We introduce the notion of polynomials with Lorentzian signature. This generalizes the notion of Lorentzian polynomials studied in \cite{BH20}. At first, we define a homogeneous multivariate polynomial $f(x)$ to be a log-concave polynomial over a closed convex cone $K$ if its Hessian $H_f(x) := (\partial_i \partial_j f)_{i,j=1}^{n}$ has at most one positive eigenvalue for any $x \in K$, and nonsingular for any $x \in \text{int} K$, the interior of $K$. A log-concave polynomial is called strictly log-concave if its Hessian $H_f(x) := (\partial_i \partial_j f)_{i,j=1}^{n}$ has Lorentzian signature, i.e., $H_f(x)$ has exactly one positive eigenvalue. Then we define a homogeneous multivariate polynomial $f(x)$ to have Lorantzian signature if $f(x)$ and all of its directional derivatives are log-concave over some cone $K$. If $f(x)$ and all of its directional derivatives are strictly log-concave over the cone $K$, that means their Hessian matrices have exactly one positive eigenvalue, we call $f(x)$ as a polynomial with strict Lorentzian signature. More precisely, by defining the polynomial with Lorentzian signature over the cone $K$, we mean $f$ and all of its directional derivatives in the directions have Hessian with at most one positive eigenvalue for all directions $e$ in $K$. Then we show that the limits of polynomials with strict Lorentzian signature are polynomials with Lorentzian signature in Theorem 20. Note that the cone $K$ is set of all the directions $e$ for which $f$ and all of its directional derivatives in the direction $e$ have the right signature of Hessian, and $f(e) > 0$ over the cone $K$ and a polynomial with Lorentzian signature over the cone $K$ satisfies Hodge-Riemann relations of degree $\leq 1$, cf. Remark 21.

Obviously, Lorentzian polynomials studied in \cite{BH20} are polynomials with Lorentzian signature where the cone $K$ is the nonnegative orthant. Besides, the coefficients of Lorentzian polynomials are nonnegative and this property of a Lorentzian polynomial induces the support of a Lorentzian polynomial to be an $M$-convex set. A recent approach to extend the theory of Lorentzian polynomials over cones other than the nonnegative orthant is explored to prove Heron-Rota-Welsh conjecture via Lorentzian polynomials, see \cite{BL21}. Here our aim is to generalize the notion of Lorentzian polynomials by not only choosing any closed convex cone $K$ but also relaxing the nonnegativity conditions of the coefficients of the polynomial. As a result, we can capture hyperbolic polynomials as a subclass of polynomials with Lorentzian signature. This allows us to develop a technique to deal with computing permanents of nonsingular matrices via hyperbolic programming. Furthermore, even though, the notion of hyperbolicity is coordinate-free, it may seem that the results in the context of stable polynomials can be derived in the language of hyperbolic polynomials using the coordinate change, but it’s worth mentioning the results in this paper (including the proof of closed-ness) are not straightforward in that manner.

Polynomials with Lorentzian signature, in particular, Lorentzian polynomials arise in matroid theory, entropy optimization, and negative dependence properties. For an excellent introduction to negative dependence properties see \cite{Pem00}. Stable polynomials, a subclass of Lorentzian polynomials can be found in Gurvits’s work \cite{Gur06a}, \cite{Gur06b} that includes solving combinatorial problems like bipartite matching problem with computing permanents,
solving Van der Waerden conjecture for doubly stochastic matrices and mixed discriminant, and mixed volume. Properties of stable polynomials and interlacing families are also used to prove Kadison-Singer conjecture [MSS15a] and existence of Ramanujan graphs, and furthermore to construct an infinite families of bipartite Ramanujan graphs of every degree [MSS18]. Completely log-concave polynomials are explored in the context of optimization [AG17] and counting on matroids (for example, strongest Mason’s conjectures on the log-concavity of the number of independent sets of a matroid) [ALGV18], [BH20].

In this paper, we provide a few equivalent characterization results in Proposition 5 for a log-concave polynomial to introduce the notion of polynomials with Lorentzian signature. We show that any hyperbolic polynomial has Lorentzian signature [Theorem 11]. However, the converse need not be true and it has been exemplified. Furthermore, we prove that any conic stable polynomial is a constant multiple of a polynomial with Lorentzian signature [Theorem 15]. Moreover, we show that every polynomial with Lorentzian signature is the limit of polynomials with strict Lorentzian signature, which implies that the space of polynomials with Lorentzian signature is closed [Theorem 20]. Next, we provide a formula for the determinant of a sum of \( k\) matrices in terms of mixed discriminant of \( k\) tuple of matrices [Proposition 32], which generalizes the matrix determinant lemma. As a consequence, we characterize the class of VdW family of stable polynomials in \( n\) variables such that the coefficient of \( x_1 \ldots x_n\) is the permanent of some nonnegative matrix \( A\) up to scaling [Proposition 39]. Furthermore, we characterize a class of nonnegative matrices for which permanents can be computed by solving a hyperbolic programming [Theorem 42]. Nonsingular \( k\)-locally singular matrices are introduced in the context of \( k\) locally psd matrices by Blekherman et. al. in [BDS20] to solve large scale positive semidefinite programs by applying psd-ness on a collection of principal submatrices. We show that the generating polynomial of nonsingular \( k\)-locally singular matrix is a polynomial with Lorentzian signature, and the permanent of a nonsingular \( k\)-locally singular matrix can be computed by solving a hyperbolic programming.

The remainder of this paper is organized as follows. In section 2, we discuss preliminary notions and results which are used in the sequel. In section 3, we investigate the properties of polynomials with Lorentzian signature. In section 4, using the notion of mixed discriminant of matrices we generalize the matrix determinant lemma. In section 5, it’s shown how hyperbolic programming can be used to compute the permanents of a class of nonsingular matrices. This special class of nonsingular matrices includes the class of nonsingular \( k\)-locally singular matrices which has been studied in section 6. In section 7, we discuss the results and conclude with some open questions.

2. Preliminaries

A homogeneous polynomial \( f\) of degree \( d\) in \( n\) variables is hyperbolic with respect to \( e \in \mathbb{R}^n\) if \( f(e) \neq 0\) and if for all \( x \in \mathbb{R}^n\) the univariate polynomial \( t \mapsto f(x + te)\) has only real roots. A polynomial \( f \in \mathbb{R}[x]\) is called a stable polynomial if it has no roots in the upper half plane, i.e., \( f(z) \neq 0\) \( \forall z \in \mathbb{C}^n\) with \( \text{im}(z) \in \mathbb{R}^n_+\), the positive orthant. Equivalently, \( f \in \mathbb{R}[x]\) is stable if for all \( v \in \mathbb{R}^n\) and \( u \in \mathbb{R}^n_+\) the univariate polynomial \( f(v + tu)\) is real-rooted. A generalization to the notion of stable polynomials is the notion of conic stable polynomials introduced and studied in [JT18a],[JTdW19]. Let \( K\) be a closed convex cone in \( \mathbb{R}^n\). For a polynomial \( f \in \mathbb{R}[z]\), its imaginary projection \( \mathcal{I}(f)\) is defined as the projection of the variety of \( f\) onto its imaginary part, i.e.,

\[
\mathcal{I}(f) = \{ \text{im}(z) = (\text{im}(z_1), \ldots, \text{im}(z_n)) : f(z) = 0 \},
\]

where \( \text{im}(\cdot)\) denotes the imaginary part of a complex number.

**Definition 1.** A polynomial \( f \in \mathbb{R}[z]\) is called \( K\)-stable, if \( f(z) \neq 0\) whenever \( \text{im}(z) \in \text{int} K\).

If \( f \in \mathbb{R}[Z]\) on the symmetric matrix variables \( Z = (z_{ij})_{n \times n}\) is \( S_n^{\text{psd}}\)-stable, then \( f\) is called positive semidefinite-stable (for short, psd-stable) where \( S_n^{\text{psd}}\) denotes the cone of psd matrices.

**Fact 2.** A polynomial \( f \in \mathbb{R}[x]\) is stable if and only if the (unique) homogenization polynomial w.r.t \( x_0\) is hyperbolic w.r.t every vector \( e \in \mathbb{R}^{n+1}\) such that \( e_0 = 0 \) and \( e_j > 0 \) for all \( 1 \leq j \leq n\).

A nice constructive way to get stable polynomials from determinantal polynomials follows from Brändén’s result.

**Proposition 3.** Let \( A_1, \ldots, A_n\) be positive semidefinite \( d \times d\)-matrices and \( A_0\) be a Hermitian \( d \times d\)-matrix. Then

\[
f(x) = \det(A_0 + \sum_{j=1}^n A_j x_j)
\]
is real stable or the zero polynomial.

On the other hand, due to Gårding’s foundational work on hyperbolic polynomials [Går59] it’s known that the hyperbolicity cone corresponding to a hyperbolic polynomial $f$ with respect to $e$, denoted by $\Lambda_{++}(f, e)$, defined by $\Lambda_{++}(f, e) = \{ x \in \mathbb{R}^n : f(x + te) = 0 \Rightarrow t < 0 \}$ is an open convex cone. Note that the hyperbolicity cones depend on the spectrum of some matrices.

**Log-Concavity.** Logarithmic concavity is a property of a sequence of real numbers, and it has a wide range of applications in algebra, geometry, and combinatorics, [Huh18]. A sequence of real numbers $a_0, \ldots, a_d$ is log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i$$

When the numbers are positive, log-concavity implies unimodality, i.e., there is an index $i$ such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d \text{ for all } i$$

**M-convex.** A subset $J \subset \mathbb{N}^n$ is said to be $M$-convex if it satisfies any one of the following equivalent conditions:

1. Exchange Property: For any $\alpha, \beta \in J$ and any index $i$ satisfying $\alpha_i > \beta_i$, there is an index $j$ satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$

2. Symmetric Exchange Property: For any $\alpha, \beta \in J$ and any index $i$ satisfying $\alpha_i > \beta_i$, there is an index $j$ satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$ and $\beta - e_j + e_i \in J$

A proof of the equivalence can be found [Mur98, Chapter 4]. We refer to [Mur98, Section 4.2], [KMT07], [Fuj05] for details on $M$-convex sets. The support of the polynomial $f$ is the subset of $\mathbb{Z}_+^n$ defined by

$$\text{supp}(f) = \{ a \in \mathbb{N}^n : c_a \neq 0 \}$$

**Multivariate interlacing.** Let $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ with $\deg(g) = \deg(f) - 1$ be hyperbolic with respect to $e \in \mathbb{R}^n$. We say that $g$ interlaces $f$ with respect to $e$ if for all $a \in \mathbb{R}^n$ the univariate restriction $g(a + te)$ interlaces $f(a + te)$. For univariate polynomials $f, g \in \mathbb{R}[t]$ with all real roots and $\deg(g) = \deg(f) - 1$, let $\alpha_1 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \cdots \leq \beta_{d-1}$ be the roots of $f$ and $g$ respectively. We say that $g$ interlaces $f$ if $\alpha_i \leq \beta_i \leq \alpha_i + 1$ for all $i = 1, \ldots, d - 1$.

