GRADIENT FLOWS FOR REGULARIZED STOCHASTIC CONTROL PROBLEMS

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Abstract. This paper studies stochastic control problems with the action space taken to be probability measures, with the objective penalised by the relative entropy. We identify suitable metric space on which we construct a gradient flow for the measure-valued control process, in the set of admissible controls, along which the cost functional is guaranteed to decrease. It is shown that any invariant measure of this gradient flow satisfies the Pontryagin optimality principle. If the problem we work with is sufficiently convex, the gradient flow converges exponentially fast. Furthermore, the optimal measure-valued control process admits a Bayesian interpretation which means that one can incorporate prior knowledge when solving such stochastic control problems. This work is motivated by a desire to extend the theoretical underpinning for the convergence of stochastic gradient type algorithms widely employed in the reinforcement learning community to solve control problems.

1. Introduction

Stochastic control problems are ubiquitous in technology and science and have been a very active area of research for over half-century [27, 5, 3, 4, 13, 7]. Two classical approaches to tackle the (stochastic) control problem are dynamic programming principle and Pontryagin’s optimality principle. Either can be used to establish existence (and possibly uniqueness) of solutions to the control problem (this, depending on the context, means either the value function or the optimal control). Either approach can form basis for the derivation of approximation methods. Numerical approximations are almost always needed in practice and rarely scale well with the dimension of the problem at hand. Indeed, the term “curse of dimensionality” (computational effort growing exponentially with dimension) was coined by R. E. Bellman when considering problems in dynamic optimisation [1].

This work provides a new perspective by establishing a connection between stochastic control problems and the theory of gradient flows on the space of probability measures and shows how they are fundamentally intertwined. The connection is reminiscent of stochastic gradient type algorithms widely used in the reinforcement learning community to solve high-dimensional control problems [11, 33]. We also refer the reader to [15, 8, 25, 23, 20] for recent work on iterative algorithms for stochastic control problems.

Our work builds on [37, 32] and [40, 14] where entropy-regularised stochastic control problems have been considered in continuous and discrete time settings. Further results on entropy regularised control problems can be found in [35] and [19].

1.1. Problem Formulation. Let \((0^W, F^W, \mathbb{W})\) be a probability space and let \((\mathcal{F}_t^W)_{t \in [0,T]}\) be a filtration on this space satisfying the usual conditions. Let \(W\) be a \(d^2\)-dimensional Wiener process on this space which is also a martingale w.r.t. the filtration \((\mathcal{F}_t^W)_{t \in [0,T]}\). The expectation with respect to the measure \(\mathbb{W}\) will be denoted \(\mathbb{E}^W\). Given some metric space \(X\) and \(0 \leq q < \infty\), let \(\mathcal{P}_q(X)\) denote the set of probability measures defined on \(X\) with finite \(q\)-th moment. Let \(\mathcal{P}_0(X) = \mathcal{P}(X)\), the set of probability measures and let \(\mathcal{M}(X)\) denote the set of measures on \(X\). Let \(\Lambda\) denote the Lebesgue measure. Throughout the paper we will abuse notation and won’t distinguish between a measure and its density (provided it exists). For \(q \geq 2\) let

\[
\mathcal{M}_q := \left\{ \nu \in \mathcal{M}([0,T] \times \mathbb{R}^p) : \Lambda([0,T]) - \text{a.e.} \exists \nu_t \in \mathcal{P}(\mathbb{R}^p) \text{ s.t. } \nu(dt, da) = \nu_t(a) \, da \, dt \right\},
\]

and

\[
\mathcal{V}_q := \left\{ \nu : \Omega^W \rightarrow \mathcal{M}_q : \mathbb{E}^W \int_0^T |a|^q \nu_t(da, dt) < \infty \right\},
\]

\[
\mathcal{V}_q^W := \left\{ \nu \in \mathcal{V}_q : (\nu_t)_{t \in [0,T]} \text{ is progressively measurable w.r.t. } (\mathcal{F}_t^W)_{t \in [0,T]} \right\},
\]

(1.1)

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where here and elsewhere, any integral without an explicitly stated domain of integration is over $\mathbb{R}^p$ and where we say that $\nu \in \mathcal{V}_q$ is progressively measurable if for every Borel set $B$ we have that $(\omega, t) \mapsto \nu_t(\omega, B)$ is progressively measurable. Note that we require that $\mathbb{P}^W \otimes \Lambda([0, T])$-almost everywhere $\nu \in \mathcal{V}_q$ has first marginal equal to the Lebesgue measure and the second marginal absolutely continuous with respect to the Lebesgue measure i.e. $\nu(\omega)(dt, da) = \nu_t(a)da \, dt$. For $\xi \in \mathbb{R}^d$ and $\nu \in \mathcal{V}^W_2$, consider the controlled process

$$X_t(\nu) = \xi + \int_0^t \Phi_r(X_r(\nu), \nu_r) \, dr + \int_0^t \Gamma_r(X_r(\nu), \nu_r) \, dW_r, \quad t \in [0, T]. \quad (1.2)$$

Fix $m' \in \mathcal{P}(\mathbb{R}^p)$ which is absolutely continuous w.r.t. the Lebesgue measure. We will use $\infty$ to denote the positive infinity. Let us now define the relative entropy $R : \mathcal{P}(\mathbb{R}^p) \times \mathcal{P}(\mathbb{R}^p) \to \mathbb{R} \cup \{\infty\}$ as follows: for $m \in \mathcal{P}(\mathbb{R}^p)$ that are not absolutely continuous with respect to the Lebesgue measure let $R(m|m') = \infty$ while for the absolutely continuous ones (so that we can write $m(da) = m(a) \, da$) let

$$R(m|m') := \int [\log m(a) - \log m'(a)] \, m(a) \, da. \quad (1.3)$$

Let $F$ and $g$ be given. Let $U = \{U_t\}_{t \in [0, T]}$ be an $\mathbb{R}^p$-valued progressively measurable process such that with $\gamma_t = e^{U_t}$ we have $\gamma = (\gamma_t)_{t \in [0, T]} \in \mathcal{V}^W_2$. We define the entropy regularized objective functional

$$J^\sigma(\nu) := \mathbb{E}^W \left[ \int_0^T \left( F_t(X_t(\nu), \nu_t) + \frac{\sigma^2}{2} R(\nu_t | \gamma_t) \right) \, dt + g(X_T(\nu)) | \nu \right]. \quad (1.4)$$

Our aim is to minimize, for some fixed $\sigma > 0$ and $\xi$, the objective functional $J^\sigma$ over $\mathcal{V}^W_2$ subject to the controlled process $X(\nu)$ satisfying (1.2). Note that if we permitted $\nu_t(\omega)$ singular w.r.t. Lebesgue measure for all $(t, \omega) \in B \in \mathcal{B}([0, T]) \otimes \mathcal{F}^W$ with $B$ of non-zero measure in the set over which we are minimizing then for such $\nu$ we would have $J^\sigma = \infty$. In other words such “singular” $\nu$ will never be optimal for the regularized problem. So not allowing such controls leads to no loss of generality. Our setup encompasses the setting of stochastic relaxed controls that dates back to the work of L.C. Young on generalised solutions of problems in the calculus of variations [38].

Historically, in control theory, measure-valued controls have been used as a mathematical tool for proving the existence of solutions to the relaxed control problems associated with the original strict control problem. On the other hand, in the theory of Markov Decision Processes (MDP) it is common to seek solutions within a class of probability measures, [6, 33] as this often improves stability and efficiency of algorithms used to solve the MDP. It is only very recently that regularised relaxed control problems have been studied in the differential control setting, see [37] where the regularised relaxed linear-quadratic stochastic control problem is studied in great detail. In [32] it is proved that for sufficiently regularised stochastic control problems, the optimal Markov control function is smooth. This is in sharp contrast to unregularized control problems for which controls are often discontinuous (only measurable) functions.

Let us now introduce the Hamiltonians

$$H^\sigma_t(x, y, z, m) := \Phi_t(x, m)y + \text{tr}(\Gamma_t \otimes (x, m)z) + F_t(x, m),$$

$$H^0_t(x, y, z, m, m') := H^\sigma_t(x, y, z, m) + \frac{\sigma^2}{2} R(m|m'). \quad (1.5)$$

We work with Pontryagin’s optimality principle and hence we use the adjoint processes

$$dY_t(\mu) = -(\nabla_x H^0_t)(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t) \, dt + Z_t(\mu) \, dW_t, \quad t \in [0, T], \quad Y_T(\mu) = (\nabla_x g)(X_T(\mu)) \quad (1.6)$$

and note that (trivially) $\nabla_x H^0 = \nabla_x H^\sigma$. Under the assumptions we postulate in this work, for each $\xi \in \mathbb{R}^d$ and $\mu \in \mathcal{V}^W_2$ a unique solution $Y(\mu)$ and $Z(\mu)$ exists, see Lemma 2.4.

The necessary condition for optimality which we formulate precisely in Theorem 2.7 says that if $\nu \in \mathcal{V}^W_2$ is (locally) optimal for $J^\sigma$, $X(\nu)$ and $Y(\nu)$, $Z(\nu)$ are the associated optimally controlled state and adjoint processes respectively, then for a.e. $(\omega, t) \in \Omega^W \times (0, T)$ $\nu_t$ locally minimises (point-wise) $H^\sigma(X_t(\nu), Y_t(\nu), Z_t(\nu), \nu, \gamma)$. In many cases the solution to this optimisation problem can be found explicitly. Motivated by the success of various gradient descent algorithms used in reinforcement learning to solve control problems [11, 33] our contribution here is to construct an appropriate gradient flow and study its convergence to the optimal solution of the stochastic control problem (1.2)–(1.4). Since the Hamiltonian $H^\sigma$ is optimised over (probability) measures we will require a notion of the linear functional derivative, see [7, Definition 5.43], and will define, for $\mu \in \mathcal{V}^W_2$, $t \in [0, T]$, $a \in \mathbb{R}^p$ and $\xi \in \mathbb{R}^d$:

$$\frac{\delta H^\sigma_t}{\delta m}(\mu, a) := \frac{\delta H^0_t}{\delta m}(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t, a). \quad (1.7)$$
Note that the stochastic process \((\frac{dH}{dt}(\mu, a))_{t \in [0,T]}\) is \((J^W_t)_{t \in [0,T]}\)-adapted for every \(\mu \in \mathcal{V}_2^W\) and \(a \in \mathbb{R}^p\). Moreover, let
\[
\frac{dH}{dt}(\mu, a) := \frac{dH}{dt}(\mu, a) + \frac{\sigma^2}{2} \left(U_t(a) + \log \mu_t(a)\right).
\] (1.8)
Note that, in (1.8), the right-hand side is not always the linear functional derivative as the entropy, being only lower semi-continuous, does not have a flat derivative on the entire \(\mathcal{P}(\mathbb{R}^p)\). However this abuse of notation makes clear what role the left-hand side of (1.8) plays and, as long as it is only applied on the appropriate gradient flow, this notational choice is justified by Lemma 3.3.

1.2. Heuristic derivation of the gradient flow. Recall that \(t \in [0, T]\) is the time associated with our control problem. We introduce a new time \(s \in [0, \infty)\) which will be the gradient flow time. Our aim is to minimize \(J^\sigma : \mathcal{V}_2^W \to \mathbb{R}\) defined in (1.4) using some gradient flow equation for the evolution of \(\nu_s \in \mathcal{V}_2^W\). More precisely, let \(E : [0, \infty) \times \Omega^W \times [0, T] \times \mathbb{R}^p \rightarrow \mathbb{R}\) be a vector field depending on the gradient flow time \(s \in [0, \infty)\), the control problem time \(t \in [0, T]\) and \(\omega^W \in \Omega^W\) and which is such that \(E_{\nu_s}(a)\) is \(\mathcal{F}_t^W\)-measurable so that \(\nu_s\) can be admissible. Dynamic representation of the Wasserstein distance due to Benamou and Brenier [2], suggest to consider the continuity equation
\[
\partial_t \nu_s = \nabla_a \cdot (E_{s,t}(\nu_s)) \quad s \in [0, \infty), \quad \nu_{0,t} \in \mathcal{P}_2(\mathbb{R}^p).
\] (1.9)
We wish to identify the vector field \(E\) so that \(J^\sigma(\nu_s)\) decreases as \(s\) increases. We do this by following the method of “free energy” dissipation studied with Otto calculus as presented in [31, Section 3] and [36, Chapter 15].

For fixed \(\varepsilon, \lambda > 0\) let \(\nu^{s,\varepsilon}_s := \nu_s + \lambda(\nu_{s+\varepsilon} - \nu_s)\). Using the notion of linear functional derivative we see that
\[
\partial_s J^\sigma(\nu^{s,\varepsilon}_s) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( J^\sigma(\nu_{s+\varepsilon}) - J^\sigma(\nu_s) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \int_0^T \int_0^T \int_0^\lambda \frac{dH}{dt}(\nu^{s,\varepsilon}_s, a)(\nu_{s+\varepsilon,t} - \nu_{s,t}) \, dt \, d\lambda \right).
\]
Due to Lemma 3.6 and ignoring momentarily that entropy is only lower semi-continuous, one can deduce that
\[
\partial_s J^\sigma(\nu_s) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \int_0^T \int_0^T \int_0^\lambda \frac{dH}{dt}(\nu^{s,\varepsilon}_s, a)(\nu_{s+\varepsilon,t} - \nu_{s,t}) \, dt \, d\lambda \right)
\]
Continuity of \(\frac{dH}{dt}\) in measure, and the fact that for all \(\lambda > 0\) we have \(\nu^{s,\varepsilon}_s \to \nu_s\) as \(\varepsilon \to 0\), yields
\[
\partial_t J^\sigma(\nu_s) = \mathbb{E} \int_0^T \left\{ \int \frac{dH}{dt}(\nu_{s,t}, a) \partial_s \nu_{s,t}(da) \right\} dt = \mathbb{E} \int_0^T \left\{ \int \frac{dH}{dt}(\nu_{s,t}, a) \nabla_a \cdot (E_{s,t}(\nu_{s,t}, \cdot)) (da) \right\} dt.
\]
Formal integration by parts implies that
\[
\partial_s J^\sigma(\nu_s) = -\mathbb{E} \int_0^T \left\{ \int \nabla_a \left( \frac{dH}{dt}(\nu_{s,t}, \cdot) \right) (E_{s,t}(\nu_{s,t}, \cdot)) (da) \right\} dt.
\]
Hence if we take \(E_{s,t} := (\nabla_a \frac{dH}{dt})(\nu_{s,t}, \cdot)\) then
\[
\partial_s J^\sigma(\nu_s) = -\mathbb{E} \int_0^T \left\{ \int \left| \nabla_a \left( \frac{dH}{dt}(\nu_{s,t}, \cdot) \right) \right|^2 \nu_{s,t}(da) \right\} dt \leq 0.
\]
From the definition of \(\frac{dH}{dt}\) in (1.8), the continuity equation (1.9) can thus be written as
\[
\partial_s \nu_{s,t} = \nabla_a \cdot \left( \left( \nabla_a \left( \frac{dH}{dt}(\nu_{s,t}, \cdot) \right) \right) + \frac{\sigma^2}{2} \left( \nabla_a U_t \right) \right) \nu_{s,t} + \frac{\sigma^2}{2} \nabla_a \nu_{s,t}.
\] (1.10)
Thus we see that for each \((\omega^W, t) \in \Omega^W \times [0, T]\) the Hamiltonian (1.5) can be viewed as the potential function of the gradient flow (1.10). If \(\nu^*\) is a stationary solution to (1.10) then clearly \((\nabla_a \frac{dH}{dt}) = 0\) \(\nu^*_s\)-a.s. and hence \(\nu^*\) satisfies the first order condition
\[
\frac{dH}{dt}(\nu^*_s, a) = \text{a constant } \nu^* - a.s.
\] (1.11)
This implies that \(\nu^*\) is a Gibbs measure in the sense that it satisfies the equation
\[
\nu^*_s(a) = Z_t^{-1} e^{-\frac{\sigma^2}{2\tau} \frac{dH}{dt}(\nu^*_{s,t}, a)} \gamma_t(a), \quad Z_t := \int e^{-\frac{\sigma^2}{2\tau} \frac{dH}{dt}(\nu^*_{s,t}, a)} \gamma_t(a) da.
\] (1.12)
In Theorem 2.6 we show that if the gradient flow converges, then the limit point satisfies the first order condition (1.12) and moreover we show that if the invariant measure is unique then \( \nu^* \) is the unique global minimiser of the objective functional (1.4). These results are proved under mild assumptions: in essence those required for the necessary conditions for optimality arising from Pontryagin’s maximum principle stated in Theorem 2.7. Theorem 2.13 tells us that for sufficiently convex stochastic control problems or if the entropic regularisation is sufficiently strong then the gradient flow converges exponentially to the unique invariant measure. This work extends the analysis in [17] and [21], that considered the gradient flow perspective for solving differential control problems with ODE dynamics and was motivated by the desire to develop a mathematical theory of deep learning.

1.3. Probabilistic representation. Next we explain how the gradient flow (1.10) relates to a familiar noisy gradient descent. For each fixed \( t \in [0,T] \) and \( \omega \in \Omega^W \) we know that the Kolmogorov–Fokker–Planck equation (2.1) has a stochastic representation which we will introduce below. Let there be \((\Omega^B, \mathcal{F}_B, \mathbb{P}^B)\) equipped with a \( \mathbb{R}^d\)-Brownian motion \( B = (B_s)_{s \geq 0} \) and the filtration \( \mathbb{F}^B = (\mathcal{F}_s^B) \) where \( \mathcal{F}_s^B := \sigma(B_u : 0 \leq u \leq s) \). Let \((\Omega^0, \mathcal{F}_0, \mathbb{P}^0)\) be a probability space on which \( \theta^0 = \theta^0(\omega^W, \cdot) \) and \( \theta^0 = \theta^0(\omega^W, \cdot) \) are random variables for each \( (\omega^W, t) \). Let \( \Omega := \Omega^W \times \Omega^0 \times \Omega^B, \mathcal{F} := \mathcal{F}^W \otimes \mathcal{F}_0 \otimes \mathcal{F}_B \) and \( \mathbb{P} := \mathbb{P}^W \otimes \mathbb{P}_0 \otimes \mathbb{P}_B \).

