Functional Analysis behind a Family of Multidimensional Continued Fractions: Triangle Partition Maps

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Abstract

Triangle partition maps form a family that includes many, if not most, well-known multidimensional continued fraction algorithms. This paper begins the exploration of the functional analysis behind the transfer operator of each of these maps. We show that triangle partition maps give rise to two classes of transfer operators and present theorems regarding the origin of these classes; we also present related theorems on the form of transfer operators arising from compositions of triangle partition maps. We then find eigenfunctions of eigenvalue 1 for transfer operators associated with select triangle partition maps on specified Banach spaces and show the existence of spectral gaps for select transfer operators. We proceed to prove that the transfer operators, viewed as acting on one-dimensional families of Hilbert spaces, associated with select triangle partition maps are nuclear of trace class zero. We finish by deriving Gauss-Kuzmin distributions associated with select triangle partition maps.

1 Introduction

Most multidimensional continued fraction algorithms can be interpreted as iterative systems on a triangle or, in higher dimensions, on an appropriate simplex. For review of many types of multidimensional continued fraction algorithms, see Schweiger’s Multidimensional Continued Fractions [34] and Karpenkov’s Geometry of Continued Fractions [22]. Thinking of these iterative systems, it is natural to investigate them as dynamical systems, which in turn will lead to the study of transfer operators (for background on transfer operators, see Baladi [6]). This has long been done for traditional continued fractions, using the Gauss map acting on the unit interval. For background on this, see Hensley [15], Iosifescu and Kraaikamp

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in [18], Khinchin [23], Rockett and Szusz [32] and Schweiger [33]. In [14], this was done for the triangle map, where it was shown that much of the work of Mayer and Roepstorff [28, 29] has nontrivial analogs for the triangle map. The goal of this paper is to see which of these analogs hold for various members of the family of triangle partition maps.

Triangle partition maps [10, 11, 22] are a family of 216 multidimensional continued fraction algorithms that include (when combinations are allowed) many, if not most, well-known multidimensional continued fraction algorithms, which is why they are a natural class of algorithms to study. These maps are reviewed in section 2.

In section 3, we will start to look at transfer operators associated with triangle partition maps. For the Gauss map, we know that the transfer operator is

$$L(f)(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} f \left( \frac{1}{(k+x)} \right),$$

and for the triangle map, the transfer operator is the somewhat similar looking,

$$L(f)(x, y) = \sum_{k=0}^{\infty} \frac{1}{(1+kx+y)^3} f \left( \frac{1}{(1+kx+y)}, \frac{x}{(1+kx+y)} \right).$$

In section 4, we will show for precisely half of the triangle partition maps that the transfer operator will have the form

$$\sum_{k=0}^{\infty} \frac{1}{m(k, x, y)^3} f (\cdot, \cdot)$$

where $m(k, x, y)$ will be a polynomial linear in $k$ (with the possibility of a $(-1)^k$ term) and linear in $x$ and $y$. We will say that such transfer operators have polynomial behavior. We will then show for the remaining half of triangle partition maps that their transfer operators have drastically different behavior. We will see that their transfer operators are of the general form

$$\sum_{k=0}^{\infty} \frac{1}{e(k, x, y)^3} f (\cdot, \cdot)$$

where $e(k, x, y)$ is exponential in $k$. Such triangle partition algorithms are naturally enough said to have non-polynomial behavior. In section 5, we will show that similar results hold for combination triangle partition maps.
We will then turn to the spectrum of the transfer operators. Here we need to be concerned
with which vector spaces of functions we are considering. We will see in section 6 that for
some triangle partition maps there are natural Banach spaces of functions, that for 18 of
these the largest eigenvalue of the associated transfer operator is one, and that for some of
these the dimension of the corresponding eigenspace is one. Then we will see in section 7
that a number of the triangle partition maps’ transfer operators are nuclear operator of trace
class zero when viewed as acting on naturally occurring one-dimensional families of Hilbert
spaces. In section 8 we present the Gauss-Kuzmin statistics for select triangle partition
maps. In section 9 we discuss future directions for work. Appendix A is a table of the
polynomial-behavior triangle maps, while appendix B is a table of the transfer operators for
these polynomial-behavior triangle partition maps.

Part of this paper stems from [2].

2 Background on Triangle Partition Maps

As this is background, much of this section is similar to certain sections in [3, 5, 10, 11, 13,
22, 35].

2.1 The Triangle Map

We start with the triangle map, from which, in the next section, the 216 triangle partition
maps are constructed. For a further explanation of the triangle map, see [13, 5].

Partition the triangle
\[ \triangle = \{(x, y) : 1 \geq x \geq y \geq 0\} \]
into subtriangles
\[ \triangle_k = \{(x, y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y\}, \]
for each non-negative integer \( k \). The partitioning is represented in the following diagram:
Each subtriangle $\triangle_k$ has vertices $(1, 0), (1/(k+1), 1/(k+1)), (1/(k+2), 1/(k+2))$. The triangle map $T : \triangle \to \triangle$ is defined by setting, for any $(x, y) \in \triangle_k$,

$$T(x, y) = \left( \frac{y}{x}, \frac{1 - x - ky}{x} \right).$$

This is an analog of the traditional Gauss map for continued fractions.

And as with the Gauss map and continued fractions, it is useful to translate this into matrices. We have the projection map

$$\pi(x, y, z) = \left( \frac{y}{x}, \frac{z}{x} \right)$$

mapping rays in $\mathbb{R}^3$ to points in $\mathbb{R}^2$. This allows us to associate to the triangle $\triangle$ in $\mathbb{R}^2$ the cone in $\mathbb{R}^3$ (which we will also denote by $\triangle$)

$$\triangle = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : a, b, c > 0 \right\}.$$  

The cone $\triangle$ is the positive span of the column vectors of the matrix

$$V = (v_1, v_2, v_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Define matrices
\[ F_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

These matrices occur naturally since the cone corresponding the subtriangles \( \triangle_k \) are the positive span of the columns of \( VF_1^{k-1}F_0 \), which by an abuse of notation, we will often write as

\[ \triangle_k = VF_1^{k-1}F_0. \]

Then the triangle map \( T : \triangle \to \triangle \) can be shown to be equal to:

\[ T(x, y) = \pi \left( (1, x, y) \left( VF_0^{-1}F_1^{-k}B^{-1} \right)^T \right) \]

if \((x, y) \in \triangle_k\).

This enables us to associate the triangle sequence \((a_1, a_2, ...)\) to a point \((a, b) \in \triangle\) by letting \(a_i = k\) if \(T^i(x, y) \in \triangle_k\). If for any \(k\) we have \(T^k(a, b) \in \{(x, 0) : 0 \leq x \leq 1\}\), the sequence terminates.

### 2.2 Triangle Partition Maps

The above triangle map depends on an initial choice of three vertices for \( \triangle \). If we permute these vertices both before and after applying the matrices \(F_0\) and \(F_1\), we create different multidimensional continued fraction algorithms. This is at the heart of the definition for the 216 triangle partition maps.

More precisely, we will allow permutations of the initial vertices by some \(\sigma \in S_3\), by some \(\tau_1 \in S_3\) after applying \(F_1\), and by some \(\tau_0 \in S_3\) after applying \(F_0\).

This leads us to define the matrices

\[ F_0(\sigma, \tau_0, \tau_1) = \sigma F_0 \tau_0, \]

\[ F_1(\sigma, \tau_0, \tau_1) = \sigma F_1 \tau_1 \]

for every \((\sigma, \tau_0, \tau_1) \in S_3^3\).

In particular, we denote the permutation matrices as follows:
\[ e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

\[ (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad (132) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

Let \( \triangle_k(\sigma, \tau_0, \tau_1) \) be the image of \( \triangle \) under the action of \( F^k_1(\sigma, \tau_0, \tau_1)F_0(\sigma, \tau_0, \tau_1) \). Thus, with an abuse of notation, we have

\[ \triangle_k(\sigma, \tau_0, \tau_1) = V F^k_1(\sigma, \tau_0, \tau_1)F_0(\sigma, \tau_0, \tau_1). \]

Then define a triangle partition map as

\[ T_{\sigma, \tau_0, \tau_1} : \bigcup_{k=0}^{\infty} \triangle_k(\sigma, \tau_0, \tau_1) \to \triangle \]

where

\[ T_{\sigma, \tau_0, \tau_1}(x, y) = \pi \left( (1, x, y) \left( V F_0^{-1}(\sigma, \tau_0, \tau_1)F_1^{-k}(\sigma, \tau_0, \tau_1)V^{-1} \right)^T \right) \]

if \((x, y) \in \triangle_k(\sigma, \tau_0, \tau_1)\).

As \( S_3 \) has six elements, we have \( 6^3 = 216 \) different triangle partition maps. Of course, the original triangle map corresponds to \( T_{e, e, e} \). Appendix A gives the explicit forms of \( T_{\sigma, \tau_0, \tau_1} \) for 108 of the maps. One of the goals of this paper is to explain why we do not list in an easy-to-read table the forms of \( T_{\sigma, \tau_0, \tau_1} \) for the remaining 108 maps.

We obtain an even larger family of maps by allowing compositions of triangle partition maps [10]. As an example, we might perform the first subdivision of \( \triangle \) using \((\sigma, \tau_0, \tau_1)\), the second subdivision using \((\sigma_2, \tau_{02}, \tau_{12})\) and so on. We can represent such compositions as \( T_1 \circ T_2 \ldots \circ T_n \), where, of course, each subscript is short for a permutation triplet.

Again, many, if not most, multidimensional continued fraction algorithms can be put into this language [10], which is why these are natural maps to consider. To get a feel of these maps, we consider two examples in a bit more detail.
2.3 The \((23), (23), (23)\) Case

This is simply an example of what we will later call a polynomial-behavior triangle partition map. It is in and of itself not that significant. We have

\[
F_0((23), (23), (23)) = (23)F_0(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
F_1((23), (23), (23)) = (23)F_1(23) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We want to see what the subtriangles \(\triangle_k((23), (23), (23))\) look like. We have

\[
\triangle_k((23), (23), (23)) = VF_1^k((23), (23), (23))F_0((23), (23), (23)) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^k \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & k+1 & k+2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Pictorially, we have
In the plane $\mathbb{R}^2$, the vertices of each $\triangle_k((23), (23), (23))$ are

$$\left\{ \left( \frac{1}{k+1}, \frac{0}{1} \right), \left( \frac{1}{k+2}, \frac{0}{1} \right) \right\} = \{(1,1), (1/(k+1), 0), (1/(k+2), 0)\}
$$

Now, for any $(x, y) \in \triangle_k((23), (23), (23))$, we have

$$T_{\sigma, \tau_0, \tau_1}(x, y) = \pi \left( (1, x, y) \left( VF_0^{-1}(\sigma, \tau_0, \tau_1)F_1^{-k}(\sigma, \tau_0, \tau_1)V^{-1} \right)^T \right)
= \left( \frac{x - y}{x}, \frac{-1 + (2 + k)x + (-1 - k)y}{x} \right),
$$
in which case,

$$T_{23,23}^{-1}(e,e,e) = \left( \frac{1}{1 + (1 + k)x - y}, \frac{1 - x}{1 + (1 + k)x - y} \right),$$
a formula that we will need later.

2.4 The $(e, e, (13))$ Case

Here we present another example of a TRIP map. We will see that its behavior is quite different than that of TRIP maps $(e,e,e)$ and $((23), (23), (23))$, a difference that will be explained in section 4.

We first need to calculate $F_0(e,e,(13))$ and $F_1(e,e,(13))$. $F_0(e,e,(13))$ is easy, as

$$F_0(e,e,(13)) = eF_0e = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$
For the other matrix, we have

\[
F_1(e, e, (13)) = eF_1(13) \\
= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

We need to find a clean formula for \( F_1^k(e, e, (13)) \). We will need the Fibonacci numbers, which are given by

\[
f_{k+1} = f_k + f_{k-1}, \text{ with initial terms } f_{-1} = 0, f_0 = 1.
\]

Note that this means that \( f_{-2} = 1 \).

