Equivalent Representations of Max-Stable Processes via $\ell^p$ Norms

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Abstract

While max-stable processes are typically written as pointwise maxima over an infinite number of stochastic processes, in this paper, we consider a family of representations based on $\ell^p$ norms. This family includes both the construction of the Reich–Shaby [21] model and the classical spectral representation by de Haan [3] as special cases. As the representation of a max-stable process is not unique, we present formulae to switch between different equivalent representations. We further provide a necessary and sufficient condition for the existence of a $\ell^p$ norm based representation in terms of the stable tail dependence function of a max-stable process. Finally, we discuss several properties of the represented processes such as ergodicity or mixing.

Keywords: extreme value theory, Reich–Shaby model, spectral representation, stable tail dependence function

1. Introduction

Arising as limits of rescaled maxima of stochastic processes, max-stable processes play an important role in spatial and spatio-temporal extremes. A stochastic process $X = \{X(s), s \in S\}$ on a countable index set $S$ is called max-stable if there exist sequences $\{a_n(\cdot)\}_{n \in \mathbb{N}}$ and $\{b_n(\cdot)\}_{n \in \mathbb{N}}$ of functions $a_n : S \to (0, \infty]$ and $b_n : S \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$
\mathcal{L}(X) = \mathcal{L}\left(\max_{i=1}^{n} \frac{X_i - b_n}{a_n}\right),
$$

where $X_i, i \in \mathbb{N}$, are independent copies of $X$ and the maximum is taken pointwise. From univariate extreme value theory, it is well-known that the marginal distributions of $X$, if non-degenerate, are necessarily Generalized Extreme Value (GEV) distributions, i.e.

$$
\mathbb{P}(X(s) \leq x) = \exp\left( - \left(1 + \xi(s) \frac{x - \mu(s)}{\sigma(s)}\right)^{-1/\xi(s)}\right), \quad 1 + \xi(s) \frac{x - \mu(s)}{\sigma(s)} > 0,
$$

with $\xi(s) \in \mathbb{R}$, $\mu(s) \in \mathbb{R}$ and $\sigma(s) > 0$. As max-stability is preserved by marginal transformations, it is common practice in extreme value theory to consider only one type of
marginal distributions, e.g. the case that the shape parameter $\xi$ is positive. In this case, the marginal distributions are of $\alpha$-Fréchet type, i.e., up to affine transformations, the marginal distribution functions are of the form

$$\Phi_\alpha(x) = \exp(-x^\alpha), \quad x > 0,$$

for some $\alpha > 0$. Here, we will focus on the case of max-stable processes with unit Fréchet margins, i.e. $X(s) \sim \Phi_1$ for all $s \in S$. In this case, $X$ is called a simple max-stable process.

By de Haan [3], the class of simple max-stable processes on $S$ can be fully characterized: A stochastic process $\{X(s), s \in S\}$ is simple max-stable if and only if it possesses the spectral representation

$$X(s) = \max_{i \in \mathbb{N}} A_i V_i(s), \quad s \in S, \quad (1)$$

where $\sum_{i \in \mathbb{N}} \delta_{A_i}$ is a Poisson point process on $(0, \infty)$ with intensity measure $\alpha^{-2} \mathrm{d}\alpha$ and $V_i = \{V_i(s), s \in S\}$ are independent copies of a stochastic process $V$ such that $\mathbb{E}(V(s)) = 1$ for all $s \in S$ [see also 10, 20]. It is important to note that this representation is not unique. As different representations of the same max-stable process might be convenient for different purposes such as estimation [see 7, 6, among others] or simulation [cf. 18, 4, 19, for instance], finding novel representations is of interest.

Recently, Reich and Shaby [21] came up with a class of max-stable processes written as a product

$$X(s) = U^{(p)}(s) \cdot \left( \sum_{l=1}^L B_l w_l(s)^p \right)^{1/p}, \quad s \in S, \quad (2)$$

where $\{U^{(p)}(s)\}_{s \in S}$ is a noise process with $U^{(p)}(s) \sim_{\text{iid}} \Phi_p$, the functions $w_l : S \rightarrow [0, \infty)$, $l = 1, \ldots, L$, are deterministic weight functions such that $\sum_{l=1}^L w_l(s) = 1$ for all $s \in S$ and, independently from $\{U^{(p)}(s)\}_{s \in S}$, the independent random variables $B_l$, $l = 1, \ldots, L$, follow a stable law given by the Laplace transform

$$\mathbb{E}\{\exp(-t \cdot B_l)\} = \exp(-t^{-1/p}), \quad t > 0.$$

The parameter $p \in (1, \infty)$ determines the strength of the effect of the noise process which – analogously to the terminology in geostatistics – is also called a nugget effect. In Reich and Shaby [21], the weight functions $w_l$ are chosen as shifted and appropriately rescaled Gaussian density functions yielding an approximation of the well-known Gaussian extreme value process [29] joined with a nugget effect. Similarly, Reich and Shaby [21] propose analogues to popular max-stable processes such as extremal Gaussian processes [25] and Brown-Resnick processes [15] by choosing appropriately rescaled realizations of Gaussian and log-Gaussian processes, respectively, as weight functions. Due to the flexibility in modeling the strength of the nugget by the additional parameter $p$ and the tractability of the likelihood which allows to embed the model in a hierarchical Bayesian model, the Reich–Shaby model (2) has found its way into several applications [cf. 28, 22, 30, 27, for instance].
While a simple max-stable process in the spectral representation (1) is written as the pointwise supremum of an infinite number of processes, i.e. the pointwise $\ell_\infty$ norm of the random sequence $\{A_i \cdot W_i(s)\}_{i \in \mathbb{N}}$, the Reich–Shaby model (2) is represented as the pointwise $p$ norm of the finite random vector $(B_i^{1/p}, w_i(s))_{i=1,...,L}$. In this paper, we will present a more general class of representations of max-stable processes by writing them as pointwise $\ell_p$ norms of sequences of stochastic processes, including both de Haan’s representation and the Reich–Shaby model as special cases. The finite-dimensional distributions of the resulting processes will turn out to be generalized logistic mixtures introduced by Fougères et al. [9] and Fougères et al. [8].

