ORBITAL STABILITY OF STANDING WAVES FOR FRACTIONAL HARTREE EQUATION WITH UNBOUNDED POTENTIALS

JIAN ZHANG, SHIJUN ZHENG, AND SHIHUI ZHU

Abstract. We prove the existence of the set of ground states in a suitable energy space \( \Sigma_s = \{ u : \int_{\mathbb{R}^N} (\Delta + m^2)^s u + V|u|^2 < \infty \}, \) for the mass-subcritical nonlinear fractional Hartree equation with unbounded potentials. As a consequence we obtain, as a priori result, the orbital stability of the set of standing waves. The main ingredient is the observation that \( \Sigma_s \) is compactly embedded in \( L^2 \). This enables us to apply the concentration compactness argument in the works of Cazenave-Lions and Zhang, namely, relative compactness for any minimizing sequence in the energy space.

1. Introduction

Consider the nonlinear fractional Hartree equation with an unbounded potential in the following form: For \( 0 < s < N/2 \) and \( (t, x) \in \mathbb{R}^{1+N} \)

\[
i u_t = (-\Delta + m^2)^s u + V(x)u - \left( \frac{1}{|x|^\gamma} \ast |u|^2 \right) u
\]

(1.1)

\[
u(0, x) = u_0 \in \Sigma^s,
\]

(1.2)

where \( u = u(t, x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \) is a complex valued function and the convolution \( \ast \) is defined by \( W \ast |u|^2 := \left( \frac{1}{|\cdot|^\gamma} \ast |u|^2 \right)(x) = \int \frac{|u(y)|^2}{|x-y|^\gamma} dy \) with \( 0 < \gamma < \min \{4s, N \} \).

The operator \((-\Delta + m^2)^s\) is defined by

\[
(-\Delta + m^2)^s u = \mathcal{F}^{-1}((|\xi|^2 + m^2)^s \mathcal{F}[u](\xi)),
\]

where \( m \geq 0, \mathcal{F} \) and \( \mathcal{F}^{-1} \) are the Fourier transform and the inverse in \( \mathbb{R}^N \), respectively. Here the potential function \( V \in C^\infty(\mathbb{R}^N) \) is bounded from below \( V(x) \geq -c_0 \) for some \( c_0 > 0 \) and satisfies

\[
V(x) \to \infty \quad \text{as} \quad |x| \to \infty.
\]

(1.3)

Such class of \( V \) includes unbounded potentials arising in physics, for instance, the harmonic potential \( V(x) = |x|^2 \), and more generally, polynomial functions that are bounded from below. Then \( H_{s,V} := (-\Delta + m^2)^s + V \) is essentially self-adjoint in the Hilbert space \( \Sigma^s := \{ v \in L^2 : (-\Delta + m^2)^{s/2} v \in L^2 \text{ and } |V|^{1/2} v \in L^2 \} \), as is given in Section 2.

The Hartree equation is non-integrable analog of the NLS system, which arises naturally in large quantum systems that describe the wave motion for the bosonic or fermionic particles. When \( s = 1 \), it can be derived from the mean-field limit of \( N \)-body GPE hierarchy, where \( W(x) = |x|^{-\gamma} \) represents the long-range two-body interactions.
interaction potential. A special feature of the Hartree equation lies in the convolution kernel $W$ that preserves the fine structure of micro two-body interactions of particles. This is in contrast to the NLS, which can be viewed as the limiting case $W \to \delta$ and where the two-body interactions are modeled from the scattering length. When $s = \frac{1}{2}$, (1.1) arises as an effective description of pseudo-relativistic boson stars in the mean field limit, where $u(t, x)$ is a complex-valued wave field. The study of such a perturbed system in the presence of external potential $V$ is both physically and mathematically very important.

When $V$ is zero or bounded, the dynamical properties of solutions for the fractional Hartree equation (1.1) have been considered in e.g., [16, 8, 13, 1, 24, 7, 12, 25]. In [24, 7], the authors studied the orbital stability for standing wave solutions, that is, ground states (the set of minimizers) for (1.1) with zero potential by profile decomposition method. Furthermore, [24] showed the strong instability in the mass-critical case. The analogous results on orbital stability for nonlinear fractional Schrödinger equation (FNLS) were obtained in [5, 11, 26]. The paper [5] studied the FNLS with power nonlinearity and a decaying potential, while [11, 26] studied FNLS with power nonlinearity in the form $|u|^{p-1}u + |u|^{q-1}u$.

