The exotic conformal Galilei algebra and nonlinear partial differential equations

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Abstract

The conformal Galilei algebra (CGA) and the exotic conformal Galilei algebra (ECGA) are applied to construct partial differential equations (PDEs) and systems of PDEs, which admit these algebras. We show that there are no single second-order PDEs invariant under the CGA but systems of PDEs can admit this algebra. Moreover, a wide class of nonlinear PDEs exists, which are conditionally invariant under CGA. It is further shown that there are systems of non-linear PDEs admitting ECGA with the realisation obtained very recently in [D. Martelli and Y. Tachikawa, \texttt{arXiv:0903.5184v2} [hep-th] (2009)]. Moreover, wide classes of non-linear systems, invariant under two different 10-dimensional subalgebras of ECGA are explicitly constructed and an example with possible physical interpretation is presented.

Keywords: partial differential equation (PDE), the conformal Galilei algebra, Lie symmetry, conditional symmetry.

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1 Introduction

Symmetries have since a long time played an important rôle in the analysis of physical systems. In this paper, we consider non-relativistic space-time symmetries. The best known of those is the Lie algebra in N spatial dimensions with the basic operators

\[ X_{\pm 1,0}, \ Y_{\pm 1/2}^{(j)}, \ M_0, \ R_0^{(jk)}, \ j, k = 1, \ldots, N \]  

(1.1)

The operators (1.1) satisfy the non-vanishing commutation relations

\[
\begin{align*}
[X_n, X_{n'}] &= (n - n')X_{n+n'}, \\
[X_n, Y^{(j)}_m] &= \left(\frac{n}{2} - m\right)Y^{(j)}_{n+m}, \\
[Y_{1/2}^{(j)}, Y_{-1/2}^{(k)}] &= \delta^{j,k} M_0, \\
[R_0^{(jk)}, Y^{(\ell)}_m] &= \delta^{j,\ell} Y^{(k)}_m - \delta^{k,\ell} Y^{(j)}_m
\end{align*}
\]  

(1.2)

where \( R_0^{(jk)} \in \mathfrak{so}(N) \), \( j, k, \ell \in \{1, \ldots, N\} \), \( n, n' \in \{\pm 1, 0\} \) and \( m = \pm \frac{1}{2} \). Since Sophus Lie [1] discovered this algebra for \( N = 1 \) as the maximal algebra of invariance (MAI) of the one-dimensional linear heat equation, this algebra has been intensively studied and today is often called Schrödinger algebra, to be denoted in this paper by \( \mathfrak{sch}(N) \). Besides being the Lie algebra of invariance of the linear heat (diffusion) and free Schrödinger equations, see e.g. [2, 3, 4] and references therein, it is also the Lie symmetry algebra of non-linear systems of evolution equations and Schrödinger type equations [5, 6, 7]. Probably the first example of a Schrödinger-invariant system of non-linear equations is given by the hydrodynamic equations of motion of a compressible fluid [2]

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \rho (\partial_t + (\mathbf{v} \cdot \nabla)) \mathbf{v} + \nabla P = 0
\]  

(1.3)

where \( \mathbf{v} = (\partial_{r_1}, \ldots, \partial_{r_N}) \) and \( \cdot \) means the scalar product, while \( \rho = \rho(t, \mathbf{r}) \) is the density, \( \mathbf{v} = \mathbf{v}(t, \mathbf{r}) \) is the velocity of the fluid. Furthermore, the pressure \( P \) satisfies a polytropic equation of state \( P = \rho^\gamma \) with a polytropic exponent \( \gamma = 1 + 2/N \). Physicists have rediscovered this result not so long ago [8].

It is well-known that the Schrödinger algebra \( \mathfrak{sch}(N) \) can be embedded into the (complexified) conformal Lie algebra \( \mathfrak{conf}(N+2) \) in \( N + 2 \) dimensions [9]. When one considers explicit space-time representations which contain a dimensional constant \( c \) with the physical units of velocity, taking the formal non-relativistic limit \( c \to \infty \) does not lead back to the Schrödinger algebra \( \mathfrak{sch}(N) \) but rather to a non-isomorphic Lie algebra, which actually is a parabolic subalgebra of \( \mathfrak{conf}(N+2) \) [10]. This algebra was identified at least as early as 1978 in [11] and can indeed be obtained from the conformal algebra \( \mathfrak{conf}(N+1) \) by a group contraction. In the physical literature, this algebra is usually called the conformal Galilei algebra \( \mathfrak{cga}(N) \) [11, 12], but the name of altern algebra [13] is also used, with reference to the physical contexts (space-time geometry and ageing phenomena, respectively), where it was identified, and it will be the object of study in this paper. The algebra \( \mathfrak{cga}(N) \) is spanned by the generators

\[ X_{\pm 1,0}, \ Y_{\pm 1/2}^{(j)}, \ R_0^{(jk)}, \ j, k = 1, \ldots, N \]  

(1.4)
and has the following non-vanishing commutators \([11, 12, 13, 14]\)

\[
\begin{align*}
[X_n, X_m] &= (n - m)X_{n+m}, \\
[R_0^{(jk)}, Y_m^{(ℓ)}] &= \delta^{j\ell}Y_m^{(k)} - \delta^{k\ell}Y_m^{(j)}
\end{align*}
\]

(1.5)

where \(j, k, \ell \in \{1, \ldots, N\}\) and \(n, m \in \{±1, 0\}\) and, again, the \(R_0^{(jk)}\) denote infinitesimal spatial rotations in \(N\) dimensions. This compact way of expressing the commutators makes the similarities and differences with \(\mathfrak{sch}(N)\) in formulae (1.2) explicit. In particular, the subalgebra \(\mathfrak{s}(2, \mathbb{R}) = \langle X_{±1,0} \rangle\) clearly appears. An explicit representation of the operators (1.4) is given by, summarising and generalising earlier partial results \([14, 15, 16, 17, 18]\)

\[
\begin{align*}
X_n &= -t^{n+1}\partial_t - (n+1)t^n\mathbf{r} \cdot \nabla - \lambda(n+1)t^n - n(n+1)t^{n-1}\gamma \cdot \mathbf{r} \\
Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n\gamma_j \\
R_0^{(jk)} &= -(r_j\partial_k - r_k\partial_j) - (\gamma_j\partial_{γ_k} - γ_k\partial_{γ_j}); \quad j \neq k
\end{align*}
\]

(1.6)

where we used the abbreviations \(\partial_j = \frac{∂}{∂r_j}\) and \(\partial_{γ_k} = \frac{∂}{∂γ_k}\). The constant \(λ\) is physically interpreted as a scaling dimension and \(γ = (γ_1, \ldots, γ_N)\) is a vector of auxiliary variables. From this, it can be seen that \(X_{-1,0,1}\), respectively, generate time-translations, space-time dilatations and projective transformations, whereas \(Y_n^{(j)}\) generate space translations \((n = -1)\), Galilei transformations \((n = 0)\) and constant accelerations \((n = 1)\). We also see that although the terms parameterised by \(γ_j\) will create phase changes in the transformed wave functions, they do not give rise to a central extension, in contrast to the ‘mass’ parameter in the Schrödinger algebra, therein related to the generator \(M_0\). Note the explicit representation (1.6) of \(\text{CGA}(N)\) can be extended to a representation of an infinite-dimensional Lie algebra \([11, 14]\), which might be called \(\text{altern-Virasoro algebra \altn(N)} := \langle X_n, Y_n^{(j)}, R_0^{(jk)} \rangle\) (here \(n \in \mathbb{Z}; j, k = 1, \ldots, N\) such that the commutation relations (1.5) remain valid. Its central extensions were studied in \([19]\). See \([20]\) for a thorough study of the geometric interpretation of these and several other algebras of non-relativistic space-time symmetries.

