DEFORMATIONS OF GRADED NILPOTENT LIE ALGEBRAS
AND SYMPLECTIC STRUCTURES

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Abstract. We study symplectic structures on filiform Lie algebras – nilpotent
Lie algebras of the maximal length of the descending central sequence. There
are two basic examples of symplectic \( \mathbb{Z}_{>0} \)-graded filiform Lie algebras defined
by their bases \( e_1, \ldots, e_{2k} \) and structure relations
1) \( m_{0}^{(2k)} : [e_1, e_i] = e_{i+1}, i = 2, \ldots, 2k-1 \).
2) \( V_{2k} : [e_i, e_j] = (j-i)e_{i+j}, i+j \leq 2k \).

Let \( g \) be a symplectic filiform Lie algebra and \( \dim g = 2k \geq 12 \). Then
\( g \) is isomorphic to some \( \mathbb{Z}_{>0} \)-filtered deformation either of \( m_{0}^{(2k)} \) or of \( V_{2k} \). In the
present article we classify \( \mathbb{Z}_{>0} \)-filtered deformations of \( V_n \), i.e., Lie algebras
with structure relations of the following form:

\[
[e_i, e_j] = (j-i)e_{i+j} + \sum_{l=1}^{n} c_{ij} l e_{i+j+l}, \quad i+j \leq n
\]

Namely we prove that for \( n \geq 16 \) the moduli space \( M_n \) of these algebras can
be identified with the orbit space of the following \( \mathbb{K}^* \)-action on \( \mathbb{K}^5 \):

\[
\alpha \ast X = (\alpha^{n-11} x_1, \alpha^{n-10} x_2, \ldots, \alpha^{n-7} x_5), \alpha \in \mathbb{K}^*, \quad X \in \mathbb{K}^5.
\]

For \( n = 2k \) the subspace \( M_{2k}^{\text{sympl}} \subset M_{2k} \) of symplectic Lie algebras is deter-
mined by equation \( x_1 = 0 \). A table with the structure constants of symplecto-
ismorphism classes in \( M_{2k}^{\text{sympl}} \) is presented.

INTRODUCTION

Nilmanifolds \( M = G/\Gamma \) (compact homogeneous spaces of nilpotent Lie groups \( G \)
over lattices \( \Gamma \)) are important examples of symplectic manifolds that do not admit
Kähler structures. An interesting family of graded symplectic nilpotent Lie alge-
bas \( V_{2k} \) (and corresponding family of nilmanifolds \( M_{2k} \)) was considered in [1], [2],
[3]. The finite dimensional Lie algebras \( V_n \), that are defined by the commutating
relations \( [e_i, e_j] = (j-i)e_{i+j}, i+j \leq n \), "came" from the infinite dimensional Vir-
soro algebra and they are examples of so-called filiform Lie algebras – nilpotent Lie
algebras \( g \) with the maximal length \( s = \dim g - 1 \) of the descending central sequence
of \( g \). The study of filiform Lie algebras was started by M. Vergne in [22], [23].

The classification of symplectic filiform Lie algebras of dimensions \( \leq 10 \) was
discussed in [10], [11]. The present paper is the continuation of [19], where a
criterion of the existence of a symplectic structure on a filiform Lie algebra \( g \) was
proposed. In particular in a symplectic filiform \( g = L^1 g \) one can find the ideal
\( L^2 g \) of codimension 1 such that the sequence of ideals \( L^i g, i = 1, \ldots, 2k \), where
\( L^i g = C^{i-1} g, i = 3, \ldots, n \) \( \{C^i g\} \) are the ideals of the descending central sequence
\( C \) of \( g \) determines a decreasing filtration \( L \) of the Lie algebra \( g \). The associated

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graded Lie algebra \( \mathfrak{g}_{L} \) is symplectic also. The graded filiform algebras of the type 
\( \mathfrak{g}_{L} \) were classified in \[18\], \[19\]. There are two one-parameter families of graded
symplectic filiform Lie algebras of dimensions 8, 10. But if \( \mathfrak{g}_{L} \) is a symplectic
filiform Lie algebra with \( \dim \mathfrak{g} = 2k \geq 12 \) then \( \mathfrak{g}_{L} \) is isomorphic either to \( \mathfrak{m}_0(2k) \)
or to \( \mathcal{V}_{2k} \). In other words in dimensions \( 2k \geq 12 \) one can obtain symplectic filiform
Lie algebras as special deformations (that we call \( \mathbb{Z}_{>0} \)-filtered deformations) of two
graded Lie algebras: \( \mathfrak{m}_0(2k) \) and \( \mathcal{V}_{2k} \).

We classify \( \mathbb{Z}_{>0} \)-filtered deformations of \( \mathcal{V}_n \), i.e., Lie algebras with the structure
relations of the following form:
\[
[e_i, e_j] = (j-i)e_{i+j} + \sum_{l=1}^{i+j} c_{ij} e_{i+j+l}, \quad i + j \leq n.
\]

We compute in the Section 6 the space \( H^2(\mathcal{V}_n, \mathcal{V}_n) \) for \( n \geq 12 \). To the
\( \mathbb{Z}_{>0} \)-filtered deformations corresponds the subspace \( \oplus_{>0} H^2(i)(\mathcal{V}_n, \mathcal{V}_n) \). The main theorem 7.9 of
the present article asserts that for \( n \geq 16 \) the moduli space \( M_n \) (i.e. the set of
isomorphism classes) of these algebras can be identified with the orb it space of the
following \( K^* \)-action on \( K^6 = \oplus_{>0} H^2(i)(\mathcal{V}_n, \mathcal{V}_n) \):
\[
\alpha \star X = (\alpha^{n-11} x_1, \alpha^{n-10} x_2, \ldots, \alpha^{n-7} x_5), \alpha \in K^*, \quad X \in K^6,
\]
where the coordinates \( x_1, \ldots, x_5 \) of the space \( K^6 = \oplus_{>0} H^2(i)(\mathcal{V}_n, \mathcal{V}_n) \) are defined by
the choice of the basic cocycles \( \psi_{n,12-i}, i = 1, \ldots, 5 \).

For \( n = 2k \) the subspace \( M^\text{sympl}_{2k} \subset M_{2k} \) of symplectic Lie algebras is determined
by equation \( x_1 = 0 \). A table with structure constants of symplecto-isomorphism
classes in \( M^\text{sympl}_{2k} \) is presented in the Section 9.

1. IN Variant SYMPLECTIC STRUCTURES ON LIE GROUPS

**Definition 1.1.** A Lie group \( G \) is said to have a left-invariant symplectic structure
if it has a left-invariant non-degenerate closed 2-form \( \omega \).

**Example 1.2.** \( G \) is a two-dimensional abelian Lie group \( \mathbb{R}^2 \) with coordinates \( x, y \)
and \( \omega = dx \wedge dy \).

**Example 1.3.** \( G \) is a direct product \( \mathcal{H}_3 \times \mathbb{R} \) of the Heisenberg group \( \mathcal{H}_3 \) of all
matrices of the form
\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}, \quad x, y, z \in \mathbb{R},
\]
and abelian \( \mathbb{R} \) (with coordinate \( t \)). The invariant symplectic form \( \omega \) is defined as
\[
\omega = dx \wedge (dz - xdy) + dy \wedge dt.
\]

**Theorem 1.4** (B.-Y. Chu, \[14\]). Any semisimple Lie group has no left-invariant
symplectic structure.

**Theorem 1.5** (B.-Y. Chu, \[14\]). A connected unimodular Lie group admitting a
left-invariant symplectic structure must to be solvable.

One can obtain examples of left-invariant symplectic structures in the framework
of Kirillov’s orbit method.

Let us consider the coadjoint action \( Ad^* \) of a Lie group \( G \) on the dual \( \mathfrak{g}^* \) of its
Lie algebra \( \mathfrak{g} \):
\[
(Ad^*(g)f)(X) = f(Ad(g^{-1})X), \quad \forall X \in \mathfrak{g},
\]
where \( g \in G, f \in g^* \). The orbit \( O_f \) of this action has a homogeneous symplectic structure \( \omega_{O_f} \) such that

\[
\pi_f^*(\omega_{O_f}) = df,
\]

where \( \pi_f \) denotes the natural mapping \( \pi_f : G \to O_f, \pi_f(g) = A_d^*(g)f \). Let the stabilizer \( G_f = \{ g \in G | A_d^*(g)f = f \} \) of \( f \in g^* \) be a normal subgroup of \( G \) then the orbit \( O_f \) can be identified with the quotient group \( G/G_f \) and the corresponding symplectic structure \( \omega_{O_f} \) is left \( G/G_f \)-invariant. If \( G \) is a nilpotent Lie group then nilpotent ones are \( G_f \) and \( G/G_f \).

**Definition 1.6.** A skew-symmetric non-degenerate bilinear form \( \omega \) on the Lie algebra \( g \) is called symplectic if it closed, i.e.

\[
\omega([x,y],z) + \omega([y,z],x) + \omega([z,y],x) = 0, \quad \forall x, y, z \in g.
\]

If \( \omega_G \) is a left-invariant symplectic form on \( G \), then \( \omega_G \) defines a symplectic structure \( \omega_g \) on the Lie algebra \( g \) of \( G \), and conversely any symplectic form \( \omega_g \) of \( g \) defines a left-invariant symplectic structure on \( G \).

**Lemma 1.7** (A. Médina, P. Revoy [15]). Let \( g \) be a symplectic Lie algebra with non-trivial center \( Z(g) \) and \( I \) be a one-dimensional subspace in \( Z(g) \) and \( I^\omega \) its symplectic complement with respect to \( \omega \). Then one can consider two following exact sequences of Lie algebras and their homomorphisms

\[
\begin{align*}
0 & \to I \to I^\omega \to I^\omega/I \to 0 \\
0 & \to I^\omega \to g \to g/I^\omega \cong I \to 0
\end{align*}
\]

where \( I^\omega/I \) is symplectic Lie algebra (the restriction of \( \omega \) to \( I^\omega \) defines symplectic form \( \tilde{\omega} \) on the quotient-algebra \( I^\omega/I \)).

In other words 2\( k \)-dimensional symplectic Lie algebra \( g \) with non-trivial center can be obtained from 2\( k \)-2-dimensional symplectic \( I^\omega/I \) by means of two consecutive operations:

1) one-dimensional central extension of \( I^\omega/I \) by \( I \);
2) semidirect product of \( I^\omega \) and one-dimensional \( g/I^\omega \cong I \).

The combination of these two operations is called the double extension of symplectic Lie algebra \( I^\omega/I \).

**Theorem 1.8** (A. Médina, P. Revoy [15]). Symplectic nilpotent Lie algebras can be obtained by means of the sequence of consecutive double extensions starting with trivial Lie algebra of zero dimension.

The even-dimensional nilpotent Lie algebras are classified for dimensions 2\( k \leq 6 \) ([20]). The classification of symplectic 6-dimensional nilpotent Lie algebras based on this classification was done in [11].

2. **Filiform Lie algebras**

The sequence of ideals of a Lie algebra \( g \)

\[
C^1 g = g \supset C^2 g = [g,g] \supset \ldots \supset C^k g = [g,C^{k-1}g] \supset \ldots
\]

is called the descending central sequence of \( g \).

A Lie algebra \( g \) is called nilpotent if there exists \( s \) such that:

\[
C^{s+1} g = [g,C^s g] = 0, \quad C^s g \neq 0.
\]
The natural number $s$ is called the nil-index of the nilpotent Lie algebra $\mathfrak{g}$, or $\mathfrak{g}$ is called $s$-step nilpotent Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra. We call a set $F$ of subspaces

$$
\mathfrak{g} \supset \cdots \supset F^i \supset F^{i+1} \supset \cdots \quad (i \in \mathbb{Z})
$$

a decreasing filtration $F$ of $\mathfrak{g}$ if $F$ is compatible with the Lie structure

$$
[F^k \mathfrak{g}, F^l \mathfrak{g}] \subset F^{k+l} \mathfrak{g}, \forall k, l \in \mathbb{Z}.
$$

Let $\mathfrak{g}$ be a filtered Lie algebra. A graded Lie algebra

$$
\text{gr}_F \mathfrak{g} = \bigoplus_{k=1} (\text{gr}_F \mathfrak{g})_k, \quad (\text{gr}_F \mathfrak{g})_k = F^k \mathfrak{g}/F^{k+1} \mathfrak{g}
$$

is called the associated graded Lie algebra $\text{gr}_F \mathfrak{g}$.

The ideals $C^k \mathfrak{g}$ of the descending central sequence define a decreasing filtration $C$ of the Lie algebra $\mathfrak{g}$

$$
C^1 \mathfrak{g} = \mathfrak{g} \supset C^2 \mathfrak{g} \supset \cdots \supset C^k \mathfrak{g} \supset \cdots; \quad [C^k \mathfrak{g}, C^l \mathfrak{g}] \subset C^{k+l} \mathfrak{g}.
$$

One can consider the associated graded Lie algebra $\text{gr}_C \mathfrak{g}$.

The finite filtration $C$ of a nilpotent Lie algebra $\mathfrak{g}$ is called the canonical filtration of a nilpotent Lie algebra $\mathfrak{g}$.

**Proposition 2.1.** Let $\mathfrak{g}$ be a $n$-dimensional nilpotent Lie algebra. Then for its nil-index we have the estimate $s \leq n - 1$.

