The modular class of a singular foliation

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Abstract

The modular class of a regular foliation is a cohomological obstruction to the existence of a volume form transverse to the leaves which is invariant under the flow of the vector fields of the foliation. It has been used in dynamical systems and operator theory, from which the name modular comes from. By drawing on the relationship between Lie algebroids and regular foliations, this paper extends the notion of modular class to the realm of singular foliations. The singularities are dealt with by replacing the singular foliations by any of their universal Lie $\mathcal{L}_\infty$-algebroids, and by picking up the modular class of the latter. The geometric meaning of the modular class of a singular foliation $\mathcal{F}$ is not as transparent as for regular foliations: it is an obstruction to the existence of a universal Lie $\mathcal{L}_\infty$-algebroid of $\mathcal{F}$ whose Berezinian line bundle is a trivial $\mathcal{F}$-module. This paper illustrates the relevance of using universal Lie $\mathcal{L}_\infty$-algebroids to extend mathematical notions from regular foliations to singular ones, thus paving the road to defining other characteristic classes in this way.

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1 Introduction

Singular foliations form a class of objects that appear in geometry, operator algebra, dynamical systems, etc. that generalize the notion of regular foliation by allowing the leaves to have non-constant dimension. This property is manifest at the infinitesimal level: a regular foliation is primarily defined as a vector subbundle of the tangent bundle, whereas a singular foliation can be considered as a particular subsheaf of the sheaf of vector fields, so that the rank of the induced distribution may not be constant. Due to the difficulty to properly handle the singularities, many notions and theories that are otherwise well established for regular foliations have not been generalized to the singular case yet. In this paper we intend to discuss the notion of modular class associated to a singular foliation, by drawing on previous work [20] that allows to ‘regularize’ the foliation.

A regular foliation induces an equivalence relation on the manifold: that of belonging to the same leaf. The quotient of the manifold by the equivalence relation – the leaf space – is not necessarily a manifold, hence the difficulty of making sense of a measure – or volume form – on it. An alternative way of characterizing such a measure is to define a measure on the transversal of the leaves (if they exist), that would be invariant with respect to the flow of the vector fields defining the (regular) foliation. Under appropriate assumptions on the leaves, such a transverse measure would then be considered as a measure on the leaf space. Transverse measures of regular foliations have proved to be very important in the study of dynamical systems on differentiable manifolds [12, 26, 27]. The existence of a globally defined transverse measure is conditioned to the vanishing of a certain cohomology class of the foliated de Rham cohomology: the modular class (or Reeb class).

The name modular originally refers to the Tomita-Takesaki theory in non-commutative geometry, which classifies von Neumann algebras from their so-called modular automorphisms. This theory dating back from the end of the 1960 has been widely fueled by examples taken from regular foliations, since any regular foliation equipped with a transverse measure canonically induces a von Neumann algebra [6, 34]. The notion of modular automorphisms has been transported to the field of Poisson geometry [32], since the notion of modular flow in Poisson geometry forms a classical limit of the notion of modular automorphism in the theory of von Neumann algebras. Due to the proximity between Poisson manifolds and Lie algebroids, the notion of modularity was extended to Lie algebroids and Lie groupoids [8, 16], generalizing the already existing and well-known notion of modular function of Lie groups which evaluates the existence of a bi-invariant Haar measure. The modular class of Lie algebroids also restricts to the well-known notion of modular class of regular foliations when the Lie algebroid in question is a foliation Lie algebroid. This allows to think about the modular class of a Lie algebroid as an obstruction to the existence of a certain invariant measure on the differentiable stack associated to this Lie algebroid [7, 33].

Given this background, it seems legitimate to generalize the notion of modular class to the singular setting, e.g. to singular foliations. The definition of the modular class of a regular foliation cannot straightforwardly apply to the singular case because the rank of the induced distribution of tangent vectors is not constant. Hence, the construction using Bott connections and transversals to the leaves cannot be imported as such. Since the modular class of a regular foliation coincides with the modular class of its associated foliation Lie algebroid, one way to overcome this difficulty is to find an analogue of the foliation Lie algebroid in the singular case, namely: a universal Lie $\infty$-algebroid associated to the singular foliation [20]. A Lie $\infty$-algebroid is to a Lie algebroid what a (non-positively graded) $L_\infty$-algebra is to a Lie algebra. A universal Lie $\infty$-algebroid of a singular foliation $\mathcal{F}$ is a projective resolution of $\mathcal{F}$ by locally free $C^\infty(M)$-modules which, by a transfer theorem, can be equipped with a Lie $\infty$-algebroid structure.
lifting the Lie bracket of vector fields. The name *universal* is justified by the observation that two such universal Lie-$\infty$ algebroids resolving the same singular foliation are homotopically equivalent [20].

In a recent work, Caseiro and Laurent-Gengoux have extended the notion of modular class from Lie algebroids to Lie $\infty$-algebroids: it is an obstruction to the existence of an invariant section of the associated Berezinian line bundle [5]. They show that the modular class neither depends on the choice of the resolution, nor on the choice of the section of its associated Berezinian, etc. This allows to obtain a well-defined and unique notion of modular class of a singular foliation by setting it to be the modular class of any of its universal Lie $\infty$-algebroids. More precisely, it is the unique cohomology class in the foliated cohomology of $\mathcal{F}$ whose representant in the cohomology of any universal Lie $\infty$-algebroid $E$ of $\mathcal{F}$ is precisely the modular class of $E$ (see Definition 3.14). In this context, the modular class of a singular foliation $\mathcal{F}$ is an obstruction the the existence of a trivial $\mathcal{F}$-module structure on the Berezinian of any of its universal Lie $\infty$-algebroids (see Proposition 3.16). In full rigor, we only define modular classes of *solvable* singular foliations, i.e. those foliations which admit at least one universal Lie $\infty$-algebroid of finite length. This is a necessary condition for the Berezinian to be well-defined.

This notion of modular class for singular foliations coincides with that of regular foliations when $\mathcal{F}$ is regular and the Lie $\infty$-algebroid $E$ is the foliation Lie algebroid or, more generally, is a resolution of the latter. However, contrary to the regular case, singular foliations preserve their specificity: what is central in singular foliations is not the set of leaves but the set of vector fields generating the singular foliation. There are indeed examples of singular foliations whose regular leaves admit a transverse volume form but which do not necessarily admit invariant sections of the Berezinian of *any* of its universal Lie $\infty$-algebroids. This curiosity can be explained by the behavior at the singularities of the vector fields generating the singular foliation. This paper thus illustrates the relevance of using universal Lie $\infty$-algebroids to extend mathematical notions from regular foliations to singular ones [20], thus paving the road to defining other characteristic classes in this way.

Part of the material of this paper is already known but is scattered in the literature. It is here presented in a unified fashion, with natural and adapted conventions susceptible to be reused for further treatment of characteristic classes of singular foliations. Section 2 is a reminder on generalities on Lie $\infty$-algebroids and their representations up to homotopy. This material is already known but has been included here for self-consistency of the paper, and because it has not been presented under such conventions. Subsection 2.3 recalls some basics about singular foliations and elaborates about the relationship between their foliated cohomology and the cohomology of their (universal) Lie $\infty$-algebroids. Section 3 is dedicated to presentation of the notion of modular class of Lie $\infty$-algebroids and singular foliations. Subsection 3.1 does not contain new important results but adapts the material of [5] to the skew-symmetric convention for Lie $\infty$-algebroids. Drawing on the material introduced in subsections 2.3 and 3.1, subsection 3.2 defines the modular class of singular foliations (Definition 3.14), its meaning (Proposition 3.16), and its main relationship with that of regular foliations (Proposition 3.17). Section 4 is entirely dedicated to examples of various kinds. In particular, several cases are investigated in order to illustrate the extent to which the notion of modular class of singular foliations is meaningful.

2 Lie $\infty$-algebroids and singular foliations

Throughout the paper, we will stay in the category of real smooth manifolds and will denote by $\mathcal{C}^\infty$ the sheaf of real valued smooth functions on a given manifold. In the present section we give the necessary mathematical background and we introduce notations that will be used later on.
2.1 Generalities on Lie $\infty$-algebroids

There exists two equivalent conventions for $L_{\infty}$-algebra brackets: the graded skew-symmetric version is the original and most natural convention [18, 19], while the graded symmetric version can be more useful for computations; the latter convention being sometimes called $L_{\infty}[1]$-algebras [24, 28]. An $L_{\infty}$-algebra structure on $V$ is equivalent to an $L_{\infty}[1]$-algebra structure on the desuspension of $V$, denoted $s^{-1}V$ or $V[1]$ [30]. In this section – and more generally in this paper – we will restrict ourselves to the use of the graded skew-symmetric convention to stick with the analogy with Lie algebras, as it seems more natural and easier to understand for newcomers. For an equivalent formulation under the symmetric convention, see [20].

**Definition 2.1.** [18, 19] An $L_{\infty}$-algebra is a graded vector space $V = \bigoplus_{j \in \mathbb{Z}} V_j$ together with a family of graded skew-symmetric $k$-multilinear maps $(l_k)_{k \geq 1}$ of degree $2-k$, called $k$-brackets, which satisfy a set of compatibility conditions called higher Jacobi identities. For all $k \geq 1$ and for every $k$-tuple of homogeneous elements $a_1, \ldots, a_k \in V$, they are given by:

$$\sum_{j=1}^{k} (-1)^{j(k-j)} \sum_{\sigma \in Un(j,k-j)} \epsilon(\sigma) l_{k-j+1}(l_j(a_{\sigma(1)}, \ldots, a_{\sigma(j)}), a_{\sigma(j+1)}, \ldots, a_{\sigma(k)}) = 0 \quad (2.1)$$

where $Un(j,k-j)$ denotes the set of $(j,k-j)$-unshuffles, and where $\epsilon(\sigma)$ is the sign induced by the permutation of the elements $a_1, \ldots, a_k$ under $\sigma$ in the exterior algebra of $V$:

$$a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(k)} = \epsilon(\sigma) a_1 \wedge \ldots \wedge a_k \quad (2.2)$$

**Remark.** The definition implies that the 1-bracket is a differential on $V$ that is compatible with the 2-bracket, i.e. it is a derivation of the 2-bracket:

$$l_1(l_2(a, b)) = l_2(l_1(a), b) + (-1)^{|a|} l_2(a, l_1(b)) \quad (2.3)$$

Moreover, the 2-bracket is a Lie bracket up to homotopy:

$$l_2(l_2(a, b), c) + \circ = -[l_1, l_3](a, b, c) \quad (2.4)$$

where on the right-hand side, $[\ldots]$ symbolizes the graded commutator of operators on $\text{End}(\wedge \bullet V)$.

We can now define what is a Lie $\infty$-algebroid:

**Definition 2.2.** A Lie $\infty$-algebroid over a smooth manifold $M$ is a triple $(E, (l_k)_{k \geq 1}, \rho)$, where $E = (E_{-i})_{i \geq 0}$ is a non-positively graded vector bundle over $M$, $(l_k)_{k \geq 1}$ is a family of $\mathbb{R}$-linear brackets on the sheaf $\Gamma(E)$ of sections of $E$ defining an $L_{\infty}$-algebra structure on $\Gamma(E)$, and $\rho : E_0 \to TM$ is a vector bundle morphism, called the anchor map, such that the following items hold true:

1. For every $k \geq 1$ the $k$-brackets $l_k$ are always $C^\infty$-linear in each of their $k$ arguments, except when $k = 2$ and at least one of its two entries has degree zero

$$l_2(a, fb) = fl_2(a, b) + \rho(a)[f]b \quad (2.5)$$

for all $a \in \Gamma(E_0), b \in \Gamma(E)$ and $f \in C^\infty$;

\[\text{In the present paper, we adopt the convention that the suspension operator} \ s \ (\text{also denoted} \ [-1]) \text{increases the degree of the associated space by 1, while the desuspension operator} \ s^{-1} \ (\text{also denoted} \ [1]) \text{decreases the degree by 1. In other words,} \ (sV)_i = V_{i-1} \text{while} \ (s^{-1}V)_i = V_{i+1}.\]
2. The anchor map satisfies, for every $a \in \Gamma(E_{-2})$:

$$\rho(l_1(a)) = 0$$  \hspace{1cm} (2.6)

A Lie $\infty$-algebroid is said to be a Lie $n$-algebroid when $E_{-i} = 0$ for all $i \geq n$. In that case we say that $E$ is of length $n$. A Lie 1-algebroid is called a Lie algebroid.

Remark. 1. Equation (2.5) together with Equation (2.4) imply that the anchor map and the 2-bracket on $\Gamma(\wedge^2 E_0)$ satisfy the same relationship as in the Lie algebroid case:

$$\rho(l_2(a, b)) = [\rho(a), \rho(b)]$$  \hspace{1cm} (2.7)

for every $a, b \in \Gamma(E_0)$, and where $[,]$ is the Lie bracket of vector fields on $M$.

2. The $\mathcal{C}^\infty$-linearity of $l_1$ together with the Jacobi identity $l_1^2 = 0$ implies that $l_1$ defines a chain complex of vector bundles:

$$\ldots \longrightarrow l_1 \longrightarrow E_{-2} \longrightarrow l_1 \longrightarrow E_{-1} \longrightarrow l_1 \longrightarrow E_0 \longrightarrow \rho \longrightarrow TM \longrightarrow 0$$

We call the triple $(E, l_1, \rho)$ the linear part of the Lie $\infty$-algebroid.

Given a Lie $\infty$-algebroid $E = \bigoplus_{i \geq 0} E_{-i}$, we can define a cohomology generalizing the Lie algebroid cohomology and the Chevalley-Eilenberg cohomology of Lie algebras. We call forms on $E$ elements of the graded vector space $\Omega^\bullet(E) = \bigoplus_{k \geq 1} \Omega^k(E)$, where $\Omega^k(E) = \Gamma(\wedge^k E^*)$. This space carries a bidegree: the form degree $k$ and the degree $p$ induced by the grading of $E^*$, which is reverse than that of $E$. More precisely, since the Lie $\infty$-algebroid $E$ is non-positively graded, the grading $p$ takes values in non-negative integers. Then, we say that a form $\eta$ is a $k$-form of degree $p$ if $\eta \in \Omega^k(E)|_p = \Gamma(\wedge^k E)|_p$. The sum $h = k + p$ is called the height of $\eta$, and we write $\Omega^k_h(E)$ instead of $\Omega^k(E)|_p$ if one wants to emphasize the use of the height instead of the degree on $E^*$ (which can always be found again by setting $p = h - k$). For every $h \geq 0$, one sets

$$\Omega_h(E) = \bigoplus_{k \geq 0} \Omega^k_h(E) \quad \text{and} \quad \Omega^\bullet(E) = \bigoplus_{h \geq 0} \Omega_h(E)$$

Remark. The height is defined so that, when $E$ is a Lie algebroid (a Lie $\infty$-algebroid concentrated in degree 0, then), the height of a $k$-form is precisely $k$. This is the correct notion to use when generalizing results on Lie algebroid cohomology to Lie $\infty$-algebroid cohomology.

We define a sequence of operators $(d^{(s)}_E : \Omega^\bullet(E) \to \Omega^{\bullet+s}(E))_{s \geq 0}$ by their action on $k$-forms. The index $s$ is called the arity and measures by how much the form-degree is increased under the action of $d^{(s)}_E$ on a $k$-form $\eta \in \Omega^k(E)$, for $k \geq 1$:

$$d^{(0)}_E \eta(a_1, \ldots, a_k) = \sum_{\sigma \in U_n(1,k-1)} \epsilon(\sigma) \eta(l_1(a_{\sigma(1)}), a_{\sigma(2)}, \ldots, a_{\sigma(k)})$$  \hspace{1cm} (2.8)

$$d^{(1)}_E \eta(a_1, \ldots, a_{k+1}) = \sum_{\sigma \in U_n(1,k)} \epsilon(\sigma) \rho(a_{\sigma(1)}) [\eta(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)})]$$

$$- \sum_{\sigma \in U_n(2,k-1)} \epsilon(\sigma) \eta(l_2(a_{\sigma(1)}), a_{\sigma(2)}, a_{\sigma(3)}, \ldots, a_{\sigma(4)})$$  \hspace{1cm} (2.9)

and, for every $s \geq 2$, as:

$$d^{(s)}_E \eta(a_1, \ldots, a_{k+s}) = (-1)^s \sum_{\sigma \in U_n(s+1,k-1)} \epsilon(\sigma) \eta(l_{s+1}(a_{\sigma(1)}, \ldots, a_{\sigma(s+1)}), a_{\sigma(s+2)}, \ldots, a_{\sigma(k+s)})$$

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On zero-forms, i.e. smooth functions on $M$, the only operator acting is $d_E^{(1)}: C^\infty(M) \to \Gamma((E_0)^*)$, via the anchor map:

$$d_E^{(1)} f(a) = \rho(a)[f] \tag{2.10}$$

for every $f \in C^\infty(M)$, and $a \in \Gamma(E_0)$.

From these formulas, the higher Jacobi identities (2.1) together with Equation (2.7) – which is itself a consequence of the former – imply that $d_E = \sum_{s \geq 0} d_E^s$ is a differential, i.e. that $d_E^0 = 0$. The degree of this operator cannot be based on the form degree alone as

$$d_E(\Omega^\bullet(E)) \subset \bigoplus_{s \geq 0} \Omega^{\bullet + s}(E)$$

but a straightforward calculation shows that $d_E$ increases the height by 1, turning $\Omega^\bullet(E)$ into a co-chain complex.

**Definition 2.3.** The cohomology of the co-chain complex $(\Omega^\bullet(E), d_E)$ is called the Lie $\infty$-algebroid cohomology of $E$, and is denoted $H^\bullet(E)$, or $\mathcal{H}(E)$ for short.

The Lie $\infty$-algebroid cohomology is analogous to the Chevalley-Eilenberg cohomology for Lie algebras, and coincides with the Lie algebroid cohomology when $E$ is a Lie algebroid. The one-to-one correspondence between Lie algebroid structures on a vector bundle $A$ and a differential graded manifold structure on $A[1]$ [29] extends to the Lie $\infty$-algebroid context:

**Theorem 2.4.** [31] Let $E = \bigoplus_{i \geq 0} E_{-i}$ be a non-positively graded vector bundle over $M$. Then there is a one-to-one correspondence between Lie $\infty$-algebroid structures on $E$ and linear operators $d_E: \Omega^\bullet(E) \to \Omega^\bullet+1(E)$ satisfying the homological condition $d_E^2 = 0$.

This correspondence provides a very efficient way of defining morphisms of Lie $\infty$-algebroids [4]:

**Definition 2.5.** Let $E$ (resp. $E'$) be a Lie $\infty$-algebroid with base manifold $M$ (resp. $M'$). Then a Lie $\infty$-morphism between $E$ and $E'$ is a height-preserving graded algebra morphism $\Phi: \Omega^\bullet(E') \to \Omega^\bullet(E)$ that intertwines $d_E$ and $d_{E'}$:

$$d_E \circ \Phi = \Phi \circ d_{E'} \tag{2.11}$$

The Lie $\infty$-morphism $\Phi$ can be decomposed into components of various arities: $\Phi = \sum_{s \geq 0} \Phi^{(s)}$ where $\Phi^{(s)}$ is a map from $\Omega^\bullet(E')$ to $\Omega^{\bullet + s}(E)$. Each of these components, for $s \geq 1$, defines a dual map $f_s: \Gamma(\wedge^s E) \to \Gamma(E')$ of degree $1 - s$ satisfying some intricate consistency conditions generalizing the Lie algebra homomorphism condition [14]. Also the restriction of $\Phi$ to $C^\infty(M')$ induces a map of smooth manifolds $\varphi: M \to M'$ so that for every $f \in C^\infty(M')$ and $\eta \in \Omega(E')$, we have:

$$\Phi(f \eta) = \varphi^*(f) \Phi(\eta) \tag{2.12}$$

When $M' = M$ and $\varphi = \text{id}_M$, we say that $\Phi$ is over $M$.

Morphisms of Lie $\infty$-algebroids admit homotopies, that admit in turn homotopies of homotopies and so on. We refer to Section 3 of [20] for a detailed discussion on this notion. We will only say that this notion of homotopy between Lie $\infty$-morphisms coincides, when the Lie $\infty$-algebroids are of finite length, with the usual cylinder homotopies (see Proposition 3.61 in [20]). Homotopy is an equivalence relation – denoted $\sim$ – between Lie $\infty$-morphisms, which allows us to define a notion of equivalence between Lie $\infty$-algebroids:
Definition 2.6. Let $E$ and $E'$ be two Lie $\infty$-algebroids over $M$ and $\Phi : \Omega_{A}(E') \to \Omega_{A}(E)$ a Lie $\infty$-algebroid morphism between them. We say that $\Phi$ is a homotopy equivalence if there exists a Lie $\infty$-morphism $\Psi : \Omega_{A}(E) \to \Omega_{A}(E')$ such that

$$\Phi \circ \Psi \sim \text{id}_{\Omega(E)} \quad \text{and} \quad \Psi \circ \Phi \sim \text{id}_{\Omega(E')}.$$ 

In such a case, the Lie $\infty$-algebroids $E$ and $E'$ are said to be homotopy equivalent.

Remark. In particular, any homotopy equivalence between $E$ and $E'$ is a quasi-isomorphism, i.e. it is an isomorphism at the cohomology level. Since moreover any two homotopy equivalence are homotopic (see Theorem 2.8 in [20]) we deduce that this isomorphism at the cohomology level is canonical.

