Bifurcation locus and branches at infinity of a polynomial \( f : \mathbb{C}^2 \to \mathbb{C} \)

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Abstract We show that the number of bifurcation values at infinity of a polynomial function \( f : \mathbb{C}^2 \to \mathbb{C} \) is at most the number of branches at infinity of a general fiber of \( f \) and that this upper bound can be diminished by one in certain cases.

1 Introduction

Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be a polynomial function in a fixed coordinate system. It is well known (as being proved originally by Thom [17]), that \( f \) is a locally trivial \( C^\infty \) fibration outside a finite subset of the target. The smallest such set is called the bifurcation set of \( f \) and will be denoted here by \( B(f) \). The set \( B(f) \) might be larger than the set of critical values \( f(\text{Sing} f) \), like for instance in the following simple example due to Broughton [1]: \( f(x, y) = x + x^2y \), where \( \text{Sing} f = \emptyset \) but \( B(f) = \{0\} \), and we say that 0 is a critical value at infinity of \( f \). The set \( B_\infty(f) \) of bifurcation values at infinity, or critical values at infinity, consists of points \( a \in \mathbb{C} \) at which the restriction of \( f \) to the complement of a large enough ball (centred at 0 \( \in \mathbb{C}^2 \)) is not a locally trivial bundle. There are several criteria to detect such a value; one may consult e.g. [2,3,5,16,18,19]. For instance: \( a \in B_\infty(f) \) if and only if there exists a sequence of
points \((p_k)_{k \in \mathbb{N}} \subset \mathbb{C}^2\) such that \(\|p_k\| \to \infty\), \(\text{grad } f(p_k) \to 0\) and \(f(p_k) \to a\) as \(k \to \infty\).

Upper bounds for \(#B_\infty(f)\) have been found in the 1990’s by Lê and Oka [12] in terms of Newton polyhedra at infinity. An estimation in terms of the degree \(d\) of \(f\) was given by Gwoździewicz and Płoski [8]: if \(\dim \text{Sing } f \leq 0\) then \(#B_\infty(f) \leq \max\{1, d - 3\}\). In the general case (dropping the condition \(\dim \text{Sing } f \leq 0\)) we have \(#B_\infty(f) \leq d - 1\), see e.g. [10,11]. Recently Gwoździewicz [9] proved the following estimation of \(#B_\infty(f)\): if \(\nu_0\) denotes the number of branches at infinity of the (reduced) fibre \(f^{-1}(0)\), then the number of critical values at infinity other than 0 is at most \(\nu_0\). Here we refine and improve this statement by using a different method, in which results by Miyanishi [13,14] and Gurjar [6] play an important role.

For \(a \in \mathbb{C}\), let us denote by \(\nu_a\) the number of branches at infinity of the reduced fiber \(f^{-1}(a)\). This number is equal to \(\nu_{\text{gen}}\) for all values \(a \in \mathbb{C}\) except finitely many for which one may have either \(\nu_a < \nu_{\text{gen}}\) or \(\nu_a > \nu_{\text{gen}}\). Let \(\nu_{\text{min}} := \inf\{\nu_a \mid a \in \mathbb{C}\}\). Let us denote by \(b\) the number of points at infinity of \(f\), i.e. \(b := \#f^{-1}(a) \cap L_\infty\), where \(L_\infty\) is the line at infinity \(\mathbb{P}^2 \setminus \mathbb{C}^2\).

Under these notations, our main result is the following:

**Theorem 1.1** Let \(f : \mathbb{C}^2 \to \mathbb{C}\) be a polynomial function of degree \(d\). Then:

(a) \(#B_\infty(f) \leq \min\{\nu_{\text{gen}}, \nu_{\text{min}} + 1\}\).

(b) \(#\{a \in \mathbb{C} \mid \nu_a < \nu_{\text{gen}}\} \leq \nu_{\text{gen}} - b\).

