We investigate a compact spherically symmetric relativistic body with anisotropic particle pressure profiles. The distribution possesses characteristics relevant to modeling compact stars within the framework of general relativity. For this purpose, we consider a spatial metric potential of Korkina and Orlyanskii [Ukr. Phys. J. 36, 885 (1991)] type in order to solve the Einstein field equations. An additional prescription we make is that the pressure anisotropy parameter takes the functional form proposed by Lake [Phys. Rev. D 67, 104015 (2003)]. Specifying these two geometric quantities allows for further analysis to be carried out in determining unknown constants and obtaining a limit of the mass-radius diagram, which adequately describes compact strange star candidates like Her X-1 and SMC X-1. Using the anisotropic Tolman-Oppenheimer-Volkoff equations, we explore the hydrostatic equilibrium and the stability of such compact objects. Then, we investigate other physical features of this models, such as the energy conditions, speeds of sound and compactness of the star in detail and show that our results satisfy all the required elementary conditions for a physically acceptable stellar model. The results obtained are useful in analyzing the stability of other anisotropic compact objects like white dwarfs, neutron stars, and gravastars.

PACS numbers:
Keywords: general relativity; embedding class one; anisotropic fluid; compact stars

I. INTRODUCTION

In a theoretical sense, stars are formed in gas and dust clouds with a nonuniform matter distribution and scattered throughout most galaxies. In astrophysics, the term compact object usually refers collectively to white dwarfs and neutron stars that form at the end of their stellar evolution. Typically, for such compact sources it is necessary to investigate the microscopic composition and properties of dense matter on extreme conditions. This is because at such extreme densities nuclear matter may consist not only of nucleons and leptons but also several exotic components in their different forms and phases such as mesons, hyperons and baryon resonances as well as strange quark matter (SQM). However, it is still not possible to find a comprehensive description of the extremely dense matter in a strongly interacting regime. Therefore it is useful to investigate an exact composition and the nature of particle interactions in the interior of this kind of object. To determine the structure of a compact star within the framework of the general theory of relativity, a widely followed route is to specify an equation of state and then solve the Einstein field equations. Customarily this avenue has proved fruitful when the law of energy conservation is used in the form of the Tolman-Oppenheimer-Volkoff (TOV) equation (see [1, 2]) or the equation of hydrodynamical equilibrium.

It is possible that anisotropic matter is an important ingredient in many astrophysical objects such as stars, gravastars etc. Historically, considerable effort has been dedicated to gaining a comprehensive understanding of the properties of anisotropic matter, with the hope of producing physically viable models of compact stars. In particular, compact stars may soon provide information about the gravitational interaction in an extreme gravitational...
environments. Their extreme internal density and strong gravity hints that pressure within such compact objects may not be in the form of a perfect fluid i.e., there exist two different kinds of interior pressures namely, the radial and tangential pressure \[ P_r - P_t = gq^2r^2 \] where \( g \) is a non-zero constant, whereas Barreto et al. \[11\] considered electrically charged matter as anisotropic matter and so on. On the other hand, Mak and Harko in \[31\] found an exact solution of Einstein’s field equations for an anisotropic fluid sphere. In a recent treatment, Herrera and Barreto studied polytropes for anisotropic matter both in the Newtonian \[13\] and the general relativistic regimes \[14, 15\]. It should be worth noting that a simple algorithm for all static spherically symmetric anisotropic solutions of Einstein’s equations have been analyzed in \[10\]. Models for charged anisotropic solutions with a quadratic equation of state have been in \[17, 18\]. Maharaj and Maartens \[19\] have critically examined models of static anisotropic fluid spheres under the assumption of uniform energy density.

Characteristically the mathematical problem of developing models of anisotropic fluid spheres amounts to solving a coupled system of three independent nonlinear partial differential equations in five geometrical and dynamical variables namely the metric potentials \( (\nu) \) and \( (\lambda) \) and the density \( (\rho) \), radial pressure \( (p_r) \) and tangential pressure \( (p_t) \). Because the system is underdetermined it is possible for any metric to solve the system of field equations. This approach is not necessarily productive as all control over the physics of the problem is relinquished. For example, an equation of state is not likely to be present and this is often viewed as a standard for perfect fluids. Note that the difference between the pressures \( p_r - p_t \) is known as the anisotropic parameter denoted by \( \Delta \). The approach we follow in this paper is to specify one of the gravitational potentials as the Vaidya-Tikekar \[20\] potential which has been shown to model superdense stars. Then we specify the behavior of the anisotropy parameter with the additional help of Lake’s potential. Finally we endeavor to solve the pressure anisotropy equation to reveal the general behavior of the remaining gravitational potential. Once the model is complete we are in a position to investigate its physical properties.