**Definition 2.2.** A nilpotent $n$-dimensional Lie algebra $\mathfrak{g}$ is called filiform Lie algebra if it has the nil-index $s = n - 1$.

**Example 2.3.** The Lie algebra $\mathfrak{m}_0(n)$ is defined by its basis $e_1, e_2, \ldots, e_n$ with commutating relations:

$$
[e_1, e_i] = e_{i+1}, \quad \forall 2 \leq i \leq n-1.
$$

**Remark.** We will omit in the sequel trivial commutating relations $[e_i, e_j] = 0$ in the definitions of Lie algebras.

**Example 2.4.** The Lie algebra $\mathfrak{m}_2(n)$ is defined by its basis $e_1, e_2, \ldots, e_n$ and commutating relations:

$$
[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1; \quad [e_2, e_j] = e_{j+2}, \quad 3 \leq j \leq n-2.
$$

**Example 2.5.** Let us define the algebra $L_k$ as the infinite-dimensional Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$ with a zero in $x = 0$ of order not less then $k + 1$.

The algebra $L_k$ can be defined by its infinite basis and commutating relations

$$
e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{N}, \quad i \geq k; \quad [e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{N}.
$$

One can consider the $n$-dimensional quotient algebra $\mathcal{V}_n = L_1/L_{n+1}$.

The Lie algebras $\mathfrak{m}_0(n)$, $\mathfrak{m}_2(n)$, $\mathcal{V}_n$ considered above are filiform Lie algebras.

**Proposition 2.6.** Let $\mathfrak{g}$ be a filiform Lie algebra and $\text{gr}_C \mathfrak{g} = \bigoplus_i (\text{gr}_C \mathfrak{g})_i$ is the corresponding associated (with respect to the canonical filtration $C$) graded Lie algebra. Then

$$
\dim(\text{gr}_C \mathfrak{g})_1 = 2, \quad \dim(\text{gr}_C \mathfrak{g})_2 = \cdots = \dim(\text{gr}_C \mathfrak{g})_{n-1} = 1.
$$
We have the following isomorphisms of graded Lie algebras:
\[ \text{gr}_C \mathfrak{m}_2(n) \cong \text{gr}_C \mathfrak{v}_n \cong \text{gr}_C \mathfrak{m}_0(n) \cong \mathfrak{m}_0(n). \]

**Theorem 2.7 (M. Vergne [23]).** Let \( \mathfrak{g} = \oplus \alpha \mathfrak{g}_\alpha \) be a graded \( n \)-dimensional filiform Lie algebra and
\[ (2) \quad \dim \mathfrak{g}_1 = 2, \quad \dim \mathfrak{g}_2 = \cdots = \dim \mathfrak{g}_{n-1} = 1. \]
then
1) if \( n = 2k + 1 \), then \( \mathfrak{g} \) is isomorphic to \( \mathfrak{m}_0(2k+1) \);
2) if \( n = 2k \), then \( \mathfrak{g} \) is isomorphic either to \( \mathfrak{m}_0(2k) \) or to the Lie algebra \( \mathfrak{m}_1(2k) \), defined by its basis \( e_1, \ldots, e_{2k} \) and commutating relations:
\[ [e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, 2k-1; \quad [e_j, e_{2k+1-j}] = (-1)^{j+1} e_{2k}, \quad j = 2, \ldots, k. \]

**Remark.** In the settings of the Theorem 2.7 the gradings of the algebras \( \mathfrak{m}_0(n) \), \( \mathfrak{m}_1(n) \) are defined as \( \mathfrak{g}_1 = \text{Span}(e_1, e_2), \mathfrak{g}_i = \text{Span}(e_{i+1}), i = 2, \ldots, n-1. \)

**Corollary 2.8 (M. Vergne [23]).** Let \( \mathfrak{g} \) be a filiform Lie algebra. Then one can choose a so-called adapted basis \( e_1, e_2, \ldots, e_n \) in \( \mathfrak{g} \):
\[ (3) \quad [e_1, e_i] = e_{i+1}, i=2, \ldots, n-1; \quad [e_i, e_j] = \begin{cases} \sum_{k=0}^{n-i-j} c_{ij}^{i+j+k} e_{i+j+k}, & i+j \leq n; \\ (-1)^{i+j+k} \alpha e_n, & i+j = n+1; \\ 0, & i+j > n+1; \\ 2 \leq i < j \leq n. \end{cases} \]
where \( \alpha = 0 \) if \( n \) is odd number.

3. **Symplectic filiform Lie algebras: filtrations and gradings**

**Definition 3.1.** A Lie algebra \( \mathfrak{g} \) is called symplectic if it admits at least one symplectic structure.

**Lemma 3.2 ([19]).** Let \( \mathfrak{g} \) be an \( 2k \)-dimensional symplectic filiform Lie algebra, then
\[ (4) \quad \text{gr}_C \mathfrak{g} \cong \mathfrak{m}_0(2k). \]

In other words: let \( \mathfrak{g} \) be a symplectic filiform Lie algebra and \( e_1, e_2, \ldots, e_{2k} \) be some adapted basis, then \( [e_i, e_j] = 0, \quad i+j = 2k+1 \).

**Proposition 3.3.** A fixed adapted basis of \( \mathfrak{g} \) \( e_1, e_2, \ldots, e_{2k} \) such that \( [e_i, e_j] = 0, \quad i+j = 2k+1 \) defines a non-canonical filtration \( L \) of \( \mathfrak{g} \):
\[ \mathfrak{g} = L^1 \mathfrak{g} \supset L^2 \mathfrak{g} \supset \cdots \supset L^{2k} \mathfrak{g} \supset \{0\}, \]
\[ L^j \mathfrak{g} = \text{Span}(e_j, \ldots, e_{2k}), \quad j = 1, \ldots, 2k. \]

**Remark.** For homogeneous components of associated graded \( \text{gr}_L \mathfrak{g} \) we have
\[ \dim(\text{gr}_L \mathfrak{g})_1 = \dim(\text{gr}_L \mathfrak{g})_2 = \cdots = \dim(\text{gr}_L \mathfrak{g})_{2k} = 1. \]

**Proposition 3.4 ([19]).** Let \( \mathfrak{g} \) be a symplectic filiform Lie algebra with a fixed adapted basis. Then the corresponding associated graded Lie algebra \( \text{gr}_L \mathfrak{g} = \oplus_i (\text{gr}_L \mathfrak{g})_i \) is symplectic also.

**Remark.** It was shown in [19] that the previous condition is necessary but not sufficient condition.
Theorem 3.5 (D. Millionschikov, [19]). Let $\mathfrak{g} = \bigoplus_{\alpha=1}^{2k} \mathfrak{g}_{\alpha}$ be a real graded symplectic Lie algebra such that

$$\dim \mathfrak{g}_{\alpha} = 1, \ 1 \leq \alpha \leq 2k; \quad [\mathfrak{g}_{1}, \mathfrak{g}_{\alpha}] = \mathfrak{g}_{\alpha+1}, \ 2 \leq \alpha \leq 2k - 1.$$ 

Then $\mathfrak{g}$ is isomorphic to the one and only one Lie algebra from the following list:

| dim | algebra | commutating relations | symplectic form |
|-----|---------|-----------------------|-----------------|
| 4   | $\mathfrak{m}_{0}(4)$ | $[e_1, e_2] = e_3, [e_1, e_3] = e_4$ | $e^1 \land e^4 - e^2 \land e^3$ |
| 6   | $\mathfrak{m}_{0}(6)$ | $[e_1, e_i] = e_{i+1}, i = 2, \ldots, 5$ | $e^1 \land e^6 - e^2 \land e^5 + e^3 \land e^4$ |
|     | $\mathfrak{V}_6$ | $[e_i, e_j] = (j - i)e_{i+j}, i+j \leq 6$ | $5e^1 \land e^6 + 3e^2 \land e^5 + e^3 \land e^4$ |
| 8   | $\mathfrak{m}_{0}(8)$ | $[e_1, e_i] = e_{i+1}, i = 2, \ldots, 7$ | $e^1 \land e^8 - e^2 \land e^7 + e^3 \land e^6 - e^4 \land e^5$ |
|     | $\mathfrak{g}_{8, \alpha}$ | $\alpha \neq -\frac{5}{2}, -2, -\frac{1}{2}, \frac{1}{2}$ | $e^1 \land e^8 + 2a^2 + 3a - 2e^2 \land e^7 + 2(2a^2 + 4a + 5)e^3 \land e^6 + \frac{3}{2(2a^2 + 4a + 5)}e^4 \land e^5$ |
| 10  | $\mathfrak{m}_{0}(10)$ | $[e_1, e_i] = e_{i+1}, i = 2, \ldots, 9$ | $e^1 \land e^{10} - e^2 \land e^9 + e^3 \land e^8 - e^4 \land e^7 + e^5 \land e^6$ |
|     | $\mathfrak{g}_{10, \alpha}$ | $\alpha \neq -\frac{5}{2}, -1, -\frac{1}{2}, -\frac{3}{2}, \alpha_1, \alpha_2; \quad \alpha_1, \alpha_2 \in \mathbb{R}, 2\alpha_1^2 + 2\alpha_2^2 + 3 = 0, 4\alpha_1^2 + 8\alpha_2^2 - 8\alpha_2 - 21 = 0$ | $e^1 \land e^{10} + 2a^2 + 3a - 2e^2 \land e^9 + \frac{3}{2(2a^2 + 4a + 5)}e^3 \land e^8 + \frac{3}{2(2a^2 + 4a + 5)}e^4 \land e^7 + \frac{3}{2(2a^2 + 4a + 5)}e^5 \land e^6$ |
| $2k \geq 12$ | $\mathfrak{m}_{0}(2k)$ | $[e_1, e_i] = e_{i+1}, i = 2, \ldots, 2k-1$ | $\frac{1}{2} \sum_{i+j=2k+1}(1)^{i+1}1^1 \land e^j$ |
|     | $\mathfrak{V}_{2k}$ | $[e_i, e_j] = (j - i)e_{i+j}, i+j \leq 2k$ | $\frac{1}{2} \sum_{i+j=2k+1}(1)^{i+1}1^1 \land e^j$ |

**Remark.** In the fourth column of the table we give only one variant of possible symplectic structure.

Corollary 3.6. Let $\mathfrak{g}$ be a symplectic filiform Lie algebra of dimension $2k \geq 12$ then one can choose a basis $e_1, \ldots, e_{2k}$ in $\mathfrak{g}$ such that the corresponding commutating relations will be either

$$[e_1, e_i] = e_{i+1} + \sum_{l=1}^{2k-i-1} c_{i1} e_{i+l+1}, \ i = 2, \ldots, 2k-1;$$

(7)

$$[e_i, e_j] = \sum_{l=1}^{2k-i-j} c_{i1} e_{i+j+l}, \ i + j \leq 2k; \ 2 \leq i < j \leq 2k;$$

or

$$[e_i, e_j] = (j - i)e_{i+j} + \sum_{l=1}^{2k-i-j} c_{i1} e_{i+j+l}, \ i + j \leq 2k; \ 1 \leq i < j \leq 2k;$$

(8)
Remark. The one-parameter family \( \mathfrak{g}_{\alpha, \rho} \) was considered in [11] as well as corresponding symplectic form \( \omega_0(\alpha) \). In [10] symplectic (over \( \mathbb{C} \)) low-dimensional \((\dim \mathfrak{g} \leq 10)\) filiform Lie algebras were classified (but this article contains some mistakes).

4. LIE ALGEBRA COHOMOLOGY

Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{K} \) and \( \rho: \mathfrak{g} \to \mathfrak{gl}(V) \) its linear representation (or in other words \( V \) is a \( \mathfrak{g} \)-module). We denote by \( C^q(\mathfrak{g}, V) \) the space of \( q \)-linear skew-symmetric mappings of \( \mathfrak{g} \) into \( V \). Then one can consider an algebraic complex:

\[
V \xrightarrow{d_0} C^1(\mathfrak{g}, V) \xrightarrow{d_1} C^2(\mathfrak{g}, V) \xrightarrow{d_2} \cdots \xrightarrow{d_{q-1}} C^q(\mathfrak{g}, V) \xrightarrow{d_q} \cdots
\]

where the differential \( d_q \) is defined by:

\[
(d_qf)(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1})) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}).
\]

The cohomology of the complex \((C^*(\mathfrak{g}, V), d)\) is called the cohomology of the Lie algebra \( \mathfrak{g} \) with coefficients in the representation \( \rho: \mathfrak{g} \to V \).

In this article we will consider two main examples:
1) \( V = \mathbb{K} \) and \( \rho: \mathfrak{g} \to \mathbb{K} \) is trivial;
2) \( V = \mathfrak{g} \) and \( \rho = ad: \mathfrak{g} \to \mathfrak{g} \) is the adjoint representation of \( \mathfrak{g} \).

The cohomology of \((C^*(\mathfrak{g}, \mathbb{K}), d)\) (the first example) is called the cohomology with trivial coefficients of the Lie algebra \( \mathfrak{g} \) and is denoted by \( H^*(\mathfrak{g}) \). Also we fix the notation \( H^*(\mathfrak{g}, \mathfrak{g}) \) for the cohomology of \( \mathfrak{g} \) with coefficients in the adjoint representation.

