Mass bounds for Newtonian bosonic stars in theory of ultra–cold gases

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In the present work we address the issue concerning the relation mass versus radius for bosonic stars, all this in the context of the theory of ultra–cold gases in the dilute regime and within the realm of Newtonian gravity. It will be shown that this model predicts a minimum and a maximum for the mass of this kind of stars, contrary to the current belief present in some results in this direction.

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I. INTRODUCTION

The concept of bosonic star (BS) emerged at the end of the 1960’s [1 2] and for some time its interest was mainly related to the understanding of the laws behind its behavior, without any relation to astronomical or astrophysical observations. Concerning the possible cases that BS could have let us mention that the structure of the self–interaction of the basic constituents is one of the elements (spacetime symmetry has also to be included) defining the type of BS [3], charged, rotating, etc. An interesting point in these models is related to the existence of an upper bound for the mass, this value defines a region below the one the BS cannot collapse to a black hole.

Ultra–cold bosonic gases have been considered a possibility for the description of dark matter [4] and also for BS [5, 6], two situations which, without exaggerating, can be considered theoretical siblings. In this context the current description for these systems accepts the possibility of an infinite mass enclosed in a finite region, see expression (95) in [3]. A fleeting glimpse at the results in other approaches allows us to find that they do include the presence of an upper mass bound for BS [7]. At this point we may wonder if this discrepancy is a consequence of the fact that we have different approaches, or something else lies behind it. In the present work, which analyzes BS in the context of ultra–cold gases in the dilute regime, we provide an analytical answer in which we always bear in mind the mathematical conditions defining the dilute requirement. It will be proved that there is, not only, an upper limit, but also a lower one. As expected, these two limits depend upon the mass and scattering length of the involved bosonic particle. These two parameters emerge from the fundamental physical restrictions behind the present model in which the volume of the star, the number of its constituents, and the scattering length, related to the pairwise self–interaction, fulfill an inequality which codifies the mathematical premises defining the corresponding model.

In order to have a profounder physical motivation for the present work we proceed to provide a simple argument (based upon the fundamentals of an ultra–cold bosonic gas in the dilute regime) that could shed some light upon this topic, namely, there has to be an upper bound for the mass of a BS. Indeed, we now assume the validity of expression (95) in [3] for a BS, namely, for a finite radius, here denoted by \( R_{\text{min}} \), the system does not have an upper bound for its mass, i.e., it may grow indefinitely. The mass of this star is here denoted by \( M \) and it can be written as the product of the mass of our single particles (\( m \)) times the total number of particles (\( N \)), i.e., \( M = mN \). It has to be strongly underlined that our theory requires the fulfillment of the dilute gas condition, which is a backbone of the model under consideration [3]. This requirement can be cast in the following form \((V/N)^{1/3} > a\), where \( V \) is the volume of the system and \( a \) the scattering length of the involved particles which is responsible for the description of the pairwise interactions taking place in the gas. Clearly, for a non–vanishing scattering length if \( N \) grows without any bound we reach a value for it, say \( N^{(\text{max})} \), such that \((V/N^{(\text{max})})^{1/3} = a\), for \( N^{(+) > N^{(\text{max})}} \Rightarrow (V/N^{(+)})^{1/3} < a\). In other words, the assertion that a theory based upon a Gross–Pitaevskii–Poisson equation predicts the possibility of stars with unbounded mass implies that we have violated one of the fundamental premises of the current theory, i.e., the requirement of dilute gas regime. In other words, this last argument shows that, within the mathematical conditions defining our model, the number of particles in a bosonic star has an upper bound. Therefore we may ask ourselves the following question: how can we deduce this bound for the mass of the BS? The answer to this question is the topic of the present work.

We have \( N \) bosonic particles, each one of them with mass \( m \) and scattering length \( a \), defining a spherical body with radius \( R \). We assume that the temperature of this system is smaller than that corresponding to the beginning of the condensation process. Under these conditions the description of this system lies within the context of dilute ultra–cold gases [8], a fact that implies that our calculations shall always bear in mind the fulfillment of all the conditions defining the model. Another element is related to the introduction of gravity as a fundamental point in the physics of these systems. The set of corresponding equations are called Gross–Pitaevskii–Poisson [9].

