Canonical Quantum Statistics
of an Isolated Schwarzschild Black Hole
with a Spectrum $E_n = \sigma \sqrt{n} E_P$

H.A. Kastrup
Institute for Theoretical Physics, RWTH Aachen
52056 Aachen, Germany

Abstract

Many authors - beginning with Bekenstein - have suggested that the energy levels $E_n$ of a quantized isolated Schwarzschild black hole have the form $E_n = \sigma \sqrt{n} E_P$, $n = 1, 2, \ldots$, $\sigma = O(1)$, with degeneracies $g^n$.

In the present paper properties of a system with such a spectrum, considered as a quantum canonical ensemble, are discussed:

Its canonical partition function $Z(g, \beta = 1/k_B T)$, defined as a series for $g < 1$, obeys the 1-dimensional heat equation. It may be extended to values $g > 1$ by means of an integral representation which reveals a cut of $Z(g, \beta)$ in the complex $g$-plane from $g = 1$ to $g \to \infty$. Approaching the cut from above yields a real and an imaginary part of $Z$. Very surprisingly, it is the (explicitly known) imaginary part which gives the expected thermodynamical properties of Schwarzschild black holes:

Identifying the internal energy $U$ with the rest energy $M c^2$ requires $\beta$ to have the value (in natural units)

$$\beta = 2M(\ln g / \sigma^2)[1 + O(1/M^2)]$$

($4\pi\sigma^2 = \ln g$ gives Hawking’s $\beta_H$) and yields the entropy

$$S = [\ln g/(4\pi\sigma^2)] A/4 + O(\ln A),$$

where $A$ is the area of the horizon.

1E-Mail: kastrup@physik.rwth-aachen.de
1 Introduction

As the (Hawking) temperature

\[ k_B T_H = \frac{E_P^2}{8\pi Mc^2} \]  

\( E_P = \sqrt{c^5\hbar/G} \) is Planck’s energy) of the radiation emitted by a black hole\(^1\) is proportional to Planck’s constant \( \hbar \), i.e. a quantum effect, it was clear from the beginning that its deeper understanding would require a quantum theory of the gravitational properties of black holes. Despite the possible lack of a convincing general quantum theory of gravity many attempts have been made to identify the quantum energy levels \( E_n \) of an isolated Schwarzschild black hole. Bekenstein\(^2\) was the first to use Bohr-Sommerfeld type quantisation arguments and suggested a spectrum

\[ E_n = \sigma \sqrt{n} E_P, \quad n = 1, 2, \ldots, \]  

where \( \sigma \) is a (model dependent) dimensionless constant of order 1. Since then a number of authors\(^3-23\) have given different arguments for a quantum black hole spectrum of the type (2).

I myself discussed in ref. [19] how such a spectrum may be understood in the framework of a stringent canonical quantisation of the purely gravitational Schwarzschild spherically symmetric system\(^24\)\(^25\). In the following I shall take the relation (2) for granted and ask what its implications for the thermodynamics of the system are if this is viewed as a canonical ensemble. Again I shall not enter into a discussion of possible conceptual problems of such a statistical framework in the context of black holes\(^26-29,31\) and shall make a few comments after the results have been presented.

As to the present status of the thermodynamics of black holes see the reviews by Wald\(^23, 30\), Brout et al.\(^31\) and Sorkin\(^32\). The present situation as to the quantum states of black holes in the framework of string theories has recently been reviewed by Horowitz\(^33\). (Strings can have a spectrum \( m_n^2 \propto n, \) too!)
2 Canonical partition function

Following the recent discussion by Bekenstein and Mukhanov [15] (and that by the same and other authors before) I assume the degeneracies $d_n$ of the levels (2) to be $g^n$ (those authors actually take $d_n = 2^{n-1}$ which is, however, not essential for the arguments below). It is convenient in the following to define $t = \ln g$.

