Weight filtration on the cohomology of complex analytic spaces

Joana Cirici and Francisco Guillén

Abstract. We extend Deligne’s weight filtration to the integer cohomology of complex analytic spaces (endowed with an equivalence class of compactifications). In general, the weight filtration that we obtain is not part of a mixed Hodge structure. Our purely geometric proof is based on cubical descent for resolution of singularities and Poincaré-Verdier duality. Using similar techniques, we introduce the singularity filtration on the cohomology of compactifiable analytic spaces. This is a new and natural analytic invariant which does not depend on the equivalence class of compactifications and is related to the weight filtration.

Introduction

The weight filtration was introduced by Deligne [Del71], [Del74] following Grothendieck’s yoga of weights, and as the key ingredient of mixed Hodge theory. This is an increasing filtration defined functorially on the rational cohomology of every complex algebraic variety, expressing the way in which its cohomology is related to cohomologies of smooth projective varieties. In Deligne’s approach, the weight filtration on the cohomology of a singular complex algebraic variety $X$, supposed compact to simplify, is the filtration induced by a smooth hypercovering $X_\bullet \to X$ of $X$. Indeed, the induced spectral sequence
\[ E_1^{p,q}(X; A) = H^q(X_p; A) \Rightarrow H^{p+q}(X; A) \]
defines a filtration on $H^n(X; A)$ for any given coefficient ring $A$. Using Hodge theory, Deligne proved that when $A = \mathbb{Q}$, the above spectral sequence degenerates at the second stage and that the filtration on the rational cohomology is well defined and functorial.

In [GS96], Gillet and Soulé gave an alternative proof of the well-definedness of the weight filtration using smooth hypercoverings and algebraic K-theory. Their more geometric approach allowed them to obtain the result with integer coefficients, at least for compact support cohomology. For a general coefficient ring $A$, the above spectral sequence does not necessarily degenerate at the second stage. However, they proved that, from the second stage onwards, the corresponding spectral sequence for cohomology with compact supports is a well defined algebraic invariant of the variety. An analogous construction yielded a weight filtration on algebraic K-theory with compact supports (see also the work of Pascual-Rubió [PR09]).

Based on cubical hypercoverings (see [GNPP88]), Guillén and Navarro-Aznar [GN02] developed a general descent theory which allows to extend contravariant functors compatible with elementary acyclic squares (smooth blow-ups) on the category of smooth schemes, to functors on the category of all schemes in such a way that the extended functor is compatible with acyclic squares (abstract blow-ups).

Totaro observed in his ICM lecture [Tot02], that using the results on cohomological descent of [GN02], the weight filtration is well defined on the cohomology with compact supports of any complex or real analytic space with a given equivalence class of compactifications, for which Hironaka’s resolution of singularities holds. Following this idea, and using Poincaré duality for
real manifolds with $\mathbb{Z}_2$-coefficients, McCrory and Parusinski obtained the weight filtration for real algebraic varieties on Borel-Moore $\mathbb{Z}_2$-homology in [MP11], and on compactly supported $\mathbb{Z}_2$-homology in [MP12]. Similar results for compactly supported $\mathbb{Z}_2$-cohomology of real varieties appear in [LP14], where Limoges and Priziac prove that products in cohomology are compatible with the weight filtration.

Let $X$ be a (compactifiable) complex analytic space. Every cubical hyperresolution $X_\bullet \to X$ induces a spectral sequence

$$E_1^{p,q}(X; A) = \bigoplus_{|\alpha|=p} H^q(X_\alpha; A) \Rightarrow H^{p+q}(X; A)$$

converging to a filtration of $H^n(X; A)$, which we call singularity filtration. In this note we prove that from the $E_2$-term onwards, this spectral sequence does not depend on the choice of the hyperresolution (Theorem 4.1). This result is a corollary of the cohomological descent of [GN02] and Poincaré-Verdier duality. In the same vein, in Theorem 5.1 we obtain a generalization of the weight filtration on the cohomology $H^n(X; A)$, when $X$ is endowed with an equivalence class of compactifications (see Definition 2.8). In particular, if $X$ is a complex algebraic variety, the weight filtration is well-defined on $H(X; A)$. For smooth manifolds, the singularity filtration is trivial, while for compact analytic spaces, it coincides with the weight filtration.

To obtain these results we give an analytic version of the extension criterion of functors of [GN02] (see Theorem 2.3). The analytic setting differs from the algebraic setting appearing in loc.cit., mainly due to the weaker formulation of Chow-Hironaka’s Lemma and certain finiteness issues, and it may find applications to study other topological invariants. For instance, it should allow to define a Hodge and a weight filtration on the rational homotopy type of complex analytic spaces, extending the filtrations obtained by Morgan [Mor78], to the analytic setting. We will present this multiplicative theory elsewhere.

It is easy to see that the previous results are also valid for the cohomology with $\mathbb{Z}_2$ coefficients of real algebraic varieties. Other cohomology theories such as Borel-Moore homology or cohomology with compact supports can also be studied using parallel techniques, allowing to recover the quoted results of Gillet-Soulé and McCrory-Parusinski, and the results announced by Totaro concerning the weight filtration.

In Section 1 we show that the category of filtered complexes over an abelian category admits a cohomological descent structure with respect to the class of weak equivalences given by $E_r$-quasi-isomorphisms: morphisms of filtered complexes inducing a quasi-isomorphism at the $r$-stage of the associated spectral sequence (see Theorem 1.13). In Section 2 we establish an extension criterion of functors from smooth to singular complex analytic spaces (Theorem 2.3) as well as a relative version (Theorem 2.9). In Section 3 we study the behavior of the cohomology functor with respect to acyclic squares of analytic spaces. We then study the Gysin complex of a smooth compactification $U \hookrightarrow X$ with $D = X - U$ a normal crossings divisor.

In Sections 4 and 5 we use the results of the previous sections to define the singularity and weight filtrations respectively, on the cohomology with coefficients in an arbitrary ring $A$, of compactifiable complex analytic spaces (Theorem 4.1 and Theorem 5.1).

1. Cohomological descent structures on the category of filtered complexes

The extension criterion of functors of [GN02] is based on the assumption that the target category is a cohomological descent category, a variant of the triangulated categories of Verdier. This is essentially a category $\mathcal{D}$ endowed with a saturated class of weak equivalences $\mathcal{E}$, and a simple functor $s$ sending every cubical codiagram of $\mathcal{D}$ to an object of $\mathcal{D}$ and satisfying certain axioms analogous to those of the total complex of a double complex. The simple functor can be viewed as the homotopy limit, and allows to define realizable homotopy limits for diagrams indexed by finite categories (see [Rod12]).
In this section we show that the category of filtered complexes over an abelian category admits a cohomological descent structure, where the weak equivalences are given by \(E_r\)-quasi-isomorphisms.

We first recall some features of cubical codiagrams and cohomological descent categories. We refer to [GN02] for the precise definitions.

**Definition 1.2.** Let \(\delta : \square \to \square'\) be a morphism of \(\Pi\). The inverse image of \(\delta\) is the functor \(\delta^* : Fun(\square',D) \to Fun(\square,D)\) defined by \(\delta^*(F) := F \circ \delta\).

**Definition 1.3.** Let \(D\) be an arbitrary category. A cubical codiagram of \(D\) is a pair \((X,\square)\), where \(\square\) is an object of \(\Pi\) and \(X\) is a functor \(X : \square \to D\). A morphism \((X,\square) \to (Y,\square')\) between cubical codiagrams is given by a pair \((a,\delta)\) where \(\delta : \square' \to \square\) is a morphism of \(\Pi\) and \(a : \delta^*X \to Y\) is a natural transformation.