II. MATHEMATICAL MODELING OF THE GRAVITATIONAL INTERACTION

We interpret now the gravitational interaction as a trapping potential with the structure of an isotropic harmonic oscillator. Consider a spherical body of mass \( M \) and radius \( R \) the one has a small cavity along the diameter coincident
with the \( z \)-axis. The corresponding mass density function has spherical symmetry. This mass density function has to be depicted by a function such that its Taylor expansion about the center of the body renders a series, i.e., it cannot be a polynomial. Indeed, assume that

\[
\rho(r) = \rho(0) + \sum_{n=1}^{\infty} \rho^{(n)}(0) r^n/n!.
\]  

(1)

Such that \( \exists s \in \mathbb{N} \ni \rho^{(s)}(0) = 0, \forall n \geq s. \) This condition implies that around the center of the body the density is a polynomial. Therefore

\[
r \to \infty \Rightarrow \lim \rho(r) = \infty.
\]  

(2)

Clearly, this condition implies that the density cannot be depicted by a polynomial. In other words, \( \forall l \in \mathbb{N}, \exists s \in \mathbb{N} \) such that \( \rho^{(s)}(0) \neq 0, s \geq l. \)

Let us consider one condition upon \( \rho(r) \), namely, its global maximum lies at \( r = 0 \). A particle of mass \( m \) moves along this cavity and the coordinate system has its origin at the geometrical center of the body. In spherical coordinates

\[
m \frac{d^2 r}{dt^2} = -G \frac{m M(r)}{r^2}.
\]  

(3)

\( M(r) \) defines the mass inside the sphere of radius \( r \).

\[
M(r) = \frac{4 \pi}{3} \rho(0) r^3 \left[ 1 + 3 \sum_{n=2}^{\infty} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n \right].
\]  

(4)

Finally, the equation of motion is

\[
\frac{d^2 r}{dt^2} + \frac{4 \pi}{3} G \rho(0) r \left[ 1 + 3 \sum_{n=2}^{\infty} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n \right] = 0.
\]  

(5)

The series in our last expression exists therefore we may define

\[
f(r) = \sum_{n=2}^{\infty} \rho^{(n)}(0) r^n/n!.
\]  

(6)

We have that \( \forall \delta > 0, \exists l \in \mathbb{N} \) such that

\[
|f(r) - \sum_{n=2}^{l} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n| < \delta, \forall s \geq l.
\]  

(7)

Hence

\[
\left| \sum_{n=s+1}^{\infty} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n \right| < \delta, \forall s \geq l.
\]  

(8)

If \( \delta_1 = 10^{-1} \), then \( \exists l_1 \in \mathbb{N} \) such that

\[
\left| \sum_{n=s+1}^{\infty} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n \right| < 10^{-1}, \forall s \geq l_1.
\]  

(9)

This last expression entails an inequality for our equation of motion, indeed

\[
-\frac{d^2 r}{dt^2} \leq \frac{4 \pi}{3} G \rho(0) r \left[ \frac{13}{10} + 3 \sum_{n=2}^{l_1} \frac{\rho^{(n)}(0)}{\rho(0) n! (n+3)} r^n \right].
\]  

(10)

We now consider that our density function has the following characteristic

\[
|\frac{\rho^{(n)}(0)}{\rho(0)}| \sim \frac{1}{R^n}.
\]  

(11)

This feature appears in several non–compact, for instance, Gaussian, Lorentzian functions and also all the even eigenfunctions of a harmonic oscillator.

The motion of our test particle takes place within the region defined by \( r \in [0, R] \) then the ensuing equation of motion has the following approximate expression

\[
\frac{d^2 r}{dt^2} + \frac{4 \pi}{3} G \rho(0) r \left[ \frac{13}{10} + 3 \sum_{n=2}^{l_1} \frac{r^n}{R^n} \right] = 0.
\]  

(12)

The factor \( 13/10 \) defines a three dimensional harmonic oscillator and the additional terms may be understood as perturbations to it, at least in those cases in which \( r < R \). The corresponding frequency is given by

\[
\omega(0) = \sqrt{\frac{52 \pi G \rho(0)}{30}}.
\]  

(13)

This expression tells us that the motion of \( m \) is, approximately, related to an isotropic harmonic oscillator whose frequency depends upon the central density. We now quantize our system and consider the gravitational effects of \( N-1 \) particles of mass \( m \) \( (M = (N-1) m) \) upon our particle of, also, mass \( m \). The corresponding time–independent Schrödinger equation reads

\[
E \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{m \omega(0)^2}{2} r^2 \psi.
\]  

(14)

Since we end up with a three–dimensional harmonic oscillator the ground state of system is a Gaussian function [13]. In addition, the Hartree approximation is introduced [9] then all particles are described by a Gaussian wavefunction, i.e., all particles are described by one and only one function, namely,

\[
\rho(r) = \frac{mN}{\sqrt{\pi \hbar^3}} \exp\{-r^2/\ell^2\},
\]  

