Integral Inequalities in Thermodynamics

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Abstract

Thermodynamic systems involving reversible and non-reversible heat transfer are used to derive integral inequalities expected from the Second of Law of Thermodynamics. Then, the inequalities are proved and generalized to higher dimensions with intended application of being used as a quick way of generating numerous other inequalities such as the Weighted Means Inequality. The proofs also serve as a way of establishing the equivalence of the Planck’s statement of the Second Law and the positivity of specific heats.

Keywords: Second Law of Thermodynamics, Integral inequalities, Weighted Means inequality
I. INTRODUCTION

A ubiquitous class of problems found in many thermodynamics textbooks is the one where the student is required to find the equilibrium temperature of a thermodynamics system and to verify that the entropy change in the equilibrating process is consistent with the Second law\(^1\). Here, we essentially turn the problem on its head and ask whether one can state and prove useful inequalities inspired by the form of the second law in specific physical situations. Thus, the Second law provides the motivation for studying certain integral inequalities which are expected to hold independently as mathematical results. Treating the inequalities as purely mathematical results we generalize them to a larger class of functions and to higher number of dimensions. Throughout this paper we will adopt a strategy of generalization illustrated by the figure below:

![Flow chart for generalizing Theorem 1](image)

**FIG. 1. Flow chart for generalizing Theorem 1**

The inequalities derived are important due to their utility as inequality generators. The generalized versions also prove to be useful in extending well known results such as the weighted means inequality and in proving results like the Jensen inequality. We then finally show how these results can be interpreted as a proof of the equivalence of the Planck statement of the Second Law and the positivity of specific heats in certain special cases.

II. MOTIVATION FOR THE REVERSIBLE ONE-DIMENSIONAL INEQUALITY

Consider a closed system of \(N\) identical blocks each with a positive heat capacity \(f(T)\) (some function of temperature) at initial temperatures \(T_i\) where the index \(i\) runs from 1 to
N. If now all pairs of blocks are connected via a Carnot engines that reside inside the system then the system will finally reach an equilibrium temperature $T_0$ dictated by the constraint of no net creation of entropy in the system.

Symbolically:

$$T_i > 0, i = 1 \text{ to } N \& f(T) > 0 \quad \forall \quad T > 0$$

$$\sum_{i=1}^{N} \int_{T_i}^{T_0} \frac{f(T)}{T}dT = 0 \quad (1)$$

The fact that the net-work output of the engines must be non-negative leads to:

$$\sum_{i=1}^{N} \int_{T_i}^{T_0} f(T)dT \geq 0 \quad (2)$$

If this result is interpreted mathematically as an inequality concerning the integral for a positive function then questions about the nature of function for which the inequality works can be asked. Also, it is reasonable to demand a proof of the result that does not use any physics whatsoever for a pre-stated class of functions $f(T)$. We must also prove that the final equilibrium temperature lies between the maximum and the minimum temperatures.

III. THE REVERSIBLE ONE-DIMENSIONAL INEQUALITY:

**Theorem 1**

Suppose $x_i, i = 1$ to $n$ is a sequence of non-decreasing positive real numbers and let $f(x) : [x_1, x_n] \rightarrow \mathbb{R}^+$ be a step-wise continuous function. Define

$$F(y) \equiv \sum_{i=1}^{N} \int_{y}^{x_i} f(x) dx; y \in [x_1, x_n]$$

Then there exists $x_0$ such that -

$$x_0 \in [x_1, x_n] \& F(x_0) = \sum_{i=1}^{N} \int_{x_0}^{x_i} f(x) dx = 0 \quad (3)$$

And the following holds -

$$\sum_{i=1}^{N} \int_{x_0}^{x_i} f(x) dx \geq 0 \quad (4)$$
Proof:

Since, every step-wise continuous function is Riemann- integrable on its domain, \( f(x) \) is also Riemann- integrable over \([x_1, x_n]\). Using the fact that indefinite integral of a Riemann-integrable function is continuous, one can say that each integral inside the sum in eq(3) and hence \( F(y) \) is a well-defined continuous function of \( y \). Since, \( f(x)/x > 0 \ \forall \ x \in [x_1, x_n] \) we have:

\[
F(x_n) \geq 0 \ \& \ F(x_1) \leq 0
\]

