Light-cone Gauge String Field Theory in Noncritical Dimensions

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Light-cone gauge SFT (closed)

\[ S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R (1, 2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - H \right) | \Phi \rangle_2 

+ \frac{2g}{3} \int d1d2d3 \langle V_3 (1, 2, 3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right] \]

- No gauge symmetry \( \longrightarrow \) no need to keep \( d = 26 \) or 10
CFT for $X^\pm$

- No Lorentz symmetry $\longrightarrow$ it should correspond to a string theory in a Lorentz noninvariant background

$$X^i + \text{ghost} + \text{nontrivial CFT for } X^\pm$$

$$c = d - 2 \quad -26 \quad 28 - d$$

- With all these variables we can construct a nilpotent BRST charge.

$$Q_B = \int \frac{dz}{2\pi i} \left( cT + bc\partial c \right) + c.c.$$
CFT for $X^{\pm}$

- No Lorentz symmetry $\rightarrow$ it should correspond to a string theory in a Lorentz noninvariant background

$$X^i + \text{ghost} + \text{nontrivial CFT for } X^{\pm}$$

$$c = d - 2 \quad -26 \quad 28 - d$$

- With all these variables we can construct a nilpotent BRST charge.

$$Q_B = \oint \frac{dz}{2\pi i} \left( cT + bc\partial c \right) + c.c.$$ 

We would like to construct the CFT for the longitudinal variables $X^{\pm}$. ($X^{\pm}$ CFT)
Motivation

Light-cone gauge SFT for superstrings (Mandelstam, S.J. Sin)

\[
S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R(1, 2) \mid \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - H \right) \mid \Phi \rangle_2 \\
+ \frac{2g}{3} \int d1d2d3 \langle V_3(1, 2, 3) \mid \Phi \rangle_1 \mid \Phi \rangle_2 \mid \Phi \rangle_3 \right]
\]

\[
\text{propagator} \quad \text{vertex} \quad \text{yields divergent results even for tree amplitudes.}
\]
The amplitude diverges when two $T_F$’s come close to each other.

$$A \sim \int dT d\theta$$

- How can one deal with the divergences in the light-cone gauge SFT?
For general $d$

\[ ds^2 = d\rho d\bar{\rho} = \partial \rho \bar{\partial} \bar{\rho} dz d\bar{z} \]

\[ \mathcal{A} \sim \int d^2 T \left\langle \prod_{I=1,2} \left[ T^{LC}_F(z_I) \tilde{T}^{LC}_F(\bar{z}_I) \right] \prod_{r=1}^{4} V^{LC}_r \right\rangle \]

\[ \times e^{-\frac{d-2}{16} \Gamma[\ln(\partial \rho \bar{\partial} \bar{\rho})]} \prod_{I=1,2} \left( \partial^2 \rho (z_I) \bar{\partial}^2 \bar{\rho} (\bar{z}_I) \right)^{-\frac{3}{4}} \]

\[ \Gamma [\phi] = -\frac{1}{\pi} \int d^2 z \partial \phi \bar{\partial} \phi: \text{Liouville action} \]
For general $d$

\[ ds^2 = d\rho d\bar{\rho} = \partial \rho \partial \bar{\rho} d\rho d\bar{\rho} \]

\[
\mathcal{A} \sim \int d^2 T \left\langle \prod_{I=1,2} \left( T_{LC}^{IF} (z_I) \bar{T}_{LC}^{IF} (\bar{z}_I) \right) \prod_{r=1}^4 V_{r \text{LC}} \right\rangle \\
\times e^{-\frac{d-2}{16} \Gamma[\ln(\partial \rho \partial \bar{\rho})]} \prod_{I=1,2} \left( \partial^2 \rho (z_I) \bar{\partial}^2 \bar{\rho} (\bar{z}_I) \right)^{-\frac{3}{4}}
\]

\[ e^{-\frac{d-2}{16} \Gamma[\ln(\partial \rho \partial \bar{\rho})]} \text{ is needed because } \hat{c} = d - 2, \]
For general $d$

\[ ds^2 = d\rho d\bar{\rho} = \partial\rho \partial\bar{\rho} dz d\bar{z} \]

\[
\mathcal{A} \sim \int d^2 T \left\langle \prod_{I=1,2} \left[ T_{F}^{\text{LC}}(z_I) \tilde{T}_{F}^{\text{LC}}(\bar{z}_I) \right] \prod_{r=1}^{4} V_{r}^{\text{LC}} \right\rangle \\
\times e^{-\frac{d-2}{16} \Gamma[\ln(\partial\rho \partial\bar{\rho})]} \prod_{I=1,2} \left( \partial^2 \rho(z_I) \bar{\partial}^2 \bar{\rho}(\bar{z}_I) \right)^{-\frac{3}{4}}
\]

$e^{-\frac{d-2}{16} \Gamma[\ln(\partial\rho \partial\bar{\rho})]}$ behaves as $|z_1 - z_2|^{-\frac{d-2}{8}}$ in the limit $z_1 \rightarrow z_2$, and this amplitude is finite for large $-d$. 
Motivation

- Dimensional regularization is possible.
- $X^\pm$ CFT $\longrightarrow$ the dimensional regularization preserves
  
  BRST on the worldsheet $\sim$ gauge symmetry of SF

- Using the CFT, one can show that the tree level (NS,NS) sector amplitudes derived from the SFT coincide with the results of the 1-st quantized formulation.