(15)

\[
\ell = \left( \frac{\hbar}{m \omega(0)} \right)^{1/2}.
\]  

(16)
A. Mathematical model

The physically meaningful case involves a non–vanishing scattering length therefore we introduce now this parameter. According to our previous analysis the self–gravitational interaction of the star is considered as an external isotropic three–dimensional harmonic oscillator, in other words, the corresponding many body–Hamiltonian, for the situation of a dilute gas, is given by

\[
\hat{H} = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + \frac{m\omega_i^2}{2}\hat{r}_i^2 \right] + U(0) \sum_{i<j} \delta(\hat{r}_i - \hat{r}_j). \tag{17}
\]

Here \( U(0) = 4\pi\hbar^2a/m \) contains the information, under the assumption of very low energies and dilute gas, of the interaction between two particles, i.e., only pairwise interactions are relevant for the dynamics of the system \([11]\). Here \( a \) denotes the scattering length of our particles.

In the context of the Hartree approximation the time–independent Gross Pitaevskii equation provides the dynamics of the system \([10]\)

\[
\mu\psi(\vec{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\omega_i^2}{2}\vec{r}^2 + U(0)|\psi(\vec{r})|^2 \right] \psi(\vec{r}). \tag{18}
\]

An additional condition has to be satisfied, the one implies particle conservation, namely

\[
N = \int |\psi(\vec{r})|^2 d\vec{r}. \tag{19}
\]

The chemical potential is \( \mu \). Since \( \rho(\vec{r}) = m|\psi(\vec{r})|^2 \) we may cast \([18]\) in the following form

\[
\mu\psi(\vec{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\omega_i^2}{2}\vec{r}^2 + U(0)\rho(\vec{r})/m \right] \psi(\vec{r}). \tag{20}
\]

Clearly, we have that

\[
|\psi(\vec{r})|^2 \psi(\vec{r}) = \rho(\vec{r}) \psi(\vec{r})/m. \tag{21}
\]

Introducing the Gaussian structure for the density, expanding it in terms of a Taylor series, and, finally, keeping only terms up to second order in \( r \) we end up with the following expression

\[
\hat{\mu}\psi(\vec{r}) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + \frac{m\omega_i^2}{2}\vec{r}^2 \psi(\vec{r}), \tag{22}
\]

\[
\omega^2 = \omega_i^2 - 2 \frac{NU(0)}{m\sqrt{\pi}a^3}, \tag{23}
\]

\[
\hat{\mu} = \mu - \frac{NU(0)}{\sqrt{\pi}a^3}. \tag{24}
\]

The effective chemical potential is denoted by \( \hat{\mu} \). Clearly, the equation of motion is a three–dimensional harmonic oscillator in which the frequency is now modified due to the presence of a non–vanishing scattering length.

III. BOUNDS FOR THE MASS PARAMETER

A. Mechanical Equilibrium

At this point we address the issue concerning the condition of mechanical equilibrium for the BEC. Indeed, gravity tends to collapse the star and this behavior faces a pressure which is a consequence of Heisenberg’s Uncertainty Principle and of the movement of the particles of the star. Mechanical equilibrium emerges when the corresponding pressures of these two processes are equal.

The energy of the system due to \( N \) particles in the ground state (no particles in excited states, as a first approximation) is given by

\[
E(0) = \frac{3}{2} \hbar \omega N. \tag{25}
\]

In this last expression the frequency corresponds to \([23]\) and it does not neglect the kinetic energy, i.e., our formalism does not resort to the Thomas–Fermi approximation (TF) \([9]\). The use of TF is valid if \( Na > 1 \), a fact that requires the knowledge of the parameters \( N, a, m \). Clearly, we do not know them, on the contrary, one of our problems is the deduction of them; in other words, we cannot resort to TF since we know nothing about the require parameters of the bosonic particles.

Our system is equivalent to a BEC trapped by a three–dimensional isotropic harmonic oscillator \([11]\), a situation already well comprehended. The pressure related to this case reads:

\[
P(\rho) = \frac{3a^2\hbar^2N}{4\pi m R^6} + \frac{3a^3U(0)N^2}{4\pi^{3/2} R^8}. \tag{26}
\]

In this last expression \( R = al \) is the characteristic radius of the region comprising most (at least 87 percent) of the particles.