Denote by \(CoDiag_{\Pi}D\) the category of cubical codiagrams of \(D\).

**Definition 1.4.** A cohomological descent category is given by a cartesian category \(D\) provided with an initial object 1, together with a saturated class of morphisms \(E\) of \(D\) which is stable by products, called weak equivalences, and a contravariant functor \(s : CoDiag_{\Pi}D \to D\), called the simple functor. The data \((D,E,s)\) must satisfy the axioms of Definition 1.5.3 of [GN02]. Objects weakly equivalent to the initial object 1 are called acyclic.

**1.5 (\(\Phi\)-rectified functors).** If \(D\) is a cohomological descent category, then for each cubical index category \(\square \in \Pi\), the simple functor induces a functor \(\text{Ho}(D^\square) \to \text{Ho}(D)\). In certain situations, we are interested in cubical diagrams in \(\text{Ho}(D)\). In general we do not have a simple functor \(\text{Ho}(D)^{\square} \to \text{Ho}(D)\). The notion of \(\Phi\)-rectified functor corresponds, roughly speaking, to functors \(F : C \to \text{Ho}(D)\) which are defined on all cubical diagrams in the form \(F^{\square} : C^\square \to \text{Ho}(D^\square)\), so that we can take the composition \(C^\square \to \text{Ho}(D^\square) \to \text{Ho}(D)\) (see 1.6 of [GN02]). For our purposes, it suffices to note that every functor \(F : C \to D\) induces a \(\Phi\)-rectified functor \(F : C \to \text{Ho}(D)\).

Let \(A\) be an abelian category. The primary example of a cohomological descent category is given by the category of complexes \(C^+(A)\), with weak equivalences being quasi-isomorphisms and the simple functor \(s\) defined via the total complex. We adapt this structure to filtered complexes and obtain a family of cohomological descent structures.

Denote by \(\mathbf{F}A\) the category of filtered objects of \(A\), and by \(C^+(\mathbf{F}A)\) the category of complexes over objects of \(\mathbf{F}A\).

**Definition 1.6.** Let \(r \geq 0\) be an integer. A morphism of filtered complexes \(f : K \to L\) is called \(E_r\)-quasi-isomorphism if the induced morphism \(E_r(f) : E_r(K) \to E_r(L)\) of the associated spectral sequences is a quasi-isomorphism (the map \(E_{r+1}(f)\) is an isomorphism).

Denote by \(E_r\) the class of \(E_r\)-quasi-isomorphisms.

**Definition 1.7.** The \(r\)-derived category of \(\mathbf{F}A\) is the localized category
\[
\mathbf{D}^+_r(\mathbf{F}A) := C^+(\mathbf{F}A)[E^{-1}_r]
\]
of filtered complexes of \(\mathbf{F}A\) with respect to the class of \(E_r\)-quasi-isomorphisms.

Let \(s > r\) and consider the functor \(E_s : C^+(\mathbf{F}A) \to C^+(A)\) defined by sending a filtered complex to the \(E_s\)-stage of its associated spectral sequence. Since it sends morphisms of \(E_r\) to isomorphisms, there is a functor \(E_s : \mathbf{D}^+_r(\mathbf{F}A) \to C^+(A)\).
Deligne introduced the décalage of a filtered complex and proved that its associated spectral sequences are related by a shift of indexing. This proves to be a key tool in the study of filtered complexes and their cohomological descent properties.

**Definition 1.8.** The décalage $\text{Dec}K$ of a filtered complex $K$ is the filtered complex defined by $(\text{Dec}W)_pK^n = W_{p-n}K^n \cap d^{-1}(W_{p-n-1}K^{n-1})$.

**Proposition 1.9** ([Del71], Prop. 1.3.4). The canonical morphism $E_0^{p,n-p}(\text{Dec}K) \to E_0^{p+n-n-p}(K)$ is a quasi-isomorphism. For all $r > 0$, the induced morphism $E_r^{p,n-p}(\text{Dec}K) \to E_r^{p+n-n-p}(K)$ is an isomorphism. In particular $\mathcal{E}_r = \text{Dec}^{-1}(\mathcal{E}_{r-1})$ for all $r > 0$.

**Definition 1.10.** The $r$-simple of a codiagram of filtered complexes $K = (K, W^\bullet)$ is the filtered complex $s^r(K) := (s(K), W(r))$ defined by

$$W(r)_p(s(K)) = \int_\alpha C^*(\Delta^{[\alpha]}) \otimes W_{p+r|\alpha}K^\alpha = \bigoplus_{|\alpha|=0} W_pK^\alpha \oplus \bigoplus_{|\alpha|=1} W_{p+r}K^\alpha[-1] \oplus \cdots$$

Note that $s^0$ and $s^1$ correspond to the filtered total complexes defined via the convolution with the trivial and the bête filtrations respectively, introduced by Deligne in [Del74]. By forgetting the filtrations on $s^r$ we recover the simple functor $s$ on complexes.

**Proposition 1.11.** Let $K$ be a codiagram of filtered complexes. Then for $r \geq 0$,

$$\text{Dec}(s^{r+1}(K)) \cong s^r(\text{Dec}K).$$

**Proof.** The category $\mathbb{C}^{+}(\mathbb{F}, \mathcal{A})$ is complete. Furthermore, since the décalage has a left adjoint defined by the shift of a filtration (see [CG13]), it commutes with pull-backs. It also commutes with $r$-translations $(K, W) \to (K[r], W(-r))$. We have:

$$\text{Dec} \int_\alpha (C^*(\Delta^{[\alpha]}) \otimes W_{p+r|\alpha}K^\alpha) \cong \int_\alpha \text{Dec}(C^*(\Delta^{[\alpha]}) \otimes W_{p+r|\alpha}K^\alpha) \cong \int_\alpha C^*(\Delta^{[\alpha]}) \otimes \text{Dec}W_pK^\alpha.$$

This gives an isomorphism of the filtrations $\text{Dec}(W(r+1))$ and $(\text{Dec}W)(r)$ on $s(K)$.

**Proposition 1.12.** Let $K$ be a codiagram of filtered complexes. For $r \geq 0$, there is a chain of quasi-isomorphisms $E_r^{s,q}(s^r(K)) \xrightarrow{\sim} sE_r^{s,q}(K)$.

**Proof.** For $r = 0$ we have an isomorphism $E_0^{s,q}(s^0(K)) \cong sE_0^{s,q}(K)$. Assume inductively that the proposition is true for $r - 1$. We then have a chain of quasi-isomorphisms $E_r^{s,q}(s^r(K)) \cong E_{r-1}^{s,q+2}(\text{Dec}(s^r(K))) \cong E_{r-1}^{s,q+2}(s^{r-1}(\text{Dec}K)) \cong sE_{r-1}^{s,q+2}(\text{Dec}K) \xrightarrow{\sim} sE_r^{s,q}(K)$, where the first and last quasi-isomorphisms follow from Proposition 1.9 and the isomorphisms follow from Proposition 1.11 and the induction hypothesis respectively.

**Theorem 1.13.** Let $\mathcal{A}$ be an abelian category. The triple $(\mathbb{C}^{+}(\mathbb{F}, \mathcal{A}), \mathcal{E}_r, s^r)$ is a cohomological descent category for all $r \geq 0$.