In collaboration with Y. Baba and K. Murakami (Riken)
arXiv:0906.3577 [hep-th] JHEP10(2009) 035
arXiv:0909.4675 [hep-th] JHEP to appear
arXiv:0911.3704 [hep-th]
arXiv:0912.****
Plan of the talk

1. $X^\pm$ CFT (bosonic)
2. Light-cone gauge amplitudes
3. $X^\pm$ CFT (super)
4. Dimensional regularization
5. Outlook
§1 $X^\pm$ CFT (bosonic)

We propose a 2d CFT with an action

$$S_{X^\pm} = -\frac{1}{2\pi} \int d^2 z \left( \partial X^+ \bar{\partial} X^- + \bar{\partial} X^+ \partial X^- \right)$$

$$+ \frac{d - 26}{24} \Gamma [\ln (\partial X^+ \bar{\partial} X^+)]$$

$\Gamma$ is the Liouville action

$$\Gamma [\phi] = -\frac{1}{\pi} \int d^2 z \partial \phi \bar{\partial} \phi$$

- We calculate the correlation functions starting from this action.
energy momentum tensor

\[ T_{X^\pm}(z) \equiv \partial X^+ \partial X^- - \frac{d-26}{12} \{X^+, z\} \]

Schwarzian derivative

\[ \{X^+, z\} \equiv \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2} \left( \frac{\partial^2 X^+}{\partial X^+} \right)^2 \]

From the correlation functions, one can see that the energy-momentum tensor satisfies the Virasoro algebra with \( c = 28 - d \).
energy momentum tensor

\[ T_{X^\pm}(z) \equiv \partial X^+ \partial X^- - \frac{d-26}{12} \{X^+, z\} \]

Schwarzian derivative

\[ \{X^+, z\} \equiv \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2} \left( \frac{\partial^2 X^+}{\partial X^+} \right)^2 \]

Later, we will show that the tree amplitude of LC gauge SFT can be described by using this CFT.
Correlation functions

$X^+$ should possess an nonzero expectation value.
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We always consider the theory in the presence of vertex operators

$$\exp (-ip^+ X^-).$$

$$\left\langle F \left[ X^+, X^- \right] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r) \right\rangle$$

$$\equiv \int \left[ dX^+ dX^- \right] e^{-S_{X^\pm}} F \left[ X^+, X^- \right] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r)$$
$X^+$ should possess an nonzero expectation value. We always consider the theory in the presence of vertex operators $\exp(-ip^+X^-)$.

\[
\left\langle F[X^+,X^-] \prod_{r=1}^{N} e^{-ip_r^+X^-} (Z_r, \bar{Z}_r) \right\rangle \\
\equiv \int [dX^+ dX^-] e^{-S_{X^\pm}} F[X^+,X^-] \prod_{r=1}^{N} e^{-ip_r^+X^-} (Z_r, \bar{Z}_r)
\]

Let us calculate the correlation function for a functional $F[X^+]$ of $X^+$. 
Correlation functions $F[X^+]$

\[
\left\langle F[X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \right\rangle
\]

\[
= \int [dX^\pm] e^{-S_{X^\pm}} F[X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r)
\]

\[
S_{X^\pm} = -\frac{1}{2\pi} \int d^2z (\partial X^+ \bar{\partial} X^- + \bar{\partial} X^+ \partial X^-) + \frac{d-26}{24} \Gamma[\ln (\partial X^+ \bar{\partial} X^+)]
\]

This should be considered as a Euclideanized version of a Lorentzian path integral.
Correlation functions $1 \ F \ [X^+]$

$$\left\langle F \ [X^+] \prod_{r=1}^{N} e^{-ip^+_r X^-} (Z_r, \bar{Z}_r) \right\rangle$$

$$= \int [dX] \exp \left( \frac{1}{\pi} \int d^2 z \, X^- \left( \partial \bar{\partial} X^+ - \pi i \sum_{r=1}^{N} p^+_r \delta^2 (z - Z_r) \right) \right)$$

$$\times F \ [X^+] \exp \left( -\frac{d - 26}{24} \Gamma \left[ \ln (\partial X^+ \bar{\partial} X^+) \right] \right)$$

$$- \frac{i}{2} \partial \bar{\partial} (\rho (z) + \bar{\rho} (\bar{z})) = \pi i \sum_{r=1}^{N} p^+_r \delta^2 (z - Z_r)$$
Correlation functions $1 \ F [X^+]$

\[
\left\langle F [X^+] \prod_{r=1}^{N} e^{-i p^+_r X^-} (Z_r, \bar{Z}_r) \right\rangle
\]

\[
\sim (\det (\partial \bar{\partial}))^{-1} F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right] \exp \left( -\frac{d - 26}{24} \Gamma [\ln (\partial \rho \bar{\partial} \bar{\rho})] \right)
\]

\[
\rho (z) = \sum_{r=1}^{N} \alpha_r \ln (z - Z_r) \ (\alpha_r \equiv 2 p^+_r)
\]
Mandelstam mapping

This implies that $X^+$ has an expectation value

$$\langle X^+ (z, \bar{z}) \rangle \sim -\frac{i}{2} \left( \rho (z) + \bar{\rho} (\bar{z}) \right)$$

$$\rho (z) = \sum_{r=1}^{N} \alpha_r \ln (z - Z_r) \quad (\alpha_r \equiv 2p_r^+)$$

$\rho (z)$ coincides with the Mandelstam mapping.
1. $\langle F \left[ X^+ \right] \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \rangle$ is used to express the tree amplitude corresponding to the light-cone diagram.
2. $e^{-ipr^+X^-} (Z_r, \bar{Z}_r)$ corresponds to a hole with length $2\pi \alpha_r$, similar to the macroscopic loop operator in the old matrix models.
Correlation functions 2 $X^-$ insertions

Correlation functions with $X^-$ insertions can be calculated using

\[ \left\langle F [X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \right\rangle \] as a generating functional.