(c) \(#\{a \in \mathbb{C} \mid \nu_a > \nu_{\text{gen}}\} \leq \nu_{\text{min}}\) (this remains true even if we count branches with multiplicities).

In case \(\nu_{\text{gen}} > \frac{d}{2}\), we moreover have:

(d) \(#B_\infty(f) \leq \min\{\nu_{\text{gen}} - 1, \nu_{\text{min}}\}\).

(e) \(#\{a \in \mathbb{C} \mid \nu_a > \nu_{\text{gen}}\} \leq \nu_{\text{min}} - 1\) (this remains true even if we count branches with multiplicities).

**Remark 1.2** Point (a) of Theorem 1.1 is equivalent to Gwoździewicz’s [9, Theorem 2.1]. His result is a by-product of the local study of pencils of curves of Yomdin-Ephraim type. Our method is totally different and allows us to prove moreover several new issues, namely (b)–(e) of Theorem 1.1.

**Remark 1.3** As Gwoździewicz remarks, his inequality [9, Theorem 2.1] is “almost” sharp, i.e. not sharp by one. Our new inequality (d) improves by one the inequality (a) under the additional condition \(\nu_{\text{gen}} > \frac{d}{2}\), thus yields the sharp upper bound, as shown by the example \(f : \mathbb{C}^2 \to \mathbb{C}, f(x, y) = x + x^2y\), where \(d = \deg f = 3\), \(\nu_{\text{min}} = \nu_{\text{gen}} = 2\), \(b = 2\) and \(B_\infty(f) = \{0\}\) with \(\nu_0 = 3\).

The same example shows that our estimations (b) and (e) are also sharp.

2 Proof of Theorem 1.1

We need here the important concept of *affine surfaces which contain a cylinder-like open subset* which was introduced by Miyanishi [13]. Let us recall it together with some properties which we shall use.
**Definition 2.1** [14] Let $X$ be a normal affine surface. We say that $X$ contains a cylinder-like open subset $U$, if there exists a smooth curve $C$ such that $U \cong \mathbb{C} \times C$.

Let $X$ be as in the above definition and let $\pi : U \to C$ be the projection. After [14, p.194], the projection $\pi$ has a unique extension to a $\mathbb{C}$-fibration $\rho : X \to \hat{C}$, where $\hat{C}$ denotes the smooth completion of the curve $C$. We have the following important result of Gurjar and Miyanishi:

**Theorem 2.2** [6,7,13] Let $X$ be a normal affine surface with a $\mathbb{C}$-fibration $f : X \to B$, where $B$ is a smooth curve. Then:

(a) $X$ has at most cyclic quotient singularities.
(b) Every fiber of $f$ is a disjoint union of curves isomorphic to $\mathbb{C}$.
(c) A component of a fiber of $f$ contains at most one singular point of $X$. If a component of a fiber occurs with multiplicity $1$ in the scheme-theoretic fiber, then no singular point of $X$ lies on this component. \(\square\)

**Corollary 2.3** Let $X$ be a normal affine surface, which contains a cylinder-like open subset $U$. Then the set $X \setminus U$ is a disjoint union of curves isomorphic to $\mathbb{C}$. Moreover, every connected component $l_i$ of this set contains at most one singular point of $X$. \(\square\)

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function in fixed affine coordinates and denote by $\tilde{f}(x, y, z)$ the homogenization of $f$ by a new variable $z$, namely $\tilde{f}(x, y, z) = f_d + zf_{d-1} + \cdots + z^df_0$. Let $X := \{(x : y : z, t) \in \mathbb{P}^2 \times \mathbb{C} | \tilde{f}(x, y, z) = te^d\}$ be the closure in $\mathbb{P}^2 \times \mathbb{C}$ of the graph $\Gamma := \text{graph}(f) \subset \mathbb{C}^2 \times \mathbb{C}$. Then $X$ is a hypersurface and the points at infinity of $X$ (i.e., points outside of $\Gamma$) forms precisely the set $\{a_1, \ldots, a_b\} \times \mathbb{C}$, where $\{a_1, \ldots, a_b\}$ are all points at infinity of the curve $f = 0$. In particular if $\rho : \mathbb{P}^2 \times \mathbb{C} \to \mathbb{P}^2$ denotes the first projection, then $\rho(X \setminus \Gamma) = \{a_1, \ldots, a_b\}$.