Our work is motivated by the following considerations.

1. Physicists have recently become interested in the altern algebra in the context of non-relativistic version of the AdS/CFT (Anti-de-Sitter/Conformal Field Theory) correspondence, see \([18, 21]\) and references therein. Furthermore, it has also been attempted to show that the equations of motions of incompressible fluids could admit the algebra \(\text{CGA}(N)\). The current state of knowledge, discussing the various mutually exclusive assertions, is nicely summarised in \([22]\). Sometimes, in order to clarify a confusing physical situation it may be helpful to look for a precise mathematical statement. It therefore seems appropriate to try to apply the well-known Lie machinery \([2, 3, 4]\) in order to construct classes of non-linear equations with the invariance with respect to the altern algebra.

2. In \(N = 2\) spatial dimensions, it was recently shown in \([23]\) that the conformal Galilei algebra does admit a so-called ‘exotic’ central extension. This is achieved by adding to the algebra (1.5) the following commutator

\[
[Y_n^{(1)}, Y_m^{(2)}] = \delta_{n+m,0} (3\delta_{n,0} - 2) \Theta, \quad n, m \in \{±1, 0\},
\]

(1.7)
where the new central generator $\Theta$ is needed for this central extension. Physicists usually call this central extension of $\text{cga}(2)$ the \textit{exotic Galilei conformal algebra}, and we shall denote it by $\text{ecga}$. A representation in terms of infinitesimal space-time transformations was recently given in [18], which we repeat here and also include the phases parameterised by the $\gamma_j$:

$$
X_n = -t^{n+1}\partial_t - (n+1)t^n r \cdot \nabla - \lambda(n+1)t^n - (n+1)n t^{n-1} \gamma \cdot r - (n+1)n h \cdot r,
$$

$$
Y_n^{(j)} = -t^{n+1}\partial_j - (n+1)t^n \gamma_j - (n+1)t^n h_j - (n+1)n(r_2 - r_1)\theta,
$$

$$
R_0^{(12)} = -(r_1\partial_2 - r_2\partial_1) - (\gamma_1\partial_{r_2} - \gamma_2\partial_{r_1}) - \frac{1}{2\theta} h \cdot h,
$$

where $n \in \{\pm 1, 0\}$ and $j, k \in \{1, 2\}$. Because of Schur’s lemma, the central generator $\Theta$ can be replaced by its eigenvalue $\theta \neq 0$. The components of the vector-operator $h = (h_1, h_2)$ are connected by the commutator $[h_1, h_2] = \Theta$. We are interested in finding systems of non-linear PDEs which are invariant under the exotic conformal Galilei algebra $\text{ecga}$ and its subalgebras.

The paper is organised as follows. In section 2, we use the known techniques for finding Lie symmetries [2, 3, 4] and conditional symmetries [4] to construct non-linear partial differential equations (PDEs) with the $\text{cga}(N)$-symmetry, although explicit results will only be written down for $N = 2$ and shown how they extend to the case $N > 2$. Since the notion of conditional symmetry requires the introduction of an auxiliary condition, we also look for \textit{pairs} of non-linear equations which are invariant under the Lie algebra $\text{cga}(2)$. In section 3, we extend these results to the case of $\text{ecga}$. Large classes of systems of non-linear PDEs, which are invariant under $\text{ecga}$ and some of its subalgebras, are found. In section 4, we illustrate the results obtained through examples of correctly-specified systems, which can be reduced to the hydrodynamic equations of motion of a two-dimensional incompressible fluid subject to certain forces. Our conclusions are given in section 5.

## 2 Conformal Galilei-invariance and nonlinear PDEs

In this section, we use the standard representation (1.6) for the space-time transformations in the conformal Galilei algebra algebra $\text{cga}(N)$, but normalise it such that $\lambda = 0$ and $\gamma_j = 0$. For the sake of simplicity, and since we plan to consider the exotic central extension later, we restrict our attention to $N = 2$ from now on. Thus, we consider here the following representation of the algebra $\text{cga}(2)$

$$
X_n = -t^{n+1}\partial_t - (n+1)t^n (r_1\partial_1 + r_2\partial_2), \quad Y_n^{(1)} = -t^{n+1}\partial_1, \quad Y_n^{(2)} = -t^{n+1}\partial_2, \quad R_0^{(12)} = -(r_1\partial_2 - r_2\partial_1),
$$

where $n = \pm 1, 0$. Our aim is to describe all second-order PDEs, which are invariant under the altern algebra in the $(1+2)$-dimensional space of independent variables $t, r_1, r_2$. The most general form of such a PDE reads

$$
H \left(t, r_1, r_2, u, u_1, u_2\right) = 0
$$

Here and afterwards, we use the notations $u = (u_t, u_1, u_2) = (\partial u/\partial t, \partial u/\partial r_1, \partial u/\partial r_2)$, $u_1 = (u_{1t}, u_{11}, \ldots, u_{12})$, $\Delta = \partial_1^2 + \partial_2^2$ is the spatial Laplacian and $H$ is an arbitrary smooth function.
First of all, we consider the subalgebra with basic operators \( \langle X_{-1}, Y_{-1,0}, R_{12}^{(0)} \rangle \), which is well-known Galilei algebra with zero mass. Hereafter, we use the notation \( \mathfrak{g} \mathfrak{a} \mathfrak{l}(0)(N) \) with \( N = 2 \) for this algebra for which the notation \( \mathcal{A} \mathcal{G} \mathcal{O}(1,N) \) is also common, see [24]. Our starting point is the following well-established result [24, Theorem 7]: an arbitrary second-order PDE \( (2.2) \) is invariant under the massless Galilei algebra \( \mathfrak{g} \mathfrak{a} \mathfrak{l}(0)(2) \) if and only if it can be written in the form

\[
H_{\text{gal}}(u, u_a u_a, \Delta u, u_a u_b u_{ab}, u_{11} u_{22} - u_{12}^2, W^I, W^{II}) = 0, \quad (2.3)
\]

where the summation convention over repeated indices \( a = 1, 2 \) and \( b = 1, 2 \) is implied and \( H_{\text{gal}} \) is an arbitrary smooth function and

\[
W^I := \det \begin{bmatrix} u_t & u_1 & u_2 \\ u_{t1} & u_{11} & u_{12} \\ u_{t2} & u_{12} & u_{22} \end{bmatrix}, \quad W^{II} := \det \begin{bmatrix} u_{tt} & u_{t1} & u_{t2} \\ u_{t1} & u_{11} & u_{12} \\ u_{t2} & u_{12} & u_{22} \end{bmatrix}. \quad (2.4)
\]

In other words, the seven arguments of the function \( H_{\text{gal}} \) form a full set of absolute differential invariants (up to the second order) of the Galilei algebra \( \mathfrak{g} \mathfrak{a} \mathfrak{l}(0)(2) \). Recall that the invariants \( u_{11} u_{22} - u_{12}^2 \) and \( W^{II} \) are nothing else but the right-hand-side (RHS) of the well-known Monge-Ampère equation in two- and three-dimensional space, respectively [25], while the invariant determinant \( W^I \) was derived for the first time in [26].