And thus by application of Bolzanos theorem to \( F(x) \) on \([x_1, x_n]\) we obtain eq(3). Now, \( x_0 \) partitions \( x_i \) such that

\[
x_p \leq x_0 \leq x_{p+1}; 1 \leq p \leq n - 1
\]

\[
F(x_0) = \sum_{i=1}^{N} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx
\]

\[
F(x_0) = \sum_{i=1}^{p} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx + \sum_{i=p+1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx
\]

\[
\Rightarrow \sum_{i=1}^{p} \int_{x_i}^{x_0} \frac{f(x)}{x} \, dx = \sum_{i=p+1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx \tag{5}
\]

\[
\therefore \sum_{i=p+1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx \leq \sum_{i=p+1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} \, dx
\]

\[
\therefore \sum_{i=1}^{p} \int_{x_i}^{x_0} \frac{f(x)}{x} \, dx \geq \sum_{i=1}^{p} \int_{x_i}^{x_0} \frac{f(x)}{x} \, dx
\]

We get from eq(5)

\[
\sum_{i=p+1}^{n} \frac{1}{x_0} \int_{x_0}^{x_i} f(x) \, dx \geq \sum_{i=1}^{p} \frac{1}{x_0} \int_{x_0}^{x_i} f(x) \, dx
\]

\[
\Rightarrow \sum_{i=p+1}^{n} \int_{x_0}^{x_i} f(x) \, dx \geq \sum_{i=1}^{p} \int_{x_0}^{x_i} f(x) \, dx
\]

\[
\Rightarrow \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) \, dx \geq 0
\]

Comments:

1. The fact that only piece-wise continuity is required for the proof to go through has a nice physical consequence. If the heat capacity of the blocks is a step-function of
temperature then blocks undergo phase transformation/s, which is a situation to which this analysis can be applied by the assumption of piece-wise continuity of \( f(x) \). Despite this eq(1) & eq(2) will continue to hold and so will eq(3) & eq(4).

2. Notice that the proof also works if we replace the \( x \) in the denominator with any continuous increasing function \( g(x) \) which leads to theorem below.

3. If the function \( g(x) \) above is decreasing the sign of the inequality in eq(7) is reversed.

**Theorem 2**

Suppose \( \{x_i\}, i = 1 \text{ to } n \) is a sequence of non-decreasing positive reals and let \( f(x) : [x_1, x_n] \rightarrow \mathbb{R}^+ \) be a step-wise continuous function.

Let \( g(x) : [x_1, x_n] \rightarrow \mathbb{R}^+ \) be a continuous function such that:

\[
g(a) \geq g(b) \quad \forall \ a, b \in [x_1, x_n] \text{ & } a > b
\]

Define

\[
F(y) \equiv \sum_{i=1}^{n} \int_{y}^{x_i} \frac{f(x)}{g(x)} dx \quad y \in [x_1, x_n]
\]

Then there exists a \( x_0 \) such that,

\[
F(x_0) = \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{g(x)} dx = 0 \quad \& \quad x_0 \in [x_1, x_n]
\]  
(6)

And the following holds:

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq 0
\]  
(7)

**IV. PROOF OF RESTRICTED JENSEN INEQUALITY**

We now present a proof of the following restricted form of Jensen’s inequality using Theorem 2.

**Restricted Jensen’s inequality**

Let \( h : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a doubly differentiable function such that

\[
h''(x), h'(x) > 0 \quad \forall \ x > 0
\]  
(8)
And let \( \{y_j\}, j = 1 \) to \( m \) be a sequence of positive reals, then the following holds:

\[
\sum_{j=1}^{m} \frac{h(y_j)}{m} \geq h \left( \frac{\sum_{j=1}^{m} y_j}{m} \right) \tag{9}
\]

**Proof**

Consider a new sequence of positive reals \( \{x_j\}, j = 1 \) to \( m \) which is just a permutation of \( \{y_j\} \), in a non-decreasing order. Choose \( f \) & \( g \) in Theorem 2 such that:

\[
f(x) = g(x) = h'(x) \quad \forall \ x \in [x_1, x_m]
\tag{10}
\]

Now \( h''(x) \) exists and is greater than 0 so that:

\[
f'(x) = g'(x) = h''(x) > 0 \quad \forall \ x \in [x_1, x_m]
\tag{11}
\]

Thus, such a choice of \( f \) & \( g \) can always be made consistent with the conditions of Theorem 2. We can say so because \( f \) & \( g \) are continuous since differentiability of a function implies continuity (from (10)) and because \( g \) is increasing (from (10)). Apply Theorem 2 to \( \{x_j\} \) with the given choice of functions.

