Genuine high-dimensional entanglement, i.e. the property of having a high Schmidt number, constitutes a resource in quantum communication, overcoming limitations of low-dimensional systems. States with a positive partial transpose (PPT), on the other hand, are generally considered weakly entangled, as they can never be distilled into pure entangled states. This naturally raises the question, whether high Schmidt numbers are possible for PPT states. Volume estimates suggest that optimal, i.e. linear, scaling in local dimension should be possible, albeit without providing an insight into the possible slope. We provide the first explicit construction of a family of PPT states that achieves linear scaling in local dimension and we prove that random PPT states typically share this feature. Our construction also allows us to answer a recent question by Chen et al. on the existence of PPT states whose Schmidt number increases by an arbitrarily large amount upon partial transposition. Finally, we link the Schmidt number to entangled sub-block matrices of a quantum state. We use this connection to prove that quantum states invariant under partial transposition on the smaller of their two subsystems cannot have maximal Schmidt number. This generalizes a well-known result by Kraus et al. We also show that the Schmidt number of absolutely PPT states cannot be maximal, contributing to an open problem in entanglement theory.

I. INTRODUCTION

Entanglement is a cornerstone of quantum information theory and an important resource for quantum and private communication [1]. Current communication experiments are often based on entanglement between two degrees of freedom [2], i.e. qubits. This is despite the fact that through recent technological advances a growing number of high-dimensional quantum systems can be controlled [8–12]. Using such high-dimensional systems can for instance increase the resistance to noise [3–7] compared to low-dimensional implementations. However, for such improvements genuine high-dimensional entanglement is needed, as opposed to merely entanglement in a high-dimensional system. One natural measure to certify this dimensionality of entanglement is given by the Schmidt number [13]: This number is easily understood (a more formal definition will follow later) as the minimal local dimension of entangled systems needed to reproduce a specific state via local operations and classical communication. In other words, certifying a high Schmidt number shows that the state could not have arisen from low-dimensional quantum systems.

Another possible way to certify the dimensionality of entanglement in high-dimensional systems is given by the positive partial transpose (PPT) criterion [14]. Unfortunately, most entangled states have a positive partial transpose [15] and the PPT criterion cannot be used to certify the dimensionality of this entanglement. This gives rise to a natural question: Can high-dimensional entanglement occur at all in systems that are noisy enough to be PPT? By letting the local dimensions grow, examples of PPT states with increasingly high Schmidt number have been constructed in [18]. However, their Schmidt number scales only logarithmically in the local dimension. It also turns out that for systems of local dimension \( d = 3 \) the maximum Schmidt number cannot be reached by states which have a positive partial transpose [17] (this had been earlier conjectured in [16]).

In our article we study the above questions. The following is an outline of our results:

• In Section III we explicitly construct a general family of PPT states with local dimension \( d \) and Schmidt number scaling as \( d/4 \). As a byproduct, we also solve a recent conjecture [18, Conjecture 36] by exhibiting PPT states whose Schmidt number increases by an arbitrarily large amount upon partial transposition.

• In Section IV we investigate, whether Schmidt numbers scaling linear in the local dimension are generic among PPT states. We first comment on results from [18, §4.3] comparing the volume of the set of PPT states with the volume of the set of states with ‘not too large’ Schmidt number. This comparison implies that indeed most PPT states have Schmidt number scaling linearly in the local dimensions. We proceed by constructing a simple family of random states that exhibit this behaviour with, asymptotically, high probability.
Finally in Section IV we refine a method from [17] to study when PPT states in bipartite systems cannot reach maximal Schmidt number. For states invariant under a partial transposition on the smaller dimensional subystem we show that their Schmidt number cannot be maximal. This generalizes a well-known result from [20]. Finally, we find that the same conclusion is true for absolutely PPT states [21]. This contributes to related open problems in entanglement theory [22].

II. PRELIMINARIES AND NOTATION

Let us fix some notation that we will be using throughout our article. We will denote the set of \(d \times d\) matrices with complex entries by \(\mathcal{M}_d\), and the subcone of positive semidefinite (commonly referred to simply as positive) matrices by \(\mathcal{M}_d^+\). Occasionally, we will write \(X \geq 0\) to abbreviate \(X \in \mathcal{M}_d^+\) when the dimension of \(X\) is clear from context. The identity matrix will be denoted by \(\mathbb{1}\), and sometimes we will write \(\mathbb{1}_d\) to emphasize its dimension. Recall that the state of a quantum system is modelled by a positive matrix with unit trace (referred to as simply a quantum state).

A quantum state \(\rho_{AB}\) on \(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\) is called separable if it can be written as

\[
\rho_{AB} = \sum_{i=1}^{m} p_i \sigma_A^{(i)} \otimes \tau_B^{(i)}
\]

for some \(m \in \mathbb{N}\), a probability distribution \(\{p_i\}_{i=1}^{m}\), and quantum states \(\sigma_A^{(i)}\) on \(\mathbb{C}^{d_A}\) and \(\tau_B^{(i)}\) on \(\mathbb{C}^{d_B}\), \(1 \leq i \leq m\). Any quantum state that is not separable is called entangled.

To quantify different degrees of entanglement various measures of entanglement have been considered. Here we will focus on one such measure called the Schmidt number, first introduced in [13]. Given a pure state \(|\psi\rangle_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\) we can define its Schmidt rank as

\[
\text{SR}(|\psi\rangle_{AB}) := \text{rk}(\text{Tr}_A(|\psi\rangle_{AB}\langle\psi|_{AB}))
\]

where \(\text{Tr}_A\) denotes the partial trace over the first tensor factor and \(|\psi\rangle_{AB}\langle\psi|_{AB}\) is the rank-1 projector onto the span of \(|\psi\rangle_{AB}\). Now, given any quantum state \(\rho_{AB}\) on \(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\) we define its Schmidt number as

\[
\text{SN}(\rho_{AB}) := \min \left\{ \max_{1 \leq i \leq m} \text{SR}(|\psi^{(i)}\rangle_{AB}) : \sum_{i=1}^{m} p_i |\psi^{(i)}\rangle_{AB} \rho_{AB} = \rho_{AB} \right\},
\]

where the minimum is over decompositions of \(\rho_{AB}\) into convex combinations of rank-1 projectors, i.e. with a probability distribution \(\{p_i\}_{i=1}^{m}\) and pure states \(|\psi^{(i)}\rangle_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\), \(1 \leq i \leq m\). It is easy to see that \(\text{SN}(\rho_{AB}) \in \{1, \ldots, \min(d_A, d_B)\}\) with \(\text{SN}(\rho_{AB}) = 1\) iff \(\rho_{AB}\) is separable.

Note that the above definitions extend more generally to positive operators (i.e. unnormalized quantum states). Sometimes it will be convenient to drop the normalization and we will do so for some of our results.

B. Positive maps

There is a powerful duality relation between the set of positive operators with Schmidt number below some value and certain cones of positive maps. A linear map \(\mathcal{L} : \mathcal{M}_{d_A} \rightarrow \mathcal{M}_{d_B}\) is called

- **positive** if \(\mathcal{L}(X) \in \mathcal{M}_{d_B}^+\) for any \(X \in \mathcal{M}_{d_A}^+\).
• **k-positive** if $\text{id}_k \otimes \mathcal{L} : \mathcal{M}_k \otimes \mathcal{M}_d \to \mathcal{M}_k \otimes \mathcal{M}_d$ is positive.

• **completely positive** if it is k-positive for any $k \in \mathbb{N}$.