Now consider the generators \( Y_1^{(1)} \) and \( Y_1^{(2)} \) of constant accelerations, which lead to the finite transformations

\[
t' = t, \quad r'_a = r_a + v_a t^2, \quad u' = u, \quad (2.5)
\]

where \( v_1 \) and \( v_2 \) are arbitrary parameters. It can be checked by direct computation that a PDE belonging to the class \( (2.3) \) is invariant under the transformations \( (2.5) \) only if the function \( H_{\text{gal}} \) does not depend on \( W^{II} \). Next, consider the scale-transformations generated by the operator \( X_0 = -t \partial_t - r_j \partial_j \) one obtains in a similar way a further restriction on the function \( H_{\text{gal}} \). Therefore, we arrive at our first result, where we consider the subalgebra of CGA(2) which is obtained when leaving out the accelerations \( Y_1^{(j)}, \ j = 1, 2 \) and projective transformations \( X_1 \).

**Theorem 1** A PDE of the form \( (2.3) \) is invariant under the following subalgebra of the conformal Galilei algebra

\[
\langle X_{-1,0}, Y_{-1,0}, R_{12}^{(12)} \rangle_{j=1,2} \subset \text{CGA}(2)
\]

with the realisation \( (2.7) \), if and only if this equation possesses the form

\[
H_0(u, Z_1, Z_2, Z_3, Z_4) = 0, \quad (2.6)
\]

where \( H_0 \) is an arbitrary smooth function and

\[
Z_1 := \Delta u \cdot (u_a u_a)^{-1}, \quad Z_2 := (u_a u_b u_{ab}) \cdot (u_a u_a)^{-2},
\]

\[
Z_3 := (u_{11} u_{22} - u_{12}^2) \cdot (u_a u_a)^{-2}, \quad Z_4 := W^I \cdot (u_a u_a)^{-5/2}. \quad (2.7)
\]
Remark 1. Theorem 1 can be easily generalised to the case $N = 3$ using paper [24]. Generalisation to higher dimensionality is also possible, but will lead to very lengthy expressions.

One observes that the invariant $Z_4$ is the only one in equation (2.6) which contains time derivatives. It follows, and can also be checked directly, that $Z_4$ cannot be modified by the invariants $u, Z_1, Z_2$ and $Z_3$ to a form that is invariant under projective transformations

$$t' = \frac{t}{(1- pt)}, \quad r'_a = \frac{r_a}{(1- pt)^2}, \quad u' = u,$$

(2.8)
generated by the operator $X_1$ from (2.1) with $N = 2$ (hereafter $p$ is the group parameter). In other words, a given PDE belonging to the class (2.6) can be invariant under the operator $X_1$ only in the case when it does not contain time derivatives, i.e., it contains the variable $t$ as a parameter. We are not interested in this case and hereafter consider only PDEs containing time derivatives.

Thus we can conclude: there are no non-trivial PDEs belonging to the class (2.2) that are invariant under the representation (2.1) of the conformal Galilei algebra $cga(2)$.

Indeed, we found earlier an analogous result for the infinite-dimensional extension of the representation (2.1) where $n \in \mathbb{Z}$ [27]. Following the same lines as in [27], we use now the concept of conditional invariance to construct PDEs that are conditionally invariant under $cga(2)$. The notion of conditional invariance was introduced in [28] as a generalisation of the well-known non-classical symmetry [29].

Definition. [4, Section 5.7] A PDE of the form

$$S(t, r, u, u_1, u_{11}) = 0$$

(2.9)
is said to be conditionally invariant under the operator

$$Q = \xi^0(t, r, u)\partial_t + \xi^a(t, r, u)\partial_a + \eta(t, r, u)\partial_u$$

(2.10)
where $r = (r_1, \ldots, r_N)$ and $\xi^a$ with $a = 0, 1, \ldots, N$ and $\eta$ are smooth functions, if it is invariant (in Lie’s sense) under this operator only together with an additional condition of the form

$$S_Q(t, r, u, u_1, u_{11}) = 0$$

(2.11)
that is, the over-determined system of equations (2.9,2.11) is invariant under a Lie group generated by the operator $Q$.

With this concept at hand, we can now prove the following.

Theorem 2 A PDE of the form (2.6) is conditionally invariant under the algebra $cga(2)$ with the realisation (2.1) if and only if the condition holds:

$$W^{III} := \det \begin{bmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{bmatrix} = 0.$$
Proof. One needs to establish a necessary and sufficient condition when a system, consisting of an arbitrary PDE of the form (2.6) and the condition in question, is invariant under the projective transformations (2.8). It is a straightforward calculation that the transformations (2.8) transform the derivatives as follows:

\[
\begin{align*}
    u'_a &= u_a (1 - pt)^2, \quad a = 1, 2, \\
    u'_a u'_b &= u_{ab} (1 - pt)^4, \quad a, b = 1, 2, \\
    u'_t &= (1 - pt)^3 (1 - pt) u_t - 2pr_b u_{ab} - 2pu_a.
\end{align*}
\]

(2.13)

So, using formulas (2.7) and (2.13) one easily shows that

\[
\begin{align*}
    Z'_1 &= Z_1, \\
    Z'_2 &= Z_2, \\
    Z'_3 &= Z_3,
\end{align*}
\]

(2.14)
i.e., all arguments of the function \(H_0\) from eq. (2.6), excepting \(Z_4\), are invariant under transformations (2.8). It turns out, \(Z_4\) is not invariant with respect to the projective transformations. In fact, substituting formulas (2.13) into the expression

\[
Z'_4 := (W^I)' \cdot (u'_a u'_a)^{-5/2},
\]

(2.15)
we obtain

\[
Z'_4 = Z_4 - 2p(1 - pt)^{-1} W^{III}.
\]

(2.16)
Thus, the projective transformations (2.8) bring the PDE (2.6) to the form

\[
H_0 \left( u', Z'_1, Z'_2, Z'_3, Z'_4 + 2p(1 - pt)^{-1} W^{III} \right) = 0.
\]

(2.17)
Since \(p\) is an arbitrary parameter, we conclude that expression (2.17) must take the form

\[
H_0 \left( u', Z'_1, Z'_2, Z'_3, Z'_4 \right) = 0
\]

(2.18)
if and only if the condition (2.12) holds. Hence, the PDE (2.6) is conditionally invariant under the algebra cga(2) and the corresponding condition must be equation (2.12). This completes the proof. ■

Remark 2. Condition (2.12) contains the time variable as a parameter and can be rewritten in the equivalent form \(u_a u_a \Delta u = u_a u_b u_{ab}\), i.e., \(Z_1 = Z_2\).

Remark 3. The simplest PDE that is conditionally invariant under cga(2) is \(W^I = 0\). Lie symmetries and a wide range of exact solutions for this equation have been constructed in [26, 30].