The canonical partition function of the system is

$$Z(t, x) = \sum_{n=0}^{\infty} e^{nt} e^{-\sqrt{n}x} = 1 + e^t \tilde{Z}(t, x), \quad t = \ln g, \quad x = \beta \sigma E_P,$$

(3)

where $\tilde{Z}$ is the partition function corresponding to the assumptions of Bekenstein and Mukhanov.

The series (3) is obviously convergent for $t < 0 (|g| < 1)$ and converges for $t = 0 (g = 1), x > 0$, according to the Maclaurin-Cauchy integral criterium [34]. For $t > 0$ the series is divergent, but the function $Z(t, x)$ can nevertheless be defined by continuation (see below).

The series (3) obeys the heat equation

$$\partial_t Z = \partial_x^2 Z,$$

(4)

with the boundary values

$$Z(t \to -\infty, x) = 1, \quad Z(t = 0, x) = \sum_{n=0}^{\infty} e^{-\sqrt{n}x} \equiv \phi(x),$$

(5)

$$Z(t, x \to \infty) = 1, \quad Z(t \neq 0, x = 0) = \frac{1}{1 - e^t} \equiv \eta(t).$$

(6)

For the series $\phi(x)$ we have the lower and upper bounds

$$\int_{1}^{\infty} d\nu e^{-\sqrt{\nu}x} = 2\left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x} \leq \phi(x) - 1 \leq \int_{0}^{\infty} d\nu e^{-\sqrt{\nu}x} = \frac{2}{x^2}. \quad (7)$$

Observing the behaviour of $e^{-\sqrt{n}x}$ between $n = k^2$, $n = (k+1)^2$, $(k + 1)^2 - k^2 = 2k + 1$ one can sharpen these bounds [34]:

$$\left(\sum_{k=1}^{\infty} 2ke^{-kx}\right) - e^{-x} = \left(\frac{2}{(1 - e^{-x})^2} - 1\right)e^{-x}$$

(8)

$$\leq \phi(x) - 1 \leq \sum_{k=1}^{\infty} 2ke^{-kx} = \frac{2e^{-x}}{(1 - e^{-x})^2}.$$
As to the physics of the system one is interested in properties of the partition function for \( t > 0 \). At a first sight one might consider to solve the heat equation for \( t \geq 0, x \geq 0 \) with the known initial values (5) and (6) in a standard manner \([36, 37]\):

\[
Z(t, x) = \int_0^\infty dy [K(t, x - y) - K(t, x + y)] \phi(y) + \int_0^t d\tau \hat{K}(t - \tau, x) \eta(\tau),
\]

where \( K \) is the "heat kernel"

\[
K(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \tag{10}
\]

and \( \hat{K} \), essentially, its \( x \)-derivative,

\[
\hat{K}(t, x) = \frac{x}{2\sqrt{\pi t^3}} e^{-x^2/(4t)} = -2\partial_x K(t, x). \tag{11}
\]

This approach does not appear to work, however, because the functions \( \phi \) and \( \eta \) are not "decent" enough \([36, 37]\): \( \eta \) becomes singular as \( 1/t \) for \( t \to 0 \) and \( \phi \) behaves like \( 2/x^2 \) for \( x \to 0 \). The latter property can be inferred from the inequalities (7) and (8).

An extension of the function \( Z(t, x) \) - actually it was \( \tilde{Z} \) - defined by the series (3) into the complex \( g \)-plane was discussed 100 years ago by the mathematician Lerch \([38]\). Using the relation \([39]\)

\[
e^{-\sqrt{n}x^2} = \frac{|x|}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2 v^2/4 - n/v^2}
\]

\[
= \frac{|x|}{2\sqrt{\pi}} \int_0^\infty d\tau \frac{e^{-x^2/(4\tau)} - n \tau}{\tau^{3/2}} = \int_0^\infty d\tau \hat{K}(\tau, x) e^{-n\tau}
\]

converts the series (3) into a geometrical one which can be summed under the integral for \( t < 0, x > 0 \):

\[
Z(t = \ln g, x) = \frac{x}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2 v^2/4} \frac{1}{1 - e^{(t - 1/v^2)}}
\]

\[
= \frac{x}{2\sqrt{\pi}} \int_0^\infty d\tau \frac{e^{-x^2/(4\tau)} - 1}{1 - e^{(t - \tau)}}
\]

\[
= \frac{x}{2\sqrt{\pi}} \int_1^\infty du \frac{e^{-x^2/(4\ln u)} - 1}{u - g}.
\]