**Proof.** Consider the functor $E_r : \mathbb{C}^{+}(\mathbb{F}, \mathcal{A}) \to \mathbb{C}^{+}(\mathcal{A})$ defined by sending every filtered complex to the $r$-stage of its associated spectral sequence. Then $\mathcal{E}_r = E_r^{-1}(\mathcal{E})$, where $\mathcal{E}$ denotes the class of quasi-isomorphisms of $\mathbb{C}^{+}(\mathcal{A})$. Furthermore, by Proposition 1.12, the complexes $E_r(s^r(K))$ and $sE_r(K)$ are isomorphic in the derived category $\mathbb{D}^{+}(\mathcal{A}) = \mathbb{C}^{+}(\mathcal{A})[\mathcal{E}^{-1}]$, for every codiagram $K$ in $\mathbb{C}^{+}(\mathbb{F}, \mathcal{A})$. This isomorphism is compatible with the morphisms $\mu$ and $\lambda$. By Proposition 1.7.2 of [GN02] the triple $(\mathbb{C}^{+}(\mathcal{A}), \mathcal{E}, s)$ is a cohomological descent category. Hence by Proposition 1.5.12 of loc.cit., this lifts to a cohomological descent structure for the triple $(\mathbb{C}^{+}(\mathbb{F}, \mathcal{A}), \mathcal{E}_r, s^r)$.

**Remark 1.14.** For all $r \geq 0$, Deligne’s décalage is compatible with the cohomological descent structures $\text{Dec} : (\mathbb{C}^{+}(\mathbb{F}, \mathcal{A}), \mathcal{E}_{r+1}, s^{r+1}) \to (\mathbb{C}^{+}(\mathbb{F}, \mathcal{A}), \mathcal{E}_r, s^r)$. Furthermore, it induces an equivalence of categories $\text{Dec} : \mathbb{D}^{+}_{r+1}(\mathbb{F}, \mathcal{A}) \to \mathbb{D}^{+}_r(\mathbb{F}, \mathcal{A})$ (see Theorem 2.19 of [CG13]).
2. Extension criterion of functors for analytic spaces

The extension criterion of functors of [GN02] allows to extend certain cohomological type functors defined on smooth schemes, to all algebraic schemes, using resolution of singularities.

Let \( \mathbf{An}(\mathbb{C}) \) denote the category of complex analytic spaces that are reduced, separated and of finite dimension. Denote by \( \mathbf{Man}(\mathbb{C}) \) the full subcategory of smooth manifolds.

**Definition 2.1.** A cartesian diagram of \( \mathbf{An}(\mathbb{C}) \)

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{j} & \tilde{X} \\
\downarrow g & & \downarrow f \\
Y & \xrightarrow{i} & X
\end{array}
\]

is said to be an *acyclic square* if \( i \) is a closed immersion, \( f \) is proper and it induces an isomorphism \( \tilde{X} - \tilde{Y} \rightarrow X - Y \). It is an *elementary acyclic square* if, in addition, all the objects in the diagram are in \( \mathbf{Man}(\mathbb{C}) \), and \( f \) is the blow-up of \( X \) along \( Y \). In the latter case, the map \( f \) is said to be an *elementary proper modification*.

**Remark 2.2.** In the analytic setting, we still have Hironaka’s resolution of singularities. However, in order to provide an extension criterion valid for analytic spaces, we need to address certain issues concerning finiteness.

The first of this issues is Chow-Hironaka’s Lemma ([Hir64], 0.5), stating that every proper birational map of irreductible schemes factors as a composition of a finite sequence of blow-ups with smooth centers. This result allows the passage from acyclic squares to elementary acyclic squares in the hypotheses of the extension criterion. In the analytic setting, the factorization is made through the composition of a possibly infinite sequence of blow-ups, which is locally finite. This is a consequence of Hironaka’s Flattening Theorem [Hir75].

The second issue concerns the finiteness of \( \nu(X) = (n, c_n(X), \cdots, c_0(X)) \), where \( c_i(X) \) is the number of irreductible components of dimension \( i \), of a variety \( X \) of dimension \( n \), which contain the singular points of \( X \). If \( X \) is an algebraic variety, then \( c_i(X) \) is finite for all \( i \). However, for an analytic space this may not be the case. For compactifiable analytic spaces, these two drawbacks disappear.

Denote by \( \widehat{\mathbf{Man}}(\mathbb{C}) \) (resp. \( \widehat{\mathbf{An}}(\mathbb{C}) \)) the full subcategory of \( \mathbf{Man}(\mathbb{C}) \) (resp. \( \mathbf{An}(\mathbb{C}) \)) of compactifiable analytic spaces. The following is an analytic version of the extension criterion of functors defined over smooth schemes (see Theorem 2.1.10 of [GN02]).

**Theorem 2.3.** Let \( \mathcal{D} \) be a cohomological descent category, and let

\[ F: \widehat{\mathbf{Man}}(\mathbb{C}) \rightarrow \text{Ho}(\mathcal{D}) \]

be a contravariant \( \Phi \)-rectified functor satisfying:

(F1) \( F(\emptyset) \) is the final object of \( \mathcal{D} \) and \( F(X \sqcup Y) \rightarrow F(X) \times F(Y) \) is an isomorphism.

(F2) If \( X_\bullet \) is an elementary acyclic square of \( \widehat{\mathbf{Man}}(\mathbb{C}) \), then \( sF(X_\bullet) \) is acyclic.

Then there exists a contravariant \( \Phi \)-rectified functor

\[ F': \widehat{\mathbf{An}}(\mathbb{C}) \rightarrow \text{Ho}(\mathcal{D}) \]

such that:

1. If \( X \) is an object of \( \widehat{\mathbf{Man}}(\mathbb{C}) \), then \( F'(X) \cong F(X) \).

2. If \( X_\bullet \) is an acyclic square of \( \widehat{\mathbf{An}}(\mathbb{C}) \), then \( sF'(X_\bullet) \) is acyclic.

In addition, the functor \( F' \) is essentially unique.

**Proof.** With the notations of 2.1.10 of [GN02], this is equivalent to prove that the inclusion functor \( \widehat{\mathbf{Man}}(\mathbb{C}) \rightarrow \widehat{\mathbf{An}}(\mathbb{C}) \) verifies the extension property. It suffices to replace \( \mathcal{M}' = \mathbf{Sm}(\mathbb{K}) \)
by $\widetilde{\text{Man}}(\mathbb{C})$ and $\mathcal{M} = \text{Sch}(k)$ by $\widetilde{\text{An}}(\mathbb{C})$ in the proof of Theorem 2.1.5 of loc.cit., which by Remark 2.2 is valid for compactifiable analytic spaces.

To prove the invariance of the weight filtration we will use a relative version of the above result.

Let $\text{An}(\mathbb{C})^2$ denote the category of pairs $(X, U)$ where $X$ is an analytic space and $U$ is an open subset of $X$ such that $D = X - U$ is a closed analytic subspace of $X$.

Likewise, let $\text{Man}(\mathbb{C})^2$ be the full subcategory of $\text{An}(\mathbb{C})^2$ of those pairs $(X, U)$ with $X$ smooth and $D = X - U$ a normal crossings divisor in $X$ which is a union of smooth divisors.

**Definition 2.4.** A commutative diagram of $\text{An}(\mathbb{C})^2$

\[
\begin{array}{ccc}
(Y, U \cap Y) & \xrightarrow{i} & (X, U) \\
\downarrow g & & \downarrow f \\
(\bar{Y}, \bar{U} \cap \bar{Y}) & \xrightarrow{j} & (\bar{X}, \bar{U})
\end{array}
\]

is said to be an *acyclic square* if $f : \bar{X} \to X$ is proper, $i : Y \to X$ is a closed immersion, the diagram of the first components is cartesian, $f^{-1}(U) = \bar{U}$ and the diagram of the second components is an acyclic square of $\text{An}(\mathbb{C})$.