One can remark that \( d_1 : C^1(\mathfrak{g}, \mathbb{K}) \to C^2(\mathfrak{g}, \mathbb{K}) \) of the \((C^*(\mathfrak{g}, \mathbb{K}), d)\) is the dual mapping to the Lie bracket \([, ] : \mathbb{K}^2 \mathfrak{g} \to \mathfrak{g} \). Moreover the condition \( d^2 = 0 \) is equivalent to the Jacobi identity for \([, ] \).

Let \( \mathfrak{g} = \bigoplus \mathfrak{g}_\alpha \) be a \( \mathbb{Z} \)-graded Lie algebra and \( V = \bigoplus \mathfrak{V}_\beta \) is a \( \mathbb{Z} \)-graded \( \mathfrak{g} \)-module, i.e., \( \mathfrak{g}_\alpha \mathfrak{V}_\beta \subset \mathfrak{V}_{\alpha+\beta} \). Then the complex \((C^*(\mathfrak{g}, V), d)\) can be equipped with the \( \mathbb{Z} \)-grading \( C^q(\mathfrak{g}, V) = \bigoplus \mu C^q_{(\mu)}(\mathfrak{g}, V) \), where a \( V \)-valued \( q \)-form \( c \) belongs to \( C^q_{(\mu)}(\mathfrak{g}, V) \) iff for \( X_1 \in \mathfrak{g}_{\alpha_1}, \ldots, X_q \in \mathfrak{g}_{\alpha_q} \) we have

\[
c(X_1, \ldots, X_q) \in \mathfrak{V}_{\alpha_1+\alpha_2+\ldots+\alpha_q+\mu}.
\]

This grading is compatible with the differential \( d \) and hence we have \( \mathbb{Z} \)-grading in cohomology:

\[
H^q(\mathfrak{g}, V) = \bigoplus_{\mu \in \mathbb{Z}} H^q_{(\mu)}(\mathfrak{g}, V).
\]

Remark. The trivial \( \mathfrak{g} \)-module \( \mathbb{K} \) has only one non-trivial homogeneous component \( \mathbb{K} = \mathbb{K}_0 \).

Example 4.1. Let \( \mathfrak{g} \) be a Lie algebra with the basis \( e_1, e_2, \ldots, e_n \) and commutating relations

\[
[e_i, e_j] = c_{ij}e_{i+j}, i + j \leq n.
\]
Let us consider the dual basis $e^1, e^2, \ldots, e^n$. One can introduce a grading (that we will call the weight) of $\Lambda^*(g^*) = C^*(g, \mathbb{K})$:

$$
\Lambda^*(g^*) = \bigoplus_{\lambda=1}^{n(n+1)/2} \Lambda^*(\alpha),
$$

where a subspace $\Lambda^*_\alpha(g^*)$ is spanned by $q$-forms $\{e^{i_1} \wedge \ldots \wedge e^{i_q}, i_1 + \ldots + i_q = \lambda\}$. For instance a monomial $e^{i_1} \wedge \ldots \wedge e^{i_q}$ has the degree $q$ and the weight $\lambda = i_1 + \ldots + i_q$.

The complex $(C^*(g, g), d)$ is $\mathbb{Z}$-graded:

$$
C^*(g, g) = \bigoplus_{\mu \in \mathbb{Z}} C^*_\mu(g, g),
$$

where $C^*_\mu(g, g)$ is spanned by monomials $\{e_1 \otimes e^{i_1} \wedge \ldots \wedge e^{i_q}, i_1 + \ldots + i_q + \mu = l\}$.

Now we consider a filtered Lie algebra $g$ with a finite decreasing filtration $F$. One can define a decreasing filtration $\tilde{F}$ of $\Lambda^*(g^*)$.

$$
\tilde{F}^\mu \Lambda^p(g^*) = \{\omega \in \Lambda^p(g^*) \mid \omega(F^{\alpha_1}g \wedge \ldots \wedge F^{\alpha_p}g) = 0, \alpha_1 + \ldots + \alpha_p + \mu \geq 0\}.
$$

**Example 4.2.** Let $g$ be a Lie algebra with the basis $e_1, e_2, \ldots, e_n$ and commutating relations

$$
[e_i, e_j] = \sum_{k=0}^{n-i-j} c_{ij} e_{i+j+k}, i + j \leq n.
$$

As it was remarked above the corresponding filtration $L$ ($L^k = \text{Span}(e_k, \ldots, e_n)$) of $g$ can be defined. The associated graded Lie algebra $\text{gr}_L g$ has the following structure relations

$$
[e_i, e_j] = c_{ij}^0 e_{i+j}, i + j \leq n.
$$

Let us consider the dual basis $e^1, e^2, \ldots, e^n$. Then $\tilde{L}^\mu \Lambda^p(g^*)$ is spanned by $p$-monomials of weights less or equal to $-\mu$, i.e. by $p$-forms $e^1 \wedge \ldots \wedge e^n$ such that $i_1 + \ldots + i_p \leq -\mu$. For instance

$$
\tilde{L}^{-5} \Lambda^2(g^*) = \text{Span}(e^1 \wedge e^2, e^1 \wedge e^3, e^1 \wedge e^4, e^2 \wedge e^3).
$$

**Remark.** One can consider the spectral sequence $E_r$ that corresponds to the filtration $\tilde{F}$ of the complex $\Lambda^*(g^*)$. We have an isomorphism (see [23] for example)

$$
E^{p,q}_1 = H^{p+q}(\text{gr}_F g).
$$

**Theorem 4.3 ( [19]).** Let $g$ be a filiform Lie algebra such that $\text{gr}_C g \cong m_0(2k)$ and $\text{gr}_L g$ is symplectic.

Then the Lie algebra $g$ is symplectic if and only if some homogeneous symplectic class $[\omega_{2k+1}] \in E_1^{-2k-1, 2k+3} = H_{2(2k+1)}^2(\text{gr}_L g)$ survives to the term $E_\infty$.

5. $\mathbb{Z}_{>0}$-FILTERED DEFORMATIONS AND $H^2(g, g)$

In this section we recall some definitions from the Nijenhuis-Richardson deformation theory (see [21]).

**Definition 5.1.** Let $g$ be a Lie algebra with a Lie bracket $[,]$ and $\Psi : g \otimes g \to g$ is a skew-symmetric bilinear map. $\Psi$ is called a deformation of $[,]$ iff $[,]' = [,] + \Psi$ is a Lie bracket on the vector space $g$. 
Also we have the following important property:

\[[x, y]', z] + [[y, z]', x]' + [[z, x]', y]' = 0

is equivalent to the so-called deformation equation:

\[(10)\]

\[\Psi([x, y], z) + \Psi([y, z], x) + \Psi([z, x], y) + [\Psi(x, y), z] + [\Psi(y, z), x] + [\Psi(y, z), x] + \Psi(z, x), y] = 0.\]

The first six terms can be rewritten in the form \(d\Psi(x, y, z)\) where \(\psi : C^2(\mathfrak{g}, \mathfrak{g}) \to C^3(\mathfrak{g}, \mathfrak{g})\) is the differential of the complex \((\mathfrak{g}, \mathfrak{g}), d)\).

Finally we have

\[(11)\]

\[d\Psi + \frac{1}{2}[\Psi, \Psi] = 0,\]

where \([,]\) denotes a symmetric bilinear function \([,] : C^2(\mathfrak{g}, \mathfrak{g}) \times C^2(\mathfrak{g}, \mathfrak{g}) \to C^3(\mathfrak{g}, \mathfrak{g})\)

\[(12)\]

\[
\begin{aligned}
[\Psi, \tilde{\Psi}](x, y, z) &= \Psi(\tilde{\Psi}(x, y), z) + \Psi(\tilde{\Psi}(y, z), x) + \Psi(\tilde{\Psi}(z, x), y) \\
&+ \tilde{\Psi}(\Psi(x, y), z) + \tilde{\Psi}(\Psi(y, z), x) + \tilde{\Psi}(\Psi(z, x), y).
\end{aligned}
\]

The last definition can be generalised in terms of Nijenhuis-Richardson bracket in \(C^*(\mathfrak{g}, \mathfrak{g})\):

\[[,] : C^p(\mathfrak{g}, \mathfrak{g}) \times C^q(\mathfrak{g}, \mathfrak{g}) \to C^{p+q-1}(\mathfrak{g}, \mathfrak{g}).\]

Namely, for \(\alpha \in C^p(\mathfrak{g}, \mathfrak{g})\) and \(\beta \in C^q(\mathfrak{g}, \mathfrak{g})\) one can define \([\alpha, \beta] \in C^{p+q-1}(\mathfrak{g}, \mathfrak{g})\):

\[(13)\]

\[\begin{aligned}
[\alpha, \beta](\xi_1, \ldots, \xi_{p+q-1}) &= \sum_{1 \leq i_1 < \ldots < i_p \leq p+q-1} \alpha(\xi_{i_1}, \ldots, \xi_{i_p})\xi_1, \ldots, \xi_{i_1}, \ldots, \xi_{i_p}, \ldots, \xi_{p+q-1}) \\
&+ (-1)^{pq+p+q} \sum_{1 \leq j_1 < \ldots < j_p \leq p+q-1} \beta(\alpha(\xi_{j_1}, \ldots, \xi_{j_p}), \xi_{j_1}, \ldots, \xi_{j_1}, \ldots, \xi_{j_p}, \ldots, \xi_{p+q-1}).
\end{aligned}\]

The Nijenhuis-Richardson bracket defines a Lie superalgebra structure in \(C^*(\mathfrak{g}, \mathfrak{g})\), i.e., if \(\alpha \in C^p(\mathfrak{g}, \mathfrak{g}), \beta \in C^q(\mathfrak{g}, \mathfrak{g})\) and \(\gamma \in C^r(\mathfrak{g}, \mathfrak{g})\) then

\[(14)\]

\[\begin{aligned}
1) \quad [\alpha, \beta] &= -(1)^{(p-1)(q-1)}[\beta, \alpha]; \\
2) \quad (-1)^{(p-1)(q-1)}[[\alpha, \beta], \gamma] + (-1)^{(q-1)(r-1)}[[\beta, \gamma], \alpha] + (-1)^{(r-1)(p-1)}[[\gamma, \alpha], \beta] = 0.
\end{aligned}\]

Also we have the following important property:

\[d[\alpha, \beta] = [d\alpha, \beta] + (-1)^p[\alpha, d\beta].\]

Thus the Nijenhuis-Richardson bracket defines a Lie superalgebra structure in cohomology \(H^*(\mathfrak{g}, \mathfrak{g})\), i.e., the set of bilinear functions

\[\mathfrak{g} \times \mathfrak{g} \to H^{p+q-1}(\mathfrak{g}, \mathfrak{g})\]

with the properties \([14]\).
Proposition 5.2. Let $\mathfrak{g} = \oplus \alpha \mathfrak{g}_\alpha$ be a $\mathbb{Z}$-graded Lie algebra. Then the $\mathbb{Z}$-grading of $C^*(\mathfrak{g}, \mathfrak{g})$ and $H^*(\mathfrak{g}, \mathfrak{g})$ is compatible with the Nijenhuis-Richardson bracket:

$$\begin{align*}
[\cdot, \cdot] : C^p(\mathfrak{g}, \mathfrak{g}) \times C^q(\mathfrak{g}, \mathfrak{g}) & \rightarrow C^{p+q}(\mathfrak{g}, \mathfrak{g}) \\
[\cdot, \cdot] : H^p(\mathfrak{g}, \mathfrak{g}) \times H^q(\mathfrak{g}, \mathfrak{g}) & \rightarrow H^{p+q}(\mathfrak{g}, \mathfrak{g})
\end{align*}$$

(15)

Definition 5.3. A deformation $\Psi$ of a $\mathbb{Z}_{>0}$-graded Lie algebra $\mathfrak{g} = \oplus \alpha \mathfrak{g}_\alpha$ is called $\mathbb{Z}_{>0}$-filtered if the following condition holds:

$$\forall X_1 \in \mathfrak{g}_{a_1}, \ldots, \forall X_q \in \mathfrak{g}_{a_q}, \quad \Psi(X_1, \ldots, X_q) \in \bigoplus_{a > a_1 + a_2 + \ldots + a_q} \mathfrak{g}_a.$$ 

Or in other words:

$$\Psi = \Psi_1 + \Psi_2 + \cdots + \Psi_i + \cdots, \quad \Psi_i \in C^2(\mathfrak{g}, \mathfrak{g}), \quad i = 1, 2, \ldots$$

Decomposing $\Psi = \Psi_1 + \Psi_2 + \ldots$ in the deformation equation (11) and comparing terms with the same grading we came to the following system of equations on homogeneous components $\Psi_i$:

$$d\Psi_1 = 0, \quad d\Psi_2 + \frac{1}{2}[\Psi_1, \Psi_1] = 0, \quad d\Psi_3 + [\Psi_1, \Psi_2] = 0, \ldots,$$

(17)

$$d\Psi_i + \frac{1}{2} \sum_{m+i} [\Psi_m, \Psi_i] = 0, \ldots$$

This is the well-known system of equations for one-parametric deformation, in our case one can associate to $\Psi$ the following one-parametric deformation $\Psi_1$:

$$\Psi_t = t\Psi_1 + t^2\Psi_2 + \cdots + t^i\Psi_i + \cdots$$

1) the first equality of (17) shows that $\Psi_1$ is a cocycle, we denote by $\bar{\Psi}_1$ its cohomology class in $H^2(\mathfrak{g}, \mathfrak{g})$.