We will use \( \mathcal{L}(\mathcal{F}_s^W) \) to denote the conditional law of a random variable on \((\Omega, \mathcal{F}, \mathbb{P})\) conditioned on \( \mathcal{F}_s^W \). Let \((\theta^0_t)_{t \in [0,T]}\) be an \((\mathcal{F}_s^W)\)-adapted, \( \mathbb{R}^d \)-valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) s.t. \( \mathbb{E} \int_0^T |\theta_t^0|^2 dt < \infty \) so that \( \mathcal{L}(\mathcal{F}_t^W) \) is a decoupled FBSDE system in the sense that at any \( s \in [0,T] \), the dependence on \( \nu \) is clear from the context. Notice that (1.14) is a decoupled FBSDE system in the sense that at any \( s \geq 0 \) the forward process \( (X_s(\nu), Y_s(\nu), Z_s(\nu), \nu_s) \) doesn’t depend on \( (X_s(\nu), Y_s(\nu), Z_s(\nu)) \) directly, only through the control \( \nu_s \). The implication of this decoupling becomes clear when one considers an explicit time discretisation of (1.13), as we shall describe below, or an implicit time discretisation of (1.13), which would correspond to a modified method of successive approximations (MSA) algorithm, see e.g. [23].

The gradient flow, together with probabilistic numerical methods, provides a basis for a new class of algorithms for solving data-dependent stochastic control problems in high dimensions. Indeed, the system (1.13)–(1.14) can be used to design a complete algorithm as follows. First one could approximate (1.13) by a discrete time particle system, say \((\theta_{s,t}^i)_{i=1, \ldots, N}\) for \( t \) fixed. One discretizes (1.13) and computes \( (\nu_{s,t}^i)_{i=1, \ldots, N} \) which \( \nu_{s,t}^i = \frac{1}{N} \sum_{k=1}^N \delta_{\theta_{s,t}^k} \). At the next step one would then solve forward process \( (X_{s,t}(\nu_{s,t}^i))_{i=1, \ldots, N} \) on \([0,T] \) and then “back-propagate” \((Y_{s,t}(\nu_{s,t}^i))_{i=1, \ldots, N}\) on \([0,T] \). Finally one would update the gradient term. At this point one can obtain the next value of the particle system approximating the control: \( \theta_{s+1,t}^i \). In this work we do not study this numerical approximation but refer a reader to [21] where this has been analysed for the deterministic control problem with gradient flow approximated by Euler scheme and usual interacting particle system. We refer the reader to [9, 34, 10] for recent progress on particle approximations for related McKean–Vlasov SDEs.

This paper is organised as follows: in Section 2 we state the assumptions and announce the main results proved in this paper. Section 3 is devoted to the proof of Pontryagin optimality condition in the setting of this paper. In Section 4 we prove that the gradient flow system has unique solution and converges to invariant measure.
2.1. Characterisation of the optimal control. We begin by formalising the definition of gradient flow PDEs.

Definition 2.1. We will say that \( b : [0, \infty) \times [0, T] \times \Omega \times \mathbb{R}^p \to \mathbb{R}^p \) is a permissible flow if for a.e. \((\omega^W, t)\) we have \( b_t(\omega^W, \cdot) \in C^{0,1}([0, \infty) \times \mathbb{R}^p; \mathbb{R}^p) \) and for all \( s \) and a.e. \((\omega^W, t)\) the function \( a \mapsto b_{s,t}(\omega^W, a) \) is of linear growth and for any \( s \geq 0 \) and \( a \in \mathbb{R}^p \) the random variable \( b_{s,t}(a) \) is \( \mathcal{F}_t^W \)-measurable.

We do not expect \( b = b_{s,t}(\omega^W, a) \) to have any regularity in \((\omega^W, t)\). Indeed from the heuristic derivation in Section 1 it is clear that this term will involve the gradient of the flat derivative of the Hamiltonian for every \((\omega^W, t)\).

Lemma 2.2. If \( b \) is a permissible flow (cf. Definition 2.1) then the linear PDE

\[
\partial_s \nu_{s,t} = \nabla_a \cdot \left( b_{s,t} \nu_{s,t} + \frac{\alpha^2}{2} \nabla_x \nu_{s,t} \right), \quad s \in [0, \infty), \quad \nu_{0,t} \in \mathcal{P}_2(\mathbb{R}^p) \tag{2.1}
\]

has unique solution \( \nu_{s,t} \in C^{1,\infty}([0, \infty) \times \mathbb{R}^p; \mathbb{R}) \) for each \( t \in [0, T] \) and \( \omega^W \in \Omega^W \). Moreover for each \( s > 0 \) and \( t \in [0, T] \) and \( \omega^W \in \Omega^W \) we have \( \nu_{s,t}(a) > 0 \) and \( \nu_{s,t}(a) \) is \( \mathcal{F}_t^W \)-measurable.

The proof of Lemma 2.2 will be given in Section 3.

Assumption 2.3 (For characterisation of the optimal control). Let \( G : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^p) \to \mathbb{R}^k \) stand for any of \( \Phi, \Gamma \) or \( F \) with \( k = d, d = d \times d' \) or \( k = 1 \) respectively. Let \( K > 0 \) be given.

i) For all \( t \in [0, T] \) we have \( |G_t(0, \delta_0)| \leq K \) and \( |g(0)| \leq K \).

ii) The function \( G \) is differentiable in \( x \) for every \( (t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^p) \). The derivatives are jointly continuous.

iii) For all \( (t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^p) \) we have \( |\nabla_x \Phi_t(x, m)| + |\nabla_x \Gamma_t(x, m)| \leq K \).

iv) For all \( (t, x, x', m) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^p) \) we have

\[
|\nabla_x \Phi_t(x, m) - \nabla_x \Phi_t(x', m)| + |\nabla_x \Gamma_t(x, m) - \nabla_x \Gamma_t(x', m)| \leq K |x - x'|.
\]

v) For each \( (t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^p) \) the linear functional derivative \( \frac{\partial G}{\partial m} \) exists and is jointly continuous.

vi) For each \( (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \) the linear functional derivatives \( \frac{\partial G}{\partial m} \) exist and for all \( (t, x, a, a') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \) we have \( |\frac{\partial G}{\partial m}(x, m, a, a')| \leq K \).

vii) The function \( g \) is twice differentiable and for all \( x \in \mathbb{R}^d \) we have \( |\nabla_x^2 g(x)| \leq K \).

viii) For all \( (t, x, m, a) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^p) \times \mathbb{R}^p \) the function \( \frac{\partial^2 G}{\partial m^2} \) is continuously differentiable in \( x \) and for all \( (t, x, m, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^p) \times \mathbb{R}^p \) we have \( |\nabla_x \frac{\partial^2 G}{\partial m^2}(x, m, a)| \leq K \).

Note that Assumption 2.3, i), ii), iii) and iv) imply that the coefficients of (1.2) are Lipschitz continuous in \( x \) uniformly in \((t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^p)\) and that they have linear growth in \( x \) uniformly in \((t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^p)\).

Lemma 2.4. Let Assumption 2.3 hold. Then for any \( q \geq 2 \) and \( \mu \in \mathcal{Y}_q^W \) and \( \xi \in \mathbb{R}^d \) the equation (1.2) has a unique solution \( X(t) \) which is adapted to \( (\mathcal{F}_t^W)_{t \in [0, T]} \) and there is \( c > 0 \) such that

\[
\mathbb{E}^W \left[ \sup_{t \in [0, T]} |X_t(\mu)|^q \right] < c(1 + |\xi|^q).
\]

Moreover (1.6) has unique solution \((Y(t), Z(t))\) which is adapted to \( (\mathcal{F}_t^W)_{t \in [0, T]} \) and \( Y(\mu) \in L^2((0, T) \times \Omega; \mathbb{R}^d) \) and \( Z(\mu) \in L^2((0, T) \times \Omega; \mathbb{R}^d \times \mathbb{R}^d) \).

We will not prove Lemma 2.4 since the existence of a unique solution to (1.2) and the moment bound state above is classical and can be found e.g. in Krylov [27]. The adjoint equation (1.6) is affine, hence the coefficients are Lipschitz continuous in \( y \) and \( z \). Due to Assumption 2.3 and utilising the moment bound for \( X \), we get existence, uniqueness and the stated integrability from e.g. from Zhang [39, Th 4.3.1].

Let us introduce the following metrics and spaces. First, for \( \mu, \mu' \in \mathcal{M}_q \) let

\[
W_q^T(\mu, \mu') := \left( \int_0^T W_q(\mu_t, \mu'_t)^q dt \right)^{1/q}, \tag{2.2}
\]
where $W_q$ denotes the usual $q$-Wasserstein metric in $P_q(\mathbb{R}^p)$. Note that $(\mathcal{M}_q,W_q^T)$ is a complete metric space. For $\mu,\mu' \in \mathcal{V}_q^\pi$ let

$$
rho_q(\mu,\mu') = (E^W[|W_q^T(\mu,\mu')|^q])^{1/q}.
$$

The next result is reminiscent of the study of “dissipation of free energy” along $W_2$-gradient flow as in [31, Section 3] and [36, Chapter 15], but in the setting of stochastic control with gradient flow (2.1) in the metric space $(\mathcal{V}_2^\pi,\rho_2)$.

**Theorem 2.5.** Fix $\sigma > 0$ and let Assumption 2.3 hold. Let $b$ be a permissible flow (c. f. Definition 2.1) such that $a \mapsto |\nabla_q b_{s,t}(a)|$ is bounded uniformly in $s,t$ and $W$. Let $\nu_{s,t}$ be the solution to (2.1). Assume that $X_s,Z_s$ are the forward and backward processes arising from control $\nu_s \in \mathcal{V}_2^\pi$ and $\xi \in \mathbb{R}^d$ given by (1.2) and (1.6). Then

$$
\frac{d}{ds} J(\nu_s) = -\mathbb{E}^W \int_0^T \left(\int (\nabla a \frac{\delta H}{\delta m})(\nu_{s,t},\cdot) + \frac{\sigma^2}{2} \nabla a \log \nu_{s,t} \right) \cdot \left( b_{s,t} + \frac{\sigma^2}{2} \nabla a \log \nu_{s,t} \right) \nu_{s,t}(da) dt.
$$

**Theorem 2.6.** Let Assumptions 2.3 hold. Let

$$
\mathcal{I}^* := \left\{ \nu \in \mathcal{V}_q^W : a \mapsto \frac{\delta H}{\delta m}(\nu,a) \text{ is constant for a.e. } a \in \mathbb{R}^p, \text{ a.e. } (t,\omega^W) \in (0,T) \times \Omega^W \right\}.
$$

Then,

i) any solution of (1.13) which satisfies $P_s \mu^* = \mu^*$ (i.e. an invariant measure) lies in $\mathcal{I}^*$ and moreover

ii) if there is a unique invariant measure $\mu^* \in \mathcal{V}_2^W$ and if for any $\mu^0 \in \mathcal{V}_2^W$ the system (1.13)-(1.14) has solution given by $P_s \mu^0$ such that $\lim_{s \to \infty} \rho_2(P_s \mu^0,\mu^*) = 0$ then $\mathcal{I}^* = \{ \mu^* \}$, i.e. $\mu^*$ is the only control which satisfies the first order condition (2.4), and for any $\mu^0 \in \mathcal{V}_2^W$ we have $J^*(\mu^*) \leq J^*(\mu^0)$.

Theorem 2.6 will be proved at the end of Section 3. To prove Theorem 2.6 we will need the following necessary condition for optimality, known as the Pontryagin optimality principle.

**Theorem 2.7 (Necessary condition for optimality).** Fix $\sigma > 0$. Fix $q > 2$. Let the Assumptions 2.3 hold. If $\nu \in \mathcal{V}_2^W$ is (locally) optimal for $J^*$ given by (1.4), $X(\nu)$ and $Y(\nu)$, $Z(\nu)$ are the associated optimally controlled state and adjoint processes given by (1.2) and (1.6) respectively, then for any other $\mu \in \mathcal{V}_2^W$ it holds that

$$
\int (\frac{\delta H}{\delta m}(X_t,Y_t,Z_t,\nu_t,a) + \frac{\sigma^2}{2}(\log \nu_t(a) - \log \gamma_t(a))) \right( \mu_t - \nu_t \right)(da) \geq 0 \text{ for a.e. } (\omega,t) \in \Omega^W \times (0,T).
$$

The proof of Theorem 2.7 is given in Section 3.
2.2. Existence and uniqueness for (1.13)-(1.14) and its convergence to invariant measure.

Assumption 2.8. Let $\nabla_a U$ be Lipschitz continuous in $a$ with the constant uniform in $(t, \omega)$, let $\nabla_a U_t(0) = 0$ for a.e. $(t, \omega)$ and moreover let there be $\kappa_a > 0$ such that for a.e. $(t, \omega)$:

$$(\nabla_a U_t(a') - \nabla_a U_t(a)) \cdot (a' - a) \geq \kappa_a |a' - a|^2, a, a' \in \mathbb{R}^p.$$ 

Additionally assume that there exists $\alpha$ which is an $\mathbb{R}^p$-valued $(F_t^W)_{t \in [0, T]}$-progressively measurable process such that $E \int_0^T |\alpha_t|^2 dt \leq K$ such that for a.e. $(t, \omega)$ we have

$$\nabla_a U_t(a) \cdot a \geq \kappa_a |a|^2 - \alpha_t.$$ 

In applications, Assumption 2.8 has natural interpretation. Imagine that one has already solved a related control problem, potentially one where a closed form solution exists, with strict open loop controls where we denote the optimal solution by $\alpha$. It is then natural to take $\gamma (a) \sim e^{-U_t} \sim e^{-\frac{K}{2} |a - a|^2}$.

We will assume that the running reward can be decomposed into a Lipschitz part and a convex part:

$F = F^L + F^C$. As an example consider $F^C(x, m) = x + \int \frac{1}{2} |a|^2 m(da)$. Then $\delta F^C x, m, a) = x + \frac{1}{2} |a|^2$, so that $\nabla_a \delta F^C x, m, a) = a$. This will satisfy Assumption 2.9 below with $\kappa_f = 1$.

Assumption 2.9. Let $\nabla_a \delta F^C_{\frac{d}{dm}}$ exist on $[0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^p) \times \mathbb{R}^p$ and there be $\kappa_f > 0$ such that for any $x \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{R}^p)$ and for all $a \in \mathbb{R}^p$ we have:

$$2 \left( \nabla_a \delta F^C_{\frac{d}{dm}}(x, m, a) \right) \cdot (a - a') \geq \kappa_f |a' - a|^2, a, a' \in \mathbb{R}^p.$$ 

Assumption 2.10. There is $K > 0$ such that for all $t \in [0, T]$, for all $x, x' \in \mathbb{R}^d$, for all $m, m' \in \mathcal{P}(\mathbb{R}^p)$ and for all $a \in \mathbb{R}^p$ we have:

i) $|\Phi_t(x, m) - \Phi_t(x', m')| + |\Gamma_t(x, m) - \Gamma_t(x', m')| \leq K \left( |x - x'| + W_2(m, m') \right)$.

ii) $|\nabla_x \Phi_t(x, m) + \sum_{j=1}^d \sum_{a'=1}^{a'} |\nabla_{x} F^U_t(x, m) + |\nabla_x g(x) + |\nabla_x F_t(x, m)| \leq K$.

iii) $|\nabla_x \Phi_t(x, m) - \nabla_x \Phi_t(x', m')| \leq K |x - x'| + W_2(m, m')$ and $|\nabla_x \Gamma_t(x, m) - \nabla_x \Gamma_t(x', m')| \leq K |x - x'|$.

iv) $|\nabla_x F_t(x, m) - \nabla_x F_t(x', m')| \leq K |x - x'| + W_2(m, m')$ and $|\nabla_x g(x) - \nabla_x g(x')| \leq K |x - x'|$.

Assumption 2.11. There is $K > 0$ such that for all $t \in [0, T]$, for all $x, x' \in \mathbb{R}^d$, for all $m, m' \in \mathcal{P}(\mathbb{R}^p)$ and for all $a, a' \in \mathbb{R}^p$ we have:

i) $|\nabla_a \frac{dG_t}{dm}(x, m, a)| + |\nabla_a \frac{dG_t}{dm}(x, m, a)| \leq K$ and $\nabla^2 \frac{dG_t}{dm} = 0$.

ii) With $G$ standing in for either $\Phi$ or $F^L$:

$$|\nabla_a \frac{dG_t}{dm}(x, m, a) - \nabla_a \frac{dG_t}{dm}(x', m', a)| \leq K (|x - x'| + W_2(m, m') + |a - a'|).$$

Lemma 2.12 (Existence and uniqueness). Let Assumptions 2.8, 2.9, 2.10 and 2.11 hold. Let $\dot{c} > 0$ be the constant arising in Lemma 4.1 and let $L > 0$ be the constant arising in Lemma 4.3. If $\sigma^2 \kappa_f + \kappa_f - 2L > 0$ then there is a unique solution to (1.13)-(1.14) for all $s \geq 0$. Moreover if $\lambda := \sigma^2 \kappa_f + \kappa_f - \dot{c} > 0$ then for any $s \geq 0$ we have

$$\int_0^T E[\theta_t, t]^2 dt \leq e^{-\lambda s} \int_0^T E[\theta_{0,t}]^2 dt + \frac{1}{\lambda} \left( \frac{\eta \sigma^2 T + \dot{c}(1 + |\xi|^2) \right).$$

Theorem 2.13. Let Assumptions 2.8, 2.9, 2.10 and 2.11 hold. Moreover, assume that $\lambda := \sigma^2 \kappa_f + \kappa_f - 2L > 0$. Then there is $\mu^* \in V_0^{F}$ such that for any $s \geq 0$ we have $P_s \mu^* = \mu^*$ and $\mu^*$ is unique. For any $\mu^0 \in V_0^{F}$ we have that

$$\rho_2 (P_s \mu^0, \mu^*) \leq e^{-\frac{\lambda s}{2} \rho_2 (\mu^0, \mu^*)}.$$ 

Theorem 2.13 will be proved in Section 4. Let us now present an example.