2.2 Representations up to homotopy of Lie $\infty$-algebroids

Representations up to homotopy of a Lie algebroid $A$ [1] are characterized by the fact that the vector bundle $K$ serving as the representation of $A$ is promoted to a graded vector bundle, and the $A$-connection $\nabla$ on $K$ becomes a family of ‘connections $k$-forms’. Then it is required that the curvatures associated to these ‘connections $k$-forms’ vanish only up to homotopy. Such notion of representation up to homotopy, although intricate, emerge in a very natural way in the definition of the adjoint representation of a Lie algebroid. Representations up to homotopy of Lie algebroids can be generalized to Lie $\infty$-algebroids [5,13], although it is a bit more involved due to the fact that a Lie $\infty$-algebroid is itself graded. For self-consistency of the paper, the present section is devoted to present this notion based on the graded skew-symmetric convention for Lie $\infty$-algebroids, building on the original notations of [1]. See also [23] for a discussion about the various notions of representations of Lie algebroids.

Let $E = \bigoplus_{i \geq 0} E_i$ be a Lie $\infty$-algebroid and let $K = \bigoplus_{j \in \mathbb{Z}} K_j$ be a graded vector bundle, both defined over a smooth manifold $M$. A $k$-form on $E$ taking values in $K$ of total degree $d$ is a finite sum of $k$-forms $\eta = \sum_{i=1}^{m} \eta_i$ of respective height $h_i \geq 0$ (as defined in Section 2.1), each of the $\eta_i$ taking values in some $K_j$, where $j_i$ is such that $d = h_i + j_i$. For example, a 3-form $\eta \in (E_a)^* \wedge (E_b)^* \wedge (E_c)^* \otimes K_j$ has a total degree of $3 + a + b + c + j$, where the integer 3 is the form degree and the height is $3 + a + b + c$. We emphasize the requirement that the sum should be finite.

One sets $\Omega^h_k(E, K_j)$ to be the space of $k$-forms of height $h$ taking values in $K_j$. For every integer $d$, one then sets $\Omega^k(E, K)_d$ to be the space of $K$-valued $k$-forms of total degree $d$:

$$\Omega^k(E, K)_d = \bigoplus_{h \geq k} \Omega^h_k(E, K_{d-h})$$

(2.13)

This notation should not be confused with the notation corresponding to singling out an element of degree $d$ of a $k$-form:

$$\Omega^k(E, K)_d \neq \Omega^k(E, K)|_d$$

(2.14)

In the former case, one takes into account the form degree $k$ to compute $d$, while in the latter case, one only sums the contributions from $E^*$ and that from $K$, but does not take into account the form degree $k$. As the black triangle in $\Omega_{A}(E)$ denotes the height of a form, in the rest of the text, the black lozenge will denote the total degree:

$$\Omega^\bullet(E, K)_\bullet = \bigoplus_{k \geq 0} \bigoplus_{d \in \mathbb{Z}} \Omega^k(E, K)_d$$

Sometimes the latter space $\Omega^\bullet(E, K)_\bullet$ is denoted $\Omega(E, K)_\bullet$ when emphasis is made on the total degree.
A Lie algebroid being a Lie ∞-algebroid concentrated in degree 0, the only degrees that enter the total degree are the form (polynomial) degree and the grading of $K$, which is actually what appears for representations up to homotopy of Lie algebroids [1]. We now have enough material to generalize the latter notion to Lie ∞-algebroids [5, 13]:

**Definition 2.7.** A representation up to homotopy of a Lie ∞-algebroid $E$ is a graded vector bundle $K$ over $M$ together with a linear map $D: \Omega(E, K)_\bullet \to \Omega(E, K)_{\bullet+1}$ of total degree $+1$ satisfying:

$$D(\eta \wedge u) = d_E(\eta) \wedge u + (-1)^m \eta \wedge D(u)$$  \hspace{1cm} (2.15)

for every $\eta \in \Omega_m(E)$ and $u \in \Omega(E, K)$, and such that $D^2 = 0$.

The differential $D$ is more intricate than in the Lie algebroid case. Here, the operator $D$ splits into a sum $D = \sum_{k \geq 0} D^{(k)}$ where each component is a $\mathbb{R}$-linear map $D^{(k)}: \Omega^*(E, K)_\bullet \to \Omega^{*+k}(E, K)_{\bullet+1}$ of arity $k$ and of total degree $+1$. Each of these operators $D^{(k)}$ can be further decomposed with respect to the height: $D^{(k)} = \sum_{h \geq k} D^{(k)}_h$, where each component is a $\mathbb{R}$-linear map:

$$D^{(k)}_h: \Omega^*_h(E, K)_\bullet \to \Omega^{*+k}_h(E, K)_{\bullet+1}$$  \hspace{1cm} (2.16)

Since the vector bundle $K$ is graded, the vector bundle of fiberwise endomorphisms of $K$ is graded as well: $\text{End}(K) = \bigoplus_{i \in \mathbb{Z}} \text{End}(K)_i$. For every $k \neq 1$, the restriction of the action of $D^{(k)}_h$ to $\Gamma(K)$ canonically induces an $\text{End}(K)_{1-h}$-valued $k$-form $\omega^{(k)}_h$ on $E$:

$$D^{(k)}_h: \Gamma(K) \longrightarrow \Omega^*_h(E, K_{1-h}) \iff \omega^{(k)}_h \in \Omega^*_h(E, \text{End}(K)_{1-h})$$ \hspace{1cm} (2.17)

where the space on the right hand side should be understood as:

$$\Omega^*_h(E, \text{End}(K)_{1-h}) = \sum_{i_1 + \ldots + i_k = h-k} \Gamma((E^*)_i \wedge \ldots \wedge (E^*)_k) \otimes \text{End}(K)_{1-h}$$

The 0-connection $\omega^{(0)} \in \Gamma(\text{End}_1(K))$, being a zero-form, has height 0. Being of total degree $+1$ implies that it is a degree $+1$ vector bundle morphism on $K$ that we denote $\partial: K_\bullet \to K_{\bullet+1}$. For $k \geq 2$, the correspondence between $D^{(k)}$ and $\omega^{(k)}$ satisfies the following rule:

$$D^{(k)}(u) = \omega^{(k)} \wedge u \text{ for } k \neq 1$$ \hspace{1cm} (2.18)

for every $u \in \Omega(E, K)$. Here, the wedge product means that we apply a concatenation on the $E$-form part of $u$, and that $\omega^{(k)}$ acts as an endomorphism on the part of $u$ sitting in $K$. We call connection $k$-form the $\text{End}(K)$-valued $k$-form $\omega^{(k)}$. For $k = 1$, the operator $D^{(1)}$ is the exterior covariant derivative associated to what we call an $E$-connection, that is to say, a family of differential operators $(\nabla^i: \Gamma(E_{-i}) \times \Gamma(K_\bullet) \to \Gamma(K_{\bullet-i}))_{i \geq 0}$ satisfying the usual axioms of a connection:

$$\nabla^i f(s) = f \nabla^i_e(s)$$ \hspace{1cm} (2.19)

$$\nabla^i_e(f s) = f \nabla^i_e(s) + \rho(e)[f] s$$ \hspace{1cm} (2.20)

for every $f \in C^\infty, e \in \Gamma(E_{-i})$ and $s \in \Gamma(K)$, and where for consistency we assume that $\rho|_{E_{-i}} = 0$ if $i \geq 1$. The exterior covariant derivative $D^{(1)}$ is also denoted $d^E$.

**Remark.** When the grading of the graded vector bundle $K$ is bounded (which will be the case in the present paper), then there exists an open covering of $M$ by open sets $U_\alpha$ trivializing $K$. Over such a covering, to every differential operator $\nabla^i$ corresponds a family of locally defined 1-forms
\[ \omega^{(1),i}_{\alpha} \in \Omega^1(E_{-i}|U_\alpha, \text{End}_{-i}(K)|U_\alpha). \] On the overlap \( U_\alpha \cap U_\beta \), the family of local 1-forms \( \omega^{(1),i}_{\alpha} \) satisfies the following conditions, depending on the value of \( i \):

\[ \omega^{(1),i}_{\beta} = g_{\alpha\beta}^{-1} \omega^{(1),i}_{\alpha} g_{\alpha\beta} \quad \text{if} \quad i \geq 1 \quad \text{and} \quad \omega^{(1),0}_{\beta} = g_{\alpha\beta}^{-1} d^{(1)}_E g_{\alpha\beta} + g_{\alpha\beta}^{-1} \omega^{(1),0}_{\alpha} g_{\alpha\beta} \quad \text{otherwise} \quad (2.21) \]

where \( g_{\alpha\beta} \) represents the transition map of \( K \) over \( U_\alpha \cap U_\beta \). A family of locally defined 1-forms \( (\omega^{(1),i}_{k,\alpha})_{i,\alpha} \) satisfying Conditions (2.21) is abusively called a \textit{connection 1-form} and denoted \( \omega^{(1)} \) in the rest of the text.

When decomposing the equation \( D^2 = 0 \) by form degree, one obtains for all \( k \geq 0 \):

\[ \sum_{l=0}^{k} D^{(l)} D^{(k-l)} = 0 \quad (2.22) \]

For \( k = 0 \), this equation means that \( \partial : K_s \rightarrow K_{s+1} \) is a differential, turning \( K \) into a chain complex. Let us extend this differential by setting \( \partial : \Omega^\bullet(E, \text{End}(K))_s \rightarrow \Omega^\bullet(E, \text{End}(K))_{s+1} \) to be the \( C^\infty \)-linear map of total degree +1 defined by:

\[ \partial(\alpha)(a_1, \ldots, a_k; u) = \partial(\alpha(a_1, \ldots, a_k; u)) - (-1)^{|\alpha| + |a_1| + \ldots + |a_k|} \alpha(a_1, \ldots, a_k; \partial(u)) - \sum_{j=1}^{k} (-1)^{|\alpha| + \ldots + |a_{j-1}|} \alpha(a_1, \ldots, l_1(a_j), \ldots, a_k; u) \quad (2.23) \]

for every \( k \geq 0 \), \( \alpha \in \Omega^k(E, \text{End}(K)), a_1, \ldots, a_k \in \Gamma(E) \) and \( u \in \Gamma(K) \). The symbol \( |\alpha| \) denotes the total degree of \( \alpha \) while the symbols \( |a_j| \) denote the degree of the homogeneous element \( a_j \in \Gamma(E) \). More precisely, if \( \alpha \) is a \( \text{End}(K) \)-valued \( k \)-form of height \( h \) and of total degree \( d \), i.e., if \( \alpha \in \Omega^k_h(E, \text{End}(K)_{d-h}) \), then \( \partial(\alpha) \in \Omega^k_h(E, \text{End}(K)_{d-h+1}) \oplus \Omega^k_{h+1}(E, \text{End}(K)_{d-h}) \). A straightforward calculation shows that \( \partial^2 = 0 \), turning \( \Omega^k(E, \text{End}(K))_s \) into a chain complex for each \( k \geq 0 \).

Next, Equation (2.22) for \( k = 1 \) means that the \( E \)-connection \( \nabla \) on \( K \) is such that, for every \( a \in \Gamma(E) \):

\[ \partial \circ \nabla_a - (-1)^{|a|} \nabla_a \circ \partial - \nabla_{l_1(a)} = 0 \quad (2.24) \]

This can be equivalently written as \( d^{(1)}_E \omega^{(0)} + \partial(\omega^{(1)}) = 0 \). For \( k = 2 \), Equation (2.22) reads:

\[ R_{\nabla} + \partial(\omega^{(2)}) = 0 \quad (2.25) \]

More generally, to every connection \( k - 1 \)-form \( \omega^{(k-1)} \) is associated a \textit{curvature} \( k \)-form \( R^{(k)} \) defined by:

\[ R^{(1)} = d^{(1)}_E \omega^{(0)} \quad (2.26) \]

\[ R^{(k)} = \sum_{1 \leq i, l \leq k-1, \sum_{s+l=k} 1 \leq s \leq l} d^{(s)}_E \omega^{(l)} + \omega^{(s)} \circ \omega^{(l)} \quad (2.27) \]

Each curvature has total degree 2, while the index \( k \) indicates the form degree of \( R^{(k)} \). Then, the system of equations (2.22) can be written in a way synthesizing that of Proposition 3.2 in [1]:

**Proposition 2.8.** There is a one-to-one correspondence between representations up to homotopy \((K, D)\) of \( E \), and graded vector bundles \( K \) endowed with a fiberwise linear differential \( \partial : K_s \rightarrow K_{s+1} \), an \( E \)-connection \( \nabla \) on \( K \), and a family of \( \text{End}(K) \)-valued \( k \)-forms on \( E \) \((\omega^{(k)})_{k \geq 2} \) of total degree +1 satisfying the following set of equations:

\[ R^{(k)} + \partial(\omega^{(k)}) = 0 \quad \text{for every} \quad k \geq 1 \quad (2.28) \]
Remark. As for Lie algebroids [1], representations up to homotopy of Lie ∞-algebroids are characterized by the fact that, for every \( k \geq 1 \), the curvature \( R^{(k)} \) corresponding to the connection \( k - 1 \)-form is flat up to homotopy, and the homotopy is the connection \( k \)-form \( \omega^{(k)} \). This implies that \( R^{(k)} \) is a 2-coboundary in the chain complex \( (\Omega^k(E, \text{End}(K)))_\bullet, \delta \).

Example 1. As an example let us now present what is the adjoint representation of a Lie ∞-algebroid \((E, l_k, \rho)\), under our notations. It is defined on the graded vector bundle \( K = E \oplus sTM \), that is to say: \( K_{-i} = E_{-i} \) for every \( i \geq 0 \), and \( K_1 = sTM = TM[-1] \) where \( s \) is the suspension operator increasing the degree of \( TM \) by +1. We can then define a differential \( \partial : K_\bullet \to K_{\bullet+1} \) by the following requirements:

\[
\partial|_{K_{-i}} = \begin{cases} 
|_E \quad & \text{for } i \geq 1 \\
\circ \rho & \text{for } i = 0 
\end{cases}
\]

This differential turns \( K = E \oplus sTM \) into a chain complex.

The possibility of defining the adjoint representation of \( E \) relies on a choice of a \( TM \)-connection \( \nabla \) on \( E \). Using \( \nabla \), let us define an \( E \)-connection \( \nabla \) on \( K \):

\[
\nabla_a(b) = l_2(a, b) + \nabla_{\rho(b)}(a) \quad (2.29)
\]

\[
\nabla_a(sX) = s[\rho(a), X] + [\partial, \nabla_X](a) \quad (2.30)
\]

for every \( a, b \in \Gamma(E) \) and \( X \in \mathfrak{X}(M) \), and where \( \rho|_{E_{-i}} = 0 \) if \( i \geq 1 \). In particular, if \( |b| \leq -1 \) then the right-hand side of Equation (2.29) reduces to the usual bracket \( l_2(a, b) \) while if \( |a| \leq -1 \) the right-hand side of Equation (2.30) reduces to \( -\nabla_X(l_1(a)) \). Following [1], we call this connection \( \nabla \) the basic connection associated to \( \nabla \). We had to emphasize the suspension of \( X \) in Equation (2.30) in order to make explicit the fact that the basic connection is a degree 0 operator.

As in the Lie algebroid case, we can define a basic curvature associated to the \( TM \)-connection \( \nabla \). The basic 2-curvature \( \overline{R}^{(2)} \) measures the ‘compatibility’ between this connection and the 2-bracket \( l_2 \), corrected by some terms that makes it \( C^\infty \)-linear [1]:

\[
\overline{R}^{(2)}(a, b)(sX) = \nabla_X(l_2(a, b)) - l_2(\nabla_X(a), b) - l_2(a, \nabla_X(b)) - \nabla_{s^{-1}\nabla_{(a)}(sX)}(a) + \nabla_{s^{-1}\nabla_{(b)}(sX)}(b) \quad (2.31)
\]

for every \( a, b \in \Gamma(E) \) and for every \( X \in \mathfrak{X}(M) \). Since the right-hand side has degree \(|a| + |b| \), while on the left-hand the degrees of the arguments sum up to \(|a| + |b| + 1 \) (because \(|sX| = 1 \)), the basic 2-curvature \( \overline{R}^{(2)} \) is a 2-form taking values in \( \text{Hom}_{-1}(sTM, E) \), i.e. it has total degree +1: \( \overline{R}^{(2)} \in \Omega^2(E, \text{Hom}(sTM, E))_{+1} \). Now recall that a Lie \( \infty \)-algebroid comes equipped with a set of graded skew-symmetric brackets \((l_k)_{k \geq 3}\). Hence, we can define a whole set of higher basic curvatures \( \overline{R}^{(k)} \) measuring the compatibility of \( \nabla \) with the \( k \)-brackets. More precisely, for every \( k \geq 3 \), and for every \( a_1, \ldots, a_k \in \Gamma(E) \) and \( X \in \mathfrak{X}(M) \), we define the \( k \)-th basic curvature \( \overline{R}^{(k)} \) as:

\[
\overline{R}^{(k)}(a_1, \ldots, a_k)(sX) = \nabla_X(l_k(a_1, \ldots, a_k)) - \sum_{i=1}^{k} l_k(a_1, \ldots, \nabla_X(a_i), \ldots, a_k) \quad (2.32)
\]

Notice that here, contrary to the basic 2-curvature \( \overline{R}^{(2)} \), we don’t need any correcting term because the higher brackets are naturally \( C^\infty \)-linear. Comparing the degree of the arguments on left-hand side and on the right-hand side, we deduce that \( \overline{R}^{(k)} \) if a \( k \)-form on \( E \) taking values in \( \text{Hom}_{-k}(sTM, E) \), i.e. it has total degree +1 as \( \overline{R}^{(2)}: \overline{R}^{(k)} \in \Omega^k(E, \text{Hom}(sTM, E))_{+1} \).
The adjoint representation of $E$ (with respect to the connection $\nabla$) will be defined on the graded vector bundle $K = E \oplus sTM$ as follows: first, we set $D^{(0)} = \partial$, $D^{(1)} = d\nabla$ and for every $k \geq 2$ we define the connection $k$-form $\omega^{(k)}$ as:

$$\omega^{(k)}(a_1, \ldots, a_k) = l_{k+1}(a_1, \ldots, a_k, \cdot) + \mathcal{R}_{(k)}(a_1, \ldots, a_k)(\cdot)$$

By definition, $\omega^{(k)}(a_1, \ldots, a_k)$ is a graded endomorphism of $K = E \oplus sTM$. The first term $l_{k+1}(a_1, \ldots, a_k, \cdot)$ (resp. $\mathcal{R}_{(k)}(a_1, \ldots, a_k)$) defines the action of $\omega^{(k)}(a_1, \ldots, a_k)$ on $E$ (resp. $sTM$). One verifies that this family $(\omega^{(k)})_{k \geq 2}$ of $\text{End}(K)$-valued $k$-forms of total degree $k$ induces the desired family of operators $D^{(k)}$ via the correspondence (2.18). This adjoint representation straightforwardly generalizes that of Lie algebroids [1], and coincides with the one defined with respect to the graded symmetric convention for Lie algebroids appearing in Proposition 3.27 of [5]. The adjoint representation of $E$ presented above depends on the choice of $TM$-connection $\nabla$ so it is not unique. However, as in Section 3.2 of [1], representations of the Lie $\infty$-algebroid $E$ forms a category and the isomorphism classes of adjoint representations of $E$ (with respect to homotopy equivalences) is the ‘true’ adjoint representation of $E$ (see also Section 5 of [13]).

### 2.3 Singular foliations and their foliated cohomology

Although most of the material of this section has been introduced in [20], we provide it again for consistency, and develop it so that it fits our current objective: to define the modular class of a singular foliation. Let $M$ be a smooth manifold, $C^\infty: U \to C^\infty(U)$ be the sheaf of smooth functions on $M$, and $\mathcal{X}: U \to \mathcal{X}(U)$ be the sheaf of smooth vector fields on $M$.

**Definition 2.9.** A singular foliation is a $C^\infty$-subsheaf $\mathcal{F}: U \to \mathcal{F}(U)$ of the sheaf of vector fields $\mathcal{X}$, which is:

1. closed under the Lie bracket of vector fields,
2. locally finitely generated as a $C^\infty$-module.

A singular foliation induces a generalized distribution, i.e. the (smooth) assignment $D: x \mapsto D_x \subset T_x M$, for every $x \in M$, of a subspace of the tangent space of $M$ at $x$. The rank of this distribution may not be constant over $M$. We define a leaf as an immersed – in fact, weakly embedded – submanifold $L$ of $M$ such that $T_x L = D_x$ for every $x \in L$. By Hermann’s theorem, finitely generated involutive generalized distributions are integrable [11], i.e. leaves of a singular foliation in the sense of Definition 2.9 form a partition of $M$ having the expected diffeological properties generalizing that of regular foliations [25]. This represents one of the generalizations of Frobenius’s integrability theorem to singular foliations [22].

A regular leaf is a leaf $L$ for which every leaf in the vicinity has the same dimension as $L$, i.e. it is such that for every $x \in L$, there exists a small neighborhood of $x$ on which the map $x \mapsto \dim(D_x)$ is locally constant. The dimension of the regular leaves may not be constant over $M$. The union of all the regular leaves forms however an open dense subset in $M$, and the rank of the distribution $D$ is constant on each connected component of this subset. On the contrary, a singular leaf is a leaf which is not a regular leaf. In particular, for any point $y$ in a singular leaf $L$, any neighborhood of $y$ admits an non-empty intersection with a regular leaf – which has dimension strictly higher than $\dim(L)$, because the function $x \mapsto \dim(D_x)$ taking values in the integers is lower semi-continuous. Most examples of singular foliations in the sense of Definition 2.9 come from almost Lie algebroids:
Definition 2.10. An almost Lie algebroid is a vector bundle $A$, together with a bilinear skew-symmetric bracket $l_2$ defined on the sections of $A$ and an anchor map $\rho: A \rightarrow TM$ satisfying Equations (2.5) and (2.7). The image by the anchor map of the sections of $A$ is a singular foliation $\mathcal{F}_A = \rho(\Gamma(A))$ called the characteristic foliation of $A$. Given a singular foliation $\mathcal{F}$, we say that an almost Lie algebroid $A$ covers $\mathcal{F}$ if $\mathcal{F}_A = \mathcal{F}$.