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Notice that the relation (14) may also be written as

\[ Z(t, x) = \int_0^\infty \hat{K}(\tau, x) \eta(t - \tau) = \int_{-\infty}^t d\tilde{\tau} \hat{K}(t - \tilde{\tau}, x) \eta(\tilde{\tau}) , \]  

where \( \tilde{\tau} = t - \tau \).

Observing that

\[ 1 = \frac{x}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2v^2/4} \]

we get

\[ Z - 1 = e^t \tilde{Z}(t, x) = e^t \frac{x}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2v^2/4} \frac{e^{-1/v^2}}{1 - e(t - 1/v^2)} \]

(17)

The expressions (13) etc. may also be obtained by inserting the representation (12) of \( \exp(-\sqrt{nx}) \) into the series (3) and Borel summing it for \( t < 0 \).

The integrals converge for all values of \( g \neq 1 \), \( t \neq 0 \) (the ones representing \( \tilde{Z} \) converge even better than those for \( Z \)). For real \( g > 1(t > 0) \) one has to take the principal value of the integrals.

The integral representations (13)-(15) for \( Z(t, x) \) are solutions of the heat equation for all (even complex) \( t \neq 0 \) as can be seen immediately, e.g., by replacing the differentiation of \( 1/(1 - e^{t - \tau}) \) in eq. (14) with respect to \( t \) by the negative one with respect to \( \tau \) and performing a partial integration afterwards, or from the eq. (16) directly, because \( \hat{K}(t, x) \) is a solution of the heat equation.

Notice that \( \tilde{Z}(t, x) \) is not a solution of the heat equation (4), only \( e^t \tilde{Z}(t, x) \) is one.

As

\[ Z(t, \lambda x) = \frac{x}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2v^2/4} \frac{1}{1 - e^{t - \lambda^2/v^2}} , \quad \lambda > 0 , \]

(18)

we see that

\[ \lim_{\lambda \to 0} Z(t, \lambda x) = \frac{1}{1 - e^t} \quad \lim_{\lambda \to \infty} Z(t, \lambda x) = 1 , \]

(19)
in accordance with eqs. (6), but now for \( t > 0 \). Correspondingly we get for \( \tilde{Z}(t, x) \):

\[
\lim_{\lambda \to 0} \tilde{Z}(t, \lambda x) = \frac{1}{1 - e^t}, \quad \lim_{\lambda \to \infty} \tilde{Z}(t, \lambda x) = e^{-\lambda x}.
\] (20)

The expressions (13)-(15) or (17) can be used to extend the function \( Z(t = \ln g, x) \) or \( \tilde{Z}(t, x) \) into the complex \( g \)- or \( t \)-planes:

According to eq. (15) \( Z(g, x) \) has a branch cut in the complex \( g \)-plane along the real axis from 1 to \( \infty \). The discontinuity of \( Z \) across the cut is given by\([38, 41]\):

\[
\lim_{\epsilon \to 0^+} [Z(g + i\epsilon) - Z(g - i\epsilon)] = 2\pi i \hat{K}(t, x),
\] (21)

and \( Z(g, x) \) is an analytic function of \( g \) except for this cut\([38, 41]\).