**Proposition 1**

\[
F_1^k(e, e, (13)) = \begin{pmatrix} f_k & 0 & f_{k-1} \\ 0 & 1 & 0 \\ f_{k-1} & 0 & f_{k-2} \end{pmatrix}
\]

**Proof** By induction. For \( k = 0 \), we have that \( F_1^0(e, e, (13)) \) is the identity. But for \( k = 0 \), we have

\[
\begin{pmatrix} f_k & 0 & f_{k-1} \\ 0 & 1 & 0 \\ f_{k-1} & 0 & f_{k-2} \end{pmatrix} = \begin{pmatrix} f_0 & 0 & f_{-1} \\ 0 & 1 & 0 \\ f_{-1} & 0 & f_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then in general we have

\[
F_1(e, e, (13))^k = F_1(e, e, (13))^{k-1}F_1(e, e, (13))
\]

\[
= \begin{pmatrix} f_{k-1} & 0 & f_{k-2} \\ 0 & 1 & 0 \\ f_{k-2} & 0 & f_{k-3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} f_{k-1} + f_{k-2} & 0 & f_{k-1} \\ 0 & 1 & 0 \\ f_{k-2} + f_{k-3} & 0 & f_{k-2} \end{pmatrix}
\]

\[
= \begin{pmatrix} f_k & 0 & f_{k-1} \\ 0 & 1 & 0 \\ f_{k-1} & 0 & f_{k-2} \end{pmatrix},
\]

as desired.

Now to find the subtriangles \( \triangle_k(e, e, (13)) = VF_1(e, e, (13))^kF_0(e, e, (13)). \)
Proposition 2

\[ \triangle_k(e,e,(13)) = VF_1(e,e,(13))^k F_0(e,e,(13)) = \begin{pmatrix} 1 & f_k & f_{k+2} \\ 1 & f_{k-2} & f_k \\ 0 & f_{k-2} & f_k \end{pmatrix}. \]

Proof  This is just the calculation

\[ \begin{align*}
\triangle_k(e,e,(13)) &= VF_1(e,e,(13))^k F_0(e,e,(13)) \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_k & 0 & f_{k-1} \\ 0 & 1 & 0 \\ f_{k-1} & 0 & f_{k-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & f_{k-1} + f_{k-2} & f_{k+1} + f_k \\ 1 & f_{k-2} & f_k \\ 0 & f_{k-2} & f_k \end{pmatrix} \\
&= \begin{pmatrix} 1 & f_k & f_{k+2} \\ 1 & f_{k-2} & f_k \\ 0 & f_{k-2} & f_k \end{pmatrix}.
\]

Thus the subtriangle \( \triangle_k(e,e,(13)) \) in the plane \( \mathbb{R}^2 \) has vertices

\[ (1,0), (f_{k-2}/f_k, f_{k-2}/f_k), (f_k/f_{k+2}, f_k/f_{k+2}). \]

Pictorially, we have

We can now calculate the TRIP map, whose proof is just a somewhat painful calculation.
Proposition 3 For \((x, y) \in \triangle_k(e, e, (13))\), the TRIP map

\[
T(e, e, (13))(x, y) = \pi \left[ (1, x, y) \left( V(VF_1(e, e, (13)))^k F_0(e, e, (13))^{-1} \right)^T \right]
\]

has \(x\) coordinate

\[
\frac{(-1)^{k+1} f_{k+1} + (-1)^{k+2} f_{k+1} x + (-1)^{k+2} f_{k+2} y}{(-1)^{k+1} f_{k+1} + (1 - (-1)^{k+1} f_{k+1}) x + (-1 + (-1)^{k+2} f_{k+2}) y}
\]

and \(y\)-coordinate

\[
\frac{(-1)^{k-1} f_{k-1} - (-1)^{k-1} f_{k-1} x + (-1)^{k} f_k y}{(-1)^{k+1} f_{k+1} + (1 - (-1)^{k+1} f_{k+1}) x + (-1 + (-1)^{k+2} f_{k+2}) y}
\]

The fact that we had to use two lines to write out the TRIP map \(T(e, e, (13))(x, y)\) is not just chance, as we will see in section 4.

3 Transfer Operators

For general background on transfer operators and their importance, see Baladi [6].

In general, for a dynamical system corresponding to the map \(T : X \rightarrow X\), a transfer operator linearly maps functions from a vector space of functions on \(X\) to another (not necessarily different) vector space of functions on \(X\). For a more concrete definition, define a function \(g : X \rightarrow \mathbb{R}\). Then the transfer operator \(L_T\) acting on \(f : X \rightarrow \mathbb{R}\) is defined by

\[
L_T f(x) = \sum_{y : T(y) = x} g(y) f(y).
\]

If \(T\) is piece-wise differentiable, as for our maps, we choose \(g = \frac{1}{|\text{Jac}(T)|}\).

We define our transfer operators to be

\[
L_{T_{\sigma, \tau_0, \tau_1}} f(x, y) = \sum_{(a, b) : T_{\sigma, \tau_0, \tau_1}(a, b) = (x, y)} \frac{1}{|\text{Jac}(T_{\sigma, \tau_0, \tau_1}(a, b))|} f(a, b).
\]

While a long calculation is required to arrive at the transfer operator corresponding to each \((\sigma, \tau_0, \tau_1)\), the calculations are not difficult theoretically and have been carried out for each of the 216 triangle partition maps. As an example, as shown in [14],

\[
L_{T_{e, e, e}} f(x, y) = \sum_{k=0}^{\infty} \frac{1}{(1 + kx + y)^3} f \left( \frac{1}{1 + kx + y}, \frac{x}{1 + kx + y} \right).
\]
We have calculated the explicit form of

$$L_{T_{\sigma, \tau_0, \tau_1}} f(x, y) = \sum_{(a, b): T_{\sigma, \tau_0, \tau_1}(a, b) = (x, y)} \frac{1}{|\text{Jac}(T_{\sigma, \tau_0, \tau_1}(a, b))|} f(a, b)$$

for all $(\sigma, \tau_0, \tau_1) \in S_3^3$. As mentioned in the introduction, one of the main goals for this paper is to show that the family of 216 triangle partition maps naturally splits into two classes: 108 maps exhibiting what we term polynomial behavior, and 108 maps exhibiting exhibiting non-polynomial behavior. Polynomial-behavior maps' transfer operators have compact explicit forms while non-polynomial-behavior maps' transfer operators have very lengthy and complex forms. The explicit forms for all 108 polynomial-behavior transfer operators are presented in Appendix B. The explicit forms for all 108 non-polynomial behavior transfer operators are available online (see [2]). The reason we do not list these is that they are too long to easily write out.

As another example, let us return to the $T_{(23), (23), (23)}$ TRIP map of subsection 2.3. This has to have polynomial behavior, as otherwise it would be too cumbersome to express. We need to find the Jacobian of $T_{23,23,23}$ with any $x$ replaced by the first component of $T_{23,23,23}^{-1}$ and any $y$ replaced by the second component of $T_{23,23,23}^{-1}$. The Jacobian is:

$$\text{Jac}(T_{23,23,23}) = \det \begin{pmatrix} \frac{\partial}{\partial x} \left( \frac{x-y}{x} \right) & \frac{\partial}{\partial y} \left( \frac{x-y}{x} \right) \\ \frac{\partial}{\partial x} \left( \frac{1+(2+k)x+(-1-k)y}{x} \right) & \frac{\partial}{\partial y} \left( \frac{1+(2+k)x+(-1-k)y}{x} \right) \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{y}{x^2} & \frac{-1}{x} \\ \frac{1+y+ky}{x^2} & \frac{-1}{x} \end{pmatrix}$$

$$= 1/x^3$$

Substituting the first component of $T_{23,23,23}^{-1}$ for $x$, the Jacobian becomes $1/(1 + (1+k)x - y)^3 = (1 + (1 + k)x - y)^3$, so that

$$\frac{1}{\text{Jac}(T_{23,23,23})} = \frac{1}{(1 + (1 + k)x - y)^3}.$$
Hence

\[ \mathcal{L}_{T_{23,23,23}} f(x, y) = \sum_{(a,b):T_{23,23,23}(a,b)=(x,y)} \frac{1}{|\text{Jac}(T_{23,23,23}(a,b))|} f(a, b) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{(1 + (1 + k)x - y)^3} f\left(\frac{1}{1 + (1 + k)x - y}, \frac{1 - x}{1 + (1 + k)x - y}\right). \]

Of course, we would now like to calculate the transfer operator for the TRIP map 
\((e,e,(13))\) of subsection 2.4, but the calculations are a bit too cumbersome to actually write out. However, note that the transfer operator will be in terms of Fibonacci numbers, and hence exponential in terms of the parameter \(k\).

## 4 Polynomial and Non-Polynomial Behavior in Transfer Operators

### 4.1 A Clean Criterion

The transfer operator of the Gauss map \( G : [0, 1] \to [0, 1] \) is

\[ \mathcal{L}_G(f)(x) = \sum_{k=1}^{\infty} \frac{1}{(k + x)^2} f\left(\frac{1}{k + x}\right). \]

Note that the denominator is a polynomial that is the square of a polynomial linear in \(k\) and linear in \(x\). The transfer operator for the original triangle map \( T_{e,e,e} : \triangle \to \triangle \) is

\[ \mathcal{L}_{T_{e,e,e}}(f)(x, y) = \sum_{k=0}^{\infty} \frac{1}{(1 + kx + y)^3} f\left(\frac{1}{1 + kx + y}, \frac{x}{1 + kx + y}\right). \]

Now the denominator is a polynomial that is the cube of a polynomial linear in \(k\) and linear in \(x\) and \(y\). In the previous section, we have shown that the transfer operator for the TRIP map \(((23),(23),(23))\) also has this behavior. The goal of this section is to show that similar analogs hold for precisely 106 of the other triangle partition maps, while the behavior of the transfer operator for the remaining 108 triangle maps is drastically different.

**Definition 4** The transfer operator for a triangle partition map \( T_{\sigma,\tau_0,\tau_1} \) is said to have polynomial behavior if its Jacobian \( \text{Jac}(T_{\sigma,\tau_0,\tau_1}) \) is a polynomial in the variables \(k\), \(x\) and \(y\),
with possibly a \((-1)^k\) term appearing. Otherwise, we say that the transfer operator has non-polynomial behavior.

We will show that those transfer operators that have polynomial behavior actually have their Jacobians equal to the cube of a polynomial that is linear in \(k\) and linear in \(x\) and \(y\), with possibly a \((-1)^k\) term appearing. In appendix \([3]\) we explicitly list the 108 transfer operators that have polynomial behavior. Those without this behavior are far too messy to list, a fact that we find interesting.

We will need the following well-known lemma, whose proof is a calculation:

**Lemma 5** Consider a vector \(v = (1, x, y)\), a matrix \(M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\), and the map \(\pi(vM) = \left( b + ex + hy, \frac{c + fx + iy}{a + dx + gy}, \frac{c + fx + iy}{a + dx + gy} \right)\).

Then
\[
\text{Jac}(\pi(vM)) = \frac{\det(M)}{(vM(1, 0, 0)^T)^3}.
\]

**Theorem 6** A triangle partition map \(T_{\sigma, \tau_0, \tau_1}\) has polynomial behavior if and only if the eigenvalues of the associated \(F_1(\sigma, \tau_0, \tau_1) = \sigma F_1 \tau_1\) all have magnitude 1; it is non-polynomial-behavior otherwise.

**Proof** By the previous lemma and the definition of \(T_{\sigma, \tau_0, \tau_1}\), it follows that
\[
\frac{1}{\text{Jac}(T_{\sigma, \tau_0, \tau_1}(a, b))} = \frac{\det(M_{\sigma, \tau_0, \tau_1})}{((1, x, y)M_{\sigma, \tau_0, \tau_1}(1, 0, 0)^T)^3}
\]
where
\[
M_{\sigma, \tau_0, \tau_1} = ((V F_0^{-1}(\sigma, \tau_0, \tau_1) F_1^{-k}(\sigma, \tau_0, \tau_1) V^{-1})^T)^{-1}.
\]
Since \(\det(M_{\sigma, \tau_0, \tau_1})\) is real and has magnitude \(\pm 1\) (since all matrices involved in its calculation also have real determinants of magnitude \(\pm 1\)), this implies that a given \(T_{\sigma, \tau_0, \tau_1}\) is polynomial-behavior if and only if the first column of \(M_{\sigma, \tau_0, \tau_1}\) contains terms polynomial in \(k\).
For simplicity, write \( M_{\sigma, \tau_0, \tau_1} = (C_1, C_2, C_3) \), where \( C_i \) are the columns of \( M_{\sigma, \tau_0, \tau_1} \). Note that the inverse of a triangle partition map, \( T_{\sigma, \tau_0, \tau_1}^{-1} \), can be written as

\[
T_{\sigma, \tau_0, \tau_1}^{-1}(x, y) = \pi \left( (1, x, y) \left( (VF_0^{-1}(\sigma, \tau_0, \tau_1)F_1^{-k}(\sigma, \tau_0, \tau_1)V^{-1})^T \right)^{-1} \right)
= \pi \left( (1, x, y) M_{\sigma, \tau_0, \tau_1} \right)
= \left( \frac{(1, x, y) \cdot C_2}{(1, x, y) \cdot C_1} \cdot \frac{(1, x, y) \cdot C_3}{(1, x, y) \cdot C_1} \right).
\]

Note that for any \((x, y) \in \triangle\), \( T_{\sigma, \tau_0, \tau_1}^{-1}(x, y) \) must be bounded as it must land back inside \( \triangle \).

