Relative Morse index theory and applications in wave equations

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Abstract: We develop the relative Morse index theory for linear self-adjoint operator equation without compactness assumption and give the relationship between the index defined in [44] and [45]. Then we generalize the method of saddle point reduction and get some critical point theories by the index, topology degree and critical point theory. As applications, we consider the existence and multiplicity of periodic solutions of wave equations.

Keywords: Relative Morse index; Periodic solutions; Wave equations

1 Introduction

Many problems can be displayed as a self-adjoint operator equation

\[ Au = F'(u), \quad u \in D(A) \subset H, \quad (O.E.), \]

where \( H \) is an infinite-dimensional separable Hilbert space, \( A \) is a self-adjoint operator on \( H \) with its domain \( D(A) \), \( F \) is a nonlinear functional on \( H \). Such as boundary value problem for Laplace’s equation on bounded domain, periodic solutions of Hamiltonian systems, Schrödinger equation, periodic solutions of wave equation and so on. By variational method, we know that the solutions of (O.E.) correspond to the critical points of a functional. So we can transform the problem of finding the solutions of (O.E.) into the problem of finding the critical points of the functional. From 1980s, begin with Ambrosetti and Rabinowitz’s famous work\textsuperscript{[5]} (Mountain Pass Theorem), many crucial variational methods have been developed, such as Minimax-methods, Lusternik-Schnirelman theory, Galerkin

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approximation methods, saddle point reduction methods, dual variational methods, convex analysis theory, Morse theory and so on (see\cite{2,3,4,8,19,22} and the reference therein).

We classified all of these variational problems into three kinds by the spectrum of $A$. For simplicity, denote by $\sigma(A)$, $\sigma_e(A)$ and $\sigma_d(A)$ the spectrum, the essential spectrum and the discrete finite dimensional point spectrum of $A$ respectively.

The first is $\sigma(A) = \sigma_d(A)$ and $\sigma(A)$ is bounded from below(or above), such as boundary value problem for Laplace’s equation on bounded domain and periodic problem for second order Hamiltonian systems. Morse theory can be used directly in this kind and this is the simplest situation.

The second is $\sigma(A) = \sigma_d(A)$ and $\sigma(A)$ is unbounded from above and below, such as periodic problem for first order Hamiltonian systems. In this kind, Morse theory cannot be used directly because in this situation the functionals are strongly indefinite and the Morse indices at the critical points of the functional are infinite. In order to overcome this difficulty, the index theory is worth to note here. By the work \cite{18} of Ekeland, an index theory for convex linear Hamiltonian systems was established. By the works \cite{11,35,36,37} of Conley, Zehnder and Long, an index theory for symplectic paths was introduced. These index theories have important and extensive applications, e.g \cite{16,20,21,33,39}. In \cite{48,38} Long and Zhu defined spectral flows for paths of linear operators and redefined Maslov index for symplectic paths. Additionally, Abbondandolo defined the concept of relative Morse index theory for Fredholm operator with compact perturbation (see\cite{1} and the references therein). In the study of the $L$-solutions (the solutions starting and ending at the same Lagrangian subspace $L$) of Hamiltonian systems, Liu in \cite{31} introduced an index theory for symplectic paths using the algebraic methods and gave some applications in \cite{31,32}. This index had been generalized by Liu, Wang and Lin in \cite{34}. In addition to the above index theories defined for specific forms, Dong in \cite{17} developed an index theory for abstract operator equations (O.E.).

The third is $\sigma_e(A) \neq \emptyset$, the most complex situation. Since lack of compactness, many classical methods can not be used here. Specially, if $\sigma_e(A) \cap (-\infty, 0) \neq \emptyset$ and $\sigma_e(A) \cap (0, \infty) \neq \emptyset$, Ding established a series of critical points theories and applications in homoclinic orbits in Hamiltonian systems, Dirac equation, Schrödinger equation and so on, he named these problems very strongly indefinite problems (see \cite{12,13}). Wang and Liu defined the index theory $(i_A(B), \nu_A(B))$ for this kind and gave some applica-
tions in wave equation, homoclinic orbits in Hamiltonian systems and Dirac equation, the methods include dual variation and saddle point reduction (see [44] and [45]). Additionally, Chen and Hu in [10] defined the index for homoclinic orbits of Hamiltonian systems. Recently, Hu and Portaluri in [30] defined the index theory for heteroclinic orbits of Hamiltonian systems.

In this paper, consider the kind of \( \sigma_c(A) \neq \emptyset \). Firstly, we develop the relative Morse index theory. Compared with Abbondandolo’s work ([1]), we generalize the concept of relative Morse index \( i^*_A(B) \) for Fredholm operator without the compactness assumption on the perturbation term (see Section 2). And we gave the relationship between the relative Morse index \( i^*_A(B) \) and the index \( i_A(B) \) defined in [44] and [45]. The bridge between them is the concept of spectral flow. As far as we know, the spectral flow is introduced by Atiyah-Patodi-Singer (see [6]). Since then, many interesting properties and applications of spectral flow have been subsequently established (see [7], [24], [40], [41] and [48]).

Secondly, we generalize the method of saddle point reduction and get some critical point theories. With the relative Morse index defined above, we will establish some new abstract critical point theorems by saddle point reduction, topology degree and Morse theory, where we do not need the nonlinear term to be \( C^2 \) continuous (see Section 3).

Lastly, as applications, we consider the existence and multiplicity of the periodic solutions for wave equation and give some new results (see Section 4). To the best of the authors’ knowledge, the problem of finding periodic solutions of nonlinear wave equations has attracted much attention since 1960s. Recently, with critical point theory, there are many results on this problem. For example, Kryszewski and Szulkin in [29] developed an infinite dimensional cohomology theory and the corresponding Morse theory, with these theories, they obtained the existence of nontrivial periodic solutions of one dimensional wave equation. Zeng, Liu and Guo in [47], Guo and Liu in [25] obtained the existence and multiplicity of nontrivial periodic solution of one dimensional wave equation and beam equation by their Morse index theory developed in [26]. Tanaka in [43] obtained the existence of nontrivial periodic solution of one dimensional wave equation by linking methods. Ji and Li in [28] considered the periodic solution of one dimensional wave equation with \( x \)-dependent coefficients. By minimax principle, Chen and Zhang in [14] and [15] obtained infinitely many symmetric periodic solutions of \( n \)-dimensional wave equation. Ji in [27] considered the periodic solutions for one dimensional wave equation with bounded
nonlinearity and $x$-dependent coefficients.

2 Relative Morse Index $i_A^*(B)$ and the relationship with $i_A(B)$

Let $H$ be an infinite dimensional separable Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $\| \cdot \|_H$. Denote by $O(H)$ the set of all linear self-adjoint operators on $H$. For $A \in O(H)$, we denote by $\sigma(A)$ the spectrum of $A$ and $\sigma_e(A)$ the essential spectrum of $A$. We define a subset of $O(H)$ as follows

$$O^0_e(a, b) = \{ A \in O(H) | \sigma_e(A) \cap (a, b) = \emptyset \text{ and } \sigma(A) \cap (a, b) \neq \emptyset \}. $$

Denote $L_s(H)$ the set of all linear bounded self-adjoint operators on $H$ and a subset of $L_s(H)$ as follows

$$L_s(H, a, b) = \{ B \in L_s(H), a \cdot I < B < b \cdot I \},$$

where $I$ is the identity map on $H$. $B < b \cdot I$ means that there exists $\delta > 0$ such that $(b-\delta) \cdot I - B$ is positive definite, $B > a \cdot I$ has the similar meaning. For any $B \in L_s(H, a, b)$, we have the index pair $(i_A(B), \nu_A(B))$(see [44, 45] for details). In this section, we will define the relative Morse index $i_A^*(B)$ and give the relationship with $i_A(B)$.

2.1 Relative Morse Index $i_A^*(B)$

As the beginning of this subsection, we will give a brief introduction of relative Morse index. The relative Morse index can be derived in different ways (see[1, 9, 23, 48]). Such kinds of indices have been extensively studied in dealing with periodic or bits of first order Hamiltonian systems. As far as authors known, the existing relative Morse index theory can be regarded as compact perturbation for Fredholm operator. Assume $A$ is a self-adjoint Fredholm operator on Hilbert space $H$, with the orthogonal splitting

$$H = H^-_A \oplus H^0_A \oplus H^+_A,$$

where $A$ is negative, zero and positive definite on $H^-_A$, $H^0_A$ and $H^+_A$ respectively. Let $P_A$ denote the orthogonal projection from $H$ to $H^-_A$. If the perturbation term $F$ is a compact self-adjoint operator on $H$, then we have $P_A - P_{A-F}$ is compact and $P_A : H^-_{A-F} \rightarrow H^-_A$ is a Fredholm operator and we can define the so called relative Morse index by the Fredholm index of $P_A : H^-_{A-F} \rightarrow H^-_A$. 
Generally, if the operator $A$ is not Fredholm operator or the perturbation $F$ is not compact, $P_A : H_{A-F} \rightarrow H_A$ will not be Fredholm operator and the concept of relative Morse index will be meaningless, but if the perturbation lies in the gap of $\sigma_e(A)$, that is to say $A \in \mathcal{O}_e^0(\lambda_a, \lambda_b)$ for some $\lambda_a, \lambda_b \in \mathbb{R}$ and the perturbation $B \in \mathcal{L}_s(H, \lambda_a, \lambda_b)$, we can also defined the relative Morse index $i^*_A(B)$ and give the relationship with the index $i_A(B)$ defined in [45]. Firstly, we need two abstract lemmas.

