COLLAPSE OF THE LYAPUNOV SPECTRUM FOR PERRON-FROBENIUS OPERATOR COCYCLES

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Abstract. In this paper, we study random Blaschke products, acting on the unit circle and consider the cocycle of Perron-Frobenius operators acting on Banach spaces of analytic functions on an annulus. We completely describe the Lyapunov spectrum of these cocycles.

We then consider a perturbation in which the Perron-Frobenius operators are composed with a convolution with a normal distribution with small variance (this corresponds to replacing a map by an annealed version where the image point is moved by a random normal amount). Combining the above, we obtain a very natural cocycle, where the unperturbed cocycle has infinitely many distinct Lyapunov exponents, but arbitrarily small natural perturbations cause a complete collapse of the Lyapunov spectrum except for the exponent 0 associated with the absolutely continuous invariant measure.

The phenomenon is superficially similar to the finite-dimensional phenomenon, discovered by Bochi [3], where a generic area-preserving diffeomorphism is either hyperbolic, or has all zero Lyapunov exponents. In this paper, however, the cocycle and its perturbation are explicitly described; and further, the mechanism for collapse is quite different.

1. Introduction

A well known technique for computing rates of decay of correlation for hyperbolic dynamical systems is based on computing the spectral gap for the Perron-Frobenius operator, acting on a suitable Banach space of functions (different function spaces give rise to different spectral gaps, and so to different rates of decay). In particular, peripheral eigenvalues (those outside the essential spectral radius) play a key role in determining rates of decay of correlation.

Dellnitz, Froyland and Sertl [5] made a conjecture that sub-level sets of eigenfunctions corresponding to these eigenvalues allow one to locate almost-invariant regions of the phase space. This technique has been applied in practice to very high-dimensional dynamical systems, leading to techniques for locating poorly mixing regions of the ocean from...
eigenvectors of matrices built from satellite measurements of transport data \((17)\).

The scope of the conjecture has subsequently been expanded to Perron-Frobenius cocycles. We let \(\sigma : \Omega \to \Omega\) be a measure-preserving base dynamical system and we then study a random dynamical system driven by \(\sigma\). Let \((T_\omega)_{\omega \in \Omega}\) be a family of maps on a phase space, \(X\). The random dynamical system is then the skew product \((\omega, x) \mapsto (\sigma(\omega), T_\omega(x))\). If \(T_\omega\) has Perron-Frobenius operator \(L_\omega\), acting on a Banach space, \(B\) then one studies the cocycle generated by \(L\), that is
\[
L_\omega^{(n)} = L_{\sigma^{n-1}\omega} \circ \ldots \circ L_\omega.
\]

The random dynamical system version of the Dellnitz-Froyland Ansatz states that sub-level sets of Oseledets spaces corresponding to peripheral Lyapunov exponents give rise to almost-equivariant subsets of the phase space. This circle of questions has led to the development of semi-invertible multiplicative ergodic theory \([9, 10, 11]\) (where the base is assumed to be invertible, but the operators are not assumed to be invertible: Perron-Frobenius operators often have a large kernel). The conclusions in this setting are much stronger than those obtained in the non-invertible setting in Oseledets’ original paper, allowing one to obtain an equivariant direct sum decomposition of the Banach space, rather than a filtration. This is essential for the interpretation of the Dellnitz-Froyland Ansatz.

In applications, of course, one measures finite-dimensional approximations of the Perron-Frobenius operators. It is therefore natural to ask how the Lyapunov exponents and Oseledets subspaces vary if one makes small perturbations to the operators. In our previous work with Froyland, \([7]\), we looked at stability of the top Oseledets subspace in various contexts (although here, the top Lyapunov exponent is always 0). Prior results on stability of the Lyapunov exponents are mixed. Based on an outline proposed by Mañé, Bochi, in his thesis \([3]\), showed that a generic area preserving diffeomorphism is either hyperbolic, or may be perturbed so that its Lyapunov exponents become zero. Similar results are established for two-dimensional matrix cocycles. Bochi and Viana \([4]\) extended this to higher-dimensional systems and cocycles, so that these results show that Lyapunov exponents are highly unstable. On the other hand, Ledrappier and Young \([13]\) showed that if one makes absolutely continuous perturbations to invertible matrix cocycles, small perturbations lead to small changes in the Lyapunov exponents. Ochs \([16]\) then showed in the finite-dimensional invertible case, small changes in Lyapunov exponents lead to small changes (in probability) in the Oseledets spaces. With Froyland \([8]\), we established
corresponding results to these in the semi-invertible matrix cocycle setting. In [6], we also gave the first result on stability of Lyapunov exponents in an infinite-dimensional setting: we consider Hilbert-Schmidt cocycles on a separable Hilbert space with exponential decay of the entries. In this case, there is no single natural notion of noise, but we consider perturbations by random matrices with faster exponential decay. Again, we recover in [6] the stability of the Lyapunov exponents and Oseledets spaces. We interpret these results collectively as saying that carefully chosen perturbations may lead to radical change to the Lyapunov spectrum, while noise-like perturbations of the cocycle tend to lead to small changes to the Lyapunov spectrum.

In the current paper, for the first time, we study perturbations of a true Perron-Frobenius operator cocycle (albeit in a relatively benign setting where the Perron-Frobenius operators are compact). We show, to our surprise, that there are natural examples of random dynamical systems, where natural perturbations of the Perron-Frobenius cocycle lead to a collapse of the Lyapunov spectrum.

The examples that we focus on are expanding finite Blaschke products, analytic maps from the unit circle to itself. The Perron-Frobenius operators for a single map of this type, acting on the Hardy space of a suitable annulus, were studied by Bandtlow, Just and Slipantschuk [2], where they used results on composition operators to obtain a precise description of the set of eigenvalues. Indeed, the eigenvalues they obtain are precisely the non-negative powers of the derivative of the underlying Blaschke product at its unique (attracting) fixed point in the unit disc and their conjugates. We study a random version, where instead of a single Blaschke product, $B$, one applies a Blaschke product $B_{\omega}$ that is selected by the base dynamics. We were able to generalize the results of [2] to this setting, so that we can give a precise description of the Lyapunov spectrum of the Perron-Frobenius cocycle as the non-negative multiples of the log of the absolute value of the Lyapunov exponent of the underlying Blaschke product cocycle at the random attracting fixed point in the unit disc (with multiplicity two for all positive multiples of this exponent).

In the final section, we focus on a particular Blaschke product cocycle (with maps $T_0$ and $T_1$ applied in an i.i.d. way, where $T_0(z) = z^2$ and $T_1(z) = [(z + \frac{1}{4})/(1 + z/4)]^2$). The Perron-Frobenius operator of $T_0$ is known to be highly degenerate [1, Exercise 2.14]. If the frequency of applying $T_0$ is $p$, then we find a phase transition: for $p \geq \frac{1}{2}$, the Lyapunov spectrum collapses, while for $p < \frac{1}{2}$, there is a complete
Lyapunov spectrum. We then consider normal perturbations (corresponding to adding random normal noise to the dynamical system), $L^0_\epsilon$ and $L^1_\epsilon$ and show that for $p \geq \frac{1}{4}$, there is collapse of the Lyapunov spectrum for all $\epsilon > 0$ (so that there is an exponent $0$ with multiplicity $1$ and all other Lyapunov exponents are $-\infty$).

In particular, for $\frac{1}{4} \leq p < \frac{1}{2}$, the unperturbed system has a complete Lyapunov spectrum, while arbitrarily small normal perturbations have a collapsed Lyapunov spectrum.

2. Statement of Theorems

In this section, we state our main theorems. We will assume standard definitions, but for completeness, some terms used here will be defined in the next section.

Theorem 1. Let $\sigma$ be an invertible ergodic measure-preserving transformation of a probability space $(\Omega, \mathbb{P})$. Let $(B_\omega)_{\omega \in \Omega}$ be a family of finite Blaschke products, depending measurably on $\omega$, such that there exist $1 < r < R$ such that $|B_\omega(z)| \geq R$ if $|z| = r$ (and hence $|B_\omega(z)| \leq 1/R$ if $|z| = 1/r$). Let $L_\omega$ denote the Perron-Frobenius operator of $B_\omega$, acting on the Banach space $H^2(A_r)$ on the annulus $A_r := \{z: 1/r < |z| < r\}$.

Then

(1) (Random Fixed Point). There exists a measurable map $x: \Omega \to D_{1/R}$ (with $x(\omega)$ written as $x_\omega$), such that $B_\omega(x_\omega) = x_\sigma(\omega)$. For all $z \in D_{1/R}$, $B^{(N)}_{\sigma^{-N}\omega}(z) := B_{\sigma^{-1}\omega} \circ \cdots \circ B_{\sigma^{-N}\omega}(z) \to x_\omega$;

(2) (Critical Random Fixed Point) If $\mathbb{P}(\{\omega: B'_\omega(x_\omega) = 0\}) > 0$, then the Lyapunov spectrum of the cocycle is $0$ with multiplicity $1$; and $-\infty$ with infinite multiplicity.