\[
\left\langle F [X^+] X^- (Z_N, \bar{Z}_N) \prod_{r=1}^{N-1} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \right\rangle
\]

\[
\sim i \partial_{p_N^+} \left| \left. F [X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \right|_{p_N^+=0} \right.
\]

\[
\sim i \partial_{p_N^+} \left( F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right] \exp \left( -\frac{d-26}{24} \Gamma \left[ \ln (\partial \rho \partial \bar{\rho}) \right] \right) \right) \bigg|_{p_N^+=0}
\]
Evaluation of $\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right]$ 

One can derive all the correlation functions from $\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right]$. 

\[ \Gamma \left[ \phi \right] = -\frac{1}{\pi} \int d^2 z \, \partial \phi \bar{\partial} \bar{\phi} \]

\[ \rho \left( z \right) = \sum_{r=1}^{N} \alpha_r \ln \left( z - Z_r \right) \]

$\partial \rho \left( z \right)$ has poles at $z \sim Z_r$ and zeros at $z \sim Z_I$.

\[ ds^2 = d\rho d\bar{\rho} = \partial \rho \bar{\partial} \bar{\rho} dz d\bar{z} \]
Evaluation of $\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \rho \right) \right]$ 

There are at least two ways to obtain $\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \rho \right) \right]$.

1. Direct evaluation regularizing the singularities (Mandelstam, lectures at “Unified String Theory”)

2. Integration of the variation $\delta \Gamma$ under $T \rightarrow T + \delta T$ (Cremmer and Gervais, Baba, Murakami and N.I.)
Evaluation of \( \Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right] \)

\[
\exp \left( -\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right] \right) = e^{-W} e^{-2 \sum_{r=1}^{N} \Re \bar{N}_{00}^{rr} \prod_{I} \left| \partial^{2} \rho(z_{I}) \right|^{-3}}
\]

\[
e^{-W} \equiv \frac{\prod_{I>J} |z_{I} - z_{J}|^{4} \prod_{r>s} |Z_{r} - Z_{s}|^{4}}{\prod_{r,I} |Z_{r} - z_{I}|^{4}}
\]

\[
\bar{N}_{00}^{rr} \equiv \frac{\rho(z_{I})}{\alpha_{r}} - \sum_{s \neq r} \frac{\alpha_{s}}{\alpha_{r}} \ln (Z_{r} - Z_{s})
\]
Evaluation of $\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right]$

\[
\exp \left( -\Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right] \right) = e^{-W} e^{-2 \sum_{r=1}^{N} \text{Re} \tilde{N}_{00}^{rr}} \prod_{I} \left| \partial^2 \rho(z_I) \right|^{-3} \]

\[
= e^{-2 \sum_{r=1}^{N} \text{Re} \tilde{N}_{00}^{rr}} \left[ \sum_{s=1}^{N} \alpha_s Z_s \right]^{-2N+6} \prod_{r>s} |Z_r - Z_s|^2 \prod_{r=1}^{N} |\alpha_r| \prod_{I>J} \left| z_I - z_J \right|^2
\]

\[
\exp \left( -\frac{d-2}{24} \Gamma \left( \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right) \right) \sim |z_I - z_J|^{-\frac{d-2}{12}} \text{ for } z_I \rightarrow z_J
\]
Energy-momentum tensor

From the correlation functions of the energy-momentum tensor, we can deduce

- $T_{X^\pm}(z)$ is regular at the points where there are no operator insertions.
- OPE

$$T_{X^\pm}(z) e^{-ip^+X^-} (Z_r, \bar{Z}_r) \sim \frac{1}{z - Z_r} \partial e^{-ip^+X^-} (Z_r, \bar{Z}_r)$$

$$ds^2 = d\rho d\bar{\rho} = \partial \rho \bar{\partial} \rho dz d\bar{z}$$
Energy-momentum tensor

OPE

\[ \partial X^+(\bar{z}) \partial X^+(\bar{z}') \sim \text{regular} \]
\[ \partial X^-(\bar{z}) \partial X^+(\bar{z}') \sim \frac{1}{(\bar{z} - \bar{z}')^2} \]
\[ \partial X^-(\bar{z}) \partial X^-(\bar{z}') \sim \frac{d - 26}{12} \partial \bar{z} \partial \bar{z}' \left[ \frac{1}{(\bar{z} - \bar{z}')^2} \frac{1}{\partial X^+(\bar{z}) \partial X^+(\bar{z}')} \right] \]
Energy-momentum tensor

OPE

$$\partial X^+(z)\partial X^+(z') \sim \text{regular}$$
$$\partial X^-(z)\partial X^+(z') \sim \frac{1}{(z - z')^2}$$
$$\partial X^-(z)\partial X^-(z') \sim -\frac{d-26}{12}\partial z\partial z' \left[ \frac{1}{(z - z')^2} \frac{1}{\partial X^+(z)\partial X^+(z')} \right]$$

From these, one can deduce

$$T_{X^\pm}(z)T_{X^\pm}(z') \sim \frac{1}{2}(28 - d)\frac{2}{(z - z')^4} + \frac{2}{(z - z')^2}T_{X^\pm}(z') + \frac{1}{z - z'}\partial T_{X^\pm}(z')$$
Energy-momentum tensor

OPE

\[
\begin{align*}
\partial X^+(z) \partial X^+(z') \sim & \quad \text{regular} \\
\partial X^-(z) \partial X^+(z') \sim & \quad \frac{1}{(z - z')^2} \\
\partial X^-(z) \partial X^-(z') \sim & \quad -\frac{d - 26}{12} \partial z \partial z' \left[ \frac{1}{(z - z')^2} \frac{1}{\partial X^+(z) \partial X^+(z')} \right]
\end{align*}
\]

From these, one can deduce

\[
T_{X^\pm}(z)T_{X^\pm}(z') \sim \frac{1}{2} \frac{(28 - d)}{(z - z')^4} + \frac{2}{(z - z')^2} T_{X^\pm}(z') + \frac{1}{z - z'} \partial T_{X^\pm}(z')
\]

\(T_{X^\pm}\) satisfies the Virasoro algebra with \(c = 28 - d\)
§2 Light-cone gauge amplitudes

We would like to show that the $X^+$ CFT can be used to describe LC string theory in $d \neq 26$ dimensions.