Remark. Contrary to Lie algebroids, the bracket $l_2$ needs not satisfy the Jacobi identity. A typical example of an almost Lie algebroid is the zeroth component $E_0$ of a Lie $\infty$-algebroid $E = \bigoplus_{i \geq 0} E_{-i}$.

Now, it may happen that a singular foliation $\mathcal{F}$ admits a geometric resolution [20]: it is a family of non-positively graded vector bundles $E = (E_{-i})_{i \geq 0}$, together with a vector bundle morphism $d = E_* \rightarrow E_{*+1}$ acting as a differential, and an anchor map $\rho: E_0 \rightarrow TM$, such that for every open set $U \subset M$, the following chain complex:

$$\cdots \xrightarrow{d} \Gamma_U(E_{-2}) \xrightarrow{d} \Gamma_U(E_{-1}) \xrightarrow{d} \Gamma_U(E_0) \xrightarrow{\rho} \mathcal{F}(U) \xrightarrow{0}$$

is an exact sequence of $C^\infty(U)$-modules. Not every singular foliation admits a geometric foliation (see Example 3.38 in [20]). We say that a singular foliation of $\mathcal{F}$ is solvable when it admits a geometric resolution of finite length. Such resolutions are a natural generalization of resolutions of integrable regular distributions of the form (3.32) to the setup of singular foliations. This justifies the introduction of the following notion:

Definition 2.11. Let $\mathcal{F}$ be a singular foliation on a smooth manifold $M$, and let $(E, l_k, \rho)$ be a Lie $\infty$-algebroid over $M$ covering $\mathcal{F}$. We say that $E$ is a universal Lie $\infty$-algebroid of $\mathcal{F}$ if its linear part is a geometric resolution of $\mathcal{F}$. The set of all universal Lie $\infty$-algebroids of $\mathcal{F}$ is denoted $\mathcal{E}_\mathcal{F}$.

The problem of existence and unicity of such universal Lie $\infty$-algebroids has been answered in [20]. For the sake of completeness, we recall here the most useful and central results:

Theorem 2.12. Existence. Let $\mathcal{F}$ be a singular foliation admitting a geometric resolution $(E, d)$. Then there exists a Lie $\infty$-algebroid structure on $E$ such that:

1. its linear part is the afore-mentioned geometric resolution, and
2. it is a universal Lie $\infty$-algebroid of $\mathcal{F}$.

Theorem 2.12 is a result about existence of universal Lie $\infty$-algebroids. As a particular case of frequent interest, we have singular foliations which are locally real analytic. A singular foliation $\mathcal{F}$ is said real analytic if there exists an open cover of $M$ such that the generators of $\mathcal{F}$ only involve real analytic functions of the local coordinates associated to each chart. Then, for every $x \in M$ there exists a universal Lie $\infty$-algebroid of $\mathcal{F}$ of length at most $n+1$ defined over some neighborhood of $x$. The following theorem is a result about the universal property of universal Lie $\infty$-algebroids:

Theorem 2.13. Unicity. Let $\mathcal{F}$ be a singular foliation on a smooth manifold $M$ and let $E$ be a universal Lie $\infty$-algebroid of $\mathcal{F}$. Then for every Lie $\infty$-algebroid $E'$ covering $\mathcal{F}$, there exists a Lie $\infty$-algebroid morphism over $M$ from $E'$ to $E$ and any two such Lie $\infty$-algebroid morphisms are homotopic.

---

2It is finitely generated as $A$ is a vector bundle, and it is involutive by Equation (2.7), see [20] for details.
Hence, universal Lie ∞-algebroids of a singular foliation are in some sense unique: any two universal Lie ∞-algebroids $E$ and $E'$ of $\mathcal{F}$ are homotopically equivalent, in the sense of Definition 2.6. This universality property found another formulation in category theory [9]; in the semi-model category of $L_\infty$-algebroids over the smooth manifold $M$ (these are mild generalizations of Lie ∞-algebroids), a universal Lie $\infty$-algebroid of the singular foliation $\mathcal{F}$ is a replacement of $\mathcal{F}$. See also [21] for another discussion about equivalence of categories between Lie-Rinehart algebras over a commutative algebra and homotopy equivalence classes of non-positively graded Lie $\infty$-algebroids over their resolutions.

The second statement of Theorem 2.13 provides another important information about equivalences of universal Lie $\infty$-algebroids: that any two homotopy equivalences between universal Lie $\infty$-algebroids are homotopic. This implies that the Lie $\infty$-algebroid cohomologies of two universal Lie $\infty$-algebroids of $\mathcal{F}$ are canonically isomorphic as graded commutative algebras [20]. For two such Lie $\infty$-algebroids $E$ and $E'$, we note $\Phi_{E,E'} : H^\bullet(E') \to H^\bullet(E)$ this canonical isomorphism. Hence, we can define an equivalence relation $\sim$ on the space $\bigcup_{E \in \mathfrak{E}_\mathcal{F}} H(E)$ by:

$$x \sim y \quad \text{if and only if} \quad x = \hat{\Phi}_{E,E'}(y)$$

where $x \in H(E)$ and $y \in H(E')$. This equivalence relation provides the singular foliation $\mathcal{F}$ with a cohomology:

**Definition 2.14.** Let $\mathcal{F}$ be a singular foliation admitting a geometric resolution. We call universal foliated cohomology of $\mathcal{F}$ and denote by $H_\mathfrak{U}(\mathcal{F})$ the set of equivalence classes of $\bigcup_{E \in \mathfrak{E}_\mathcal{F}} H(E)$ with respect to the equivalence relation $\sim$.

This definition respects the graduation defined by the height, i.e. for every $h \geq 0$, the space $H^h_\mathfrak{U}(\mathcal{F})$ is the $\mathbb{R}$-vector space freely generated by the set of equivalence classes of $\bigcup_{E \in \mathfrak{E}_\mathcal{F}} H^h(E)$. This is well defined because for any universal Lie $\infty$-algebroids $E$ and $E'$, the canonical isomorphism $\hat{\Phi}_{E,E'} : H^\bullet(E') \to H^\bullet(E)$ preserves the height. For every $h \geq 0$ and every $E \in \mathfrak{E}_\mathcal{F}$, there is a canonical isomorphism between $H^h_\mathfrak{U}(\mathcal{F})$ and $H^h(E)$ which corresponds to picking up the unique representant $x^E \in H^h(E)$ of the equivalence class $[x] \in H^h_\mathfrak{U}(\mathcal{F})$. Hence, $H^h_\mathfrak{U}(\mathcal{F})$ is finite dimensional as it has the same dimension as $H^h(E)$, which does not depend on the choice of Lie $\infty$-algebroid $E \in \mathfrak{E}_\mathcal{F}$. The universal foliated cohomology $H_\mathfrak{U}(\mathcal{F})$ inherits a canonical graded commutative algebra structure from the one on $H(E)$.

The modular class of a regular foliation is an element of the first group of the foliated de Rham cohomology of the corresponding involutive regular distribution. We would like to reproduce such a statement for singular foliations, but the presence of singularities compels us to make a detour via the universal foliated cohomology. Let us call forms on $\mathcal{F}$ and denote by $\Omega^\bullet(\mathcal{F})$ or simply $\Omega(\mathcal{F})$ the space of $C^\infty$-multilinear skew-symmetric assignments from $\mathcal{F}$ to $C^\infty$:

$$\Omega^\bullet(\mathcal{F}) = \text{Hom}_{C^\infty}(\wedge^\bullet \mathcal{F}, C^\infty) = \bigoplus_{k \geq 0} \text{Hom}_{C^\infty}(\wedge^k \mathcal{F}, C^\infty).$$

Note that 0-forms on $\mathcal{F}$ are just functions on $M$. One can define a foliated de Rham differential $d_{\text{dR}}$ on $\Omega(\mathcal{F})$ from its action on any $k$-form $\alpha \in \Omega^k(\mathcal{F})$:

$$d_{\text{dR}}(\alpha)(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i X_i [\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)] + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k),$$

with the understanding that $\widehat{X_i}$ means that the term $X_i$ is omitted. We call the cohomology of this operator the foliated de Rham cohomology of $\mathcal{F}$ and denote it by $H^\bullet_{\text{dR}}(\mathcal{F})$. Obviously, the
0-th group of foliated de Rham cohomology corresponds to the smooth functions that are $\mathcal{F}$-invariant. When the singular foliation is regular and integrates an involutive regular distribution $F$, the foliated cohomology coincides with the foliated de Rham cohomology of $F$ or, equivalently, the Lie algebroid cohomology of the foliation Lie algebroid $F$.

Let $\mathcal{F}$ be a singular foliation on $M$, and let $(E, l_k, \rho)$ be a Lie $\infty$-algebroid covering $\mathcal{F}$ (i.e. not necessarily a resolution). The anchor map induces a map $\rho^*$ from $\Omega(\mathcal{F})$ to $\Omega(E)$ given by associating to each $\alpha \in \Omega^k(\mathcal{F})$ the element $\rho^* \alpha \in \Gamma(\wedge^k E^0_0) = \Omega^k_0(E)$ defined by:

$$\rho^* \alpha(a_1, \ldots, a_k) = \alpha(\rho(a_1), \ldots, \rho(a_k)) \quad (2.35)$$

for every $a_1, \ldots, a_k \in \Gamma(E_0)$. One can check that $\alpha \mapsto \rho^*(\alpha)$ is an injective chain map and a graded commutative algebra morphism (Lemma 4.5 in [20]), inducing therefore an algebra morphism, still denoted $\rho^*$, from $H_{\text{dR}}(\mathcal{F})$ to $H(E)$. The grading in $H_{\text{dR}}(\mathcal{F})$ is the form-degree, whereas in $H(E)$ it is the height, and the map $\rho^*$ preserves this grading. The link between the foliated de Rham cohomology and the universal foliated cohomology is induced by this map:

**Proposition 2.15.** Let $\mathcal{F}$ be a singular foliation on a smooth manifold admitting a geometric resolution. Then, there exists a canonical injective morphism of graded commutative algebras:

$$\rho^*_\mathcal{F} : H^\bullet_{\text{dR}}(\mathcal{F}) \longrightarrow H^\bullet_\mathcal{F}(\mathcal{F})$$

that is an isomorphism in degree 1.

**Proof.** Let $(E, l_k, \rho)$ and $(E', l'_k, \rho')$ be two universal Lie $\infty$-algebroids of the singular foliation $\mathcal{F}$. Then by Theorem 2.13, there exists a Lie $\infty$-morphism over $M$ from $E$ to $E'$, denoted $\Phi_{E,E'}$. Equation (2.11), applied to functions on $M$, implies that $\rho' = \Phi_{E,E'} \circ \rho^*$. Since this is true for every such Lie $\infty$-morphism from $E$ to $E'$, and since they are all homotopic to one another, the canonical isomorphism $\Phi_{E,E'} : H^\bullet(E') \rightarrow H^\bullet(E)$ that they induce at the cohomology level makes the following diagram commutative:

$\begin{tikzcd}
H_{\text{dR}}^\bullet(\mathcal{F}) \arrow[hookrightarrow]{r}{\rho^*_\mathcal{F}} \arrow[hookrightarrow]{d}{\tilde{\Phi}_{E,E'}} & H_\mathcal{F}^\bullet(\mathcal{F}) \arrow[hookrightarrow]{d}{\rho^*} \\
H^\bullet(E') \arrow[hookrightarrow]{r}{\rho^*} & H^\bullet(E)
\end{tikzcd}$

From this diagram, we deduce that for any $\alpha \in H_{\text{dR}}^\bullet(\mathcal{F})$, $\rho^*(\alpha)$ and $\rho'^*(\alpha)$ induce the same element in the universal foliated cohomology $H_\mathcal{F}(\mathcal{F})$. We call this element $\rho^*_\mathcal{F}(\alpha)$. This assignment defines uniquely the height-preserving map $\rho^*_\mathcal{F} : H_{\text{dR}}^\bullet(\mathcal{F}) \rightarrow H_\mathcal{F}^\bullet(\mathcal{F})$ that we are looking for. Now, the map $\rho^* : \Omega^k(\mathcal{F}) \rightarrow \Omega^k_0(E)$ is injective since $\rho$ is surjective. This result being true for every universal Lie $\infty$-algebroid of $\mathcal{F}$, we conclude that the maps $\rho^*_\mathcal{F}$ is injective.

Let us now show that it is an isomorphism at level 1. The fact that $\rho^* : \Omega^1(\mathcal{F}) \rightarrow \Omega^1_0(E)$ is injective implies that it is as well at the cohomology level. Let us now show that it is surjective on cohomology. Let $u \in \Omega^1_0(E) = \Gamma(E^0_0)$ be a $d_E$-cocycle, then in particular it means that $d_E^{(0)}(u) = 0$, that is: $u|_{l_1(\Gamma(E_{-1}))} = 0$. Since we have the equality $\text{Im} (l_1(\Gamma(E_{-1}))) = \text{Ker}(\rho)$, then $u$ vanishes on the kernel of the anchor map. Then it comes from a 1-form on $\mathcal{F}$, i.e. there exists some $\alpha \in \Omega^1(\mathcal{F})$ such that:

$$u = \rho^*(\alpha) \quad (2.36)$$

This element is unique since $\rho^*$ is injective on $\Omega^1(\mathcal{F})$. Since $u$ is a $d_E$-cocycle, we also have that $d_E^{(1)}u = 0$, that is: $u([a,b]) = \rho(a)[u(b)] - \rho(b)[u(a)]$ which, by virtue of Equation (2.36), can
also be written as $\alpha([\rho(a), \rho(b)]) = \rho(a)[\alpha(\rho(b))] - \rho(b)[\alpha(\rho(a))]$. This is precisely the condition that $\alpha$ is a cocycle in $\Omega^1(\mathcal{F})$. Then, since $\rho^*$ is a chain map, and since it is injective, we deduce that the closed 1-forms on $\mathcal{F}$ are in one-to-one correspondence with the closed 1-forms on $E$ of degree 0. For the same reasons, exact 1-forms are in one-to-one correspondence as well. This means that the map $\rho^* : \Omega^1(\mathcal{F}) \to \Omega^1_1(E)$ is bijective at the cohomology level. This implies that the canonically induced morphism $\rho_F^*$ is an isomorphism in degree 1.

**Remark.** Actually, $\rho_\mathcal{F}$ is also bijective at level 0, and it is even the identity map. This is rather obvious since in both case the zero-th cohomology group consists of the functions that are $\mathcal{F}$-invariant.

The newly defined map $\rho_\mathcal{F}^*$ actually has the following universal property:

**Proposition 2.16.** Let $\mathcal{F}$ be a singular foliation admitting a geometric resolution on a smooth manifold $M$, and let $E$ be a Lie $\infty$-algebroid covering $\mathcal{F}$, with anchor map $\rho$. Then the map $\rho^* : H_{dR}(\mathcal{F}) \to H(E)$ defined by Equation (2.35) factors through $H_\mathcal{U}(\mathcal{F})$:

\[
\begin{array}{ccc}
H_{dR}(\mathcal{F}) & \xrightarrow{\rho^*} & H(E) \\
\downarrow & & \downarrow \rho^* \\
H_\mathcal{U}(\mathcal{F}) & \xrightarrow{\rho_\mathcal{F}^*} & H(E)
\end{array}
\]

**Proof.** Let $E'$ (resp. $E''$) be a universal Lie $\infty$-algebroid of $\mathcal{F}$, and let $\Phi_{E,E'} : \Omega(E') \to \Omega(E)$ (resp. $\Phi_{E,E''} : \Omega(E'') \to \Omega(E)$) be an Lie $\infty$-morphism over $M$ between $E$ and $E'$ (resp. $E''$), whose existence is guaranteed by Theorem 2.13. The same theorem insures that any two such morphisms induce a unique graded commutative algebra morphism $\tilde{\Phi}_{E,E'} : H(E') \to H(E)$ (resp. $\tilde{\Phi}_{E,E''} : H(E'') \to H(E)$) at the cohomology level. They are such that they make the following diagram commutative:

\[
\begin{array}{ccc}
H_{dR}(\mathcal{F}) & \xrightarrow{\rho^*} & H(E) \\
\downarrow & & \downarrow \rho^* \\
H(E) & \xrightarrow{\Phi_{E,E'}} & H(E') \\
\downarrow & & \downarrow \Phi_{E',E''} \\
H(E) & \xrightarrow{\Phi_{E,E''}} & H(E'')
\end{array}
\]

Since $\tilde{\Phi}_{E,E''} = \tilde{\Phi}_{E,E'} \circ \tilde{\Phi}_{E',E''}$, when passing to the universal foliated cohomology by identifying $H(E')$ and $H(E'')$ through $\tilde{\Phi}_{E',E''}$, the graded commutative algebra morphisms $\Phi_{E,E'}$ and $\tilde{\Phi}_{E,E''}$ induce a canonical graded commutative algebra morphism $\tilde{\Phi} : H_\mathcal{U}(\mathcal{F}) \to H(E)$. By commutativity of the diagram and by definition of the map $\rho_\mathcal{F}$, this morphism satisfies the identity $\rho^* = \Phi \circ \rho_\mathcal{F}^*$. \qed
3 A perspective on modular classes

3.1 The modular class of a Lie $\infty$-algebroid

In this section we adapt the definition of modular classes of Lie algebroids [8] to the Lie $\infty$-algebroid context. Modular classes of Lie algebroids have been defined as the natural counterpart of the notion of modular vector fields of Poisson manifolds [32]. It is conjectured that it measures the obstruction of the existence of a volume form on the differentiable stack associated to the groupoid integrating the Lie algebroid [7,33]. The generalization of the notion of modular class to the Lie $\infty$-algebroid context has already been investigated in [5] under the graded symmetric convention for $L_\infty$ brackets, and we merely summarize some of their results here, with the graded skew-symmetric convention.

Let us first apply the notion of representations up to homotopy to line bundles. Let $E$ be a Lie $\infty$-algebroid over $M$ and let $L$ be a line bundle over $M$ and a representation of $E$, that we assume to be concentrated in degree 0. Then, for degree reasons, the only sub-bundle of $E$ that acts on $L$ is the almost Lie algebroid $E_0$. The notion of Lie algebroid connection straightforwardly extends to almost Lie algebroids when an additional condition is added:

**Definition 3.1.** Given a (possibly graded) vector bundle $K \to M$, and an almost Lie algebroid $A$ over $M$, a (degree 0) differential operator $\nabla: \Gamma(A) \times \Gamma(K) \to \Gamma(K)$ satisfying axioms (2.19) and (2.20) is called an $A$-connection on $K$. Such a connection is flat if the corresponding curvature vanishes:

$$R_\nabla(a,b) = [\nabla_a, \nabla_b] - \nabla_{l_2(a,b)} = 0$$

for every $a,b \in \Gamma(A)$ (3.1)

Graded vector bundles $K \to M$ with flat $A$-connections are called representations of $A$ or $A$-modules\(^3\). We say that a representation $K$ is trivial when there exists a global frame of $K$ on which the action of $A$ is zero.

The covariant derivative associated to the connection $\nabla$ satisfies the following behavior:

$$(d\nabla)^2 f(a,b) = R_\nabla(a,b)(f)$$

$$(d\nabla)^3 \alpha(a,b,c) = R_\nabla(a,b)(\alpha(c)) + \circ + \alpha(l_2(a,l_2(b,c)) + l_2(b,l_2(c,a)) + l_2(c,l_2(a,b)))$$

for $f \in \Gamma(K)$ and $\alpha \in \Omega^1(E,K)$. The last equation is expected because the 2-bracket $l_2$ does not satisfy the Jacobi identity and so there is no reason that $(d\nabla)^2 = R_\nabla$. This curvature then satisfies the following Bianchi-like identity:

$$d\nabla R_\nabla(a,b,c) = [\nabla_a, [\nabla_b, \nabla_c]] + \circ - \nabla_{l_2(a,l_2(b,c)) + l_2(b,l_2(c,a)) + l_2(c,l_2(a,b))}$$ (3.2)

The first term on the right-hand side (including the circular permutations) is necessarily zero as a Jacobiator of operators while the second term vanishes if and only if $d\nabla R_\nabla = 0$, which is the case at least when $\nabla$ is flat.

The notion of representations of almost Lie algebroids obviously gives back the usual notion of representations of Lie algebroids when $A$ is a Lie algebroid. Trivial representations of $A$ are characterized by the existence of a global frame which is invariant under the action of $A$. However, the action of $A$ has no reason to be trivial on another global frame, even on a constant one. It is then important to distinguish trivialness as a vector bundle and trivialness as a $A$-module. Examples of representations of almost Lie algebroids can be induced by representations of Lie $\infty$-algebroids on line bundles:

\(^3\)The latter denomination may not be standard – see e.g. [23] – but we introduce and use it for its convenience.
Lemma 3.2. Any representation up to homotopy of the Lie ∞-algebroid $E = \bigoplus_{i \geq 0} E_{-i}$ over $M$ on a line bundle $L \to M$ canonically induces a representation of the almost Lie algebroid $E_0$ on $L$.

Proof. Assume that $L$ is a representation up to homotopy of $E$ concentrated in degree 0. For degree reasons, the only space of $E$ acting on $L$ is $E_0$, and the only connection $k$-form that is not zero is the connection 1-form $\omega^{(1)}$. Thus, there exists an $E_0$-connection $\nabla : \Gamma(E_0) \times \Gamma(L) \to \Gamma(L)$ satisfying the following set of equations:

\begin{align}
\nabla_{l_1(u)} &= 0 \quad (3.3)

[\nabla_a, \nabla_b] &= \nabla_{l_2(a,b)} \quad (3.4)
\end{align}

for every $a, b \in \Gamma(E_0)$ and $u \in \Gamma(E_{-1})$. The connection is flat because of Equation (3.4).

