Quantizing field theories in noncommutative geometry and the correspondence between anti de Sitter space and conformal field theory

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Abstract

By using the approach of non-commutative geometry, we study spinors and scalars on the two layers $\text{AdS}_{d+1}$ space. We have found that in the boundary of two layers $\text{AdS}_{d+1}$ space, by using the AdS/CFT correspondence, we have a logarithmic conformal field theory. This observation propose a way to get the quantum field theory in the context of non-commutative geometry.
1 Introduction

Various aspects of the correspondence between field theories in \((d+1)\)-dimensional Anti de Sitter space (AdS) and \(d\)-dimensional conformal field theories (CFT’s) has been studied in the last few months. An important example is the conjectured correspondence between the large \(N\) limits of certain conformal field theories in \(d\)-dimensions and supergravity on the product of a \((d+1)\)-dimensional AdS space with a compact manifold \([1]\). This suggested correspondence was made more precise in \([2, 3]\).

The general correspondence between a theory on an AdS and a conformal theory on the boundary of AdS is the following. Consider the partition function of a field theory on AdS, subjected to the constraint

\[
\phi \big|_{\partial \text{AdS}} = \phi_0,
\]

that is \(Z_{\text{AdS}}[\phi_0] = \int_{\phi_0} D\phi \exp\{iS[\phi]\}\), where the functional integration is over configurations satisfying (1.1). It is well known that the symmetry algebra of a \((d+1)\)-dimensional AdS is \(O(d,2)\), which is the same as the conformal algebra on a \(d\)-dimensional Minkowski space. From this, it is seen that \(Z_{\text{AdS}}[\phi_0]\) is invariant under conformal transformations, and this is the root of the analogy between theories on AdS and conformal theories on \(\partial\text{AdS}\). In fact, if \(\phi \rightarrow O\phi\), is a space–time symmetry of the theory on AdS, it is seen that \(Z_{\text{AdS}}[O^{-1}\phi_0] = \int_{\phi_0} D(O\phi) \exp\{iS[O\phi]\}\), = \(\int_{\phi_0} D\phi \exp\{iS[\phi]\}\), which means that \(Z_{\text{AdS}}[O^{-1}\phi_0] = Z_{\text{AdS}}[\phi_0]\). So, one can use \(Z_{\text{AdS}}[\phi_0]\) as the generating function of a conformally invariant theory on the boundary of AdS, with \(\phi_0\) as the current.

The length element in AdS\(_{d+1}\) takes the form

\[
ds^2 = -(dx^0)^2 + \sum_{i=1}^{d}(dx^i)^2.
\]

and almost all of the boundary is now contained in \(x^d = 0\).

The above mentioned correspondence have been studied for various cases, e.g. a free massive scalar field and a free U(1) gauge theory \([3]\), an interacting massive scalar field theory \([4]\), free massive spinor field theory \([5]\), and interacting massive spinor-scalar field theory \([6]\). Also, the group-theoretic interpretation of correspondence has been studied in \([7]\). Our aim in this article is to shed light, in one hand, to the correspondence between theories on the non-commutative AdS spaces and logarithmic conformal field theories (LCFT’s), and on the other hand, to the construction of quantum field theory in the non-commutative geometry. In this context, we give a geometrical interpretation of the derived theories which are considered in \([8]\). In this article, we will build the toy models for spinors and scalars in AdS\(_{d+1}\) space by using the generalized Dirac operator and we will show that on the boundary of the AdS\(_{d+1}\), there will be a logarithmic conformal quantum field theory. In section 2 and 3, we review in brief, the LCFT’s and non-commutative geometry. In section 4 and 5, we study the spinor and scalar fields in two layers AdS space respectively.
2 A brief review of logarithmic conformal field theories