- We consider bosonic closed string field theory for $d \neq 26$.
- We show that the tree amplitude can be written in a BRST invariant form using the $X^\pm$ CFT.
String field action for $d \neq 26$

\[
S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R(1, 2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - H \right) | \Phi \rangle_2 + \frac{2g}{3} \int d1d2d3 \langle V_3(1, 2, 3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right]
\]
String field action for $d \neq 26$

$$S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R (1, 2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - H \right) | \Phi \rangle_2 
+ \frac{2g}{3} \int d1d2d3 \langle V_3 (1, 2, 3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right]$$

$$dr = \frac{\alpha_r d\alpha_r}{4\pi} \frac{d^{d-2} p_r}{(2\pi)^{d-2}}$$

$$\langle R (1, 2) | = \frac{1}{\alpha_1} 4\pi \delta (\alpha_1 + \alpha_2) (2\pi)^{d-2} \delta (p_1 + p_2)_{12} \langle 0 | e^{E(1,2)}$$

$$E (1, 2) = - \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_n^{i(1)} \alpha_n^{i(2)} + \tilde{\alpha}_n^{i(1)} \tilde{\alpha}_n^{i(2)} \right)$$

$$H = \frac{L^{LC(2)}_0 + \tilde{L}^{LC(2)}_0 - \frac{d-2}{12}}{\alpha_2}$$
String field action for $d \neq 26$

\[
S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R (1, 2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - H \right) | \Phi \rangle_2 
+ \frac{2g}{3} \int d1d2d3 \langle V_3 (1, 2, 3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right]
\]

\[
\langle V_3 (1, 2, 3) \rangle = 4\pi \delta \left( \sum_{r=1}^{3} \alpha_r \right) (2\pi)^{d-2} \delta^{d-2} \left( \sum_{r=1}^{3} p_r \right)
\times \langle V_3^{\text{LPP}} (1, 2, 3) \rangle \ e^{-\Gamma_{[3]}(1,2,3)}
\]

\[
e^{-\Gamma_{[3]}(1,2,3)} = \text{sgn} (\alpha_1 \alpha_2 \alpha_3) \left| \frac{e^{-2\hat{\tau}_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}}}{\alpha_1 \alpha_2 \alpha_3} \right|^{\frac{d-2}{24}}
\]
Light-cone gauge amplitudes

With this choice of the action, tree amplitudes in the LC gauge SFT is given as

\[ A \sim \int \prod_I d^2 T_I \left\langle \prod_{r=1}^{N} V_r^{\text{LC}} \right\rangle e^{-\frac{d-2}{24} \Gamma \left[ \ln (\partial \rho \bar{\partial} \bar{\rho}) \right]} \]

\[ ds^2 = d\rho d\bar{\rho} = \partial \rho \bar{\partial} \bar{\rho} dz d\bar{z} \]

- We would like to recast the amplitude into a BRST invariant form using the \( X^\pm \) CFT.
\[ e^{-\frac{d-2}{24}} \Gamma \left[ \ln (\partial \rho \bar{\partial} \bar{\rho}) \right] \]

\[ = e^{-\frac{d-24}{24}} \Gamma \left[ \ln (\partial \rho \bar{\partial} \bar{\rho}) \right] \]

\[ \times \prod_{r=1}^{N} |\alpha_r|^{-1} e^{-\frac{1}{2} W} \left| \sum_{s=1}^{N} \alpha_s Z_s \right|^{2} e^{-2 \sum_{r=1}^{N} \text{Re} \tilde{N}_{00}^{rr} \prod_{I} |\partial^2 \rho(z_I)|^{-2}} \]
$X^\pm$ and ghost

$$e^{-\frac{d-2}{24}\Gamma[\ln(\partial \rho \bar{\rho})]} = e^{-\frac{d-2}{24}\Gamma[\ln(\partial \rho \bar{\rho})]}$$

$$\times \prod_{r=1}^{N} |\alpha_r|^{-1} e^{-\frac{1}{2} W} \left| \sum_{s=1}^{N} \alpha_s Z_s \right|^2 e^{-2 \sum_{r=1}^{N} \text{Re} \tilde{N}_{00}^{rr} \prod_{I} |\partial^2 \rho(z_I)|^{-2}}$$

$$e^{-\frac{1}{2} W} = \frac{\prod_{I>J} |z_I - z_J|^2 \prod_{r>s} |Z_r - Z_s|^2}{\prod_{r,I} |Z_r - z_I|^2}$$

$$\sim \int [d \text{ (ghost)}] e^{-S_{bc}} \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\bar{z}) \right)$$

$$\times \prod_{I} \left( b(z_I) \tilde{b}(\bar{z}_I) \right) \prod_{r} \left( c(Z_r) \tilde{c}(\bar{Z}_r) \right)$$
\[ e^{-\frac{d-2}{24} \Gamma \left[ \ln (\partial \rho \bar{\rho}) \right]} = e^{-\frac{d-2}{24} \Gamma \left[ \ln (\partial \rho \bar{\rho}) \right]} \times \prod_{r=1}^{N} |\alpha_r|^{-1} e^{-\frac{1}{2} W} \left| \sum_{s=1}^{N} \alpha_s Z_s \right|^{2} e^{-2 \sum_{r=1}^{N} \text{Re} \bar{N}^{rr}_{00} \prod_{I} \partial^2 \rho(z_I)} \right|^{-2} \]