(3) (Generic case) If $\mathbb{P}(\{\omega: B'_\omega(x_\omega) = 0\}) = 0$, then define $\Lambda = \int \log |B'_\omega(x_\omega)| \, d\mathbb{P}(\omega)$. This satisfies $\Lambda \leq \log(r/R) < 0$.

If $\Lambda > -\infty$, then the Lyapunov spectrum of the cocycle is $0$ with multiplicity $1$; and $-\infty$ with infinite multiplicity.

If $\Lambda > -\infty$, then the Lyapunov spectrum of the cocycle is $0$ with multiplicity $1$; and $n\Lambda$ with multiplicity $2$ for each $n \in \mathbb{N}$. The Oseledets vector with exponent $0$ is $1/(z - x_\omega) - 1/(z - 1/x_\omega)$. Of the two Oseledets vectors with exponent $j\Lambda$, one has a pole of order $j$ at $x_\omega$, and is a linear combination of $1/(z - x_\omega)^2, \ldots, 1/(z - x_\omega)^{j+1}$; the other has a pole of order $j$ at $1/x_\omega$ and is a linear combination of $z^{k-1}/(1 - x_\omega z)^{k+1}$ for $k = 1, \ldots, j$. 
The condition that a finite Blaschke product maps the disc of radius $1/r$ inside the disc of radius $1/R$ (for $1 < r < R$) implies by a result of Tischler [20] that the Blaschke product is expanding on the unit circle.

**Theorem 2.** Let $\Omega = \{0,1\}^\mathbb{Z}$, $\sigma$ be the shift map, and $\mathbb{P}_p$ be the Bernoulli measure where $\mathbb{P}([0]) = p$. Let $\mathcal{L}_0$ be the Perron-Frobenius operator of $T_0$: $z \mapsto z^2$ and $\mathcal{L}_1$ be the Perron-Frobenius operator of $T_1$: $z \mapsto [(z + \frac{1}{4})/(1 + \frac{z}{4})]^2$ and consider the operator cocycle generated by $\mathcal{L}_\omega := \mathcal{L}_{\omega_0}$, acting on $H^2(A_r)$. Then

(a) The cocycle is compact;
(b) If $p < \frac{1}{2}$, then the cocycle has countably infinitely many finite Lyapunov exponents;
(c) If $p \geq \frac{1}{2}$, then $\lambda_0 = 0$ and all remaining Lyapunov exponents are $-\infty$.

We define an operator on $H^2(A_r)$ by

$$(\mathcal{N}_\epsilon f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ze^{-2\pi i t})e^{-2\pi i t-\epsilon t^2/2} dt.$$ 

We show below that this operator corresponds to the operator on functions on $\mathbb{R}/\mathbb{Z}$ given by $\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}} f(x) = \mathbb{E} f(x + \epsilon N)$, where $N$ is a standard normal random variable. That is, $\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}}$ is convolution with a Gaussian with variance $\epsilon^2$. Let $\mathcal{L}_\omega$ be the cocycle in the theorem above and consider the perturbation $\mathcal{L}_\omega^\epsilon$ of $\mathcal{L}_\omega$ given by $\mathcal{L}_\omega^\epsilon = \mathcal{N}_\epsilon \circ \mathcal{L}_\omega$.

**Theorem 3.** Let $\Omega = \{0,1\}^\mathbb{Z}$, equipped with the map $\sigma$ and measure $\mathbb{P}_p$ as above. If $p \geq \frac{1}{4}$, the perturbed cocycle $(\mathcal{L}_\omega^\epsilon)_{\omega \in \Omega}$ has $\lambda_1 = 0$ and $\lambda_j = -\infty$ for all $j > 0$.

**Corollary 4.** In particular, if $\frac{1}{4} \leq p < \frac{1}{2}$, then the unperturbed cocycle has a complete Lyapunov spectrum, but for each $\epsilon > 0$, the Lyapunov spectrum collapses.

### 3. Background

Recall that a (finite) Blaschke product is a map from $\hat{\mathbb{C}}$ to itself of the form:

$$B(z) = \zeta \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z},$$

where the $a_j$'s lie in $D$, the unit disc and $|\zeta| = 1$. It is easy to verify that Blaschke products map the unit circle to itself. They also map the interior of the unit circle, the unit disc, to itself; as well as the exterior to itself. Indeed, Blaschke products satisfy $B(1/\bar{z}) = 1/B(z)$, so that Blaschke products commute with the inversion map $I(z) = 1/\bar{z}$. We
are interested in finite Blaschke products whose restriction to the unit circle is expanding. A simple sufficient condition for this, namely that \( \sum_{j=1}^{n} \frac{1-|a_j|}{1+|a_j|} > 1 \), may be found in the work of Martin [15].

Let \( \pi(x) = e^{2\pi i x} \) be the natural bijection between \( \mathbb{R}/\mathbb{Z} \) and \( C_1 \), the unit circle in the complex plane. Let \( S \) be an orientation-preserving expanding real analytic map from \( \mathbb{R}/\mathbb{Z} \) to \( \mathbb{R}/\mathbb{Z} \) and let \( T = \pi S \pi^{-1} \) be its conjugate mapping \( C_1 \) to \( C_1 \). Then there exists \( 1 < r < R \) such that \( T \) maps the annulus \( A_r = \{ z : 1/r < |z| < r \} \) over \( A_R \) in a \( k \)-to-1 way (where \( k \) is the absolute value of the degree of \( S \)). (Put differently, \( T \) maps \( A_r \) into \( A_R \), \( T(D_{1/r}) \subset D_{1/R} \) and \( T(D_r) \subset D_R \)). This \( r \) may be chosen by ensuring that \( |T(z)| \geq R \) whenever \( |z| = r \) and \( |T(z)| \leq 1/R \) whenever \( |z| \leq 1/r \). We will work with a family of expanding analytic maps of \( C_1 \), all mapping the same annulus \( A_r \) into \( A_R \). We consider the Hardy-Hilbert space \( H^2(A_r) \) of analytic functions on \( A_r \) with an \( L^2 \) extension to \( \partial A_r \). An orthonormal basis for the Hilbert space is \( \{ d_n z^n : n \in \mathbb{Z} \} \), where \( d_n = r^{-|n|}(1 + r^{-4|n|})^{-1/2} = r^{-|n|}(1 + o(1)) \).

Details, including a description of the (separable) dual can found in [2].

Let \( (\sigma_i)_{i=1,2} \) be a family of inverse branches of \( S \) and \( (\tau_i)_{i=1,2} \) be a family of inverse branches of \( T|_{C_1} \).

If \( f \in C(C_1) \) and \( g \in C(\mathbb{R}/\mathbb{Z}) \), define

\[
\mathcal{L}_T f(z) = \sum_{i=1}^{n} f(\tau_i(z)) \tau_i'(z); \quad \text{and} \quad \mathcal{L}_S g(x) = \sum_{i=1}^{n} g(\sigma_i(x)) \sigma_i'(x)
\]

In fact, rather than acting on \( C(C_1) \), we think of \( \mathcal{L}_T \) as acting on \( H^2(A_r) \). This corresponds to an action of \( \mathcal{L}_S \) on the strip \( \mathbb{R}/\mathbb{Z} \times (-\log r/(2\pi), \log r/(2\pi)) \). Note that there is a one–one correspondence between elements of \( H^2(A_r) \) and their restrictions to \( C_1 \) (which are necessarily continuous).

We record the following lemma which is a version of [18, Remark 4.1]

**Lemma 5** (Correspondence between Perron-Frobenius Operators). The operators \( \mathcal{L}_S \) and \( \mathcal{L}_T \) defined above are conjugate by the map \( Q : C(C_1) \to C(\mathbb{R}/\mathbb{Z}) \) given by

\[
(Qf)(x) = f(e^{2\pi i x}) e^{2\pi i x}.
\]

In particular, the spectral properties of \( \mathcal{L}_S \) and \( \mathcal{L}_T \) are the same, so that even though \( \mathcal{L}_S \) is the more standard object in dynamical systems, we will study \( \mathcal{L}_T \) since this will allow us to directly apply tools of complex analysis. Further, if \( S_{i_1}, \ldots, S_{i_n} \) are expanding
maps of $\mathbb{R}/\mathbb{Z}$ and $T_{i_{1}}, \ldots, T_{i_{n}}$ are their conjugates, acting on $C_{1}$, then $\mathcal{L}_{S_{i_{n}}} \circ \cdots \circ \mathcal{L}_{S_{i_{1}}} = Q^{-1} \mathcal{L}_{T_{i_{n}}} \circ \cdots \circ \mathcal{L}_{T_{i_{1}}} Q$, so the Lyapunov exponents of a cocycle of $\mathcal{L}_{S}$ operators are the same as the Lyapunov exponents of the corresponding cocycle of $\mathcal{L}_{T}$ operators (provided that $Q$ is an isomorphism of the two Banach spaces on which the operators are acting).