The second projection $\pi : X \to \mathbb{C}$, $(x, t) \mapsto t$, is a proper extension of $f$. Let $\nu : X' \to X$ be the normalization of $X$. Composing $\nu$ with $\pi$ yields $\pi' : X' \to \mathbb{C}$, which is also a proper extension of $f$. We shall denote it by $\tilde{f}$ in the following.

On the other side composing $\nu$ with $\rho$ yields $\rho' : X' \to \mathbb{P}^2$ and $\rho'(X' \setminus \Gamma) = \{a_1, \ldots, a_b\}$, i.e., the points at infinity of $X'$ lie over the points $\{a_1, \ldots, a_b\}$.

**Lemma 2.4** The set $X' \setminus \Gamma$ is a disjoint union of affine curves, $l_1, \ldots, l_r$, each curve $l_i$ is isomorphic to $\mathbb{C}$. On each line $l_i$ there is at most one singular point of $X'$. Moreover, $b \leq r \leq \nu_{\min}$.

**Proof** Let us choose a line $l \subset \mathbb{P}^2$ such that $l \cap \{a_1, \ldots, a_b\} = \emptyset$. Let $X_1 := (\mathbb{P}^2 \setminus l) \times \mathbb{C} \cap X$. The surface $X_1$ is affine and $X_1 \setminus \Gamma = \bigcup_{i=1}^r l_i$, where $X_1'$ denotes the normalization of $X_1$. The surfaces $X'$ and $X_1'$ have the same points at infinity since there is no points at infinity of $X'$ which belongs to the line $l$.

Since the surface $X_1'$ contains a cylinder-like open subset $U := \text{graph}(f|_{\mathbb{C}^2 \setminus l}) \cong \mathbb{C} \times \mathbb{C}^*$ and $X_1' \setminus U = \bigcup_{i=1}^r l_i$, the first part of our claim follows from Corollary 2.3. Next, the map $\tilde{f}$ restricted to $l_i$ is finite, hence surjective. This implies that every fiber of $\tilde{f}$ has a branch at infinity which intersects $l_i$. In particular $r \leq \nu_{\min}$. The inequality $r \geq b$ is obvious. \(\square\)
Denote by $f_i : l_i \cong \mathbb{C} \to \mathbb{C}$ the restriction of $\tilde{f}$ to $l_i$. It can be identified with a one variable polynomial, the degree of which is equal to the number $v_i$ of branches of a generic fiber of $\tilde{f}$ which intersect $l_i$. In particular $\sum_{i=1}^r v_i = v_{\text{gen}}$.