Eventually notice that Equation (3.2) is satisfied since the higher Jacobi identity (2.1) for three elements of $E_0$ is:

\begin{align}
l_2(a, l_2(b, c)) + l_2(b, l_2(c, a)) + l_2(c, l_2(a, b)) &= l_1(l_3(a, b, c)) \quad (3.5)
\end{align}

so that, for $u = l_3(a, b, c)$, Equation (3.3) implies that the last term of Equation (3.2) is zero, as should be since the left-hand side is zero too.

Assume for a time that the line bundle $L$ is trivial as a vector bundle, and let $\Omega$ be a nowhere vanishing global section. The action of $E$ on $\Omega$ – in fact only $E_0$ is acting – is again proportional to $\Omega$, so there exists a 1-form $\theta_\Omega \in \Gamma(E^*_0)$, called the modular 1-form (w.r.t. $\Omega$), satisfying:

\begin{align}
\nabla_a(\Omega) &= \theta_\Omega(a) \Omega \quad (3.6)
\end{align}

for every $a \in \Gamma(E_0)$. Equation (3.3) implies that:

\begin{align}
\theta_\Omega|_{\Gamma(l_1(E_{-1}))} &= 0 \quad (3.7)
\end{align}

Furthermore, Equation (3.4) is equivalent to:

\begin{align}
\rho(a)(\theta_\Omega(b)) - \rho(b)(\theta_\Omega(a)) - \theta_\Omega(l_2(a, b)) &= 0 \quad (3.8)
\end{align}

Using the definition of the differential $d_E$ as given in Equations (2.8) and (2.9), Equations (3.7) and (3.8) are equivalent, respectively, to:

\begin{align}
d_E^{(0)} \theta_\Omega &= 0 \quad \text{and} \quad d_E^{(1)} \theta_\Omega = 0 \quad (3.9)
\end{align}

By definition of the differential $d_E$, these two identities are in turn equivalent to the fact that $\theta_\Omega$ is $d_E$-closed. In particular it means that this 1-form is a cocycle and defines a class in $H^1(E)$. As in the Lie algebroid case, this cohomology class is invariant under the choice of section $\Omega$ of $L$; we denote it by $\theta_L$. When $L$ is not a trivial line bundle, we cannot define $\theta_L$ as above. However, following [8], the line bundle $L^2 = L \otimes L$ is trivial so that one can associate the following distinguished cohomology class to $L$:

\begin{align}
\theta_L &= \frac{1}{2} \theta_{L^2} \quad (3.10)
\end{align}

Hopefully, both definitions coincide when $L$ is trivial.

Definition 3.3. We call characteristic class of the line bundle $L$ the class $\theta_L \in H^1(E)$. 17
This characteristic class is thus attached to a particular representation of $E$ on $L$. The following proposition shows that it captures only the action of the almost Lie algebroid $E_0$, as no other vector bundle $E_{-i}$ acts on $L$.

**Proposition 3.4.** Let $L$ be a trivial line bundle and a representation up to homotopy of a Lie $\infty$-algebroid $E$. Then $L$ is a trivial $E_0$-module if and only if $\theta_L = 0$.

**Proof.** Assume that the line bundle $L$ is trivial as a vector bundle and that it admits a nowhere vanishing global section $\Omega$. Moreover suppose that the characteristic class $\theta_L$ is zero, i.e. that there exists a function $f \in \mathbb{C}^\infty$ such that $\theta_L = d_E(f)$. Then, one can check from Equation (3.6) that the nowhere vanishing section $e^{-f}\Omega$ is invariant under the action of any section $a$ of $E_0$:

$$\nabla_a(e^{-f}\Omega) = \rho(a)(e^{-f})\Omega + e^{-f}\nabla_a(\Omega) = (-\rho(a)(f) + d_E(f)(a))e^{-f}\Omega = 0 \quad (3.11)$$

The action of $E_0$ being zero on this section, we conclude that the trivial line bundle $L$ is a trivial representation of $E_0$. Conversely, if $L$ is a trivial $E_0$-module, then there exists a global section $\Omega$ of $L$ invariant under the action of $E_0$, meaning that the left-hand side of Equation (3.6) vanishes. This implies that the modular cocycle $\theta_\Omega$ vanishes, hence the result. \boxed{}

Thus, the characteristic class of the line bundle $L$ measures the trivialness of the representation $L$, not only as a vector bundle, but as a $E_0$-module. In particular a trivial line bundle $L$ which is not a trivial representation of $E_0$ admits nowhere vanishing global sections, but none of them is zero under the action of $E_0$. We will now apply this result to a line bundle of particular interest, which is associated to any Lie $n$-algebroid $E$:

**Definition 3.5.** Let $E$ be a Lie $n$-algebroid, for some $n \geq 1$. The determinant line bundle of $E$ is called the Berezinian line bundle of $E$ and is defined, depending on the parity of $n$, as:

- $n$ even: $\text{Ber}(E) = \wedge^{\text{top}} T^* M \otimes \wedge^{\text{top}} E_0 \otimes \wedge^{\text{top}} (E_{-1})^* \otimes \wedge^{\text{top}} (E_{-2}) \otimes \ldots \otimes \wedge^{\text{top}} (E_{-n+1})^*$  \quad (3.12)
- $n$ odd: $\text{Ber}(E) = \wedge^{\text{top}} T^* M \otimes \wedge^{\text{top}} E_0 \otimes \wedge^{\text{top}} (E_{-1})^* \otimes \wedge^{\text{top}} (E_{-2}) \otimes \ldots \otimes \wedge^{\text{top}} E_{-n+1}$  \quad (3.13)

Here, we consider the spaces $E_{-i}$ as vector spaces and do not keep track of the graduation.

In full rigor, the original formula of the Berezinian uses symmetric products for odd graded vector spaces, and alternating products for even graded vector spaces. Once we do not take graduation into consideration and use only alternating products, the original formula of the Berezinian is canonically isomorphic to the right hand side of Equations (3.12), (3.13). We decided not to keep track of the graduation because it is not used further in the definition of modular classes as well as in computations.

**Proposition 3.6.** The Berezinian of the Lie $n$-algebroid $E$ comes with a canonical structure of representation up to homotopy of $E$ (and hence, by Lemma 3.2, with a $E_0$-module structure).

**Proof.** Given a frame $e_1^{(i)}, \ldots, e_{\dim(E_{-i})}^{(i)}$ of $E_{-i}$, and a section $a \in \Gamma(E_0)$, we define the Lie derivative of $a$ as the unique differential operator $\mathcal{L}_a: \Gamma(E_{-i}) \to \Gamma(E_{-i})$ which coincides with the 2-bracket:

$$\mathcal{L}_a(e_k^{(i)}) = l_2(a, e_k^{(i)}) \quad (3.14)$$

It induces a dual differential operator on $(E_{-i})^*$, satisfying the usual formula $\mathcal{L}_a = d_E + \iota_a d_E$. The action of the Lie derivative straightforwardly extends as a derivation to the line bundle $\wedge^{\dim(E_{-i})} E_{-i}$. Setting $\mu^{(i)} = e_1^{(i)} \wedge \ldots \wedge e_{\dim(E_{-i})}^{(i)}$, we define the smooth function $\text{div}_{\mu^{(i)}}(a) \in \mathcal{C}^\infty(M)$ as the following proportionality coefficient:

$$\mathcal{L}_a(\mu^{(i)}) = -\text{div}_{\mu^{(i)}}(a) \cdot \mu^{(i)} \quad (3.15)$$
By duality, the action of $a$ on the dual line bundle $\wedge^{\dim(E)}(E^*)$ with dual volume form $\mu^{(i)*}$ defines a proportionality coefficient which is minus the latter:

$$\mathcal{L}_a(\mu^{(i)*}) = \text{div}_{\mu^{(i)}}(a) \cdot \mu^{(i)*}$$

(3.16)

The convention has been chosen so that if $E = E_0 = TM$ then we find the usual divergence of a vector field. Moreover, $E_0$ naturally acts on $\wedge^n T^* M$ with the (usual) Lie derivative $\mathcal{L}_{\rho(a)}$:

$$\mathcal{L}_{\rho(a)}(\omega) = \text{div}_\omega(\rho(a)) \cdot \omega$$

(3.17)

for any section $\omega$ of $\wedge^n T^* M$.

Let $\Omega = \omega \otimes \mu^{(1)} \otimes \mu^{(2)*} \otimes \mu^{(3)} \otimes \ldots$ be a section of $\text{Ber}(E)$. Given the above discussion, the action of $E_0$ on the Berezinian of $E$ via the Lie derivative amounts to:

$$\mathcal{L}_a(\Omega) = \left( \text{div}_\omega(\rho(a)) - \sum_{i=0}^{n-1} (-1)^i \text{div}_{\mu^{(i)}}(a) \right) \cdot \Omega$$

(3.18)

The sums stops at $i = n - 1$ because $E$ is a Lie $n$-algebroid. The term in parenthesis on the right-hand side of Equation (3.18) corresponds to the supertrace of the operator $\mathcal{L}_a: \Gamma(E \oplus sTM) \to \Gamma(E \oplus sTM)$, where $\mathcal{L}_a$ is understood to act as $\mathcal{L}_{\rho(a)}$ on $sTM$. Moreover, any vector bundle endomorphism $T: \Gamma(E \oplus sTM) \to \Gamma(E \oplus sTM)$ induces an endomorphism $\tilde{T}$ of $\text{Ber}(E)$ and is such that:

$$\tilde{T}(\Omega) = \text{Str}_\Omega(T) \cdot \Omega$$

(3.19)

As expected, the supertrace, when evaluated on the commutator of two operators, is zero:

$$\text{Str}_\Omega([S, T]) = 0$$

(3.20)

Here, the commutator is understood in the space of vector bundle endomorphisms of the graded vector bundle $E \oplus sTM$.

The 1-form $\theta_\Omega$ associated to the action of $E_0$ on Ber($E$) (see Equation (3.6)) thus satisfies:

$$\theta_\Omega(a) = \text{Str}_\Omega(\mathcal{L}_a)$$

(3.21)

First, by the higher Jacobi identity (2.3), we have that, for any $u \in \Gamma(E_{-1})$, $\mathcal{L}_{l_1(u)} = [l_1, \mathcal{L}_u]$. Then, by Equation (3.20), we deduce that:

$$d_E^{(0)} \theta_\Omega(u) = \theta_\Omega(l_1(u)) = \text{Str}_\Omega([l_1, \mathcal{L}_u]) = 0$$

(3.22)

Second, by the Jacobi identity (2.4) we deduce that the Lie derivative satisfies the following identity on $\Gamma(E \oplus sTM)$:

$$\mathcal{L}_{l_2(a,b)} - [\mathcal{L}_a, \mathcal{L}_b] = -[l_1, l_3(a, b, \ldots)]$$

(3.23)

By Equation (3.19), together with the trace property (3.20) of the supertrace, we deduce that:

$$\left( \mathcal{L}_{l_2(a,b)} - [\mathcal{L}_a, \mathcal{L}_b] \right)(\Omega) = -\text{Str}_\Omega([l_1, l_3(a, b, \ldots)]) \cdot \Omega = 0$$

(3.24)

But, if one expands the left-hand side by using the definition of the Lie derivative, then one obtains:

$$\left( \mathcal{L}_{l_2(a,b)} - [\mathcal{L}_a, \mathcal{L}_b] \right)(\Omega) = (\rho(a)(\theta_\Omega(b)) - \rho(b)(\theta_\Omega(a)) - \theta_\Omega(l_2(a, b))) \cdot \Omega$$

(3.25)

So we conclude that:

$$d_E^{(1)} \theta_\Omega(a, b) = -\text{Str}_\Omega([l_1, l_3(a, b, \ldots)]) = 0$$

(3.26)

Equations (3.22) and (3.26) show that $\theta_\Omega$ is $d_E$-closed, so that Ber($E$) is indeed a representation (up to homotopy) of the Lie $\infty$-algebroid $E$, and then also of $E_0$ by Lemma 3.2. □
The authors in [5] define the Berezinian line bundle of any representation $K$ of a Lie $\infty$-algebroid $E$, so that the representation of $E$ on $K$ canonically descends to this Berezinian. This allows them to define the characteristic class of such a representation $K$ as the characteristic class of the associated Berezinian bundle. In particular, when $K = E \oplus sTM$, and the corresponding action is the adjoint action defined in Example 1, the Berezinian of $E$ inherits a representation of $E$. A priori, this representation would differ from the canonical representation of Proposition 3.6 by some contribution coming from the last terms of Equations (2.29) and (2.30). But one can show that in both cases the summation (3.18) implies that these contributions cancel out and we are left with the canonical representation of $E$ on $\text{Ber}(E)$ induced by Equations (3.14) and (3.17). In other words, the representation of $E$ on $\text{Ber}(E)$ induced by any adjoint representation of $E$ coincides with the canonical representation of $E$ on $\text{Ber}(E)$ defined in Proposition 3.6. Accordingly, the modular class of the Lie $n$-algebroid $E$ is the characteristic class of any adjoint representation of $E$ or, equivalently, the characteristic class of the Berezinian of $E$ equipped with the canonical representation defined in Proposition 3.6:

**Definition 3.7.** [5] Let $E$ be a Lie a $n$-algebroid over $M$, then the characteristic class $\theta_{\text{Ber}(E)}$ associated to the canonical representation of $E$ on the Berezinian line bundle $\text{Ber}(E)$, as defined in Proposition 3.6, is called the modular class of $E$ and is denoted $\theta^E$.

**Remark.** By Proposition 3.4, as a characteristic class of a line bundle, the modular class measures the trivialness of the Berezinian as a representation of $E_0$: i.e. $\theta^E = 0$ if and only if there exists a nowhere vanishing section of $\text{Ber}(E)$, which is zero under the action of $E_0$. The notion of modular class of Lie $n$-algebroids coincides with that of Lie algebroids when $n = 1$.

**Example 2.** By Frobenius’ integrability theorem, a regular foliation on a smooth manifold can be understood as an involutive regular distribution $F$ of the tangent bundle $TM$. By abuse of denomination and by analogy with singular foliations, we call $F$ regular foliation too. This sub-bundle becomes a Lie algebroid when equipped with the inclusion $\iota: F \rightarrow TM$ as anchor map. The modular class of $F$ of this foliation Lie algebroid is a class in the first cohomology group of the foliated cohomology and is precisely the modular class of the regular foliation. That is to say, the obstruction to the existence of transverse measures to the leaves, invariant under the flow of vector fields tangent to the leaves (see Section 3.2 and [15]).

We now need to be able to compare the modular classes of homotopy equivalent Lie $\infty$-algebroids (see Definition 2.6). The notion of modular class of a morphism of Lie algebroids [17] straightforwardly generalizes to Lie $\infty$-algebroids. Indeed, given a Lie $\infty$-morphism $\Phi: \Omega_\bullet(E') \rightarrow \Omega_\bullet(E)$, it induces a morphism of algebras $\tilde{\Phi}: H^\bullet(E') \rightarrow H^\bullet(E)$ at the level of cohomologies. Then the modular class of $\Phi$ is given by:

$$\theta^\Phi = \theta^E - \tilde{\Phi}(\theta^{E'})$$  \hspace{1cm} (3.27)

In the Lie algebroid context, if $\varphi: A \rightarrow A'$ is a Lie algebroid isomorphism, the modular class $\theta^\varphi$ is zero. However, for Lie $\infty$-algebroids, the notion of isomorphism is weakened to that of homotopy equivalence. In that context, the morphism $\tilde{\Phi}$ is an isomorphism, which turns out to be canonical; it does not depend on the homotopy equivalence, see Section 2.3. Then the following result holds [5]:

**Proposition 3.8.** Let $E$ and $E'$ be two Lie $\infty$-algebroids of finite length. If $\Phi$ is a homotopy equivalence between $E$ and $E'$, then $\theta^\Phi = 0$, i.e. $\Phi$ intertwines the modular classes of $E$ and $E'$.

**Corollary 3.9.** Let $E$ and $E'$ be two homotopy equivalent Lie $\infty$-algebroids of finite length. Then $E$ and $E'$ are simultaneously unimodular, i.e. $E$ is unimodular if and only if $E'$ is unimodular.

**Example 3.** Let $A$ be a regular Lie algebroid over $M$, i.e. such that the anchor map $\rho$ has constant rank. In that case its kernel is a vector subbundle of $A$ denoted $\text{Ker}(\rho)$ and called
isotropy Lie algebra bundle. The image of the anchor map defines a regular foliation $F = \rho(A)$. One can equip $E = A \oplus s^{-1}\text{Ker}(\rho)$ with a (strict) Lie 2-algebroid structure, compatible with the Lie algebroid structure on $A$. The 1-bracket on $E$ is the inclusion of $s^{-1}\text{Ker}(\rho)$ into $A$:

$$l_1 = \iota \circ s : s^{-1}\text{Ker}(\rho) \longrightarrow A$$

and it induces an exact sequence of vector bundles:

$$0 \longrightarrow s^{-1}\text{Ker}(\rho) \overset{l_1}{\longrightarrow} A \overset{\rho}{\longrightarrow} F \longrightarrow 0$$

The 2-bracket between two elements of $\Gamma(A)$ is the Lie algebroid bracket on $A$, and the 2-bracket between a section $a$ of $A$ and a section $b$ of $s^{-1}\text{Ker}(\rho)$ is defined as:

$$l_2(a, b) = s^{-1}[a, sb]_A$$

(3.28)

The 2-bracket $l_2$ satisfies the usual Leibniz rule:

$$l_2(a, f b) = f l_2(a, b) + \rho(a)[f, b]$$

(3.29)

for every $a \in \Gamma(A)$, $b \in \Gamma(E)$ and $f \in C^\infty$. There is no 3-bracket because $A$ is a Lie algebroid.

The quotient vector bundle $A/\text{Ker}(\rho)$ is a well-defined vector bundle over $M$, canonically isomorphic to $F$. Then, the map $\varphi : E \rightarrow F$ acting as the quotient map $A \rightarrow A/\text{Ker}(\rho)$ on $E_0 = A$ and as the zero map on $E_{-1} = s^{-1}\text{Ker}(\rho)$ induces a canonical homotopy equivalence between $E$ and the foliation Lie algebroid $F$:

$$0 \longrightarrow s^{-1}\text{Ker}(\rho) \overset{l_1}{\longrightarrow} A \overset{\rho}{\longrightarrow} F \longrightarrow 0$$

$$\downarrow \varphi \downarrow \varphi \downarrow \text{id}$$

$$0 \longrightarrow 0 \longrightarrow A/\text{Ker}(\rho) \overset{\sim}{\longrightarrow} F \longrightarrow 0$$

$$\downarrow \sim \downarrow \text{id} \downarrow \text{id}$$

$$0 \longrightarrow 0 \longrightarrow F \overset{\text{id}}{\longrightarrow} F \longrightarrow 0$$

Then Corollary (3.9) applies so the Lie 2-algebroid $E$ and the Lie algebroid $F$ are simultaneously unimodular: the modular class of $E$ vanishes if and only if that of $F$ vanishes. Two reasons explain this: first, the Berezinian Line bundle $\text{Ber}(E) = \wedge^nT^*M \otimes \wedge^\text{top}A \otimes \wedge^\text{top}\text{Ker}(\rho)$ is canonically isomorphic to the line bundle $Q_F = \wedge^nT^*M \otimes \wedge^\text{top}F$. The respective action of $A$ on $\text{Ber}(E)$ and that of $F$ on $Q_F$ give the same value of the parenthesis in Equation (3.18).

Second, there is a one-to-one correspondence between closed one-forms on $E$ of height 1 and closed one-forms on $F$. More precisely, a closed one-form on $E$ of height 1, say $\theta$, is a section of $A^*$, that satisfies the following two conditions:

$$d^{(0)}_E(\theta) = 0 \iff \theta|_{\text{Ker}(\rho)} = 0$$

(3.30)

$$d^{(1)}_E(\theta) = 0 \iff \theta([a, b]) = \rho(a)(\theta(b)) - \rho(b)(\theta(a))$$

(3.31)

for every $a, b \in \Gamma(A)$. Identity (3.31) illustrates the fact that $\theta$ is closed in the Lie algebroid cohomology of $A$. Notice that not every closed form in the Lie algebroid cohomology of $A$ vanish on $\text{Ker}(\rho)$: only those that are closed with respect to the differential on $E$ do so. Identity (3.30) shows that such a one form canonically defines a differential one-form $\theta'$ on $F \simeq A/\text{Ker}(\rho)$ because it vanishes on the kernel of the anchor map. Since exact one-forms are also in one to one correspondence, we deduce that $H^1(E)$ and $H^1(F)$ are isomorphic, proving the desired correspondence between modular classes.
The argument presented in Example 3 can be generalized to Lie $n$-algebroids defining resolutions of the regular foliation $F$:

**Proposition 3.10.** Let $F$ be a regular foliation on $M$, and let $E$ be any Lie $n$-algebroid such that its linear part defines the following exact sequence of vector bundles:

$$
0 \longrightarrow E_{-n+1} \overset{l_1}{\longrightarrow} \ldots \overset{l_i}{\longrightarrow} E_{-1} \overset{l_0}{\longrightarrow} E_0 \overset{\rho}{\longrightarrow} F \longrightarrow 0 \quad (3.32)
$$

Then $F$ and $E$ are simultaneously unimodular.