We record the following well known lemma.

**Lemma 6 (Duality Relations).** Let $T$ be an expanding analytic map from $C_{1}$ to $C_{1}$ and let $\mathcal{L}_{T}$ be as above. If $f \in C(C_{1})$, $g \in L^{\infty}(C_{1})$, then

$$
\frac{1}{2\pi i} \int f(z)g(Tz) \, dz = \frac{1}{2\pi i} \int \mathcal{L}_{T} f(z)g(z) \, dz.
$$

That is, if a linear functional $\theta$ is defined by integrating against $g$, then $\mathcal{L}_{T}^{\ast} \theta$ is given by integrating against $g \circ T$.

Let $\sigma$ be an ergodic measure-preserving transformation of $(\Omega, \mathbb{P})$. Let $X$ be a Banach space and suppose that $(\mathcal{L}_{\omega})_{\omega \in \Omega}$ is a family of linear operators on $X$ that is strongly measurable, that is for any fixed $x \in X$, $\omega \mapsto \mathcal{L}_{\omega}(x)$ is $(\mathcal{F}_{\Omega}, \mathcal{F}_{X})$-measurable, where $\mathcal{F}_{\Omega}$ is the $\sigma$-algebra on $\Omega$ and $\mathcal{F}_{X}$ is the Borel $\sigma$-algebra on $X$. In this case we say that the tuple $(\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ is a linear dynamical system and we define $\mathcal{L}_{\omega}^{(n)} = \mathcal{L}_{\sigma^{-n} \omega} \circ \cdots \circ \mathcal{L}_{\omega}$. 

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**Figure 1.** Schematic diagram showing the annulus $A_r$ (shaded); the inner boundary, $C_{1/r}$ (blue); its image under $T$ (dashed, blue); the outer boundary $C_{r}$ (red); and its image (dashed, red).
A cocycle analogue to the (logarithm of the) spectral radius of a single operator is the quantity
\[ \lambda_1(\omega) = \lim_{n \to \infty} \frac{1}{n} \log \| L^{(n)}_\omega \|. \]

If one assumes that \( \int \log \| L_\omega \| \, d\mathbb{P} < \infty \), then the Kingman sub-additive ergodic theorem guarantees the \( \mathbb{P} \)-a.e. convergence of this limit to a value in \([\frac{1}{2}, \infty)\). Ergodicity also ensures that \( \lambda_1(\cdot) \) is almost everywhere constant, so that we just write \( \lambda_1 \).

A second quantity of interest is the analogue of the (logarithmic) essential spectral radius:
\[ \kappa(\omega) = \lim_{n \to \infty} \frac{1}{n} \log \alpha(L^{(n)}_\omega), \]

where \( \alpha(L) \) is the index of compactness of an operator \( L \), the infimum of those real numbers \( r \) such that the image of the unit ball in \( X \) under \( L \) may be covered by finitely many balls of radius \( r \), so that \( L \) is a compact operator if and only if \( \alpha(L) = 0 \). The quantity \( \alpha(L) \) is also sub-multiplicative, so that Kingman’s theorem again implies \( \kappa(\omega) \) exists for \( \mathbb{P} \)-a.e. \( \omega \) and is independent of \( \omega \), so that we just write \( \kappa \).

The cocycle will be called quasi-compact if \( \kappa < \lambda_1 \). The first Multiplicative Ergodic Theorem in the context of quasi-compact cocycles of operators on Banach spaces was proved by Thieullen [19]. We require a semi-invertible version (that is: although the base dynamical system is required to be invertible, the operators are not required to be injective) of a result of Lian and Lu [14].

**Theorem 7** ([12]). Let \( \sigma \) be an invertible ergodic measure-preserving transformation of a probability space \( (\Omega, \mathbb{P}) \) and let \( \omega \mapsto L_\omega \) be a quasi-compact strongly measurable cocycle of operators acting on a Banach space \( X \) with a separable dual.

Then there exist \( 1 \leq \ell \leq \infty \), exponents \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq \kappa \geq -\infty \), finite multiplicities \( m_1, m_2, \ldots, m_\ell \) and subspaces \( V_1(\omega), \ldots, V_\ell(\omega), R(\omega) \) such that

1. \( \dim(V_i(\omega)) = m_i \);
2. \( L_\omega V_i(\omega) = V_i(\sigma(\omega)) \) and \( L_\omega R(\omega) \subset R(\sigma(\omega)) \);
3. \( V_1(\omega) \oplus \ldots \oplus V_\ell(\omega) \oplus R(\omega) = X \);
4. for \( x \in V_i(\omega) \setminus \{0\} \), \( \lim \frac{1}{n} \log \| L^{(n)}_\omega x \| \to \lambda_i \);
5. for \( x \in R(\omega) \), \( \limsup \frac{1}{n} \log \| L^{(n)}_\omega x \| \leq \kappa \).
For a bounded linear operator $A$ from $X$ to itself, we defined in [12] the following crude notion of volume growth:

$$D_k(A) = \sup_{x_1, \ldots, x_k} \prod_{j=1}^k d(Ax_j, \text{lin}\{Ax_i : i < j\}),$$

where the supremum is taken over $x$’s of norm 1; $\text{lin}\{y_1, \ldots, y_n\}$ denotes the linear span of the vectors $y_1, \ldots, y_n$; the linear span of the empty set is taken to be 0; and $d(x, S) := \inf_{y \in S} \|x - y\|$.

Lemma 8. Let $\sigma$, $(\Omega, \mathbb{P})$ and $\omega \mapsto L_\omega$ be as in the statement of Theorem 7. Let $\mu_1 \geq \mu_2 \geq \ldots$ be the sequence of $\lambda$’s in decreasing order with repetition, so that $\lambda_i$ occurs $m_i$ times in the sequence.

(a) $D_k$ is sub-multiplicative: $D_k(AB) \leq D_k(A)D_k(B)$ if $A$ and $B$ are bounded linear operators on $X$;

(b) There exists a constant $c_k$ such that if $Y$ is a closed subspace of $X$ of co-dimension 1 and $A$ is a linear operator on $X$, then $D_k(A) \leq c_k \|A\| \|A|_Y\|^{k-1}$.

(c) $\frac{1}{n} \log D_k(L_\omega^{(n)}) \to \mu_1 + \ldots + \mu_k$ for $\mathbb{P}$-almost every $\omega$.

The proof of this lemma is in lemmas 1,8 and 12 of [12].

4. Lyapunov Spectrum for Expanding Blaschke Products

Lemma 9. Let $r < R$ and $T$ be a Blaschke product such that $|T(z)| \leq 1/R$ whenever $|z| = 1/r$. Let $d_r$ be the hyperbolic metric on $D_{1/r}$: $d_r(z,w) = d_H(z/r, w/r)$, where $d_H$ is the standard hyperbolic metric on the unit disc. Then

$$d_r(T(z), T(w)) \leq \frac{r}{R} d_r(z, w) \text{ for all } z, w \in D_{1/r}.$$

Proof. We may write $T$ as $Q \circ S$ where $Q(z) = rz/R$ and $S(z) = RT(z)/r$, so that $S$ maps $D_{1/r}$ to itself. By the Schwartz-Pick theorem, $d_r(S(z), S(w)) \leq d_r(z, w)$ for all $z, w \in D_{1/r}$, so it suffices to show that $d_r(Q(z), Q(w)) \leq \frac{r}{R} d_r(z, w)$ for all $z, w \in D_{1/r}$. The metric $d_r$ is given, up to a constant multiple by

$$d_r(z, w) = \inf_\gamma \int_{\gamma} \frac{|dz|}{1 - r^2|z|^2}$$

where the infimum is taken over paths $\gamma$ from $z$ to $w$. Given $z$ and $w$, let $\gamma$ be the geodesic joining them. Now $(r/R)\gamma(t)$ is a path (generally not a geodesic) joining $Q(z)$ and $Q(w)$. The length element is scaled by a factor of $r/R$ and the integrand is decreased, so that $d_r(Q(z), Q(w)) \leq \frac{r}{R} d_r(z, w)$ as claimed. □
Corollary 10. Let \((B_\omega)\) be a measurable cocycle of expanding finite Blaschke products, with \(|B_\omega(z)| \leq 1/R\) for all \(z\) such that \(|z| = 1/r\). Then there exists a measurable random fixed point \(x_\omega\) (that is a point such that \(B_\omega(x_\omega) = x_\sigma(\omega)\)) in \(\overline{D}_{1/R}\) such that for all \(\epsilon > 0\), there exists \(n\) such that for all \(z \in \overline{D}_{1/R}\) and all \(\omega \in \Omega\), \(|B^{(n)}_{\sigma^{-\omega}}(z) - x_\omega| < \epsilon\).