Recall that a linear map $\mathcal{L} : \mathcal{M}_{d_A} \to \mathcal{M}_{d_C}$ is completely positive iff its Choi matrix $C_{\mathcal{L}} \in \mathcal{M}_{d_C} \otimes \mathcal{M}_{d_B}$ given by

$$C_{\mathcal{L}} := (\mathcal{L}_A \otimes \text{id}_B)(\omega_{AB})$$

is positive (see [33]). Here $\omega_{AB} = |\Omega_{AB}\rangle\langle\Omega_{AB}|$, with $d_A = d_B$, is the maximally entangled state on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, i.e. the rank-1 projector on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ corresponding to the pure state

$$|\Omega\rangle_{AB} = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i\rangle_A \otimes |i\rangle_B.$$

A paradigmatic example of a positive map that is not completely positive is given by the transposition map $\vartheta_d : \mathcal{M}_d \to \mathcal{M}_d$, defined by $\vartheta_d(|i\rangle\langle j|) = |j\rangle\langle i|$ in the computational basis (here $|i\rangle\langle j|$ in $\mathcal{M}_{d}$ denote the matrix units having a single 1 in the $(i,j)$-entry). Recall that the Choi matrix of the transposition map is given by $C_{\vartheta_d} = F/d$ where $F \in \mathcal{M}_d \otimes \mathcal{M}_d$ denotes the flip operator defined as $F|ij\rangle = |ji\rangle$ in the computational basis.

It turns out [13, Theorem 1] that a quantum state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ satisfies $\text{SN}(\rho_{AB}) \leq n$ iff $(\text{id}_A \otimes P_B)(\rho_{AB}) \geq 0$ for any $n$-positive map $P : \mathcal{M}_{d_B} \to \mathcal{M}_{d_A}$. In particular it is separable iff $(\text{id}_A \otimes P_B)(\rho_{AB}) \geq 0$ for any positive map $P : \mathcal{M}_{d_B} \to \mathcal{M}_{d_A}$ (see [23]). It is a hard problem in general to decide whether a given state is separable [10] and the set of positive maps has a very complicated structure. However, if the dimensions are low enough positive maps have a simple characterization [11]: If $d_Ad_B \leq 6$ a linear map $\mathcal{L} : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ is positive iff

$$\mathcal{L} = T + \vartheta_d \circ T'$$

for completely positive maps $T, T' : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$. The structure of low dimensional positive maps implies that for $d_Ad_B \leq 6$ a quantum state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is separable iff its partial transpose is positive, i.e. $\rho^{T_A}_{AB} := (\text{id}_A \otimes \vartheta_B)(\rho_{AB}) \geq 0$. In higher dimensions this statement turns out to be false, and in general there exist quantum states on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ for $d_A d_B > 6$ that are entangled despite having positive partial transpose (see [23] for some examples). In the following we will refer to quantum states having positive partial transpose simply as PPT states.

The subset of positive maps $\mathcal{P} : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ satisfying (1) for completely positive maps $T, T' : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ are called decomposable. It turns out that a positive map $P$ is decomposable iff $(\text{id}_A \otimes P_B)(\rho_{AB}) \geq 0$ for any PPT state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ (see [42]). Conversely, this means that the entanglement of PPT states can only be detected by positive maps that are non-decomposable.

**C. What is the maximal Schmidt number of a PPT state?**

Recall that for dimensions $d_A, d_B \in \mathbb{N}$ satisfying $d_A d_B \leq 6$ the set of PPT states on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ coincides with the set of separable states [23], i.e. the states with Schmidt number 1. Motivated by this result, it has been conjectured in [16] that for $d_A = d_B = 3$ the Schmidt number of any PPT state is less than 2. Recently this conjecture has been proven by Yang et al. [17]. Thus, it is a natural question to ask how large the Schmidt number $\text{SN}(\rho_{AB})$ of a PPT state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be, for fixed local dimensions $d_A, d_B > 3$. We will be particularly interested in the case where $d_A = d_B$.

A partial answer to this question has been given in the asymptotic setting [19, §4.3] by comparing the volumes of the sets of PPT states and states of Schmidt number below a given value. Specifically, it has been shown that for sufficiently large $d = d_A = d_B$ there exist PPT states $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ with $\text{SN}(\rho_{AB}) \geq \alpha d$ for some constant $\alpha > 0$ that is independent of the dimension $d$. It should be noted that the constant $\alpha$ in these results is hard to compute exactly, and probably quite small. Moreover, these methods do not yield explicit examples of PPT states exhibiting such Schmidt numbers.

Recently, an explicit construction of a PPT state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ with $d = d_A = d_B$ and Schmidt number SN$(\rho_{AB}) \sim \log d$ has been given in [18, Proposition 20]. This is still far away from the linear dependence observed using volume estimates. A simple argument achieving a power law scaling SN$(\rho_{AB}) \sim d^\gamma$ can be given using any convex, faithful and additive entanglement measure. For simplicity we will only state this argument for the squashed entanglement: Given a quantum state $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ its squashed entanglement is defined as [24, 25]

$$E_{sq}(\rho_{AB}) := \inf \left\{ \frac{1}{2} I(A : B|C)_\sigma : d_C \in \mathbb{N}, \sigma_{ABC} \in (\mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B} \otimes \mathcal{M}_{d_C})^+, \text{Tr}_Cting the conditional mutual information of $\sigma_{ABC}$. It has been shown that the squashed entanglement satisfies

$$I(A : B|C)_\sigma := S(AC)_\sigma + S(BC)_\sigma - S(ABC)_\sigma - S(C)_\sigma$$

where $I(A : B|C)_\sigma := S(AC)_\sigma + S(BC)_\sigma - S(ABC)_\sigma - S(C)_\sigma$ denotes the conditional mutual information of $\sigma_{ABC}$. It has been shown that the squashed entanglement satisfies

$$E_{sq}(\rho_{AB}) := \inf \left\{ \frac{1}{2} I(A : B|C)_\sigma : d_C \in \mathbb{N}, \sigma_{ABC} \in (\mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B} \otimes \mathcal{M}_{d_C})^+, \text{Tr}_C(\sigma_{ABC}) = \rho_{AB} \right\}$$

where $I(A : B|C)_\sigma := S(AC)_\sigma + S(BC)_\sigma - S(ABC)_\sigma - S(C)_\sigma$ denotes the conditional mutual information of $\sigma_{ABC}$. It has been shown that the squashed entanglement satisfies
1. \( E_{\text{sq}}(\rho_{AB}) > 0 \) iff \( \rho_{AB} \) is entangled.

2. \( E_{\text{sq}}(\rho_{AB}^n) = nE_{\text{sq}}(\rho_{AB}) \) for any \( n \in \mathbb{N} \).

3. \( 2E_{\text{sq}}(\rho_{AB}) \leq \text{SN}(\rho_{AB}) \).

See [20] for a proof of the first, and [25, Proposition 4] for a proof of the second property. The third property is a simple consequence of the convexity of squashed entanglement (see [25, Proposition 3]). For any entangled PPT state \( \rho_{AB} \) on \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) combining these properties of squashed entanglement yields

\[
\text{SN}(\rho_{AB}^n) \geq 2^n E_{\text{sq}}(\rho_{AB}) \quad \xrightarrow{n \rightarrow \infty} 0.
\]

Therefore, setting \( d_A = d_B \geq 3 \) and defining

\[
\gamma = \frac{E_{\text{sq}}(\rho_{AB})}{\log(d_A)} > 0
\]

yields a PPT state \( \sigma_{A^*B^n} = \rho_{AB}^n \) with local dimension \( d = d_A^3 \) and Schmidt number \( \text{SN}(\sigma_{A^*B^n}) \geq d^3 \). Note that \( 0 < \gamma \leq 1 \), and it is not known whether \( \gamma = 1 \) can be achieved using PPT states.