Remark 4. Theorem 2 can be generalised to the case \(N = 3\). The condition needed has the form

\[
W^{III} := \det \begin{bmatrix}
0 & u_1 & u_2 & u_3 \\
1 & u_{11} & u_{12} & u_{13} \\
2 & u_{12} & u_{22} & u_{23} \\
3 & u_{31} & u_{32} & u_{33}
\end{bmatrix} = 0.
\]

Theorems 1 and 2 state that although there are no second-order PDEs admitting cga(2)-symmetry in the Lie sense, there is a wide class of such PDEs possessing this symmetry algebra when the concept
of conditional invariance is used. In order to understand where this result comes from, we consider now the system of PDEs

\[
\begin{align*}
H_1 \left( t, r_1, r_2, u, v, u_1, u_{11}, v_1, v_{11} \right) &= 0 \\
H_2 \left( t, r_1, r_2, u, v, u_1, u_{11}, v_1, v_{11} \right) &= 0,
\end{align*}
\]

where \( H_1 \) and \( H_2 \) are arbitrary smooth functions while \( u(t, r_1, r_2) \) and \( v(t, r_1, r_2) \) are unknown functions.

**Theorem 3** A system of PDEs of the form (2.19) is invariant under the algebra \( \text{CGA}(2) \) with the realization (2.1) if it takes the form

\[
\begin{align*}
H_{\text{CGA}} (u, v, Z_{uv}, Z_{u_1}, \ldots, Z_{u_4}, Z_{v_1}, \ldots, Z_{v_4}) &= 0 \\
\lambda_1 W_{III}(u, u, u) + \lambda_2 W_{III}(v, u, u) + \lambda_3 W_{III}(u, v, v) + \lambda_4 W_{III}(v, v, v) &= 0
\end{align*}
\]

where \( H_{\text{CGA}} \) is an arbitrary smooth function, \( \lambda_1, \ldots, \lambda_4 \) are arbitrary constants (at least one of them must be non-zero) and the notations are used:

\[
\begin{align*}
Z_{u_1} &= u_a u_a / u_a v_a, & Z_{v_1} &= v_a v_a / u_a v_a \\
Z_{u_2} &= \Delta u / u_a v_a, & Z_{v_2} &= \Delta v / u_a v_a \\
Z_{u_3} &= (u_a u_b u_{ab}) / (u_a v_a)^2, & Z_{v_3} &= (v_a v_b v_{ab}) / (u_a v_a)^2 \\
Z_{u_4} &= (u_{11} u_{22} - u_{12}^2) / (u_a v_a)^2, & Z_{v_4} &= (v_{11} v_{22} - v_{12}^2) / (u_a v_a)^2,
\end{align*}
\]

\[
Z_{uv} = (u_a u_a)^{-5/2} \left( \lambda_1 W_I(u, u, u) + \lambda_2 W_I(v, u, u) + \lambda_3 W_I(u, v, v) + \lambda_4 W_I(v, v, v) \right); \quad (2.22)
\]

\[
W_I(u, v, w) := \det \begin{bmatrix} u_t & u_1 & u_2 \\
                  u_1 & v_1 & v_2 \\
                  w_{12} & w_{12} & w_{22} \end{bmatrix}, \quad W_{III}(u, v, w) := \det \begin{bmatrix} 0 & u_1 & u_2 \\
                           v_1 & v_1 & v_2 \\
                           w_2 & w_2 & w_{22} \end{bmatrix}. \quad (2.23)
\]

**Proof:** This result is readily derived by direct application of the continuous transformations generated by the basic operators (2.1) to the system (2.20).

The system (2.20) contains the time derivatives only in the case if the function \( H_{CGA} \) explicitly depends on \( Z_{uv} \). The second equation of the system contains the variable \( t \) as a parameter and reproduces the condition (2.12) if one sets \( \lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0 \) (simultaneously \( Z_{uv} \) is reduced to \( Z_4 \)).

### 3 Systems of non-linear PDEs invariant under the exotic conformal Galilei algebra

In this section we study possibilities to construct systems of PDEs that are invariant under \( \text{ECGA} \). This can only be done in \( N = 2 \) space dimensions, since only then the exotic central extension exists [23]. The application of the Lie scheme requires an explicit realization in terms of linear first-order
differential operators, which have been constructed very recently \cite{18} and listed in (1.8). It means that the operators \(h_1, h_2\) and \(\Theta\) must be correctly specified. According to the paper \cite{18}, the possible explicit realization of the algebra ECGA reads as

\[
X_{-1} := -\partial_t, \quad Y_{-1}^{(1)} := -\partial_t, \quad Y_{-1}^{(2)} := -\partial_2, \quad \Theta := \partial_w \tag{3.1}
\]

\[
Y_0^{(1)} := -t\partial_1 + \partial_{v^1} - \frac{v^2}{2} \partial_{w}, \quad Y_0^{(2)} := -t\partial_2 + \partial_{v^2} + \frac{v^1}{2} \partial_{w} \tag{3.2}
\]

\[
Y_1^{(1)} := -t^2 \partial_1 + 2t \partial_{v^1} - (2r_2 + tv^2) \partial_{w}, \quad Y_1^{(2)} := -t^2 \partial_2 + 2t \partial_{v^2} + (2r_1 + tv^1) \partial_{w} \tag{3.3}
\]

\[
X_0 := -t\partial_t - r_1 \partial_1 - r_2 \partial_2 \tag{3.4}
\]

\[
X_1 := -t(t\partial_t + 2r_1 \partial_1 + 2r_2 \partial_2) + r_1 \partial_{v^1} + r_2 \partial_{v^2} - (r_1 v^2 - r_2 v^1) \partial_{w} \tag{3.5}
\]

\[
R_{0}^{(12)} := -r_1 \partial_2 + r_2 \partial_1 - v^1 \partial_{v^1} + v^2 \partial_{v^2}, \tag{3.6}
\]

where \(\partial_w = \partial/\partial w, \ \partial_{v^1} = \partial/\partial v^1, \ \partial_{v^2} = \partial/\partial v^2\).

Clearly, one has the massless Galilei subalgebra with the basic operators \(\mathfrak{gal}^{(0)}(2) = \{X_{-1}, Y_{-1,0}^{(j)}, R^{(12)}\}\). Because of the additional variables \(v^1, v^2\) and \(w\), however, this realisation cannot be reduced to the one studied in sections 1 and 2. The case when the generators \(Y_0^{(j)}\) of Galilei transformations depend only on the variables \(v^1\) and \(v^2\) (set formally \(\partial_w = 0\) in (3.2)) is rather typical \cite{24}.

To prepare for the construction of PDEs that admit ECGA with the realisation (3.1–3.6) one should choose dependent and independent variables. The example given in \cite{18} suggests that all the variables \(t, r_1, r_2, v^1, v^2\) and \(w\) are independent. Furthermore, a linear second-order PDE on the function \(\Psi(t, r_1, r_2, v^1, v^2, w)\), which is invariant under ECGA, is constructed. That example is very simple but rather artificial. In fact, the equation (3.11) \cite{18} reads as

\[
\frac{\partial^2 \Psi}{\partial t \partial w} - \frac{\partial^2 \Psi}{\partial r_1 \partial v^1} + \frac{\partial^2 \Psi}{\partial r_2 \partial v^1} - \frac{v^1}{2} \frac{\partial^2 \Psi}{\partial r_1 \partial w} - \frac{v^2}{2} \frac{\partial^2 \Psi}{\partial r_2 \partial w} = 0. \tag{3.7}
\]

Using Maple, we have established that this equation is invariant under an 28-dimensional Lie algebra containing the 11-dimensional algebra ECGA with realisation (3.1–3.6) as a subalgebra. Since the standard 6-dimensional wave equation is also invariant under this 28-dimensional Lie algebra, which is the conformal algebra with the standard realization, equation (3.7) can be nothing else but the wave equation in an unusual form.