**Lemma 2.1.** Let $A : H \rightarrow H$ be a bounded self-adjoint operator. Let $W, V$ be closed spaces of $H$. Denote the orthogonal projection $H \rightarrow Y$ by $P_Y$ for any closed linear subspace $Y$ of $H$. Assume that

1. $(Ax, x)_H < -\epsilon_1 \|x\|^2_H$, $\forall x \in W \setminus \{0\}$, with some constant $\epsilon_1 > 0$,
2. $(Ax, x)_H > 0$, $\forall x \in V^\perp \setminus \{0\}$,
3. $(Ax, y)_H = 0$, $\forall x \in V, y \in V^\perp$.

Then $P_V|_W$ is an injection and $P_V(W)$ is a closed subspace of $H$. Furthermore, if we assume

4. $(Ax, x)_H \leq 0$, $\forall x \in V \setminus \{0\}$,

and there is a closed subspace $U$ of $W^\perp$ such that

5. $W^\perp / U$ is finite dimensional,
6. $(Ax, x)_H > 0$, $\forall x \in U \setminus \{0\}$.

Then $P_V : W \rightarrow V$ and $P_W : V \rightarrow W$ are both Fredholm operators and

$$\text{ind}(P_W : V \rightarrow W) = -\text{ind}(P_V : W \rightarrow V).$$

**Proof.** Note that ker $P_V|_W = \ker P_V \cap W = V^\perp \cap W$. From condition (1) and (2), we have $V^\perp \cap W = \{0\}$, so $P_V|_W$ is an injection. For $x \in W$, from condition (2) and (3), we have

$$-\|A\|\|P_V x\|^2_H \leq (AP_V x, P_V x)_H$$

$$= (Ax, x)_H - (A(I - P_V)x, (I - P_V)x)_H$$

$$\leq (Ax, x)_H$$

$$< -\epsilon_1 \|x\|^2_H$$

It follows that

$$\|P_V x\|_H \geq \sqrt{\frac{\epsilon_1}{\|A\|}} \|x\|_H, \forall x \in W;$$

so $P_V(W)$ is a closed subspace of $H$. 

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For any $x \in (P_V(W))^\perp \cap V$, that is to say $x \perp P_V(W)$ and $x \perp (I - P_V)(W)$, so we have $x \perp W$ and

$$P_V(W)^\perp \cap V \subset W^\perp. \quad (2.4)$$

From condition (4) and (6),

$$((P_V(W))^\perp \cap V) \cap U \subset V \cap U = \{0\}. \quad (2.5)$$

From (2.4), (2.5) and condition (5), $(P_V(W))^\perp \cap V$ is finite dimensional. It follows that $P_V : W \to V$ is a Fredholm operator. From (2.3), we have

$$\| (I - P_V)x \|^2 = \|x\|^2 - \|P_Vx\|^2 \leq (1 - \epsilon_1/\|A\|)\|x\|^2, \forall x \in W.$$

It follows that $\| I - P_V \|_W < 1$. So the operator $P_W P_V = P_W - P_W (I - P_V) : W \to W$ is invertible. It follows that $P_W : V \to W$ is surjective, and

$$\ker P_W \cap P_V(W) = 0. \quad (2.6)$$

Note that $V$ has the following decomposition

$$V = P_V(W) \bigoplus ((P_V(W))^\perp \cap V),$$

from (2.6) and $\dim((P_V(W))^\perp \cap V) < \infty$, we have $\ker P_W \cap V$ is finite dimensional. So the operator $P_W : V \to W$ is a Fredholm operator. Since $P_W P_V : W \to W$ is invertible, we have

$$0 = \text{ind}(P_W P_V : W \to W) = \text{ind}(P_W : V \to W) + \text{ind}(P_V : W \to V).$$

Thus we have proved the lemma.

\[\square\]

**Lemma 2.2.** Let $V_1 \subset V_2, W_1 \subset W_2$ be linear closed subspaces of $H$ such that $V_2/V_1$ and $W_2/W_1$ are finite dimensional linear spaces. Let $P_{V_i}$, $P_{W_j}$ be the orthogonal projections onto $V_i$ and $W_j$ and respectively, $i, j = 1, 2$. Assume that $P_{W_{j^*}} : V_{i^*} \to W_{j^*}$ is a Fredholm operator for some fixed $i^*, j^* \in \{1, 2\}$. Then $P_{W_j} : V_i \to W_j$, $i, j = 1, 2$ are all Fredholm operators. Furthermore, we have

$$\text{ind}(P_{W_j} : V_i \to W_j) = \text{ind}(P_{V_{i^*}} : V_i \to V_{i^*}) + \text{ind}(P_{W_{j^*}} : V_{i^*} \to W_{j^*}) + \text{ind}(P_{W_j} : W_{j^*} \to W_j).$$
Proof. Since $V_2/V_1$ and $W_2/W_1$ are finite dimensional linear spaces, $P_{W_j} - P_{V_j}$ and $P_{V_i} - P_{V_i}$ are both compact operator. So $P_{W_j} P_{V_i} - P_{W_j} P_{V_i}$ is also compact operator. Note that on $V_i$:
$$ (P_{W_j} - P_{W_j} P_{V_i} P_{V_i})|_{V_i} = P_{W_j}(P_{W_j} P_{V_i} - P_{W_j} P_{V_i})|_{V_i}. $$
It follows that $P_{W_j} - P_{W_j} P_{V_i} P_{V_i} : V_i \rightarrow W_j$ is compact. Then we can conclude that
$$ \text{ind}(P_{W_j} : V_i \rightarrow W_j) = \text{ind}(P_{W_j} P_{V_i} P_{V_i} : V_i \rightarrow W_j) $$
$$ = \text{ind}(P_{V_i} : V_i \rightarrow V_i) + \text{ind}(P_{W_j} : V_i \rightarrow W_j) $$
$$ + \text{ind}(P_{W_j} : W_j \rightarrow W_j). $$
We have proved the lemma. □

With these two lemmas, we can define the relative Morse index. We consider a normal type that is $A \in C_\epsilon^0(-1, 1)$ for simplicity. Let $B \in L_s(\mathbb{H}, -1, 1)$ with its norm $\|B\| = c_B$, so we have $0 \leq c_B < 1$. Then $A - tB$ is a self-adjoint Fredholm operator for $t \in [0, 1]$. We have $\sigma_{ss}(A - tB) \cap (-1 + tc_B, 1 - tc_B) = 0$. Let $E_{A-tB}(z)$ be the spectral measure of $A - tB$. Denote
$$ P(A - tB, U) = \int_U dE_{A-tB}(z), \quad (2.7) $$
with $U \subset \mathbb{R}$, and rewrite it as $P(t, U)$ for simplicity. Let
$$ V(A - tB, U) = \text{im}P(t, U) $$
and rewrite it as $V(t, U)$ for simplicity. For any $c_0 \in \mathbb{R}$ satisfying $c_B < c_0 < 1$, we have
$$ ((A - B)x, x)_\mathbb{H} > (c_0 - c_B)\|x\|_{\mathbb{H}}^2, \quad x \in V(0, (c_0, +\infty)) \cap D(A). $$
So there is $\epsilon > 0$, such that
$$ ((A - B)x, x)_\mathbb{H} > \epsilon((|A - B| + I)x, x)_\mathbb{H}, \forall x \in V(0, (c_0, +\infty)) \cap D(A) \quad (2.8) $$
Similarly, we have
$$ ((A - B)x, x)_\mathbb{H} < -\epsilon((|A - B| + I)x, x)_\mathbb{H}, \forall x \in V(0, (-\infty, -c_0)) \cap D(A) \quad (2.9) $$
Denote
$$ P_{s,a}^{t,b} := P(t, (-\infty, b))|_{V(s,(-\infty,a))}, \forall t, s \in [0, 1] \text{ and } a, b \in \mathbb{R}. $$
Clearly, we have $P(t, (-\infty, b)) = P_{s,\infty}^{t,b}, \forall s \in [0, 1]$. 

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Lemma 2.3. For any $a \in [-c_0, c_0]$, the map $P_{0,a}^{1,0} P_{1,0}^{0,a}$ are both Fredholm operators. Furthermore, we have $\text{ind}(P_{0,a}^{1,0}) = -\text{ind}(P_{1,0}^{0,a})$.

Proof. From (2.8) and (2.9), there is $\epsilon > 0$ such that

$$((A - B)(|A - B| + I)^{-1} x, x) > \epsilon \|x\|^2, \forall x \in V(0, (c_0, +\infty)), $$

and

$$((A - B)(|A - B| + I)^{-1} x, x) < -\epsilon \|x\|^2, \forall x \in V(0, (-\infty, -c_0)).$$

Now, let the operator $(A - B)(|A - B| + I)^{-1}$, the spaces $V(0, (c_0, +\infty))$, $V(0, (c_0, +\infty))$ be the operator $A$ and the spaces $W, V$ and $U$ in Lemma 2.1 correspondingly. It’s easy to verify that condition (1), (2), (3), (4) and (6) are satisfied, and since $A \in \mathcal{O}^0(1, 1)$, $V(0, [-c_0, c_0])$ is finite dimensional, so condition (5) is satisfied.