2) the second one shows that the Nijenhuis-Richardson product $[\bar{\Psi}_1, \bar{\Psi}_1]$ defines a trivial element in $H^3(\mathfrak{g}, \mathfrak{g})$. If $[\bar{\Psi}_1, \bar{\Psi}_1] = 0$ then $\Psi_2$ is determined not uniquely but up to some closed element in $Z^2(\mathfrak{g}, \mathfrak{g})$.

3) hence one can find $\bar{\Psi}_3$ iff $[\bar{\Psi}_1, \bar{\Psi}_2]$ is trivial in $H^3(\mathfrak{g}, \mathfrak{g})$ for some choice of $\Psi_2$.

The subset in $H^3(\mathfrak{g}, \mathfrak{g})$ formed by elements $-\bar{\Psi}_1, \bar{\Psi}_2$, where $\bar{\Psi}_2$ is a solution in $C^2(\mathfrak{g}, \mathfrak{g})$ of the equation $d\Psi_2 + \frac{1}{2}[\bar{\Psi}_1, \bar{\Psi}_1] = 0$ is called a triple Massey product $<\bar{\Psi}_1, \bar{\Psi}_1, \bar{\Psi}_1>$ and it is defined iff $[\bar{\Psi}_1, \bar{\Psi}_1] = 0$.

Definition 5.4. A Massey product of the $k$-th order $<\bar{\Psi}_1, \bar{\Psi}_1, \ldots, \bar{\Psi}_1>$ is a subset in $H^3(k)(\mathfrak{g}, \mathfrak{g})$ formed by classes $-\frac{1}{2} \sum_{m+i=k} [\Psi_m, \Psi_i]$ where $\Psi_j, j = 2, \ldots, k-1$ are solutions of the first $k-1$ equations of the system (17) and $\Psi_1$ is a representative of $\bar{\Psi}_1$. A Massey product $<\bar{\Psi}_1, \bar{\Psi}_1, \ldots, \bar{\Psi}_1>$ is called trivial if it contains zero.

Remark. A Massey product $<\bar{\Psi}_1, \bar{\Psi}_1, \ldots, \bar{\Psi}_1>$ of the the $k$-th order is defined iff all Massey products $<\bar{\Psi}_1, \bar{\Psi}_1, \ldots, \bar{\Psi}_1>$ of orders less than $k$ are trivial. One can easily show that it does not depend on the choice of $\Psi_1$ in $\bar{\Psi}_1$. A Lie product $[\bar{\Psi}_1, \bar{\Psi}_1]$ is called a Massey product of the second order.

Proposition 5.5. Let be $\bar{\Psi}_1$ an element of $H^3(\mathfrak{g}, \mathfrak{g})$. One can construct a deformation $\Psi$ of $\mathfrak{g}$ with a first term equal to $\Psi_1 \in H^3(\mathfrak{g}, \mathfrak{g})$ if and only if all Massey products $<\bar{\Psi}_1, \bar{\Psi}_1, \ldots, \bar{\Psi}_1>$ are trivial.
Let $\Psi$ and $\tilde{\Psi}$ be two deformations of a Lie algebra $g$. The question is: whether they define non-isomorphic Lie algebras or not? Or does there exist a non-degenerate linear transformation $\varphi : g \to g$ such that

$$\varphi([x, y] + \tilde{\Psi}(x, y)) = [\varphi(x), \varphi(y)] + \Psi(\varphi(x), \varphi(y))$$

Let us consider an arbitrary cocycle in $Z^2_{(1)}(g, g)$ cohomologous to $\Psi$. Then the Lie algebra with $[,] + \Psi$ is isomorphic to some filtered deformation $[,] + \tilde{\Psi}, \tilde{\Psi} = \Phi_1$.

### Proposition 6.1

$E^p_1 = g_q \otimes H^{p+q}(g)$.

We have the following natural isomorphisms:

$$C^{p+q}(g, g) = g \otimes \Lambda^{p+q}(g^\ast)$$

$$E^p_0 = F^qC^{p+q}(g, g) / F^{q+1}C^{p+q}(g, g) = g_q \otimes \Lambda^{p+q}(g^\ast).$$

Now the proof follows from the formula for the $d^{p, q}_0 : E^{p, q}_0 \to E^{p+1, q}_0$:

$$d_0(X \otimes f) = X \otimes df,$$

where $X \in g, f \in \Lambda^{p+q}(g^\ast)$ and $df$ is the standard differential of the cochain complex of $g$ with trivial coefficients.

### Theorem 6.2

Let $n \geq 12$ then

1) $\dim H^{0}(V_n, V_n) = \dim H^{0}_{(n)}(V_n, V_n) = 1$;

2) $\dim H^{1}(V_n, V_n) = 4$, namely

$$\dim H^{1}_{(\mu)}(V_n, V_n) = \begin{cases} 1, & \mu = 0, n-4, n-3, n-2; \\ 0, & \text{otherwise}. \end{cases}$$

3a) Let $n \geq 16$, then $\dim H^{2}(V_n, V_n) = 10$, more precisely:

$$\dim H^{2}_{(\mu)}(V_n, V_n) = \begin{cases} 2, & \mu = -2; \\ 1, & \mu = -4, -3, -1, n-11, n-10, n-9, n-8, n-7; \\ 0, & \text{otherwise}. \end{cases}$$
3b) Let $12 \leq n \leq 15$, then $\dim H^2(\mathcal{V}_n, \mathcal{V}_n) = 11$, more precisely:

$$
\dim H^2(\mu, \mathcal{V}_n) = \begin{cases}
2, & \mu = -2; \\
1, & \mu = -4, -3, -1, 1, n-11, n-10, n-9, n-8, n-7; \\
0, & \text{otherwise}.
\end{cases}
$$

Proof. For the proof we will use the spectral sequence considered above. Let us recall some results from [17], namely:

1) the basis of $H^1(\mathcal{V}_n)$ consists of two classes $[e^1]$ and $[e^2]$.

2) the space $H^2(\mathcal{V}_n)$ is 3-dimensional and generated by classes

$$
g_5 = [e^2 \wedge e^3],
    g_7 = [e^2 \wedge e^5 - 3e^3 \wedge e^4],
    [\Omega_{n+1}] = \frac{1}{2} \left( \sum_{i+j=n+1} (j-i)e^i \wedge e^j \right)
$$

of weights 5, 7, $n+1$ respectively.

3) $H^3(\mathcal{V}_n)$ is 5-dimensional and generated by elements $g_{12}, g_{15}$ and

$$
\{ [e^2 \wedge \Omega_{n+1}], [e^2 \wedge \Omega_{n+2} - ne^3 \wedge \Omega_{n+1}] \}
$$

of weights 12, 15, $n+3, n+4, n+5$ respectively. Where

$$
\Omega_{n+2} = \frac{1}{2} \sum_{i+j=n+2, i,j > 1} (j-i)e^i \wedge e^j,
    \Omega_{n+3} = \frac{1}{2} \sum_{i+j=n+3, i,j > 2} (j-i)e^i \wedge e^j
$$

are projections of non-existing $de^{n+2}$ and $de^{n+3}$ to $\Lambda^2(\mathcal{V}_n)$. We have also

$$
\Omega_{n+2} = d(e^{n+2} - ne^{n+1}) = ne^1 \wedge \Omega_{n+1},
    \Omega_{n+3} = d(e^{n+3} - (n+1)e^1 \wedge e^{n+2} - (n-1)e^2 \wedge e^n) = (n+1)e^1 \wedge \Omega_{n+2} + (n-1)e^2 \wedge \Omega_{n+1}.
$$

Remark. The classes $g_{12}, g_{15}$ are generators of $H^3(L_1)$ of weights 12, 15 respectively, they arise in Gontcharova’s theorem (see [9] for details), as well as $g_5, g_7$ in $H^3(L_1)$ and $[e^1], [e^2]$ in $H^1(L_1)$. We will need the following formula for $g_{12}$:

$$
g_{12} = [2e^2 \wedge e^7 - 5e^2 \wedge e^4 \wedge e^6 + 20e^3 \wedge e^4 \wedge e^5].
$$

Proposition 6.3. Let $n \geq 16$, then we have only a finite number of non-trivial differentials $d_i^{j-1}, d_i^{j-1}, d_i^{j-2-j}$ of the spectral sequence $E_i$. They are:

$$
\begin{align*}
    d_1^{j-1} &: e_j \mapsto (1-j)e_{j+1} \otimes [e^1], j \neq 1; \\
    d_2^{1,1} &: e_1 \mapsto e_3 \otimes [e^2], \\
    d_2^{j,2-j} &: e_j \otimes [\Omega_{n+1}] \mapsto \left(j-2 - \frac{(n-1)(j-1)}{(n+1)n}\right)e_{j+2} \otimes [e^2 \wedge \Omega_{n+1}]; \\
    d_3^{1,1-j} &: e_j \otimes [e^2] \mapsto -\frac{1}{6}(j-3)(j^2-3j+8)e_{j+3} \otimes [e^2 \wedge e^3], j \neq 3; \\
    d_4^{j-1} &: e_2 \otimes [e^1] \mapsto -\frac{4}{3}e_6 \otimes [e^2 \wedge e^3]; \\
    d_5^{2,2-j} &: e_j \otimes [e^2 \wedge e^5 - 3e^3 \wedge e^4] \mapsto -\frac{1}{5544}(j-8)(j^2-4j+27)(j^2-13j+48)e_{j+5} \otimes g_{12}, j \neq 8; \\
    d_8^{8-j} &: \text{Span}(e_8 \otimes [e^2 \wedge e^5 - 3e^3 \wedge e^4]) \mapsto \text{Span}(e_{16} \otimes g_{15}).
\end{align*}
$$
If $12 \leq n \leq 15$ then only the differential $d_8^{8-6}$ becomes trivial.

**Corollary 6.4.** The following classes in $E_1^{p,q}, p+q=0, 1, 2$ survive to $E_\infty$ for $n \geq 16$:

\[ e_n; \ e_1 \otimes [e^1], e_n \otimes [e^2], e_{n-1} \otimes [e^2], e_{n-2} \otimes [e^2]; \]
\[ e_{n-1} \otimes [\Omega_{n+1}], \ e_n \otimes [\Omega_{n+1}], \]
\[ e_j \otimes [e^2 \wedge e^5 - 3e^3 \wedge e^4], \ j = n-4, n-3, n-2, n-1, n. \]

If $12 \leq n \leq 15$ then one have to add $e_8 \otimes [e^2 \wedge e^5 - 3e^3 \wedge e^4]$ to the list above.

**Remark.** 1) the element $e_n$ spans the center $Z(V_n) = H^0(V_n, V_n)$;
2) the class $e_1 \otimes [e^1]$ corresponds to the inner derivation $\sum_{j=1}^n i e_j \otimes e^1 = ad(e_0)$ in the solvable Lie algebra $L_0/L_{n+1}$ restricted to the nilpotent ideal $V_n$.

Hence the proof of our theorem follows from Corollary 6.4. It’s time to say that the Proposition 6.3 it is finite-dimensional version of the following theorem

**Theorem 6.5** (A. Fialowski, [7, 8]). $\dim H^2(L_1, L_1) = 3$, more precisely:

\[ \dim H^2_{(\mu)}(L_1, L_1) = \begin{cases} 1, & \mu = -2, -3, -4; \\ 0, & \text{otherwise.} \end{cases} \]

The differentials $d_1^{i-1}, d_2^{i-1}, d_3^{i-1}, d_4^{i-1}, d_5^{i-2}, d_8^{8-6}$ of Proposition 6.3 came from the corresponding spectral sequence for $H^*(L_1, L_1)$. Their non-triviality follows from the more general result of B. Feigin, D. Fuchs [6]. But it is a very complicate task to follow all details of the proof in [6]. So it appears to be useful to calculate $d_1^{i-1}, d_2^{i-1}, d_3^{i-2}$ explicitly. From the other hand explicit formulae will give us a possibility to write down structure relations of all deformations of $V_n$.

**Remark.** In the infinite-dimensional case only classes $e_i \otimes [e^2 \wedge e^3], \ i = 1, 2, 3$ survive to $E_\infty$ and they correspond to generators in $H^2(L_1, L_1)$ of weights $\mu = i - (2+3) = -4, -3, -2$ that were found by A. Fialowski.

The proof of Proposition 6.3 consists of direct calculations.