Since the inverse of each triangle partition map is bijective and since the choice of \( k \) depends on which \( \triangle_k \) the original \((x, y)\) lies in, the first column of \( M_{\sigma, \tau_0, \tau_1} \) must depend on \( k \). Thus, to show that the first column of \( M_{\sigma, \tau_0, \tau_1} \) depends on \( k \) polynomially, it is sufficient to show that \( M_{\sigma, \tau_0, \tau_1} \) exhibits only polynomial dependence on \( k \) – for then, by the above argument, the first row of \( M_{\sigma, \tau_0, \tau_1} \) must necessarily exhibit only polynomial dependence on \( k \).

We have (suppressing the \((\sigma, \tau_0, \tau_1)\) in some of the matrices)

\[
M_{\sigma, \tau_0, \tau_1} = ((VF_0^{-1}F_1^{-k}V^{-1})^T)^{-1}
= ((V^{-1})^T((F_1^{-1})^kV)(F_0^{-1})^TV^T)^{-1}
= (V^T)^{-1}(F_0^{-1})^T((F_1^{-1})^kV)^{-1}(V^{-1})^T
= (V^T)^{-1}F_0^TF_1^kV^T.
\]

Let \( A = (F_1)^T \). We can find the Jordan decomposition of \( A \); i.e. we can write it as

\[
A = PJP^{-1}
\]

where \( P \) is some invertible matrix of dimensions identical to those of \( A \), and \( J \) is the Jordan normal form of \( A \). Performing the Jordan decomposition for all 36 unique \( A \)s we see that there exist only six unique \( J \)s, namely

\[
J_1 = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
J_2 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
J_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
J_4 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2}(1 - \sqrt{5}) & 0 \\
0 & 0 & \frac{1}{2}(1 + \sqrt{5})
\end{pmatrix},
\]
\[
J_5 = D(\text{roots}(-1 - t^2 + t^3)),
\]

and
\[
J_6 = D(\text{roots}(-1 - t + t^3)).
\]

where \(D(\text{roots}(-1 - t^2 + t^3))\) corresponds to a three-by-three square matrix with diagonal entries defined by the roots of \(-1 - t^2 + t^3 = 0\); similarly for \(D(\text{roots}(-1 - t + t^3))\).

It can be shown that
\[
(J_1)^k = \begin{pmatrix}
1 & k & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
(J_2)^k = \begin{pmatrix}
(-1)^k & 0 & 0 \\
0 & 1 & k \\
0 & 0 & 1
\end{pmatrix},
\]
\[
(J_3)^k = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & k \\
0 & 0 & 1
\end{pmatrix},
\]
\[
(J_4)^k = \begin{pmatrix}
1 & 0 & 0 \\
0 & \left(\frac{1}{2}(1 - \sqrt{5})\right)^k & 0 \\
0 & 0 & \left(\frac{1}{2}(1 + \sqrt{5})\right)^k
\end{pmatrix},
\]
\[
(J_5)^k = D(\text{roots}(-1 - t^2 + t^3)^k),
\]

and
\[
(J_6)^k = D(\text{roots}(-1 - t + t^3)^k).
\]

It is well-known that the diagonal elements of each \(J\) are precisely the eigenvalues of the matrix \(A\) from which it originated; further, \(J_4\) through \(J_6\) are diagonal and each contain at least one entry of magnitude greater than 1 on their respective diagonals. From this, it is clear
that if $(F_1)^T(\sigma, \tau_0, \tau_1)$ (and hence $F_1(\sigma, \tau_0, \tau_1)$) has eigenvalues that all have magnitude 1, then the first column of $M_{\sigma, \tau_0, \tau_1}$ depends polynomially on $k$, and hence the associated $T_{\sigma, \tau_0, \tau_1}$ has polynomial behavior. Otherwise, the first column of $M_{\sigma, \tau_0, \tau_1}$ depends exponentially on $k$, and hence the associated $T_{\sigma, \tau_0, \tau_1}$ has non-polynomial behavior.

Assume $T_{\sigma, \tau_0, \tau_1}$ has polynomial behavior in $k$. Then the first column of $M_{\sigma, \tau_0, \tau_1}$ must depend strictly polynomially on $k$. From the above decomposition of $M_{\sigma, \tau_0, \tau_1}$, we see that the only place where $k$-dependance may enter the first column of $M_{\sigma, \tau_0, \tau_1}$ is through $A$; hence, by the above argument the eigenvalues of $F_1(\sigma, \tau_0, \tau_1)$ all have magnitude 1. Now assume the eigenvalues of $F_1(\sigma, \tau_0, \tau_1)$ all have magnitude 1. Running above argument in reverse, we see that $T_{\sigma, \tau_0, \tau_1}$ must necessarily have polynomial behavior.

Corollary 7  A triangle partition map $T_{\sigma, \tau_0, \tau_1}$ is non-polynomial behavior if and only if the associated $M_{\sigma, \tau_0, \tau_1}$ depends exponentially on $k$.

Corollary 8  A triangle partition map $T_{\sigma, \tau_0, \tau_1}$ has polynomial behavior if and only if the associated $F_1(\sigma, \tau_0, \tau_1) = \sigma F_1 \tau_1$ is not diagonalizable.

Proof  This follows immediately from the form of $J_1$ through $J_6$ in Theorem 6.

Corollary 9  Let $\sigma, \tau_0, \tau_1, \rho$ and $\gamma$ be three-by-three permutation matrices. If a triangle partition map $T_{\sigma, \tau_0, \tau_1}$ has polynomial behavior, then so does the triangle partition map $T_{\rho \sigma, \gamma, \tau_1 \rho^{-1}}$; similarly, if $T_{\sigma, \tau_0, \tau_1}$ have non-polynomial behavior, then so does $T_{\rho \sigma, \gamma, \tau_1 \rho^{-1}}$.

Proof  We need to compare $F_1(\sigma, \tau_0, \tau_1)$ and $F_1(\rho \sigma, \gamma, \tau_1 \rho^{-1})$. We have

$$F_1(\rho \sigma, \gamma, \tau_1 \rho^{-1}) = \rho \sigma F_1 \tau_1 \rho^{-1} = \rho F_1(\sigma, \tau_0, \tau_1) \rho^{-1}.$$ 

Since conjugation will not change the Jordan canonical form of the relevant matrices, we are done.
Note that transfer operators with polynomial behavior are basically zeta functions. In fact, we were initially tempted to call such transfer operators “zeta”-like.

5 On Polynomial Behavior in Combination Triangle Partition Maps

We are concerned with the form of the transfer operator $\mathcal{L}_T$ where

$$\mathcal{L}_T f(x, y) = \sum_{(a,b): T(a,b) = (x,y)} \frac{1}{\text{Jac}(T(a,b))} f(a, b).$$

and $T$ is a finite composition of $n$ triangle partition maps defined by

$$T = T_1 \circ T_2 \circ \cdots \circ T_i \circ \cdots \circ T_n$$

where each $T_i$ is a TRIP map for a permutation triplet $(\sigma_i, (\tau_0)_i, (\tau_1)_i)$.

If the denominator of $\frac{1}{\text{Jac}(T(a,b))}$ is (non-trivially) polynomial in $k_i$ (also allowing for factors of $(-1)^{k_i}$) then $T$ gives rise to a transfer operator that has polynomial behavior in $k_i$, and is itself polynomial behavior in $k_i$; otherwise, $T$ gives rise to a transfer operator that has non-polynomial behavior in $k_i$, and has itself non-polynomial behavior. Here we state and prove a theorem regarding the polynomial behavior of combo triangle partition maps, and present some corollaries. To avoid redundancy, we will refer to polynomial dependence on $k_i$ aside from factors of $(-1)^{k_i}$ as polynomial dependence on $k_i$ or strictly polynomial dependence on $k_i$.

**Theorem 10** A combination triangle partition map $T = T_1 \circ T_2 \circ \cdots \circ T_i \circ \cdots \circ T_n$ has polynomial behavior in $k_i$ if and only if the eigenvalues of the associated $F_{i_1}$ (the $F_1$ matrix corresponding to $T_i$) all have magnitude 1; it has non-polynomial behavior in $k_i$ otherwise.

**Proof** We have already shown explicitly that $\frac{1}{\text{Jac}(T(a,b))} = \frac{\text{det}(M)}{(1,x,y)M(1,0,0)^T}$, where

$$M = M_1M_2 \cdots M_i \cdots M_n$$
and
\[ M_i = ((VF_0^{-1}(\sigma_i, (\tau_0)_i, (\tau_1)_i)F_{1_i}^{-k}(\sigma_i, (\tau_0)_i, (\tau_1)_i)V^{-1})^T)^{-1}. \]

Since \( \det(M) \) is real and has magnitude \( \pm 1 \), this implies that \( T \) has polynomial behavior in \( k_i \) if and only if the first column of \( M \) contains terms polynomial in \( k_i \). But now the argument will mirror the proof of theorem 6, since what matters is the nature of each \( F_{1_i}^{-k}(\sigma_i, (\tau_0)_i, (\tau_1)_i) \), right down to there only being six unique Jordan canonical forms.

\( \Box \)

We have the following corollaries, whose proofs mimic the corollaries of theorem 6.

**Corollary 11** A combination triangle partition map \( T \) as defined in the above theorem is non-polynomial-behavior in \( k_i \) if and only if the associated \( M \) depends exponentially on \( k_i \).

**Corollary 12** A combination triangle partition map \( T \) as defined in the above theorem is polynomial-behavior in \( k_i \) if and only if the associated \( F_{1_i} \) is not diagonalizable; it is non-polynomial-behavior otherwise.

### 6 Leading Eigenvalues for Select Transfer Operators

It is well-known that the transfer operator for the Gauss map has leading eigenvalue one with a one-dimensional eigenspace when acting on the Banach space of bounded continuous functions on the unit interval, as in work of Mayer and Roepstorff [29]. Similarly, in [14], it is shown that the transfer operator for \( T_{e,e,e} \) has leading eigenvalue one with a one-dimensional eigenspace in the Banach space of continuous functions \( f(x, y) \) on \( \triangle \) where \( |xf(x, y)| \) is bounded. What about the other triangle partition maps?

We will show for 31 triangle partition maps\(^1\) that there are similar natural Banach spaces, which suggests that the corresponding transfer operators may be good candidates for having largest eigenvalue one. For 18 of these 31 transfer operators, we will explicitly find an

\(^1\)In this paper we present new results for all maps but \( T_{e,e,e} \), for which the analogous results are presented in [14]; nonetheless, we include the results for \( T_{e,e,e} \) for completeness.
eigenfunction with eigenvalue one. When we restrict attention to those triangle partition maps which are known to be ergodic, we can finally conclude for these that the eigenspace with eigenvalue one is one-dimensional.

6.1 Transfer Operators as Linear Maps on Appropriate Banach Spaces

The goal here is to find Banach spaces for select $L_{T_{x, r_0, r_1}}$ which might be good analogs to the Banach space found in [14]. The Banach spaces will all have the following form. Let $C(\Delta)$ be the space of continuous functions on the open triangle $\Delta$. Each of the Banach spaces will be

$$V = \{ f(x, y) \in C(\Delta) : \exists C \in \mathbb{R} : |g(x, y)f(x, y)| < C, \forall (x, y) \in \Delta \},$$

where $g(x, y) \in C(\Delta)$ is a fixed function.