The polynomial $f_i$ of degree $v_i$ can have at most $v_i - 1$ critical points. If a fiber $\tilde{f}^{-1}(a)$ does not contain critical points of any $f_i$ and does not contain singular points of $X'$, then the point $a \notin B_{\infty}(f)$. This follows from general arguments concerning Whitney stratifications and Thom Isotopy Lemma, like in [3, 15, 19], but let us outline a short proof here. Firstly, the fiber $\tilde{f}^{-1}(a)$ cannot contain multiple components since otherwise, for some $i$, the fiber $f_i^{-1}(a)$ will also have a multiple component, thus a singularity, which contradicts our assumption. Therefore the fiber $\tilde{f}^{-1}(a)$ is non-singular outside some large ball $B(0, R) \subset \mathbb{C}^2$. By the Sard Theorem there is a real value $R' > R$ such that the sphere $\partial B(0, R')$ is transversal to $\tilde{f}^{-1}(a)$. In particular there is a small disc $U(a, \rho)$ such that for every $b \in U(a, \rho)$ the fiber $\tilde{f}^{-1}(b)$ is smooth outside $B(0, R)$ and it is transversal to $\partial B(0, R')$. We can also assume that $\rho$ is so small that $\tilde{f}^{-1}(b)$ does not contain critical points of any of the polynomials $f_i$, for $i = 1, \ldots, r$, and it does not contain any singular point of $X'$. This means in particular that all these fibers are transversal to all curves $l_i$, $i = 1, \ldots, r$. Now take $Y = \tilde{f}^{-1}(U(a, \rho)) \setminus \text{Int}(B(0, R'))$. It is a smooth manifold with boundary, where the boundary $\partial Y = \partial B(0, R') \cap \tilde{f}^{-1}(U(a, \rho))$. The set $V := (\bigcup_{i=1}^r l_i) \cap Y$ is a smooth submanifold of $Y$. The mapping $g := \tilde{f}_| Y : Y \to U(a, \rho)$ is proper and all fibers of $g$ are transversal to $V$ and to $\partial V$. By the Ehresmann Theorem [4] there is a trivialization of $g$ which preserves $V$ and $\partial V$. This proves our claim that $a \notin B_{\infty}(f)$.

Finally we conclude that the bifurcation values at infinity for $f$ can be only images by $\tilde{f}$ of critical points of $f_i$, $i = 1, \ldots, r$ and images of singular point of $X'$. Summing up, we get that $f$ can have at most $v_{\text{gen}}$ critical values at infinity, which shows one of the inequalities of point (a). Moreover, the inequality $v_a < v_{\text{gen}}$ is possible only if $a$ is a critical value of some polynomial $f_i$. This means that $\{a \in \mathbb{C} \mid v_a < v_{\text{gen}}\} \subseteq \sum_{i=1}^r (v_i - 1) \leq v_{\text{gen}} - r \leq v_{\text{gen}} - b$, which proves (b).

Let us assume now $v_a = v_{\text{min}}$. We have $v_a \geq \sum_{i=1}^r \# \{x \in l_i \mid f_i(x) = a\}$ since in every such point $x$ there is at least one branch at infinity of the fiber $f^{-1}(a)$. Note that if $f_i(x) = a$ then $\text{ord}_x(f_i - a) = \text{ord}_x f_i' + 1$. Thus:

$$\# \{x \in l_i \mid f_i(x) = a\} = \sum_{x \in l_i, f_i(x) = a} [\text{ord}_x(f_i - a) - \text{ord}_x f_i']$$

We have clearly the equality $\sum_{x \in l_i} \text{ord}_x(f_i - a) = v_i$. Hence

$$\sum_{x \in l_i, f_i(x) = a} [\text{ord}_x(f_i - a) - \text{ord}_x f_i'] = v_i - \sum_{x \in l_i, f_i(x) = a} \text{ord}_x f_i'.$$

Since $\sum_{x \in l_i} \text{ord}_x f_i' = v_i - 1$ we have:

$$v_i - \sum_{x \in l_i, f_i(x) = a} \text{ord}_x f_i' = 1 + \sum_{x \in l_i, f_i(x) \neq a} \text{ord}_x f_i'.$$
Note that:

\[ 1 + \sum_{x \in l_i, f_i(x) \neq a} \text{ord}_x f_i' \geq \# \{ x \in l_i \mid f(x) \neq a \}, \text{ and either } f_i'(x) = 0 \text{ or } x \in \text{Sing}(X'). \]

The number at the right side is greater or equal to the number of critical values at infinity of \( f \) different from \( a \). Finally, taking the sum over all \( i \in \{1, \ldots, r\} \) we get \( \#B_\infty(f) \leq v_{\min} + 1 \), which completes the proof of (a).