Example 2.14. Consider the controlled SDE

$$dX_t(\nu) = b(X_t(\nu)) dt + \left( \int \phi(a) \nu_t(da) \right) dt + \sigma(X_t(\nu)) dW_t + \left( \int a \nu_t(da) \right) dW_t,$$

where $b$ and $\sigma$ are differentiable, Lipschitz continuous functions with bounded and Lipschitz continuous derivatives. Moreover $\phi$ is differentiable with bounded derivatives.
To define the objective let \( \zeta(x) := \frac{1}{2}|x|^2 \) for \( |x| \leq 1 \) and \( \zeta(x) = \frac{1}{2}|x| \) for \( |x| > 1 \). This function is differentiable with bounded derivative. Our objective functional is

\[
J^\rho(\nu) := \mathbb{E}^W \left[ \int_0^T \left[ \zeta(X_t(\nu)) + \frac{\kappa_1}{2} \int |a|^2 \nu_t(da) + \frac{\kappa_2}{2} R(\nu_t|\gamma_t) \right] dt + \zeta(X_T(\nu)) \right] X_0(\nu) = \zeta.
\]

We can see that in this setting Assumptions 2.3, 2.9, 2.10 and 2.11 and all hold. We are free to choose any prior which will satisfy Assumption 2.8. A possible prior would be of the form \( e^{-\frac{1}{2}a^2 + \alpha t} \) with \( \alpha \) the solution of an associated linear-quadratic control problem.

3. Pontryagin Optimality for Entropy-Regularized Stochastic Control

We start by giving the proof of Lemma 2.2 as this will be independent of all the results concerning Pontryagin’s optimality principle.

**Proof of Lemma 2.2.** The linear PDE (2.1) has unique solution \( \nu_{s,t} \in C^{1,\infty}_{\mathbb{P}}(0,\infty) \) for each \( s \geq 0 \) and \( \omega^W \in \Omega^W \) due to e.g. Ladyzenskaja, Solomnikov and Ural’ceva [29, Chapter IV]. The \( \mathbb{F}^W \) measurability of \( \nu_{s,t}(a) \) is a question of measurability of an explicitly defined function and this is proved e.g. in [12, Lemma 3.2]. Consider, for each \( t \in [0,T] \), the stochastic process \((\theta_{s,t})_{s \geq 0}\), solving

\[
d\theta_{s,t} = -b_{s,t}(\theta_{s,t}) ds + \sigma dB_s.
\]

Let \( \mu_{s,t} \) denote the law of \( \theta_{s,t} \) given \( \omega \in \Omega^W \). From Girsanov’s theorem we see that the \( \mu_{s,t} \) has, for each \( s > 0 \) and \( t \in [0,T] \), smooth density and moreover \( \mu_{s,t}(a) > 0 \). Applying Itô’s formula to \( \varphi \in C^2_b(\mathbb{R}^p) \), taking expectation (over \( \Omega^B \)) and using \( \mu_{s,t} \) to denote the law of \( \theta_{s,t} \) given \( \omega \in \Omega^W \) we can check that if \( \mu_{s,t} \) satisfies

\[
\int \varphi(a) \mu_{s,t}(da) = \int \varphi(a) \mu_s(da) + \int^s_t \left[ -b_{s,t}(a) \nabla \varphi(a) \mu_{r,t}(da) + \frac{1}{2} \sigma^2 \Delta \varphi(a) \mu_{r,t}(da) \right] ds.
\]

Integrating by parts we see that this is a solution to (2.1). As the solutions are unique we conclude that \( \mu_{s,t} = \mu_s \), and so \( \nu_{s,t}(a) > 0 \) for all \( s > 0 \) and \( t \in [0,T] \) and \( \Omega^W \)-a.s.

We know that relative entropy is only lower semi-continuous on \( \mathcal{P}_2(\mathbb{R}^p) \) and thus we wouldn’t expect even the directional derivative to exists (in the sense that the limit of the difference quotient is a finite number) everywhere on \( \mathcal{P}_2(\mathbb{R}^p) \). The following lemma gives two useful estimates on the difference quotient.

**Lemma 3.1 (Difference quotient estimates for relative entropy).** Let \( \nu_s, \mu_t \in \mathcal{V}_W^2 \) and let \( \nu^\varphi = \nu + \varepsilon (\mu - \nu) \). Then a.s.

i) for any \( \varepsilon \in (0,1) \) we have

\[
\frac{1}{\varepsilon} \int_0^T \left[ R(\nu^\varphi_t|\gamma_t) - R(\nu_t|\gamma_t) \right] dt \geq \int_0^T \int \left[ \log \nu_t(a) - \log \gamma_t(a) \right] (\mu_t - \nu_t)(da) dt,
\]

ii) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \left[ R(\nu^\varphi_t|\gamma_t) - R(\nu_t|\gamma_t) \right] dt \leq \int_0^T \int \left[ \log \nu_t(a) - \log \gamma_t(a) \right] (\mu_t - \nu_t)(da) dt.

**Proof.** This proof is similar to the one given in [18, Proposition 2.4] but extended to the setting of this paper. For i) we begin by observing that

\[
\frac{1}{\varepsilon} \left( R(\nu^\varphi_t|\gamma_t) - R(\nu_t|\gamma_t) \right) = \frac{1}{\varepsilon} \int \left[ \log \frac{\nu^\varphi_t(a)}{\gamma_t(a)} \log \frac{\nu_t(a)}{\gamma_t(a)} \right] da.
\]

Since \( x \log x \geq x - 1 \) for \( x \in (0,\infty) \) we get

\[
\frac{1}{\varepsilon} \int \nu_t^\varphi(a) \log \frac{\nu_t^\varphi(a)}{\nu_t(a)} \nu_t(a) da \geq \frac{1}{\varepsilon} \int \left[ \log \frac{\nu_t^\varphi(a)}{\nu_t(a)} \right] \nu_t(a) da = \frac{1}{\varepsilon} \int \left[ \nu_t^\varphi(a) - \nu_t(a) \right] da = 0.
\]
Hence
\[ \frac{1}{\varepsilon} (R(\nu^\varepsilon_t|\gamma_t) - R(\nu_t|\gamma_t)) \geq \int [\log \nu_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da), \]
which proves i).

To prove ii) start by noting that one may write
\[ \frac{1}{\varepsilon} (R(\nu^\varepsilon_t|\gamma_t) - R(\nu_t|\gamma_t)) = \frac{1}{\varepsilon} \int [\log \nu^\varepsilon_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da) \]
and let Assumption 2.8 hold. Let \( \sigma \) be s.t.
\[ \int [\log \nu^\varepsilon_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da) \leq \sigma(B_{s,t}) \]
where \( B_{s,t} \) are bounded uniformly in \( s > 0 \).

Moreover, since the map \( x \mapsto x \log x \) is convex for \( x > 0 \) and by definition of \( \nu^\varepsilon \), we have
\[ \frac{1}{\varepsilon} [\log \nu^\varepsilon_t(a) - \log \gamma_t(a)] \leq \mu(a) \log \mu(a) - \nu(a) \log \nu(a). \]

Hence
\[ \frac{1}{\varepsilon} (R(\nu^\varepsilon_t|\gamma_t) - R(\nu_t|\gamma_t)) \leq R(\mu_t|\gamma_t) - R(\nu_t|\gamma_t). \]

Since \( \mu, \nu \in V_t^W \) the right hand side is finite. Finally, by the reverse Fatou’s lemma,
\[ \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} (R(\nu^\varepsilon_t|\gamma_t) - R(\nu_t|\gamma_t)) \leq \int \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} [\log \nu^\varepsilon_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da). \]

Calculating the derivative of \( x \mapsto x \log x \) for \( x > 0 \) leads to
\[ \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} [\log \nu^\varepsilon_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da) \leq \int [\log \nu_t(a) - \log \gamma_t(a)](\mu_t - \nu_t)(da). \]
This completes the proof. \( \square \)

The next Lemma is proved in Lemmas 6.1, 6.2 and 6.3 in [17].

**Lemma 3.2 (Properties of the Gradient Flow).** Let \( b \) be a permissible flow such that \( a \mapsto |\nabla_a b_{s,t}(a)| \) is bounded uniformly in \( s > 0 \), \( t \in [0,T], \omega^W \in \Omega^W \). Then

i) For all \( s > 0 \), \( t \in [0,T], \omega^W \in \Omega^W \) and \( a \in \mathbb{R}^p \) we have \( \nu_{s,t}(a) > 0 \) and \( R(\nu_{s,t}|\gamma_t) < \infty \).

ii) For all \( s > 0 \), \( t \in [0,T] \) and \( \omega^W \in \Omega^W \) we have \( \int |\nabla_a \log \nu_{s,t}(a)|^2 \nu_{s,t}(a)(da) < \infty \).

iii) For all \( s > 0 \), \( t \in [0,T] \) and \( \omega^W \in \Omega^W \) we have
\[ \int |\nabla_a \nu_{s,t}(a)|^2 da + \int |a \cdot \nabla_a \nu_{s,t}(a)| da + \int |\Delta_a \nu_{s,t}(a)| da < \infty. \]

The following lemma proves that along the gradient flow (2.1) the function \( s \mapsto R(\nu_{s,t}|\gamma_t) \) is in fact differentiable and we have an explicit expression for the derivative.

**Lemma 3.3 (Derivative of entropy along a gradient flow).** Fix \( \sigma \geq 0 \) and let Assumption 2.8 hold. Let \( b \) be a permissible flow (c.f. Definition 2.1) such that \( a \mapsto |\nabla_a b_{s,t}(a)| \) is bounded uniformly in \( s,t \) and \( \omega^W \in \Omega^W \). Let \( \nu_{s,t} \) be the solution to (2.1). Then
\[ dR(\nu_{s,t}|\gamma_t) = -\int \left( \nabla_a \log \nu_{s,t} + \nabla_a U \right) \cdot \left( b_{s,t} + \frac{\sigma}{2} \nabla_a \log \nu_{s,t} \right) \nu_{s,t}(da)ds. \]

Proof. Let
\[ d\theta_{s,t} = -b_{s,t}(\theta_{s,t})ds + \sigma dB_s, \]
where \( B \) is an \( \mathbb{R}^p \)-Wiener process on some \((\Omega^B, \mathcal{F}^B, \mathbb{P}^B)\), where the filtration \( \mathbb{F}^B = (\mathcal{F}^B_t) \) where \( \mathcal{F}^B_t := \mathcal{F}^B_{\theta_{s,t}} : \sigma(B_u : 0 \leq u \leq s). \) \( \Omega := \Omega^W \times \Omega^B, \mathcal{F} := \mathcal{F}^W \boxdot \mathcal{F}^B \) and \( \mathbb{P} := \mathbb{P}^W \boxdot \mathbb{P}^B \). Let \( \theta_{0,t} \) be s.t. \( \mathcal{L}(\theta_{0,t}, \mathcal{F}^W_t) = \nu_{0,t} \). It is easy to check, as in the proof of Lemma 2.2, that if \( b \) a permissible flow (cf.
Definition 2.1) then $\nu_{s,t} = \mathcal{L}(\theta_{s,t}| F^W_t)$. From Lemma 3.2 we know that $\nu_{s,t} > 0$ and so from Itô’s formula we get that

$$
\begin{aligned}
d\Bigl( \log(\nu_{s,t}(\theta_{s,t})) + U_t((\theta_{s,t})) \Bigr) &= \left( \frac{\partial \nu_{s,t}}{\partial s_t}(\theta_{s,t}) - \frac{\nabla \nu_{s,t}}{\nu_{s,t}}(\theta_{s,t}) \cdot b_{s_t}(\theta_{s,t}) - \frac{\sigma^2}{2} \left| \nabla \nu_{s,t} \right|^2 + \frac{\sigma^2}{2} \frac{\delta \nu_{s,t}}{\nu_{s,t}}(\theta_{s,t}) \right) ds + dM^B_t \\
&\quad - b_{s_t}(\theta_{s,t}) \cdot \nabla \nu_{s,t}(\theta_{s,t}) + \frac{\sigma^2}{2} \Delta \nu_{s,t}(\theta_{s,t}) \Bigr) ds + dM^B_t,
\end{aligned}
$$

where $M^B_t = \sigma \nabla \log(\nu_{s,t}(\theta_{s,t})) dB_s$ is a $\mathcal{F}^B$-martingale starting from 0 due to Lemma 3.2. From (2.1) we get that $\theta_{s,t} = b_{s_t} \nabla \nu_{s,t} = \nu \nabla \nu_{s,t} \frac{\delta \nu_{s,t}}{\nu_{s,t}}$ and so

$$
\begin{aligned}
d\Bigl( \log(\nu_{s,t}(\theta_{s,t})) + U_t((\theta_{s,t})) \Bigr) &= \left( \frac{\nabla \cdot b_{s_t}(\theta_{s,t})}{\nu_{s,t}} \right) ds + dM^B_t.
\end{aligned}
$$

Now observe that $\frac{\sigma^2}{2} \frac{\delta \nu_{s,t}}{\nu_{s,t}} = \sigma \nabla \nu_{s,t} \cdot \left( \nabla^2 \log(\nu_{s,t}) \right) \nu_{s,t}\partial s_t$. Hence

$$
\begin{aligned}
d\Bigl( \log(\nu_{s,t}(\theta_{s,t})) + U_t((\theta_{s,t})) \Bigr) &= \left( \frac{\nabla \cdot b_{s_t}(\theta_{s,t})}{\nu_{s,t}} + \frac{\sigma^2}{2} \frac{\nabla \nu_{s,t}}{\nu_{s,t}} \right) ds + dM^B_t.
\end{aligned}
$$

Taking expectation w.r.t. $\mathbb{P}^B$ we get

$$
\begin{aligned}
\mathbb{E}^B \left[ \log(\nu_{s,t}(\theta_{s,t})) + U_t((\theta_{s,t})) \right] &= \int \nabla \cdot b_{s_t} + \frac{\sigma^2}{2} \frac{\nabla^2 \nu_{s,t}}{\nu_{s,t}} \right) ds + dM^B_t.
\end{aligned}
$$

Integrating by parts (see Lemma 3.2) and noting that $\int \nabla \cdot (\nabla \log(\nu_{s,t})) \nu_{s,t} ds = -\int \left| \nabla \log(\nu_{s,t}) \right|^2 \nu_{s,t} ds$ this becomes

$$
\begin{aligned}
d\nu_{t|s} &= \left( -b_{s,t} \cdot \nabla \nu_{s,t} \frac{\delta \nu_{s,t}}{\nu_{s,t}} - \frac{\sigma^2}{2} \left| \nabla \nu_{s,t} \right|^2 \right) ds + dM^B_t.
\end{aligned}
$$

This completes the proof.

Next, start working towards the proof of Theorem 2.7, which is the necessary part of Pontryagin’s optimality principle in the setting of this paper. To that end we need an expression for the directional derivative of $J^0$ and an estimate for the directional derivative of $J^\sigma$. We will write $(X_t)_{t\in[0,T]}$ for the solution of (1.2) driven by $\nu \in \mathcal{Y}^B$. We will work with an additional control $\mu \in \mathcal{Y}^W$ and define $\nu^\mu_t := \nu_t + \varepsilon(\mu_t - \nu_t)$ and $(X^\mu_t)_{t\in[0,T]}$ for the solution of (1.2) driven by $\nu^\mu$. First, however we need a directional derivative for the forward process (1.2). Let $V_0 = 0$ be fixed and consider

$$
\begin{aligned}
\frac{dV_t}{dt} &= \left( \nabla \Phi_t(X_t, \nu_t) V_t + \frac{\partial \Phi_t}{\partial s_t}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right) dt \\
&\quad + \left( \nabla \Gamma_t(X_t, \nu_t) V_t + \frac{\partial \Gamma_t}{\partial s_t}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right) dW_t.
\end{aligned}
$$

We observe that this is a linear equation and so, under Assumption 2.3 its solution is unique and has all the moments i.e. for any $p^* \geq 1$ we have $\mathbb{E}^W \sup_{t\leq T} |V_t|^p^* < \infty$. In the following lemma we will prove that the process $(V^\mu_t)_{t\in[0,T]}$ given by (3.1) is the “variation process” for (1.2) in that it is an $L^2$-directional derivative of $(X^\mu_t)_{t\in[0,T]}$ in the direction $\mu - \nu$.

**Lemma 3.4** (Variation process is a directional derivative of the forward process). Assume that $\nabla \Phi_t$ and $\nabla \Gamma_t$ exist and that there is $K > 0$ such that for all $(t, m) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)$ and for all $x, x' \in \mathbb{R}^d$ we have

$$
\begin{aligned}
|\nabla \Phi_t(x, m)| + |\nabla \Gamma_t(x, m)| &\leq K, \\
|\nabla \Phi_t(x, m) - \nabla \Phi_t(x', m)| + |\nabla \Gamma_t(x, m) - \nabla \Gamma_t(x', m)| &\leq K|x - x'|.
\end{aligned}
$$
Further assume that \( \frac{\partial \Phi}{\partial m}, \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial a} \) exists and moreover there is \( K > 0 \) such that for all \( (t, x, m, a, a') \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^p) \times \mathbb{R}^p \times \mathbb{R}^p \) we have \( |\frac{\partial \Phi}{\partial m}(x, m, a, a')| + |\frac{\partial \Phi}{\partial t}(x, m, a, a')| \leq K. \) Then

\[
\lim_{\varepsilon \to 0} \mathbb{E}^W \left[ \sup_{t \leq T} \frac{|X_t^\varepsilon - X_t|}{\varepsilon} \right] = 0.
\]

Proof. Let

\[ V_t^\varepsilon := \frac{X_t^\varepsilon - X_t}{\varepsilon}, \] i.e. \( X_t^\varepsilon = X_t + \varepsilon (V_t^\varepsilon + V_t) \).