In Section III we will present a construction of PPT states \( \rho_{AB} \) on \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) with \( d = d_A = d_B \) and Schmidt numbers \( \text{SN}(\rho_{AB}) \geq \lceil (d-1)/4 \rceil \), where \( \lceil x \rceil := \min \{ n \in \mathbb{Z} : n \geq x \} \) is the ceiling function. This is the first explicit construction of a family of PPT states achieving a Schmidt number that scales linearly in the local dimension. We leave it as an open problem to determine the best possible Schmidt number for PPT states, with any given local dimensions.

### III. Explicit Constructions of PPT States with High Schmidt Number

In this section we present an explicit construction of a family of PPT states whose Schmidt number scales linearly in the local dimension. We will start with a general construction and then discuss the properties of specific examples. It should be emphasized that our family contains examples answering a question recently posed by Chen et al. [18, Conjecture 36]: Is there a PPT state \( \rho \) with an arbitrary large difference \( \text{SN}(\rho) - \text{SN}(\rho^\Gamma) \)? We will discuss such an example in the second subsection.

#### A. A Family of States which are PPT and have High Schmidt Number

Let us start with a basic lemma introducing the general form of the states we want to consider.

**Lemma III.1.** Consider an operator \( Z \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2} \otimes \mathcal{M}_{d_2} \) of the form

\[
Z_{A_1B_1A_2B_2} = X_{A_1B_1} \otimes (1 - \omega)_{A_2B_2} + Y_{A_1B_1} \otimes \omega_{A_2B_2},
\]

where \( X, Y \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2} \) and where we label the first two tensor factors (of dimension \( d_1 \)) as \( A_1, B_1 \) and the second two tensor factors (of dimension \( d_2 \)) as \( A_2, B_2 \). Then \( Z_{A_1B_1A_2B_2} \) is:

(a) positive semidefinite iff \( X, Y \geq 0 \);

(b) PPT with respect to the bipartition \( A_1A_2 : B_1B_2 \) iff \( (d_2 - 1)X^\Gamma \geq -Y^\Gamma \) and \( (d_2 + 1)Y^\Gamma \geq Y^\Gamma \).

**Proof.** Claim (a) is clear. To prove claim (b) we compute

\[
Z_{A_1B_1A_2B_2}^{\Gamma_{B_1B_2}} = X_{A_1B_1}^{\Gamma_{B_1B_2}} \otimes \left( 1 - \frac{F}{d_2} \right)_{A_2B_2} + Y_{A_1B_1}^{\Gamma_{B_1B_2}} \otimes \left( \frac{F}{d_2} \right)_{A_2B_2}
\]

\[
= \left( X + \frac{Y - X}{d_2} \right)_{A_1B_1} \otimes \left( 1 + \frac{F}{d_2} \right)_{A_2B_2} + \left( X - \frac{Y - X}{d_2} \right)_{A_1B_1} \otimes \left( \frac{1 - F}{d_2} \right)_{A_2B_2},
\]

which is positive iff the stated conditions hold. \( \square \)

The previous lemma characterizes the cases where a matrix \( Z_{A_1B_1A_2B_2} \) of the form (2) is positive and PPT (i.e. an unnormalized PPT state). To characterize the entanglement of these states the following lemma will be useful.
Let $X, Y \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ be two Hermitian operators such that
\begin{equation}
\exists |\alpha\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1} : \langle \alpha|X|\alpha\rangle = 0 \text{ and } \langle \alpha|Y|\alpha\rangle > 0 .
\end{equation}

Let $Z_{A_1B_1A_2B_2} \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2} \otimes \mathcal{M}_{d_2}$ denote the matrix from (2) with these $X, Y$. Then, for any $d'_2 \in \mathbb{N}$, any linear map $\mathcal{L} : \mathcal{M}_{d_2} \to \mathcal{M}_{d'_2}$ that is not completely positive satisfies
\begin{equation}
(id_{A_1} \otimes id_{B_1} \otimes \mathcal{L}_{A_2} \otimes id_{B_2})(Z_{A_1B_1A_2B_2}) \not\geq 0 .
\end{equation}

**Proof.** Note that
\begin{equation}
(id_{A_1} \otimes id_{B_1} \otimes \mathcal{L}_{A_2} \otimes id_{B_2})(Z_{A_1B_1A_2B_2}) = X_{A_1B_1} \otimes (\mathcal{L}(1) \otimes 1 - C_{2B_2})_{A_2B_2} + Y_{A_1B_1} \otimes (C_{2A_2})_{A_2B_2} .
\end{equation}

Since $\mathcal{L}$ is not completely positive we have that the Choi matrix $C_{\mathcal{L}}$ is not positive semidefinite, i.e. there exists $|\beta\rangle$ such that
\begin{equation}
\langle \beta|C_{\mathcal{L}}|\beta\rangle < 0 .
\end{equation}

With the vector $|\alpha\rangle$ from the assumptions of the lemma we get
\begin{equation}
(\langle \alpha|A_1B_1 \otimes \langle \beta|A'_2B_2\rangle)(id_{A_1} \otimes id_{B_1} \otimes \mathcal{L}_{A_2} \otimes id_{B_2})(Z_{A_1B_1A_2B_2})(\langle \alpha|A_1B_1 \otimes |\beta\rangle A'_2B_2) = \langle \alpha|Y|\alpha\rangle \langle \beta|C_{\mathcal{L}}|\beta\rangle < 0 .
\end{equation}

With the previous lemmas we can now present the main result of this section.

**Theorem III.1** (Constructing PPT states with high Schmidt number). Let $X, Y \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ be such that
\begin{enumerate}
  
  \item[(i)] $X, Y \geq 0$;
  
  \item[(ii)] $(d_2 - 1)X^T \geq -Y^T$ and $(d_2 + 1)X^T \geq Y^T$;
  
  \item[(iii)] $\exists |\alpha\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1} : \langle \alpha|X|\alpha\rangle = 0$ and $\langle \alpha|Y|\alpha\rangle > 0$.
\end{enumerate}

Then the operator $Z_{AB} = Z_{A_1B_1A_2B_2}$ (with joint labels $A = A_1A_2$ and $B = B_1B_2$) defined in (2) is positive, PPT and satisfies
\begin{equation}
\text{SN}(Z_{AB}) \geq \left\lfloor \frac{d_2}{d_1} \right\rfloor .
\end{equation}

**Proof.** By conditions (i) and (ii), applying Lemma [III.1] shows that $Z_{AB}$ is positive and PPT.

We can assume without loss of generality that $2d_1 \leq d_2$ since otherwise the statement about the Schmidt number becomes trivial. Consider the Choi map $\mathcal{P} : \mathcal{M}_{d_2} \to \mathcal{M}_{d_2}$ given by
\begin{equation}
\mathcal{P} := 1_{d_2} \text{Tr} - \frac{1}{d_2 - 1} \text{id}_{d_2} .
\end{equation}

It is well known (see [39]) that $\mathcal{P}$ is $(d_2 - 1)$-positive, but not completely positive. Applying Lemma [III.2] with $d'_2 = d_2$ and $\mathcal{L} = \mathcal{P}$ shows that
\begin{equation}
(id_{A_1} \otimes id_{B_1} \otimes \mathcal{P}_{A_2} \otimes id_{B_2})(Z_{A_1B_1A_2B_2}) \not\geq 0 .
\end{equation}