It has been shown by Gurarie [8], that conformal field theories whose correlation functions exhibit logarithmic behaviour, can be consistently defined. It is shown that if in the OPE of two local fields, there exist at least two fields with the same conformal dimension, one may find some special operators, known as logarithmic operators. As discussed in [8], these operators with the ordinary operators form the basis of a Jordan cell for the \( L_i \)'s (the generators of the conformal algebra). In some interesting physical theories, one can naturally find logarithmic terms in the correlators of theories. [9]. Logarithmic conformal field theories for \( d \)-dimensional case \((d > 2)\) has also been studied [10]. The basic properties of logarithmic operators are that they form a part of the basis of the Jordan cell for \( L_i \)'s, generators of the Virasoro algebra, and in the correlator of such fields there is a logarithmic singularity [8]. In [11] and [12] assuming conformal invariance two- and three-point functions for the case of one or more logarithmic fields in a block, and one or more sets of logarithmic fields have been explicitly calculated. Regarding logarithmic fields formally as derivatives of ordinary fields with respect to their conformal dimension, \( n \)-point functions containing logarithmic fields have been calculated in terms of those of ordinary fields. These have been done when conformal weights belong to a discrete set. In [13], logarithmic conformal field theories with continuous weights have been considered. It is shown in [11] and [12] that if the set of weights is discrete, when the Jordan cell for \( L_i \) is two dimensional, there are two fields \( \mathcal{O} \) and \( \mathcal{O}' \), with the following two-point functions

\[
\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = 0, \quad (2.1)
\]

\[
\langle \mathcal{O}(x)\mathcal{O}'(y) \rangle = \frac{c}{|x-y|^{2\Delta}}, \quad (2.2)
\]

\[
\langle \mathcal{O}'(x)\mathcal{O}'(y) \rangle = \frac{1}{|x-y|^{2\Delta}} (c' - 2c \ln |x-y|). \quad (2.3)
\]

In [14] another type of derivation is also introduced. The main idea of this construction is based on the formal derivation of the entities of the original system with respect to a parameter, which may or may not explicitly appear in the original theory. One way of viewing this is through the concept of contraction: consider two systems with parameters \( \lambda \) and \( \lambda + \delta \). These two systems are independent to each other. One can write an action as the difference of the actions of the two system divided by \( \delta \) to describe both systems. One can use one of these degrees of freedom and the difference of them divided by \( \delta \) a new set of variables. This system, however, is equivalent to two copies of the original system. But if one lets \( \delta \) tend to zero, a well-defined theory of double number of variables is obtained, which no longer can be decomposed to two independent parts. One can, however, solve this theory in terms of the solution of the original theory. This procedure is nothing but a contraction. It has been shown in [15] that any symmetry, and any constant of motion of the original theory, results in a symmetry and a constant of motion of the derived one and any theory derived from an integrable theory is integrable. At last, it has also been shown that this technique is applicable to classical field theories as well. This technique is applicable to quantum systems as well. Here, however, a novel property arises: the derived quantum
theory is *almost classical*: that is, in the derived theory there are only one–loop quantum corrections to
the classical action [15]. Using this property, one can calculate all of the Green functions of the derived
theory exactly, even though this may be not the case for the original theory.

Using the AdS/CFT correspondence, in [16], a correspondence between field theories in $(d+1)$–dimensional
AdS space and $d$–dimensional logarithmic conformal field theories has been obtained. It has been showed
that by a suitable choice of action in AdS$_{d+1}$, one gets a LCFT on the $\partial$AdS. In general, using any field
theory on AdS, which corresponds to a CFT on $\partial$AdS, one can systematically construct other theories
on AdS corresponding to LCFT’s on $\partial$AdS.