\[ e^{-\frac{d-24}{24} \Gamma \left[ \ln (\partial \rho \bar{\rho}) \right]} \sim \int [dX^\pm] e^{-S_{X^\pm}} \prod_{r=1}^{N} e^{-ip_r^+ X^-} (Z_r, \bar{Z}_r) \]
\[ e^{-\frac{d-2}{24}\Gamma[\ln(\partial\rho\bar{\partial}\bar{\rho})]} = e^{-\frac{d-2}{24}\Gamma[\ln(\partial\rho\bar{\partial}\bar{\rho})]} \times \prod_{r=1}^{N} |\alpha_r|^{-1} e^{-\frac{1}{2}W} \left| \sum_{s=1}^{N} \alpha_s Z_s \right|^2 e^{-2\sum_{r=1}^{N} \text{Re} \tilde{N}_{00}^{rr}} \prod_{I} \left| \partial^2 \rho(z_I) \right|^{-2} \]

\[ \sim \int [dX^\pm] e^{-S_{X^\pm}} \prod_{r=1}^{N} \left( e^{-ip_r^+ X^- (Z_r, \bar{Z}_r)} |\alpha_r|^{-1} e^{-2\text{Re} \tilde{N}_{00}^{rr}} \right) \]

\[ \times \int [d \text{ (ghost)}] e^{-S_{bc}} \left| \sum_r \alpha_r Z_r \right|^2 \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\bar{z}) \right) \]

\[ \times \prod_{I} \left( \frac{\partial}{\partial^2 \rho}(z_I) \right) \left( \frac{\partial \rho}{\partial^2 \rho}(\bar{z}_I) \right) \prod_{r} \left( c(Z_r) \tilde{c}(\bar{Z}_r) \right) \]
Substituting this, we obtain

\[
\left\langle \prod_{r=1}^{N} V_{r}^{\text{LC}} \right\rangle e^{-\frac{d-2}{24} \Gamma[\ln(\partial \rho \bar{\partial} \rho)]} X^i
\]

\[
\sim \int \left[ dX^\mu d(\text{ghost}) \right] e^{-S_X - S_{bc}} \left| \sum_r \alpha_r Z_r \right|^2 \left( \lim_{z \to \infty} \frac{1}{|z|^4} e(z) \tilde{c}(\bar{z}) \right)
\]

\[
\times \prod_I \left( \frac{b}{\partial^2 \rho}(z_I) \frac{\tilde{b}}{\partial^2 \bar{\rho}}(\bar{z}_I) \right)
\]

\[
\times \prod_{r=1}^{N} \left( c \tilde{c} \frac{V^{\text{LC}}_r}{\alpha_r} e^{-ip_r^+ X^- - 2 \Re \bar{N}^r_{00}} \right) (Z_r, \bar{Z}_r)
\]
Substituting this, we obtain

\[
\left\langle \prod_{r=1}^{N} V_r^{LC} \right\rangle_{X^i} \sim \left| \sum_r \alpha_r Z_r \right|^2 \left\langle \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\tilde{z}) \right) \prod_I \left( \frac{b}{\partial^2 \rho} (zI) \frac{\tilde{b}}{\partial^2 \tilde{\rho}} (\tilde{z}I) \right) \right.
\]

\[
\times \prod_{r=1}^{N} \left( c\tilde{c} \frac{V_r^{LC}}{\alpha_r} e^{-i p_r^{+} X^-} - 2 \text{Re} \tilde{N}_{00}^r \right) (Z_r, \tilde{Z}_r) \right\rangle_{X^{\mu}, b, c}
\]
BRST invariant form

\[ A = \int \prod_I d^2 T_I \left\langle \prod_{r=1}^N V^\text{LC}_r \right\rangle_{X^i} e^{-\frac{d-2}{24} \Gamma \left[ \ln (\partial \bar{\partial} \bar{\rho}) \right]} \]

\[ \approx \int \prod_I d^2 T_I \left| \sum_r \alpha_r Z_r \right|^2 \]

\[ \times \left\langle \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\bar{z}) \right) \prod_I \left( \frac{b}{\partial^2 \rho} (z_I) \frac{\tilde{b}}{\bar{\partial}^2 \bar{\rho}} (\bar{z}_I) \right) \right. \]

\[ \times \left. \prod_{r=1}^N \left( c \tilde{c} \frac{V^\text{LC}_r}{\alpha_r} e^{-ip_r^+ X - 2 \Re \tilde{N}_{00}^{rr}} \right) (Z_r, \bar{Z}_r) \right\rangle_{X^\mu, b, c} \]
BRST invariant form

\[ A = \int \prod_I d^2 T_I \left< \prod_{r=1}^N V_{r}^{LC} \right> e^{-\frac{d-2}{24} \Gamma \left[ \ln (\partial \rho \bar{\rho}) \right]} \]

\[ \sim \int \prod_I d^2 T_I \left| \sum_r \alpha_r Z_r \right|^2 \]

\[ \times \left< \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\bar{z}) \right) \prod_I \left( \frac{b}{\partial^2 \rho} (z_I) \frac{\tilde{b}}{\partial^2 \bar{\rho}} (\bar{z}_I) \right) \right. \]

\[ \times \left. \prod_{r=1}^N \left( c \tilde{c} \frac{V_{r}^{LC}}{\alpha_r} e^{-ip_r^+ X - 2 \text{Re } \bar{N}_{00}^{rr}} \right) (Z_r, \bar{Z}_r) \right>_{X^\mu, b, c} \]