Since the map $id_{A_1} \otimes \mathcal{P}_{A_2}$ is [$(d_2 - 1)/d_1$]-positive we find that, with respect to the bipartition $A : B = A_1A_2 : B_1B_2$,
\begin{equation}
\text{SN}(Z_{AB}) \geq \left\lfloor \frac{d_2 - 1}{d_1} \right\rfloor + 1 = \left\lfloor \frac{d_2}{d_1} \right\rfloor .
\end{equation}

We have not shown so far that there actually are operators $X, Y \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ satisfying conditions (i),(ii) and (iii) in Theorem [III.1]. A simple construction of such operators is shown in the following theorem.
Theorem III.2 (Concrete family of PPT states with high Schmidt number). Let \( d_1, d_2 \in \mathbb{N} \) with \( d_1 \leq d_2 \) and let \( X, Y \in M_{d_1} \otimes M_{d_1} \) be given by
\[
X_{A_1B_1} = (1 - \omega)_{A_1B_1} \quad \text{and} \quad Y_{A_1B_1} = (d_1 - 1)(d_2 + 1)\omega_{A_1B_1}.
\]
Then the operator \( Z_{AB} = Z_{A_1B_1A_2B_2} \) (with joint labels \( A = A_1A_2 \) and \( B = B_1B_2 \)) defined in (2) is positive, PPT and satisfies
\[
\text{SN}(Z_{AB}) \geq \left\lceil \frac{d_2}{d_1} \right\rceil.
\]

Proof. By a straightforward computation the operators \( X, Y \) satisfy conditions (i),(ii) of Theorem III.1 and condition (iii) follows by choosing \( |\alpha\rangle = |\Omega\rangle \). \( \square \)

The previous theorem provides a concrete family of PPT states with high Schmidt number. Note that these states appeared previously in [27, 28] and [29]. However, in these works their Schmidt number is not estimated.

To further clarify the scaling of the Schmidt number with the local dimension we state the following simple corollary.

Corollary III.3. For any \( d \in \mathbb{N} \) with \( d \geq 2 \) there exists a PPT state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \) satisfying
\[
\text{SN} (\rho) \geq \left\lceil \frac{d-1}{4} \right\rceil.
\]

Proof. If \( d = 2d_2 \) we can directly apply Theorem III.2 with \( d_1 = 2 \). If \( d = 2d_2 + 1 \) again apply Theorem III.2 for dimensions \( d_2 \) and \( d_1 = 2 \) to obtain \( Z_{AB} \in M_{2d_2} \otimes M_{2d_2} \), which we can embed into the higher dimension by setting \( Z_{AB} = Z_{AB} \otimes 0 \). \( \square \)

The above Corollary III.3 constitutes an improvement over previous results, in particular over the explicit examples of [18] (\( \text{SN} \sim \log d \)) and over the implicit construction of [19, §4.3] (\( \text{SN} \sim \alpha d \) for some unknown universal constant \( \alpha \)).

To conclude this section we will briefly discuss some consequences of the above constructions to the theory of positive maps between matrix algebras. The following corollary has previously been obtained in [28]. We restate it here together with a slightly different proof, since it is not as well known as it should be.

Corollary III.1 (Non-decomposability from tensoring with identity [28]). Consider \( k, d, d' \in \mathbb{N} \) with \( k \geq 2 \). For any linear map \( L : M_d \to M_{d'} \) that is not completely positive, the linear map \( \text{id}_k \otimes L \) is not decomposable.

Proof. Note that for \( k \geq d \) the map \( \text{id}_k \otimes L \) cannot be positive, and hence it is not decomposable. Therefore, we can assume that \( k < d \). Consider the operator \( Z_{AB} = Z_{A_1B_1A_2B_2} \in M_k \otimes M_k \otimes M_d \otimes M_d \) obtained from Theorem III.2 (i.e., setting \( d_1 = k \) and \( d_2 = d \)). By construction \( Z_{AB} \) is positive and PPT with respect to the bipartition \( A : B = A_1A_2 : B_1B_2 \). Now applying Lemma III.2 for the linear map \( L \) shows that
\[
(\text{id}_{A_1} \otimes \text{id}_{B_1} \otimes L_{A_2} \otimes \text{id}_{B_2})(Z_{A_1B_1A_2B_2}) \not\geq 0.
\]
By [42] this shows that \( \text{id}_{A_1} \otimes L_{A_2} = \text{id}_k \otimes L \) is not decomposable. \( \square \)

The previous corollary gives an easy method to construct non-decomposable positive maps. Starting from an \( n \)-positive map \( P : M_{d_1} \to M_{d_2} \) that is not completely positive, the map \( \text{id}_k \otimes P \) for \( k \leq n \) will be positive and not decomposable. Any state from the general family constructed in Theorem III.1 will serve as a witness (in the sense of [42]) for this property.

B. Large variation of Schmidt number under partial transposition

In this section we will answer the following question, recently raised in the literature [18, Conjecture 36]: Are there PPT states \( \rho \) with an arbitrarily large difference \( \text{SN}(\rho) - \text{SN}(\rho^T) \)? With the following theorem we demonstrate that this difference can be arbitrarily large as the local dimension grows.
Theorem III.4. For any \( d_1, d_2 \in \mathbb{N} \) with \( d_2 \geq d_1 \) the operator \( Z_{AB} = Z_{A_1B_1A_2B_2} \in M_{d_1} \otimes M_{d_2} \otimes M_{d_1} \otimes M_{d_2} \) from Theorem III.3 is positive, PPT and satisfies
\[
\text{SN}(Z_{AB}^\Gamma) \leq 4, 
\]
and hence
\[
\text{SN}(Z_{AB}) - \text{SN}(Z_{AB}^\Gamma) \geq \left\lceil \frac{d_2}{d_1} \right\rceil - 4. 
\]

Proof. By Theorem III.2 the operator \( Z_{AB} \) is positive, PPT and satisfies
\[
\text{SN}(Z_{AB}) \geq \left\lceil \frac{d_2}{d_1} \right\rceil.
\]
Taking the partial transpose with respect to the \( B \) system we obtain
\[
Z_{AB}^\Gamma = \mathbb{I}_{A_1B_1} \otimes \mathbb{I}_{A_2B_2} - \frac{1}{d_2} \mathbb{I}_{A_1B_1} \otimes F_{A_2B_2} - \frac{1}{d_1} F_{A_1B_1} \otimes \mathbb{I}_{A_2B_2} + \frac{d_1 d_2 + d_1 - d_2}{d_1 d_2} F_{A_1B_1} \otimes F_{A_2B_2}.
\]
The four summands appearing in the above expression are mutually commuting operators, and therefore it is easy to diagonalise \( Z_{AB}^\Gamma \). The eigenvectors of \( Z_{AB}^\Gamma \) are of the form
\[
|\psi_{jkm}\rangle := |\chi_{jk}\rangle_{A_1B_1} \otimes |\chi_{lm}\rangle_{A_2B_2},
\]
with \( 1 \leq j, k \leq d_1, 1 \leq l, m \leq d_2 \), and where
\[
|\chi_{pq}\rangle := \begin{cases} 
\frac{\sqrt{2}}{2} (|pq\rangle + |qp\rangle) & \text{if } p < q, \\
|pp\rangle & \text{if } p = q, \\
\frac{\sqrt{2}}{2} (|pq\rangle - |qp\rangle) & \text{if } p > q.
\end{cases}
\]
It is easy to see that the Schmidt rank of each vector \( |\psi_{jkm}\rangle \) is upper bounded by 4 with respect to the bipartition \( A : B = A_1A_2 : B_1B_2 \). Since \( Z_{AB}^\Gamma \) is positive semidefinite this shows that
\[
\text{SN}(Z_{AB}^\Gamma) \leq 4.
\]

\[ \square \]

We obtain the following immediate corollary.