In this paper, we are concerned with the orbital stability of standing wave solutions for (1.1) in the mass-subcritical regime $\gamma < 2s$. Note that when $V = 0$, $\gamma = 2s$ corresponds to the mass-critical case and $\gamma = 4s$ corresponds to the energy-critical case by scaling invariance argument. The presence of an unbounded $V$ brings in technical difficulties and the methods in [5, 11, 24] do not directly apply to (1.1). Motivated by the treatment for the classical NLS in [22, 23], we prove a new compactness lemma in Section 3, adapted to the energy space for (1.1), which leads to the solution to the variational problem 4.1 in Proposition 4.1. Thus follows the main result Theorem 4.3. The proof in Section 4 can be viewed as an adaptation to the potential case, also see [3, 2] for the original treatment of NLS type equations via variational method.

2. Preliminaries

We will use the notations $L^q := L^q(\mathbb{R}^N)$, $\| \cdot \|_q := \| \cdot \|_{L^q(\mathbb{R}^N)}$, $H^s := H^s(\mathbb{R}^N)$ and $\dot{H}^s := \dot{H}^s(\mathbb{R}^N)$, the latter two denoting the usual Sobolev space and its homogeneous version. The various positive constants will be simply denoted by $C$. From now on throughout the paper, without loss of generality we may assume there is some positive constant $c_1$ such that for all $x$,

$$V(x) \geq c_1 > 0 \quad \text{and} \quad V(x) \to \infty \quad \text{as} \ |x| \to \infty.$$  

If otherwise, the lower bound of $V$ is $-c_0 < 0$, then one can always apply the substitution $u \to e^{itC}u$ for any $C \geq c_0 + c_1$ to convert equation (1.1) into one with $V$ bounded from below by a positive constant; see also Remark 4.4.

For $V$ satisfying (2.1) define the energy space $\Sigma^s$ as

$$\Sigma^s := \{v \in L^2 \mid \int |(\nabla - m^2)^s v + V|v|^2 < \infty \}.$$  

Then $\Sigma^s$ is a Hilbert space equipped with the norm

$$\|v\|_{\Sigma^s} := \left( \int |(\nabla - m^2)^s v + V|v|^2 \right)^{\frac{1}{2}},$$
which means that $H_{s,V} = (-\Delta + m^2)^s + V$ is essentially self-adjoint in the quadratic form on $\Sigma^s \times \Sigma^s$ given by

$$q(u,v) := \int (-\Delta + m^2)^{s/2} v (-\Delta + m^2)^{s/2} u + \int \nabla V u.$$ 

Note that $\|v\|_{\Sigma^s} < \infty$ implies $\|v\|_2$ is finite. When $s \in (0,1)$ and $m = 0$, one can also express

$$(-\Delta)^s u = c_{s,N} p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

where $c_{s,N} = \frac{2^{2s}\pi^{s-N}}{|\Gamma(-s)|}$.

Define the energy functional $E : \Sigma^s \to \mathbb{R}$ as

$$(2.4) \quad E(v) = \frac{1}{2} \int \nabla (-\Delta + m^2)^s v + \frac{1}{2} \int V|v|^2 - \frac{1}{4} \int (\frac{1}{|x|^s} * |v|^2)|v|^2.$$ 

From the Hardy inequality (3.1), the functional $E(v)$ is well-defined in $\Sigma^s$. Note that if $V = 0$ and $m = 0$, $\gamma = 4s < N$ corresponds to the energy-critical case due to that the scaling $u \mapsto \frac{2}{(2s-N\rho)^\frac{1}{2}} u(x, \frac{x}{\rho})$ leaves invariant the energy and the solution to (1.1). When $\gamma = 2s < N$, it corresponds to the mass-critical case due to that the scaling $u \mapsto \frac{2}{(2s-N\rho)^\frac{1}{2}} u(x, \frac{x}{\rho})$ leaves invariant the mass $M(u) := \int |u|^2$ and (1.2).

