Two-fluid hydrodynamic model for superfluids in fractal dimensions

D.A. Tayurskii, Yu.V. Lysogorski and D. Yu. Zvezdov
Department of Physics, Kazan State University, 18 Kremlevskaya st., Kazan, 420018, Russia.
E-mail: dtayursk@gmail.com

Abstract. It was recognized last years that the state of quantum liquids at nanoscales could be considered as some new state of quantum matter. Nanoporous media found to show the fractal geometry behavior so the problem of the correct description of quantum liquids in the space with fractal dimensions is a rather interesting. In the present work we develop two-hydrodynamic model for superfluids in fractal geometry and investigate the wave propagation and phenomena like to "sound transformations" in superfluids in this case.

1. Introduction
It is well known that near temperature $T_\lambda=2.17\text{K}$ the second-order phase transition occurs in liquid $^4\text{He}$ and the superfluidity phenomenon is observed. The number of physical properties of liquid $^4\text{He}$ below $T_\lambda$ (He-II) can be understood in terms of two-fluid hydrodynamic model (see, f.e. [1], [2]). Properties of normal component are similar to ordinary liquid helium, but superfluid component has no viscosity and its flow is potential. It follows from two-fluid model that in He-II several types of waves can propagate. There are compressional waves (the first sound, normal and superfluid components move in phase) and temperature waves (second sound, normal and superfluid components move in opposite). In a porous medium where the normal component is clamped by its viscosity and only superfluid component can move, fourth sound (relative motion of the superfluid and normal fluids) propagates.

In recent year the intensive studies of the properties of liquid $^4\text{He}$ confined in high-porosity and nanoporous media (silica aerogels [3], [4], [5], [6], Vycor glass [7], [8], mesoporous substrate FSM [9], [10]) revealed a crucial role of the disorder introduced by the porous structure on the superfluid transition, solidification, sound propagation etc.

It is recognized that nanoporous media shows the fractal geometry behavior [11]. For example the silica aerogel structure consists of randomly located clusters of fractal mass dimension in the range of typical lengths between 10 and 100 \text{Å} [12]. So it is reasonable to propose that the fractal geometry of nanoporous media could be "seen" by the confined liquid $^4\text{He}$ and the problem of the correct description of quantum liquids in the space with fractal dimensions appears. In the present work we develop two-hydrodynamic model for superfluids in fractal geometry and investigate the wave propagation and phenomena like to "sound transformations" in superfluids in this case.
2. Two-fluid hydrodynamic model
The usual two-fluid hydrodynamic model [1], [2] is described by the system of four differential equations (1,2,3,4)

\[ \frac{\partial \rho}{\partial t} + \nabla j = 0 \] (1)

\[ \frac{\partial \rho \sigma}{\partial t} + \nabla (\rho \sigma v_n) = 0 \] (2)

\[ \rho_s \left[ \frac{\partial v_s}{\partial t} + (v_s \nabla) v_s \right] = -\frac{\rho_s}{\rho} \nabla p + \rho_s \sigma \nabla T \] (3)

\[ \rho_n \left[ \frac{\partial v_n}{\partial t} + (v_n \nabla) v_n \right] = -\frac{\rho_n}{\rho} \nabla p - \rho_s \sigma \nabla T, \] (4)

where \( \rho \) – He-II density, \( \rho_s, \rho_n \) – superfluid and normal component density, \( v_s, v_n \) – superfluid and normal component velocity, \( j = \rho_s v_s + \rho_n v_n \) – He-II flow density. Equations (1) and (2) represent conservation laws of mass and entropy accordingly. Equations (3) and (4) correspond to Euler equations written for superfluid and normal components. Interaction between the fluid components, forces of viscous friction and energy dissipation are neglected because of the small velocities of both components.

3. Fractional two-fluid hydrodynamic model
One can imagine aerogel in the range of typical lengths between 10 and 100 Å as a cluster with fractional dimension \( D \) [13] consisting of pores with a radius \( r_0 \), which can be filled in by He-II. Remind that the number of clusters inside a sphere of radius \( r \) is determined by the ratio (5)

\[ N(r) \propto \left( \frac{r}{r_0} \right)^D. \] (5)

The concentration of particles in a spherical layer with thickness \( r_0 \) and a radius \( r \) is written as

\[ n(r) \propto \left( \frac{r}{r_0} \right)^{D-3}. \] (6)

Assuming that the density of He-II in the pores depends only on the distance to the center of the cluster and is equal to \( \rho_0(r) \), one can determine the density of spherical layer of cluster as

\[ \rho(r) \propto \rho_0(r) \left( \frac{r}{r_0} \right)^{D-3}. \] (7)

The non-uniform particle concentration like (6) can exist only if the pressure inside cluster is also inhomogeneous and obeys the same factorization rule:

\[ p(r) \propto p_0(r) \left( \frac{r}{r_0} \right)^{D-3}. \] (8)

From the equations (7) and (8) one can see that density and pressure depend on \( r^{D-3} \). If \( D = 3 \), then density and pressure do not depend on sphere radius as it should be in the case of Euclidean geometry.