In the present paper, our main motivation is to obtain an exact solution for a static anisotropic fluid sphere to the Einstein equations, employing the Korkina and Orlyanskii \[21\] ansatz for the metric potential. The outline of the paper will be as follows: Following a brief introduction in Sec. \( \text{I} \), we consider a spherical symmetric metric and present the structure equations for anisotropic fluid distributions in Sec. \( \text{II} \). Paying particular attention to solving the system of equations analytically, we assume a particular form of metric potential and obtain the expression for density and pressures in Sec. \( \text{III} \). Next, in Sec. \( \text{IV} \), we discuss some physical features of the model maintaining the regularity and matching conditions for the solution and obtained results compared with observational data. Finally, Sec. \( \text{V} \), is devoted to closing remarks.

### II. METRIC AND THE EINSTEIN FIELD EQUATIONS

Consider the metric for static fluid distributions with spherical symmetry is given by

\[
\text{d}s^2 = e^{\nu(r)} \text{d}t^2 - e^{\lambda(r)} \text{d}r^2 - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),
\]

where \( \nu = \nu(r) \) and \( \lambda = \lambda(r) \) are the two unknown metric functions of the radial coordinate \( r \) alone. These potentials uniquely determine the surface redshift and gravitational mass functions, respectively. The matter content is assumed to be that of an anisotropic fluid. Such a stress-energy tensor can be written as

\[
T_{\mu\nu} = (\rho + p_t) u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t) \chi^\mu \chi_\nu,
\]
Consequently, $\Delta = p - t$ is the pressure anisotropy of the fluid. Using Eqs. (4) and (5), one can obtain the simple form of the anisotropic factor

$$\Delta = \kappa (p_t - p_r) = e^{-\lambda} \left[ \frac{\nu''}{2} - \frac{\lambda' \nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{1}{r^2} \right] + \frac{1}{r^2},$$

where the prime denotes a derivative with respect to the radial variable, r. Here, $\rho$ is the energy density, while $p_r$ and $p_t$ are radial and transverse pressures of the fluid distribution. We consider this discussion by $p_t \neq p_r$. Consequently, $\Delta = p_t - p_r$ is denoted as the anisotropy factor according to Herrera and Leon [22], and its measure the pressure anisotropy of the fluid. It is to be noted that at the origin $\Delta = 0$ is a particular case of an isotropic pressure. Using Eqs. (4) and (5), one can obtain the simple form of the anisotropic factor

$$\Delta = \kappa (p_t - p_r) = e^{-\lambda} \left[ \frac{\nu''}{2} - \frac{\lambda' \nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{1}{r^2} \right] + \frac{1}{r^2}.$$

However, a force due to the anisotropic nature is represented by $\Delta/r$, which is repulsive, if $p_t > p_r$, and attractive if $p_t < p_r$ of the stellar model. Throughout the discussion we assume the natural units $\kappa = 8\pi$ and $G = c = 1$.

### III. EXACT SOLUTION OF THE MODELS FOR ANISOTROPIC STARS

In seeking solutions to Einsteins field equations for an anisotropic fluid matter we have five unknown functions of r, namely, $\rho(r)$, $p_r(r)$, $p_t(r)$, $\nu(r)$ and $\lambda(r)$. In place of the pressure functions, we may invoke the anisotropy parameter $\Delta$ expressing the difference between the tangential and radial pressures. In either case we have 3 equations and 5 unknown functions. For this reason, to solve these equations analytically one has to specify two variables a priori. We will demonstrate this feature in an explicit manner for physically acceptable stellar models.

Introducing the metric ansatz

$$e^{-\lambda(r)} = \frac{1 + Cr^2}{1 + 2Cr^2},$$

and the redefinition $y(x) = e^{x^2/2}$ where $x = Cr^2$ for some constant $C$, the field equations may be transformed to an equivalent form conducive to locating exact solutions more efficiently. This form of the metric potential was initially considered by Vaidya and Tikekar [20] in studying spheroidal spacetimes governing the behavior of superdense stars and subsequently utilized in the work of Korkina and Orlyanskii [21]. The Korkina-Orlyanskii model was extensively studied in [23]. Observe that this choice of metric potential yields a singularity free solution at $r = 0$ and the metric coefficient is $e^{\lambda(0)} = 1$. This will become relevant in the physical analysis later. Using Eq. (6) and Eq. (7), we obtain

$$\frac{d^2 y}{dx^2} - \frac{1}{2(1 + x)(1 + 2x)} \frac{dy}{dx} + \frac{(1 + 2x)^2}{4x(1 + 3x + 2x^2)} \left[ \frac{2x}{(1 + 2x)^2} - \frac{\Delta}{C} \right] y = 0,$$

in our transformed coordinates. The above Eq.(8) incorporates the anisotropy factor $\Delta$. Note that in the case of an isotropic pressure $\Delta = 0$ then at the centre $y = (1 + x)^{1/2}$ and we have a particular solution of Eq.(7). This form of $y$ has physical relevance as discussed by Lake [24].