Proof. The set \(\overline{D}_{1/R}\) has has bounded diameter, \(L\) say, in the \(d_r\) metric and by assumption, the sets \(B^{(n)}_{\sigma^{-\omega}}(\overline{D}_{1/R})\) are nested. By Lemma 9, \(B^{(n)}_{\sigma^{-\omega}}(\overline{D}_{1/R})\) has \(d_r\)-diameter at most \(L(r/R)^n\). By completeness, \(\bigcap B^{(n)}_{\sigma^{-\omega}}(\overline{D}_{1/R})\) is a singleton, \(\{x_\omega\}\). Since \(x_\omega = \lim_{n \to \infty} B^{(n)}_{\sigma^{-\omega}}(0)\), and so is the limit of measurable functions, we see that \(x_\omega\) depends measurably on \(\omega\). This equality also implies that \(x_\sigma(\omega) = B_\omega(x_\omega)\). Since on \(D_{1/R}\), \(d_r\) is within a bounded factor of Euclidean distance, we obtain the required uniform convergence in the Euclidean distance. \(\square\)

We introduce a non-standard definition of order of singularity for meromorphic functions on the Riemann sphere: if \(x \in \mathbb{C}\), \(\text{ord}_x(f)\) is \(n\) if \(f(z) \sim a/(z-x)\) as \(z \to x\) for some \(n \geq 1\); or 0 otherwise. If \(f(z) \sim bz^{n-2}\) for some \(n \geq 1\) as \(z \to \infty\) then \(\text{ord}_\infty(f) = n\) or \(\text{ord}_\infty(f) = 0\) otherwise. In particular, with this definition \(\text{ord}_\infty(1) = 2\). If \(f\) is a non-zero meromorphic function on the unit sphere, then \(\sum_{z \in \mathbb{C}} \text{ord}_z(f) \geq 2\) if \(f\) is a non-zero constant function, it has a singularity of order 2 at \(\infty\); otherwise \(f\) must have at least one pole by Liouville’s theorem. If \(f\) has exactly one pole, of order 1 at \(x\) say, then \(f - a/(z-x)\) (where \(a\) is the residue) is bounded and hence constant, but \(f(z) = c + a/(z-x)\) has order 1 at \(\infty\) if \(c = 0\) or order 2 at \(\infty\) otherwise.

Lemma 11. Suppose \(N\) is an open subset of \(\mathbb{C}\) such that \(T: N \to \mathbb{C}\) is analytic and injective. If \(f\) is meromorphic on \(N\), then \(\mathcal{L}_T f\) is meromorphic on \(T(N)\); \(\text{ord}_{T(x)}(\mathcal{L}_T f) = \text{ord}_x f\) for all \(x \in N\).

If \(Q\) is a Möbius transformation, then \(\text{ord}_{Q(x)}(\mathcal{L}_Q f) = \text{ord}_x f\) for all \(x \in \mathbb{C}\). In particular, \(\mathcal{L}_Q\) maps the collection of meromorphic functions on \(\mathbb{C}\) to themselves.

Proof. Let \(T\) and \(N\) be as in the statement of the lemma. Since \(T\) is injective, its derivative is non-zero on \(N\). As \(\mathcal{L}_T f(z) = f(T^{-1}z)/T'(T^{-1}z)\), the first statement is clear. This establishes the claim for Möbius transformations, \(Q\), at all points \(x\) such that \(x\) and \(Q(x)\) are not \(\infty\).

If \(x \in \mathbb{C}\) and \(Q(x) = \infty\), then \(Q(z) = a/(z-x) + b\) and \(Q'(z) = -a/(z-x)^2\) as \(z \to x\); and \(Q^{-1}(z) = x + a/z + O(z^{-2})\) as \(z \to \infty\). So if \(f(z) = c/(z - x)^n + O(z - x)^{-(n-1)}\) in a neighbourhood of \(x\), then the leading term of \(\mathcal{L}_Q f(z)\) as \(z \to \infty\) is \((c/(a/z)^n)/(-a/(a/z)^2) = -(c/a^{n-1})z^{n-2}\), so that \(\text{ord}_\infty \mathcal{L}_Q f = \text{ord}_x f\) as required.
If \( Q(\infty) = y \), then \( Q(z) = y + a/z + O(z^{-2}) \) and \( Q'(z) = -a/z^2 + O(z^{-3}) \) as \( z \to \infty \). Also, \( Q^{-1}(z) = a/(z - y) + O(1) \) as \( z \to y \). So if \( f(z) = b z^{-n} + O(z^{-n-1}) \) as \( z \to \infty \), then the leading term of \( \mathcal{L}_Q f(z) \) as \( z \to y \) is \( b(a/(z - y)) z^{-n}/(-a/[a/(z - y)]^2) = -ba^{n-1}/(z - y)^n \), so \( \text{ord}_y \mathcal{L}_Q f = \text{ord}_\infty f \).

The case \( Q(\infty) = \infty \) is where \( Q \) is a linear map. This trivially preserves the order at \( \infty \). \( \square \)

**Theorem 12.** Let \( T \) be a rational function. Then \( \mathcal{L}_T \) maps the collection of meromorphic functions on \( \hat{\mathbb{C}} \) into themselves. \( \mathcal{L}_T \) does not increase orders of singularities (measured as above) and may decrease them: \( \text{ord}_x \mathcal{L}_T f \leq \max_{y \in T^{-1}(x)} \text{ord}_y f \) for each \( x \in \mathbb{C} \).

Further, if \( T \) has a critical point at \( x \), \( f \) has a singularity of order greater than 1 at \( x \) and no singularity at any other point of \( T^{-1}(Tx) \), then \( \text{ord}_{T(x)} \mathcal{L}_T f < \text{ord}_x f \).

The inequality is an equality except at points \( x \) such that \( T^{-1}(Tx) \) contains a critical point at which \( f \) has a singularity, or contains multiple singularities of \( f \).

**Proof.** Let \( f \) be a meromorphic function on the Riemann sphere. In order to show that \( \mathcal{L}_T f \) is meromorphic, it suffices to work locally, showing that for each \( z \), \( \mathcal{L}_T f \) is meromorphic on a neighbourhood of \( z \).

Fix \( w \in \mathbb{C} \). By the lemma above, and the fact that \( \mathcal{L}_{ST} = \mathcal{L}_S \circ \mathcal{L}_T \), we see that \( \mathcal{L}_T(f) \) is meromorphic on a neighbourhood of \( w \) if and only if \( \mathcal{L}_{Q_1 \circ T \circ Q_2} \) is meromorphic on a neighbourhood \( Q_2^{-1}(w) \) for some Möbius transformations \( Q_1 \) and \( Q_2 \). This allows us to assume without loss of generality that \( w \neq \infty \) and \( T(\infty) \neq w \).

Let the preimages of \( w \) be \( y_1, \ldots, y_n \) and suppose they have multiplicities \( m_1, \ldots, m_n \). By the local mapping theorem, there exist disjoint neighbourhoods \( N_1, \ldots, N_n \) of \( y_1, \ldots, y_n \) and injective holomorphic functions \( g_i : N_i \to \hat{\mathbb{C}} \) such that \( g_i(y_i) = 0 \) and for \( z \in N_i \), \( T(z) = k(h_i(g_i(z))) \), where \( h_i(z) = z^{m_i} \) and \( k(z) = z + w \). Let \( \delta \) be such that \( g_i^{-1}(B_{\delta/(m_i)}(0)) \subset N_i \) for each \( i \).