**Derivative Relaxation.** It’s a well-known fact that if $f$ has degree $d$ and is hyperbolic with respect to $e$, then for $k = 0, 1, \ldots, d$, the $k$-th directional derivative in the direction $e$, i.e.,

$$D^{(k)}_e f(x) = \frac{d^k}{dt^k} f(x + te)|_{t=0}$$

is also hyperbolic with respect to $e$. This is one familiar way to produce new hyperbolic polynomials by taking directional derivatives of hyperbolic polynomials in directions of hyperbolicity [ABG70, Section 3.10]. A beautiful construction can be found by Renegar [Ren04] in the context of optimization. Moreover, the hyperbolicity cones of the directional derivatives form a sequence of relaxations of the original hyperbolicity cone as follows.

$$\lambda_+(D^{(k)}_e f, e) \supseteq \lambda_+(D^{(k-1)}_e f, e) \supseteq \cdots \supseteq \lambda_+(f, e).$$

Someone interested in this topic can see [Ren04], [SP15],[Sau18] for the details.

3. **Polynomials with Lorentzian Signature.**

Gårding in his pioneering work [Går59] has shown that if $f(x)$ is a hyperbolic of degree $d$ w.r.t $e$, log($f(x)$) is concave, by showing that $f(x)^{1/d}$ is concave and homogeneous of degree 1 in $\Lambda_{++}(f, e)$, and vanishes on the boundary of $\Lambda_{++}(f, e)$, see [Gül97] for a compact proof.
3.1. The space of Log-Concave Polynomials. Let $n$ and $d$ be nonnegative integers, and $\mathbb{R}_d^n$ denote the set of degree $d$ homogeneous polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. The Hessian of $f \in \mathbb{R}[x_1, \ldots, x_n]$ is the symmetric matrix

$$\mathcal{H}_f(x) = (\partial_i \partial_j f)_{i,j=1}^n,$$

where $\partial_i$ denotes the partial derivative $\frac{\partial}{\partial x_i}$. Motivated by Gårding’s result and inspired by the ideas to define the notion of the log-concave polynomials with non-negative coefficients in [AGV18], we define that $f$ is log-concave if the Hessian of $f$ at a point in $K$ is continuous functions in the coefficients of $f$. Thus, a homogeneous polynomial $f$ is log-concave if and only if the Hessian of $f$ evaluated at $x \in K$ and at most one positive eigenvalue. For simplicity, we assume that the zero polynomial, constant polynomials and linear polynomials are log-concave. Then, equivalently, $f(x) \in \mathbb{R}[x]$ is log-concave if for any two points $a, b \in K$, and $\lambda \in [0, 1]$ we have

$$f(\lambda a + (1 - \lambda) b) \geq f(a)^\lambda \cdot f(b)^{1-\lambda}.$$

In order to show an analogous equivalent characterization result for the notion of log-concavity over $K \subset \mathbb{R}^n$ we use the same line of thoughts mentioned in [AGV18]. Note that if a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ has exactly one positive eigenvalue, $PQP^T$ has exactly one positive eigenvalue where $P \in \mathbb{R}^{m \times n}$, and let $R \in \mathbb{R}^{(n-1) \times (n-1)}$ be homogeneous of degree $d$ such that $a^T Qb > 0$ with $Q$ having at most one positive eigenvalue.

**Lemma 4.** Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric matrix with at most one positive eigenvalue. Then for any $a \in \mathbb{R}^n$ such that $a^T Qa > 0$, the $n \times n$ matrix $(a^T Qa) \cdot Q - t(Qa)(Qa)^T$ is negative definite for all $t \geq 1$. More precisely, $Q$ has exactly one positive eigenvalue if and only if there exists $a \in \mathbb{R}^n$ such that $a^T Qa > 0$ with $Q$ having at most one positive eigenvalue.

**Proof.** Note that if $a^T Qa \leq 0$ for all $a \in \mathbb{R}^n$, $Q$ is negative definite. If there exists $a \in \mathbb{R}^n$ such that $a^T Qa > 0$, it’s sufficient to prove the result for $t = 1$. Let $b \in \mathbb{R}^n$ and consider the $2 \times n$ matrix $P$ with rows $a^T$ and $b^T$. Then

$$(3) \quad PQP^T = \begin{bmatrix} a^T Qa & a^T Qb \\ b^T Qa & b^T Qb \end{bmatrix}$$

Since $Q$ has exactly one eigenvalue, by using the Cauchy’s interlacing theorem mentioned above, we can conclude that $PQP^T$ has exactly one positive eigenvalue. That implies $\det(PQP^T) = a^T Qa \cdot b^T Qb - a^T Qb \cdot b^T Qa \leq 0$. Therefore, $b^T ((a^T Qa) \cdot Q - (Qa)(Qa)^T) b \leq 0$ for all $b \in \mathbb{R}^n$.

**Proposition 5.** Let $f \in \mathbb{R}[x]$ be homogeneous of degree $d \geq 2$. Fix a point $a \in K$ with $f(a) > 0$, and let $H_f(a) := Q$. The following are equivalent.

1. $f$ is strictly log-concave at $x = a$.
2. $Q$ has the Lorentzian signature $(+, -, \ldots, -)$, equivalently, exactly one positive eigenvalue.
3. $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional space.
4. The matrix $(a^T Qa) Q - (Qa)(Qa)^T$ is negative definite.

**Proof.** Euler’s Identity states that for a homogeneous polynomial $f$ of degree $d$, $\langle \nabla f \cdot x \rangle = \sum_{i=1}^n x_i \partial_i f = d \cdot f(x)$. Using this on $f$ and $\partial_i f$ one can easily get the following identities.

$$H_f x = (d - 1) \cdot \nabla f(x), \quad x^T H_f x = d(d - 1) \cdot f(x).$$

Besides, the Hessian of $\log(f)$ at $x = a$ equals

$$H_{\log f}(a) = \left( \frac{f \cdot H_f - \nabla f \nabla f^T}{f^2} \right) \big|_{x=a} = d(d-1) \frac{a^T Qa \cdot Q - \frac{d}{d-1} (Qa)(Qa)^T}{(a^T Qa)^2}$$

$$(1 \Leftrightarrow 2)$$

Thus, by eq. (4) negative definiteness of the Hessian of $\log(f)$ at $a \in K$ is equivalent to the fact that the Hessian $H_f(a) =: Q$ of $f$ evaluated at $a \in K$ has the Lorentzian signature.
(1 $\Rightarrow$ 3) If $f$ is strictly log-concave at $x = a$, then the Hessian of $\log(f(x))$ at $x = a$ is negative definite. Then by eq. (4) \((a^T Q a) \cdot Q - \frac{d}{dt} (Q a)(Q a)^T\) is negative definite. Therefore, since $a^T Q a > 0$, so its restriction to the linear space $(Q a)^\perp = \{x \in \mathbb{R}^n | x^T Q a = 0\}$ implies that $H_{\log f}(a) = \frac{d(d-1)}{a^T Q a} Q$ is negative definite on this linear space. That means the quadric $x \mapsto x^T Q x$ is negative definite on $(Q a)^\perp$. Since $(Q a)^\perp$ is $n - 1$ dimensional for $Q a \neq 0$, therefore, $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional space.

(3 $\Rightarrow$ 4) Since $f(a) > 0$ implies $a^T Q a > 0$ and $Q$ is nonsingular, so by Equation (3) in Lemma 4 det $PQP^T < 0$. Hence the result follows from Lemma 4.

(4 $\Rightarrow$ 1) $(a^T Q a) \cdot Q - (Q a)(Q a)^T$ is negative definite implies $(a^T Q a) \cdot Q - \frac{d}{a^T Q a} (Q a)(Q a)^T$ is negative definite. Then it follows from eq. (4).

(2 $\Rightarrow$ 5) Consider a $b \in K \cap \mathbb{R}^n$ such that $Q b \neq 0$. Then construct a $2 \times n$ matrix $P$ with rows $a^T$ and $b^T$. Since $a^T Q a > 0$ and $Q$ has Lorentzian signature, so by Equation (3) in Lemma 4 det $PQP^T < 0$, i.e., $(a^T Q b)^2 > (a^T Q a)(b^T Q b)$. This implies $Q$ is negative definite on the hyperplane $\{b \in \mathbb{R}^n : a^T Q b = 0\}$ for $Q b \neq 0$.

(5 $\Rightarrow$ 3) Since $Q b \neq 0$ and $x \mapsto x^T Q x$ is negative definite on $(Q b)^\perp$ for every $b \in K$, so $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional linear space. □

**Corollary 6.** Polynomial $f(x)$ is log-concave at $x = a$ in $K$ if and only if $H_f(a)$ has at most one positive eigenvalue if and only if $x \mapsto x^T Q x$ is negative semidefinite on $n - 1$ dimensional space.

**Proof.** Since $K$ is a closed convex cone, so it follows from Proposition 5.

**Remark 7.** Note that the sign of the coefficients of $f(x)$ remains unchanged by taking derivatives until the coefficients vanish since the coefficient of monomial $\prod_i x_i^{k_i}$ in any $f$ is a positive multiple of $\partial_1^{k_1} \cdots \partial_n^{k_n} f(x) |_{x = 0}$.

**Proposition 8.** If $f(x) \in \mathbb{R}_d^n$ is log-concave at $x = a$, then $f(Ax)$ is log-concave at some point (need not be $a$) for any nonsingular $n \times n$ matrix $A$.

**Proof.** Since $f$ is log-concave at $x = a$, so $H_f(a)$ has at most one positive eigenvalue by Corollary 6. Using Euler’s Identity we have $x^T H_f x = d(d - 1) \cdot f(x)$. Therefore, $x^T A^T H_f A x = d(d - 1) \cdot f(Ax)$. Say $H_f(a) =: Q$. By Sylvester’s law of inertia, the signatures of $Q$ and $A^T Q A$ are the same as they are congruent to each other. Hence, the result. □

### 3.2. Polynomials with Lorentzian signature

In this subsection we introduce the notion of a polynomial with Lorentzian signature. It’s shown in [DP18, Proposition 3.1] that a homogeneous quadratic polynomial $f(x) = x^T Q x$ is hyperbolic w.r.t. $x \in \mathbb{R}^n$ such that $f(x) > 0$ if and only if its Hessian $H_f = 2Q$ can have at most one positive eigenvalue. Moreover, it’s shown in [DP18, Theorem 4.5] that the hyperbolicity cone associated with a hyperbolic quadratic polynomial is a full dimensional cone if and only if the matrix representation $Q$ is not negative semidefinite. Thus, the interesting case is when a quadratic hyperbolic polynomial has a full dimensional hyperbolicity cone, i.e., a quadratic polynomial such that its Hessian has the Lorentzian signature. Up to normalization (since every quadric in $\mathbb{R}^n$ is affinely equivalent to a quadric given by one of the three normal forms) it’s of the form $x_1^2 - (x_2^2 + \cdots + x_n^2)$, cf. [DGT21] for details. In this case, there are two full dimensional hyperbolicity cones

$$\{x \in R_+ \times \mathbb{R}^n : x_1^2 > \sum_{j=2}^n x_j^2\} \text{ and } \{x \in R_- \times \mathbb{R}^n : x_1^2 > \sum_{j=2}^n x_j^2\}$$

containing $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$ respectively.