This paper is structured as follows: In Section 2, we will introduce the spectral representation based on $\ell_p$ norms. As a single max-stable process might allow for equivalent $\ell_p$ norm based representations for different $p \in (1, \infty]$, we give formulae to switch between them in Section 3. Section 4 provides a full characterization of the resulting class of processes whose properties are finally discussed in Section 5.

2. Generalization of the Spectral Representation

Denoting by

$$\|A \circ V(s)\|_p = \begin{cases} \left[ \sum_{i \in \mathbb{N}} (A_i \cdot V_i(s))^p \right]^{1/p}, & p \in (1, \infty), \\ \max_{i \in \mathbb{N}} A_i \cdot V_i(s), & p = \infty, \end{cases}$$

the $\ell_p$ norm of the Hadamard product of the sequences $A = \{A_i\}_{i \in \mathbb{N}}$ and $V(s) = \{V_i(s)\}_{i \in \mathbb{N}}, s \in S$, the spectral representation (1) can be rewritten as

$$X(s) = \|A \circ V(s)\|_\infty, \quad s \in S.$$  

We present a more general representation replacing the $\ell_\infty$ norm by a general $\ell_p$ norm, $p \in (1, \infty]$, and multiplication by an independent noise process with $\Phi_p$ marginal distributions. Here, we use the convention that $\Phi_\infty$ denotes the weak limit of $\Phi_p$ as $p \to \infty$, i.e. $\Phi_\infty(x) = \mathbb{1}_{[1, \infty)}(x)$ is a degenerate distribution function.

**Theorem 1.** Let $p \in (1, \infty]$ and $\{U^{(p)}(s)\}_{s \in S}$ be a collection of independent $\Phi_p$ random variables. Further, let $\sum_{i \in \mathbb{N}} \delta_{A_i}$ be a Poisson process on $(0, \infty)$ with intensity $a^{-2}da$ and $W_i^{(p)}, i \in \mathbb{N}$, be independent copies of a stochastic process $\{W^{(p)}(s), s \in S\}$ with $\mathbb{E}\{W^{(p)}(s)\} = 1$ for all $s \in S$. Then, the process $X$, defined by

$$X(s) = \frac{U^{(p)}(s)}{\Gamma(1-p^{-1})} \|A \circ W^{(p)}(s)\|_p, \quad s \in S,$$

is simple max-stable.

**Proof.** For $p = \infty$, we have $U^{(p)}(s) = 1$ a.s. and, thus, representation (3) is of the same form as representation (1). Consequently, max-stability follows from de Haan [3].
For $p \in (1, \infty)$, we first show that $\|A \circ W(s)\|_p < \infty$ a.s. According to Campbell’s Theorem [cf. 16, p.28], this holds true if and only if
\[
\mathbb{E} \left( \int_0^\infty \min\{ |a W(s)|^p, 1 \} a^{-2} \, da \right) < \infty.
\] (4)
Substituting $v = aW(s)$, we can easily see that the left-hand side of (4) equals
\[
\mathbb{E} \left( W(s)^p \right) \cdot \int_0^\infty \min\{ |v|^p, 1 \} v^{-2} \, dv = 1 + \frac{1}{p-1}.
\]
Thus, $\|A \circ W(s)\|_p < \infty$ a.s. Then, for $s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n > 0$, $n \in \mathbb{N}$, we obtain
\[
\mathbb{P}(X(s_i) \leq x_i, i = 1, \ldots, n) = \mathbb{E} \left( \mathbb{P} \left( U(s_i) \leq \frac{\Gamma(1-p^{-1})x_i}{\|A \circ W(s_i)\|_p^{1/p}}, i = 1, \ldots, n \mid A, W(s_i) \right) \right)
\]
\[
= \mathbb{E} \left( \exp \left( -\sum_{i=1}^n \frac{\Gamma(1-p^{-1})x_i}{\|A \circ W(s_i)\|_p^{1/p}} \right) \right).
\]
Using well-known results on the Laplace functional of Poisson point processes, this yields
\[
\mathbb{P}(X(s_i) \leq x_i, i = 1, \ldots, n) = \exp \left( \mathbb{E} \left( \left\| \left( \frac{W(s_i)}{x_i} \right)^n \right\|_p^{1/p} \right) \right) \cdot \frac{1}{p \Gamma(1-p^{-1})} \cdot \int_0^\infty (e^{-a} - 1) a^{-1-p^{-1}} \, da
\]
\[
= \exp \left( -\mathbb{E} \left( \left\| \left( \frac{W(s_i)}{x_i} \right)^n \right\|_p^{1/p} \right) \right)
\] (5)
where we used Formula 3.478.2 in Gradshteyn and Ryzhik [11]. Thus, for $m$ independent copies $X_1, \ldots, X_m$ of $X$, $m \in \mathbb{N}$, the homogeneity of the $\ell^p$ norm yields
\[
\mathbb{P} \left( \frac{1}{m} \max_{j=1}^m X_j(s_i) \leq x_i, i = 1, \ldots, n \right) = \mathbb{P} (X(s_i) \leq x_i, i = 1, \ldots, n),
\]
i.e. $Z$ is simple max-stable.

\begin{remark}
Theorem [11] could alternatively be verified by observing that the process $T(s) = \|A \circ W(s)\|_p^{1/p}$, $s \in S$, is $\alpha$-stable with $\alpha = 1/p$ (see also the proof of Theorem 3). Thus, all the finite-dimensional distributions of $X$ are generalized logistic mixtures [cf. 9, 8] and, consequently, are max-stable distributions.
\end{remark}
Noting that the finite-dimensional distributions of the Reich–Shaby model (1) are given by
\[ P(X(s_i) \leq x_i, i = 1, \ldots, n) = \exp\left(-\sum_{j=1}^{L} \left\| \frac{w_j(s_i)}{x_i} \right\|_p^n \right), \]
it can be easily seen that (2) is a special case of representation (3) where \( p = 1 \). Thus, any \( \ell^p \) norm based representation (3) with \( p < \infty \) of a simple max-stable process \( X \) is not unique. Furthermore, there might be representations of type (3) with different \( p \) for the same process \( X \). Such equivalent representations are discussed in the following section.