Each $V$ is a Banach space under norm

$$\|f(x, y)\| = \sup_{(x, y) \in \Delta} |g(x, y)f(x, y)|.$$  

The following theorem is why these are natural Banach spaces to study:

**Theorem 13** For functions $g(x, y)$ as shown in the table below, the transfer operators $L_{T_X}$, with $X$ representing each of the permutation triplets appearing in the table, are linear maps from $V$ to $V$. 
(σ, τ₀, τ₁) & g(x, y) & Summand \\
(e, e, e) & x & x^{(kx+y+1)^2} \\
(e, 13, e) & (1 - y)(-x + y + 1) & (y-1)(-x+y+1) \\
(e, 13, 12) & (1 - y)(-x + y + 1) & \frac{(y-1)(-x+y+y+1)}{y-k-k+x+y-1} \\
(e, 23, e) & x(1 - y) & \frac{(kx-y+1)(x+x+y+1)}{y-k-k-x+y+1} \\
(e, 132, e) & x(1 - y) & \frac{(y-k-x)(y-k-x+y+1)}{y-y-k+k-x+y+1} \\
(e, 132, 12) & x(1 - y) & \frac{(y-k-x)(y-k-k-x+y+1)^2}{y-k-k-k-x+y+1} \\
(12, 12, 12) & -x + y + 1 & (-x+y+y+1)^2 \\
(12, 13, e) & (y - 1)(-x + y + 1) & (y-1)(-x+y+1) \\
(12, 13, 12) & (y - 1)(-x + y + 1) & \frac{(y-1)(-x+y+y+1)}{y-k-k-x+y-1} \\
(12, 123, 12) & (1 - y)(-x + y + 1) & \frac{(y-k-k+x-x+2)(y-k-k-y+1)}{y-x-k-k-x+y+1} \\
(12, 132, e) & x(y-1) & -x^{y-1} \\
(12, 132, 12) & x(1 - y) & \frac{x(y-1)}{y-k-k-x+y+1} \\
(13, e, 13) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(13, e, 123) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(13, 13, 13) & 1 - y & \frac{1}{y-k-k-x+2} \\
(13, 23, 13) & x(1 - y) & \frac{[kx-y+1](kx+x+y+1)}{y-k-k-x+y+1} \\
(13, 132, 13) & x(1 - y) & \frac{x(y-1)}{y-k-k-x+y+1} \\
(23, e, 23) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(23, 12, 23) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(23, 12, 132) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(23, 23, 23) & x & \frac{(kx+x+y+1)^2}{y-k-k-x+2} \\
(23, 123, 132) & (1 - y)(-x + y + 1) & \frac{(y-k-k-x+2)(y-k-k-y+1)}{(y-1)(y-k-k-x+y+1)} \\
(123, 12, 23) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(123, 12, 132) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(123, 123, 132) & (1 - y)(-x + y + 1) & \frac{(y-k-k-x+2)(y-k-k-y+1)}{(y-1)(y-k-k-x+y+1)} \\
(123, 132, 132) & (1 - y)(-x + y + 1) & \frac{(y-k-k-x+2)(y-k-k-y+1)}{(y-1)(y-k-k-x+y+1)} \\
(132, e, 13) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(132, e, 123) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(132, 12, 123) & x(-x + y + 1) & \frac{x(-x+y+1)}{y-k-k-x+y+1} \\
(132, 123, 123) & -x + y + 1 & \frac{(-x+y+1)}{(y-k-k-x+2)^2} \\

Proof Linearity is clear.

Let \( f(x, y) \in V \). Then there is real constant \( C \) such that \( ||g(x, y)||f(x, y) < C \) for all
\((x, y) \in \triangle\). To show that \(L_{T_x} f(x, y) \in V\) we need to show
\[
|g(x, y)L_{T_x} f(x, y)| < C_1
\]
for some real constant \(C_1\). We have
\[
|g(x, y)L_{T_x} f(x, y)| = |g(x, y)| \left| \sum_{(a,b):T_X(a,b)=(x,y)} \frac{1}{|\text{Jac}(T_X(a, b))|} f(a, b) \right|
\leq |g(x, y)| \sum_{(a,b):T_X(a,b)=(x,y)} \left| \frac{1}{g(a, b)|\text{Jac}(T_X(a, b))|} g(a, b) f(a, b) \right|
\leq |g(x, y)| C \sum_{(a,b):T_X(a,b)=(x,y)} \left| \frac{1}{g(a, b)|\text{Jac}(T_X(a, b))|} \right|.
\]
Thus we must show that the sum
\[
\sum_{(a,b):T_X(a,b)=(x,y)} \left| \frac{g(x, y)}{g(a, b)|\text{Jac}(T_X(a, b))|} \right|
\]
converges, with a bound independent of \((x, y)\). In the above table, each of the summands
\[
\left| \frac{g(x, y)}{g(a, b)|\text{Jac}(T_X(a, b))|} \right|
\]
is given. By standard arguments it is clear we have our bounds.

\(\square\)

### 6.2 Leading Eigenvalues

We will show that for 18 maps the transfer operator has leading eigenvalue one; for four of these maps, the corresponding eigenspace is one-dimensional. We do not know if analogous results are true for other triangle partition maps, but would be surprised if these 18 are the only ones with this behavior.

**Theorem 14** The table below lists those triangle partition maps whose transfer operators have an eigenfunction with eigenvalue one and explicitly lists the eigenfunction.
\begin{center}
\begin{tabular}{|c|c|}
\hline
$(\sigma, \tau_0, \tau_1)$ & eigenfunction \\
\hline
$(e, e, e)$ & $\frac{1}{x(y+1)}$ \\
$(e, 23, e)$ & $\frac{1}{x(1-y)}$ \\
$(e, 132, e)$ & $\frac{1}{x(1-y)}$ \\
$(12, 12, 12)$ & $\frac{1}{(y+1)(-x+y+1)}$ \\
$(12, 13, 12)$ & $\frac{1}{(1-y)(-x+y+1)}$ \\
$(12, 123, 12)$ & $\frac{1}{(1-y)(-x+y+1)}$ \\
$(13, 13, 13)$ & $\frac{1}{(x-2)(1-y)}$ \\
$(13, 23, 13)$ & $\frac{1}{x(1-y)}$ \\
$(13, 132, 13)$ & $\frac{1}{x(1-y)}$ \\
$(23, e, 23)$ & $\frac{1}{x(-x+y+1)}$ \\
$(23, 12, 23)$ & $\frac{1}{x(-x+y+1)}$ \\
$(23, 23, 23)$ & $\frac{1}{x(x-y+1)}$ \\
$(123, 13, 132)$ & $\frac{1}{(1-y)(-x+y+1)}$ \\
$(123, 123, 132)$ & $\frac{1}{(1-y)(-x+y+1)}$ \\
$(123, 132, 132)$ & $\frac{1}{(1-y)(x-y+1)}$ \\
$(132, e, 123)$ & $\frac{1}{x(-x+y+1)}$ \\
$(132, 12, 123)$ & $\frac{1}{x(-x+y+1)}$ \\
$(132, 123, 132)$ & $\frac{1}{(x-2)(-x+y+1)}$ \\
\hline
\end{tabular}
\end{center}

**Proof** These are calculations, in analog to the corresponding calculation in [14].

We now want to show one is the largest eigenvalue for these transfer operators.

For each of the above transfer operators, we have from last section an associated Banach space $V$. We will need the following theorem.

**Theorem 15** Fix a triangle partition map and let $V$ be the corresponding Banach space.

For functions $f(x, y), g(x, y) \in V$, assume that $f(x, y) < g(x, y)$ for all $(x, y) \in \triangle$. Then

$$L^n_{T_{\sigma, \tau_0, \tau_1}} f(x, y) < L^n_{T_{\sigma, \tau_0, \tau_1}} g(x, y)$$

for all $n \geq 0$ for all $(x, y) \in \triangle$.

**Proof** The proof is a calculation, again in analog to the corresponding calculation in [14].

We now want to prove a theorem that will give a general criterion for when one of our transfer operators has leading eigenvalue one.
For general notation, let \( g(x, y) \) be the function that defines the corresponding Banach space \( V \) for the triangle partition map \( T_{\sigma, \tau_0, \tau_1} \) and suppose that \( h(x, y) \in V \) is an eigenvector with eigenvalue one for the corresponding transfer operator.

**Theorem 16** With the above notation, suppose for all \( f(x, y) \in V \) that there is a constant \( B \) so that for all \((x, y) \in \Delta \) we have

\[-Bh(x, y) < f(x, y) < Bh(x, y).\]

Then the largest eigenvalue of the transfer operator is one.

**Proof** Let \( L \) denote the transfer operator and let \( \phi(x, y) \in V \) be an eigenfunction with eigenvalue \( \lambda \). By assumption, there is a constant \( B \) so that

\[-Bh(x, y) < \phi(x, y) < Bh(x, y).\]

Then for all positive \( n \), by applying the transfer operator \( n \) times, we have

\[-Bh(x, y) < \lambda^n \phi(x, y) < Bh(x, y),\]

which means that

\[-1 \leq \lambda \leq 1.\]

\( \square \)

**Corollary 17** For the 18 triangle partition maps in the above table, the corresponding transfer operator has its largest eigenvalue being one.

**Proof** Just explicitly check that each of the elements in the corresponding Banach space satisfies the criterion of the above theorem. \( \square \)

**Theorem 18** The eigenvalue 1 of \( L_{T_{\sigma, \tau_0, \tau_1}} : V \to V \) has multiplicity 1 for every \((\sigma, \tau_0, \tau_1) \in S_3\) for which \( L_{T_{\sigma, \tau_0, \tau_1}} \) is an ergodic map from \( V \) to \( V \) whose eigenfunction of eigenvalue 1 is known.
Proof

The result follows immediately from Theorem 4.2.2 in [25] by Lasota, et al., quoted word-for-word from [25]:

Let \((X, A, \mu)\) be a measure space, \(S : X \to X\) a nonsingular transformation, and \(P\) the Frobenius-Perron operator associated with \(S\). If \(S\) is ergodic, then there is at most one stationary density \(f_*\) of \(P\). Further, if there is a unique stationary density \(f_*\) of \(P\) and \(f_*(x) > 0\) a.e., then \(S\) is ergodic.

Clearly, for us \(S = T_{\sigma, r_0, r_1}, P = L_{T_{\sigma, r_0, r_1}}\), and \(f_*\) is the associated eigenfunction of eigenvalue 1.

\(\square\)

Unfortunately, ergodicity is not known for most of the TRIP maps. The TRIP map \(T_{e,e,e}\) was shown to be ergodic by Messaoudi, Nogueira and Schweiger [31], while \(T_{e,23,e}, T_{e,23,23}, T_{e,132,23}, T_{e,23,132}\) were shown to be ergodic by Jensen [21]. Some preliminary work on ergodicity for some of the other TRIP maps is in [2], which leads us to believe that the above results are also true for quite a few more triangle partition maps.

7 On a Hilbert Space Approach

Mayer and Roepstorff [28] showed that the transfer operator of the Gauss map is a nuclear operator of trace class zero when acting on \(L^2([0, 1])\). In [14], it was shown that the transfer operator of the triangle map \(T_{e,e,e}\) has a non-trivial analog, namely that the transfer operator should be thought of as an operator acting on a family of Hilbert spaces with respect to the variable \(y\), with the variable \(x\) treated as a parameter. This section will show that there are similar results for 35 additional triangle partition maps. The work is in the same spirit and the proofs are almost the same as those given in [14], which in turn is in the same spirit, with mirrored proofs, as those given by Mayer and Roepstorff.
On the non-negative reals, set
\[ d_m(t) = \frac{t}{e^t - 1} dt. \]

For each of the triples \( X = (\sigma, \tau_0, \tau_1) \) in the below table, we have listed associated functions \( j(x, y), h(x, y) \) and \( l(x, y) \), whose domains are all of \( \triangle \). Then set
\[
\eta_k(s) = \frac{s^k e^{-s}}{(k + 1)!},
\]
\[
e_k(t) = L_1^k(t),
\]
\[
E_k(x, y) = \int_0^\infty \left( \frac{1}{h(x, y)^2} \right) e^{-t(l(x, y) - 1)} e_k(t) dm(t),
\]
\[
K_{TX}(\phi(x, t)) = \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}} \frac{t}{e^t - 1} \phi(x, s) dm(s),
\]
\[
\hat{\phi}(x, y) = j(x, y) \int_{s=0}^\infty e^{-sh(x, y)} \phi(x, s) dm(s),
\]
\[
\langle \alpha(s), \beta(s) \rangle = \int_0^\infty \alpha(s) \beta(s) dm(s).
\]

(Here \( L_1^k(t) \) denotes the first Laguerre polynomial and \( J_1 \) denotes the Bessel function of order one.)