We wish to show that \( \mathbb{E} \sup_{s \leq t} |V_s^\varepsilon|^2 \leq c_T \varepsilon^2 \to 0 \) as \( \varepsilon \to \infty \). To that end we start calculating the terms appearing in the difference quotient. First we note that

\[
\Phi(X_t^\varepsilon, \nu_t^\varepsilon) = \Phi(X_t^\varepsilon, \nu_t) + \Phi(X_t^\varepsilon, \nu_t) - \Phi(X_t, \nu_t) + \Phi(X_t, \nu_t) - \Phi(X_t, \nu_t)\]

\[
= \varepsilon \int_0^1 (\nabla_x \Phi)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) (V_t^\varepsilon + V_t) d\lambda + \varepsilon \int_0^1 \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, (1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t, a) (\mu_t - \nu_t) (da) d\lambda.
\]

Hence, rearranging, for the drift we get,

\[
\frac{1}{\varepsilon} \left[ \Phi(X_t^\varepsilon, \nu_t^\varepsilon) - \Phi(X_t, \nu_t) - \varepsilon (\nabla_x \Phi)(X_t, \nu_t) V_t - \varepsilon \int \frac{\partial \Phi}{\partial m}(X_t, \nu_t, a) (\mu_t - \nu_t) (da) \right]
\]

\[
= \int_0^1 (\nabla_x \Phi)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) V_t^\varepsilon d\lambda + \int_0^1 \left[ (\nabla_\nu \Phi)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) - (\nabla_\nu \Phi)(X_t, \nu_t) \right] V_t \lambda d\lambda + \int_0^1 \left[ \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, (1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t, a) - \frac{\partial \Phi}{\partial m}(X_t, \nu_t, a) \right] (\mu_t - \nu_t) (da) d\lambda := I_t^{(0)} + I_t^{(1)} + I_t^{(2)} =: I_t.
\]

Similarly, for the diffusion coefficient we get,

\[
\frac{1}{\varepsilon} \left[ \Gamma(X_t^\varepsilon, \nu_t^\varepsilon) - \Gamma(X_t, \nu_t) - \varepsilon (\nabla_\nu \Gamma)(X_t, \nu_t) V_t - \varepsilon \int \frac{\partial \Gamma}{\partial m}(X_t, \nu_t, a) (\mu_t - \nu_t) (da) \right]
\]

\[
= \int_0^1 (\nabla_\nu \Gamma)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) V_t^\varepsilon d\lambda + \int_0^1 \left[ (\nabla_\nu \Gamma)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) - (\nabla_\nu \Gamma)(X_t, \nu_t) \right] V_t \lambda d\lambda + \int_0^1 \left[ \frac{\partial \Gamma}{\partial m}(X_t^\varepsilon, (1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t, a) - \frac{\partial \Gamma}{\partial m}(X_t, \nu_t, a) \right] (\mu_t - \nu_t) (da) d\lambda := J_t^{(0)} + J_t^{(1)} + J_t^{(2)} =: J_t.
\]

Note that

\[
dV_t^\varepsilon = \varepsilon^{-1} [dX_t^\varepsilon - dX_t] - dV_t
\]

and so we then see that for any \( p' \geq 1 \) we have

\[
|V_t^\varepsilon|^{p'} \leq c_p \left[ \int_0^t I_r \, dr \right]^{p'} + c_p \left[ \int_0^t J_r \, dW_r \right]^{p'}.
\]

Hence, due to Burkholder–Davis–Gundy inequality we have, with a constant depending also on \( d' \), that

\[
\mathbb{E} \left[ \sup_{s \leq t} |V_s^\varepsilon|^{p'} \right] \leq c_{p,T} \mathbb{E} \left[ \int_0^t |I_r|^{p'} \, dr + \left( \int_0^t |J_r|^{p'} \, dr \right)^{p'/2} \right]. \tag{3.2}
\]

Let \( \nu_t^{\lambda, \lambda'} := (1 - \lambda')((1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t) + \lambda' \nu_t. \) Due to the differentiability assumptions

\[
\int_0^1 \int \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, (1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t, a) - \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, \nu_t^\varepsilon, a) (\mu_t - \nu_t) (da) d\lambda
\]

\[
= \int_0^1 \int_0^1 (1 - \lambda') \int \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, \nu_t^{\lambda, \lambda'}, a, a') (\nu_t^\varepsilon - \nu_t) (da) (\mu_t - \nu_t) (da) d\lambda d\lambda'
\]

\[
= \varepsilon \int_0^1 \int_0^1 (1 - \lambda') \int \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, \nu_t^{\lambda, \lambda'}, a, a') (\mu_t - \nu_t) (da) (\mu_t - \nu_t) (da) d\lambda d\lambda'.
\]

Hence, using the assumption of uniform bound on \( \frac{\partial \Phi}{\partial m}, \) we have

\[
\left| \int_0^1 \int \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, (1 - \lambda) \nu_t^\varepsilon + \lambda \nu_t, a) - \frac{\partial \Phi}{\partial m}(X_t^\varepsilon, \nu_t^\varepsilon, a) (\mu_t - \nu_t) (da) d\lambda \right| \leq \varepsilon c.
\]
Using this and our assumptions again yields
\[
|I_i^{(2)}|' = \left| \int_0^1 \int \frac{d}{d\alpha} \left( X_i^\alpha, (1 - \lambda) \nu^\alpha_t + \lambda \nu_t, a \right) - \frac{d}{d\alpha} \left( X_i^\alpha, \nu_t \right) (da) d\lambda \right|'
\leq c_p' \left| \int_0^1 \int \frac{d}{d\alpha} \left( X_i^\alpha, (1 - \lambda) \nu^\alpha_t + \lambda \nu_t, a \right) - \frac{d}{d\alpha} \left( X_i^\alpha, \nu_t \right) (da) d\lambda \right|'
\leq c_p' \left( \int_0^1 \int \frac{d}{d\alpha} \left( X_i^\alpha, \nu_t \right) (da) d\lambda \right)'
\leq c_p' \left( \int_0^1 \int |X_i^\alpha - X_t| |\mu_t - \nu_t| (da) d\lambda \right)'
\leq c_p' \epsilon' + c_p' |X_i^\alpha - X_t|'.
\]
Hence
\[
|I_i^{(2)}|' \leq c_p' \epsilon' + c_p' |X_i^\alpha - X_t|' = c_p' \epsilon' + c_p' |X_i^\alpha - X_t|'.
\] (3.3)
and similarly
\[
|J_i^{(2)}|' \leq c_p' \epsilon' + c_p' |X_i^\alpha - X_t|' = c_p' \epsilon' + c_p' |X_i^\alpha - X_t|'.
\] (3.4)
By our hypothesis $\nabla_x \Phi$ and $\nabla_x \Gamma$ are bounded uniformly in $(t, x, m)$ which implies that
\[
|I_i^{(1)}|' \leq c_p' |V_t^\alpha|', \quad |J_i^{(1)}|' \leq c_p' |V_t^\alpha|'.
\]
Applying Hölder’s inequality in (3.2) for $p' \geq 2$ and $\epsilon \leq 1$ we get that
\[
E \left[ \sup_{t \leq t} |V_t^\alpha|^p \right] \leq c_{p', T} E \left[ \int_0^t |L_r|^p^' dr + \int_0^t |J_r|^p^' dr \right].
\] (3.5)
Thus, for $\epsilon \leq 1$ and $p' \geq 2$, we see that
\[
E \left[ \sup_{t \leq t} |V_t^\alpha|^p \right] \leq c_{p', T} E \left[ \int_0^t |L_r|^p^' dr + \int_0^t |J_r|^p^' dr \right]
\leq c_{p', T} E \left[ \int_0^t |V_t^\alpha|^p^' dr + c_{p', T} E \int_0^t |V_t|^p^' dr \right] \leq c_{p', T} E \left[ \sup_{t \leq t} |V_t^\alpha|^p^' dr + c_{p', T} \right].
\]
Gronwall’s lemma yields, for any $p' \geq 2$, that
\[
\sup_{t \leq T} E \left[ \sup_{t \leq t} |V_t^\alpha|^p \right] < \infty.
\] (3.6)
By our hypothesis we have that $\nabla_x \Phi$ and $\nabla_x \Gamma$ are Lipschitz continuous in $x$ uniformly in $t$ and $m$. Consequently, with Young’s inequality, one sees that
\[
E \int_0^T \left[ |J_i^{(1)}|^2 + |J_i^{(1)}|^2 \right] dt \leq \epsilon^2 c E \int_0^T E \left[ |V_t^\alpha + V_t|^2 |V_t|^2 \right] dt \leq \epsilon^2 c E \int_0^T E \left[ |V_t^\alpha + V_t|^4 + |V_t|^4 \right] dt
\leq \epsilon^2 c E \int_0^T E \left[ |V_t|^4 + |V_t|^4 + |V_t|^4 \right] dt.
\]
With (3.6) used with $p' = 4$ and with the observation made earlier that $V_t$ has all the moments bounded we get that
\[
E \int_0^T \left[ |I_i^{(1)}|^2 + |J_i^{(1)}|^2 \right] dt \leq c \epsilon^2.
\] (3.7)
Since we already established (3.3)-(3.4) and since we may apply (3.6) with $p' = 2$ and since $V_t$ has all the moments bounded
\[
E \int_0^T \left[ |I_i^{(2)}|^2 + |J_i^{(2)}|^2 \right] dt \leq c \epsilon^2 + c \epsilon^2 E \int_0^T |V_t^\alpha + V_t|^2 dt \leq c \epsilon^2.
\] (3.8)
From (3.7) and (3.8), together with (3.5) it follows that
\[
E \sup_{t \leq t} |V_t^\alpha|^2 \leq c \left( \int_0^t E \sup_{t' \leq t} |V_{t'}^\alpha|^2 dr + \epsilon^2 \right).
\]
and by Gronwall’s lemma $E\sup_{s\leq T}|\nabla_{\epsilon} f(s)|^2 \leq c_T \epsilon^2 \to 0$ as $\epsilon \to \infty$.

\textbf{Lemma 3.5} (Directional derivative of $J^0$ in terms of the variation process). Let the hypothesis of Lemma 3.4 hold and additionally that $\nabla_x F$ and $\nabla_x g$, $\frac{\delta^2 F}{\delta m^2}$, $\frac{\delta^2 g}{\delta m^2}$ exist and are jointly continuous in $(t, x, m)$ or $(t, x, m, a)$ respectively and there is $K > 0$ such that for all $(t, x, m, a, \alpha')$ we have

$$\left|\frac{\delta^2 F}{\delta m^2}(x, m, a, \alpha')\right| + \left|\nabla_x \frac{\delta^2 F}{\delta m^2}(x, m, a)\right| + \left|\nabla^2 g(x)\right| \leq K \text{ and } \left|\nabla x g(x)\right| \leq K(1 + |x|).$$

Then for any $\nu, \mu \in V^T_2$ the mapping $\nu \mapsto J^0(\nu)$ defined by (1.4) satisfies

$$\frac{d}{dt}J^0((\nu_t + \epsilon(\mu_t - \nu_t))_{\epsilon \in [0, T]}, \xi)\bigg|_{\epsilon = 0} = E\int_0^T \left[\int \frac{\delta^2 F}{\delta m^2}(X_t, \nu_t, a) (\mu_t - \nu_t)(da) + (\nabla_x F)(X_t, \nu_t) V_t \right] dt + (\nabla_x g)(X_T) V_T.$$

\textbf{Proof.} We need to show that $I_\epsilon := I^{(1)}_\epsilon + I^{(2)}_\epsilon \to 0$ as $\epsilon \to 0$, where

$$I^{(1)}_\epsilon := \int_0^T \left[\epsilon^{-1} (F(X_t^\epsilon, \nu_t^\epsilon) - F(X_t, \nu_t)) + \int \frac{\delta^2 F}{\delta m^2}(X_t, \nu_t, a) (\mu_t - \nu_t)(da) - (\nabla_x F)(X_t, \nu_t) V_t \right] dt$$

and

$$I^{(2)}_\epsilon := \int_0^T \epsilon^{-1} (g(X_t^\epsilon) - g(X_T)) - (\nabla_x g)(X_T) V_T.$$

We will start with the simpler case of $I^{(2)}_\epsilon$. First we note that

$$\epsilon^{-1} [g(X_T^\epsilon) - g(X_T)] = \epsilon^{-1} \int_0^1 \frac{d}{d\lambda} g(X_T + \lambda (X_T^\epsilon - X_T)) d\lambda = \epsilon^{-1} \int_0^1 \frac{d}{d\lambda} (X_T + \lambda \epsilon (V_T^\epsilon + V_T)) d\lambda$$

and hence

$$I^{(2)}_\epsilon = \int_0^1 \left|\nabla_x g(X_T + \lambda \epsilon (V_T^\epsilon + V_T)) - \nabla_x g(X_T)\right| (V_T^\epsilon + V_T) d\lambda + \nabla_x g(X_T) V_T$$

$$\leq K \epsilon E[V_T^\epsilon + V_T^2] + \left(\mathbb{E}[(1 + |X_T|)^2]^{1/2} (\mathbb{E}|V_T^\epsilon|^2)^{1/2}\right),$$

where we used the assumption that $\|\nabla^2 g\| \leq K$, that $\forall x \in \mathbb{R}^d$ we have $|\nabla_x g(x)| \leq K(1 + |x|)$ and Hölder’s inequality. Due to Lemma 3.4 we know that $\mathbb{E}|V_T^\epsilon + V_T|^2$ is bounded uniformly in $\epsilon$ and moreover $(\mathbb{E}|V_T^\epsilon|^2)^{1/2} \to 0$ as $\epsilon \to 0$. Hence $I^{(2)}_\epsilon \to 0$.

To handle $I^{(1)}_\epsilon$ first note that

$$\epsilon^{-1} \left[F(X_t^\epsilon, \nu_t^\epsilon) - F(X_t, \nu_t) + F(X_t, \nu_t) - F(X_t, \nu_t)\right] = \int_0^1 \int \frac{\delta^2 F}{\delta m^2}(X_t^\epsilon, 1 - \lambda) \nu_t^\epsilon + \lambda \nu_t, a)(\mu_t - \nu_t)(da) d\lambda + \int_0^1 (\nabla_x F_t)(X_t + \lambda \epsilon (V_T^\epsilon + V_T), \nu_t)(V_T^\epsilon - V_t) d\lambda.$$

Hence we will write $I^{(1)}_\epsilon = I^{(1,1)}_\epsilon + I^{(1,2)}_\epsilon + I^{(1,3)}_\epsilon$, where

$$I^{(1,1)}_\epsilon := \int_0^T \left[\int_0^1 \int \frac{\delta^2 F}{\delta m^2}(X_t^\epsilon, 1 - \lambda) \nu_t^\epsilon + \lambda \nu_t) (\mu_t - \nu_t)(da) d\lambda - \int \frac{\delta^2 F}{\delta m^2}(X_t, \nu_t, a) (\mu_t - \nu_t)(da) d\lambda\right] dt,$$

$$I^{(1,2)}_\epsilon := \int_0^T \left[\int_0^1 \int \frac{\delta^2 F}{\delta m^2}(X_t, \nu_t, a) (\mu_t - \nu_t)(da) d\lambda - \int \frac{\delta^2 F}{\delta m^2}(X_t, \nu_t, a) (\mu_t - \nu_t)(da) d\lambda\right] dt$$

and

$$I^{(1,3)}_\epsilon := \int_0^T \left[\int_0^1 (\nabla_x F_t)(X_t + \lambda \epsilon (V_T^\epsilon + V_T), \nu_t)(V_T^\epsilon - V_t) d\lambda - (\nabla_x F_t)(X_t, \nu_t) V_t\right] dt.$$
Next, using the assumption that $\nabla_x \frac{\delta F}{\delta \mu}$ exists and is uniformly bounded together with Hölder’s inequality we get

$$I^{(1,3)}_t \leq \mathbb{E} \int_0^T \int_0^T K \lambda \varepsilon |V_t^\varepsilon + V_t| |\mu_t - \nu_t| (da) d\lambda \leq \varepsilon 2KT \left( \mathbb{E} \sup_{t \in [0,T]} |V_t^\varepsilon + V_t|^2 \right)^{1/2}$$

Thus due to Lemma 3.4 we get that $I^{(1)}_t \to 0$ as $\varepsilon \to 0$ which concludes the proof. \hfill \Box

**Lemma 3.6.** [Directional derivative of $J^0$ in terms of the linear functional derivative of the Hamiltonian]
Under the hypothesis of Lemma 3.5 for any $\nu, \mu \in \mathcal{V}_2^W$ we have that

$$\frac{d}{d\varepsilon} J^0 \left( \nu_t + \varepsilon (\mu_t - \nu_t) \right) \bigg|_{\varepsilon = 0} = \mathbb{E} \left[ \int_0^T \left[ \int \frac{\delta F}{\delta \mu}(X_t, Y_t, Z_t, \nu_t, a)(\mu_t - \nu_t)(da) \right] dt \right].$$

**Proof of Lemma 3.6.** We first observe that due to (1.6), (3.1) and the fact that $V_0 = 0$, we have

$$Y_T V_T = \int_0^T Y_t dV_t + \int_0^T V_t dY_t + \int_0^T d(V_t, Y_t)$$

$$= \int_0^T Y_t \left[ (\nabla_x \Phi)(X_t, \nu_t) V_t + \int \frac{\delta \Phi}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right] dt - \int_0^T V_t (\nabla_x H^\varepsilon_t)(X_t, Y_t, Z_t, \nu_t) dt$$

$$+ \int_0^T \text{tr} \left( Z_t^T (\nabla_x \Gamma)(X_t, \nu_t) V_t + Z_t^T \int \frac{\delta \Gamma}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right) dt + M_T,$$

where $M$ is a local martingale. Using the definition of the Hamiltonian (1.5) we get

$$Y_T V_T = \int_0^T Y_t \left[ (\nabla_x \Phi)(X_t, \nu_t) V_t + \int \frac{\delta \Phi}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right] dt$$

$$+ \int_0^T \text{tr} \left( Z_t^T (\nabla_x \Gamma)(X_t, \nu_t) V_t + Z_t^T \int \frac{\delta \Gamma}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right) dt$$

$$- \int_0^T \left[ Y_t (\nabla_x \Phi)(X_t, \nu_t) + \text{tr} [Z_t^T (\nabla_x \Gamma)(X_t, \nu_t) V_t] + V_t (\nabla_x F_t)(X_t, \nu_t) \right] dt + M_T$$

and so

$$Y_T V_T = \int_0^T Y_t \left[ \frac{\delta \Phi}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) + \text{tr} \left[ Z_t^T \frac{\delta \Gamma}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right] \right] dt$$

$$- \int_0^T V_t (\nabla_x F)(X_t, \nu_t) \right] dt + M_T.$$

From this, Lemma 3.5, noting that $(\nabla_x g)(X_T) V_T = Y_T V_T$ and a usual stopping time argument we see that

$$\frac{d}{d\varepsilon} J^0 \left( \nu_t + \varepsilon (\mu_t - \nu_t) \right) \bigg|_{\varepsilon = 0} = \mathbb{E} \int_0^T \left[ \int \frac{\delta F}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) + (\nabla_x F_t)(X_t, \nu_t) V_t \right]$$

$$+ \int_0^T Y_t \frac{\delta \Phi}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) + \text{tr} \left( Z_t^T \frac{\delta \Gamma}{\delta \mu}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right) - \int_0^T V_t (\nabla_x F)(X_t, \nu_t) \right] dt$$

$$= \mathbb{E} \left[ \int_0^T \left[ \frac{\delta F}{\delta \mu}(X_t, Y_t, Z_t, \nu_t, a) \right] (\mu_t - \nu_t)(da) \right].$$