Now for each \( z \in B_\delta(w) \), we have

\[
\mathcal{L}_T f(z) = \sum_{i=1}^{n} \mathcal{L}_h \mathcal{L}_{g_i} f(z).
\]

If \( h(z) = z^m \) for \( m > 1 \), we check by explicit calculation that if \( f(z) \) has Laurent expansion \( \sum_{j \in \mathbb{Z}} a_j z^j \), then \( \mathcal{L}_h f = \sum_{j \in \mathbb{Z}} a_{jm-1} z^{j-1} \). In particular if \( \text{ord}_0(f) > 1 \) and \( m > 1 \), then \( \text{ord}_0(\mathcal{L}_h f) < \text{ord}_0 f \), while if \( m = 1 \), \( \text{ord}_0(\mathcal{L}_h f) = \text{ord}_0(f) \). Since \( g_i \) is injective and analytic on \( N_i \); \( \text{ord}_0(\mathcal{L}_{g_i} f) = \text{ord}_{g_i}(f) \); \( \mathcal{L}_h \) clearly preserves orders; and summing
singularities can only reduce orders, we see that $L_T f$ is meromorphic in a neighbourhood of each $w \in \hat{C}$. Hence $L_T f$ is a meromorphic function on $\hat{C}$ as required. □

**Corollary 13.** Let $T$ be a rational polynomial and let $x \in \mathbb{C}$ satisfy $T(x) \in \mathbb{C}$. If $f(z) = 1/(z - x)^{n+1}$ for some $n \geq 1$, then $L_T f$ is a linear combination of $\{1/(z - T(x))^{j+1} : 1 \leq j \leq n\}$.

**Proof.** Since $f$ has only one singular point, at $x$, of order $n$, then $L_T f$ is a meromorphic function of the sphere with only one singular point, at $T(x)$, of order at most $n$. In particular, $L_T f = O(z^{-2})$ in a neighbourhood of $\infty$, which ensures that there is no $1/(z - T(x))$ term in $L_T f$. □

It is well known that finite Blaschke products commute with inversion. We shall exploit this by introducing an operator that commutes with the Perron-Frobenius operators of Blaschke products, performing this inversion at the level of meromorphic functions. The advantage of this will be that if we check that order $n$ poles at points $x \in D_1/r$ converge under application of the Perron-Frobenius cocycle to order $n$ poles at the random fixed point $x_\omega$, then a parallel evolution will take place exterior to the unit circle, removing the need to make deal separately with the case when $x_\omega = 0$ (where the corresponding exterior pole is $\infty$).

Define $I(z) = 1/\bar{z}$ and

$$L_I f(z) = \frac{\bar{f}(I(z))}{z^2}.$$  

It is a standard fact that $L_I$ maps meromorphic functions to meromorphic functions. We can check that $\text{ord}_{I(z)} L_I(f) = \text{ord}_z f$ for every $z \in \hat{C}$.

**Lemma 14.** Let $B$ be a finite Blaschke product. Then $L_B L_I = L_I L_B$. Also $L_I \circ L_I(f) = f$.

**Proof.** Using the identity $B(z) = 1/B(1/\bar{z})$ (that is $I \circ B = B \circ I$), we have

$$B'(z) = \lim_{h \to 0} \frac{B(z + h) - B(z)}{h} = \lim_{h \to 0} \frac{1}{B(1/(\bar{z} + \bar{h}))} \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\bar{B}(1/\bar{z}) - \bar{B}(1/(\bar{z} + \bar{h}))}{h \bar{B}(1/\bar{z})}$$

$$= \frac{\bar{B}'(1/\bar{z})}{z^2 \bar{B}(1/\bar{z})^2}.$$
We deduce \( B'(I(y)) = (y^2 B'(y)/B(y)^2) \). Now we have

\[
\mathcal{L}_B \mathcal{L}_I f(z) = \sum_{y \in B^{-1}z} \frac{\mathcal{L}_I f(y)}{B'(y)} = \sum_{y \in B^{-1}z} \frac{\bar{f}(I(y))}{y^2 B'(y)};
\]

and

\[
\mathcal{L}_I \mathcal{L}_B f(z) = \frac{\mathcal{L}_B f(I(z))}{z^2} = \frac{1}{z^2} \sum_{y \in B^{-1}(I(z))} \frac{\bar{f}(y)}{B'(y)} = \frac{1}{z^2} \sum_{y \in B^{-1}(z)} \frac{\bar{f}(I(y))}{B'(I(y))} = \frac{1}{z^2} \sum_{y \in B^{-1}(z)} \frac{\bar{f}(I(y))B(y)^2}{y^2 B'(y)^2} = \mathcal{L}_B \mathcal{L}_I f(z).
\]

That \( \mathcal{L}_I \) is an involution is straightforward.

**Lemma 15.** Let \( B \) be a finite Blaschke product, let \( x \in \mathbb{C} \) and let \( f(z) = 1/(z - x) \). Then

\[
\mathcal{L}_B f(z) = \frac{1}{z - B(x)} - \frac{1}{z - B(\infty)},
\]

where \( 1/(z - \infty) \) is interpreted as the constant 0 function.

**Proof.** It follows from Theorem 12 that \( \mathcal{L}_B f \) is a meromorphic function with singularities of order 1 at \( B(x) \) and \( B(\infty) \). We deal first with the case \( B(0) \neq 0 \), \( B(x) \neq \infty \). Computing the leading order term of the Laurent expansion of \( \mathcal{L}_B f(z) \) at \( B(x) \) and \( B(\infty) \), we see that \( \mathcal{L}_B f(x) - [1/(z - B(x)) - 1/(z - B(\infty))] \) has no poles in \( \hat{\mathbb{C}} \), and hence is constant by Liouville’s theorem. Since \( \mathcal{L}_B f \) has no singularity at \( \infty \), the constant is 0.

The other cases are handled similarly.

**Corollary 16.** Let \( (B_\omega) \) be a cocycle of expanding finite Blaschke products. Let \( x_\omega \in D_{1/R} \) be the random fixed point. The space \( E_0(\omega) \), spanned by \( f_{0,\omega} \) where

\[
f_{0,\omega}(z) = \frac{1}{z - x_\omega} - \frac{1}{z - I(x_\omega)}
\]

is a one-dimensional equivariant subspace with Lyapunov exponent 0.  

If \( x \in D_{1/R} \) and \( f(z) = 1/(z - x) \), then \( \|L_\omega^{(n)} f - f_{0,\sigma^n \omega}\| \to 0 \).
Proof. By Lemma 15 and the fact that \( \mathcal{L}_{B_1 \circ B_2} = \mathcal{L}_{B_1} \circ \mathcal{L}_{B_2} \), we see

\[
\mathcal{L}_{\omega}^{(n)} f(z) = \frac{1}{z - B_{\omega}^{(n)}(x)} - \frac{1}{z - B_{\omega}^{(n)}(\infty)}.
\]

The claimed equivariance follows. Since \(|B_{\omega}^{(n)}(x) - x_{\sigma^n \omega}| \to 0\), we see that

\[
\left\| \frac{1}{z - B_{\omega}^{(n)}(x)} - \frac{1}{z - x_{\sigma^n \omega}} \right\| \to 0.
\]

Similarly, since \( B_{\omega}^{(n)}(\infty) = I(B_{\omega}^{(n)}(0)) \), we see from Corollary 10 that \( d_{\hat{C}}(B_{\omega}^{(n)}(\infty), I(x_{\sigma^n \omega})) \to 0 \), where \( d_{\hat{C}} \) is the standard metric on the Riemann sphere. It follows that

\[
\left\| \frac{1}{z - B_{\omega}^{(n)}(\infty)} - \frac{1}{z - I(x_{\sigma^n \omega})} \right\| \to 0,
\]

as required. \( \square \)

**Lemma 17.** Let \( \sigma \) be an invertible ergodic measure-preserving transformation of the probability space \((\Omega, \mathbb{P})\), let \( X \) be a Banach space with separable dual, and let \( (\mathcal{L}_\omega)_{\omega \in \Omega} \) be a quasi-compact cocycle with Lyapunov exponents \( \lambda_1, \ldots, \lambda_l \) (with \( 1 \leq l \leq \infty \)) with multiplicities \( m_1, \ldots, m_l \). Let \( E_j(\omega) \) be the \( j \)th “fast space” (the space of dimension \( M = m_1 + \cdots + m_j \) spanned by the top \( j \) Oseledets subspaces). There exists a measurable function \( \epsilon(\omega) > 0 \) such that if \( V \) is a subspace of \( X \) such that for all \( x \in E_j(\omega) \cap S(X) \), there exists \( v \in V \) such that \( \|v - x\| \leq \epsilon \), then

\[
\sup_{x \in E_j(\sigma^n \omega) \cap S(X)} d(x, \mathcal{L}_\omega^{(n)} V) \to 0 \text{ as } n \to \infty.
\]

Proof. From the proof of the Multiplicative Ergodic Theorem, there exists a measurable family of projections \( \Pi_{E_j(\omega)} : X \to E_j(\omega) \) such that vectors in the range of \( \text{Id} - \Pi_{E_j(\omega)} \) lie in \( F_j(\omega) \), the “slow space” of vectors whose expansion rate is at most \( \lambda_{j+1} \).