In this paper, we assume that the Cauchy problem (1.1)-(1.2) is well-posed in $\Sigma^s$, namely, the following global in time existence, uniqueness and conservation laws hold in the mass-subcritical regime $\gamma < 2s$.

**Hypothesis 1.** Let $N \geq 1$ and $\frac{N}{2} < s < N/2$. If the initial data $u_0 \in \Sigma^s$, then there exists a unique solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) on $\mathbb{R}$ such that $u \in C(\mathbb{R}; \Sigma^s) \cap C^1(\mathbb{R}; H^s)$. Moreover, for all $t \in \mathbb{R}$, $u(t,x)$ satisfies the following conservation laws:

(i) (mass) \hspace{1cm} $M(u(t)) = M(u_0)$.

(ii) (energy) \hspace{1cm} $E(u(t)) = E(u_0)$.

**Remark 2.1.** For local well-posedness in the above conjectured proposition, so far we only know about numerical result on cubic FNLS with a quadratic potential by Zhang and Kirkpatrick [15], where is suggested the long-time existence along with its dynamics for some special initial data. The theoretical proof has remained an open question concerning FNLS and Hartree equations with harmonic potentials. The main difficulty is the lack of proper dispersive or Strichartz estimates because the fractional Laplacian for $0 < s < 1$ does not hold a control over the harmonic potential, which is shown in the deformed trajectories for the associated hamiltonian, see [12] and [13]. This is in sharp contrast to the classical NLS ($s = 1$) with a harmonic potential, where the existence and stability problem has been studied quite extensively [14, 17, 18, 22, 23, 13, 6].

Heuristically the Laplacian $-\Delta$ and $V = |x|^2$ have balanced strength or effect so the $L^1 \to L^\infty$ time decay $t^{-N/2}$ holds locally for $e^{it(-\Delta-V)}$. However, for $0 < s < 1$, in the phase space the bound energy for fractional laplacian $(-\Delta + m^2)^s$ is less than $|x|^2$, or, $H_{s,0}$ relative to $V$ is like the laplacian vs. anharmonic potential of higher order. It falls within the quantum situation “a particle at a higher altitude
falls down to the bottom of the potential in a shorter time than one at a lower altitude”, which obstructs Fujiwara’s theorem, cf. [17].

3. Proof of main result

For estimating the Hartree nonlinearity we will need Hardy’s inequality, see e.g., [20]: If $s \in (0, N/2)$

$$\sup_n \int \frac{|u(y)|^2}{|x - y|^{2s}} dy \leq c(s, N)||u||_{H^s}^2.$$  (3.1)

**Lemma 3.1.** Let $v \in H^s$ and $0 < s < N/2$. If $0 < \gamma < 2s$, then there exists a positive constant $C > 0$ such that

$$\int \left(\frac{1}{|x|^\gamma} \ast |v|^2\right)|v|^2 dx \leq C\|v\|_{H^s}^2 \|v\|_{L^\infty}^{2-\gamma}.$$  (3.2)

**Proof.** Note that

$$\int \left(\frac{1}{|x|^\gamma} \ast |v|^2\right)|v|^2 dx = \int \int \frac{|v(x)|^2}{|x - y|^{2s}} |v(y)|^2 dx dy \leq \| \int \frac{|v(x)|^2}{|x - y|^{2s}} dx \|_{L^\infty} ||v(y)||_{L^2}^2.$$  (3.3)

Using Hölder inequality with $1 = \frac{\gamma}{2s} + \frac{2s - \gamma}{2s}$, we obtain

$$\int \frac{|v(x)|^2}{|x - y|^{\gamma}} dx = \int \frac{|v(x)|^{\frac{\gamma}{2s}}}{|x - y|^{\gamma - \frac{\gamma}{2s}}} \cdot \frac{|v(x)|^{\frac{2s - \gamma}{2s}}}{|x - y|^{\frac{2s - \gamma}{2s}}} dx \leq C\|x - y|^{-\frac{\gamma}{2s}}v(x)||^2 \|v\|_{L^\infty}^{2-\gamma} \leq C\|v\|_{H^s}^2 \|v\|_{L^\infty}^{2-\gamma}.$$  (3.4)

In the last step of the above, we have employed (3.1). Thus, (3.2) follows from (3.3) and (3.4).