By Theorem 2:

\[
F(y) \equiv \sum_{j=1}^{m} \int_{y}^{x_j} \frac{f(x)}{g(x)} \, dx = \sum_{j=1}^{m} (x_j - y); \ y \in [x_1, x_n]
\tag{12}
\]

Thus

\[
x_0 = \frac{\sum_{j=1}^{m} y_j}{m}
\]

Now,

\[
\sum_{j=1}^{m} \int_{x_0}^{x_j} f(x) \, dx \geq 0 \tag{13}
\]

But, from eq(10)

\[
\sum_{j=1}^{m} \int_{x_0}^{x_j} f(x) \, dx = \sum_{j=1}^{m} (h(x_j) - h(x_0))
\tag{14}
\]

Therefore, from eq(12) and eq(13), we get

\[
\frac{\sum_{j=1}^{m} h(y_j)}{m} \geq h \left( \frac{\sum_{j=1}^{m} y_j}{m} \right)
\]

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V. THE ODD-DIMENSIONAL VERSION OF THE REVERSIBLE ONE-DIMENSIONAL INEQUALITY

Theorem 3

Suppose \( \{x_i\}, i = 1 \text{ to } n \) is a sequence of non-decreasing positive reals. Consider \( n \) \( k \)-tuples of reals:

\[
(x_i, x_i, \ldots, x_i); i = 1 \text{ to } n \text{ & } k \text{ is odd as well as positive}
\]

And let \( f : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \rightarrow \mathbb{R}^+ \) be a continuous function.

Let \( g : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \rightarrow \mathbb{R}^+ \) be a continuous function that is non-decreasing with respect to each of its arguments.

Define

\[
F(y) \equiv \sum_{i=1}^{n} \int_{y}^{x_i} \ldots \int_{y}^{x_i} \int_{y}^{x_i} f dz_1 \, dz_2 \ldots dz_k; \ y \in [x_1, x_n]
\]

Then there exists \( x_0 \) such that

\[
F(x_0) = 0 \text{ & } x_0 \in [x_1, x_n]
\]  \hspace{1cm} (15)

And the following holds:

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} \ldots \int_{x_0}^{x_i} \int_{x_0}^{x_i} fdz_1 \, dz_2 \ldots dz_k \geq 0
\]  \hspace{1cm} (16)

Proof:

The proof of Theorem 3 is identical to the proof of Theorem 2 except for the fact that multiple integrals replace the single integral. The oddness of \( k \) is exploited while writing the expression corresponding to eq (6).

Comments:

1. The theorem does not hold for even values of \( k \).

2. The sign of inequality is reversed if \( g \) is decreasing with respect to each of its arguments.
Counter-example in even dimensions

Take \( k = n = 2 \) and \( f = g = z_1 \).

Then the equation for \( x_0 \) turns out to be:

\[
2x_0^2 - 2x_0(x_1 + x_2) + (x_1^2 + x_2^2) = 0
\]

Which has no real roots if \( x_2 \neq x_1 \). Thus \( x_0 \) fails to exist and so the theorem collapses.

VI. GENERALIZATION OF WEIGHTED MEANS INEQUALITY

We propose the following generalization of weighted means inequality:\[6\]:

**Corollary 1**

Suppose \( \{x_i\}, i = 1 \) to \( n \) is a sequence of non-decreasing positive reals. Let \( k \) be an odd positive number and \( b \& a \) be any two reals such that \( a \geq b \& a, b \neq 0, -1 \). Then the following holds:

\[
\exists x_0 \in \mathbb{R}^+ \text{ such that } \sum_{i=1}^{n} (x_i^b - x_0^b)^k = 0
\]

And

\[
\sum_{i=1}^{n} (x_i^a - x_0^a)^k \geq 0
\]

The sign of inequality flips if \( a \leq b \).