**Definition 2.5.** A morphism $f : (\bar{X}, \bar{U}) \to (X, U)$ in $\text{Man}(\mathbb{C})^2$ is called *proper elementary modification* if $f : \bar{X} \to X$ is the blow-up of $X$ along a smooth center $Y$ which has normal crossings with the complementary of $U$ in $X$, and if $U = f^{-1}(U)$.

**Definition 2.6.** An acyclic square of objects of $\text{Man}(\mathbb{C})^2$ is said to be an *elementary acyclic square* if the map $f : (\bar{X}, \bar{U}) \to (X, U)$ is a proper elementary modification, and the diagram of the second components is an elementary acyclic square of $\text{Man}(\mathbb{C})$.

Let $\text{An}(\mathbb{C})^2_{\text{comp}}$ denote the full subcategory of $\text{An}(\mathbb{C})^2$ given by those pairs $(X, U)$ such that $X$ is compact. Define $\text{Man}(\mathbb{C})^2_{\text{comp}}$ similarly. In particular, if $(X, U) \in \text{An}(\mathbb{C})^2_{\text{comp}}$ we have that both $X$ and $U$ are objects of $\widetilde{\text{An}}(\mathbb{C})$.

**2.7.** Denote by $\gamma : \text{An}(\mathbb{C})^2_{\text{comp}} \to \text{An}(\mathbb{C})$ the forgetful functor $(X, U) \mapsto U$, and let $\Sigma$ be the class of morphisms $s$ of $\text{An}(\mathbb{C})^2_{\text{comp}}$ such that $\gamma(s)$ is an isomorphism. Then $\gamma$ induces a functor

\[
\eta : \text{An}(\mathbb{C})^2_{\text{comp}}[\Sigma^{-1}] \to \text{An}(\mathbb{C}).
\]

In the algebraic situation, Nagata’s Compactification Theorem implies that the functor $\eta^{\text{alg}}$ induces an equivalence of categories

\[
\eta^{\text{alg}} : \text{Sch}(\mathbb{C})^2_{\text{comp}}[\Sigma^{-1}] \xrightarrow{\sim} \text{Sch}(\mathbb{C}).
\]

This does not hold in the analytic case. However, the localized category $\text{An}(\mathbb{C})^2_{\text{comp}}[\Sigma^{-1}]$ is equivalent to the category $\text{An}(\mathbb{C})_{\infty}$ defined as follows.

**Definition 2.8** ([GN02], 4.7). An *object* of $\text{An}(\mathbb{C})_{\infty}$ is given by an equivalence class of objects $(X, U)$ of $\text{An}(\mathbb{C})^2_{\text{comp}}$, where $(X, U)$ and $(X', U')$ are said to be *equivalent* if $U = U'$ and there exists a third compactification of $U$ that dominates $X$ and $X'$.

Two compactifications $f_1 : X_1 \to X'_1$ and $f_2 : X_2 \to X'_2$ of a morphism of analytic spaces $f : U \to U'$ are said to be *equivalent* if there exists a third compactification $f_3 : X_3 \to X'_3$ which dominates them. A *morphism* of $\text{An}(\mathbb{C})_{\infty}$ is given by an equivalence class of morphisms of $\text{An}(\mathbb{C})^2_{\text{comp}}$.

An *acyclic square* in $\text{An}(\mathbb{C})_{\infty}$ is a square induced by an acyclic square of $\text{An}(\mathbb{C})^2_{\text{comp}}$. The following is an analytic version of Theorem 2.3.6 of [GN02].
Theorem 2.9. Let $\mathcal{D}$ be a cohomological descent category, and let

$$F : \text{Man}(\mathbb{C})_{\text{comp}}^2 \to \text{Ho}(\mathcal{D})$$

be a contravariant $\Phi$-rectified functor satisfying:

(F1) $F(\emptyset, \emptyset)$ is the final object of $\mathcal{D}$ and $F((X, U) \sqcup (Y, V)) \to F(X, U) \times F(Y, V)$ is an isomorphism.

(F2) If $(X_\bullet, U_\bullet)$ is an elementary acyclic square of $\text{Man}(\mathbb{C})_{\text{comp}}^2$, then $sF(X_\bullet, U_\bullet)$ is acyclic.

Then there exists a contravariant $\Phi$-rectified functor

$$F' : \text{An}(\mathbb{C})_\infty \to \text{Ho}(\mathcal{D})$$

such that:

1. If $(X, U)$ is an object of $\text{An}(\mathbb{C})_{\text{comp}}^2$, then $F'(U_\infty) \cong F(X, U)$.
2. If $U_{\infty^\bullet}$ is an acyclic square of $\text{An}(\mathbb{C})_\infty$, then $sF'(U_{\infty^\bullet})$ is acyclic.

In addition, the functor $F'$ is essentially unique.

Proof. If $(X, U)$ is an object of $\text{An}(\mathbb{C})_{\text{comp}}^2$, then both $U$ and $X$ are objects of $\overline{\text{An}}(\mathbb{C})$. Hence by Remark 2.2, the proof of Theorem 2.3.3 of [GN02] applies to show that the inclusion functor $\text{Man}(\mathbb{C})_{\text{comp}}^2 \to \text{An}(\mathbb{C})_{\text{comp}}^2$ verifies the extension property 2.1.10 of loc. cit.. Therefore there exists a $\Phi$-rectified functor $F' : \text{An}(\mathbb{C})_{\text{comp}}^2 \to \text{Ho}(\mathcal{D})$ satisfying:

1. If $(X, U)$ is an object of $\text{Man}(\mathbb{C})_{\text{comp}}^2$, then $F'(X, U) \cong F(X, U)$.
2. If $(X_\bullet, U_\bullet)$ is an acyclic square of $\text{An}(\mathbb{C})_{\text{comp}}^2$, then $sF'(X_\bullet, U_\bullet)$ is acyclic.

Furthermore, the functor $F'$ is essentially unique. The remaining of the proof follows analogously to that of Theorem 2.3.6 of loc. cit., via the equivalence of categories

$$\eta : \text{An}(\mathbb{C})_{\text{comp}}^2[\Sigma^{-1}] \overset{\sim}{\to} \text{An}(\mathbb{C})_\infty$$

given in 4.7 of loc. cit..

□

3. Acyclic squares and Gysin complex

In this section we study the behavior of the cohomology functor with respect to certain acyclic squares of smooth analytic spaces. We then introduce the Gysin complex of a pair $(X, U)$, where $U \hookrightarrow X$ is a smooth compactification with $D = X - U$ a normal crossings divisor and describe its behavior with respect to elementary acyclic squares.

Let $X$ be a complex analytic space. Given a commutative ring $A$ we will denote by $\underline{A}_X$ the constant sheaf over $X$ associated to $A$ and by $H^q(X; A)$ the singular cohomology of $X$ with coefficients in $A$. For a continuous map $f : X \to Y$ we will denote $RF_* := f^*C^*_{Gdm}$, where $C_{Gdm}$ is the Godement resolution.

Proposition 3.1. For every acyclic square of $\text{Man}(\mathbb{C})$ as in Definition 2.1, the sequence

$$0 \to H^q(X; A) \to H^q(\tilde{X}; A) \to H^q(Y; A) \to H^{q+1}(\tilde{Y}; A) \to 0$$

is exact for all $q$.