The master equation (8) is a second order differential equation and is difficult to solve by standard techniques. Moreover it is still under-determined. We elect to prescribe the anisotropic parameter $\Delta$ with the help of a slightly modified version of Lake’s [24] potential in the form

$$y = (1 + x - \beta x)^{1/2}, \quad \beta > 0, \quad \beta > 0,$$

where $u_\mu$ is the four-velocity and $\chi_\mu$ is the unit spacelike vector in the radial direction. Thus, the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ provides the following gravitational field equations ($G_{\mu\nu}$ is the Einstein tensor)

$$\kappa \rho(r) = \frac{\lambda}{r} e^{-\lambda} + \frac{(1 - e^{-\lambda})}{r^2}, \quad (3)$$

$$\kappa p_r(r) = \frac{\nu}{r} e^{-\lambda} - \frac{(1 - e^{-\lambda})}{r^2}, \quad (4)$$

$$\kappa p_t(r) = \left[ \frac{\nu''}{2} - \frac{\lambda' \nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{1}{r^2} \right] e^{-\lambda}, \quad (5)$$

and

$$G_{\mu\nu} = \frac{8\pi}{\kappa} T_{\mu\nu}, \quad \mu = t, \quad \nu = t$$

where $\mu$ and $\nu$ are radial and transverse directions. The above Eq.(8) incorporates the anisotropy factor $\Delta$. Note that in the case of an isotropic pressure $\Delta = 0$ then at the centre $y = (1 + x)^{1/2}$ and we have a particular solution of Eq.(7). This form of $y$ has physical relevance as discussed by Lake [24].

The master equation (8) is a second order differential equation and is difficult to solve by standard techniques. Moreover it is still under-determined. We elect to prescribe the anisotropic parameter $\Delta$ with the help of a slightly modified version of Lake’s [24] potential in the form

$$y = (1 + x - \beta x)^{1/2}, \quad \beta > 0, \quad (9)$$
for a positive constant $\beta$. This approach is not novel (as can be verified in the work in [25]). Substituting the modified Lake potential for $y$ from Eq.(9) in Eq.(8) we obtain

$$\Delta = \frac{\beta C x [3 - \beta - 4 \beta x + 4 x]}{(1 + 2 x)^2 (1 + x - \beta x)^2},$$

(10)

for the measure of pressure anisotropy. Obviously $\Delta = 0$ when $\beta = 0$ thus regaining the neutral isotropic perfect fluid solution of solution of Korkina and Orlyanskii [21]. However for a physical anisotropic solution the anisotropy factor $\Delta$ must be finite and positive. Then using the value of $\Delta$ from Eq.(10) in Eq.(8), we have

$$\frac{d^2 y}{dx^2} - \frac{1}{2 (1 + x) (1 + 2 x)} \frac{dy}{dx} + \frac{(1 + 2 x)^2}{4 x (1 + 3 x + 2 x^2)} \left[ \frac{2 x}{(1 + 2 x)^2} - \frac{\beta x [3 - \beta + 4 (1 - \beta) x]}{(1 + 2 x)^2 (1 + x - \beta x)^2} \right] y = 0,$$

(11)

which is a second order linear differential equation in $y$. Reverting to the original coordinates the exact solution of the above differential equation may be expressed as

$$y = (1 + C r^2 - \beta C r^2)^{1/2} \left[ A + B \left( \sqrt{2} \ln \left| \frac{(1 - \beta) (4 C r^2 + 3)}{4} + \frac{F(r)}{\sqrt{2}} \right| + G(r) \right) \right],$$

(12)

where we denote $x = C r^2$, $F(r) = \sqrt{(1 - \beta)^2 (2 C^2 r^4 + 3 C r^2 + 1)}$, $G(r) = -\sqrt{\frac{\beta + 1}{\beta}} \ln \left| 1 + \frac{2 \beta (\beta + 1) + 2 \beta (\beta + 1) r}{(3 \beta + 1) (1 + C r^2 - \beta C r^2)} \right|.$