3. Equivalent Representations

By de Haan [3], the class of simple max-stable processes is fully covered by the class of processes which allow for the spectral representation (1), i.e. representation (3) with \( p = \infty \). Thus, any \( \ell^p \) norm based representation (3) with \( p < \infty \) of a simple max-stable process can be transformed to an equivalent representation of type (1). This transformation is presented in the following proposition. Even more generally, it is shown how a \( \ell^q \) norm based representation can be derived from a \( \ell^p \) norm based representation with \( p < q < \infty \).

**Proposition 2.** Let \( X \) be a simple max-stable process with representation (3) for some \( p \in (1, \infty) \). Then, the following holds:

1. The process \( X \) allows for the spectral representation (1) with
\[ V(\cdot) =_d \frac{U^{(p)}(\cdot)}{\Gamma(1 - p^{-1})} W^{(p)}(\cdot). \]

2. For \( q \in (p, \infty) \), the process \( X \) satisfies
\[ X(\cdot) =_d \frac{U^{(q)}(\cdot)}{\Gamma(1 - q^{-1})} \|A \circ W^{(q)}(\cdot)\|_q, \]

where \( \{U^{(q)}(s)\}_{s \in S} \) is a collection of independent \( \Phi_q \) random variables and \( W^{(q)}(\cdot) \), \( i \in \mathbb{N} \), are independent copies of a stochastic process \( \{W^{(q)}(s), s \in S\} \) given by
\[ W^{(q)}(s) = \frac{\Gamma(1 - q^{-1})}{\Gamma(1 - p^{-1})} (T_{(p/q)}(s))^{p/q} \cdot W^{(p)}(s), \quad s \in S. \]
Remark 2. Even though the transformation in the second part of the proposition requires $p < q < \infty$, the two cases $p = q$ and $q = \infty$ can be regarded as limiting cases. As $q \searrow p$, we obtain that $U^{(q)}(\cdot) \rightarrow_d U^{(p)}(\cdot)$ and $\{T_{(p/q)}(s)\}_{s \in S}$ converges in distribution to a collection of random variables which equal 1 a.s. Thus, in the limit $p = q$, there is no transformation.

Proof. 1. By comparing the finite-dimensional distributions of the processes defined via (1) and (3), it suffices to show that

$$\frac{1}{\Gamma(1 - p^{-1})} \mathbb{E} \left( \left\| \left( \frac{U^{(p)}(s_i)W^{(p)}(s_i)}{x_i} \right)_{i=1}^{p} \right\|_{\infty}^{n} \right) = \mathbb{E} \left( \left\| \left( \frac{W^{(p)}(s_i)}{x_i} \right)_{i=1}^{n} \right\|_{p}^{n} \right),$$

(8)

for all $s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n > 0$, $n \in \mathbb{N}$. To this end, we first note that

$$\mathbb{P} \left( \left\| \left( \frac{U^{(p)}(s_i)W^{(p)}(s_i)}{x_i} \right)_{i=1}^{n} \right\|_{\infty} \leq y \right| W^{(p)}(\cdot) = \exp \left( -\frac{1}{y^p} \sum_{i=1}^{n} \left( \frac{W(s_i)}{x_i} \right)^p \right), \quad y > 0,$$

that is, conditionally on $W^{(p)}(\cdot)$, the norm $\left\| (U^{(p)}(s_i)W^{(p)}(s_i)/x_i)_{i=1}^{n} \right\|_{\infty}$ follows a $p$-Fréchet distribution with scale parameter $\left\| (W^{(p)}(s_i)/x_i)_{i=1}^{n} \right\|_{p}$. Thus,

$$\mathbb{E} \left( \left\| \left( \frac{U^{(p)}(s_i)W^{(p)}(s_i)}{x_i} \right)_{i=1}^{n} \right\|_{\infty}^{n} \right) = \mathbb{E}_{W} \left\{ \mathbb{E} \left( \left\| \left( \frac{U^{(p)}(s_i)W^{(p)}(s_i)}{x_i} \right)_{i=1}^{n} \right\|_{\infty} \right| W^{(p)}(\cdot) \right\} \right. \right.$$

$$= \left. \mathbb{E}_{W} \left\{ \Gamma(1 - p^{-1}) \left\| \left( \frac{W^{(p)}(s_i)}{x_i} \right)_{i=1}^{n} \right\|_{p}^{n} \right\} \right.$$

i.e. Equation (8).

2. From the first part of the proposition, it follows that the right-hand side of (7) allows for a spectral representation (1) where the spectral functions are independent copies of the process $\tilde{V}$ given by

$$\tilde{V}(\cdot) = \frac{U^{(q)}(\cdot) \cdot (T_{(p/q)}(\cdot))^{1/q}}{\Gamma(1 - p^{-1})} \cdot W^{(p)}(\cdot),$$

while the spectral functions of the process $X$ on the left-hand side of (7) are independent copies of the process $V$ given in (3). Conditioning on the value of the stable random variable $T_{(p/q)}(s)$, it can be shown that the product $U^{(q)}(s) \cdot T_{(p/q)}(s)$ has the distribution function $\Phi_p$ for all $s \in S$ [cf. 9] and, thus, $\tilde{V}(\cdot) \equiv_d V(\cdot)$. 

\[\square\]
As \( q \to \infty \), we have that \( \Gamma(1 - q^{-1}) \to 1 \) and each \( U^{(q)}(s), s \in S \), converges to 1 a.s. Further, by Thm. 1.4.5 in Samorodnitsky and Taqqu [24], for each \( s \in S \), the random variable \( T_{p/q}(s) \) can be represented as \( \frac{1}{\Gamma(1 - p/q)} \sum_{i \in \mathbb{N}} (\hat{A}_i Y_i)^{q/p} \) where \( \{\hat{A}_i\}_{i \in \mathbb{N}} \) are the points of a Poisson point process on \((0, \infty)\) with intensity \( \tilde{a}^{-2} \) and \( Y_i, i \in \mathbb{N} \), are independently and identically distributed non-negative random variables with expectation 1. Thus, as \( q \to \infty \),
\[
\left( T_{p/q}(s) \right)^{1/q} \to_d \frac{1}{\Gamma(1 - p/q)} \max_{i \in \mathbb{N}} (\hat{A}_i Y_i)^{1/p}
\]
which has the distribution function \( \Phi_p \). Consequently, \( \left( T_{p/q}(\cdot) \right)^{1/q} \to_d U^{(p)}(\cdot) \).