First of all by definition of $d: C^0(g, g) = g \to C^1(g, g) = \text{Hom}(g, g) = g \otimes g^*$ we have

\[ d(e_j) = (j-1)e_{j+1} \otimes e^1 + (j-2)e_{j+2} \otimes e^2 + (j-3)e_{j+3} \otimes e^3 + \ldots \]

Hence

\[ d_1^{i-1}(e_j) = e_{j+1} \otimes (j-1)[e^1], \ j \neq 1. \]

Now we go to the differential $d_2$.

\[ d(e_j \otimes \Omega_{n+1}) = e_{j+1} \otimes (j-1)e^1 \wedge \Omega_{n+1} + \ldots \]

where dots stand instead of terms of higher filtration. As we know $e^1 \wedge \Omega_n = \frac{1}{n!} \Omega_{n+2}$ and we now can take new representative $e_j \otimes \Omega_n + \frac{j-1}{n} e_{j+1} \otimes \Omega_n$ such that:

\[ d(e_j \otimes \Omega_{n+1} + \frac{j-1}{n} e_{j+1} \otimes \Omega_{n+2}) = e_{j+2} \otimes \left( \frac{j(j-1)}{n} e^1 \wedge \Omega_{n+2} + (j-2)e^2 \wedge \Omega_{n+1} \right) + \ldots \]

But $d\Omega_{n+3} = (n+1)e^1 \wedge \Omega_{n+2} + (n-1)e^2 \wedge \Omega_{n+1}$ and therefore $\frac{j(j-1)}{n} e^1 \wedge \Omega_{n+2} + (j-2)e^2 \wedge \Omega_{n+1}$ is cohomologous to $\left( j-2 - \frac{(n-1)j(j-1)}{(n+1)n} \right) e^2 \wedge \Omega_{n+1}$. Hence we conclude
Now the most complicated case: computation of \( d_2(e_j \otimes [e^2 \wedge e^5 - 3e^3 \wedge e^4]) \). We have to find \( \xi_1, \xi_2, \xi_3, \xi_4 \) such that

\[
d(e_j \otimes e^2 \wedge e^5 - 3e^3 \wedge e^4) + \sum_{p=1}^{4} e_{j+p} \otimes \xi_p = e_{j+5} \otimes \xi + \ldots
\]

for some \( \xi \in C^3(\mathcal{V}_n, \mathcal{V}_n) \). We recall the notation \( g_7 = e^2 \wedge e^5 - 3e^3 \wedge e^4 \). We have the following system of equations on \( \xi_1, \xi_2, \xi_3, \xi_4 \):

\[
d\xi_1 = (j-1)e^1 \wedge g_7; \\
d\xi_2 = je^1 \wedge \xi_1 + (j-2)e^2 \wedge g_7; \\
d\xi_3 = (j+1)e^1 \wedge \xi_2 + (j-2)e^2 \wedge \xi_1 + (j-3)e^3 \wedge g_7; \\
d\xi_4 = (j+2)e^1 \wedge \xi_3 + je^2 \wedge \xi_2 + (j-2)e^2 \wedge \xi_1 + (j-4)e^4 \wedge g_7.
\]

Taking \( \xi_1, \xi_2, \xi_3, \xi_4 \) as homogeneous 2-forms of weights 8, 9, 10, 11 one can remark that the right parts of these equations are exact forms because \( H^3_{\nu p}(\mathcal{V}_n) = 0, p \leq 11 \). \( \xi_1, \xi_2, \xi_3, \xi_4 \) are defined uniquely by the condition \( \xi_p \in \Lambda^2(e^2, \ldots, e^n), p = 1, 2, 3, 4 \).

The answer is:

\[
\xi_1 = \frac{j-1}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5); \\
\xi_2 = P_1(j)e^2 \wedge e^7 + P_2(j)e^3 \wedge e^6 + P_3(j)e^4 \wedge e^5; \\
\xi_3 = Q_1(j)e^2 \wedge e^8 + Q_2(j)e^3 \wedge e^7 + Q_3(j)e^4 \wedge e^6; \\
\xi_4 = Z_1(j)e^2 \wedge e^9 + Z_2(j)e^3 \wedge e^8 + Z_3(j)e^4 \wedge e^7 + Z_4(j)e^5 \wedge e^6,
\]

where polynomials \( P_1(j), Q_1(j), Z_1(j) \) are defined by

\[
\begin{align*}
P_1(j) &= \frac{(5(j)+3(j)-6)}{21}; & P_2(j) &= \frac{(4(j)+15(j)-30)}{21}; & P_3(j) &= -\frac{(13(j)-30(j)+60)}{21}; \\
Q_1(j) &= \frac{(3(j+1)+4(j+1)^2-4(j+1))}{28}; & Q_2(j) &= \frac{(j+1)^2-8(j+1)^2+8(j+1)}{14}; \\
Q_3(j) &= \frac{(-13(j+1)+20(j+1)^2-20(j+1))}{28}; \\
Z_1(j) &= \frac{1}{22}(j+2) + \frac{23}{231}(j+2)(j+2) - \frac{7}{198}(j+2)(j+2) - \frac{59}{693}(j+2)(j+2) + \frac{37}{231}; \\
Z_2(j) &= \frac{17}{154}(j+2) - \frac{62}{231}(j+2)(j+2) - \frac{53}{1386}(j+2)(j+2) - \frac{59}{99}(j+2)(j+2) + \frac{37}{33}; \\
Z_3(j) &= \frac{29}{154}(j+2) - \frac{4}{77}(j+2)(j+2) + \frac{317}{462}(j+2)(j+2) - \frac{59}{33}(j+2)(j+2) + \frac{37}{11}; \\
Z_4(j) &= \frac{47}{154}(j+2) + \frac{185}{231}(j+2)(j+2) - \frac{2245}{1386}(j+2)(j+2) + \frac{295}{99}(j+2)(j+2) - \frac{185}{33}.
\end{align*}
\]

Now one can calculate \( \xi \) from equation (24):

\[
\xi = (j+3)e^1 \wedge \xi_4 + (j+1)e^2 \wedge \xi_3 + (j-1)e^3 \wedge \xi_2 + (j-3)e^4 \wedge \xi_1 + (j-5)e^5 \wedge g_7.
\]
The space $H^3_{(12)}(V_n)$ is one-dimensional and we have for the cohomology class $[\xi]$:

$$[\xi] = -\frac{1}{5544} (j-8)(j^2-4j+27)(j^2-13j+48) \left[ 2e^2 \wedge e^3 \wedge e^7 - 5e^2 \wedge e^4 \wedge e^6 + 20e^3 \wedge e^4 \wedge e^5 \right].$$

In the same way one can remark that

$$d(e_j \otimes e^2 + e_{j+1} \otimes (j-1)e^3 + e_{j+2} \otimes \frac{j(j-1)}{2} e^4) = e_{j+3} \otimes \left( \frac{(j+1)(j-1)}{2}e^1 \wedge e^4 + ((j-1)^2 - (j-3)) e^2 \wedge e^3 \right) + \ldots$$

As $3e^1 \wedge e^4$ is cohomologous to $e^2 \wedge e^3$ it follows that

$$d_3(e_j \otimes [e^2]) = e_{j+3} \otimes \left( (j-1)^2 - (j-3) - \frac{(j+1)(j-1)}{6} \right) [e^2 \wedge e^3]$$

Now we choose the following basic cocycles $\psi_{k,i}$ in $\oplus_{\mu \geq 0} H^2_{(\mu)}(V_n, V_n)$, $n \geq 16$:

| $H^2_{(n-7)}(V_n, V_n)$ | $\psi_{n,7} = e_n \otimes (e^2 \wedge e^5 - 3e^3 \wedge e^4)$ |
|---------------------------|--------------------------------------------------|
| $H^2_{(n-8)}(V_n, V_n)$ | $\psi_{n,8} = e_{n-1} \otimes (e^2 \wedge e^5 - 3e^3 \wedge e^4) + e_n \otimes \frac{n-3}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5);$ |
| $H^2_{(n-9)}(V_n, V_n)$ | $\psi_{n,9} = e_{n-2} \otimes (e^2 \wedge e^5 - 3e^3 \wedge e^4) + e_{n-1} \otimes \frac{n-4}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5) +$ $+ e_n \otimes (P_1(n-2)e^2 \wedge e^7 + P_2(n-2)e^3 \wedge e^6 + P_3(n-2)e^4 \wedge e^5) ;$ |
| $H^2_{(n-10)}(V_n, V_n)$ | $\psi_{n,10} = e_{n-3} \otimes (e^2 \wedge e^5 - 3e^3 \wedge e^4) + e_{n-2} \otimes \frac{n-5}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5) +$ $+ e_{n-1} \otimes (P_1(n-3)e^2 \wedge e^7 + P_2(n-3)e^3 \wedge e^6 + P_3(n-3)e^4 \wedge e^5) +$ $+ e_n \otimes (Q_1(n-3)e^2 \wedge e^8 + Q_2(n-3)e^3 \wedge e^7 + Q_3(n-3)e^4 \wedge e^6);$ |
| $H^2_{(n-11)}(V_n, V_n)$ | $\psi_{n,11} = e_{n-4} \otimes (e^2 \wedge e^5 - 3e^3 \wedge e^4) + e_{n-3} \otimes \frac{n-7}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5) +$ $+ e_{n-2} \otimes (P_1(n-4)e^2 \wedge e^7 + P_2(n-4)e^3 \wedge e^6 + P_3(n-4)e^4 \wedge e^5) +$ $+ e_{n-1} \otimes (Q_1(n-4)e^2 \wedge e^8 + Q_2(n-4)e^3 \wedge e^7 + Q_3(n-4)e^4 \wedge e^6) +$ $+ e_n \otimes (Z_1(n-4)e^2 \wedge e^9 + Z_2(n-4)e^3 \wedge e^8 + Z_3(n-4)e^4 \wedge e^7 + Z_4(n-4)e^5 \wedge e^6);$ |

7. Moduli space of $\mathbb{Z}_{>0}$-filtered deformations.

In this section we classify up to an isomorphism the Lie algebras over $\mathbb{K}$ defined by the basis $e_1, \ldots, e_n$, $n \geq 16$ and commuting relations of the following form:

$$[e_i, e_j] = (j-i)e_{i+j} + \sum_{l=1}^{n-i-j} c_{ij}^l e_{i+j+l}.$$

A Lie algebra $\mathfrak{g}$ with the commutating relations (29) is a $\mathbb{Z}_{>0}$-filtered deformation of the $\mathbb{Z}_{>0}$-graded Lie algebra $V_n$, i.e. $\mathfrak{g} = (V_n, [\cdot, \cdot] + \Psi)$, where

$$\Psi = \Psi_1 + \Psi_2 + \cdots + \Psi_{n-3}, \quad \Psi_l \in C^2_{(l)}(V_n, V_n), \quad \Psi_l(e_i, e_j) = \begin{cases} c_{ij}^l e_{i+j+l}, & i+j \leq n-l, \\ 0, & \text{otherwise;} \end{cases} \quad l = 1, 2, \ldots, n-3.$$
satisfying to the system \( [14] \) of deformation equations:

\[
d\Psi_1 = 0, \quad d\Psi_2 + \frac{1}{2} [\Psi_1, \Psi_1] = 0, \quad \ldots, \quad d\Psi_{n-6} + \frac{1}{2} \sum_{i+j=n-6} [\Psi_i, \Psi_j] = 0.
\]

**Lemma 7.1.** Let \( \Psi, \bar{\Psi} \) be two \( \mathbb{Z}_{\geq 0} \)-filtered deformations of \( \mathcal{V}_n, n \geq 5 \) and

\[
\varphi : g = (\mathcal{V}_n, [,] + \Psi) \to \mathfrak{g} = (\mathcal{V}_n, [,] + \bar{\Psi})
\]
is a Lie algebra isomorphism.

Then

1) \( \varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots + \varphi_{n-1}, \quad \varphi_j \in C^1_{(j)}(\mathcal{V}_n, \mathcal{V}_n), \ j = 0, 1, 2, \ldots, n-1, \)

2) \( \varphi_0(e_i) = \alpha^i e_i, \quad \alpha \in \mathbb{K}^*, \quad i = 1, 2, \ldots, n. \)

**Proof.** We will study the dual situation. The mapping

\[
\varphi^* : g^* = (\mathcal{V}_n^*, d\varphi = d + \Psi^*) \to \mathfrak{g}^* = (\mathcal{V}_n^*, d\bar{\varphi} = d + \bar{\Psi}^*)
\]
is an isomorphism of \( d \)-algebras.

For the dual basis \( e^1, e^2, \ldots, e^n \) of \( (\mathcal{V}_n^*, d\varphi) \) we have the following structure relations:

\[
d\varphi e^k = \frac{1}{2} \sum_{i+j=k} (j-i)e^i \wedge e^j + \frac{1}{2} \sum_{m+p<k} c_{mp}^k e^m \wedge e^p, \quad k = 3, \ldots, n.
\]

Let us write down some of them.

\[
d\varphi e^1 = d\varphi e^2 = 0, \quad d\varphi e^3 = e^1 \wedge e^2, \quad d\varphi e^4 = 2e^1 \wedge e^3 + c_{12}^1 e^1 \wedge e^2,
\]

\[
d\varphi e^5 = 3e^1 \wedge e^4 + e^2 \wedge e^3 + c_{13}^1 e^1 \wedge e^3 + c_{12}^2 e^1 \wedge e^2, \ldots
\]

The dual mapping \( \varphi^* : \mathfrak{g}^* \to g^* \) is the isomorphism of \( d \)-algebras, i.e.

\[
d\varphi^* = \varphi^* d\varphi.
\]

1) \( d\varphi \varphi^* e^1 = d\varphi \varphi^* e^2 = 0, \) thus

\[
\varphi^* e^1 = \alpha_1 e^1 + \alpha_2 e^2; \quad \varphi^* e^2 = \alpha_1 e^1 + \alpha_2 e^2.
\]

2) \( d\varphi \varphi^* e^3 = \varphi^* e^1 \wedge \varphi^* e^2 = (\alpha_1 \alpha_2 - \alpha_2 \alpha_1) e^1 \wedge e^2, \) and we have

\[
\varphi^* e^3 = (\alpha_1 \alpha_2 - \alpha_2 \alpha_1) e^1 \wedge e^2 + \alpha_1 e^1 \wedge e^3 + \alpha_2 e^2 \wedge e^3.
\]

3) The form \( \varphi^* d\varphi^* e^4 \) is cohomologous to \( 2\alpha_2 (\alpha_1 \alpha_2 - \alpha_2 \alpha_1) e^2 \wedge e^3 \) and it is exact iff \( \alpha_2 = 0, \) thus

\[
\varphi^* e^4 = \alpha_1^2 \alpha_2 e^4 + (\alpha_1 \alpha_3 + c_{12}^1 \alpha_1 \alpha_2 - c_{12}^1 \alpha_1 \alpha_2) e^3 + \alpha_4 e^1 + \alpha_2 e^2.
\]

4) \( \varphi^* d\varphi^* e^5 \sim 3\alpha_2 \alpha_1 \alpha_3 e^1 \wedge e^4 + \alpha_2 \alpha_1 e^2 \wedge e^3 \) and the last one is exact iff \( \alpha_1^2 = 0, \) and for the moment we have

\[
\varphi^* e^1 = \alpha_1 e^1; \quad \varphi^* e^2 = \alpha_1^2 e^2 + \alpha_1 e^1,
\]

\[
\varphi^* e^3 = \alpha_1^3 e^3 + \alpha_1 e^1 + \alpha_2 e^2,
\]

\[
\varphi^* e^4 = \alpha_1^4 e^4 + \ldots; \quad \varphi^* e^5 = \alpha_1^5 e^5 + \ldots
\]

Going on and using an obvious inductive assumption we have for the operator \( \varphi^* : \)

\[
\varphi^* e^i = \alpha^i e^i + \sum_{l<i} \alpha_{il} e^l, \quad i = 1, \ldots, n,
\]

for some \( \alpha \neq 0, \alpha_{ii} \in \mathbb{K}. \) \( \square \)
From the other hand it is evident that changing the canonical basis of an arbitrary Lie algebra \( g \) of the type \([21]\) by an operator \( \varphi \) with property \([14]\) we will get again the commutating relations of the same type.