The obtained metric function $y = e^{\nu/2}$ is finite and monotonically increasing throughout the stellar interior as depicted in Fig. 1 (second row, left panel). Accordingly we obtain the energy density, radial pressure and transverse pressures in the form

$$\rho, p_r, p_t.$$
\[
\frac{8\pi p_r}{C} = \frac{(1 - 2\beta - \beta Cr^2 + 2Cr^2)}{(1 + 2Cr^2)(1 + Cr^2 - \beta Cr^2)} + \frac{4(1 - \beta)B}{y} \left[ \frac{2 + (1 + \beta) f(r)}{(1 + Cr^2 - \beta Cr^2) g(r)} - 2\sqrt{\beta(\beta + 1)}(\beta + 1)h(r) \right],
\]
\[
\frac{8\pi \rho}{C} = \frac{3 + 2Cr^2}{(1 + 2Cr^2)^2},
\]
\[
\frac{8\pi p_t}{C} = \frac{8\pi p_r}{C} + \frac{\beta Cr^2 [3 - \beta - 4\beta Cr^2 + 4Cr^2]}{(1 + 2Cr^2)^2 (1 + Cr^2 - \beta Cr^2)^2},
\]

where we have set

\[
f(r) = (1 - \beta)[1 + Cr^2 + \beta(2 + 3Cr^2)],
g(r) = (1 - 3\beta)(1 + Cr^2 - \beta Cr^2) + 2\sqrt{\beta(\beta + 1)}\left[ \sqrt{\beta(\beta + 1)} + h(r) \right],
\]

\[
h(r) = \sqrt{1 + \beta^2} [2Cr^2 + 3Cr^2 + 1] \text{ for simplicity.}
\]

Observe again from Eq.\((15)\) that we have \(p_t = p_r\), when \(r = 0\). This is expected at the center of the star. To further examine the physical character of these solutions for physical admissibility it is required that

- the energy density is positive definite and its gradient is negative everywhere within the stellar interior,
- for an anisotropic fluid distribution radial and tangential pressures are positive definite and the radial pressure gradient is negative within the radii.

To examine the consequences more closely we take the first order differentiation with respect to the radial coordinate and obtain

\[
8\pi \frac{dp_r}{Cdr} = 2Cr \left[ -\frac{4P_1(r)}{(1 + 2Cr^2)^2} + \frac{4(1 + Cr^2)[P_2(r) - P_3(r) - P_4(r)]}{(1 + 2Cr^2)^2} + \frac{2}{(1 + 2Cr^2)^2} \right],
\]
\[
8\pi \frac{dp_t}{Cdr} = 8\pi \frac{dp_r}{Cdr} + \frac{2\beta Cr [-3 + Cr^2 + 18Cr^2r^4 + 16Cr^2(1 + 6Cr^2 + 16Cr^5r^4) - \beta P_5(r)]}{(1 + 2Cr^2)^3 (1 + Cr^2 - \beta Cr^2)^3},
\]

and

\[
8\pi \frac{d\rho}{Cdr} = -\frac{4Cr(5 + 2Cr^2)}{(1 + 2Cr^2)^3}.
\]

For notational simplicity we choose

\[
P_1(r) = \frac{1 - \beta}{y(1 + Cr^2 - \beta Cr^2)} - \frac{B(1 - \beta)(1 + 2Cr^2)}{\sqrt{1 + Cr^2 - \beta Cr^2} \sqrt{1 + 2Cr^2 + 2Cr^4}},
\]
\[
P_2(r) = \frac{B(1 - \beta)^2(1 + 2Cr^2) - \beta(1 - \beta)(1 + 2Cr^2)}{\sqrt{1 + Cr^2 - \beta Cr^2} \sqrt{1 + 2Cr^2 + 2Cr^4} \sqrt{1 + 3Cr^2 + 2Cr^4}},
\]
\[
P_3(r) = \frac{B(1 - \beta)(1 + 2Cr^2)}{\sqrt{1 + Cr^2 - \beta Cr^2} \sqrt{1 + 2Cr^2 + 2Cr^4} \sqrt{1 + 3Cr^2 + 2Cr^4}},
\]
\[
P_4(r) = \frac{B(1 - \beta)(1 + 2Cr^2)}{\sqrt{1 + Cr^2 - \beta Cr^2} \sqrt{1 + 2Cr^2 + 2Cr^4} \sqrt{1 + 3Cr^2 + 2Cr^4}},
\]
\[
P_5(r) = \frac{1 - 2Cr^2 + 24Cr^2 + 32Cr^5}{1 + 2Cr^2 + 2Cr^4}.
\]

which convey information on the maximum value of the central density and central pressure. Interestingly, the radial pressure \(p_r\) vanish but the tangential pressure \(p_t\) does not vanishes at the boundary (see Fig.\((1)\)).