This concludes the proof. \hfill \Box

**Proof of Theorem 2.5.** Let us write $\nu_t^\varepsilon := \nu_t + \varepsilon (\mu_t - \nu_t)$ and $\mu_t^\varepsilon := \mu_t + \varepsilon (\mu_t - \nu_t)$. Note that $\mu_t^\varepsilon = \mu_t - \nu_t + \nu_t^\varepsilon$ and so $\mu_t^\varepsilon - \nu_t^\varepsilon = \mu_t - \nu_t$. From the Fundamental Theorem of Calculus we get

$$J^0(\mu) - J^0(\nu) = \int_0^1 \lim_{\delta \to 0} \frac{1}{\delta} \left( J^0(\nu + \varepsilon (\mu - \nu), \xi) - J^0(\nu + \varepsilon (\mu - \nu), \xi) \right) d\varepsilon$$

$$= \int_0^1 \lim_{\delta \to 0} \frac{1}{\delta} \left( J^0(\nu^\varepsilon + \delta (\mu^\varepsilon - \nu^\varepsilon), \xi) - J^0(\nu^\varepsilon, \xi) \right) d\varepsilon.$$
Due to Lemma 3.6 and using the notation introduced in (1.7) we have
\[
\lim_{\delta \to 0} \frac{1}{\delta} \left( J^0(\nu^\varepsilon + \delta(\mu^\varepsilon - \nu^\varepsilon), \xi) - J^0(\nu^\varepsilon, \xi) \right) = \mathbb{E} \left[ \int_0^T \left[ \int \frac{\delta H_s}{\delta \mu} (\nu^\varepsilon, a)(\mu^\varepsilon - \nu^\varepsilon)(da) \right] dt \right].
\]
Hence
\[
J^0(\mu) - J^0(\nu) = \int_0^1 \mathbb{E} \left[ \int_0^T \left[ \int \frac{\delta H_s}{\delta \mu} (\nu^\varepsilon, a)(\mu_t - \nu_t)(da) \right] dt \right] d\varepsilon.
\]
Take \( \mu_t = \nu_{s+h,t} \) and \( \nu_t = \nu_{s,t} \) and write \( \nu^\varepsilon_{s,t} = \nu_{s,t} + \varepsilon(\nu_{s+h,t} - \nu_{s,t}) \). Note that \( \nu^\varepsilon_{s,t} \to \nu_t \) as \( h \to 0 \). Hence, using (2.1), we get
\[
\frac{d}{ds} J^0(\nu_s) = \lim_{h \to 0} h^{-1} \left( J^0(\nu_{s+h,t}) - J^0(\nu_{s,t}) \right) = \int_0^1 \mathbb{E} \left[ \int_0^T \lim_{h \to 0} \left[ \int \frac{\delta H_s}{\delta \mu} (\nu^\varepsilon_{s,t}, a) \frac{1}{h} (\nu_{s+h,t} - \nu_{s,t})(da) \right] dt \right] d\varepsilon.
\]
Due to Lemma 3.2 we have that \( \int \nu_a \log \nu_{a,t}(a) da < \infty \) and due to standard moment estimates for SDEs that \( \int a^m \nu_{a,t}(a) da < \infty \) for any \( m \geq 2 \) and for all \( s > 0, t \in [0,T] \) and \( \omega^W \in \Omega^W \). The continuity \( a \mapsto \nu_{a,t}(a) \) then implies that \( \nabla_a \nu_{a,t}(a) \) and \( |a|^2 \nu_{a,t}(a) \) → 0 as \( |a| \to \infty \). Thus, upon integration by parts there are no boundary terms left, and we get that
\[
\frac{d}{ds} J^0(\nu_s) = \mathbb{E} \left[ \int_0^T \left[ \frac{\delta H_s}{\delta \mu}(\nu_{s,t}, \cdot) \right] dt \right] (b_{s,t} + \frac{\alpha^2}{2} \nabla_a \log \nu_{s,t})(\nu_{s,t}(da)) dt.
\]
Recalling the definition of \( J^\sigma \) and combining (3.9) and Lemma 3.3 we get
\[
\frac{d}{ds} J^\sigma(\nu_s) = -\mathbb{E} \left[ \int_0^T \left[ \frac{\delta H_s}{\delta \mu}(\nu_{s,t}, \cdot) \right] dt \right] (b_{s,t} + \frac{\alpha^2}{2} \nabla_a \log \nu_{s,t})(\nu_{s,t}(da)) dt.
\]
This completes the proof. \( \square \)

**Proof of Theorem 2.7.** Let \((\mu_t)\) be an arbitrary relaxed control. Since \((\nu_t)\) is optimal we know that \( J^\sigma(\nu_t + \varepsilon(\mu_t - \nu_t)) \geq J^\sigma(\nu) \) for any \( \varepsilon > 0 \). From this, Lemma 3.6 and 3.1 we get that
\[
0 \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( J^\sigma(\nu_t + \varepsilon(\mu_t - \nu_t)) \right) - J^\sigma(\nu_t)
\]
\[
\leq \mathbb{E} \int_0^T \left[ \int \frac{\delta H_s}{\delta \mu} (X_t, Y_t, Z_t, \nu_t, a) + \frac{\alpha^2}{2} (\log \nu_t(a) - \log \gamma(a)) \right] (\mu_t - \nu_t)(da) dt.
\]
Now assume there is \( S \in \mathcal{F} \otimes \mathcal{B}([0,T]) \) with strictly positive \( \mathbb{P} \otimes \lambda \) (with \( \lambda \) denoting the Lebesgue on \( \mathcal{B}([0,T]) \)) measure such that
\[
\mathbb{E} \int_0^T 1_S \int \left[ \frac{\delta H_s}{\delta \mu} (X_t, Y_t, Z_t, \nu_t, a) + \frac{\alpha^2}{2} (\log \nu_t(a) - \log \gamma(a)) \right] (\mu_t - \nu_t)(da) dt < 0
\]
Define \( \tilde{\mu}_t := \mu_t 1_S + \nu_t 1_{S^c} \). Then by the same argument as above
\[
0 \leq \mathbb{E} \int_0^T \left[ \int \frac{\delta H_s}{\delta \mu} (X_t, Y_t, Z_t, \nu_t, a) + \frac{\alpha^2}{2} (\log \nu_t(a) - \log \gamma(a)) \right] (\tilde{\mu}_t - \nu_t)(da) dt
\]
\[
= \mathbb{E} \int_0^T 1_S \int \left[ \frac{\delta H_s}{\delta \mu} (X_t, Y_t, Z_t, \nu_t, a) + \frac{\alpha^2}{2} (\log \nu_t(a) - \log \gamma(a)) \right] (\mu_t - \nu_t)(da) dt < 0
\]
leading to a contradiction. \( \square \)

**Proof of Theorem 2.6.** Fix \( t \in [0,T] \) and \( \omega^W \in \Omega^W \). Let \( b_{s,t}(a) := (\nabla_a \frac{\delta H}{\delta \mu})(\mu_{s,t}, a) + \frac{\alpha^2}{2} (\nabla_a U)(a) \) and \( \mu_{s,t} = \mathcal{L}(\theta_{s,t} | \mathcal{F}_s^W) \). As in the proof of Lemma 2.2 we see that \( \mu_{s,t} \) is a solution to
\[
\partial_s \mu_{s,t} = \nabla_a \cdot \left( b_{s,t} \mu_{s,t} + \frac{\alpha^2}{2} \nabla_a \mu_{s,t} \right), \quad s \geq 0, \quad \mu_{s,0} = \mu^0_s := \mathcal{L}(\theta_{0,t} | \mathcal{F}_{s}^W) \quad (3.10)
\]
Due to Lemma 2.2 we know that the solution is unique and moreover for each \( t \in [0,T] \) and \( \omega^W \in \Omega^W \) fixed we have \( \mu_{s,t} \in C^{1,\infty}((0,\infty) \times \mathbb{R}^p; \mathbb{R}) \). Also, \( P_{s,t}^{\mu} = \mu_{s,t} \) so \( P_s \) is the solution operator for (3.10).
Since $\mu^*$ is an invariant measure $0 = \rho(P_s\mu^*, \mu^*) = \langle \mathbb{E}^W [W_2^2(P_s\mu^*, \mu^*)] \rangle^{1/2}$ we get that for almost all $t \in [0, T]$ and $\omega^W \in \Omega^W$ we have $(P_s\mu^*_t) = \mu^*$ and $\partial_s \mu^*_s = 0$. Hence for almost all $t \in [0, T]$ and $\omega^W \in \Omega^W$ we have that $\mu^*_t$ is a solution to the stationary Kolmogorov–Fokker–Planck equation

$$0 = \nabla a \cdot \left( \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(\mu^*, \cdot) + \frac{\sigma^2}{2} (\nabla a \mu^*_t) \right) \mu^*_t + \frac{\sigma^2}{2} \nabla a \mu^*_t \right). \quad (3.11)$$

This implies that $\mu^* \in \mathcal{I}^\sigma$. This proves item i).

Next we will show that if the invariant measure is unique then $\{\mu^*\} = \mathcal{I}^\sigma$. To that end we will first show by contradiction that $\mu^*$ is at least (locally) optimal. Assume that $\mu^*$ is not the (locally) optimal control for $J^\sigma$ defined in (1.4). Then for some $\mu^0 \in \mathcal{Y}^W$ it holds that $J^\sigma(\mu^0) < J^\sigma(\mu^*)$. We have by assumption that $\lim_{t \to \infty} P_s \mu^0 = \mu^*$. From this, Theorem 2.5 and from lower semi-continuity of $J^\sigma$ we get

$$J^\sigma(\mu^*) - J^\sigma(\mu^0) \leq \liminf_{s \to \infty} [J^\sigma(P_s \mu^0) - J^\sigma(\mu^0)]$$

$$= -\liminf_{s \to \infty} \int_0^s E^W \left[ \int_0^T \left( \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(P_s \mu^0_t, a) \right)^2 (P_s \mu^0_t)_{t}(da) \right) dt \right. \left. ds \right] \leq 0.$$ \quad (3.12)

This is a contradiction and so $\mu^*$ must be (locally) optimal.

On the other hand for any (locally) optimal control $\nu^* \in \mathcal{Y}^W$ we have for any $\nu \in \mathcal{Y}^W$, due to Theorem 2.7 that

$$0 \leq \mathbb{E}^W \left[ \int_0^T \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(\nu^*, a)(\nu_t - \nu^*_t)(da) \right) dt \right].$$

This together Lemma A.3 implies that $\nu^*$ is a constant function of $a$ and so $\nu^* \in \mathcal{I}^\sigma$, where $\mathcal{I}^\sigma$ is defined in (2.4). From the uniqueness of $\mu^*$ as the invariant measure we get that the set of local minimizers is a singleton. Consider now any $\mu^0 \in \mathcal{Y}^W$ s.t. $\mu^0 \notin \mathcal{I}^\sigma$. We know $\mu^0$ is not a local minimizer. Moreover from (3.12) we know that $J^\sigma(\mu^0) \leq J^\sigma(\mu^*)$. This completes the proof. \hfill \square

4. Existence and Uniqueness of Solutions to the Mean-field system and exponential convergence to the invariant measure

**Lemma 4.1** (System dissipativity). Let Assumptions 2.9, 2.10 and 2.11 hold. Let $(\theta_{s,t})_{s,t \geq 0, t \in [0,T]}$ be a solution to (1.13)-(1.14) and let $\mu_{s,t} := \mathcal{L}(\theta_{s,t} | \mathcal{F}^W_t)$.

If $\int_0^T |\theta_{0,t}|^2 dt < \infty$ then there is $\hat{c} > 0$ such that for any $s \geq 0$ we have

$$-\mathbb{E} \int_0^T \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(\mu_{s,t}, \theta_{s,t}) \right) dt \leq \frac{\kappa_f T}{2} \mathbb{E} \int_0^T \left| \theta_{s,t} \right|^2 dt + \hat{c} \left( 1 + |\xi|^2 + \mathbb{E} \int_0^T |\theta_{s,t}|^2 dt \right).$$

The constant $\hat{c}$ is independent of $\kappa_f$, $\kappa_u$ and $\sigma$ but depends on $T$, $K$ and the constants arising in Lemmas 5.2 and 2.4.

**Proof.** Recall that

$$\left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(\mu_{s,t}, \theta_{s,t}) \right) = \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(X_t(\mu_{s,t}), \mu_{s,t}, \theta_{s,t}(\mu)) Y_t(\mu_{s,t}), \right) + \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(X_t(\mu_{s,t}), \mu_{s,t}, \theta_{s,t}(\mu)) Z_{s,t}(\mu_{s,t}) \right)$$

Then due to Assumption 2.9 we have

$$-\mathbb{E} \int_0^T \left( \nabla_{\frac{\partial H^*}{\partial \sigma}}(\mu_{s,t}, \theta_{s,t}) \right) dt \leq \mathbb{E} \int_0^T \left| \theta_{s,t} \right|^2 \left( |I_{s,t}^{(1)}| + |I_{s,t}^{(2)}| + |I_{s,t}^{(3)}| \right) dt - \frac{\kappa_f T}{2} \mathbb{E} \int_0^T |\theta_{s,t}|^2 dt, \quad (4.1)$$

where

$$I_{s,t}^{(1)} = \nabla_{\frac{\partial H^*}{\partial \sigma}}(X_t(\mu_{s,t}), \mu_{s,t}, \theta_{s,t}(\mu)) Y_t(\mu_{s,t}), \quad I_{s,t}^{(2)} = \nabla_{\frac{\partial H^*}{\partial \sigma}}(X_t(\mu_{s,t}), \mu_{s,t}, \theta_{s,t}(\mu)) Z_{s,t}(\mu_{s,t}),$$

$$I_{s,t}^{(3)} = \nabla_{\frac{\partial H^*}{\partial \sigma}}(X_t(\mu_{s,t}), \mu_{s,t}, \theta_{s,t}(\mu)).$$

We now observe that Assumption 2.11 ii) yields

$$|I_{s,t}^{(1)}| \leq K \left( 1 + |X_t(\mu_{s,t})| + W_2(\mu_{s,t}, \delta_0) + |\theta_{s,t}|^2_1 \right) |Y_t(\mu_{s,t})|.$$
With Lemmas 5.2 and 2.4 and Young’s inequality we observe that
\[
E \int_0^T |\theta_{s,t}||I_{s,t}^{(1)}| dt \leq K||Y(\mu(\nu_{s,.})))||_{\mathcal{H}^{s\cdot}} E \int_0^T \left[ \frac{1}{2} + \frac{1}{2} |X_t(\mu_{s,.})|^2 + \frac{1}{2} W_2^2(\mu_{s,t}, \delta_0) + \frac{1}{2} |\theta_{s,t}|^2 \right] dt \\
\leq cT + cT(1 + |\xi|^2) + cE \int_0^T |\theta_{s,t}|^2 dt.
\]
(4.2)

With Assumption 2.11 i), Corollary 5.4 and Lemma 5.2 we see that
\[
E \int_0^T |\theta_{s,t}||I_{s,t}^{(2)}| dt \leq K E \int_0^T |\theta_{s,t}||Z_t(\mu_{s,.})| dt \leq K \sqrt{2}||Z(\mu_{s,.})||_{W^{1,\infty}} E \left[ \left( \int_0^T |\theta_{s,t}|^2 dt \right)^{1/2} \right]
\leq c \left( 1 + E \int_0^T |\theta_{s,t}|^2 dt \right).
\]
(4.3)

Finally, again with Lemma 2.4 and Young’s inequality, we obtain
\[
E \int_0^T |\theta_{s,t}||I_{s,t}^{(3)}| dt \leq cT + cT(1 + |\xi|^2) + cE \int_0^T |\theta_{s,t}|^2 dt.
\]

Combining this, with (4.1), (4.2) and (4.3) we conclude that
\[
-E \int_0^T (\nabla_a \frac{4H}{3m})(\mu_{s,.}, \theta_{s,t}) \theta_{s,t} dt \leq -\frac{cT}{2} \int_0^T |\theta_{s,t}|^2 dt + c + cT + cT(1 + |\xi|^2) + cE \int_0^T |\theta_{s,t}|^2 dt.
\]
Choosing \( \hat{c} > 0 \) appropriately concludes the proof. \( \square \)

**Lemma 4.2 (A priori estimate).** Let Assumptions 2.8, 2.9, 2.10 and 2.11 hold. Let \((\theta_{s,t}), s \geq 0, t \in [0,T] \) be a solution to (1.13)-(1.14). Let \( E \int_0^T |\theta_{0,t}|^2 dt < \infty \). If \( \lambda := \sigma^2 \kappa_u + \kappa_f - \hat{c} > 0 \) then for any \( s \geq 0 \) we have
\[
\int_0^T E|\theta_{s,t}|^2 dt \leq e^{-\lambda s} \int_0^T E|\theta_{0,t}|^2 dt + \frac{1}{\lambda} (p\sigma^2 T + \hat{c}(1 + |\xi|^2)).
\]
(4.4)

**Proof.** Let \( \mu_{s,.} := \mathcal{L}(\theta_{s,t}|_{\mathcal{F}_t^W}) \). Applying Itô’s formula in (1.13), first with \( t \in [0,T] \) fixed, then integrating over \([0,T] \), leads to
\[
e^{\lambda s} \int_0^T |\theta_{s,t}|^2 dt = \int_0^T |\theta_{0,t}|^2 dt + \lambda \int_0^s e^{\lambda v} \int_0^T |\theta_{v,t}|^2 dt dv + p\sigma^2 T \int_0^s e^{\lambda v} dv \\
- 2 \int_0^s e^{\lambda v} \int_0^T \left( \frac{\sigma^2}{2} \nabla_a U_t(\theta_{v,t}) + (\nabla_a \frac{4H}{3m})(\mu_{v,.}, \theta_{v,t}) \right) \theta_{v,t} dt dv + 2\sigma \int_0^T \int_0^s e^{\lambda v} \theta_{v,s} dB_v dv.
\]