Replacing \( \rho + \bar{\rho} \) by \( 2iX^+ \) in the braces.
BRST invariant form

\[ \mathcal{A} = \int \prod_I d^2 T_I \left\langle \prod_{r=1}^N V_r^{\text{LC}} \right\rangle \left. \prod_i e^{-\frac{d-2}{24}} \Gamma[\ln(\partial \rho \bar{\partial} \bar{\rho})] \right|_{X^i} \]

\[ \sim \int \prod_I d^2 T_I \left| \sum \alpha_r Z_r \right|^2 \]

\[ \times \left\langle \left( \lim_{z \to \infty} \frac{1}{|z|^4} c(z) \tilde{c}(\tilde{z}) \right) \prod_I \left( \frac{b}{\partial^2 \rho} (z_I) \frac{\tilde{b}}{\partial^2 \bar{\rho}} (\tilde{z}_I) \right) \prod_{r=1}^N \left( \frac{c\tilde{c}}{\alpha_r} V_r^{\text{LC}} e^{-ip^+ X - 2 \text{Re} \bar{N}^{rr}_{00}} \right) (Z_r, \bar{Z}_r) \right\rangle \]

rewriting \( \frac{b}{\partial^2 \rho} (z_I) \) as \( \oint_{z_I} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) \) and deforming the contour we obtain
BRST invariant form

\[ \mathcal{A} \sim \prod_{r=4}^{N} \int d^2 Z_r \left\langle \prod_{r=1}^{3} (c\tilde{c}V_{r}'^{DDF}) (Z_r, \bar{Z}_r) \prod_{r=4}^{N} V_{r}'^{DDF} (Z_r, \bar{Z}_r) \right. \\
\left. \times \prod_{r=1}^{N} \exp \left( i \frac{d - 26}{24} \frac{X^+}{p_r^+} \right) \left( z^{(r)}_I, \bar{z}^{(r)}_I \right) \right\rangle \]
BRST invariant form

\[ \mathcal{A} \sim \prod_{r=4}^{N} \int d^2 Z_r \left\langle \prod_{r=1}^{3} (c \tilde{c} V_r^{\text{DDF}}) (Z_r, \bar{Z}_r) \prod_{r=4}^{N} V_r^{\text{DDF}} (Z_r, \bar{Z}_r) \right. \\
\left. \times \prod_{r=1}^{N} \exp \left( i \frac{d - 26}{24} \frac{X^+}{p_r^+} \right) \left( \bar{z}_I^{(r)}, \bar{z}_I^{(r)} \right) \right\rangle \]

\[ V_r^{\text{DDF}} \equiv : V_r^{\text{DDF}} \exp \left( -i \frac{d - 26}{24} \frac{X^+}{p_r^+} \right) : \]

\( V_r^{\text{DDF}} \) is the DDF operator which corresponds to \( V_r^{\text{LC}} \).
BRST invariant form

\[ \mathcal{A} \sim \prod_{r=1}^{N} \left\langle \prod_{r=1}^{3} (c\tilde{c}V_r^{DDF}) (Z_r, \bar{Z}_r) \prod_{r=1}^{N} V_r^{DDF} (Z_r, \bar{Z}_r) \right. \]

\[ \times \prod_{r=1}^{N} \exp \left( i \frac{d-26}{24} \frac{X^+}{p_r^+} \right) \left( z^{(r)}_I, \bar{z}^{(r)}_I \right) \right] \]

\[ V_r^{DDF} : \text{ primary field of weight } \left( \frac{d-2}{24}, \frac{d-2}{24} \right) \]

\[ V_r^{DDF} : \text{ primary field of weight } (1, 1) \]
BRST invariant form

\[ A \sim \prod_{r=4}^{N} \int d^2 Z_r \left\langle \prod_{r=1}^{3} (c\tilde{c} V'_r^{DDF}) (Z_r, \bar{Z}_r) \prod_{r=4}^{N} V'_r^{DDF} (Z_r, \bar{Z}_r) \right. \\
\left. \times \prod_{r=1}^{N} \exp \left( i \frac{d-26}{24} \frac{X^+}{p^+_r} \right) \left( z^{(r)}_I, \bar{z}^{(r)}_I \right) \right\rangle \\
\exp \left( i \frac{d-26}{24} \frac{X^+}{p^+_r} \right) \left( z^{(r)}_I, \bar{z}^{(r)}_I \right) \text{ commutes with } T_{X^\pm}, Q_B. \]
BRST invariant form

\[ \mathcal{A} \sim \prod_{r=4}^{N} \int d^2 Z_r \left\langle \prod_{r=1}^{3} (c\bar{c}V_r^{DDF}) (Z_r, \bar{Z}_r) \prod_{r=4}^{N} V_r^{DDF} (Z_r, \bar{Z}_r) \right. \]

\[ \times \prod_{r=1}^{N} \exp \left( i \frac{d - 26}{24} \frac{X^+}{p_r^+} \right) \left( \bar{z}_I^{(r)}, \bar{z}_I^{(r)} \right) \left\rangle \right. \]

This form of the amplitude is BRST invariant.
BRST invariant form

\[ \mathcal{A} \sim \prod_{r=4}^{N} \int d^{2}Z_{r} \left\langle \prod_{r=1}^{3} (c\bar{c}V_{r}^{\text{DDF}}) (Z_{r}, \bar{Z}_{r}) \prod_{r=4}^{N} V_{r}^{\text{DDF}} (Z_{r}, \bar{Z}_{r}) \right. \\
\left. \times \prod_{r=1}^{N} \exp \left( i \frac{d - 26}{24} \frac{X^{+}}{p_{r}^{+}} \right) \left( z_{I}^{(r)}, \bar{z}_{I}^{(r)} \right) \right\rangle \]

The LC gauge SFT in \( d \) dimensions is described by the worldsheet theory

\( X^{i} + \text{ghost} + X^{\pm}\text{CFT} \)
§3 $X^\pm$ CFT (super)

Let us consider the supersymmetric generalization of the results so far.