**IV. PHYSICAL FEATURES AND STABILITY OF ANISOTROPIC COMPACT STARS**

To confirm that we are not losing essential physics for a stellar structure at the interior and outer radius, we perform some analytical calculations. Then we discuss how the equilibrium structure and stability of strange stars are affected due to anisotropic pressure. In essence, this is done by studying general physical properties and plotting several figures for some of the compact star candidates. The solutions found in this paper may be used to study relativistic compact stellar objects.
TABLE I: The approximate values of the masses $M$, radii $R$, and the constants $A$, $B$ and $C$ for the compact stars

| Compact Stars | $CR^2$ | $\beta$ | $A$ | $B$ | $C$ (Km. $^{-2}$) | $M(M_\odot)$ | $R$ | $M/R$ |
|---------------|--------|---------|-----|-----|------------------|---------------|-----|-------|
| Her X-1       | 0.65087| 0.74391 | 0.2846647| -0.11649 | $8.276327 \cdot 10^{-3}$ | 0.8505 | 8.8605 | 0.14146 |
| SMC X-1       | 0.66755| 0.73157 | 0.282161 | -0.114127 | $5.800099 \cdot 10^{-3}$ | 1.04 | 10.7506 | 0.142956 |

A. Boundary Condition

It is known that all astrophysical objects are immersed in vacuum or almost vacuum spacetime and at the juncture interface we match the interior spacetime ($\mathcal{M}_-$) to an appropriate exterior vacuum region ($\mathcal{M}_+$). In the case at hand, the exterior is described by the Schwarzschild geometry, i.e.,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(19)

where $M$ is the mass within a sphere of radius $R$. In order to match smoothly on the boundary surface $r = R$, we impose the Israel-Darmois junction conditions for this system which are tantamount to the following two conditions [20]:

$$e^{-\lambda(R)} = 1 - \frac{2M}{R} \quad \text{and} \quad e^{\nu(R)} = 1 - \frac{2M}{R},$$

(20a)

$$p_r(r = R) = 0,$$

(20b)

from which the constants $A$, $B$ and $C$ may be determined. Furthermore, using the expression [13], we obtain

$$\frac{A}{B} = -\frac{4 (1 - \beta) (1 + 2CR^2) \Phi(R)}{(1 - 2\beta + \beta CR^2 + 2CR^2)} \left[ 2 + \frac{(1 + \beta) F(R)}{\Phi^2(R) G(R)} + \frac{2\sqrt{\beta (\beta + 1) (\beta + 1) H(R)}}{\Phi^2(R) G(R)} \right] - \Psi(R),$$

(21)

and using the condition [20a], we obtain

$$B = \sqrt{\frac{(1 + CR^2)}{(1 + 2CR^2) (1 + CR^2 - \beta CR^2)}} \times \left[ \frac{1}{2} + \left(\sqrt{2} \ln \left[ \frac{(1 - \beta) (4CR^2 + 3)}{\sqrt{2}} \right] \right) + \Phi_2(R) \right],$$

(22)

for notational simplicity we introduce

$$\Phi(R) = (1 + CR^2 - \beta CR^2)^{1/2}, \quad F(R) = (1 - \beta) \left[ 1 + CR^2 + \beta (2 + 3CR^2) \right],$$

$$G(R) = -(3\beta + 1) \left[ 1 + CR^2 - \beta CR^2 \right] + 2\sqrt{\beta (\beta + 1)} \left[ \sqrt{\beta (\beta + 1)} + H(R) \right],$$

$$H(R) = \sqrt{(\beta - 1)^2 \left( 2CR^2 + 3CR^2 + 1 \right)}, \quad \Psi_1(R) = \left[ \left( \sqrt{2} \ln \left[ \frac{(1 - \beta) (4CR^2 + 3)}{\sqrt{2}} \right] \right) + \Phi_2(R) \right],$$

$$\Psi_2(R) = -\sqrt{\frac{\beta + 1}{\beta}} \ln \left[ 1 - \frac{2\beta (\beta + 1) + 2\sqrt{\beta (\beta + 1)} \Psi_1(R)}{(3\beta + 1)(1 + CR^2 - \beta CR^2)} \right].$$

For a given radius $R$, one can determine the total mass $M$ of the star and vice-versa. It may be mentioned here that bounds on stellar structures, including the mass-radius ratio as proposed by the Buchdahl-Bondi inequality [27, 28], exists and is given by $\frac{2M}{R} \leq \frac{\lambda_0}{2}$. This serves as an upper bound on the total compactness of a static spherically symmetric isotropic fluid sphere (the geometric units $c = G = 1$ have been used). In fact, this bound has been updated in the presence of charged gravitational fields [30, 31] and for a non-zero cosmological constant [29, 31].