Proof. We have a Mayer-Vietoris long exact sequence

$$\cdots \to H^q(X; A) \to H^q(\tilde{X}; A) \oplus H^q(Y; A) \to H^q(\tilde{Y}; A) \to H^{q+1}(X; A) \to \cdots$$

Therefore it suffices to see that the map $f^* : H^q(X; A) \to H^q(\tilde{X}; A)$ is injective. This is a well known consequence of Poincaré-Verdier duality, which gives the existence of a trace morphism $f^*$ such that $f_*f^* = 1$. We recall the proof. The map $f^*$ is induced by a morphism of sheaves $\underline{A}_X \to RF_*\underline{A}_\tilde{X}$.

Since $f$ is proper, $RF_* = RF$, and we have an adjunction $RF_* \dashv f^!$. Since $\tilde{X}$ and $X$ are smooth and of the same pure dimension, there is a quasi-isomorphism
that this is a map of complexes (see \([G]q\)). Equivalently, we have a short exact sequence

\[\text{Hom}(Rf_!A_X, A_Y) \to \text{Hom}(A_X, f^!A_X) \cong \text{Hom}(A_Y, A_X).\]

Lastly, since \(f\) is birational, the composition \(f_! \circ f^* : A_X \to Rf_!A_Y \to A_X\) is the identity, since it coincides with the identity over an open dense subset of \(X\). Hence \(f_!\) induces a left inverse of \(f^*\), and \(f^*\) is injective.

We shall also use the following blow-up formula for cohomology. A proof can be found in Theorem VI.4.5 of [FL85], which is an axiomatization of Theorem VII.3.7 of [SGA71].

**Proposition 3.2.** Consider an elementary acyclic square of \(\text{Man}(\mathbb{C})\) as in Definition 2.1. Let \(m = \text{codim}_X Y\), and let \(g^* = c_{m-1}(E) \cdot g^* : H^{s-2m}(Y; A) \to H^{s-2}(\tilde{Y}; A)\), where \(c_{m-1}(E) \in H^{2m-2}(Y; A)\) denotes the \((m-1)\)-th-Chern class of the normal bundle \(E = N_{Y/X}\) of \(Y\) in \(X\).

For all \(q \geq 0\), there is a commutative acyclic square

\[
\begin{array}{ccc}
H^{q-2}(\tilde{Y}; A) & \xrightarrow{j_*} & H^q(\tilde{X}; A) \\
\bar{g}^* \downarrow & & \downarrow j^* \\
H^{q-2m}(Y; A) & \xrightarrow{i_*} & H^q(X; A)
\end{array}
\]

Equivalently, we have a short exact sequence

\[0 \to H^{q-2m}(Y; A) \xrightarrow{(\bar{g}^*, i_*)} H^{q-2}(\tilde{Y}; A) \oplus H^q(X; A) \xrightarrow{j_* + j^*} H^q(\tilde{X}; A) \to 0.\]

**3.3 (Gysin complex).** Let \((X, U) \in \text{Man}(\mathbb{C})^2\). We may write \(D := X - U = D_1 \cup \cdots \cup D_N\) as the union of irreducible smooth divisors meeting transversally. Let \(D^{(0)} = X\) and for \(0 < p \leq N\) let \(D^{(p)}\) be the disjoint union of all \(p\)-fold intersections \(D_I = D_{i_1} \cap \cdots \cap D_{i_p}\) with \(I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}\). Since \(D\) is a normal crossings divisor, \(D^{(p)}\) is smooth. For all \(q \geq 0\), the **Gysin complex** \(G^q(X, U)\) is the cochain complex defined by

\[G^q(X, U)^p := H^q(D^{(p)}; A)[2p],\]

with \(d^p : G^q(X, U)^p \to G^q(X, U)^{p+1}\) defined by the alternated sum of Gysin morphisms

\[i_s(I, J) : H^{q+2p}(D_J; A) \to H^{q+2(p+1)}(D_I; A),\]

where \(I \subset J \subset \{1, \ldots, N\}\) and \(|J| = |I| + 1 = -p\).

**Lemma 3.4.** For all \(q \geq 0\), the **Gysin complex** defines a contravariant functor

\[G^q : \text{Man}(\mathbb{C})^2 \to \mathbb{C}^\oplus (A\text{-mod}).\]

**Proof.** Let \(f : (X', U') \to (X, U)\) be a morphism in \(\text{Man}(\mathbb{C})^2\). Let \(D = X - U = D_1 \cup \cdots \cup D_N\) and \(D' = X' - U' = D'_1 \cup \cdots \cup D'_M\). For every irreducible component \(D_i\) of \(D\), its inverse image divisor is a sum

\[f^{-1}(D_i) = \sum_{i=1}^M m_{ij} D'_j\]

of irreducible components of \(D'\). Let \(M_f = \{m_{ij}\}\) denote the matrix of multiplicities of \(f\). We next define \(G^q(f)^* : G^q(X', U')^* \to G^q(X, U)^*\). Let \(G^q(f)^0 = f^*\). Let \(I \subset \{1, \ldots, N\}\) and \(J \subset \{1, \ldots, M\}\) be two sets with \(|I| = |J| = p > 0\). Let \(m_{I,J}\) denote the determinant of the minor of \(M_f\) of indices \((I, J)\).

If \(I\) and \(J\) are such that \(f(D'_j) \subset D_i\), we define a morphism \(G^q(f)_{I,J} : H^q(D_I) \to H^q(D'_j)\) by letting \(G^q(f)_{I,J} := m_{IJ} f_{I,J}^*\), where \(f_{I,J} : D'_j \to D_I\) denotes the restriction of \(f\). If \(I\) and \(J\) are such that \(f(D'_j) \not\subset D_I\), we let \(G^q(f)_{I,J} = 0\). Then the morphisms \(G^q(f)_{I,J}\) are the components of \(G^q(f)^p : G^q(X', U')^p \to G^q(X, U)^p\). It follows from the decomposition property of determinants, that this is a map of complexes (see [GN02], pag. 84).
If \( g : (X'', U'') \to (X', U') \) is a morphism of \( \text{Man}(\mathbb{C})^2 \), then the matrix of multiplicities \( M_{f\circ g} \) of \( f \circ g \) is the product of the multiplicity matrices \( M_f \) and \( M_g \) of \( f \) and \( g \). The functoriality of \( G^q \) then follows from the functoriality of the determinants. \( \square \)

**Proposition 3.5.** Consider an elementary acyclic square of \( \text{Man}(\mathbb{C})^2 \) as in Definition 2.4.

1. If \( Y \not\subseteq D \) then the simple of the double complex

\[
0 \to G^q(X, U) \to G^q(\tilde{X}, \tilde{U}) \oplus G^q(Y, U \cap Y) \to G^q(\tilde{Y}, \tilde{U} \cap \tilde{Y}) \to 0
\]

is acyclic for all \( q \).

2. If \( Y \subset D \) then the map \( G^q(X, U) \to G^q(\tilde{X}, \tilde{U}) \) is a quasi-isomorphism for all \( q \).

**Proof.** We adapt the proof of Proposition 5.9 of [GN02] in the motivic setting (see also [MP12], Sections 5 and 6).

Assume that \( Y \not\subseteq D \). We proceed by induction on the number \( N \) of smooth irreducible components of \( D \). If \( N = 0 \) then \( G^q(X, U) = H^q(X; A) \) is concentrated in degree 0 and the sequence \((*)\) becomes that of Proposition 3.1.