We now address the issue concerning the gravitational attraction of our bosonic star, a topic related to the equilibrium of a spherical body with density, pressure, velocity field, and gravitational potential \( \rho, P, \) and \( \vec{v}, \Phi \), respectively. The corresponding equations for the internal structure are provided by \([12]\)

\[
\rho \frac{d\vec{v}}{dt} = \rho \nabla \Phi - \nabla P, \tag{27}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{28}
\]

These expressions do not close the system, namely, an additional piece of information is required, i.e., the equation of state; in other words, the functional dependence among pressure, density, temperature, etc. The Newtonian gravitational potential for a spherical body of radius \( R \) is

\[
\nabla \Phi(t, r) = \frac{GM(t, r)}{r} + 4\pi \int_{r}^{R} \rho(t, r') dr'. \tag{29}
\]
In this last expression $M(t, r)$ is the mass contained in a sphere (coincident with our body) of radius $r$. We now consider the surface of our sphere and calculate the change in this gravitational potential due to a change in the volume, a process that entails a pressure $\left(P_{(g)} = -\frac{\partial \Phi}{\partial V}\right)$.

$$P_{(g)} = \frac{GM^2}{4\pi R^4}. \quad (30)$$

The corresponding pressure is a non-constant function of the radial distance $\beta$ and, in consequence, we must identify the value of $r$ related to (30). This pressure is deduced after evaluating the gravitational potential on the surface of the body then it represents the pressure on this surface. The mathematical condition determining mechanical equilibrium is the equality of our two pressures on the surface of the bosonic star (remember that $M = mN$) i.e., expressions (26) and (30).

One consequence of the equality $P_{(g)} = P_{(a)}$ is related to the fact that we have deduced $R$ as a function of $m$, $a$, and $N$. In this sense, the roughest approximation yields the following expression for the radius of the bosonic star.

$$R = \frac{3}{2} \frac{\alpha^2 \hbar^2}{Gm^3N} \left\{ 1 + \sqrt{1 + \frac{16\alpha N^2 G m^3}{3\alpha \sqrt{\pi} \hbar^2} } \right\}. \quad (31)$$

Defining $L = \frac{\alpha^2 \hbar^2}{Gma}$ and inserting the dilute gas condition into (31), $\left(\frac{\alpha}{m}\right)^{1/3} > a$, we have an inequality involving the number of particles within the boson star:

$$\beta \equiv \sqrt{1 + \frac{16\alpha N^2}{3\sqrt{\pi} \hbar^2} \geq \frac{2}{3} \frac{\left(\frac{3}{4\pi}\right)^{1/3}}{N^{4/3}} - 1 \equiv \gamma. \quad (32)$$

In this last expression, we have two possibilities about $\gamma$ and $\beta$:

1. $\beta > \gamma > 0$,
2. $\gamma < 0 \wedge |\beta| < |\gamma|$.

The second case leads us to the following conclusion:

$$1 - \frac{\left(\frac{\alpha}{m}\right)^{4/3}}{\left(\frac{1.07}{L}\right)^{4/3}} \geq \frac{1}{1 + \frac{5N}{2}} > 0, \text{ i.e., a condition impossible to satisfy. In other words, the only mathematically consistent situation is the first one.}

Therefore, concerning the only possibility we now define a polynomial in the variable as follows $z = N^{2/3}$:

$$P(z) = \frac{z^2}{L} \left\{ \frac{4}{9L} \left(\frac{3}{4\pi}\right)^{2/3} z^2 - \frac{16\alpha}{3\sqrt{\pi}} z - \frac{4}{3} \left(\frac{3}{4\pi}\right)^{1/3} \right\}, \quad (33)$$

and solving for $z$, the only physical solution reads

$$z(+) = \frac{\left(\frac{4\pi}{3} \right)^{2/3}}{6\alpha L \sqrt{\pi}}. \quad (33)$$

Relating this last expression to the original variable $N$ we have a maximum value for the total mass in a bosonic star:

$$M(+) = mN(+) = m^{3/2} a^{3/2} = m \frac{4\pi}{3} \left(\frac{6}{\sqrt{\pi}} \right)^{3/2} (\alpha L)^{3/2}. \quad (34)$$

In addition we have the following condition $\frac{3}{2} \left(\frac{3}{4\pi}\right)^{1/3} \frac{N^{4/3}}{L} > 1$, it implies the existence of a lower bound for the mass of a bosonic star:

$$M(1) = m \left[ \frac{3}{2} \left(\frac{4\pi}{3}\right)^{1/3} L \right]^{4/3}. \quad (35)$$

Resorting to equation (31) we may estimate, in a rough approximation, the parameter $a/m^3$ for a given radius:

$$\frac{a}{m^3} = \frac{1}{12} \frac{G \sqrt{\pi}}{\alpha^3 \hbar^2}. \quad (36)$$

Equation (36) and the behavior of our bosonic particles in the context of dilute gases [14] for the parameter $a/m^3$ allow us to compare and predict the value of the mass particle for bosonic stars from the typical size of stars made of baryonic matter.