The impact of the mass-radius ratio on the equation of state has been considered by Carvalho et al [35] for white dwarfs and neutron stars, by Swift et al for exoplanets and for the nuclear centre by Lattimer [30].

Here, we demonstrate that for some particular values of the parameters and by plugging in the true values for $c$ and $G$ at appropriate places as given in Table 1, we can generate specific masses and radii of some well known pulsars given by Gangopadhyay et al (2013), for the objects Her X-1 and SMC X-1. Some possibilities of such types are tabulated in Table I.

Let us start by considering the surface gravitational redshift $z_S$ of this compact objects with help of the definition $z_S = \frac{\Delta \lambda}{\lambda_c} = \frac{\lambda_0 - \lambda_c}{\lambda_c}$, where $\lambda_c$ is the emitted wavelength at the surface of a nonrotating star and the observed
wavelength $\lambda_0$ received at radial coordinate $r$. Thus, the gravitational redshift, $z_S$ from the surface of the star as measured by a distant observer $(g_{tt}(r) \to -1)$, is given by

$$z_S = -1 + |g_{tt}(r)|^{-1/2} = -1 + \left(1 - \frac{2M}{R}\right)^{-1/2}, \quad (23)$$

where $g_{tt}(r) = e^{\nu(R) - \lambda(R)}$ is the metric function [37]. According to Buchdahl [27] and Straumann [38] a constraint on the gravitational redshift for perfect fluid spheres is given by $z_S < 2$ for isotropic stars. However, we may have a situation for an anisotropic star to admit higher redshifts such as $z_S = 3.84$, as given in Ref. [39]. We have summarized our results in Table II for the stellar structures Her X-1 and SMC X-1 by taking the same values of the constant as mentioned in Table I.

### B. Energy conditions

It is reasonable to expect that models of anisotropic fluids satisfy the energy conditions within the framework of general relativity. There often exists a linear relationship between energy density and pressure of the matter obeying certain restrictions. In view of the above situation, we examine (i) the Null energy condition (NEC), (ii) Weak energy condition (WEC) and (iii) Strong energy condition (SEC) to enhance our investigation of the structure of relativistic spacetimes. More precisely, we have the following proposition:

- **NEC:** $\rho(r) + p_r \geq 0$,
- **WEC** $r$: $\rho + p_t \geq 0$, and $\rho(r) \geq 0$,
- **WEC** $t$: $\rho + p_t \geq 0$ and $\rho(r) \geq 0$,
- **SEC:** $\rho + p_r + 2p_t \geq 0$. \quad (24a-d)

Using the above expression for all the terms in this inequality, one can easily justify the nature of energy condition for the specific stellar configuration Her X-1 and SMC X-1. To further interpret these results we use graphical representation of the energy Conditions, as can be seen from the Fig. 2. For the complicated expression given in equations ((24a-24d)), we only write down the inequalities and plotted the graphs as a function of the radius. As a result, this is eminently clear from Fig 2, that all the energy conditions are satisfied for our proposed model.

### C. Generalized Tolman-Oppenheimer-Volkov Equation

In order to investigate the hydrostatic equilibrium under different forces of compact star for a physically acceptable model we investigate the gravitational and other fluid forces. By considering the generalized Tolman-Oppenheimer-Volkoff (TOV) [10], equation one can clarify the situation for an anisotropic fluid distribution, which is

$$-\frac{MG(r)(\rho + p_t)}{r^2} e^{\frac{\lambda - \nu}{2}} - \frac{dp_t}{dr} + \frac{2}{r}(p_t - p_r) = 0,$$  

(25)
where the effective gravitational mass $M_G(r)$ is defined by

$$M_G(r) = \frac{1}{2} r^2 e^{\frac{-\nu}{r^2}} \nu'. \quad (26)$$

Then Eq. (26) may be simply obtained as

$$-\frac{\nu'}{r^2} (\rho + p_r) - \frac{dp_r}{dr} + \frac{2}{r} (p_t - p_r) = 0, \quad (27)$$