If one, therefore, approaches the cut along the real axis from above, the limit

\[
\lim_{\epsilon \to 0^+} Z(g + i\epsilon, x), \quad g > 1,
\]

is no longer a real-valued function of \( g \) but has a nonvanishing imaginary part \( Z_i(t, x) \). Standard procedures used in the field of dispersion relations\([11]\) yield as the real part \( Z_r(t, x) \) the principal value integral

\[
Z_r(t, x) = \text{p.v.} \int_0^\infty d\tau \hat{K}(\tau, x)\eta(t - \tau),
\] (22)

and for the imaginary part \( Z_i \) the expression

\[
Z_i(t, x) = \pi \hat{K}(t, x) = \frac{\sqrt{\pi x}}{2t^{3/2}} e^{-x^2/(4t)}. \tag{23}
\]

Obviously \( Z_r \) and \( Z_i \) are solutions of the heat equation separately. \( Z_r(g, \cdot) \) is the Hilbert transform\([11, 12]\) of \( \hat{K}(g, \cdot) \).

We shall see that, strangely enough, it is this imaginary part of the partition function which gives exactly the thermodynamical properties expected for black holes!

The formal reasons for this can be seen from the limits (19) which obviously are those of \( Z_r \): The limit of \( Z_r(t, x) \) for \( x \to 0 \) is negative for \( t > 0 \), whereas \( Z_r \) is positive for large \( x \)! Notice that the integrand in eq. (22) is negative for \( t > \tau > 0 \) and that, therefore, \( Z_r \) may be negative for small \( x \). A negative
partition function can, however, hardly be interpreted thermodynamically where the logarithm has to be taken. On the other hand the imaginary part (23) is positive for all \( t > 0, x > 0 \) and does not have the ”sign-desease” of \( Z_i \)!

### 3 Thermodynamics

In the thermodynamics of black holes one is especially interested in the behaviour of the system for large \( \beta \) (low temperatures), because the inverse Hawking temperature \( \beta_H = 1/(k_B T_H) \) is very large for macroscopic black holes (see eq. (1)).

According to eq. (20) the real part \( \tilde{Z}_r \) of the partition function \( \tilde{Z}(t, x) \) behaves like \( \exp(-x) \) for large \( x \) and therefore the associated internal energy \( U = -\partial \ln Z_r / \partial \beta \) is just \( E_P \), i.e. the lowest possible energy level, as one would expect naively from the conventional paradigms of statistical mechanics.

The main reason for not using the real part \( Z_r \), however, is its property to become negative for small \( x \) if \( t > 0 \) as discussed above in connection with eq. (22).

The situation becomes surprisingly interesting and unconventional if we take the imaginary part \( Z_i \), eq. (23), as the partition function for calculating the thermodynamical properties of the system:

We first calculate the internal energy

\[
U = \bar{E} = -\frac{\partial \ln Z_i}{\partial \beta} = -\frac{\partial Z_i / \partial \beta}{Z_i} = Mc^2 ,
\]

which we identify with the total rest energy \( Mc^2 \) of the black hole and get

\[
U = \frac{\sigma^2 E_P^2}{2t} \beta - \frac{1}{\beta} = Mc^2 .
\]  

(24)

Solving this equation for \( \beta \) and discarding the negative root gives

\[
\beta = \frac{M c^2 t}{\sigma^2 E_P^2} [1 + (\frac{2\sigma^2 E_P^2}{M^2 c^4 t})^{1/2}] ,
\]

(25)

which for \( Mc^2 \gg E_P \) leads to

\[
\beta = \frac{2t Mc^2}{\sigma^2 E_P} \left(1 + \frac{\sigma^2 E_P^2}{2tMc^4}\right) .
\]

(26)
For $\sigma^2 = t/(4\pi)$ we get Hawking’s $\beta_H$, plus a small correction of order $(E_P^2/(M^2c^4))\beta_H$!