Our aim is to construct PDEs in the (1+2)-dimensional space of the variables \(t, r_1, r_2\). This immediately leads to the need to consider systems of PDEs with the unknown functions \(v^1, v^2\) and \(w\). Hereafter the notations

\[
u^1 = v^1, \quad u^2 = v^2, \quad u^3 = w \tag{3.8}
\]

are used to simplify the corresponding formulæ.

We start from the most general system of first-order PDEs

\[
B^k = 0, \quad k = 1, 2, 3 \tag{3.9}
\]
where $B^1, B^2$ and $B^3$ are arbitrary sufficiently smooth functions independent variables $t, r_1, r_2$, dependent variables (3.8) and their first-order derivatives. Obviously, to construct systems with the ECGA-invariance, one straightforwardly obtains that all the functions in (3.9) cannot depend on the variables $t, r_1, r_2$ and $u^3$ (see the generators (3.1) of the exotic conformal Galilei algebra). Hence $B^1, B^2$ and $B^3$ are assumed to be functions on

$$u^1, u^2, u^3,$$  
(3.10)

where $u^k = (u^k_0, u^k_1, u^k_2), k = 1, 2, 3$. To avoid possible misunderstanding, we also assume that system (3.9) doesn’t contain algebraic equation(s) and cannot be reduced to one containing algebraic equation(s).

It will be helpful to proceed step by step and to consider several subalgebras of the algebra ECGA. Specifically, we shall consider the following cases:

1. the 8-dimensional subalgebra $\mathfrak{ca}_1 := \langle X_{-1}, Y^{(j)}_{-1,0,1}, \Theta \rangle j = 1, 2$, which contains the operators of time- and space-translations, the Galilei transformations, accelerations and the generator $\Theta$ for the central extension. The Lie algebra $\mathfrak{ca}_1$ is semidirect sum of the Weil algebra and the Abelian algebra generated by the two 'exotic' operators (3.3) and the operator $\Theta$.

2. the 10-dimensional subalgebra $\mathfrak{ca}_2 := \langle X_{-1,0,1}, Y^{(j)}_{-1,0,1}, \Theta \rangle$, where with respect to $\mathfrak{ca}_1$ the dilatation operator $X_0$ and the projective operator $X_1$ are added.

3. the 10-dimensional subalgebra $\mathfrak{ca}_3 := \langle X_{-1,0}, Y^{(j)}_{-1,0,1}, \Theta, R^{(12)}_0 \rangle$, where with respect to $\mathfrak{ca}_1$ the dilatation operator $X_0$ and the rotation operator $R^{(12)}_0$ are added.

4. the 11-dimensional ECGA with the basic operators (3.1–3.6).

**Theorem 4** A system of PDEs system of the form (3.9) is invariant under the Lie algebra $\mathfrak{ca}_1$ if and only if it is of the form

$$b^k \left(u^1_1, u^2_1, u^2_2, W_1, W_2, W_3\right) = 0, \quad k = 1, 2, 3$$

(3.11)

where $b^1, b^2$ and $b^3$ are arbitrary smooth functions of the variables $u^1_1, u^2_1, u^2_2, W_1, W_2$ and $W_3$. Here, the notations

$$W_1 := 2u^1_0 + 2u^3_2 - 2u^1_1 + u^1_2u^2_1 - 3u^2_1,$$  
(3.12)

$$W_2 := 2u^2_0 - 2u^3_1 - 2u^2_2 + u^2_1u^1_2 - 3u^1_2,$$  
(3.13)

$$W_3 := 2u^3_0 - u^2_1u^2_1 + u^1_0u^2_2 - 2u^1_1u^3_2 - 2u^2_1u^3_1 + u^1_1u^1_1 - u^1_2u^2_2 - u^1_1u^2_1 + u^2_2u^1_2$$  
(3.14)

are used.
Proof. Consider the algebra \( \mathfrak{ca}_1 \). A straightforward analysis shows that system (3.9) will automatically be invariant under the Galilei operators (3.2) if it is invariant with respect to the acceleration operators (3.3). Thus, to construct systems with \( \mathfrak{ca}_1 \)-symmetry we need to find among systems of the form (3.9) those, which are invariant under the operators \( Y^{(j)}_1, j = 1, 2 \), i.e. under the operator

\[
X \equiv -\alpha_1 Y^{(1)}_1 - \alpha_1 Y^{(1)}_1 = \alpha_1 t^2 \partial_t + \alpha_2 t^2 \partial_t - 2\alpha_1 t \partial_{u_1} - 2\alpha_2 t \partial_{u_2} + (\alpha_1 (2r_2 + tu_1^2) - \alpha_2 (2r_1 + tu_1^1)) \partial_{u_3},
\]

where \( \alpha_1 \) and \( \alpha_2 \) are arbitrary parameters.

The construction is based on a direct application of the classical Lie algorithm \([2, 3, 4]\). According to this algorithm, we consider system (3.9) as the manifold \( S \). To construct the set mentioned above one needs to use the invariance conditions

\[
\bar{X} B^k |_{S_j = 0, j = 1, 2, 3} = 0, \quad k = 1, 2, 3,
\]

where \( \bar{X} \) is the first prolongation of the operator \( X \) (3.15). The explicit form of \( \bar{X} \) is calculated by the well-known prolongation formulae (see, e.g. \([2, 3, 4]\)). In the case of of the operator \( \bar{X} \) (3.15), these formulae lead to the operator

\[
\bar{X} = X - 2(\alpha_1 + \alpha_1 t u_1 + \alpha_2 t u_2) \partial_{u_1} - 2(\alpha_2 + \alpha_2 t u_2 + \alpha_1 t u_1^2) \partial_{u_2} + \\
+ (\alpha_1 u_2 - \alpha_2 u_1 + \alpha_1 t (u_2 - 2u_1^2) - \alpha_2 t (u_1^2 + 2u_2^3)) \partial_{u_3} + (-2\alpha_2 - \alpha_2 t u_1 + \alpha_1 t u_1^2) \partial_{u_3} + (2\alpha_1 - \alpha_2 t u_2 + \alpha_1 t u_2^2) \partial_{u_3}. 
\]

Substituting operator (3.18) into system (3.17) and carrying out the relevant calculations, we obtain a system of linear first-order PDEs for finding the functions \( B^k \). Firstly, we aim to find the set of absolute zero- and first-order differential invariants of the algebra \( \mathfrak{ca}_1 \) and then to show that the functions \( B^k \) can depend only on those invariants. To construct the set mentioned above one needs to substitute operator (3.18) into system (3.17) without the conditions \( S_j = 0, j = 1, 2, 3 \), and to solve three linear first-order PDEs obtained (they are omitted here). Since those PDEs contain two arbitrary parameters and the unknown functions don’t depend on the time-variable, each PDE can be splitted with respect to the \( \alpha_1, \alpha_2, \alpha_1 t \) and \( \alpha_2 t \). Thus, we arrive at the system of 12 PDEs:

\[
-2\partial_{u_0} B^k + u_2^2 \partial_{u_3} B^k + 2\partial_{u_2} B^k = 0
\]

\[
2\partial_{u_0} B^k + u_1^2 \partial_{u_0} B^k + 2\partial_{u_1} B^k = 0
\]

\[
2\partial_{u_1} B^k + 2u_1^2 \partial_{u_0} B^k + 2u_1^2 \partial_{u_0} B^k + (2u_1^3 - u_0^2) \partial_{u_0} B^k - u_1^2 \partial_{u_1} B^k - u_2^2 \partial_{u_2} B^k = 0
\]
\[ 2\partial_{u^2} B^k + 2u_2 \partial_{u_0} B^k + 2u_2 \partial_{u_0} B^k + (2u_2^2 + u_0^2) \partial_{u_0} B^k + u_1 \partial_{u_1} B^k + u_2 \partial_{u_2} B^k = 0, \quad (3.22) \]

where \( k = 1, 2, 3 \). Now one notes that this system consists of three separate subsystems for \( k = 1, k = 2 \) and \( k = 3 \) so that its general solution can be found by solving these subsystems. A straightforward application of the standard technics leads to the following general solution:

\[ B^k = b^k \left( u_1^k, u_2^k, u_0^k, u_1^2, W_1, W_2, W_3 \right), \quad (3.23) \]

where \( W_j, j = 1, 2, 3 \) are defined in \( (3.12-3.14) \). Thus, there are exactly seven absolute first-order differential invariants (no zero-order invariants!) of the algebra \( \mathfrak{ca}_1 \) so that an arbitrary system of the form \( (3.11) \) admits this algebra.