**Corollary 7.2.** The matrix Lie group \( G_n \) of lower-triangular matrices \( \varphi \) of the following type

\[
\varphi = \begin{pmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \alpha^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \alpha_{n-1} & \cdots & \alpha^n
\end{pmatrix}, \quad \alpha_{ij} \in \mathbb{K}, \alpha \neq 0,
\]

acts on the set \( V_n \) of \( \mathbb{Z}_{>0} \)-filtered deformations of \( \mathcal{V}_n \) as the group of changes of canonical basis:

\[
(\varphi \ast \Psi)(x,y) = \varphi^{-1}([\varphi x, \varphi y] + \Psi(\varphi x, \varphi y)) - [x,y], \forall x, y \in V_n, \varphi \in G_n.
\]

Two \( \mathbb{Z}_{>0} \)-filtered deformations \((\mathcal{V}_n, [\cdot, \cdot] + \Psi)\) and \((\mathcal{V}_n, [\cdot, \cdot] + \tilde{\Psi})\) are isomorphic as Lie algebras if and only if they are in the same orbit \( O_\Psi \) of the \( G_n \)-action.

Hence we have proved the following

**Theorem 7.3.** Let \( n \geq 5 \), then there is a one-to-one correspondence between the orbit space \( O(G_n, V_n) \) of the action \( G_n \) on the set \( V_n \) of \( \mathbb{Z}_{>0} \)-filtered deformations of \( \mathcal{V}_n \) and the moduli space \( \mathcal{M}_n \) (the set of isomorphism classes of the \( \mathbb{Z}_{>0} \)-filtered deformations of \( \mathcal{V}_n \)).

**Proposition 7.4.** The matrix Lie group \( G_n \) is the semi-direct product \( \mathbb{K}^* \ltimes UT_n \) where \( UT_n \) denotes the group of unitriangular matrices:

\[
G_n = \mathbb{K}^* \ltimes UT_n = \mathbb{K}^* \ltimes \begin{bmatrix}
1 & 0 & \cdots & 0 \\
* & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & 1
\end{bmatrix}, \quad \mathbb{K}^* \cong \begin{bmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \alpha^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha^n
\end{bmatrix}.
\]

**Remark** (see also section [5]). If \( \Psi = \Psi_1 + \Psi_2 + \cdots + \Psi_{n-3} \) is a solution of the system \([17]\) of deformation equations then \( \Psi = t\Psi_1 + t^2\Psi_2 + \cdots + t^{n-3}\Psi_{n-3}, \forall t \in \mathbb{K} \) also satisfies to the system \([17]\). It follows that the space \( V_n \) can be retracted over itself to \( \mathcal{V}_n \). From another hand one can define \( \mathbb{K}^* \)-action on \( V_n \):

\[
\rho_n(\alpha)(\Psi_1, \Psi_2, \ldots, \Psi_{n-3}) = (\alpha \Psi_1, \alpha^2 \Psi_2, \ldots, \alpha^{n-3} \Psi_{n-3}), \alpha \in \mathbb{K}^*.
\]

Evidently this action coincides with the action on \( V_n \) of the subgroup \( \mathbb{K}^* \) of diagonal matrices in \( G_n \).

**Proposition 7.5.** There are the following bijections:

\[
O_\rho_n(\mathbb{K}^*, O(UT_n, V_n)) \to O(G_n/UT_n, O(UT_n, V_n)) \to O(G_n, V_n).
\]

**Proposition 7.6.** Let \( \Psi \) be a \( \mathbb{Z}_{>0} \)-filtered deformation of \( \mathcal{V}_n, n \geq 12 \). Then there exists an element \( \tilde{\Psi} \) in the \( UT_n \)-orbit \( O_\Psi \) of \( \Psi \) such that

\[
\tilde{\Psi}_1 = \cdots = \tilde{\Psi}_{n-12} = 0.
\]

For an arbitrary \( \mathbb{Z}_{>0} \)-filtered deformation \( \Psi \) the first equation is \( d\Psi_1 = 0 \). We recall now

\[
H^2_{(i)}(\mathcal{V}_n, \mathcal{V}_n) = 0, \quad i \leq n-12.
\]
and hence $\Psi_1 = d\varphi_1$ for some $\varphi_1$ of $C^1(\mathcal{V}_n, \mathcal{V}_n)$. Acting by $g = id + \varphi_1 \in UT_n$ we get $\tilde{\Psi} = g \ast \Psi$ such that $\tilde{\Psi}_1 = 0$. Now the second equation of the system for this new element $\tilde{\Psi}$. will be $d\tilde{\Psi}_2 = [\tilde{\Psi}_1, \tilde{\Psi}_1] = 0$. We act on $\tilde{\Psi}$ by $g = id + \varphi_2$, where $\tilde{\Psi}_2 = d\varphi_2$ and continue the procedure step by step.

Now we suppose to be constructed an element $\Psi$ such that $\tilde{\Psi}_1 = \cdots = \tilde{\Psi}_{n-12} = 0$ satisfying

$$d\tilde{\Psi}_{n-11} = 0.$$  

We recall that in the section we found the basic cocycles $\psi_{n,i}$, such that

$$\text{Span}([\psi_{n,i}, \psi_{n,j}]) = H^2(n-1)(\mathcal{V}_n, \mathcal{V}_n), \ i = 7, \ldots, 11.$$  

Hence $\tilde{\Psi}_{n-11} = x_1\psi_{n,11} + d\varphi_{n-11}, x_1 \in \mathbb{K}$, and $\varphi_{n-11} \in C^1(n-11)(\mathcal{V}_n, \mathcal{V}_n)$. Then acting on $\tilde{\Psi}$ by $id + \varphi_{n-11}$ we get again a new element in the orbit $O_{\Psi}$ (we keep the same notation $\Psi$ for it) with the property $\tilde{\Psi}_{n-11} = x_1\psi_{n,11}$.

**Proposition 7.7.** Let $n \geq 14$ then all Nijenhuis-Richardson products of basic cocycles $\psi_{n,i}$ are trivial elements in $C^3(\mathcal{V}_n, \mathcal{V}_n)$:

$$[\psi_{n,i}, \psi_{n,j}](x, y, z) = 0 \ \forall x, y, z \in \mathcal{V}_n.$$  

The proof follows from the two properties of $\psi_{n,i}$:

1) $\text{Im}\psi_{n,i} = \text{Span}(e_n, \ldots, e_{n-4})$

2) $\psi_{n,i}(x, y) = 0$ if $x \wedge y \not\in \Lambda^2(e_2, \ldots, e_9)$.

Hence $\psi_{n,i}(x, y, z) = 0, \ \forall x, y, z$ if $n-4 > 9$.

**Proposition 7.8.** Let $n \geq 14$. There is a one-to-one correspondence between the orbit space $O(UT_n, \mathcal{V}_n)$ and the 5-dimensional vector space $\oplus_{i>0} H^2(n)(\mathcal{V}_n, \mathcal{V}_n)$.

This proposition follows from the previous one. Namely in an arbitrary $UT_n$-orbit $O_{\Psi}$ one can choose the unique representative $\tilde{\Psi}$ such that

$$\tilde{\Psi} = x_1\psi_{n,11} + x_2\psi_{n,10} + x_3\psi_{n,9} + x_4\psi_{n,8} + x_5\psi_{n,7},$$  

where $x_i \in \mathbb{K}, i = 1, \ldots, 5$. We will call $\tilde{\Psi}$ the canonical element of the orbit $O_{\Psi}$ and the set $\{x_1, \ldots, x_5\}$ is called the homogeneous coordinates of the orbit $O_{\Psi} \in O(UT_n, \mathcal{V}_n)$.

**Theorem 7.9.** Let $n \geq 16$. There is a one-to-one correspondence between the moduli space $\mathcal{M}_n = O(G_n, \mathcal{V}_n)$ of $\mathbb{Z}_{>0}$-filtered deformations of $\mathcal{V}_n$ and the orbit space $O_{\tilde{\rho}_n}(K^*, K^5)$ where the action $\tilde{\rho}_n$ of $K^*$ on $K^5$ is defined in coordinates $x^i$:

$$\tilde{\rho}_n(\alpha)(x_1, x_2, \ldots, x_5) = (\alpha^{n-11}x_1, \alpha^{n-10}x_2, \ldots, \alpha^{n-7}x_5), \ \alpha \in K^*.$$

We have to verify only the formula for $K^*$-action on $O(UT_n, \mathcal{V}_n)$. But

$$\tilde{\rho}_n(\alpha)(x_1, x_2, \ldots, x_5) = \rho_n(\alpha)(x_1\psi_{n,11} + x_2\psi_{n,10} + \cdots + x_5\psi_{n,17}) =$$

$$= \alpha^{n-11}x_1\psi_{n,11} + \alpha^{n-10}x_2\psi_{n,10} + \cdots + \alpha^{n-7}x_5\psi_{n,7}.$$  

8. **Affine variety of $\mathbb{Z}_{>0}$-filtered deformations**

One can regard the set $\mathcal{V}_n$ of $\mathbb{Z}_{>0}$-filtered deformation of $\mathcal{V}_n$ as a affine variety in the affine space $\oplus_{i>0} C^2(\mathcal{V}_n, \mathcal{V}_n)$. Namely one can define a mapping (polynomial in the coordinates $\{e_{ij}^k\}$):

$$\mathcal{F} : \oplus_{i>0} C^2(\mathcal{V}_n, \mathcal{V}_n) \rightarrow \oplus_{i>0} C^3(\mathcal{V}_n, \mathcal{V}_n),$$
where
\[ F(\Psi_1, \Psi_2, \ldots, \Psi_{n-3}) = (d\Psi_1, d\Psi_2 + \frac{1}{2}[\Psi_1, \Psi_1], \ldots, d\Psi_{n-6} + \frac{1}{2} \sum_{i+j=n-6} [\Psi_i, \Psi_j]). \]

Then
\[ V_n = \{ \Psi \in \oplus_{i>0} C^2_{(i)}(V_n, V_n) \mid F(\Psi) = 0 \}. \]

**Theorem 8.1.**

1. The variety \( V_n \) has no singular points and
\[ \dim V_n = \begin{cases} \frac{n(n-3)}{2} + 3, & n \geq 16; \\ \frac{n(n-3)}{2} + 4, & 12 \leq n \leq 15. \end{cases} \]

2. The group \( UT_n \) acts on \( V_n \) with constant rank and we have for the dimension of an arbitrary orbit \( O_\Psi \)
\[ \dim O_\Psi = \frac{n(n-3)}{2} - 2. \]

One can identify the Lie algebra of \( UT_n \) with \( \oplus_{i>0} C^1_{(i)}(V_n, V_n) \). Let \( \Psi \) be some fixed element of \( V_n \), then decomposing \( \varphi = \exp \alpha = 1 + \alpha + \alpha^2 + \ldots \) in the formula and taking linear terms with respect to \( \alpha \) one can get the following formula for the differential \( D \) of the \( UT_n \)-action \((x, y) \in V_n)\):
\[ D\Psi(\alpha)(x, y) = [\alpha(x), y] + [x, \alpha(y)] - \alpha([x, y]) + \Psi(\alpha(x), y) + \Psi(x, \alpha(y)) - \alpha(\Psi(x, y)). \]

Hence
\[ D\Psi(\alpha) = -d(\alpha) + [\Psi, \alpha] \]

where \( d \Psi : \oplus_{i>0} C^1_{(i)}(V_n, V_n) \to \oplus_{i>0} C^2_{(i)}(V_n, V_n) \) is a deformed differential of the cochain complex of \( V_n \). But in the same time \( d \Psi \) defines the differential of the cochain complex of the Lie algebra \( g = (V_n, [\cdot, \cdot]) + \Psi \).