Assume that \( N > 0 \). Let \( D = D'' \cup X' \) and \( D' = D'' \cap X' \), where \( X' \) is a component of \( D \). From the definition of the Gysin complex we obtain an exact sequence

\[
0 \to G(X, X - D'') \to G(X, X - D) \to G(X', X' - D')[1] \to 0.
\]

Denote by \((X_*, X_* - D''_*)\) the commutative square

\[
\begin{array}{ccc}
(Y, Y - E'') & \longrightarrow & (\tilde{X}, \tilde{X} - \tilde{D}'') \\
\downarrow & & \downarrow \\
(Y, Y - E'') & \longrightarrow & (X, X - D'')
\end{array}
\]

where \( E'' = D'' \cap Y \), \( \tilde{D}'' = f^{-1}(D'') \) and \( \tilde{E}'' = \tilde{D}'' \cap \tilde{Y} \). Consider the blow-up \( \tilde{X}' \) of \( X' \) along \( Y' = Y \cap X' \), and denote by \((X'_*, X'_* - D'_*)\) the commutative square

\[
\begin{array}{ccc}
(Y', Y' - E') & \longrightarrow & (\tilde{X}', \tilde{X}' - \tilde{D}') \\
\downarrow & & \downarrow \\
(Y', Y' - E') & \longrightarrow & (X', X' - D')
\end{array}
\]

where \( E' = D' \cap Y' \), \( \tilde{Y}' = \tilde{X}' \cap \tilde{Y} \), \( \tilde{D}' = \tilde{X}' \cap f^{-1}(D') \) and \( \tilde{E}' = \tilde{D}' \cap \tilde{Y}' \). We then have a short exact sequence

\[
0 \to sG(X_*, X_* - D''_*) \to sG(X_*, X_* - D_*) \to sG(X'_*, X'_* - D'_*)[1] \to 0.
\]

By induction hypothesis, both \( sG(X_*, X_* - D''_*) \) and \( sG(X'_*, X'_* - D'_*) \) are acyclic complexes. Therefore the middle complex is acyclic, as desired. This proves (1).

Assume that \( Y \subset D \). We proceed by induction over the number of components \( r \) of \( D \) which contain \( Y \), and the number \( s \) of components which do not contain \( Y \).

Assume that \((r, s) = (1, 0)\), so that \( D \) is smooth irreducible and \( Y \subset D \). Then \( G^q(X, X - D) \) is the simple of the morphism \( H^{q+2}(D; A) \to H^q(X; A) \). Denote by \( \tilde{D} \) the proper transform of \( D \), and let \( \tilde{E} = \tilde{Y} \cap \tilde{D} \). Denote by

\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{i_{E, \tilde{D}}} & \tilde{D} \\
\gamma \downarrow & & \downarrow \gamma \\
Y & \xrightarrow{i_{Y, D}} & D
\end{array}
\]
the induced diagram. Then \( \tilde{D} = \tilde{Y} \cup \tilde{D} \), and \( G^0(\tilde{X}, \tilde{X} - \tilde{D}) \) is the simple of the square

\[
\begin{array}{ccc}
H^{q-4}(\tilde{E}; A) & \xrightarrow{i_{\tilde{E}, \tilde{D}*}} & H^{q-2}(\tilde{D}; A) \\
\downarrow^{i_{\tilde{E}, \tilde{Y}*}} & & \downarrow^{i_{\tilde{D}, \tilde{X}*}} \\
H^{q-2}(\tilde{Y}; A) & \xrightarrow{i_{\tilde{D}, \tilde{X}*}} & H^q(\tilde{X}; A)
\end{array}
\]

Therefore it suffices to show that the following complex is acyclic:

\[
H^{q-2}(D; A) \oplus H^{q-4}(\tilde{E}; A) \xrightarrow{\alpha} H^{q-2}(\tilde{D}; A) \oplus H^{q-2}(\tilde{Y}; A) \oplus H^q(X; A) \xrightarrow{\beta} H^q(\tilde{X}; A),
\]

where \( \beta = i_{\tilde{D}, \tilde{X}*} + i_{\tilde{Y}, \tilde{X}*} + f^* \) and \( \alpha \) is given by the matrix

\[
\begin{pmatrix}
-\hat{f}^* & -i_{\tilde{E}, \tilde{D}*} \\
-(i_{\tilde{Y}, \tilde{D}*} g^*) & i_{\tilde{E}, \tilde{Y}*} \\
i_{\tilde{D}, \tilde{X}*} & 0
\end{pmatrix}.
\]

After adding the acyclic complex \( H^{q-2m}(Y; A) \longrightarrow H^{q-2m}(Y; A) \) and rearranging factors, we obtain the following complex

\[
\begin{array}{ccc}
H^{q-2}(\tilde{Y}; A) & \longrightarrow & H^q(\tilde{X}; A) \\
\downarrow & & \downarrow \\
H^{q-2m}(Y; A) & \longrightarrow & H^q(X; A) \\
\downarrow & & \downarrow \\
H^{q-4}(\tilde{E}; A) & \longrightarrow & H^{q-2}(\tilde{D}; A) \\
\downarrow & & \downarrow \\
H^{q-2m}(Y; A) & \longrightarrow & H^{q-2}(D; A)
\end{array}
\]

where the top and bottom faces of the cube are squares of blow-up type which are acyclic by Proposition 3.2. Therefore the total complex of this complex is acyclic. This proves (2) for the case \( (r, s) = (1, 0) \).

Assume that \( r = 1 \) and \( s > 0 \). Let \( D = D'' \cup X' \), where \( Y \subseteq X' \), and \( D' = D'' \cap X' \). We have a commutative diagram with acyclic rows

\[
\begin{array}{ccc}
0 & \longrightarrow & G(X, X - D'') \\
\downarrow^{f''*} & & \downarrow^{f'^*} \\
0 & \longrightarrow & G(\tilde{X}, \tilde{X} - \tilde{D}'')
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & G(X, X - D') [1] \\
\downarrow^{f'^*} & & \downarrow^{f'^*} \\
0 & \longrightarrow & G(\tilde{X}, \tilde{X} - \tilde{D'}) [1]
\end{array}
\]

By induction hypothesis, the maps \( f''* \) and \( f'^* \) are quasi-isomorphisms. Therefore \( f^* \) is a quasi-isomorphism.

Assume that \( r > 1 \) and consider a decomposition \( D = D'' \cup X' \) such that \( Y \subset X' \). An argument parallel to the previous case, by induction over \( r \), shows that (2) is satisfied in the general case.

\[ \square \]

4. Singularity filtration

The singularity filtration is an analytic invariant that appears naturally when we extend the functor of singular chains with the trivial filtration, from smooth to singular analytic spaces, using the \( E_1 \)-cohomological descent structure on filtered complexes.

Let \( X \) be a complex manifold. Given a commutative ring \( A \) denote by \( S^* (X; A) \) the complex of singular cochains of \( X \) with values in \( A \), so that \( H^n(S^* (X; A)) = H^n(X; A) \). Together with the trivial filtration defined on \( S^* (X; A) \) this defines a functor \( S : \mathbf{Man}(\mathbb{C}) \longrightarrow \mathbf{C}^+(\mathbf{FA}\text{-mod}) \).

Any extended functor \( \widehat{\mathbf{An}}(\mathcal{C}) \rightarrow \mathbf{Ho}(\mathcal{D}) \) defined via Theorem 2.3 depends strongly on the cohomological descent structure that we consider on the category \( \mathcal{D} \). In our case of interest, we may
extend the functor $S$ to a functor $\widehat{\text{An}}(\mathbb{C}) \to D^+_{\text{H}(\mathcal{A})}$, using the cohomological descent structure on $C^+(\mathcal{A})$ associated with the class of $E_0$-quasi-isomorphisms. It is easy to see that this is an empty exercise: the extended filtration of the trivial filtration is also trivial. However, if we consider the cohomological descent structure associated with $E_1$-quasi-isomorphisms, we obtain a non-trivial filtration which for compact spaces coincides with the weight filtration.