To prove that system \( (3.11) \) is the most general system with \( \mathfrak{ca}_1 \)-symmetry, one needs to solve \( (3.17-3.18) \), i.e. to take into account the conditions \( S_j = 0, j = 1, 2, 3 \). We assumed from the very beginning that system \( (3.9) \) doesn’t contain algebraic equation(s) and \( B^k \) are arbitrary sufficiently smooth functions, hence system \( (3.9) \) can be solved with respect to three different first-order derivatives. Assuming that these derivatives are smooth functions, hence system \( (3.9) \) can be solved with respect to three different first-order derivatives.

\[ S_k \equiv D^k \left( u_1^k, u_2^k, u_0^k, u_1^2, u_0^3, u_2^3 \right) - u_0^k = 0 \quad k = 1, 2, 3 \quad (3.24) \]

where \( D^k \) are smooth functions, which can be assumed arbitrary. Substituting operator \( (3.18) \) into system \( (3.17) \), applying \( (3.24) \) to eliminate the variables \( u_0^1, u_0^2 \) and \( u_0^3 \) and carrying out the relevant calculations, we obtain a system of linear first-order PDEs for finding the functions \( D^k \). Each PDE can be again splitted with respect to the \( \alpha_1, \alpha_2, \alpha_1 t \) and \( \alpha_2 t \). Finally, we arrive at the system of 12 PDEs, which is nothing else but system \( (3.19)-(3.22) \) with

\[ B^k = D^k \left( u_1^k, u_2^k, u_0^k, u_1^2, u_0^3, u_2^3 \right) - u_0^k, \quad k = 1, 2, 3. \quad (3.25) \]

Setting \( k = 1 \) one obtains the subsystem

\[ \partial_{u_2} D^1 = -1, \quad \partial_{u_1} D^1 = 0 \quad (3.26) \]

\[ 2\partial_{u^1} D^1 = 2u_1 - u_2, \quad 2\partial_{u^2} D^1 = 3u_2 \quad (3.27) \]

to find the function \( D^1 \). Solution of this system is rather trivial and gives

\[ D^1 = \frac{1}{2} d^1(u_1, u_2, u_1^2, u_2^2) - \frac{1}{2}(2u_2^3 - 2u_1 u_1^2 + u_1 u_2^2 - 3u_2^2), \quad (3.28) \]

where \( d^1 \) is an arbitrary function. Substituting \( D^1 \) into \( (3.24) \) for \( k = 1 \), we immediately obtain that the first equation of system \( (3.9) \) must possess the form

\[ W_1 = d^1(u_1, u_1^2, u_2, u_2^2). \quad (3.29) \]

In a quite similar way, the second and third equation were found:

\[ W_2 = d^2(u_1, u_1^2, u_2, u_2^2), \quad W_3 = d^3(u_1, u_1^2, u_2, u_2^2). \quad (3.30) \]
However, it is easily seen that system (3.29-3.30) is a particular case of the \( \mathfrak{ca}_1 \)-invariant system (3.11).

If system (3.9) cannot be solved with respect to the variables \( u_0^1, u_0^2 \) and \( u_0^3 \) then one must be solved with respect to another triplet of derivatives. Nevertheless, making the relevant calculations, a system of 12 PDEs is obtained, which will be again a particular case of system (3.19)-(3.22). Thus, its solution will lead to another particular case of system (3.11). This completes the proof.

\[\blacksquare\]

**Remark 5.** Formulae (3.12–3.14) and \( u_1^1, u_1^2, u_2^1, u_2^2 \) present the full set of absolute first-order differential invariants of the algebra \( \mathfrak{ca}_1 \).

**Theorem 5** A system of PDEs of the form (3.11) is invariant under the Lie algebra \( \mathfrak{ca}_2 \) if and only if it can be written in the form

\[ h^k(U^1, U^2, W^*_1, W^*_2, W^*_3) = 0, \quad k = 1, 2, 3, \]  

(3.31)

where \( h^1, h^2 \) and \( h^3 \) are arbitrary smooth functions of five variables

\[ W^*_k := \frac{W_k}{u_2^k - u_1^k}, \quad k = 1, 2, 3, \quad \frac{u_1^1 - u_2^1}{u_2^k - u_1^k}, \quad \frac{u_2^1}{u_2^1 - u_1^1}. \]  

(3.32)

**Proof:** To construct all PDEs’ systems with \( \mathfrak{ca}_2 \)-invariance, we need to find among systems of the form (3.11) those, which are invariant under the operator (3.5). Note that scale-invariance (see operator (3.4)) will automatically be obtained if the invariance with respect to the projective operator (3.5) holds true.

The operator (3.5) produces the projective transformations

\[ t \mapsto t' = \frac{t}{(1 - pt)}, \quad r_a \mapsto r'_a = \frac{r_a}{(1 - pt)^2} \]  

(3.33)

for the independent variables and

\[ u^a \mapsto (u^a)' = u^a - \frac{2pr_a}{1 - pt}, \quad a = 1, 2, \quad u^3 \mapsto (u^3)' = u^3 - \frac{p(r_1u^2 - r_2u^1)}{1 - pt} \]  

(3.34)

for the dependent variables.

One may directly check that the transformations (3.33) (3.34) transform the set of absolute first-order differential invariants of the algebra \( \mathfrak{ca}_1 \) as follows:

\[ u^a \mapsto (u^a)'_a = (1 - pt)^2u^a_a - 2p(1 - pt), \quad u^a_b \mapsto (u^a)'_b = (1 - pt)^2u^a_b, \quad a, b = 1, 2, \quad b \neq a \]  

(3.35)

\[ W_k \mapsto (W_k)' = (1 - pt)^2W_k, \quad k = 1, 2, 3. \]  

(3.36)

Now it is easily seen that at most five independent absolute first-order differential invariants of the algebra \( \mathfrak{ca}_2 \) can be constructed. The form of these invariants can be taken in different ways, however
the resulting sets of invariants will be equivalent. In the particular case, one can suggest the form (3.32) assuming \( u_2^1 - u_1^2 \neq 0 \).