**Lemma 3.2.** Let $0 < \gamma < 2s$, $0 < s < N/2$ and $V$ satisfy (2.1). Suppose $\{v_n\}_{n=1}^\infty$ converges weakly to $U$ in $\Sigma^s$. Then there exists a subsequence (still denoted by $\{v_n\}$) such that

$$\|v_n\|_{L^2}^2 \to \|U\|_{L^2}^2 \quad \text{as} \quad n \to \infty,$$

and

$$\iint \frac{|v_n(x)|^2|v_n(y)|^2}{|x - y|^{\gamma}} dx dy \to \iint \frac{|U(x)|^2|U(y)|^2}{|x - y|^{\gamma}} dx dy \quad \text{as} \quad n \to \infty.$$  (3.5)

**Remark 3.3.** Equation (3.5) indeed implies that for $s > 0$, $\Sigma^s$ is compactly embedded in $L^2$.

**Proof.** Since $\Sigma^s \subset L^2$, it is easy to see that for $U \in \Sigma^s \cap L^2$, $v_n \to U$ weakly in $\Sigma^s$ and

$$v_n \to U \quad \text{weakly in} \quad L^2$$

as $n \to \infty$. For an elementary proof of (3.7), see Proposition 5.1 in the Appendix.

Since $\{v_n\}$ is weakly convergent in $\Sigma^s$, $\|v_n\|_{\Sigma^s}$ is uniformly bounded. So there is a positive constant $K$ such that

$$\sup_n \int V(x)|v_n(x)|^2 dx \leq \sup_n \|v_n\|_{\Sigma^s}^2 < K.$$  (3.8)
Furthermore, we will show that there is a subsequence of \( \{v_n\} \) that strongly converges in \( L^2 \) and satisfies (3.10).

(1) First, we consider the case where \( U = 0 \). From (3.8) with \( V \) satisfying (2.1), we have for arbitrary \( \epsilon > 0 \), there exists a constant \( B = B_\epsilon > 0 \) (large enough) such that
\[
\frac{1}{V(x)} \leq \epsilon \quad \text{when} \quad |x| > B.
\]
We see that
\[
\int_{|x| > B} |v_n|^2 \, dx = \int_{|x| > B} \frac{1}{V(x)} |v_n|^2 \, dx \leq K \epsilon.
\]
For the fixed \( B \), compact embedding property for Sobolev space on bounded domain gives that one can extract a subsequence (still denoted by \( \{v_n\} \)) such that
\[
v_n \to 0 \quad \text{strongly in} \quad L^2(\{|x| \leq B\}).
\]
Hence, there exists \( L = L_\epsilon \) such that for all \( n > L \), \( \int_{|x| \leq B} |v_n|^2 \, dx \leq \epsilon \). Thus, we have, if \( n > L \),
\[
\int |v_n|^2 \, dx = \int_{|x| \leq B} |v_n|^2 \, dx + \int_{|x| > B} |v_n|^2 \, dx \leq (K + 1) \epsilon.
\]
Now, taking \( \epsilon = \epsilon_k \to 0 \) as \( k \to \infty \), a standard diagonal argument shows that there exists a subsequence of \( \{v_n\} \), still denoted by \( \{v_n\} \), such that (3.6) is true in the case \( U = 0 \). Then, it follows from (3.2) in Lemma 3.1 that
\[
\int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x - y|^{\gamma}} \, dx \, dy \leq C \|v_n\|_{H^\gamma} \|v_n\|_{2^{-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.
\]
This proves (3.6) when \( U = 0 \).