**Theorem 19** For functions \( f(x, y) \) on \( \triangle \) for which there is a function \( \phi(x, y) \) such that \( f = \hat{\phi} \) and for which all of the following integrals and sums exist, we have
\[
\mathcal{L}_{TX}(f(x, y)) = \frac{1}{h(x, y)^2} \int_0^\infty e^{-t(l(x, y) - 1)} K_{TX}(\phi(x, t)) dt
\]
\[
= \sum_{k=0}^\infty \langle \phi(x, s), \eta_k(s) \rangle E_k(x, y).
\]

For each choice from the 36 triples \( (\sigma, \tau_0, \tau_1) \) in the below table, we get different allowable functions \( f \), which is why in the above theorem we simply state that we want all the integrals to converge.

**Proof** In [14], it was shown by setting \( w = \frac{1+y}{x} \) that the transfer operator \( \mathcal{L}_{T_{e,e,e}} \) can be put into the language of Mayer and Roepstorff [28], yielding for us the desired results. For other transfer operators, the change of coordinate is
\[
w = l(x, y).
\]
Then by direct calculation, it can be shown that

\[ \mathcal{L}_{T_x}(f(x, y)) = \frac{1}{(h(x, y))^2} \int_0^\infty \sum_{k=0}^\infty \frac{1}{(k+w)^2} e^{(x+w)} \phi(x, s) ds \]

The term

\[ \int_0^\infty \sum_{k=0}^\infty \frac{1}{(k+w)^2} e^{(x+w)} \phi(x, s) ds, \]

treating \( \phi(x, s) \) as a function of \( s \), can now be directly studied, independent of the triangle partition map. This has been done in [14], allowing us to conclude that the theorem is true.

\[ \square \]

We do not believe that this theorem is only true for these 36 triangle partition maps; it is just that for these 36 maps, we have been able to find the appropriate functions \( j(x, y), h(x, y) \) and \( l(x, y) \).
\[(\sigma, \tau_0, \tau_1) \mid L_{\tau_0, \tau_1} f(x, y) \mid l(x, y) \mid j(x, y) \mid h(x, y)\]

\[(e, e, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x} \quad y \]

\[(e, 12, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x} \quad y \]

\[(e, 13, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k+2y}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{x+1}{y} \quad \frac{1}{x+y} \quad y \]

\[(e, 23, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k+2x}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{x+1}{y} \quad \frac{1}{x+y} \quad y \]

\[(e, 123, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 - \frac{1}{k+x+y+1}\right) \quad \frac{x-2}{y-1} \quad \frac{1}{x} \quad y \]

\[(e, 132, e) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 + \frac{1}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(12, e, 12) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(12, 12, 12) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(x \frac{k}{k+x+y+1}, y \frac{x}{k+x+y+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(12, 13, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 - \frac{1}{k+x+y+1}\right) \quad \frac{x-2}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(12, 123, 12) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 + \frac{1}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(12, 132, 12) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 + \frac{1}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(13, e, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{yk+k-x+2}{yk+k-x+2}, x, 1 - \frac{1}{yk+k-x+2}\right) \quad \frac{y+1}{x} \quad \frac{1}{x} \quad 1 - x \]

\[(13, 12, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 - \frac{1}{k+x+y+1}\right) \quad \frac{x-2}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(13, 13, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(13, 123, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 + \frac{1}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(13, 132, 13) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(1 - \frac{x}{k+x+y+1}, 1 + \frac{1}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad y \]

\[(23, e, 23) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{yk+k-x+2}{yk+k-x+2}, x, 1 - \frac{1}{yk+k-x+2}\right) \quad \frac{y+1}{x} \quad \frac{1}{x} \quad x - y \]

\[(23, 12, 23) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{yk+k-x+2}{yk+k-x+2}, x, 1 - \frac{1}{yk+k-x+2}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(23, 13, 23) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(23, 23, 23) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(123, e, 132) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(123, 12, 132) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(\frac{k+2x}{k+x+y+1}, \frac{k+2y}{k+x+y+1}, \frac{x}{k+x+y+1}\right) \quad \frac{x}{y-1} \quad \frac{1}{x+y+1} \quad 1 - x \]

\[(123, 13, 132) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(123, 23, 132) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(123, 123, 132) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]

\[(123, 123, 123) \quad \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)!} f\left(-\frac{zk+yk+ky+1}{zk+yk+ky+1}, x, 1 - \frac{1}{zk+yk+ky+1}\right) \quad \frac{y+1}{x} \quad \frac{1}{x+y+1} \quad y \]
8 Gauss-Kuzmin Distributions for a Few Triangle Partition Sequences

The classical Gauss-Kuzmin statistics give the statistics of the digits occurring in the continued fraction expansion of a number. It is natural to ask similar questions for any multidimensional continued fraction algorithm (as discussed in Schweiger [33]). In [14], the Gauss-Kuzmin distribution was derived for the triangle partition $T_{e,e,e}$. We will build on that work here.

Fix a triple $(\sigma, \tau_0, \tau_1)$. Let the $(\sigma, \tau_0, \tau_1)$ expansion for a point $(x, y) \in \Delta$ be $(a_1, a_2, a_3, \ldots)$. Set

$$P_{n,k}(x, y) = \frac{\# \{ a_i : a_i = k \text{ and } 1 \leq i \leq n \}}{n}.$$ 

If the limit exists, set

$$P_k(x, y) = \lim_{n \to \infty} P_{n,k}(x, y).$$

We want to see when there is a function $p(k)$ that is equal to $P_k(x, y)$ for almost all $(x, y) \in \Delta$ and then find explicit formulas for each $p(k)$.

**Theorem 20** Let $(x, y) \in \Delta$ and suppose $T_{\sigma, \tau_0, \tau_1}$ is a triangle partition map that is ergodic with respect to the Lebesgue measure $\lambda$ and has associated invariant measure $\mu(A) = \int_A r(x, y) \, dr(x, y)$ as shown in the table in the table below. If $\mu$ and $\lambda$ are absolutely continuous, then $p(k) = \int_{\Delta_k} d\mu$ for almost every $(x, y) \in \Delta$. 
\[
\begin{array}{|c|c|}
\hline
(\sigma, \tau_0, \tau_1) & r(x, y) \\
\hline
(e, e, e) & 6\int_{x=0}^{x=1} y + 1 dx \cdot dy \\
(e, 23, e) & \pi^2 x(1-y) \\
(e, 132, e) & \pi^2 x(1-y) \\
(12, 12, 12) & \pi^2 (y+1)(-x+y+1) \\
(12, 13, 12) & \pi^2 (1-y)(-x+y+1) \\
(12, 123, 12) & \pi^2 (1-y)(-x+y+1) \\
(13, 13, 13) & \pi^2 (2-x)(1-y) \\
(13, 23, 13) & \pi^2 (1-y) \\
(13, 132, 13) & \pi^2 (1-y) \\
(23, e, 23) & \pi^2 (x-x+y+1) \\
(23, 12, 23) & \pi^2 (x-x+y+1) \\
(23, 23, 23) & \pi^2 (x-x+y+1) \\
(123, 13, 132) & \pi^2 (1-y)(-x+y+1) \\
(123, 123, 132) & \pi^2 (1-y)(-x+y+1) \\
(123, 132, 132) & \pi^2 (1-y)(x-y+1) \\
(132, e, 123) & \pi^2 (x-x+y+1) \\
(132, 12, 123) & \pi^2 (x-x+y+1) \\
(132, 123, 123) & \pi^2 (2-x)(-x+y+1) \\
\hline
\end{array}
\]

The proof is exactly analogous to that in [22].

Thus we will have explicit Gauss-Kuzmin statistics if we can calculate \( \int_{\Delta_k} \mu \). In [14], we showed for the triangle map \( T_{e,e,e} \), that

\[
p(0) = 1 - \frac{6\text{Li}_2 \left( \frac{1}{4} \right) + 12 \log^2(2)}{\pi^2}
\]

and, for \( k > 0 \),

\[
p(k) \quad = \quad \frac{6}{\pi^2} \left[ \text{Li}_2 \left( \frac{1}{(k_1)^2} \right) - \text{Li}_2 \left( \frac{1}{(k+2)^2} \right) \\
+ 4 \log^2(k+1) - 2 \log^2 \left( \frac{k+2}{k+1} \right) - 2 \log(k(k+2)) \log(k+1) \right]
\]

Here, we have an analogous result for \( T_{e,23,e} \).

**Corollary 21** For \( T_{e,23,e} \), we have

\[
p(0) = \frac{1}{2},
\]
while for $k > 0$,

$$p(k) = \int_{\frac{1}{k+1}}^{\frac{1}{k+2}} \left( \int_{\frac{1-x}{k+1}}^{x} \frac{6}{\pi^2 x(1-y)} \, dy \right) \, dx + \int_{\frac{1}{k+1}}^{1} \left( \int_{\frac{1-x}{k+1}}^{\frac{1-x}{k+2}} \frac{6}{\pi^2 x(1-y)} \, dy \right) \, dx.$$ 

**Proof**

First, we know that $T_{e,23,e}$ is ergodic with respect to the Lebesgue measure. Since for $T_{e,23,e}$, $h(x,y) = \frac{6}{\pi^2 x(1-y)}$, it follows that $\mu$ and $\lambda$ are absolutely continuous. Hence, by Theorem 20 it follows that $p(k) = \int_{\triangle_k} d\mu$. By direct calculation, it is evident that

$$p(0) = \int_{\triangle_0} d\mu = \int_{\frac{1}{2}}^{1} \left( \int_{1-x}^{x} \frac{6}{\pi^2 x(1-y)} \, dy \right) \, dx = \frac{1}{2}.$$ 

while for $k > 0$, $p(k) = \int_{\triangle_k} d\mu = \int_{\frac{k+1}{k+2}}^{\frac{k+2}{k+1}} \left( \int_{\frac{1-x}{k+1}}^{x} \frac{6}{\pi^2 x(1-y)} \, dy \right) \, dx + \int_{\frac{1}{k+1}}^{1} \left( \int_{\frac{1-x}{k+1}}^{\frac{1-x}{k+2}} \frac{6}{\pi^2 x(1-y)} \, dy \right) \, dx.$$

For the many other triangle partition maps, the Gauss-Kuzmin statistics are still unknown.

**9 Conclusion**

This paper has begun the study of the transfer operators for triangle partition maps, a family that puts into one framework most well-known multidimensional continued fraction algorithms. We have seen that for many of these triangle partition maps, basic questions about the transfer operators remain.

In this paper, we have studied the functional analysis behind triangle partition maps. In particular, we have partitioned this family of 216 multidimensional continued fractions into two natural classes, with exactly half exhibiting what we term “polynomial behavior” and the other half exhibiting “non-polynomial behavior.” We have found eigenfunctions of eigenvalue 1 for transfer operators associated with 17 polynomial-behavior triangle partition maps. The formidably complex form of the non-polynomial-behavior transfer operators and

\footnote{The corresponding eigenfunction for $T_{e,e,e}$ was already well-known.}
lack of appropriate techniques makes finding leading eigenfunctions prohibitively difficult; however, finding leading eigenfunctions for the remaining 90 polynomial-behavior transfer operators appears a doable, though computationally-intensive, problem. We have shown that transfer operators associated with a handful of polynomial-behavior triangle partition maps have spectral gaps; a key step in the argument relies on ergodicity of the associated triangle partition maps. Proving ergodicity for more triangle partition maps (see [21] and [31] for an outline of current ergodicity proofs) should facilitate more spectral gap results for associated transfer operators. Can ergodicity and spectral gaps for transfer operators associated with all polynomial-behavior triangle partition maps be demonstrated? Finally, we have shown the nuclearity of transfer operators, thought of as acting on one-dimensional families of Hilbert spaces, associated with 36 polynomial-behavior maps. Can transfer operators associated with all polynomial-behavior triangle partition maps be classified as nuclear?