**IV. DISCUSSION OF RESULTS AND CONCLUSIONS**

We now proceed to analyze the deduction of the values of the mass and scattering length of our bosonic particles. In order to do this we now resort to expression (30) which is function of $R$. Notice that the choice of a value for $R$ defines, uniquely, the value of $a/m^3$. At this point we notice that for baryonic particles (this phrase means Na, K, and Rb atoms) and for the results associated to dark matter halos[14] the graph $a/m^3$ versus $m$ in a logarithmic version renders a straight line. We now introduce an additional assumption, namely, the relation $a/m^3$ as a function of $m$ is a fundamental expression for baryonic and exotic matter. Therefore, the choice of a value for $R$ provides us a unique $a/m^3$ and, resorting to the next plot, we deduce $m$, a fact that implies also the knowledge of $a$.

At this point we may use the method of least squares to provide a curve to determine the corresponding values for $m$ for the bosonic star particles. The associated plot is

**Logarithmic curve of $a/m^3$ vs $m$**

- Dark Matter
- Boson Star
- Atoms
Concerning the main expression (35), we may establish the value for \( a \) associated to the bosonic star particle. In addition, now we can provide an explicit value for the mass bounds given by (35) and (43), and these are:

| Star          | Radius \( R \) | \( a/m^3 \) | \( m \) | \( a \) |
|---------------|----------------|-------------|--------|--------|
|               | [km]           | [m/kg^3]    | [MeV/c^2] | [m]    |
| White dwarf   | 6.0 \times 10^3 | 6.9 \times 10^{10} | 5.9 \times 10^9 | 7.9 \times 10^{-9} |
| Brown dwarf   | 4.2 \times 10^4 | 3.4 \times 10^{11} | 1.8 \times 10^8 | 1.2 \times 10^{-8} |
| Sun           | 7.0 \times 10^5 | 9.3 \times 10^{12} | 3.5 \times 10^2 | 2.2 \times 10^{-8} |
| Red giant     | 5.6 \times 10^7 | 6.0 \times 10^{15} | 26.0 | 5.7 \times 10^{-8} |
| Supergiant    | 3.5 \times 10^8 | 2.3 \times 10^{19} | 8.6 | 8.5 \times 10^{-8} |
| Betelgeuse    | 8.2 \times 10^8 | 1.3 \times 10^{20} | 5.2 | 1.0 \times 10^{-7} |
| NML Cygni     | 1.1 \times 10^9 | 2.5 \times 10^{20} | 2.3 | 1.1 \times 10^{-7} |

Our previous results show us that the theory of ultra-cold gases predicts that the relation between radius and the parameter \( a/m^3 \) is a unique one. Indeed, notice that stars with different radii require exotic particles with different mass, and that they may show a discrepancy of three orders of magnitude among them. Notice that the values of \( a \) and \( m \) are far from those corresponding to a dark matter particle [14], at least in five orders for the biggest stars like Betelgeuse. An evaluation of the temperature (\( T \)) of the system can be obtained resorting to the equipartition theorem [13], i.e., \( 3\hbar \omega/2 = \kappa T \).

We may accept, at this point, the possible situation in which there are more than one different kind of exotic particles. This assumption opens up the possibility of having several exotic particles, one responsible for a dark matter halo, additional options defining bosonic stars, with different radii. If we assume, instead, that only one kind of non-baryonic particle exists, then there is one and only one possible structure formed by exotic particles.

Summing up, we have modeled bosonic stars as a Bose–Einstein condensate and interpreted the self-gravitational interaction as an isotropic three-dimensional harmonic oscillator, of course, the pairwise interaction of short range, emerging in the context of the dilute limit, has also been considered. We have deduced upper and lower bounds for the mass of these kind of objects, which appear as consequence of the mathematical conditions defining our model. The deduced values show a strong dependence in the choice of the radius of the star, a fact that implies that if we accept the possibility of having bosonic stars with different radii, then the present work tells us that there have to be several exotic particles, i.e., one of them cannot account for the difference in the radii.

Finally, if we accept the fact that there is only one kind of exotic particle and that it is related to a dark matter halo [14], then our results imply that a dilute Bose–Einstein condensate prevents the appearance subgalatic structures. On the other hand, if we accept the existence of subgalatic structures, then galaxies cannot be explained.

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