Analogously one can show that the differential \( D F \Psi \) coincides with the differential \( d \Psi : \oplus_{i>0} C^1_{(i)}(V_n, V_n) \to \oplus_{i>0} C^2_{(i)}(V_n, V_n) \) of the cochain complex with coefficients in the adjoint representation of \( g = (V_n, [\cdot, \cdot]) + \Psi \).

**Remark.** Let us denote by \( d_l : C^*_{(i)}(V_n, V_n) \to C^{*+1}_{(i)}(V_n, V_n) \) the restriction of the differential \( d \) of the cochain complex \( (C^*(V_n, V_n), d) \) to homogeneous components. Then the matrix of \( d \Psi \) for an arbitrary \( \Psi \) is a block-triangular for some natural number \( l \):
\[ d \Psi = \begin{pmatrix} d_1 & * & \ldots & * \\ 0 & d_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{n-l} \end{pmatrix} \]

And we have the following estimates
\[ \text{rank } d \Psi \geq \text{rank } d, \quad \dim \ker d \Psi \leq \dim \ker d. \]

In order to compute these ranks one may assume that \( \Psi = \tilde{\Psi} \), where \( \tilde{\Psi} \in \mathbb{R}^5 \) is the canonical representative of the orbit \( O_\Psi \).

**Proposition 8.2.**

\[ \dim \ker \left( d \Psi : \oplus_{i>0} C^1_{(i)}(V_n, V_n) \to \oplus_{i>0} C^2_{(i)}(V_n, V_n) \right) = \dim \ker \left( d : \oplus_{i>0} C^1_{(i)}(V_n, V_n) \to \oplus_{i>0} C^2_{(i)}(V_n, V_n) \right) = n + 2. \]
One can easily verify that operators

\[
\begin{align*}
  ad_\psi(e_1), \quad ad_\psi(e_2), \quad \ldots, \quad ad_\psi(e_{n-1}), \quad e_n \otimes e^2, \quad e_{n-1} \otimes e^2 + (n-2)e_n \otimes e^3, \\
  e_{n-2} \otimes e^2 + (n-3)e_n \otimes e^3 + \frac{(n-2)(n-3)}{2}e_n \otimes e^3
\end{align*}
\]  

(39)

give the basis of \( \ker d_\psi : \oplus_{>0} C^1_{(i)}(\mathcal{V}_n, \mathcal{V}_n) \rightarrow \oplus_{>0} C^2_{(i)}(\mathcal{V}_n, \mathcal{V}_n) \).

Hence

\[
\dim O_\psi = \dim \text{Im} \left( d_\psi : \oplus_{>0} C^1_{(i)}(\mathcal{V}_n, \mathcal{V}_n) \rightarrow \oplus_{>0} C^2_{(i)}(\mathcal{V}_n, \mathcal{V}_n) \right) = \frac{n(n-1)}{2} - n - 2.
\]

From the other hand the cocycles \( \{ \psi_{n,i} \mid i = 7, \ldots, 11 \} \) span the tangent space to \( \mathbb{K}^5 \) in an arbitrary \( \tilde{\Psi} \), moreover \( \{ \psi_{n,i} \} \) are linearly independent modulo \( \text{Im} d_\psi \) because \( \{ |\psi_{n,i}| \mid i = 7, \ldots, 11 \} \) is the basis in \( \oplus_{>0} H^2_{(i)}(g, g) \).

The variety \( V_n \) can be regarded as a subvariety of the affine variety \( \mathcal{N}_n \) of \( n \)-dimensional nilpotent Lie algebras. There exists a \( \text{GL}_n \)-action on \( \mathcal{N}_n \) by basis changes. Let us consider the set \( V'_n = \{ y \in \mathcal{N}_n \mid \exists \psi \in V_n, \exists g \in \text{GL}_n, y = g \star \psi \} \).

Remark. In fact the Zariski closure of \( V'_n \) in \( \mathcal{N}_n \) coincides with one of irreducible components of \( \mathcal{N}_n \) that were discussed by Yu.Khakimdjanov in [13].

The \( \text{GL}_n \)-action on \( V'_n = \{ y \in \mathcal{N}_n \mid \exists x \in V_n, \exists g \in \text{GL}_n, y = g \star x \} \) has singularities. Namely by the lemma 7.1 the stabilizer \( (GL_n)_\psi \) of a point \( \psi \in \mathbb{K}^5 \subset V_n \) coincides with the stabilizer \( (G_n)_\psi \) and we have

\[
\dim(GL_n)_\psi = \begin{cases} 
  n+2, & \text{if } \tilde{\Psi} \neq 0; \\
  n+3, & \text{if } \tilde{\Psi} = 0.
\end{cases}
\]

Remark. If \( \tilde{\Psi} = 0 \) then one have to add the operator \( \sum_{i=1}^n i e_i \otimes e^i \) to (39) in order to get a basis of the Lie algebra of the stabilizer \( (GL_n)_\psi \).

Moreover, even for non-zero points \( \tilde{\Psi} \) there are some particularities. Let \( \tilde{\Psi} \) be a generic point in \( \mathbb{K}^5 \), i.e. \( x_i \neq 0, \forall i \), then

\[
(GL_n)_\psi = (G_n)_\psi = (UT_n)_\psi.
\]

But if \( \mathbb{K} = \mathbb{C} \) then for instance for \( \tilde{\psi} = (1, 0, 0, 0, 0) \) we have

\[
(GL_n)_\psi = (G_n)_\psi = \mathbb{Z}_{n-11} \ltimes (UT_n)_\psi,
\]

where \( \mathbb{Z}_{n-11} \) stands for the group of the roots of the unit:

\[
\mathbb{Z}_{n-11} = \{ \alpha \in \mathbb{C} \mid \alpha^{n-11} = 1 \}.
\]

It is interesting here to remark some parallels between our discussions and the theory of non-singular deformations, that was introduced in [8].

**Definition 8.3 (8).** Let \( g \) be a Lie algebra with the comutator \([ , ]\). Consider a formal one-parameter deformation

\[
[x, y]_t = [x, y] + \sum_{k \geq 1} \alpha_k(x, y)t^k
\]

of \( g \). A deformation is called non-singular if there exists a formal one-parameter family of linear transformations

\[
\phi_t(x) = x + \sum_{l \geq 1} \beta_l(x)t^l
\]
of \( \mathfrak{g} \) and a formal (not necessarily invertible) parameter change \( u = u(t) \) which transform the deformation \( [x, y]_1 \) into a deformation
\[
[x, y]_u = [x, y] + \sum_{k \geq 1} \alpha_k(x, y)u^k, \quad \varphi_t^{-1}[\varphi_t(x), \varphi_t(y)]_t = [x, y]_u(t)
\]
with the cocycle \( \alpha' \in C^2(\mathfrak{g}, \mathfrak{g}) \) being not cohomologous to 0. Otherwise the deformation is called singular.

We have already remarked that one can associate to an arbitrary \( \mathbb{Z}_{>0} \)-deformation \( \Psi = \Psi_1 + \Psi_2 + \Psi_3 + \ldots \) of \( \mathcal{V}_n \) the following one-parametric deformation:
\[
[x, y]^{\Psi} = [x, y] + t\Psi_1(x, y) + t^2\Psi_2(x, y) + \cdots + t^k\Psi_k(x, y) + \ldots.
\]
One can also remark that in our case a formal deformation \( [x, y]^{\Psi}_t \) is in fact a polynomial on \( t \).

**Proposition 8.4.** Let fix the isomorphism \( O(UT_n, \mathcal{V}_n) = \mathbb{K}^5 \), \( n \geq 16 \), then a formal deformation \( [x, y]^{\Psi}_t \) corresponding to some \( \mathbb{Z}_{>0} \)-deformation \( \Psi \) of \( \mathcal{V}_n \) is non-singular if and only if its orbit \( O_\Psi \) belongs to the union \( \bigcup_{i=1}^5 O_{x_i} \) of coordinate lines \( O_{x_i} \) in \( \mathbb{K}^5 \).

Acting by corresponding \( \varphi_t = id + t\varphi_1 + t^2\varphi_2 + \ldots \) we reduce our problem to a parameter change in
\[
\Psi_t = t^{n-11}x_1\psi_{n,11} + t^{n-10}x_2\psi_{n,10} + \cdots + t^{n-7}x_5\psi_{n,7}.
\]
If \( x_1 \neq 0 \) then a parameter change \( u(t) = t^{n-11} \) is possible iff \( x_2 = x_3 = x_4 = x_5 = 0 \) the other cases are treated in the same way.

**Remark.** If \( n \geq 14 \) one can associate to an arbitrary deformation of the form \( \Psi = \sum_{i=1}^5 x_i\psi_{n,12-i} \) a linear one-parameter deformation \([\cdot, \cdot]^{\tau}_t \) of \( \mathcal{V}_n \):
\[
[x, y]^{\tau}_t = [x, y] + t\Psi(x, y).
\]
It follows from \([\Psi, \Psi] = 0\) in \( C^5(\mathcal{V}_n, \mathcal{V}_n) \). Evidently a linear one-parameter deformation \([\cdot, \cdot]^{\tau}_t \) is non-singular iff \( \Psi \neq 0 \).

### 9. Symplectic structures and one-dimensional central extensions

First of all we are going to calculate \( H^2(\mathfrak{g}) \) of an arbitrary \( \mathbb{Z}_{>0} \)-deformation \( \mathfrak{g} = (\mathcal{V}_n, [\cdot, \cdot] + \Psi) \). As we remarked in the section 4 the canonical basis \( e_1, e_2, \ldots, e_n \) of \( \mathfrak{g} = (\mathcal{V}_n, [\cdot, \cdot] + \Psi) \) defines the filtration \( L \) of \( \mathfrak{g} \):
\[
\mathfrak{g} = L^1\mathfrak{g} \supset L^2\mathfrak{g} = \text{Span}(e_2, \ldots, e_n) \supset \cdots \supset L^n\mathfrak{g} = \text{Span}(e_n) \supset \{0\},
\]
and \( L \) defines the filtration \( \tilde{L} \) of the cochain complex \( (C^*(\mathfrak{g}), d) \). For the simplicity one can assume that \( \Psi = x_1\psi_{n,11} + \cdots + x_5\psi_{n,7} \). Let us consider the corresponding to \( \tilde{L} \) spectral sequence \( E^{p,q}_{i} \) that converges to \( H^*(\mathfrak{g}) \). We recall also that
\[
E^{p,q}_{1} = H^p_{(-p)}(\text{gr}_L\mathfrak{g}) = H^p_{(-p)}(\mathcal{V}_n).
\]
We recall that the space \( H^2(\mathcal{V}_n) \) is 3-dimensional and it is spanned by classes
\[
g_5 = [e^2 \wedge e^3], \quad g_7 = [e^2 \wedge e^5 - 3e^3 \wedge e^4], \quad [\Omega_{n+1}] = \frac{1}{2} \sum_{i+j=n+1} (j-i)e^i \wedge e^j.
\]
of weights 5, 7, \(n+1\) respectively. It is evident that the classes \([g_5]\) and \([g_7]\) survive to the term \(E_\infty\) because \(d_5 e^i = d e^i, i = 1, \ldots, 7, \forall \Psi\). And the question is: does \([\Omega_{n+1}] \in E^{2k-1, 2k+3}_1\) survive to \(E_\infty\)?

**Proposition 9.1.** Let \(g = (\mathcal{V}_n, [\cdot, \cdot] + \Psi)\) be a Lie algebra defined by the following deformation of \(\mathcal{V}_n\):

\[
\hat{\Psi} = \tilde{\Psi}_{n-11} + \cdots + \tilde{\Psi}_{n-7} = x_1 \psi_{n,11} + \cdots + x_5 \psi_{n,7}.
\]

Then in the spectral sequence \(E^{p,q}_r\) the following properties hold on:

1) \(d_r \equiv 0, r = 1, \ldots, n-12\);
2) \(E^{p,q}_r = E^{p,q}_{11}, r = 2, \ldots, n-11\);
3) \(d_{n-11}(\Omega_{n+1}) = 0\) if and only if \(x_1 = 0\);
4) If \(x_1 = 0\) then \(d_r(\Omega_{n+1}) = 0\) \(\forall r \geq 1\), i.e. the class \([\Omega_{n+1}]\) survive to the term \(E_\infty\).