Let \( \epsilon = (2M^2 \|\Pi_{E_j(\omega)}\|)^{-1} \). Let \( u_1, \ldots, u_M \) be an Auerbach basis of \( E_j(\omega) \), that is a collection of vectors of norm 1 such that there exist functionals \( \theta_1, \ldots, \theta_M \) in \( X^* \), also of norm 1 such that \( \theta_i(u_m) = \delta_{lm} \) for each \( 1 \leq l, m \leq M \).

Now suppose the subspace \( V \) of \( X \) is such that for all \( x \in E_j(\omega) \cap S(X) \), there exists \( v \in V \) with \( \|v - x\| < \epsilon \). Let \( v_1, \ldots, v_M \) in \( V \) satisfy \( \|v_i - u_i\| < \epsilon \) for \( i = 1, \ldots, M \) and set \( w_i = \Pi_{E_j(\omega)} v_i \). By assumption, \( \|w_i - u_i\| = \|\Pi_{E_j(\omega)} (v_i - u_i)\| < 1/(2M^2) \) for each \( i \). Since \( w_i \in E_j(\omega) \) and \( (\theta_k) \) is a dual basis to \( (u_i) \), we have \( w_i = \sum_{k=1}^M \theta_k (w_i) u_k \). By the
above, \(|\theta_k(w_i) - \delta_{ik}| \leq 1/(2M^2)| for each \(i, k\). It follows that the matrix expressing \(w_1, \ldots, w_M\) in terms of \(u_1, \ldots, u_M\) is invertible, so that \(w_1, \ldots, w_M\) is a basis for \(E_j(\omega)\). Since all norms on a finite-dimensional space are equivalent, there exists \(c > 0\) such that \(\left\|\sum_{i=1}^{M} a_i w_i\right\| \geq c \sum_{i=1}^{M} |a_i|\) for all \((a_1, \ldots, a_M) \in \mathbb{R}^M\).

Let \(\delta = \frac{1}{3}(\lambda_j - \lambda_{j+1})\). By [12, Theorem 16], there exists an \(N\) such that \(\|L_\omega^{(n)} x\| \geq e^{(\lambda_j - \delta)n}\|x\|\) for all \(x \in E_j(\omega)\) and all \(n \geq N\) and similarly \(\|L_\omega^{(n)} z\| \leq e^{(\lambda_{j+1} + \delta)n}\|z\|\) for all \(z \in F_j(\omega)\) and \(n \geq N\). Finally, let \(x \in E_j(\sigma^n\omega) \cap S(X)\) for some \(n > N\). Then \(x = \sum_{i=1}^{M} a_i L_\omega^{(n)} w_i\) (since the \(L_\omega^{(n)} w_i\) form a basis for \(E_j(\sigma^n\omega)\)). Let \(y = \sum_{i=1}^{M} a_i v_i\) and \(z = \sum_{i=1}^{M} a_i w_i\).

We have \(1 = \|x\| = \|L_\omega^{(n)} z\| \geq e^{(\lambda_j - \delta)n}\|z\| \geq c e^{\lambda_j - \delta} \sum_{i=1}^{M} |a_i|\), so that \(\sum |a_i| \leq (1/c) e^{-(\lambda_j - \delta)n}\). Next,

\[
\|L_\omega^{(n)} y - x\| = \|L_\omega^{(n)} (y - z)\| \\
\leq \sum_{i=1}^{M} |a_i| \|L_\omega^{(n)} (v_i - w_i)\| \\
\leq \left( \sum_{i=1}^{M} |a_i| \right) e^{(\lambda_{j+1} + \delta)n}/M^2 \\
\leq e^{-\delta n}/cM^2,
\]

where in the third line, we used the facts that \(v_i - w_i \in F_j(\omega)\) and \(\|v_i - w_i\| \leq \|v_i - u_i\| + \|u_i - w_i\| < 1/M^2\). Hence, we have shown that for all \(n \geq N\) and all \(x \in E_j(\sigma^n\omega) \cap S(X)\), there exists \(v \in L_\omega^{(n)} V\) such that \(\|v - x\| \leq C e^{-\delta n}\), where \(C = 1/(cM^2)\).

**Lemma 18.** Let the measure-preserving transformation and cocycles be as in the statement of Theorem 1. Let \(V\) be the subspace of \(H^2(A_\omega)\) spanned by Laurent polynomials \(z^{-(j+1)}\) for \(j = 1, \ldots, N\). Then \(L_\omega^{(n)} V\) is spanned by \(1/(z - B_\omega^{(n)}(0))^{j+1}\) for \(j = 1, \ldots, N\).

In particular, the sequence \(L_\omega^{(n)} V\) approaches the equivariant sequence of subspaces

\[
P_N^+(\sigma^n\omega) := \operatorname{lin} \left\{ \frac{1}{(z - x_{\sigma^n\omega})^{j+1}} : 1 \leq j \leq N \right\}.
\]

**Proof.** It suffices to show that if \(f(z) = 1/(z - x)^{j+1}\) with \(x \in D\), then for any finite expanding Blaschke product, \(L_\omega f\) is a linear combination
Proof. Notice that an sequence of subspaces, $\mathcal{B}_1, \ldots, \mathcal{B}_N$ of $1$, be as above. Let $W$ be the subspace of $H^2(A_r)$ spanned by Laurent polynomials $z^{j-1}$ for $j = 1, \ldots, N$. Then $\mathcal{L}_\omega^{(n)}W$ is spanned by $z^{j-1}/(1 - B^{(n)}_j(0)z)^{j+1}$ for $j = 1, \ldots, N$.

In particular, the sequence $\mathcal{L}_\omega^{(n)}W$ approaches the equivariant sequence of subspaces,

$$P_N(\sigma^n\omega) = \text{lin} \left\{ \frac{z^{j-1}}{(1 - \bar{x}_{\sigma^n\omega}z)^{j+1}} : 1 \leq j \leq N \right\}.$$  

Proof. Notice that $\mathcal{L}_I$ maps $z^{-(j+1)}$ to $z^{j-1}$ (and vice versa), and is a continuous operator on $H^2(A_r)$. So $\mathcal{L}_I(V) = W$, where $V$ and $W$ are as in the statements of Lemma 18 and Corollary 19. Since $\mathcal{L}_\omega$ and $\mathcal{L}_I$ commute, we see $\mathcal{L}_\omega^{(n)}W = \mathcal{L}_\omega^{(n)}\mathcal{L}_I(V) = \mathcal{L}_I(\mathcal{L}_\omega^{(n)}(V))$. A computation shows that if $f(z) = 1/(z-x)^{n+1}$, then $\mathcal{L}_If(z) = z^{n-1}/(1-\bar{x}z)^{n+1}$. □

Corollary 20. Let the dynamical system, Blaschke product cocycle and family of Perron-Frobenius operators be as above. Let $E(\omega)$ be an equivariant family of finite-dimensional fast spaces for the cocycle. Then there exists an $N$ such that for $\mathbb{P}$-a.e. $\omega$,

$$E(\omega) \subset \text{lin}(E_0(\omega), P_N^+(\omega), P_N^-(\omega))$$

Proof. Let $\omega \in \Omega$. Let $\epsilon(\omega)$ be as in the statement of Lemma 17. Then, since $E(\omega) = \mathcal{L}_{\sigma^{-1}\omega}(E(\sigma^{-1}\omega)) \subset H^2(A_r)$, elements of $E(\omega)$ have Laurent series with coefficients satisfying $a_n = O(R^{-|n|})$. It follow that elements of $E(\omega)$ may be approximated in $H^2(A_r)$ by Laurent polynomials of sufficiently large degree: the truncated Laurent series gives a uniformly convergent sequence of approximations on $A_r$, which therefore gives a convergent sequence of approximations in $H^2(A_r)$. Hence there exists an $N$ such that $V = \text{lin}\{z^{-(j+1)} : |j| \leq N\}$ satisfies the hypothesis of Lemma 17. By Lemma 18 and Corollaries 16 and 19, we see

$$\sup_{x \in E(\sigma^n\omega) \cap S(X)} d(x, \text{lin}(E_0(\sigma^n\omega), P_N^+(\sigma^n\omega), P_N^-(\sigma^n\omega))) \to 0.$$  

For a fixed $N$, let $A_N$ be the set of $\omega$ for which (1) is satisfied and notice that $A_N$ is a $\sigma$-invariant measurable subset of $\Omega$. Hence there exists an $N > 0$ such that for $\mathbb{P}$-a.e. $\omega$,

$$\sup_{x \in E(\sigma^n\omega) \cap S(X)} d(x, \text{lin}(E_0(\sigma^n\omega), P_N^+(\sigma^n\omega), P_N^-(\sigma^n\omega))) \to 0.$$
It follows from the Poincaré recurrence theorem that if \( \kappa : \Omega \to [0, \infty) \) is a measurable function such that \( \kappa(\sigma^n \omega) \to 0 \) for \( \mathbb{P} \)-a.e. \( \omega \), then \( \kappa(\omega) = 0 \) for almost every \( \omega \). We apply this to
\[
\kappa(\omega) = \sup_{x \in E(\omega) \cap S(\chi)} d(x, \text{lin}(E_0(\omega), P_N^+(\omega), P_N^-(\omega))),
\]
to deduce that \( E(\omega) \subset \text{lin}(E_0(\omega), P_N^+(\omega), P_N^-(\omega)) \) for \( \mathbb{P} \)-a.e. \( \omega \), as required.