**Theorem 4.1.** There exists a $\Phi$-rectified functor $S^t: \widehat{\text{An}}(\mathbb{C}) \to D^+_{\text{H}(\mathcal{A})}$ such that:

1. If $X \in \widehat{\text{An}}(\mathbb{C})$ then $H^n(S^t(X)) \cong H^n(X; A)$.
2. If $X$ is a smooth manifold then $S^t(X) = (S^t(X; A), L)$, where $L$ is the trivial filtration.
3. For every $p, q \in \mathbb{Z}$ and every acyclic square of $\widehat{\text{An}}(\mathbb{C})$ as in Definition 2.1 there is a long exact sequence

\[
\cdots \to E^{p,q}_2(S^t(X)) \to E^{p,q}_2(S^t(Y)) \oplus E^{p,q}_2(S^t(Y)) \to E^{p+1,q}_2(S^t(Y)) \to \cdots
\]

4. If $X$ is a compact complex algebraic variety and $A = \mathbb{Q}$ then the filtration induced in cohomology coincides with Deligne’s weight filtration after décalage.

**Proof.** By Theorem 1.13 the triple $(C^+(\mathcal{A}), E_1, s^t)$ is a cohomological descent category. Therefore it suffices to show that the functor

\[
\widehat{\text{Man}}(\mathbb{C}) \xrightarrow{S} C^+(\mathcal{A}) \xrightarrow{\tilde{S}} D^+_{\text{H}(\mathcal{A})}
\]

given by $S(X) = (S^t(X; A), t)$, where $t$ denotes the trivial filtration, satisfies properties (F1) and (F2) of Theorem 2.3. Property (F1) is trivial.

Let us prove (F2). This is equivalent to the condition that for every elementary acyclic square $X_\bullet \to X$ of $\widehat{\text{Man}}(\mathbb{C})$, the map $S(X) \to s^t(S(X_\bullet))$ is an $E_1$-quasi-isomorphism. By Proposition 1.12, given a codiagram of filtered complexes $K^\bullet$, we have a chain of quasi-isomorphisms $E_1^t(s^t(K^\bullet)) \xrightarrow{\sim} sE_1^t(K^\bullet)$. Hence it suffices to check that for all $q \in \mathbb{Z}$, the sequence

\[
\cdots \to E_2^p,q(S(X)) \to E_2^p,q(S(X)) \oplus E_2^p,q(S(Y)) \to E_2^p,q(S(Y)) \to \cdots
\]

is exact. Since the filtrations are trivial, we have $E_1^t(S(\cdot)) = H^t(S^t(\cdot; A)) = H^t(\cdot; A)$ and $E_1^p,q(S(\cdot)) = 0$ for $p \neq 0$. Therefore it suffices to see that the sequence

\[
0 \to H^q(X; A) \to H^q(X; A) \oplus H^q(Y; A) \to H^q(Y; A) \to 0
\]

is exact. This follows from Proposition 3.1. \hfill $\square$

**Definition 4.2.** Let $X$ be a compactificable complex analytic space. The *singularity spectral sequence* is the spectral sequence associated with the filtered complex $S^t(X)$ of Theorem 4.1. Let $L'$ denote the increasing filtration induced on $H^*(X; A)$. The *singularity filtration* $L_p$ on $H^*(X; A)$ is defined by $L_pH^n(X; A) := L_{p-n}H^n(X; A)$.

**Corollary 4.3.** Let $X$ be a compactificable complex analytic space. Then for every $n \geq 0$, its cohomology $H^n(X; A)$ with values in any commutative ring $A$ carries a singularity filtration

\[
0 = L_{-1} \subset L_0 \cdots \subset L_n = H^n(X; A)
\]

which is functorial for morphisms in $\widehat{\text{An}}(\mathbb{C})$ and satisfies:

1. If $X$ is smooth then $L$ is the trivial filtration.
2. If $X$ is a complex projective variety and $A = \mathbb{Q}$ then $L$ coincides with Deligne’s weight filtration.

Note that by Theorem 4.1, the $E_2$-term of the singularity spectral sequence is well-defined. The first term $L_1E_2^p$, which is well-defined up to quasi-isomorphism, admits a description in terms of resolutions as follows: let $X_\bullet \to X$ be a resolution of a compactificable complex analytic space $X$. Then:

\[
\cdots \to L_1E_2^{p,q}(X; A) = \bigoplus_{|\alpha|=p} H^q(X_\alpha; A) \to H^{p+q}(X; A).
\]
If $X$ is a projective complex variety and $A = \mathbb{Q}$ this corresponds to the analogue formula for the weight filtration appearing in Theorem 8.1.15 of [Del74], (see also IV.3 of [GNPP88]).

**Remark 4.4.** The same arguments give a filtration $L$ on the homology with compact supports and on the Borel-Moore homology of a variety $X$. In [Gui87], Deligne’s weight filtration $W$ and Zeeman’s filtration $S$ are compared in the homology of a compact variety, giving the relation $S^{2N-i-q} \subset W^{i-q}$ on $H_i(X; \mathbb{Q})$, where $N = \dim X$. The same proof of [Gui87] gives the relation $S^{2N-i-q} \subset L^{i-q}$ for the singularity filtration on the Borel-Moore homology $H_i^{BM}(X; A)$.

5. **Weight filtration**

Recall that the **canonical filtration** $\tau$ is defined on any given complex $K$ by truncation:

$$\tau_{\leq p}K = \{ \cdots \to K^{p-1} \to \ker d \to 0 \to 0 \to \cdots \}.$$  

Given $(X, U) \in \text{Man}(\mathbb{C})^2_{\text{comp}}$, let $j : U \hookrightarrow X$ denote the inclusion, and $(\mathcal{R}j_!A_U, \tau)$ the filtered complex of sheaves given by the direct image of the constant sheaf $A_U$, together with the canonical filtration. Taking the right derived functor of global sections we obtain a $\Phi$-rectified functor

$$\mathcal{W} : \text{Man}(\mathbb{C})^2_{\text{comp}} \to D^+_1(F \text{A-mod})$$

with values in the 1-derived category of filtered complexes of $A$-modules (see Definition 1.7), given by

$$\mathcal{W}(X, U) = R\Gamma(X, (\mathcal{R}j_!A_U, \tau)).$$

By the properties of the global sections functor and the derived direct image functor, we have $H^n(\mathcal{W}(X, U)) \cong H^n(U; A)$.

**Theorem 5.1.** There exists a $\Phi$-rectified functor $\mathcal{W} : \text{An}(\mathbb{C})_{\text{infty}} \to D^+_1(F \text{A-mod})$ such that:

1. If $U_{\infty} \in \text{An}(\mathbb{C})_{\text{infty}}$ then $H^n(\mathcal{W}(U_{\infty})) \cong H^n(U; A)$.
2. If $(X, U)$ is an object of $\text{Man}(\mathbb{C})^2_{\text{comp}}$ then $\mathcal{W}(U_{\infty}) \cong \mathcal{W}(X, U)$.
3. For every $p, q \in \mathbb{Z}$ and every acyclic square of $\text{An}(\mathbb{C})_{\text{infty}}$

$$\begin{array}{ccc}
\mathcal{Y}_{\infty} & \to & \mathcal{X}_{\infty} \\
\downarrow & & \downarrow \\
\mathcal{Y}_\infty & \to & \mathcal{X}_\infty
\end{array}$$

there is a long exact sequence

$$\cdots \to E^2_2(\mathcal{W}(\mathcal{X}_{\infty})) \to E^2_2(\mathcal{W}(\mathcal{X}_\infty)) \oplus E^2_2(\mathcal{W}(\mathcal{Y}_{\infty})) \to E^2_2(\mathcal{W}(\mathcal{Y}_\infty)) \to E^2_2(\mathcal{W}(\mathcal{X}_\infty)) \to \cdots$$

4. If $X$ is a complex algebraic variety and $A = \mathbb{Q}$ then the filtration induced in cohomology coincides with Deligne’s weight filtration after décalage.