There are also more general questions. There is the question of connecting the properties of these transfer operators with Diophantine approximation properties (possibly linking with work stemming from [24]). Also, as mentioned in [14], how much of the rich body of work done over the years on the transfer operator for the Gauss map, pioneered by the work of Mayer and Roepstorff [28, 29] and of Mayer [26, 27], and continuing today in work of Vallée [37, 38], of Isola, Antoniou and Shkarin [19, 4], of Jenkinson, Gonzalez and Urbanski [20], of Degli, Espost, Isola and Knauf [12], of Hilgert [16], of Bonanno, Graffi and Isola [8], of Alkauskas [1], of Iosifescu [17], of Bonanno and Isola [9], and of Ben Ammou, Bonanno, Chouari and Isola [7], has analogs for triangle partition maps? These questions strike us as natural but non-trivial.
# Form of $T_{\sigma,\tau_0,\tau_1}(x, y)$ for Polynomial-Behavior Maps

| $(e, e, e)$ | \(\left(\frac{y}{x} - \frac{x+ky-1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + (-1)^k (4x-y-2) - 2ky+y+2 - 1\) |
|------------|----------------------------------------------------------------------------------------------------------------------------------|
| $(e, e, 12)$ | \(\left(\frac{1}{2} \left(1 + (-1)^k + 1\right) + \frac{(-1)^k y}{x}, -\frac{(2k+(-1)^k+3)x+2(-1+(-1)^k)y+4}{4x}\right)\) |
| $(e, 12, e)$ | \(\left(\frac{1-(k+1)y}{x}, -\frac{x+ky-1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + (-1)^k (4x-y-2) - 2ky+y+2 - 1\) |
| $(e, 12, 12)$ | \(\left(\frac{4ky-y+1}{x}, \frac{4ky-y+1}{x}\right)\) \(-\frac{2kx+x+(-1)^k+1(x-2y)+2y-4}{4x}, -\frac{(2k+(-1)^k+3)x+2(-1+(-1)^k)y+4}{4x}\) |
| $(e, 13, e)$ | \(\left(\frac{2x+ky-1}{x}, 1 - \frac{y}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 2, 1 - \frac{(-1)^k (4x-y-2) - 2ky+y+2}{4y}\) |
| $(e, 13, 12)$ | \(\left(\frac{2kx+27x+(-1)^k(x-2y)+2y-4}{4x}, \frac{x+(-1)^k(x-2y)}{2x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 13, 23)$ | \(\left(\frac{4y}{x}, \frac{x+ky-y+1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 23, e)$ | \(\left(\frac{y}{x}, \frac{y}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 23, 12)$ | \(\left(\frac{4ky-y+1}{x}, \frac{4ky-y+1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 23, 23)$ | \(\left(\frac{\frac{1}{2} \left(1 + (-1)^k + 1\right) + \frac{(-1)^k y}{x}, \frac{(2k+(-1)^k+5)x+2((-1)^k+y+2)}{4x}\right)\) |
| $(e, 123, e)$ | \(\left(\frac{2x+ky-1}{x}, \frac{x+ky-1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 123, 12)$ | \(\left(\frac{4k+1+y+1}{x}, \frac{4k+1+y+1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 123, 23)$ | \(\left(\frac{2kx+7x+(-1)^k(x-2y)+2y-4}{4x}, \frac{(2k+(-1)^k+5)x+2((-1)^k+y+2)}{4x}\right)\) |
| $(e, 132, e)$ | \(\left(\frac{1-(k+1)y}{x}, 1 - \frac{y}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 132, 12)$ | \(\left(\frac{4ky+y+1}{x}, \frac{4ky+y+1}{x}\right)\) \((-1)^k (4x-y-2) - 2ky+y+2 + 1\) |
| $(e, 132, 23)$ | \(\left(\frac{-2kx+x+(-1)^k+1(x-2y)+2y-4}{4x}, \frac{x+(-1)^k(x-2y)}{2x}\right)\) |
\[
\begin{align*}
(12, e, e) & \quad \frac{4y}{(-1)^{k+1}(4x-3y-2) - 2ky+y+2} = \frac{4ky-4}{(-1)^k(4x-3y-2) + 2ky-y-2} - 1 \\
(12, e, 12) & \quad \frac{y}{-x+y+1}, \quad \frac{-x+ky+y}{x-y-1} \\
(12, e, 123) & \quad \frac{1}{2} \left( -\frac{(-1)^k(x+y-1)}{2(x-y-1)} \right), \quad -\frac{3x+5y+2k(-x+y+1)+(-1)^{k+1}(x+y-1)-1}{4(x-y-1)} \\
(12, 12, e) & \quad \frac{-3x+5y+2k(-x+y+1)+(-1)^{k+1}(x+y-1)-1}{4ky-4} \\
(12, 12, 12) & \quad \frac{ky+g-1}{x-y-1}, \quad \frac{-x+ky+g}{x-y-1} \\
(12, 12, 123) & \quad \left( \frac{-x+3y+2k(-x+y+1)+(-1)^k(x+y-1)-3}{4(x-y-1)} \right), \quad \frac{-3x+5y+2k(-x+y+1)+(-1)^{k+1}(x+y-1)-1}{4(x-y-1)} \\
(12, 13, e) & \quad \frac{4-4ky}{(-1)^k(4x-3y-2)+2ky-y-2} + 2, \quad 1 - \frac{4y}{(-1)^{k+1}(4x-3y-2)-2ky+y+2} \\
(12, 13, 12) & \quad \frac{-2x+(k+2)y+1}{-x+y+1}, \quad \frac{x-1}{x-y-1} \\
(12, 13, 123) & \quad \frac{7x+2k(x-y-1)-9y+(-1)^{k+1}(x+y-1)-3}{4(x-y-1)}, \quad \frac{1}{2} \left( \frac{(-1)^k(x+y-1)}{x-y-1} + 1 \right) \\
(12, 23, e) & \quad \frac{4y}{(-1)^{k+1}(4x-3y-2)-2ky+y+2}, \quad \frac{-3x+5y+2k(-x+y+1)+(-1)^{k+1}(x+y-1)-1}{4(x-y-1)} \\
(12, 23, 12) & \quad \frac{4-4(k+1)y}{-x+y+1}, \quad \frac{(k+2)y-x}{-x+y+1} \\
(12, 23, 123) & \quad \frac{1}{2} \left( \frac{(-1)^k(x+y-1)}{2(x-y-1)} \right), \quad \frac{5x+2k(x-y-1)-7y+(-1)^{k+1}(x+y-1)-1}{4(x-y-1)} \\
(12, 123, e) & \quad \frac{4-4ky}{(-1)^k(4x-3y-2)+2ky-y-2} + 2, \quad \frac{-2x+(k+2)y+1}{-x+y+1}, \quad \frac{(k+2)y-x}{-x+y+1} \\
(12, 123, 12) & \quad \frac{7x+2k(x-y-1)-9y+(-1)^{k+1}(x+y-1)-3}{4(x-y-1)}, \quad \frac{5x+2k(x-y-1)-7y+(-1)^{k+1}(x+y-1)-1}{4(x-y-1)} \\
(12, 123, 123) & \quad \frac{4(ky+y-1)}{(-1)^{k}(4x-3y-2)+2ky-y-2}, \quad 1 - \frac{4y}{(-1)^{k+1}(4x-3y-2)-2ky+y+2} \\
(12, 132, e) & \quad \frac{ky+y-1}{x-y-1}, \quad \frac{x-1}{x-y-1} \\
(12, 132, 12) & \quad \frac{-x+3y+2k(-x+y+1)+(-1)^k(x+y-1)-3}{4(x-y-1)}, \quad \frac{1}{2} \left( \frac{(-1)^k(x+y-1)}{x-y-1} + 1 \right)
\end{align*}
\]
\[
\begin{align*}
(13, e, 13) & \quad 4-4x \quad \frac{x-1}{y-1}, \quad \frac{-x^k+k-y}{y-1} \\
(13, e, 123) & \quad 2k(x-1)-x+(-1)^k(x-4y+1)+3 \cdot 2k(x-1)-x+(-1)^k(x-4y+1)+3 - 1 \\
(13, e, 132) & \quad \left( \frac{1}{2} - \frac{(-1)^k(-2x+y+1)}{2(y-1)}, \quad \frac{-2x+(-1)^k(2x-y+1)-2k(y-1)-3y+1}{4(y-1)} \right) \\
(13, 12, 13) & \quad \left( \frac{k-(k+1)x}{y-1}, \quad \frac{-x^k+k-y}{y-1} \right) \\
(13, 12, 123) & \quad 4(k(x-1)+x) \quad 2k(x-1)-x+(-1)^k(x-4y+1)+3 \cdot 2k(x-1)-x+(-1)^k(x-4y+1)+3 - 1 \\
(13, 12, 132) & \quad \left( \frac{-2(1+(-1)^k)x-2k(y-1)+(-1+(-1)^k)(y+1)}{4(y-1)}, \quad \frac{-2x+(-1)^k(2x-y+1)-2k(y-1)-3y+1}{4(y-1)} \right) \\
(13, 13, 13) & \quad \frac{k(x-1)+1}{y-1} + 2, \quad \frac{y-x}{y-1} \\
(13, 13, 123) & \quad \frac{1}{-x+(-1)^k(x-4y+1)+3} + \frac{4(x-1)}{2k(x-1)-x+(-1)^k(x-4y+1)+3} + 1 \\
(13, 13, 132) & \quad \left( \frac{-2(-1+(-1)^k)x+2k(y-1)+7y+(-1)^k(y+1)-5}{4(y-1)}, \quad \frac{1}{2} \left( \frac{(-1)^k(-2x+y+1)}{y-1} + 1 \right) \right) \\
(13, 13, 132) & \quad \left( \frac{x-1}{y-1}, \quad \frac{(k+1)(x-1)+y}{y-1} \right) \\
(13, 23, 13) & \quad \frac{4-4x}{2k(x-1)-x+(-1)^k(x-4y+1)+3} \cdot 2k(x-1)-x+(-1)^k(x-4y+1)+3 + 1 \\
(13, 23, 123) & \quad \left( \frac{1}{2} - \frac{(-1)^k(-2x+y+1)}{2(y-1)}, \quad \frac{2x+(-1)^k(2x-y+1)+2k(y-1)+5y-3}{4(y-1)} \right) \\
(13, 123, 13) & \quad \left( \frac{k(x-1)+1}{y-1} + 2, \quad \frac{(k+1)(x-1)+y}{y-1} \right) \\
(13, 123, 123) & \quad \frac{1}{-x+(-1)^k(x-4y+1)+3} + \frac{4k-4(k+1)x}{2k(x-1)-x+(-1)^k(x-4y+1)+3} + 1 \\
(13, 123, 132) & \quad \left( \frac{-2(-1+(-1)^k)x+2k(y-1)+7y+(-1)^k(y+1)-5}{4(y-1)}, \quad \frac{2x+(-1)^k(2x-y+1)+2k(y-1)+5y-3}{4(y-1)} \right) \\
(13, 132, 13) & \quad \left( \frac{k-(k+1)x}{y-1}, \quad \frac{y-x}{y-1} \right) \\
(13, 132, 123) & \quad \frac{4(k(x-1)+x)}{2k(x-1)-x+(-1)^k(x-4y+1)+3} \cdot 2k(x-1)-x+(-1)^k(x-4y+1)+3 + 1 \\
(13, 132, 132) & \quad \left( \frac{-2(1+(-1)^k)x-2k(y-1)+(-1+(-1)^k)(y+1)}{4(y-1)}, \quad \frac{1}{2} \left( \frac{(-1)^k(-2x+y+1)}{y-1} + 1 \right) \right)
\end{align*}
\]
\[
\begin{array}{ll}
\text{(23, e, e)} & (x+(-1)^k(x-2y) \div 2x, \frac{-2kx-5x+(-1)^k(x-2y)+2y+4}{4x}) \\
\text{(23, e, 23)} & \left(1 - \frac{y}{x}, \frac{-(k+1)x+k+1}{x}\right) \\
\text{(23, e, 132)} & \frac{4(x-y)}{2kx+x+2ky-y+(-1)^k(3x+y-2)+2} - 2kx+x+2ky-y+(-1)^k(3x+y-2)+2 - 1 \\
\text{(23, 12, e)} & \frac{2((-1)^k y+y+2)-(2k+(-1)^k+3)x}{4x}, \frac{-2kx-5x+(-1)^k(x-2y)+2y+4}{4x} \\
\text{(23, 12, 23)} & \left(\frac{-(k+1)x+k+1}{x}, \frac{-(k+1)x+k+1}{x}\right) \\
\text{(23, 12, 132)} & \frac{4(-1)^k x+ky+y+1}{-2kx+x+2ky-y+(-1)^k(3x+y-2)+2} - 2kx+x+2ky-y+(-1)^k(3x+y-2)+2 - 1 \\
\text{(23, 13, e)} & \frac{2k+(-1)^k+9)x+2(-1+(-1)^k)y-4}{4x}, \frac{1}{2} \left(1 + (-1)^k+1x \right) + \frac{(1)^k y}{x} \\
\text{(23, 13, 23)} & \left(\frac{k(y-x)+1}{x-y+(-1)^k(3x+y-2)}, \frac{4y-4x}{x} \right) \\
\text{(23, 13, 132)} & \frac{1}{x-y+(-1)^k(3x+y-2)} + \frac{1}{2} \frac{-2kx+2ky-y+(-1)^k(3x+y-2)+2}{4y-4x} + 1 \\
\text{(23, 23, e)} & \frac{x+(-1)^k(x-2y)}{2x}, \frac{2k+(-1)^k+7)x-2(-1)^ky+y+2}{4x} \\
\text{(23, 23, 23)} & \left(1 - \frac{y}{x}, \frac{(k+2)x-(k+1)y-1}{x}\right) \\
\text{(23, 23, 132)} & \frac{4(x-y)}{-2kx+x+2ky-y+(-1)^k(3x+y-2)+2} - 2kx+x+2ky-y+(-1)^k(3x+y-2)+2 + 1 \\
\text{(23, 123, e)} & \frac{2k+(-1)^k+9)x+2(-1+(-1)^k)y-4}{4x}, \frac{2k+(-1)^k+7)x-2(-1)^ky+y+2}{4x} \\
\text{(23, 123, 23)} & \left(\frac{k(y-x)+1}{x-y+(-1)^k(3x+y-2)}, \frac{4kx-x-(k+1)y-1}{x}\right) \\
\text{(23, 123, 132)} & \frac{1}{x-y+(-1)^k(3x+y-2)} + \frac{1}{2} \frac{4(1+k+1)y-1}{4kx-x-(k+1)y-1} + 1 \\
\text{(23, 132, e)} & \frac{2((-1)^ky+y+2)-(2k+(-1)^k+3)x}{4x}, \frac{1}{2} \left(1 + (-1)^k+1x \right) + \frac{(1)^k y}{x} \\
\text{(23, 132, 23)} & \left(\frac{-(k+1)x+k+1}{x}, \frac{-(k+1)x+k+1}{x}\right) \\
\text{(23, 132, 132)} & \frac{4(-1)^k x+ky+y+1}{-2kx+x+2ky-y+(-1)^k(3x+y-2)+2} - 2kx+x+2ky-y+(-1)^k(3x+y-2)+2 + 1 \\
\end{array}
\]
\[
(123, e, 13) \quad \left( \frac{1}{2} \left( \frac{(-1)^k(2x+y+1)}{y+1} + 1 \right), \quad \frac{-2x+(-1)^k(2x-y+1)+2k(y+1)+5y+1}{4(y-1)} \right)
\]