**Proof of Theorem 2** Part 1 follows from Corollary 10.

In the light of the Corollary 20, it suffices to evaluate the Lyapunov exponents when the system is restricted to the finite-dimensional equivariant subspaces \( P_N(\omega) = \text{lin}(E_0(\omega), P_N^+(\omega), P_N^-(\omega)) \). Notice that since each of \( E_0(\omega) \) and \( P_N^\pm(\omega) \) is equivariant, the Lyapunov exponents of the cocycle restricted to \( P_N \) are simply the combination of the Lyapunov exponents of \( E_0(\omega), P_N^+(\omega) \) and \( P_N^-(\omega) \). Since \( L_I(P_N^+(\omega)) = P_N^+(\omega), L_I \) is a bounded involution, and \( L_B(\omega) \circ L_I = L_I \circ L_B(\omega) \), we deduce the Lyapunov exponents of the restriction of the cocycle to \( P_N^+(\omega) \) are the same as those of the restriction to \( P_N^-(\omega) \). As noted above, the exponent of the cocycle restricted to the equivariant space \( E_0(\omega) \) is 0. This will turn out to be the leading exponent. Hence it suffices to compute the Lyapunov exponents of the restriction of the cocycle to \( P_N^+(\omega) \). Each of these Lyapunov exponents will then have multiplicity two for the full cocycle, being repeated as a Lyapunov exponent in the restriction to \( P_N^-(\omega) \).

It follows from Corollary 13 that the matrix representing the restriction of the cocycle to \( P_N^+(\omega) \) is upper triangular with respect to the natural family of bases, \((z - x_\omega)^{-j+1}\) for \( j = 1, \ldots, N \).

If \( B'(x_\omega) = 0 \), then all the diagonal terms of the matrix are 0 by Theorem 12. Hence if \( B'_o(x_\omega) = 0 \) for a set of \( \omega \)'s of positive measure, we see that the Lyapunov spectrum is 0 with multiplicity 1 and \(-\infty\) with infinite multiplicity.

Otherwise, we compute the leading term of \( L_\omega f(z) \) near \( x_{\sigma,\omega} \), where \( f(z) = 1/(z - x_\omega)^{j+1} \). Let \( \alpha = B'_o(x_\omega) \). Then we have
\[
L_\omega f(x_{\sigma,\omega} + h) = \frac{f(x_\omega + h/\alpha)}{B'_o(x_\omega)} + O(h^{-j})
= (\alpha/h)^{j+1}/\alpha + O(h^{-j}) = \alpha^j/h^{j+1} + O(h^{-j}).
\]
That is, the diagonal entry of the matrix is \( (B'_o(x_\omega))^j \). We also verify that the off-diagonal elements of the matrix are bounded: If \( i < j \), then the \((i,j)\) entry of the matrix is given by \( \frac{1}{2\pi i} \int L_\omega f(z)(z - x_{\sigma,\omega})^{j-i} \, dz \), where \( f(z) = (z - x_\omega)^{-j} \) and the integral is over the unit circle. Since
the operators $L_\omega$ are a uniformly bounded family on $H^2(A_r)$ (see [2, Lemmas 3.2 and 3.3] and [20, Theorem 2]), we see that the entries of the matrix are uniformly bounded.

The Lyapunov exponents of the cocycle restricted to $P^+_N(\omega)$ are therefore given by
\[
\lim_{n \to \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |B'_{\sigma^k \omega}(x_{\sigma^k \omega})|^j \]
\[
= j \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |B'_{\sigma^k \omega}(x_{\sigma^k \omega})| \]
\[
= j \int \log |B'_{\omega}(x_{\omega})| \, d\mathbb{P}(\omega) =: j \Lambda,
\]
where $j$ ranges from 1 to $N$, and we used the Birkhoff ergodic theorem in the last line.

Finally, to show that $\Lambda \leq \log \frac{r}{R}$, notice that the restriction of $d_r$ to $D_{1/R}$ agrees with Euclidean distance up to a bounded factor. The above shows that $\Lambda = \lim_{n \to \infty} \log |B^{(n)}_{\omega}(x_{\omega})|$. Lemma 9 shows that $d_r(B^{(n)}_{\omega}(x_{\omega} + h), B^{(n)}_{\omega}(x_{\omega})) \leq a|h|(r/R)^n$, where $a = d_r(x_{\omega} + h, x_{\omega})/|h|$ is a uniformly bounded quantity. Hence $|B^{(n)}_{\omega}(x_{\omega} + h) - B^{(n)}_{\omega}(x_{\omega})|/h \leq c(r/R)^n$. The fact that $\Lambda \leq \log \frac{r}{R}$ follows.

5. Spectrum Collapse

In this section, we focus on an example. Let $T_0(z) = z^2$ and $T_1(z) = \left(\frac{z + 1/4}{1 + z/4}\right)^2$, so that both $T_0$ and $T_1$ are expanding degree 2 maps of the unit circle, mapping the unit disc to itself in a two-to-one way. We take the base dynamical system to be the full shift $\sigma$ on $\Omega = \{0, 1\}^\mathbb{Z}$ with invariant measure $\mathbb{P}_p$, the Bernoulli measure where each coordinate takes the value 0 with probability $p$ and 1 with probability $1 - p$.

We let $\mathcal{L}_0$ and $\mathcal{L}_1$ be the Perron-Frobenius operators corresponding to $T_0$ and $T_1$ acting on the unit circle with respect to the signed measure, $dz$, and consider the cocycle $\mathcal{L}_\omega := \mathcal{L}_\omega_0$ and study the properties of $\mathcal{L}^{(n)}_{\omega} := \mathcal{L}_{\omega_{n-1}} \circ \cdots \circ \mathcal{L}_{\omega_0}$.

Lemma 21. Let $T_0$ and $T_1$ be defined as above. Then
(a) $T_0$ fixes 0 and $T_1$ fixes $a = \frac{1}{2}(7 - 3\sqrt{5}) \approx 0.146$;
(b) $T_0$ and $T_1$ both map the subset $[0, a]$ of the unit disk in a monotonically increasing way into itself (with disjoint ranges);
(c) both maps act as contractions on $[0, a]$: $\frac{15}{32} \leq T_1' \leq \frac{2}{3}$ on $[0, a]$ and $0 \leq T_0' \leq 2a$ on $[0, a]$. 
For \( \omega \in \Omega \), let \( x_\omega \) denote the random fixed point as described in Theorem 1.

(d) If \( \omega = \ldots 10^n \cdot 0 \ldots \), then \( 2b^{2^n} \leq T'_{\omega_0}(x_\omega) \leq 2a^{2^n} \), where \( b = T_1(0) \).

Proof. We just prove statement (d). Let \( L \) replaced by \( \mathbb{N} \), where \( \mathbb{N} \) is a standard normal random variables. That is one takes \( N(0) \). Then since \( x_{\sigma-n\omega} = T_n(x_{\sigma-(n+1)\omega}) \), we have \( b \leq x_{\sigma-n\omega} \leq a \). Since \( x_\omega = T_0^n(x_{\sigma-n\omega}) \), we have \( b^{2^n} \leq x_\omega \leq a^{2^n} \) and \( 2b^{2^n} \leq T'(x_\omega) \leq 2a^{2^n} \).

Proof of Theorem 2 For (a) it is shown in [2] that \( L_\omega \) is compact for each \( \omega \).