In other words, for this case, the TOV equation (27) expresses the equilibrium condition for anisotropic fluid spheres subject to gravitational, hydrostatic plus another force due to the anisotropic pressure. Combined with the above expressions we can write

$$F_g + F_h + F_a = 0. \quad (28)$$

Let us now attempt to explain the Eq. (28) from an equilibrium point of view, where three different forces are the gravitational force ($F_g$), hydrostatics force ($F_h$) and anisotropic force ($F_a$) with the expressions:

$$F_g = -\frac{\nu' (\rho + p_r)}{2} = \frac{2 C \Gamma (1-\beta)}{2 (1+C r^2 - \beta C r^2)} \left[ 1 + \frac{2 C B r \sqrt{1+2 C r^2}}{\sqrt{1+C r^2 - \beta C r^2}} \right] (\rho + p_r), \quad (29)$$

$$F_h = -\frac{dp_r}{dr} = -\frac{C^2 r}{4 \pi} \left[ \frac{4 P_r^2(\rho_t - P_3)}{(1+2C r^2)^2} + \frac{(1+C r^2) [P_2(r) - P_3(r) - P_4(r)]}{(1+2C r^2)^2} \right], \quad (30)$$

$$F_a = \frac{2}{r} (p_t - p_r) = \frac{2 \beta C^2 r [3-\beta - 4 \beta C r^2 + 4 C r^2]}{(1+2C r^2)^2 (1+C r^2 - \beta C r^2)^2}. \quad (31)$$

To simplify the above equations we also draw two figures for the compact star candidates Her X-1 and SMC X-1 (Fig. 3). Therefore, the claim is that the gravitational force ($F_g$) dominates the hydrostatic ($F_h$) and anisotropic ($F_a$) forces to maintain the equilibrium condition. In other words, the static equilibrium is attainable due to pressure anisotropy, gravitational and hydrostatic forces, which is evident from Fig. 3.

**FIG. 3**: We have plotted different forces, namely, gravitational force ($F_g$), hydrostatics force ($F_h$) and anisotropic force ($F_a$) for studying the effect of the anisotropy in the stability of compact stars. See the subsection C, for details about the stable configuration mode.

### D. Stability Analysis

A very important point that has to be analyzed is the speed of sound propagation $v_s^2$, which is given by the expression $v_s^2 = dp/d\rho$. Naturally the velocity of sound does not exceed the velocity of light. Thus, the behavior of the sound speed is always less than unity, as we fix here $c = 1$. To analyze the situation we investigate the speed of sound along a radial as well as transverse direction. For an anisotropic fluid distribution and for a stable equilibrium
configuration this should always satisfy $0 < v_r^2 = \frac{dp_r}{d\rho} < 1$ and $0 < v_t^2 = \frac{dp_t}{d\rho} < 1$, as in ref. [41] for a subluminal sound speed. In looking for charged solutions, Canuto in [42] argued that the speed of sound should decrease monotonically towards the surface of the star for an ultra-high distribution of matter.

In our case, the sound velocity has been studied with a graphical representation for an anisotropic fluid distribution. To see this we have plotted figures 4, for strange star candidates SMC X-1 and Her X-1. As the resulting expressions are very complicated, we illustrate the causality conditions without mathematical explanation and the values of parameters are tabulated in Table-1. Turning to the case, we see that both $0 \leq v_r^2 \leq 1$ and $0 \leq v_t^2 \leq 1$ everywhere within the anisotropic fluid and monotonic increasing function, which is evident for other compact objects. In addition, it is important to mention the stability of local anisotropic matter distribution, using the concept of Herrera in [41].

According to this $0 < |v_r^2 - v_t^2| \leq 1$, for stable potential. In our case, Fig. 4 (right panel) indicates that there is no sign change for the term $v_t^2 - v_r^2$ within the stellar interior. Therefore, we conclude that our chosen stellar model is stable for our choice of parameters.

V. DISCUSSION

In this paper, we have investigated the nature of anisotropic compact stars. Beginning with the Korkina and Orlyanskii [21] ansatz that $e^{-\lambda(r)} = (1+x)/(1+2x)$ where $x = Cr^2$ together with the choice of an anisotropy function $\Delta$ inspired by a prescription of Lake [24], it has been shown that a number of compact objects are compatible with observational data and we have cited Her X-1 and SMC X-1 as specific examples of this kind of star. Next, by employing the chosen metric functions we simplify the Einstein field equations and study the structure of compact stars.

Based on physical requirements, we matched the interior solution to an exterior vacuum Schwarzschild spacetime and fixed the constants $A$, $B$, and $C$ (see Table-1 for more details). Then using the values of constant parameters it is also possible to determine masses and radii for compact stars. To refine the model further, we show that energy density and pressures are finite at the center and monotonically decreasing towards the boundary which is illustrated in Fig. 1. For an isotropic compact spheres Buchdahl has provided a bound for spherical mas distributions satisfying the inequality $R > (9/8) R_s = (9/4)G M/c^2$ [13] which is stricter than the Schwarzschild bound. From Fig. 1 (first row extreme right), it can be seen that radial pressure vanishes at the boundary, whilst the tangential one is non-vanishing at the stellar surface.