**Proof.** By Theorem 1.13 the triple $\left( \mathcal{C}^+(F \text{A-mod}), \mathcal{E}_1, s_1 \right)$ is a cohomological descent category. Therefore by Theorem 2.9 it suffices to show that the functor

$$\mathcal{W} : \text{Man}(\mathbb{C})^2_{\text{comp}} \to D^+_1(F \text{A-mod})$$

given by $\mathcal{W}(X, U) = R\Gamma(X, (\mathcal{R}j_!A_U, \tau))$ satisfies properties (F1) and (F2). Property (F1) is trivial.

Condition (F2) is equivalent to the condition that the map $\mathcal{W}(X, U) \to s_1(\mathcal{W}(X, U))$ is an $E_1$-quasi-isomorphism for every elementary acyclic square $(X_a, U_a) \to (X, U)$ of $\text{Man}(\mathbb{C})^2_{\text{comp}}$.

As in the proof of Theorem 4.1, it suffices to check that for all $q \in \mathbb{Z}$, the sequence

$$\cdots \to E^q_{2}(\mathcal{W}(X, U)) \to E^q_{2}(\mathcal{W}(X, U)) \oplus E^q_{2}(\mathcal{W}(Y, Y \cap Y)) \to E^q_{2}(\mathcal{W}(Y, Y \cap Y)) \to \cdots$$

is exact. Since $E^q_{2}(\mathcal{W}(X, U))$ is the shifted Leray spectral sequence of the inclusion $j : U \hookrightarrow X$, it is isomorphic to the Gysin complex $G^q(X, U)^*$ for all $q$ (see [Del71], 3.1.9 and 3.2.4, see also Section 4.3 of [PS08]). Hence the exactness of this sequence follows from Proposition 3.5. \qed
**Definition 5.2.** Let \( X_\infty \) be a complex analytic space with an equivalence class of compactifications. The *weight spectral sequence* is the spectral sequence associated with the filtered complex \( W(X_\infty) \). If \( W' \) denotes the induced filtration on \( H^*(X; A) \), the weight filtration \( W_p \) on \( H^*(X; A) \) is defined by \( W_p H^n(X; A) := W'_{p-n} H^n(X; A) \).

**Corollary 5.3.** Let \( X_\infty \) be a complex analytic space with an equivalence class of compactifications. Then for every \( n \geq 0 \), its cohomology \( H^n(X; A) \) with values in any commutative ring \( A \) carries a weight filtration

\[
0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2n} = H^n(X; A)
\]

which is functorial for morphisms in \( \text{An}(\mathbb{C})_\infty \) and satisfies:

1) If \( X \) is smooth then \( 0 = W_{n-1} \subset H^n(X; A) \).
2) If \( X \) is compact then \( W_n = H^n(X; A) \).
3) If \( X \) is a complex algebraic variety and \( A = \mathbb{Q} \) then \( W \) is Deligne’s weight filtration.

Note that by Theorem 5.1 the weight spectral sequence of \( X_\infty \) is well-defined from the \( E_2 \)-term onwards. The first term \( W E_1 \) of the weight spectral sequence admits a description in terms of compactifications and resolutions as follows:

1) Assume that \( X_\infty \) is smooth. Choose a representative \( X \hookrightarrow \overline{X} \) of the compactification class \( X_\infty \) with \( D = \overline{X} - X \) a normal crossings divisor. Denote by \( D^{(p)} \) the disjoint union of all \( p \)-fold intersections of the smooth irreducible components of \( D \). Then:

\[
W E_1^{p-q}(X_\infty; A) = H^{q-2p}(D^{(p)}; A) \Rightarrow H^{q-p}(X; A).
\]

If \( X \) is algebraic and \( A = \mathbb{Q} \) we recover Deligne’s formula 3.2.4 of [Del71].

2) Assume that \( X \) is compact. Let \( X_\bullet \hookrightarrow X \) be a cubical hyperresolution of \( X \). Then:

\[
W E_1^{p-q}(X; A) = \bigoplus_{|\alpha| = p} H^q(X_\alpha; A) \Rightarrow H^{p+q}(X; A).
\]

3) For the general case, let \( X \hookrightarrow \overline{X} \) be a representative of \( X_\infty \), and let \( (\overline{X}_\bullet, X_\bullet) \rightarrow (\overline{X}, X) \) be a resolution of \( (\overline{X}, X) \). These are resolutions \( X_\bullet \hookrightarrow X \) and \( \overline{X}_\bullet \hookrightarrow \overline{X} \) such that the complement \( D_\alpha = \overline{X}_\alpha - X_\alpha \) is a normal crossings divisor for each \( \alpha \). Then:

\[
W E_1^{p-q}(X_\infty; A) = \bigoplus_{\alpha} W E_1^{p-|\alpha|q}(X_\alpha; A) = \bigoplus_{\alpha} H^{q-2p-2|\alpha|}(D^{(p+|\alpha|)}_\alpha; A) \Rightarrow H^{q-p}(X; A).
\]

If \( X \) is algebraic and \( A = \mathbb{Q} \) this corresponds to the analogue formula appearing in Theorem 8.1.15 of [Del74] (see also IV.3 of [GNPP88]).

**Remarks 5.4.** Let us end with a few remarks.

1) Note that in general, \( W \) depends on the class of \( X_\infty \), and it is functorial for morphisms in \( \text{An}(\mathbb{C})_\infty \). As a corollary of the above theorem we obtain a functorial weight filtration on the cohomology \( H(X; A) \) for every complex algebraic variety \( X \) and every coefficient ring \( A \), which by Nagata’s Theorem, is independent of the compactification, and is functorial for morphisms of algebraic varieties.

2) As noted by Gillet and Soulé, for a general ring \( A \), the weight spectral sequence does not necessarily degenerate at the second stage, even for complex algebraic varieties. Therefore the term \( E_2(X; A) \) is an analytic invariant of \( X \), which may differ from \( \text{Gr}_W^* H(X; A) \).

3) The results on the weight filtration for complex algebraic varieties have their analogues for the cohomology with coefficients in \( \mathbb{Z}_2 \) of real algebraic varieties, as remarked by Totaro in [Tot02], thus giving a cohomological version of the results on homology of McCrory-Parusinski [MP12]. Indeed, the results of Section 3 also valid in this case, since Poincaré-Verdier duality applies. Note that one needs to adjust the degree of the cohomology groups appearing in Proposition 3.2 and in the definition of the Gysin complex (as done by in [MP12]), since a closed immersion \( f : X \hookrightarrow Y \) of real algebraic varieties induces a Gysin map \( f_* : H^k(X; A) \rightarrow H^{k+m}(Y; A) \), where \( m = \text{codim}_Y X \).
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(J. Cirici) Fachbereich Mathematik und Informatik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin
E-mail address: jcirici@math.fu-berlin.de

(F. Guillén) Departament d’Àlgebra i Geometria, Universitat de Barcelona, Gran Via 585, 08007 Barcelona
E-mail address: fguillen@ub.edu