3 A brief review of non-commutative geometry

In 1989, Alain Connes and John Lott obtained the lagrangian of the standard model by an algebraic-
geometrical approach which is called non-commutative geometry or spectral geometry [17]. They consid-
ered a two layers Minkowskian space-time, which each layer endowed by a suitable bundle for describing
the $U(1) \times SU(2)$ gauge symmetry. What they really has been obtained was a generalized Yang-Mills
theory in which the Higgs potential appears naturally. This Higgs field seems to be the component of
the generalized gauge field in the direction of discretnes of space. But there was some difficulties in their
approach such as their fermions Hilbert space does not match with the phenomenological standard model
lagrangian [18] and another problem was due to the existance of some relations between some parameters
of the theory such as the mass of the Higgs and Top quark [19]. These relations are not consistent
with the flow of renormalization of this parameters [20]. The reason for the latter problem is said to be
due to taking the incorrect geometrical space which is a two layers commutative Minkowski space (or in
other words the classical space). By commutative space, we mean that the algebra which describes the
space-time is $C^\infty(M)$ which is the commutative $C^*$-algebra. Now if one modifies this algebra to some
non-commutative or quantum algebra, he or she can derives standard model which is in accordance with
phenomenological standard model with the same number of free parameters.

In non-commutative geometry, we express the topological space (which is assumed to be compact) by
a unital $C^*$-algebra $\mathcal{A}$ which is commutative in the case of ordinary manifolds. Then for endowing this
algebra with a differential structure, we need the K-cycle $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{H}$ is the Hilbert space for
representating the elements of $\mathcal{A}$ as the linear operators and $\mathcal{D}$ is called the generalized Dirac opera-
tor. The role of K-cycle for non-commutative geometry is similar to the role of differential structure in
ordinary differential geometry. Having a K-cycle, one can develop a differential algebra $\Omega(\mathcal{A})$ for the
non-commutative space which is equivalent to differential geometry for the manifolds.

In correspondence to ordinary lagrangian for spin $\frac{1}{2}$ particles, one can take the following statement
as the generalized spin $\frac{1}{2}$ action in non-commutative geometry,

$$\mathcal{I} = \langle \psi | \mathcal{D} | \psi \rangle = \int d^{d+1}x \sqrt{g} \bar{\psi} \mathcal{D} \psi.$$  \hspace{1cm} (3.1)

where $\psi$ is the spinor field in $\mathcal{H}$. In fact, one can take the Dirac operator $\mathcal{D}$ such that the suitable mass term emerges naturally in the above statement. In the case of spin 0 particles in accordance to Klein-Gordon action, one can define the non-commutative geometric action as follows,

$$S = \frac{1}{4} \text{Tr}_{\omega} \{ [\mathcal{D}, \phi] \Gamma [\mathcal{D}, \phi] + M^2 \phi \Gamma \phi \},$$ \hspace{1cm} (3.2)

where $\phi$ is an element of the algebra $\mathcal{A}$, and by $\text{Tr}_{\omega}$, we mean the Dixmier trace [21]. In the above statement, $\Gamma$ is some gradation operator.

### 4 Two layers AdS space and spin $\frac{1}{2}$ lagrangian

At this stage, let us take a two layers space as our topological space. A suitable algebra which describes it, is $\mathcal{A} = C^\infty(AdS) \otimes \mathbb{M}_2(C)$, where by $C^\infty(AdS)$, we mean the algebra of complex smooth functions defined on AdS, and by $\mathbb{M}_2(C)$, we mean the algebra of $2 \times 2$ complex matrices. Then as a representation for the elements of $\mathcal{A}$, we choose the following representation, $\pi(a) = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix}; \ a \in \mathcal{A}$. In fact, this representation among the others has a simple interpretation, simply $f_1(x)$ and $f_2(x)$ can be interpreted as functions on each AdS spaces. Here we can split our Hilbert space $\mathcal{H}$ into two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, which each of them corresponds to one AdS layer. So it is logical to define a gradation operator $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for showing the splitting of the Hilbert space $\mathcal{H}$, and we can represent our spinors as $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Now, it is the time to define our Dirac operator to complete the definition of our K-cycle. We take it as follows,

$$\mathcal{D} = \begin{pmatrix} \partial + m & M \\ M & -(\partial + m) \end{pmatrix}; \ m, M \in C \otimes 1_{d \times d}. \hspace{1cm} (4.1)$$

So according to Eq. (3.1) for spin $\frac{1}{2}$ lagrangian, we have,

$$\mathcal{L} = \bar{\psi} \mathcal{D} \psi = (\bar{\psi}_1, \bar{\psi}_2) \begin{pmatrix} \partial - m & M \\ M & -(\partial + m) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \bar{\psi}_1 (\partial + m) \psi_1 + \bar{\psi}_1 M \psi_2 - \bar{\psi}_2 (\partial + m) \psi_2 + \bar{\psi}_2 M \psi_1.$$