(2) Secondly, we consider the case \( U \neq 0 \). Take \( w_n = v_n - U \). We have \( w_n \rightharpoonup 0 \) weakly in \( \Sigma^\ast \) and in \( L^2 \) as \( n \to \infty \). Then, from the above discussion, we see that there is a subsequence (still denoted by \( \{w_n\} \)) such that \( w_n \to 0 \) strongly in \( L^2 \). In other words, we have
\[
\|v_n - U\|_2 \to 0 \quad \text{as} \quad n \to \infty.
\]
To prove (3.6), we will apply (3.4). Indeed, if \( 0 < \gamma < 2s \), we deduce that
\[
\int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x - y|^{\gamma}} \, dx \, dy - \int \int \frac{|U(x)|^2 |U(y)|^2}{|x - y|^{\gamma}} \, dx \, dy \leq \\
\int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x - y|^{\gamma}} \, dx \, dy - \int \int \frac{|v_n(x)|^2 |U(y)|^2}{|x - y|^{\gamma}} \, dx \, dy \\
+ \int \int \frac{|v_n(x)|^2 |U(y)|^2}{|x - y|^{\gamma}} \, dx \, dy - \int \int \frac{|U(x)|^2 |U(y)|^2}{|x - y|^{\gamma}} \, dx \, dy \\
\leq \|v_n\|^2_{2^{-\gamma}} \|v_n\|_{2^{-\gamma}} \|v_n\|_{2^{-\gamma}} \|v_n\|_{2^{-\gamma}} \|u_n\|_{2^{-\gamma}} \|u_n\|_{2^{-\gamma}} \|U\|_{2^{-\gamma}} \|U\|_{2^{-\gamma}} \\
\leq C \|v_n\|^2_{H^\gamma} \|v_n\|_{2^{-\gamma}} \|v_n\|_{2^{-\gamma}} \|U\|_{2^{-\gamma}} \|U\|_{2^{-\gamma}} \|v_n\|_{2^{-\gamma}} \|U\|_{2^{-\gamma}} \\
\leq C \|v_n\|^2_{2} \|U\|_{2} \to 0 \quad \text{as} \quad n \to \infty.
\]
Therefore, (3.6) is true for \( U \neq 0 \), which completes the proof. \( \square \)
4. Orbital Stability of Standing Waves

Given \(0 < \gamma < 2s, s \in (0, N/2), N \geq 1,\) let \(m \geq 0\) and \(M > 0.\) Consider the following variational problem

\[
d_M := \inf_{\{v \in \Sigma^s \mid \|v\|^2_{\Sigma^s} = M\}} E(v),
\]

where \(E(v) = \frac{1}{2}\|v\|^2_{\Sigma^s} - \frac{1}{4} \int \frac{\ln|v|^2|v|^2}{|x-y|^s}dxdy,\) as defined in (4.1). The following proposition constructs a minimizer, called ground state, to the problem (4.1).

**Proposition 4.1.** Let \(M > 0.\) Suppose \(0 < \gamma < 2s\) and \(V\) satisfies (2.1), then the infimum in the variational problem can be attained. That is, there exists \(U \in \Sigma^s\) such that

\[
E(U) = d_M = \min_{\{v \in \Sigma^s \mid \|v\|^2_{\Sigma^s} = M\}} E(v).
\]

Moreover, any minimizing sequence to (4.1) must be relatively compact in \(\Sigma^s.\)

**Proof.** Firstly, we prove the variational problem (4.1) is well-defined. That is, \(E(v)\) has a lower bound in \(\{v \in \Sigma^s \mid \|v\|^2_{\Sigma^s} = M\}.\) From Lemma 3.1 we deduce that

\[
E(v) \geq \frac{1}{2} \int (|\xi|^2 + m^2)|\tilde{v}|^2d\xi + \frac{1}{2} \int V|v|^2dx - C\|v\|_{\dot{H}^s}^{\gamma/2s} \|v\|_{\Sigma^s}^{4s-\gamma}
\]

\[
\geq \frac{1}{2} \int |\xi|^2|\tilde{v}|^2d\xi - C_M M \left( \int |\xi|^2|\tilde{v}|^2d\xi \right)^{\frac{s}{2s}}
\]

\[
\geq -C_M.
\]

In the last step, we have used the elementary inequality \(\frac{1}{2}X - C_M X^{\gamma/2s} \geq -C_M\) for all \(X > 0\) and some constants \(C_M > 0.\) Thus, \(E(v)\) is bounded from below and the infimum \(d_M\) exists.