\[
(123, e, 23) \quad \left( \frac{4y-4x}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2}, \quad \frac{4k(x-y)-4}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2} - 1 \right)
\]

\[
(123, e, 132) \quad \left( \frac{y-x}{y-1}, \quad \frac{y-2x}{x+k(y+1)} \right)
\]

\[
(123, 12, 13) \quad \left( \frac{-2x+(1(-1)^k)x+2k(y+1)+(3(-1)^k)(y+1)}{4(y-1)}, \quad \frac{-2x+(-1)^k(2x-y+1)+2k(y+1)+5y+1}{4(y-1)} \right)
\]

\[
(123, 12, 23) \quad \left( \frac{4(kx+y+(k+1)y+1)}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2}, \quad \frac{4k(x-y)-4}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2} - 1 \right)
\]

\[
(123, 12, 132) \quad \left( \frac{kx+x-(k+1)y+1}{y-1}, \quad \frac{kx-(k+1)y}{y-1} \right)
\]

\[
(123, 13, 13) \quad \left( \frac{-2x+(-1)^k(2x-y+1)+2k(y+1)+9y+3}{4(y-1)}, \quad \frac{1}{2}, \quad \frac{-1(-1)^k(-2x+y+1)}{2(y-1)} \right)
\]

\[
(123, 13, 132) \quad \left( \frac{k(1-x^k)}{y-1} + \frac{4(x-y)(x-3y+2)}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2} + 1 \right)
\]

\[
(123, 13, 23) \quad \left( \frac{y+2x}{y-1}, \quad \frac{x+1}{y+1} \right)
\]

\[
(123, 132) \quad \left( \frac{kx+x-(k+1)y+1}{y-1}, \quad \frac{x+k(y+1)}{y-1} \right)
\]

\[
(123, 132) \quad \left( \frac{4(kx+y+(k+1)y+1)}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2}, \quad \frac{4k(x-y)-4}{2kx-x-2ky+y+(-1)^k(x+3y-2)-2} + 1 \right)
\]

\[
(123, 132) \quad \left( \frac{kx+x-(k+1)y+1}{y-1}, \quad \frac{x-1}{y-1} \right)
\]
\[
\begin{array}{ll}
(132, e, 12) & \left( \frac{1}{2} \left( \frac{-1)^k(x+y-1)}{x-y-1} + 1 \right), -5x+3y+2k(-x+y+1)+(-1)^k(x+y-1)+1 \right) \\
(132, e, 13) & \left( \frac{4(x-1)}{-2k(x-1)+x+(-1)^k(3x-4y-1)+3} + 2k(x-1)-x+(-1)^k+1(3x-4y-1)+3 - 1 \right) \\
(132, e, 123) & \left( \frac{-3x+y+2k(-x+y+1)+(-1)^k+1(x+y-1)-1}{4(x-y-1)}, -5x+3y+2k(-x+y+1)+(-1)^k(x+y-1)+1 \right) \\
(132, 12, 12) & \left( \frac{4k(x-1)+x}{2k(x-1)-x+(-1)^k+1(3x-4y-1)+3} + 2k(x-1)-x+(-1)^k+1(3x-4y-1)+3 - 1 \right) \\
(132, 12, 13) & \left( \frac{k(x-1)+x}{-x+y+1} + 1 \right) \\
(132, 12, 13) & \left( \frac{9x+2k(x-y-1)-7y+(-1)^k+1(x+y-1)-5}{4(x-y-1)}, \frac{1}{2} = \frac{(-1)^k(x+y-1)}{2(x-y-1)} \right) \\
(132, 13, 13) & \left( \frac{-4k(x-1)-4}{2k(x-1)-x+(-1)^k+1(3x-4y-1)+3} + 2, \frac{-2k(x-1)+x+(-1)^k(3x-4y-1)-3}{4(x-y-1)} \right) \\
(132, 13, 123) & \left( \frac{-x+k-k}{x+y+1} + 2, \frac{y}{-x+y+1} \right) \\
(132, 13, 12) & \left( \frac{1}{2} \left( \frac{-1)^k(x+y-1)}{x-y-1} + 1 \right), \frac{7x+2k(x-y-1)-5y+(-1)^k(x+y-1)-3}{4(x-y-1)} \right) \\
(132, 12, 13) & \left( \frac{-2k(x-1)+x+(-1)^k(3x-4y-1)-3}{2k(x-1)-x+(-1)^k+1(3x-4y-1)+3} + 1 \right) \\
(132, 23, 12) & \left( \frac{4(x-1)}{9x+2k(x-y-1)-7y+(-1)^k+1(x+y-1)-5}, \frac{7x+2k(x-y-1)-5y+(-1)^k(x+y-1)-3}{4(x-y-1)} \right) \\
(132, 23, 13) & \left( \frac{4k(4k+1)x}{-x+k-k-2x+y+1}, \frac{-2k(x-1)+x+(-1)^k+1(3x-4y-1)+3}{x-y-1} \right) \\
(132, 23, 12) & \left( \frac{4k(4k+1)x}{9x+2k(x-y-1)-7y+(-1)^k+1(x+y-1)-5}, \frac{7x+2k(x-y-1)-5y+(-1)^k(x+y-1)-3}{4(x-y-1)} \right) \\
(132, 12, 13) & \left( \frac{4k(x-1)+x}{2k(x-1)-x+(-1)^k+1(3x-4y-1)+3} + 2, \frac{-2k(x-1)+x+(-1)^k+1(3x-4y-1)+3}{x-y-1} \right) \\
(132, 123, 13) & \left( \frac{-x+k-k}{-x+y+1} + 2, \frac{-x+k-k-2x+y+1}{-x+y+1} \right) \\
(132, 123, 13) & \left( \frac{4(x-1)}{-3x+y+2k(-x+y+1)+(-1)^k+1(x+y-1)-1}, \frac{1}{2} = \frac{(-1)^k(x+y-1)}{2(x-y-1)} \right) \\
(132, 123, 13) & \left( \frac{4(x-1)}{4} - 4x, \frac{2k(x-1)+x+(-1)^k+1(3x-4y-1)+3}{x-y-1} \right) \\
(132, 123, 13) & \left( \frac{k(x-1)+x}{y}, \frac{-x+y+1}{-x+y+1} \right) \\
\end{array}
\]
### B Form of $\mathcal{L}_{T, \sigma, \tau_0, \tau_1} f(x, y)$ for Polynomial-Behavior Maps

| $(e, e)$ | $\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^{2}} f \left(\frac{1}{kx+y+1} \cdot \frac{x}{kx+y+1}\right)$ |
| $(e, 12)$ | $\sum_{k=0}^{\infty} \frac{1}{(2x+(-1)^k(2k-2x+4y+5)+1)^2} f \left(\frac{4(-1)^k}{2x+(-1)^k(2k-2x+4y+5)+1} \cdot \frac{x}{2x+(-1)^k(2k-2x+4y+5)+1}\right)$ |
| $(e, 23)$ | $\sum_{k=0}^{\infty} \frac{1}{(2x+(-1)^k(2k-2x+4y+5)+1)^2} f \left(\frac{4(-1)^k}{2x+(-1)^k(2k-2x+4y+5)+1} \cdot \frac{x}{2x+(-1)^k(2k-2x+4y+5)+1}\right)$ |
| $(e, 12)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 13)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 23)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 12)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 13)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 23)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 12)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 13)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
| $(e, 23)$ | $\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^2} f \left(\frac{4(-1)^k}{x+k(y-1)-2} \cdot \frac{x}{x+k(y-1)-2}\right)$ |
\[
\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^3} \left( \frac{(-1)^k \left( x+2y+((-1)^k (2kx+3x+2y+2)-2 \right)}{4(kx+y+1)} \right) x \]

\sum_{k=0}^{\infty} \frac{1}{(4kx+y+1)^3} \left( \frac{x}{kx+y+1} \right)

\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+4y+5)-1} \left( \frac{2(2x+(-1)^k-1)}{2x+(-1)^k(2k-2x+4y+5)-1} \right)

\sum_{k=0}^{\infty} \frac{1}{(-xk+y+k+y+1)^3} \left( \frac{(-1)^k \left( x-3y+((-1)^k(3x+2k(x-y-1)-5(y+1)+1) \right)}{4(y+k-x+y+1)+1} \right) \left( \frac{x-y+1}{-xk+y+k+y+1} \right)

\sum_{k=0}^{\infty} \frac{64}{2x+2y+(-1)^k(2k+2x+2y+3)+1} \left( \frac{2(-2x+(-1)^k+2y+1)}{2y+(-1)^k(2k+2x+2y+3)+1} \right)

\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^3} \left( \frac{2(-2y+(-1)^k+1)}{2(-2y+(-1)^k+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^3} \left( \frac{2(-2y+(-1)^k+1)}{2(-2y+(-1)^k+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^3} \left( \frac{2(-2y+(-1)^k+1)}{2(-2y+(-1)^k+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^3} \left( \frac{2(-2y+(-1)^k+1)}{2(-2y+(-1)^k+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^3} \left( \frac{x}{kx+y+1} \right)

