A NOTE ON BAND-LIMITED MINORANTS OF AN EUCLIDEAN BALL

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Abstract. We study the Beurling-Selberg problem of finding band-limited \(L^1\)-functions that lie below the indicator function of an Euclidean ball. We compute the critical radius of the support of the Fourier transform for which such construction can have a positive integral.

1. Introduction

For a given \(r > 0\) we denote by \(B^d(r)\) the closed Euclidean ball in \(\mathbb{R}^d\) centered at the origin with radius \(r > 0\). We simply write \(B^d\) when \(r = 1\). Define the following quantity

\[
\beta(d, r) = \sup_F \int_{\mathbb{R}^d} F(x) dx,
\]

where the supremum is taken among functions \(F \in L^1(\mathbb{R}^d)\) such that:

1. The Fourier Transform of \(F(x)\),
\[
\hat{F}(\xi) = \int_{\mathbb{R}^d} F(x) e^{2\pi ix \cdot \xi} dx,
\]

is supported in \(B^d(r)\);

2. \(F(x) \leq 1_{B^d}(x)\) for all \(x \in \mathbb{R}^d\).

We call such a function \(\beta(d, r)\)-admissible. A trivial observation is that \(F \equiv 0\) is \(\beta(d, r)\)-admissible, hence \(\beta(d, r) \geq 0\). Heuristically, such function \(F(x)\) should exist and its mass should be close to \(\text{vol}(B^d)\) when \(r\) is large. On the other hand, if \(r\) is small, the mass of \(F(x)\) should be close to zero and a critical \(r_d > 0\) should exist such that no function can beat the identically zero function for \(r \leq r_d\). For this reason we define

\[
r_d = \inf\{r > 0 : \beta(d, r) > 0\}
\]

and it is this critical radius that we want to study in this paper.

The problem stated in (1.1) has its origins with Beurling and Selberg which studied one-sided band-limited approximations for many different functions other than indicator functions with the purpose of using them to derive sharp estimates in analytic number theory (see the introduction of [9] for a nice first view). Although Selberg was one of the first to study the higher dimensional problem, it was first
systematically analyzed by Holt and Vaaler in the remarkable paper [7]. They were able to construct non-zero \( \beta(d,r) \)-admissible functions for any \( r > 0 \) and, most importantly, they established a fascinating connection of the \( d \)-dimensional problem with the theory of Hilbert spaces of entire functions constructed by de Branges (see [1]). They reduced the higher dimensional problem, after a radialization argument, to a weighted one-dimensional problem where the weight was given by a special function of Hermite-Biehler class, which in turn allowed them to use the machinery of homogeneous de Branges spaces to attack the problem. This new connection established by Holt and Vaaler started a new way of thinking about these kind of problems and ultimately inspired Littmann to completely solve the one-dimensional problem in [8] by using a clever argument based on a special structure of certain de Branges spaces. Finally, using the ideas introduced by Littmann in [8], the problem of minorizing the indicator function of a symmetric interval was completely solved in [2] in the de Branges space setting.

This paper was mainly motivated by the related problem where balls are substituted by boxes \( Q(r) = [-r,r]^d \) and where practically nothing is known (see [3]). The box minorant problem is harder since it is a truly higher dimensional problem, whereas for the ball we can make radial reductions that transform it in a one-dimensional problem. Another interesting similar question, connected with upper bounds for sphere packings in \( \mathbb{R}^d \), is studied in [6] (see also [4]), where the author constructs a minorant \( F(x) \leq 1_{B^d}(x) \) with Fourier transform non-negative and supported in \( B^d(\pi d/2,1) \) and such that it maximizes \( \hat{F}(0) \) among all functions with these properties.

1.1. Main Result. For any given parameter \( \nu > -1 \) let \( J_\nu \) denote the classical Bessel function of the first kind. We also denote by \( \{ j_{\nu,n} \}_{n \geq 1} \) its positive zeros listed in increasing order. The Bessel function of the first kind \( J_\nu \) can be defined in a number of ways. We follow the treatise [10] and define it for \( \nu > -1 \) and \( \Re(z) > 0 \) by

\[
J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n}}{n! \Gamma(\nu + n + 1)}.
\]  

For these values of \( \nu \), one can check that on the half space \( \{ \Re(z) > 0 \} \) the Bessel functions defined by (1.2) satisfy the differential equation

\[
z^2J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0,
\]

and that the following recursion relations hold

\[
J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z),
\]

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1 It is not the intention of this paper to give a survey of related articles on the subject, which is very rich and full of subtleties, the purpose here is to draw a straight line between what he have so far and what we want to show.
\[ J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z). \]

In particular we have \( J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \) and \( J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \), which implies that \( j_{-1/2} = \frac{\pi}{2} \) and \( j_{1/2} = \frac{\pi}{2} \). The following is the main result of this paper.