**Proof:**

Choose \( f \& g \) in Theorem 3 as follows:

\[
f(x_1, x_2, \ldots, x_n) = x_1^{a-1}x_2^{a-1} \ldots x_n^{a-1}
\]

And

\[
g(x_1, x_2, \ldots, x_n) = x_1^{a-b}x_2^{a-b} \ldots x_n^{a-b}
\]

Clearly \( f \) and \( g \) have all the properties required for Theorem 3 to work and thus we obtain Corollary 1.
If \(a \leq b\) then \(g\) turns decreasing in each of its arguments and thus the sign of inequality flips as mentioned in Comment number 2 on the proof of Theorem 3.

The case where either \(a\) or \(b\) equals 0 or -1 can be handled similarly as above however because the integrals now involve natural logarithms instead on just polynomials the final form of the result changes from Corollary 1. The case where \(k\) is unity corresponds to the Weighted means inequality.

VII. GENERALIZATIONS OF THEOREM 1 TO CASES WITH IRREVERSIBLE HEAT EXCHANGE

Theorem 4

Suppose \(\{x_i\}, \ i = 1\ to\ n\) is a sequence of non-decreasing positive reals and let \(f(x) : [x_1, x_n] \to \mathbb{R}^+\) be a step-wise continuous function and \(x_0\) be positive real

Then the following holds:

\[
\frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx
\]  

(17)

Proof:

We split the theorem into 3 cases.

• Case 1: \(x_0 < x_1\)

\[
\int_{x_0}^{x_i} \frac{f(x)}{x} dx \geq 0
\]

\[
\Rightarrow x_0 \int_{x_0}^{x_i} \frac{f(x)}{x} dx \geq x_0 \int_{x_0}^{x_i} \frac{f(x)}{x} dx \ \forall \ i = 1\ to\ n
\]

\[
\Rightarrow \frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx
\]  

(18)
• **Case 2:** $x_n < x_0$

\[
\int_{x_0}^{x_i} \frac{f(x)}{x} dx \leq 0 \\
\Rightarrow x_0 \int_{x_0}^{x_i} \frac{f(x)}{x} dx \geq x_0 \int_{x_0}^{x_i} \frac{f(x)}{x} dx \land i = 1 \text{ to } n
\]

\[
\Rightarrow \frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \quad (19)
\]

• **Case 3:** $x_1 < x_0 < x_n$

Now, $x_0$ partitions \{x_i\} such that

\[
x_p \leq x_0 \leq x_{p+1} \, ; \, 1 \leq p \leq n - 1
\]

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \quad = \quad \sum_{i=1}^{p} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \quad + \quad \sum_{i=p+1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx
\]

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \quad \leq \quad \frac{1}{x_0} \sum_{i=1}^{p} \int_{x_0}^{x_i} f(x) dx \quad + \quad \frac{1}{x_0} \sum_{i=p+1}^{n} \int_{x_0}^{x_i} f(x) dx \quad (By \ (18) \ & \ (19))
\]

\[
\Rightarrow \frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \quad (20)
\]

**Physical interpretation**

This describes the physical situation where the N identical blocks in the system are connected to one another and allowed to attain thermal equilibrium. It is not necessary that they be connected by Carnot engines. Since this process is irreversible, the entropy of the universe must increase. This can be proven form eq(17) as follows:

\[
\Delta Q = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx = 0
\]

Using eq(17),

\[-\Delta S = \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{x} dx \leq 0\]
Comments:

1. Notice that the proof also works if we replace the x in the denominator with any continuous increasing function g(x) which leads to theorem below.

2. If the function g(x) above is decreasing then sign of the inequality in eq(21) is reversed.

**Theorem 5**

Suppose \( \{x_i\}, i = 1 \text{ to } n \) is a sequence of non-decreasing positive reals and let \( f(x) : [x_1, x_n] \rightarrow \mathbb{R}^+ \) be a step-wise continuous function and \( x_0 \) be positive real. Let \( g(x) : [x_{min}, x_{max}] \rightarrow \mathbb{R}^+ \) be a continuous function such that:

\[
g(a) \geq g(b) \forall a > b
\]

In its domain

Where \( x_{min} = \min \{x_0, x_n\} \) & \( x_{max} = \max \{x_0, x_n\} \)

Then the following holds:

\[
\frac{1}{g(x_0)} \sum_{i=1}^{n} \int_{x_0}^{x_i} f(x) dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f(x)}{g(x)} dx
\]

(21)

The above theorem can be generalized to odd dimensions as follows:

**Theorem 6**

Suppose \( \{x_i\}, i = 1 \text{ to } n \) is a sequence of non-decreasing positive reals. Consider n k-tuples of reals:

\[(x_i, x_i, \ldots, x_i) ; i = 1 \text{ to } n \text{ & } k \text{ is odd as well as positive}\]

And let \( f : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \rightarrow \mathbb{R}^+ \) be a continuous function.