The observation made in Proposition 3.10 is the central idea behind the extension of the notion of modular class to the context of singular foliations. Indeed, if one finds a replacement of a singular foliation $F$ by a Lie $\infty$-algebroid $E$ generalizing the Lie 2-algebroid $A \oplus s^{-1}\text{Ker}(\rho)$, one may define (a representant of) the modular class of $F$ as the modular class of $E$. The natural candidates of such structures are the universal Lie $\infty$-algebroids of $F$.

### 3.2 The modular class of a singular foliation

Let us first describe the geometric meaning of the modular class of a regular foliation, in order to make sense later of that of a singular foliation. A Lie algebroid representation of a regular foliation $F$ is the annihilator bundle $F^\circ \subset T^*M$, which is generated by every covector which is zero when evaluated on $F$. The annihilator bundle is canonically isomorphic to the conormal bundle of $F$, which is the vector bundle dual to the normal bundle $\nu(F)$. Then there exists a distinguished flat connection on $F^\circ$, called the Bott connection \([15]\), and defined by:

$$\nabla^\text{Bott}_u(\xi) = \mathcal{L}_u(\xi) = \iota_u d\xi \quad (3.33)$$

where $u \in \Gamma(F)$, $\xi \in \Gamma(F^\circ)$ and $\mathcal{L}_u$ is the Lie derivative. Here, flatness can be seen as a consequence of the fact that the Lie derivative $\mathcal{L} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a Lie algebra morphism.

Using the derivation property of the Lie derivative, the Bott connection can be extended to $\Omega^\cdot(F^\circ) = \bigoplus_{1 \leq i \leq \text{codim}(F)} \Gamma(\wedge^i F^\circ)$. We denote $\wedge^\top F^\circ$ the top power $\wedge^{\text{codim}(F)} F^\circ$, which is then a one-dimensional representation of $F$.

**Definition 3.11.** We say that $F$ is transversely orientable if $\wedge^\top F^\circ$ is a trivializable vector bundle or, equivalently, if it admits a nowhere vanishing global section. The modular class $\theta^F$ of a transversely orientable regular foliation $F$ is the characteristic class of the $F$-module $\wedge^\top F^\circ$.

The definition of the modular class of a regular foliation as given in Definition 3.11 is consistent with the definition of the modular class presented in Example 2. This is a consequence of the fact that a short exact sequence of vector bundles:

$$
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
$$

implies that there is a canonical isomorphism between $\wedge^\top B$ and $\wedge^\top A \otimes \wedge^\top C$. From the following short exact sequence:

$$
0 \longrightarrow F \overset{i}{\longrightarrow} TM|_U \longrightarrow \nu(F) \longrightarrow 0
$$

we deduce that $\wedge^\top TM|_U$ is canonically isomorphic to $\wedge^\top F \otimes \wedge^\top (\nu(F))$. Multiplying both sides by $\wedge^\top T^*M|_U \otimes \wedge^\top (\nu(F))^*$ implies in turn that the line bundle $\wedge^\top (\nu(F))^* \simeq \wedge^\top F^\circ$ is canonically isomorphic to the Berezinian bundle of the foliation Lie algebroid $F$:

$$
\wedge^\top F^\circ \simeq \wedge^\top T^*M|_U \otimes \wedge^\top F \quad (3.34)
$$

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Obviously, $F$ will be transversely orientable when $F^o$ is a rank zero vector bundle, i.e. if $F = TM$, for in that case the line bundle $\Lambda^{\top}F^o = \Lambda^0F^o = M \times \mathbb{R}$ is trivial. In other cases, the top exterior power of $F^o$ is a rank one subbundle of $\wedge^{\text{codim}(F)}TM$, and even if the latter might be a trivial vector bundle, it may well happen that $\Lambda^{\top}F^o$ is not trivial as a vector bundle, but only trivializable. The modular class of $F$ measures the trivialness – as a representation of $F$ – of the line bundle $\Lambda^{\top}(F^o)$. In general, the existence of a nowhere vanishing global section of the determinant bundle $\Lambda^{\top}F^o$ does not necessarily entail that this section is invariant under the flow of vector fields of $F$. At least, we deduce from Equation (3.33) that if a chosen transverse volume form is exact, then it is necessarily $F$-invariant. The vanishing of the modular class of the foliation is precisely equivalent to the existence of an \textit{invariant} volume form transverse to the regular foliation $F$, this is why we originally require that $F$ is transversely orientable. The following examples are providing some situations illustrating the notion of invariant transverse volume forms and the subsequent vanishing of the modular classes:

\textbf{Example 4.} The regular foliation consisting of concentric circles on the punctured plane $\mathbb{R}^2 - \{(0,0)\}$ is induced by the irrotational vector field $v = -y \partial_x + x \partial_y$. It admits a transverse measure $\omega = d(x^2 + y^2)$ which is actually invariant because it is exact.

\textbf{Example 5.} The regular foliation of $\mathbb{R}^2$ corresponding of horizontal lines in the lower half-plane, and of parabolas of equations $y = a(x^2 + 1)$ (for $a \geq 0$) in the upper half-plane, is generated by the following vector field:

\[
v(x, y) = \begin{cases} 
\partial_x & \text{whenever } y \leq 0 \\
\partial_x + 2ax \partial_y & \text{whenever there exists } a \geq 0 \text{ such that } y = a(x^2 + 1)
\end{cases}
\]  

(3.35)

Notice that for $a = 0$ both definitions coincide. Then it turns out that the differential one-form $\omega = d(y - a(x^2 + 1))$ is an invariant transverse volume form to the foliation.

\textbf{Example 6.} In $\mathbb{R}^2 \setminus B^2$, where $B^2$ is the closed 2-ball, let us define the following set of parametrized curves:

\[\forall \ \varphi \in [0, 2\pi] \quad L_{\varphi} = \{(e^t \cos(\varphi + t), e^t \sin(\varphi + t)) \mid t > 0\}
\]  

(3.36)

These curves foliate the open set $\mathbb{R}^2 \setminus B^2$ into spirals. The tangent vector to a leaf at a given point $(x, y)$ is the one-dimensional subspace $F_{(x,y)}$ of $T_{(x,y)}M$ generated by the element $v = (x-y)\partial_x + (x+y)\partial_y$, which is the velocity vector tangent to the leaf at $(x, y)$. The fiber of the conormal bundle $F^o \subset T^*M$ at the same point $(x, y)$ is generated by the covector $\omega = (x+y)dx - (x-y)dy$. This plays the role of a transverse measure to the foliation since $F^o$ is one-dimensional. Notice that $\omega$ is not exact hence the Bott connection evaluated on the vector field $v$ does not vanish on it:

\[
\nabla^{\text{Bott}}_v (\omega) = -2 \iota_v dx \wedge dy = 2 \omega
\]  

(3.37)

Since the vector bundle $F$ is of rank one, then every vector field tangent to the leaves is colinear to $v$, so that the modular 1-form $\theta_\omega$ can be directly read on Equation (3.37):

\[
\theta_\omega(v) = 2
\]  

(3.38)

The differential 1-form $\theta_\omega$ is not zero, but it may occur that its cohomology class in $H^1(F)$ is zero, i.e. that it could be written as an exact 1-form: $\theta_\omega = df$ for some function $h$. In particular this function would have to satisfy the following first-order partial differential equation:

\[
(x - y) \frac{\partial h}{\partial x} + (x + y) \frac{\partial h}{\partial y} = 2
\]  

(3.39)

A solution of this equation on $\mathbb{R}^2 \setminus B^2$ is $h = \frac{1}{2} \ln(x^2 + y^2)$. Hence the differential one-form $\theta_\omega$ is exact so the modular class $\theta^F$ of the regular foliation $F$ vanishes, which means that there exists
an invariant transverse measure to the foliation. Letting \( r^2 = x^2 + y^2 \), one can check that the Bott connection indeed vanishes on the following measure:

\[
\omega_{\text{inv}} = \frac{1}{r} \omega
\]  
(3.40)

We cannot straightforwardly reproduce Definition 3.11 for singular foliations because the normal bundle and the conormal bundle is not well-defined everywhere in that case. Indeed, since the leaves of a singular (non-regular) foliation \( \mathcal{F} \) have various dimensions – and thus so have their transversals – the counterpart of the normal bundle is not a vector bundle. The wise reader would notice that defining the normal bundle is not necessary, as one only needs to find the counterpart of the line bundle \( \wedge^{\top} F^o \) for singular foliations in order to define modular classes of the latter. The simplest idea would be to pick up the line bundle \( \wedge^{\top} F^o \) generated by the regular distribution \( F \) induced from \( \mathcal{F} \), but then we are confronted with the problem of extending this line bundle at singularities. More precisely, assume that the singular \( \mathcal{F} \) has regular leaves of the same, maximal dimension. It is a fact that the union of the regular leaves form a dense open subset \( U \) of \( M \). Thus, on \( U \), the singular foliation \( \mathcal{F} \) induces a regular distribution \( F \) admitting a conormal bundle canonically isomorphic to \( F^o \subset T^* M|_U \). However, this vector subbundle does not necessarily extends to the singular leaves of \( \mathcal{F} \), implying that there is no straightforward way of defining the modular class of the singular foliation \( \mathcal{F} \) in the sense of Definition 3.11, as the following discussion shows.

**Example 7.** On the punctured plane \( U = \mathbb{R}^2 - \{(0,0)\} \), the vector bundle \( F^o \) of Example 4 is a subbundle of \( T^* \mathbb{R}_U^2 \), which cannot be extended at the origin. Indeed, at each point \((x,y)\) of the punctured plane, \( F^o(x,y) \) is the one dimensional subspace of \( T^*_{(x,y)} \mathbb{R}^2 \cong \mathbb{R}^2 \) colinear to the vector \((x,y)\), so it has no unequivocal definition at the origin. Although the vector bundle \( F^o \) – or equivalently \( \wedge^{\top} F^o \), because \( F^o \) has rank one – may not be extended at the origin, it admits a nowhere vanishing global section \( \omega = d(x^2 + y^2) \) on the punctured plane.

Although this is a problem to make sense of invariant transverse volume at singularities, we can nonetheless get inspired by Definition 3.11 to define the modular class of \( \mathcal{F} \). From the notion of Bott connection, we will keep the idea that a singular foliation acts on a distinguished line bundle. We then need only define what is a representation of a singular foliation, by extending the notion of connection of almost Lie algebroids.

**Definition 3.12.** Let \( \mathcal{F} \) be a singular foliation, then an \( \mathcal{F} \)-connection on a (graded) vector bundle \( K \) is an operator \( \nabla : \mathcal{F} \times \Gamma(K) \rightarrow \Gamma(K) \), satisfying the usual axioms (2.19)-(2.20). The connection is said flat if its curvature vanishes. In that case \( K \) is said to be a representation of \( \mathcal{F} \), or a \( \mathcal{F} \)-module. We say that a section \( s \in \Gamma(K) \) is \( \mathcal{F} \)-invariant when

\[
\nabla_u(s) = 0, \text{ for every } u \in \mathcal{F}.
\]  
(3.41)

A representation \( K \) of \( \mathcal{F} \) is trivial when there exists a global frame of \( K \) which is \( \mathcal{F} \)-invariant.

**Example 8.** Let \( \mathcal{F} = \mathfrak{X}(M) \) and \( K = M \times \mathbb{R} \), so sections of \( K \) are smooth functions on \( M \). Let \( \nabla \) be the \( TM \)-connection on \( K \) defined as the standard action of vector fields:

\[
\nabla_X(f) = X(f)
\]  
(3.42)

for every vector field \( X \) and smooth function \( f \). This is a flat connection and the constant functions are \( \mathcal{F} \)-invariant so \( K \) is a trivial \( \mathcal{F} \)-module.

Given a trivial line bundle \( L \), which is additionally a \( \mathcal{F} \)-module, the characteristic class of \( L \) with respect to the action of \( \mathcal{F} \) is the cohomology class – in the foliated de Rham cohomology
of \( \mathcal{F} \) – of the foliated one form \( \theta_{\Omega} \in \Omega^1(\mathcal{F}) \), defined as in identity (3.6). This may provide us with a definition of modular class for singular foliations once we find the correct line bundle. Proposition 3.10 has shown that the modular class of a regular foliation \( F \) can be equivalently computed from any Lie \( n \)-algebroid forming a geometric resolution of \( F \). It is thus natural to extend this notion to singular foliations. Since we want to work with Lie \( n \)-algebroids in order to define their Berezinian, from now on we will only consider solvable singular foliations. By Theorem 2.12, such a foliation admits at least one universal Lie \( \infty \)-algebroid \( E \) of finite length. The fact that the singular foliation admits a universal Lie \( \infty \)-algebroid of finite length implies that all regular leaves of \( \mathcal{F} \) have the same, maximal dimension (see Proposition 2.5 in [20]).

We set \( U \) to be the dense open subset of \( M \) consisting of the union of such regular leaves, and the regular distribution induced by \( \mathcal{F} \) on \( U \) is denoted \( F \). Since \( F|_U \subset \Gamma(F)|_U \) as sheaves over \( U \), the Bott connection associated to the action of \( F \) on \( F^0 \) induces an \( F|_U \)-connection on \( F^0 \) (over \( U \))

\[
\nabla^\text{Bott}_X(\xi) = \iota_X d\xi
\]

for every \( X \in \mathcal{F}|_U \) and \( \xi \in \Gamma(F^0) \). This action canonically extends to \( \wedge^{\text{top}} F^0 \), turning it into a \( \mathcal{F}|_U \)-module.

**Proposition 3.13.** Given the above assumptions and notations, the Berezinian line bundle of \( E \) satisfies the following two criteria:

1. it is canonically isomorphic to \( \wedge^{\text{top}} F^0 \) on \( U \), via an isomorphism \( \varphi: \text{Ber}(E)|_U \to \wedge^{\text{top}} F^0 \);
2. it is a \( \mathcal{F} \)-module, and \( \varphi \) is a morphism of \( \mathcal{F}|_U \)-modules.

**Proof.** Our goal is to replace \( \wedge^{\text{top}} F \) by the alternating powers of the Lie \( n \)-algebroid \( E \), following the same argument leading to Equation (3.34). On the open dense subset \( U \), the linear part of the Lie \( n \)-algebroid \( E \) is a geometric resolution of \( F \) (for brevity, we omit to write the restriction to \( U \) in the following):

\[
0 \rightarrow E_{n+1} \xrightarrow{l_1} \cdots \xrightarrow{l_1} E_1 \xrightarrow{l_1} E_0 \xrightarrow{\rho} F \rightarrow 0
\]

Since the rank of each map is constant, one can split this long exact sequence of vector bundles into various short exact sequences of vector bundles:

\[
0 \rightarrow E_{n+1} \rightarrow E_{n+2} \rightarrow l_1(E_{n+2}) \rightarrow 0
\]

\[
0 \rightarrow l_1(E_{n+2}) \rightarrow E_{n+3} \rightarrow l_1(E_{n+3}) \rightarrow 0
\]

\[
\cdots
\]

\[
0 \rightarrow l_1(E_{-2}) \rightarrow E_{-1} \rightarrow l_1(E_{-1}) \rightarrow 0
\]

\[
0 \rightarrow l_1(E_{-1}) \rightarrow E_0 \rightarrow F \rightarrow 0
\]

Then one has \( \wedge^{\text{top}} E_0 \simeq \wedge^{\text{top}} l_1(E_{-1}) \otimes \wedge^{\text{top}} F \), which in turn implies that \( \wedge^{\text{top}} F \simeq \wedge^{\text{top}} E_0 \otimes \wedge^{\text{top}} (l_1(E_{-1}))^* \). Let us find a replacement for \( \wedge^{\text{top}} (l_1(E_{-1}))^* \) now. We have that \( \wedge^{\text{top}} E_{-1} \simeq \wedge^{\text{top}} l_1(E_{-2}) \otimes \wedge^{\text{top}} l_1(E_{-1}) \) so finally \( \wedge^{\text{top}} (l_1(E_{-1}))^* \simeq \wedge^{\text{top}} l_1(E_{-2}) \otimes \wedge^{\text{top}} (E_{-1})^* \). This implies that:

\[
\wedge^{\text{top}} F \simeq \wedge^{\text{top}} E_0 \otimes \wedge^{\text{top}} (E_{-1})^* \otimes \wedge^{\text{top}} l_1(E_{-2})
\]

(3.44)
where the right-hand side is assumed to be restricted to $U$ of course. Continuing the process up to $E_{-n+1}$, and plugging $\wedge^\top F$ into Equation (3.34), we deduce that there is canonical isomorphism of line bundles:

$$\wedge^\top F^\top \simeq \text{Ber}(E)|_U$$  \hspace{1cm} (3.45)

We denote $\varphi: \text{Ber}(E)|_U \to \wedge^\top F^\top$ the corresponding isomorphism.

By Proposition 3.6, the Berezinian is a $E_0$-module. This structure canonically induces a $\mathcal{F}$-module structure via the anchor map. Indeed, for a given $X \in \mathcal{F}$, let $a \in \Gamma(E_0)$ be a preimage of $X$ through $\rho$, i.e. such that $\rho(a) = X$, so we define the action of $X$ on a section $\Omega$ of $\text{Ber}(E)$ as:

$$\nabla_X^\mathcal{F}(\Omega) = \mathcal{L}_a(\Omega)$$  \hspace{1cm} (3.46)

where the right-hand side is defined in Equation (3.18). The connection $\nabla^\mathcal{F}$ does not depend on the preimage of $X$, for the following reason: if one had chosen another preimage $a' \in \Gamma(E_0)$ of $X$, then $\rho(a - a') = 0$ so there exists $b \in \Gamma(E_{-1})$ such that $l_1(b) = a - a'$ and $\mathcal{L}_{l_1(b)}(\Omega) = 0$ by Equation (3.22). The reason why $l_1(b) = a - a'$ is explained by the fact that $E$ is a universal Lie $\infty$-algebroid of $\mathcal{F}$, which means that its linear part is a geometric resolution of $\mathcal{F}$, at the sheaf level. The fact that $\text{Ber}(E)$ is a $E_0$-module implies that the connection $\nabla^\mathcal{F}$ is flat, turning $\text{Ber}(E)$ into a $\mathcal{F}$-module.

The line bundle $\wedge^\top T^*M|_U \otimes \wedge^\top F$ is a representation of $F$ under the canonical action of the Lie derivative of vector fields. Since the Bott connection on the normal bundle $\nu(F)$ is the quotient representation induced by the canonical action of $F$ on $TM$ via the Lie derivative, it is straightforward to check that the line bundles $\wedge^\top \nu(F)^*$ and $\wedge^\top T^*M|_U \otimes \wedge^\top F$ are not only isomorphic as line bundles, but also as $F$-modules, and hence as $\mathcal{F}|_U$-modules as well. From this, we deduce that Equation (3.34) also holds at the level of $\mathcal{F}|_U$-modules.

Now, the regular foliation $F$ is canonically isomorphic to the quotient vector bundle $E_0/l_1(E_{-1})|_U$ and the identification can be made via the anchor map $\rho$. Since $l_1(E_0) = 0$, the Jacobi identity (2.3) implies that $l_1(E_{-1})$ is stable under the adjoint action of $E_0$ on itself: $l_2(E_0, l_1(E_{-1})) \subset l_1(E_{-1})$. This action then passes to the quotient and, by Equation (2.7), the induced action of $E_0$ on $F \simeq E_0/l_1(E_{-1})|_U$ coincides with the canonical action of $F$ on itself:

$$a \cdot \rho(b) = \rho(l_2(a, b)) = [\rho(a), \rho(b)]$$  \hspace{1cm} (3.47)

It means in particular that the canonical isomorphism between $\wedge^\top F$ and $\wedge^\top E_0 \otimes \wedge^\top (l_1(E_{-1}))^*$ intertwines the action of $E_0$ on both sides. One step further, we know that the vector bundle $l_1(E_{-1})$ is canonically isomorphic to the quotient vector bundle $E_{-1}/l_1(E_{-2})|_U$, and the identification can be made via $l_1$. By the identities $l_1(E_0) = 0$ and $l_1 \circ l_2 = l_2 \circ l_1$, all terms of the short exact sequence:

$$0 \longrightarrow l_1(E_{-2}) \longrightarrow E_{-1} \longrightarrow l_1(E_{-1}) \longrightarrow 0$$

are stable under the adjoint action of $E_0$. From this we deduce that the isomorphism $\wedge^\top E_{-1} \simeq \wedge^\top l_1(E_{-2}) \otimes \wedge^\top l_1(E_{-1})$ intertwines the action of $E_0$. In turn this implies that the isomorphism of line bundles presented in Equation (3.44) intertwines the action of $E_0$.

By going up the ladder of short exact sequences we deduce that the canonical isomorphism:

$$\wedge^\top F \simeq \wedge^\top E_0 \otimes \wedge^\top (E_{-1})^* \otimes \wedge^\top E_{-2} \otimes \wedge^\top (E_{-3})^* \otimes \cdots |_U$$  \hspace{1cm} (3.48)

intertwines the action of $E_0$ on both sides. Since the action of $E_0$ is a lift of the action of $\mathcal{F}$, we deduce that the action of $\mathcal{F}|_U$ on both sides of Equation (3.48) commute with the corresponding
canonical isomorphism. Then, the isomorphism between \(\wedge^{\top}T^*M|_U \otimes \wedge^{\top}F\) and \(\Ber(E)|_U\) intertwines the action of \(\mathcal{F}|_U\). Together with the observation that \(\wedge^{\top}\nu(F)^*\) and the Berezinian \(\wedge^{\top}T^*M|_U \otimes \wedge^{\top}F\) are isomorphic \(\mathcal{F}|_U\)-modules, we deduce that \(\wedge^{\top}\nu(F)^*\) and \(\Ber(E)|_U\) are not only canonically isomorphic as line bundles, but also as \(\mathcal{F}|_U\)-modules. \(\square\)

Item 2. of Proposition 3.13 states that the Berezian of any universal Lie \(\infty\)-algebroid of a solvable singular foliation \(\mathcal{F}\) inherits a \(\mathcal{F}\)-module structure. We use this result to define the modular class of the singular foliation \(\mathcal{F}\).