Secondly, take any minimizing sequence \(\{v_n\}\) of Problem (4.1) satisfying

\[
E(v_n) \to d_M \quad \text{and} \quad \|v_n\|^2_{\Sigma^s} \to M, \quad n \to \infty.
\]

We see that there exists an \(L\) such that for all \(n \geq L,\)

\[
E(v_n) < d_M + 1.
\]

This, together with (4.3) implies for all \(n \geq L,\)

\[
\frac{1}{2}\|v_n\|^2_{\Sigma^s} \leq C_M \|v_n\|^\gamma/2s_{\dot{H}^s} + d_M + 1.
\]

Hence, \(\{v_n\}\) must be bounded in \(\Sigma^s\) by virtue of the condition \(\frac{\gamma}{s} < 2.\)

Thirdly, from the boundedness of \(\{v_n\},\) we know there exists a subsequence (still denoted by \(\{v_n\}\)) and \(U \in \Sigma^s\) such that

\[
v_n(x) \to U(x) \quad \text{weakly in} \quad \Sigma^s.
\]

By the lower semi-continuity of norm \(\Sigma^s,\) we have

\[
\|U\|^2_{\Sigma^s} \leq \lim inf_{n \to \infty} \|v_n\|^2_{\Sigma^s}.
\]

Combining (4.8), Lemma 5.2 and (4.4) we obtain that \(\|v_n\|^2 \to \|U\|^2 = M\) and

\[
E(U) \leq \lim inf_{n \to \infty} E(v_n) = d_M.
\]
But from the definition of \( d_M \), we must have \( E(U) = d_M \). That is, \( U \) is a minimizer of (4.1). To prove the statement on relative compactness, observe that the last argument shows

\[
E(U) = \lim_{n \to \infty} E(v_n).
\]

This, along with Lemma 3.2 implies there exists a subsequence \( \{v_{n_k}\} \) such that \( \lim_{k \to \infty} \|v_{n_k}\|_{\Sigma^s} = \|U\|_{\Sigma^s} \). Therefore, in view of (4.7), the strong convergence \( v_{n_k} \to U \) in \( \Sigma^s \) follows. \( \square \)

**Remark 4.2.** The existence of ground states for FNLS with unbounded \( V \) was obtained in [4] via Nehari’s manifold approach. The existence and symmetry of ground state solutions were studied in [21] for Hartree equation with zero potential.

Define the set

\[
S_M := \{ v \in \Sigma^s \mid v \text{ is the minimizer of Problem (4.1)} \}.
\]

From the Euler-Lagrange Theorem (see [2]), for any \( v \in S_M \subset \Sigma^s \), there exists \( \omega \in \mathbb{R} \) such that

\[
(-\Delta + m^2)^s v + V(x)v + \omega v - \left( \frac{1}{|x|^\gamma} \right)^s |v|^2 v = 0.
\]

Moreover, \( u(t, x) = e^{i\omega t}v(x) \) is a standing wave solution for (1.1). Thus, each \( v \) in \( S_M \) is called an orbit. It is easy to check that for any \( t > 0 \), if \( v \in S_M \), then \( e^{i\omega t}v(x) \in S_M \). Applying Proposition 4.1 we now prove the following orbital stability for (1.1). More precisely, assuming Hypothesis 1, we show that if the initial data is close to an orbit \( v \in S_M \), then the solution of evolution system (1.1)-(1.2) remains close to \( S_M \), the set of ground states, for all time.

**Theorem 4.3** (orbital stability of standing waves). Let \( M > 0 \) and \( V \) satisfy (2.1). Let \( 0 < s < N/2 \) and \( 0 < \gamma < 2s \). Then, for arbitrary \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if the initial data \( u_0 \) in \( \Sigma^s \) satisfies

\[
\inf_{v \in S_M} \|u_0 - v\|_{\Sigma^s} < \delta,
\]

it holds for the solution \( u \) of the Cauchy problem (1.1)-(1.2),

\[
\inf_{v \in S_M} \|u(t, x) - v(x)\|_{\Sigma^s} < \varepsilon
\]

for all \( t \), where \( S_M \) is defined in (4.9).

**Proof.** We prove the theorem by contradiction following the standard method for NLS (see [2] [22] [23]). Let \( u_0 \in \Sigma^s \) and \( u \) be the unique solution for (1.1). From the proof of Proposition 4.1 and the conservation laws in Hypothesis 1 we obtain that for all \( t \),

\[
\frac{1}{2} \|u(t, \cdot)\|_{\Sigma^s}^2 \leq E(u_0) + C(\|u_0\|_2)\|u(t, \cdot)\|_{\Sigma^s}^\gamma.
\]

This suggests that the \( \Sigma^s \)-norm of \( u \) is uniformly bounded for all \( t \).