Corollary III.2. For any \( d \in \mathbb{N} \) with \( d \geq 2 \) there exist a PPT entangled state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \) satisfying
\[
\text{SN}(\rho) - \text{SN}(\rho^\Gamma) \geq \left\lceil \frac{d - 1}{4} \right\rceil - 4.
\]

Proof. The statement follows by combining Theorem III.4 with Corollary III.3 \[ \square \]

C. States which are invariant under partial transpose and have high Schmidt number

The results obtained in the previous subsection show that the Schmidt numbers of the states constructed in Theorem III.3 can change dramatically by applying a partial transposition. It is therefore a natural question to ask what values the Schmidt number can take on states that are invariant under partial transposition. Note for a bipartite state there are two possibilities to define invariance under partial transposition depending on which subsystem we choose the transposition to act on. Since we will aim at states on \( \mathbb{C}^d \otimes \mathbb{C}^d \) (i.e. where the two subsystems have the same dimension) we will use the term PT-invariant and it will be clear from context which of the two possibilities we choose.

In Section IV we show that PT-invariant states, where the transposition is taken on the smaller subsystem, the Schmidt number cannot reach the highest possible value. In this section, we show that nevertheless there do exist states that are invariant under partial transposition and have a Schmidt number which scales linearly with the local dimension.
In the following we denote by $| + i⟩, | - i⟩ \in \mathbb{C}^2$ the pure states

$$| + i⟩ = \frac{1}{\sqrt{2}}(|0⟩ + i|1⟩) \text{ and } | - i⟩ = \frac{1}{\sqrt{2}}(|0⟩ - i|1⟩)$$

where $\{0, 1\}$ denotes the computational basis of $\mathbb{C}^2$ (the standard basis for our transposition). Note that $| + i⟩$ and $| - i⟩$ are orthonormal. The following lemma is probably well known.

**Lemma III.1.** Given a PPT state $ρ_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, the state $\tilde{ρ}_{ABB'}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^2$ given by

$$\tilde{ρ}_{ABB'} = ρ_{AB} \otimes | + i⟩⟨ + i|_{B'} + ρ_{AB}^{Γ_B} \otimes | - i⟩⟨ - i|_{B'}$$

is invariant under partial transposition on $BB'$. 

**Proof.** Note that

$$Θ(| + i⟩⟨ + i|) = | - i⟩⟨ - i|$$

And it thus easily follows that

$$\tilde{ρ}_{ABB'}^{Γ_{BB'}} = ρ_{AB}^{Γ_B} \otimes Θ(| + i⟩⟨ + i|_{B'}) + ρ_{AB} \otimes Θ(| - i⟩⟨ - i|_{B'}) = \tilde{ρ}_{ABB'}$$

With the previous lemma we can easily modify the family of states constructed in Theorem III.1 to make it invariant under partial transposition. To keep the discussion simple we will only state the result on the maximal Schmidt number that can be obtained using this construction.

**Theorem III.3.** For any $d \geq 4$ there exists a PT-invariant state $ρ$ on $\mathbb{C}^d \otimes \mathbb{C}^d$ satisfying

$$SN(ρ) ≥ \left\{ \begin{array}{ll} \left\lceil \frac{d-2}{8} \right\rceil, & \text{if } d \text{ is even,} \\ \left\lceil \frac{d-3}{8} \right\rceil, & \text{else.} \end{array} \right.$$ 

**Proof.** Consider first the case where $d$ is even. For $d' = d/2$ consider the PPT state $σ_{AB}$ on $\mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$ constructed in Corollary III.3 with

$$SN(σ_{AB}) ≥ \left\lceil \frac{d'-1}{4} \right\rceil.$$

By Lemma III.1 the state

$$\tilde{σ}_{ABB'} = σ_{AB} \otimes | + i⟩⟨ + i|_{B'} + σ_{AB}^{Γ_B} \otimes | - i⟩⟨ - i|_{B'},$$

on $\mathbb{C}^{d'} \otimes \mathbb{C}^{d'} \otimes \mathbb{C}^2$ is invariant under partial transposition on $BB'$. Since the Schmidt number is decreasing under the application of local completely positive maps (see [13]), and

$$σ_{AB} = (I_{AB} \otimes ⟨ + i|_{B'})σ_{ABB'} (I_{AB} \otimes | + i⟩_{B'})$$

we have that, with respect to the bipartition $A : BB'$,

$$SN(\tilde{σ}_{ABB'}) ≥ \left\lceil \frac{d'-1}{4} \right\rceil.$$

Now in order to make the two local dimensions equal we just consider the state

$$ρ_{AA'BB'} = \tilde{σ}_{ABB'} \otimes |0⟩⟨0|_{A'}$$

on $\mathbb{C}^{d'} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{d'} \otimes \mathbb{C}^2$. With respect to the bipartition $AA' : BB'$, we have

$$SN(ρ_{AA'BB'}) ≥ \left\lceil \frac{d'-1}{4} \right\rceil.$$

Since $d = 2d'$ the statement of the theorem follows.

In the case where $d$ is odd we can consider the state $ρ_{AA'BB'}$ on $\mathbb{C}^{d'-1} \otimes \mathbb{C}^{d'-1}$ obtained from the above construction. Then the embedding $ρ_{AA'BB'} ⊕ 0$ of this state into $M_d ⊗ M_d$ (extending each of the local dimensions by 1) satisfies the Schmidt number bound stated in the theorem.

$\square$
IV. TYPICAL SCHMIDT NUMBER OF PPT STATES IN HIGH DIMENSIONS

Having constructed examples of PPT states requiring high dimensions it is now natural to ask: Is this phenomenon generic? In other words we want to find out whether PPT states with high Schmidt numbers are just an oddity that may physically not be relevant. We will approach this question from two angles: First, we compare the relative volumes occupied by PPT states and states with bounded Schmidt number respectively in the set of all states. Then, we derive probabilities for random states to be both PPT and of high Schmidt number.

To start this section, let us fix some further notation. For any $1 \leq k \leq d$, we denote by $\text{SN}_k(\mathbb{C}^d : \mathbb{C}^d)$ the set of states on $\mathbb{C}^d \otimes \mathbb{C}^d$ which have Schmidt number at most $k$. In particular, $\text{SN}_1(\mathbb{C}^d : \mathbb{C}^d)$ is just the set of separable states and $\text{SN}_d(\mathbb{C}^d : \mathbb{C}^d)$ is the set of all states on $\mathbb{C}^d \otimes \mathbb{C}^d$. We also denote by $\text{PPT}(\mathbb{C}^d : \mathbb{C}^d)$ the set of states on $\mathbb{C}^d \otimes \mathbb{C}^d$ which are PPT.

A. Sizes of the sets of bounded Schmidt number states vs PPT states

To begin with, we would like to know how the sizes of the sets $\text{SN}_k(\mathbb{C}^d : \mathbb{C}^d)$ and $\text{PPT}(\mathbb{C}^d : \mathbb{C}^d)$ compare to one another. Here, we are interested in the high-dimensional regime, i.e. when $d$ is large. There are two size parameters that we will be looking at: the mean width and the volume radius. Informally, given a convex body $K$, its mean width is defined as the distance between an origin point and the tangent hyperplane to $K$ in some direction, averaged over all directions, while its volume radius is defined as the radius of the Euclidean ball which would have the same volume as $K$. The precise mathematical definitions appear below.