Furthermore, for the average of the energies squared we have

$$\overline{E^2} = \frac{\partial^2 Z_i}{\partial x^2} \sigma^2 E_P^2 / Z_i = \partial_t Z_i \sigma^2 E_P^2 / Z_i = (\frac{x^2}{4t^2} - \frac{3}{2t})\sigma^2 E_P^2 ,$$

yielding the mean square fluctuations

$$\overline{E^2} - \bar{E}^2 = -\frac{\sigma^2 E_P^2}{2t} - \frac{1}{\beta^2} .$$

We see that for $t > 0$ the right hand side is negative, as expected, corresponding to a negative specific heat. Considering the fact that the left hand side of eq. (28) is the difference of two very large numbers, both of the order $M^2c^4$, the mean square fluctuations are actually very small and essentially universal, because the right hand side depends on $M$ itself only through the term $1/\beta^2$, which is negligible for very large $\beta$!

If we define the average level number $\bar{N}$ by (see eq. (3))

$$\bar{N} = \frac{\partial^2 Z_i}{\partial x^2} / Z_i ,$$

then we get

$$\bar{N} = (\frac{x^2}{4t^2} - \frac{3}{2t}) = \frac{E^2}{\sigma^2 E_P^2} / (\sigma^2 E_P^2) \approx (M^2c^2)^2 / (\sigma^2 E_P^2) .$$

The last approximate equality follows from the fact that the fluctuations (28) are so small.

Thus we have for large $\bar{N}$

$$x \equiv \sigma E_P \beta = 2t\sqrt{\bar{N}} = 2t \frac{Mc^2}{\sigma E_P} .$$

Finally we come to the entropy

$$S/k_B = \ln Z_i + \beta U ,$$

for which we get the exact result

$$S/k_B = \frac{x^2}{4t} + \ln x - \frac{3}{2} \ln t + \ln(\sqrt{\pi}/(2e)) ,$$

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or, using the relation (31) and ignoring terms of order $O(1)$,

$$S/k_B = t\bar{N} + \frac{1}{2} \ln \bar{N}$$

$$= \frac{t}{4\pi \sigma^2} \frac{A}{4l_P^2} + \frac{1}{2} \ln \left(\frac{A}{l_P^2}\right),$$

where $A = 4\pi R_S^2 = 16\pi G^2 M^2 / c^4$ is the area of the horizon with Schwarzschild radius $R_S$ and $l_P^2 = G\hbar / c^3$ the Planck length squared.

For $\sigma^2 = t/(4\pi)$ the leading term of $S/k_B$ has the Bekenstein-Hawking value $A/(4l_P^2)$, for $\sigma = 1/2$, as discussed in ref. [19], it is slightly smaller if $t = \ln 2$ etc.

Notice that the factor $\exp(-x^2/4t)$ - typical for the heat equation - is the decisive one for providing the essential features of the thermodynamical properties just discussed.

Up to now we have been dealing mainly with the large $\beta$-(low temperature)-behaviour of the system. There are in addition some features for small $\beta$ (high temperature) worth mentioning. Whether they have any physical significance - e.g. for the big bang era - remains to be seen.

The internal energy $U$ in eq. (24) vanishes for $x = x_1 = \sqrt{2t}$ or $k_B T_1 = \sigma E_P / \sqrt{2t}$. $E^2$ becomes negative (eq. (27)): $E^2_1 = -\sigma^2 E_P^2 / t$! The entropy (33) at this temperature takes the value $S_1/k_B = (1/2) \ln[\pi/(2et^2)]$ which for $t = \ln 2$ is equal to $0.092 \cdots$!

According to eq. (27) the quantity $E^2$ vanishes for $x = x_2 = \sqrt{6t}$, $k_B T_2 = \sigma E_P / \sqrt{6t} < k_B T_1$, with $U = U_2 = 2\sigma E_P / \sqrt{6t}, S_2/k_B = (1/2) \ln[3\pi/(2t^2)] > S_1/k_B$.