Then $P_{0, -c_0}^{1,0}$ and $P_{1,0}^{0,-c_0}$ are both Fredholm operators. We also have

$$\text{ind}(P_{0,-c_0}^{1,0}) = -\text{ind}(P_{1,0}^{0,-c_0}).$$

By Lemma 2.2, $P_{0,a}^{1,0}$ and $P_{1,0}^{0,a}$ are both Fredholm operators with $a \in [-c_0, c_0]$, and we have

$$\text{ind}(P_{0,a}^{1,0}) = -\text{ind}(P_{1,0}^{0,a}), a \in [-c_0, c_0].$$

Remark 2.4. Generally, we have $P_{s,a}^{t,b}$ and $P_{s,b}^{t,a}$ are both Fredholm operators with $a \in (-1 + sc_B, 1 - sc_B)$, $b \in (-1 + tc_B, 1 - tc_B)$ and we have

$$\text{ind}(P_{s,a}^{t,b}) = -\text{ind}(P_{s,b}^{t,a}).$$

Here we replace $A, B$ by $A' = A - sB$ and $B' = (t - s)B$ respectively in Lemma 2.3, then all the proof will be same, so we omit the proof here.

Definition 2.5. Define the relative Morse index by

$$i_A^*(B) := \text{ind}(P_{1,0}^{0,0}), \forall B \in L_s(H, -1, 1).$$

2.2 The relationship between $i_A^*(B)$ and $i_A(B)$

Now, we will prove that $i_A^*(B) = i_A(B)$ by the concept of spectral flow. We need some preparations. There are some equivalent definitions of spectral flow. We use the Definition
2.1, 2.2 and 2.6 in [48]. Let $A_s$ be a path of self-adjoint Fredholm operators. The APS projection of $A_s$ is denoted by $Q_{A_s} = P(A_s, [0, +\infty))$. Recall that locally, the spectral flow of $A_s$ is the $s$-flow of $Q_{A_s}$. Choose $\epsilon > 0$ such that $V(A_{s_0}, [0, +\infty)) = V(A_{s_0}, [-\epsilon, +\infty))$. Then $\epsilon \notin \sigma(A_{s_0})$. Let $P_{A_s} = P(A_s, [-\epsilon, +\infty))$. Then there is $\delta > 0$ such that $P_{A_s}$ is continuous on $(s_0 - \delta, s_0 + \delta)$ and $P_{A_s} - Q_{A_s}$ is compact for $s \in (s_0 - \delta, s_0 + \delta)$. The $s$-flow of $Q_{A_s}$ on $[s_0, b] \subset (s_0 - \delta, s_0 + \delta)$ can be calculated as

$$sfl(Q_{A_s}, [s_0, b]) = -\text{ind}(P_{A_{s_0}} : V(A_{s_0}, [0, +\infty)) \to V(A_{s_0}, [0, +\infty))$$

$$+ \text{ind}(P_{A_{s_b}} : V(A_{s_b}, [0, +\infty)) \to V(A_{s_b}, [-\epsilon, +\infty))$$

$$= -\dim(V(A_{s_b}, [-\epsilon, 0]))$$

$$= \text{ind}(\text{Id} - P_{A_{s_b}} : V(A_{s_b}, (-\infty, -\epsilon)) \to V(A_{s_b}, (-\infty, 0))).$$

If $A_s = A - sB$, with $\epsilon$ and $\delta$ chosen like above, we have $sfl\{A - sB, [s_0, s_1]\} = \text{ind}P_{s_1,-\epsilon}^{s_1,0}$ for $[s_1, s_2] \subset [s_1, s_1 + \delta]$.

**Lemma 2.6.** Let $t_0 \in [0, 1]$. Let $a \in (-1 + t_0cB, 1 - t_0cB) \setminus \sigma(A - t_0B)$. Then we have

$$\lim_{s \to t_0} \|P_{t_0,a}^{s,0} - P_{s,a}^{t_0,0}\| = 0, \forall t \in [0, 1].$$

and

$$\lim_{s \to t_0} \text{ind}(P_{t_0,a}^{s,0}) = \text{ind}(P_{t_0,a}^{t_0,0}),$$

$$\lim_{s \to t_0} \text{ind}(P_{s,a}^{t_0,0}) = \text{ind}(P_{t_0,a}^{t_0,0}).$$

**Proof.** Since $a \notin \sigma(A - t_0B)$, there is $\delta_1 > 0$ such that $P(\cdot, (\infty, a))$ is a continuous path of operators on $(t_0 - \delta_1, t_0 + \delta_1)$, and

$$\|(P(s, (\infty, a)) - P(t_0, (\infty, a)))\| < 1$$

with $s \in (t_0 - \delta_1, t_0 + \delta_1)$. Then $P_{s,a}^{t_0,0}$ and $P_{s,a}^{t_0,0}$ are both homeomorphisms. Note that on $V(t_0, (-\infty, a))$, we have

$$P_{t_0,a}^{t_0,0} - P_{s,a}^{t_0,0}P_{t_0,a}^{t_0,0} = P(t, (\infty, 0))(P(t_0, (\infty, a)) - P(t, (\infty, a)))V(t_0, (\infty, a)).$$

By the continuity of $P(s, (\infty, a))$, it follows that

$$\lim_{s \to t_0} \|P_{t_0,a}^{t_0,0} - P_{s,a}^{t_0,0}P_{t_0,a}^{t_0,0}\| = 0.$$
Then we have

\[
\begin{align*}
\text{ind}(P_{t_0,a}) &= \lim_{s \to t_0} \text{ind}(P_{s,a}^{t_0,a}) \\
&= \lim_{s \to t_0} \text{ind}(P_{s,a}^{t_0}) + \text{ind}(P_{t_0,a}^{s,a}) \\
&= \lim_{s \to t_0} \text{ind}(P_{s,a}^{t}).
\end{align*}
\]

By remark 2.4, we get

\[
\begin{align*}
\text{ind}(P_{t_0,a}^{t}) &= -\text{ind}(P_{s,a}^{t_0}) \\
&= -\lim_{s \to t_0} \text{ind}(P_{s,a}^{t_0}) \\
&= \lim_{s \to t_0} \text{ind}(P_{s,a}^{s,a}).
\end{align*}
\]

\[\square\]

**Lemma 2.7.** For each \(t_1 \in [0,1]\), there is \(\delta > 0\) such that

\[\text{ind}(P_{t_0,a}^{t_2}) = \text{sf}\{A - t_1 B - s(t_2 - t_1) B, [0,1]\}
\]

with \(|t_2 - t_1| < \delta\).

**Proof.** Since \(A - t_1 B\) is a Fredholm operator, there is \(\epsilon > 0\) such that \(P(t_1, (-\infty, 0)) = P(t_1, (-\infty, -\epsilon))\). It follows that \(\epsilon \notin \sigma(A - t_1 B)\), and we have

\[P_{t_1,-\epsilon}^{t_2,0} = P_{t_1,0}^{t_2,0}.
\]

By lemma 2.6 we have

\[\lim_{t_2 \to t_1} \text{ind}(P_{t_1,-\epsilon}^{t_2,0}) = \text{ind}(P_{t_1,-\epsilon}^{t_1,0}) = 0.
\]

It follows that

\[\begin{align*}
\lim_{t_2 \to t_1} \text{ind}(P_{t_1,-\epsilon}^{t_2,0}) &= \lim_{t_2 \to t_1} \text{ind}(P_{t_1,-\epsilon}^{t_2,0} P_{t_2,-\epsilon}^{t_2,0}) \\
&= \lim_{t_2 \to t_1} \text{ind}(P_{t_2,-\epsilon}^{t_2,0}) + \lim_{t_2 \to t_1} \text{ind}(P_{t_1,-\epsilon}^{t_2,-\epsilon}) \\
&= \lim_{t_2 \to t_1} \text{ind}(P_{t_2,-\epsilon}^{t_2,0}).
\end{align*}
\]

So there is \(\delta > 0\) such that \(\text{ind}(P_{t_1,-\epsilon}^{t_2,0}) = \text{ind}(P_{t_2,-\epsilon}^{t_2,0})\) with \(|t_2 - t_1| < \delta\), and \(P(t, (-\infty, -\epsilon))\) is continuous on \((t_1 - \delta, t_1 + \delta)\). Note that \(\text{sf}\{A - t_1 B - s(t_2 - t_1) B, [0,1]\} = \text{ind}(P_{t_2,-\epsilon}^{t_2,0})\) by continuation of \(P(t, (-\infty, -\epsilon))\). Then the lemma follows. \[\square\]

**Lemma 2.8.** \(\text{ind}(P_{0,0}^{t,0}) = \text{ind}(P_{t,0}^{1,0}) + \text{ind}(P_{t,0}^{t,0})\) with \(\forall t \in [0,1]\).
Proof. By Lemma 2.2 and Lemma 2.4, for any $t_0 \in [0, 1]$

$$\text{ind}(P_{t_0,0}^1) + \text{ind}(P_{0,0}^{t_0}) = \text{ind}(P_{t_0,a}^1) + \text{ind}(P_{t_0,0}^{a_t,0}) + \text{ind}(P_{0,0}^{t_0,a})$$

$$= \text{ind}(P_{t_0,a}^1) + \text{ind}(P_{0,0}^{a_t,0}), \forall a \in (-1 + sc_B, 1 - sc B).$$

Choose $a_{t_0} \in (-1 + t_0c_B, 1 - t_0c_B)$ and $a_{t_0} \not\in \sigma(A - t_0 B)$. By lemma 2.6,

$$f : t \to \text{ind}(P_{t,a_{t_0}}^1) + \text{ind}(P_{0,0}^{t,a_{t_0}})$$

is continuous at $t_0$. So the function $f : t \to \text{ind}(P_{t,0}^{1,0}) + \text{ind}(P_{0,0}^{t,0})$ is continuous on $[0, 1]$. So it must be a constant function. It follows that

$$\text{ind}(P_{0,0}^{1,0}) = f(1) = f(t) = \text{ind}(P_{t,a}^1) + \text{ind}(P_{0,0}^{t,a}).$$

\[\Box\]

Remark 2.9. In fact, we have

$$\text{ind}(P_{0,0}^{a,b}) = \text{ind}(P_{b,0}^{a}) + \text{ind}(P_{a,0}^{s})$$

with $s \in [0, 1]$.