The mean width is defined as follows: Given $K(\mathbb{C}^n)$ a convex set of states on $\mathbb{C}^n$,

$$w(K(\mathbb{C}^n)) := \mathbb{E} \sup \{ \text{Tr}(X\sigma) : \sigma \in K(\mathbb{C}^n) \},$$

for $X$ uniformly distributed on the Hilbert–Schmidt unit sphere of the set of trace-0 Hermitian operators on $\mathbb{C}^n$. Equivalently, we can rewrite

$$w(K(\mathbb{C}^n)) = \frac{1}{\mathbb{E} \|G\|_2} \mathbb{E} \sup \{ \text{Tr}(G\sigma) : \sigma \in K(\mathbb{C}^n) \} \sim \frac{1}{n} \mathbb{E} \sup \{ \text{Tr}(G\sigma) : \sigma \in K(\mathbb{C}^n) \},$$

for $G$ a trace-0 matrix from the Gaussian Unitary Ensemble (GUE) on $\mathbb{C}^n$ (see e.g. [30, Chapter 2] for a proof of the last asymptotic estimate). Since we will be manipulating repeatedly such random matrix in the remainder of this section, let us recall here its precise definition: Start from $\tilde{G}$ an $n \times n$ matrix whose entries are independent complex Gaussian variables (with mean 0 and variance 1). Then, $G = (\tilde{G} + G^\dagger)/\sqrt{2}$ is an $n \times n$ GUE matrix and $G = G' - (\text{Tr}G')\mathbb{I}/n$ is a trace-0 $n \times n$ GUE matrix. In other words, $G'$, resp. $G$, is simply the standard Gaussian vector in the space of Hermitian operators, resp. trace-0 Hermitian operators, on $\mathbb{C}^n$.

The volume radius is defined as follows: Given $K(\mathbb{C}^n)$ a convex set of states on $\mathbb{C}^n$,

$$\text{vrad}(K(\mathbb{C}^n)) := \left( \frac{\text{Vol}(K(\mathbb{C}^n))}{\text{Vol}(B^{n^2-1})} \right)^{1/(n^2-1)},$$

where $B^{n^2-1}$ stands for the Hilbert–Schmidt unit ball of the set of trace-1 Hermitian operators on $\mathbb{C}^n$ (which can be identified with the real Euclidean unit ball of dimension $n^2 - 1$) and $\text{Vol}(\cdot)$ for the $(n^2 - 1)$-dimensional Lebesgue measure.

By Urysohn’s inequality (see e.g. [34, Corollary 1.4]), we know that

$$\text{vrad}(K(\mathbb{C}^n)) \leq w(K(\mathbb{C}^n)),$$

and in many cases, these two quantities are actually of the same order.

The mean width and the volume radius of the sets of bounded Schmidt number states and PPT states were estimated in [19] and [17], respectively. We will restate these results for completeness:

**Theorem IV.1** (Size of the set of bounded Schmidt number states [19, Section 4]). There exist universal constants $c, C > 0$ such that, for any $d \in \mathbb{N}$ and $1 \leq k \leq d$, we have

$$c \frac{\sqrt{k}}{d^{\sqrt{d}}} \leq \text{vrad}(\text{SN}_k(\mathbb{C}^d : \mathbb{C}^d)) \leq w(\text{SN}_k(\mathbb{C}^d : \mathbb{C}^d)) \leq C \frac{\sqrt{k}}{d^{\sqrt{d}}}.$$
and define the associated ‘maximally mixed + Gaussian noise’ state on C⊗Cd as defined by equation (9). Let α small enough (depending on the unknown constants c and C) the theorems show that for d large enough most PPT states on C⊗Cd have Schmidt numbers higher than αd.

B. Random construction of PPT states with high Schmidt number

Let us now explain how one could exhibit bipartite quantum states which have both properties of being PPT and having a high Schmidt number. These states will be constructed at random, in such a way that one can argue that, with high probability, they meet these two conditions simultaneously. Our random state model is basically the same as the one considered in (33, Section V), which we recall here. Let G be a trace-0 GUE matrix on C⊗Cd, and define the associated ‘maximally mixed + Gaussian noise’ state on C⊗Cd as

$$\rho := \frac{1}{d^2} \left(1 + \frac{\alpha}{d} G\right)$$ ,

(9)

where 0 < α < 1/2 is a fixed parameter.

Theorem IV.3. Let ρ be a random state on C⊗Cd, as defined by equation (9). Then,

$$\mathbb{P} \left(\rho \in \text{PPT}(\mathbb{C}^d : \mathbb{C}^d) \right) \geq 1 - 2e^{-c_\alpha d^2}$$ ,

where c_\alpha > 0 is a constant depending only on the parameter α.

Proof. The argument is exactly the same as in the proof of (33, Proposition V.5). We briefly repeat it here for the sake of completeness. G and G^T are both trace-0 GUE matrices on C⊗Cd. Hence, we know from (31), that for H being either G or G^T, we have

$$\forall \varepsilon > 0, \mathbb{P}(\lambda_{\text{min}}(H) < -(2 + \varepsilon)) \leq e^{-c_\varepsilon d^2}$$ .

Applying this deviation probability estimate to ε = 1/α − 2 > 0, we get by the union bound that

$$\mathbb{P}(\rho \geq 0 \text{ and } \rho^T \geq 0) \geq 1 - 2e^{-c(1/\alpha-2)^3/2d^2}$$ ,

which is precisely the advertised result.

Theorem IV.4. Let ρ be a random state on C⊗Cd, as defined by equation (9). Then, for any 1 ≤ k ≤ c'_\alpha d, we have

$$\mathbb{P} \left(\rho \notin \text{SN}_k(\mathbb{C}^d : \mathbb{C}^d) \right) \geq 1 - 2e^{-cd}$$ ,

where c'_\alpha > 0 is a constant depending only on the parameter α and c > 0 is a universal constant.

Proof. The argument follows the exact same lines as in the proof of (33, Proposition V.6). We might thus skip a few details here. The strategy is to exhibit a Hermitian operator M on C⊗Cd which is with probability greater than 1 − 2e^{-cd} a witness of the fact that ρ has Schmidt number larger than k.

To begin with observe that

$$E \text{Tr}(\rho G) = \frac{\alpha}{d^3} E \text{Tr}(G^2) = \alpha d$$ ,

while Theorem IV.1 gives us

$$E \sup \{\text{Tr}(\sigma G) : \sigma \in \text{SN}_k(\mathbb{C}^d : \mathbb{C}^d)\} \leq C \sqrt{k} \sqrt{d}$$ .

With the two average estimates above at our disposal, we are now in position to apply the following Gaussian deviation inequality: If f is a function such that, for any Gaussian variables H, H', |f(H) - f(H')| ≤ L(H, H')\|H -
Theorem V.1 proved in [17]. We will need the following corollary of this result:

Indeed such maps exhibit a behaviour that is not completely positive only on a subspace of their input space \( M_d \). We will relate this in the following to the Schmidt number.