Denoting by \( \mathcal{MS} \) the class of all simple max-stable processes and by \( \mathcal{MS}_p \) the class of simple max-stable processes allowing for a \( \ell^p \) norm based spectral representation [3], Proposition [2] yields
\[
\mathcal{MS}_p \subset \mathcal{MS}_q \subset \mathcal{MS}_\infty = \mathcal{MS}, \quad 1 < p < q < \infty.
\]
a full characterization of the class \( \mathcal{MS}_p \) is given in the following section.

### 4. Existence of \( \ell^p \) Norm Based Representations

In the following, we will present a necessary and sufficient criterion for the existence of a \( \ell^p \) norm based representation of a simple max-stable process \( X \) in terms of the stable tail dependence functions of its finite-dimensional distributions. For a simple max-stable distribution \((X(s_1), \ldots, X(s_n))^\top\), its stable tail dependence function \( l_{s_1,\ldots,s_n} \) is defined via
\[
l_{s_1,\ldots,s_n} : [0, \infty)^n \to [0, \infty)
\]
\[
(x_1, \ldots, x_n) \mapsto -\log \left\{ \mathbb{P} \left( X(s_1) \leq \frac{1}{x_1}, \ldots, X(s_n) \leq \frac{1}{x_n} \right) \right\}.
\]
From the spectral representation (11), we obtain the form
\[
l_{s_1,\ldots,s_n}(x) = \mathbb{E} \left( \max_{i=1,\ldots,n} x_i W(s_i) \right), \quad x \in [0, \infty)^n.
\]
(9)
The stable tail dependence function is homogeneous and convex [cf. 1, among others]. Further, from Equation (9) together with dominated convergence, we can deduce that the stable tail dependence function is continuous.

**Theorem 3.** Let \( \{X(s), s \in S\} \) a simple max-stable process and \( p \in (1, \infty) \). Then, the following statements are equivalent:

(i) \( X \) possesses a \( \ell^p \) norm based representation [3].
Further, Thm. 4.4.7 in Berg et al. [2], there exists a unique finite measure for all $x \in \mathbb{R}^n$. Because of the Laplace transform being non-negative and continuous, $e^{x f}$ is conditionally negative definite on the additive semigroup $[0, \infty)^n$, i.e. for all $x^{(1)}, \ldots, x^{(m)} \in [0, \infty)^n$ and $a_1, \ldots, a_m \in \mathbb{R}$ such that $\sum_{i=1}^m a_i = 0$, we have
\[
\sum_{i=1}^m \sum_{j=1}^m a_i a_j f_{s_1, \ldots, s_n}^{(p)}(x^{(i)} + x^{(j)}) \leq 0.
\] (10)

Proof. Firstly, we show that (i) implies (ii). To this end, let $X$ be a simple max-stable process with representation [3]. Then, from [5], we obtain that
\[
f_{s_1, \ldots, s_n}^{(p)}(x) = -\log \left\{ \mathbb{P} \left( X(s_1) \leq \frac{1}{x_1^{1/p}}, \ldots, X(s_n) \leq \frac{1}{x_n^{1/p}} \right) \right\} = \mathbb{E} \left\{ \left( \sum_{i=1}^n x_i W^{(p)}(s_i)^{1/p} \right)^{1/p} \right\},
\] x = (x_1, \ldots, x_n) \in [0, \infty)^n.

Now, let $w(s_1), \ldots, w(s_n) \geq 0$ be fixed. Then, by a straightforward computation, it can be seen that the function $x \mapsto \sum_{k=1}^n x_k w(s_k)^p$ is conditionally negative definite on $[0, \infty)^n$. As the function $y \mapsto y^{1/p}$ is a Bernstein function and the composition of a conditionally negative definite and a Bernstein function yields a conditionally negative definite function [2], Thm. 3.2.9], the function $x \mapsto (\sum_{k=1}^n x_k w(s_k)^p)^{1/p}$ is conditionally negative definite, as well. Being a mixture, the same is true for $f_{s_1, \ldots, s_n}^{(p)}$.

Secondly, we show that (ii) implies (i). From the conditionally negative definiteness of $f_{s_1, \ldots, s_n}^{(p)}$, it follows that $e^{-f_{s_1, \ldots, s_n}^{(p)}}$ is positive definite on $[0, \infty)^n$ [2, Thm. 3.2.2]. As $l_{s_1, \ldots, s_n}$ is non-negative and continuous, $e^{-f_{s_1, \ldots, s_n}^{(p)}}$ is further bounded by 1 and continuous. Thus, by Thm. 4.4.7 in Berg et al. [2], there exists a unique finite measure $\mu_{s_1, \ldots, s_n}$ on $[0, \infty)^n$ with Laplace transform
\[
\mathcal{L} \mu_{s_1, \ldots, s_n}(x) = \int_{[0, \infty)^n} \exp \left( -\langle x, a \rangle \right) \mu(da) = \exp \left( -f_{s_1, \ldots, s_n}(x) \right), \quad x \in [0, \infty)^n.
\] (11)

Because of $\mu_{s_1, \ldots, s_n}([0, \infty)^n) = \exp(-l_{s_1, \ldots, s_n}(0, \ldots, 0)) = 1$, $\mu_{s_1, \ldots, s_n}$ is a probability measure. Further,
\[
l_{s_1, \ldots, s_n}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = l_{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\] (12)
for all $x = (x_1, \ldots, x_n) \in [0, \infty)^n$ and $i \in \{1, \ldots, n\}$ implies that
\[
\mu_{s_1, \ldots, s_n}(A_1 \times \ldots \times A_{i-1} \times [0, \infty) \times A_{i+1} \times \ldots \times A_n)
= \mu_{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n}(A_1 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n)
\]
for all Borel sets \( A_1, \ldots, A_n \subset [0, \infty) \) and \( i \in \{1, \ldots, n\} \), that is, the family \( \{\mu_{s_1, \ldots, s_n} : s_1, \ldots, s_n \in S, n \in \mathbb{N}\} \) of probability measures satisfies the consistency conditions from Kolmogorov’s existence theorem. Thus, there exists a stochastic process \( \{T(s), s \in S\} \) with finite-dimensional distributions \( \mu \).