Consider the homogeneous bivariate cubic polynomial $f = x_1^3 - x_1^2 x_2 + x_3^3$. In this case, $\partial_1(f)$ and $\partial_2(f)$ are log-concave at $x \in \mathbb{R}^n$, but for example, neither $\sum_{i=1}^2 \partial_i f$ nor $f$ is log-concave at $a = (1, 1)$. Also note that $f$ is not a hyperbolic polynomial. In this example, we see that though $\partial_1 f$ and $\partial_2 f$ are log-concave at any $x \in \mathbb{R}^2$ and there exist $b = (0, 1), c = (1, 0) \in \mathbb{R}^2$ such that $D_b \partial_1 f = D_c \partial_2 f \neq 0$, but $\partial_1 f + \partial_2 f$ is not log-concave at $(1, 1)$. Thus, the observation about quadratic polynomials; besides, the notion of Lorentzian polynomials [BH20] motivates us to define the notion of polynomials with Lorentzian signature which is preserved under all directional derivatives. In [Gur09], Gurvits defines $f$ to be strongly log-concave if for all $a \in \mathbb{N}^n$, $\partial^a f$ is identically zero or $\partial^a f$ is log-concave on $\mathbb{R}_{>0}$, the positive orthant. In [AGV18], Anari et al. define $f$ to be completely log-concave if
for all $m \in \mathbb{N}$ and any $m \times n$ matrix $A = (a_{ij})$ with nonnegative entries,
$$
(\prod_{i=1}^{m} D_i) f \text{ is identically zero or } (\prod_{i=1}^{m} D_i) f \text{ is log-concave on } \mathbb{R}_{>0}^n
$$
where $D_i$ is the differential operator $\sum_{j=1}^{n} a_{ij} \partial_j$.

On the other hand, Brändén et al. [BH20] define the notion of Lorentzian polynomials whose supports are $M$-convex, and show that the notion of Lorentzian polynomials is equivalent [cf. [BH20, Theorem 2.30]] to other two notions of strongly log-concave introduced in [Gur09] and completely log-concave introduced in [AGV18]. The proof is based on showing that the supports of completely log-concave and strongly log-concave are $M$-convex. Note that the support of hyperbolic polynomials need not be $M$-convex. Also note that if $f$ is hyperbolic w.r.t $e$, it is hyperbolic w.r.t $-e$. If $f$ is irreducible, there are only two hyperbolicity cones, one is $\Lambda_{++}(f, e)$ containing $e$ and other one is the negative of $\Lambda_{++}(f, e)$ containing $-e$, see [LPR05, Prop. 2]. Let $H^2_n$ denote the space of hyperbolic polynomials $f$ of degree $d$ such that $f(a) > 0$ for all $a \in \Lambda_{++}(f, e)$. The space $H^2_n$ can be associated with the set of $n \times n$ symmetric matrices that have at most one positive eigenvalue. More precisely, a homogeneous quadratic polynomial $f$ is a hyperbolic polynomial w.r.t a point $e$ such that $f(e) > 0$ if and only if its Hessian has at most one positive eigenvalue. A homogeneous quadratic polynomial $f$ is a strictly hyperbolic polynomial w.r.t a point $e$ such that $f(e) > 0$ if and only if it has Lorentzian signature or in other words, it’s strongly log-concave. Besides, if we assume that quadratic $f$ has only positive coefficients and its Hessian has Lorentzian signature (i.e., exactly one positive eigenvalue), then $f$ is a univariate quadratic polynomial. Thus, quadratic Lorentzian polynomials are univariate quadratic polynomials.

Thus, by exploiting the nature of quadratic hyperbolic polynomials we define the notion of polynomials with Lorentzian signature as follows. First, we define a topology on the space $\mathbb{R}^d_n$ of degree $d$ homogeneous polynomials using the Euclidean norm for the coefficients, cf. [Nui68].

**Definition 9.** A homogeneous polynomial $f \in \mathbb{R}^d_n$ is said to be a polynomial with Lorentzian signature over a closed convex cone $K$ if for all $a_3, \ldots, a_d \in K$,

(1) $D_{a_1} \ldots D_{a_d} f > 0$, and

(2) the symmetric bilinear form

$$
(x, y) \mapsto D_x D_y D_{a_3} \ldots D_{a_d} f
$$

has at most one positive eigenvalue.

Here $D_a$ denotes the differential operator $\sum_{i=1}^{n} a_i \partial_i$. By convention, we say that zero polynomial is a polynomial with Lorentzian signature. Since the Hessian of $f$ at $a$ is the matrix $H_f(a) = (\partial_i \partial_j f(a))_{i,j=1}^{d}$, the eq. (5) implies that the Hessian of $D_{a_3} \ldots D_{a_d} f$ has at most one positive eigenvalue. Equivalently, we set $SL^0_n = H^0_n, SL^1_n = H^1_n$, and $SL^2_n = H^2_n$. For $d$ larger than 2, we define

$$
SL^d_n = \{ f \in \mathbb{R}^d_n : D_a f \in SL^{d-1}_n \text{ for all } a \in K \}
$$

A polynomial $f \in SL^d_n$ is a polynomial with strict Lorentzian signature if its Hessian evaluated at any $a \in \text{int } K$ has the Lorentz signature and nonsingular, or equivalently, $D_{a_3} \ldots D_{a_d} f$ is strictly hyperbolic for all $a_3, \ldots, a_d \in \text{int } K$.

**Remark 10.** If the cone $K$ is nonnegative orthant, and all the coefficients of the polynomial $f \in \mathbb{R}^d_n$ are nonnegative, the notion of polynomial with Lorentzian signature coincides with the notion of Lorentzian polynomials studied in [BH20].

Here we list a couple of results about hyperbolic polynomials to derive analogous results about polynomials with Lorentzian signature.

(1) If $f(t)$ is hyperbolic w.r.t $e$, for every real number $s$ the polynomial $f(t) + sf'(t)$ is hyperbolic w.r.t $e$ [Nui68].

(2) The directional derivatives of hyperbolic polynomials in the directions of hyperbolicity are hyperbolic, cf. [ABG70], [Ren04], i.e., if $f \in \mathbb{R}^d_n$ is hyperbolic with respect to $e \in \mathbb{R}^n$, the directional derivative $D_e(f) = \sum_{i=1}^{n} e_i \partial_i f$ is hyperbolic with respect to $e$ and interlaces $f$. 
Theorem 11. Any hyperbolic polynomial \( f(x) \) is a polynomial with Lorentzian signature.

Proof. Let \( f \in \mathcal{H}_n^d \). That means \( f \) is a hyperbolic polynomial w.r.t \( e \) such that \( f(e) > 0 \). Then \( f \) is hyperbolic w.r.t all \( a \in \Lambda_+^d(f,e) \) and \( f(a) > 0 \). Gårding [Går59] has shown that hyperbolic polynomial \( f \) is log-concave on its hyperbolicity cone \( \Lambda_+^d(f,e) \). So, the Hessian \( H_f(a) \) for all \( a \in \Lambda_+^d(f,e) \) has at most one positive eigenvalue by Corollary 6. Since the hyperbolicity is preserved by directional derivatives in the directions of hyperbolicity, so using eq. (2) \( D_a(f) \) is hyperbolic w.r.t all \( a \in \Lambda_+^d(f,e) \). Thus, we conclude that hyperbolic \( f \) is a polynomial with Lorentzian signature where the cone \( K \) is the hyperbolicity cone \( \Lambda_+^d(f,e) \).

The following theorem in [JT18a] reveals the connection between \( K \)-stable polynomials and hyperbolic polynomials.

Theorem 12. For a homogeneous polynomial \( f \in \mathbb{C}[z] \), the following are equivalent.

1. \( f \) is \( K \)-stable.
2. \( \mathcal{I}(f) \cap \text{int} \ K = \emptyset \).
3. \( f \) is hyperbolic w.r.t every point in \( \text{int} \ K \).

The following lemma in [JT18a] establish a reduction of multivariate \( K \)-stability to univariate stable polynomials.

Lemma 13. A multivariate polynomial \( f \in \mathbb{C}[z] \setminus \{0\} \) is \( K \)-stable if and only if for all \( x, y \in \mathbb{R}^n \) with \( y \in \text{int} \ K \) the univariate polynomial \( t \mapsto f(x + ty) \) is stable.

Note that a polynomial need not be stable in order to be a \( K \)-stable polynomial. For example, \((x_1 + x_2)^2 - x_2^2\) is not a stable polynomial but \( \text{psd-stable} \) where \( K \) is the cone of positive semidefinite matrices. See [DGT21] for a comparison among stable, \( \text{psd-stable} \) and determinantal polynomials. By [JT18b], the hyperbolicity cones of a homogeneous polynomial \( f \) coincide with the components of \( \mathcal{I}(f)^c \), where \( \mathcal{I}(f)^c \) denotes the complement of \( \mathcal{I}(f) \). This implies:

Corollary 14. A hyperbolic polynomial \( f \in \mathbb{C}[z] \) is \( K \)-stable if and only if \( \text{int} \ K \subseteq \Lambda_+^d(f,e) \) for some hyperbolicity direction \( e \) of \( f \).

The proof is based on the observation that a hyperbolic polynomial \( f \in \mathbb{C}[z] \) is \( K \)-stable if and only if \( \text{int} \ K \subseteq \mathcal{I}(f)^c \). See [DGT21] for the details.

Note that the nonzero coefficients of a homogeneous stable polynomial have the same sign [COSW04, Theorem 6.1]. Using this result any homogeneous stable polynomial with nonnegative coefficients is Lorentzian polynomial, cf. [BH20]. Here we show an analogous result to the fact any homogeneous stable polynomial is a constant multiple of a Lorentzian polynomial. Let \( K_n \) denote the space of degree \( d \), \( K \) stable \( n \)-variate polynomials such that \( f(a) > 0 \) for all \( a \in \text{int} \ K \).

Theorem 15. Any \( K \)-stable polynomial \( f \in \mathbb{R}^n \) is a constant multiple of a polynomial with Lorentzian signature.

Proof. Let \( f \in \mathbb{R}^n \) be a \( K \)-stable polynomial. Then by Theorem 12 and Corollary 14 \( f \) is a hyperbolic polynomial w.r.t all \( a \) in \( \text{int} \ K \subseteq \Lambda_+^d(f,e) \) such that \( f(e) > 0 \) (w.l.o.g one can assume it). Thus, we need to consider the point \(-e\). Say \( H_f(e) := Q \). Gårding [Går59] has shown that hyperbolic polynomial \( f \) is log-concave on its hyperbolicity cone \( \Lambda_+^d(f,e) \). So, the Hessian \( H_f(a) \) for all \( a \in \text{int} \ K \) has at most one positive eigenvalue by Corollary 6. Since the hyperbolicity is preserved by directional derivatives in the directions of hyperbolicity, so \( K \)-stability is preserved.
by directional derivatives \( a \in \text{int } K \). Thus, using Corollary 14 and eq. (2), \( D_a \) is an open map sending \( \mathcal{K}^d_n \) to \( \mathcal{K}^{d-1}_n \). Thus, \( \mathcal{K}^d_n \) is contained in the space of polynomials with Lorentzian signature.

Moreover, Hermite and Kakeya [RS+02, Theorem 6.3.8] proved that \( f \) and \( g \) have strictly interlacing zeros if and only if, for all nonzero real \( \lambda, \mu \), the polynomial \( \lambda f(z) + \mu g(z) \) has simple, real zeros for any non-constant univariate polynomials \( f, g \in \mathbb{R}[x] \). For multivariate analogs of HKO Theorem see [Wag11, Section 2.4]. A conic generalization of Hermite-Kakeya-Obreschkoff Theorem can be found [JT18a, Theorem 4.3].