**Theorem 1.** We have \( r_d = \frac{j_{d/2-1,1}}{\pi} \).

Moreover, if \( j_{d/2-1,1} < \pi r < j_{d/2,1} \) then

\[ \beta(d, r) = \frac{(2/r)^d}{|S^{d-1}|} \frac{\gamma_{\pi r}}{1 + \gamma_{\pi r}/d}, \]

where \( \gamma_{\pi r} = -\frac{\pi r J_{d/2-1}(\pi r)}{J_{d/2}(\pi r)} > 0 \). In particular we have

\[ \beta(d, r) = \frac{\pi^{2d}}{r^{d-1}|S^{d-1}|} \left( r - \frac{j_{d/2-1,1}}{\pi} \right) + O_d \left( r - \frac{j_{d/2-1,1}}{\pi} \right)^2 \]

for \( r \) close to \( \frac{j_{d/2-1,1}}{\pi} \).

**Remarks.**

(1) It is known that \( j_{\nu,1} = \nu + 1.855757 \nu^{1/3} + O(\nu^{-1/3}) \) as \( \nu \to \infty \) (see [5, Section 1.3]). This implies that \( r_d = \frac{d}{2\pi} + \frac{1.855757}{27/4} d^{1/3} + O(d^{-1/3}) \) as \( d \to \infty \).

Heuristically, this means that if one wishes to non-trivially minorate (that is, beat the zero function) the indicator function of a ball of radius of order \( \sqrt{d} \) then one needs frequencies of order at least \( \sqrt{d} \).

(2) The first 5 values of \( r_d \) rounded up to 4 significant digits are the following: \( r_1 = 1/2, r_2 = 0.7655, r_3 = 1, r_4 = 1.220 \) and \( r_5 = 1.431 \).

(3) Explicit expressions for \( \beta(d, r) \) can also be tracked from [2, Theorem 5], but they involve sums of Bessel functions evaluated at Bessel zeros that can be quite complicated to grasp. Moreover, this is the case only when \( \pi r \) is a zero of \( J_{d/2-1}(z) \) or \( J_{d/2}(z) \). If that is not the case, then writing a formula for \( \beta(d, r) \) becomes pointless, since it will involve zeros of more complicated functions related to Bessel functions and this is not the purpose here.

## 2. Proof of Theorem 1

**Step 1.** The first step is to reduce the higher dimensional by considering only radial functions. We can apply [7, Lemmas 18 and 19] to reduce the \( d \)-dimensional problem to the following weighted one-dimensional problem

\[ \beta(d, r) = \frac{|S^{d-1}|}{2} \sup_{F} \int_{\mathbb{R}} F(x)|x|^{d-1}dx, \]  

where \( |S^{d-1}| \) denotes the surface area of the unit sphere in \( \mathbb{R}^d \) and the supremum is taken among functions \( F \in L^1(\mathbb{R}, |x|^{d-1}dx) \) such that:
(1) \( F(x) \) is the restriction to the real axis of an even entire function \( F(z) \) of exponential type at most \( 2\pi r \), that is,
\[ |F(z)| \leq Ce^{2\pi r|\text{Im } z|}, \quad z \in \mathbb{C} \]
for some constant \( C > 0 \);

(2) \( F(x) \leq 1_{[-1,1]}(x) \) for all \( x \in \mathbb{R} \).

In this framework the problem becomes treatable with the theory of de Branges spaces of entire functions. The latter generalize the well known Paley-Wiener spaces by using weighted norms given by Hermite-Biehler functions. In what follows we briefly review the construction of a special family of de Branges spaces called homogeneous spaces which were introduced by de Branges (see [1, Section 50] and [7, Section 5]). We refer to [7, Section 3] for a brief description of the general theory and also to [1, Chapter 2] for the full theory.

**Step 2.** Let \( \nu > -1 \) be a parameter and consider the real entire functions \( A_\nu(z) \) and \( B_\nu(z) \) given by
\[
A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n}}{n!(\nu + 1)(\nu + 2)\ldots(\nu + n)} = \Gamma(\nu + 1) \left( \frac{1}{2}z \right)^{-\nu} J_\nu(z)
\]
and
\[
B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n+1}}{n!(\nu + 1)(\nu + 2)\ldots(\nu + n + 1)} = \Gamma(\nu + 1) \left( \frac{1}{2}z \right)^{-\nu} J_{\nu+1}(z).
\]
If we write
\[ E_\nu(z) = A_\nu(z) - iB_\nu(z), \]
then \( E_\nu(z) \) is a Hermite–Biehler function, that is, it satisfies the following fundamental inequality
\[ |E_\nu(z)| < |E_\nu(z)| \]
for all \( z \in \mathbb{C} \) with \( \text{Im } z > 0 \). It is also known that this function does not have real zeros, that \( E(iy) \in \mathbb{R} \) for all real \( y \) (that is, \( B_\nu(z) \) is odd and \( A_\nu(z) \) is even), that \( E_\nu(z) \) is of bounded type in the upper-half plane (that is, \( \log |E_\nu(z)| \) has a positive harmonic majorant in the upper-half plane) and \( E_\nu(z) \) is of exponential type 1. We also have that
\[ c|x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq C|x|^{2\nu+1} \]
for all \( |x| \geq 1 \) and for some \( c, C > 0 \). The homogeneous space \( \mathcal{H}(E_\nu) \) is then defined as the space of entire functions \( F(z) \) of exponential type at most 1 and such that
\[ \int_{\mathbb{R}} |F(x)|^2 |E_\nu(x)|^{-2} dx < \infty. \]

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2As an historical note, de Branges originally defined this space in another way but, in [7, Lemma 16], the authors showed that this is an equivalent definition.
Using standard asymptotic expansions for Bessel functions one can show that
\( A_\nu, B_\nu \notin \mathcal{H}(E_\nu) \). As a particular case, observing that \( E_{-1/2} = e^{-iz} \) we can deduce that \( \mathcal{H}(E_{-1/2}) \) coincides with the Paley-Wiener space of square integrable entire functions of exponential type at most 1.