Theorem 2.10. We have

$$sf\{A - tB, [a, b]\} = \text{ind}(P_{b,0}^{a}) = -\text{ind}(P_{a,0}^{b})$$

with $[a, b] \subset [0, 1]$.

Proof. It is a direct consequence of Lemma 2.7 and Lemma 2.8. \[\Box\]

Now by the property of $i_A(B)$ (see [44, Lemma 2.9], [45, Lemma 2.3]) and Theorem 2.10, we have the following result.

Proposition 2.11. $i_A^*(B) = i_A(B)$, $A \in O_e^0(-1, 1), B \in L_s(H, -1, 1)$.

Generally, with the same method we can define the relative Morse index $i_A^*(B)$ for $A \in O_e^0(\lambda_a, \lambda_b), B \in L_s(H, \lambda_a, \lambda_b)$ and we can prove the index $i_A^*(B)$ coincide with $i_A(B)$ by the concept of spectral flow, we omit them here.
3 Saddle point reduction of (O.E.) and some abstract critical points Theorems

Now for simplicity, let $b > 0$ and $a = -b$, for $A \in \mathcal{O}_e^0(-b, b)$, we consider the following operator equation

$$Az = F'(z), \ z \in D(A) \subset H,$$

where $F \in C^1(H, \mathbb{R})$. Assume

$$(F_1) \ F \in C^1(H, \mathbb{R}), \ F' : H \to H \text{ is Lipschitz continuous}$$

with its Lipschitz constant $l_F < b$.

3.1 Saddle point reduction of (O.E.)

In this part, assume $A \in \mathcal{O}_e^0(-b, b)$ and $F$ satisfies condition $(F_1)$, we will consider the method of saddle point reduction without assuming the nonlinear term $F \in C^2(|A|^{1/2}))$, then we will give some abstract critical point theorems. Let $E_A(z)$ the spectrum measure of $A$, since $\sigma_e(A) \cap (-b, b) = \emptyset$, we can choose $l \in (l_F, b)$, such that

$$-l, l \notin \sigma(A).$$

Different from the above section, in this section, consider projection map $P(A, U)$ defined in (2.7) on $H$, for simplicity, we rewrite them as

$$P^-_A := P(A, (-\infty, -l)), \ P^+_A := P(A, (l, \infty)), \ P^0_A := P(A, (-l, l)),$$

in this section. Then we have the following decomposition which is different from (2.2),

$$H = \hat{H}^-_A \oplus \hat{H}^+_A \oplus \hat{H}^0_A,$$

where $\hat{H}_A := P_A^\perp H(\ast = \pm, 0)$ and $\hat{H}^0_A$ is finite dimensional subspace of $H$, for simplicity we rewrite $H^\ast := \hat{H}^\ast_A$. Denote $A^\ast$ the restriction of $A$ on $H^\ast(\ast = \pm, 0)$, thus we have $(A^\pm)^{-1}$ are bounded self-adjoint linear operators on $H^\pm$ respectively and satisfying

$$\|(A^\pm)^{-1}\| \leq \frac{1}{l}, \quad (3.3)$$

Then (OE) can be rewritten as

$$z^\pm = (A^\pm)^{-1} P^\pm_A F'(z^+ + z^- + z^0), \quad (3.4)$$
and

\[ A^0z^0 = \tilde{P}^0_A F'(z^+ + z^- + z^0), \]  

(3.5)

where \( z^* = P_A^*z(\ast = \pm, 0) \), for simplicity, we rewrite \( x := z^0 \). From (3.1) and (3.3), we have \((A^\pm)^{-1}P_A^\pm F'\) is contraction map on \( H^+ \oplus H^- \) for any \( x \in H^0 \). So there is a map \( z^\pm(x) : H^0 \to H^\pm \) satisfying

\[ z^\pm(x) = (A^\pm)^{-1}P_A^\pm F'(z^\pm(x) + x), \forall x \in H, \]  

(3.6)

and the following properties.

**Proposition 3.1.** (1) The map \( z^\pm(x) : H^0 \to H^\pm \) is continuous, in fact we have

\[ \|(z^+ + z^-)(x + h) - (z^+ + z^-)(x)\|_H \leq \frac{l_F}{l - l_F} \|h\|_H, \forall x, h \in H^0. \]  

(2) \( \|(z^+ + z^-)(x)\|_H \leq \frac{l_F}{l - l_F} \|x\|_H + \frac{1}{l - l_F} \|F'(0)\|_H. \)

**Proof.** (1) For any \( x, h \in H^0 \), here we write \( z^\pm(x) := z^+(x) + z^-(x) \) and \((A^\pm)^{-1}P_A^\pm := (A^+)^{-1}P_A^+ + (A^-)^{-1}P_A^-\) for simplicity, we have

\[ \|z^\pm(x + h) - z^\pm(x)\|_H = \|(A^\pm)^{-1}P_A^\pm F'(z^\pm(x + h) + x + h) - (A^\pm)^{-1}P_A^\pm F'(z^\pm(x) + x)\|_H \]

\[ \leq \frac{1}{l} \|F'(z^\pm(x + h) + x + h) - F'(z^\pm(x) + x)\|_H \]

\[ \leq \frac{l_F}{l} \|z^\pm(x + h) - z^\pm(x) + h\|_H \]

\[ \leq \frac{l_F}{l} \|z^\pm(x + h) - z^\pm(x)\|_H + \frac{l_F}{l} \|h\|_H. \]

So we have \( \|z^\pm(x + h) - z^\pm(x)\|_H \leq \frac{l_F}{l - l_F} \|h\|_H \) and the map \( z^\pm(x) : H^0 \to H^\pm \) is continuous. (2) Similarly,

\[ \|z^\pm(x)\|_H = \|(A^\pm)^{-1}P_A^\pm F'(z^\pm(x) + x)\|_H \]

\[ \leq \frac{1}{l} \|F'(z^\pm(x) + x)\|_H \]

\[ \leq \frac{1}{l} \|F'(z^\pm(x) + x) - F'(0)\|_H + \frac{1}{l} \|F'(0)\|_H \]

\[ \leq \frac{l_F}{l} (\|z^\pm(x)\|_H + \|x\|_H) + \frac{1}{l} \|F'(0)\|_H. \]

So we have \( \|z^\pm(x)\|_H \leq \frac{l_F}{l - l_F} \|x\|_H + \frac{1}{l - l_F} \|F'(0)\|_H. \)

\[ \Box \]

**Remark 3.2.** Denote \( E = D(|A|^\frac{1}{2}) \), with its norm

\[ \|z\|_E := \|\frac{1}{2}(z^+ + z^-)\|_H^2 + \|x\|_H^2, \ u \in E. \]
From (3.6), we have \( z^\pm(x) \in D(A) \subset E \), and we have

1. The map \( z^\pm(x) : H^0 \to E \) is continuous, and

\[
\| (z^+ + z^-)(x + h) - (z^+ + z^-)(x) \|_E \leq \frac{l_F \cdot l^2}{l - l_F} \| h \|_H, \ \forall x, h \in H^0. \tag{3.7}
\]

2. \( \| (z^+ + z^-)(x) \|_E \leq \frac{l^2}{l - l_F} (l_F \cdot \| x \|_H + \| F'(0) \|_H) \).

**Proof.** The proof is similar to Proposition 3.1, we only prove (1).

\[
\| z^\pm(x + h) - z^\pm(x) \|_E = \| (A^\pm)^{-1}[z^\pm(x + h) - z^\pm(x)] \|_H
\]

\[
= \| (A^\pm)^{-1} [P_A^\pm F'(z^\pm(x + h) + x + h) - P_A^\pm F'(z^\pm(x) + x)] \|_H
\]

\[
\leq \frac{1}{l^2} \| F'(z^\pm(x + h) + x + h) - F'(z^\pm(x) + x) \|_H
\]

\[
\leq \frac{l_F}{l^2} \| z^\pm(x + h) - z^\pm(x) + h \|_H
\]

\[
\leq \frac{l_F}{l^2} \| z^\pm(x + h) - z^\pm(x) \|_H + \frac{l_F}{l^2} \| h \|_H
\]

\[
\leq \frac{l_F}{l} \| z^\pm(x + h) - z^\pm(x) \|_E + \frac{l_F}{l^2} \| h \|_H,
\]

where the last inequality depends on the fact that \( \| z^\pm \|_E \geq \frac{l^2}{l} \| z^\pm \|_H \), so we have (3.7).