The differential \(d_\Psi\) of the cochain complex \((C^*(g^*), d)\) has the form:

\[
d = d_0 + \Psi_{n-11}^* + \cdots + \Psi_{n-7}^* = d_0 + x_1 \psi_{n,11}^* + \cdots + x_5 \psi_{n,7}^*,
\]

where by \(\Psi_i^*\) and \(\psi_{n,j}^*\) we denote the dual applications to \(\Psi_i\) and \(\psi_{n,j}\) respectively and \(d_0\) stands for the differential of \((C^*(\mathcal{V}_n), d_0)\). The items 1) and 2) of the proposition are evident and

\[
d_{n-11}(\Omega_{n+1}) = [\Psi_{n-11}^*(\Omega_{n+1})] = x_1 [\psi_{n,11}^*(\Omega_{n+1})],
\]

On another hand one can compute \(\psi_{n,11}^*(\Omega_{n+1})\) explicitly:

\[
(-1) e^1 \wedge e^n + (n-3) e^2 \wedge e^{n-1} + \cdots + (n-9) e^5 \wedge e^{n-4} = (n-1) e^1 \wedge (Z_1(n-4) e^2 \wedge e^9 + Z_2(n-4) e^3 \wedge e^8 + Z_3(n-4) e^4 \wedge e^7 + Z_4(n-4) e^5 \wedge e^6) + (n-3) e^2 \wedge (Q_1(n-4) e^2 \wedge e^8 + Q_2(n-4) e^3 \wedge e^7 + Q_3(n-4) e^4 \wedge e^6) + (n-5) e^3 \wedge (P_1(n-4) e^2 \wedge e^7 + P_2(n-4) e^3 \wedge e^6 + P_3(n-4) e^4 \wedge e^5) + (n-7) e^4 \wedge \frac{n-5}{2} (e^2 \wedge e^6 - 2 e^3 \wedge e^5) + (n-9) e^5 \wedge (e^2 \wedge e^5 - 3 e^3 \wedge e^4)
\]

As it was shown in the Section \(\S\) this element is cohomologous to

\[-\frac{1}{5544} (n-12)(n^2-12n+59)(n^2-21n+116)[2e^2 \wedge e^3 \wedge e^7 - 5e^2 \wedge e^4 \wedge e^6 + 20e^3 \wedge e^4 \wedge e^5]
\]

and it defines a non-trivial element in \(H^3_{(12)}(\mathcal{V}_n)\) for \(n > 12\).

We can finish the proof by remark that

\[
E_{r}^{-n-1+r,n-r+4} = E_{1}^{-n-1+r,n-r+4} = H_{(n+1-r)}^3(\mathcal{V}_n) = 0, \quad r > n-11,
\]

because \(H_{(i)}^3(\mathcal{V}_n) = 0\) for all \(i < 12\). Hence all other differentials of the spectral sequence

\[
d_r : E_{r}^{-n-1,n+3} \rightarrow E_{r}^{-n-1+r,n-r+4}, \quad r > n-11
\]

are trivial.

**Corollary 9.2.** Let \(g = (\mathcal{V}_{2k}, [\cdot, \cdot] + \Psi)\) be a \(\mathbb{Z}_{>0}\)-deformation of \(\mathcal{V}_{2k}\) and \((x_1, x_2, \ldots, x_5)\) be the set of homogeneous coordinates of the \(UT_n\)-orbit \(O_\Psi\). \(g\) admits a symplectic structure if and only if \(x_1 = 0\).
Corollary 9.3. If the orbit $O_\Psi$ of a $\mathbb{Z}_{>0}$-deformation $g = (V_n, [,] + \Psi$ has a non-zero first coordinate $x_1 \neq 0$ then:

$$\dim H^1(g) = \dim H^2(g) = 2.$$ 

Hence for a generic deformations $g$ of $V_n$ the Dixmier inequalities (see [5])

$$\dim H^i(g) \geq 2$$

for $i = 1, 2$.

Recall that a one-dimensional central extension of a Lie algebra $g$ is an exact sequence

$$(41) \quad 0 \to K \to \tilde{g} \to g \to 0$$

of Lie algebras and their homomorphisms, in which the image of the homomorphism $K \to \tilde{g}$ is contained in the center of the Lie algebra $g$. To the cocycle $c \in \Lambda^2(g^*)$ corresponds the extension

$$0 \to K \to K \oplus g \to g \to 0$$

where the Lie bracket in $K \oplus g$ is defined by the formula

$$[(\lambda, g), (\mu, h)] = (c(g, h), [g, h]).$$

It can be checked directly that the Jacobi identity for this Lie bracket is equivalent to $c$ being cocycle and that to cohomologous cocycles correspond equivalent (in a obvious sense) extensions.

Now let $\tilde{g}$ be a filiform Lie algebra, it has one-dimensional center $Z(\tilde{g})$ and we have the following one-dimensional central extension:

$$0 \to K = Z(\tilde{g}) \to \tilde{g} \to \tilde{g}/Z(\tilde{g}) \to 0$$

where the quotient Lie algebra $\tilde{g}/Z(\tilde{g})$ is also filiform Lie algebra. Moreover, let $e_1, \ldots, e_{n+1}$ be some adapted basis of $\tilde{g}$ ($Z(\tilde{g}) = \text{Span}(e_{n+1})$) and $e_1, \ldots, e_{n+1} \in g^*$ its dual basis. Then the forms $e_1, \ldots, e_n$ can be regarded as the dual basis to the basis $e_1 + Z(\tilde{g}), \ldots, e_n + Z(\tilde{g})$ of $\tilde{g}/Z(\tilde{g})$ and the cocycle $\Omega_{n+1} = de_{n+1}$ determines this one-dimensional central extension.

Proposition 9.4. Let $g$ be a filiform Lie algebra and let its center $Z(g)$ be spanned by some $\xi \in Z(g)$. Then $\tilde{g}$ taken from a one-dimensional central extension

$$0 \to K = Z(\tilde{g}) \to \tilde{g} \to g \to 0$$

with cocycle $c \in \Lambda^2(g)$ is a filiform Lie algebra if and only if the restricted function $f(\cdot) = c(\cdot, \xi)$ is non-trivial in $g^*$.

Corollary 9.5. Let $g$ be a symplectic filiform Lie algebra and $\omega$ its symplectic form, then the one-dimensional central extension $\tilde{g}$ with the cocycle $c = \omega$ will be also a filiform Lie algebra.

Theorem 9.6. Let $\tilde{g} = (V_{2k+1}, [,] + \tilde{\Psi}$ be $\mathbb{Z}_{>0}$-filtered deformation of $V_{2k+1}$, where

$$\tilde{\Psi} = x_1 \psi_{2k+1,11} + \cdots + x_5 \psi_{2k+1,17}.$$ 

Then $\tilde{g}$ can be represented as a one-dimensional central extension of $\mathbb{Z}_{>0}$-filtered deformation $g_X = (V_{2k}, [,] + \Phi$ of $V_{2k}$, where

$$\Phi = x_1 \psi_{2k,10} + x_2 \psi_{2k,5} + x_3 \psi_{2k,8} + x_4 \psi_{2k,7}, \ X = (x_1, x-2, x_3, x_4).$$
The cocycle $\Omega_{X,x_5}$ that determines this one-dimensional central extension is equal to

\[
\Omega_{X,x_5} = \frac{1}{2} \sum_{i+j=2k+1} (j-i)e^i \wedge e^j + x_1 \Omega_{2k,11} + x_2 \Omega_{2k,10} + x_3 \Omega_{2k,9} + x_4 \Omega_{2k,8} + x_5 \Omega_{2k,7},
\]

\[
\Omega_{2k,11} = Z_1(2k-3)e^2 \wedge e^9 + Z_2(2k-3)e^3 \wedge e^8 + Z_3(2k-3)e^4 \wedge e^7 + Z_4(2k-3)e^5 \wedge e^6,
\]

\[
\Omega_{2k,10} = Q_1(2k-2)e^2 \wedge e^8 + Q_2(2k-2)e^3 \wedge e^7 + Q_3(2k-2)e^4 \wedge e^6,
\]

\[
\Omega_{2k,9} = P_1(2k-1)e^2 \wedge e^7 + P_2(2k-1)e^3 \wedge e^6 + P_3(2k-1)e^4 \wedge e^5,
\]

\[
\Omega_{2k,8} = \frac{2k-1}{2}(e^2 \wedge e^6 - 2e^3 \wedge e^5), \quad \Omega_{2k,7} = e^2 \wedge e^6 - 3e^3 \wedge e^4.
\]

**Definition 9.7.** Let $\mathfrak{g}, \mathfrak{g}$ be two symplectic Lie algebras, $\omega_\mathfrak{g}, \omega_\mathfrak{g}$ be corresponding symplectic structures. A Lie algebras isomorphism $f : \mathfrak{g} \to \mathfrak{g}$ is called a symplecto-isomorphism $f : (\mathfrak{g}, \omega_\mathfrak{g}) \to (\mathfrak{g}, \omega_\mathfrak{g})$ if and only if $\omega_\mathfrak{g} = f^*(\omega_\mathfrak{g})$.

**Theorem 9.8.** 1) Let $\mathfrak{g}$ be a symplectic filtered deformation of $\mathcal{V}_{2k}$, $\omega_\mathfrak{g}$ its symplectic structure. Then there exists a vector $X = (x_1, x_2, x_3, x_4) \in \mathbb{K}^4$, $x_5 \in \mathbb{K}$ such that the pair $(\mathfrak{g}, \omega_\mathfrak{g})$ is symplecto-isomorphic to $(\mathfrak{g}_X, \Omega_{X,x_5})$, where $\mathfrak{g}_X$ is one of the Lie algebras from the table below.

| $i + j$ | $\mathfrak{g}_X$: commutating relations $[e_i, e_j]$, $i < j$ |
|---------|-------------------------------------------------|
| $3 \leq i + j \leq 6$ | $[e_1, e_6] = 5e_7,$ |
| | $[e_2, e_3] = 3e_7 + x_1 e_{2k-3} + x_2 e_{2k-2} + x_3 e_{2k-1} + x_4 e_n,$ |
| | $[e_3, e_4] = e_7 - 3x_1 e_{2k-3} - 3x_2 e_{2k-2} - 3x_3 e_{2k-1} - 3x_4 e_n,$ |
| $7$ | $[e_1, e_7] = 6e_8,$ |
| | $[e_2, e_6] = 4e_8 + x_1 \frac{(2k-3)}{2} e_{2k-2} + x_2 \frac{(2k-3)}{2} e_{2k-1} + x_3 \frac{(2k-2)}{2} e_{2k},$ |
| | $[e_3, e_5] = 2e_8 + x_1 (2k-4)e_{2k-2} + x_2 (2k-3)e_{2k-1} + x_3 (2k-2)e_{2k};$ |
| $8$ | $[e_1, e_8] = 7e_9,$ |
| | $[e_2, e_7] = 5e_9 + x_1 \frac{(2^{2k-2} - 2^{(2k-3)})}{2^{(2k-1)}} e_{2k-1} + x_2 \frac{(2^{2k-2} - 2^{(2k-3)})}{2^{(2k-1)}} e_{2k},$ |
| | $[e_3, e_6] = 3e_9 - x_1 \frac{(4^{2k-2} - 15^{(2k-3)})}{2^{(2k-1)}} e_{2k-1} - x_2 \frac{(4^{2k-2} - 15^{(2k-3)})}{2^{(2k-1)}} e_{2k},$ |
| | $[e_4, e_5] = 8e_9 - x_1 \frac{(13^{2k-3}) - 30^{(2k-3)}}{2^{(2k-1)}} e_{2k-1} - x_2 \frac{(13^{(2k-2)} - 30^{(2k-2)})}{2^{(2k-1)}} e_{2k};$ |
| $10$ | $[e_1, e_9] = 8e_{10},$ |
| | $[e_2, e_8] = 6e_{10} + x_1 \frac{(3^{(2k-2)} + 3^{(2k-3)})}{2^{(2k-1)}} e_{2k},$ |
| | $[e_3, e_7] = 4e_{10} + x_1 \frac{(2^{2k-2} - 8^{(2k-3)})}{2^{(2k-1)}} e_{2k},$ |
| | $[e_4, e_6] = 2e_{10} + x_1 \frac{(13^{(2k-2)} + 20^{(2k-3)})}{2^{k-1}} e_{2k},$ |
| $11 \leq i + j \leq 2k$ | $[e_1, e_j] = (j-i)e_{i+j}$ |

and its corresponding symplectic form $\Omega_{X,x_5}$ is equal to

\[
\Omega_{X,x_5} = \frac{1}{2} \sum_{i+j=2k+1} (j-i)e^i \wedge e^j + x_1 \Omega_{2k,11} + x_2 \Omega_{2k,10} + x_3 \Omega_{2k,9} + x_4 \Omega_{2k,8} + x_5 \Omega_{2k,7}.
\]
2) A pair \((\mathfrak{g}_X, \Omega_{X,x})\) is symplecto-isomorphic to \((\mathfrak{g}_Y, \Omega_{Y,y})\) if and only if there exist an \(\alpha \in K^*\) such that 
\[
y_1 = \alpha^{n-11}x_1, \quad y_2 = \alpha^{n-10}x_2, \quad y_3 = \alpha^{n-9}x_3, \quad y_4 = \alpha^{n-8}x_4, \quad y_5 = \alpha^{n-7}x_5.
\]

Remark. The previous theorem shows that all \(Z_{>0}\)-deformations \(\tilde{\mathfrak{g}}\) of \(\mathcal{V}_{2k+1}\) are contact Lie algebras and gives their complete classification. An arbitrary symplectic \(Z_{>0}\)-deformations \(\mathfrak{g}\) of \(\mathcal{V}_{2k}\) can be obtained as quotient Lie algebra \(\tilde{\mathfrak{g}}/Z(\tilde{\mathfrak{g}})\). This method of classification of symplecto-isomorphism classes of low-dimensional filiform Lie algebras was considered in [10].

Taking rational coordinates \((x_1, x_2, \ldots, x_5)\) in the table above one will get a nilpotent Lie algebra \(\mathfrak{g}_X\) with rational structure constants and hence due to the Malcev theorem ([14]) the corresponding simply connected nilpotent Lie group \(G\) has a cocompact lattice \(\Gamma\). Thus one will get a family of examples of symplectic nilmanifolds \(M = G/\Gamma\).

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