Now, let \( \{U^{(p)}(s)\}_{s \in S} \) be a collection of independent \( \Phi_p \) random variables and

\[
\tilde{X}(s) = U^{(p)}(s)T(s)^{1/p}, \quad s \in S.
\]

Then, for all pairwise distinct \( s_1, \ldots, s_n \in S \) and \( x_1, \ldots, x_n > 0 \), we have

\[
\begin{align*}
\mathbb{P}(\tilde{X}(s_1) \leq x_1, \ldots, \tilde{X}(s_n) \leq x_n) &= \mathbb{E}\left\{ \mathbb{P}\left( U^{(p)}(s_1) \leq \frac{x_1}{T_1^{1/p}(s_1)}, \ldots, U^{(p)}(s_n) \leq \frac{x_n}{T_1^{1/p}(s_n)} \right | T(s_1), \ldots, T(s_n) \right\} \\
&= \mathbb{E}\left\{ \exp\left( -\sum_{i=1}^{n} \frac{T(s_i)}{x_i^p} \right) \right\},
\end{align*}
\]

By Equation (11), we obtain

\[
\begin{align*}
\mathbb{P}(\tilde{X}(s_1) \leq x_1, \ldots, \tilde{X}(s_n) \leq x_n) &= \exp\left( -f^{(p)}_{s_1, \ldots, s_n}(x_1^p, \ldots, x_n^p) \right) \\
&= \mathbb{P}(X(s_1) \leq x_1, \ldots, X(s_n) \leq x_n).
\end{align*}
\]

Thus, \( X \) allows for the spectral representation

\[
X(s) = U(s)T^{1/p}(s), \quad s \in S.
\] (13)

Now, let \( T^{(1)}, \ldots, T^{(m)} \) be \( m \) independent copies of \( T \) for \( m \in \mathbb{N} \). Then, for all \( s_1, \ldots, s_n \in S \) and \( x = (x_1, \ldots, x_n) \in [0, \infty)^n \), we have

\[
\begin{align*}
\mathbb{E}\left\{ \exp\left( -\langle x, \left( \sum_{k=1}^{m} T^{(k)}(s_i) \right)^n \right) \right\} &= \mathbb{E}\left\{ \exp\left( -\langle x, (T(s_i))_{i=1}^{n} \right) \right\} \\
&= \mathbb{E}\left\{ \exp\left( -m \cdot l_{s_1, \ldots, s_m}(x_1^{1/p}, \ldots, x_n^{1/p}) \right) \right\} \\
&= \mathbb{E}\left\{ \exp\left( \langle x, m^p(T(s_i))_{i=1}^{n} \right) \right\},
\end{align*}
\]

where we used the homogeneity of the stable tail dependence function. Hence, for all \( s_1, \ldots, s_n \in S \), the vectors \( \left( \sum_{k=1}^{m} T^{(k)}(s_i) \right)_{i=1}^{n} \) and \( m^p(T(s_i))_{i=1}^{n} \) have the same distribution, i.e. \( \{T(s), s \in S\} \) is a \( \alpha \)-stable process with \( \alpha = 1/p \). Thus, from Thm. 13.1.2 and Thm. 3.10.1 in Samorodnitsky and Taqqu [24], we can deduce that \( \{T(s), s \in S\} \) allows for the representation

\[
T(s) = \frac{1}{\Gamma(1 - p^{-1})} \sum_{i \in \mathbb{N}} A_i^p \tilde{W}_i(s), \quad s \in S,
\] (14)

where \( \{A_i\}_{i \in \mathbb{N}} \) are the points of a Poisson point process on \([0, \infty)\) with intensity \( \alpha^{-2}da \) and \( \{\tilde{W}_i(s), s \in S\} \) are independent and identically distributed stochastic processes which are
independent from \( \{A_i\}_{i \in \mathbb{N}} \) and satisfy \( \mathbb{E}(\tilde{W}_i(s)^{1/p}) = l_i(1) = 1 \) for all \( s \in S \). Defining \( W_i^{(p)}(s) = \tilde{W}_i(s)^{1/p}, s \in S, i \in \mathbb{N} \), and plugging Equation (14) into Equation (13), we obtain Equation (3). 

**Remark 3.** Note that Theorem 3 assumes that, for each \( s_1, \ldots, s_n \in S \), \( l_{s_1, \ldots, s_n} \) is the stable tail dependence function of the simple max-stable vector \((X(s_1), \ldots, X(s_n))^{\top}\). The conditional negative definiteness of the function \( f^{(p)}_{s_1, \ldots, s_n} \) is an additional condition. In particular, it is always satisfied for \( p = \infty \) – i.e. any simple max-stable process allows for de Haan’s [3] spectral representation (11) – as \( f^{(\infty)}_{s_1, \ldots, s_n} = l_{s_1, \ldots, s_n}(1, \ldots, 1) \) is always conditionally negative definite.

In order to check whether a function \( l_{s_1, \ldots, s_n} \) is the stable tail dependence function of some process \( X \) with an \( \ell^p \) norm based representation, we first need to ensure that \( l_{s_1, \ldots, s_n} \) is a valid stable tail dependence function. This can be done by checking necessary and sufficient conditions given in Molchanov [17] and Ressel [23], for instance.