Let \( g : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \rightarrow \mathbb{R}^+ \) be a continuous function that is non-decreasing with respect to each of its arguments. Let \( x_0 \) be positive real, then following holds:

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} \cdots \int_{x_0}^{x_i} \int_{x_0}^{x_i} \frac{f}{g(x_0, x_0, \ldots, x_0)} dz_1 \, dz_2 \cdots dz_k \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \cdots \int_{x_0}^{x_i} \int_{x_0}^{x_i} \frac{f}{g} \, dz_1 \, dz_2 \cdots dz_k
\]

(22)
Proof:

The proof of Theorem 6 is identical to the proof of Theorem 4 except for the fact that multiple integrals replace the single integral. The oddness of \( k \) is exploited while writing the expression corresponding to \( \int_{x_0}^{x_i} \frac{f(x)}{g(x)} \, dx \leq 0 \) in the second case where \( x_n < x_0 \).

Comments

1. The theorem does not hold for even values of \( k \).
2. The sign of inequality is reversed if \( g \) is decreasing with respect to each of its arguments.

Theorem 7

Suppose \( \{x_i\}, i = 1 \text{ to } n \) is a sequence of non-decreasing positive reals and \( x_0 \) be positive real let \( f_i(x) : [\min(x_0, x_i), \max(x_0, x_i)] \to \mathbb{R}^+ \) be n step-wise continuous functions.

Then the following holds:

\[
\frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f_i(x) \, dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f_i(x)}{x} \, dx \quad (23)
\]

If \( g(x) : [x_{\min}, x_{\max}] \to \mathbb{R}^+ \) be a continuous function such that:

\( g(a) \geq g(b) \forall \ a > b \) In its domain

Where \( x_{\min} = \min \{x_0, x_n\} \) & \( x_{\max} = \max \{x_0, x_n\} \)

Then the following holds:

\[
\frac{1}{g(x_0)} \sum_{i=1}^{n} \int_{x_0}^{x_i} f_i(x) \, dx \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f_i(x)}{g(x)} \, dx \quad (24)
\]

Proof:

The result follows from the fact that

\[
\int_{x_0}^{x_i} \frac{f_i(x)}{x_0} \, dx \geq \int_{x_0}^{x_i} \frac{f_i(x)}{x} \, dx
\]

\&

\[
\int_{x_0}^{x_i} \frac{f_i(x)}{g(x)} \, dx \geq \int_{x_0}^{x_i} \frac{f_i(x)}{g(x_0)} \, dx \quad \forall \ i = 1 \text{ to } n
\]

Which holds irrespective of whether \( x_i \) is greater than, less than or equal to \( x_0 \).
Physical interpretation

Physically this corresponds to the situation in which the different blocks labelled by index \(i\) and having different heat capacities \((f_i(x))\) and initial temperatures \((x_i)\) are connected to an infinite reservoir at temperature \(x_0\). The net entropy of universe is expected to increase from the second law of thermodynamics and this can be proven using eq(23).

\[
\Delta S \text{ for reservoir} = \frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f_i(x) \, dx
\]

\& \Delta S \text{ for all blocks} = - \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f_i(x)}{x} \, dx

\implies \Delta S \text{ for universe} = \frac{1}{x_0} \sum_{i=1}^{n} \int_{x_0}^{x_i} f_i(x) \, dx - \sum_{i=1}^{n} \int_{x_0}^{x_i} \frac{f_i(x)}{x} \, dx \geq 0 \text{ (from 23)}

Theorem 8

Suppose \(\{x_i\}, \ i = 1 \text{ to } n\) is a sequence of non-decreasing positive reals. Consider \(n\) \(k\)-tuples of reals:

\((x_i, x_i, \ldots, x_i) ; i = 1 \text{ to } n \& k \text{ is odd as well as positive}\)

And let \(f_i : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \to \mathbb{R}^+\) be \(n\) continuous functions.