The proposed model of cluster with a fractal dimension we call a porous model of fractional cluster. When the non-linear terms involving \( v_n \) and \( v_s \) are omitted, and after the substitution of (7) and (8) to the equations (1,2,3,4), the fractionalized two-fluid hydrodynamic model of He-II can be obtained

\[ \frac{\partial \rho_0}{\partial t} + \nabla j_0 + \frac{(D-3)\rho_0}{r^2} = 0 \] (9)
\[
\frac{\partial \rho_0}{\partial t} + \nabla (\rho_0 \sigma \mathbf{v}_n) + \rho_0 \sigma \frac{(D-3)\mathbf{r} \cdot \mathbf{v}_n}{r^2} = 0
\]  
(10)

\[
\rho_0 \frac{\partial \mathbf{v}_s}{\partial t} = -\rho_0 \frac{\partial p_0}{\rho_0} \mathbf{v}_0 + \rho_0 \sigma \nabla T - \frac{\rho_0 \sigma (D-3)\mathbf{r}}{r^2} p_0
\]  
(11)

\[
\rho_{0n} \frac{\partial \mathbf{v}_n}{\partial t} = -\rho_{0n} \frac{\partial p_{0n}}{\rho_{0n}} \mathbf{v}_{0n} - \rho_0 \sigma \nabla T - \frac{\rho_0 \sigma (D-3)\mathbf{r}}{r^2} p_{0n}
\]  
(12)

The equations for the pressure and temperature waves oscillation obtained from the system (9,10,11,12) have the following form:

\[
\frac{\partial^2 \rho_0}{\partial t^2} = \nabla^2 p_0 + 2\frac{(D-3)\mathbf{r}}{r^2} \nabla p_0 + \frac{(D-3)(D-2)p_0}{r^2},
\]  
(13)

\[
\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_{0s} \sigma^2}{\rho_{0n}} \nabla^2 T + \frac{\rho_0 \sigma (D-3)\mathbf{r}}{r^2} \nabla T.
\]  
(14)

4. Fractional sound wave solutions

Linearizing the system of hydrodynamics equations with respect to small deviations of pressure and temperature, and neglecting the thermal expansion, one can get the following system of equations:

\[
\frac{\partial \rho_0}{\partial t} \frac{\partial^2 \rho_0}{\partial t^2} = \nabla^2 p_0 + 2\frac{(D-3)\mathbf{r}}{r^2} \nabla p_0 + \frac{(D-3)(D-2)p_0}{r^2},
\]  
(15)

\[
\frac{\partial \sigma}{\partial T} \frac{\partial^2 T}{\partial t^2} = \frac{\rho_{0s} \sigma^2}{\rho_{0n}} \nabla^2 T + \frac{\rho_0 \sigma (D-3)\mathbf{r}}{r^2} \nabla T.
\]  
(16)

One can assume

\[
p_0(r, t) = p_0^{(0)}(r) + p_0'(r, t),
\]  
(17)

\[
T(r, t) = T^{(0)}(r) + T'(r, t),
\]  
(18)

where \(p_0^{(0)}(r)\) and \(T^{(0)}(r)\) stand for stationary spatial pressure and temperature distributions, \(p_0'(r, t)\) and \(T'(r, t)\) represent the small pressure and temperature deviations from the stationary values correspondingly. It is clear that \(p_0'(r, t)\) and \(T'(r, t)\) are satisfied the same equations as \(p_0(r, t)\) and \(T(r, t)\). The solutions of the obtained equations for \(p_0'(r, t)\) and \(T'(r, t)\) have the following forms:

\[
p_0'(r, t) = \frac{A_1 \sin (k_1 r) \cos (\omega_1 t + \varphi_1)}{\sqrt{k_1}},
\]  
(19)

\[
T'(r, t) = A_2 r^{1-\frac{D}{2}} J_{\frac{D-2}{2}}(k_2 r) \cos (\omega_2 t + \varphi_2),
\]  
(20)

with dispersion laws \(\omega_1 = \sqrt{\left(\frac{\rho_0 \sigma^2}{\rho_{0n}}\right) \frac{1}{k_1}}\), \(\omega_2 = \sqrt{\rho_{0s} \sigma^2 / \rho_{0n} (\partial \sigma / \partial T)p_0} k_2\) for the first and second sounds respectively. Here \(k\) denotes a wavenumber, \(A\) - amplitude, \(\varphi\) - initial phase. It follows from this solution that under assumptions made above the fractionalizing of hydrodynamic equations does not lead to the appearance of the coupled modes of pressure and temperature. As an example the profiles of temperature waves obtained from this set of equations for different dimensions between 2D and 3D cases are shown in Fig. 1.
5. Conclusion

Thus, mixing between sound modes (first and second sounds) in the complex system “aerogel + He-II” in the case of fractionalization of only intensive values (density and pressure) doesn’t occur. Consequently, the observable peculiarities of He-II behavior in media with noninteger dimension of nanopores could be attributed to the nonextensivity of thermodynamic quantities [14] (such as entropy). Because of strong influence of the surface effects the total entropy of two connected nanopores does not equal to the sum of entropies for each pore. Therefore, any further improvement of the proposed model should be accompanied by the complete relevant thermodynamics, not to mention a better handling of the entropy (and similarly the temperature).

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