For (b) using Theorem 1 it suffices to prove \( \Lambda > -\infty \), where \( \Lambda = \int \log |T'_\omega(x_\omega)| \, d\mathbb{P}(\omega) \). \( \Omega \) may be partitioned up into a set of measure 0 into \([1] := \{ \omega \in \Omega : \omega_0 = 1 \} \) and the sets \([10^n \cdot 0] := \{ \omega \in \Omega : \omega_{-n+1} = 1, \omega_{-1} = \ldots = x_0 = 0 \} \) for \( 0 \leq n < \infty \). On \([1] \), by Lemma 21(c), \( \log T'_\omega(x_\omega) \geq \log 15 \). On \([10^n \cdot 0] \), \( \log T'_\omega(x_\omega) \geq 2^n \log b \) by Lemma 21(d). Since \( \mathbb{P}([10^n \cdot 0]) = (1 - p)^2 p^n \), we see

\[
\int \log T'_\omega(x_\omega) \, d\mathbb{P}(\omega) = \int_{[1]} \log T'_\omega(x_\omega) \, d\mathbb{P} + \sum_{n=0}^{\infty} \int_{[10^n \cdot 0]} \log T'_\omega(x_\omega) \, d\mathbb{P} \\
\geq p \log 15 + (1 - p)^2 \log b \sum_{n=0}^{\infty} (2p)^n > -\infty.
\]

For (c) using Theorem 1 it suffices to show that \( \lambda_2 = -\infty \). Arguing as above, we see that on \([10^n \cdot 0] \), \( \log |T'_\omega(x_\omega)| \leq 2^n \log a + 2 \) (where \( \log a \approx -1.925 \)). Hence

\[
\Lambda \leq \mathbb{P}_p([1]) (\log \frac{2}{3}) + \sum_{n=0}^{\infty} (2^n \log a + 2) \mathbb{P}_p([10^n \cdot 0]) \\
= (1 - p) \log \frac{2}{3} + p \log 2 + p(1 - p) \log a \sum_{n=0}^{\infty} (2p)^n = -\infty.
\]

We now consider a perturbed version of the cocycle, where \( L_\epsilon \) is replaced by \( L_\epsilon^\epsilon := \mathcal{N}_\epsilon \circ L_\epsilon \), where \( \mathcal{N}_\epsilon \) has the effect of convolving with a Gaussian with mean 0 and variance \( \epsilon^2 \). On \( \mathbb{R}/\mathbb{Z} \), we have

\[
(\mathcal{N}_\epsilon^\mathbb{R}/\mathbb{Z} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \epsilon t) e^{-t^2/2} \, dt = \mathbb{E} f(x + \epsilon N),
\]

where \( N \) is a standard normal random variables. That is one takes convolves the densities with a Gaussian. The corresponding conjugate
operator on $C(C_1)$ is $\mathcal{N}_\epsilon := \mathcal{N}_\epsilon^{C_1} = Q^{-1}N_{\mathbb{Z}}^{Q}/Q$, where $Q$ is as in Lemma 5. A calculation using that lemma shows

$$(\mathcal{N}_\epsilon f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ze^{-2\pi i\epsilon t})e^{-2\pi i\epsilon t-t^2/2} \, dt.$$ 

Proof of Theorem 3. From the definition of $L_0$, we check

$$L_0(f)(z) = \frac{1}{2} \left( f(\sqrt{z}) + f(-\sqrt{z}) \right),$$

where $\pm \sqrt{z}$ are the two square roots of $z$. We define $e_n(z) = z^{n-1}$ and verify that

$$L_0(e_n) = \begin{cases} e_{n/2} & \text{if } n \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that $\lambda_2 = -\infty$. We compute

$$\mathcal{N}_\epsilon(e_n)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e_n(z e^{-2\pi i\epsilon t})e^{-2\pi i\epsilon t-t^2/2} \, dt$$

$$= z^{n-1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\pi i\epsilon t-t^2/2} \, dt$$

$$= e^{-2\pi^2 n^2 \epsilon^2} e_n(z) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(t+2\pi \epsilon \epsilon)^2) \, dt$$

$$= e^{-2\pi^2 n^2 \epsilon^2} e_n(z).$$

Combining the two, we have

$$(L_0^\epsilon)^n f(z) = \begin{cases} \exp(-2\pi^2 \epsilon^2 m^2 (4^{n-1} + \ldots + 4 + 1)) e_m & \text{if } \ell = 2^m m; \\ 0 & \text{otherwise.} \end{cases}$$

We let $H_0^2(A_r)$ be the subspace of $H^2(A_r)$ consisting of those functions whose Laurent expansions have a vanishing $z^{-1}$ term. Let $f \in H_0^2(A_r)$ be of norm 1 and let $f = \sum_{n \in \mathbb{Z}} a_n z^n$ be its Laurent expansion. We recall $|a_n| \leq r^{-|n|} \leq 1$ for all $n \in \mathbb{Z}$ and $a_{-1} = 0$.

Now

$$(L_0^\epsilon)^n f(z) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \exp(-2\pi^2 \epsilon^2 m^2 (4^{n-1} + \ldots + 4 + 1)) a_{2^m m-1} z^{m-1}.$$ 

so that for $z \in A_r$,

$$|(L_0^\epsilon)^n f(z)| \leq \sum_{m \in \mathbb{Z} \setminus \{0\}} \exp(-2\pi^2 \epsilon^2 m^2 (4^{n-1} + \ldots + 4 + 1)) r^{-2^m m-1} |r|^{m-1}$$

$$\leq \frac{r}{r-1} \exp(-2\pi^2 \epsilon^2 4^{n-1}).$$
Since if $g$ is a bounded analytic function on $A_r$, $\|g\|_{H^2(A_r)} \leq 2\|g\|_{\infty}$, we see $\|(L_0^n)\|_{H^2(A_r)} \leq 2r \exp\left(-2\pi^2 \epsilon^2 4^{n-1}\right)$. By Lemma 8(b),

$$D_2((L_0^n)L_1(n)) \leq A \exp\left(-2\pi^2 \epsilon^2 4^{n-1}\right),$$

where $A = 2r\|L_1\|^2 c_2/(r - 1)$.

Now let $N(\omega) = \min\{n > 0: \omega_n = 1\}$. We consider the induced map on $[1]$: $\tilde{\sigma}(\omega) = \sigma^{N(\omega)}(\omega)$. The induced cocycle is defined for $\omega \in [1]$ by $\tilde{L}_\omega^\epsilon = L_\omega^{(N(\omega))}$, so that $\tilde{L}_\omega^\epsilon = (L_0^n)_{N(\omega)}^{-1} L_1^\epsilon$. By $\tilde{L}_\omega^\epsilon(n)$, we mean $\tilde{L}_{\tilde{\sigma}n-1(\omega)} \circ \cdots \circ \tilde{L}_\omega$ and by $\bar{P}$, we mean the normalized restriction of $P$ to $[1]$ (so the convention is that quantities marked with tildes refer to the induced system).

We define the return times for $\omega \in [1]$ by $N_1(\omega) = N(\omega)$ and $N_{n+1}(\omega) = N_n(\omega) + N(\sigma^{N_n(\omega)}(\omega))$ for $n \geq 1$. Now we have, using Lemma 8

$$\frac{1}{N_n(\omega)} \log D_2(L_\omega^\epsilon(N_n(\omega))) = \frac{1}{N_n(\omega)} \log D_2(\tilde{L}_\omega^\epsilon(n))$$

$$= \frac{n}{N_n(\omega)} \frac{1}{n} \log D_2(\tilde{L}_{\tilde{\sigma}n-1(\omega)} \circ \cdots \circ \tilde{L}_\omega)$$

$$\leq \frac{n}{N_n(\omega)} \frac{1}{n} \log \left(D_2(\tilde{L}_{\tilde{\sigma}n-1(\omega)}) \cdots D_2(\tilde{L}_\omega)\right)$$

$$= \frac{n}{N_n(\omega)} \frac{1}{n} \sum_{i=0}^{n-1} \log D_2(\tilde{L}_{\tilde{\sigma}i(\omega)})$$

$$\leq \left(\frac{n}{N_n(\omega)}\right) \frac{1}{n} \sum_{i=0}^{n-1} (2\pi^2 \epsilon^2 4^{N(\tilde{\sigma} \omega) - 1} + \log A).$$

Since $\int_{[1]} 4^{N(\omega)} d\bar{P}_p(\omega) = \sum_{n=1}^{\infty} \frac{1}{n} p^{n-1}(1 - p) = \infty$, we see the average $\frac{1}{n} \sum_{i=0}^{n-1} (2\pi^2 \epsilon^2 4^{N(\tilde{\sigma} \omega) - 1} + \log A)$ in the last line converges to $-\infty$ almost surely by Birkhoff’s theorem applied to the ergodic transformation $\tilde{\sigma}$ of $([1], \bar{P}_p)$). As $n/N_n(\omega) \to 1/P_{\bar{P}}([1])$ for $\bar{P}_p$-almost every $\omega \in [1]$, we see $\frac{1}{n} \log D_2(L_\omega^\epsilon(N_n(\omega))) \to -\infty$ for $\bar{P}_p$-a.e. $\omega \in [1]$. Since this is a subsequence of the convergent sequence $\frac{1}{n} \log D_2(L_\omega^\epsilon(n))$, we see that $\frac{1}{n} \log D_2(L_\omega^\epsilon(n)) \to -\infty$ for $\bar{P}_p$-a.e. $\omega$. \hfill $\square$

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