If one assumes that \( u_2^1 - u_1^2 = 0 \) then the system of PDEs’ in question must contain this equation (otherwise the system of four PDEs will be obtained). To construct the other two equations, we again use the formulæ (3.33) (3.36). Since \( u_2^1 - u_1^2 = 0 \), one may assume that \( u_1^1 - u_1^2 \neq 0 \), hence four invariants

\[
\frac{W_k}{u_1^1 - u_2^1}, \quad k = 1, 2, 3, \quad \frac{u_1^1}{u_1^1 - u_2^1}
\]

(3.37)

are obtained. Finally, we arrive at the system of the form

\[
g^1 = 0, \quad g^2 = 0, \quad u_1^1 - u_1^2 = 0,
\]

(3.38)

where \( g^1 \) and \( g^2 \) are arbitrary smooth functions of invariants (3.37). Now one easily notes that system (3.38) can formally be derived from (3.31) as a particular case.

Assuming \( u_1^1 - u_1^2 = 0 \), we again obtain a particular case of the system (3.31). This completes the proof.

Now we want to find systems belonging to the class of systems (3.11), which are invariant under the Lie algebra \( \mathfrak{ca}_3 \). Such systems are described by the following statement.

**Theorem 6** A PDEs’ system of the form (3.11) is invariant under the Lie algebra \( \mathfrak{ca}_3 \) if and only if it can be reduced to the form

\[
\mathcal{B}^1 = 0, \quad \mathcal{B}^2 = 0, \quad \mathcal{B}^3 = 0
\]

(3.39)

where \( \mathcal{B}^k, \ k = 1, 2, 3 \) are arbitrary smooth functions of the variables

\[
W_{12}^* := \frac{W_1^2 + W_2^2}{(u_2^1 - u_1^2)^2}, \quad W_3^* := \frac{W_3}{u_2^1 - u_1^2}, \quad W^* := \frac{u_1^1 W_1^2 + u_2^2 W_2^2 + (u_1^1 + u_1^2) W_1 W_2}{(u_2^1 - u_1^2)^3},
\]

\[
\frac{u_1^1 + u_2^2}{u_2^1 - u_1^2}, \quad \frac{U}{(u_2^1 - u_1^2)^2}
\]

(3.40)

with the abbreviation \( U = (u_1^1)^2 + (u_2^1)^2 + (u_1^2)^2 + (u_2^2)^2 \).

**Proof.** This is shown in a way very similar with respect to the previous theorem and therefore omitted here. In fact, the direct application of the transformations

\[
t \mapsto t' = t, \quad r_1 \mapsto r_1' = r_1 \cos p + r_2 \sin p, \quad r_1 \mapsto r_2' = -r_1 \sin p + r_2 \cos p
\]

(3.41)

\[
r_1 \mapsto (u^1)' = u_1^1 \cos p + u_2^1 \sin p, \quad r_1 \mapsto (u^2)' = -u_1^1 \sin p + u_2^1 \cos p, \quad u^3 \mapsto (u^3)' = u_3
\]

(3.42)

generated by the operator (3.6) to the system of equations (3.11) leads to the \( \mathfrak{ca}_3 \)-invariant systems of the form (3.39) (3.40).

Finally, using theorems 5 and 6 we can prove the theorem giving a complete description of PDEs systems of the form (3.39), which are invariant under ECGA with the basic operators (3.1) (3.6).
Theorem 7  A PDEs’ system of the form (3.4) is invariant under the Lie algebra ECGA if and only if it possesses the form

\[ \mathcal{H}^1 = 0, \quad \mathcal{H}^2 = 0, \quad \mathcal{H}^3 = 0, \]  

(3.43)

where \( \mathcal{H}^k \) are arbitrary smooth functions of the four variables \( W_{12}^*, \ W_3^* \) and

\[ U^* := \frac{(u_1^1 - u_2^2)^2 + 2(u_2^1)^2 + 2(u_1^3)^2}{(u_2^1 - u_1^2)^2}, \quad V^* := 2W^* - \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2}W_{12}^* \]

Proof

is based on theorem 6 and formulae (3.33) (3.36). We need to find necessary and sufficient conditions when the given system of the form (3.39) (3.40) admits transformations (3.33) (3.34) generated by the projective operator.

All equations in (3.39) have the same structure so that we can consider them together. Since the functions \( \mathcal{B}^1, \mathcal{B}^2 \) and \( \mathcal{B}^3 \) may depend on five variables at maximum, one needs to find how these arguments are transformed by the projective transformations (3.33) (3.34). Using formulae (3.35) (3.36) one easily establishes that

\[ (W_{12}^*)' = \left( \frac{W_1^2 + W_2^2}{(u_2^1 - u_1^2)^2} \right)' = \frac{W_1^2 + W_2^2}{(u_2^1 - u_1^2)^2} - W_{12}^*, \quad (W_3^*)' = \left( \frac{W_3}{u_2^1 - u_1^2} \right)' = \frac{W_3}{u_2^1 - u_1^2} = W_3^* \]  

(3.44)

so that \( W_{12}^* \) and \( W_3^* \) are absolute first-order differential invariants of ECGA. Three other variables are transformed as follows

\[ (W^*)' = W^* - \frac{2pW_{12}^*}{(1 - pt)(u_2^1 - u_1^2)}, \]

(3.45)

\[ \left( \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2} \right)' = \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2} - \frac{4p}{(1 - pt)(u_2^1 - u_1^2)}, \]

(3.46)

\[ \left( \frac{U}{(u_2^1 - u_1^2)^2} \right)' = \frac{U}{(u_2^1 - u_1^2)^2} + \frac{8p^2}{(1 - pt)^2(u_2^1 - u_1^2)^2} - \frac{4p(u_1^1 + u_2^2)}{(1 - pt)(u_2^1 - u_1^2)^2}. \]

(3.47)

One observes that there is the possibility to construct the third and fourth absolute first-order differential invariants of ECGA using formulae (3.35) (3.47):

\[ U^* := 2U \left( \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2} \right)^2 - \left( \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2} \right)^2 + 2(u_2^1)^2 + 2(u_1^3)^2 \]

(3.48)

and

\[ V^* := 2W^* - \frac{u_1^1 + u_2^2}{u_2^1 - u_1^2}W_{12} = \frac{(u_1^1 - u_2^2)(W_1^1 - W_2^2) + 2(u_2^1 + u_2^1)W_1W_2}{(u_2^1 - u_1^2)^2}. \]

(3.49)

Thus, to be invariant under transformations (3.33) (3.34) the equations (3.39) must contain the functions \( \mathcal{B}^k = \mathcal{H}^k(W_{12}^*, W_3^*, U^*, V^*) \), where \( \mathcal{H}^k, k = 1, 2, 3 \) are smooth functions. In the case of arbitrary functions \( \mathcal{H}^k, k = 1, 2, 3 \), we obtain the most general form of the first-order PDEs’ system that admits ECGA.

This completes the proof. \( \square \)

Remark 6. ECGA can be treated as a highly non-trivial extension of the ‘massless’ Schrödinger algebra \( \text{sch}^{(0)}(2) \) by the ‘exotic’ operators (3.3), and the ‘mass’ operator \( \Theta \). Note the last operator is produced only by the ‘exotic’ operators because of the commutation relation (1.7).
4 Examples of non-linear systems with invariances related to the ECGA

We now illustrate the content of the general theorems presented in section 4 through a few examples.