**Definition 3.14.** Let \(\mathcal{F}\) be a solvable singular foliation and let \(E\) be a universal Lie \(\infty\)-algebroid of \(\mathcal{F}\) of finite length. Let \(\theta^E_U\) be the cohomology class induced by the modular class of \(E\) in the universal foliated cohomology of \(\mathcal{F}\). Then the modular class of \(\mathcal{F}\) is the unique element \(\theta^\mathcal{F} \in H^1_{dR}(\mathcal{F})\) defined by:

\[
\theta^\mathcal{F} = (\rho^*_\mathcal{F})^{-1}\theta^E_U \quad (3.49)
\]

The choice of cohomology – the foliated de Rham cohomology instead of the universal foliated cohomology – comes from the fact that we want the modular cocycle to be applied to elements of \(\mathcal{F}\), rather than sections of any of its universal Lie \(\infty\)-algebroids. Doing so, we mimic the definition of the modular class for regular foliations, which is defined on the foliated de Rham cohomology. However, in practice, it is often more useful to find a universal Lie \(\infty\)-algebroid \(E\) and to work with the modular class of \(E\), which then appears to be a particular representant of \(\theta^\mathcal{F}\). Indeed, Definition 3.14 makes sense only if the modular class does not depend on the choice of the universal Lie \(\infty\)-algebroid:

**Proposition 3.15.** The modular class of the solvable singular foliation \(\mathcal{F}\) does not depend on the choice of universal Lie \(\infty\)-algebroid from which it is defined.

**Proof.** Assume that \(E\) (resp. \(E'\)) is a universal Lie \(\infty\)-algebroid of \(\mathcal{F}\) of finite length, and let \(\theta^E\) (resp. \(\theta^{E'}\)) be its modular class (see Definition 3.7). We know by Theorem 2.13 that \(E\) and \(E'\) are homotopically equivalent, and that any two choices of homotopy equivalences are homotopic. Let us pick one, say \(\Phi_{E,E'}: \Omega(E') \rightarrow \Omega(E)\), then by Proposition 3.8 it has vanishing modular class. This implies that, at the level of cohomologies, the canonical isomorphism \(\tilde{\Phi}_{E,E'}\) induced by the homotopy equivalence \(\Phi_{E,E'}\) intertwines the respective modular classes of \(E\) and \(E'\). By definition of the universal foliated cohomology, these modular classes define the same element in \(H^1_{dR}(\mathcal{F})\), that we note \(\theta^\mathcal{F}\). Pulling back this cohomology class in the de Rham foliated cohomology using Proposition 2.15 gives the modular class associated to \(\mathcal{F}\). \(\square\)

By construction, the vanishing or non-vanishing of the modular class of a singular foliation has a straightforward meaning:

**Proposition 3.16.** Let \(\mathcal{F}\) be a solvable singular foliation. Then \(\theta^\mathcal{F} = 0\) if and only if, for any universal Lie \(\infty\)-algebroid of finite length \(E\) of \(\mathcal{F}\), the Berezinian \(\Ber(E)\) is a trivial \(\mathcal{F}\)-module.

**Remark.** The main motivation behind Definition 3.14 is the following observation: defining the modular class of a singular foliation from that of a Lie algebroid from which it descends has several drawbacks. First, assume that one finds several different, non-isomorphic, Lie algebroids covering the same singular foliation: then we do not have a unique way of defining a modular class for the latter. This problem is solved in the Lie \(\infty\)-algebroid realm by Proposition 3.15. Moreover, even for Lie algebroids for which the anchor has constant rank on an open dense subset, the modular class of the Lie algebroid may not coincide with the modular class of the induced regular foliation (see Examples 4.1 and 4.2). This is somewhat problematic. Even worse, there are examples of singular foliations which we do not know if they descend from a
Lie algebroid or not (this is still a conjecture [3]), but of which we know some universal Lie
∞-algebroids (see Example 4.3). Thus, relying on Lie algebroids raises many problems and
justifies that we use the notion of universal Lie ∞-algebroids to define modular class of singular
foliations.

We will now understand the relationship between the modular class of singular foliations
and that of regular foliations. Recall that the regular leaves of a solvable singular foliation have
the same, maximal dimension (see Proposition 2.5 in [20]) on a dense open set $U \subset M$, hence
forming a regular foliation $F$. From Example 7 we know that the conormal bundle $F^\circ$ may,
however, not be extendable to the singular leaves $M \setminus U$. If it is extendable, we call $F^\circ$ the
vector bundle on $M$ whose restriction to the open dense subset $U$ is $F^\circ$. The action of $F^\circ|_U$
on $F^\circ$ extends at the singularities as an action of $F$ on $F^\circ$. By construction, the line bundle
$\wedge^{\text{top}}F^\circ$ restricts to $\wedge^{\text{top}}F^\circ$ on $U$. Moreover, it inherits a $F$-module structure from that of $F^\circ$,
which extends that of $F^\circ|_U$ on $\wedge^{\text{top}}F^\circ$. By a similar argument as in item 1. of Proposition 3.13,
the line bundle $\wedge^{\text{top}}F^\circ$ is isomorphic, as a vector bundle, to the Berezinian of any universal Lie
∞-algebroid of finite length $E$ of $F$. While item 2. of Proposition 3.13 states that the restriction
on $U$ of these two line bundles are isomorphic as $F^\circ|_U$-modules, it does not mean that over $M$
they are isomorphic as $F$-modules, precisely because the action of $F^\circ|_U$ coincides with that of
the regular foliation $F$ and forgets everything about the behavior of the vector fields at the
singularities. To better understand the situation, let us explore the following example:

**Example 9.** Let $M = \mathbb{R}^n$ with standard coordinates $x_1, \ldots, x_n$. Let $f$ be a smooth function on
$\mathbb{R}$ vanishing only at 0 and let $g$ be a smooth function on $\mathbb{R}^{n-1}$ vanishing only at the origin. We
use them to define two vector fields:

$$X_1 = g(x_2, \ldots, x_n)\partial_{x_1} \quad \text{and} \quad X_2 = f(x_1)\partial_{x_1}$$

Let $F_1$ be the singular foliation generated by the vector field $X_i$ ($i = 1, 2$). The regular foliation
$F_1$ induced by $X_1$ is defined over the open dense subset $U_1 = \mathbb{R}^n - \{(x_1, \ldots, x_n) | x_2 = \ldots = x_n = 0\}$
while the regular foliation $F_2$ induced by $X_2$ is defined over $U_2 = \mathbb{R}^n - \{(x_1, \ldots, x_n) | x_1 = 0\}$.
Both regular leaves consist of parallel lines along the $x_1$-direction, but in the first case the
singular leaves are the points forming the $x_1$-axis, while in the second case the singular leaves
consist of points on the $x_1 = 0$ plane

The annihilator bundle $F^\circ_1$ is the rank $n-1$ subbundle of $T^*\mathbb{R}^n|_{U_1}$ spanned by the one-
forms $dx_2, dx_3, \ldots, dx_n$, and the determinant line bundle $\wedge^{n-1}F^\circ_1$ is the rank one subbundle of
$\wedge^{n-1}T^*\mathbb{R}^n|_{U_1}$ generated by the $n-1$-form $dx_2 \wedge \ldots \wedge dx_n$. The vector bundles $F^\circ_1$ and $\wedge^{n-1}F^\circ_1$
(resp. $F_2$ and $\wedge^{n-1}F^\circ_2$) straightforwardly extend along the $x_1$-axis (resp. at the plane $x_1 = 0$),
and so does the section $dx_2 \wedge \ldots \wedge dx_n$. We denote by $L$ the rank one trivial subbundle of
$\wedge^{n-1}T^*\mathbb{R}^n$ generated by the global section $dx_2 \wedge \ldots \wedge dx_n$ (over the whole space $\mathbb{R}^n$ then). The
vector fields $X_1$ and $X_2$ act on $L$ via the Bott connection (see Equation (3.43)) and their action
on the global section $dx_2 \wedge \ldots \wedge dx_n$ is trivial. Thus, $L$ is both a trivial $F_1$ and a trivial $F_2$ line
bundle. However, we will now see that these singular foliations have different modular classes.

A universal Lie ∞-algebroid of $F_1$ can be chosen to be the rank one trivial Lie algebroid
$A = \mathbb{R}^n \times \mathbb{R}$ such that the anchor map $\rho_A$ sends the constant section 1 to $X_1$. By construction it
is injective on sections so there is no additional term in the resolution. The Berezinian of $A$ is
$\text{Ber}(A) = \wedge^nT^*\mathbb{R}^n \otimes A$, and admits as global section the following form: $\Omega = dx_1 \wedge \ldots \wedge dx_n \otimes 1$.
The action of the constant section 1 $\in \Gamma(A)$ on $\Omega$ then gives:

$$\nabla_1(\Omega) = L_{X_1}(dx_1 \wedge \ldots \wedge dx_n) \otimes 1 = \text{div}(X_1)\Omega$$

(3.50)

In the first case, $\text{div}(X_1) = 0$, while in the second case, $\text{div}(X_2) = f'(x_2)$. So, in the first
case, the modular class $\theta^{F_1} = 0$. However, for the second case, the modular 1-form $\theta \Omega$ from
Equation (3.50) is $\theta_\Omega(1) = f'(x_1)$. Although it is not zero as a cochain element, it can be written as a $d_\mathcal{A}$-exact one-form on the open dense subset $U_2$:

$$\theta_\Omega|_{U_2} = \frac{1}{2} d_\mathcal{A} \ln(f^2)$$

(3.51)

Then, $\theta_\Omega$ is not exact on the whole of $\mathbb{R}^n$ (or equivalently, on any neighborhood of the hyperplane $x_1 = 0$) because the function $x_1 \mapsto \frac{1}{2} \ln(f(x_1)^2)$ diverges at $x_1 = 0$. It implies in turn that the modular class $\theta^{\mathcal{F}_2}$ is not zero (at least when $f'(0) \neq 0$, see Example 4.1). To conclude: $\text{Ber}(\mathcal{A})$ is a trivial $\mathcal{F}_1$-module, but not a trivial $\mathcal{F}_2$-module, while in both cases $L$ is a trivial module. We interpret this observation as follows: as line bundles, $L$ and $\text{Ber}(\mathcal{A})$ are isomorphic. However, as $\mathcal{F}_1$-modules they are isomorphic, while as $\mathcal{F}_2$-modules they are not. This statement extends to the Berezinian of any universal Lie $\infty$-algebroid of finite length of the singular foliations $\mathcal{F}_i$.

From Example 9, we understand that, when the regular foliation $F$ induced from the singular foliation $\mathcal{F}$ has a vanishing modular class (as a regular foliation, on the open dense subset of $M$ made of regular leaves), the modular class of the singular foliation $\mathcal{F}$ tells us to what extent item 2. of Proposition 3.13 extends to the singular leaves. More precisely, assume that the modular class of this regular foliation is zero on $U$, and that the vector bundle $F^\circ$ extends at singularities. We denote by $\mathcal{F}^\circ$ this vector bundle and by $\wedge^{\text{top}} F^\circ$ the unique line bundle which restricts to the determinant line bundle $\wedge^{\text{top}} F^\circ$ on $U$. Then, drawing on Proposition 3.16, we have the following statement:

**Proposition 3.17.** Let $\mathcal{F}$ be a solvable singular foliation. Assume that the regular foliation $F$ induced by $\mathcal{F}$ admits a global invariant transverse measure, and that $F^\circ$ extends at singularities. Then, $\theta^{\mathcal{F}} = 0$ if and only if there exists an isomorphism of $\mathcal{F}$-modules between $\wedge^{\text{top}} F^\circ$ and the Berezinian of any universal Lie $\infty$-algebroid of finite length of $\mathcal{F}$.

We assumed that the modular class of the regular foliation $F$ on $U$ vanishes, because otherwise that of the singular foliation $\mathcal{F}$ will not vanish either, not telling us anything that is not already captured by the modular class of $F$. For the same reason, if the vector bundle $F^\circ$ cannot extend at singular leaves, the modular class of the singular foliation $\mathcal{F}$ still does exist, but does not say anything on $\mathcal{F}$ that is not already known about $F$. On the contrary, if those two conditions are met, Proposition 3.17 establishes that the modular class of the singular foliation $\mathcal{F}$ is an obstruction to the existence of an invariant global section of a particular family of line bundles – the Berezinians of the universal Lie $\infty$-algebroids of finite length – which, on the regular leaves, can be interpreted as the existence of an invariant transverse measure to $F$.

The use of universal Lie $\infty$-algebroids to define modular classes of singular foliations illustrates how to extend other characteristic classes or geometric notions from the regular foliations realm to the singular foliations one. For prospective researches, it would be important to investigate the behaviour of the modular class (or other characteristic classes) under Hausdorff Morita equivalences of singular foliations [10]. Also, based on the idea that the modular class of a Lie algebroid can be seen as an obstruction to the existence of a measure on the differentiable stack associated to this Lie algebroid [7, 33], it would be certainly fruitful to investigate the relationship between the modular class of a singular foliation and the existence of an invariant measure on its associated holonomy groupoid [2].

4 Examples

The first example 4.1 and the second example 4.2 are raw calculations of modular classes of two solvable singular foliations: the Euler vector field and the characteristic foliation of a Poisson
manifold. In both cases, the annihilator bundle $F^\circ$ of the induced regular foliation $F$ cannot be extended at the singular leaves. The modular classes of these singular foliations then do not bring any new knowledge that is not already captured by the modular class of the induced regular foliations. Examples 4.3 and 4.4 study solvable singular foliations whose conormal bundles can be extended at the origin. In both cases the modular class vanishes, proving that the line bundle $\wedge^{\text{top}} F^\circ$ is a trivial module of the singular foliation. Example 4.5 presents a case of a solvable singular foliation whose conormal bundle cannot be extended at the origin, but if we add the Euler vector field, it can. However, even if the action of the newly defined singular foliation on the determinant line bundle $\wedge^{\text{top}} F^\circ$ is trivial, we show that its modular class is not. This case should then be interpreted under the light of Proposition 3.17.

4.1 The Euler vector field

On $\mathbb{R}^n$, for $n \geq 2$, the Euler vector field $\epsilon = \sum_{i=1}^n x_i \partial_{x_i}$ defines a singular foliation $F_\epsilon$ whose leaves are straight lines (regular leaves) escaping the origin (singular leaf). The punctured space $\mathbb{R}^n \setminus \{0\}$ is the union of regular leaves, whose transversals consist of the $(n-1)$-spheres centered at 0 (and of any non-zero radius). Let us denote by $F$ the regular distribution induced by $F_\epsilon$ on the punctured space $\mathbb{R}^n \setminus \{0\}$. Sections of the annihilator vector bundle $F^\circ \subset T^*(\mathbb{R}^n \setminus \{0\})$ are generated by elements of the form $x_i dx_j - x_j dx_i$ for $1 \leq i < j \leq n$. The codistribution they generate consists of concentric spheres, so the annihilator bundle does not extend at the origin.

A distinguished global section of the line bundle $\wedge^{\text{top}} F^\circ$ on the punctured space is the $(n-1)$-dimensional spherical volume form:

$$\omega_{F^\circ} = \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n,$$  

(4.1)

showing that $\wedge^{\text{top}} F^\circ$ is a trivializable line bundle over the punctured space. The action of the regular foliation $F$ on $\omega_{F^\circ}$ is made through the Bott connection defined as in Equation (3.33). Since the restriction of $\epsilon$ to the punctured space generates all the sections of $F$, it is sufficient to compute the action of the Bott connection on $\epsilon$, so we obtain:

$$\nabla^{\text{Bott}}_\epsilon (\omega_{F^\circ}) = n \omega_{F^\circ}$$  

(4.2)

One can straightforwardly read the modular 1-form $\theta_{\omega_{F^\circ}} \in \Omega^1(F)$ from this equation; evaluated on the Euler vector field it reads: $\theta_{\omega_{F^\circ}}(\epsilon) = n$. Although the modular 1-form $\theta_{\omega_{F^\circ}}$ is not zero as a cochain element of $\Omega^1(F)$, it is exact since it can be written as:

$$\theta_{\omega_{F^\circ}} = n d_F \left( \ln \sqrt{x_1^2 + \ldots + x_n^2} \right)$$  

(4.3)

It is well defined because the sum in the square root never vanishes on the punctured space $\mathbb{R}^n \setminus \{0\}$. Then, the modular class of the regular foliation $F$ induced by the Euler vector field vanishes. It means that there exists a transverse measure invariant under the action of the Euler vector field, which can be chosen to be:

$$\omega_{\text{inv}} = \frac{1}{\sqrt{x_1^2 + \ldots + x_n^2}} \omega_{F^\circ}$$  

(4.4)

One can check that the action of the Bott connection is trivial on $\omega_{\text{inv}}$, showing that $\wedge^{\text{top}} F^\circ$ is a trivial $F$-module.

Now let us turn to the modular class of the singular foliation $F_\epsilon$. A universal Lie $\infty$-algebroid of $F_\epsilon$ is the rank 1 trivial vector bundle $E = \mathbb{R}^n \times \mathbb{R}$, whose anchor map $\rho$ sends the constant
section 1 to the Euler vector field $\epsilon$. The Berezinian $\text{Ber}(E)$ is a trivial vector bundle and by Proposition 3.13, its restriction to the punctured space can be canonically identified with $\wedge^\top F^\circ$. The latter line bundle does not extend to the origin however, so the identification cannot be extended there. A global non-vanishing section of the Berezinian is $\Omega = \omega \otimes 1$, where $\omega = dx_1 \wedge \ldots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$. Since the Lie algebroid bracket on $E$ is zero, the action of $E$ on $\Omega$ is only given by the Lie derivative of the Euler vector field on $\omega$:

$$\nabla_1(\Omega) = \mathcal{L}_\epsilon(\omega) \otimes 1 = \text{div}(\epsilon) \Omega = n \Omega \hspace{1cm} (4.5)$$

One obtains an equation similar to Equation (4.2). Then we know that the modular 1-form $\theta_\Omega \in \Omega^1(E)$ satisfying $\theta_\Omega(1) = n$ is exact when restricted to the punctured space, and

$$\theta_\Omega\big|_{\mathbb{R}^n \setminus \{0\}} = \theta_{\omega_F} \hspace{1cm} (4.6)$$

However, as seen from Equation (4.3), the right-hand side cannot be extended at the origin. One can show that in fact, there is no smooth function $g$ such that $\theta_\Omega = d_E g$ on any neighborhood of the origin. Indeed, if it were the case, by Equation (2.10) we would have:

$$\theta_\Omega(1) = d_E g(1) = \epsilon(g) = \sum_{i=1}^n x_i \partial_i g \hspace{1cm} (4.7)$$

Since the derivative of $g$ is continuous at the origin it is bounded and then the right-hand side of Equation (4.7) vanishes. However, by Equation (4.5), the left-hand side of Equation (4.7) should be equal to $\epsilon(n)$ everywhere, in particular at the origin. This is a contradiction.

Then, the modular 1-form $\theta_\Omega$ is not an exact 1-form on any neighborhood of the origin, and so we conclude that the modular class of the universal Lie $\infty$-algebroid $E$ is not zero. This implies in turn that the modular class of the singular foliation $\mathcal{F}_\epsilon$ is not zero. This not only means that the Berezinian of $E$ is not a trivial $\mathcal{F}_\epsilon$-module, but that this statement holds for the Berezinian of any universal Lie $\infty$-algebroid of $\mathcal{F}_\epsilon$. This result is clearly related to the behavior of the Euler vector field at the origin. Since in any case the conormal bundle $F^0$ could not be extended at the origin, the non-vanishing modular class of $\mathcal{F}_\epsilon$ does not tell us anything more about the regular foliation $F$ that we already knew from its own modular class.

### 4.2 The characteristic foliation of a degenerate Poisson manifold

Poisson manifolds form a nice class of foliated manifolds. Indeed, for any Poisson manifold $M$ with Poisson bivector $\pi \in \mathfrak{X}^2(M)$, the cotangent bundle $T^*M$ is a Lie algebroid — called the \textit{cotangent Lie algebroid} — with anchor the Poisson bivector $\pi^\sharp : T^*M \rightarrow TM$, and bracket the following:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta)) \hspace{1cm} (4.8)$$

for any two differential 1-forms $\alpha, \beta \in \Omega^1(M)$. Thus, the image of $\Omega^1(M)$ through the anchor map defines a singular foliation $\mathcal{F}_{\pi^\sharp}$, called the \textit{characteristic foliation} of the Poisson manifold. In particular, a property of such a foliation is that its leaves are symplectic manifolds, hence necessarily even dimensional.

For the present example — due to Vladimir Rubstov — let us pick up $M = \mathbb{R}^3$ and the following Poisson bivector:

$$\pi = (x \partial_x + y \partial_y) \wedge \partial_z \hspace{1cm} (4.9)$$

The cotangent bundle $T^*\mathbb{R}^3$ is trivial and admit three canonical generators, the constant sections $dx, dy, dz$. The Lie algebroid structure on $E_0 = T^*\mathbb{R}^3$ is such that the anchor map reads:

$$X_x = \pi^\sharp(dx) = x \partial_z, \quad X_y = \pi^\sharp(dy) = y \partial_z \text{ and } \quad X_z = -x \partial_x - y \partial_y \hspace{1cm} (4.10)$$
while the Lie bracket reads, because of the identity 
\[ [df, dg]_\pi = d\{f, g\} \]:

\[
\begin{align*}
[dx, dy]_\pi &= 0, \quad [dx, dz]_\pi = dx \\
[dy, dz]_\pi &= dy
\end{align*}
\] (4.11)

The characteristic foliation \( F_\pi \) induced by the three vector fields \( X_x, X_y, X_z \) consists of the union of 2-dimensional half-planes escaping radially from the vertical axis \( (x = 0, y = 0, z) \), itself constituted of the singular points: points.

