij}$, can be introduced. By substituting $\tilde{\Gamma}_k_{il}^{m}$ in the expression for the "tilde" Ricci tensor $\tilde{R}_{ij}$, and requiring the equality of the "tilde" scalar curvature $\tilde{R}$ with the usual one $R$, i.e., $\tilde{R} = R$ (assuming also that $\tilde{R}_{ij} = R_{ij}$), one can obtain the s.c. "cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian" [10, 11]

$$dX^i dX^l \left( p \Gamma^m_{ki} g_{mk} dX^k - \Gamma^m_{ki} g_{ml} dX^m - \Gamma^r_{ij} g_{ir} dX^r \right) - dX^i dX^l R_{ij} = 0 .$$

(2.1)

In the same way, assuming the contravariant metric tensor components to be equal to the "tilde" ones, the Einstein's equations in vacuum were derived in the general case for arbitrary $\tilde{g}^{ij}$, when the assumption $\tilde{g}^{ij} = dX^i dX^j$ is no longer implemented [11].

Now we shall briefly explain the essence of the s.c. method of "embedded sequence of cubic algebraic equations", proposed for the first time in the paper [11] and enabling to find solutions of multivariable cubic equations in terms of (non-elliptic) functions, depending on the Weierstrass function and its derivative. The method is based on representing (the three-dimensional case is taken as a model example) the initial cubic algebraic equation (2.1) as a cubic equation with respect to the variable $dX^3$ only and applying with respect to it the linear - fractional transformation. Thus a cubic algebraic equation with respect to the two-dimensional algebraic variety of the (remaining) variables $dX^1$ and $dX^2$ is derived (further $\alpha$, $\beta$, $\gamma = 1, 2$) $p \Gamma^r_{\gamma(a) g_{(3)}} dX^r dX^a dX^b + K^{(3)}_{a,b} dX^a dX^b + K^{(2)}_{a} dX^a + 2p \left( \frac{\rho(z)}{z^3} \right)^3 \Gamma_{3,3} g_{33} = 0$. From the last equation the solutions of the initially given multivariable equation (2.1) (called "the embedding equation of the preceeding one") are found to be [11]

$$dX^3 = \frac{2a}{c_3} + \frac{\rho(z)}{c_3} \left( \frac{z}{\sqrt{a}} \right) - L_{1}^{(3)} \frac{2a}{c_3} - L_{2}^{(3)} \rho(z), \quad dX^2 = \frac{A}{B} .$$

(2.2)
\[
dX^1 = \frac{1}{\sqrt{\rho'(z)}} \rho(z) \sqrt{F_1 \rho^2 + F_2 \rho(z) + K_1^{(2)} + f_1 \rho^3 + f_2 \rho^2 + f_3 \rho + f_4},
\]

where \( A := \frac{1}{\sqrt{\rho'(z)}} \rho(z) \sqrt{C_1} \) and \( B := \frac{1}{\sqrt{\rho'(z)}} \rho(z) \sqrt{C_2} + k_1(dX^1)^2 + h_2(dX^1)^2 + h_3 \). The found solutions do not represent elliptic functions, since they cannot be represented in the form \( dX^1 = K_1(\rho) + \rho'(z)K_2(\rho) \), where \( \rho \) is the Weierstrass elliptic function (1.1). Also, since the solution \( dX^2 \) contains itself \( dX^1 \), it is called "the embedding solution" of \( dX^1 \) [11]. Similarly, \( dX^3 \) is the embedding solution of \( dX^1 \) and \( dX^2 \).

3. "Compactification + Rescaling" and "Rescaling + Compactification" in Type I String Theory. Tensor Length Scale.

The standard approach in type I string theory in ten dimensions is based on the low-energy action [14, 15, 16]

\[
S = \int d^{10}x \left( \frac{m_8^8}{(2\pi)^7} R + \frac{1}{4} \frac{m_6^6}{(2\pi)^7} F^2 + \ldots \right) = \int d^4x V_6 L,
\]

where \( L \) is the expression inside the bracket. After compactification to 4 dimensions on a manifold of volume \( V_6 \), one can identify the resulting coefficients in front of the \( R \) and \( \frac{1}{4} F^2 \) terms with \( M_4^2 \) and \( \frac{1}{g_4^2} \) and obtain as a result the relations [14] \( M_4^2 = \frac{(2\pi)\xi^7}{V_6 m_8^8} \), \( \lambda = \frac{(2\pi)^7}{V_6 m_8^8} \). The essence of the proposed new approach in the paper [13] is that the operation of compactification is "supplemented" by the additional operation of "rescaling" of the contravariant metric tensor components in the sense, clarified in Section 2. This means that since the contraction of the covariant metric tensor \( g_{ij} \) with the contravariant one \( g^{jk} = dX^j dX^k \) gives exactly (when \( i = k \)) the length interval \( l = ds^2 = g_{ij} dX^j dX^i \), then naturally for \( i \neq k \) the contraction will give a (mixed) tensor \( l^i^j = g_{ij} dX^j dX^k \), which can be interpreted as a "tensor length scale" for the different directions. Further the case of general contravariant tensor components \( \tilde{g}^i^k \) had been assumed when \( \tilde{g}^i^k g_{im} \equiv f^n(x) := t^n_m := \delta^n_m \), from where \( \tilde{g}^i^k \) and the "rescaled" scalar quantities \( \tilde{R} \) and \( \tilde{F}^2 \) can easily be expressed [13]. In the following one can discern two cases:

1st case - "compactification + rescaling". One starts from the "unrescaled" ten-dimensional action (3.1), then performs a compactification to a five-dimensional manifold and afterwards a transition to the usual "rescaled" scalar quantities \( \tilde{R} \) and \( \tilde{F}^2 \). Then it is required that the "unrescaled" ten-dimensional effective action (3.1) is equivalent to the five-dimensional effective action after compactification, but in terms of the rescaled quantities \( \tilde{R} \) and \( \tilde{F}^2 \) in the right-hand side (R. H. S.) of (3.1). This can be expressed as follows

\[
S = \int d^{10}x \left( \frac{m_8^8}{(2\pi)^7} \tilde{R} + \frac{1}{4} \frac{m_6^6}{(2\pi)^7} \tilde{F}^2 \right) = \int d^4x \left( M_4^2 \tilde{R} + \frac{1}{4} \frac{1}{g_4^2} \tilde{F}^2 \right).
\]

Note also that since \( R^{(5)} = R^{(4)} \) (\( R^{(5)} \) means the curvature of the 5D spacetime), the compactification is in fact to four dimensions and consequently the integration in the R. H. S. of (3.2) is over a 4D-volume. Expressing the uilda (rescaled) quantities \( \tilde{R} \) and \( \tilde{F}^2 \) in the right-hand side of (3.2) through the unrescaled ones \( R \) and \( F^2 \) by means of the relation \( \tilde{g}^{i^k} = \frac{1}{g^{i^k}} \) and identifying the expressions in front of the "unrescaled" scalar quantities \( F^2 \) and \( R \) in both sides of (3.2), one obtains an algebraic relation and a quasilinear differential equation in partial derivatives with respect to the length function \( l(x) \) [13].

2nd case - "rescaling + compactification". This case is just the opposite to the previous one in the sense that the "rescaled" components become unrescaled ones and vice versa. In an analogous way, an algebraic relation can be obtained again after comparing the coefficient functions [13], from where after
introducing the notation $\beta \equiv \left[ \frac{(2\pi)^7}{V_{\text{euclidean}}} - M_{(4)}^2 \right] m^4 V_6 (2\pi)^{-7}$, and assuming a small deviation from the relation $M_{(4)}^2 = \frac{(2\pi)^7}{V_{\text{euclidean}}} \cdot \bar{g}_{44}$, i.e. $\beta \ll 1$, one can express the length scale $l(x)$ as $l^2 = \frac{1}{1 - \beta \frac{\partial P}{\partial \beta}} \approx 1 + \beta \frac{\partial P}{\partial \beta}$. In the last expression $P$ denotes the term with the second partial derivatives of the metric tensor, i.e. $P := (g_{AB,BC} + g_{BC,AB} - g_{AC,BD} - g_{BD,AC})$. For $l = 1$ (when $\bar{g}_{ij} = g_{ij}$ and $\bar{g}_{ij} \bar{g}_{lm} \equiv \delta_{ij}$), as should be expected, we can obtain the usual relation $M_{(4)}^2 = \frac{(2\pi)^7}{V_{\text{euclidean}}}$. The above result may also have an important physical meaning - any deviations from the relation $M_{(4)}^2 = \frac{(2\pi)^7}{V_{\text{euclidean}}}$ may be attributed to deviations from the usual scale $l = 1$ for the standard gravity theory.

3rd case - simultaneous fulfillment of "rescaling + compactification" and "compactification + rescaling". This means that it does not matter in what sequence the two operations are performed, i.e. the process of compactification is accompanied by rescaling. From the simultaneous fulfillment of the two differential equations one obtains a cubic algebraic equation with respect to $m$, from where under the assumption about the positivity of the square of the length function $l(x)$ (consequently - positive roots of the cubic equation), the following two inequalities (written for compactness as one - the upper and lower signs in $\pm$ mean two separate cases), relating the parameters in the low energy type I string theory action are obtained

$$p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b \sqrt{\frac{b^2}{2} + \frac{a^3}{27}} \approx \left[ \frac{a_1 + 6a_6}{18} \pm \frac{a_1}{18} \sqrt{\frac{a_1^2}{4} + 12a_6} \right]^3,$$

(3.3)

where $a_1 \equiv a_2 - \frac{a_3^2}{9}$, $b \equiv 2\pi^3 - \frac{a_6a_2}{3} + a_3$, $Q \equiv \frac{g^{AC} g^{BD} (2\pi)^7 \bar{g}_{P} P}{g^{AC} g^{BD} (2\pi)^7 \bar{g}_{P} - 2K(2\pi)^7 - M_{(4)}^2 V_6 (2\pi)^{-7}}$, $a_1 \equiv -\frac{(2\pi)^7}{M_4^2 V_6 (2\pi)^{-7}}$, $a_2 \equiv \frac{(2\pi)^7}{M_4^2 V_6 (2\pi)^{-7}}$, and $a_3 \equiv -\frac{a_2}{Q}$. Therefore, we do not consider here the case of an imaginary length function $l(x)$ in the case of the imaginary Lobachevsky space [17], realized by all the straight lines outside the absolute cone (on which the scalar product is zero, i.e. $[x,x] = 0$). The last two relations (3.3) are new (although rather complicated) inequalities between the parameters in the low energy type I string theory action, which cannot be obtained in the framework of the usual gravity theory. Also, it is important to stress that since the scale function $l(x)$ does not enter in them, they are valid for the standard gravity theory approach in theories with extra dimensions.

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