These spaces are relevant to our problem since we have the following magical identity

\[
a_\nu \int_{\mathbb{R}} |F(x)|^2 |E_\nu(x)|^{-2} \, dx = \int_{\mathbb{R}} |F(x)|^2 |x|^{2\nu+1} \, dx \tag{2.2}
\]

for each \( F \in \mathcal{H}(E_\nu) \), where \( a_\nu = 2^{2\nu+1} \Gamma(\nu+1)^2/\pi. \) For our purposes we will need an identity analogous of (2.2), but which allow us to compute integrals instead of \( L^2 \)-norms. It can be derived as follows. Let \( F(z) \) be an entire function of exponential type at most 2 such that \( F(x) \leq 1_{[-t,t]}(x) \) for some \( t > 0 \) and \( F \in L^1(\mathbb{R}, |x|^{2\nu+1} \, dx). \)

Since \( G_n(x) = \left( \frac{\sin(x/n)}{x/n} \right)^n \) belongs to \( \mathcal{H}(E_\nu) \) for large \( n \) and it converges to 1 uniformly in compact sets as \( n \to \infty \), we have that \( 4G_n(x)^2 - F(x) \geq 0 \) for all real \( x \) (if \( n \) is large and even) and this function has exponential type at most 2. This implies that \( 4G_n(x)^2 - F(x) = H_n(x)E_n(x) \) for all \( z \in \mathbb{C} \), for some entire function \( H_n(z) \) of exponential type at most 1 (see [1 Theorem 13]). We conclude that \( H_n \in \mathcal{H}(E_\nu) \) and we obtain

\[
a_\nu \int_{\mathbb{R}} F(x) |E_\nu(x)|^{-2} \, dx = a_\nu \int_{\mathbb{R}} \left\{ 4G_n(x)^2 - |H_n(x)|^2 \right\} |E_\nu(x)|^{-2} \, dx
= \int_{\mathbb{R}} \left\{ 4G_n(x)^2 - |H_n(x)|^2 \right\} |x|^{2\nu+1} \, dx \tag{2.3}
= \int_{\mathbb{R}} F(x)|x|^{2\nu+1} \, dx.
\]

**Step 3.** Taking \( \nu = d/2 - 1 \), we can apply the change of variables \( x \mapsto x/(\pi r) \) in (2.1) and use identity (2.2) to reduce the problem of minorizing the indicator function of an Euclidean ball to the following final form

\[
\beta(d, r) = \frac{(2/r)^d}{\pi^{d/2-1}} \Lambda^r_{E_{d/2-1}}(\pi r),
\]

where

\[
\Lambda^r_{E_{d/2-1}}(\pi r) = \sup_F \int_{\mathbb{R}} F(x)|E_{d/2-1}(x)|^{-2} \, dx
\]

and the supremum is taken among functions \( F \in L^1(\mathbb{R}, |E_{d/2-1}(x)|^{-2} \, dx) \) such that:

1. \( F(x) \) is the restriction to the real axis of an even entire function \( F(z) \) of exponential type at most 2;
2. \( F(x) \leq 1_{[-\pi r, \pi r]}(x) \) for all \( x \in \mathbb{R} \).

The above problem was completely solved in [2]. By all the previous discussion in Step 2, we can apply [2 Theorem 5 (i) and (iv)] to the function \( E_{d/2-1}(z) \) (it actually can be applied to any \( E_\nu(z) \)) to derive that \( \pi r_d = j_{d/2-1, 1} \). Moreover, if
$jd_{d/2-1,1} < \pi r < jd_{d/2,1}$ then [2] Theorem 5 (iv) also give us that

$$\Lambda_{E_{d/2-1}}^- (\pi r) = \pi \gamma_{\pi r} \frac{1}{1 + \gamma_{\pi r}/d},$$

where $\gamma_{\pi r} = -\frac{\pi r J_{d/2-1}(\pi r)}{J_{d/2}(\pi r)} > 0$. A simple Taylor expansion leads to

$$\Lambda_{E_{d/2-1}}^- (\pi r) = \pi^2 r (\pi r - jd_{d/2-1,1}) + O((\pi r - jd_{d/2-1,1})^2)$$

and we finally obtain that

$$\beta(d, r) = \frac{\pi^2 2d}{\pi^d-1 |S^{d-1}|} \left( r - \frac{jd_{d/2-1,1}}{\pi} \right) + O\left( \frac{jd_{d/2-1,1}}{\pi} \right)^2.$$

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