(4.2)

It seems to be logical to take the physics of both AdS spaces identical, which means $\psi_1$ and $\psi_2$ to be the same. So at this stage, we take $\psi_1$ and $\psi_2$ as follows, $\psi_1 = \frac{1}{2}(\psi_+ + \epsilon \psi_-)$, $\psi_2 = \frac{1}{2}(\psi_+ - \epsilon \psi_-)$, where $\epsilon$ is an infinitesimal number. Now in terms of these new spinors, the lagrangian can be written as follows,

$$\mathcal{L} = \bar{\psi}_+ (\partial + m) \psi_- + \bar{\psi}_- (\partial + m) \psi_+ + \bar{\psi}_+ M \psi_+,$$

(4.3)
which we have redefined the fields $\sqrt{2}\psi_\pm$ as $\psi_\pm$. This is exactly the fermionic singleton lagrangian. However, as observed in Ref. [5], to obtain the non-vanishing on-shell spinor action in the usual AdS space, one must add to the massive free spinor action a boundary term. For such a boundary term in the non-commutative geometry, we take the following boundary lagrangian,

$$L_{\text{boundary}} = \bar{\psi}_{\text{boundary}} \Gamma \psi_{\text{boundary}}, \quad (4.4)$$

where $\psi_{\text{boundary}}$ is the projection of $\psi$ on the boundary of AdS $d+1$. Hence the total lagrangian of spinors system on the two layers AdS space is,

$$L_{\text{tot}} = \bar{\psi}_+ (\partial + m) \psi_+ + \bar{\psi}_- (\partial + m) \psi_- + \bar{\psi}_+ M \psi_+$$

$$+ \{ \bar{\psi}_+(\text{boundary}) \psi_-(\text{boundary}) + \bar{\psi}_-(\text{boundary}) \psi_+(\text{boundary}) \} \delta (x^d). \quad (4.5)$$

By using the same method of Ref. [16], one can show that the projection of the fields $\psi_-$ and $\psi_+$ to the boundary are currents for the pseudo-conformal operators $\lambda_+^\alpha$ and $\lambda_-^\alpha$ respectively. The two point correlation function $\langle \bar{\lambda}_+^\alpha (x) \lambda_-^\beta (y) \rangle$ vanishes. The two point correlation function $\langle \bar{\lambda}_+^\alpha (x) \lambda_+^\beta (y) \rangle$ has a scaling behaviour and $\langle \bar{\lambda}_-^\alpha (x) \lambda_+^\beta (y) \rangle$ has a logarithmic behaviour.

### 5 Two layers AdS space and scalar field lagrangian

In this section, we consider the action (3.2) which we have introduced it for the scalar field as the generalization of massive Klein-Gordon action in the non-commutative geometry. For the two layers AdS space, we take the Dirac operator as same as (4.1). As it was mentioned in section 3, the scalar fields are elements of the algebra $\mathcal{A}$, and again we take the following representation for the elements of $\mathcal{A}$ in Hilbert space $\mathcal{H}$, $\pi(\phi) = \begin{pmatrix} \phi_1 (x) & 0 \\ 0 & \phi_2 (x) \end{pmatrix}$, where $\phi_1 (x)$ and $\phi_2 (x)$ are functions in $C^\infty (AdS)$. By using $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the gradation operator, it would be straightforward to obtain,

$$S = \frac{1}{4} \int d^{d+1} x \sqrt{g} \text{tr} \{ Tr \{ [D, \pi(\phi)] \Gamma [D, \pi(\phi)] + m^2 \phi \Gamma \phi \} \}$$

$$= \frac{1}{4} \int d^{d+1} x \sqrt{g} \text{tr} \{ (\partial \phi_1 \phi_2 \phi_2 + 2M^2 (\phi_2 - \phi_1) - \phi \phi_2 \phi_2 + m^2 (\phi_1^2 - \phi_2^2)) \}, \quad (5.1)$$