Assume that the conclusion in the theorem is false, then there exist \( \varepsilon_0 > 0 \) and a sequence of initial data \( \{u_{0, n}\}_{n=1}^\infty \) such that

\[
\inf_{v \in S_M} \|u_{0, n} - v\|_{\Sigma^s} < \frac{1}{n}.
\]
and there exists a sequence of time \( \{t_n\}_{n=1}^{\infty} \) such that for all \( n \),
\begin{equation}
\inf_{v \in \mathcal{S}_M} \|u_n(t_n, x) - v\|_{\Sigma^s} \geq \varepsilon_0.
\end{equation}
But from (4.13), the conservation laws and Lemma 3.2 it follows that there exist \( w \in \Sigma^s \) and a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)) such that as \( n \to \infty \)
\[ \|u_n(t_n, x)\|_2^2 = \|u_{0,n}\|_2^2 \to \|w\|_2^2 = M \]
and
\[ E(u_n(t_n, x)) = E(u_{0,n}) \to E(w) = d_M. \]
Hence, \( \{u_n(t_n, \cdot)\}_{n=1}^{\infty} \) is a minimizing sequence of (4.1). According to Proposition 4.1 \( w \in \mathcal{S}_M \) is a minimizer such that, when passing to a subsequence if necessary,
\begin{equation}
\|u_n(t_n, x) - w(x)\|_{\Sigma^s} \to 0 \text{ as } n \to \infty.
\end{equation}
This contradicts (4.14), which proves the theorem.
\[ \square \]

Remark 4.4. The results in Theorem 4.3 and Proposition 4.1 continue to hold if \( V \) satisfies the slightly more general condition (1.3). To see this, it suffices to observe the following properties. Let \( E_V(u) := E(u) = \frac{1}{2}\|u\|_{\Sigma^s}^2 - \frac{1}{4} \int \int \frac{|u(x)|^2 |u(y)|^2}{|x-y|^s} \, dx \, dy \) be given as in (2.4) and \( \Sigma_V := \Sigma^s \) as in (2.2). Let \( C \geq c_0 + c_1 \) be a fixed constant. Then we have:
\begin{enumerate}
\item \( M(e^{i\theta}u) = M(u) \)
\item \( E_V(e^{i\theta}u) = E_V(u) \)
\item \( e^{i\theta} \mathcal{S}_M = \mathcal{S}_M \)
\item \( E_{V+C}(u) = E_V(u) + C M(u) \)
\item The set of minimizers \( \mathcal{S}_M \) is independent of \( V + C \)
\item Denote by \( u_V \) the solution of (1.1)-(1.2), then we have \( u_{V+C}(t, x) = e^{-itC}u_V(t, x) \).
\end{enumerate}

Remark 4.5. The case \( s = 1, \gamma = 2 \) (mass-critical) was studied in [14], where the threshold for the stability of standing waves for (1.1) are obtained. Earlier result on the l.w.p. and mass and energy conservation laws for (1.1) can be found in [2] for \( s = 1 \), and \( s \leq 2 \). Let \( Q_{0} \) be the unique radial positive ground state solution of (1.1), where \( V = 0 \). The paper [13] Theorem 4.1 followed a quite standard variational approach and proved that if \( s = 1, \gamma = 2 \) and \( M < \|Q_0\|_{\Sigma^s}^2 \), then there exist ground state solutions of the minimization problem (4.1), where \( V = |x|^2 \). Moreover, these ground state solutions are orbitally stable.

In the absence of \( V \), when \( \gamma = 2s \) \((0 < s < 1 \text{ and } N \geq 2)\), Zhang and Zhu [24] proved the orbital stability via profile decomposition method. They also showed strong instability of (1.1) by constructing blowup solutions with initial data arbitrarily close to \( Q_0 \) in \( H^s \).

5. Appendix: Uniqueness of weak convergence

For the proof of Lemma 3.2 we need the following.