The union of regular leaves forms an open dense subset \( U = \mathbb{R}^3 \setminus \{ z \text{-axis} \} \), on which \( F_\pi \) induces a regular foliation \( F \). The conormal bundle is the rank one subbundle \( F^0 \) of \( T^*\mathbb{R}^3 \) generated by the differential form \( ydx - xdy \), so it does not extend at the singular leaves. Another way of seeing this is to notice that the transversals to the regular leaves consist of concentric circles around the vertical axis. The determinant line bundle \( \wedge^{top} F^0 \), being of rank one, coincides with \( F^0 \), so a transversal volume form to the regular leaves is given by:

\[ \omega_{F^0} = ydx - xdy \] (4.12)

The action of \( X_x, X_y \) and \( X_z \) on this volume form is computed via the Lie derivative as in Equation (3.33). One can check that only \( X_z \) has a non trivial action on \( \omega_{F^0} \) since

\[ \nabla_{X_z}^{\text{Bott}} (\omega_{F^0}) = -2 \omega_{F^0} \] (4.13)

The modular one-form \( \theta_{\omega_{F^0}} \) can be read directly from this equation:

\[ \theta_{\omega_{F^0}}(X_x) = \theta_{\omega_{F^0}}(X_y) = 0 \quad \text{and} \quad \theta_{\omega_{F^0}}(X_z) = -2 \] (4.14)

It turns out to be exact on \( U \), because one can write:

\[ \theta_{\omega_{F^0}} = d_F \left( \ln(x^2 + y^2) \right) \] (4.15)

From this we deduce that the modular class of the regular foliation \( F \) is zero. Indeed, the latter admits an invariant transverse measure to the regular leaves given by:

\[ \omega_{\text{inv}} = \frac{1}{x^2 + y^2} (ydx - xdy) \] (4.16)

Notice that it does not extend to the vertical axis because the function at the denominator diverges.

Now let us turn to the modular class of the singular foliation \( F_\pi \). The kernel of the anchor map \( \pi^\sharp \) is generated by the differential one-form \( ydx - xdy \). A universal Lie \( \infty \)-algebroid of the characteristic foliation \( F_\pi \) can then be built on the following geometric resolution:

\[
0 \rightarrow \Gamma(\mathbb{R}^3 \times \mathbb{R}) \xrightarrow{\delta} \Omega^1(\mathbb{R}^3) \xrightarrow{\pi^\sharp} F_\pi
\]

where the vector bundle morphism \( \delta \) is such that it sends the constant section \( 1 \) of the trivial line bundle \( E_{-1} = \mathbb{R}^3 \times \mathbb{R} \) to the one-form \( ydx - xdy \). Then, we indeed have \( \text{Im}(\delta) = \text{Ker}(\pi^\sharp) \). The 2-bracket \( l_2 \) on \( E_0 \) coincides with the Lie algebroid bracket \( [\ldots]_\pi \). The 2-bracket \( l_2 \) between a section of \( E_0 = T^*\mathbb{R}^3 \) and a section of \( E_{-1} \) should satisfy Equation (2.3), for \( l_1 = \delta \). So, we deduce that:

\[ l_2(dx, 1) = 0, \quad l_2(dy, 1) = 0 \quad \text{and} \quad l_2(dz, 1) = -2 \quad 1 \] (4.17)

The Berezinian of \( E \) is given by:

\[ \text{Ber}(E) = \wedge^3 T^*\mathbb{R}^3 \otimes \wedge^3 T^*\mathbb{R}^3 \otimes E_{-1}^* \] (4.18)
where the second slot corresponds to $\wedge^3 E_0$. The Berezinian admits a global section given by $
abla = dx \wedge dy \wedge dz \otimes dx \wedge dy \wedge dz \otimes 1^*$. The constant section $1^* \in \Gamma(E_{-1}^*)$ is the section dual to $1$.

We will compute the modular class of $E$ by using Formula (3.18), i.e. by computing the action of $E_0$ on each of the three terms of the tensor product in $\text{Ber}(E)$. Having Formulas (4.10), (4.11) and (4.17) at hands, one straightforwardly notices that the only a priori non-trivial action on $\text{Ber}(E)$ is that of $dz$. To compute the explicit action of the element $dz$ on $E_{-1}^*$, we will in fact compute it on $E_{-1}$ and revert the sign, as is explained in Equations (3.15) and (3.16). The action of $dz$ on the first term $\wedge^3 T^* \mathbb{R}^3$ is given by the divergence of $X$, which is:

$$\text{div}(X) = -2 \quad (4.19)$$

The action of $dz$ on $\wedge^3 E_0$ is given via the Lie algebroid bracket (4.11) and gives another $-2$ contribution. The action of $dz$ on the constant section $1$ of $E_{-1}$ gives $-2 1$. Taking the opposite sign, we deduce that the section $dz$ acts on $E_{-1}^*$ as a multiplication by $+2$. Summing all the numbers, we obtain that the modular one-form $\theta_\Omega \in \Omega_1(E)$ satisfies:

$$\theta_\Omega(X_x) = \theta_\Omega(X_y) = 0 \quad \text{and} \quad \theta_\Omega(X_z) = -2 \quad (4.20)$$

Following the same lines of argument as in the second part of Example 4.1, we deduce that the modular class of the singular foliation of $\mathcal{F}_\pi$ is not zero, i.e. that the Berezinian of any universal Lie $\infty$-algebroid is not a trivial $\mathcal{F}_\pi$-module.

### 4.3 Vector fields that quadratically vanish at 0

This example illustrates that a solvable singular foliation of which we do not know if it descends from a Lie algebroid, can nonetheless be attributed a modular class because we know a universal Lie $\infty$-algebroid associated to it. Let $\mathcal{F}$ be the sheaf of vector fields on $\mathbb{R}^2$ that quadratically vanish at the origin. A typical element of $\mathcal{F}$ can be written as $X = (a x^2 + 2 b x y + c y^2) (u \partial_x + v \partial_y)$, where $a, b, c$ are functions that are possibly locally defined, and $u, v$ are smooth functions. The leaves of $\mathcal{F}$ consist of the punctured plane (the regular leaf) and the origin (singular leaf). Since on the punctured plane the conormal bundle is the zero vector bundle, it is extendable at the origin and is denoted $\mathcal{F}^\circ$. We deduce that the determinant line bundle $\wedge^{\text{top}} \mathcal{F}^\circ$ actually involves the zeroth power of $\mathcal{F}^\circ$ and is the trivial vector bundle: $\wedge^0 \mathcal{F}^\circ = \mathbb{R} \times \mathbb{R}$. The sections of this line bundle consists of the algebra of smooth functions on $\mathbb{R}^n$ and any constant function is a global nowhere-vanishing section on which any vector field acts trivially. It means that $\wedge^0 \mathcal{F}^\circ$ is a trivial $\mathcal{F}$-module.

Following Examples 3.35 and 3.100 in [20], there exists a universal Lie $\infty$-algebroid of $\mathcal{F}$ that has length 2:

$$0 \longrightarrow \Gamma(E_{-1}) \overset{d}{\longrightarrow} \Gamma(E_0) \overset{\rho}{\longrightarrow} \mathcal{F}$$

where the fiber of $E_0$ is $S^2(\mathbb{R}^2) \otimes \mathbb{R}^2$ and the fiber of $E_{-1}$ is $\mathbb{R}^2 \otimes \mathbb{R}^2$. At the point $(x, y) \in \mathbb{R}^2$, the anchor map is defined as:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \otimes (u, v) \longmapsto (a x^2 + 2 b x y + c y^2) (u \partial_x + v \partial_y) \quad (4.21)$$

whereas the map $d$ is defined as:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes (u, v) \longmapsto \begin{pmatrix} 2 y \alpha & -x \alpha + y \beta \\ -x \alpha + y \beta & -2 x \beta \end{pmatrix} \otimes (u, v) \quad (4.22)$$
For clarity, in the following, we will denote a generic constant section of $E_0$ as $A \otimes X$ where one has to understand that $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a $2 \times 2$ symmetric matrix, and where $X = (u, v)$ is a vector of $\mathbb{R}^2$. The quadratic function $ax^2 + bxy + cy^2$ will be denoted $A_{(x,y)}$, and the vector field $\partial_x u + v \partial_y$, which corresponds to $X$ is denoted $\vec{X}$. Then, at any point $(x, y) \in \mathbb{R}^2$, the 2-brackets and 3-bracket defining the $L_\infty$-algebra are given (on constant sections) by:

\begin{align*}
  l_2(A \otimes X, B \otimes Y) &= \vec{X}(B_{(x,y)}) \otimes Y - \vec{Y}(A_{(x,y)}) \otimes X \\
  l_2\left( A \otimes X, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes Y \right) &= \begin{pmatrix} \alpha(axu + bxy + cyv) + \beta(ayu - axv) \\ \beta(axu + byu + cyv) + \alpha(cxv - cyu) \end{pmatrix} \otimes Y \\
  l_3(A_1 \otimes X_1, A_2 \otimes X_2, A_3 \otimes X_3) &= \vec{X}_1 \circ \vec{X}_2(A_{3,(x,y)}) \begin{pmatrix} (b_1a_2 - a_1b_2)x + \frac{1}{2}(c_1a_2 - a_1c_2)y \\ (c_1b_2 - b_1c_2)y + \frac{1}{2}(c_1a_2 - a_1c_2)x \end{pmatrix} \otimes X_3 + \circ
\end{align*}

where the circle arrow $\circ$ means that we proceed to a circular permutation. Then the 2-brackets are extended to every section by applying the Leibniz rule (2.5). The brackets have been precisely defined so that they satisfy the three equations: $\rho \circ l_2 = l_2 \circ \rho$, $d \circ l_2 = l_2 \circ d$ and $l_2 \circ l_2 = -d \circ l_3$.

The Berezinian of $E$ is given by:

$$\text{Ber}(E) = \wedge^2 T^* \mathbb{R}^2 \otimes \wedge^6 E_0 \otimes \wedge^4 E_{-1}^*$$

We will compute the modular class of $E$ by using Formula (3.18). The action of an element $A \otimes X$ on the first factor is given by the divergence of the vector field $\rho(A \otimes X)$. On the second factor, it is given by the adjoint action $l_2(A \otimes X, -)$ as given in Equation (4.23), and on the third factor by the dual of action (4.24). One can check that the action of every basis element of $E_0$:

\begin{align*}
  e_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1, 0), & e_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (1, 0), & e_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1, 0) \\
  e_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (0, 1), & e_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (0, 1), & e_6 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (0, 1)
\end{align*}

is zero on $\text{Ber}(E)$. For example, by taking $e_1$, we have $\text{div}(\rho(e_1)) = \text{div}(x^2 \partial_x) = 2x$, $\mathcal{L}_{e_1}|_{\wedge^6 E_0} = -2x \text{id}_{\wedge^6 E_0}$ and $\mathcal{L}_{e_1}|_{\wedge^4 (E_{-1})^*} = 0$. Let us show the two last results in more detail since this may not be obvious to the reader.

Let $\lambda = e_1 \wedge \ldots \wedge e_6$ be the canonical section of $\wedge^6 E_0$, then the action of $e_1$ on $\lambda$ is performed through the adjoint action (4.23), and it defines a factor $a_1$: $\mathcal{L}_{e_1}(\lambda) = a_1 \lambda$. Thus we have to compute the bracket of $e_1$ with every basis elements of $E_0$, by using Equation (4.23), and sum the contributions to obtain $a_1$:

\begin{align*}
  [e_1, e_1] = 0, & \quad [e_1, e_2] = 2y e_1 - 2x e_2, & \quad [e_1, e_3] = -2x e_3, \\
  [e_1, e_4] = 2x e_4, & \quad [e_1, e_5] = 2y e_4, & \quad [e_1, e_6] = 0.
\end{align*}

Not every terms on the right hand side will contribute to $a_1$ since for example the $e_1$ contribution in $[e_1, e_2]$ will be cancelled out because $e_1 \wedge e_1 = 0$ so that $e_1 \wedge [e_1, e_2] \wedge e_3 \wedge \ldots \wedge e_6 = -2x e_1 \wedge \ldots \wedge e_6$. Hence we have: $a_1 = -2x - 2x + 2x = -2x$, so that $a_1$ and $\text{div}(e_1)$ cancel one another.
To conclude the calculation, we have to show that the action of \( e_1 \) on \( \wedge^4 E_{-1} \) is trivial. To do this, we will instead compute the action of \( e_1 \) on the dual \( \wedge^4 E_{-1} \) and show that it is zero. Let \( \mu = f_1 \wedge \ldots \wedge f_4 \) be the canonical section of \( \wedge^4 E_{-1} \), where \( f_1, \ldots, f_4 \) are the basis elements of \( E_{-1} \):

\[
f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (1, 0), \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (1, 0)
\]

\[
f_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0, 1), \quad f_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (0, 1)
\]

Then the action of \( e_1 \) on \( \mu \) is defined by: \( \mathcal{L}_{e_1}(\mu) = a_\mu \mu \). We compute \( a_\mu \) by summing all contributions of the \( \mathcal{L}_{e_1} \) on the \( f_i \). We have, by using Equation (4.24):

\[
\begin{align*}
[e_1, f_1] &= x f_1 - 2 x f_1 = -x f_1, & [e_1, f_2] &= x f_2 - 2 x f_2 = -x f_2, \\
[e_1, f_3] &= x f_3, & [e_1, f_4] &= x f_4.
\end{align*}
\]

Then the action of \( e_1 \) on \( \mu \) consists of the sum of the contributions on each basis vector, which sums up to zero. Hence the result: the modular one form \( \theta_\Omega \), when applied on \( e_1 \), is zero. One can check that this result is true for the other basis vectors \( e_i \), proving the nullity of the modular class of \( E \), and thus that of the singular foliation \( \mathcal{F} \). Then, by Proposition 3.17, the Berezinian \( \text{Ber}(E) \) is isomorphic – as a \( \mathcal{F} \)-module – to the trivial line bundle \( \wedge^0 \mathcal{F}^\circ = \mathbb{R}^n \times \mathbb{R} \). In particular, the standard constant section \( \Omega \) of the former can be sent to the constant section \( 1 \) of the latter.

As a side remark, notice that although \( \wedge^4 E_{-1} \) does not contribute to the modular class, the vector bundle \( E_{-1} \) is nonetheless necessary to make the Jacobi identity on \( E_0 \) close up to homotopy. As a matter of fact, it is not known yet if \( \mathcal{F} \) descends from a Lie algebroid. This illustrates the importance of relying on universal Lie \( \infty \)-algebroids to define modular classes of singular foliations.

### 4.4 The action of \( \mathfrak{gl}_n(\mathbb{R}) \) on \( \mathbb{R}^n \)

We study the singular foliation \( \mathcal{F}_{\mathfrak{gl}_n} \) induced by the action of \( \mathfrak{gl}_n(\mathbb{R}) \) on \( \mathbb{R}^n \), for \( n \geq 2 \). As for Example 4.3, the leaves of \( \mathcal{F}_{\mathfrak{gl}_n} \) consist of the punctured plane (the regular leaf) and the origin (singular leaf). Since on the punctured plane the conormal bundle is the zero vector bundle, it is extendable at the origin and is denoted \( \mathcal{F}^0_{\mathfrak{gl}_n} \). We deduce that the determinant line bundle \( \wedge^{\text{top}} \mathcal{F}^0_{\mathfrak{gl}_n} \) actually involves the zeroth power of \( \mathcal{F}^0_{\mathfrak{gl}_n} \) and is the trivial vector bundle: \( \wedge^0 \mathcal{F}^0_{\mathfrak{gl}_n} = \mathbb{R}^n \times \mathbb{R} \). The sections of this line bundle consists of the algebra of smooth functions on \( \mathbb{R}^n \) and any constant function is a global nowhere-vanishing section on which any vector field acts trivially. It means that \( \wedge^0 \mathcal{F}^0_{\mathfrak{gl}_n} \) is a trivial \( \mathcal{F}^0_{\mathfrak{gl}_n} \)-module. Computing the modular class of \( \mathcal{F}^0_{\mathfrak{gl}_n} \) will show us if the trivial line bundle \( \wedge^0 \mathcal{F}^0_{\mathfrak{gl}_n} \) is isomorphic to the Berezinian of any universal Lie \( \infty \)-algebroid of \( \mathfrak{gl}_n \) (see Proposition 3.17).

Let \( x_1, \ldots, x_n \) be the canonical coordinates on \( \mathbb{R}^n \) and let \( (e_{i,j}^{(0)})_{1 \leq i,j \leq n} \) be the canonical basis of \( \mathfrak{gl}_n(\mathbb{R}) \), i.e. \( e_{i,j}^{(0)} \) is the \( n \times n \)-matrix with zero everywhere and a 1 at line \( i \) and column \( j \). Let \( E_0 = \mathbb{R}^n \times \mathfrak{gl}_n \) be the action Lie algebroid associated to the action of \( \mathfrak{gl}_n(\mathbb{R}) \) on \( \mathbb{R}^n \). The anchor and brackets are defined on constant sections by:

\[
\rho(e_{i,j}^{(0)}) = x_i \partial_{x_j} \quad \text{ (4.35)}
\]

\[
[e_{i,j}^{(0)}, e_{k,l}^{(0)}] = \delta_{jk} e_{i,l}^{(0)} - \delta_{il} e_{k,j}^{(0)} \quad \text{ (4.36)}
\]
for every $1 \leq i, j, k, l \leq n$. The bracket is the usual bracket on $\mathfrak{gl}_n(\mathbb{R})$, that we extend to every sections of $E_0$ by the Leibniz rule (2.5). The anchor map has a non trivial kernel, spanned by elements of the following kind:

$$x_i e^{(0)}_{j,k} - x_j e^{(0)}_{i,k}$$

(4.37)

Notice that the $i,j$-indices on the left hand side have a skew-symmetric property, hence showing that the minimal number of generators of Ker$(\rho)$ is $\frac{d(n-1)}{2}$. Let us set $E_{-1} = \mathbb{R}^n \times \wedge^2 \mathbb{R}^n \oplus \ldots \oplus \wedge^2 \mathbb{R}^n$, where the tensor product is made over $n$ copies of $\wedge^2 \mathbb{R}^n$ and where, for clarity, we omitted the suspension operator that shifts the degree of the fiber by $-1$. Let $(e^{(1)}_{ij,k})_{1 \leq i < j \leq n}$ be the canonical basis of the $k$-th factor in the product $\wedge^2 \mathbb{R}^n \oplus \ldots \oplus \wedge^2 \mathbb{R}^n$. For $j < i$ we set $e^{(1)}_{ij,k} = -e^{(1)}_{ji,k}$ so that we have a set of vectors $(e^{(1)}_{ij,k})_{1 \leq i,j,k \leq n}$ that have this anti-symmetry property on the $i,j$-indices. Then, define a vector bundle map $d^{(0)} : E_{-1} \to E_0$ by the following relations, for every $1 \leq i,j,k \leq n$:

$$d^{(0)}(e^{(1)}_{ij,k}) = x_i e^{(0)}_{j,k} - x_j e^{(0)}_{i,k}$$

(4.38)

The bracket between a section of $E_0$ and a section of $E_{-1}$ can be first defined on constant sections as:

$$[e^{(0)}_{i,j}, e^{(1)}_{kl,m}] = \delta_{jk} e^{(1)}_{il,m} + \delta_{jl} e^{(1)}_{ki,m} - \delta_{im} e^{(1)}_{kl,j}$$

(4.39)

Then it is extended to every sections of $E_{-1}$ by the Leibniz rule (2.5).

At the level of sections, the kernel of the map $d^{(0)} : E_{-1} \to E_0$ is not zero, since elements of the form:

$$x_i e^{(1)}_{jk,l} + x_j e^{(1)}_{ki,l} + x_k e^{(1)}_{ij,l}$$

(4.40)

span this kernel. Let us set $E_{-2} = \mathbb{R}^n \times \wedge^3 \mathbb{R}^n \oplus \ldots \oplus \wedge^3 \mathbb{R}^n$, where the tensor product is made over $n$ copies of $\wedge^3 \mathbb{R}^n$ and where, for clarity, we omitted the suspension operator that shifts the degree of the fiber by $-1$. Let $(e^{(2)}_{ijk,l})_{1 \leq i < j < k \leq n}$ be the canonical basis of the $l$-th factor in the tensor product $\wedge^3 \mathbb{R}^n \oplus \ldots \oplus \wedge^3 \mathbb{R}^n$. Then for any $1 \leq i, j, k, l \leq n$ we set $e^{(2)}_{ijk,l} = (-1)^{\sigma} e^{(2)}_{\sigma(i)\sigma(j)\sigma(k),l}$ where $\sigma$ is the unique permutation such that $\sigma(i) < \sigma(j) < \sigma(k)$. We define a vector bundle map $d^{(1)} : E_{-2} \to E_{-1}$ by:

$$d^{(1)}(e^{(2)}_{ijk,l}) = x_i e^{(1)}_{jk,l} + x_j e^{(1)}_{ki,l} + x_k e^{(1)}_{ij,l}$$

(4.41)

The bracket between a section of $E_0$ and a section of $E_{-2}$ can be first defined on constant sections as:

$$[e^{(0)}_{i,j}, e^{(2)}_{klm,r}] = \delta_{jk} e^{(2)}_{ilm,r} + \delta_{jl} e^{(2)}_{kim,r} + \delta_{lm} e^{(2)}_{kji,r} - \delta_{ir} e^{(2)}_{klm,j}$$

(4.42)

Then it is extended to every sections of $E_{-2}$ by the Leibniz rule (2.5). We do not need to define the bracket between two elements of $E_{-1}$ because is not used to compute the modular class.