**Theorem 16.** (Conic HKO Theorem). Let \( f, g \in \mathbb{R}[x] \) be multivariate polynomials. Then \( \lambda f + \mu g \) is either \( K \)-stable or the zero polynomial for all \( \lambda, \mu \in \mathbb{R} \) if and only if \( f + ig \) or \( g + if \) is \( K \)-stable or \( f \equiv g \equiv 0 \) if and only if \( f \) interlaces \( g \) or \( g \) interlaces \( f \).

Based on the above result we show the following result.

**Proposition 17.** If a degree \( d \) polynomial \( f \) admits Lorentzian signature over the cone \( K \) such that \( (1,0,\ldots,0) \in K \), then \( (1 + sx_1 \partial_1) f \) is also a polynomial with Lorentzian signature over \( K \) for any real \( s \).

**Proof.** When \( d = 2 \), it is straightforward as this statement is true for hyperbolic polynomials [Nui68]. Suppose \( d \geq 3 \), and set \( g := (1 + sx_1 \partial_1)f \). Therefore, it is enough to prove that the Hessian of \( \partial_d^2 g \) has at most one positive eigenvalue. Since \( f \) is a polynomial with Lorentzian signature, the Hessian of \( \partial_d^2 f \) has at most one positive eigenvalue. Note that

\[
\partial_d^2 g = \partial_d^2 f + sx_1 \partial_1 \partial_1^{-1} f
\]

Using the fact that all quadratic polynomials with Lorentzian signature are hyperbolic polynomials, and also using Nuij’s result on \( \partial_d^2 f \) which states that for a hyperbolic polynomial \( f \) w.r.t \( (1,0,\ldots,0) \), \( f + sx_1 \frac{\partial f}{\partial x_1} \) is hyperbolic w.r.t \( (1,0,\ldots,0) \). □

Next, we show that any linear change of variables preserves \( \text{SL}_n^d \).

**Proposition 18.** If \( f(x) \) is a polynomial with Lorentzian signature over the cone \( K \) such that \( (1,0,\ldots,0) \in K \), then \( f(Ax) \) is a polynomial with Lorentzian signature for any \( m \times n \) nonzero matrix \( A \) over the cone \( \tilde{K} \) such that

\[
A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \tilde{K}.
\]

**Proof.** Note that the desired result follows from the facts how polynomials with Lorentzian signature are preserved by three basic operations:

- the elementary operation: \( f(x_1 + x_{n+1}, \ldots, x_{n-1}, x_n) \in \text{SL}_n^d \) - this follows from Proposition 17 as a special case since \( \lim_{k \to \infty} \left( 1 + \frac{x_1 + x_{n+1}}{k} \right)^k f = f(x_1 + x_{n+1}, \ldots, x_n) \in \text{SL}_n^d \)
- the scaling/dilation: \( f(sx_1, \ldots, x_{n-1}, x_n) \in \text{SL}_n^d \) for any real \( s \). It’s straightforward.
- the diagonalization: \( f(x_2, x_2, \ldots, x_{n-1}, x_n) \in \text{SL}_n^d \). This is a combination of the first two properties. By an elementary operation \( f(x_1 + x_2, x_2, \ldots, x_{n-1}, x_n) \in \text{SL}_n^d \). Then we can get the desired result by substituting \( x_1 = 0 \).

□

**Lemma 19.** The directional derivatives of a polynomial with Lorentzian signature over \( K \) is also a polynomial with Lorentzian signature over the cone \( K \) provided the direction vectors are chosen from the cone \( K \).

**Proof.** We want to show that if \( f \) is a polynomial with Lorentzian signature over the cone \( K \), then \( \sum_{i=1}^n a_i \partial_i f \) is so for all \( a \in K \), i.e., if \( f \in \text{SL}_n^d \), then \( \sum_{i=1}^n a_i \partial_i f \in \text{SL}_n^{d-1} \) for all \( a \in K \). Observe that for all \( a \in K \), \( \tilde{f} := f(x_1 + a_1 x_{n+1}, \ldots, x_{n-1} + a_{n-1} x_{n+1}, x_n + a_n x_{n+1}) \in \text{SL}_n^{d-1} \) over the cone \( \tilde{K} \) such that \( (0,\ldots,0,1) \in \tilde{K} \). This can be obtained by applying Proposition 18 to \( f \) where the column vectors of the matrix \( A \) are \( e_i \), \( i \)-th standard basis vectors and \( \sum_{i=1}^n a_i e_i \). Since \( \tilde{f} \in \text{SL}_n^{d-1} \), a polynomial with Lorentzian signature in \( n + 1 \) variables, so
\[ \partial_{n+1} \hat{f} = \frac{\partial f}{\partial x_{n+1}} \in \text{SL}_{n+1}^d. \]

Then apply Proposition 18 to \( \partial_{n+1} \hat{f} \) where the column vectors of the matrix \( A \) are \( e_i \) and 0 to get

\[ \partial_{n+1} \hat{f} \big|_{x_{n+1}=0} = \sum_{i=1}^{n} a_i \partial_i f \in \text{SL}_n^d. \]  

(6)

The observation mentioned in eq. (6) can be found in [BH20, Corollary 2.11]. See Nuij’s brilliant piece of work in the context of hyperbolic polynomials in [Nui68] for the following theorem.

**Theorem 20.** Every polynomial with Lorentzian signature over a closed cone \( K \) is the limit of polynomials with strict Lorentzian signature over \( K \). Thus the space of polynomials with Lorentzian signature over \( K \) is closed.

**Proof.** Let \( f(x) \) be a polynomial with Lorentzian signature over a closed cone \( K \). That means the Hessian \( H_f(x)|_{x=a} \) of \( f \) has at most one positive eigenvalue for any \( a \in K \). By Proposition 18 we know that the polynomial with Lorentzian signature is invariant under the linear change of variables. So, wlog, we can assume that the cone \( K \) contains at least one of \( j \)-th coordinates, otherwise, we need to change the coordinates to get a suitable cone. Say \( e_j \in K \). For a fixed \( j \in [n] \) we define a linear operator \( T_{i,j} \) by

\[ T_{i,j}(s, f(x)) = (1 + sx_i \partial_j)f(x), i \in [n] \setminus j \text{ and } s \text{ is real} \]

Then by Proposition 17 we know that \( T_{i,j}(s, f) \in \text{SL}_d^n \) since \( f(x) \in \text{SL}_n^d \). Moreover, for a fixed \( j, T_{i,j}(s, f) \) reduces the multiplicity of zeros of \( f(x + te_j) \) except \( sx_i = 0 \) along \( j \)-th direction. Therefore, by repeated actions of \( T_{i,j} \) we have

\[ F_j f(x + te_j) = \prod_{i=1, i \neq j}^{n} T_{i,j}^d f(x + te_j) \]

which is polynomial with strict Lorentzian signature over the cone \( K \) which contains \( e_j \). Indeed, \( F_j f(x + te_j) \) lies in the interior of the space of polynomials with Lorentzian signature if \( s \neq 0 \). Note that \( F_j f \) converges to \( f \) if \( s \) approaches zero. Thus, every polynomial with Lorentzian signature is the limit of polynomials with strict Lorentzian signature. Another way to think is that the entries of the Hessian of \( \log(f) \) at a point in \( K \) are continuous functions in the coefficients of \( f \), and the cone \( K \) is closed, thus the set of polynomials with Lorentzian signature is closed in the space of polynomials of degree \( \leq d \) under a topology on \( \mathbb{R}_d^n[x] \) defined by the Euclidean norm for the coefficients.

**Remark 21.** \( f \in \mathbb{R}_d^n \) is a polynomial with Lorentzian signature over the cone \( K \) \( \Rightarrow \) the triple \( (A_f, \text{deg int } K) \) satisfies Hodge-Riemann relations of degree \( \leq 1 \), abbreviated as \( \mathcal{HR}^{\leq 1} \Rightarrow f \) is strictly log-concave on \( K \). Here \( A_f := \mathbb{R}[x_1, \ldots, x_n]/\text{Ann}(f) = \bigoplus_{i=0}^{d} A^i \), a graded algebra; \( \text{Ann}(f) = \{ p \in \mathbb{R}[\partial_1, \ldots, \partial_n] : p(\partial_1, \ldots, \partial_n)f = 0, \partial_i := \frac{\partial}{\partial x_i} \} \), degree map \( \text{deg} : A^d \rightarrow \mathbb{R} \) a linear isomorphism, and \( K := \{ a_1 \partial_1 + \cdots + a_n \partial_n | a_1, \ldots, a_n \in \mathbb{R}^d \} \subset A^1 \), see [BH20]) and references therein.

**Remark 22.** Not all polynomials with Lorentzian signature are hyperbolic polynomials. For example, \( f(x) = -2x_1^4 + 12x_1^2 x_2 + 18x_1 x_2^2 - 8x_2^3 \) is not hyperbolic w.r.t \( (1, \ldots, 1) \) although it’s log-concave at \( e = (1, \ldots, 1) \) and moreover, since all of its directional derivatives \( D_e f \) are hyperbolic quadratic polynomials with Lorentzian signature for any \( a \in \mathbb{R}_d^n \), so it’s a polynomial with Lorentzian signature over a cone \( K \subset \mathbb{R}_d^n \) containing \( e \). In fact, this example also shows that hyperbolicity of \( D_e f \) does not imply that hyperbolicity of \( f(x) \).

Let \( c \) be a fixed positive real number, \( K \) be a cone, and let \( f(x) \in \mathbb{R}[x] \) be the polynomial of degree \( d \), not necessarily homogeneous.

**Definition 23.** We say that \( f \) is \( c \)-Rayleigh if it satisfies the following inequality.

\[ \partial^\alpha f(x) \partial^{\alpha - e_i + e_j} f(x) \leq c\partial^{\alpha + e_i} f(x) \partial^{\alpha + e_j} f(x) \quad \forall i, j \in [n], \alpha \in \mathbb{Z}_+^n, x \in K \]

For an multi-affine (means degree of each variable at most one) polynomial \( f \), \( c \)-Rayleigh condition is equivalent to

\[ f(a) \partial_i \partial_j f(a) \leq c \partial_i f(a) \partial_j f(a) \text{ for all } a \in K, \text{ and } i, j \in [n] \]
It’s shown in [BH20] that if $f$ is a Lorentzian polynomial, then $f$ is $2(1 - \frac{1}{d})$-Rayleigh. This can be seen as an analog of the Hodge-Riemann relations for homogeneous stable polynomials, cf. [Huh18]. Here we show that the $2(1 - 1/d)$ Rayleigh condition is not true in general for polynomials with Lorentzian signature.

Example 24. Consider the polynomial $f = -\sum_{i=1}^{4} x_i^4 + 2(x_2^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_3^2 x_4^2) + 8x_1 x_2 x_3 x_4$. This is a polynomial with Lorentzian signature. In fact, it’s a hyperbolic polynomial w.r.t some direction $e = (1, \ldots, 1)$. Thus, $H_f(a) = Q$ has exactly one positive eigenvalue for all $a \in \Lambda_{++}(f, e)$, and the hyperbolicity cone contains $e = (1, \ldots, 1)$ and lines, see Fig 1. Note that the hyperbolicity cone does not contain $e_i, i = 1, \ldots, 4$. By Theorem 11 $f$ is a polynomial with Lorentzian signature over its hyperbolicity cone. Although, $f$ is not a $2(1 - \frac{1}{3})$-Rayleigh, but it’s 2-Rayleigh since $16 \cdot 8 \leq 2(1 - 1/4)64$ but $16 \cdot 8 \leq 2 \cdot 64$.