Using an integral representation of continuous conditionally negative definite functions on \([0, \infty)^n\) [cf. 2, Paragraph 4.4.6], condition (ii) in Theorem 3 can be reformulated yielding the following corollary.

**Corollary 4.** For a simple max-stable process \( \{X(s), s \in S\} \) and \( p \in (1, \infty) \), the following statements are equivalent:

(i) \( X \) possesses a \( \ell^p \) norm based representation (3).

(ii) For all pairwise distinct \( s_1, \ldots, s_n \in S \) and \( n \in \mathbb{N} \), there exist a vector \( c(s_1, \ldots, s_n) = (c_1(s_1, \ldots, s_n), \ldots, c_n(s_1, \ldots, s_n))^\top \in [0, \infty)^n \) and a Radon measure \( \mu_{s_1, \ldots, s_n} \) on \([0, \infty)^n\) such that the stable tail dependence function \( l_{s_1, \ldots, s_n} \) satisfies

\[
l_{s_1, \ldots, s_n}(x) = \sum_{i=1}^{n} c_i(s_1, \ldots, s_n) \cdot x_i^p + \int_{[0, \infty)^n} \left\{ 1 - \exp \left( -\sum_{i=1}^{n} a_i x_i^p \right) \right\} \mu_{s_1, \ldots, s_n}(da),
\]

for all \( x = (x_1, \ldots, x_n)^\top \in [0, \infty)^n \).

From the characterization given in Theorem 3, we can deduce necessary conditions on the dependence structure of a max-stable process with \( \ell^p \) norm based representation (3) in terms of its extremal coefficients: For a general simple max-stable process \( \{X(s), s \in S\} \) and a finite set \( \hat{S} = \{s_1, \ldots, s_n\} \subset S \), let the extremal coefficient \( \theta(\hat{S}) \) be defined via

\[
\mathbb{P}\left( \max_{s \in \hat{S}} X(s) \leq x \right) = \exp\left( -\frac{\theta(\hat{S})}{x} \right), \quad x > 0
\]
Then, we necessarily have $\theta(\tilde{S}) \in [1, n]$ where $\theta(\tilde{S}) = n$ if and only if $X(s_1), \ldots, X(s_n)$ are independent and $\theta(\tilde{S}) = 1$ if and only if $X(s_1) = X(s_2) = \ldots = X(s_n)$ a.s. The extremal coefficient is closely connected to the stable tail dependence function via the relation

$$\theta(\{s_1, \ldots, s_n\}) = l_{s_1, \ldots, s_n}(1, \ldots, 1).$$

If $X$ further allows for an $\ell^p$ norm based representation, we obtain the following condition.

**Proposition 5.** Let $\{X(s), s \in S\}$ be a simple max-stable process with representation (3) and $S_1, S_2 \subset S$ finite and disjoint. Then, we have

$$\theta(S_1 \cup S_2) \geq 2^{1/p} \frac{\theta(S_1) + \theta(S_2)}{2}.$$  

**Proof.** Let $S_1 = \{s_1, s_2, \ldots, s_{k_1}\}$ and $S_2 = \{s_{k_1+1}, \ldots, s_{k_1+k_2}\}$ and let $\{e_1, \ldots, e_{k_1+k_2}\}$ denote the standard basis in $\mathbb{R}^{k_1+k_2}$. As the function $(x_1, \ldots, x_{k_1+k_2}) \mapsto l_{s_1, \ldots, s_{k_1+k_2}}(x_1^{1/p}, \ldots, x_{k_1+k_2}^{1/p})$ is conditionally negative definite by Theorem 3, inequality (10) particularly holds true for $n = 2$, $a_1 = 1$, $a_2 = -1$, $x^{(1)} = \sum_{i=1}^{k_1} e_i$ and $x^{(2)} = \sum_{i=k_1+1}^{k_1+k_2} e_i$, i.e.

$$l_{s_1, \ldots, s_{k_1+k_2}} \left( \frac{1}{p} \sum_{i=1}^{k_1} e_i \right) + l_{s_1, \ldots, s_{k_1+k_2}} \left( \frac{1}{p} \sum_{i=k_1+1}^{k_1+k_2} e_i \right) - 2l_{s_1, \ldots, s_{k_1+k_2}} \left( \sum_{i=1}^{k_1+k_2} e_i \right) \leq 0.$$  

Using the homogeneity and property (12) of the stable tail dependence function, we obtain

$$2^{1/p} l_{s_1, \ldots, s_{k_1}}(1, \ldots, 1) + 2^{1/p} l_{s_{k_1+1}, \ldots, s_{k_1+k_2}}(1, \ldots, 1) - 2l_{s_1, \ldots, s_{k_1+k_2}}(1, \ldots, 1) \leq 0.$$  

As $\theta(\tilde{S}) = l_{\tilde{S}}(1, \ldots, 1)$ for any finite $\tilde{S} \subset S$, this yields the assertion. \qed

Of particular interest in extreme value analysis is the case of the pairwise extremal coefficient function [cf. 29, 26] where $\tilde{S} = \{s_1, s_2\}$. Then, Proposition 5 provides the lower bound

$$\theta(\{s_1, s_2\}) \geq 2^{1/p} \quad \text{for all } s_1 \neq s_2 \in S.$$  \hspace{1cm} (15)

For the particular case of model (2), this bound has already been found by Reich and Shaby [21] motivating their interpretation of model (2) as a max-stable process with nugget effect in analogy to the Gaussian case.

The bound (15) and the characterization of simple max-stable processes with a $\ell^p$ norm based representation given in Theorem 4 can be used to show the existence of a minimal $\ell^p$ norm based representation of a simple max-stable process $X$, i.e. the existence of some $p_{\min}(X)$ such that $X \in MS_p$ if and only if $p \geq p_{\min}(X)$.