The same process repeats itself up to order $n$, since the dimension of $\wedge^n \mathbb{R}^n$ is one. In the end, we will have a geometric resolution:

$$0 \longrightarrow \Gamma(E_{-n+1}) \xrightarrow{d^{(n-2)}} \ldots \xrightarrow{d^{(2)}} \Gamma(E_{-2}) \xrightarrow{d^{(1)}} \Gamma(E_{-1}) \xrightarrow{d^{(0)}} \Gamma(E_0) \xrightarrow{p} \mathcal{F}_{\mathfrak{gl}_n}$$

where, for every $0 \leq p \leq n-1$, $E_{-p} = \mathbb{R}^n \times \wedge^{p+1} \mathbb{R}^n \oplus \ldots \oplus \wedge^{p+1} \mathbb{R}^n$, such that there are $n$ copies of $\wedge^{p+1} \mathbb{R}^n$. The differential is defined as:

$$d^{(p-1)}(e^{(p)}_{i_1 \ldots i_{p+1},k}) = \sum_{\sigma \in \mathfrak{S}_n(1,p)} (-1)^{\sigma} x_{\sigma(i_1)} e^{(p-1)}_{\sigma(2),\ldots,\sigma(i_{p+1}),k}$$

(4.43)
where $Un(1,p)$ designates the $(1,p)$-unshuffles, i.e. the permutations of $p + 1$ elements such that $\sigma(i_2) < \ldots < \sigma(i_{p+1})$. The sign $(-1)^n$ is the signature of the permutation. The bracket between $E_0$ and $E_{-p}$ is given by the following formula:

$$[e_{i,j}^{(0)}, e_{i_1 \ldots i_{p+1}, k}^{(p)}] = \sum_{l=1}^{p+1} \delta_{ij} e_{i_1 \ldots i_{p+1}, k}^{(p)} - \delta_{ik} e_{i_1 \ldots i_{p+1}, j}^{(p)}$$ \hspace{1cm} (4.44)

where the index $i$, in the term under the sum, is replacing the $l$-th index $i_l$. There are other brackets, such as the two bracket between two elements of degree lower than 0, but these are not necessary to compute the modular class.

The Berezinian associated to the geometric resolution is

for $n$ even \hspace{1cm} $\text{Ber}(E) = \wedge^n T^* \mathbb{R}^n \otimes \wedge^n E_0 \otimes \wedge^n (E_{-1})^* \otimes \wedge^n (E_{-2}) \otimes \ldots \otimes \wedge^n (E_{-n+1})^*$

for $n$ odd \hspace{1cm} $\text{Ber}(E) = \wedge^n T^* \mathbb{R}^n \otimes \wedge^n E_0 \otimes \wedge^n (E_{-1})^* \otimes \wedge^n (E_{-2}) \otimes \ldots \otimes \wedge^n (E_{-n+1})$

For $n \geq 1$ even (resp. odd), let $\Omega = \omega \otimes \lambda \otimes \mu^{(1)*} \otimes \mu^{(2)} \otimes \ldots \otimes \mu^{(n-1)*}$ (resp. $\omega \otimes \lambda \otimes \mu^{(1)*} \otimes \mu^{(2)} \otimes \ldots \otimes \mu^{(n-1)}$) be a section of the Berezinian, where $\mu^{(p)}$ is a nowhere vanishing section of $\wedge^n (E_{-p})$, and where $\mu^{(p)*}$ is a nowhere vanishing section of $\wedge^n (E_{-p})^*$. Since the modular class does not depend on the chosen volume form, we could pick up the one defined by the following constant sections:

$$\omega = dx_1 \wedge \ldots \wedge dx_n$$ \hspace{1cm} (4.45)

$$\lambda = \bigwedge_{1 \leq k \leq n} e_{1,k}^{(0)} \wedge \bigwedge_{1 \leq k \leq n} e_{2,k}^{(0)} \wedge \ldots \wedge \bigwedge_{1 \leq k \leq n} e_{n,k}^{(0)}$$ \hspace{1cm} (4.46)

$$\mu^{(p)} = \mu_1^{(p)} \wedge \ldots \wedge \mu_n^{(p)}$$ \hspace{1cm} (4.47)

where, for any $1 \leq k, p \leq n$, we set:

$$\mu_k^{(p)} = \bigwedge_{2 \leq l_2 < \ldots < l_{p+1} \leq n} e_{l_2 \ldots l_{p+1}, k}^{(p+1)} \wedge \bigwedge_{3 \leq l_3 < \ldots < l_{p+1} \leq n} e_{l_3 \ldots l_{p+1}, k}^{(p+1)}$$ \hspace{1cm} (4.48)

We will then compute the modular class of $E$ by using Formula (3.18). The computation of the value of $\theta_\Omega$ is performed in several steps: we pick up a constant section $e_{i,j}^{(0)}$ of $E_0$, and we compute independently the value of the action of $e_{i,j}^{(0)}$ on each term $\omega, \lambda$ and $\mu^{(p)}$, for every $1 \leq p \leq n - 1$. Then we add up all the contributions to obtain the value of $\theta_\Omega(e_{i,j}^{(0)})$.

First, the action of $e_{i,j}^{(0)}$ on $\wedge^n T^* \mathbb{R}^n$ gives the divergence of $\rho(e_{i,j}^{(0)})$:

$$\mathcal{L}_{e_{i,j}^{(0)}}(\omega) = \text{div}(x_i \partial_{x_j}) \omega = \delta_{ij} \omega$$ \hspace{1cm} (4.49)

Second, the action of $e_{i,j}^{(0)}$ on $\wedge^n E_0$ is obtained from the adjoint action $a \rightarrow [e_{i,j}^{(0)}, a]$ defined in Equation (4.36). Since $\lambda$ is a totally alternating tensor, the only part of $[e_{i,j}^{(0)}, e_{k,l}^{(0)}]$ that will contribute to the value of $\theta_\Omega(e_{i,j}^{(0)})$ is the part that is again proportional to $e_{k,l}^{(0)}$. For this reason, if $i \neq j$, one can check that the action of $e_{i,j}^{(0)}$ on $\lambda$ is null. Now if $i = j$, it is still trivial, but for another reason: the only elements that will contribute will be $(e_{i,l}^{(0)})_{1 \leq l \leq n}$ and $(e_{l,i}^{(0)})_{1 \leq l \leq n}$, but for every $l$, $[e_{i,l}^{(0)}, e_{i,l}^{(0)}] = e_{i,l}^{(0)}$ and $[e_{l,i}^{(0)}, e_{l,i}^{(0)}] = -e_{i,l}^{(0)}$, so that the sum of the contributions of $e_{i,l}^{(0)}$ and of $e_{l,i}^{(0)}$ will always cancel. Hence:

$$\mathcal{L}_{e_{i,j}^{(0)}}(\lambda) = 0$$ \hspace{1cm} (4.50)
Now, let us compute the contribution of the action of \( e_{i,j}^{(0)} \) on \( \mu^{(1)*} \) to the value of the modular class. We will proceed as follows: we will compute the action of \( e_{i,j}^{(0)} \) on \( \mu^{(1)} \), and then, by duality, we will revert the sign of the result to obtain the action of \( e_{i,j}^{(0)} \) on \( \mu^{(1)*} \). Applying the same argument as above to the bracket defined in Equation (4.39), we notice that if \( i \neq j \) then the action of \( e_{i,j}^{(0)} \) on \( \mu^{(1)} \) is trivial. In the case that \( i = j \), the only basis elements that will contribute are \( e_{d,m}^{(1)} \), \( e_{k,m}^{(1)} \) and \( e_{kl,i}^{(1)} \), for some values of \( k, l \neq i \) and \( m \neq i \), for if we have simultaneously e.g. \( k = i \) and \( m = i \), from Equation (4.39), we obtain:

\[
[e_{i,j}^{(0)}, e_{d,m}^{(1)}] = e_{d,m}^{(0)} - e_{d,m}^{(0)} = 0
\]

Then one can suppose that, when they appear in the subsequent formulas, both \( k, l \) and \( m \) are different than \( i \). Then we have:

\[
[e_{i,j}^{(0)}, e_{d,m}^{(1)}] = e_{d,m}^{(1)}
\]

\[
[e_{i,j}^{(0)}, e_{k,m}^{(1)}] = e_{k,m}^{(1)}
\]

\[
[e_{i,j}^{(0)}, e_{kl,i}^{(1)}] = -e_{kl,i}^{(1)}
\]

The first equation is valid for \( i < l \leq n \) and \( m \neq i \), and hence contribute with a factor of \( (n - i)(n - 1) \) to the modular class. The second equation is valid for \( 1 \leq k < i \) and \( m \neq i \), thus contributing to a factor \( (i - 1)(n - 1) \) to the modular class. Finally the last equation is valid for those \( k, l \) that are not equal to \( i \). Since we have the strict inequality \( k < l \), there are \( \frac{n(n - 1)}{2} - (i - 1)(n - i) = \frac{n(n - 1)}{2} - (n - 1) \) such terms. Due to the minus sign, these terms would contribute to a factor \( (n - 1) - \frac{n(n - 1)}{2} \) to the modular class. Adding all these contributions together we deduce how \( e_{i,j}^{(0)} \) acts on \( \mu^{(1)} \):

\[
\mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(1)}) = (n - i)(n - 1) + (i - 1)(n - 1) + (n - 1) - \frac{n(n - 1)}{2} = \frac{n(n - 1)}{2}
\]

Reverting the sign, we deduce that \( \mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(1)*}) = -\frac{n}{2} \mu^{(1)*} \).

Now turning to the action of \( e_{i,j}^{(0)} \) on \( \mu^{(2)} \). The above discussion implies that \( \mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(2)}) = 0 \) if \( i \neq j \). Then, when \( i = j \), several terms contribute to the value of the modular class, for Equation (4.42) gives:

\[
[e_{i,j}^{(0)}, e_{d,m,r}^{(2)}] = e_{d,m,r}^{(2)}
\]

\[
[e_{i,j}^{(0)}, e_{k,m,r}^{(2)}] = e_{k,m,r}^{(2)}
\]

\[
[e_{i,j}^{(0)}, e_{kl,i}^{(2)}] = e_{kl,i}^{(2)}
\]

\[
[e_{i,j}^{(0)}, e_{kl,m}^{(2)}] = -e_{kl,m}^{(2)}
\]

when \( k, l, m, r \neq i \). More precisely, terms of the kind \( e_{d,m,r}^{(2)} \) contribute for a value of \( (n - 1)(n - 1) \) times, terms of the kind \( e_{k,m,r}^{(2)} \) contribute for a value of \( (n - 1)(i - 1)(i - 2) \), terms of the kind \( e_{kl,i}^{(2)} \) contribute for a value of \( (n - 1)(n - i) \) times, and those of the kind \( e_{kl,m}^{(2)} \) contribute for a value of \( \left( \frac{n}{2} - \frac{(n - i)(n - 1)(i - 1)}{2} - \frac{(i - 1)(i - 2)}{2} + (i - 1)(n - i) \right) \). A short calculation then shows that adding them up gives \( \mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(2)}) = 2\mu^{(2)} \). More generally, one can check that for \( 1 \leq p \leq n - 1 \), we have:

\[
\mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(p)}) = p \binom{n}{p + 1} \mu^{(p)}
\]
and that $\mathcal{L}_{e_{i,j}^{(0)}}(\mu^{(p)}) = 0$ if $i \neq j$.

From this, we first deduce that if $i \neq j$ then $\Theta_{\Omega}(e_{i,j}^{(0)}) = 0$ since the action of $e_{i,j}^{(0)}$ on $\Omega$ is trivial. When $i = j$ however, we can compute the value of $\Theta_{\Omega}(e_{i,j}^{(0)})$ by adding up all previous contributions. By keeping in mind that there are minus sign coming up when acting on $\mu^{(p)}$, we obtain:

$$\Theta_{\Omega}(e_{i,j}^{(0)}) = 1 + 0 + \sum_{p=1}^{n-1} (-1)^p p \left( \begin{array}{c} n \\ p+1 \end{array} \right)$$

$$= 1 + \sum_{p=1}^{n-1} (-1)^p (p + 1) \left( \begin{array}{c} n \\ p+1 \end{array} \right) - \sum_{p=1}^{n-1} (-1)^p \left( \begin{array}{c} n \\ p+1 \end{array} \right)$$

$$= 1 + \sum_{p=1}^{n-1} (-1)^p \left( \begin{array}{c} n-1 \\ p \end{array} \right) + \sum_{p=2}^{n} (-1)^p \left( \begin{array}{c} n \\ p \end{array} \right)$$

$$= 1 + \sum_{p=1}^{n-1} (-1)^p \left( \begin{array}{c} n-1 \\ p \end{array} \right) + (1 - 1)^n + n - 1$$

$$= n \sum_{p=0}^{n-1} (-1)^p \left( \begin{array}{c} n-1 \\ p \end{array} \right)$$

$$= n(1 - 1)^{n-1}$$

Since $n \geq 2$, we obtain that $\Theta_{\Omega}(e_{i,j}^{(0)}) = 0$ for every $1 \leq i, j \leq n$, so the modular class $\Theta^{\mathfrak{gl}_n}$ of the singular foliation $\mathcal{F}_{\mathfrak{gl}_n}$ is zero.

As in Example 4.3 then, the Berezinian is isomorphic – as a $\mathcal{F}_{\mathfrak{gl}_n}$-module – to the trivial line bundle $\bigwedge^0 \mathcal{F}_{\mathfrak{gl}_n} = \mathbb{R}^n \times \mathbb{R}$. In particular, the standard constant section $\Omega$ of the former can be sent to the constant section 1 of the latter. However this should not be taken as a general rule: at the end of Example 4.5 we will have the same determinant line bundle but the modular class of the singular foliation there would not vanish. This proves that the generators of the foliation carry more informations than the mere leaves.

In the case where $n = 1$, the singular foliation $\mathcal{F}_{\mathfrak{gl}_1}$ on $\mathbb{R}$ is generated by the vector field $x \partial_x$, which is the Euler vector field on $\mathbb{R}$. The following argument shows that its modular class does not vanish: the singular foliation $\mathcal{F}_{\mathfrak{gl}_1}$ admits a universal Lie $E_0 = \infty$-algebroid $\mathbb{R} \times \mathbb{R}$ with anchor $\rho(e_{1,1}^{(0)}) = x \partial_x$, with no kernel. The divergence of this vector field is 1, and the action of $e_{1,1}^{(0)}$ on $\bigwedge^1 E_0$ is trivial, so the modular differential 1-form of this foliation is not zero, and rather satisfies $\Theta_{\Omega}(e_{1,1}^{(0)}) = 1$. This proves nonetheless that Formula (4.61)-(4.66) is still valid for $n = 1$, and can be summarized as follows:

$$\Theta_{\Omega}(e_{i,j}^{(0)}) = n 0^{n-1}$$

for every $1 \leq i, j \leq n$ (4.67)

However, based on the discussion following Equation (4.7), the modular class of $\mathcal{F}_{\mathfrak{gl}_1}$ is not zero.

4.5 The action of $\mathfrak{so}_n(\mathbb{R})$ on $\mathbb{R}^n$

Let us now turn to the action of $\mathfrak{so}_n(\mathbb{R})$ on $\mathbb{R}^n$, for $n \geq 2$. It formalizes Example 4 and will then meet the same obstructions found in Example 7. The singular foliation $\mathcal{F}_{\mathfrak{so}_n}$ consists of all the vector fields that preserve the quadratic function:

$$\varphi(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

(4.68)
The leaves consist of concentric $n-1$-dimensional spheres (regular leaves) and the origin (singular leaf). The function defined in Equation (4.68) is homogeneous with isolated singularity (the origin). Then a geometric resolution of the induced singular foliation $\mathcal{F}_{\mathcal{F}_0}$ is known and has been given by Koszul, see Example 3.36 in [20]:

$$0 \longrightarrow \Gamma(\wedge^n T\mathbb{R}^n) \xrightarrow{\text{id}_{\varphi}} \ldots \xrightarrow{\text{id}_{\varphi}} \Gamma(\wedge^3 T\mathbb{R}^n) \xrightarrow{\text{id}_{\varphi}} \Gamma(\wedge^2 T\mathbb{R}^n) \xrightarrow{\text{id}_{\varphi}} \mathcal{F}_{\mathcal{F}_0} \xrightarrow{\text{id}_{\varphi}} 0$$

We then set $E_{-1} = \wedge^{i+2} T\mathbb{R}^n$. The Lie $\infty$-algebroid structure is given in Example 3.101 of [20]. In particular, the 2-bracket between a constant section $\partial_{\{k,l\}} = \partial_{x_k} \land \partial_{x_l}$ of $E_0$ and a section $\partial_{I} = \partial_{i_1} \land \ldots \land \partial_{i_{|I|}}$ of any other $E_{-|I|+2}$ (where $|I| \geq 2$ is the length of the multi-index $I = \{i_1, \ldots, i_{|I|}\}$) is given by:

$$l_2(\partial_{\{k,l\}}, \partial_{I}) = \sum_{j=1}^{|I|} \delta_{k,i_j}(-1)^j \partial_{x_{i_j}} \land \partial_{I \setminus \{i_j\}} - \delta_{l,i_j}(-1)^j \partial_{x_{i_j}} \land \partial_{I \setminus \{i_j\}}$$

(4.69)

where the multi-vector field $\partial_{I \setminus \{i_j\}}$ is the multi-vector field $\partial_I$ in which the $j$-th term $\partial_{i_j}$ has been removed. The Kronecker deltas come from the fact that, e.g. the term $\frac{\partial_{x_k}}{\partial_{x_k} \partial_{x_{i_j}}}$ is not vanishing only when $k = i_j$, due to the very specific form of the function $\varphi$.

From Equation (4.69), one can check that $l_2(\partial_{\{k,l\}}, \partial_I)$ does not give any contribution proportional to $\partial_I$, since for it to happen one should have, e.g. some $i_j = k$, and at the same time, $l = i_j$ so that $\delta_{k,i_j}(-1)^j \partial_{x_{i_j}} \land \partial_{I \setminus \{i_j\}} = \pm \partial_{I}$. But this would imply that $k = l$ and thus that $\partial_{\{k,l\}} = 0$. Hence, the adjoint action of $E_0$ on $\wedge^{\text{top}} E_{-|I|+2}$ is necessarily trivial. The anchor map acts on $\partial_{\{k,l\}}$ as follows:

$$\rho(\partial_{\{k,l\}}) = \partial_{x_k}(\varphi)\partial_{x_l} - \partial_{x_l}(\varphi)\partial_{x_k} = x_k \partial_{x_l} - x_l \partial_{x_k}$$

(4.70)

Since $k \neq l$, this vector field has a null divergence. As a conclusion, the action of $E_0$ on the Berezinian $\text{Ber}(E)$ is trivial, and hence so is the action of $\mathcal{F}_{\mathcal{F}_0}$ on $\text{Ber}(E)$. Thus, the modular class of $\mathcal{F}_{\mathcal{F}_0}$ is zero.

On the union of regular leaves (the punctured space), the set of vector fields defined in Equation (4.70) induces a regular distribution $F$. The conormal bundle of this distribution is a rank 1 vector bundle $F^0 \subset T^* (\mathbb{R}^n \setminus \{0\})$ generated by the radial one-form $d\rho^2 = \sum_{i=1}^n x_i dx_i$, where we set $\rho^2 = x_1^2 + \ldots + x_n^2$. This is a transverse volume form and it is easy to check that it is invariant under the action of $F$ because it is exact. The conormal vector bundle does not extend at the origin so the vanishing of the modular class of $\mathcal{F}_{\mathcal{F}_0}$ does not bring any new informations on $F$ that was not already known from the study of the regular leaves.

However, adding the Euler vector field $\epsilon = \sum_{i=1}^n x_i \partial_{x_i}$ to the singular foliation $\mathcal{F}_{\mathcal{F}_0}$ induces a new singular foliation $\mathcal{F}$ because the bracket (4.69) with the Euler vector field is zero. The singular leaf does not change, but there is now only one regular leaf: the entire punctured space. Then, the conormal bundle of the regular leaf is a rank zero vector bundle as in Example 4.4, so it extends at the origin as a rank zero vector bundle $F^0$. The determinant line bundle of the latter is then a trivial line bundle $\wedge^0 F^0 = \mathbb{R}^n \times \mathbb{R}$, and a trivial $F$-module. Although the situation is similar as the one in Example 4.4, so that one could naively expect that there is an isomorphism of $F$-modules between this trivial line bundle and the Berezinian of any universal Lie $\infty$-algebroid of $\mathcal{F}$, the following argument proves that it is not the case. Indeed we can chose as a universal Lie $\infty$-algebroid of $\mathcal{F}$ the sum of that of $\mathcal{F}_{\mathcal{F}_0}$ and that of Example 4.1. This latter example implies that the modular class of this new universal Lie $\infty$-algebroid is not zero on any neighborhood of the origin. Then this Berezinian, and any other Berezinian of other universal Lie $\infty$-algebroids of $\mathcal{F}$, are not trivial $F$-modules. Hence, although the foliation $\mathcal{F}$ possesses
exactly the same leaves as that of Example 4.4, and induces the same trivial representation on the trivial line bundle $\wedge^0 F^\circ$, we see that the two foliations have quite a different behavior on their respective Berezinians. This proves again that the some information about a singular foliation is not contained in the leaves, but in the family of vector fields generating it.

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