Now, define the map \( z : H^0 \to H \) by

\[
z(x) = x + z^+(x) + z^-(x).
\]

Define the functional \( a : H^0 \to \mathbb{R} \) by

\[
a(x) = \frac{1}{2}(Az(x), z(x))_H - F(z(x)), \ x \in H^0. \tag{3.8}
\]

With standard discussion, the critical points of \( a \) correspond to the solutions of (O.E.), and we have

**Lemma 3.3.** Assume \( F \) satisfies (F\(_1\)), then we have \( a \in C^1(H^0, \mathbb{R}) \) and

\[
a'(x) = Az(x) - F'(z(x)), \ \forall x \in H^0. \tag{3.9}
\]

Further more, if \( F \in C^2(H, \mathbb{R}) \), we have \( a \in C^2(H^0, \mathbb{R}) \), for any critical point \( x \) of \( a \), \( F''(z(x)) \in \mathcal{L}_s(H, -b, b) \) and the morse index \( m^-_a(x) \) satisfies the following equality

\[
m^-_a(x_2) - m^-_a(x_1) = i^-_A(F''(z(x_2))) - i^-_A(F''(z(x_1))), \ \forall x_1, x_2 \in H^0. \tag{3.10}
\]
Proof. For any $x, h \in \mathbf{H}^0$, write

$$\eta(x, h) := z^+(x + h) + z^-(x + h) - z^+(x) - z^-(x) + h$$

for simplicity, that is to say

$$z(x + h) = z(x) + \eta(x, h), \ \forall x, h \in \mathbf{H}^0,$$

and from (3.7), we have

$$\|\eta(x, h)\|_{\mathbf{H}} \leq C\|h\|_{\mathbf{H}}, \ \forall x, h \in \mathbf{H}^0,$$

(3.11)

where $C = \frac{l + l_F}{l - l_F}$. Let $h \to 0$ in $\mathbf{H}^0$, and for any $x \in \mathbf{H}^0$, we have

$$a(x + h) - a(x) = \frac{1}{2}[(Az(x + h), z(x + h)] - (Az(x), z(x)] - [F(z(x + h)) - F(z(x))]$$

$$= (Az(x), \eta(x, h)_H + \frac{1}{2}(A\eta(x, h), \eta(x, h))_H$$

$$- (F'(z(x)), \eta(x, h))_H + o(\|\eta(x, h)\|_H).$$

From (3.11) we have

$$a(x + h) - a(x) = (Az(x) - F'(z(x)), \eta(x, h))_H + o(\|h\|_H), \ \forall x \in \mathbf{H}^0, \ \text{and } \|h\|_H \to 0.$$

Since $z^\pm(x)$ is the solution of (3.6) and from the definition of $\eta(x, h)$, we have

$$(Az(x) - F'(z(x)), \eta(x, h))_H = (Az(x) - F'(z(x)), h)_H, \ \forall x, h \in \mathbf{H}^0,$$

so we have

$$a(x + h) - a(x) = (Az(x) - F'(z(x)), h)_H + o(\|h\|_H), \ \forall x \in \mathbf{H}^0, \ \text{and } \|h\|_H \to 0,$$

and we have proved (3.9). If $F \in C^2(\mathbf{H}, \mathbb{R})$, from (3.6) and by Implicit function theorem, we have $z^\pm \in C^1(\mathbf{H}_0, \mathbf{H}_\pm)$. From (3.6) and (3.9), we have

$$a'(x) = Ax - P_0 F(z(x))$$

and

$$a''(x) = A|_{\mathbf{H}_0} - P_0 F''(z(x))z'(x),$$

that is to say $a \in C^2(\mathbf{H}_0, \mathbb{R})$. Finally, from Theorem 2.10 received above, Definition 2.8 and Lemma 2.9 in [44], we have (3.10). \qed
3.2 Some abstract critical points Theorems

In this part, we will give some abstract critical points Theorems for (O.E.) by the method of saddle point reduction introduced above. Since we have Proposition 2.11, we will not distinguish \( i_A^*(B) \) from \( i_A(B) \). Beside condition \((F_1)\), assume \( F \) satisfying the following condition.

\((F_2)\) There exist \( B_1, B_2 \in L_s(H, -b, b) \) and \( B : H \to L_s(H, -b, b) \) satisfying

\[
B_1 \leq B_2, \; i_A(B_1) = i_A(B_2), \quad \text{and} \quad \nu_A(B_2) = 0,
\]

such that

\[
B_1 \leq B(z) \leq B_2, \forall z \in H,
\]

such that

\[
F'(z) - B(z)z = o(\|z\|_H), \|z\|_H \to \infty.
\]

Before the following Theorem, we need a Lemma.

**Lemma 3.4.** Let \( B_1, B_2 \in L_s(H, -b, b) \) with \( B_1 \leq B_2, \; i_A(B_1) = i_A(B_2), \) and \( \nu_A(B_2) = 0, \) then there exists \( \varepsilon > 0, \) such that for all \( B \in L_s(H) \) with

\[
B_1 \leq B \leq B_2,
\]

we have

\[
\sigma(A - B) \cap (-\varepsilon, \varepsilon) = \emptyset.
\]

**Proof.** For the property of \( i_A(B) \), we have \( \nu_A(B_1) = 0. \) So there is \( \varepsilon > 0, \) such that

\[
i_A(B_1, \varepsilon) = i_A(B_1) = i_A(B_2) = i_A(B_2, \varepsilon),
\]

with \( B_{*,\varepsilon} = B_* + \varepsilon \cdot I, (* = 1, 2). \) Since \( B_{1,\varepsilon} \leq B - \varepsilon I < B + \varepsilon I \leq B_2', \) it follows that

\[
i_A(B - \varepsilon I) = i_A(B + \varepsilon I).
\]

Note that

\[
i_A(B + \varepsilon) - i_A(B - \varepsilon) = \sum_{-\varepsilon < t \leq \varepsilon} \nu_A(B - t \cdot I).
\]

We have \( 0 \notin \sigma(A - B - \eta), \forall \eta \in (-\varepsilon, \varepsilon), \) thus the proof is complete. \( \square \)

**Theorem 3.5.** Assume \( A \in \mathcal{O}^0_e(-b, b). \) If \( F \) satisfies conditions \((F_1)\) and \((F_2)\), then (O.E.) has at least one solution.
Proof. Firstly, for $\lambda \in [0, 1]$, consider the following equation

$$Az = (1 - \lambda)B_1z + \lambda F'(z).$$

(O.E.)$_\lambda$

We claim that the set of all the solutions $(z, \lambda)$ of (O.E.)$_\lambda$ are a priori bounded. If not, assume there exist $\{(z_n, \lambda_n)\}$ satisfying (O.E.)$_\lambda$ with $\|z_n\|_H \to \infty$. Without lose of generality, assume $\lambda_n \to \lambda_0 \in [0, 1]$. Denote by

$$F_\lambda(z) = \frac{1 - \lambda}{2} (B_1z, z)_H + \lambda F(z), \forall z \in H.$$

Since $F$ satisfies condition (F1) and $B_1 \in \mathcal{L}_a(H, -b, b)$, we have $F'_\lambda : H \to H$ is Lipschitz continuous with its Lipschitz constant less than $b$, that is to say there exists $\hat{l} \in [l_F, b)$ such that

$$\|F'_\lambda(z + h) - F'_\lambda(z)\|_H \leq \hat{l}\|h\|_H, \forall z, h \in H, \lambda \in [0, 1].$$

Now, consider the projections defined in (3.2), choose $l \in (\hat{l}, b)$ satisfying $-\hat{l}, l / \in \sigma(A)$, from (3.4) and (3.5), we decompose $z_n$ by

$$z_n = z_n^+ + z_n^- + x_n,$$

with $z_n^* \in H^*(\ast = \pm, 0)$ and $z_n^\pm$ satisfies Proposition 3.1 with $l_F$ replaced by $\hat{l}$. So we have $\|x_n\|_H \to \infty$. Denote by

$$y_n = \frac{z_n}{\|z_n\|_H},$$

and $\tilde{B}_n := (1 - \lambda_n)B_1 + \lambda_n B(z_n)$, we have

$$Ay_n = \tilde{B}_n y_n + \frac{o(\|z_n\|_H)}{\|z_n\|_H}.$$ 

Decompose $y_n = y_n^+ + y_n^0$ with $y_n^* = z_n^*/\|z_n\|_H$, we have

$$\|y_n^0\|_H = \|\frac{x_n}{\|z_n\|_H}\|_H \geq \frac{\|x_n\|_H}{\|x_n\|_H + \|z_n^+ + z^-\|_H} \geq \frac{l\|x_n\|_H}{l\|x_n\|_H + \|F_\lambda'(0)\|_H}.$$ 

That is to say

$$\|y_n^0\|_H \geq c > 0$$

for some constant $c > 0$ and $n$ large enough. Since $B_1 \leq B(z) \leq B_2$, we have $B_1 \leq \tilde{B}_n \leq B_2$. Let $H = H^+_A \oplus H^-_A$ with $A - \tilde{B}_n$ is positive and negative define on $H^+_A$ and
\( \textbf{H}_{n_{A-B}} \) respectively. Re-decompose \( y_n = \bar{y}_n^+ + \bar{y}_n^- \) respect to \( \textbf{H}_{n_{A-B}}^+ \) and \( \textbf{H}_{n_{A-B}}^- \). From Lemma 3.4 and (3.12), we have

\[
\|y_0^0\|_H^2 \leq \|y_n\|_H^2 \leq \frac{1}{\varepsilon}((A - \bar{B}_n)y_n, \bar{y}_n^+ + \bar{y}_n^-)_H \leq \frac{1}{\varepsilon}o(\|z_n\|_H)\|y_n\|_H.
\] (3.14)

Since \( \|z_n\|_H \to \infty \) and \( \|y_n\| = 1 \), we have \( \|y_0^0\|_H \to 0 \) which contradicts to (3.13), so we have \( \{z_n\} \) is bounded.