superspace coordinate

$$z = (z, \theta)$$

superfield

$$X^\pm (z, \bar{z}) = x^\pm + i\theta \psi^\pm + i\bar{\theta} \bar{\psi}^\pm + i\theta \bar{\theta} F^\pm$$

covariant derivative

$$D = \partial_\theta + \theta \partial_z$$

It is convenient to introduce

$$\Theta^+ (z) = \frac{DX^+}{(\partial X^+)^{1/2}} (z)$$

so that the map $z = (z, \theta) \mapsto X^+_L (z) = (X^+_L (z), \Theta^+ (z))$ is a superconformal mapping.
We propose a superconformal field theory with an action

\[ S_{X^\pm} = -\frac{1}{2\pi} \int d^2z \left( \bar{D} X^+ DX^- + \bar{D} X^- DX^+ \right) + \frac{d - 10}{8} \Gamma_{\text{super}}[\Phi] \]

\( \Gamma_{\text{super}}[\Phi] \) is the super Liouville action

\[ \Gamma_{\text{super}}[\Phi] = -\frac{1}{2\pi} \int d^2z \bar{D} \Phi D \Phi \]

\[ \Phi (z, \bar{z}) = \ln \left( -4 (D \Theta^+)^2 (z) (\bar{D} \Theta^+)^2 (\bar{z}) \right) \]

- We calculate the correlation functions starting from this action.
energy-momentum tensor

\[ T_{X^\pm}(z) = \frac{1}{2} DX^+ \partial X^- + \frac{1}{2} DX^- \partial X^+ - \frac{d - 10}{4} S(z, X_L^+) \]

super Schwarzian derivative

\[ S(z, X_L^+) = \frac{D^4 \Theta^+}{D \Theta^+} - 2 \frac{D^3 \Theta^+ D^2 \Theta^+}{(D \Theta^+)^2} \]

From the correlation functions, one can see that the energy-momentum tensor satisfies the Virasoro algebra with \( \hat{c} = 12 - d \).
Correlation functions

Calculations are essentially the same as those in the bosonic case.

\[
\left\langle F \left[ X^+ \right] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r) \right\rangle \\
\sim F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right] \exp \left( -\frac{d - 10}{8} \Gamma_{\text{super}} \left[ \ln \left( (D\xi)^2 (\bar{D}\bar{\xi})^2 \right) \right] \right)
\]

super Mandelstam mapping

\[
\rho (z) = \sum_{r=1}^{N} \alpha_r \ln (z - Z_r)
\]

\[
\xi (z) = \frac{D\rho}{(\partial \rho)^{\frac{1}{2}}} (z)
\]

where

\[
z - z' = z - z' - \bar{\theta} \theta'
\]
Much more complicated compared with the bosonic case.

interaction points $z_I$

\[
\partial \rho(z_I) - \frac{1}{2} \frac{\partial^2 D \rho D \rho}{\partial^2 \rho}(z_I) = 0, \quad \partial D \rho(z_I) - \frac{1}{6} \frac{\partial^3 \rho D \rho}{\partial^2 \rho}(z_I) = 0
\]

$z_I$ is different from the naive generalization $\tilde{z}_I$ satisfying $\partial \rho(\tilde{z}_I) = 0, \partial D \rho(\tilde{z}_I) = 0$. 
Evaluation of $\Gamma_{\text{super}}$

Much more complicated compared with the bosonic case.

This is because of the presence of the odd supermoduli $\xi_I = \frac{D\rho}{(\partial^2 \rho)\frac{1}{4}}(z_I) \neq 0$. 
Evaluation of $\Gamma_{\text{super}}$

Much more complicated compared with the bosonic case.

Although it is very complicated, it is possible to evaluate $\Gamma_{\text{super}}$. (Berkovits, Baba-Murakami-N.I.)
Evaluation of $\Gamma_{\text{super}}$

\[-\Gamma_{\text{super}} = -W_{\text{super}} - \frac{1}{2} \sum_r \tilde{N}^{rr}_{00} \]

\[-\frac{3}{4} \sum_I \ln \left( \partial^2 \rho - \frac{13}{9} \frac{\partial^3 D \rho D \rho}{\partial^2 \rho} + \frac{8}{3} \frac{\partial^3 \rho \partial^2 D \rho D \rho}{(\partial^2 \rho)^2} \right) (\tilde{z}_I) \]

\[+ \text{c.c. .} \]

\[-W_{\text{super}} \equiv \sum_{r>s} \ln (Z_r - Z_s) + \sum_{I>J} P_I P_J \ln (\tilde{z}_I - \tilde{z}_J) \]

\[-\sum_r \sum_I P_I \ln (Z_r - \tilde{z}_I) \]

\[P_I \equiv 1 + \frac{\partial^2 D \rho D \rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \tilde{\partial}_I + \frac{D \rho}{\partial^2 \rho} (\tilde{z}_I) \tilde{\partial}_I \tilde{D}_I \]
Evaluation of $\Gamma_{\text{super}}$

\[-\Gamma_{\text{super}} = -W_{\text{super}} - \frac{1}{2} \sum_r \bar{N}_{00}^{rr} \]

\[-\frac{3}{4} \sum_I \ln \left( \frac{\partial^2 \rho - \frac{13}{9} \frac{\partial^3 \rho}{\partial^2 \rho} D^\rho}{\partial^3 \rho} + \frac{8}{3} \frac{\partial^3 \rho \partial^2 \rho}{(\partial^2 \rho)^2} \right) (\tilde{z}_I) \]

\[+ \text{c.c.} \]

\[\bar{N}_{00}^{rr} \equiv \frac{\rho(\tilde{z}_I^{(r)})}{\alpha_r} - \sum_{s \neq r} \frac{\alpha_s}{\alpha_r} \ln (\mathcal{Z}_r - \mathcal{Z}_s)\]
Energy-momentum tensor

In principle, one can evaluate all the correlation functions starting from $\Gamma_{\text{super}}$. From the correlation functions, we can see