Taking expectation (after employing the usual stopping time arguments) yields that
\[
e^{\lambda s} \int_0^T E|\theta_{s,t}|^2 dt = \int_0^T E|\theta_{0,t}|^2 dt + p\sigma^2 T \int_0^s e^{\lambda v} dv \\
+ \int_0^s e^{\lambda v} \int_0^T E \left[ |\theta_{v,t}|^2 - 2 \left( \frac{\sigma^2}{2} \nabla_a U_t(\theta_{v,t}) + (\nabla_a \frac{4H}{3m})(\mu_{v,.}, \theta_{v,t}) \right) \theta_{v,t} \right] dt dv.
\]

With Assumptions 2.8 and Lemma 4.1 we can obtain that
\[
e^{\lambda s} \int_0^T E|\theta_{s,t}|^2 dt \leq \int_0^T E|\theta_{0,t}|^2 dt + p\sigma^2 T \int_0^s e^{\lambda v} dv \\
+ \int_0^s e^{\lambda v} \int_0^T E \left[ (\lambda - \sigma^2 \kappa_u - \kappa_f - \hat{c}) |\theta_{v,t}|^2 \right] dt dv + \int_0^s e^{\lambda v} \hat{c}(1 + |\xi|^2) dv.
\]

Taking \( \lambda = \sigma^2 \kappa_u + \kappa_f - \hat{c} > 0 \) we then have
\[
\int_0^T E|\theta_{s,t}|^2 dt \leq e^{-\lambda s} \int_0^T E|\theta_{0,t}|^2 dt + (p\sigma^2 T + \hat{c}(1 + |\xi|^2)) \frac{1}{\lambda} (1 - e^{-\lambda s}).
\]

This concludes the proof. \( \square \)
Fix \( S > 0, I := [0, S] \), \( C(I; \mathcal{V}_2^W) := \{ \nu = (\nu_s)_{s \in I} : \nu_s \in \mathcal{V}_2^W \text{ and } \lim_{s \to -\infty} \rho_2(\nu_s) = 0 \forall s \in I \} \).
Consider \( \mu \in C(I; \mathcal{V}_2^W) \). For each \( \mu_s \in \mathcal{V}_2^W \), \( s \geq 0 \) we obtain unique solution to (1.2) and (1.6) which we denote \((X_s, Y_s, Z_s)\). Moreover, for each \( t \in [0, T] \) the SDE
\[
d\theta_{s,t}(\mu) = -\left( (\nabla_a \frac{1}{2V}((\mu_s, \theta_{s,t}(\mu))) + a_s^2(\nabla_a U)(\theta_{s,t}(\mu)) \right) ds + \sigma dB_s, \quad s \geq 0
\]
has a unique strong solution, see e.g. [28, Theorem 3.1]. We denote the measure in \( \mathcal{V}_2^W \) induced by \( \theta_{s,t} \) conditioned on \( F_t^W \) for each \( t \in [0, T] \) as \( \mathcal{L}(\theta_{s,t}| \mathcal{W}) \). Our aim will be to obtain a contraction based on the map \( \Psi \) given by \( C(I; \mathcal{V}_2^W) \ni \mu \mapsto \{ \mathcal{L}(\theta_{s,t}| W(\omega^W)) : \omega^W \in \Omega^W, s \in I \} \). Before we do that we need the following estimate.

**Lemma 4.3** (Linear flow monotonicity). Let Assumptions 2.9, 2.10 and 2.11 hold. Let \( \mu, \mu' \in C(I; \mathcal{V}_2^W) \) and let \( \theta(\mu), \theta(\mu') \) be the two solutions to (4.5) arising from \( \mu \) and \( \mu' \). Then there is \( L > 0 \) such that for any \( s \geq 0 \),
\[
-2E^W \int_0^T \left( \nabla_a \frac{\delta H}{\delta \theta}(\mu_s, \theta_{s,t}(\mu)) - \nabla_a \frac{\delta H}{\delta \theta}(\mu'_s, \theta_{s,t}(\mu')) \right) \left( \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right) dt \\
\leq -\kappa_f E^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt + LE^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt + L\rho_2(\mu_s, \mu'_s)^2.
\]
The constant \( L \) is independent of \( \kappa_f \) and \( \sigma \) but depends on \( T, K \) and the constants arising in Lemmas 5.1, 5.2 and 5.5.

**Proof.** We see that due to Assumption 2.9 we have
\[
-2E^W \int_0^T \left( \nabla_a \frac{\delta H}{\delta \theta}(\mu_s, \theta_{s,t}(\mu)) - \nabla_a \frac{\delta H}{\delta \theta}(\mu'_s, \theta_{s,t}(\mu')) \right) \left( \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right) dt \\
= -2E^W \int_0^T \left( \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu), \mu'_s, \theta_{s,t}(\mu)) - \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu'), \mu'_s, \theta_{s,t}(\mu')) \right) \left( \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right) dt \\
-2E^W \int_0^T \left( \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right) \left( \Gamma_{s,t}^{(1)} + \Gamma_{s,t}^{(2)} + \Gamma_{s,t}^{(3)} + \Gamma_{s,t}^{(4)} \right) dt \\
\leq -\kappa_f E^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt + 2E^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')| \left( \Gamma_{s,t}^{(1)} + \Gamma_{s,t}^{(2)} + \Gamma_{s,t}^{(3)} + \Gamma_{s,t}^{(4)} \right) dt,
\]
where
\[
\Gamma_{s,t}^{(1)} := \left( \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu), \mu'_s, \theta_{s,t}(\mu)) Y_t(\mu_s) - \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu'), \mu'_s, \theta_{s,t}(\mu')) Y_t(\mu'_s), \right) \\
\Gamma_{s,t}^{(2)} := \frac{d}{ds} \left[ (\nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu), \mu'_s, \theta_{s,t}(\mu)) Z_t(\mu) - \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu'), \mu'_s, \theta_{s,t}(\mu')) Z_t(\mu') \right], \\
\Gamma_{s,t}^{(3)} := \left( \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu), \mu'_s, \theta_{s,t}(\mu)) - \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu'), \mu'_s, \theta_{s,t}(\mu')) \right) \\
\Gamma_{s,t}^{(4)} := \left( \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu), \mu'_s, \theta_{s,t}(\mu)) - \nabla_a \frac{\delta H}{\delta \theta}(X_{s,t}(\mu'), \mu'_s, \theta_{s,t}(\mu')) \right).
\]
Assumption 2.11 ii) and Lemma 5.2 yield
\[
E^W \int_0^T \left| \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right| |\Gamma_{s,t}^{(1)}| dt \leq C E^W \int_0^T \left( |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')| |X_t(\mu) - X_t(\mu')| \\
+ |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')| |\mathcal{W}_2(\mu_s, \mu'_s, \theta_{s,t}(\mu)) - |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 \right) dt \\
\leq C E^W \int_0^T \left( |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 + |X_t(\mu) - X_t(\mu')|^2 + |\mathcal{W}_2(\mu_s, \mu'_s, \theta_{s,t}(\mu)) - |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 \right) dt. 
\]
With Assumption 2.11 i) and Lemma 5.1 we can observe that
\[
E^W \int_0^T \left| \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right| |\Gamma_{s,t}^{(1)}| dt \leq C^{(1)} E^W \int_0^T \left| \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right|^2 dt + C^{(1)} \rho_2(\mu_s, \mu'_s)^2.
\]
Due to Assumption 2.11 and Lemma 5.5 we have
\[
E^W \int_0^T \left| \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right| |\Gamma_{s,t}^{(2)}| dt \leq C^{(2)} E^W \int_0^T \left| \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right|^2 dt + C^{(2)} \rho_2(\mu_s, \mu'_s)^2.
\]
Assumption 2.11 ii) and Lemma 5.1 allow us to conclude that
\[ \mathbb{E}^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 ds + C(3) \rho_2(\lambda, \mu_s, \mu_{s'}^2). \]
Letting \( L := C(1) + C(2) + C(3) \) completes the proof.

**Proof of Lemma 2.12.** **Step 1.** We need to show that \( \{ \mathcal{L}(\theta_{s,t}, \mu) \in \Omega W, s \in I \} \in \mathcal{C}(\mathcal{I}, W^2) \).
This amounts to showing that we have the appropriate integrability and continuity. Integrability follows from the same argument as in the proof Lemma 4.2 with \( \theta_{s,t} \) replaced by \( \theta_{s,t}(\mu) \). To establish the continuity property note that for \( s' \geq s \) we have
\[ \theta_{s',t}(\mu) - \theta_{s,t}(\mu) = - \int_s^{s'} \left( \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu)) - \frac{\sigma^2}{2} \nabla a U(\theta_{s,t}(\mu)) \right) dr + \sigma(B_{s'} - B_s). \]
From Lemma 4.2 we know that there is \( c > 0 \) independent of \( r \in [s', s] \) such that \( \mathbb{E}^W \int_0^T |\theta_{s,t}(\mu)|^2 dr < c \) and hence \( \mathbb{E} \int_0^T \int_0^T (\theta_{s',t}(\mu) - \theta_{s,t}(\mu)) dB_r = 0 \). Thus using Itô’s formula and integrating we get
\[ \mathbb{E} \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt \]
\[ = \int_s^{s'} \mathbb{E} \int_0^T (\theta_{r,t}(\mu) - \theta_{s,t}(\mu)) \left( \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu)) - \frac{\sigma^2}{2} \nabla a U(\theta_{s,t}(\mu)) \right) dt dr + T \sigma^2 \int_s^{s'} \int_0^T \mathbb{E} \int_0^T (\theta_{r,t}(\mu) - \theta_{s,t}(\mu)) \left( \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu)) - \frac{\sigma^2}{2} \nabla a U(\theta_{s,t}(\mu)) \right) dt dr \]
\[ \leq \mathbb{E} \int_0^T \left( \lambda - \sigma^2 K_U \right) |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 \]
\[ + 2(\theta_{s,t}(\mu) - \theta_{s,t}(\mu')) \left( \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu)) - \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu')) \right) ds. \]
Notice that the difference has no martingale term since the same process \( (B_s)_{s \in I} \) is used regardless of the initial condition since it appears in (4.5) as an additive term. Integrating over \([0, T] \times \Omega W\) we get
\[ d \left( e^{\lambda s} \mathbb{E}^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt \right) \leq e^{\lambda s} \mathbb{E}^W \int_0^T \left( \lambda - \sigma^2 K_U \right) |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt \]
\[ - e^{\lambda s} 2 \mathbb{E}^W \int_0^T \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu)) - \nabla a \frac{\partial L}{\partial \mu} (\mu_{s'}, \theta_{s,t}(\mu')) \left( \theta_{s,t}(\mu) - \theta_{s,t}(\mu') \right) dt ds. \]
Lemma 4.3 leads us to
\[ d \left( e^{\lambda s} \mathbb{E}^W \int_0^T |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt \right) \]
\[ \leq e^{\lambda s} \mathbb{E}^W \int_0^T (\lambda - \sigma^2 K_U - \kappa_f + L) |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 dt ds + Le^{\lambda s} \rho_2(\mu_s, \mu_{s'}^2) ds. \]
Taking \( \lambda := \sigma^2 \kappa_U + \kappa_f - L \geq 0 \), using the definition of 2-Wasserstein distance and integrating over from 0 to s find that
\[
e^{\lambda s} \rho_2(\Psi(s), \Psi(s)) \leq \int_0^s e^{\lambda v} \rho_2(\mu_v, \mu_v')^2 \, dv, \quad s \geq 0.
\]

**Step 3.** Thus
\[
\rho_2(\Psi(s), \Psi(s)) \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, dv, \quad s \geq 0.
\]

So for any \( s \geq 0 \) we see that
\[
\rho_2(\Psi(s), \Psi(s)) \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, dv \leq L^2 \int_0^s \rho_2(\mu_v, \mu_v')^2 \, ds_1 \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, ds_1.
\]

Let \( \Psi^k \) denote the \( k \)-th composition of the mapping \( \Psi \) with itself. Then for any \( s \geq 0 \),
\[
\rho_2(\Psi^k(s), \Psi^k(s)) \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, ds_1 \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, ds_1 \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, ds_1.
\]

Hence
\[
\rho_2(\Psi^k(s), \Psi^k(s)) \leq \rho_2(\mu_v, \mu_v'), \quad s \geq 0.
\]

So
\[
\sup_{0 \leq s \leq \rho_2(\Psi^k(s), \Psi^k(s)) \leq \left( \frac{L^k s^k}{k!} \right)^{1/2} \sup_{0 \leq s \leq \rho_2(\mu_v, \mu_v'), \quad s \geq 0}.
\]

Thus for any \( s \in J \) there is \( k \geq 1 \) such that \( \Psi^k \) is a contraction.

**Step 4.** Since \( \Psi^k \) is a contraction we get existence and uniqueness of solution to the system from Banach’s fixed point theorem. The estimate (2.5) follows from Lemma 4.2. \( \square \)

Recall that \( P_{\lambda, \mu}^0 := \mathcal{L}(\theta_t^0, T^W) \), where \( \theta_{s,t} \geq 0 \) is the unique solution (cf. Lemma 2.12) to the system (1.13)–(1.14) started with the initial condition \( \theta_0^0 \in [0, T] \) such that \( \mathcal{L}(\theta_0^0, \omega^W) = \text{exp}\theta_0^0 \omega^W \) for \( \omega^W \in \Omega^W \). Let \( P_{\lambda, \mu}^0 := (P_{\lambda, \mu}(s,t))_{s \in [0, T]} \) and note that \( P_{\lambda, \mu}^0 \in \mathcal{W}_q^W \) for every \( s \geq 0 \). Moreover note that due to uniqueness of solutions to (1.13)–(1.14) we have \( P_{\lambda, \mu}^0 = P_{s+t}(P_{\lambda, \mu}^0) \).

**Lemma 4.4.** Let Assumptions 2.8, 2.9, 2.10 and 2.11 hold. If \( \lambda = 2L - \sigma^2 \kappa_U - \kappa_f \geq 0 \) and if \( \mu_0, \mu_0^0 \in \mathcal{W}_q^W \), then for all \( s \geq 0 \) we have
\[
\rho_2(P_{\lambda, \mu}^0, P_{\lambda, \mu}^0) \leq e^{-\lambda s} \rho_2(\mu_0, \mu_0^0). \tag{4.7}
\]

**Proof.** Let \( \varepsilon > 0 \) be fixed. We claim that there are \( \theta_{s,t}^0 \geq 0 \) such that \( \mathcal{L}(\theta_{s,t}^0, \omega^W) = \mu_0^0 \omega^W \) and \( \theta_{s,t}^0 \geq 0 \) such that \( \mathcal{L}(\theta_{s,t}^0, \omega^W) = \mu_0 \omega^W \) for all \( \omega^W \in \Omega^W \) and such that \( \mathcal{W}_q^W \int_0^T \theta_{s,t}^0 - \theta_{s,t}^0 \omega^W \) for all \( \omega^W \in \Omega^W \). This follows from the definition of Wasserstein distance and from the definition of \( \rho_2 \), see (2.2). Let \( \theta_{s,t}^0 \geq 0 \) denote two solutions to (1.13)–(1.14) with initial conditions \( \theta_{s,t}^0 \geq 0 \) and \( \theta_{s,t}^0 \geq 0 \). For \( \lambda = 2L - \sigma^2 \kappa_U - \kappa_f \geq 0 \), the same computation as in the Step 2 of the proof of Lemma 2.12, see (4.6), gives for all \( s \geq 0 \) that
\[
\rho_2(\Psi(s), \Psi(s)) \leq \int_0^s \rho_2(\mu_v, \mu_v')^2 \, dv.
\]

From the properties of Wasserstein norm and the fact that \( \lambda = 2L - \sigma^2 \kappa_U - \kappa_f \geq 0 \) we get for all \( s \geq 0 \) that
\[
\mathcal{W}_q^W \int_0^T \theta_{s,t}^0 - \theta_{s,t}^0 \omega^W \leq e^{-\lambda s} \mathcal{W}_q^W \int_0^T \theta_{s,t}^0 - \theta_{s,t}^0 \omega^W.
\]

Hence
\[
\rho_2(P_{\lambda, \mu}^0, P_{\lambda, \mu}^0) \leq e^{-\lambda s} \rho_2(\mu_0, \mu_0^0), \quad s \geq 0.
\]

Since \( \varepsilon > 0 \) was arbitrary we get the desired conclusion. \( \square \)
Proof of Theorem 2.13. We follow a similar argument as in the proof of [30, Corollary 1.8] or in [26, Section 3]. Choose $s_0 > 0$ such that $e^{-\frac{\lambda}{2} s_0} < 1$. Then $P_{s_0} : V_q^W \to V_q^W$ is a contraction due to Lemma 4.4. By Banach's fixed point theorem and Lemma A.5 there is a (unique) $\tilde{\mu} \in V_q^W$ such that $P_{s_0} \tilde{\mu} = \tilde{\mu}$. Note that $\tilde{\mu}$ depends on the above choice of $s_0$.

Let $\mu^* := \int_0^s P_t \tilde{\mu} \, dt$. Now take an arbitrary $r \leq s_0$. Then

$$P_r \mu^* = P_r \int_0^{s_0} P_t \tilde{\mu} \, dt = \int_0^{s_0} P_{r+s} \tilde{\mu} \, ds = \int_s^{s+s_0} P_r \tilde{\mu} \, ds + \int_0^s P_r \tilde{\mu} \, ds.$$

Since $\tilde{\mu} = P_{s_0} \tilde{\mu}$ we get that

$$P_r \mu^* = \int_0^{s_0} P_r \tilde{\mu} \, ds + \int_{s+s_0}^{s+r} P_s \tilde{\mu} \, ds = \int_0^{s+r} P_s \tilde{\mu} \, ds + \int_0^s P_r \tilde{\mu} \, ds = \mu^*.$$

For $r > s_0$ choose $k \in \mathbb{N}$ so that $k s_0 < r$ and $(k + 1) s_0 \geq r$. Then $P_r \mu^* = (P_k)^k P_{(r-k) s_0) \mu^* = \mu^*$. To see that $\mu^*$ is unique consider $\nu^* \neq \mu^*$ such that $P_r \nu^* = \nu^*$ for any $r \geq 0$. Then from Lemma 4.4 we have, for any $r > s_0$, that

$$\rho_q(\mu^*, \nu^*) = \rho_q(P_r \mu^*, P_r \nu^*) \leq e^{-\frac{\lambda}{2} r} \rho_q(\mu^*, \nu^*)$$

which is a contradiction as $e^{-\frac{\lambda}{2} r} < 1$. From Lemma 4.4 we immediately get (2.6). \qed

5. BSDE estimates

Let $\mathbb{E}^W[\cdot] := \mathbb{E}[\cdot | F_t^W]$. Let $\|X\|_\infty := \text{ess sup}_{t \in [0,T]} |X(\omega)|$ and $\|Z\|_{\mathcal{H}^\infty} := \text{ess sup}_{t \in [0,T]} |Z_t(\omega)|$ for any r.v. $X$ and for any progressively measurable process $Z$ on $(\Omega^W, \mathcal{F}_t^W, (\mathcal{F}_t^W)_{t \in [0,T]}, \mathbb{P}^W)$. For a uniformly integrable martingale $M$ such that $M_0 = 0$ let $\|M\|_{\text{BMO}} := \sup_{t \geq 0} \|E^W[(M)_t^2 - (M)_r^2]\|_{\mathcal{H}^\infty}$, where the supremum is taken over all $(F_t^W)_{t \in [0,T]}$ stopping times. For a process $Z$ adapted to $(F_t^W)_{t \in [0,T]}$ let $(Z \cdot W)_t := \int_0^t Z_s \, dW_s$. We start by recalling a well known lemma about controlled SDEs with Lipschitz coefficients.