Secondly, we apply the topological degree theory to complete the proof. Since the solutions of (O.E.) are bounded, there is a number \( R > 0 \) large enough, such that all of the solutions \( z_\lambda \) of (O.E.) are in the ball \( B(0, R) \): \( \{z \in H \|z\|_H < R\} \). So we have the Brouwer degree

\[
\text{deg}(a'_1, B(0, R) \cap H^0, 0) = \text{deg}(a'_0, B(0, R) \cap H^0, 0) \neq 0,
\]

where \( a_\lambda(x) = \frac{1}{2}(Az_\lambda(x), z_\lambda(x))_H - F_\lambda(z_\lambda(x)), \lambda \in [0, 1]. \) That is to say (O.E.) has at least one solution. \( \square \)

In Theorem 3.5, the non-degeneracy condition of \( B(z) \) is important to keep the boundedness of the solutions. The following theorem will not need this non-degeneracy condition, the idea is from [27].

**Theorem 3.6.** Assume \( A \in \mathcal{O}_e^0(-b, b) \). If \( F \) satisfies conditions (\( F_1 \)) and the following condition.

(\( F_2^+ \)) There exists \( M > 0, B_\infty \in L_s(H, -b, b) \), such that

\[
F'(z) = B_\infty z + r(z),
\]

with

\[
\|r(z)\|_H \leq M, \forall z \in H,
\]

and

\[
(r(z), z)_H \to \pm \infty, \|z\|_H \to \infty.
\] (3.15)

Then (O.E.) has at least one solution.

**Proof.** If \( 0 \notin \sigma(A - B_\infty) \), then with the similar method in Theorem 3.5, we can prove the result. So we assume \( 0 \in \sigma(A - B_\infty) \) and we only consider the case of (\( F_2^- \)). Since 0
is an isolate eigenvalue of $A - B_\infty$ with finite dimensional eigenspace (see [44] for details), there exists $\eta > 0$ such that

$$(-\eta, 0) \cap \sigma(A - B_\infty) = \emptyset.$$  

For any $\varepsilon \in (0, \eta)$, we have $0 \not\in \sigma(\varepsilon + A - B_\infty)$. Thus, with the similar method in Theorem 3.5, we can prove that there exists $z_\varepsilon \in H$ satisfying the following equation

$$\varepsilon z_\varepsilon + (A - B_\infty) z_\varepsilon = r(z_\varepsilon).$$  

(3.16)

In what follows, We divide the following proof into two steps and $C$ denotes various constants independent of $\varepsilon$.

**Step 1.** We claim that $\|z_\varepsilon\|_H \leq C$. Since $z_\varepsilon$ satisfies the above equation, we have

$$\varepsilon(z_\varepsilon, z_\varepsilon)_H = -((A - B_\infty) z_\varepsilon, z_\varepsilon)_H + (r(z_\varepsilon), z_\varepsilon)_H$$

$$\leq \frac{1}{\eta} \|(A - B_\infty) z_\varepsilon\|_H^2 + M \|z_\varepsilon\|_H$$

$$= \frac{1}{\eta} \|\varepsilon z_\varepsilon - r(z_\varepsilon)\|_H^2 + M \|z_\varepsilon\|_H$$

$$\leq \frac{\varepsilon^2}{\eta} \|z_\varepsilon\|_H^2 + C \|z_\varepsilon\|_H + C.$$  

So we have

$$\varepsilon \|z_\varepsilon\|_H \leq C.$$  

Therefore

$$\|(A - B_\infty) z_\varepsilon\|_H = \|\varepsilon z_\varepsilon - r(z_\varepsilon)\|_H \leq C.$$  

(3.17)

Now, consider the orthogonal splitting as defined in (2.2),

$$H = H^0_{A - B_\infty} \oplus H^*_{A - B_\infty},$$

where $A - B_\infty$ is zero definite on $H^0_{A - B_\infty}$; $H^*_{A - B_\infty}$ is the orthonormal complement space of $H^0_{A - B_\infty}$. Let $z_\varepsilon = u_\varepsilon + v_\varepsilon$ with $u_\varepsilon \in H^0_{A - B_\infty}$ and $v_\varepsilon \in H^*_{A - B_\infty}$. Since $0$ is an isolated point in $\sigma(A - B_\infty)$, from (3.17), we have

$$\|v_\varepsilon\|_H \leq C.$$  

(3.18)

Additionally, since $r(z)$ and $v_\varepsilon$ are bounded, we have

$$(r(z_\varepsilon), z_\varepsilon)_H = (r(z_\varepsilon), v_\varepsilon)_H + (r(z_\varepsilon), u_\varepsilon)_H$$

$$= (r(z_\varepsilon), v_\varepsilon)_H + (\varepsilon z_\varepsilon + (A - B_\infty) z_\varepsilon, u_\varepsilon)_H$$

$$= (r(z_\varepsilon), v_\varepsilon)_H + \varepsilon (u_\varepsilon, u_\varepsilon)_H$$

$$\geq C.$$  

(3.19)
Therefor, from (3.15), $\|u_\varepsilon\|_H$ are bounded in $H$ and we have proved the boundedness of $\|z_\varepsilon\|_H$.

**Step 2.** Passing to a sequence of $\varepsilon_n \to 0$, there exists $z \in H$ such that

$$\lim_{\varepsilon_n \to 0} \|z_{\varepsilon_n} - z\|_H = 0.$$  

Different from the above splitting, now, we recall the projections $P^-_A$, $P^0_A$ and $P^+_A$ defined in (3.2) and the splitting $H = H^- \oplus H^0 \oplus H^+$ with $H^* = P^*_A(\ast = \pm, 0)$. So $z_\varepsilon$ has the corresponding splitting

$$z_\varepsilon = z^+_\varepsilon + z^-_\varepsilon + z^0_\varepsilon,$$

with $z^\pm_\varepsilon \in H^\pm$ respectively. Since $H^0$ is a finite dimensional space and $\|z_\varepsilon\|_H \leq C$, there exists a sequence $\varepsilon_n \to 0$ and $z^0 \in H^0$, such that

$$\lim_{n \to \infty} z^0_{\varepsilon_n} = z^0.$$  

For simplicity, we rewrite $z^*_n := z^*_{\varepsilon_n}$, $A_n := \varepsilon_n + A$ and $A^\pm_n := A_n|_{H^\pm}$. Since $z_\varepsilon$ satisfies (3.16), we have

$$z^\pm_n = (A^\pm_n)^{-1} P^\pm_A F'(z^\pm_n + z^-_n + z^0_n).$$

Since $F$ satisfies $(F_1)$, with the similar method used in Proposition 3.1, for $n$ and $m$ large enough, we have

$$\|z^\pm_n - z^\pm_m\|_H \leq \|(A^\pm_n)^{-1} P^\pm_A F'(z_n) - (A^\pm_m)^{-1} P^\pm_A F'(z_m)\|_H$$

$$\leq \|(A^\pm_n)^{-1} P^\pm_A F'(z_n) - F'(z_m)\|_H + \|(A^\pm_n)^{-1} - (A^\pm_m)^{-1}\) P^\pm_A F'(z_m)\|_H$$

$$\leq \frac{l_F}{l} \|z_n - z_m\|_H + \|(A^\pm_n)^{-1} - (A^\pm_m)^{-1}\) P^\pm_A F'(z_m)\|_H.$$  

Since $(A^\pm_n)^{-1} - (A^\pm_m)^{-1} = (\varepsilon_m - \varepsilon_n)(A^\pm_n)^{-1}(A^\pm_m)^{-1}$ and $z_n$ are bounded in $H$, we have

$$\|(A^\pm_n)^{-1} - (A^\pm_m)^{-1}\) P^\pm_A F'(z_m)\|_H = o(1), \ n, m \to \infty.$$  

So we have

$$\|z^\pm_n - z^\pm_m\|_H \leq \frac{l_F}{l - l_F} \|z^0_n - z^0_m\|_H + o(1), \ n, m \to \infty,$$

therefore, there exists $z^\pm \in H^\pm$, such that $\lim_{n \to \infty} \|z^\pm_n - z^\pm\|_H = 0$. Thus, we have

$$\lim_{n \to \infty} \|z_{\varepsilon_n} - z\|_H = 0,$$

with $z = z^- + z^+ + z^0$. Last, let $n \to \infty$ in (3.16), we have $z$ is a solution of (O.E.). □
Theorem 3.7. Assume \( A \in \mathcal{O}_e^0(-b, b) \), \( F \) satisfies (\( F_1 \)) with \( \pm l_F \not\in \sigma(A) \) and the following condition:

(\( F_3^+ \)) There exist \( B_3 \in \mathcal{L}_s(H, -b, b) \) and \( C \in \mathbb{R} \), such that

\[
B_3 > \beta := \max \{ \lambda | \lambda \in \sigma(A) \cap (-\infty, l_F) \},
\]

with

\[
F(z) \geq \frac{1}{2}(B_3z, z)_H - C, \quad \forall z \in H.
\]

Or (\( F_3^- \)) There exist \( B_3 \in \mathcal{L}_s(H, -b, b) \) and \( C \in \mathbb{R} \), such that

\[
B_3 < \alpha := \min \{ \lambda | \lambda \in \sigma(A) \cap (-l_F, \infty) \},
\]

with

\[
F(z) \leq \frac{1}{2}(B_3z, z)_H + C, \quad \forall z \in H.
\]

Then (O.E.) has at least one solution. Furthermore, assume \( F \) satisfies (\( F_4^+ \)) \( F \in C^2(H, \mathbb{R}) \), \( F'(0) = 0 \) and there exists \( B_0 \in \mathcal{L}_s(H, -b, b) \) with

\[
\pm (i_A(B_0) + \nu_A(B_0)) < \pm i_A(B_3), \quad (3.20)
\]

such that

\[
F'(z) = B_0z + o(\|z\|_H), \quad \|z\|_H \to 0.
\]

Then (O.E.) has at least one nontrivial solution. Additionally, if

\[
\nu_A(B_0) = 0 \quad (3.21)
\]

then (O.E.) has at least two nontrivial solutions.