Our main results show such limitations for states invariant under partial transposition. We will now focus on the converse question: Can we infer limitations on the Schmidt number from positive partial transpose? These results are all based on our main results show such limitations for states invariant under partial transposition. We will now focus on the converse question: Can we infer limitations on the Schmidt number from positive partial transpose? These results are all based on a general technique relating the Schmidt number to entangled principal sub-blocks of a quantum state.

\[
\text{Proposition IV.4 is increasing with } \alpha \quad \text{and decreasing with } \alpha
\]

\[
\text{Definition V.1}\quad \text{A linear map } L : M_{d_1} \rightarrow M_{d_2} \text{ is called an } S\text{-trivial lifting for a set } S \subseteq \{1, \ldots, d_1\} \text{ iff } L(|i\rangle|j\rangle) = 0 \text{ whenever } i \in S \text{ or } j \in S.
\]

Note that in the above definition we do not care about the map that gets lifted to a particular \( S\)-trivial lifting \( L : M_{d_1} \rightarrow M_{d_2} \). This omission simplifies our presentation slightly, and we refer to [17] for a more elaborate presentation. A remarkable decomposition technique for \( k\)-positive maps (called ‘Choi decomposition’) has been proved in [17]. We will need the following corollary of this result:

**Theorem V.1 ([17]).** For \( k \in \{2, \ldots, \min(d_1, d_2)\} \) any \( k\)-positive map \( P : M_{d_1} \rightarrow M_{d_2} \) can be written as

\[
P = Q + T,
\]

where \( T : M_{d_1} \rightarrow M_{d_2} \) is completely positive and \( Q : M_{d_1} \rightarrow M_{d_2} \) is a positive \( S\)-trivial lifting for some \( S \subseteq \{1, \ldots, d_1\} \) with \( |S| = k - 1 \).

The main consequence of the previous theorem is a dimension reduction for \( k\)-positive maps \( P : M_{d_1} \rightarrow M_{d_2} \). Indeed such maps exhibit a behaviour that is not completely positive only on a subspace of their input space \( M_{d_1} \). We will relate this in the following to the Schmidt number.
A. Schmidt numbers imply entanglement in principal sub-blocks

The following theorem is the main technical tool of this section.

**Theorem V.2** (Entangled sub-blocks from Schmidt number). For \( d_1 \leq d_2 \) consider a matrix \( X \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+ \) written as
\[
X = \sum_{i,j=1}^{d_1} |i\rangle\langle j| \otimes X_{ij},
\]
with blocks \( X_{ij} \in \mathcal{M}_{d_2} \) for \( i, j \in \{1, \ldots, d_1\} \). If \( 2 \leq \text{SN}(X) = k \), then there exists a set
\[
\{m_1, \ldots, m_{d_1-k+2}\} \subseteq \{1, \ldots, d_1\}
\]
such that the principal sub-block matrix
\[
Y = \sum_{s,t=1}^{d_1-k+2} |s\rangle\langle t| \otimes X_{m_s,m_t} \in (\mathcal{M}_{d_1-k+2} \otimes \mathcal{M}_{d_2})^+
\]
is entangled.

**Proof.** Since \( \text{SN}(X) = k \) there exists a \((k-1)\)-positive map \( \mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2} \) such that
\[
(\mathcal{P} \otimes \text{id}_{d_2})(X) \geq 0.
\]
By Theorem V.1 there exists a set \( S \subset \{1, \ldots, d_1\} \) with \( |S| = k - 2 \) such that
\[
\mathcal{P} = \mathcal{Q} + \mathcal{T},
\]
where \( \mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2} \) is completely positive and \( \mathcal{Q} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2} \) is a positive \( S \)-trivial lifting. Setting
\[
\{m_1, \ldots, m_{d_1-k+2}\} = \{1, \ldots, d_1\} \setminus S,
\]
we can conclude from (10) that
\[
(\mathcal{Q} \otimes \text{id}_{d_2})(X) = \sum_{i,j=1}^{d_1} \mathcal{Q}(|i\rangle\langle j|) \otimes X_{ij} = \sum_{s,t=1}^{d_1-k+2} \mathcal{Q}(|m_s\rangle\langle m_t|) \otimes X_{m_s,m_t} \geq 0,
\]
where we used that \( \mathcal{Q}(|i\rangle\langle j|) = 0 \) whenever \( i \in S \) or \( j \in S \). Since the map \( \mathcal{Q} \) is positive we have that the positive matrix
\[
\sum_{s,t=1}^{d_1-k+2} |m_s\rangle\langle m_t| \otimes X_{m_s,m_t} = (V \otimes \text{I}_{d_2})Y(V^\dagger \otimes \text{I}_{d_2})
\]
is entangled. Here \( V : \mathbb{C}^{d_1-k+2} \to \mathbb{C}^{d_1} \) is the isometry defined by \( V|s\rangle = |m_s\rangle \) for \( s \in \{1, \ldots, d_1-k+2\} \), and since the application of a local isometry preserves separability the proof is finished.

For convenience we restate as a corollary the special case where the previous theorem is applied to a positive matrix with maximal Schmidt number.

**Corollary V.1.** For \( d_1 \leq d_2 \) consider a matrix \( X \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+ \) written as
\[
X = \sum_{i,j=1}^{d_1} |i\rangle\langle j| \otimes X_{ij},
\]
with blocks \( X_{ij} \in \mathcal{M}_{d_2} \) for \( i, j \in \{1, \ldots, d_1\} \). If \( \text{SN}(X) = d_1 \), then there exists \( \{k_1, k_2\} \subset \{1, \ldots, d_1\} \) such that the principal sub-block matrix
\[
Y = \sum_{s,t=1}^{2} |s\rangle\langle t| \otimes X_{k_s,k_t} \in (\mathcal{M}_2 \otimes \mathcal{M}_{d_2})^+
\]
is entangled.

It is a simple consequence of the previous corollary that a state \( \rho \) on \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \) with \( d_1 \leq d_2 \) has Schmidt number \( \text{SN}(\rho) \leq d_1 - 1 \) (i.e. the Schmidt number is not maximal) if none of the principal sub-block matrices from (11) is entangled. We will use this technique in the following to show that the Schmidt number of certain subsets of PPT states is not maximal.
B. States which are invariant under partial transpose

The following theorem shows that a state which is PT-invariant with respect to the smaller of the two subsystems cannot have maximal Schmidt number. In the case where the smaller system has dimension 2 this had been shown in [20].

**Theorem V.3** (Schmidt number of PT-invariant states). If a quantum state ρ on \( C^{d_1} \otimes C^{d_2} \) with \( 2 \leq d_1 \leq d_2 \) satisfies \((\vartheta_{d_1} \otimes \text{id}_{d_2})(\rho) = \rho\), then \( \text{SN}(\rho) \leq d_1 - 1 \).

**Proof.** We write \( \rho \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+ \) as

\[
\rho = \sum_{i,j=1}^{d_1} |i\rangle \langle j| \otimes X_{ij}
\]

with matrices \( X_{ij} \in \mathcal{M}_{d_2} \) for \( i, j \in \{1, \ldots, d_1\} \). Since \( \rho \) is invariant under a partial transpose we have that

\[
\rho = \sum_{i,j=1}^{d_1} |i\rangle \langle j| \otimes X_{ij} = \sum_{i,j=1}^{d_1} |i\rangle \langle j| \otimes X_{ji} = (\vartheta_{d_1} \otimes \text{id}_{d_2})(\rho),
\]

and thus \( X_{ij} = X_{ji} \) for any \( i, j \in \{1, \ldots, d_1\} \). For any \( k_1, k_2 \in \{1, \ldots, d_1\} \) the matrix

\[
\rho^{k_1 k_2} = \sum_{s,t=1}^{2} |s\rangle \langle t| \otimes X_{k_s k_t}
\]

is positive as a principal sub-block matrix of the positive matrix \( \rho \).