**Corollary 6.** Let $\{X(s), s \in S\}$ be a simple max-stable process such that not all random variables $\{X(s)\}_{s \in S}$ are independent. Then, there exists a number $p_{\min}(X) \in (1, \infty]$ such that $X \in MS_p$ if and only if $p \geq p_{\min}(X)$.
Proof. By de Haan [3], any simple max-stable process $X$ satisfies $X \in \mathcal{MS}_\infty$. Thus, the assertion follows directly if

$$p_{\min}(X) = \inf \{ p > 1 : X \in \mathcal{MS}_p \} = \infty.$$  

Thus, we restrict ourselves to the case that $p_{\min}(X) < \infty$. As not all the random variables \{X(s)\}_{s \in S} are independent, there exist $s_1, s_2 \in S$ and $\varepsilon > 0$ such that $\theta(\{s_1, s_2\}) < 2^{1/(1+\varepsilon)}$. Hence, by Equation (13), we obtain that $p_{\min}(X) \geq 1 + \varepsilon$. Using the fact that $\mathcal{MS}_p \subseteq \mathcal{MS}_q$ for $p < q$, it remains to show that $X \in \mathcal{MS}_{p_{\min}(X)}$. By Theorem 3 for all pairwise distinct $s_1, \ldots, s_n \in S$, $n \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{R}$ such that $\sum_{i=1}^m a_i = 0$, $x^{(1)}, \ldots, x^{(m)} \in [0, \infty)^n$ and $m \in \mathbb{N}$ we have that

$$\sum_{i=1}^m \sum_{j=1}^n a_ia_j l_{s_1,\ldots,s_n}(x^{(i)}_1 + x^{(j)}_1)^{1/p}, \ldots, (x^{(i)}_n + x^{(j)}_n)^{1/p} \leq 0$$

for all $p > p_{\min}(X)$. By the continuity of $l_{s_1,\ldots,s_m}$, the same holds true for $p = p_{\min}(X)$, and, thus, by Theorem 3, $X \in \mathcal{MS}_{p_{\min}(X)}$. \qed

For any $p \in (1, \infty]$, we now give an example for a simple max-stable process $X^{(p)}$ such that $p_{\min}(X^{(p)}) = p$. Thus, we will also see that

$$\mathcal{MS}_p \subseteq \mathcal{MS}_q \subset \mathcal{MS}_\infty = \mathcal{MS}, \quad 1 < p < q < \infty.$$  

We consider the process $X^{(p)}_{\log} \in \mathcal{MS}_p$ which possesses an $\ell^p$ norm based representation (3) with $W(s) = 1$ a.s. for all $s \in S$. From Equation (3), for pairwise distinct $s_1, \ldots, s_n \in S$, we obtain the finite-dimensional distributions

$$\mathbb{P} \left( X^{(p)}_{\log}(s_i) \leq x_i, 1 \leq i \leq n \right) = \exp \left\{ - \left( \sum_{i=1}^n x_i^{-p} \right)^{1/p} \right\}, \quad x_1, \ldots, x_n > 0,$$

i.e. all the multivariate distributions are multivariate logistic distributions [12]. Thus, the process $X^{(p)}_{\log}$ has pairwise extremal coefficients $\theta(s, t) = 2^{1/p}$ for all $s, t \in S$, $s \neq t$. From Equation (13), it follows that $X^{(p)}_{\log} \notin \mathcal{MS}_{p'}$ for $p' < p$. Consequently, we have $p_{\min}(X^{(p)}_{\log}) = p$.

5. Properties of Processes with $\ell^p$ Norm Based Representation

In this section, we will analyze several properties of simple max-stable processes with an $\ell^p$ norm based representation in more detail. We will particularly focus on properties related to the dependence structure of the process such as stationarity, ergodicity and mixing. A characteristic feature of a process $X$ with $\ell^p$ norm based representation (3) is the additional noise introduced via the process \{U^{(p)}(s), s \in S\}. Thus, we will compare the process $X$ to a “denoised” reference process

$$\overline{X}(s) = \max_{i \in \mathbb{N}} A_i W_i^{(p)}(s), \quad s \in S,$$

i.e. the simple max-stable process constructed via the same spectral functions used in the original ($\ell^\infty$ norm based) spectral representation (1).
Proposition 7. Let \( \{X(s), \ s \in S\} \) be a simple max-stable process with \( \ell^p \) norm based representation (3) with \( p \in (1, \infty) \). Then, for the pairwise extremal coefficients \( \theta(\{s_1, s_2\}) \), we obtain the bounds:

\[
\mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right) \leq \theta(\{s_1, s_2\}) \leq 2^{1/p} \left[ \mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right) \right]^{1-p^{-1}}.
\]

Proof. In the case \( p = \infty \), we have

\[
\theta(\{s_1, s_2\}) = \mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right),
\]

which equals both the lower and the upper bound given in the assertion.

Now, let \( p \in (1, \infty) \). Then, we have the lower bound

\[
\theta(\{s_1, s_2\}) = \mathbb{E} \left\{ \left( W^{(p)}(s_1)^p + W^{(p)}(s_2)^p \right)^{1/p} \right\} \geq \mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right).
\]

Further, for any \( p < r < \infty \) and \( w \in [0, \infty)^2 \), we obtain

\[
\|w\|^p \leq \|w\|^p_{\frac{r}{p}} \cdot \|w\|^p_{\frac{1-r}{1-p}}
\]

[cf. 13, Thm. 18], or equivalently

\[
\|w\|_p \leq \|w\|^{\frac{1-r}{1-p}}_1 \cdot \|w\|^{\frac{1-p^{-1}}{1-r}}_\infty.
\]

As \( r \to \infty \), this yields

\[
\|w\|_p \leq \|w\|^{1/p}_1 \cdot \|w\|^{1-p^{-1}}_\infty.
\]

Taking the expectation of \( w \) with respect to the joint distribution of \( W^{(p)}(s_1) \) and \( W^{(p)}(s_2) \) and applying Hölder’s inequality, we obtain the upper bound

\[
\theta(\{s_1, s_2\}) = \mathbb{E} \left\{ \left( W^{(p)}(s_1)^p + W^{(p)}(s_2)^p \right)^{1/p} \right\}
\leq \mathbb{E} \left\{ \left( W^{(p)}(s_1) + W^{(p)}(s_2) \right)^{1/p} \cdot \max\{W^{(p)}(s_1), W^{(p)}(s_2)\}^{1-p^{-1}} \right\}
\leq \left[ \mathbb{E} \left\{ W^{(p)}(s_1) + W^{(p)}(s_2) \right\}^{1/p} \right] \left[ \mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right) \right]^{1-p^{-1}}.
\]