Then there is the value $x = x_0(t)$ for which the entropy (33) vanishes. The resulting equation

$$x_0^2 + 4t \ln \left(\frac{\sqrt{\pi} x_0}{2e t^{3/2}}\right) = 0$$

cannot be solved for $x_0(t)$ explicitly, but the inequality $x_0(t) < 2 e t^{3/2} / \sqrt{\pi}$ follows immediately.

Strange things seem to happen at temperatures of the order of the Planck energy!
4 Remarks

The most surprising feature of the ”canonical” quantum statistical mechanics of the level spectrum (2) is that the expected thermodynamical properties of a Schwarzschild black hole are associated with the imaginary part of the partition function for real \( g > 1 \! \). Superficially this appears to question the canonical approach to the thermodynamics of black holes as possibly inappropriate. On the other hand the thermodynamical properties associated with \( Z_i \) are too intriguing and too interesting in order to dismiss them. The formal reasons for preferring \( Z_i \) compared to \( Z_r \) as the partition function relevant for the thermodynamics of the system have already been stressed in connection with the eqs. (22) and (23). The ”physical” thermodynamical consequences fully justify the more formal conclusions!

There is in addition an interesting heuristic consistency argument why the imaginary part (23) of the partition function is the physically relevant one: Suppose the sum (3) does not extend up to \( \infty \) but up to a very large number \( N \). Then we get

\[
Z_N(t, x) = \frac{x}{2\sqrt{\pi}} \int_0^\infty \frac{d\tau}{\tau^{3/2}} e^{-x^2/(4\tau)} \frac{1 - e^{N(t - \tau)}}{1 - e(t - \tau)}
\]  

instead of eq. (14) and the integrand is not singular anymore for \( \tau = t \) (the fraction resulting from the finite geometrical series just has the value \( N \) for \( \tau = t \) and there is no cut! If we now exploit eq. (31) and replace the \( x^2 \) under the integral sign by \( 4t^2N \) we get

\[
Z_N = \frac{x}{2\sqrt{\pi}} \int_0^\infty e^{Nh(\tau)}k(\tau),
\]

\[
h(\tau) = t - \tau - \frac{t^2}{\tau}, \quad k(\tau) = \frac{1}{\tau^{3/2}} \frac{e^{-N(t - \tau)} - 1}{1 - e(t - \tau)}.
\]

Evaluating the integral for large \( N \) by means of a saddle point approximation\(^{13}\) yields a saddle point for \( \tau = t \) and the result

\[
Z_N \approx \frac{x}{2t} \sqrt{N} e^{-tN} = \frac{x^2}{4t^2} e^{-x^2/(4t)}
\]

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where again the relation $x^2 = 4t^2 N$ has been used.
The approximation (38) is not equal to the function (23) and not a solution of the heat equation - there is mainly one additional factor $x/\sqrt{t}$ due to the approximations involved - but it contains the most essential factor $\exp(-x^2/(4t))$ which is so important for the qualitative structure of the thermodynamics.

Furthermore, the smallness of the thermal fluctuations (28) show that the thermal interactions of the black hole with the heat bath are small and, therefore, a "canonical" statistical treatment seems plausible and may not be too far off a microcanonical one.

In any case, the above "canonical" results have to be interpreted as requirements on the properties of a heat bath if it is to be in thermal equilibrium with the black hole.

If the whole approach discussed above is not unsound then the level spectrum (2) has to be taken seriously and a convincing justification of its validity is desirable[44].

A final remark: The thermodynamics discussed would be quite different if we would not interpret $g = \exp(t)$ as a fixed number but as the temperature dependent fugacity $g = z = \exp(\mu\beta)$, $\mu$: chemical potential, of a grand canonical ensemble[45].

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Finally I thank my wife Dorothea for her support, her understanding and her patience while this paper was being prepared.

Note added:

After this paper was submitted as an e-print M. Perry kindly drew my attention to refs.[46, 47] where the imaginary part of the partition function is related to the metastable states of the system. Especially the last paper[47] is of considerable interest in the context of the present article.
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