Proposition 5.1. Let \( s > 0 \) and \( \Sigma^s \subset L^2 \) be defined as in (2.2) with \( V \) satisfying (2.1). Let \( \{v_n\} \) be a sequence in \( \Sigma^s \). If \( \{v_n\} \) converges weakly to \( f \) in \( \Sigma^s \) and \( \{v_n\} \) converges weakly to \( g \) in \( L^2 \). Then \( f \) is identical to \( g \) in \( \Sigma^s \cap L^2 \).
The domain of $H = H_{s,V}$ is given by

$$D(H) = \{ \phi \in L^2 : (-\Delta + m^2)^s \phi \in L^2, V\phi \in L^2 \}.$$ 

It is easy to verify that $D(H)$ is a complete Hilbert subspace of $L^2$ with respect to the norm

$$\|\phi\|_{D(H)} := (\|(-\Delta + m^2)^s \phi\|_2^2 + \|V\phi\|_2^2)^{1/2}.$$ 

The form domain of $H$ is defined as $v \in Q(H) \iff v \in L^2(\mathbb{R}^N)$ and

$$(-\Delta + m^2)^{s/2} v \in L^2 \quad \text{and} \quad V^{1/2} v \in L^2,$$

which is complete with respect to the norm $\sqrt{q(v,v)} = \|v\|_{\Sigma^*}$ given in (2.3). We have $D(H) \subset Q(H)$. The form definition for $H$ is equivalent to the following weak form definition.

**Definition 5.2.** A function $f \in \Sigma^* = H^s \cap D(\sqrt{V})$ is in $Q(H)$ if and only if for all $\phi \in H^s(\mathbb{R}^N)$, there exists an $h \in L^2 \cap H^{-s}$ s.t.

$$\int_{\mathbb{R}^N} f(x) H \phi \, dx = \int_{\mathbb{R}^N} h(x) \phi(x) \, dx \quad \forall \phi \in H^s$$

$$= \int f(x)(-\Delta + m^2)^s \phi \, dx + \int f(x)V\phi \, dx.$$ 

We see that $Q(H) = D(H) = H^{2s} \cap D(V)$. Since on $D(H)$ it holds

$$((-\Delta + m^2)^s \phi, \phi) = (V\phi, \phi) \geq (m^{2s} + c_1) \langle \phi, \phi \rangle,$$

we obtain that the spectrum $\sigma(H) \subset [c_1, \infty)$, which implies 0 is in the resolvent set for $H$. Hence $H^{-1} = (H - 0)^{-1} : L^2 \to D(H) \subset \Sigma^*$ exists and is a continuous mapping.

**Proof of Proposition 5.1.** For all $\phi \in D(H) \subset \Sigma^* \subset L^2$, $s > 0$, the inner product on $\Sigma^*$

$$\langle f, \phi \rangle_{\Sigma^*} = \lim_n \langle v_n, \phi \rangle_{\Sigma^*} = \lim_n (\langle (-\Delta + m^2)^{s/2} v_n, (-\Delta + m^2)^{s/2} \phi \rangle_{L^2} + \langle V^{1/2} v_n, V^{1/2} \phi \rangle_{L^2} = \lim_n q(v_n, \phi) = \lim_n (v_n, H \phi)_{L^2} = \langle g, H \phi \rangle_{2},$$

where $H = H_{s,V}$ is a positive self-adjoint operator in $L^2$, cf. (2.3) and the quadratic form defined there. On the other hand, observe that

$$\langle f, \phi \rangle_{\Sigma^*} = \langle (-\Delta + m^2)^{s/2} f, (-\Delta + m^2)^{s/2} \phi \rangle_{L^2} + \langle V^{1/2} v_n, V^{1/2} \phi \rangle_{L^2} = \langle f, H \phi \rangle_{2}.$$ 

Now as functions (or distributions) it follows that $f = g$ in $L^2$ by taking $\phi = H^{-1} \varphi$ for all $\varphi \in L^2$. This proves Proposition 5.1.

**Remark 5.3.** The proof here relies on the fact that the spectrum of $H_{s,V}$ is bounded below by zero, whence one sees that $H_V = (-\Delta + m^2)^s + V$ has an inverse that continuously maps $L^2$ onto $D(H)$. 


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(Jian Zhang) School of Mathematical Science, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, China

E-mail address: zhangjiancdv@sina.com

(Shijun Zheng) Department of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia 30460-8093, USA

E-mail address: szheng@GeorgiaSouthern.edu

(Shihui Zhu) Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan, 610066, China.

E-mail address: shihuizhumath@163.com