The assertion follows from \( \mathbb{E}\{W^{(p)}(s_1)\} = \mathbb{E}\{W^{(p)}(s_2)\} = 1 \). \( \Box \)

Note that Proposition 7 relates the extremal coefficients \( \theta(\{s_1, s_2\}) \), \( s_1, s_2 \in S \), to the terms \( \mathbb{E} \left( \max\{W^{(p)}(s_1), W^{(p)}(s_2)\} \right) \) which are the extremal coefficients of the process

\[
\overline{X}(s) = \max_{i \in \mathbb{N}} A_i W^{(p)}(s), \quad s \in S.
\]

As the processes \( X \) and \( \overline{X} \) just differ by the Fréchet noise process \( U^{(p)} \), we will call \( \overline{X} \) the denoised max-stable process associated to \( X \). From Proposition 7 we obtain that extremal
dependence of the process $X$ is always weaker than dependence of the associated denoised process – as expected.

In the following, we will consider the case that $S = \mathbb{Z}$. In this case, properties such as stationarity, ergodicity or mixing are of interest. For a simple max-stable \( \{X(s), s \in \mathbb{Z}\} \) with representation (1), necessary and sufficient conditions for these properties can be expressed in terms of the distribution of the spectral function $V$: By Kabluchko et al. [15], $X$ is stationary if and only if

\[
\mathbb{E} \{V(s_1)^{u_1} \cdots V(s_n)^{u_n}\} = \mathbb{E} \{V(s_1 + s)^{u_1} \cdots V(s_n + s)^{u_n}\}
\]  

(17)

for all $n \in \mathbb{N}$, $s, s_1, \ldots, s_n \in \mathbb{Z}$ and $u_1, \ldots, U_n \in [0, 1]$ such that $\sum_{i=1}^n u_i = 1$. For stationary simple max-stable processes, Kabluchko and Schlather [14] give conditions for ergodicity and mixing in terms of the pairwise extremal coefficients $\theta(\{s_1, s_2\}) = \mathbb{E}(\max\{V(s_1), V(s_2)\})$, stating that $X$ is mixing if and only if

\[
\lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} \theta(\{0, k\}) = 2,
\]

and $X$ is ergodic if and only if

\[
\lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} \theta(\{0, k\}) = 2,
\]

respectively.

Now, we transfer these results to a max-stable process $X$ with $\ell^p$ norm based representation (3) giving necessary and sufficient conditions in terms of $W^{(p)}$. For the associated denoised process $\overline{X}$, Equations (17)–(19) depend on the distribution $W^{(p)} = V$ only, while the structure of the process $X$ is more difficult as we have $V(\cdot) = [\Gamma(1 - p^{-1})]^{-1}U^{(p)}(\cdot)W^{(p)}(\cdot)$ (cf. Proposition 2). The following result, however, shows that those conditions simplify to the conditions for the associated denoised process $\overline{X}$.

**Proposition 8.** Let $\{X(s), s \in \mathbb{Z}\}$ be a simple max-stable process with $\ell^p$ norm based representation (3) and let $\overline{X}$ be the denoised process associated to $X$. Then, the following holds:

1. $X$ is stationary if and only if $\overline{X}$ is stationary.

If $X$ is stationary, we further have

2. $X$ is mixing if and only if $\overline{X}$ is mixing.

3. $X$ is ergodic if and only if $\overline{X}$ is ergodic.

**Proof.** 1. By Kabluchko et al. [15] and Proposition 2 the process $X$ is stationary if and only if (17) holds for $V(\cdot) = [\Gamma(1 - p^{-1})]^{-1}U^{(p)}(\cdot)W^{(p)}(\cdot)$. The left-hand side of (17) equals

\[
\mathbb{E} \{V(s_1)^{u_1} \cdots V(s_n)^{u_n}\} = \frac{1}{\Gamma(1 - p^{-1})} \mathbb{E} \left\{ \prod_{i=1}^n U^{(p)}(s_i)^{u_i} W^{(p)}(s_i)^{u_i} \right\}
\]

\[
= \frac{1}{\Gamma(1 - p^{-1})} \mathbb{E} \left\{ \prod_{i=1}^n U^{(p)}(s_i)^{u_i} \right\} \mathbb{E} \left\{ \prod_{i=1}^n W^{(p)}(s_i)^{u_i} \right\}
\]

\[
= \left( \prod_{i=1}^n \frac{\Gamma(1 - u_ip^{-1})}{\Gamma(1 - p^{-1})} \right) \mathbb{E} \left\{ \prod_{i=1}^n W^{(p)}(s_i)^{u_i} \right\},
\]


The mixing properties of a stochastic process

Remark 4. Precisely by its mixing coefficients. For two subsets \( C \) and \( \tilde{C} \)

where, for each \( \tilde{S} \subset S \), the probability measure \( \mathcal{P}_\tilde{S} \) denotes the distribution of the restricted process \( \{X(s), s \in \tilde{S}\} \) on the space of non-negative functions on \( \tilde{S} \) endowed with the Borel-\( \sigma \) algebra \( \mathcal{C}_\tilde{S} \).

For the case of a max-stable process, Dombry and Éyi-Minko \( \text{(5)} \) provide the upper bound

\[
\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)].
\]
Applying Proposition 7, we obtain
\[
\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)] \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \mathbb{E}(\max\{W^{(p)}(s_1), W^{(p)}(s_2)\})],
\]

i.e. the upper bound for a process with \( \ell^p \) norm based representation \( \text{(3)} \) is lower than the bound for the associated denoised process.

As Proposition 8 states, a max-stable process with \( \ell^p \) norm based representation \( \text{(3)} \) shares properties such as stationarity, ergodicity and mixing with the associated denoised process. In particular, the “noisy” analogues of well-studied max-stable processes might be used without changing any of these properties.

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