Lemma 5.1. Assume there exists $K > 0$ such that for all $t \in [0,T]$, for all $x, x' \in \mathbb{R}^d$ and for all $m, m' \in \mathcal{P}_2(\mathbb{R}^p)$ we have

$$\|\Phi(t, x, m) - \Phi(t, x', m') + |\Gamma_t(x, m) - \Gamma_t(x', m')| \leq K|x - x'| + W_2(m, m').$$

Then there is $C = C_{T, K}$ such that for any $\mu, \mu' \in \mathcal{V}^W_f$ we have

$$\mathbb{E}^W \sup_{0 \leq t \leq T} |X_t(\mu) - X_t(\mu')|^2 \leq C \rho_2(\mu, \mu')^2.$$

The following lemma is proved in [24] which uses results proved in [16]. It is the first key stability result needed to show that the gradient flow system (1.13)-(1.14) has a solution and that it converges to invariant measure.

Lemma 5.2. Assume there exists $K > 0$ such that for all $x \in \mathbb{R}^d$, all $m \in \mathcal{P}_2(\mathbb{R}^p)$ and all $t \in [0,T]$ we have $|\nabla_x \Phi_t(x, m)| + \sum_{i=1}^d \sum_{j=1}^d |\nabla_x \Gamma_{ij}^t(x, m)| + |\nabla_x g(x)| + |\nabla_x f_t(x, m)| \leq K$. Then

$$\sup_{\mu \in \mathcal{V}_f^W} \|Y(\mu)\|_{\mathcal{H}^\infty} < \infty \quad \text{and} \quad \sup_{\mu \in \mathcal{V}_f^W} \|Z(\mu) \cdot W\|_{\text{BMO}} < \infty.$$

To proceed we need to recall the following deep result about BMO martingales that is proved in [22, Theorem 2.5].

Theorem 5.3. If $M \in \text{BMO}(\Omega^W)$ and $N$ is an $(\mathcal{F}_t^W)_{t \in [0,T]}$-martingale such that $\sup_{0 \leq t \leq T} |N_t| \in L_1(\Omega^W)$, then $\mathbb{E}^W \left[\int_0^T |d(M, N)|_t\right] \leq \sqrt{3} \|M\|_{\text{BMO}}\mathbb{E}^W \left[\|N\|_{\text{BMO}}^2\right].$

We will mainly use Theorem 5.3 to give us the following corollary.

Corollary 5.4. Let $\varphi, \psi$ be progressively measurable such that $\varphi, \psi \in L^2((0, T) \times \Omega^W; \mathbb{R}^k)$ and $\varphi \cdot W \in \text{BMO}(\Omega^W)$. Then $\mathbb{E}^W \int_0^T |\varphi_t| |\psi_t| \, dt \leq \sqrt{2} \|\varphi \cdot W\|_{\text{BMO}} \left(\mathbb{E}^W \int_0^T |\psi_t|^2 \, dt\right)^{1/2}$.
Proof. Observe that \( \mathbb{E}^W \int_0^T |\varphi_t| |\psi_t| \, dt = \mathbb{E}^W \int_0^T |d(|\varphi| \cdot W, |\psi| \cdot W)| \). Using Theorem 5.3 and Hölder’s inequality we get
\[
\mathbb{E}^W \int_0^T |d(|\varphi| \cdot W, |\psi| \cdot W)|_1 \\
\leq \sqrt{2} \|\varphi \cdot W\|_{\text{BMO}} \mathbb{E}^W \left( \left( \int_0^T |\psi_t|^2 \, dt \right)^{1/2} \right) \leq \sqrt{2} \|\varphi \cdot W\|_{\text{BMO}} \left( \mathbb{E}^W \int_0^T |\psi_t|^2 \, dt \right)^{1/2}.
\]
□

We now will now state and prove the second key stability result on the backward equation that is needed to show that the gradient flow system (1.13)-(1.14) has a solution and that it converges to an invariant measure.

Lemma 5.5. Assume that there is \( K > 0 \) such that for all \( t \in [0, T] \), for all \( x, x' \in \mathbb{R}^d \) and for all \( m, m' \in \mathcal{P}_2(\mathbb{R}) \) we have:

i) \( |\Phi_t(x, m) - \Phi_t(x', m')| + |\Gamma_t(x, m) - \Gamma_t(x', m')| \leq K|x - x'| + W_2(m, m') \).

ii) \( |\nabla_x \Phi_t(x, m)| + \sum_{\alpha=1}^d \sum_{\beta=1}^d |\nabla_{x \alpha} \Gamma_t^{ij}(x, m)| + |\nabla_x g(x)| + |\nabla_x F_t(x, m)| \leq K \).

iii) \( |\nabla_x \Phi_t(x, m) - \nabla_x \Phi_t(x', m')| \leq K|x - x'| + W_2(m, m') \) and \( |\nabla_x \Gamma_t(x, m) - \nabla_x \Gamma_t(x', m')| \leq K|x - x'| \).

iv) \( |\nabla_x F_t(x, m) - \nabla_x F_t(x', m')| \leq K|x - x'| + W_2(m, m') \) and \( |\nabla_x g(x) - \nabla_x g(x')| \leq K|x - x'| \).

Then there is a constant \( C = C_{d, \beta, K, T} > 0 \) such that if \( \mu, \mu' \in \mathcal{V}_W \) then
\[
\mathbb{E}^W \sup_{0 \leq t \leq T} |Z_t(\mu) - Z_t(\mu')|^2 + \mathbb{E}^W \int_0^T |Z_t(\mu) - Z_t(\mu')|^2 \, dt \leq C \rho_2(\mu, \mu')^2.
\]

Proof. Given a constant \( \beta \geq 1 \) to be fixed later, we compute by Itô’s formula that:
\[
e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 = e^{\beta T} |(\nabla_x g)(X_T(\pi)) - (\nabla_x g)(X_T(\pi'))|^2 \leq \int_0^T \frac{e^{\beta t} |Z_t(\mu) - Z_t(\mu')|^2}{\sqrt{2}} \, dt.
\]

By taking expectation and using the fact that \( Z(\mu), Z(\mu') \in L^2((0, T) \times \Omega^W) \) and Lemma 5.2 we get
\[
\mathbb{E}^W \int_0^T e^{\beta t} \left( \frac{1}{2} |Z_t(\mu) - Z_t(\mu')|^2 + |Z_t(\mu) - Z_t(\mu')|^2 \right) \, dt \leq e^{\beta T} \mathbb{E}^W \left( (\nabla_x g)(X_T(\mu)) - (\nabla_x g)(X_T(\mu')) \right)^2
\]
\[
+ \mathbb{E}^W \int_0^T 2e^{\beta t} \left( (\nabla_x H_0^0)(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t) + (\nabla_x H_0^0)(X_t(\mu'), Y_t(\mu'), Z_t(\mu'), \mu_t') \right) \, dt.
\]

We now wish to decompose the term involving the Hamiltonian as follows
\[
- (\nabla_x H_0^0)(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t) + (\nabla_x H_0^0)(X_t(\mu'), Y_t(\mu'), Z_t(\mu'), \mu_t')
\]
\[
= -(\nabla_x \Phi_t)(X_t(\mu), \mu_t) (Y_t(\mu) - Y_t(\mu')) + Y_t(\mu') \left( \nabla_x \Phi_t(X_t(\mu'), \mu_t') - (\nabla_x \Phi_t)(X_t(\mu), \mu_t) \right)
\]
\[
- (\nabla_x \Gamma_t)(X_t(\mu), \mu_t) (Z_t(\mu) - Z_t(\mu')) + Z_t(\mu') \left( (\nabla_x \Gamma_t)(X_t(\mu'), \mu_t') - (\nabla_x \Gamma_t)(X_t(\mu), \mu_t) \right)
\]
\[
- (\nabla_x F_t)(X_t(\mu), \mu_t) + (\nabla_x F_t)(X_t(\mu'), \mu_t').
\]

Carrying this through we obtain
\[
\mathbb{E}^W \int_0^T e^{\beta t} \left( Y_t(\mu) - Y_t(\mu') \right) \left( (\nabla_x H_0^0)(X_t(\mu'), Y_t(\mu'), Z_t(\mu'), \mu_t') - (\nabla_x H_0^0)(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t) \right) \, dt \leq I_1 + I_2 + I_3 + I_4 + I_5,
\]
where, estimating the first term using assumption ii),

\[
I_1 := \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 |\nabla_x \Phi_t(X_t(\mu), \mu_t)| \, dt \leq K \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt,
\]

\[
I_2 := \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')| |Z_t(\mu) - Z_t(\mu')| |\nabla_x \Gamma_t(X_t(\mu), \mu_t)| \, dt,
\]

\[
I_3 := \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')| |\nabla_x F_t(X_t(\mu), \mu_t) - \nabla_x F_t(X_t(\mu'), \mu_t')| \, dt,
\]

\[
I_4 := \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')| |\nabla_x \Gamma_t(X_t(\mu), \mu_t) - \nabla_x \Gamma_t(X_t(\mu), \mu_t')| \, dt,
\]

\[
I_5 := \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')| |\nabla_x \Phi_t(X_t(\mu), \mu_t) - \nabla_x \Phi_t(X_t(\mu), \mu_t')| \, dt.
\]

Using assumption ii) and Young’s inequality

\[
I_2 \leq 4K \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt + \frac{1}{4} \mathbb{E}^W \int_0^T e^{\beta t} |Z_t(\mu) - Z_t(\mu')|^2 \, dt.
\]

Next we note that due to assumption iii) and iv) and due to Lemma 5.1 we obtain, with G standing in for any of Φ, Γ or F, that

\[
\mathbb{E}^W \int_0^T e^{\beta t} |\nabla_x G_t(X_t(\mu), \mu_t) - \nabla_x G_t(X_t(\mu'), \mu_t')|^2 \, dt \leq C_K \mathbb{E}^W \int_0^T e^{\beta t} W_2(\mu_t, \mu_t')^2 \, dt \leq C \rho_2(\mu, \mu')^2.
\]

With this in mind and using Young’s inequality we get

\[
I_3 \leq \frac{1}{2} \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt + \frac{1}{2} \mathbb{E}^W \int_0^T e^{\beta t} |\nabla_x F_t(X_t(\mu), \mu_t) - \nabla_x F_t(X_t(\mu'), \mu_t')|^2 \, dt
\]

\[
\leq \frac{1}{2} \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt + C e^{\beta T} \rho_2(\mu, \mu')^2.
\]

To estimate the next term we will employ Theorem 5.3 as follows:

\[
I_4 = \mathbb{E}^W \int_0^T \left| d\left( Z(\mu') \cdot W, e^{\beta t} |Y_t(\mu) - Y_t(\mu')| |\nabla_x \Gamma_t(X_t(\mu'), \mu_t') - \nabla_x \Gamma_t(X_t(\mu), \mu_t)| \cdot W \right) \right| \leq \sqrt{2} \| Z(\mu') \cdot W \|_{\text{BMO}^W} \left( \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 |\nabla_x \Gamma_t(X_t(\mu'), \mu_t') - \nabla_x \Gamma_t(X_t(\mu), \mu_t)| \cdot W \|_T^{1/2} \right).
\]

Lemma 5.2 yields

\[
I_4 \leq C \mathbb{E}^W \left[ \left( \int_0^T e^{2\beta t} |Y_t(\mu) - Y_t(\mu')|^2 |\nabla_x \Gamma_t(X_t(\mu'), \mu_t') - \nabla_x \Gamma_t(X_t(\mu), \mu_t)| \, dt \right)^{1/2} \right]
\]

\[
\leq C \mathbb{E}^W \left[ \sup_{0 \leq t \leq T} e^{\sqrt{2t}} |\nabla_x \Gamma_t(X_t(\mu'), \mu_t') - \nabla_x \Gamma_t(X_t(\mu), \mu_t)| \left( \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt \right)^{1/2} \right] + C \rho_2(\mu, \mu')^2.
\]

Thus with Young’s inequality, assumption iii) and Lemma 5.1 we have

\[
I_4 \leq \mathbb{E}^W \int_0^T e^{\beta t} |Y_t(\mu) - Y_t(\mu')|^2 \, dt + C \rho_2(\mu, \mu')^2.
\]

To estimate the final term we use Lemma 5.2, Young’s and Hölder’s inequalities to see that

\[
I_5 \leq \mathbb{E}^W \int_0^T |Y_t(\mu) - Y_t(\mu')|^2 \, dt + C e^{2\beta T} \rho_2(\mu, \mu')^2.
\]

Note that assumption iv) and Lemma 5.1 allow us to see that

\[
e^{\beta T} \mathbb{E}^W \left( |\nabla_x g(X_T(\pi)) - (\nabla_x g)(X_T(\pi'))|^2 \right) \leq e^{\beta T} C \rho_2(\mu, \mu')^2.
\]
Thus (5.2) and can be estimated as

\[
\mathbb{E}^W \int_0^T e^{bt} \left( \beta |Y_t(\mu) - Y_t(\mu')|^2 + |Z_t(\mu) - Z_t(\mu')|^2 \right) dt \\
\leq (10K + 4) \mathbb{E}^W \int_0^T e^{bt} |Y_t(\mu) - Y_t(\mu')|^2 dt + \frac{1}{2} \mathbb{E}^W \int_0^T e^{bt} |Z_t(\mu) - Z_t(\mu')|^2 dt + C e^{2bT} \rho_2(\mu, \mu')^2.
\]

Let us take \( \beta = 10K + 5 \). Then

\[
\mathbb{E}^W \int_0^T e^{bt} \left( |Y_t(\mu) - Y_t(\mu')|^2 + \frac{1}{2} |Z_t(\mu) - Z_t(\mu')|^2 \right) dt \leq C e^{2bT} \rho_2(\mu, \mu')^2.
\]

Now let us return to (5.1), take \( \beta = 0 \), supremum over \( t \in [0, T] \) and expectation:

\[
\mathbb{E}^W \sup_{0 \leq t \leq T} |Y_t(\mu) - Y_t(\mu')|^2 \leq e^{bT} \mathbb{E}^W \left( (\nabla_x g)(X_T(\pi)) - (\nabla_x g)(X_T(\pi')) \right)^2 \\
+ 2 \mathbb{E}^W \left[ \sup_{0 \leq t \leq T} \int_t^T (Y_{t'}(\mu) - Y_{t'}(\mu'))^\top (Z_{t'}(\mu) - Z_{t'}(\mu')) dt \right] \\
+ 2 \mathbb{E}^W \int_0^T |Y_t(\mu) - Y_t(\mu')| \left( (\nabla_x H_t^0)(X_t(\mu), Y_t(\mu), Z_t(\mu), \mu_t) - (\nabla_x H_t^0)(X_t(\mu'), Y_t(\mu'), Z_t(\mu'), \mu_t') \right) dt.
\]

Using the estimates we obtained for \( I_1, \ldots, I_3 \) together with Lemma 5.1 and (5.4) thus yields

\[
\mathbb{E}^W \sup_{0 \leq t \leq T} |Y_t(\mu) - Y_t(\mu')|^2 \leq 2 \mathbb{E}^W \left[ \sup_{0 \leq t \leq T} \int_t^T (Y_{t'}(\mu) - Y_{t'}(\mu'))^\top (Z_{t'}(\mu) - Z_{t'}(\mu')) dt \right] + C \rho_2(\mu, \mu')^2.
\]

Applying the inequality of Burkholder–Davis–Gundy and finally Young’s inequality we obtain that

\[
\mathbb{E}^W \left[ \sup_{0 \leq t \leq T} \int_t^T (Y_{t'}(\mu) - Y_{t'}(\mu'))^\top (Z_{t'}(\mu) - Z_{t'}(\mu')) dt \right] \\
\leq C \mathbb{E}^W \left( \int_0^T |Y_{t'}(\mu) - Y_{t'}(\mu')|^2 |Z_{t'}(\mu) - Z_{t'}(\mu')|^2 dt \right)^{1/2} \\
\leq C \mathbb{E}^W \left[ \sup_{0 \leq t \leq T} |Y_{t'}(\mu) - Y_{t'}(\mu')| \left( \int_0^T |Z_{t'}(\mu) - Z_{t'}(\mu')|^2 dt \right)^{1/2} \right] \\
\leq C \gamma \mathbb{E}^W \sup_{0 \leq t \leq T} |Y_{t'}(\mu) - Y_{t'}(\mu')|^2 + C C_\gamma \mathbb{E}^W \int_0^T |Z_{t'}(\mu) - Z_{t'}(\mu')|^2 dt.
\]

Hence, taking \( \gamma > 0 \) sufficiently small and recalling (5.4) we get

\[
\mathbb{E}^W \sup_{0 \leq t \leq T} |Y_t(\mu) - Y_t(\mu')|^2 \leq C \rho_2(\mu, \mu')^2.
\]

This concludes the proof. \( \square \)

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**Appendix A. Measure derivatives**

We first define flat derivative on \( P_2(\mathbb{R}^p) \). See e.g. [7, Section 5.4.1] for more details.