Clearly, we have

\[
(\vartheta_{d_1} \otimes \text{id}_{d_2})(\rho^{k_1 k_2}) = \sum_{s,t=1}^{2} |k_s\rangle \langle k_t| \otimes X_{k_s k_t} = \sum_{s,t=1}^{2} |k_s\rangle \langle k_t| \otimes X_{k_s k_t} = \rho^{k_1 k_2}.\]

Now [20, Theorem 2] implies that \( \rho^{k_1 k_2} \in (\mathcal{M}_2 \otimes \mathcal{M}_{d_2})^+ \) has to be separable (for a simple proof of this fact, see [37, §6]). Therefore, \( \text{SN}(\rho) \leq d_1 - 1 \) since otherwise Corollary V.1 would provide \( k_1, k_2 \in \{1, \ldots, d_1\} \) such that \( \rho^{k_1 k_2} \) is entangled. \( \square \)

C. States which are absolutely positive under partial transpose

We now give an application of our techniques to the problem of classifying the absolutely PPT set. Let \( \mathcal{U}(C^d) \subset \mathcal{M}_d \) denote the set of unitary matrices acting on \( C^d \). A quantum state \( \rho \) on \( C^{d_1} \otimes C^{d_2} \) is called

- **absolutely PPT (APPT)** if \( U \rho U^\dagger \) is PPT for any unitary \( U \in \mathcal{U}(C^{d_1} \otimes C^{d_2}) \).
- **absolutely separable (ASEP)** if \( U \rho U^\dagger \) is separable for any unitary \( U \in \mathcal{U}(C^{d_1} \otimes C^{d_2}) \).

It has been shown in [43] that there exists a ball of ASEP states around the maximally mixed state. Despite this not much is known about the structure of the set of ASEP states. In contrast the set of APPT states can be characterized in terms of a semidefinite program [35]. It is an open problem whether the sets of APPT states and ASEP states coincide for all dimensions \( d_1 \) and \( d_2 \) [35]. Here we prove that, at least, APPT states cannot have maximal Schmidt number.

**Theorem V.4** (Schmidt number of APPT states). Any APPT state \( \rho \) on \( C^{d_1} \otimes C^{d_2} \) satisfies

\[
\text{SN}(\rho) \leq \min(d_1, d_2) - 1.
\]

**Proof.** Without loss of generality we assume that \( d_1 \leq d_2 \). We can write

\[
\rho = \sum_{i,j=1}^{d_1} |i\rangle \langle j| \otimes X_{ij},
\]
with matrices $X_{ij} \in \mathcal{M}_{d_2}$ for $i, j \in \{1, \ldots, d_1\}$. For any $k_1, k_2 \in \{1, \ldots, d_1\}$ with $k_1 \neq k_2$ the matrix

$$\rho^{k_1, k_2} = \sum_{s, t = 1}^{2} |s\rangle\langle t| \otimes X_{k_s, k_t}$$

is positive as a principal sub-block matrix of the positive matrix $\rho$. We will show that $\rho^{k_1, k_2}$ is separable for any $k_1, k_2 \in \{1, \ldots, d_1\}$. Once this is done, the statement of our theorem follows from Corollary V.3.

To show that $\rho^{k_1, k_2}$ for fixed $k_1, k_2 \in \{1, \ldots, d_1\}$ is separable consider a unitary $\hat{U} \in \mathcal{U}(\mathbb{C}^2 \otimes \mathbb{C}^{d_2})$. We can write

$$\hat{U} = \sum_{s, t = 1}^{2} |s\rangle\langle t| \otimes V_{st},$$

where $V_{st} \in \mathcal{M}_{d_2}$ for $s, t \in \{1, 2\}$. Now consider

$$U = \sum_{s, t = 1}^{2} |k_s\rangle\langle k_t| \otimes V_{st} + \sum_{i \in \{1, \ldots, d_1\} \setminus \{k_1, k_2\}} |i\rangle\langle i| \otimes 1_{d_2} \in \mathcal{U}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}).$$

Since $\rho$ is APPT we have that

$$U \rho U^\dagger = \sum_{s, t = 1}^{2} |k_s\rangle\langle k_t| \otimes \sum_{lm} V_{st} X_{k_s, k_l} V_{tl}^\dagger + \sum_{i, j \in \{1, \ldots, d_1\} \setminus \{k_1, k_2\}} |i\rangle\langle j| \otimes \rho_{ij}$$

is PPT. This implies that the principal sub-block matrix

$$\sum_{s, t = 1}^{2} |k_s\rangle\langle k_t| \otimes \sum_{lm} V_{st} X_{k_s, k_l} V_{tl}^\dagger = (W \otimes 1_{d_2}) \hat{U} \rho^{k_1, k_2} \hat{U}^\dagger (W \otimes 1_{d_2})^\dagger$$

is PPT as well. Here $W : \mathbb{C}^2 \rightarrow \mathbb{C}^{d_1}$ denotes the isometry defined by $W|s\rangle = |k_s\rangle$ for $s = 1, 2$. Finally, since partial applications of isometries preserve positivity we have that $\hat{U} \rho^{k_1, k_2} \hat{U}^\dagger$ is PPT. Since the unitary $\hat{U} \in \mathcal{U}(\mathbb{C}^2 \otimes \mathbb{C}^{d_2})$ was chosen arbitrarily we find that $\rho^{k_1, k_2}$ is an (unnormalized) APPT state. By Theorem [56, Theorem 1] this implies that $\rho^{k_1, k_2}$ is ASEP and therefore in particular separable. This finishes the proof.

**VI. CONCLUSION AND OUTLOOK**

We have constructed explicit examples of PPT states on $\mathbb{C}^d \otimes \mathbb{C}^d$ with Schmidt numbers at least $[(d - 1)/4]$. By modifying our construction slightly we even obtained states on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ invariant under partial transpose with Schmidt numbers at least $[(d - 3)/8]$. We continued by studying a family of random PPT states also achieving such a linear scaling in the local dimension, although without being able to specify the exact constants. Finally, we introduced a technique to derive upper bounds on the Schmidt number of general quantum states. Using this technique we show that for both PT-invariant and APPT states the Schmidt number cannot reach its maximal value. There are a number of open problems, and directions for further research to be explored from here:

- What is the maximal Schmidt number of a PPT state (or a PT-invariant state) on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ for general $d_1$ and $d_2$? We only know the answer when $\min(d_1, d_2) = 2$ or when $d_1 = d_2 = 3$. It is well-known that there exist entangled PPT states on $\mathbb{C}^2 \otimes \mathbb{C}^d$ for any $d \geq 4$, a concrete example being the Tang-Horodecki state $[44, 45]$ (which is even PT-invariant $[20]$) for $d = 4$. For $d_1 = d_2 = 3$ it has been shown $[17]$ that the Schmidt number of a PPT state can only reach 2.

- What can be said about the *absolute Schmidt number* (i.e. Schmidt number from spectrum)? Also, can one estimate the size of bounded Schmidt number balls around the identity?

- What are the implications of positive partial transposition constraints on the Schmidt number in the multipartite setting? There exist states on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ that are genuine multipartite entangled despite being PPT across every bipartition $[33]$, but similar examples for higher Schmidt numbers are not known. A possible starting point could be the following questions: *Is there a PPT state on a $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ with Schmidt number 3 across every bipartition? Or what is the maximum genuine multipartite Schmidt number possible for PPT states?* In this and related questions, an extension of positive map mixers $[32]$ to $k$-positive map mixers (or of the corresponding positive maps $[47]$ to $k$-positive maps) might allow for an SDP/map based characterization. It would also be interesting to explore this question via probabilistic techniques in high dimensions.
• Despite being non-distillable, PPT states can be useful in other tasks. For instance, using data hiding (see [52, 53] for more details) PPT states arbitrarily close to private states [52, 53] can be constructed. Such states have a high key rate. In multipartite systems PPT states can still yield metrological advantages [54]. Could one show that all PPT states with high Schmidt number are useful in some operational sense?

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