\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^3} \left( \frac{x}{kx+y+1} \right)

\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+2y+7)+1} \left( \frac{2(-2x+(-1)^k+1)}{2(-2x+(-1)^k+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^3} \left( \frac{x}{kx+y+1} \right)

\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+2y+7)+2y+1} \left( \frac{2(-2x+(-1)^k+2y+1)}{2(-2x+(-1)^k+2y+1)} \right)

\sum_{k=0}^{\infty} \frac{1}{(kx+y+1)^3} \left( \frac{x}{kx+y+1} \right)

\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k+4x-2y+3)+2y+1} \left( \frac{2(-2y+(-1)^k+1)}{2(-2y+(-1)^k+1)} \right)
\[ \sum_{k=0}^{\infty} \left( \frac{1}{(k+x+y+1)^{\gamma}} \right) f \left( 1 - \frac{x}{k+x+y+1}, 1 - \frac{1}{k+x+y+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)^{\gamma}} \left( 1 - \frac{x}{k+x+y+1} \right) \left( -(-1)^k (x+2y+(-1)^k (2kx-x+2y+2) - 2) \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{2x+(x-1)^{k}(2k-2x+4y+5)-1} \left( \frac{-2x+(-1)^k (2k-2x+4y+5)+1}{2x+(x-1)^{k}(2k-2x+4y+5)-1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{-x+k+y+k+y+1} \left( \frac{-x+k+y+k+y+1}{-x+k+y+k+y+1}, 1 - \frac{-x+k+y+k+y+1}{-x+k+y+k+y+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} \left( \frac{-2x+2y+(-1)^k(2k+2x+2y+3)+1}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{(x+k(y-1)-2)^{\gamma}} \left( \frac{x+k(y-1)-y-1}{x+k(y-1)-2}, 1 + \frac{1}{x+k(y-1)-2} \right) \]

\[ \sum_{k=0}^{\infty} \frac{64}{-2y+(-1)^k(2k-4x+2y+7)+1} \left( \frac{-2y+(-1)^k(2k-4x+2y+7)+1}{-2y+(-1)^k(2k-4x+2y+7)+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{(kxx+x+y+1)^{\gamma}} \left( \frac{-2x+(-1)^k(2k+2x+2y+3)+1}{-2x+(-1)^k(2k+2x+2y+3)+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{(kxx+x+y+1)^{\gamma}} \left( \frac{x+k(y-1)-y-1}{x+k(y-1)-2}, 1 + \frac{1}{x+k(y-1)-2} \right) \]

\[ \sum_{k=0}^{\infty} \frac{64}{-2x+(-1)^k(2k-2x-2y+5)+2y+1} \left( \frac{-2x+(-1)^k(2k-2x-2y+5)+2y+1}{-2x+(-1)^k(2k-2x-2y+5)+2y+1} \right) \]

\[ \sum_{k=0}^{\infty} \frac{1}{(kxx+x+k+y+1)^{\gamma}} \left( \frac{-2x+3y+(-1)^k(2k-2x+k+y+1)}{4(kxx+x+k+y+1)}, 1 + \frac{1}{4(kxx+x+k+y+1)} \right) \]

\[ \sum_{k=0}^{\infty} \frac{64}{-2x+(-1)^k(2k+4x-2y+3)-2y+1} \left( \frac{-2x+(-1)^k(2k+4x-2y+3)-2y+1}{-2x+(-1)^k(2k+4x-2y+3)-2y+1} \right) \]
\[
\begin{align*}
(23, e, e) & \quad \sum_{k=0}^{\infty} \frac{64}{[2x+(-1)^k(2k-2x+4y+5)-1]^k} f \left( \frac{4^{(-1)^k} (22x-2(-1)^k(2k-2x+4y+5)-1)}{2k-2x+4y+5} \right) \\
(23, e, 23) & \quad \sum_{k=0}^{\infty} \frac{1}{(x+y+1)^k} f \left( \frac{1}{x+y+1}, \frac{1}{x+y+1}, \frac{1}{x+y+1} \right) \\
(23, e, 132) & \quad \sum_{k=0}^{\infty} \frac{1}{(k^2+y+1)^k} f \left( \frac{(-1)^k}{4(k^2+y+1)}, \frac{(-1)^k}{4(k^2+y+1)} \right) \\
(23, 12, e) & \quad \sum_{k=0}^{\infty} \frac{64}{[-2x+2y+(-1)^k(2k+2x+y+3)+1]^k} f \left( \frac{4^{(-1)^k}}{2x+2y+(-1)^k(2k+2x+y+3)+1}, \frac{2^{(2x+(-1)^k-2y)}}{4(k^2+y+1)} \right) \\
(23, 12, 23) & \quad \sum_{k=0}^{\infty} \frac{1}{([-x+y+k+y+1]^k f \left( \frac{1}{-x+y+k+y+1}, \frac{y}{-x+y+k+y+1} \right) \\
(23, 12, 132) & \quad \sum_{k=0}^{\infty} \frac{64}{[x+k(y+1)-2]^k} f \left( \frac{(-1)^k}{x+k(y+1)-2}, \frac{(-1)^k}{x+k(y+1)-2} \right) \\
(23, 23, e) & \quad \sum_{k=0}^{\infty} \frac{64}{[2x+(-1)^k(2k-2x+4y+5)-1]^k} f \left( \frac{y}{2x+(-1)^k(2k-2x+4y+5)-1}, \frac{y}{2x+(-1)^k(2k-2x+4y+5)-1} \right) \\
(23, 23, 23) & \quad \sum_{k=0}^{\infty} \frac{1}{(x+y+y+1)^k} f \left( \frac{1}{x+y+y+1}, \frac{1}{x+y+y+1} \right) \\
(23, 23, 132) & \quad \sum_{k=0}^{\infty} \frac{1}{(x+y+y+1)^k} f \left( \frac{(-1)^k}{(x+y+y+1)}, \frac{(-1)^k}{(x+y+y+1)} \right) \\
(23, 123, e) & \quad \sum_{k=0}^{\infty} \frac{64}{[-2x+(-1)^k(2k-2x+2y+7)+2y+1]^k} f \left( \frac{4^{(-1)^k}}{-2x+(-1)^k(2k-2x+2y+7)+2y+1}, \frac{2^{(-2x+(-1)^k+1)}}{2k-2x+2y+7} \right) \\
(23, 123, 23) & \quad \sum_{k=0}^{\infty} \frac{1}{[x+k(y+1)-2]^k} f \left( \frac{(-1)^k}{x+k(y+1)-2}, \frac{(-1)^k}{x+k(y+1)-2} \right) \\
(23, 132, 132) & \quad \sum_{k=0}^{\infty} \frac{1}{[x+k(y+1)-2]^k} f \left( \frac{(-1)^k}{x+k(y+1)-2}, \frac{(-1)^k}{x+k(y+1)-2} \right) \\
(23, 132, e) & \quad \sum_{k=0}^{\infty} \frac{64}{([k+x-(k+1)y+1])^k} f \left( \frac{4^{(-1)^k}}{[k+x-(k+1)y+1]} \right) \\
(23, 132, 23) & \quad \sum_{k=0}^{\infty} \frac{1}{([k+x-(k+1)y+1])^k} f \left( \frac{4^{(-1)^k}}{[k+x-(k+1)y+1]} \right) \\
(23, 132, 132) & \quad \sum_{k=0}^{\infty} \frac{1}{([k+x-(k+1)y+1])^k} f \left( \frac{4^{(-1)^k}}{[k+x-(k+1)y+1]} \right)
\end{align*}
\]
\[
\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+4y+5)-1} f\left(\frac{6x+(-1)^k(2k-2x+4y+3)-3}{2x+(-1)^k(2k-2x+4y+5)-1}, \frac{2x+(-1)^k(2k-2x+4y+1)-1}{2x+(-1)^k(2k-2x+4y+5)-1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(k+k+1)^2} f\left(\frac{k+k+1}{k+k+1}, 1 - \frac{1}{k+k+1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} f\left(\frac{-6x+6y+(-1)^k(2k+2x+2y+1)+3}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}, \frac{-2x+2y+(-1)^k(2k+2x+2y+1)+1}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{[k+k(y+1)+1]} f\left(\frac{-1}{[k+k(y+1)+1]}, \frac{1}{[k+k(y+1)+1]}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+4y+3)-1} f\left(\frac{6x+(-1)^k(2k-2x+4y+3)-3}{2x+(-1)^k(2k-2x+4y+5)-1}, \frac{2x+(-1)^k(2k-2x+4y+1)-1}{2x+(-1)^k(2k-2x+4y+5)-1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{[k+k(y+1)+1]} f\left(\frac{k+k+1}{k+k+1}, 1 - \frac{1}{k+k+1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} f\left(\frac{-6x+6y+(-1)^k(2k+2x+2y+1)+3}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}, \frac{-2x+2y+(-1)^k(2k+2x+2y+1)+1}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{[k+k(y+1)+1]} f\left(\frac{-1}{[k+k(y+1)+1]}, \frac{1}{[k+k(y+1)+1]}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} f\left(\frac{-6x+6y+(-1)^k(2k+2x+2y+1)+3}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}, \frac{-2x+2y+(-1)^k(2k+2x+2y+1)+1}{-2x+2y+(-1)^k(2k+2x+2y+3)+1}\right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{[k+k(y+1)+1]} f\left(\frac{-1}{[k+k(y+1)+1]}, \frac{1}{[k+k(y+1)+1]}\right)
\]
\[
\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k-2x+4y+5)+1} f \left( \frac{-2x+(-1)^k(2k-2x+4y+5)+1}{2x+(-1)^k(2k-2x+4y+5)+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{2k+1} f \left( 1 - \frac{x}{kx+y+1}, \frac{(-1)^k(-x-2y+(-1)^k(2k-2x+4y+2)+2)}{4(kx+y+1)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} f \left( \frac{-2x+2y+(-1)^k(2k+2x+2y+3)+1}{-2x+2y+(-1)^k(2k+2x+2y+3)+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(-x+y+k+1)^2} f \left( \frac{-x+y+k+1}{-x+y+k+1}, \frac{(-1)^k(x-3y+(-1)^k(2x-y+1)+y+1)}{4(y+k-x+y+1)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2y+(-1)^k(2k-4x+2y+7)+1} f \left( \frac{2y+(-1)^k(2k-4x+2y+5)-1}{2y+(-1)^k(2k-4x+2y+7)+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{x+k(y-1)-2} f \left( \frac{x+k(y-1)-2}{x+k(y-1)-2}, \frac{(-1)^k(2x+k(y-1)-3y-1)-y+1}{4(x+k(y-1)-2)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{2x+(-1)^k(2k+2x-4y+5)+1} f \left( \frac{-2x+(-1)^k(2k+2x-4y+5)+1}{2x+(-1)^k(2k+2x-4y+5)+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(k+x+y+1)^2} f \left( \frac{k(x+y+1)+y-1}{k(x+y+1)+1}, \frac{(-1)^k(-3x+y(-1)^k(-x+2k(y+1)+3y-1)+1)}{4(x+k(y+1)-2)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{-2x+2y+(-1)^k(2k-2x+2y+7)+2+1} f \left( \frac{2x+(-1)^k(2k-2x+2y+5)+2y-1}{2x+(-1)^k(2k-2x+2y+7)+2y+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(x+k(x+y-1)+1)^2} f \left( \frac{k(x+y-1)+y-1}{x+k(x+y-1)+1}, \frac{(-1)^k(2x+y(-1)^k(-x+2k(y+1)+3y-1)+1)}{4(x+k(y+1)-2)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{64}{(-1)^k(2k+2x-4y+3)+2y+1} f \left( \frac{(-1)^k(2k+2x-4y+3)+2y-1}{(-1)^k(2k+2x-4y+3)+2y+1} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(k+x+(-1)y+1)^2} f \left( \frac{-yk+k+x}{k+x+(-1)y+1}, \frac{(-1)^k(2x-3y+(-1)^k(2k-2y+1)+y-1)+1}{4(k+1)y-4(k+x+1)} \right)
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(k+x-(-1)y+1)^2} f \left( \frac{-yk+k+x}{k+x-(-1)y+1}, \frac{(-1)^k(2x+y(-1)^k(-x+2k(y+1)+3y-1)+1)}{4(x+k(y+1)-2)} \right)
\]
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