**Definition A.1.** A functional \( U : P_2(\mathbb{R}^p) \to \mathbb{R} \) is said to admit a linear derivative if there is a (continuous on \( P_2(\mathbb{R}^p) \)) map \( \frac{dU}{dm} : \mathcal{P}(\mathbb{R}^p) \times \mathbb{R}^d \to \mathbb{R} \), such that \( |\frac{dU}{dm}(a, \mu)| \leq C(1 + |a|^2) \) and, for all \( m, m' \in P_2(\mathbb{R}^p) \), it holds that

\[
U(m) - U(m') = \int_0^1 \int \frac{dU}{dm}(m + \lambda(m' - m), a) (m' - m)(da) d\lambda.
\]

Since \( \frac{dU}{dm} \) is only defined up to a constant we make a choice by demanding \( \int \frac{dU}{dm}(m, a) m(da) = 0 \).
We will also need the linear functional derivative on
\[ \mathcal{M}_2 := \left\{ \nu \in \mathcal{M}([0, T] \times \mathbb{R}^p) : \nu(dt, da) = \nu_t(da)dt, \nu_t \in \mathcal{P}_2(\mathbb{R}^p), \int_0^T |a|^2 \nu_t(da)dt < \infty \right\}, \]
which provides a slight extension of the one introduced above in Definition A.1.

**Definition A.2.** A functional \( F : \mathcal{Y}_2^W \rightarrow \mathbb{R}^q \), is said to admit a first order linear derivative, if there exists a functional \( \frac{\delta F}{\delta \nu} \) : \( \mathcal{Y}_2^W \times \mathbb{R}^q \rightarrow \mathbb{R}^q \), such that

i) For all \( (\omega, t, a) \in \Omega^W \times (0, T) \times \mathbb{R}^p \), \( \mathcal{Y}_2^W \ni \nu \mapsto \frac{\delta F}{\delta \nu}(\nu, \omega, t, a) \) is continuous (for \( \mathcal{Y}_2^W \) endowed with the weak topology of \( \mathcal{M}_0^+(\Omega^W \times (0, T) \times \mathbb{R}^p) \)).

ii) For any \( \nu \in \mathcal{Y}_2^W \) there exists \( C = C_{\nu, t, d, p} > 0 \) such that for all \( a \in \mathbb{R}^p \) we have that
\[ |\frac{\delta F}{\delta \nu}(\nu, \omega, t, a)| \leq C(1 + |a|^2). \]

iii) For all \( \nu, \nu' \in \mathcal{Y}_2^W \),
\[ F(\nu') - F(\nu) = \int_0^1 \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(1 - \lambda)\nu + \lambda \nu', t, a) (\nu' - \nu)(da)dt d\lambda. \quad (A.1) \]
The functional \( \frac{\delta F}{\delta \nu} \) is then called the linear (functional) derivative of \( F \) on \( \mathcal{Y}_2^W \).

The linear derivative \( \frac{\delta F}{\delta \nu} \) is here also defined up to the additive constant \( \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a) \nu(da)dt \).

By a centering argument, \( \frac{\delta F}{\delta \nu} \) can be generically defined under the assumption that \( \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a) \nu(da)dt = 0 \). Note that if \( \frac{\delta F}{\delta \nu} \) exists according to Definition A.2 then
\[ \forall \nu, \nu' \in \mathcal{Y}_2^W, \quad \lim_{\epsilon \rightarrow 0^+} \frac{F(\nu + \epsilon(\nu' - \nu)) - F(\nu)}{\epsilon} = \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a) (\nu' - \nu)(da)dt. \quad (A.2) \]
Indeed (A.1) immediately implies (A.2). To see the implication in the other direction take \( \nu^\lambda := \nu + \lambda(\nu' - \nu) \) and \( \nu^\lambda := \nu' - \nu + \nu^\lambda \) and notice that (A.2) ensures for all \( \lambda \in [0, 1] \) that
\[ \lim_{\epsilon \rightarrow 0^+} \frac{F(\nu^\lambda + \epsilon(\nu' - \nu^\lambda)) - F(\nu^\lambda)}{\epsilon} = \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(\nu^\lambda, t, a) (\nu' - \nu^\lambda)(da)dt \]
By the fundamental theorem of calculus
\[ F(\nu') - F(\nu) = \int_0^1 \lim_{\epsilon \rightarrow 0^+} \frac{F(\nu^{\lambda + \epsilon}) - F(\nu^{\lambda})}{\epsilon} d\lambda = \int_0^1 \mathbb{E}^W \int_0^T \int \frac{\delta F}{\delta \nu}(\nu^\lambda, t, a)(\nu' - \nu^\lambda)(da)dt d\lambda. \]

**Lemma A.3.** Fix \( m \in \mathcal{P}(\mathbb{R}^p) \). Let \( u : \mathbb{R}^p \rightarrow \mathbb{R} \) be such that for all \( m' \in \mathcal{P}(\mathbb{R}^p) \) we have that
\[ 0 \leq \int u(a) (m' - m)(da). \]
Then for all \( a \in \mathbb{R}^p \) we have \( u(a) = \int u(a')(m'da') \).

**Proof of Lemma A.3.** Let \( M := \int u(a) m(da) \). Fix \( \varepsilon > 0 \). Assume that \( m(\{a : u(a) - M \leq -\varepsilon\}) > 0 \). Take \( dm' := \frac{1}{m(\{u-M\leq-\varepsilon\})} \mathbb{1}_{\{u-M\leq-\varepsilon\}} dm \). Then
\[ 0 \leq \int u(a) (m' - m)(da) = \int [u(a) - M] m'(da) \]
\[ = \int \mathbb{1}_{\{u-M\leq-\varepsilon\}} [u(a) - M] m'(da) + \int \mathbb{1}_{\{u-M\geq\varepsilon\}} [u(a) - M] m'(da) \]
\[ = \int \mathbb{1}_{\{u-M\leq-\varepsilon\}} [u(a) - M] \frac{1}{m(\{u-M\leq-\varepsilon\})} m(da) \leq -\varepsilon. \]
As this is a contradiction we get \( m(\{u-M\leq-\varepsilon\}) = 0 \) and taking \( \varepsilon \rightarrow 0 \) we get \( m(\{u-M<0\}) = 0 \). On the other hand assume that \( m(\{u-M\geq\varepsilon\}) > 0 \). Then, since \( u-M \geq 0 \) holds m-a.s., we have
\[ 0 = \int [u(a) - M] m(da) \geq \int \mathbb{1}_{\{u-M\geq\varepsilon\}} [u(a) - M] m(da) \geq \varepsilon m(u-M \geq \varepsilon) > 0 \]
which is again a contradiction meaning that for all \( \varepsilon > 0 \) we have \( m(u-M \geq \varepsilon) = 0 \) i.e. \( u = M \) m-a.s. \( \square \)
Lemma A.4. Let $F : \mathcal{P}_2(\mathbb{R}^p) \to \mathbb{R}$. Let $m^* \in \arg \min_{m \in \mathcal{P}_2(\mathbb{R}^p)} F(m)$. Assume that $\frac{\delta F}{\delta m}$ exists and for all $m$ and all $a$ we have $\frac{\delta F}{\delta m}(m, a) \leq |a|^2$. Then $\frac{\delta F}{\delta m}(m^*, \cdot)$ is a constant function.

Proof. Let $m \in \mathcal{P}_2(\mathbb{R}^p)$ be arbitrary. Let $m^* := (1-\varepsilon)m + \varepsilon m$. Clearly $0 \leq F(m^*) - F(m^*)$. Hence

$$0 \leq \frac{1}{\varepsilon} \left( F(m^*) - F(m^*) \right) = \int_0^1 \frac{\delta F}{\delta m}((1-\lambda)m^* + \lambda m^*, a)(m^*-m^*)(da) \, d\lambda.$$ 

By reverse Fatou’s lemma

$$0 \leq \limsup_{\varepsilon \to 0} \int_0^1 \frac{\delta F}{\delta m}((1-\lambda)m^* + \lambda m^*, a)(m^*-m^*)(da) \, d\lambda \leq \int \frac{\delta F}{\delta m}(m^*, a)(m^*-m^*)(da).$$

Using Lemma A.3 we conclude $\frac{\delta F}{\delta m}(m^*, \cdot)$ is a constant function. \hfill \Box

Lemma A.5. Let $(X, d)$ be a complete metric space and let $(\Omega, F, \mathcal{P})$ be a probability space. Let $p \geq 1$. Let $\| \cdot \|_p$ denote the norm in $L^p(\Omega; \mathbb{R})$. Let $\mu_0 : \Omega \to X$ be a random variable. Let $S := \{ \mu : \Omega \to X \text{ r.v.} : \| d(\mu, \mu_0) \|_p < \infty \}$.

Let $p(\mu, \mu') := \| d(\mu, \mu') \|_p$. Then $(S, p)$ is a complete metric space.

Proof of Lemma A.5. Consider a Cauchy sequence $(\mu_n)_{n \in \mathbb{N}} \subset S$. Then there exists a subsequence $(\mu_{n(k)})_{k \in \mathbb{N}}$ such that $p(\mu_{n(k)}, \mu_{n(k+1)}) \leq 4^{-k/p}$. Hence $\mathbb{E}[d(\mu_{n(k)}, \mu_{n(k+1)})^p] \leq 4^{-k}$.

Let $A_k := \{ \omega : d(\mu_{n(k)}, \mu_{n(k+1)}) \geq 2^{-k} \}$. Then, due to Chebychev’s inequality,

$$\mathbb{P}(A_k) \leq 2^k \mathbb{E}[d(\mu_{n(k)}, \mu_{n(k+1)})^p] \leq 2^{-k}.$$ 

By the Borel-Cantelli lemma $\mathbb{P}(\limsup_{k \to \infty} A_k) = 0$. This means that for almost all $\omega \in \Omega$ there is $K(\omega)$ such that for all $k \geq K(\omega)$ we have $d(\mu_{n(k)}(\omega), \mu_{n(k+1)}(\omega)) \leq 2^{-k}$. Hence for any $k \geq K(\omega)$ and any $j \geq 1$, by the triangle inequality,

$$d(\mu_{n(k)}(\omega), \mu_{n(k+j)}(\omega)) \leq \sum_{j=0}^{j-1} d(\mu_{n(k+j-1)}(\omega), \mu_{n(k+j+1)}(\omega)) \leq 2^{-k} \sum_{j=0}^{j-1} 2^{-j} \leq 2 \cdot 2^{-k}.$$ 

This means that almost surely $(\mu_{n(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence in the complete space $(X, d)$ and thus it has a limit $\mu$.

Recall that $(\mu_n)_{n \in \mathbb{N}} \subset S$ is Cauchy in $(S, p)$. Hence for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for all $m, n > N$ we have $p(\mu_m, \mu_n) < \varepsilon$. Moreover for $n > N$ we have, due to Fatou’s Lemma, that

$$\mathbb{E}[d(\mu, \mu_n)^p] = \mathbb{E}\left[ \liminf_{k \to \infty} d(\mu_{n(k)}, \mu_n)^p \right] \leq \liminf_{k \to \infty} \mathbb{E}[d(\mu_{n(k)}, \mu_n)^p] < \varepsilon^p.$$ 

Thus for all $n > N$ and so $p(\mu, \mu_n) = \| d(\mu, \mu_n) \|_p < \varepsilon$. In other words $p(\mu, \mu_n) \to 0$ as $n \to \infty$.

Finally, we will show that $\mu \in S$. Indeed for sufficiently large $N$ we have $\| d(\mu, \mu_N) \|_p \leq 1$ and so

$$\| d(\mu_0, \mu) \|_p \leq \| d(\mu_0, \mu_N) \|_p + \| d(\mu_N, \mu) \|_p \leq \| d(\mu_0, \mu_N) \|_p + 1 < \infty.$$ 

Hence $(S, p)$ is complete. \hfill \Box

Appendix B. Sufficient condition for optimality

The main results of the article do not use the following Pontryagin sufficient condition for optimality but we include it for completeness.

Theorem B.1 (Sufficient condition for optimality). Fix $\sigma \geq 0$. Assume that $g$ and $H^0$ are continuously differentiable in the $x$ variable. Assume that $\nu \in V^0_p$, $X, Y, Z$, are a solution to (1.2)-(1.6) such that

$$\nu_t \in \arg \min_{\mu \in \mathcal{P}_2(\mathbb{R}^p)} H^\sigma(X_t, Y_t, Z_t, m, \gamma_t).$$

Finally assume that

i) the map $x \mapsto g(x)$ is convex and

ii) the map $(x, m) \mapsto H^0(x, Y_t, Z_t, m, \gamma_t)$ is convex for a.e. $(t, \omega)$, in the sense that for all $x, x' \in \mathbb{R}^d$ and all $m, m' \in \mathcal{P}(\mathbb{R}^p)$ (absolutely continuous w.r.t. the Lebesgue measure if $\sigma > 0$) it holds that

$$H^\sigma(x, Y_t, Z_t, m, \gamma_t) - H^\sigma(x', Y_t, Z_t, m', \gamma_t) \leq (\nabla_x H^\sigma)(x, Y_t, Z_t, m)(x-x') + \int \frac{\delta H^0}{\delta m}(x, Y_t, Z_t, m, a)(m-m')(da) + \frac{\sigma^2}{2} \int (\log m(a) - \log \gamma_t(a))(m-m')(da).$$
Then the control $\nu \in \mathcal{V}_2^W$ is an optimal control (and if $g$ or $H^\sigma$ are strictly convex then it is the optimal control).

Proof of Theorem B.1. Let $(\tilde{\nu}_t)_{t \in [0,T]}$ be another control with the associated family of forward and backward processes $\tilde{X}, \tilde{Y}, \tilde{Z}, \xi \in \mathbb{R}^d$. Of course $X_0 = \tilde{X}_0$. First, we note that due to convexity of $x \mapsto g(x)$ we have
\[
E^W \left[ g(X_t - g(\tilde{X}_T)) \right] \leq E^W \left[ (\nabla_x g)(X_T) (X_T - \tilde{X}_T) \right]
\]

\[
= E^W \left[ Y_T (X_T - \tilde{X}_T) \right]
\]

\[
= E^W \left[ \int_0^T (X_t - \tilde{X}_t) dY_t + \int_0^T \dot{Y}_t (dX_t - d\tilde{X}_t) + \int_0^T d\langle Y, X_t - \tilde{X}_t \rangle \right]
\]

\[
= -E^W \int_0^T (X_t - \tilde{X}_t) (\nabla_x H)(X_t, Y_t, \nu_t) dt + E^W \int_0^T Y_t \left( \Phi(X_t, \nu_t) - \Phi(\tilde{X}_t, \tilde{\nu}_t) \right) dt
\]

\[+ E^W \int_0^T \left( \Gamma(X_t, \nu_t) - \Gamma(\tilde{X}_t, \tilde{\nu}_t) \right)^T Z_t dt. \]

Moreover, since $F(x, \nu) + \frac{\kappa^2}{2} R(\nu|x) = H^\sigma(x, y, z, \nu) - \Phi(x, \nu) y - \text{tr}[\Gamma(x, \nu)^T z]$ we have
\[
\int_0^T \left[ F(X_t, \nu_t) - F(\tilde{X}_t, \tilde{\nu}_t) + \frac{\kappa^2}{2} \gamma_t - \frac{\kappa^2}{2} R(\nu_t|x_t) \right] dt
\]

\[= \int_0^T \left[ H^\sigma(X_t, Y_t, Z_t, \nu_t) - \Phi(X_t, \nu_t) Y_t - \text{tr}[\Gamma(X_t, \nu_t)^T Z_t] - H^\sigma(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{\nu}_t) + \Phi(\tilde{X}_t, \tilde{\nu}_t) \tilde{Y}_t + \text{tr}[\Gamma(\tilde{X}_t, \tilde{\nu}_t)^T \tilde{Z}_t] \right] dt.
\]

Hence
\[
J^\sigma(\nu) - J^\sigma(\tilde{\nu}) \leq -E^W \int_0^T \left( X_t - \tilde{X}_t \right) (\nabla_x H^\sigma)(X_t, Y_t, \nu_t) dt + E^W \int_0^T \left[ H^\sigma(X_t, Y_t, \nu_t) - H^\sigma(\tilde{X}_t, \tilde{Y}_t, \tilde{\nu}_t) \right] dt. \tag{B.1}
\]

We are assuming that $(x, \mu) \mapsto H^\sigma(x, y, z, \mu)$ is jointly convex in the sense of flat derivatives and so we have
\[
H^\sigma(X_t, Y_t, Z_t, \nu_t) - H^\sigma(\tilde{X}_t, \tilde{Y}_t, \tilde{\nu}_t)
\]

\[
\leq (\nabla_x H^\sigma)(X_t, Y_t, Z_t, \nu_t)(X_t - \tilde{X}_t) + \int \frac{\partial H^\sigma}{\partial \mu_m}(X_t, Y_t, Z_t, \nu_t, a)(\nu_t - \tilde{\nu}_t) (da) + \frac{\kappa^2}{2} \int (\log \nu_t(a) - \log \gamma_t(a))(\nu_t - \tilde{\nu}_t) (da). \]

We thus have
\[
J^\sigma(\nu) - J^\sigma(\tilde{\nu}) \leq E^W \int_0^T \left( \frac{\partial H^\sigma}{\partial \mu_m}(X_t, Y_t, Z_t, \nu_t, a) + \frac{\kappa^2}{2} (\log \nu_t(a) - \log \gamma_t(a)) \right) (\nu_t - \tilde{\nu}_t) (da) dt.
\]

The assumption $\nu_t = \arg \min_m H^\sigma(X_t, Y_t, Z_t, m, \gamma_t)$ together with Lemma A.4 implies that
\[
a \mapsto \left( \frac{\partial H^\sigma}{\partial \mu_m}(X_t, Y_t, Z_t, \nu_t, a) + \frac{\kappa^2}{2} (\log \nu_t(a) - \log \gamma_t(a)) \right)
\]

is a constant function and hence $\int \left( \frac{\partial H^\sigma}{\partial \mu_m}(X_t, Y_t, Z_t, \nu_t, a) + \frac{\kappa^2}{2} (\log \nu_t(a) - \log \gamma_t(a)) \right) (\nu_t - \tilde{\nu}_t) (da) = 0$. This implies that $J^\sigma(\nu) - J^\sigma(\tilde{\nu}) \leq 0$ and so $\nu$ is an optimal control. We note that if either $g$ or $H$ are strictly convex then $\nu$ is the optimal control. \qed

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