Let \(g : [(x_1, x_1, \ldots, x_1), (x_n, x_n, \ldots, x_n)] \to \mathbb{R}^+\) be a continuous function that is non-decreasing with respect to each of its arguments. Let \(x_0\) be positive real, then following holds:

\[
\sum_{i=1}^{n} \int_{x_0}^{x_i} \ldots \int_{x_0}^{x_i} \frac{f_i}{g(x_0, x_0, \ldots, x_0)} \, dz_1 \, dz_2 \ldots dz_k \geq \sum_{i=1}^{n} \int_{x_0}^{x_i} \ldots \int_{x_0}^{x_i} \frac{f_i}{g} \, dz_1 \, dz_2 \ldots dz_k
\]

Proof:

The proof of Theorem 8 is identical to the proof of Theorem 7 except for the fact that multiple integrals replace the single integral. The oddness of \(k\) is exploited while writing the expression corresponding to

\[
\int_{x_0}^{x_i} \frac{f_i(x)}{g(x)} \, dx \geq \int_{x_0}^{x_i} \frac{f_i(x)}{g} \, dx
\]

for the case where \(x_i < x_0\).
VIII. EQUIVALENCE OF POSITIVITY OF SPECIFIC HEATS AND THE SECOND LAW OF THERMODYNAMICS:

We will now prove the equivalence of the positivity of the specific heats and of the Planck’s statement of the second law of thermodynamics in some special cases.

Consider the two statements P & Q as follows:

P: all statistical systems have positive specific heat.

Q: Every process occurring in nature proceeds in the sense in which the sum of the entropies of all bodies taking part in the process is increased.

Proof of Equivalence of P & Q:

Proof that \( P \rightarrow Q \):

The theorems until now imply that in the scenarios considered the statement P implies the statement Q.

Proof that \( Q \rightarrow P \):

Instead of proving that \( Q \rightarrow P \) we will prove that \( \sim P \rightarrow \sim Q \) by performing a thought experiment as follows:

Since P is not true consider a block with negative heat capacity, let say \(-C\) (where \( C > 0 \)) at initial temperature \( T_1 \). Put it in thermal contact with a block of heat capacity 2C at initial temperature \( T_2 \). Now close the system consisting of two blocks using adiabatic walls. The system will reach an equilibrium temperature \( T_{eq} \) which can be calculated using the first law of thermodynamics:

Heat lost/gained by block with heat capacity 2C: \( 2C(T_{eq} - T_2) \)

Heat lost/gained by block with heat capacity -C: \( -C(T_{eq} - T_1) \)

Therefore, we get from the first law of thermodynamics:

\[
T_{eq} = 2T_2 - T_1
\] (25)

Change in entropy of the block with heat capacity -C:

\[
\Delta S_1 = \int_{T_1}^{T_{eq}} \frac{-C}{\bar{T}} d\bar{T}
\] (26)

Change in entropy of the block with heat capacity 2C:

\[
\Delta S_2 = \int_{T_2}^{T_{eq}} \frac{2C}{\bar{T}} d\bar{T}
\] (27)
Thus, the total change in entropy of the system is:

\[ \Delta S = C \text{ln}(\frac{2T_2 - T_1}{T_1/2^2}) \]

Which is negative when \( T_2 > (\sqrt{2} - 1)T_1 \). Therefore, the negation of statement P implies the negation of statement Q, which completes the proof of equivalence of P and Q.

IX. DISCUSSION

We would like to remark that the Theorems work even if the function f is just Lebesgue integrable and thus the given versions are a restricted form of the inequalities; however the fact that the function might be discontinuous over a set of measure zero hardly makes any difference as regard to its intended use.

We have been unable to figure out any physical interpretation of either the odd dimensional inequalities or of the Lebesgue integrable form of Theorems suggested above. We would like to welcome everyone to tackle this issue of interpretation.

In each of the above results, we have assumed that: ‘All statistical systems have positive specific heat’\(^9\) \(^{10}\), holds and then proved what was expected on the basis of the Planck’s statement of Second Law which states: ‘Every process occurring in nature proceeds in the sense in which the sum of the entropies of all bodies taking part in the process is increased’. Thus, the results derived here can be interpreted as a mathematical proof of the equivalence of Second law of thermodynamics and of positivity of specific heats in various cases. The theorems are also a remarkable way of churning out new inequalities by feeding them with the appropriate functions as has been demonstrated in the paper.

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