3.3. Generating Polynomials with Lorentzian Signature. In this subsection, we provide a means to create polynomials with Lorentzian signature which need not be Lorentzian.

Proposition 25. The polynomial $f_A(x) = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \sum_{j=1}^{n} a_{ij} E_j, E_j = e_j e_j^T$, $e_j$ are the standard canonical basis of $\mathbb{R}^n$, is a hyperbolic polynomial w.r.t some direction $e \in \mathbb{R}^n$ if $A = (a_{ij})$ is nonsingular.

Proof. Note that $A_i = ID_i I^T$ where $I = \begin{bmatrix} e_1 & e_2 & \ldots & e_n \end{bmatrix}$, the identity of matrix of size $n$ and $D_i = \text{Diag}(a_{i1}, \ldots, a_{in})$. Thus, $\det(\sum_{i=1}^{n} x_i A_i) = \det(I(\sum_{i=1}^{n} x_i D_i) I^T)$. Thus, $\det(\sum_{i=1}^{n} x_i A_i) = \det(\text{Diag}(\sum_{i=1}^{n} x_i a_{ij})).$ A determinantal polynomial is a hyperbolic polynomial if there exists a direction in which the linear span of its coefficient matrices is positive definite as it is shown that any non-empty semidefinite slice is a hyperbolicity cone, see [LPR05]. So, it is sufficient to find such a direction. In particular, we find a direction $e \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_i a_{ij} E_j = I \Rightarrow \sum_{i=1}^{n} e_i \text{Diag}(a_{i1}, \ldots, a_{in}) = I$$

Then it can be translated into a linear system of equations such that $\sum_{j=1}^{n} e_j a_{ji} = 1$ for all $1 \leq i \leq n \Rightarrow A^T e = 1$ where $A^T = (a_{ji}), e = (e_1, \ldots, e_n)$ and $1$ is the all ones vector. Therefore, if $A$ is an invertible matrix, the linear system is consistent and one can find the direction $e$ in $\mathbb{R}^n$. Thus, $f_A$ is a hyperbolic polynomial w.r.t some direction $e \in \mathbb{R}^n$.

Special Cases

1. If $A$ is a nonsingular matrix with nonnegative entries, the generating polynomial $f_A$ is a stable polynomial. A doubly stochastic matrix is an example of a nonnegative matrix.
2. If $A$ is an $n \times n$ (symmetric) positive definite matrix, the generating polynomial $f_A$ is a hyperbolic polynomial where the monomial $x_1 \ldots x_n$ appears with positive coefficient.

The hyperbolicity cone associated with $f_A$ is given by

$$\Lambda_{++}(f_A, e) = \{ x \in \mathbb{R}^n : t \mapsto f_A(x + te) \text{ has negative roots} \} = \{ x \in \mathbb{R}^n : f_A(x + te) = 0 \Rightarrow t \leq 0 \}$$
Fig 1: The surface defined by $f_A$ and the hyperbolicity cone $\Lambda_{++}(f_A, 1)$, see Example 24.

It’s clear from the picture that the hyperbolicity cone contains lines, thus, its dual cone has empty interior.

**Corollary 26.** $f_A$ is a polynomial with strict Lorentzian signature, and the $\mathcal{H}_{f_A}(e)$ and for any $v \in \Lambda_{++}(f, e)$, $\mathcal{H}_{D_v}(e)$ have exactly one positive eigenvalue, or the Lorentz signature.

### 4. Mixed Discriminant

In this section, we generalize the matrix determinant lemma using the notion of mixed discriminant of matrices. At first, we report a result of [Dey21] which uniquely express each coefficient of a determinantal multivariate polynomial $\det(\sum_{i=1}^{\alpha_i}(x_i A_i))$ in terms of the mixed discriminant of coefficient matrices $A_i, i = 1, \ldots, n$. There are various instructive ways to define the notion of mixed discriminant of $n$-tuple of $n \times n$ matrices, see [Gur06b], [BR97] for details. A clever use of mixed discriminant of $k(\leq n)$-tuple of $n \times n$ matrices can be found in [MSS15b] to establish a connection between the mixed characteristic polynomials with diagonal matrices and matching polynomials introduced by Heilmann and Leib [HL72] of a bipartite graph. We follow the constructive definition of the notion of mixed discriminant of $k(\leq n)$-tuple of $n \times n$ matrices (need not be distinct) introduced in [Dey20] to avoid the scalar factors appearing in other definitions. Notice that the mixed discriminant of matrices is called the generalized mixed discriminant in [Dey20].

**Definition 27.** Consider the $n \times n$ matrices $A^{(l)} = (a_{ij}^{(l)})$ for $l = 1, \ldots, n$. The mixed discriminant (MD) of a tuple of matrices $\left(\begin{array}{cccc} A^{(1)} & \cdots & A^{(2)} & \cdots & A^{(n)} \end{array}\right)$ is defined as

$$D\left(A^{(1)}_{k_1}, \ldots, A^{(2)}_{k_2}, \ldots, A^{(n)}_{k_n}\right) = \sum_{\alpha \in \tilde{S}[k]} \sum_{\sigma \in \hat{S}} \left| \begin{array}{ccc} a_{\alpha_1 \alpha_1}^{(\sigma(1))} & \cdots & a_{\alpha_1 \alpha_k}^{(\sigma(1))} \\ \vdots & \ddots & \vdots \\ a_{\alpha_k \alpha_1}^{(\sigma(k))} & \cdots & a_{\alpha_k \alpha_k}^{(\sigma(k))} \end{array} \right|$$

where $k_j \in \{0, 1, \ldots, n\}$ with $k = \sum_{j=1}^{n} k_j$ and $S[k]$ is the order preserving $k$-cycles in the symmetric group $S_n$ i.e.,

$$\alpha = (\alpha_1, \ldots, \alpha_k) \in S[k] \Rightarrow \alpha_1 < \alpha_2 < \cdots < \alpha_k,$$

and $\tilde{S}$ is the set of all distinct permutations of $\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{n, \ldots, n}_{k_n}$.

It’s proved in [Dey20]
Theorem 28. (Mixed Discriminant Theorem) The coefficients of a multivariate determinantal polynomial \( \det(I + \sum_{i=1}^{n} x_i A_i) \in \mathbb{R}[x] \) of degree \( d \) are uniquely determined by the generalized mixed discriminants of the coefficient matrices \( A_i \) as follows. If the degree of a monomial \( x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \) is \( k(k_1 + k_2 + \ldots + k_n = k) \leq d \), then the coefficient \( f_{k_1 \ldots k_n} \) of \( (x_1^{k_1} \cdots x_n^{k_n}) \) is given by

\[
D(A_1, \ldots, A_{d-k}, A_{d+1}, \ldots, A_n) = D(\underbrace{I, \ldots, I}_{d-k}, A_1, A_2, \ldots, A_n)
\]

A couple of well-known facts about mixed discriminant of a tuple of \( d \times d \) matrices.

1. \( \text{tr} A_i = D(I, \ldots, I, A_i) \) is the coefficient of \( x_i \) for all \( i = 1, \ldots, n \) in \( \det(I + \sum_{i=1}^{n} x_i A_i) \).

2. \( \det A_i = D(A_i, \ldots, A_i) \) is the coefficient of \( x_i^d \) for all \( i = 1, \ldots, n \) where \( d \) is the degree of the polynomial \( \det(I + \sum_{i=1}^{n} x_i A_i) \), equal to the size \( d \) of the coefficient matrix.

Another basic fact about mixed discriminant which will be used later, see [Gur06b].

Fact 29. If \( X, Y; A_i, 1 \leq i \leq n \) are \( n \times n \) complex matrices and \( \alpha_i, i \leq i \leq n \) are complex numbers then the following identity holds:

\[
D(X\alpha_1 A_1 Y, \ldots, X\alpha_i A_i Y, \ldots, X\alpha_n A_n Y) = \det(X) \cdot \det(Y) \prod_{i=1}^{n} \alpha_i D(A_1, \ldots, A_n)
\]

In a homogeneous setting, as a consequence of Theorem 28 we have the following results.

Corollary 30. Consider any \( k \leq n \)-tuple of \( n \times n \) matrices \( A_i \) for \( i = 1, \ldots, n \).

\[
(7) \quad \det(\sum_{i=1}^{k} x_i A_i) = \sum_{i=1}^{k} \det A_i x_i^k + \sum_{k \in \mathbb{N}_d^k, k_j \in \{0, \ldots, d-1\}} D(\underbrace{A_1, \ldots, A_1}_{k_1}, \ldots, \underbrace{A_d, \ldots, A_d}_{k_d}) x_1^{k_1} \ldots x_n^{k_n}
\]

where \( k \in \mathbb{N}_d^k := \{ (k_1, \ldots, k_n) \in \mathbb{N}^k | \sum_{j=1}^{n} k_j = d \} \) and \( d \leq n \) is the degree of the polynomial \( \det(\sum_{i=1}^{k} x_i A_i) \). As a special case for a 2-tuple of \( n \times n \) matrices we have

\[
\det(x_1 A_1 + x_2 A_2) = \det(A_1) x_1^k + \det(A_2) x_2^k + \sum_{k \in \mathbb{N}_d^k, k_j \in \{0, \ldots, d-1\}} D(\underbrace{A_1, \ldots, A_1}_{k_1}, \ldots, \underbrace{A_d, \ldots, A_d}_{k_d}) x_1^{k_1} x_2^{k_2}
\]

Remark 31. If none of the terms in the r.h.s of Equation (7) is nonzero, the degree of the polynomial must not be equal to the size of the matrices. If any of \( A_i \)'s is invertible, the degree of the polynomial \( d = n \).

Moreover, we generalize the matrix determinant lemma and provide another proof for the matrix determinant lemma by substituting \( x_i = 1 \) for all \( 1 \leq i \leq k \leq n \) in Equation (7).

Proposition 32.

\[
\det(\sum_{i=1}^{k} A_i) = \sum_{i=1}^{k} \det A_i + \sum_{k \in \mathbb{N}_d^k, k_j \in \{0, \ldots, d-1\}} D(\underbrace{A_1, \ldots, A_1}_{k_1}, \ldots, \underbrace{A_d, \ldots, A_d}_{k_d})
\]

\[
\det(A_1 + A_2) = \det(A_1) + \det(A_2) + \sum_{k \in \mathbb{N}_d^k, k_j \in \{0, \ldots, d-1\}} D(\underbrace{A_1, \ldots, A_1}_{k_1}, \ldots, \underbrace{A_d, \ldots, A_d}_{k_d})
\]

Another Proof of Matrix Determinant Lemma:

\[
\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)
\]

for a nonsingular matrix \( A \) and column vectors \( u, v \).
Minc’s conjecture: Let

\[ \text{Theorem 34.} \]

The Van der Waerden conjecture: If \( A = (a_{ij}) \) is an \( n \times n \) doubly stochastic matrix, \( \text{Per}(A) \geq \frac{n^l}{n!} \).

Remark 33. This result provides an explicit formula for the determinant of the sum of any finite (in particular, two) matrices which is not equal to the sum of the determinant of the matrices.