Proof. We only consider the case of (\( F_3^+ \)). According to the saddle point reduction, since \( \pm l_F \not\in \sigma(A) \), we can choose \( l \in (l_F, b) \) in (3.2) satisfying

\[
[-l, -l_F] \cap \sigma(A) = \emptyset = [l_F, l] \cap \sigma(A).
\]

We turn to the function

\[
a(x) = \frac{1}{2}(Ax, z(x)) - F(z(x)),
\]

where \( z(x) = x + z^+(x) + z^-(x), \quad x \in H^0 \) and \( z^\pm \in H^\pm \). Denote by \( w(x) = x + z^-(x) \) and write \( z = z(x), \quad w = w(x) \) for simplicity. Since

\[
a(x) = \left\{ \frac{1}{2}(Aw, w) - F(w) \right\} + \left\{ \frac{1}{2}[(Az, z) - (Aw, w)] - [F(z) - F(w)] \right\}. \quad (3.22)
\]
By condition \( (F'^+_3) \), we obtain
\[
\frac{1}{2}(Aw, w) - F(w) \leq \frac{1}{2}((\beta - B_3)w, w)_{\mathbf{H}} + C, \quad (3.23)
\]
and the terms in the second bracket are equal to
\[
\frac{1}{2}(Az^+, z^+) - \int_0^1 (F'(sz^+ + w), z^+)ds \\
= \frac{1}{2}(Az^+, z^+) - (F'(z^+ + w), z^+) + \int_0^1 (F'(z^+ + w) - F'(sz^+ + w), z^+)ds \\
= - \frac{1}{2}(Az^+, z^+) + \int_0^1 (F'(z^+ + w) - F'(sz^+ + w), z^+)ds \\
\leq - \frac{1}{2}(Az^+, z^+) + \int_0^1 (1 - s)ds \cdot l_F \cdot \|z^+\|_{\mathbf{H}}^2 \\
\leq - \frac{l - l_F}{2}\|z^+\|_{\mathbf{H}}^2, \quad (3.24)
\]
where the last equality is from the fact that \( Az^+ = P^+F'(z^+ + w) \). From (3.22),(3.23) and (3.24) we have
\[
a(x) \leq \frac{1}{2}((\beta - B_3)w, w)_{\mathbf{H}} - \frac{l - l_F}{2}\|z^+\|_{\mathbf{H}}^2 + C \\
\rightarrow -\infty, \text{ as } \|x\| \rightarrow \infty.
\]
Thus the function \(-a(x)\) is bounded from below and satisfies the (PS) condition. So the maximum of \(a\) exists and the maximum points are critical points of \(a\).

In order to prove the second part, similarly, we only consider the case of \((F'^+_3)\) and \((F'^+_4)\). We only need to realize that 0 is not a maximum point from (3.20), so the maximum points discovered above are not 0. In the last, if (3.21) is satisfied, we can use the classical three critical points theorem, since 0 is neither a maximum nor degenerate and the proof is complete.

**Remark 3.8.** (A). Theorem 3.7 is generalized from [8, IV, Theorem 2.3]. In the first part of our Theorem, we do not need \(F\) to be \(C^2\) continuous.

(B). Theorem 3.7 is different from our former result in [44, Theorem 3.6]. Here, we need the Lipschitz condition to keep the method of saddle point reduction valid, where, in [44, Theorem 3.6], in order to use the method of dual variation, we need the convex property.
4 Applications in one dimensional wave equation

In this section, we will consider the following one dimensional wave equation

\[
\left\{
\begin{array}{ll}
\Box u = u_{tt} - u_{xx} = f(x, t, u), \\
u(0, t) = u(\pi, t) = 0, \quad \forall (x, t) \in [0, \pi] \times S^1, \\
u(x, t + T) = u(x, t),
\end{array}
\right.
\]  
(W.E.)

where \( T > 0, \) \( S^1 := \mathbb{R}/T\mathbb{Z} \) and \( f : [0, \pi] \times S^1 \times \mathbb{R} \rightarrow \mathbb{R} \). In what follows we assume systematically that \( T \) is a rational multiple of \( \pi \). So, there exist coprime integers \((p, q)\), such that

\[ T = \frac{2\pi q}{p}. \]

Let

\[ L^2 := \left\{ u, u = \sum_{j \in \mathbb{N}^+, k \in \mathbb{Z}} u_{j,k} \sin jx \exp ik\frac{p}{q} t \right\}, \]

where \( i = \sqrt{-1} \) and \( u_{j,k} \in \mathbb{C} \) with \( u_{j,k} = \bar{u}_{j,-k} \), its inner product is

\[ (u, v)_2 = \sum_{j \in \mathbb{N}^+, k \in \mathbb{Z}} (u_{j,k}, \bar{v}_{j,k}), \quad u, v \in L^2, \]

the corresponding norm is

\[ \|u\|_2^2 = \sum_{j \in \mathbb{N}^+, k \in \mathbb{Z}} |u_{j,k}|^2, \quad u, v \in L^2. \]

Consider \( \Box \) as an unbounded self-adjoint operator on \( L^2 \). Its’ spectrum set is

\[ \sigma(\Box) = \{(p^2 k^2 - q^2 j^2)/q^2 | j \in \mathbb{N}^+, k \in \mathbb{Z}\}. \]

It is easy to see \( \Box \) has only one essential spectrum \( \lambda_0 = 0 \). Let \( \Omega := [0, \pi] \times S^1 \), assume \( f \) satisfying the following conditions.

\((f_1)\) \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \), there exist \( b \neq 0 \) and \( l_F \in (0, |b|) \), such that

\[ |f_b(x, t, u + v) - f_b(x, t, u)| \leq l_F |v|, \quad \forall (x, t) \in \Omega, \ u, v \in \mathbb{R}, \]

where

\[ f_b(x, t, u) := f(x, t, u) - bu, \quad \forall (x, t, u) \in \Omega \times \mathbb{R}. \]

Let the working space \( H := L^2 \) and the operator \( A := \Box - b \cdot I \), with \( I \) the identity map on \( H \). Thus we have \( A \in \mathcal{O}_e^0(-|b|, |b|) \). Denote \( L^\infty := L^\infty(\Omega, \mathbb{R}) \) the set of all essentially
bounded functions. For any \( g \in L^\infty \), it is easy to see \( g \) determines a bounded self-adjoint operator on \( L^2 \), by
\[
u(x,t) \mapsto g(x,t)u(x,t), \quad \forall u \in L^2,
\]
without confusion, we still denote this operator by \( g \), that is to say we have the continuous embedding \( L^\infty \hookrightarrow \mathcal{L}_a(H) \). Thus for any \( g \in L^\infty \cap \mathcal{L}_s(H, -|b|, |b|) \), we have the index pair \((i_A(g), \nu_A(g))\). Besides, for any \( g_1, g_2 \in L^\infty \), \( g_1 \leq g_2 \) means that
\[
u(x,t) \leq g(x,t), \quad \forall (x,t) \in \Omega.
\]
(f2) There exist \( g_1, g_2 \in L^\infty \cap \mathcal{L}_s(H, -|b|, |b|) \) and \( g \in L^\infty (\Omega \times \mathbb{R}, \mathbb{R}) \), with
\[
u(x,t) \leq g(x,t) \leq g_2(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R},
\]
such that
\[
u_b(x,t,u) - g(x,t,u)u = o(|u|), \quad |u| \to \infty, \quad \text{uniformly for} (x,t) \in \Omega.
\]

We have the following results.

**Theorem 4.1.** Assume \( T \) is a rational multiple of \( \pi \), \( f \) satisfying (f1) and (f2), then (W.E.) has a weak solution.

**Proof of Theorem 4.1.** Let
\[
u_b(x,t,u) := \int_0^u f_b(x,t,s)ds, \quad \forall (x,t,u) \in \Omega \times \mathbb{R},
\]
and
\[
u(x,t,u) := \int_\Omega \nu_b(x,t,u(x,t))dxdt, \quad \forall u \in H. \tag{4.1}
\]
It is easy to verify that \( F \) will satisfies condition \((F_1)\) and \((F_2)\) if \( f \) satisfies condition \((f_1)\) and \((f_2)\). Thus, by Theorem 3.5, the proof is complete. \(\square\)

Here, we give an example of Theorem 4.1.