where $tr$ and $Tr$ are traces on the Clifford and matrix structure respectively. We notice that in the first line of eq. (5.1), the meaning of $Tr$ is integrating over the discrete dimension of space. So, the resultant relation (5.1) could be interpreted as the effective action in the ordinary AdS$_{d+1}$ space. By introducing the new fields $\phi_+$ and $\phi_-$ as follows, $\phi_1 = \epsilon \phi_+ + \phi_-$, $\phi_2 = \epsilon \phi_+ - \phi_-$, and substituting them in (5.1), and preserving the terms up to the first order in $\epsilon$, we will obtain,

$$S = \frac{1}{4} \int d^{d+1} x \sqrt{g} \text{tr} \{ 4 \epsilon \partial \phi_+ \phi_- + 2M^2 \phi_-^2 + 4m \epsilon \phi_+ \phi_- \}. \quad (5.2)$$
Now, if one redefines $\sqrt{d+1} \phi_+$ as $\phi_+$ and $\sqrt{d+1} \phi_-$ as $\phi_-$, then the last relation becomes,

$$S = \int d^{d+1}x \sqrt{g} \left\{ \partial_+ \phi_+ \partial_+ \phi_- + \frac{M^2 \phi_-^2}{2} + m \phi_+ \phi_- \right\}. \quad (5.3)$$

This action in the literature is known as bosonic singleton action. In the Ref. [16], by an explicit calculation, it has been showed that the projection fields $\phi_-(0)$ and $\phi_+(0)$ obtained through the fields $\phi_-$ and $\phi_+$ by restricting them to the boundary of the AdS$_{d+1}$, are the currents for the conformal operators, $O_+$ and $O_-$ respectively. In fact, the two–point correlation functions of the operators $O_+$ and $O_-$ which live in the boundary of AdS$_{d+1}$ are in the following form; The correlation function of $O_-(x)$ and $O_-(y)$ vanishes as the relation (2.1). The correlation function of $O_-(x)$ and $O_+(y)$ is in the scaling form (2.2) and the correlation function of $O_+(x)$ and $O_+(y)$ is in the logarithmic form (2.3). The constants $c$ and $c'$ which appear in the correlation functions (2.2) and (2.3) are related to the parameters $m$ and $M$ appearing in (5.3). This shows that in the boundary of two layers AdS space, which is in fact a non-commutative double sheets $d$–dimensional Minkowski space, we have a quantum field theory. Also, we note that by using a generalized action for the interacting scalar field in the two layers AdS space, one naturally get the $n$–point correlation functions of a quantum field theory in the $d$–dimensional non-commutative Minkowski spaces [16].

6 Conclusions

In this article, we have shown how one can use the approach of non-commutative geometry to obtain a classical field theory for the spinor and scalar fields in the classical or commutative spaces (i.e. the spaces which can be expressed by a commutative algebra). We have used a two layers Anti de Sitter space and by the above mentioned approach, we have obtained a classical field theory for spinors and scalars in the bulk space. However by considering the AdS/CFT correspondence, our goal is to correspond the generating functions of these theories to the quantum correlation functions of a logarithmic conformal field theory on the boundaries of the AdS space. On the other hand, we have presented a geometrical model for the fermionic and bosonic singleton theories, i.e. equations (4.3) and (5.3). In fact, in [16], an algebraic model was established for these theories by taking the derivative of the ordinary Klein-Gordon and Dirac actions with respect to some parameter. However, in this article, we have shown that one can obtain the singleton actions from a generalized Dirac or Klein-Gordon theory in a non-commutative space.

This consideration makes an idea for solving the parameters restriction in Connes-Lott model for the geometrization of standard model. In fact if someone tries to obtain the Connes-Lott model in the boundary of a two layer space with the appropriate Higgs potential by considering a suitable theory in the bulk, then the theory on the boundary is a quantum theory, without any parameters restriction. This is what we are going to show explicitly in near future.
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