Example 1. One of the simplest systems belonging to the class systems, which are invariant under the 10-dimensional Lie algebra $\mathfrak{e}_3$, is read off from theorem 6

$$W_1 = 0, \quad W_2 = 0, \quad u_1^1 + u_2^2 = 0. \quad (4.1)$$

This system can formally be derived from $(3.39)$ if one sets $B_a = W^*_{12}, a = 1, 2$ and $B_3 = \frac{u_1^1 + u_2^2}{u_2^1 - u_1^1}$. Note that the system $(4.1)$ is not invariant under the projective transformations because the last equation is incompatible with theorem 5.

The first and second PDEs in $(4.1)$ can be simplified using the last equations from this system. The change of variables $r_1 \mapsto \frac{\sqrt{2}}{2} x, r_2 \mapsto \frac{\sqrt{2}}{2} y$ and $u^3 \mapsto \frac{\sqrt{2}}{2} w$ with $q \in \mathbb{R}$ brings this system to the form

$$u_1^1 t + u_1^1 u_1^1 + u_2^2 u_1^1 - qw_y = 0$$
$$u_2^2 + u_1^2 u_2^2 + u_2^2 u_2^2 + qw_x = 0$$
$$u_1^1 x + u_2^2 y = 0. \quad (4.2)$$

It may be more appealing to restate this in a vector notation

$$\nabla \cdot v = 0, \quad (\partial_t + v \cdot \nabla)v - q\nabla \wedge \omega = 0 \quad (4.3)$$

where $v = (u^1, u^2, 0)$ describes the velocity of a two-dimensional incompressible flow, $\wedge$ means vector product, $\omega = (0, 0, w)$ and $\nabla = (\partial_x, \partial_y, 0)$.

Remark 7. Equations $(4.3)$ can be formally obtained from the Navier-Stokes equations, generalised to include rotational forces [31, eq. (2.39)], when restricting them to a planar motion of an incompressible fluid of density $\rho$. Because of theorem 6, we have in addition $\nabla \wedge v = 0$ and if one identifies $q = 2\eta_{\text{rot}}/\rho$, where $\eta_{\text{rot}}$ is the rotational viscosity, system $(4.3)$ is recovered.

We point out that MAI of $(4.2)$ is infinite-dimensional because the system does not explicitly contain $w$ and $w_t$. For example, the system admits the operator $X_{\infty} = \phi(t)\partial_w$ with the arbitrary given smooth function $\phi(t)$. In fact, this operator generate the transformations

$$t \mapsto t', x \mapsto x' = x, \quad y' = y, \quad u^a \mapsto (u^a)' = u^a, \quad a = 1, 2, \quad u^3 \mapsto (u^3)' = u^3 + p\phi(t), \quad (4.4)$$

which preserve the form of system $(4.2)$.

In order to appreciate better this example, we recall briefly the well-known shallow-water equations, of the form [32][33]

$$u_1^1 + u_1^1 u_1^1 + u_2^2 u_1^1 + qw_x = 0$$
$$u_2^2 + u_1^2 u_2^2 + u_2^2 u_2^2 + qw_y = 0$$
$$w_t + (u^1 w)_x + (u^2 w)_y = 0. \quad (4.5)$$
The equivalent vector form is

\[ \partial_t w + \nabla \cdot (w \mathbf{v}) = 0, \quad (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + q \nabla w = 0 \]  

(4.6)

with the same notations as above and where \( \mathbf{v} \) is the fluid velocity, \( w \) is the free surface height over the flat bottom and the constant \( q \) describes the effect of gravity. It should be noted that the model for the two-dimensional polytropic gas dynamics has also form (4.5) (see [2] and the references cited therein).

The systems (4.2) and (4.5) have a very similar structure and trivially coincide if \( w = \text{const.} \).

However, the MAI of (4.5) is nine-dimensional with the basic generators [2]

\[
X_{-1} = -\partial_t, \quad Y_{-1}^{(1)} = -\partial_x, \quad Y_{-1}^{(2)} = -\partial_y
\]

(4.7)

\[
Y_0^{(1)} = -t\partial_x - \partial_{w^1}, \quad Y_0^{(2)} = -t\partial_y - \partial_{w^2}
\]

(4.8)

\[
X_0 = -t\partial_t - x\partial_x - y\partial_y + u^1\partial_{u^1} + u^2\partial_{u^2} + 2w\partial_w
\]

(4.9)

\[
X_1 = -t(t\partial_t + x\partial_x + x\partial_y) - (x - tu^1)\partial_{w^1} - (y - tu^2)\partial_{w^2} + 2tw\partial_w
\]

(4.10)

\[
R_0^{(12)} = -x\partial_y + y\partial_x - u^1\partial_{w^1} + u^2\partial_{w^2},
\]

(4.11)

\[
D = t\partial_t + x\partial_x + y\partial_y.
\]

(4.12)

This Lie algebra, which we denote by \( \tilde{\mathfrak{sch}}^{(0)}(2) \), is the semi-direct sum of the massless Schrödinger algebra \( \mathfrak{sch}^{(0)}(2) \) and a further dilatation generator \( D \), which belongs to the Cartan subalgebra of the conformal algebra in four dimensions into which \( \mathfrak{sch}^{(0)}(2) \) is imbedded [10].

Thus, the systems (4.2) and (4.5) have essentially different symmetry properties, although at first sight they appear to have a similar structure. This structural difference implies that these two systems should correspond to different physical situations.

**Example 2.** The simplest examples of non-linear PDEs possessing eCGA-invariance may be read off from theorem 7. We now give two of them. The first one reads

\[
W_1 = 0, \quad W_2 = 0, \quad u_2^1 - u_1^2 = 0
\]

(4.13)

which in the same vector notation as in Example 1 can be written as

\[
\nabla \wedge \mathbf{v} = 0, \quad \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{2}(\mathbf{v} \wedge \nabla) \wedge \mathbf{v} = q \nabla \wedge \omega.
\]

(4.14)

The MAI of (4.13) is also infinite-dimensional because this system is invariant under the transformations (4.4).
The second example of an ECGA-invariant system simply is

\[ W_1 = 0, \quad W_2 = 0, \quad W_3 = 0. \] (4.15)

but apparently cannot be rendered in a simple vectorial form. The MAI of the system (4.15) is finite-dimensional, in contrast to the systems listed above.

5 Conclusions

In this paper, the Lie and conditional symmetry methods were applied to find non-linear PDEs admitting the conformal Galilei algebra CGA. Theorems 1 and 2 state that a single PDE of either first or second order can possess this algebra only in the sense of a conditional symmetry. However, we have constructed a wide class of systems of PDEs, which are invariant under the CGA and theorem 3 gives the structure of such systems.

The main part of work is devoted to the, so-called ‘exotic’ conformal Galilei algebra, abbreviated here by ECGA. We remind the reader that the explicit realisation of this algebra in terms of the first-order linear operators was found very recently [18] so that we restricted ourselves to this. To the best of our knowledge, there are not yet any papers devoted to mathematically rigorous deductions of PDEs with ECGA-symmetry. By studying the invariance of systems of second-order PDEs under several subalgebras of ECGA (see theorems 4-7), the rôle of the several possible extensions of the massless Galilei algebra which is the common subalgebra, can be appreciated. We believe that the most significant of our results is presented in theorem 7. If fact, we have constructed the most general form of the system of the first-order PDEs that admits the exotic conformal Galilei algebra.

Finally, a few examples of systems of PDEs’ invariant under ECGA were presented, which illustrate the theorems obtained, and the similarities and differences with respect to the well-known shallow-water equations was discussed. The form of new invariant systems suggests that they might be of interest in physical applications, for instance in magnetohydrodynamics.

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