5. Permanent

In this section, we propose a novel technique to express the permanent of a nonsingular matrix (need not be a nonnegative matrix) via some hyperbolic polynomial. This point of view enables us to identify the VdW stable family for which the coefficient of \( x_1 \ldots x_n \) is the permanent of some nonnegative matrix. Moreover, we characterize the class \( \mathcal{M} \) of nonsingular matrices for which computing the permanent of a matrix \( A \in \mathcal{M} \) is equivalent to solving a hyperbolic programming (a special type of convex programming for which self-concordant barrier is known and the interior point method can be applied). The permanent of a square \( n \times n \) matrix \( A = (a_{ij}) \) is

\[
\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}
\]

where \( S_n \) is the symmetric group of degree \( n \). This is almost the same as the definition of the determinant of a matrix except the signatures of the permutations are not taken into account in the definition of the permanent of a matrix. Computing the permanent of a matrix is \#P-hard (Valiant 1979) even if the matrix entries are all either 0 or 1.

The problem of finding the upper and lower bounds of the permanents of various types of matrices including \((1,0), (1,0,-1)\), doubly stochastic, and nonnegative matrices have been studied by a large group of people from various areas of mathematics and theoretical computer science since 20th century. Here is a list of a few interesting articles on this fascinating topic [Sch78], [MM65], [Gly10].

Theorem 34. Minc’s conjecture: Let \( A = (a_{ij}) \in \{0,1\}^{n \times n} \). Then \( \text{Per}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}} \) where \( r_i := \sum_{j=1}^{n} a_{ij} \).

It was proved by Brégman, 1973 [Brè73]. A concise proof of this result is due to Schrijver [Sch78] out of several known proofs.

Theorem 35. The Van der Waerden conjecture: If \( A = (a_{ij}) \) is an \( n \times n \) doubly stochastic matrix, \( \text{Per}(A) \geq \frac{n!}{n^n} \).

The conjecture was proved by D.I.Falikman [Fal81] and the uniqueness part of the conjecture about the bound which claims that the bound is attained uniquely at \( A = J_n \) where each entry of \( J_n \) is 1/n was proved by G.P.Egorychev [Ego81].

The permanent of a matrix has attracted a lot of attention due to its connections with combinatorial objects which are in general difficult to study. The permanent of an \( n \times n \) matrix \( A = (a_{ij}) \) is equal to the number of weighted directed cycle covers of a digraph \( G \) on \( n \) vertices labelled by \( \{1, \ldots, n\} \) and \( (a_{ij}) \) be the weighted adjacency matrix for \( G \). The permanent of a \((0,1)\) matrix counts the number of perfect matchings in a bipartite graph. Gurvits’s pioneering work in [Gur06a] has first expressed \( \text{Per}(A) \) via stable polynomials as follows.

Theorem 36. Let \( A = (a_{ij}), a_{ij} \geq 0 \) be the adjacency matrix of some bipartite graph \( G \) and consider the following polynomial \( f_A(x_1, \ldots, x_n) = \prod_{i=1}^{n} (\sum_{j=1}^{n} a_{ij}x_j) \). Then

\[
\text{Per}(A) = \prod_{k=1}^{n} \partial_{x_k}|_{x_k=0} f_A = \# \text{perfect matchings in } G.
\]

Moreover, using Stirling’s formula \( \frac{n!}{\sqrt{2\pi n}n^ne^{-n}} \) tends to unity for \( n \) tends to \( \infty \), cf. [Fel68], see also [Fri79] for the lower bound result. Linial et.al. in [LSW98] proposed a deterministic strongly polynomial algorithm to approximate the permanent of a nonnegative \( n \times m \) matrix to within a multiplicative factor of \( e^n \). Gurvits’s way
of looking at the permanent of a matrix as the coefficient of certain repetitive derivatives of a stable polynomial enables him to approximate the permanent of a nonnegative matrix within a factor $\frac{e^n}{n^n}$ (for any fixed integer $n$) by a deterministic polynomial time oracle algorithm, see [Gur06a].

**Fact 37.** Van-der-Waerden conjecture: For a $n \times n$ doubly stochastic matrix, $\text{Per}(A) \geq \frac{n!}{n^n} \geq (1/e)^n$.

Indeed, this implies Schrijver’s perfect matching inequality which was conjectured by Schrijver in [Sch98] and proved by Schrijver and Valliant in [SV80]. In fact, Gurvits introduced the notion of VdW-family in [Gur06a] to unify Van der Waerden / Schrijver-Valliant/ Bapat conjectures in terms of homogeneous polynomials with nonnegative coefficients.

Gurvits introduced the notion of polynomial capacity. Consider an $n$-variate homogeneous polynomial $f \in \mathbb{R}_n^+[x]$ of degree $n$ with nonnegative real coefficients. Define the capacity of $f$ as

$$\text{Cap}_\alpha(f(x)) := \inf_{x > 0} \frac{f(x)}{x^\alpha}, \text{ for } \alpha \in \mathbb{R}_n^+.$$  

Here are some known facts about polynomial capacity for a nonnegative polynomial which will be used in the sequel. For $\mu \in \text{supp}(f)$, we have $f(1, \ldots, 1) \geq \text{Cap}_\mu f(x) = \inf_{x > 0} \frac{f(x)^\mu}{x^\mu} \geq f_\mu$ and thus, $\text{Cap}_1(f)$ could be a good approximation to $f_1$ (all-ones coefficient). Based on the following crucial observation

$$\log \text{Cap}_\alpha(f) = \inf_{y \in \mathbb{R}^n} \{-\langle y, \alpha \rangle + \log \sum \mu f_\mu e^{\langle y, \mu \rangle}\} \text{ via } x \rightarrow e^y.$$  

It turns out that computing polynomial capacity is a convex programming (or geometric programming) problem since $f_\mu > 0$. In particular, $\log(\text{Cap}_1(f)) = \inf_{1 \leq i \leq n, y_i = 0} \log(f(e^y)))$, see [Gur06a] for details.

**Definition 38.** VdW Stable Family is a stratified set of homogeneous polynomials, i.e., a class $F = \cup_{1 \leq n < \infty} F_d$, where $F_d \subset \mathbb{R}_d^+[x]$ such that

- if $f \in F_d, d > 1$, then for all $1 \leq i \leq n$ the polynomials $\partial_i f \in F_{d-1}$ and
- $\text{Cap}_1(\partial x_k | x_k = 0 f) \geq (\frac{n_k - 1}{n_k})^{n_k - 1} \text{Cap}_1(f)$ where $n_k$ is the maximum degree of $x_k$ in $f$.

**Questions:** Are there any other stable polynomials whose coefficients of $x_1 \ldots x_n$ are the permanents of some nonnegative matrices?

**Proposition 39.** Consider the determinantal polynomial $\det(\sum_{i=1}^n x_i A_i)$ where each $A_i$ is in the nonnegative linear span of $n$ rank one matrices, i.e., $A_i = \sum_{j=1}^n a_{ij} v_j v_j^T$ for all $i = 1, \ldots, n$ and $a_{ij} \geq 0$. The coefficient $x_1 \ldots x_n$ is $D(A_1, \ldots, A_n) = \det(V) \text{ Per}(A) \det(V^T)$ where $A = (a_{ij})$, nonnegative matrix.

**Proof.** Note that $A_i = V D_i V^T$ where $V = [v_1, v_2, \ldots, v_n]$ and $D_i = \text{Diag}(a_{i1}, \ldots, a_{in})$ with $a_{ij} \geq 0$. Thus each coefficient matrix $A_i$ is positive semidefinite. Then by Proposition 3, $\det(\sum_{i=1}^n x_i A_i)$ is stable polynomial for any choice of $V \in GL_n$. On the other hand, polynomial $\det(\sum_{i=1}^n x_i A_i) = \det(V(\sum_{i=1}^n x_i D_i)V^T) = \det(V) \det(\sum_{i=1}^n x_i D_i) \det(V^T)$. Then the rest follows from Theorem 28 and Fact 29. □

**Remark 40.** When $A_i$’s are diagonal matrices, $D(A_1, \ldots, A_n) = \text{Per}(A)$ and when $A_i$’s are simultaneously diagonalizable matrices, $D(A_1, \ldots, A_n) = \text{Per}(A)$.

Thus, we characterize the class of VdW family of stable polynomials in $n$ variables such that the coefficient of $x_1 \ldots x_n$ which is the mixed discriminant $D(A_1, \ldots, A_n)$ of $n$-tuple symmetric matrices is the permanent of some $n \times n$ nonnegative matrix $A$ up to scaling.

However, Gurvits’s clever way to view the permanent of a nonnegative matrix as the repetitive derivatives of a stable polynomial evaluated at zero leads to a way to visualize the permanent of any nonsingular matrix (the entries could be negative and complex numbers). In this paper, following the similar concepts we propose a technique to compute the permanent of any nonsingular matrix by using a special type of hyperbolic polynomials.
Proposition 41. For a nonsingular matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, where the generating polynomial $f_A(x) = \det(\sum_{i=1}^{n} x_i A_i)$ with $A_i = \sum_{j=1}^{n} a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$.

Proof. The permanent of $A = (a_{ij}), a_{ij} \in \mathbb{R}$ is the coefficient of $x_1 \ldots x_n$ of the determinantal polynomial $\det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \sum_{j=1}^{n} a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$. The coefficient of $x_1 \ldots x_n$ is the mixed discriminant $D(A_1, \ldots, A_n)$ by Theorem 28. Observe that $\det(\sum_{i=1}^{n} x_i A_i) = \det(D(\sum_{i=1}^{n} x_i a_{ij}, 1 \leq j \leq n))$. Since the coefficient matrices $A_i$ are diagonal, $D(D(1, \ldots, D_n) = \Per(A^T) = \Per(A)$. The mixed discriminant of diagonal matrices is the permanent of matrix $A$ follows from the definitions of the permanent and mixed discriminant. \hfill \Box

Here we characterize the class of nonsingular matrices for which the permanents can be computed by solving a hyperbolic programming. Note that we cannot use polynomial capacity because $f_A$ in Proposition 41 need not have nonnegative coefficients. Thus, we cannot directly use Gurvits’s machinery. First, we need to make sure the decision problem works, i.e., the coefficient of $x_1 \ldots x_n$ is positive, equivalently, $D(A_1, \ldots, A_n) > 0$. Thus, if the hyperbolic polynomial $f(x) > 0$ for some $x \in \Lambda_{++}(f,e)$ with $x_i > 0$ for all $1 \leq i \leq n$, then we define the polynomial capacity of $f(x)$ over a convex cone as follows.

$$\text{Cap}_\alpha(f(x)) := \inf_{x > 0 \& x \in \Lambda_{++}(f,e)} \frac{f(x)}{x^\alpha}, \text{ for } \alpha \in \mathbb{R}_+^n.$$ 

Note that this is a conic geometric programming if the coefficients of $f(x)$ are nonnegative, cf [CS14], [BLNW20]. However, since in the context of $f(x)$ being hyperbolic, the coefficients of $f(x)$ need not be nonnegative, thus, a natural question arises here that how we can make sure that the hyperbolicity cone $\Lambda_{++}(f,e)$ contains such a direction $x \in \Lambda_{++}(f,e)$ such that $x_i > 0$ for all $1 \leq i \leq n$. More precisely, it would be interesting to characterize the class $\mathcal{M}$ of nonsingular matrices for which the corresponding hyperbolicity cones intersect the positive orthant.