- $T_{X^\pm}$ is regular at the points where no operators are inserted.
- OPE

\[
T_{X^\pm}(z)e^{-ip^+_r X^-}(Z_r, \bar{Z}_r) \\
\sim \frac{1}{z - Z_r} \frac{1}{2} De^{-ip^+_r X^-}(Z_r, \bar{Z}_r) + \frac{\theta - \Theta_r}{z - Z_r} \partial e^{-ip^+_r X^-}(Z_r, \bar{Z}_r)
\]
Energy-momentum tensor

$T_{X^\pm}$ satisfies super Virasoro algebra with $\hat{c} = 12 - d$

$$T_{X^\pm}(z) T_{X^\pm}(z') \sim \frac{12 - d}{4 (z - z')^3} + \frac{\theta - \theta'}{(z - z')^2} \frac{3}{2} T_{X^\pm}(z')$$

$$+ \frac{1}{z - z'} \frac{1}{2} DT_{X^\pm}(z') + \frac{\theta - \theta'}{z - z'} \partial T_{X^\pm}(z')$$

With the ghost superfield $B(z) = \beta(z) + \theta b(z), C(z) = c(z) + \theta \gamma(z)$ and the transverse variables $X^i(z, \bar{z})$, one can construct a nilpotent BRST charge

$$Q_B = \int \frac{dz}{2\pi i} \left[ -C \left( T_{X^\pm} - \frac{1}{2} DX^i \partial X^i \right) + \left( C \partial C - \frac{1}{4} (DC)^2 \right) B \right]$$
§4 Dimensional regularization

Tree amplitudes for superstrings with (NS,NS) external lines

\[ A \sim \int \prod_I d^2 T_I \left\langle \prod_I \left[ T_{\text{F}}^{\text{LC}}(z_I) \tilde{T}_{\text{F}}^{\text{LC}}(\bar{z}_I) \right] \prod_{r=1}^{N} V_r^{\text{LC}} \right\rangle_{X^i,\psi^i} \times \prod_I \left( \partial^2 \rho(z_I) \bar{\partial}^2 \bar{\rho}(\bar{z}_I) \right)^{-\frac{3}{4}} e^{-\frac{d-2}{16} \Gamma[\ln(\partial \rho \bar{\partial} \bar{\rho})]} \]

\[ A \sim \int dT d\theta \]

Diagram:

- \( T_F \) and \( \bar{T}_F \) are Fock states.
- \( T \) is the time parameter.
- \( \theta \) is the angular parameter.
Dimensional regularization

Longitudinal variables and ghosts are introduced and

\[ A_N \sim \left< \prod_{r=1}^{3} \left[ c\tilde{c}e^{-\phi-\bar{\phi}} V^r_{\text{DDF}} (Z_r, \bar{Z}_r) \right] \right. \]

\[ \times \prod_{r=4}^{N} \int d^2 Z_r \prod_{r=4}^{N} e^{-\phi-\bar{\phi}} V^r_{\text{DDF}} (Z_r, \bar{Z}_r) \]

\[ \times \prod_I \left[ X(z_I) \tilde{X}(\bar{z}_I) \right] \prod_{r=1}^{N} e^{\frac{d-10}{16} \frac{i}{p_r^+} X^+} (z_I^{(r)}, \bar{z}_I^{(r)}) \]

- \( X(z) = \{ Q_B, \xi(z) \} \) : the picture changing operator
- \( V^r_{\text{DDF}} =: V^r_{\text{DDF}} e^{-\frac{d-10}{16} \frac{i}{p_r^+} X^+} \) : a superconformal primary with weight \( \left( \frac{1}{2}, \frac{1}{2} \right) \)
Everything is BRST invariant. By standard procedure

\[ A_N \sim \left\langle \prod_{r=1}^{2} \left[ c\tilde{c}e^{-\phi - \tilde{\phi}}V_r^{\text{DDF}} \right] \right. \]

\[ \times \left\{ Q_B, \xi \left\{ Q_B, \tilde{\xi}c\tilde{c}e^{-\phi - \tilde{\phi}}V_3^{\text{DDF}} \right\} \right\} \]

\[ \times \prod_{r=4}^{N} \int d^2Z_r \prod_{r=4}^{N} \left\{ Q_B, \xi \left\{ Q_B, \tilde{\xi}e^{-\phi - \tilde{\phi}}V_3^{\text{DDF}} \right\} \right\} \]

\[ \times \prod_{r=1}^{N} e^{\frac{d-10}{16}} \frac{i}{pr^+} X^+ (z_I^{(r)}, \bar{z}_I^{(r)}) \right\rangle \]

For sufficiently large \(-d\), we do not encounter any divergences.
In this form, the amplitude is not divergent even in the limit $d \to 10$ and we obtain

$$A_N \sim \left< \prod_{r=1}^{2} \left[ c\tilde{c} e^{-\phi - \tilde{\phi}} V_r^{\text{DDF}} \right] \right. $$

$$\times \left\{ Q_B, \xi \left\{ Q_B, \tilde{c}\tilde{c} e^{-\phi - \tilde{\phi}} V_3^{\text{DDF}} \right\} \right\} $$

$$\times \left( \prod_{r=4}^{N} \int d^2 Z_r \left\{ Q_B, \xi \left\{ Q_B, \tilde{c} e^{-\phi - \tilde{\phi}} V_r^{\text{DDF}} \right\} \right\} \right) $$

which coincides with the result of the first quantized formalism. Therefore we get the right answer without adding any contact term interactions as counterterms.
§5 Outlook

- We have invented yet another way to realize string theories in noncritical dimensions.
- Dimensional regularization works without any contact term interactions.
- Ramond sector
- multi-loop amplitudes
- BRST invariant formulation $\alpha = 2p^+$ HKKO, covariantized light-cone,...