In order to investigate the relevance of our model we consider the masses and radii for some well known pulsars

![Graph](image_url)

**Table II**: The central density, surface density and central pressure for compact star candidates.

| Compact Stars | Central Density (gm/cm$^3$) | Surface Density (gm/cm$^3$) | Central Pressure (dyne/cm$^2$) | Surface Redshift $Z_0$ |
|---------------|-----------------------------|-----------------------------|-------------------------------|------------------------|
| Her X-1       | $1.333316 \times 10^{15}$   | $3.6684 \times 10^{14}$     | $9.2297127 \times 10^{15}$   | 0.180787               |
| SMC X-1       | $9.343815 \times 10^{14}$   | $2.4753015 \times 10^{14}$  | $3.77957 \times 10^{14}$     | 0.183396               |
Her X-1, and SMC X-1 given by Gangopadhyay et al. [14] to fit into the observational data. We demonstrate this by using suitable choices of the constant parameters $A, B,$ and $C$ in Table-I. In the Table-I we have displayed the surface density of the star Her X-1 and SMC X 1 as $3.60864 \times 10^{14} \& 2.4753015 \times 10^{14}$ gm/cm$^3$, which is very high and consistent with ultra compact stars [7, 45, 46]. We also note that the gravitational redshift satisfied $z_s \leq 2$ i.e. it is bounded from above as shown in Table-II.

As a future work, we envisage investigating other forms of metric potentials could exhibit more general behaviour and thereby describe other types of compact objects such as superdense stars for which there exists ample observational data.

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Appendix: Solution generating technique for anisotropic fluids: Details

Since the back drop of such varied developments, Herrera et al. [16] provided a solution generating technique to construct all possible types of solutions of the Einstein field equations for a static spherically symmetric locally anisotropic fluids source in terms of two generating functions. To see this, let us analyze by using the Eq. (4) and Eq. (5), one finds

$$8 \pi (p_r - p_t) = e^{-\lambda} \left[ -\frac{\nu''}{2} + \frac{\lambda' \nu'}{4} - \frac{\nu'^2}{4} + \frac{\nu' + \lambda'}{2r} + \frac{1}{r^2} \right] - \frac{1}{r^2},$$

(32)

To facilitate computations, we introduce new variables as

$$e^\nu(r) = e^{\left( \int 2 \Psi(r) - \frac{z}{r} \right) dr} \quad \text{and} \quad e^{-\lambda(r)} = Z(r)$$

(33)

Putting Eqs. (33) into the Eq. (32), yields

$$Z' + Z \left[ \frac{2Z'}{Z} + 2Z - \frac{6}{r} + \frac{4}{r^2} \frac{1}{Z} \right] = -\frac{2}{Z} \left[ \frac{1}{r^2} + \Pi(r) \right],$$

(34)

where $\Pi(r) = 8 \pi (p_r - p_t)$. Integrating Eq. (34) we obtain $e^\lambda$

$$e^{\lambda(r)} = \frac{\Psi^2 e^{\int \left[ \frac{4 + 2r^2 \Phi^2(r)}{r^2 \Psi^2(r)} \right] dr}}{r^6 \left[ -2 \int \Psi(r) [1 + \Pi(r) r^2 e^{\int \left[ \frac{4 + 2r^2 \Phi^2(r)}{r^2 \Psi^2(r)} \right] dr} dr + D \right]},$$

(35)

which is obtained be Herrera et al. [16]. According to them the the two generating functions are $\Psi(r)$ and $\Pi$. Thus, in our case when we introduce $y(r) = e^{\nu/2}$, the above Eq. (33) turns out to be $\Psi(r) = \left[ y'(r) y + \frac{1}{r} \right]$.

As a consequence of the following algorithm, the generating functions in the present case for anisotropic fluid distribution as follows (using the Eq. (10) and Eq. (12)):

$$\Psi(r) = \frac{C r (1 - \beta)}{(1 + C r^2 - \beta C r^2)} + \frac{2 \tilde{B} C r \sqrt{1 + 2 C r^2}}{y \sqrt{1 + C r^2} \sqrt{1 - \beta C r^2 + C r^2}} + \frac{1}{r},$$

(36)

$$\Pi = -\frac{8 \pi \beta C^2 r^2 \left[ 3 - \beta - 4 \beta C r^2 + 4 C r^2 \right]}{(1 + 2 C r^2)^2 (1 + C r^2 - \beta C r^2)^2},$$

(37)

where $\beta$ is positive constant as indicated in Eq. (9) with $\tilde{B}$ as a constant of integration.

[1] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. 55, 374 (1939).