Indeed, this class contains nonnegative matrices, in particular doubly stochastic matrices since the corresponding polynomials are stable polynomials in this case. Although it’s worthy to mention the class $\mathcal{M}$ is certainly bigger than the class of nonnegative matrices. For example, we study the class of nonsingular locally singular matrices in the next section in details to show that the corresponding hyperbolic polynomials for this class of matrices don’t have all the coefficients with the same sign but their hyperbolicity cones have nonempty intersection with the positive orthant. In this context, for a given direction $e \in \mathbb{R}_+^n$ and matrix $A$ one needs to solve a system of linear equations to check whether the vector $e$ lies in the range space of $A$ which is easy to verify.

Consider the class $\mathcal{M}$ of nonsingular matrices $A$ such that $D(A_1, \ldots, A_n) > 0$ and the generating polynomials $f_A(x)$ are hyperbolic w.r.t some $x$ with $x_i > 0$, for all $1 \leq i \leq n$. Then the problem of approximating the permanent of such a nonsingular matrix $A \in \mathcal{M}$ can be translated into computing the capacity of the generating polynomial $f_A$ which is equivalent to solving a hyperbolic programming problem.

Theorem 42. $\Per(A)$ of a nonsingular matrix with $\Per(A) > 0$ can be approximated by computing the $\text{Cap}_\alpha(f_A(x))$ where $f_A$ is the generating polynomial associated with $A$.

Proof. By Proposition 25 the polynomial $f_A(x) = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \sum_{j=1}^{n} a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$, is a hyperbolic polynomial w.r.t some direction $e \in \mathbb{R}^n$ if $A$ is nonsingular. Thus, $f_A$ in Proposition 41 is a hyperbolic polynomial but not a stable polynomial. Using eq. (8) for $\alpha = 1$ we have

$$\text{Cap}_1(f(x)) := \inf_{x_i > 0 \& x \in \Lambda_{++}(f,e)} \frac{f(x)}{x_1 \cdots x_n},$$

$$\Rightarrow \log \text{Cap}_1(f) = \inf_{x_i > 0 \& x \in \Lambda_{++}(f,e)} \left[ -\sum_{i=1}^{n} \log x_i + \log f(x) \right],$$

$$\Rightarrow \log \text{Cap}_1(f) = \inf_{x_i > 0 \& x \in \Lambda_{++}(f,e)} \left[ -\sum_{i=1}^{n} \log x_i + \log f(x) \right].$$
Remark 43. Recall that Gårding’s result shows that the function $\log f(x)$ is concave for a hyperbolic polynomial $f(x)$, cf. [Gar59], [GüI97]. Therefore, one can use interior point method and Ellipsoid method, and furthermore, G"uler has shown in [GüI97] that long step interior point method ([cf. [NN94],[BGLS01], [Nes03, Chapter 4],[NT08] for excellent survey on this topic) can be applied to hyperbolic barrier function $-\log f(x)$ while solving hyperbolic programming.

It would be interesting to find a deterministic polynomial time algorithm for approximating the permanent of a nonsingular matrix $A \in M$ such that $D(A_1, \ldots, A_n) > 0$. This can be a topic for future work.

6. k-locally PSD matrices

In this section we show that the class $M$ is nonempty. In order to identify the class $M$ we need to characterize the set of matrices whose entries are negative but the permanents are positive. Interestingly, the set of $k$-locally psd matrices, introduced and studied by Blekherman et.al. [BDSS20] is a prominent candidate for the class $M$.

(9) $S(n, k) = \{X \in S^n : \text{all } k \times k \text{ principal submatrices of } X \text{are PSD}\}$

be the set of $k$-locally PSD matrices. However, a matrix $M$ is $(n,k)$-locally singular if it lies in $S(n,k)$ and all of the $k \times k$ minors of $M$ are singular. An interesting class of locally singular matrices in $S(n,k)$ are nonsingular locally singular matrices, abbreviated as NLS. We show that $\text{Per}(M) > 0$ when $M$ is a NLS using a beautiful structure theorem proved in [BDSS20]. Another important fact is $DMD \in S(n,k)$ where $D$ is a diagonal matrix with non-zero diagonal entries and $DMD$ is called diagonally congruent to $M$, see [BDSS20] for details.

Theorem 44. Let $n-1 > k > 2$ with $k$ being close to $n$ or $(n,k) = (4,2)$. Suppose that $M$ is a NLS in $S(n,k)$. Then $M$ must be diagonally congruent to $G(n,k)$ where $G(n,k) = \frac{k}{k-1} I - \frac{1}{k-1} 11^T$.

We prove the following result to show the permanent of NLS is positive.

Lemma 45. The permanent $\text{Per}(D'AD')$ is $\det(D')^2 \text{Per}(A)$ for any complex matrix $A$ and diagonal matrix $D'$ with nonzero entries.

Proof. Say $A = (a_{ij})$ is a $n \times n$ matrix, then $\text{Per}(A)$ is the coefficient of $x_1 \ldots x_n$ in the determinantal polynomial $\det(\sum_{i=1}^{n} x_i D_i)$ where $D_i = \text{Diag}(a_{ij})$. Thus, by Proposition 41 $\text{Per}(A) = D(D_1, \ldots, D_n)$. Then it follows from Fact 29 as $\text{Per}(D'AD') = D(D'D_1D', \ldots, D'D_nD') = \det(D')^2 \text{Per}(A)$.

Proposition 46. When $n$ is an even integer, $\text{Per}(M)$ is positive for any NLS $M \in S(n,k)$ where $n-1 > k > 2$ or $(n,k) = (4,2)$. When $n$ is an odd integer, $\text{Per}(M)$ is positive for any NLS $M \in S(n,k)$ where $n-1 > k > \sqrt{2(n-1)}$.

Proof. Using Theorem 44 we know that $M$ is diagonally congruent to $G(n,k)$. On the other hand, we derive that

$$\text{Per}(G(n,k)) = \left(\frac{k}{k-1}\right)^n \sum_{i=0}^{n} \left(\frac{-1}{k}\right)^i \frac{n!}{(n-i)!} = \begin{cases} \frac{1}{(k-1)^n} \sum_{i=0}^{n} (-1)^{n-i} k^{n-i} \frac{n!}{(n-i)!} & \text{when } n \text{ is even} \\ \frac{1}{(k-1)^n} \sum_{i=0}^{n} (-1)^i k^{n-i} \frac{n!}{(n-i)!} & \text{when } n \text{ is odd} \end{cases}$$

When $n$ is an even integer, $\text{Per}(G(n,k)) = \frac{n!}{(k-1)^n} \sum_{i=0}^{n} (-1)^{n-i} k^{n-i} \frac{n!}{(n-i)!} = \frac{n!}{(k-1)^n} \sum_{i=0}^{n} (-1)^i \frac{k^{n-i}}{n!}$. The r.h.s summation is always a finite positive number. In fact, when $n \to \infty$, it converges to $e^{-k}$ for any $k$ since it is the Maclaurin
when $n$ is odd,

$$\Per(G(n, k)) > 0 \iff k[k^{n-1} + n(n-1)k^{n-3} + \cdots + n!] > n[k^{n-1} + (n-1)(n-2)k^{n-3} + \cdots + (n-1)!]$$

$$\Leftrightarrow \frac{k^{n-1} + \sum_{j=1}^{n} k^{n-2j+1} \prod_{i=1}^{j} (n-2i+2)(n-2i+1)}{k^{n-1} + \sum_{j=1}^{n} k^{n-2j+1} \prod_{i=1}^{j} (n-2i+1)(n-2i)} > \frac{n}{k}$$

$$\Leftrightarrow \frac{1 + \sum_{j=1}^{n} k^{-2j} [n^2 + 3n + 2 + 4j]^2 - (4n+6)j]}{1 + \sum_{j=1}^{n} k^{-2j} [n^2 + 4j]^2 - (4n+2)j]} > \frac{n}{k}$$

$$\Leftrightarrow \frac{\sum_{j=1}^{n} k^{-2j} (n-1)(n-2j+1)}{1 + \sum_{j=1}^{n} k^{-2j} \prod_{i=1}^{j} (n-2i+1)(n-2i)} > \frac{n-k}{k}$$

$$\Leftrightarrow \frac{2(n-1) [k^{n-3} + \sum_{j=2}^{n} k^{n-2j-1} (n-2)(n-2j+1)]}{1 + \sum_{j=1}^{n} k^{n-2j-1} \prod_{i=1}^{j} (n-2i+1)(n-2i)} > \frac{n-k}{k}$$

The above inequality is valid if the l.h.s. is a finite number when $n$ is large. Note that the coefficients of the likewise terms of the expression inside the parentheses of the numerator are greater equal to the coefficients of likewise terms of the expression inside the parentheses of the denominator. Thus, $\Per(G(n, k)) > 0$ if $k > \sqrt{2(n-1)}$ when $n$ is odd. Then irrespective of $n$ being even or odd the rest of the theorem follows from Lemma 45 that $\Per(M) = \Per(D \Per(G(n, k)) D) = \det(D)^2 \Per(G(n, k))$ where the diagonal entries of a diagonal matrix $D$ are nonzero. Hence the proof.

**Remark 47.** When $n$ is even, $\Per(-G(n, k)) = (-1)^n \Per(G(n, k)) = \Per(G(n, k)) > 0$.

**Remark 48.** It’s noticed while doing experiments in Mathematica that $\Per(G(n, k)) > 0$ even for some $k < \sqrt{2(n-1)}$. For example, $(n, k) = (9, 3), \Per(G(n, k)) < 0$, but $\Per(G(n, k)) > 0$ for $k \geq 4 = \sqrt{2(9-1)}; (n, k) = (7, 3), \Per(G(n, k)) > 0, k \geq 3 < \sqrt{2(7-1)}$; and $(n, k) = (19, 6), \Per(G(n, k)) < 0$, but $\Per(G(n, k)) > 0$ for $k \geq 7 > \sqrt{2(19-1)}$.

**Corollary 49.** For any integer $k \in \{1 + \sqrt{2(n-1)}, \ldots, n-1\}$, $\Per(G(4, 2)) > \cdots > \Per(G(n, k)) > \Per(G(n, k+1)) > \cdots > \Per(G(n, n-1))$

Following [BKS+21], consider $\mathcal{T}$ be the class of functions $T : S^n \to \mathbb{R}$ so that $T$ is a unitarily invariant matrix norm, that means the norm depends entirely on the eigenvalues or in particular, the trace function. The Schatten $p$-norms, $\|M\|_p = \left(\sum_{i=1}^{n} |\lambda_i(M)|^p\right)^{\frac{1}{p}}$ for $p \geq 1$, including the Frobenius norm as a special case of the Schatten $p$-norm when $p = 2$ are examples of unitarily invariant matrix norms.

**Theorem 50.** Let $k \in \{2, \ldots, n\}$. For any $M \in S(n, k)$ such that $\tr(M) = 1$,

$$\Per(M) \leq \Per(\tilde{G}(n, k))$$

The upper bound of the permanent of this class of matrices is $1/32$. The upper bound is attained at $\tilde{G}(4, 2)$.