To prove (c), note that if the fiber \( \tilde{f}^{-1}(a) \) does not contain a singular point of \( X' \), which lies on some \( l_i \), then the intersection multiplicity \( \mathcal{T}_i \cdot f^{-1}(a) \) is equal to \( v_i = \text{deg} f_i \), where we consider here \( \tilde{f}^{-1}(a) \) as a scheme-theoretic fiber of \( \tilde{f} \). Hence the fiber \( \tilde{f}^{-1}(a) \) has at most \( v_i \) branches on \( l_i \) (even if counted with multiplicity). This implies \( v_a \leq v_{\text{gen}} \). Therefore \( \# \{ a \in \mathbb{C} \mid v_a > v_{\text{gen}} \} \leq r \leq v_{\min} \).

To prove (d) and (e) it is enough to show that if \( v_{\text{gen}} > \frac{d}{2} \), then at least one line \( l_i \) does not contain singular points of \( X' \). Let \( d_i \) be the smallest positive integer such that \( d_i/l_i \) is a Cartier divisor in \( X' \) (such a number exists because \( X' \) has only cyclic singularities). Since \( l_i \) is smooth, we have that \( d_i = 1 \) if and only if the line \( l_i \) does not contain any singular point of \( X' \), by the following lemma, the proof of which is left to the reader:

**Lemma 2.5** Let \( X^n \) be an algebraic variety and let \( Z^r \subset X^n \) be a subvariety which is a complete intersection in \( X^n \). If a point \( z \in Z^r \) is nonsingular on \( Z^r \), then it is nonsingular on \( X^n \). \( \square \)

Now let \( Z \) be the closure of \( \Gamma \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \) and let \( Z' \) denote its normalization. We have clearly the inclusion \( X' \subset Z' \). Let \( \Pi : Z' \to \mathbb{P}^2 \) the first projection, where the second projection \( Z' \to \mathbb{P}^1 \) is an extension of \( \tilde{f} \) which we will denote by \( \tilde{f}' \). Note that for \( a \neq \infty \) fibers \( \tilde{f}^{-1}(a) \) and \( (\tilde{f}')^{-1}(a) \) coincide.

Let \( (\tilde{f}')^{-1}(\infty) = S_1 \cup \cdots \cup S_k \) (where \( S_i \) irreducible and taken with reduced structure). Recall that \( L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2 \) is the line at infinity. We have \( \Pi^*(L_\infty) = \sum_{i=1}^k m_i S_i + \sum_{i=1}^r e_i \mathcal{T}_i \). Since \( \Pi^*(L_\infty) \) is a Cartier divisor we have \( e_i = n_i d_i \), where \( n_i \) is a positive integer.

Let us assume that every line \( l_i \) contains some singular point of \( X' \), i.e., that \( d_i > 1 \) for any \( i \). Denoting by \( F \subset \mathbb{P}^2 \) the closure of a general fiber of \( f \), since \( \Pi \) is a birational morphism, we have:

\[
 d = F \cdot L_\infty = \Pi^*(F) \cdot \Pi^*(L_\infty) = \left( \tilde{f}' \right)^* (a) \cdot \left( \sum_{i=1}^k m_i S_i + \sum_{i=1}^r e_i \mathcal{T}_i \right).
\]

Note that \( \Pi^*(F) \cdot \sum_{i=1}^k m_i S_i = 0 \) since \( |(\tilde{f}')^*(a) \cap \sum_{i=1}^k m_i S_i| = |(\tilde{f}')^*(a) \cap (\tilde{f}')^*(\infty)| = 0 \). Moreover we have \( v_i = (\tilde{f}')^*(a) \cdot \mathcal{T}_i \). Thus:

\[
 d = \sum_{i=1}^r n_i d_i v_i \geq \sum_{i=1}^r 2v_i = 2v_{\text{gen}}
\]

and this ends our proof. \( \square \)
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