**Example 4.1.** For any \( b \neq 0 \), assume \( \alpha, \beta \in (-|b|, |b|) \) and \( [\alpha, \beta] \cap \sigma(\square - b) = \emptyset \). Let
\[
u(x,t,u) := \frac{\beta - \alpha}{2} \sin(\epsilon_1 \ln(|x| + |t| + |u| + 1)) + \frac{\alpha + \beta}{2},
\]
and \( h \in C(\mathbb{R}, \mathbb{R}) \) is Lipschitz continuous with
\[
u(h) = o(|u|), \quad |u| \to \infty.
\]
then
\[ f(x, t, u) := bu + g(x, t, u)u + \varepsilon h(u) \]
will satisfies condition \((f_1)\) and \((f_2)\) for \(\varepsilon_1\) and \(\varepsilon_2 > 0\) small enough.

**Theorem 4.2.** Assume \(T\) is a rational multiple of \(\pi\), \(f\) satisfies \((f_1)\) and the following condition,
\[ (f_2^+) \text{ There exists } g_\infty(x, t) \in L^\infty \cap L_s(H, -|b|, |b|) \text{ with } \]
\[ |f_b(x, t, u) - g_\infty(x, t)u| \leq M_1 \quad \forall (x, t, u) \in \Omega \times \mathbb{R}, \]
and
\[ \pm (f_b(x, t, u) - g_\infty(x, t)u, u) \geq c|u|, \quad \forall (x, t, u) \in \Omega \times \mathbb{R}/[-M_2, M_2], \]
where \(M_1, M_2, c > 0\) are constants. Then \((W.E.)\) has a weak solution.

**Proof.** We only consider the case of \(f_2^+\). Let \(r(x, t, u) := f_b(x, t, u) - g_\infty(x, t)u\), then \(r\) is bounded in \(H\). Generally speaking, from \((4.2)\), we cannot prove \((3.15)\), so we cannot use Theorem 3.6 directly. By checking the proof of Theorem 3.6, in step 1, when we got \((3.18)\), \((3.15)\) was only used to get the boundedness of \(z^0\). Now, with \((4.2)\), we can also get the boundedness of \(z^0\) from \((3.18)\). Recall that \(H = L^2(\Omega)\) in this section, from the boundedness of \(z^\pm\) in \(H\), we have the boundedness of \(z^\pm\) in \(L^1(\Omega)\). On the other hand, since ker\((A - g_\infty)\) is a finite dimensional space, if \(\|z^0\|_H \to \infty\), we have \(\|z^0\|_{L^1} \to \infty\), thus \(\|z\|_{L^1} \to \infty\). Therefor, we have the contradiction from \((3.18)\) and \((4.2)\). So we have gotten the boundedness of \(z^0\). The rest part of the proof is similar to Theorem 3.6, we omit it here.

**Example 4.2.** Here we give an example of Theorem 3.6. For any \(b \neq 0\), and \(g_\infty \in C(\Omega)\) with
\[ \|g_\infty\|_{C(\Omega)} < |b|. \]
Let \(r(u) = \varepsilon \arctan u\), then
\[ f(x, t, u) := bu + g_\infty(x, t)u \pm r(u) \]
will satisfies the conditions in Theorem 3.6 for \(\varepsilon > 0\) small enough.

Now, in order to use Theorem 3.7, we assume \(f\) satisfies the following conditions.
\[ (f_3^+) \text{ There exists } g_3(x, t) \in L^\infty \cap L_s(H, -|b|, |b|), \text{ with } \]
\[ \pm g_3(x, t) > \max\{\lambda|\lambda \in \sigma(\pm A) \cap (-\infty, l_F)\}, \]
such that
\[ \pm F_b(x, t, u) \geq \frac{1}{2} (g_3(x, t)u, u) + c, \ \forall (x, t, u) \in \Omega \times \mathbb{R}, \]
for some \( c \in \mathbb{R} \).

\((f_4^+)\) \( f \in C^1(\Omega \times \mathbb{R}, \mathbb{R}) \), \( f(x, t, 0) \equiv 0, \ \forall (x, t) \in \Omega \) and
\[ g_0(x, t) := f'_b(x, t, u), \ \forall (x, u) \in \Omega, \]
with
\[ \pm (i_A(g_0) + \nu_A(g_0)) < \pm i_A(g_3). \]

We have the following result.

**Theorem 4.3.** Assume \( T \) is a rational multiple of \( \pi \).

(A.) If \( f \) satisfies condition \((f_1)\), \((f_3^+)\) (or \((f_3^-)\)), then \((W.E.)\) has at least one solution.

(B.) Further more, if \( f \) satisfies condition \((f_4^+)\) (or \((f_4^-)\)), then \((W.E.)\) has at least one nontrivial solution. Additionally, if \( \nu_A(g_0) = 0 \), then \((W.E.)\) has at least two nontrivial solutions.

The proof is to verify the conditions in Theorem 3.7, we only verify the smoothness of \( F(u) \) defined in (4.1). From condition \((f_1)\) and \( f \in C^1(\Omega \times \mathbb{R}) \), we have the derivative \( f'_b(x, t, u) \) of \( f_b \) with respect to \( u \), satisfying
\[ |f'_b(x, t, u)| \leq l_F, \ \forall (x, t, u) \in \Omega \times \mathbb{R}. \]  
(4.3)

For any \( u, v \in H \),
\[ F''(u + v) - F'(u) = f_b(x, t, u + v) - f_b(x, t, u) = f'_b(x, t, u)v + (f'_b(u + \xi v) - f'_b(u))v. \]

From (4.3), we have \( f'_b(u + \xi v) - f'_b(u) \in H \) and
\[ \lim_{\|v\|_H \to 0} \|f'_b(u + \xi v) - f'_b(u)\|_H = 0, \ \forall u \in H. \]

That is to say \( F''(u) = f'_b(x, t, u) \) and \( F \in C^2(H, \mathbb{R}) \).

**Example 4.3.** In order to give an example for Theorem 4.3, assume
\[ \sigma(\square) = \bigcup_{n \in \mathbb{Z}} \{\lambda_n\}, \]  
(4.4)
with \( \lambda_0 = 0 \) and \( \lambda_n < \lambda_{n+1} \) for all \( n \in \mathbb{Z} \). Choose any \( k \in \{2, 3, \cdots \} \). Let

\[
g_0(x, t) \in C(\Omega, [\alpha, \beta]), \text{ with } [\alpha, \beta] \in (0, \lambda_k),
\]

and \( h \in C(\mathbb{R}, \mathbb{R}) \) defined above. Define

\[
g(x, t, u) := g_0(x, t) + (\lambda_k - g_0(x, t) - \varepsilon_1) \frac{2}{\pi} \arctan(\varepsilon_1 u^2),
\]

then

\[
f(x, t, u) := g(x, t, u)u + \varepsilon_2 h(u)
\]

will satisfies condition \((f_1^+)\) and \((f_{\varepsilon_1}^+)\) with \( b = \frac{\lambda_k}{2} \) and \( \varepsilon_1, \varepsilon_2 > 0 \) small enough. Further more, if \( g_0, h \) are \( C^1 \) continuous and \( \beta < \lambda_{k-1} \), we have condition \((f_{\varepsilon_1}^+)\) is satisfied. Additionally, if \([\alpha, \beta] \cap \sigma(\Box) = \emptyset\), then \( \nu_A(g_0) = 0 \).

**Remark 4.4.** We can also use Theorem 3.5, Theorem 3.6 and Theorem 3.7 to consider the radially symmetric solutions for the n-dimensional wave equation:

\[
\begin{aligned}
\Box u &\equiv u_{tt} - \Delta_x u = h(x, t, u), & t \in \mathbb{R}, & x \in B_R, \\
u(x, t) & = 0, & t \in \mathbb{R}, & x \in \partial B_R, \\
u(x, t + T) & = \nu(x, t), & t \in \mathbb{R}, & x \in B_R.
\end{aligned}
\]

\((\text{n-W.E.})\)

where \( B_R = \{x \in \mathbb{R}^n, |x| < R\} \), \( \partial B_R = \{x \in \mathbb{R}^n, |x| = R\} \), \( n > 1 \) and the nonlinear term \( h \) is \( T \)-periodic in variable \( t \). Restriction of the radially symmetry allows us to know the nature of spectrum of the wave operator. Let \( r = |x| \) and \( S^1 := \mathbb{R}/T \), if \( h(x, t, u) = h(r, t, u) \) then the n-dimensional wave equation \((\text{n-W.E.})\) can be transformed into:

\[
\begin{aligned}
A_0u &:= u_{tt} - u_{rr} - \frac{n-1}{r}u_r = h(r, t, u), \\
u(R, t) & = 0, \\
u(r, 0) & = u(r, T), \quad (r, t) \in \Omega := [0, R] \times S^1. 
\end{aligned}
\]

\((\text{RS-W.E.})\)

\( A_0 \) is symmetric on \( L^2(\Omega, \rho) \), where \( \rho = r^{n-1} \) and

\[
L^2(\Omega, \rho) := \left\{ \left. u \right| \left\| u \right\|_{L^2(\Omega, \rho)}^2 := \int_\Omega |u(t, r)|^2 r^{n-1} dt \, dr < \infty \right\}. 
\]

By the asymptotic properties of the Bessel functions (see[46]), the spectrum of the wave operator can be characterized (see[42, Theorem 2.1]). Under some more assumption, the self-adjoint extension of \( A_0 \) has no essential spectrum, and we can get more solutions of \((\text{RS-W.E.})\).
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