**Proof.** We obtain that $\Per(\tilde{G}(n, k)) = \frac{n!}{n^n} \left(\frac{k}{k-1}\right)^n \sum_{i=0}^{n} \left(\frac{-1}{k}\right)^i \frac{1}{(n-i)!}$. Thus, for any integer $n$, and $k > \sqrt{2(n-1)}$, $\Per(\tilde{G}(n, k)) \geq \Per(\tilde{G}(n+1, k))$. On the other hand, using Lemma 45 $\Per(D^2 AD') = \det(D)^2 \Per(A)$ for any complex matrix $A$ and diagonal matrix $D'$ with nonzero entries. Thus, when we normalize the matrix to get $\tr(M) = 1$, $\Per(M) \leq \Per(\tilde{G}(n, k))$. Then using Corollary 49 we have the following result.

$$\Per(\tilde{G}(4, 2)) > \Per(\tilde{G}(n, k)) > \Per(M)$$

**Remark 51.** $\Per(\tilde{G}(4, 2)) = \frac{\Per(G(4, 2))}{4^4} = \frac{8}{4^4} = \frac{1}{32} > \frac{1}{32}$.

**Conjecture 52.** Let $k \in \{2, \ldots, n\}$. Let $T \in \mathcal{T}$ and $G(n, k) = \frac{G(n, k)}{T(G(n, k))}$. For any $M \in S(n, k)$ with $T(M) = 1$, the permanent of $M$ is at most as large the permanent of $\tilde{G}(n, k)$, i.e.,

$$\Per(M) \leq \Per(G(n, k)), \text{ for all } M \in S(n, k) \text{ s.t } T(M) = 1$$
The bound on $\text{Per}(M)$ is tight since $G(n,k) \in S(n,k)$ achieves the bound.

However, all diagonal entries of $G(n,k)$ are identically 1, and all off-diagonal entries are identically $\frac{-1}{k-1}$. In particular, when $k = 2$, all diagonal entries of $G(n,k)$ are identically $-1$, and all off-diagonal entries are identically $-1$. So, it’s a $(1,-1)$ matrix. It’s interesting to compare the matrix $C(n,n)$ studied in [Sei84] with $G(n,2)$. In fact, $C(n,n) = G(n,2)$. It’s shown in [Sei84] that $\text{Per}(C(n,m)) > 0$ for all $n \geq 4$ and $0 \leq m \leq n$ where $n \times m$ matrix $C(n,m) = c_{ij}, 0 \leq m \leq n$ is given by

$$C(n,m) = \begin{cases} c_{ij} = -1, & \text{for } n - m < i = j \\ 1 & \text{otherwise} \end{cases}$$

Following the above result we have

**Proposition 53.** $\text{Per}(-G(n,2))$ is positive for $n \geq 4$ where $-G(n,2)$ is the polar cone of $G(n,2)$.

**Remark 54.** Note that locally singularity conditions may not be necessary to show that the permanent of any nonsingular $k$ locally psd matrix is positive. But as of now it’s not clear how to classify any nonsingular matrix $M \in S(n,k)$ without the structure theorem 44.

### 6.1 Why NLS matrices.

**Proposition 55.** Consider a NLS $A = (a_{ij})$ in $S(n,k)$, where $n - 1 > k > 2$ or $(n,k) = (4,2)$ and the generating polynomial $f_A = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \text{Diag}(a_{ij})$ for all $1 \leq j \leq n$. Then $f_A$ is a hyperbolic polynomial, and the decision problem of membership of the monomial $x_1 \ldots x_n$ in $f_A$, i.e., $(1,\ldots,1) \in \text{supp}(f_A)$ is true. Furthermore, $D(A_1,\ldots,A_n) > 0$ if $n - 1 > k > \sqrt{2(n-1)}$ when $n$ is odd or if $n - 1 > k > 2$ when $n$ is even or $(n,k) = (4,2)$.

**Proof.** By the structure theorem 44 $A = (a_{ij}) = D'G(n,k)D'$ in $S(n,k)$ for $n - 1 > k > 2$ or $(n,k) = (4,2)$ where $D'$ is a diagonal matrix with nonzero diagonal entries. Note that $f_A = \det(\sum_{i=1}^{n} x_i A_i) = \det\left(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} E_j \right)$ where $E_j = e_j e_j^T$ and $e_j$ form the standard canonical basis of $\mathbb{R}^n$. Then by Proposition 25 the generating polynomial $f_A$ is a hyperbolic polynomial w.r.t $e = (1,\ldots,1)$, all-ones vector since $A$ is nonsingular. By Proposition 46 $\text{Per}(G(n,k))$ is non-zero. So, $(1,\ldots,1) \in \text{supp}(f_A)$. Then $D(A_1,\ldots,A_n) > 0$ follows from Proposition 41 and Proposition 46.

**Lemma 56.** Consider a NLS $A = (a_{ij})$ such that $-A \in S(n,2)$, $n \geq 4$ and the generating polynomial $f_A = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \text{Diag}(a_{ij})$ for all $1 \leq j \leq n$. Then $f_A$ is a hyperbolic polynomial, and the decision problem of membership of the monomial $x_1 \ldots x_n$ in $f_A$, i.e., $(1,\ldots,1) \in \text{supp}(f_A)$ is true. Furthermore, $D(A_1,\ldots,A_n) > 0$ if $n \geq 4$.

Recall that Güler [Gü97] showed that the associated hyperbolicity cone is open, convex and may contain the entire lines. Following the terminology mentioned in [Gü97] a polynomial is said to be a complete polynomial if the linearity space $L(f) = L(\Lambda_+(f,e)) = \{0\}$ where

$$L(f) = \{x \in \mathbb{R}^n : f(y + tx) = f(y), t \in \mathbb{R}, y \in \mathbb{R}^n\}$$

$$L(\Lambda_+(f,e)) = \{x \in \mathbb{R}^n : \Lambda_+(f,e) + x = \Lambda_+(f,e)\}$$

Then, he showed that $f$ is a complete polynomial if and only if the associated hyperbolicity cone is regular.

**Remark 57.** Consider the hyperbolicity cone $\Lambda_+(f_A,e)$ of the generating polynomial $f_A$ where $A \in S(n,k)$ is nonsingular; $n$ is even and $n - 1 > k > 2$, or $(n,k) = (4,2)$. Then $\Lambda_+(f_A,e)$ is not a regular cone, equivalently, $f_A$ is not a complete polynomial.

**Example 58.** Consider $A = (a_{ij}) = G(4,2)$. Then the generating polynomial is

$$f_A = -\sum_{i=1}^{4} x_i^4 + 2 \sum_{i,j \in \{1,2,3,4\}, i < j} x_i^2 x_j^2 + 8x_1x_2x_3x_4.$$
Note that $f_A$ is the same polynomial mentioned in Example 24 and the Hessian of $f_A$ has one positive eigenvalue.

$$H_{f_A}(1,\ldots,1) = 16 \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =: 16B$$

where $B$ is the adjacency matrix of the complete graph $K_4$.

7. Conclusions and Open Questions

The crux of the article is to propose and study homogeneous polynomials with Lorentzian signature that generalizes Lorentzian polynomials. We show that hyperbolic polynomials and $K$-(conic) stable polynomials are members of the proposed class of polynomials. As an immediate consequence of generating polynomials with Lorentzian signature, we establish a connection with mixed discriminant of matrices and permanents of nonsingular (need not be nonnegative) matrices via hyperbolic polynomials. Another insightful result that we obtain in this article is the characterization of nonsingular matrices for which permanents can be computed by solving hyperbolic programming using long-step interior point methods, which include nonsingular $k$-locally singular matrices.

We contemplate that the notion of polynomials with Lorentzian signature will unlock many possibilities for extending several results of Lorentzian polynomials into a general framework. It would be interesting to propose a deterministic polynomial time algorithm to approximate permanents of the special class of nonsingular matrices. An immediate question is to investigate more specific criteria on matrix $A$ such that the hyperbolicity cone $\Lambda^+_+(f_A)$ intersects the positive orthant.

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References

[ABG70] Michael Francis Atiyah, Raoul Bott, and Lars Gårding. Lacunas for hyperbolic differential operators with constant coefficients i. Acta mathematica, 124(1):109–189, 1970.

[AG17] Nima Anari and Shayan Oveis Gharan. A generalization of permanent inequalities and applications in counting and optimization. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 384–396, 2017.

[AGV18] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 35–46. IEEE, 2018.

[ALGV18] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials iii: Mason’s ultra-log-concavity conjecture for independent sets of matroids. arXiv preprint arXiv:2012.04031, 2020.

[BGLS01] Heinz H Bauschke, Osman Güler, Adrian S Lewis, and Hristo S Sendov. Hyperbolic polynomials and convex analysis. Canadian Journal of Mathematics, 53(3):470–488, 2001.

[BH20] Petter Brändén and June Huh. Lorentzian polynomials. Annals of Mathematics, 192(3):821–891, 2020.

[BKS+21] Grigoriy Blekherman, Mario Kummer, Raman Sanyal, Kevin Shu, and Shengding Sun. Linear principal minor polynomials: Hyperbolic determinantal inequalities and spectral containment. arXiv preprint arXiv:2112.15321, 2021.

[BL21] Peter Brändén and Jonathan Leake. Lorentzian polynomials on cones and the heron-rotla-welsh conjecture. arXiv preprint arXiv:2110.00487, 2021.

[BLNW20] Peter Bürgisser, Yinan Li, Harold Nieuwboer, and Michael Walter. Interior-point methods for unconstrained geometric programming and scaling problems. arXiv preprint arXiv:2008.12110, 2020.

[BR97] R. B. Bapat and T.E.S. Raghavan. Nonnegative matrices and applications. Number 64. Cambridge university press, 1997.

[Bro73] Lev Meerovich Brégman. Some properties of nonnegative matrices and their permanents. In Doklady Akademii Nauk, volume 211, pages 27–30. Russian Academy of Sciences, 1973.

[COSW04] Young-Bin Choe, James G Oxley, Alan D Sokal, and David G Wagner. Homogeneous multivariate polynomials with the half-plane property. Advances in Applied Mathematics, 32(1-2):88–187, 2004.

[CS14] Venkat Chandrasekaran and Parikshit Shah. Conic geometric programming. In 2014 48th Annual Conference on Information Sciences and Systems (CISS), pages 1–4. IEEE, 2014.

[Dey20] Papri Dey. Definite determinantal representations of multivariate polynomials. Journal of Algebra and Its Applications, 19(07):2050129, 2020.

[Dey21] Papri Dey. Definite determinantal representations via orthostochastic matrices. Journal of Symbolic Computation, 104:15–37, 2021.
[Sch98] Alexander Schrijver. Counting 1-factors in regular bipartite graphs. *Journal of Combinatorial Theory, Series B*, 72(1):122–135, 1998.

[Sei84] Norbert Seifter. Upper bounds for permanents of (1,-1)-matrices. *Israel Journal of Mathematics*, 48(1):69–78, 1984.

[SP15] James Saunderson and Pablo A Parrilo. Polynomial-sized semidefinite representations of derivative relaxations of spectrahedral cones. *Mathematical Programming*, 153(2):309–331, 2015.

[SV80] Alexander Schrijver and WG Valiant. On lower bounds for permanents. In *Indagationes Mathematicae (Proceedings)*, volume 83, pages 425–427. Elsevier, 1980.

[Wag11] David Wagner. Multivariate stable polynomials: theory and applications. *Bulletin of the American Mathematical Society*, 48(1):53–84, 2011.

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